ANALYSIS OF MINIMAL REPRESENTATIONS OF SL($n, \mathbb{R}$)

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Abstract Some minimal representations of SL($n, \mathbb{R}$) can be realized on a Hilbert space of holomorphic functions. This is the analogue of the Brylinski-Kostant model. They can also be realized on a Hilbert space of homogeneous functions on $\mathbb{R}^n$. This is the analogue of the Kobayashi-Orsted model. We will describe the two realizations and a transformation which maps one model to the other. It can be seen as an analogue of the classical Bargmann transform.

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Introduction

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References

Introduction. The construction of an analogue of the Bargmann-transform between Fock-type space and Schrödinger-type space of minimal representations is given in a uniform manner, in the case of simple real Lie groups of Hermitian type in [HKMO12]. We consider here a Lie group of non Hermitian type, SL($n, \mathbb{R}$), and two minimal representations whose Fock-type spaces and Schrödinger-type spaces are described explicitly. We first recall the theory in [A11], [A12] and [AF12] to describe the analogue of the Brylinski-Kostant model ([BK94], [B97], [B98]), in terms of holomorphic functions on a covering of the $K$-minimal nilpotent orbit. The analogue of the Kobayashi-Orsted model ([KO03a], [KO03b], [KO03c]), in terms of homogeneous functions on $\mathbb{R}^n$, will be given in this paper.
In [A11] a general construction for a simple complex Lie algebra \( \mathfrak{g} \) has been given, starting from a pair \((V,Q)\) where \( V \) is a semi-simple Jordan algebra of rank \( \leq 4 \) and \( Q \) a polynomial on \( V \), homogeneous of degree 4. The Lie algebra \( \mathfrak{g} \) is of non-Hermitian type.

In [AF12], the manifold \( \Xi \) is the orbit of \( Q \) under the conformal group \( \text{Conf}(V,Q) \) acting on a space of polynomials \( \mathcal{W} \) by an irreducible representation \( \kappa \). The Fock space \( \mathcal{F}(\Xi) \) is a Hilbert space of holomorphic functions on the complex manifold \( \Xi \). A spherical minimal representation of \( \mathfrak{g} \) is realized (when it exists) in \( \mathcal{F}(\Xi) \).

In [A12], using the decomposition \( V = \bigoplus_{i=1}^{s} V_i \) of \( V \) into simple summands, which means that \( Q = \prod_{i=1}^{s} \Delta_i(z_i)^{k_i} \) gets factored as a product of powers of Jordan determinants \( \Delta_i \) of \( V_i \), and considering irreducible representations \( \kappa_i^{(k_i)} \) of the conformal groups \( \text{Conf}(V_i,\Delta_i) \), which act on spaces \( \mathcal{W}_i^{(k_i)} \) of the polynomials generated by the \( \Delta_i^{k_i}(z_i - a_i) \) for \( a_i \in V_i \), the manifold \( \Xi \) is the orbit of \( Q \), under the tensor product representation \( \kappa^{(k_1,\ldots,k_s)} := \bigotimes_{i=1}^{s} \kappa_i^{(k_i)} \) of the group \( \prod_{i=1}^{s} \text{Conf}(V_i,\Delta_i) \) in \( \mathfrak{p} = \bigotimes_{i=1}^{s} \mathcal{W}_i^{(k_i)} \).

Then one gets a quotient map \( \Xi \rightarrow \Xi, (\xi_1,\ldots,\xi_s) \mapsto \xi_1^{k_1} \cdots \xi_s^{k_s} \) of \( \Xi \), where \( \Xi = \prod_{i=1}^{s} \Xi_i \) is the product of the orbits \( \Xi_i \) of the \( \Delta_i \) under \( \text{Conf}(V_i,\Delta_i) \), acting on spaces \( \mathcal{W}_i \) of the polynomials generated by the \( \Delta_i(z_i - a_i) \) by irreducible representations \( \kappa_i \). The \( \mathcal{F}_q(\Xi) \) are Hilbert spaces of holomorphic functions on the complex manifold \( \Xi \), and \( \mathcal{F}_q(\Xi) = \mathcal{F}(\Xi) \) iff \( q = 0 \).

Two minimal representations of \( \mathfrak{g} \) are realized in the Fock-type spaces \( \mathcal{F}_q(\Xi) \), for some suitable multi-indices \( q \in \mathbb{N}^s \). This is the analogue of the Brylinski-Kostant model.

In this paper we consider the special case where \( V = \mathbb{C}^p \), \( Q \) is the square of a quadratic form and the construction leads to the Lie algebra \( \mathfrak{sl}(p+2,\mathbb{R}) \) with \( p \geq 3 \). The two minimal representations \( \rho_0 \) and \( \rho_1 \) are respectively realized in \( \mathcal{F}_0(\Xi) \) and \( \mathcal{F}_1(\Xi) \).

On another hand, we realize two minimal representations \( \omega_0 \) and \( \omega_1 \) of \( \text{SL}(p+2,\mathbb{R}) \) on spaces \( \mathcal{V}_0(\mathbb{R}^{p+2}) \) and \( \mathcal{V}_1(\mathbb{R}^{p+2}) \) of homogeneous functions on \( \mathbb{R}^{p+2} \). This is the analogue of one of the Kobayashi-Orsted models.

We also give explicit integral operators \( \mathcal{B}_0 \) from the space \( \mathcal{V}_0(\mathbb{R}^{p+2}) \) onto the space \( \mathcal{F}_0(\Xi) \) which intertwines the representations \( \rho_0 \) and \( d\omega_0 \) of \( \text{SL}(p+2,\mathbb{R}) \), and \( \mathcal{B}_1 \) from the space \( \mathcal{V}_1(\mathbb{R}^{p+2}) \) onto the space \( \mathcal{F}_1(\Xi) \) which intertwines the representations \( \rho_1 \) and \( d\omega_1 \) of \( \text{SL}(p+2,\mathbb{R}) \). These unitary operators are bijective and can be seen as analogues of the classical Bargmann transform.
1. The analogue of the Brylinski-Kostant model. General case

Let $V$ be a semi-simple Jordan algebra and $Q$ a homogeneous polynomial on $V$. Let $L = \text{Str}(V, Q)$ be the structure group

$$\text{Str}(V, Q) = \{ g \in GL(V) \mid \exists \gamma(g), Q(g \cdot z) = \gamma(g)Q(z) \}.$$ 

The conformal group Conf($V, Q$) is the group of rational transformations $g$ of $V$ generated by: the translations $z \mapsto z + a$ ($a \in V$), the dilations $z \mapsto \ell \cdot z$ ($\ell \in L$), and the inversion $\sigma : z \mapsto -z^{-1}$.

Let $\mathcal{W}$ be the space of polynomials on $V$ generated by the translated $Q(z - a)$ of $Q$, with $a \in V$. Let $\kappa$ be the cocycle representation of Conf($V, Q$) or of a covering of order two of it on $\mathcal{W}$, defined in [A11] as follows:

**Case 1.** In case there exists a character $\chi$ of $\text{Str}(V, Q)$ such that $\chi^2 = \gamma$, then let $K = \text{Conf}(V, Q)$. Define the cocycle $\mu(g, z) = \chi((Dg(z))^{-1})$ ($g \in K$, $z \in V$), and the representation $\kappa$ of $K$ on $\mathcal{W}$,

$$\kappa(g)p(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

The cocycle $\mu(g, z)$ is a polynomial in $z$ of degree $\leq \deg Q$ and

$$\kappa(\tau_a)p(z) = p(z - a) \quad (a \in V),$$
$$\kappa(\ell)p(z) = \chi(\ell)p(\ell^{-1} \cdot z) \quad (\ell \in \text{Str}(V, Q)),$$
$$\kappa(\sigma)p(z) = Q(z)p(-z^{-1}).$$

**Case 2.** Otherwise the group $K$ is a covering of Conf($V, Q$), defined as the set of pairs $\hat{g} = (g, \mu)$, with $g \in \text{Conf}(V, Q)$, and $\mu(\hat{g}, \cdot) = \mu$ is a function on $V$ such that $\mu(z)^2 = \gamma(Dg(z))^{-1}$, and $\kappa(\hat{g})p(z) = \mu(\hat{g}^{-1}, z)p(\hat{g}^{-1} \cdot z)$.

We recall the construction of the Lie algebra $\mathfrak{g}$ in [A11]. The Jordan algebra is assumed to be of rank $\leq 4$ and the polynomial $Q$ is assumed to be homogeneous of degree 4. We use the same notation $\mathfrak{t} = \text{Lie}(K), \mathfrak{p} = \mathcal{W}$. There is $H \in \mathfrak{l} = \text{Lie}(L)$ which defines gradings of $\mathfrak{k}$ and $\mathfrak{p}$:

$$\mathfrak{k} = \mathfrak{k}_{-2} + \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1 + \mathfrak{k}_2,$$
$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2,$$

with

$$\mathfrak{k}_j = \{ X \in \mathfrak{k} \mid \text{ad}(H)X = jX \}, \quad \mathfrak{p}_j = \{ p \in \mathfrak{p} \mid \text{d}\kappa(H)p = jp \},$$
$$\mathfrak{k}_{-1} \simeq V, \quad \mathfrak{k}_0 = \text{Lie}(L), \quad \text{Ad}(\sigma) : \mathfrak{k}_j \to \mathfrak{k}_{-j},$$
$$\mathfrak{p}_{-2} = \mathbb{C}, \quad \mathfrak{p}_2 = \mathbb{C} Q, \quad \mathfrak{p}_{-1} \simeq V, \quad \mathfrak{p}_1 \simeq V, \quad \kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}.$$ 

Define $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $E := Q$, $F := 1$. Then $[H, E] = 2E$, $[H, F] = -2F$ and (see Theorem 3.1 in [A11] and [AF12]):

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Theorem 1.1. There is a unique Lie algebra structure on $\mathfrak{g}$ satisfying

(i) $[X, X'] = [X, X']_t$ \hspace{1em} ($X, X' \in \mathfrak{t}$),

(ii) $[X, p] = d\kappa(X)p$ \hspace{1em} ($X \in \mathfrak{t}, p \in \mathfrak{p}$),

(iii) $[E, F] = H$.

We recall also the real form $\mathfrak{g}_R$ of $\mathfrak{g}$ which we consider. We fix a Euclidean real form $V_R$ of the complex Jordan algebra $V$, denote by $z \mapsto \bar{z}$ the conjugation of $V$ with respect to $V_R$, and then consider the involution $g \mapsto \tilde{g}$ of $\text{Conf}(V, Q)$ given by: $\tilde{g} \cdot z = \overline{g \cdot z}$. For $(g, \mu) \in K$ define $(g, \mu) = (\tilde{g}, \tilde{\mu})$, where $\tilde{\mu}(z) = \overline{\mu(z)}$. The involution $\alpha$ defined by $\alpha(g) = \sigma \circ \tilde{g} \circ \sigma^{-1}$ is a Cartan involution of $K$ (see [P.02] Proposition 1.1), and $K_R = \{g \in K \mid \alpha(g) = g\}$ is a compact real form of $K$. For $\mathfrak{u}$ the compact real form of $\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{u} = \mathfrak{t}_R$ and $\mathfrak{p}_R := \mathfrak{p} \cap (\mathfrak{iu})$, one has $\mathfrak{p}_R = \{p \in \mathfrak{p} \mid \kappa(\sigma)p = p\}$, where $\bar{p}(z) = p(\bar{z})$, and $\mathfrak{g}_R := \mathfrak{t}_R + \mathfrak{p}_R$ is a real form of $\mathfrak{g}$ and the decomposition $\mathfrak{t}_R + \mathfrak{p}_R$ is its Cartan decomposition.

