The embedding dimension of Laplacian eigenfunction maps

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Abstract

Any closed, connected Riemannian manifold $M$ can be smoothly embedded by its Laplacian eigenfunction maps into $\mathbb{R}^m$ for some $m$. We call the smallest such $m$ the maximal embedding dimension of $M$. We show that the maximal embedding dimension of $M$ is bounded from above by a constant depending only on the dimension of $M$, a lower bound for injectivity radius, a lower bound for Ricci curvature, and a volume bound. We interpret this result for the case of surfaces isometrically immersed in $\mathbb{R}^3$, showing that the maximal embedding dimension only depends on bounds for the Gaussian curvature, mean curvature, and surface area. Furthermore, we consider the relevance of these results for shape registration.

Keywords: spectral embedding, eigenfunction embedding, eigenmap, diffusion map, global point signature, heat kernel embedding, shape registration, nonlinear dimensionality reduction, manifold learning

1. Introduction

Let $M = (M, g)$ be a closed (compact, without boundary), connected Riemannian manifold; we assume both $M$ and $g$ are smooth. The Laplacian of $M$ is a differential operator given by $\Delta := -\text{div} \circ \text{grad}$, where $\text{div}$ and $\text{grad}$ are the Riemannian divergence and gradient, respectively. Since $M$ is compact and connected, $\Delta$ has a discrete spectrum $\{\lambda_j\}_{j \in \mathbb{N}}$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$. We may choose an orthonormal basis for $L^2(M)$ of eigenfunctions $\{\varphi_j\}_{j \in \mathbb{N}}$ of $\Delta$, where $\Delta \varphi_j = \lambda_j \varphi_j$, $\varphi_j \in C^\infty(M)$, $\varphi_0 \equiv V(M)^{-1/2}$. Here, $V(M)$ denotes the volume of $M$ with respect to the canonical Riemannian measure $V = V_{(M,g)}$.

We consider maps of the form

$$\Phi^m : M \rightarrow \mathbb{R}^m$$
$$x \mapsto \{\varphi_j(x)\}_{1 \leq j \leq m}.$$  \hspace{1cm} (1)

If $\Phi^m : M \rightarrow \mathbb{R}^m$ happens to be a smooth embedding, then we call it an $m$-dimensional eigenfunction embedding of $M$. The smallest number $m$ for which $\Phi^m$ is an embedding for some choice of basis $\{\varphi_j\}_{j \in \mathbb{N}}$ will herein be called the embedding dimension of $M$, and the smallest number $m$ for
which $\Phi^m$ is an embedding for every choice of basis $\{\varphi_j\}_{j \in \mathbb{N}}$ will be called the maximal embedding dimension of $M$. Our aim is to establish a (qualitative) bound for the maximal embedding dimension of a given Riemannian manifold in terms of basic geometric data.

That finite eigenfunction maps of the form (1) yield smooth embeddings for large enough $m$ appears in a few papers in the spectral geometry literature. Abdallah [1] traces this fact back to Béard [2]. To our knowledge, the latest embedding result is given in Theorem 1.3 in Abdallah [1], who shows that when $(M, g(t))$ is a family of Riemannian manifolds with $g(t)$ analytic in a neighborhood of $t = 0$, then there are $\epsilon > 0, m \in \mathbb{N}$, and eigenfunctions $\{\varphi_j(t)\}_{1 \leq j \leq m}$ of $\Delta_{g(t)}$ such that

$$(M, g(t)) \rightarrow \mathbb{R}^m$$

$$x \mapsto \{\varphi_j(x; t)\}_{1 \leq j \leq m}$$

is an embedding for all $t \in (-\epsilon, \epsilon)$. The proof does not suggest how topology and geometry determine the embedding dimension, however.

Jones, Maggioni, and Schult [3, 4] have studied local properties of eigenfunction maps, and their results are essential to the proof of our main result. In particular, they show that at $z \in M$, for an appropriate choice of weights $a_1, \ldots, a_n \in \mathbb{R}$ and eigenfunctions $\varphi_j, \ldots, \varphi_n$, one has a coordinate chart $(U, \Phi_z)$ around $z \in M$, where $\Phi_z(x) := (a_1\varphi_j(x), \ldots, a_n\varphi_n(x))$, satisfying $\|\Phi_z(x) - \Phi_z(y)\|_{L^2} \sim d_M(x, y)$ for all $x, y \in U$. A more explicit statement of this result is given below.

Minor variants of such eigenfunction maps have been used in a variety of contexts. For example, spectral embeddings

$M \rightarrow \ell^2$

$$x \mapsto \{e^{-t^{1/2}\varphi_j(x)}\}_{j \in \mathbb{N}} \quad (t > 0)$$

have been used to embed closed Riemannian manifolds into the Hilbert space $\ell^2$ (i.e. square summable sequences with the usual inner product) in Béard, Besson, and Gallot [5, 6]; Fukaya [7]; Kasue and Kumura, e.g. [8, 9]; Kasue, Kumura, and Ogura [10]; Kasue, e.g. [11, 12]; and Abdallah [1].

Relatives of the eigenfunction maps, or a discrete counterpart, have been studied for data parametrization and dimensionality reduction, e.g. [13–18] for shape distances, e.g. [19–22]; and for shape registration, e.g. [23–29]. In particular, in the data analysis community, (1) is known as the eigenmap [13, 14] is known as the diffusion map [15, 16], and $x \mapsto \{1^{1/2}\varphi_j(x)\}$ is known as the global point signature [18]. These maps are all equivalent up to an invertible linear transformation. Hence, any embedding result applies to all of them. For an overview of spectral geometry in shape and data analysis, we refer the reader to Mémoli [22].

