A Note on Best Constants for Weighted Integral Hardy Inequalities on Homogeneous Groups

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Abstract. The main aim of this note is to prove sharp weighted integral Hardy inequality and conjugate integral Hardy inequality on homogeneous Lie groups with any quasi-norm for the range $1 < p \leq q < \infty$. We also calculate the precise value of sharp constants in respective inequalities, improving the result of Ruzhansky and Verma (Proc R Soc A 475:20180310, 2019) in the case of homogeneous groups.

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1. Introduction

In his seminal papers [8,9], G. H. Hardy stated and proved in 1920 and 1925, respectively, the following inequality, which is now known as “(integral) Hardy inequality”: For a non-negative and measurable function $f$ on $(0, \infty)$, we have

$$\int_{0}^{\infty} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^{p} \, dx \leq \left( \frac{p}{p-1} \right)^{p} \int_{0}^{\infty} f^{p}(x) \, dx, \quad p > 1. \quad (1.1)$$

The first generalization of (1.1) was presented by Hardy [10] in 1927, which is a weighted form of (1.1) and state that: If $f$ is non-negative and measurable function on $(0, \infty)$, then

$$\int_{0}^{\infty} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^{p} x^{\alpha} \, dx \leq \left( \frac{p}{p-1-\alpha} \right)^{p} \int_{0}^{\infty} f^{p}(x) x^{\alpha} \, dx, \quad p \geq 1, \quad \alpha < p - 1. \quad (1.2)$$

The further developments and improvements of (1.1) and (1.2) are discussed in several books, monographs and papers; we refer [2,11,13–17] and
references therein for more details. Let us first mention the following modern form of weighted version of Hardy’s original inequality:

\[
\left( \int_0^\infty \left( \int_0^x f(t) \, dt \right)^q u(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p(x) v(x) \, dx \right)^{\frac{1}{p}}, \tag{1.3}
\]

where \( f(x) \geq 0 \), \( u \) and \( v \) are weight functions and \( 1 \leq p, q < \infty \). In this paper, we consider the case \( 1 < p \leq q < \infty \). The constants in (1.1) and (1.2) are sharp. To find a best constant \( C \) in (1.3) is an intriguing and challenging problem. In the case of power weights, the analysis of the best constant of (1.3) is given in \([13,17]\).

Since then a lot of work has been done on Hardy inequalities in different forms and in different settings on higher dimensional Euclidean space. It is clearly impossible to give a complete overview of the literature, so let us only refer to books and surveys \([2,3,12,16]\) and references therein. The sharp constants in Hardy type inequalities on the Euclidean space are known only in a few cases. More precisely, Persson and Samko \([17]\) derived sharp weighted integral Hardy inequalities on Euclidean space with power weights.

In this article, our main objects are homogeneous (Lie) groups of homogeneous dimension \( Q \) equipped with a quasi-norm \( | \cdot | \). By definition, a homogeneous Lie group is a Lie group equipped with a family of dilations compatible with the group law. For the general description of the set-up of homogeneous groups, we refer to \([4–6,19]\). Particular examples of homogeneous groups are the Euclidean space \( \mathbb{R}^n \) (in which case \( Q = n \)), the Heisenberg group, as well as general stratified groups (homogeneous Carnot groups) and graded groups. Recently, Hardy type inequalities and their best constants have been extensively investigated in non-commutative settings (e.g. Heisenberg groups, graded groups, homogeneous groups); we cite \([7,18,19,22]\) just to mention a few of them.

More generally, Ruzhansky and Verma \([20]\) obtained several characterizations of weights for two-weight Hardy inequalities to hold on general metric measure spaces possessing polar decompositions for the range \( 1 < p \leq q < \infty \) (see, \([21]\) for the case \( 0 < q < p \) and \( 1 < p < \infty \)). As a consequence, one deduced new weighted Hardy inequalities on \( \mathbb{R}^n \), on homogeneous groups, on hyperbolic spaces and on Cartan-Hadamard manifolds \([20]\). In particular, one proved the following integral Hardy inequality \([20\text{, Corollary 3.1}]\) on homogeneous groups.

