COHERENT INFRARED REPRESENTATIONS IN
NON-RELATIVISTIC QED

THOMAS CHEN AND JÜRГ FRÖHLICH

ABSTRACT. We consider dressed 1-electron states in a translation-invariant model of non-relativistic QED. To start with a well-defined model, the interaction Hamiltonian is cutoff at very large photon energies (ultraviolet cutoff) and regularized at very small photon energies (infrared regularization). The infrared regularization is then removed, and the representations of the canonical commutation relations of the electromagnetic field operators determined by the dressed 1-electron states are studied using operator-algebra methods. A key ingredient in our analysis is a bound on the renormalized electron mass uniform in the infrared regularization. Our results have important applications in the scattering theory for infraparticles.

Dedicated to Barry Simon on the occasion of his 60th birthday, in admiration and friendship.

CONTENTS

1. Introduction 2
2. Definition of the model 3
3. Statement of the main Theorems 6
3.1. Estimates on the expected photon number 6
3.2. Infrared representations 7
4. Infraparticle scattering 8
4.1. Step 1: Control of the norm 11
4.2. Step 2: Strong convergence 11
4.3. Modifications of [13] for QED 12
5. Proofs of Theorems 3.1 and 3.2 13
5.1. Proof of Theorem 3.1 14
5.2. Proof of Theorem 3.2, Part 1 14
5.3. Proof of Theorem 3.2, part 2. 14
5.4. Proof of Theorem 3.2, part 3. 15
5.5. Proof of Theorem 3.2, part 4. 16
6. Proof of Proposition 5.1 17
Acknowledgements 22
References 22
1. Introduction

In this note, we consider a translation-invariant model of non-relativistic Quantum Electrodynamics (QED) describing a non-relativistic Pauli (spin $\frac{1}{2}$) electron interacting with the quantized electromagnetic field. An infrared regularization (parametrized by a number $\sigma \ll 1$) and a fixed ultraviolet cutoff are imposed on the interaction Hamiltonian. Let $H(p, \sigma)$ denote the cutoff fiber Hamiltonian corresponding to the conserved momentum $p$ on the fiber Hilbert space $\mathcal{H}_p$. This space is isomorphic to $\mathbb{C}^2 \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the photon Fock space, and $\mathbb{C}^2$ accounts for the spin of the electron. It is proved in [2] and [4] that, for sufficiently small values of the finestructure constant, $H(p, \sigma)$ possesses a ground state eigenvalue $E(p, \sigma)$ (of multiplicity two for spin $\frac{1}{2}$) at the bottom of its essential spectrum. Let $E_{p, \sigma}$ denote the corresponding ground state eigenspace. The unit rays determined by the eigenvectors $\Psi_u(p, \sigma) \in E_{p, \sigma}$, $\|\Psi_u(p, \sigma)\| = 1$, can be parametrized by $u \in S^2 \subset \mathbb{R}^3$, with $\langle \Psi_u(p, \sigma), \tau \Psi_u(p, \sigma) \rangle = u$ ($\tau$ is the vector of Pauli matrices, see (2.18)).

Let $K_\rho := \{ k \in \mathbb{R}^3 | |k| \geq \rho \}$ be the set of photon momenta corresponding to photon energies $\geq \rho$. (We choose units such that $\hbar = c = 1$. The finestructure constant is $\alpha = e^2$.) By $\mathfrak{F}_\rho$, we denote the symmetric Fock space over the one-photon Hilbert space $L^2(K_\rho, d^3k) \otimes \mathbb{C}^2$ of wave functions describing the pure states of a photon of energy $\geq \rho$; the factor $\mathbb{C}^2$ accounts for the two possible polarizations of a photon. Let $\mathcal{B}(\mathfrak{F}_\rho)$ denote the algebra of all bounded operators on $\mathfrak{F}_\rho$. We define a $C^*$-algebra, $\mathcal{A}$, by setting

$$\mathcal{A} := \bigvee_{\rho > 0} \mathcal{B}(\mathfrak{F}_\rho),$$

where the closure is taken in the operator norm. We are interested in the representations of $\mathcal{A}$ determined by dressed 1-electron states via the GNS construction. We define the infrared-regularized states

$$\omega_{p, \sigma}(A) := \langle \Psi_u(p, \sigma), A \Psi_u(p, \sigma) \rangle, \quad A \in \mathcal{A},$$

for a fixed choice of $u \in S^2$. We prove that, for momenta $p$ with $0 \leq |p| < \frac{1}{\sigma}$ and any sequence $\sigma_n \searrow 0$ ($n \to \infty$), there exists a state $\omega_p$ on $\mathcal{A}$ given by $\omega_p(A) = \lim_{n \to \infty} \omega_{p, \sigma_n}(A)$, for all $A \in \mathcal{A}$, for some subsequence $(\sigma_{n_j})$. By the GNS construction, the state $\omega_p$ determines a representation of $\mathcal{A}$. For $p \neq 0$, this representation turns out to be quasi-equivalent to a coherent state representation of $\mathcal{A}$ unitarily inequivalent to the Fock representation. It will be determined explicitly.

For Nelson’s model, similar results were proven in [9, 10]. However, the more complicated coupling structure of the Hamilton operator of non-relativistic QED makes a key argument in [9] inapplicable. The difficulty arises from the fact that the interaction term in QED is of minimal substitution type and hence quadratic in creation- and annihilation operators, while, in Nelson’s model, it is linear. We arrive at our main result by making use of the uniform bounds on the renormalized electron mass recently derived in [4] and [2].

An important application of our results concerns infraparticle scattering theory, in particular Compton scattering. Recently, some significant progress in scattering
theory was made by A. Pizzo in [13], where infraparticle scattering states are constructed for Nelson’s model after a complete removal of the infrared regularization. The proof uses, and significantly extends, ideas proposed in [9, 10]. A bound on the renormalized particle mass uniform in the infrared regularization \( \sigma \geq 0 \) is assumed in [13] without proof.

The construction of an infraparticle scattering state in [9, 13] crucially involves a dressing transformation. To construct the latter, it is necessary to identify a coherent state representation that is quasi-equivalent to the GNS representation determined by \( \omega_p \). This was achieved in [9] for Nelson’s model, but has not been accomplished for non-relativistic QED, due to the difficulties noted above. This is the main reason why attempts to construct an infraparticle scattering theory for non-relativistic QED have been unsuccessful, so far, even after the appearance of Pizzo’s work. With Theorem 3.2 of the present paper, we provide this important missing ingredient. Further modifications necessary to adapt Pizzo’s analysis to non-relativistic QED are outlined, but a detailed discussion of these matters is beyond the scope of the present paper.

2. Definition of the model

We consider an electron of spin \( \frac{1}{2} \) coupled to the quantized electromagnetic field, with a fixed ultraviolet cutoff imposed on the interaction Hamiltonian.

The Hilbert space of one-electron states is given by
\[
\mathcal{H}_{el} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2.
\]
The Fock space of the quantized electromagnetic field in the Coulomb gauge is given by
\[
\mathfrak{F} = \bigoplus_{n \geq 0} \mathfrak{F}^{(n)}, \quad \mathfrak{F}^{(0)} = \mathbb{C},
\]
where the fully symmetrized \( n \)-fold tensor product space
\[
\mathfrak{F}^{(n)} = \text{Sym}_n(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^\otimes n
\]
denotes the \( n \)-photon Hilbert space. The factor \( \mathbb{C}^2 \) accounts for the two transverse polarization modes of a photon, and \( \text{Sym}_n \) symmetrizes the \( n \) factors in the tensor product, in accordance with the fact that photons are bosons.

A vector \( \Phi \in \mathfrak{F} \) corresponds to a sequence
\[
\Phi = (\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n)}, \ldots), \quad \Phi^{(n)} \in \mathfrak{F}^{(n)},
\]
where \( \Phi^{(n)} = \Phi^{(n)}(k_1, \lambda_1, \ldots, k_n, \lambda_n) \), \( k_j \in \mathbb{R}^3 \) is the momentum, and \( \lambda_j \in \{+, -\} \) labels the two possible helicities of the \( j \)-th photon. The scalar product on \( \mathfrak{F} \) is given by
\[
\langle \Phi_1, \Phi_2 \rangle = \sum_{n \geq 0} \langle \Phi_1^{(n)}, \Phi_2^{(n)} \rangle_{\mathfrak{F}^{(n)}}.
\]
Let \( \hat{f} \) denote the Fourier transform of \( f \). For \( \lambda \in \{+, -\} \) and \( f \in L^2(\mathbb{R}^3) \), we introduce annihilation operators
\[
a_\lambda(f) : \mathfrak{F}^{(n)} \to \mathfrak{F}^{(n-1)}
\]
defined by
\[
(a_\lambda(f) \Phi)^{(n-1)}(k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1}) = \sqrt{n} \int dk_n \hat{f}(k_n) \Phi^{(n)}(k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1}, k_n, \lambda)
\] (2.5)

and creation operators
\[
a_\lambda^*(f) : \mathcal{F}^{(n)} \to \mathcal{F}^{(n+1)} , \text{ with } a_\lambda^*(f) = (a_\lambda(f))^* .
\] (2.6)

These operators satisfy the canonical commutation relations
\[
[a_\lambda(f), a_{\lambda'}^*(g)] = (f, g)_{L^2} \delta_{\lambda, \lambda'}
\]
\[
[a_\lambda^x(f), a_{\lambda'}^x(g)] = 0 ,
\] (2.7)

for all \(f, g \in L^2(\mathbb{R}^3)\), where \(a^x\) denotes either \(a_\lambda\) or \(a_\lambda^*\). The Fock vacuum is the unique unit vector
\[
\Omega_f = (1, 0, 0, \ldots)
\] (2.8)
in \(\mathcal{F}\) with the property that
\[
a_\lambda(f) \Omega_f = 0 ,
\] (2.9)

for all \(f \in L^2(\mathbb{R}^3)\).

