Some results on the target set selection problem

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Abstract In this paper we consider a fundamental problem in the area of viral marketing, called TARGET SET SELECTION problem. We study the problem when the underlying graph is a block-cactus graph, a chordal graph or a Hamming graph. We show that if $G$ is a block-cactus graph, then the TARGET SET SELECTION problem can be solved in linear time, which generalizes Chen’s result (Discrete Math. 23:1400–1415, 2009) for trees, and the time complexity is much better than the algorithm in Ben-Zwi et al. (Discrete Optim., 2010) (for bounded treewidth graphs) when restricted to block-cactus graphs. We show that if the underlying graph $G$ is a chordal graph with thresholds $\theta(v) \leq 2$ for each vertex $v$ in $G$, then the problem can be solved in linear time. For a Hamming graph $G$ having thresholds $\theta(v) = 2$ for each vertex $v$ of $G$, we precisely determine an optimal target set $S$ for $(G, \theta)$. These results partially answer an open problem raised by Dreyer and Roberts (Discrete Appl. Math. 157:1615–1627, 2009).

Keywords Target set selection · Viral marketing · Tree · Block graph · Block-cactus graph · Chordal graph · Hamming graph · Social networks · Diffusion of innovations · Viral marketing · Dynamic monopoly · Irreversible spread of influence


1 Introduction and preliminaries

A graph $G$ consists of a set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of vertices called edges. We use $uv$ for an edge $\{u, v\}$. Two vertices $u$ and $v$ are adjacent to each other if $uv \in E(G)$. In this paper, all graphs are finite and have no loops or multiple edges. For $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is the graph $G[S]$ with vertex set $S$ and edge set $\{uv \in E(G) : u, v \in S\}$. Denote by $G - S$ the subgraph of $G$ induced by $V(G) \setminus S$ and, for convenience, we write $G - v$ for $G - \{v\}$ when $v \in V(G)$. The neighborhood of a vertex $v$ in $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree $d_G(v)$ of $v$ is defined by $d_G(v) = |N_G(v)|$. The distance $d_G(x, y)$ of two vertices $x$ and $y$ in $G$ is defined to be the length of the shortest path from $x$ to $y$ in $G$. A complete graph is a graph in which every two distinct vertices are adjacent. The complete graph on $n$ vertices is denoted by $K_n$. The $n$-cycle is the graph $C_n$ with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. The $n$-path is the graph $P_n$ with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

The topology of a person-to-person recommendation social network is usually modeled by a graph $G$ in which the vertices $V(G)$ represent customers, and edges $E(G)$ connect people to their friends. Consider the following scenario: A company wishes to market a new product. The company has at hand a description of the social network $G$ formed among a sample of potential customers. The company wants to target key potential customers $S \subseteq V(G)$ of the social network and persuade them into adopting the new product by handing out free samples. We assume that individuals in $S$ will be convinced to adopt the new product after they receive a free sample, and the friends of customers in $S$ would be persuaded into buying the new product, which in turn will recommend the product to other friends. The company hopes that by word-of-mouth effects, convinced vertices in $S$ can trigger a cascade of further adoptions, and many customers will ultimately be persuaded. This advertising technique of spreading commercial message via social networks $G$ is called viral marketing by analogy with computer viruses. But now how to find a good set of potential customers $S$ to target?

In general, each vertex $v$ is assigned a threshold value $\theta(v)$. The thresholds represent the different latent tendencies of vertices (customers) to buy the new product when their neighbors (friends) do. To be precise, let $G$ be a connected undirected graph equipped with thresholds $\theta : V(G) \to \mathbb{Z}$. Denote by $(G, \theta)$ the social network $G$ equipped with thresholds $\theta$. When $\theta$ is a constant function such that $\theta(v) = k$ for all vertices $v$, $(G, \theta)$ will be written as $(G, k)$ for short. Vertices $v$ of $G$ are in one of two states, active or inactive, which indicate whether $v$ is persuaded into buying the new product. We call a vertex $v$ active if it has been convinced to adopt the new product and assume that vertex $v$ becomes active if $\theta(v)$ of its neighbors have adopted the new product.

In this paper we consider the following repetitive process, called activation process in $(G, \theta)$ starting at target set $S \subseteq V(G)$, which unfolds in discrete steps. Initially (at time 0), set all vertices in $S$ to be active (with all other vertices inactive). After that, at each time step, the states of vertices are updated according to following rule:
Parallel updating rule: All inactive vertices \( v \) that have at least \( \theta(v) \) already-active neighbors become active.

The activation process terminates when no more vertices can get activated. Let \( \bigcup_{\theta}^{G} \) denote the set of vertices that are active at the end of the process. If \( F \subseteq \bigcup_{\theta}^{G} \), then we say that the target set \( S \) influences \( F \) in \( (G, \theta) \). Notice that if \( v \) has threshold \( \theta(v) > d_G(v) \) and \( v \in \bigcup_{\theta}^{G} \) for some target set \( S \), then it must be \( v \in S \). We also note that, according to our rule, if an inactive vertex \( v \) has threshold \( \theta(v) \leq 0 \) at time step \( t \), then it becomes active automatically at the next time step. We are interested in the following optimization problem:

**TARGET SET SELECTION:** Finding a target set \( S \) of smallest possible size that influences all vertices in the social network \( (G, \theta) \), that is \( \bigcup_{\theta}^{G} = V(G) \).

We define \( \text{min-seed}(G, \theta) \) to be the minimum size of a target set that guarantees that all vertices in \( (G, \theta) \) are eventually active at the end of the activation process, that is, \( \text{min-seed}(G, \theta) = \min\{ |S| : S \subseteq V(G) \text{ and } \bigcup_{\theta}^{S} = V(G) \} \). For \( S \subseteq V(G) \), if \( \bigcup_{\theta}^{S} = V(G) \) and \( |S| = \text{min-seed}(G, \theta) \), then we call \( S \) an optimal target set for \( (G, \theta) \).

Domingos and Richardson (2001) considered TARGET SET SELECTION problem in a probabilistic setting and presented heuristic solutions. Kempe et al. (2003) considered probabilistic thresholds, called linear threshold model, and focused on the maximization version of the TARGET SET SELECTION problem—for any given \( k \), find a target set \( S \) of size \( k \) to maximize the expected number of active vertices at the end of the activation process. They showed that this problem is NP-hard and proved that a hill-climbing algorithm can efficiently obtain an approximation solution that is 63 \% of optimal.

