Irreducibility criterion for tensor products of Yangian evaluation modules

A. I. MOLEV

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexm@maths.usyd.edu.au

Abstract

The evaluation homomorphisms from the Yangian $Y(gl_n)$ to the universal enveloping algebra $U(gl_n)$ allow one to regard the irreducible finite-dimensional representations of $gl_n$ as Yangian modules. We give necessary and sufficient conditions for irreducibility of tensor products of such evaluation modules.

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1 Introduction

The Yangian $Y(\mathfrak{gl}_n)$ for the general linear Lie algebra $\mathfrak{gl}_n$ is a deformation of the universal enveloping algebra $U(\mathfrak{gl}_n[x])$ in the class of Hopf algebras; see Drinfeld \[5\]. A theorem of his [6] (see also Tarasov [23]) provides a complete discription of finite-dimensional irreducible representations of $Y(\mathfrak{gl}_n)$ in terms of their highest weights. Recently, Arakawa [2] has found a character formula for each of these representations with the use of the Kazhdan–Lusztig polynomials; see also Vasserot [24] for the case of quantum affine algebras. Nazarov and Tarasov [20] (see also Cherednik [3]) have given an explicit construction for a class of the so-called tame representations. However, the structure of the general finite-dimensional irreducible $Y(\mathfrak{gl}_n)$-module (with $n \geq 3$) still remains unknown. In this paper we establish an irreducibility criterion for tensor products of the Yangian evaluation modules which thus solves this problem for a wide class of representations of $Y(\mathfrak{gl}_n)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple of complex numbers such that $\lambda_i - \lambda_{i+1}$ is a non-negative integer for each $i$. Denote by $L(\lambda)$ the irreducible finite-dimensional representation of the Lie algebra $\mathfrak{gl}_n$ with the highest weight $\lambda$. For each $a \in \mathbb{C}$ there is an evaluation homomorphism $\varphi_a$ from the Yangian $Y(\mathfrak{gl}_n)$ to the universal enveloping algebra $U(\mathfrak{gl}_n)$: see Section 2 for the definitions. Using $\varphi_a$ we make $L(\lambda)$ into a Yangian module and denote it by $L_a(\lambda)$. We keep the notation $L(\lambda)$ for the evaluation module $L_a(\lambda)$ with $a = 0$. The Hopf algebra structure on $Y(\mathfrak{gl}_n)$ allows one to regard tensor products of the type

$$L_{a_1}(\lambda^{(1)}) \otimes L_{a_2}(\lambda^{(2)}) \otimes \cdots \otimes L_{a_k}(\lambda^{(k)})$$

as Yangian modules. Our main result is a criterion of irreducibility of these modules: see Theorems 1.1 and 1.2 below. To formulate the result we first note that the problem can be reduced to the particular case where all the parameters $a_i$ in (1.1) are equal to zero. This is done by using the composition of the module (1.1) with an appropriate automorphism of the Yangian: see Proposition 2.4 below.

We give first an irreducibility criterion for the tensor product $L(\lambda) \otimes L(\mu)$ of two evaluation modules. It is well-known (see e.g. [9] and Theorem 3.1 below) that this module is irreducible if the differences $\lambda_i - \mu_j$ are not integers. Furthermore, for any $c \in \mathbb{C}$ the simultaneous shifts $\lambda_i \mapsto \lambda_i + c$ and $\mu_j \mapsto \mu_j + c$ for all $i$ and $j$ do not affect the irreducibility of $L(\lambda) \otimes L(\mu)$; see Proposition 2.3. Thus, we may assume without loss of generality that all the entries of $\lambda$ and $\mu$ are integers.

We shall be using the following definition. Two disjoint finite subsets $A$ and $B$ of $\mathbb{Z}$ are crossing if there exist elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that either $a_1 < b_1 < a_2 < b_2$, or $b_1 < a_1 < b_2 < a_2$. Otherwise, $A$ and $B$ are called non-crossing.

Given a highest weight $\lambda$ with integer entries introduce the following subset of $\mathbb{Z}$:

$$A_{\lambda} = \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1\}.$$
**Theorem 1.1** The module $L(\lambda) \otimes L(\mu)$ is irreducible if and only if the sets $A_{\lambda} \setminus A_{\mu}$ and $A_{\mu} \setminus A_{\lambda}$ are non-crossing.

Using the argument of Kitanine, Maillet and Terras [10, 14], Nazarov and Tarasov [22, Theorem 4.9] demonstrated that the irreducibility criterion for the multiple tensor product (1.1) can be obtained from the particular case of $k = 2$ tensor factors. Namely, the following “binary property” holds. Here we let $\lambda^{(1)}, \ldots, \lambda^{(k)}$ be $n$-tuples of complex numbers such that $\lambda^{(p)}_i - \lambda^{(p)}_{i+1}$ is a non-negative integer for each $i$ and $p$.

**Theorem 1.2** The module

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(k)})$$

is irreducible if and only if all the modules $L(\lambda^{(p)}) \otimes L(\lambda^{(q)})$ with $p < q$ are irreducible.

Note that the “only if” part of this theorem is well known. It is implied by Proposition 2.6 (see below); cf. [3], [16]. If the module (1.2) is irreducible, its highest weight is easy to find. Therefore, together with Theorem 1.1 the binary property of Theorem 1.2 allows one to determine whether a given irreducible $Y(gl_n)$-module can be realized in a tensor product (1.2).

For the proof of Theorem 1.1 we use the Gelfand–Tsetlin bases of the $gl_n$-modules $L(\lambda)$ and $L(\mu)$. The key role is played by the formulas for the action of the Drinfeld generators of $Y(gl_n)$ in these bases as well as by the quantum minor formulas for the Yangian lowering operators [15]; cf. Nazarov and Tarasov [19, 20].

In the case of the Yangian $Y(gl_2)$ the criterion coincides with the one obtained by Chari and Pressley [3] and it is also implicitly contained in Tarasov’s paper [23]; see also [16]. Nazarov and Tarasov [21] found a criterion of irreducibility of (1.1) in the case where each highest weight $\lambda^{(p)}$ has the form $(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ with $\alpha - \beta \in \mathbb{Z}_+$. This generalized earlier results by Akasaka and Kashiwara [1] and Zelevinsky [25].

Leclerc, Nazarov and Thibon [12] have found an irreducibility criterion for the induction products of evaluation modules over the affine Hecke algebras of type $A$ with the use of the canonical bases; see also Leclerc and Thibon [11]. Leclerc and Zelevinsky [13]. The application of the Drinfeld functor [3] (see also [2]) leads to an irreducibility criterion for the Yangian modules (1.1) (equivalent to Theorems 1.1 and 1.2), when the highest weights $\lambda^{(p)}$ satisfy some extra conditions. Namely, assuming that the $\lambda^{(p)}$ are partitions (we may do this without loss of generality), one should require that the sum of their lengths does not exceed $n$.

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He kindly informed me about the results of [12, 22] prior to their publication, and
the present form of Theorem 1.1 was inspired by the irreducibility criterion in [12].
Originally, the author obtained this result in the form of Theorems 3.1 and 4.1: see
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2 Preliminaries

We refer the reader to the expository papers [16, 18] where the results on the structure
theory and representations of the Yangians are collected.

The Yangian $Y(n) = Y(\mathfrak{gl}_n)$ [3, 7] is the complex associative algebra with the
generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $1 \leq i, j \leq n$, and the defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)), \quad (2.1)$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots \in Y(n)[[u^{-1}]]$$

and $u$ is a formal (commutative) variable. The Yangian $Y(n)$ is a Hopf algebra with
the coproduct $\Delta : Y(n) \to Y(n) \otimes Y(n)$ defined by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u). \quad (2.2)$$

The series $t_{b_1 \cdots b_r}(u)$ is skew symmetric under permutations of the indices $a_i$, or $b_i$.

The Poincaré–Birkhoff–Witt theorem for the Yangian $Y(n)$ (see e.g. [18, Corollary 1.23]) implies that given a subset of indices $\{a_1, \ldots, a_r\} \subseteq \{1, \ldots, n\}$ the corresponding quantum minor of the matrix $[t_{ij}(u)]$ is defined by the following equivalent formulas:

$$t_{b_1 \cdots b_r}(u) = \sum_{\sigma \in \mathfrak{S}_r} \text{sgn } \sigma \cdot t_{a_{\sigma(1)}b_1}(u) \cdots t_{a_{\sigma(r)}b_r}(u - r + 1) \quad (2.3)$$

$$= \sum_{\sigma \in \mathfrak{S}_r} \text{sgn } \sigma \cdot t_{a_1b_{a(1)}}(u - r + 1) \cdots t_{a_rb_{a(r)}}(u). \quad (2.4)$$

The following proposition is proved in [17, Proposition 1.1] by using the $R$-matrix
form of the defining relations (2.1).
Proposition 2.1 We have the relations
\[
[t_{a_1 \cdots a_k}^{b_1 \cdots b_k}(u), t_{c_1 \cdots c_l}^{d_1 \cdots d_l}(v)] = \sum_{p=1}^{\min(k,l)} \frac{(-1)^{p-1} p!}{(u - v - k + 1) \cdots (u - v - k + p)} \sum_{i_1 < \cdots < i_p \atop j_1 < \cdots < j_p} \left( t_{a_1 \cdots a_k}^{b_1 \cdots \cdots b_k}(u) t_{c_1 \cdots a_k}^{d_1 \cdots \cdots d_l}(v) - t_{c_1 \cdots c_l}^{d_1 \cdots b_k \cdots b_k}(v) t_{b_1 \cdots d_1 \cdots d_l}(v) t_{b_1 \cdots d_1 \cdots d_l}(u) \right).
\]
Here the \(p\)-tuples of upper indices \((a_1, \ldots, a_p)\) and \((c_1, \ldots, c_p)\) are respectively interchanged in the first summand on the right hand side while the \(p\)-tuples of lower indices \((b_1, \ldots, b_p)\) and \((d_1, \ldots, d_p)\) are interchanged in the second summand.  

We note the following particular case of these relations:
\[
[t_{a_1 \cdots a_p}^{c_1 \cdots c_l}(u), t_{d_1 \cdots d_l}^{c_1 \cdots c_l}(v)] = \frac{1}{u - v} \left( \sum_{i=1}^l t_{c_i \cdots}^{c_i \cdots} t_{d_1 \cdots d_l}^{c_1 \cdots c_l}(v) - \sum_{i=1}^l t_{d_i \cdots}^{c_1 \cdots c_l} t_{d_1 \cdots d_l}(u) t_{d_1 \cdots d_l}(u) \right). \quad (2.6)
\]
This implies the well-known property of the quantum minors: for any indices \(i, j\) we have
\[
[t_{c_i d_j}^{c_i d_j}(u), t_{d_i \cdots d_l}(v)] = 0. \quad (2.7)
\]

We shall frequently use the following result proved in [20].

Proposition 2.2 The images of the quantum minors under the coproduct are given by
\[
\Delta(t_{b_1 \cdots b_r}^{a_1 \cdots a_r}(u)) = \sum_{c_1 < \cdots < c_r} t_{c_1 \cdots c_r}^{a_1 \cdots a_r}(u) \otimes t_{b_1 \cdots b_r}(u),
\]
summed over all subsets of indices \(\{c_1, \ldots, c_r\}\) from \(\{1, \ldots, n\}\).  

For \(m \geq 1\) introduce the series \(a_m(u), b_m(u)\) and \(c_m(u)\) by
\[
a_m(u) = t_{1 \cdots m}^{1 \cdots m}(u), \quad b_m(u) = t_{1 \cdots m-1,m+1}^{1 \cdots m-1,m+1}(u), \quad c_m(u) = t_{1 \cdots m-1,m+1}(u). \quad (2.8)
\]
The coefficients of these series generate the algebra \(Y(n)\) [7], they are called the Drinfeld generators.

By a theorem of Drinfeld [8] every finite-dimensional irreducible representation of the Yangian \(Y(n)\) is a highest weight representation. That is, it contains a unique, up to a scalar factor, nonzero vector \(\zeta\) (the highest vector) which is annihilated by
all upper triangular elements \( t_{ij}(u), \ i < j \), and \( \zeta \) is an eigenvector for the diagonal generators \( t_{ii}(u) \),

\[
t_{ii}(u) \zeta = \lambda_i(u) \zeta, \quad i = 1, \ldots, n.
\]

Here the \( \lambda_i(u) \) are formal series in \( u^{-1} \) with complex coefficients. We call the collection \( (\lambda_1(u), \ldots, \lambda_n(u)) \) the highest weight of the representation. Equivalently, \( \zeta \) is annihilated by \( b_1(u), \ldots, b_{n-1}(u) \) and it is an eigenvector for each of the operators \( a_1(u), \ldots, a_n(u) \) \([7]\), so that

\[
a_m(u) \zeta = \lambda_1(u) \lambda_2(u-1) \cdots \lambda_m(u-m+1) \zeta, \quad m = 1, \ldots, n.
\]

If \( L \) is any \( Y(n) \)-module, then a nonzero element \( \zeta \in L \) is called a singular vector if \( \zeta \) is annihilated by all upper triangular generators \( t_{ij}(u), \ i < j \), and \( \zeta \) is an eigenvector for the diagonal elements \( t_{ii}(u) \). Such a vector \( \zeta \) generates a highest weight submodule in \( L \). The following proposition if proved by a standard argument; see e.g. \([16]\).

