STABILITY OF THE BLASCHKE-SANTALÓ INEQUALITY IN THE PLANE

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Abstract. We give a stability version of the Blaschke-Santaló inequality in the plane.

1. Introduction

The setting of this paper is the n-dimensional Euclidean space. A compact convex subset of \( \mathbb{R}^n \) with non-empty interior is called a convex body. The set of convex bodies in \( \mathbb{R}^n \) is denoted by \( K^n \). Write \( K^n_e \) for the set of origin-symmetric convex bodies and \( K^n_0 \) for the set of convex bodies whose interiors contain the origin.

The support function of \( K \in K^n \), \( h_K : S^{n-1} \to \mathbb{R} \), is defined by

\[
h_K(u) = \max_{x \in K} \langle x, u \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) stands for the usual inner product of \( \mathbb{R}^n \). The polar body, \( K^* \), of \( K \in K^n_0 \) is defined by

\[
K^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.
\]

For \( x \in \text{int} \ K \), let \( K^x := (K - x)^* \). The Santaló point of \( K \), denoted by \( s \), is the unique point in \( \text{int} \ K \) such that

\[
V(K^s) \leq V(K^x)
\]

for all \( x \in \text{int} \ K \). For a body \( K \in K^n_e \), the Santaló point is at the origin. The Blaschke-Santaló inequality [4, 21] states that

\[
V(K^s)V(K) \leq \omega_n^2,
\]

with equality if and only if \( K \) is an ellipsoid. Here \( \omega_n \) is the volume of \( B \), the unit ball of \( \mathbb{R}^n \). The equality condition was settled by Saint Raymond [20] in the symmetric case and Petty [19] in the general case.

A natural tool in the affine geometry of convex bodies is the Banach-Mazur distance which for two convex bodies \( K, \tilde{K} \in K^n \) is defined by

\[
d_{BM}(K, \tilde{K}) = \min\{\lambda \geq 1 : (K-x) \subseteq \Phi(\tilde{K}-y) \subseteq \lambda(K-x), \ \Phi \in \text{GL}(n), x, y \in \mathbb{R}^n\}.
\]

It is easy to see that \( d_{BM}(K, \Phi \tilde{K}) = d_{BM}(K, \tilde{K}) \) for all \( \Phi \in \text{GL}(n) \). Moreover, the Banach-Mazur distance is multiplicative. That is, for \( K_1, K_2, K_3 \in K^n_e \) the following inequality holds:

\[
d_{BM}(K_1, K_3) \leq d_{BM}(K_1, K_2)d_{BM}(K_2, K_3).
\]

The main result of the paper is stated in the following theorem.

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Theorem. There exist constants $\gamma$, $\varepsilon_0 > 0$, such that the following holds: If $0 < \varepsilon < \varepsilon_0$ and $K$ is a convex body in $\mathbb{R}^2$ such that $V(K^*)V(K) \geq \frac{\pi}{1 + 2\varepsilon}$, then $d_{BM}(K, B) \leq 1 + \gamma \varepsilon^2$. Furthermore, if $K$ is an origin-symmetric body, then $d_{BM}(K, B) \leq 1 + \gamma \varepsilon^2$. In $\mathbb{R}^n$, $n \geq 3$, the stability of the Blaschke-Santaló inequality was first proved by K.J. Böröczky [6], and then by K. Ball and K.J. Böröczky [2] with a better order of approximation (see also [3] for the stability of functional forms of the Blaschke-Santaló inequality). In $\mathbb{R}^2$, a result has been obtained by K.J. Böröczky and E. Makai [7] where the order of approximation in the origin-symmetric case is $1/3$ and in the general case is $1/6$. Therefore, our main theorem provides a sharper stability result. Moreover, stability of the $p$-affine isoperimetric inequality also follows from the stability of the Blaschke-Santaló inequality (See [17, 22] for definitions of the $p$-affine surface areas, and for the statements of the $p$-affine isoperimetric inequalities, and see also [13, 14] for their generalizations in the context of the Orlicz-Brunn-Minkowski theory, basic properties, and affine isoperimetric inequalities they satisfy.). Stability of the $p$-affine isoperimetric inequality, in the Hausdorff distance, for bodies in $K^2$ was established by the author in [12] via the affine normal flow with the order of approximation equal to $3/10$. Therefore, the main theorem here replaces $3/10$ by $1/2$ and extends that result, if $p > 1$, to bodies with the Santaló points or centroids at the origin, and if $p = 1$, to any convex body in $K^2$. An application of such a stability result to some Monge-Ampère functionals is given by Ghilli and Salani [9].

