GLOBAL SOLUTION TO ENSKOG EQUATION WITH EXTERNAL FORCE IN INFINITE VACUUM

ZHENGLU JIANG

Abstract. We first give hypotheses of the bicharacteristic equations corresponding to the Enskog equation with an external force. Since the collision operator of the Enskog equation is more complicated than that of the Boltzmann equation, these hypotheses are more complicated than those given by Duan et al. for the Boltzmann equation. The hypotheses are very related to collision of particles of moderately or highly dense gases along the bicharacteristic curves and they can be used to make the estimation of the so-called gain and loss integrals of the Enskog integral equation. Then, by controlling these integrals, we show the existence and uniqueness of the global mild solution to the Enskog equation in an infinite vacuum for moderately or highly dense gases. Finally, we make some remarks on the locally Lipschitz assumption of the collision factors in the Enskog equation.

1. Introduction

This paper is to consider the existence and uniqueness of the global mild solution to the Enskog equation with an external force in an infinite vacuum for moderately or highly dense gases. Throughout this paper, \( \mathbb{R}_+ \) represents the positive side of the real axis including its origin and \( \mathbb{R}^3 \) denotes a three-dimensional Euclidean space. In the presence of external forces depending on the time and space variables, the Enskog equation is as follows (see [7] or [10]):

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + E(t,x) \cdot \frac{\partial f}{\partial v} = Q(f)
\]  

where \( f = f(t,x,v) \) is a one-particle distribution function that depends on the time \( t \in \mathbb{R}_+ \), the position \( x \in \mathbb{R}^3 \) and the velocity \( v \in \mathbb{R}^3 \), \( E(t,x) \) is a vector-valued function which belongs to \( \mathbb{R}^3 \) and represents an external force with respect to the time and space variables, and \( Q \) is the Enskog collision operator whose form will be explained below.

The collision operator \( Q \) is expressed by the difference between the gain and loss terms respectively defined by

\[
Q^+(f)(t,x,v) = \int_{\mathbb{R}^3 \times S_+^2} F^+(f) f(t,x,v') f(t,x-a\omega,w') B(v-w,\omega) d\omega dw,
\]

\[
Q^-(f)(t,x,v) = \int_{\mathbb{R}^3 \times S_+^2} F^-(f) f(t,x,v) f(t,x+a\omega,w) B(v+w,\omega) d\omega dw.
\]

Here and below everywhere, \( S_+^2 = \{ \omega \in S^2 : \omega(v-w) \geq 0 \} \) is a subset of a unit sphere surface \( S^2 \) in \( \mathbb{R}^3 \), \( a \) is a positive constant that represents a diameter of hard sphere, \( \omega \) is a unit vector along the line passing through the centers of the spheres at their interaction, \( B(v-w,\omega) = (v-w)\omega \) is the collision kernel and \( F^\pm \) are the collision factors. \( F^\pm \) are usually assumed to be two functionals of \( f \), more precisely speaking, they depend on the density \( \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv \) at the time \( t \) and the point \( x \).

Date: May 31, 2008.
2000 Mathematics Subject Classification. 76P05; 35Q75.
Key words and phrases. The Enskog equation; global solution.
This work was supported by NSFC 10271121 and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, the Education Ministry of China, and sponsored by joint grants of NSFC 10511120278/10611120371 and RFBR 04-02-39026.
In equations (1.2) and (1.3), \((v, w)\) and \((v', w')\) are velocities before and after the collision, respectively. As for the Boltzmann equation, the conservation of both kinetic momentum and energy of two colliding particles gives

\[
v + w = v' + w', \quad v^2 + w^2 = v'^2 + w'^2,
\]

which leads to their velocity relations

\[
v' = v - [(v - w)\omega], \quad w' = w + [(v - w)\omega],
\]

where \(\omega \in S^2_+\). We denote \(u = v - w, \quad u\parallel = (u\omega)\omega \) and \(u\perp = u - u\parallel\) (see \([9]\) or \([11]\)), thus getting another expression of (1.5) as follows:

\[
v' = v - u\parallel, \quad w' = v - u\perp.
\]

Then the gain and loss terms (1.2) and (1.3) can be rechanged as

\[
Q^+(f)(t, x, v) = \int_{\mathbb{R}^3 \times S^2_+} F^+(f) f(t, x, v - u\parallel) f(t, x - a\omega, v - u\perp) B(u, \omega) d\omega du, \tag{1.7}
\]

\[
Q^-(f)(t, x, v) = \int_{\mathbb{R}^3 \times S^2_+} F^-(f) f(t, x, v) f(t, x + a\omega, v - u) B(u, \omega) d\omega du. \tag{1.8}
\]

If the factors \(F^\pm\) are set to be the same positive constant and the diameter \(a\) equal to zero in the density variables, then the Enskog equation becomes the Boltzmann one that provides a successful description for dilute gases. The Boltzmann equation is no longer valid for moderately or highly dense gases. As a modification of the Boltzmann equation, the Enskog equation proposed by Enskog \([10]\) in 1922 is usually used to explain the dynamical behavior of the density profile of moderately or highly dense gases.

As we know, there are global solutions for the Boltzmann equation in the absence of external forces not only in an infinite vacuum but also with large initial data. Existence of such vacuum solutions was first considered by Illner & Shinbrot \([12]\) and later by Bellomo & Toscani \([4]\). The global existence of solution was shown by DiPerna & Lions \([8]\) for the Boltzmann equation with the large data. There are also some similar results about the Enskog equation without any effect of external forces. For example, Polewczak \([15]\) and Arkeryd \([1]\) gave their different existence proofs of global-in-time solutions to the Enskog equation without external forces for both near-vacuum and large data, respectively. It is worth mentioning that some early works on the existence of global solutions to the Enskog equation were given by Cercignani and/or Arkeryd (e.g. \([2]\), \([3]\), \([5]\), \([6]\)). Recently, Duan et al. \([9]\) proved the existence and uniqueness of a global mild solutions for the Boltzmann equation with external forces in an infinite vacuum and many relevant works can be found in the reference. Now there is not yet such similar result for the Enskog equation in the presence of external forces. The aim of this paper is to extend the result to the case of the Enskog equation, that is, to show the existence and uniqueness of such vacuum solution to the Enskog equation (1.1) with (1.2) and (1.3). In Section 2 hypotheses of the external forces are given and a Banach space and its operators are constructed. These hypotheses are very related to collision of particles of moderately or highly dense gases along the bicharacteristic curves and they are much more complicated than those given by Duan et al. mentioned above for the Boltzmann equation since the collision operator of the Enskog equation is fairly more complicated than that of the Boltzmann equation. In spite of this, the two examples shown by Duan et al. in \([9]\) satisfy our hypotheses. Then the estimation of the so-called gain and loss integrals is made in Section 3. An existence and uniqueness theorem of global mild solution to the Enskog equation in an infinite vacuum is given in Section 4 and some remarks on the assumption of the factors \(F^\pm\) are finally made in Section 5.
2. Hypotheses and Operators