Using the decomposition of the Jordan algebra $V$ is a direct sum of simple ideals: $V = \sum_{i=1}^n V_i$, and $Q(z) = \prod_{i=1}^n \Delta_i(z_i)^{k_i}$, where $\Delta_i$ is the determinant polynomial of the simple Jordan algebra $V_i$ and the $k_i$ are positive integers such that the degree of $Q$ is equal to $\sum_{i=1}^n k_i r_i$, where $r_i$ is the rank of $V_i$. One can consider the structure groups $L_i = \text{Str}(V_i, \Delta_i)$, the conformal groups $\text{Conf}(V_i, \Delta_i)$, the corresponding groups $K_i$ analogous of $K$, the spaces $\mathcal{W}_i$ of the polynomials on $V_i$ generated by the $\Delta_i(z_i - a_i)$ with $a_i \in V_i$ and the cocycle representations $\kappa_i$ of $K_i$ analogous of $\kappa$, on the spaces $\mathcal{W}_i$ given by $(\kappa_i(g_i)p)(z_i) = \mu_i(g_i^{-1}, z_i)p(g_i^{-1} \cdot z_i)$, given. If $\chi_i$ exists, by $(\kappa_i(g_i)p)(z_i) = \mu_i(g_i^{-1}, z_i)p(g_i^{-1} \cdot z_i)$ and, when $\chi_i$ doesn’t exist, by $(\kappa_i(g_i)p)(z_i) = \mu_i(g_i^{-1}, z_i)p(g_i^{-1} \cdot z_i)$.

Moreover, if $\mathcal{W}_i^{(k_i)}$ is the vector space generated by the polynomials $\Delta_i^{k_i}(z_i - a_i)$ for $a_i \in V_i$. Then, the group $K_i$ acts on $\mathcal{W}_i^{(k_i)}$ by the representation $\kappa_i^{(k_i)}$ given by $(\kappa_i^{(k_i)}(g_i)p)(z_i) = \mu_i(g_i, z_i)^{k_i}p(g_i^{-1} \cdot z_i)$.

The product $\tilde{L} := \prod_{i=1}^n L_i$ acting on $V$ by $g \cdot z = (g_i \cdot z_i)$ for $g = (g_i)$, $z = (z_i)$, is a subgroup of $L$, $\text{Lie}(\tilde{L}) = \text{Lie}(L) = \mathfrak{l} = \sum_{i=1}^n \mathfrak{l}_i$, $\gamma(g) = \prod_{i=1}^n \gamma_i^{k_i}(g_i)$. The product $\tilde{K} := \prod_{i=1}^n K_i$ and $K$ have the same Lie algebra $\mathfrak{k} = \sum_{i=1}^n \mathfrak{k}_i$, where $\mathfrak{k}_i = \text{Lie}(K_i) \simeq V_i \oplus \mathfrak{l}_i \oplus V_i$.

Furthermore, the vector space $\mathfrak{p}$ is the tensor product of the $\mathcal{W}_i^{(k_i)}$ and the tensor product representation $\kappa^{(k_1, \ldots, k_s)} := \otimes_{i=1}^s \kappa_i^{(k_i)}$ of $\tilde{K}$ in $\mathfrak{p}$ and the representation $\kappa$ of $K$ in $\mathfrak{p}$ have the same differential. Then, the structure of simple Lie algebra on $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ can be obtained by considering $\tilde{K}$ (instead of $K$) and $\kappa^{(k_1, \ldots, k_s)}$ (instead of $\kappa$).
The orbit $\Xi$ of $Q$ under the group $\tilde{K} = \prod_{i=1}^{s} K_i$ acting on $p$ by $\kappa(k_1,\ldots,k_s)$, is conical. It is the $\tilde{K}$-minimal nilpotent orbit in $p$.

Let $\Xi_i = \{\kappa_i(g_i)\Delta_i \mid g_i \in K_i\}$ be the $K_i$-orbit of $\Delta_i$ in $p_i$. A polynomial $\xi_i \in W_i$ can be written $\xi_i(v) = w_i \Delta_i(v) + \text{terms of degree} < r_i \quad (w_i \in \mathbb{C})$, and $w_i = w_i(v_i)$ is a linear form on $W_i$ which is semi-invariant under the preimage in $K_i$ of the maximal parabolic subgroup $P_{\max}^{(i)} = L_i \ltimes N_i$, where $N_i$ is the group of translations $z_i \in V_i \mapsto z_i + a_i$, for $a_i \in V_i$. The set $\Xi_{i,0} = \{\xi_i \in \Xi_i \mid w_i(\xi_i) \neq 0\}$ is open and dense in $\Xi_i$. A polynomial $\xi_i \in \Xi_{i,0}$ can be written $\xi_i(v_i) = w_i \Delta_i(v_i - z_i) \quad (w_i \in \mathbb{C}^*, z_i \in V_i)$. Hence we get a coordinate system $(w_i,z_i) \in \mathbb{C}^* \times V_i$ for $\Xi_{i,0}$.

In this coordinate system, the cocycle action of $K_i$ is given by

$$\kappa_i(g_i) : (w_i, z_i) \mapsto (\mu_i(g_i,z_i)w_i, g_i \cdot z_i).$$

Consider $\tilde{\Xi} = \prod_{i=1}^{s} \Xi_i$ and the equivalence relation

$$(\xi_1,\ldots,\xi_s) \sim (\lambda_1 \xi_1,\ldots,\lambda_s \xi_s), \quad \text{for} \quad (\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^s, \lambda_1^{k_1} \ldots \lambda_s^{k_s} = 1.$$ 

The map $\tilde{\Xi}/ \sim \to \Xi, [(\xi_1,\ldots,\xi_s)] \mapsto \xi_1^{k_1} \ldots \xi_s^{k_s}$ is a diffeomorphism.

The group $K_i$ acts on the space $O(\Xi_i)$ of holomorphic functions by:

$$(\pi_i(g_i)f_i)(\xi_i) = f_i(\kappa_i(g_i^{-1})\xi_i).$$

If $\xi_i(v_i) = w_i \Delta_i(v_i - z_i)$, and $f_i \in O(\Xi_i)$, we denote by $f_i(\xi_i) = \phi_i(w_i,z_i)$ the restriction of $f_i$ to $\Xi_{i,0}$. Then $\phi_i \in O(\mathbb{C}^* \times V_i)$. In the coordinates $(w_i, z_i)$, the representation $\pi_i$ is given by

$$(\pi_i(g_i)\phi_i)(w_i,z_i) = \phi_i(\mu_i(g_i^{-1},z_i)w_i, g_i^{-1} \cdot z_i)$$

The group $\tilde{K}$ acts on the tensor product space $\otimes_{i=1}^{s} O(\Xi_i)$ by the tensor product representation $\pi = \otimes_{i=1}^{s} \pi_i$, which is given by, for $g = (g_i)$:

$$(\pi(g)(f_1 \otimes \ldots \otimes f_s))(\xi_1,\ldots,\xi_s) = f_1(\kappa_1(g_1^{-1})\xi_1) \ldots f_s(\kappa_s(g_s^{-1})\xi_s)$$

and in coordinates, by

$$(\pi(g)(\phi_1 \otimes \ldots \otimes \phi_s))(w_1,z_1,\ldots, w_s,z_s) = \prod_{i=1}^{s} \phi_i(\mu_i(g_i^{-1},z_i)w_i, g_i^{-1} \cdot z_i).$$

For $m_i \in \mathbb{Z}$, the space $O_{m_i}(\Xi_i)$ of holomorphic functions $f_i$ on $\Xi_i$, homogeneous of degree $m_i$, is invariant under the representation $\pi_i$. If $f_i \in O_{m_i}(\Xi_i)$, then its restriction $\phi_i$ to $(\Xi_i)_{0}$ can be written $\phi_i(w_i,z_i) = w_i^{m_i} \psi_i(z_i)$ where $\psi_i$ is a holomorphic function on $V_i$. We write $O_{m_i}(V_i)$ for the space of the functions $\psi_i$ corresponding to the functions $f_i \in O_{m_i}(\Xi_i)$, and denote by $\tilde{\pi}_{i,m_i}$ the representation of $K_i$ on $O_{m_i}(V_i)$ corresponding to the restriction $\pi_{i,m_i}$ of $\pi$ to $O_{m_i}(\Xi_i)$. It is given by

$$(\tilde{\pi}_{i,m_i}(g_i)\psi_i)(z_i) = \mu_i(g_i^{-1},z_i)^{m_i} \psi_i(g_i^{-1} \cdot z_i).$$

In particular

$$(\tilde{\pi}_{i,m_i}(\sigma_i)\psi_i)(z_i) = \Delta_i^{m_i}(z_i) \psi_i(-z_i^{-1}).$$

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The spaces $O_{m_i}(\Xi_i) (= \{0\}$ if $m_i < 0$), are finite dimensional, and the representations $\pi_{i,m_i}$ are irreducible.

