On webs in quantum type C

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Abstract. We study webs in quantum type C, focusing on the rank three case. We define a linear pivotal category $\text{Web}(sp_4)$ diagrammatically by generators and relations, and conjecture that it is equivalent to the category $\text{FundRep}(U_q(sp_4))$ of quantum $sp_4$ representations generated by the fundamental representations, for generic values of the parameter $q$. We prove a number of results in support of this conjecture, most notably that there is a full, essentially surjective functor $\text{Web}(sp_4) \rightarrow \text{FundRep}(U_q(sp_4))$, that all Hom-spaces in $\text{Web}(sp_4)$ are finite-dimensional, and that the endomorphism algebra of the monoidal unit in $\text{Web}(sp_4)$ is one-dimensional. The latter corresponds to the statement that all closed webs can be evaluated to scalars using local relations; as such, we obtain a new approach to the quantum $sp_4$ link invariants, akin to the Kauffman bracket description of the Jones polynomial.

1 Introduction

Given a classical algebraic object, such as a group or ring, one of the basic questions one can ask is for a presentation of the object via generators and relations. The advent of quantum invariants in low-dimensional topology suggested the investigation of a related problem, one “categorical dimension” higher: is it possible to find a presentation of a given monoidal category via generators and relations? Indeed, work of Reshetikhin and Turaev [39, 40] shows that suitable monoidal categories lead to invariants of links and 3-manifolds, and, among other applications, such presentations elucidate various properties of these invariants and related structures.

One of the first results in this direction is a folklore theorem that describes the category $\text{Rep}(U_q(sl_2))$ of finite-dimensional representations of the quantum group $U_q(sl_2)$ (see [43] for a very early incarnation). Specifically, the full subcategory $\text{FundRep}(U_q(sl_2))$ of $\text{Rep}(U_q(sl_2))$ monoidally generated by the fundamental representation is equivalent to the $\mathbb{C}(q)$-linear pivotal category freely generated by a single object, modulo a single relation in the endomorphism algebra of the monoidal unit. Translated via the diagrammatic formalism for monoidal categories, this equivalently states that $\text{FundRep}(U_q(sl_2))$ is equivalent to the Temperley–Lieb category, wherein objects are non-negative integers and morphisms $n \rightarrow m$ are $\mathbb{C}(q)$-linear combinations of planar $(m, n)$-tangles, modulo isotopy and the local relation

$$
\begin{array}{c}
\circ
\end{array} = -q - q^{-1}.
$$

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Note that this, in effect, gives a diagrammatic description of the entire category $\text{Rep}(U_q(\mathfrak{sl}_2))$ of finite-dimensional representations, as this category can be recovered from $\text{FundRep}(U_q(\mathfrak{sl}_2))$ via idempotent completion. Furthermore, $\text{Rep}(U_q(\mathfrak{sl}_2))$ is braided, and explicit diagrammatic formulae for the braiding in $\text{FundRep}(U_q(\mathfrak{sl}_2))$ recover the Kauffman bracket formulation [18] of the Jones polynomial [17]. This diagrammatic description of $\text{FundRep}(U_q(\mathfrak{sl}_2))$ in terms of generators and relations thus serves as both the basis for applications of the Jones polynomial to classic problems in knot theory [18, 31, 49] and for the development of Khovanov homology [20] and related constructions in categorical representation theory [9, 10].

Pioneering work of Kuperberg [23] poses the problem of finding a similar description for the category $\text{FundRep}(U_q(\mathfrak{g}))$ tensor-generated by the fundamental representations of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a complex simple Lie algebra. He answers this question in the case that $\mathfrak{g}$ is rank 2, showing that $\text{FundRep}(U_q(\mathfrak{g}))$ admits a description as the $\mathbb{C}(q)$-linear pivotal category freely generated by a finite set of morphisms depicted graphically as trivalent vertices, modulo a finite list of local relations. It follows that morphisms therein are given by $\mathbb{C}(q)$-linear combinations of webs, certain labeled trivalent graphs, modulo planar isotopy and local relations. Again, diagrammatic formulae for the braiding on these categories give an explicit, local description for the $U_q(\mathfrak{g})$ link invariants, that again serve as the basis for further study of these invariants and their categorified analogues.

The problem of extending Kuperberg’s results to higher rank resisted various attempts [22, 27], although related descriptions were obtained for the corresponding link polynomials [28, 29]. However, in breakthrough work [11], Cautis, Kamnitzer, and Morrison found a web-based description of $\text{FundRep}(U_q(\mathfrak{sl}_n))$ using a quantized version of skew Howe duality. This technique was adapted to give new diagrammatic descriptions of various other categories of representations of quantum groups [8, 15, 37, 41, 51]. Nevertheless, it remains an open problem to give such a description of $\text{FundRep}(U_q(\mathfrak{g}))$ for simple complex $\mathfrak{g}$ of rank $\geq 3$ outside type A. Complicating matters, the work of Sartori and Tubbenhauer [44] interestingly suggests that skew Howe duality cannot be used in a straightforward way to solve this problem in quantum types $BCD$, thus new ideas are required.

In this paper, we take the first steps toward solving this problem for the quantum group $U_q(\mathfrak{sp}_{2n})$, focusing on the rank 3 case. Specifically, in Definition 2.1 below, we define the category $\text{Web}(\mathfrak{sp}_6)$ via generators and relations, and we conjecture that it is equivalent to $\text{FundRep}(U_q(\mathfrak{sp}_6))$. Although our conjecture remains open at the moment, we provide ample evidence for its validity: among other results, we prove that there is a full, essentially surjective functor $\Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6))$ of ribbon categories, that all Hom-spaces in $\text{Web}(\mathfrak{sp}_6)$ are finite-dimensional, and that the endomorphism algebra of the monoidal unit in $\text{Web}(\mathfrak{sp}_6)$ is isomorphic to $\mathbb{C}(q)$. The latter result equivalently states that all closed webs in $\text{Web}(\mathfrak{sp}_6)$ can be evaluated to scalars, and our proof provides an explicit algorithm; this result pairs with diagrammatic formulae for the braiding to give an explicit, local, diagrammatic description of the (colored) $U_q(\mathfrak{sp}_6)$ link invariant, à la the Kauffman bracket description of the Jones polynomial. As such, we expect the category $\text{Web}(\mathfrak{sp}_6)$ to form the basis for an explicit construction of the $\mathfrak{sp}_6$ link homologies, in the spirit of [1, 21, 26, 34]. In follow-up work [7], we plan to investigate faithfulness of the functor.
Quantum groups and their representation theory

In Section 2, we recall the relevant background on quantum groups and web categories, and we collect the main results of the present work. Section 3 is devoted to the definition and study of the functor \( \Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6)) \). In Section 4, we prove the finite-dimensionality of Hom-spaces and results about closed webs in the plane and the annulus; algebraically, these latter results correspond to the aforementioned result about the endomorphism algebra of the monoidal unit and to a computation of the “trace decategorification” of \( \text{Web}(\mathfrak{sp}_6) \). To do so, we introduce a category of ladder-like webs as a technical tool. We believe this category will play an important role in the eventual resolution of our conjecture. Finally, in Section 5, we briefly discuss our approach to the quantum \( \mathfrak{sp}_6 \) link invariant, which gives a fresh perspective on the \( n = 3 \) case of the invariants in [28].

2 Background and statement of results

In this section, we begin by reviewing quantum groups and their representation theory, along the way establishing notation and conventions. We then recall known results about webs in type \( C \), and finally state the main results of the present work.

2.1 Quantum groups and their representation theory

We begin with some background on quantum groups and their representations; almost all of this material is standard, and can be found, e.g., in [12], except when noted otherwise.

Let \( \mathfrak{g} \) be a simple complex Lie algebra of rank \( n \) with corresponding Cartan matrix \( (a_{ij}) \) and let \( q \) be a (generic) indeterminant. Recall that the quantum group \( U_q(\mathfrak{g}) \) associated to \( \mathfrak{g} \) is the unital \( \mathbb{C}(q) \)-algebra generated by \( X_i^+, X_i^- \), \( K_i, K_i^{-1} \) \( (1 \leq i \leq n) \) modulo the relations:

- \( K_i K_i^{-1} = 1, K_i^{-1} K_i = 1; \)
- \( K_i K_j = K_j K_i; \)
- \( K_i X_j^+ K_i^{-1} = q_i^{a_{ij}} X_j^+; \)
- \( X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} K_i K_i^{-1}; \)
- \( \sum_{m=0}^{\infty} (-1)^m [1-a_{ij}]_{q_i} (X_i^+)^{1-a_{ij}-m} X_j^+ (X_i^+)^m = 0. \)

Here, \( q_i := q^{d_i} \), where \( \{d_i\}_{i=1}^n \) are the relatively prime positive integers such that \( (d_i, a_{ij}) \) is symmetric, and \( [n]_{q_i} = [n]_{q_i^{-1}} \) is the quantum binomial coefficient, defined using the quantum integers \( [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}. \)

Recall that \( U_q(\mathfrak{g}) \) is a Hopf algebra with comultiplication, counit, and antipode defined by

- \( \Delta(K_i) = K_i \otimes K_i, \Delta(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+, \Delta(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-; \)
- \( \varepsilon(K_i) = 1, \varepsilon(X_i^+) = \varepsilon(X_i^-) = 0; \)
- \( S(K_i) = K_i^{-1}, S(X_i^+) = -X_i^+ K_i^{-1}, S(X_i^-) = -K_i X_i^-; \)
The quantum group can be understood as a Hopf algebra deformation of the enveloping algebra $U(g)$. Furthermore, there exists a universal $R$-matrix $R \in U_q(g) \otimes U_q(g)$ and ribbon element $\nu \in U_q(g)$, so that the triple $(U_q(g), R, \nu)$ is a (topological) ribbon Hopf algebra.

The representation theory of $U_q(g)$ closely parallels that of $g$. Specifically, let $\text{Rep}(U_q(g))$ be the category of (type I) finite-dimensional representations of $U_q(g)$, then, as for $g$, $\text{Rep}(U_q(g))$ is semisimple, and there is a unique simple object $\Gamma_\lambda$ corresponding to each dominant weight $\lambda$ for $g$. The decomposition of tensor products of irreducible representations into direct sums of irreducibles in $\text{Rep}(U_q(g))$ (i.e., the fusion rules) are identical to those for $g$. Furthermore, the ribbon Hopf structure on $U_q(g)$ endows $\text{Rep}(U_q(g))$ with the structure of a ribbon (also called tortile) category. In particular, $\text{Rep}(U_q(g))$ is a braided pivotal category.

To expand on the latter, the ribbon Hopf algebra structure produces a group-like element $g := \nu^{-1}u \in U_q(g)$ (here, $u$ is an element satisfying $u^2 = uS(u)$), and for $\Gamma \in \text{Rep}(U_q(g))$, the pivotal isomorphism $\Gamma \cong \Gamma^{**}$ is given by $\nu \mapsto (f \mapsto f(\nu v))$ for $v \in \Gamma$ and $f \in \Gamma^{**}$. This element $g$ is also relevant, as it determines the quantum dimension of $\Gamma \in \text{Rep}(U_q(g))$, defined explicitly by

$$\dim_q(\Gamma) = \text{Tr}(g|_{\Gamma}).$$

**Remark 2.1** Work of Snyder and Tingley classifies the choices of ribbon element for $U_q(g)$; see [47] for complete details. To summarize, they show that there exists a “half-ribbon” element $X \in U_q(g)$, so that all ribbon elements are given by $s(\phi)X^{-2}$. Here, $\phi$ is a character of the weight lattice modulo the root lattice with $|\phi| \leq 2$ and $s(\phi)$ is the corresponding element in $\overline{U_q(g)}$.

The standard choice of ribbon element for $U_q(g)$ is the so-called quantum Casimir element $c \in U_q(g)$, which acts on the irreducible representation $\Gamma_\lambda$ by $q^{-(\lambda,\lambda + 2\rho)}$. Here, $\rho$ is the half sum of the positive roots (or equivalently, the sum of the fundamental weights) and $(\cdot, \cdot)$ is the standard symmetric bilinear form. This ribbon element gives a pivotal structure on $\text{Rep}(U_q(g))$ for which the Frobenius–Schur indicators agree with those for $g$.

However, for the choice of ribbon element $X^{-2}$ (and the corresponding pivotal structure), Snyder and Tingley show that all nonzero Frobenius–Schur indicators equal one. For $g$ of type $C$, this implies that this choice of ribbon element differs from the quantum Casimir. Since the weight lattice modulo the root lattice is isomorphic to $\mathbb{Z}/2$ in type $C$, the quantum Casimir element $c$ corresponds to the choice of nontrivial character $\phi$ (i.e., for this $\phi$, $s(\phi)$ acts by $-1$ on irreducible representations with highest weight not in the root lattice). Hence,

$$X^{-2} = s(\phi^2)X^{-2} = s(\phi)s(\phi)X^{-2} = s(\phi)c,$$

which acts by $q^{-(\lambda,\lambda + 2\rho)}$ on irreducible representations whose highest weight lies in the root lattice, and by $-q^{-(\lambda,\lambda + 2\rho)}$ otherwise. In this paper, we work with this latter, nonstandard, ribbon element $\nu = X^{-2}$ and the corresponding pivotal structure.
We now further detail the above in the case \( \mathfrak{g} = \mathfrak{sp}_6 \) of most interest in this work. We have

\[
(a_{ij}) = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}
\]

and \( d_1 = d_2 = 1, d_3 = 2 \). Fixing elements \( \{ \varepsilon_i \}_{i=1}^3 \) in the dual of the Cartan that satisfy \( (\varepsilon_i, \varepsilon_j) = \delta_{i,j} \), the simple roots are given by \( \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = 2\varepsilon_3 \), and the fundamental weights are \( \omega_1 = \varepsilon_1, \omega_2 = \varepsilon_1 + \varepsilon_2, \omega_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \). It follows that \( \rho = 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 \).

