FROM GAUGE ANOMALIES TO GERBES AND GERBAL ACTIONS

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ABSTRACT The purpose of this contribution is to point out connections between recent ideas about gerbes and gerbal actions (as higher categorical extension of representation theory) and old discussion in quantum field theory on commutator anomalies, gauge group extensions, and 3-cocycles. The unifying concept is the classical obstruction theory for group extensions as explained in the reference [ML]. [Based on talk given at the meeting “Motives, Quantum Field Theory, and Pseudodifferential operators”, Boston University, June 2-13, 2008]

1. INTRODUCTION

It was first realized through perturbative analysis of gauge theories that the gauge symmetry is broken in presence of chiral fermions, [ABJ]. Later, it was found that this phenomenon is related to index theory of (families) of Dirac operators. In particular, the effective action functional, defined as a regularized determinant of the Dirac operator, is not always gauge invariant and the lack of invariance can be formulated as the curvature of a complex line bundle, the determinant line bundle, over the moduli space of gauge connections, [AS].

In the hamiltonian formulation of gauge theory the symmetry breaking manifests itself as a modification of the commutation relations of the Lie algebra of infinitesimal gauge transformations. The gauge algebra (in case of trivial vector bundles) is the Lie algebra of functions $M_g$ from the physical space $M$ to a finite-dimensional Lie algebra $g$. The commutation relations of the modified algebra can be written as

$$[(X,a),(Y,b)] = ([X,Y], c(A; X,Y))$$
where \([X,Y]\) is the point-wise commutator in \(M\mathfrak{g}\) and \(a, b\) are complex valued functions of the gauge potential \(A\). \(c\) is a Lie algebra 2-cocycle determining an extension of \(M\mathfrak{g}\). In the case when \(M\) is the unit circle \(S^1\) it turns out that \(c\) is independent of \(A\) and we have a central extension defining (when \(\mathfrak{g}\) is simple) an affine Kac-Moody algebra.

When \(\dim M > 1\) the cocycle \(c\) depends explicitly on \(A\). It is still an open question whether this algebra has interesting faithful Hilbert space representations, analogous to the highest weight representations of affine Lie algebras extensively used in string theory and constructions of quantum field theory models in 1+1 space-time dimensions. What is known at present that there are natural unitary Hilbert bundle actions of the extended gauge Lie algebras and groups. These come from quantizing chiral fermions in background gauge fields. For each gauge connection \(A\) there is a fermionic Fock space \(\mathcal{F}_A\) where the quantized Dirac Hamiltonian \(\hat{D}_A\) acts. This family of (essentially positive) Dirac operators transforms equivariantly with respect to the action of the extension of the group \(MG\). The gauge transformations are defined as projective unitary operators between the fibers \(\mathcal{F}_A\) and \(\mathcal{F}_{A\nu}\) of the Fock bundle, corresponding to a true action of the gauge group extension. As a consequence, the Fock bundle is defined only as a projective bundle over the moduli space \(A/MG\). Actually, in order that the moduli space is a smooth manifold, one has to restrict \(MG\) to the based gauge transformations which are functions on \(M\) taking the value \(e \in G\) at a fixed base point \(x_0 \in M\).

A projective bundle is completely determined, up to equivalence, by the Dixmier-Douady class which is an element of \(H^3(A/MG, \mathbb{Z})\). This is the origin of gerbes in quantum field theory, [CMM]. Topologically a gerbe on a space \(X\) is just an equivalence class of \(PU(H)\) bundles over \(X\). In terms of Čech cohomology subordinate to a good cover \(\{U_\alpha\}\) of \(X\), the gerbe is given as a \(\mathbb{C}^\times\) valued cocycle \(\{f_{\alpha\beta\gamma}\}\),

\[
f_{\alpha\beta\gamma}f_{\alpha\beta\delta}^{-1}f_{\alpha\gamma\delta}f_{\beta\gamma\delta}^{-1} = 1
\]

on intersections \(U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta\). This cocycle arises from the lifting problem: A \(PU(H)\) bundle is given in terms of transition functions \(g_{\alpha\beta}\) with values in \(PU(H)\). After lifting these to \(U(H)\) one gets a family of functions \(\hat{g}_{\alpha\beta}\) which satisfy the 1-cocycle condition up to a phase,

\[
\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = f_{\alpha\beta\gamma}\mathbf{1}.
\]

