Successive Over Relaxation Q-Learning*

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Abstract—In a discounted reward Markov Decision Process (MDP) the objective is to find the optimal value function, i.e., the value function corresponding to an optimal policy. This problem reduces to solving a functional equation known as the Bellman equation and a fixed point iteration scheme known as the value iteration is utilized to obtain the solution. In [1], a successive over-relaxation based value iteration scheme is proposed to speed up the computation of the optimal value function. They propose a modified Bellman equation and prove faster convergence to the optimal value function. However, in many practical applications, the model information is not known and we resort to Reinforcement Learning (RL) algorithms to obtain optimal policy and value function. One such popular algorithm is Q-Learning. In this paper, we propose Successive Over Relaxation (SOR) Q-Learning. We first derive a fixed point iteration for optimal Q-values based on [1] and utilize stochastic approximation to derive a learning algorithm to compute the optimal value function and an optimal policy. We then prove the convergence of the SOR Q-Learning to optimal Q-values. Finally, through numerical experiments, we show that SOR Q-Learning is faster compared to the standard Q-Learning algorithm.

I. INTRODUCTION

In a discounted reward Markov Decision Process (MDP), the objective is to find optimal value function and a corresponding optimal policy. If the model information is known, the optimal value function can be computed by finding the fixed points of the Bellman equation [2]. The contraction factor for this fixed point scheme is shown to be the discount factor of the MDP. It determines the rate of convergence of the value function estimates to the optimal solution. In [1], a modified Bellman equation using the concept of Successive Over Relaxation (SOR) is proposed that is shown to have a contraction factor less than or equal to the discount factor. More specifically, under a special structure for MDPs, it can be shown that the contraction factor is strictly less than the discount factor. The special structure for the MDP is as follows. For each action in the action space, there is a positive probability of self loop for every state in the state space.

Reinforcement Learning algorithms are used to obtain the optimal policy and value function when the full model of the MDP is not known. These algorithms make use of the state and reward samples and compute the optimal policy.

One of the popular Reinforcement learning algorithms is the Q-learning algorithm. The Q-learning algorithm combined with the Deep Learning framework has gained popularity in recent times and has been successfully applied to solve many problems [3], [4]. In this paper, we propose a generalized Q-Learning algorithm based on Successive Over Relaxation technique. First, we derive a Q-value based modified Bellman equation and show that the contraction factor of this equation is less than or equal to the discount factor. We then utilize the stochastic approximation technique to derive the generalized Q-learning algorithm that we call as SOR Q-learning.

We now discuss some of the works in the literature that propose variants of the standard Q-learning algorithm. In [5], the $Q(\lambda)$ algorithm has been proposed that combines ideas of Q-learning and eligibility traces. In [6], the Double Q-learning algorithm has been proposed to mitigate the problem of overestimation in Q-learning. The Double Q-learning makes use of two Q-value functions in the update equation. In [7], speedy Q-learning has been proposed for improving the convergence of the Q-estimates. The speedy Q-learning algorithm makes use of the current and the previous Q-value estimates in the update equation. More recently, the zap Q-learning algorithm has been proposed in [8] that imitates the stochastic Newton-Raphson method and it is shown that zap Q-learning exhibits faster convergence to the optimal solution.

Note that unlike [6], [7] our algorithm utilizes only the current Q-value estimates in the update equation and unlike [8] our algorithm does not use matrix-valued step-sizes. Our key contributions in this paper are as follows:

- We construct modified Q-Bellman equation using the SOR technique.
- We derive a generalized Q-learning algorithm (SOR Q-Learning) using stochastic approximation.
- We show the convergence of SOR Q-Learning to the optimal Q-values.
- We prove that the contraction factor of the modified Q-Bellman equation is less than or equal to the contraction factor for standard Q-learning.
- Through numerical evaluation we demonstrate the effectiveness of our algorithm.

The rest of the paper is organized as follows. In section II we introduce the necessary background. We propose our algorithm in section III. Section IV describes the convergence analysis. Section V presents the results of our numerical experiments. Finally section VI presents concluding remarks and future research directions.