All (type I) finite-dimensional irreducible representations of \( U_q(\mathfrak{sp}_6) \) take the form \( \Gamma_{a,b,c} := \Gamma_{a\omega_1 + b\omega_2 + c\omega_3} \) for \( a, b, c \in \mathbb{N} \), and we abbreviate

\[
V := \Gamma_{\omega_1}, \quad W := \Gamma_{\omega_2}, \quad U := \Gamma_{\omega_3}.
\]

We record here that

\[
V \otimes V \cong \Gamma_{2,0,0} \oplus W \oplus \mathbb{C}(q),
\]

\[
V \otimes W \cong \Gamma_{1,1,0} \oplus U \oplus V,
\]

\[
V \otimes U \cong \Gamma_{1,0,1} \oplus W,
\]

which, in particular, imply that every irreducible representation appears as a summand of \( V^{\otimes n} \) for some \( n \), and the parity of this \( n \) is uniquely determined by the irreducible representation.

By Remark 2.1, the nonstandard ribbon element \( X^{-2} \) acts on the irreducible representation \( \Gamma_{\lambda} \) by \( (-1)^n q^{-(\lambda, \lambda + 2\rho)} \), where \( \Gamma_{\lambda} \subset V^{\otimes n} \). The corresponding group-like element \( g \) acts on \( \Gamma_{\lambda} \) by \( (-1)^n K^{2\rho} := (-1)^n K_1^K_{10} K_2 K_3 \), with \( n \) as before (here, we use that \( 2\rho = 6\alpha_1 + 10\alpha_2 + 6\alpha_3 \)). Because \( K_1 \) acts on the \( \mu \)-weight space of \( \Gamma_{\lambda} \) as \( q^{(\alpha_i, \mu)} \), this implies that

\[
\dim_q(V) = -(q^6 + q^4 + q^2 + q^{-2} + q^{-4} + q^{-6}) = - \frac{[3][8]}{[4]},
\]

\[
\dim_q(W) = q^{10} + q^8 + q^6 + q^4 + 2q^2 + 2 + 2q^{-2} + q^{-4} + q^{-6} + q^{-8} + q^{-10}
\]

\[
= \frac{[7][8]}{[4]},
\]

\[
\dim_q(U) = -(q^{12} + q^8 + q^6 + 2q^4 + q^2 + 2 + q^{-2} + 2q^{-4} + q^{-6} + q^{-8} + q^{-12})
\]

\[
= - \frac{[6][7][8]}{[2][3][4]}.
\]

Note that these formulae can also be obtained via the quantum Weyl dimension formula (see, e.g., [12]).

### 2.2 Web categories

Recall from above that Kuperberg [23] extended the Temperley–Lieb description of \( \text{Rep}(U_q(\mathfrak{sl}_2)) \) to \( U_q(\mathfrak{g}) \) for \( \mathfrak{g} \) of rank 2. We now summarize his construction in the case \( \mathfrak{g} = \mathfrak{sp}_4 \), as it is the most relevant to the present work. To be precise, we actually
work with a category equivalent to Kuperberg’s, in which his trivalent vertex has been rescaled by $\sqrt{-1}$.

The type $C_2$ spider, denoted here by $\text{Web}(\mathfrak{sp}_4)$, is obtained from the $\mathbb{C}(q)$-linear ribbon category pivotally generated by self-dual objects $\{1, 2\}$ and the morphism

$$
\begin{array}{c}
\begin{array}{c}
\mathbf{1} \\
\mathbf{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathbf{2}
\end{array}
\end{array}
\in \text{Hom}_{\text{Web}(\mathfrak{sp}_4)}(1 \otimes 1, 2)
$$

by taking the quotient by the tensor-ideal generated by the following relations:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
&= -\frac{[2][6]}{[3]}, \\
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
&= \frac{[5][6]}{[2][3]}, \\
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathbf{1}
\end{array}
\end{array}
&= 0, \\
\begin{array}{c}
\begin{array}{c}
\mathbf{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathbf{2}
\end{array}
\end{array}
&= 0.
\end{align*}
$$

As in these relations, we will follow the conventions that we will not label the (co)domain in our morphisms, electing instead to color our web edges. For the duration, black denotes $1$-labeled edges and blue denotes $2$-labeled edges. Furthermore, as can be inferred from the above, we read all webs as mapping from bottom to top.

Denote the full subcategory of $\text{Rep}(U_q(\mathfrak{sp}_4))$ tensor-generated by the fundamental representations by $\text{FundRep}(U_q(\mathfrak{sp}_4))$. Kuperberg’s results then give the following.

**Theorem 2.2** There is an equivalence of pivotal categories $\text{Web}(\mathfrak{sp}_4) \cong \text{FundRep}(U_q(\mathfrak{sp}_4))$.

Indeed, this is the $\mathfrak{g} = \mathfrak{sp}_4$ case of his more-general result, while holds for all rank 2 Lie algebras (i.e., additionally for $\mathfrak{sl}_3$ and $\mathfrak{g}_2$). Furthermore, he extends this result to an equivalence of ribbon categories by giving explicitly formulae for the braidings in the web categories. As a consequence, Kuperberg obtains an explicit construction of the colored $U_q(\mathfrak{g})$ link invariant in the spirit of the Kauffman bracket formulation of the Jones polynomial, which is thus amenable to combinatorial study.

2.3 Type $\mathfrak{sp}_6$ webs and statement of results

We now take the first steps toward giving a web-based description of $\text{Rep}(U_q(\mathfrak{g}))$ in type $C$ and higher rank. We focus on the case $\mathfrak{g} = \mathfrak{sp}_6$, but believe our results should extend to higher rank as well. Specifically, we define a category $\text{Web}(\mathfrak{sp}_6)$ and conjecture that it gives a description of the category of quantum $\mathfrak{sp}_6$ representations (see Conjecture 2.5 below). At the time of writing, this remains a conjecture; however, we provide ample evidence for its validity in Theorems 2.6–2.10 below.
**Definition 2.1** Let $\text{Web}(\mathfrak{s}_6)$ be the strict pivotal $\mathbb{C}(q)$-linear category generated by the self-dual objects \{1, 2, 3\} and with morphisms generated by

\[
\begin{align*}
\vcenter{\hbox{\includegraphics[scale=0.5]{web1}}}, & \quad \in \text{Hom}_{\text{Web}(\mathfrak{s}_6)}(1 \otimes 1, 2), \\
\vcenter{\hbox{\includegraphics[scale=0.5]{web2}}}, & \quad \in \text{Hom}_{\text{Web}(\mathfrak{s}_6)}(1 \otimes 2, 3)
\end{align*}
\]

modulo the tensor-ideal generated by the following relations:

\[
\begin{align*}
\vcenter{\hbox{\includegraphics[scale=0.5]{relations1}}}, & = -\frac{[3][8]}{[4]}, \\
\vcenter{\hbox{\includegraphics[scale=0.5]{relations2}}}, & = 0, \\
\vcenter{\hbox{\includegraphics[scale=0.5]{relations3}}}, & = [2][3], \\
\vcenter{\hbox{\includegraphics[scale=0.5]{relations4}}}, & = [2][3], \\
\vcenter{\hbox{\includegraphics[scale=0.5]{relations5}}}, & = [2][3]
\end{align*}
\]

In these relations, and for the duration, we extend our color conventions by denoting 3-labeled web edges in **green**. (Later, we also depict edges with arbitrary labels in thick **gray**.) Furthermore, we employ the shorthand

\[
\begin{align*}
\vcenter{\hbox{\includegraphics[scale=0.5]{shorthand1}}}, & := \frac{1}{[2][3]}
\end{align*}
\]

**Remark 2.3** The reader will notice that, with the exception of the final relation, these relations agree with the $n = 3$ case of the relations in [28]. Indeed, as the proof of Theorem 3.1 shows, up to rescaling our generating morphisms, these relations are the only ones possible between the constituent webs. Our choice of normalization (which differs from that taken in a previous incarnation of this work [48]) was greatly influenced by the notes [33], which, in particular, correct a typo in [28, Lemma 2.7] that makes their “square relation” agree with the above.

**Remark 2.4** We record the following useful consequences of the above:

\[
\begin{align*}
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences1}}}, & = \frac{[7][8]}{[4]}, \\
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences2}}}, & = -\frac{[6][7][8]}{[2][3][4]}, \\
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences3}}}, & = 0, \\
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences4}}}, & = -[2][7], \\
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences5}}}, & = [6][7][3], \\
\vcenter{\hbox{\includegraphics[scale=0.5]{consequences6}}}, & = -[6]
\end{align*}
\]

We posit the following extension of Theorem 2.2.
Conjecture 2.5 There is an equivalence of ribbon categories \( \text{Web}(\mathfrak{sp}_e) \cong \text{FundRep}(U_q(\mathfrak{sp}_e)) \).

While Conjecture 2.5 remains open at the moment, in the subsequent sections of this paper, we prove a number of results concerning \( \text{Web}(\mathfrak{sp}_e) \) that serve as strong evidence for its validity. Explicitly, we show the following.

**Theorem 2.6** The category \( \text{Web}(\mathfrak{sp}_e) \) is ribbon, and there is a full, essentially surjective, braided, monoidal functor \( \Psi : \text{Web}(\mathfrak{sp}_e) \to \text{FundRep}(U_q(\mathfrak{sp}_e)) \).

Thus, the outstanding portion of Conjecture 2.5 is to show that the functor \( \Psi \) is faithful, or, equivalently, to show that for objects \( \tilde{k}, \tilde{\ell} \) in \( \text{Web}(\mathfrak{sp}_e) \), we have

\[
\dim \left( \text{Hom}_{\text{Web}(\mathfrak{sp}_e)}(\tilde{k}, \tilde{\ell}) \right) \leq \dim \left( \text{Hom}_{\mathfrak{sp}_e}(\Psi(\tilde{k}), \Psi(\tilde{\ell})) \right).
\]

Recall that \( V \) denotes the standard representation of \( U_q(\mathfrak{sp}_e) \). It is well known that \( \text{End}_{\mathfrak{sp}_e}(V^\otimes k) \) is a quotient of the \( k \)-strand Birman–Murakami–Wenzl (BMW) algebra \( \text{BMW}_k(r, z) \) [6, 16, 30], after specializing parameters (see Section 3 below). We show the analogue of this result for \( \text{Web}(\mathfrak{sp}_e) \).

**Theorem 2.7** There is an algebra homomorphism \( \text{BMW}_k(-q^7, q-q^{-1}) \to \text{End}_{\text{Web}(\mathfrak{sp}_e)}(1^{\otimes k}) \).

Because all Hom-spaces in \( \text{Web}(\mathfrak{sp}_e) \) can be identified with subspaces of \( \text{End}_{\text{Web}(\mathfrak{sp}_e)}(1^{\otimes k}) \) for some \( k \), we suspect that equation (4), and thus Conjecture 2.5, can be deduced by showing that the map in Theorem 2.7 is surjective, and comparing its kernel to that of \( \text{BMW}_k(1^{\otimes k}) \).

Although we do not prove (4) in general (this will be pursued further in [7]), we show that it holds for the endomorphism algebra of the monoidal unit, and that the left-hand side is indeed finite (which is not guaranteed a priori):

**Theorem 2.8** \( \text{End}_{\text{Web}(\mathfrak{sp}_e)}(\emptyset) \cong \mathbb{C}(q) \). Furthermore, for all objects \( \tilde{k}, \tilde{\ell} \) in \( \text{Web}(\mathfrak{sp}_e) \), the \( \mathbb{C}(q) \)-vector space \( \text{Hom}_{\text{Web}(\mathfrak{sp}_e)}(\tilde{k}, \tilde{\ell}) \) is finite-dimensional.

This first statement is equivalent to the fact that every closed web in \( \text{Web}(\mathfrak{sp}_e) \) can be evaluated to an element of \( \mathbb{C}(q) \). Our proof provides an explicit algorithm for this evaluation.

Another possible approach to the resolution of Conjecture 2.5 is via work of Tuba and Wenzl [50]. Therein, they classify the semisimple braided monoidal categories whose Grothendieck ring agrees with that of \( \text{Rep}(U_q(\mathfrak{sp}_{2n})) \), in terms of the eigenvalues of the braiding. The Grothendieck ring of a semisimple linear category \( \mathcal{C} \) agrees with its categorical trace \( \text{Tr}(\mathcal{C}) \) (also known as its zeroth Hochschild–Mitchell homology). The latter is amenable to diagrammatic methods, as it has an interpretation as the corresponding skein module of the annulus. To this end, we show the following.