In this section we first give some constructive hypotheses of the external forces with the help of the bicharacteristic equations of the Enskog equation with these forces and then build a Banach space and its operators relative to the Enskog integral equation.

Let us begin with considering the bicharacteristic equations of the Enskog equation (1.1)

\[
\frac{dX}{ds} = V, \quad \frac{dV}{ds} = E(s, X), \quad (X, V)|_{s=t} = (x, v).
\]

Suppose that such a vector-valued force function \(E(t, x)\) allows the above system (2.1) to have a global-in-time smooth solution denoted by

\[
[X(s; t, x, v), V(s; t, x, v)]
\]

for any fixed \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\), and that there exist three functions \(\alpha_i(s; t, x, v) (i = 1, 2, 3)\) such that the solution (2.2) satisfies the following conditions:

\[
\alpha_1(s; t, x, v) > 0 \quad \text{as} \quad s > 0, \quad \alpha_1(0; t, x, v) \geq 0, \quad \alpha_2(s; t, x, v) > 0, \quad \alpha_3(s; t, x, v) \geq 0,
\]

(2.3)

\[
X(0; s, X(t; x, v) + \xi, V(t; x, v) - \eta) = X(0; t, x, v) + \alpha_1(s; t, x, v)\xi + \alpha_2(s; t, x, v)\eta,
\]

(2.4)

\[
V(0; s, X(t; x, v) + \xi, V(t; x, v) - \eta) = V(0; t, x, v) - \alpha_2(s; t, x, v)\eta - \alpha_3(s; t, x, v)\xi,
\]

(2.5)

either \(\alpha_3(s; t, x, v) \equiv 0\) or \(\max \{\alpha_1(s; t, x, v), \alpha_3(s; t, x, v)\}/\alpha_2(s; t, x, v) \leq \tau_0\),

(2.6)

\[
\min \{(\alpha_2(s; t, x, v))^2\alpha(s; t, x, v), \alpha_2(s; t, x, v)\alpha(s; t, x, v), \alpha_2(s; t, x, v)\} \geq \alpha_0 > 0,
\]

(2.7)

for any \(s \in \mathbb{R}_+\) and \((\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3\) when any point \((t, x, v)\) is fixed in \(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\), where \(\alpha_0, \epsilon_0\) and \(\tau_0\) are three fixed positive constants independent of \(s\) and \((t, x, v)\), and \(\alpha(s; t, x, v)\) is denoted by

\[
\alpha(s; t, x, v) \equiv \alpha_1'(s; t, x, v)\alpha_2(s; t, x, v) - \alpha_1(s; t, x, v)\alpha_2'(s; t, x, v),
\]

(2.8)

here \(\alpha_i'(s; t, x, v) (i = 1, 2)\) represent the derivative with respect to \(s\). For their understanding of their hypotheses of the external force in the Boltzmann equation, Yuan et al. [9] took the following two examples: \(E(t, x) = E_0(t)\) and \(E(t, x) = \epsilon_0^2 x + E_0(t)\) with \(\epsilon_0\) being a positive constant. Since the Enskog equation is much more complicated than the Boltzmann one, our hypotheses of the external forces for the Enskog equation are much more complicated than those given by Yuan et al. mentioned above for the Boltzmann equation. In spite of this, the above two examples satisfy the above hypotheses of the external force. These examples are obviously suitable for our explanation of the corresponding constructive conditions whether the Boltzmann equation or the Enskog one is considered.

We give the five conditions (2.3) - (2.7) in order to get the following inequalities

\[
|X(0; s, X(s; t, x, v), V(s; t, x, v) - u)|^2 + |X(0; s, X(s; t, x, v) - a\omega, V(s; t, x, v) - u_{\perp})|^2
\]

\[
\geq |X(0; t, x, v)|^2 + |X(0; t, x, v) + \alpha_1(s; t, x, v)u - a\alpha_2(s; t, x, v)\omega|^2
\]

and

\[
|V(0; s, X(s; t, x, v), V(s; t, x, v) - u)|^2 + |V(0; s, X(s; t, x, v) - a\omega, V(s; t, x, v) - u_{\perp})|^2
\]

\[
\geq |V(0; t, x, v)|^2 + |V(0; t, x, v) - \alpha_2(s; t, x, v)u + a\alpha_3(s; t, x, v)\omega|^2
\]

along their bicharacteristic curves after collision of particles of moderately or highly dense gases. The above two inequalities can be used to control the gain and loss integral terms of the Enskog integral equation along the bicharacteristic curves. Therefore this form of the external forces is pertinent to collision of particles of moderately or highly dense gases along the bicharacteristic curves so that the gain and loss integral terms can be estimated.

We below give a representation of mild solution to the Enskog equation. Let us first introduce a notation \(f^\#\) defined as

\[
f^\#(s; t, x, v) = f(s, X(s; t, x, v), V(s; t, x, v))
\]
for any measurable function \( f \) on \( \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \). Obviously, it can be found that \( f^\#(t; t, x, v) = f(t, x, v) \) and that \( f^\#(0; t, x, v) = f(0, X(0; t, x, v), V(0; t, x, v)) \).