Since \(a_\lambda^*(f)\) is linear and \(a_\lambda(f)\) is antilinear in \(f\), one can write
\[
a_\lambda^*(f) = \int_{\mathbb{R}^3} dk \ a_\lambda^*(k) \hat{f}(k) , \ a_\lambda(f) = \int_{\mathbb{R}^3} dk \ \hat{f}^*(k) a_\lambda(k)
\] (2.10)

where \(a_\lambda^x(k)\) are operator-valued distributions also referred to as creation- and annihilation operators. They satisfy the commutation relations
\[
[a_{\lambda'}(k'), a_\lambda^x(k)] = \delta_{\lambda, \lambda'} \delta(k - k')
\]
\[
[a_{\lambda'}^x(k'), a_\lambda^x(k)] = 0
\] (2.11)

for all \(k, k' \in \mathbb{R}^3\) and \(\lambda, \lambda' \in \{+, -\}\), and
\[
a_\lambda(k) \Omega_f = 0
\] (2.12)

for all \(k, \lambda\).

The Hilbert space of the system consisting of a single Pauli electron and the quantized radiation field is given by the tensor product space
\[
\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F} .
\] (2.13)

The Hamiltonian is given by
\[
H(\sigma) = \frac{1}{2} \left( i \nabla_x \otimes 1_f - \sqrt{\alpha} A_\sigma(x) \right)^2 + \sqrt{\alpha} \tau \cdot B_\sigma(x) + 1_{el} \otimes H_f ,
\] (2.14)

where
\[
H_f = \sum_{\lambda=1,2} \int dk |k| a_\lambda^*(k) a_\lambda(k)
\] (2.15)
is the free-field Hamiltonian, and

\[
A_\sigma(x) = \sum_{\lambda=+,-} \int \frac{dk}{\sqrt{|k|}} \kappa_\sigma(|k|) \left[ \epsilon_\lambda(k) e^{-ikx} \otimes a_\lambda(k) + h.c. \right]
\]

\[
B_\sigma(x) = \sum_{\lambda=+,-} \int \frac{dk}{\sqrt{|k|}} \kappa_\sigma(|k|) \left[ (-ik) \wedge \epsilon_\lambda(k) e^{-ikx} \otimes a_\lambda(k) + h.c. \right]. \tag{2.16}
\]

denote the (ultraviolet-cutoff) quantized electromagnetic vector potential in the Coulomb gauge, and the magnetic field operator, respectively. The function \(\kappa_\sigma\) imposes an ultraviolet cutoff and an infrared regularization parametrized by \(\sigma \ll 1\). One may choose it to satisfy

\[
\kappa_\sigma(x) = \begin{cases} 
\frac{x}{\sigma} & \text{for } x \in [0, \sigma] \\
1 & \text{for } x \in [\sigma, \frac{1}{2}], C^\infty \text{ and positive}, \text{for } x \in (\frac{1}{2}, 1), \\
0 & \text{for } x > 1. 
\end{cases}
\tag{2.17}
\]

Moreover, \(\tau = (\tau_1, \tau_2, \tau_3)\), with \(\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(\tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\), \(\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), denotes the vector of Pauli matrices. We remark that \(H(\sigma)\) defines a selfadjoint operator on \(H\) bounded from below; see e.g. [14].

For a charged particle of spin 0, the Zeeman term proportional to \(B_\sigma(x)\) is absent.

The momentum operator of the system is given by

\[
P_{\text{tot}} = i \nabla_x \otimes 1_f + 1_{el} \otimes P_f, \tag{2.19}
\]

where

\[
P_f = \sum_{\lambda=+,-} \int dk \, k a_\lambda^*(k) a_\lambda(k) \tag{2.20}
\]
is the momentum operator of the electromagnetic field.

The model under consideration is translation invariant in the sense that

\[
[H(\sigma), P_{\text{tot}}] = 0. \tag{2.21}
\]

We decompose the Hilbert space into a direct integral,

\[
H = \int_{\mathbb{R}^3} dp \, H_p, \tag{2.22}
\]

where \(H_p\), the fiber Hilbert space corresponding to a total momentum \(p\), is isomorphic to \(\mathbb{C}^2 \otimes \mathfrak{g}\). Since \(H_p\) is invariant under \(\text{exp}[-itH(\sigma)]\), we may consider the restriction of \(H(\sigma)\) to \(H_p\),

\[
H(p, \sigma) = H(\sigma) \big|_{H_p} = \frac{1}{2} \left( p - P_f - \sqrt{\alpha} A_\sigma \right)^2 + \sqrt{\alpha} \tau \cdot B_\sigma + H_f, \tag{2.23}
\]

where, henceforth, \(A_\sigma \equiv A_\sigma(0)\) and \(B_\sigma \equiv B_\sigma(0)\).

We will use results established in [4] and [2] on the nature of the infimum of the spectrum of \(H(p, \sigma)\), for \(|p|\) sufficiently small. We define

\[
E(p, \sigma) := \text{infspec}\{H(p, \sigma)\}. \tag{2.24}
\]

The following theorem is proved in [4, 5].
**Theorem 2.1.** Assume that \(|p| < \frac{1}{3}\). There exists a small positive constant \(\alpha_0\) independent of \(\sigma\) such that, for all \(\alpha < \alpha_0\), the following holds: For every \(\sigma > 0\), \(E(p, \sigma)\) is an eigenvalue at the bottom of the essential spectrum, and, by rotation symmetry, is a function only of \(|p|\). The corresponding eigenspace \(E_{p,\sigma}\) has dimension 2 for spin \(\frac{1}{2}\).

The functions \(E(p, \sigma)\), \(\partial_{|p|} E(p, \sigma)\) and \(\partial_{|p|}^2 E(p, \sigma)\) are uniformly bounded in \(\sigma \geq 0\). There is a constant \(c_0 > 0\) independent of \(\sigma\) and \(\alpha\) such that the second derivative

\[
\partial_{|p|}^2 E(p, \sigma) = 1 - 2 \left( \nabla_p \Psi_u(p, \sigma) ; (H(p, \sigma) - E(p, \sigma)) \nabla_p \Psi_u(p, \sigma) \right),
\]

(2.25)

where \(\Psi_u(p, \sigma) \in \mathcal{E}_{p,\sigma}\), \(\|\Psi_u(p, \sigma)\| = 1\), \(u \in S^2\) (see Section 1), satisfies

\[
1 - c_0 \alpha < \partial_{|p|}^2 E(p, \sigma) < 1,
\]

(2.26)

and

\[
\left| E(p, \sigma) - \frac{p^2}{2} - \frac{\alpha}{2} \left( \Omega_f , A^2_{\sigma, \Omega_f} \right) \right| < \frac{c_0 \alpha p^2}{2},
\]

\[
\left| \nabla_p E(p, \sigma) - p \right| < c_0 \alpha |p|.
\]

(2.27)

The renormalized electron mass,

\[
m_{\text{ren}}(p, \sigma) := \frac{1}{\partial_{|p|}^2 E(p, \sigma)},
\]

(2.28)

is bounded by

\[
1 < m_{\text{ren}}(p, \sigma) < 1 + c_0 \alpha,
\]

(2.29)

uniformly in \(\sigma \geq 0\), i.e., the radiative corrections increase the mass of the electron by an amount of \(O(\alpha)\).

In [2], a convergent, finite algorithm is devised to determine \(m_{\text{ren}}(0, 0)\) to any given precision, with rigorous error bounds.

We remark that (2.27) implies that \(\nabla_p E(p, \sigma) = 0\) if and only if \(p = 0\) if and only if \(p = 0\), for all momenta \(p\), with \(|p| < \frac{1}{3}\), all \(\sigma \geq 0\), and \(\alpha\) sufficiently small.

### 3. Statement of the main Theorems

In this paper, we prove accurate upper and lower bounds on the expected photon number in the dressed one-electron state \(\Psi_u(p, \sigma)\) and study the GNS representation determined by \(\Psi_u(p, \sigma)\) in the limit \(\sigma \downarrow 0\), for momenta \(p\) with \(|p| < \frac{1}{3}\).