**Proposition 2.3** If \( L \) is an irreducible highest weight \( Y(n) \)-module and \( \zeta \in L \) is annihilated by all operators \( t_{ij}(u) \) with \( i < j \) then \( \zeta \) is proportional to the highest vector of \( L \). \( \square \)

Let the \( E_{ij}, i, j = 1, \ldots, n \) denote the standard basis elements of the Lie algebra \( \mathfrak{gl}_n \). For any \( a \in \mathbb{C} \) the mapping

\[
\varphi_a : t_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u-a}
\]

defines an algebra epimorphism from \( Y(n) \) to the universal enveloping algebra \( U(\mathfrak{gl}_n) \) so that any \( \mathfrak{gl}_n \)-module can be extended to a \( Y(n) \) module via \((2.9)\). In particular, let \( \lambda \) be an \( n \)-tuple of complex numbers \( \lambda = (\lambda_1, \ldots, \lambda_n) \) such that \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \) for all \( i \) (we call such \( n \)-tuples \( \mathfrak{gl}_n \)-highest weights). Consider the irreducible finite-dimensional \( \mathfrak{gl}_n \)-module \( L(\lambda) \) with the highest weight \( \lambda \) with respect to the upper triangular Borel subalgebra. The corresponding \( Y(n) \)-module is denoted by \( L_a(\lambda) \), and we call it the evaluation module. We keep the notation \( L(\lambda) \) for the module \( L_a(\lambda) \) with \( a = 0 \). The coproduct \( \Delta \) defined by \((2.2)\) allows one to consider the tensor products \((1.1)\) as \( Y(n) \)-modules.

Let us denote by \( I \) the \( n \)-tuple \((1,1,\ldots,1)\).

**Proposition 2.4** The \( Y(n) \)-module \((1.1)\) is irreducible if and only if the module

\[
L(\lambda^{(1)} - a_1 I) \otimes L(\lambda^{(2)} - a_2 I) \otimes \cdots \otimes L(\lambda^{(k)} - a_k I)
\]

is irreducible.
Proof. Suppose that (1.1) is irreducible. Let \( \xi^{(p)} \) denote the highest vector of the \( \mathfrak{gl}_n \)-module \( L(\lambda^{(p)}) \). We derive from (2.2) and (2.9) that \( \zeta = \xi^{(1)} \otimes \cdots \otimes \xi^{(k)} \) is the highest vector of the \( Y(n) \)-module (1.1) with the highest weight \( (\lambda_1(u), \ldots, \lambda_n(u)) \) where

\[
\lambda_i(u) = \left( 1 + \frac{\lambda_i^{(1)}}{u - a_1} \right) \cdots \left( 1 + \frac{\lambda_i^{(k)}}{u - a_k} \right), \quad i = 1, \ldots, n. \tag{2.11}
\]

Consider the automorphism of \( Y(n) \)

\[
t_{ij}(u) \mapsto f(u) t_{ij}(u), \tag{2.12}
\]

where \( f(u) \) is the formal series in \( u^{-1} \) given by

\[
f(u) = (1 - a_1 u^{-1}) \cdots (1 - a_k u^{-1}).
\]

The composition of the module (1.1) with this automorphism is an irreducible \( Y(n) \)-module \( \tilde{L} \) with the highest weight \( (\tilde{\lambda}_1(u), \ldots, \tilde{\lambda}_n(u)) \) where

\[
\tilde{\lambda}_i(u) = \left( 1 + \frac{\lambda_i^{(1)} - a_1}{u} \right) \cdots \left( 1 + \frac{\lambda_i^{(k)} - a_k}{u} \right), \quad i = 1, \ldots, n. \tag{2.13}
\]

On the other hand, the tensor product of the highest vectors of the \( \mathfrak{gl}_n \)-modules \( L(\lambda^{(p)} - a_p I) \) is a singular vector of the \( Y(n) \)-module (2.10) with the weight given by (2.13). Therefore, \( \tilde{L} \) is isomorphic to a subquotient of (2.10). However, these two modules have the same dimension and hence, they are isomorphic. In particular, the module (2.10) is irreducible. The proof is completed by reversing the argument. \( \square \)

**Proposition 2.5** Given \( c \in \mathbb{C} \), the simultaneous shifts

\[
\lambda_i^{(p)} \mapsto \lambda_i^{(p)} + c, \quad p = 1, \ldots, k, \quad i = 1, \ldots, n
\]

of the parameters of the module (1.2) do not affect its irreducibility.

**Proof.** It the module (1.2) is irreducible then so is the module \( L^c \) which is the composition of (1.2) with the automorphism of \( Y(n) \) given by

\[
t_{ij}(u) \mapsto t_{ij}(u + c), \quad i, j = 1, \ldots, n.
\]

The highest weight of \( L^c \) is \( (\lambda_1^c(u), \ldots, \lambda_n^c(u)) \) with

\[
\lambda_i^c(u) = \left( 1 + \frac{\lambda_i^{(1)}}{u + c} \right) \cdots \left( 1 + \frac{\lambda_i^{(k)}}{u + c} \right), \quad i = 1, \ldots, n.
\]

The proof is completed by repeating the argument of the proof of Proposition 2.4 with the use of the automorphism (2.12) of \( Y(n) \) where \( f(u) = (1 + cu^{-1})^k \). \( \square \)
Proposition 2.6 Suppose that the \(Y(n)\)-module (1.2) is irreducible. Then any permutation of the tensor factors in (1.2) gives an isomorphic representation of \(Y(n)\).

Proof. Denote the tensor product (1.2) by \(L\). Note that \(L\) is a representation with the highest weight \((\lambda_1(u), \ldots, \lambda_n(u))\) given by (2.11) with \(a_1 = \cdots = a_k = 0\). Consider a representation \(L'\) obtained by a certain permutation of the tensor factors in (1.2). The tensor product \(\zeta'\) of the highest vectors of the representations \(L(\lambda_i)\) is a singular vector in \(L'\) whose weight is given by the same formulas (2.11). This implies that \(\zeta'\) generates a highest weight submodule in \(L'\) such that its irreducible quotient is isomorphic to \(L\). However, \(L\) and \(L'\) have the same dimension which implies that \(L\) and \(L'\) are isomorphic. \(\square\)

We shall use a version given in \([15]\) (cf. \([19]\)) of the construction of a basis of the \(\mathfrak{gl}_n\)-module \(L(\lambda)\) which is originally due to Gelfand and Tsetlin \([8]\). We equip \(L(\lambda)\) with a \(Y(n)\)-module structure by using the epimorphism

\[Y(n) \to U(\mathfrak{gl}_n), \quad t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}, \quad (2.14)\]

see (2.9). A pattern \(\Lambda\) (associated with \(\lambda\)) is a sequence of rows \(\Lambda_n, \Lambda_{n-1}, \ldots, \Lambda_1\), where \(\Lambda_r = (\lambda_{r,1}, \ldots, \lambda_{r,r})\) is the \(r\)-th row from the bottom, the top row \(\Lambda_n\) coincides with \(\lambda\), and the following betweenness conditions are satisfied: for \(r = 2, \ldots, n\)

\[\lambda_{r,i} - \lambda_{r-1,i} \in \mathbb{Z}_+, \quad \lambda_{r-1,i} - \lambda_{r,i+1} \in \mathbb{Z}_+, \quad \text{for} \quad i = 1, \ldots, r - 1. \quad (2.15)\]

For any pattern \(\Lambda\) introduce the vector \(\xi_{\Lambda} \in L(\lambda)\) by

\[\xi_{\Lambda} = \prod_{r=2}^{n-1} \prod_{i=1}^{r-1} \tau_{r,i}(-\lambda_{r-1,i} - 1) \cdots \tau_{r,i}(-\lambda_{r,i} + 1) \tau_{r,i}(-\lambda_{r,i}) \xi, \quad (2.16)\]

where \(\xi\) is the highest vector of \(L(\lambda)\) and

\[\tau_{r,i}(u) = u(u-1) \cdots (u-r+i+1) t_i^{r+1 \cdots r}(u)\]

is the lowering operator; see also Section \([4]\). The vectors \(\xi_{\Lambda}\), where \(\Lambda\) runs over all patterns associated with \(\lambda\), form a basis of \(L(\lambda)\). The \(\tau_{r,i}(u)\) essentially coincide with the standard lowering operators arising from the transvector algebras; cf. \([17]\). We find from (2.14) that the operators

\[B_m(u) = u(u-1) \cdots (u-m+1) b_m(u), \quad A_m(u) = u(u-1) \cdots (u-m+1) a_m(u)\]

in \(L(\lambda)\) are polynomials in \(u\); see (2.8). Their action in the basis \(\{\xi_{\Lambda}\}\) of \(L(\lambda)\) is given by the following formulas; see \([15]\). They can also be deduced from Lemmas \([4,3,4.5]\); see Section \([4]\). We use the notation \(l_{ri} = \lambda_{ri} - i + 1\).
Proposition 2.7 We have
\begin{align}
A_m(u) \xi_\Lambda &= (u + l_{m1}) \cdots (u + l_{mm}) \xi_\Lambda, \\
B_m(-l_{mj}) \xi_\Lambda &= -\prod_{i=1}^{m+1} (l_{mi+1} - l_{mj}) \xi_{\Lambda+\delta_{mj}} \text{ for } j = 1, \ldots, m,
\end{align}
where \( \Lambda + \delta_{mj} \) is obtained from \( \Lambda \) by replacing the entry \( \lambda_{mj} \) with \( \lambda_{mj} + 1 \), and \( \xi_{\Lambda+\delta_{mj}} \) is supposed to be equal to zero if \( \Lambda + \delta_{mj} \) is not a pattern.

Applying the Lagrange interpolation formula we can find the action of \( B_m(u) \) for any \( u \). Note that the polynomial \( B_m(u) \) has degree \( m-1 \) with the leading coefficient \( E_{m,m+1} \). This therefore implies the Gelfand–Tsetlin formulas \([8]\) for the action of the elements \( E_{m,m+1} \):
\begin{align}
E_{m,m+1} \xi_\Lambda &= -\sum_{j=1}^{m} \frac{(l_{m+1,1} - l_{mj}) \cdots (l_{mi+1,m+1} - l_{mj})}{(l_{m1} - l_{mj}) \cdots \Lambda_j \cdots (l_{mm} - l_{mj})} \xi_{\Lambda+\delta_{mj}},
\end{align}
where \( \Lambda_j \) indicates that the \( j \)-th factor is skipped.

We conclude this section with an equivalent form of the conditions of Theorem 1.1.

Given complex \( \mathfrak{gl}_n \)-highest weights \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_n) \) we shall use the notation
\[ l_i = \lambda_i - i + 1, \quad m_i = \mu_i - i + 1, \quad i = 1, \ldots, n. \]
For a pair of indices \( i < j \) we shall denote
\begin{align}
\langle l_j, l_i \rangle &= \{l_j, l_j + 1, \ldots, l_i\} \setminus \{l_j, l_{j-1}, \ldots, l_i\}, \\
\langle m_j, m_i \rangle &= \{m_j, m_j + 1, \ldots, m_i\} \setminus \{m_j, m_{j-1}, \ldots, m_i\}.
\end{align}
In particular, if \( \lambda_i = \lambda_{i+1} = \cdots = \lambda_j \) then \( \langle l_j, l_i \rangle = \emptyset \).

We shall assume now that \( \lambda \) and \( \mu \) are \( \mathfrak{gl}_n \)-highest weights with integer entries.

Proposition 2.8 The sets \( A_\lambda \setminus A_\mu \) and \( A_\mu \setminus A_\lambda \) are non-crossing if and only if for all pairs of indices \( 1 \leq i < j \leq n \) we have
\begin{align}
m_j, m_i \notin \langle l_j, l_i \rangle \quad \text{or} \quad l_j, l_i \notin \langle m_j, m_i \rangle.
\end{align}

Proof. Let us write \( \text{Cond}(A_\lambda, A_\mu) \) for the condition that \( A_\lambda \setminus A_\mu \) and \( A_\mu \setminus A_\lambda \) are non-crossing. We use induction on \( n \). In the case \( n = 2 \) the statement is obviously true. Let \( n \geq 3 \). Suppose first that \((2.20)\) holds. Set
\[ A^-_\lambda = \{l_1, \ldots, l_{n-1}\} \quad \text{and} \quad A^+_\lambda = \{l_2, \ldots, l_n\} \]
and similarly define $A^-_m$ and $A^+_m$. By the induction hypothesis, both conditions $\text{Cond}(A^-_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\lambda, A^+_\mu)$ are satisfied. If $m_1 = l_1$ then $\text{Cond}(A^-_\lambda, A^-_\mu)$ obviously holds. We may assume without loss of generality that $m_1 > l_1$. Let

$$A^-_\lambda \setminus A^-_\mu = \{i_1, \ldots, i_k\}, \quad A^-_\mu \setminus A^-_\lambda = \{j_1, \ldots, j_l\},$$

where $1 \leq i_1 < \cdots < i_k \leq n$ and $1 = j_1 < \cdots < j_l \leq n$. We must have for some $a \in \{1, \ldots, k\}$ that

$$m_{j_{a+1}} < l_k < \cdots < l_{i_1} < m_{j_{a}},$$

where the leftmost inequality is ignored when $a = k$. If $2 \leq a \leq k - 1$ then together with $\text{Cond}(A^-_\lambda, A^-_\mu)$ this clearly ensures $\text{Cond}(A^-_\lambda, A^-_\mu)$. Similarly, this is also true when $a = 1$ and $i_1 \geq 2$. So, if $a = 1$ then the only case where both $\text{Cond}(A^-_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\lambda, A^+_\mu)$ hold but $\text{Cond}(A^-_\lambda, A^-_\mu)$ does not, is the one with the following inequalities between the elements of $A^-_\lambda \setminus A^-_\mu$ and $A^-_\mu \setminus A^-_\lambda$:

$$l_n < m_n < m_{j_k} < \cdots < m_{j_2} < l_{i_k} < \cdots < l_{i_2} < l_1 < m_1.$$

However, in this case $m_n \in \langle l_n, l_1 \rangle$ and $l_1 \in \langle m_n, m_1 \rangle$, so that (2.20) is violated for $i = 1$ and $j = n$. An analogous argument shows that if $a = k$ then the only case where both $\text{Cond}(A^-_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\lambda, A^+_\mu)$ hold but $\text{Cond}(A^-_\lambda, A^-_\mu)$ does not, is

$$l_n < l_{i_k} < \cdots < l_{i_2} < m_n < l_1 < m_{j_k} < \cdots < m_{j_2} < m_1.$$

But then $m_n \in \langle l_n, l_1 \rangle$ and $l_1 \in \langle m_n, m_1 \rangle$ which contradicts (2.20) again.

