Linear Universal Decoding for Compound Channels: a Local to Global Geometric Approach

Emmanuel Abbe  
Massachusetts Institute of Technology  
Laboratory for Information and Decision Systems  
Cambridge, MA 02139  
eabbe@mit.edu

Lizhong Zheng  
Massachusetts Institute of Technology  
Laboratory for Information and Decision Systems  
Cambridge, MA 02139  
lizhong@mit.edu

Abstract—Over discrete memoryless channels (DMC), linear decoders (maximizing additive metrics) afford several nice properties. In particular, if suitable encoders are employed, the use of decoding algorithm with manageable complexities is permitted. Maximum likelihood is an example of linear decoder. For a compound DMC, decoders that perform well without the channel's knowledge are required in order to achieve capacity. Several such decoders have been studied in the literature. However, there is no such known decoder which is linear. Hence, the problem of finding linear decoders achieving capacity for compound DMC is addressed, and it is shown that under minor concessions, such decoders exist and can be constructed.

This paper also develops a local geometric analysis, which allows in particular, to solve the above problem. By considering very noisy channels, the original problem is reduced, in the limit, to an inner product space problem, for which insightful solutions can be found. The local setting can then provide counterexamples to disproof claims, but also, it is shown how in this problem, results proven locally can be “lifted” to results proven globally.

I. INTRODUCTION

We consider a discrete memoryless channel with input alphabet \( \mathcal{X} \) and output alphabet \( \mathcal{Y} \). The channel is described by the probability transition matrix \( W \), each row of which is the conditional distribution of the output symbol \( Y \) conditioned on a particular input \( X = x \in \mathcal{X} \). We are interested in the compound channel, where the exact value of \( W \) is not known, either at the transmitter or the receiver. Such problems can often be motivated by the wireless applications with unknown fading realizations. Here, instead of assuming the channel \( W \) to be known at the receiver and transmitter, we assume that a set \( S \) of possible channels is known at the receiver and transmitter; and our goal is to design encoders and decoders that support reliable communication, no matter which channel in \( S \) actually takes place.

Compound channels have been extensively studied in the literature. In particular, Blackwell et.al. [2] shown that the highest achievable rate is given by the following expression:

\[
C(S) \triangleq \max_{P} \inf_{W \in S} I(P, W),
\]

where the maximization is over all probability distributions \( P \) on \( \mathcal{X} \). Thus, \( C(S) \) is referred to as the compound channel capacity. To achieve the capacity, i.i.d. (or fixed composition) random codes from the optimal input distribution, i.e. the distribution maximizing \( I(P, W) \), are used. The random coding argument is commonly employed to prove achievability for a single given channel, such as in Shannon’s original paper. By showing that the error probability averaged over the random ensemble can be made arbitrarily small, one can conclude that there exists “good” codes with low enough error probability. This argument is strengthened in [2] to show that with the random coding argument, we can indeed prove the existence of codes that are good for all possible channels. Adopting this view, in this paper, we will not be concerned about constructing the code, or even finding the optimal input distribution, but rather simply assume that one of the above mentioned universally good code is used, and focus on the designs of efficient decoding algorithms.

In [2], a decoder that maximizes a uniform mixture of likelihoods over most possible channels is used, and shown to achieve capacity. The most general universal decoder is the maximum mutual information (MMI) decoder [4], which computes the empirical mutual information between each codeword and the received word and picks the highest one. The practical difficulty of implementing MMI decoders is obvious. As empirical distributions are used in computing the “score” of each codeword, it becomes challenging to efficiently store the exponentially many scores, and update the scores as symbols being received sequentially. Conceptually, when the empirical distribution of the received signals is computed, one can in principle estimate the channel \( W \), making the assumption of lack in channel knowledge less meaningful. There has been a number of different universal decoders proposed in the literature, including the LZ based algorithm [10], or merged likelihood decoder [6]. In this paper, we try to find universal decoders in a class of particularly simple decoders: linear decoders.

Here, linear (or additive) decoders are defined to have the following structure. Upon receiving the \( n \)-symbol word \( y \), the decoder compute a score/decoding metric \( d^n(x_m, y) \) (note that the score of a codeword does not depend on other codewords) for each codeword \( x_m, m = 1, 2, \ldots, 2^n R \), and decodes to the one codeword with the highest score (ties can be resolved arbitrarily). Moreover, the \( n \)-symbol decoding metric has the
following additive structure

\[ d^a(x_m, y) = \sum_{i=1}^{n} d(x_m(i), y(i)) \]

where \( d : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is a (single-letter) decoding metric. Such decoders are called linear since the decoding metric it computes is indeed linear in the joint empirical distribution between the codeword and the received word, since

\[ d^a(x_m, y) = n \cdot \sum_{a \in \mathcal{X}, b \in \mathcal{Y}} \hat{P}(x_m, y)(a, b) \cdot d(a, b) \]

where \( \hat{P}(x_m, y) \) denotes the joint empirical distribution of \( (x_m, y) \). We call such a decoder a linear decoder induced by the single-letter metric \( d \).

Linear decoders have been widely studied in [5], [11]. An additive decoding metric has some obvious advantages. First, when used with appropriate codes, it allows the decoding complexity to be reduced. Note that maximum likelihood (ML) decoder is by definition a linear decoder, with single-letter complexity to be reduced. Note that maximum likelihood (ML) additive decoding metric has some obvious advantages. First, that linear universal decoder does not exist. In [5], [11], it is shown decoders is revealed, allowing the effects of “mismatched” decoding maps, and decodes to the codeword with the highest \( K \) scores. In order such a generalized linear decoder to have a manageable complexity, we emphasize the restriction that \( K \) has to be finite. In particular, it should not increase with the codeword length \( n \). For example, the decoder proposed in [2], a mixture of likelihoods over all possible channels, in general might require averaging over polynomial(\( n \)) channels. In addition, optimizing the mixture of additive metrics, i.e. \( \arg \max_m 1/K \sum_{k=1}^{K} d_k(x_m, y) \), cannot be solved by computing \( K \) parallel additive metric optimizations: the codewords having the best scores for each of the \( K \) metrics may not be the only candidates for the best score of the mixture of the metrics; on the other hand, if we consider a generalized linear decoder, the codewords having the best score for each of the \( K \) metrics are the only one to be considered for the maximum of the \( K \) metrics.

The main result of this paper is the construction of generalized linear decoders that achieve compound channel capacity on most compound sets. As to be shown in Section III, this construction requires solving some rather complicated optimization problems involving the Kullback-Leibler (KL) divergence (like almost every other information theoretical problem). To obtain insights to this problem, we introduced in Section III a special tool: local geometric analysis. In a nutshell, we focus on the special cases where the two distributions in the KL divergence are “close” to each other, which can be thought in this context as approximating the given compound channels by very noisy channels. In this local setting, information theoretical quantities can be naturally understood as quantities in an inner product space, where conditional distributions and decoding metrics correspond to vectors; divergence and mutual information correspond to squared norms and the data rate with mismatched linear decoders can be understood with projections. The relation between these quantities can thus be understood intuitively. While the results from such local approximations only apply to the special very noisy cases, we show in Section IV that some of these results can be “lifted” to the naturally corresponding statements about general cases. Using this approach, we derive the following main results of the paper.

- First we derive a new condition on \( S \) to be “one-sided”, cf. Definition 2, under which a linear decoder, which decodes using the log likelihood of the worst channel over the compound set, achieves capacity. This condition is more general than the previously known one, which requires \( S \) to be convex;
- Then, we show in our main result, that if the compound set \( S \) can be written as a finite union of one sided sets, then a generalized linear decoder using the log posteriori distribution of the worst channels of each one-sided subset achieves the compound capacity; in contrast, GLRT using these worst channels is not a universal decoder.

Besides the specific results on the compound channels, we also like to emphasize the use of the local geometric analysis. As most of multi-terminal information theory problems
involve optimizations of K-L divergences, often between distributions with high dimensionality, we believe the localization method used in this paper can be a generic tool to simplify these problems. Focusing on certain special cases, this method is obviously useful in providing counterexamples to disprove conjectures. However, we also hope to convince the readers that the insights provided by the geometric analysis can be also valuable in solving the general problem. For example, our definition of one-sided sets and the use of log a posteriori distributions as decoding metrics can be seen as “naturally” suggested by the local analysis.

In the next section, we will start with the precise problem formulations and notations.

II. LINEARITY AND UNIVERSALITY

We consider discrete memoryless channels with input and output alphabets \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. The channel is often written as a probability transition matrix, \( W \), of dimension \( |\mathcal{X}| \times |\mathcal{Y}| \), each row of which denotes the conditional distribution of the output, conditioned on a specific value of the input. We are interested in the compound channel, where \( W \) can be any elements of a given set \( S \), referred to as the set of possible channels, or the compound set. For convenience, we assume \( S \) to be compact. The value of the true channel is then the compound channel capacity for a compound set \( S \). However, the results in this paper can be stated for every possible channel have non-empty intersection, has been used as a standard method to study compound channels. In this paper, we are focused on designing efficient decoders, which is interesting since the optimal maximum likelihood decoder is voided by the channel’s law ignorance. We will not be particularly concerned about finding a good codebook, or even the optimal input distribution. To simplify our discussions, we will, for most of our results, only show that the ensemble average error probability can be made small, when decoders discussed in the paper are used. Arguments similar to that of [2] can be used to show that the error probability can be made small when appropriately chosen codes are used.

Now before we proceed to define decoders, we need to define some notations:

- We always assume that we are working with the optimal input distribution \( P_X \) for the considered compound set \( S \), i.e.

\[
P_X = \arg \max_{P} \inf_{W \in S} I(P, W)
\]

(if the maximizers were not to be unique, we pick arbitrarily one of them). Therefore, \( \inf_{W \in S} I(P_X, W) \) is the compound channel capacity for a compound set \( S \). However, the results in this paper can be stated for arbitrary input distributions (not necessarily optimal), the only difference would then be that we would talk about mutual informations instead of capacities.

- For convenience, we assume that \( S \) is compact. We define

\[
W_S = \arg \min_{I \in S} I(P_X, W)
\]

and call it the worst channel of \( S \) when the minimizer is unique; \( I(P_X, W_S) \) is then the compound channel capacity for a compound set \( S \). We make the convention that each time a worst channel is considered throughout the paper for any set, the set in question is compact.

