AN ACCELERATED SPLITTING-UP METHOD FOR PARABOLIC EQUATIONS

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Abstract. We approximate the solution \( u \) of the Cauchy problem

\[
\frac{\partial}{\partial t} u(t, x) = L u(t, x) + f(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d,
\]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d \]

by splitting the equation into the system

\[
\frac{\partial}{\partial t} v_r(t, x) = L_r v_r(t, x) + f_r(t, x), \quad r = 1, 2, ..., d_1,
\]

where \( L, L_r \) are second order differential operators, \( f, f_r \) are functions of \( t, x \), such that \( L = \sum_r L_r, f = \sum_r f_r \). Under natural conditions on solvability in the Sobolev spaces \( W^{m,p} \), we show that for any \( k > 1 \) one can approximate the solution \( u \) with an error of order \( \delta^k \), by an appropriate combination of the solutions \( v_r \) along a sequence of time discretization, where \( \delta \) is proportional to the step size of the grid. This result is obtained by using the time change introduced in \([7]\), together with Richardson’s method and a power series expansion of the error of splitting-up approximations in terms of \( \delta \).

1. Introduction

In this paper we are interested in the rate of convergence of splitting-up approximations to the solution of the parabolic, possibly degenerate, differential equation with time dependent coefficients

\[
\frac{\partial}{\partial t} u(t, x) = L u(t, x) + f(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d
\]

with initial condition

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d \]

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where $L$ is a differential operator of the form

$$L = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + a^i(t,x)\frac{\partial}{\partial x^i} + a(t,x)$$

and $f$ is a function of $t \geq 0$ and $x \in \mathbb{R}^d$. The first step in the splitting methods is to choose suitable decompositions $L = L_1 + L_2 + ... + L_{d_1}$ and $f = f_1 + f_2 + ... + f_{d_1}$ for the operator $L$ and the free term $f$, such that each equation

$$\frac{\partial}{\partial t} v_r(t,x) = L_r v_r(t,x) + f_r(t,x) \quad (1.3)$$

$r = 1, 2, ..., d_1$ is integrable exactly or approximately.

Assume for simplicity that the operators $L_r$ and the free terms $f_r$, $r = 1, 2, ..., d_1$, are time independent, and, for fixed $T > 0$ and integer $n \geq 1$, consider the uniform step

$$T_n := \{t^n_i := iT/n, i = 0, 1, 2, ..., n\} \quad (1.4)$$

of step size $\delta := T/n$. Then a splitting-up approximation $u^{(n)}$ for the solution $u$ of (1.1)-(1.2) is defined by

$$u^{(n)}(t^n_i) := (S^{(d_1)}_{\delta} ... S^{(2)}_{\delta} S^{(1)}_{\delta})^i u_0, \quad i = 0, 1, ..., n \quad (1.5)$$

at the grid points. Here $S^{(r)}_{\delta}$ denotes the solution operator of equation (1.3), i.e., $S^{(r)}_{\delta} \varphi$ is the solution of (1.3) at time $t$ with initial condition $\varphi$ at $t = 0$. Formula (1.5) means that we take $u^{(n)}(0) = u_0$, and we calculate the approximation at a grid point $t + \delta$ from the approximation $u^{(n)}(t)$ at the previous grid point $t$, by solving equations (1.3) for $r = 1, 2, ..., d_1$ on the same time interval $[0, \delta]$ successively. First we solve the first equation ($r = 1$) on $[0, \delta]$ with initial condition $v_1(0) = u^{(n)}(t)$, and then we solve the second equation, third equation, and so on, on the same interval $[0, \delta]$, by taking always the value at $\delta$ of the solution of the previous equation as the initial value for the following equation. Finally we solve the last equation ($r = d_1$) on the interval $[0, \delta]$ with initial condition $v_{d_1}(0) = v_{d_1-1}(\delta)$, and the value $v_{d_1}(\delta)$ is the value of the splitting-up approximation at $t + \delta$.

This kind of approximations is well-known in numerical analysis, and it has been successfully applied to various types of PDE problems. They are often combined with other numerical methods, as finite differences, finite elements, etc. Pioneering applications to the heat equation, to hyperbolic equations, to nonlinear PDEs are presented, for example, in [16], [4], [2], in [29], [9] and in [3], [1], [25], respectively. Many applications and modifications of the splitting-up method have been developed in various applied fields of linear and nonlinear PDEs.
and ODEs, under a variety of different names, like dimensional splitting, operator splitting, predictor-corrector method, method of alternating directions, fractional step method, Lie-Trotter-Kato formula, Baker-Campbell-Hausdorff formula, Chernoff formula, split Hamiltonian, split-steps, leapfrog. For guidance in the huge varieties of methods, names and references we refer to the survey article [13] and books [11], [12].

In the context of semigroups the splitting-up method first appears as Trotter’s formula [25], which can be formulated as follows:

$$\lim_{n \to \infty} (e^{tA_1/n} \ldots e^{tA_2/n} e^{tA_1/n})^n z = e^{tA} z, \quad \forall z \in \mathcal{B},$$

where $A = A_1 + A_2 + \ldots + A_{d_1}$ and $A_r$ are infinitesimal generators of $C_0$-semigroups of contractions $\{e^{tA} : t \geq 0\}$ and $\{e^{tA_r} : t \geq 0\}$ on a Banach space $\mathcal{B}$, such that the intersection of the domains of the generators is dense in $\mathcal{B}$. Clearly, in the context of Cauchy problems Trotter’s formula states the convergence of the splitting-up approximations defined by the splitting

$$\frac{\partial v_r}{\partial t} = A_r v_r(t), \quad r = 1, 2, \ldots, d_1$$

to the solution of the abstract Cauchy problem

$$\frac{\partial u}{\partial t} = Au(t), \quad t \geq 0, \quad u(0) = z.$$

Our main interest in the present paper is to increase the accuracy of the splitting-up approximations for equation (1.1). It is known that the error of the splitting-up approximations is proportional to $\delta$, the step-size. There are, however, modifications of these approximations which are more accurate. A celebrated example is the Strang symmetric scheme

$$u_i(t^{(n)}) := (S^{(1)}_{\delta/2} S^{(2)}_{\delta/2} \ldots S^{(d_1)}_{\delta/2} \ldots S^{(2)}_{\delta/2} S^{(1)}_{\delta/2})^i u_0, \quad i = 0, 1, \ldots, n,$$

whose error is proportional to $\delta^2$. This approximation scheme is presented in [17], [19]. Other symmetric schemes and their generalizations, are given in [17], [18], and [6]. All these schemes are of second order accuracy. Inspired by the above example, for given $k \geq 2$ one looks for a composition of splittings

$$\prod_{i=1}^{m} \prod_{j=1}^{d_i} S^{(j)}_{\epsilon^{ij} \delta}, \quad (1.6)$$
with real numbers $c^{ij}$ and integer $m \geq 1$ to be determined, such that

$$u(\delta) - \prod_{i=1}^{m} \prod_{j=1}^{d_1} S_{c^{ij}}^{(j)} u_0,$$

the local error of the corresponding approximation is proportional to $\delta^{k+1}$ in appropriate norms. Such local error leads to a global error, proportional to $\delta^k$, i.e., composition (1.6), represents a method of (at least) order $k$. The conditions on the numbers $c^{ij}$ and $m$ which lead to splitting methods of high order have been studied intensively in the literature. Such methods are obtained in [15] for Hamiltonian systems by the Baker-Campbell-Hausdorff formula. Variations of the Trotter formula and the Baker-Campbell-Hausdorff formula are used for linear and for nonlinear equations, respectively, to show the existence of methods of any order (see [13], [21], [20], [28] and the literature therein). An adaptation of the method of rooted trees from the theory of Runge-Kutte approximations is used in [14]. By [20] and [27], however, the numbers $c^{ij}$ in each scheme (1.6) of order $k \geq 3$ cannot be all non-negative. Thus, by [20] and [27], the above splitting methods of order greater than or equal to 3 cannot be used to approximate the solution of partial differential equations of parabolic type. As R.I. McLachlan and G.R.W. Quispel write on page 392 of [13]: “...splitting was proposed as a cheap way to retain unconditional stability. Methods with backward time steps can only be conditionally stable; this stumbling block held up the development of high-order compositions for years.”

Then the natural question arises, as to whether there exists, in the case of parabolic equations, a different way from the multiplicative one to accelerate the convergence to a higher order. One of our main results consists of showing that using the step size of order $\delta$, but organizing the computations differently, it is indeed possible to achieve the accuracy of order $\delta^k$ for any $k$, even if $A_r$ are (degenerate) elliptic operators with coefficients depending on time. In a subsequent article we intend to show that our method is much more universal in the sense that it covers very many situations in which method (1.6) works and requires approximately the same amount of work.

In the present paper we use linear combinations of splittings of type (1.5) with different step-sizes, to achieve arbitrary high accuracy. We prove that for any given $k \geq 0$ there exist absolute constants $b_0, b_1, ..., b_k$ expressed by simple formulas such that the accuracy of the approximation

$$v_n := b_0 u_n + b_1 u_{2n} + b_2 u_{4n} + ... + b_k u_{2^k n}$$

(1.7)
is of order $\delta^{k+1}$ (see Theorem 2.2 below). Here $u_{2j/n}$ is the splitting-up approximation along the grid, with $2^j/n$ in place of $n$. In particular, if $k = 1$, we have to deal with two step sizes: $\delta$ and $\delta/2$, and we get the order of accuracy $\delta^2$. The Strang formula giving the same order of accuracy, generally, also requires working with step size $\delta/2$. By the way, if $A = A_1 + A_2$ and we construct our splitting-up scheme according to $A = (1/2)A_1 + A_2 + (1/2)A_1$, then our approximations just coincide with the Strang one and there is no need to use linear combinations to get the error of order $\delta^2$. It is also worth noting that the above coefficients $b = (b_0, ..., b_k)$ are given by $b := e_1V^{-1}$, where $e_1 := (1, 0, 0, ..., 0)$ and $V^{-1}$ is the inverse of the $k+1 \times k+1$ Vandermonde matrix $V_{ij} = 2^{-(i-1)(j-1)}$.

Our work is of purely theoretical nature and, as the referees pointed out, much work yet needs to be done before our results could be used in practical applications. We restricted ourselves to making the first step in attacking Problem 10 on page 492 of [13]: “For systems that evolve in a semigroup, such as the heat equation, develop effective methods of order higher than 2”. However, our results show that each time when one has any algorithm of implementing standard splitting-up method to approximating the solutions of the Cauchy problem for degenerate parabolic equations with sufficiently smooth coefficients and free terms, one can improve the rate of convergence to any degree. For instance, we believe that usually in practice one is not doing computations with only one step size, and we show that having, say, three different step sizes each of which is of order $\delta$ of accuracy, and just taking a linear combination of the results, one gets an approximation with error of order $\delta^3$.

We have to admit that we do not know if our methods can be carried over to quasilinear equations or to equations in domains. In this connection we note that there is a very active area of developing and applying in practice splitting-up methods for degenerate nonlinear convection-diffusion equations (see, for instance, [5] and 185 references therein). Our equations can be viewed as belonging to this area only if $a^{ij}$ are constant. However, it is perhaps worth mentioning that our methods can be applied to solving systems of (nonlinear) ODEs and we are in the process of working on this subject.

Inspired by Richardson’s method we obtain our results by expanding the error $u - u_n$ of the splitting-up approximation in powers of $\delta = T/n$. This is Theorem 2.1, the main theorem of our paper. We use this expansion with $\delta = T/2^j/n$, $j = 0, 1, 2, ..., k$, and choose the above
coefficients $b_0, b_1, \ldots, b_k$ to eliminate the terms of order less than $k + 1$ in the linear combination (1.7).