Conversely, suppose that $\text{Cond}(A^-_\lambda, A^-_\mu)$ holds. This condition clearly implies both $\text{Cond}(A^-_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\lambda, A^+_\mu)$ and so, by the induction hypothesis, (2.20) holds for all pairs $i < j$ with the possible exception for $(i, j) = (1, n)$. If the latter condition fails then we have

$$\begin{cases} m_n \in \langle l_n, l_1 \rangle \\ l_1 \in \langle m_n, m_1 \rangle \end{cases} \quad \text{or} \quad \begin{cases} l_n \in \langle m_n, m_1 \rangle \\ m_1 \in \langle l_n, l_1 \rangle. \end{cases}$$

However, in each of the two cases this contradicts $\text{Cond}(A^-_\lambda, A^-_\mu)$.

\[\square\]

### 3 Sufficient conditions

Our aim in this section is to prove the following.

**Theorem 3.1** Let $\lambda$ and $\mu$ be complex $\mathfrak{gl}_n$-highest weights. Suppose that for each pair of indices $1 \leq i < j \leq n$ we have

$$m_j, m_i \notin \langle l_j, l_i \rangle \quad \text{or} \quad l_j, l_i \notin \langle m_j, m_i \rangle. \quad (3.1)$$

Then the $Y(n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible.

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We give the proof of Theorem 3.1 as a sequence of lemmas. Let $\xi$ and $\xi'$ denote the highest vectors of the $\mathfrak{gl}_n$-modules $L(\lambda)$ and $L(\mu)$, respectively. Let $N$ be a nonzero $Y(n)$-submodule of $L(\lambda) \otimes L(\mu)$. A standard argument (see e.g. [16]) shows that $N$ must contain a singular vector $\zeta$. The key part of the proof of the theorem is to show by induction on $n$ that

$$\zeta = \text{const} \cdot \xi \otimes \xi'. \quad (3.2)$$

Then considering dual modules we also show that the vector $\xi \otimes \xi'$ is cyclic.

By Proposition 2.6, exchanging $\lambda$ and $\mu$ if necessary, we may assume that for the pair $(i, j)$ with $i = 1$ and $j = n$ the condition

$$m_1, m_n \notin \langle l_n, l_1 \rangle \quad (3.3)$$

is satisfied. Consider the Gelfand–Tsetlin basis $\{\xi_\Lambda\}$ of the $\mathfrak{gl}_n$-module $L(\lambda)$; see Section 2. The singular vector $\zeta$ is uniquely written in the form

$$\zeta = \sum_\Lambda \xi_\Lambda \otimes \eta_\Lambda, \quad (3.4)$$

summed over all patterns $\Lambda$ associated with $\lambda$, and $\eta_\Lambda \in L(\mu)$.

For the diagonal Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_n$, we denote by $\varepsilon_i$ the basis vector of $\mathfrak{h}^*$ dual to the element $E_{ii}$ so that the $n$-tuple $\lambda$ can be identified with the element $\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in \mathfrak{h}^*$. We shall be using a standard partial ordering on the weights of $L(\lambda)$. Given two weights $v, w \in \mathfrak{h}^*$, we shall write $v \preceq w$ if $w - v$ is a $\mathbb{Z}_+$-linear combination of the simple roots $\varepsilon_a - \varepsilon_{a+1}$. Equivalently, $v \preceq w$ if and only if

$$w - v = \sum_{a=1}^n p_a \varepsilon_a, \quad (3.5)$$

with the conditions

$$p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_{n-1} \in \mathbb{Z}_+, \quad p_1 + \cdots + p_n = 0.$$

The embedding

$$U(\mathfrak{gl}_n) \hookrightarrow Y(n), \quad E_{ij} \mapsto t_{ij}^{(1)}, \quad (3.6)$$

defines the natural $U(\mathfrak{gl}_n)$-module structure on $L(\lambda) \otimes L(\mu)$. We shall usually identify the operators $E_{ij}$ and $t_{ij}^{(1)}$. The vector $\zeta$ is clearly a $\mathfrak{gl}_n$-singular vector. In particular, it is a weight vector. Since the basis $\{\xi_\Lambda\}$ consists of weight vectors, each element $\eta_\Lambda \in L(\mu)$ in (3.4) is also a $\mathfrak{gl}_n$-weight vector. Moreover, all elements $\xi_\Lambda \otimes \eta_\Lambda$ in (3.4) have the same $\mathfrak{gl}_n$-weight.
We shall denote the weight of the vector \( \xi_\Lambda \), or, the weight of the pattern \( \Lambda \), by \( w(\Lambda) \). It is well known \cite{8}, and can be deduced e.g. from (2.17) that

\[
    w(\Lambda) = w_1 \varepsilon_1 + \cdots + w_n \varepsilon_n, \quad w_k = \sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i}. \tag{3.7}
\]

We shall say that a pattern \( \Lambda \) occurs in the expansion (3.4) if \( \eta_\Lambda \neq 0 \). Consider the set of patterns occurring in (3.4) and suppose that \( \Lambda_0 \) is a minimal element of this set with respect to the partial ordering on the weights \( w(\Lambda) \). In other words, if \( \Lambda \) occurs in (3.4) and \( w(\Lambda) \preceq w(\Lambda_0) \) then \( w(\Lambda) = w(\Lambda_0) \).

**Lemma 3.2** The vector \( \eta_{\Lambda_0} \) coincides with \( \zeta' \), up to a constant factor.

**Proof.** We have \( b_m(u) \zeta = 0 \) for \( m = 1, \ldots, n - 1 \). Therefore, by Proposition 2.2,

\[
    \sum_{c_1 < \cdots < c_m} \sum_{\Lambda} t_{c_1 \cdots c_m}^{1 \cdots m}(u) \xi_\Lambda \otimes t_{c_1 \cdots c_m}^{1 \cdots m-1,m+1}(u) \eta_\Lambda = 0. \tag{3.8}
\]

Write the elements \( t_{c_1 \cdots c_m}^{1 \cdots m}(u) \xi_\Lambda \) as linear combinations of the basis vectors \( \xi_\Lambda \) and take the coefficient at \( \xi_{\Lambda_0} \) in the relation (3.8). The weight of the vector \( t_{c_1 \cdots c_m}^{1 \cdots m}(u) \xi_\Lambda \) (or, to be more precise, the weight of each of the coefficients of these series) equals

\[
    w' = w(\Lambda) + \varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{c_1} - \cdots - \varepsilon_{c_m}.
\]

Therefore \( w' \succeq w(\Lambda) \), and \( w' = w(\Lambda) \) if and only if \( c_i = i \) for each \( i \). Since \( \Lambda_0 \) is a pattern of a minimal weight, the vector \( \xi_{\Lambda_0} \) can only occur in the expansion of

\[
    t_{c_1 \cdots c_m}^{1 \cdots m}(u) \xi_{\Lambda_0} = a_m(u) \xi_{\Lambda_0}. \tag{3.9}
\]

By (2.17) this implies that

\[
    t_{c_1 \cdots c_m}^{1 \cdots m}(u) \eta_{\Lambda_0} = b_m(u) \eta_{\Lambda_0} = 0
\]

for each \( m \). Thus, \( \eta_{\Lambda_0} \) is a singular vector of \( L(\mu) \) and so, \( \eta_{\Lambda_0} = \text{const} \cdot \zeta' \). \( \square \)

**Lemma 3.3** The pattern \( \Lambda_0 \) is determined uniquely. Moreover, if a pattern \( \Lambda \) occurs in (3.4) then \( w(\Lambda) \succeq w(\Lambda_0) \).

**Proof.** We use the fact that \( \zeta \) is an eigenvector for the generators \( a_1(u), \ldots, a_n(u) \).

By Proposition 2.2,

\[
    a_m(u) \zeta = \sum_{c_1 < \cdots < c_m} \sum_{\Lambda} t_{c_1 \cdots c_m}^{1 \cdots m}(u) \xi_\Lambda \otimes t_{c_1 \cdots c_m}^{1 \cdots m}(u) \eta_\Lambda. \tag{3.10}
\]
This vector equals $\alpha_m(u) \zeta$ for a formal series $\alpha_m(u)$. As we have seen in the proof of Lemma 3.2, the vector $\xi_{\lambda'}$ can only occur in (3.10) in the expansion of (3.9). By (2.17) and Lemma 3.2, comparing the coefficients at $\xi$ of Lemma 3.2, the vector $\xi = \alpha_r \zeta$ to the pattern $\Lambda^0$. For each entry $\lambda_{ka}$ of a pattern $\Lambda$ occurring in (3.4) we have

$$\lambda_{ka} - \lambda^0_{ka} \in \mathbb{Z}_+, \quad \text{for} \quad 1 \leq a \leq k \leq n - 1.$$  

**Proof.** If $\Lambda$ occurs in (3.4) then $w(\Lambda) \geq w(\Lambda^0)$ by Lemma 3.3. We use induction on $w(\Lambda)$. Fix $\Lambda \neq \Lambda^0$. Then there exists another pattern $\Lambda'$ occurring in (3.4) such that $w(\Lambda^0) \leq w(\Lambda') < w(\Lambda)$, and for some $m$ and some indices $c_1 < \cdots < c_m$ the expansion of $t_{c_1 \ldots c_m}^m(u) \xi_{\Lambda'}$ contains $\xi_{\Lambda}$ with a nonzero coefficient. Indeed, if this is not the case, then considering the coefficient at $\xi_{\Lambda}$ in (3.8) we come to the conclusion that $b_m(u) \eta_{\Lambda} = 0$ for all $m = 1, \ldots, n - 1$, and so, $\eta_{\Lambda}$ is, up to a constant, the highest vector of $L(\mu)$: see the proof of Lemma 3.2. This implies that $\Lambda$ and $\Lambda^0$ must have the same weight. Due to Lemma 3.3, we have to conclude that $\Lambda = \Lambda^0$, contradiction.

By (2.6) the operator $t_{c_1 \ldots c_m}^m(u)$ can be represented as the commutator

$$t_{c_1 \ldots c_m}^m(u) = [\ldots [t_{c_1 \ldots c_m}^m(u), E_{m_{c_m}}], E_{m-1,c_{m-1}}, \ldots, E_{p_{c_p}}],$$

where $p$ is the minimum of the indices $i$ such that $c_i \neq i$. Here, as before, we identify the elements $E_{ij}$ and $t_{ij}^{(1)}$ using the embedding (3.6). The operator $t_{c_1 \ldots c_m}^m(u)$ acts on the basis vectors $\xi_{\Lambda}$ by scalar multiplication; see (2.17). Furthermore, $E_{ij}$ with $i < j$ is a commutator in the generators $E_{k,k+1}$ with $k = 1, \ldots, n - 1$. By the Gelfand–Tsetlin formulas (2.18), $E_{k,k+1} \xi_{\Lambda'}$ is a linear combination of the basis vectors $\xi_{\Lambda' + b_{ka}}$, where $a = 1, \ldots, k$. The proof is completed by the application of the induction hypothesis to the pattern $\Lambda'$.

**Lemma 3.4** For each entry $\lambda_{ka}$ of a pattern $\Lambda$ occurring in (3.4) we have

$$\lambda_{ka} - \lambda^0_{ka} \in \mathbb{Z}_+, \quad \text{for} \quad 1 \leq a \leq k \leq n - 1.$$  

**Proof.** If $\Lambda$ occurs in (3.4) then $w(\Lambda) \geq w(\Lambda^0)$ by Lemma 3.3. We use induction on $w(\Lambda)$. Fix $\Lambda \neq \Lambda^0$. Then there exists another pattern $\Lambda'$ occurring in (3.4) such that $w(\Lambda^0) \leq w(\Lambda') < w(\Lambda)$, and for some $m$ and some indices $c_1 < \cdots < c_m$ the expansion of $t_{c_1 \ldots c_m}^m(u) \xi_{\Lambda'}$ contains $\xi_{\Lambda}$ with a nonzero coefficient. Indeed, if this is not the case, then considering the coefficient at $\xi_{\Lambda}$ in (3.8) we come to the conclusion that $b_m(u) \eta_{\Lambda} = 0$ for all $m = 1, \ldots, n - 1$, and so, $\eta_{\Lambda}$ is, up to a constant, the highest vector of $L(\mu)$: see the proof of Lemma 3.2. This implies that $\Lambda$ and $\Lambda^0$ must have the same weight. Due to Lemma 3.3, we have to conclude that $\Lambda = \Lambda^0$, contradiction.

