TENSOR CATEGORIES ATTACHED TO EXCEPTIONAL CELLS
IN WEYL GROUPS

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Abstract. Using truncated convolution of perverse sheaves on a flag variety
Lusztig associated a monoidal category to a two sided cell in the Weyl group.
We identify this category in the case which was not decided previously.

1. Introduction

Let $W$ be a finite irreducible crystallographic Coxeter group and let $\text{Irr}(W)$ be
the set of isomorphism classes of irreducible representations of $W$ over $\mathbb{Q}$. It is
well known that two irreducible representations of dimension 512 for $W$ of type $E_7$
and four irreducible representations of dimension 4096 for $W$ of type $E_8$ behave
differently from other elements of $\sqcup_W \text{Irr}(W)$ in many ways, see e.g. [Cu], [BL], [L1,
Theorem 4.23] (see [L1, p. 109] for the definition of function $\Delta$). For this reason
the representations above are called exceptional. In this note we give one more
example of unusual behavior of exceptional representations or rather of associated
geometric objects.

Recall that the group $W$ is partitioned into two sided cells, $W = \sqcup c$, see e.g. [L1,
p. 137]. We say that a two sided cell $c$ is exceptional if the corresponding family
(see [L1, 5.15, 5.25]) consists of exceptional representations of $W$. Thus there are
just three exceptional two sided cells, one for $W$ of type $E_7$ and two for $W$ of type
$E_8$, see [L1, Chapter 4].

We fix an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $G$ be a simple
algebraic group over $\mathbb{C}$ with the Weyl group $W$. Using truncated convolution of
perverse $\mathbb{K}$–sheaves (in the classical topology) on the flag variety of $G$, Lusztig
associated to each two sided cell $c \subset W$ a semisimple monoidal category $P_c$
(over $\mathbb{K}$), see [L4] and [2.3 below]. Moreover, Lusztig conjectured that there is a tensor
equivalence $P_c \simeq \text{Coh}_{A(c)}(Y \times Y)$ where $A(c)$ is a finite group associated with family
corresponding to $c$ in [L1, Chapter 4], $Y$ is a finite $A(c)$-set and $\text{Coh}_{A(c)}(Y \times Y)$
is the category of $A(c)$-equivariant sheaves on the set $Y \times Y$ with convolution as a
tensor product and a natural associativity constraint, see [L4, §3.2]. This conjecture
was verified in [BFO1] for all non-exceptional two sided cells.

Let $c$ be an exceptional cell. Then $A(c) = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order
2. The Lusztig’s conjecture from [L4] predicts that we have a tensor equivalence
$P_c \simeq \text{Coh}_{A(c)}(Y \times Y)$ where $Y$ is a finite set with free $A(c) = \mathbb{Z}/2\mathbb{Z}$-action (the set
$Y$ should be of cardinality 1024 if $W$ is of type $E_7$ and of cardinality 8192 if $W$ is
of type $E_8$). Our main goal is to show that one needs to change the associativity
constraint a little bit in order to make this statement correct.
Let $Y'$ be a finite set of cardinality 512 if $W$ is of type $E_7$ and of cardinality 4096 if $W$ is of type $E_8$ (one can identify $Y'$ with the set of $Z/2Z$-orbits on the set $Y$). Let $\text{Vec}_{Z/2Z}$ be the monoidal category of finite dimensional $Z/2Z$-graded spaces. Then there is a tensor equivalence $\text{Coh}_{Z/2Z}(Y \times Y) \simeq \text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$. The category $\text{Vec}_{Z/2Z}$ has two simple objects, the unit object $\text{1}$ and one more simple object $\delta$. Let $\text{Vec}_{Z/2Z}$ be the same as category $\text{Vec}_{Z/2Z}$ but with modified associativity constraint: for simple objects $X, Y, Z$ the associativity constraint $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ is the same as in the category $\text{Vec}_{Z/2Z}$ if at least one of $X, Y, Z$ is isomorphic to $\text{1}$ and the associativity constraint $(\delta \otimes \delta) \otimes \delta \simeq \delta \otimes (\delta \otimes \delta)$ in $\text{Vec}_{Z/2Z}$ differs by sign from the one in $\text{Vec}_{Z/2Z}$. Here is our main result:

**Theorem 1.1.** For an exceptional two sided cell $c$ there is a tensor equivalence $\mathcal{P}^c \simeq \text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$.