In this paper we only consider the TARGET SET SELECTION problem with deterministic, explicitly given, thresholds. Peleg (2002) showed that this problem is NP-hard for majority thresholds, that is \( \theta(v) = \lceil d_G(v)/2 \rceil \) for each vertex \( v \) in \( G \). Dreyer and Roberts (2009) showed that the problem is NP-hard for constant thresholds—given a fixed \( k \geq 3 \), \( \theta(v) = k \) for each vertex \( v \) in \( G \), and Chen (2009) proved that it is NP-hard for bounded bipartite graphs \( G \) with thresholds at most 2.

In general, the TARGET SET SELECTION problem is not just NP-hard but also extremely hard to approximate. Kempe et al. (2003) showed that a maximization version of TARGET SET SELECTION with constant thresholds cannot be approximated within any non-trivial factor, unless \( P = NP \). Chen (2009) proved that given any \( n \)-vertices regular graph with thresholds \( \theta(v) \leq 2 \) for any vertex \( v \), the TARGET SET SELECTION problem can not be approximated within the ratio of \( O(2^{\log^{1+\epsilon} n}) \), for any fixed constant \( \epsilon > 0 \), unless \( NP \subseteq DTIME(n^{\omega\log(\omega)}) \).

Very little is known about \( \text{min-seed}(G, \theta) \) for specific classes of graphs \( G \). Dreyer and Roberts (2009) showed that when \( G \) is a tree, the TARGET SET SELECTION problem can be solved in linear time for constant thresholds. Chen (2009) showed that when the underlying graph is a tree, the problem can be solved in polynomial-time under a general threshold model. Ben-Zwi et al. (2010) showed that for \( n \)-vertices graph \( G \) with treewidth bounded by \( \omega \), the TARGET SET SELECTION problem can be solved in \( n^{O(\omega)} \) time. In Dreyer and Roberts (2009), Flocchini et al. (2004), \( \text{min-seed}(G, \theta) \) is computed for paths, cycles and for different kinds of grids \( G \) under constant threshold model.
The objective of this paper is to study the TARGET SET SELECTION problem when the underlying graph is a block-cactus graph, a chordal graph or a Hamming graph. In Sect. 2, we show that if \( G \) is a block-cactus graph, then the problem can be solved in linear time, which generalizes Chen’s (2009) result for trees, and the time complexity is much better than the algorithm in Ben-Zwi et al. (2010) (for bounded treewidth graphs) when restricted to block-cactus graphs. In Sect. 3, we show that if the underlying graph \( G \) is a chordal graph with thresholds \( \theta(v) \leq 2 \) for each vertex \( v \) in \( G \), then the TARGET SET SELECTION problem can be solved in linear time. Our results partially answer an open problem raised by Dreyer and Roberts at the end of their paper (Dreyer and Roberts 2009), where the authors say that they would like a characterization of irreversible minimum conversion sets for chordal graphs, or efficient algorithms for finding such sets on chordal graphs. In Sect. 4, for a Hamming graph \( G \) having thresholds \( \theta(v) = 2 \) for each vertex \( v \) of \( G \), we precisely determine an optimal target set \( S \) for \( (G, \theta) \).

In order to study min-seed\((G, \theta)\) we introduce a sequential version of the above activation process, called sequential activation process, which employs the following rule instead of the parallel updating rule:

**Sequential updating rule:** At each time step \( t \), exactly one of inactive vertices that have at least \( \theta(v) \) already-active neighbors becomes active.

The proof of the following lemma is straightforward and so is omitted. In the sequel, Lemma 1 will be used without explicit reference to it.

**Lemma 1** For a social network \((G, \theta)\), an optimal target set under sequential updating rule is also an optimal target set under parallel updating rule, and vice versa.

Let \( \mathcal{P} \) be a sequential activation process on \((G, \theta)\) starting out from a target set \( S \). In this process, if \( v_1, v_2, \ldots, v_r \) is the order that vertices in \( [S]_G \setminus S \) are convinced, then \((v_1, v_2, \ldots, v_r)\) is called the convinced sequence of \( \mathcal{P} \), and we say that target set \( S \) has a convinced sequence \((v_1, v_2, \ldots, v_r)\) on \((G, \theta)\).

\[2\text{ Block-cactus graphs}\]

A vertex \( v \) of a graph is called a cut-vertex if removal of \( v \) and all edges incident to it increases the number of connected components. A block of a graph \( G \) is a maximal connected induced subgraph of \( G \) that has no cut-vertices. A graph \( G \) is a block graph if every block of \( G \) is a complete graph. A block \( B \) of a graph \( G \) is called a pendent block of \( G \) if \( B \) has at most one cut-vertex of \( G \). A graph \( G \) is a block-cactus graph if every block of \( G \) is either a complete graph or a cycle. Let \( v \) be a cut-vertex of \( G \). If \( G - v \) consists of two disjoint graphs \( W_1 \) and \( W_2 \) and let \( G_i \) (\( i = 1, 2 \)) be the subgraph of \( G \) induced by \( \{v\} \cup V(W_i) \), then \( G \) is called the vertex-sum at \( v \) of the two graphs \( G_1 \) and \( G_2 \), and denoted by \( G = G_1 \oplus v G_2 \).

In the following Theorem 2, let \( G_1 \oplus v G_2 \) be a social network equipped with threshold function \( \theta \). Let \( \theta_1 \) be a threshold function of \( G_1 - v \) which is the same as the function \( \theta \), except that \( \theta_1(x) = \theta(x) - 1 \) for every \( x \in N_{G_1}(v) \). Let \( S_1 \) be...
an optimal target set for \((G_1 - v, \theta_1)\) that maximizes the cardinality of the set 
\(N_{G_1}(v) \cap [S_1]^{G_1}_{\theta_1}\), where, by slight abuse of notation, \(\theta\) also means the threshold function of \(G_1\) by restricting the threshold \(\theta\) of \(G_1 \oplus_v G_2\) to the set \(V(G_1)\). Let \(\theta_2\) be a threshold function of \(G_2\) which is the same as the function \(\theta\), except that 
\(\theta_2(v) = \theta(v) - |N_{G_1}(v) \cap [S_1]^{G_1}_{\theta_1}|\). Let \(S_2\) be an optimal target set for \((G_2, \theta_2)\). Now, with the definitions and notation introduced in this paragraph, we prove the following theorem.

