SU(2) GAUGE THEORY IN COVARIANT (MAXIMAL) ABELIAN GAUGES

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The local covariant continuum action of an SU(2) gauge theory in covariant Abelian gauges is investigated. It describes the critical limit of an Abelian Lattice Gauge Theory (LGT) with an equivariant BRST-symmetry. This Abelian LGT has previously been proven to be physically equivalent to the SU(2)-LGT. Renormalizability requires a quartic ghost interaction in these non-linear gauges (also in maximal Abelian gauge). Arguments that a certain global SL(2,R) symmetry is dynamically broken by ghost-antighost condensation in a BCS-like mechanism are presented. The scenario can be viewed as a dynamical Higgs mechanism in the adjoint that gives massive off-diagonal gluons and a BRST quartet of Goldstone bosons that decouples from physical observables. The gap parameter is related to the expectation value of the trace anomaly and the consistency of this scenario with the Operator Product Expansion is discussed.

1 Introduction

An SU(2) Lattice Gauge Theory (LGT) on a finite lattice is invariant under the compact group $\mathcal{G}$

$$\mathcal{G} = \otimes_{\text{sites}} SU(2).$$

(1)

$\mathcal{G}$ does not have a smooth continuum limit and it is not entirely clear whether some of the effects observed on the lattice, such as absolute confinement, are due to the compact nature of $\mathcal{G}$ and are absent in a local Quantum Field Theory (QFT) with non-compact SU(2) gauge invariance. The fact that compact (lattice) QED does differ markedly from the continuum theory due to lattice "artefacts" makes this question all the more relevant. These "artefacts" are Abelian monopoles specific to lattice QED – their existence is intimately related to the compactness of the (Abelian) lattice gauge group and they have no continuum analogues. In the Abelian case, one can remove these lattice artefacts by imposing the constraint

$$A_\mu = \Delta^{-1} \partial_\mu F_{\mu\nu},$$

(2)

where $P_{\mu\nu}(x) = e^{i F_{\mu\nu}(x)}$ is the plaquette variable and $U_\mu(x) = e^{i A_\mu(x)}$ is the $U(1)$ link variable. On the lattice, Eq. (2) is not just a gauge fixing condition, but in addition eliminates the monopoles associated with harmonic one forms.
The continuum limit of this (projected) Abelian LGT is free QED in Landau gauge. If Eq. (2) was not imposed, the gauge invariant transverse photon correlation function of lattice QED was found to differ markedly from what one expects in the continuum.

In view of this example, the question whether the critical limit of a non-Abelian LGT can be described by “QCD” with a non-compact gauge group is legitimate. It has been conjectured that a non-Abelian LGT may not be asymptotically free and exhibit a Kosterlitz-Thouless phase transition at a finite value of the coupling constant.

In resolving the issue of the critical limit of a non-Abelian LGT it may be useful to have a physically equivalent local LGT with an equivariant BRST-symmetry whose structure group has been reduced to the maximal Abelian subgroup of the original LGT. The two LGT’s in question are physically equivalent because the expectation values of gauge-invariant operators (i.e. Wilson loops and their linked generalizations) are the same. Although this equivalent Abelian LGT has only been constructed for an SU(2)-LGT, the method can be generalized to any SU(n)-LGT. The lattice group \( G \) of an SU(2)-LGT is reduced to the maximal Abelian subgroup \( H \) by a local Topological Lattice Theory (TLT) that computes the Euler number of the coset manifold \( G/H \),

\[
\chi(G/H) = \otimes_{\text{sites}} \chi(SU(2)/U(1) \sim S_2) = 2^{\#\text{sites}},
\]

on each orbit of a lattice configuration using Morse Theory. I should stress that this construction of a TLT is conceptually quite different from the usual Faddeev-Popov procedure and does not require the uniqueness of the solution to a “gauge condition” – the Euler number of the manifold would in fact have to be 1 for this to be the case. There are at least \( 2^{\#\text{sites}} \) gauge equivalent Gribov copies that contribute to \( \chi(G/H) \) on any orbit of the (finite) lattice. The construction of the TLT is mathematically rigorous, because the coset manifold \( G/H \) is compact and finite-dimensional on a finite lattice (albeit of rather large dimension) and the orbit-space of the original LGT is connected. One cannot reduce the full lattice group \( G \) in this manner because \( \chi(G) = 0 \). The best one can do is to reduce the lattice gauge group to the smallest subgroup \( H \) for which the Euler number of the coset manifold \( G/H \) does not vanish. In the case of compact SU(n) the smallest subgroup which satisfies this requirement is the maximal Abelian one.

