QCD coupling below 1 GeV from quarkonium spectrum

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Abstract

In this paper we exploit calculations within a Bethe-Salpeter (BS) formalism adjusted for QCD, in order to extract an “experimental” strong coupling \( \alpha_{\exp}(Q^2) \) below 1 GeV by comparison with the meson spectrum. The BS potential follows from a proper ansatz on the Wilson loop to encode confinement and is the sum of a one-gluon-exchange and a confinement term. Besides, the common perturbative strong coupling is replaced by the ghost-free expression \( \alpha_{E}(Q^2) \) according to the prescription of Analytic Perturbation Theory (APT).

The agreement of \( \alpha_{\exp}(Q^2) \) with the APT coupling \( \alpha_{E}(Q^2) \) turns out to be reasonable from 1 GeV down to the 200 MeV scale, thus confirming quantitatively the validity of the APT prescription. Below this scale, the experimental points give a hint on the vanishing of \( \alpha_{s}(Q^2) \) as \( Q \) approaches zero. This infrared behavior would be consistent with some lattice results and a “massive” generalization of the APT approach.

As a main result, we claim that the combined BS-APT theoretical scheme provides us with a rather satisfactory correlated understanding of very high and rather low energy phenomena from few hundreds MeV to few hundreds GeV.

1 Introduction

As it is well known, a rather consistent picture of many strong high energy processes was obtained by perturbative QCD, if the running coupling \( \tilde{\alpha}_{s}(Q^2) \), as derived from the renormalization group equation, is used. Inversely, if the
QCD scale value $\Lambda_{n_f=5} \sim 200$ MeV is taken, the values of $\alpha_s$ extracted from the data for the appropriate $Q$ or $\sqrt{s}$ fit rather well the theoretical $\bar{\alpha}_s(Q^2)$ curve, with few exceptions. Quite important, the 2-loop level of approximation for $\bar{\alpha}_s$ seems to be sufficient [1] for practical size of data errors. Unhappily, $\bar{\alpha}_s(Q^2)$ develops (at any loop level) unphysical singularities for $Q \sim \Lambda_{QCD} \sim 400$ MeV, that makes the expression useless in the infrared region. This is a particularly serious difficulty in any type of potential model in which $Q$ should be identified with the momentum transfer, that takes typically values below 1 GeV, according to the state and to the mass of the quarks implied.

Among various proposals to eliminate these singularities (see, e.g., Section 3 in Ref. [2]), we mention here two particular ones, i.e., the freezing hypothesis, that simply consists in assuming that $\bar{\alpha}_s(Q^2)$ freezes to a certain maximal value $H$ in the infrared (IR) region as a consequence of a non-perturbative effect, and the analyticization prescription of the APT approach [3]. The latter imposes $\bar{\alpha}_s(Q^2)$ to satisfy a dispersion relation with the only unitary cut for $-\infty < Q^2 < 0$ (throughout this paper we assume the momentum scale to be spacelike $q^2 = -Q^2$) and uses perturbation theory to evaluate the spectral function.

On the other hand, in the last years a Bethe-Salpeter (BS) like formalism [4] (second order BS formalism) has been developed and applied with a certain success to the calculation of the quarkonium (meson) spectrum in the light and in the heavy quark sectors. The formalism is essentially derived from the QCD Lagrangian taking advantage of a Feynman-Schwinger representation for the solution of the iterated Dirac equation in an external field. Confinement is encoded through an ansatz on the Wilson loop correlator; indeed the expression $i \ln W$ is written as the sum of a one-gluon exchange (OGE) and an area term

$$ i \ln W = (i \ln W)_{\text{OGE}} + \sigma S. $$

By means of a three dimensional reduction, the original BS equation takes the form of the eigenvalue equation for a squared bound state mass

$$ M^2 = M_0^2 + U_{\text{OGE}} + U_{\text{Conf}}, $$

where $M_0$ is the kinematic term $M_0 = w_1 + w_2 = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2}$, $k$ being the c.m. momentum of the quark, $m_1$ and $m_2$ the quark and the antiquark constituent masses, and $U = U_{\text{OGE}} + U_{\text{Conf}}$ the resulting potential. As a consequence of ansatz [1], the perturbative part of the potential $U_{\text{OGE}}$
turns out to be proportional to $\alpha_s(Q^2)$, where in a sense $\alpha_s(Q^2)$ should be identified as an effective charge of the type proposed in [5] and denoted in [2] with $\alpha_{\text{SGD}}(Q^2)$.

Calculations have been performed in Refs. [6, 7] by using both a frozen and the 1-loop analytic coupling $\alpha^{(1)}_E(Q^2)$ with an effective scaling constant $\Lambda^{(1,\text{eff})}_{n_f=3} \simeq 200$ MeV (see Eq. (9) below), which is equivalent at the 3-loop level to $\Lambda^{(3)}_{n_f=3} \simeq 400$ MeV or to the world average $\Lambda^{(3)}_{n_f=5} \simeq 200$ MeV.

The results of the two sets of calculations are relatively similar for the heavy-heavy quark states. However, for the 1S states involving light and strange quarks, quite different results have been obtained in the two cases. In the case of a frozen coupling the $\pi$ and $K$ masses turn out to be too high, independently of how small the light quark mass is taken (see Fig. 1); e.g., if we fit the light and the strange quark masses to the $\rho$ and the $\phi$ masses, we find $m_\pi \sim 500$ MeV and $m_K \sim 700$ MeV, respectively. On the contrary, if appropriate values for the quark masses are chosen, the $\pi$, $\rho$, $K$, $K^*$, $\phi$ masses can be rather well reproduced when the analytic coupling $\alpha^{(1)}_E(Q^2)$ is used. This occurrence strongly supports the use of the analytic coupling in the BS framework.

In this paper we reverse somewhat the point of view. For every quark-antiquark state we compare our theoretical results, obtained for a certain choice of the parameters and the analytic coupling, with the results of a similar calculation for the same values of the quark masses and string tension but using a fixed value of $\alpha_s$. We denote by $\alpha^{\text{th}}_s$ the value that reproduces the same theoretical result as obtained with $\alpha^{(1)}_E(Q^2)$ and by $\alpha^{\exp}_s$ the corresponding value that reproduces the experimental mass. The value $\alpha^{\text{th}}_s$ is then used to identify an effective $Q$ pertaining to that particular state, which is to be understood as the argument of the related “experimental” coupling $\alpha^{\exp}_s(Q^2)$.

Since only the leading perturbative contribution in the BS kernel has been included, a rough estimate of NLO effects on the $\alpha^{\text{exp}}_s$ value leads to a relative theoretical error which spans from 20% to much less than 1% throughout the spectrum according to the quark masses involved. Furthermore, since coupling among different quark-antiquark channels has not been taken into account, the theoretical masses are expected to reproduce the experimental ones within the half width $\Gamma/2$ of the state. In the framework of the BS formalism these are the most relevant sources of theoretical error and overwhelm all other errors, like those related to the three dimensional reduction.
or to the ansatz (1). When relevant, the experimental error, related to the uncertainty of the experimental mass is added to the theoretical one. If compared with the 3-loop analytic curve $\alpha_E^{(3)}(Q^2)$, the values of $\alpha_s^{\exp}(Q^2)$ fit it rather well within error bars in all the region $\Lambda_{n_f=3}^{(3)}/2 < Q < 3\Lambda_{n_f=3}^{(3)}$ (being $\Lambda_{n_f=3}^{(3)} \simeq 400$ MeV according to normalization and threshold matching adopted). On the other hand, for $Q < \Lambda_{n_f=3}^{(3)}/2$ our data fall below the original $\alpha_E^{(3)}(Q^2)$ and give a hint on the vanishing of $\alpha_s(Q^2)$ as $Q$ approaches zero. Actually the experimental situation is particularly unclear in this region and the theoretical understanding is perhaps poor. The deep IR behavior above, nonetheless, is consistent with a recently developed “massive” version of analytic approach for the QCD coupling [8] (see Sec. 2), in which the effect of a nonvanishing mass of the lightest hadron state is taken into account, and moreover agrees with some results from lattice simulations.

Finally let us stress that the choice to compare $\alpha_s^{\exp}(Q^2)$ with the 3-loop expression $\alpha_E^{(3)}(Q^2)$ was to stay as close as possible to the usual practice in perturbation theory. In APT, however, when the appropriate small change of scale is made ($\Lambda_{n_f=5}^{(2)} \simeq 258$ MeV rather than $\Lambda_{n_f=5}^{(3)} \simeq 236$ MeV if both normalized at the $Z$ mass) the 2-loop coupling $\alpha_E^{(2)}(Q^2)$ is practically indistinguishable from $\alpha_E^{(3)}(Q^2)$ in the entire Euclidean range and can be used instead of the latter for all practical purposes.

The layout of the paper is as follows. In Sec. 2 the ghost-pole problem is discussed and an overview of the key points of analytic approach to QCD is given. In Sec. 3 an explicit expression for the BS potential $U$ is given and the mathematical method to treat the eigenvalue problem for the squared mass operator $M^2$ is described. Sec. 4 is devoted to the strategy for extracting $\alpha_s^{\exp}(Q^2)$ from the data and to errors estimate. Finally, in Sec. 5 our results are discussed and a match of the QCD coupling data at high energy scales is attempted via Analytic Perturbation Theory.

