ON THE CAUCHY PROBLEM FOR THE SCHRÖDINGER-HARTREE EQUATION

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Abstract. In this paper, we undertake a comprehensive study for the Schrödinger-Hartree equation

\[ iu_t + \Delta u + \lambda (I_\alpha * |u|^p)|u|^{p-2}u = 0, \]

where \( I_\alpha \) is the Riesz potential. Firstly, we address questions related to local and global well-posedness, finite time blow-up. Secondly, we derive the best constant of a Gagliardo-Nirenberg type inequality. Thirdly, the mass concentration is established for all the blow-up solutions in the \( L^2 \)-critical case. Finally, the dynamics of the blow-up solutions with critical mass is in detail investigated in terms of the ground state.

1. Introduction. In this paper, we consider the Cauchy problem for the following Schrödinger-Hartree equation

\[
\begin{aligned}
&iu_t + \Delta u + \lambda (I_\alpha * |u|^p)|u|^{p-2}u = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \\
&u(0, x) = \varphi(x),
\end{aligned}
\]

(1)

where \( u : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{C} \) is the complex valued function, \( N \geq 3, \varphi \in H^1 \) and \( 0 < T \leq \infty, \lambda \in \mathbb{R}, 2 \leq p < \frac{N+\alpha}{N-2}, I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R} \) is the Riesz potential defined by

\[
I_\alpha(x) = \frac{\Gamma(N-\alpha)}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}},
\]

with \( \max\{0, N-4\} < \alpha < N \) and \( \Gamma \) is the Gamma function.

Equation (1) has several physical origins. In the physical case \( N = 3, p = 2 \) and \( \alpha = 2 \), equation (1) was originally proposed by Pekar around 1954 in the framework of quantum mechanics (see [14] for references). Subsequently, in 1976, P. Choquard [19] used this equation to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma. In 1996, R. Penrose [25] proposed this equation to describe self-gravitating matter in a programme in which quantum state reduction is understood as a gravitational phenomenon.

When \( p = 2 \), equation (1) reduces to the well-known Hartree equation. The Cauchy problem has been extensively studied in [4, 5, 7, 20, 22, 23, 1]. The local well-posedness and asymptotical behavior of the solutions for the Cauchy problem

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The soliton dynamics was investigated in [1]. They proved the local and global well-posedness, the existence of blow-up solutions, and considered the problem (1). They showed the existence of the standing wave solutions, the existence of blow-up solutions, and the problem (1) were established in [4, 7]. Lions [20] showed the existence of the standing waves. The global well-posedness and scattering for the defocusing, energy critical Hartree equation were discussed by Miao et al. [22]. Miao et al. [23] investigated the dynamics of the blow-up solutions with minimal mass for the focusing $L^2$-critical Hartree equation.

When $N \geq 3$, $\alpha = 2$, $2 \leq p < \frac{N+2}{N-2}$, Genev and Venkov [8] studied the Cauchy problem (1). They proved the local and global well-posedness, the existence of standing wave solutions, the existence of blow-up solutions, and considered the dynamics of the blow-up solutions in the $L^2$-critical case, i.e., $p = 1 + \frac{4}{N}$. Notice that these assumptions on $N$, $\alpha$ and $p$ imply $N \in \{3, 4, 5\}$. In particular, when $p = 1 + \frac{4}{N}$, $N$ only belongs to $\{3, 4\}$.

For the general case $0 < \alpha < N$ and $1 + \frac{\alpha}{N} \leq p < \frac{N+2}{N-2}$, there are few results known for (1). Under the assumption that (1) admits a unique solution $u \in C([0, T), H^1)$ for some $T > 0$, Chen and Guo [5] studied the existence of blow-up solutions and the strong instability of standing waves. When $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p < 1 + \frac{2+\alpha}{N}$, the soliton dynamics for (1) has been investigated in [2] under the assumption that the solution $u$ of (1) is in $C([0, \infty), H^2) \cap C^1((0, \infty), L^2)$.

In general, equation (1) admits special regular solutions, which are called solitary waves. More precisely, these solutions have the form $u(t, x) = e^{i\omega t}Q(x)$, where $\omega > 0$ is the frequency and the profile $Q(x)$ is time-independent and solves the following elliptic equation

$$-\Delta Q + \omega Q - (I_\alpha * |Q|^p)|Q|^{p-2}Q = 0. \quad (2)$$

The solutions of (2) have recently been an object of various deep investigations from regularity, qualitative properties such as symmetry and asymptotic behavior and concentration properties of semiclassical states. We refer the reader to [24]. When $\alpha = 2$, the uniqueness of the ground state solutions of (2) has been proved in [8] by using Lieb’s method, which strongly depends on the specific nonlocal character of the nonlinear elliptic equation. However, $\alpha \neq 2$, the uniqueness is an open problem.

In this paper, following the framework for the study of nonlinear Schrödinger equation, we will systematically study the Cauchy problem (1) for general $\max\{0, N-4\} < \alpha < N$ and $2 \leq p < \frac{N+2}{N-2}$. We are interested in the local and global well-posedness, finite time blow-up and the dynamics of blow-up solutions. Our results give some answers to the questions in [2, 5]. In addition, compared with the results in [8, 23], there are some essential difficulties in investigating the dynamics of blow-up solutions to (1). For example, the arguments in [8, 23] strongly depend on the uniqueness of ground state solutions. But the uniqueness of the ground state solutions to (2) is not known, we have to use different techniques in the proofs of our results.