Of interest here will be the continuum theory that describes the critical limit of this “partially gauge fixed” LGT. The equivariant BRST-symmetry and \( U(1) \)-invariance together with locality and power-counting renormalizability determine the continuum limit up to lattice artefacts associated with the compactness of the residual \( U(1) \)-structure group. Assuming these Abelian
artefacts have been removed in a manner similar to the one prescribed above, the critical limit of this LGT is unique because the BRST-invariance of the LGT is a global one. The continuum model is then described by the local action given below.

Because physical correlation functions are the same in the reduced Abelian LGT and can be shown to satisfy reflection positivity in the original SU(2)-LGT, the physical states of the “partially gauge fixed” Abelian LGT also have positive norm. By proving the equivalence of the two LGT’s for gauge-invariant correlation functions one thus also verifies the unitarity of the partially gauge-fixed Abelian LGT. The continuum theory describing the critical limit of this LGT should therefore be unitary as well. Note that this proof of the unitarity of the continuum theory is valid non-perturbatively and not just to all orders in perturbation theory. Instead of investigating the critical limit of the original SU(2)-LGT, we thus will consider the critical limit of the (physically) equivalent LGT with an Abelian structure group and an equivariant BRST-symmetry.

2 Continuum SU(2) Gauge Theory in Abelian Gauges

Up to Abelian lattice artifacts mentioned above, the continuum theory describing the critical limit of the “partially gauge fixed” Abelian LGT is completely specified by the global symmetries and power counting. It is described by the Lagrangian

\[ \mathcal{L} = \mathcal{L}_{\text{inv.}} + \mathcal{L}_{\text{AG}} + \mathcal{L}_{\text{aGF}} \, . \] (4)

Here \( \mathcal{L}_{\text{inv.}} \) is the usual SU(2)-invariant Lagrangian with the SU(2)-connection \( \tilde{V}_\mu = (W_1^1, W_2^2, A_\mu) \) written in terms of two (real) vector bosons \( W \), and an Abelian “photon” \( A \),

\[ \mathcal{L}_{\text{inv.}} = \mathcal{L}_{\text{matter}} + \frac{1}{4} (G_{\mu\nu} G_{\mu\nu} + G_{a\mu}^a G_{\mu\nu}^a) \, , \] (5)

with

\[ G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g\varepsilon^{ab} W_\mu^a W_\nu^b \]
\[ G_{\mu\nu}^a = D_\mu W_\nu^b - D_\nu W_\mu^b = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{ab} (A_\mu W_\nu^b - A_\nu W_\mu^b) \, . \] (6)

\( \mathcal{L}_{\text{AG}} \) reduces the invariance to the maximal Abelian subgroup \( U(1) \) of SU(2) in a covariant manner,

\[ \mathcal{L}_{\text{AG}} = \frac{F^a F^a}{2\alpha} - \bar{c}^a M c^b - \frac{\alpha}{2} \bar{c}^a \varepsilon^{ab} c^b \, , \] (7)

Latin indices take values in \( \{1, 2\} \) only, Einstein’s summation convention applies and \( \varepsilon^{12} = -\varepsilon^{21} = 1 \), vanishing otherwise. All results are given in the \( \overline{MS} \) renormalization scheme.
with

\[ F^a = D_\mu W^b_\mu = \partial_\mu W^a_\mu + g A_\mu \varepsilon^{ab} W^b_\mu \quad \text{and} \quad M^{ab} = D_\mu D^\mu + g^2 \varepsilon^{ac} \varepsilon^{bd} W^c_\mu W^d_\mu . \quad (8) \]

Like the corresponding Abelian LGT, \( L_{U(1)} = L_{\text{inv}} + \mathcal{L}_{\text{AG}} \) is invariant under \( U(1) \)-gauge transformations and an on-shell BRST symmetry \( s \) and anti-BRST symmetry \( \bar{s} \), whose action on the fields is

\[
\begin{align*}
   s A_\mu &= g \varepsilon^{ab} c^a W^b_\mu \\
   s W^a_\mu &= D_\mu c^b \varepsilon^{ab} \\
   s c^a &= 0 \quad \bar{s} c^a = 0 \\
   s F^a &= F^a / \alpha \\
   \bar{s} F^a &= -F^a / \alpha , \\
\end{align*}
\]

(9)

with an obvious extension to include matter fields. On the connections \( A_\mu \) and \( W^a_\mu \) this BRST-variation effects an infinitesimal transformation in the coset \( G / H \) parameterized by the two ghosts \( c^a(x) \). Note that \( s c^a = 0 \) here, because the coset is not a group manifold.