Some technical material is exposed in Appendices. A brief review of the derivation of the second order BS formalism and of the expression of $M^2$ from the ansatz (1) is given in App. A. Numerical tables in App. B display all results in details. In App. C an useful formula for 3-loop analytic coupling (the spectral density) is explicitly given and compared with the usual 3-loop

\footnote{Note that these values of the scale constant turn out to be somewhat larger than the perturbative values, as given e.g., by Bethke [1], with the same normalization for $\alpha_s$.}
perturbative coupling.

2 Ghost-free APT coupling below 1 GeV

The renormalization group (RG) method is an inherent part of theoretical description of strong interaction processes. It is usually employed to improve the results of Feynman perturbation theory in the high energy region. However, as mentioned, a straightforward application of the RG method to perturbative expansion eventually gives rise to unphysical singularities of both the RG-invariant coupling function\(^2\) and physical observables. The presence of these singularities contradicts the general principles of the local QFT and severely complicates theoretical analysis of hadron dynamics in the IR domain. At the same time, the results of lattice simulation testify to the absence of spurious singularities of the QCD coupling at low energies (see, e.g., a recent overview in Sec. 2 of paper [9] as well as original papers [10]).

There is a number of nonperturbative tricks to handle this singularity problem (for a recent review of this issue see Sec. 3 in paper [2] and references therein). Among those that employ properties of perturbative power series are: the method of effective charges [11], the “optimal conformal mapping” method [12] (see also Ref. [13]), the “optimized perturbation theory” [14], and some others. There are also few methods which impose some nonperturbative constraints either on the strong coupling (see, e.g., Refs. [15, 16]) or on the RG beta function (see, e.g., Refs. [17, 18, 19]). In this paper we will exploit the so-called Analytic Perturbation Theory (APT) approach to QCD [3] and its recent “massive” modification [8], which both are overviewed in the next two subsections.

2.1 Analytic running coupling

The analytic approach has been first devised [20] in the QED context, and then extended to the QCD case about ten years ago [3]. After the RG-summation, the analytic approach constitutes the next step in improving

\(^2\)For example, the one-loop QCD invariant coupling [1] possesses the so-called Landau (or spurious) pole in the physical IR region. The inclusion of higher loop corrections does not resolve the trouble.
the QFT perturbative results. Specifically, in addition to the property of renormalizability this method retains a general feature of local QFT, the property of causality. The basic merits of the analytic approach to QCD are the absence of unphysical singularities in the invariant coupling and in matrix elements and the enhanced stability of outcoming results with respect to both higher loop effects and the choice of a renormalization scheme [21]. Besides, this method enables one to process the spacelike and timelike data in a congruent way [22]. A fresh review of APT in QCD and its applications can be found in paper [23] (a generalization of APT for fractional powers of $\alpha_s$ was implemented in Ref. [24]).

Consider a QCD observable $D(Q^2)$ depending on single kinematic variable $Q^2 = -q^2 \geq 0$, the space-like momentum transfer squared. In the framework of RG-improved perturbation theory it is represented as power series in the strong coupling $\bar{\alpha}_s(Q^2)$:

$$D_{pt}(Q^2) = 1 + \sum_{n \geq 1} d_n \left[ \bar{\alpha}_s(Q^2) \right]^n ,$$

(3)

where $d_n$ are the relevant Feynman coefficients. In the IR domain, this expansion is inapplicable due to spurious singularities of the $\bar{\alpha}_s(Q^2)$. For example, the one-loop expression

$$\bar{\alpha}_s^{(1)}(Q^2) = \frac{1}{\beta_0} \frac{1}{\ln(Q^2/\Lambda^2)} ,$$

(4)

besides the physical cut along the negative real semiaxis $Q^2 \leq 0$, possesses unphysical pole at $Q^2 = \Lambda^2$.

In the framework of the APT, the power series (3) for an “Euclidean” observable is replaced [25] by the nonpower expansion

$$D_{\text{APT}}(Q^2) = 1 + \sum_{n \geq 1} d_n A_n(Q^2) ,$$

(5)

over the set of functions

$$A_n(Q^2) = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2} d\sigma ,$$

(6)

3At the same time, a “Minkowskian” observable $R(s)$ of the c.m. energy squared $s = q^2 > 0$ is represented by the nonpower expansion similar to Eq. (5) over a set of functions $A_n(s) = \int_s^\infty d\sigma \rho_n(\sigma)/\sigma$ related to $A_n(Q^2)$ by appropriate integral transformation.
see Ref. [23] for details. The spectral function $\rho(\sigma)$ in Eq. (6) is defined as the discontinuity of the relevant power of the common QCD coupling $\bar{\alpha}_s(Q^2)$ across the physical cut, namely

$$\rho_n(\sigma) = \frac{1}{\pi} \text{Im}[\bar{\alpha}_s(-\sigma - i\varepsilon)]^n. \quad (7)$$

The first function $A_1(Q^2)$ (6) plays the role of the effective Euclidean QCD coupling:

$$\alpha_E(Q^2) \equiv A_1(Q^2) = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2} d\sigma. \quad (8)$$

In the one-loop case, Eqs. (6) and (8) produce explicit expressions [3]

$$\alpha^{(1)}_E(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right], \quad A^{(1)}_{n+1}(Q^2) = -\frac{1}{n\beta_0} \frac{dA^{(1)}_n(Q^2)}{d\ln Q^2}. \quad (9)$$

At the higher loop levels Eq. (8) can be integrated, generally, only numerically\(^4\) since the spectral functions $\rho_n(\sigma)$ (7) become rather involved. Extensive computations of the analytic coupling (8) and its “effective powers” (6) at 2- and 3-loop levels were tabulated in the paper [27]. It is worthwhile to note that contribution of every subsequent term of the APT expansion (5) is substantially suppressed with respect to the preceding one. Ultimately, this leads to an enhanced stability of the expansion (5) (in comparison with the perturbative one (3)) with respect to both, higher loop corrections and choice of the renormalization scheme, see Ref. [23] for details.

Nevertheless, for practical applications one can also use simple explicit expressions of the form of Eq. (9), with special model argument. According to Ref. [28], 2- and 3-loop Euclidean coupling can be approximated with reasonable accuracy by

$$\alpha^{(3)}_{\text{appr}}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{L_2(Q^2)} + \frac{1}{1 - \exp[L_2(Q^2)]} \right], \quad (10)$$

where

$$L_2(Q^2) = \ln\left(\frac{Q^2}{\Lambda^2}\right) + B_1 \ln \sqrt{\ln^2\left(\frac{Q^2}{\Lambda^2}\right) + 2\pi^2}, \quad B_1 = \frac{\beta_1}{\beta_0^2} \quad (11)$$

\(^4\)Besides the 2-loop case, where result is expressed via special, Lambert function [26].
with \( \beta \) function coefficients \( \beta_0 = (11 - 2n_f/3)/(4\pi) \), \( \beta_1 = (102 - 38n_f/3)/(4\pi)^2 \) taken in the Bethke normalization. The relative difference between the exact two-loop analytic coupling (Eq. (8) with appropriate spectral density) and its approximation (10) is less than 3% for \( Q \geq 500 \text{ MeV} \) (see Table 1 in Ref. [28] for the details). However, this error could reach 5-10% in the deep IR domain we are interested in. Therefore, in what follows we will compare our results with more accurate analytic coupling (8) (see App. C). Meanwhile, in the region of our interest, the expression (11) can be further simplified. Specifically, in the interval \( 1/2 \lesssim Q/\Lambda \lesssim 2 \) one can neglect the first term under the sign of the square root in Eq. (11):

\[
L_2(Q^2) \simeq \ln\left(\frac{Q^2}{\Lambda_{\text{eff}}^2}\right), \quad \Lambda_{\text{eff}} = \Lambda \exp\left[\frac{-B_1}{4} \ln(2\pi^2)\right] \simeq 0.555 \Lambda, \tag{12}
\]

that gives less than 1% accuracy in \( \alpha_{\text{appr}}(Q^2) \) (10).

## 2.2 Massive analytic perturbation theory

In view of a possibility for the extracted \( \alpha_s^{\exp}(Q^2) \) values to vanish in the IR limit (see Sec. 5), it is plausible to mention a recently devised “massive” modification for the QCD analytic coupling [8]. The point is that the representation of the form of Eq. (6) does not hold for every QCD quantity. For example, the Adler function, being the logarithmic derivative of the hadronic vacuum polarization function, satisfies the dispersion relation [29]

\[
D(Q^2) = Q^2 \int_{4m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds, \tag{13}
\]

where \( m = m_\pi \) is the pion mass. Here, \( R(s) \) stands for the cross-section ratio

\[
R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons}; s)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-; s)} \tag{14}
\]

and \( s \) is the c.m. energy squared. Now, the expression (13) can be expanded over the set of \( \mathcal{A}_n(Q^2) \) (6) only in the massless limit \( (m = 0) \).

In the APT, the effect of the nonvanishing mass \( m \) could play a substantial role. In particular, the integral representations for the Adler function and
$R$-ratio of the $e^+e^-$ annihilation are affected by the mass. Specifically, in this case the Adler function \(13\) can be expanded

\[
D_{\text{MAPT}}(Q^2, m^2) = \frac{Q^2}{Q^2 + 4m^2} + \sum_{n \geq 1} d_n A_n(Q^2, m^2) \tag{15}
\]

over the set of the “massive” functions

\[
A_n(Q^2, m^2) = \frac{Q^2}{Q^2 + 4m^2} \int_{4m^2}^{\infty} \rho_n(\sigma) \frac{\sigma - 4m^2 d\sigma}{\sigma^2}, \tag{16}
\]

with one adjustable parameter \(m\), see Ref. \[8\] for details.