Firstly, consider the local well-posedness. There are two types of techniques for studying the local well-posedness for the following nonlinear Schrödinger equation:

$$\begin{cases}
iu_t + \Delta u + g(u) = 0, \\ u(0, x) = \varphi(x),
\end{cases} \quad (3)$$

where $\varphi \in H^1$. One is Kato’s method, based on a fixed point argument and Strichartz’s estimates, see [11] or Theorems 4.4.1 and 4.4.6 in [4]. This method requires that the nonlinearity $g(u)$ satisfies $W^{1,q}$ regularity for some $q \in (\frac{N+2}{2N}, 2)$, see (4.4.22) in [4]. The other is Cazenave’s method, based on a compactness argument, see Chapter 3 in [4]. This method requires that equation (3) has the Hamiltonian...
structure. However, the second method does not require that the nonlinearity \(g(u)\) satisfies \(W^{1,q}\) regularity. Therefore, we use the second method to show the local well-posedness for (1). According to Theorem 3.3.9 in [4], we need only check that the nonlinearity \((I_\alpha * |u|^p)|u|^{p-2}u\) satisfies conditions (3.3.5)-(3.3.8) in [4]. Then, we prove the uniqueness by using Strichartz’s estimates, see 4.2 in [4]. Our results are as follows:

**Theorem 1.1.** Let \(\max\{0, N-4\} < \alpha < N, 2 \leq p < \frac{N+\alpha}{N-2}\), and \(\varphi \in H^1\). Then, there exists \(T = T(||\varphi||_{H^1})\) such that (1) admits a unique solution \(u \in C([−T,T), H^1) \cap C^1(−T,T, H^{-1})\). Let \((-T_*, T_*)\) be the maximal time interval on which the solution \(u\) is well-defined. If \(T_* < \infty\), then \(||u(t)||_{H^1} \to \infty\) as \(t \uparrow T_*\); similarly, if \(T_* < \infty\), then \(||u(t)||_{H^1} \to \infty\) as \(t \downarrow T_*\). Moreover, \(u\) depends continuously on the initial data \(\varphi\) in the following sense: There exists \(T = T(||\varphi||_{H^1})\) such that if \(\varphi_n \to \varphi\) in \(H^1\) and if \(u_n\) is the solution of (1) with initial data \(\varphi_n\), then \(u_n\) is defined on \([−T,T]\) for sufficiently large and \(u_n \to u\) in \(C([−T,T], H^1)\). Finally, the solution \(u\) satisfies the following mass and energy conservation laws

\[
M(u(t)) = \int_{\mathbb{R}^N} |u|^2 dx = M(\varphi),
\]

\[
E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p} dx = E(\varphi).
\]

**Remark.** When \(\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}\), it follows from the inequality (12) that \(\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p} dx\) is well-defined for \(u \in H^1\). Therefore, we guess the results of this theorem also hold for \(\frac{N+\alpha}{N} < p < 2\). However, we cannot prove these results since the nonlinearity \((I_\alpha * |u|^p)|u|^{p-2}u\) is singular when \(\frac{N+\alpha}{N} < p < 2\). Consequently, the case of \(\frac{N+\alpha}{N} < p < 2\) will be the object of a future investigation.

Secondly, we investigate the global well-posedness. According to the local well-posedness, the time of existence for local solutions to (1) depends only on the \(H^1\) norm of the initial data. Therefore, in order to prove the global well-posedness of (1), it suffices to obtain a priori estimate on the kinetic energy. By the following generalized Gagliardo-Nirenberg inequality

\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p} dx \leq C_{\alpha,p} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{Np-N-n}{2p}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{N+n-Np+2p}{2}}
\]

and some standard arguments, we obtain the following theorem.

**Theorem 1.2.** Let \(\max\{0, N-4\} < \alpha < N, \varphi \in H^1\). Then, there exists a unique global solution \(u\) to (1) in each of the following cases:

**Case (1).** \(\lambda < 0, 2 \leq p < \frac{N+\alpha}{N-2}\);

**Case (2).** \(\lambda > 0, 2 \leq p < 1 + \frac{2+\alpha}{N}\);

**Case (3).** \(\lambda > 0, p = 1 + \frac{2+\alpha}{N} \geq 2\) and \(\lambda^{\frac{1}{p-1}}||\varphi||_{L^2} < ||Q||_{L^2}\), where \(Q\) is the ground state of elliptic equation (2);

**Case (4).** \(\lambda > 0, \max\{1 + \frac{2+\alpha}{N}, 2\} < p < \frac{N+\alpha}{N-2}\), \(||\varphi||_{H^1}\) is small enough.

Next, we consider the existence of blow-up solutions. Without loss of generality, we may assume \(\lambda = 1\). To prove the existence of blow-up solutions, we will follow the convexity method of Glassey [10]. More precisely, for strong \(H^1\) solution with
initial data \( \varphi \in \Sigma := \{ u : u \in H^1 \text{ and } |x|u \in L^2 \} \), we will consider the variance

\[
V(t) = \int_{\mathbb{R}^N} |x| |u(t,x)|^2 dx
\]

and show that as a function of \( t > 0 \), it is decreasing and concave, which suggests the existence of a blow-up time \( T^* \) at which \( V(T^*) = 0 \). We can prove the following theorem.

**Theorem 1.3.** Assume that \( \max\{0, N - 4\} < \alpha < N \) and \( \max\{1 + \frac{2 + \alpha}{N}, 2\} \leq p < \frac{N + \alpha}{N - 2} \). Let \( \varphi \in \Sigma \) and satisfy one of the following condition:

(i) \( E(\varphi) < 0 \);

(ii) \( E(\varphi) = 0 \) and \( \text{Im} \int_{\mathbb{R}^N} \varphi_x \cdot \nabla \varphi dx < 0 \);

(iii) \( E(\varphi) > 0 \) and \( \int_{\mathbb{R}^N} \varphi_x \cdot \nabla \varphi dx \leq -\sqrt{2E(\varphi)}\|x\varphi\|_{L^2} \).

Then the solution \( u \) of (1) blows up in finite time.

Next, we consider the \( L^2 \)-critical case, i.e., \( p = 1 + \frac{2 + \alpha}{N} \geq 2 \). In this case, much more information can be derived about the blow-up solutions and the ground state of (2) plays a crucial role in the study of the dynamics of the blow-up solutions.