The BRST algebra Eq. (9) closes on-shell on the set of \( U(1) \)-invariant functionals: on functionals that depend only on \( W, A, c \) and the matter fields, \( s^2 \) for instance effects an infinitesimal \( U(1) \)-transformation with the parameter \( \varepsilon^{ab} c^a \). The algebra Eq. (9) thus defines an equivariant cohomology. It was derived from a more extensive nilpotent (off-shell) BRST-algebra on the lattice by integrating out some of the additional fields. As mentioned in the introduction, the renormalizability and unitarity of this continuum theory is guaranteed because it describes the critical limit of an Abelian LGT that was proven to have the same gauge invariant correlation functions as the original \( SU(2) \)-LGT. Note that the physical sector comprises states created by composite operators of \( A, W \) and the matter fields in the equivariant cohomology of \( s \) (or \( \bar{s} \)). They are BRST closed, \( U(1) \)-invariant and do not depend on the ghosts.

For \( \alpha > 0 \), Eq. (9) could be considered a “soft” gauge fixing to the Maximal Abelian Gauge (MAG). It differs from what one naively obtains using a Faddeev-Popov procedure by a quartic ghost interaction proportional to \( \alpha \). Eq. (7) also does not implement the non-linear constraint \( F^a = 0 \) exactly. However, setting \( \alpha = 0 \) and perturbatively solving the constraint \( F^a = 0 \) is not consistent and not the same as taking the limit \( \alpha \to 0 \). One could have inferred the highly singular nature of this limit from the fact that the 4-ghost interaction diverges at one loop even when the photon- and vector boson propagators are transverse. A 4-ghost counterterm thus is required even in the (formal) limit \( \alpha \to 0 \) and the leading term in the anomalous dimension of the gauge parameter in fact is \( -3g^2 / (8\pi^2 \alpha) \) in this limit. The physical reason for the singular behavior of the limit \( \alpha \to 0 \) is inherently non-perturbative and nicely

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exhibited by the lattice calculation\[1\]: without the quartic ghost interaction, Gribov copies of a configuration conspire to give vanishing expectation values for all physical observables. No matter how small, the quartic ghost interaction is required to have a normalizable partition function and expectation values of physical observables that are identical with those of the original SU(2)-LGT. From a perturbative point of view, \(\alpha \rightarrow 0\) at finite coupling \(g^2\) corresponds to a strong coupling limit that is not accessible perturbatively\[5\].

\(L_{aGF}\) in Eq. (4) fixes the remaining \(U(1)\) gauge invariance and thus defines the perturbative series unambiguously. I will consider a conventional covariant gauge-fixing of the form,

\[
L_{aGF} = \delta \left[ \bar{\omega} \left( \partial_\mu A_\mu - \frac{\xi}{2} b \right) \right] = b \partial_\mu A_\mu - \frac{\xi}{2} b^2 + \bar{\omega} \Delta \omega \equiv \frac{1}{2\xi} (\partial_\mu A_\mu)^2 ,
\]

where the last equivalence is obtained by decoupling the Abelian ghosts \(\omega\) and \(\bar{\omega}\) and the Nakanishi-Lautrup field \(b\). \(\delta\) is a BRST-symmetry defined on the fields as

\[
\begin{align*}
\delta A_\mu &= -\partial_\mu \omega \\
\delta c^a &= g \omega \varepsilon^{ab} c^b \\
\delta \bar{c}^b &= g \omega \varepsilon^{ab} b^a \\
\delta \omega &= 0 \\
\delta \bar{\omega} &= b \\
\delta b &= 0
\end{align*}
\]

(11)

