Global large solutions and incompressible limit for the compressible Navier-Stokes equations

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Abstract

The present paper is dedicated to the global large solutions and incompressible limit for the compressible Navier-Stokes system in $\mathbb{R}^d$ with $d \geq 2$. Motivated by the $L^2$ work of Danchin and Mucha [Adv. Math. 320, 904–925, 2017] in critical Besov spaces, we extend the solution space into an $L^p$ framework. The result implies the existence of global large solutions initially from large highly oscillating velocity fields.

Keywords: Compressible Navier-Stokes equations; incompressible limit; Besov spaces; global well-posedness

Mathematics Subject Classification (2010): 35Q35, 76N10

1. Introduction

In this paper, we study the global well-posedness of the compressible Navier-Stokes equations in the following form:

$$
\begin{aligned}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) - \mu \Delta v - (\mu + \lambda) \nabla \text{div} v + \nabla P(\rho) &= 0, \\
(\rho, v)|_{t=0} &= (\rho_0, v_0),
\end{aligned}
$$

(1.1)

where $\rho$ is density, $v$ is velocity, $\mu$ is shear viscosity coefficient and $\lambda$ is volume viscosity coefficient. Here $\mu$ and $\lambda$ are subject to the standard strong parabolicity assumption:

$$
\mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.
$$

The pressure $P = P(\rho)$ is smooth function such that $P' > 0$ and that $P(\bar{\rho}) = 0$ for some positive constant reference density $\bar{\rho}$.

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As one of the most popular fluid motion model in the field of analysis and applications, the compressible Navier-Stokes equations system has attracted much attention and there is a large literature important to mathematical analysis and fluid mechanics. The local well-posedness for the system (1.1) was proved by Nash [31] for the smooth initial data being away from vacuum. The existence of global smooth solutions was obtained by Matsumura and Nishida [29], when the initial data is close to the equilibrium in $H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. In general, whether a smooth solution blows up in finite time is an open problem.

The global existence of weak solutions was proved by Hoff [22, 23] assuming discontinuous initial data with small energy. The global existence of large weak solution was established by Lions [27] under the isentropic assumption, i.e. $P = A\rho^\gamma$ for $\gamma \geq \frac{9}{5}$. This $\gamma$ restriction domain was enlarged by Feireisl et al. [17] to that of $\gamma > \frac{3}{2}$. Motivated by Hoff [22, 23], Huang et al. [26] obtained the existence of global strong solutions with small energy. However, the question of the regularity and uniqueness of weak solution is generally open even in the case of two dimensional space.

As given by Fujita and Kato [19], the classical incompressible Navier-Stokes equations has the scaling invariance property and gives rise to critical spaces. This observation was introduced to the compressible Navier-Stokes equations by Danchin [6, 7, 8] with respect to the scaling transformation

$$
\begin{align*}
(\rho_0, v_0) &\rightarrow (\rho_0(\ell x), \ell v_0(\ell x)), \\
(\rho(t, x), v(t, x)) &\rightarrow (\rho(\ell^2 t, \ell x), \ell v(\ell^2 t, \ell x)), \\
\ell &> 0
\end{align*}
$$

(1.2)

by neglecting the pressure term $P = P(\rho)$. Here a function space being critical with respect to (1.1) means that the norm of the space is invariant with respect to the scaling transformation (1.2). For example, the product space $B^{d}_{p,1}(\mathbb{R}^d) \times B^{-1+d}_{q,1}(\mathbb{R}^d), 1 \leq p, q \leq \infty$, is critical for the system (1.1).

In the critical space framework, a breakthrough was made by Danchin [6], showing the local well-posedness of (1.1) for the initial data $(\rho_0 - \bar{\rho}, v_0)$ in the critical Besov space $B^{d}_{2,1}(\mathbb{R}^d) \times B^{-1+d}_{2,1}(\mathbb{R}^d)$ and the global existence of strong solutions initially in the vicinity of an equilibrium in the space $(B^{d}_{2,1}(\mathbb{R}^d) \cap B^{-1+d}_{2,1}(\mathbb{R}^d)) \times B^{-1+d}_{2,1}(\mathbb{R}^d)$. Inspired by Danchin [6], Charve and Danchin [3] and Chen et al. [4] obtained the global well-posedness of (1.1)
in the critical $L^p$ framework. The critical Besov space, used by Charve and Danchin [3] and Chen et al. [4], seems the largest one in which the system (1.1) is well-posed. Indeed, Chen et al. [5] proved the ill-posedness of (1.1) in $\dot{B}^{3/p}_{p,1}(\mathbb{R}^d) \times \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^d)$ for $p > 6$. An alternative proof to the results of [3, 4] was further obtained by Haspot [20] by using the viscous effective flux. Moreover, Danchin and He [10] generalized the previous results by allowing the incompressible part of the velocity in the space $\dot{B}^{3/p}_{p,1}(\mathbb{R}^d)$ with $p \in [2, 4]$. Interested readers may also refer to [13, 14, 15, 16, 18, 21, 28, 32] for stability, decay estimate and zero Mach number limit of system (1.1).

1.1. The main result and its motivation

Recently, Danchin and Mucha [12] obtained the global existence of regular solutions to system (1.1) with arbitrary large initial velocity $v_0$, almost constant density $\rho_0$, and large volume viscosity $\lambda$. This result strongly relies on the fact that the limit velocity for $\lambda \to +\infty$ satisfies the incompressible Navier-Stokes equations:

$$
\begin{align*}
V_t + V \cdot \nabla V - \mu \Delta V + \nabla \Pi &= 0, \\
\text{div } V &= 0, \\
V|_{t=0} &= \mathcal{P}v_0,
\end{align*}
$$

(1.3)

with the Leray projection $\mathcal{P} = \mathcal{I} - Q$ with $Q = \nabla \Delta^{-1} \text{div}$.

More precisely, the present study is motivated by the following result in $\mathbb{R}^2$:

**Theorem 1.1.** (Danchin and Mucha [12]) Let $\nu \geq \mu$, $v_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ and $a_0 := \rho_0 - 1 \in \dot{B}^0_{2,1}(\mathbb{R}^2) \cap \dot{B}^1_{2,1}(\mathbb{R}^2)$ such that

$$
C e^{C(\tilde{M} + \tilde{M}^2)} (\|a_0\|_{\dot{B}^0_{2,1}} + \nu \|a_0\|_{\dot{B}^1_{2,1}} + \|Qv_0\|_{\dot{B}^0_{2,1}} + \tilde{M}^2 + \mu^2) \leq \sqrt{\mu \nu}
$$

for a large constant $C$ and

$$
\tilde{M} = C \|\mathcal{P}v_0\|_{\dot{B}^0_{2,1}} \exp\left(\frac{C}{\mu^4 \|\mathcal{P}v_0\|^4_{L^2}}\right).
$$

Then there exists a unique global regular solution $(\rho, v)$ to (1.1) such that

$$
v \in C([0, \infty); \dot{B}^0_{2,1}(\mathbb{R}^2)), \quad v_t, \nabla^2 v \in L^1(0, \infty; \dot{B}^0_{2,1}(\mathbb{R}^2)),
$$

$$
a := \rho - 1 \in C([0, \infty); \dot{B}^0_{2,1}(\mathbb{R}^2) \cap \dot{B}^1_{2,1}(\mathbb{R}^2)) \cap L^2([0, \infty); \dot{B}^1_{2,1}(\mathbb{R}^2)).
$$
In addition, there holds the following estimate

\[ \|Qv\|_{L^\infty(0,\infty;B^0_{2,1})} + \|a\|_{L^\infty(0,\infty;B^0_{2,1})} + \nu \|a\|_{L^\infty(0,\infty;B^0_{2,1})} \leq C e^{C(M+M^2)} \left( \|a_0\|_{B^0_{2,1}} + \nu \|a_0\|_{B^0_{2,1}} + \|Qv_0\|_{B^0_{2,1}} + \tilde{M}^2 + \mu^2 \right). \]

As a byproduct, Danchin and Mucha [12] obtained the convergence \((\rho, v) \to (1, V)\) at the order of \(v^{-\frac{1}{2}}\). This result in the high-dimensional case \(d \geq 3\) was additionally given in [12] under the condition that the incompressible Navier-Stokes equations (1.3) admit a global large regular solution. Moreover, they [12] predicted that those results in \(L^2\) Besov spaces be improved to the critical framework of \(L^p\) Besov spaces. The purpose of the present paper is to give a positive answer to the prediction.

For stating our main result, a homogeneous tempered distribution \(z = \sum_{j \in \mathbb{Z}} \hat{\Lambda}_j z \in \mathcal{S}'(\mathbb{R}^d)\) is truncated by lower and higher oscillation parts in the following sense:

\[
z^\ell := \sum_{2^\nu \leq 1} \hat{\Lambda}_j z \quad \text{and} \quad z^h := \sum_{2^\nu > 1} \hat{\Lambda}_j z \quad \text{for} \quad z \in \mathcal{S}'(\mathbb{R}^d). \tag{1.4}
\]

Sometimes, for convenience, we will use the notation:

\[
\|z\|_{B^\ell_{p,1}} := \|z^\ell\|_{B^\ell_{p,1}} \quad \text{and} \quad \|z\|_{B^h_{p,1}} := \|z^h\|_{B^h_{p,1}}. \tag{1.5}
\]

The main result of the present paper reads:

**Theorem 1.2.** Let \(2 \leq p \leq \min\{4, 2d/(d - 2)\}\) for \(d \geq 2\), and \(2 \leq p < 4\) for \(d = 2\). Assume \(a_0 \in B^{-1+\frac{d}{2}}_{2,1}(\mathbb{R}^d)\), \(a_0^h \in B^{-1+\frac{d}{2}}_{p,1}(\mathbb{R}^d)\), \(Pv_0 \in B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)\), \(Qv_0^\ell \in B^{-1+\frac{d}{2}}_{2,1}(\mathbb{R}^d)\) and \(Qv_0^h \in B^{-1+\frac{d}{2}}_{p,1}(\mathbb{R}^d)\). Suppose that (1.3) admits a unique global solution

\[ V \in C([0, \infty); B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^1(0, \infty; B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)). \]

Denote

\[ M := \|V\|_{L^\infty(\mathbb{R}^d; B^{-1+\frac{d}{p}}_{p,1})} + \mu \|V\|_{L^1(\mathbb{R}^d; B^{-1+\frac{d}{p}}_{p,1})} + \|V_1\|_{L^1(\mathbb{R}^d; B^{-1+\frac{d}{p}}_{p,1})}. \]

Assume \(\nu \geq \mu\) and the existence of a (large) generic constant \(C\) such that

\[
\|a_0\|_{B^{-1+\frac{d}{2}}_{2,1}} + \nu \|a_0\|_{B^{-1+\frac{d}{p}}_{p,1}} + \|Qv_0^\ell\|_{B^{-1+\frac{d}{2}}_{2,1}} + \|Qv_0^h\|_{B^{-1+\frac{d}{2}}_{p,1}} + M^2 + \mu^2 \leq C \sqrt{\mu \nu \exp \left( -C(M+M^2) \right)}. \tag{1.6}
\]
Remark 1.3. It should be noted that the zero index Besov space $\dot{B}^{0}_{2,1}(\mathbb{R}^2)$ in Theorem 1.1 has now been extended by the Besov space $\dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$, which may have the negative index $-1 + \frac{d}{p} < 0$. This implies the global well-posedness of compressible Navier-Stokes equations with highly oscillatory initial velocity field $v_0$, of which a typical example (see [4, Proposition 2.9]) is

$$v_0(x) = \sin\left(\frac{x_1}{\varepsilon}\right)\phi(x), \quad \phi(x) \in \mathcal{S}(\mathbb{R}^d), \quad p > d \text{ and } \varepsilon > 0.$$ 

This function is subject to the estimate:

$$\|v^\ell\|_{\dot{B}^{-1+\frac{d}{2}}_{2,1}} + \|v^h\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \leq Ce^{1-\frac{d}{p}},$$

for $C$ a constant independent of $\varepsilon > 0$. 

