Killing-Yano tensors and surface terms

Dumitru Baleanu\textsuperscript{1}, Özlem Defterli\textsuperscript{2}

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara , Turkey

Abstract

New geometries were obtained by adding a suitable surface term involving the components of the angular momentum to the corresponding free Lagrangians. Killing vectors, Killing-Yano and Killing tensors of the obtained manifolds were investigated.

1 Introduction

Killing-Yano (KY) tensors introduced by Yano \cite{1} play an important role in the Dirac theory on the curved spacetimes \cite{2}. A new supersymmetry corresponding to (KY) tensor was found in black-hole solutions of the Kerr-Newman type \cite{3}. (KY) tensors are crucial in supersymmetric extension of charged point particle’s motion in investigation of the symmetries of gravitational and electromagnetic fields \cite{4}. (KY) tensors of order three are special cases of Lax tensors introduced by Rosquist \cite{5}. A (KY) tensor of order two generates a Killing tensor and, in some cases, it produces new constants of motion on a curved manifold \cite{6, 7, 8}. In the last years a huge effort was devoted for analyzing the importance of (KY) tensors \cite{9, 10, 11, 12, 13, 14, 15} in several areas but there are relatively few manifolds of physical interest admitting these tensors. This drawback is mainly because (KY) tensors are antisymmetric and their equations impose restrictions on the manifold structure \cite{8}. Despite of this fact there were some attempts to find new geometries admitting (KY) tensors of order two or three. For example, the three-particle open Toda lattice was geometrized by a suitable canonical transformation and it was realized as the geodesic system of a certain Riemannian geometry \cite{16}. Adding a time-like dimension a four-dimensional space-times admitting two Killing vector fields were found \cite{16}.

Motivated by the above results we decided to construct integrable geometries admitting (KY) and Killing tensors by adding a surface term \cite{17}, \cite{18} to a known free Lagrangian.

The main aim of this paper is to add, using suitable Lagrangian multipliers, the components of the angular momenta to a free Lagrangian in two or three dimensions and to study the hidden symmetries of the induced manifolds.

The plan of the paper is as follows:

\textsuperscript{1}On leave of absence from Institute of Space Sciences, P.O BOX, MG-23, R 76900 Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr
\textsuperscript{2}E-Mail: defterli@cankaya.edu.tr
In Sec. 2 generic and non-generic symmetries of the extended Lagrangians are investigated. Sec. 3 deals with symmetries of new geometries induced by the motion on a sphere. Our conclusions are given in Sec. 4.

2 Extended Lagrangians and their corresponding geometries

Let us assume that a given free Lagrangian \( L(\dot{q}^i, q^i) \) admits a set of constants of motion denoted by \( L_i, i = 1, \ldots, 3 \). If we add the components of the angular momentum corresponding to \( L \), the extended Lagrangian \( L' = L + \lambda^i L_i, i = 1, \ldots, 3 \) can be rewritten as \( L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j \). Since \( a_{ij} \) is symmetric by construction, the issue is to find a way to construct induced manifolds. In other words we are looking to find whether \( a_{ij} \) is singular or not. If the matrix \( a_{ij} \) is singular \( L' \) corresponds to a singular system [20]. Assuming that \( a_{ij} \) is a singular \( nxn \) matrix of rank \( n-1 \) we obtain non-singular symmetric matrices of order \( (n-1) \times (n-1) \), where \( n \) will be 3, 5 and 6. The final step is to consider the obtained matrices as metrics on the extended space and to investigate their generic and non-generic symmetries.

2.1 The nonsingular case

2.1.1. As a starting point let us consider the following Lagrangian

\[
L' = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \lambda_3 (x \dot{y} - y \dot{x})
\]

From (1) we obtain \( L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j \), where \( a_{ij} \) is given by

\[
a_{ij} = \begin{pmatrix}
1 & 0 & -y \\
0 & 1 & x \\
-y & x & 0
\end{pmatrix}.
\]

The corresponding Killing vector is \( V = (y, -x, 0) \).