There seem to be no rules for choosing the number of eigenfunctions to use for a given application. While not all applications require an (injective) embedding of data, many eigenfunction-based shape registration methods do, e.g. [24–29], as we explain in Section 1.1 below. In the discrete setting one can write an algorithm to determine the smallest $m$ for which $\Phi^m : M \rightarrow \mathbb{R}^m$ is an embedding, although such an approach may become computationally intensive. For example, if $M$ is represented as a polyhedral surface, one may write an algorithm to check for self-intersections of polygon faces in the image $\Phi^m : M \rightarrow \mathbb{R}^m$. The fail-proof approach is to use all eigenfunctions, in which case one is assured an embedding. This approach is mentioned for point cloud data in Coifman and Lafon [16]. Specifically, they bound the maximal embedding dimension from above by the size of the full point sample. This becomes computationally...
demanding, however, especially in applications where one must solve an optimization problem over all eigenspaces, e.g. [21, 24, 25, 28], as we discuss in Section 1.1. Under the assumption that the shape or data is a sample drawn from some Riemannian manifold, we expect the embedding dimension of the sample to depend only on the topology and geometry of the manifold and the quality of the sample (e.g. covering radius). In this note we consider what topological and geometric data influence the embedding dimension of the underlying manifold.

The 3D image $\Phi^3 : M \to \mathbb{R}^3$ of a hippocampus is plotted in Figure 1. It is not clear from inspection whether the 3D image has self-intersections. To use the $N$-D image for registration as in [23–29], it would help to have an a priori estimate for the number of eigenfunctions necessary to embed the hippocampus by its eigenfunctions into Euclidean space. As the hippocampus is initially embedded in Euclidean space, the reason for re-embedding it by its eigenfunctions is geometric, as explained in Section 1.1 below. The 3D images $\Phi^3 : M \to \mathbb{R}^3$ of a few human model surfaces are plotted in Figure 2. From this figure, one may get a sense of why eigenfunction embeddings have been used to find point correspondences between shapes, as the arms and legs are better aligned in the image. The eigenfunctions in these examples are computed using the normalized graph Laplacian with Gaussian weights (cf. [30–32] and references therein).

We now recall some relevant notions from differential geometry. Let $M, M'$ be smooth manifolds. A smooth map $F : M \to M'$ is called an immersion if rank $dF_x = \dim M$ for every $x \in M$. A smooth map $F : M \to M'$ is called a (smooth) embedding if $F$ is an immersion and a homeomorphism onto its image $F(M)$. Recall that for a compact manifold $M$, if $F : M \to M'$ is
an injective immersion, then it is a smooth embedding.

Suppose now that \( M = (M, g) \) and \( M' = (M', g') \) are Riemannian manifolds. We write the corresponding geodesic distance metrics as \( d_M \) and \( d_{M'} \). For \( M \) and \( M' \) to be isometric means that there is a diffeomorphism \( F : M \to M' \) such that \( F^*g' = g \). Such a map \( F : M \to M' \) is called an isometry. In particular, if \( F : M \to M' \) is an isometry, then \( d_M(x, y) = d_{M'}(F(x), F(y)) \) for all \( x, y \in M \).

Let \( M = (M, g) \) be a complete \( n \)-dimensional Riemannian manifold. Herein, \( B(x, r) \) will denote the geodesic ball of radius \( r \) centered at \( x \in M \), and \( B(r) \) will denote the Euclidean ball of radius \( r \) centered at the origin of \( \mathbb{R}^n \). As \( M \) is complete, the domain of the exponential map is \( T_xM \cong \mathbb{R}^n \), i.e. \( \exp_x : \mathbb{R}^n \to M \). The injectivity radius of \( M \), denoted \( \text{inj}(M) \), is the largest number for which the restriction \( \exp_x : B(r) \subseteq \mathbb{R}^n \to B(x, r) \) is a diffeomorphism for all \( x \in M \), \( r \leq \text{inj}(M) \).

Let \( x \in M \), and let \( P \) be a 2-plane in \( T_xM \). The circle of radius \( r < \text{inj}(M) \) centered at \( 0 \) in \( P \) is mapped by \( \exp_x : \mathbb{R}^n \to M \) to the geodesic circle \( C_P(r) \), whose length we denote \( l_p(r) \). Then

\[
    l_p(r) = 2\pi r (1 - \frac{r^2}{6} K(P) + O(r^3)) \quad \text{as} \quad r \to 0^+.
\]

The number \( K(P) \) is called the sectional curvature of \( P \). If \( \dim M = 2 \), then \( K(x) = K(T_xM) \) is equivalent to the Gaussian curvature at \( x \).

Next, we use \( V \) to denote the canonical Riemannian measure associated with \((M, g)\). Let \( x \in M \). The pulled-back measure \( \exp_x^*(V) \) has a density with respect to the Lebesgue measure in \( T_xM \cong \mathbb{R}^n \). Let \((r, u) \in [0, \infty) \times S^{n-1} \) be polar coordinates in \( T_xM \). For \( r < \text{inj}(M) \), we may write \( \exp_x^*(V) = \theta_x(r, u) dr du \). Then

\[
    \theta_x(r, u) = r^{n-1} (1 - \frac{r^2}{6} \text{Ric}_x(u, u) + O(r^3)) \quad \text{as} \quad r \to 0^+.
\]

The term \( \text{Ric}_x(u, u) \) is a quadratic form in \( u \), whose associated symmetric bilinear form is called the Ricci curvature at \( x \). If \( \dim M = 2 \), then \( \text{Ric}_x(u, u) = K(x)g(u, u) \), where \( K(x) \) is the Gaussian curvature at \( x \).

Heat flow on a closed Riemannian manifold \((M, g)\) is modeled by the heat equation

\[
    (\partial_t + \Delta) u(t, x) = 0,
\]

where \( \Delta \) is the Laplacian of \( M \) applied to \( x \in M \). Any initial distribution \( f \in L^2(M) \) determines a unique smooth solution \( u(t, x), t > 0 \), to (6) such that \( u_t \to f \) as \( t \to 0^+ \). This solution is given by

\[
    u(t, x) = \int_M p(t, x, y) f(y) dV(y),
\]

where \( p \in C^\infty(\mathbb{R}^+ \times M \times M) \) is called the heat kernel of \( M \). For example, the heat kernel of \( \mathbb{R}^n \) (with Euclidean metric) is the familiar Gaussian kernel. Lastly, the heat kernel may be expressed in the eigenvalues-functions as

\[
    p(t, x, y) = \sum_{j=0}^\infty e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).
\]

For more on the Laplacian, heat kernel, and Riemannian geometry, we refer the reader to, e.g., [33,36].
We are now ready to state the results of this note. Let $\kappa_0 \geq 0, i_0 > 0$ be fixed constants, $n \geq 2,$ and consider the class of closed, connected $n$-dimensional Riemannian manifolds
\[
\mathcal{M} := \{ (M, g) \mid \text{dim} M = n, \ \text{Ric}_M \geq -(n-1)\kappa_0 g, \ \text{inj}(M) \geq i_0, \ V(M) = 1 \}.
\]
Note that $\text{Ric}_M \geq -(n-1)\kappa_0 g$ means
\[
\text{Ric}(\xi, \xi) \geq -(n-1)\kappa_0 g(\xi, \xi) \quad (\forall \xi \in TM).
\]
If $M$ is a surface and $K$ denotes its Gaussian curvature, then $\text{Ric}_M \geq -(n-1)\kappa_0 g$ is equivalent to $K \geq -\kappa_0$.