**Theorem 1.1.** Let \( G \) be a homogeneous group of homogeneous dimension \( Q \), equipped with a quasi norm \( | \cdot | \). Let \( 1 < p \leq q < \infty \) and let \( \alpha, \beta \in \mathbb{R} \), then the inequality

\[
\left( \int_G \left( \int_{B(0,|x|)} |u(y)| \, dy \right)^q |x|^\beta \, dx \right)^{\frac{1}{q}} \leq C \left( \int_G |u(x)|^q |x|^\alpha \, dx \right)^{\frac{1}{p}} \tag{1.4}
\]
holds for all measurable functions $u$ on $\mathbb{G}$ if and only if

$$\beta + Q < 0, \quad \alpha < Q(p - 1) \quad \text{and} \quad q(\alpha + Q) - p(\beta + Q) = pqQ. \quad (1.5)$$

Moreover, the constant $C$ for (1.4) satisfies

$$\frac{\mathcal{S}^{\frac{1}{q} + \frac{1}{p'}}}{|\beta + Q|^\frac{1}{q} (\alpha (1 - p') + Q)^{\frac{1}{p'}}} \leq C \leq \frac{\mathcal{S}^{\frac{1}{q} + \frac{1}{p'}}}{|\beta + Q|^\frac{1}{q} (\alpha (1 - p') + Q)^{\frac{1}{p'}}} \quad (1.6)$$

where $\mathcal{S}$ is the area of the unit sphere in $\mathbb{G}$ with respect to the quasi-norm $|\cdot|$.\\

In the above Theorem 1.1, the constant appearing in inequality (1.4) is bounded by upper and lower bound of certain quantities as given in (1.6) but may not be sharp, in general. The main aim of this paper is to fill this gap and to obtain a sharp version of (1.4) with the precise value of the sharp constant. We also prove a sharp weighted conjugate Hardy inequality on Homogeneous (Lie) groups. For the proof, we follow the method developed in [1,17] in the (isotropic and abelian) setting of Euclidean space. We note that also in the abelian (both isotropic and anisotropic) cases of $\mathbb{R}^n$, our results provide new insights in view of the arbitrariness of the quasi-norm $|\cdot|$ which does not necessarily have to be the Euclidean norm.

In this next section, we present basic definition, notation and terminologies related with homogeneous Lie groups and some basic result concerning to the one dimensional sharp Hardy inequalities. In the last section, we discuss our main results on sharp weighted Hardy inequalities on homogeneous groups and obtain the precise value of the sharp constant.

Throughout this paper, the symbol $A \asymp B$ means $\exists C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

2. Preliminaries

In this section, we recall the basics of homogeneous groups and Hardy inequalities on Euclidean space. For more detail on homogeneous groups as well as several functional inequalities on homogeneous groups, we refer to monographs [4,6,19] and references therein. For Hardy inequalities on Euclidean space, one can see [12,13,17] and references therein.

2.1. Basics on homogeneous groups

A Lie group $\mathbb{G}$ (identified with $(\mathbb{R}^N, \circ)$) is called a homogeneous group if it is equipped with a dilation mapping

$$D_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \lambda > 0,$$

defined as

$$D_\lambda(x) = (\lambda^{v_1} x_1, \lambda^{v_2} x_2, \ldots, \lambda^{v_N} x_N), \quad v_1, v_2, \ldots, v_N > 0, \quad (2.1)$$
which is an automorphism of the group $G$ for each $\lambda > 0$. At times, we will denote the image of $x \in G$ under $D_\lambda$ by $\lambda(x)$ or, simply $\lambda x$. The homogeneous dimension $Q$ of a homogeneous group $G$ is defined by

$$Q = v_1 + v_2 + \cdots + v_N.$$ 

It is well known that a homogeneous group is necessarily nilpotent and uni-

module. The Haar measure $dx$ on $G$ is nothing but the Lebesgue measure on $\mathbb{R}^N$.