Then there exists a unique global regular solution $(\rho, v)$ to (1.1) such that

$$\mathcal{P}v \in C(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{p}}_{p,1}),$$

$$a^\ell \in C(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{2}}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{2}}_{2,1}), \quad a^h \in C(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{-1+\frac{d}{p}}_{p,1})$$

In addition, the following estimate holds true:

$$\|a^\ell\|_{L^\infty(0,\infty; \dot{B}^{-1+\frac{d}{2}}_{2,1})} + \nu \|a^\ell\|_{L^\infty(0,\infty; \dot{B}^{-1+\frac{d}{2}}_{2,1})} + \|a^h\|_{L^\infty(0,\infty; \dot{B}^{-1+\frac{d}{p}}_{p,1})} + \nu \|a^\ell\|_{L^1(0,\infty; \dot{B}^{-1+\frac{d}{2}}_{2,1})}$$

$$+ \nu \|Qv^\ell\|_{L^\infty(0,\infty; \dot{B}^{-1+\frac{d}{2}}_{2,1})} + \|Qv^h\|_{L^\infty(0,\infty; \dot{B}^{-1+\frac{d}{p}}_{p,1})} + \nu \|Qv^\ell\|_{L^1(0,\infty; \dot{B}^{-1+\frac{d}{2}}_{2,1})} + \nu \|Qv^h\|_{L^1(0,\infty; \dot{B}^{-1+\frac{d}{p}}_{p,1})}$$

$$\leq C \exp\left(C(M + M^2)\right) \left(\|a^\ell\|_{B^{-1+\frac{d}{2}}_{2,1}} + \nu \|a^\ell\|_{B^{-1+\frac{d}{2}}_{2,1}} + \nu \|a^h\|_{B^{-1+\frac{d}{p}}_{p,1}}

+ \|Qv^\ell\|_{B^{-1+\frac{d}{2}}_{2,1}} + \|Qv^h\|_{B^{-1+\frac{d}{p}}_{p,1}} + M^2 + \mu^2\right).$$
Remark 1.4. If \( p = d = 2 \), Theorem 1.2 is identical to [12, Theorem 1.1]. Especially, when \( d = 2 \) and \( 2 \leq p < 4 \), according to [25, Proposition 3.1], the quantity \( M \) in Theorem 1.2 can be expressed precisely as
\[
M = C\|Pv_0\|_{\dot{B}^{-1+\frac{d}{p},1}_{p,1}} \left(1 + \|Pv_0\|_{\dot{B}^{-1+\frac{d}{p},1}_{p,1}}\right) \exp \left(\frac{C}{\mu^2} \|Pv_0\|_{\dot{B}^{-1+\frac{d}{p},1}_{p,1}}^2\right).
\]
If \( d \geq 3 \), we can construct some examples of large initial data for (1.3) generating global smooth solutions. One can refer for instance to [2, 30, 34] and citations therein.

Remark 1.5. Recently, Danchin and Mucha [11] derived the large volume viscosity limit to the inhomogeneous incompressible Navier-Stokes equations from (1.1) in the two dimensional torus \( T^2 \). In particular, they can handle large variations of density.

1.2. Decomposition of (1.1) by the Leray projection

Without loss of generality, we fix the shear viscosity \( \mu = 1 \) throughout the paper.

Theorem 1.2 is based on a decomposition of (1.1) by using (1.3). Employ the Leray projection to decompose the velocity solution into the compressible part \( Qv \) and the incompressible part \( P\nu + V \) as
\[
v = Qu + P\nu + V \quad \text{for} \quad u := v - V
\]
with \( V \) the global solution of (1.3). A simple computation implies
\[
Qu = Qv, \quad \text{div } Qu = \text{div } u. \tag{1.7}
\]

For \( \rho = 1 + a \), we rewrite the second equation of (1.1) as
\[
v_t + (1 + a)(v \cdot \nabla v) - \Delta v - (\lambda + 1) \nabla \text{div } v + P' \nabla a = -av_t. \tag{1.8}
\]
Applying \( Q \) to (1.8) and using (1.7) and the assumption \( P'(1) = 1 \), we get compressible part of system (1.1):
\[
\begin{cases}
a_t + \text{div } Qu = -\text{div}(a(u + V)), \\
(Qu)_t - vAQu + \nabla a = -QH_1, \\
a|_{t=0} = a_0, \quad Qu|_{t=0} = Qv_0,
\end{cases} \tag{1.9}
\]
with

\[ H_1 := a \left( V_t + Pu_t + (Qu_t + \nabla a) \right) + (1 + a)(u + V) \cdot \nabla (u + V) + (k(a) - a) \nabla a, \]

and

\[ k(a) = P'(1 + a) - P'(1) = P'(1 + a) - 1. \]

Applying \( P \) to the first equation of (1.3) and (1.8), respectively, and then taking the difference between the two resultant equations, we have

\[ (Pu)_t - \Delta Pu = -PH_2, \quad Pu\big|_{t=0} = 0, \quad \text{ (1.10)} \]

with

\[ H_2 := a \left( V_t + Pu_t + (Qu_t + \nabla a) \right) + (1 + a)Pu \cdot \nabla (V + Qu) + a(u + V) \cdot \nabla Pu \]

\[ + (u + V) \cdot \nabla Pu + (1 + a)(V \cdot \nabla Qu + Qu \cdot \nabla V) + a(Qu \cdot \nabla Qu + V \cdot \nabla V). \]

1.3. Scheme for the proof of Theorem 1.2

The proof of Theorem 1.2 lies on the observation that the flow becomes incompressible if the volume viscosity \( \lambda \) is sufficiently large. This mechanism comes from strong dissipation on the potential part of the velocity when \( \lambda \) is large.

In the proof of the main result, we separate the original system (1.1) into incompressible part and compressible part. As the incompressible part satisfies a freedom heat equations, the estimates are standard. However, much effort has to be made in the examination of the compressible part. By careful analysis of the linear system of the compressible part, we can find the density \( a \) has different smoothing effect in low frequency and high frequency parts. Compared to the \( L^2 \) case obtained by Danchin and Mucha [12], the \( L^p \) setting gives rise to an extra difficulty to be overcome. In fact, we cannot get the smoothing effect of density in the low frequency and the damping effect of density in the high frequency by employing the traditional energy argument, which relies heavily on a cancelation, because the cancelation is only valid in the \( L^2 \) framework. To get rid of this difficulty, we shall derive
the smoothing effect of density in the $L^2$ setting in low frequency and the damping effect of density in the $L^p$ setting in the high frequency, respectively. In the low frequency, one can follow the method used in [12] to derive the desired estimates, whereas in the high frequency, we follow an elementary energy approach in terms of effective velocity developed by Haspot [20], using Hoff’s viscous effective flux in [24].

As the solutions of incompressible part of (1.1) are constructed in the $L^p$ setting, all the estimates involved in incompressible part must work in the $L^p$ framework, even for us to derive the smoothing effect of density in the low frequency. For example, the nonlinear term $v \cdot \nabla v$ is decomposed into the following four parts:

$$v \cdot \nabla v = \mathcal{P}v \cdot \nabla \mathcal{P}v + \mathcal{P}v \cdot \nabla \mathcal{Q}v + \mathcal{Q}v \cdot \nabla \mathcal{P}v + \mathcal{Q}v \cdot \nabla \mathcal{Q}v. \quad (1.11)$$

As the incompressible part of (1.1) lies in an $L^p$-type space yet $v \cdot \nabla v$ is estimated in the $L^2$ setting. Hence, we need some product laws and commutators estimates (see Lemma 2.10, (3.54) and (3.58) ) in the Besov spaces to deal with the terms involved in $\mathcal{P}v$ in (1.11). A similar difficulty arises for the compressible part in the high frequency, for example the term $\mathcal{Q}v^h \cdot \nabla \mathcal{Q}v^h$.

The global solution to be obtained in Theorem 1.2 will be extended from the local solution given in the following:

**Theorem 1.6.** ([9, Theorem 2.2]) Let $2 \leq d$, $1 < p < 2d$ and $\nu_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$. Assume $a_0 = \rho_0 - 1 \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $\inf_x \rho_0(x) > 0$. Then there exists a maximal time $T^* > 0$ so that (1.1) has a unique solution $(a, v)$ on $[0, T^*)$ satisfying, for any $T \in (0, T^*)$,

$$a \in C(\lbrack 0, T]; \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^\infty_T (\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)),$$

$$v \in C(\lbrack 0, T]; \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^\infty_T (\dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^1_T (\dot{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)).$$

Moreover, continuation beyond $T^*$ is possible if

$$\int_0^{T^*} \|\nabla v\|_{L^\infty} \, dt < \infty, \quad \|a\|_{L^\infty(0,T^*; \dot{B}^{\frac{d}{p}}_{p,1})} < \infty \quad \text{and} \quad \inf_{(t,x) \in [0,T^*) \times \mathbb{R}^d} \rho(t,x) > 0.$$

Thus the existence of the global solution in Theorem 1.2 becomes to show the maximal time $T^*$ to be $\infty$. This requires global-in-time a priori estimates. Based on the decomposition
in the previous subsection, the estimation will be divided into three subsections in Section 3 through the estimate of the incompressible part in the critical $L^p$ framework, the estimate of the high frequency of the compressible part of (1.9) and the estimate of the low frequency of the compressible part of (1.9).

2. Littlewood-Paley theory and preliminaries

In this section, we recall some basic facts on Littlewood-Paley theory (see [1] for instance). Let $\chi$ and $\varphi$ be two smooth radial functions valued in the interval $[0, 1]$ so that the support of $\chi$ is the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\}$, the support of $\varphi$ is the annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \neq 0.$$ 

Let $\mathcal{F}$ be the Fourier transform. The homogeneous dyadic blocks $\Delta_j$ and the homogeneous low-frequency cutoff operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), \quad \dot{S}_j u = \mathcal{F}^{-1}(\chi(2^{-j} \cdot) \mathcal{F} u).$$

Denote by $\mathcal{S}'_h(\mathbb{R}^d)$ the space of tempered distributions subject to the condition

$$\lim_{j \to -\infty} \dot{S}_j u = 0.$$

Then we have the decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \quad \forall u \in \mathcal{S}'_h(\mathbb{R}^d).$$

**Definition 2.1.** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ consists of all the distributions $u \in \mathcal{S}'_h(\mathbb{R}^d)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| (2^{js} \|\dot{\Delta}_j u\|_{L^p})_{j \in \mathbb{Z}} \right\|_{\ell^r} < \infty.$$ 

This definition implies the following properties (see [1, p71]).

**Lemma 2.2.** Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $u \in \mathcal{S}'_h(\mathbb{R}^d)$. Then $u$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^d)$ if and only if there exists a sequence $\{c_{j,r}\}_{j \in \mathbb{Z}}$ with $c_{j,r} \geq 0$ and $\|c_{j,r}\|_{\ell^r} = 1$ such that

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}},$$

for a constant $C > 0$. If $r = 1$, we set $d_j = c_{j,1}$. 

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Lemma 2.3. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then there hold the estimates

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \|\nabla u\|_{\dot{B}^{s-1}_{p,r}}, \quad \|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p,r}}.$$

Here and in what follows, $a \lesssim b$ means the inequality $a \leq Cb$ for a generic constant $C$.

Moreover, if $1 \leq p_1 < p_2 \leq \infty$ and $1 \leq r_1 < r_2 \leq \infty$, then we have

$$\dot{B}^{s}_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-d/p_2}_{p_2,r_2}(\mathbb{R}^d).$$

Definition 2.4. Let $s \in \mathbb{R}$ and $0 < T \leq \infty$. The norm of the Chemin-Lerner type Besov space is defined as

$$\|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})} := \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{\Delta}_j u\|_{L^q_T(0,T;\mathbb{L}^p(\mathbb{R}^d))}$$

for $1 \leq p \leq \infty$ and $1 \leq q < \infty$.

This definition implies the inequality

$$\|u\|_{L^q_T(\dot{B}^s_{p,1})} \leq \|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})}, \quad \text{for } q, p \geq 1,$$

and the following interpolation property.