The notion of gerbal action was introduced in the recent paper [FZ]. This is to be viewed as the next level after projective actions related to central extensions of groups and is given in terms of third group cohomology. In fact, the appearance of third cohomology in this context is not new and is related to group extensions as explained in [ML]. In the simple form, the problem is the following. Let \(F\) be an extension of \(G\) by the group \(N\),

\[
1 \to N \to F \to G \to 1
\]
an exact sequence of groups. Suppose that $1 \to a \to \hat{N} \to N \to 1$ is a central extension by the abelian group $a$. Then one can ask whether the extension $F$ of $G$ by $N$ can be prolonged to an extension of $G$ by the group $\hat{N}$. An obstruction to this is an element in the group cohomology $H^3(G, a)$ with coefficients in $a$. In case of Lie groups, there is a corresponding Lie algebra cocycle representing a class in $H^3(g, a)$. We shall demonstrate this in detail for an example arising from quantization of gauge theory. It is closely related to the idea in [Ca], further elaborated in [CGRS], which in turn was a response to a discussion in the 80’s on breaking of Jacobi identity for the field algebra in Yang-Mills theory [GJJ].

The paper is organized as follows. In Section 2 we explain the gauge group extensions arising from action on bundles of fermionic Fock spaces over background gauge fields and the corresponding Lie algebra cocycles. Section 3 consists of a general discussion how gerbal action arises from the group of outer automorphisms of an associative algebra, how this leads to a 3-cocycle on the symmetry group, and finally we give an example coming from Yang-Mills theory in 1+1 space-time dimensions. Section 4 contains a generalization to Yang-Mills theory in higher space-time dimensions. Finally in Section 5 we explain an application to twisted K-theory on moduli space of gauge connections.

2. BUNDLES OF FOCK SPACES OVER GAUGE CONNECTIONS

A basic problem in quantum field theory in higher than two space-time dimensions is that the representations of canonical anticommutation relations algebra (CAR) are not equivalent in different background gauge fields, and this leads to various divergencies in perturbation theory. However, in the case of the linear problem of quantizing fermions in a background gauge field one can construct the hamiltonian and the Hilbert space in a nonperturbative way. One can actually avoid the divergencies by taking systematically into account the need of dealing with a family of nonequivalent CAR algebra representations.

The method introduced in [Mi93] and generalized in [LM] is based on the observation that for each gauge connection $A$ in the family $\mathcal{A}$ of all gauge connections on a vector bundle $E$ over a compact spin manifold, one can choose a unitary operator $T_A$ in the Hilbert space $H$ of $L^2$ sections in the tensor product of the spin bundle and the vector bundle $E$ such that the Dirac hamiltonian $D_A$ is conjugated to $\tilde{D}_A = T_A D_A T_A^{-1}$ such that the equivalent hamiltonian $\tilde{D}_A$ can be quantized in the ”free” Fock space, the Fock space for a fixed background connection $A_0$. In the case of a trivial bundle $E$ one can take as $A_0$ the globally defined gauge connection represented by 1-form equal to zero.

The action of the group $\mathcal{G}$ of smooth gauge transformations $A \mapsto A^g = g^{-1} A g +$
$g^{-1}dg$ on the family $\tilde{D}_A$ is then given by

$$\tilde{D}_A \mapsto \omega(A; g)^{-1} \tilde{D}_A \omega(A; g)$$

corresponding to $D_A \mapsto g^{-1}D_A g$ where $\omega(A; g) = T_A g T_A^{-1}$ satisfies the 1-cocycle relation

$$\omega(A; gg') = \omega(A; g)\omega(A g; g').$$

Furthermore, the cocycle satisfies the condition $[\epsilon, \omega(A; g)]$ is Hilbert-Schmidt. Here $\epsilon$ is the sign $D_{A_0}/|D_{A_0}|$ of the free Dirac operator. This means that the operators $\omega(A; g)$ belong to the restricted unitary group $U_{res}(H_+ \oplus H_-)$ where $H = H_+ \oplus H_-$ is the polarization of $H$ with respect to the sign operator $\epsilon$. [PS].

The quantization of the operator $\tilde{D}_A$ is obtained in a fermionic Fock space $\mathcal{F}$ which carries an irreducible representation of the CAR algebra $\mathcal{B}$ which is a completion of the algebra defined by generators and relations according to

$$a^*(u)a(v) + a(v)a^*(u) = 2 <v, u>$$

and all other anticommutators equal to zero. Here $u, v \in H$ and $<\cdot, \cdot>$ is the Hilbert space inner product (antilinear in the first argument). The representation is fixed (up to equivalence) by the requirement that there exists a vacuum vector $|0> \in \mathcal{F}$ such that

