Sharpening The Leading Singularity

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Abstract

We show how studying leading singularities of Feynman diagrams, when all momenta are complex, gives a simple way of writing multi-loop and multi-particle scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills. The simplicity of the method is equivalent to that of the quadruple cut technique introduced in [hep-th/0412103] at one-loop. The new technique only involves the computation of residues and the solution of linear equations. In our technique both parity even and parity odd pieces of a coefficient are computed simultaneously and it is only at the end that a separation can be made if desired. We explain the procedure via examples. The main example, which we compute in detail, is the five-particle two-loop amplitude first given in [hep-th/0604074]. Another feature of our method is that the helicity structure of the amplitude only enters in the inhomogeneous part of the linear equations. In other words, the homogeneous part is universal. We illustrate this feature by presenting the linear equations which determine a large class of terms for MHV and next-to-MHV six-particle two-loop amplitudes.

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I. INTRODUCTION

The idea of determining the $S$-matrix of a theory from the structure of its singularities was intensively studied in the 60’s. The main idea was to consider the $S$-matrix as an analytic function of the kinematical invariants with singularities determined by physical input. Most efforts were directed towards the study of hadronic interactions. Nowadays we know that understanding analytically the strong interactions is a very hard problem.

A more modern version, similar in spirit, was developed in the 90’s, where extensive use of branch cut singularities was shown to be a powerful tool in the computation of scattering amplitudes in Yang-Mills theories. These techniques came to be known as the unitarity based method.

More recently, the highest codimension singularities, called leading singularities already in the 60’s, were shown to be a powerful tool in computing amplitudes. At one-loop, the problem of computing amplitudes is reduced to that of computing tree amplitudes for theories where no triangles or bubbles appear. One such theory is $\mathcal{N} = 4$ super Yang-Mills (SYM) and it has been hypothesized that so too is $\mathcal{N} = 8$ supergravity.

In this paper we show that at higher loop orders the power of the leading singularity has only been partially unleashed. By carefully studying all leading singularities we propose a method which reduces the computation of multi-loop and multi-particle amplitudes in $\mathcal{N} = 4$ SYM to the computation of residues (which end up being related to tree amplitudes) and to the solution of linear equations.

The discontinuity across the leading singularity at one loop is computed by collecting all Feynman diagrams that share four given propagators and then cutting the propagators to turn the sum into simply the product of four tree-level amplitudes. Cutting means removing the principal value part of a Feynman propagator, i.e. $1/(\ell^2 + i\epsilon) \rightarrow \delta^{(+)}(\ell^2)$. Thus the integral over the four-vector $\ell$ is completely localized by the four delta functions. The value of the integral for each solution $\ell_s$ is given by the jacobian of the change of variables from $\ell$ to the argument of the delta functions evaluated at $\ell_s$. The final answer is then the sum over all contributions. In massless theories there are two solutions $\ell_s^{(1)}$ and $\ell_s^{(2)}$. In general, the product of the four tree-level amplitudes gives different answers at the different points.
\{\ell_s^{(1)}, \ell_s^{(2)}\}. An important exception is when the number of particles is four. In this case, the product of amplitudes is equal to \(A_{\text{tree}}^{\text{st}}\) for both values of \(\ell\).

In general, the support of the delta functions is outside the region where \(\ell\) is a real vector. This means that the computation of the leading singularity\(^1\) is more naturally interpreted in terms of a contour integral in \(\mathbb{C}^4\) where \(\ell\) is now a complex vector. There are two distinct leading singularities and the prescription of [14] is equivalent to choosing a contour that picks up both residues. A natural question is what the role of each isolated leading singularity is. Naively, one could try to expand the one-loop amplitude in scalar boxes and then compute their coefficients by matching the residues at a given leading singularity. If one did that one would find different answers for the same coefficient which would be a contradiction.

In this paper we show that the resolution to this puzzle is that at one-loop, reduction formulas that express \(e.g.\) pentagons in terms of boxes, derived in [21, 22, 23], are not valid for generic complex integration contours. This means that in order to write an expression in terms of scalar integrals which reproduces all singularities of Feynman diagrams correctly one has to allow for higher point integrals. This explains why four particles is special and why the first non-trivial case is that of five particles, where one should allow for a pentagon.

At higher loops, the original approach of [15] is to sum over all solutions. In [16], it was stated that one should be able to work with individual singularities, but this was in the context of four-particles where there is no distinction.

In this paper we show that requiring agreement between Feynman diagrams and scalar (or generalized scalar) integrals on each individual leading singularity provides enough linear equations to determine higher-loop and multi-particle amplitudes.

In an nutshell, one starts collecting all Feynman diagrams with chosen propagators that give rise to topologies with only boxes. By writing an ansatz of scalar integrals (\(i.e.,\) with numerator one) one requires agreement at all leading singularities independently. This gives a set of linear equations. Sometimes the system of equations does not have solutions which means that generalized scalar integrals (\(i.e.,\) with numerators that are propagators with negative power) must be added. The numerators serve as zeroes to cancel poles so that the new integrals have zero residue on leading singularities where they are not needed.

\(^1\) Here and throughout the paper we will abuse terminology and refer to the discontinuity across a given leading singularity as the leading singularity itself.
We explain the method via examples. Our main example is the five-particle two-loop amplitude. This amplitude was first computed in [24]. In this case, our computation only requires solving linear equations in two variables!

An interesting new feature already found in [25], as compared to the four-particle amplitude, is that after normalizing by the corresponding tree-level amplitude one finds terms that are parity even and some that are parity odd. Using our technique, it becomes very transparent why the answer does not have definite parity. The reason is that leading singularities come in pairs. For real external momenta, one of them would be located at the complex conjugate value of the other. For four particles, the value of Feynman diagrams on each leading singularity is the same and hence one gets a parity even answer. For five or more particles, the values differ and one gets complex solutions.

It is important to mention that the actual helicity structure of the amplitude only enters as the inhomogeneous part of the linear equations that determine the coefficients of the integrals. In this sense, the homogeneous part of the linear system of equations is universal. Since for five particles, all non-trivial helicity configurations are MHV (or MHV) the feature just described is not very surprising. This is why we present the linear equations which determine a large class of terms contributing to six-particle MHV and next-to-MHV two-loop amplitudes. Solving the equations could allow a comparison to the recent computation of the parity even part of the six-particle MHV amplitude in [27].

This paper is organized as follows. In section II we establish our conventions. In section III we give a detailed explanation of the leading singularity technique at one-loop. The discussion of how the scalar basis must be extended is given for five particles since the results there can be directly applied to the two-loop case. In section IV we introduce the leading singularity technique at two loops in the simplest case, i.e., the four-particle amplitude. In section V we present the main example of the paper, the five-particle two-loop amplitude, in complete detail. In section VI we present the linear equations for coefficients in MHV and next-to-MHV six-particle two-loop amplitudes. In section VII we give conclusions and future directions.
II. PRELIMINARIES

Scattering amplitudes of on-shell particles in $\mathcal{N} = 4$ SYM with gauge group $U(N)$ can be written as a sum over color-stripped partial amplitudes using the color decomposition [10, 29, 30]. Each partial amplitude admits a large $N$ expansion. More explicitly,

$$A_n(1, 2, \ldots, n) = \delta^{(4)}(p_1 + p_2 + \ldots + p_n) \ Tr(T^{a_1} T^{a_2} \ldots T^{a_n}) A_n(1, 2, \ldots, n)$$

$$+ \text{permutations} + \ldots$$

where the sum is over non-cyclic permutations of the external states (cyclic ones being symmetries of the trace) and the ellipsis represents terms with double and higher numbers of traces. $A_n$ may be expanded in perturbation theory and we denote the $L$-loop planar partial amplitude by $A_n^{(L)}$. We also use $A_n^{(0)} = A_n^{\text{tree}}$.