The main theorem of the present paper is proved by exploiting a new approach of [7] and [8] to splitting-up methods. As we discussed above, the splitting-up approximation (1.5) means that to get the approximation at $t + \delta$ from that at $t$, one goes back and forth in time $d_1$-times while solving equations (1.3), $r = 1, 2, \ldots, d_1$, successively. A basic idea of [7] is to arrange the splitting continuously in forward time direction, and to synchronize with it the original equation by time scaling. In this way we have differential equations for the rearranged splitting-up approximations and for the time-scaled solution of the original equation, which enables us to use methods of the theory of partial differential equations and not semigroup theory and get an expansion for their difference in terms of powers of $\delta$ even if the coefficients depend on time. The method of [7] and [8] appeared in connection with splitting-up for stochastic partial differential equations. It is worth mentioning that most likely it is impossible to accelerate the splitting-up method in this more complicated situation.

The paper is organized as follows. In the next section we introduce our general setting but state the results, Theorems 2.2 and 2.1 only for the case of time independent data for the sake of simplicity of presentation. Theorem 2.1 is proved immediately after its formulation on the basis of Theorem 2.2 which in turn is proved in Sec. 4 after we prepare some auxiliary facts in Sec. 3. In Sec. 5 we generalize Theorems 2.2 and 2.1 for time dependent data and derive some consequences valid in the time-homogeneous case as well.

In conclusion we introduce some notation used everywhere below. Throughout the paper $d \geq 1, d_1 \geq 2$ are fixed positive integers, $K, T$ are fixed finite positive constants, and

$$D_i := \partial/\partial x^i, \quad D_{ij} := \partial^2/\partial x^i \partial x^j, \quad D_t := \partial/\partial t.$$ 

We denote by $W_{p}^{m}$ the Sobolev space defined as the closure of $C^\infty_0$ functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ in the norm

$$\|\varphi\|_{m,p} := \left( \sum_{|\gamma| \leq m} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^p \, dx \right)^{1/p},$$

where $D^\gamma := D_1^{\gamma_1} \ldots D_d^{\gamma_d}$ for multi-indices $\gamma = (\gamma_1, \ldots, \gamma_d)$ of length $|\gamma| := \gamma_1 + \gamma_2 + \ldots + \gamma_d$. Unless otherwise indicated, we use the summation convention with respect to repeated indices.

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2. FORMULATION OF THE MAIN RESULTS. THE CASE OF TIME INDEPENDENT COEFFICIENTS

We consider the problem

\[ D_t u(t, x) = Lu(t, x) + f(t, x), \quad t \in (0, T], \quad x \in \mathbb{R}^d, \]  
\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \]

where \( L \) is an operator of the form

\[ L = a^{ij}(t, x) D_{ij} + a^i(t, x) D_i + a(t, x), \]

\( f \) and \( u_0 \) are real functions of \((t, x) \in (0, T] \times \mathbb{R}^d\) and of \( x \in \mathbb{R}^d \), respectively. We assume that the coefficients \( a^{ij}, a^i, a \) and the derivatives \( a^{ij}_{x_k} \) of \( a^{ij} \) are bounded Borel functions of \((t, x) \). We fix \( p \geq 2 \) and assume that \( u_0 \) and \( f \) are measurable and \(|u_0|^p\) and \(|f|^p\) are integrable over \( \mathbb{R}^d \) and over \([0, T] \times \mathbb{R}^d\), respectively.

**Definition 2.1.** By a solution of problem (2.1)-(2.2) we mean a \( W^{1,p} \)-valued weakly continuous function \( u(t) = u(t, \cdot) \) defined on \([0, T]\) such that for all \( \phi \in C_0(\mathbb{R}^d) \) and \( t \in [0, T] \)

\[ (u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t [- (a^{ij} D_i u(s), D_j \phi) + ((a^i - a^{ij}_{x_j}) D_i u(s) + a u(s) + f_r(s, \phi))] ds, \]

where \((, \cdot)\) denotes the usual inner product in \( L^2(\mathbb{R}^d) \). Quite often we write equation (2.1) and similar equations in the form

\[ du(t) = (Lu(t) + f(t)) dt \]

bearing in mind the differential of \( u \) in \( t \) only.

Suppose that we split equation (2.1) into the equations

\[ D_t v(t, x) = L_r v(t, x) + f_r(t, x), \quad t \in (0, T], \quad x \in \mathbb{R}^d \]  
\[ L_r := a^{ij}_r(t, x) D_{ij} + a^i_r(t, x) D_i + a_r(t, x), \quad L = \sum_{r=1}^{d_1} L_r, \quad f = \sum_{r=1}^{d_1} f_r, \]

such that these equations are more pleasant from point of view of numerical methods than the original one. This motivates the multi-stage splitting method, which we describe below. First we need some assumptions.

Fix an integer \( l \geq 1 \).
Assumption 2.1 (ellipticity of $L_r$). For each $r = 1, 2, \ldots, d_1$ for $dt \times dx$-almost every $(t, x) \in [0, T] \times \mathbb{R}^d$
\[ a_{ij}^{r}(t, x)\lambda^i \lambda^j \geq 0 \]
for all $(\lambda^1, \lambda^2, \ldots, \lambda^d) \in \mathbb{R}^d$.

Assumption 2.2. (i) The partial derivatives
\[ D^s_t D^p \rho a_{ij}^r, \quad D^s_t D^p a_{i}^r, \quad D^s_t D^p a_r \]
for $i, j = 1, 2, \ldots, d, \quad r = 1, 2, \ldots, d_1$
exist and by magnitude are bounded by $K$ for all integers $s \geq 0$ and multi-indices $\rho$, satisfying $2s + |\rho| \leq l$.

(ii) For every integer $s \in [0, l/2]$
\[ \sup_{t \in [0, T]} \| D^s_t f_r(t) \|_{l-2s,p} \leq K. \]

(iii) We have $u_0 \in W^l_p$ and $\| u_0 \|_{l,p} \leq K$.

It is well-known that under the above conditions equations (2.1) and (2.3) with initial condition
\[ u(0) = u_0 \]
admit a unique generalized solution $u$ and $v$, respectively, which are $W^l_p$-valued weakly continuous functions of $t \geq 0$ (see, for instance, Theorem 3.1 below). We want to approximate the solution $u$, by using the splitting-up method, i.e., by solving equations (2.3) successively with appropriate initial conditions on appropriate time intervals. Let us formulate now our splitting-up scheme in the case when the coefficients $a_{ij}^r, a_{i}^r, a_r$ and free terms $f_r$ are independent of the time variable $t$.

Set $T_n := \{ t_i := iT/n : i = 0, 1, 2, \ldots, n \}$, $\delta := T/n$ for an integer $n \geq 1$. Then for fixed $n$ we approximate the solution $u$ of (2.1) - (2.2) at $t_i = iT/n$ recursively by $u_n(0) := u_0$,
\[ u_n(t_{i+1}) := S_{\delta}^{(d_1)} \ldots S_{\delta}^{(2)} S_{\delta}^{(1)} u_n(t_i), \quad i = 0, 1, 2, \ldots, n-1 \]
where $S_{\delta}^{(\rho)} \psi := v(t)$ denotes the solution of equation (2.3) for $t \geq 0$ with initial condition $v(0) = \psi$.

It is known that if Assumptions 2.1, 2.2 are satisfied with $l = m + 4$, then
\[ \max_{i \in T_n} \| u(t) - u_n(t) \|_{m,p} \leq N/n \]
for all $n \geq 1$, where $N$ depends only on $d, d_1, T, K, p, m$. Moreover, this rate of convergence is sharp (see [8], where this result is a special case of the rate of convergence estimates for stochastic PDEs). In the present paper we want to show that by suitable combinations of splitting-up approximations we can achieve as fast convergence as we wish. We show this by the aid of the following theorem on expansion of $u_n$ in powers of the step-size $\delta$. 
Theorem 2.1. Let $m \geq 0$ and $k \geq 0$ be integers. Let Assumptions 2.1 and 2.2 hold with
\begin{equation}
    l \geq 4 + m + 4k. \tag{2.5}
\end{equation}
Suppose that the coefficients $a^{ij}, a^{i}, a^{r}$ and the free terms $f_{r}$ do not depend on $t$. Then for all $n \geq 1$ and $t \in T_{n}$ and $x \in \mathbb{R}^{d}$, the following representation holds
\begin{equation}
    u_{n}(t, x) = u(t, x) + \delta u^{(1)}(t, x)
    \quad \quad + \delta^{2} u^{(2)}(t, x) + \ldots + \delta^{k} u^{(k)}(t, x) + R^{(k)}_{n}(t, x), \tag{2.6}
\end{equation}
where the functions $u^{(1)}, \ldots, u^{(k)}$ and $R^{(k)}_{n}$, defined on $[0, T]$, are $W^{m}_{p}$-valued and weakly continuous. Furthermore, $u^{(j)}$, $j = 1, 2, \ldots, k$, are independent of $n$, and
\begin{equation}
    \sup_{t \in T_{n}} \|R^{(k)}_{n}(t)\|_{m,p} \leq N\delta^{k+1} \tag{2.7}
\end{equation}
for all $n$, where $N$ depends only on $l, k, d, d_{1}, K, m, p, T$.

Remark 2.1. If $k = 0$ and $p = 2$, the result holds under a weaker restriction on $l$: $l \geq 3 + m$ (see, for instance [7]). For general $p \geq 2$ and $k = 0$ the result is proved in [8].

We prove Theorem 2.1 in Section 4. Now we deduce from it a result on the acceleration of the splitting-up method. Let $V$ denote the square matrix defined by
\begin{equation}
    V_{ij} := 2^{-(i-1)(j-1)} \quad i, j = 1, \ldots, k + 1.
\end{equation}
Notice that the determinant of $V$ is the Vandermonde determinant, generated by $1, 2^{-1}, \ldots, 2^{-k}$, and hence it is different from 0. Thus $V$ is invertible. Set $b := (b_{0}, b_{1}, \ldots, b_{k}) := (1, 0, 0, \ldots, 0)V^{-1}$, and define
\begin{equation}
    v_{n}(t) := \sum_{j=0}^{k} b_{j} u_{2^{j}n}(t), \quad t \in T_{n} := \{iT/n : i = 0, 1, \ldots, n\},
\end{equation}
where $u_{2^{j}n}$ is the splitting-up approximation based on the grid $T_{2^{j}n} := \{iT/(2^{j}n) : i = 0, 1, \ldots, 2^{j}n\}$.

Theorem 2.2. Let $m \geq 0$ and $k \geq 0$ be any integers. Let Assumptions 2.1 and 2.2 hold with $l$ satisfying (2.5). Suppose that the coefficients $a^{ij}, a^{i}, a^{r}$ and the free terms $f_{r}$ do not depend on $t$. Then
\begin{equation}
    \max_{t \in T_{n}} \|v_{n}(t) - u(t)\|_{m,p} \leq N\delta^{k+1},
\end{equation}
where $N$ is a constant, depending only on $l, k, d, d_{1}, K, m, p, T$. 
Proof. By Theorem 2.1
\[ u_{2;n} = u + \sum_{i=1}^{k} \frac{\delta^i}{2^{ij}} u^{(i)} + R^{(k)}_{2;n}, \quad j = 0, 1, \ldots, k. \]
Therefore for all \( n \geq 1 \)
\[ v_n = \sum_{j=0}^{k} b_j u_{2;n} = (\sum_{j=0}^{k} b_j) u + \sum_{j=0}^{k} \sum_{i=1}^{k} \frac{\delta^i}{2^{ij}} u^{(i)} + \sum_{j=0}^{k} b_j R^{(k)}_{2;n} \]
\[ = u + \sum_{i=1}^{k} \delta^i u^{(i)} + \sum_{j=0}^{k} b_j R^{(k)}_{2;n} = u + \sum_{j=0}^{k} b_j R^{(k)}_{2;n}, \]
since \( \sum_{j=0}^{k} b_j = 1 \) and \( \sum_{j=0}^{k} b_j 2^{-ij} = 0 \) for \( i = 1, 2, \ldots, k \) by the definition of \( (b_0, \ldots, b_k) \). Hence \( v_n - u = \sum_{j=0}^{k} b_j R^{(k)}_{2;n} \), and
\[ \max_{t \in T_n} \| v_n(t) - u(t) \|_{m,p} = \max_{t \in T_n} \| \sum_{j=0}^{k} b_j R^{(k)}_{2;n}(t) \|_{m,p} \leq \sum_{j=0}^{k} |b_j| \max_{t \in T_n} \| R^{(k)}_{2;n}(t) \|_{m,p} \leq N \delta^{k+1}, \]
by (2.7), where \( N \) is a constant depending only on \( l, T, K, d, d_1, m, p, k \).

