Mathematical analysis on the cosets of subgroup in the group of E-convex sets

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Abstract. In this work, analyzing the cosets of the subgroup in the group of \( L \) – convex sets is presented as a new and powerful tool in the topics of the convex analysis and abstract algebra. On \( L \) – convex sets, the properties of these cosets are proved mathematically. Most important theorem on a finite group of \( L \) – convex sets theory which is the Lagrange's Theorem has been proved. As well as, the mathematical proof of the quotient group of \( L \) – convex sets is presented.

Keywords: \( L \) – convex sets, Group of \( L \) – convex sets, Subgroup of \( L \) – convex sets, Cosets of \( L \) – convex sets, Quotient group of \( L \) – convex sets.

1. Introduction
The convex sets and convex functions are studied by many researchers. These concepts have been extended into \( L \) – convex sets and \( L \) – convex functions respectively by Youness [1]. In addition to, Suneja, Lalitha and Govil [2] follow the Youness study through characterizing the \( L \) – convex sets and proving some certain inequalities. On these sets, some properties are discussed and proved [2]. On the set of the \( L \) – convex sets, the axioms of the group are proved under the addition operation [3]. With addition operation, the set of the \( L \) – convex sets also formed a subgroup. One can observe that the intersection on the subgroups of the \( L \) – convex sets is also a subgroup of the \( L \) – convex sets. On the other hand, the union of two subgroups of the \( L \) – convex sets is also a subgroup of the \( L \) – convex sets with some conditions [3]. Our work in this paper will focus on using these sets, namely the \( L \) – convex sets [1], to build new algebraic structures. In this work, first the \( L \) – convex sets and their operations, the group law on the set of \( L \) – convex sets and subgroup of this group have been studied. On the subgroup of the group on the \( L \) – convex sets, the left (right) cosets have been analyzed mathematically as the main points in this paper. The Lagrange's Theorem on a finite group of \( L \) – convex sets is also proved. Finally, the concept of the quotient group has been displayed as well in this work. The outline of this paper shows: Section 2 gives a summary of the mathematical background to clarify the \( L \) – convex sets, the algebraic properties and the operations on \( L \) – convex sets, the group law on the \( L \) – convex sets and the subgroup of the \( L \) – convex sets. Section 3 explains the proposed idea of the cosets of a subgroup of the \( L \) – convex sets. New proposed definition and theorems related to these cosets of a subgroup on the \( L \) – convex sets are presented and proved. Section 4 reviews the concept of the normal subgroup of the \( L \) – convex sets and proves a new important theorem related to this concept. Section 5 displays the
defintion of quotient group on the \( L \)-convex sets and how to prove this group on the \( L \)-convex sets. Finally, Section 6 draws the conclusions.

2. Mathematical Background

This section presents some important basic facts which are starting with the \( L \)-convex sets and their operations, the group law on the \( L \)-convex sets and the subgroup of the \( L \)-convex sets. The explanations of these concepts have been presented as follows.

Definition 2.1. Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) be two elements in \( \mathbb{R}^n \). The inner product \([4,5,6]\) between \( a \) and \( b \) can be defined by \( \langle a, b \rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n \).

Definition 2.2. Let \( L \) be a map from \( \mathbb{R}^n \) into \( \mathbb{R}^n \). A set \( W \subseteq \mathbb{R}^n \) is called \( L \)-convex set if \((1 - \kappa)L(a) + \kappa L(b) \in W \) for \( a, b \in W \) with \( 0 \leq \kappa \leq 1 \) [8].

Remark 2.1. Based on the study of Youness [1], one can prove:

- Every convex set is \( L \)-convex set with \( L \) is an identity map.
- If \( W \) is \( L \)-convex then \( L(W) \subseteq W \).
- If \( W \) is convex and \( L(W) \subseteq W \) for a map \( L \) then \( W \) is \( L \)-convex.
- The intersection of the \( L \)-convex sets is also \( L \)-convex.
- The union of \( L \)-convex sets is not necessarily \( L \)-convex; see the following example.

