A BK'M'S CRITERION OF SMOOTH SOLUTION TO THE INCOMPRESSIBLE VISCOELASTIC FLOW

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Abstract. In this paper, we study the regularity criterion of smooth solution to the Oldroyd model in \( \mathbb{R}^n (n = 2, 3) \). We obtain a Beale-Kato-Majda-type criterion in terms of vorticity in two and three space dimensions, namely, the solution \((u(t, x), F(t, x))\) does not develop singularity until \( t = T \) provided that
\[ \nabla \times u \in L^1(0, T; \dot{B}^0_{\infty, \infty}(\mathbb{R}^n)) \] in the case \( n = 2, 3 \).

1. Introduction. In this paper, our study is concerned with the following incompressible Oldroyd model in \( \mathbb{R}^n (n = 2, 3) \) describing an incompressible non-Newtonian fluid:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P &= \nabla \cdot (FF^T), \\
\partial_t F + u \cdot \nabla F &= \nabla u F, \\
\nabla \cdot u &= 0,
\end{align*}
\]

for any \( t > 0, x \in \mathbb{R}^n \), where \( u(t, x) \) denotes the fluid velocity vector field, \( P = P(t, x) \) is the scalar pressure, \( F = F(t, x) \in \mathbb{R}^n \times \mathbb{R}^n \) the deformation tensor, \( \mu > 0 \) is the constant kinematic viscosity, while \( u_0 \) is the given initial velocity with \( \nabla \cdot u_0 = 0 \). The above system (1) is one of the basic macroscopic model for viscoelastic flows (see [5] and references therein). This system corresponds to the so-called Hookean linear elasticity at a microscopic level. For more physical background to this system, see [7, 13, 23].

Significant progress has been made as regards viscoelastic flow by many physicists and mathematicians in the last few decades. The authors of [21] proved global existence in the two-dimensional case by introducing an auxiliary vector field to replace the transport variable \( F \), while Lei and Zhou [19] proved the same results via the incompressible limit working directly on the deformation tensor \( F \) rather than exploiting a subtle damping effect of the deformation tensor. Then, the authors of [16] proved the existence of both local and global smooth solutions to the Cauchy problem in the whole space and the periodic problem in the \( n \)-dimensional torus.

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(n = 2, 3) in the case of near equilibrium initial data, while Lin and Zhang [22] established the global well-posedness of the initial-boundary value problem of the viscoelastic fluid system of the Oldroyd model with Dirichlet conditions. Further discussions of these topics can be referred to [3, 4, 6, 8, 10, 11, 14, 15, 16, 17, 18, 19, 21, 22, 25, 26, 27, 28, 29, 31, 34, 35].

Before stating our main result, let us start with the standard description of general mechanical evolution to introduce some notations and definitions. In the context of hydrodynamics, the basic variable is the particle trajectory \(x(t, X)\), where \(X\) is the original labelling (Lagrangian coordinate) of the particle and referred to as the reference coordinate. For a given velocity field \(v(x, t)\), the flow map \(x(t, X)\) is defined by the ordinary differential equation: \(\frac{dx(t, X)}{dt} = u(t, x(t, X))\), \(x(0, X) = X\). The deformation tensor is then defined by \(F(t, X) = \frac{\partial x(t, X)}{\partial X}\). In the Eulerian coordinate, the corresponding deformation tensor \(F(t, x)\) is defined as \(F(t, x(t, X)) = \tilde{F}(t, X)\). Using the chain rule, one can see that \(F(t, x)\) satisfies the following transport equation, i.e. the second equation of (1):

\[
\partial_t F + u \cdot \nabla F = \nabla u F.
\]

If \(\nabla \cdot F^T(0, x) = 0\), then we have from the second equations of (1):

\[
\partial_t (\nabla \cdot F^T) + u \cdot \nabla (\nabla \cdot F^T) = 0. \tag{2}
\]

Therefore, if \(\nabla \cdot F^T(0, x) = 0\), it will remain so for later times, namely, \(\nabla \cdot F^T = 0\) for any time \(t > 0\). In what follows, we will make this assumption. Denote \(F_k = F_{ek}\) as the columns of \(F\), then \(\nabla \cdot (FF^T) = \sum_{k=1}^{n} F_k \cdot \nabla F_k\) by the fact \(\nabla \cdot F_k = 0\). Hence the system (1) takes the following form equivalently:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = \sum_{k=1}^{n} F_k \cdot \nabla F_k, \\
\partial_t F_k + u \cdot \nabla F_k = F_k \cdot \nabla u, \\
\nabla \cdot u = \nabla \cdot F_k = 0,
\end{cases} \tag{3}
\]

with \(k = 1, \ldots, n\), and \(u(0, x) = u_0, F_k(0, x) = F_{k, 0}\).