Along the bicharacteristic curves, the Enskog equation (1.11) can be also written as

\[
\frac{d}{ds} f^\#(s; t, x, v) = Q(f)^\#(s; t, x, v),
\]

which leads to the following integral equation

\[
f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) + \int_0^t Q(f)^\#(s; t, x, v)ds,
\]

(2.9)

where \( f_0(x, v) \equiv f(0, x, v) \). A function \( f(t, x, v) \) is called global mild solution to the Enskog equation (1.11) if \( f(t, x, v) \) satisfies the above integral equation (2.9) for almost every \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\).

Then we construct a subset \( M \) of a Banach space \( C(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3) \), which has the property that every element \( f = f(s, x, v) \in M \) if and only if there exists a positive constant \( c \) such that \( f \) satisfies

\[
|f(t, x, v)| \leq ch(X(0; t, x, v))m(V(0; t, x, v))
\]

where

\[
h(x) = e^{-p|x|^2}, \quad m(v) = e^{-qv^2},
\]

(2.10)

for any fixed \( p \) and \( q \) in \((0, +\infty)\). It follows that \( M \) is a Banach space when it has a norm of the following form

\[
||f|| = \sup_{t, x, v} \{ |f(t, x, v)|h^{-1}(X(0; t, x, v))m^{-1}(V(0; t, x, v)) \}.
\]

In particular, \( ||f(0, x, v)|| = \sup_{x, v} \{ |f(0, x, v)|h^{-1}(x)m^{-1}(v) \} \). The initial data \( f_0 \equiv f(0, x, v) \) is bounded in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). This implies that the total mass of the system is finite. Hence the mean free path is sufficiently large if the finite total mass is sufficiently small. This is exactly the requirement on the Enskog equation with external forces in an infinite vacuum, which is similar to one considered by Illner & Shinbrot [12] for the Boltzmann equation. It is worth mentioning that in other cases there are many different classes of functions which can be taken as the choice of \( h(x) \) and \( m(v) \) (see [9], [11]). For example, in the case of the external forces depending only on the time \( t \), one can also choose \( h(x) = (1 + |x|^2)^{-p} \) and \( m(v) = e^{-qv^2} \) for any fixed \( p \in (1/2, +\infty) \) and \( q \in (0, +\infty) \); when \( p > 3/2 \), the initial data \( f_0 \equiv f(0, x, v) \) is bounded in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \); when \( 1/2 < p \leq 3/2 \), the initial total mass might be infinite; this choice of both \( h(x) \) and \( m(v) \) is not suitable for using our method considered in this paper to deal with this existence problem of vacuum solutions to the Enskog equation with external forces depending on the time and space variables, however, this choice can be applied to the case of the Boltzmann equation in the presence of external forces [9].

To give global existence, it is necessary to study the properties of the collision operator in a Banach space. To do this, by (1.7) and (1.8), \( Q(f^\#)(s; t, x, v) \) can be first rewritten as the difference between the gain and loss terms of other two forms

\[
Q^+(f^\#)(s; t, x, v) = \int_{\mathbb{R}^3 \times S^2_+} F^+(f)f(s, X(s; t, x, v), V(s; t, x, v) - u_\parallel) \\
\times f(s, X(s; t, x, v) - a_\omega, V(s; t, x, v) - u_\parallel)B(u, \omega)d\omega du,
\]

(2.11)

\[
Q^-(f^\#)(s; t, x, v) = \int_{\mathbb{R}^3 \times S^2_+} F^-(f)f(s, X(s; t, x, v), V(s; t, x, v)) \\
\times f(s, X(s; t, x, v) + a_\omega, V(s; t, x, v) - u)B(u, \omega)d\omega du.
\]

(2.12)
Estimation of the collision integrals can be then made by use of a similar argument to that developed in the previous work (see \[9\], \[15\], \[18\]). According to (2.11) and (2.12), we in fact have to estimate the following two integrals:

\[
I_g \equiv \int_0^t \int_{\mathbb{R}^3 \times S^2_+} h(X(0; s, X(s; t, x, v), V(s; t, x, v) - u_\parallel)) \times m(V(0; s, X(s; t, x, v), V(s; t, x, v) - u_\parallel))h(X(0; s, X(s; t, x, v) - a\omega, V(s; t, x, v) - u_\perp))
\times m(V(0; s, X(s; t, x, v) - a\omega, V(s; t, x, v) - u_\perp))B(u, \omega)d\omega du ds,
\]

\[
I_l \equiv \int_0^t \int_{\mathbb{R}^3 \times S^2_+} h(X(0; s, X(s; t, x, v), V(s; t, x, v))) \times m(V(0; s, X(s; t, x, v)))h(X(0; s, X(s; t, x, v) + a\omega, V(s; t, x, v) - u))
\times m(V(0; s, X(s; t, x, v) + a\omega, V(s; t, x, v) - u))B(u, \omega)d\omega du ds.
\]

Here \(I_g\) and \(I_l\) are called the gain and loss integrals respectively. Once the estimation of the integrals is finished, the global existence result may be obtained by constructing a contractive map from a Banach space to itself. Therefore it is one of the best important to estimate the above two integrals, especially the gain one. It will be discussed in the next section.

We finally denote an operator \(J\) on \(M\) by

\[
J(f) = f_0(X(0; t, x, v), V(0; t, x, v)) + \int_0^t Q(f)#(s; t, x, v)ds.
\]

It will be proved in Section 4 that \(J\) is indeed a contractive map on a Banach space. This is what we need.

3. Estimation of the Gain and Loss Integrals

In this section the so-called gain and loss integrals are estimated by use of a similar device to one given in \[15\] for the Enskog equation in the absence of external forces. To do this, we first introduce the preliminary lemmas which will be used below.

**Lemma 3.1.** Let any \(z \in \mathbb{R}^3\), \(s \in \mathbb{R}_+\), \((u_\parallel, u_\perp) \in \mathbb{R}^3 \times \mathbb{R}^3\) with \(u_\parallel u_\perp = 0\) and \(\omega \in S^2\) with \(u_\parallel \omega \geq 0\). Then

\[
|z \pm su_\parallel|^2 + |z \pm su_\perp \mp a\omega|^2 \geq |z|^2 + |z \pm s(u_\parallel + u_\perp) \mp a\omega|^2
\]

for any fixed real number \(a \in \mathbb{R}_+\).

