STRICT WEAK MIXING OF SOME $C^*$–DYNAMICAL SYSTEMS BASED ON FREE SHIFTS

FRANCESCO FIDALEO AND FARRUKH MUKHAMEDOV

Abstract. We define a stronger property than unique ergodicity with respect to the fixed–point subalgebra firstly investigated in [1]. Such a property is denoted as $F$–strict weak mixing ($F$ stands for the Markov projection onto the fixed–point operator system). Then we show that the free shifts on the reduced $C^*$–algebras of RD–groups, including the free group on infinitely many generators, and amalgamated free product $C^*$–algebras, considered in [1], are all strictly weak mixing and not only uniquely ergodic.

Mathematics Subject Classification: 37A30, 46L55, 60J99, 20E06.

Key words: Ergodic theory, $C^*$–dynamical systems, Markov operators, Free products with amalgamation.

1. Introduction

Recently, the investigation of the ergodic properties of quantum dynamical systems had a considerable growth. In quantum setting, the matter is more complicated than the classical case. For example, some differences between classical and quantum situations are pointed out in [11]. It is then natural to address the study of the possible generalizations to quantum case of the various ergodic properties known for classical dynamical systems.

A very strong ergodic property for a classical system is the unique ergodicity. Namely, let $(\Omega, T)$ be a classical dynamical systems based on a compact Hausdorff space $\Omega$ and a homeomorphism $T$ of $\Omega$. It is said to be uniquely ergodic if there exists a unique invariant Borel measure $\mu$ for $T$. It is seen that the ergodic average $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$ converges uniformly to the constant function $\int f \, d\mu$. The pivotal example of classical uniquely ergodic dynamical system is the irrational rotations on the unit circle, see e.g. [8]. In quantum setting, the last property is formulated as follows. Let $(\mathfrak{A}, \alpha)$ be a $C^*$–dynamical systems based on the $C^*$–algebra $\mathfrak{A}$ and the automorphism $\alpha$. The unique ergodicity


for \((\mathfrak{A}, \alpha)\) is equivalent (cf. [1, 10]) to the norm convergence
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) = E(a),
\]
where \(E\) is the conditional expectation, given by \(E = \varphi(\cdot)\mathbf{1}\), onto the fixed–point subalgebra of \(\alpha\) consisting of the constant multiples of the identity. Here, \(\varphi \in \mathcal{S}(\mathfrak{A})\) is the unique invariant state for \(\alpha\). A natural generalization of unique ergodicity is to require that the ergodic mean in (1.1) converges to a conditional expectation \(E\) (necessarily unique) projecting onto the fixed–point subalgebra \(\mathfrak{A}^\alpha\) which, in general, is supposed to be nontrivial. This property, denoted as unique ergodicity with respect to the fixed–point subalgebra, has been investigated in [1]. In that paper, it is proven that free shifts based on reduced \(C^*\)–algebras of RD–groups (including the free group on infinitely many generators), and amalgamated free product \(C^*\)–algebras, are uniquely ergodic w.r.t. the fixed–point subalgebra. This provides nontrivial examples of quantum dynamical systems based on automorphisms, exhibiting very strong ergodic properties.

A stronger property than unique ergodicity, called strict weak mixing, was investigated in [10]. In order to achieve quantum probability, this was done in the more general situation of \(C^*\)–dynamical systems \((\mathfrak{A}, T)\), where \(T\) is a Markov (i.e. completely positive and identity–preserving) operator acting on \(\mathfrak{A}\). The last property simply means that
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \psi(T^k(a)) - \varphi(a) \right| = 0
\]
for each \(\psi \in \mathcal{S}(\mathfrak{A})\), \(\varphi\) being the unique invariant state for \(T\). Notice that the irrational rotations on the unit circle provides an example of uniquely ergodic dynamical system which is not strictly weak mixing, see [10], Example 2. Other examples of uniquely ergodic, non strictly weak mixing quantum dynamical systems can be easily constructed by using the algebra of all the \(n \times n\) matrices \(\mathbb{M}_n(\mathbb{C})\), see [3].

