Dot product invariant valuations on Lip($S^{n-1}$)

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Abstract

We provide an integral representation for continuous, rotation invariant and dot product invariant valuations defined on the space Lip($S^{n-1}$) of Lipschitz continuous functions on the unit $n$–sphere.

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1 Introduction

A valuation on a family $\mathcal{F}$ of sets is a function $\nu : \mathcal{F} \to \mathbb{R}$ such that

$$\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B),$$

for every $A, B \in \mathcal{F}$ with $A \cup B, A \cap B \in \mathcal{F}$. The previous relation has a clear geometric meaning, being in fact a finite additivity condition. In particular, roughly speaking, every measure is a valuation. Of particular interest (in general and for the purposes of the present paper), is the theory of valuations on the family $K^n$ of convex bodies of $\mathbb{R}^n$ (a convex body is a compact convex subset of $\mathbb{R}^n$). The importance of this theory is due not only to its role in the solution of Hilbert’s third problem, but also to the beauty and the profundity of some of the results that it comprises. We mention, as instances, the Hadwiger characterisation of rigid motion invariant valuations, the McMullen homogeneous decomposition for translation invariant valuations, and the Alesker irreducibility theorem, along with its consequences. For details concerning these results we refer to Chapter 6 of the monograph [22] by Schneider, which contains an up-to-date account on the state of the art of this theory.
The notion of valuation can be transferred to a different context, where the domain is a family of functions. Let $X$ be a space of real-valued functions; a \textit{valuation} on $X$ is a functional $\mu : X \to \mathbb{R}$ such that

$$\mu(u \lor v) + \mu(u \land v) = \mu(u) + \mu(v),$$

for every $u, v \in X$ such that $u \lor v$ and $u \land v \in X$, where $\lor$ and $\land$ denote the pointwise maximum and pointwise minimum, respectively. That (1.2) be the natural functional counterpart of (1.1) can be motivated, for example, observing that if $I_A$ denotes the characteristic function of a general set $A$, then

$$I_{A_1 \cup A_2} = I_{A_1} \lor I_{A_2} \quad \text{and} \quad I_{A_1 \cap A_2} = I_{A_1} \land I_{A_2}.$$ 

The study of valuations on spaces of functions started quite recently, mainly under the impulse of the rich theory of valuations on convex bodies. A branch of this area of research is focused on spaces of functions related to convexity, such as convex functions (see [3, 6, 7, 8, 2]), log-concave functions (see [18, 19]), quasi-concave theory of valuations on convex bodies. A branch of this area of research is focused on spaces of functions related to convexity, such as convex functions (see [3, 6, 7, 8, 2]), log-concave functions (see [18, 19]), quasi-concave functions (see [4, 5]).

A different line of investigation in the theory of valuations on function spaces traces back to the papers \cite{10, 11} by Klain, concerning valuations on star-shaped sets. A subset $S$ of $\mathbb{R}^n$ is \textit{star-shaped} (with respect to the origin) if for every $x \in S$, the segment joining $x$ and the origin is contained in $S$. To a star-shaped set $S$ we can associate its \textit{radial function $\rho_S : S^{n-1} \to [0, +\infty)$ defined by:

$$\rho_S(u) = \sup\{\lambda \geq 0 : \lambda u \in S\}.$$ 

Note that for every $S_1$ and $S_2$ star-shaped

$$\rho_{S_1 \cup S_2} = \rho_{S_1} \lor \rho_{S_2} \quad \text{and} \quad \rho_{S_1 \cap S_2} = \rho_{S_1} \land \rho_{S_2}. \quad (1.3)$$

In \cite{10, 11}, Klain considered the family $\mathcal{F}$ of star-shaped sets having radial functions of class $L^p(S^{n-1})$, endowed with the topology induced by the $L^p(S^{n-1})$ norm. He obtained a complete characterisation of continuous and rotation invariant valuations $\nu$ on $\mathcal{F}$, proving that they are all of the form

$$\nu(S) = \int_{S_{n-1}} F(\rho_S) dH^{n-1}, \quad \forall S \in \mathcal{F},$$

where $F$ is a function in $C(\mathbb{R})$ verifying a suitable growth condition at infinity.

The family $\mathcal{F}$ can be identified with $L^p_+(S^{n-1})$ (non-negative functions in $L^p(S^{n-1})$) and, in view of (1.3), to every valuation on $\mathcal{F}$ it corresponds a valuation on $L^p_+(S^{n-1})$. Therefore the result of Klain can be rephrased as a characterisation of continuous and rotation invariant valuations on $L^p_+(S^{n-1})$. In this spirit, Tsang in \cite{25} extended the result of \cite{10, 11} to the spaces $L^p(S^{n-1})$, with $1 \leq p < \infty$.

As a natural continuation of the previous results, the third and fourth author established in \cite{26} and \cite{23} a characterisation of continuous and rotation invariant valuations defined on star-shaped sets with continuous radial function (see also [24] for further extensions). The functional counterpart of their results reads as follows.

\textbf{Theorem 1.1 (Tradacete, Villanueva).} A function $\mu : C(S^{n-1}) \to \mathbb{R}$ is a continuous (w.r.t. uniform convergence) and rotation invariant valuation if and only if there exists $F \in C(\mathbb{R})$ such that

$$\mu(u) = \int_{S^{n-1}} F(u) dH^{n-1}$$

for every $u \in C(S^{n-1})$.

In the present paper we consider valuations defined on a smaller space, namely the space Lip($S^{n-1}$) consisting of Lipschitz function on $S^{n-1}$ (equipped with a topology $\tau$ defined in section 2). The main novelty of this space is that its elements are differentiable $H^{n-1}$-a.e. on $S^{n-1}$ (by Rademacher’s theorem), and therefore we expect that derivatives will come into play. Given $u \in \text{Lip}(S^{n-1})$, we denote by $\nabla u(x)$ the spherical gradient of $u$ at a point $x \in S^{n-1}$, if $u$ is differentiable at $x$ (see section 2 for definitions).

It can be shown that if $F: \mathbb{R} \times \mathbb{R}^n \times S^{n-1} \to \mathbb{R}$ is continuous, then the application $\mu: \text{Lip}(S^{n-1}) \to \mathbb{R}$ defined by

$$\mu(u) = \int_{S^{n-1}} F(u(x), \nabla u(x), x) dH^{n-1}(x), \quad \forall u \in \text{Lip}(S^{n-1})$$

is a continuous and rotation invariant valuation.
is a continuous valuation. If we consider, as a special case, functionals of the form

$$\mu(u) = \int_{S^{n-1}} F(u(x), \|\nabla u(x)\|) dH^{n-1}(x), \quad \forall u \in \text{Lip}(S^{n-1})$$

(1.4)

with \( F \in C(\mathbb{R} \times [0, \infty)) \), then we have a whole family of continuous and rotation invariant valuations on \( \text{Lip}(S^{n-1}) \). We are then led to the problem of characterising all possible continuous and rotation invariant valuations on \( \text{Lip}(S^{n-1}) \).

Here we solve a special case of this problem, namely we characterise valuations \( \mu \) which are additionally \textit{dot product invariant}, i.e. invariant under the addition of linear functions:

$$\mu(u + l) = \mu(u)$$

(1.5)

for every \( u \in \text{Lip}(S^{n-1}) \) and for every \( l: S^{n-1} \to \mathbb{R} \) which is the restriction to \( S^{n-1} \) of a linear function on \( \mathbb{R}^n \).

We can give a geometric interpretation of the previous assumption. Note that \( \text{Lip}(S^{n-1}) \) contains the family \( \mathcal{H}(S^{n-1}) \) of \textit{support functions} of convex bodies (see section 2 for definitions). If \( h \in \mathcal{H}(S^{n-1}) \) is the support function of a convex body \( K \), and \( l \) is the restriction to \( S^{n-1} \) of a linear function, then \( h + l \) is the support function of a translated copy of \( K \). Hence if \( \mu \) is dot product invariant, its restriction to support functions is “translation invariant”. This observation is crucial for the proof of our characterisation result; indeed, dot product and rotation invariance will allow us to apply the McMullen homogeneous decomposition and the Hadwiger characterisation theorem to the restriction to \( \mathcal{H}(S^{n-1}) \) of the valuations under consideration. In particular we may say that our argument relies heavily on the theory of valuations on convex bodies.

The main result of this paper is the following theorem.

**Theorem 1.2.** A functional \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) is a continuous, rotation invariant and dot product invariant valuation if and only if there exist constants \( c_0, c_1, c_2 \in \mathbb{R} \) such that

$$\mu(u) = c_0 + c_1\int_{S^{n-1}} u(x) dH^{n-1}(x) + c_2\int_{S^{n-1}} [(n - 1)u^2(x) - \|\nabla u(x)\|^2] dH^{n-1}(x),$$

(1.6)

for every \( u \in \text{Lip}(S^{n-1}) \).

This result indicates that dot product invariance is a very strong condition. Indeed, the space of continuous, rotation and dot product invariant valuations has finite dimension, similarly to what happens for continuous and rigid motion invariant valuations on convex bodies, by Hadwiger theorem. This is far from being true without dot product invariance, as shown by the class of examples (1.4).

The right hand side of (1.6) deserves some explanation. First, note that it is a continuous and rotation invariant valuation, by (1.4). Concerning dot product invariance, the constant term \( c_0 \) is clearly invariant, the second addend is invariant as well, as the integral over the sphere of the restriction to \( S^{n-1} \) of a linear function is zero. Moreover, if \( u \) is sufficiently regular, by the divergence theorem we have

$$\int_{S^{n-1}} [(n - 1)u^2(x) - \|\nabla u(x)\|^2] dH^{n-1}(x) = \int_{S^{n-1}} u[(n - 1)u + \Delta u(x)]dH^{n-1}(x),$$

where \( \Delta u \) denotes the spherical Laplacian of \( u \). Dot product invariance of this term follows now from the fact that restrictions of linear functions are eigenfunctions of the \( \Delta \) operator, with eigenvalue \(- (n - 1)\).

We also remark that if \( u = h \in \mathcal{H}(S^{n-1}) \) is the support function of a convex body \( K \), then the right hand side of (1.6) is just a linear combination of \( V_0(K) \), \( V_1(K) \) and \( V_2(K) \), the first three intrinsic volumes of \( K \). Therefore (1.6) appears as a truncated version of Hadwiger’s theorem; the reason why the remaining intrinsic volumes are not involved is that they do not admit an extension from \( \mathcal{H}(S^{n-1}) \) to \( \text{Lip}(S^{n-1}) \). This will be clarified in the proof of Theorem 1.2 (see in particular Proposition 5.1).

The rest of the paper is devoted to the proof of Theorem 1.2. However, we will obtain in section 3 an approximation result along the way (see Proposition 3.1) which could be used to deal also with valuations which are not (rotation invariant nor) dot product invariant, and, in section 4, a homogenous decomposition for continuous and dot-product invariant valuations on \( \text{Lip}(S^{n-1}) \).
2 Preliminaries

2.1 Basic notions

For \( n \in \mathbb{N}, n \geq 2 \), we denote by \( S^{n-1} \) the unit \( n \)-sphere, that is,
\[
S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \},
\]
where \( \| \cdot \| \) represents the Euclidean norm. We will use the \((n-1)\)-dimensional Hausdorff measure \( H^{n-1} \) on the sphere.

Even though we will mainly be interested in what happens on \( S^{n-1} \), it will often be useful to reason on the whole space \( \mathbb{R}^n \), where we will use the standard \( n \)-dimensional Lebesgue measure, so that every “a.e.” referred to functions defined on \( \mathbb{R}^n \) is to be understood with respect to said measure. The standard basis of \( \mathbb{R}^n \) will be denoted by \( \{ e_1, \ldots, e_n \} \).

Let \( \text{Lip}(S^{n-1}) \) be the space of Lipschitz continuous maps defined on \( S^{n-1} \), i.e., the set of functions \( u : S^{n-1} \to \mathbb{R} \) for which there exists a constant \( L \geq 0 \) such that
\[
|u(x) - u(y)| \leq L \|x - y\|,
\]
for every \( x, y \in S^{n-1} \). The smallest constant for which this inequality holds is called the \textit{Lipschitz constant} associated with \( u \) and is denoted by \( L(u) \):
\[
L(u) = \inf \left\{ \frac{|u(x) - u(y)|}{\|x - y\|} : x, y \in S^{n-1}, x \neq y \right\}.
\]

Let \( u \in \text{Lip}(S^{n-1}) \); as an application of the Rademacher theorem, \( u \) is differentiable \( H^{n-1}\text{-a.e} \) on \( S^{n-1} \). Let \( x \in S^{n-1} \) be a point of differentiability; the differential of \( u \) at \( x \) is a linear application from \( T_x(S^{n-1}) \), the tangent space to \( S^{n-1} \) at \( x \), to \( \mathbb{R} \), and hence it can be represented by a vector \( \nabla u(x) \in T_x(S^{n-1}) \) that we will call the \textit{spherical gradient} of \( u \) at \( x \).

Let us consider on \( \text{Lip}(S^{n-1}) \) the topology \( \tau \) induced by the following convergence: we say that a sequence \( \{ u_i : i \in \mathbb{N} \} \subseteq \text{Lip}(S^{n-1}) \) converges to \( u \in \text{Lip}(S^{n-1}) \) with respect to \( \tau \), in symbols \( u_i \xrightarrow{\tau} u \), as \( i \to \infty \), if

- \( \| u_i - u \|_{\infty} \to 0 \), that is, \( u_i \to u \) uniformly on \( S^{n-1} \);
- \( \nabla u_i \to \nabla u \) \textit{almost uniformly}, i.e. \( \nabla u_i \to \nabla u \ H^{n-1}\text{-a.e.} \) in \( S^{n-1} \) and there exists a suitable constant \( C > 0 \) such that
  \[
  \| \nabla u_i(x) \| \leq C,
  \]
  for every \( i \in \mathbb{N} \) and \( H^{n-1}\text{-a.e.} \) \( x \in S^{n-1} \).