#### 3.1. Estimates on the expected photon number.

Our first main result is the following theorem:

**Theorem 3.1.** Assume that \(|p| < \frac{1}{3}\), and let

\[
N_f = \sum_\lambda \int dk \ a_\lambda^*(k) a_\lambda(k)
\]

denote the photon number operator. Then, for all \(\alpha < \alpha_0\) (where \(\alpha_0\) is the same constant as in Theorem 2.1), and independently of \(u \in S^2\), the following holds.
For \( p \neq 0 \), so that \( \nabla_p E(p, \sigma) \neq 0 \),

\[
\left( -c \alpha + c' \alpha |\nabla_p E(p, \sigma)|^2 \log \frac{1}{\sigma} \right)_+ \leq \left\langle \Psi_u(p, \sigma), N_f \Psi_u(p, \sigma) \right\rangle \\
\leq C \alpha + C' \alpha |\nabla_p E(p, \sigma)|^2 \log \frac{1}{\sigma}
\] (3.1)

for non-negative constants \( c, C \) and \( 0 < c' < C' \) independent of \( p, \alpha \) and \( \sigma \geq 0 \); (here \( r_+ := \max\{0, r\} \)).

For \( p = 0 \) (with \( \nabla_p E(0, \sigma) = 0 \)),

\[
\left\langle \Psi_u(0, \sigma), N_f \Psi_u(0, \sigma) \right\rangle \leq C \alpha,
\] (3.2)

uniformly in \( \sigma \geq 0 \).

### 3.2. Infrared representations.

For \( \rho > 0 \), let

\[
\mathfrak{A}_\rho := B(\mathfrak{F}_\rho)
\] (3.3)
denote the algebra of bounded operators on

\[
\mathfrak{F}_\rho := \bigoplus_{n \geq 0} \mathfrak{F}^{(n)}_{\rho}
\] (3.4)

where

\[
\mathfrak{F}^{(n)}_{\rho} := \text{Sym}_n(L^2(K_\rho, dk) \otimes \mathbb{C}^2)^{\otimes n},
\] (3.5)

with

\[
K_\rho := \left\{ k \in \mathbb{R}^3 \big| |k| \geq \rho \right\}.
\] (3.6)

As indicated above, we define a \( C^* \) algebra \( \mathfrak{A} \) as the direct limit

\[
\mathfrak{A} := \bigvee_{\rho > 0} \mathfrak{A}_\rho \| \cdot \|,
\] (3.7)

where \( \| \cdot \| \) denotes the closure with respect to the operator norm.

We define a state \( \omega_{p, \sigma} \) on \( \mathfrak{A} \) by

\[
\omega_{p, \sigma}(A) = \left\langle \Psi_u(p, \sigma), A \Psi_u(p, \sigma) \right\rangle,
\] (3.8)

for \( A \in \mathfrak{A} \), corresponding to a vector \( \Psi_u(p, \sigma) \in \mathcal{E}_{p, \sigma} \) (the space of dressed 1-electron states, i.e., ground states of the fiber Hamiltonian \( H(p, \sigma) \)). The choice of \( u \in S^2 \) is arbitrary but fixed; our results will not depend on \( u \).

We prove that, in the limit \( \sigma \searrow 0 \), \( \omega_{p, \sigma} \) tends to a well-defined state, \( \omega_p \), on \( \mathfrak{A} \) which determines a GNS representation that is quasi-equivalent to a coherent state representation.

**Theorem 3.2.** Assume that \( 0 \leq |p| < \frac{1}{3} \), and let \( \omega_{p, \sigma} \) be as defined above. Then, for all \( \alpha < \alpha_0 \) (where \( \alpha_0 \) is the same constant as in Theorem 2.1), the following holds.

1. Let \( \{ \sigma_i \} \) denote an arbitrary sequence with \( \lim_{i \to \infty} \sigma_i = 0 \). Then there exists a subsequence \( \{ \sigma_{i_j} \} \) and a state \( \omega_p \) on \( \mathfrak{A} \) such that

\[
\lim_{j \to \infty} \omega_{p, \sigma_{i_j}}(A) = \omega_p(A),
\] (3.9)
for all $A \in \mathfrak{A}$. The state $\omega_p$ is normal on the subalgebras $\mathfrak{A}_\rho$, for $\rho > 0$.

2. The state $\omega_{p,\sigma}$ satisfies

$$\left| \omega_{p,\sigma}(a_\lambda(k)^* a_\lambda(k)) - |\omega_{p,\sigma}(a_\lambda(k))|^2 \right| \leq c \alpha \frac{\kappa_\sigma^2(|k|)}{|k|^2}$$

(3.10)

uniformly in $\sigma \geq 0$, where $\kappa_\sigma$ is the cutoff function (2.17), and

$$\int dk \left| \omega_p(a_\lambda(k)^* a_\lambda(k)) - |\omega_p(a_\lambda(k))|^2 \right| \leq C \alpha ,$$

(3.11)

in the limit $\sigma \searrow 0$, for some finite constants $c, C$.

3. Let $\pi_p$ denote the representation of $\mathfrak{A}$, $H_{\omega_p}$ the Hilbert space, and $\Omega_p \in H_{\omega_p}$ the cyclic vector corresponding to $(\omega_p, \mathfrak{A})$ by the GNS construction, (with $\omega_p(A) = \langle \Omega_p, \pi_p(A) \Omega_p \rangle$, for all $A \in \mathfrak{A}$). Moreover, let

$$v_{p,\sigma,\lambda}(k) := - \sqrt{\alpha} \varepsilon_\lambda(k) \cdot \nabla_p E(p, \sigma) \frac{\kappa_\sigma(|k|)}{|k|} \frac{1}{|k|^2} \left| k \cdot \nabla_p E(p, \sigma) \right| ,$$

(3.12)

and

$$v_{p,\lambda}(k) := \lim_{\sigma \searrow 0} v_{p,\sigma,\lambda}(k) .$$

(3.13)

Then, $\pi_p$ is quasi-equivalent to $\pi_{Fock} \circ \alpha_p$ (where $\pi_{Fock}$ is the Fock representation of $\mathfrak{A}$), and $\alpha_p$ is the $^*$-automorphism of $\mathfrak{A}$ determined by

$$\alpha_p(a_\lambda^\dagger(k)) = a_\lambda^\dagger(k) + v_{p,\lambda}^\dagger(k) .$$

(3.14)

4. The Fock representation and $\pi_p$ are related to each other as follows.

(i) If $p = 0$

$$| \lim_{\sigma \searrow 0} \omega_0,\sigma(N_f) | < c \alpha ,$$

(3.15)

and $\pi_0$ is (quasi-)equivalent to $\pi_{Fock}$.

(ii) If $p \neq 0$, $\pi_p$ is unitarily inequivalent to the Fock representation, and

$$\lim_{\sigma \searrow 0} \omega_{p,\sigma}(N_f) = \infty .$$

(3.16)

However, $\omega_p$ has the following "local Fock property":

(a) For every $p > 0$, the restriction of $\omega_p$ to $\mathfrak{A}_p$ determines a GNS representation which is quasi-equivalent to the Fock representation.

(b) For every bounded region $B$ in physical $x$-space, the restriction of $\omega_p$ to the local algebra $\mathfrak{A}(B)$ determines a GNS representation which is quasi-equivalent to the Fock representation of $\mathfrak{A}(B)$.

Similar results also hold for a charged particle with spin 0.

4. IFRAPARTICLE SCATTERING

In this section, we comment on the significance and implications of our results for the scattering theory of infraparticles, more precisely Compton scattering, in view of recent work of A. Pizzo.
A framework for an infraparticle scattering theory in Nelson’s model was outlined in [9], and the existence of one-electron scattering states for $\sigma > 0$ was established. The existence of scattering states in the limit $\sigma \downarrow 0$ has only recently been proven by Pizzo for Nelson’s model in [13], using results in [12].

The only unproven hypothesis in [13] is that the renormalized electron mass satisfies $m_{\text{ren}}(p, \sigma) < c$, uniformly in $\sigma > 0$, for $|p| < \frac{1}{20}$ (in our units). Uniform bounds on the renormalized electron mass in non-relativistic QED are proven in [4, 5] and [2], and also hold for Nelson’s model; (but it has to be assumed there that the infrared regularization $\kappa_\sigma(|k|)$ is non-zero in an open neighborhood of $|k| = 0$). The infrared regularization in [13] is implemented by a sharp cutoff $\chi(|k| > \sigma)$, because [13] uses results of [9, 12] (where this choice is technically convenient). Replacing $\chi(|k| > \sigma)$ by $\kappa_\sigma(|k|)$ in [13] can be implemented with minor modifications. The methods of [4, 5] then yield the bound $\partial^2_{|p|^2} E(p, \sigma) > c$, uniformly in $\sigma \geq 0$.