Let us denote the Haar volume of a measurable set $\omega \subset G$ by $|\omega|$. Then we have the following consequences: for $\lambda > 0$

$$|D_\lambda(\omega)| = \lambda^Q|\omega| \quad \text{and} \quad \int_G f(\lambda x)dx = \lambda^{-Q}\int_G f(x)dx. \quad (2.2)$$

A quasi-norm on $G$ is any continuous non-negative function $|\cdot|: G \to [0, \infty)$ satisfying the following conditions:

(i) $|x| = |x^{-1}|$ for all $x \in G$,
(ii) $|\lambda x| = \lambda|x|$ for all $x \in G$ and $\lambda > 0$,
(iii) $|x| = 0 \iff x = 0$.

If $S = \{x \in G : |x| = 1\} \subset G$ is the unit sphere with respect to the quasi-norm, then there is a unique Radon measure $\sigma$ on $S$ such that for all $f \in L^1(G)$, we have the following polar decomposition

$$\int_G f(x)dx = \int_0^\infty \int_S f(ry)r^{Q-1}d\sigma(\omega)dr. \quad (2.3)$$

Firstly, let us consider the metric space $(X,d)$ with a Borel measure $dx$ allowing for the following polar decomposition at $a \in X$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(X)$ we have

$$\int_X f(x)dx = \int_0^\infty \int_\Sigma f(r,\omega)\lambda(r,\omega)d\omega dr, \quad (2.4)$$

for some set $\Sigma = \{x \in X : d(x,a) = r\} \subset X$ with a measure on it denoted by $d\omega$, and $r \rightarrow a$ as $r \rightarrow 0$.

Similar to the Euclidean spaces, we define the weighted Hardy operator and the weighted conjugate Hardy operator on a homogeneous group $G$ by

$$Hu(x) = |x|^{-Q}w(|x|)\int_{B(0,|x|)} \frac{u(y)}{w(|y|)} dy, \quad (2.5)$$

$$H^*u(x) = |x|^{-Q}w(|x|)\int_{G \setminus B(0,|x|)} \frac{u(y)}{|y|^Q w(|y|)} dy,$$

where $\alpha \geq 0$ and $w$ is a radial weight on $G$. In particular, for $\mu \in \mathbb{R}$ the function $w(x) := |x|^\mu, \forall x \in G$, defines a weight function on $G$.

Now, we recall the one dimensional sharp weighted Hardy and conjugate Hardy inequalities established in [17, Theorem 2.7]. This result will be used to prove our main result.
Theorem 2.1. Let $1 < p \leq q < \infty$. Then the following statements (a) and (b) hold and are equivalent:

(a) The inequality

$$\left( \int_0^\infty \left( \int_0^x u(t)dt \right)^q x^\beta dx \right)^{\frac{1}{q}} \leq D_{p,q,\alpha} \left( \int_0^\infty u^p(x)x^\alpha dx \right)^{\frac{1}{p}}$$

(2.6)

holds for all measurable functions $u$ on $(0, \infty)$ if and only if

$$\alpha < p - 1 \text{ and } q(\alpha + 1) - p(\beta + 1) = pq$$

(2.7)

(b) The inequality

$$\left( \int_0^\infty \left( \int_x^\infty u(t)dt \right)^q x^{\beta_0}dx \right)^{\frac{1}{q}} \leq D_{p,q,\alpha} \left( \int_0^\infty u^p(x)x^{\alpha_0}dx \right)^{\frac{1}{p}}$$

(2.8)

holds for all measurable functions $u$ on $(0, \infty)$ if and only if

$$\alpha_0 > p - 1 \text{ and } q(\alpha_0 + 1) - p(\beta_0 + 1) = pq,$$

(2.9)

and also

(c) The relation between the parameters $\alpha$ and $\alpha_0$ is

$$\alpha_0 = -\alpha - 2 + 2p$$

(2.10)

and the best constants in (2.6) and (2.8) are same.