$$a(u)|0> = a^*(v)|0>,$$

for $u \in H_+, v \in H_-.$

The group $U_{res}(H)$ has a central extension by $S^1$ such that the Lie algebra central extension is given by the 2-cocycle $c(X, Y) = \frac{1}{4} \text{tr} [\epsilon, X][\epsilon, Y]$. The central extension $\hat{U}_{res}$ has a unitary representation $g \mapsto \hat{g}$ in $\mathcal{F}$ fixed by the requirement

$$\hat{g}a^*(u)\hat{g}^{-1} = a^*(gu)$$

for all $u \in H.$

If we choose a lift $\hat{\omega}(A; g)$ of the element $\omega(A, g)$ to unitaries in the Fock space $\mathcal{F}$ we can write

$$\hat{\omega}(A; gg') = \Phi(A; g, g')\hat{\omega}(A; g)\hat{\omega}(A^g; g')$$

where $\Phi$ takes values in $S^1$. It is a 2-cocycle by construction,

$$\Phi(A; g, g')\Phi(A; gg', g'') = \Phi(A; g, g'g'')\Phi(A^g; g', g''),$$

which is smooth in an open neighborhood of the neutral element in $\mathcal{G}$. This just reflects the associativity in the group multiplication in the central extension $\hat{U}_{res}$. 
Taking the second derivative
\[
\frac{d^2}{dt\,ds}\bigg|_{t=s=0}\Phi(A; e^{tX}, e^{sY}) = \frac{1}{2}c(A; X, Y)
\]
gives a 2-cocycle \(c\) for the Lie algebra of \(G\) with coefficients in the ring of complex functions of the variable \(A\).

The cocycle depends on the lift \(\omega \mapsto \hat{\omega}\) but two lifts are related by a multiplication by a circle valued function \(\psi(A; g)\) and the corresponding 2-cocycles are related by a coboundary,
\[
\Phi'(A; g, g') = \Phi(A, g, g')\psi(A; gg')\psi(A; g)^{-1}\psi(A^g; g')^{-1}.
\]

The Lie algebra cocycle \(c\) satisfies
\[
c(A; X, [Y, Z]) + \mathcal{L}_X c(A; Y, Z) + \text{cyclic permutations} = 0,
\]
where \(\mathcal{L}_X\) is the Lie derivative acting on functions \(f(A)\) through infinitesimal gauge transformation, \((\mathcal{L}_X f)(A) = Df(A) \cdot ([A, X] + dX)\).

Explicit expressions for the cocycle \(c\) have been computed in the literature, for example if the physical space is a circle we get the central extension of a loop algebra (affine Kac-Moody algebra),
\[
(2.1) \quad c(A; X, Y) = \frac{1}{2\pi} \int_{S^1} \text{tr} \, XdY,
\]
where the trace is evaluated in a finite dimensional representation of \(G\). In this case \(c\) does not depend on \(A\) and the abelian extension reduces to a central extension. This reflects the fact that elements of \(LG\) act in the Hilbert space \(H\) through an embedding \(LG \to U_{res}\) and we can simply choose \(T_A \equiv 1\) for all gauge connections \(A\).

In three dimensions the simplest expression for the cocycle is, [Fa], [Mi85],
\[
(2.2) \quad c(A; X, Y) = \frac{1}{24\pi^2} \int_M \text{tr} \, A[dX, dY].
\]

3. GERBAL ACTIONS AND 3-COCYCLES

Let \(\mathcal{B}\) be an associative algebra and \(G\) a group. Assume that we have a group homomorphism \(s : G \to \text{Out}(\mathcal{B})\) where \(\text{Out}(\mathcal{B})\) is the group of outer automorphims
of $B$, that is, $\text{Out}(B) = \text{Aut}(B)/\text{In}(B)$, all automorphisms modulo the normal subgroup of inner automorphisms. If one chooses any lift $\tilde{s} : G \to \text{Aut}(B)$ then we can write
\[
\tilde{s}(g)\tilde{s}(g') = \sigma(g, g') \cdot \tilde{s}(gg')
\]
for some $\sigma(g, g') \in \text{In}(B)$. From the definition follows immediately the cocycle property
\[
(3.1) \quad \sigma(g, g')\sigma(gg', g'') = [\tilde{s}(g)\sigma(g', g'')\tilde{s}(g)^{-1}\sigma(g, g'')^{-1}]\sigma(g, g'') \text{ for all } g, g', g'' \in G.
\]