Similarly to the case of the usual APT \([5]\), the first function \(A_1(Q^2, m^2)\) \([10]\) plays the role of an effective “massive” coupling

\[
\alpha(Q^2, m^2) \equiv A_1(Q^2, m^2) = \frac{Q^2}{Q^2 + 4m^2} \int_{4m^2}^{\infty} \rho_1(\sigma) \frac{\sigma - 4m^2 d\sigma}{\sigma^2}. \tag{17}
\]

Irrespective of the loop level this coupling possesses the universal IR limiting value \(\alpha(Q^2, m^2) \to 0\) at \(Q^2 \to 0\), that will be important for discussion in Sec. 5.

3 BS-model for $q\bar{q}$ spectrum

As mentioned, in \([6, 7]\) the meson spectrum is obtained by solving the eigenvalue equation for the squared mass operator \([2]\), where the perturbative and confinement part of the potential are respectively

\[
(k| U_{\text{OGE}} |k') =
\]

\[
\frac{4}{3} \frac{\alpha_s(Q^2)}{\pi^2} \sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1 w_2 w'_1 w'_2}} \left[ -\frac{1}{Q^2} \left( q_{10} q_{20} + q^2 - \frac{(Q \cdot q)^2}{Q^2} \right) + \frac{i}{2Q^2} k \times k' \cdot (\sigma_1 + \sigma_2) + \frac{1}{2Q^2} [q_{20}(\alpha_1 \cdot Q) - q_{10}(\alpha_2 \cdot Q)] + \frac{1}{6} \sigma_1 \cdot \sigma_2 + \frac{1}{4} \left( \frac{1}{3} \sigma_1 \cdot \sigma_2 - \frac{(Q \cdot \sigma_1)(Q \cdot \sigma_2)}{Q^2} \right) + \frac{1}{4Q^2} (\alpha_1 \cdot Q)(\alpha_2 \cdot Q) \right] \tag{18}
\]
\[ \langle k | U_{\text{conf}} | k' \rangle = \frac{\sigma}{(2\pi)^3} \sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1 w_2 w'_1 w'_2}} \int d^3 r \, e^{iQ \cdot r} J^{\text{inst}}(r, q_{10}, q_{20}) \]

with

\[ J^{\text{inst}}(r, q_{10}, q_{20}) = \frac{r}{q_{10} + q_{20}} \left[ \frac{q_{20}^2}{q_{10}^2 - q_{10}^2} q_{10}^2 \right] \left( \arcsin \frac{|q_{10}|}{q_{10}} + \arcsin \frac{|q_{20}|}{q_{20}} \right) \]

\[ -\frac{1}{r} \left[ \frac{q_{20}}{\sqrt{q_{10}^2 - q_{10}^2}} (r \times q \cdot \sigma_1 + iq_{10}(r \cdot \alpha_1)) \right] \]

\[ + \frac{q_{10}}{\sqrt{q_{20}^2 - q_{10}^2}} (r \times q \cdot \sigma_2 - iq_{20}(r \cdot \alpha_2)) \].

(20)

Here \( \alpha_j^k \) denote the usual Dirac matrices \( \gamma_j^k \), \( \sigma_j^k \) the \( 4 \times 4 \) Pauli matrices \( \left( \begin{array}{cc} \sigma_j^k & 0 \\ 0 & \sigma_j^k \end{array} \right) \) and \( q = \frac{k + k'}{2} \), \( Q = k - k' \), \( q_{j0} = \frac{w_j + w'_j}{2} \), \( m_1 \) and \( m_2 \) are constituent masses.

Eqs. (18-20) follow from the ansatz (1) and a 3-dimensional reduction of our Bethe-Salpeter equation (see Refs. [4, 6, 7] and App. A for the details). Actually in the calculation of [6, 7] only the center of gravity (c.o.g.) masses of the fine multiplets were considered as a rule, and the spin dependent terms in (18-20) (spin-orbit and tensorial terms) were neglected with the exception of the hyperfine separation term in (18) proportional to \( \frac{1}{6} \sigma_1 \cdot \sigma_2 \).

Within this limitation a generally good reproduction of the spectrum was obtained for appropriate values of the parameters, as apparent from Fig. [1]. Here the results of three sets of calculations are displayed. Diamonds refer to the usual perturbative 1-loop coupling (to be replaced in Eq. (18)), freeze at a maximum value \( H \), which has been taken as an additional adjustable parameter. Squares and circles both refer to the 1-loop APT coupling as given by the first of Eq. (9), repeated here for convenience

\[ \alpha_E^{(1)}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right], \]

(21)
with $\Lambda \simeq 200$ MeV. For light quarks a running constituent mass was used too.

We stress that only with the choice (21) the $1^1S_0$ state has been correctly reproduced when light and strange quarks were involved, as in the case of $\pi$ and the $K$ mesons.

In the present work a similar calculation with the input (21) and a slight different choice of the parameters is made (preliminary results were given in [30]). First, we have fixed the string tension to the value $\sigma = 0.18$ GeV$^2$ (consistent with other phenomenology and lattice simulations) and the scale constant to $\Lambda_{\chi_f=3} = 193$ MeV. The whole set of remaining parameters, all the quark masses, are then determined by fitting the $\pi$, $\phi$, $J/\psi$ and $\Upsilon$ masses. It turns out $m_u = m_d = 196$ MeV, $m_s = 352$ MeV, $m_c = 1.516$ GeV and $m_b = 4.854$ GeV. The results for the meson spectrum are given in the fourth column of tables in App. B.

The chosen value for $\Lambda_{\chi_f=3}^{(1,\text{eff})}$ has been dictated by the comparison with the 3-loop analytic coupling normalized at the Z boson mass (see App. C) according to the world average. As displayed in Fig. 2 the relative difference between the two curves is no more than 1% in the region $0.5 < Q < 1.2$ GeV, to which the states used as an input in the calculation belong.

Furthermore, as already noted, our equations refer to a single definite quark-antiquark channels. So, having correct relativistic kinematics, they do not include coupling with other channels like any potential model (see App. A). Then we can not expect to have any insight into the splitting of over-threshold complicated multiplets which involve mixture of different states. Even the position of the c.o.g. mass is expected to be reproduced only within one-half of the width of the state. This has been taken into account in the estimate of the theoretical error (see Sec. 4).

The resolution method of the eigenvalue equation for the operator (2, 18–20) we have used in [6, 7] and in the present work can be summarized in the following way.

a) In the static limit the problem can be reduced to the corresponding one

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5Circles refer to a phenomenological running mass as function of the c.m. quark momentum $m_u^2 = m_d^2 = 0.17|k| - 0.025|k|^2 + 0.15|k|^4$. Squares refer to a running constituent mass resulting from a solution of the DS equation with an analytic RG running current mass (see App. A for the details) which was, however, more in line with an attempt to define an analytic $\alpha_{E}^{(1)}(Q^2)$ singular at $Q^2 \to 0$ and including confinement [17, 18] than to (21).
for the center of mass Hamiltonian (see App. A)

\[ H_{\text{CM}} = w_1 + w_2 - \frac{4}{3} \frac{\alpha_s}{r} + \sigma r. \]  

(22)

b) The eigenvalue equation for (22) is solved for a convenient fixed \( \alpha_s \) by the Rayleigh-Ritz method, using the three dimensional harmonic oscillator basis and diagonalizing a \( 30 \times 30 \) matrix.

c) The square of the meson mass is evaluated as \( \langle \phi_a | M^2 | \phi_a \rangle \), \( \phi_a \) being the eigenfunction obtained in step b) (with \( a \) the whole set of quantum numbers) and the operator \( M^2 \) given by Eq. (2).

d) Prescription c) is equivalent to treat \( M^2 - H_{\text{CM}}^2 \) as a first order perturbation. Consistently the hyperfine separation should be given by

\[
(3 m_{nl})^2 - (1 m_{nl})^2 = \frac{32}{9 \pi} \int_0^\infty dk \frac{k^2}{2} \int_0^\infty dk' \frac{k'^2}{2} \varphi_{nl}^*(k) \varphi_{nl}(k') \times \\
\sqrt{\frac{w_1 + w_2}{w_1 w_2}} \sqrt{\frac{w'_1 + w'_2}{w'_1 w'_2}} \int_{-1}^1 d\xi \alpha_s(Q^2) P_l(\xi),
\]

(23)

where \( \varphi_{nl} \) is the radial part of the complete eigenfunction \( \phi_a \). However, in the case of the states involving light and strange quarks the quantity is further corrected to the second order of perturbation theory.

For the quark masses and string tension \( \sigma \) in (22) we have used the same values listed above and for what regards \( \alpha_s \), that is supposed to be a constant in (22), we have taken \( \alpha_s = 0.35 \), which is the typical value used in non-relativistic calculations and also the freezing value adopted in [6].

4 Extracting \( \alpha_s^{\text{exp}}(Q^2) \) from the data

One focus now on the reversed problem, i.e., the determination of the \( \alpha_s^{\text{exp}}(Q^2) \) values at the characteristic scales of a selected number of ground and excited states.