Trivially extending \(s\) and \(\bar{s}\) to the additional fields

\[
\begin{align*}
s \omega &= s \bar{\omega} = sb = s \bar{s} = \bar{s} \omega = \bar{s} \bar{\omega} = \bar{s} b = 0 ,
\end{align*}
\]

(12)

one can verify that \(\delta\) is nil-potent and anticommutes with \(s\) and \(\bar{s}\)

\[
\delta^2 = s \delta + \delta s = \bar{s} \delta + \delta \bar{s} = 0
\]

(13)

As far as algebraic renormalization is concerned, the action Eq. (4) thus is composed of a term in the cohomology of \(\delta\) that is invariant under the global symmetries \(s\) and \(\bar{s}\) given in Eq. (9) and a \(\delta\)-exact term. The latter is not invariant under \(s\) nor \(\bar{s}\). Since the global symmetries commute with \(\delta\), the situation is the same as in any gauge-fixing that breaks some of the global symmetries (or supersymmetries) of the theory. There is a well-defined procedure to handle this case\[6\] in algebraic renormalization. From a more heuristic point of view, we already know that the \(s\) and \(\bar{s}\) symmetries of the theory are not anomalous from the lattice regularization\[3\] of this model. I will therefore not give the algebraic proof here.

What has been gained compared to conventional covariant gauge fixing? Since the present gauge fixing in a sense is “hierarchical”, we are able to single out the maximal Abelian subgroup. As emphasized before, the \(s\) and \(\bar{s}\)
Symmetries can be implemented on the lattice and the resulting Abelian LGT shown to be physically equivalent to one with an SU(2) structure group. It may eventually be possible to construct the dual of this Abelian LGT. In addition, the theory described by Eq. (4) does not suffer from a generic Gribov problem due to zero modes of the ghosts. Since the global $s$ and $\bar{s}$ symmetries are preserved by the lattice regularization, Eq. (4) probably describes the critical limit of a LGT better than any other set of covariant gauges. Finally, because the Abelian $\omega$-ghost and $\bar{\omega}$-antighost as well as the Nakanishi-Lautrup field $b$ decouple, we effectively end up with a local and covariant gauge-fixed theory with fewer ghosts. This reduction in the number of fields is at the expense of a quartic ghost interaction that gives the ghosts some interesting dynamics of their own.

### 3 The Dynamically Broken SL(2,R) Symmetry

The Lagrangian Eq. (4) also exhibits a global bosonic SL(2,R) symmetry that is generated by

$$\Pi^+ = \int c^a(x) \frac{\delta}{\delta \bar{c}^a(x)} \quad \Pi^- = \int \bar{c}^a(x) \frac{\delta}{\delta c^a(x)} ,$$

and the ghost number $\Pi = [\Pi^+, \Pi^-]$. This SL(2,R) symmetry is preserved by the regularization (for instance dimensional) and thus is not anomalous. The conserved currents corresponding to $\Pi^\pm$ are $U(1)$-invariant and BRST, respectively anti-BRST exact,

$$j^\mu_+ = c^a D_\mu c^b = sc^a W_\mu^a \quad j^\mu_- = \bar{c}^a D_\mu \bar{c}^b = \bar{s}c^a W_\mu^a .$$

I will argue that the global $SL(2, R)$ symmetry of the theory is spontaneously broken to the Abelian subgroup generated by the ghost number $\Pi$. An order parameter for the spontaneous breakdown of the SL(2,R) symmetry thus is

$$\langle \bar{c}^a \varepsilon^{ab} c^b \rangle = \frac{1}{2} \langle \Pi^- (c^a \varepsilon^{ab} c^b) \rangle = -\frac{1}{2} \langle \Pi^+ (\bar{c}^a \varepsilon^{ab} \bar{c}^b) \rangle .$$

Because the currents Eq. (15) are (anti)-BRST exact, a spontaneously broken SL(2,R) symmetry is accompanied by a BRST-quartet of massless Goldstone states with ghost numbers 2, 1, −1 and −2. They are $U(1)$-invariant $c - c$, $c - W$, $\bar{c} - W$ and $\bar{c} - \bar{c}$ bound states. It is important to note in this context that BRST quartets do not contribute to physical quantities.$^b$ The spontaneous breaking of the $SL(2, R)$ symmetry in a sense is similar to a dynamical

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$^b$This is analogous to the decoupling of the Goldstone quartets of the weak interaction in renormalizable $R_\xi$ gauges.
Higgs mechanism in the adjoint. The vector bosons $W$ acquire a mass (see below) but in contrast to a conventional Higgs mechanism in the adjoint, this mass is not a free parameter of the theory, but can be determined in terms of $\Lambda_{\overline{MS}}$.