Lemma 2.5. (see [1]) For $0 < s_1 < s_2$, $0 \leq \theta \leq 1$ and $1 \leq p_1, q_1, q_2 \leq \infty$, then we have

$$\|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})} \leq \|u\|_{\tilde{L}^{\theta q_1}_T(\dot{B}^{s_1}_{p_1,1})} \|u\|^{1-\theta}_{\tilde{L}^{q_2}_T(\dot{B}^{s_2}_{p_2,1})} \quad \text{with } \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta)s_2.$$

Lemma 2.6. (Bernstein inequalities [1]) Let $B$ be a ball and $C$ an annulus of $\mathbb{R}^d$ centered at the origin. For an integer $0 \leq k \leq 2$ and reals $1 \leq p \leq q \leq \infty$, there hold

$$\sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \lesssim \sigma^{k+d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p}, \quad \text{if } \text{Supp } F u \subset \sigma B,$$

$$\sigma^k \|u\|_{L^p} \lesssim \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^p} \lesssim \sigma^k \|u\|_{L^p}, \quad \text{if } \text{Supp } F u \subset \sigma C$$

with respect to scaling parameter $\sigma > 0$.

The Bony decomposition is very effective in the estimate of nonlinear terms in fluid motion equations. Here, we recall the decomposition in the homogeneous context:

$$uv = \dot{T}uv + \dot{T}_u u + \dot{R}(u,v),$$
where
\[ T_u v := \sum_{j \in \mathbb{Z}} \tilde{S}_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) := \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v \quad \text{with} \quad \tilde{\Delta}_j v := \sum_{|j-j'| \leq 1} \Delta_{j'} v. \]

The estimates of nonlinear terms of the compressible and incompressible equations are essentially based on the following lemmas.

**Lemma 2.7. (see [1])** Let \( s, s_1, s_2 \in \mathbb{R}, \quad 1 \leq p_1, p_2, r_1, r_2 \leq \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \) and \( \tau < 0 \). Then we have
\[
\| T_u v \|_{B^s_{p,r}} \lesssim \| u \|_{L^{p_1}} \| v \|_{B^{s_2}_{p_2,r_2}}, \quad \| T_u v \|_{B_{p,r}^{s+t}} \lesssim \| u \|_{B^{s_1}_{p_1,\infty}} \| v \|_{B^s_{p_2,r_2}},
\]
\[
\| R(u, v) \|_{B^{s_1}_{p_1,r_1}} \lesssim \| u \|_{B^{s_1}_{p_1,r_1}} \| v \|_{B^{s_2}_{p_2,r_2}} \quad \text{for} \quad s_1 + s_2 > 0,
\]
\[
\| R(u, v) \|_{B^{s_1}_{p_1,r_1}} \lesssim \| u \|_{B^{s_1}_{p_1,r_1}} \| v \|_{B^{s_2}_{p_2,r_2}} \quad \text{for} \quad s_1 + s_2 = 0.
\]

**Lemma 2.8. (see [13, Proposition A.1])** Let \( d \geq 2, \quad 1 \leq p, q \leq \infty, \quad s_1 \leq \frac{d}{q}, \quad s_2 \leq d \min\{\frac{1}{p'}, \frac{1}{q}\} \) and \( s_1 + s_2 > d \max\{0, \frac{1}{p} + \frac{1}{q} - 1\} \). Then we have, for \( (u, v) \in B^{s_1}_{q,1}(\mathbb{R}^d) \times B^{s_2}_{p,1}(\mathbb{R}^d) \),
\[
\| uv \|_{B^{s_1+s_2-\frac{d}{q}}_{p,1}} \lesssim \| u \|_{B^{s_1}_{q,1}} \| v \|_{B^{s_2}_{p,1}}.
\]

**Lemma 2.9. (see [1, Lemma 2.100])** Let \( d \geq 2, \quad 1 \leq p, q \leq \infty, \quad s \leq 1 + d \min\{\frac{1}{p'}, \frac{1}{q}\}, \quad v \in B^{s}_{q,1}(\mathbb{R}^d) \) and \( u \in B_{p,1}^{d+1}(\mathbb{R}^d) \). Assume that
\[ s > -d \min\left\{ \frac{1}{p}, 1 - \frac{1}{q} \right\}, \quad \text{or} \quad s > -1 - d \min\left\{ \frac{1}{p'}, 1 - \frac{1}{q} \right\} \quad \text{if} \quad \text{div} \ u = 0.
\]
Then there holds the commutator estimate
\[
\| [u \cdot \nabla, \Delta_j] v \|_{L^q} \lesssim d j 2^{-js} \| u \|_{B^{d+1}_{p,1}} \| v \|_{B^s_{q,1}},
\]
where and in what follows, we use the commutator symbol \([A, B] = AB - BA\) of operators \( A \) and \( B \).

The following estimates are implied from [35, Lemma 2.16].

**Lemma 2.10.** Let \( 2 \leq p \leq \min\{4, 2d/(d-2)\} \) for \( d > 2 \) and \( 2 \leq p < 4 \) for \( d = 2 \). Assume \( A(D) \) a zero-order Fourier multiplier. For \( v^l \in B_{2,1}^{-\frac{d+2}{2}}(\mathbb{R}^d), \quad v^h \in B_{p,1}^{1-d/p}(\mathbb{R}^d) \) and
\( \nabla u \in B^\frac{d}{p^1_1}(\mathbb{R}^d) \), we have

\[
\sum_{j \leq j_0} 2^{(-1 + \frac{d}{p^1})j} \| \hat{A}_j([A(D), u \cdot \nabla]v) \|_{L^2} \leq C(\| \nabla u \|_{B^\frac{d}{p^1}_{p^1,1}} + \| \nabla u^h \|_{B^\frac{d}{p^1}_{p^1,1}})(\| \nabla v \|_{B^{-1+\frac{d}{p^1}}_{p^1,1}} + \| v^h \|_{B^{-1+\frac{d}{p^1},1}_{p^1,1}}),
\]

\[
\sum_{j \leq j_0} 2^{(-1 + \frac{d}{p^1})j} \| \hat{A}_j([A(D), u \cdot \nabla]v) \|_{L^2} \leq C(\| \nabla u \|_{B^\frac{d}{p^1}_{p^1,1}})(\| \nabla v \|_{B^{-1+\frac{d}{p^1}}_{p^1,1}} + \| v^h \|_{B^{-1+\frac{d}{p^1},1}_{p^1,1}}), \quad \text{if } \text{div } u = 0,
\]

for a constant dependent on \( j_0 \).

The following lemma will be used to get appropriate estimates of the solutions.

**Lemma 2.11.** (see [10, Lemma 6.1]) Let \( A(D) \) be a zero-order Fourier multiplier. Let \( j_0 \in \mathbb{Z}, \tau \in \mathbb{R}, 1 \leq p_1, p_2 \leq \infty \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then we have

\[
\| [\hat{A}_j A(D), T_f]g \|_{B^\tau_{p^1,1}} \leq C \| \nabla f \|_{B^{-1+\tau}_{p^1,1}} \| g \|_{B^{1+\tau}_{p^1,1}}, \quad s < 1,
\]

\[
\| [\hat{A}_j A(D), T_f]g \|_{B^{1+\tau}_{p^1,1}} \leq C \| \nabla f \|_{L^p_I} \| g \|_{B^{1+\tau}_{p^1,1}}, \quad s=1,
\]

for a constant \( C \) dependent on \( j_0 \).

Finally, we recall a composition result and a heat flow optimal regularity estimate.

**Proposition 2.12.** (see [1]) Let \( G \) with \( G(0) = 0 \) be a smooth function defined on an open interval \( I \) of \( \mathbb{R} \) containing 0. Then the following estimates

\[
\| G(f) \|_{B^{s}_{p^1,1}} \lesssim \| f \|_{B^{s}_{p^1,1}} \quad \text{and} \quad \| G(f) \|_{L^q_I(B^{s}_{p^1,1})} \lesssim \| f \|_{L^q_I(B^{s}_{p^1,1})}
\]

hold true for \( s > 0, 1 \leq p, q \leq \infty \) and \( f \) valued in a bounded interval \( J \subset I \).

**Proposition 2.13.** (see [1]) Let \( \tau \in \mathbb{R}, \mu > 0, T > 0, 1 \leq p \leq \infty \) and \( 1 \leq q_2 \leq q_1 \leq \infty \). Let \( u \) satisfy the heat equation

\[
\begin{align*}
\partial_t u - \mu \Delta u &= f, \\
|_{t=0} u &= u_0.
\end{align*}
\]

Then the following a priori estimate

\[
\mu^{\frac{1}{\tau+\frac{d}{2}}} \| u \|_{L^\tau_I(B^{\frac{d}{p^1}+\tau+\frac{d}{2}}_{p^1,1})} \lesssim \| u_0 \|_{B^{\frac{d}{p^1}+\tau+\frac{d}{2}}_{p^1,1}} + \mu^{\frac{1}{\tau+\frac{d}{2}}} \| f \|_{L^\tau_I(B^{\frac{d}{p^1}+\tau+\frac{d}{2}}_{p^1,1})}
\]

holds true.
3. Proof of Theorem 1.2

The proof is to be completed through three subsections with respect to the incompressible part, the compressible part and their combination.

Let us begin with the notation:

\[
\mathcal{X} := \| (a^\ell, v\nabla a^\ell, Qu^\ell) \|_{L^\infty(0,T;\dot{B}^{-1-\frac{d}{p}}_{p,1})} + \| va^h \|_{L^\infty(0,T;\dot{B}^{-1-\frac{d}{p}}_{p,1})} + \| Qu^h \|_{L^\infty(0,T;\dot{B}^{-1-\frac{d}{p}}_{p,1})},
\]

\[
\mathcal{Y} := \| (va^\ell, v^2\nabla a^\ell, vQu^\ell) \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} + \| a^h \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} + \| vQu^h \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})},
\]

\[
\mathcal{Z} := \| P u \|_{L^\infty(0,T;\dot{B}^{-1-\frac{d}{p}}_{p,1})},
\]

\[
\mathcal{V} := \| P u_t \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} + \| P u \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})},
\]

\[
\mathcal{W} := \| V \|_{L^\infty(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} + \| V_t \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} + \| V \|_{L^1(0,T;\dot{B}^{-1+\frac{d}{p}}_{p,1})} \leq M.
\]

As assumed in Theorem 1.2, the system (1.3) has a unique global solution for \( P v_0 \in \dot{B}^{1+\frac{d}{p}}_{p,1}(\mathbb{R}^d) \). Thus the bound \( M \) is known.

We claim that if \( v \) is large enough then one may find some (large) \( \eta \) and (small) \( \delta \) so that for all \( T < T^* \), the following bounds are valid:

\[
\mathcal{X} + \mathcal{Y} \leq \eta \quad \text{and} \quad \mathcal{Z} + \mathcal{W} \leq \delta. \tag{3.1}
\]

3.1. Estimates for the incompressible part of (1.1)

Applying \( \dot{\Delta}_j \) to both side of (1.10) gives

\[
(\dot{\Delta}_j P u)_t - \Delta \dot{\Delta}_j P u = -\dot{\Delta}_j PH_2.
\]

Taking \( L^2 \) inner product with \( |\dot{\Delta}_j P u|^{p-2} \dot{\Delta}_j P u \) to the above equation, we have

\[
\frac{1}{p} \frac{d}{dt} \| \dot{\Delta}_j P u \|_{L^p}^p + C_1 2^{2j} \| \dot{\Delta}_j P u \|_{L^p}^p \lesssim \| \dot{\Delta}_j PH_2 \|_{L^p} \| \dot{\Delta}_j P u \|_{L^p}^{p-1}, \tag{3.2}
\]

in which we have used the following fact [7, Appendix]:

\[
- \int_{\mathbb{R}^d} \dot{\Delta}_j P u \cdot |\dot{\Delta}_j P u|^{p-2} \dot{\Delta}_j P u dx \geq C_1 2^{2j} \| \dot{\Delta}_j P u \|_{L^p}^p.
\]
for some positive constant $C_1 > 0$. Integrating from 0 to $T$ and using the Hölder inequality, we get from (3.2) that

$$\|\mathcal{P}u\|_{L^\infty(\mathcal{B}_{p,1}^{-1+\frac{d}{p}})} + \|\mathcal{P}u\|_{L^1(\mathcal{B}_{p,1}^{1+\frac{d}{p}})} \lesssim \int_0^T \|\mathcal{P}H_2\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt. \quad (3.3)$$

In the following, we will deal with each terms in $\mathcal{P}H_2$.