Remark 2.2. Assume that \( u^{(1)} = 0 \) in expansion (2.6). This happens, for example, for Strang’s splitting, which is a special case of our splitting-up scheme, as it is explained in the Introduction. In this case we need only take \( k \) terms in the linear combination to achieve accuracy of order \( k + 1 \). Namely, we define now \( v_n(t) \) by
\[ v_n(t) := \sum_{j=0}^{k-1} \lambda_j u_{2;n}(t), \quad t \in T_n, \]
where
\[ (\lambda_0, \lambda_1, \ldots, \lambda_{k-1}) := (1, 0, \ldots, 0) V^{-1}, \]
and \( V \) is now a \( k \times k \) Vandermonde matrix with entries \( V_{i1} := 1, V_{i,j} := 2^{-(i-1)}j \) for \( i = 1, 2, \ldots, k \) and \( j = 2, \ldots, k \). Then Theorem 2.2 remains valid, what one can prove in the same way as Theorem 2.2 is proved. For example,
\[ v_n(t) := -\frac{1}{3} u_n(t) + \frac{4}{3} u_{2;n}(t), \quad t \in T_n \]
is an approximation of accuracy \( \delta^3 \) in the case of Strang’s splitting.
3. Auxiliary Results

Let us consider the partial differential equation
\[ du(t, x) = (Lu(t, x) + f(t, x)) dA(t), \quad t \in (0, T], \ x \in \mathbb{R}^d, \quad (3.1) \]
\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad (3.2) \]
where \( L \) is an operator of the form
\[ L = a^{ij}(t, x)D_{ij} + a^i(t, x)D_i + a(t, x), \]
\( A = A(t) \) is a continuous increasing function starting from 0, \( f \) and \( u_0 \) are real functions of \((t, x) \in (0, T] \times \mathbb{R}^d \) and of \( x \in \mathbb{R}^d \), respectively. Fix an integer \( l \geq 0 \) and a real number \( p \geq 2 \). We understand the solution in the spirit of Definition 2.1 and make the following assumptions.

**Assumption 3.1** (smoothness of the coefficients). The coefficients of \( L \) are measurable. The derivatives in \( x \in \mathbb{R}^d \) of the coefficients \( a^{ij} \) up to order \( 2 \lor l \), of the coefficients \( a^i(t, x) \) up to order \( 1 \lor l \), and of \( a(t, x) \) up to order \( l \) exist for any \( t \in (0, \infty) \), and by magnitude are bounded by \( K \).

**Assumption 3.2.** We have
\[ u_0 \in W^l_p, \quad f \in L^p([0, T], W^l_p). \]

**Assumption 3.3** (ellipticity of \( L \)). For all \( t \geq 0, x \in \mathbb{R}^d, \) and \( \lambda \in \mathbb{R}^d \), we have
\[ a^{ij}(t, x)\lambda^i\lambda^j \geq 0. \]

**Assumption 3.4.** The function \( A \) is absolutely continuous and
\[ \dot{A}(t) := \frac{d}{dt}A(t) \leq K \]
for \( dt \)-almost every \( t \geq 0 \).

The following result is well-known in PDE theory (after replacing \( dA \) in (3.1) with \( \dot{A} dt \) we easily get it, for instance, from \([10]\) or from Theorem 3.1 in \([7]\)).

**Theorem 3.1.** Under Assumptions 3.1, 3.2, 3.3, and 3.4 with \( l \geq 1 \) the Cauchy problem (3.1)-(3.2) has a unique generalized solution \( u \). If Assumptions 3.1, 3.2, 3.3, and 3.4 hold with \( l \geq 0 \), and \( u \) is a generalized solution of (3.1)-(3.2), then for every integer \( l_1 \in [0, l] \)
\[ \sup_{t \in [0, T]} \|u(t)\|_{l_1,p}^p \leq N \left\{ \|u_0\|_{l_1,p}^p + \int_0^T \|f(t)\|_{l_1,p}^p \, dt \right\}, \]
where \( N \) is a constant depending only on \( T, K, l, p, d \).
Under the assumptions of Theorem 3.1 let $\mathcal{R}f$ denote the solution of equation (3.1) with initial data $u_0 = 0$. Then by virtue of Theorem 3.1

$$\mathcal{R} : L_p([0, T], W^l_p) \rightarrow C_w([0, T], W^l_p)$$

is a bounded linear operator, where $C_w([0, T], W^l_p)$ denotes the Banach space of weakly continuous $W^l_p$-valued functions $u = u(t), ~ t \in [0, T]$ with the norm $\sup_{t \in [0, T]} \| u(t) \|_{l,p}$.

Let us now consider the equation

$$du(t, x) = Lu(t, x) \, dA(t) + g(t, x) \, dH(t), \quad (t, x) \in (0, T] \times \mathbb{R}^d,$$

where $g$ is a real-valued function of $(t, x) \in [0, T] \times \mathbb{R}^d$ and $H$ is an absolutely continuous function of $t \in [0, T]$.

Assumption 3.5. We have $g \in L_p([0, T], W^{l+2}_p)$, and there exists $g' \in L_p([0, T], W^l_p)$ such that

$$d(g(t), \phi) = (g'(t), \phi) \, dA(t), \quad t \in [0, T]$$

for all $\phi \in C_0(\mathbb{R}^d)$.

Lemma 3.2. Under Assumptions 3.1, 3.3, 3.4, 3.5 with $l \geq 1$ equation (3.3) with zero initial data has a unique generalized solution $u$. Moreover,

$$u = \mathcal{R}(H(Lg - g')) + Hg =: Q(H, g).$$

If Assumptions 3.1, 3.3, 3.4, 3.5 hold with $l \geq 0$ and equation (3.3) with zero initial data admits a generalized solution $u$, then for every integer $l_1 \in [0, l]$

$$\sup_{t \in [0, T]} \| u(t) \|_{l_1,p} \leq N \sup_{t \in [0, T]} \| H(t) \| \left( \sup_{t \in [0, T]} \| g(t) \|_{l_1,p} \right. \left. + \left\{ \int_0^T \| g(t) \|_{l_1+2,p} + \| g'(t) \|_{l_1,p} \right\} \right)^{1/p},$$

where $N$ is a constant depending only on $p, d, K, l, T$.

Proof. Note that

$$d((g(t), \phi)H(t)) = (g(t), \phi) \, dH(t) + (g'(t), \phi)H(t) \, dA(t),$$

for all $\phi \in C_0(\mathbb{R}^d)$. Therefore $u$ solves equation (3.3), when $w(t, x) := u(t, x) - H(t)g(t, x)$ solves

$$dw(t, x) := \{Lw(t, x) + H(t)(Lg(t, x) - g'(t, x))\} \, dA(t).$$

Hence equality (3.4) follows by Theorem 3.1 and it implies (3.5). \qed
**Remark 3.1.** We often consider equation (3.3) when $dH(t)/dt$ is bounded. Then under Assumptions 3.1, 3.3, 3.4, 3.5 with $l \geq 1$, equation (3.3) with zero initial data has a unique generalized solution $u$ by Theorem 3.1. By Theorem 3.1 this solution belongs to $C_w([0, T], W^l_p)$ and its norm in this space admits an estimate with a constant depending on the bound for $\dot{H}$ and only the $L^p_w([0, T], W^l_p)$-norm of $g$. It is important that the function $H$ enters (3.5) only through $\sup |H|$ and not any characteristic of its derivative, however for that we pay a price requiring $g$ to have more derivatives.

### 4. Proof of Theorem 2.1

Throughout this section the assumptions of Theorem 2.1 are supposed to be satisfied. In particular, $l \geq 4$. Fix $n$ and introduce $\delta = T/n$. We use the idea from [7] and [8] of rearranging the splitting method in forward time. We achieve this by considering the equation

$$dw(t,x) = \sum_{r=1}^{d_1} (L_r w(t,x) + f_r) dA_r(t), \quad w(0,x) = u_0(x),$$

(4.1)

where the time change $A_r$, $r = 1, ..., d_1$, is defined by the requirements that $A_r(0) = 0$, $A_r(t)$ be absolutely continuous, and its derivative in time $\dot{A}_r$ be periodic with period $d_1 \delta$ and

$$\dot{A}_r(t) = 1_{[r-1,r]}(t/\delta), \quad t \in [0, d_1 \delta] \text{ (a.e.)}.$$  (4.2)

Instead of the original Cauchy problem (2.1)-(2.2) we consider

$$dv(t,x) = (L v(t,x) + f) dA_0(t), \quad v(0,x) = u_0(x),$$

where

$$A_0(t) := t/d_1.$$  (4.3)

Clearly, $v(t) = u(A_0(t))$, and

$$v(d_1 t) = u(t), \quad w(d_1 t) = u_n(t) \quad \text{for all } t \in T_n.$$

Therefore our aim is to show that Theorem 2.1 holds with $v$ and $w$ in place of $u$, and $u_n$, respectively, for all $t = id_1 \delta, i = 0, 1, ..., n$. To this end first we introduce some notation. We call a sequence of numbers $\alpha = \alpha_1 \alpha_2 ... \alpha_i$ a multi-number of length $|\alpha| := i$, if $\alpha_j \in \{0, 1, 2, ..., d_1\}$. The reader should notice the difference between multi-numbers and multi-indices. The set of all multi-numbers is denoted by $\mathcal{N}$. For every
multi-number $\alpha$ we define a function $B_\alpha : [0, \infty) \to \mathbb{R}$ and a number $c_\alpha$ recursively starting as follows:

$$B_\gamma := \delta^{-1}(A_\gamma - A_0), \quad c_\gamma = 0 \quad \text{for } \gamma = 0, 1, 2, \ldots d_1. \quad (4.4)$$

If for every multi-number $\beta = \beta_1 \ldots \beta_i$ of length $i$ the function $B_\beta$ and the number $c_\beta$ are defined, then

$$c_{\beta \gamma} := \delta^{-1} \int_0^{d_1 \delta} B_\beta(s) \dot{A}_\gamma(s) \, ds, \quad (4.5)$$

$$B_{\beta \gamma}(t) := \delta^{-1} \int_0^t (B_\beta(s) \dot{A}_\gamma(s) - c_{\beta \gamma} \dot{A}_0(s)) \, ds \quad (4.6)$$

for $\gamma = 0, 1, 2, \ldots, d_1$, where $\dot{A}_\gamma(s) := dA_\gamma(s)/ds$.

Notice that by (4.6) we have

$$B_\beta(t) \, dA_\gamma(t) = c_{\beta \gamma} \, dA_0(t) + \delta \, dB_{\beta \gamma}(t) \quad (4.7)$$

for all multi-numbers $\beta$ and $\gamma = 0, 1, 2, \ldots, d_1$. We will often make use of this equality and of the following lemma.

**Lemma 4.1.** For every $\alpha \in \mathcal{N}$ the function $B_\alpha$ is $d_1 \delta$-periodic, i.e., $B_\alpha(t + d_1 \delta) = B_\alpha(t)$ for all $t \geq 0$, and $B_\alpha(id_1 \delta) = 0$ for all integer $i \geq 0$. Moreover, the numbers $c_\alpha$, the functions $C_\alpha(t) := B_\alpha(\delta t)$, and

$$\sup_{t \geq 0} |B_\alpha(t)| = \sup_{t \geq 0} |C_\alpha(t)|$$

are finite and do not depend on $\delta$.

**Proof.** That the first assertion is true for $\alpha = 0, \ldots, d_1$ is almost obvious. If it is true for $\alpha = \beta$, where $\beta$ is a multi-number, then the integrand in (4.6) is $d_1 \delta$-periodic and by definition of $c_{\beta \gamma}$ its integral over the period is zero. It follows that the first assertion holds for $\alpha = \beta \gamma$, so the induction on the length $|\alpha|$ finishes the proof of the first assertion.