Yet, although much progress has been made on multi-dimensional Oldroyd model as mentioned above, the global existence of smooth solutions to (1) with general large initial data is still an outstanding open problem. More precisely, since the Oldroyd system (1) can be regarded as a combination between Navier-Stokes equations and with the source term \(\nabla \cdot (FF^T)\) and the second equation of (1), we expect the local smooth solutions to blow up in a finite time. In three space dimension case, the authors [9] established a Beale-Kato-Majda [1] type criterion for smooth solutions as \(\mu = 0\) in (3), which reads that the smooth solution \((u(t, x), F(t, x))\) is smooth for \(0 \leq t \leq T\) provided that

\[
\int_0^T \|\nabla \times u(t)\|_{L^\infty} dt + \sum_{k=1}^{3} \int_0^T \|\nabla \times F_k(t)\|_{L^\infty} dt < \infty. \tag{4}
\]

In the case \(\mu > 0\), Yuan [32] obtained the blowup criterion of smooth solution to the Oldroyd model (3) in two and three space dimensions by means of only \(\|\nabla u\|_{L^\infty}\), namely, the smooth solution \((u(t, x), F(t, x))\) does not appear breakdown until \(t = T\) provided that

\[
\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^n)). \tag{5}
\]
In [30], the author obtained that the smooth solution \((u(t, x), F(t, x))\) is smooth at \(0 \leq t \leq T\) provided that \(\nabla \times u \in L^1(0, T; BMO)\) and \(\nabla F \in L^1(0, T; BMO)\), while in [33], Yuan and Li give the similar regularity criteria at the same time.

In this paper, we consider the blowup criterion of smooth solution to the Oldroyd system \((3)\) and establish a Beale-Kato-Majda-type criterion of smooth solutions to the system \((3)\) in the space \(B^0_{\infty, \infty}\). This is motivated by Lei and Zhou’s work [20], which they established a Beale-Kato-Majda-type criterion to MHD system under the norm \(BMO\) in the case of zero viscosity.

Now, our main result is stated as follows.

**Theorem 1.1.** Suppose that \((u_0, F_{k,0}) \in H^m(\mathbb{R}^n) (n = 2, 3), m \geq 3\) with \(\nabla \cdot u_0 = \nabla \cdot F_{k,0} = 0\) for \(k = 1, \ldots, n\), and \((u(t, x), F_k(t, x))\) is the smooth solution to the Oldroyd system \((3)\) with the initial data \((u_0, F_{k,0})\) for \(0 \leq t < T\). Then the solution \((u(t, x), F_k(t, x))\) remains smooth at time \(t = T\) provided that

\[
\int_0^T \|\nabla \times u(t)\|_{B^m_{\infty, \infty}} dt < \infty. \tag{6}
\]

We have the following corollary immediately.

**Corollary 1.** Suppose that \((u_0, F_{k,0}) \in H^m(\mathbb{R}^n) (n = 2, 3), m \geq 3\) with \(\nabla \cdot u_0 = \nabla \cdot F_{k,0} = 0\) for \(k = 1, \ldots, n\), and \((u(t, x), F_k(t, x))\) is the smooth solution to the Oldroyd system \((3)\) with the initial data \((u_0, F_{k,0})\) for \(0 \leq t < T\). Suppose that \(T\) is the maximal existence time, then

\[
\int_0^T \|\nabla \times u(t)\|_{B^m_{\infty, \infty}} dt = \infty. \tag{7}
\]

**Remark 1.** Since \(L^\infty \subset BMO \subset \dot{B}^0_{\infty, \infty}\), then the result (6) is an extension to (4) in some sense, although they showed (4) for the Oldroyd system describing the motion of inviscid fluids. The result (6) improves the result (7) of Theorem 1.2 in [30].

