BLOW-UP RESULTS FOR AN INHOMOGENEOUS PSEUDO-PARABOLIC EQUATION

MEIIRKHA N B. BORIKHANOV AND BERIKBOL T. TOREBEK

Abstract. In the present paper, we study an inhomogeneous pseudo-parabolic equation with nonlocal nonlinearity

\[ u_t - k \Delta u_t - \Delta u = I^\gamma_0(|u|^p) + \omega(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \]

where \( p > 1, k \geq 0, \omega(x) \neq 0 \) and \( I^\gamma_0 \) is the left Riemann-Liouville fractional integral of order \( \gamma \in (0, 1) \). Based on the test function method, we have proved the blow-up result for the critical case \( \gamma = 0, p = p_c \) for \( N \geq 3 \), which answers an open question posed in [13], and in particular when \( k = 0 \) it improves the result obtained in [2]. An interesting fact is that in the case \( \gamma > 0 \), the solution of the problem blows up for any \( p > 1 \).

1. Introduction

Recently, Zhou in [13] has investigated the inhomogeneous pseudo-parabolic equation

\[
\left\{
\begin{array}{ll}
u_t - k \Delta u_t - \Delta u = |u|^p + \omega(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{array}
\right.
\]

(1.1)

where \( p > 0, k > 0 \) and \( u_0, \omega \in C_0(\mathbb{R}^N) \).

There was studied the effect of the inhomogeneous term \( \omega(x) \) on the critical exponent \( p_c \) of the problem (1.1), and it was proven that for

\[
p_c = \begin{cases} 
\infty & \text{if } N = 1, 2, \\
\frac{N}{N-2} & \text{if } N \geq 3,
\end{cases}
\]

(a) if \( 1 < p < p_c, u_0 \geq 0 \) and \( \int_{\mathbb{R}^N} \omega(x)dx > 0 \), then the solution of (1.1) blows up in finite time.

(b) if \( p > p_c \), then there exist \( u_0 \geq 0 \) and \( \omega \geq 0 \) such that the problem (1.1) admits global solutions.

Note that the critical case \( p = p_c \) was left open (see [13, Remark 4(b)]).

At first, the problem (1.1) for \( \omega(x) \equiv 0 \) has studied in [3, 10]. It is shown that there exists the critical exponent \( p_F = 1 + \frac{2}{N} \) for the pseudo-parabolic equation. This exponent coincides with the Fujita critical exponent of the semilinear heat equations, which was first introduced by Fujita in [6].

The problem (1.1) with \( k = 0 \) is considered by Bandle et. al. [2]. Namely, it was studied the cases (a), (b) and

(c) if \( N \geq 3, p = p_c, \int_{\mathbb{R}^N} \omega(x)dx > 0, \omega(x) = O(|x|^{-\varepsilon-N}) \) as \( |x| \to \infty \) for some \( \varepsilon > 0 \), and either \( u \geq 0 \) or

\[
\int_{|x|>R} \frac{\omega^-(y)}{|x-y|^{N-2}}dy = \frac{o(1)}{|x|^{N-2}}, \omega^- = \max\{-\omega, 0\}
\]

when \( R \) is enough large, then (1.1) has no global solutions.
Later on, Jleli et al. [8] generalized these results with the forcing term $t^\sigma \omega(x)$, $\sigma > -1$, and showed the effects of forcing term on the critical exponents.

In this paper, we study the semilinear pseudo-parabolic equation with a forcing term depending on the space

$$
\begin{cases}
  u_t - k\Delta u_t - \Delta u = I_{0+}^\gamma(|u|^p) + \omega(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
$$

(1.2)

where $p > 1$, $k \geq 0$, $\omega(x) \neq 0$ and $I_{0+}^\gamma$ is the left Riemann-Liouville fractional integral of order $\gamma \in [0, 1)$.

We note that the problem (1.2) for $k = 0$ and $\omega(x) \equiv 0$, was considered in [4, 5, 12].

The main purpose of this paper is to prove a blow-up result for the critical case $p = p_c$ for $N \geq 3$, thereby answering the open question proposed in [13]. In addition, to study the effect of nonlocal nonlinearity in time on the critical exponent.