A (KY) is an antisymmetric tensor define as

\[
D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0.
\]

Solving (3) corresponding to (2) we obtained the following (KY) tensor

\[
f_{13} = 0, f_{23} = -Cx \sqrt{x^2 + y^2}, f_{13} = Cy \sqrt{x^2 + y^2},
\]

where \( C \) is a constant.

If a (KY) tensor exists, then a Killing tensor of order is generated as

\[
K_{\mu\nu} = f_{\mu\lambda} f_{\nu}^\lambda.
\]
Using (4) and (5) a Killing tensor is constructed as
\[
K_{ij} = \begin{pmatrix}
y^2 & -xy & -y(y^2 + x^2) \\
-xy & x^2 & x(x^2 + y^2) \\
-y(y^2 + x^2) & x(x^2 + y^2) & 0
\end{pmatrix}
\] (6)

Another method to obtain a Killing tensor is to solve the corresponding equations
\[
D_\mu k_{\nu\lambda} + D_\nu k_{\lambda\mu} + D_\lambda k_{\mu\nu} = 0,
\] (7)
where \( k_{\mu\nu} \) is a symmetric tensor. Solving (7) corresponding to (2) we obtain a class of solutions given by
\[
k_{11} = \frac{1}{4} y^2 (C_2 z + C_3) + C_1, \quad k_{12} = -\frac{1}{2} xy (C_2 z + C_3)
\]
\[
k_{13} = -\frac{1}{4} [(x^2 + y^2)(C_2 \arctan(\frac{y}{x}) - 4C_4) + 4C_1], \quad k_{22} = \frac{1}{2} x^2 (C_2 z + C_3) + C_1
\]
\[
k_{23} = \frac{1}{4} [(x^2 + y^2)(C_2 \arctan(\frac{y}{x}) - 4C_4) + 4C_1], \quad k_{33} = 0.
\] (8)

We observed that if \( C_2 = C_1 = 0, C_3 = \frac{1}{2} \) we reobtain the solution from (6). Choosing the appropriate values of the constants \( C_i, i = 1, \cdots 4 \), we obtain a set of non-singular Killing tensors. These Killing tensors can be considered as manifolds and we have so called geometric duality (for more details see Refs. [10, 19]). If \( C_2 = 0 \) the dual metrics have the following forms
\[
k_{11}^{-1} = \frac{x^2}{(x^2+y^2)[C_4(x^2+y^2)-C_1]}, \quad k_{12}^{-1} = \frac{xy}{(x^2+y^2)[C_4(x^2+y^2)-C_1]}, \quad k_{13}^{-1} = \frac{y}{(x^2+y^2)[C_4(x^2+y^2)-C_1]}, \quad k_{22}^{-1} = \frac{x^2}{(y^2+y^2)[C_4(x^2+y^2)-C_1]},
\]
\[
k_{23}^{-1} = \frac{x}{(x^2+y^2)[C_4(x^2+y^2)-C_1]}, \quad k_{33}^{-1} = -\frac{1}{2} \frac{x^2}{(x^2+y^2)[C_4(x^2+y^2)-C_1]}
\] (9)
and the scalar curvatures corresponding to (9) are
\[
R = \frac{2C_1 C_4 [5C_4 (x^2 + y^2) + 2C_1]}{[-C_1 + C_4 (x^2 + y^2)]^2}.
\] (10)

2.1.2 Let us add two components of the angular momentum at a free, three-dimensional Lagrangian. The extended Lagrangian becomes
\[
L' = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda_1 (y \dot{z} - z \dot{y}) + \lambda_2 (z \dot{x} - x \dot{z})
\] (11)
and (11) we identify \( a_{ij} \) as the following non-singular matrix
\[
a_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 & z \\
0 & 1 & 0 & -z & 0 \\
0 & 0 & 1 & y & -x \\
0 & -z & y & 0 & 0 \\
z & 0 & -x & 0 & 0
\end{pmatrix}
\] (12)
The metric (12) admits three Killing vectors
\[
V_1 = (y, -x, 0, 0, 0), \quad V_2 = (0, -z, y, 0, 0), \quad V_3 = (z, 0, -x, 0, 0).
\] (13)
In this case (KY) tensors components are given by