By (2.6) the operator $t_{c_1 \ldots c_m}^m(u)$ can be represented as the commutator

$$t_{c_1 \ldots c_m}^m(u) = [\ldots [t_{c_1 \ldots c_m}^m(u), E_{m_{c_m}}], E_{m-1,c_{m-1}}, \ldots, E_{p_{c_p}}],$$

where $p$ is the minimum of the indices $i$ such that $c_i \neq i$. Here, as before, we identify the elements $E_{ij}$ and $t_{ij}^{(1)}$ using the embedding (3.6). The operator $t_{c_1 \ldots c_m}^m(u)$ acts on the basis vectors $\xi_{\Lambda}$ by scalar multiplication; see (2.17). Furthermore, $E_{ij}$ with $i < j$ is a commutator in the generators $E_{k,k+1}$ with $k = 1, \ldots, n - 1$. By the Gelfand–Tsetlin formulas (2.18), $E_{k,k+1} \xi_{\Lambda'}$ is a linear combination of the basis vectors $\xi_{\Lambda' + b_{ka}}$, where $a = 1, \ldots, k$. The proof is completed by the application of the induction hypothesis to the pattern $\Lambda'$.

**Lemma 3.5** The $(n - 1)$-th row of the pattern $\Lambda^0$ is $(\lambda_1, \ldots, \lambda_{n-1})$.

**Proof.** We shall be proving the following property of $\Lambda^0$ which clearly implies the statement. For every $r = 1, \ldots, n - 1$ we have: if $i \geq r$ and

$$\lambda^0_{n-1,i} = \lambda^0_{n-2,i-1} = \cdots = \lambda^0_{n-r,i-r+1},$$

then

$$\lambda^0_{n-1,i} = 0.$$
then either $\lambda^0_{n-1,i} = \lambda_i$, or $i \geq r + 1$ and

$$
\lambda^0_{n-1,i} = \lambda^0_{n-2,i} = \cdots = \lambda^0_{n-r,i} = \lambda^0_{n-r-1,i}.
$$

Suppose the contrary, and let $i \geq r$ take the minimum value for which the property fails. That is, there exists $i \geq r$ such that $\Lambda' := \Lambda^0 + \delta_{n-1,i} + \cdots + \delta_{n-r,i}$ is a pattern. Since $\zeta$ is a singular vector, we have

$$
t^{1 \cdots n-r}_{1 \cdots n-r-1,n}(u) \zeta = 0. \tag{3.12}
$$

By Proposition 2.2,

$$
\sum_{c_1 \leq \cdots \leq c_n} t^{1 \cdots n-r}_{c_1 \cdots c_n}(u) \xi_{\Lambda} \otimes t^{1 \cdots c_n}_{1 \cdots n-r-1,n}(u) \eta_{\Lambda} = 0. \tag{3.13}
$$

The coefficient of the vector $\xi_{\Lambda'} \otimes \eta_{\Lambda^0}$ in the expansion of the left hand side of (3.13) must be 0. Let us determine which patterns $\Lambda$ yield a nontrivial contribution to this coefficient. Considering the weight of $t^{1 \cdots n-r}_{c_1 \cdots c_n}(u) \xi_{\Lambda}$ we come to the relation

$$
w(\Lambda) + \varepsilon_1 + \cdots + \varepsilon_{n-r} - \varepsilon_{c_1} - \cdots - \varepsilon_{c_n} = w(\Lambda^0) + \varepsilon_{n-r} - \varepsilon_n,
$$

and hence

$$
w(\Lambda) - w(\Lambda^0) = \varepsilon_{c_1} + \cdots + \varepsilon_{c_n} - \varepsilon_{c_1} - \cdots - \varepsilon_{n-r} - \varepsilon_n.
$$

Since $w(\Lambda) \geq w(\Lambda^0)$ by Lemma 3.3, we obtain from (3.5) that $c_a = a$ for $a = 1, \ldots, n-r-1$, and $c_{n-r} \in \{n-r, \ldots, n\}$. Then $w(\Lambda) = w(\Lambda^0) + \varepsilon_{c_{n-r}} - \varepsilon_n$. By Lemma 3.4, $\Lambda$ should be obtained from $\Lambda^0$ by increasing exactly one entry by 1 in each of the rows $c_n, c_{n-r} + 1, \ldots, n-1$. On the other hand, by the minimality of $r$ and the betweenness conditions (2.15), the array $\Lambda$ would not be a pattern, unless $c_{n-r} = n - r$ or $c_{n-r} = n$. Thus, the coefficient in question can only have a contribution from two summands in (3.13), namely,

$$
t^{1 \cdots n-r}_{1 \cdots n-r-1,n}(u) \xi_{\Lambda^0} \otimes t^{1 \cdots n-r-1,n}_{1 \cdots n-r-1,n}(u) \eta_{\Lambda^0} \tag{3.14}
$$

and

$$
t^{1 \cdots n-r}_{1 \cdots n-r}(u) \xi_{\Lambda'} \otimes t^{1 \cdots n-r}_{1 \cdots n-r-1,n}(u) \eta_{\Lambda'} \tag{3.15}
$$

We consider (3.14) first. By (2.6),

$$
t^{1 \cdots n-r}_{1 \cdots n-r-1,n}(u) = [t^{1 \cdots n-r}_{1 \cdots n-r}(u), E_{n-r,n}]. \tag{3.16}
$$

It has been observed above that the minimality of $r$ and the betweenness conditions (2.13) imply that $E_{cn} \xi_{\Lambda^0} = 0$ for $n - r < c \leq n - 1$. Hence using the relation

$$
E_{n-r,n} = [E_{n-r,n-r+1}, \cdots [E_{n-3,n-2}, [E_{n-2,n-1}, E_{n-1,n}]] \cdots]
$$
we get
\[ E_{n-r,n} \xi_{A^0} = (-1)^{r-1}E_{n-1,n}E_{n-2,n-1} \cdots E_{n-r,n-r+1} \xi_{A^0}. \]

Therefore, by (2.18) the expansion of \( E_{n-r,n} \xi_{A^0} \) in terms of the basis vectors \( \xi_A \) contains \( \xi_A' \) with a nonzero coefficient \( C \). It will now be convenient to use polynomial quantum minor operators defined by
\[ T_{b_1 \cdots b_m}^{a_1 \cdots a_m}(u) = u(u-1) \cdots (u-m+1) t_{b_1 \cdots b_m}^{a_1 \cdots a_m}(u), \]
see (2.3). Using (2.17) we find from (3.16) that the coefficient of \( \xi_A' \) in the expansion of \( T_{1 \cdots n-r-1,n}^{1 \cdots n}(u) \xi_{A^0} \) equals
\[ C(u+l_0^{0-r-1}) \cdots \Lambda_j \cdots (u+l_0^{0-n-r}) = (u+1) \cdots (u+m_{n-r-1})(u+m+n+r) \Lambda_{A^0}. \]

where \( C \) is a nonzero constant. For the second factor in (3.14) we find from (2.3) and Lemma 3.2 that
\[ T_{1 \cdots n-r-1,n}^{1 \cdots n}(u) \eta_{A^0} = (u+m_1) \cdots (u+m_{n-r-1})(u+m+n+r) \eta_{A^0}. \]

Consider now the expression (3.15). By (2.17) we have
\[ T_{1 \cdots n-r}^{1 \cdots n}(u) \xi_A' = (u+l_0^{0-r-1}) \cdots (u+l_0^{0-n-r}) \xi_A'. \]

Since \( \zeta \) is a \( \mathfrak{gl}_n \)-weight vector and \( w(M) = w(A^0) + \varepsilon_{n-r} - \varepsilon_n \), the vector \( \eta_{A'} \) is a linear combination of the \( \xi_A' \) with \( w(M) = \mu - \varepsilon_{n-r} + \varepsilon_n \), where \( \{ \xi_A' \} \) is the Gelfand–Tsetlin basis of \( L(\mu) \). This implies that the \( (n-r) \)-th row of each of the patterns \( M \) is \( (\mu_1, \ldots, \mu_{n-r-1}, \mu_{n-r} - 1) \). We therefore have \( E_{n-r,n} \eta_{A'} = \text{const} \cdot \zeta' \) and so, by (2.17) and (3.16)
\[ T_{1 \cdots n-r-1,n}^{1 \cdots n}(u) \eta_{A'} = \text{const} \cdot (u+m_1) \cdots (u+m_{n-r-1}) \xi_A'. \]

Combining the results of the above calculations, and taking the coefficient of the vector \( \xi_A' \otimes \eta_{A^0} \) in (3.12), we obtain
\[ C(u+m+r) \cdots (u+l_0^{0-n-r})(u+m_1) \cdots (u+m_{n-r-1})(u+m+n+r) \]
\[ + \text{const} \cdot (u+l_0^{0-r-1}) \cdots (u+l_0^{0-n-r}) \Lambda_{A^0} = 0. \]

Deleting the common factors gives
\[ C(u+m+r) + \text{const} \cdot (u+l_0^{0-r-1}) = 0. \]

Put \( u = -l_0^{0-r-1} - 1 \) in this relation. Since \( C \) is nonzero we get \( m_n = l_0^{0-n-r} - r + 1 \) and thus \( m_n = l_0^{0-n-1} \). By the betweenness conditions for \( \Lambda^0 \) and \( \Lambda' \),
\[ \lambda_i - \lambda_{n-1,i} > 0 \quad \text{and} \quad \lambda_{n-1,i} - \lambda_{i+1} \geq 0, \]

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which implies that both differences $l_i - m_n$ and $m_n - l_{i+1}$ are positive integers. Thus $m_n \in (l_{i+1}, l_i) \subseteq (l_n, l_1)$ which contradicts (3.3). Therefore, our assumption that $\Lambda'$ is a pattern must be wrong. 

By Lemma 3.4 we can now conclude that all vectors $\xi_\Lambda$ which occur in (3.4) belong to the $U(\mathfrak{gl}_{n-1})$-span of the highest vector $\xi$ of $L(\lambda)$. This span is isomorphic to the irreducible representation $L(\lambda_-)$ of $\mathfrak{gl}_{n-1}$ with the highest weight $\lambda_- = (\lambda_1, \ldots, \lambda_{n-1})$. In particular, $E_m \xi_\Lambda = \lambda_n \xi_\Lambda$ for each $\Lambda$. Furthermore, if $\zeta$ is a linear combination of vectors $\xi_\Lambda \otimes \xi'_M$ then by Lemma 3.2, for the corresponding patterns we have

$$w(\Lambda) + w(M) = \mu.$$

Therefore, $E_m \xi'_M = \mu_n \xi'_M$ for all $M$ which implies that the $(n-1)$-th row of each pattern $M$ coincides with $(\mu_1, \ldots, \mu_{n-1})$. In other words, each vector $\xi'_M$ belongs to the $U(\mathfrak{gl}_{n-1})$-span of $\xi'$ which is isomorphic to $L(\mu_-)$ where $\mu_- = (\mu_1, \ldots, \mu_{n-1})$. Thus, $\zeta$ belongs to

$$L(\lambda_-) \otimes L(\mu_-). \quad (3.17)$$

By (2.1) and the defining relations (2.2), the $Y(n-1)$-module structure on (3.17) coincides with the one obtained by restriction from $Y(n)$ to the subalgebra generated by the $t_{ij}(u)$ with $1 \leq i, j \leq n - 1$. The vector $\zeta$ is annihilated by the operators $b_1(u), \ldots, b_{n-2}(u)$. By the assumption of the theorem, for each pair $(i, j)$ such that $1 \leq i < j \leq n - 1$ the condition (3.1) is satisfied. Therefore, the $Y(n-1)$-module (3.17) is irreducible by the induction hypothesis, and we may finally conclude from Proposition 2.3 that (3.2) holds.

Next, we derive a similar result for the singular lowest vectors under the assumptions of Theorem 3.1. As we pointed out, the condition (3.3) can also be assumed due to Proposition 2.6.

**Lemma 3.6** If $\zeta' \in L(\lambda) \otimes L(\mu)$ and $t_{ij}(u) \zeta' = 0$ for all $1 \leq j < i \leq n$ then

$$\zeta' = \text{const} \cdot \eta \otimes \eta',$$

where $\eta$ and $\eta'$ are the lowest vectors of $L(\lambda)$ and $L(\mu)$, respectively.