Theorem 2.2. Let $1 < p \leq q < \infty$ and let the parameters $\alpha$ and $\beta$ satisfy (2.7). Then the sharp constant $D_{p,q,\alpha}$ in (2.6) is given by

$$D_{p,q,\alpha} = \left( \frac{p - 1}{p - 1 - \alpha} \right)^{\frac{1}{p} + \frac{1}{q}} \left( \frac{p'}{q} \right)^{\frac{1}{p}} \left( \frac{\Gamma\left(\frac{p}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)} \right)^{\frac{1}{p} - \frac{1}{q}},$$

(2.11)

when $p < q$ and

$$D_{p,p,\alpha} = \lim_{q \to p} D_{p,q,\alpha} = \frac{p}{p - 1 - \alpha},$$

(2.12)

when $p = q$.

3. Main results

This section is devoted to establishing sharp weighted Hardy and conjugate Hardy inequalities on homogeneous groups. We begin with the following sharp weighted integral Hardy inequality on homogeneous groups for the range $1 < p \leq q < \infty$. 
Theorem 3.1. Let $G$ be a homogeneous group with homogeneous dimension $Q$ equipped with quasi norm $|\cdot|$. Let $1 < p \leq q < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then, the following inequality
\[
\left( \int_{G} \left( \int_{B(0,|x|)} u(y) dy \right)^{q} |x|^\beta dx \right)^{\frac{1}{q}} \leq C(p, q, Q, \alpha) \left( \int_{G} |u(x)|^{p} |x|^\alpha dx \right)^{\frac{1}{p}}
\] (3.1)
holds for all measurable functions $u$ on $G$ if and only if
\[
\alpha < Q(p - 1) \quad \text{and} \quad q(\alpha + Q) - p(\beta + Q) = pqQ.
\] (3.2)
Moreover, the sharp constant $C(p, q, Q, \alpha)$ is given by
\[
C(p, q, Q, \alpha) = \left| S \right|^{1 + \frac{1}{q} - \frac{1}{p}} \left( \frac{p - 1}{Q(p - 1) - \alpha} \right)^{\frac{1}{p'}} \left( \frac{p'}{q} \right)^{\frac{1}{p'}} \left( \frac{q - p}{\Gamma \left( \frac{pq}{q - p} \right) \Gamma \left( \frac{p(q - 1)}{q - p} \right) \Gamma \left( \frac{p(q - 1)}{q - p} \right)} \right)^{\frac{1}{p} - \frac{1}{q}}
\] (3.3)
when $q > p$ and
\[
C(p, p, Q, \alpha) = \lim_{q \to p} C(p, q, Q, \alpha) = \frac{p|S|}{Q(p - 1) - \alpha},
\] (3.4)
when $p = q$, where $S = \{ x \in G : |x| = 1 \} \subset G$ is the unit sphere with respect to the quasi-norm $|\cdot|$, and $|S|$ denotes the measure of the unit sphere $S$ in the homogeneous group $G$ with respect to the quasi-norm $|\cdot|$.

Remark 3.2. The condition (3.2) in Theorem 3.1 automatically implies that $\beta + Q < 0$ in (1.5) of Theorem 1.1, so the conditions on indices are consistent in both theorems.

Proof. For the proof of the sharp Hardy inequality (3.1) we use the weighted Hardy operator on a homogeneous group $G$ with quasi norm $|\cdot|$, given by (2.5), in the following form
\[
(Hu)(x) = |x|^{\lambda + \mu - Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} dy, \quad \lambda \geq 0,
\] (3.5)
where $\lambda = \frac{1}{p} - \frac{1}{q}$ and $\mu \in \mathbb{R}$ will be chosen later accordingly. We will calculate the $L^p$-$L^q$ norm of the weighted Hardy operator $H$ in the range $1 < p \leq q < \infty$ and eventually conclude the proof of (3.1). For this purpose, let us estimate the following quantity:
\[
\|Hu\|_{L^q(G)} = \left( \int_{G} \left( \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} dy \right)^{q} dx \right)^{\frac{1}{q}}.
\] (3.6)
Using the polar decomposition (2.3) in the setting of homogeneous groups (3.6) will take the following form with \( x = r\sigma \) and \( y = \rho\sigma \):

\[
\left( \int_G |x|^{\lambda+\mu-Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^{\mu}} dy \right)^{\frac{1}{q}} = |G|^{\frac{1}{q}} \left( \int_0^\infty r^{\frac{Q-1}{q} - \mu} \int_0^r \int_S u(\rho\sigma) \rho^{Q-1-\mu} d\rho d\sigma \right)^{\frac{1}{q}}. \tag{3.7}
\]