**Remark 1.2.** (i) It will be clear from the proof that the category $\text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$ is not equivalent to $\text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$.

(ii) Using the methods from [BFO1] one can show that for an exceptional two sided cell $c$ one has either $\mathcal{P}^c \simeq \text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$ or $\mathcal{P}^c \simeq \text{Vec}_{Z/2Z} \boxtimes \text{Coh}(Y' \times Y')$. Thus our main result is just a computation of the sign in the associativity constraint.

(iii) Lusztig’s construction of the category $\mathcal{P}^c$ and our arguments extend with trivial changes to the settings of $D$-modules ($G$ is defined over an algebraically closed field of characteristic zero) or $\ell$-adic sheaves ($G$ defined over an algebraically closed field in which $\ell$ is invertible).

(iv) There is an alternative definition of the category $\mathcal{P}^c$ based on idea of A. Joseph. Namely, it is proved in [BFO2, Corollary 4.5(b)] that ($D$-module counterpart of) the category $\mathcal{P}^c$ is tensor equivalent to certain subquotient of the category of Harish-Chandra bimodules with tensor product coming from the usual tensor product of bimodules.

(v) The arguments in the proof of Theorem 1.1 extend to the case of monodromic sheaves on the flag variety as in [BFO1, §5]. Thus we get a proof of statement in [BFO1, Remark 5.7].

A direct computation of the associativity constraint in the category $\mathcal{P}^c$ seems to be difficult. Thus our proof of Theorem 1.1 is indirect. Our main tool is the theory of (unipotent) character sheaves developed by Lusztig [L3]. We use a commutator functor (see [BFO1] §6]) from $\mathcal{P}^c$ to the category of sheaves on the group $G$ (values of this functor are direct sums of character sheaves on $G$). Our main observation is that a commutator functor carries a canonical automorphism and keeping track of the order of this automorphism is sufficient in order to distinguish between categories $\text{Vec}_{Z/2Z}$ and $\text{Vec}_{Z/2Z}$.

2. Proofs

2.1. Commutator functors. Let $\mathcal{C}$ be a monoidal category with associativity isomorphism $a$ and let $\mathcal{A}$ be a category.

**Definition 2.1.** ([BFO1] §6]) A commutator functor $F : \mathcal{C} \to \mathcal{A}$ is a functor endowed with a natural isomorphism $u_{X,Y} : F(X \otimes Y) \simeq F(Y \otimes X)$ such that the
follow ing diagram commutes:

\[
\begin{array}{ccc}
F(X \otimes (Y \otimes Z)) & \xrightarrow{F(a_{X,Y,Z})} & (X \otimes Y) \otimes Z \\
\downarrow u_{X,Y,Z} & & \downarrow u_{X,Y,Z} \\
F(Z \otimes (X \otimes Y)) & \xrightarrow{F(a^1_{Z,X,Y})} & F((Z \otimes X) \otimes Y) \\
\downarrow u_{Z \otimes X,Y} & & \downarrow u_{Z \otimes X,Y} \\
F(Y \otimes (Z \otimes X)) & \xrightarrow{F(a_{Y,Z,X})} & F((Y \otimes Z) \otimes X) \\
\end{array}
\]

**Remark 2.2.** It follows immediately from definition that \( u_{X,1} : F(X) \to F(X) \) is an idempotent and hence an identity map.

**Definition 2.3.** The map \( u_{1,X} : F(X) \to F(X) \) is called **canonical automorphism** of the commutator functor \( F \).