Our conventions are:

- A tree-level MHV amplitude has two particles of negative helicity and the rest of positive helicity.
- A null vector $p_\mu$ is written as a bispinor as $p_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}$.
- The Lorentz invariant inner product of two null vectors $p_\mu$ and $q_\mu$ is given by $2p \cdot q = \langle \lambda_p, \lambda_q \rangle [\bar{\lambda}_p, \bar{\lambda}_q]$. We also use $\langle \lambda_p, \lambda_q \rangle = \langle p, q \rangle$ and $[\bar{\lambda}_p, \bar{\lambda}_q] = [p, q]$.
- All external momenta in an amplitude are outgoing and usually denoted by $k_i$.
- Some useful Lorentz invariant combinations are $s_{ij} = (k_i + k_j)^2$, $t_{ijl} = (k_i + k_j + k_l)^2$, and $[a|b + c|d] = [a, b] \langle b, d \rangle + [a, c] \langle c, d \rangle$.

III. LEADING SINGULARITY AT ONE-LOOP

Any one-loop amplitude in $\mathcal{N} = 4$ SYM may be expressed both as a sum over Feynman diagrams, and also in terms of scalar box integrals [7]:

$$A_n^{(1)} = \sum \{1\text{-loop Feynman diagrams} \} = \sum_\mathcal{I} B_\mathcal{I} \times I(K_1^T, K_2^T, K_3^T, K_4^T)$$

where the second sum is over all partitions $\mathcal{I}$ of $\{1, 2, \ldots, n\}$ into four non-empty sets, $K_i^T$ equals the sum of the momenta in the $i^{th}$ subset of partition $\mathcal{I}$ and the $B_\mathcal{I}$ are coefficients to
FIG. 1: Computation of a coefficient using the leading singularity of a box. The lines circling the propagators represent the $T^4$ contour of integration. The left hand side of the figure represents the sum of all 1-loop Feynman diagrams - note that only those Feynman diagrams that contain the displayed propagators actually contribute to this particular contour integral.

be determined. Since we are working with color ordered Feynman diagrams, one considers only partitions that respect the color ordering.

Scalar box integrals are of the form

$$I(K_1, K_2, K_3, K_4) := \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2}.$$  \hspace{1cm} (3)

Since scalar integrals are known explicitly (see e.g. [28]), the problem of computing any one-loop amplitude is reduced to that of determining the coefficients $B_T$.

The amplitude, defined in terms of Feynman diagrams, possesses many singularities. Some of them are branch cut singularities and comparing them on both sides is a way of obtaining information about the coefficients and in many cases it allows their determination like for MHV amplitudes [7]. The main difficulty is that a given branch cut is generically shared by several scalar integrals and therefore several coefficients appear at the same time and the information must be disentangled.

It turns out that the highest codimension singularities of Feynman diagrams, called leading singularities, receive contributions from a single scalar box integral. Thus, using leading singularities of Feynman diagrams one can determine all coefficients, $B_T$, one by one [14].

If we let $f_i(\ell)$ with $i = 1, \ldots, 4$ correspond to the four factors in the denominator of (3), then the leading singularity is computed by replacing $1/f_i(\ell)$ by $\delta(f_i(\ell))$. Applying this to
both sides of (2) one finds that

\[ \sum_{\ell^*} \det \left( \frac{\partial f_i}{\partial \ell^*_\mu} \right)^{-1} \sum_{\text{Multiplet } i=1}^4 A_{\text{tree}}^{(i)} \bigg|_{\ell = \ell^*} = B_J \times \sum_{\ell^*} \det \left( \frac{\partial f_i}{\partial \ell^*_\mu} \right)^{-1} \bigg|_{\ell = \ell^*}, \quad (4) \]

where the sum over \( \ell^* \) means a sum over all solutions to the equations \( f_i(\ell) = 0 \) for \( i = 1, \ldots, 4 \). In general there are two solutions. The second sum on the l.h.s. is over all members of the \( \mathcal{N} = 4 \) supermultiplet as choices of internal particles.

When the number of external particles is four, i.e. \( n = 4 \), both solutions give a jacobian factor equal to \( 1/(st) \) and the sum over the product of tree-level amplitudes equals \( A_{\text{tree}}^{st} \). This implies that \( B = A_{\text{tree}}^{st} \). For five or more particles the sum over tree amplitudes is not the same for both solutions. In fact, for five particles the product of tree amplitudes vanishes in one solution and it gives \( A_{\text{tree}}^{st} \) in the other. Here \( s = (K_1 + K_2)^2 \) and \( t = (K_2 + K_3)^2 \) with \( K_i \)'s as in (3). Using this in (4) one finds that \( B = A_{\text{tree}}^{st}/2 \) which is the correct answer (30). Note the factor of half coming from the fact that on the r.h.s. of (4) one gets a factor of two.

### A. Sharpening The Leading Singularity

The solutions, \( \ell^* \), to the equations \( f_i(\ell) = 0 \) are complex in general. This makes the mathematical interpretation of the leading singularity more transparent if defined as a contour integral in \( \mathbb{C}^4 \). The contour integral is obtained by simply taking (3) and allowing \( \ell \) to be a complex vector. The contour of integration has the topology of a four-torus, \( \mathbb{T}^4 \cong (S^1)^4 \), around each isolated singularity, and it is given by \( \Gamma = \{ \ell : |f_i(\ell)| = \epsilon_i \} \) with \( \epsilon_i \) some small positive number. The integral is computed by a generalization of Cauchy’s theorem to higher dimensions (see for example section 5.1 of [31]). This can be seen by a change of variables to local coordinates \( z_i \)'s in terms of which the singular point is located at the origin, i.e. \( z_i = f_i(\ell) = 0 \). Now it is clear that the contour of integration, \( \Gamma = \{ \ell : |z_i| = \epsilon_i \} \), when projected on each \( z_i \)-plane is just a circle of radius \( \epsilon_i \). The integral becomes

\[ I = \left( \prod_{i=1}^4 \oint_{z_i=0} \frac{dz_i}{z_i} \right) \det \left( \frac{\partial z_i}{\partial \ell^*_\mu} \right)^{-1}. \quad (5) \]

\[ ^2 \text{Here and throughout the paper we use the convention that the symbol } \oint \text{ contains a factor of } 1/(2\pi i). \]
A natural question that comes from this formalism is how to compare the sum over Feynman diagrams and the sum over scalar integrals when evaluated on each leading singularity independently. In other words, we would like to make sense of (4) term by term in the sum over solutions. The reader might anticipate a problem from the fact that the sum over tree amplitudes gives different answers for $n > 4$. Before giving the resolution to the puzzle explicitly for $n = 5$ let us discuss in detail the four-particle case.

1. **Four Particles**

For four particles there is a single planar configuration with no triangles or bubbles. This is depicted in figure 2A. The location of the two leading singularities is: $\ell_{a\bar{a}} = \alpha \lambda_a^{(1)} \bar{\lambda}_{\bar{a}}^{(4)}$ and $\ell_{a\bar{a}} = \tilde{\alpha} \lambda_a^{(4)} \bar{\lambda}_{\bar{a}}^{(1)}$ with $\alpha = [1, 2]/[4, 2]$ and $\tilde{\alpha} = (1, 2)/\langle 4, 2 \rangle$.

It is straightforward to compute the product of the four three-particle tree-level amplitudes on each of the solutions. In each case we find the same answer $A^{\text{tree}\, \text{st}}$. In order to reproduce this with scalar integrals we start with a single box and a coefficient to be determined.

Comparing both sides evaluated on each of the two leading singularities we find the same condition:

$$B = A^{\text{tree}\, \text{st}}.$$  \hspace{1cm} (6)

This indeed gives the correct answer. As we will see, the fact that we got the same condition by requiring the two sides to match on each leading singularity independently is only a peculiarity of $n = 4$ and it will not be true for any $n > 4$.

2. **Five Particles**

Five particles is the first non-trivial example where the conditions coming from the two leading singularities are different. At this point there seem to be a puzzle since requiring agreement on each singularity independently leads to distinct coefficients for the same scalar box integral which is indeed a contradiction.

