ON VERTICALLY-RECURRENT MATRICES AND THEIR ALGEBRAIC PROPERTIES

HOSSEIN TEIMOORI FAAL

ABSTRACT. In this paper, we first introduce the new class of vertically-recurrent matrices, using a generalization of “the Hockey stick and Puck theorem” in Pascal’s triangle. Then, we give an interesting formula for the lower triangular decomposition of these matrices. We also deal with the $m$-th power of these matrices in some special cases. Furthermore, we present two important applications of these matrices for decomposing admissible matrices and matrices which arise in the theory of ladder networks. Finally, we pose some open problems and conjectures about these new kind of matrices.

1. INTRODUCTION

In theory of linear algebra, the general problem of the classification of the integral matrices which have simpler decompositions is the key in many theoretical and applied fields. In this paper, we intend to find the lower-triangular decomposition for some new kind of matrices which we call them matrices with vertically recurrence relation into Toeplitz block matrices.

Fortunately, several interesting classes of the integer-valued Toeplitz matrices can be nicely factorized into Pascal matrices [3]. The important point in finding this matrix decomposition is the well-known property of the Pascal triangle which is called the hockey stick and puck theorem. Using the generalization of the above theorem, one can construct a new linearly recurrence relation which we call it the generalized hockey stick and puck theorem. This later one is the principle of the multiplicative decomposition of the above matrices.

In section 2, we start by the definition of the matrix with vertically-recurrent relation associated with the sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$, then we investigate some of it’s properties, specially its multiplicative decomposition and also we find its inverse matrix. In section 3, we deal with the power of this matrix and find it’s associated sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$. Furthermore, we present two important applications of this new kind of matrix for factorization of admissible matrices and in ladder networks. Finally, we propose some open problems and conjectures about these matrices.
2. The Vertically-Recurrent Matrices

We start by motivating the main idea behind these new integral matrices. Consider the two dimensional linear recurrence relation among entries of the well-known Pascal’s triangle, as follows

\[ u \quad v \]
\[ \quad w \]

Figure 1. \( w = u + v \)

Now if we consider the two consecutive columns of the left-justified Pascal’s triangle; i.e., the \( k - 1 \) and the \( k \)th columns, we have

\[ u_{l-1} \quad v_{l-1} = w_{l-1} \]
\[ u_{l-2} \quad v_{l-2} = w_{l-2} \]
\[ \vdots \]
\[ u_2 \quad v_2 = w_2 \]
\[ u_1 \quad v_1 = w_1 \]
\[ w = w_0 \]

Figure 2. Hockey Stick and Puck Theorem

Therefore, at any step, after computing the \( w_i \)'s with respect to \( u_i \)'s and \( v_i \)'s, fix the \( u_i \) and just rewrite \( v_i \), as the next \( w_i \), with respect to \( u_i + 1 \) and \( v_i + 1 \). Continuing this process until to get the main diagonal. It can be easily seen that \( w \) is expressible as the sum of the entries \( u_1, u_2, \ldots, u_l \). More precisely, we have

\[ w_0 = w = u_1 + v_1 \]
\[ w_1 = v_1 = u_2 + v_2 \]
\[ \vdots \]
\[ w_{l-2} = v_{l-2} = u_{l-1} + v_{l-1} \]
\[ w_{l-1} = v_{l-1} = u_l \]

and consequently,

\[ w = u_1 + u_2 + \cdots + u_l. \]

Now, if we translate the above relation into the language of recurrence relations, we obtain

\[ a_{n,k} = \sum_{l=k-1}^{n-1} a_{l,k-1}, \quad (n \geq k \geq 1). \]

The above equality is known as the \textit{vertically-recurrent} relation. In special case for the left-justified Pascal’s triangle; that is \( a_{n,k} = \binom{n-1}{k-1} \), we get the well-known \textit{hockey stick and puck theorem} \[4\] as follows (see Figure 2)

\[ \binom{n}{k} = \sum_{l=k-1}^{n-1} \binom{l}{k-1}, \quad (n \geq k \geq 1). \]
Thus, it is natural to generalize the relation (2.2) associated with an arbitrary sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$, $\lambda_0 = 1$, as follows:

$$a_{n,k} = \sum_{l=k-1}^{n-1} \lambda_{n-1-l} a_{l,k-1}, \quad (n \geq k \geq 1).$$

We call the equation (2.4) the **generalized hockey stick and puck theorem**. Now we are at the position to define our new class of matrices that we call them **vertically-recurrent** matrices.