To prove the second one we again use the induction on the length $i = |\alpha|$. This statement is true when $|\alpha| = 1$. Assume that it is true for all multi-numbers $\beta$ of length $i$ and notice that according to (4.2) and (4.3) $\dot{A}_\gamma(\delta s)$ are $d_1$-periodic in $s$ and independent of $\delta$. Therefore,

$$c_{\beta \gamma} = \frac{1}{\delta} \int_0^{d_1 \delta} B_\beta(s) \dot{A}_\gamma(s) \, ds = \int_0^{d_1} C_\beta(s) \dot{A}_\gamma(\delta s) \, ds$$

is independent of $\delta$ by the induction hypothesis. Similar argument works for $C_\alpha(t)$. \qed
We use the notation $\mathcal{R}f$ and $\mathcal{Q}_\alpha g$, $\alpha \in \mathcal{N}$, for the solutions of equations (3.1) and (3.3), respectively, with zero initial condition and $A_0$ and $B_\alpha$ in place of $A$ and $H$, respectively. Notice that, unlike in the case of uniformly parabolic operators, $\mathcal{R}$ and $\mathcal{Q}_\alpha$ do not increase regularity.

The following lemma exhibits our two main technical tools: centering $B_\alpha$ and integrating by parts with respect to $t$.

**Lemma 4.2.** Take some functions $h \in L^p([0,T], W^1_p)$, $h_r \in L^p([0,T], W^1_p)$, $r = 1, \ldots, d_1$, $h_0 = 0$. Let $u$ be a solution of the “equation”

$$du = \sum_{r=1}^{d_1} h_r \, dA_r$$

$u(0) \in W^1_p$, which is a particular case of equation (4.1) when $L_r \equiv 0$. Finally, let $Lu \in L^p([0,T], W^1_p)$. Then for any $\alpha \in \mathcal{N}$

$$\mathcal{R}(B_\alpha h) = c_{\alpha 0} \mathcal{R} h + \delta \mathcal{Q}_{\alpha 0} h, \quad (4.8)$$

$$\mathcal{Q}_\alpha u = \mathcal{R}(c_{\alpha 0} Lu - c_{\alpha r} h_r) + \delta \mathcal{Q}_{\alpha 0} Lu - \delta \mathcal{Q}_{\alpha r} h_r + B_\alpha u. \quad (4.9)$$

**Proof.** To prove (4.8) it suffices to use the definitions of $\mathcal{R}$ and $\mathcal{Q}_\beta$ (see Theorem 3.1 and Lemma 3.2) and use that by virtue of (4.7) for $\varphi = \mathcal{R}(B_\alpha h)$ we have

$$d\varphi = L \varphi \, dA_0 + B_\alpha h \, dA_0 = L \varphi \, dA_0 + c_{\alpha 0} h \, dA_0 + \delta h \, dB_{\alpha 0}. \quad (4.10)$$

To prove (4.9) observe that by definition $\theta := \mathcal{Q}_\alpha u$ satisfies

$$d\theta = L \theta \, dA_0 + u \, dB_\alpha, \quad \theta(0) = 0. \quad (4.11)$$

This and (4.7) imply that $\psi := \theta - uB_\alpha$ satisfies $\psi(0) = 0$ and

$$d\psi = L \psi \, dA_0 + LuB_\alpha \, dA_0 - h_r B_\alpha \, dA_r = L \psi \, dA_0 + c_{\alpha 0} Lu \, dA_0 + \delta Lu \, dB_{\alpha 0} - c_{\alpha r} h_r \, dA_0 - \delta h_r \, dB_{\alpha r}. \quad (4.12)$$

Now (4.9) follows from the definitions of $\mathcal{R}$ and $\mathcal{Q}_\beta$. The proof of the lemma is complete. \hfill \Box

Now we introduce some differential operators $L_\gamma$ and functions $f_\gamma$ defined for multi-numbers $\gamma$ as follows: $L_0 := 0$, $f_0 := 0$,

$$L_\gamma := L_r, \quad f_\gamma := f_r$$

for $\gamma = r \in \{1, 2, \ldots, d_1\}$, and

$$L_{\gamma 0} := LL_\gamma, \quad L_{\gamma r} := -L_\gamma L_r$$

$$f_{\gamma 0} := Lf_\gamma, \quad f_{\gamma r} := -L_\gamma f_r$$
for $r = 1, 2, \ldots, d_1$.

In this notation we have the following.

**Lemma 4.3.** Let $\alpha, \beta \in \mathcal{N}$ and $2|\beta| + 3 \leq l$. Then

$$Q_\alpha(L_\beta w + f_\beta) = c_{\alpha r} R(L_{\beta r} w + f_{\beta r}) + \delta Q_{\alpha r}(L_{\beta r} w + f_{\beta r}) + B_\alpha(L_{\beta} w + f_{\beta}). \quad (4.10)$$

**Proof.** It follows from formula (4.9) applied to $u := L_\beta w + f_\beta$, when $h_r = L_\beta L_r w + L_\beta f_r$ (remember $f_r$ are independent of $t$), that the left part of (4.10) equals

$$R[\sum_{|\alpha| = i} \delta^i \sum_{|\beta| = 1} c_{\alpha \beta} R(L_{\alpha \beta} w + f_{\alpha \beta}) + \delta Q_{\alpha r}(L_{\alpha \beta} w + f_{\alpha \beta}) + \sum_{|\alpha| = i+1} B_\alpha(L_{\beta} w + f_{\beta})],$$

which is easily seen to be equal to the right-hand side of (4.10).

We derive from (4.10) one of the most important formulas.

**Proposition 4.4.** Let $\kappa \geq 0$ be an integer and $l \geq 2\kappa + 3$. Then

$$w = v + \sum_{i=1}^{\kappa} \delta^i \sum_{|\alpha| = i} B_\alpha(L_\alpha w + f_\alpha)$$

$$+ \sum_{i=1}^{\kappa} \delta^i \sum_{|\alpha| = i+1} c_{\alpha r} R(L_{\alpha r} w + f_{\alpha r}) + \delta^{\kappa+1} r^{(\kappa)} \quad (4.11)$$

for all $t \in [0, d_1 T]$, where

$$r^{(\kappa)} = \sum_{|\alpha| = \kappa+1} Q_{\alpha}(L_{\alpha} w + f_{\alpha}).$$

**Proof.** First notice that for $\varphi_0 := w - v$ we have

$$d\varphi_0 = d(w - v) = L\varphi_0 \, dA_0 + \delta(L_r w + f_r) \, dB_r,$$

which proves (4.11) for $\kappa = 0$.

Next we fix a $\kappa \geq 1$ and transform $r^{(i)}$, for $i = 0, \ldots, \kappa-1$, by applying (4.10) with $\alpha = \beta$ and $|\alpha| = i + 1$ when $f_r \in W^{2|\beta|+3}_p$. Then we get

$$r^{(i)} = \sum_{|\alpha| = i+1, |\beta| = 1} c_{\alpha \beta} R(L_{\alpha \beta} w + f_{\alpha \beta}) + \delta \sum_{|\alpha| = i+1, |\beta| = 1} Q(L_{\alpha \beta} w + f_{\alpha \beta})$$

$$+ \sum_{|\alpha| = i+1} B_\alpha(L_\alpha w + f_\alpha) = \sum_{|\alpha| = i+1} B_\alpha(L_\alpha w + f_\alpha)$$

$$+ \sum_{|\alpha| = i+2} c_{\alpha} R(L_{\alpha} w + f_{\alpha}) + \delta r^{(i+1)}.$$
This shows how $r^{(0)}$, $r^{(1)}$, ..., $r^{(κ)}$ are related to each other and certainly proves the proposition.

Decomposition (4.11) looks very much like (2.6) the only difference being that the factors of $δ^j$ depend on the approximating function $w$ and the coefficients of $δ^j$ in the second term on the right contain $B_α$ which is no power series in $δ$. However, observe that we have to estimate the difference $v − w$ only at the points $id_1 δ$ at which all $B_α$ vanish.

Our next step is to “solve” (4.11) with respect to $w$ by the method of successive iterations, that is by substituting $w$ given by (4.11) into the right-hand side of the same equation. In the process of doing so we encounter only one difficulty when the second term on the right is plugged into the third one and we have to develop expressions like $R(B_α,u)$ into power series in $δ$. We transform these terms by using (4.8) and (4.10).

First we introduce the notation

$$w_β = L_β w + f_β,$$

observe that in these terms (4.10) is rewritten as

$$Q_α w_β = c_α r R w_β r + B_α w_β + δ Q_α r w_β r,$$

and note the following.

**Lemma 4.5.** If $κ ≥ 0$ is an integer and $α$, $β ∈ N$ and $2(|β| + κ) + 1 ≤ l$, then

$$R(B_α w_β) = \sum_{i=0}^{κ} δ^i \sum_{|γ|=i} c_α_γ r R w_β γ$$

$$+ \sum_{i=1}^{κ} δ^i \sum_{|γ|=i-1} B_α_γ w_β γ + δ^{κ+1} \sum_{|γ|=κ} Q_α_γ w_β γ,$$

(4.13)

where for any multi-numbers $μ, ν$

$$\sum_{|γ|=0} c_γ w_μ γ \overset{:=}{=} c_γ R w_μ, \quad \sum_{|γ|=0} B_γ w_μ γ \overset{:=}{=} B_γ w_μ,$$

$$\sum_{|γ|=0} Q_γ w_μ γ \overset{:=}{=} Q_γ w_μ.$$

**Proof.** If $κ = 0$, (4.13) follows from (4.8). If $κ ≥ 1$, by applying repeatedly (4.12) as in the proof of Proposition 4.4 we find

$$Q_α w_β = \sum_{i=0}^{κ-1} δ^i \sum_{|γ|=i+1} c_α_γ R w_β γ.$$
\[ + \sum_{i=0}^{\kappa-1} \delta^i \sum_{|\gamma|=i} B_{\alpha\gamma} w_{\beta\gamma} + \delta^\kappa \sum_{|\gamma|=\kappa} Q_{\alpha\gamma} w_{\beta\gamma}. \]

We use this formula for \( \alpha 0 \) in place of \( \alpha \) and finish the proof by referring to (4.8).

Let \( \mathcal{M} \) denote the set of multi-numbers \( \gamma_1 \gamma_2 \cdots \gamma_i \) with \( \gamma_j \in \{1, 2, \ldots, d_1\} \), \( j = 1, 2, \ldots, i \), and integers \( i \geq 1 \).