Let next $H$ be any central extension of $\text{In}(B)$ by an abelian group $a$. That is, we have an exact sequence of groups,
\[
1 \to a \to H \to \text{In}(B) \to 1.
\]
Let $\hat{\sigma}$ be a lift of the map $\sigma : G \times G \to \text{In}(B)$ to a map $\hat{\sigma} : G \times G \to H$. We have then
\[
\hat{\sigma}(g, g')\hat{\sigma}(gg', g'') = [\tilde{s}(g)\hat{\sigma}(g', g'')\tilde{s}(g)^{-1}\hat{\sigma}(g, g'') \cdot \alpha(g, g', g'') \text{ for all } g, g', g'' \in G.
\]
where $\alpha : G \times G \times G \to a$. Here the action of the outer automorphism $s(g)$ on $\hat{\sigma}(\ast)$ is defined by $s(g)\hat{\sigma}(\ast)s(g)^{-1} = \text{the lift of } s(g)\sigma(\ast)s(g)^{-1} \in \text{In}(B)$ to an element in $H$. One can show that $\alpha$ is a 3-cocycle [S. MacLane, Lemma 8.4],
\[
\alpha(g_2, g_3, g_4)\alpha(g_1g_2, g_3, g_4)^{-1}\alpha(g_1, g_2g_3, g_4)\alpha(g_1, g_2, g_3g_4)^{-1}\alpha(g_1, g_2, g_3) = 1.
\]

Next we construct an example from quantum field theory. Let $G$ be a compact Lie group and $P$ the space of smooth paths $f : [0, 1] \to G$ with initial point $f(0) = e$, the neural element, and quasiperiodicity condition $f^{-1}df$ a smooth function.

$P$ is a group under point-wise multiplication but it is also a principal $\Omega G$ bundle over $G$. Here $\Omega G \subset P$ is the loop group with $f(0) = f(1) = e$ and $\pi : P \to G$ is the projection to the end point $f(1)$. Fix an unitary representation $\rho$ of $G$ in $\mathbb{C}^N$ and denote $H = L^2(S^1, \mathbb{C}^N)$.

For each polarization $H = H_- \oplus H_+$ we have a vacuum representation of the CAR algebra $\mathcal{B}(H)$ in a Hilbert space $\mathcal{F}(H_+)$. Denote by $\mathcal{C}$ the category of these representations. Denote by $a(v), a^*(v)$ the generators of $\mathcal{B}(H)$ corresponding to a vector $v \in H$,
\[
a^*(u)a(v) + a(v)a^*(u) = 2 < v, u >
\]
and all the other anticommutators equal to zero.

Any element $f \in P$ defines a unique automorphism of $\mathcal{B}(H)$ with $\phi_f(a^*(v)) = a^*(f \cdot v)$, where $f \cdot v$ is the function on the circle defined by $\rho(f(x))v(x)$. These automorphisms are in general not inner except when $f$ is periodic. We have now a
map $s : G \rightarrow \text{Aut}(B)/\text{In}(B)$ given by $g \mapsto F(g)$ where $F(g)$ is an arbitrary smooth quasiperiodic function on $[0, 1]$ such that $F(g)(1) = e$. Any two such functions $F(g), F'(g)$ differ by an element $\sigma$ of $\Omega G$, $F(g)(x) = F'(g)(x)\sigma(x)$. Now $\sigma$ is an inner automorphism through a projective representation of the loop group $\Omega G$ in $\mathcal{F}(H_+)$. In an open neighborhood $U$ of the neutral element $e$ in $G$ we can fix in a smooth way for any $g \in U$ a path $F(g)$ with $F(g)(0) = e$ and $F(g)(1) = g$. Of course, for a connected group $G$ we can make this choice globally on $G$ but then the dependence of the path $F(g)$ would not be a continuous function of the end point. For a pair $g_1, g_2 \in G$ we have $\sigma(g_1, g_2)F(g_1g_2) = F(g_1)F(g_2)$ with $\sigma(g_1, g_1) \in \Omega G$.

For a triple of elements $g_1, g_2, g_3$ we have now

$$F(g_1)F(g_2)F(g_3) = \sigma(g_1, g_2)F(g_1g_2)F(g_3) = \sigma(g_1, g_2)\sigma(g_1g_2, g_3)F(g_1g_2g_3).$$

In the same way,

$$F(g_1)F(g_2)F(g_3) = F(g_1)\sigma(g_2, g_3)F(g_2g_3) = [g_1\sigma(g_2, g_3)g_1^{-1}]/F(g_1)F(g_2g_3)$$

$$= [g_1\sigma(g_2, g_3)g_1^{-1}]\sigma(g_1, g_2g_3)F(g_1g_2g_3)$$

which proves the cocycle relation (3.1).

