AN INFINITE ANTICHAIN OF PLANAR TANGLEGRAMS

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Abstract. Contrary to the expectation arising from the tanglegram Kuratowski theorem of É. Czabarka, L. A. Székely and S. Wagner [SIAM J. Discrete Math. 31(3): 1732–1750, (2017)], we construct an infinite antichain of planar tanglegrams with respect to the induced subtanglegram partial order. R.E. Tarjan, R. Laver, D.A. Spielman and M. Bóna, and possibly others, showed that the partially ordered set of finite permutations ordered by deletion of entries contains an infinite antichain, i.e. there exists an infinite collection of permutations, such that none of them contains another as a pattern. Our construction adds a twist to the construction of Spielman and Bóna [Electr. J. Comb. Vol. 7. N2.]

1. Introduction

Informally, a tanglegram is a specific kind of graph, consisting of two rooted binary trees of the same size and a perfect matching joining their leaves. Tanglegrams are drawn under specific rules, such drawings are called tanglegram layouts. (Formal definitions are postponed to Section 2) The tanglegram crossing number of a tanglegram is the minimum crossing number (i.e. the minimum number of unordered crossing edge-pairs) among its layouts. The tanglegram is planar, if it has a layout without crossings. Tanglegrams play a major role in phylogenetics, especially in the theory of cospeciation [17]. The first binary tree is the phylogenetic tree of hosts, while the second binary tree is the phylogenetic tree of their parasites, e.g. gopher and louse [10]. The matching connects the host with its parasite. The tanglegram crossing number has been related to the number of times parasites switched hosts [10], or, working with gene trees instead of phylogenetic trees, to the number of horizontal gene transfers ([5], pp. 204–206). Tanglegrams are well-studied objects in phylogenetics and computer science (see e.g. [1, 2, 4, 6, 7, 9, 11, 13, 16, 21]).

Czabarka, Székely and Wagner [8] discovered a Kuratowski-type theorem that characterized planar tanglegrams by two excluded induced subtanglegrams. They asked

Problem 1. Are there similar characterizations

(i) for tanglegrams with tangle crossing number at most $k$?

(ii) for tanglegrams that have a layout without $k$ pairwise crossing edges?

Were the induced subtanglegram partial order a well-quasi-ordering, the answer to these questions would immediately be in the affirmative, delivering a number of algorithmic
consequences. To be a well-quasi-ordering, there should not be an infinite antichain in the well-founded partially order.

Whether a well-founded partially ordered set has an infinite antichain has been well studied (e.g. [12, 14, 19, 20]). In particular, Kruskal’s Tree Theorem [14] would give one hope that the induced subtanglegram relation would be a well-quasi-ordering as well. However, tanglegrams, where the two trees are caterpillars, are closely related to permutations and permutation patterns (see Section 2). Laver [15], Pratt [18], Tarjan [22], and Speilman and Bóna [3] constructed infinite antichains of permutations for the partial order defined by permutation patterns.

While the antichain of permutations in [3] does not immediately yield an infinite antichain of tanglegrams (in fact, it defines a chain, as will be explained at the end of Section 3), when we turn these permutations “upside down” (i.e. in a permutation of \([n]\) we replace every entry \(j\) by \(n + 1 - j\)), we manage to obtain an infinite antichain of tanglegrams with respect to the induced tanglegram relation. Furthermore, the elements of the antichain are planar tanglegrams (shown in Section 4), making Problem 1 even more intriguing. An algorithmic consequence of a positive answer to Problem 1 would be fixed-parameter tractability of computing the tanglegram crossing number, a result that is already known [4].

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2. Definitions and basic setup

As customary, \([n]\) denotes the set \(\{1, 2, 3, \ldots, n\}\), \(S_n\) denotes the symmetric group acting on \([n]\). For \(\pi \in S_n\), we use the notation \(\pi = (a_1, \ldots, a_n)\), if \(\pi(i) = a_i\) for all \(i \in [n]\).

Definition 1. A rooted tree \(T\) is a tree with a distinguished vertex called the root. Given a vertex \(v\) in a rooted tree, and a neighbor \(y\) of \(v\), \(y\) is the parent of \(v\), if \(y\) is on the path from \(v\) to the root; otherwise \(y\) is a child of \(v\). The rooted tree \(T\) is binary, if every vertex has zero or two children.

Definition 2. For \(n \geq 2\), the rooted caterpillar \(C_n\) with \(n\) leaves is the rooted binary tree, whose \(n - 2\) internal vertices form a path, and the root is an endvertex of this path.