This paper is organized as follows. Section 2 gives the preliminaries. Section 3 details the regularity criterion (6).

2. **Preliminaries.** This section is devoted to providing some lemmas and some basic facts on Littlewood-Paley theory, which will be used in the proofs of our main results.

Let \(\mathcal{S}(\mathbb{R}^n)\) be the Schwartz class of rapidly decreasing functions. Given \(f \in \mathcal{S}(\mathbb{R}^n)\), its Fourier transformation \(\mathcal{F}f = \hat{f}\) is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx,
\]

and for any given \(g \in \mathcal{S}(\mathbb{R}^n)\), its inverse Fourier transform \(\mathcal{F}^{-1}g = \check{g}\) is defined by

\[
\check{g}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(x) dx.
\]

Now let us recall the Littlewood-Paley decomposition (see [2]). Choose a non-negative radial function \(\phi \in \mathcal{S}(\mathbb{R}^n)\) supported in \(\mathcal{C} = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{4}{3}\}\), such that

\[
\sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1, \xi \in \mathbb{R}^n \setminus \{0\}.
\]
Let $h = F^{-1} \phi$. The frequency localization operator is defined by
\[ \Delta_j f = \phi(2^{-j} D) f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y) f(x - y) dy, \quad -\infty \leq j \leq +\infty. \]

We now introduce the homogeneous Besov spaces.

**Definition 2.1.** Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by
\[ \dot{B}^s_{p,q} = \{ f \in Z'(\mathbb{R}^n); \| f \|_{\dot{B}^s_{p,q}} < \infty \}, \]
where
\[ \| f \|_{\dot{B}^s_{p,q}} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \Delta_j f \|_{L^p}^q \right)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^p}, & \text{for } q = \infty, \end{cases} \]
and $Z'(\mathbb{R}^n)$ can be identified by the quotient space of $S'/\mathcal{P}$ with the polynomials space $\mathcal{P}$, the set $S'(\mathbb{R}^n)$ of temperate distributions is the dual set of $S$ for the usual pairing.

Let us recall the well-known Gagliardo-Nirenberg inequality.

**Lemma 2.2.** Suppose that $f \in L^q(\mathbb{R}^n) \cap W^{m,r}(\mathbb{R}^n), 1 \leq q, r \leq \infty$. Then for $0 \leq j \leq m, \frac{j}{m} \leq \theta \leq 1, 1 \leq p \leq \infty$ and
\[ \frac{1}{p} = \frac{j}{n} + \theta \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \theta) \frac{1}{q}, \]
there exists a constant $C$ such that
\[ \| \nabla^j f \|_{L^p} \leq C \| f \|_{L^q}^{1-\theta} \| \nabla^m f \|_{L^r}^\theta. \]

The following lemma follows from Majda-Bertozzi [24].

**Lemma 2.3.** The following inequality holds
\[ \| \nabla^m (u \cdot \nabla v) - u \cdot \nabla v \|_{L^2} \leq C (\| \nabla u \|_{L^\infty} \| \nabla^m v \|_{L^2} + \| \nabla v \|_{L^\infty} \| \nabla^m u \|_{L^2}) \]
for $m \geq 1$.

The following lemma is well known (for example, see [2]).

**Lemma 2.4** (Bernstein inequality). The following inequalities hold
\[ \begin{cases} c 2^{km} \| \Delta_k f \|_{L^p} \leq \| \nabla^m \Delta_k f \|_{L^p} \leq C 2^{km} \| \Delta_k f \|_{L^p}, \\ \| \Delta_k f \|_{L^q} \leq C 2^{kn} \left( \frac{k}{2} - \frac{1}{2} \right) \| \Delta_k f \|_{L^p}, \end{cases} \]
for any $m \in \mathbb{N}, 1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Here $c$ and $C$ are positive constants independent of $f$ and $k$.

At last, let us recall the following logarithmic Sobolev inequality which is proved by Kozono-Ogawa-Taniuchi [12]. For completeness, the proof will be also sketched here.