According to Theorem 1.2, if \( \|\varphi\|_{L^2} < \|Q\|_{L^2} \), then the solution of (1) exists globally. On the other hand, after some basic calculations, it follows that if \( u(t, x) \) is a solution of (1), then so does

\[
\hat{u}(t, x) = \frac{1}{T-t} e^{-\frac{|x|^2}{4(T-t)}} u\left(\frac{1}{t} x, \frac{T}{T-t} \right).
\]

Therefore, the pseudo-conformal transformation also holds for (1). Applying this transformation to the standing waves solution \( e^{it}Q(x) \), we obtain for all positive \( T \) that

\[
\hat{u}(t, x) = \frac{1}{(T-t)^{\frac{3}{2}}} e^{\frac{-|x|^2}{4(T-t)}} Q\left(\frac{x}{T-t} \right)
\]

is a solution blowing up at the time \( T \) and \( \|\hat{u}\|_{L^2} = \|Q\|_{L^2} \). Moreover, Case (3) in Theorem 1.2 implies that this is the minimal mass solution blowing up in finite time. In addition, this implies that the result of Case (3) in Theorem 1.2 is sharp, in the sense that for any \( \rho \geq \|Q\|_{L^2} \), there exists \( \varphi \in H^1 \) such that \( \|\varphi\|_{L^2} = \rho \) and such that \( u \) blows up in finite time. Indeed, we deduce from (15) and (16) that \( E(Q) = 0 \). Let \( \rho > \|Q\|_{L^2} \), set \( \gamma = \rho/\|Q\|_{L^2} > 1 \), and consider \( \varphi_\rho = \gamma Q \). It follows that \( \|\varphi_\rho\|_{L^2} = \rho \) and

\[
E(\varphi_\rho) = \gamma^2 E(Q) + \frac{\gamma^2 - \gamma^{2p}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |Q|) |Q|^p dx < 0,
\]

and so the corresponding solution \( u \) of (1) with the initial data \( \varphi_\rho \) blows up in finite time.

In the following theorem, we study the mass concentration of the blow-up solutions for (1) in the \( L^2 \)-critical case. The main result reads

**Theorem 1.4.** Assume that \( N - 2 \leq \alpha < N \) and \( p = 1 + \frac{2 + \alpha}{N} \). Let \( u \) be a blow-up solution of (1) and \( T^* \) its blow-up time. Let \( \lambda(t) \) be a real-valued nonnegative function defined on \([0, T^*)\) satisfying \( \lim_{t \to T^*} \lambda(t) \|\nabla u(t)\|_{L^2} = \infty \). Then there is \( x(t) \in \mathbb{R}^N \) such that

\[
\liminf_{t \to T^*} \int_{|x-x(t)| \leq \lambda(t)} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2,
\]

where \( Q \) is the ground state of (2).
The corresponding result for the Schrödinger equation was proved by M. Weinstein in [21]. T. Hmidi and S. Keraani gave a direct and simplified proof in [13]. The corresponding result for the homogeneous and inhomogeneous Hartree equation was proved by Miao in [23] and Cao in [3] respectively. The corresponding result for the Davey-Stewartson system was proved by Li et al. in [15]. In addition, as a direct consequence, this theorem implies that the Case (3) in Theorem 1.2 holds.

Finally, we shall concentrate on the further analysis of the blow-up solution with critical mass \( \| \varphi \|_{L^2} = \| Q \|_{L^2} \). Note that when \( p = 2 \) or \( \alpha = 2 \), the characterization of the blow-up solutions with minimal mass depends heavily on the uniqueness of the ground state of equation (2) in [8, 23]. However, the uniqueness of ground states of (2) is not known, we cannot follow the method in [8, 23] to characterize the dynamics of the blow-up solutions. In the following theorem, we obtain that the blow-up solution \( \| u(t,x) \|_{L^2} \) like a \( \delta \)-function as \( t \to T^* \) at the point \( x = x_0 \), which implies that the point \( x_0 \) concentrates all mass of the blow-up solution of (1).

**Theorem 1.5.** Assume that \( N - 2 \leq \alpha < N \) and \( p = 1 + \frac{2 + \alpha}{N} \). Let \( \varphi \in \Sigma \) and \( u \) be the corresponding solution of (1), which blows up at finite time \( T^* \). If \( \| \varphi \|_{L^2} = \| Q \|_{L^2} \), then there exists \( x_0 \in \mathbb{R}^N \) such that

\[
|u(t,x)|^2 \to \| Q \|^2_{L^2} \delta_{x_0}
\]

in the sense of distribution as \( t \to T^* \).

The following theorem gives the lower bound for the blow-up rate of the blow-up solutions with critical mass \( \| \varphi \|_{L^2} = \| Q \|_{L^2} \).

**Theorem 1.6.** Assume that \( N - 2 \leq \alpha < N \) and \( p = 1 + \frac{2 + \alpha}{N} \). Let \( \varphi \in \Sigma \) and \( u \) be the corresponding solution of (1), which blows up at finite time \( T^* \). If \( \| \varphi \|_{L^2} = \| Q \|_{L^2} \), then there exists a constant \( C > 0 \) such that

\[
\| \nabla u(t) \|_{L^2} \geq \frac{C}{T^* - t}, \quad \forall t \in [0, T^*).
\]

**Remark.** Contrarily to the results obtained for example in [8, 23], we have no information about the limiting profile of the blow-up solutions of (1) with critical mass. This is due to the fact that we use no information about the uniqueness of the ground states, so we cannot conclude that the blow-up solution is close to some specific function up to scaling, translation and phase parameters.

This paper is organized as follows: in Section 2, we will collect some lemmas such as the Hardy-Littlewood-Sobolev inequality, and some properties of the ground state to (2). Next, we obtain the best constant for the generalized Gagliardo-Nirenberg inequality (6) by the ground state solutions. Finally, we establish a compactness Lemma, which is important to study the concentration of the blow-up solutions to (1). In section 3, we will prove the local and global well-posedness of (1). In addition, we obtain the \( H^2 \) regularity of the solutions to (1) in the \( L^2 \)-subcritical case, which give the answer to the question in [2]. In section 4, we will give the proof of the existence of the blow-up solutions and then give their concentration property in the \( L^2 \)-critical case. In addition, the dynamics of the blow-up solutions with critical mass will be investigated.