In the present paper we generalize this mixing–like property to the situation of [1], by considering \(C^*\)–dynamical systems based on Markov operators. Namely, for the dynamical system \((\mathfrak{A}, T)\), we require that there exists a linear map \(F : \mathfrak{A} \mapsto \mathfrak{A}\) (necessarily a Markov projection projecting onto the fixed–point operator system of \(T\)) such that
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \psi(T^k(x)) - \psi(F(x)) \right| = 0
\]
for every \( \varphi \in \mathcal{S}(\mathfrak{A}) \). Such a mixing–like property is denoted as \( F \)-strict weak mixing. It is immediate to see that, if there exists a unique invariant state \( \varphi \) for \( T \), then \( F = \varphi(\cdot)1 \) and (1.3) reduces itself to (1.2).

Our interest for the above mentioned property is to prove that all the dynamical systems based on free shifts considered in [1] are strictly weak mixing and not merely uniquely ergodic.

2. TERMINOLOGY, NOTATIONS AND BASIC RESULTS

Let \( \mathfrak{A} \) be a \( C^* \)-algebra with identity \( 1 \). A closed selfadjoint subspace \( \mathfrak{R} \subset \mathfrak{A} \) containing \( 1 \) is said to be an operator system. By considering the inclusion \( \mathbb{M}_n(\mathfrak{R}) \subset \mathbb{M}_n(\mathfrak{A}) \), \( \mathbb{M}_n(\mathfrak{R}) \) is also an operator system for each \( n \). A linear map \( T : \mathfrak{R} \mapsto \mathfrak{S} \) between operator systems is said to be completely positive if \( T_n := T \otimes \text{id}_{\mathbb{M}_n} : \mathbb{M}_n(\mathfrak{R}) \mapsto \mathbb{M}_n(\mathfrak{S}) \) is positive for each \( n = 1, 2, \ldots \). It is well–known (cf. [13]) that sup \( n \| T_n \| = T(1) \) for completely positive maps. Let \( T : \mathfrak{A} \mapsto \mathfrak{A} \) be completely positive and identity–preserving, \( T \) is called a Markov operator. For such a Markov operator, the fixed–point subspace

\[
\mathfrak{A}^T := \{ x \in \mathfrak{A} : T(x) = x \}
\]

is an operator system.

Recall that a conditional expectation \( E : \mathfrak{A} \mapsto \mathfrak{B} \subset \mathfrak{A} \) is a norm–one projection of the \( C^* \)-algebra \( \mathfrak{A} \) onto a \( C^* \)-subalgebra (with the same identity \( 1 \)) \( \mathfrak{B} \). The map \( E \) is automatically a completely positive identity–preserving \( \mathfrak{B} \)-bimodule map, see e.g. [14].

Let \( T \) be a Markov operator. It is seen in [2] that \( \mathfrak{A}^T \) is a \( * \)-subalgebra if there exists a faithful invariant state for \( T \). It is readily seen (the same proof as in [2]) that \( \mathfrak{A}^T \) is also a \( * \)-subalgebra if there exists a set of invariant states for \( T \) which separate the cone of the positive elements \( \mathfrak{A}_+ \). In general, \( \mathfrak{A}^T \) is not a \( * \)-subalgebra. At the same way, a Markov projection \( F : \mathfrak{A} \mapsto \mathfrak{A} \) is not necessarily a conditional expectation, see [4], Corollary 7.2.

A (discrete) \( C^* \)-dynamical system is a pair \( (\mathfrak{A}, T) \) consisting of a \( C^* \)-algebra and a Markov operator \( T \).

The following theorem is a generalization of Theorem 3.2 of [1] to our context, see also [10], Theorem 3.2 for similar results.

**Theorem 2.1.** Let \( (\mathfrak{A}, T) \) be a \( C^* \)-dynamical system. Then the following assertions are equivalent.

(i) Every bounded linear functional on \( \mathfrak{A}^T \) has a unique bounded, \( T \)-invariant linear extension to \( \mathfrak{A} \).
(ii) Every state on $\mathcal{A}^T$ has a unique bounded, $T$–invariant state extension to $\mathcal{A}$.

(iii) $\mathcal{A}^T + \{x - T(x) : x \in \mathcal{A}\} = \mathcal{A}$.

(iv) The ergodic averages $\frac{1}{n} \sum_{k=0}^{n-1} T^k$ converge pointwise in norm.

(v) The ergodic averages $\frac{1}{n} \sum_{k=0}^{n-1} T^k$ converge pointwise in the weak topology to a linear map $F : \mathcal{A} \mapsto \mathcal{A}$.

(vi) $\mathcal{A}^T + \{x - T(x) : x \in \mathcal{A}\} = \mathcal{A}$.