For combinatorial reasons, we mainly concentrate on the class of matrices in which their associate sequences have only integer values.

**Definition 2.1.** Suppose $n$ and $k$ are positive integers. We define the **vertically-recurrent** matrix $V_n[\Lambda]$ associated with the sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$, $\lambda_0 = 1$, of order $(n + 1) \times (n + 1)$ in the following form:

$$(V_n[\Lambda])_{ij} = \begin{cases} 
\lambda_i & \text{if } i \geq 0, j = 0, \\
a_{i,j} & \text{if } j \geq i \geq 1, \\
0 & \text{if } i < j,
\end{cases}$$

in which the entries $a_{i,j}$’s satisfy the relation (2.4).

**Example 2.2.** For $\Lambda = \{\lambda_n = 1\}_{n \geq 0}$, we have:

$$V_3[\Lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1 \end{bmatrix},$$

where the above matrix is called the Pascal matrix $P_n$, for $n = 3$ (see [5]).

**Example 2.3.** For $\Lambda = \{\lambda_n = 2^n\}_{n \geq 0}$, we have:

$$V_3[\Lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
4 & 4 & 1 & 0 \\
8 & 12 & 6 & 1 \end{bmatrix}.$$ 

We note that the above matrix is the **Pascal functional matrix** $P_n[x]$ for $n = 3$ and $x = 2$ (see [4]).

Next, we start to obtain a **multiplicative decomposition** of $V_n[\Lambda]$. To do this, we first define the lower triangular **Teoplitz** matrix by $T_n[\Lambda] = [\lambda_{i-j}]_{0 \leq i \leq j \leq n}$ and the Teoplitz-block matrix $\overline{T}_k[\Lambda]$, as follows

$$\overline{T}_k[\Lambda] = \begin{bmatrix} I_k & 0 \\
0 & T_{n-k}[\Lambda] \end{bmatrix}.$$

By convention, $\overline{T}_0[\Lambda] = T_n$ and $\overline{T}_n[\Lambda] = I_{n+1}$, where $I_{n+1}$ is the identity matrix of order $n + 1$.

**Theorem 2.4.** Suppose $n$ is a natural number. Then, we have

$$V_n[\Lambda] = T_n[\Lambda]([1] \oplus V_{n-1}[\Lambda]),$$

in which the symbol $\oplus$ denotes the direct sum of two matrices.
Proof. For each \( i \) and \( j \) with \( i \geq j \geq 0 \), since the \( (i,j) \)-entry of \([1] \oplus V_{n-1}[A]\) is \((V_n[A])_{i-1,j-1}\), from the definition of matrix product and the relation (0.4), we get

\[
(T_n[A]([1] \oplus V_{n-1}[A]))_{i,j} = \sum_{l=j}^{i} (T_n[A])_{i,l}(V_n[A])_{l-1,j-1}
\]

\[
= \sum_{l=j}^{i-1} (T_n[A])_{i,l+1}(V_n[A])_{l,j-1}
\]

\[
= \sum_{l=j}^{i-1} \lambda_{i-l-1}a_{l,j-1}
\]

\[
= a_{i,j} = (V_n[A])_{i,j}
\]

\[\square\]

Now, as an immediate consequence of Theorem 2.4, we have the following results:

**Corollary 2.5.**

(i). \( V_n[A] = T_n[A]T_{n-1}[A] \cdots T_1[A] \).  

(ii). \( V_n^{-1}[A] = T^{-1}_1[A]T^{-1}_2[A] \cdots T^{-1}_k[A] \).

In [3], the Toeplitz matrices with integer entries are investigated. Considering these matrices, we can calculate the inverse of \( T_n[A] \) by means of the Pascal functional matrices [3] in some important special cases. We leave the general case as an open question.