Lifting the loop group elements $\sigma$ to inner automorphims $\hat{\sigma}$ through a projective representation of $\Omega G$ we can write

$$\hat{\sigma}(g_1, g_2)\hat{\sigma}(g_1g_2, g_3) = \text{Aut}(g_1)[\hat{\sigma}(g_2, g_3)]\hat{\sigma}(g_1, g_2g_3)\alpha(g_1, g_2, g_3),$$

where $\alpha : G \times G \times G \rightarrow \mathbb{S}^1$ is some phase function arising from the fact that the projective lift is not necessarily a group homomorphism.

An equivalent point of view to the construction of the 3-cocycle $\alpha$ is this: We are trying to construct a central extension $\tilde{P}$ of the group $P$ of paths in $G$ (with initial point $e \in G$) as an extension of the central extension over the subgroup $\Omega G$. The failure of this central extension is measured by the cocycle $\alpha$, as an obstruction to associativity of $\tilde{P}$. On the Lie algebra level, we have a corresponding cocycle $c_3 = d\alpha$ which is easily computed. The cocycle $c$ of $\Omega g$ extends to the path Lie algebra $Pg$ as

$$c(X, Y) = \frac{1}{4\pi i} \int_{[0, 2\pi]} \text{tr} (X dY - Y dX).$$

This is an antisymmetric bilinear form on $Pg$ but it fails to be a Lie algebra 2-cocycle. The coboundary is given by

$$(\delta c)(X, Y, Z) = c(X, [Y, Z]) + c(Y, [Z, X]) + c(Z, [X, Y])$$

$$= -\frac{1}{4\pi} \text{tr} X[Y, Z] |_{2\pi} = d\alpha(X, Y, Z).$$
Thus $\delta c$ reduces to a 3-cocycle of the Lie algebra $\mathfrak{g}$ of $G$ on the boundary $t = 2\pi$. This cocycle defines by left translations on $G$ the left-invariant de Rham form $-\frac{1}{12\pi^3}\text{tr} (g^{-1}dg)^3$; this is normalized as $2\pi i$ times an integral 3-form on $G$.

Let $f_1, f_2 \in P$ and $f_{12} \in P$ with the property $f_1(2\pi)f_2(2\pi) = f_{12}(2\pi)$. Then we have a fiber $S^1$ over the loop $\phi_{12} = f_1(t)f_2(t)f_{12}(t)^{-1}$ coming from the central extension $\widehat{\Omega} G \to \Omega G$. Concretely, this fiber can be realized geometrically as a pair $(f, \lambda)$ where $f : D \to G$, $D$ is the unit disk with boundary $S^1$ such that the restriction of $f$ to $S^1$ is the loop above, and $\lambda \in S^1$. Two pairs $(f, \lambda), (f', \lambda')$ are equivalent if the restrictions of $f, f'$ to the boundary are equal and $\lambda' = \lambda e^{W(f,f')}$ where

$$W(f,f') = \frac{1}{12\pi^3} \int_B \text{tr} (g^{-1}dg)^3,$$

where $B$ is a unit ball with boundary $S^2 = S^2_+ \cup S^2_-$ and $g$ is any extension to $B$ of the map $f \cup f'$ on $S^2$ obtained by joining $f, f'$ on the boundary circle $S^1$ of the unit disk $D$. The product in the central extension of the full loop group $LG$ is then defined as

$$[(f, \lambda)] \cdot [(f', \lambda')] = [(ff', \lambda' e^{\gamma(f,f')})],$$

where

$$\gamma(f,f') = \frac{1}{4\pi i} \int_D \text{tr} f^{-1}df \wedge df'^{-1},$$

see [Mi87] for details; here the square brackets mean equivalence classes of pairs, subject to the equivalence defined above.

The 3-cocycle $\alpha$ can now be written in terms of the local data as

$$\alpha = \exp[\gamma(\phi_{12}, \phi_{12,3}) - \gamma(\text{Aut} f_1, \phi_{12,3}) + W(h)]$$

where $\phi_{12,3}$ is an extension to the disk of the loop composed from the paths $f_1f_2$ and $f_3$ and the path $f_{12,3}$ joining the identity $e$ to $g_1g_2g_3$. $h$ is the function on $D$, equal to the neutral element on the boundary, such that $\phi_{12}\phi_{12,33} = \phi_{12}\phi_{12,3}h$. The value of $\alpha$ depends now, besides on the paths $f_i$, on the extensions $\phi$ to the disk $D$ of the boundary loops determined by $f_i, f_{12}, f_{23}, f_{12,3}, f_{1,23}$. However, different choices of extensions are related by phase factors which can be obtained from the equivalence relation

$$(\phi, \lambda) \equiv (\phi h, e^{\gamma(\phi,h)+W(h)})$$

defining the central extension of the loop group.