Furthermore, if one (and hence all) of the above statements holds, then there exists a unique Markov projection $F$ of $\mathcal{A}$ onto $\mathcal{A}^T$. It is given by

$$F(x) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k(x). \quad (2.1)$$

Proof. (iv)$\Rightarrow$(v): If the ergodic averages $\frac{1}{n} \sum_{k=0}^{n-1} T^k$ converge in norm, they define a linear map on $\mathcal{A}$ given in (2.1). Then (v) easily follows. We then prove (v)$\Rightarrow$(ii) as the remaining implications follow the same lines of Theorem 3.2 of [1].

It is readily seen that if there exists a linear map $F : \mathcal{A} \mapsto \mathcal{A}$ such that, for $x \in \mathcal{A}$,

$$w\lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) = F(x),$$

then $F$ is automatically a completely positive and identity–preserving, hence bounded. In addition, if $f \in \mathcal{A}^*$,

$$f(F(T(x))) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{k+1}(x)) = f(F(x)) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = f(T(F(x))).$$

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1. A state $\varphi$ on a operator system is a norm one positive linear functional. In this situation, $\|\varphi\| = \varphi(1)$.
2. It is seen in the proof that such a linear map $F$ is necessarily a Markov projection onto $\mathcal{A}^T$, satisfying $TF = F = FT$. 


This leads to $TF = F = FT$. Thus,

$$f(F^2(x)) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(F(x))) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(F(x)) \equiv f(F(x)),$$

that is $F^2 = F$. Now, if $x = F(x)$ then $T(x) = T(F(x)) = F(x) = x$. If $x = T(x)$ then

$$f(F(x)) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f(x),$$

that is $x = F(x)$. Namely, $F$ projects onto $\mathcal{A}^T$.

Let now $\varphi_j$, $j = 1, 2$ invariant state extensions of the state $\omega$ on $\mathcal{A}^T$. Then

$$\varphi_j(x) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_j(T^k(x)) = \varphi_j(F(x)) = \omega(F(x)),$$

that is any state on $\mathcal{A}^T$ has a unique invariant state extension on $\mathcal{A}$. □

**Definition 2.2.** The $C^*$-dynamical system $(\mathcal{A}, T)$ is said to be $F$–uniquely ergodic if one of the equivalent properties (i)–(vi) of Theorem 2.1 holds true.$^3$

The $C^*$-dynamical system $(\mathcal{A}, T)$ is said to be $F$–strictly weak mixing if there exists a linear map $F : \mathcal{A} \to \mathcal{A}$ such that

$$\lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(T^k(x)) - \varphi(F(x))| = 0 \quad (2.2)$$

whenever $\varphi \in S(\mathcal{A})$.

**Proposition 2.3.** If the $C^*$-dynamical system $(\mathcal{A}, T)$ is $F$–strictly weak mixing, then it is $F$–uniquely ergodic.

**Proof.** Let $\varphi \in S(\mathcal{A})$. We get

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} (\varphi(T^k(x)) - \varphi(F(x))) \right| \leq \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(T^k(x)) - \varphi(F(x))| \to 0$$

whenever $n \to +\infty$, as $(\mathcal{A}, T)$ is $F$–strictly weak mixing. By using the Jordan decomposition of bounded linear functionals (cf. [15]), we conclude that (v) of Theorem 2.1 is satisfied. □

**Remark 2.4.** By taking into account Proposition 2.3 and Theorem 2.1, the map $F$ in (2.2) is a Markov projection projecting onto $\mathcal{A}^T$.

$^3$In [1], the analogous property relative to $C^*$-dynamical systems based on automorphisms is denoted as unique ergodicity w.r.t. its fixed–point subalgebra.
In many interesting situations, the ergodic behavior of dynamical systems are connected with some spectral properties, see e.g. [3, 9, 11]. It is not possible to extend such results to the full generality. However, a $F$–strictly weak mixing map $T$ cannot have eigenvalues on the unit circle $\mathbb{T}$ except $z = 1$. Namely, for $z$ in $\mathbb{C}$ denote

$$\mathfrak{A}_z = \{ x \in \mathfrak{A} : T(x) = zx \}.$$ 

Of course, $\mathfrak{A}_1 = \mathfrak{A}^T$. Furthermore,

**Proposition 2.5.** Let $(\mathfrak{A}, T)$ be a $F$–strictly weak mixing $C^*$-dynamical system. Then $z \in \mathbb{T}\backslash\{1\}$ implies $\mathfrak{A}_z = \{0\}$.