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II. BACKGROUND AND PRELIMINARIES

A Markov Decision Process (MDP) is defined by a tuple \( (S, A, p, r, \alpha) \) where \( S := \{1, 2, \ldots, i, j, \ldots, M\} \) is the set of states, \( A \) is the set of actions, \( p \) denotes the transition probability rule i.e., \( p(j|i, a) \) denotes the probability of transition to state \( j \) from state \( i \) when action \( a \) is chosen. \( r(i, a, j) \) denotes the single-stage reward obtained in state \( i \) when action \( a \) is chosen and the system transitions to state \( j \). Also, \( 0 \leq \alpha < 1 \) denotes the discount factor. The goal of the MDP is to learn an optimal policy i.e., \( \pi : S \to A \), where \( \pi(i) \) indicates the action to be taken in state \( i \) that maximizes the discounted reward objective:

\[
V(i) = \max_{a \in A} \left\{ \sum_{j=1}^{M} p(j|i, a)(r(i, a, j) + \alpha V(j)) \right\},
\]

where \( s_t \) is the state of the system at time \( t \) and \( E[.] \) is the expectation taken over the states obtained over time \( t = 1, \ldots, \infty \). We denote \( V(i) \) to be the optimal value function associated with state \( i \). We assume for simplicity that all actions are feasible in every state. It can be shown that the optimal value function is the solution to the Bellman equation [9]:

\[
V(i) = \max_{a \in A} \left\{ \sum_{j=1}^{M} p(j|i, a)(r(i, a, j) + \alpha V(j)) \right\}.
\]

Let \( \zeta \) denote the set of all bounded functions from \( S \) to \( \mathbb{R} \). Then equation (2) can be viewed as a fixed point equation given by:

\[
V = TV,
\]

where the operator \( T : \zeta \to \zeta \) is a function given by

\[
(TV)(i) = \max_{a \in A} \left\{ r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a)V(j) \right\},
\]

and \( r(i, a) = \sum_{j=1}^{M} p(j|i, a)r(i, a, j) \).

Value iteration is a well-known fixed point iteration scheme employed to solve (3). In the value iteration scheme, an initial \( V_0 \) is selected and a sequence of \( V_n, n \geq 1 \) is obtained as follows:

\[
V_n = TV_{n-1}, \quad n \geq 1.
\]

It can be shown that the optimal value function:

\[
V = \lim_{n \to \infty} V_n = TV.
\]

In this way, we can compute the optimal value function when the model information is known. However, in many practical applications, we do not have access to the model information. Instead, the states visited and reward samples are available to us and the objective is to find the optimal value function and a corresponding policy from the samples. One of the popular algorithms for computing the optimal policy and value function from samples is Q-Learning [10].

We now briefly discuss the derivation of the Q-learning update rule from the fixed point iteration discussed above. Let \( Q(i, a) \) be defined as

\[
Q(i, a) := r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a)V(j).
\]

Here \( Q(i, a) \) is the optimal Q-value function associated with state \( i \) and action \( a \). Then from (2), it is clear that

\[
V(i) = \max_{a \in A} Q(i, a).
\]

Therefore, the equation (6) can be re-written as follows:

\[
Q(i, a) = r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a) \max_{b \in A} Q(j, b).
\]

This is a Bellman equation involving Q-values \( Q(i, a) \) instead of the value function \( V \). We obtain the optimal policy by letting

\[
\pi(i) = \arg\max_{a \in A} Q(i, a).
\]

The corresponding optimal value function is then

\[
V(i) = \max_{a \in A} Q(i, a).
\]

It is easy to see that the contraction factor for Q-value iteration is \( \alpha \), the discount factor [9]. The contraction factor indicates how fast the Q-value estimates converge to the optimal Q-values. Finally, the Q-learning update can be obtained from equation (8) by applying the stochastic fixed point iteration scheme [11] as follows:

\[
Q_{n+1}(i, a) = (1 - \gamma_n(i, a))Q_n(i, a) + \gamma_n(i, a)(r(i, a, j) + \max_{b \in A} Q_n(j, b)),
\]

where \( Q_n(i, a) \) is the current estimate of the Q-values, \( \gamma_n(i, a) \) is the step-size and \( (i, a, r, j) \) is the current (state, action, reward, next state) sample. The convergence of Q-learning to the optimal Q-values is established in [10]. A comprehensive discussion on MDPs and Reinforcement Learning algorithms can be found in [9], [12].

In this work, we derive a new Q-Bellman equation that has a contraction factor less than or equal to \( \alpha \). To this end, we utilize the Successive Over Relaxation (SOR) technique proposed in [1] for the optimal value function. We propose our SOR Q-learning algorithm based on the modified Bellman equation involving Q-values.