An inequality similar to (3.10) for Nelson’s model plays a central role in [9] and [13], since it explicitly identifies a coherent state representation which is quasi-equivalent to the GNS representation defined by $\omega_p$. This coherent state representation determines the correct choice of a “dressing transformation” for the asymptotic (free) comparison dynamics, which is an essential ingredient for the construction of infraparticle scattering states. With (3.10), we provide such a dressing transformation for non-relativistic QED. However, due to the more complicated structure of the interaction Hamiltonian in non-relativistic QED, as compared to Nelson’s model, there are some additional modifications which we sketch without detailed proofs.

We start by recalling some basic results in [13], but formulated for QED. Let

$$\Sigma := \left\{ p \in \mathbb{R}^3 \left| |p| < \frac{1}{3} \right. \right\}$$

(4.1)

denote the ball of admissible infraparticle momenta (in [13], the bound $|p| < \frac{1}{20}$ is used). Let

$$W_{p, \sigma}(t) := e^{-itH_f} \exp \left[ i \sum_{\lambda = +, -} \Pi_\lambda(u_{p, \sigma, \lambda}) \right] e^{itH_f} ,$$

(4.2)

where $t$ denotes time. The function $u_{p, \sigma, \lambda}(k)$ is defined in (3.12), and $\Pi_\lambda(f) := i(a_\lambda(f) - a_\lambda^*(f))$. As proposed in [9], a natural candidate for the asymptotic, freely moving comparison state is given by

$$\int_{\Sigma} dp \ W_{p, \sigma}(t) e^{-i(p-P_f)x} e^{i\gamma_\sigma(\nabla_p E(p, \sigma), t)} h(p) e^{-itE(p, \sigma)} \Psi_u(p, \sigma) \in \mathcal{H}_{el} \otimes \mathcal{F} .$$

(4.3)

Here, $h(p) e^{-itE(p, \sigma)} \Psi_u(p, \sigma)$ describes a freely moving electron with wave function $h$ (in Schwartz space, and supported in $\Sigma$). The operator $W_{p, \sigma}(t)$ describes a freely time-evolving cloud of physical soft photons surrounding the electron. The integral over $p$ and the factor $e^{-i(p-P_f)x}$ implement the inverse Fourier transform. The purpose of adding a scalar phase factor $\gamma_\sigma(\nabla_p E(p, \sigma), t)$ (which we do not specify in detail here) is similar as in Dollard’s classical construction of modified wave operators for Coulomb scattering, [7]. While the limit $\sigma \downarrow 0$ of the one-electron
states $\Psi_u(p, \sigma)$ does not define vectors in the Fock spaces $\mathcal{H}_p$, the limit $\sigma \searrow 0$ of the vectors (4.3) defines vectors in the physical Hilbert space $\mathcal{H}_{cl} \otimes \mathfrak{F}$.

Next, we sketch the main construction in [13].

In [13], a discretized (Riemann sum) version of (4.3) is used as the free comparison state, where the resolution of the discretization becomes arbitrarily fine, as $t \to \infty$.

Let $T_n^{(\varepsilon)} := 2^n/\varepsilon$, for some $0 < \varepsilon \ll 1$ (to be fixed appropriately). Let

$$ N(t) = \left(2^n\right)^3, \quad \text{for} \quad T_n^{(\varepsilon)} \leq t < T_{n+1}^{(\varepsilon)}.$$

(4.5)

The analysis in [13] assumes that

$$ \partial^2 p E(p, \sigma) = \left(m_{\text{ren}}(p, \sigma)\right)^{-1} > c > 0,$$

(4.6)

uniformly in $\sigma \geq 0$.

A key element of the construction in [13] is to render the infrared cutoff $\sigma_t$ time-dependent, with $\sigma_t$ converging to 0 at a prescribed rate, as $t \to \infty$.

Accordingly, let

$$ \psi_{h, \sigma_t}(t, x) := \sum_{j=1}^{N(t)} \psi_{h, \sigma_t, j}(t, x) $$

(4.7)

with

$$ \psi_{h, \sigma_t, j}(t, x) := e^{itH(\sigma_t)} \int_{\Gamma_j(t)} dp W_{\sigma_t}(V_j, t) e^{-i(p - P_f)x} e^{i\tilde{\gamma}_{\sigma_t}(V_j, \nabla p E(p, \sigma_t), t) h(p)} e^{-itE(p, \sigma_t)} \Psi_u(p, \sigma_t),$$

(4.8)

and

$$ V_j := \nabla p E(p_j, \sigma),$$

(4.9)

where $p_j$ is the center of the cell $\Gamma_j(t)$. Here, $W_{\sigma_t}(V_j, t)$ is defined as the operator obtained after replacing $\nabla p E(p, \sigma)$ by $V_j$ in $W_{p, \sigma_t}(t)$, and $\tilde{\gamma}_{\sigma_t}(V_j, \nabla p E(p, \sigma_t), t)$ is a scalar phase factor.

The main result of [13], formulated for the model of non-relativistic QED studied here, can be stated as follows.

Let

$$ \sigma_t \sim t^{-\beta}, $$

(4.10)

for $\beta > 1$ sufficiently large, and $N(t)$, $T_n^{(\varepsilon)}$ as in (4.5), with $\varepsilon$ sufficiently small. Then the limit

$$ \psi_h^{(out)} = s - \lim_{t \to \infty} \psi_{h, \sigma_t}(t) $$

(4.11)
exists in the one-particle Hilbert space $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$, for the model of non-relativistic QED defined in Section 2. A similar result holds for $t \to -\infty$, yielding a state $\psi_h^{(\text{in})}$.

The vectors $\psi_h^{(\text{in/out})}$ are infraparticle scattering states.

The proof strategy of [13] comprises two main steps, which can be sketched as follows.

4.1. **Step 1: Control of the norm.** This step consists in proving that the norm $\|\psi_{h,\sigma_t}(t)\|_{\mathcal{H}}$ is uniformly bounded in $t$, and that, in fact,

$$\lim_{t \to \infty} \|\psi_{h,\sigma_t}(t)\|_{\mathcal{H}} = \|\mathbf{h}\|_{L^2} .$$  \hspace{1cm} (4.12)

Introducing the matrix elements

$$M_{i,j}(t) := \langle \psi_{h,\sigma_t,i}(t), \psi_{h,\sigma_t,j}(t) \rangle_{\mathcal{H}} ,$$  \hspace{1cm} (4.13)

one easily sees that the sum over diagonal terms, $i = j$, yields the right hand side of (4.12), in the limit $t \to \infty$. The off-diagonal matrix elements are shown to satisfy

$$|M_{i,j}(t)| < c(t) , \quad i \neq j ,$$  \hspace{1cm} (4.14)

where

$$c(t) N(t)^2 \searrow 0 ,$$  \hspace{1cm} (4.15)

as $t \to \infty$, so that

$$\lim_{t \to \infty} \sum_{i \neq j} |M_{i,j}(t)| = 0 .$$  \hspace{1cm} (4.16)

Since the centers of the cells $\Gamma_j(t)$ label distinct asymptotic velocities of infraparticle states, this result implies that, asymptotically, the latter become mutually orthogonal, for $i \neq j$. One uses here dispersive estimates for the free infraparticle propagation, which are derived from the uniform bounds on $\partial_p^2 E(p, \sigma)$, for $\sigma \geq 0$. For further details, see [13].

4.2. **Step 2: Strong convergence.** In this step, one proves that $\{\psi_{h,\sigma_t}(t)\}_{t}$ defines a Cauchy sequence in the one-particle Hilbert space $\mathcal{H}$. To this end, let $t_2 > t_1 \gg 1$. The main result of [13] is an estimate of the form

$$\|\psi_{h,\sigma_{t_2}}(t_2) - \psi_{h,\sigma_{t_1}}(t_1)\|_{\mathcal{H}} < t_1^{-\delta} , \quad \delta > 0 .$$  \hspace{1cm} (4.17)

The proof in [13] is organized as follows.

Let $\psi_{h,\sigma_t,\Gamma(t')}(s)$ denote the vector obtained from $\psi_{h,\sigma_{t'}}(t')$ by first replacing $\sigma_{t'} \to \sigma_t$ and then $t' \to s$, while keeping the cell decomposition

$$\Gamma(t') := \{\Gamma_j(t')\}_{j=1}^{N(t')}$$  \hspace{1cm} (4.18)

corresponding to time $t'$ fixed.

Assuming $t_2 > t_1 \gg 1$, the left hand side of (4.17) is estimated by

$$\|\psi_{h,\sigma_{t_2},\Gamma(t_2)}(t_2) - \psi_{h,\sigma_{t_1},\Gamma(t_1)}(t_1)\|_{\mathcal{H}} \leq (I) + (II) + (III)$$  \hspace{1cm} (4.19)

with the following definitions.
The term \( (I) := \| \psi_{h,\sigma,t_2,\Gamma(t_2)}(t_2) - \psi_{h,\sigma,t_1,\Gamma(t_1)}(t_2) \|_{\mathcal{H}} \) (4.20) is the error made by replacing \( \Gamma(t_2) \) by the coarser cell decomposition \( \Gamma(t_1) \) in \( \psi_{h,\sigma,t_2,\Gamma(t_2)}(t_2) \), while keeping the infrared cutoff and the argument \( t_2 \) fixed. One can control \( (I) \) similarly as the off-diagonal terms in (4.14).