The tensor product $O_{(m_1,\ldots,m_s)}(\Xi) = \bigotimes_{i=1}^s O_{m_i}(\Xi_i)$ is invariant under the representation $\pi = \bigotimes_{i=1}^s \pi_i$. For $f = f_1 \otimes \ldots \otimes f_s \in O_{(m_1,\ldots,m_s)}(\Xi)$, let $\phi_i(w_i,z_i)$ be the restriction of $f_i$ to $\Xi_i,0$, then $\phi_i(w_i,z_i) = w_i^{m_i} \psi_i(z_i)$ with $\psi_i \in \tilde{O}_{m_i}(V_i)$, and $f$ is in coordinates given by

$$(\phi_1 \otimes \ldots \otimes \phi_s)((w_1,z_1),\ldots,(w_s,z_s)) = w_1^{m_1} \ldots w_s^{m_s} \psi_1(z_1) \ldots \psi_s(z_s).$$

For $\tilde{O}_{(m_1,\ldots,m_s)}(V) = \bigotimes_{i=1}^s \tilde{O}_{m_i}(V_i)$ and $\tilde{\pi}_{(m_1,\ldots,m_s)} = \bigotimes_{i=1}^s \tilde{\pi}_{m_i}$,

$$(\tilde{\pi}_{(m_1,\ldots,m_s)}(g)(\psi_1 \otimes \ldots \otimes \psi_s))(z_1,\ldots,z_s) = \prod_{i=1}^s \mu_i(g_i^{-1},z_i)^{m_i} \psi_i(g_i^{-1}.z_i).$$

We consider a Euclidean real form of each $V_i$, and the compact real form $(K_i)_{\mathbb{R}}$ of $K_i$ analogous of $K_{\mathbb{R}}$. Then $\tilde{K}_{\mathbb{R}} := \prod_{i=1}^s (K_i)_{\mathbb{R}}$ is a compact real form of $\tilde{K}$. There is a $(K_i)_{\mathbb{R}}$-invariant norm on $\tilde{O}_{m_i}(V_i)$, which is unique up to a positive factor. It is given by

$$\|\psi_i\|_{i,m_i}^2 = \frac{1}{a_{i,m_i}} \int_{V_i} |\psi_i(z_i)|^2 H_i(z_i)^{-m_i} dV_i(dz_i),$$

and there is a $\tilde{K}_{\mathbb{R}}$-invariant norm on $O_{(m_1,\ldots,m_s)}(\Xi)$, which is unique up to a positive factor, it is given by $\|f\|_{(m_1,\ldots,m_s)}^2 = \prod_{i=1}^s \|\psi_i\|_{i,m_i}^2$, where $H_i(z_i) = H_i(z_i,z_i')$ with $H_i(z_i,z_i')$ holomorphic in $z_i$, anti-holomorphic in $z_i'$ such that for $x_i \in (V_i)_{\mathbb{R}}$, $H_i(x_i,x_i) = \Delta_i(e_i + x_i^2)$, $m_i,0(dz_i) = \frac{1}{c_{i,0}} H_i(z_i)^{-\frac{m_i}{2}} m(dz_i)$, with

$$C_{i,0} = \int_{V_i} H_i(z_i)^{-\frac{m_i}{2}} m(dz_i),$$

where $n_i = \dim(V_i)$ and the Lebesgue measure $m(dz_i)$ is chosen such that $C_{i,0} = 1$ (see [A12] Proposition 2.3).

First, the multi-index $q = (q_1,\ldots,q_s) \in \mathbb{N}^s$ is determined in such a way that the space of finite sums $O_q(\Xi)_{\text{fin}}$, which is $\tilde{K}$-invariant, carries a representation $\rho_q$ of the Lie algebra $\mathfrak{g}$ (see [A12a], Theorem 3.4). For the existence of this representation one has to suppose that there is $i$ such that $q_i < k_i$, and to add a condition (T):

$$\exists \eta_q \text{ such that } \eta_q = \frac{q_i}{k_i} + \frac{n_i}{k_i r_i} \quad (\forall i) \quad (T)$$

For $X \in \mathfrak{k}$, $\rho_q(X) = d\pi(X)$. In particular

$$\rho_q(H) = d\pi(H) = \frac{d}{dt} |_{t=0} \pi(\exp tH).$$
For $X \in \mathfrak{p}$, $\rho_q(X)$ is a differential operator of degree $\leq 4$. The representation $\rho_q$ is determined by the operators $\rho_q(E)$ which involves the differential operator $Q^{\delta \rho_q D}$, and $\rho_q(F) = -\rho_q(E)^*$. More precisely,

$$\rho_q(E) = \mathcal{M} - \delta \circ \mathcal{D} \quad \text{and} \quad \rho_q(F) = \mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma$$

where $\mathcal{M}$ and $\mathcal{M}^\sigma$ are multiplication operators which map the space $\mathcal{O}_m(\widetilde{\Xi}) := \mathcal{O}(k_1 m + q_1, \ldots, k_s m + q_s)(\widetilde{\Xi})$ into $\mathcal{O}_{m+1}(\widetilde{\Xi})$, and $\mathcal{D}$ and $\mathcal{D}^\sigma$ are differential operators which map the space $\mathcal{O}_m(\widetilde{\Xi})$ into $\mathcal{O}_{m-1}(\widetilde{\Xi})$. Their restrictions to $\mathcal{O}_m(\widetilde{\Xi})$ are: for $\phi \in \mathcal{O}_m(\widetilde{\Xi})$, given by

$$(\mathcal{M}\phi)((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^{s} w_i^{k_i (m+1) + q_i} \psi_i(z_i),$$

$$(\mathcal{M}\phi)((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^{s} w_i^{k_i (m+1) + q_i} \psi_i(z_i),$$

$$(\mathcal{D}\phi)((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^{s} w_i^{k_i (m+1) + q_i} \Delta_i(z_i) \psi_i(z_i),$$

$$(\mathcal{D}\phi)((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^{s} w_i^{k_i (m+1) + q_i} \Delta_i(z_i) \psi_i(z_i),$$

$$(\mathcal{D}\phi)((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^{s} w_i^{k_i (m+1) + q_i} ((D_1)^{(i)}) \psi_i(z_i),$$

where $D_1^{(i)}$ is the Maass differential operator

$$D_1^{(i)} = \Delta_i(z_i)^{k_i + \alpha} \Delta_i \left( \frac{\partial}{\partial z_i} \right) \Delta_i(z_i)^{-\alpha},$$

and $(D_1^{(i)})^*$ is the adjoint of $(D_1^{(i)})$ given by $(D_1^{(i)})^* = J \circ D_1^{(i)} \circ J$, with

$$Jf(z_i) = f \circ \sigma_i(z_i) = f(-z_i^{-1}),$$

and

$$\delta_{(k_1 m + q_1, \ldots, k_s m + q_s)}(z_i) = (\delta_{(k_1 m + q_1, \ldots, k_s m + q_s)}(z_i),$$

and for $m = 0$, by a case by case calculation (see [A12], Theorem 3.4).

Furthermore, we consider for a sequence $(c_m) := (c_{(k_1 m + q_1, \ldots, k_s m + q_s)})$ of positive numbers, an inner product on $\mathcal{O}_q(\widetilde{\Xi})\text{fin}$ such that, if $f = \sum_m f_m$, with $f_m \in \mathcal{O}_m(\widetilde{\Xi})$, $\|f\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|f_m\|^2$. This inner product is invariant under $K_R$. The completion of $\mathcal{O}_q(\widetilde{\Xi})\text{fin}$ for this inner product is a Hilbert subspace $\mathcal{F}_q(\widetilde{\Xi})$ of $\mathcal{O}_q(\widetilde{\Xi})$.

The sequence $(c_m) = (c_{(k_1 m + q_1, \ldots, k_s m + q_s)})$ is determined in such a way that the representation $\rho_q$ is unitary (see [A.12], Theorem 4.1). It is given, if $1 - \eta_q$ is a root of the Bernstein polynomial, by

$$c_m = c_{(k_1 m + q_1, \ldots, k_s m + q_s)} = \frac{(\eta_q + 1)_m}{(\eta_q + a_0)_m (\eta_q + b_0)_m} \frac{1}{m!}$$

and if not, by

$$c_m = c_{(k_1 m + q_1, \ldots, k_s m + q_s)} = \frac{(\eta_q + 1)_m (1)_m}{(\eta_q + a'_0)_m (\eta_q + b'_0)_m (\eta_q + c'_0)_m} \frac{1}{m!},$$

where the constants $a_0, b_0$ are given in Table 3, and $a'_0, b'_0$ are given in Table 4 of that paper.
2. Some harmonic analysis related to a quadratic form

We consider on $V = \mathbb{C}^p$ the quadratic form
\[
\Delta(x) = x_1^2 + \ldots + x_p^2.
\]
Then the structure group is
\[
\text{Str}(V, \Delta) = \mathbb{C}^* \times O(p, \mathbb{C})
\]
(quotiented by \(\{(1, I_p), (-1, -I_p)\}\)). The conformal group \(\text{Conf}(V, \Delta)\), generated by the translations, the structure group, and the inversion
\[
\sigma(z) = -\frac{z}{\Delta(z)},
\]
is a complex Lie group isomorphic to \(O(p + 2, \mathbb{C})\).

The vector space \(W\) generated by the translated \(\Delta(x - z)\) \((z \in V)\)
\[
\Delta(x - z) = \Delta(x) - 2 \sum_{j=1}^p a_j x_j + \Delta(z),
\]
has dimension \(p + 2\).

A transformation \(l \in \text{Str}(V, \Delta)\) is of the form \(l \cdot z = \lambda u \cdot z\) with \(\lambda \in \mathbb{C}^*\), \(u \in \text{SO}(p, \mathbb{C})\) and \(\Delta(l \cdot z) = \lambda^2 \Delta(z)\). Hence the characters \(\gamma\) and \(\chi\) of \(\text{Str}(V, \Delta)\) are given by \(\gamma(l) = \lambda^2\), \(\chi(l) = \lambda\) and the cocycle \(\mu\) by, for \(g \in \text{Conf}(V, \Delta)\) and \(z \in V\), \(\mu(g, z) = \chi(Dg(z)^{-1})\).

The representation \(\kappa\) of \(K = \text{Conf}(V, \Delta)\) on \(W\) is defined by
\[
(\kappa(g)\xi)(z) = \mu(g^{-1}, z)\xi(g^{-1} \cdot z).
\]
In particular
\[
(\kappa(\sigma)\xi)(z) = \Delta(z)\xi(-\frac{z}{\Delta(z)}),
\]
\[
(\kappa(\sigma)1)(z) = \Delta(z), \quad (\kappa(\sigma)\Delta)(z) = 1.
\]

We consider the following basis \(\mathcal{B}\) of \(W\):
\[
e_j(x) = x_j (1 \leq j \leq p), e_{p+1}(x) = \frac{1}{2}(\Delta(x) - 1), e_{p+2}(x) = \frac{i}{2}(\Delta(x) + 1).
\]
We write an element \(\xi \in W\) as
\[
\xi(v) = \sum_{j=1}^{p+2} \xi_j e_j(v).
\]
In particular
\[
\Delta = e_{p+1} - ie_{p+2}, \quad 1 = -e_{p+1} - ie_{p+2}.
\]
Associated to this basis we consider the bilinear form on \(W\) given by
\[
\langle \xi, \eta \rangle = \sum_{j=1}^{p+2} \xi_j \eta_j.
\]
For \( g \in \text{Conf}(V, \Delta) \) the matrix of \( \kappa(g) \) with respect to this basis belongs to \( O(p + 2, \mathbb{C}) \), and this defines an isomorphism from \( \text{Conf}(V, \Delta) \) onto \( O(p + 2, \mathbb{C}) \).

