Electrodynamic theory for the operation principle of a superconducting kinetic inductance stripline detector

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Abstract. Generation and transmission of voltage signals in a kinetic inductance detector made of a superconducting nanowire stripline is investigated on the basis of superconducting electromagnetism. We show that electric signals traveling along the stripline in this detector can be regarded as a Swihart mode. An equation which can describe the generation and transmission of voltage pulses in this detector is derived. A pair of voltage pulses with opposite polarities are created when a spatiotemporal variation in the density of the superconducting condensate occurs in a small region of this superconducting nanowire. The voltage pulses propagate along the stripline in the directions opposite to each other with the Swihart velocity. The characteristic of the radiation detector named current-biased kinetic inductance detector (CB-KID) is well described by our model.

1. Introduction

Superconductors are widely utilized in various high-performance devices with low energy consumption. It is also known that radiation detectors using superconducting circuits exhibit high sensitivity and resolution. Up to now several types of superconducting radiation detectors such as transition edge sensor (TES)$^{[1, 2]}$, superconducting nanowire single-photon detector (SNSPD)$^{[3, 4]}$, microwave kinetic inductance detector (MKID)$^{[5]}$ and also the one using S-I-S tunnel junctions$^{[6, 7, 8]}$ have been developed as the high-performance radiation detectors. Recently, a new type of detector using a superconducting stripline, which is named current-biased kinetic inductance detector (CB-KID), has been fabricated for application to a radiation detector that makes it possible to image the intensity distribution of incident radiations$^{[9, 10, 11, 12, 13, 14]}$. The stripline detector is formed by a superconducting meander deposited on a substrate composed of a stack of insulating and superconducting layers, that is, the stripline has a S-I-S waveguide structure. Voltage pulses traveling on the current-biased stripline in CB-KID can be generated by a variation of the kinetic inductance of the stripline when the nanowire is locally heated. It has also been observed that a voltage pulse generated at a small hot spot on the current-biased stripline is split into two voltage pulses with opposite polarities and propagate toward the opposite directions with a constant velocity and reach the two ends of the stripline at different time depending on the place where the voltage pulse was created. Hence, by measuring the difference in the arrival times of the two voltage pulses at both
ends of the stripline one can image the hot spot at which a variation of the superconducting current occurred. Since such a pair of voltage pulses can be generated by radiation incident on the detector as shown by Ishida et al.[9], CB-KID can be utilized as a radiation detector that can image incident radiation with high resolution. In fact CB-KID with an area of 1 cm$^2$ available now attains a resolution of $10^6$ pixels[11].

In this paper the workings of CB-KID is theoretically investigated on the basis of superconducting electromagnetism. We show that the kinetic inductance variation caused by a transient hot spot on the current-biased superconducting nanowire can be described by an equation for the superconducting phase difference between the superconducting nanowire and the superconducting ground plane. The equation has a solution that two voltage pulses with opposite polarities excited by a local temporal variation of the density of the superconducting condensate propagate with the Swihart velocity[15] in the directions toward opposite electrodes.

2. Current-biased kinetic inductance detector

The radiation detector CB-KID, which is composed of a superconducting stripline, probes the time derivative of the kinetic inductance on the occasion that the superconducting nanowire in the stripline is locally heated by radiation. Since the kinetic inductance of a superconducting wire, $L_k$, is inversely proportional to the density of superconducting condensate $n_s$ and the thickness $s$ as $L_k \propto 1/n_s s$, a decrease in $n_s$ by heating causes a large variation of the kinetic inductance in a thin superconducting wire. The workings of CB-KID have been analyzed by Shishido et al[11] on the basis of a superconducting circuit theory permitting the time variation of the kinetic inductance. In this circuit theory the voltage appearing across a hot spot in the current biased stripline is given as $V \simeq (dL_k/dt) I$, $I$ being the bias current. In this paper we clarify the electrodynamic properties of a superconducting stripline on the basis of superconducting electromagnetism and provide a theoretical basis for the variation of the kinetic inductance by heating in CB-KID.