Now, by setting

\[
U(\rho) = \int_S u(\rho\sigma) d\sigma
\]

in the above identity (3.7), we obtain

\[
\left( \int_G |x|^{\lambda+\mu-Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^{\mu}} dy \right)^{\frac{1}{q}} = |G|^{\frac{1}{q}} \left( \int_0^\infty r^{\frac{Q-1}{q} - \mu} \int_0^r U(\rho) \rho^{Q-1-\mu} d\rho d\sigma \right)^{\frac{1}{q}}, \tag{3.8}
\]

where \( p' \) is the Lebesgue conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). Thus, from (3.8) we have

\[
\left( \int_G |x|^{\lambda+\mu-Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^{\mu}} dy \right)^{\frac{1}{q}} = |G|^{\frac{1}{q}} \left( \int_0^\infty r^{\frac{Q-1}{q} + \lambda+\mu-Q} \int_0^r U(\rho) \rho^{Q-1-\mu} d\rho d\sigma \right)^{\frac{1}{q}}. \tag{3.9}
\]
By denoting \( \delta = \frac{\lambda}{Q} \) and \( \mu = \gamma + \frac{Q-1}{p'} \) and performing simple calculations

\[
\frac{Q-1}{q} + \lambda + \mu - Q = \frac{Q-1}{q} + \lambda + \gamma + \frac{Q-1}{p'} - Q
\]
\[
= (Q - 1) \left( \frac{1}{p} - \frac{\lambda}{Q} \right) + \lambda + \gamma + \frac{Q-1}{p'} - Q \quad \text{(since } \frac{\lambda}{Q} = \frac{1}{p} - \frac{1}{q} \text{)}
\]
\[
= (Q - 1) \left( \frac{1}{p} + \frac{1}{p'} \right) - \frac{\lambda(Q-1)}{Q} + \lambda + \gamma - Q
\]
\[
= Q - 1 - \lambda + \frac{\lambda}{Q} + \lambda + \gamma - Q \quad \text{(as } \frac{1}{p} + \frac{1}{p'} = 1 \text{)}
\]
\[
= \delta + \gamma - 1.
\]

So that we have

\[
\frac{Q-1}{q} + \lambda + \mu - Q = \delta + \gamma - 1,
\]

and therefore, from (3.9) we get

\[
\left( \int_{G} |x|^\lambda + \mu - Q \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} dy \right)^{\frac{1}{q}}
\]
\[
= |\mathcal{G}|^{\frac{1}{q}} \left( \int_{0}^{\infty} r^{\delta + \gamma - 1} \int_{0}^{r} \rho^{-\gamma} \rho^{\frac{Q-1}{p}} U(\rho) d\rho \right)^{\frac{1}{q}}.
\]

(3.10)

It follows from the relation \( \mu = \gamma + \frac{Q-1}{p'} \) that

\[
\mu < \frac{Q}{p'} \iff \gamma < \frac{1}{p'}.
\]

(3.11)

Now, we recall the one dimensional Hardy inequality from Theorem 2.1:

For \( 1 < p \leq q < \infty \), the inequality

\[
\left( \int_{0}^{\infty} \left( \int_{0}^{x} u(t)dt \right)^{q} x^{\beta'} dx \right)^{\frac{1}{q}} \leq D_{p,q,\alpha'} \left( \int_{0}^{x} u^p(x)x^{\alpha'} dx \right)^{\frac{1}{p}}
\]

(3.12)

holds for all measurable functions \( u \) on \((0, \infty)\) if and only if

\[
\alpha' < p - 1 \quad \text{and} \quad q(\alpha' + 1) - p(\beta' + 1) = pq
\]