**Notation.** Throughout this paper, we use the following notation. \( C > 0 \) will stand for a constant that may be different from line to line when it does not cause any confusion. Since we exclusively deal with \( \mathbb{R}^N \), We often use the abbreviations \( L^r = L^r(\mathbb{R}^N) \) and \( W^{1,p} = W^{1,p}(\mathbb{R}^N) \). Given any interval \( I \subset \mathbb{R} \), the norms of
mixed spaces \( L^q(I, L^p(\mathbb{R}^N)) \) and \( L^q(I, W^{1,p}(\mathbb{R}^N)) \) are denoted by \( \| \cdot \|_{L^q(I, L^p)} \) and \( \| \cdot \|_{L^q(I, W^{1,p})} \) respectively. For any \( x \in \mathbb{R}^N, r > 0, B(x, r) \) denotes the ball in \( \mathbb{R}^N \) centered at \( x \) with radius \( r \).

2. Preliminaries. In this section, we recall some useful estimates. Firstly, we have the following Hardy-Littlewood-Sobolev inequality.

**Lemma 2.1.** Let \( 0 < \lambda < N \) and \( s, r > 1 \) be constants such that

\[
\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2.
\]

Assume that \( f \in L^r \) and \( g \in L^s \). Then

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) |x - y|^{-\lambda} g(y) dxdy \right| \leq C(N, s, \lambda) \| f \|_{L^r} \| g \|_{L^s}.
\]

See Lieb [18] for details of the proof.

Because the ground state of (2) plays an important role in the study of the dynamics of blow-up solutions to (1), in the following lemma we summarize some results obtained in [2] and [24].

**Lemma 2.2.** Let \( \alpha \in (0, N) \) and \( 1 + \frac{\alpha}{N} < p < 1 + \frac{2+\alpha}{N-2} \). It follows that (2) admits a ground state solution \( Q \) in \( H^1 \). Every ground state \( Q \) of (2) is in \( L^1 \cap C^\infty \), it has fixed sign and there exist \( x_0 \in \mathbb{R}^N \) and a monotone real function \( v \in C^\infty(0, \infty) \) such that for every \( x \in \mathbb{R}^N, Q(x) = v(|x - x_0|) \). Moreover, the \( L^2 \) norm of any ground state \( Q \) of (2) is the same.

Following the idea of Weinstein [26], the best constant for the generalized Gagliardo-Nirenberg inequality (6) can be obtained by considering the existence of the minimizer of the following functional

\[
J_{\alpha,p}(u) = \frac{\left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{2p-N-N-\alpha}}{\left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{2p-2N+p+2p}}.
\]

More precisely, we have the following theorem.

**Theorem 2.3.** Let \( Q \) be the ground state of the elliptic equation (2). It follows that the best constant in the generalized Gagliardo-Nirenberg inequality (6) is

\[
C_{\alpha,p} = \frac{2p}{2p - Np + N + \alpha} \left( \frac{2p - Np + N + \alpha}{Np - N - \alpha} \right)^{\frac{Np-N-\alpha}{2}} \| Q \|_{L^2}^{2-2p}.
\]

In particular, in the \( L^2 \)-critical case \( p = 1 + \frac{2+\alpha}{N} \), \( C_{\alpha,p} = p \| Q \|_{L^2}^{2-2p} \).

**Proof.** Firstly, from the Hardy-Littlewood-Sobolev inequality, the interpolation inequality and Sobolev imbedding, we have

\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p dx \leq C \| u \|_{L^2}^{2p-N} \| u \|_{L^\infty}^{2N-N-\alpha} \int_{\mathbb{R}^N} |\nabla u|^2 dx \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{Np-N-\alpha}{2}},
\]

which implies that the functional (13) is well-defined. Thus, we consider the variational problem

\[
J := \inf \{ J_{\alpha,p}(u), u \in H^1 \setminus \{0\} \}.
\]
The existence of a minimizer has been proved in [24]. In particular, the minimizer \( u \) satisfies
\[
\frac{dJ_{\alpha,p}(u + \varepsilon w)}{d\varepsilon}
|_{\varepsilon = 0} = 0, \text{ for all } w \in H^1.
\]

By some basic calculations, \( u \) satisfies the following elliptic equation
\[
- \frac{Np - N - \alpha}{2} \Delta u + \frac{N + \alpha - p(N - 2)}{2} u = pJ(I_\alpha * |u|^p)|u|^{p-2} u.
\]

Setting \( u = \mu v(\lambda x) \), where \( \lambda = (\frac{N+\alpha-p(N-2)}{Np-N-\alpha})^{1/2} \) and \( \mu = (\frac{(N+\alpha-pN+2p)^{\frac{\alpha}{2}+1}}{2pJ(I_\alpha * |u|^p)^{\frac{\alpha}{2}-\frac{1}{2}}} N_p^{\frac{\alpha}{2}} \).

Then, \( v \) satisfies (2).

Multiplying (2) by \( Q \) and by \( x \cdot \nabla Q \), and integrating by parts, we obtain the following Pohozaev’s identities
\[
\int_{\mathbb{R}^N} |\nabla Q|^2 dx + \int_{\mathbb{R}^N} |Q|^2 dx = \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p dx,
\]
and
\[
(N - 2) \int_{\mathbb{R}^N} |\nabla Q|^2 dx + N \int_{\mathbb{R}^N} |Q|^2 dx = \frac{N + \alpha}{p} \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p dx.
\]

These identities lead to relations
\[
\int_{\mathbb{R}^N} |\nabla Q|^2 dx = \frac{Np - N - \alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p dx = \frac{Np - N - \alpha}{2p - Np + N + \alpha} \int_{\mathbb{R}^N} |Q|^2 dx,
\]
and
\[
\int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p dx = \frac{2p}{2p - Np + N + \alpha} \int_{\mathbb{R}^N} |Q|^2 dx.
\]

Therefore, we derive the best constant \( C_{\alpha,p} \)
\[
C_{\alpha,p} = \frac{1}{J} = \frac{2p}{2p - Np + N + \alpha} \left( \frac{Np - N - \alpha}{Np - N - \alpha} \right)^{\frac{Np - N - \alpha}{2}} \|Q\|_{L^2}^{2-2p}.
\]

\[\square\]

Next, we shall give the profile decomposition of bounded sequences in \( H^1 \) proposed by Gérard [9], Hmidi and Keraani [13], which is important to study the concentration of the blow-up solutions to (1).

