Parametrization of the angular distribution of Cherenkov light in air showers

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Abstract

The Cherenkov light produced in air showers largely contributes to the signal observed in gamma-ray and cosmic-ray observatories. Yet, no description of this phenomenon is available covering both small and large angular regions. To fill this gap, a parametrization of the angular distribution of Cherenkov photons is performed in terms of a physically motivated parametric function. Model parameters are constrained using simulated gamma-ray and proton showers with energies in the TeV to EeV region. As a result, a parametrization is obtained that overcomes in precision previous works. Results presented here can be used in the reconstruction of showers with imaging Cherenkov telescopes as well as the reconstruction of shower profiles in fluorescence detectors.

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I. INTRODUCTION

Large amount of Cherenkov light is produced in atmospheric air showers and several experimental techniques have been proposed to explore this signal to study astroparticles. The generation of light in the cascade is highly dominated by electrons. The emission of Cherenkov light by relativistic electrons including geometry, intensity, and wavelength is explained by classical electrodynamics, what has been used as an inspiration for the development of robust detection techniques with minimum bias and reduced systematic uncertainties.

The total signal produced by all particles in air showers evolves with depth due to the multiplication of particles, to the division of the primary energy, and to changes of the atmosphere. The description of this evolution is mandatory to understand the signal and to reconstruct the properties of the primary particle. In order to extract physical results from measurements, the collaborations running Imaging Atmospheric Cherenkov Telescopes (IACT), ground-based detectors, fluorescence detectors and proposing space experiments have to understand the properties of Cherenkov light production in air showers, such as the longitudinal distribution, the lateral distribution, and the angular distribution. In this paper, special attention is given to the description of the angular distribution of Cherenkov photons in air showers.

IACTs are at the very foundation of contemporary gamma-ray astronomy. The identification and the reconstruction of the primary gamma-ray is done by the interpretation of the Cherenkov light detected by telescopes at ground. Current observatories are equipped with some (< 5) telescopes with few degrees (< 10°) of field of view installed a hundred meters apart from each other. The Cherenkov Telescope Array (CTA) is the next generation of IACTs presently under development. The CTA baseline design calls for 118 telescopes to be installed at two sites covering areas of 0.6 km² in La Palma, Spain and 4 km² in Paranal, Chile. The angular distribution of Cherenkov photons in an air shower determines the image shape detected by IACTs and therefore is a key aspect in many reconstruction techniques.

Fluorescence Detectors (FD) have been long used to study Ultra-High Energy Cosmic Rays (UHERC). These telescopes have been optimized to measure the isotropic fluorescence light emitted by nitrogen molecules due to the passage of the particles in the
atmosphere. The telescopes in operation \cite{12, 13} have large aperture (≈30°) and cover a detection area of thousands km². The emitted fluorescence light spectrum lays in the same wavelength band of the transmitted Cherenkov light (300-450 nm) making it impossible for FDs to filter it out. Traditionally Cherenkov light was considered noise in the FD measurements \cite{13, 14}, but recently the Cherenkov light seen by FD has been used as signal to detect showers with energies down to 2 PeV \cite{15}. Direct Cherenkov light is also used to study UHECR with ground detectors \cite{16, 17} and is proposed as an important signal source in future space experiments \cite{18}. The angular distribution of Cherenkov photons in an air shower is an important feature for all UHECRs experiments because it determines the lateral spread of light and the balance between fluorescence and Cherenkov signal measured by FD including large angles (>10°) and great distances (several km) from shower axis.

The number of Cherenkov photons produced in an air shower reaching a detector at a given distance from the shower axis can be calculated only if the angular distribution of photons is known. Reversely, the reconstruction of the primary particle properties is only possible if the measured amount of light in each detector is converted into the amount of light emitted by the particles in the shower. The angular distribution of Cherenkov photons is determined by the convolution of the longitudinal development of electrons, energy distribution of the electrons, angular distribution of the electrons, scattering of the electrons, refractive index, geomagnetic effects, and scattering of the photons \cite{19–23}.