(3.13)

with the sharp constant \( D_{p,q,\alpha'} \) (when \( p < q \) as well as \( p = q \)) given by Theorem 2.2.
For our convenience we rewrite (3.12) with both the weight functions on the left hand side:

\[
\left( \int_0^\infty \left| x^{\beta'/q} \int_0^x \frac{t^{\alpha'/p} u(t)}{t^{\alpha'/p}} \, dt \right|^q \, dx \right)^{\frac{1}{q}} \leq D_{p,q,\gamma} \left( \int_0^\infty (x^{\alpha'/p} u(x))^p \, dx \right)^{\frac{1}{p}}. 
\]

(3.14)

For that we change the notation: \( \frac{\beta'}{q} = \delta + \gamma - 1 \), \( \alpha' = \gamma p \), and \( u(t)^{t^{\gamma}} = g(t) \) to get

\[
\left( \int_0^\infty \left| x^{\delta + \gamma - 1} \int_0^x g(t) \frac{dt}{t^{\gamma}} \right|^q \, dx \right)^{\frac{1}{q}} \leq D_{p,q,\gamma} \left( \int_0^\infty g^p(x) \, dx \right)^{\frac{1}{p}} ; \quad \gamma < \frac{1}{p},
\]

(3.15)

where \( 1 < p < q < \infty \) and \( \delta = \frac{1}{p} - \frac{1}{q} \) as \( \delta = \frac{\lambda}{Q} \). Here the sharp constant \( D_{p,q,\gamma} \) is given by \( D_{p,q,\alpha'} \) as in (2.11) by replacing \( \alpha' \) by \( \gamma p \).

Next, the one dimensional sharp Hardy inequality (3.15) is applicable on right hand side of identity (3.10) with the function \( g(\rho) = \rho^{Q-1} U(\rho) \) and the relation (3.11) by recalling the relation here for the inequalities in (3.2):

\[ \alpha < Q(p-1) \quad \text{and} \quad q(\alpha + Q) - p(\beta + Q) = pqQ \]

holds if and only if

\[ \alpha' < p - 1 \quad \text{and} \quad q(\alpha' + 1) - p(\beta' + 1) = pq \]

with \( \alpha' = \alpha/Q \) and \( \beta' = \beta/Q \). Therefore, from (3.10) we obtain the following sharp inequality

\[
\left( \int_G \left| x^{\lambda + \mu - Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} \, dy \right|^q \, dx \right)^{\frac{1}{q}} \leq D_{p,q,\gamma} |G|^{\frac{1}{q}} \left( \int_0^\infty \rho^{Q-1} U^p(\rho) \, d\rho \right)^{\frac{1}{p}}.
\]

(3.16)

By the Hölder inequality, we calculate

\[
U^p(\rho) = \left( \int_G |u(\rho \sigma)|^p \, d\sigma \right)^{\frac{1}{p}} \leq \left( \int_G 1 \, d\sigma \right)^{\frac{1}{p-1}} \int_G |u(\rho \sigma)|^p \, d\sigma = |G|^{\frac{1}{p-1}} \int_G |u(\rho \sigma)|^p \, d\sigma,
\]

(3.17)

and use this in (3.16) to get

\[
\left( \int_G \left| x^{\lambda + \mu - Q} \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} \, dy \right|^q \, dx \right)^{\frac{1}{q}} \leq D_{p,q,\gamma} |G|^{\frac{1}{q}} \frac{1}{q-1} \left( \int_0^\infty \rho^{Q-1} \int_G |u(\rho \sigma)|^p \, d\sigma \, d\rho \right)^{\frac{1}{p}}
\]

\[
= D_{p,q,\gamma} |G|^{1 + \frac{1}{q} - \frac{1}{p}} \left( \int_0^\infty \rho^{Q-1} \int_G |u(\rho \sigma)|^p \, d\sigma \, d\rho \right)^{\frac{1}{p}}.
\]