- \( W_0 \in S \) denotes the true channel;

- For a joint distribution \( \mu \) on \( \mathcal{X} \times \mathcal{Y} \); \( \mu_X \) and \( \mu_Y \) denote respectively the \( X \) and \( Y \) marginal distributions; and \( \mu^P = \mu_X \times \mu_Y \) the induced product distribution. Note that \( \{ \mu_X = P_X, \mu_Y = (\mu_0)_Y \} \Leftrightarrow \mu^P = \mu_0^P \)

Lemma 1: Compound Channel Capacity [2]

\[
C(S) = \max_{P_X} \inf_{W \in S} I(P_X, W).
\]

Remark: The random coding argument is often used in proving the coding theorem for a fixed channel. By showing that the error probability, averaged over the ensemble of random codes, approaches 0 as \( n \) increases, one can draw the conclusion that there exists at least one sequence of codes, for which the probability of error, averaged over the specific codes, is driven to 0. A similar argument is used in compound channels. Here, it is however not enough to show that the ensemble average error probability is small for every \( W \). Since the “good” codes for different channels can in principle be different, this is not enough to guarantee the existence of a single code that is universally good for all possible channels. The random coding argument is strengthened in [2] to show that universally good code indeed exists. The approach used in [2], to show that the sets of good codes corresponding to every possible channel have non-empty intersection, has been used as a standard method to study compound channels. In this paper, we are focused on designing efficient decoders, which is interesting since the optimal maximum likelihood decoder is voided by the channel’s law ignorance. We will not be particularly concerned about finding a good codebook, or even the optimal input distribution. To simplify our discussions, we will, for most of our results, only show that the ensemble average error probability can be made small, when decoders discussed in the paper are used. Arguments similar to that of [2] can be used to show that the error probability can be made small when appropriately chosen codes are used.
• $\mu = P_X \circ W$ denotes the joint distribution with $P_X$ as the $X$ marginal distribution and $W$ as the conditional distribution. For example, the mutual information

$$I(P_X, W) = D(P_X \circ W \| (P_X \circ W)^p)$$

where $D(\cdot \| \cdot)$ is the Kullback-Leibler divergence.

The decoders we consider has the following form. Upon receiving $y$, it computes, for each codeword $x_m$, a score $d^n(x_m, y)$, and decodes to the message corresponding to the highest score. Here, $d^n : X^n \times Y^n \mapsto \mathbb{R}$ is also called a decoding metric. Note the restriction here is that the score for codeword $x_m$ does not depend on other codewords. Such decoders are called $\alpha$-decoders in [5]. As an example, the maximum mutual information (MMI) decoder has a score defined as

$$d^n_{\text{MMI}}(x_m, y) = I(\hat{P}_{(x_m, y)})$$

where $\hat{P}$ denotes the empirical distribution. To be specific, $\forall a \in \mathcal{X}, b \in \mathcal{Y}$

$$\hat{P}_{(x_m, y)}(a, b) = \frac{1}{n} | \{ i : (x_m(i), y(i)) = (a, b) \}|,$$

and $I(\mu)$ denotes the mutual information, as a function of the joint distribution $\mu$ on $\mathcal{X} \times \mathcal{Y}$.

It is well known that the MMI decoder is universal; when used with the optimal code, it achieves the compound channel capacity on any code set. In fact, there are other advantages of the MMI decoder: it does not require the knowledge of $S$; and it achieves universally the random coding error exponent [4]. Despite these advantages, the practical difficulties to implement an MMI decoder prevents it from becoming a real “universally used” decoder. As empirical distributions are used in computing the scores, it is difficult to store and update the scores, even when a structured codebook is used. The main goal of the current paper is to find linear decoders that can, like the MMI decoder, be capacity achieving on compound channels.

**Definition 1: Linear Decoder**

We refer to a map

$$d : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$$

as a single-letter metric. A linear decoder induced by $d$ is defined by the decoding mapping:

$$G_n(y) = \arg \max_{a \in \mathcal{X}} d^n(x_m, y)$$

where $d^n(x_m, y) = \frac{1}{n} \sum_{i=1}^n d(x_m(i), y(i)) = E_{\hat{P}_{(x_m, y)}}[d]$.

Note that the reason why such decoders are called linear decoders ($d$-decoders in [5]) is to underline the fact that the decoding metric is additive, i.e., is a linear function of the empirical distribution $\hat{P}_{(x_m, y)}$. The decoding metric $d^n$ for any $n$ of a linear decoder is naturally defined by the single-letter metric $d$ through the additive structure.

The advantages of using linear decoders have been discussed thoroughly in [5], [11], [8], and also briefly in the introduction. In short, when used with structured codes, one can replace the log likelihood metric in a conventional decoder by a well designed single-letter metric. This way, with little changes in the decoder designs, one can have a decoder for the compound channel with much less complexity.

Unfortunately, there are some examples for which no linear decoder can achieve the compound capacity. The most well-known example is the compound set with two binary symmetric channels, with crossover probabilities of $1/4$ and $3/4$, respectively. To address the decoding challenge of these cases, we will need a slightly more general version of linear decoders.

**Definition 2: Generalized Linear Decoder**

Let $d_1, d_2, \ldots, d_K$ be $K$ single-letter metrics, where $K$ is a finite number. A generalized linear decoder induced by these metrics is defined by the decoding map:

$$G_n(y) = \arg \max_{m} \sum_{k=1}^K d_k(x_m(i), y(i))$$

Note that $\vee$ denotes the maximum, and it is crucial that $K$ is a finite number, which does not depend on the code length $n$.

As an example, the maximum likelihood decoder, of a given channel $W$, is a linear decoder induced by

$$d_{\text{ML}}(a, b) = \log W(b|a), \quad \forall a \in \mathcal{X}, b \in \mathcal{Y}.$$
to $\mu_0$, and thus has a score
\[ d^n(x_1, y) > E_{\mu_0}[d] - \delta := \gamma \]
for an arbitrarily small $\delta > 0$ with a high probability when
$n$ is large enough. Now an error occurs only if there is an
incorrect codeword, whose score is above $\gamma$. For a particular
codeword, $x_2$, this occurs with probability
\[ P(d^n(x_2, y) > \gamma) \leq \exp \left[-n \left( \min_{\mu : E_{\mu}[d] \geq \gamma} D(\mu||\mu_0^p) - \delta \right) \right], \]
using the fact that $x_2$ is independent of $y$ with an i.i.d. $P_X$
distribution. The optimization is over the joint distributions $\mu$
with the correct $X$ and $Y$ marginal distributions. Now applying
union bound, the probability
\[ P(\exists i \neq 1, \text{ s.t. } d^n(x_i, y) > \gamma) \leq 2^nR \cdot P(d^n(x_2, y) > \gamma). \]

Moreover, the empirical distribution of $x_2, y$ is arbitrarily close
to $\mu_0^p$ with probability one. Hence, if $R < R(P_X, W_0, d)$ as
defined in the lemma’s statement, the above probability can
be made arbitrarily small by taking $\delta$ small enough.

With a similar proof as for previous result, the following
lemma can also be proved.

**Lemma 3:** When the true channel is $W_0$ and a generalized
linear decoder induced by the single-letter metrics $\{d_k\}_{k=1}^K$ is
used, we can achieve the following rate
\[ R(P_X, W_0, \{d_k\}_{k=1}^K) = \min_{\mu \in \mathcal{A}} D(\mu||\mu_0^p) \]
where
\[ \mathcal{A} = \{\mu : \mu_X = P_X, \mu_Y = (\mu_0)_Y, \forall k \sum_{d_k} E_{\mu}[d_k] > \sum_{d_k} E_{\mu_0^p}[d_k]\}. \]

Note that $R(P_X, W_0, \{d_k\}_{k=1}^K)$ can equivalently be expressed as
\[ R(P_X, W_0, \{d_k\}_{k=1}^K) = \min_{\mu \in \mathcal{A}_1} D(\mu||\mu_0^p) \wedge \ldots \wedge \min_{\mu \in \mathcal{A}_K} D(\mu||\mu_0^p) \]
where
\[ \mathcal{A}_k = \{\mu : \mu_X = P_X, \mu_Y = (\mu_0)_Y, E_{\mu}[d_k] > \sum_{d_k} E_{\mu_0^p}[d_k]\}, \quad \forall 1 \leq k \leq K. \]

Now we are ready for the main problem studied in this
paper. For any given compound set $S$, let the compound
channel capacity be $C(S)$ and the corresponding optimal input
distribution be $P_X$. We would like to find $K$ and $d_1, \ldots, d_K$,
such that
\[ R(P_X, W_0, \{d_k\}_{k=1}^K) \geq C(S) \]
for every $W_0 \in S$.

If this holds, the generalized decoder induced by the metrics
$\{d_k\}_{k=1}^K$ is capacity achieving on the compound set $S$ (i.e.,
using analogue arguments as for the achievability proof of the
compound capacity in [2], there exists a code book that makes
the overall coding scheme capacity achieving).

### III. The Local Geometric Analysis

We know that the divergence is not a distance between two
distributions. However, if its two arguments are close enough,
the divergence is approximately a squared norm, namely for
any probability distribution $p$ on $\mathcal{Z}$ (where $\mathcal{Z}$ is any alphabet)
and for any $v$ s.t. $\sum_z v(z)p(z) = 0$, we have
\[ D(p(1 + \varepsilon v)||p) = \frac{1}{2} \varepsilon^2 \sum_{z \in \mathcal{Z}} v^2(z)p(z) + o(\varepsilon^2). \]

This is the main tool used in this section. For convenience, we define
\[ \|v\|_p^2 = \sum_{z \in \mathcal{Z}} v^2(z)p(z) \]
which is the squared $l_2$-norm of $v$, with weight measure $p$.
Similarly, we can define the weighted inner product,
\[ \langle u, v \rangle_p = \sum_{z \in \mathcal{Z}} u(z)v(z)p(z) \]
With these notations, one can write the approximation (7) as
\[ D(p(1 + \varepsilon v)||p) = \frac{\varepsilon^2}{2} \|v\|_p^2 + o(\varepsilon^2) \]
Ignoring the higher order term, the above approximation can
greatly simplify many optimization problems involving K-L
divergences. In information theoretic problems dealing with
 discrete channels, such approximation is tight for some special
cases such as when the channel is very noisy.

In general, very noisy channel means that the channel output
weakly depends on the input. If the conditional probability of
observing any output does not depend on the input (i.e. the
transition probability matrix has constant columns), we have
a “pure noise” channel. So a very noisy channel should be somehow
close to such a pure noise channel. Formally, we
consider the following family of channels:
\[ W_\varepsilon(b|a) = P_N(b)(1 + \varepsilon L(a, b)), \]
where $L$ satisfies for any $a \in \mathcal{X}$
\[ \sum_{b \in \mathcal{Y}} L(a, b)P_N(b) = 0. \]
We say that $W_\varepsilon$ is a very noisy channel if $\varepsilon \ll 1$. In this
case, the conditional distribution of the output, conditioned on
any input symbol, is close to a distribution $P_N$ (on $\mathcal{Y}$), which
can be thought as the distribution of pure noise. Each of these
channels, $W_\varepsilon(\cdot|\cdot)$, can be viewed as a perturbation from a pure
noise channel $P_N$, along the direction specified by $L(\cdot, \cdot)$.

This way of defining very noisy channel can be found in [9], [7]. In fact, there are many other possible ways to
describe a perturbation of distribution. For example, readers
familiar with [1] might feel it natural to perturb distributions
along exponential families. Since we are interested only in
small perturbations, it is not hard to verify that these different
definitions are indeed equivalent.
When an input distribution $P_X$ is chosen, the corresponding output distribution, over the very noisy channel, can be written as, $\forall b \in Y$,

$$P_{Y,\varepsilon}(b) = \sum_{a \in \mathcal{X}} P_X(a) W_\varepsilon(b|a)$$

$$= P_N(b) \left(1 + \varepsilon \sum_a P_X(a) L(a,b)\right)$$

$$= P_N(b)(1 + \varepsilon \bar{L}(b))$$

where $\bar{L}(b) = \sum_a P_X(a) L(a,b)$, $\forall a \in \mathcal{X}$. Hence, a codeword which is sent and the received output have components which are i.i.d. from the following distribution

$$P_X \circ W_\varepsilon = P_X P_N(1 + \varepsilon L),$$

and similarly, the codeword which is not sent and the received output have components which are i.i.d. from the following distribution

$$(P_X \circ W_\varepsilon)^p = P_X P_N(1 + \varepsilon \bar{L}).$$

Therefore, the mutual information for very noisy channels is given by

$$I(P_X, W_\varepsilon) = D(P_X P_N(1 + \varepsilon L)\|P_N P_N(1 + \varepsilon \bar{L}))$$

$$= \frac{\varepsilon^2}{2} \|\bar{L}\|^2 + o(\varepsilon^2),$$

where

$$\| \cdot \| = \| \cdot \|_{P_X \times P_N}$$

and

$$\bar{L}(a,b) \triangleq L(a,b) - \bar{L}(b),$$

which we call the centered directions.