![Figure 1. Schematical view of the stripline in CB-KID. The bias current $I$ flows in the superconducting nanowire. The origin $z = 0$ is taken at the top surface of the superconducting ground layer.](image)

2.1. S-I-S waveguide model

Let us construct a theory for the electrodynamic response of the superconducting stripline in CB-KID. The schematic view of a part of the stripline in this detector is depicted in Fig.1. It is seen that the present stripline has a S-I-S waveguide structure. Then, we first investigate the propagating mode in a S-I-S waveguide system. Suppose that the thickness of the superconducting nanowire $s$ in this system is less than the London penetration depth $\lambda_L$, considering the fact that the thinner the thickness $s$ is the better to obtain a large variation in the kinetic inductance in this system, while the insulating and superconducting layers forming
the substrate are thick, for instance, \( d > \lambda_L \) and \( D \gg \lambda_L \), where \( d \) and \( D \) are, respectively, the thicknesses of the insulating and superconducting ground layers. Noticing this system has a S-I-S layer structure, we define the gauge-invariant phase difference \( \theta(x,t) \) between the two superconductors as follows,

\[
\theta(x,t) = \varphi(x, d, t) - \varphi(x, z_s, t) - \frac{\pi}{\hbar c} \int_{z_s}^{d} dz A_z(x, z, t),
\]

(1)

where \( \varphi(x, z, t) \) is the superconducting phase at \((x, z, t)\), \( A_z(x, z, t) \) is the \( z \) component of the vector potential and \( e^* = 2e \) is the charge of a Cooper pair. The coordinate axes in the present system are taken as shown in Fig.1. For simplicity the spatial dependence along the \( y \)-direction is neglected in this paper. Note that \( \theta(x, t) \) is defined as the gauge-invariant phase difference between a point on the lower surface of the superconducting wire \((z = d)\) and a point deeply inside the superconducting substrate\((z = z_s < 0\) and \( \lambda_L < |z_s| < D \)). Following the procedure which is well-known in the theory of Josephson effect, one can derive the equation of motion for \( \theta(x, t) \) as

\[
\frac{\lambda_L + d}{d} \frac{\hbar c}{e^*} \partial_t^2 \theta(x, t) - \partial_x^2 \theta(x, t) = \left( \frac{e^*}{\hbar c} \right) \partial_x [A_x(x, d, t) - \frac{\hbar c}{e^*} \partial_x \varphi(x, d, t)].
\]

(2)

Here, the term on the right-hand-side of Eq.(2) arises from the screening current flowing on the lower surface of the superconducting nanowire. To calculate this term one can utilize the London equation for the superconducting current flowing inside the superconducting nanowire, i.e.,

\[
\mathbf{j}(x, z, t) = - \frac{c}{4\pi\lambda_L^2} (A(x, z, t) - \frac{\hbar c}{e^*} \nabla \varphi(x, z, t)).
\]

(3)

Substituting Eq.(3) into the Maxwell equations, one can derive the equation for the \( y \)-component of the magnetic field, \( B_y(x, z, t) \), as

\[
(\partial_x^2 + \partial_z^2) B_y(x, z, t) = \frac{1}{\lambda_L^2} B_y(x, z, t).
\]

(4)

Consider the case where the scale of the spatial variation along the \( x \)-direction is much larger than \( \lambda_L \). In this case Eq.(4) yields the solution in the presence of a bias current \( I \) as

\[
B_y(x, z, t) = h(z) \tilde{B}_y(x, d, t) + \frac{e^{(2d+s-2z)/\lambda_L} - e^{-(2d+s-2z)/\lambda_L}}{e^{s/\lambda_L} - e^{-s/\lambda_L}} \frac{2\pi I}{c},
\]

(5)

where

\[
h(z) = \frac{e^{(d+s-z)/\lambda_L} - e^{-(d+s-z)/\lambda_L}}{e^{s/\lambda_L} - e^{-s/\lambda_L}}.
\]

(6)

In Eq.(5) \( \tilde{B}_y(x, d, t) \) stands for the magnetic field component generated by the traveling signals at the boundary \( z = d \). Using the above solution, one can derive the relation,

\[
\partial_x (A_x(x, z, t) - \frac{\hbar c}{e^*} \partial_x \varphi(x, z, t)) = \lambda_L^2 h'(d) \partial_x B_y(x, d, t),
\]

(7)

with \( h'(d) = [\partial_z h(z)]|_{z=d} = -\lambda_L^{-1} \coth(s/\lambda_L) \). Furthermore, since \( B_y(x, z, t) \) is continuous at the interface \( z = d \), the right-hand-side of Eq.(7) can be rewritten as

\[
\lambda_L^2 h'(d) \partial_x B_y(x, d, t) = \lambda_L^2 h'(d) \frac{e}{c} \partial_t E_z(x, d, t) = -\left( \frac{\hbar c}{e^*} \right) \lambda_L \coth(s/\lambda_L) \frac{e}{d} \frac{c^2}{e^*} \partial_t^2 \theta(x, t),
\]