To see that ghost condensation will almost invariably occur at weak coupling in the model described by Eq. (4) it is illustrative to compare with the BCS-theory of superconductivity. In BCS-theory, an at low momentum transfers attractive 4-fermion interaction leads to the condensation of certain fermion pairs and the formation of a gap in the quasi-particle spectrum near the Fermi surface. An analogous phenomenon occurs here for ghost and anti-ghost modes corresponding to small eigenvalues of the FP-Operator $M^{ab}$ defined by Eq. (8): for zero modes, the bilinear term in Eq. (7) vanishes and the quartic ghost interaction selects the channel in which condensation occurs. The quartic ghost interaction in Eq. (7) is attractive when the color of the ghost and anti-ghost are opposite: it thus leads to the formation of a $\langle \bar{c}^a \varepsilon^{ab} c^b \rangle$ condensate at arbitrarily weak coupling $\alpha g^2$ by the BCS-mechanism. [The two spin states of a fermion here has an analog in the two color orientations of the ghosts. We choose ghost number to be conserved and observe $\bar{c} - c$, rather than $c - c$ or $\bar{c} - \bar{c}$ condensation, as would be the case if we chose $\Pi^-$ or $\Pi^+$ as unbroken generators.] The analogy with BCS-theory is particularly appealing because the operator $M^{ab}$ has small eigenvalues whenever the gauge field configuration is in the vicinity of a Gribov horizon. That the ground state may be dominated by such configurations was previously suggested in an attempt to restrict the functional integral to the fundamental modular region of Landau gauge. In the present context, gauge field configurations with non-Abelian monopoles are on the Gribov horizon, since the failure of the gauge fixing condition to an Abelian subgroup is necessary for the presence of monopoles.

Thus, if monopoles are relevant in describing the ground state of the theory, it is not inconceivable that the ghosts will condense. The converse is not necessarily true because $M^{ab}$ can have arbitrarily small eigenvalues at field configurations with vanishing monopole number. We will see below that the ghosts already condense in the vicinity of the trivial gauge field configuration. The analogy with BCS-theory suggests that they condense for any value of the quartic coupling $\alpha g^2$, with a gap that depends exponentially on $1/(\alpha g^2)$.

To perturbatively investigate the consequences of $\langle \bar{c}^a \varepsilon^{ab} c^b \rangle \neq 0$, the quartic ghost interaction in Eq. (4) is linearized by introducing an auxiliary scalar field $\rho(x)$ of canonical dimension two. Adding the quadratic term

$$L_{\text{aux}} = \frac{1}{2g^2} (\rho - g^2 \lambda c^a \varepsilon^{ab} c^b)^2$$

(17)
to the Lagrangian of Eq. (1), the tree level quartic ghost interaction vanishes at $\lambda^2 = \alpha$ and is then formally of $O(g^4)$, proportional to the difference $Z_\alpha^2 - Z_\lambda$ of the renormalization constants of the two couplings.

We will see that the perturbative expansion about a non-trivial solution $\langle \rho \rangle = v \neq 0$ to the gap equation

$$\frac{v}{g^2} = \sqrt{\alpha} \left( \langle c^a(x) \epsilon^{ab} c^b(x) \rangle \right)_{\langle \rho \rangle = v}, \quad (18)$$

is stable and much better behaved in the infrared.

Defining the quantum part $\sigma(x)$ of the auxiliary scalar $\rho$ by

$$\rho(x) = v + \sigma(x) \quad \text{with} \quad \langle \sigma \rangle = 0, \quad (19)$$

the momentum representation of the Euclidean ghost propagator at tree level becomes

$$\langle c^a \bar{c}^b \rangle_p = \frac{p^2 \delta^{ab} + \sqrt{\alpha v} \epsilon^{ab}}{p^4 + \alpha v^2} = \int_0^\infty d\omega e^{-\omega p^2} \left[ \delta^{ab} \cos(\omega v \sqrt{\alpha}) + \epsilon^{ab} \sin(\omega v \sqrt{\alpha}) \right]. \quad (20)$$

Feynman’s parameterization of this propagator leads to an evaluation of loop integrals using dimensional regularization that is only slightly more complicated than usual.