The orbit \( \tilde{\Xi} \) of \( \Delta \) under \( \text{Conf}(V, \Delta) \) is isomorphic to the isotropic cone
\[
\tilde{\Xi} = \{ \xi \in \mathbb{C}^{p+2} | \sum_{j=1}^{p+2} \xi_j^2 = 0, \xi \neq 0 \}.
\]

It is of dimension \( p + 1 \), isomorphic to the homogeneous space
\[
O(p + 2, \mathbb{C})/O(p, \mathbb{C}) \ltimes \mathbb{C}^p.
\]

An element \( \xi \in \mathcal{W} \) is a polynomial of degree \( \leq 2 \) which can be written
\[
\xi(v) = w\Delta(v) + \text{lower order terms}.
\]

Hence, \( \xi \mapsto w(\xi) \) is a linear form on \( \mathcal{W} \). The set
\[
\tilde{\Xi}_0 = \{ \xi \in \tilde{\Xi} | w(\xi) \neq 0 \}
\]
is open and dense in \( \tilde{\Xi} \), and
\[
w(\xi) = \frac{1}{2}(\xi_{p+1} + i\xi_{p+2}).
\]

An element \( \xi \in \tilde{\Xi}_0 \) can be written \( \xi(v) = w\Delta(v - z) \), with \( w \in \mathbb{C}^* \), \( z \in V \), and the map \( \mathbb{C}^* \times V \to \tilde{\Xi}_0 \), \( (w, z) \mapsto \xi(v) = w\Delta(v - z) \) is a diffeomorphism. Explicitly
\[
\xi(v) = w\Delta(v - z) = w(\Delta(v) - 2\sum_{j=1}^{p} z_j v_j + \Delta(z)) = \sum_{j=1}^{p+2} \xi_j e_j(v),
\]
with
\[
\xi_j = -2wz_j \quad (1 \leq j \leq p),
\]
\[
\xi_{p+1} = w \frac{1}{2}(1 - \Delta(z)),
\]
\[
\xi_{p+2} = w \frac{1}{2i}(1 + \Delta(z)).
\]

One checks that
\[
\sum_{j=1}^{p+2} \xi_j^2 = 0,
\]
and also
\[
\sum_{j=1}^{p+2} \|\xi_j\|^2 = \frac{1}{2}|w|^2 H(z),
\]
with
\[
H(z) := 1 + 2\sum_{j=1}^{p} |z_j|^2 + |\Delta(z)|^2.
\]

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Recall that, for \( m \geq 0 \), \( \mathcal{O}_m(\mathbb{S}) \) denotes the space of holomorphic functions on \( \mathbb{S} \) which are homogeneous of degree \( m \). It is isomorphic to the quotient \( \mathcal{P}_m(\mathbb{C}^{p+2})/\Delta \mathcal{P}_{m-2}(\mathbb{C}^{p+2}) \), where \( \mathcal{P}_m(\mathbb{C}^{p+2}) \) denotes the space of polynomials in \( p + 2 \) variables, homogeneous of degree \( m \).

The restriction to \( \mathbb{S}_0 \) of a polynomial \( f \in \mathcal{O}_m(\mathbb{S}) \) can be written

\[
\psi_m(z_1, \ldots, z_p) = \frac{1}{\omega_m} u_m(-2wz_1, \ldots, -2wz_p, \frac{w}{2}(1 - \Delta(z)), \frac{w}{2}(1 + \Delta(z))) = u_m(-2z_1, \ldots, -2z_p, 1 - \Delta(z), 1 + \Delta(z)).
\]

We get an isomorphism of \( \mathcal{O}_m(\mathbb{S}) \) to the space \( \mathcal{O}_m(V) \) of holomorphic polynomials \( \psi_m \) on \( V = \mathbb{C}^p \) such that there exists an homogeneous harmonic polynomial \( u_m \) of degree \( m \) on \( \mathbb{R}^{p+2} \) such that

\[
\psi_m(z_1, \ldots, z_p) = u_m(-2z_1, \ldots, -2z_p, 1 - \Delta(z), 1 + \Delta(z)).
\]

The \( O(p + 2) \)-invariant Hilbert norm on \( \mathcal{O}_m(\mathbb{S}) \) is given, for \( f_m(\xi) = \phi_m(w, z) = u^m \psi_m(z) \), by

\[
\|f_m\|^2_m = \frac{1}{\omega_m} \int_V |\psi_m(z)|^2 H(z)^{-m-p} m(dz),
\]

with

\[
a_m = \int_V H(z)^{-m-p} m(dz).
\]

Let \( \mathcal{Y}_m(\mathbb{R}^{p+2}) \) be the space of spherical harmonics of degree \( m \) on \( \mathbb{R}^{p+2} \); harmonic polynomials which are homogeneous of degree \( m \) on \( \mathbb{R}^{p+2} \). Observe that, for \( \xi \in \mathbb{S} \), the function \( x \mapsto \langle \xi, x \rangle^m \) belongs to \( \mathcal{Y}_m \). The operator

\[
\mathcal{A}_m : \mathcal{Y}_m(\mathbb{R}^{p+2}) \to \mathcal{O}_m(\mathbb{S}),
\]

\[
(\mathcal{A}_m u)(\xi) = \int_S \langle \xi, x \rangle^m u(x)s(dx),
\]

intertwines the representations of \( O(p + 2) \) on both spaces. (\( S \) is the unit sphere in \( \mathbb{R}^{p+2} \), and \( s(dx) \) is the uniform measure on \( S \) with total measure equal to one). In fact, it follows from next Proposition and its corollary (see [F.15], Section 2).
Proposition 2.1 — For \( u \in \mathcal{V}_m \),
\[
\int_S \langle \xi, x \rangle^m u(x) s(dx) = m! \frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+2}{2} + m)} u(\xi).
\]

Corollary 2.2 — For \( u \in \mathcal{V}_m \), \( \|A_m u\|_m^2 = m! \frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+2}{2} + m)} \|u\|_{L^2(S)}^2 \).

We consider now the case of the square of a quadratic form: \( V = \mathbb{C}^p \), and \( Q(z) = \Delta^2(z) \), where
\[
\Delta(z) = z_1^2 + \ldots + z_p^2.
\]
Then the vector space \( \mathcal{W}^{(2)} \) generated by the translated \( \Delta^2(x-z) \) with \((z \in V)\) of the polynomial \( Q = \Delta^2 \),
\[
\Delta^2(x-z) = (\Delta(x) - 2 \sum_{j=1}^{p} x_j z_j + \Delta(z))^2
= \Delta^2(x) + \Delta^2(z) + 4\langle x, z \rangle^2 + 2\Delta(x)\Delta(z) - 4\langle x, z \rangle \Delta(x) - 4\langle x, z \rangle \Delta(x),
\]
has dimension \( \frac{(p+4)(p+1)}{2} \). We consider the following basis \( \mathcal{B}^{(2)} \) of \( \mathcal{W}^{(2)} \):
\[
e_0 = \frac{1}{2}(\Delta^2(x) - 1), \quad \tilde{e}_0 = \frac{i}{2}(\Delta^2(x) + 1),
e_j(x) = x_j^2 \quad (1 \leq j \leq p), \quad e_{jk}(x) = x_j x_k \quad (1 \leq j < k \leq p),
e_j(x) = x_{j-p} \quad (p+1 \leq j \leq 2p), \quad e_j(x) = x_{j-2p} \quad (2p+1 \leq j \leq 3p).
\]
We write \( \xi \in \mathcal{W}^{(2)} \) as
\[
\xi(x) = \sum_{j=1}^{3p} \xi_j e_j(x) + \sum_{1 \leq j < k \leq p} \xi_{jk} e_{jk}(x) + \xi_0 e_0(x) + \tilde{\xi}_0 \tilde{e}_0(x).
\]
For \( \xi(x) = wQ(x-z) \), one has
\[
\xi_0 = w(1 - Q(z)), \quad \tilde{\xi}_0 = w(-i - iQ(z)),
\]
\[
\xi_j = 2w\Delta(z) + 4wz_j^2 \quad (1 \leq j \leq p), \quad \xi_{jk} = 8wz_j z_k \quad (1 \leq j < k \leq p),
\]
\[
\xi_j = -4w\Delta(z)z_{j-p} \quad (p+1 \leq j \leq 2p), \quad \xi_j = -4wz_{j-2p} \quad (2p+1 \leq j \leq 3p).
\]
In particular,
\[
Q = e_0 - ie_0, \quad 1 = -e_0 - ie_0.
\]
Denote by \( \Xi \) the orbit of \( Q = \Delta^2 \) under the action of the representation \( \kappa^{(2)} \). Then \( \Xi/\{\pm 1\} \approx \Xi \). In coordinates, the open and dense subset \( \Xi_0 \) (resp. \( \Xi_0 \)) of \( \Xi \) (resp. \( \Xi \)) is diffeomorphic to \( \mathbb{C}^* \times \mathbb{C}^p \) (resp. \((\mathbb{C}^*\{\pm 1\}) \times \mathbb{C}^p\)). An element \( \xi \in \Xi_0 \) can be written
\[
\xi(v) = w\Delta^2(v-z)
\]
with
\[
w = w(\xi) = \frac{1}{2}(\xi_0 + i\tilde{\xi}_0).
\]
One has $\mathcal{O}_m(\Xi) \simeq \mathcal{O}_{2m}(\tilde{\Xi})$, and, formally, $\mathcal{O}_{m+\frac{1}{2}}(\Xi) \simeq \mathcal{O}_{2m+1}(\tilde{\Xi})$, then $\mathcal{F}_0(\tilde{\Xi}) = \mathcal{F}(\Xi)$ consists in functions on the orbit $\Xi$, but the space $\mathcal{F}_1(\tilde{\Xi})$ does not.

The integral operators given by

$$\mathcal{A}_{2m} : \mathcal{Y}_{2m}(\mathbb{R}^{p+2}) \rightarrow \mathcal{O}_{2m}(\tilde{\Xi}),$$

$$(\mathcal{A}_{2m} u)(\xi) = \int_{S} \langle \xi, x \rangle^{2m} u(x) s(dx),$$

and

$$\mathcal{A}_{2m+1} : \mathcal{Y}_{2m+1}(\mathbb{R}^{p+2}) \rightarrow \mathcal{O}_{2m+1}(\tilde{\Xi}),$$

$$(\mathcal{A}_{2m+1} u)(\xi) = \int_{S} \langle \xi, x \rangle^{2m+1} u(x) s(dx),$$

are isomorphisms. In particular, for the functions $U_{2m} \in \mathcal{Y}_{2m}(\mathbb{R}^{p+2})$ and $U_{2m+1} \in \mathcal{Y}_{2m+1}(\mathbb{R}^{p+2})$

$$U_{2m}(x) = (x_{p+1} + ix_{p+2})^{2m},$$

and

$$U_{2m+1}(x) = (x_{p+1} + ix_{p+2})^{2m+1},$$

we get $\mathcal{A}_{2m} U_{2m} = \gamma(2m) F_{2m}$, and $\mathcal{A}_{2m+1} U_{2m+1} = \gamma(2m+1) F_{2m+1}$ where

$$F_{2m}(\xi) = (\xi_{p+1} + i\xi_{p+2})^{2m},$$

and

$$F_{2m+1}(\xi) = (\xi_{p+1} + i\xi_{p+2})^{2m+1},$$

with

$$\gamma(2m) = 2^{-2m}(2m)! \frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(\frac{p}{2} + 2 + 2m)}$$

and

$$\gamma(2m+1) = 2^{-2m-1}(2m+1)! \frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(\frac{p}{2} + 2 + 2m + 1)}.$$
3. The Lie algebra $g$ and its isomorphism with $sl(p+2, \mathbb{C})$

We describe the Lie algebra construction in the case for $Q$ to be the square of a quadratic form, $Q = \Delta^2$. Let $H \in \mathfrak{k}$ be the generator of one parameter group of dilations $h_t$ of $V$: $h_t \cdot z = e^{-t}z$, \quad H = \frac{d}{dt} |_{t=0} h_t$.