**Theorem 2** \(S_1 \cup S_2\) is an optimal target set for \((G_1 \oplus_v G_2, \theta)\).

**Proof** Consider a sequential activation process in \((G_1 \oplus_v G_2, \theta)\) starting at target set \(S_1 \cup S_2\). Clearly \(N_{G_1}(v) \cap [S_1]^{G_1}_{\theta_1} \subseteq [S_1]^{G_1 \oplus_v G_2}_{\theta_1}\), and hence \(V(G_2) \subseteq [S_1 \cup S_2]^{G_1 \oplus_v G_2}_{\theta_1}\), which implies \(V(G_1) \subseteq [S_1 \cup S_2]^{G_1 \oplus_v G_2}_{\theta_1}\). That is the target set \(S_1 \cup S_2\) influences all vertices in \((G_1 \oplus_v G_2, \theta)\). To prove the theorem it remains to show that 
\(|S_1| + |S_2| = \min\text{-seed}(G_1 \oplus_v G_2, \theta)\).

Let \(S\) be an optimal target set for \((G_1 \oplus_v G_2, \theta)\) that minimizes the size of the set 
\(S \cap V(G_1 - v)\). Since \((S \cap V(G_1 - v)) \cup \{v\}\) influences all vertices in \((G_1, \theta)\), we have that \(S \cap V(G_1 - v)\) influences all vertices in \((G_1 - v, \theta_1)\). It now follows that \(|S \cap V(G_1 - v)| = |S_1|\) since if not, then we have \(|S \cap V(G_1 - v)| \geq |S_1| + 1\), and hence \((S_1 + v) \cup (S \cap V(G_2))\) is an optimal target set for \((G_1 \oplus_v G_2, \theta)\), a contradiction to the choice of \(S\). Therefore \(S \cap V(G_1 - v)\) is an optimal target set for \((G_1 - v, \theta_1)\). By the choice of \(S_1\), we see that \(|N_{G_1}(v) \cap [S \cap V(G_1 - v)]^{G_1}_{\theta_1}| \leq |N_{G_1}(v) \cap [S_1]^{G_1}_{\theta_1}|\). This implies that \(S_1 \cup [S \cap V(G_2)]\) is an optimal target set for \((G_1 \oplus_v G_2, \theta)\), and hence \(S \cap V(G_2)\) influences all vertices in \((G_2, \theta_2)\), which implies \(|S \cap V(G_2)| \geq |S_2|\). We conclude that 
\(|S_1| + |S_2| = |S \cap V(G_1 - v)| + |S_2| \leq |S \cap V(G_1 - v)| + |S \cap V(G_2)| = |S|\). Therefore \(S_1 \cup S_2\) is an optimal target set for \((G_1 \oplus_v G_2, \theta)\). \(\square\)

**Corollary 3** \(\min\text{-seed}(G_1 \oplus_v G_2, \theta) = \min\text{-seed}(G_1 - v, \theta_1) + \min\text{-seed}(G_2, \theta_2)\).

**Lemma 4** Let \(v\) be a vertex in the social network \((G, \theta)\). If \(G \in \{K_n, C_n\}\), then an optimal target set \(S\) for \((G - v, \theta_1)\) that maximizes the size of the set \(N_G(v) \cap [S]_\theta^G\) can be found in linear time, where \(\theta_1\) is the threshold function of \(G - v\) which is the same as the function \(\theta\), except that \(\theta_1(x) = \theta(x) - 1\) for every \(x \in N_G(v)\). Moreover, the size of the set \(N_G(v) \cap [S]_\theta^G\) can also be determined in linear time.

**Proof** Let \(\mathcal{F}\) be the set of optimal target sets \(S\) for \((G - v, \theta_1)\) such that \(S\) maximizes the size of the set \(N_G(v) \cap [S]_\theta^G\).

We first consider the case that \(G = K_n\). Let \(V(G - v) = \{v_1, v_2, \ldots, v_{n-1}\}\) such that \(\theta_1(v_1) \leq \theta_1(v_2) \leq \cdots \leq \theta_1(v_{n-1})\). Let \(S\) be an optimal target set for \((G - v, \theta_1)\). Since any two vertices \(v_i, v_{i+1}\) in \(G - v\) have \(\{v_i\} \cup N_{G - v}(v_i) = \{v_i, v_{i+1}\} \cup N_{G - v}(v_{i+1})\) and \(\theta_1(v_i) \leq \cdots \leq \theta_1(v_{n-1})\), we give the following simple observation without proof.

**Observation** If \(v_i \in S\) and \(v_{i+1} \notin S\), then \((S \setminus \{v_i\}) \cup \{v_{i+1}\}\) is an optimal target set for \((G - v, \theta_1)\) and \(|[(S \setminus \{v_i\}) \cup \{v_{i+1}\}]_\theta^G| \geq |[S]_\theta^G|\).
Since $G$ is a complete graph, the above observation says that if min-seed($G - v, \theta_1$) = $s$, then the target set $\{v_{n-1}, v_{n-2}, \ldots, v_{n-s}\} \in \mathcal{F}$. Moreover, such a target set has a convinced sequence $(v_1, v_2, \ldots, v_{n-s-1})$ on $(G - v, \theta_1)$. Now we are in a position to show that Algorithm $K$ outputs an optimal target set $S$ for $(G - v, \theta_1)$ such that $S \in \mathcal{F}$.

In steps 2–3 of the algorithm we see that min-seed($G - v, \theta_1$) $\geq \max \{v_i : \theta_1(v_i) > n - 2 \text{ and } 1 \leq i \leq n - 1\} = \ell$. In steps 4–8, we want to find the value $s$ such that \{v_{n-1}, v_{n-2}, \ldots, v_{n-\ell}\} $\cup$ \{v_{n-\ell-1}, v_{n-\ell-2}, \ldots, v_{n-s}\} $\in$ $\mathcal{F}$. During the $i$th iteration of the for loop in step 4, we have $(v_1, v_2, \ldots, v_{i-1}) \subseteq \{v_{n-1}, v_{n-2}, \ldots, v_{n-s}\}$. In step 6, when $\theta_1(v_i) > s + i - 1$, in order to influence vertex $v_i$ in $(G - v, \theta_1)$ we need to add another $\theta_1(v_i) - (s + i - 1)$ vertices to the set $\{v_{n-1}, v_{n-2}, \ldots, v_{n-s}\}$. Note that in step 5 we have $\theta_1(v_i) \leq n - 2$. If follows that after step 5 and before step 6 we have $n - (s + \theta_1(v_i) - (s + i - 1)) > i$. Therefore in step 7 if $n-s = i + 1$, then it must be min-seed($G - v, \theta_1$) = $s$, and hence $\{v_{n-1}, v_{n-2}, \ldots, v_{n-s}\} \in \mathcal{F}$. Clearly, the time complexity of Algorithm $K$ takes linear time, where the bucket sort algorithm is used to sort vertices by their thresholds.