**Proof.** Assume that $T(x_0) = zx_0$ for some $z \neq 1$. Then $F(x_0) = F(T(x_0)) = zF(x_0)$ which means $F(x_0) = 0$. In addition, from the $F$-strict weak mixing we infer

$$0 = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(T^k(x_0)) - \varphi(F(x_0))| = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} |z^k \varphi(x_0)|$$

$$= \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(x_0)| = |\varphi(x_0)|.$$ 

Namely, $\varphi(x_0) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{A})$, hence $x_0 = 0$. \hfill $\Box$

Finally, we recall the following result relative to the bounded sequences which are weakly mixing to zero. For the definitions of (lower) density of a subset of the natural numbers, or relatively dense sequences of natural numbers, we refer the reader to [17].

**Theorem 2.6.** (cf. [17], Theorem 2.3) Let $\{x_n\}_{n \geq 1}$ be a bounded sequence in the Banach space $X$. The following assertions are equivalent.

(i) $\lim_n \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} |f(x_k)| : f \in X^*, \|f\| \leq 1 \right\} = 0$.

(ii) $\lim_n \left\| \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\| = 0$ for each sequence $k_1 < k_2 < \cdots$ of strictly positive lower density.

(iii) $\lim_n \left\| \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\| = 0$ for each relatively dense sequence $k_1 < k_2 < \cdots$. 
3. STRICT WEAK MIXING OF LENGTH–PRESERVING AUTOMORPHISMS OF RD–GROUPS

In [1], it has been proved that some automorphisms of the reduced $C^*$–algebra of RD–groups are $E$–uniquely ergodic. Here, we prove that such automorphisms are $E$–strictly weak mixing.

Proposition 3.1. Let $\beta$ be a length–preserving automorphism of a RD–group $G$ for the length–function $L$, such that its orbits are infinite or singletons. Then the automorphism $\alpha$ induced by $\beta$ on $C_r^*(G)$ is $E$–strictly weak mixing.

Proof. Let $H := \{g \in G : \beta(g) = g\}$. As $\alpha$ is $E$–uniquely ergodic (cf. [1], Proposition 3.5), the pointwise limit in norm

$$E := \frac{1}{n} \sum_{k=1}^{n} \alpha^k$$

exists and gives rise a conditional expectation projecting onto the fixed–point algebra $C_r^*(H) \subset C_r^*(G)$. By a standard density argument, it is enough to prove that the sequence $\{\alpha^n(\lambda_g)\}_{n \geq 1}$ is weakly mixing to zero whenever $\beta(g) \neq g$, that is

$$\frac{1}{n} \sum_{k=1}^{n} |f(\alpha^k(\lambda_g))| \to 0$$

for each $f \in C_r^*(G)^*$. On the other hand (cf. [7]), for each sequence $\{k_j\}$ of natural numbers,

$$\lim_{n} \left\| \frac{1}{n} \sum_{j=1}^{n} \alpha^{k_j}(\lambda_g) \right\| \leq C(1 + L(g))^s \left\| \frac{1}{n} \sum_{j=1}^{n} \delta_{\beta^{k_j}(g)} \right\|_{L^1(G)}$$

$$\equiv \frac{C(1 + L(g))^s}{\sqrt{n}}.$$  

The assertion follows by Theorem 2.6

Finally, we have the case of the automorphism generated by the shift on the free group on infinitely many generators.

Corollary 3.2. Let $F_\infty$ be the free group on infinitely many generators $\{g_i\}_{i \in \mathbb{Z}}$. The automorphism $\alpha$ induced on $C_r^*(F_\infty)$ by the free shift of the generators is $E$–strictly weak mixing with $E = \tau(\cdot)\mathbb{1}$, $\tau$ being the canonical trace on $C_r^*(F_\infty)$.

$^4$The RD–groups are defined and studied in [7]. Notice that the RD–groups include the Gromov hyperbolic groups, see [6].

$^5$The $C^*$–dynamical system $\left( C_r^*(F_\infty), \alpha, \tau \right)$ is indeed strictly weak mixing in the language of [10].
Proof. By taking into account the Haagerup inequality (cf. [5], Lemma 1.4), the situation under consideration is a particular case of Proposition 3.1. □

4. STRICT WEAK MIXING OF THE FREE-SHIFT ON THE REDUCED FREE PRODUCT $C^*$–ALGEBRAS

The present section is devoted to show that the shift on the reduced amalgamated free product $C^*$–algebra $(A, \phi) = (\ast_B)_{i \in I}(A_i, \phi_i)$ is indeed $\phi$–strictly weak mixing, the last being stronger than unique ergodicity w.r.t. its fixed subalgebra, proven in [1]. Also this proof relies on the analogue of the Haagerup inequality proven in [1] (cf. Proposition 5.1).