**Case 1.** Let \( \Lambda = \{\lambda_n = \lambda\}_{n \geq 0} \), then we clearly have

\[
T_n[A] = \lambda S_n[1],
\]

where

\[
(S_n[x])_{i,j} = \begin{cases} 
  x^{i-j} & \text{if } i \geq j \geq 0, \\
  0 & \text{if } i < j.
\end{cases}
\]

We also have \( S_n[x] = P_{n,1}[x]P_{n,-1} \), in which the matrices \( P_{n}[x] \) and \( P_{n,1}[x] \) are Pascal functional and Pascal k-eliminated functional matrices, respectively [5,6]. Thus, we get

\[
T_n[A] = \lambda P_{n,1}[1]P_n[-1].
\]

**Case 2.** Let \( \Lambda = \{\lambda_n = \lambda^n\}_{n \geq 0} \). Now, it is clearly the generalization of the above case, since in this case \( T_n[A] = S_n[\lambda] \) and consequently

\[
T_n[A] = P_{n,1}[\lambda]P_n[-\lambda].
\]

3. The Powers of Vertically-Recurrent Matrices

If we consider the Pascal functional matrix for the values 1, 2, . . ., \( l \), then we observe that all of these matrices are vertically-recurrent. Indeed, for \( P_n[l] \) in general, the associated sequence is \( \Lambda = \{\lambda_n = l^n\}_{n \geq 0} \). On the other hand, we know that the Pascal matrix \( P_n[x] \) has an exponential property (see [5]) therefore the matrix \( P_n[l] \) is just the \( l \)-th power of the matrix \( P_n[1] \). For the above reason, the following challenging question naturally arises in the context of vertically-recurrent matrices.

*If the associated sequence of the matrix \( V_n[A] \) is \( \Lambda = \{\lambda_n\}_{n \geq 0} \), then what is the associated sequence of the matrix \( (V_n[A])^m \) with respect to the sequence \( \lambda_n \)?*
Lemma 3.2. Suppose \( a_n \) is a sequence we just need to consider the following result from [7].

\[
(3.1) \quad a_n = \lambda a_{n-1, k-1} + (\frac{\lambda^m - 1}{\lambda - 1}) n \ a_{n-1, k}.
\]

Since, considering the above identity and mathematical induction we are able to conclude that

\[
(3.2) \quad a_n = \lambda a_{n-1, k-1} + (\frac{\lambda^m - 1}{\lambda - 1}) n \ a_{n-1, k}.
\]

The above statement is equal to prove that the recurrence relation for entries \( (V_n[A])^m \) is as follows:

(3.3)

\[
\begin{align*}
    a_{n,k} &= \lambda^m a_{n-1,k-1} + (\frac{\lambda^m - 1}{\lambda - 1}) n a_{n-1,k}, \\
    a_{n,0} &= 1, \\
    a_{n,k} &= 0, \\
    b_{n,k} &= \alpha' b_{n-1,k-1} + \beta' b_{n-1,k}, \\
    b_{n,0} &= 1, \\
    b_{n,k} &= 0,
\end{align*}
\]

Case 1. Suppose \( V_n[A] \) is a matrix with constant associated sequence \( \Lambda = \{ \lambda_n = \lambda \}_{n \geq 0} \). We have the following interesting result.

Proposition 3.1. Let \( V_n[A] \) be a vertically-recurrent matrix with its associated sequence \( \Lambda = \{ \lambda_n = \lambda \}_{n \geq 0} \). Then, the associated sequence of \( (V_n[A])^m \) is \( \lambda^m (\frac{\lambda^m - 1}{\lambda - 1})^n \)

Proof. The above statement is equal to prove that the recurrence relation for entries of \( (V_n[A])^m \) is, as follows:

(3.4)

\[
(3.3) \quad \sum_{k=1}^{n-1} \left[ \lambda^m \left( \frac{\lambda^m - 1}{\lambda - 1} \right)^{n-1-t} a_{l,k-1} \right] a_{n-k, l}.
\]

Namely, \( \lambda^m (\frac{\lambda^m - 1}{\lambda - 1})^n \) is the associated sequence for \( (V_n[A])^m \). For proving the equivalence statement we just need to consider the following result from [7].