**Remark 2.4.** It follows from definition that \( u_{Y,X} \circ u_{X,Y} = u_{1,X} \otimes Y \).

Recall that a **central functor** \( G : \mathcal{A} \to \mathcal{C} \) is a functor endowed with functorial isomorphism \( v_{A,X} : G(A) \otimes X \simeq X \otimes G(A) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G(A) \otimes (X \otimes Y) & \xrightarrow{a_{G(A),X,Y}} & (G(A) \otimes X) \otimes Y \\
\downarrow v_{A,X} \otimes \text{id}_Y & & \downarrow v_{A,X} \otimes \text{id}_Y \\
(X \otimes G(A)) \otimes Y & \xrightarrow{a_{X,G(A),Y}} & X \otimes (Y \otimes G(A)) \\
\downarrow \text{id}_X \otimes v_{A,Y} & & \downarrow \text{id}_X \otimes v_{A,Y} \\
X \otimes (G(A) \otimes Y) & & \\
\end{array}
\]

Equivalently, the structure of central functor on \( G \) is the same as factorization \( \mathcal{A} \to Z(C) \to \mathcal{C} \) where \( Z(C) \) is the Drinfeld center of \( \mathcal{C} \) and \( Z(C) \to \mathcal{C} \) is the forgetful functor, see e.g. [ENO] §2.3.

**Proposition 2.5.** ([BFOTI] §6) Assume that the category \( \mathcal{C} \) is rigid and has a pivotal structure (that is an isomorphism of tensor functors \( \text{Id} \to ** \)). Let \( (G,F) \) be an adjoint pair of functors between \( \mathcal{C} \) and \( \mathcal{A} \). Then the structures of commutator functor on \( F \) are in natural bijection with structures of central functor on \( G \).

**Proof.** Assume that \( G \) has a central structure. Then

\[
\begin{align*}
\text{Hom}(?, F(X \otimes Y)) &= \text{Hom}(G(?), X \otimes Y) = \text{Hom}(G(?), *Y, X) \\
&= \text{Hom}(Y \otimes G(?), X) = \text{Hom}(G(?), *Y \otimes X) = \text{Hom} (?, F(Y \otimes X))
\end{align*}
\]

and it is straightforward to check that the resulting isomorphism \( F(X \otimes Y) \simeq F(Y \otimes X) \) satisfies the definition of commutator functor.
Conversely, assume that $F$ has a commutator structure. Then
$$\text{Hom}(G(A) \otimes X, ?) = \text{Hom}(G(A), ? \otimes X^*) = \text{Hom}(A, F(? \otimes X^*))$$
$$= \text{Hom}(A, F(X^* \otimes ?)) = \text{Hom}(G(A), X^* \otimes ?) = \text{Hom}(X \otimes G(A), ?)$$
and it is straightforward to check that the resulting isomorphism $G(A) \otimes X \simeq X \otimes G(A)$ satisfies the definition of central functor.

Finally, one verifies that the two constructions above are mutually inverse. $\square$

Let $C$ be a pivotal category. Assume that the forgetful functor $Z(C) \to C$ has a right adjoint functor $\text{Ind} : C \to Z(C)$. It follows from Proposition 2.5 that the functor $\text{Ind}$ has a canonical structure of commutator functor. Clearly, for any functor $Z(C) \to A$ the composition $C \xrightarrow{\text{Ind}} Z(C) \to A$ has a structure of commutator functor. Moreover, we have the following universal property:

**Corollary 2.6.** Let $F : C \to A$ be a commutator functor such that left adjoint of $F$ exists. Then $F$ factorizes as $C \xrightarrow{\text{Ind}} Z(C) \to A$. $\square$

Let $D$ be a ribbon category, for example Drinfeld center $Z(C)$ of a spherical category $C$. Recall that there is a canonical automorphism $\theta$ of the identity functor called *twist* defined as following composition:

$$X \to **X \otimes ^*X \otimes X \to **X \otimes X \otimes ^*X \to **X = X.$$ 

One can rewrite this definition in terms of functor represented by $X$ as follows:

$$\text{Hom}(?, X) = \text{Hom}(? \otimes ^*X, 1) = \text{Hom}(^*X \otimes ?, 1) = \text{Hom}(?, **X).$$

Thus we have the following

**Corollary 2.7.** The canonical automorphism of the commutator functor $\text{Ind} : C \to Z(C)$ equals the twist automorphism. $\square$

**Example 2.8.** (a) Let $C = \text{Vec}_{\hat{Z}/2\hat{Z}}$. It is well known that the category $C$ is spherical (in two different ways) and the twist of any simple object of $Z(C)$ is $\pm 1$ (for any choice of the spherical structure). It follows that for any commutator functor $F : C \to A$ such that left adjoint of $F$ exists the square of the canonical automorphism is the identity map.