Influenced by the main techniques detecting Cherenkov light (IACT and FD), the study of the angular distribution of Cherenkov photons has been divided respectively in two regimes: a) gamma-ray primaries, small angles < 10°, and TeV energies and b) cosmic ray primaries, large angles > 10°, and highest energies (10^{17} eV). Experiments have measured the angular distribution of Cherenkov photons \cite{24} in regime b). Since the pioneering work \cite{20}, the angular distribution was simulated for regime a) \cite{8} and b) \cite{23, 25}.

In this paper, the angular distribution of Cherenkov photons is simulated using the most updated simulations software and a new parametrization based on shower physics is proposed. The new parametrization presented here improves the precision of the angular distribution in comparison to previous proposals \cite{8, 23, 25}. Besides the needed update of the parametrizations concerning the new shower models made available after the previous works, this paper aims at the improvement of the precision requested by the new generation of experiments \cite{7, 18} and at the refinement demanded by the new uses of Cherenkov light.
as the main signal in fluorescence telescope analysis \cite{15}. Moreover, a unified view of the two regimes is presented for the first time.

This paper is organized as follows. In section II an exact model to compute the angular distribution of Cherenkov models is derived. This model is simplified in section III to obtain a simple form in terms of free parameters. The parameters of the model are constrained by Monte Carlo simulations in section IV. A discussion of the results and a comparison to previous works is presented in section V and some final remarks are given in section VI.

II. EXACT MODEL FOR THE CHERENKOV ANGULAR DISTRIBUTION

A mathematical description of the number of Cherenkov photons emitted in a given angular interval as a function of the shower development in the atmosphere \( \frac{d^2N_\gamma}{d\theta dX} \) is presented in this section. Each element that contributes to this quantity is identified and explained below.

It is known that electrons\footnote{The term \textit{electrons} here refer to both electrons and positrons.} are responsible for more than 98\% of the Cherenkov photon content in a shower \cite{23}, therefore it is assumed here that all photons are emitted by electrons. Figure I depicts the composition of angles determining the final angular distribution of Cherenkov photons. Electrons are emitted in the development of the shower and are scattered in the atmosphere making an angle \( \theta_p \) with the shower axis. These electrons emit Cherenkov photons in a cone of aperture angle \( \theta_{em} \) around the direction of propagation of the electron. The direction of each photon is determined by \( \theta_{em} \) and \( \phi_{em} \), measured in the plane perpendicular to the moving electron direction.

The number of Cherenkov photons emitted by electrons with energy \( E \) and angle \( \theta_p \) in a shower per interval of depth \( dX \) is given by\footnote{The dependency on the primary particle energy \( (E_0) \) is omitted here for brevity and discussed in terms of simulated showers in the following sections. For the purpose of this section, \( E_0 \) may be regarded as fixed.}

\[
\frac{dN_\gamma}{dX} = N_e(s) \frac{dN_e}{dE}(E, s) dE \frac{dN_e}{d\theta_p}(\theta_p, E) d\theta_p Y_\gamma(E, h) \frac{dX}{\cos \theta_p} \frac{d\phi_{em}}{2\pi},
\]

where \( s \) is the shower age\footnote{\( s = 3X/(X + 2X_{max}) \) where \( X_{max} \) is the depth in which the shower reaches the maximum number of particles.} and \( h \) is the emission height above sea level. \( N_e(s) \) is the total number of electrons, \( \frac{dN_e}{dE} \) is the energy distribution of electrons and \( \frac{dN_e}{d\theta_p} \) is the angular distribution of electrons. The function \( Y_\gamma(E, h) \) represents the number of photons emitted

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3 \( s = 3X/(X + 2X_{max}) \) where \( X_{max} \) is the depth in which the shower reaches the maximum number of particles.
by one electron per depth interval (yield) and the factor of $1/\cos \theta_p$ takes into account the correction in the length of the electron track due to its inclined trajectory. Photons are uniformly distributed in $\phi_{em}$ (factor of $1/2\pi$). According to reference [23], $Y_\gamma(E,h)$ is given by:

$$Y_\gamma(E,h) \approx 4\pi \alpha \frac{n(h) - 1}{\rho(h)} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \left( 1 - \frac{E_{\text{thr}}^2(h)}{E^2} \right),$$

(2)

in which $\alpha \approx 1/137$ is the fine-structure constant, $n(h)$ is the refractive index of the medium, $\rho(h)$ is the atmospheric density, and $\lambda_i$ the wavelength interval of the emitted photons. The threshold energy $E_{\text{thr}}$ for an electron to produce Cherenkov light is $E_{\text{thr}}(h) = m_e c^2 / \sqrt{1 - n^{-2}(h)}$, where $m_e$ is the electron rest mass.