We get the matrix

$$
\begin{pmatrix}
I_p & 0 & 0 \\
0 & \cosh t & i \sinh t \\
0 & -i \sinh t & \cosh t
\end{pmatrix},
$$

and

$$
d\kappa(H) =
\begin{pmatrix}
0_p & 0 & 0 \\
0 & 0 & i \\
0 & -i & 0
\end{pmatrix}.
$$

We consider the elements $E, F \in \mathfrak{p} := \mathcal{W}$ given by $E := Q, F := 1$.

By Theorem 1.1. there exists on $g$ a unique Lie algebra structure such that for $X, X' \in \mathfrak{k}$, $p \in \mathfrak{p}$,

$$
[X, X'] = [X, X'], \quad [X, p] = d\kappa(X)p, \quad [E, F] = H.
$$

**Theorem 3.1**— If $Q = \Delta^2$ is the square of a quadratic form,

$$
\Delta(z) = (z_1)^2 + \ldots + (z_p)^2,
$$

then $(g, \mathfrak{t})$ is isomorphic to $(sl(p+2, \mathbb{C}), \mathfrak{o}(p+2, \mathbb{C}))$ and $(g_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$ is isomorphic to $(sl(p+2, \mathbb{R}), \mathfrak{o}(p+2))$.

**Proof.** From the isomorphism $\text{Conf}(V, \Delta) \to O(p+2, \mathbb{C})$, we get an isomorphism from the Lie algebra $\mathfrak{t}$ of $\text{Conf}(V, Q)$ onto the Lie algebra $\mathfrak{o}(p+2, \mathbb{C})$. Using the basis $\mathcal{B}^{(2)}$ of $\mathcal{W}^{(2)}$, we define an isomorphism from $\mathfrak{p} = \mathcal{W}^{(2)}$ onto $\{X \in \text{Sym}(p+2, \mathbb{C}), \text{tr}(X) = 0\}$, given by

$$
\xi \mapsto \mu(\xi) = \begin{pmatrix}
\alpha(\xi) \\
\beta(\xi) \\
\delta(\xi)
\end{pmatrix}
$$

with

$$
\alpha(\xi) = \frac{1}{4}
\begin{pmatrix}
\xi_{11} & \xi_{12} & \ldots & \xi_{1p} \\
\xi_{12} & \xi_{22} & \ldots & \xi_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{1p} & \xi_{2p} & \ldots & \xi_{pp}
\end{pmatrix}
$$

$$
(\xi_{jj} = 2\xi_j, 1 \leq j \leq p),
$$

$$
\beta(\xi) = -\frac{1}{4}
\begin{pmatrix}
\xi_{p+1} & \xi_{2p+1} \\
\vdots & \vdots \\
\xi_{2p} & \xi_{3p}
\end{pmatrix},
$$

$$
\delta(\xi) = \begin{pmatrix}
\xi_0 \\
\tilde{\xi}_0 \\
\xi_0 - \text{tr}(\alpha(\xi)) - \text{tr}(\beta(\xi))
\end{pmatrix}.
$$

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In the isomorphism described above, the images of the elements of the \( \mathfrak{sl}(2) \)-triple \( \{H, E, F\} \) are the following matrices

\[
H \mapsto \tilde{H} = \begin{pmatrix}
0_{pp} & 0 & i \\
0 & -i & 0
\end{pmatrix},
\]

\[
E \mapsto \tilde{E} = \begin{pmatrix}
0_{pp} & 1 & -i \\
1 & -i & -1
\end{pmatrix},
\]

\[
F \mapsto \tilde{F} = \begin{pmatrix}
0_{pp} & 1 & i \\
1 & i & -1
\end{pmatrix}.
\]

One checks that \( \{\tilde{H}, \tilde{E}, \tilde{F}\} \) is an \( \mathfrak{sl}(2) \)-triple:

\[
[\tilde{H}, \tilde{E}] = 2\tilde{E}, \quad [\tilde{H}, \tilde{F}] = -2\tilde{F}, \quad [\tilde{E}, \tilde{F}] = \tilde{H}.
\]

By using Theorem 1.1, this proves that we have obtained an explicit Lie algebra isomorphism from \( g \) onto \( \mathfrak{sl}(p + 2, \mathbb{C}) \).

One can notice that

\[
\mathfrak{k} = \{X \in \mathfrak{sl}(p + 2, \mathbb{C}), X^t = -X\}
\]

and

\[
\mathfrak{p} = \{X \in \mathfrak{sl}(p + 2, \mathbb{C}), X^t = X\}.
\]

In particular, for \( \xi(x) = Q(x - z) \), one has

\[
\alpha(\xi) = 2zz^t + \Delta(z)I_p =: a(z) \quad (\xi_{jj} = 2\xi_j, 1 \leq j \leq p),
\]

\[
\beta(\xi) = \begin{pmatrix}
\Delta(z)z_1 & z_1 \\
\vdots & \vdots \\
\Delta(z)z_p & z_p
\end{pmatrix} =: b(z),
\]

and \( \delta(\xi) = \)

\[
\begin{pmatrix}
1 - Q(z) & -i - iQ(z) \\
-i - iQ(z) & -1 + Q(z) + (p - 2)\Delta(z)
\end{pmatrix} =: \begin{pmatrix}
d_0(z) & \bar{d}_0(z) \\
\bar{d}_0(z) & -d_0(z) + (p - 2)\Delta(z)
\end{pmatrix}.
\]

Then the orbit \( \Xi \) which we recall has dimension \( p + 1 \), consists in the matrices

\[
m(z) = \begin{pmatrix}
\alpha(z) & \beta(z) \\
\beta(z)^t & \delta(z)
\end{pmatrix}
\]

where

\[
d(z) = \begin{pmatrix}
d_0 & -2i + id_0 \\
-2i + id_0 & -d_0 + (p - 2)\Delta(z)
\end{pmatrix}, \quad z \in \mathbb{C}^p, d_0 \in \mathbb{C}.
\]
4. The analogue of the Kobayashi-Orsted model for the minimal representations of the group $\text{SL}(p + 2, \mathbb{R})$

Let $\Gamma$ be the open cone in $\mathbb{R}^{p+2}$:

$$\Gamma = \{ x \in \mathbb{R}^{p+2} | |x| \neq 0 \},$$

and $S$ be the unit sphere

$$S = \{ x \in \mathbb{R}^{p+2} : |x| = 1 \}.$$

The group $G_\mathbb{R} = \text{SL}(p + 2, \mathbb{R})$ acts on $\mathbb{R}^{p+2}$ by the natural representation, denoted by $L_g : x \mapsto gx$ ($g \in G_\mathbb{R}, x \in \mathbb{R}^{p+2}$). This action stabilizes the cone $\Gamma$. The multiplicative group $\mathbb{R}_+^*$ acts on $\Gamma$ as a dilation and the quotient space $M = \Gamma / \mathbb{R}_+^*$ is identified with $S$. This defines an action of $G_\mathbb{R}$ on $S$, which leads to a $G_\mathbb{R}$-equivariant principal $\mathbb{R}_+^*$-bundle:

$$\Phi : \Gamma \to S, x \mapsto \frac{x}{|x|}.$$ 

For $\lambda \in \mathbb{C}$, let $\mathcal{E}_\lambda(\Gamma)$ be the space of $C^\infty$-functions on $\Gamma$ homogeneous of degree $\lambda$:

$$\mathcal{E}_\lambda(\Gamma) = \{ u \in C^\infty(\Gamma) | u(tx) = t^\lambda u(x), \ x \in \Gamma, t > 0 \}.$$

The group $\text{SL}(p + 2, \mathbb{R})$ acts naturally on $\mathcal{E}_\lambda(\Gamma)$, and, under the action of the subgroup $O(p + 2)$, the space $\mathcal{E}_\lambda(\Gamma)$ decomposes as:

$$\mathcal{E}_\lambda(\Gamma) |_S \simeq \bigoplus_{m=0}^{\infty} \mathcal{V}_m(\mathbb{R}^{p+2}).$$

Furthermore, for $\epsilon = \pm 1$, we put

$$\mathcal{E}_{\lambda, \epsilon}(\Gamma) := \{ u \in \mathcal{E}_\lambda(\Gamma) : u(-x) = \epsilon \cdot u(x), \ x \in \Gamma \}.$$

Then we have a direct sum decomposition into two $G_\mathbb{R}$-invariant subspaces:

$$\mathcal{E}_\lambda(\Gamma) = \mathcal{E}_{\lambda, 1}(\Gamma) + \mathcal{E}_{\lambda, -1}(\Gamma)$$

and, under the action of the subgroup $O(p + 2)$, each of them decomposes

$$\mathcal{E}_{\lambda, +1}(\Gamma) |_S \simeq \bigoplus_{m=0}^{\infty} \mathcal{V}_m(\mathbb{R}^{p+2})$$

and

$$\mathcal{E}_{\lambda, -1}(\Gamma) |_S \simeq \bigoplus_{m=0}^{\infty} \mathcal{V}_{2m+1}(\mathbb{R}^{p+2}).$$

For $\lambda = \lambda_0 = -(p + 2)$, we denote by:

$$\mathcal{V}_0(\Gamma) = \mathcal{E}_{\lambda_0, +1}, \ \mathcal{V}_1(\Gamma) = \mathcal{E}_{\lambda_0, -1}$$

and by $\omega_0$ and $\omega_1$ the restrictions to $\mathcal{V}_0(\Gamma)$ and $\mathcal{V}_1(\Gamma)$ of the natural action

$$(\omega(g)u)(x) := u(g^{-1}x).$$

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Lemma 4.1— For \( u \in \mathcal{E}_{\lambda_0}(\Gamma) \),

\[-(d\omega)(\widetilde{E})u(x) = (x_{p+1} - ix_{p+2}) \left( \frac{\partial u}{\partial y_{p+1}} - i \frac{\partial u}{\partial y_{p+2}} \right),\]

\[-(d\omega)(\widetilde{F})u(x) = (x_{p+1} + ix_{p+2}) \left( \frac{\partial u}{\partial y_{p+1}} + i \frac{\partial u}{\partial y_{p+2}} \right).\]

**Proof.** For \( M \in \mathfrak{sl}(p + 2, \mathbb{C}) \) and, for \( u \in \mathcal{E}_{\lambda_0}(\Gamma) \),

\[(d\omega(M)u)(x) = \frac{d}{dt}|_{t=0} u(\exp(tM)) \cdot (x) = -Du(x)(Mx),\]

where \( D \) denotes the differential of the function \( u \). For the matrix \( M = F_0 \) introduced in section 3 we get the formula of the lemma. \( \Box \)