III. PROPOSED ALGORITHM

In this section we describe our SOR Q-Learning algorithm. We assume that we have a trajectory \( \{(i_n, a_n, r(i_n, a_n, i_{n+1}), i_{n+1})\}^{\infty}_{n=1} \) in which each tuple \( (i, a) \in S \times A \) appears infinitely often. At each time step \( n \) the input to the algorithm is an over relaxation parameter \( w \), current single-stage reward \( r(i_n, a_n, i_{n+1}) \), the next state \( i_{n+1} \) and the current SOR Q-Learning estimates \( Q_n \). The algorithm proceeds to calculate \( d_{n+1} \) and \( Q_{n+1} \) as given by steps 2 and 3 in Algorithm [4]. The procedure terminates
Hence, using the subsequent analysis, we have obtained. Note the difference in the estimation of \(d\) by \(\gamma_n(i, a_n)\). Algorithm 1 Successive Over Relaxation Q-Learning

**Input:**
- \(w\): over relaxation parameter
- \(i_n, a_n, i_{n+1}\): current state, action and next state
- \(r(i_n, a_n, i_{n+1})\): single stage reward
- \(Q_n(i_n, a_n)\): current estimate of \(Q(i_n, a_n)\)

**Output:** Updated Q-values \(Q_{n+1}\) estimated after \(n\) iterations of the algorithm

1. **procedure** SOR Q-LEARNING:
2. \[d_{n+1} = w \left(r(i_n, a_n, i_{n+1}) + \alpha \max_{b \in A} Q_n(i_{n+1}, b)\right) + (1 - w) \max_{c \in A} Q_n(i_n, c) - Q_n(i_n, a_n)\]
3. \[Q_{n+1}(i_n, a_n) = Q_n(i_n, a_n) + \gamma_n(i_n, a_n)d_{n+1}\]
4. **return** \(Q_{n+1}\)

Recall that for a given MDP \((S, A, p, r, \alpha)\) the optimal value function \(V^*\) satisfies [13], [14] the Bellman equation

\[V^*(i) = \max_{a \in A} \left\{r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a)V^*(j)\right\}. \quad (12)\]

It can be seen [15] that \(T\) is a contraction under the max-norm \(|x| := \max_{1 \leq i \leq M} |x(i)|\) with contraction factor \(\alpha\). Let \(w^*\) be given by

\[w^* = \min_{i, a} \left\{1 \over 1 - \alpha p(i|i, a)\right\}. \quad (13)\]

Note that \(w^* \geq 1\). For \(0 < w \leq w^*\) define \(T_w: \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}\) as follows:

\[(T_wV)(i) = wTV(i) + (1 - w)V(i),\]

where \(w\) represents a prescribed relaxation factor. Observe that the optimal value function \(V^*\) is also the unique fixed point of \(T_w\). Moreover it is shown [1] that \(T_w\) is a contraction with contraction factor \(\xi(w)\) and \(\xi(w^*) \leq a\). Now we have

\[(T_wV)(i) = \max_{a \in A} \left\{w(r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a)V^*(j)\right\} + (1 - w)V(i). \quad (14)\]

Let \(Q^*(i, a)\) be defined as follows:

\[Q^*(i, a) := w \left(r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a)V^*(j)\right) + (1 - w)V^*(i). \quad (14)\]

Since \(V^*\) is the unique fixed point of \(T_w\) clearly it can be seen that

\[V^*(i) = (T_wV^*)(i) = \max_{a \in A} Q^*(i, a) \quad \forall i \in S.\]

Hence the equation [14] can be rewritten as follows:

\[Q^*(i, a) = w \left(r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a) \max_{b \in A} Q^*(j, b)\right) + (1 - w) \max_{c \in A} Q^*(i, c). \quad (15)\]

Let \(H_w: \mathbb{R}^{|S| \times |A|} \rightarrow \mathbb{R}^{|S| \times |A|}\) be defined as follows.

\[(H_wQ)(i, a) := w \left(r(i, a) + \alpha \sum_{j=1}^{M} p(j|i, a) \max_{b \in A} Q(j, b)\right) + (1 - w) \max_{c \in A} Q(i, c). \quad (15)\]

**Lemma 2:** \(H_w: \mathbb{R}^{|S| \times |A|} \rightarrow \mathbb{R}^{|S| \times |A|}\) is a max-norm contraction and \(Q^*\) is the unique fixed point of \(H_w\).