Note that the following Theorems 1, 2, and 3 are independent of the choice of eigenfunction basis. We first show that the eigenfunction maps $\Phi^m$ are well-controlled immersions in the sense that the neighborhoods on which they are embeddings cannot be too small.

**Theorem 1.** There is a positive integer $m$ and constant $\epsilon > 0$ such that, for any $M \in \mathcal{M}$, for all $z \in M$,
\[
\Phi^m_z : B(z, \epsilon) \to \mathbb{R}^m
\]
\[
x \mapsto (\varphi_1(x), \ldots, \varphi_m(x))
\]
is a smooth embedding.

The proofs are deferred to the sections following. Our main goal is to prove the following result.

**Theorem 2 (Uniform maximal embedding dimension).** There is a positive integer $d$ such that, for all $M \in \mathcal{M}$,
\[
\Phi^d : M \to \mathbb{R}^d
\]
\[
x \mapsto (\varphi_1(x), \ldots, \varphi_d(x))
\]
is a smooth embedding.

We lastly consider closed, connected surfaces isometrically immersed in $\mathbb{R}^3$. We denote mean curvature by $H$, Gaussian curvature by $K$, and surface area by $V$. Let $H_0, \kappa_0, A$ be fixed positive constants and consider the class of surfaces
\[
\mathcal{S} := \{ (M, g) \mid \text{dim} M = 2, |K| \leq \kappa_0, |H| \leq H_0, V(M) \leq A, \xi : M \to \mathbb{R}^3 \text{ is an isometric immersion} \}.
\]

**Theorem 3 (Uniform maximal embedding dimension for surfaces).** There is a positive integer $d$ such that, for all $M \in \mathcal{S}$,
\[
\Phi^d : M \to \mathbb{R}^d
\]
\[
x \mapsto (\varphi_1(x), \ldots, \varphi_d(x))
\]
is a smooth embedding.
Before continuing, we consider the natural question of whether the eigenfunction maps are stable under perturbations of the metric. This has been answered in [6].

**Theorem 4** (Bérard-Besson-Gallot [6]). Let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold, \(\epsilon_0 > 0\), and \(m \in \mathbb{N}\). Let \(g'\) be any metric on \(M\) such that \((1 - \epsilon)g \leq g' \leq (1 + \epsilon)g\), \(\epsilon \in [0, \epsilon_0)\). We assume that all metrics under consideration satisfy \(\text{Ric}(M, g') \geq -(n - 1)\kappa_0 g'\) for some constant \(\kappa_0 \geq 0\). There exist constants \(\eta_{\epsilon, j, \alpha}(\epsilon)\), \(1 \leq j \leq m\), which go to 0 with \(\epsilon\), such that to any orthonormal basis \(\{\varphi_j'\}\) of eigenfunctions of \(\Delta_{g'}\) one can associate an orthonormal basis \(\{\varphi_j\}\) of eigenfunctions of \(\Delta_g\) satisfying \(\|\varphi_j' - \varphi_j\|_{L^\infty} \leq \eta_{\epsilon, j, \alpha}(\epsilon)\) for \(j \leq m\).

### 1.1. Motivations from eigenfunction-based shape registration methods

Here we consider the significance of a uniform maximal embedding dimension from the perspective of the shape registration methods in [24–29]. In shape registration, we begin with two closed, connected Riemannian manifolds \(M = (M, g)\) and \(M' = (M', g')\), and our goal is to find a correspondence between them given by \(\alpha : M \to M'\). (Note some use a looser notion of correspondence, e.g. [22], allowing for many-many matches between points of the “shapes”.) Moreover, if \(M\) and \(M'\) are isometric, we require the correspondence \(\alpha : M \to M'\) to be an isometry. This correspondence may be established using eigenfunction maps, followed by closest point matching as follows. Here we must be precise regarding the choice of eigenfunction basis, and we let \(\mathcal{B}(M)\) denote the set of orthonormal bases of real eigenfunctions of the Laplacian of \(M\). For \(m \in \mathbb{N}\) and \(b \in \mathcal{B}(M)\), \(b = \{\varphi_j^b\}_{j \in \mathbb{N}}\), let \(\Phi^b_m\) denote the corresponding eigenfunction map, i.e. \(x \mapsto \{\varphi_j^b(x)\}_{j \in \mathbb{N}}\). Given \(b \in \mathcal{B}(M), b' \in \mathcal{B}(M')\), and \(m \in \mathbb{N}\), we consider as a potential correspondence the map \(\alpha(b, b', m) : M \to M'\) given by

\[
\alpha(x; b, b', m) := \arg \inf_{x' \in M'} \|\Phi^b_m(x') - \Phi^{b'}_m(x)\|_{L^\infty},
\]

ties being broken arbitrarily. We first consider the sense in which \(\alpha\) yields the desired correspondence for isometric shapes, and then the sense in which \(\alpha\) is stable.

**Proposition 1.** If \(M\) and \(M'\) are isometric and \(m \geq \max\\{n, m'\}\), then \(\alpha(b, b', m) : M \to M'\) is an isometry for some choice of \(b \in \mathcal{B}(M), b' \in \mathcal{B}(M')\).