In order to estimate \( \alpha_s^{\text{exp}}(Q^2) \) at low scales one needs first to assign an effective \( Q \)-value to each state. To this end one first rewrites the squared mass, as given by point c) in Sec. 3, more explicitly as the sum of the unperturbed part, the perturbative and the confinement one respectively
\[ m_a^2 = \langle \phi_a | M_0^2 | \phi_a \rangle + \langle \phi_a | U_{\text{OGE}} | \phi_a \rangle + \langle \phi_a | U_{\text{Conf}} | \phi_a \rangle. \]  

(24)

Here \( U_{\text{OGE}} \) is given by the second line of (18) and \( U_{\text{Conf}} \) by Eq. (19) and first two lines of (20). From the OGE contribution we then extract for each state the fixed coupling value \( \alpha_a^{\text{th}} \) which leads to the same theoretical mass as by using \( \alpha E (Q^2) \) given by Eq. (21). This can be done by means of the relation

\[ \langle \phi_a | U_{\text{OGE}} | \phi_a \rangle \equiv \langle \phi_a | \alpha E (Q^2) \mathcal{O}(q; Q) | \phi_a \rangle = \alpha_a^{\text{th}} \langle \phi_a | \mathcal{O}(q; Q) | \phi_a \rangle, \]

(25)

where \( \mathcal{O}(q; Q) \) can be drawn again by the second line of Eq. (18). The effective momentum transfer \( Q_a \) associated to each bound state is then identified by equating

\[ \alpha E (Q_a^2) = \alpha_a^{\text{th}}. \]

(26)

The next step is to search for the correct (fixed) value of the coupling that exactly reproduces the experimental mass of each state. This is defined by the relation

\[ \langle \phi_a | M_0^2 | \phi_a \rangle + \alpha_s^{\text{exp}}(Q_a^2) \langle \phi_a | \mathcal{O}(q; Q) | \phi_a \rangle + \langle \phi_a | U_{\text{Conf}} | \phi_a \rangle = m_a^2, \]

(27)

so that, by combining Eqs. (24), (25) and (27) we finally obtain

\[ \alpha_s^{\text{exp}}(Q_a^2) = \frac{m_a^2 - m_a^2 + \alpha_a^{\text{th}} \langle \phi_a | \mathcal{O}(q; Q) | \phi_a \rangle}{\langle \phi_a | \mathcal{O}(q; Q) | \phi_a \rangle}. \]

(28)

This procedure has been applied to a number of light-light, light-heavy and heavy-heavy ground as well as excited states.

Note that, apart from the particular ansatz made in (1) to take into account confinement, the other relevant approximations are:

i) only the leading perturbative contribution is included in (1) and so in the potential;
ii) quark antiquark annihilations and couplings with other channels have been ignored;
iii) an instantaneous approximation is involved in deriving the eigenvalue equation for (2) from the original BS equation.

As even the experience with QED suggests, retardation corrections are expected to be relevant for the hyperfine and possibly fine splitting, but of minor importance for the positions of the c.o.g. of the multiplets that we essentially use to evaluate $\alpha_s^{\exp}(Q^2)$.

Thus, as we told, the main sources of theoretical error in the whole procedure are expected to arise from neglecting the NLO contribution to the BS kernel as well as the coupling with other channels. For what concerns the former (point i)), it is worth noting that the next to leading contribution to the perturbative part of the BS-kernel comes from four diagrams with two-gluon exchange; two triangular graphs containing a four-line vertex of the type $g^2\phi^*\phi A_\mu A^\mu$ and two three-line vertices $g\phi^*\partial_\mu\phi A^\mu$ (the spin independent part of our second order formalism is quite similar to scalar QED), one fish diagram with two four-line vertices, and a crossing box with four three line vertices. If the renormalization scale is identified with the momentum transfer $Q$ the fish graphs contribution is completely reabsorbed in the renormalization. On the other hand, a somewhat crude estimate of the contribution of each of the two triangular graphs gives

$$I_{\text{triang}} \sim 4 \left( \frac{4}{3} \alpha_s \right)^2 \frac{9m^2}{4Q^2 + 2m^2}$$

and for the crossing box graph, similarly

$$I_{\text{crsbox}} \sim \frac{64}{3} \left( \frac{4}{3} \alpha_s \right)^2 \frac{m^4}{(Q^2 + m^2 + k^2)^2}. \quad (30)$$

These expressions have to be compared with the leading one-gluon term we have used (see Eq. (43) of App. A)

$$I_{\text{OGE}} \sim 16\pi \frac{4}{3} \alpha_s \frac{m^2}{Q^2}. \quad (31)$$

Putting all things together, the overall error due to the omission of such NLO contributions to the BS kernel is then

$$\frac{\Delta I}{I} = \sqrt{\left( \frac{2I_{\text{triang}}}{I_{\text{OGE}}} \right)^2 + \left( \frac{I_{\text{crsbox}}}{I_{\text{OGE}}} \right)^2}, \quad (32)$$
and this produces

\[ \frac{\Delta O}{O} \sim \frac{\Delta I}{I}. \]  

(33)

By using Eqs. (25,28), after some algebra it is easy to recognize that the NLO effects on \( \alpha_s^{\text{exp}} \) turns out to be of the same order, that is

\[ \Delta_{\text{NLO}} \alpha_s \sim \alpha_a^{\text{th}} \frac{\Delta I}{I}, \]  

(34)

which is what is assumed in the foregoing. The NLO errors do not exceed 5% for heavy quark states while they are enhanced up to 20% when light and strange quarks are involved.

Finally, since the strength of the neglected coupling with other channels (OC) is obviously measured by the width \( \Gamma_a \) of the state, one roughly estimates an error of the order of \( \Delta m_a \sim \Gamma_a/2 \) in the evaluation of \( m_a \). On this ground, for each determination of \( \alpha_s^{\text{exp}}(Q^2_a) \) the related theoretical error is given by

\[ \Delta \Gamma \alpha_s = \frac{m_a}{\langle \phi_a | O(q; Q) | \phi_a \rangle} \Gamma_a. \]  

(35)

Usually the error \( \Delta m_{\text{exp}} \) on the experimental mass \( m_{\text{exp}} \) is much smaller than \( \Gamma_a/2 \). When, however, this is not the case one has to consider also the experimental error \( \Delta_{\text{exp}} \alpha_s \), obtained from (35) by replacing \( m_a \Gamma_a \) with \( 2m_{\text{exp}} \Delta m_{\text{exp}} \).

Before discussing our results some comments are in order. First note that in the evaluation of \( Q_a \) in (25) one has neglected the hyperfine splitting which however was taken into account in (28), bringing possibly to different values of \( \alpha_s^{\text{exp}}(Q^2_a) \) for the singlet and the triplet states (when there are reliable data for both).

Furthermore, the sensitivity of the effective \( Q \)’s determined from the specific coupling (21), has been checked by analyzing the deviation of each \( Q_a \) for a 25% shift of \( \Lambda_{n_f=3}^{(1,\text{eff})} \) around the value 193 MeV, and one finds that the average change in the momentum scale amounts to 3%. This makes the resulting \( \alpha_s^{\text{exp}}(Q^2) \) reliable, at least qualitatively, even in the deep IR region \( (Q < 0.2 \text{ GeV}) \), where the discrepancy with respect to massless \( \alpha_E^{(1)}(Q^2) \) is sizable.

There is a subtle point concerning the choice of the “unperturbed” \( \alpha_s \) involved in the static Hamiltonian (22). Actually, the value adopted is very near to the \( \alpha_a^{\text{th}} \) pertaining to the \( bb(1S) \) state, but definitively smaller than the
typical $\alpha^\text{th}_a$’s reported in Tab. I-VII, App. B. The point is that the hyperfine splitting is much more sensible than the c.o.g. mass to the behavior of the unperturbed wave function at small distance (large momentum), which is specifically controlled by the value of the unperturbed $\alpha_a$. As a result, the effective fixed value $\alpha^\text{spl}_s$ in Eq. (22) that reproduces the same splitting as by using the coupling $\alpha_E(Q^2)$ turns out to be significantly smaller than $\alpha^\text{th}_a$ calculated from the c.o.g. mass. Essentially, it was chosen a phenomenological value for the unperturbed $\alpha_s$ in order to have a good reproduction of the hyperfine splitting so as to reasonably reconstruct the c.o.g. of the doublet when one component is missing. It was then used the position of the c.o.g. (which is rather stable w.r.t. the unperturbed $\alpha_s$) to extract $\alpha^\text{exp}_s(Q^2_a)$.

We finally stress again that the 1-loop analytic coupling with the above mentioned value of the scaling constant used in our computation, Eq. (21), differs by no more than 1% from the 3-loop analytic coupling in the region $0.5 < Q < 1.2 \text{ GeV}$ where all the input states ($\pi, \phi, J/\psi$ and $\Upsilon$) fall.

5 BS-model results: concert of low and high energy data via APT

All results are displayed in details in tables I-VII of App. B, and pictorially in Fig. 3. The first three columns specify the state and its experimental mass as given by [31]. The fourth column gives our theoretical results for the meson masses, and the last three give the effective $Q$ and the relative theoretical and experimental coupling with errors (theoretical and experimental).

In Fig. 3 values of $\alpha^\text{exp}_s$ at the same $Q$ from triplet and singlet states have been combined through a weighted average according to their errors (both experimental and theoretical). The c.o.g. of the light-heavy states (that are interpreted according to the j-j scheme) are also reported in the above figure. As can be seen, the agreement between the 3-loop analytic coupling with $\Lambda^{(3)}_{n_f=3} = 417 \text{ MeV}$, and the points representing our experimental values for the QCD coupling is quite good within the errors down to 200 MeV.

At energies below 200 MeV a consistent tendency of $\alpha^\text{exp}_s(Q^2)$ to diminish with $Q$ seems to exist, although the errors are rather large and some of the multiplets incomplete. As already noted, this deep IR behavior can be theoretically described within the “massive” modification of APT in Sec. 2.2.