Firstly, by Lemma 2.8, we have

$$\int_0^T \|\mathcal{P}H_2^{(1)}\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt \lesssim \int_0^T \|a(V_t + \mathcal{P}u_t + Qu_t + \nabla a)\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt$$

$$\lesssim v^{-1}\|va\|_{L^\infty(\mathcal{B}_{p,1}^{d})} \|((Qu_t + \nabla a, \mathcal{P}u_t, V_t)|_{L^1(\mathcal{B}_{p,1}^{-1+\frac{d}{p}})}$$

$$\lesssim v^{-1} \left(\|va^h\|_{L^\infty(\mathcal{B}_{p,1}^{d})} + \|va^f\|_{L^\infty(\mathcal{B}_{p,1}^{d})} \right) \left(\|((Qu_t + \nabla a)^h, \mathcal{P}u_t, V_t)|_{L^1(\mathcal{B}_{p,1}^{-1+\frac{d}{p}})} \right)$$

$$\lesssim v^{-1} \mathcal{X}(\mathcal{Y} + \mathcal{W} + \mathcal{V}). \quad (3.4)$$

Similarly, we have

$$\int_0^T \|\mathcal{P}H_2^{(2)}\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt \lesssim (1 + \|a\|_{L^\infty(\mathcal{B}_{p,1}^{d})}) \int_0^T \left(\|\nabla V\|_{\mathcal{B}_{p,1}^{d}} + \|\nabla Qu\|_{\mathcal{B}_{p,1}^{d}} \right) \|\mathcal{P}u\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt, \quad (3.5)$$

and

$$\int_0^T \|\mathcal{P}H_2^{(3)}\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt \lesssim v^{-1} \mathcal{X}(\mathcal{Z} + \mathcal{X} + \mathcal{V})\mathcal{W}. \quad (3.6)$$

By the interpolation inequality in Lemma 2.8, we get

$$\int_0^T \|\mathcal{P}H_2^{(4)}\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt \lesssim \int_0^T \|(u + V) \cdot \nabla \mathcal{P}u\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt$$

$$\lesssim \int_0^T \left(\|u\|_{\mathcal{B}_{p,1}^{d}} + \|V\|_{\mathcal{B}_{p,1}^{d}} \right) \|\nabla \mathcal{P}u\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt$$

$$\lesssim \int_0^T \left(\|u\|_{\mathcal{B}_{p,1}^{d}} + \|V\|_{\mathcal{B}_{p,1}^{d}} \right) \|\mathcal{P}u\|_{\mathcal{B}_{p,1}^{d}} \|\mathcal{P}u\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt$$

$$\lesssim \varepsilon \|\mathcal{P}u\|_{L^1(\mathcal{B}_{p,1}^{1+\frac{d}{p}})} + \int_0^T \left(\|u\|^2_{\mathcal{B}_{p,1}^{d}} + \|V\|^2_{\mathcal{B}_{p,1}^{d}} \right) \|\mathcal{P}u\|_{\mathcal{B}_{p,1}^{-1+\frac{d}{p}}} dt, \quad (3.7)$$
\[
\int_0^T \| \mathcal{P}H_2^{(5)} \|_{B_{p,1}^{-1+\frac{d}{p}}} dt 
\]
\[
\lesssim \int_0^T (1 + \| a \|_{B_{p,1}^{-1+\frac{d}{p}}}^\frac{d}{p}) \| Qu \|_{B_{p,1}^{-1+\frac{d}{p}}} \| Qu \|_{B_{p,1}^{1+\frac{d}{p}}} \| V \|_{B_{p,1}^{1+\frac{d}{p}}} \| V \|_{B_{p,1}^{1+\frac{d}{p}}} dt 
\]
\[
\lesssim (1 + \| a \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})}^\frac{d}{p}) \| Qu \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})} \| Qu \|_{L_t^\infty(B_{p,1}^{1+\frac{d}{p}})} \| V \|_{L_t^\infty(B_{p,1}^{1+\frac{d}{p}})} \| V \|_{L_t^\infty(B_{p,1}^{1+\frac{d}{p}})} 
\]
\[
\lesssim (1 + v^{-1}X) v^{-\frac{1}{2}} X^\frac{1}{2} \mathcal{V}^2. \tag{3.8} \]

Assuming from now on that 
\[
v^{-1} \eta \ll 1. \tag{3.10} \]

After a simple computation, it follows from definition of \(X\) and (3.1) that
\[
\| a \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})} \lesssim v^{-1} \| Va^\xi \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})} + v^{-1} \| Va^k \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})} \lesssim v^{-1} \eta \ll 1. \tag{3.11} \]

Inserting the estimates (3.4)–(3.9) into (3.3) and choosing \(\varepsilon\) small enough, we have
\[
\| \mathcal{P}u \|_{L_t^\infty(B_{p,1}^{-1+\frac{d}{p}})} + \| \mathcal{P}u \|_{L_t^1(B_{p,1}^{1+\frac{d}{p}})} 
\]
\[
\lesssim \int_0^T \left( \| (\nabla V, \nabla Qu) \|_{B_{p,1}^\frac{d}{p}} + \| u \|_{B_{p,1}^\frac{d}{p}}^2 + \| V \|_{B_{p,1}^\frac{d}{p}}^2 \right) \| \mathcal{P}u \|_{B_{p,1}^{-1+\frac{d}{p}}} dt + v^{-\frac{1}{2}} X^\frac{1}{2} \mathcal{V}^\frac{1}{2}
\]
\[
+ v^{-1} X (X + \mathcal{V}) \mathcal{W} + v^{-1} (\mathcal{V} + \mathcal{W} + \mathcal{V}) X + v^{-1} X (X^2 + v^{-1} X \mathcal{V}). \tag{3.12} \]

By (1.10), we have
\[
\| \mathcal{P}u \|_{L_t^1(B_{p,1}^{1+\frac{d}{p}})} \lesssim \| \mathcal{P}u \|_{L_t^1(B_{p,1}^{1+\frac{d}{p}})} + \int_0^T \| \mathcal{P}H_2 \|_{B_{p,1}^{-1+\frac{d}{p}}} dt. \tag{3.13} \]

The combination of (3.12) and (3.13) with the Gronwall inequality produces that
\[
\mathcal{Z} + \mathcal{W} \leq \exp \left( C \int_0^T \left( \| (\nabla V, \nabla Qu) \|_{B_{p,1}^\frac{d}{p}} + \| u \|_{B_{p,1}^\frac{d}{p}}^2 + \| V \|_{B_{p,1}^\frac{d}{p}}^2 \right) dt \right) \left\{ v^{-\frac{1}{2}} X^\frac{1}{2} \mathcal{V}^\frac{1}{2}
\]
\[
+ v^{-1} X (X + \mathcal{V}) \mathcal{W} + v^{-1} (\mathcal{V} + \mathcal{W} + \mathcal{V}) X + v^{-1} X (X^2 + v^{-1} X \mathcal{V}) \right\}. \tag{3.14} \]
3.2. High frequencies for the compressible part of (1.1)

To estimate the high frequencies of \((a, Qu)\), we follow the approach of [20] and introduce the following “effective” velocity field

\[ w = Qu + v^{-1}(-\Delta)^{-1}\nabla a. \]

Multiplying by \(v^{-1}(-\Delta)^{-1}\nabla\) on the first equation in (1.9) and then adding the resultant equation to the second one in (1.9), we deduce that

\[
\begin{align*}
\omega_t - v\Delta w &= v^{-1}Qu + v^{-1}Q(a(u + V)) - QH_1 \\
&= v^{-1}w - v^{-2}(-\Delta)^{-1}\nabla a + v^{-1}Q(a(u + V)) - QH_1. \\
\end{align*}
\]

(3.15)

Applying the operator \(\dot{\Lambda}_j\) on (3.15) and multiplying by \(|\dot{\Lambda}_j w|^{p-2}\dot{\Lambda}_j w\) to the resultant equation, we get that

\[
\frac{1}{p} \frac{d}{dt} \|\dot{\Lambda}_j w\|_{L^p}^p + Cv^{2j} \|\dot{\Lambda}_j w\|_{L^p}^p \lesssim v^{-1} \|\dot{\Lambda}_j w\|_{L^p}^p + \|v^{-2}(-\Delta)^{-1}\nabla a\|_{L^p}^p \|\dot{\Lambda}_j w\|_{L^p}^{p-1} \\
&\quad + \|(v^{-1}Q(a(u + V)) - QH_1)\|_{L^p} \|\dot{\Lambda}_j w\|_{L^p}^{p-1}. \\
\]

(3.16)

Hence multiplying (3.16) by \(2^{-1}\frac{d}{dt} \|\dot{\Lambda}_j w\|_{L^p}^{p-1}\), then integrating with respect to \(t\) and summing up the resultant equations for the high frequencies \(\dot{\Lambda}_j w\) only, we get

\[
\begin{align*}
\|w\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + v\|w\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} &\lesssim \|w_0\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + \|v^{-1}w\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + \|v^{-2}(-\Delta)^{-1}\nabla a\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} \\
&\quad + \|(v^{-1}Q(a(u + V)))\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + \|QH_1\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} \\
&\lesssim \|w_0\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + v^{-1}\|w\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + v^{-2}2^{-2j_0}\|a^h\|_{L^d_t(B^{-\frac{d}{p}}_{p,1})} \\
&\quad + \|v^{-1}a(u + V)\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})} + \|QH_1\|_{L^h_t(B^{-1+\frac{d}{p}}_{p,1})}. \\
\end{align*}
\]

(3.17)

By Lemma 2.8 and Young’s inequality, we have

\[
\begin{align*}
\|v^{-1}a(u + V)\|_{L^1_t(B^{-1+\frac{d}{p}}_{p,1})} &\lesssim v^{-1}\|a\|_{L^2_t(B^{-\frac{d}{p}}_{p,1})} \|u + V\|_{L^2_t(B^{-\frac{d}{p}}_{p,1})} \\
&\lesssim \|a\|^2_{L^2_t(B^{-\frac{d}{p}}_{p,1})} + v^{-2}\|(u, V)\|^2_{L^2_t(B^{-\frac{d}{p}}_{p,1})} \\
&\lesssim \int_0^T \left( \|v^{-1}a^h\|_{B^{-\frac{d}{p}}_{p,1}} + v^{-1}\|va^f\|_{B^{-1+\frac{d}{p}}_{2,1}} \right) \left( \|va^h\|_{B^{\frac{d}{p}}_{p,1}} + \|a^f\|_{B^{-1+\frac{d}{p}}_{2,1}} \right) dt \\
&\quad + v^{-2}ZW + v^{-3}XY + v^{-2}V^2, \\
\end{align*}
\]

(3.18)
where we have used the following estimate:

$$
\| (u, V) \|^2_{L^2_T(B_{p,1}^{d, \frac{d}{2}})} \\
\lesssim \int_0^T \left( \| Pu \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| Qu^h \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| Qu^\ell \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| V \|^2_{B_{p,1}^{-1, \frac{d}{2}}} \right) dt \\
\lesssim \int_0^T \left( \| Pu \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| Qu^h \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| Qu^\ell \|^2_{B_{p,1}^{-1, \frac{d}{2}}} + \| V \|^2_{B_{p,1}^{-1, \frac{d}{2}}} \right) dt \\
\lesssim \left\| \mathcal{P} u \right\|_{L^\infty_T(B_{p,1}^{-1, \frac{d}{2}})} + \| Qu^\ell \|_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} + \| v Qu^h \|_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} + \| V \|_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} \\
\lesssim Z \mathcal{W} + v^{-1} \mathcal{W} + \mathcal{V}^2. \tag{3.19}
$$

Thanks to Lemma 2.8, we obtain that

$$
\| QH_1^{(1)} \|^h_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} \lesssim a (V_t + \mathcal{P} u_t + (Qu_t + \nabla a)) \| L^1_T(B_{p,1}^{-1, \frac{d}{2}}) \\
\lesssim a \| (P u_t, V_t, (Qu_t + \nabla a)) \|_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} \\
\lesssim v^{-1} \mathcal{W} (\mathcal{W} + \mathcal{W} + \mathcal{V}). \tag{3.20}
$$

Similarly, we have

$$
\| QH_1^{(2)} \|^h_{L^1_T(B_{p,1}^{-1, \frac{d}{2}})} \lesssim (1 + a) \| (\nabla u, \nabla V) \|_{L^\infty_T(B_{p,1}^{-1, \frac{d}{2}})} \\
\lesssim (1 + a) \left\{ \int_0^T \| (\nabla Pu, \nabla Qu^h, \nabla V) \|_{B_{p,1}^{-1, \frac{d}{2}}} + \| \nabla Qu^\ell \|_{B_{p,1}^{-1, \frac{d}{2}}} \right\} dt \\
\lesssim (\mathcal{W} + \mathcal{W})(\mathcal{W} + \mathcal{W}) + v^{-1} \mathcal{W}(\mathcal{W} + \mathcal{W}) \\
+ \int_0^T \| (\nabla Pu, \nabla Qu^h, \nabla V) \|_{B_{p,1}^{-1, \frac{d}{2}}} + \| \nabla Qu^\ell \|_{B_{p,1}^{-1, \frac{d}{2}}} dt, \tag{3.21}
$$

after the use of the smallness assumption (3.11) on $a$. 