1.1. Preliminaries.

**Definition 1.1** ([7], p. 69). The left and right Riemann-Liouville fractional integrals of order $\gamma \in (0, 1)$ for an integrable function $u(t)$, $t \in (0, T)$ are given by

$$I_{0+}^\gamma u(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} u(s) \, ds \quad \text{and} \quad I_{T-}^\gamma u(t) = \int_t^T \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} u(s) \, ds.$$

**Definition 1.2** (Weak solution). Let $u_0, w \in L^1(\mathbb{R}^N)$. A function $u \in L^p((0, T); L^\infty_{loc}(\mathbb{R}^N))$ is called a local weak solution of (1.2), if it holds

$$\int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} \omega \varphi \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} u_0(\varphi(x, 0) - k\Delta \varphi(x, 0)) \, dx \, dt$$

$$= - \int_0^T \int_{\mathbb{R}^N} u_0 \varphi_t \, dx \, dt + k \int_0^T \int_{\mathbb{R}^N} u \Delta \varphi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u \Delta \varphi \, dx \, dt,$$

(1.3)

for any test function $\varphi = \varphi(t, x) \in C^1_{loc}([0, T], \mathbb{R}^N)$, $\varphi \geq 0$ and $\varphi = 0$, $t \geq T$.

If $T = +\infty$, then $u$ is a global in time weak solution of (1.2).

**Lemma 1.3.** [11, Lemma 3.1] Let $\omega \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \omega(x) \, dx > 0$. Then there exists a test function $0 \leq \phi \leq 1$ such that $\int_{\mathbb{R}^N} \omega \phi \, dx > 0$.

2. Main results

In this section, we will show the blow-up of the solution to (1.2) using the test function method.

**Theorem 2.1.** Let $u_0, \omega \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \omega(x) \, dx > 0$. Then

(i) if $\gamma > 0$, then for any $p > 1$ the problem (1.2) admits no global weak solution.

(ii) if $\gamma = 0$ and $p = p_c = \frac{N}{N-2}$, $N \geq 3$, then the problem (1.2) admits no global weak solution.

**Remark 2.2.** Note that the part (ii) of Theorem 2.1 answers to the open question posed by Zhou in [13].

**Remark 2.3.** When $k = 0$ the equation (1.2) coincides with the heat equation considered in [2], then our results remain true for the heat equation. Note that the part (i) of Theorem 2.1 in the case $k = 0$ improves the result in [1], since we do not assume that $u_0$ is positive. The part (ii) of Theorem 2.1, in case $k = 0$ improves the result in [2]. Since we do not
assume some asymptotic properties of the function \( \omega(x) \) as in [2], our result improves part (b) of Theorem 2.1 from [2].

**Proof of Theorem 2.1.** We present the proofs of the cases (i) and (ii) separately.

**Case** **(i) The case** \( \gamma > 0 \) **and** \( p > 1 \). The proof is done by contradiction.

Assume that \( u \) is a global weak solution to problem (1.2). We choose the test function in the following form
\[
\varphi(t,x) = \psi(t) \xi(x),
\]
with
\[
\psi(t) = \left(1 - \frac{t}{T}\right)^m, \quad m = \left\lfloor \frac{1}{p-1} \right\rfloor + 1, \quad t \in [0,T], \quad T \in (0, \infty),
\]
and
\[
\xi(x) = \Phi \left( \frac{|x|^2}{R^2} \right), \quad R \gg 1, \quad x \in \mathbb{R}^N,
\]
where \( \left\lfloor \frac{1}{p-1} \right\rfloor \) means the largest integer not greater than \( \frac{1}{p-1} \).

Let \( \Phi(z) \in C_0^\infty(\mathbb{R}_+) \) be a nonincreasing function
\[
\Phi(z) = \begin{cases} 
1 & \text{if } 0 \leq z \leq 1, \\
\gamma & \text{if } 1 < z < 2, \\
0 & \text{if } z \geq 2.
\end{cases}
\]

Then, from (1.3) it follows that
\[
\int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) dx dt + \int_0^T \int_{\mathbb{R}^N} \omega \varphi dx dt + \int_{\mathbb{R}^N} u_0(\varphi(0,x) - k \Delta \varphi(0,x)) dx \\
\leq \int_0^T \int_{\mathbb{R}^N} |u| |\varphi_t| dx dt + k \int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi_t| dx dt + \int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi| dx dt.
\]