**Lemma 3.2.** Let \(p > 0\) and \((z, u) \in \mathbb{R} \times \mathbb{R}\) with \(u \neq 0\). Then

\[
\int_0^{+\infty} e^{-p|z + su|^2} ds \leq \frac{\sqrt{\pi}}{\sqrt{p}|u|}.
\]

**Lemma 3.3.** Let \(q > 0\), \(-2 < \gamma \leq 1\) and \(z \in \mathbb{R}^3\). Then

\[
\int_{\mathbb{R}^3} |u|^\gamma e^{-q|z - u|^2} du \leq \frac{4\pi}{\gamma + 2} + \frac{\pi}{q^{3/2}}.
\]

**Lemma 3.4.** Let any \((s, z, u) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3\) and \(\omega_\perp\) be a unit vector perpendicular to \(\omega \in S^2\). Then

\[
|z + su + h\omega| \geq |z\omega_\perp + su\omega_\perp|
\]

for any \(h \in \mathbb{R}\).

The proof of Lemmas 3.1 and 3.4 is easily given. Lemmas 3.2 and 3.3 can be obtained from the transformation of integral variables.
Lemma 3.5. Assume that three functions \(\alpha_i(s;t,x,v)\) \((i = 1, 2, 3)\) satisfy the external force conditions \((2.3)\) and \((2.7)\), and that

\[
\tilde{I}_l(z_1, z_2, t, x, v) \equiv \int_0^1 \int_{\mathbb{R}^3 \times S_+^2} \left| u\omega \right| e^{-p|z_1 + \alpha_1(s;t,x,v)u + \alpha_2(s;t,x,v)\omega|^2} \\
\times e^{-q|z_2 - \alpha_2(s;t,x,v)u + \alpha_3(s;t,x,v)\omega|^2} \, du\omega ds
\]

for any fixed real number \(a \in \mathbb{R}_+\). Then

\[
\tilde{I}_l(z_1, z_2, t, x, v) \leq \tilde{I}_{lpq}
\]

(3.5)

for any \((z_1, z_2) \in \mathbb{R}^3 \times \mathbb{R}^3\), \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\), where \(\tilde{I}_{lpq}\) is a positive constant depending only on \(p\) and \(q\), \(p > 0, q > 0\).

Proof. First let us fix \((z_1, z_2) \in \mathbb{R}^3 \times \mathbb{R}^3\) and \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\). Put \(\tilde{u} = \alpha_2(s;t,x,v)u + \alpha_3(s;t,x,v)\omega\). Then \(\tilde{u}\omega = \alpha_2(s;t,x,v)(u\omega) + \alpha_3(s;t,x,v)\omega\) for \(\omega \in S_+^2\). By \((2.3)\), it thus follows that

\[
\tilde{I}_l(z_1, z_2, t, x, v) \leq \int_0^1 \int_{\mathbb{R}^3 \times S_+^2} \left| \tilde{u}\omega \right| (\alpha_2(s;t,x,v))^{-1} \\
\times e^{-p|z_1 + \alpha_1(s;t,x,v)\tilde{u} + \alpha_2(s;t,x,v)\omega|^2} e^{-q|z_2 - \tilde{u}|^2} \, d\tilde{u}\omega ds,
\]

(3.6)

where \(\tilde{\alpha}(s;t,x,v) = \alpha_2(s;t,x,v) - \alpha_1(s;t,x,v)\omega\). Take \(\tau = \frac{\alpha_1(s;t,x,v)}{\alpha_2(s;t,x,v)}\). Then, by \((2.7)\),

\[
\frac{d\tau}{ds} = \frac{\alpha(s;t,x,v)}{(\alpha_2(s;t,x,v))^2} > 0.
\]

By replacing the integral variable \(s\) with the new variable \(\tau\) and using Lemma 3.4 the estimation of the integral on the right side of \((3.6)\) thus gives

\[
\tilde{I}_l(z_1, z_2, t, x, v) \leq \int_0^{+\infty} \int_{\mathbb{R}^3 \times S_+^2} \left| \tilde{u}\omega \right| (\alpha_2(s;t,x,v))^{-2} (\alpha(s;t,x,v))^{-1} \\
\times e^{-p|z_1 + \omega_{\perp} + \tau\tilde{u}\omega_{\perp}|^2} e^{-q|z_2 - \tilde{u}|^2} \, d\tilde{u}\omega d\tau
\]

\[
\leq \frac{1}{\alpha_0} \int_{\mathbb{R}^3 \times S_+^2} \left| \tilde{u}\omega \right| \left\{ \int_0^{+\infty} e^{-p|z_1 + \omega_{\perp} + \tau\tilde{u}\omega_{\perp}|^2} d\tau \right\} e^{-q|z_2 - \tilde{u}|^2} \, d\tilde{u}\omega, \tag{3.7}
\]

where \(\omega_{\perp}\) is a unit vector perpendicular to \(\omega\). The last inequality in \((3.7)\) comes from \((2.7)\) and Fubini’s theorem. By estimation of the integral on the right side of the last inequality, it then follows that

\[
\tilde{I}_l(z_1, z_2, t, x, v) \leq \frac{1}{\alpha_0} \frac{\sqrt{\pi}}{\sqrt{p}} \int_{\mathbb{R}^3 \times S_+^2} \left| \tilde{u}\omega \right| e^{-q|z_2 - \tilde{u}|^2} \, d\tilde{u}\omega \leq \frac{1}{\alpha_0} \frac{4\pi^{5/2}}{\sqrt{p}} \left( \frac{4}{3} + \frac{1}{q^{3/2}} \right), \tag{3.8}
\]

where Lemmas 3.2 and 3.3 are used. The proof of Lemma 3.5 is hence completed. \(\square\)