Lemma 3.2. Suppose \( \alpha, \alpha', \beta, \beta' \) are four real numbers. Also let \( A = [a_{ij}] \), \( B = [b_{ij}] \) be two lower triangular matrices where their entries satisfy the following recurrence relations respectively,

(3.2)

\[
\begin{align*}
    a_{n,k} &= \alpha a_{n-1,k-1} + \beta a_{n-1,k}, \quad (n \geq k \geq 1), \\
    a_{n,0} &= 1, \\
    a_{n,k} &= 0, \\
    b_{n,k} &= \alpha' b_{n-1,k-1} + \beta' b_{n-1,k}, \quad (n \geq k \geq 1), \\
    b_{n,0} &= 1, \\
    b_{n,k} &= 0.
\end{align*}
\]

If \( AB = [c_{ij}] \), then, there are real numbers \( \alpha'' = \alpha \alpha' \) and \( \beta'' = \beta + \alpha \beta' \), such that

(3.3)

\[
\begin{align*}
    c_{n,k} &= \alpha'' c_{n-1,k-1} + \beta'' c_{n-1,k}, \quad (n \geq k \geq 1), \\
    c_{n,0} &= \sum_{i=0}^{n} a_{n,i} a_{n,i} \quad (n \geq 0), \\
    c_{n,k} &= 0 \quad (k > n).
\end{align*}
\]

Proof. Let \( C = AB = [c_{ij}] \). By the definition of the product of two matrices, we conclude that \( c_{n,k} = \sum_{i=0}^{n} a_{n,i} a_{n,i} (n \geq k \geq 1) \). Hence, this immediately implies that

\[
(3.5) \quad c_{n,0} = \sum_{l=0}^{n} a_{n,l} a_{n,l} = \sum_{l=0}^{n} a_{n,l} (n \geq 0) \quad (k \geq n).
\]

Moreover, we have

\[
(3.5) \quad c_{n-1,k-1} = \sum_{l=k-1}^{n-1} a_{n-1,l} a_{n,k}, \quad c_{n-1,k} = \sum_{l=k}^{n-1} a_{n-1,l} a_{n,k}.
\]

Put \( I_{n,k} = \alpha \alpha' c_{n-1,k} + (\beta + \alpha \beta') c_{n-1,k} \). Then, we have
\[ I_{n,k} = \alpha \alpha' \sum_{l=k-1}^{n-1} a_{n-1,l}b_{l,k-1} + (\beta + \alpha \beta') \sum_{l=k}^{n-1} a_{n-1,l}b_{l,k}, \]
\[
= \left[ \alpha \alpha' \sum_{l=k-1}^{n-1} a_{n-1,l}b_{l,k-1} + \alpha \beta' \sum_{l=k}^{n-1} a_{n-1,l}b_{l,k} \right]
+ \beta \sum_{l=k}^{n} a_{n-1,l}b_{l,k},
\]
\[
= \alpha \left[ \sum_{l=k}^{n-1} \left( \alpha' b_{l,k-1} + \beta' b_{l,k} \right) + a_{n-1,k-1}b_{k-1,k-1} \right]
+ \beta \sum_{l=k}^{n} a_{n-1,l}b_{l,k},
\]
\[
= \alpha \left[ \sum_{l=k}^{n-1} b_{l+1,k} + a_{n-1,k-1}a_{n-1,k-1} \right]
+ \beta \sum_{l=k+1}^{n} a_{n-1,l}b_{l,k} + \beta \left( a_{n-1,k-1}b_{k,k} - a_{n-1,n}b_{n,k} \right)
\]
\[
= \sum_{l=k+1}^{n} \left( \alpha a_{n-1,l-1} + \beta a_{n-1,l} \right) b_{l,k} + \left( \alpha a_{n-1,l-1} + \beta a_{n-1,k} \right)
\]
\[
= \sum_{l=k+1}^{n} a_{n,l}b_{l,k} + a_{n,k}
\]
\[
= \sum_{l=k}^{n} a_{n,l}b_{l,k} = c_{n,k},
\]
as required.

Now the equivalence statement is easily proved considering the above lemma and the mathematical induction.