The term \( (II) := \| \psi_{h,\sigma,t_2,\Gamma(t_1)}(t_2) - \psi_{h,\sigma,t_1,\Gamma(t_1)}(t_2) \|_{\mathcal{H}} \) (4.21) is the error made by subsequently changing the infrared cutoff from \( \sigma_{t_2} \) to \( \sigma_{t_1} \) in \( \psi_{h,\sigma,t_2,\Gamma(t_1)}(t_2) \). It admits a bound that involves a positive power of \( \sigma_{t_1} = t_1^{-\beta} \).

The term \( (III) := \| \psi_{h,\sigma,t_1,\Gamma(t_1)}(t_2) - \psi_{h,\sigma,t_1,\Gamma(t_1)}(t_1) \|_{\mathcal{H}} \) (4.22) is the left hand side of (4.17) with \( \psi_{h,\sigma,t_2,\Gamma(t_2)}(t_2) \) replaced by \( \psi_{h,\sigma,t_1,\Gamma(t_1)}(t_2) \). To bound \( (III) \), one applies Cook’s argument to

\[
\psi_{h,\sigma,t_{1,j_1}}(t_2) - \psi_{h,\sigma,t_{1,j_1}}(t_1) = \int_{t_1}^{t_2} ds \partial_s \psi_{h,\sigma,t_{1,j_1}}(s). \quad (4.23)
\]

This is the most involved part of the analysis, and the integrand on the right hand side of (4.23) must be subdivided into many different terms for which one can either prove rapid decay in \( s \) or (asymptotically) precise cancellations.

### 4.3. Modifications of [13] for QED

Most of the constructions in [13] can be adopted directly to yield the corresponding ones in non-relativistic QED. The following minor modifications are necessary.

- The infrared regularization is implemented by a sharp cutoff \( \chi(|k| > \sigma) \) in [13]. It must be replaced by an infrared regularization \( \kappa_{\sigma}(|k|) \) which is zero at \( |k| = 0 \), but non-zero in an open neighborhood of \( |k| = 0 \). Implementing this modification in [13] (invoking results of [4, 5], instead of [9, 12]) is straightforward.

- The dressing transformations in [13] are slightly different from the ones used in non-relativistic QED. In [13], the integral kernel corresponding to \( \psi_{p,\sigma,\lambda}(k) \) has the form

\[
\sqrt{\alpha} \chi(|\sigma < |k| < 1) \frac{1}{|k|^{\frac{1}{2}}} \frac{1}{|k| - k \cdot \nabla_F(p,\sigma)},
\]

while, here, there is an additional factor \( \nabla_F(p,\sigma) \cdot \varepsilon_\lambda(k) \); see (3.12). This does not lead to any non-trivial changes of the considerations in [13].

However, some other modifications are less straightforward, due to the more complicated interaction term of non-relativistic QED.

- In the application of Cook’s method, there is a derivative

\[
\partial_s \left( e^{isH(\sigma_1)}\Gamma_{\sigma_1}(V_j,s)e^{-isH(\sigma_1)} \right)
\]
which contains a term of the form
\[ ie^{i\sigma t}[H(\sigma_t) - H_f, W_{\sigma_t}(V_j, s)]e^{-i\sigma t} \] (4.26)

(we recall that the interaction term in \( H(\sigma) \) depends on \( x \)). Due to the linear coupling in Nelson’s model, the above commutator is given by
\[ [H(\sigma_t) - H_f, W_{\sigma_t}(V_j, s)] = W_{\sigma_t}(V_j, s) \phi_{\sigma_t, V_j}(x, s), \] (4.27)

where \( \phi_{\sigma_t, V_j}(x, s) \) is a scalar function that has rapid decay in \( s \).

For QED, \( \phi_{\sigma_t, V_j}(x, s) \) is replaced by an operator linear in \( \nabla p H(p, \sigma_t) \) (for total momentum \( p \)). The modifications arising here are technically somewhat demanding and involve an application of the uniform bounds on the renormalized electron mass.

A more detailed analysis of scattering theory along the lines of [13] would be appropriate.

5. Proofs of Theorems 3.1 and 3.2

Our proofs follow closely [9], where the statements of Theorems 3.1 and 3.2 were established for Nelson’s model.

In our proofs of Theorem 3.1 and part 2 of Theorem 3.2, the first step is to employ the usual “pull-through formula”, which yields an explicit expression for \( a_\lambda(k)\Psi_u(p, \sigma) \) in terms of \( \Psi_u(p, \sigma) \). However, this is not the end of the story, in contrast to [9], where the result corresponding to Theorem 3.2 for Nelson’s model was established. In non-relativistic QED, the different coupling structure in the Hamiltonian \( H(p, \sigma) \) poses considerable difficulties. Our method involves application of the uniform bounds (2.29) on the renormalized electron mass, which has only recently become available.

Our main technical result is formulated in the following proposition.

**Proposition 5.1.** Under the hypotheses of Theorem 2.1, the vector \( a_\lambda(k)\Psi_u(p, \sigma) \) can be decomposed into
\[ a_\lambda(k)\Psi_u(p, \sigma) = \Phi_1(p, \sigma; k, \lambda) + \Phi_2(p, \sigma; k, \lambda), \] (5.1)

where
\[ \Phi_1(p, \sigma; k, \lambda) = -\sqrt{\alpha} \xi_\lambda(k) \cdot \nabla_p E(p, \sigma) \frac{\kappa_\sigma(|k|)}{|k|^{5/2}} \frac{1}{|k| - k \cdot \nabla_p E(p, \sigma)} \Psi_u(p, \sigma) \] (5.2)

and
\[ \|\Phi_2(p, \sigma; k, \lambda)\| \leq c \sqrt{\alpha} \frac{\kappa_\sigma(|k|)}{|k|}, \] (5.3)

for a constant \( c \) that is independent of \( \sigma \) and \( \alpha \).

The uniform bound on the renormalized electron mass (2.29) enters the estimate for the vector \( \Phi_2(p, \sigma; k) \). (We recall that \( \kappa_\sigma \) denotes the cutoff function in (2.16).)
5.1. **Proof of Theorem 3.1.** The statement of Theorem 3.1 is an immediate consequence of Proposition 5.1.

5.2. **Proof of Theorem 3.2, Part 1.** For the existence of a convergent subsequence, we refer to [9]. The proof comprises the following main steps.

Let $K_\rho := \{ k \in \mathbb{R}^3 \mid |k| \geq \rho \}$ for $0 < \rho < 1$, and let $\mathcal{F}_\rho$ denote the Fock space over the one-photon Hilbert space $L^2(K_\rho) \otimes \mathbb{C}^2$. Let $\mathfrak{A}_\rho$ denote the $C^*$-algebra of bounded operators on $\mathcal{F}_\rho$.

One first establishes the existence of an operator $C_\rho$ affiliated with $\mathfrak{A}_\rho$, which has a compact resolvent on $\mathcal{F}_\rho$, and which satisfies

$$\omega_{p,\sigma}(C_\rho) < M(\rho) < \infty$$

uniformly in $\sigma > 0$. For instance, the operator

$$C_\rho := \sum_{\lambda=\pm} \int_{|k| \geq \rho} dk\, a_\lambda^*(k) \left[ -\Delta_k + |k|^2 \right] a_\lambda(k)$$

has these properties in the present case (see also [9, 10] and [11]).

It follows that $\{ \omega_{p,\sigma} \mid \mathfrak{A}_\rho \}_{\sigma > 0} \subset \mathfrak{A}_\rho^*$ is norm compact, see [11]. The dual $\mathfrak{A}_\rho^*$ of $\mathfrak{A}_\rho$ is a Banach space, because $\mathfrak{A}_\rho$ is a von Neumann algebra. Hence, for any sequence $\{ \sigma_j \}_{j=0}^\infty$ converging to zero, there exists a subsequence $\{ \sigma_{j_i} \}_{i=0}^\infty$ converging to zero such that $\{ \omega_{p,\sigma_{j_i}} \}_{i=0}^\infty$ converges to a normal state $\omega_p^{(\rho)}$ on $\mathfrak{A}_\rho$.

Choosing $\rho_n = \frac{1}{n}$ for $n \in \mathbb{N}$, we get, by Cantor’s diagonal procedure, a subsequence $\{ \sigma_{j_i} \}_{i=0}^\infty$ converging to zero such that $\{ \omega_{p,\sigma_{j_i}} \}_{i=0}^\infty$ converges on $\mathfrak{A}_\rho^*$, for all $n < \infty$. Hence, $\{ \sigma_{p,\sigma_{j_i}} \}_{i=0}^\infty$ converges on $\bigvee_n \mathfrak{A}_n^*$, and thus on $\mathfrak{A}$, to a state $\omega_p$ on $\mathfrak{A}^*$. $\omega_p^{(\rho)} = \omega_p \bigg|_{\mathfrak{A}_\rho}$ is a normal state.