**Proof.** Let \(F : M \to M'\) be an isometry, and let \(m \geq \max\\{n, m'\}\) be the maximal embedding dimensions of \(M\) and \(M'\). Note that there are \(b \in \mathcal{B}(M), b' \in \mathcal{B}(M')\) such that \(\varphi_j^b = \varphi_j^{b'} \circ F\) for all \(j \in \mathbb{N}\) (cf. [24]). In particular, \(\Phi^b_m(x) = \Phi^{b'}_m(F(x))\) for all \(x \in M\). Since \(\Phi^b_m\) is injective (as it is an embedding), the infimum in (12) is uniquely realized for each \(x \in M\). Hence \(\alpha(b, b', m) = F\).

Now let \(M = (M, g, \epsilon_0 > 0, g_\epsilon, \epsilon \in [0, \epsilon_0)\) be the family of Riemannian metrics on \(M\) such that \((1 - \epsilon)g \leq g_\epsilon \leq (1 + \epsilon)g\) for all \(\epsilon \in [0, \epsilon_0)\). We assume that there exist \(\epsilon_0 \geq 0, \kappa_0 > 0\) for which, with \(M\) as defined in (9), \(M_\epsilon := (M, g_\epsilon) \in M\) for all \(\epsilon \in (0, \epsilon_0)\). For each \(\epsilon \in [0, \epsilon_0)\), let \(b'_\epsilon \in \mathcal{B}(M_\epsilon)\) be arbitrary. The following proposition is an immediate consequence of Theorem 4, the triangle inequality, and the definition of \(\alpha\).

**Proposition 2.** Let \(m \in \mathbb{N}\). There is a constant \(\eta_m(\epsilon)\), which goes to 0 with \(\epsilon\), and \(b : \epsilon \in [0, \epsilon_0) \mapsto b_\epsilon \in \mathcal{B}(M)\) such that, for all \(\epsilon \in [0, \epsilon_0)\),

\[
\sup_{x \in M} \|\Phi^{b_\epsilon}_m(\alpha(x; b_\epsilon, b'_\epsilon, m)) - \Phi^{b'_\epsilon}_m(x)\|_{L^\infty} \leq \eta_m(\epsilon),
\]

where \(\alpha(b_\epsilon, b'_\epsilon, m)\) is defined as in (12).
The size of the search space of potential correspondences \( \{ \sigma(b,b',\mathcal{M}) | b \in \mathcal{B}(\mathcal{M}), b' \in \mathcal{B}(\mathcal{M}') \} \) grows at least exponentially in \( m \). To see this, note that we may arbitrarily flip the sign of any eigenfunction, and so \( |\{ \Phi^m_b | b \in \mathcal{B}(\mathcal{M}) \}| \geq 2^m \). Consequently, to find the isometry asserted by Proposition[1] with minimal computational demands, it would be useful to know the maximal embedding dimensions of \( \mathcal{M} \) and \( \mathcal{M}' \).

1.2. Examples: the embedding dimensions of the sphere and stretched torus

We now compute the embedding dimensions of the standard sphere and a “stretched torus” using formulas for their eigenfunctions. One usually cannot derive the embedding dimension in this way, however, as, to paraphrase from [37], there are only a few Riemannian manifolds for which we have explicit formulas for the eigenfunctions.

Identifying the standard sphere \( S^n = (S^n, \text{can}) \) with the Riemannian submanifold

\[
\begin{aligned}
\left\{ (x^1, \ldots, x^{n+1}) \mid \|x\|_{\mathbb{R}^{n+1}} = 1 \right\}
\end{aligned}
\]

of \( \mathbb{R}^{n+1} \), the eigenfunctions of \( \Delta_{S^n} \) are restrictions of harmonic homogeneous polynomials on \( \mathbb{R}^{n+1} \) [34, 37]. A polynomial \( P(x^1, \ldots, x^{n+1}) \) on \( \mathbb{R}^{n+1} \) is called (1) homogeneous (of degree \( k \)) if \( P(rx) = r^k P(x) \) and (2) harmonic if \( \Delta_{\mathbb{R}^{n+1}} P(x) = 0 \). Moreover, if \( P(x) \) is a harmonic homogeneous polynomial of degree \( k \), then its corresponding eigenvalue is \( \lambda = k(n+k-1) \), whose multiplicity is

\[
\frac{n+k}{k} - \frac{n+k-2}{k-2}.
\]

One may show that an \( L^2(S^n) \)-orthogonal basis of the eigenspace corresponding to \( \lambda(S^n) = n \) is given by the restriction of the coordinate functions \( x^1, \ldots, x^{n+1} \) on \( \mathbb{R}^{n+1} \) to \( S^n \) (cf. Proposition 1, p. 35, [34]). We immediately have

**Proposition 3.** The embedding dimension of \( S^n \) is \( d = n + 1 \).

Although we get an explicit answer for the sphere, it does not reveal how geometry influences the embedding dimension. Let us look at another space.

Explicit formulas are also available for the eigenfunctions of products of spheres, e.g. tori, by virtue of the decomposition \( \Delta_{M \times N} = \Delta_M + \Delta_N \). We consider stretching a flat torus to have a given injectivity radius and volume, and then explicitly compute the embedding dimension. We see that the embedding dimension depends on both injectivity radius and volume, and thus cannot be bounded using only curvature and volume bounds, or curvature and injectivity radius bounds. In particular, let \( 0 < a < b \), \( n \geq 2 \), and consider the flat \( n \)-torus \( T \) constructed by gluing the rectangle

\[
\begin{aligned}
\left\{ (x^1, \ldots, x^n) \mid 0 \leq x^j \leq a \ (j \neq n), \ 0 \leq x^n \leq b \right\}
\end{aligned}
\]

as usual. Note \( \text{Ric}_T = 0 \), \( \text{inj}(T) = a/2 \), and \( \text{V}(T) = a^{n-1} b \).

**Proposition 4.** The embedding dimension of \( T \) is

\[
\begin{aligned}
d = 2([a^{-1}b] + n - 2) \\
\geq 2^{1-n} \frac{\text{V}(T)}{\text{inj}(T)^n},
\end{aligned}
\]

where \( [x] \) is the smallest integer greater than or equal to \( x \).
Remark 1.