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From Proposition 2.12, we get
\[
\| (k(a) - a) \nabla a \|_{L^1(B_{p,1}^{-1} + \frac{p}{q})}^h
\leq \int_0^T \left( \nu^{-1} \| a^h \|_{B_{p,1}^{-1} + \frac{p}{q}} + \nu^{-1} \| va^\ell \|_{B_{2,1}^{-1} + \frac{p}{q}} \right) \left( \| va^h \|_{B_{p,1}^p} + \| a^\ell \|_{B_{2,1}^{-1} + \frac{p}{q}} \right) dt. \tag{3.22}
\]

Thus, the combination of (3.18)–(3.22) with (3.17) implies that
\[
\| w \|_{L^p(B_{p,1}^{-1} + \frac{p}{q})}^h + \nu \| w \|_{L^p(B_{p,1}^{-1} + \frac{p}{q})}^h
\leq \| w_0 \|_{B_{p,1}^{-1} + \frac{p}{q}} + \nu^{-1} \| w \|_{L^p(B_{p,1}^{-1} + \frac{p}{q})}^h + \nu^{-2} \| a^h \|_{L^1(B_{p,1}^p)} + (\mathcal{W} + \mathcal{W})(\mathcal{Z} + \mathcal{V}) + v^{-1} \mathcal{Y}(\mathcal{Z} + \mathcal{V}) + v^{-1} \mathcal{X}(\mathcal{W} + \mathcal{V} + \mathcal{Y}) + v^{-2} \mathcal{Z} \mathcal{W} + v^{-3} \mathcal{X} \mathcal{Y} + v^{-2} \mathcal{Y}^2
+ \int_0^T \| (\nabla u, \nabla V) \|_{B_{p,1}^p} \left( \| Q u^h \|_{B_{p,1}^{-1} + \frac{p}{q}} + \| Q u^\ell \|_{B_{2,1}^{-1} + \frac{p}{q}} \right) dt
+ \int_0^T \left( \nu^{-1} \| a^h \|_{B_{p,1}^p} + \nu^{-1} \| va^\ell \|_{B_{2,1}^{-1} + \frac{p}{q}} \right) \left( \| va^h \|_{B_{p,1}^p} + \| a^\ell \|_{B_{2,1}^{-1} + \frac{p}{q}} \right) dt. \tag{3.23}
\]

We now detect damping phenomenon on the high frequencies of the density.

As \( \text{div } Q u = \text{div } w + v^{-1} a \), we deduce from the first equation in (1.9) that
\[
\partial_t a + (u + V) \cdot \nabla a + v^{-1} a = -a \text{ div } u - \text{div } w. \tag{3.24}
\]

To bound the high frequencies of \( a \), we write
\[
\partial_t \hat{\Delta} a + (u + V) \cdot \nabla \hat{\Delta} a + v^{-1} \hat{\Delta} a = [(u + V) \cdot \nabla, \hat{\Delta}] a - \hat{\Delta}(a \text{ div } u + \text{div } w). \tag{3.25}
\]

Taking \( L^2 \) inner product with \( \hat{\Delta} a |^{p-2} \hat{\Delta} a \), using integrating by part and the Hölder inequality, we thus get, for \( t \geq 0 \),
\[
\| \hat{\Delta} a(t) \|_{L^p} + v^{-1} \| \hat{\Delta} a \|_{L^p} dt
\leq \| \hat{\Delta} a_0 \|_{L^p} + \frac{1}{p} \int_0^T \| \text{div}(u + V) \|_{L^\infty} \| \hat{\Delta} a \|_{L^p} dt
+ \int_0^T \| [(u + V) \cdot \nabla, \hat{\Delta}] a \|_{L^p} dt + \int_0^T \| \hat{\Delta}(a \text{ div } u + \text{div } w) \|_{L^p} dt. \tag{3.26}
\]

By Lemmas 2.8 and 2.9, we get
\[
\| a \text{ div } u \|_{B_{p,1}^p} \lesssim \| a \|_{B_{p,1}^p} \| \text{div } u \|_{B_{p,1}^p},
\]
\[
\sum_{j \in \mathbb{Z}} 2^{|j/2|} \| [(u + V) \cdot \nabla, \hat{\Delta}] a \|_{L^p} \leq C \| \nabla (u + V) \|_{B_{p,1}^p} \| a \|_{B_{p,1}^p}.\]
Multiplying (3.26) by $2^d$, using the embedding relation $B^d_{p,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, we have

$$
\|a^h\|_{L^\infty_t(B^d_{p,1})} + \nu^{-1}\|a^h\|_{L^1_t(B^d_{p,1})} \leq \|a_0\|_{B^d_{p,1}}^h + \|w\|_{L^1_t(B^d_{p,1} + 1^{1/d})} + C \int_0^T \|(\nabla u, \nabla V)\|_{B^d_{p,1}} \|a\|_{B^d_{p,1}}^d dt,
$$

which implies that

$$
\|va\|_{L^\infty_t(B^d_{p,1})}^h + \|a^h\|_{L^1_t(B^d_{p,1})} \leq \|va_0\|_{B^d_{p,1}}^h + \nu \|w\|_{L^1_t(B^d_{p,1} + 1^{1/d})} + C \int_0^T \|(\nabla u, \nabla V)\|_{B^d_{p,1}} \|va^h\|_{B^d_{p,1}}^d + \|va^\ell\|_{B^d_{p,1}}^d dt. \tag{3.27}
$$

By (3.23) and (3.27) and since $\nu$ is large enough (here we use $\nu > 1$), we get

$$
\|va\|_{L^\infty_t(B^d_{p,1})}^h + \|a^h\|_{L^1_t(B^d_{p,1})} \leq \|va_0\|_{B^d_{p,1}}^h + \|w_0\|_{B^{-1+1/d}_{p,1}}^h + (\nu + W)(Z + V) + \nu^{-1}Y(Z + V)
$$

$$
+ \nu^{-1}X(W + V + Y) + \nu^{-2}ZW + \nu^{-3}XY + \nu^{-2}Y^2
$$

$$
+ \int_0^T \left( \nu^{-1}\|a^h\|_{B^{-1}_{p,1}} + \nu^{-1}\|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) \left( \|va^h\|_{B^{-1+1/d}_{p,1}} + \|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) dt
$$

$$
+ \int_0^T \|(\nabla u, \nabla V)\|_{B^{-1+1/d}_{p,1}} \left( \|Qu^h\|_{B^{-1+1/d}_{p,1}} + \|Qu^\ell\|_{B^{-1+1/d}_{2,1}} + \|va^h\|_{B^{-1+1/d}_{p,1}} + \|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) dt. \tag{3.28}
$$

Recalling that

$$
w = Qu + v^{-1}(-\Delta)^{-1}\nabla a,
$$

we get

$$
\|Qu\|_{L^\infty_t(B^{-1+1/d}_{p,1})} \leq \|w\|_{L^\infty_t(B^{-1+1/d}_{p,1})} + \|va\|_{L^1_t(B^{-1+1/d}_{p,1})} \leq \nu \|w\|_{L^1_t(B^{-1+1/d}_{p,1})} + \|a^h\|_{L^1_t(B^{-1+1/d}_{p,1})}
$$

and so, by (3.28) and (3.29),

$$
\|va\|_{L^\infty_t(B^{-1+1/d}_{p,1})} + \|a^h\|_{L^1_t(B^{-1+1/d}_{p,1})} + \|Qu\|_{L^\infty_t(B^{-1+1/d}_{p,1})} + \nu \|Qu\|_{L^1_t(B^{-1+1/d}_{p,1})} \leq \nu \|w\|_{L^1_t(B^{-1+1/d}_{p,1})} + \|a^h\|_{L^1_t(B^{-1+1/d}_{p,1})}
$$

$$
+ \int_0^T \left( \nu^{-1}\|a^h\|_{B^{-1}_{p,1}} + \nu^{-1}\|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) \left( \|va^h\|_{B^{-1+1/d}_{p,1}} + \|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) dt
$$

$$
+ \int_0^T \|(\nabla u, \nabla V)\|_{B^{-1+1/d}_{p,1}} \left( \|Qu^h\|_{B^{-1+1/d}_{p,1}} + \|Qu^\ell\|_{B^{-1+1/d}_{2,1}} + \|va^h\|_{B^{-1+1/d}_{p,1}} + \|va^\ell\|_{B^{-1+1/d}_{2,1}} \right) dt. \tag{3.30}
$$
We employ the second equation of (1.9) to produce that
\[
\| (Q u_t + \nabla a)^{h} \|_{L^1_t(B^0_{p,1})} \lesssim \| v Q u^{h} \|_{L^1_t(B^1_{p,1})} + \| Q \mathcal{H}_t \|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}. \tag{3.31}
\]
Thus, the combination of (3.30) and (3.31) implies that
\[
\begin{align*}
\| v a^{h} \|_{L^T_t(B^0_{p,1})} + \| a^{h} \|_{L^T_t(B^0_{p,1})} + \| Q u^{h} \|_{L^T_t(B^{1+\frac{3}{p}}_{p,1})} + v \| Q u^{h} \|_{L^T_t(B^{1+\frac{3}{p}}_{p,1})} + \| (Q u_t + \nabla a)^{h} \|_{L^T_t(B^{1+\frac{3}{p}}_{p,1})} \\
\lesssim \| v a^{0} \|_{B^{0}_{p,1}} + \| a^{0} \|_{B^{0}_{p,1}} + (\mathcal{V} + \mathcal{W})(\mathcal{Z} + \mathcal{V}) + v^{-1}\mathcal{Y}(\mathcal{Z} + \mathcal{V}) \\
+ v^{-1}\mathcal{X}(\mathcal{V} + \mathcal{V} + \mathcal{Y}) + v^{-2}\mathcal{W} + v^{-3}\mathcal{X}\mathcal{Y} + v^{-2}\mathcal{Y}^2 \\
+ \int_0^T (v^{-1}\|a^{h}\|_{B^{0}_{p,1}} + v^{-1}\|v a^{\ell}\|_{B^{1+\frac{3}{p}}_{2,1}}) \left( \|v a^{h}\|_{B^{0}_{p,1}} + \|a^{\ell}\|_{B^{1+\frac{3}{p}}_{2,1}} \right) dt \\
+ \int_0^T \| (\nabla u, \nabla V) \|_{B^{0}_{p,1}} \left( \|Q u^{h}\|_{B^{1+\frac{3}{p}}_{p,1}} + \|Q u^{\ell}\|_{B^{1+\frac{3}{p}}_{2,1}} + \|va^{h}\|_{B^{0}_{p,1}} + \|va^{\ell}\|_{B^{0}_{2,1}} \right) dt. \tag{3.32}
\end{align*}
\]

3.3. Low frequencies for the compressible part of (1.1)

Now we estimate the low frequencies for the compressible part of (1.1) or (1.9), which is rewritten as
\[
\begin{align*}
\begin{cases}
    a_t + (\mathcal{P} u + V) \cdot \nabla a + \text{div } Q u = -\text{div} (a Q u), \\
    (Q u)_t - v \Delta Q u + Q ((u + V) \cdot \nabla Q u) + \nabla a = Q f, \tag{3.33}
\end{cases}
\end{align*}
\]
with
\[
f = a (V_t + \mathcal{P} u_t + (Q u_t + \nabla a)) + (1 + a)(u + V) \cdot \nabla \mathcal{P} u \\
+ (1 + a)(u + V) \cdot \nabla V + a(u + V) \cdot \nabla Q u + (k(a) - a) \nabla a.
\]