**Lemma 2.5.** There exists a uniform positive constant $C$ such that
\[ \| \nabla f \|_{L^\infty} \leq C \left( 1 + \| f \|_{L^2} + \| \nabla \times f \|_{B_{t, \infty}^n} \ln(1 + \| f \|_{H^3}) \right) \]
holds for all vectors $f \in H^3(\mathbb{R}^n)(n = 2, 3)$ with $\nabla \cdot f = 0$. 
Proof. First of all, it follows from the Littlewood-Paley decomposition and Lemma 2.4 that
\[
\| \nabla f \|_{L^\infty} \leq \left\| \sum_{j=-\infty}^{0} \Delta_j \nabla f \|_{L^\infty} + \sum_{j=1}^{N} \Delta_j \nabla f \|_{L^\infty} + \sum_{j=N+1}^{+\infty} \Delta_j \nabla f \|_{L^\infty}
\]
\[
\leq C \left( \sum_{j=-\infty}^{0} 2^{j\left(1+\frac{n}{2}\right)} \| \Delta_j f \|_{L^2} + N \max_{1 \leq j \leq N} \| \Delta_j \nabla f \|_{L^\infty} + \sum_{j=N+1}^{+\infty} 2^{-j\left(2-\frac{n}{2}\right)} \| \Delta_j \nabla f \|_{L^2} \right)
\]
\[
\leq C \left( \| f \|_{L^2} + N \| \nabla f \|_{B^0_{\infty,\infty}} + 2^{-\left(2-\frac{n}{2}\right)N} \| \nabla^3 f \|_{L^2} \right).
\]
Taking
\[
N = \left\lfloor \frac{1}{(2 - \frac{n}{2}) \ln 2} \mathrm{ln}(e + \| f \|_{H^3}) \right\rfloor + 1.
\]
Consequently, it follows from the above estimates and Calderon-Zygmand theory that Lemma 2.5 holds. We have proved Lemma 2.5.

3. Proof of Theorem 1.1. In this section we give the proof of Theorem 1.1. For ease of notation, \( \sum_k \) is denoted as \( \sum_{k=1}^{n} (n = 2, 3) \). Multiplying (3)1 by \( u \) and (3)2 by \( F_k \) respectively, and integrating on \( \mathbb{R}^n \), then we have
\[
\frac{1}{2} \frac{d}{dt} \| u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 = \sum_k \int_{\mathbb{R}^n} (F_k \cdot \nabla F_k) \cdot u \, dx,
\]
\[
\frac{1}{2} \frac{d}{dt} \| F_k \|_{L^2}^2 = \int_{\mathbb{R}^n} (F_k \cdot \nabla u) \cdot F_k \, dx,
\]
where the fact
\[
\int_{\mathbb{R}^n} (u \cdot \nabla u) \cdot u \, dx = \int_{\mathbb{R}^n} (u \cdot \nabla F_k) \cdot F_k \, dx = 0
\]
is used by \( \nabla \cdot u = \nabla \cdot F_k = 0 \).

Noticing that
\[
\int_{\mathbb{R}^n} (F_k \cdot \nabla F_k) \cdot u \, dx + \int_{\mathbb{R}^n} (F_k \cdot \nabla u) \cdot F_k \, dx = 0.
\]
Summing over \( k \) for (9) and combining with (8), we have
\[
\frac{1}{2} \frac{d}{dt} (\| u \|_{L^2}^2 + \sum_k \| F_k \|_{L^2}^2) + \| \nabla u \|_{L^2}^2 = 0,
\]
which follows that
\[
\| u \|_{L^2}^2 + \sum_k \| F_k \|_{L^2}^2 + 2 \int_0^t \| \nabla u \|_{L^2}^2 \, d\tau = \| u_0 \|_{L^2}^2 + \sum_k \| F_{k,0} \|_{L^2}^2.
\]
Then
\[
\| u \|_{L^\infty(0,T;L^2)} + \| u \|_{L^2(0,T;H^1)} \leq \| u_0 \|_{L^2} + \sum_k \| F_{k,0} \|_{L^2}
\]
(10)
and
\[
\sum_k \| F_k \|_{L^\infty(0,T;L^2)} \leq \| u_0 \|_{L^2} + \sum_k \| F_{k,0} \|_{L^2}.
\]
(11)
Now, applying $\nabla$ to (3), multiplying the resulting equations by $(\nabla u, \nabla F_k)$, and integrating on $\mathbb{R}^n$, then it follows that by summing them up