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2. Background material

A convex body is said to be of class $C^k$, for some $k \geq 2$, if its boundary hypersurface is $k$-times continuously differentiable, in the sense of differential geometry, and the Gauss map $\nu : \partial K \to \mathbb{S}^{n-1}$, which takes $x$ on the boundary of $K$ to its unique outer unit normal vector $\nu(x)$, is well-defined and a $C^{k-1}$-diffeomorphism.

Let $K, L$ be two convex bodies and $0 < a < \infty$, then the Minkowski sum $K + aL$ is defined by $h_{K + aL} = h_K + ah_L$ and the mixed volume $V_1(K, L)$ ($V(K, L)$ for planar convex bodies) of $K$ and $L$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{a \to 0^+} \frac{V(K + aL) - V(K)}{a}.$$ 

A fundamental fact is that corresponding to each convex body $K$, there is a unique Borel measure $S_K$ on the unit sphere such that

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L dS_K$$

for any convex body $L$. The measure $S_K$ is called the surface area measure of $K$.

A convex body $K$ is said to have a positive continuous curvature function $f_K$, defined on the unit sphere, provided that for each convex body $L$

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L f_K d\sigma,$$
where \( \sigma \) is the spherical Lebesgue measure on \( \mathbb{S}^{n-1} \). A convex body can have at most one curvature function; see [5, p. 115]. If \( K \) is of class \( C^2 \), then \( S_K \) is absolutely continuous with respect to \( \sigma \), and the Radon-Nikodym derivative \( dS_K/d\sigma : \mathbb{S}^{n-1} \to \mathbb{R} \) is the reciprocal Gauss curvature of \( \partial K \) (viewed as a function of the outer unit normal vectors). For every \( K \in \mathcal{K}^n \), \( V(K) = V_1(K, K) \).

Of significant importance in convex geometry is the Minkowski mixed volume inequality. Minkowski’s mixed volume inequality states that for \( K, L \in \mathcal{K}^n \),

\[
V_1(K, L)^n \geq V(K)^{n-1}V(L).
\]

In the class of origin-symmetric convex bodies, equality holds if and only if \( \Lambda \)

**Remark**

the centroid of \( \Lambda \)

**Theorem 1.** [10] Let \( K, L \in \mathcal{K}^n \) and set \( D(K) = 2\max_{\mathbb{S}^1} h_K \), then

\[
\frac{V(K, L)^2}{V(K)V(L)} - 1 \geq \frac{V(K)}{4D^2(K)} \max_{u \in \mathbb{S}^1} \left| \frac{h_K(u)}{V(K)\frac{3}{2}} - \frac{h_L(u)}{V(L)\frac{3}{2}} \right|^2.
\]

The Santaló point of \( K \) is characterized by the following property

\[
\int_{\mathbb{S}^{n-1}} u \frac{h_{K-s}^{n+1}(u)}{h_K^{n+1}(u)} d\sigma(u) = 0.
\]

Thus for an arbitrary convex body \( K \), the indefinite \( \sigma \)-integral of \( h_{K-s}^{(n+1)} \) satisfies the sufficiency condition of Minkowski’s existence theorem in \( \mathbb{R}^n \) (see, for example, Schneider [22, Theorem 8.2.2]). Hence, there exists a unique convex body (up to translation) with curvature function

\[
f_{\Lambda K} = \frac{V(K)}{V(K_s)} h_{K-s}^{-(n+1)}.
\]

Moreover, \( \Lambda \Phi K = \Phi \Lambda K \) (up to translation) for \( \Phi \in GL(n) \), by [16, Lemma 7.12]. Finally, we remark that by the Minkowski inequality for all \( L \in \mathcal{K}^2 \) there holds \( V^2(L) = V(\Lambda L, L)^2 \geq V(L)V(\Lambda L) \). Therefore \( V(L) \geq V(\Lambda L) \) for all \( L \in \mathcal{K}^2 \), with equality if and only if \( \Lambda L \) is a translate of \( L \). In this paper we always assume that the centroid of \( \Lambda K \) is the origin of the plane.