\[ f_{15} = -Gxy, f_{14} = G(z^2 + y^2), f_{24} = -Gxy, f_{34} = -Gxz, \]
\[ f_{25} = G(x^2 + z^2), f_{35} = \frac{G_{xxy}}{x}, f_{12} = Cz, f_{13} = -Cy, \] (14)

others zero. Here C and G are constants. The corresponding Killing tensor has

the following form

\[ K = \begin{pmatrix}
  G(-2C + G)(z^2 + y^2) & GDx & G & 0 & G^2 r^2 z \\
  GDx & -GD(x^2 + z^2) & GDy & -r^2 z G^2 & 0 \\
  GDy & GDz & -GD(y^2 + x^2) & G^2 r^2 y & -G^2 r^2 x \\
  0 & -G^2 z r^2 & G^2 y r^2 & 0 & 0 \\
  G^2 z r^2 & 0 & -G^2 x r^2 & 0 & 0 \\
\end{pmatrix}, \] (15)

where \( D = 2C + G \) and \( r^2 = x^2 + y^2 + z^2 \).

### 2.2 The singular case

2.2.1. The final step is to add all angular momentum components at the Lagrangian of the free particle in three-dimensional. In this case \( L' \) is given by

\[ L' = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{x} \dot{z} - \dot{y} \dot{z}) + \dot{z}(\dot{z} - \dot{x} \dot{z}) + \dot{x}(\dot{x} - \dot{y} \dot{y}) \] (16)

In compact form (16) is written as \( L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j \), where \( a_{ij} \) is singular having the form

\[ a_{ij} = \begin{pmatrix}
  1 & 0 & 0 & 0 & z & -y \\
  0 & 1 & 0 & -z & 0 & x \\
  0 & 0 & 1 & y & -x & 0 \\
  0 & -z & y & 0 & 0 & 0 \\
  z & 0 & -x & 0 & 0 & 0 \\
  -y & x & 0 & 0 & 0 & 0 \\
\end{pmatrix}. \] (17)

Since the rank of (17) is 5 we obtained three non-singular symmetric matrices corresponding to three non-zero minors. The first one is given by (12) and the other two are as follows

\[ b_{\mu \nu}^{(2)} = \begin{pmatrix}
  1 & 0 & 0 & 0 & -y \\
  0 & 1 & 0 & -z & x \\
  0 & 0 & 1 & y & 0 \\
  0 & -z & y & 0 & 0 \\
  -y & x & 0 & 0 & 0 \\
\end{pmatrix}. \] (18)
and

\[ b^{(3)}_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & z & -y \\
0 & 1 & 0 & 0 & x \\
0 & 0 & 1 & -x & 0 \\
z & 0 & -x & 0 & 0 \\
-y & x & 0 & 0 & 0
\end{pmatrix} \]  \quad (19)

By direct calculations we observed that (18) and (19) admit three Killing vectors given by (13) and a (KY) tensor having the following non-zero components

\[ f_{12} = z, \quad f_{13} = -y, \quad f_{23} = x. \]  \quad (20)

3  The motion on a sphere and its induced geometries

It was proved in [21] that the motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum. In the following using the surface term we will generate four-dimensional manifolds. In this case the Lagrangian is given by

\[ L' = \frac{1}{2} \left( 1 + \frac{x^2}{u} \right) \dot{x}^2 + \frac{1}{2} \left( 1 + \frac{y^2}{u} \right) \dot{y}^2 + \frac{xy}{u} \dot{x} \dot{y} - \frac{xy}{\sqrt{u}} \dot{\lambda}_1 \dot{x} + \left( \frac{x^2}{\sqrt{u}} + \sqrt{u} \right) \dot{\lambda}_2 \dot{x} \]  \quad (21)

\[ - \left( \frac{y^2}{\sqrt{u}} + \sqrt{u} \right) \dot{\lambda}_1 \dot{y} + \frac{xy}{\sqrt{u}} \dot{\lambda}_2 \dot{y} + x \dot{\lambda}_3 \dot{y} - y \dot{\lambda}_3 \dot{x}, \]  \quad (22)

where \( u = 1 - x^2 - y^2 \). From (23) we identify the singular matrix \( a_{ij} \) as

\[ a_{ij} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} & -y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{xy}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & x \\
-\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 & 0 \\
\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & x & 0 & 0 \\
-y & x & 0 & 0 & 0
\end{pmatrix}. \]  \quad (23)