Let $S$ be the output of the Algorithm $K$ and $|S| = s$. Let $V(G) \setminus S = \{u_1, u_2, \ldots, u_{n-s}\}$ such that $\theta(u_1) \leq \theta(u_2) \leq \cdots \leq \theta(u_{n-s})$. Let $U = \{i : \theta(u_i) > s + i - 1 \text{ and } 1 \leq i \leq n - s\}$. We define the value $r$ by

$$r = \begin{cases} 
\min U - 1, & \text{if } U \neq \emptyset; \\
n - s, & \text{if } U = \emptyset.
\end{cases}$$

(1)

Since $G$ is a complete graph, it can be seen that $[S]_{\theta}^G = S \cup \{u_1, u_2, \ldots, u_r\}$. Therefore the size of the set $N_G(v) \cap [S]_{\theta}^G$ can also be determined in linear time.

**Algorithm $K$**

Begin
1 $s \leftarrow 0$;
2 for $i = 1$ to $n - 1$ do if $\theta_1(v_i) > n - 2$ then $s \leftarrow s + 1$;
3 $\ell \leftarrow s$;
4 for $i = 1$ to $n - \ell - 1$ do
5 begin
6 if $\theta_1(v_i) > s + i - 1$ then $s \leftarrow s + [\theta_1(v_i) - (s + i - 1)]$;
7 if $n - s = i + 1$ then STOP and output $S = \{v_{n-1}, v_{n-2}, \ldots, v_{n-s}\}$;
8 end
End.

Finally, consider the remaining case that $G = C_n$. Let $E(G) = \{vv_1, vv_{n-1}\} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 2\}$. Thus $V(G - v) = \{v_1, v_2, \ldots, v_{n-1}\}$. Let $\mathcal{H}$ be the set of optimal target sets $S$ for $(G - v, \theta_1)$. First we consider the following Algorithm $C$ which computes $S_1$ and $S_2$. Clearly, $S_1 \subseteq S$ for each $S \in \mathcal{H}$ and $S_2 \subseteq [S_1]_{\theta}^{G-v}$.

**Algorithm $C$**

Begin
1 Find the set $S_1 = \{v_i : \theta_1(v_i) > d_{G-v}(v_i) \text{ and } 1 \leq i \leq n - 1\}$.

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Theorem 5 End.

Now Theorem 5 follows from Theorem 2 and Lemma 4 immediately.

**Theorem 5** If $G$ is a block-cactus graph, then an optimal target set for $(G, \theta)$ can be found in linear time.

3 Chordal graphs

A graph is called **chordal** if it does not have an induced cycle of length greater than three. A vertex $v$ in $G$ is called **simplicial** if the subgraph of $G$ induced by the neighbors of $v$ is complete. Let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of $V(G)$. We say that...
σ is a perfect elimination order if each \( v_i \) is a simplicial vertex of the subgraph \( G[v_i, v_{i+1}, \ldots, v_n] \). Fulkerson and Gross (1965) showed that every chordal graph has a perfect elimination order. In Rose et al. (1976), Tarjan and Yannakakis (1984) it was shown that if \( G \) is a chordal graph, then there is a linear time algorithm which receives the adjacency sets of \( G \) and outputs a perfect elimination order \( \sigma \) of \( V(G) \). For nonadjacent vertices \( u \) and \( v \) of a graph \( G \), a subset \( S \subseteq V(G) \) is called a \( u-v \) separator if the removal of \( S \) from \( G \) separates \( u \) and \( v \) into distinct connected components. If no proper subset of \( S \) is a \( u-v \) separator, then \( S \) is called a minimal \( u-v \) separator.

**Lemma 6** (Dirac 1961) Every chordal graph \( G \) has a simplicial vertex. Moreover, if \( G \) is not complete, then it has two nonadjacent simplicial vertices.

**Lemma 7** (Fulkerson and Gross 1965) For nonadjacent vertices \( u \) and \( v \) of a chordal graph \( G \), if \( S \) is a minimal \( u-v \) separator of \( G \), then \( S \) induces a complete subgraph of \( G \).

**Lemma 8** For \( t \geq 2 \), let \( G \) be a \( t \)-connected chordal graph with \( \theta(x) \leq t \) for all vertices \( x \). If \( S \subseteq V(G) \) induces a complete subgraph of size \( t \) in \( G \), then the target set \( S \) influences all vertices in \( (G, \theta) \).

**Proof** Without loss of generality, we may assume that \( G \) is not complete. Let \( |V(G)| = n \). To prove this theorem, we want to demonstrate a sequence of distinct vertices \( [v_1, v_2, \ldots, v_\ell] \) in \( G \) such that \( G - \{v_1, v_2, \ldots, v_\ell\} \) is a complete graph that contains all vertices of \( S \). Moreover, for \( 1 \leq i \leq \ell \), vertex \( v_i \) is adjacent to at least \( t \) vertices in the \( G - \{v_1, v_2, \ldots, v_\ell\} \). It is clear that if such a sequence exists, then the target set \( S \) influences all vertices in \( (G, \theta) \), since \( \theta(x) \leq t \) for all vertices \( x \) in \( G \).

To construct such a sequence, by Lemma 6, we can pick a simplicial vertex \( v_1 \) of \( G \) such that \( v_1 \notin S \). Note that \( G - v_1 \) is \( t \)-connected, since otherwise there is a set \( U \subseteq V(G - v_1) \) with \( |U| \leq t - 1 \) such that \( G - v_1 - U \) is disconnected. Since \( v_1 \) is a simplicial vertex of \( G \), it follows that \( G - U \) is disconnected, a contradiction to \( G \) is \( t \)-connected. Next, if \( G - v_1 \) is not complete, then by Lemma 6 again, we can pick a simplicial vertex \( v_2 \) of \( G - v_1 \) such that \( v_2 \notin S \). It can also be seen that \( G - v_1 - v_2 \) is \( t \)-connected. If we continue in this way, we eventually have a desired sequence of distinct vertices \( [v_1, v_2, \ldots, v_\ell] \) such that the graph \( G - \{v_1, v_2, \ldots, v_\ell\} \) is \( t \)-connected for each \( i \in \{1, 2, \ldots, \ell - 1\} \) and \( G - \{v_1, v_2, \ldots, v_\ell\} \) is a complete graph that contains all vertices of \( S \). Which completes the proof of the lemma.