Specifically, as displayed in Fig. 3 the one-loop coupling $\alpha(Q^2, m^2)$ (17) with
an effective mass $m_{\text{eff}} \simeq (38 \pm 10) \text{ MeV}$ reasonably fits all experimental points down to the very low $Q$ region.

At this point it is worthwhile to comment on the dependence of the results on the renormalization scheme. First of all, our definition of the coupling is implicitly contained in ansatz (1). Specifically, here one assumes both that $i \ln W$ is dominated by the OGE term after the subtraction of the area term, and that the OGE term is represented as the corresponding tree-level expression, the fixed coupling $\alpha_s$ being replaced with the running one $\alpha_s(Q^2)$. The latter assumption amounts to the embodying all the dressing effects into $\alpha_s(Q^2)$ (see, e.g., Sec. 3.2 of Ref. [2] and references cited therein). It is worth noting that the coupling defined in such way is free of unphysical singularities by construction. At the same time, the analytic running coupling $\alpha_E(Q^2)$, which is involved in our calculations, is remarkably stable with respect to both the higher loop corrections and the choice of renormalization scheme (see Sec. 2.1 and detailed discussion of this issue in Ref. [23]). Thus one might expect that the same situation should also occur for $\alpha_{\exp}(Q^2)$, with the possible exception for the deep infrared region (see, e.g., Sec. 4.5 of Ref. [2] and references cited therein), where other nonperturbative effects could be relevant.

Notice that in our selection of states as a rule we have excluded irregular and incomplete multiplets. Of this type, e.g., in the light quark sector, are the $3S$ states ($m_{3S_1} - m_{3S_0}$ is anomalously large and about twice as $m_{2S_1} - m_{2S_0}$), $1^3P$ ($m_{1^3P_0}$ being larger than $m_{1^3P_1}$), $1^3D, F, G, H$ (incomplete). If however included in the analysis, all these states would bring the results in agreement with the general tendency outlined. On the contrary, some of the most relevant discrepancies relate to $c\bar{c}$ states.

Finally, Fig. 4 displays a synthesis of results for $\alpha_s(Q^2)$ defined from bound states in the BS framework with high energy data. Here, low energy results are reported in a logarithmic scale from 100 MeV to 220 GeV together with a sample of high energy data as given by S. Bethke [1], against the 3-loop analytic coupling $\alpha^{(3)}_E(Q^2)$ [8] and its massive modification (17). Also shown in the figure is the common perturbative 3-loop coupling with IR singular behavior that is definitively ruled out by the data. As can be seen, the BS-APT theoretical scheme allows a rather satisfactory correlated understanding of very high and rather low energy phenomena.
6 Conclusive remarks

To summarize, we have exploited calculations within the Bethe-Salpeter formalism adjusted for QCD, in order to extract an “experimental” strong coupling $\alpha_s^{\exp}(Q^2)$ below 1 GeV by comparison with the meson spectrum. A key point is the comparison of $\alpha_s^{\exp}$ with the analytic coupling $\alpha_E(Q^2)$ which avoids the hurdle of the unphysical singularities in the IR region [3]. The method consists in solving the eigenvalue equation for the squared mass operator as given by Eq. (2), obtained by a three dimensional reduction of the original BS equation. The relativistic potential $U$ then follows from a proper ansatz (1) on the Wilson loop to encode confinement, and is the sum (2) of a one-gluon-exchange term $U_{OGE}$ and a confining term $U_{Conf}$. The coupling occurring in the perturbative part of the potential needs to be IR finite since its argument is to be identified with the momentum transfer in the $q\bar{q}$ interaction, and this typically takes values down to few hundreds MeV. The usual perturbative running coupling $\bar{\alpha}_s$ has then been replaced by the 1-loop analytic expression $\alpha_E^{(1)}$ Eq. (21) with an effective QCD scale $\Lambda^{(1,eff)}_{n_f=3} = 193$ MeV. This value reasonably reproduces the 3-loop analytic coupling, normalized at the Z boson mass along with world average [31] (i.e., $\Lambda^{(3)}_{n_f=5} = 236$ MeV which leads to $\Lambda^{(3)}_{n_f=3} = 417$ MeV by continuous threshold matching, see App. C).

Thus we have taken advantage of our BS results for the meson spectrum, both in the light and heavy quark sector, to infer within this framework the fixed coupling value for each state that exactly matches the theoretical and experimental mass. Our results are twofold. On the one hand, as expected (given the good agreement of theoretical and experimental meson data), the 3-loop analytic coupling reasonably fits $\alpha_s^{\exp}(Q^2)$ from 1 GeV down to 200 MeV within the estimated theoretical and experimental errors, with very few exceptions. This confirms and yields a quantitative estimate of the relevance of the APT to IR phenomena down to 200 MeV.

On the other hand, below this scale, the experimental points exhibit a tendency to fall under the APT curve, giving a hint on the vanishing of $\alpha_s(Q^2)$ as $Q$ approaches zero, in concert with some results from lattice simulations [10]. Moreover, this deep IR behavior can be theoretically understood in the framework of a recent “massive” modification [8] of the APT algorithm, which takes into account effects of a finite threshold in the dispersion relation. Since in the extremely low $Q$ region confinement forces play the dominant role, the
reasonable agreement between the “massive” APT model and the results of the BS formalism would suggest a relation between the linear potential, arising from the area term in the ansatz [I], and the thresholds effects in the analytic properties of the QCD coupling, to be further investigated.

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A Second order Bethe-Salpeter formalism

In the QCD framework a second order four point quark-antiquark function and full quark propagator can be defined as

\[ H_4(x_1, x_2; y_1, y_2) = -\frac{1}{3} \text{Tr}_{\text{color}} \langle \Delta_1(x_1, y_1; A) \Delta_2(y_2, x_2; A) \rangle \]  \hspace{1cm} (36)

and

\[ H_2(x - y) = \frac{i}{\sqrt{3}} \text{Tr}_{\text{color}} \langle \Delta(x, y; A) \rangle, \]  \hspace{1cm} (37)

where

\[ \langle f[A] \rangle = \int DA M_F[A] e^{iS_G[A]} f[A], \]  \hspace{1cm} (38)

\[ M_F[A] = \text{Det} \Pi_j \{1 + g\gamma^\mu A_\mu (i\gamma^\nu \partial_\nu - m_{\text{curr}})^{-1}\} \] and \( \Delta(x, y; A) \) is the second order quark propagator in an external gauge field.

The quantity \( \Delta \) is defined by the second order differential equation

\[ (D_\mu D^\mu + m_{\text{curr}}^2 - \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu}) \Delta(x, y; A) = -\delta^4(x - y), \]  \hspace{1cm} (39)

\( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \) and \( D_\mu = \partial_\mu + igA_\mu \) and it is related to the corresponding first order propagator by \( S(x, y; A) = (i\gamma^\nu D_\nu + m_{\text{curr}}) \Delta(x, y; A) \), \( m_{\text{curr}} \) being the so-called current mass of the quark.

The advantage of considering second order quantities is that the spin terms are more clearly separated and it is possible to write for \( \Delta \) a generalized
Feynman-Schwinger representation, i.e., to solve Eq. (39) in terms of a quark path integral \[4, 6\]. Using the latter in (36) or (37) a similar representation can be obtained for \(H_4\) and \(H_2\).

The interesting aspect of this final representation is that the gauge field appears in it only through a Wilson line correlator \(W\). In the limit \(x_2 \to x_1, y_2 \to y_1\) or \(y \to x\) the Wilson lines close in a single Wilson loop \(\Gamma\) and if \(\Gamma\) stays on a plane, \(i \ln W\) can be written according to (1) as

\[
i \ln W = \frac{16\pi}{3} \alpha_s \oint dz^\mu d z' D_{\mu\nu}(z - z') + \\
\sigma \oint dz^0 d z'^0 \delta(z^0 - z'^0)|z - z'| \int_0^1 d\lambda \left\{ 1 - \frac{dz_\perp}{dz^0} + (1 - \lambda) \frac{dz_\perp'}{dz'^0} \right\}^{\frac{1}{2}}.
\]

The area term here is written as the algebraic sum of successive equal time strips and \(dz_\perp = dz - (dz \cdot r)r/r^2\) denotes the transversal component of \(dz\). The basic assumption now is that in the center of mass frame (40) remains a good approximation even in the general case, i.e., for non flat curves and when \(x_2 \neq x_1, y_2 \neq y_1\) or \(y \neq x\). Then, by appropriate manipulations on the resulting expressions, an inhomogeneous Bethe-Salpeter equation for the 4-point function \(H_4(x_1, x_2; y_1, y_2)\) and a Dyson-Schwinger equation for \(H_2(x-y)\) can be derived in a kind of generalized ladder and rainbow approximation. This should appear plausible, even from the point of view of graph resummation, for the analogy between the perturbative and the confinement terms in (40).

