No Swiss-cheese universe on the brane

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We study the possibility of brane-world generalization of the Einstein-Straus Swiss-cheese cosmological model. We find the modifications induced by the brane-world scenario. At a first glance only the motion of the boundary is modified and the fluid in the exterior region is allowed to have pressure. The general relativistic Einstein-Straus model emerges in the low density limit. By imposing that the brane is static, a combination of the junction conditions and modified cosmological evolution leads to the conclusion that the brane is flat. Thus no static Swiss-cheese universe can exist on the brane. The conclusion is not altered by the introduction of a cosmological constant in the FLRW regions. This result mimics a similar general relativistic result: static Einstein-Straus universes do not exist.

I. INTRODUCTION

Brane-world scenarios introduced by Arkani-Hamed, Dimopoulos and Dvali [1] and Randall and Sundrum (RS) [2] are motivated by string theory, where open strings end on branes. In the generalized RS scenario our 4-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) universe is a hypersurface of codimension one (the brane) embedded into a 5-dimensional charged Vaidya-Anti de Sitter or Reissner-Nordström-Anti de Sitter space-time (the bulk). The radiation in the bulk, the charge and mass of the central bulk black hole can be switched off independently (for a detailed discussion see [3]), however the bulk will always have a negative cosmological constant, which balances the brane tension in such a way that the 4-dimensional (4D) cosmological constant can be fine-tuned. Branes with various other symmetries were also examined, like the Einstein brane [4], a Kantowski-Sachs type brane [5] and the Gödel brane [6]. All these cases bear in common unusual properties or exotic matter.

There were also various attempts to find black hole solutions on the brane (for a recent review see [7]). For example in [8] the charge term of the 4D Reissner-Nordström black hole solution was interpreted as arising from a tidal mass in [9] the charge term of the 4D Reissner-Nordström-Straus Swiss-cheese model [10], in which Schwarzschild black holes and surrounding vacuum regions are immersed into a FLRW background. This model was employed to show that cosmic expansion has no influence on planetary orbits and also to give corrections to the luminosity-redshift relation [11]. The Einstein-Straus model however is unstable against perturbations. (It is also unsuitable for other type of symmetry then spherical, like cylindrical [12] or axial [13].) A more realistic model suitable for encompassing (spherically symmetric) inhomogeneities would be the McVittie solution [14], which allows for both the FLRW and the Schwarzschild limits. According to this model, the cosmic expansion drives the planetary orbits into an outward spiraling and galactic clusters into expansion, which however happen more slowly than the expansion of the cosmic background and consequently these effects are undetectable. An excellent review of these topics can be found in the book of Krasinski [15]. Recently, the McVittie solution was generalized both to arbitrary dimensions [16] and to include charge in 4D [17].

Such cosmological models with local inhomogeneities were not studied before in the context of the generalized RS scenario. It is the aim of this paper to study the simplest such model, the Swiss-cheese on the brane. The difficulty in such an approach is threefold. First, one has to join (according to the Lanczos-Sen-Darmois-Israel junction conditions [18]-[21]) black hole solutions with the cosmological background on the brane, both being solutions of a modified Einstein equation [22], such that there is no distributional matter on the junction surface (Fig. 1). Second, one has to extend somehow these black hole solutions into the bulk, which is far from trivial. Indeed, the simplest 4D Schwarzschild solution could be embedded in the bulk only by extending the singularity into the fifth dimension [23], obtaining a black string. In order to obtain a Schwarzschild brane black hole with regular AdS horizon, exotic matter has to be introduced in the bulk [24]. If the horizon is not compactified through the fifth dimension, gravitons from the black hole will escape into the fifth dimension even if there is an event horizon surrounding the hole on the brane. The third task is to interpret geometrically such an inhomogeneous brane as a junction hypersurface. Obviously, for different matter content $\tau_{ab} = -\lambda g_{ab} + T_{ab}$ in the voids and outside them (where $\lambda$ is the brane tension, $g_{ab}$ the brane metric and $T_{ab}$ represents the energy-momentum tensor of ordinary matter on the brane), the Lanczos equation

$$\Delta K_{ab} = -\kappa^2 \left( \tau_{ab} - \frac{\tau}{3} g_{ab} \right)$$  (1)
with $\Delta K_{ab} = -2K_{ab}$ for $Z_2$-symmetric embedding) gives rise to different extrinsic curvatures $K_{ab}$ for the voids and for the rest of the brane. Although the embedding (given by the 1-form $n = dy$, the brane being at $y = 0$) and the extrinsic curvature $K_{ab} = g^c_d \nabla_c n_d$ are closely interrelated, the latter also depends on the bulk metric $\tilde{g}_{ab} = g_{ab} + n_a n_b$ and the associated connection $\tilde{\nabla}$. Some of the difference in $K_{ab}$ at voids and expanding background can be explained by the difference in $\tilde{g}_{ab}$ due to how near or far the voids are, however the rest should be interpreted as humps and bumps in the brane embedding (Fig. 2).