**Proposition 1.** Let \( \{u_n\}_{n=1}^\infty \) be a bounded sequence in \( H^1 \). Then, there exist a subsequence of \( \{u_n\}_{n=1}^\infty \) (still denoted by \( \{u_n\}_{n=1}^\infty \)), a family \( \{x_n^k\}_{k=1}^\infty \) of sequences in \( \mathbb{R}^N \) and a sequence \( \{J^j\}_{j=1}^\infty \) in \( H^1 \) such that
(i) for every \( k \neq j \), \( |x_n^k - x_n^j| \to +\infty \), as \( n \to \infty \);
(ii) for every \( l \geq 1 \) and every \( x \in \mathbb{R}^N \), we have
\[
u_n(x) = \sum_{j=1}^l U_j^l(x - x_n^j) + r_n^l, \tag{17}
\]
with \( \limsup_{n \to \infty} \|r_n^l\|_{L^q} \to 0 \) as \( l \to \infty \) for every \( q \in (2, \frac{2N}{N-2}) \). Moreover,
\[
\|u_n\|_{L^2}^2 = \sum_{j=1}^l \|U_j^l\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o(1), \tag{18}
\]
\[
\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^l \|\nabla U_j^l\|_{L^2}^2 + \|\nabla r_n^l\|_{L^2}^2 + o(1), \tag{19}
\]
Therefore, (20) follows.

\[ \int_{\mathbb{R}^N} I_\alpha \ast \left[ \sum_{j=1}^{l} |U_j(x - x_n^j)|^p \right] \sum_{j=1}^{l} |U_j(x - x_n^j)|^p \, dx \]

\[ = \sum_{j=1}^{l} \int_{\mathbb{R}^N} I_\alpha \ast |U_j(x - x_n^j)|^p |U_j(x - x_n^j)|^p \, dx + o(1), \quad (20) \]

where \( o(1) = o_n(1) \to 0 \) as \( n \to \infty \).

**Proof.** For the proof of (17)-(19), we refer the reader to [13]. We need only prove (20).

Using the elementary inequality

\[ \left| \sum_{j=1}^{l} a_j \right|^q - \left| \sum_{j=1}^{l} a_j \right|^q \leq C \sum_{j \neq k} |a_j| |a_k|^{q-1}, \]

and the pairwise orthogonality of the family \( \{x_n^j\}_{j=1}^{\infty} \), we can estimate as follows:

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \sum_{j=1}^{l} U_j(x - x_n^j) \right|^p \left| \sum_{j=1}^{l} U_j(y - x_n^j) \right|^p \frac{dx \, dy}{|x - y|^{\alpha}} \]

\[ \leq \sum_{j=1}^{l} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_j(x - x_n^j)|^p |U_j(y - x_n^j)|^p}{|x - y|^{\alpha}} \, dx \, dy \quad (21) \]

\[ + \sum_{j=1}^{l} \sum_{k \neq j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_j(x - x_n^j)| |U_k(x - x_n^j)|^{p-1} |U_j(y - x_n^j)|^p}{|x - y|^{\alpha}} \, dx \, dy \quad (22) \]

\[ + \sum_{j=1}^{l} \sum_{k \neq j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_j(y - x_n^j)| |U_k(y - x_n^j)|^{p-1} |U_j(x - x_n^j)|^p}{|x - y|^{\alpha}} \, dx \, dy \quad (23) \]

\[ + \sum_{j=1}^{l} \sum_{k \neq j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_j(x - x_n^j)|^p |U_k(y - x_n^j)|^p}{|x - y|^{\alpha}} \, dx \, dy. \quad (24) \]

By orthogonality, we have

\[ (211) \leq \sum_{j=1}^{l} \sum_{k \neq j} \left\| U_j(x - x_n^j) \right\| \left| U_j(y - x_n^j) \right|^{p-1} \left\| U_j(x - x_n^j) \right\| \left\| U_j(y - x_n^j) \right\| \to 0, \]

as \( n \to \infty \). (212) can be similarly estimated. Finally, we estimate

\[ (213) = \sum_{j=1}^{l} \sum_{k \neq j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_j(x)|^p |U_k(y)|^p}{|x - y - x_n^j + x_n^k|^{\alpha}} \, dx \, dy \]

\[ \leq \sum_{j=1}^{l} \sum_{k \neq j} C \frac{|U_j|^p |U_k|^p}{|x_n^j - x_n^k|^{\alpha}} \to 0, \quad as \ n \to \infty. \quad (25) \]

Therefore, (20) follows. \( \square \)

**Theorem 2.4.** Let \( p = 1 + \frac{2 \alpha}{N} \), \( \{u_n\}_{n=1}^{\infty} \) be a bounded sequence in \( H^1 \) and satisfy

\[ \limsup_{n \to \infty} \| \nabla u_n \|_{L^2}^2 \leq M, \quad \limsup_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha \ast |u_n|^p) |u_n|^p \, dx \geq m. \]
Then, there exists \( \{ x_n \}_{n=1}^{\infty} \subset \mathbb{R}^N \), such that, up to a subsequence
\[
u_n (\cdot + x_n) \to U
\]
with \( \| U \|_{L^2} \geq (\frac{m}{\rho M})^{\frac{1}{p-2}} \| Q\|_{L^2} \).