Proof. Put \( f_1(x) := \cos(2\pi x), f_2(x) := \sin(2\pi x) \). The unnormalized real eigenfunctions of \( T \) are

\[
f_k(a^{-1}m_1x) \cdots f_k(a^{-1}m_{n-1}x^{n-1})f_k(b^{-1}m_nx^n) \quad (m_i \in \mathbb{N}, k_i \in \{1, 2\}),
\]

(18)

with corresponding eigenvalues

\[
\lambda(m_1, \ldots, m_n) = (2\pi)^2(a^{-2}m_1^2 + \cdots + a^{-2}m_{n-1}^2 + b^{-2}m_n^2).
\]

(19)

We denote \( \lambda(m, j) = \lambda(0, \ldots, m, \ldots, 0) \).

First, suppose \( a^{-1}b \) is not an integer, and put \( p := \lfloor a^{-1}b \rfloor \). One may check that the initial sequence of eigenvalues corresponds to

\[
0 < \lambda(1, n) < \lambda(2, n) < \cdots < \lambda(p, n)
\]

\[
< \lambda(1, 1) = \lambda(1, 2) = \cdots = \lambda(1, n-1)
\]

(20)

\[
< \cdots
\]

The eigenvalues \( \lambda(k, n), k \leq p, \) each have multiplicity 2; for example, the eigenspace corresponding to \( \lambda(k, n) \) has as a basis \( \{f_1(b^{-1}kx^n), f_2(b^{-1}kx^n)\} \). It follows that \( \Phi^{2p} : T \to \mathbb{R}^{2p} \) depends only on \( x^n \). It is readily verified that \( x^n \mapsto \Phi^{\lambda}(x) \) is injective since, up to phase, \( \Phi^{\lambda}(x) = (f_1(b^{-1}kx^n), f_2(b^{-1}kx^n)) \). Thus \( x^n \mapsto \Phi^{\lambda}(x) \) is injective. Put \( F(x^n) = (f_1(a^{-1}x^n), f_2(a^{-1}x^n)) \). Then, up to phase and up to a permutation of the last \( 2(n-1) \) coordinates,

\[
\Phi^{2p+2(n-1)}(x) = (\Phi^{2p}(x), F(x^2), F(x^3), \ldots, F(x^{n-1})).
\]

(21)

Noting \( x^i \mapsto F(x^i) \) is an embedding of \( [0, a]/(0 \sim a) \) into \( \mathbb{R}^2 \), we deduce that \( \Phi^{2p+2(n-1)} : T \to \mathbb{R}^{2p+2(n-1)} \) is an embedding and, furthermore, that if any one of the last \( 2(n-1) \) coordinates are removed, then the map is no longer injective. It follows that \( d = 2p + 2(n-1) = 2\lfloor a^{-1}b \rfloor + n-2 \) is the embedding dimension of \( T \) when \( a^{-1}b \) is not an integer.

Now suppose that \( a^{-1}b \) is an integer; put \( p := a^{-1}b \). One may check that the initial sequence of eigenvalues is

\[
0 < \lambda(1, n) < \cdots < \lambda(p-1, n)
\]

\[
< (2\pi)^2a^{-2} = \lambda(1, 1) = \cdots = \lambda(1, n-1) = \lambda(p, n)
\]

(22)

\[
< \cdots
\]

Following the preceding arguments, we see that \( \Phi^{2p+2(n-1)} : T \to \mathbb{R}^{2p+2(n-1)} \) is an embedding when the eigenfunctions are ordered according to the sequence suggested by (22), where the two eigenfunctions corresponding to \( \lambda(p, n) \) are not included as coordinates. \( \square \)

Remark 1. Note the stretched torus example shows that the embedding dimension of \( \mathcal{M}(n, k_0, i_0) \) is bounded below by \( 2^{1-n} \).

2. Proof of Theorem 1

We first show that the manifolds of \( \mathcal{M} \) have uniformly bounded diameter. That is, there is a \( D > 0 \) such that diameter \( d(M) \leq D \) for all \( M \in \mathcal{M} \). Recall \( d(M) := \sup_{x,y \in M} d_M(x, y) \).

To see this, let \( M \in \mathcal{M} \). By the Theorem of Hopf-Rinow, we may take a unit speed geodesic \( \gamma : \mathbb{R} \to M \) that realizes the diameter, say, \( d(\gamma(0), \gamma(d(M))) = d(M) \). Stack geodesic balls of
radius \( i_0 / 2 \) end-to-end along \( \gamma \). It is a simple exercise in proof by contradiction to show these balls are disjoint. The volumes of these balls are uniformly bounded below by Croke’s estimate (see below). Finally, the volume requirement \( V(M) = 1 \) limits the number of such balls, hence the diameter of \( M \).

We now recall a few function norms (cf., e.g., [38]). Let \( \Omega \subseteq \mathbb{R}^n \) be open, \( 0 < \alpha \leq 1 \), \( k \) a nonnegative integer, \( 1 \leq p < \infty \). In this note, the following norms and seminorms will be used with a smooth function \( f : \Omega \to \mathbb{R} \). We write

\[
\| f \|_{C^\alpha(\Omega)} := \sup_{x \in \Omega} |f(x)| (23)
\]

\[
[ f ]_{C^\alpha(\Omega)} := \sup_{x, y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^{\alpha}_{\mathbb{R}^n}} (24)
\]

\[
\| f \|_{C^\alpha(\Omega)} := \| f \|_{C^0(\Omega)} + [ f ]_{C^\alpha(\Omega)} (25)
\]

\[
\| f \|_{W^{1,p}(\Omega)} := \left( \sum_{i | i \leq k} \int_{\Omega} |D^i f|^p \, dx \right)^{1/p} . (26)
\]

Theorem 5 is an adaptation of the following local embedding result.

**Theorem 5** (Jones-Maggioni-Schul [3]; see also [4]). Assume \( V(M) = 1 \). Let \( z \in M \) and suppose \( u : U \to \mathbb{R}^n \) is a chart satisfying the following properties.