(8)
where $\varepsilon$ is the dielectric constant of the insulating layer. Finally, from Eqs.(2), (7) and (8) it follows the wave equation,

$$\frac{1}{v^2} \partial^2_x \theta(x,t) - \partial^2_t \theta(x,t) = 0, \quad (9)$$

where the velocity $v$, which corresponds to the Swihart velocity in a Josephson junction [15], is given as

$$v = \frac{c}{\sqrt{\varepsilon}} \sqrt{\frac{d}{d + \lambda_L(1 + \coth(s/\lambda_L))}}. \quad (10)$$

Note that $v$ is a function of the London penetration depth, which is dependent on temperature, that is, $v$ is expected to vary with temperature. In Fig.2 we plot the velocity $v(T)/v(0)$ as a function of temperature $T$ together with the experimental one, assuming $\lambda_L(T)$ obeys the temperature dependence in the two-fluid model, i.e., $\lambda_L(T) = \lambda_L(0)/\sqrt{1 - (T/T_c)^4}$. It is seen that the temperature variation of the velocity of the voltage signal observed in the experiment is in good agreement with that predicted by Eq.(10). Hence, one may conclude that the electromagnetic propagation modes in CB-KID can be identified with those in the nanoscale S-I-S waveguide.

![Figure 2. Temperature variation of the normalized velocity $v(t)/v(0)$. The circles indicate the experimental data obtained by Iizawa et al.[16] in the CB-KID using a Nb meander with a thickness $s = 40$nm which is deposited on an insulating layer with a thickness $d = 240$nm. The value $\lambda_L(0) = 150$nm is chosen for fitting the theoretical result (solid line) to the experimental one.]

2.2. Response to a local heating

Let us clarify the workings of CB-KID when a heat pulse is injected into this detector. Suppose that an incident heat pulse causes a local heating in CB-KID and then brings about a decrease in the density of superconducting condensate in a small heated region inside the current-biased superconducting nanowire. In the following the region in which the superconducting condensate is decreased is called a hot spot. Noting that $\lambda_L^{-2}$ is proportional to the density of the condensate $n_s$, one may incorporate phenomenologically the effect of a hot spot into a theory by the replacement, $n_s \rightarrow n_s(x,t)$, i.e.,

$$\frac{1}{\lambda_L^2} \cdot \frac{4\pi e^2}{mc^2} n_s \rightarrow \frac{4\pi e^2}{mc^2} n_s(x,t) = \frac{1}{\lambda_L^2} \cdot \frac{n_s(x,t)}{n_s} = \frac{\zeta(x,t)}{\lambda_L^2}, \quad (11)$$

where the function $\zeta(x,t) = n_s(x,t)/n_s$ is less than 1 only inside the hot spot. It is assumed that the hot spot is created at $t = 0$ and disappears after a short time interval $\tau$. In this paper we investigate an ideal case in which the hot spot has a symmetric shape with a length $x_h$ ($\gg \lambda_L$) along the $x$ direction and its center is located at $x = 0$, i.e., $\zeta(-x,t) = \zeta(x,t)$, and
thus $\zeta(x, t) < 1$ for $|x| < x_h/2$ and $0 < t < \tau$. On the basis of the above phenomenological consideration one can generalize the Maxwell-London equation as

$$
\partial_x B_y(x, z, t) = \frac{\zeta(x, t)}{\lambda_L} [A_x(x, z, t) - \frac{hc}{e^s} \partial_x \varphi(x, z, t)],
$$

(12)
in the presence of a hot spot. Here, the quasi-particle current is neglected, which is justified in the low temperature region where the density of thermally excited quasi-particles is negligibly small. From Eq.(12) one can easily obtain the relation,

$$
\partial_x[A_x(x, d, t) - \frac{hc}{e^s} \partial_x \varphi(x, d, t)] = -\frac{\lambda_L}{\zeta(x, t)} \coth\left(\frac{s}{\lambda_L}\right) \partial_x B_y(x, d, t)
$$

+ \frac{4\pi \lambda_L^2}{c} \partial_x \zeta(x, t) \left[ \frac{c}{4\pi \lambda_L} \coth\left(\frac{s}{\lambda_L}\right) \tilde{B}_y(x, d, t) + I \right].
$$

(13)