Note that \(C_n\) has two leaves at distance \(n - 1\) from the root, and for all \(i\) (\(1 \leq i \leq n - 2\)) it has precisely one leaf at distance \(i\) from the root. These properties characterize \(C_n\).

Definition 3. Given a rooted binary tree \(T\) with root \(r\) and a non-empty subset \(B\) of its leaves, the rooted binary subtree induced by \(B\), \(T[B]\), is obtained as follows: Take the smallest subtree \(T'\) of \(T\) containing all vertices of \(B\), and designate the vertex \(\rho \in V(T')\) closest to \(r\) in \(T\) as the root of \(T'\). This rooted tree is not necessarily binary—suppress all vertices of degree 2 (except \(\rho\)) in \(T'\) to make it binary. The resulting rooted binary tree is \(T[B]\).

Definition 4. A tanglegram of size \(n\) is an ordered triplet \((T_1, T_2, M)\), where \(T_1\) and \(T_2\) are rooted binary trees with \(n\) leaves each, and \(M\) is a perfect matching between the two leaf sets. \(T_1\) is called the left tree and \(T_2\) is the right tree of the tanglegram. Two tanglegrams
are considered the same, if there is a graph isomorphism between them, which fixes the
roots of the left tree and the right tree.

Definition 5. Given a tanglegram \( T = (T_1, T_2, M) \) and an \( \emptyset \neq M' \subseteq M \), the sub-tanglegram induced by \( M' \) is \( T[M'] = (T_1[B_1], T_2[B_2], M') \), where \( B_i \) is the set of leaves in \( T_i \) matched by \( M' \). We say that \( T' \) is an induced sub-tanglegram of \( T \) (in notation: \( T' \preceq T \)), if there
is an \( M^* \subseteq M \) such that \( T^* = T[M^*] \).

Note that \( \preceq \) is a partial order on the set of tanglegrams, and \( \preceq \) is well-founded, i.e. it has no infinite strictly decreasing chains.

Definition 6. Given a tanglegram \( T = (T_1, T_2, M) \), where the root of \( T_i \) is \( r_i \), the multiset of distance pairs, \( \mathcal{D}(T) \), contains exactly \( k \) copies of \( (d_1, d_2) \) if and only if there exists exactly \( k \) matching edges of the form \( (x_1, x_2) \in M \) such that \( x_i \) is a leaf of \( T_i \) at distance \( d_i \) from \( r_i \).

From now on we restrict ourselves to tanglegrams, in which both the left and right trees are rooted caterpillars. Note that in this case, if two tanglegrams have the same distance pair multiset, then they are the same.

Definition 7. For \( n \geq 2 \), the distance labeling of the leaves of \( C_n \) is the following: for each \( i \), \( 1 \leq i \leq n-2 \), the leaf labeled \( i \) is the one at distance \( i \) from the root, and the two leaves at distance \( n-1 \) are labeled arbitrarily by \( n-1 \) and \( n \). For \( n \geq 2 \) and \( \pi \in S_n \), the catergram \( T_{\pi} \) is the tanglegram \((C_n, C_n, M_{\pi})\), where \( M_{\pi} \) is defined as follows. Use the distance labeling of the leaves of both caterpillars, match the leaf on the left tree labeled \( i \) with the leaf on the right tree labeled \( j \) if and only if \( \pi(i) = j \).

Note that every tanglegram, in which both the left tree and right tree are rooted caterpillars, does arise as a catergram, but the permutation that defines it is not unique.

Definition 8. Assume \( n \geq 2 \). Given a \( \pi = (a_1, \ldots, a_n) \in S_n \), we define the (not necessarily different) permutations \( \widehat{\pi}, \tilde{\pi} \) as

\[
\widehat{\pi}(i) = \begin{cases} 
    a_i, & \text{if } i \leq n-2 \\
    a_n, & \text{if } i = n-1 \\
    a_{n-1}, & \text{if } i = n
\end{cases}
\]

and finally let \( \pi^* = (\widehat{\widehat{\pi}}) \). We define the set \( \Pi = \{ \pi, \widehat{\pi}, \tilde{\pi}, \pi^* \} \).