The resolution to this puzzle is simple but subtle. It has to do with the fact that when momenta are complex, the integrand of loop amplitudes is invariant under the complexification of the (double cover) of the Lorentz group, i.e. $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. However,
FIG. 2: Sum over Feynman diagrams with no triangles or bubbles. (A) Unique four-particle configuration. (B) Unique five-particle topology with some choice of external labels. For this particular choice of external helicities there is a single configuration of internal helicities.

the integral might be invariant under different subgroups of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ depending on the choice of contour\textsuperscript{3}. The physical choice of contour requires the loop momentum to be a real vector and hence it breaks the group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ down to the diagonal $SL(2, \mathbb{C})_D$ which is a double cover of the physical Lorentz group $SO(3, 1)$. In particular, on the physical contour, scalar integrals are parity invariant. This is what fails on the individual $T^4$ contours.

Let us illustrate this by computing a five-particle MHV amplitude at one-loop.

Once again there is a single topology of a planar configuration of Feynman diagrams with only boxes. This is shown in figure 2B with a particular choice of labels. In this case, the location of the two leading singularities is

$$q_{aa}^{(1)} = \lambda^{(1)}_a \left( \tilde{\lambda}^{(1)}_a + \frac{\langle 2, 3 \rangle}{\langle 1, 3 \rangle} \tilde{\lambda}^{(2)}_a \right), \quad q_{aa}^{(2)} = \left( \lambda^{(1)}_a + \frac{[2, 3]}{[1, 3]} \lambda^{(2)}_a \right) \tilde{\lambda}^{(1)}_a,$$

(7)

Evaluating the product of the four tree-level amplitudes is again straightforward and gives

$$\sum_{\text{Multiplet}} \prod_{i=1}^{4} A^{\text{tree}}_{i} \bigg|_{q=q^{(1)}} = A^{\text{tree}}_5 s_{12}s_{23}, \quad \sum_{\text{Multiplet}} \prod_{i=1}^{4} A^{\text{tree}}_{i} \bigg|_{q=q^{(2)}} = 0.$$

We want to reproduce this behavior using scalar integrals. The natural candidate is the scalar box integral in figure 3A. It is easy to see that on each of the leading singularities the scalar box integral gives the same residue, \textit{i.e.}, $1/(s_{12}s_{23})$. If this box were the only contribution on the scalar integral side we would find that by comparing to the first equation\textsuperscript{3},

\textsuperscript{3} A similar situation was encountered in the derivation of MHV diagrams from twistor string theory in \[32\].
FIG. 3: (A) Scalar box integral for five particles. The particular choice of labels corresponds to the integral $I^{(a):1}$ in the text. (B) Scalar pentagon integral. This integral is denoted by $I^{(b)}$. As in the rest of the paper, external momenta are taken to be outgoing.

In (8) the coefficient would be $B = A_5^\text{tree} s_{12} s_{23}$, which is twice the known answer. If, instead, we use the second equation in (8) we would find $B = 0$ which is also wrong.

This is the puzzle mentioned earlier. In order to discover the resolution, let us assume we did not know that $\mathcal{N} = 4$ SYM amplitudes can be written purely in terms of scalar boxes\textsuperscript{4}. The natural starting point to reproduce the behavior of the collection of the Feynman diagrams is the scalar box integral in figure 3. However, as we have seen, this is not enough. Therefore we need to expand the basis. The only other scalar integral that can contribute to the same leading singularities, but with different residues, is a pentagon (see figure 3B).

Denoting the coefficient of the pentagon by $C$ we find that reproducing (8) means

$$B + \frac{C}{(q^{(1)} + k_5)^2} = A_5^\text{tree} s_{12} s_{23}, \quad B + \frac{C}{(q^{(2)} + k_5)^2} = 0.$$ (9)

where $(q + k_5)^2$ is the only propagator of the pentagon not shared by the box.

These equations can easily be solved. A convenient way to express the solution is obtained by making the following definitions

$$\beta_i := \left(1 + \frac{\langle i + 2, i + 3 \rangle [i + 2, i]}{\langle i + 1, i + 3 \rangle [i + 1, i]}\right)^{-1}, \quad \tilde{\beta}_i := \left(1 + \frac{\langle i + 2, i \rangle [i + 2, i + 3]}{\langle i + 1, i \rangle [i + 1, i + 3]}\right)^{-1}.$$ (10)

Note that for real momenta, $\tilde{\beta}_i$ is the complex conjugate of $\beta_i$. This will be important when identifying odd and even pieces under parity.

\textsuperscript{4}In dimensional regularization this is true only to order $\epsilon^0$ except for $n = 4$ when is true to all orders in $\epsilon$. 
The solution to the equations in (9) is

\[ B = -A_{\text{tree}}^{s_{12}s_{23}} \frac{\tilde{\beta}_5}{\beta_5 - \tilde{\beta}_5}, \quad C = A_{\text{tree}}^{s_{51}s_{12}s_{23}} \frac{\beta_5 - \tilde{\beta}_5}{\beta_5 - \tilde{\beta}_5}. \] (11)

The attentive reader might anticipate another possible contradiction: If we have fixed the coefficient of the pentagon by using the leading singularity in figure 2B, it is hard to believe that it will simultaneously solve the equations coming from figure 2B after a cyclic permutation of the labels. For this to be true the same coefficient of the pentagon should work in all cases, i.e., \( C/A_{\text{tree}} \) in (11) must be invariant under cyclic permutations of the labels of the external particles!

Indeed, an explicit computation reveals that

\[ C := \frac{C}{A_{\text{tree}}} = \frac{s_{i,i+1}s_{i+1,i+2}s_{i+2,i+3}}{\beta_i - \tilde{\beta}_i}. \] (12)

is the same quantity for all \( i \in \{1, 2, 3, 4, 5\} \). How to write \( C \) in a manifestly invariant form will become clear in the next section.

We are ready to write down the final answer for the amplitude. In order to compare with the known result it is convenient to separate contributions into parity even and parity odd pieces. Note that the coefficient of the pentagon has definite parity; it is parity odd. The coefficient of the box does not have definite parity. In order to decompose it we write in the numerator of \( B \), \( \tilde{\beta}_5 = (\tilde{\beta}_5 - \beta_5)/2 + (\tilde{\beta}_5 + \beta_5)/2 \). The amplitude is then given as

\[ \frac{A_5^{(1)MHV}}{A_5^{\text{treeMHV}}} = \sum_{cyclic} \left( \frac{1}{2}s_{12}s_{23}I^{(a):1} - \frac{1}{2} \left( \frac{\beta_5 + \tilde{\beta}_5}{\beta_5 - \tilde{\beta}_5} \right)s_{12}s_{23}I^{(a):1} \right) + C I^{(b)}. \] (13)

where \( I^{(a):i} \) is a scalar box integral with three massless legs given by the momenta of the \( i^{th} \), \((i+1)^{th}\) and \((i+2)^{th}\) particles while \( I^{(b)} \) is a pentagon with all massless legs as shown in figure 3B.

We claim that this is the correct answer. In order to prove it we have to go back to the usual contour of integration where \( \ell \) is a real vector and where integrals must be dimensionally regulated. In that case we have to show that (13) reduces to

\[ \frac{A_5^{(1)MHV}}{A_5^{\text{treeMHV}}} \bigg|_{\text{Dim.Reg.}} = \sum_{cyclic} \left( \frac{1}{2}s_{12}s_{23}I^{(a):1} \right). \] (14)

\(^5\) Note that the pentagon is invariant under a cyclic permutation of the labels.
Comparing our answer (13) with the answer for real $\ell$ and in dimensional regularization we conclude that the scalar box and pentagon integrals when dimensionally regulated must satisfy the following identity

$$I^{(b)} = \frac{1}{2} \sum_{i=1}^{5} \left( \frac{\beta_{i-1} + \tilde{\beta}_{i}}{s_{i-1,i}} \right) I^{(a):i}. \tag{15}$$

In other words, a pentagon can be written as a sum of five boxes with particular coefficients.