The dependency of $dN_\gamma$ on the angle between the Cherenkov photon and the shower axis directions ($\theta$) is found after a change of variable from $\phi_{em}$ to $\theta$ (see figure 1):

$$\cos \theta = \cos \theta_p \cos \theta_{em} - \sin \theta_p \sin \theta_{em} \cos \phi_{em}.$$

(3)

which leads to

$$d\phi_{em} = 2 \left| \frac{d\phi_{em}}{d\theta} \right| d\theta = \frac{2 \sin \theta \, d\theta}{\sqrt{\sin^2 \theta_p \sin^2 \theta_{em} - (\cos \theta_p \cos \theta_{em} - \cos \theta)^2}},$$

(4)

in which a factor of 2 was added to account for the fact that there are always two values of $\phi_{em}$ resulting in the same value of $\theta$ (see figure 2). The Cherenkov cone emission angle ($\theta_{em}$) relates to the particle velocity $\beta$ by the usual relation

$$\cos \theta_{em} = \frac{1}{\beta n}.$$

(5)

Substitution of equation 4 into equation 1 gives:

$$dN_\gamma = N_e(s) \frac{dN_e}{dE}(E,s) \frac{dN_e}{d\theta_p}(\theta_p, E) d\theta_p Y_\gamma(E,h) \frac{dX}{\cos \theta_p} \times \frac{1}{\pi \sqrt{\sin^2 \theta_p \sin^2 \theta_{em} - (\cos \theta_p \cos \theta_{em} - \cos \theta)^2}}.$$
Finally, to obtain the desired angular distribution of Cherenkov photons \( \frac{d^2N_\gamma}{d\theta dX} \) it is necessary to integrate equation 6 over all possible values of electron energies \( E \) and angles \( \theta_p \). Integration over \( E \) must assert that relation 5 is satisfied, therefore \( E \) takes values for which \( E > E_{\text{thr}}(h) \). Limits of the integral over electron angles \( \theta_p \) should take only values that contribute to \( \theta \). From figure 2 and equation 3, it is found that this interval is \( |\theta - \theta_{em}| < \theta_p < \theta + \theta_{em} \). Thus, the exact angular distribution of Cherenkov photons is given by

\[
\frac{d^2N_\gamma}{d\theta dX}(\theta, s, h) = \frac{1}{\pi} N_e(s) \sin \theta \int_{E_{\text{thr}}(h)}^{\infty} dE \frac{dN_e}{dE}(E, s) \times \int_{0}^{\theta + \theta_{em}} \frac{dN_e}{d\theta_p}(\theta_p, E) \frac{d\theta_p}{\cos \theta_p \sqrt{\sin^2 \theta_p \sin^2 \theta_{em} - (\cos \theta_p \cos \theta_{em} - \cos \theta)^2}}.
\]

(7)

III. APPROXIMATED MODEL FOR THE CHERENKOV ANGULAR DISTRIBUTION

In this section an approximation of the above equation is going to be proposed in order to obtain a simpler yet meaningful description of the angular distributions of Cherenkov light. The idea is to summarize the angular distribution to a minimum set of parameters allowing its parametrization.