### A. Very Noisy with Mismatched Decoder

As stated in Lemma 2, for an input distribution $P_X$, a mismatched linear decoder induced by the metric $d$, when the true channel is $W_0$, can achieve the following rate

$$\inf_{\mu \in \mathcal{A}} D(\mu \| \mu_0^d)$$

where

$$\mathcal{A} = \{\mu : \mu_X = P_X, \mu_Y = (\mu_0)_Y, E_\mu[d] \geq E_{\mu_0}[d]\}.$$ 

Now, if the channels are very noisy, this achievable rate can be expressed in the following simple form.

**Proposition 1:** Let $W_{0,\varepsilon} = P_N(1 + \varepsilon L_0)$ and $d_\varepsilon = \log W_{1,\varepsilon}$, where $W_{1,\varepsilon} = P_N(1 + \varepsilon L_1)$. For a given input distribution $P_X$, we can achieve the following rate

$$\lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} R(P_X, W_{0,\varepsilon}, d_\varepsilon) = \begin{cases} \frac{(\bar{L}_0, \bar{L}_1)^2}{\|L_1\|^2}, & \text{when } \langle \bar{L}_0, \bar{L}_1 \rangle \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Note that it is w.l.o.g. to consider the single-letter metric to be the log of a channel, however, we do restrict all channels to be around a common $P_N$ distribution.

Previous result says that the mismatched mutual information obtained when decoding with the linear decoder induced by the mismatched metric $\log W_{1,\varepsilon}$, whereas the true channel is $W_{0,\varepsilon}$, is approximately the projections’ squared norm of the true channel centered direction $\bar{L}_0$ onto the mismatched centered direction $\bar{L}_1$. This result gives an intuitive picture of the mismatched mutual information, as expected, if the decoder is matched, i.e. $\bar{L}_0 = \bar{L}_1$, the projections’ squared norm is $\|\bar{L}_0\|^2$, which is the very noisy mutual information of $\bar{L}_0$; and the more orthogonal $\bar{L}_1$ is to $\bar{L}_0$, the more mismatched the decoder is, with a lower achievable rate (eventually 0).

**Proof:** For each $\varepsilon$, the minimizer $\mu_\varepsilon$ can be expressed as

$$\mu_\varepsilon = P_X P_N(1 + \varepsilon L)$$

where $L$ is a function on $\mathcal{X} \times Y$, satisfying

$$\sum_{a \in \mathcal{X}, b \in Y} P_X(a) P_N(b) L(a,b) = 0$$

and the two marginal constraints, resp.

$$(\mu_\varepsilon)_X = P_X \iff \sum_{b \in Y} P_N(b) L(a,b) = 0, \forall a \in \mathcal{X}$$

$$(\mu_\varepsilon)_Y = (\mu_0)_Y \iff \sum_{a \in \mathcal{X}} P_X(a) L_0(a,b) = \sum_{a \in \mathcal{X}} P_X(a) L_0(a,b), \forall b \in Y$$

Now the constraint $E_\mu[\log W_{1,\varepsilon}] \geq E_{\mu_0}[\log W_{1,\varepsilon}]$ can be written as

$$\sum_{a \in \mathcal{X}, b \in Y} P_X(a) P_N(b)(1 + \varepsilon L(a,b))$$

$$\geq \sum_{a \in \mathcal{X}, b \in Y} P_X(a) P_N(b)(1 + \varepsilon L_0(a,b))$$

$$\geq \sum_{a \in \mathcal{X}, b \in Y} P_X(a) P_N(b)(1 + \varepsilon L_0(a,b))$$

Using a first order Taylor expansion for the two log terms, and the marginal constraint (10), we have that previous constraint is equivalent to

$$\langle L, L_1 \rangle \geq \langle L_0, L_1 \rangle + o(1),$$

where

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{P_X \times P_N}.$$  

(12)

Finally, we can write the objective function as

$$D(\mu_\varepsilon \| \mu_0^d) = D(P_X P_N(1 + \varepsilon L)\|P_X P_N(1 + \varepsilon \bar{L}_0))$$

$$= \frac{\varepsilon^2}{2} \|L - \bar{L}_0\|^2_{P_X \times P_N} + o(\varepsilon^2)$$

So we have transformed the original optimization problem into the very noisy setting

$$\lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} \inf_{\mu \in \mathcal{A}} D(\mu \| \mu_0^d) = \inf_{L : \langle L, L_1 \rangle \geq \langle L_0, L_1 \rangle} \|L - \bar{L}_0\|^2.$$ 

(13)
where the optimization on the RHS is over \( L \) satisfying the marginal constraints (9) and (10).

Now this optimization can be further simplified. By noticing that (10) implies \( L = L_0 \), we have that \( L - L_0 = L - \bar{L} \), which we defined to be \( \bar{L} \). So \( L \) satisfies both marginal constraints and the constraint in (13) becomes
\[
\langle L, L_1 \rangle \geq \langle L_0, L_1 \rangle \quad \Leftrightarrow \quad \langle \bar{L}, L_1 \rangle \geq \langle L_0, L_1 \rangle \quad \Leftrightarrow \quad \langle \bar{L}, L_1 \rangle \geq \langle L_0, \bar{L}_1 \rangle
\]
That is, both the objective and the constraint functions are now written in terms of centered directions, \( \bar{L} \). Hence, (13) becomes
\[
\inf_{L : \langle \bar{L}, L_1 \rangle \geq \langle L_0, \bar{L}_1 \rangle} \| \bar{L} \|^2
\]
and we can simply recognize that, if \( \langle L_0, \bar{L}_1 \rangle \geq 0 \), the minimizer of this expression is obtained by the projection of \( L_0 \) onto \( \bar{L}_1 \), with a minimum given by the projections’ squared norm:
\[
\frac{(\bar{L}_0, \bar{L}_1)^2}{\| \bar{L}_1 \|^2},
\]
otherwise, if \( \langle L_0, \bar{L}_1 \rangle < 0 \), the minimizer is \( \bar{L} = 0 \), leading to a zero rate.

Remark: We have just seen two examples where in the very noisy limit, information theoretic quantities have a natural geometric meaning, in the previously described inner product space. The cases treated in this section are the ones relevant for the paper’s problem, however, following similar expansions, other information theoretic problems, in particular multi-user ones (e.g. broadcast or interference channels) can also be treated in this geometrical setting. To simplify the notation, since the very noisy expressions scale with \( \varepsilon^2 \) and have a factor \( \frac{1}{2} \) in the limit, we denote by \( \text{VN} \) the following operator:
\[
T(\varepsilon) \xrightarrow{\text{VN}} \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} T(\varepsilon).
\]

We use the abbreviation VN for very noisy. Note that the main reason why we use the VN limit in this paper is similar somehow to the reason why we consider infinite block length in information theory: it gives us a simpler model to analyze and helps us understanding the more complex (not necessarily very noisy) general model. This makes the VN limit more than just an approximation for a specific regime of interest, it makes it an analysis tool of our problems, by setting them in a geometric framework where notion of distance and angles are this time well defined. Moreover, as we will show in section \ref{sec:comp-geo} in some cases, results proven in the VN limit can in fact be “lifted” to results proven in the general cases.

IV. \textbf{Linear Decoding for Compound Channel: The Very Noisy Case}

In this section, we will study a special case of the compound channel, the very noisy case. The local geometric analysis introduced in the previous section can be immediately applied to such problems. Throughout this process, we will develop a few important concepts that will be used in solving the general compound channel problems, in section \ref{sec:comp-geo} In the following, we first make clear of our assumptions, and introduce some notations.

- All the channels are very noisy, with the same pure noise distribution. That is, all considered channels are of the form
\[
W_\varepsilon(b|a) = P_N(b)(1 + \varepsilon L(a, b)), \quad \forall a \in \mathcal{X}, b \in \mathcal{Y}
\]
where \( L \) satisfies \( \sum_b P_N(b)L(a, b) = 0, \forall a \). The compound set is hence depending on \( \varepsilon \), and is expressed as
\[
S_\varepsilon = \{ P_N(1 + \varepsilon L)L \in \mathcal{S} \},
\]
where \( \mathcal{S} \) is the set of all possible directions. Hence, \( \mathcal{S} \) together with the pure noise distribution \( P_N \), completely determine the compound set for any \( \varepsilon \). We refer to \( \mathcal{S} \) as the compound set in the VN setting. Note that \( \mathcal{S} \) being convex, resp. compact, is the sufficient and necessary condition that \( S_\varepsilon \) is convex, resp. compact, for all \( \varepsilon \).
- \( P_X \) is fixed (it is the optimal input distribution) and we write
\[
\mu_\varepsilon = P_X P_N(1 + \varepsilon L), L \in \mathcal{S}
\]
as the joint distribution of the input and output over a particular channel. For a given channel \( W_\varepsilon \), the output distribution is \( P_N(1 + \varepsilon L) \), where
\[
\bar{L}(b) = \sum_{a \in \mathcal{X}} L(a, b) P_X(a), \quad \forall b \in \mathcal{Y}
\]
and as before, \( \bar{L} = L - \bar{L} \). We then denote \( \bar{S} = \{ \bar{L} : L \in \mathcal{S} \} \). Again, the convexity and compactness of \( \mathcal{S} \) is equivalent to those of \( \bar{S} \). The only difference is that \( \mathcal{S} \) depends on the channels only, whereas \( \bar{S} \) depends on the input distribution as well. As we fix \( P_X \) in this section, we use the conditions \( L \in \mathcal{S} \) and \( \bar{L} \in \bar{S} \) exchangeably.
- As a convention, we often give an index, \( j \), to the possible channels, and we naturally associate the channel index (the joint distribution index) and the direction index, i.e.
\[
W_{j,\varepsilon} = P_N(1 + \varepsilon L_j) \quad \text{and} \quad \mu_{j,\varepsilon} = P_X P_N(1 + \varepsilon L_j).
\]
In particular, we reserve \( W_{0,\varepsilon} = P_N(1 + \varepsilon L_0) \) for the true channel and use other indices, \( L_1, L_2 \), etc. for other specific channels.
- If one considers the metrics to be the log of some channels, i.e., \( d_j = \log W_{j,\varepsilon} \),
\[
d_j,\varepsilon = \log W_{j,\varepsilon} = \log(P_N) + \log(1 + \varepsilon L_j).
\]
In general, the single-letter decoding metric \( d \) does not have to be the log likelihood of a channel; and even if it is, the channel \( W_{j,\varepsilon} \) does not have to be in the compound set.
- We write all inner products and norms as weighted by \( P_X \times P_N \), and omit the subscript:
\[
\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{P_X \times P_N}.
\]
- Finally,
\[
\min_{W \in \mathcal{S}_\varepsilon} I(P_X, W) = \frac{\varepsilon^2}{2} \min_{L \in \mathcal{S}} \| \bar{L} \|^2 + o(\varepsilon^2)
\]
and we define
\[ L_S = \arg \min_{L \in \mathcal{S}} \|\tilde{L}\|^2, \]

to be the worst direction and \(\|\tilde{L}_S\|^2\) is referred to as the very noisy compound channel capacity (on \(\mathcal{S}\)).

