On correlation functions of characteristic polynomials for chiral Gaussian Unitary Ensemble

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Abstract

We calculate a general spectral correlation function of products and ratios of characteristic polynomials for a \( N \times N \) random matrix taken from the chiral Gaussian Unitary Ensemble (chGUE). Our derivation is based upon finding a Itzykson-Zuber type integral for matrices from the non-compact manifold \( \text{Gl}(n,\mathbb{C})/\text{U}(1) \times \ldots \times \text{U}(1) \) (matrix Macdonald function). The correlation function is shown to be always represented in a determinant form generalising the known expressions for only positive moments. Finally, we present the asymptotic formula for the correlation function in the large matrix size limit.

1 Introduction

A class of random matrices that has attracted a considerable attention recently \cite{1,2,3,4,5,6,7,8,9,10,11,12,13} is the so-called chiral GUE, also known as the Laguerre ensemble. The corresponding matrices are of the form \( \hat{\mathcal{D}} = (0 \hat{\mathcal{J}} \hat{\mathcal{J}}^\dagger 0) \), where \( \hat{\mathcal{J}} \) stands for a complex matrix, with \( \hat{\mathcal{J}}^\dagger \) being its Hermitian conjugate.

The off-diagonal block structure is characteristic for systems with chiral symmetry. The chiral ensemble was introduced to provide a background for calculating the universal part of the microscopic level density for the Euclidian QCD Dirac operator, see \cite{15} and references therein. Independently and simultaneously it was realised that the same chiral ensemble is describing a new group structure associated with scattering in disordered mesoscopic wires \cite{1}.

One of the main objects of interest in QCD is the so-called Euclidean partition function used to describe a system of quarks characterized by \( n_f \) flavors and quark masses \( m_f \) interacting with the Yang-Mills gauge fields. At the level of Random Matrix Theory the true partition function is replaced by the matrix integral:

\[
Z_{n_f}(\hat{\mathcal{M}}_f) = \int \mathcal{D}\hat{\mathcal{J}} \prod_{k=1}^{n_f} \det\{i\hat{\mathcal{D}} + m_f^{(k)} 1_{2N}\} e^{-N\text{Tr} V(\hat{\mathcal{J}}^\dagger \hat{\mathcal{J}})}
\]

where \( V(z) \) is a suitable potential. Here the integration over complex \( \hat{\mathcal{J}} \) serves to mimic the functional integral over gauge field configurations \cite{15}. Then the calculation of the partition function amounts to performing the ensemble average of the product of characteristic polynomials of \( i\hat{\mathcal{D}} \) over the probability density \( P(\hat{\mathcal{J}}) \propto e^{-N\text{Tr} V(\hat{\mathcal{J}}^\dagger \hat{\mathcal{J}})} \). To model the simplest case of the sector with zero topological charge the matrices \( \hat{\mathcal{J}} \) have to be chosen general \( N \times N \) complex, and the probability distribution can be chosen Gaussian as defined by the formula:

\[
P(\hat{\mathcal{J}})d\hat{\mathcal{J}}d\hat{\mathcal{J}}^\dagger = \text{const} \exp\left(-N\text{Tr} \left( \hat{\mathcal{J}}^\dagger \hat{\mathcal{J}} \right) \right) \prod_{i,j=1}^{N} (d\text{Re} J_{ij})(d\text{Im} J_{ij})
\]
In a more general case of non-zero topological charge the matrices \( \hat{J} \) have to be chosen rectangular [5], see also [6]. One may also wish to consider a more general type of potentials \( V \) [11, 13]. We do not consider these modifications in the present paper.

The characteristic feature of the chiral ensemble is the presence of a particular point \( \lambda = 0 \) in the spectrum, also called the "hard edge" [3]. The eigenvalues of chiral matrices appear in pairs \( \pm \lambda_k \), \( k = 1, ..., N \). Far from the hard edge the statistics of eigenvalues is practically the same as for usual GUE matrices without chiral structure, but in the vicinity of the edge eigenvalues behave very differently.

Define the following characteristic polynomials for chiral matrices:

\[
Z(\hat{J}, m) = \det (m1_{2N} + i\hat{D}) = \det \left( \begin{array}{cc} m1_N & i\hat{J} \\ i\hat{J}^\dagger & m1_N \end{array} \right). \tag{3}
\]

To calculate the eigenvalue density \( \rho(\lambda) \) one can use the trace of the resolvent for the matrix \(-i\hat{D}\) which is given in terms of the eigenvalues by

\[
R_{\pm}(m) = \frac{1}{2m1_{2N} \pm i\hat{D}} = 2m \sum_{k=1}^{N} \frac{1}{m^2 + \lambda_k^2}.
\]

The eigenvalue density is then extracted in the usual way as

\[
\rho(\lambda) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Re \frac{1}{2} \frac{\Tr (m1_{2N} + i\hat{D})}{\lambda_{m=\epsilon+i\lambda}}.
\]

In turn, the trace of the resolvent can be most easily found via the identity: \( \frac{\partial}{\partial m_j} Z_{\pm}(\hat{J}, m_j) \big|_{m_j=m_k} \). We therefore see that the interest in spectral properties of chiral matrices suggests to consider a general correlation function containing both products and ratios of the characteristic polynomials:

\[
\mathcal{K}(\hat{M}_f, \hat{M}_b) = \left\langle \prod_{k=1}^{n_f} \mathcal{Z}(\hat{J}, m_f^{(k)}) \right\rangle_{\hat{J}} \tag{4}
\]

where \( \hat{M}_f = \text{diag} \left( m_f^{(1)}, ..., m_f^{(n_f)} \right) \), \( \hat{M}_b = \text{diag} \left( m_b^{(1)}, ..., m_b^{(n_b)} \right) \) and the average is taken over the ensemble \( \text{Eq.}(3) \). In fact, correlation functions of such a type contain the most detailed information about the spectra of random matrices and can be used in various contexts and applications [17].