**Remark 2.** If \( K \in \mathcal{K}^n \) is of class \( C^\infty \), then \( h_K \in C^\infty \). In fact, by definition of the class \( C^\infty \), the Gauss map \( \nu \) is a diffeomorphism of class \( C^\infty \) and so \( h_K(\cdot) = \langle \nu^{-1}(\cdot), \cdot \rangle \) is of class \( C^\infty \). In this case, since \( \Lambda K \) is a solution to the Minkowski problem (2.2) with positive \( C^\infty \) prescribed data \( \frac{V(K)}{V(K_s)} h_{K-s}^{-(n+1)} \), \( \Lambda K \) is of class \( C^\infty \); see Cheng and Yau [8, Theorem 1].

**Theorem 3.** [11] Suppose that \( K \in \mathcal{K}^n \) is of class \( C^\infty \). If \( m \leq h_K \langle \cdot, \cdot \rangle^{1/3} \leq M \) for some positive numbers \( m \) and \( M \), then there exist two ellipses \( E_{in} \) and \( E_{out} \) such that \( E_{in} \subseteq K \subseteq E_{out} \) and

\[
\left( \frac{V(E_{in})}{\pi} \right)^{2/3} = m, \quad \left( \frac{V(E_{out})}{\pi} \right)^{2/3} = M.
\]

**Corollary 4.** Suppose that \( K \in \mathcal{K}^n \) is of class \( C^\infty \). If \( m \leq h_K \langle \cdot, \cdot \rangle^{1/3} \leq M \) for some positive numbers \( m \) and \( M \) and \( V(K) = \pi \), then \( m \leq 1 \leq M \). Moreover, without
any assumption on the area of $K$, we have

$$d_{BM}(K, B) \leq \left( \frac{M}{m} \right)^{\frac{2}{n}}.$$  

**Proof.** Let $E_{in}$ and $E_{out}$ be the ellipses from Theorem 3. Since $V(E_{out}) \geq \pi$ and $V(E_{in}) \leq \pi$, the first claim follows (For another proof by Andrews, see [1, Lemma 10] in which he does not assume that $K$ is origin-symmetric.). To prove the bound on the Banach-Mazur distance, we may first apply a special linear transformation $\Phi \in SL(2)$ such that $\Phi E_{out}$ is a disk. Then it is easy to see that

$$\Phi E_{out} \subseteq \frac{V(E_{out})}{V(E_{in})} \Phi E_{in},$$

and

$$d_{BM}(K, B) \leq \frac{V(E_{out})}{V(E_{in})}.$$  

\[ \square \]

Let $K$ be a convex body with Santaló point at the origin. In [15], by using the affine isoperimetric inequality, Lutwak proved

$$V(K)V(K^*) \leq \omega^2_n \left( \frac{V(\Lambda K)}{V(K)} \right)^{n-1}.  \tag{2.3}$$

We will use this inequality for $n = 2$ in the proof of the main theorem.

3. **Proof of the main theorem**

We shall begin by proving the claim for bodies in $K^2_2$ that are of class $C^\infty$. By John’s ellipsoid theorem, we may assume without losing any generality, after applying a $GL(2)$ transformation, that

$$1 \leq h_K \leq \sqrt{2}.  \tag{3.1}$$

In view of inequality (2.3), inequality $V(K)V(K^*) \geq n^2 \pi^{-\frac{n}{2}}$ gives

$$1 \geq \frac{V(\Lambda K)}{V(K)} \geq \frac{1}{1 + \varepsilon}.  \tag{3.2}$$

We will rewrite (3.2) as the following equivalent expression

$$\frac{V(K, \Lambda K)}{V(\Lambda K)V(K)} - 1 \leq \varepsilon.$$

Therefore, by Groemer’s stability theorem, (2.1), we obtain

$$\frac{V(K)}{4D^2(K)} \max_{u \in S^1} \left| \frac{h_K(u)}{V(K)^{\frac{1}{2}}} - \frac{h_{\Lambda K}(u)}{V(\Lambda K)^{\frac{1}{2}}} \right|^2 \leq \varepsilon.$$