**Theorem 2.9** The functor \( \Psi \) induces an isomorphism \( \text{Tr}(\text{Web}(\mathfrak{sp}_e)) \cong \text{Tr}(\text{Rep}(U_q(\mathfrak{sp}_e))) \) of commutative \( \mathbb{C}(q) \)-algebras.

Using the aforementioned results of Tuba and Wenzl, Conjecture 2.5 would follow from showing that all endomorphism algebras in \( \text{Web}(\mathfrak{sp}_e) \) are semisimple \( \mathbb{C}(q) \)-algebras.
As a final piece of supporting evidence for Conjecture 2.5, we note that restricting along the inclusion $U_q(\mathfrak{sp}_4) \to U_q(\mathfrak{sp}_6)$ gives a functor $\text{Rep}(U_q(\mathfrak{sp}_6)) \to \text{Rep}(U_q(\mathfrak{sp}_4))$. In [48], it is shown that there is an analogous functor between web categories, that we now state. Specifically, let $\mathbb{C}^\oplus$ denote the additive closure of a category $\mathbb{C}$, then we have the following.

**Theorem 2.10** There is a functor $\text{Web}(\mathfrak{sp}_6) \to \text{Web}(\mathfrak{sp}_4)^\oplus$.

**Remark 2.11** Our interest in a graphical description for $\text{Rep}(U_q(\mathfrak{sp}_{2n}))$ is partially motivated by considerations in low-dimensional topology. Indeed, given any simple Lie algebra $\mathfrak{g}$, there is a corresponding Reshetikhin–Turaev invariant $P_\mathfrak{g}(\mathcal{L}) \in \mathbb{Z}[q,q^{-1}]$ of framed links $\mathcal{L} \subset S^3$, defined using the ribbon structure on $\text{Rep}(U_q(\mathfrak{g}))$ (see [39]). A graphical description for $\text{Rep}(U_q(\mathfrak{g}))$ thus provides a graphical description for the corresponding link polynomials. In type $A$, such descriptions underly generators-and-relations constructions of Khovanov and Khovanov–Rozansky link homology via cobordisms and foams [1, 21, 26, 34]. In turn, deformations of these foam/cobordism 2-categories [3, 42] play a crucial role in the definition and study of associated concordance invariants and slice-genus lower bounds [24, 25, 38, 53]. As such, we view a graphical description of $\text{Rep}(U_q(\mathfrak{g}))$ as the first step in a long program to provide topological applications for link homology theories outside type $A$. Such homological invariants of links are known to exist by work of Webster [52].

To this end, we note that Theorem 2.8 (or Theorem 2.9) implies that $\text{Web}(\mathfrak{sp}_6)$ indeed gives a generators-and-relations graphical setting for the $\mathfrak{sp}_6$ quantum link polynomial. To wit, given a link diagram (or braid closure, if using Theorem 2.9), explicit formulae assign to it a $\mathbb{C}(q)$-linear combination of webs, which can be evaluated to an element in $\mathbb{C}(q)$ using the aforementioned theorems. As usual, a skein-theoretic argument shows that this invariant actually takes values in $\mathbb{Z}[q,q^{-1}] \subset \mathbb{C}(q)$. As such, we believe that a manifestly integral version of Theorem 2.8 should hold. See Section 5 for full details.

### 3 The functor from $\text{Web}(\mathfrak{sp}_6)$ to $\text{FundRep}(U_q(\mathfrak{sp}_6))$

In this section, we define a functor $\Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6))$ and prove that it is full and essentially surjective. Along the way, we endow $\text{Web}(\mathfrak{sp}_6)$ with the structure of a ribbon category that is compatible with the ribbon structure on $\text{FundRep}(U_q(\mathfrak{sp}_6))$ and establish a relation between $\text{Web}(\mathfrak{sp}_6)$ and the BMW algebra. As such, we deduce Theorems 2.6 and 2.7 above.

We begin by defining the functor $\Psi$. A direct approach—e.g., by presenting $\text{Web}(\mathfrak{sp}_6)$ by generators and relations as a monoidal category, explicitly specifying the images of generating morphisms, and checking monoidally generating relations—is possible, but tedious (see [48] for the relevant formulae). To avoid the numerous computations, such an approach would require, we instead proceed via an indirect, more-conceptual approach that exploits the ribbon structure on $\text{Rep}(U_q(\mathfrak{sp}_6))$.

**Theorem 3.1** There exists an essentially surjective functor $\Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6))$. 
Proof Let $\text{FundRep}^+(U_q(\mathfrak{sp}_6))$ denote the full subcategory of $\text{Rep}(U_q(\mathfrak{sp}_6))$ monoidally generated by the fundamental representations and their duals. By the coherence theorem for pivotal categories (see, e.g., [4] or [32]), this is equivalent, as a pivotal category, to a strict pivotal category. Furthermore, because we work with the nonstandard pivotal structure from Remark 2.1, all nonzero Frobenius–Schur indicators are $+1$. This implies that the full subcategory $\text{FundRep}(U_q(\mathfrak{sp}_6))$ inherits the structure of a strict pivotal category with self-duality structure, in the sense of [45]. The latter consists of a choice of coherent isomorphism $X \cong X^*$ for all objects $X$ in $\text{FundRep}(U_q(\mathfrak{sp}_6))$ that is compatible with the pivotal structure.

Next, equation (1) implies that both $\text{Hom}_{\mathfrak{sp}_6}(V \otimes V, W)$ and $\text{Hom}_{\mathfrak{sp}_6}(V \otimes W, U)$ are one-dimensional. Choosing a nonzero morphism in each of these Hom-spaces, it follows (e.g., from the results in [46]) that there exists a pivotal functor $\Psi$ from the strict pivotal category freely generated by self-dual objects $\{1, 2, 3\}$ and the morphisms

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
\]

to $\text{FundRep}(U_q(\mathfrak{sp}_6))$ that sends $1 \mapsto V$, $2 \mapsto W$, and $3 \mapsto U$, and sends these webs to the chosen morphisms. By construction, $\Psi$ is essentially surjective.

It remains to show that $\Psi$ descends to the quotient obtained by imposing the relations in (3). First, note that

\[
\Psi\left(\begin{array}{c}
\end{array}\right) = 0 \quad \text{and} \quad \Psi\left(\begin{array}{c}
\end{array}\right) = 0,
\]

because $\text{Hom}_{\mathfrak{sp}_6}(W, \mathbb{C}(q))$ and $\text{Hom}_{\mathfrak{sp}_6}(U \otimes U, W)$ are both trivial, and that

\[
\Psi\left(\begin{array}{c}
\end{array}\right) = \delta \cdot \text{id}_W,
\]

for some $\delta \in \mathbb{C}(q)$, because $\text{Hom}_{\mathfrak{sp}_6}(W, W) \cong \mathbb{C}(q)$. Furthermore, equation (2) implies that

\[
\begin{array}{ccc}
\frac{[3][8]}{4} &\frac{[7][8]}{4} &\frac{[6][7][8]}{2[3][4]}
\end{array}
\]

Equation (1) implies that the images under $\Psi$ of

\[
\begin{array}{ccc}
\end{array}
\]

are linearly independent, hence give a basis for $\text{End}_{\mathfrak{sp}_6}(V \otimes V)$. This, together with the self-duality structure, implies that there exist $\alpha \in \mathbb{C}(q)$ and $\gamma \in \{-1, 1\}$, so that

\[
\Psi\left(\begin{array}{c}
\end{array}\right) + \gamma \cdot \Psi\left(\begin{array}{c}
\end{array}\right) = \alpha \cdot \left(\Psi\left(\begin{array}{c}
\end{array}\right) + \gamma \cdot \Psi\left(\begin{array}{c}
\end{array}\right)\right).
\]
Taking a braid-like closure, and applying the above relations, we see that
\[ \gamma \delta \begin{bmatrix} 7 & 8 \\ 4 \end{bmatrix} = \alpha \begin{bmatrix} 3 & 8 \\ 4 \end{bmatrix} \left( \begin{bmatrix} 3 & 8 \\ 4 \end{bmatrix} - \gamma \right), \]
and simplifying gives
\[ \delta[7] = [3] \alpha \left( \gamma([7] - 1) - 1 \right). \]
Here, we use that \[ [3] [8] \begin{bmatrix} 4 \end{bmatrix} = [7] - 1 \text{ and } \gamma^2 = 1. \]

Next, recall that \( \text{FundRep}(U_q(\mathfrak{sp}_6)) \) inherits a braiding \( \beta \) from \( \text{Rep}(U_q(\mathfrak{sp}_6)) \). It follows that
\[ \beta_{V, V} = \kappa \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \rule[0.5em]{1em}{0.5pt} \end{array} \right) + \lambda \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) + \mu \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right), \]
for some \( \kappa, \lambda, \mu \in \mathbb{C}(q) \). The self-duality structure then implies that
\[ \beta^{-1}_{V, V} = \kappa \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) + \lambda \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) + \mu \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right), \]
\[ = (\kappa + \mu \alpha \gamma) \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) + (\lambda + \mu \alpha) \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) - \mu \gamma \cdot \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right). \]

The equality \( \beta^{-1}_{V, V} \beta_{V, V} = \text{id}_{V \otimes V} \), together with linear independence of the images of (6), implies that
\[ \kappa (\lambda + \mu \alpha) = 1, \]
\[ \kappa (\kappa + \mu \alpha \gamma) + \lambda (\lambda + \mu \alpha) = \frac{[3][8]}{4} \lambda (\kappa + \mu \alpha \gamma), \]
\[ \lambda + \mu \alpha = \gamma (\kappa + \mu \delta). \]
In particular, we deduce that
\[ \kappa \gamma (\kappa + \mu \delta) = 1. \]

Setting
\[ \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} = \kappa \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} + \lambda \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} + \mu \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array}, \]
we have that \footnote{We follow the convention, as in [47], that the ribbon element \( v \) acts as the negative full twist.}
\[ \left( \kappa - \lambda([7] - 1) \right) \text{id}_V = \Psi \left( \begin{array}{c} \rule[0.5em]{1em}{0.5pt} \end{array} \right) = v \text{id}_V = -q^{-2(\omega_1 \cdot \omega_1 + \omega_1 \omega_2)} \text{id}_V = -q^{-1} \text{id}_V; \]
thus,
\[ \lambda = \frac{\kappa + q^{-7}}{[7] - 1}. \]
Similarly, we compute that
\[ \nu|_W \cdot \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right) = \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right) = \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right) = \nu|_V^{-2} (\kappa + \mu \delta)^2 \cdot \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right), \]

and because \( \nu|_W = q^{-12} \), this gives
\[ (12) \quad \kappa + \mu \delta = \pm q^{-1}. \]

Equations (10) and (12) then imply that
\[ \kappa = \pm q. \]

We further claim that we have \( \kappa \neq -q \). Indeed, if \( \kappa = -q \), then (11) implies that \( \lambda|_{q=1} = 0 \) and \( \mu \alpha|_{q=1} = -1 \) by (9). Equation (9) also gives that \( \mu \delta|_{q=1} = 1 - \gamma \). Multiplying (8) by \( \mu \) and evaluating at \( q = 1 \) then give that
\[ 7(1 - \gamma) = -3(6\gamma - 1), \]
which implies that \( \gamma = -4/11 \), contradicting that \( \gamma = \pm 1 \).

Thus, we have that \( \kappa = q \). Equation (11) then implies that
\[ \lambda = q^{-3} \frac{(q^4 + q^{-4})[4]}{[3][8]} = \frac{q^{-3}}{[3]}, \]
so
\[ \mu \alpha = \kappa^{-1} - \lambda = q^{-1} - \frac{q^{-3}}{[3]} = \frac{2}{[3]}. \]

Equation (9) then implies that \( \mu \delta = \gamma q^{-1} - q \), so, in particular, \( \mu \delta|_{q=1} = \gamma - 1 \). Multiplying (8) by \( \mu \) and evaluating at \( q = 1 \) as above then give
\[ 7(\gamma - 1) = 2(6\gamma - 1), \]
so \( \gamma = -1 \). This, in turn, implies that \( \mu \delta = -[2] \) and thus \( \delta = -[3] \alpha \).

Hence, if we can deduce the value of \( \delta \), we will have identified all unknown coefficients and deduced that all relations in (3) that do not involve 3-labeled edges hold (We have also already deduced one relation involving a 3-labeled edge holds.) Note that we do not expect to explicitly identify \( \delta \) at this point; indeed, there is some flexibility in our choice for this parameter, because changing our choice of \( \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right) \) will change \( \delta \) by the square of an element in \( \mathbb{C}(q) \). We thus can compute \( \delta \) by choosing \( M \in \text{Hom}_{sp_n}(V \otimes V, W) \) and explicitly comparing the maps
\[ \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right) \text{ and } \Psi \left( \begin{array}{c} \gamma \\ \cdot \end{array} \right). \]

However, we now note that we have a choice in the value of the latter. Indeed, we can rescale the unit morphism for \( W \) by any nonzero element in \( \mathbb{C}(q) \), provided we also
rescale its counit by the inverse. Hence, we can choose any nonzero value for \( \delta \), and we let\(^2\) \( \delta = [2][3] \). It then follows that \( \alpha = -[2] \) and \( \mu = -\frac{1}{[3]} \).