A functional \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) is said to be a \textit{valuation} if
\[
\mu(u \vee v) + \mu(u \wedge v) = \mu(u) + \mu(v),
\]
for every \( u, v \in \text{Lip}(S^{n-1}) \), where \( \vee \) and \( \wedge \) are the pointwise maximum and pointwise minimum respectively. Note that, since \( \text{Lip}(S^{n-1}) \) is closed with respect to these operations, all the terms in (2.1) are well defined.

We will say that a valuation \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) is \textit{continuous} if it is continuous with respect to the topology \( \tau \), unless otherwise stated.

Besides continuity, we will be interested in other properties, namely the rotation invariance, dot product invariance and homogeneity. These concepts are defined below.

A valuation \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) is \textit{rotation invariant} if for every \( u \in \text{Lip}(S^{n-1}) \) and \( \varphi \in \mathcal{O}(n) \) we have
\[
\mu(u \circ \varphi) = \mu(u),
\]
where \( \mathcal{O}(n) \) is the orthogonal group.

\( \mu \) is called \textit{dot product invariant} if, for every \( u \in \text{Lip}(S^{n-1}) \) and \( x \in \mathbb{R}^n \),
\[
\mu(u + \langle , x \rangle) = \mu(u),
\]
where \((\cdot, \cdot)\) denotes the standard scalar product in \(\mathbb{R}^n\). In other words, \(\mu\) is dot product invariant if it is invariant under the addition of linear functions restricted to \(S^{n-1}\).

For \(\alpha \geq 0\), \(\mu\) is \(\alpha\)-homogeneous if 
\[
\mu(\lambda u) = \lambda^\alpha \mu(u),
\]
for every \(\lambda \geq 0\) and \(u \in \text{Lip}(S^{n-1})\).

We will also work with valuations defined on the space \(K^n\) of convex bodies of \(\mathbb{R}^n\), namely compact and convex subsets of \(\mathbb{R}^n\). We recall some definitions in this context.

A valuation on \(K^n\) is a function \(\nu : K^n \to \mathbb{R}\) such that 
\[
\nu(K \cup L) + \nu(K \cap L) = \nu(K) + \nu(L),
\]
for every \(K, L \in K^n\) satisfying \(K \cup L \in K^n\). Such \(\nu\) is rotation invariant if 
\[
\nu(\varphi(K)) = \nu(K),
\]
for every \(K \in K^n\) and \(\varphi \in O(n)\). It is called translation invariant if 
\[
\nu(K + x) = \nu(K),
\]
for every \(K \in K^n\) and \(x \in \mathbb{R}^n\). For \(0 \leq i \leq n\), \(\nu\) is \(i\)-homogeneous if 
\[
\nu(\lambda K) = \lambda^i \nu(K),
\]
for every \(\lambda \geq 0\) and \(K \in K^n\).

The Hausdorff metric on \(K^n\) is defined by 
\[
d_H(K, L) = \max \left\{ \sup_{x \in K} \inf_{y \in L} \|x - y\|, \sup_{y \in L} \inf_{x \in K} \|x - y\| \right\},
\]
for every non-empty \(K, L \in K^n\).

The following set is dense in \(K^n\) (see e.g. [22]):
\[
C^{2,+} = \{ K \in K^n : \partial K \in C^2\text{ and has strictly positive Gaussian curvature at every point} \}.
\]

Let us now recall the definition of support function: for every non-empty \(K \in K^n\), its support function is \(h_K : \mathbb{R}^n \to \mathbb{R}\) defined by 
\[
h_K(x) = \max_{y \in K} (x, y), \ x \in \mathbb{R}^n.
\]
Support functions are convex and \(1\)-homogeneous, that is, \(h_K(\lambda x) = \lambda h_K(x)\) for every \(\lambda \geq 0\) and \(x \in \mathbb{R}^n\). Moreover, \(\|h_K - h_L\|_{\infty} = d_H(K, L)\), for every non-empty \(K, L \in K^n\).

The notion of piecewise linear function will also be useful. A continuous function \(f : \mathbb{R}^n \to \mathbb{R}\) is said to be a piecewise linear function if there exist closed convex cones \(C_1, \ldots, C_m\) with vertex at the origin and pairwise disjoint interiors such that 
\[
\bigcup_{i=1}^m C_i = \mathbb{R}^n,
\]
and linear functions \(L_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m\), such that \(f = L_i\) on \(C_i\), for \(i = 1, \ldots, m\).

We denote by \(\mathcal{H}(S^{n-1})\) and \(\mathcal{L}(S^{n-1})\) the sets of the restrictions to \(S^{n-1}\) of support functions and piecewise linear functions respectively. When considering these same functions defined on the whole space \(\mathbb{R}^n\) we will use the symbols \(\mathcal{H}(\mathbb{R}^n)\) and \(\mathcal{L}(\mathbb{R}^n)\) instead. Note that, since support functions are convex, they are locally Lipschitz continuous, hence \(\mathcal{H}(S^{n-1}) \subseteq \text{Lip}(S^{n-1})\). We also have the inclusion \(\mathcal{L}(S^{n-1}) \subseteq \text{Lip}(S^{n-1})\).

We introduce one last notation, which will come in handy in the future, denoting by 
\[
\hat{\mathcal{H}}(S^{n-1}) = \left\{ \bigwedge_{i=1}^m h_{K_i} : m \in \mathbb{N}, \ h_{K_i} \in \mathcal{H}(S^{n-1}) \text{ for } i = 1, \ldots, m \right\}
\]
the space of finite minima of support functions.
2.2 The theorems of Hadwiger and McMullen for valuations on convex bodies

In section 5 we will prove a McMullen-type decomposition result for continuous and dot product invariant valuations; we hereby recall the original McMullen decomposition theorem for valuations defined on \( K^n \), which will be used in the proof of our result in section 5.

Theorem 2.1 (McMullen, [22]). Let \( \nu : K^n \to \mathbb{R} \) be a translation invariant valuation which is continuous with respect to the Hausdorff metric. Then there exist continuous and translation invariant valuations \( \nu_0, \ldots, \nu_n : K^n \to \mathbb{R} \) such that \( \nu_i \) is \( i \)-homogeneous, for \( i = 0, \ldots, n \), and

\[
\nu(\lambda K) = \sum_{i=0}^{n} \lambda^i \nu_i(K),
\]

(2.2)

for every \( K \in K^n \) and \( \lambda \geq 0 \).

The Inclusion-Exclusion Principle will also play a crucial role in the proof of the aforementioned McMullen-type result. Every valuation on a lattice satisfies this principle, as it can be proved by induction (see [26] for the analogous statement for valuations on \( C(S^{n-1})^+ \)).

Proposition 2.2 (Inclusion-Exclusion Principle). Let \( (L, \vee, \wedge) \) be a lattice and \( \mu : L \to \mathbb{R} \) a valuation. Then for every \( u_1, \ldots, u_m \in L \) we have

\[
\mu \left( \bigvee_{j=1}^{m} u_j \right) = \sum_{1 \leq j \leq m} \mu(u_j) - \sum_{1 \leq j_1 < j_2 \leq m} \mu(u_{j_1} \wedge u_{j_2}) + \sum_{1 \leq j_1 < j_2 < j_3 \leq m} \mu(u_{j_1} \wedge u_{j_2} \wedge u_{j_3}) - \ldots + (-1)^{m-1} \mu \left( \bigwedge_{j=1}^{m} u_j \right),
\]

(2.3)

The proof of Theorem 1.2 is based upon the famous Hadwiger theorem, recalled below.

Theorem 2.3 (Hadwiger, [22]). A map \( \nu : K^n \to \mathbb{R} \) is a rotation and translation invariant valuation which is continuous with respect to the Hausdorff metric, if and only if there exist constants \( c_0, \ldots, c_n \in \mathbb{R} \) such that

\[
\nu(K) = \sum_{i=0}^{n} c_i V_i(K),
\]

(2.4)

for every \( K \in K^n \), where \( V_i \) denotes the \( i \)th intrinsic volume.

For the definition and properties of the intrinsic volumes, see [24].

2.3 McShane’s lemma

Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \), and assume that it is differentiable at a point \( x \in \mathbb{R}^n \); we denote by \( \nabla_e \varphi(x) \) the standard Euclidean gradient of \( \varphi \) at \( x \).

Given a function \( u : S^{n-1} \to \mathbb{R} \), it will be often convenient to extend it to \( \mathbb{R}^n \) as 1-homogeneous function. Let us denote by \( \tilde{u} \) this extension:

\[
\tilde{u}(x) = \begin{cases} 
\|x\| \ u \left( \frac{x}{\|x\|} \right) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

Let \( x \in S^{n-1} \) be a point where \( u \) is differentiable; then \( \tilde{u} \) is differentiable at \( x \) as well. If we denote by \( \nabla_e \tilde{u}(x) \) the standard Euclidean gradient of \( \tilde{u} \) at \( x \), by the Euler relation we obtain the following equality:

\[
\| \nabla_e \tilde{u}(x) \|^2 = \| \nabla u(x) \|^2 + u^2(x).
\]

(2.5)
Let $u \in \text{Lip}(S^{n-1})$; formula (2.5) can be applied to get the following bound on the spherical gradient:

$$
\|\nabla u(x)\| \leq \sqrt{n} \cdot L(u),
$$

(2.6)

for $H^{n-1}$-a.e. $x \in S^{n-1}$.

There is another way of extending Lipschitz functions defined on $S^{n-1}$, which we will be interested in; it is stated in the following theorem.

**Theorem 2.4 (McShane, [17]).** Let $S \subseteq \mathbb{R}^n$ and $u : S \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $L$. Then the map $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$
\tilde{u}(x) = \sup_{z \in S} [u(z) - L\|x - z\|],
$$

for $x \in \mathbb{R}^n$, is still Lipschitz continuous with the same Lipschitz constant $L$.

We conclude this paragraph by recalling that the gradient of a support function possesses the following property (see [22]).

**Proposition 2.5.** Let $K \in \mathbb{K}^n$. If $h_K$ is differentiable at $x \in S^{n-1}$, then $\nabla_e h_K(x) \in \partial K$, and $\nabla_e h_K(x)$ is the only point of $\partial K$ with outer normal vector $x$.

### 2.4 Some remarks

In this paragraph we collect some technical remarks which will be useful throughout the paper. It is convenient to study the behaviour of support functions with respect to the operators $\lor$ and $\land$. The result hereby presented is well-known, we include the proof for completeness.

**Lemma 2.6.** Let $K, L \in \mathbb{K}^n$. Then $h_K \lor h_L = h_{\text{conv}(K \cup L)}$, where $\text{conv}(K \cup L)$ denotes the convex hull of $K \cup L$. Moreover, if $K \cup L \in \mathbb{K}^n$ we have

$$
h_K \lor h_L = h_{K \cup L}, \quad h_K \land h_L = h_{K \cap L}.
$$

**Proof.** Clearly $\text{conv}(K \cup L)$ contains $K$ and $L$, thus $h_{\text{conv}(K \cup L)} \geq h_K$ and $h_{\text{conv}(K \cup L)} \geq h_L$. This implies the inequality

$$
h_{\text{conv}(K \cup L)}(x) \geq h_K \lor h_L(x),
$$

for every $x \in \mathbb{R}^n$. Vice versa, if $x \in \mathbb{R}^n$, then

$$
h_{\text{conv}(K \cup L)}(x) = \max_{z \in \text{conv}(K \cup L)} (x, z),
$$

where the maximum will be attained in correspondence of a certain element $z = \sum_{i=1}^{m} \lambda_i x_i$, with $m \in \mathbb{N}$, $\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = 1$, $x_i \in K \cup L$. By reordering the elements, we can assume that $\{x_1, \ldots, x_l\} \subseteq K$ and $\{x_{l+1}, \ldots, x_m\} \subseteq L$. Therefore, we have

$$
h_{\text{conv}(K \cup L)}(x) = \left< x, \sum_{i=1}^{m} \lambda_i x_i \right> = \sum_{i=1}^{l} \lambda_i \left< x, x_i \right> + \sum_{i=l+1}^{m} \lambda_i \left< x, x_i \right>
$$

$$
\leq \sum_{i=1}^{l} \lambda_i h_K(x) + \sum_{i=l+1}^{m} \lambda_i h_L(x)
$$

$$
\leq \left( \sum_{i=1}^{l} \lambda_i + \sum_{i=l+1}^{m} \lambda_i \right) h_K \lor h_L(x) = h_K \lor h_L(x).
$$

We now work under the hypothesis $K \cup L \in \mathbb{K}^n$. We first prove that

$$
(K \cup L) + (K \cap L) = K + L.
$$

(2.7)
Note that 
\[(K \cup L) + (K \cap L) \subseteq K + L.\]
Vice versa, let \(x + y \in K + L.\) If either \(x \in K \cap L\) or \(y \in K \cap L,\) then we are done; suppose now \(x \in K \setminus L\) and \(y \in L \setminus K.\) Because of this assumption, there exists \(t \in (0, 1)\) such that
\[z := tx + (1 - t)y \in K \cap L,\]
since \(K \cup L \in \mathcal{K}^n.\) Therefore,
\[x + y = (1 - t)x + ty + z \in (K \cup L) + (K \cap L),\]
using the convexity of \(K \cup L\) again. Formula (2.7) follows. By the properties of support functions (see \([22, \text{Section 1.7}]\) we obtain
\[
h_{K \cup L} + h_{K \cap L} = h_{(K \cup L) + (K \cap L)} = h_{K + L} = h_K + h_L
\]
\[= h_K \vee h_L + h_K \wedge h_L = h_{\operatorname{conv}(K \cup L)} + h_K \wedge h_L = h_{K \cup L} + h_K \wedge h_L,
\]
where the last equality follows from the hypothesis \(K \cup L \in \mathcal{K}^n.\) This proves the lemma.

We are now going to state a topological result concerning the continuity of a functional \(\mu : \mathcal{H}(S^{n-1}) \to \mathbb{R}.\) By definition of \(\tau,\) we have that if such a functional is continuous with respect to \(\| \cdot \|_\infty,\) then it is also continuous with respect to \(\tau.\) The converse is also true.