**Proposition 4.2—** Consider the functions \( U_{2m} \in \mathcal{V}_0(\Gamma), U_{2m+1} \in \mathcal{V}_1(\Gamma) \) defined by

\[U_{2m}(x) = \frac{1}{|x|^{2m+p+2}} (x_{p+1} + ix_{p+2})^{2m},\]

\[U_{2m+1}(x) = \frac{1}{|x|^{2m+1+p+2}} (x_{p+1} + ix_{p+2})^{2m+1}.\]

Then

\[d\omega_0(\widetilde{E})U_{2m} = (2m - p - 2)U_{2(m-1)},\]

\[d\omega_1(\widetilde{E})U_{2m+1} = (2m + 1 - p - 2)U_{2(m-1)+1}\]

and

\[d\omega_0(\widetilde{F})U_{2m} = (2m + p + 2)U_{2(m+1)},\]

\[d\omega_1(\widetilde{F})U_{2m+1} = (2m + 1 + p + 2)U_{2(m+1)+1}.\]

**Proof.** One writes

\[U_\lambda(x) = \frac{1}{|x|\lambda} (x_{p+1} + ix_{p+2})^\lambda.\]

and uses the formula

\[\left( \frac{\partial}{\partial x_{p+1}} + i \frac{\partial}{\partial x_{p+2}} \right)(x_{p+1} + ix_{p+2})^\lambda = 0,\]

and

\[\left( \frac{\partial}{\partial x_{p+1}} + i \frac{\partial}{\partial x_{p+2}} \right)|x|^\lambda = \lambda |x|^{\lambda - 2} (x_{p+1} + ix_{p+2}).\]
Theorem 4.3—
1) (Irreducibility) The representations \((\omega_0, \mathcal{V}_0(\Gamma))\) and \((\omega_1, \mathcal{V}_1(\Gamma))\) of \(\text{SL}(p + 2, \mathbb{R})\) are irreducible.

2) \((K\text{-type decomposition})\) The underlying \((g, K)\)-modules, \((\omega_0)_K\) and \((\omega_1)_K\), for \(K = O(p + 2, \mathbb{C})\), have the following \(K\)-type formulas
\[
(\omega_0)_K = \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m}(\mathbb{R}^{p+2}), \quad (\omega_1)_K = \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m+1}(\mathbb{R}^{p+2}).
\]

3) (Unitarity) The representations \(\omega_0\) and \(\omega_1\) of \(\text{SL}(p + 2, \mathbb{R})\) on \(\mathcal{V}_0(\Gamma)\) and respectively \(\mathcal{V}_1(\Gamma)\) are unitary for the Hilbert norms on \(\mathcal{V}_0(\Gamma)\) and \(\mathcal{V}_1(\Gamma)\) defined as follows: if
\[
u^{(0)}(x) = \sum_{m=0}^{\infty} \frac{1}{|x|^{2m+p+2}} u_{2m}(x), \quad \nu^{(1)}(x) = \sum_{m=0}^{\infty} \frac{1}{|x|^{2m+1+p+2}} u_{2m+1}(x),
\]
with \(u_{2m} \in \mathcal{Y}_{2m}(\mathbb{R}^{p+2}), \ u_{2m+1} \in \mathcal{Y}_{2m+1}(\mathbb{R}^{p+2})\), then
\[
\|\nu^{(0)}\|^2 = \sum_{m=0}^{\infty} \frac{1}{2^{2m}} \int_S |u_{2m}(x)|^2 s(dx), \quad \|\nu^{(1)}\|^2 = \sum_{m=0}^{\infty} \frac{1}{2^{2m}} \int_S |u_{2m+1}(x)|^2 s(dx)
\]

This Theorem is less or more known: the representations \((\omega_0, \mathcal{V}_0(\Gamma))\) and \((\omega_1, \mathcal{V}_1(\Gamma))\) correspond respectively to the degenerate principal series \(\pi^\text{GL}(p+2,\mathbb{R})_{\mu,0}\) and \(\pi^\text{GL}(p+2,\mathbb{R})_{\mu,1}\), for \(\mu = 0\), as they are given in [KOP11] Section 4, in the case of \(\text{GL}(2n, \mathbb{R})\). We give a proof in our context.

**Proof.** 1) In fact, let \(U_0\) be an invariant subspace of \(\mathcal{V}_0\) for \(\omega_0\) and \(U_1\) an invariant subspace of \(\mathcal{V}_1\) for \(\omega_1\) and assume that \(U_0 \neq \{0\}\) and \(U_1 \neq \{0\}\). Since \(U_0\) and \(U_1\)-invariant, and since the subspaces \(\mathcal{Y}_{2m}(\mathbb{R}^{p+2})\) and \(\mathcal{Y}_{2m+1}(\mathbb{R}^{p+2})\) are irreducible for the restriction representations \((\omega_0)_K\) and \((\omega_1)_K\) respectively, then there exist \(\mathcal{I}_0 \subset \mathbb{N}\) and \(\mathcal{I}_1 \subset \mathbb{N}\) such that \(U_0 = \sum_{m \in \mathcal{I}_0} \mathcal{Y}_{2m}\) and \(U_1 = \sum_{m \in \mathcal{I}_1} \mathcal{Y}_{2m+1}\). Furthermore, since \(d\omega_0(\bar{F})(U_{2m}) = (2m + p + 2)U_{2(m+1)}\), \(d\omega_1(\bar{F})(U_{2m+1}) = (2m + p + 3)U_{2(m+1)+1}\), then \(d\omega_0(\bar{F})(\mathcal{Y}_{2m}) = \mathcal{Y}_{2(m+1)}\) and \(d\omega_1(\bar{F})(\mathcal{Y}_{2m+1}) = \mathcal{Y}_{2(m+1)+1}\), which means that if \(m\) belongs to \(\mathcal{I}_0\) then so does \(m + 1\) and if \(m'\) belongs to \(\mathcal{I}_1\) then so does \(m' + 1\). Moreover, since \(d\omega_0(\bar{E})U_{2m} = (2m - p - 2)U_{2(m-1)}\) and \(d\omega_1(\bar{E})U_{2m+1} = (2m + 1 - p - 2)U_{2(m-1)+1}\) then \(d\omega_0(\bar{E})(\mathcal{Y}_{2m}) = \mathcal{Y}_{2(m-1)}\) and \(d\omega_1(\bar{E})(\mathcal{Y}_{2m+1}) = \mathcal{Y}_{2(m-1)+1}\), which means that if \(m\) belongs to \(\mathcal{I}_0\) then so does \(m - 1\) and if \(m'\) belongs to \(\mathcal{I}_1\) then so does \(m' - 1\). It follows that \(\mathcal{I}_0 = \mathcal{I}_1 = \mathbb{N}\), i.e. \(U_0 = \mathcal{V}_0\) and \(U_1 = \mathcal{V}_1\).

2) This follows from the decomposition to \(K_{\mathbb{R}}\)-irreducible spaces:
\[
L^2(S^{p+1}) \simeq \sum_{k=0}^{\infty} \mathcal{Y}_k(\mathbb{R}^{p+2}).
\]
3) Since the inner products on $\mathcal{V}_0$ and $\mathcal{V}_1$ are $K_\mathbb{R}$-invariant, then they are $\mathfrak{sl}(p + 2, \mathbb{R})$-invariant if and only if, for every $X \in \mathfrak{p}$, $d\omega(X)^* = -d\omega(\bar{X})$. This is equivalent to the single condition

$$d\omega(\bar{E})^* = -d\omega(\bar{F}).$$

In fact, assume these conditions are satisfied. Then, for $X = g\bar{E}g^{-1}$ ($g \in K_\mathbb{R}$),

$$d\omega(X) = \omega(g)d\omega(\bar{E})\omega(g)^{-1},$$

and

$$d\omega(X)^* = -\omega(g)^{-1}d\omega(\bar{F})\omega(g)^* = -\omega(g)d\omega(\bar{F})\omega(g)^{-1} = -d\omega(\bar{X}).$$

It remains to check that for every $u, u' \in \mathcal{V}$, one has

$$\langle d\omega(\bar{E})u, u' \rangle_{\mathcal{V}} = -\langle u, d\omega(\bar{F})u' \rangle_{\mathcal{V}}$$

which is equivalent to checking that for every $u_k \in \mathcal{Y}_k, v_{k'} \in \mathcal{Y}_{k'}$, one has

$$\langle d\omega(\bar{E})u_k, v_{k'} \rangle_{\mathcal{V}} = -\langle u_k, d\omega(\bar{F})v_{k'} \rangle_{\mathcal{V}}.$$

For every $u_k \in \mathcal{Y}_k, v_{k'} \in \mathcal{Y}_{k'}$, using integration by parts, one gets

$$\langle d\omega(\bar{E})u_k, v_{k'} \rangle_{\mathcal{V}} = c_{k,k'} \int_{S} (x_{p+1} - ix_{p+2})(\frac{\partial u_k}{\partial x_{p+1}} - i \frac{\partial u_k}{\partial x_{p+2}})v_{k'}(x)s(dx)$$

$$= c_{k,k'} \int_{S} \left( \frac{\partial u_k}{\partial x_{p+1}} - i \frac{\partial u_k}{\partial x_{p+2}} \right)(x_{p+1} + ix_{p+2})v_{k'}(x)s(dx)$$

$$= -c_{k,k'} \int_{S} u_k(x)(x_{p+1} + ix_{p+2})\left( \frac{\partial v_{k'}}{\partial x_{p+1}} + i \frac{\partial v_{k'}}{\partial x_{p+2}} \right)s(dx)$$

$$= -\langle u_k, d\omega(\bar{F})v_{k'} \rangle_{\mathcal{V}}$$

where $c_{k,k'}$ is a constant.

Observe that the scalar products on $\mathcal{V}_0(\Gamma)$ and $\mathcal{V}_1(\Gamma)$ are given by

$$\langle u^{(0)}, v^{(0)} \rangle_{\mathcal{V}_0} = \int_{S} D_p^{(0)} u^{(0)}(x)D_p^{(0)} v^{(0)}(x)s(dx)$$

and

$$\langle u^{(1)}, v^{(1)} \rangle_{\mathcal{V}_1} = \int_{S} D_p^{(1)} u^{(1)}(x)D_p^{(1)} v^{(1)}(x)s(dx)$$

where $D_p^{(0)}$ and $D_p^{(1)}$ are the diagonal operators given by

$$(D_p^{(0)} u^{(0)})(x) = \sum_{m=0}^{\infty} \frac{1}{2^m |x|^{2m+p+2}} u_{2m}(x).$$

and

$$(D_p^{(1)} u^{(1)})(x) = \sum_{m=0}^{\infty} \frac{1}{2^m |x|^{2m+1+p+2}} u_{2m+1}(x).$$
5. The analogue of the Brylinski-Kostant model for the minimal representations of the group $\text{SL}(p + 2, \mathbb{R})$.