**Theorem 9** Suppose that \( G \) is a \( t \)-connected chordal graph with \( t \geq 2 \). (a) \( \text{min-seed}(G, t) = t \). (b) If \( \theta(x) \leq t \) for each vertex \( x \) of \( G \) and \( \theta(v) < t \) for some vertex \( v \). Then \( \text{min-seed}(G, \theta) < t \).

**Proof** (a) By Lemma 6, the fact that \( G \) is a \( t \)-connected chordal graph implies that \( G \) contains a complete subgraph \( H \) of \( t \) vertices. By Lemma 8, we see that the target set \( V(H) \) influences all vertices in the social network \( (G, t) \), and hence
min-seed\((G, t) \leq t\). Note that an inactive vertex \(v\) in \((G, t)\) become active only if \(v\) has at least \(t\) already-active neighbors. It follows that min-seed\((G, t) \geq t\), which completes the proof of part (a).

(b) If \(v\) is adjacent to all other vertices of \(G\), then, by Lemma 6, \(G - v\) contains a complete subgraph \(H\) of size \(t - 1\), since \(G - v\) has a simplicial vertex and \(G\) is \(t\)-connected. It follows that, by Lemma 8, the target set \(V(H)\) influences all vertices in \((G, \theta)\), and hence min-seed\((G, \theta) < t\). Next consider the case that \(v\) is not adjacent to some vertex \(u\) in \(G\). Clearly there is a minimal \(v\)-\(u\) separator \(S\) such that \(v\) adjacent to all vertices of \(S\). Note that \(|S| \geq t\), since \(G\) is \(t\)-connected. Let \(S' \subseteq S\) with \(|S'| = t - 1\). By Lemma 7, \(S' \cup \{v\}\) induces a complete subgraph of size \(t\) in \(G\). It follows that, by Lemma 8 and the fact that \(\theta(v) \leq t - 1\), the target set \(S'\) influences all vertices of \((G, \theta)\). We conclude that min-seed\((G, \theta) < t\).

Corollary 10 Let \(G\) be a 2-connected chordal graph with thresholds \(\theta(v) \leq 2\) for every vertex \(v\) of \(G\). Then min-seed\((G, \theta) = 2\) if and only if \(\theta(v) = 2\) for each vertex \(v\) of \(G\).

In the sequel, for convenience, we write \(S \propto (G, \theta)\) to mean that the target set \(S\) influences all vertices in \((G, \theta)\). The following simple fact, which we state without proof, will be used implicitly and frequently in Lemma 12.

Claim 11 Let \(v\) be a vertex in the social network \((G, \theta)\) and let \(\theta_1\) be the threshold function of \(G - v\) which is the same as the function \(\theta\), except that \(\theta_1(x) = \theta(x) - 1\) for every \(x \in N_G(v)\). Then for \(S \subseteq V(G - v)\), we have \(S \propto (G - v, \theta_1)\) if and only if \(S \cup \{v\} \propto (G, \theta)\).

We state Lemma 12 using the same notation and conventions as in Claim 11.

Lemma 12 Let \(G\) be a 2-connected chordal graph with thresholds \(\theta(u) \leq 2\) for every \(u \in V(G)\). For a vertex \(v\) in \(G\), let \(F\) be the set of optimal target sets \(S\) for \((G - v, \theta_1)\) such that \(S\) maximizes the size of the set \(N_G(v) \cap [S]_G\). Let \(I = \{u \in V(G - v): \theta_1(u) \leq 0\}\), \(J = \{u \in V(G): \theta(u) < 2\}\) and \(J_0 = \{u \in V(G): \theta(u) \leq 0\}\). Let \(P_1\) (resp. \(Q_1\)) be the property that there are two distinct vertices \(x, y \in I\) (resp. \(x, y \in J_0\)) such that \(d_G(x, y) \leq 2\). Let \(P_2\) (resp. \(Q_2\)) be the property that there is an edge \(xy\) in \(G - v\) (resp. \(G\)) with \(x \in I\) (resp. \(x \in J_0\)) and \(\theta_1(y) = 1\) (resp. \(\theta(y) = 1\)). Then we have:

(a) If \(I \cap N_G(v) \neq \emptyset\), then \(\emptyset \in F\).
(b) If \(I \cap N_G(v) = \emptyset\) and \(P_1\) holds, then \(\emptyset \in F\).
(c) If \(I \cap N_G(v) = \emptyset\) and \(P_2\) holds, then \(\emptyset \in F\).
(d) If \(J = \emptyset\), then \([x] \in F\) and \([x]_G = \{x\}\) for every \(x \in N_G(v)\).
(e) If \(J \neq \emptyset\), then \(I \cap N_G(v) = \emptyset\) and neither \(P_1\) nor \(P_2\) holds, then \(\emptyset \in F\) and \([x]_G = V(G)\) for every vertex \(x\) adjacent to some vertex \(w \in J\).
(f) If \(Q_1\) or \(Q_2\) holds, then \([\emptyset]_G = V(G)\).
(g) If neither \(Q_1\) nor \(Q_2\) holds, then \([\emptyset]_G = J_0\).

Proof (a) Let \(w \in I \cap N_G(v)\). By the facts \(vw \in E(G), \theta(w) \leq 1\) and by Lemma 8, we see that \(\{v\} \propto (G, \theta)\), and hence \(\emptyset \propto (G - v, \theta_1)\).
Let

Theorem 13 If \( G \) and prove the following theorem.

\((G, \theta)\) then an optimal target set for \( NG(z) \) in terms of the size of \( V(G) \). Given two graphs \( G \) and \( H \), their Cartesian product is the graph \( G \Box H \) with vertex set \( V(G) \times V(H) \) and edge set \( \{(g, h)(g', h') : gg' \in E(G) \text{ with } h = h'\} \), or

\( \{v \} \sim (G, \theta) \), and hence \( \emptyset \sim (G - v, \theta_1) \).

(c) Since \( x \notin N_G(v) \), it can be seen that \( ([v])_\emptyset ^G \supseteq \{x, y, v\} \). By Lemma 8, it follows that \( \{v\} \sim (G, \theta) \), and hence \( \emptyset \sim (G - v, \theta_1) \).

(d) For each \( x \in N_G(v) \), by Lemma 8, \( \{x, v\} \sim (G, \theta) \), and hence \( \{x\} \sim (G - v, \theta_1) \). Clearly \( \text{min-seed}(G - v, \theta_1) \geq 1 \). It follows that \( \{x\} \) is an optimal target set for \( (G - v, \theta_1) \). Since \( \theta(u) = 2 \) for each \( u \in V(G) \), we have \( ([S])_\emptyset ^G = 1 \) for any optimal target set \( S \) for \( (G - v, \theta_1) \). Therefore \( \{x\} \in \mathcal{F} \) and \( ([x])_\emptyset ^G = \{x\} \).