There exist positive constants \( r, C_1, C_2 \) such that

1. \( u(z) = 0 \);
2. \( u(U) = B \), where \( B := B(r) \) is the ball of radius \( r \) in \( \mathbb{R}^n \) centered at the origin;
3. for some \( \alpha > 0 \), the coefficients \( g^{ij}(u) = g(du^i, du^j) \) of the metric inverse satisfy \( g^{ij}(0) = \bar{g}^{ij} \) and are controlled in the \( C^\alpha \) topology on \( B \):

\[
C_1^{-1} \| \xi \|_{\mathbb{R}^n}^2 \leq \sum_{i,j} \xi_i \xi_j g^{ij}(u) \leq C_1 \| \xi \|_{\mathbb{R}^n}^2 \quad (\forall u \in B, \forall \xi \in \mathbb{R}^n); (27)
\]

\[
\| g^{ij} \|_{C^\alpha} \leq C_2 \quad (\forall i, j). (28)
\]

Then there are constants \( \nu = \nu(n, C_1, C_2) > 1, a_j > 0, j = 1, \ldots, n, \) and integers \( j_1, \ldots, j_n \) such that the following hold.

(a) The map

\[
\Phi_u : B(z, \nu^{-1} r) \to \mathbb{R}^n
\]

\[
x \mapsto (a_{j_1} \varphi_{j_1}(x), \ldots, a_{j_n} \varphi_{j_n}(x))
\]

satisfies, for all \( x, y \in B(z, \nu^{-1} r) \),

\[
\frac{\nu^{-1}}{r} \, d_M(x, y) \leq \| \Phi_u(x) - \Phi_u(y) \|_{\mathbb{R}^n} \leq \frac{\nu}{r} \, d_M(x, y) ; (29)
\]

(b) the associated eigenvalues satisfy \( \nu^{-1} r^2 \leq \lambda_{j_1}, \ldots, \lambda_{j_n} \leq \nu r^2 \).

We point out that this result (Theorem 2.2.1 in [4]) is stated for \( g \in C^n, \alpha > 0, \) and \( M \) possibly having a boundary. We now invoke an eigenvalue bound to use with (b) in Theorem 5.

**Theorem 6** (Bérard-Besson-Gallot [6]). Let \( M \) be a closed, connected Riemannian manifold such that \( \dim M = n \), \( \text{Ric}_M \geq -(n - 1)g \), and \( d(M) \leq D \). There is a constant \( C_1 = C_1(n, \kappa_0, D) \) such that

\[
C_1 \, f^{2/n} \leq \lambda_j(M) \quad (\forall j \geq 0).
\]
Finally, we must choose a coordinate system satisfying the hypotheses of Theorem 5. We use harmonic coordinates. By definition, a coordinate chart \((U, x')\) of \(M = (M^n, g)\) is harmonic if \(\Delta_M x^i = 0\) on \(U\) for \(i = 1, \ldots, n\) (cf., e.g., [39] [40]). All necessary properties of harmonic coordinates for this note are contained in the following result, which follows from the proof of Theorem 0.3 in Anderson-Cheeger [41].

**Lemma 1.** Let \(\kappa_0 \geq 0\) and \(i_0 > 0\), let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold satisfying

\[
\text{Ric}_M \geq -(n-1)\kappa_0 g, \quad \text{inj}(M) \geq i_0,
\]

and let \(\alpha \in (0, 1)\) and \(Q > 1\) be fixed. Then there exist constants \(r_0, C_h, \) both depending only on \(n, \kappa_0, i_0, \alpha, Q\), such that for all \(z \in M\) there is a harmonic coordinate chart \(u : U \rightarrow \mathbb{R}^n\) satisfying

1. \(u(z) = 0\);
2. \(u(U) = B\), where \(B := B(r_0)\) is the ball of radius \(r_0\) in \(\mathbb{R}^n\) centered at the origin;
3. the coefficients \(g^{ij}(u) = g(du^i, du^j)\) of the metric inverse satisfy \(g^{ij}(0) = \delta^{ij}\) and are controlled in the \(C^\alpha\) topology on \(B\):

\[
Q^{-1}||\xi||_{C^\alpha}^2 \leq \sum_{ij} \xi_i \xi_j g^{ij}(u) \leq Q||\xi||_{C^\alpha}^2, \quad (\forall u \in B, \forall \xi \in \mathbb{R}^n); \quad (31)
\]

\[
|g^{ij}|_{C^\alpha} \leq C_h \quad (\forall i, j). \quad (32)
\]

In deriving Lemma 1 we will use the following Sobolev-type estimate (cf. Theorem 5.6.5 in Evans [38]).

**Proposition 5** (Morrey’s inequality). Let \(\Omega \subset \mathbb{R}^n\) be open, bounded, and with \(C^1\) boundary. Assume \(p > n\) and \(u \in W^{1, p}(\Omega)\) is continuous. Then \(u \in C^\alpha(\overline{\Omega})\), for \(\alpha = 1 - n/p\), with

\[
||u||_{C^\alpha(\overline{\Omega})} \leq C||u||_{W^{1, p}(\Omega)}, \quad (33)
\]

where \(C\) is a constant depending only on \(n, \alpha, \Omega\).

**Proof of Lemma 1**. Theorem 0.3 in Anderson and Cheeger [41] asserts that under the given hypotheses there is a harmonic coordinate chart \(u : B(z, r_0) \rightarrow \mathbb{R}^n\), \(E := u(B(z, r_0))\), such that

1. \(u(z) = 0\);
2. \(r_0 = r_0(n, \kappa_0, i_0, \alpha, Q)\);
3. the coefficients \(g_{ij}(u) = g(\frac{\partial u^i}{\partial x^j}, \frac{\partial u^j}{\partial x^i})\) of the Riemannian metric satisfy \(g_{ij}(0) = \delta_{ij}\) and, with \(p\) defined by \(\alpha = 1 - n/p\),

\[
Q^{-1}||\xi||_{C^\alpha}^2 \leq \sum_{ij} \xi_i \xi_j g_{ij}(u) \leq Q||\xi||_{C^\alpha}^2, \quad (\forall u \in E, \forall \xi \in \mathbb{R}^n); \quad (34)
\]

\[
r_0^p||\nabla g_{ij}||_{L^p(E)} \leq Q - 1 \quad (\forall i, j). \quad (35)
\]

First, we put \(r_0 := r_0/\sqrt{Q}\) and show that \(B = B(r_0) \subseteq E\). Fix a unit vector \(v \in \mathbb{R}^n\), and put \(\gamma(t) = tv\). Note \(||\gamma'(\gamma(0))||_{C^\alpha} \leq Q\) by (34). Let \(L(\cdot)\) denote the length function on curves in \(M\). Then \(t < r_0/\sqrt{Q}\) implies \(d_L(\gamma(0), \gamma(t)) \leq L(\gamma|_{[0,t]}) \leq t \sqrt{Q} < r_0\).