4. THE CASE OF HIGHER DIMENSIONS

The construction of the gerbal action of $G$ has a generalization which comes from a study of gauge anomalies in higher dimensions. Fix again a compact Lie group
$G$ and denote now by $MG$ the group of smooth maps from a compact manifold $M$ to $G$, which is an infinite-dimensional Lie group under point-wise multiplication of maps. Assume also that $M$ is a boundary of a compact manifold $N$ and denote by $NG$ the group of maps from $N$ to $G$ such that the normal derivatives at the boundary $M$ vanish in all orders. Finally, let $G$ be the normal subgroup of $NG$ consisting of maps equal to the constant $e \in G$ on the boundary. This will play the role of $\Omega G$ in the previous section. Now $NG/G = MG$.

We also assume that $M$ is a boundary of a contractible manifold $N'$ and denote by $\overline{N}$ the manifold obtained from $N, N'$ by gluing along the common boundary. We also assume a spin structure and Riemann metric given on $\overline{N}$.

We may view elements $f \in NG$ as $g$ valued vector potentials on the space $\overline{N}$. This correspondence is given by $f \mapsto A = f^{-1}df$ on $N$ and $A'$ is fixed from the boundary values $A|_M$ and from a contraction of $N'$ to one point. This construction gives a map from the group $MG$ to the moduli space of gauge connections in a trivial vector bundle over $\overline{N}$.

**Example** Let $M = S^n$, viewed as the equator in the sphere $S^{n+1}$. Fix a path $\alpha$ from the South Pole of $S^{n+1}$ to the North Pole. Then for any great circle joining the North Pole to the South Pole we can take the union with $\alpha$, giving a loop starting from the South Pole and traveling via the North Pole. For given vector potential $A$ let $g_A$ be the holonomy around this loop. The great circles are parametrized by points on the equator $S^n$ and thus we obtain a map $S^n \to g_A$. The group $G$ of *based* gauge transformations, those which are equal to the identity on the South Pole, does not affect the holonomy, thus we obtain a map from the moduli space $A/G$ of gauge connections to $S^n G$. The same construction can be made for any $n$-sphere $S^n$ around the South Pole, and we may view $S^{n+1}$ as the $(n+1)$-dimensional solid ball $N$ with boundary $M = S^n$ contracted to one point, the South Pole of $S^{n+1}$. Here $G$ is viewed as the group $G$ valued maps on $N$ equal to the constant $e$ on the boundary. In this case one can actually show that the gauge moduli space $A/G$ is homotopy equivalent to $MG$.

The main difference as compared to the case of loop group is that the transformations $f \in G$ are not in general implementable Bogoliubov automorphisms. However, as explained in Section 2, they are well-defined automorphism of a Hilbert bundle $\mathcal{F}$ over $\mathcal{A}$, the space of $g$ valued connections on $N$. The fibers of this bundle are fermionic Fock spaces, each of them carries an (inequivalent) representation of the canonical anticommutation relations, with a Dirac vacuum which depends on the background field $A \in \mathcal{A}$. The group $G$ acts on this bundle through an abelian extension $\widehat{G}$.

The 3-cocycle is constructed in essentially the same way as in Section 3. So for an element $g \in MG$ select $f \in NG$ such that the restriction of $f$ to the boundary is equal to $g$. For any pair $g_1, g_2 \in MG$ we have then $f_1 f_2 f_1^{-1} \in G$ where again $f_{12} \in NG$ such that $f_{12}|_M = g_1 g_2$. For a triple $g_1, g_2, g_3 \in MG$ we then construct
\[ \alpha(A; g_1, g_2, g_3) \in S^1 \] as before, but now it depends on the connection \( A \) since the operators \( \hat{\sigma} \) now all depend on \( A \).

\[ \hat{\sigma}_{A'}(g_1, g_2)\hat{\sigma}_A(g_1 g_2, g_3) = \text{Aut}(g_1)[\hat{\sigma}_{A''}(g_2, g_3)]\hat{\sigma}_A(g_1, g_2 g_3)\alpha(A; g_1, g_2, g_3), \]

where \( A' = A^{\sigma(g_1 g_2, g_3)} \) is the gauge transform of \( A \) by \( \sigma(g_1 g_2, g_3) \in G \) and similarly \( A'' \) is the gauge transform of \( A \) by \( \sigma(g_1, g_2 g_3) \).