Example 2.3. Consider \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined as \( L(a,b) = (2b/3 - a/3, b/3 + 4a/3) \) and consider the two sets

\[
W_1 = \{(a,b) \in \mathbb{R}^2 : (a,b) = \kappa_1(0,0) + \kappa_2(2,1) + \kappa_3(0,3)\},
\]

\[
W_2 = \{(a,b) \in \mathbb{R}^2 : (a,b) = \kappa_1(0,0) + \kappa_2(0,3) + \kappa_3(-2,-1)\}
\]

With \( \kappa_1, \kappa_2, \kappa_3 \geq 0 \) and \( \sum_{i=1}^{3} \kappa_i = 1 \). The two sets \( W_1 \) and \( W_2 \) are \( L \)-convex, but \( W_1 \cup W_2 \) is not \( L \)-convex.

Theorem 2.4. The union of two \( L \)-convex sets is also \( L \)-convex set if and only if one is contained in another one.

On the \( L \)-convex sets , the mathematical operations have been studied through the following propositions [2].

Theorem 2.5. Let \( W_1 \) and \( W_2 \) be \( L \)-convex sets. Let \( L : \mathbb{R}^n \to \mathbb{R}^n \) be linear. Then \( \eta W_1 + \eta W_2 \) is an \( L \)-convex set convex for all \( \eta \) which is a real number [2].

A special case of Theorem (2.5) can be obtained through the following corollary.

Corollary 2.6. If \( \eta = 0 \) and \( W \) is \( L \)-convex. Then \( \eta W \) is \( L \)-convex set.

Remark 2.7. In Corollary (2.5), the \( L \)-convex set, namely \( 0W \), will be considered as an identity element in the additive group \( G \) of the \( L \)-convex sets.

On the other hand, some definitions on the group of the set of \( L \)-convex sets are presented. Further theorems which have been proved to show that the axioms of the additive group are verified on the set \( L \)-convex sets is displayed [3]. These definitions and theorems are discussed as follows.
Definition 2.8. Let G be a collection of all L – convex sets $W_i$ with a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $i = 1, 2, ..., n$. The binary operation $+$ on G is a function $+ : G \times G \rightarrow G$. That is a rule that assigns to each order pair $(W_i, W_j) \in G \times G$ into an element $W_i + W_j \in G$ [3].

Definition 2.9. Let G be a nonempty L – convex sets with a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let + be a binary operation on G. Then the mathematical system $(G, +)$ is called a group if the following axioms are holds [3]:

- If $W_1, W_2 \in G$, where $W_1, W_2$ are L – convex sets then $W_1 + W_2$ is an L – convex sets and $W_1 + W_2 \in G$.
- If $W$ is an element in G then there exists an element $0W = e$ in G such that $W + 0W = W$. The element $0W$ is called identity element of G.
- For each $W \in G$, there exists $-W \in G$ such that $W + (-W) = 0W$. An element $-W$ is called an inverse element of $W$ and it is denoted by $-W = (W)^{-1}$.

Theorem 2.10. The group G is called abelian if $W_1 + W_2 = W_2 + W_1$, for all $W_1, W_2$.

Theorem 2.11. For all $W_1 + W_2 \in G$, where $W_1$ and $W_2$ are E – convex sets, then $W_1 + W_2 = W_2 + W_1$.

Theorem 2.12. For all $W_1, W_2$ and $W_3$ in a group G, where $W_1, W_2$ and $W_3$ are L – convex sets, then $(W_1 + W_2) + W_3 = W_1 + (W_2 + W_3)$ (that is an associative law of the addition operation) [3].

Theorem 2.13. Suppose $W_1, W_2$ and $W_3$ are elements (that is, L – convex sets) in a group G. If $W_1 + W_2 = W_3 + W_2$. Then $W_1 = W_3$. And if $W_2 + W_1 = W_2 + W_3$. Then $W_1 = W_3$ [3].