**Lemma 4.6.** The following statements hold.

(i) Let \( \gamma = \gamma_1 \gamma_2 \cdots \gamma_i \in \mathcal{M} \) be such that \( |\gamma| = i \leq 1 + l/2 \). Then

\[
L_\gamma = (-1)^{|\gamma|-1} L_{\gamma_1} \cdots L_{\gamma_i}, \quad f_\gamma = (-1)^{|\gamma|-1} L_{\gamma_1} \cdots L_{\gamma_{i-1}} f_{\gamma_i},
\]

(ii) Let \( \beta, \gamma \in \mathcal{M} \) be such that \( |\beta| + |\gamma| \leq 1 + l/2 \). Then

\[
L_\beta L_\gamma = -L_{\beta \gamma}, \quad L_\beta f_\gamma = -f_{\beta \gamma}.
\]

(iii) Let \( \alpha \in \mathcal{N} \) be such that \( \rho := |\alpha| \leq 1 + l/2 \). Then there exist constants \( c(\gamma) = c(\alpha, \gamma) \in \{0, \pm 1\} \) defined for all \( \gamma \in \mathcal{M} \) with \( |\gamma| = \rho \), such that

\[
L_\alpha = \sum_{\gamma \in \mathcal{M}, |\gamma| = \rho} c(\gamma) L_\gamma, \quad f_\alpha = \sum_{\gamma \in \mathcal{M}, |\gamma| = \rho} c(\gamma) f_\gamma. \tag{4.14}
\]

**Proof.** Part (i) follows immediately from the definition of \( L_\gamma, f_\gamma \) by induction on \( |\gamma| \). Part (i) obviously implies Part (ii). Part (iii) clearly holds for \( \alpha = 0 \) and \( \alpha = r \in \{1, \ldots, d_1\} \). Assume that equations (4.14) hold for some \( \alpha \in \mathcal{N}, |\alpha| < 1 + l/2 \). Then

\[
L_{\alpha r} = -L_\alpha L_r = -\sum_{|\gamma|=|\alpha|} c(\gamma) L_\gamma L_r = \sum_{|\gamma|=|\alpha|} c(\gamma) L_{\gamma r},
\]

\[
f_{\alpha r} = -L_\alpha f_r = -\sum_{|\gamma|=|\alpha|} c(\gamma) L_\gamma f_r = \sum_{|\gamma|=|\alpha|} c(\gamma) f_{\gamma r}
\]

for \( r \in \{1, 2, \ldots, d_1\} \), and

\[
L_{\alpha 0} = LL_\alpha = \sum_{r=1}^{d_1} L_r \sum_{|\gamma|=\rho} c(\gamma) L_\gamma = -\sum_{r=1}^{d_1} \sum_{\gamma \in \mathcal{M}, |\gamma| = \rho} c(\gamma) L_{\gamma r},
\]

\[
f_{\alpha 0} = Lf_\alpha = \sum_{r=1}^{d_1} L_r \sum_{|\gamma|=\rho} c(\gamma) f_\gamma = -\sum_{r=1}^{d_1} \sum_{\gamma \in \mathcal{M}, |\gamma| = \rho} c(\gamma) f_{r \gamma},
\]

which prove (iii) by induction on \( |\alpha| \). \( \Box \)
We introduce sequences \( \sigma = (\beta_1, \beta_2, ..., \beta_i) \) of multi-numbers \( \beta_j \in \mathcal{M} \), where \( i \geq 1 \) is any integer, and set \( |\sigma| := |\beta_1| + |\beta_2| + ... + |\beta_i| \). We consider also the ‘empty sequence’ \( e \) of length \( |e| = 0 \), and denote the set of all these sequences by \( \mathcal{J} \). For \( \sigma = (\beta_1, \beta_2, ..., \beta_i), i \geq 1 \), we define

\[
S_{\sigma} = R L_{\beta_1} \cdots R L_{\beta_i}
\]

and for \( \sigma = e \) we set \( S_{e} = R \).

Notice that \( S_{\sigma} \) involves \( 2|\sigma| \) derivatives with respect to \( x \) and certain number of operators \( R \) which do not increase regularity. Therefore, basically, \( S_{\sigma} \) has the power of a differential operator of \( 2|\sigma| \)’th order.

If we have a collection of functions \( g_{\nu} \) indexed by a parameter \( \nu \) taking values in a set \( A \), then we use the notation

\[
\sum^*_{\nu \in A} g_{\nu}
\]

for any linear combination of \( g_{\nu} \) with coefficients independent of the argument of \( g_{\nu} \) and of \( \delta \). For instance,

\[
\sum^*_{A} S_{\sigma} w_{\gamma} = \sum^*_{(\sigma, \gamma) \in A} S_{\sigma} w_{\gamma} = \sum_{(\sigma, \gamma) \in A} c(\sigma, \gamma) S_{\sigma} w_{\gamma},
\]

where \( c(\sigma, \gamma) \) are certain constants independent of \( \delta \). These constants are allowed to change from one occurrence to another.

For functions \( u = u(t, x) = u(\delta, t, x) \) depending on the parameter \( \delta \) we write \( u = O_{m}(\delta^{\kappa}) \), if

\[
\sup_{\delta} \delta^{-\kappa} \sup_{t \in [0, d_{1}T]} \|u(t)\|_{m, p} < \infty.
\]

We also use the following sets

\[
A(i) = \{(\sigma, \beta) : \sigma \in \mathcal{J}, \beta \in \mathcal{M}, |\sigma| + |\beta| \leq i\},
\]

\[
B(i, j) = \{ (\alpha, \beta) : \alpha \in \mathcal{N}, \beta \in \mathcal{M}, |\alpha| = i, |\beta| \leq j \}.
\]

**Lemma 4.7.** Let \( \kappa, \mu \geq 0 \) be integers and \( \alpha \in \mathcal{N}, \beta \in \mathcal{M}, \sigma \in \mathcal{J} \). Assume that

\[
2(|\sigma| + |\beta| + \kappa) + \mu + 2 \leq l.
\]

Then

\[
S_{\sigma}(B_{\alpha} w_{\beta}) = \sum^{\kappa}_{i=0} \delta^{i} \sum^*_{A(|\sigma|+|\beta|+i)} S_{\sigma_{1}} w_{\beta_{1}}
\]

\[
+ \sum^{\kappa}_{i=1} \delta^{i} \sum^*_{B(|\alpha|+i, |\sigma|+|\beta|+i-1)} B_{\alpha_{1}} w_{\beta_{1}} + O_{\mu}(\delta^{\kappa+1}).
\]
Proof. For $\sigma = e$, when $S_\sigma = R$, equation (4.16) turns out to be just a different form of (4.13), which is applicable since $2(\|\beta\| + \kappa) + 1 \leq l$.

Indeed, owing to Lemma 4.6 (iii)

$$\sum_{|\gamma| = i} c_{\alpha\gamma} R w_{\beta\gamma} = \sum_{A(\|\sigma\| + \|\beta\| + i)} S_{\sigma_1} w_{\beta_1},$$

$$\sum_{|\gamma| = i - 1} B_{\alpha_0\gamma} w_{\beta\gamma} = \sum_{B(\|\alpha\| + i) + (\|\beta\| + i)}^* B_{\alpha_1} w_{\beta_1}.$$  

Furthermore, for $|\gamma| = \kappa$ (see Remark 3.1)

$$Q_{\alpha_0\gamma} w_{\beta\gamma} = O_\mu(1),$$ since $2(\|\beta\| + \kappa) + \mu + 2 \leq l$.

For $|\sigma| \geq 1$ we proceed by induction on the length $\ell(S_\sigma)$ of $S_\sigma = RL_{\beta_1} \cdot \cdots \cdot RL_{\beta_j}$, which we define to be $j$. If $\ell(S_\sigma) = 1$, then $S_\sigma = RL_\nu$ for a $\nu \in M$ with $\nu = \sigma$ and it suffices to notice that

$$S_\sigma(B_\alpha w_\beta) = RL_\nu(B_\alpha w_\beta) = -R(B_\alpha w_\nu) = -S_{\epsilon}(B_\alpha w_\nu), \quad (4.17)$$

where $\beta' = \nu \beta \in M$ and $2(\|\beta'\| + \kappa) + \mu + 2 = 2(\|\sigma\| + \|\beta\| + \kappa) + \mu + 2 \leq l$.

Assume that (4.16) holds whenever $\ell(S_\sigma) = s$, and take an $S_{\sigma'}$ such that $\ell(S_{\sigma'}) = s + 1$. Then $S_{\sigma} = RL_\nu S_{\sigma'}$, where $\nu, \sigma' \in M, |\nu| + |\sigma'| = |\sigma|$ and $\ell(S_{\sigma'}) = s$. Furthermore,

$$2(|\sigma'| + |\beta| + \kappa) + \mu' + 2 \leq l,$$

where $\mu' = \mu + 2|\nu|$. By the induction hypothesis

$$S_{\sigma'}(B_\alpha w_\beta) = \sum_{i=0}^{\kappa} \delta^i \sum_{A(\|\sigma'\| + \|\beta\| + i)}^* S_{\sigma_1} w_{\beta_1}$$

$$+ \sum_{i=1}^{\kappa} \delta^i \sum_{B(\|\alpha\| + i) + (\|\sigma'\| + \|\beta\| + i - 1)}^* B_{\alpha_1} w_{\beta_1} + O_\mu(\delta^{\kappa+1}).$$

We apply $RL_\nu$ to both parts of this equality and take into account that $L_\nu w_{\beta_1} = -w_{\nu_1}$ and $|\nu| + |\sigma'| = |\sigma|$. Then similarly to (4.17) we get that

$$S_{\sigma}(B_\alpha w_\beta) = \sum_{i=0}^{\kappa} \delta^i \sum_{A(\|\sigma\| + \|\beta\| + i)}^* S_{\sigma_1} w_{\beta_1}$$

$$+ \sum_{i=1}^{\kappa} \delta^i \sum_{B(\|\alpha\| + i) + (\|\sigma\| + \|\beta\| + i - 1)}^* S_{\epsilon}(B_{\alpha_1} w_{\beta_1}) + O_\mu(\delta^{\kappa+1}). \quad (4.18)$$

Now we transform the second term on the right. Take $(\alpha_1, \beta_1) \in B(\|\alpha\| + i, \|\sigma\| + \|\beta\| + i - 1)$ and notice that then $\|\beta_1\| \leq |\sigma| + |\beta| + i - 1$. Hence by assumption (4.13)

$$2(|\beta_1| + \kappa - i) + \mu + 2 < l.$$
Therefore, by the result for $\sigma = e$

$$S_e(B_{\alpha_1}w_{\beta_1}) = \sum_{j=0}^{\kappa-i} \delta^j \sum_{A(\beta_1+j)}^* S_{\sigma_2} w_{\beta_2}$$

$$+ \sum_{j=1}^{\kappa-i} \delta^j \sum_{B(\alpha_1+j, \beta_1+j-1)}^* B_{\alpha_2}w_{\beta_2} + O_\mu(\delta^{\kappa-i+1}).$$

We substitute this result into (4.18) and obtain (4.16) after collecting the coefficients of $\delta^i$, and noticing that, if $(\alpha_1, \beta_1) \in B(|\alpha| + i, |\sigma| + |\beta| + i - 1)$ and $(\alpha_2, \beta_2) \in B(|\alpha_1| + j, |\beta_1| + j - 1)$, then

$$(\alpha_2, \beta_2) \in B(|\alpha| + i + j, |\sigma| + |\beta| + i + j - 1).$$

This justifies the induction and finishes the proof of the lemma. \qed

We remind the reader that throughout this section the assumptions of Theorem 2.1 are supposed to be satisfied and in the following proposition use the notation $B^*(i, j) = \bigcup_{i_1=1}^{i} B(i_1, j)$.

**Proposition 4.8.** For any $j = 0, 1, ..., k$ we have

$$w = v + \sum_{i=1}^{j} \delta^i \sum_{A(2i)}^* S_{\sigma} v_{\beta} + \sum_{i=j+1}^{k} \delta^i \sum_{A(i+j+1)}^* S_{\sigma_1} w_{\beta_1}$$

$$+ \sum_{i=1}^{k} \delta^i \sum_{B^*(i,i+j)}^* B_{\alpha_1}w_{\beta_1} + O_m(\delta^{k+1}),$$

(4.19)

where $v_{\beta} := L_{\beta}v + f_{\beta}$.