Lemma 3.6. Assume that three functions \(\alpha_i(s;t,x,v)\) \((i = 1, 2, 3)\) satisfy the external force conditions \((2.3)\), \((2.6)\) and \((2.7)\), and put

\[
\tilde{I}_g(z_1, z_2, t, x, v) \equiv \int_0^1 \int_{\mathbb{R}^3 \times S_+^2} \left| u\omega \right| e^{-p|z_1 + \alpha_1(s;t,x,v)u + \alpha_2(s;t,x,v)\omega|^2} \\
\times e^{-q|z_2 - \alpha_2(s;t,x,v)u + \alpha_3(s;t,x,v)\omega|^2} \, du\omega ds,
\]

where \(a \geq 0\). Then

\[
\tilde{I}_g(z_1, z_2, t, x, v) \leq \tilde{I}_{gpq}, \tag{3.9}
\]

for any \((z_1, z_2) \in \mathbb{R}^3 \times \mathbb{R}^3\), \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\), where \(\tilde{I}_{gpq}\) is a positive constant depending only on \(p\) and \(q\), \(p > 0, q > 0\).
It follows from Lemma 3.5 that can be rewritten as

Let us first estimate the loss integral. By using (2.4) and (2.5), the loss integral (2.14)

Proof. Let us fix \((z_1, z_2) \in \mathbb{R}^3 \times \mathbb{R}^3\) and \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\). Note that \(\alpha_2(s; t, x, v) > 0\) in (2.3). Put \(\hat{u} = \hat{u}_\parallel + \hat{u}_\perp\), where \(\hat{u}_\parallel = \alpha_2(s; t, x, v)u_\parallel - \alpha_3(s; t, x, v)\omega\) and \(\hat{u}_\perp = \alpha_2(s; t, x, v)u_\perp\). Then \(\hat{u}_\omega = \alpha_2(s; t, x, v)(\omega) - \alpha_3(s; t, x, v)\) for \(\omega \in S^2_+\). Thus

\[
\hat{I}_g(z_1, z_2, t, x, v) \leq \int_0^t \int_{\mathbb{R}^3 \times S^2_+} (|\hat{u}_\omega| + \alpha_3(s; t, x, v))(\alpha_2(s; t, x, v))^{-1}
\times e^{-p|z_1 + \alpha_1(s; t, x, v)| - \alpha_1(s; t, x, v)\omega|} e^{-q|z_2 - \hat{u}_\parallel^2} d\hat{u}_\omega ds,
\]

where \(\alpha(s; t, x, v) = \alpha_2^2(s; t, x, v) - \alpha_1(s; t, x, v)\alpha_3(s; t, x, v)\). To prove this lemma, it suffices to consider the case of \(\max\{\alpha_1(s; t, x, v), \alpha_3(s; t, x, v)\}/\alpha_2(s; t, x, v) \leq \tau_0\) in (2.3). By repeating a similar integral estimation to one given in Lemma 3.5, the estimate of the integrals on the right side of the above inequality gives

\[
\hat{I}_g(z_1, z_2, t, x, v) \leq \int_0^t \int_{\mathbb{R}^3 \times S^2_+} (|\hat{u}_\omega| + \alpha_3(s; t, x, v))(\alpha_2(s; t, x, v))^{-1}
\times e^{-p|z_1 + \tau_0 \hat{u}_\parallel^2|} e^{-q|z_2 - \hat{u}_\parallel^2} d\hat{u}_\omega ds,
\]

where \(\omega\) is a unit vector perpendicular to \(\omega\), (a) is obtained by first making the transformation \(\tau = \frac{\alpha_1(s; t, x, v)}{\alpha_3(s; t, x, v)}\) and then using (2.6) and Lemma 3.4, (b) is given by (2.7), (c) is obtained by Lemma 3.2 and (d) results from Lemma 3.3. This hence completes the proof of Lemma 3.6.

By Lemmas 3.5 and 3.6, we can hence give the estimates of the gain and loss integrals as follows.

**Lemma 3.7.** Let \(I_g\) and \(I_l\) be the same integrals as defined by (2.13) and (2.14), respectively. In the integrals, \(a\) is a positive constant. Suppose that all the five external force conditions (2.3)–(2.7) hold for these functions \(X(s; t, x, v)\) and \(V(s; t, x, v)\) defined by the solution (2.2) to the system (2.1), and that \(h(x)\) and \(m(v)\) are the same as in (2.10). Then it follows that

\[
I_g \leq Kh(X(0; t, x, v))m(V(0; t, x, v)),
\]

\[
I_l \leq Kh(X(0; t, x, v))m(V(0; t, x, v)),
\]

for any \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\) and some positive constant \(K\).

**Proof.** Let us first estimate the loss integral. By using (2.4) and (2.5), the loss integral (2.14) can be rewritten as

\[
I_l = \int_0^t \int_{\mathbb{R}^3 \times S^2_+} |\hat{u}_\omega| e^{-p|X(0; t, x, v)|^2} e^{-q|V(0; t, x, v)|} d\hat{u}_\omega ds,
\]

It follows from Lemma 3.5 that

\[
I_l \leq h(X(0; t, x, v))m(V(0; t, x, v))\hat{I}_{pq},
\]
Then the estimation of the gain integral will be made below. Similarly, by using (2.4) and (2.5), the gain integral (2.13) is as follows:

\[
I_g = \int_0^t \int_{\mathbb{R}^3 \times S^2_+} |u\omega|e^{-p}|X(0,t,x,v) + \alpha_1(s,t,x,v)u||^2 e^{-p}|X(0,t,x,v) + \alpha_1(s,t,x,v)u| - a\alpha_2(s,t,x,v)\omega|^2 \\
\quad \times e^{-q}|V(0,t,x,v) - \alpha_2(s,t,x,v)u||^2 e^{-q}|V(0,t,x,v) - \alpha_2(s,t,x,v)u + a\alpha_3(s,t,x,v)\omega|^2 d\omega ds. \tag{3.15}
\]

Using Lemma 3.1, we have

\[
I_g \leq e^{-p}|X(0,t,x,v)||^2 e^{-q}|V(0,t,x,v)||^2 \int_0^t \int_{\mathbb{R}^3 \times S^2_+} |u\omega|e^{-p}|X(0,t,x,v) + \alpha_1(s,t,x,v)u - a\alpha_2(s,t,x,v)\omega|^2 \\
\quad \times e^{-q}|V(0,t,x,v) - \alpha_2(s,t,x,v)u + a\alpha_3(s,t,x,v)\omega|^2 d\omega ds \\
= h(X(0, t, t, v))m(V(0, t, x, v)) I_g(X(0, t, t, v), V(0, t, t, v), t, x, v). \tag{3.16}
\]