**Lemma 2.7.** Let \(\mu : \mathcal{H}(S^{n-1}) \to \mathbb{R}.\) Then \(\mu\) is continuous with respect to \(\tau\) if and only if it is continuous with respect to \(\| \cdot \|_\infty.\)

**Proof.** Consider a \(\tau-\)continuous functional \(\mu : \mathcal{H}(S^{n-1}) \to \mathbb{R}\) and a sequence \(\{h_{K_i}\} \subseteq \mathcal{H}(S^{n-1})\) of support functions such that \(\|h_{K_i} - h_K\|_\infty \to 0,\) as \(i \to \infty,\) where \(K \in \mathcal{K}^n.\) It is enough to prove that \(h_{K_i} \stackrel{\tau}{\to} h_K.\)

Define
\[D_i = \{x \in S^{n-1} : h_{K_i} \text{ is differentiable at } x\},\]
for \(i \in \mathbb{N},\) and
\[D_0 = \{x \in S^{n-1} : h_K \text{ is differentiable at } x\}.
\]
We also set
\[D = \bigcap_{i=0}^{\infty} D_i.
\]
Note that \(H^{n-1}(S^{n-1}\setminus D) = 0,\) because of Rademacher’s theorem.

For every \(x \in D\) we have
\[
\nabla h_{K_i}(x) \to \nabla h_K(x). \tag{2.8}
\]
Indeed, consider a subsequence \(\{h_{K_{i_j}}\} \subseteq \{h_{K_i}\}.\) For every \(j \in \mathbb{N},\) the differentiability of \(h_{K_{i_j}}\) at \(x\) implies (see, e.g., \([22, \text{Section 1.5}]\)
\[h_{K_{i_j}}(y) \geq h_{K_{i_j}}(x) + \langle \nabla e h_{K_{i_j}}(x), y - x \rangle, \tag{2.9}
\]
for every \(y \in \mathbb{R}^n.\) The condition \(\|h_{K_{i_j}} - h_K\|_\infty \to 0\) implies \(K_{i_j} \to K\) with respect to the Hausdorff metric, hence there is a convex body \(\tilde{K}\) such that \(K_{i_j} \subseteq \tilde{K},\) for every \(j \in \mathbb{N}\) (see \([22, \text{Section 1.8}]\)). From Proposition \(2.5\) we have that
\[
\nabla e h_{K_{i_j}}(x) \in \partial K_{i_j} \subseteq K_{i_j} \subseteq \tilde{K},
\]
thus there is a subsequence \(\{h_{K_{i_{j_l}}}\} \subseteq \{h_{K_{i_j}}\}\) such that \(\lim_{l \to \infty} \nabla e h_{K_{i_{j_l}}}(x)\) exists, by the Bolzano-Weierstrass theorem. Writing (2.9) for this subsequence and letting \(l \to \infty\) we obtain
\[
h_K(y) \geq h_K(x) + \left\langle \lim_{l \to \infty} \nabla e h_{K_{i_{j_l}}}(x), y - x \right\rangle,
\]
for every $y \in \mathbb{R}^n$. Recalling the uniqueness of the subgradient at differentiability points for convex functions (Section 1.5), the last inequality implies
\[
\lim_{l \to \infty} \nabla_e h_{K_{i_j}}(x) = \nabla_e h_K(x).
\]
This, together with relation (2.5) and the arbitrariness of \(\{h_{K_{i_j}}\} \subseteq \{h_{K_i}\}\), proves (2.8).

Moreover, for every $x \in D$ we have
\[
\|\nabla h_{K_j}(x)\| \leq \|\nabla_e h_{K_j}(x)\| \leq \max\{\|y\| : y \in \widehat{K}\},
\]
where we have used Proposition 2.5 again. Hence we have a uniform bound of $\|\nabla h_{K_i}\|$ on $D$. Thus $h_{K_i} \tau \to h_K$, as desired.

Theorems 2.1 and 2.3 concern valuations on convex bodies, and since we will be interested in studying valuations on support functions using these results, it would be nice to know that we can “move” valuations from $\mathcal{H}(S^{n-1})$ to $\mathcal{K}^n$ without losing any property. This is stated precisely in the next result.

**Lemma 2.8.** Let $\mu : \mathcal{H}(S^{n-1}) \to \mathbb{R}$ be a continuous valuation. Define $\nu : \mathcal{K}^n \to \mathbb{R}$ by setting
\[
\nu(K) = \mu(h_K),
\]
for every $K \in \mathcal{K}^n$. Then
i) if $\mu$ is a valuation, $\nu$ is too;
ii) if $\mu$ is $\tau$–continuous, $\nu$ is continuous with respect to the Hausdorff metric;
iii) if $\mu$ is rotation invariant, $\nu$ is too;
iv) if $\mu$ is dot product invariant, $\nu$ is translation invariant;
v) if $\mu$ is $i$–homogeneous for some $i \in \{0, \ldots, n\}$, $\nu$ is too.

Assertions i) and ii) are consequences of lemmas 2.6 and 2.7 respectively. The rest of the statement follows from the properties of support functions, which can be found in [22].

**3 Approximation**

The idea behind the proof of Theorem 1.2 is that of using an approximation result to narrow down the study of our valuations from the space $\text{Lip}(S^{n-1})$ to its subset $\mathcal{H}(S^{n-1})$, which is in bijection with $\mathcal{K}^n$, where Hadwiger’s theorem can be applied.

More precisely, our goal is to prove that continuous valuations on $\text{Lip}(S^{n-1})$ are uniquely determined by the values they attain at support functions, as stated in the following proposition.

**Proposition 3.1.** Let $\mu_1, \mu_2 : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be continuous valuations. If $\mu_1 = \mu_2$ on $\mathcal{H}(S^{n-1})$, then $\mu_1 = \mu_2$ on $\text{Lip}(S^{n-1})$.

The proof is split into four main steps, which will be detailed in the next paragraphs.

**3.1 $L(S^{n-1}) \subseteq \widehat{H}(S^{n-1})$**

First of all, we are going to prove that piecewise linear functions can be written as finite minima of support functions.

**Lemma 3.2.** Let $f \in \mathcal{L}(\mathbb{R}^n)$. Then there exist $m \in \mathbb{N}$ and $h_{K_1}, \ldots, h_{K_m} \in \mathcal{H}(\mathbb{R}^n)$ such that
\[
f = \bigwedge_{i=1}^{m} h_{K_i}.
\]

In particular, $\mathcal{L}(S^{n-1}) \subseteq \widehat{H}(S^{n-1})$. 

Proof. For \( f \in \mathcal{L}(\mathbb{R}^n) \), there are closed convex cones \( C_1, \ldots, C_m \) with vertex at the origin and pairwise disjoint interiors such that
\[
\bigcup_{i=1}^{m} C_i = \mathbb{R}^n,
\]
and \( f = L_i \) is linear on \( C_i \), for \( i = 1, \ldots, m \).

We will now focus on the cone \( C_1 \). Consider \( \tilde{f} = f - L_1 \). Let \( P_{C_1} : \mathbb{R}^n \to \mathbb{R}^n \) denote the so called metric projection onto \( C_1 \): for every \( x \in \mathbb{R}^n \), \( P_{C_1} (x) \) is the unique point in \( C_1 \) such that
\[
\|x - P_{C_1} (x)\| = \min_{y \in C_1} \|x - y\|.
\]
As \( C_1 \) is closed and convex, this function is well defined. We also define the function \( g : \mathbb{R}^n \to \mathbb{R} \) by
\[
g(x) = \|x - P_{C_1} (x)\| = \min_{z \in C_1} \|x - z\|,
\]
for every \( x \in \mathbb{R}^n \); this is the distance function from the cone \( C_1 \). As \( C_1 \) is a convex cone with apex at the origin, \( g \) is 1-homogeneous and subadditive, hence it is also convex. These properties imply the existence of a convex body \( K \in \mathcal{K}^n \) such that \( g = h_K \) (see [22, Section 1.8]).

We prove that there exists a suitable constant \( c > 0 \) such that
\[
cg(x) \geq \tilde{f}(x),
\]
for every \( x \in \mathbb{R}^n \). Suppose this to be false; then for every \( c > 0 \) there exists a point \( x_c \in \mathbb{R}^n \) such that \( cg(x_c) < \tilde{f}(x_c) \). Choosing \( c = i, i \in \mathbb{N} \), we construct a sequence \( \{x_i\} \subseteq \mathbb{R}^n \) satisfying
\[
g(x_i) < \frac{1}{i} \tilde{f}(x_i),
\]
for every \( i \in \mathbb{N} \). Because this inequality is strict, we have that \( x_i \neq 0 \) for every \( i \in \mathbb{N} \). From the 1-homogeneity we get
\[
g\left( \frac{x_i}{\|x_i\|} \right) < \frac{1}{i} \tilde{f}\left( \frac{x_i}{\|x_i\|} \right).
\]
This means that, since \( S^{n-1} \) is compact, in (3.2) we may assume that \( \{x_i\} \subseteq S^{n-1} \) and \( x_i \to x \) as \( i \to \infty \), for some \( x \in S^{n-1} \).

We observe that \( x \in \partial C_1 \). In fact, if \( x \in \mathbb{R}^n \setminus C_1 \), then letting \( i \to \infty \) in (3.2) we would have \( 0 < g(x) \leq 0 \), a contradiction. Thus \( x \in C_1 \), but since \( \{x_i\} \subseteq \mathbb{R}^n \setminus C_1 \) (by (3.2) and the non-negativity of \( g \)), \( x \) can not be in the interior of \( C_1 \), and so it belongs to \( \partial C_1 \).

Let \( \tilde{x}_i = P_{C_1} (x_i) \); by continuity of the projection, \( \tilde{x}_i \to x \), as \( i \to \infty \).

Note that there exist an index \( k \in \{2, \ldots, m\} \) and a subsequence \( \{x_{i_j}\} \subseteq \{x_i\} \) such that \( \{x_{i_j}\} \subseteq C_k \). For the sake of simplicity, assume \( \{x_i\} \subseteq C_k \). We write
\[
L_j(y) = \langle a_j, y \rangle, \ y \in \mathbb{R}^n,
\]
with \( a_j \in \mathbb{R}^n \), for \( j = 1, \ldots, m \). From (3.2) and using the fact that \( \tilde{f}(\tilde{x}_i) = 0 \) (since \( \tilde{x}_i \in C_1 \)) and the Cauchy-Schwarz inequality, we get
\[
\|x_i - \tilde{x}_i\| = g(x_i) < \frac{1}{i} \left[ \tilde{f}(x_i) - \tilde{f}(\tilde{x}_i) \right] = \frac{1}{i} \langle a_k - a_1, x_i - \tilde{x}_i \rangle \leq \frac{1}{i} \|a_k - a_1\| \cdot \|x_i - \tilde{x}_i\|.
\]
Since \( x_i \notin C_1 \) whereas \( \tilde{x}_i \in C_1 \), we have \( x_i \neq \tilde{x}_i \), and so the last inequality yields \( i < \|a_k - a_1\| \); letting \( i \to \infty \) we obtain a contradiction. Then there must be a constant \( c > 0 \) such that (3.1) holds for every \( x \in \mathbb{R}^n \).

Therefore,
\[
f_1 := cg + L_1 \geq \tilde{f} + L_1 = f
\]
on \( \mathbb{R}^n \), with \( f_1 = L_1 = f \) on the cone \( C_1 \). Furthermore,
\[
f_1 = cg + L_1 = ch_K + h_{a_1} = h_{cK + a_1} =: h_{K_1}
\]
is a support function.

We repeat the process for each cone $C_i$, $i = 2, \ldots, m$, building support functions $h_{K_i} : \mathbb{R}^n \to \mathbb{R}$ such that $h_{K_i} \geq f$ on $\mathbb{R}^n$ and $h_{K_i} = f$ on $C_i$. Thus we can write

$$f = \bigwedge_{i=1}^m h_{K_i}.$$ 

3.2 Approximation of $C^1$ functions by piecewise linear functions

Piecewise linear functions can be used to approximate $C^1$ functions with respect to the topology $\tau$, as stated in the following lemma.

Lemma 3.3. Let $u \in C^1(S^{n-1})$. Then there exists a sequence $\{f_i\} \subseteq \mathcal{L}(S^{n-1})$ such that $f_i \to u$.

To prove this we will need a preliminary observation, which in turn requires a definition: a partition $\mathcal{P}$ of a set $Q \subseteq \mathbb{R}^n$ is called a simplicial partition if it is made up of simplices such that for every two of them, their intersection is either empty or a face (of any dimension between 0 and $n-1$) of both simplices. We can now point out the following fact.

Remark 3.4. Let $Q \subseteq \mathbb{R}^n$ be an $n$–dimensional hypercube. Then there exists a simplicial partition $\mathcal{P}$ of $Q$ in $n$-dimensional simplices which induces isometric simplicial partitions in $(n-1)$-dimensional simplices on the facets of $Q$.

Proof. We proceed by induction on the dimension $n$. If $n = 2$, then $Q$ is a square, and we can choose $\mathcal{P}$ to be the partition made up of the four simplices obtained by connecting the center of the square to each of the four vertices.

Let $n \geq 3$. Since the facets of an $n$–cube are $(n-1)$–cubes, we can apply the inductive hypothesis to a facet $F$ of $Q$ to obtain a simplicial partition of it. We replicate this same partition on each facet of $Q$. This gives a simplicial partition of the boundary $\partial Q$. We now pick a point in the interior of $Q$ and connect it through a segment to each vertex of every simplex in the aforementioned partition of $\partial Q$. This gives a simplicial partition of $Q$ with the required property.

This allows us to prove the lemma stated above.