**Example** Again, passing to the Lie algebra cocycles one gets reasonably simple expressions. For example, in the case \( \dim M = 2 \) the Lie algebra extension of \( \text{Lie}(G) \) is given by the 2-cocycle (2.2) and for a manifold \( N \) with boundary \( M \) this formula is not a cocycle but its coboundary is the Lie algebra 3-cocycle

\[(4.1) \quad d\alpha(X, Y, Z) = -\frac{1}{8\pi^2} \int_M \text{tr} X[dY, dZ].\]

In this case the cocycle does not depend on the variable \( A \) but when \( \dim M > 2 \) it does.

### 5. Twisted K-Theory on Moduli Spaces of Gauge Connections

Let \( P \) be a principal bundle over a space \( X \) with model fiber equal to the projective unitary group \( \text{PU}(H) = U(H)/S^1 \) of a complex Hilbert space. It is known that equivalence classes of such bundles are classified by elements in \( H^3(X, \mathbb{Z}) \), the Dixmier-Douady class of the bundle, [DD].

K-theory of \( X \) twisted by \( P \), denoted by \( K^*(X, P) \), is defined as the abelian group of homotopy classes of sections of a bundle \( Q \), defined as an associated bundle with fiber equal to the \((\mathbb{Z}_2 \text{ graded})\) space of Fredholm operators in \( H \) with \( \text{PU}(H) \) action given by the conjugation \( T \mapsto gTg^{-1} \). The grading is as in ordinary complex K theory: The even sector is defined by the space of all Fredholm operators whereas the odd sector is defined by self-adjoint operators with both positive and negative essential spectrum. As a model, one can use either bounded Fredholm operators, or unbounded operators for example with the graph topology, [AtSe].

If \( X = G \) is a compact Lie group one can construct elements of \( K^*(G, P) \) in terms of highest weight representation of the central extension \( \hat{\text{LG}} \), [Mi04]. Actually, these come as \( G \) equivariant classes, under the conjugation action of \( G \) on itself. In the equivariant case the construction of \( K^*(G, P) \) is related to the Verlinde algebra in conformal field theory, [FHT]. Although for simple compact Lie groups there exists classification theorems [Do], [Br] also in the nonequivariant case it is still an open problem how to give explicit constructions for all classes in the nonequivariant
case, in terms of families of Fredholm operators, using representation theory even for unitary groups $SU(n)$ when $n > 3$.

Let $\omega : A \times \mathbb{G} \rightarrow U_{res}(H_+ \oplus H_-)$ be the 1-cocycle constructed in Section 2. Let $Y$ be a family of Fredholm operators in $\mathcal{F}$ which is mapped onto itself under a projective representation $g \mapsto \hat{g}$ of $U_{res}$ in $\mathcal{F}$, $T \mapsto \hat{g}T\hat{g}^{-1} \in Y$ for any $T \in Y$.

Now we have an action of a central extension of the groupoid $(A, \mathbb{G})$ on $Y$ by

$$(A, g) : Y \rightarrow Y, T \mapsto \hat{\omega}(A; g)T\hat{\omega}(A; g)^{-1}. $$

We can also view this as a central extension of the transformation groupoid defined by the action of the gauge group $\mathbb{G}$ on the space $A \times Y$. If $\mathbb{G}$ is the group of based gauge transformations then it acts freely on $A$ and therefore also freely on $A \times Y$. If furthermore $Y$ is contractible then $(A \times Y)/\mathbb{G} \simeq A/\mathbb{G}$ is the gauge moduli space.

A system of Fredholm operators transforming covariantly under $U_{res}$ can be constructed from a Dirac operator on the infinite-dimensional Grassmann manifold $Gr_{res} = U_{res}(H_+ \oplus H_-)/(U(H_+) \times U(H_-))$, [Tä]. The members of the family are parametrized by a gauge connection on a complex line bundle $L$ over $Gr_{res}$. The line bundle $L$ is used as twisting of the spin bundle over $Gr_{res}$, and can be viewed as defining a spin$_{C}$ structure on $Gr_{res}$.

In the case when $A$ is the space of gauge connections on the circle and $\mathbb{G} = LG$ is a loop group the cocycle $\omega(A; g)$ does not depend on $A$ and it gives a unitary representation of $LG$ in the Hilbert space $H$ and $g \mapsto \hat{g}$ is given by a representation of a central extension $\widehat{LG}$ in $\mathcal{F}$. As $Y$ we can take the family $Q_A$ of supercharges in [Mi04] parametrized by points in $A$. This means that in the notation above, we can identify $A$ as the diagonal in $A \times Y$ and we have a natural identification of the groupoid moduli space with the moduli space of gauge connections on the circle. Here the groupoid action defines an element in the twisted $G$-equivariant K-theory on $G$. In fact, in the case of the circle, one can directly work with highest weight representations of the loop group without using the embedding $LG \subset U_{res}$.