5.3. **Proof of Theorem 3.2, part 2.** This is an immediate consequence of Proposition 5.1.

Indeed, we have that

$$\left| \langle \Psi_u(p,\sigma), \Phi_1(p,\sigma;k,\lambda) \rangle \right|^2 = \langle \Phi_1(p,\sigma;k,\lambda), \Phi_1(p,\sigma;k,\lambda) \rangle,$$  

since $\Phi_1(p,\sigma;k,\lambda)$ is a scalar multiple of $\Psi_u(p,\sigma)$, and $\| \Psi_u(p,\sigma) \| = 1$. Therefore,

$$\langle \Psi_u(p,\sigma), a_\lambda^*(k) a_\lambda(k) \Psi_u(p,\sigma) \rangle$$

$$= \langle \Phi_1(p,\sigma;k,\lambda), \Phi_1(p,\sigma;k,\lambda) \rangle + \rho_1(p,\sigma;k,\lambda)$$

$$= \left| \langle \Psi_u(p,\sigma), \Phi_1(p,\sigma;k,\lambda) \rangle \right|^2 + \rho_1(p,\sigma;k,\lambda)$$

$$= \left| \langle \Psi_u(p,\sigma), a_\lambda^*(k) \Psi_u(p,\sigma) \rangle \right|^2 + \rho_1(p,\sigma;k,\lambda) - \rho_2(p,\sigma;k,\lambda)$$

where

$$\rho_1(p,\sigma;k,\lambda) = \langle \Phi_1(p,\sigma;k,\lambda), \Phi_2(p,\sigma;k,\lambda) \rangle + \langle \Phi_2(p,\sigma;k,\lambda), \Phi_1(p,\sigma;k,\lambda) \rangle$$

$$+ \langle \Phi_2(p,\sigma;k,\lambda), \Phi_2(p,\sigma;k,\lambda) \rangle$$

(5.8)
and
\[\rho_2(p, \sigma; k, \lambda) = \langle \Psi_u(p, \sigma), \Phi_1(p, \sigma; k, \lambda) \rangle \langle \Phi_2(p, \sigma; k, \lambda), \Psi_u(p, \sigma) \rangle + \langle \Psi_u(p, \sigma), \Phi_2(p, \sigma; k, \lambda) \rangle \bigg| \langle \Phi_1(p, \sigma; k, \lambda), \Psi_u(p, \sigma) \rangle \bigg|^{-2}. \tag{5.9}\]

Clearly,
\[|\rho_1(p, \sigma; k, \lambda)|, |\rho_2(p, \sigma; k, \lambda)| \leq 2 ||\Phi_1(p, \sigma; k, \lambda)|| ||\Phi_2(p, \sigma; k, \lambda)|| + ||\Phi_1(p, \sigma; k, \lambda)||^2 \leq c\alpha \kappa^2 \sigma^2 \bigg| k \bigg|^5 + c' \alpha \kappa^2 \sigma^2 \bigg| k \bigg|^2. \tag{5.10}\]

This proves the claim.

5.4. Proof of Theorem 3.2, part 3. We sketch the proof, and refer to Lemma 3.1 in [9] for details (see also [3, 6, 8]).

We consider the coherent \(*\)-automorphisms
\[\alpha_{p, \sigma}(A) = W_{p, \sigma} A W_{p, \sigma}^*, \quad A \in \mathfrak{A}, \tag{5.11}\]
where
\[W_{p, \sigma} = \exp \left[ i \sum_{\lambda=+,-} \Pi_{\lambda}(v_{p, \sigma, \lambda}) \right], \tag{5.12}\]
see (3.14), and \(\Pi_{\lambda}(f) = i(a_{\lambda}(f) - a_{\lambda}^*(f))\). In the limit \(\sigma \searrow 0\), the states
\[\mu_{p, \sigma} := \omega_p(\alpha_{p, \sigma}(\cdot)), \tag{5.13}\]
converge to
\[\mu_p = \omega_p(\alpha_p(\cdot)), \tag{5.14}\]
where \(\alpha_p(A) = n - \lim_{\sigma \searrow 0} \alpha_{p, \sigma}(A), \quad A \in \mathfrak{A}; \text{ see } [9].\)

Next, one proves that the representation \(\pi_{\mu_p} = \pi_p \circ \alpha_p\) admits a positive, self-adjoint number operator. This implies that \(\pi_{\mu_p}\) is quasi-equivalent to the Fock representation, for \(0 \leq |p| < \frac{1}{3}\), [6]. To this end, we define the local number operators
\[N_p := \sum_{\lambda=+,-} \int_{|k|>\rho} dk a_{\lambda}^*(k) a_{\lambda}(k), \quad \text{for } \rho > 0, \tag{5.15}\]
where \(\exp[i t N_p] \in \mathfrak{A}_p\). Let \(\mathcal{H}_{\mu_p}\) denote the Hilbert space and \(\Omega_{\mu_p} \in \mathcal{H}_{\mu_p}\) the cyclic vector corresponding to \(\mu_p\) by GNS construction.

One can show that \(\pi_{\mu_p}(\exp[it N_p]) \pi_{\mu_p}(A) \Omega_{\mu_p}\) converges strongly, as \(\rho \searrow 0\), for all \(A \in \bigvee_{\rho>0} \mathfrak{A}_p\), and all \(t \in \mathbb{R}\). The limit of \(\pi_{\mu_p}(\exp[it N_p])\), as \(\rho \searrow 0, t \in \mathbb{R}\), defines a strongly continuous unitary group on \(\mathcal{H}_{\mu_p}\). Its generator defines a positive, selfadjoint number operator on \(\mathcal{H}_{\mu_p}\).
Since $A \in \bigvee_{\rho > 0} \mathfrak{A}_\rho$, there is some $\tilde{\rho} > 0$ such that $A \in \mathfrak{A}_{\tilde{\rho}}$. Let $\rho' \leq \rho \leq \tilde{\rho}$, and let

$$N_{\rho', \rho} := \sum_{\lambda = +, -} \int_{\rho' \leq |k| \leq \rho} dk \ a^*_\lambda(k) a_\lambda(k).$$  \quad (5.16)

Then,

$$\|\pi_{\mu_p}(e^{itN_{\rho'}} - e^{itN_{\rho}})A\Omega\|_{\mu_p}^2 = 2\mu_p(A^*A) - \mu_p(A^*e^{itN_{\rho'}}A) - \mu_p(A^*e^{-itN_{\rho'}}A).$$  \quad (5.17)

Using that

$$\pi_{\mu_p}(1 - e^{itN_{\rho'}}) = -i \int_0^t \pi_{\mu_p}(e^{isN_{\rho'}}) N_{\rho', \rho}$$  \quad (5.18)

a straightforward calculation shows that

$$|\mu_{\mu_p}(A^*A) - \mu_{\mu_p}(A^*e^{itN_{\rho'}}A)| \leq |t| \|A\|^2 \sum_{\lambda} \int_{|k| \leq \rho} dk \left| \omega_p(a^*_\lambda(k)a_\lambda(k)) - |\omega_p(a_\lambda(k))|^2 \right| \leq |t| \|A\|^2 |\rho - \rho'|^{\frac{1}{2}},$$  \quad (5.19)

which tends to zero as $\rho \searrow 0$. In the last step, we used (3.10).

Our results imply that $\pi_p$ is quasi-equivalent to the coherent representation corresponding to (3.14) by the GNS construction.

5.5. **Proof of Theorem 3.2, part 4.** Theorem 2.1 implies that $\nabla_p E(p, \sigma) \neq 0$ if and only if $p \neq 0$, for $|p| < \frac{1}{3}$. Thus if $p \neq 0$, (3.1) in Theorem 3.1 implies that

$$\omega_{p, \sigma}(N_f) = \langle \Psi_u(p, \sigma), N_f \Psi_u(p, \sigma) \rangle \geq c_{\alpha}(1 + |\nabla_p E(p, \sigma)|^2 \log \frac{1}{\sigma})$$  \quad (5.20)

which diverges to $\infty$ as $\sigma \searrow 0$. Hence, (3.16) follows. However, if $p = 0$, one gets (3.15) from $\nabla_p E(p, \sigma) = 0$.

The local Fock properties of $\omega_p$ are derived from the following considerations.

Using Proposition 5.1, it is easy to see that

$$\omega_{p, \sigma}(N_p) \leq 2 \sum_{\lambda} \int_{|k| > \rho} dk \left[ \|\Phi_1(p, \sigma; k, \lambda)\|^2 + \|\Phi_2(p, \sigma; k, \lambda)\|^2 \right] < C(\rho),$$  \quad (5.21)

uniformly in $\sigma \geq 0$. By similar considerations as in the proof of part 3 of Theorem 3.2, one concludes that the representation of $\mathfrak{A}_p$ corresponding to $\omega_p$ by GNS construction is quasi-equivalent to the Fock representation for every $\rho > 0$.