**Proof.** According to the profile decomposition (Proposition 2.4), we have
\[
u_n (x) = \sum_{j=1}^{l} U^j (x - x^j_n) + r^j_n,
\]
with \( \limsup_{n \to \infty} \| r^j_n \|_{L^2} \to 0 \) as \( l \to \infty \). By the inequality (12), we have
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} (I_{\alpha} * |r^j_n|^p) |r^j_n|^p \, dx \leq \limsup_{n \to \infty} C \| r^j_n \|_{L^{2p} U_{\rho M}}^{2p} \to 0, \text{ as } l \to \infty.
\]
From (20) and Theorem 2.3, we obtain
\[
m \leq \limsup_{n \to \infty} \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^p) |u_n|^p \, dx
\]
\[
\leq \limsup_{n \to \infty} \int_{\mathbb{R}^N} (I_{\alpha} * \left| \sum_{j=1}^{l} U^j (x - x^j_n) \right|^p) \left| \sum_{j=1}^{l} U^j (x - x^j_n) \right|^p \, dx
\]
\[
\leq \sum_{j=1}^{l} \int_{\mathbb{R}^N} (I_{\alpha} * |U^j (x - x^j_n)|^p) |U^j (x - x^j_n)|^p \, dx
\]
\[
\leq \sum_{j=1}^{\infty} \frac{p}{\| Q \|^{2p-2}_{L^2}} \| \nabla U^j \|^2_{L^2} \| U^j \|^{2p-2}_{L^2}
\]
\[
\leq \frac{p}{\| Q \|^{2p-2}_{L^2}} \sup \{ \| U^j \|^{2p-2}_{L^2}, j \geq 1 \} \sum_{j=1}^{\infty} \| \nabla U^j \|^2_{L^2}.
\]
(27)

On the other hand, we observe that
\[
\sum_{j=1}^{\infty} \| \nabla U^j \|^2_{L^2} \leq \limsup_{n \to \infty} \| \nabla u_n \|^2_{L^2} \leq M,
\]
which, together with (27), implies
\[
\sup \{ \| U^j \|^{2p-2}_{L^2}, j \geq 1 \} \geq \frac{m \| Q \|^{2p-2}_{L^2}}{\rho M}.
\]
Since the series \( \| U^j \|^{2p-2}_{L^2} \) is convergent, then the supremum above is attained. In particular, there exists \( j_0 \) such that
\[
\| U^{j_0} \|^{2p-2}_{L^2} \geq \frac{m \| Q \|^{2p-2}_{L^2}}{\rho M}.
\]
On the other hand, a change of variables gives
\[
u_n (x + x^{j_0}_n) = U^{j_0} (x) + \sum_{j \neq j_0} U^j (x + x^{j_0}_n - x^j_n) + r^j_n (x + x^{j_0}_n).
\]
Using the pairwise orthogonal \( \{ x^j_n \}_{j=1}^{\infty} \), we have
\[
U^j (\cdot + x^{j_0}_n - x^j_n) \to 0, \text{ weakly in } H^1, \text{ for every } j \neq j_0.
\]
Hence, we have
\[
u_n (\cdot + x^{j_0}_n) \to U^{j_0} + r^j, \text{ weakly in } H^1,
\]
where \( \bar{r}^l \) denotes the weak limit of \( r_n^l(x + x_n^{j_0}) \). However, we have

\[
\int_{\mathbb{R}^N} (I_\alpha * |\bar{r}^l|^p) |\bar{r}^l|^p \, dx \leq \| \bar{r}^l \|_{L^{2p/N}}^{2p} \leq \limsup_{n \to \infty} \| r_n^l \|_{L^{2p/N}}^{2p} \to 0.
\]

Thus, from uniqueness of weak limit, we have \( \bar{r}^l = 0 \) for all \( l \geq j_0 \). Therefore,

\[
u_n(\cdot + x_n^{j_0}) \to U^{j_0}, \text{ weakly in } H^1.
\]

This completes the proof. \( \square \)

3. The local and global well-posedness. In this section, we will prove the local and global well-posedness for (1).

Proof of Theorem 1.1. In order to apply Theorem 3.3.9 in [4], we need only check the nonlinearity \( g(u) = (I_\alpha * |u|^p)|u|^{p-2}u \) satisfies assumptions of Theorem 3.3.9 in [4].

We deduce from (12) and Hölder’s inequality that

\[
\| (I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v \|_{L^{r'}} \leq (\| u \|_{L^{2p-2}} + \| v \|_{L^{2p-2}}) \| u - v \|_{L^r},
\]

where \( r = \frac{2Np}{N-\alpha} \). In addition, let \( G(u) = \frac{\lambda}{2p} \int (I_\alpha * |u|^p)|u|^{p} \, dx \), it is easy to check that the conditions of Theorem 3.3.9 in [4] hold. Applying Strichartz’ estimates as that in 4.2 in [4], uniqueness follows. This completes the proof of Theorem 1.1. \( \square \)

In the following theorem, we study the \( H^2 \) regularity of the solutions in the \( L^2 \) subcritical case. Our result suggests that the assumption on the \( H^2 \) regularity of the solutions in [2] is right.

Theorem 3.1. Assume that \( N - 2 \leq \alpha < N \) and \( 2 \leq p < 1 + \frac{2+\alpha}{N} \). Let \( \varphi \in L^2 \) and \( u \in C((-T_*, T^*), L^2) \) be the maximal solution of (1). If \( \varphi \in H^2 \), then \( u \in C((-T_*, T^*), H^2) \cap C^1((-T_*, T^*), L^2) \).

In order to prove this theorem, it suffices to check that the nonlinearity \( (I_\alpha * |u|^p)|u|^{p-2}u \) satisfies the assumptions of Theorem 5.4.3 in [4]. Since the calculation is analogous to that of Theorem 1.1, we omit it.

Next, we shall prove the global well-posedness of (1). The proof is standard. Nevertheless, for the sake of clarity and completeness, we sketch the main steps.

Proof of Theorem 1.2. Case (1) follows from the conservation of mass and energy.