Next, because \( \text{Hom}_{sp}(V^{\otimes 3}, U) \) is one-dimensional, we must have

\[
\Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = \tau \cdot \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right),
\]

for some \( \tau \), and an explicit computation verifies that \( \tau \neq 0 \). We thus set

\[
\vdash := \tau^{-1},
\]

which implies that

(13) \[
\Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right).
\]

Given this, we then compute that

\[
\Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = [2][3] \cdot \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right),
\]

so \( \tau = [2][3] \), as desired.

Because \( \text{End}_{sp}(U) \) is one-dimensional, we have

\[
\Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = \delta' \cdot \text{id}_U,
\]

and we can determine \( \delta' \) as before. Indeed, because we can rescale the image of a 3-labeled cup (and/or the image of the \((1, 2) \to 3\) trivalent vertex), we are free to choose any value for \( \delta' \), and we again set\(^3\) \( \delta' = [2][3] \).

For our final two relations, observe that equation (1) implies that the images of

\[
\begin{array}{c}
\text{ } \\
\end{array}
\]

give a basis for \( \text{End}_{sp}(V \otimes W) \). This implies that

(14) \[
\Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) = a \cdot \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) + b \cdot \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right) + c \cdot \Psi \left( \begin{array}{c}
\text{ } \\
\end{array} \right),
\]

\(^2\)Note that the different choice of \( \delta = -[2][3] \) is used in [48], which follows from the choices of (co)unit used there.

\(^3\)In [48], the different choice of \( \delta' = -[3]^2 \) is used.
for some \(a, b, c \in \mathbb{C}(q)\). Closing the 1-labeled strand and using the relations

\[
\Psi\left( \left\langle \right\rangle \right) = -[2][7] \cdot \text{id}_V, \quad \Psi\left( \left\langle \right\rangle \right) = -[6] \cdot \text{id}_W
\]

(which themselves follow by observing that \(\text{End}_{\text{sp}}(V)\) and \(\text{End}_{\text{sp}}(U)\) are one-dimensional, taking a closure, and applying (5)), we find that

\[
[2]^2[7] = \frac{[8]}{[4]} a - [2][7] b + \frac{[6]}{[3]} c.
\]

Similarly, composing (14) with \(\Psi\left( \left\langle \right\rangle \right)\) and using the relation

\[
\Psi\left( \left\langle \right\rangle \right) = [4] \cdot \Psi\left( \left\langle \right\rangle \right)
\]

(which itself follows by composing (7) with \(\Psi\left( \left\langle \right\rangle \right)\)), we obtain the equation

\[
[4]^2 = a - [2][7] b,
\]

and composing (14) with \(\Psi\left( \left\langle \right\rangle \right)\) gives

\[
[2]^2[3]^2 = a + [2][3] c.
\]

Thus, we have that

\[
a = [3]^2, \quad b = -\frac{1}{[2]}, \quad c = \frac{[3]^2}{[2]},
\]

as desired.

Finally, similar glueing/closure arguments show that

\[
\Psi\left( \left\langle \right\rangle \right), \quad \Psi\left( \left\langle \right\rangle \right), \quad \Psi\left( \left\langle \right\rangle \right)
\]

are linearly independent, thus span \(\text{Hom}_{\text{sp}}(V \otimes W, W \otimes V)\). As above, this implies that we must have a relation of the form

\[
\Psi\left( \left\langle \right\rangle \right) = x \cdot \Psi\left( \left\langle \right\rangle \right) + y \cdot \Psi\left( \left\langle \right\rangle \right) + z \cdot \Psi\left( \left\langle \right\rangle \right),
\]

for \(x, y, z \in \mathbb{C}(q)\). Splitting the 2-labeled endpoints (by composing with trivalent vertices), applying (13), and using the self-duality structure, we find that \(z = \pm 1\). Furthermore, composing with \(\Psi\left( \left\langle \right\rangle \right)\) and simplifying show that \(y = -zx\), so we have

\[
(15) \quad \Psi\left( \left\langle \right\rangle \right) = x \cdot \left( \Psi\left( \left\langle \right\rangle \right) - z \cdot \Psi\left( \left\langle \right\rangle \right) \right).
\]
Next, splitting the bottom-right 2-labeled endpoint, glueing it to the bottom-left endpoint, and simplifying give the equation

\[ x = \frac{[6]}{[2][7] + z[4]}; \]

hence, the proof is complete once we verify that \( z = 1 \).

To do so, we first note that

\[ \Psi \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) = \Psi \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right), \]

which follows from (14). Using this, we see that composing (15) with \( \Psi \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) \) and simplifying give a multiple of (14) when \( z = 1 \) and a contradictory equation when \( z = -1 \). \( \blacksquare \)

This proof suggests that we can define a braided structure on \( \text{Web}(\mathfrak{sp}_6) \) by setting

\[
\beta_{1,1} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle = q \left\langle \begin{array}{c} + q^{-3} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle \quad \text{and} \quad \beta_{1,1}^{-1} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle = q^{-1} \left\langle \begin{array}{c} + q^3 \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle.
\]

The proof of Theorem 3.1 shows that we indeed have \( \beta_{1,1}^{-1} \beta_{1,1} = 1 \). Furthermore, naturality implies that we must then set

\[
\beta_{1,2} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle, \quad \beta_{1,3} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle.
\]

\[
\beta_{2,1} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle, \quad \beta_{2,2} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle, \quad \beta_{2,3} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle.
\]

\[
\beta_{3,1} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle, \quad \beta_{3,2} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle, \quad \beta_{3,3} := \left\langle \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle := \frac{1}{[2][3]} \left\langle \begin{array}{c} \bigcap \frac{1}{[3]} \bigcup \frac{1}{[3]} \end{array} \right\rangle.
\]

These assignments then determine \( \beta_{k,l}^{-1} \) for \( k, l \in \{1, 2, 3\} \) using the pivotal structure. Explicit formulae for the crossings in (17) can be found in Section 5.

**Theorem 3.2** The formulae in equations (16) and (17) endow \( \text{Web}(\mathfrak{sp}_6) \) with the structure of a ribbon category.
**Proof** To begin, we extend the definition of $\beta$ to all objects $\vec{k} = (k_1, \ldots, k_m)$ and $\vec{l} = (l_1, \ldots, l_n)$ in $\text{Web}(\mathfrak{sp}_6)$ by setting

$$\beta_{\vec{k}, \vec{l}} := \begin{array}{c}
\vdots \\
\vdots \\
\hline
k_1 \\
\cdots \\
\hline
k_m \\
\cdots \\
\hline
l_1 \\
\cdots \\
\hline
l_n
\end{array},$$

which also determines $\beta^{-1}$ using the pivotal structure. To see that $\beta$ indeed gives a braiding on $\text{Web}(\mathfrak{sp}_6)$, it suffices to see that $\beta$ is natural (with respect to morphisms in $\text{Web}(\mathfrak{sp}_6)$) and that the braid relations are satisfied.

The former follows via explicit computations that show that

$$\beta = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array},$$

For example, a computation shows that

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = \frac{1}{[2]} - \frac{q}{[3]} - \frac{q^{-2}}{[2][3]}$$

which gives the first relation in (18) for these colors, by composing with $\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array}$.

The braid relations then follow from the 1-labeled case, i.e., from the relations

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array},$$

The first (Reidemeister II) of these relations holds, because the coefficients in (16) satisfy (9), whereas the second (Reidemeister III) is given as follows:

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = q + \frac{q^{-3}}{[3]} - \frac{1}{[3]}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} = q + \frac{q^{-3}}{[3]} - \frac{1}{[3]} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\hline
\end{array}
\end{array}.$$
Finally, a computation shows that

\[
\begin{align*}
\mathcal{B} &= -q^7 \quad \text{and} \quad \mathcal{B} = -q^7;
\end{align*}
\]

hence,

\[
\begin{align*}
\mathcal{B} &= -q^7 \quad \text{and} \quad \mathcal{B} = -q^7;
\end{align*}
\]

so \( \text{Web}(\mathfrak{sp}_6) \) is ribbon.

Furthermore, it is clear from the proof of Theorem 3.1 that the functor \( \Psi \) defined therein is ribbon. Thus, Theorem 2.6 will follow once we have shown that \( \Psi \) is full.

To aid in this task, we first establish the relation between \( \text{Web}(\mathfrak{sp}_6) \) and the BMW algebra \([6, 30]\). We now recall its definition, following the conventions from \([16]\).

**Definition 3.1** The \( k \)-strand BMW algebra \( \text{BMW}_k(r, z) \) is the unital associative \( \mathbb{C}(q) \)-algebra generated by \( e_i, g_i, g_i^{-1} \) for \( 1 \leq i \leq k - 1 \), with relations:

1. \( g_i - g_i^{-1} = z(1 - e_i) \);
2. \( e_i^2 = (1 + \frac{r - r^{-1}}{z}) e_i \);
3. \( g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \) for \( 1 \leq i \leq k - 2 \);
4. \( g_i g_j = g_j g_i \) for \( |i - j| > 1 \);
5. \( e_i e_{i+1} e_i = e_i \) and \( e_{i+1} e_i e_{i+1} = e_{i+1} \) for \( 1 \leq i \leq k - 2 \);
6. \( g_i g_{i+1} e_i = e_{i+1} e_i \) and \( g_i g_{i+1} e_{i+1} = e_i e_{i+1} \) for \( 1 \leq i \leq k - 2 \);
7. \( e_i e_i = 1 \);
8. \( e_{i+1} e_i = e_i \).

It is known that the BMW algebra is “quantum Schur–Weyl” dual to \( U_q(\mathfrak{sp}_{2n}) \); see \([16]\) and references therein. In particular, there is a surjective homomorphism

\[
\text{BMW}_k(-q^7, q - q^{-1}) \to \text{End}_{\mathfrak{sp}_n}(V^\otimes k).
\]

We show a partial analogue for \( \text{Web}(\mathfrak{sp}_6) \).

**Proposition 3.3** There is a homomorphism \( \text{BMW}(-q^7, q - q^{-1}) \to \text{End}_{\text{Web}(\mathfrak{sp}_6)}(V^\otimes k) \) and (21) factors through this.

**Proof** The homomorphism in (21) is defined using the braiding and (co)unit morphisms in \( \text{Rep}(U_q(\mathfrak{sp}_6)) \); thus, it suffices to show that analogous formulae determine a homomorphism to \( \text{End}_{\text{Web}(\mathfrak{sp}_6)}(V^\otimes k) \), i.e., that

\[
\begin{align*}
g_i &\mapsto \begin{array}{c}
\cdots \bigspadesuit \cdots,
\end{array} \\
g_i^{-1} &\mapsto \begin{array}{c}
\cdots \heartsuit \cdots,
\end{array} \\
e_i &\mapsto \begin{array}{c}
\cdots \bigcirc \cdots.
\end{array}
\end{align*}
\]
gives a homomorphism. (Here, the crossings and caps/cups occur between the \( i \)th and \((i + 1)\)st positions.) A direct computation shows that this is indeed the case; we now summarize the computation.

1. Equation (16) implies that
\[
\begin{pmatrix}
\circ & \circ \\
\bullet & \bullet
\end{pmatrix} - \begin{pmatrix}
\bullet & \bullet \\
\circ & \circ
\end{pmatrix} = (q - q^{-1}) \begin{pmatrix}
1 & 0 \\
1 & -1
\end{pmatrix} + \frac{q^{-3} - q^3}{[3]} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = (q - q^{-1}) \left( \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \right).
\]

2. Because \( 1 + \frac{r - r^{-1}}{z} = 1 - [7] = -\frac{[3][8]}{[4]} \), this relation follows by observing that \(-\frac{[4]}{[3][8]} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}\) is an idempotent in \(\text{Web}(\mathfrak{sp}_6)\).

3. This follows from the Reidemeister III move in (19).

4. This holds by a height exchange isotopy in \(\text{Web}(\mathfrak{sp}_6)\), i.e., because this category is monoidal.

5. This holds via (planar) isotopy in \(\text{Web}(\mathfrak{sp}_6)\), i.e., because this category is pivotal.

6. This follows from the Reidemeister II move in (19) and planar isotopy.

7. This and relation (8) hold by (20). \(\square\)

We now complete the proof of Theorem 2.6, by showing the following.