Proposition 9. The following facts are obvious for any \( \pi = (a_1, \ldots, a_n) \):

(a) We have \( \mathcal{D}(T_{\pi}) = \{(1, a_1^*), (2, a_2^*), \ldots, (n-1, a_{n-1}^*), (n-1, a_n^*)\} \), where

\[
a_i^* = \begin{cases} 
    a_i, & a_i < n \\
    n-1, & a_i = n
\end{cases}
\]

(b) \( (\widehat{\pi}) = (\tilde{\pi}), \pi = (\widehat{\tilde{\pi}}) = (\pi), \) and \( \pi \not\in \{\widehat{\pi}, \tilde{\pi}\} \).

(c) \( \rho \in \pi \iff \rho = \pi \).

(d) \( \pi = \pi^* \iff \{a_{n-1}, a_n\} = \{n-1, n\} \ iff \pi = \pi^* ; \) consequently \( |\pi| \in \{2, 4\} \).
(e) $\mathcal{T}_\rho = \mathcal{T}_{\pi} \iff \mathbb{D}(\mathcal{T}_\rho) = \mathbb{D}(\mathcal{T}_{\pi}) \iff \rho \in \pi$.

**Definition 10.** We say that two sequences of $n$ numbers, $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$, are order isomorphic, if for all $i, j \in [n]$, we have $a_i < a_j \iff b_i < b_j$. Given a $\pi \in S_n$ and a non-empty $A \subseteq [n]$, where $a_1, \ldots, a_k$ lists the elements of $A$ in increasing order, we denote by $\pi[A]$ the permutation in $S_{|A|}$ that is order isomorphic to $(\pi(a_1), \pi(a_2), \ldots, \pi(a_k))$. If $\rho \in S_m$ and $\pi \in S_n$, then we say that $\rho$ is a pattern in $\pi$ (in notation $\rho \leq \pi$), if $\pi[A] = \rho$ for some $A \subseteq [n]$.

**Definition 11.** Assume $\pi \in S_n$ and $\emptyset \neq A \subseteq [n]$. Then (with a slight abuse of notation) we denote by $\mathcal{T}_{\pi}[A]$ the induced subtanglegram $\mathcal{T}_{\pi}[M^*]$, where $M^*$ is the matching containing edges of $M$ incident upon leaves of the left tree that are labeled with elements of $A$.

**Proposition 12.** The following statements are true:

(a) Let $v$ be a leaf of $C_n$ at distance $i$ from the root $r$ of $C_n$, and $y \neq v$ be another leaf that is at distance $j$ from $r$. Let $T$ be the binary tree induced by all leaves except $v$ (so $T = C_{n-1}$) with root $r^*$. Then $y$ is a leaf in $T$, and the distance of $y$ from $r^*$ is $j$ if $j < i$, and $j - 1$ otherwise.

(b) For any $\pi \in S_n$ and non-empty $A \subseteq [n]$, we have $\mathcal{T}_{\pi}[A] = \mathcal{T}_{\pi[A]}$. (This follows from (a)).

(c) For $\rho \in S_m$ and $\pi \in S_n$, we have $\mathcal{T}_\rho \preceq \mathcal{T}_\pi$ iff $\mathcal{T}_\rho = \mathcal{T}_{\pi[A]}$ for some $A \subseteq [n]$ iff $\sigma \leq \pi$ for some $\sigma \in \mathcal{P}$. (This follows from (b) and Proposition 9(e)).

**Definition 13.** For $i \in \mathbb{Z}^+$, we set $\rho_i \in S_{|2i|}$ as $(\rho_i(1), \rho_i(2), \rho_i(3), \rho_i(4)) = (2, 3, 5, 1), (\rho_i(9 + 2i), \rho_i(10 + 2i), \rho_i(11 + 2i), \rho_i(12 + 2i)) = (10 + 2i, 11 + 2i, 12 + 2i, 8 + 2i)$ and for $j : 5 \leq j \leq 8 + 2i$

$$\rho_i(j) = \begin{cases} j + 2, & \text{if } j \text{ is odd} \\ j - 2, & \text{if } j \text{ is even}. \end{cases}$$

So for example, the first two permutations in our sequence will be

$$\rho_1 = (2, 3, 5, 1, 7, 4, 9, 6, 11, 8, 12, 13, 14, 10)$$
$$\rho_2 = (2, 3, 5, 1, 7, 4, 9, 6, 11, 8, 13, 10, 14, 15, 16, 12).$$

Spielman and Bóna showed that if $\pi_i$ is $\rho_i$ turned “upside down”, then $\{\pi_i : i \in \mathbb{Z}^+\}$ is an antichain for the pattern partial order of permutations. We are now ready to show our result:

**Theorem 14.** $\{\mathcal{T}_{\rho_i} : i \in \mathbb{Z}^+\}$ is an antichain with respect to the relation $\preceq$.

**Proof.** In the proof we will use the fact that for any $k$ and any $\gamma \in \mathcal{P}_k$, the permutation $\gamma$ has exactly two entries that are preceded by at least 3 larger elements: the entry 1 and the entry $8 + 2k$; moreover, if $\gamma \in \{\rho_k, \tilde{\rho}_k\}$ then $8 + 2k$ is preceded by exactly 4 larger elements, but these 4 elements are not order isomorphic in $\rho_k$ and $\tilde{\rho}_k$. 