Applying $\dot{\lambda}_j$ to both side of the first two equations of (3.33), we get
\[
\partial_t \dot{\lambda}_j a + (\mathcal{P} u + Q u + V) \cdot \nabla \dot{\lambda}_j a + \text{div } Q \dot{\lambda}_j u = \dot{\lambda}_j g, \tag{3.34}
\]
\[
\partial_t \dot{\lambda}_j Q u + Q ((u + V) \cdot \nabla \dot{\lambda}_j u) - v \Delta \dot{\lambda}_j u + \nabla \dot{\lambda}_j a = \dot{\lambda}_j Q f. \tag{3.35}
\]

with
\[
\dot{\lambda}_j g := \dot{\lambda}_j (-a \text{ div } Q u) + [(\mathcal{P} u + Q u + V) \cdot \nabla, \dot{\lambda}_j] a,
\]
\[
\dot{\lambda}_j f := - [\dot{\lambda}_j, u + V] \cdot \nabla Q u + \dot{\lambda}_j \left( a (V_t + \mathcal{P} u_t + (Q u_t + \nabla a)) + (1 + a) Q u \cdot \nabla (\mathcal{P} u + V) \\
+ (1 + a)(\mathcal{P} u + V) \cdot \nabla (\mathcal{P} u + V) + a(\mathcal{P} u + V) \cdot \nabla Q u + a Q u \cdot \nabla Q u + (k(a) - a) \nabla a \right).
\]
Applying the gradient operator \( \nabla \) with a relation involving which involves the highest order derivative. Actually, it is convenient to combine equation identity
\[
\Delta 1 2 1 2 d dt = d dt + \int R_d \nabla a dx = 1 2 \int R_d (\nabla u) (\Delta a)^2 dx + \int R_d \Delta a dx \quad (3.36)
\]
and
\[
\frac{1}{2} d dt \int R_d |Q\Delta a|^2 dx + \int R_d Q\Delta a \cdot \nabla \Delta a dx
\]
\[
= 1 2 \int R_d (\nabla u) |Q\Delta a|^2 dx + \int R_d (\nabla a \cdot \nabla \Delta a) \cdot \nabla \Delta a dx. \quad (3.37)
\]

Applying the gradient operator \( \nabla \) on both side of (3.34) gives
\[
\nabla a_j, + (P u + Q u + V) \cdot \nabla \nabla \Delta a + \nabla \nabla \Delta a = \nabla \Delta a - \nabla (P u + Q u + V) \cdot \nabla \Delta a.
\]
(3.38)

Taking \( L^2 \) inner product with \( \nabla \Delta a \) to the above equation implies
\[
\frac{1}{2} d dt \int R_d |\nabla \Delta a|^2 dx + \int R_d \nabla \Delta a \cdot \nabla \Delta a dx
\]
\[
= 1 2 \int R_d (\nabla u) |\nabla \Delta a|^2 dx + \int R_d (\nabla \Delta a \cdot \nabla \Delta a) \cdot \nabla \Delta a dx. \quad (3.39)
\]

The second term on the left hand side of (3.39) is troublesome in our further estimation. We have to use some special technique to overcome the difficulty by eliminating this term, which involves the highest order derivative. Actually, it is convenient to combine equation (3.39) with a relation involving \( \int R_d \nabla \Delta a \cdot \nabla \Delta a dx \). Now testing (3.38) by \( Q\Delta a \) and the momentum equation (3.35) by \( \nabla \Delta a \), we get
\[
\frac{d}{dt} \int R_d Q\Delta a \cdot \nabla \Delta a dx + \int R_d (u + V) \cdot \nabla (Q\Delta a \cdot \nabla \Delta a) dx - \nu \int R_d \Delta Q\Delta a \cdot \nabla a dx
\]
\[
+ \int R_d |\nabla \Delta a|^2 dx + \int R_d \nabla \Delta a \cdot \nabla \Delta a dx
\]
\[
= \int R_d (\nabla \Delta a \cdot \nabla (P u + Q u + V) \cdot \nabla \Delta a) \cdot Q\Delta a dx + \int R_d \Delta a \cdot \nabla \Delta a dx. \quad (3.40)
\]

Hence, multiplying (3.39) by \( \nu \) and adding the resultant equation to (3.40), we use the identity \( \Delta Q\Delta a \equiv \nabla \nabla Q\Delta a \) to cancel the highest order terms to obtain that
\[
\frac{1}{2} d dt \int R_d (\nu |\nabla \Delta a|^2 + 2 Q\Delta a \cdot \nabla \Delta a) dx + \int R_d (|\nabla \Delta a|^2 - |\nabla Q\Delta a|^2) dx
\]
\[
= \int R_d \left( \frac{\nu}{2} |\nabla \Delta a|^2 + Q\Delta a \cdot \nabla \Delta a \right) dx + \nu \int R_d (\nabla \Delta a \cdot \nabla (u + V) \cdot \nabla \Delta a) \cdot \nabla \Delta a dx
\]
\[
+ \int R_d (\nabla \Delta a \cdot \nabla (u + V) \cdot \nabla \Delta a) \cdot Q\Delta a dx + \int R_d \Delta a \cdot \nabla \Delta a dx. \quad (3.41)
\]
After multiplying (3.41) by \( \nu \) and adding the resultant equation to (3.36) and (3.37) respectively, we get

\[
\frac{1}{2} \frac{d}{dt} L_j^2 + \nu \int_{\mathbb{R}^d} \left( |\nabla \hat{Q} \hat{j} u|^2 + |\nabla \hat{j} a|^2 \right) dx \\
= \int_{\mathbb{R}^d} (2 \Delta_j g \Delta_j a + 2 \Delta_j Q \cdot \nabla \hat{j} a + v^2 \nabla \Delta_j \hat{j} \cdot \nabla \hat{j} a + v \nabla \Delta_j \hat{j} \cdot Q \Delta_j u + v \Delta_j Q f \cdot \nabla \hat{j} a) dx \\
+ \frac{1}{2} \int_{\mathbb{R}^d} (2(\Delta_j a)^2 + 2|Q \Delta_j u|^2 + 2vQ \Delta_j u \cdot \nabla \hat{j} a + |vQ \Delta_j a|^2) \operatorname{div} Q u \, dx \\
- \nu \int_{\mathbb{R}^d} (\nabla(u + V) \cdot \nabla \hat{j} a) \cdot (v \nabla \hat{j} a + Q \Delta_j u) \, dx,
\]

where

\[
L_j := \left( \int_{\mathbb{R}^d} (2(\Delta_j a)^2 + 2|Q \Delta_j u|^2 + 2vQ \Delta_j u \cdot \nabla \hat{j} a + |vQ \Delta_j a|^2) \, dx \right)^{1/2}.
\]

It is readily seen that

\[
C^{-1} \|(Q \Delta_j u, \Delta_j a, v \nabla \Delta_j a)\|_{L^2} \leq L_j \leq C \|(Q \Delta_j u, \Delta_j a, v \nabla \Delta_j a)\|_{L^2}, \quad j \in \mathbb{Z},
\]

and

\[
\nu \int_{\mathbb{R}^d} \left( |\nabla \hat{j} a|^2 + |\nabla \hat{j} a|^2 \right) dx \geq C_2 v^{2j} L_j, \quad j \leq j_0,
\]

for some constants \( C, C_2 > 0 \). Therefore it follows from (3.42), (3.44) and (3.45) that, for \( j \leq j_0 \),

\[
\frac{1}{2} \frac{d}{dt} L_j^2 + v^{2j} L_j^2 \leq \left( \frac{1}{2} \| \operatorname{div} Q u \|_{L^\infty} + C \| \nabla (u + V) \|_{L^\infty} \right) L_j^2 + C \| [\Delta_j g, \Delta_j Q f, v \nabla \Delta_j g] \|_{L^2} L_j.
\]

Hence, after integration in time, we have

\[
L_j(T) + v^{2j} \int_0^T L_j \, dt \\
\leq L_j(0) + C \int_0^T \| \nabla (u + V) \|_{L^\infty} L_j \, dt + C \int_0^T \| [\Delta_j g, \Delta_j Q f, v \nabla \Delta_j g] \|_{L^2} \, dt.
\]

Summing up (3.46) with respect to \( j < j_0 \), we have

\[
\left\| (a^\ell, v \nabla a^\ell, Q u^\ell) \right\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}})} + \left\| (v a^\ell, v^2 \nabla a^\ell, v Q u^\ell) \right\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \\
\leq \left\| (a_0, v \nabla a_0, Q u_0) \right\|_{B_{2,1}^{1+\frac{d}{2}}} + \int_0^T \| \nabla (u + V) \|_{L^\infty} \left\| (a^\ell, v \nabla a^\ell, Q u^\ell) \right\|_{B_{2,1}^{1+\frac{d}{2}}} \, dt \\
+ \int_0^T \sum_{j \in \mathbb{Z}} 2^{(1+\frac{d}{2}j)} \|(\Delta_j g, v \nabla \Delta_j g)\|_{L^2} \, dt + \int_0^T \sum_{j \in \mathbb{Z}} 2^{(1+\frac{d}{2}j)} \|\Delta_j Q f\|_{L^2} \, dt.
\]
In the following, we estimate last two terms on the right hand side of the previous equality.

By Lemma 2.10, we have

\[
\int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{4}{d})j} \| \hat{\Delta}_j (\nabla P) \|_{L^2}^\ell dt \\
\lesssim \int_0^T (\| \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) (\| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} + \| a^h \|_{B^0_{\frac{d}{p},1}}) dt \\
\lesssim \int_0^T \nu^{-1} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) \| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} dt \\
+ \int_0^T \nu^{-2} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) \| v a^h \|_{B^0_{\frac{d}{p},1}} dt.
\] (3.48)

By Lemma 2.10, we have

\[
\int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{4}{d})j} \| \nabla \hat{\Delta}_j (\nabla P + \nabla Q) \cdot \nabla, \hat{\Delta}_j a \|_{L^2}^\ell dt \\
\lesssim \int_0^T (\| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}} + \| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}}) (\| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} + \| a^h \|_{B^0_{\frac{d}{p},1}}) dt \\
\lesssim \int_0^T (\nu^{-1} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) + \| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}}) \| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} dt \\
+ \int_0^T (\nu^{-2} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) + \nu^{-1} (\| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}}) \| v a^h \|_{B^0_{\frac{d}{p},1}} dt. \) (3.49)

From the above two estimates, we find that

\[
\int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{4}{d})j} \| \hat{\Delta}_j (\nabla P + \nabla Q + V) \cdot \nabla, \hat{\Delta}_j a \|_{L^2}^\ell dt \\
\lesssim \int_0^T (\nu^{-1} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) + \| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}}) \| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} dt \\
+ \int_0^T (\nu^{-2} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) + \nu^{-1} (\| \nabla \nabla P, \nabla Q \|_{B^{1+\frac{4}{d}}_{\frac{d}{p},1}}) \| v a^h \|_{B^0_{\frac{d}{p},1}} dt. \) (3.50)

Similarly, from Lemma 2.10, we get

\[
\nu \int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{4}{d})j} \| \nabla \hat{\Delta}_j (\nabla P + \nabla Q + V) \cdot \nabla, \hat{\Delta}_j a \|_{L^2}^\ell dt \\
\lesssim \int_0^T (\nu^{-1} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) \| a^\ell \|_{B^{-1+\frac{4}{d}}_{\frac{d}{p},1}} + \| v a^h \|_{B^0_{\frac{d}{p},1}}) dt \\
\lesssim \int_0^T \nu^{-1} (\| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}} + \| \nabla \nabla P \|_{B^{1+\frac{4}{d}}_{2,1}}) \| v a^h \|_{B^0_{\frac{d}{p},1}} dt. \) (3.51)
and

\[
\int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{j}{2})}\|\nabla (\{(\mathcal{P}u + \mathcal{Q}u + V) \cdot \nabla, \Delta_j\}va)\|_{L^2}^\ell dt \\
\lesssim \int_0^T \| (\nabla \mathcal{P}u, \nabla V)\|_{B_{2,1}^{\frac{d}{p}}}(\|va^\ell\|_{B_{2,1}^{\frac{d}{p}}} + \|va^h\|_{B_{2,1}^{\frac{d}{p}}}) dt \\
+ \int_0^T v^{-1}(\|v \nabla \mathcal{Q}u^h\|_{B_{2,1}^{\frac{d}{p}}} + \|v \nabla \mathcal{Q}u^\ell\|_{B_{2,1}^{\frac{d}{p}}})(\|va^\ell\|_{B_{2,1}^{\frac{d}{p}}} + \|va^h\|_{B_{2,1}^{\frac{d}{p}}}) dt. \quad (3.52)
\]