It follows from Lemma 3.6 that

\[
I_g \leq h(X(0, t, t, v))m(V(0, t, x, v)) I_{gq}. \tag{3.17}
\]

The proof of Lemma 3.7 is hence completed. \qed

4. Existence and Uniqueness

In this section we show a result about the existence and uniqueness of such vacuum solution to the Enskog equation in presence of external forces. To do this, we first define a set \(M_R\) by

\[
M_R = \{ f : ||f|| \leq R, f \in C(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3) \} \tag{4.1}
\]

and then assume that \(F^\pm\) are two functionals on \(M_R\) such that the so-called locally Lipschitz condition

\[
|F^\pm(f) - F^\pm(g)| \leq L(R)||f - g|| \tag{4.2}
\]

holds for any \(f, g \in M_R\) where \(M_R\) is defined by (4.1) and \(L(R)\) is a positive nondecreasing function on \(\mathbb{R}_+\). Thus we can get the following lemma.

**Lemma 4.1.** Suppose that all the five conditions (2.3)–(2.7) hold for any \(X(s,t,x,v)\) and \(V(s,t,x,v)\) defined by the solution (2.2) to the system (2.1), and that the factors \(F^\pm\) in the collision integrals \(Q^\pm(f)\) of (2.10) are two functionals satisfying the inequality (4.2). Let \(h(x)\) and \(m(v)\) be the same as in (2.10). In the collision integrals, \(a\) is a positive constant. Then the following inequalities hold:

\[
\int_0^t |Q^+(f)^\#(s,t,x,v)| ds \leq C(R)h(X(0, t, t, v))m(V(0, t, t, v)||f||^2, \tag{4.3}
\]

\[
\int_0^t |Q^-(f)^\#(s,t,x,v)| ds \leq C(R)h(X(0, t, t, v))m(V(0, t, t, v)||f||^2
\]

for any \(f \in M_R\), where \(C(R)\) is a positive nondecreasing function on \(\mathbb{R}_+\).

**Proof.** It can be first found from the assumption (4.2) of the two functionals \(F^\pm\) that \(\tilde{L}(R) = L(R)R + |F^+(0)| + |F^-(0)|\) such that \(|F^\pm(f)| \leq \tilde{L}(R)\) for any \(f \in M_R\). It follows from (2.11) and (2.12) that

\[
Q^+(f)^\#(s,t,x,v) ds \leq \tilde{L}(R) \int_0^t \int_{\mathbb{R}^3 \times S^2_+} ||f||^2 h(X(0, s, X(s,t,x,v), V(s,t,x,v) - u_\perp)) \\
\times m(V(0, s, X(s,t,x,v), V(s,t,x,v) - u_\perp))h(X(0, s, X(s,t,x,v) - a\omega, V(s,t,x,v) - u_\perp)) \times m(V(0, s, X(s,t,x,v) - a\omega, V(s,t,x,v) - u_\perp)) d\omega d\omega ds, \tag{4.3}
\]

\[
Q^-(f)^\#(s,t,x,v) ds \leq \tilde{L}(R) \int_0^t \int_{\mathbb{R}^3 \times S^2_+} ||f||^2 h(X(0, s, X(s,t,x,v), V(s,t,x,v)))h(X(0, s, X(s,t,x,v) + a\omega, V(s,t,x,v) - u))
\]

\[
\times m(V(0, s, X(s,t,x,v), V(s,t,x,v)))h(X(0, s, X(s,t,x,v) + a\omega, V(s,t,x,v) - u))
\]

\[
\times m(V(0, s, X(s,t,x,v), V(s,t,x,v)))h(X(0, s, X(s,t,x,v) + a\omega, V(s,t,x,v) - u))
\]

\[
\times m(V(0, s, X(s,t,x,v), V(s,t,x,v)))d\omega d\omega ds.
\]
global solution to Enskog equation

\[ \times m(V(0; s, X(s; t, x, v) + aw, V(s; t, x, v) - u)) \omega \omega d\omega dud. \]  

(4.4)

By (3.11) and (3.12), (4.3) and (4.4) give

\[ \int_0^t Q^+ (f) dx d\tau \leq \bar{R}(R) Kh(X(0; t, x, v))m(V(0; t, x, v))\| f \|^2, \]

\[ \int_0^t Q^- (f) dx d\tau \leq \bar{R}(R) Kh(X(0; t, x, v))m(V(0; t, x, v))\| f \|^2. \]

Take \( C(R) = \bar{R}(R)K \). It obviously follows that Lemma 4.1 holds.

Then we can get the following theorem.

**Theorem 4.2.** Suppose that all the five external force conditions (2.3)–(2.7) hold for these functions \( X(s; t, x, v) \) and \( V(s; t, x, v) \) defined by the solution (2.2) to the system (2.1), and that the factors \( F^\pm \) in the collision integrals \( Q^\pm (f)(t, x, v) \) defined by (1.2) and (1.3) are two functionals satisfying the inequality (4.2). In the collision integrals, \( a \) is a positive constant. Then there exists a positive constant \( R_0 \) such that the Enskog equation (1.1) with (1.2) and (1.3) has a unique non-negative global mild solution \( f = f(t, x, v) \in M_{R_0} \) through a non-negative initial data \( f_0 = f_0(x, v) \) when

\[ \sup_{t, x, v} \{ f_0(X(0; t, x, v), V(0; t, x, v))h^{-1}(X(0; t, x, v))m^{-1}(V(0; t, x, v)) \} \]

is sufficiently small, where \( h(x) \) and \( m(v) \) are the same as in (2.10).

Theorem 4.2 shows that there exists a unique global mild solution to the Enskog equation (1.1) given by (1.2) and (1.3) with the initial data near vacuum if a suitable assumption of the external force is given. As in [13], we below give our proof of Theorem 4.2.