Here, use was made of the relation $B_y(x, z, t) = h(z) B_y(x, d, t)$ and Eq.(5). Since the spatial derivative, $\partial_x B_y(x, d, t)$, in Eq.(13) is expressed in terms of the phase difference as

$$
\partial_x B_y(x, d, t) = \frac{hc}{c^2 e^s} \partial^2 \theta(x, t),
$$

(14)

one can rewrite Eq.(2) as

$$
\left[ \frac{1}{v(x,t)^2} \partial^2_t - \partial^2_x \right] \theta(x, t) = F(x, t).
$$

(15)
in the presence of a hot spot, where $v(x, t)$ is the Swihart velocity,

$$
v(x, t) = \frac{c}{\sqrt{\pi}} \sqrt{\frac{d}{d + \lambda_L(1 + \coth(s/\lambda_L)/\zeta(x, t))}},
$$

(16)

which is dependent on $(x, t)$, and

$$
F(x, t) = \frac{4\pi e^s \lambda_L^2}{hc^2} \partial_x \zeta(x, t) \left[ \frac{c}{4\pi \lambda_L} \coth\left(\frac{s}{\lambda_L}\right) \tilde{B}_y(x, d, t) + I \right].
$$

(17)

Note that the function $F(x, t)$ has a finite value only inside the hot spot confined in a small spatiotemporal region, that is, $F(x, t)$ can be regarded as a source term which induces electromagnetic excitations propagating along the stripline, the Swihart mode. Hence, one may conclude that the emergence of a hot spot in the superconducting wire in CB-KID excites the Swihart mode inductively.

Let us now solve Eq.(15) in the presence of a hot spot, i.e., $F(x, t) \neq 0$ for $|x| < x_h/2$ and $0 < t < \tau$. Since we are interested in the asymptotic behavior of the solution for $|x| \to \infty$ and $t \to \infty$, we use the approximation, $v(x, t) \sim v$, Then, Eq.(15) is approximated as

$$
\left[ \frac{1}{v^2} \partial^2_t - \partial^2_x \right] \theta(x, t) = F(x, t).
$$

(18)

Suppose that a hot spot appears at $t = 0$. In this case the solution of Eq.(18) can be expressed as

$$
\theta(x, t) = \theta_0(x, t) + \int_{-\infty}^\infty dx' \int_0^\infty dt' G(x - x', t, t') F(x', t'),
$$

(19)
where $\theta_0(x,t)$ is a solution of the homogenous equation, i.e., $((1/v^2)\partial_t^2 - \partial_x^2)\theta_0(x,t) = 0$, and $G(x;t,t')$ is the Green function satisfying the equation,

$$\left[\frac{1}{v^2}\partial_t^2 - \partial_x^2\right]G(x;t,t') = \delta(x)\delta(t-t').$$

In the present case we assume that $\theta_0(x,t)$ is the phase difference originating from the magnetic field induced by a constant bias current, that is, $\theta_0(x,t) = \theta_0(x) \propto x$, and $G(x;0,t') = 0$ because a hot spot is supposed to appear at $t = 0$. It is easy to solve Eq.(20) by using the Fourier transformation for the spatial variable $x$, i.e., $G(x,t,t') = \sum_k G_k(t,t') \exp(ikx)$. Here, the Fourier component, $G_k(t,t')$, is obtained as

$$G_k(t,t') = -\frac{v^2}{\omega_k} \left\{ \Theta(t' - t) \sin \omega_k t \cos \omega_k t' + \Theta(t - t') \cos \omega_k t \sin \omega_k t' \right\},$$

where $\omega_k = v|k|$ and $\Theta(t)$ is the Heaviside step function. Furthermore, using the Josephson relation $V(x,t) = -(\hbar/e^*)\partial_t \theta(x,t)$ (see Appendix), one can also derive the voltage difference from Eq.(19) as

$$V(x,t) = \frac{\hbar}{e^*} \int_{-\infty}^{\infty} dx' \int_0^\infty dt' G(x - x', t, t') F(x', t').$$

The integral by $x'$ in Eq.(22) can be performed analytically and then Eq.(22) yields the expression,

$$V(x,t) = V(-)(x,t) + V(+)\left(x,t\right),$$

where

$$V(-)(x,t) = \frac{\hbar v^2}{4e^*} \left\{ \int_0^t dt' [F(x + vt + vt', t') - F(x + vt - vt', t')] + \int_t^\infty dt' [F(x + vt + vt', t') + F(x + vt - vt', t')] \right\}$$

and

$$V(+)(x,t) = -\frac{\hbar v^2}{4e^*} \left\{ \int_0^t dt' \left[ F(x - vt + vt', t') - F(x - vt - vt', t') \right] + \int_t^\infty dt' \left[ F(x - vt + vt', t') + F(x - vt - vt', t') \right] \right\}.$$ 