Hence

\[
v \int_0^T \sum_{j \in \mathbb{Z}} 2^{(-1+\frac{j}{2})}\|\nabla \Delta_j g\|_{L^2}^\ell dt \\
\lesssim \int_0^T \| (\nabla \mathcal{P}u, \nabla V)\|_{B_{2,1}^{\frac{d}{p}}}(\|va^\ell\|_{B_{2,1}^{\frac{d}{p}}} + \|va^h\|_{B_{2,1}^{\frac{d}{p}}}) dt \\
+ \int_0^T v^{-1}(\|v \nabla \mathcal{Q}u^h\|_{B_{2,1}^{\frac{d}{p}}} + \|v \nabla \mathcal{Q}u^\ell\|_{B_{2,1}^{\frac{d}{p}}})(\|va^\ell\|_{B_{2,1}^{\frac{d}{p}}} + \|va^h\|_{B_{2,1}^{\frac{d}{p}}}) dt. \quad (3.53)
\]

Now we claim that, for any \( p \) given in Theorem 1.2, there holds

\[
\| \mathcal{Q}(bc) \|_{B_{2,1}^{\frac{d}{p}}}^\ell \lesssim (\|b\|_{B_{p,1}^{-1+\frac{d}{p}}} + \|b^\ell\|_{B_{2,1}^{-1+\frac{d}{p}}}) \|c\|_{B_{p,1}^{\frac{d}{p}}}. \quad (3.54)
\]

Indeed, denoting \( \mathcal{Q}^\ell := \mathcal{S}_{j_{0}+1} \mathcal{Q} \), we get

\[
\mathcal{Q}^\ell (bc) = \mathcal{Q}^\ell (\mathcal{T}_b c + \mathcal{R}(b,c)) + \mathcal{T}_c \mathcal{Q}^\ell b + [\mathcal{Q}^\ell, \mathcal{T}_c] b. \quad (3.55)
\]

By Lemma 2.7, we obtain

\[
\| \mathcal{T}_c \mathcal{Q}^\ell b \|_{B_{2,1}^{\frac{d}{p}}} \lesssim \|c\|_{L^\infty} \|b\|_{B_{2,1}^{\frac{d}{p}}} \lesssim \|b\|_{B_{2,1}^{\frac{d}{p}}} \|c\|_{B_{p,1}^{\frac{d}{p}}},
\]

\[
\| \mathcal{Q}^\ell (\mathcal{T}_b c + \mathcal{R}(b,c)) \|_{B_{2,1}^{\frac{d}{p}}} \lesssim \|b\|_{B_{p,1}^{-1+\frac{d}{p}}} \|c\|_{B_{p,1}^{\frac{d}{p}}} \lesssim \|b\|_{B_{p,1}^{-1+\frac{d}{p}}} \|c\|_{B_{p,1}^{\frac{d}{p}}}. \quad (3.56)
\]

By Lemma 2.11, we have

\[
\| \mathcal{Q}^\ell, \mathcal{T}_c b \|_{B_{2,1}^{\frac{d}{p}}} \lesssim \|\nabla c\|_{B_{p,1}^{-1+\frac{d}{p}}} \|b\|_{B_{p,1}^{-1+\frac{d}{p}}} \lesssim \|b\|_{B_{p,1}^{-1+\frac{d}{p}}} \|c\|_{B_{p,1}^{\frac{d}{p}}} \cdot \frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}. \quad (3.57)
\]

Thus, the combination of (3.55)–(3.57) shows the validity of (3.54).

Moreover, if \( \text{div } b = 0 \), then we have \( \mathcal{Q}^\ell b = 0 \) and thus

\[
\| \mathcal{Q}(bc) \|_{B_{2,1}^{\frac{d}{p}}} \lesssim \|b\|_{B_{p,1}^{-1+\frac{d}{p}}} \|c\|_{B_{p,1}^{\frac{d}{p}}}. \quad (3.58)
\]
Since \( \text{div}(V_t + Pu_t) = 0 \), taking \( b = V_t + Pu_t \) and \( c = a \) in (3.58), we have

\[
\| Q(a(V_t + Pu_t)) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim \| a \|_{L^\infty(0,T;B^{\frac{d}{p}}_{p,1})} \| (V_t, Pu_t) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim v^{-1} \mathcal{X}(\mathcal{V} + \mathcal{W}). \tag{3.59}
\]

Similarly, using \( \| a \|_{L^\infty(0,T;B^{\frac{d}{p}}_{2,1})} \ll 1 \), we have

\[
\| Q((1 + a)(Pu + V) \cdot \nabla (Pu + V)) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim (\mathcal{V} + \mathcal{W})(\mathcal{Z} + \mathcal{V}), \tag{3.60}
\]

\[
\| Q(a(Pu + V) \cdot \nabla Pu) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim v^{-2} \mathcal{X}(\mathcal{Z} + \mathcal{V}). \tag{3.61}
\]

Taking \( b = Qu_t + \nabla a \) and \( c = a \) in (3.54) gives

\[
\| Q(a(Qu_t + \nabla a)) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim \| a \|_{L^\infty(0,T;B^{\frac{d}{p}}_{2,1})} \left( \| (Qu_t + \nabla a)^h \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} + \| (Qu_t + \nabla a)^e \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \right) \lesssim v^{-1} \mathcal{X}' \mathcal{Y}. \tag{3.62}
\]

Similarly, we obtain that

\[
\| Q((1 + a)Qu \cdot \nabla (Pu + V)) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim \int_0^T \| (\nabla Pu, \nabla V) \|_{B^{\frac{d}{p}}_{p,1}} \left( \| Qu^h \|_{B^{-1+\frac{d}{p}}_{2,1}} + \| Qu^e \|_{B^{-1+\frac{d}{p}}_{2,1}} \right) dt, \tag{3.63}
\]

and

\[
\| Q(aQu \cdot \nabla Qu) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim v^{-2} \mathcal{X}'^2 \mathcal{Y}. \tag{3.64}
\]

By Proposition 2.12, we have

\[
\| Q((k(a) - a) \nabla a) \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim \int_0^T \| \nabla a \|_{B^{-1+\frac{d}{p}}_{2,1}} + \| \nabla a^e \|_{B^{-1+\frac{d}{p}}_{2,1}} \| a \|_{B^{\frac{d}{p}}_{p,1}} dt \lesssim \| a \|_{L^2(0,T;B^{\frac{d}{p}}_{p,1})}^2 + \| a \|_{L^2(0,T;B^{\frac{d}{p}}_{p,1})} \| a^e \|_{L^2(0,T;B^{\frac{d}{p}}_{p,1})} \lesssim \| a^h \|_{L^\infty(0,T;B^{\frac{d}{p}}_{p,1})} \| a^h \|_{L^1(0,T;B^{\frac{d}{p}}_{p,1})} + \| a^e \|_{L^\infty(0,T;B^{-1+\frac{d}{p}}_{2,1})} \| a^e \|_{L^1(0,T;B^{-1+\frac{d}{p}}_{2,1})} \lesssim \int_0^T \left( v^{-1} \| a^h \|_{B^{\frac{d}{p}}_{p,1}} + v^{-1} \| va^e \|_{B^{-1+\frac{d}{p}}_{2,1}} \right) \left( \| va^h \|_{B^{\frac{d}{p}}_{p,1}} + \| a^e \|_{B^{-1+\frac{d}{p}}_{2,1}} \right) dt. \tag{3.65}
\]
It follows from Lemma 2.10 that

\[
\int_0^T \sum_{j \in \mathbb{Z}} 2^{-\left(1 + \frac{d}{2}\right)j} \left\| [\Delta_j, u + V] \cdot \nabla Qu \right\|_{L^2} dt 
\approx \int_0^T \left( \left\| (\nabla Pu, \nabla Qu^h, \nabla V) \right\|_{B_{p,1}^{\frac{d}{2}}} + \left\| \nabla Qu^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| Qu^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt. \quad (3.66)
\]

Now we assume \( v^{-1} \ll 1 \). Inserting (3.50), (3.53), (3.59)–(3.66) into (3.47) gives

\[
\left\| (a^\ell, v \nabla a^\ell, Qu^\ell) \right\|_{L^\infty(0,T;B_{Z,1}^{\frac{d}{2}})} + \left\| (va^\ell, v^2 \nabla a^\ell, vQu^\ell) \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} 
\approx \left\| (a_0, v \nabla a_0, Qu_0) \right\|_{B_{Z,1}^{\frac{d}{2}}} + \int_0^T \int_0^T \left( (\nabla Pu, \nabla V) \right) \left( \left\| a^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ \int_0^T \left( (Qu^\ell) \left( \left\| a^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ \int_0^T \left( (\nabla Pu, \nabla Qu^h, \nabla V) \right) \left( \left\| a^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ \int_0^T \left( (Qu^\ell) \left( \left\| a^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ v^{-1} \nabla V(\nabla V) + v^{-1} \nabla V(\nabla V) \left( \nabla V + \nabla V \right) \left( \nabla V + \nabla V \right) + v^{-2} \nabla V(\nabla V + \nabla V) + v^{-2} \nabla V(\nabla V + \nabla V). \quad (3.67)
\]

Form the second equation in (3.33), we deduce that

\[
\left\| (Qu_t + \nabla a)^\ell \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} 
\approx \left\| vQu^\ell \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} + \left\| Q((u + V) \cdot \nabla Qu) \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} + \left\| Qf \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})}. \quad (3.68)
\]

Following the derivation of (3.61) and (3.64), we get

\[
\left\| Q((u + V) \cdot \nabla Qu) \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} 
\approx \left( \left\| Qu^h \right\|_{L^\infty(0,T;B_{p,1}^{\frac{d}{2}})} + \left\| Qu^\ell \right\|_{L^\infty(0,T;B_{p,1}^{\frac{d}{2}})} + \left\| (Pu, V) \right\|_{L^\infty(0,T;B_{p,1}^{\frac{d}{2}})} \right) \left\| \nabla Qu \right\|_{L^1(0,T;B_{p,1}^{\frac{d}{2}})} 
\approx v^{-1} \nabla V(\nabla V + \nabla V). \quad (3.69)
\]

Combining the estimates (3.59)–(3.66), we find that

\[
\left\| Qf \right\|_{L^1(0,T;B_{Z,1}^{\frac{d}{2}})} \approx \int_0^T \left( v^{-1} \left\| a^h \right\|_{B_{p,1}^{\frac{d}{2}}} + v^{-1} \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| a^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| va^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ \int_0^T \left( \left\| (\nabla Pu, \nabla Qu^h, \nabla V) \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| \nabla Qu^\ell \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) \left( \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} + \left\| Qu^h \right\|_{B_{Z,1}^{\frac{d}{2}}} \right) dt 
+ v^{-1} \nabla V(\nabla V + \nabla V) + v^{-1} \nabla V(\nabla V + \nabla V) \left( \nabla V + \nabla V \right) + v^{-2} \nabla V(\nabla V + \nabla V) + v^{-2} \nabla V(\nabla V + \nabla V). \quad (3.70)
\]
By (3.67)--(3.70), we have

\[
\begin{align*}
&\| (a^\ell, v \nabla a^\ell, Qu^\ell) \|_{L^\infty(0,T;B^{-1+\frac{3}{2}}_{2,1})} + \| (va^\ell, v^2 \nabla a^\ell, vQu^\ell) \|_{L^1(0,T;B^{1+\frac{3}{2}}_{2,1})} \\
&\quad + \| (Qu_t + \nabla a)^\ell \|_{L^1(0,T;B^{-1+\frac{3}{2}}_{2,1})} \\
&\lesssim \| (a_0, v \nabla a_0, Qu_0) \|_{B^{-1+\frac{3}{2}}_{2,1}} + v^{-1} \mathcal{X}(\mathcal{V} + \mathcal{W}) + v^{-1} \mathcal{Y}(\mathcal{X} + Z + \mathcal{V}) + (\mathcal{V} + \mathcal{W})(\mathcal{Z} + \mathcal{V}) + v^{-2} \mathcal{X}\mathcal{Y}(\mathcal{Z} + \mathcal{V}) + v^{-2} \mathcal{X}^2 \mathcal{Y} \\
&\quad + \int_0^T \left( v^{-1} \| a^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d + v^{-1} \| va^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \| (\nabla P u, \nabla V) \|_{B^{1+\frac{3}{2}}_{p,1}}^d \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \left( \| Qu^\ell \|_{B^{1+\frac{3}{2}}_{2,1}}^d + \| Qu^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \left( \| (\nabla P u, \nabla Q u^h, \nabla V) \|_{B^{1+\frac{3}{2}}_{p,1}}^d + \| \nabla Qu^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) \left( \| Qu^h \|_{B^{-1+\frac{3}{2}}_{p,1}}^d + \| Qu^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) dt. \\
\end{align*}
\]