First, note that the integration in \( \theta_p \) is done in a very narrow interval given that \( \theta_{em} < 1.5^\circ \). Therefore it is possible to consider that: \( \frac{1}{\cos \theta_p} \frac{dN_e(\theta_p, E)}{d\theta_p} \) varies little within integration limits and, in a first approximation, can be taken as constant and calculated in the mean angle \( \langle \theta_p \rangle \) of the range in between the limits of the integration:

\[
\frac{d^2N_\gamma}{d\theta dX}(\theta, s, h) \approx \frac{1}{\pi} N_e(s) \sin \theta \int_{E_{\text{thr}}(h)}^{\infty} dE \frac{dN_e}{dE}(E, s) \times \frac{1}{\cos \langle \theta_p \rangle} \frac{dN_e(\langle \theta_p \rangle, E)}{d\theta_p} \times \int_{|\theta - \theta_{em}|}^{\theta + \theta_{em}} \frac{d\theta_p}{\sqrt{\sin^2 \theta_p \sin^2 \theta_{em} - (\cos \theta_p \cos \theta_{em} - \cos \theta)^2}}.
\]

(8)

where

\[
\langle \theta_p \rangle = \begin{cases} 
\theta_{em}, & \text{if } \theta < \theta_{em} \\
\theta, & \text{if } \theta > \theta_{em}
\end{cases}
\]

(9)
The remaining integral over \( \theta_p \) is a complete elliptic integral of the first kind and can be approximated by a logarithmic function:

\[
\int_{|\theta - \theta_{em}|}^{\theta + \theta_{em}} \frac{d\theta_p}{\sqrt{\sin^2 \theta_p \sin^2 \theta_{em} - (\cos \theta_p \cos \theta_{em} - \cos \theta)^2}} \approx \frac{1}{\sin(\theta_p)} \times \begin{cases} 
\pi - \log \left(1 - \frac{\theta}{\theta_{em}}\right), & \text{if } \theta < \theta_{em} \\
\pi - \log \left(1 - \frac{\theta_{em}}{\theta}\right), & \text{if } \theta > \theta_{em}
\end{cases}
\]

The abbreviation below is introduced:

\[I(\theta, \theta_{em}, E) = \frac{1}{\sin(\theta_p)} \times \begin{cases} 
\pi - \log \left(1 - \frac{\theta}{\theta_{em}}\right), & \text{if } \theta < \theta_{em} \\
\pi - \log \left(1 - \frac{\theta_{em}}{\theta}\right), & \text{if } \theta > \theta_{em}
\end{cases}\]

and by noting that \( \cos \theta_{em} = 1/\beta n \) rapidly converges to \( 1/n \) as the electron energy increases, it is reasonable to assume that \( \cos \theta_{em} = 1/n \) for all electrons. With this assumption the function \( I(\theta, \theta_{em}, E) \sim I(\theta, \theta_{em}) = I(\theta, h) \) becomes independent of the electron energy.

\[
\frac{d^2 N_\gamma}{d\theta \ dX}(\theta, s, h) \approx \frac{1}{\pi} N_\gamma(s) \times \sin \theta \times I(\theta, h) \times \int_{E_{thr}(h)}^{\infty} dE Y_\gamma(E, h) \frac{dN_e}{dE}(E, s) \frac{1}{\cos(\theta_p)} \frac{dN_e}{d\theta_p}(\langle \theta_p \rangle, E).
\]

The validity of approximations done until here were tested using Monte Carlo simulations of air showers as shown in [A].

The remaining integral over electron energies:

\[
\int_{E_{thr}(h)}^{\infty} dE Y_\gamma(E, h) \frac{dN_e}{dE}(E, s) \frac{1}{\cos(\theta_p)} \frac{dN_e}{d\theta_p}(\langle \theta_p \rangle, E)
\]

has been studied before in references [23, 25]. A parametric form to describe this quantity is proposed here:

\[
K(\theta, s, h) = C \langle \theta_p \rangle^{\nu-1} e^{-\langle \theta_p \rangle/\theta_1} \left(1 + \epsilon e^{(\langle \theta_p \rangle/\theta_2)}\right),
\]

where \( \nu, \theta_1, \theta_2, \) and \( \epsilon \) are parameters varying with shower age, height (or refractive index), and, possibly, the primary energy. The constant \( C \) is intended to normalize equation (14).

\[4\] From now on \( \theta_{em} = \arccos(1/n) \).
according to equation (13). In the next section the parameters of this function are going to be studied and the quality of the description is going to be tested. The approximated model is summarized as:

\[
\frac{d^2 N_c}{d\theta \, dX}(\theta, s, h) = \frac{1}{\pi} N_c(s) \times \sin \theta \times I(\theta, h) \times K(\theta, s, h).
\] (15)