Second, by Morrey’s inequality, there is a constant \(C = C(n, \alpha, r_0)\) for which \(||g_{ij}||_{C^\alpha(B)} \leq C||g_{ij}||_{W^{1, p}(B)}\). Then, by (34) and (35), there is a constant \(C = C(n, \alpha, r_0, Q)\) such that \(||g_{ij}||_{C^\alpha(B)} \leq ||g_{ij}||_{C^\alpha(B)} \leq C\) for all \(i, j\).

Third, note that bounds (34) and (31) on the metric and its inverse are equivalent.
Fourth, we show that $[g^{ij}]_{C^1}$ is bounded. For $x, y \in B$, put $A := (g_{ij}(x))$ and $B := (g_{ij}(y))$. We use $\| \cdot \|_2$ to denote the induced 2-norm on matrices in $\mathbb{R}^{n \times n}$, and $\| \cdot \|_{\max}$ to denote the largest magnitude over entries of a matrix in $\mathbb{R}^{n \times n}$. Note $\| \cdot \|_{\max} \leq \| \cdot \|_2 \leq n \| \cdot \|_{\max}$, $\|A^{-1}\|_2 \leq Q$, $\|B^{-1}\|_2 \leq Q$, and $A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$. Hence

$$
|g^{ij}(x) - g^{ij}(y)| \leq \|A^{-1} - B^{-1}\|_{\max}
$$

(36)

$$
\leq \|A^{-1} - B^{-1}\|_2
$$

(37)

$$
= \|A^{-1}(A - B)B^{-1}\|_2
$$

(38)

$$
\leq \|A^{-1}\|_2 \|A - B\|_2 \|B^{-1}\|_2
$$

(39)

$$
\leq nQ^2 \cdot \max_{ij}|g_{ij}(x) - g_{ij}(y)|
$$

(40)

It follows that $[g^{ij}]_{C^\alpha} \leq nQ^2 C$ for all $i, j$. \hfill $\square$

Using harmonic coordinates and the eigenvalue bound with Theorem 5, we finish the proof.

**Proof of Theorem 2** Fix $Q > 1$ and $\alpha < 1$. Our choice of $n, \kappa_0, i_0$ then fixes the constants $r_b, n$ for harmonic coordinates. Use harmonic coordinates in Theorem $5$ with $M = M$ and $B = B$, respectively. Let $m + 1$ be the smallest integer such that $C_3(m + 1)^{2/n} > \nu r_b^2$. Now, for any $M \in \mathcal{M}, \lambda_{m+1}(M) \geq C_3(m+1)^{2/n} \nu r_b^2$. It follows from Theorem 5 that $\Phi_\varepsilon^m(M) : B(\varepsilon, \delta) \to \mathbb{R}^m$ is an embedding with $\varepsilon = r_b^{-1} r_b$.

\hfill $\square$

### 3. Proof of Theorem 2

The proof of Theorem 2 builds on Theorem 1, extending injectivity to the whole manifold via heat kernel estimates. In particular, a Gaussian bound for the heat kernel will be extended to the partial sum

$$
p^k(t, x, y) := \sum_{j=0}^k e^{-\lambda_j t} \varphi_j(x) \varphi_j(y)
$$

(41)

through a universal bound for the remainder term.

#### 3.1. Off-diagonal Gaussian upper bound for the heat kernel

**Theorem 7** (Li-Yau [42]). Let $M$ be a complete $n$-dimensional Riemannian manifold without boundary and with $\text{Ric}_M \geq -(n-1)\kappa_0 g$ ($\kappa_0 \geq 0$). Put $V_t(r) = V(B(x, r))$. Then, for $0 < \delta < 1$, the heat kernel satisfies

$$
p(t, x, y) \leq \frac{C(n, \delta)}{V_t^{1/2}(\sqrt{t}) V_y^{1/2}(\sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{4 + \delta} + C(n)\kappa_0 t \right\}
$$

for all $t > 0$ and $x, y \in M$. Moreover, $C(n, \delta) \to \infty$ as $\delta \to 0$.

**Theorem 8** (Croke [43]). Let $M$ be an $n$-dimensional Riemannian manifold. Then there is a constant $C_n$ depending only on $n$ such that, for all $x \in M$, for all $r \leq \frac{1}{2} \text{inj}(M)$,

$$
V(B(x, r)) \geq C_n r^n.
$$
Corollary 1. There is a constant \( C_U > 0 \) such that, for any \( M \in \mathcal{M} \), for any \( x, y \in M \), for any \( t \in (0, i_0/2] \),

\[
p(t, x, y; M) \leq \frac{C_U}{p^{n/2}} \exp \left\{- \frac{d^2(x, y)}{(4 + \delta)t} \right\},
\]

where \( 0 < \delta < 1 \).

Proof. Put \( \delta = 1/2 \). Applying Croke’s estimate to the Li-Yau heat kernel bound,

\[
p(t, x, y; M) \leq \frac{C(n, \delta)}{C_n p^{n/2}} \exp \left\{- \frac{d^2(x, y)}{(4 + \delta)t} + C(n)\kappa_0 i_0/2 \right\}.
\]

\( \square \)

3.2. Truncating the heat kernel sum

We consider control over \( M \in \mathcal{M} \) of the remainder term

\[
R_k(t; M) := \sup_{x \in \mathcal{M}} \sum_{j \geq k} e^{-s/2} \varphi_j^2(x).
\]

Lemma 2. For all \( k \in \mathbb{N} \), there is \( E_k : \mathbb{R}^+ \to \mathbb{R}^+ \) such that, for all \( M \in \mathcal{M} \),

\[
R_k(t; M) \leq E_k(t) t^{-n/2},
\]

and \( \lim_{k \to \infty} E_k(t) = 0 \) for fixed \( t > 0 \).