Theorem 2.14. For each element $W$ in a group G of the L – convex sets, there is a unique inverse element $W$ in G such that $W + (-W) = 0W$. The mathematical proofs of the above theorems can be seen in [3].

On a set of all L – convex sets with addition operation, new mathematical structure can be created. This structure is called subgroup (see [9,10,11]) of the set of all L – convex sets. New definitions and theorems are presented as follows.

Definition 2.17. Let G be a group of the L – convex sets. A subset H of a group G is a subgroup [3] if

- $0W = e \in H$
- If $W_1, W_2 \in H$, then $W_1 + W_2 \in H$;
- If $W \in H$, then $-W \in H$.
Example 2.18. The set $\{W\}$ and G are always subgroups of a group G [3].

Definition 2.19. A subgroup H of a group G is called a proper subgroup if $H \neq G$. Whereas, a subgroup H of a group G is called a nontrivial if $H \neq \{W\}$ [3].

Theorem 2.20. A subset H of a group G is a subgroup if and only if H is a nonempty set and if $W_1, W_2 \in H$, then $W_1 + (-W_2) \in H$ [3].

Theorem 2.21. Let $H_1$ and $H_2$ be two subgroups of a group G. Then $H_1 \cap H_2$ is a subgroup of G [3].

Theorem 2.22. Let $H_1$ and $H_2$ be two subgroups of the L-convex sets of a group G of the L-convex sets. Then $H_1 \cup H_2$ is a subgroup of L-convex sets of G if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. The Cosets of a subgroup H of the L-convex sets [3]. The full proofs of the previous theorems, namely Theorems 2.10, 2.11 and 2.12 have been presented in [3].

3. The cosets of a subgroup of the L-convex sets
In this section, the mathematical concepts on the cosets of a subgroup of the L-convex sets are proposed as a new contribution in this work. These concepts are discussed as follows.

Definition 3.1. Let H be a subgroup of the group G of the L-convex sets. For any $W \in G$. The set $W + H = \{W + h : h \in H\}$ is called the left coset of H in G which contains W and $H + W = \{h + W : h \in H\}$ is called the right Coset of H in G containing W. The element W is called the Coset representative of $W + H$.

Theorem 3.2. Suppose G is a group of the L-convex sets W. Let H is a subgroup of G. Let $L : R^n \rightarrow R^n$ is a linear map with $L(W) = W$ for some $W \in R^n$. Then the left Coset $W + H$ is a L-convex set.

Proof. Let $W + h_1, W + h_2 \in W + H$, where $h_1, h_2$ are the L-convex sets in subgroup H and $0 \leq \kappa \leq 1$. Then

$\kappa[L(W + h_1)] + (1 - \kappa)L(W + h_2) = \kappa[L(W) + L(h_1)] + (1 - \kappa)[L(W) + L(h_2)]$

$= \kappa L(W) + \kappa L(h_1) + (1 - \kappa)L(W) + (1 - \kappa)L(h_2)$

$= \kappa L(W) + (1 - \kappa)L(W) + \kappa L(h_1) + (1 - \kappa)L(h_2)$

$= \kappa L(W) + [\kappa L(h_1) + (1 - \kappa)L(h_2)] \in W + H$

So, the left coset $W + H$ is an L-convex set. ■

Theorem 3.3. Let G be a group of the L-convex sets. Suppose H is a subgroup of G. Let $W_1 \in G$ then $W_1 \in W_1 + H$.

Proof. The element $W_1 + W_1 \in W_1 + H$ and $W_1 + W_1 \in W_1 + H$. Since $W_1 + W_1 = W_1 \in W_1 + H$. So $W_1 \in W_1 + H$. ■

Theorem 3.4. Suppose G is a group of the L-convex sets. Let H be a subgroup of G and $W_1 \in G$. Then $W_1 + H = H$ if and only if $W_1 \in H$. ■
Proof. Assume that \( W_1 + H = H \). From Theorem (2), \( W_1 \in W_1 + H \). Since \( W_1 + H = H \), then \( W_1 \in H \). Conversely, assume \( W_1 \in H \). Again, from Theorem (2) \( W_1 \in W_1 + H \). Then \( W_1 + H \subseteq H \).\hspace{1cm}(1)

On the other hand, \( W_1 \in H \) then \(-W_1 \in H \). Let \( W \in H \), then \(-W_1 + W \in H \). So, \(-W_1 + W = h \), for some \( h \in H \). Hence \( W = W_1 + h \in W_1 + H \). Therefore,
\[
H \subseteq W_1 + H.\hspace{1cm}(2)
\]

From (1) and (2), one can obtain \( W_1 + H = H \).