**Theorem 3.4** The functor \(\Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6))\) is full.

**Proof** As noted above, the homomorphism \(\text{BMW}_k(-q^7, q - q^{-1}) \to \text{End}_{\mathfrak{sp}_6}(V^\otimes k)\) is known to be surjective. Because this factors through \(\Psi : \text{End}_{\text{Web}(\mathfrak{sp}_6)}(1^\otimes k) \to \text{End}_{\mathfrak{sp}_6}(V^\otimes k)\), this latter homomorphism is surjective as well. Furthermore, the unit/counit morphisms in \(\text{Web}(\mathfrak{sp}_6)\) and \(\text{FundRep}(U_q(\mathfrak{sp}_6))\) give the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Web}(\mathfrak{sp}_6)}(1^\otimes r, 1^\otimes s) & \xrightarrow{\Psi} & \text{Hom}_{\mathfrak{sp}_6}(V^\otimes r, V^\otimes s) \\
\downarrow \cong & & \downarrow \cong \\
\text{End}_{\text{Web}(\mathfrak{sp}_6)}(1^\otimes r + s) & \xrightarrow{\Psi} & \text{End}_{\mathfrak{sp}_6}(V^\otimes r + s)
\end{array}
\]

for any \(r, s \geq 0\) with \(r + s\) even, which gives surjectivity for these Hom-spaces. In addition, note that if \(r + s\) is odd, then
\[
\text{Hom}_{\text{Web}(\mathfrak{sp}_6)}(1^\otimes r, 1^\otimes s) = 0,
\]

because the parity of the sum of the entries in the domain and codomain are equal for all morphisms in \(\text{Web}(\mathfrak{sp}_6)\). Because \(\text{Hom}_{\mathfrak{sp}_6}(V^\otimes r, V^\otimes s) = 0\) for \(r + s\) odd as well, we have thus shown that
\[
\Psi : \text{Hom}_{\text{Web}(\mathfrak{sp}_6)}(1^\otimes r, 1^\otimes s) \to \text{Hom}_{\mathfrak{sp}_6}(V^\otimes r, V^\otimes s)
\]
is surjective for all \(r, s \geq 0\).
It remains to extend this to the remaining Hom-spaces in \( \mathbf{Web}(\mathfrak{sp}_6) \). Let \( \vec{k} = (k_1, \ldots, k_m) \) and \( \vec{l} = (l_1, \ldots, l_n) \) be objects in \( \mathbf{Web}(\mathfrak{sp}_6) \). Consider the webs

\[
W_b = \bigotimes_{i=1}^{m} W_{b,i} : (k_1, \ldots, k_m) \to 1^{\otimes \Sigma k_i}
\quad \text{and} \quad
W_t = \bigotimes_{j=1}^{n} W_{t,j} : 1^{\otimes \Sigma l_j} \to (l_1, \ldots, l_n)
\]

defined by

\[
W_{b,i} = \begin{cases} 
\text{if } k_i = 1 \\
\text{if } k_i = 2 \\
\text{if } k_i = 3 
\end{cases} 
\quad \text{and} \quad
W_{t,j} = \begin{cases} 
\text{if } l_j = 1 \\
\text{if } l_j = 2 \\
\text{if } l_j = 3 
\end{cases}
\]

We then obtain a \( \mathbb{C}(q) \)-linear map

\[
\text{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes \Sigma k_i}, 1^{\otimes \Sigma l_j}) \xrightarrow{W_{b,i} \circ (-) \circ W_{t,j}} \text{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(\vec{k}, \vec{l}),
\]

which is surjective by the relations in the first line of (3). This, in turn, implies that

\[
\text{Hom}_{\mathfrak{sp}_6}(V^{\otimes \Sigma k_i}, V^{\otimes \Sigma l_j}) \xrightarrow{\Psi(W_i) \circ (-) \circ \Psi(W_b)} \text{Hom}_{\mathfrak{sp}_6}(\Psi(\vec{k}), \Psi(\vec{l})),
\]

is surjective as well, and the result then follows from the diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes \Sigma k_i}, 1^{\otimes \Sigma l_j}) & \xrightarrow{\Psi} & \text{Hom}_{\mathfrak{sp}_6}(V^{\otimes \Sigma k_i}, V^{\otimes \Sigma l_j}) \\
\downarrow W_{b,i} \circ (-) \circ W_{t,j} & & \downarrow \Psi(W_i) \circ (-) \circ \Psi(W_b) \\
\text{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(\vec{k}, \vec{l}) & \xrightarrow{\Psi} & \text{Hom}_{\mathfrak{sp}_6}(\Psi(\vec{k}), \Psi(\vec{l}))
\end{array}
\]

\[\blacksquare\]

4 Closed webs and ladders

In this section, we prove Theorems 2.8 and 2.9. Save for the second statement in Theorem 2.8, these results can be interpreted topologically as saying that closed webs, living in the plane and annulus, respectively, can be expressed in terms of the simplest such webs (the empty web and nested essential circles, respectively). We begin by discussing our approach and introducing the requisite structures.

4.1 Strategy and \( \text{Lad}(\mathfrak{g} \mathfrak{sp}_6) \)

In [44], a web formalism is used to describe morphisms between representations of certain type \( BCD \) coideal subalgebras of \( U_q(\mathfrak{g} \mathfrak{l}_n) \) and to prove quantum analogues of Howe dualities in these types. However, their results suggest that Howe dualities cannot be used (at least in a straightforward way) to give descriptions of \( \text{Rep}(U_q(\mathfrak{g})) \) in types \( BCD \). Our main observation is that several salient structures used in the study of type \( A \) webs can be exploited independently of their connection to Howe duality.
As such, we abandon Howe duality and instead attempt to parallel these aspects of the type A story.

The Cautis–Kamnitzer–Morrison approach to $\text{Web}(\mathfrak{sl}_n)$ proceeds by considering several auxiliary categories that are related to $\text{Rep}(U_q(\mathfrak{sl}_n))$. Specifically, for fixed $n > 0$, they consider the category $\text{Lad}(\mathfrak{gl}_n)$ of “ladder-like” $\mathfrak{gl}_n$-webs, and show that a certain full subcategory $\text{Lad}_m(\mathfrak{gl}_n)$ is equivalent to the quotient of the idempotent form $U_q(\mathfrak{gl}_m)$ by the kernel of the map $U_q(\mathfrak{gl}_m) \to \text{Rep}(U_q(\mathfrak{gl}_n))$ induced via quantum skew Howe duality. (Recall that the latter is the quantum analogue of the duality between $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ induced by their commuting actions on $\wedge^*(\mathbb{C}^m \otimes \mathbb{C}^n)$.) Their construction of $\text{Web}(\mathfrak{sl}_n)$ then follows by analyzing the diagram:

$$
\begin{array}{ccc}
\text{Lad}(\mathfrak{gl}_n) & \longrightarrow & \text{Web}(\mathfrak{gl}_n) \\
\text{Rep}(U_q(\mathfrak{gl}_n)) & \longrightarrow & \text{Rep}(U_q(\mathfrak{sl}_n))
\end{array}
$$

where $\text{Web}(\mathfrak{gl}_n)$ is a $\mathfrak{gl}_n$ analogue of $\text{Web}(\mathfrak{sl}_n)$. Most importantly, for our considerations, there exist explicit proofs of the type A analogues of Theorems 2.8 and 2.9 using properties of $\text{Lad}(\mathfrak{gl}_n)$ that can be formulated without reference to skew Howe duality (see [14, 35, 36]).

This suggests that we should introduce type $C_3$ versions of $\text{Lad}(\mathfrak{gl}_n)$ and $\text{Web}(\mathfrak{gl}_n)$, denoted $\text{Lad}(\mathfrak{gsp}_6)$ and $\text{Web}(\mathfrak{gsp}_6)$, respectively. These categories not only admit functors to $\text{Web}(\mathfrak{sp}_6)$ possessing certain fullness properties, but also have a rigidity of structure that allows for the proof of Theorems 2.8 and 2.9.

**Remark 4.1** The category $\text{Web}(\mathfrak{gsp}_6)$ is not strictly used in this work, i.e., all of our proofs bypass it, using only the relation between $\text{Lad}(\mathfrak{gsp}_6)$ and $\text{Web}(\mathfrak{sp}_6)$. However, $\text{Web}(\mathfrak{gsp}_6)$ serves to motivate the definition of $\text{Lad}(\mathfrak{gsp}_6)$, which otherwise would seem much more ad hoc, so we briefly (and informally) discuss it now.

Paralleling the type A case, the category $\text{Web}(\mathfrak{gsp}_6)$ is related to the category of representations of $\mathfrak{gsp}_6$. The latter is the Lie algebra of the Lie group

$$GSp(6) = \{ A \in M_6(\mathbb{C}) \mid A^TJA = \lambda_A J \},$$

where $J \in M_n(\mathbb{C})$ is nondegenerate and skew symmetric, and $\lambda_A \in \mathbb{C}$. The tensor square of the vector representation no longer contains the trivial representation as a summand, but rather an irreducible one-dimensional representation corresponding to $A \mapsto \lambda_A$. We thus append a new generating objection $0'$ to the generators of $\text{Web}(\mathfrak{sp}_6)$, and let $\text{Web}(\mathfrak{gsp}_6)$ be generated by objects $\{0', 1, 2, 3\}$ and morphisms

\[
\begin{array}{cccc}
\begin{array}{c|c}
0' & 1 \\
\hline
1 & 2 \\
\end{array} &
\begin{array}{c|c}
1 & 2 \\
\hline
1 & 3 \\
\end{array} &
\begin{array}{c|c}
2 & 1 \\
\hline
1 & 3 \\
\end{array}
\end{array}
\]

\footnote{Rigid in the colloquial, noncategory theoretic sense.}
together with vertical and horizontal reflections thereof. We then impose analogues of the Web(\(\mathfrak{sp}_6\)) relations, e.g.,

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} &= -\frac{[3][8]}{[4]},
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} = 0, \\
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} &= [2][3], \\
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} &= [2][3], \\
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} &= [2][3].
\end{align*}
\]

(22)

The astute reader will note that rescaling both of the generating morphisms

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} \quad \text{and} \quad \\
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}
\end{align*}
\]

by \(\sqrt{-1}\) removes all minus signs from the above relations. As such, save for a troubling denominator, Web(\(\mathfrak{gsp}_6\)) seems ripe for categorification. We plan a detailed study of Web(\(\mathfrak{gsp}_6\)), in the context of type C link homologies, in future work. Note, however, that the Web(\(\mathfrak{sp}_6\)) relation

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} - \begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} = [3]\left(\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}\right)
\end{align*}
\]

has no obvious analogue in Web(\(\mathfrak{gsp}_6\)). We would like to impose the relation

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} - \begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture} = [3]\left(\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\coordinate (A) at (0,0);
\coordinate (B) at (0,-1);
\coordinate (C) at (1,0);
\coordinate (D) at (1,-1);
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}\right)
\end{align*}
\]

(23)

as an analogue, but this requires vertices, which are not (yet) defined in Web(\(\mathfrak{gsp}_6\)). Thus, the above informal definition of Web(\(\mathfrak{gsp}_6\)) should be expanded to include them. (The use of such vertices is also suggested by the following.)

We now introduce the \(\mathfrak{gsp}_6\) ladder category. To motivate the definition, note that we would like to define morphisms in this category to be Web(\(\mathfrak{gsp}_6\)) webs written in the ladder form of [11]. We would then impose ladder analogues of the relations in
However, we eventually wish to employ PBW style arguments to prove Theorems 2.8 and 2.9, where negatively and positively sloped “ladder rungs” play the roles of the Chevalley generators $E$ and $F$ in the PBW theorem for $\mathfrak{sl}_2$, respectively. The second term on the right-hand side of relation (24) is then problematic. However, we note that the “middle slice” of this term corresponds to the object $1 \otimes 0'$ and, by introducing a new object $1' \equiv 1 \otimes 0'$ (together with further generating morphisms and relations), we can replace this by the ladder

After doing so, relation (24) is amenable to PBW style arguments. Similar considerations, and equation (23) above, suggest that we should also introduce new objects $k^{(i)}$ that correspond to the tensor product $k \otimes 0' \otimes \cdots \otimes 0'$. 

**Definition 4.1** The category $\text{Lad}(\mathfrak{gsp}_6)$ is the $\mathbb{C}(q)$-linear monoidal category with objects generated by:

$$
\{0^{(i)}, 1^{(j)}, 2^{(s)}, 3^{(t)} \mid i, j, s, t \geq 0\}.
$$

Defining the **mass** of an object by $\mu(k^{(i)}) = k + 2i$, the morphisms are then generated by the **rung morphisms**:

$$
\begin{align*}
&\begin{array}{c}
x^{(s)} \quad y^{(t)} \\
a^{(i)} & b^{(j)}
\end{array} \\
&\begin{array}{c}
y^{(t)} \quad x^{(s)} \\
b^{(j)} & a^{(i)}
\end{array}
\end{align*}
$$

and

$$
\begin{align*}
&\begin{array}{c}
x^{(s)} \quad y^{(t)} \\
\quad c^{(r)}
\end{array} \\
&\begin{array}{c}
y^{(t)} \quad x^{(s)} \\
\quad c^{(r)}
\end{array}
\end{align*}
$$

where the labels satisfy all of the following conditions:

1. **Rung mass**: $\mu(c^{(r)}) > 0$;
2. **Upper vertex**: exactly one of the following holds for the elements $a, c, x$:
   - two are equal and the other is 0,
   - two are 1 and the other is 2, or
   - all are nonzero and distinct;
(3) **Lower vertex:** exactly one of the following holds for the elements $b$, $c$, $y$:
- two are equal and the other is 0,
- two are 1 and the other is 2, or
- all are nonzero and distinct;

(4) **Mass preservation:** $\mu\left(a^{(i)}\right) + \mu\left(c^{(r)}\right) = \mu\left(x^{(t)}\right)$ and $\mu\left(b^{(j)}\right) = \mu\left(y^{(t)}\right) + \mu\left(c^{(r)}\right)$.

We adopt terminology from [11]. Specifically, we will refer to compositions of tensor products of generating morphisms as *ladders*, so morphisms in $\text{Lad}(\text{gsp}_q)$ are $\mathbb{C}(q)$-linear combinations of ladders. The vertical line segments in ladders are called *uprights*, and the segments passing between the uprights are called *rungs*. As indicated therein, the generating rungs in (25) are called $E$-*rungs* and $F$-*rungs*, respectively. Finally, we will write the object (or edge label) $\ell^{(k)}$ as $\ell$ with $k$ primes, when $k$ is small, e.g., $2' = 2^{(1)}$ and $3 = 3^{(0)}$.