Using the \( \varepsilon \)-Young inequality in the right-side of (2.1) with \( \varepsilon = \frac{p}{3} \), we obtain
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\varphi_t| dx dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) dx dt \\
+ \frac{p-1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} dx dt.
\]

Similarly, one obtains
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi_t| dx dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) dx dt \\
+ \frac{p-1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} dx dt
\]
and
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi| dx dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) dx dt \\
+ \frac{p-1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} dx dt.
\]
Therefore, we can rewrite the inequality (2.1) in the following form
\[
\int_0^T \int_{\mathbb{R}^N} \omega \varphi dxdt + \int_{\mathbb{R}^N} u_0(\varphi(0,x) - k\Delta \varphi(0,x))dx \leq C(p)\left(I_1 + kI_2 + I_3\right),
\]
(2.2)
where \(C(p) = \frac{p - 1}{p}\left(\frac{p}{3}\right)^{-\frac{1}{p-1}}\).

Next, we estimate the integrals \(I_1, I_2, I_3\). At this stage, inserting the equality
\[
(I_T^\gamma - \psi)(t) = \frac{\Gamma(m + 1)}{\Gamma(\gamma + m + 1)} T^\gamma \left(1 - \frac{t}{T}\right)^{m+\gamma}, \quad t \in [0,T),
\]
to the term of the above integrals and changing the variable \(y = xR\), we obtain
\[
I_1 \leq CT^\frac{\gamma-1}{p-1} R^N,
\]
\[
I_2 \leq CT^\frac{\gamma-1}{p-1} R^{N - \frac{2p}{p+1}},
\]
\[
I_3 \leq CT^1 T^\frac{\gamma-1}{p-1} R^{N - \frac{2p}{p+1}}.
\]
(2.4)
On the other hand, it follows from a simple calculation that
\[
\int_0^T \psi(t)dt = \int_0^T \left(1 - \frac{t}{T}\right)^m dt = C(m)T.
\]
(2.5)
Combining (2.2)-(2.5) we arrive at
\[
\int_{\mathbb{R}^N} \omega \xi dx + C(m)T^{-1} \int_{\mathbb{R}^N} u_0(\xi - k\Delta \xi)dx
\leq C(p,m)\left(CT^\frac{\gamma-1}{p-1} R^N + kCT^\frac{\gamma-1}{p-1} R^{N - \frac{2p}{p+1}} + CT^{-\frac{\gamma}{p-1}} R^{N - \frac{2p}{p+1}}\right).
\]
Finally, fixing \(R\) and passing \(T \to +\infty\) in the last inequality and using Lemma 1.3, we deduce that \(\int_{\mathbb{R}^N} \omega \xi dx \leq 0\), which is a contradiction.

(ii) The critical case \(\gamma = 0\) and \(p = p_c = \frac{N}{N-2}, N \geq 3\). The proof also will be done by contradiction. Suppose that \(u\) is a global weak solution to (1.2).

Now, following the idea of [9], we set the test function as
\[
\varphi(t,x) = \eta(t) \phi(x),
\]
for large enough \(R,T\)
\[
\eta(t) = \nu \left(\frac{t}{T}\right), \quad t > 0,
\]
(2.6)
and
\[
\phi(x) = \mathcal{F} \left(\frac{\ln \left(\frac{|x|}{\sqrt{R}}\right)}{\ln \left(\sqrt{R}\right)}\right), \quad x \in \mathbb{R}^N.
\]
(2.7)
Let \(\nu \in C^\infty(\mathbb{R})\) be such that \(\nu \geq 0; \nu \not\equiv 0; \text{ supp}(\nu) \subset (0,1)\), and \(\mathcal{F} : \mathbb{R} \to [0,1]\) be a smooth function satisfying
\[
\mathcal{F}(s) = \begin{cases} 
1, & \text{if } -\infty < s \leq 0, \\
0, & \text{if } s \geq 1.
\end{cases}
\]
(2.8)
and there exist positive constants \(\theta_1, \theta_2\) such that
\[
|\phi''(x)| \leq \theta_1 |\phi(x)|, \quad |\phi'(x)| \leq \theta_2 |\phi(x)|.
\]
(2.9)
Using the fact that \( \text{supp}(\nu) \subset (0, 1) \), we can easily get
\[
\int_{\mathbb{R}^N} u_0(\varphi(0, x) - k\Delta \varphi(0, x))dx = \nu(0) \int_{\mathbb{R}^N} u_0(\phi(x) - k\Delta \phi(x))dx = 0. \tag{2.10}
\]