**Proof:** Observe that \(Q^*\) is a fixed point of \(H_w\) from [15]. It is enough to show that \(H_w\) is a max-norm contraction.
For $P, Q \in \mathbb{R}^{[S] \times [A]}$, we have
\[
\left| (H_w P - H_w Q)(i, a) \right| = w\alpha \sum_{j=1}^{M} p(j|i, a) \max_{b \in A} (P(j, b) - \max_{b \in A} Q(j, b))
+ (1 - w) \max_{c \in A} (P(i, c) - \max_{c \in A} Q(i, c))
\]
\[
\leq w\alpha \sum_{j=1, j \neq i}^{M} p(j|i, a) \max_{b \in A} (P(j, b) - \max_{b \in A} Q(j, b))
+ (1 - w + w\alpha p(i|i, a)) \max_{c \in A} (P(i, c) - \max_{c \in A} Q(i, c))
\]
\[
\leq w\alpha \sum_{j=1}^{M} p(j|i, a) \max_{b \in A} (P(j, b) - \max_{b \in A} Q(j, b))
+ (1 - w + w\alpha p(i|i, a)) \max_{b \in A} (P(i, b) - Q(i, b))
\]
\[
\leq (w\alpha + 1 - w)\|P - Q\|.
\]

Hence,
\[
\max_{(i, a)} \| (H_w P - H_w Q)(i, a) \| \leq (w\alpha + 1 - w)\|P - Q\|,
\]
or
\[
\| (H_w P - H_w Q) \| \leq (w\alpha + 1 - w)\|P - Q\|.
\]

Note that in equation [16], we make use of the assumption $0 \leq w \leq w^*$ (refer equation [13]) that ensures the term $(1 - w + w\alpha p(i|i, a)) \geq 0$ to arrive at equation (17). Also note the application of Lemma 3 in equation (17) to arrive at equation (18). Finally it is easy to see from the assumptions on $w$ and discount factor $\alpha$ that $0 < (w\alpha + 1 - w) < 1$. Therefore $H_w$ is max-norm contraction with contraction factor $(w\alpha + 1 - w)$ and $Q^*$ is the unique fixed point.

**Lemma 3:** For $1 \leq w \leq w^*$ the contraction factor for the map $H_w$,
\[
1 - w + \alpha w \leq \alpha.
\]

**Proof:** Define $f(w) = 1 - w + \alpha w$. Clearly $f'(w) = -(1 - \alpha) < 0$. Hence $f$ is decreasing. Also observe that $f(1) = \alpha$. Hence for $1 \leq w \leq w^*$, $1 - w + \alpha w = f(w) \leq f(1) = \alpha$.

This is one of the key results in this paper. This lemma shows that, SOR Q-learning asymptotically tracks the optimal Q-values faster than the standard Q-learning.

We apply the following theorem [16] to show the convergence of the iterates of SOR Q-Learning to the optimal Q-values.

**Theorem 1:** The $p$-dimensional random process $\{\Delta_n\}$ taking values in $\mathbb{R}^p$ and defined as
\[
\Delta_{n+1}(l) = (1 - \gamma_n(l))\Delta_n(l) + \gamma_n(l) F_n(l), 1 \leq l \leq p,
\]
converges to zero with probability 1 under the following assumptions:

- $0 \leq \gamma_n(l) \leq 1, \sum_{n=1}^{\infty} \gamma_n(l) = \infty$ and $\sum_{n=1}^{\infty} \gamma_n^2(l) < \infty$;
- $\|E[F_n | F_n]\| \leq \beta \|\Delta_n\|$, with $\beta < 1$;
- $\text{var}[F_n(l)|F_n] \leq C(1 + \|\Delta_n\|^2)$, for $C > 0$.

where $F_n = \{\Delta_n, \Delta_{n-1}, \ldots, F_{n-1}, \ldots, \gamma_n, \ldots\}$ stands for the past at time $n$.

**Theorem 2:** Given a finite MDP $(S, A, p, r, \alpha)$ with bounded rewards i.e. $|r(i, a, j)| \leq B \forall (i, a, j) \in S \times A \times S$, the SOR Q-Learning algorithm given by the update rule
\[
Q_{n+1}(i, a) = Q_n(i, a) + \gamma_n(i, a) \left( w(r(i, a, j) + \alpha \max_{a \in A} Q_n(j, a)) 
+ (1 - w) \max_{a \in A} Q_n(i, a) - Q_n(i, a) \right)
\]
converges with probability 1 to the optimal Q-values as long as
\[
\sum_{n} \gamma_n(i, a) = \infty, \quad \sum_{n} \gamma_n^2(i, a) < \infty,
\]
for all $(i, a) \in S \times A$.