(e) Note that \( I \cap N_G(v) = \emptyset \) implies that \( \theta(y) = 2 \) for each \( y \in N_G(v) \). We claim that \( \{v\} \) can not influence all vertices in \( (G, \theta) \). If not, then it must be that either \( \mathcal{P}_1 \) or \( \mathcal{P}_2 \) holds, a contradiction. Thus \( \text{min-seed}(G - v, \theta_1) \geq 1 \). Now let \( w \in J \) and \( x \in N_G(w) \). Note that \( x \neq v \). Clearly \( ([x])_\emptyset ^G \supseteq \{x, w\} \) and hence, by Lemma 8, \( \{x\} \sim (G, \theta) \). It follows that, by Claim 11, \( \{x\} \sim (G - v, \theta_1) \). Moreover, we have \( ([x])_\emptyset ^G = V(G) \), and hence \( \{x\} \in \mathcal{F} \). This completes the proof of (e).

Finally, by similar arguments as in the proofs of (c), (d) and (e), it is easy to prove (f) and (g), so we omit the proofs of (f) and (g).

Using the same notation and conventions as in Claim 11 and Lemma 12, we state and prove the following theorem.

Theorem 13 If \( G \) is a chordal graph with thresholds \( \theta(x) \leq 2 \) for each vertex \( x \in G \), then an optimal target set for \( (G, \theta) \) can be found in linear time.

Proof Let \( G_1 \) be a block of \( G \) which contains exactly one cut vertex \( v \) of \( G \). If \( G \) is not 2-connected, then \( G \) can be written as the following form: \( G = G_1 \oplus_v G_2 \), where \( G_2 \) is an induced subgraph of \( G \) and is also chordal. To prove the theorem, we omit the easy case \( G_1 = K_2 \), which follows from Lemma 4. We only consider the case that \( G_1 \) is a 2-connected chordal graph. By using Lemma 12, we can in linear time in terms of the size of \( G_1 \) find an optimal target sets \( S_1 \) for \( (G_1 - v, \theta_1) \) such that \( S_1 \) maximizes the size of the set \( N_{G_1}(v) \cap [S_1]_\emptyset ^{G_1} \) and compute \( |N_{G_1}(v) \cap [S_1]_\emptyset ^{G_1}| \).

Next, we want to find an optimal target set \( S_2 \) for \( (G_2, \theta_2) \), where \( \theta_2 \) is a threshold function of \( G_2 \) which is the same as the function \( \theta \), except that \( \theta_2(v) = \theta(v) - |N_{G_1}(v) \cap [S_1]_\emptyset ^{G_1}| \). If \( G_2 \) is a 2-connected chordal graph, then \( S_2 \) can be found in linear time in terms of the size of \( G_2 \) by using Lemma 8 and Corollary 10, and hence an optimal target set \( S_1 \cup S_2 \) for \( (G, \theta) \) can be found in linear time by using Theorem 2.

If \( G_2 \) has a cut vertex \( v' \) and a pendant block \( G_{21} \) such that \( G_2 = G_{21} \oplus_{v'} G_{22} \), then we can repeat the arguments in the previous paragraphs and use Theorem 2 to find the desired \( S_2 \) in linear time in terms of the size of \( G_2 \), and hence an optimal target set for \( (G, \theta) \) can be found in linear time.

4 Hamming graphs

Given two graphs \( G \) and \( H \), their Cartesian product is the graph \( G \Box H \) with vertex set \( V(G) \times V(H) \) and edge set \( \{(g, h)(g', h') : gg' \in E(G) \text{ with } h = h'\} \), or
$g = g'$ with $hh' \in E(H)$. The Cartesian product is commutative and associative (see page 29 of Imrich and Klavžar 2000). A Hamming graph is a Cartesian product of nontrivial complete graphs, i.e., of the form $K_{n_1} \square K_{n_2} \square \cdots \square K_{n_t}$ for some integers $n_1, \ldots, n_t \geq 2, t \geq 1$, which is also denoted as $\prod_{i=1}^{t} K_{n_i}$. Note that $\prod_{i=1}^{t} K_{n_i}$ has vertex set $V(K_{n_1}) \times V(K_{n_2}) \times \cdots \times V(K_{n_t})$.

Let $u = (u_1, \ldots, u_t)$ and $v = (v_1, \ldots, v_t)$ be two vertices of $\prod_{i=1}^{t} K_{n_i}$. The Hamming distance $H(u, v)$ between $u$ and $v$ is the number of coordinate positions in which $u$ and $v$ differ. Note that there is an edge between $u$ and $v$ if and only if $H(u, v) = 1$. For $S_1, S_2 \subseteq V(\prod_{i=1}^{t} K_{n_i})$, denote by $d(S_1, S_2)$ the value $\min\{H(u, v) : u \in S_1, v \in S_2\}$. Let $[i, j]$ denote the set of integers $k$ such that $i \leq k \leq j$. For $A \subseteq [1, t]$, if $u_i = v_i$ for all $i \in A$, then we write $u_A = v_A$. Let $u_A$ denote the set of vertices $x$ in $\prod_{i=1}^{t} K_{n_i}$ such that $x|A = u|A$. The proof of the following claim is straightforward and hence omitted.

**Claim 14** Let $u, v, w$ be three distinct vertices of $\prod_{i=1}^{t} K_{n_i}$ and $u_A = v_A$ for some set $A \subseteq [1, t]$. If $w$ is adjacent to both $u$ and $v$, then $w|A = u|A = v|A$.

**Lemma 15** Suppose $G = (V, E)$ is the Hamming graph $\prod_{i=1}^{t} K_{n_i}$. Let $x, y \in V$, $i, j \in [1, t]$, and $A, B \subseteq [1, t]$. The following properties hold.

(a) If $xy \in E$ and $x_i \neq y_i$, then $[x_A \cup \{y\}]_2^G = x_A \setminus \{i\}$.
(b) If $x \in A \cap B$.
(c) If $xy \in E$ and $x_i \neq y_i$, then $[x_A \cup y_B]_2^G = x(A \cap B) \setminus \{i\}$.
(d) If $H(x, y) = 2$, $x_i \neq y_i$, $x_j \neq y_j$ and $i \neq j$, then $[x_A \cup y_B]_2^G = x(A \cap B) \setminus \{i, j\}$.
(e) If $d(x_A, y_B) \geq 3$, then $[x_A \cup y_B]_2^G = x_A \cup y_B$.