$$\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \sum_k \|\nabla F_k\|_{L^2}^2) + \|\nabla^2 u\|_{L^2}^2$$

$$= - \int_{\mathbb{R}^n} \nabla (u \cdot \nabla u) \nabla u \, dx + \sum_k \int_{\mathbb{R}^n} \nabla (F_k \cdot \nabla F_k) \nabla u \, dx$$

$$- \sum_k \int_{\mathbb{R}^n} \nabla (u \cdot \nabla F_k) \nabla F_k \, dx + \sum_k \int_{\mathbb{R}^n} \nabla (F_k \cdot \nabla u) \nabla F_k \, dx$$

$$= - \int_{\mathbb{R}^n} (\nabla u \cdot \nabla) u \nabla u \, dx$$

$$+ \sum_k \int_{\mathbb{R}^n} [(\nabla F_k \cdot \nabla) F_k \nabla u + (\nabla u \cdot \nabla) F_k \nabla F_k + (\nabla F_k \cdot \nabla) u \nabla F_k] \, dx$$

$$\leq C \|\nabla u\|_{L^\infty} (\|\nabla u\|_{L^2}^2 + \sum_k \|\nabla F_k\|_{L^2}^2),$$

where the following fact

$$\int_{\mathbb{R}^n} [(F_k \cdot \nabla^2) F_k \nabla u + (F_k \cdot \nabla^2) u \nabla F_k] \, dx = 0,$$

$$\int_{\mathbb{R}^n} (u \cdot \nabla) \nabla u \, dx = \int_{\mathbb{R}^n} (u \cdot \nabla) \nabla F_k \nabla F_k \, dx = 0,$$

is used in the second equality of (12) by $\nabla \cdot u = \nabla \cdot F_k = 0$.

From the Gronwall inequality, we get

$$\|\nabla u(t)\|_{L^2}^2 + \sum_k \|\nabla F_k(t)\|_{L^2}^2 + 2 \int_{t_1}^t \|\nabla^2 u(\tau)\|_{L^2}^2 \, d\tau$$

$$\leq (\|\nabla u(t_1)\|_{L^2}^2 + \sum_k \|\nabla F_k(t_1)\|_{L^2}^2) \exp \left( C \int_{t_1}^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right).$$

(13)

In terms of (6), we know that for any small constant $\epsilon > 0$, there exists $T^* < T$, such that

$$\int_{T^*}^T \|\nabla \times u(t)\|_{B^0_{\infty,\infty}} \leq \epsilon.$$

(14)

Let

$$\Phi(t) = \sup_{T^* \leq \tau \leq T} \left( \|\nabla^3 u(\tau)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(\tau)\|_{L^2}^2 \right), \quad T^* \leq t < T.$$

(15)

From (13), (14), (15) and Lemma 2.5, we get

$$\|\nabla u(t)\|_{L^2}^2 + \sum_k \|\nabla F_k(t)\|_{L^2}^2 + 2 \int_{t_1}^t \|\nabla^2 u(\tau)\|_{L^2}^2 \, d\tau$$

$$\leq C_1 \exp \left( C_0 \int_{T^*}^T \|\nabla \times u(\tau)\|_{B^0_{\infty,\infty}} \ln(e + \|u(\tau)\|_{H^3}) \, d\tau \right)$$

$$\leq C_1 \exp \left( C_0 e \ln(e + \Phi(t)) \right)$$

$$\leq C_1 (e + \Phi(t))^{C_0 e},$$

(16)

Here $C_0$ is an absolute positive constant, $C_1$ depends on $T^* \leq t < T$ and $\|\nabla u(T^*)\|_{L^2}^2 + \sum_k \|\nabla F_k(T^*)\|_{L^2}^2.$
Now, applying $\nabla^m$ to the first equation of (3), taking $L^2$-inner product of the resulting equation with $\nabla^m u$, and integrating over $\mathbb{R}^n$, we have

$$\frac{1}{2} \frac{d}{dt} \| \nabla^m u(t) \|_{L^2}^2 + \| \nabla^{m+1} u(t) \|_{L^2}^2$$

$$= - \int_{\mathbb{R}^n} \nabla^m (u \cdot \nabla u) \nabla^m u \, dx + \sum_k \int_{\mathbb{R}^n} \nabla^m (F_k \cdot \nabla F_k) \nabla^m u \, dx.$$