(b) Let $C = \text{Vec}_{\hat{Z}/2\hat{Z}}$. The category $C$ has two spherical structures and the twists of simple objects of $Z(C)$ are $1$ and $\pm \sqrt{-1}$. Thus there exists a commutator functor $F : C \to A$ such that left adjoint of $F$ exists and the square of the canonical automorphism is not the identity map, for example $F = \text{Ind}$.

### 2.2. Commutator functor $\Gamma$ and its canonical automorphism.

Let $G$ be a simple algebraic group over an algebraically closed field and let $B \subset G$ be a Borel subgroup. The group $B \times B$ acts on $G$ via left and right translations. We consider the subcategories $C_{B \times B}$ and $C_G$ of the bounded derived category of constructible sheaves on $G$ consisting of complexes that are isomorphic to a finite direct sum of shifts of simple, $B \times B$-equivariant (respectively, $G$-equivariant with respect to the adjoint action) perverse sheaves on $G$.

We recall now that the category $C_{B \times B}$ has a natural monoidal structure with tensor product given by the convolution, see e.g. [L4, Section 1]. The group $B$ acts freely on $G \times G$ as follows: $b \cdot (x, y) = (xb, b^{-1}y)$. Let $G \times_B G$ denote the quotient of $G \times G$ by this action and let $\pi : G \times G \to G \times_B G$ be the natural projection. Clearly, the multiplication map $m(x, y) = xy$ is a well defined map.
m : G × B G → G. For F₁, F₂ ∈ ℂ_{B × B} let F₁ ⊠ F₂ be the unique complex of sheaves on G × B G such that π⁺F₁ ⊠ F₂ = F₁ × F₂. By definition, the convolution of F₁ and F₂ is F₁ * F₂ := m₁F₁ ⊠ F₂. It follows from the Decomposition Theorem (see [BBD]) that F₁ * F₂ ∈ ℂ_{B × B} and it is clear that the convolution is a bifunctor.

Let G × B G × B G be the quotient of G × G × G by the free B × B-action via (b₁, b₂) · (x, y, z) = (b₁⁻¹b₂⁻¹y, b₂⁻¹z) with obvious projection π₂ : G × G × G → G × B G × B G and multiplication map m₂ : G × B G × B G → G, m₂(x, y, z) = xyz. For F₁, F₂, F₃ ∈ ℂ_{B × B} the convolutions (F₁ * F₂) * F₃ and F₁ * (F₂ * F₃) are both canonically isomorphic to (m₂₁(F₁ ⊠ F₂) ⊠ F₃) where F₁ ⊠ F₂ ⊠ F₃ is the unique complex on G × B G × B G such that π₂₁(F₁ ⊠ F₂) ⊠ F₃ = F₁ ⊠ F₂ ⊠ F₃. Identifying (F₁ * F₂) * F₃ and F₁ * (F₂ * F₃) via these isomorphisms we endow the category ℂ_{B × B} with associativity constraint satisfying the pentagon axiom. Moreover, ℂ_{B × B} is a monoidal category with unit object given by the constant sheaf on B ∈ G.

**Remark 2.9.** The indecomposable objects of the category ℂ_{B × B} are IC_w[i] where IC_w is the intersection cohomology complex (see [BBD]) of the Bruhat cell corresponding to w ∈ W and [i] stands for the shift. It is well known (see e.g., [L4 §1.4]) that the (split) Grothendieck ring of the category ℂ_{B × B} identifies with Hecke algebra; under this identification the objects IC_w[i] correspond to the elements of Kazhdan-Lusztig basis multiplied by i-th power of the Hecke algebra parameter.

There is an equivariantization functor Γ : ℂ_{B × B} → ℂ_G defined as follows (see [MV] Section 1). Let G × B G be the quotient of G × G by the following free B-action: b ◦ (g, x) = (b⁻¹g, b⁻¹xb). We have a canonical projection ˜π : G × G → G × B G and the adjoint action map a : G × B G → G, a(g, x) = g⁻¹xg. For F ∈ ℂ_{B × B} let ˜F be a unique complex on G × B G such that ˜π⁺ ˜F = k ⊠ F. We set Γ(F) : = a( ˜F). It is clear that Γ(F) is G-equivariant complex on G (with respect to the adjoint action) and the Decomposition Theorem (see [BBD]) implies that Γ(F) is semisimple. In other words, Γ is a functor ℂ_{B × B} → ℂ_G.