There are several analytical techniques for dealing with the correlation function \( \left\langle \prod_{k=1}^{n_f} \mathcal{Z}(\hat{J}, m_k) \right\rangle_{\hat{J}} \) of only products of characteristic polynomials. The majority of the results obtained so far resorted to exploitation of the orthogonal system of Laguerre polynomials, see [1, 2, 3, 4, 10, 11, 12, 13]. Another important tool in QCD applications of the chiral ensemble (as well as in studies on closely related non-Hermitian random matrices [15]) proved to be the following generalisation of the Itzykson-Zuber type integral

\[
\int \exp \left[ -\frac{1}{2} \Tr \left( \hat{X}_d \left[ \hat{U} \hat{Y}_d \hat{V}^\dagger + \hat{V} \hat{Y}_d \hat{U}^\dagger \right] \right) \right] d\mu(\hat{U})d\mu(\hat{V}) = \text{const} \frac{\det |I_0(x_i, y_j)|_{1 \leq i, j \leq n}}{\Delta(x_1^2, ..., x_n^2) \Delta(y_1^2, ..., y_n^2)} \tag{5}
\]

Here the integration in [15] goes over unitary \( n \times n \) matrices \( \hat{U}, \hat{V} \in U(n) \) with \( d\mu(\hat{U}, \hat{V}) \) being the corresponding invariant measures. The matrices \( \hat{X}_d \) and \( \hat{Y}_d \) are diagonal:

\[
\hat{X}_d = \text{diag} (x_1, x_2, ..., x_n), \quad \hat{Y}_d = \text{diag} (y_1, y_2, ..., y_n) \tag{6}
\]
and $I_0(x)$ denotes the Bessel function of imaginary argument and 0th order. In fact, the integral formula (6) was known already to Berezin and Karpelevich [19], and rediscovered very recently by Guhr and Wettig [20] and by Jackson, Sener and Verbaarschot [6].

A more general correlation function Eq.(4) can be studied for the particular case of the Gaussian measure (chiral Gaussian Unitary Ensemble, eq.(2)) by the procedure known in the literature as the supermatrix (or "supersymmetry") approach. The method was pioneered by Efetov [21] in the theory of disordered systems. First, one represents each of the characteristic polynomials in the numerator as the Gaussian integral over anticommuting (Grassmann) variables. Similarly, characteristic polynomials in the denominator are represented by the Gaussian integrals over usual commuting complex vectors. This allows to average the resulting expressions in a simple way. The resulting multiple non-Gaussian integral is brought to a tractable form by the so-called Hubbard-Stratonovich transformation. The latter transformation has, in general, a quite complicated analytical structure which mixes commuting and anticommuting as well as compact and non-compact integration variables. Such a trick allows one to expose the integration variables amenable to the saddle-point treatment in the limit of large matrix size $N$.

The first applications of the supersymmetry technique to the chiral case [5, 7] showed that the computation was quite technically involved even for the lowest correlation functions. In fact, any explicit calculation beyond the two-point correlation function of the resolvents seemed to be problematic.

In our paper [23] we demonstrated that an alternative method suggested in [22] enabled us to calculate general spectral correlation functions for random Hermitian matrices without chiral structure in a systematic and efficient way. Our starting point was the same as for the supersymmetry approach. At the same time, we integrated out Grassmann variables at a very early stage and avoided the complicated Hubbard-Stratonovich transformation thus seriously departing from the general spirit of supersymmetry.

In the present paper we show that the methods developed in [22, 23] can be successfully applied to calculate the general correlation function of products and ratios of characteristic polynomials for chiral GUE matrices (for the simplest case $n_b = n_f = 1$ this fact was already demonstrated in [22]).

Our main result is the formula:

$$K_N(\hat{X}) = \text{const} \frac{e^{-\frac{1}{4} \text{Tr} \hat{X}^2}}{\Delta \{ \hat{X}_B^2 \} \Delta \{ \hat{X}_F^2 \}} \det \left( \int_0^\infty dR e^{-\frac{1}{4} N N R^2} R^{n_b + i - 1} J_k(X^{(k)} \sqrt{R}) \right)_{i,k=1}^n$$

valid for any values of the parameters $N, n_b, n_f, \hat{X}_B = 2N \hat{M}_b, \hat{X}_F = 2N \hat{M}_f$, provided $N \geq n_b$.

Here we introduced the matrix $\hat{X} = \text{diag} (\hat{X}_F, \hat{X}_B)$ of the size $n = n_f + n_b$ and denoted

$$J_k(z) = \begin{cases} I_0(z) & , 1 \leq k \leq n_f \\ K_0(z) & , n_f + 1 \leq k \leq n \end{cases}$$

where $K_0(x)$ denotes the Macdonald function of 0th order.

Quite remarkably, the final expression is represented in a compact determinantal form reminiscent of that arising from the orthogonal polynomial method for positive moments. In a separate publication [24] we show that such a determinantal form is in no way accidental and is not specific for the chiral case. In fact, we managed to derive it in full generality not only for the Gaussian distribution, but for arbitrary unitary-invariant potential. For the present case of chiral GUE one can perform the asymptotic analysis of Eq.(6) in the "chiral limit" $N \to \infty$ with fixed $\hat{X}_B$ and $\hat{X}_F$ and obtain more compact determinant expression Eq.(36). This is also a new result going beyond
the known asymptotic expressions for the product of positive \[10, 6, 12\] as well as only negative \[22\] moments.

The key technical tool for our analysis is the matrix integral:

\[
\int \exp \left[ -\frac{1}{2} \text{Tr} \left( \hat{X}_d \left( \hat{Y}_d \hat{T}^\dagger + (\hat{T}^\dagger)^{-1} \hat{Y}_d \hat{T}^{-1} \right) \right) \right] d\mu(\hat{T}, \hat{T}^\dagger) = \text{const} \det \left[ K_0(x_i y_j) \right]_{1 \leq i,j \leq n} \triangle(x_1^2, \ldots, x_n^2) \triangle(y_1^2, \ldots, y_n^2) \quad (8)
\]

Now \(\hat{X}_d\) and \(\hat{Y}_d\) are two positive definite diagonal matrices:

\[
\hat{X}_d = \text{diag} (x_1, x_2, \ldots, x_n) > 0, \quad \hat{Y}_d = \text{diag} (y_1, y_2, \ldots, y_n) > 0 \quad (9)
\]

The integration in (8) goes over \(\hat{T} \in \text{Gl}(n, \mathbb{C})/U(1) \times \ldots \times U(1)\), i.e. over arbitrary \(n \times n\) complex matrices with positive diagonal elements, with \(d\mu(\hat{T}, \hat{T}^\dagger)\) being the corresponding invariant measure.

The integrals (8) and (9) can be looked at as certain matrix Bessel and Macdonald functions. In fact, second integral (8) is a very natural non-compact counterpart of the integral (3).

The structure of the paper is as follows. First we follow the (improved) version of the methods suggested in \[22\] and derive a convenient representation of the general correlation function in terms of a matrix integral. Then we integrate out irrelevant degrees of freedom exploiting Eqs.(5) and (8) and reveal a simple determinantal structure of the resulting expression. As a by-product we expose variables amenable to saddle-point treatment in the limit of large matrix size \(N\). Finally we calculate the resulting correlation functions in the ”chiral scaling” limit \(N \to \infty, mN = x < \infty\). In the Appendices we present a derivation of the Itzykson-Zuber-like integral (8) using standard ”diffusion equation” arguments, as well as discuss some technical details of matrix parametrisations used in our calculation.