**Proof.** By Proposition 4.4 (notice that, due to (2.5), $l \geq 2k + 3$ and $2(k + 1) + m + 2 \leq l$) we have

$$w = v + \sum_{i=1}^{k} \delta^i \sum_{|\beta|=i} B_{\beta} w_{\beta} + \sum_{i=1}^{k} \delta^i \sum_{|\beta|=i+1} c_{\beta} R w_{\beta} + O_m(\delta^{k+1}).$$

which means that (4.19) holds for $j = 0$, since by Lemma 4.6 (iii)

$$\sum_{|\beta|=i} B_{\beta} w_{\beta} = \sum_{|\beta|=i} B_{\beta} \sum_{\gamma \in M, |\gamma|=i} c(\beta, \gamma) w_{\gamma} = \sum_{B^*(i,i)}^* B_{\alpha_1} w_{\beta_1},$$

$$\sum_{|\beta|=i+1} c_{\beta} R w_{\beta} = \sum_{|\beta|=i+1} c_{\beta} \sum_{\gamma \in M, |\gamma|=i+1} c(\beta, \gamma) R w_{\gamma} = \sum_{A(i+1)}^* S_{\sigma_1} w_{\beta_1}.$$
Next, assume that $k \geq 1$ and (4.13) holds for a $j \in \{0, \ldots, k-1\}$. Transform the first term with $i = j + 1$ in the second sum on the right in (4.10) by using Lemma 4.7. To prepare the transformation take $(\sigma_1, \beta_1) \in A(2i) = A(i + j + 1)$ so that $|\sigma_1| + |\beta_1| \leq 2i$ and apply the operator $S_{\sigma_1}L_{\beta_1}$ to both parts of equation (4.11) with $k - i$ in place of $\kappa$. Then we obtain

$$S_{\sigma_1}w_{\beta_1} = S_{\sigma_1}v_{\beta_1} + \sum_{i_1=1}^{k-i} \delta^{i_1} \sum_{|\alpha_i|=i_1} S_{\sigma_1}(B_{\alpha_1}L_{\beta_1}w_{\alpha_1})$$

$$+ \sum_{i_1=1}^{k-i} \delta^{i_1} \sum_{|\alpha_1|=i_1+1} c_{\alpha_1}S_{\sigma_1}L_{\beta_1}Rw_{\alpha_1} + \delta^{k-i+1}r^{(k-i)},$$

where

$$r^{(k-i)} = \sum_{|\alpha|=k-i+1} S_{\sigma_1}L_{\beta_1}Q_\alpha w_\alpha.$$

Owing to

$$l - 2(k - i + 1 + |\beta_1| + |\sigma_1|) \geq l - 2(k + i + 1) \geq l - 2(2k + 1) \geq m + 2,$$

we have $r^{(k-i)} = O_m(1)$. By the way, this is the only place where we need $l$ to be not smaller than $4 + m + 4k$. Hence by Lemma 4.6 (iii)

$$S_{\sigma_1}w_{\beta_1} = S_{\sigma_1}v_{\beta_1} + \sum_{i_1=1}^{k-i} \delta^{i_1} \sum_{(\alpha_2, \beta_2) \in B(i_1, |\beta_1|+i_1)} S_{\sigma_1}(B_{\alpha_2}w_{\beta_2})$$

$$+ \sum_{i_1=1}^{k-i} \sum_{A(|\sigma_1|+|\beta_1|+i_1+1)} S_{\sigma_2}w_{\beta_2} + O_m(\delta^{k-i+1}) =: J_1 + \ldots + J_4. \quad (4.20)$$

Now Lemma 4.7 with $k - i - i_1$ in place of $\kappa$ and $m$ in place of $\mu$ allows us to transform terms entering $J_2$. For $|\beta_2| \leq |\beta_1| + i_1$ we have (remember that $(\sigma_1, \beta_1) \in A(2i)$)

$$2(|\sigma_1| + |\beta_2| + k - i - i_1) + m + 2 \leq 2(|\sigma_1| + |\beta_1| + k - i) + m + 2$$

$$\leq 2(i + k) + m + 2 \leq 4k + m + 2 < l.$$

Therefore

$$S_{\sigma_1}(B_{\alpha_2}w_{\beta_2}) = \sum_{i_2=0}^{k-i-i_1} \delta^{i_2} \sum_{A(|\sigma_1|+|\beta_2|+i_2)} S_{\sigma_3}w_{\beta_3}$$

$$+ \sum_{i_2=1}^{k-i-i_1} \delta^{i_2} \sum_{B(|\alpha_2|+i_2, |\sigma_1|+|\beta_2|+i_2-1)} B_{\alpha_3}w_{\beta_3} + O_m(\delta^{k-i-i_1+1}).$$
We plug this result into $J_2$ and in order to collect the coefficients of $\delta^{i_1+i_2}$ notice that, for $(\sigma_3, \beta_3) \in A(|\sigma_1| + |\beta_2| + i_2)$ and $(\alpha_2, \beta_2) \in B(i_1, |\beta_1| + i_1)$ it holds that

$$|\sigma_3| + |\beta_3| \leq |\sigma_1| + |\beta_2| + i_2 \leq |\sigma_1| + |\beta_1| + i_1 + i_2.$$  

Furthermore, if $(\alpha, \beta_3) \in B(|\alpha_2| + i_2, |\sigma_1| + |\beta_2| + i_2 - 1)$, then

$$|\alpha_3| = |\alpha_2| + i_2 = i_1 + i_2, \quad |\beta_3| \leq |\sigma_1| + |\beta_2| + i_2 - 1 < |\sigma_1| + |\beta_1| + i_1 + i_2.$$  

It follows that $J_2$ is written as

$$\sum_{i_1=1}^{k-i} \delta^{i_1} \left( \sum_{A(|\sigma_1|+|\beta_1|+i_1)} S_{\sigma_2} w_{\beta_2} + \sum_{B(i_1,|\sigma_1|+|\beta_1|+i_1)} B_{\alpha_2} w_{\beta_2} \right) + O_m(\delta^{k-i+1}),$$

which just amounts to saying that visually in the definition of $J_2$ one can erase $S_{\sigma_1}$, carry all differentiations in it onto $w_{\beta_2}$, and still preserve (4.20).

Then we see that

$$\delta^{i_1} \sum_{A(2j+2)} S_{\sigma_1} w_{\beta_1} = O_m(\delta^{k+1}) + \delta^{i_1} \sum_{A(2j+2)} S_{\sigma_1} v_{\beta_1}$$

$$+ \sum_{i_1=1}^{k-j-1} \delta^{i_1+j+1} \left( \sum_{A(|\sigma_1|+|\beta_1|+i_1+1)} S_{\sigma_2} w_{\beta_2} + \sum_{B(i_1,|\sigma_1|+|\beta_1|+i_1)} B_{\alpha_2} w_{\beta_2} \right).$$

Next we notice again that, for $(\sigma_1, \beta_1) \in A(2j+2)$ and $|\sigma_2| + |\beta_2| \leq |\sigma_1| + |\beta_1| + i_1 + 1$, we have $|\sigma_2| + |\beta_2| \leq j + 2 + i_1 + j + 1$, whereas if $|\beta_2| \leq |\sigma_1| + |\beta_1| + i_1$, then $|\beta_2| \leq j + 1 + i_1 + j + 1$. Therefore, after changing $i_1 + j + 1 \to i \geq j + 2$ we get

$$\delta^{i_1} \sum_{A(2j+2)} S_{\sigma_1} w_{\beta_1} = O_m(\delta^{k+1}) + \delta^{i_1} \sum_{A(2j+2)} S_{\sigma_1} v_{\beta_1}$$

$$+ \sum_{i=j+2}^{k} \delta^{i} \left( \sum_{A(i+j+2)} S_{\sigma_2} w_{\beta_2} + \sum_{B^*(i,j+1)} B_{\alpha_2} w_{\beta_2} \right).$$

This shows that the term with $i = j + 1$ in the second sum on the right in (4.19) can be eliminated on the account of changing other terms with simultaneous shift $j \to j + 1$. Thus the induction on $j$ proves the proposition indeed.

Now we finish the proof of Theorem 4.1. By taking $j = k$ in Proposition 4.8, we find

$$w = v + \sum_{i=1}^{k} \delta^{i} w^{(i)} + \sum_{B^*(k,2k)} c(\alpha, \beta, \delta) B_{\alpha} w_{\beta} + O_m(\delta^{k+1}), \quad (4.21)$$
where
\[ w^{(i)} := \sum_{A(2i)}^{*} S_{\sigma} v_{\beta} \in C_{w}([0, T], W_{p}^{m}), \quad i = 1, 2, ..., k, \]
are independent of \( \delta \), and \( c(\alpha, \beta, \delta) \) are certain constants. It is not hard to follow our computations in order to see that
\[ \sup_{t \in [0, d_{1}T]} \sup_{n, \delta = T/n} \delta^{-(k+1)}\|O_{m}(\delta^{k+1})(t, \cdot)\|_{m, p} \leq N, \]
where the constant \( N \) depends only on \( d, d_{1}, T, K, k, m, p, l \). After that to finish the proof it only remains to recall that \( B_{\alpha}(jd_{1}\delta) = 0 \) for all integers \( j \geq 0 \) and \( v(d_{1}t) = u(t), \quad w(d_{1}t) = u_{n}(t) \quad \forall t \in T_{n} = \{iT/n : i = 0, 1, 2, ..., n\} \).

5. The Case of Time Dependent Coefficients

We consider here the Cauchy problem (1.1)-(1.2) for time dependent coefficients. We split, as before, the coefficients and the free terms into \( d_{1} \) terms,
\[ a^{ij} = \sum_{r=1}^{d_{1}} a_{r}^{ij}, \quad a^{i} = \sum_{r=1}^{d_{1}} a_{r}^{i}, \quad a = \sum_{r=1}^{d_{1}} a_{r}, \quad f = \sum_{r=1}^{d_{1}} f_{r}, \]
define \( \delta = T/n, \quad t_{i} = t_{i}^{n} = \delta i, \quad T_{n} = \{t_{i} : i = 0, 1, ..., n\}, \) and keep Assumptions 2.1, 2.2 As before, we also denote \( L_{r} := a_{r}^{ij}D_{ij} + a_{r}^{i}D_{i} + a_{r} \).

One of splitting-up approximations \( u_{n}(t) \) for \( t \in T_{n} \) is defined as follows. Let \( S_{st}^{(r)} \) be the operator mapping each function \( \varphi \) of an appropriate class into the solution of the problem
\[ D_{t}v(t, x) = d_{1}L_{r}v(t, x) + d_{1}f_{r}(t, x), \quad t > s, \quad v(s, x) = \varphi(x). \]
Then the approximations are introduced according to
\[ u_{n}(0) := u_{0}, \]
\[ u_{n}(t_{i+1}) := S_{t_{i}^{d_{1}}}^{(d_{1})} ... S_{t_{i}^{d_{1}}}^{(2)} S_{t_{i}^{d_{1}}}^{(1)} u_{n}(t_{i}), \quad i = 0, 1, 2, ..., n, \quad (5.1) \]
where \( t_{ij} := t_{i} + j\delta/d_{1} \), for \( j = 1, 2, ..., d_{1} - 1, \bar{d} := d_{1} - 1 \).

There are many other ways to extend the splitting-up approximations (2.4) to PDEs with time dependent data. Along with (5.1) we also consider another approximation, which has the advantage that in each step we need to solve a time independent PDE, which is usually more convenient in practice than solving time dependent PDEs. This time we define the approximation \( u_{n} \) by
\[ u_{n}(0) := u_{0}, \]
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\[ u_n(t_{i+1}) := S^{(d_1)}_\delta (t_{i+1}^n) \ldots S^{(2)}_\delta (t_{i+1}^n) S^{(1)}_\delta (t_{i+1}^n) u_n(t_i^n), \quad i = 0, 1, 2, \ldots, n, \]  

(5.2)

where \( S^{(r)}(s) \varphi \) denotes the solution of the problem

\[ D_t v(t) = L_r(s) v(t) + f(s), \quad t \geq 0, \quad v(0) = \varphi, \]  

(5.3)

with

\[ L_r(s) := a_{ij}^r(s,x) D_{ij} + a_i^r(s,x) D_i + a_r^i(s,x), \]

for \( r = 1, 2, \ldots, d_1 \). Notice that the coefficients of the operator \( L_r(s) \) and \( f(s) \) are “frozen” at time \( s \), thus (5.3) is a Cauchy problem with time independent data.

We extend Theorem 2.1 as follows.

**Theorem 5.1.** Let \( m \geq 0 \) and \( k \geq 0 \) be any integers. Let Assumptions 2.1 and 2.2 hold with \( l \geq 4 + m + 4k \). Let the splitting-up approximation \( u_n \) be defined by (5.1) or by (5.2). Then there exist functions \( u^{(j)} \in C_w([0,T], W^m_p) \), \( j = 1, 2, \ldots, k \), \( R_n^{(k)} \in C_w([0,T], W^m_p) \), such that

\[ u_n(t,x) = u(t,x) + \delta u^{(1)}(t,x) \]

\[ + \delta^2 u^{(2)}(t,x) + \ldots + \delta^k u^{(k)}(t,x) + R_n^{(k)}(t,x) \]  

(5.4)

for all \( t \in T_n, x \in \mathbb{R}^d \), and \( n \geq 1 \). The functions \( u^{(j)} \), \( j = 1, 2, \ldots, k \), are independent of \( n \), and

\[ \sup_{t \in [0,T]} \| R_n^{(k)}(t) \|_{m,p} \leq N \delta^{k+1} \]

for all \( n \), where \( N \) depends only on \( k, d, d_1, K, m, p, T \).