Let $B \subset \mathbb{R}^3$ denote a bounded region in physical $x$-space, and let $\mathfrak{A}(B)$ denote the corresponding local algebra. Then, the restriction of $\omega_p$ to $\mathfrak{A}(B)$ defines a GNS representation of $\mathfrak{A}(B)$ which is quasi-equivalent to the Fock representation. This can be shown by a straightforward adaptation of results in [11] to the present model.
6. Proof of Proposition 5.1

It remains to prove Proposition 5.1, our key analytical result in this paper. To this end, we first derive the following representation of \( a_\lambda(k)\Psi_u(p,\sigma) \).

**Lemma 6.1.** Assume that \( 0 < |k| < 1, \ |p| < \frac{1}{3}, \) and \( \alpha < \alpha_0 \) (where \( \alpha_0 \) denotes the same constant as in Theorem 2.1). Let \( E(p,\sigma) \) denote the ground state eigenvalue of \( H(p,\sigma) \), and let \( \Psi_u(p,\sigma) \) be an eigenvector in the corresponding two-dimensional eigenspace.

Then the operator \( H(p - k) + |k| - E(p,\sigma) \) is invertible, and

\[
a_\lambda(k)\Psi_u(p,\sigma) = -\frac{1}{H(p - k,\sigma) + |k| - E(p,\sigma)} \left[ \sqrt{\alpha k_\sigma(|k|)} \varepsilon_\lambda(k) \cdot \nabla_p H(p,\sigma) \right. \\
\left. + \sqrt{\alpha k_\sigma(|k|)} \tau \cdot (k \wedge \varepsilon_\lambda(k)) \right] \Psi_u(p,\sigma) .
\] (6.1)

In the scalar case, \( E(p,\sigma) \) is a simple eigenvalue, and the magnetic term (proportional to \( \tau \)) is absent.

Moreover, the a priori bound

\[
\|a_\lambda(k)\Psi_u(p,\sigma)\| \leq c\sqrt{\alpha k_\sigma(|k|)} \left[ \sqrt{p^2 + c'\alpha + |k|} \right]
\] (6.2)

holds, where the constants \( c \) and \( c' \) are independent of \( \alpha \) and \( \sigma \).

**Proof.** We recall the definition of the fiber Hamiltonian

\[
H(p,\sigma) = \frac{1}{2} (p - P_f - \sqrt{\alpha} A_\sigma)^2 + \sqrt{\alpha} \tau \cdot B_\sigma + H_f .
\] (6.3)

The ”pull-through formula” says that

\[
a_\lambda(k)H(p,\sigma) = \left( \frac{1}{2} (p - P_f - k - \sqrt{\alpha} A_\sigma)^2 + \sqrt{\alpha} \tau \cdot B_\sigma + H_f + |k| \right) a_\lambda(k) \\
- \sqrt{\alpha k_\sigma(|k|)} \varepsilon_\lambda(k) \cdot (p - P_f - \sqrt{\alpha} A_\sigma) \\
+ \sqrt{\alpha k_\sigma(|k|)} \tau \cdot (ik \wedge \varepsilon_\lambda(k)) ,
\] (6.4)

where \( \tau = (\tau_1,\tau_2,\tau_3) \) is the vector of Pauli matrices. We observe that

\[
\nabla_p H(p,\sigma) = p - P_f - \sqrt{\alpha} A_\sigma ,
\] (6.5)

and that

\[
\varepsilon_\lambda(k) \cdot \nabla_p H(p - k,\sigma) = \varepsilon_\lambda(k) \cdot \nabla_p H(p,\sigma)
\] (6.6)
since \(\varepsilon_\lambda(k) \cdot k = 0\), by the Coulomb gauge condition. Hence

\[
a_\lambda(k)E(p,\sigma)\Psi_u(p,\sigma) = a_\lambda(k)H(p,\sigma)\Psi_u(p,\sigma) = \left[ a_\lambda(k) + |k| \right] a_\lambda(k)
\]

\[
+ \sqrt{\alpha \kappa_\sigma(|k|)} \varepsilon_\lambda(k) \cdot \nabla_p H(p,\sigma)
\]

\[
+ \sqrt{\alpha \kappa_\sigma(|k|)} \tau \cdot (ik \wedge \varepsilon_\lambda(k)) \Psi_u(p,\sigma),
\]

so that

\[
\left[ H(p - k,\sigma) + |k| - E(p,\sigma) \right] a_\lambda(k) \Psi_u(p,\sigma)
\]

\[
= - \left[ \sqrt{\alpha \kappa_\sigma(|k|)} \varepsilon_\lambda(k) \cdot \nabla_p H(p,\sigma) + \sqrt{\alpha \kappa_\sigma(|k|)} \tau \cdot (ik \wedge \varepsilon_\lambda(k)) \right] \Psi_u(p,\sigma).
\]

Furthermore, the bounds

\[
H(p - k,\sigma) + |k| - E(p,\sigma) \geq E(p - k,\sigma) + |k| - E(p,\sigma) > \frac{|k|}{10}
\]

follow from (6.19) below. Hence, \(H(p - k,\sigma) + |k| - E(p,\sigma)\) (see left side of (6.8)) is invertible, for any \(0 < |k| < 1\) and \(|p| < \frac{1}{3}\).

We conclude that

\[
a_\lambda(k)\Psi_u(p,\sigma) = \frac{1}{H(p - k,\sigma) + |k| - E(p,\sigma)}
\]

\[
\left[ \sqrt{\alpha \kappa_\sigma(|k|)} \varepsilon_\lambda(k) \cdot \nabla_p H(p,\sigma) + \sqrt{\alpha \kappa_\sigma(|k|)} \tau \cdot (ik \wedge \varepsilon_\lambda(k)) \right] \Psi_u(p,\sigma),
\]

as claimed.

Moreover, (6.9) immediately implies the a priori bound (6.2). \(\square\)

Proof of Proposition 5.1.

Proof. We note that

\[
(\nabla_p H)(p,\sigma)\Psi_u(p,\sigma) = \nabla_p (H(p,\sigma)\Psi_u(p,\sigma)) - H(p,\sigma)\nabla_p \Psi_u(p,\sigma)
\]

\[
= \nabla_p (E(p,\sigma)\Psi_u(p,\sigma)) - H(p,\sigma)\nabla_p \Psi_u(p,\sigma)
\]

\[
= (\nabla_p E)(p,\sigma)\Psi_u(p,\sigma) - (H(p,\sigma) - E(p,\sigma))\nabla_p \Psi_u(p,\sigma).
\]

From (6.1), we get

\[
a_\lambda(k)\Psi_u(p,\sigma) = (I) + (II),
\]
where

$$
\begin{align*}
(I) &= -\sqrt{\frac{\alpha\sigma}{|k|^2}}(\varepsilon_\lambda(k) \cdot \nabla_p E(p, \sigma)) \frac{1}{H(p-k, \sigma) + |k| - E(p, \sigma)} \Psi_u(p, \sigma) \\
(II) &= \sqrt{\frac{\alpha\sigma}{|k|^2}} \frac{1}{H(p-k, \sigma) + |k| - E(p, \sigma)} \\
&\quad \left[ (H(p, \sigma) - E(p, \sigma))\varepsilon_\lambda(k) \cdot \nabla_p \Psi_u(p, \sigma) \\
&\quad - \tau \cdot (i k \wedge \varepsilon_\lambda(k)) \Psi_u(p, \sigma) \right].
\end{align*}
$$

(6.13)

Let us first bound \((II)\).