**Remark 2.10.** We recall that irreducible constituents of perverse cohomology of complexes Γ(F) are by definition character sheaves (with trivial central character, or unipotent) on G, see [L3] Sections 2, 11 and [MV] Lemma 2.3.

We now define commutator structure on the functor Γ. Let G × B(G × B G) be the quotient of G × G × G with respect to the following free B × B-action: (b₁, b₂) · (g, x, y) = (b⁻¹₁b⁻¹₂xb₁⁻¹y, b⁻¹₂yb₁). We have an obvious projection ˜π₂ : G × G → G × B(G × B G) and a well defined map a₂ : G × B(G × B G) → G, a₂(g, x, y) = g⁻¹xyg. We have an obvious isomorphism Γ(F₁ * F₂) = (a₂)!(F₁ ⊠ F₂) where F₁ ⊠ F₂ is a unique complex on G × B(G × B G) such that ˜π₂(F₁ ⊠ F₂) = k ⊠ F₁ ⊠ F₂.

Consider the following (well defined) map ρ : G × B(G × B G) → G × B(G × B G), ρ(g, x, y) = (yg, y, x). For F₁, F₂ ∈ ℂ_{B × B} there is a unique isomorphism ˜u_{F₁, F₂} : ρ(F₁ ⊠ F₂) ⊠ F₁ ∼ F₁ ⊠ F₂ inducing the identity map in every stalk (clearly, the stalks of both ρ(F₁ ⊠ F₂) and F₁ ⊠ F₁ at (g, x, y) ∈ G × B(G × B G) are canonically isomorphic to (F₂)x ⊠ (F₁)y). Now observe that a₂ ◦ ρ = a₂. Thus we have an isomorphism u_{F₁, F₂} : Γ(F₁ * F₂) ∼ Γ(F₂ * F₁) defined as composition

Γ(F₁ * F₂) = (a₂)!(F₁ ⊠ F₂) = (a₂ ◦ ρ)!(F₁ ⊠ F₂) ∼ (a₂)(F₁ ⊠ F₂) = Γ(F₂ * F₁).

The following result is straightforward:
Proposition 2.11. The functor $\Gamma$ together with isomorphism $u_{\bullet, \bullet}$ is a commutator functor $\mathcal{C}_{B \times B} \to \mathcal{C}_G$. □

There is a canonical automorphism $\Theta$ of the identity functor of the category $\mathcal{C}_G$ defined as follows (see [O2 Definition 2.1]): for any $F \in \mathcal{C}_G$ by definition of equivariance there is an isomorphism $ad^*(F) \simeq p^*(F)$ where $p : G \times G \to G$ is the second projection and $ad : G \times G \to G$ is the adjoint action map, $ad(g, x) = g^{-1}xg$; taking pullback of this isomorphism with respect to the diagonal map $\Delta : G \to G \times G$ and noticing that $\Delta \circ ad = \Delta \circ p = \text{Id}$ we get an automorphism $\Theta_F : F \to F$. In other words, the automorphism $\Theta_F$ at the stalk $F_x$ is precisely $ad(x) : F_x \simeq F_{x^{-1}x}$. We have the following

Proposition 2.12. The canonical automorphism of the commutator functor $\Gamma$ equals $\Theta$, that is, $u_{1, F} = \Theta_{\Gamma(F)}$. □

Remark 2.13. Our construction of commutator structure on the functor $\Gamma$ is just a more explicit version of construction in [BFO1 §6] (we will not need this fact in what follows). The advantage of the present version is that it makes Proposition 2.12 easy.

2.3. Truncated convolution and equivariantization. We recall (see e.g. [L1 Chapter 5]) that the Weyl group $W$ is partitioned into two sided cells, $W = \mathcal{C} \mathcal{E}$. Remind that there is a partial order $\leq_{LR}$ on the set of two sided cells. Let $a(c)$ denote the common value of Lusztig’s $a$-function on $w \in W$, see [L4 §2.3].