IV. PARAMETRIZATION OF THE CHERENKOV ANGULAR DISTRIBUTION

Monte Carlo simulations of air showers are done using CORSIKA 7.6900 package [26]. Gamma-ray and proton showers are simulated with energies between 100 GeV and 1 EeV in intervals of 1 in \(\log_{10}(E_0/eV)\). For each combination of primary type and energy, at least 120 showers are simulated. Simulations are performed for vertical showers and showers inclined at 20°. QGSJetII.04 [27] and urqmd [28] are used as high- and low-energy hadronic interaction models, respectively. The U.S. standard atmosphere model is used in the simulations and the refractive index is considered to be independent of the wavelength (180 nm ≤ \(\lambda\) ≤ 700 nm) of the emitted photons. Cherenkov photons are produced in bunches of maximum five. The COAST option is used to store the angle between the Cherenkov photons and the shower axis directions (\(\theta\)). \(X_{\text{max}}\), which is used to compute the shower age, is extracted from the longitudinal development of charged particles by fitting a Gaisser-Hillas function [29].

The approximated model summarized in equation (15) suggests that the angular distribution of Cherenkov photons should vary with age and height. The dependency on shower age and height is made clear in the upper plots of figure 3, where the angular distribution of Cherenkov photons of five randomly chosen gamma-ray-induced air showers for different values of \(s\) and \(h\) are shown. From cascade theory [30-32], the angular distributions of Cherenkov light in gamma-ray showers are expected to be energy independent and this is confirmed in the bottom-left plot of figure 3. In the case of proton showers, some dependency on the primary energy is observed in the bottom-right plot of the same figure. These plots also reiterate the fact that distributions with common age, height, and primary energy are similar.

Taking this dependency into account, the angular distribution of Cherenkov photons in a given interval with mean age \(\bar{s}\) and height \(\bar{h}\) in a shower of energy \(E_0\) can be described by:
\[
\frac{dN}{d\theta}(\theta, \bar{s}, \bar{h}, E_0) = N \times \sin \theta \times I(\theta, \bar{h}) \times K(\theta, \bar{s}, \bar{h}, E_0),
\]  

in which \(N\) (different from \(N_e(s)\)) is a normalization constant that depends on the parameters of \(K(\theta, \bar{s}, \bar{h}, E_0)\).

The parameters of \(K(\theta, \bar{s}, \bar{h}, E_0)\) are considered to be:

\[
\nu(s,n) = p_{0,\nu} (n - 1)^{p_{1,\nu}} + p_{2,\nu} \log(s), \\
\theta_1(s,n,E_0) = p_{0,\theta_1} (n - 1)^{p_{1,\theta_1}} (E_0/\text{TeV})^{p_{2,\theta_1}} + p_{3,\theta_1} \log(s), \\
\theta_2(s,n) = \theta_1(s,n) \times (p_{0,\theta_2} + p_{1,\theta_2} s), \\
\epsilon(E_0) = p_{0,\epsilon} + p_{1,\epsilon} (E_0/\text{TeV})^{p_{2,\epsilon}}.
\]  

The coefficients \(p_{i,\mu}\) are the parameters of the model to be fitted. In these equations, the dependence in height \(h\) was changed by the dependence in the refractive index \(n\) to make the parametrization independent of the atmospheric model used in the simulations.

The simulated angular distributions of Cherenkov photons are fitted with this model. For that, a multinomial likelihood function \((L_{\text{MLE}})\) is built taking into account every simulated distribution from shower ages in the interval \(0.8 \leq s \leq 1.2\). A single value of refractive index \(n\) is associated with each distribution according to the emission height. Histograms are weighted by the inverse of the primary energy in TeV, so the contribution from showers of distinct energies to \(L_{\text{MLE}}\) are of the same order of magnitude. Gamma-ray and proton showers are fit separately, as distributions strongly depend on the primary particle type in lower energies. All coefficients \(p_{i,\mu}\) are allowed to vary in the fit procedure. In the case of gamma showers, however, the energy dependency is dropped \((p_{2,\theta_1}, p_{1,\epsilon}, p_{2,\epsilon} = 0)\). Fitted values of \(p_{i,\mu}\) and their associated confidence intervals are found in tables I and II.