Using Eq. (20) in Eq. (18), the gap equation to one-loop in $d = 4 - 2\varepsilon$ dimensions is,

$$\frac{v}{g^2} = 2\mu^4 \sqrt{\alpha} \int_0^\infty \frac{d\omega}{(4\pi \mu^2 \omega)^{2-\varepsilon}} \sin(\omega v \sqrt{\alpha})$$

$$= \frac{\alpha v}{8\pi^2} \left[ \frac{1}{\varepsilon} - \ln \frac{\pi g^2 T^2}{\mu^2 e^{1-\gamma_E}} + O(\varepsilon) \right]. \quad (21)$$

Here $\gamma_E$ is Euler’s constant. Including the counterterm

$$\frac{v}{g^2} \rightarrow \frac{v}{g^2} Z_v Z_g^{-2} = \frac{v}{g^2} + \frac{\alpha v}{8\pi^2 \varepsilon} + O(g^2) \quad (22)$$

on the left-hand side of Eq. (21) cancels the $1/\varepsilon$ divergence of the right-hand side of Eq. (21). The renormalized (non-trivial) solution to the gap equation in four dimensions thus is,

$$\alpha v^2 = \varepsilon^2 \Lambda^4, \quad \text{with} \quad \Lambda^2(\alpha, g, \mu) = 4\pi \mu^2 e^{-\gamma_E - \frac{\alpha v^2}{8\pi^2}}. \quad (23)$$

\(^c\)The discrete symmetry $c^a \rightarrow \bar{c}^a$, $\bar{c}^a \rightarrow -c^a$, $\rho \rightarrow -\rho$ relating $s$ and $\bar{s}$ also ensures that $\rho$ only mixes with $\bar{\epsilon}^{ab} c^b$.\(^d\)
Note the exponential dependence of the gap $v$ on the quartic coupling $\alpha g^2$ expected from BCS-theory. One can show that the solution Eq. \((23)\) corresponds to the global minimum of the effective potential by either directly computing the (one-loop renormalized) effective potential,

$$V(v, \mu, g, \alpha) = \frac{\alpha v^2}{32\pi^2} \ln \frac{\alpha v^2}{e^3\Lambda^4} + O(g^2),$$

or by integrating Eq. \((21)\) and noting that $V(\Lambda, 0) = 0$. Consistency requires that the 1PI two-point function of the scalar $\sigma$ is positive definite for all Euclidean momenta when Eq. \((18)\) is satisfied. From Eq. \((22)\) one obtains for the anomalous dimension of $v$ to one loop

$$\gamma_v = -\frac{d\ln Z_v}{d\ln \mu} = \frac{g^2}{16\pi^2}(2\alpha - \beta_0) + O(g^4)$$

where $\beta_0$ is the lowest order coefficient of the $\beta$-function. At $\alpha = \beta_0/2$, the anomalous dimension of $v$ is of order $g^4$ and corrections to the asymptotic $v(g \to 0)$ solution in this particular critical gauge therefore are analytic in $g^2$ and may be computed order by order in perturbation theory. In this critical gauge we thus have that the scale $\Lambda$ describing the minimum of the effective potential $V(v, \mu, g, \alpha)$ is analytically related to $\Lambda_{\overline{MS}}$:

$$\Lambda(\alpha = \beta_0/2, g, \mu) = \Lambda_{\overline{MS}}(1 + O(g^2))$$

In other gauges $v \neq 0$ at weak coupling is either much larger than $\Lambda_{\overline{MS}}$ (for $\alpha \gg \beta_0/2$), or much smaller (for $\alpha \ll \beta_0/2$). To leading order in the loop expansion, the anomalous dimension Eq. \((23)\) does not vanish in these cases and higher order loop corrections to $v \neq 0$ are of comparable magnitude. [As noted before, the extreme limit $\alpha \to 0$ in which the non-trivial solution Eq. \((23)\) becomes degenerate with the trivial one, in particular corresponds to a strong coupling problem.] In the critical gauge $\alpha = \beta_0/2$, the perturbative 1-loop calculation of $v \neq 0$ is self-consistent in the sense that all higher order corrections to the expectation value are of order $g^2$ because the anomalous dimension Eq. \((23)\) is of order $g^4$ at this point. I wish to emphasize that this does not imply that physical effects associated with ghost condensation in this gauge are themselves gauge dependent. It only implies that a non-trivial solution to the gap equation is perturbatively consistent at $\alpha = \beta_0/2$. [That some gauges are better suited than others for a non-perturbative evaluation of gauge invariant quantities is well known from QED; the gauge invariant hydrogen spectrum is qualitatively best obtained in Coulomb gauge. Below we relate $\alpha v^2$ to the vacuum expectation value of the trace of the energy-momentum tensor.]
4 The Vector Boson Mass

Recent lattice simulations indicate that the $W$-bosons are massive in maximal Abelian gauges. At least qualitatively, this may be explained by the mechanism discussed here. Although the tree level contribution to $m_W^2$ vanishes by Bose symmetry, ghost condensation induces a finite mass $m_W^2 = g^2 \sqrt{\alpha v^2 / (16\pi)}$ at one loop as shown in Fig. 1.

$$m_W^2 = g^2 \sqrt{v^2 \alpha / (16\pi)} \delta_{\mu\nu} \delta^{ab}. \quad (27)$$

Fig. 1. The finite one-loop contribution to the $W$ mass.

Technically, the one-loop contribution is finite because the integral in Eq. (27) involves only the $\delta^{ab}$-part of the ghost propagator Eq. (20). Since $p^2/(p^4 + \alpha v^2) = -\alpha v^2/(p^2(p^4 + \alpha v^2)) + 1/p^2$, the $v$-dependence of the loop integral is IR- and UV-finite. The quadratic UV-divergence of the $1/p^2$ subtraction at $v = 0$ is canceled by the other, $v$-independent, quadratically divergent one-loop contributions – (in dimensional regularization this scale-invariant integral vanishes by itself). $m_W^2$ furthermore is positive due to the overall minus sign of the ghost loop. The sign of $m_W^2$ is crucial. It is a further indication that the model is stable and (as far as the loop expansion is concerned) does not develop tachyonic poles at Euclidean $p^2$ for $v \neq 0$. Conceptually, the local mass term proportional to $\delta_{\mu\nu} \delta^{ab}$ is finite due to the BRST symmetry Eq. (9), which excludes a mass counter-term. The latter argument implies that contributions to $m_W^2$ are finite to all orders of the loop expansion.

Although this “mass”-term regulates the IR-behavior of the $W$-propagator perturbatively, the 1-loop calculation above should quantitatively describe the behavior of the $W$-propagator at high momenta, where $g^2(p^2)$ is a small parameter and this calculation is consistent. Ghost condensation thus should lead to a leading power correction $\propto v/p^2$ to the $W$-propagator at high momenta. The consistency of this behavior with the Operator Product Expansion (OPE) is examined below.

5 Discussion of Physical Consequences

We have seen that ghost condensation in covariant Abelian gauges is associated with the spontaneous breaking of a global $\text{SL}(2,R)$ symmetry whose diagonal generator is the ghost number. The currents of the broken symmetries are (anti-)BRST exact and the Goldstone states form a BRST-quartet that de-
couples from physical observables. What are observable consequences? We gain some insight by computing the contribution to the expectation value of the trace of the energy momentum tensor. From the effective potential Eq. (24) one obtains:

$$\langle \theta_{\mu} \rangle = -\frac{\alpha v^2}{8\pi^2} = -\frac{e^2}{8\pi^2} \Lambda^4,$$

At the minimum of the effective potential, $v \neq 0$ thus lowers the vacuum energy density and ghost condensation may be interpreted as a low-energy manifestation of the trace anomaly.

Let me also comment on the consistency of the approach from the point of view of the Operator Product Expansion (OPE). In generic covariant gauges, the OPE implies that power corrections to physical correlators (but also to Green functions) are at least suppressed by an order $M^4/p^4$ relative to the perturbative behavior at high momenta (in the absence of quarks). This is simply because the operator of lowest dimension in the BRST-cohomology of these gauges has canonical dimension four. Consistency of the present approach requires that one explain

- why the ground state in the present case can support a vacuum expectation value $v \propto \langle \bar{c}^{a} e^{ab} \rangle \neq 0$ of canonical dimension two and simultaneously,
- why power corrections of order $v/p^2$ are absent in physical correlation functions.