### 2 Correlation Functions for Chiral GUE

As we discussed in the Introduction the starting point of our method is similar to the standard supersymmetry approach \[21\] and is based on a simultaneous exploitation of commuting (or bosonic) and anticommuting Grassmann (or fermionic) variables. We introduce \(2n_f\) fermionic and \(2n_b\) bosonic vectors, each of them with \(N\) components:

- \(\chi_k, \varphi_k\) - fermionic vectors, \(k = 1, 2, \ldots, n_f\)
- \(s_l, p_l\) - bosonic vectors, \(l = 1, 2, \ldots, n_b\)

These vectors enable us to rewrite the ratio of products of characteristic polynomials in Eq. (4) as the following integral:

\[
\frac{\prod_{k=1}^{n_f} \mathcal{Z} \left( J, m^{(k)}_f \right)}{\prod_{l=1}^{n_b} \mathcal{Z} \left( J, m^{(l)}_b \right)} = \text{const} \int \mathcal{D}\mathcal{F} \mathcal{D}\mathcal{B} \exp \left\{ -\mathcal{A}_\mathcal{F}(J) - \mathcal{A}_\mathcal{B}(J) \right\} \quad (10)
\]

where we have denoted:

\[
\mathcal{D}\mathcal{F} \equiv \prod_{k=1}^{n_f} d\chi_k^\dagger d\chi_k d\varphi_k^\dagger d\varphi_k, \quad \mathcal{D}\mathcal{B} \equiv \prod_{l=1}^{n_b} d\sigma_l^\dagger d\sigma_l d\pi_l^\dagger d\pi_l \quad (11)
\]
yielding the following representation for the correlation function:

\[
A_F(J) = \sum_{k=1}^{n_f} \left[ m_f^{(k)} \left( \chi_k^\dagger \chi_k + \varphi_k^\dagger \varphi_k \right) + i \varphi_k^\dagger \hat{J}^\dagger \chi_k + i \chi_k^\dagger \hat{J} \varphi_k \right]
\]

(12)

\[
A_B(J) = \sum_{l=1}^{n_b} \left[ m_b^{(l)} \left( s_l^\dagger s_l + p_l^\dagger p_l \right) + i p_l^\dagger \hat{J}^\dagger s_l + i s_l^\dagger \hat{J} p_l \right]
\]

(13)

The result of the above manipulation is that we can easily perform the ensemble average over the matrices \( J \) in Eq.(4) using the identity:

\[
\langle e^{-Tr(J^\dagger \hat{C} + \hat{D} J)} \rangle_J = e^{\frac{1}{2} Tr \hat{C} \hat{D}}
\]

(14)

where in our case:

\[
\hat{C} = i \sum_l s_l^\dagger p_l^\dagger - i \sum_k \varphi_k^\dagger \chi_k^\dagger,
\]

\[
\hat{D} = i \sum_l p_l^\dagger s_l^\dagger - i \sum_k \chi_k^\dagger \varphi_k^\dagger
\]

After inserting Eq.(11) into Eq. (4) and performing the average in the way described above we find it convenient to introduce two matrices \( \hat{X}, \hat{Y} \) of the size \( n_f \times n_f \) with entries:

\[
X_{k_1,k_2} = \chi_{k_1}^\dagger \chi_{k_2},
\]

\[
Y_{k_1,k_2} = \varphi_{k_1}^\dagger \varphi_{k_2},
\]

(15)

as well as two matrices \( \hat{Q}_{B1}, \hat{Q}_{B2} \) of the size \( n_b \times n_b \):

\[
Q_{B1}^{l_1l_2} = s_{l_1}^\dagger s_{l_2},
\]

\[
Q_{B2}^{l_1l_2} = p_{l_1}^\dagger p_{l_2}
\]

(16)

Then the result of the average can be written in a compact form as

\[
K(\hat{M}_f, \hat{M}_b) = \text{const} \int \mathcal{D} \hat{F} \mathcal{D} \hat{B} \exp \left\{ -W_F - W_B - W_{FB} \right\}
\]

(17)

where

\[
W_F = \text{Tr} \left\{ M_f(\hat{X} + \hat{Y}) - \frac{1}{N} \hat{X} \hat{Y} \right\}
\]

(18)

\[
W_B = \text{Tr} \left\{ M_b(\hat{Q}_{B1} + \hat{Q}_{B2}) - \frac{1}{N} \hat{Q}_{B1} \hat{Q}_{B2} \right\}
\]

(19)

\[
W_{FB} = \frac{1}{N} \sum_{k,l} \left[ \chi_k^\dagger (s_l^\dagger \otimes p_l^\dagger) \phi_k + \phi_k^\dagger (p_l^\dagger \otimes s_l^\dagger) \chi_k \right]
\]

(20)

In contrast to the standard supersymmetric approach from now on we deal with fermionic and bosonic integration separately. Let us first integrate out fermionic variables in Eq.(17). To remove the non-gaussian terms in the exponent we employ the simplest version of the Hubbard-Stratonovich transformation (cf. (14)):

\[
\exp \left[ \frac{1}{N} \text{Tr} \left( \hat{X} \hat{Y} \right) \right] = \text{const} \int d\hat{Q}_F d\hat{Q}_F^\dagger \exp \left\{ -N \text{Tr}(\hat{Q}_F^\dagger \hat{Q}_F) - \text{Tr}(\hat{Q}_F \hat{X}) - \text{Tr}(\hat{Q}_F^\dagger \hat{Y}) \right\}
\]

(21)

where \( \hat{Q}_F \) is a \( n_f \times n_f \) complex matrix. We insert the above formula to the integral Eq.(17) and change the order of integrations. Then the integration over fermionic vectors can be done explicitly yielding the following representation for the correlation function:

\[
K(\hat{M}_f, \hat{M}_b) = \int \mathcal{D} \hat{B} d\hat{Q}_F d\hat{Q}_F^\dagger \det \hat{A} \exp \left\{ -W_B - N \text{Tr}(\hat{Q}_F^\dagger \hat{Q}_F) \right\}
\]