V. RESULTS

The parametrization proposed in the previous section is compared to the Monte Carlo distributions and to previous works. Figure 4 shows the simulated angular distribution of Cherenkov photons in comparison to four models for one single gamma-ray and one single proton shower. The ability of the model to be adjusted to simulated data both around the peak of the distributions and at the small and large \(\theta\) regions is evident.
Figure 5 shows the overall quality of the models by comparing the average relative deviation of the models to the simulated distribution for four combinations of primary type and energy. It is clear that the model proposed here has many advantages. The model developed here has an acceptable deviation from the simulations for a large angular range. For gamma showers, the deviation is very small (< 5%) for angles smaller than 25°. For proton showers, the deviation of the model proposed here improves with energy.

Results presented in this work are optimized with simulated showers having energies from 100 GeV (1 TeV, in case of proton) to 1 EeV. The quality of the model measured with respect to shower energy can be assessed in figure 6 where the average relative deviation is shown at $s = 1.0$ for all studied energies. It is seen that this deviation is smaller than 10% in a large angular and energy interval for both primaries. This confirms again the quality of the proposed model and ensures its adequacy to be employed in both the aforementioned regimes (a) and (b).

VI. CONCLUSION

To understand the nature and describe the angular distribution of Cherenkov photons in air showers is of great importance in current experimental astrophysics. An exact model has been derived in section II to describe the angular distribution photons in terms of the unknown energy-angular distribution of electrons. In section III, successive approximations to this exact model have lead to a factorized form for the angular distributions of Cherenkov photons: one term ($I$) depending on the maximum Cherenkov emission angle and a second term ($K$) depending on the energy-angular distribution of electrons. A simple parametric form has been proposed to describe this second term, overcoming the necessity of describing the two-dimensional energy-angular distribution of electrons. Parameters of this model have been obtained by fitting Monte Carlo simulations in section IV.

The direct comparison of the parametrization and Monte Carlo simulations in section V has shown the excellent capability of the model to describe the angular distributions of Cherenkov photons. The use of this model has many advantages as it is able to: 1) cover both small and large angular regions, including the peak around $\theta_{em}$ and 2) cover a large energy interval, from hundreds of GeV to EeV energies. The parametrization presented here is therefore adequate to be employed in both the reconstruction of gamma-rays and
cosmic-rays in IACT systems and also in the study of extensive air showers with fluorescence detectors.

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Appendix A: Validation of approximations

In this appendix the models presented in section II (exact) and section III (approximated) are compared to a direct simulation of the angular distribution of Cherenkov photons. This comparison is shown in figure 7 for the case of a vertical 1 PeV gamma-ray air shower at three different shower ages. The small inset plot shows the region of large angles ($\theta > 5^\circ$). The angular distribution of Cherenkov photons directly extracted from the simulation (reference) is shown as a black solid line. The dashed blue (exact model) and solid orange (approximated model) lines show the computation of the angular distribution of Cherenkov photons using equations 7 and 15, respectively. For these computations, the energy and angular distribution of electrons ($dN_e/dE$ and $dN_e/d\theta_p$) were extracted from the same simulation.

From figure 7 it is seen that in the region of $\theta > 5^\circ$ (inset plot) the curves of both models appear superimposed with the reference distribution for the three ages being shown. Further insight about the quality of these models in the region of smaller angles can be obtained by inspection of figure 8 where the relative deviations between both models and the reference distribution are studied.

The exact model presents no deviation with respect to the reference distribution, except in the region around the peak of this distribution where a deviation of < 10% is found, as can be seen in figure 8. However, this may be attributed as a side effect of binning the electron distributions ($dN_e/dE$ and $dN_e/d\theta_p$) used as input in equation 7.
The approximated model of section III on the other hand, deviates less than 10% from the reference distribution at \( s = 1.0 \) (center plot). At the ages of \( s = 0.8 \) (upper plot) and \( s = 1.2 \) (lower plot), on the other hand, this deviation is typically smaller than 20%, except at the peak. While this approximation is not as good as the exact model, it validates the idea that it is possible to approximately reproduce the shape of the angular distribution of Cherenkov photons as a product of two functions, as claimed in section III.