To address the first question, we construct a $U(1)$-invariant local operator of dimension two whose vacuum expectation value manifestly is invariant under the BRST-algebra of Eq. (9). Note that

$$O_2 := [\bar{c}^{a} e^{a} + \frac{1}{2} W^{a}_{\mu} W^{a}_{\mu}]$$ is not in the equivariant cohomology of the on-shell BRST-algebra Eq. (9), because $s^2 \sim 0$ only on $U(1)$-invariant operators that do not depend on $\bar{c}$ whereas $O_2$ does. Nevertheless, the zero-momentum component of $O_2$ is invariant under $s$, $\bar{s}$ and $\delta$. $\langle O_2 \rangle \neq 0$ therefore is an invariant statement as far as the algebra of symmetries is concerned. We have seen that $\langle O_2 \rangle = \frac{\sqrt{2\pi}}{16\pi} \neq 0$ in fact appears to be the dynamically favored possibility. The existence of an operator of dimension two whose zero-momentum component is invariant explains the leading power corrections we find for the $W$-, $A$- and ghost- propagators from the point of view of the OPE.
Fig. 2. Schematic representation of leading contributions that give rise to power corrections at large momenta in the gauge invariant correlators Eq. (30). For reasons given in the main text, the power leading correction $\propto g^2v$ of the propagators cancels in this gauge invariant combination.

On the other hand, since $O_2$ is not invariant under global $SU(2)$ transformations, we do not expect $\langle O_2 \rangle$ to appear in the OPE of gauge invariant correlators. The leading power correction $\propto v$ in the propagators therefore should cancel in gauge invariant correlation functions such as

$$\langle (G_{\mu\nu}G_{\rho\sigma} + G^a_{\mu\nu}G^a_{\rho\sigma})(G_{\alpha\beta}G_{\gamma\delta} + G^b_{\alpha\beta}G^b_{\gamma\delta}) \rangle_{p^2 \sim \infty}$$

To leading order in the loop expansion this can be verified explicitly. To this order, the three diagrams of Fig. 2 with two (transverse) photons and two (transverse) vector bosons as intermediate states lead to power corrections. In the limit $p^2 \sim \infty$ at least one of the photons, respectively vector bosons in the loop integrals has momentum much larger than $v$. The leading power correction $\propto v$ in Eq. (30) from the vector bosons and the photon thus cancel if and only if the photon polarization,

$$\Gamma^{AA}_{\mu\nu}(p, v) = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\Pi^{AA}(p^2, v)$$

for $p^2 \gg v$ has the asymptotic power expansion

$$\Pi^{AA}(p^2 \gg v, v) \sim 1 - 2\frac{m_W^2}{p^2} + O(g^2 \ln p^2, v^2/p^4) \ .$$

Here $m_W^2 = g^2 \frac{\sqrt{\alpha \bar{v}}}{16\pi}$ is the W-boson mass Eq. (27). Evaluating the one ghost loop contributions to the photon polarization at high external momentum one indeed verifies Eq. (32). Note that the factor $-2$ in Eq. (32) is essential for the cancellation of the leading power correction in Eq. (30), since twice as many $W$’s as photons contribute. Power corrections of order $\alpha v^2/p^4$ in the
asymptotic expansion of Eq. (30) do not cancel and are related to \( \langle \Theta^\mu_\mu \rangle \) by Eq. (28).

We have here proposed a mechanism by which the \( W \)-bosons of an SU(2) gauge theory in Abelian gauges essentially become massive while the leading power corrections to gauge-invariant correlation functions nevertheless are of order \( \Lambda^4_{\overline{MS}}/p^4 \) only. Although numerical lattice simulations show similar effects for the off-diagonal gluons, the numerical gauge fixing to MAG is not described by a local effective action and the results therefore cannot be directly compared with the ones presented here. One unfortunately cannot even extract the anomalous dimension of \( m_W^2 \) from the present numerical studies due to their rather narrow range of couplings. Although lattice simulations presently do not unambiguously confirm the mechanism of mass generation by ghost condensation I discussed, the simplicity and inherent consistency of this approach may warrant further study.

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