Likewise,

$$\frac{1}{2} \frac{d}{dt} \left( \sum_k \| \nabla^m F_k(t) \|_{L^2}^2 \right)$$

$$= - \sum_k \int_{\mathbb{R}^n} \nabla^m (u \cdot F_k) \nabla^m F_k \, dx + \sum_k \int_{\mathbb{R}^n} \nabla^m (F_k \cdot \nabla u) \nabla^m F_k \, dx.$$

By using $\nabla \cdot u = \nabla \cdot F_k = 0$, integrating by parts the above two equalities infers that

$$\frac{1}{2} \frac{d}{dt} \left( \| \nabla^m u(t) \|_{L^2}^2 + \sum_k \| \nabla^m F_k(t) \|_{L^2}^2 \right) + \| \nabla^{m+1} u(t) \|_{L^2}^2$$

$$= - \int_{\mathbb{R}^n} [\nabla^m (u \cdot \nabla u) - u \cdot \nabla \nabla^m u] \nabla^m u \, dx$$

$$+ \sum_k \int_{\mathbb{R}^n} [\nabla^m (F_k \cdot \nabla F_k) - F_k \cdot \nabla \nabla^m F_k] \nabla^m u \, dx$$

$$- \sum_k \int_{\mathbb{R}^n} [\nabla^m (u \cdot \nabla F_k) - u \cdot \nabla \nabla^m F_k] \nabla^m F_k \, dx$$

$$+ \sum_k \int_{\mathbb{R}^n} [\nabla^m (F_k \cdot \nabla u) - F_k \cdot \nabla \nabla^m u] \nabla^m F_k \, dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

For simplicity, we shall set $m = 3$ in what follows. First of all, using the Hölder inequality and Lemma 2.3, we have

$$I_1 \leq C \| \nabla u(t) \|_{L^\infty} \| \nabla^3 u(t) \|_{L^2}^2.$$

Then by (15), we have

$$I_1 \leq C \| \nabla u(t) \|_{L^\infty} (e + \Phi(t)).$$

For the term $I_3$, by integrating by parts and the Hölder inequality, one has

$$I_3 \leq \sum_k \| \nabla u(t) \|_{L^\infty} \| \nabla^3 F_k(t) \|_{L^2}^2 + 4 \| \nabla u(t) \|_{L^\infty} \| \nabla^2 F_k(t) \|_{L^2} \| \nabla^4 F_k(t) \|_{L^2}$$

$$+ \| \nabla^2 u(t) \|_{L^\infty} \| \nabla F_k(t) \|_{L^\infty} \| \nabla^4 F_k(t) \|_{L^2}.$$
By virtue of Lemma 2.2, the Young inequality and (16), we have

\[
\sum_k 4\|\nabla u(t)\|_\infty \|\nabla^2 F_k(t)\|_{L^2} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq C \sum_k \|\nabla u(t)\|_\infty \|\nabla F_k(t)\|_{L^2}^{\frac{3}{2}} \|\nabla^4 F_k(t)\|_{L^2}^{\frac{1}{2}} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_\infty^3 \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^2 \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^2 \left(\epsilon + \Phi(t)\right)^{\frac{3C_0}{4}} \Phi^\frac{3}{2}(t)
\]

for any \(\delta > 0\) in 2D case, and in 3D case:

\[
\sum_k 4\|\nabla u(t)\|_\infty \|\nabla^2 F_k(t)\|_{L^2} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq C \sum_k \|\nabla u(t)\|_\infty \|\nabla F_k(t)\|_{L^2}^{\frac{3}{2}} \|\nabla^4 F_k(t)\|_{L^2}^{\frac{1}{2}} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_\infty^3 \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^2 \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^2 \left(\epsilon + \Phi(t)\right)^{\frac{3C_0}{4}} \Phi^\frac{3}{2}(t)
\]

for any \(\delta > 0\). Next, using Lemma 2.2, the Young inequality and (16), it follows that