Proof of Lemma 3.3. Let $u \in C^1(S^{n-1})$ and consider its 1-homogeneous extension $\tilde{u}$ to $\mathbb{R}^n$:

$$\tilde{u}(x) := \|x\| \cdot u \left( \frac{x}{\|x\|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}, \quad \tilde{u}(0) = 0.$$ 

Then $\tilde{u} \in C^1(\mathbb{R}^n \setminus \{0\})$. In particular, $\tilde{u} \in C^1(D)$, where

$$D = \{ x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq \sqrt{n} \}.$$ 

Fix $\varepsilon > 0$. Since $\tilde{u}$ and its Euclidean gradient $\nabla_e \tilde{u}$ are uniformly continuous on $D$, there exists $\delta > 0$ such that for every $x, y \in D$ with $\|x - y\| \leq \delta$ we have

$$|\tilde{u}(x) - \tilde{u}(y)| < \varepsilon \quad (3.3)$$

and

$$\|\nabla_e \tilde{u}(x) - \nabla_e \tilde{u}(y)\| < \varepsilon. \quad (3.4)$$

Consider now the hypercube $\Omega = [-1,1]^n$ centered at the origin with edge of length 2. For each coordinate axis, we draw hyperplanes orthogonal to such axis so that $\Omega$ is cut into hypercubes with edges of length $\frac{1}{N}$, where $N = \left\lceil \frac{\sqrt{n}}{\delta} \right\rceil$ (\lceil \cdot \rceil denotes the ceiling function). Note that these hypercubes all have the same diameter $d \leq \delta$. 

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We apply Remark 3.4 to the facets of these hypercubes (which are \((n - 1)\)-dimensional hypercubes) that are contained in \(\partial \Omega\): this determines a simplicial partition \(\{\Delta_1, \ldots, \Delta_m\}\) in \((n - 1)\)-simplices of \(\partial \Omega\). For every \(i = 1, \ldots, m\), let
\[
C_i = \{tx : t \geq 0, x \in \Delta_i\}.
\]
Then \(C_1, \ldots, C_m\) are closed convex cones with pairwise disjoint interiors, which form a partition of the whole space \(\mathbb{R}^n\). Note that since the annulus \(D\) contains all the simplices \(\Delta_i\) and \(d \leq \delta\), formulas (3.3) and (3.4) are satisfied for every \(x\) and \(y\) belonging to the same simplex.

We consider linear maps \(L_i : C_i \to \mathbb{R}, i = 1, \ldots, m\), such that \(L_i\) coincides with \(\bar{u}\) on each of the \(n\) vertices of \(\Delta_i\); these maps are uniquely determined. Let \(f \in C(\mathbb{R}^n)\) be the continuous function such that \(f = L_i\) on \(C_i\), for \(i = 1, \ldots, m\), and define \(\psi = \bar{u} - f\).

For a fixed \(x \in D\), we have that \(x \in C_k\) for some \(k \in \{1, \ldots, m\}\), and we can write \(x = \lambda x'\), with \(\lambda > 0\) and \(x' \in \Delta_k\). Choose an arbitrary vertex \(v\) of \(\Delta_k\). Since \(\Delta_k\) is compact, there is a \(w \in \Delta_k\) such that
\[
|L_k(v) - L_k(w)| = \max_{z \in \Delta_k} |L_k(v) - L_k(z)|.
\]
Since convex functions on convex polytopes attain their maximum at the vertices, \(w\) must be a vertex of \(\Delta_k\).

Given that both \(\bar{u}\) and \(f\) are 1-homogeneous, using the triangular inequality, (3.3) and the fact that \(\bar{u} = L_k\) on the vertices of \(\Delta_k\), we get
\[
|\psi(x)| = \lambda|\bar{u}(x') - f(x')| = \lambda|\bar{u}(x') - L_k(x')| < \lambda \varepsilon + \lambda|L_k(v) - L_k(x')| \\
\leq \lambda \varepsilon + \lambda|L_k(v) - L_k(w)| = \lambda \varepsilon + \lambda|\bar{u}(v) - \bar{u}(w)| < 2\lambda \varepsilon = 2\varepsilon \left\| \frac{x}{\|x\|} \right\| \varepsilon \\
\leq 2\sqrt{n} \varepsilon.
\]
Therefore,
\[
\|\psi\|_{\infty, D} < 2\sqrt{n} \varepsilon, \quad (3.5)
\]
where \(\| \cdot \|_{\infty, D}\) denotes the uniform norm on \(D\).

We now turn to the gradient \(\nabla e \psi\). Fix arbitrarily \(k \in \{1, \ldots, m\}\) and \(x \in \text{relint}(\Delta_k)\), the relative interior of \(\Delta_k\) (i.e. the interior of \(\Delta_k\) as a subset of an \((n - 1)\)-dimensional affine hyperplane of \(\mathbb{R}^n\)). Then \(\nabla e \psi(x)\) exists. We choose a vertex of \(\Delta_k\) and consider the \(n - 1\) edges \(l_1, \ldots, l_{n-1}\) incident to it; since \(\Delta_k\) is a simplex, these edges lie on linearly independent directions \(\nu_1, \ldots, \nu_{n-1}\). Note that the restriction of \(\psi\) to each \(l_i\) can be seen as a function of one variable which is continuous on \(l_i\), differentiable in its relative interior and satisfies \(\psi = 0\) at the ends of \(l_i\). Hence there exist points \(z_i \in l_i\) such that
\[
\frac{\partial \psi}{\partial \nu_i}(z_i) = 0,
\]
for every \(i = 1, \ldots, n - 1\). Using the fact that \(f|_{C_k}\) is linear and the Cauchy-Schwarz inequality, for every \(i = 1, \ldots, n - 1\) we get
\[
\left| \frac{\partial \psi}{\partial \nu_i}(x) \right| = \left| \frac{\partial \psi}{\partial \nu_i}(x) - \frac{\partial \psi}{\partial \nu_i}(z_i) \right| = \left| \frac{\partial \bar{u}}{\partial \nu_i}(x) - \frac{\partial \bar{u}}{\partial \nu_i}(z_i) \right| + \left| f(\nu_i) \right| \\
= \left| \langle \nabla e \bar{u}(x) - \nabla e \bar{u}(z_i), \nu_i \rangle \right| \leq \|\nabla e \bar{u}(x) - \nabla e \bar{u}(z_i)\| \varepsilon, \quad (3.6)
\]
where the last inequality follows from (3.3).

Let \(H\) be the hyperplane passing through the \(n\) vertices of \(\Delta_k\). Since \(\Delta_k \subseteq \partial \Omega\), the exterior unit normal vector \(N\) to \(H\) is of the form \(N = \pm e_{i_k}\), for some \(i_k \in \{1, \ldots, n\}\); for the sake of simplicity, let us assume \(N = e_{i_k}\), since the general case can be dealt with analogously. We now observe that both \(\{\nu_1, \ldots, \nu_{n-1}\}\) and \(\{e_1, \ldots, e_{n-1}\}\) are bases of \(H\); in particular, there exist numbers \(\alpha_{ij} \in \mathbb{R}\), \(i, j = 1, \ldots, n - 1\), such that
\[
e_i = \sum_{j=1}^{n-1} \alpha_{ij} \nu_j.
\]
for every $i = 1, \ldots, n-1$. Therefore, for $i = 1, \ldots, n-1$ we have
\[
\left| \frac{\partial \psi}{\partial x_i}(x) \right| = |\langle \nabla_e \psi(x), e_i \rangle| \leq \sum_{j=1}^{n-1} |\alpha_{ij}| \cdot \left| \frac{\partial \psi}{\partial \nu_j}(x) \right| < M \varepsilon, \tag{3.7}
\]
where we have used (3.6) and we have set
\[
M = \max_{i \in \{1, \ldots, n-1\}} \sum_{j=1}^{n-1} |\alpha_{ij}|.
\]
Note that the $\alpha_{ij}$'s, and thus $M$, do not depend on $\varepsilon$, since they are determined by the $\nu_i$'s, which are in turn determined by the simplicial partitions of the $(n-1)$-cubes contained in $\partial \Omega$; the length $\frac{1}{N}$ of the edge of such cubes depends on $\varepsilon$, but the angles appearing in the aforementioned partitions, hence the $\nu_i$'s, do not.

We now write
\[
\nabla_e \psi(x) = \langle \nabla_e \psi(x), e_1 \rangle + \ldots + \langle \nabla_e \psi(x), e_{n-1} \rangle e_{n-1} + \langle \nabla_e \psi(x), e_n \rangle e_n,
\]
which, together with (3.7), implies
\[
\| \nabla_e \psi(x) \| < M(n-1) \varepsilon + \left| \frac{\partial \psi}{\partial x_n}(x) \right|. \tag{3.8}
\]
Let us consider the radial direction $r_x = \frac{x}{\|x\|}$. We have
\[
\frac{\partial \psi}{\partial r_x}(x) = \langle \nabla_e \psi(x), r_x \rangle = \frac{\psi(x)}{\|x\|},
\]
thanks to Euler’s formula for homogeneous functions. This yields
\[
\left| \frac{\partial \psi}{\partial r_x}(x) \right| = \frac{|\psi(x)|}{\|x\|} \leq |\psi(x)| < 2 \sqrt{n} \varepsilon, \tag{3.9}
\]
because of (3.5). On the other hand, $r_x = \langle r_x, e_1 \rangle e_1 + \ldots + \langle r_x, e_n \rangle e_n$, hence
\[
\frac{\partial \psi}{\partial r_x}(x) = \langle \nabla_e \psi(x), r_x \rangle = \sum_{i=1}^{n-1} \left[ \langle r_x, e_i \rangle \cdot \frac{\partial \psi}{\partial x_i}(x) \right] + \langle r_x, e_n \rangle \cdot \frac{\partial \psi}{\partial x_n}(x),
\]
so that
\[
|\langle r_x, e_n \rangle| \cdot \left| \frac{\partial \psi}{\partial x_n}(x) \right| \leq \left| \frac{\partial \psi}{\partial r_x}(x) \right| + \sum_{i=1}^{n-1} |\langle r_x, e_i \rangle| \cdot \left| \frac{\partial \psi}{\partial x_i}(x) \right| < 2 \sqrt{n} \varepsilon + \sum_{i=1}^{n-1} M \varepsilon = [2 \sqrt{n} + M(n-1)] \varepsilon,
\]
where we have used (3.9), (3.7) and the Cauchy-Schwarz inequality. But since
\[
|\langle r_x, e_n \rangle| = |\langle r_x, N \rangle| = \frac{1}{\|x\|} \geq \frac{1}{\sqrt{n}},
\]
we obtain
\[
\left| \frac{\partial \psi}{\partial x_n}(x) \right| < \left[ 2n + M \sqrt{n}(n-1) \right] \varepsilon.
\]
From (3.8) we conclude that
\[
\| \nabla_e \psi(x) \| < C \varepsilon, \tag{3.10}
\]
where
\[ C = 2n + M(\sqrt{n} + 1)(n - 1). \]

Formula (3.10) holds for every \( x \in \text{relint}(\Delta_k) \). By 0-homogeneity of \( \nabla \psi \), this extends to every \( x \) in the interior of \( C_k \). Since \( k \in \{1, \ldots, m\} \) was arbitrary, using (25) we have that
\[ \|\nabla \psi(x)\| < C\varepsilon, \]
for \( H^{n-1}\)-a.e. \( x \in S^{n-1} \).

In particular, we have proved that for every \( i \in \mathbb{N} \) we can find a piecewise linear function \( f_i \in \mathcal{L}(S^{n-1}) \) such that
\[ \|u - f_i\|_\infty < \frac{2\sqrt{n}}{i}, \]
and
\[ \|\nabla u(x) - \nabla f_i(x)\| < \frac{C}{i}, \]
for \( H^{n-1}\)-a.e. \( x \in S^{n-1} \). Therefore, \( f_i \rightarrow u \) uniformly on \( S^{n-1} \) and \( \nabla f_i \rightarrow \nabla u \) \( H^{n-1}\)-a.e. in \( S^{n-1} \). Besides, for every \( i \in \mathbb{N} \) and \( H^{n-1}\)-a.e. \( x \in S^{n-1} \) we have
\[ \|\nabla f_i(x)\| < \frac{C}{i} + \|\nabla u(x)\| \leq C + \max_{S^{n-1}} \|\nabla u\|, \]
so that \( f_i \rightharpoonup u \). \( \Box \)

### 3.3 Approximation of Lipschitz functions by \( C^1 \) functions

Functions in \( C^1(S^{n-1}) \) can in turn be used to \( \tau \)–approximate Lipschitz functions defined on the sphere.

**Lemma 3.5.** Let \( u \in \text{Lip}(S^{n-1}) \). Then there exists a sequence \( \{u_i\} \subseteq C^1(S^{n-1}) \) such that \( u_i \rightharpoonup u \).

**Proof.** Let \( u \in \text{Lip}(S^{n-1}) \) be a Lipschitz function with Lipschitz constant \( L \). As stated in Theorem 2.4, such a function can be extended to a map \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by
\[ u(x) = \max_{z \in S^{n-1}} [u(z) - L\|x - z\|], \]
for \( x \in \mathbb{R}^n \), which is still Lipschitz continuous with the same Lipschitz constant \( L \). To simplify the notations, the extension will still be denoted by the same symbol \( u \).

Consider the annuli
\[ C_0 = \left\{ x \in \mathbb{R}^n : \frac{1}{3} \leq \|x\| \leq \frac{5}{3} \right\}, \]
and
\[ C_1 = \left\{ x \in \mathbb{R}^n : \frac{2}{3} \leq \|x\| \leq \frac{4}{3} \right\}, \]
and let \( \eta : \mathbb{R}^n \rightarrow [0, 1] \) be a \( C^\infty \) function with \( \text{supp}(\eta) \subseteq C_0 \) such that \( \eta \equiv 1 \) on \( C_1 \).

The function \( \tilde{u} := u \cdot \eta \) is Lipschitz continuous; let \( L \) be its Lipschitz constant.