3. Constructing the antichain of tanglegrams
By Proposition 12(c) it is sufficient to show that for any $i < j$ and for any $\sigma \in \rho_i$, $\sigma \not\leq \rho_j$. By our starting remark, if $\sigma < \rho_j$, then the entries 1 and $8 + 2i$ in $\sigma$ should map to the entries 1 and $8 + 2j$ in $\rho_j$, and the preceding larger elements must map to preceding larger entries; consequently $\hat{\rho}_i \not\leq \rho_j$. As $8 + 2j$ is the last entry of $\rho_j$, but not of $\hat{\rho}_i$ or $\rho_i^*$ (unless $\rho_i^* = \rho_i$), we get that $\hat{\rho}_i \not\leq \rho_j$ and $\rho_i^* \not\leq \rho_j$. So what remains to be shown is $\rho_i \not\leq \rho_j$, which was essentially stated and proved in [3], but for completeness, we include a (somewhat different) proof here.

Suppose for contrary that $\rho_i < \rho_j$, i.e. entries of $\rho_i$ map to entries of $\rho_j$ in an order preserving fashion. By our earlier remarks, the first 4 elements of $\rho_i$ must map to the first 4 elements of $\rho_j$ and the last 6 elements of $\rho_i$ must map to the last 6 elements of $\rho_j$, so we must map the sequence $(7, 4, 9, 6, \ldots, 7 + 2i, 4 + 2i)$ to $(7, 4, 9, 6, \ldots, 7 + 2j, 4 + 2j)$ by leaving out $2(j - i) \geq 2$ elements.

Let $x$ be an entry of the contiguous subsequence $(7, 4, 9, 6, \ldots, 7 + 2k, 4 + 2k)$ of $\rho_k$. If $x$ is even, then there are no entries that appear after $x$ in $\rho_k$ that are smaller than $x$, and $x$ is preceeded by the entry $x + 1$. If $x$ is odd, then there are exactly two entries in $\rho_k$ that follow $x$ and are smaller than $x$, and they are both even.

Let $x$ now be the first entry that is erased from $\rho_j$. The entries before $x$ in $\rho_i$ are mapped to the same entries, respectively, in $\rho_j$, and the entry $x$ in $\rho_i$ is mapped to a different entry that appears after $x$ in $\rho_j$.

If $x$ is even, then, as the entry $x + 1$ is before $x$ in $\rho_i$, $x$ must map to an entry smaller than $x + 1$ but is after $x$ in $\rho_j$. As such an entry does not exist, $x$ must be odd.

As $x$ is odd, it is immediately followed by the even entry $x - 3$ in both $\rho_i$ and $\rho_j$, and preceeded by the entry $x - 2$, which was not erased from $\rho_j$. As entry $x - 2$ in $\rho_i$ maps to entry $x - 2$ in $\rho_j$, and entry $x$ in $\rho_i$ maps to an entry after $x$ in $\rho_j$, it follows that entry $x - 3$ in $\rho_i$ must map to an entry that is after $x - 3$ in $\rho_j$ and is smaller than $x - 3$. Since such an entry does not exist, $\rho_i \not\leq \rho_j$. □

We remark here that in the infinite antichain of permutations $\{\pi_i : i \in \mathbb{Z}^+\}$ of [3], $\pi_i$ is our $\rho_i$ is turned “upside down”. For example,

\[
\begin{align*}
\pi_1 &= (13, 12, 10, 14, 8, 11, 6, 9, 4, 7, 3, 2, 1, 5) \\
\pi_2 &= (15, 14, 12, 16, 10, 13, 8, 11, 6, 9, 4, 7, 3, 2, 1, 5).
\end{align*}
\]

One can easily check that for $A = [16] \setminus \{2, 4\}$ we get $\pi_1 = \pi_2[A]$, showing that $\mathcal{T}_{\pi_1} \preceq \mathcal{T}_{\pi_2}$. Moreover, for every $i \in \mathbb{Z}^+$, setting $A_i = [14 + 2i] \setminus \{2, 4\}$, we observe that $\hat{\pi}_i = \pi_{i+1}[A_i]$, showing that

**Proposition 15.** $\{\mathcal{T}_{\pi_i} : i \in \mathbb{Z}^+\}$ is an infinite chain in the induced subtanglegram partial order.