**Proof.** Let $\omega$ be the permutation of the indices $1, \ldots, n$ such that $\omega(i) = n - i + 1$. The mapping

$$Y(n) \to Y(n), \quad t_{ij}(u) \mapsto t_{\omega(i)\omega(j)}(u) \quad (3.19)$$

defines an automorphism of the Yangian $Y(n)$. This follows easily from the defining relations (2.7). We equip the space $L = L(\lambda) \otimes L(\mu)$ with another structure of $Y(n)$-module which is obtained by pulling back through the automorphism (3.19). Denote
this new representation by \( L^\omega \). Similarly, the mapping

\[
U(\mathfrak{g} l_n) \to U(\mathfrak{g} l_n), \quad E_{ij} \mapsto E_{\omega(i) \omega(j)}
\]
defines an automorphism of \( U(\mathfrak{g} l_n) \). Denote by \( L(\lambda)^\omega \) the representation of \( U(\mathfrak{g} l_n) \) obtained from \( L(\lambda) \) by pulling back through this automorphism and extend it to \( Y(n) \) using (2.14). It follows from (2.2) that the \( Y(n) \)-module \( L^\omega \) is isomorphic to the tensor product \( L(\lambda)^\omega \otimes L(\mu)^\omega \). The weight of the lowest vector \( \eta \) of \( L(\lambda) \) is \( (\lambda_n, \ldots, \lambda_1) \). Therefore \( \eta \), when regarded as an element of \( L(\lambda)^\omega \), is the highest vector of the weight \( \lambda \). In particular, the \( U(\mathfrak{g} l_n) \)-module \( L(\lambda)^\omega \) is isomorphic to \( L(\lambda) \). Now, \( \zeta' \) is a singular vector of the \( Y(n) \)-module \( L^\omega \). By the proved above claim for the singular vectors, \( \zeta' \) is, up to a constant factor, the tensor product of the highest vectors of \( L(\lambda)^\omega \) and \( L(\mu)^\omega \), that is, (3.18) holds.

To complete the proof of the theorem, we need to show that the submodule of \( L = L(\lambda) \otimes L(\mu) \) generated by the tensor product of the highest vectors \( \zeta = \xi \otimes \xi' \) coincides with \( L \). For this we introduce a \( Y(n) \)-module structure on the space \( L^* \) dual to \( L \). It follows immediately from the defining relations (2.1) that the mapping

\[
\sigma : Y(n) \to Y(n), \quad t_{ij}(u) \mapsto t_{ij}(-u),
\]
defines an anti-automorphism of \( Y(n) \). Now, \( L^* \) becomes a \( Y(n) \)-module if we set

\[
(yf)(v) = f(\sigma(y)v), \quad y \in Y(n), \quad f \in L^*, \quad v \in L.
\]

Similarly, the dual space \( L(\lambda)^* \) of the \( \mathfrak{g} l_n \)-module \( L(\lambda) \) can be regarded as a \( \mathfrak{g} l_n \)-module with the action defined by

\[
(E_{ij} f)(v) = f(-E_{ij}v), \quad f \in L(\lambda)^*, \quad v \in L(\lambda).
\]

We obtain easily from (2.2) that the \( Y(n) \)-module \( L^* \) is isomorphic to the tensor product \( L(\lambda)^* \otimes L(\mu)^* \), where \( L(\lambda)^* \) and \( L(\mu)^* \) are extended to \( Y(n) \) by (2.14). The vector \( \xi^* \in L(\lambda)^* \), dual to the highest vector \( \xi \), is the lowest vector with the weight \(-\lambda\). The highest weight of \( L(\lambda)^* \) will be therefore \(-\lambda^\omega = (-\lambda_n, \ldots, -\lambda_1)\). Thus, we have

\[
L^* \simeq L(-\lambda^\omega) \otimes L(-\mu^\omega).
\]

If we assume that the vector \( \zeta \) generates a proper submodule \( N \) in \( L \) then its annihilator

\[
\text{Ann } N = \{ f \in L^* \mid f(v) = 0 \quad \text{for all} \quad v \in N \}
\]
is a nonzero submodule in $L^*$. Hence, $\text{Ann} \, N$ must contain a vector $\zeta'$ which is annihilated by the generators $t_{ij}(u)$ with $i > j$. However, the condition (3.3) remains satisfied when $\lambda$ and $\mu$ are respectively replaced with $-\lambda^\omega$ and $-\mu^\omega$. So, by Lemma 3.6, the vector $\zeta'$ must be, up to a constant factor, the tensor product of the lowest vectors of the representations $L(-\lambda^\omega)$ and $L(-\mu^\omega)$. But the vector $\xi^* \otimes \xi'^*$ does not belong to $\text{Ann} \, N$. This makes a contradiction and so, the submodule generated by $\zeta$ must coincide with $L$. This completes the proof of Theorem 3.1.

4 Necessary conditions

We keep using the notation (2.19). As in the previous section, we assume that $\lambda$ and $\mu$ are complex $\mathfrak{gl}_n$-highest weights.

Theorem 4.1 Suppose that the $Y(n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible. Then for each pair of indices $1 \leq i < j \leq n$ we have

$$m_j, m_i \notin \langle l_j, l_i \rangle \quad \text{or} \quad l_j, l_i \notin \langle m_j, m_i \rangle. \tag{4.1}$$

The proof will follow from a sequence of lemmas. We use induction on $n$. Given a $\mathfrak{gl}_n$-highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ we set

$$\lambda_- = (\lambda_1, \ldots, \lambda_{n-1}) \quad \text{and} \quad \lambda_+ = (\lambda_2, \ldots, \lambda_n).$$

Lemma 4.2 If the $Y(n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible then both $Y(n-1)$-modules $L(\lambda_-) \otimes L(\mu_-)$ and $L(\lambda_+) \otimes L(\mu_+)$ are irreducible.

Proof. We shall identify $L(\lambda_-)$ and $L(\mu_-)$ with the $U(\mathfrak{gl}_{n-1})$-spans of the highest vectors $\xi$ in $L(\lambda)$ and $\xi'$ in $L(\mu)$, respectively. Any generator $E_{in}$ of $\mathfrak{gl}_n$ with $i < n$ annihilates $L(\lambda_-)$ and $L(\mu_-)$. Hence, by (2.2) and (2.14), the subspace $L(\lambda_-) \otimes L(\mu_-)$ or $L(\lambda_+) \otimes L(\mu_+)$ of $L(\lambda) \otimes L(\mu)$ is invariant with respect to the action of the subalgebra $Y(n-1)$ of $Y(n)$, and this action coincides with the one defined in Section 2.

Suppose that there is a nonzero submodule in $L(\lambda_-) \otimes L(\mu_-)$ which does not contain the vector $\xi \otimes \xi'$. Then this submodule contains a $Y(n-1)$-singular vector $\zeta$. However, $\zeta$ must also be a $Y(n)$-singular vector. Indeed, $t_{in}(u) \zeta = 0$ for any $i < n$ which easily follows from (2.1) and (2.2). This implies that $L(\lambda) \otimes L(\mu)$ is not irreducible, contradiction.

Suppose now that the $Y(n-1)$-submodule of $L_- = L(\lambda_-) \otimes L(\mu_-)$ generated by $\xi \otimes \xi'$ is proper. It follows from the defining relations (2.1) that the mapping

$$\tau : Y(n-1) \to Y(n-1), \quad t_{ij}(u) \mapsto t_{ji}(u),$$

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defines an anti-automorphism of $Y(n - 1)$. The dual space $L^*$ becomes a $Y(n - 1)$-module if we set

$$(yf)(v) = f(\tau(y)v), \quad y \in Y(n - 1), \quad f \in L^*, \quad v \in L_-. $$

We easily derive from (2.2) that the module $L^*$ is isomorphic to $L(\mu_-) \otimes L(\lambda_-)$. Since $\xi \otimes \xi'$ generates a proper submodule in $L_-$, its annihilator in $L^*$ is a nonzero submodule which does not contain the vector $\xi' \otimes \xi \in L(\mu_-) \otimes L(\lambda_-)$. However, the $Y(n)$-module $L(\mu) \otimes L(\lambda)$ is irreducible by Proposition 2.6. Thus, our assumption leads again to a contradiction due to the previous argument. This proves that the $Y(n - 1)$-module $L(\lambda_-) \otimes L(\mu_-)$ is irreducible.

Now consider the $Y(n - 1)$-module $L(\lambda_+) \otimes L(\mu_+)$. If the $Y(n)$-module $L = L(\lambda) \otimes L(\mu)$ is irreducible then so is the module $L^*$ defined in (3.20). To complete the proof we apply the isomorphism $(3.21)$ and the above argument. \[\square\]

Due to Lemma 4.2, if the $Y(n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible then, by the induction hypothesis, both conditions $\text{Cond}(A^+_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\mu, A^-_\lambda)$ hold. Therefore, by Proposition 2.8 the conditions (4.1) are satisfied for all pairs $(i, j) \neq (1, n)$. Suppose that they are violated for the pair $(1, n)$. Then, as was shown in the proof of Proposition 2.8, the condition $(2.21)$ should hold. Using Proposition 2.6, if necessary, we may assume that $m_n \in \langle l_n, l_1 \rangle$ and $l_1 \in \langle m_n, m_1 \rangle$. Therefore, there exist indices $p, q \in \{1, \ldots, n - 1\}$ such that

$$m_n \in \langle l_{p+1}, l_p \rangle \quad \text{and} \quad l_1 \in \langle m_{q+1}, m_q \rangle. \quad (4.2)$$

If $p = n - 1$ then by $\text{Cond}(A^+_\lambda, A^-_\mu)$ we must have $q = 1$; cf. the proof of Proposition 2.8. Thus, $l_1 \in \langle m_2, m_1 \rangle$. Moreover, using also the condition $\text{Cond}(A^+_\mu, A^-_\lambda)$, we conclude that there should exist indices $r$ and $s$ such that

$$m_2, \ldots, m_r \in \{l_2, \ldots, l_s\}, \quad l_{s+1}, \ldots, l_{n-1} \in \{m_{r+1}, \ldots, m_{n-1}\};$$

as shown in the picture:

In particular, this implies that

$$l_i - m_i \in \mathbb{Z}_+ \quad \text{for all} \quad i = 2, \ldots, n - 1. \quad (4.3)$$
Now let $p \leq n - 2$ in (4.2). The conditions $\text{Cond}(A^-_\lambda, A^-_\mu)$ and $\text{Cond}(A^+_\lambda, A^+_\mu)$ imply that

$$l_{p-i+1} = m_{n-i} \quad \text{for} \quad i = 1, \ldots, p - 1$$

(4.4)

while

$$l_1 \in \langle m_{n-p+1}, m_{n-p} \rangle,$$

(4.5)

as illustrated:

$$
\begin{array}{cccccccc}
&m_n & m_{n-1} & m_{n-p+1} & m_{n-p} & & & m_1 \\
l_n & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
l_{p+1} & m_n & & & & & & \\
l_p & & & & & & & \\
l_2 & & & & & & & \\
l_1 & & & & & & & \\
\end{array}
$$

Let $L$ be a highest weight module over $Y(n)$ generated by a vector $\zeta$ such that

$$T_{ii}(u) \zeta = (u + \lambda_i)(u + \mu_i) \zeta, \quad i = 1, \ldots, n,$$

(4.6)

where $T_{ij}(u) = u^2 t_{ij}(u)$. We shall also suppose that the elements $t_{ij}^{(r)}$ with $r \geq 3$ act trivially on $L$ so that the operators $T_{ij}(u)$ are polynomials in $u$. In other words, $L$ is a module over the quotient algebra $Y(n)/I$ where $I$ is the ideal generated by the elements $t_{ij}^{(r)}$ with $r \geq 3$.

Given sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ of elements of $\{1, \ldots, n\}$ we denote by $T^{a_1 \cdots a_m}_{b_1 \cdots b_m}(u)$ the corresponding quantum minor defined by (2.3) or (2.4) with $t_{ij}(u)$ respectively replaced by $T_{ij}(u)$. Similarly, we define the operators $B_m(u)$ and $A_m(u)$ by (2.8) with the same replacement.