We will use mollifiers to build our approximating sequence. Consider \( \varphi : \mathbb{R}^n \rightarrow [0, +\infty) \) defined, for \( z \in \mathbb{R}^n \), by
\[ \varphi(z) = \begin{cases} c \cdot e^{-\frac{1}{\|z\|^{n-1}}} & \text{if } \|z\| < 1, \\ 0 & \text{if } \|z\| \geq 1, \end{cases} \]
where \( c > 0 \) is a constant such that
\[ \int_{\mathbb{R}^n} \varphi(z)dz = 1. \]

The function \( \varphi \) is \( C^\infty \) with support \( \overline{B_1(0)} \). For \( \varepsilon > 0 \), we set
\[ \varphi_\varepsilon(z) = \frac{1}{\varepsilon^n} \varphi\left( \frac{z}{\varepsilon} \right), \ z \in \mathbb{R}^n. \]

Note that, for every \( \varepsilon > 0 \), \( \varphi_\varepsilon \in C^\infty(\mathbb{R}^n) \), its support is the set \( \overline{B_\varepsilon(0)} \) and the following properties hold:
\[ \int_{\mathbb{R}^n} \varphi_\varepsilon(z) dz = 1; \]

\[ \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon(z) dz = 0, \text{ for every } \delta > 0. \]

We now consider the \( C^\infty \) functions

\[ \tilde{u}_\varepsilon(x) = \varphi_\varepsilon \ast \tilde{u}(x) = \int_{\mathbb{R}^n} \tilde{u}(x - y) \varphi_\varepsilon(y) dy, \ x \in \mathbb{R}^n. \]

For \( x, y \in \mathbb{R}^n \), we have

\[ |\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)| = \left| \int_{\mathbb{R}^n} \varphi_\varepsilon(z)[\tilde{u}(x - z) - \tilde{u}(y - z)] dz \right| \leq \tilde{L} \|x - y\|. \]

Remembering (2.6), this yields

\[ \|\nabla \tilde{u}_\varepsilon(x)\| \leq \sqrt{n} \cdot \tilde{L}, \quad (3.11) \]

for every \( x \in S^{n-1} \).

We also have that \( \tilde{u}_\varepsilon \to u \) uniformly on \( S^{n-1} \), as \( \varepsilon \to 0 \). Indeed, since \( \tilde{u} \) is uniformly continuous, for a fixed \( \varepsilon' > 0 \) there is a \( \delta > 0 \) such that

\[ |\tilde{u}(z_1) - \tilde{u}(z_2)| < \frac{\varepsilon'}{2}, \]

for every \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( \|z_1 - z_2\| \leq \delta \). Therefore, for \( x \in S^{n-1} \) we get

\[ |\tilde{u}_\varepsilon(x) - u(x)| = \left| \int_{\mathbb{R}^n} \tilde{u}(x - y) \varphi_\varepsilon(y) dy - \int_{\mathbb{R}^n} \tilde{u}(x) \varphi_\varepsilon(y) dy \right| \leq \int_{\mathbb{R}^n} \varphi_\varepsilon(y)|\tilde{u}(x - y) - \tilde{u}(x)|dy < \frac{\varepsilon'}{2} + \int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon(y)|\tilde{u}(x - y) - \tilde{u}(x)|dy. \]

Since

\[ \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon(y) dy = 0, \]

there exists \( 0 < \varepsilon_0 < 1 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \) we have that, for \( x \in S^{n-1} \),

\[ |\tilde{u}_\varepsilon(x) - u(x)| < \frac{\varepsilon'}{2} + \int_{B_1(0) \setminus B_\delta(0)} \varphi_\varepsilon(y)|\tilde{u}(x - y) - \tilde{u}(x)|dy \leq \frac{\varepsilon'}{2} + 2M \int_{B_1(0) \setminus B_\delta(0)} \varphi_\varepsilon(y) dy = \frac{\varepsilon'}{2} + 2M \int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon(y) dy < \varepsilon', \]

where

\[ M = \max_{z \in B_\delta(0)} |\tilde{u}(z)|. \]

This proves that \( \tilde{u}_\varepsilon \to u \) uniformly on \( S^{n-1} \).

To show the \( H^{n-1} \)-a.e. convergence of the gradients, for an arbitrary \( k \in \{1, \ldots, n\} \) we evaluate, for \( x \in \mathbb{R}^n \),

\[ \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(x) = \lim_{h \to 0} \frac{\tilde{u}_\varepsilon(x + he_k) - \tilde{u}_\varepsilon(x)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \frac{\tilde{u}(x - y + he_k) - \tilde{u}(x - y)}{h} dy \]

\[ = \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \frac{\partial \tilde{u}}{\partial x_k}(x - y) dy = \varphi_\varepsilon \ast \frac{\partial \tilde{u}}{\partial x_k}(x), \]

where we have used the Dominated Convergence Theorem, which can be applied because of the Lipschitz continuity of \( \tilde{u} \) and the Lebesgue integrability of \( \varphi_\varepsilon \). The Lipschitz continuity of \( \tilde{u} \), together with the fact that \( \eta \) is compactly supported, also implies \( \frac{\partial \tilde{u}}{\partial x_k} \in L^1(\mathbb{R}^n) \). The properties of the \( \varphi_\varepsilon \)'s then guarantee that

\[ \frac{\partial \tilde{u}_\varepsilon}{\partial x_k} = \varphi_\varepsilon \ast \frac{\partial \tilde{u}}{\partial x_k} \to \varphi_\varepsilon \ast \frac{\partial \tilde{u}}{\partial x_k} \text{ in } L^1(\mathbb{R}^n), \text{ as } \varepsilon \to 0^+. \]
for \( k = 1, \ldots, n \).

We now turn the family \( \{ \tilde{u}_\varepsilon \}_{\varepsilon > 0} \) into a sequence \( \{ \tilde{u}_i \}_{i \in \mathbb{N}} \) by choosing \( \varepsilon = 1/i \), \( i \in \mathbb{N} \), and renaming \( \tilde{u}_i := \tilde{u}_{i/\varepsilon} \) for the sake of simplicity. Every sequence of functions which converges in \( L^1 \) possesses a subsequence which converges a.e. to the same limit, hence there is a subsequence

\[
\left\{ \tilde{u}_{i_j}^{(1)} \right\}_{j \in \mathbb{N}} \subseteq \{ \tilde{u}_i \}_{i \in \mathbb{N}}
\]

such that

\[
\frac{\partial \tilde{u}_{i_j}^{(1)}}{\partial x_1}(x) \to \frac{\partial \tilde{u}}{\partial x_1}(x) \quad \text{as} \quad j \to \infty,
\]

for a.e. \( x \in \mathbb{R}^n \), and

\[
\frac{\partial \tilde{u}_{i_j}^{(1)}}{\partial x_k}(x) \to \frac{\partial \tilde{u}}{\partial x_k} \quad \text{in} \quad L^1(\mathbb{R}^n) \quad \text{as} \quad j \to \infty,
\]

for \( k = 2, \ldots, n \). We repeat the process for every \( 2 \leq j \leq n \), finding sequences

\[
\left\{ \tilde{u}_{i_j}^{(n)} \right\}_{j \in \mathbb{N}} \subseteq \left\{ \tilde{u}_{i_j}^{(n-1)} \right\}_{j \in \mathbb{N}} \subseteq \ldots \subseteq \left\{ \tilde{u}_{i_j}^{(1)} \right\}_{j \in \mathbb{N}} \subseteq \{ \tilde{u}_i \}_{i \in \mathbb{N}}.
\]

If we set \( u_j := \tilde{u}_{i_j}^{(n)} \), \( j \in \mathbb{N} \), we have that

\[
\frac{\partial u_j}{\partial x_k}(x) \to \frac{\partial \tilde{u}}{\partial x_k}(x) \quad \text{as} \quad j \to \infty,
\]

for a.e. \( x \in \mathbb{R}^n \) and for all \( k = 1, \ldots, n \). This implies

\[
\nabla u_j(x) \to \nabla u(x) \quad \text{as} \quad j \to \infty,
\]

(3.12) for \( H^{n-1}\text{-a.e.} \ x \in S^{n-1} \), using (2.5) again.

Recalling that \( \{ u_j \}_{j \in \mathbb{N}} \subseteq \{ \tilde{u}_\varepsilon \}_{\varepsilon > 0} \), from the fact that \( \tilde{u}_\varepsilon \to u \) uniformly on \( S^{n-1} \) and the properties (3.11), (3.12) we conclude that \( u_j \to u \).

We have actually proved that \( C^\infty(S^{n-1}) \) is \( \tau \)-dense in \( \text{Lip}(S^{n-1}) \). However, for our purposes Lemma 3.5 will be sufficient.

### 3.4 Density of \( \mathcal{L}(S^{n-1}) \) in \( \text{Lip}(S^{n-1}) \)

Putting things together, we obtain the following density result.

**Lemma 3.6.** The space \( \mathcal{L}(S^{n-1}) \) is \( \tau \)-dense in \( \text{Lip}(S^{n-1}) \).

**Proof.** We have already noted that \( \mathcal{H}(S^{n-1}) \subseteq \text{Lip}(S^{n-1}) \), and since \( \text{Lip}(S^{n-1}) \) is closed with respect to the pointwise minimum, from Lemma 3.2 we have that \( \mathcal{L}(S^{n-1}) \subseteq \mathcal{H}(S^{n-1}) \subseteq \text{Lip}(S^{n-1}) \).

Let \( u \in \text{Lip}(S^{n-1}) \). Because of Lemma 3.5 there is a sequence \( \{ u_i \} \subseteq C^1(S^{n-1}) \) such that \( u_i \to u \). If \( U \subseteq \text{Lip}(S^{n-1}) \) is an open neighbourhood of \( u \) (with respect to \( \tau \)), then there exists \( I \in \mathbb{N} \) such that \( u_i \in U \) for every \( i \geq I \). Now, for every fixed \( i \geq I \) we can apply Lemma 3.3 to get a sequence \( \{ f^i_j \}_{j \in \mathbb{N}} \subseteq \mathcal{L}(S^{n-1}) \) such that \( f^i_j \to u_i \), as \( j \to \infty \). Since \( U \) is an open neighbourhood of \( u_i \), for every \( i \geq I \) there is a \( J_i \in \mathbb{N} \) such that \( f^i_j \in U \) for every \( j \geq J_i \); if \( J_i \leq J_{i-1} \), we replace \( J_i \) by \( J_{i-1} + J_i \), so that \( \{ J_i \}_{i \in \mathbb{N}} \) is a strictly increasing sequence.

The sequence \( \{ f^i_{J_i} \}_{i \in \mathbb{N}} \subseteq \mathcal{L}(S^{n-1}) \) satisfies \( f^i_{J_i} \to u \), as \( i \to \infty \). \( \square \)

The tools developed throughout this section allow us now to prove Proposition 3.1 whom the proof of Theorem 1.2 is based upon.
We obtain that $u$ is continuous with respect to the Hausdorff metric. From Theorem 2.1 we obtain continuous and translation invariant valuations $\nu$. Actually, $\nu$ is homogeneous, for $i = 0, \ldots, n$, and

$$
\mu(\lambda u) = \sum_{i=0}^{n} \lambda^i \mu_i(u),
$$

for every $u \in \text{Lip}(S^{n-1})$ and $\lambda \geq 0$.

**Proof.** Let $\mu : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be as in the hypothesis. Consider the map $\nu : \mathcal{K}^{n} \to \mathbb{R}$ defined by $\nu(K) = \mu(h_K)$, for $K \in \mathcal{K}^{n}$; because of Lemma 2.8 this is a valuation on $\mathcal{K}^{n}$ which is translation invariant and continuous with respect to the Hausdorff metric. From Theorem 2.1 we obtain continuous and translation invariant valuations $\nu_0, \ldots, \nu_n : \mathcal{K}^{n} \to \mathbb{R}$ such that each $\nu_i$ is $i$--homogeneous and (2.2) holds.

Define now $\mu_i : \mathcal{H}(S^{n-1}) \to \mathbb{R}$, $i = 0, \ldots, n$, by setting $\mu_i(h_K) = \nu_i(K)$, for $h_K \in \mathcal{H}(S^{n-1})$. Reading McMullen’s formula (2.2) in the support functions’ setting, we have that, for every $h_K \in \mathcal{H}(S^{n-1})$ and $\lambda \geq 0$,

$$
\mu(\lambda h_K) = \nu(\lambda K) = \sum_{i=0}^{n} \lambda^i \nu_i(K) = \sum_{i=0}^{n} \lambda^i \mu_i(h_K).
$$

This is the desired decomposition formula stated for support functions; we would now like to extend it to all Lipschitz functions $u \in \text{Lip}(S^{n-1})$. To do that, we must first extend each $\mu_i$ to $\text{Lip}(S^{n-1})$.

In what follows, for simplicity we write $h$ to denote a generic element of $\mathcal{H}(S^{n-1})$. We write (4.2) for an arbitrary $h$ and for $\lambda = k = 1, \ldots, n + 1$:

$$
\mu(kh) = \sum_{i=0}^{n} k^i \mu_i(h).
$$

We see it as a system of $n + 1$ equations in the $n + 1$ unknowns $\mu_0(h), \mu_1(h), \ldots, \mu_n(h)$. The matrix associated with this system is

$$
M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2^2 & \cdots & 2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & n & n^2 & \cdots & n^n \\
1 + n & (n + 1)^2 & \cdots & (n + 1)^n
\end{pmatrix},
$$

which is a Vandermonde matrix, hence

$$
\det M = \prod_{1 \leq i < j \leq n+1} (j - i) \neq 0,
$$

# 4 The homogeneous decomposition for continuous and dot product invariant valuations

This section is devoted to the proof of the following result.

**Theorem 4.1.** Let $\mu : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be a continuous and dot product invariant valuation. Then there exist continuous and dot product invariant valuations $\mu_0, \ldots, \mu_n : \text{Lip}(S^{n-1}) \to \mathbb{R}$ such that $\mu_i$ is $i$--homogeneous, for $i = 0, \ldots, n$, and

$$
\mu(\lambda u) = \sum_{i=0}^{n} \lambda^i \mu_i(u),
$$

for every $u \in \text{Lip}(S^{n-1})$ and $\lambda \geq 0$. 

**Proof.** Let $\mu : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be as in the hypothesis. Consider the map $\nu : \mathcal{K}^{n} \to \mathbb{R}$ defined by $\nu(K) = \mu(h_K)$, for $K \in \mathcal{K}^{n}$; because of Lemma 2.8 this is a valuation on $\mathcal{K}^{n}$ which is translation invariant and continuous with respect to the Hausdorff metric. From Theorem 2.1 we obtain continuous and translation invariant valuations $\nu_0, \ldots, \nu_n : \mathcal{K}^{n} \to \mathbb{R}$ such that each $\nu_i$ is $i$--homogeneous and (2.2) holds.