The morphisms are subject to relations, all of which take the following forms, or a reflection thereof. Here, we allow some rungs to have zero mass, with the understanding that such a rung is simply the corresponding identity morphism:

- **Rung explosion:** If $\mu\left(c^{(r)}\right) > 2$ or $c^{(r)} = 2^{(0)}$, we have that

$$c^{(r)} = \sum_{i} f_i a_i^{(s_i)} b_i^{(t_i)}$$

with $f_i \in \mathbb{C}(q)$ and $\mu\left(c^{(r)}\right) > \max\left(\mu\left(a_i^{(s_i)}\right), \mu\left(b_i^{(t_i)}\right)\right)$.

- **Rung swap:** For $a^{(s)}$, $b^{(t)} \in \{1, 0'\}$, we have

$$\begin{cases} 
\sum_{i} f_i a_i^{(s_i)} b_i^{(t_i)} + g & \text{if } a^{(s)} = 1 = b^{(t)} \\
1 & \text{if } a^{(s)} = 0' \text{ or } b^{(t)} = 0' 
\end{cases}$$

with $f_i, g \in \mathbb{C}(q)$. 

---

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• **Square relation:** For \( a^{(s)}, b^{(t)} \in \{1, 0\} \), we have

\[
\begin{align*}
\begin{array}{c|c|c|c}
\hline
a^{(s)} & f & + \sum_i g_i x_i^{(e_i)} y_i^{(r_i)} \\hline
\end{array}
\end{align*}
\]

with \( c^{(r)}, d^{(u)} \in \{0, 1\} \) and \( f, g_i \in \mathbb{C}(q) \).

The precise relations (i.e., the values of the above coefficients) depend on the labels and masses of the rungs and uprights, and are recorded in Appendix A. There, we continue our edge-coloring conventions, denoting \( 0^{(i)} \)-labeled edges in black, \( 1^{(i)} \)-labeled edges in blue, and \( 2^{(i)} \)-labeled edges in green. We denote \( 1^{(i)} \)-labeled edges in thin, dashed black, as in Remark 4.1. All such relations are "ladderized" versions of relations that hold in \( \text{Web}(\mathfrak{sp}_6) \).

**Remark 4.2** We content ourselves here with only displaying the general form of the relations in \( \text{Lad}(\mathfrak{osp}_6) \), as the specific forms of the relations are only used to show that the functor in Proposition 4.4 below is well-defined. After that, the general forms of the relations presented above suffice for the proofs of the remaining results from Section 2.3.

**Remark 4.3** Ladders were introduced as a tool in type BCD representation theory in [44], where they are used to prove Howe dualities between quantum groups in these types and certain coideal subalgebras of \( U_q(\mathfrak{gl}_n) \). In that setup, ladders describe morphisms between representations of these coideal subalgebras, rather than the quantum groups. Hence, we do not expect a direct relation to our ladder category, except in the classical \( q \to 1 \) limit. It would be interesting to compare our categories in that case; however, we note that their edge labels correspond to (skew)symmetric tensors, as opposed to fundamental representations.

**Proposition 4.4** There is a monoidal functor \( \Phi : \text{Lad}(\mathfrak{osp}_6) \to \text{Web}(\mathfrak{sp}_6) \) given by

\[
\Phi \left( \begin{array}{cc}
\begin{array}{c}
\hline
a^{(i)} & c^{(r)} \\
\hline
\end{array}
\end{array}
\begin{array}{cc}
\begin{array}{c}
\hline
x^{(i)} & y^{(i)} \\
\hline
\end{array}
\end{array}
\right) = \begin{array}{c}
\hline
x \quad y \quad c \\quad a \\quad b \\
\hline
\end{array}
\]

\[
\text{and} \quad \Phi \left( \begin{array}{cc}
\begin{array}{c}
\hline
b^{(j)} & a^{(i)} \\
\hline
\end{array}
\end{array}
\begin{array}{cc}
\begin{array}{c}
\hline
x^{(s)} & y^{(t)} \\
\hline
\end{array}
\end{array}
\right) = \begin{array}{c}
\hline
y \quad x \\
\hline
\end{array}
\]

**Proof** This follows via a direct computation, which shows that each of the ladder web relations in Appendix A is sent by \( \Phi \) either to a relation in (3), or to a relation easily deduced from these relations.

\[ \blacksquare \]
We now show that all webs in \textbf{Web}(\text{sp}_6) have “preimages” in \textbf{Lad}(\text{gsp}_6), hence can be studied using the latter category. To begin, we have:

\textbf{Lemma 4.5}  Let \( L_1 \in \text{Hom}_{\text{Lad}(\text{gsp}_6)}(\bar{x}, \bar{y}_c) \) and \( L_2 \in \text{Hom}_{\text{Lad}(\text{gsp}_6)}(\bar{y}_d, \bar{z}) \) be ladders such that \( \Phi(\bar{y}_c) = \Phi(\bar{y}_d) \), then there exist ladders \( \bar{L}_1 \in \text{Hom}_{\text{Lad}(\text{gsp}_6)}(\bar{x}, \bar{y}) \) and \( \bar{L}_2 \in \text{Hom}_{\text{Lad}(\text{gsp}_6)}(\bar{y}, \bar{z}) \) so that \( \Phi(\bar{L}_1) = \Phi(\bar{L}_2) \).

Informally, this lemma says that if we have ladders with composable images under \( \Phi \) (but are perhaps not composable themselves), then we can find composable ladders with the same images under \( \Phi \).

\textbf{Proof}  We repeatedly postcompose \( L_1 \) and precompose \( L_2 \) with (mutually inverse) rungs of the form:

\[
\begin{array}{c}
\begin{array}{|c|}
\hline
\cdot \\
\hline
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{|c|}
\hline
\cdot \\
\hline
\end{array}
\end{array}
\]

respectively, to obtain ladders \( \bar{L}_1 \) and \( \bar{L}_2 \) with the codomain of \( \bar{L}_1 \) equal to \( \bar{y}_c = y_1^{(n)} \cdots y_k^{(r_k)} 0^{(i_1)} \cdots 0^{(i_l)} \) and the domain of \( \bar{L}_2 \) equal to \( \bar{y}_d = y_1^{(s_1)} \cdots y_k^{(s_k)} 0^{(j_1)} \cdots 0^{(j_m)} \) with \( y_i \neq 0 \) for all \( i = 1, \ldots, k \). By construction, we have that \( \Phi(\bar{L}_1) = \Phi(\bar{L}_2) \).

Now, if \( l \geq m \), then define \( \bar{L}_2 = \bar{L}_2 \otimes \text{id}_{0^{l-m}} \), which has domain \( \bar{y}_d = y_1^{(s_1)} \cdots y_k^{(s_k)} 0^{(j_1)} \cdots 0^{(j_l)} \) with \( j_t = 0 \) for \( m + 1 \leq t \leq l \). Finally, we define \( \bar{L}_1 \) and \( \bar{L}_2 \) as follows. For \( 1 \leq i \leq k \), add \( |r_i - s_i| \) to the exponent of every label in the \( i \)th upright of \( \bar{L}_1 \) if \( r_i < s_i \), and in the \( i \)th upright of \( \bar{L}_2 \) otherwise. Similarly, for \( 1 \leq t \leq l \), add \( |i_t - j_t| \) to the exponent of every label in the \( (k + t) \)th upright of \( \bar{L}_1 \) if \( i_t < j_t \), and in the \( (k + t) \)th upright of \( \bar{L}_2 \) otherwise. It follows that the codomain of \( \bar{L}_1 \) and the domain of \( \bar{L}_2 \) are both equal to \( \bar{y} = y_1^{(\max(r_1,s_1))} \cdots y_k^{(\max(r_k,s_k))} 0^{(\max(i_1,j_1))} \cdots 0^{(\max(i_l,j_l))} \), and \( \Phi(\bar{L}_1) = \Phi(\bar{L}_2) \) by construction. The case \( l \leq m \) can be handled similarly.

\textbf{Example 4.6}  Let

\[
L_1 = \quad L_2 =
\]

where we omit the explicit exponents in the (co)domains. Note that \( L_2 \circ L_1 \) is not defined, whereas \( \Phi(L_2) \circ \Phi(L_1) \) is defined. Lemma 4.5 then produces the composable ladders

\[
\bar{L}_1 = \quad \bar{L}_2 =
\]

with the same image under \( \Phi \) as \( L_1 \) and \( L_2 \).

\textbf{Corollary 4.7}  Let \( \mathcal{W} \) be a web in \textbf{Web}(\text{sp}_6), then there exists a ladder \( L_\mathcal{W} \) in \textbf{Lad}(\text{gsp}_6) with \( \Phi(L_\mathcal{W}) = \mathcal{W} \).
Proof  By applying a planar isotopy, we can assume that all trivalent vertices and horizontal tangencies in \( W \) occur at distinct heights. The result then follows by inductively applying Lemma 4.5, using the fact that \( \Phi \) is monoidal and that for all \( a, b, c \), we can find \( i, j, s, t \), so that

\[
\Phi \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \right) = a, \quad \Phi \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \right) = b, \quad \Phi \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \right) = c,
\end{array}
\right)
\]

4.2 Closed web evaluation

In this section, we show that every closed web, i.e., every endomorphism of the monoidal unit (empty sequence) in \( \text{Web}(\mathfrak{sp}_6) \), is equal to a \( \mathbb{C}(q) \)-multiple of the empty web. We begin with the following observation.

Lemma 4.8  Any ladder \( L \in \text{Lad}(\mathfrak{gsp}_6) \) can be expressed as a \( \mathbb{C}(q) \)-linear combination of ladders in which no E-rungs appear above F-rungs.

Proof  Repeated use of the run explosion relation (26) expresses \( L \) as a linear combination of ladders in which every rung is 1- or 0′-labeled. The rung swap relation (27) and square relation (28) can then be inductively used to express each ladder in terms of ladders in which fewer F-rungs appear below E-rungs. □

Theorem 4.9  \( \text{End}_{\text{Web}(\mathfrak{sp}_6)}(\varnothing) = \mathbb{C}(q) \).

Proof  Existence of the functor \( \Psi : \text{Web}(\mathfrak{sp}_6) \to \text{FundRep}(U_q(\mathfrak{sp}_6)) \) implies that

\[
\text{dim}_{\mathbb{C}(q)} \left( \text{End}_{\text{Web}(\mathfrak{sp}_6)}(\varnothing) \right) \geq 1,
\]

thus it suffices to show the opposite inequality, i.e., that every closed web \( W \in \text{End}_{\text{Web}(\mathfrak{sp}_6)}(\varnothing) \) is equal to a multiple of the empty web.

Given such \( W \), Corollary 4.7 gives a ladder \( \widetilde{W} \) with \( \Phi(\widetilde{W}) = W \), and further, by pre- and postcomposing with 0(1)-labeled rungs, we can assume that

\[
\widetilde{W} \in \text{End}_{\text{Lad}(\mathfrak{gsp}_6)}(0^{\otimes k} \otimes 0^{\otimes l}),
\]

where \( 0^{\otimes k} = 0' \cdots 0' \) and \( 0^{\otimes l} = 0 \cdots 0 \). We induct on the minimal such \( k \) for which such \( \widetilde{W} \) exists, noting that all morphisms in

\[
\text{End}_{\text{Lad}(\mathfrak{gsp}_6)}(0 \cdots 0)
\]

are multiples of the identity (i.e., the empty ladder).
For the inductive step, suppose that \( \tilde{\mathcal{W}} \in \text{End}_{\text{Lad}(\mathfrak{osp}_k)}(0' \otimes k \otimes 0^\otimes \ell) \) for \( k \geq 1 \). Lemma 4.8 then implies that

\[
\tilde{\mathcal{W}} = \sum_{i=1}^{m} f_i(q) \tilde{\mathcal{W}}_i,
\]

where, in each \( \tilde{\mathcal{W}}_i \), no E-rungs appear above any F-rungs. Because the first entry of both the domain and codomain is \( 0' \), this implies that every label on the left-most upright of \( \tilde{\mathcal{W}}_i \) takes the form \( k^{(i)} \) for \( i > 0 \).