For $1 \leq a < r \leq n$ introduce the raising operators $\tau_{ar}(v)$ and lowering operators $\tau_{ra}(v)$ [15] by

$$
\tau_{ar}(v) = T^{1 \cdots a}_{a+1 \cdots r}(v), \quad \tau_{ra}(v) = T^{a+1 \cdots r}_{a \cdots r-1}(v),
$$

where $v$ is a variable. We also set $\tau_{ra}(v) \equiv 1$ for $r \leq a$. For any non-negative integer $k$ introduce the product of the lowering operators

$$
T_{ra}(v, k) = \tau_{ra}(v + k - 1) \cdots \tau_{ra}(v + 1) \tau_{ra}(v).
$$

(4.7)

Suppose now that $\eta \in L$ is a $Y(n - 1)$-singular vector. That is, $\eta$ is annihilated by $T_{ij}(u)$ for $1 \leq i < j \leq n - 1$ and

$$
T_{ii}(u) \eta = \nu_i(u) \eta, \quad i = 1, \ldots, n - 1,
$$

(4.8)

for some polynomials $\nu_i(u)$ of degree two. The next three lemmas will provide a basis for our calculations.
Lemma 4.3 We have the following relations in $L$:

$$T_{ii}(u) \ T_{na}(v, k) \eta = \nu_i(u) \ T_{na}(v, k) \eta, \quad (4.9)$$

if $1 \leq i \leq n - 1$ and $i \neq a$, while

$$T_{aa}(u) \ T_{na}(v, k) \eta = \frac{(u - v - k) \nu_a(u)}{u - v} \ T_{na}(v, k) \eta \quad (4.10)$$

$$+ \frac{k}{u - v} \sum_{c=a+1}^{n} \nu_a(v) \nu_{a+1}(v - 1) \cdots \nu_{c-1}(v - c + a + 1)$$

$$\times T_{ca}(u) \ T_{na}(v + 1, k - 1) \tau_{nc}(v - c + a) \eta.$$

Moreover,

$$T_{i,i+1}(u) \ T_{na}(v, k) \eta = 0, \quad (4.11)$$

if $1 \leq i < n - 1$ and $i \neq a$, while for $a < n - 1$

$$T_{a,a+1}(u) \ T_{na}(v, k) \eta = \frac{k}{u - v} \sum_{c=a+1}^{n} \nu_a(v) \nu_{a+1}(v - 1) \cdots \nu_{c-1}(v - c + a + 1)$$

$$\times T_{c,a+1}(u) \ T_{na}(v + 1, k - 1) \tau_{nc}(v - c + a) \eta. \quad (4.12)$$

In particular, if $\nu_a(-\rho) = 0$ for some $\rho$ then

$$T_{aa}(u) \ T_{na}(-\rho, k) \eta = \frac{(u + \rho - k) \nu_a(u)}{u + \rho} \ T_{na}(-\rho, k) \eta, \quad (4.13)$$

and for $a < n - 1$

$$T_{a,a+1}(u) \ T_{na}(-\rho, k) \eta = 0. \quad (4.14)$$

Proof. Note that the coefficients of $T_{na}(v, k)$ are linear combinations of monomials in the generators $t^{(j)}_{rs}$ with $a \leq s \leq n - 1$. Suppose that $i < a$. We have $T_{il}(u) \eta = 0$ for $a \leq l \leq n - 1$. Therefore, applying (2.1) to the commutators $[T_{ii}(u), T_{rs}(v)]$ and $[T_{il}(u), T_{rs}(v)]$, we conclude by an easy induction that $T_{il}(u) \ T_{na}(v, k) \eta = 0$. This proves (4.9) and (4.11) for $i < a$. For $i > a$ both these relations are immediate from (2.7). Further, by (2.6),

$$T_{aa}(u) \ T_{na}(v, k) = \frac{u - v - k}{u - v - k + 1} \tau_{na}(v + k - 1) T_{aa}(u) \ T_{na}(v, k - 1)$$

$$+ \frac{1}{u - v - k + 1} \sum_{c=a+1}^{n} T_{ca}(u) \ T_{a+\cdots+n-1}(v + k - 1) \ T_{na}(v, k - 1) (-1)^{c-a-1}.$$
The subalgebra \( Y_a \) of \( Y(n) \) generated by \( t_{rs}(u) \) with \( a \leq r, s \leq n \) is naturally isomorphic to the Yangian \( Y(n - a + 1) \). Applying the automorphism (2.7) to this subalgebra, we derive from the defining relations (2.1) that

\[
T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v + 1) T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v) = T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v + 1) T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v)
\]

for every \( c = a + 1, \ldots, n \). Hence,

\[
T_{a \cdots \tilde{n}}^{a \cdots \tilde{n}}(v + k - 1) T_{na}(v, k - 1) = T_{na}(v + 1, k - 1) T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v).
\]

Note that \( T_{ca}(u) \) commutes with \( T_{na}(v - k - 1) \) by (2.7). Therefore, an easy induction on \( k \) gives

\[
T_{aa}(u) T_{na}(v, k) = \frac{u - v - k}{u - v} T_{na}(v, k) T_{aa}(u) + \frac{k}{u - v} \sum_{c=a+1}^{n} T_{ca}(u) T_{na}(v + 1, k - 1) T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v) (-1)^{c-a-1}.
\]

(4.15)

The same argument proves the following counterpart of (4.15): for \( a < n - 1 \)

\[
T_{a,a+1}(u) T_{na}(v, k) = \frac{u - v - k}{u - v} T_{na}(v, k) T_{a,a+1}(u) + \frac{k}{u - v} \sum_{c=a+1}^{n} T_{c,a+1}(u) T_{na}(v + 1, k - 1) T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v) (-1)^{c-a-1}.
\]

(4.16)

By (2.4),

\[
T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v) = \sum_{\sigma \in S_{n-a}} \text{sgn } \sigma \cdot T_{n,\sigma(n-1)}(v - n + a + 1) \cdots T_{c+1,\sigma(c)}(v - c + a) \times T_{c-1,\sigma(c-1)}(v - c + a + 1) \cdots T_{a,\sigma(a)}(v).
\]

Since \( T_{ij}(u) \eta = 0 \) for \( 1 \leq i < j \leq n - 1 \) we conclude from (4.16) that

\[
T_{a \cdots \tilde{n}-1}^{a \cdots \tilde{n}}(v) \eta = (v - c + a) \nu_a(v) \nu_{a+1}(v - 1) \cdots \nu_{c-1}(v - c + a + 1) \eta.
\]

The proof is completed by using (4.13) and (4.18).

\[\square\]

**Lemma 4.4** Let \( 1 \leq a < n - 1 \). Then we have the relations in \( Y(n) \):

\[
[E_{n-1,n} T_{na}(v, k)] = -k T_{na}(v, k - 1) T_{a \cdots \tilde{n}-2,n}^{a \cdots \tilde{n}}(v + k - 1).
\]

(4.17)

Moreover,

\[
T_{a \cdots \tilde{n}-2,n}^{a \cdots \tilde{n}}(u) T_{a+1 \cdots \tilde{n}-1}(u) = T_{n-1,a}^{n-1}(u) T_{a+1 \cdots \tilde{n}-1}(u) + T_{na}(u) T_{a+1 \cdots \tilde{n}-2,n}(u).
\]

(4.18)
Proof. By (2.6), \( [E_{n-1,n}, \tau_{na}(v)] = -T_{a \cdots n-2,n}^{a+1 \cdots n}(v) \). Since the elements \( \tau_{na}(u) \) and \( \tau_{na}(v) \) commute, we can write

\[
[E_{n-1,n} \tau_{na}(v, k)] = - \sum_{i=1}^{k} \tau_{na}(v) \cdots T_{a \cdots n-2,n}^{a+1 \cdots n}(v + i - 1) \cdots \tau_{na}(v + k - 1). \tag{4.19}
\]

Applying the automorphism (2.5) to the subalgebra \( Y_a \) introduced in the proof of Lemma 4.3, we derive from (2.1) that

\[
T_{a \cdots n-2,n}^{a+1 \cdots n}(v + 1) = \tau_{na}(v) T_{a \cdots n-2,n}^{a+1 \cdots n}(v + 1). \tag{4.20}
\]

Together with (4.19) this proves (4.17) by an easy induction.

To prove (4.18) consider the expression provided by Proposition 2.1 for the commutator \( [\tau_{a,n-1}(u), T_{a+1 \cdots n}(v)] \). Multiply both sides of the relation by \( u - v \) and put \( u = v \). The terms which do not vanish after this operation correspond to the maximum value of the summation parameter in the formula, which implies (4.18). \( \square \)

We keep using the notation of Lemma 4.3. As before, \( \zeta \) is the highest vector of \( L \) satisfying (4.6). We also regard \( L \) as a \( \mathfrak{gl}_n \)-module using (3.6).

Lemma 4.5 We have the following relations in \( L \).

\[
E_{in} \tau_{na}(v, k) \zeta = 0 \quad \text{if} \quad i < a. \tag{4.21}
\]

If \( a < i \leq n - 1 \) then

\[
E_{in} \tau_{na}(v, k) \zeta = (-1)^{n-i} k \prod_{j=i+1}^{n} (v' + l_j)(v' + m_j) \tau_{ia}(v + k - 1) \tau_{na}(v, k - 1) \zeta, \tag{4.22}
\]

where \( v' = v + a + k - 1 \). Moreover, if \( 1 \leq a \leq n - 1 \) then

\[
E_{an} \tau_{na}(v, k) \zeta = (-1)^{n-a-1} k \times \left( \tau_{na}(v + 1, k - 1) T_{a \cdots n-1}^{a \cdots n-1}(v) - \tau_{na}(v, k - 1) T_{a+1 \cdots n}^{a+1 \cdots n}(v + k - 1) \right) \zeta. \tag{4.23}
\]

Proof. The relation (4.21) follows from the fact that the coefficients of \( \tau_{na}(v, k) \) belong to the subalgebra \( Y_a \): see the proof of Lemma 4.3. We easily deduce from (2.6) that

\[
[E_{an}, \tau_{na}(v)] = (-1)^{n-a-1} T_{a \cdots n-1}^{a \cdots n-1}(v) - (-1)^{n-a-1} T_{a+1 \cdots n}^{a+1 \cdots n}(v).
\]
Therefore,
\[ E_{an} T_{na}(v, k) \zeta = (-1)^{n-a-1} \]
\[ \times \sum_{i=1}^{k} \tau_{na}(v) \cdots (T_{a \cdots n-1}^{a \cdots n-1}(v + i - 1) - T_{a+1 \cdots n}^{a+1 \cdots n}(v + i - 1)) \cdots \tau_{na}(v + k - 1) \zeta. \]
(4.24)

Furthermore, by analogy with (4.20) we get
\[ T_{a+1 \cdots n}^{a+1 \cdots n}(v) \tau_{na}(v + 1) = \tau_{na}(v) T_{a+1 \cdots n}^{a+1 \cdots n}(v + 1). \]
The expression (4.24) now takes the form
\[ (-1)^{n-a-1} \sum_{i=1}^{k} \tau_{na}(v) \cdots \tau_{na}(v + i - 2) T_{a \cdots n-1}^{a \cdots n-1}(v + i - 1) T_{na}(v + i, k - i) \zeta 
- (-1)^{n-a-1} k T_{na}(v, k - 1) T_{a+1 \cdots n}^{a+1 \cdots n}(v + k - 1) \zeta. \]

Similarly, applying again (2.5) to \( Y_a \) we bring the sum here by an easy induction to the form
\[ \sum_{i=1}^{k} \left( (k - i + 1) \tau_{na}(v + k - 1) \cdots \tau_{na}(v + i - 1) \cdots \tau_{na}(v) T_{a \cdots n-1}^{a \cdots n-1}(v + i - 1) \right. 
- (k - i) \tau_{na}(v + k - 1) \cdots \tau_{na}(v + i) \cdots \tau_{na}(v) T_{a \cdots n-1}^{a \cdots n-1}(v + i) \zeta, \]
which simplifies to \( k T_{na}(v + 1, k - 1) T_{a \cdots n-1}^{a \cdots n-1}(v) \zeta \) thus proving (4.23).

By (2.6), if \( i > a \) then \([E_{in}, \tau_{na}(v)] = (-1)^{n-i} T_{a \cdots n}^{a+1 \cdots n}(v)\). As in the proof of (4.17), this brings the left hand side of (4.22) to the form
\[ (-1)^{n-i} k T_{na}(v, k - 1) T_{a \cdots n}^{a+1 \cdots n}(v + k - 1) \zeta. \]
Using (2.3), we get
\[ T_{a \cdots i \cdots n}^{a+1 \cdots n}(v + k - 1) \zeta = \prod_{j=i+1}^{n} (v' + l_j)(v' + m_j) \tau_{ia}(v + k - 1) \zeta, \]
which completes the proof. \(\square\)

We now consider the irreducible highest weight module \( V(\lambda, \mu) \) over \( Y(n) \) generated by the highest vector \( \zeta \) satisfying (4.6). It follows easily from (2.1) and the irreducibility of \( V(\lambda, \mu) \) that all elements \( t_{ij}^{(r)} \) with \( r \geq 3 \) act trivially in this module. Furthermore, \( V(\lambda, \mu) \) is isomorphic to the irreducible quotient of the submodule of
$L(\lambda) \otimes L(\mu)$ generated by the vector $\xi \otimes \xi'$; see (2.2) and (2.14). With the parameter $p$ defined in (4.2), the numbers

$$k_i = l_i - m_{n-p+i}, \quad i = 1, \ldots, p$$

are positive integers by (4.3) and (4.4). Introduce the vector $\theta \in V(\lambda, \mu)$ by

$$\theta = T_{n-p+1,1}(-\lambda_1, k_1) T_{n-p+2,2}(-\lambda_2, k_2) \cdots T_{np}(-\lambda_p, k_p) \zeta,$$

where $\zeta$ is the highest vector of $V(\lambda, \mu)$, and $T_{ra}(v, k_a)$ denotes the derivative of the polynomial $T_{ra}(v, k_a)$; see (4.7). Our aim is to prove that the vector $\theta$ is zero. We shall do this in Lemma 4.7 below. The idea of the proof is to show that $\theta$ is annihilated by the operators $B_i(u)$ for all $i = 1, \ldots, n - 1$ and then apply Proposition 2.3 noting that $\theta$ is obviously not proportional to $\zeta$. This works directly in the case $p = 1$. However, if $p \geq 2$ then applying the operators $B_i(u)$ to $\theta$, we come to a more general problem to prove that all vectors parametrized by a certain finite family of pattern-like arrays $\Lambda$ associated with $\lambda$ are zero. We prove a preliminary lemma first which describes the properties of these vectors. The arrays $\Lambda$ which arise in this way will be called admissible. They are defined as follows. Each $\Lambda$ is a sequence of rows $\Lambda_r = (\lambda_{r1}, \ldots, \lambda_{rr})$ with $r = 1, \ldots, n$ of the form described in Section 2. The top row $\Lambda_n$ coincides with $\lambda$ and for all $r$ the following conditions hold

$$\lambda_{ri} - \lambda_{r-1,i} \in \mathbb{Z}_+ \quad \text{for} \quad i = 1, \ldots, r - 1.$$  