Thus for every $u \in S^1$ there holds

$$\frac{h_K^2(u)}{V(K)} \left| \frac{V(\Lambda K)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}} - \frac{h_{\Lambda K}(u)}{h_K(u)} \right|^2 \leq \frac{h_K^2(u)}{V(\Lambda K)^{\frac{1}{2}}} \left| \frac{V(\Lambda K)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}} - \frac{h_{\Lambda K}(u)}{h_K(u)} \right|^2 \leq \frac{32}{\pi^2} \varepsilon.  \tag{3.3}$$
Using (3.1) we can estimate the left-hand side of (3.3) to obtain

\[(3.4) \qquad \max_{u \in S^1} \left| \frac{V(\Lambda K)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}} - \frac{h_{\Lambda K}(u)}{h_K(u)} \right|^2 \leq 64\varepsilon.\]

Recall from (2.2) that

\[h_K = \left( \frac{V(K)}{V(K^*)} \right)^{\frac{1}{2}} \frac{1}{f_{\Lambda K}^K}.
\]

Plugging this into (3.4) gives

\[\left( \frac{V(K^*)}{V(K)} \right)^{\frac{1}{2}} \max_{u \in S^1} \left| \frac{V(\Lambda K)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}} \left( \frac{V(K)}{V(K^*)} \right)^{\frac{1}{2}} - \frac{h_{\Lambda K}f_{\Lambda K}^K(u)}{h_K(u)} \right|^2 \leq 64\varepsilon.
\]

On the other hand, as (3.1) also implies \(\frac{1}{\sqrt{2}} \leq h_{K^*} \leq 1\), we deduce that

\[\max_{u \in S^1} \left| \frac{V(\Lambda K)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}} \left( \frac{V(K)}{V(K^*)} \right)^{\frac{1}{2}} - \frac{h_{\Lambda K}f_{\Lambda K}^K(u)}{h_K(u)} \right|^2 \leq (64)^{\frac{1}{2}} \varepsilon.
\]

In particular, this last inequality leads us to

\[(3.5) \qquad \max_{u \in S^1} (h_{\Lambda K}f_{\Lambda K}^K(u)) - \min_{u \in S^1} (h_{\Lambda K}f_{\Lambda K}^K(u)) \leq 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}}.
\]

By multiplying \(\Lambda K\) with \(\sqrt{\frac{\pi}{V(\Lambda K)}}\) we have \(V\left(\sqrt{\frac{\pi}{V(\Lambda K)}} \Lambda K\right) = \pi\). So by Remark 2, Corollary 4, and (3.5) we get

\[2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} \left( \frac{\pi}{V(\Lambda K)} \right)^{\frac{2}{3}} + 1 \geq \left( \frac{\pi}{V(\Lambda K)} \right)^{\frac{2}{3}} \max_{S^1} (h_{\Lambda K}f_{\Lambda K}^K),
\]

and

\[1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} \left( \frac{\pi}{V(\Lambda K)} \right)^{\frac{2}{3}} \leq \left( \frac{\pi}{V(\Lambda K)} \right)^{\frac{2}{3}} \min_{S^1} (h_{\Lambda K}f_{\Lambda K}^K).
\]

Furthermore, notice that by (3.1) and (3.2) the following inequality holds:

\[1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} \left( \frac{\pi}{V(\Lambda K)} \right)^{\frac{2}{3}} \geq 1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}}.
\]

Take \(\varepsilon\) small enough such that

\[1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}} > 0.
\]

So far we have proved: If \(\varepsilon\) is small enough, then

\[\max_{S^1} (h_{\Lambda K}f_{\Lambda K}^K) \leq \left( 1 + 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}} \right) \left( \frac{\pi}{V(\Lambda K)} \right)^{-\frac{2}{3}},
\]

and

\[\min_{S^1} (h_{\Lambda K}f_{\Lambda K}^K) \geq \left( 1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}} \right) \left( \frac{\pi}{V(\Lambda K)} \right)^{-\frac{2}{3}} > 0.
\]