(22)
where the \((2n_fN) \times (2n_fN)\) matrix \(\hat{A}\) has the following structure:

\[
\hat{A} = \begin{pmatrix}
\hat{m}_F \otimes 1_N & \frac{1}{N}(1_{n_f} \otimes \hat{B}) \\
\frac{1}{N}(1_{n_f} \otimes \hat{B})^\dagger & \hat{m}_F \otimes 1_N
\end{pmatrix}
\]  \hspace{1cm} (23)

with \(\hat{m}_F = \hat{M}_f + \hat{Q}_F\), and \(\hat{B} = \sum_{i=1}^{n_h} s_i \otimes (\hat{p}_i^\dagger)\). By simple algebraic manipulations one can show that:

\[
\det \left( \hat{a} \otimes 1_N \begin{pmatrix} 1_{n_f} & \hat{b} \\ \hat{b}^\dagger & 1_N \end{pmatrix} \right) = \det \left( \left[ \hat{a}^{\dagger} \hat{a} \right] \otimes 1_N - 1_{n_f} \otimes \left[ \hat{b} \hat{b}^\dagger \right] \right)
\]

for any \(n_f \times n_f\) matrix \(\hat{a}\) and \(N \times N\) matrix \(\hat{b}\). Moreover, it is easy to see that

\[
\det \left( \hat{R}_a \otimes 1_N - 1_n \otimes \hat{R}_b \right) = \prod_{i=1}^n \det \left( r_a^{(i)} 1_N - \hat{R}_b \right) = \prod_{i=1}^n \prod_{j=1}^{N} \left( r_a^{(i)} - r_b^{(j)} \right)
\]

for any two Hermitian matrices \(\hat{R}_a\) and \(\hat{R}_b\) with eigenvalues \(r_a^{(1)}, ..., r_a^{(n)}\) and \(r_b^{(1)}, ..., r_b^{(N)}\), respectively. Finally, we notice that the matrix \(\hat{R}_B = \hat{B} \hat{B}^\dagger\) has obviously rank \(n_b\), i.e. \(n_b\) nonzero eigenvalues, and verify that \(\text{Tr} \hat{R}_B = \text{Tr} \left( \hat{Q}_{B1} \hat{Q}_{B2} \right)^p\) for any positive integer \(p\). As a consequence, for \(n_b \leq N\) we have

\[
\det \left( r_a^{(i)} 1_N - N^{-2} \hat{R}_B \right) = \left[ r_a^{(i)} \right]^{N-n_b} \det \left( r_a^{(i)} 1_{n_a} - N^{-2} \hat{Q}_{B1} \hat{Q}_{B2} \right)
\]

and immediately obtain the following formula for the determinant entering the equation Eq.(22):

\[
\det \hat{A} = \det \left[ \hat{m}_F \hat{m}_F^\dagger \right]^{N-n_b} \det \left( \hat{m}_F \hat{m}_F^\dagger \otimes 1_{n_f} - \frac{1}{N^2} \left( 1_{n_f} \otimes \left[ \hat{Q}_{B1} \hat{Q}_{B2} \right] \right) \right)
\]  \hspace{1cm} (24)

We see now that the integrand depends on vectors \(s_l, p_l, l = 1, ..., n_h\) only via the scalar products \(s_l^\dagger s_{l'}\) and \(p_l^\dagger p_{l'}\) forming the matrices \(\hat{Q}_{B1(1,2)}, \hat{Q}_{B2}\), see Eq.(16). Then one can convert integration over vectors into integration over the Hermitian matrices \(\hat{Q}_{B1} > 0, \hat{Q}_{B2} > 0\) using the integration theorem, see [22, 23]:

\[
\int \mathcal{D} \mathcal{B} \mathcal{F} \left( \hat{Q}_{B1}, \hat{Q}_{B2} \right) \propto \int_{\hat{Q}_{B1} > 0} d\hat{Q}_{B1} \int_{\hat{Q}_{B2} > 0} d\hat{Q}_{B2} \mathcal{F} \left( \hat{Q}_{B1}, \hat{Q}_{B2} \right) \left[ \det \left( \hat{Q}_{B1} \right) \det \left( \hat{Q}_{B2} \right) \right]^{N-n_b}
\]

and to write down the correlation function under consideration as a matrix integral:

\[
\mathcal{K}_N(\hat{X}_F, \hat{X}_B) \propto e^{-\frac{1}{4} \text{Tr} \hat{X}_F^2} \int d\hat{Q}_F \int d\hat{Q}_B e^{\frac{i}{2} \text{Tr} \hat{X}_F \left( \hat{Q}_F + \hat{Q}_B \right) - N \text{Tr} \left( \hat{Q}_F \hat{Q}_B \right)} \left[ \det \left( \hat{Q}_F \hat{Q}_B \right) \right]^{N-n_b}
\]

\[
\times \int_{\hat{Q}_{B1} > 0} d\hat{Q}_{B1} \int_{\hat{Q}_{B2} > 0} d\hat{Q}_{B2} e^{-\frac{1}{4} \text{Tr} \hat{X}_B \left( \hat{Q}_{B1} + \hat{Q}_{B2} \right) - N \text{Tr} \left( \hat{Q}_{B1} \hat{Q}_{B2} \right)} \left[ \det \left( \hat{Q}_{B1} \hat{Q}_{B2} \right) \right]^{N-n_b}
\]

\[
\times \det \left[ \left( \hat{Q}_F \hat{Q}_B \right) \otimes 1_{n_b} - 1_{n_f} \otimes \left( \hat{Q}_{B1} \hat{Q}_{B2} \right) \right]
\]

where we denoted \(\hat{X}_B = 2N \hat{M}_b\) and \(\hat{X}_F = 2N \hat{M}_f\), correspondingly.
Next natural step is to use the singular value decomposition $\hat{Q}_F = \hat{U}^\dagger \hat{q} \hat{V}$, where $\hat{U}, \hat{V} \in U(n_f)$ are two different $n_f \times n_f$ unitary matrices, and $\hat{q} = \text{diag}(q_1, \ldots, q_{n_f})$ are singular values, i.e. positive square roots of the eigenvalues of $\hat{Q}_F^\dagger \hat{Q}_F > 0$. The integration measure in the new variables is given by $d\hat{Q}_F d\hat{Q}_F^\dagger \propto \prod_{k=1}^{n_f} q_k dq_k \Delta^2(q^2) d\mu(\hat{U}) d\mu(\hat{V})$, with $d\mu(\hat{U}), d\mu(\hat{V})$ being the corresponding Haar’s measure on the group $U(n_f)$ and $\Delta(q^2) = \prod_{k_1 < k_2} (q^2_{k_1} - q^2_{k_2})$.