**Proof** (a) First let us consider the case of $i \notin A$. From Claim 14 and the fact $x|A = y|A$, we see that $[x_A \cup \{y\}]_2^G = x_A$. Now we consider the remaining case $i \in A$. To prove this case it suffices to consider the case that $i = 1$ and $A = [1, j]$. We want to prove, by induction on $j$, that $[x_{[1, j}] \cup \{y\}]_2^G = x_{[2, j]}$ for $j = t, t - 1, \ldots, 1$. For $j = t$, we see that $[x_{[1, j}] \cup \{y\}]_2^G = \{(x, y)\}$. Since $x_{[2, t]} = y_{[2, t]}$, it follows from Claim 14 that if $w \in \{(x, y)\}$ then $w_{[1, 2]} = x_{[2, t]} = y_{[2, t]}$, and hence $w \in x_{[2, t]}$. That is $\{(x, y)\} \subseteq x_{[2, t]}$. Since any vertex in $x_{[2, t]} \setminus \{(x, y)\}$ is adjacent to both $x$ and $y$, it follows that $\{(x, y)\} \supseteq x_{[2, t]}$. Therefore $\{(x, y)\} = x_{[2, t]}$.

Next, we assume that $[x_{[1, j]} \cup \{y\}]_2^G = x_{[2, j]}$ holds for some $j \in [2, t]$. From this induction hypothesis it follows that $x_{[2, j-1]} \subseteq [x_{[1, j-1]} \cup \{y\}]_2^G$. For any vertex $w$ in $x_{[2, j-1]}$, either $w \in x_{[2, j]} \cup x_{[1, j-1]}$ or $w$ is adjacent to at least one vertex in $x_{[2, j]}$ and at least one vertex in $x_{[1, j-1]}$. Thus $x_{[2, j-1]} \subseteq [x_{[1, j-1]} \cup \{y\}]_2^G$. On the other hand, by the fact $x_{[2, j-1]} = y_{[2, j-1]}$ and Claim 14, we also see that $x_{[2, j-1]} \supseteq [x_{[1, j-1]} \cup \{y\}]_2^G$. Therefore $[x_{[1, j-1]} \cup \{y\}]_2^G = x_{[2, j-1]}$, this completes the proof of Lemma 15(a).

(b) Since for any $i \in A \setminus B$, there exists a vertex $v \in E$ such that $xy \in E$ and $x_i \neq y_i$, by Lemma 15(a), it follows that $[x_A \cup y_B]_2^G \supseteq x_A \setminus (A \cap B) = x_{A \cap B}$. We note that if a vertex $w$ is adjacent to at least two vertices in $x_A \cup y_B$, then, by Claim 14, it must be the case that $w_{A \cap B} = x_{A \cap B}$. Therefore $[x_A \cup y_B]_2^G \supseteq x_{A \cap B}$. We conclude that $[x_A \cup y_B]_2^G = x_{A \cap B}$.
(c) Lemma 15(a) shows that \([x_A \cup y_B]_2^G \supseteq [x_A \cup \{y\}]_2^G = x_A \setminus \{i\}\), and hence \([x_A \cup y_B]_2^G = [x_A \setminus \{i\} \cup y_B]_2^G\), since \(x_A \subseteq x_A \setminus \{i\}\). It follows that \([x_A \cup y_B]_2^G = [y_A \setminus \{i\} \cup y_B]_2^G = y_A \setminus B = x_A \cap B \setminus \{i\}\), by Lemma 15(b) and the fact that \(x_A \setminus \{i\} = y_A \setminus \{i\}\).

(d) Without loss of generality, consider only the case \([i, j] = \{1, 2\}\). Clearly there exist two vertices \(w, z\) in \(G\) such that \((w_1, w_2) = (y_1, x_2), (z_1, z_2) = (x_1, y_2)\) and \(w_1[3, i] = z_1[3, i] = x_1[3, i] = y_1[3, i]\). Since \(w\) and \(z\) are each adjacent to both \(x\) and \(y\), we see that \(x_A \cup y_B\) influences \((w, z)\) in the social network \((G, 2)\). It follows that \([x_A \cup y_B]_2^G \supseteq [x_A \cup \{w\} \cup \{z\}]_2^G\) and \([x_A \cup y_B]_2^G \supseteq \{y_B \cup \{w\} \cup \{z\}\}_2^G\). By using Lemma 15(a) twice, we see that \([x_A \cup \{w\} \cup \{z\}]_2^G \supseteq x_A \setminus \{1, 2\}\), and hence \([y_B \cup \{w\} \cup \{z\}]_2^G \supseteq B\). By the fact \(y_B \setminus \{1, 2\} = x_B \setminus \{1, 2\}\) and using Lemma 15(b), we get that \([x_A \cup y_B]_2^G \supseteq [x_A \setminus \{1, 2\} \cup y_B \setminus \{1, 2\}]_2^G = x_A \cap B \setminus \{1, 2\}\). Since \((x_A \cup y_B) \subseteq x_A \cap B \setminus \{1, 2\}\), we conclude that \([x_A \cup y_B]_2^G = x_A \cap B \setminus \{1, 2\}\).

(e) For a vertex \(w\) in \(V \setminus (x_A \cup y_B)\), by Claim 14, we see that \(w\) cannot be adjacent to two distinct vertices in \(x_A\) (resp. \(y_B\)). Note that since \(d(x_A, y_B)\geq 3\) there is no vertex \(w\) in \(V \setminus (x_A \cup y_B)\) that is adjacent to one vertex in \(x_A\) and is also adjacent to one vertex in \(y_B\). This completes the proof of (e). \(\square\)

For \(\mathcal{U} \subseteq [1, t]\), let \(\overline{\mathcal{U}}\) denote the set \([1, t] \setminus \mathcal{U}\). Using the notation and results in Lemma 15, we immediately obtain the following:

**Claim 16** (a) If \(x_A \cap y_B \neq \emptyset\), then \([x_A \cup y_B]_2^G = x_A \cap B\).

(b) If \(d(x_A, y_B) = 1\), then \([x_A \cup y_B]_2^G = x_A \cap B \setminus \{i\}\) for some \(i \in [1, t]\).

(c) If \(d(x_A, y_B) = 2\), then \([x_A \cup y_B]_2^G = x_A \cap B \setminus \{i, j\}\) for some \(i, j \in [1, t]\).

**Theorem 17** Suppose \(G = (V, E)\) is the Hamming graph \(\prod_{i=1}^t K_{n_i}\) and \(S\) is a non-empty set of vertices.