By the conservation of energy and the inequality (6), we can estimate as follows:

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{\lambda}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx + 2E(\varphi)
\]

\[
\leq C_{\alpha, p} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{Np-N-\alpha}{2}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{N+\alpha-Np+2p}{2}} + 2E(\varphi)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + C_\varepsilon,
\]

for some \( \varepsilon < 1 \). Thus, Case (2) follows.

Similarly, using the Theorem 2.3, Case (3) follows.

Case (4) follows from Theorem 3.4.3 in [4]. \( \square \)
4. Blow-up solutions, mass concentration and rate of collapse. In this section, we will study the existence of the blow-up solutions and their dynamics. First of all, by the similar argument as that of Proposition 6.5.1 in [4], we have the following lemma, which is essential to establish the existence of blow-up solutions.

**Lemma 4.1.** Assume that \( \max\{0, N - 4\} < \alpha < N \) and \( 2 \leq p < \frac{N + 2}{N - 2} \). Let \( \varphi \in \Sigma \) and \( u \) be the solution of (1). Then the function \( t \to |u(t, \cdot)| \) belongs to \( C((-T_*, T^*), L^2) \). Furthermore, the function \( V(t) \) belongs to \( C^2(-T_*, T^*) \), and we have

\[
V''(t) = 8 \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{4N + 4\alpha - 4pN}{p} \int_{\mathbb{R}^N} (I\alpha * |u|^p)|u|^p dx
\]

\[
= 8(Np - N - \alpha)E(u) + (8 + 4N + 4\alpha - 4Np)\int_{\mathbb{R}^N} |\nabla u|^2 dx,
\]

for all \( t \in (-T_*, T^*) \).

As a direct result of this lemma, we have

**Lemma 4.2.** Let \( \varphi \in \Sigma \) and the corresponding solution \( u \) blow up in finite time. Then, there is a constant \( c_0 > 0 \) such that

\[
\int_{\mathbb{R}^N} |x|^2|u(t, x)|^2 dx \leq c_0.
\]

**Proof of Theorem 1.3.** In view of Lemma 4.1, we can prove Theorem 1.3. Since the proof is standard, we refer the readers to Theorem 6.5.4 in [4] or Theorem 3.3 in [8].

**Proof of Theorem 1.4.** Set \( \rho(t) = \left\| \nabla Q \right\|_{L^2}^2 \) and \( v(t, \cdot) = \rho(t)^{N/2}u(t, \rho(t) \cdot) \). Let \( \{t_n\}_{n=1}^\infty \) be an any time sequence such that \( t_n \to T^* \), \( \rho_n = \rho(t_n) \) and \( v_n(x) = v(t_n, x) \). Therefore, we have

\[
\|v_n\|_{L^2} = \|\varphi\|_{L^2}, \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.
\]

On the other hand, it follows from the conservation of energy that

\[
E(v_n) = \rho_n^2E(u(t_n)) = \rho_n^2E(\varphi) \to 0, \text{ as } n \to \infty.
\]

This implies that

\[
\int (I\alpha * |v_n|^p)|v_n|^p dx \to p\|\nabla Q\|^2_{L^2}.
\]

Set \( M = \|\nabla Q\|^2_{L^2} \) and \( m = p\|\nabla Q\|^2_{L^2} \), we deduce from Theorem 2.5 that there exist \( V \in H^1 \) and \( \{x_n\}_{n=1}^\infty \) such that, up to a subsequence,

\[
v_n(\cdot + x_n) = \rho_n^{N/2}u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^1
\]

with \( \|V\|_{L^2} \geq \|Q\|_{L^2} \). Thus, for every \( A > 0 \),

\[
\liminf_{n \to \infty} \int_{|x| \leq A} \rho_n^N |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq A} |V(x)|^2 dx.
\]

In view of the assumption \( \lambda(t_n)/\rho_n \to \infty \), this implies immediately

\[
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x - y| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V(x)|^2 dx,
\]

for every \( A > 0 \), which means that

\[
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x - y| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{\mathbb{R}^N} |V(x)|^2 dx \geq \int_{\mathbb{R}^N} |Q(x)|^2 dx,
\]
Therefore, for some \( x(t) \in \mathbb{R}^N \). This completes the proof. \( \Box \)

**Proof of Theorem 1.5.** According to Theorem 1.4, it follows that for all \( R > 0 \)

\[
\liminf_{t \to T^*} \int_{|x-x(t)| < R} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.
\] (30)

On the other hand,

\[
\|u(t,x)\|_{L^2}^2 = \|\varphi\|_{L^2}^2 = \|Q\|_{L^2}^2.
\]

So for all \( R > 0 \)

\[
\liminf_{t \to T^*} \int_{|x-x(t)| < R} |u(t,x)|^2 dx = \|Q\|_{L^2}^2,
\]

and

\[
|u(t,x + x(t))|^2 \to \|Q\|_{L^2}^2 \delta_{x=0}.
\] (31)

By using the Gagliardo-Nirenberg inequality (6), for any \( \varepsilon > 0 \) and any real-valued function \( \theta \) defined on \( \mathbb{R}^N \), we have

\[
E(e^{\pm i \varepsilon \theta} u) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (e^{\pm i \varepsilon \theta} u)|^2 dx \left( 1 - \frac{\left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{p-1}}{\left(\int_{\mathbb{R}^N} |Q|^2 dx\right)^{p-1}} \right) = 0.
\]

Therefore,

\[
0 \leq E(e^{\pm i \varepsilon \theta} u) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |u|^2 |\nabla \theta|^2 dx + \varepsilon Im \int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \theta dx + E(u),
\]

which implies that

\[
\left| Im \int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \theta dx \right| \leq \left( 2E(u) \int_{\mathbb{R}^N} |u|^2 |\nabla \theta|^2 dx \right)^{1/2}.
\] (32)

For any \( j = 1, 2, \ldots, N \), it follows from (32) that

\[
\left| \frac{d}{dt} \int_{\mathbb{R}^N} |u(t,x)|^2 x_j dx \right| = 2 \int_{\mathbb{R}^N} \bar{u} \partial_j u dx = 2 \int_{\mathbb{R}^N} \bar{u} \nabla u \nabla x_j dx \leq 2 \left( 2E(u) \int_{\mathbb{R}^N} |u|^2 |\nabla x_j|^2 dx \right)^{1/2} \leq C.
\]