Theorem 3.5. Suppose \( G \) is a group of the \( L \)-convex sets. Let \( H \) be a subgroup of \( G \). For any \( W_1, W_2 \in G \), then \( W_1 + H = W_2 + H \) if and only if \( W_1 \in W_2 + H \).

Proof. Assume \( W_1 + H = W_2 + H \). Based on Theorem (3.2), one can get \( W_1 \in W_2 + H \). Conversely, assume \( W_1 \in W_2 + H \). Then \( W_1 = W_2 + h \), for some \( h \in H \).

So \( W_1 + H = (W_2 + h) + H \). Hence \( W_1 + H = W_2 + (h + H) \). Therefore \( W_1 + H = W_2 + H \).

Theorem 3.6. Suppose \( G \) is a group of the \( L \)-convex sets. Let \( H \) be a subgroup of \( G \) and let \( W_1, W_2 \in G \). Then either \( W_1 + H = W_2 + H \) or \( W_1 + H \cap W_2 + H = \phi \).

Proof. Assume that \( (W_1 + H \cap W_2 + H) \neq \phi \). Let \( W \in (W_1 + H \cap W_2 + H) \). Then \( W \in W_1 + H \) and \( W \in W_2 + H \). So \( W = W_1 + h_1 \) and \( W = W_2 + h_2 \). Therefore, \( W_1 + h_1 = W_2 + h_2 \). Hence \( (W_1 + h_1) + H = (W_2 + h_2) + H \). This leads to \( W_1 + (h_1 + H) = W_2 + (h_2 + H) \). So \( W_1 + H = W_2 + H \).

Theorem 3.7. Suppose \( G \) is a group of the \( L \)-convex sets. A subgroup \( H \) of \( G \) and let \( W_1, W_2 \in G \). Then \( W_1 + H = W_2 + H \) if and only if \(-W_1 + W_2 \in H \).

Proof. Assume that \( W_1 + H = W_2 + H \) if and only if \(-W_1 + (W_1 + H) = -W_1 + (W_2 + H) \) if and only if \((-W_1 + W_1) + H = (-W_1 + W_2) + H \) if and only if \( H = (-W_1 + W_2) + H \) if and only if \(-W_1 + W_2 \in H \) based on Theorem (3.3).

Theorem 3.8. Let \( H \) be a subgroup of a group \( G \) which has elements are the \( L \)-convex sets. Suppose \( W_1, W_2 \in G \). Then \( |W_1 + H| = |W_2 + H| \).

Proof. Let \( \varphi : W_1 + H \rightarrow W_2 + H \) which is defined by \( \varphi(W_1 + h) = W_2 + h \), for all \( E \)-convex set \( h \in H \). Suppose \( h_1, h_2 \) are two \( E \)-convex set in \( H \). we need first to prove that \( \varphi \) is one to one function. Assume that \( \varphi(W_1 + h_1) = \varphi(W_1 + h_2) \). Then \( W_2 + h_1 = W_2 + h_2 \), so \( h_1 = h_2 \). Thus, \( W_1 + h_1 = W_1 + h_2 \). So \( \varphi \) is a one to one function. Now, it is clear that the function \( \varphi \) is onto. Let \( W \in W_2 + H \). Then \( W = W_2 + h \) for some \( h \in H \). So \( W = \varphi(W_1 + h) \). Hence \( \varphi \) is one to one and onto function. Therefore, \( |W_1 + H| = |W_2 + H| \).
Theorem 3.9. Let \( H \) be a subgroup of a group \( G \) which has elements are the \( L^- \) convex sets. Suppose \( W_1 \in G \). Then \( W_1 + H = H + W_1 \) if and only if \( H = W_1 + H + (-W_1) \).