We conclude this section with the following lemma, which will be frequently used in the subsequent.

**Lemma 4:** Let \(L_i, L_j, L_k\) and \(L_l\) be four directions and assume that \(\sum_a P_X(a)L_i(a) = \sum_a P_X(a)L_k(a)\). We then have
\[
E_{\mu_{i,\varepsilon}} \log W_{j,\varepsilon} > E_{\mu_{k,\varepsilon}} \log W_{l,\varepsilon}
\]
\[
\frac{\nu_{\text{VN}}}{\|\langle L_i, L_j \rangle\|} - \frac{1}{2} \|L_j\|^2 > \langle L_k, L_l \rangle - \frac{1}{2} \|L_l\|^2.
\]

**Proof:** Using a second order Taylor expansion for \(\log(1 + \varepsilon L_j)\), we have
\[
E_{\mu_{i,\varepsilon}} \log W_{j,\varepsilon} = \sum P_X P_N(1 + \varepsilon L_i) \log(P_N(1 + \varepsilon L_i))
\]
= \(\sum P_X P_N \log P_N + \varepsilon \sum P_X P_N L_i + \varepsilon^2 \sum P_X P_N L_i L_j - \varepsilon \frac{1}{2} \sum P_X P_N L_i^2\)
(14)

The only term which is zero in previous summation is the third term, namely \(\sum P_X P_N L_i = 0\), which is a consequence of the fact that \(L_j\) is a direction (i.e. \(\sum P_N L_j = 0\)). Now, when we look at the inequality \(E_{\mu_{i,\varepsilon}} \log W_{j,\varepsilon} > E_{\mu_{k,\varepsilon}} \log W_{l,\varepsilon}\), we can surely simplify the term \(\sum P_X P_N \log P_N\), since it appears both on the left and right hand side. Moreover, using the assumption that \(\sum P_X(a)L_i(a) = \sum P_X(a)L_k(a)\), we have \(\sum P_X P_N L_i \log P_N = \sum P_X P_N L_i \log P_N\). Hence, the only terms that survive in (14), when computing \(E_{\mu_{i,\varepsilon}} \log W_{j,\varepsilon} > E_{\mu_{k,\varepsilon}} \log W_{l,\varepsilon}\), are the terms in \(\varepsilon^2\), which proves the lemma.

### A. One-sided Sets

We consider for now the use of linear decoder (i.e., induced by only one metric). We recall that, as proved in previous section, for \(W_{0,\varepsilon} = P_N(1 + \varepsilon L_0)\) and \(d_\varepsilon = \log W_{1,\varepsilon}\), where \(W_{1,\varepsilon} = P_N(1 + \varepsilon L_1)\), we have
\[
\lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} R(P_X, W_{0,\varepsilon}, d_\varepsilon) = \begin{cases} \frac{\langle \tilde{L}_0, \tilde{L}_1 \rangle^2}{\|\tilde{L}_1\|^2}, & \text{when } \langle \tilde{L}_0, \tilde{L}_1 \rangle \geq 0 \\ 0, & \text{otherwise} \end{cases}
\]
This picture of the mismatched mutual information directly suggests a first result. Assume \(\mathcal{S}\), hence \(\tilde{\mathcal{S}}\), to be convex. By using the worse channel to be the only decoding metric, it is then clear that the VN compound capacity can be achieved. In fact, no matter what the true channel \(L_0 \in \tilde{\mathcal{S}}\) is, the mismatched mutual information given by the projections’ squared norm of \(L_0\) onto \(L_S\) cannot be shorter than \(\|L_S\|^2\), which is the very noisy compound capacity of \(\mathcal{S}\) (cf. Figure 1). This agrees with a result proved in [5].

However, with this picture we understand that the notion of convexity is not necessary. As long as the compound set is such that its projection in the direction of the minimal vector stays on one side, i.e., if the compound set is entirely contained in the half space delimited by the normal plan to the minimal vector, i.e., if for any \(L_0 \in \mathcal{S}\), we have \(\langle \tilde{L}_0, L_S \rangle \geq 0\) and :
\[
\frac{\langle \tilde{L}_0, L_S \rangle^2}{\|L_S\|^2} \geq \|L_S\|^2,
\]
we will achieve compound capacity by using the linear decoder induced by the worst channel metric (cf. figure 1) where \(\mathcal{S}\) is not convex but still verifies the above conditions. We call such sets one-sided sets, as defined in the following.

**Definition 3:** VN One-sided Set

A VN compound set \(\mathcal{S}\) is one-sided iff for any \(L_0 \in \mathcal{S}\), we have
\[
\langle \tilde{L}_0, L_S \rangle \geq 0,
\]
\[
\frac{\langle \tilde{L}_0, L_S \rangle^2}{\|L_S\|^2} \geq \|L_S\|^2.
\]
Equivalently, a VN compound set \(\mathcal{S}\) is one-sided iff for any \(L_0 \in \mathcal{S}\), we have
\[
\|\tilde{L}_0\|^2 - \|L_S\|^2 - \|\tilde{L}_0 - L_S\|^2 \geq 0.
\]

**Proposition 2:** In the VN setting, the linear decoder induced by the worst channel metric \(\log L_S\) is capacity achieving for one-sided sets.

The very noisy picture also suggests that the one-sided property is indeed necessary in order to be able to achieve the compound capacity with a single linear decoder. However, our main goal here is not motivated by results of this kind and we will not discuss this in more details. We now investigate whether we can still achieve compound capacity on non one-sided compound sets, by using generalized linear decoders.
B. Finite Sets

Let us consider a simple case of non one-sided set, namely when \( S \) contains only two channels that are not satisfying the one-sided property in (17). We denote the set by

\[
S = \{ W_0, W_1 \}.
\]

and it contains the true channel \( W_0 \) and an arbitrary other channel \( W_1 \). A first idea is to use a generalized decoder induced by the two metrics \( d_1 = \log W_0 \) and \( d_2 = \log W_1 \), i.e. decoding with the GLRT test using both channels, which defines the following decoding map

\[
\arg \max_{x_m} W_0^n(y|x_m) \lor W_1^n(y|x_m).
\]

The maximization of \( W_0^n(y|x_m) \) corresponds to the maximization of an optimal ML decoder with the true channel, whereas the maximization of \( W_1^n(y|x_m) \) corresponds to the maximization of ML decoder with a mismatched metric, which may have nothing to do with the true channel metric. So we need to estimate how probable it is that a codeword which has may have nothing to do with the true channel metric. So we define

\[
W = \{ W : \hat{L} = \bar{L}_0, \langle L, L_1 \rangle > \frac{1}{2}((\| L_0 \|^2 + \| L_1 \|^2)) \}
\]

Putting pieces together we get

\[
R_{1, \epsilon} \longrightarrow \bigcap C(S) \longrightarrow \| \bar{L}_0 \|^2 \land \| \bar{L}_1 \|^2.
\]

Finally, using lemma [4] we have

\[
E_{\mu_\epsilon} \log W_{1, \epsilon} > E_{\mu_{0, \epsilon}} \log W_{0, \epsilon} \quad \text{ VN } \langle L, L_1 \rangle \rightarrow \frac{1}{2} \| L_1 \|^2 > \frac{1}{2} \| L_0 \|^2.
\]

Hence

\[
\mathcal{A}_{1, \epsilon} \longrightarrow \{ L : \bar{L} = \bar{L}_0, \langle L, L_1 \rangle > \frac{1}{2}((\| L_0 \|^2 + \| L_1 \|^2)) \}
\]

\[
= \{ \bar{L} : \langle \bar{L}, \bar{L}_1 \rangle > \frac{1}{2}((\| \bar{L}_0 \|^2 + \| L_1 \|^2) - \langle \bar{L}_0, \bar{L}_1 \rangle) \}.
\]

Note that we used \( \bar{L} = \bar{L}_0 \) to get (20) from its previous line. Putting pieces together we get

\[
R_{1, \epsilon} \longrightarrow \bigcap C(S) \longrightarrow \| \bar{L}_0 \|^2 \land \| \bar{L}_1 \|^2.
\]

Therefore, the inequality which allows us to verify locally if the proposed decoding rule achieves compound capacity, i.e.

if \( R_1 \geq C(S) \) in the VN setting, is given by

\[
R_{1, \epsilon} \geq C(S) \quad \text{ VN } \quad \frac{1}{2}((\| L_0 \|^2 + \| L_1 \|^2) - \langle L_0, L_1 \rangle^2) \land \| \bar{L}_0 \|^2 \land \| \bar{L}_1 \|^2.
\]

But

\[
\frac{1}{2}((\| L_0 \|^2 + \| L_1 \|^2) - \langle L_0, L_1 \rangle^2) \geq \| \bar{L}_0 \|^2 \land \| \bar{L}_1 \|^2,
\]

hence, (21) is equivalent to

\[
\frac{1}{2}((\| L_0 \|^2 + \| L_1 \|^2) - \langle L_0, L_1 \rangle \geq \| \bar{L}_0 \|^2 \land \| \bar{L}_1 \|^2,
\]

which clearly holds no matter what \( L_0 \) and \( L_1 \) are. This can be directly generalized to any finite sets and we have the following result.

**Proposition 3:** In the VN setting, GLRT with all channels in the set is capacity achieving for finite compound sets, and generalized linear.

C. Finite Union of One-sided Sets

1) Using ML Metrics: In the previous sections, we have found linear, or generalized linear, decoders that are capacity achieving for one-sided sets and for finite sets. Next we consider compound sets that are finite unions of one-sided sets and hope to combine our results in these two cases. Assume

\[
S = S_1 \cup S_2,
\]
where \( S_1 \) and \( S_2 \) are one-sided: in this section we consider only the VN setting, hence saying that \( S_1 \) is one sided really means that the VN compound set \( S_1 \) corresponding to \( S_{1,c} \) is one-sided according to Definition 4.

For a fixed input distribution \( P_X \), let \( W_1 = W_{S_1} \) and \( W_2 = W_{S_2} \) be the worst channel of \( S_1, S_2 \), respectively (cf. figure 2). A plausible candidate for a generalized linear universal decoder the GLRT with metrics \( d_1 = \log W_1 \) and \( d_2 = \log W_2 \), hoping that a combination of earlier results for finite and one-sided sets would make this decoder capacity achieving. Say w.l.o.g. that \( W_0 \in S_1 \). Using (6), the following rate can be achieved with the proposed decoding rule:

\[
R(P_X, W_0, \{d_k\}_{k=1}^K) = R_1 \land R_2
\]

where

\[
R_k = \inf_{\mathcal{A}_k} D(\mu || \mu_0^k), \quad k = 1, 2
\]

and for \( k = 1, 2 \),

\[
\mathcal{A}_k = \{ \mu : \mu = \mu_0^k, E_\mu \log W_k \geq \vee_{i=1}^2 E_{\mu_0} \log W_i \},
\]

(22)

Note that we are using similar notations for this section as for the previous one, although the sets \( \mathcal{A}_k \) and rates \( R_k \) are now given by different expressions. We also use \( \mu^0 = \mu_0^0 \) to express in a more compact way that the marginals of \( \mu \) and \( \mu_0 \) are the same.