(a) There exist vertices \(x^1, x^2, \ldots, x^k \in V\) and sets \(A_1, A_2, \ldots, A_k \subseteq [1, t]\) such that \([S]_2^G = \bigcup_{i=1}^k x^i_{A_i}\) with \(d(x^i_{A_i}, x^j_{A_j}) \geq 3\) for any \(1 \leq i < j \leq k\).

(b) If \([S]_2^G = \bigcup_{i=1}^k x^i_{A_i}\) for some vertices \(x^1, \ldots, x^k \in V\) and some sets \(A_1, \ldots, A_k \subseteq [1, t]\) with \(d(x^i_{A_i}, x^j_{A_j}) \geq 3\) for any \(1 \leq i < j \leq k\), then the following inequality holds:

\[
\sum_{i=1}^k |A_i| \geq (2 + t)k - 2|S|.
\]

**Proof** (a) Note that \(S = \bigcup_{x \in S} x_{[1, t]}\) and \([S']_2^G = [S]_2^G = [S']_2^G = S''_2^G\) for any \(S', S'' \subseteq V\). By using several times Claim 16 and Lemma 15(e), we can get vertices \(x^1, x^2, \ldots, x^k \in V\) and sets \(A_1, A_2, \ldots, A_k \subseteq [1, t]\) such that \([S]_2^G = \bigcup_{i=1}^k x^i_{A_i}\) with \(d(x^i_{A_i}, x^j_{A_j}) \geq 3\) for any \(1 \leq i < j \leq k\).

(b) To prove this part we use induction on the size of \(S\). When \(|S| = 1\) (say \(S = \{x^1\}\)), since in this scenario \([S]_2^G = x^1_{[1, t]}\), it can be seen that the inequality (\(\ast\)) clearly holds. Now assume that the statement of Theorem 17(b) holds for any \(S \subseteq V\) having \(|S| < \ell\).

When \(|S| = \ell \geq 2\), the proof is divided into cases according to the value of \(k\).
Case 1. $k = 1$. In this case, pick $x \in S$ and let $S' = S \setminus \{x\}$. Note that $S'$ is not empty. By Theorem 17(a) we see that there are vertices $y^1, y^2, \ldots, y^r \in V$ and sets $B_1, B_2, \ldots, B_r \subseteq [1, t]$ such that $[S']^G = \bigcup_{i=1}^r y^i_{B_i}$ having $d(y^i_{B_i}, y^j_{B_j}) \geq 3$ for any $1 \leq i < j \leq r$. We have $x^1_{A_1} = [S]^G = [x_{1,1}] \cup [S']^G = [x_{1,1}] \cup \bigcup_{i=1}^r y^i_{B_i}$. Then from Claim 16 and Lemma 15(e) we see that $A_1 = (\bigcap_{i=1}^r B_i) \cap \overline{U}$ for some set $U \subseteq [1, t]$ having $|U| \leq 2r$. Since $|S'| < \ell$, by the induction hypothesis we have $\overline{S'}^\star \cap \bigcup_{i=1}^r y^i_{B_i} 
eq \emptyset$. Clearly $[S'^\star]^G = x^1_{A_1}$ and $(S')^G = \bigcup_{i=2}^k x^i_{\tilde{A}_i}$. By the induction hypothesis we see that $|A_1| \geq (2 + t) - 2|S'|$ and $\sum_{i=2}^k |A_i| \geq (2 + t)(k - 1) - 2|S'|$. It follows immediately that inequality $(\star)$ holds in this case. This completes the proof of the theorem.

Theorem 18. If $G = (V, E)$ is the Hamming graph $\prod_{i=1}^t K_{n_i}$, then min-seed$(G, 2) = 1 + \lceil \frac{\ell}{2} \rceil$.

Proof. Note that $V = V(K_{n_1}) \times V(K_{n_2}) \times \cdots \times V(K_{n_t})$. For each $i = 1, 2, \ldots, t$, we pick two distinct vertices $x_i, y_i \in V(K_{n_i})$. Let $x = (x_1, x_2, \ldots, x_t)$. For $1 \leq j \leq t$, let $p^j = (p^j_1, \ldots, p^j_t)$ be a vertex in $V$ such that $p^j_i = x_i$ when $i \neq j$, and $p^j_j = y_j$. For $1 \leq j \leq t - 1$, let $q^j = (q^j_1, \ldots, q^j_t)$ be a vertex in $V$ such that $q^j_i = x_i$ when $i \notin \{j, j + 1\}$, and $q^j_j = y_j, q^j_{j+1} = y_{j+1}$.

First, we want to show that $1 + \lceil \frac{\ell}{2} \rceil$ is an upper bound for min-seed$(G, 2)$. The proof is divided into two cases according to the parity of $t$.

Case 1. $t = 2\ell$. Let $S = \{p^1, p^2\} \cup \{q^3, q^5, q^7, \ldots, q^{t-1}\}$. By Lemma 15(d) it can be seen that $[\{p^1, p^2\}]^G = [p^1_{3,1}] = x_{3,1}, [\{p^1, p^2, q^3\}]^G = [x_{3,1}] \cup q^3_{1,1}]^G = x_{3,1}$, and $[\{p^1, p^2, q^3, q^5\}]^G = [x_{5,1}] \cup q^5_{1,1}]^G = x_{5,1}$. Continue in this way, we obtain $[S]^G = [x_{t-1,1}] \cup q^{t-1}_{1,1}]^G = x_\emptyset = V$, which means that min-seed$(G, 2) \leq |S| = \ell + 1 = 1 + \lceil \frac{\ell}{2} \rceil$.

Case 2. $t = 2\ell + 1$. Let $S = \{p^1, p^2, p^3\} \cup \{q^4, q^6, q^8, \ldots, q^{t-1}\}$. By Lemma 15(d) and the same arguments as above, we obtain $[S]^G = [x_{t-1,1}] \cup q^{t-1}_{1,1}]^G = V$, and hence min-seed$(G, 2) \leq |S| = \ell + 2 = 1 + \lceil \frac{\ell}{2} \rceil$.

To show that $1 + \lceil \frac{\ell}{2} \rceil$ is also a lower bound for min-seed$(G, 2)$, let $S$ be an optimal target set for $(G, 2)$. Since $[S]^G = V = x_\emptyset$, by Theorem 17(b), we have $|S| \geq (2 + t) - 2|S'|$, that is $|S| \geq 1 + \frac{\ell}{2}$. Which completes the proof of the theorem.

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