With the aid of these last inequalities and Corollary 4 we deduce that

\[(3.6) \qquad d_{BM}(\Lambda K, B) \leq \left( \frac{1 + 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}}}{1 - 2^{\frac{24}{2}} \varepsilon^{\frac{1}{2}} (1 + \varepsilon)^{\frac{1}{2}}} \right)^{\frac{3}{2}}.
\]
We return to inequality (3.4) and combine it with (3.2) to get

\[-8\varepsilon^+ + \frac{1}{(1 + \varepsilon)^{1/2}} \leq -8\varepsilon^+ + \frac{V(\Lambda K)^{1/2}}{h_K} \leq 8\varepsilon^+ + \frac{V(\Lambda K)^{1/2}}{V(K)^{1/2}} \leq 1 + 8\varepsilon^+.

Furthermore, take \(\varepsilon\) small enough such that \(-8\varepsilon^+ + \frac{1}{(1 + \varepsilon)^{1/2}} > 0\). Consequently

(3.7) \quad d_{BM}(K, \Lambda K) \leq \frac{1 + 8\varepsilon^+}{-8\varepsilon^+ + \frac{1}{(1 + \varepsilon)^{1/2}}}.

Taking into account (3.6), (3.7), and the multiplicativity of the Banach-Mazur distance results in the desired estimate:

\[d_{BM}(K, B) \leq \left(\frac{1 + 2\pi^2\varepsilon^+ (1 + \varepsilon)^{2/3}}{1 - 2\pi^2\varepsilon^+ (1 + \varepsilon)^{2/3}}\right)^{3/2} \left(\frac{1 + 8\varepsilon^+}{-8\varepsilon^+ + \frac{1}{(1 + \varepsilon)^{1/2}}}\right) \leq 1 + \gamma \varepsilon^+,
\]

for some universal \(\gamma > 0\), provided that \(\varepsilon\) is small enough.

It follows from [22, Section 3.4] that the class of \(C_+\) origin-symmetric convex bodies is dense in \(K^n\). Therefore, an approximation argument will prove that the claim of the main theorem, in fact, holds for any origin-symmetric convex body. To get the more general result, for bodies in \(K^2\), we will first need to recall Theorem 1.4 of Böröczky from [6] and a theorem of Meyer and Pajor from [18]:

**Theorem** (Böröczky, [6]). For any convex body \(K\) in \(\mathbb{R}^n\) with \(d_{BM}(K, B) \geq 1 + \varepsilon\) for \(\varepsilon > 0\), there exists an origin-symmetric convex body \(C\) and a constant \(\gamma' > 0\) depending on \(n\) such that \(d_{BM}(C, B) \geq 1 + \gamma' \varepsilon^+\) and \(C\) results from \(K\) as a limit of subsequent Steiner symmetrizations and affine transformations.

**Theorem** (Meyer, Pajor, [18]). Let \(K\) be a convex body in \(\mathbb{R}^n\), \(H\) be a hyperplane, and let \(K_H\) be the Steiner symmetral of \(K\) with respect to \(H\). If \(s\) and \(s'\) denote the Santaló points of \(K\) and \(K_H\), respectively, then \(s' \in H\), and \(V(K^s) \leq V((K_H)^s')\).

Now we give the proof in the general case by contraposition. Let \(K\) be a convex body such that

\[d_{BM}(K, B) > 1 + \left(\frac{\gamma}{\gamma'}\right)^{1/2} \varepsilon^+\,
\]

where \(\gamma'\) is the constant in Böröczky’s theorem. So by the last two theorems, there exists an origin-symmetric convex body \(C\), such that \(V(C)V(C^s) \geq V(K)V(K^s)\) and \(d_{BM}(C, B) > 1 + \gamma \varepsilon^+\). Moreover, \(d_{BM}(C, B) > 1 + \gamma \varepsilon^+\) implies that

\[V(C)V(C^s) < \frac{\pi^2}{1 + \varepsilon}.
\]

Therefore

\[V(K)V(K^s) \leq \frac{\pi^2}{1 + \varepsilon}.
\]

The argument is complete.
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