Since \( W_1 \) and \( W_2 \) are the worst channel for \( P_X \) in each component, the compound capacity over \( S = S_1 \cup S_2 \) is

\[
C(S) = I(P_X, W_1) \land I(P_X, W_2).
\]

In the finite compound set case of previous section, we further simplified the expression of the \( \mathcal{A}_k \)'s, since we the maximum in \( \vee_{i=1}^2 E_{\mu_0} \log W_i \) could be identified. This is no longer the case here, and we have to consider both cases, i.e.:

Case 1: \( E_{\mu_0} \log W_1 \geq E_{\mu_0} \log W_2 \) \hspace{1cm} (23)

Case 2: \( E_{\mu_0} \log W_1 \leq E_{\mu_0} \log W_2 \). \hspace{1cm} (24)

In order to verify that the decoder is capacity achieving, we need to check if both \( R_1 \) and \( R_2 \) are greater than or equal to the compound capacity \( C(S) \), no matter which of case 1 or case 2 occurs. Thus, there are totally 4 inequalities to check. While checking these cases is somewhat tedious, we will, in the following, go through each of them carefully and point out a specific case that is problematic, before giving a counterexample where GLRT with the worst channels is in fact not capacity achieving. Later when we propose a capacity achieving decoder, we will go through a similar procedure in a more concise way.

Note that under case 1,

For case 1: \( \mathcal{A}_1 = \{ \mu : \mu = \mu_0^0, E_\mu \log W_1 \geq E_{\mu_0} \log W_1 \} \),

which has the form of the constraint set for \( R(P_X, W_0, d_1) \) expressed in (4). Hence we have

\[
R_1 = R(P_X, W_0, \log W_1).
\]

(25)

As shown in section [IV-A] \( R(P_X, W_0, \log W_1) \) becomes in the VN limit:

\[
R(P_X, W_0, \log W_1, \epsilon) \xrightarrow{\text{VN}} \frac{\langle \vec{L}_0, \vec{L}_1 \rangle^2}{\|\vec{L}_1\|^2}
\]

(26)

(note that since \( S_1 \) is one-sided, \( \langle \vec{L}_0, \vec{L}_1 \rangle \geq 0 \)). Also, in the VN limit, \( C(S_\epsilon) \) becomes \( \|\vec{L}_1\|^2 \land \|\vec{L}_2\|^2 \), hence

\[
\text{For case 1: } R_{1,\epsilon} \geq C(S_\epsilon) \xrightarrow{\text{VN}} \frac{\langle \vec{L}_0, \vec{L}_1 \rangle^2}{\|\vec{L}_1\|^2} \geq \|\vec{L}_1\|^2 \land \|\vec{L}_2\|^2.
\]

(27)

But we assumed that \( S_1 \) is one-sided and that \( L_1 \) is the worst direction of \( S_1 \). Moreover, we assumed that \( W_0 \in S_1 \), i.e. \( L_0 \in S_1 \). Hence, (27) holds by definition of one-sided sets, cf. def. 17 (with this definition, (27) holds with \( \|\vec{L}_1\|^2 \) on the right hand side, hence it holds for \( \|\vec{L}_1\|^2 \land \|\vec{L}_2\|^2 \).

For case 2, i.e. when \( E_{\mu_0} \log W_1 \leq E_{\mu_0} \log W_2 \), we have

\[
R_1 = \inf_{\mu \in A_1} D(\mu || \mu_0^0), \quad \text{where this time } A_1 \text{ is given by}
\]

For case 2: \( A_1 = \{ \mu : \mu = \mu_0^0, E_\mu \log W_1 \geq E_{\mu_0} \log W_2 \} \)

Note that, by definition of case 2, the constraint set \( A_1 \) is smaller than the constraint set \( B \) given below:

\[
A_1 = \{ \mu : \mu = \mu_0^0, E_\mu \log W_1 \geq E_{\mu_0} \log W_2 \}
\subset
B = \{ \mu : \mu = \mu_0^0, E_\mu \log W_1 \geq E_{\mu_0} \log W_1 \}
\]

(29)

hence,

\[
\inf_{\mu \in A_1} D(\mu || \mu_0^0) \geq \inf_{\mu \in B} D(\mu || \mu_0^0).
\]

But \( B \) is the constraint set appearing in \( R(P_X, W_0, \log W_1) \), which means that

\[
\inf_{\mu \in B} D(\mu || \mu_0^0) = R(P_X, W_0, \log W_1),
\]

therefore, under case 2, we showed that \( R_1 \geq R(P_X, W_0, \log W_1) \). Now, as shown before, \( R(P_X, W_0, \log W_1) \) is locally lower bounded by \( I(P_X, W_1) \geq C(S) \), by the one-sided assumption on \( S_1 \).

![Fig. 2. A VN Compound set which is the union of two one-sided components, \( S_1 \) and \( S_2 \), drawn in the space of centered directions (tilde vectors).]
Hence, we have just shown that $R_1 \geq C(S)$, both under case 1 and 2.

Next, we check whether $R_2 = \inf_{\mu \in A_2} D(\mu \| \mu_0) \geq C(S)$ holds or not. We have again to check this for case 1 and 2. This time we start with case 2. Note that the expression of $R_2$ in case 2 is perfectly symmetric to the expression of $R_1$ in case 1, we just have to swap the indices 1 and 2, hence

For case 2: $R_2 = R(P_X, W_0, \log W_2)$

and the inequality we need to check in the very noisy case is

For case 2: $R_{2,\epsilon} \geq C(S) \epsilon \rightarrow \inf_{V} \frac{\langle \tilde{L}_0, \tilde{L}_2 \rangle^2}{\| L_2 \|^2} \geq \| \tilde{L}_1 \|^2 \land \| \tilde{L}_2 \|^2$.

(30)

However, the one-sided property does not apply anymore, since we assumed that $L_0$ belongs to $S_1$ and not $S_2$. Indeed, if we have no restriction on the positions of $L_0$ and $L_2$, (30) can be zero. Comparing this with the case of a single one-sided set, we see this is exactly the difficulty of analyzing generalized linear decoders. Using multiple metrics, especially $d_2 = \log W_2$, which does not have any one-sided relation with the actual channel $W_0$, causes an extra chance of making errors: an incorrect codeword can appear very plausible according to metric $d_2$. The probability for this to happen is captured by the rate $R_2$. On the other hand, there is also a lower target: (30) should not hold for any possible $\tilde{L}_0$, $\tilde{L}_1$ and $\tilde{L}_2$, (30) should hold when these centered directions are satisfying case 2. Moreover, the compound capacity is now the minimum between the mutual informations $\| \tilde{L}_1 \|^2$ and $\| \tilde{L}_2 \|^2$. One might hope that the combination of all these effects leads to $R_2 > C(S)$ and hence a capacity achieving decoder design. Unfortunately, this is not the case.

**Proposition 4:** In the VN setting and for compound sets having a finite number of one-sided components, GLRT with the worst channel of each component is not capacity achieving.

**Counterexample:** Let $X = Y = \{0,1\}$, $P_X = P_N = \{1/2, 1/2\}$,

$L_0 = \begin{pmatrix} -2 \\ -7 \end{pmatrix}, L_1 = \begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$ and $L_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

The achievable rate can be easily checked with this counterexample, and in fact there are many other examples that one can construct. We will, in following, discuss the geometric insights that leads to these counterexamples (and check that it is indeed a counterexample). This will also be valuable in constructing better decoders in the next section.

We first use Lemma 4 to write

$E_{\mu_0,\epsilon} \log W_{1,\epsilon} \leq E_{\mu_0,\epsilon} \log W_{2,\epsilon} \rightarrow \inf_{V} \| L_0 - L_2 \| \leq \| L_0 - L_1 \|$, which can be use to rewrite (23) and (24) in the very noisy setting as

Case 1: $\| L_0 - L_2 \| \geq \| L_0 - L_1 \|$ \hspace{1cm} (31)

Case 2: $\| L_0 - L_2 \| \leq \| L_0 - L_1 \|$. \hspace{1cm} (32)

Now to construct a counterexample, we consider the special case where $\| L_0 - L_2 \| = \| L_0 - L_1 \|$ and $\| L_1 \| = \| L_2 \|$. These assumptions are used to simplify our discussion, and are not necessary in constructing counterexamples. One can check that the above example satisfies both assumptions. Now (30) holds if and only if

$$\frac{\langle \tilde{L}_0, \tilde{L}_2 \rangle^2}{\| L_2 \|^2} \geq \| L_2 \|,$$

which is equivalent to

$$\| L_0 \|^2 - \| L_2 \|^2 - \| L_0 - L_2 \|^2 \geq 0.$$

It is easy to check that the last inequality does not hold for the given counterexample, which completes the proof of Proposition 4. Indeed, one can write

$$\| L_0 \|^2 - \| L_2 \|^2 - \| L_0 - L_2 \|^2 = \| L_0 \|^2 - \| L_1 \|^2 - \| L_0 - L_1 \|^2 + \| L_0 - L_1 \|^2 - \| L_0 - L_2 \|^2$$

(33)

The term on the second line above is always positive (by the one-sided property), but we have a problem with the term on the last line: we assumed that $\| L_0 - L_2 \| = \| L_0 - L_1 \|$, and this does not imply that $\| L_0 - L_1 \|^2 = \| L_0 - L_2 \|^2$. The problem here is that when using log likelihood functions as decoding metrics, the constraints in (22), (23) and (24) are, in the very noisy case, given in terms of the perturbation directions $L_i, i = 0, 1, 2$, while the desired statement about achievable rates and the compound capacity are given in terms of the centered directions $\tilde{L}_i$’s. Thus, counterexamples can be constructed by carefully assign $L_i$’s to be different, hence the constraints on $L_i$’s cannot effectively regulate the behavior of $\tilde{L}_i$’s (35) can be made negative). Figure 5 gives a pictorial illustration of this phenomenon. The above discussion also suggests a fix to the problem. If one could replace the constraints on $L_i$’s in (22), (23) and (24), by the corresponding constraints on $\tilde{L}_i$’s, that might at least allow better controls over the achievable rates. This is indeed possible by making a small change of the decoding metrics, as done in the following section.

2) Using MAP Metrics: We now use different metrics than the one used in previous section, instead of the ML metrics given by $\log W_k$, we use the metrics

$$\log \frac{W_k}{(\mu_k)_Y},$$

which we call the MAP metrics for maximum a posteriori and which may also be referred as the Fano metrics in the literature.