**Proof.** By (2.15) and Lemma 4.1 we have

\[ |J(f)|h^{-1}(X(0; t, x, v))m^{-1}(V(0; t, x, v)) \]

\[ \leq |f_0(X(0; t, x, v), V(0; t, x, v))h^{-1}(X(0; t, x, v))m^{-1}(V(0; t, x, v)) + 2C(R)|f| |^2 \]

\[ \leq R/2 + 2C(R)R^2 \]

for any \( f \in M_R \) and \( f_0 \) with \( \| f_0 \| \leq R/2 \). Since \( C(R) \) is a positive nondecreasing function on \( \mathbb{R}_+ \), it follows that \( \| J(f) \| \leq R \) for sufficiently small \( R \). Therefore \( J \) is an operator on \( M_R \) to itself for sufficiently small \( R \). Similarly, it can be also found that \( J \) is a contractive operator on \( M_R \) for some suitably small \( R \). Thus there exists a unique element \( f \in M_R \) such that \( f = J(f) \), i.e., (2.9) holds. It then follows from the same argument as the one in [12] (or see [14, 19]) that if \( f_0(x, v) \geq 0 \) then \( f(t, x, v) \geq 0 \). Hence the proof of Theorem 4.2 is finished. \( \square \)

5. Remarks on the Assumption of the Factors \( F^\pm \)

In this section we make some remarks on the locally Lipschitz assumption (4.2) of the factors of \( F^\pm \) appearing in Theorem 4.2 given in the previous section.

We begin with a different kind of locally Lipschitz condition of \( F^\pm \). It was originally given by Polewczak [15] as follows:

\[ |F^\pm (f) - F^\pm (g)| \leq L_0(R) \left| \int_{\mathbb{R}^3} f(t, x, v) dv - \int_{\mathbb{R}^3} g(t, x, v) dv \right| \]

(5.1)

holds for any \( f = f(t, x, v), g = g(t, x, v) \in M_R \) where \( M_R \) is defined by (4.1) and \( L_0(R) \) is a positive nondecreasing function on \( \mathbb{R}_+ \). Note that assumptions (2.4) and (2.5) have the following properties:

\[ X(0; t, x + \xi, v) - X(0; t, x, v) = \alpha_2(s; t, x, v) \xi \]

(5.2)

and

\[ V(0; t, x + \eta, v) - V(0; t, x, v) = \alpha_2(s; t, x, v) \eta \]

(5.3)
for any \((\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3\) when any point \((t, x, v)\) is fixed in \(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\). By (5.2) and (5.3), we have
\[
\frac{\partial X(0; t, x, v)}{\partial x} = \frac{\partial V(0; t, x, v)}{\partial v} = \alpha_2(s; t, x, v),
\]
thus giving
\[
\frac{\partial X(0; t, x, v)}{\partial x} = \frac{\partial V(0; t, x, v)}{\partial v} \geq \alpha_0 > 0
\]
because of assumption (2.4). Put \(L(R) = L_0(R) \int_{\mathbb{R}^3} m(v)dv/\alpha_0\). Then, by use of (5.3), (5.1) gives (4.2). This means that (5.1) is a stronger assumption than (4.2) when the external forces satisfy the assumptions (2.3)-(2.7). Therefore the locally Lipschitz assumption (4.2) is at least mathematically very useful in a more general case.

Now let us recall the Enskog equation for our further understanding the locally Lipschitz conditions (4.2) and (5.1) of the external forces. The Enskog equation can be roughly divided into two classes: the standard and the revised one. In the standard Enskog equation (see [7], [10], [15]), \(F^\pm\) are defined by a geometrical factor \(Y\) which is a contact-value pair correlation function of the hard-sphere system at uniform equilibrium and depends on the density \(\rho(t, x)\), i.e., they are given by \(F^+ = Y(\rho(t, x - a\omega/2))\) and \(F^- = Y(\rho(t, x + a\omega/2))\). For a fairly rare uniform gas of one particle with mass \(m\), it can be found in [7] that the value of \(Y\) is approximatively expressed by \(Y(\rho(t, x)) = [1 - 11b\rho(t, x)/8]/[1 - 2b\rho(t, x)]\) where \(b = 2\pi a^3/(3m)\). It can be easily shown that in this case the factor \(F^\pm = Y\) satisfies (5.1) and is locally Lipschitz as defined in (4.2) with the external forces of the assumptions (2.3)-(2.7) for \(R \in (0, R_0]\), where \(R_0\) is some suitably small positive constant. Generally, the dependence of the function \(Y\) on the local density \(\rho(t, x)\) is of the form
\[
Y(\rho(t, x)) = 1 + \sum_{i=1}^{+\infty} b_i [2\pi a^3\rho(t, x)/3]^i,
\]
where \(b_i\) are given in terms of the virial coefficients \(B_i\) appearing in the equation of state for the hard sphere system. We cannot know whether the series (5.6) converges when one of the two different locally Lipschitz conditions (4.2) and (5.1) is satisfied. Even if this series converges, we cannot yet know whether one of the assumptions (4.2) and (5.1) of \(F^\pm\) holds for any factor \(Y\) of the above form. Of course, if \(F^\pm = Y\) is a factor defined by the form (5.6) with finite terms, then (5.1) holds and so \(F^\pm\) satisfy (4.2) when the external forces are of the assumptions (2.3)-(2.7).