Now we have to find an appropriate parameterisation for the pair of positive definite Hermitian matrices $\hat{Q}_{B1} > 0$ and $\hat{Q}_{B2} > 0$ of the size $n_b \times n_b$. One can prove that such a pair can always be uniquely represented as

$$\hat{Q}_{B1} = \left[\hat{T}\right]^\dagger -1 \hat{P} \left[\hat{T}\right]^{-1}, \quad \hat{Q}_{B2} = \hat{T} \hat{P} \hat{T}^\dagger$$

in terms of a positive definite diagonal $\hat{P} > 0$ and a general complex matrix $\hat{T}$ with real positive diagonal entries: $\hat{T} \in \text{GL}(n, \mathbb{C})/U(1) \times \ldots \times U(1)$. The corresponding integration measure is:

$$d\hat{Q}_{B1} d\hat{Q}_{B2} \propto \prod_{l=1}^{n_b} \prod_{i \leq m} (p^2_l - p^2_m) d\mu(\hat{T}, \hat{T}^\dagger), \quad d\mu(\hat{T}, \hat{T}^\dagger) = d\hat{T} d\hat{T}^\dagger \text{det} \left[\hat{T} \hat{T}^\dagger\right]^{-n_b + n_f}$$

The proof of the above statements is presented in the Appendix A.

The only terms in the integrand Eq. (27) which depend on the matrices $\hat{U}, \hat{V}$ and $\hat{T}, \hat{T}^\dagger$ are $e^{\frac{1}{2} \text{Tr} X_F (\hat{q} + \hat{Q}_F)}$ and $e^{\frac{1}{2} \text{Tr} X_B (\hat{q} + \hat{Q}_B)}$, respectively. It is immediately evident that the integration over $d\mu(\hat{U}, \hat{V})$ can be performed using the matrix Bessel function Eq. (28), whereas its counterpart Eq. (29) allows us to integrate out the matrices $\hat{T}$ and $\hat{T}^\dagger$. Moreover, the Vandermonde determinant factors arising in the denominator after that integration cancel away one of such factors coming from the integration measure. As the result, the integrand becomes proportional to fully antisymmetric Vandermonde factor. This antisymmetry can be used to show that the integral of the determinant of Bessel functions is equal to the integral of the product of the diagonal elements of the corresponding $n_f \times n_f$ matrix, multiplied with the overall $n_f!$ factor. (This factor counts the number of terms in the determinant). The same procedure is applicable to the integral of the $n_b \times n_b$ determinant containing Macdonald functions. Denoting $\hat{P}^2 = \hat{R}_B$, $\hat{q}^2 = \hat{R}_F$ with eigenvalues $R_B^{(l)}$ and $R_F^{(k)}$, respectively, the resulting expression is given by:

$$K_N(X_F, X_B) \propto \frac{e^{-\frac{1}{2} \text{Tr} \hat{X}_F^2}}{\Delta(X_B^2) \Delta(X_F^2)} \times \int_{\hat{R}_B > 0} d\hat{R}_B \Delta(\hat{R}_B) \left[\text{det} \hat{R}_B\right]^{N-n_b} e^{-N \text{Tr} R_B} \prod_{l=1}^{n_f} I_0\left(X_F^{(l)}/\sqrt{R_F^{(l)}}\right)$$

$$\times \int_{\hat{R}_F > 0} d\hat{R}_F \Delta(\hat{R}_F) \left[\text{det} \hat{R}_F\right]^{N-n_b} e^{-N \text{Tr} R_F} \prod_{k=1}^{n_f} K_0\left(X_B^{(k)}/\sqrt{R_B^{(k)}}\right) \prod_{l=1}^{n_f} \prod_{k=1}^{n_b} (R_F^{(k)} - R_B^{(l)})$$

We further introduce a matrix $\hat{X} = \text{diag} (\hat{X}_F, \hat{X}_B)$ of the size $n = n_f + n_b$ and another matrix $\hat{X} = \text{diag} (\hat{X}_F, \hat{X}_B)$ of the same size. Then

$$K_N(\hat{X}) = C_N \frac{e^{-\frac{1}{2} \text{Tr} \hat{X}_F^2}}{\Delta(X_B^2) \Delta(X_F^2)} \int_{\hat{R} > 0} d\hat{R} \Delta(\hat{R}) \left[\text{det} \hat{R}\right]^{N-n_b} e^{-N \text{Tr} \hat{R}} \prod_{k=1}^{n} J_k\left(X^{(k)}/\sqrt{R^{(k)}}\right)$$

where

$$J_k(z) = \begin{cases} I_0(z), & 1 \leq k \leq n_f \\ K_0(z), & n_f + 1 \leq k \leq n \end{cases}$$
Finally, the presence of the Vandermonde determinant in Eq. (28) suggests to rewrite the correlation function in the form of a $n \times n$ determinant. This leads us to the determinant formula Eq. (7) which is the principal result of the present paper. Our formula generalises known expressions for the product of positive moments [10]. It is valid for any values of the parameters $N, n_b, n_f, X_B, X_F$, provided $N \geq n_b$.

The proportionality constant $C_N$ in the above formula is fixed by the obvious normalisation condition $\mathcal{K}_N(\hat{X})|_{X_B=X_F}=1$ following from the very definition (4). In fact, we find it easier to restore this constant considering the limit $X_B \to \infty, X_F \to \infty$, when obviously $\mathcal{K}_N(\hat{X}) \to (2N)^{2N(n_b-n_f)}/\prod_{k=1}^{n_b}[X^{(k)}_F]^{2N}$. On the other hand, inspecting the integral Eq. (29) one notices that in the discussed limit one can effectively put $R^{(l)}_B \ll R^{(k)}_F$ in the integrand. Then the integral decouples into the product of two integrals: $\mathcal{K}_N(\hat{X})|_{X_B,X_F=\infty} = \mathcal{K}_B(\hat{X}_B)|_{X_B=\infty} \mathcal{K}_F(\hat{X}_F)|_{X_F=\infty}$, where

\[
\mathcal{K}_B(\hat{X}_B) = \frac{1}{\Delta \left( \hat{X}_B^2 \right)} \int_{\hat{R}_B>0} d\hat{R}_B \Delta \left( \hat{R}_B \right) \left[ \det \hat{R}_B \right]^{N-n_b} e^{-N \text{Tr} \hat{R}_B} \prod_{l=1}^{n_b} K_0 \left( X_B^{(l)} \sqrt{R^{(l)}_B} \right) \tag{30}
\]

\[
\mathcal{K}_F(\hat{X}_F) = \frac{e^{-\frac{1}{4N} \text{Tr} \hat{X}_F^2}}{\Delta \left( \hat{X}_F^2 \right)} \int_{\hat{R}_F>0} d\hat{R}_F \Delta \left( \hat{R}_F \right) \left[ \det \hat{R}_F \right]^N e^{-N \text{Tr} \hat{R}_F} \prod_{k=1}^{n_f} I_0 \left( X_F^{(k)} \sqrt{R^{(k)}_F} \right) \tag{31}
\]