As before, let us consider $W_0$, $W_1$ and $W_2$ such that $W_1$ and $W_2$ are the worst channels of two one-sided components $S_1$ and $S_2$, and $W_0$ belongs to $S_1$. Using (6), with $d_1 = \log \frac{W_1}{(\mu_1)_Y}$ and $d_2 = \log \frac{W_2}{(\mu_2)_Y}$, the proposed generalized linear decoder can achieve

$$R(P_X, W_0, \{d_k\}_{k=1}) = R_1 \land R_2.$$
follows since (36) gives a more stringent constraint in \( \mathcal{W} \) as before, and thus the achievable rate is the mismatched we see that the optimization problem is exactly the same 
\[ \mu \]
Noticing that 
\[ \mu \]
As we did for (23) and (24), we consider separately two 
\[ \mu = \mu_0, \mu = \mu_0, E \mu \log \frac{W_{k}}{(\mu_k)^{Y}} \geq \sum_{i=1}^{2} E \mu_0 \log \frac{W_{i}}{(\mu_i)^{Y}} \right \}, \]
Note that again, we use same notations for this section as for the previous one, although the sets \( \mathcal{A}_k \) and rates \( R_k \) are now given by different expressions. Since \( W_1 \) and \( W_2 \) are the worst channel for \( P_X \) in each component, the compound capacity over \( S = S_1 \cup S_2 \) is still given by 
\[ C(S) = I(P_X, W_1) \land I(P_X, W_2). \]
As we did for (23) and (24), we consider separately two cases:

**Case 1:** 
\[ E_{\mu_0} \left( \frac{W_{1}}{(\mu_1)^{Y}} \right) \geq E_{\mu_0} \left( \frac{W_{2}}{(\mu_2)^{Y}} \right) \]  \quad (35)

**Case 2:** 
\[ E_{\mu_0} \left( \frac{W_{2}}{(\mu_2)^{Y}} \right) \leq E_{\mu_0} \left( \frac{W_{1}}{(\mu_1)^{Y}} \right). \]  \quad (36)

Following the same argument as in the last section, we verify that \( R_1 \geq C(S) \) under both cases. Note that in case 1, the constraint in \( \mathcal{A}_1 \) is 
\[ E \mu \log \frac{W_{1}}{(\mu_1)^{Y}} \geq E \mu_0 \log \frac{W_{1}}{(\mu_1)^{Y}} \],
Comparing this with its counterpart for in ML decoding, the only difference is the extra \( E \log(\mu_1)^{Y} \) terms on both sides. Noticing that \( \mu \) and \( \mu_0 \) have the same \( Y \) marginal distribution, we see that the optimization problem is exactly the same as before, and thus the achievable rate is the mismatched rate \( R(P_X, W_0, \log W_1) \), which by the one-sided assumption \( W_0 \in S_1 \) is higher than \( I(P_X, W_1) \). In case 2, \( R_1 \geq C(S) \) follows since (36) gives a more stringent constraint in \( \mathcal{A}_1 \), and hence a higher achievable rate (conf. (29)). Hence, just like it was the case for the ML decoding metrics, \( R_1 \geq C(S) \) is easily checked with the one-sided property. We now show that as opposed to the ML case, with the MAP metrics, we also have \( R_2 \geq C(S) \).

The main difference between the proposed MAP decoding metric and the ML metric used in the previous section can be seen clearly from the very noisy setting. Using a similar argument as in Lemma 4 we have

\[ E_{\mu_0, e} \log \frac{W_{1, e}}{(\mu_1, e)^{Y}} \]
\[ \stackrel{\text{VN}}{\rightarrow} \langle \tilde{L}_0, \tilde{L}_1 \rangle - \frac{1}{2} \|	ilde{L}_1\|^2 = \frac{1}{2} (\|	ilde{L}_0\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2). \]  \quad (37)

Thus, the optimization in \( R_k \) are over the sets

\[ \mathcal{A}_{k, e} = \{ L : \tilde{L} = \tilde{L}_0 : \]
\[ \langle \tilde{L}, \tilde{L}_k \rangle - \frac{1}{2} \|	ilde{L}_k\|^2 \geq \sqrt{\frac{1}{2} (\|	ilde{L}_0\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2)} \]
and the two cases to be considered are

Case 1: 
\[ \|	ilde{L}_0 - \tilde{L}_1\|^2 \leq \|	ilde{L}_0 - \tilde{L}_2\|^2 \]  \quad (39)

Case 2: 
\[ \|	ilde{L}_0 - \tilde{L}_1\|^2 \geq \|	ilde{L}_0 - \tilde{L}_2\|^2. \]  \quad (40)

These expressions are almost the same as the ones for the ML metric, the very noisy version of (22), (31), and (32), except now we have the conditions on the centered directions (tilde vectors). As discussed in the proof of Proposition 4, this change is precisely what is needed to avoid the counter example. It turns out that this change is also sufficient for the decoder to be capacity achieving.

Now what remains to be proved is that \( R_2 \geq C(S) \). Using (37), and noticing the marginal constraints, we have for case 1

\[ R_{2, e} \stackrel{\text{VN}}{\rightarrow} \min_{L: \tilde{L} = \tilde{L}_0, \langle \tilde{L}, \tilde{L}_2 \rangle \geq \frac{1}{2} (\|	ilde{L}_0\|^2 + \|	ilde{L}_2\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2)} \|	ilde{L}\|^2 \]
and for case 2

\[ R_{2, e} \stackrel{\text{VN}}{\rightarrow} \min_{L: \tilde{L} = \tilde{L}_0, \langle \tilde{L}, \tilde{L}_2 \rangle \geq \frac{1}{2} (\|	ilde{L}_0\|^2 + \|	ilde{L}_2\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2)} \|	ilde{L}\|^2. \]

These optimizations can be explicitly solved as projections:

For Case 1: 
\[ R_{2, e} \stackrel{\text{VN}}{\rightarrow} \frac{1}{4} \frac{(\|	ilde{L}_0\|^2 + \|	ilde{L}_2\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2)^2}{\|	ilde{L}_2\|^2}. \]

For Case 2: 
\[ R_{2, e} \stackrel{\text{VN}}{\rightarrow} \frac{1}{4} \frac{(\|	ilde{L}_0\|^2 + \|	ilde{L}_2\|^2 - \|	ilde{L}_0 - \tilde{L}_1\|^2)^2}{\|	ilde{L}_2\|^2}. \]

Recalling that the compound capacity is given by
\[ C(S) \stackrel{\text{VN}}{\rightarrow} \|	ilde{L}_1\|^2 \land \|	ilde{L}_2\|^2, \]
we have

Case 1: \( R_{2,c} \geq C(S_c) \)
\[
\frac{1}{2} \left( \| \bar{L}_0 \|^2 + \| \bar{L}_2 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \right) \geq ||\bar{L}_1|| \wedge ||\bar{L}_2|| \tag{41}
\]

Case 2: \( R_{2,c} \geq C(S_c) \)
\[
\frac{1}{2} \left( \| \bar{L}_0 \|^2 + \| \bar{L}_2 \|^2 - \| \bar{L}_0 - \bar{L}_2 \|^2 \right) \geq \| \bar{L}_1 \| \wedge \| \bar{L}_2 \| \tag{42}
\]

and we now check that inequalities (41) and (42) hold with \( \| \bar{L}_1 \| \) instead of \( \| \bar{L}_1 \| \wedge \| \bar{L}_2 \| \) on the right hand side.

Starting with (42), we write
\[
\frac{1}{2} \left( \| \bar{L}_0 \|^2 + \| \bar{L}_2 \|^2 - \| \bar{L}_0 - \bar{L}_2 \|^2 \right) - \| \bar{L}_1 \| =
\]
\[
\frac{1}{2} \left( \| \bar{L}_0 \|^2 + \| \bar{L}_2 \|^2 - 2 \| \bar{L}_0 \| \| \bar{L}_2 \| - \| \bar{L}_0 - \bar{L}_2 \|^2 \right)
\]
\[
= \frac{1}{2} \left( \| \bar{L}_1 \| - \| \bar{L}_2 \| \right)^2 + \| \bar{L}_0 \|^2 - \| \bar{L}_1 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \geq \frac{1}{2} \left( \| \bar{L}_1 \| - \| \bar{L}_2 \| \right)^2 + \| \bar{L}_0 \|^2 - \| \bar{L}_1 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \geq 0
\]

where last inequality follows from the one-sided property
\[
\frac{\langle \bar{L}_0, \bar{L}_1 \rangle}{\| \bar{L}_1 \|^2} \geq \| \bar{L}_0 \|^2 - \| \bar{L}_1 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \geq 0
\]

For (41), the same expansion gets us directly to
\[
\frac{1}{2} \left( \| \bar{L}_0 \|^2 + \| \bar{L}_2 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \right) - \| \bar{L}_1 \| =
\]
\[
\frac{1}{2} \left( \| \bar{L}_1 \| - \| \bar{L}_2 \| \right)^2 + \| \bar{L}_0 \|^2 - \| \bar{L}_1 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \geq \frac{1}{2} \left( \| \bar{L}_1 \| - \| \bar{L}_2 \| \right)^2 + \| \bar{L}_0 \|^2 - \| \bar{L}_1 \|^2 - \| \bar{L}_0 - \bar{L}_1 \|^2 \geq 0
\]

again by the one-sided property. Now combining these results, we get that the GMAP decoder is capacity achieving for the VN case. The result can be easily generalized to cases with more than two one-sided components.

**Discussions:**

The above derivations can also be viewed from a pictorial way. We take case 2 for \( R_2 \) for example. The one-sided constraint \( \langle \bar{L}_0, \bar{L}_1 \rangle \geq \| \bar{L}_1 \|^2 \) says that \( \bar{L}_0 \) lies on the right side of \( \bar{L}_1 \); but the constraint for case 2, \( \| \bar{L}_0 - \bar{L}_1 \| \geq \| \bar{L}_0 - \bar{L}_2 \| \), precisely implies that \( \bar{L}_2 \) can only lie in the smaller circle centered at \( \bar{L}_0 \), as in Figure 4, but the small circle intersect the large circle only in the hatched region, where
\[
\frac{\langle \bar{L}_0, \bar{L}_2 \rangle}{\| \bar{L}_2 \|} \geq ||\bar{L}_1|| \wedge ||\bar{L}_2||, \tag{43}
\]

holds. On the other hand, if we work with the ML metrics, the constraint for case 2 is given by \( \| \bar{L}_0 - \bar{L}_1 \| \geq \| \bar{L}_0 - \bar{L}_2 \| \), and how we showed it in the counterexample of section [V C.1] this does no longer force \( \bar{L}_2 \) to lie inside the smaller circle centered at \( \bar{L}_0 \), hence inside the hatched region, as Figure 5 illustrates it.
It is insightful to try to understand the reason that the GMAP decoder works well while the GLRT fails. For a linear decoder with a single metric \( d : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \), if one forms a different test by picking \( d'(x, y) = d(x, y) + f(y) \), for some function \( f : \mathcal{Y} \to \mathbb{R} \), it is not hard to see that the resulting decision is exactly the same, for every possible received signal \( y \). This is why the ML decoder and the MAP decoder, from the same mismatched channel \( W_1 \), are indeed equivalent, as they differ by a factor of \( f = \log(P_X \circ W_1) \). For a generalized linear decoder with multiple metrics, \( d_1, d_2, \ldots, d_K \), if one changes the metrics to \( d_1 + f, d_2 + f, \ldots, d_K + f \), for the same function \( f \) on \( \mathcal{Y} \), again the resulting decoder is the same. Things are different, however, if one changes these metrics by different functions, to have \( d_1 + f_1, \ldots, d_K + f_K \). The problem is that this changes the balance between the metrics, which as we observed in the GMAP story, is critical for the generalized linear decoder to work properly. For example, if one adds a big number on one of the metrics to make it always dominate the others, the purpose of using multiple metrics is defeated. However, if one changes these metrics by different functions, \( \bar{d} \), 

\[
D(\mu_0||\mu_S^K) \geq D(\mu_0||\mu_S) + D(\mu_S||\mu_S^K), \quad \forall W_0 \in S.
\]  

(44)

where

\[
W_S = \arg \min_{W \in S} I(P_X, W).
\]

(45)

and \( \mu_0 = P_X \circ W_0, \mu_S = P_X \circ W_S \), are the joint distribution over the channel \( W_0 \) and \( W_S \), respectively. Note that in order for (44) to hold, the minimizer in (45) must be unique.