Define now $\mu_i : \mathcal{H}(S^{n-1}) \to \mathbb{R}$, $i = 0, \ldots, n$, by setting $\mu_i(h_K) = \nu_i(K)$, for $h_K \in \mathcal{H}(S^{n-1})$. Reading McMullen’s formula (2.2) in the support functions’ setting, we have that, for every $h_K \in \mathcal{H}(S^{n-1})$ and $\lambda \geq 0$,

$$
\mu(\lambda h_K) = \nu(\lambda K) = \sum_{i=0}^{n} \lambda^i \nu_i(K) = \sum_{i=0}^{n} \lambda^i \mu_i(h_K).
$$

This is the desired decomposition formula stated for support functions; we would now like to extend it to all Lipschitz functions $u \in \text{Lip}(S^{n-1})$. To do that, we must first extend each $\mu_i$ to $\text{Lip}(S^{n-1})$.

In what follows, for simplicity we write $h$ to denote a generic element of $\mathcal{H}(S^{n-1})$. We write (4.2) for an arbitrary $h$ and for $\lambda = k = 1, \ldots, n + 1$:

$$
\mu(kh) = \sum_{i=0}^{n} k^i \mu_i(h).
$$

We see it as a system of $n + 1$ equations in the $n + 1$ unknowns $\mu_0(h), \mu_1(h), \ldots, \mu_n(h)$. The matrix associated with this system is

$$
M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2^2 & \cdots & 2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & n & n^2 & \cdots & n^n \\
1 + n & (n + 1)^2 & \cdots & (n + 1)^n
\end{pmatrix},
$$

which is a Vandermonde matrix, hence

$$
\det M = \prod_{1 \leq i < j \leq n+1} (j - i) \neq 0,
$$
and then $M$ is nonsingular. Therefore, the system (4.3) is invertible and we can find coefficients $a_{ij}$, $i = 0, \ldots, n$, $j = 1, \ldots, n + 1$, such that

$$\mu_i(h) = \sum_{j=1}^{n+1} a_{ij} \mu(jh); \quad (4.4)$$

note that the coefficients are independent of $h$. This allows us to extend the $\mu_i$’s to $\text{Lip}(S^{n-1})$: for $i = 0, \ldots, n$ we set

$$\mu_i(u) := \sum_{j=1}^{n+1} a_{ij} \mu(ju), \quad (4.5)$$

for every $u \in \text{Lip}(S^{n-1})$. We observe that for every $j \in \{1, \ldots, n + 1\}$, the function defined on $\text{Lip}(S^{n-1})$ by $\mu(ju)$ inherits all the properties of $\mu$, i.e. it is a continuous and dot product invariant valuation on $\text{Lip}(S^{n-1})$ as well.

Let

$$\bar{\mu} = \sum_{i=0}^{n} \mu_i.$$

By (4.2) and (4.4), $\mu$ and $\bar{\mu}$ coincide on $\mathcal{H}(S^{n-1})$; hence, by Proposition 5.1, $\mu = \bar{\mu}$ on $\text{Lip}(S^{n-1})$.

It remains to be shown that $\mu_i$ is $i$-homogeneous on $\text{Lip}(S^{n-1})$. Let $\lambda \geq 0$, and define $\mu', \mu''$: $\text{Lip}(S^{n-1}) \to \mathbb{R}$ by

$$\mu'(u) = \mu_i(\lambda u), \quad \mu''(u) = \lambda^i \mu_i(u).$$

These are continuous valuations and as they coincide on $\mathcal{H}(S^{n-1})$, they coincide on $\text{Lip}(S^{n-1})$. This proves that $\mu_i$ is $i$-homogeneous. \hfill \Box

5 Characterisation of dot product and rotation invariant valuations

5.1 Dot product invariant valuations on $\text{Lip}(S^{n-1})$

The proof of Theorem 1.2 requires this last result.

**Proposition 5.1.** Let $n \geq 3$ and $3 \leq k \leq n$. Let $\mu : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be a continuous, rotation invariant, dot product invariant and $k$–homogeneous valuation. Then $\mu \equiv 0$ on $\text{Lip}(S^{n-1})$.

To ease the reading, we have stated some of the steps of the proof of this result as lemmas. Their proofs are provided along the way.

**Proof.** Let $n$, $k$, and $\mu$ be as in the hypothesis. Define $\nu : K^n \to \mathbb{R}$ by setting

$$\nu(K) = \mu(h_K),$$

for $K \in K^n$. The functional $\nu$ is a $k$–homogeneous, translation and rotation invariant valuation which is continuous with respect to the Hausdorff metric, thanks to Lemma 2.8. From Theorem 2.3 we have that there exists a constant $c \in \mathbb{R}$ such that

$$\mu(h_K) = \nu(K) = cV_k(K),$$

for every $K \in K^n$, where $V_k$ is the $k$th intrinsic volume.

If $c = 0$, then $\mu = 0$ on $\mathcal{H}(S^{n-1})$, and from Proposition 3.1 we have the assertion.

Suppose now $c \neq 0$. We will show that this leads to a contradiction. Since the functional $\frac{1}{c} \mu$ retains all of $\mu$’s properties, up to dividing by $c$ we can assume that

$$\mu(h_K) = V_k(K), \quad (5.1)$$

for every $K \in K^n$.

For $x \in \mathbb{R}^n$ we write $x = (\xi, \eta)$, with $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^{n-k}$. Fix $\zeta \in S^{k-1}$ and define $w_{\zeta} : \mathbb{R}^n \to \mathbb{R}$ by setting

$$w_{\zeta}(x) = w_{\zeta}(\xi, \eta) = \|\xi - (\xi, \zeta, \zeta)\|,$$
for \( x \in \mathbb{R}^n \). Consider the \((k-1)\)-dimensional disk in \( \mathbb{R}^n \) defined by

\[
D_\xi = \left\{ (\xi, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \langle \xi, \bar{\xi} \rangle = 0, \| \xi \| \leq 1 \right\}.
\]

The map \( u_\xi \) is the support function of \( D_\xi \). In fact, up to a change of coordinate system, we can assume \( \bar{\xi} = (1, 0, \ldots, 0) \); from the definition of support function, for every \((\xi, \eta) \in \mathbb{R}^n\) we have

\[
h_{D_\xi}(\xi, \eta) = \max_{(\xi', 0) \in D_\xi} \langle \xi, \xi' \rangle = \max_{(\xi', 0) \in D_\xi} \langle (\xi_2, \ldots, \xi_k), (\xi'_2, \ldots, \xi'_k) \rangle = \| (\xi_2, \ldots, \xi_k) \| = u_\xi(\xi, \eta).
\]

Define now \( v_\xi : \mathbb{R}^n \to \mathbb{R} \) by setting

\[
v_\xi(x) = v_\xi(\xi, \eta) = \langle \xi, \bar{\xi} \rangle,
\]

for \( x \in \mathbb{R}^n \); \( v_\xi \) is the support function of the convex body (in fact, a singleton) \( \{ (\bar{\xi}, 0) \} \).

For \( \lambda \geq 1 \), consider \( w_{\lambda, \xi} : S^{n-1} \to \mathbb{R} \) defined by

\[
w_{\lambda, \xi} = (\lambda u_\xi - v_\xi) \wedge \Omega,
\]

where \( \Omega \) denotes the function which is identically zero on \( S^{n-1} \). Note that \( w_{\lambda, \xi} = h_{\lambda D_\xi - (\bar{\xi}, 0) \wedge \Omega} \in \text{Lip}(S^{n-1}) \), being a minimum of Lipschitz functions. Therefore, \( \mu \) can be evaluated at \( w_{\lambda, \xi} \), and we do that in the following lemma.

**Lemma 5.2.** We have

\[
\mu(w_{\lambda, \xi}) = -\frac{\omega_{k-1}}{k} \lambda^{k-1},
\]

where \( \omega_{k-1} \) denotes the measure of the unit ball of \( \mathbb{R}^{k-1} \).

**Proof.** From the valuation property we get

\[
\mu(w_{\lambda, \xi}) = \mu((\lambda u_\xi - v_\xi) \wedge \Omega) = \mu(\lambda u_\xi - v_\xi) + \mu(\Omega) - \mu((\lambda u_\xi - v_\xi) \vee \Omega),
\]

since \( \mu(\Omega) = 0 \), because of the homogeneity.

As we have already pointed out, \( \lambda u_\xi - v_\xi = h_{\lambda D_\xi - (\bar{\xi}, 0)} \), and remembering (5.1) we obtain

\[
\mu(\lambda u_\xi - v_\xi) = V_k(\lambda D_\xi - (\bar{\xi}, 0)) = V_k(\lambda D_\xi) = \lambda^k V_k(D_\xi) = 0,
\]

where the last equality follows from the fact that \( D_\xi \) has dimension \( k-1 \).

Now, \( \lambda u_\xi - v_\xi \vee \Omega \) is the support function of \( \text{conv} \left( (\lambda D_\xi - (\bar{\xi}, 0)) \cup \{0\} \right) \) (see Lemma 2.6), which is a cone with vertex at the origin, base \( \lambda D_\xi - (\bar{\xi}, 0) \) and height 1, since \( \| \bar{\xi} \| = 1 \). From (5.2), (5.3) and (5.1) we get

\[
\mu(w_{\lambda, \xi}) = -\mu((\lambda u_\xi - v_\xi) \vee \Omega) = -V_k \left( \text{conv} \left( (\lambda D_\xi - (\bar{\xi}, 0)) \cup \{0\} \right) \right) = -\frac{\omega_{k-1}}{k} \lambda^{k-1}.
\]

The next lemma concerns the support set \( \text{supp}(w_{\lambda, \xi}) \) of the function \( w_{\lambda, \xi} \).

**Lemma 5.3.** For every \((\xi, 0) \in \text{supp}(w_{\lambda, \xi})\) we have

\[
\| \xi - \bar{\xi} \| < \frac{\sqrt{2}}{\lambda}.
\]
Proof. Like before, we assume $\xi = (1, 0, \ldots, 0)$. Thus, for every $(\xi, \eta) \in S^{n-1}$,

$$w_{\lambda, \xi}(\xi, \eta) = \left( \lambda \sqrt{\xi_2^2 + \cdots + \xi_k^2} - \xi_1 \right) \land 0.$$ 

If $(\xi, 0) \in \text{supp}(w_{\lambda, \xi})$, we have $\|\xi\| = 1$ and $\lambda \sqrt{\xi_2^2 + \cdots + \xi_k^2} - \xi_1 \leq 0$, hence

$$\sqrt{\xi_2^2 + \cdots + \xi_k^2} \leq \frac{\xi_1}{\lambda}. \quad \text{(5.4)}$$

In particular, this implies $\xi_1 \geq 0$.

We write $\xi = (\xi_1, \xi')$, with $\xi' \in \mathbb{R}^{k-1}$. Since $\|\xi\| = 1$ and $\xi_1 \geq 0$, we have $\xi_1 = \sqrt{1 - \|\xi'\|^2}$. Using this last equality in (5.4) we obtain

$$\|\xi'\| \leq \frac{\sqrt{1 - \|\xi'\|^2}}{\lambda},$$

which in turn gives

$$\|\xi'\|^2 \leq \frac{1}{1 + \lambda^2} < \frac{1}{\lambda^2}.$$

We can also estimate

$$|\xi_1 - 1| = 1 - \xi_1 = 1 - \sqrt{1 - \|\xi'\|^2} = \frac{\|\xi'\|^2}{1 + \sqrt{1 - \|\xi'\|^2}} \leq \|\xi'\|^2 < \frac{1}{\lambda^2}.$$

From these inequalities we get

$$\|\xi - \xi_1\|^2 = \|\xi - (1, 0, \ldots, 0)\|^2 = |\xi_1 - 1|^2 + \|\xi'\|^2 < \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \leq \frac{2}{\lambda^2},$$

since $\lambda \geq 1$. The assertion follows. \hfill \Box

This result yields the following one.

Lemma 5.4. For every $\xi_1, \xi_2 \in S^{k-1}$ such that $\|\xi_1 - \xi_2\| \geq \frac{4}{\lambda}$ we have

$$w_{\lambda, \xi_1} \cdot w_{\lambda, \xi_2} = \emptyset.$$

Proof. Take $\xi_1, \xi_2$ as in the hypothesis. Suppose the result to be false. Then there is a point $(\tilde{\xi}, \tilde{\eta}) \in S^{n-1}$ such that

$$w_{\lambda, \xi_1}(\tilde{\xi}, \tilde{\eta}) \cdot w_{\lambda, \xi_2}(\tilde{\xi}, \tilde{\eta}) \neq 0.$$

Note that $w_{\lambda, \xi_1}(0, \tilde{\eta}) = w_{\lambda, \xi_2}(0, \tilde{\eta}) = 0$, hence $\tilde{\xi} \neq 0$.

For $i = 1, 2$, the function

$$w_{\lambda, \xi_i}(\xi, \eta) = [\lambda \|\xi - (\xi, \xi_i)\| \cdot (\xi, \xi_i)] \land 0$$

is 1–homogeneous with respect to $\xi$, and since $w_{\lambda, \xi_1}(\tilde{\xi}, \tilde{\eta}) \neq 0$, we also have $w_{\lambda, \xi_1}(\tilde{\xi}, \tilde{\eta}) \neq 0$, where $\tilde{\xi} = \frac{\xi}{\|\xi\|}$.

This means that $(\tilde{\xi}, \tilde{\eta}) \in \text{supp}(w_{\lambda, \xi_1})$, hence $(\tilde{\xi}, 0) \in \text{supp}(w_{\lambda, \xi_1})$ too (since $w_{\lambda, \xi_i}$ does not depend on $\eta$), and from the previous lemma we have

$$\|\tilde{\xi} - \xi_i\| < \frac{\sqrt{2}}{\lambda},$$

for $i = 1, 2$. Therefore,

$$\|\xi_1 - \xi_2\| \leq \|\xi_1 - \tilde{\xi}\| + \|\tilde{\xi} - \xi_2\| < \frac{2\sqrt{2}}{\lambda},$$

which contradicts the hypothesis. \hfill \Box

Iterating, the previous result can be extended to any finite number of points.
Lemma 5.6. Let $N \in \mathbb{N}$ and $\xi_1, \ldots, \xi_N \in S^{k-1}$ be such that $\|\xi_i - \xi_j\| \geq \frac{1}{N}$, for every $i \neq j$. Then

$$w_\lambda, \xi_i \cdot w_\lambda, \xi_j = 0,$$

for every $i \neq j$.