Note that in the asymptotic region $t \to \infty$ the solution (23) is reduced to

$$V(x,t) = Y(x + vt) - Y(x - vt),$$

where

$$Y(z) = \frac{\hbar v^2}{4e^*} \int_0^\infty dt' \left[ F(z + vt', t') - F(z - vt', t') \right].$$

From this result one may conclude that a pair of voltage pulses with opposite polarities are created at a hot spot and they propagate in the directions opposite to each other with the Swihart velocity $v$. To clarify the polarity of the voltage pulses consider a case where the hot spot has a symmetric shape, i.e., $\zeta(-x,t) = \zeta(x,t)$, and the bias current $I$ is large enough, i.e., $F(x,t) \simeq (4\pi c^*\lambda^2 I/\hbar c^2 s)[\partial_x \zeta(x,t)/\zeta(x,t)^2]$ (see Eq.(17)). In this case the spatial dependence of $F(x,t)$ satisfies the condition, $F(x,t) = -F(-x,t)$ and $F(x,t) \geq 0$ for $x > 0$. Thus, one finds

$$Y(0) = \frac{\hbar v^2}{4e^*} \int_0^\infty dt' \left[ F(vt', t') - F(-vt', t') \right] = \frac{\hbar v^2}{2e^*} \int_0^\infty dt' F(vt', t') > 0,$$
which indicates that a positive voltage (negative) pulse travels in the negative (positive) direction with a constant velocity when the bias current flows in the positive direction. This result agrees with the polarities observed in CB-KID by Miyajima et al. [12], that is, our S-I-S waveguide model can correctly describe the operation of CB-KID. It is noted that the phase velocity derived from Eq.(15) shows no dispersion outside a hot spot, that is, a voltage pulse traveling in the stripline keeps its shape in the region outside the hot spot. This remarkable feature holds for a voltage pulse whose width is much greater than the inverse pair-breaking frequency, i.e., $\frac{\hbar}{2\Delta}$ with $\Delta$ and $\hbar$ being, respectively, the superconducting energy gap and the Planck constant, as discussed in [19]. Finally, we also mention that the voltage given in Eq.(22) can be rewritten as

$$V \sim \frac{dL_k}{dt} \cdot I,$$

when the bias current is large. In Eq.(29) $L_k$ is understood to be the kinetic inductance of the stripline and its explicit expression is given as

$$L_k(x, t) = \frac{4\pi\lambda^2}{c^2} \int_{-\infty}^{\infty} dx' \int_{0}^{\infty} dt' G(x - x', t - t') \frac{\partial_x \zeta(x', t')}{\zeta(x', t')}.$$  

Thus, our theory also provides a microscopic basis of the circuit theory for CB-KID.

In conclusion we have constructed an electrodynamic theory for the generation and transmission of voltage pulses in the stripline in CB-KID, when the superconducting wire in the stripline is locally heated by irradiation. Electromagnetic signals traveling in the stripline have been identified with the Swihart mode in a S-I-S waveguide. The dynamical equation for the phase difference derived in this paper well describes the workings of CB-KID.

Acknowledgments

This work was partially supported by Grant-in Aid for Scientific Research (S) (No. 23226019) and (A) (No. 16H02450) from JSPS, Japan.

Appendix

Taking the time derivative of Eq.(1), we obtain the relation,

$$\partial_t \theta(x, t) = \partial_t \phi(x, d, t) - \partial_t \phi(x, z_s, t) - \frac{e^*}{\hbar c} \int_{z_s}^{d} dz \partial_z A_z(x, z, t)$$

$$= \frac{e^*}{\hbar} \int_{z_s}^{d} dz E_z(x, z, t) + \frac{e^*}{\hbar} [A^\phi_0(x, d, t) - A^\phi_0(x, z_s, t)],$$

where

$$A^\phi_0(x, z, t) = A_0(x, z, t) + \frac{\hbar}{e^*} \partial_t \phi(x, z, t),$$

with $A_0(x, z, t)$ being the scalar potential. As discussed in [17], the gauge-invariant scalar potential $A^\phi_0$ can be neglected in conventional junction systems where the superconducting layers are much thicker than the Thomas-Fermi screening length. Then, defining the voltage difference as

$$V(x, t) = -\int_{z_s}^{d} dz E_z(x, z, t),$$

one finds the Josephson relation [18],

$$V(x, t) = -\frac{\hbar}{e^*} \partial_t \theta(x, t).$$
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