In the limit of large $X^{(l)}_B \gg N^{1/2}$ one can use the asymptotic expression for the Macdonald function $K_0(x \gg 1) = \sqrt{2x} e^{-x}$ in the integrand of (30) and safely neglect the terms $NR^{(l)}_B$ in the exponent. The resulting integral is of the form

\[
\int_0^\infty \cdots \int_0^\infty \Delta \{R_1, ..., R_n\} \prod_{j=1}^n R_j^a e^{-X_j \sqrt{R_j}} dR_j = 2^n \frac{\Delta \{X_1^2, ..., X_n^2\}}{\prod_{j=1}^n X_j^{2(a+n)} \prod_{j=1}^n \Gamma[2(j+a)]} \tag{32}
\]

In our case $a = N - n_b - 1/4$ which yields:

\[
\mathcal{K}_B(\hat{X}_B)|_{X_B=\infty} = (2\pi)^{n_b/2} \prod_{l=1}^{n_b} X_B^{(l)}{-2N} \prod_{l=1}^{n_b} \Gamma[2(N - n_b + l) - 1/2] \tag{33}
\]

For the second integral we should use the asymptotic expression for the modified Bessel function $I_0(x \gg 1) = \sqrt{\frac{\pi}{2x^2}} e^x$. Here, however, we can not neglect the terms $NR^{(k)}_F$ in the exponent since it ensures the convergence of the integral. In fact, it is easy to see that for $X^{(k)}_F \gg 2N$ the integral is dominated by small vicinity of the value $R^{(k)}_F = X^{(k)}_F / 2N$. Taking into account small Gaussian fluctuations around those points we find that

\[
\mathcal{K}_F(\hat{X}_F)|_{X_F=\infty} = \prod_{k=1}^{n_f} \left( X^{(k)}_F \right)^{2N} \frac{1}{2^{2(Nn_j+n_f^2-n_f)} N^{2Nn_f+n_f^2}} \tag{34}
\]

The equations (33)-(34) yield the following value for the normalisation constant:

\[
C_N = \frac{2^{2(Nn_b+n_f^2-n_f)} N^{n_f^2+2Nn_b}}{(2\pi)^{n_b/2} \prod_{l=1}^{n_b} \Gamma[2(N - n_b + l) - 1/2]}
\]
As usual, we would like to extract the leading behavior of the above correlation function in the scaling limit $N \to \infty$. Here we are mostly interested in the so-called “chiral scaling” $K^{\text{chir}}(X) = \lim_{N \to \infty} K_N(X)$ which amounts to keeping the parameters $X$ fixed when treating the corresponding integrals by the saddle-point method. The saddle-points are extremals of the function $\mathcal{L}(R_k) = R_k - \ln R_k$, $k = 1, \ldots, n$ in the domain $R_k > 0$, which are given by $R_k = 1$. To pick up a nonvanishing contribution one has to take accurately into account Gaussian fluctuations around the saddle point. It is done in the most straightforward way by exploiting the expression (29) and putting there $R_k = 1 + \xi_k$, with $-\infty < \xi_k < \infty$ parameterizing the fluctuations. Introducing $\Xi = (\xi_1, \ldots, \xi_n)$ we have:

$$K^{\text{chir}}(X) \propto \frac{1}{\Delta(X^2_B) \Delta(X^2_F)} \int d\Xi \Delta(\Xi) e^{-2N \text{Tr} \Xi} \prod_{k=1}^n J_k(1 + \xi_k)$$

(35)

Evaluating the limit we finally find:

$$K^{\text{chir}}(X) \propto \Delta(X^2_B) \Delta(X^2_F) \times$$

$$\det \left[
\begin{array}{cccc}
I_0 \left[ X_F^{(1)} \right] & X_F^{(1)} t_0^{(1)} \left[ X_F^{(1)} \right] & \cdots & (X_F^{(1)})_{n-1} t_0^{(n-1)} \left[ X_F^{(n_f)} \right] \\
I_0 \left[ X_F^{(n_f)} \right] & X_F^{(n_f)} t_0^{(n_f)} \left[ X_F^{(n_f)} \right] & \cdots & (X_F^{(n_f)})_{n-1} t_0^{(n-1)} \left[ X_F^{(n_f)} \right] \\
K_0 \left[ X_B^{(1)} \right] & X_B^{(1)} K_0^{(1)} \left[ X_B^{(1)} \right] & \cdots & (X_B^{(1)})_{n-1} K_0^{(n-1)} \left[ X_B^{(1)} \right] \\
K_0 \left[ X_B^{(n_b)} \right] & X_B^{(n_b)} K_0^{(n_b)} \left[ X_B^{(n_b)} \right] & \cdots & (X_B^{(n_b)})_{n-1} K_0^{(n-1)} \left[ X_B^{(n_b)} \right]
\end{array}\right]$$

(36)

where $I_0^{(l)}(z)$ and $K_0^{(l)}(z)$ stand for $l$-th derivatives of the Bessel and Macdonald functions.

Such an expression provides us with the most general correlation function for chiral GUE and generalises earlier results known for $n_f > 0$, $n_b = 0$ [10], $n_f = 0$, $n_b > 0$ [22], and $n_f = n_b = (1, 2)$ [3]. For example, for $n_f = n_b = 1$ the equation (36) amounts to

$$K^{\text{chir}}(X_F, X_B) \propto [X_F I_1(X_F) K_0(X_B) + X_B I_0(X_F) K_1(X_B)]$$

and is simply related to the formula found in [3].

2.1 Conclusion

We have demonstrated that the general spectral correlation functions containing both products and ratios of the characteristic polynomials of chiral GUE matrices can be calculated in a closed form. Our method amounted to representing the characteristic polynomials in terms of the Gaussian integrals and exploiting the Itzykson-Zuber type integration formulae Eqs. (38), both over compact and non-compact domains. The results were shown to have an attractively simple determinantal structure reminiscent of that arising in the method of orthogonal polynomials. In a separate publication [23] we demonstrate that indeed there exists a way to calculate the general correlation function by a method resorting to the orthogonal polynomial technique. This way makes clear that the determinantal structure observed in the present paper has, in fact, a very profound origin.
and allows one to extend our results to other unitary-invariant ensembles, with or without chiral structure. As interesting prospects for future research we would like to mention a challenging problems of extending the suggested methods to other symmetry classes (see [13, 25]), as well as to ensembles of non-Hermitian random matrices. The latter are important for QCD applications (see e.g. [26] and references therein) and for the problems of quantum chaotic scattering [27].