Each entry $\lambda_{ri}$ of $\Lambda$ is equal to $\lambda_i$ unless

$$i = 2, \ldots, p \quad \text{and} \quad r < n - p + i. \quad (4.26)$$

Moreover, we also require that if $i \in \{2, \ldots, p\}$ then

$$l_{ii} - m_{n-p+i} \in \mathbb{Z}_+, \quad (4.27)$$

where we denote $l_{ri} = \lambda_{ri} - i + 1$. This condition implies that $0 \leq \lambda_{ri} - \lambda_{r-1,i} \leq k_i$ for all $i$. By definition, only a part of an admissible array can vary with the remaining entries fixed, as illustrated:

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Given such an array, we set
\[ \theta_\Lambda = \prod_{r=3}^{n} \left( \prod_{i=2}^{p} \tilde{T}_{ri}(-\lambda_{ri}, \lambda_{ri} - \lambda_{r-1,i}) \right) \zeta, \]
where the polynomials \( \tilde{T}_{ri}(v, k) \) are defined by
\[ \tilde{T}_{ri}(v, k) = \begin{cases} T'_{ri}(v, k) & \text{if } r = n - p + i \text{ and } k = k_i \\ T_{ri}(v, k) & \text{otherwise}. \end{cases} \] \hfill (4.28)

Recall also that \( T_{ri}(v, k) \equiv 1 \) if \( r \leq i \). Note that the factors in the brackets commute for any index \( r \) due to (2.7). We have \( \zeta = \theta_{\Lambda^0} \) for the array \( \Lambda^0 \) with \( \lambda_{ri}^0 = \lambda_i \) for all \( r \) and \( i \). Furthermore, \( \theta = T_{n-p+1,1}(-\lambda_1, k_1) \theta_\Lambda \) for the array \( \Lambda \) with \( l_{ri} = m_{n-p+i} \) for all indices \( r \) and \( i \) satisfying (4.26). We define the weight \( w(\Lambda) \) of an admissible array \( \Lambda \) by (3.7). We use the ordering on the weights described in Section 3. Given \( \Lambda \), take the minimum index \( r = r(\Lambda) \) such that for some \( 2 \leq a \leq p \) the difference \( \lambda_{ra} - \lambda_{r-1,a} \) is a positive integer. The following relations for the admissible arrays \( \Lambda \) with \( r(\Lambda) \geq n - p + 2 \) will be used in Lemma 4.7.

**Lemma 4.6** We have the following relations in \( V(\lambda, \mu) \):
\[
\begin{align*}
T_{ii}(u) \theta_\Lambda &= (u + \lambda_{r-1,i})(u + \mu_i) \theta_\Lambda \quad \text{for } 1 \leq i \leq r - 1, \quad \text{(4.29)} \\
T_{ij}(u) \theta_\Lambda &= 0 \quad \text{for } 1 \leq i < j \leq r - 1, \quad \text{(4.30)} \\
B_i(u) \theta_\Lambda &= \sum_{j=1}^{i} \beta_{ij}(u, \Lambda) \theta_{\Lambda + \delta_{ij}} \quad \text{for } r - 1 \leq i < n, \quad \text{(4.31)}
\end{align*}
\]
where \( \beta_{ij}(u, \Lambda) \) are some polynomials in \( u \), and we suppose that \( \theta_\Lambda = 0 \) if \( \Lambda \) is not an admissible array.
Proof. Let $2 \leq a \leq p$ be the least index such that $k = \lambda_{ra} - \lambda_{r-1,a} > 0$. Then

$$\theta_\Lambda = \tilde{T}_{ra}(-\lambda_{ra},k) \eta, \quad \eta := \theta_{\Lambda'},$$

where $\Lambda'$ is the array obtained from $\Lambda$ by increasing each entry $\lambda_{aa}, \ldots, \lambda_{r-1,a}$ by $k$. We shall use a (reverse) induction on the pairs $(r,a)$ ordered lexicographically, with the base $\theta_\Lambda = \zeta$. Note that by the definition of admissible arrays we must have $r \leq n - p + a$.

Identify the subalgebra of $\mathcal{Y}(n)$ generated by $t_{ij}(u)$ with $1 \leq i, j \leq r$ with the Yangian $\mathcal{Y}(r)$. By the induction hypothesis $\eta$ is a $\mathcal{Y}(r-1)$-singular vector such that (4.8) holds with $n$ replaced by $r$, where

$$\nu_i(u) = \begin{cases} (u + \lambda_{ri})(u + \mu_i) & \text{if } i \leq a, \\ (u + \lambda_{r-1,i})(u + \mu_i) & \text{if } i > a. \end{cases}$$

Therefore, if $\tilde{T}_{ra}(-\lambda_{ra},k) = T_{ra}(-\lambda_{ra},k)$ then (4.29) and (4.30) are immediate from Lemma 4.3.

Suppose now that $\tilde{T}_{ra}(-\lambda_{ra},k) = T_{ra}(-\lambda_{ra},k)$. Then $r = n - p + a$ and $k = k_a$. Therefore, $\lambda_{ra} = \lambda_a$ by (4.27). It is clear from Lemma 4.3 that (4.29) and (4.30) hold for $i \neq a$ so that we may assume $i = a$. In this case, due to (4.10) and (4.12), it suffices to show that

$$T_{ra}(-\lambda_a,k_a) \eta = 0$$

and that for every $c = a + 1, \ldots, r$ the polynomial

$$\nu_a(v) \nu_{a+1}(v-1) \cdots \nu_{c-1}(v-c + a + 1) T_{ra}(v+1,k_a-1) \tau_{rc}(v-c+a) \eta$$

has zero of multiplicity at least two at $v = -\lambda_a$.

Suppose first that $p < n - 1$ and consider $T_{ra}(-\lambda_a,k_a) \eta$. By (2.3) we have

$$\tau_{ra}(v) = \sum_{\sigma \in S_{r-a}} \text{sgn } \sigma \cdot T_{\sigma(a+1),a}(v) \cdots T_{\sigma(r),r-1}(v-r+a+1).$$

By the induction hypothesis we have $T_{\sigma(r),r-1}(u) \eta = 0$ if $\sigma(r) < r-1$, while

$$T_{r-1,r-1}(u) \eta = (u + \lambda_{r-1,r-1})(u + \mu_{r-1}) \eta.$$

The factor $u + \mu_{r-1}$ is zero if $u = -\lambda_a - r + a + 1$ by (4.4). Therefore, if $v = -\lambda_a$ then we may assume that $\sigma(r) = r$ in (4.33) which gives

$$\tau_{ra}(-\lambda_a) \eta = \tau_{r-1,a}(-\lambda_a) T_{r,r-1}(-\mu_{r-1}) \eta.$$

Since $T_{r,r-1}(v) = \tau_{r,r-1}(v)$ is a lowering operator, we verify by an easy induction with the use of (2.7) and (4.13) that

$$T_{ra}(-\lambda_a,k_a) \eta = T_{r-1,a}(-\lambda_a,k_a) T_{r,r-1}(-\mu_{r-1},k_a) \eta.$$
We have $k_a = m_{r-1} - m_r$ by (1.4) and so, the equality $T_{ra}(-\lambda_a, k_a) \eta = 0$ in the case $p < n - 1$ will be implied by the fact that the vector
\[
\tilde{\eta} = T_{r,r-1}(-\mu_r) \cdots T_{r,r-1}(-\mu_{r-1}) \eta
\] (4.36)
is zero in $V(\lambda, \mu)$. Since the $Y(n)$-module $V(\lambda, \mu)$ is irreducible, it will be sufficient to show, due to Proposition 2.3, that the vector $\tilde{\eta}$ is annihilated by all operators $B_i(u)$ with $i = 1, \ldots, n - 1$. Since $T_{r,r-1}(u)$ commutes with the lowering operators $\tau_{ri}(v)$ we may assume that the array $\Lambda$ satisfies $\lambda_{ri} = \lambda_{r-1,i}$ for all $i \neq a$. In other words, $\eta = \theta_{\Lambda'}$ is a $Y(r)$-singular vector such that
\[
T_{ii}(u) \eta = (u + \lambda_{ri})(u + \mu_i) \eta \quad \text{for} \quad i = 1, \ldots, r.
\]
By (4.11), $\tilde{\eta}$ is annihilated by the operators $T_{ii+1}(u)$ for $i = 1, \ldots, r - 2$, and by (4.3) and (4.13), $\tilde{\eta}$ is an eigenvector for the operators $T_{ii}(u)$ with $i = 1, \ldots, n - 1$. We have $T_{r-1,r}(u) = [T_{r-1,r-1}(u), E_{r-1,r}]$. By (2.23), $E_{r-1,r} \tilde{\eta} = 0$ and therefore $T_{r-1,r}(u)\tilde{\eta} = 0$. On the other hand, if $i \geq r$ then $B_i(u)$ commutes with the elements $T_{r,r-1}(v)$ by (2.7). Hence, by the induction hypothesis, $B_i(u) \tilde{\eta}$ is a linear combination of the vectors
\[
T_{r,r-1}(-\mu_r) \cdots T_{r,r-1}(-\mu_{r-1}) \theta_{\Lambda' + \delta_{ij}}
\]
with $2 \leq j \leq p$. We conclude by induction on the weight of $\Lambda'$ that $B_i(u) \tilde{\eta} = 0$ thus proving (4.33).

Consider now the polynomial (4.34). Suppose first that $c = r$. Note that $r - 1 > a$ since $p < n - 1$. We have
\[
\nu_a(v) = (v + \lambda_a)(v + \mu_a), \quad \nu_{r-1}(v) = (v + \lambda_{r-1,r-1})(v + \mu_{r-1}).
\] (4.37)
However, $\mu_{r-1} - r + a + 1 = \lambda_a$ by (4.4). This shows that the coefficient of $\eta$ in (1.34) is divisible by $(v + \lambda_a)^2$. Let now $a + 1 \leq c < r$. By (2.3),
\[
\tau_{rc}(-\lambda_a - c + a) = \sum_{\sigma \in \mathcal{S}_{r-c}} \text{sgn} \sigma \cdot T_{\sigma(c+1),c}(-\lambda_a - c + a) \cdots T_{\sigma(r),r-1}(-\lambda_a - r + a + 1).
\]
We can repeat the argument which we have applied to the expression (1.35) to show that the polynomial $T_{ra}(v + 1, k_a - 1) \tau_{rc}(v - c + a) \eta$ has zero at $v = -\lambda_a$. Together with (1.34), this completes the proof of (1.29) and (1.30) in the case $p < n - 1$.

In the case $p = n - 1$ we have $a = r - 1$ and
\[
T_{r,r-1}(-\lambda_{r-1}, k_{r-1}) \eta = T_{r,r-1}(-\mu_r) \cdots T_{r,r-1}(-\lambda_{r-1}) \eta.
\]
Note that the operators $T_{r,r-1}(u)$ and $T_{r,r-1}(v)$ commute. Therefore, due to (4.3) it suffices to show that the vector (4.36) is zero. The argument used in the case $p < n - 1$ works here as well.
For \( p = n - 1 \) the polynomial (4.34) equals
\[
\nu_{r-1}(v) T_{r,r-1}(-\lambda_{r-1} + 1, k_{r-1} + 1) \eta. \tag{4.38}
\]
If \( \lambda_{r-1} = \mu_{r-1} \) then \( \nu_{r-1}(v) = (v + \lambda_{r-1})^2 \). If \( \lambda_{r-1} - \mu_{r-1} > 0 \) then
\[
T_{r,r-1}(-\lambda_{r-1} + 1, k_{r-1} + 1) \eta = T_{r,r-1}(-\mu_r) \cdots T_{r,r-1}(-\lambda_{r-1} + 1) \eta.
\]
Using again the fact that the vector (4.36) is zero we conclude that this vector is also zero. In the both cases (4.38) has zero of multiplicity at least two at \( v = -\lambda_{r-1} \) proving (4.29) and (4.30).