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TABLE I. Coefficients describing the angular distribution of Cherenkov photons in gamma-ray air showers.

| µ   | $p_{0,\mu}$ (±err) | $p_{1,\mu}$ (±err) | $p_{2,\mu}$ (±err) | $p_{3,\mu}$ (±err) |
|-----|-------------------|-------------------|-------------------|-------------------|
| $\nu$ | 0.34329 (0.00006) | -0.10683 (0.00002) | 1.46852 (0.00004) | -                 |
| $\theta_1$ | 1.4053 (0.0002) | 0.32382 (0.00002) | 0 | -0.048841 (0.000003) |
| $\theta_2$ | 0.95734 (0.00008) | 0.26472 (0.00005) | - | - |
| $\epsilon$ | 0.0031206 (0.0000006) | 0 | 0 | - |

TABLE II. Coefficients describing the angular distribution of Cherenkov photons in proton air showers.

| µ   | $p_{0,\mu}$ (±err) | $p_{1,\mu}$ (±err) | $p_{2,\mu}$ (±err) | $p_{3,\mu}$ (±err) |
|-----|-------------------|-------------------|-------------------|-------------------|
| $\nu$ | 0.21155 (0.00006) | -0.16639 (0.00003) | 1.21803 (0.00006) | -                 |
| $\theta_1$ | 4.513 (0.001) | 0.45092 (0.00003) | -0.008843 (0.000002) | -0.058687 (0.000006) |
| $\theta_2$ | 0.90725 (0.00008) | 0.41722 (0.00005) | - | - |
| $\epsilon$ | 0.009528 (0.000002) | 0.022552 (0.000007) | -0.4207 (0.0002) | - |
FIG. 1. Definition of the relevant angles for Cherenkov light emission in air showers. The blue cone represents the emission of Cherenkov photons around the emitting electron trajectory, in orange. The final angle between each Cherenkov photon (blue trajectory) and the shower axis is denoted by $\theta$.

FIG. 2. Depiction of the intersecting region between the Cherenkov cone (blue ring) and the ring of width $d\theta$ around the angle $\theta$ (grey ring) in the unit sphere. There are two intersection points whenever $|\theta - \theta_p| < \theta_{em}$ and none otherwise.
FIG. 3. Examples of simulated angular distributions of Cherenkov photons highlighting its dependency with respect to shower age (top left), emission height (top right), primary energy in gamma-ray showers (bottom left), and primary energy in proton showers (bottom left).
FIG. 4. Angular distribution of Cherenkov photons from a single gamma-ray (top) and proton (bottom) shower. CORSIKA simulations are compared to the parametrized distributions (thick, solid lines) at $s = 0.8$ (red), 1.0 (blue), and 1.2 (green). Predictions from Refs. [8, 23, 25] are shown for comparison (see legend). Curves of a common shower age are vertically displaced all together for better visualization.
FIG. 5. Average relative deviation between parametrized and simulated angular distributions of Cherenkov photons at $s = 0.8$ (red), 1.0 (blue), and 1.2 (green). Each box depicts a single primary-energy combination. Parametrization of this work (solid, thick lines) is compared to predictions from Refs. 8, 23, 25 (see legend in the upper right box).
FIG. 6. Average relative deviation between parametrized and simulated angular distributions of Cherenkov photons at $s = 1.0$ for gamma-ray (left) and proton (right) showers at various primary energies.

FIG. 7. Comparison of the exact (blue dashed) and the approximated (orange solid) models of sections II and III to a simulated angular distribution of Cherenkov photons at three shower ages. The inset plot shows the region $\theta > 5^\circ$. 

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FIG. 8. Relative deviation between models of sections II (blue dashed) and III (orange solid) and the simulated angular distribution of Cherenkov photons in the region of small angles. Curves are shown for three shower ages, indicated in the boxes.