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3 Appendix A. A parametrisation for the pair of positive definite matrices $Q_B^1$ and $Q_B^2$

Given two positive definite complex Hermitian matrices $Q_B^1 > 0$ and $Q_B^2 > 0$ of the size $n_b \times n_b$ let us introduce $\tilde{Q}_B^1 = [Q_B^1]^{-1} > 0$, so that

$$dQ_B^1 dQ_B^2 = d\tilde{Q}_B^1 dQ_B^2 \left(\det \tilde{Q}_B^1\right)^{-2n_b}$$

Now we use that any pair of complex Hermitian $Q_B^1 > 0$ and $\tilde{Q}_B^2 > 0$ can be parameterized as (see, for example [28])

$$\tilde{Q}_B^1 = TT^\dagger, \quad Q_B^2 = TP T^\dagger$$

where the matrix $P$ is a diagonal positive definite: $P = \text{diag} (p_1, \ldots, p_{n_b}) > 0$ and the matrix $T$ is general complex, with the only restriction that its diagonal entries are real and positive: $T \in \mathbb{G}(n_b, \mathbb{C})/U(1) \times \cdots \times U(1)$. The latter condition is necessary to ensure one-to-one correspondence between the parametrisations. Our goal is to calculate the related Jacobian.

For this goal we use the following relation between the matrix differentials:

$$d\tilde{Q}_B^1 = T \left[\delta T + \delta T^\dagger\right] T^\dagger$$

where we denoted $\delta T = T^{-1} dT$ and $\delta T^\dagger = dT^\dagger \left[ T^\dagger \right]^{-1}$. Analogously

$$dQ_B^2 = T \left[\delta TP + dP + P \delta T^\dagger\right] T^\dagger$$

Further denoting $\delta Q_B^1 = T^{-1} d\tilde{Q}_B^1 \left[ T^\dagger \right]^{-1}$, $\delta Q_B^2 = T^{-1} dQ_B^2 \left[ T^\dagger \right]^{-1}$ we first calculate the Jacobian of the transformation from the set of variables $[\delta Q_B^1]_{ii}, [\delta Q_B^2]_{ii}, [\delta Q_B^1]_{i<j}, [\delta Q_B^1]_{i<j}^*, [\delta Q_B^2]_{i<j}$ to a new set of independent variables $dp_i, (\delta T)_{ii}, (\delta T)_{i<j}, (\delta T)^*_{i<j}, (\delta T)_{i<j}^*$ where $1 < i, j < n_b$ and $\ast$ stands for the complex conjugation. The calculation is simple and yields the factor that we symbolically write as

$$\frac{\delta (Q_B^1, Q_B^2)}{\delta (P, T, T^\dagger)} \propto \prod_{i<j} (p_i - p_j)^2$$

We need also the following intermediate Jacobians:

$$\frac{d (\tilde{Q}^1_B, Q^2_B)}{\delta (Q_B^1, Q_B^2)} = \det (TT^\dagger)^{2n_b}$$
complex matrices as:

\[
K
\]

where \(T\)

Then all the factors taken together yield the full Jacobian:

\[
\frac{d(Q_{B1}, Q_{B2})}{d(P, T, T^\dagger)} = \frac{d(Q_{B1}, Q_{B2})}{d(P, T, T^\dagger)} \times \frac{d(Q_{B1}, Q_{B2})}{d(P, T, T^\dagger)} \times \frac{d(T, T^\dagger)}{d(T, T^\dagger)} \quad (37)
\]

\[
= \left(\prod_{i<j} (p_i - p_j)^2 \det (TT^\dagger)^{(n_b+1)/2}\right)
\]

To arrive to the parametrisation Eq.(26) used in the main text of the paper we change \(P \rightarrow P^2\) so that \(dP \rightarrow 2^{n_b}dP\) in the measure and then change \(T \rightarrow T P^{-1/2}\) so that \(T^\dagger \rightarrow P^{-1/2}T^\dagger\).

Then \(Q_{B1} = \hat{Q}_{B1}^{-1} = [TP^{-1}T^\dagger]^{-1} = [T^\dagger]^{-1}PT^{-1}\) as required and the matrix \(Q_{B2}\) stays equal to \(TP^{\dagger}\). The transformation amounts to change in the measure \(dTdT^\dagger \rightarrow det P^{-n_b/2}dTdT^\dagger\).

Finally, taking into account the factor

\[
\det \left[\hat{Q}_{B1}\right]^{-2n_b} = det P^{-2n_b} \det (TT^\dagger)^{2n_b}
\]

we arrive to the expression for the measure given in Eq.(27). Let us note that \(d\mu(T, T^\dagger)\) is exactly the invariant measure on the manifold \(\text{Gl}(n, \mathbb{C})/U(1) \times \ldots \times U(1)\).

**Appendix B. Matrix Macdonald functions associated with integrals over complex matrices**

The matrix Bessel functions that correspond to Itzykson-Zuber-like integrals over unitary matrices were considered in details by Guhr and Wettig, Guhr and Kohler [20]. Below we consider the matrix Macdonald functions that are associated with integrals over arbitrary complex matrices with real positive diagonal elements.

Let \(X_d\) and \(Y_d\) be two diagonal matrices with real positive elements,

\[
X_d = \text{diag}\,(x_1, x_2, \ldots, x_n), \quad Y_d = \text{diag}\,(y_1, y_2, \ldots, y_n) \quad (38)
\]

Our aim is to show that the integral:

\[
\Phi(X_d, Y_d) = \int \exp \left[ -\frac{1}{2} \text{Tr} \left( X_d \left[ TY_d T^\dagger + (T^\dagger)^{-1} Y_d T^{-1}\right]\right) \right] d\mu(T) \quad (39)
\]

(where \(T\) are arbitrary \(n \times n\) complex matrices) can be considered as the matrix Macdonald function. We are going to demonstrate the following expression for the function \(\Phi(X_d, Y_d)\):

\[
\Phi(X_d, Y_d) = \text{const} \left| \frac{\det \left[ K_0(x_i y_j) \right] |_{1 \leq i, j \leq n}}{\Delta(x_1, \ldots, x_n)^2 \Delta(y_1, \ldots, y_n)^2} \right| (40)
\]

where \(K_0(x)\) is the Bessel function of an imaginary argument of \(0^{th}\) order.