Because (23) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we treat these minors as a metric we observed that they are not conformally flat but their scalar curvatures are zero.

The first metric is given by

\[ g^{(1)}_{\mu\nu} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & \sqrt{u} + \frac{x^2}{\sqrt{u}} & -y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\
\sqrt{u} + \frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\
-y & x & 0 & 0
\end{pmatrix}. \]  \quad (24)
The Killing vectors of (24) has the following components

\[
V_1 = (y, -x, 0, 0), \quad V_2 = \left( \frac{x}{\sqrt{1 - x^2 - y^2}}, \frac{y}{\sqrt{1 - x^2 - y^2}}, 0, 0 \right), \\
V_3 = \left( -\frac{xy}{\sqrt{1 - x^2 - y^2}}, \frac{xy}{\sqrt{1 - x^2 - y^2}}, 0, 0 \right). 
\]  

(25)

The next step is to investigate its (KY) tensors. Solving (3) we obtain the following set of solutions:

\begin{itemize}
\item[a.] One-solution is \( f_{21} = \frac{C_1}{\sqrt{1 - x^2 - y^2}} \), others zero.
\item[b.] Two-by-two solution has the form: \( f_{31} = f_{42} = C \),
\item[c.] Three by three solution is \( f_{21} = \frac{C_1}{\sqrt{1 + x^2 + y^2}} \) and \( f_{31} = f_{42} = C \), where \( C \) and \( C_1 \) are constants.
\end{itemize}

From (23) another two metrics can be identified as

\[
g^{(2)}_{\mu\nu} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & x \\
-\frac{xy}{\sqrt{u}} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & 0 & 0 \\
-y & x & 0 & 0
\end{pmatrix} 
\]  

(26)

and

\[
g^{(3)}_{\mu\rho} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} \\
-\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 \\
\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0
\end{pmatrix} 
\]  

(27)

respectively. By direct calculations we obtained that (26) and (27) admit the same Killing vector as in (25). Solving (3) corresponding to (26) and (27) we find one non-zero component of (KY) tensor as

\[
f_{21} = \frac{C_1}{\sqrt{1 - x^2 - y^2}}. 
\]  

(28)

4 Conclusions

Integrable geometries were reported by adding a surface term involving the components of the angular momentum to a given free Lagrangian. The existence of Killing vectors, (KY) and Killing tensors is investigated and in all cases a solution is presented.

The first step was to add, to a free two-dimensional Lagrangian, a surface term involving the third component of the angular momentum. In this case a three-dimensional metric was induced. This metric is conformally flat but its duals are not.
Increasing the number of dimensions to three and adding a surface term involving two components of the angular momentum we obtained geometries, in four and five dimensions. The obtained induced manifolds are not conformally flat but all of them have Ricci scalar zero.

If we add a surface term involving all components of the angular momentum to a three dimensional free Lagrangian we observed that a singular matrix $a_{ij}$ arises. We identify three symmetric minors of this metric and we investigated the existence of Killing vectors, $(KY)$ and Killing tensors corresponding to those induced manifolds. We observed that the obtained manifolds admit the same Killing vectors but different $(KY)$ solutions.

Finally, the geometries induced by the motion on a sphere are investigated and a four dimensional induced manifolds were obtained. As in the previous case the manifolds admit the same Killing vectors but different $(KY)$ tensors.

5 Acknowledgments

One of the authors (D. B.) would like to thank M. Henneaux, B. Edgar and M. Montesinos for their helpful discussions. This work is partially supported by the Scientific and Technical Research Council of Turkey.

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