Let $\mathcal{P}_{B \times B}$ denote the full subcategory of $\mathcal{C}_{B \times B}$ consisting of perverse sheaves. Let $\mathcal{P}^c_{B \times B}$ denote the full subcategory of $\mathcal{P}_{B \times B}$ consisting of direct sums of perverse sheaves $IC_{w}$, $w \in c$ and let $pr^c : \mathcal{P}_{B \times B} \to \mathcal{P}^c_{B \times B}$ be the obvious projection functor.

Let $^pH$ denote the perverse cohomology functor. Consider the following bifunctor

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := pr^c((^pH^a(c))(\mathcal{F}_1 \ast \mathcal{F}_2)).$$

It is explained in [L4 Section 2] that the associativity constraint of the convolution category $\mathcal{C}_{B \times B}$ restricts to the associativity constraint for the bifunctor $\otimes$. Moreover, there exists the unit object $1_c \in \mathcal{P}^c_{B \times B}$ (see [L4 §2.9]), so the category $\mathcal{P}^c_{B \times B}$ has a monoidal structure. We have the following result

Proposition 2.14. ([BFO1 p. 222]) The monoidal category $\mathcal{P}^c_{B \times B}$ is rigid. Hence $\mathcal{P}^c_{B \times B}$ is a multi-fusion category in a sense of [ENO §2.4]. □

Remark 2.15. The Grothendieck ring of the category $\mathcal{P}^c_{B \times B}$ identifies with Lusztig’s asymptotic Hecke ring $\mathcal{H}_c$, see [L4 §2.6]. This together with [L1 Corollary 12.16] implies that the multi-fusion category $\mathcal{P}^c_{B \times B}$ is indecomposable in the sense of [ENO §2.4].

Let $\mathcal{P}_G$ be the full subcategory of $\mathcal{C}_G$ consisting of direct sums of (unshifted) unipotent character sheaves, see Remark 2.10. It is known (see [L3 Section 16], [G §3.4]) that there is a unique direct sum decomposition (or, equivalently, partition of the set of isomorphism classes of unipotent character sheaves) $\mathcal{P}_G = \bigoplus_c \mathcal{P}^c_G$ with the following properties:

(i) For $w \in c$ and $i \in \mathbb{Z}$ we have $^pH^i(\Gamma(\mathcal{C}_w)) \in \bigoplus_{c' \leq_{LR} c} \mathcal{P}^c_G$; moreover $^pH^i(\Gamma(\mathcal{C}_w)) \in \bigoplus_{c' \leq_{LR} c} \mathcal{P}^c_G$ if $|i| > a(c)$. 

Lemma 2.18. We see that this category has precisely 2 isomorphism classes of simple objects. The order of the canonical automorphism of \( \Gamma_c \) is 4. The Grothendieck ring of the category \( \mathcal{P}_G \) is isomorphic to \( \text{Vec}_{Z/2} \). We fix such \( i \).

Clearly \( 1_1 \otimes \mathcal{P}_c \otimes 1_i \subset \mathcal{P}_c \) is a tensor subcategory with unit object \( 1_i \) and the restriction \( \Gamma^c|_{1_1 \otimes \mathcal{P}_c \otimes 1_i} \) is a commutator functor. The choice of \( i \) guarantees that the order of the canonical automorphism of \( \Gamma^c|_{1_1 \otimes \mathcal{P}_c \otimes 1} \) is 4. The Grothendieck ring of the category \( \mathcal{P}_c \) is isomorphic to \( \text{Vec}_{Z/2} \). We see that this category has precisely 2 isomorphism classes of simple objects \( 1_j \) and \( \delta_i \), furthermore \( \delta_i \otimes \delta_i \simeq 1_i \). The following result is well known; it is a special case of results of [S].

Lemma 2.18. Let \( \mathcal{C} \) be a fusion category with two (isomorphism classes of) simple objects: the unit object \( 1 \) and one more object \( \delta \) such that \( \delta \otimes \delta \simeq 1 \). Then either \( \mathcal{C} \simeq \text{Vec}_{Z/2} \) or \( \mathcal{C} \simeq \text{Vec}_{Z/2} \).