Consider the web \( \tilde{\mathcal{W}}_{i}^{(-1,\delta)} \in \text{End}_{\text{Lad}(\mathfrak{osp}_k)}(0 \otimes 0' \otimes k^{-1} \otimes 0^\otimes \ell) \) that is obtained from the web \( \tilde{\mathcal{W}}_i \) by replacing every label \( k^{(i)} \) on the left-most upright by the label \( k^{(i-1)} \). Finally, let

\[
\tilde{\mathcal{W}}_{i} \in \text{End}_{\text{Lad}(\mathfrak{osp}_k)}(0' \otimes k^{-1} \otimes 0^\otimes \ell+1)
\]

be obtained from \( \tilde{\mathcal{W}}_{i}^{(-1,\delta)} \) by repeatedly pre- and postcomposing with rungs of the form

\[
\begin{array}{c}
\text{and} \\
\text{and}
\end{array}
\]

respectively. Note that \( \Phi(\tilde{\mathcal{W}}_i) = \Phi(\tilde{\mathcal{W}}_{i}^{(-1,\delta)}) = \Phi(\tilde{\mathcal{W}}_i) \). We then have

\[
\mathcal{W} = \Phi(\tilde{\mathcal{W}}) = \sum_{i=1}^{m} f_i(q) \Phi(\tilde{\mathcal{W}}_i) = \sum_{i=1}^{m} f_i(q) \Phi(\tilde{\mathcal{W}}_i) \in \mathbb{C}(q),
\]

as desired. \( \blacksquare \)

**Example 4.10** To exhibit the web evaluation given by the proof of Theorem 4.9, we apply the steps to a lift of the web

\[
\mathcal{W} = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw[thick, blue] (0,0) -- (1,0);
  \draw[thick, blue] (1,0) -- (2,0);
  \draw[thick, blue] (2,0) -- (1,1);
  \draw[thick, blue] (1,1) -- (0,1);
\end{tikzpicture}
\]

as follows:

\[
\tilde{\mathcal{W}} = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw[thick, blue] (0,0) -- (1,0);
  \draw[thick, blue] (1,0) -- (2,0);
  \draw[thick, blue] (2,0) -- (1,1);
  \draw[thick, blue] (1,1) -- (0,1);
\end{tikzpicture} = -[6]
\]

remove priming on left upright

\[
\tilde{\mathcal{W}}^{(-1,\delta)} = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw[thick, blue] (0,0) -- (1,0);
  \draw[thick, blue] (1,0) -- (2,0);
  \draw[thick, blue] (2,0) -- (1,1);
  \draw[thick, blue] (1,1) -- (0,1);
\end{tikzpicture}
\]

pre- and post-compose

\[
\tilde{\mathcal{W}} = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw[thick, blue] (0,0) -- (1,0);
  \draw[thick, blue] (1,0) -- (2,0);
  \draw[thick, blue] (2,0) -- (1,1);
  \draw[thick, blue] (1,1) -- (0,1);
\end{tikzpicture}
\]

In the first step, we applied a relation from Appendix A. Using further such relations, we find that \( \tilde{\mathcal{W}} \) is identical to \( \tilde{\mathcal{W}}^{(-1,\delta)} \), but with the (invisible) 0-labeled upright now
on the right. Denoting this ladder by $\widetilde{\mathcal{V}}$, we now iterate:

$$\begin{align*}
\widetilde{\mathcal{V}} &\rightarrow \underbrace{\mathcal{V}(-1,0)}_{\text{remove priming on left upright}} \rightarrow \vdots \rightarrow \underbrace{\mathcal{V}(0,\delta)}_{\text{pre- and post-compose}} \rightarrow \underbrace{\mathcal{V}(-1,0)}_{\text{compose}} \rightarrow \vdots
\end{align*}$$

Thus, we have that $\mathcal{W} = \Phi(\widetilde{\mathcal{W}}) = -[6] \Phi(\mathcal{V}) = -[3][6][8][4]$.

Theorem 4.9 can be equivalently formulated as saying that the linear map

$$\text{End}_{\text{Lad}(gsp)}(\emptyset) \rightarrow \text{End}_{\text{Web}(sp)}(\emptyset)$$

is surjective. In fact, a variation of the above proof gives the following.

**Proposition 4.11** Let $k_i \in \{1, 2, 3\}$ for $1 \leq i \leq m$, then

$$\text{End}_{\text{Lad}(gsp)}(k_1 \cdots k_m) \rightarrow \text{End}_{\text{Web}(sp)}(k_1 \cdots k_m)$$

is surjective.

**Proof** The argument closely parallels the proof of Theorem 4.9. Let $\mathcal{W} \in \text{End}_{\text{Web}(sp)}(k_1 \cdots k_m)$ be a web, then we can find $\widetilde{\mathcal{W}} \in \text{End}_{\text{Lad}(gsp)}(0^{\otimes k} \otimes k_1 \cdots k_m \otimes 0^{\otimes \ell})$, so that $\Phi(\widetilde{\mathcal{W}}) = \mathcal{W}$. Again using Lemma 4.8, we can write

$$\widetilde{\mathcal{W}} = \sum_{i=1}^{m} f_i(q) \widetilde{W}_i$$

with no $E$-rungs appearing above any $F$-rungs in each $\widetilde{W}_i$. The same procedure as above then implies that

$$\mathcal{W} = \sum_{i=1}^{m} f_i(q) \Phi(\widetilde{W}_i)$$

with each $\widetilde{W}_i \in \text{End}_{\text{Lad}(gsp)}(0^{\otimes k-1} \otimes k_1 \cdots k_m \otimes 0^{\otimes \ell+1})$. Repeating this, we find that

$$\mathcal{W} = \sum_{j=1}^{p} g_j(q) \Phi(\widetilde{W}_j)$$

with each $\widetilde{W}_j \in \text{End}_{\text{Lad}(gsp)}(k_1 \cdots k_m \otimes 0^{\otimes r})$. Furthermore, again using Lemma 4.8, we can assume that in each $\widetilde{W}_j$ has no $E$-rungs appearing above any $F$-rungs. However, this then implies that the webs $\widetilde{W}_j$ take the form

$$\widetilde{W}_j = \widetilde{W}_j \otimes \text{id}_{0^{\otimes r}}$$

with $\widetilde{W}_j \in \text{End}_{\text{Lad}(gsp)}(k_1 \cdots k_m)$, and the result follows, because

$$\mathcal{W} = \sum_{j=1}^{p} g_j(q) \Phi(\widetilde{W}_j).$$

∎
Proposition 4.11 suggests a clear strategy for the resolution of Conjecture 2.5: we should deduce the bound in equation (4) by finding explicit bases for the endomorphism algebras in $\text{Lad}(\mathfrak{sp}_n)$ and comparing dimensions to those in $\text{Rep}(U_q(\mathfrak{sp}_n))$. This approach will be pursued in follow-up work [7].

**Corollary 4.12** For all objects $\tilde{k}, \tilde{\ell}$ in $\text{Web}(\mathfrak{sp}_n)$, the $C(q)$-vector space $\text{Hom}_{\text{Web}(\mathfrak{sp}_n)}(\tilde{k}, \tilde{\ell})$ is finite-dimensional.

**Proof** Recall from the proof of Theorem 3.4 that there is a surjective linear map

$$\text{Hom}_{\text{Web}(\mathfrak{sp}_n)}(1^\otimes \Sigma k_i, 1^\otimes \Sigma \ell_j) \xrightarrow{W_i \circ W_k} \text{Hom}_{\text{Web}(\mathfrak{sp}_n)}(\tilde{k}, \tilde{\ell})$$

and an isomorphism

$$\text{Hom}_{\text{Web}(\mathfrak{sp}_n)}(1^\otimes \Sigma k_i, 1^\otimes \Sigma \ell_j) \cong \text{End}_{\text{Web}(\mathfrak{sp}_n)}(1^\otimes \frac{1}{2}(\Sigma k_i + \Sigma \ell_j)).$$

The result then follows from Proposition 4.11, because Lemma 4.8 shows that $\text{End}_{\text{Lad}(\mathfrak{sp}_n)}(1^\otimes \frac{1}{2}(\Sigma k_i + \Sigma \ell_j))$ is spanned by ladders in which no $E$-rungs appear above any $F$-rungs. It is easy to see that there are only finitely many such ladders. ■

### 4.3 Decategorification of $\text{Web}(\mathfrak{sp}_n)$

In this section, we compute a “decategorification” of $\text{Web}(\mathfrak{sp}_n)$. Typically, the decategorification of an abelian or additive category $\mathcal{C}$ is taken to be the Grothendieck group $K_0(\mathcal{C})$, i.e., the quotient of the free abelian group on the isomorphism classes $[X]$ of objects $X \in \text{Ob}(\mathcal{C})$ by relations corresponding to an appropriate notion of exact sequence. Unfortunately, computing $K_0$ for (the additive closure of) diagrammatic categories, such as $\text{Web}(\mathfrak{sp}_n)$, is typically a difficult endeavor, requiring detailed knowledge of the endomorphism algebras therein.

In [5], the following is proposed as an alternative notion of decategorification.

**Definition 4.2** Let $\mathbb{K}$ be a field and let $\mathcal{C}$ be a $\mathbb{K}$-linear category. The categorical trace of $\mathcal{C}$ is the $\mathbb{K}$-vector space

$$\text{Tr}(\mathcal{C}) := \bigoplus_{X \in \text{Ob}(\mathcal{C})} \text{End}_{\mathcal{C}}(X) \big/ (fg - gf),$$

where $f \in \text{Hom}_\mathcal{C}(X, Y)$ and $g \in \text{Hom}_\mathcal{C}(Y, X)$ range over all $X, Y \in \text{Ob}(\mathcal{C})$.

We will denote the equivalence class in $\text{Tr}(\mathcal{C})$ of an endomorphism $h \in \text{End}_\mathcal{C}(X)$ by $\text{tr}(h)$. We now record some standard facts about the categorical trace:

- $\text{Tr}(\text{Kar}(\mathcal{C})) = \text{Tr}(\mathcal{C})$, where $\text{Kar}(\mathcal{C})$ denotes the Karoubi (i.e., idempotent) completion of $\mathcal{C}$.
- If $\mathcal{C}$ is additive linear, there is a generalized Chern character map $\mathcal{X} : K_0(\mathcal{C}) \to \text{Tr}(\mathcal{C})$ defined by $\mathcal{X}([X]) = \text{tr}(\text{id}_X)$.
- If $\mathcal{C}$ is semisimple, then $\mathcal{X}$ induces an isomorphism $\mathbb{K} \otimes_{\mathbb{Z}} K_0(\mathcal{C}) \to \text{Tr}(\mathcal{C})$.
- If $\mathcal{C}$ is (braided) monoidal, then $\text{Tr}(\mathcal{C})$ is a (commutative) algebra, with product induced via tensor product, and $\mathcal{X}$ is an algebra homomorphism.

Combining these facts, we observe that

$$\text{Tr}(\text{FundRep}(U_q(\mathfrak{sp}_n))) \cong \text{Tr}(\text{Rep}(U_q(\mathfrak{sp}_n))) \cong C(q) \otimes_{\mathbb{Z}} K_0(\text{Rep}(U_q(\mathfrak{sp}_n)))$$

$$\cong C(q)[\chi_1, \chi_2, \chi_3].$$

(29)
where we have used the standard identification of the representation ring $K_0(\text{Rep}(U_q(\mathfrak{sp}_6)))$ with polynomials in the classes (or characters) $\chi_i$ of the fundamental representations. We now prove the analogue of this result for $\text{Web}(\mathfrak{sp}_6)$ using the fact that the categorical trace is well-suited to diagrammatically presented pivotal algebra. We now prove the analogue of this result for $\text{Web}(\mathfrak{sp}_6)$ using the fact that the categorical trace is well-suited to diagrammatically presented pivotal algebra.

**Theorem 4.13** There is an isomorphism of algebras

$$\tau : \mathbb{C}(q)[\chi_1, \chi_2, \chi_3] \rightarrow \text{Tr}(\text{Web}(\mathfrak{sp}_6))$$
determined by $\tau(\chi_i) = \text{tr}(\text{id}_i)$.

**Proof** Theorem 3.2 implies that $\text{Web}(\mathfrak{sp}_6)$ is braided, so $\text{Tr}(\text{Web}(\mathfrak{sp}_6))$ is a commutative algebra. The assignment $\tau(\chi_i) = \text{tr}(\text{id}_i)$ then determines an algebra homomorphism $\tau : \mathbb{C}(q)[\chi_1, \chi_2, \chi_3] \rightarrow \text{Tr}(\text{Web}(\mathfrak{sp}_6))$. Furthermore, injectivity of $\tau$ follows from the commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}(q)[\chi_1, \chi_2, \chi_3] & \xrightarrow{\tau} & \text{Tr}(\text{Web}(\mathfrak{sp}_6)) \\
\cong & & \downarrow \text{Tr}(\Psi) \\
K_0(\text{Rep}(U_q(\mathfrak{sp}_6))) & \xrightarrow{\cong} & \text{Tr}(\text{FundRep}(U_q(\mathfrak{sp}_6)))
\end{array}$$

where the composition of the indicated isomorphisms is the inverse of the isomorphism given in (29).

It remains to show surjectivity, i.e., that every class in $\text{Tr}(\text{Web}(\mathfrak{sp}_6))$ can be expressed as a linear combination of classes of identity morphisms. First, note that the functor $\Phi : \text{Lad}(\mathfrak{sp}_6) \rightarrow \text{Web}(\mathfrak{sp}_6)$ induces a homomorphism $\text{Tr}(\Phi) : \text{Tr}(\text{Lad}(\mathfrak{sp}_6)) \rightarrow \text{Tr}(\text{Web}(\mathfrak{sp}_6))$ and Corollary 4.7 implies that $\text{Tr}(\Phi)$ is surjective. It thus suffices to show that every element in $\text{Tr}(\text{Lad}(\mathfrak{sp}_6))$ can be expressed as a linear combination of classes of identity morphisms. To do so, we adapt the argument from [36, Theorem 3.2] to our setting.