This is an example of what is known as a reduction formula [21, 22, 23]. Indeed, one can check that the coefficients in (15) agree with those obtained in [23] by using differential equations, i.e.,

$$\frac{\beta_{i} + \tilde{\beta}_{i}}{s_{i+1,i+1}} = (\alpha_{i-2} - \alpha_{i-1} + \alpha_{i} - \alpha_{i+1} + \alpha_{i+2}) \alpha_{1} \alpha_{2} \alpha_{3} \alpha_{4} s_{i+1,i+2}s_{i+2,i+3} \tag{16}$$

with $\alpha_{i} = s_{i+1,i+2}s_{i+2,i+3}/\Xi$ and $\Xi = \sqrt{-s_{12}s_{23}s_{34}s_{45}s_{51}}$. If we had known the existence of reductions formulas but not their form then the leading singularities would have given an elementary way of finding them.

Let us conclude this section by explaining why reduction formulas do not hold on the $T^4$ contours used to compute independent leading singularities. First observe that the choice of one such contour breaks the $\mathbb{Z}_2$ symmetry that exchanges the two factors in $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ which is the symmetry of the integrand. This is nothing but a parity transformation. The reason this affects the reduction formulas is that their derivation relies on the identity [22]

$$\int_{\Gamma} d^4 \ell \frac{\epsilon_{\mu\nu\rho\sigma} R_{1}^{\mu} R_{2}^{\nu} R_{3}^{\rho}}{\ell^2 (\ell + R_{1})^2 (\ell + R_{2})^2 (\ell + R_{3})^2} = 0 \tag{17}$$

with $\Gamma$ a real contour.

A simple way to prove this is by noting that

$$I^{\mu} = \int_{\Gamma} d^4 \ell \frac{\ell^{\mu}}{\ell^2 (\ell + R_{1})^2 (\ell + R_{2})^2 (\ell + R_{3})^2} = A_{1} R_{1}^{\mu} + A_{2} R_{2}^{\mu} + A_{3} R_{3}^{\mu} \tag{18}$$

for some scalar functions $A_{i}$. This is obviously true by Lorentz invariance including parity invariance. In general any four vector, in particular, $I^{\mu}$ can be expanded in a basis of vectors given by $R_{1}^{\mu}$, $R_{2}^{\mu}$, $R_{3}^{\mu}$ and $\epsilon_{\mu\nu\rho\sigma} R_{1}^{\nu} R_{2}^{\rho} R_{3}^{\sigma}$. The latter does not contribute in (18) because it is not parity invariant.

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6 In comparing with [23], we had to shift the index $i$ because what we call $I^{(a):i}$ is called $I^{(a):i-1}$ in [23].

7 If any of the $R^{2}_{i}$ vanishes this integral is divergent. The analysis in dimensional regularization was performed in [23]. Out $T^4$ contour renders all integrals finite and this is why we do not need any regulators.
Now it is clear that on a given $T^4$ contour corresponding to a single leading singularity the integral is not parity invariant and (18) does not hold due to the presence of the extra vector $\epsilon^{\mu\nu\rho\sigma} R_{1\nu} R_{2\rho} R_{3\sigma}$ in the expansion of $I^\mu$. This also shows why in the original quadruple cut technique introduced in [14], which is equivalent to summing over both contributions, reduction formulas are valid. Summing over both contours preserves parity since a parity transformation corresponds to exchanging the two $T^4$’s.

3. Higher Point Amplitudes

Repeating the procedure we applied to the five-particle case to higher point amplitudes one should find reduction formulas for higher point scalar integrals since we know that in dimensional regularization all $n$-particle amplitudes are given in terms of boxes. A case of special interest in the following is $n = 6$. We postpone the discussion to section VI where several coefficient of two-loop six-particles amplitudes are studied.

IV. LEADING SINGULARITY AT TWO LOOPS

The physical meaning of leading singularities at higher loops was given in [16]. It was shown that Feynman diagrams arrange themselves so that a meaningful $T^{4L} \cong (S^1)^{4L}$ contour can be defined at $L$ loops. A generalization of what was done at one-loop would require interpreting the integral over $L$ loop momenta as a contour integral in $C^{4L}$. However, combining Feynman diagrams that share the topology of a graph with only boxes generically gives rise to less than $4L$ propagators. Therefore defining a $T^{4L}$ contour is more subtle than in the one-loop case. As shown in [15], once a partial integration is done on one of the loop variables, new propagator-like singularities appear which can be used to define the $T^{4L}$ contour iteratively. The physical meaning given in [16] is that after the partial integration, one produces a set of Feynman diagrams at one loop order less than the original ones and which possesses new tree-level factorization channels. Those channels are the hidden singularities needed to define the $T^{4L}$ contour. In this section we restrict our attention to $L = 2$. We start with the $n = 4$ case in order to clarify the concepts just explained.
FIG. 4: Basis of scalar two-loop integrals for four-particle amplitudes in $\mathcal{N} = 4$ SYM.

A. Four Particles

We will illustrate the use of the leading singularity technique on a well known case: Four-particle two-loop amplitudes. These were first computed in [25, 33] and are given by

$$A_4^{(2)} = A_4^{\text{tree}} (st^2 I^{(1)} + s^2 t I^{(2)})$$

(19)

where the integrals $I^{(1)}$ and $I^{(2)}$ are shown in figure 4.

We now proceed to reproduce this result using the leading singularities.

In $\mathcal{N} = 4$ SYM if we sum over Feynman diagrams and consider contour of integrations where all legs attached to a one-loop subdiagram are on-shell, such a subdiagram does not contain triangles or bubbles. This means that in order to study all leading singularities, we only need to consider sums of Feynman diagrams that contain only boxes. In the case at hand, these are Feynman diagrams with the topology of a 2-loop ladder.

A particular choice of external particles is shown in figure 5. Performing the integration over the $p$ momentum we find that the product over the four three-particle tree-level amplitudes, including the jacobian, gives rise to a four-particle tree-level amplitude [16]. Therefore, we are left with the sum over one-loop Feynman diagrams shown on the upper right of figure 5. Note that we have used that for $n = 4$ the product of amplitudes gives the same answer when evaluated on the two solutions $p^{(1)}$ and $p^{(2)}$.

The tree-level four particle amplitude has two factorization channels. One of them is in the limit when $(k_1 + k_2)^2 \to 0$ while the other is in the limit when $(q - k_1)^2 \to 0$. It must now be clear how to define the remaining $T^4$ in order to perform the $q$ integration; one uses the original three propagators together with the new propagator $1/(q - k_1)^2$. On this contour
FIG. 5: Computation of the amplitude using the leading singularity of Feynman diagrams. The residue of the $p$ integral on any of the two $T^4$s represented by the lines circling the propagators is a tree-level four particle amplitude. Choosing a $T^4$ in the $q$ variable which induces a factorization in the $(q - k_1)^2$ channel of the tree amplitude one finds the diagram enclosed by dashed lines. The residue on the final $T^4$ is again a four-particle tree-level amplitude.

the tree amplitude factorizes and gives rise to the diagram in the bottom left of figure 5. This is identical to the one-loop case. Once again there are two solutions $q^{(1)}$ and $q^{(2)}$. On both solutions the diagrams evaluate to $8A_{\text{tree}}^4$.

On the scalar integral side, we start with an integral of the form $I^{(2)}$ shown in figure 4. After performing the integration over $p$ one finds that the jacobian gives rise to a factor of $1/(s(q - k_1)^2)$. This means that the scalar integral has a non-zero residue on the same $T^8$ used for the evaluation of the Feynman diagrams. One finds that the total integration gives $1/(s^2t)$.

Comparing both sides we conclude that the coefficient of $I^{(2)}$ is $s^2tA_{\text{tree}}^4$.

By using a cyclic permutation of the labels we find that the coefficient of the integral $I^{(1)}$ must be $st^2A_{\text{tree}}^4$. Combining the results we reproduce the known answer (19).

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8 The computation also involves the jacobian. In other words, we are computing the full residue.
V. TWO-LOOP FIVE-PARTICLE AMPLITUDE

Five-particle MHV two-loop amplitudes in $\mathcal{N} = 4$ SYM were computed in \cite{24} using the unitarity based method. Here we write the known answer and then show how to re-derive it using the leading singularity technique. The answer we find comes out in an strikingly different form. It is important to make a distinction between MHV and $\overline{\text{MHV}}$ as the two amplitudes, when normalized by $A_5^\text{tree}$, are different.