To prove (4.31) we note that \( B_i(u) \) commutes with \( T_{sa}(v, k) \) for \( i \geq s \) by (2.7). Therefore, it suffices to consider the case \( i = r - 1 \). We derive from (2.6) that \( B_{r-1}(u) = [A_{r-1}(u), E_{r-1,r}] \). Suppose first that \( p < n - 1 \). By (4.29) and (4.30) the operator \( A_{r-1}(u) \) acts on \( \theta_{\Lambda} \) as multiplication by a polynomial in \( u \). So it suffices to prove that
\[
E_{r-1,r} \theta_{\Lambda} = \sum_{j=1}^{r-1} \beta_j(\Lambda) \theta_{\Lambda + \delta_{r-1,j}}, \tag{4.39}
\]
where \( \beta_j(\Lambda) \) are some constants. Write \( \theta_{\Lambda} \) in the form (4.32) and assume that \( a < r - 1 \). We now use Lemma 4.4. By (4.17) we have
\[
E_{r-1,r} T_{ra}(v, k) \theta_{\Lambda'} = T_{ra}(v, k) E_{r-1,r} \theta_{\Lambda'} - k T_{ra}(v, k - 1) T^{a+1 \cdots r}_{a \cdots r-2,r}(v + k - 1) \theta_{\Lambda'}. \tag{4.40}
\]
Further, (4.18) gives
\[
T^{a+1 \cdots r}_{a \cdots r-2,r}(u) T^{a+1 \cdots r-1}_{a+1 \cdots r-1}(u) \theta_{\Lambda'} = T^{a+1 \cdots r}_{a \cdots r-2}(u) \theta_{\Lambda'} + T_{ra}(u) T^{a+1 \cdots r-1}_{a+1 \cdots r-2,r}(u) \theta_{\Lambda'}. \tag{4.41}
\]
By (2.6), \( T^{a+1 \cdots r-1}_{a+1 \cdots r-2,r}(u) = [T^{a+1 \cdots r-1}_{a+1 \cdots r-1}(u), E_{r-1,r}] \). Using the induction hypothesis we obtain
\[
E_{r-1,r} \theta_{\Lambda'} = \sum_{j=a+1}^{r-1} \beta_j(\Lambda') \theta_{\Lambda' + \delta_{r-1,j}}.
\]
On the other hand, using (4.29) and (4.30) we derive from (2.3) that
\[
T^{a+1 \cdots r-1}_{a+1 \cdots r-1}(u) \theta_{\Lambda'} = \prod_{i=a+1}^{r-1} (u + l_{r-1,i} + a)(u + m_i + a) \theta_{\Lambda'},
\]
while
\[
T^{a+1 \cdots r}_{a+1 \cdots r}(u) \theta_{\Lambda'} = \prod_{i=a+1}^{r} (u + l_{r,i} + a)(u + m_i + a) \theta_{\Lambda'},
\]
since \( T^{a+1}_{a+1,1}(u) \) commutes with the lowering operators \( \tau_b(v) \). Thus, by (4.41)

\[
T^{a+1}_{a+1,1}(u) \theta_{\Lambda'} = (u + l_{r,i} + a)(u + m_r + a) \prod_{i=a+1}^{r-1} \frac{u + l_{r,i} + a}{u + l_{r-1,i} + a} \tau_{r-1,a}(u) \theta_{\Lambda'} + \sum_{j=a+1}^{r-1} \beta_j(\Lambda') \frac{u + l_{r-1,j} + a}{u + l_{r-1,j} + a} \tau_{ra}(u) \theta_{\Lambda' + \delta_{r-1,j}}.
\]

(4.42)

Now put \( v = -\lambda_r \) and \( k = \lambda_r - \lambda_{r-1} \) into (4.40). The denominator \( u + l_{r-1,i} + a \) in (4.42) becomes \( l_{r-1,i} - l_{r-1,a} \) at \( u = v + k - 1 \). Due to the conditions (4.4) and (4.27) the difference \( l_{r-1,i} - l_{r-1,a} \) can only be zero if \( i = a + 1 \). Moreover, in this case \( l_{r-1,a+1} = l_{r,a+1} = l_{a+1} \). Then \( \Lambda' + \delta_{r-1,a+1} \) is not an admissible array so that the summand with \( j = a + 1 \) does not occur in the sum in (4.42). The denominator \( u + l_{r,a+1} + a \) does not occur in the product either, since it cancels with \( u + l_{r,a+1} + a \). Thus the substitution \( u = v + k - 1 \) into (4.42) is well defined. Using the fact that \( \tau_{r-1,l}(v) \) commutes with \( \tau_{ra}(u) \) if \( b \geq a \) we complete the proof of (1.33) for the case \( \widetilde{T}_ra(v,k) = T_{ra}(v,k) \) in (4.32).

Assume now that \( \widetilde{T}_ra(v,k) = T_{ra}(v,k) \). Then by (4.28) we must have \( r = n - p + a \) and \( k = a = l_a - m_r \). Moreover, we also have \( \lambda_r = \lambda_a \) by (4.27). Take the derivative with respect to \( v \) in (4.40) and put \( v = -\lambda_a \). Note that the factor \( u + m_r + a \) in (4.42) vanishes at \( u = -\lambda_a + k_a - 1 \). Furthermore, as has been shown above, \( T_{ra}(-\lambda_a, k_a) \theta_{\Lambda' + \delta_{r-1,j}} = 0 \); see (1.33). The application of the induction hypothesis finally proves (1.39) in the case \( a < r - 1 \).

If \( a = r - 1 \) in (4.32) then \( \eta = \theta_{\Lambda'} \) is a \( Y(r) \)-singular vector, so that we may use the relation (1.28) to prove (1.39). The same relation applies in the case \( p = n - 1 \), where we also use the fact that the polynomial (4.38) has zero of multiplicity at least two at \( v = -\lambda_{r-1} \). \( \square \)

Consider the vector \( \theta \in V(\lambda, \mu) \) defined in (1.23).

**Lemma 4.7** \( \theta = 0 \).

**Proof.** We shall be proving by induction on the weight of \( \Lambda \) that

\[
T_{n-p+1,1}(-\lambda_1, k_1) \theta_{\Lambda} = 0
\]

(4.43)

for all admissible arrays \( \Lambda \) such that the parameter \( r = r(\Lambda) \) satisfies \( r \geq n - p + 2 \). For the induction base we note that

\[
T_{n-p+1,1}(-\lambda_1, k_1) \xi = 0.
\]

Indeed, using (2.7) we find that the vector on the left hand side is annihilated by the operators \( B_i(u) \) with \( i = n - p + 1, \ldots, n - 1 \). On the other hand, by (1.14) it
is also annihilated by $T_{i,i+1}(u)$ with $i = 1, \ldots, n - p - 1$. Further, $T_{n-p,n-p+1}(u) = [T_{n-p,n-p}(u), E_{n-p,n-p+1}]$ and we find from (4.9) and (4.22) that it is also annihilated by $E_{n-p,n-p+1}$. By Proposition 2.3 the vector must be zero.

Suppose now that $w(\Lambda) < \lambda$. Denote the left hand side of (4.43) by $\tilde{\theta}_\Lambda$. We shall show that $B_i(u) \tilde{\theta}_\Lambda = 0$ for all $i = 1, \ldots, n - 1$. By Lemma 4.6, the $Y(n - p + 1)$-span of the vector $\theta_\Lambda$ is a highest weight module with the highest weight defined from (4.23) with $r = n - p + 2$. Exactly as above we find that $T_{n-p,n-p+1}(u) \theta_\Lambda$ is zero. Furthermore, the operators $B_i(u)$ with $i = n - p + 1, \ldots, n - 1$ commute with $T_{n-p+1,1}(-\lambda_1, k_1)$. Therefore by (4.31) $B_i(u) \tilde{\theta}_\Lambda$ is a linear combination of the vectors $T_{n-p+1,1}(-\lambda_1, k_1) \theta_{\Lambda+\delta_{ij}}$. If $i \geq n - p + 2$ then the arrays $\Lambda + \delta_{ij}$ satisfy the condition $r \geq n - p + 2$ on the parameter $r = r(\Lambda)$ used in Lemma 4.6 and we complete the proof in this case applying the induction hypothesis.

If $i = n - p + 1$ then

$$\theta_{\Lambda+\delta_{n-p+1,j}} = \tau_{n-p+1,j}(-\lambda_{n-p+1,j} - 1) \theta_{\Lambda'},$$

for some array $\Lambda'$ for which the corresponding parameter $r' = r(\Lambda')$ satisfies $r' \geq n - p + 2$. However, $\tau_{n-p+1,j}(v)$ is permutable with $T_{n-p+1,1}(-\lambda_1, k_1)$ which again ensures that $B_i(u) \tilde{\theta}_\Lambda = 0$ by the induction hypothesis.

By (2.2) and (2.14) the operators $T_{ij}(u) = u^2 t_{ij}(u)$ in $L(\lambda) \otimes L(\mu)$ are polynomials in $u$. Therefore, we can introduce the vector $\tilde{\theta} \in L(\lambda) \otimes L(\mu)$ by

$$\tilde{\theta} = T_{n-p+1,1}(-\lambda_1, k_1) T'_{n-p+2,2}(-\lambda_2, k_2) \cdots T'_{np}(-\lambda_p, k_p) (\xi \otimes \xi'),$$

cf. (4.23). Here $\xi$ and $\xi'$ are the highest vectors of the $\mathfrak{gl}_n$-modules $L(\lambda)$ and $L(\mu)$, respectively.

**Lemma 4.8** $\tilde{\theta} \neq 0$.

**Proof.** Write the vector $\tilde{\theta}$ in the form

$$\tilde{\theta} = \sum_{\Lambda, M} c_{\Lambda,M} \xi_\Lambda \otimes \xi'_M,$$

where $\xi_\Lambda$ and $\xi'_M$ are the Gelfand–Tsetlin basis vectors in $L(\lambda)$ and $L(\mu)$. It suffices to show that at least one coefficient $c_{\Lambda,M}$ is nonzero. We shall calculate these coefficients for the case $\xi'_M = \xi'_M^0$ is the highest vector $\xi'$ of $L(\mu)$. That is, the $(k, i)$-th entry of the pattern $M^0$ coincides with $\mu_i$ for all $k$ and $i$. We prove by a (reverse) induction on $a$ that for any $a = 2, \ldots, p$ we have

$$T'_{n-p+a,a}(-\lambda_a, k_a) \cdots T'_{np}(-\lambda_p, k_p) (\xi \otimes \xi') = \sum_{\Lambda, M} c^{(a)}_{\Lambda,M} \xi_\Lambda \otimes \xi'_M,$$

\[ (4.45) \]
where for each $M$ occurring in this expansion,

$$\mu - w(M) = \sum_{i=a}^{n-1} q_i (\varepsilon_i - \varepsilon_{i+1}), \quad q_i \in \mathbb{Z}_+,$$

(4.46)

and $c_{\Lambda^{(a)},M^0} \neq 0$ for the pattern $\Lambda^{(a)}$ defined for each $a = 1, \ldots, p$ as follows. The entry $\lambda_{s_i}^{(a)}$ of $\Lambda^{(a)}$ coincides with $\lambda_i$ unless $i = a, \ldots, p$ and $s < n - p + i$. For the entries with these $s$ and $i$ we have $\lambda_{s_i}^{(a)} - i + 1 = m_{n-p+i}$. The betweenness conditions (2.15) for $\Lambda^{(a)}$ are guaranteed by the assumptions (1.3) and (4.4).

Suppose that $a \leq p$ and denote the left hand side of (4.45) by $\theta^{(a)}$. By Proposition 2.2, we have

$$\Delta(\tau_a(v)) = \sum_{b_1 < \cdots < b_{n-p}} T_{b_1 \cdots b_{n-p}}^{a+1 \cdots r}(v) \otimes T_{a \cdots r-1}^{b_1 \cdots b_{n-p}}(v), \quad r := n - p + a,$$

where the quantum minor operators in $L(\lambda)$ and $L(\mu)$ are defined by the formulas (2.3) where the $t_{ij}(u)$ are replaced with the polynomial operators $T_{ij}(u) = u t_{ij}(u)$. If $w$ is a weight of the $\mathfrak{gl}_n$-module $L(\mu)$ then $w \leq \mu$. Therefore, by the induction hypothesis, if $b_1 < a$ then

$$T_{a \cdots r-1}^{b_1 \cdots b_{n-p}}(v) \xi_M^\prime = 0$$

for each pattern $M$ occurring in the expansion (4.45) for $\theta^{(a+1)}$. This proves (4.46). Furthermore, for any $k \in \mathbb{Z}_+$ the tensor products of the form $\xi_{\Lambda} \otimes \xi_{M^0}^\prime$ which occur in the expansion of $T_{ra}(v,k) \theta^{(a+1)}$ should have the form

$$T_{ra}(v,k) \xi_{\Lambda(a+1)} \otimes T_{a \cdots r-1}(v+k-1) \cdots T_{a \cdots r-1}(v) \xi^\prime.$$

(4.47)

The coefficient of $\xi^\prime$ equals

$$\prod_{j=1}^{k} (v + \mu_a + j - 1)(v + \mu_a + j - 2) \cdots (v + \mu_{r-1} + j - n + p).$$

The conditions (4.3) and (4.4) imply that for $k = k_a$ there is a unique factor in this product which vanishes at $v = -\lambda_a$. Therefore, the derivative of (4.47) with $k = k_a$ at $v = -\lambda_a$ is, up to a nonzero constant factor,

$$T_{n-p+a,a}( -\lambda_a, k_a ) \xi_{\Lambda(a+1)} \otimes \xi^\prime.$$

By the definition (2.16), this coincides with $\xi_{\Lambda(a)} \otimes \xi^\prime$ which proves (4.45).

Similarly, the application of the operator $T_{n-p+1,1}( -\lambda_1, 1 )$ to the vector $\theta^{(2)}$ produces a linear combination (4.44). Here the coefficient $c_{\Lambda^{(2)},M^0}$ is the product of $c_{\Lambda^{(2)},M^0}^{(2)}$ and the factor

$$\prod_{j=1}^{k} (-\lambda_1 + \mu_1 + j - 1)(-\lambda_1 + \mu_2 + j - 2) \cdots (-\lambda_1 + \mu_{n-p} + j - n + p)$$
with \( k = l_1 - m_{n-p+1} \) which comes from the expansion of the coefficient of \( \xi' \) in (4.14) for \( a = 1 \). It remains to note that by (4.3) (which holds for all \( p \leq n-1 \)) this factor is nonzero.

If the \( Y(n) \)-module \( L(\lambda) \otimes L(\mu) \) is irreducible then by (2.2) and (2.14) it is isomorphic to the highest weight module \( V(\lambda, \mu) \). Lemmas 4.7 and 4.8 therefore imply that this contradicts to the assumption (4.2), thus proving Theorem 4.1.

Theorem 1.1 is now implied by Theorems 3.1 and 4.1 due to Proposition 2.8.

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