Then, acting in the same way as in the above case, we get the following estimate
\[
\int_0^T \int_{\mathbb{R}^N} \omega \varphi dx dt \leq C(p) \left( J_1 + kJ_2 + J_3 \right), \tag{2.11}
\]
with
\[
J_1 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\varphi|^{\frac{p}{p-1}} dx dt,
\]
\[
J_2 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} dx dt,
\]
\[
J_3 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} dx dt.
\]

In view of (2.6) and (2.7), let us calculate the next integral
\[
J_2 = \left( \int_0^T \eta^{-\frac{1}{p-1}} |\eta|^{\frac{p}{p-1}} dt \right) \left( \int_{\mathbb{R}^N} \phi^{-\frac{1}{p-1}} |\Delta \phi|^{\frac{p}{p-1}} dx \right). \tag{2.12}
\]

Indeed, the function \( \phi \) is a radial, and remaining (2.9) we arrive at
\[
|\Delta \phi| = \frac{d^2 \phi}{dr^2} + \frac{N - 1}{r} \frac{d \phi}{dr} = \phi'' \frac{1}{r^2 \ln^2 \sqrt{R}} + \phi' \frac{N - 2}{r^2 \ln \sqrt{R}}.
\]

By combining (2.11)-(2.15), we can rewrite (2.12) as
\[
\int_{\mathbb{R}^N} \phi^{-\frac{1}{p-1}} |\Delta \phi|^{\frac{p}{p-1}} dx \leq C (\ln R)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |\phi|^{\frac{N}{N-2}} dx.
\]

Using (2.7) and (2.8), we get
\[
\int_{\mathbb{R}^N} \phi^{-\frac{1}{p-1}} |\Delta \phi|^{\frac{p}{p-1}} dx \leq C (\ln R)\frac{2N}{2-N}. \tag{2.13}
\]

Similarly, from (2.6) one obtains
\[
\int_0^T \eta^{-\frac{1}{p-1}} |\eta|^{\frac{p}{p-1}} dt = C T^{-\frac{N}{2}}. \tag{2.14}
\]

By combining (2.13)-(2.14), we can rewrite (2.12) as
\[
J_2 \leq kCT^{-\frac{p}{p-1}} (\ln R)^{\frac{2-N}{2}}. \tag{2.15}
\]

Consequently, we will estimate the integrals \( J_1 \) and \( J_3 \), respectively, in the following form
\[
J_1 \leq CT^{-\frac{p}{p-1}} R^N \quad \text{and} \quad J_3 \leq CT^1 (\ln R)^{\frac{2-N}{2}}.
\]

Finally, we deduce that
\[
\int_{\mathbb{R}^N} \omega \phi dx \leq C(p) \left( CT^{-\frac{p}{p-1}} R^N + kCT^{-\frac{p}{p-1}} (\ln R)^{\frac{2-N}{2}} + C (\ln R)^{\frac{2-N}{2}} \right). \tag{2.15}
\]
Now for $T = R^j, j > 0$, we get
\[
\int_{\mathbb{R}^N} \omega \phi dx \leq C \left( C R^{-\frac{N(j-2)}{2}} + k C R^{-\frac{N(j-2)}{2}} (\ln R)^{\frac{2-N}{2}} + C (\ln R)^{\frac{2-N}{2}} \right).
\]
Taking $j > 2$ and passing to the limit as $R \to \infty$ in the above inequality and in view of Lemma 1.3, we deduce that $\int_{\mathbb{R}^N} \omega \phi dx \leq 0$, which is a contradiction. □

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**Meirikhan B. Borikhanov**

Khoja Akhmet Yassawi International Kazakh–Turkish University
Sattarkhanov ave., 29, 161200 Turkistan, Kazakhstan

Department of Mathematics: Analysis, Logic and Discrete Mathematics

Ghent University, Belgium

*Email address:* meirikhan.borikhanov@ayu.edu.kz, meirikhan.borikhanov@ugent.be

**Berikbol T. Torebek**

Institute of Mathematics and Mathematical Modeling
125 Pushkin str., 050010 Almaty, Kazakhstan

Department of Mathematics: Analysis, Logic and Discrete Mathematics

Ghent University, Belgium

*Email address:* torebek@math.kz, berikbol.torebek@ugent.be