REFERENCES

[ABJ] S.L. Adler: Axial-vector vertex in spinor electrodynamics. Phys. Rev. 177, p. 2426 (1969). W.A. Bardeen: Anomalous Ward identities in spinor field theories. Phys. Rev. 184, p.1848 (1969) J. Bell and R. Jackiw: The PCAC puzzle: $\pi^0 \rightarrow \gamma \gamma$ in the $\sigma$-model. Nuevo Cimento A60, p. 47 (1969).

[AS] M.F. Atiyah and I.M. Singer: Dirac operators coupled to vector potentials. Proc. Natl. Acad. Sci. USA, 81, pp. 2597-2600 (1984)

[AtSe] M. Atiyah and G. Segal: Twisted K-theory. Ukr. Mat. Visn. 1, no. 3, 287–330 (2004); translation in Ukr. Math. Bull. 1, no. 3, pp. 291–334 (2004)
[Br] V. Braun: Twisted $K$-theory of Lie groups. J. High Energy Phys. no. 3, 029, 15 pp (2004) (electronic)

[Ca] A. L. Carey: The origin of three-cocycles in quantum field theory. Phys. Lett. B 194, pp. 267-270 (1987)

[CGRS] A.L. Carey, H. Grundling, I. Raeburn, and C. Sutherland: Group actions on $C^*$-algebras, 3-cocycles and quantum field theory. Commun. Math. Phys. 168, pp. 389-416 (1995)

[DD] J. Dixmier and A. Douady: Champs continus d’espaces hilbertiens et de $C^*$-algébres. Bull. Soc. Math. France 91, pp. 227–284 (1963)

[Do] C. Douglas: On the twisted $K$-homology of simple Lie groups. Topology 45, pp. 955–988 (2006)

[Fa] L.D. Faddeev: Operator anomaly for the Gauss law. Phys. Lett. B 145, p.81 (1984)

[FHT] D. Freed, M. Hopkins, and C. Teleman: Twisted $K$-theory and loop group representations, arXiv:math/0312155. Twisted equivariant $K$-theory with complex coefficients. J. Topol. 1 (2008), no. 1, 16–44.

[FZ] E. Frenkel and Xinwen Zhu: Gerbal representations of double loop groups. arXiv.math/0810.1487

[GJ] B. Grossman. The meaning of the third cocycle in the group cohomology of nonabelian gauge theories. Phys. Lett. B 160, pp. 94–100 (1985). R. Jackiw, R: Three-cocycle in mathematics and physics. Phys. Rev. Lett. 54, pp.159–162 (1985).

[Jo] S.G. Jo: Commutator of gauge generators in nonabelian chiral theory. Nuclear Phys. B 259, pp. 616–636 (1985)

[LM] E. Langmann and J. Mickelsson: Scattering matrix in external field problems. J. Math. Phys. 37, pp. 3933–3953 (1996)

[Mi85] J. Mickelsson: Chiral anomalies in even and odd dimensions. Comm. Math. Phys. 97 (1985), pp. 361–370 (1985)

[Mi87] J. Mickelsson: Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization. Comm. Math. Phys. 110, pp. 173–183 (1987)

[Mi90] J. Mickelsson: Commutator anomalies and the Fock bundle. Comm. Math. Phys. 127, pp. 285–294 (1990)

[Mi93] J. Mickelsson: Hilbert space cocycles as representations of $(3 + 1)$-D current algebras. Lett. Math. Phys. 28, pp.97–106 (1993). Wodzicki residue and anomalies of current algebras. Integrable models and strings (Espoo, 1993), pp. 123–135, Lecture Notes in Phys., 436, Springer, Berlin (1994)

[Mi04] J. Mickelsson: Gerbes, (twisted) $K$-theory, and the supersymmetric WZW model. Infinite dimensional groups and manifolds, 93–107, IRMA Lect. Math. Theor. Phys., 5, de Gruyter, Berlin (2004)

[ML] Saunders Mac Lane: *Homology*. Die Grundlehren der Mathematischen Wissenschaften, Band 114. Springer Verlag (1963)

[PS] A. Pressley and G. Segal: *Loop Groups*. Clarendon Press, Oxford (1986)
[Tä] V. Tähtinen: The Dirac operator on restricted Grassmannian, in progress