In the simplest Swiss-cheese model on the brane we will see that fulfilling the first task is restrictive enough to make the second and third ones redundant.

II. A NO-GO RESULT

We suppose the bulk contains nothing but a cosmological constant and its Weyl curvature is such that it has no electric part. The embedding is $Z_2$-symmetric. Under these assumptions the modified Einstein equation is

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + \kappa^4 S_{ab} , \quad (2)$$

where $S_{ab}$ denotes a quadratic expression in $T_{ab}$:

$$S_{ab} = \frac{1}{12} TT_{ab} - \frac{1}{4} T_{ac} T_{b}^{c} + \frac{1}{24} g_{ab} \left(3T_{cd} T^{cd} - T^2 \right). \quad (3)$$

The brane gravitational constant $\kappa^2$ and the brane cosmological constant $\Lambda$ are related to the bulk gravitational constant $\tilde{\kappa}^2$, bulk cosmological constant $\tilde{\Lambda}$ and the (positive) brane tension $\lambda$ through

$$\begin{align*}
6\kappa^2 &= \tilde{\kappa}^4 \lambda , \\
2\Lambda &= \kappa^2 \lambda + \tilde{\kappa}^2 \tilde{\Lambda} . \quad (5)
\end{align*}$$

The brane is a 4-dimensional FLRW space-time (characterized by scale factor $a (\tau)$). It contains perfect fluid (characterized by energy density $\rho$ and pressure $p$, both depending only on cosmological time $\tau$). The inhomogeneities on the brane are introduced as nonintersecting Schwarzschild voids with arbitrary radii. Let us study the junction of one of these voids to the rest of the brane. The interior solution in curvature coordinates $(T, R)$ is

$$ds^2_S = - \left(1 - \frac{2m}{R} \right) dT^2 + \left(1 - \frac{2m}{R} \right)^{-1} dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) . \quad (6)$$

This is a pure vacuum solution (with $\Lambda = 0$) of the modified Einstein equation (2) in the void regions. The exterior solution in comoving coordinates is

$$ds^2_{FLRW} = -d\tau^2 + a^2 (\tau)$$

$$\times \left[ d\chi^2 + H^2 (\chi; k) \left(d\varphi^2 + \sin^2 \theta d\varphi^2 \right) \right] , \quad (7)$$

$$H(\chi; k) = \begin{cases} 
\sin \chi , & k = 1 , \\
\chi , & k = 0 , \\
\sinh \chi , & k = -1 .
\end{cases}$$

Here the function $H$ has the properties:

$$\left( \frac{dH}{d\chi} \right)^2 = 1 - kH^2 , \quad (8)$$

$$\frac{d^2 H}{d\chi^2} = -kH . \quad (9)$$

The modified Einstein equations (2) reduce to a generalized Friedmann and a generalized Raychaudhuri equation:

$$\frac{\dot{a}^2 + k}{a^2} = \frac{\Lambda}{3} + \frac{\kappa^2 \rho}{3} \left(1 + \frac{\rho}{2\lambda} \right) , \quad (10)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{\kappa^2}{6} \left[ \rho \left(1 + \frac{2\rho}{\lambda} \right) + 3p \left(1 + \frac{\rho}{\lambda} \right) \right] . \quad (11)$$

We have kept the cosmological constant in the exterior region for later convenience. In the limit $\rho/\lambda \to 0$ the corresponding general relativistic equations are recovered. We have identified the angular coordinates $\theta$ and $\varphi$ in the two regions, as in principle the comoving coordinate systems can be centered on any chosen Schwarzschild hole.