In momentum representation, the corresponding homogeneous BS-equation becomes

\[
\Phi_P(k) = -i \int \frac{d^4u}{(2\pi)^4} \hat{I}_{ab} \left( k - u; \frac{1}{2} P + \frac{k + u}{2}, \frac{1}{2} P - \frac{k + u}{2} \right) \times \\
\times \hat{H}_2^{(1)} \left( \frac{1}{2} P + k \right) \sigma^a \Phi_P(u) \sigma^b \hat{H}_2^{(2)} \left( -\frac{1}{2} P + k \right),
\]

where \(\sigma^0 = 1; a, b = 0, \mu\nu;\) the c.m. frame has to be understood, \(P = (m_B, 0)\); \(\Phi_P(k)\) denotes the appropriate second order wave function, that in terms of the second order field \(\phi(x) = (i\gamma^\mu D_\mu + m_{\text{curr}})^{-1}\psi(x)\) can be defined as the Fourier transform of \(\langle 0|\phi(\frac{x}{2})\psi(-\frac{x}{2})|P\rangle\).
Similarly, in terms of the irreducible self-energy, defined by \( \hat{H}_2(k) = (k^2 - m_{\text{curv}}^2)^{-1} + i(k^2 - m_{\text{curv}}^2)^{-1} i \Gamma(k) \hat{H}_2(k) \), the Dyson-Schwinger equation can be written

\[
\hat{\Gamma}(k) = \int \frac{d^4l}{(2\pi)^4} \hat{I}_{ab}(k - l; \frac{k + l}{2}, \frac{k + l}{2}) \sigma^a \hat{H}_2(l) \sigma^b .
\] (42)

The kernels are the same in the two Eqs. (41) and (42), consistently with the requirement of chiral symmetry limit \([32]\), being given by

\[
\hat{I}_{0:0}(Q; p, p') = 16\pi \frac{4}{3} \alpha_s p^a p^b \hat{D}_{ab}(Q) +
+4\sigma \int d^3 \zeta e^{-iQ\cdot\zeta} |\zeta(0)| e(p_0') e(p_0') \int_0^1 d\lambda \{ p_0'^2 - 2Q_{\rho}p_{\rho'} - [\lambda p_0'^2 + (1 - \lambda)p_0 p_0']^2 \}^{1/2},
\]

\[
\hat{I}_{\mu\nu:0}(Q; p, p') = 4\pi i \frac{4}{3} \alpha_s (\delta_\mu^\nu Q_\nu - \delta_\nu^\nu Q_\mu) p'_{\rho} \hat{D}_{ab}(Q) -
-\sigma \int d^3 \zeta e^{-iQ\cdot\zeta} |\zeta(0)| \frac{\epsilon_\mu p_\nu - \epsilon_\nu p_\mu}{|\zeta| \sqrt{p_0'^2 - p_T^2}},
\]

\[
\hat{I}_{0:ab}(Q; p, p') = -4\pi i \frac{4}{3} \alpha_s p^a p^b \hat{D}_{ab}(Q) +
+\sigma \int d^3 \zeta e^{-iQ\cdot\zeta} |\zeta(0)| \frac{\epsilon_\rho p'_{\sigma} - \epsilon_\sigma p'_{\rho}}{|\zeta| \sqrt{p_0'^2 - p_T^2}},
\]

\[
\hat{I}_{\mu\nu:ab}(Q; p, p') = \pi \frac{4}{3} \alpha_s (\delta_\mu^\nu Q_\nu - \delta_\nu^\nu Q_\mu) (\delta_\sigma^\sigma Q_\sigma - \delta_\rho^\rho Q_\rho) \hat{D}_{ab}(Q),
\] (43)

where in the second and in the third equation \( \zeta_0 = 0 \) has to be understood. Notice that, due to the privileged role given to the c.m. frame, the terms proportional to \( \sigma \) in (43) formally are not covariant.

In fact, it can be checked that \( \Gamma(k) \) can be consistently assumed to be spin independent and Eq. (42) can be rewritten in the simpler form

\[
\Gamma(k) = i \int \frac{d^4l}{(2\pi)^4} \frac{R(k, l)}{l^2 - m^2 + \Gamma(l)},
\] (44)

with

\[
R(k, l) = 4\pi \frac{4}{3} \alpha_s \left[ (k + l)^\mu (k + l)^\nu D_{\mu\nu}(k - l) +
(k - l)^\nu (k - l)^\mu D_{\mu\nu}(k + l) - (k - l)^\mu (k + l)^\nu D_{\mu\nu}(k - l) \right] +
+\sigma \int d^3 r e^{-i(k - l) \cdot r} \left( k_0 + l_0 \right)^2 \sqrt{1 - \left( \frac{r_\perp + l_\perp}{k_0 + l_0} \right)^2},
\] (45)
\(\mathbf{k}_\perp\) and \(\mathbf{l}_\perp\) denoting as above the transversal part of \(\mathbf{k}\) and \(\mathbf{l}\). Eq. (44) can be solved by iteration resulting in an expression of the form \(\Gamma(k^2, k^2)\), since (45) is not formally covariant. Then the constituent (pole) mass \(m\) is defined by the equation
\[
m^2 - m_{\text{curr}}^2 + \Gamma(m^2, k^2) = 0
given by extremizing \(m(k^2)\) in \(k^2\).

The 3-dimensional reduction of Eq. (41) can be obtained by a usual procedure of replacing \(H_2(k)\) with \(i(k^2 - m^2)^{-1}\) and \(\hat{I}_{ab}\) with its so-called instantaneous approximation \(\hat{I}_{ab}^{\text{inst}}(\mathbf{k}, u)\). In this way, one can explicitly integrate over \(u_0\) and arrive to a 3-dimensional equation in the form of the eigenvalue equation for a squared mass operator Eq. (2), with

\[
\langle \mathbf{k}|U|\mathbf{k}' \rangle = \frac{1}{(2\pi)^3} \sqrt{\frac{w_1 + w_2}{2w_1w_2}} \hat{I}_{ab}^{\text{inst}}(\mathbf{k}, \mathbf{k}') \sqrt{\frac{w'_1 + w'_2}{2w'_1w'_2}} \sigma_1^a \sigma_2^b.
\]

Finally by using Eq. (43) one obtains Eqs. (18-20).

Alternatively, in more usual terms, one could look for the eigenvalue of the mass operator or center of mass Hamiltonian \(H_{\text{CM}} \equiv M = M_0 + V\) with \(V\) defined by \(M_0 V + VM_0 + V^2 = U\). Neglecting term \(V^2\) the linear form potential \(V\) can be obtained from \(U\) by the replacement \(\sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1w_2w'_1w'_2}} \rightarrow \frac{1}{2\sqrt{w_1w_2w'_1w'_2}}\). The resulting expression is particularly useful for a comparison with models based on potential. In particular, in the static limit \(V\) reduces to the Cornell potential
\[
V_{\text{stat}} = -\frac{4\alpha_s}{3} r + \sigma r.
\]

Note that it is necessary to introduce a cut-off \(B\) in Eq. (44). As a consequence the constituent mass turns out to be a function of the current mass and of \(B\), \(m = m(m_{\text{curr}}, B)\). Then if one uses a running current mass \(m_{\text{curr}}(Q^2)\) we obtain a running constituent mass \(m(Q^2)\) as it has been done in [7] (see also Ref. [17]). However the singular expression used there

\[
m_{\text{curr}}(Q^2) = \hat{m} \left( \frac{Q^2/\Lambda^2 - 1}{Q^2/\Lambda^2 \ln(Q^2/\Lambda^2)} \right)^{\gamma_{\mathrm{I}}/\gamma_0}.
\]
is not consistent with Eq. (21), and if a more consistent assumption is taken, e.g.,

$$m_{\text{curr}}(Q^2) = \hat{m} \left( \alpha^{(1)}_{E}(Q^2) \right)^{\gamma_0/2\beta_0}.$$  \hfill (50)

or the other resulting from the analytization of a similar expression with $\alpha^{(1)}_{E}(Q^2)$ replaced with the ordinary perturbative $\alpha^{(1)}_{s}(Q^2)$, the dependence of $m$ on $Q^2$ is strongly reduced. For this reason even the light quark mass is here treated as a constants to be adjusted with the the data\textsuperscript{6}.

\textsuperscript{6}In this way the only role that is left to the DS equation is to justify the difference between the constituent and the current masses.
B Numerical results

The tables below display the complete set of results as explained in Sec. 5. We recall the values of all the parameters: \( \sigma = 0.18 \text{ GeV}^2 \), \( \Lambda_{n_f=3}^{(1,\text{eff})} = 193 \text{ MeV} \), \( m_q = 196 \text{ MeV} \) (\( q = u, d \)), \( m_s = 352 \text{ MeV} \), \( m_c = 1.516 \text{ GeV} \) and \( m_b = 4.854 \text{ GeV} \). Meson masses are given in MeV. The last column displays the experimental coupling \( \alpha_s^{\exp}(Q^2) \) with the theoretical error \( \Delta_{\text{NLO}} \) coming from the missing of the next to leading order diagrams, the theoretical error \( \Delta_{\Gamma} \) coming from the half width \( \Gamma/2 \) and the experimental error \( \Delta_{\text{exp}} \) respectively.

† Center of gravity masses of the incomplete multiplet estimated in analogy with other multiplets.