Proof. \( W_1 + H = H + W_1 \) if and only if \( (W_1 + H) + (-W_1) = (H + W_1) + (-W_1) \) if and only if \( W_1 + H + (-W_1) = H + (W_1 + (-W_1)) \) if and only if \( W_1 + H + (-W_1) = H \).

Theorem 3.10. Let \( H \) be a subgroup of a group \( G \) which has elements are the \( L^- \) convex sets. Suppose \( W_1 \in G \). Then \( W_1 + H \) is a subgroup of \( G \) if and only if \( W_1 \in H \).

Proof. Assume that \( W_1 + H \) is a subgroup of \( G \) Then \( W_1 \in W_1 + H \) and \( W_1 + H \in H = W_1 + H \). So \( W_1 \in W_1 + H \) and \( W_1 + H \). Then \( W_1 + H \cap W_1 + H \neq \phi \). By Theorem (3.5) we have \( W_1 + H = W_1 + H \). So \( W_1 + H = H \). By Theorem (3.3) \( W_1 \in H \). Conversely, assume that \( W_1 \in H \). So \( W_1 + H = H \). Since \( W_1 \) is a subgroup of a group \( G \) So \( W_1 + H \) is a subgroup of a group \( G \).

Theorem 3.11. (Lagrange's Theorem). If \( G \) is a finite group of \( L^- \) convex sets and \( H \) be a subgroup of a group \( G \). Then \( |G| \) is divisible by \( |H| \) and the number of the distinct cosets of \( H \) in \( G \) is \( |G|/|H| \).

Proof. Let \( W_1 + H, W_2 + H, \ldots, W_r + H \) be the distinct left cosets of \( H \) in \( G \). Then, for each \( W \in G \), then \( W + H = W_i + H \) for some \( i \). Also, \( W \in W + H \). Thus, \( \forall \) element in \( G \), so this element is in one of the cosets \( W_i + H \).

In other words,

\[
G = W_1 + H \cup W_2 + H \cup \ldots \cup W_r + H.
\]

But \( W_i + H \neq W_j + H \) for \( i \neq j \). So this union is disjoint, so that

\[
|G| = |W_1 + H| + |W_2 + H| + \ldots + |W_r + H|.
\]

Since \( |W_i + H| = |H| \) \( \forall \) \( i \), this leads to

\[
|G| = |H| + |H| + \ldots + |H|.
\]

In other words, \( |G| = r \cdot |H| \). Then \( |H| \) divides \( |G| \). Thus, \( \exists \) \( r \) of distinct cosets of \( H \) in \( G \), namely

\[
r = \frac{|G|}{|H|}.
\]

4. A normal subgroup of \( L^- \) convex sets
In the following section, a normal subgroup of \( L^- \) convex sets will be discussed as follows.

Definition 4.1. A subgroup \( H \) of a group \( G \) of the \( L^- \) convex sets is called a normal subgroup of \( G \) if \( W + H = H + W \), for all \( W \in G \). A normal subgroup \( H \) of \( G \) can be denoted by \( H \triangleleft G \).

The notation of the \( W + H = H + W \) means that \( W + H = H + W \).
Theorem 4.2. A subgroup \( H \) of a group \( G \) of the \( L \)-convex sets is normal in \( G \) if and only if \( W + H + (-W) \subseteq H \), for all \( W \in G \).

Proof. Assume that \( H \trianglelefteq G \) then \( W + H = H + W \), for all \( W \in G \). Since \( W + H + (-W) = H \). Then \( W + H + (-W) \subseteq H \).