The expression given in \cite{24} is

$$A_5^{(2)\text{MHV}} = \frac{1}{8}A_5^\text{tree, MHV} \sum_{\text{cyclic}} (s_{12}^2 s_{23} I^{(a)}(\epsilon) + s_{12}^2 s_{15} I^{(b)}(\epsilon) + s_{12} s_{34} s_{45} I^{(c)}(\epsilon))$$

$$+ R \left[ 2I^{(d)}(\epsilon) - 2s_{12} I^{(e)}(\epsilon) + \frac{s_{12}}{s_{34}s_{45}} (\frac{s_{23}}{s_{51}} I^{(b)}(\epsilon) - \frac{s_{23}}{s_{51}} I^{(a)}(\epsilon)) + \frac{s_{23}}{s_{51}s_{45}} I^{(c)}(\epsilon) \right]$$

where

$$R = \epsilon_{1234} s_{12} s_{23} s_{34} s_{45} s_{51} / G_{1234},$$

$$\delta_{abc} = s_{51} s_{12} + a s_{12} s_{23} + b s_{23} s_{34} - s_{45} s_{51} + c s_{34} s_{45}$$

and

$$\epsilon_{1234} = 4i\epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = \text{tr} [\gamma_5 k_1 k_2 k_3 k_4],$$

$$G_{1234} = \det \begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{34} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{24} & s_{34} & 0 \end{pmatrix}$$

All integrals $I^{(a)}$, $I^{(b)}$, $I^{(c)}$, $I^{(d)}$ and $I^{(e)}$ are defined for a particular choice of external particles as shown in figure \[\text{6}\].

A. Computation Using The Leading Singularity Technique

We start by following the same steps as in the four-particle case. There are three different topologies of diagrams with only boxes. We have to analyze all of them and built an expression in terms of scalar integrals which reproduces the behavior of Feynman diagrams when evaluated in all possible leading singularities.

The three topologies are depicted in figure \[\text{7}\] where a particular choice of external labels was made. The total set of configurations is then obtained by cyclic permutations of the labels.
1. **First Topology**

Consider the set of Feynman diagrams in figure 7A. This is the first of the three different topologies that contain only boxes for five particles.

Carrying out the integration over a $T^4$ contour in the $p$ variables we find that for any of the two solutions we get a four-particle tree-level amplitude with 1 and 2 as external legs (see figure 8). In order to continue, we choose a $T^4$ contour in the $q$ variables that induces a factorization of the four-particle amplitude that contains the external particles 1 and 2. Note that in this case there is a second four-particle amplitude; the one that contains external particles 4 and 5. A $T^4$ which induces a factorization of the second four-particle amplitude must also be considered and we postpone its analysis to section V A 4. Note that this second possibility was not present in the four-particle case.

From the bottom left of figure 8 we find that the problem at hand is identical to the one-loop five-particle amplitude discussed in the previous section. The difference between the two cases comes in analyzing the scalar integrals. By analogy with the one loop case, we start by assuming that the basis contains integrals $I^{(a)}$ and $I^{(c)}$ in figure 6. After carrying
FIG. 7: All inequivalent topologies of sums of Feynman diagrams with only boxes for five particles at two loops. A particular choice of external labels is shown. All cyclic permutations of the labels must also be considered.

Out the integration over the first $\mathcal{T}^4$ in the scalar integrals, we find that the problem also reduces to that of the one-loop case except that the coefficients are multiplied by an extra factor of $1/s_{12}$. In other words, if we denote the coefficients of the two-loop integrals $I^{(a)}$ and $I^{(e)}$ by $B^{2\text{-loop}}$ and $C^{2\text{-loop}}$ respectively, then the equations are

$$\frac{B^{2\text{-loop}}}{s_{12}} + \frac{C^{2\text{-loop}}}{s_{12}(q^{(1)} + k_5)^2} = A^{\text{tree}} s_{12} s_{23}, \quad \frac{B^{2\text{-loop}}}{s_{12}} + \frac{C^{2\text{-loop}}}{s_{12}(q^{(2)} + k_5)^2} = 0.$$  \hspace{1cm} (23)

The solution is obtained by making the substitution $B \rightarrow B^{2\text{-loop}}/s_{12}$ and $C \rightarrow C^{2\text{-loop}}/s_{12}$ in (11), i.e.,

$$B^{2\text{-loop}} = -A^{\text{tree}} s_{12} s_{23} \frac{\tilde{\beta}_5}{\beta_5 - \tilde{\beta}_5}, \quad C^{2\text{-loop}} = A^{\text{tree}} s_{51} s_{12} s_{23} \frac{\beta_5 - \tilde{\beta}_5}{\beta_5 - \tilde{\beta}_5}.$$  \hspace{1cm} (24)

Recall that the coefficient of the pentagon $C$ at one loop was invariant under cyclic permutations of the labels. Here the symmetry has been explicitly broken by the extra factor of $s_{12}$ and hence the integral $I^{(e)}$ must be inside the sum over cyclic permutations.
FIG. 8: Evaluation of the sum over Feynman diagrams on two different $T^8$ contours. The residue of the $p$ integral on any of the two $T^4$'s represented by the lines circling the propagators is a tree-level four particle amplitude. For $n = 5$, as opposed to $n = 4$, there are two choices for the $T^4$ contour in the $q$ variable. Here we choose the one that induces a factorization in the $(q - k_1)^2$ channel of the tree amplitude. The other $T^4$ is also important and is considered in figure 10. The residue on the final $T^4$ is the five-particle tree-level amplitude on $q^{(1)}$ and zero on $q^{(2)}$.

2. Second Topology

The next topology of Feynman diagrams is shown in figure 7B. The contributing integrals are clearly $I^{(b)}$ and $I^{(e)}$ in figure 6. After performing the integration over $p$ on a $T^4$ contour and choosing the remaining $T^4$ in the $q$ variables to induce a factorization on the four-particle amplitude with 1 and 2, we go back to a one-loop calculation related to the one done in the previous case by a cyclic permutation.

The solution for the coefficients is obtained by performing a cyclic permutation on the one-loop coefficients and then multiplying by the $1/s_{12}$ factor to convert them into two-loop coefficients. The explicit form of the coefficients is

$$B^{2\text{-loop}} = -A^{\text{tree}} \frac{s_{51}s_{12}^2}{\beta_4 - \tilde{\beta}_4}, \quad C^{2\text{-loop}} = A^{\text{tree}} \frac{s_{45}s_{51}s_{12}^2}{\beta_4 - \tilde{\beta}_4}. \quad (25)$$
FIG. 9: Scalar integrals contributing to the third kind of topology. In the computation of the residues the propagators not used as poles remain and must be evaluated at the location of the singularity. There is one propagator left in each pentagon-box integral and they are enclosed by dashed lines in the figure.

Note that we have computed the coefficient of the pentagon-box integral, $I^{(e)}$, once again and the answer looks very different. Using the same identity (12) as in the previous section it is clear that the two formulas are the same since all we have done is to multiply them by the same $s_{12}$ factor.

3. Third Topology

Let us consider the final topology shown in figure 7C. The possible scalar diagrams that contribute to the $T^8$ which computes the natural leading singularity of this topology are shown in 9. Let us denote their coefficients by $B$, $C_5$ and $C_3$ respectively. The coefficients $C_3$ and $C_5$ have already been computed. So in practice one would need a single equation to determine $B$. Let us however consider the two equations that arise and check that they can be consistently solved.

The situation here is different from the situation in the previous cases since the sum over Feynman diagrams has zero residue on the leading singularities we will consider. From that point of view, we can think of the integral in figure 9A as canceling unphysical singularities in the integrals in figures 9B and 9C which were already shown to be present in the amplitude.