Let \( \{t_m\}, \{t_k\} \subseteq (0, T^*) \) be any two sequences satisfying \( \lim_{m \to \infty} t_m = \lim_{k \to \infty} t_k = T^* \). Then for any \( j = 1, 2, \ldots, N \), we have

\[
\left| \int_{\mathbb{R}^N} |u(t_m, x)|^2 x_j dx - \int_{\mathbb{R}^N} |u(t_k, x)|^2 x_j dx \right| \leq C|t_m - t_k| \to 0 \text{ as } m, k \to \infty,
\]
which implies that
\[\lim_{t \to T^*} \int_{\mathbb{R}^N} |u(t,x)|^2 x_j dx \text{ exists for any } j = 1,2,\ldots,N.\]
Set
\[x_0 = \|Q\|_{L^2}^2 \lim_{t \to T^*} \int_{\mathbb{R}^N} |u(t,x)|^2 x dx.\]
Then
\[\lim_{t \to T^*} \int_{\mathbb{R}^N} |u(t,x)|^2 x dx = x_0 \|Q\|_{L^2}^2.\] (33)
On the other hand, we infer from Lemma 4.2 that
\[\int_{\mathbb{R}^N} |x|^2 |u(t,x+x(t))|^2 dx \leq 2 \int_{\mathbb{R}^N} |x+x(t)|^2 |u(t,x+x(t))|^2 dx\]
\[+ 2|x(t)|^2 \int |u(t,x+x(t))|^2 dx \leq C + 2|x(t)|^2 \|\varphi\|_{L^2}^2.\] (34)
We infer from (31) and Lemma 4.2 that
\[\limsup_{t \to T^*} |x(t)|^2 \|Q\|_{L^2}^2 = \limsup_{t \to T^*} \int_{|x|<1} |x+x(t)|^2 |u(t,x+x(t))|^2 dx\]
\[\leq \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2 dx \leq c_0.\]
Thus,
\[\limsup_{t \to T^*} |x(t)| \leq \frac{\sqrt{c_0}}{\|Q\|_{L^2}}.\] (35)
Combining (35) and (34), we obtain
\[\limsup_{t \to T^*} \int |x|^2 |u(t,x+x(t))|^2 dx \leq C.\]
Hence, for any \(\varepsilon > 0\), there exists \(R_0 = R_0(\varepsilon)\) such that
\[\limsup_{t \to T^*} \int_{|x| \geq R_0} x |u(t,x+x(t))|^2 dx \leq \frac{C}{R_0} < \varepsilon.\]
It follows from (31) that
\[\limsup_{t \to T^*} \left| \int_{\mathbb{R}^N} |u(t,x)|^2 x dx - x(t) \|Q\|_{L^2}^2 \right| = \limsup_{t \to T^*} \left| \int_{\mathbb{R}^N} |u(t,x)|^2 (x-x(t)) dx \right|\]
\[\leq \limsup_{t \to T^*} \int_{|x| \leq R_0} |u(t,x+x(t))|^2 x dx + \varepsilon \leq \varepsilon,\] (36)
which, together with (33) implies that \(\lim_{t \to T^*} x(t) = x_0\). Therefore,
\[\limsup_{t \to T^*} \int_{\mathbb{R}^N} |u(t,x)|^2 x dx = \|Q\|_{L^2}^2 x_0,\] (37)
and
\[|u(t,x)|^2 \to \|Q\|_{L^2}^2 \delta_{x=x_0} \text{ in the sense of distribution as } t \to T^*.\]

\textbf{Proof of Theorem 1.6.} Let \(h \in C^\infty_0(\mathbb{R}^N)\) be a nonnegative radial function such that
\[h(x) = h(|x|) = |x|^2, \text{ if } |x| < 1 \text{ and } |\nabla h(x)|^2 \leq Ch(x).\]
For $A > 0$, we define $h_A(x) = A^2 h(x/A)$ and $g_A(t) = \int h_A(x - x_0)|u(t, x)|^2 dx$ with $x_0$ defined by (37).

From (32), for every $t \in [0, T^*)$, we have

$$
\left| \frac{d}{dt} g_A(t) \right| = 2 \left| \operatorname{Im} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \bar{u}(t, x) \nabla u(t, x) \nabla h_{A}(x - x_0) dx \right| \\
\leq 2 \sqrt{E(\varphi)} \left( \int_{\mathbb{R}^N} |u(t, x)|^2 |\nabla h_{A}(x - x_0)|^2 dx \right)^{1/2} \\
\leq C \sqrt{g_A(t)},
$$

which implies

$$
\left| \frac{d}{dt} \sqrt{g_A(t)} \right| \leq C.
$$

Integrating on both sides, we obtain

$$
\left| \sqrt{g_A(t)} - \sqrt{g_A(t_1)} \right| \leq C |t - t_1|.
$$

(39)

It follows from (10) that

$$
g_A(t_1) \to \|Q\|_{L^2} h_A(0) = 0 \text{ as } t_1 \to T^*.
$$

Therefore, letting $t_1 \to T^*$ in (39), it follows

$$
g_A(t) \leq C(T^* - t)^2.
$$

Now fix $t \in [0, T^*)$ and let $A \to \infty$, we have

$$
\int_{\mathbb{R}^N} |x - x_0|^2 |u(t, x)|^2 dx \leq C(T^* - t)^2.
$$

Then the uncertainty principle

$$
\left( \int_{\mathbb{R}^N} |u(t, x)|^2 dx \right)^2 \leq \left( \int_{\mathbb{R}^N} |x - x_0|^2 |u(t, x)|^2 dx \right) \left( \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 dx \right),
$$

implies a lower bound of the blow-up rate

$$
\|\nabla u(t)\|_{L^2} \geq \frac{C}{T^* - t}, \forall t \in [0, T^*).
$$

\[\square\]

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