Proposition 5: For one-sided sets \( S \), the linear decoder induced by the metric \( d = \log W_S \) is capacity achieving.

Note that in [5], the same linear decoder is proved to be capacity achieving for the case where \( S \) is convex.

Proposition 6: Convex sets are one-sided and there exist one-sided sets that are not convex.

Proposition 7: For any set \( S \), the decoder maximizing the score function \( G_n = \sup_{W \in S} \log W^n \) is capacity achieving, but generalized linear only if \( S \) is finite.

Proposition 8: For \( S = \bigcup_{k=1}^K S_k \), where \( \{S_k\}^K_{k=1} \) are one-sided sets, the generalized linear decoder induced by the metrics \( d_k = \log W_{S_k} \), for \( 1 \leq k \leq K \), is capacity achieving (in general).

The following Theorem is the main result of the paper.

Theorem 1: For \( S = \bigcup_{k=1}^K S_k \), where \( \{S_k\}^K_{k=1} \) are one-sided sets, the generalized linear decoder induced by the metrics \( d_k = \log W_{S_k} \), for \( 1 \leq k \leq K \), is capacity achieving.

V. LINEAR DECODING FOR COMPOUND CHANNEL: THE GENERAL CASE

A. The Results

The previous section gives us a series of results regarding linear decoders on different kinds of compound sets, in the very noisy setting. While focusing on special channels, the geometric insights we developed in the previous section is clearly helpful in understanding the problem in general. In this section, we will show that indeed most of the results reported in the previous section have “natural” counterparts in the general not very noisy cases. Moreover, the proofs of these general results often proceed in a step by step correspondence with that for the very noisy case. We often refer to such procedure of generalizing the results from the very noisy case to the general cases, as “lifting”. In the following, we will first list all the general results, and give proofs in section [IV-B].

Recall the optimal input distribution of a set \( S \) by

\[
P_X = \arg \max_{P \in \mathcal{M}_1(\mathcal{X})} \inf_{W \in S} I(P, W),
\]

and if the maximizers are not unique, we define \( P_X \) to be any arbitrary maximizer.

Definition 4: One-sided Set

A set \( S \) is one-sided, if

\[
D(\mu_0||\mu_S^K) \geq D(\mu_0||\mu_S) + D(\mu_S||\mu_S^K), \quad \forall W_0 \in S.
\]

where

\[
W_S = \arg \min_{W \in S} I(P_X, W).
\]

(45)

and \( \mu_0 = P_X \circ W_0, \mu_S = P_X \circ W_S \), are the joint distribution over the channel \( W_0 \) and \( W_S \), respectively. Note that in order for (44) to hold, the minimizer in (45) must be unique.

Proposition 5: For one-sided sets \( S \), the linear decoder induced by the metric \( d = \log W_S \) is capacity achieving.

Note that in [5], the same linear decoder is proved to be capacity achieving for the case where \( S \) is convex.

Proposition 6: Convex sets are one-sided and there exist one-sided sets that are not convex.

Proposition 7: For any set \( S \), the decoder maximizing the score function \( G_n = \sup_{W \in S} \log W^n \) is capacity achieving, but generalized linear only if \( S \) is finite.

Proposition 8: For \( S = \bigcup_{k=1}^K S_k \), where \( \{S_k\}^K_{k=1} \) are one-sided sets, the generalized linear decoder induced by the metrics \( d_k = \log W_{S_k} \), for \( 1 \leq k \leq K \), is capacity achieving (in general).

The following Theorem is the main result of the paper.

Theorem 1: For \( S = \bigcup_{k=1}^K S_k \), where \( \{S_k\}^K_{k=1} \) are one-sided sets, the generalized linear decoder induced by the metrics \( d_k = \log W_{S_k} \), for \( 1 \leq k \leq K \), is capacity achieving.

B. Proofs: Lifting Local to Global Results

In this section, we illustrate how the results and proofs obtained in section [IV] in the very noisy setting can be lifted to results and proofs in the general setting. We first consider the case of one-sided sets. By revisiting the definitions made in section [IV-A] we will try to develop a “naturally” corresponding notion of one-sidedness for the general problems.

By definition of a VN one-sided set, \( S \) is such that

\[
\|L_0\|^2 - \|L_S\|^2 - \|L_0 - L_S\|^2 \geq 0, \quad \forall L_0 \in S.
\]  

(46)

Next, we find the divergences, for the general problems, whose very noisy representations are these norms: recall that

\[
D(\mu_0||\mu_0^0) \xrightarrow{VN} \|L_0 - \bar{L}_0\|^2 = \|L_0\|^2
\]

and

\[
D(\mu_S||\mu_S^K) \xrightarrow{VN} \|L_S - \bar{L}_S\|^2 = \|L_S\|^2.
\]  

(47)

(48)

On the other hand, we also have

\[
D(\mu_0||\mu_S) \xrightarrow{VN} \|L_0 - L_S\|^2
\]

and

\[
D(\mu_0^0||\mu_S^K) \xrightarrow{VN} \|\bar{L}_0 - \bar{L}_S\|^2.
\]
and hence 

\[ D(\mu_0||\mu_S) - D(\mu_0^p||\mu_S^p) \geq 0 \]  

where the last equality simply uses the projection principle, i.e., that the projection of \( L \) onto the centered directions given by \( \bar{L} = L - \bar{L} \), is orthogonal to the projection’s height \( \bar{L} \), implying 

\[ ||\bar{L}||^2 = ||L||^2 - ||\bar{L}||^2. \]

Now, by reversing the very noisy approximation in (47), (48) and (49), we get that 

\[ D(\mu_0||\mu_S^p) \geq D(\mu_0||\mu_S) + D(\mu_0^p||\mu_S^p), \quad \forall W_0 \in S. \]  

Therefore, we use this as the definition of the general one-sided sets, as expressed in Definition 4.

Clearly, as we mechanically generalized the notion of one-sided sets from a special very noisy case to the general problem, there is no reason to believe at this point that the resulting one-sided sets will have the same property in the general setting, than their counterparts in the very noisy case; namely, that the linear decoder induced from the worst channel achieves the compound capacity. However, this turns out to be true, and the proof again follows closely the corresponding proof of the very noisy special case.

Proof: of Proposition 5

Recall that in the VN case, when the actual channel is \( W_{0,a} \), and the decoder uses metric \( d_e = \log W_{1,a} \), the achievable rate, in terms of the corresponding centered directions \( \bar{L}_0, \bar{L}_1 \), is given by, cf. (13), 

\[ \lim_{\epsilon \to 0} \frac{2}{\epsilon^2} R(P_X, W_{0,a}, d_e) = \inf_{\bar{L} : ||\bar{L}||^2 - ||\bar{L} - \bar{L}_1||^2 \geq ||\bar{L}_0||^2 - ||\bar{L}_0 - \bar{L}_1||^2} \]  

The constraint of the optimization can be rewritten in norms as 

\[ \bar{L} : ||\bar{L}||^2 - ||\bar{L} - \bar{L}_1||^2 \geq ||\bar{L}_0||^2 - ||\bar{L}_0 - \bar{L}_1||^2 \]  

Now if \( \bar{L}_0 \) lies in a one-sided set \( S_2 \) and we use decoding metric as the worst channel \( \bar{L}_1 = L_S \), by using definition (46), and recognizing that \( ||\bar{L} - \bar{L}_1||^2 \) is non-negative, this constraint implies 

\[ ||\bar{L}||^2 \geq ||\bar{L}_0||^2 - ||\bar{L}_0 - \bar{L}_S||^2 \geq ||\bar{L}_S||^2, \quad \forall \bar{L}_0 \in S, \]  

form which we conclude that the compound capacity is achievable. The proof of Proposition 5 replicates these steps closely.

First, we write in the general setting, the mismatched mutual information is given by 

\[ R(P_X, W_0, \log W_S) = \inf_{\mu \in \mathcal{A}_S} D(\mu||\mu^p) \]  

where 

\[ \mathcal{A}_S = \{ \mu_X = P_X, \mu_Y = (\mu_0)_Y, E_\mu \log W_S \geq E_{\mu_0} \log W_S \}. \]

Since we consider here a linear decoder, i.e. induced by only one single-letter metric, we can consider equivalently the ML or MAP metrics. We then work with the MAP metric and the constraint set is equivalently expressed as: 

\[ \mathcal{A}_S = \{ \mu_X = P_X, \mu_Y = (\mu_0)_Y, E_\mu \log \frac{W_S}{(\mu_S)_Y} \geq E_{\mu_0} \frac{W_S}{(\mu_S)_Y} \}. \]

Expressing the quantities of interest in terms of divergences, we write 

\[ E_\mu \left[ \frac{W_S}{(\mu_S)_Y} \right] = E_\mu \left[ \log \frac{W_S}{(\mu_S)_Y} \right] = D(\mu||\mu^p) - D(\mu||\mu_S) + D(\mu^p||\mu_S^p) \]

Similarly we have 

\[ E_{\mu_0} \frac{W_S}{(\mu_S)_Y} = D(\mu_0||\mu_0^p) - D(\mu_0||\mu_S) + D(\mu_0^p||\mu_S^p). \]

Thus we can rewrite \( \mathcal{A}_S \) as 

\[ \mathcal{A}_S = \{ \mu : \mu_X = P_X, \mu_Y = (\mu_0)_Y, D(\mu||\mu^p) - D(\mu||\mu_S) + D(\mu^p||\mu_S^p) \geq D(\mu_0||\mu_0^p) - D(\mu_0||\mu_S) + D(\mu_0^p||\mu_S^p) \} \]

It worth noticing that this is precisely the lifting of (52).

Now, in the VN limit, \( D(\mu||\mu_S) - D(\mu||\mu_S^p) \) is given by 

\[ ||L - L_S||^2 - ||\bar{L} - \bar{L}_S||^2 = ||L - \bar{L}||^2, \]

which is clearly positive. Here, we have that 

\[ D(\mu||\mu_S) - D(\mu||\mu_S^p) \geq 0, \]

is a direct consequence of log-sum inequality, and with this, we can write for all \( \mu \in \mathcal{A}_S, \)

\[ D(\mu||\mu^p) \geq D(\mu_0||\mu_0^p) - D(\mu_0||\mu_S) + D(\mu_0^p||\mu_S^p) \]

which is in turn lower bounded by \( D(\mu||\mu_S^p) = I(P_X, W_S) \), provided that the set \( S \) is one-sided, cf. (4) (note that last lines are again a lifting of (53)). Thus, the compound capacity is achieved.

This general proof can indeed be shortened. Here, we emphasize the correspondence with the proof for the very noisy case, in order to demonstrate the insights one obtains by using the local geometric analysis.