Clearly Theorem 5.1 implies that we can again accelerate the convergence of the splitting-up approximations by considering

\[ v_n(t,\cdot) := \sum_{j=0}^{k} b_j u_{2jn}(t,\cdot), \quad t \in T_n, \]

where \( u_{2jn} \) for all \( j = 0, 1, \ldots, k \) are defined by either (5.1) or by (5.2).

**Theorem 5.2.** Let \( m \geq 0 \) and \( k \geq 0 \) be any integers. Let Assumptions 2.1 and 2.2 hold with \( l \geq 4 + m + 4k \). Then for all \( n \geq 1 \)

\[ \max_{t \in T_n} \| v_n(t) - u(t) \|_{m,p} \leq N \delta^{k+1}, \]

where \( N \) is a constant, depending only on \( k, d, d_1, K, m, p, T \).

Hence by Sobolev’s Theorem on embedding \( W^m_p \) into \( C^s \) we immediately get the following result.
Theorem 5.3. Let \( m \geq 0 \) and \( k \geq 0 \) be any integers. Let Assumptions 2.1 and 2.2 hold with \( l \geq 4 + m + 4k \). Let \( s \geq 0 \) be an integer such that \( m \geq s + d/p \). Then for all \( n \geq 1 \)

\[
\max_{t \in T_n} \sup_{x \in \mathbb{R}^d} \sum_{|\rho| \leq s} |D^\rho v_n(t, x) - D^\rho u(t, x)| \leq N\delta^{k+1},
\]

where \( N \) is a constant, depending only on \( k, d, d_1, K, m, s, p, T \).

We prove Theorem 5.1 by adapting the proof of Theorem 2.1 to the time dependent case. If \( u_n \) is defined by (5.1), then we consider the problems

\[
dv(t) = (L(A_0(t))v(t) + f(A_0(t))) \, dA_0(t), \quad v(0) = u_0, \quad (5.5)
\]

\[
dw(t) = \sum_{r=1}^{d_1} (L_r(A_0(t))w(t) + f_r(A_0(t))) \, dA_r(t), \quad w(0) = u_0, \quad (5.6)
\]

where \( A_0(t), A_1(t), A_2(t), \ldots, A_{d_1}(t) \) are defined by (4.2) and (4.3), and \( L(A_0(t)), L_r(A_0(t)) \) mean that we substitute \( A_0(t) \) in place of the time variable \( t \) of the coefficients of \( L, L_r \).

If \( u_n \) is defined by (5.2) then we consider problems (5.5) and (5.6) with absolutely continuous functions \( A_0, A_1, \ldots, A_{d_1} \), defined by the following requirements:

\[
A_r(0) = 0, \quad \dot{A}_r \text{ is periodic with period } (d_1 + 1)\delta,
\]

\[
\dot{A}_r(t) = 1_{[r, r+1]}/\delta, \quad t \in [0, (d_1 + 1)\delta] \quad \text{for } r = 0, 1, \ldots, d_1. \quad (5.7)
\]

By virtue of Theorem 3.1 equations (5.5) and (5.6) admit unique solutions \( v \) and \( w \), respectively. Clearly \( v, w \in C_w([0, dT], W^s_{p'}) \), and

\[
v(d't) = u(t), \quad w(d't) = u_n(t) \quad \text{for all } t \in T_n,
\]

where \( d' = d_1 \) if \( A_0, A_1, \ldots, A_{d_1} \) are defined by (4.2) and (4.3), and \( d' = d_1 + 1 \) if \( A_0, A_1, \ldots, A_{d_1} \) are defined by (5.7). Therefore, our aim is to get an equality like (5.4) with \( v \) and \( w \) in place of \( u \) and \( u_n \), respectively.

We treat the cases of two approximations simultaneously and warn the reader that, in order not to repeat the same arguments twice, we are going to use the same notation for some objects that have different meaning in each case. From now on \( d' \) denotes \( d_1 \) if we consider the splitting-up approximations \( u_n \) defined by (5.1), and it denotes \( d_1 + 1 \) in the case of \( u_n \) defined by (5.2). We keep the notation \( \mathcal{N} \) for the set
of all multi-numbers \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_j \) for \( \alpha_i \in \{0, 1, 2, \ldots, d_1\} \) and integers \( j \geq 1 \). We use also the numbers \( c_\alpha \) and the functions \( B_\alpha \), defined by (4.4), (4.5), and (4.6), with \( d' \) in place of \( d_1 \) in (4.5). Observe that as is easy to see Lemma 4.1 still holds with \( d' \) in place of \( d_1 \) in its formulation.

Let \( \mathcal{R} f \) and \( \mathcal{R} f \) denote the solutions of the problems

\[
du(t) = (Lu(t) + f(t)) \, dt, \quad u(0) = 0,
\]

and

\[
dv(t) = (\bar{L}v(t) + f(t)) \, dA_0(t), \quad v(0) = 0,
\]

respectively, where \( \bar{L} := L(A_0(t)) \), the operator obtained from \( L \) by the substitution of \( A_0(t) \) in place of \( t \) in the coefficients of \( L \). Notice that \( \mathcal{R} \) depends on \( n \) when \( A_0 \) is defined by (5.7). Notice also that \( \mathcal{R} f \) depends on \( \cdot \) when \( A_0 \) is defined by (5.7) in its formulation.

We modify the definition of \( L_\alpha, f_\alpha \), used for time independent operators and free term, as follows: \( L_0 := 0, f_0 := 0, \)

\[
L_\gamma := L_r, \quad f_\gamma := f_r
\]

for \( \gamma = r \in \{1, 2, \ldots, d_1\} \), and

\[
L_{\gamma 0} := LL_\gamma - \dot{L}_\gamma, \quad L_{\gamma r} := -L_\gamma L_r
\]

\[
f_{\gamma 0} := Lf_\gamma - \dot{f}_\gamma, \quad f_{\gamma r} := -L_\gamma f_r
\]

for \( r = 1, 2, \ldots, d_1 \), where \( \dot{f}_\gamma := D_t f_\gamma \), and \( \dot{L}_\gamma \) denotes the differential operator which we obtain from \( L_\gamma \) by taking the derivative in \( t \) of its coefficients. As is easy to see, \( L_\gamma \) and \( f_\gamma \) are well defined if \( 2(|\gamma| - 1) \leq l \).

We use the notation \( \bar{L}_\gamma \) and \( \bar{f}_\gamma \) for the operator which we obtain from \( L_\gamma \) by substituting \( A_0(t) \) in place of \( t \) in its coefficients, and for the function obtained from \( f_\gamma \) by the same substitution, respectively. Then we have the following counterpart of Lemma 4.2.

**Lemma 5.4.** Take some functions

\[
h \in L_p([0, d'T], W^1_p), \quad h_r \in L_p([0, d'T], W^1_p), \quad r = 0, 1, \ldots, d_1.
\]

Let \( u \) be a solution of the “equation”

\[
du = h_r \, dA_r = \sum_{r=0}^{d_1} h_r \, dA_r
\]
with $u(0) \in W^1_p$. Assume that $Lu \in L^p([0, d'T], W^1_p)$. Then for any $\alpha \in \mathcal{N}$

$$\mathcal{R}(B_\alpha h) = c_{\alpha 0} \mathcal{R} h + \delta \mathcal{Q}_{\alpha 0} h,$$

$$\mathcal{Q}_\alpha u = \mathcal{R}(c_{\alpha 0} Lu - c_{\alpha r} h_r) + \delta \mathcal{Q}_{\alpha 0} Lu - \delta \mathcal{Q}_{\alpha r} h_r + B_\alpha u.$$

The proof of this lemma is an obvious modification of that of Lemma 4.2.

Next, let us use the notation

$$w_\beta = \bar{L}_\beta w + \bar{f}_\beta.$$

Since $w \in C_w([0, d'T], W^1_p)$, the functions $w_\beta$ are well defined for $2|\beta| \leq l$. Under the same condition the coefficients of $\bar{L}_\beta$ and $\bar{f}_\beta$ have the first derivative in time and these derivatives are under control. Furthermore, $dw_\beta = h_r dA_r$, where, as long as $2|\beta| + 3 \leq l$, the functions

$$h_0 = (\bar{L}\bar{L}_\beta - \bar{L}_{\beta 0}) w + \bar{L}\bar{f}_\beta - \bar{f}_{\beta 0},$$

$$h_r = \bar{L}(\bar{L}_r w + \bar{f}_r), \quad r = 1, ..., d_1,$$

are bounded $W^1_p$-valued functions on $[0, d'T]$.

Then in the same way as Lemma 4.3, Proposition 4.4 and Lemma 4.5 are obtained by the aid of Lemma 4.2, using Lemma 5.4 we get their counterparts, formulated as follows.

**Lemma 5.5.** Let $\alpha, \beta \in \mathcal{N}$. If $2|\beta| + 3 \leq l$ then

$$\mathcal{Q}_\alpha w_\beta = c_{\alpha r} \mathcal{R} w_{\beta r} + \delta \mathcal{Q}_{\alpha r} w_{\beta r} + B_\alpha w_\beta.$$

**Proposition 5.6.** Let $\kappa \geq 0$ be an integer and $l \geq 2\kappa + 3$. Then

$$w = v + \sum_{i=1}^{\kappa} \delta_i \sum_{|\alpha| = i} B_\alpha w_\alpha + \sum_{i=1}^{\kappa} \delta_i \sum_{|\alpha| = i+1} c_\alpha \mathcal{R} w_\alpha + \delta^{\kappa+1} \sum_{|\alpha| = \kappa+1} \mathcal{Q}_\alpha w_\alpha.$$

(5.10)

**Lemma 5.7.** If $\kappa \geq 0$ is an integer and $\alpha, \beta \in \mathcal{N}$ and $2(|\beta| + \kappa + 1) + 1 \leq l$, then

$$\mathcal{R}(B_\alpha w_\beta) = \sum_{i=0}^{\kappa} \delta^i \sum_{|\gamma| = i} c_{\alpha \gamma} \mathcal{R} w_{\beta \gamma}$$

$$+ \sum_{i=1}^{\kappa} \delta^i \sum_{|\gamma| = i-1} B_{\alpha \gamma} w_{\beta \gamma} + \delta^{\kappa+1} \sum_{|\gamma| = \kappa} \mathcal{Q}_{\alpha \gamma} w_{\beta \gamma},$$

where for any multi-numbers $\mu, \nu$

$$\sum_{|\gamma| = 0} c_{\nu \gamma} \mathcal{R} w_{\mu \gamma} := c_{\nu} \mathcal{R} w_{\mu}, \quad \sum_{|\gamma| = 0} B_{\nu \gamma} w_{\mu \gamma} := B_{\nu} w_{\mu},$$

$$\sum_{|\gamma| = 0} \mathcal{Q}_{\nu \gamma} w_{\mu \gamma} := \mathcal{Q}_{\nu} w_{\mu}.$$
In order to iterate equation (5.10) we introduce the following class of indices. We say that
\[
\beta = \gamma^\nu := \gamma_1^{\nu_1} \gamma_2^{\nu_2} ... \gamma_j^{\nu_j}
\] (5.11)
is a graded multi-number of length \(|\beta| := j + \nu_1 + \nu_2 + ... + \nu_j\), if \(\gamma_i \in \{1, 2, ..., d_i\}\), \(\nu_i \geq 0\) is any integer for \(i = 1, 2, ..., j\), where \(j \geq 1\) is any integer. If \(\nu_i = 0\) for some \(i\), then we also write \(\gamma_i\) in place of \(\gamma_i^0\) in (5.11). Let \(K\) denote the set of all graded multi-numbers. For each \(\beta = \gamma^\nu = \gamma_1^{\nu_1} \gamma_2^{\nu_2} ... \gamma_j^{\nu_j} \in K\) of length \(|\beta| \leq 1 + l/2\) we introduce the following operators and functions:
\[
L_\beta = L_{\gamma^\nu} := (-1)^{|\beta|-1}L_{\gamma_1^{\nu_1}} L_{\gamma_2^{\nu_2}} ... L_{\gamma_j^{\nu_j}},
\]
(5.12)
where \(f_\alpha^{(s)} := D^n f_r\), and \(L_\beta^{(s)}\) denotes the operator which we obtain from \(L_\beta\) by applying the derivation \(D^n\) to each of its coefficients. By definition \(f_r^{(0)} = f_r\) and \(L_\beta^{(0)} = L_\beta\). It is easy to see that for \(\beta \in N\), when \(\beta = \beta^0 \in K\), definitions (5.12) are consistent with (5.9).