We assume $p \geq 3$. Following the method in [A12], recalled in Section 1, we will construct the representations $\rho_0$ and $\rho_1$ on the spaces of finite sums

$$\mathcal{F}_0(\Xi)_{\text{fin}} = \bigoplus_{m=0}^{\infty} \mathcal{O}_{2m}(\Xi) \quad \text{and} \quad \mathcal{F}_1(\Xi)_{\text{fin}} = \bigoplus_{m=0}^{\infty} \mathcal{O}_{2m+1}(\Xi),$$

More precisely, we consider the case where $V = \mathbb{C}^p$ with $p \geq 3$, $Q$ is the square of a quadratic form $\Delta$:

$$\Delta(v) = v_1^2 + \ldots + v_p^2.$$

In this case, $g = \mathfrak{sl}(p + 2, \mathbb{C})$, $K = \mathcal{O}(p + 2, \mathbb{C})$, $s = 1$, $k = 2$, $r = 2$. The representations exist iff $q = 0$ or $q = 1$ (see [A12], Theorem 3.4 and Table 2). In fact, condition (T) is fulfilled with $\eta_q = \frac{q}{2} + \frac{p + 1}{4}$ and, since one of the $q_i$ must be $< k = 2$, then $q = 0$ or $q = 1$.

The Hilbert spaces $\mathcal{F}_0(\Xi)$ and $\mathcal{F}_1(\Xi)$ decompose

$$\mathcal{F}_0(\Xi) = \bigoplus_{m=0}^{\infty} \mathcal{O}_{2m}(\Xi), \quad \mathcal{F}_1(\Xi) = \bigoplus_{m=0}^{\infty} \mathcal{O}_{2m+1}(\Xi),$$

where the functions $f_{2m}$ in $\mathcal{O}_{2m}(\Xi)$ and $f_{2m+1}$ in $\mathcal{O}_{2m+1}(\Xi)$ are, in coordinates, given by:

$$\phi_{2m}(w, z) = w^{2m} \psi_{2m}(z) \quad (\psi_{2m} \in \tilde{\mathcal{O}}_{2m}(\mathbb{C}^p), w \in \mathbb{C}^*)$$

$$\phi_{2m+1}(w, z) = w^{2m+1} \psi_{2m+1}(z) \quad (\psi_{2m+1} \in \tilde{\mathcal{O}}_{2m+1}(\mathbb{C}^p), w \in \mathbb{C}^*).$$

where $\tilde{\mathcal{O}}_{2m}(\mathbb{C}^p)$ and $\tilde{\mathcal{O}}_{2m+1}(\mathbb{C}^p)$ are respectively the spaces of holomorphic homogeneous polynomials $\psi_{2m}$ and $\psi_{2m+1}$ on $\mathbb{C}^p$, such that there exist harmonic polynomials $u_{2m}$ of degree $2m$ and $u_{2m+1}$ of degree $2m + 1$ on $\mathbb{R}^{p+2}$ such that

$$\psi_{2m}(z_1, \ldots, z_p) = u_{2m}(-2z_1, \ldots, -2z_p, \frac{1-\Delta(z)}{2}, \frac{1+\Delta(z)}{2}),$$

$$\psi_{2m+1}(z_1, \ldots, z_p) = u_{2m+1}(-2z_1, \ldots, -2z_p, \frac{1-\Delta(z)}{2}, \frac{1+\Delta(z)}{2}).$$

The Euler operator $\mathcal{E}$ is defined as

$$(\mathcal{E}\phi)(w, z) = \frac{d}{dt} \big|_{t=0} \phi(w, e^t z).$$

One gets

$$\rho_0(H)\phi_{2m} = \mathcal{E}\phi_{2m} - 2m\phi_{2m},$$

$$\rho_1(H)\phi_{2m+1} = \mathcal{E}\phi_{2m+1} - (2m + 1)\phi_{2m+1}.$$
Introduce the diagonal operator $\delta$: if $f = \sum_n f_n$ with $f_n \in \mathcal{O}_n(\widetilde{\Xi})$, then $\delta f = \sum_n \delta_n f_n$, where $(\delta_n)$ is a sequence of real numbers.

The operators $\rho_0(F)$ and $\rho_0(E)$ are given by

$$\rho_0(F)\phi_{2m}(w, z) = w^{2(m+1)}\psi(z) - \delta_2(m-1)w^{2(m-1)}\Delta^2(\frac{\partial}{\partial z})\psi(z),$$

$$\rho_0(E)\phi_{2m}(w, z) = w^{2(m+1)}\Delta^2(z)\psi(z) - \delta_2(m-1)w^{2(m-1)}D^*_{2m}\psi(z),$$

and the operators $\rho_1(F)$ and $\rho_1(E)$ are given by

$$\rho_1(F)\phi_{2m+1}(w, z) = w^{2(m+2)}\psi(z) - \delta_2(m+1)w^{2(m+1)}\Delta^2(\frac{\partial}{\partial z})\psi(z),$$

$$\rho_1(E)\phi_{2m+1}(w, z) = w^{2(m+2)}\Delta^2(z)\psi(z) - \delta_2(m+1)w^{2(m+1)}D^*_{2m+1}\psi(z),$$

where $D_\alpha$ is the Maass operator

$$D_\alpha = \Delta(z)^{1+\alpha}\Delta(\frac{\partial}{\partial z})\Delta(z)^{-\alpha},$$

$$D^* = J \circ D \circ J$$

with $Jf(z) = f(-z^{-1})$.

and the sequences $(\delta_{2m})$ and $(\delta_{2m+1})$, are here given for $m \neq 0$ by

$$\delta_{2m} = \frac{1}{10(m+\eta_0)(m+\eta_0+1)}$$

and

$$\delta_{2m+1} = \frac{1}{10(m+\eta_1)(m+\eta_1+1)},$$

$$\delta_0 = 1, \quad \delta_1 = \frac{1}{4}.$$ 

**Theorem 5.1—** Assume $p \geq 3$. Then $\rho_0$ and $\rho_1$ are representations of the $\mathfrak{sO}_2$-triple \{F, E, H\} and extend as irreducible representations of g on $\mathcal{F}_0(\widetilde{\Xi})_{\text{fin}}$ and $\mathcal{F}_1(\widetilde{\Xi})_{\text{fin}}$.

This is a special case of Theorem 3.4 and Theorem 3.8 in [A12].

We consider for a sequence $(c_n)$ of positive numbers, an inner product on $\mathcal{F}_0(\widetilde{\Xi})_{\text{fin}}$ and an inner product on $\mathcal{F}_1(\widetilde{\Xi})_{\text{fin}}$ such that, if $f = \sum_m f_{2m}$, with $f_{2m} \in \mathcal{O}_{2m}(\widetilde{\Xi})$, and if $h = \sum_m h_{2m+1}$, with $h_{2m+1} \in \mathcal{O}_{2m+1}(\widetilde{\Xi})$,

$$\|f\|^2_{\mathcal{F}_0} = \sum_{m=0}^{\infty} \frac{1}{c_{2m}} \|f_{2m}\|^2_{2m}$$

and

$$\|h\|^2_{\mathcal{F}_1} = \sum_{m=0}^{\infty} \frac{1}{c_{2m+1}} \|h_{2m+1}\|^2_{2m+1}.$$

These inner products are invariant under $K_\mathbb{R} = SO(p+2)$. The completion of $\mathcal{F}_0(\widetilde{\Xi})_{\text{fin}}$ for the first inner product and of $\mathcal{F}_1(\widetilde{\Xi})_{\text{fin}}$ for the second one are Hilbert subspaces $\mathcal{F}_0(\widetilde{\Xi})$ and $\mathcal{F}_1(\widetilde{\Xi})$ of $\mathcal{O}(\widetilde{\Xi})$ whose reproducing kernels are given by

$$\mathcal{K}_0(\xi, \eta) = \sum_{m=0}^{\infty} c_{2m} \langle \xi, \eta \rangle^{2m}$$

and

$$\mathcal{K}_1(\xi, \eta) = \sum_{m=0}^{\infty} c_{2m+1} \langle \xi, \eta \rangle^{2m+1}.$$

**Theorem 5.2—** We fix

$$c_{2m} = \frac{(\eta_{0+\frac{1}{2}})_{m}}{(\eta_{0+\frac{1}{2}})_{m} (\eta_{0+\frac{1}{2}})_{m}} \frac{1}{m!}$$

and

$$c_{2m+1} = \frac{(\eta_{1+\frac{1}{2}})_{m}}{(\eta_{1+\frac{1}{2}})_{m} (\eta_{1+\frac{1}{2}})_{m}} \frac{1}{m!}.$$

Then, restricted to the real Lie algebra $\mathfrak{g}_\mathbb{R}$, $\rho_0$ is a unitary representation on the space $\mathcal{F}_0(\widetilde{\Xi})_{\text{fin}}$ and $\rho_1$ is a unitary representation on $\mathcal{F}_1(\widetilde{\Xi})_{\text{fin}}$.

This is a special case of Theorem 4.1 in [A12].
6. The intertwining operator

Define the functions \( b^{(0)} \) and \( b^{(1)} \) in one complex variable by

\[
\begin{align*}
  b^{(0)}(t) &= \sum_{m=1}^{\infty} b_{2m} t^{2m} \\
  b^{(1)}(t) &= \sum_{m=0}^{\infty} b_{2m+1} t^{2m+1}
\end{align*}
\]

and the operators \( \mathcal{B}_0 : \mathcal{V}_0(\Gamma)_\text{fin} \to \mathcal{F}_0(\tilde{\Xi})_\text{fin} \) and \( \mathcal{B}_1 : \mathcal{V}_1(\Gamma)_\text{fin} \to \mathcal{F}_1(\tilde{\Xi})_\text{fin} \): for \( x \in S, \xi \in \tilde{\Xi} \),

\[
\begin{align*}
  (\mathcal{B}_0 u)(\xi) &= \int_S b^{(0)}(\langle \xi, x \rangle) u(x) s(dx) \\
  (\mathcal{B}_1 u)(\xi) &= \int_S b^{(1)}(\langle \xi, x \rangle) u(x) s(dx).
\end{align*}
\]

By Proposition 2.1, if the constants \( b_{2m} \neq 0 \) and \( b_{2m+1} \neq 0 \), then the restriction \( (\mathcal{B}_0)^{2m} = b_{2m} A_{2m} \) of \( \mathcal{B}_0 \) to \( \mathcal{V}_2m(\mathbb{R}^{p+2}) \) and the restriction \( (\mathcal{B}_1)^{2m+1} = b_{2m+1} A_{2m+1} \) of \( \mathcal{B}_1 \) to \( \mathcal{V}_2m+1(\mathbb{R}^{p+2}) \), are isomorphisms

\[
\begin{align*}
  (\mathcal{B}_0)^{2m} : \mathcal{V}_2m(\mathbb{R}^{p+2}) &\to \mathcal{O}_{2m}(\Xi), \\
  (\mathcal{B}_1)^{2m+1} : \mathcal{V}_2m+1(\mathbb{R}^{p+2}) &\to \mathcal{O}_{2m+1}(\Xi).
\end{align*}
\]

which intertwine the action of \( O(p+2) \) by \( \omega_0 \) and \( \rho_0 \) respectively and \( \omega_1 \) and \( \rho_1 : \) for \( X \in \mathfrak{k} \) with image \( \tilde{X} \in o(p+2, \mathbb{C}), \)

\[
\mathcal{B}_0 d\omega_0(\tilde{X}) = \rho_0(X) \mathcal{B}_0 \quad \text{and} \quad \mathcal{B}_1 d\omega_1(\tilde{X}) = \rho_1(X) \mathcal{B}_1.
\]

**Theorem 6.1—** For

\[
\begin{align*}
  b_{2m} &= \frac{(\frac{p}{2}+1)^{2m}}{2^m(\frac{p+1}{2})_m (2m)!} \quad \text{and} \quad b_{2m+1} = \frac{(\frac{p}{2}+1)^{2m+1}}{2^m(\frac{p+1}{2})_m (2m+1)!}
\end{align*}
\]

the operator \( \mathcal{B}_0 \) intertwines the representations \( d\omega_0 \) and \( \rho_0 \), and the operator \( \mathcal{B}_1 \) intertwines the representations \( d\omega_1 \) and \( \rho_1 \), and they are unitary and bijective.