Table 1: \( q\bar{q} \ (q = u, d) \)

| States | \( m_{\text{exp}} \) (MeV) | \( m_{\text{th}} \) | \( Q \) | \( \alpha_s^{\text{th}} \) | \( \alpha_s^{\exp} \pm \Delta_{\text{NLO}} \pm \Delta_{\Gamma} \pm \Delta_{\text{exp}} \) |
|--------|-----------------|----------|------|-----------------|---------------------------------------------------------------------|
| \( 1^1S_0 \) \( \{ \pi^0 \ \pi^\pm \} \) | \( 134.9766 \pm 0.0006 \) | 138 | 136 | 401 | 0.534 | 0.534 \( \pm 0.122 \pm - \pm - \) |
| \( 1^3S_1 \) | \( 775.5 \pm 0.4 \) | 748.9 | 0.517 | 0.122 \( \pm 0.048 \pm - \pm - \) |
| \( 1^2S_0 \pi(1300) \) | \( 1300 \pm 100 \) | 1223 | 448 | 0.511 | 0.451 \( \pm 0.114 \pm 0.152 \pm 0.081 \) |
| \( 2^3S_1 \) | \( 1459 \pm 11 \) | 1363 | 0.427 | 0.114 \( \pm 0.062 \pm 0.010 \) |
| \( 2^1D_0 \) | 159 | 139 | & & & &
| \( 1^1P_1 \) \( b_1(1235) \) | \( 1229.5 \pm 3.2 \) | 1234.4 | 209 | 0.679 | 0.688 \( \pm 0.155 \pm 0.124 \pm 0.006 \) |
| \( 1^1D_2 \) \( \pi_2(1670) \) | \( 1672.4 \pm 3.2 \) | 1595.4 | 144 | 0.766 | 0.544 \( \pm 0.151 \pm 0.364 \pm 0.009 \) |

Table 2: \( s\bar{s} \)

| States | \( m_{\text{exp}} \) (MeV) | \( m_{\text{th}} \) | \( Q \) | \( \alpha_s^{\text{th}} \) | \( \alpha_s^{\exp} \pm \Delta_{\text{NLO}} \pm \Delta_{\Gamma} \pm \Delta_{\text{exp}} \) |
|--------|-----------------|----------|------|-----------------|---------------------------------------------------------------------|
| \( 1^3S_1 \) \( \phi(1020) \) | \( 1019.460 \pm 0.019 \) | 1019.06 | 418 | 0.525 | 0.525 \( \pm 0.098 \pm 0.002 \pm - \pm - \) |
| \( 2^3S_1 \) \( \phi(1680) \) | \( 1680 \pm 20 \) | 1601.6 | 454 | 0.508 | 0.435 \( \pm 0.096 \pm 0.068 \pm 0.019 \) |
| \( 1^1P_1 \) \( h_1(1380) \) | \( 1386 \pm 19 \) | 1472.4 | 216 | 0.672 | 0.824 \( \pm 0.098 \pm 0.083 \pm 0.032 \) |
| \( 1^3P_2 \) \( f_2(1525) \) | \( 1525 \pm 5 \) | 1518 | 5 & & &
| \( 1^3P_1 \) \( f_1(1510) \) | \( 1518 \pm 5 \) | 1483.6 | & | & &
| \( 1^3P_0 \) \( f_0(1500) \) | \( 1507 \pm 5 \) | & & & &
| \( 1^1D_2 \) \( \eta_2(1870) \) | \( 1842 \pm 8 \) | 1806.96 | 149 | 0.758 | 0.658 \( \pm 0.079 \pm 0.318 \pm 0.023 \) |
| \( 1^3F_4 \) | \( 2165 \) | 2069.8 | 118 | 0.810 | 0.452 \( \pm 0.064 \pm 0.137 \pm 0.024 \) |
| \( 1^3F_3 \) | \( 2156 \pm 11 \) | & & & &
| \( 1^3F_2 \) \( f_2(2150) \) | \( 2165 \) | 2069.8 | 118 | 0.810 | 0.452 \( \pm 0.064 \pm 0.137 \pm 0.024 \) |
Table 3: \( q\bar{s} \) \((q = u, d)\)

| States | (MeV) \( m_{\text{exp}} \) | \( m_{\text{th}} \) | \( Q \) | \( \alpha_{\text{th}} \) | \( \alpha_{\text{exp}} \pm \Delta_{\text{NLO}} \pm \Delta_{\Gamma} \pm \Delta_{\text{exp}} \) |
|--------|-----------------|--------|-----|--------|------------------|
| \( 1^1S_0 \) | \{ \( K^0 \) \( 497.648 \pm 0.022 \) \} \( 495 \) | 491.2 | 0.530 | 0.529 \( \pm 0.122 \) \( \pm - \) \( \pm - \) |
| \( 1^3S_1 \) | \{ \( K^*(892)^0 \) \( 896.00 \pm 0.25 \) \} \( 893.1 \) | 887.36 | 0.526 \( \pm 0.122 \) \( \pm 0.017 \) \( \pm - \) |
| \( 1\Delta S \) | 398 | 398 | 396.2 | 396.2 | 396.2 |
| \( 2^3S_1 \) | \( K^*(1410) \) \( 1414 \pm 15 \) | 1484.7 | 451 | 0.510 | 0.571 \( \pm 0.117 \) \( \pm 0.102 \) \( \pm 0.013 \) |
| \( 1^1P_1 \) | \( K_1(1270) \) \( 1272 \pm 7 \) | 1355 | 213 | 0.676 | 0.820 \( \pm 0.129 \) \( \pm 0.081 \) \( \pm 0.012 \) |
| \( 1^3P_2 \) | \{ \( K_2^0(1430) \) \( 1432.4 \pm 1.3 \) \} | \( 1425.6 \pm 1.5 \) | 1425.6 | 1425.6 | 1425.6 |
| \( 1^3P_1 \) | \( K_1(1400) \) \( 1402 \pm 7 \) | \( 1417.7 \) | \( 1367.2 \) | \( 0.583 \) \( \pm 0.129 \) \( \pm 0.133 \) \( \pm 0.007 \) |
| \( 1^3P_0 \) | \( K_0^*(1430) \) \( 1414 \pm 6 \) | 1414 | 1414 | 1414 | 1414 |
| \( 1^3D_3 \) | \( K_3^0(1780) \) \( 1776 \pm 7 \) | 1773 | 8 | 1773 | 1773 | 1773 |
| \( 1^3D_2 \) | \( K_2(1770) \) \( 1773 \pm 8 \) | 1763 | 1712.2 | 150 | 0.757 | 0.617 \( \pm 0.113 \) \( \pm 0.273 \) \( \pm 0.031 \) |
| \( 1^3D_1 \) | \( K^*(1680) \) \( 1717 \pm 27 \) | 1717 | 1717 | 1717 | 1717 | 1717 |
| \( 1^3F_4 \) | \( K_4^0(2045) \) \( 2045 \pm 9 \) | 2045 | 2045 | 2045 | 2045 | 2045 |
| \( 1^3F_3 \) | \( K_3(2320) \) \( 2324 \pm 24 \) | 2121 | 1973 | 116 | 0.814 | 0.248 \( \pm 0.095 \) \( \pm 0.413 \) \( \pm 0.071 \) |
| \( 1^3F_2 \) | \( K_2^0(1980) \) \( 1973 \pm 25 \) | 1973 | 1973 | 1973 | 1973 | 1973 |
Table 4: \( c\bar{c} \)

| States   | \( m_{\text{exp}} \) (MeV) | \( m_{\text{th}} \) (MeV) | \( Q \) | \( \alpha_{\chi}^{\text{th}} \) | \( \alpha_{\chi}^{\exp} \pm \Delta_{\text{NLO}} \pm \Delta_{\gamma} \pm \Delta_{\text{exp}} \) |
|----------|-----------------------------|-----------------------------|-------|-----------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| \( 1^1S_0 \) | \( \eta_c(1S) \) 2980.4 ± 1.2 | 2979.7 | 561 | 0.467 | 0.467 ± 0.025 ± 0.008 ± 0.001 |
| \( 1^3S_1 \) | \( J/\psi(1S) \) 3096.916 ± 0.011 | 3097.271 | 500 | 0.489 | 0.466 ± 0.023 ± 0.007 ± 0.004 |
| \( 1^3S_0 \) | \( \eta_c(2S) \) 3638 ± 4 | 3595 | 500 | 0.489 | 0.455 ± 0.023 ± 0.004 ± 0.001 |
| \( 2^3S_1 \) | \( \psi(2S) \) 3686.093 ± 0.034 | 3653 | 500 | 0.496 | 0.485 ± 0.022 ± 0.009 ± 0.001 |
| \( 2^3S_0 \) | \( \psi(4040) \) 4039 ± 1 | 4030 | 483 | 0.496 | 0.384 ± 0.022 ± 0.006 |
| \( 4^3S_1 \) | \( \psi(4415) \) 4421 ± 4 | 4336.6 | 474 | 0.500 | 0.631 ± 0.012 ± 0.001 ± 0.001 |

Quantum numbers of \( h_c(1P) \) and \( X(3872) \) mesons are not well established.