(3.18)
Again using polar decomposition we note that
\[
\int_0^\infty \rho^{Q-1} \int_{\mathcal{S}} |u(\rho \sigma)|^p \, d\sigma \, d\rho = \int_{\mathcal{G}} |u(y)|^p \, dy,
\]
and consequently, (3.18) yields the following inequality
\[
\left( \int_{\mathcal{G}} |x|^{\lambda+\mu+\gamma} \int_{B(0,|x|)} \frac{u(y)}{|y|^\mu} \, dy \right)^{\frac{q}{q'}} \leq D_{p,q,\gamma} |\mathcal{S}|^{\frac{1}{q'} - \frac{1}{p'}} \left( \int_{\mathcal{G}} |u(y)|^p \, dy \right)^{\frac{1}{p'}}
\]
(3.19)
with \( \gamma = \mu - \frac{Q-1}{p'} \). Finally, by choosing \( \mu = \frac{\alpha}{p} \), we note that \( \frac{\beta'}{q} = \delta + \gamma - 1 \) and \( \alpha' = \gamma p \) holds, if and only if \( \lambda + \mu - Q = \frac{\beta}{q} \) holds. Indeed, by recalling that \( \alpha' = \frac{\alpha}{q}, \beta' = \frac{\beta}{q} \) and \( \delta = \frac{\lambda}{q} \), we get
\[
\frac{\beta}{Qq} = \frac{\beta'}{q} = \delta + \gamma - 1 \iff \frac{\beta}{q} = \delta Q + \gamma Q - Q = \lambda + \mu - Q,
\]
(3.20)
where we have used that \( \gamma Q = \frac{\alpha'}{p} Q = \frac{\alpha}{q} p Q = \frac{\alpha}{p} = \mu \iff \gamma Q = \frac{\alpha}{p} \iff \gamma p = \frac{\alpha}{q} = \alpha' \).

Next, by replacing \( \frac{u(y)}{|y|^\mu} = f(y) \), we obtain \( \gamma = \frac{\alpha}{p} - \frac{Q-1}{p'} \) and the sharp inequality
\[
\left( \int_{\mathcal{G}} \left( \int_{B(0,|x|)} f(y) \, dy \right)^q |x|^\beta \, dx \right)^{\frac{1}{q'}} \leq C(p,q,Q,\alpha) \left( \int_{\mathcal{G}} |f(y)|^p |y|^\alpha \, dy \right)^{\frac{1}{p'}}
\]
(3.21)
for \( 1 < p < q < \infty \) with the sharp constant
\[
C(p,q,Q,\alpha) := D_{p,q,\gamma} |\mathcal{S}|^{\frac{1}{q'} - \frac{1}{p'}}
\]
\[
= |\mathcal{S}|^{\frac{1}{q'} - \frac{1}{p'}} \left( \frac{p-1}{p-1-\gamma p} \right)^{\frac{1}{p'}} \left( \frac{\gamma}{p} \right)^{\frac{1}{p'}} \left( \frac{q-1}{p} \right)^{\frac{1}{p'}} \left( \frac{1}{\Gamma \left( \frac{p}{q-p} \right) \Gamma \left( \frac{p(q-1)}{q-p} \right)} \right)^{\frac{1}{p'}}.
\]
By using the value of \( \gamma \), i.e., \( \gamma = \frac{\alpha}{p} - \frac{Q-1}{p'} \), we calculate
\[
p - 1 - \gamma p = p - 1 - \left( \frac{\alpha}{p} - \frac{Q-1}{p'} \right) p = Q(p-1) - \alpha.
\]
Therefore, by putting the value of \( p - 1 - \gamma p \), we get the sharp constant
\[
C(p,q,Q,\alpha) = |\mathcal{S}|^{\frac{1}{q'} - \frac{1}{p'}} \left( \frac{p-1}{Q(p-1) - \alpha} \right)^{\frac{1}{p'}} \left( \frac{\gamma}{p} \right)^{\frac{1}{p'}} \left( \frac{q-1}{p} \right)^{\frac{1}{p'}} \left( \frac{1}{\Gamma \left( \frac{p}{q-p} \right) \Gamma \left( \frac{p(q-1)}{q-p} \right)} \right)^{\frac{1}{p'}}
\]
(3.22)
when \( 1 < p < q < \infty \).
For the case $p = q$, the proof follows verbatim to the proof of the case $p < q$ above. The value of the best constant $C(p, p, Q, \alpha)$ is given by