To this end, let $L \in \text{End}_{\text{Lad}(\mathfrak{sp}_6)}(a_1^{i_1} \cdots a_m^{i_m})$ be a ladder. Using the rung explosion relation (26), we can assume that all rungs are 1- and 0-labeled, and further, we can assume that all rungs appear at distinct heights. The element $\text{tr}(L)$ then corresponds to a “cyclic sequence” $\tilde{\mu}_1, \ldots, \tilde{\mu}_{\#r(L)}$ of tuples $\tilde{\mu}_i \in \mathbb{N}^m$ that is obtained by taking a horizontal slice in between the rungs of $\text{tr}(L)$ and recording the masses of the corresponding labels. Here, $\#r(L)$ denotes the number$^5$ of rungs in $L$. For example, taking a slice at the “bottom” (or equivalently, the “top”) of $\text{tr}(L)$ gives the tuple

$$\tilde{\mu} = (\mu(a_1^{i_1}), \ldots, \mu(a_m^{i_m})).$$

Note that the sum of the entries in the tuple $\tilde{\mu}_i$ is independent of the choice of $1 \leq i \leq \#r(L)$, hence determines an invariant of $\text{tr}(L)$ that we denote by $\text{Tot}_\mu(L)$.

Now, suppose $\#r(L) \geq 1$ and consider a tuple in $\tilde{\mu}_1, \ldots, \tilde{\mu}_{\#r(L)}$ that is minimal with respect to the lexicographic order on $\mathbb{N}^m$. It follows that the rung immediately “below”

$^5$If $\#r(L) = 0$, the cyclic sequence only has one entry, thus we slightly abuse notation.
the corresponding slice in $\text{tr}(L)$ is an $F$-rung and the rung immediately “above” is an $E$-rung. We thus can apply a rung swap (27) or square (28) relation to express $\text{tr}(L)$ as a linear combination of classes of ladders in which the minimal tuple is strictly larger than that in $\text{tr}(L)$, or in which it appears a fewer number of times.

Repeated application of this procedure expresses $\text{tr}(L)$ as a linear combination of ladders without rungs, i.e., identity ladders. This follows from the following observations:

- We can apply the above procedure, provided $\#r(L) \geq 1$.
- If $\#r(L) = 0$, then $L$ is an identity ladder.
- For a fixed value of $\text{Tot}_\mu(L)$ (which remains constant when we apply rung swap and square relations), for minimal $\bar{\mu}$, sufficiently large, we must have $\#r(L) = 0$.

**Example 4.14** We will apply the proof of Theorem 4.13 to compute the class of

$$W = \begin{array}{c}
\end{array}$$

in $\text{Tr}(\text{Web}(\mathfrak{sp}_6))$. This is particularly enlightening, because $W$ itself cannot be simplified in $\text{Web}(\mathfrak{sp}_6)$. Let

$$L = \begin{array}{c}
\end{array}$$

then we will compute $\text{tr}(L)$ using the trace relation and the relations in Appendix A. In each step of the simplification, we indicate the labels at the lexicographically minimal horizontal slice (all other labels can be inferred from these and the definition of $L$). We have

$$\text{tr}(L) = \text{tr} \left( \begin{array}{c}
\end{array} \right) = \text{tr} \left( \begin{array}{c}
\end{array} \right) = \text{tr} \left( \begin{array}{c}
\end{array} \right)$$

$$= -2[3][6] \text{tr} \left( \begin{array}{c}
\end{array} \right) + \frac{6}{3} \text{tr} \left( \begin{array}{c}
\end{array} \right) + 2[4] \text{tr} \left( \begin{array}{c}
\end{array} \right)$$

$$= -2[3][6] \text{tr} \left( \begin{array}{c}
\end{array} \right) + \frac{6}{3} \text{tr} \left( \begin{array}{c}
\end{array} \right) + 2[4] \text{tr} \left( \begin{array}{c}
\end{array} \right)$$

$$= -2[3][6] \text{tr} \left( \begin{array}{c}
\end{array} \right) + 2[6] \text{tr} \left( \begin{array}{c}
\end{array} \right) - 2[3][8] \text{tr} \left( \begin{array}{c}
\end{array} \right).$$

Thus, we have $\text{tr}(W) = \tau(-2[3][6]\chi^2 + 2[6]\chi - 2[3][8])$. Note that, for the final two elements in the above computation, the minimal $\bar{\mu} = (4, 0)$ is sufficiently
large relative to \( \text{Tot}_\mu(L) = 4 \) (so rungs cannot, and indeed do not, appear in these ladders).

5 \( \mathfrak{sp}_6 \) link invariants

Our results thus far assemble to give an explicit description of the colored \( U_q(\mathfrak{sp}_6) \) invariant of framed links. Recall that in the uncolored case, this link invariant can be recovered skein-theoretically as an appropriate evaluation of the 2-variable Kauffman polynomial \([19]\), or using the \( n = 3 \) case of the state-sum model from \([28]\). We emphasize that our construction has the following important features, that are not present in those formulations:

- It describes the colored \( U_q(\mathfrak{sp}_6) \) link invariant, where link components are colored by fundamental representations. Furthermore, there should exist Jones–Wenzl-like recursions for highest weight projectors in \( \text{Web}(\mathfrak{sp}_6) \) that extend this invariant to links with components colored by arbitrary irreducible representations.
- It is local, i.e., it assigns a linear combination of \( \mathfrak{sp}_6 \) webs to tangles, as well as links. The invariant of a link can thus be computed via the “divide and conquer” approach described in \([2]\), i.e., by splitting the link into constituent tangles, computing and simplifying the invariant of these tangles, then assembling the link invariant from these constituent pieces. See \textit{loc. cit.} for a discussion of the efficiency of this approach.

We now describe the link invariant. First, straightforward (but tedious!) computations show that the crossing formulae from equations (16) and (17) are explicitly given as follows:

\[
\begin{align*}
\times &= q \left( \left( + \frac{q^3}{3} \right) \right) - \frac{1}{3} \\
\times &= \frac{1}{2} - \frac{q}{3} - \frac{q^2}{2[3]} \\
\times &= q^3 \left( \left( + \frac{q^3}{3} \right) \right) - \frac{1}{3} \\
\times &= \frac{q}{2} + \frac{q^{-1}}{2} \\
\times &= \frac{q^2}{2} + \frac{1}{2[3]} \\
\times &= q^3 \left( \left( + \frac{q^3}{3} \right) \right) + \frac{q}{2^2} + \frac{q^{-1}}{2^2} \\
\times &= \frac{1}{[3]} \\
\end{align*}
\]

In these formulae, we use a new trivalent vertex defined as follows:

\[
\begin{align*}
\times &= \frac{1}{[3]} \\
\end{align*}
\]
On webs in quantum type C

Now, suppose that $\mathcal{L} \subset S^3$ is a framed link with components colored by elements in $\{1, 2, 3\}$, let $\mathcal{D}_\mathcal{L}$ be any diagram for $\mathcal{L}$, and let $P_{sp_e}(\mathcal{L})$ be the element of $C(q)$ obtained by applying the formulae in (30) to $\mathcal{D}_\mathcal{L}$ and evaluating the closed webs using Theorem 4.9.

**Proposition 5.1** $P_{sp_e}(\mathcal{L})$ is an invariant of framed colored links (i.e., is independent of the choice of diagram) and is equal to the $U_q(sp_e)$ Reshetikhin–Turaev invariant.

**Proof** Via the Reshetikhin–Turaev construction [39], the link diagram $\mathcal{D}_\mathcal{L}$ determines an endomorphism of the trivial representation $C(q)$ in $\text{Rep}(U_q(sp_e))$, and this scalar is independent of the choice of diagram. The result then follows, because $P_{sp_e}(\mathcal{L})$ is equal to the endomorphism of the monoidal unit $\emptyset$ in $\text{Web}(sp_e)$ determined by $\mathcal{D}_\mathcal{L}$, which is taken to the Reshetikhin–Turaev invariant via the isomorphism

$$\text{End}_{\text{Web}(sp_e)}(\emptyset) \xrightarrow{\Psi} \text{End}_{sp_e}(C(q)).$$

**Remark 5.2** Suppose that $\mathcal{L}$ is 1-colored. The invariant $P_{sp_e}(\mathcal{L})$ can then be computed from a framed link diagram using the skein relation

$$\begin{array}{c}
\begin{array}{c}
\quad - \\
\end{array}
\end{array} = (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\quad - \\
\end{array}
\end{array},$$

together with the relations

$$\begin{array}{c}
\begin{array}{c}
\quad - \\
\end{array} = -q^{7}, \quad \begin{array}{c}
\begin{array}{c}
\quad - \\
\end{array}
\end{array} = -q^{7}, \quad \begin{array}{c}
\begin{array}{c}
\quad - \\
\end{array}
\end{array} = -\frac{3}{8} - 1 = 7.[2].
\end{array}$$

This implies that $P_{sp_e}(\mathcal{L}) \in \mathbb{Z}[q, q^{-1}]$.

We also obtain a refined invariant for framed links in the solid torus/thickened annulus. Indeed, the formulae in (30) assign an element $P^A_{sp_e}(\mathcal{L})$ of the $\text{Web}(sp_e)$ skein algebra of the annulus to the diagram of any such link. Theorem 4.13 shows that this skein algebra is isomorphic to $C(q)[\chi_1, \chi_2, \chi_3]$. Because $\text{Web}(sp_e)$ is ribbon, we immediately have the following.

**Proposition 5.3** $P^A_{sp_e}(\mathcal{L}) \in C(q)[\chi_1, \chi_2, \chi_3]$ is an invariant of framed annular links.

### A Relations in $\text{Lad}(gsmp_6)$

The relations in $\text{Lad}(gsmp_6)$ are as follows, together with those obtained from these via horizontal and vertical reflection.

- **Rung explosion** If $r \geq 1$, we have

  $$\begin{array}{c}
  \quad = \quad = \quad = \quad =
  \end{array}$$
and for $r = 0$ and $c = 2$ or $3$, the relations are recorded in the following tables:

| $w^{(r)}$ | $z^{(r)}$ | $y=0$ | $y=1$ | $y=2$ | $y=3$ |
|-----------|-----------|-------|-------|-------|-------|
| $x=0$     |           |       |       |       |       |
| $x=1$     |           |       |       |       |       |
| $x=2$     |           |       |       |       |       |
| $x=3$     |           |       |       |       |       |
- Rung swap: If $a^{(s)} = 0'$ or $b^{(t)} = 0'$, then

\[
\begin{array}{ccc}
\begin{array}{c}
\text{Rung swap:}
\end{array}
\end{array}
\]

If $j > 0$, then
and otherwise

\[
\begin{align*}
\text{if } d = 2, & \quad x^{(k)} = 1' \\
\text{if } d = 3, & \quad x^{(k)} = 2'
\end{align*}
\]

\[
\begin{align*}
\text{otherwise.}
\end{align*}
\]

- **Square relation:** We have

\[
\begin{align*}
0' & = 0', \\
0' & = 1', \\
0' & = 1
\end{align*}
\]

and the values of

\[
\begin{align*}
w^{(s)} & = z^{(t)} \\
a^{(k)} & = b^{(t)} \\
x^{(i)} & = y^{(j)}
\end{align*}
\]
are recorded in the following tables:

| (x, α, w) = (0, 1, 0) | (0, 1, 2) | (1, 0, 1) | (1, 2, 1) | (1, 2, 3) |
|----------------------|----------|----------|----------|----------|
| (y, h, z) = (0, 1, 0) | \(-\frac{1+i}{4}\) | 0        | \(-2\)   | 0        |
| (0, 1, 2)           | 0        | \([4]\)  |          |          |
| (1, 0, 1)           |          |          |          |          |
| (1, 2, 1)           | \(-2\)   | \(-2\)   | \([4]\)  |          |
| (1, 2, 3)           | 0        | \([4]\)  |          |          |
| (2, 1, 0)           | 0        | \([4]\)  |          |          |
| (2, 1, 2)           | \([2]\)  | \([3]\)  | \([4]\)  |          |
| (2, 3, 2)           | \(-6\)   | \(-6\)   | \([4]\)  |          |
| (3, 2, 1)           | 0        | \([4]\)  |          |          |
| (3, 2, 3)           | \([2]\)  | 0        | \([2]\)  | 0        |
| \((x, \alpha, w) = (0, 1, 0)\) | \((y, b, z) = (2, 1, 0)\) | \((2, 1, 2)\) | \((2, 3, 2)\) | \((3, 2, 1)\) | \((3, 2, 3)\) |
|---|---|---|---|---|---|
| 0 | 0 | \(\frac{[2][3]}{3^3}\) | \(-\frac{[6]}{3}\) | 0 | \(\frac{[2][3]}{3}\) |
| \((0, 1, 2)\) | | | | | |
| \((1, 0, 1)\) | | | \(-\frac{[2]}{3}\) | \(\frac{[3]}{3^3}\) | \(\frac{[6]}{3}\) | |
| \((1, 2, 1)\) | \([4]\) | | \(-\frac{[3][5]}{3}\) | \(\frac{[3]}{3^3}\) | \(\frac{[2]}{3}\) | |
| \((1, 2, 3)\) | | | | | |
| \((2, 1, 0)\) | | | | | |
| \((2, 1, 2)\) | | | | | |
| \((2, 3, 2)\) | | | \(-\frac{[2][4]}{3}\) | \(\frac{[4]}{3^3}\) | \(\frac{[6]}{3}\) | |
| \((3, 2, 1)\) | | | | | |
| \((3, 2, 3)\) | \(0\) | | \(\frac{[2][3]}{3^3}\) | \(-\frac{1}{3}\) | 0 | |

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