Proof: of Lemma 6

Let \( C \) a convex set, then for any input distribution \( P_X \) the set \( D = \{ \mu : \mu(a,b) = P_X(a)W(b|a), W \in C \} \) is a convex set as well. For \( \mu \) such that \( \mu(a,b) = P_X(a)W(b|a) \), we have 

\[ D(\mu||\mu^p) = I(P_X, W) + D(\mu_Y||\mu_C), \]

hence we obtain, by definition of \( W_C \) being the worse channel of \( \text{cl}(C) \), 

\[ \mu_C = \min_{\mu \in \text{cl}(D)} D(\mu||\mu^p). \]
Therefore, we can use theorem 3.1. in [3] and for any $\mu_0 \in D$, we have the pythagorean inequality for convex sets
\[
D(\mu_0\|\mu^0_C) \geq D(\mu_0\|\mu_C) + D(\mu_C\|\mu^0_C). \tag{56}
\]
This concludes the proof of the first claim of the Proposition. Now to construct a one-sided set that is not convex, one can simply take a convex set and remove one point in the interior, to create a "hole". This does not affect the one-sidedness, but makes the set non-convex. It also shows that there are sets that are one-sided (and not convex) for all input distributions, so the one-sidedness does not have to depend on which input distribution is chosen.

Proposition 6 says that our definition of one-sided sets is strictly more general than convex sets. This generalizes the known result [5] on when does linear receiver achieve compound capacity, but more importantly, our definition leads to the meaningful use of generalized linear decoders with finite number of metrics: it is easy to construct an example of compound set with an infinite number of disconnected convex components; but the notion of finite unions of one-sided sets is general enough to include most compound sets that one can be exposed to.

In the next proofs, we no longer give explicitly the analogy with the VN setting.

Proof: of Proposition 7.

We need to show the following
\[
\wedge W_1 \in S_{\mu: \mu^p = \mu^0_P, E_\mu \log W_1 \geq \vee W \in S_{E_{\mu_0} \log W}} D(\mu\|\mu^0_P) 
\geq \wedge W \in I(P_X, W),
\]
and we will see that the left hand side of this inequality is equal to $I(P_X, W_0)$. Note that $\vee W \in S_{E_{\mu_0} \log W} = E_{\mu_0} \log W_0 = I(P_X, W_0)$. Thus, the desired inequality is equivalent to
\[
\inf_{\mu: \mu^p = \mu^0_p, E_\mu \log W_1 \geq E_{\mu_0} \log W_0} D(\mu\|\mu^0_P) \geq \wedge W \in I(P_X, W). \tag{57}
\]
Using the marginal constraint $\mu^p = \mu^0_P$, we have
\[
E_\mu \log W_1 \geq E_{\mu_0} \log W_0 
\Leftrightarrow \quad E_\mu \left[ \log \frac{\mu^1}{\mu^0} \right] \geq E_{\mu_0} \left[ \log \frac{\mu_0}{\mu^0_P} \right] 
\Leftrightarrow \quad D(\mu\|\mu^p) - D(\mu\|\mu_1) \geq D(\mu_0\|\mu^0_P) \tag{58}
\]
using the fact that $D(\mu\|\mu_1) \geq 0$, we have
\[
\inf_{\mu: \mu^p = \mu^0_P, E_\mu \log W_1 \geq E_{\mu_0} \log W_0} D(\mu\|\mu^0_P) = \inf_{\mu: \mu^p = \mu^0_P, D(\mu\|\mu^p) - D(\mu\|\mu_1) \geq D(\mu_0\|\mu^0_P)} D(\mu\|\mu^p) \tag{59}
\geq D(\mu_0\|\mu^0_P) = I(P_X, W_0). \tag{60}
\]
This concludes the proof of the Proposition. In fact, one could get a tighter lower bound by expressing (58) as
\[
E_\mu \log W_1 \geq E_{\mu_0} \log W_0 \Leftrightarrow 
D(\mu\|\mu^p) - D(\mu\|\mu_1) - D(\mu^p\|\mu^0_P) \geq D(\mu_0\|\mu^0_P) + D(\mu_0\|\mu^1) 
\]
and using the log-sum inequality to show that $D(\mu\|\mu_1) - D(\mu^p\|\mu^1) \geq 0$, (59) is lower bounded by
\[
D(\mu_0\|\mu^0_P) + D(\mu_0\|\mu^1). \tag{61}
\]
Figure 7 illustrates this gap.

Proof: of Proposition 8.

We found a counter-example for the very noisy setting in section IV-C, therefore the negative statement holds in the general setting.

Proof: of Theorem 9.

We need to show
\[
\inf_{\mu \in A} D(\mu\|\mu^0_P) \geq \wedge_{k=1}^K I(P_X, W_k), \tag{61}
\]
where $A$ contains all joint distributions $\mu$ such that
\[
\mu_X = P_X, \quad \mu_Y = (\mu_0)_Y, \tag{62}
\]
and
\[
\wedge_{k=1}^K E_{(\mu^0)_Y} \log W_k \geq \wedge_{k=1}^K E_{\mu_0} \log (\mu_k)_Y. \tag{63}
\]
We can assume w.l.o.g. that $W_0 \in C_1$. We then have
\[
D(\mu || \mu_{0}^\text{p}) \overset{(A)}{=} D(\mu || \mu_{0}^\text{p}) \\
\overset{(B)}{\geq} \forall k = 1 \log \frac{W_k}{(\mu_k)_Y} \\
\overset{(C)}{\geq} E_{\mu_0} \log \frac{W_k}{(\mu_1)_Y} \\
\overset{(D)}{\geq} E_{\mu_1} \log \frac{W_k}{(\mu_1)_Y} \\
= I(P_{X}, W_1) \\
\geq \land_{k = 1} G_{I}(P_{X}, W_k),
\]
where (A) uses (62), (B) uses the log-sum inequality:
\[
E_{\mu} \log \frac{W_k}{(\mu_k)_Y} = D(\mu || \mu^p) + E_{\mu} \log \frac{W_k}{(\mu_k)_Y} - D(\mu || \mu^p) \\
= D(\mu || \mu^p) - (D(\mu || \mu_k) - D(\mu^p || \mu_k)),
\]
(C) is simply (63) and (D) follows from the one-sided property:
\[
E_{\mu_0} \log \frac{W_1}{(\mu_1)_Y} - E_{\mu_1} \log \frac{W_1}{(\mu_1)_Y} = D(\mu_0 || \mu_1^p) - D(\mu_0 || \mu_1) - D(\mu_1 || \mu_1^p) \geq 0.
\]

C. Discussions

We raised the question whether it is possible for a decoder to be both linear and capacity achieving on compound channels. We showed that if the compound set is a union of one-sided sets, a generalized linear which is capacity achieving decoder exists. We constructed it as follows: if $W_1, \ldots, W_K$ are the worst channels of each component (cf. figure 8), use the generalized linear decoder induced by the MAP metrics $\log \frac{W_1}{(\mu_1)_Y}, \ldots, \log \frac{W_K}{(\mu_K)_Y}$, i.e., decode with
\[
G_{n}(y) = \arg \max_{m \in [1, \ldots, M]} \log \frac{W_k}{(\mu_k)_Y},
\]
where $\mu_k = P_{X} \circ W_k$, $P_{X}$ is the optimal input distribution on $S$, and $P_{(x_m, y)}$ is the joint empirical distribution of the $m$th codeword $x_m$ and the received word $y$. We denote this decoder by GMAP($W_1, \ldots, W_K$). We also found that using the ML metrics, instead of the MAP metrics $W_1, \ldots, W_K$, i.e. GLRT($W_1, \ldots, W_K$), is not capacity achieving.

It is instrumental to compare our receiver with the MMI receiver. We observe that if the codeword $x_m$ is chosen from a fixed composition $P_X$ code, the empirical mutual information
\[
I(\tilde{P}(x_m, y)) = \sup_{W} \log \frac{W}{(P_X \circ W)_Y}
\]
where the maximization is taken over all possible DMC $W$, which means that the MMI is actually the GMAP decoders taking into account all DMC’s. Our result says that we do not need to enumerate all DMC metrics to achieve capacity, for a given compound set $S$, we can restrict ourself to selecting carefully a subset of all metrics and yet achieve the compound capacity. Those important metrics are found by extracting the one-sided components of $S$, and taking the MAP metrics induced by the worst channel of these components. When $S$ has a finite number of one-sided components, this decoder is generalized linear. The key step is to understand the structure of the space of decoding metrics. The geometric insights gives rise to a notion of which channels are dominated by which (with the one-sided property) and how to combine the dominant representatives of each components (Generalized MAP metrics).

We argued that the family of sets that can be written as finite unions of one-sided sets covers a large variety of sets, even larger than the family of sets having finite unions of convex components. This means that the generalized linear decoders with finitely many metrics can be found to achieve capacity for a large family of compound sets. Yet, there do exist compound sets that are not even a finite union of one-sided components. To see this, we can go back to the local geometric picture and imagine a compound set with infinitely many worst channels, for which the procedure shown in Figure 8 has to go through an infinite number of steps. We argue, however, that such examples are pedagogical, in the sense that if one is willing to give up a small fraction of the capacity, then a finite collection of linear decoding metrics would suffice. Moreover, there is a graceful tradeoff between the number of metrics used, and the loss in achievable rate.

Even more interestingly, one can develop a notion of a “blind” generalized linear decoder, which does not even require the knowledge of the compound set, yet guarantees...
to achieve a fraction of the compound capacity. We describe here such decoders in the VN setting. As illustrated in Figure 9, such decoders are induced by a set of metrics chosen in a "uniform" fashion. For a given compound set, we can then grow a polytope whose faces are the hyperplane orthogonal to these metrics and there will be a largest such polytope, that contains the entire compound set in its complement. This determines the rate that can be achieved with such a decoder on a given compound set, cf. $C_{\text{poly}}$ in Figure 9. In general $C_{\text{poly}}$ is strictly less than the compound capacity, denoted by $C$ in Figure 9, the only cases where $C = C_{\text{poly}}$ is if by luck, one of the uniform direction is along the worst channel (and if there are enough metrics to contain the whole compound set). Now, for a number $K$ of metrics, no matter what the compound set looks like, and not matter what its capacity is, the ratio between $C_{\text{poly}}$ and $C$ can be estimated: in the VN geometry, this is equivalent to picking a sphere with radius $C$ and to compute the ratio between $C$ and the “inner radius” of a $K$-polytope inscribed in the sphere. It is also clear that the higher the number of metrics is, the closer $C_{\text{poly}}$ to $C$ is, and this controls the tradeoff between the computational complexity and the achievable rate. Again, as suggested by the very noisy picture, there is a graceful tradeoff between the number of metrics used, and the loss in achievable rate.

VI. CONCLUSION

Many Information Theoretic problems evaluate the limiting performance of a communication scheme by an expression optimizing divergences under constrained probability distributions. The divergence is not a formal distance, however, when the distributions are close to each other, which we had by considering channels to be very noisy, we are able to make local computations and the divergence can be approximated by a squared norm. We showed that the geometry governing this local setting is the one of an inner product space, where notions of angles and distances are well defined. This geometric insight simplifies greatly the problems. Rather than getting a good approximation per-se, it provides a simplified problem, for which we have a better insight and which points out solutions to the original problem. It is also a powerful tool for finding counter-examples. Finally, we showed how in this problem, we could “lift” the results proven locally to results proven globally.

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