Case 2. \( \lambda_n = \lambda^n \). In this case, we use the above lemma again to obtain the following theorem,

**Theorem 3.3.** Let \( V_n[\Lambda] \) be a matrix with vertically recurrent relation and it’s associated sequence \( \lambda_n = \lambda^n \). Then, the associated sequence of \( (V_n[\Lambda])^m \) is

\[
(3.6) \quad \lambda_n = (\lambda m)^n.
\]

**Proof.** Considering the same argument in theorem 0.4, It is necessary to prove the following recurrent relation for \( (V_n[\Lambda])^m \):

\[ a_{n,k} = a_{n-1,k-1} + (2m)\lambda a_{n-1,k}. \]

But, applying the lemma 0.5, it is just necessary to prove the special case \( m = 1 \). Namely,

\[ a_{n,k} = a_{n-1,k-1} + 2\lambda a_{n-1,k}, \]
Now it can be easily seen that, 

\[ a_{n,k} = \sum_{k-1}^{l-1} (2\lambda)^{n-1-l} a_{l,k-1} \]

\[ \square \]

4. Applications

In this section, we present two applications of vertically-recurrent matrices in the area of integral matrices and electrical engineering.

As our first application, we mention an interesting class of integral matrices which are called admissible matrices. To do so, we consider infinite matrices \( A = (a_{n,k}) \), indexed by \( \{0, 1, 2, \ldots\} \), and denote it by \( r_m = \{a_{m,0}, a_{m,1}, \ldots\} \) the \( m \)th row.

**Definition 4.1.** \( A = (a_{n,k}) \) is called admissible [8] if

1. \( a_{n,k} = 0 \) for \( n < k \), \( a_{n,n} = 1 \) for all \( n \) (that is, \( A \) is lower triangular with main diagonal equal to 1).
2. \( r_m . r_n = (a_{m+n,0}) \) for all \( m, n \), where \( r_m . r_n = \sum_k a_{mk} a_{nk} \) is the usual inner product.

Here, we consider some few examples of these matrices and show that they are indeed vertically-recurrent matrices.

An interesting theorem in [8], states that all admissible matrices are characterized by sequence \( s_0 = b_0; s_n = b_n - b_{n-1}, n \geq 1 \) in which \( b_n = a_{n+1,n} \). Thus, any sequence \( s = \{s_0, s_1, \ldots, s_n, \ldots\} \) will present an admissible matrix \( A = (a_{n,k}) \).

**Proposition 4.2.** Let \( A = (a_{n,k}) \) be an admissible matrix with \( a_{n+1,n} = b_n \) for all \( n \). Set \( s_0 = b_0, s_1 = b_1 - b_0, \ldots, s_n = b_n - b_{n-1}, \ldots \). Then, we have

\[ a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + a_{n-1,k+1} \quad (n \geq 1) \]
\[ a_{0,0} = 1, \quad a_{0,k} = 1 \quad \text{for} \quad (k > 0). \]

Conversely, if \( a_{n,k} \) is given by the recursion (4.1), then \( (a_{n,k}) \) is an admissible matrix with \( a_{n+1,n} = s_0 + \cdots + s_n \).

For example the corresponding admissible matrix for sequence \( s = \{1,1,1,\ldots\} \) is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 \\
4 & 5 & 3 & 1 \\
9 & 12 & 9 & 4 \\
\end{bmatrix}
\]

and for the sequence \( s = \{1,2,2,\ldots\} \) is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 \\
5 & 9 & 5 & 1 \\
14 & 28 & 20 & 7 \\
\end{bmatrix}
\]

Clearly the first matrix is a vertically recurrence matrix with associated sequence \( \Lambda = \{1, 1, 2, 4, 9, \ldots\} \) and the second one with \( \Lambda = \{1, 2, 5, 14, \ldots\} \) and these are just the first columns of the above matrices.
5. Ladder Networks

The transfer ratio \( T_k \) \((k = 0, 1, 2, \ldots, n)\) of the output to input signal (voltage or current) along the network (Figure 3) is determined by a polynomial in \( x \) of the corresponding degree, in which \( x \) determined by the product of impedance of a longitudinal branch and admittance of transversal branch \([9]\).