The location of the unphysical leading singularities is given by

$$p^{(1)} = \frac{[1, 2]}{[5, 2]} \lambda_1 \tilde{\lambda}_5, \quad q^{(1)} = -\frac{\langle 5, 4 \rangle}{\langle 1, 4 \rangle} \lambda_1 \tilde{\lambda}_5,$$  \hspace{1cm} (26)
and
\[ p^{(2)} = \langle 1, 2 \rangle \lambda_5 \bar{\lambda}_1, \quad q^{(2)} = -\langle 5, 4 \rangle \lambda_5 \bar{\lambda}_1. \]  

(27)

In order to understand how these arise note that the middle propagator, \( 1/(q - p)^2 \), in figure \( \text{A} \) has two poles on the locus where \( p^2 = q^2 = 0 \). These are \( \langle p, q \rangle = 0 \) and \( [p, q] = 0 \). Using both of them at the same time gives the \( T^8 \) used to obtain (26) and (27).

Since \( p \) and \( q \) are proportional to each other in both solutions it is clear that the sum over Feynman diagrams shown in figure \( \text{C} \) vanishes.

The equations we have to solve take the form
\[ B + \frac{C_5}{(q^{(i)} - k_1 - k_2)^2} + \frac{C_3}{(p^{(i)} - k_2 - k_3)^2} = 0 \]  

(28)

for \( i = 1, 2 \). The consistency condition that the known coefficients \( C_3 \) and \( C_5 \) must satisfy for these equations to have a solution is
\[ \frac{C_5}{(q^{(1)} - k_1 - k_2)^2} + \frac{C_3}{(p^{(1)} - k_2 - k_3)^2} = \frac{C_5}{(q^{(2)} - k_1 - k_2)^2} + \frac{C_3}{(p^{(2)} - k_2 - k_3)^2}. \]  

(29)

Using the explicit form of the coefficients
\[ C_3 = A^{\text{tree}} \frac{s_{34}s_{45}s_{51}}{\beta_3 - \bar{\beta}_3}, \quad C_5 = A^{\text{tree}} \frac{s_{51}s_{12}s_{23}}{\beta_5 - \bar{\beta}_5}, \]  

(30)

one can check that (29) is indeed satisfied. Therefore any of the two equations determine \( B \). Choosing \( i = 1 \) gives
\[ B = -\frac{C_5}{(q^{(1)} - k_1 - k_2)^2} - \frac{C_3}{(p^{(1)} - k_2 - k_3)^2}. \]  

(31)

This expression can be dramatically simplified if (29) is used to solve for \( C_3 \) in terms of \( C_5 \). The answer turns out to be
\[ B = -\frac{C_5}{s_{12}}, \]  

(32)

This is the coefficient of integral \( I^{(d)} \) in figure \( \text{D} \).

4. Additional Leading Singularities

As mentioned in the discussion of the first kind of topology, once the integration over the \( T^4 \) corresponding to the \( p \) variables is carried out, one is left with a sum over Feynman diagrams that possesses two four-particle amplitudes (see figure \( \text{E} \); one which contains
particles 1 and 2 and another which contains particles 4 and 5. Previously, we chose the remaining $T^4$ contour such that the amplitude with 1 and 2 factorizes. Now we have to check that the leading singularity corresponding to 4 and 5 factorizing also works.

Here we get four equations corresponding to any combination of $q^{(1)}$, $q^{(2)}$ and $p^{(1)}$, $p^{(2)}$. Note that none of the scalar integrals with the topology of two boxes, i.e. $I^{(a)}$, $I^{(b)}$, $I^{(d)}$ contributes to these leading singularities. At this point, we have a single integral that contributes, i.e., $I^{(e)}$. Not surprisingly, it is not possible to solve all the equations with the coefficient of a single integral. This means that at least one more integral is missing.

The requirements for the new integral are:

- It must be non-zero on all the new $T^8$ contours.
- It must vanish on all the previous $T^8$ contours since they have already been accounted for.

The only possibility is to start with an integral of the form $I^{(e)}$ which ensures the first requirement and then introduce a zero in the numerator with removes the pole $(q - k_1)^2$ which enters in the definition of all other $T^8$'s already studied. Removing the pole at $(q - k_1)^2$ guarantees that the new integral has zero residue in all previous leading singularities thus fulfilling the second requirement.

The new integral has the topology of $I^{(e)}$ and a numerator factor $(q - k_1)^2$. This integral is precisely $I^{(e)}$ in figure 6.

Let us show that by adding $I^{(e)}$ to the basis all remaining leading singularities can be accounted for. Let its coefficient be $D$.

The four leading singularities are located at

$$p^{(1)}(q) = \frac{[1, 2]}{[q, 2]} \lambda_1 \lambda_q, \quad p^{(2)}(q) = \frac{\langle 1, 2 \rangle}{\langle q, 2 \rangle} \lambda_q \lambda_{\bar{1}},$$

and

$$q^{(1)} = -\lambda_5 \left( \lambda_5 + \frac{\langle 3, 4 \rangle}{\langle 3, 5 \rangle} \lambda_4 \right), \quad q^{(2)} = -\lambda_5 \left( \lambda_5 + \frac{[3, 4]}{[3, 5]} \lambda_4 \right) \lambda_5.$$

The value of $p$ does not enter in the computation and we only have to consider the two equations coming from the two values of $q$'s. The equations are obtained in a completely analogous way from the previous calculations following the steps in figure 10. The two
FIG. 10: Evaluation of the sum over Feynman diagrams over the new $T^8$ contours. In figure 8 we chose the second $T^4$ to induce a factorization in the $(q - k_1)^2$ channel, here we choose it to induce a factorization of the second four-particle amplitude in the $(q + k_5)^2$ channel.

The equations are

$$\frac{C}{s_{12}s_{34}s_{45}(q^{(1)} - k_1)^2} + \frac{D}{s_{34}s_{45}s_{12}} = A_{\text{tree}}, \quad \frac{C}{(q^{(2)} - k_1)^2} + D = 0. \quad (35)$$

The coefficient $C$ has already been computed so in practice one would simply use the second equation to determine $D$. Let us not do that and solve the equations for $D$ and $C$.

Defining

$$\gamma := \left(1 + \frac{\langle 3, 4 \rangle [4, 1]}{\langle 3, 5 \rangle [5, 1]}\right)^{-1}, \quad \tilde{\gamma} := \left(1 + \frac{\langle 4, 1 \rangle [3, 4]}{\langle 5, 1 \rangle [3, 5]}\right)^{-1} \quad (36)$$

the solution is given as follows

$$C = A_{\text{tree}}^{\text{tree}} \frac{s_{34}s_{45}s_{51}s_{12}}{(\gamma - \tilde{\gamma})}, \quad D = -A_{\text{tree}}^{\text{tree}} \frac{s_{12}s_{34}s_{45}}{\gamma - \tilde{\gamma}}. \quad (37)$$

An explicit computation reveals that the new expression for $C$ agrees with that of $C^{2\text{-loop}}$. found earlier.
5. Final Result

We now collect all the results obtained by requiring that all leading singularities are reproduced correctly. Using the scalar integrals defined in figure 6 the two-loop amplitude is

\[
\frac{A_5^{(2)}}{A_5^{(0)}} = \sum_{\text{cyclic}} s_{12} \left( -\frac{s_{12}s_{23}\tilde{\beta}_5}{\beta_5 - \beta_5} I^{(a)} - \frac{s_{51}s_{12}\tilde{\beta}_4}{\beta_4 - \beta_4} I^{(b)} - \frac{s_{34}s_{45}\tilde{\gamma}}{\gamma - \tilde{\gamma}} I^{(c)} - \frac{s_{51}s_{23}}{\beta_5 - \beta_5} I^{(d)} + \frac{s_{51}s_{12}s_{23}}{\beta_5 - \beta_5} I^{(e)} \right) \tag{38}
\]

where

\[
\beta_i := \left( 1 + \frac{\langle i + 2, i + 3 \rangle[i + 2, i]}{\langle i + 1, i + 3 \rangle[i + 1, i]} \right)^{-1}, \quad \gamma := \left( 1 + \frac{\langle 3, 4 \rangle[4, 1]}{\langle 3, 5 \rangle[5, 1]} \right)^{-1}, \tag{39}
\]

and \(\tilde{\beta}_i\) and \(\tilde{\gamma}\) the parity conjugated expressions of \(\beta_i\) and \(\gamma\) respectively.