**Proof.** a) Consider the functions

\[
\begin{align*}
  F_{2m}(\xi) &= \left( \frac{\xi_0 + i\xi_\alpha}{2} \right)^{2m} \quad \text{and} \quad F_{2m+1}(\xi) = \left( \frac{\xi_0 + i\xi_\alpha}{2} \right)^{2m+1}
\end{align*}
\]

which correspond, in coordinates, to the functions

\[
\Phi_{2m}(w, z) = w^{2m} \quad \text{and} \quad \Phi_{2m+1}(w, z) = w^{2m+1}.
\]

Then

\[
\rho_0(F) F_{2m} = F_{2(m+1)} \quad \text{and} \quad \rho_1(F) F_{2m+1} = F_{2(m+1)+1}.
\]
Furthermore,

\[
(B_0 U_{2m})(\xi) = b_{2m} \int_S \langle \xi, x \rangle^{2m} (x_{p+1} + ix_{p+2})^{2m} s(dx)
\]

\[
= b_{2m} (2m)! \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + 1 + 2m\right)} f_{2m}(\xi)
\]

\[
= \beta_{2m} F_{2m}(\xi).
\]

and

\[
(B_1 U_{2m+1})(\xi) = b_{2m+1} \int_S \langle \xi, x \rangle^{2m+1} (x_{p+1} + ix_{p+2})^{2m+1} s(dx)
\]

\[
= b_{2m+1} (2m + 1)! \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + 2 + 2m\right)} f_{2m+1}(\xi)
\]

\[
= \beta_{2m+1} F_{2m+1}(\xi)
\]

with

\[
\beta_{2m} = b_{2m} (2m)! \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + 1 + 2m\right)} \quad \text{and} \quad \beta_{2m+1} = b_{2m+1} (2m + 1)! \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + 2 + 2m\right)}.
\]

The intertwining relations

\[ B_0 d\omega_0(\tilde{F}) U_{2m} = \rho_0(F) B_0 U_{2m} \quad \text{and} \quad B_1 d\omega_1(\tilde{F}) U_{2m+1} = \rho_1(F) B_1 U_{2m+1} \]

give respectively the conditions

\[
(2m + p + 2)\beta_{2(m+1)} = \beta_{2m} \quad \text{and} \quad (2m + 1 + p + 2)\beta_{2(m+1)+1} = \beta_{2m+1}
\]

and if we fix \( \beta_0 = 1 \) and \( \beta_1 = 1 \), then

\[
\beta_{2m} = \frac{1}{2^m (\frac{p}{2} + \frac{1}{2})_m}, \quad \beta_{2m+1} = \frac{1}{2^m (\frac{p}{2} + \frac{1}{2} + 1)_m}.
\]

It follows that

\[
b_{2m} = \frac{\Gamma\left(\frac{p}{2} + 1 + 2m\right)}{\Gamma\left(\frac{p}{2} + 1\right)} \frac{1}{2^m (\frac{p}{2} + \frac{1}{2})_m (2m)!} = \frac{(\frac{p}{2} + 1)_m}{2^m (\frac{p}{2} + \frac{1}{2})_m (2m)!}
\]

and

\[
b_{2m+1} = \frac{\Gamma\left(\frac{p}{2} + 2 + 2m\right)}{\Gamma\left(\frac{p}{2} + 1\right)} \frac{1}{2^m (\frac{p}{2} + \frac{1}{2} + 1)_m (2m+1)!} = \frac{(\frac{p}{2} + 1)_m}{2^m (\frac{p}{2} + \frac{1}{2} + 1)_m (2m+1)!}.
\]

One can also check the intertwining relations

\[ B_0 d\omega_0(\tilde{E}) U_{2m} = \rho_0(E) B_0 U_{2m} \quad \text{and} \quad B_1 d\omega_1(\tilde{E}) U_{2m+1} = \rho_1(E) B_1 U_{2m+1}. \]
b) We will check that, for \( m \) and \( \rho \) given by
\[
\xi = \int_s (x_{p+1}^2 + x_{p+2}^2)\lambda s(dx) = \lambda! \Gamma\left(\frac{p}{2} + 1\right) \Gamma\left(\frac{p}{2} + 1 + \lambda\right).
\]
Therefore, the norm of \( U_{2m} \) in \( V_0(\Gamma) \) and the norm of \( U_{2m+1} \) in \( V_1(\Gamma) \) are given by
\[
\|U_{2m}\|_{V_0}^2 = \frac{1}{2^{2m}} (2m)! \Gamma\left(\frac{p}{2} + 1\right) \Gamma\left(\frac{p}{2} + 1 + 2m\right)
= \frac{(2m)!}{2^{2m} \left(\frac{p}{2} + 1\right)_{2m}} = \frac{(2m)!}{2^{4m} \frac{p}{2} + 1 \frac{p}{2} + 1 + m}
\]
and
\[
\|U_{2m+1}\|_{V_1}^2 = \frac{1}{2^{2m+1}} (2m + 1)! \Gamma\left(\frac{p}{2} + 1\right) \Gamma\left(\frac{p}{2} + 2 + 2m\right)
= \frac{(2m + 1)!}{2^{2m+1} \left(\frac{p}{2} + 1\right)_{2m+1}} = \frac{(2m + 1)!}{2^{4m+1} \frac{p}{2} + 1 \frac{p}{2} + \frac{3}{2} + m}
\]

The norm of \( F_{2m} \) in the Hilbert space \( E(\Xi) \) and the norm of \( F_{2m+1} \) in the Hilbert space \( E_1(\Xi) \) are given by
\[
\|F_{2m}\|_{E_0}^2 = \frac{1}{c_{2m}} \quad \text{and} \quad \|F_{2m+1}\|_{E_1}^2 = \frac{1}{c_{2m+1}}
\]
with
\[
c_{2m} = \frac{(\frac{p}{2} + 1)_m}{(\frac{p}{2} + \frac{1}{2})_m \frac{1}{m}} \quad \text{and} \quad c_{2m+1} = \frac{(\frac{p}{2} + \frac{3}{2})_m}{(\frac{p}{2} + 1)_m \frac{1}{m}}.
\]
In fact, the sequence \((c_m)\) is given in Theorem 4.1, table 3 and table 4 in [A12]
\[
c_{2m} = \frac{(q + 1)_m}{(q + \frac{1}{2})_m \frac{1}{m}} \quad \text{and} \quad c_{2m+1} = \frac{(q + 1)_m}{(q + \frac{1}{2})_m \frac{1}{m}}.
\]
In our case,
\[ \eta_0 = \frac{p}{4}, \quad a_0 = \frac{1}{2}, \quad b_0 = \frac{1}{2} - \frac{p}{4} \]
and
\[ \eta_1 = \frac{p}{4} + \frac{1}{2}, \quad a_0 = a_0' = \frac{1}{2}, \quad b_0 = b_0' = \frac{1}{2} - \frac{p}{4} \]
then
\[ c_{2m} = \frac{(\frac{p}{4} + \frac{1}{2})_m}{(\frac{p}{4} + \frac{3}{2})_m}\frac{1}{m!} \quad \text{and} \quad c_{2m+1} = \frac{(\frac{p}{4} + \frac{3}{2})_m}{(\frac{p}{4} + 1)_m}\frac{1}{m!}. \]

(one has to check that \(1 - \eta_0\) is a root of the Bernstein polynomial but \(1 - \eta_1\) isn’t). Therefore

\[ \|B_0 U_{2m}\|_2^2 \omega_0 = (\beta_{2m})^2\|f_{2m}\|_2^2 \omega_0 = \frac{\beta^2_{2m}}{c_{2m}} \]
\[ = \frac{\beta^2(\frac{p}{4} + \frac{1}{2})_m(\frac{1}{2})_m m!}{2^{2m}(\frac{p}{4} + \frac{1}{2})_m(\frac{p}{4} + 1)_m} \]
\[ = \frac{\beta^2(\frac{p}{4} + \frac{1}{2})_m(2m)!}{2^{4m}(\frac{p}{4} + \frac{1}{2})_m(\frac{p}{4} + 1)_m} \]
\[ = \frac{\beta^2(2m)!}{2^{4m}(\frac{p}{4} + \frac{1}{2})_m(\frac{p}{4} + 1)_m} \]

and

\[ \|B_1 U_{2m+1}\|_2^2 \omega_1 = (\beta_{2m+1})^2\|f_{2m+1}\|_2^2 \omega_1 = \frac{\beta^2_{2m+1}}{c_{2m+1}} \]
\[ = \frac{\beta'^2(\frac{p}{4} + 1)_m(\frac{3}{2})_m m!}{2^{2m}(\frac{p}{4} + 1)_m(\frac{3}{2})_m} \]
\[ = \frac{\beta'^2(\frac{p}{4} + 1)_m(2m)!}{2^{4m}(\frac{p}{4} + 1)_m(\frac{3}{2})_m} \]
\[ = \frac{\beta'^2(2m)!}{2^{4m}(\frac{p}{4} + 1)_m(\frac{3}{2})_m} \]

For \(\beta = \beta' = 1\), one gets the unitarity of the operators \(B_0\) and \(B_1\).

c) Since the representations \(d\omega_0\) and \(d\omega_1\) are irreducible then the intertwining operators \(B_0\) and \(B_1\) are injective and since the representations \(\rho_0\) and \(\rho_1\) are irreducible then \(B_0\) and \(B_1\) are surjective.

**Theorem 6.2**— For

\[ b_{2m} = \frac{(\frac{p}{4} + 1)_{2m}}{2^{2m}(\frac{p}{4} + \frac{3}{2})_m(2m)!} \quad \text{and} \quad b_{2m+1} = \frac{(\frac{p}{4})_{2m+1}}{2^{2m}(\frac{p}{4} + \frac{3}{2})_m(2m+1)!} \]

the operators \(B_0\) and \(B_1\) are unitary isomorphisms of \((g, \mathfrak{t})\) modules.

The irreducible unitary representation \(\omega_0\) is equivalent to \(\rho_0\) and the irreducible unitary representation \(\omega_1\) is equivalent to \(\rho_1\).
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