$$C(p, p, Q, \alpha) = D_{p,p,\gamma}|\mathcal{S}|,$$

(3.23)

where $D_{p,p,\gamma} = \lim_{q \to p} D_{p,q,\gamma} = \frac{p}{p - 1 - \gamma p}$. Therefore,

$$C(p, p, Q, \alpha) = \lim_{q \to p} C(p, q, Q, \alpha) = |\mathcal{S}| \lim_{q \to p} D_{p,q,\gamma} = |\mathcal{S}| \frac{p}{p - 1 - \gamma p}$$

(3.24)

where $\gamma = \frac{\alpha}{p} - \frac{Q - 1}{p'}$.

By calculating the value of $p - 1 - \gamma p$ and using $\gamma = \frac{\alpha}{p} - \frac{Q - 1}{p'}$, we get $p - 1 - \gamma p = Q(p - 1) - \alpha$ as above. Therefore, we obtain

$$C(p, p, Q, \alpha) = \lim_{q \to p} C(p, q, Q, \alpha) = |\mathcal{S}| \frac{p}{Q(p - 1) - \alpha},$$

(3.25)

completing the proof of the Theorem 3.1. \hfill \Box

**Remark 3.3.** It is clear from the proof of Theorem 3.1 that the constant $C(p, q, Q, \alpha)$ in (3.1) is sharp because the constant $D_{p,q,\gamma}$ in the one dimensional inequality (3.15) is sharp.

The next result is the sharp weighted conjugate integral Hardy inequality on homogeneous groups.

**Theorem 3.4.** Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q$ equipped with a quasi norm $|\cdot|$. Let $1 < p \leq q < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then the following inequality

$$\left( \int_{\mathbb{G}} \left( \int_{\mathbb{G} \setminus B(0, |x|)} u(y) \, dy \right)^{q \beta} |x|^{\alpha \beta} \, dx \right)^{\frac{1}{q}} \leq C(p, q, Q, \alpha) \left( \int_{\mathbb{G}} |u(x)|^{p} |x|^{\alpha} \, dx \right)^{\frac{1}{p}}$$

(3.26)

holds for all measurable functions $u$ on $\mathbb{G}$ if and only if

$$\alpha > Q(p - 1) \quad \text{and} \quad q(\alpha + Q) - p(\beta + Q) = pqQ.$$  

Moreover, the constant $C(p, q, Q, \alpha)$ is sharp and given by

$$C(p, q, Q, \alpha) = |\mathcal{S}|^{1 + \frac{1}{q} - \frac{1}{p}} \left( \frac{p - 1}{\alpha - Q(p - 1)} \right)^{\frac{1}{p'} + \frac{1}{q}} \left( p' \right)^{\frac{1}{p'}} \left( \frac{q - p}{q} \Gamma \left( \frac{pq}{q - p} \right) \Gamma \left( \frac{p(1 - Q)}{q - p} \right) \right)^{1 - \frac{1}{q}}$$

when $q > p$ and

$$C(p, p, Q, \alpha) = \lim_{q \to p} C(p, q, Q, \alpha) = \frac{p|\mathcal{S}|}{\alpha - Q(p - 1)}$$

when $p = q$, where $\mathcal{S} = \{ x \in \mathbb{G} : |x| = 1 \} \subset \mathbb{G}$ is the unit sphere with respect to the quasi-norm $|\cdot|$, and $|\mathcal{S}|$ denotes the measure of the unit sphere $\mathcal{S}$ in the homogeneous group $\mathbb{G}$ with respect to the quasi-norm $|\cdot|$. 

Proof. The proof is similar to the proof of Theorem 3.1 using one dimensional sharp weighted conjugate Hardy inequality (2.8) from Theorem 2.1 instead of one dimensional sharp Hardy inequality (2.6) from Theorem 2.1.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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