Conversely, \( W + H + (-W) \subseteq H \), for all \( W \in G \). So \(-W \in G \) and \(-W + H + (-(-W)) \subseteq H \). Then
\[
-W + H + W \subseteq H. \tag{3}
\]
So \( W + (-W) + H + W \subseteq W + H \). We have \( H + W \subseteq W + H \). Hence
\[
H \subseteq W + H + (-W). \tag{4}
\]
From (3) and (4), we have \( W + H + (-W) = H \). Hence \( W + H = H + W \). Consequently, \( H \trianglelefteq G \). \( \blacksquare \)

Remark 4.3. The above theorem can be written as \( H \trianglelefteq G \) if and only if \( W + h + (-W) \in H \), for all \( W \in G \).

5. A quotient group of the \( L \)-convex sets

On the set of \( L \)-convex sets, a quotient group has been created. The important concepts of this group are discussed as follows.

Definition 5.1. If \( H \trianglelefteq G \) then the set of left (or right) cosets of \( H \) in a group \( G \) of the \( L \)-convex sets is given by the set
\[
G \setminus H = \{ W + H : W \in G \}, \tag{5}
\]
which is called the quotient group of \( G \) by \( H \) (or the factor group of \( G \) by \( H \)).

Theorem 5.2. (Quotient group). Let \( G \) be a group of the \( L \)-convex sets and let \( H \trianglelefteq G \). Then the set \( G \setminus H = \{ W + H : W \in G \} \) is a group under the operation
\[
(W_1 + H) + (W_2 + H) = W_1 + W_2 + H.
\]

Proof. Firstly, we want to show the given operation is well defined. For some elements \( W_1, W_1', W_2, W_2' \in G \), assume \( W_1 + H = W_1' + H \) and \( W_2 + H = W_2' + H \).

Since \( W_1' \in W_1 + H \) and \( W_1' + H = W_1 + H \), then \( W_1' \in W_1 + H \) and \( W_1' = W_1 + h_1 \), for some \( h_1 \in H \).

Similarly, \( W_2' \in W_2 + H \) and \( W_2' + H = W_2 + H \), then \( W_2' \in W_2 + H \) and \( W_2' = W_2 + h_2 \), for some \( h_2 \in H \).

\[
W_1' + W_2' + H = (W_1 + h_1) + (W_2 + h_2) + H = (W_1 + h_1) + (W_2 + H) = W_1 + h_1 + H + W_2
\]
\[ W_1 + H + W_2 = W_1 + W_2 + H. \]

Now, let \( W_1 + H, W_2 + H, W_3 + H \in G \setminus H \). Then

\[
(W_1 + H) + (W_2 + H) + (W_3 + H) = (W_1 + W_2 + H) + (W_3 + H) = W_1 + (W_2 + W_3) + H = \]
\[
(W_1 + H) + (W_2 + W_3) + H = W_1 + (W_2 + H) + (W_3 + H).
\]

The element \( W_e + H = H \) is the identity element because

\[
(W_e + H) + (W + H) = (W_e + W) + H = W + H,
\]

for all \( W \in G \). Similarly,

\[
(W + H) + (W_e + H) = W + H,
\]

for all \( W \in G \). The inverse of \( W + H \) is \( -(W + H) = (-W) + H \), because

\[
(-W) + H + (W + H) = (-W + W) + H = W_e + H = H.
\]

Similarly, \( (W + H) + (-W) + H = W + (-W) + H = H \).

Thus \( G \setminus H \) is a group under the operation \( (W_1 + H) + (W_2 + H) = W_1 + W_2 + H \).

6. Conclusions

A new contribution on the set of the \( L \)-convex sets with addition operation is proposed. This contribution focuses on analysing the cosets of the subgroup of \( L \)-convex sets. This analysis considers as a new and powerful tool which connects the convex analysis with abstract algebra. On the cosets of the subgroup of \( L \)-convex sets, the fundamental properties have been proved mathematically. The Lagrange's Theorem on a finite group of \( L \)-convex sets is proved. A normal subgroup of \( L \)-convex sets and a quotient group of \( L \)-convex sets are discussed as well to give new bright spots in the topics of the convex analysis and abstract algebra.

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