To this end, we first prove that for \(0 < |k| < 1\) and \(|p| < \frac{1}{3}\),

$$
\left\| \frac{1}{H(p-k, \sigma) + |k| - E(p, \sigma)} \right\|_{op} \leq 3.
$$

(6.14)

We note that

$$
H(p-k, \sigma) = H(p, \sigma) + \frac{k^2}{2} - k \cdot \nabla_p H(p, \sigma)
$$

(6.15)

so that

$$
\begin{align*}
H(p-k, \sigma) - E(p, \sigma) &= H(p, \sigma) - E(p, \sigma) - k \cdot \nabla_p H(p, \sigma) + \frac{k^2}{2} \\
&\geq H(p, \sigma) - E(p, \sigma) - \frac{k^2}{28} - \frac{\delta}{2} (\nabla_p H(p, \sigma))^2 + \frac{k^2}{2} \\
&\geq (1 - \frac{2}{3}|k|)(H(p, \sigma) - E(p, \sigma)) + \frac{k^2}{2} - \frac{3}{4}|k| + \frac{2}{3}|k|(H_f + \sqrt{\alpha\sigma} \cdot B_\sigma - E(p, \sigma)),
\end{align*}
$$

(6.16)

using the Schwarz inequality with \(\delta = \frac{2}{3}|k| \leq \frac{2}{3}\). From

\(|B_\sigma| \leq c\sqrt{1 + H_f}\),

(6.17)

the operator on the last line is bounded by

$$
\begin{align*}
\frac{2}{3}|k| \left[ \chi(H_f \geq 1)(H_f - \sqrt{\alpha c}\sqrt{1 + H_f}) - c\sqrt{\alpha} - E(p, \sigma) \right] \\
&\geq -\frac{2}{3}|k| \left[ \frac{1}{2} \left( \frac{1}{3} \right)^2 + \alpha \right],
\end{align*}
$$

(6.18)

using \(E(p, \sigma) \leq \frac{c^2}{2} + \alpha c \) for \(|p| < \frac{1}{3}\). Therefore

$$
\begin{align*}
H(p-k, \sigma) - E(p, \sigma) + |k| &\geq (1 - \frac{2}{3}|k|)(H(p, \sigma) - E(p, \sigma)) + \frac{k^2}{2} \\
&\quad + |k|(1 - \frac{3}{4} - \frac{1}{2} \left( \frac{1}{3} \right)^2 - \alpha c) \\
&\geq \frac{1}{3}(H(p, \sigma) - E(p, \sigma)) + \frac{|k|}{10},
\end{align*}
$$

(6.19)

for \(|k| < 1\). This implies (6.14).
It is then easy to see that

\[
\| (II) \| \leq c \sqrt{\alpha \kappa (|k|)} \left[ \left\| (H(p, \sigma) - E(p, \sigma)) - \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} \right\|_{\text{op}}^{\frac{1}{2}} \right]
\]

\[
+ \left[ \left\| (H(p, \sigma) - E(p, \sigma)) \nabla p \Psi_u(p, \sigma) \right\|_{\text{op}}^{\frac{1}{2}} \right]
\]

\[
\leq c \sqrt{\alpha \kappa (|k|)} \left[ \frac{1}{|k|^2} \left| \frac{1}{m_{\text{ren}}(p, \sigma)} - 1 \right|^{\frac{1}{2}} + 1 \right]
\]

\[
\leq c \sqrt{\alpha \kappa (|k|)} \frac{|k|}{|k|^2},
\]  

(6.20)

where

\[
m_{\text{ren}}(p, \sigma) = \left[ 1 - 2 \langle \nabla_p \Psi_u(p, \sigma), (H(p, \sigma) - E(p, \sigma)) \nabla_p \Psi_u(p, \sigma) \rangle \right]^{-1}
\]

(6.21)

is the renormalized electron mass.

We recall from (2.29) that \(|m_{\text{ren}}(p, \sigma) - 1| < c\alpha\), for \(|p| < \frac{1}{3}\), uniformly in \(\sigma \geq 0\).

Next, we discuss the term \((I)\). We use the resolvent identity and (6.15) for

\[
\frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} = \frac{1}{H(p, \sigma) - E(p, \sigma) + |k| + \frac{k^2}{2}}
\]

\[
- \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} k \cdot \nabla_p H(p, \sigma)
\]

(6.22)

\[
\frac{1}{H(p, \sigma) - E(p, \sigma) + |k| + \frac{k^2}{2}}.
\]

Accordingly,

\[
(I) = (I_1) + (I_2),
\]

(6.23)

where

\[
(I_1) = - \sqrt{\alpha \kappa (|k|)} \left( \varepsilon_\lambda(k) \cdot \nabla_p E(p, \sigma) \right) - \frac{1}{H(p, \sigma) - E(p, \sigma) + |k| + \frac{k^2}{2}} \Psi_u(p, \sigma)
\]

\[
= - \sqrt{\alpha \kappa (|k|)} \left[ \frac{1}{|k|^2} (\varepsilon_\lambda(k) \cdot \nabla_p E(p, \sigma)) \Psi_u(p, \sigma) \right].
\]

(6.24)

We note that the \(L^2\)-norm of this term diverges logarithmically in the limit \(\sigma \searrow 0\).
Moreover,

\[ (I_2) = - \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|^2} \left( \frac{1}{|k| + \frac{k^2}{2}} \right) (\varepsilon(k) \cdot \nabla_p E(p, \sigma)) \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} H(p, \sigma) - E(p, \sigma) + |k| + \frac{k^2}{2} \Psi_u(p, \sigma) \]

\[ = - \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|^2} \left( \frac{1}{|k| + \frac{k^2}{2}} \right) (|k| \cdot \nabla_p E(p, \sigma)) \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} (k \cdot \nabla_p H(p, \sigma)) \Psi_u(p, \sigma) \]

\[ = (I_{21}) + (I_{22}) \quad \text{(6.25)} \]

with

\[ (I_{21}) = \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|^2} \left( \frac{1}{|k| + \frac{k^2}{2}} \right) (\varepsilon(k) \cdot \nabla_p E(p, \sigma)) \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} (k \cdot \nabla_p E(p, \sigma)) \Psi_u(p, \sigma) \]

\[ = \frac{k \cdot \nabla_p E(p, \sigma)}{|k| + \frac{k^2}{2}} \cdot (I) \quad \text{(6.26)} \]

and

\[ (I_{22}) = \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|^2} \left( \frac{1}{|k| + \frac{k^2}{2}} \right) (\varepsilon(k) \cdot \nabla_p E(p, \sigma)) \]

\[ \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} (H(p, \sigma) - E(p, \sigma)) k \cdot \nabla_p \Psi_u(p, \sigma). \]

We find that

\[
\| (I_{22}) \| \leq 3 \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|} \left( \frac{1}{|k| + \frac{k^2}{2}} \right) \| \nabla_p E(p, \sigma) \| \left\| \frac{1}{H(p - k, \sigma) + |k| - E(p, \sigma)} \right\|_{op}^{\frac{1}{2}} \left\| \frac{1}{H(p, \sigma) - E(p, \sigma)} \right\|_{op}^{\frac{1}{2}} \left\| (H(p, \sigma) - E(p, \sigma))^{\frac{1}{2}} \nabla_p \Psi_u(p, \sigma) \right\| \leq c \sqrt{\alpha} \frac{\kappa(\lambda)}{|k|} \| \nabla_p E(p, \sigma) \| \left( \frac{1}{m_{ren}(p, \sigma)} - 1 \right)^{\frac{1}{2}} \leq c \alpha \frac{\kappa(\lambda)}{|k|} \| \nabla_p E(p, \sigma) \|, \quad \text{(6.28)}
\]

using (6.14).

Hence, solving for \( I \) (recalling that (6.26) is a multiple of (I)),

\[
(I) = \left[ 1 - \frac{k \cdot \nabla_p E(p, \sigma)}{|k| + \frac{k^2}{2}} \right]^{-1} \left( (I_{1}) + (I_{22}) \right), \quad \text{(6.29)}
\]
where
\[ | k \cdot \nabla_p E(p, \sigma) | < |k| |\alpha| (1 + c \alpha) < \frac{|k|}{2}, \]
for \(|p| < \frac{1}{4}\) and \(\alpha\) sufficiently small, see (2.27). Noting that
\[
\left\| \left[ 1 - \frac{k \cdot \nabla_p E(p, \sigma)}{|k| + \frac{k^2}{2}} \right]^{-1} (I_1) + \sqrt{\alpha} \frac{\kappa_{\alpha}(|k|)}{|k|^2} \frac{\varepsilon_{\lambda}(k) \cdot \nabla_p E(p, \sigma)}{|k| - k \cdot \nabla_p E(p, \sigma)} \Psi_u(p, \sigma) \right\|
\leq c \sqrt{\alpha} \frac{\kappa_{\alpha}(|k|)}{|k|^2} \| \Psi_u(p, \sigma) \|, \tag{6.31}
\]
we find that
\[ a_{\lambda}(k) \Psi_u(p, \sigma) = \Phi_1(p, \sigma; k, \lambda) + \Phi_2(p, \sigma; k, \lambda), \tag{6.32} \]
where
\[ \Phi_1(p, \sigma; k, \lambda) = -\sqrt{\alpha} \varepsilon_{\lambda}(k) \cdot \nabla_p E(p, \sigma) \frac{\kappa_{\alpha}(|k|)}{|k|^2} \frac{1}{|k| - k \cdot \nabla_p E(p, \sigma)} \Psi_u(p, \sigma) \]
and
\[ \| \Phi_2(p, \sigma; k, \lambda) \| \leq c \sqrt{\alpha} \frac{\kappa_{\alpha}(|k|)}{|k|}. \tag{6.34} \]

This establishes Proposition 5.1. \(\square\)

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Department of Mathematics, Princeton University, 807 Fine Hall, Washington Road, Princeton, NJ 08544, U.S.A.  
E-mail address: tc@math.princeton.edu

Institute for Theoretical Physics, ETH Hönggerberg, 8093 Zürich, Switzerland.  
E-mail address: juerg@itp.phys.ethz.ch