An even simpler expression can be obtained if one uses the different identities found in the previous subsections to write

\[
\frac{1}{\gamma - \tilde{\gamma}} = \frac{s_{12}s_{23}}{s_{34}s_{45}} \frac{1}{(\beta_5 - \tilde{\beta}_5)}, \quad \frac{1}{\beta_4 - \tilde{\beta}_4} = \frac{s_{23}}{s_{45}} \frac{1}{(\tilde{\beta}_5 - \beta_5)}.
\]

Using (40) in (38) one finds

\[
\frac{A_5^{(2)}}{A_5^{(0)}} = \sum_{\text{cyclic}} \frac{s_{12}s_{23}}{\beta_5 - \beta_5} \left( -s_{12}\tilde{\beta}_5 I^{(a)} - \frac{s_{51}s_{12}\tilde{\beta}_4}{s_{45}} I^{(b)} - s_{12}\tilde{\gamma} I^{(c)} - s_{51} I^{(d)} + s_{51}s_{12} I^{(e)} \right). \tag{41}
\]

In order to compare with the known answer it is convenient to split the coefficients in parity even and odd terms. Note that using our technique both pieces are computed simultaneously and equally straightforwardly. The separation into definite parity terms is easily done by recalling that for real momenta, \(\tilde{\beta}, \tilde{\gamma}\) are the complex conjugate to \(\beta\) and \(\gamma\) respectively.

This means that we can write, for example,

\[
-\frac{s_{12}s_{23}\tilde{\beta}_5}{\beta_5 - \beta_5} I^{(a)} = \frac{1}{2} s_{12}^2 s_{23} \left( 1 - \frac{\beta_5 + \beta_5}{\beta_5 - \beta_5} \right) I^{(a)}. \tag{42}
\]

Note that some coefficients are naturally parity odd. Like those of \(I^{(e)}\) and \(I^{(d)}\).

It is easy to check analytically using a symbolic manipulation program like Mathematica that, up to an overall normalization of \(1/4\), (38) is exactly equal to (20).

\[24\]
VI. A PEEK AT TWO-LOOP SIX-PARTICLE AMPLITUDES: MHV AND NEXT-TO-MHV

One of the advantages of our technique is that the homogenous part of the linear equations that determine the coefficients of the integrals is helicity independent! All the helicity information enters in the inhomogeneous part.

In order to illustrate this feature the first non-trivial case is that of six particles. This is the first case where next-to-MHV configurations are possible. In this section we choose a particular subset of Feynman diagrams contributing to two-loop six-particle amplitudes. Studying the leading singularities one can write down linear equations that determine the complete coefficient of all pentagon-pentagon integrals and of a certain class of pentagon-box integrals. The determination of the complete set of linear equations which gives the full amplitude is outside the scope of this paper and we leave it for future work.

The set of Feynman diagrams we consider generates the five topologies shown in figure [11]. We write down explicitly the equations coming from (A) and (B), as the remaining three, (C), (D), and (E), are completely analogous to (B).

A. Topology (A)

The integrals contributing to the first kind of topology are shown in figure [12]. If we were to follow the same steps as in the previous section we would start by taking only the first two integrals in the figure and then realize that it is not possible to solve the four equations that arise by comparing all leading singularities. From the experience with the five-particle case, we start by adding eight integrals with numerators such that they will contribute to the four leading singularities under consideration but they will not contribute to other singularities where they are not needed just like in the case of $I^{(c)}$ for five particles.

Using the labels in the figure, the four leading singularities are found by choosing any combination of $p_*$’s and $q_*$’s from

$$p^{(1)} = \frac{\langle 2,3 \rangle}{\langle 1,3 \rangle} \lambda_1 \bar{\lambda}_2, \quad p^{(2)} = \frac{[2,3]}{[1,3]} \lambda_2 \bar{\lambda}_1, \quad q^{(1)} = -\frac{\langle 5,6 \rangle}{\langle 4,6 \rangle} \lambda_4 \bar{\lambda}_5, \quad q^{(2)} = -\frac{\langle 5,6 \rangle}{\langle 4,6 \rangle} \lambda_5 \bar{\lambda}_4. \quad (43)$$

Let us denote the evaluation of the sum over Feynman diagrams (see figure [11A]) in a
FIG. 11: Topologies of sums over Feynman diagrams which can be used to determine the coefficients of all pentagon-pentagon and some class of pentagon-box integrals in a six-particle two-loop amplitude. Some choice of external labels has been made. No helicities have been assigned since all choices, MHV and next-to-MHV, can be treated simultaneously.

particular solution \((p,q)\) by \(F_{p,q}\). Then the equations are

\[
\begin{align*}
8_{12}8_{23}8_{45}8_{56} F_{p,q} &= \left[ B + \frac{1}{(p+q-k_{234})^2} \left( C + (E_1 + D_1(q-k_{234}))^2(p+k_{61})^2 + 
\right.\right. \\
&\left.\left. (E_3 + D_2(p-k_{234}))^2(q-k_{234})^2 + (E_4 + D_3(p+k_{61}))^2(q-k_{34})^2 + 
\right.\right. \\
&\left.\left. (E_2 + D_4(q-k_{34}))^2(p-k_{234})^2 \right) \right]
\end{align*}
\]

The labeling of the coefficients is explained in the caption of figure [12]

The system at hand involves four equations and ten unknown coefficients. In order to find a set of equations sufficient to completely determine the coefficients we have to consider the other topologies in figure [11]

Before going to the next topology, lead us compute \(F(p,q)\) in some cases in order to illustrate the procedure which turns out to be fairly simple since the computation is reduced to that at one-loop.
FIG. 12: Scalar and generalized scalar integrals that can contribute to the first kind of topology. Their coefficients are $B, C, E_1, E_2, E_3, E_4, D_1, D_2, D_3, D_4$ respectively. In order to avoid cluttering of the figure we have added the minimal amount of information needed to determine the labels. In the double box diagram, other external legs are labeled following the color ordering. All pentagon-pentagon diagram naturally inherit their labeling from the double-box diagram.

All the steps are shown in figure 13. Consider the lower box in the two-loop diagram. These are the Feynman diagrams in a one-loop five-particle amplitude where two of the external momenta are actually internal legs in the two-loop diagram. From the one-loop discussion in the previous section we know that depending on the helicities, one solution, $p_*$, gives zero while the other gives $A_5^{\text{tree}}$. Plugging this into the two-loop diagram one gets, in the non-zero case, a one-loop six particle diagram. If the amplitude we are trying to compute is MHV or $\overline{\text{MHV}}$ then the answer is either zero or $A_6^{\text{tree}}$.

As an explicit example take helicities to be $\{1^-, 2^-, 3^+, 4^+, 5^+, 6^+\}$. Then we find

$$F(p^{(1)}, q^{(1)}) = A_6^{\text{tree}}, \quad F(p^{(1)}, q^{(2)}) = F(p^{(2)}, q^{(1)}) = F(p^{(2)}, q^{(2)}) = 0.$$  

(45)
FIG. 13: Computation of functions $F_{p,q}$ in two cases: MHV with helicity configuration \{1^-, 2^-, 3^+, 4^+, 5^+, 6^+\} and next-to-MHV (split), i.e. \{1^-, 2^-, 3^-, 4^+, 5^+, 6^+\}

If the amplitude is next-to-MHV then the answer is more interesting as it corresponds to the coefficient of a particular one-mass integral in a six-particle one-loop amplitude. All six-particle one-loop next-to-MHV amplitudes where computed in the early 90’s in \textcite{8}. They are all given in terms of a quantity defined in eq. 6.13 of \textcite{8} (we have parity conjugated the expression in \textcite{8})

$$B_0 := \frac{1|2 + 3|4\langle 3|1 + 2|6\rangle t_{123}^3}{[1, 2][2, 3]\langle 4, 5\rangle\langle 5, 6\rangle(t_{123t_{45}} - s_{12s_{45}})(t_{123t_{234}} - s_{23s_{56}}).} \tag{46}$$

As a explicit example take helicities to be \{1^-, 2^-, 3^-, 4^+, 5^+, 6^+\}. Then we find

$$F(p^{(1)}, q^{(1)}) = F(p^{(2)}, q^{(1)}) = F(p^{(2)}, q^{(2)}) = 0, \quad F(p^{(1)}, q^{(2)}) = B_0. \tag{47}$$

It should be clear that any other helicity configurations can be treated analogously.