**Proof:** Upon rewriting the update rule we have
\[
Q_{n+1}(i, a) = (1 - \gamma_n(i, a)) Q_n(i, a) 
+ \gamma_n(i, a) \left( w(r(i, a, j) + \alpha \max_{a \in A} Q_n(i, b)) 
+ (1 - w) \max_{a \in A} Q_n(i, c) \right).
\]

Define $\Delta_n(i, a) = Q_n(i, a) - Q^*(i, a)$ and $F_n = \sigma(\{Q_0, \gamma_j(i, j, a) \forall j \leq n, n \geq 0\})$ be the filtration. We have
\[
\Delta_{n+1}(i_n, a_n) = (1 - \gamma_n(i_n, a_n))\Delta_n(i_n, a_n) 
+ \gamma_n(i_n, a_n) \left( w(r_n + \alpha \max_{b \in A} Q_n(i_{n+1}, b) - Q^*(i_n, a_n)) 
+ (1 - w) \max_{c \in A} Q_n(i_n, c) - Q^*(i_n, a_n) \right)
\]

Let
\[
F_n(i, a) = w(r(i, a, \eta_i, a) + \alpha \max_{b \in A} Q_n(\eta_i, a, b) - Q^*(i, a)) 
+ (1 - w) \left( \max_{c \in A} Q_n(i, c) - Q^*(i, a) \right),
\]
where $\eta_i, a$ is a random variable having the distribution $p(. | i, a)$.
We have
\[
\mathbb{E}[F_n(i, a) | \mathcal{F}_n] = \sum_{j=1}^{M} p(j|i, a) \left[ w(r(i, a, j) + \alpha \max_{b \in A} Q_n(j, b) - Q^*(i, a)) \right. \\
+ (1 - w) \left( \max_{c \in A} Q_n(i, c) - Q^*(i, a) \right] \\
= (H_w Q_n)(i, a) - Q^*(i, a).
\]

Since \(Q^* = H_w Q^*\) from Lemma 2, we have
\[
\left\| \mathbb{E}[F_n(i, a) | \mathcal{F}_n] \right\| \leq (\alpha w + 1 - w) \| Q_n - Q^* \| \\
= (\alpha w + 1 - w) \| \Delta_n \|.
\]

Finally,
\[
\text{var} \left[ F_n(i, a) | \mathcal{F}_n \right] = \mathbb{E} \left[ \left( w(r(i, a, \eta, a) + \alpha \max_{b \in A} Q_n(\eta, a, b) - Q^*(i, a)) \right. \\
+ (1 - w) \left( \max_{c \in A} Q_n(i, c) - Q^*(i, a) \right)^2 \\
- H_w Q_n(i, a) + Q^*(i, a) \right] \\
\leq \mathbb{E} \left[ \left( w(r(i, a, \eta, a) + \alpha \max_{b \in A} Q_n(\eta, a, b)) \right. \\
+ (1 - w) \left( \max_{c \in A} Q_n(i, c) \right)^2 \\
- H_w Q_n(i, a) + Q^*(i, a) \right] \\
\leq 3 \left( w^2 B^2 + \alpha^2 w^2 \| Q_n \|^2 + (1 - w)^2 \| Q_n \|^2 \right) \\
\leq 3 \left( w^2 B^2 + 2(\alpha^2 w^2 + (1 - w)^2) (\| Q^* \|^2 + \| \Delta_n \|^2) \right) \\
\leq C(1 + \| \Delta_n \|^2),
\]
where \(C = \max \left\{ 3w^2 B^2 + 6(\alpha^2 w^2 + (1 - w)^2) \| Q^* \|^2, 6(\alpha^2 w^2 + (1 - w)^2) \right\}.\)

The first inequality follows from the fact:
\[
\mathbb{E}[Z]^2 \leq \mathbb{E}[Z^2].
\]

The second inequality follows from:
\[
|r(i, a, j)| \leq B, \\
\|v\| = \max_{i} |v(i)|, \\
\forall a, b, c (a + b + c)^2 \leq 3(a^2 + b^2 + c^2).
\]

and the third inequality from the properties:
\[
\forall a, b, (a + b)^2 \leq 2(a^2 + b^2),
\]
as well as the triangle inequality of the norm. Hence
\[
\text{var} \left[ F_n(i, a) | \mathcal{F}_n \right] \leq C(1 + \| \Delta_n \|^2).
\]

Therefore by Theorem 1, \(\Delta_n\) converges to zero with probability 1 i.e., \(Q_n\) converges to \(Q^*\).