This is why we had to put a twist on the construction of [3].

4. **Planarity of the tanglegrams in the antichain**

Lastly, we show that the tanglegrams $\mathcal{T}_{\rho_i}$ are planar. For this we need to define layouts first.
Definition 16. A plane binary tree is a rooted binary tree, in which the children of internal vertices are specified as left and right children. A plane binary tree is easy to draw on one side of a line, without edge crossings, such that only the leaves of the tree are on the line. We will say that the plane binary tree \( P \) is a plane tree of the rooted binary tree \( T \), if \( P \) is isomorphic to \( T \) as a graph.

Note that if we label all vertices of a rooted binary tree with \( n \) leaves, then there are \( 2^{n-1} \) labeled plane trees whose underlying labeled graph is this labeled rooted binary tree.

Definition 17. A layout \((L, R, M)\) of the tanglegram \( T = (T_1, T_2, M) \) is given by a left plane binary tree \( L \) isomorphic to \( T_1 \), drawn in the halfplane \( x \leq 0 \), having its leaves on the line \( x = 0 \), a right plane binary tree \( R \) isomorphic to \( T_2 \) drawn in the halfplane \( x \geq 1 \), having its leaves on the line \( x = 1 \), and the perfect matching \( M \) between their leaves drawn in straight line segments. (See Figure 1.)

![Figure 1](image1.png)

**Figure 1.** Two layouts of the same tanglegram. The leaf labels help showing that the two tanglegrams are identical.

Definition 18. A tanglegram is planar if it has a layout without crossing edges.

Theorem 19 (Czabarka, Székely, Wagner [8]). Every non-planar tanglegram contains one of the two tanglegrams in Figure 2 as an induced subtanglegram.

![Figure 2](image2.png)

**Figure 2.** The two tanglegrams excluded from planar tanglegrams. The tanglegram on the left is the catergram \( T_{(3,2,1,4)} \), but the tanglegram on the right is not a catergram, as the trees are not caterpillars.

Now we are ready to show:

Proposition 20. For every \( i \in \mathbb{Z}^+ \) the catergram \( T_{\rho_i} \) is planar.
The following facts are obvious:

**Proposition 22.** The following facts are obvious:

(a) The tanglegram $(T_1, T_2, M)$, where the leaves of $T_1$ and $T_2$ are labeled, is planar iff there are permutations $\pi_1 = (a_1, \ldots, a_n)$ and $\pi_2 = (b_1, \ldots, b_n)$ of the leaf labels of $T_i$, such that $\pi_i$ is consistent with $T_i$ for $i = 1, 2$, and $M = \{a_ib_i : i \in [n]\}$.

(b) The catergram $T_\sigma$ is planar iff there is a cater-good permutation $(\sigma(a_1), \ldots, \sigma(a_n))$ is also cater-good. A planar layout is obtained by these permutations, putting leaves in their order on the lines $x = 0$ and $x = 1$.

(c) If a permutation $(c_1, \ldots, c_n)$ of $[n]$ is unimodal, then it is cater-good.
(d) For every \( i \in \mathbb{Z}^+ \), a planar drawing of \( T_{\rho_i} \) is given by the permutation \( (a_1, \ldots, a_{12+2i}) \) where \( a_1 = 4 \), \((a_1, a_2, a_3) = (1, 2, 3)\), \((a_{4+i}, a_{9+i}, a_{10+i}, a_{11+i}) = (9 + 2i, 10 + 2i, 11 + 2i, 12 + 2i)\), and for \( j \in [4 + i] \), \( a_{3+j} = 3 + 2j \) and \( a_{12+2i-j} = 4 + 2j \).

Note that the permutation \( (a_1 = 1, \ldots, a_{12+2i}) \) in \((d)\) is unimodal, and consequently so is \( (\rho_i(a_1), \ldots, \rho_i(a_{12+2i})) = (a_2, a_3, \ldots, a_{12+2i}, 1) \). Figure 3 gives the planar drawing of \( T_{\rho_i} \) determined by the permutation given in this Proposition.
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