\[
\sum_k \|\nabla^2 u(t)\|_{L^4} \|\nabla F_k(t)\|_{L^4} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq \sum_k \|\nabla u(t)\|_L^{\frac{5}{2}} \|\nabla^3 u(t)\|_{L^2}^{\frac{3}{2}} \|\nabla F_k(t)\|_{L^2}^{\frac{5}{2}} \|\nabla^4 F_k(t)\|_{L^2}^{\frac{1}{2}} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^\frac{3}{2} \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^\frac{1}{2} \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C \|\nabla u(t)\|_L^2 \left(\epsilon + \Phi(t)\right)^{\frac{3C_0}{4}} \Phi^\frac{3}{2}(t)
\]
for any $\delta > 0$ in 2D case, and in 3D case:
\[
\sum_k \|\nabla^2 u(t)\|_{L^4} \|\nabla F_k(t)\|_{L^4} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq \sum_k \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u(t)\|_{L^2} \|\nabla F_k(t)\|_{L^2}^{\frac{3}{2}} \|\nabla^4 F_k(t)\|_{L^2}^{\frac{3}{2}} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} \|\nabla^3 u(t)\|_{L^2} \sum_k \|\nabla F_k(t)\|_{L^2}^2 \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t))^{\frac{25C_0}{24}} \Phi^\frac{10}{24} (t)
\]

for any $\delta > 0$.

Hence, if we take
\[
\epsilon < \frac{1}{5C_0},
\]
then we obtain
\[
\sum_k 4\|\nabla u(t)\|_{L^\infty} \|\nabla^2 F_k(t)\|_{L^2} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t))
\]
and
\[
\sum_k \|\nabla^2 u(t)\|_{L^4} \|\nabla F_k(t)\|_{L^4} \|\nabla^4 F_k(t)\|_{L^2} \\
\leq \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t))
\]

for any $\delta > 0$. Noting that the first term in right hand of \((19)\)
\[
\sum_k \|\nabla u(t)\|_{L^\infty} \|\nabla^3 F_k(t)\|_{L^2} \leq C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t)) .
\]

Inserting the above estimates \((20), (21)\) and \((22)\) into \((19)\) gives that
\[
I_3 = \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t)) .
\]

Similarly,
\[
I_2 = \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t)),
\]

\[
I_4 = \delta \sum_k \|\nabla^4 F_k(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t)).
\]

The combination of \((17), (18), (23), (24)\) and \((25)\) gives that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla^3 u(t)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(t)\|_{L^2}^2 \right) + \|\nabla^4 u(t)\|_{L^2}^2 \leq C\|\nabla u(t)\|_{L^\infty} (e + \Phi(t))
\]

\]
by taking $\delta$ small enough for all $T^* \leq t < T$. We deduce that
\[
e + \|\nabla^3 u(\tau)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(\tau)\|_{L^2}^2
\]
\[
\leq e + \|\nabla^3 u(T^*)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(T^*)\|_{L^2}^2
\]
\[
+ C \int_{T^*}^T \left( 1 + \|u(s)\|_{L^2} + \|\nabla \times u(s)\|_{B^1_{\infty, \infty}} \ln(e + \Phi(s)) (e + \Phi(s)) \right) ds,
\]
which implies that
\[
e + \Phi(t) \leq e + \|\nabla^3 u(T^*)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(T^*)\|_{L^2}^2
\]
\[
+ C \int_{T^*}^T \left( 1 + \|u(\tau)\|_{L^2} + \|\nabla \times u(\tau)\|_{B^1_{\infty, \infty}} \ln(e + \Phi(\tau)) (e + \Phi(\tau)) \right) d\tau.
\]
By using the Gronwall inequality, for all $T^* \leq t < T$, we have
\[
e + \|\nabla^3 u(t)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(t)\|_{L^2}^2 \leq C, \tag{26}
\]
where $C$ is a constant dependent on $\|\nabla^3 u(T^*)\|_{L^2}^2 + \sum_k \|\nabla^3 F_k(T^*)\|_{L^2}^2$. From (10), (11) and (26), we know that $(u(T, \cdot), F_k(T, \cdot)) \in H^3(\mathbb{R}^3)$. Hence, the proof of Theorem 1.1 is complete.

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