In view of Example 2.13 we see that \( 1_i \otimes \mathcal{P}_c \otimes 1_i \) can not be equivalent to \( \text{Vec}_{Z/2} \) (notice that the commutator functor \( \Gamma^c|_{1_1 \otimes \mathcal{P}_c \otimes 1_i} \) has the left adjoint since it is a functor between semisimple categories). Hence \( 1_i \otimes \mathcal{P}_c \otimes 1_i \simeq \text{Vec}_{Z/2} \).
Let $\mathcal{C}$ be a multi-fusion category and let $e \in \mathcal{C}$ be a direct summand of the unit object. Then $e \otimes \mathcal{C} \otimes e$ is a tensor subcategory of $\mathcal{C}$ and $e \otimes \mathcal{C}$ is a module category over $e \otimes \mathcal{C} \otimes e$ via the left multiplication, see \cite{O1}. Each object $X \in \mathcal{C}$ gives rise to a functor $\otimes X : e \otimes \mathcal{C} \to e \otimes \mathcal{C}$ commuting with the module structure above. Thus we get a tensor functor from $e$ to the category $\text{Fun}_{e \otimes \mathcal{C} \otimes e}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$ of $e \otimes \mathcal{C} \otimes e$-module endofunctors of $e \otimes \mathcal{C}$. The following result is well known:

**Lemma 2.19.** Assume that multi-fusion category $\mathcal{C}$ is indecomposable. The functor above is an equivalence $\mathcal{C} \simeq \text{Fun}_{e \otimes \mathcal{C} \otimes e}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$.

**Proof.** The category $e \otimes \mathcal{C}$ is a right module category over $\mathcal{C}$ and the obvious functor $e \otimes \mathcal{C} \otimes e \to \text{Fun}_{e \otimes \mathcal{C} \otimes e}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$ is an equivalence with the inverse functor $F \mapsto F(e)$. In the language of \cite{O1} §4.2, this means that the category $e \otimes \mathcal{C} \otimes e$ is dual to the category $\mathcal{C}$. Now, the result follows by duality, see \cite{O1} Corollary 4.1. $\square$

Thus we can reconstruct multi-fusion category $\mathcal{C}$ from smaller category $e \otimes \mathcal{C} \otimes e$ and module category $e \otimes \mathcal{C}$. We apply this to the case $\mathcal{C} = P^e$ and $e = 1_i$. It is well known (see e.g. \cite{O1} Remark 6.2(iii)) that the fusion category $1_i \otimes P^e \otimes 1_i \simeq \text{Vec}_Z^{(2 \mathbb{Z})}$ has only one indecomposable semisimple module category, namely the regular module category $\text{Vec}_Z^{(2 \mathbb{Z})}$. Hence $P^e \simeq \text{Fun}_{\text{Vec}_Z^{(2 \mathbb{Z})}}(M, M)$ where $M$ is a direct sum of several copies of this module category (the number of indecomposable summands in $M$ equals the number of irreducible summands in $1_i$). The same argument applies to the multi-fusion category $\text{Vec}_Z^{(2 \mathbb{Z})} \boxtimes \text{Coh}(Y' \times Y')$ (in this case the irreducible summands of the unit object are naturally labeled by the elements of the set $Y'$). Thus choosing the set $Y'$ of appropriate size we get

$$P^e \simeq \text{Fun}_{\text{Vec}_Z^{(2 \mathbb{Z})}}(M, M) \simeq \text{Vec}_Z^{(2 \mathbb{Z})} \boxtimes \text{Coh}(Y' \times Y')$$

and Theorem 1.1 is proved.

**Remark 2.20.** In \cite{BFO2} Section 4] the category $P^c_G$ is endowed with tensor structure; moreover \cite{BFO2} Theorem 5.3 states that $P^c_G \simeq Z(P^e)$ and the functor $\Gamma^e$ is isomorphic to the induction functor $\text{Ind} : P^e \to Z(P^e) \simeq P^c_G$ (one can use it to give a shorter proof of Theorem 1.1). This implies that for an exceptional cell $c$ we have $P^c \simeq Z(\text{Vec}_Z^{(2 \mathbb{Z})})$ (see \cite{BFO2} Corollary 5.4(b)] for the case when $c$ is not exceptional).

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