By the estimates (3.32) and (3.71), we get

\[
\begin{align*}
&\| (a^\ell, v \nabla a^\ell, Qu^\ell) \|_{L^\infty(0,T;B^{-1+\frac{3}{2}}_{2,1})} + \| (va^\ell, v^2 \nabla a^\ell, vQu^\ell) \|_{L^1(0,T;B^{1+\frac{3}{2}}_{2,1})} \\
&\quad + \| va^h \|_{L^\infty(0,T;B^{1+\frac{3}{2}}_{p,1})}^d + \| a^h \|_{L^1(0,T;B^{1+\frac{3}{2}}_{p,1})}^d + \| Qu^h \|_{L^\infty(0,T;B^{1+\frac{3}{2}}_{p,1})}^d + v \| Qu^h \|_{L^1(0,T;B^{1+\frac{3}{2}}_{p,1})}^d \\
&\quad + \| (Qu_t + \nabla a)^\ell \|_{L^1(0,T;B^{-1+\frac{3}{2}}_{2,1})} + \| (Qu_t + \nabla a)^h \|_{L^1(0,T;B^{-1+\frac{3}{2}}_{2,1})} \\
&\lesssim \| (a_0, v \nabla a_0, Qu_0) \|_{B^{-1+\frac{3}{2}}_{2,1}} + \| va_0 \|_{B^{1+\frac{3}{2}}_{p,1}}^d + \| v^{-1} a_0 \|_{B^{1+\frac{3}{2}}_{p,1}}^d + \| Qu_0 \|_{B^{1+\frac{3}{2}}_{p,1}}^d \\
&\quad + (\mathcal{V} + \mathcal{W})(\mathcal{Z} + \mathcal{V}) + v^{-2} \mathcal{X}\mathcal{Y}(\mathcal{Z} + \mathcal{V}) + v^{-2} \mathcal{X}^2 \mathcal{Y} + v^{-1} \mathcal{X}(\mathcal{W} + \mathcal{V} + \mathcal{Y}) + v^{-2} \mathcal{Z}\mathcal{W} + v^{-3} \mathcal{X}\mathcal{Y} + v^{-2} \mathcal{Y}^2 \\
&\quad + \int_0^T \left( v^{-1} \| a^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d + v^{-1} \| va^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \| (\nabla P u, \nabla V) \|_{B^{1+\frac{3}{2}}_{p,1}}^d \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \left( \| Qu^\ell \|_{B^{1+\frac{3}{2}}_{2,1}}^d + \| Qu^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) \left( \| a^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d + \| va^h \|_{B^{1+\frac{3}{2}}_{p,1}}^d \right) dt \\
&\quad + \int_0^T \left( \| (\nabla P u, \nabla Q u^h, \nabla V) \|_{B^{1+\frac{3}{2}}_{p,1}}^d + \| \nabla Qu^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) \left( \| Qu^h \|_{B^{-1+\frac{3}{2}}_{p,1}}^d + \| Qu^\ell \|_{B^{-1+\frac{3}{2}}_{2,1}}^d \right) dt. \\
\end{align*}
\]

(3.72)
Recalling that 

\[ \nu \]

This together with the assumption \( \nu^{-1} \ll 1 \) and (3.1) yields

\[
\begin{align*}
\mathcal{X} + \mathcal{Y} & \leq \exp \left( C(M + \delta + \nu^{-1} \eta) \right) \left\{ \mathcal{X}(0) + (M + \delta)^2 + v^{-1} \eta (\delta + M) \mathcal{X} + v^{-1} \eta^2 \mathcal{X} \
+ v^{-1} (M + \delta + \eta) \mathcal{X} + v^{-1} (M + \delta) \mathcal{Y} \right\}. 
\end{align*}
\] (3.75)

3.4. Existence of the global-in-time solution

We deduce from (3.73) that

\[
\begin{align*}
\mathcal{X} + \mathcal{Y} & \leq \exp \left( C(W + V + v^{-1} \mathcal{Y}) \right) \left\{ \mathcal{X}(0) + (V + W)(Z + V) + v^{-2} \mathcal{X} \mathcal{Y}(Z + V) \
+ v^{-2} \mathcal{X}^2 \mathcal{Y} + v^{-1} \mathcal{Y}(Z + V) + v^{-1} \mathcal{X}(W + V + \mathcal{Y}) + v^{-2} \mathcal{Z} \mathcal{W} + v^{-3} \mathcal{X} \mathcal{Y} + v^{-2} \mathcal{Y}^2 \right\}. 
\end{align*}
\] (3.74)

This together with the assumption \( \nu^{-1} \ll 1 \) and (3.1) yields

\[
\begin{align*}
\mathcal{X} + \mathcal{Y} & \leq \exp \left( C(M + \delta + \nu^{-1} \eta) \right) \left\{ \mathcal{X}(0) + (M + \delta)^2 + v^{-1} \eta (\delta + M) \mathcal{X} + v^{-1} \eta^2 \mathcal{X} \
+ v^{-1} (M + \delta + \eta) \mathcal{X} + v^{-1} (M + \delta) \mathcal{Y} \right\}. 
\end{align*}
\] (3.75)

Recalling that

\[
\begin{align*}
\mathcal{Z} + \mathcal{W} & \leq \exp \left( C \int_0^T \left( \| (\nabla V, \nabla Q u) \|_{B^p_{p,1}}^d + \| u \|_{B^p_{p,1}}^2 + \| V \|_{B^p_{p,1}}^2 \right) dt \right) \left\{ v^{-2} \mathcal{X} \mathcal{Y}^2 \mathcal{Y} \
+ v^{-1} \mathcal{X}(Z + \mathcal{X} + V) \mathcal{W} + v^{-1} (\mathcal{Y} + \mathcal{W} + \mathcal{Y}) \mathcal{X} + v^{-1} \mathcal{X} \mathcal{Y}^2 \right\}, 
\end{align*}
\] (3.76)

and employing an embedding inequality and the interpolation inequality, we have

\[
\begin{align*}
\| u \|_{B^p_{p,1}}^2 + \| V \|_{B^p_{p,1}}^2 & \lesssim \| \mathcal{P} u \|_{B^p_{p,1}}^2 + \| Q u^h \|_{B^p_{p,1}}^2 + \| Q u^\ell \|_{B^p_{p,1}}^2 + \| V \|_{B^p_{p,1}}^2 \\
& \lesssim \| \mathcal{P} u \|_{B^p_{p,1}}^{1 + \delta} \| \mathcal{P} u \|_{B^p_{p,1}}^{1 + \delta} + v^{-1} \| Q u^h \|_{B^p_{p,1}}^{1 + \delta} \| v Q u^h \|_{B^p_{p,1}}^{1 + \delta} \\
& + v^{-1} \| Q u^\ell \|_{B^p_{p,1}}^{1 + \delta} \| v Q u^\ell \|_{B^p_{p,1}}^{1 + \delta} + \| V \|_{B^p_{p,1}}^{1 + \delta} \| V \|_{B^p_{p,1}}^{1 + \delta}, 
\end{align*}
\]
and hence, by (3.1),

\[
\int_0^T \left( \| \nabla V, \nabla Q u \|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \| u \|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2 + \| V \|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2 \right) dt \\
\lesssim \int_0^T \left( \| \nabla V, \nabla Q u^h \|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \| u \|_{\dot{B}_{p,1}^{\frac{1}{2}}} + v^{-1} \| Q u^h \|_{\dot{B}_{p,1}^{\frac{1}{2}}} + \| \mathcal{P} u \|_{\dot{B}_{p,1}^{\frac{1}{2}}} + v^{-1} \| Q u^\ell \|_{\dot{B}_{p,1}^{\frac{1}{2}}} + \| V \|_{\dot{B}_{p,1}^{\frac{1}{2}}} \right) dt \\
\lesssim M + M^2 + \delta^2 + v^{-1} \eta + v^{-1} \eta^2. \tag{3.77}
\]

Inserting (3.77) into (3.76) and using (3.1) imply that

\[
\mathcal{Z} + \mathcal{W} \leq \exp \left( C (M + M^2 + \delta^2 + v^{-1} \eta + v^{-1} \eta^2) \right) \left\{ v^{-\frac{1}{2}} \eta M + v^{-1} \eta (M + \eta) \mathcal{W} + v^{-1} (M + \eta + \delta) \eta + v^{-1} \eta M^2 \right\}. \tag{3.78}
\]

Hence, assuming in addition that

\[
v^{-1} \eta + v^{-1} \eta^2 \leq M + M^2 \quad \text{and} \quad \delta \leq \max\{M, 1\}. \tag{3.79}
\]

We get from (3.75) and (3.78) that

\[
\mathcal{Z} + \mathcal{W} \leq \exp(C(M + M^2)) \left\{ v^{-\frac{1}{2}} \eta M + v^{-1} \eta (M + \eta) \mathcal{W} + v^{-1} \eta^2 + v^{-1} \eta M^2 \right\} \tag{3.80}
\]

and

\[
\mathcal{X} + \mathcal{Y} \leq \exp(C(M + M^2)) \left\{ \mathcal{X}(0) + M^2 + 1 + v^{-1}(1 + \eta)(\eta + M) \mathcal{X} + v^{-1} M \mathcal{Y} \right\}. \tag{3.81}
\]

Therefore, assume that

\[
v^{-1}(1 + \eta)(\eta + M)e^{CM} \ll 1. \tag{3.82}
\]

Then we get from (3.80) and (3.81) that

\[
\mathcal{Z} + \mathcal{W} \leq \eta \exp(C(M + M^2)) \left\{ v^{-\frac{1}{2}} M + v^{-1}(\eta + M^2 + 1) \right\}, \tag{3.83}
\]

\[
\mathcal{X} + \mathcal{Y} \leq \exp(C(M + M^2)) (\mathcal{X}(0) + M^2 + 1). \tag{3.84}
\]

Actually, it is natural to set

\[
\eta = C \exp(C(M + M^2)) (\mathcal{X}(0) + M^2 + 1) \quad \text{and} \quad \tag{3.85}
\]

\[
\eta = C \exp(C(M + M^2)) (\mathcal{X}(0) + M^2 + 1). \tag{3.85}
\]
\[ \delta = C \exp(C(M + M^2))(\mathcal{X}(0) + M^2)\left(\nu^{-1/2}M + \nu^{-1}(\mathcal{X}(0) + M^2 + 1)\right). \] (3.86)

Then take a suitably large constant \( C \) such that

\[ C \exp(C(M + M^2))(\mathcal{X}(0) + 1 + M^2) \leq \sqrt{\nu}, \] (3.87)

which implies the validity of the assumptions (3.10), (3.79) and (3.82).

Consequently, due to (3.87), we define \( \eta \) and \( \delta \) according to (3.85) and (3.86) so that (3.1) is valid for all \( T < T^* \). With the use of the global estimates of \( a \) and \( \nu \), we conclude that \( T^* = +\infty \) and hence (3.1) is satisfied for all time.

The proof of Theorem 1.2 is complete.

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