The junction conditions require the continuity of both the first and second fundamental forms (induced metric and extrinsic curvature) of the junction hypersurface. In the Swiss-cheese model the junction is made at some constant comoving $\chi = \chi_0$. Therefore it is obvious to

FIG. 1: (Color online.) A Swiss-cheese brane-world.
FIG. 2: A qualitative picture showing that the inhomogeneous extrinsic curvature of an inhomogeneous brane leads to humps and bumps of the brane embedding in the bulk.

introduce \((\tau, \theta, \varphi)\) as coordinates on the junction hypersurface. The induced metrics in the two regions become:

\[
ds^2_{\text{int}} = \left[1 - \left(1 - \frac{2m}{R_0}\right)T_0^2 + \left(1 - \frac{2m}{R_0}\right)^{-1} \dot{R}_0^2\right] d\tau^2 + R_0^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + R_0 \left( \frac{d\theta}{\sin \theta} \right)^2 ,
\]

\[
ds^2_{\text{ext}} = -d\tau^2 + a^2 (\tau) \left[ H^2_0 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] ,
\]

where \(R_0 = R(\tau, \chi_0)\), \(T_0 = T(\tau, \chi_0)\) and \(H_0 = H(\chi_0; k)\). Continuity of the induced metric implies

\[
R_0 = a(\tau) H_0 ,
\]

\[
\left(1 - \frac{2m}{a(\tau) H_0}\right)^2 T_0^2 = 1 - \frac{2m}{a(\tau) H_0} + \dot{a}(\tau) H_0^2 .
\]

These equations characterize the motion of the boundary region of the Schwarzschild void and formally they are the same as in the Einstein-Straus model. However due to the modified cosmological evolution Eqs. (10)-(11), the motion of the boundary is changed accordingly.

The extrinsic curvature of the junction hypersurface evaluated from the exterior region has only two nonvanishing components:

\[
K^\text{ext}_{\theta \theta} = a(\tau) H_0 \left(1 - kH_0^2\right)^{1/2} ,
\]

\[
K^\text{ext}_{\varphi \varphi} = K_{\theta \theta} \sin^2 \theta .
\]

(We have employed Eq. (8) in the derivation). The corresponding extrinsic curvature components evaluated from the interior region are:

\[
K^\text{int}_{\theta \theta} = \left(1 - \frac{2m}{R_0}\right) R_0 \dot{T}_0 ,
\]

\[
K^\text{int}_{\varphi \varphi} = K_{\theta \theta} \sin^2 \theta .
\]

From continuity of these components, by employing Eq. (14) we obtain a simple equation for \(\dot{T}_0\)

\[
\dot{T}_0 = \left(1 - \frac{2m}{a(\tau) H_0}\right)^{-1} \left(1 - kH_0^2\right)^{1/2} .
\]

Comparison of Eqs. (15) and (18) gives

\[
\dot{a}^2(\tau) + k = \frac{2m}{a(\tau) H_0^2} .
\]

Finally the condition \(K^\text{int}_{\alpha \alpha} = 0\) implies

\[
\ddot{a}(\tau) = - \frac{m a^2(\tau) H_0^2}{\kappa^2 (1 + \frac{a^2(\tau)}{\lambda})} - \frac{\rho(\tau)^2}{2 [\rho(\tau) + \Lambda]} .
\]

Due to cosmological symmetries, these equations derived on the boundary are valid everywhere in the FLRW brane. Therefore Eq. (21) can be also regarded as a relation between the mass and the comoving radius of the Schwarzschild void.

There are two major differences in comparison to the general relativistic model. The first is, that the cosmological perfect fluid is not dust. In the general relativistic limit \(\kappa^2 \rho = \Lambda\). If the cosmological constant is chosen to vanish, similarly as in the interior region, the fluid is dust. By contrast, in the brane-world scenario the nonlinear source term of Eq. (2) implies a \(\tau\)-dependent pressure (or tension). This agrees with the general relativistic limit in the low density regime \(\rho << \lambda\), however differs considerably in the high energy limit \(\rho >> \lambda\), where \(\rho \approx -\rho/2\). While in this later limit the classical condition for dark energy \(\rho + 3p \approx -\rho/2 < 0\) is obeyed, still the perfect fluid with equation of state (22) cannot drive inflation. Indeed, the cosmic acceleration is given by the modified Raychaudhuri equation (11), which becomes:

\[
\frac{\ddot{a}}{a} = - \frac{\Lambda}{6} - \frac{\kappa^2}{6} \rho \left(1 + \frac{\rho}{2\lambda}\right) ,
\]

giving deceleration for any positive \(\Lambda\).