**B. Topology (B)**

The scalar integrals that contribute to these leading singularities are shown in figure 14.

The location of the leading singularities is slightly more involved as $p$ is determined as a function of $q$. Let us give the two solutions for $q$ and then give the two $q$-dependent solutions for $p$.

$$q^{(1)} = -\lambda_1 \left( \bar{\lambda}_1 + \frac{[2, 3]}{[1, 3]} \bar{\lambda}_2 \right), \quad q^{(2)} = -\left( \lambda_1 + \frac{[2, 3]}{[1, 3]} \lambda_2 \right) \bar{\lambda}_1. \tag{48}$$
FIG. 14: Scalar and generalized scalar integrals that can contribute to the second kind of topology. Their corresponding coefficients are $F, G, C, E_1, E_2, E_3, E_4, D_1, D_2, D_3, D_4$ respectively. Just as in figure 12, the labels in the diagram are determine from the those of the first integral.

The solutions for $p$ are given as $p^{(1)} (q) = (\alpha \lambda_5 + \beta \lambda_6) \bar{\lambda}_q$ and $p^{(2)} (q) = \lambda_q (\bar{\alpha} \bar{\lambda}_5 + \bar{\beta} \bar{\lambda}_6)$ with

$$\alpha = \frac{(6,4)[5,6][4,q] + (s_{45} + s_{46})[5,q]}{[q,4][q|5 + 6|4]}, \quad \beta = \frac{(5,4)[5,6][q,4] + (s_{45} + s_{46})[q,6]}{[q,4][q|5 + 6|4]}$$

and $\bar{\alpha}, \bar{\beta}$ the parity conjugate of $\alpha, \beta$.

Once again we can easily write down the linear equations coming from imposing the correct behavior at the four leading singularities. Let us denote by $H_{p,q}$ the value of the sum over Feynman diagrams on one of the four solutions $p^*, q^*$. Then the equations are

$$t_{456}s_{12}s_{23}(q - k_{56})^2 H_{p,q} = F + G(q - k_{56})^2 + \frac{1}{(p+k_0)^2} (C + (E_1 + D_1(p - k_1)^2)(q - k_6)^2 + (E_3 + D_2(q - k_{56})^2)(p - k_1)^2 + (E_4 + D_3(q - k_6)^2)(p - k_{12})^2 + (E_2 + D_4(p - k_{12})^2)(q - k_{56})^2).$$

The labels of the coefficients are explained in the caption of figure 14.

Combining the two systems of equations, (44) and (50), one finds eight equations.

Repeating the same procedure for the three remaining topologies in figure 11 which are completely analogous to topology $(B)$ one finds four equations in each case. This gives a total of twenty equations for seventeen coefficients, i.e., eight pentagon-box integrals and nine pentagon-pentagon integrals.
Using seventeen equations to determine the coefficients leaves three more as consistency checks.

We end this section by mentioning that the next natural set of equations to consider comes from double-box topologies where one of the boxes is only a four-particle one-loop amplitude. For a given choice of the two external legs to the four-particle amplitude, say \( \{1, 2\} \), there are three topologies. These correspond to the possible ways of distributing the remaining external legs; in our example we would find \( \{3, 4, 5; 6\} \), \( \{3, 4; 5, 6\} \), \( \{3; 4, 5, 6\} \). By looking at the leading singularity which uses the propagator coming from the jacobian of the four-particle box, these give rise to only two equations each. Hence we find six equations in total. There are however seven scalar integrals that contribute. One of them is a hexagon-box.

Following this line of ideas it would be interesting to continue and write down the complete set of linear equations which determines all two-loop six particle amplitudes.

Already with the results presented here, it would be interesting to compare the parity even part of the coefficient of all pentagon-pentagon integrals to those obtained recently in [27] for MHV amplitudes.

VII. CONCLUSIONS

The program of determining the S-matrix of physical theories by studying the structure of its singularities when analytically continued into complex values of kinematical invariants was very ambitious. Now we know that one of the main goals of the program which was to understand the strong interactions has proven to be a formidable problem.

In the past 20 years we have learned that \( \mathcal{N} = 4 \) super Yang-Mills (SYM) can serve as a laboratory where new techniques and ideas can be tested. Based on the results presented in this paper we would like to propose that perhaps \( \mathcal{N} = 4 \) SYM, at least in the large \( N \) limit, can be used to realize the basic ideas of the S-matrix program. Of course, \( \mathcal{N} = 4 \) SYM in four dimensions is a conformal theory and no S-matrix can be defined. However, if IR divergencies are regulated using dimensional regularization then a sensible S-matrix can be defined. It is very intriguing that on the contours which computer the discontinuity across leading singularities, all integrals are finite and hence make perfect sense in four dimensions. It is natural to expect the amplitudes on the torus contours can have physical meaning. It
would be interesting to explore this further.

In general, Feynman diagrams possess many singularities; from poles, related to resonances, to branch cuts, related to unitarity. When a given scattering amplitude in theories with spin is considered as a function of independent complex variables $\lambda^{(i)}_a$ and $\tilde{\lambda}^{(i)}_a$ the problem of reconstructing it from the structure of its singularities is certainly out of hand. At tree-level we learned few years ago [34, 35] that one can solve a simpler problem by concentrating on a single complex variable deformation of the amplitude. The main simplification comes from the fact by matching only a very small subset of all poles is enough to determining the amplitude in terms of smaller ones. This idea led to recursion relations between on-shell scattering amplitudes.

At loop level, Feynman diagrams posses an intricate structure of nested branch cuts. It turns out that the discontinuity across a given branch cut possesses branch cuts itself. The nested structure gets more an more complicated the higher the loop order. The problem of finding functions which reproduce all such discontinuities is clearly very hard. Out of all these singularities, the special class studied in this paper, called leading singularities, are the ones which have the highest codimension. We have argued that the discontinuities associated to them are determined by simple computations of residues. We have shown that quite remarkably, the problem of finding a function which reproduces all leading singularities, which only implies the solution to linear equations, is enough to completely determine the amplitude in $\mathcal{N} = 4$ SYM.

Summarizing, computing multi-loop, multi-particle amplitudes in $\mathcal{N} = 4$ SYM can be reduced to the computation of residues and the solution of systems of linear equations. The residues only involve products of tree-level amplitudes. These possess all the helicity information and only enter in the inhomogeneous part of the linear equations. The homogeneous part is universal. If we think about the amplitude and the linear equations as being objects with the same information, then it might be that the matrix which determines the homogenous part of the equations is the right object to study properties like strong coupling expansions, collinear limits, IR consistency equations, etc.

A natural question that arises is whether this is property is unique to $\mathcal{N} = 4$ SYM. The most promising theories are those for which one-loop amplitudes can be written in terms of only boxes, i.e., bubbles and triangles are absent. $\mathcal{N} = 8$ supergravity has been hypothesized to have such property [17, 18, 19, 20]. This means that one could try to apply the technique
presented here to that case. Already, an important step in this direction was given in [16]. It would be very interesting to compute more examples.

In [16], higher loop four-particle amplitudes in $\mathcal{N} = 4$ SYM were considered. In cases where four-particle one-loop amplitudes were present as subdiagrams it was shown that one could related the computation to that at one less loop order. It would be interesting to explore the consequences that matching all leading singularities would impose on the generalized scalar integrals. Perhaps a formal derivation of the idea of corrections introduced in [16] can be found.

As a final note one can say that the power of matching each individual leading singularities comes from the fact that the number of conditions which are naturally linear grows much faster with the number of loops than those of previous approaches.

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