It can be determined from a solution of the following recurrence equation,

\[
a_{k+1} - (2 + x)a_k + a_{k-1} = 0;
\]

\[
a_1 = (1 + x)a_0,
\]

where \( a_0 \) denotes a known signal at the input port of the first cell and \( a_k \) is the corresponding signal at the \( k \)-port of the network (e.g., \( a_k = V_k \) as shown in Figure 3).

\[\text{Figure 3. Electrical Ladder Network}\]

The ratio \( T_k \) follows from the relation

\[ T_k = \frac{a_k}{a_0}, \ k = 0, 1, 2, \ldots, n. \]

It is easy to see that \( T_k \) is determined by a polynomial in \( x \) of the \( k \)th degree, so we can write

\[ T_k = \sum_{m=0}^{k} p_{k,m} x^m, \ k = 0, 1, 2, \ldots, n. \]

From the direct inspection of the above expression, we have that

\[ T_0 = 1, \]
\[ T_1 = 1 + x, \]
\[ T_2 = 1 + 3x + x^2, \]
\[ T_3 = 1 + 6x + 5x^2 + x^3, \]
\[ T_4 = 1 + 10x + 15x^2 + 7x^3 + x^4, \]
\[ T_5 = 1 + 15x + 35x^2 + 28x^3 + 9x^4 + x^5. \]

Now, if we define the matrix \( MNT, \) modified numerical triangle \([9]\), as follows

\[
(MNT)_{ij} = \begin{cases} 
  p_{i,j} & i \geq j \geq 0; \\
  0 & i < j,
\end{cases}
\]

it is not hard to prove (by mathematical induction) that the entries \( p_{n,k} \)'s have the following formula

\[
p_{n,k} = \binom{n+k}{2k+1}, \quad (n \geq k \geq 0).
\]
Then, we observe that the above array is a vertically-recurrent matrix with associated sequence $\Lambda = \{\lambda_n = n + 1\}$. Indeed, it is equivalent to prove the following binomial identity:

\[
\binom{n+k}{2k+1} = \sum_{l=k-1}^{n-1} \binom{n-l}{2k-1} \binom{l+k-1}{2k-1} \quad (n \geq k \geq 1).
\]

For example

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 6 & 5 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

It is interesting to note that one can observe that there is also another modified triangle that we denote it by $MNT_2$ which can be defined as follows

\[
(MNT_2)_{ij} = \begin{cases} 
\binom{i+j+1}{i} & i \geq j \geq 0; \\
0 & i < j,
\end{cases}
\]

Now, it is easy to see that $MNT_2$ is also a vertically-recurrent matrix with associate sequence $\Lambda = \{\lambda_n = (\frac{n+2}{2})\}_{n \geq 0}$. Indeed, this claim is equivalent to prove the following combinatorial identity:

\[
\binom{n+2k}{3k+1} = \sum_{l=k-1}^{n-1} \binom{n-l+1}{2k-1} \binom{l+2k-2}{3k-2} \quad (n \geq k \geq 1).
\]

6. Open Problems and Conjectures

Considering the previous discussions, we pose the following open problems and conjectures.

Open Problem 6.1. Consider the matrix with vertically recurrence relation $V_n[\Lambda]$ with associated sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$. Find the associated sequence of the matrix $(V_n[\Lambda])^m$ with respect to the sequence $\lambda_n$.

Open Problem 6.2. For any matrix with vertically recurrence sequence $V_n[\Lambda]$ with associated sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$ ($\lambda_n \in \mathbb{Z}; \quad n = 0, 1, 2, \cdots$), find its minimal polynomial in the field of $\mathbb{Z}_p$ (see [10]).

Let $A = [a_{n,k}]$ be an integral arrays (array with only integer entries) which can be defined recursively, as follows

\[
\begin{cases}
    a_{n,k} = a_{n-1,k-1} + a_{n-1}a_{n-1,k}, & (n \geq k \geq 1), \\
    a_{n,0} = 1, & n \geq 0, \\
    a_{n,k} = 0, & k > n,
\end{cases}
\]

We also come up with the following conjectures.

Conjecture 6.3. The triangular array $A = [a_{n,k}]$ is a vertically-recurrent matrix with associated sequence $\lambda_0 = 1$ and $\lambda_i = \prod_{j=1}^{i} \alpha_j$.

Conjecture 6.4. The Catalan array $C_n$ is a vertically-recurrent matrix.
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Department of Mathematics and Computer Science, Allameh Tabataba’i University, Tehran, Iran

Email address: Hossein.teimoori@atu.ac.ir