We will need a couple more results. The first one concerns the behaviour of our valuation on non-positive orthogonal functions.

**Corollary 5.5.** Let $N \in \mathbb{N}$ and $u_1, \ldots, u_N \in \text{Lip}(S^{n-1})$. If $u_i \leq 0$ for every $i = 1, \ldots, N$ and $u_i \cdot u_j = \emptyset$ for $i \neq j$, then

$$\mu \left( \bigwedge_{i=1}^{N} u_i \right) = \sum_{i=1}^{N} \mu(u_i).$$

**Proof.** We proceed by induction on $N$. If $N = 1$ the formula is satisfied. Suppose the result to be true for $N-1$. From the valuation property, we get

$$\mu \left( \bigwedge_{i=1}^{N} u_i \right) = \mu \left( \bigwedge_{i=1}^{N-1} u_i \right) + \mu(u_N) - \mu \left( \left( \bigwedge_{i=1}^{N-1} u_i \right) \lor u_N \right) = \sum_{i=1}^{N} \mu(u_i) - \mu \left( \left( \bigwedge_{i=1}^{N-1} u_i \right) \lor u_N \right),$$

where the last equality follows from the inductive hypothesis.

Fix $x \in S^{n-1}$. If $u_N(x) = 0$, then

$$\left[ \left( \bigwedge_{i=1}^{N-1} u_i \right) \lor u_N \right](x) = 0. \quad (5.5)$$

If $u_N(x) < 0$, from the hypothesis we have $u_i(x) = 0$ for $i = 1, \ldots, N-1$. This implies $\bigwedge_{i=1}^{N-1} u_i(x) = 0$, hence $(5.5)$ follows once again. Since $x \in S^{n-1}$ was arbitrary,

$$\left( \bigwedge_{i=1}^{N-1} u_i \right) \lor u_N = \emptyset$$

and, being $\mu(\emptyset) = 0$, we are done. \hfill \Box

The next lemma allows us to find sufficiently many points on the unit sphere which are not too close to each other.

**Lemma 5.7.** Let $N \in \mathbb{N}$, $N \geq 2$. For every $\nu \in \mathbb{N}$ there are $N_\nu = \nu^{N-1}$ points $x_1, \ldots, x_{N_\nu} \in S^{n-1}$ such that

$$\|x_i - x_j\| \geq \frac{1}{\sqrt{N\nu}},$$

for $i \neq j$.

**Proof.** Fix $N \in \mathbb{N}$, $N \geq 2$, and take $\nu \in \mathbb{N}$. For $a = (a_1, \ldots, a_{N-1})$, with $a_1, \ldots, a_{N-1} \in \{0, 1, \ldots, \nu - 1\}$, we define

$$x'_a = \frac{1}{\sqrt{N}} \left( \frac{a_1}{\nu}, \ldots, \frac{a_{N-1}}{\nu} \right) \in \mathbb{R}^{n-1}.$$ 

These are $\nu^{N-1}$ points, and they satisfy

$$\|x'_a - x'_b\| \geq \frac{1}{\sqrt{N\nu}},$$

for every $a \neq b$. Moreover, $\|x'_a\| < 1$ for every $a$.

Consider now

$$x_a = (x'_a, \sqrt{1 - \|x'_a\|^2}) \in S^{n-1},$$

for $a = (a_1, \ldots, a_{N-1})$ with $a_1, \ldots, a_{N-1} \in \{0, 1, \ldots, \nu - 1\}$. These are $\nu^{N-1}$ points on the sphere, and we have

$$\|x_a - x_b\| \geq \|x'_a - x'_b\| \geq \frac{1}{\sqrt{N\nu}},$$

for every $a \neq b$. \hfill \Box
We will now use these results to build a sequence of Lipschitz functions which will yield the contradiction we are looking for. Choose \( N = k \) in the last lemma and take \( \nu \in \mathbb{N} \). Then we have \( N_\nu = \nu^{k-1} \) points \( x_1, \ldots, x_{N_\nu} \in S^{k-1} \) such that

\[
\|x_i - x_j\| \geq \frac{1}{\sqrt{k\nu}},
\]

for every \( i \neq j \). Let

\[
\lambda_\nu = 4\sqrt{k\nu};
\]

note that \( \lambda_\nu \geq 1 \). Since \( \frac{2k-2}{k} \geq \frac{4}{3} > 1 \), we can pick a number

\[
1 < p < \frac{2k-2}{k}
\]

and define the function \( \psi_\nu : S^{n-1} \to \mathbb{R} \),

\[
\psi_\nu = \frac{1}{\nu^p} \sum_{i=1}^{N_\nu} w_{\lambda_\nu, x_i}.
\]

From the \( k \)-homogeneity of \( \mu \), Lemma \( 5.6 \) (which can be applied because the fact that \( \|x_i - x_j\| \geq \frac{1}{\sqrt{k\nu}} \) allows us to use Corollary \( 5.3 \) and Lemma \( 5.2 \) we get

\[
\mu(\psi_\nu) = \frac{1}{\nu^{kp}} \sum_{i=1}^{N_\nu} \mu(w_{\lambda_\nu, x_i}) = \frac{1}{\nu^{kp}} \omega_{k-1} \lambda_\nu^{k-1} N_\nu = -c_k \nu^{2k-2-kp},
\]

where

\[
c_k = 4^{k-1} \omega_{k-1} \frac{k-3}{2} > 0.
\]

Given how \( p \) was chosen, \( 2k - 2 - kp > 0 \), hence

\[
\mu(\psi_\nu) \to -\infty \quad \text{(5.6)}
\]
as \( \nu \to \infty \).

We would now like to prove that \( \psi_\nu \to \emptyset \), as \( \nu \to \infty \).

For every \( i = 1, \ldots, N_\nu \) and \( (\xi, \eta) \in S^{n-1} \), from the triangular and Cauchy-Schwarz inequalities we have

\[
|w_{\lambda_\nu, x_i}(\xi, \eta)| = |\lambda_\nu u_{x_i}(\xi, \eta) - v_{x_i}(\xi, \eta)| \leq |\lambda_\nu u_{x_i}(\xi, \eta) - v_{x_i}(\xi, \eta)| + |\lambda_\nu u_{x_i}(\xi, \eta) - v_{x_i}(\xi, \eta)|
\]

where

\[
x_i \in S^{n-1} \text{. This yields } \|w_{\lambda_\nu, x_i}\| \leq 2\lambda_\nu + 1 \text{ and consequently}
\]

\[
\|\psi_\nu\| \leq \frac{2\lambda_\nu + 1}{\nu^p} = \frac{8\sqrt{k}}{\nu^{p-1}} + \frac{1}{\nu^p},
\]

and since \( p > 1 \), this implies that \( \psi_\nu \to \emptyset \) uniformly on \( S^{n-1} \) as \( \nu \to \infty \).

We now look for a uniform bound on \( L(\psi_\nu) \), the Lipschitz constant of \( \psi_\nu \). For \( i \in \{1, \ldots, N_\nu\} \), consider \( \bar{w}_{\lambda_\nu, x_i} = \lambda_\nu u_{x_i} - v_{x_i} \). For \( (\xi_1, \eta_1), (\xi_2, \eta_2) \in S^{n-1} \), consider

\[
|\bar{w}_{\lambda_\nu, x_i}(\xi_1, \eta_1) - \bar{w}_{\lambda_\nu, x_i}(\xi_2, \eta_2)| \leq \lambda_\nu |u_{x_i}(\xi_1, \eta_1) - u_{x_i}(\xi_2, \eta_2)| + |v_{x_i}(\xi_1, \eta_1) - v_{x_i}(\xi_2, \eta_2)|
\]

where

\[
\lambda_\nu |\xi_1 - \xi_2 - \langle \xi_1 - \xi_2, x_i\rangle| + |\langle \xi_1 - \xi_2, x_i\rangle|
\]

\[
\leq \lambda_\nu |\xi_1 - \xi_2| + |\xi_1 - \xi_2| \cdot |x_i| + |\xi_1 - \xi_2| \cdot |x_i|,
\]

\[
\leq (2\lambda_\nu + 1)|\xi_1 - \xi_2| + (2\lambda_\nu + 1)|\xi_1 - \xi_2| \cdot |x_i| + |\xi_1 - \xi_2| \cdot |x_i|.
\]

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This yields
\[ L(w_{\lambda, x_i}) \leq L(\tilde{w}_{\lambda, x_i}) \leq 2\lambda_i + 1. \] (5.7)
We prove that
\[ |\psi(x) - \psi(y)| \leq \frac{2(8\sqrt{k} + 1)}{\nu^{p-1}} \|x - y\|, \] (5.8)
for every \( x, y \in S^{n-1} \). Fix \( x, y \in S^{n-1} \). If \( \psi(x) = \psi(y) = 0 \), then we are done. Without loss of generality, suppose now \( \psi(x) \neq 0 \), and let \( j \in \{1, \ldots, N_\nu\} \) be such that
\[ \psi(x) = \frac{1}{\nu^p} w_{\lambda, x_j}(x). \]
If \( w_{\lambda, x_j}(y) = 0 \) for every \( i \neq j \), then
\[ \psi(y) = \frac{1}{\nu^p} w_{\lambda, x_j}(y), \]
being \( w_{\lambda, x_j}(y) \leq 0 \). Thus, from (5.7) we get
\[ |\psi(x) - \psi(y)| = \frac{|w_{\lambda, x_j}(x) - w_{\lambda, x_j}(y)|}{\nu^p} \leq \frac{2\lambda_i + 1}{\nu^p} \|x - y\| \leq \frac{8\sqrt{k} + 1}{\nu^{p-1}} \|x - y\|. \]
Suppose now \( w_{\lambda, x_j}(y) \neq 0 \) for some \( i \neq j \). From Corollary 5.5 we have that
\[ w_{\lambda, x_i} \cdot w_{\lambda, x_l} = \emptyset, \] (5.9)
for every \( l \neq i \). This implies \( w_{\lambda, x_i}(y) = 0 \) for every \( l \neq i \), hence
\[ \psi(y) = \frac{1}{\nu^p} w_{\lambda, x_i}(y). \]
Formula (5.9) also gives \( w_{\lambda, x_i}(y) = 0 \), being \( w_{\lambda, x_i}(y) \neq 0 \), and \( w_{\lambda, x_i}(x) = 0 \), otherwise we would have \( \psi(x) = \frac{1}{\nu^p} w_{\lambda, x_j}(x) = 0 \), against the assumption. Putting things together, from (5.7) we get
\[
\begin{align*}
|\psi(x) - \psi(y)| &= \frac{|w_{\lambda, x_j}(x) - w_{\lambda, x_j}(y)|}{\nu^p} \\
&\leq \frac{|w_{\lambda, x_j}(x) - w_{\lambda, x_j}(y)|}{\nu^p} + \frac{|w_{\lambda, x_i}(x) - w_{\lambda, x_i}(y)|}{\nu^p} \\
&\leq \frac{4\lambda_i + 2}{\nu^p} \|x - y\| \\
&\leq \frac{2(8\sqrt{k} + 1)}{\nu^{p-1}} \|x - y\|.
\end{align*}
\]
Either way, (5.8) holds, and then
\[ L(\psi) \leq \frac{2(8\sqrt{k} + 1)}{\nu^{p-1}}. \]
This, together with (2.5), implies that
\[ \|\nabla \psi(x)\| \leq \frac{2\sqrt{n}(8\sqrt{k} + 1)}{\nu^{p-1}}, \]
for every \( \nu \in \mathbb{N} \) and \( H^{n-1}\text{-a.e. } x \in S^{n-1} \). The last inequality both tells us that \( \nabla \psi \to 0 \) \( H^{n-1}\text{-a.e. in } S^{n-1}, \) as \( \nu \to \infty \), and that \( \|\nabla \psi\| \) is uniformly bounded by
\[ C = 2\sqrt{n}(8\sqrt{k} + 1). \]
Therefore, \( \psi \to \emptyset \) as \( \nu \to \infty \). Since \( \mu \) is continuous, this gives \( \mu(\psi) \to \mu(\emptyset) = 0 \), which is in contradiction with (5.6). This concludes the proof of Proposition 5.1.
We are finally ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Assume the functional \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) to be defined by (1.6) for some constants \( c_0, c_1, c_2 \in \mathbb{R} \), and write

\[
\mu(u) = \int_{S^{n-1}} F(u, ||\nabla u||)dH^{n-1}(x),
\]

for \( u \in \text{Lip}(S^{n-1}) \), where \( F : \mathbb{R} \times [0, +\infty) \to \mathbb{R} \) is the \( C^\infty \) function given by

\[
F(x, y) = \frac{c_0}{H^{n-1}(S^{n-1})} + c_1 x + c_2[(n-1)x^2 - y^2],
\]

for every \((x, y) \in \mathbb{R} \times [0, +\infty)\). To prove that \( \mu \) is a valuation, we take \( u, v \in \text{Lip}(S^{n-1}) \) and estimate

\[
\mu(u \vee v) + \mu(u \wedge v) = \int_{S^{n-1}} F(u \vee v, ||\nabla (u \vee v)||)dH^{n-1}(x) + \int_{S^{n-1}} F(u \wedge v, ||\nabla (u \wedge v)||)dH^{n-1}(x) \]

\[
= \int_U F(u \vee v, ||\nabla (u \vee v)||)dH^{n-1}(x) + \int_V F(v, ||\nabla (u \vee v)||)dH^{n-1}(x) + \int_U F(u, ||\nabla (u \vee v)||)dH^{n-1}(x) + \int_V F(v, ||\nabla (u \wedge v)||)dH^{n-1}(x) + \int_E F(u \vee v, ||\nabla (u \wedge v)||)dH^{n-1}(x) + \int_E F(u \wedge v, ||\nabla (u \wedge v)||)dH^{n-1}(x),
\]

where

\[
U = \{ x \in S^{n-1} \mid u(x) > v(x) \}, \quad V = \{ x \in S^{n-1} \mid u(x) < v(x) \}, \quad E = \{ x \in S^{n-1} \mid u(x) = v(x) \}.
\]