The second crucial difference is the evolution of the fluid energy density, which differs from the general relativistic case. To see this, we integrate the continuity equation:

\[
\dot{\rho} + \frac{\dot{a}}{a} (\rho + p) = 0
\]
The cosmological fluid (which is an integrability condition of the system of Eqs. (10)-(11)). After employing Eq. (22) we obtain by integration
\[ a^3 = C \frac{k^2}{2\lambda} \left[ \Lambda + \kappa^2 \rho \left(1 + \frac{\rho}{2\Lambda}\right) \right]^{-1}, \tag{25} \]
with \( C \) an integration constant. Eq. (21) gives then the mass of the black holes
\[ m = \frac{\kappa^2 C}{12\lambda} H_0^3. \tag{26} \]
in the voids of the Einstein-Straus type Swiss-cheese branes.

Case \( \Lambda = 0 \). This is the standard choice in the Swiss-cheese model. In the low density limit \( \rho \sim a^{-3} \) emerges, as in the general relativistic case. In the special case of static branes (\( a = a_1 \) constant), the continuity Eq. (24) implies \( \rho = \rho_1 = \text{const} \) and Eqs. (23) and (29) do not arise any more. Then Eq. (22) gives \( p = \rho_1 = \text{const} \) as well. Moreover Eq. (23) implies \( \rho_1 = 0 \) and then Eq. (22) gives \( p_1 = 0, \) thus the FLRW region must be empty. Further, Eq. (20) gives \( m = 0 \) and then Eq. (19) \( k = 0, \) thus the whole brane becomes trivial, a flat hypersurface.

Case \( \Lambda \neq 0 \). Again, for static branes it is evident from the continuity Eq. (24) that \( \rho = \rho_2 = \text{const} \) should hold, while from Eq. (22) \( p = p_2 = \text{const} \) emerges. The Raychaudhuri equation (23) gives for the energy density of the fluid:
\[ \rho_2 = -\lambda \pm \sqrt{\lambda (\lambda - 2\Lambda/k^2)} \tag{27} \]
and also \( \Lambda < 0. \) (Therefore for a positive energy density we have to choose the + sign in Eq. (27).) From Eq. (22) after some straightforward algebra we find the pressure:
\[ p_2 = -\rho_2. \tag{28} \]
The cosmological fluid \( T_{ab} = -\rho_2 g_{ab} \) turns out to be just another contribution to the cosmological constant. The central mass is vanishing by virtue of Eq. (27) and then Eq. (19) implies once more \( k = 0 \). The brane is again flat, the 4-dimensional Minkowski space-time. Both a cosmological constant and a nonvanishing perfect fluid are present on the FLRW region of the brane, and they do not cancel, however the third, quadratic source term of the modified Einstein equation
\[ \kappa^4 S_{ab} = -\frac{\kappa^2 \rho_2^2}{2\lambda} g_{ab} \tag{29} \]
cancels both by virtue of Eq. (27)
\[ -\Lambda g_{ab} + \kappa^2 T_{ab} + \kappa^4 S_{ab} = -\kappa^2 \left[ \Lambda/k^2 + \rho_2 \left(1 + \frac{\rho_2}{2\Lambda}\right) \right] g_{ab} = 0. \tag{30} \]

### III. CONCLUDING REMARKS

We have found that as in the general relativistic case (in the Einstein-Straus model), the voids on a Swiss-cheese type brane-world reside in a dynamical universe. The density of the fluid in the Swiss-cheese brane evolves cf. Eq. (25) in a different way as compared to general relativity. A static Swiss-cheese universe cannot exist neither on the brane, nor in general relativity, apart from the trivial empty and flat universe.

Our no-go theorem for static Swiss-cheese branes adds to various results concerning the staticity of brane stellar models [26]-[29]. In particular, in [25] it was shown that for an Oppenheimer-Snyder collapse on the brane the exterior space-time (on the brane) cannot be static. Similarly it was shown that the vacuum exterior of a spherical star (with vanishing pressure at the surface) is in general not Schwarzschild [28]. In [26] the condition of vanishing pressure at the junction surface was lifted, as in our discussion.

It remains to see what type of inhomogeneous brane cosmological model can be found by lifting some of the original assumptions of the Swiss-cheese model. An obvious way to do this is by allowing a non-vanishing electric part of the bulk Weyl curvature, which would contribute as source of the modified Einstein equation. In such a scenario the voids would be non-vacuum solutions in the classical general relativistic sense. Investigations of these issues are under way.

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