V. EXPERIMENTS AND RESULTS

In this section, we present the experimental evaluation of our proposed algorithm. First we numerically establish the convergence of our algorithm to the optimal Q-values. Next, we show the comparison between SOQ Q-learning and standard Q-learning when we select the optimal \(w^*\) (refer Section IV). Finally, we show the comparison between various feasible \(w\) values that can be used in our algorithm.

For our experiments, we construct 100 independent and random MDPs with 10 states and 5 actions that satisfy the assumption i.e. \(p(i | a) > 0, \forall (i, a).\) Note that this condition makes sure that \(w^* > 1\) which in turn ensures that the contraction factor of \(H_w\) is strictly less than \(\alpha\) (refer Lemma 3). However any \(0 \leq w \leq w^*\) ensures convergence of SOQ Q-learning algorithm. We use python MDP toolbox [17] to generate the MDPs. For both SOQ Q-learning and standard Q-learning algorithms, we maintain the same step-size and run the algorithms for the same number of iterations. Implementation of our SOQ Q-learning is available here [1].

In Figure 1, we plot the average error as a function of number of iterations. Average error is calculated as follows. For each of 100 runs, we collect the error between the optimal value function and the Q-value estimate at every iteration. Then, the average error is calculated as the average of the errors collected, i.e., average error at iteration \(k\) is as follows:
\[
e(k) = \frac{1}{100} \sum_{m=1}^{100} ||V_m^* - \max_{a} Q_m^k(\cdot, a)||,
\]
where \(V_m^*\) is the optimal value function of the \(m^{th}\) MDP and \(Q_m^k\) is the Q-value estimate of the \(m^{th}\) MDP at iteration \(k.\)

We can see that, in Figure 1, \(e(k)\) decreases as the number of iterations increases.

In Figure 2, we show the comparison between SOQ Q-learning and the standard Q-learning over 10\(^5\) iterations. In this experiment, we select optimal \(w^*\) for our SOQ Q-learning. We can see that the average error for SOQ Q-learning is less than that of standard Q-learning during the learning process. In Table 1, we show the performance of the converged policies in both the cases. We observe that SOQ Q-learning outperforms standard Q-learning in 69 runs out of 100 and on average, gives lower error. This shows that, in most of the cases, SOQ Q-learning gives better performance than standard Q-learning numerically.

[1] https://github.com/raghudiddigi/SOR-Q-Learning
Note that in the above experiment, we have selected optimal $w^*$ in SOR Q-Learning. However, in Section IV, we showed that any $w$ satisfying $1 \leq w \leq w^*$ would suffice for faster convergence than standard Q-learning. In Figure 3, we show the performance of SOR Q-learning for different $w$ values. Note that $w = 1$ corresponds to the standard Q-learning algorithm. We can see that the performance improves as $w$ increases from 1 to $w^*$. Nonetheless, any feasible value of $w$ performs better than $w = 1$ case, which corresponds to the Q-learning algorithm.

In conclusion, we showed that, on average, SOR Q-learning learns faster than the standard Q-learning algorithm.

VI. CONCLUSIONS AND FUTURE WORK

In this work, we proposed SOR Q-learning, a generalization of Q-learning that makes use of the concept of Successive Over Relaxation. We showed that the contraction factor of SOR Q-learning is less than or equal to $\alpha$ (discount factor), which is the contraction factor of standard Q-learning. We then proved the convergence of SOR Q-learning. Finally, we numerically established that, on average, SOR Q-learning learns the optimal value function faster than standard Q-learning. In future, we would like to extend the concept of SOR to the average cost and risk-sensitive MDPs. It would also be interesting to derive the rate of convergence of SOR Q-learning. Another line of research would be to develop function approximation versions of SOR Q-learning.

| Algorithm     | Average Error | Average Policy Difference |
|---------------|---------------|---------------------------|
| SOR Q-Learning| 0.3052        | 0.95                      |
| Q-Learning    | 0.3962        | 0.97                      |

TABLE I
PERFORMANCE OF CONVERGED POLICIES

REFERENCES

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