We briefly recall some facts on the reduced amalgamated free product. For an exhaustive treatment of the subject see [1, 16].

Let $D$ be a unital $C^*$–algebra with identity $\mathds{1}$, and $E^D_B : D \mapsto B$ a conditional expectation onto the unital $C^*$–subalgebra $B$ with the same identity $\mathds{1}$. For each integer $i \in \mathds{Z}$ consider a copy $(A_i, \phi_i)$ of $(D, E^D_B)$, together with the reduced amalgamated free product

$$(A, \phi) = (\ast_B)_{i \in I}(A_i, \phi_i). \quad (4.1)$$

The $C^*$–algebra $A$ naturally acts on a Hilbert right $B$–module $E$ and it is generated by $\{\lambda^i_a : a \in A_i, i \in \mathds{Z}\}$, $\lambda^i$ being the embedding of $A_i$ in $\mathcal{B}_B(E)$, the space of all the bounded $B$–linear maps acting on $E$. The conditional expectation $\phi$ is given by

$$\phi(a) = \langle \mathds{1}a, \mathds{1} \rangle, \quad a \in A,$$

being the $B$–valued inner product of $E$ which is supposed be linear w.r.t. the first variable.\(^6\)

The free–shift automorphism $\alpha$ on $A$ is the automorphism of $A$ given by $\alpha(\lambda^i_a) = \lambda^{i+1}_a$ for all $a \in A$ and $i \in \mathds{Z}$.

**Theorem 4.1.** Let $\alpha$ be the free–shift automorphism on the reduced amalgamated free product $C^*$–algebra $A$ given in (4.1). Then $\alpha$ is $\phi$–strictly weakly mixing.

**Proof.** It was proven in [1] that $\alpha$ is $\phi$–uniquely ergodic, that is

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) = \phi(a)$$

for every $a \in A$. By a standard density argument, it is enough to prove that the sequence $\{\alpha^n(a)\}_{n \geq 1}$ is weakly mixing to zero whenever $a$ has the form $a = w$ for a word $w = \lambda^{m(1)}_{a_1} \lambda^{m(2)}_{a_2} \cdots \lambda^{m(p)}_{a_p}$, with $p \geq 1$.

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\(^6\)Relatively to the Hilbert $C^*$–modules, see e.g. [12].
\(a_i \in A^\circ_{m(i)}\), and \(m(i) \in \mathbb{Z}\) fulfilling \(m(i) \neq m(i + 1), \ i = 1, \ldots, p - 1\).
Here,

\[A_i^\circ := \{a - \phi_i(a) : a \in A_i\}, \quad i \in \mathbb{Z}.
\]

Let us take any increasing sequence \(\{k_j\} \subset \mathbb{N}\). Notice that

\[\alpha^{k}(w) = \lambda_{a_1}^{m(1)+k} \lambda_{a_2}^{m(2)+k} \cdots \lambda_{a_p}^{m(p)+k},
\]

that is \(\alpha^{k}(w)\) is a word satisfying the same properties as \(w\) with \(m'(i) = m(i) + k\).\(^7\)

Then we can apply the estimation in Proposition 5.1 of \([1]\) to the element

\[f := \sum_{j=1}^{n} \alpha^{k_j}(w)
\]

obtaining

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \alpha^{k_j}(w) \right\| \leq \frac{2p + 1}{n^{1/2}} \prod_{i=1}^{p} \|a_i\|.
\]

Now, by applying Theorem 2.6, we obtain the assertion. \(\square\)

ACKNOWLEDGMENTS

The second named–author (F. M.) thanks FCT (Portugal) grant SFRH/ BPD/17419/2004, and Prof. L. Accardi for kind hospitality from 18–22 June 2006 at “Università di Roma Tor Vergata”.

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\(^7\) This simply means that

\[\phi(w^* \alpha^{k}(w)) = \delta_{k,0}(2p + 1)^2 \prod_{i=1}^{p} \|a_i\|^2
\]

or equivalently, \(\{1 \alpha^{k}(w)\}_{k \in \mathbb{Z}} \subset E\) is an orthogonal set in the \(B\)–module \(E\).
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