Let \( x \in S^{n-1} \) be such that \( u, v, u \vee v \) and \( u \wedge v \) are differentiable at \( x \). Then clearly

\[
\nabla (u \vee v)(x) = \begin{cases} 
\nabla u(x) & \text{if } x \in U, \\
\nabla v(x) & \text{if } x \in V,
\end{cases}
\]

and

\[
\nabla (u \wedge v)(x) = \begin{cases} 
\nabla v(x) & \text{if } x \in U, \\
\nabla u(x) & \text{if } x \in V.
\end{cases}
\]

On the other hand if \( u(x) = v(x) \) it is not hard to prove (see also [21]) that

\[
\nabla u(x) = \nabla v(x) = \nabla (u \vee v)(x) = \nabla (u \wedge v)(x).
\]

Hence we can reassemble the integrals in (5.10) so that

\[
\mu(u \vee v) + \mu(u \wedge v) = \int_{S^{n-1}} F(u, ||\nabla u||)dH^{n-1}(x) + \int_{S^{n-1}} F(v, ||\nabla v||)dH^{n-1}(x) = \mu(u) + \mu(v).
\]

We now prove that \( \mu \) is continuous. Let \( \{u_i\} \subseteq \text{Lip}(S^{n-1}) \) be such that \( u_i \to u \in \text{Lip}(S^{n-1}) \). Then \( ||u_i - u||_{\infty} \to 0 \), hence there exists \( I \in \mathbb{N} \) such that \( ||u_i||_{\infty} < ||u||_{\infty} + 1 \) for every \( i > I \). Set

\[
M = \max\{||u_1||_{\infty}, \ldots, ||u_I||_{\infty}, ||u||_{\infty} + 1\}.
\]

From the almost uniform convergence of the gradients, there is a \( C > 0 \) such that

\[
(u_i(x), ||\nabla u_i(x)||) \in K := [-M, M] \times [0, C],
\]

for every \( i \in \mathbb{N} \) and \( H^{n-1}\text{a.e. } x \in S^{n-1} \). Let \( D = \max_K |F| \), thus \( F(u_i, ||\nabla u_i||) = F|_K(u_i, ||\nabla u_i||) \) is dominated by the constant function \( D \), which is integrable on \( S^{n-1} \) since the sphere has finite measure. From the Dominated Convergence Theorem we get

\[
\lim_{i \to \infty} \mu(u_i) = \lim_{i \to \infty} \int_{S^{n-1}} F(u_i, ||\nabla u_i||)dH^{n-1}(x) = \int_{S^{n-1}} F(u, ||\nabla u||)dH^{n-1}(x) = \mu(u).
\]
For what concerns rotation invariance, we have that for every \( u \in \text{Lip}(S^{n-1}) \) and \( \varphi \in O(n) \)
\[
\mu(u \circ \varphi) = \int_{S^{n-1}} F(u(\varphi(x)), \|\nabla (u \circ \varphi)(x)\|) dH^{n-1}(x)
\]
\[
= \int_{S^{n-1}} F(u(\varphi(x)), \|(D\varphi(x))^T \nabla u(\varphi(x))\|) dH^{n-1}(x)
\]
\[
= \int_{S^{n-1}} F(u(\varphi(x)), \|\nabla u(\varphi(x))\|) dH^{n-1}(x) = \mu(u),
\]
where we have used the fact that the matrix \((D\varphi(x))^T\), being orthogonal, preserves the norm, and we applied the change of variables \( y = \varphi(x) \).

It remains to be seen that \( \mu \) is dot product invariant. This can be proved with a direct computation, but it is easier to show it via a trick which also gives us the chance to recall how some intrinsic volumes can be written, something that will be useful during the second part of the proof too. It is known that (see for instance [22]) for every \( K \in \mathcal{K}^n \), the intrinsic volumes \( V_0, V_1, V_2 \) can be expressed as follows:
\[
V_0(K) = 1, \tag{5.11}
\]
\[
V_1(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} h_K dH^{n-1}(x), \tag{5.12}
\]
where \( \omega_{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure of the unit \((n-1)\)-ball, and, if \( K \in C^{2,+} \),
\[
V_2(K) = \int_{S^{n-1}} h_K \cdot \text{tr} (M(h_K)) dH^{n-1}(x), \tag{5.13}
\]
where \( \text{tr} (M(h_K)) \) denotes the trace of the matrix \( M(h_K) \) given by
\[
M(h_K) = h_K \cdot \text{Id}_{n-1} + (h_K^{ij}),
\]
\((h_K^{ij})\) being the \((n-1) \times (n-1)\) matrix of the second covariant derivatives of \( h_K \) with respect to a local orthonormal frame, on the sphere. From (5.13) we get
\[
V_2(K) = \int_{S^{n-1}} h_K [(n-1)h_K + \Delta h_K] dH^{n-1}(x) \tag{5.14}
\]
\[
= \int_{S^{n-1}} [(n-1)h^2_K + h_K \text{div} (\nabla h_K)] dH^{n-1}(x)
\]
\[
= \int_{S^{n-1}} [(n-1)h^2_K - \|\nabla h_K\|^2] dH^{n-1}(x),
\]
where the last equality follows from the Divergence Theorem (here \( \Delta \) denotes the spherical Laplace operator). Therefore,
\[
\mu(h_K) = c_0 V_0(K) + c_1 \omega_{n-1} V_1(K) + c_2 V_2(K), \tag{5.15}
\]
for every convex body \( K \in C^{2,+} \).

For \( x \in \mathbb{R}^n \), consider the functional \( \mu_x : \text{Lip}(S^{n-1}) \to \mathbb{R} \) defined by \( \mu_x(u) = \mu(u + \langle \cdot, x \rangle) \), for \( u \in \text{Lip}(S^{n-1}) \). This is still a continuous valuation on \( \text{Lip}(S^{n-1}) \) and, because of (5.15), it satisfies
\[
\mu_x(h_K) = \mu(h_K + \langle \cdot, x \rangle)
\]
\[
= \mu(h_{K+x}) = c_0 V_0(K + x) + c_1 \omega_{n-1} V_1(K + x) + c_2 V_2(K + x)
\]
\[
= c_0 V_0(K) + c_1 \omega_{n-1} V_1(K) + c_2 V_2(K) = \mu(h_K),
\]
for every convex body \( K \in C^{2,+} \), since the intrinsic volumes are translation invariant.

Now, the integral in (5.13) only makes sense for support functions of \( C^{2,+} \) bodies, but its rewritten form (5.14) is well-defined for every support function \( h_K \in \mathcal{H}(S^{n-1}) \). Since \( C^{2,+} \) bodies are dense in \( \mathcal{K}^n \) with respect to the Hausdorff metric, for an arbitrary \( h_K \in \mathcal{H}(S^{n-1}) \) we can find a sequence \( \{h_{K_i}\} \subseteq \mathcal{H}(S^{n-1}) \) with \( \{K_i\} \subseteq C^{2,+} \) such that \( \|h_{K_i} - h_K\|_{\infty} \to 0 \). Then we also have \( h_{K_i} \to h_K \) (see the proof of Lemma 27).
and since $\mu_x$ and $\mu$ are continuous with respect to $\tau$ we get $\mu_x(h_K) = \mu(h_K)$. From Proposition 3.1 it follows that they coincide on the whole space $\text{Lip}(S^{n-1})$, hence $\mu$ is dot product invariant.

Vice versa, let $\mu : \text{Lip}(S^{n-1}) \to \mathbb{R}$ be a continuous, rotation invariant and dot product invariant valuation. As we previously did, let us consider $\nu : K^n \to \mathbb{R}$ defined by

$$\nu(K) = \mu(h_K),$$

for $K \in K^n$, which is a translation and rotation invariant valuation that is continuous with respect to the Hausdorff metric, because of Lemma 2.8. From Theorem 2.3 there are real constants $c_0, c_1, \ldots, c_n$ such that

$$\mu(h_K) = \nu(K) = \sum_{i=0}^n c_i V_i(K),$$

(5.16)

for every $K \in K^n$. From Proposition 4.1 there exist continuous and dot product invariant valuations $\mu_0, \mu_1, \ldots, \mu_n : \text{Lip}(S^{n-1}) \to \mathbb{R}$ such that $\mu_i$ is $i$-homogeneous, for $i = 0, 1, \ldots, n$, and

$$\mu(\lambda u) = \sum_{i=0}^n \lambda^i \mu_i(u),$$

for every $\lambda \geq 0$ and $u \in \text{Lip}(S^{n-1})$. Moreover, if we go back to (4.5) we deduce that the $\mu_i$’s are rotation invariant too, since $\mu$ is. Applying Proposition 5.1 to $\mu_i$, for $i = 3, \ldots, n$, we get

$$\mu(\lambda u) = \mu_0(u) + \lambda \mu_1(u) + \lambda^2 \mu_2(u),$$

(5.17)

for every $\lambda \geq 0$ and $u \in \text{Lip}(S^{n-1})$.

Combining (5.16) and (5.17) we have that, for every $\lambda \geq 0$ and $K \in K^n$,

$$\mu_0(h_K) + \lambda \mu_1(h_K) + \lambda^2 \mu_2(h_K) = \mu(\lambda h_K) = \mu(h_{\lambda K}) = \sum_{i=0}^n c_i V_i(\lambda K) = \sum_{i=0}^n c_i \lambda^i V_i(K),$$

(5.18)

where the last equality follows from the $i$-homogeneity of the $i^{th}$ intrinsic volume. This implies $\mu_0(h_K) = c_0 V_0(K), \mu_1(h_K) = c_1 V_1(K), \mu_2(h_K) = c_2 V_2(K)$ and $c_3 = \ldots = c_n = 0$. Therefore, taking $\lambda = 1, u = h_K$ in (5.17) and remembering (5.11), (5.12), (5.14), we find

$$\mu(h_K) = c_0 + c_1 \int_{S^{n-1}} h_K dH^{n-1}(x) + c_2 \int_{S^{n-1}} [(n-1)h_K^2 - \|\nabla h_K\|^2] dH^{n-1}(x),$$

for every $K \in C^{2,+}$, where we have renamed $c_1 := c_1/\omega_{n-1}$. From the first part of the proof, the functional $\tilde{\mu} : \text{Lip}(S^{n-1}) \to \mathbb{R}$ defined by

$$\tilde{\mu}(u) = c_0 + c_1 \int_{S^{n-1}} u dH^{n-1}(x) + c_2 \int_{S^{n-1}} [(n-1)u^2 - \|\nabla u\|^2] dH^{n-1}(x),$$

for $u \in \text{Lip}(S^{n-1})$, is a continuous valuation like $\mu$, and they coincide on the set of support functions of $C^{2,+}$ bodies, hence on $\mathcal{H}(S^{n-1})$, by density. We conclude the proof from Proposition 5.1.

6 An improved version of Theorem 4.1

In this final section we refine Theorem 4.1 as follows.
Theorem 6.1. Let \( n \geq 3 \) and \( \mu : \text{Lip}(S^{n-1}) \to \mathbb{R} \) be a continuous and dot product invariant valuation. Then there exist continuous and dot product invariant valuations \( \mu_0, \ldots, \mu_{n-1} : \text{Lip}(S^{n-1}) \to \mathbb{R} \) such that \( \mu_i \) is homogeneous of degree \( i \), for \( i = 0, \ldots, n - 1 \), and
\[
\mu(\lambda u) = \sum_{i=0}^{n-1} \lambda^i \mu_i(u),
\] (6.1)
for every \( u \in \text{Lip}(S^{n-1}) \) and \( \lambda \geq 0 \). In particular, \( \mu = \mu_0 + \ldots + \mu_{n-1} \).

For the proof we will need the following result (see [22, Theorem 6.4.8]).

Theorem 6.2. Let \( \nu : \mathcal{K}^n \to \mathbb{R} \) be a continuous and translation invariant valuation which is homogeneous of degree \( n \). Then there exists \( c \in \mathbb{R} \) such that \( \nu(K) = cV_n(K) \), for every \( K \in \mathcal{K}^n \).

Proof of Theorem 6.1. We use the notations introduced in the proof of Theorem 4.1 by the latter result we only need to prove that \( \mu_n \equiv 0 \). By Theorem 6.2 there exists \( c \in \mathbb{R} \) such that \( \nu_n(K) = cV_n(K) \), for every \( K \in \mathcal{K}^n \).

In particular, \( \nu_n \) is rotation invariant, hence \( \mu_n \) is rotation invariant on \( \mathcal{H}(S^{n-1}) \).

Let us prove that \( \mu_n \) is rotation invariant on the whole space \( \text{Lip}(S^{n-1}) \). For a fixed \( \varphi \in O(n) \), consider \( \mu_n^\varphi : \text{Lip}(S^{n-1}) \to \mathbb{R} \) defined by
\[
\mu_n^\varphi(u) = \mu_n(u \circ \varphi) - \mu_n(u),
\]
for \( u \in \text{Lip}(S^{n-1}) \). Such functional is a continuous valuation on \( \text{Lip}(S^{n-1}) \); because \( \mu_n \) is rotation invariant on \( \mathcal{H}(S^{n-1}) \), \( \mu_n^\varphi = 0 \) on \( \mathcal{H}(S^{n-1}) \). From Proposition 5.1, \( \mu_n^\varphi = 0 \) on \( \text{Lip}(S^{n-1}) \), so that \( \mu_n(u \circ \varphi) = \mu_n(u) \), for every \( u \in \text{Lip}(S^{n-1}) \) and \( \varphi \in O(n) \). Therefore, \( \mu_n \) is a continuous, rotation invariant, dot product invariant and \( n \)–homogeneous valuation on \( \text{Lip}(S^{n-1}) \), hence \( \mu_n \equiv 0 \), using Proposition 5.1.

\[ \square \]

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