In the case of the revised Enskog equation (see [1], [16], [20]), \(F^\pm\) are expressed by a contact-value pair correlation function \(G\) of the hard-sphere system at non-uniform equilibrium. The form of \(G\) is given by the Mayer cluster expansion in terms of local density \(\rho(t, x)\). The function \(G\) depends on the position \(x\), the vector \(x \pm a\omega\) and the density \(\rho(t, x)\), i.e., \(F^+ = G(x, x - a\omega, \rho(t, x))\) and \(F^- = G(x, x + a\omega, \rho(t, x))\). In fact, \(G(x, y, \rho(t, x)) = \exp(-\beta\Phi(|x - y|))\tilde{G}(\rho(t, x))\), where \(\beta\) is a positive constant, \(\Phi(|x - y|)\) is a potential of two interaction spheres at the positions \(x\) and \(y\), and \(\tilde{G}\) is a functional of the following form [16]
\[
\tilde{G}(\rho(t, x)) = 1 + \int V(12|3)\rho(3)dx_3 + \frac{1}{2} \int \int V(12|34)\rho(3)\rho(4)dx_3dx_4 + \cdots + \frac{1}{(k - 2)!} \int \int \cdots \int V(12|34 \cdots k)\rho(3)\rho(4) \cdots \rho(k)dx_3dx_4 \cdots dx_k + \cdots,
\]
here \(\rho(k) = \rho(t, x_k)\) and \(V(12|34 \cdots k)\) is the sum of all the graphs of \(k\) labeled points which are biconnected when the Mayer factor \(\exp(-\beta\Phi(|x_i - x_j|)) - 1\) are added. In contrast to the standard Enskog equation, the revised Enskog equation possesses an H-function [17]. It can be also known that the revised Enskog theory for mixtures is consistent with irreversible thermodynamics, including Onsager reciprocity relations (see [21], [22], [23]). But we cannot know whether the above series (5.7) converges under the assumption (4.2) or (5.1). We cannot
yet know whether one of the assumptions (4.2) and (5.1) of $F^\pm$ holds for any functional $\tilde{G}$ of the above form.

The assumption (4.2) or (5.1) of $F^\pm$ is satisfied by some geometric factors present in the truncated Enskog equations in both standard and revised cases. In the standard case the geometric factor $Y$ considered above can be assumed to be of the truncated form (5.6) with finite terms while in the revised case an obvious example is that the functional $\tilde{G}$ can be truncated to be a positive constant. It can be found that these factors satisfy the two assumptions of $F^\pm$ when one assumes that both $L(R)$ and $L_0(R)$ are two positive constant functions on $R_+$. Therefore the above two assumptions are completely suitable for our understanding evolutions of moderately or highly dense gases by use of our investigation of the properties of the Enskog equation.

Acknowledgement. The author would like to thank the referees of this paper for their valuable comments and suggestions on this work.

References

[1] Arkeryd L., On the Enskog equation with large initial data, SIAM Journal on Mathematical Analysis, 1990, 21: 631-646.
[2] Arkeryd L., Cercignani C., On the Convergence of Solutions of the Enskog Equation to Solutions of the Boltzmann Equation, Comm. in PDE, 1989, 14: 1071-1089.
[3] Arkeryd L., Cercignani C., Global Existence in $L^1$ for the Enskog Equation and Convergence of the Solutions to Solutions of the Boltzmann Equation, J. Stat. Phys., 1990, 59: 845-867.
[4] Bellomo N., Toscani G., On the Cauchy problem for the nonlinear Boltzmann equation. Global existence, uniqueness and asymptotic stability, J. Math. Phys., 1985, 26: 334-338.
[5] Cercignani C., Existence of Global Solutions for the Space Inhomogeneous Enskog Equation, Transp. Theory Stat. Phys., 1987, 16: 213-221.
[6] Cercignani C., Small Data Existence for the Enskog equation in $L^1$, J. Stat. Phys., 1988, 51: 291-297.
[7] Chapman S., Cowling T. G., The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Third Edition, 1970.
[8] DiPerna R. J., Lions P. L., On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. Math., 1989, 130: 321-366.
[9] Duan R., Yang T., Zhu C., Global existence to Boltzmann equation with external force in infinite vacuum, Journal of Mathematical Physics, 2005, 46, 053307.
[10] Enskog D., Kinetiche Theorie der Wärmeleitung, Reibung und Selbstdiffusion in gewissen verdichteten Gasen und Flüsigkeiten, Kungl. Sv. Vetenskapsakademiens Handl. 63 (1922), 3-44, English Transl. in Brush, S. G., Kinetic Theory, vol 3, Pergamon, New York 1972.
[11] Guo Y., The Vlasov-Poisson-Boltzmann system near vacuum, Comm. Math. Phys., 2001, 218: 293-313.
[12] Illner R., Shinbrot M., The Boltzmann Equation, global existence for a rare gas in an infinite vacuum, Comm. Math. Phys., 1984, 95: 217-226.
[13] Jiang Z., Global Solution to the Relativistic Enskog Equation with Near-Vacuum Data, Journal of Statistical Physics, 2007, 127: 805-812.
[14] Kaniel S., Shinbrot M., The Boltzmann Equation I. Uniqueness and local existence, Comm. Math. Phys., 1978, 58: 65-84.
[15] Polewczak J., Global existence and asymptotic behavior for the nonlinear Enskog equation, SIAM Journal on Applied Mathematics, 1989, 49: 952-959.
[16] Polewczak J., On some open problems in the revised Enskog equation for dense gases, in Proceedings “WASCOM 99” 10th Conference on Waves and Stability in Continuous Media, Vulcano (Eolie Islands), Italy, 7-12 June 1999, V. Ciancio, A. Donato, F. Oliveri, and S. Rionero, Eds., World Scientific Publishing, London, 2001, 382-396.
[17] Resibois P., H-Theorem for the (Modified) Nonlinear Enskog Equation, J. Stat. Phys., 1978, 19: 593-609.
[18] Toscani G., On the non-linear Boltzmann equation in unbounded domains, Arch. Rational Mech. Anal., 1986, 95: 37-49.
[19] Ukai S., Solutions of the Boltzmann Equation, Studies in Math., Appl., 1986, 18: 37-96.
[20] van Beijeren H., Ernst M. H., The modified Enskog equation, Physica, 1973, 68: 437-456.
[21] van Beijeren H., Ernst M. H., The modified Enskog equation for mixtures, Physica, 1973, 70: 225-242.
[22] van Beijeren H., Equilibrium distribution of hard-sphere systems and revised Enskog theory, Phys. Rev. Lett., 1983, 70: 1503-1505.
[23] van Beijeren H., Kinetic theory of dense gases and liquids, in *Fundamental Problems in Statistical Mechanics VII*, H. van Beijeren Ed., Elsevier, 1990, 357-380.

Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China

E-mail address: mcsjzl@mail.sysu.edu.cn