**Lemma 5.8.** The following statements hold.
(i) Let \(\beta, \gamma \in K\) be such that \(|\beta| + |\gamma| \leq 1 + l/2\). Then
\[
L_\beta L_\gamma = -L_{\beta \gamma}, \quad L_\beta f_\gamma = -f_{\beta \gamma}.
\]
(ii) Let \(\alpha \in N\) be such that \(\rho := |\alpha| \leq 1 + l/2\). Then there exist constants \(c(\gamma) = c(\alpha, \gamma) \in \{0, \pm 1, \pm 2, ..\}\) defined for all \(\gamma \in K\) with \(|\gamma| = \rho\), such that
\[
L_\alpha = \sum_{\gamma \in K, |\gamma| = \rho} c(\gamma) L_\gamma, \quad f_\alpha = \sum_{\gamma \in K, |\gamma| = \rho} c(\gamma) f_\gamma.
\] (5.13)

**Proof.** Part (i) follows immediately from the definition (5.12) of \(L_\beta, f_\beta\). Part (ii) clearly holds for \(\alpha = 0\) and \(\alpha = r \in \{1, ..., d_1\}\). Assume that equations (5.13) hold for some \(\alpha \in N, |\alpha| < 1 + l/2\). Then
\[
L_{\alpha r} = -L_\alpha L_r = -\sum_{\beta \in K, |\beta| = |\alpha|} c(\beta) L_\beta L_r = \sum_{\beta \in K, |\beta| = |\alpha|} c(\beta) L_{\beta r},
\]
\[
f_{\alpha r} = -L_\alpha f_r = -\sum_{\beta \in K, |\beta| = |\alpha|} c(\beta) L_\beta f_r = \sum_{\beta \in K, |\beta| = |\alpha|} c(\beta) f_{\beta r}
\]
for \(r \in \{1, 2, ..., d_1\}\), and
\[
L_{\alpha 0} = LL_\alpha - \dot{L}_\alpha = \sum_{r=1}^{d_1} \sum_{\beta \in K, |\beta| = \rho} c(\beta) L_r L_\beta - \sum_{\gamma \in N, |\gamma| = \rho} c(\gamma^\nu) L_{\gamma^\nu},
\]
\[
= \sum_{r=1}^{d_1} \sum_{\beta \in M, |\beta| = \rho} c(\beta)L_r f_{\beta} - \sum_{\gamma^\prime \in M, |\gamma^\prime| = \rho} c(\gamma^\prime)\dot{f}_{\gamma^\prime}.
\]

Hence by using assertion (i) and noticing that
\[
\dot{L}_{\gamma^\prime} = \sum_{|\mu| = 1} L_{\gamma^\prime + \mu}, \quad \dot{f}_{\gamma^\prime} = \sum_{|\mu| = 1} f_{\gamma^\prime + \mu},
\]
we get equations (5.13) for \(\alpha r\) with \(r = 0, 1, ..., d_1\). Thus the induction on the length of \(\alpha\) completes the proof. \(\square\)

For \(\beta \in K\) we write \(\bar{L}_{\beta}\) and \(\bar{f}_{\beta}\), when the time change \(A_0(t)\) is done in the coefficients of \(L_{\beta}\) and in \(f_{\beta}\). We set \(w_{\gamma} := \bar{L}_{\beta} w + \bar{f}_{\beta}\) for \(\gamma \in K\), \(|\gamma| \leq 1 + l/2\). Notice that Lemma 5.8 has an obvious translation in terms of these functions. Namely, by Lemma 5.8 (ii) for every \(\alpha \in N\) such that \(\rho := |\alpha| \leq 1 + l/2\) there exist constants \(c(\alpha, \gamma) \in \{0, \pm 1, \pm 2, ..\}\) defined for all \(\gamma \in K\) with \(|\gamma| = \rho\), such that
\[
= \sum_{\gamma \in K, |\gamma| = \rho} c(\gamma)w_{\gamma}.
\]

For every integer \(i \geq 1\) we introduce finite sequences \(\sigma := (\beta_1, \beta_2, ..., \beta_i)\) of graded multi-numbers \(\beta_i \in K\), and we set \(|\sigma| := |\beta_1| + |\beta_2| + ... + |\beta_i|\). We also introduce the empty sequence \(e\) of length \(|e| := 0\). The set of all these sequences is denoted by \(I\). For \(\sigma = (\beta_1, \beta_2, ..., \beta_i)\) with \(|\sigma| \leq 1 + l/2\) we define
\[
S_{\sigma} := RL_{\beta_1} RL_{\beta_2} \cdot ... \cdot RL_{\beta_i}, \quad \bar{S}_{\sigma} := \bar{R}L_{\beta_1} \bar{R}L_{\beta_2} \cdot ... \cdot \bar{R}L_{\beta_i},
\]
and for \(|\sigma| = 0\) we set
\[
S_{e} := R, \quad \bar{S}_{e} := \bar{R}.
\]

Notice that for any \(g \in L_p([0, T], W^2_p|\sigma|)\)
\[
\bar{S}_{\sigma} \bar{g}(t, \cdot) = (S_{\sigma} g)(A_0(t), \cdot), \quad (5.14)
\]
where \(\bar{g}(t, \cdot) := g(A_0(t), \cdot)\). This follows from (5.8) by induction on \(|\sigma|\).

In order to formulate the counterparts of Lemma 4.7 and Proposition 4.8 we use the following sets
\[
A(i) = \{ (\sigma, \beta) : \sigma \in I, \beta \in K, |\sigma| + |\beta| \leq i \},
\]
\[
B(i, j) = \{ (\alpha, \beta) : \alpha \in N, \beta \in K, |\alpha| = i, |\beta| \leq j \},
\]
\[
B^*(i, j) = \bigcup_{i_1=1}^{i} B(i_1, j).
\]
Remember that if $g\nu$ is a collection of functions indexed by a parameter $\nu$ taking values in a set $A$, then $\sum^*_\nu \in A g\nu$ means any linear combination of $g\nu$ with coefficient independent of the argument of $g\nu$ and of $\delta$.

**Lemma 5.9.** Let $\sigma \in \mathcal{I}$, $\kappa, \mu \geq 0$ be integers, and $\alpha \in \mathcal{N}, \beta \in \mathcal{K}$. Assume that

$$2(|\sigma| + |\beta| + \kappa) + \mu + 2 \leq l.$$ 

Then

$$S_\sigma(B_\alpha w_\beta) = \sum^\kappa_{i=0} \delta^i A(|\sigma| + |\beta| + i) S_{\sigma, i} w_{\beta, i}$$

$$+ \sum^\kappa_{i=1} \delta^i A(|\sigma| + |\beta| + i - 1) B_{\alpha, i} w_{\beta, i} + O(\delta^{k+1}).$$

**Proof.** We can derive this lemma from Lemma 5.7 in the same way as Lemma 4.7 is proved. We need only use the sets $\mathcal{K}$ and $\mathcal{I}$ in place of $\mathcal{M}$ and $\mathcal{J}$, and the operators $\bar{R}, \bar{L}_\nu, \bar{S}_\sigma$ for $\nu \in \mathcal{K}, \sigma \in \mathcal{I}$, in place of $\mathcal{R}, L_\nu S_\sigma$, for $\nu \in \mathcal{M}, \sigma \in \mathcal{J}$, respectively. \hfill \Box

**Proposition 5.10.** Let $k, m \geq 0$ be integers, and

$$4 + m + 4k \leq l.$$ 

Then for any $j = 0, 1, ..., k$ we have

$$w = v + \sum^j_{i=1} \delta^i \sum^*_{A(2i)} \bar{S}_\sigma v_\beta + \sum^k_{i=j+1} \delta^i A(i+j+1) S_{\sigma, i} w_{\beta, i}$$

$$+ \sum^k_{i=1} \delta^i B^*(i, i+j) B_{\alpha, i} w_{\beta, i} + O(m(\delta^{k+1}),$$

where $v_\beta := \bar{L}_\beta v + \bar{f}_\beta$.

**Proof.** The proof of this proposition is a straightforward translation of the proof of the corresponding proposition, Proposition 4.8, in the time independent case. To make this translation we use the sets $\mathcal{K}$ and $\mathcal{I}$ in place of $\mathcal{M}$ and $\mathcal{J}$, and the operators $\mathcal{R}, \bar{L}_\nu, \bar{S}_\sigma$ for $\nu \in \mathcal{K}, \sigma \in \mathcal{I}$, in place of $\mathcal{R}, L_\nu S_\sigma$, for $\nu \in \mathcal{M}, \sigma \in \mathcal{J}$, respectively. \hfill \Box

Now we can finish the proof of Theorem 5.1 as follows. Taking $j = k$ in Proposition 5.10 we get

$$w = v + \sum^j_{i=1} \delta^i \sum^*_{A(2i)} c(\sigma, \beta) \bar{S}_\sigma v_\beta + \sum_{B^*(k, 2k)} c(\alpha, \beta, \delta) B_{\alpha, i} w_{\beta, i} + r\delta, \quad (5.15)$$
where \( c(\sigma, \beta), c(\alpha, \beta, \delta) \) are certain constants, and \( r_\delta \) is a function in \( C_w([0, T], W^m_p) \) for each \( \delta \), and

\[
\sup_{t \in [0, d'T]} \sup_{n, \delta = T/n} \delta^{-(k+1)}\|r_\delta(t, \cdot)\|_{m,p} \leq N.
\]

Observe that in contrast with (4.21) the functions \( v \) and \( \bar{S}_\sigma v_\beta \) in (5.15) may depend on \( \delta \). To proceed further, define \( R_n^{(k)}(t, x) := r_\delta(d't, x) \), and

\[
u^{(i)} := \sum_{A(2i)} c(\sigma, \beta) S_\sigma u_\beta, \quad i = 1, 2, \ldots, k,
\]

where \( u_\beta := L_\beta u + f_\beta \). Then by virtue of equality (5.14) and the fact that \( v(t) = u(A_0(t)) \) from equation (5.15) we get

\[
w(t, \cdot) = u(A_0(t), \cdot) + \sum_{i=1}^j \delta^i u^{(i)}(A_0(t), \cdot)
\]

\[
+ \sum_{B^{(k,2k)}} c(\alpha, \beta, \delta) B_{\alpha_1}(t) w_{\beta_1}(t, \cdot) + R_n^{(k)}(t/d', \cdot).
\]

Substituting here \( d't \) in place of \( t \) we get the required representation (5.4) by taking into account that \( w(d't) = u_n(t), \quad A_0(d't) = t, \quad B_{\alpha_1}(d't) = 0 \quad \forall t \in T_n \).

**Remark 5.1.** Let \( 1 \leq j \leq d_1 \). Then Theorems 5.1, 5.2, and 5.3 hold also when the operator \( S^{(r)}_\delta(t_{i+1}) \) is replaced with \( S^{(r)}_\delta(t_i) \) for every \( r = 1, 2, \ldots, j \) in the definition (5.2) of the splitting-up approximation \( u_n \). To see this we need only repeat the proof of the previous theorem with \( A_j \) in place of \( A_0 \) in equation (5.6) and with \( A_0 \) and \( A_j \) interchanged in (5.5).

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