Proof. From the proof of Theorem 17 in [6] (p. 393), there exists \( E_0 = E_0(n, \kappa_0, D) \) such that

\[
R_k(t; M) \leq E_0 t^{-n/2} \int_{t\kappa}^{\infty} s^{n/2} e^{-s} ds.
\]

Now recall from Theorem [6] above that \( C_{4} \leq \lambda_k \), where \( C_{4} = C_{4}(n, \kappa_0, D) \). Put

\[
E_k(t) := E_0 \int_{C_{4} \lambda_k}^{\infty} s^{n/2} e^{-s} ds.
\]

Hence \( R_k(t; M) \leq E_k(t) t^{-n/2} \) and \( \lim_{k \to \infty} E_k(t) = 0 \) for fixed \( t > 0 \). \( \square \)

3.3. Final steps

Now take \( \epsilon > 0 \) and \( m \in \mathbb{N} \) from Theorem [1]. Put

\[
g(t) := 1 - \frac{C_U}{p^{n/2}} \exp \left\{ -\frac{\epsilon^2}{(4 + \delta)t} \right\}.
\]

Let \( M \in \mathcal{M} \), and let \( p \) be its heat kernel. Note the bound \( p(t, x, x) \geq \varphi_0^2(x) = V(M)^{-1} = 1 \), which follows from the series expansion [5] of the heat kernel. Then, combined with Corollary [1] for \( t \in (0, i_0/2] \),

\[
g(t) \leq \inf_{d \in (t, x) \geq \epsilon} p(t, x, x) - p(t, x, y),
\]

and \( g(t) \to 1 \) as \( t \to 0^+ \). Choose \( T \in (0, i_0/2] \) to satisfy \( g(T) \geq 4/5 \); then choose \( d \geq m \) satisfying \( E_{d+1}(T) T^{-n/2} \leq 1/5 \). We now complete the proof.
Proof of Theorem 2. By Theorem 1 since \( d \geq m \), we already know that \( \Phi^d \) is an immersion and that it distinguishes points within distance \( \epsilon \) of one another. Suppose \( d_M(x, y) \geq \epsilon \). Then, noting
\[
\sup_{x', y' \in M} |p(T, x', y') - p^d(T, x', y')| \leq R_{d+1}(T; M) \leq 1/5,
\]
we have
\[
4/5 \leq g(t) \leq p(T, x, x) - p(T, x, y) \leq p^d(T, x, x) - p^d(T, x, y) + 2/5,
\]
hence \( p^d(T, x, x) > p^d(T, x, y) \). Finally, observe that \( p^d(T, x, x) \neq p^d(T, x, y) \) implies \( \Phi^d(x) \neq \Phi^d(y) \).

Remark 2. Note that were we able to explicitly compute \( \epsilon \) and \( m \) in Theorem 1 we could also write an explicit bound for the maximal embedding dimension \( d \) as follows. The foregoing proof reduces to finding the smallest \( d \geq m \) for which \( g(t) > 2E_{d+1}(t) r^{-n/2} \) is satisfied for some \( t \in (0, i_0/2] \). Moreover, to achieve a tighter bound, we could improve the lower bound \( g(t) \) from (46) above to
\[
\frac{C_L}{t^{n/2}} \leq \frac{C_U}{t^{n/2}} \exp \left\{ \frac{-\epsilon^2}{(4 + \delta)t} \right\},
\]
where \( p(t, x, x) \geq C_L t^{n/2} \) for all \( t \in (0, i_0/2] \), \( C_L = C_L(n, \kappa_0) \), follows from the on-diagonal Gaussian lower bound for the heat kernel (cf. 44, 45). Explicit computation of the maximal embedding dimension would then reduce to writing out the constants \( C_L, C_U, E_0, \) and \( C_1 \). One can use the formulas in [46] to compute \( C_L \), the formulas in [33] to compute \( C_U \), and the formulas in [6, 33], along with Croke’s estimate to establish the uniform diameter bound \( D \), to compute \( E_0 \) and \( C_1 \).

However, in this note, both \( \epsilon \) and \( m \) depend on the scaled “harmonic radius” \( r_h \) of Lemma 4 whose dependency on injectivity radius and Ricci curvature is established by indirect means (proof by contradiction) in Anderson and Cheeger [41], and the author of this note has not pursued deriving a formula for \( r_h \) in terms of injectivity radius and Ricci curvature.

4. Proof of Theorem 3

The last theorem derives from the following two results.

Theorem 9 (Cheeger [47]). Let \( K \) denote the sectional curvature of a complete Riemannian manifold \( M \). If \( |K| \leq \kappa_0 \), \( \text{V}(M) \geq V_0 \), \( d(M) \leq D \), then \( \text{inj}(M) \geq i_0 \) for some \( i_0 = i_0(n, \kappa_0, V_0, D) \).

Theorem 10 (Topping [58]). Let \( M \) be a closed \( n \)-dimensional Riemannian manifold isometrically immersed in \( \mathbb{R}^l \) with mean curvature vector \( H \). There is a constant \( C = C(n) \) such that
\[
d(M) \leq C \int_M |H|^{n-1} dV.
\]
Recall $\mathcal{S} := \{ (M, g) \mid \dim M = 2, |K| \leq \kappa_0, |H| \leq H_0, V(M) \leq A, \iota : M \hookrightarrow \mathbb{R}^3 \text{ is an isometric immersion} \}.$

Note that if $(M, g) \in \mathcal{S}$ and we scale $g$ by $a > 0$ so that $V(M, a^2 g) = 1$, then $1 = V(M, a^2 g) = a^2 V(M, g) \leq a^2 A$, or, $a^{-1} \leq A^{1/2}$. Noting $K(M, a^2 g) = a^{-2} K(M, g)$ and $H(M, a^2 g) = a^{-1} H(M, g)$, we have

$$(M, a^2 g) \in \{ (M, g) \mid \dim M = 2, |K| \leq A \kappa_0, |H| \leq A^{1/2} H_0, V(M) = 1, \iota : M \hookrightarrow \mathbb{R}^3 \text{ is an isometric immersion} \}. \quad (54)$$

For surfaces, note that Gaussian curvature $K$ and sectional curvature coincide, and $K$ is related to Ricci curvature by $\text{Ric}_M = K g$. Applying Theorem 10 then Theorem 9 reduces the present case to that of Theorem 2. It follows that $\mathcal{S}$ has a uniform embedding dimension.

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