Given three \(n \times n\) complex matrices \(X, A, B\) we consider the Laplace operator \(D_X\) acting on complex matrices as:

\[
D_X = \sum_{i \leq j \leq n} \left( \frac{\partial^2}{\partial (\text{Re} X_{ij})^2} + \frac{\partial^2}{\partial (\text{Im} X_{ij})^2} \right) \quad (41)
\]
Then we construct a function $W(X, A, B)$ with the property

$$D_X W(X, A, B) = \text{Tr}(AB)W(X, A, B)$$  \hspace{1cm} (42)

In particular, the following function

$$W(X, A, B) = \exp\left[-\frac{1}{2}\text{Tr}\left(XA + BX^\dagger\right)\right]$$  \hspace{1cm} (43)

satisfies the Eq.(42), as can be checked by direct calculations. Let us put specifically $A = TY_dT^\dagger$ and $B = (T^\dagger)^{-1}Y_dT^{-1}$ in Eq.(12). We obtain:

$$D_X \exp\left[-\frac{1}{2}\text{Tr}\left(X[TY_dT^\dagger + (T^\dagger)^{-1}Y_dT^{-1}]\right)\right] = \text{Tr}(Y_d^2) \exp\left[-\frac{1}{2}\text{Tr}\left(X[TY_dT^\dagger + (T^\dagger)^{-1}Y_dT^{-1}]\right)\right]$$  \hspace{1cm} (44)

As soon as the integration over complex matrices $T$ commutes with the Laplace operator $D_X$ we conclude that the matrix function $\Phi(X_d, Y_d)$ defined by the integral Eq.(39) satisfies the following differential equation:

$$D_X \Phi(X_d, Y_d) = \text{Tr}(Y_d^2)\Phi(X_d, Y_d)$$  \hspace{1cm} (45)

To derive the explicit formula (Eq.(39)) for the matrix function $\Phi(X_d, Y_d)$ we apply the method proposed by Guhr and Wettig [20]. Using the singular value decomposition for an arbitrary complex matrix $X$:

$$X = U^\dagger X_d V, \quad U^\dagger U = 1_n, \quad V^\dagger V = 1_n$$  \hspace{1cm} (46)

the radial part $D_{X_d}$ of the Laplace operator $D_X$ must have the following expression:

$$D_{X_d} = \frac{1}{J(x)} \sum_{i=1}^n \partial_i J(x) \partial_i, \quad J(x) = \Delta^2(x_1^2, \ldots, x_n^2) \prod_{i=1}^n x_i$$  \hspace{1cm} (47)

Guhr and Wettig noted [20] that the radial part $D_{X_d}$ of the Laplace operator $D_X$ is separable. It means that for an arbitrary function $f(x_1, x_2, \ldots, x_n)$ the following identity holds:

$$D_{X_d} \frac{f(x_1, \ldots, x_n)}{\Delta(x_1^2, \ldots, x_n^2)} = \frac{1}{\Delta(x_1^2, \ldots, x_n^2)} \sum_{k=1}^n \left( \frac{\partial^2}{\partial x_k^2} + \frac{1}{x_k} \frac{\partial}{\partial x_k} \right) \frac{f(x_1, \ldots, x_n)}{\Delta(x_1^2, \ldots, x_n^2)}$$  \hspace{1cm} (48)

The separability of the operator $D_{X_d}$ enables us to solve the differential equation (Eq.45) and to prove formula (39). We use the following ansatz for the solution $\Phi(X_d, Y_d)$ of differential equation (45):

$$\Phi(X_d, Y_d) = \frac{\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)}{\Delta(x_1^2, \ldots, x_n^2) \Delta(y_1^2, \ldots, y_n^2)}$$  \hspace{1cm} (49)

We insert the above ansatz to the differential equation (Eq.45). Applying the property (Eq.48) of the radial part $D_{X_d}$ of the Laplace operator $D_X$ we find:

$$\sum_{k=1}^n \left( \frac{\partial^2}{\partial x_k^2} + \frac{1}{x_k} \frac{\partial}{\partial x_k} \right) \Psi(x_1, \ldots, x_n; y_1, \ldots, y_n) = (y_1^2 + y_2^2 + \ldots + y_n^2) \Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)$$  \hspace{1cm} (50)
The form of the above differential equation suggests to look for a solution as a sum over permutations of the set \((1, 2, \ldots, n)\). Each term in the sum can be taken as a product of \(n\) multipliers and each multiplier is represented by a (kernel) function taken at different \(x_i\) and \(y_j\), i.e.

\[
\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n) = \sum_{\sigma \in S_n} C_\sigma \phi(x_1, y_{\sigma(1)}) \phi(x_2, y_{\sigma(2)}) \ldots \phi(x_n, y_{\sigma(n)})
\]  

(51)

where \(C_\sigma\) is some constant dependent on a particular permutation \(\sigma\). To determine \(C_\sigma\) we explore the symmetry properties of the function \(\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)\). The function \(\Phi(X_d, Y_d)\) is symmetric under permutations inside the set \(\{y_1, y_2, \ldots, y_n\}\) as can be seen from the integral representation (Eq.39). Indeed, the result of any such permutation can be represented by a permutation matrix \(P\) as

\[
Y_d \rightarrow Y_d' = PY_dP^\dagger
\]

(52)

Let us replace the matrix \(Y_d\) by \(Y_d'\) in the integral (Eq.39). Changing the variables of integration, \(T_1 = TP\), we see that \(\Phi(X_d, PY_dP^\dagger) = \Phi(X_d, Y_d)\), i.e. the integral (Eq.39) is symmetric under permutations inside the set \(\{y_1, y_2, \ldots, y_n\}\). As soon as \(\Phi(X_d, Y_d)\) and \(\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)\) are related by Eq.(49) the function \(\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)\) is antisymmetric under those permutations. It gives the coefficient \(C_\sigma\) in the formula (Eq.51):

\[
C_\sigma = (-1)^{\nu_\sigma}
\]

(53)

where \(\nu_\sigma = 1\) for odd permutations and \(\nu_\sigma = 0\) for even permutations. We then obtain the following expression for the function \(\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)\) from the Eq.(51):

\[
\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det (\phi(x_i, y_j)) |_{1 \leq i, j \leq n}
\]

(54)

We insert the expression Eq.(51) for \(\Psi(x_1, \ldots, x_n; y_1, \ldots, y_n)\) to the differential equation (Eq.50). As a result we find that the kernel \(\phi(x, y)\) satisfies the differential equation:

\[
\left[ \partial_x^2 + \frac{1}{x} \partial_x \right] \phi(x, y) = y^2 \phi(x, y)
\]

(55)

Both Bessel functions of an imaginary argument \(I_0(xy)\) and the Macdonald functions \(K_0(xy)\) satisfy the above differential equation. However the integral representation (Eq.39) dictates the choice for the kernel function \(\phi(x, y)\). Indeed, we note that for \(n = 1\) the integral (Eq.39) is proportional to the kernel function \(\phi(x, y)\). This integral representation coincides with that for the Macdonald function \(K_0(xy)\) and we conclude that \(\phi(x, y) = \text{const } K_0(xy)\). Then equations (14) and (54) yield the desired formula (Eq.40).

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