Table 5: \( b\bar{b} \)

| States   | \( m_{\text{exp}} \) (MeV) | \( m_{\text{th}} \) (MeV) | \( Q \) | \( \alpha_{\chi}^{\text{th}} \) | \( \alpha_{\chi}^{\exp} \pm \Delta_{\text{NLO}} \pm \Delta_{\gamma} \pm \Delta_{\text{exp}} \) |
|----------|-----------------------------|-----------------------------|-------|-----------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| \( 1^3S_1 \) | \( \Upsilon(1S) \) 9460.30 ± 0.26 | 9460.6 | 951 | 0.378 | 0.378 ± 0.006 ± 0.001 ± 0.001 |
| \( 2^3S_1 \) | \( \Upsilon(2S) \) 10023.26 ± 0.31 | 9987 | 630 | 0.446 | 0.416 ± 0.004 ± 0.001 ± 0.001 |
| \( 3^3S_1 \) | \( \Upsilon(3S) \) 10355.2 ± 0.5 | 10321.4 | 552 | 0.470 | 0.433 ± 0.003 ± 0.001 ± 0.001 |
| \( 4^3S_1 \) | \( \Upsilon(4S) \) 10579.4 ± 1.2 | 10588 | 517 | 0.482 | 0.493 ± 0.003 ± 0.003 ± 0.002 |
| \( 5^3S_1 \) | \( \Upsilon(10860) \) 10865 ± 8 | 10819.86 | 497 | 0.490 | 0.424 ± 0.003 ± 0.007 ± 0.001 |
| \( 6^3S_1 \) | \( \Upsilon(11020) \) 11018 ± 8 | 11034.2 | 506 | 0.486 | 0.508 ± 0.003 ± 0.057 ± 0.012 |

\[ \text{[Quantum numbers for } b\bar{b} \text{ mesons are not well established.]} \]

\[ \text{[Additional data points for } b\bar{b} \text{ mesons are not well established.]} \]

\[ \text{[Further data points for } b\bar{b} \text{ mesons are not well established.]} \]
Table 6: Light-heavy quarkonium systems

| States   | (MeV) | $m_{\text{exp}}$ | $m_{\text{th}}$ | $Q$ | $\alpha_s^{\text{th}}$ | $\alpha_s^{\text{exp}} \pm \Delta_{\text{NLO}} \pm \Delta_{\Gamma} \pm \Delta_{\text{exp}}$ |
|----------|-------|-----------------|----------------|-----|-----------------|-----------------------------------------------------------------|
| $q^c$    |       |                 |                |     |                 |                                                                    |
| $1^1S_0$ |       | $D^\pm$         | $1869.3 \pm 0.4$ | 1867.7 | 1843.4 | 459 | 0.506 | 0.488 $\pm$ 0.082 $\pm$ $\pm$ $\pm$ |
|          |       | $D^0$           | $1864.5 \pm 0.4$ |         |                   |                                                                 |
| $1^3S_1$ |       | $D^*(2010)^\pm$ | $2010.0 \pm 0.4$ | 2008.9 | 1999.7 |               | 0.499 $\pm$ 0.082 $\pm$ $\pm$ $\pm$ |
|          |       | $D^*(2007)^0$   | $2006.7 \pm 0.4$ |         |                   |                                                                 |
| $1\Delta S$ |   |                 |                 |     |                 |                                                                 |
|          |       |                 |                 |     |                 | 141 $\pm$ 1 | 157 |                                           |
| $1P_2$   |       | $D_2^s(2460)^\pm$ | 2459 $\pm$ 4 | |              |                                                          |
|          |       | $D_2^s(2460)^0$ | 2461.1 $\pm$ 1.6 |         |                   |                                                                 |
| $1P_1$   |       | $D_1(2420)^\pm$ | 2423.4 $\pm$ 3.1 | |              |                                                          |
|          |       | $D_1(2420)^0$   | 2422.3 $\pm$ 1.3 |         |                   |                                                                 |

Table 7: Light-heavy quarkonium systems

| States   | (MeV) | $m_{\text{exp}}$ | $m_{\text{th}}$ | $Q$ | $\alpha_s^{\text{th}}$ | $\alpha_s^{\text{exp}} \pm \Delta_{\text{NLO}} \pm \Delta_{\Gamma} \pm \Delta_{\text{exp}}$ |
|----------|-------|-----------------|----------------|-----|-----------------|-----------------------------------------------------------------|
| $s^c$    |       |                 |                |     |                 |                                                                 |
| $1^1S_0$ |       | $D^\pm$         | $1968.2 \pm 0.5$ | 1959.3 | 472 | 0.500 | 0.494 $\pm$ 0.055 $\pm$ $\pm$ $\pm$ |
|          |       | $D^0$           | $2112.0 \pm 0.6$ |         |                   |                                                                 |
| $1^3S_1$ |       | $D^*_s$         | $144 \pm 1$ | |              |                                                          |
|          |       |                 |                 |     |                 |                                                                 |
| $1\Delta S$ |   |                 |                 |     |                 |                                                                 |
|          |       |                 |                 |     |                 | 2573.5 $\pm$ 1.7 | 2545.25 | 236 | 0.651 | 0.626 $\pm$ 0.036 $\pm$ 0.009 $\pm$ 0.002 |
| $1P_1$   |       | $D_{s2}(2573)^\pm$ | 2535.35 $\pm$ 0.34 $\pm$ 0.5 | |              |                                                          |
|          |       | $D_{s1}(2536)^\pm$ | 5412.8 $\pm$ 1.7 | 5408.4 | | 0.473 $\pm$ 0.024 $\pm$ $\pm$ $\pm$ |
|          |       |                 |                 |     |                 |                                                                 |
| $s^b$    |       |                 |                |     |                 |                                                                 |
| $1^1S_0$ |       | $B^0_s$         | $5367.5 \pm 1.8$ | 5343.3 | 535 | 0.476 | 0.457 $\pm$ 0.024 $\pm$ $\pm$ $\pm$ |
|          |       | $B^s$           | $5412.8 \pm 1.7$ | 5408.4 | | 0.473 $\pm$ 0.024 $\pm$ $\pm$ $\pm$ |
| $1\Delta S$ |   |                 |                 |     |                 |                                                                 |
|          |       |                 |                 |     |                 | 47 $\pm$ 4 | 65.2 |                                           |
| $1P$     |       | $B_{sJ}^*(5850)$ | $5853 \pm 15$ | $5829.7$ | 255 | 0.634 | 0.592 $\pm$ 0.013 $\pm$ 0.042 $\pm$ 0.027 |
C 3-loop analytic coupling

As mentioned, we compare our results with the 3-loop analytic coupling (8), the difference with respect to the 4-loop approximation being negligible [23]. The exact 3-loop spectral density (7) has a rather cumbersome structure in terms of the Lambert function (see Ref. [27]). However, for practical purposes, it can be approximated with a high accuracy by the discontinuity of the 3-loop perturbative expression (as given by PDG [31]) across the physical cut [15]:

\[
\rho_1^{(3)}(\sigma) = \frac{1}{\beta_0 (t^2 + \pi^2)^3} \left[ t(3\pi^2 - t^2) J(t) - (3t^2 - \pi^2) R(t) \right].
\]

(51)

In this equation \( t = \ln(\sigma/\Lambda^2) \),

\[
J(t) = 2t - B_1[tG_1(t) + G_2(t)] + B_1^2 G_1(t)[2G_2(t) - 1],
\]

(52)

\[
R(t) = t^2 - \pi^2 - B_1[tG_2(t) - \pi^2 G_1(t)] + B_1^2 [G_2^2(t) - \pi^2 G_1^2(t) - G_2(t) - 1] + B_2,
\]

(53)

\[
G_1(t) = \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{t}{\pi} \right), \quad G_2(t) = \frac{1}{2} \ln(t^2 + \pi^2)
\]

(54)

with \( B_j = \beta_j / \beta_0^{j+1} \) being the combination of the \( \beta \) function expansion coefficients, \( \beta_0 \) and \( \beta_1 \) given in Sec. 2.1 and

\[
\beta_2^{\text{MS}} = \frac{1}{(4\pi)^3} \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right).
\]

(55)

In Fig. 5 the 3-loop analytic coupling (8) is compared to the same level perturbative expression, both normalized at the Z boson mass to the world average value \( \alpha_s(M_Z^2) = 0.1176 \pm 0.0020 \) [31] and evolved at the heavy quark thresholds crossing by continuous matching conditions. This gives for the analytic coupling the scaling constant \( \Lambda_{n_f=3}^{(3)} \simeq 417 \pm 42 \text{ MeV} \) in the IR domain.
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Figure 1: Quarkonium spectrum, three different calculations. Diamonds refer to the freezing prescription for the running coupling, squares and circles refer to the calculation with the 1-loop analytic coupling [21] and two different expressions for running constituent masses of light quarks. Horizontal lines represent experimental data.
Figure 2: Relative difference between the one-loop analytic running coupling $\alpha_E^{(1)}(Q^2)$ with $\Lambda_{n_f=3}^{(1,\text{eff})} = 193$ MeV and three-loop $\alpha_E^{(3)}(Q^2)$ with $\Lambda_{n_f=3}^{(3)} = 417$ MeV in the range $0 < Q < 1.2$ GeV.
Figure 3: Comparison between points extracted from Tables I-VII in App. B and the three-loop analytic coupling \( \Lambda_{n_f=3}^{(3)} = (417 \pm 42) \) MeV (solid curves). The “massive” one-loop analytic running coupling \( \Lambda_{n_f=3}^{(1,\text{eff})} = 204 \) MeV and the effective mass \( m_{\text{eff}} = (38 \pm 10) \) MeV. Three-loop perturbative coupling (dot-dashed curve) corresponds to \( \Lambda_{n_f=3}^{(3)} = 318 \) MeV. Circles, stars and squares refer respectively to \( q\bar{q}, s\bar{s} \) and \( q\bar{s} \) with \( q = u, d \) (light-light states). Heavy-heavy states, \( c\bar{c} \) and \( b\bar{b} \), are represented by diamonds and crosses. Finally in the light-heavy sector, asterisks stay for \( q\bar{c} \) and \( q\bar{b} \), whereas plus signs stand for \( s\bar{c} \) and \( s\bar{b} \). Data for triplet and singlet states referring to the same multiplet are combined in a weighted average according to their errors. Error bars include both theoretical and experimental uncertainty and are drawn only if relevant.
Figure 4: Summary of low (○) and high energy (○) data against the three-loop analytic coupling (solid curve) and its perturbative counterpart (dot-dashed curve) both normalized at the Z boson mass. Also shown is the “massive” one-loop analytic coupling (dashed curve) same as in Fig. 3.
Figure 5: Three-loop QCD coupling below 2 GeV: the analytic expression (solid curve) versus the perturbative one (dot-dashed curve), both normalized at the Z boson mass according to the world average [31].