Tree-level renormalization

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Abstract

It is shown in the framework of the $N$-component scalar model that the saddle point structure may generate non-trivial renormalization group flow. The spinodal phase separation can be described in this manner and a flat action is found as an exact result which is valid up to any order of the loop expansion. The correlation function is computed in a mean-field approximation.

I. INTRODUCTION

The strategy of the renormalization group method is a successive elimination of the degrees of freedom, a gradual simplification of the dynamical systems \cite{1}. In Statistical and Quantum Physics the quantity to consider is the partition function or the path integral, and the method consists of the successive integration over the dynamical variables. The renormalization group is usually viewed in this manner as a technical device to sum up the contributions of the thermal or the quantum fluctuations. The result, the renormalized trajectory is used to assess the importance and the effects of the fluctuations of a given scale. The goal of this paper is to show that the renormalization group can be non-trivial even in the absence of the fluctuations. The fluctuations and the mean values are the easiest to separate by means of the saddle-point expansion. It will be shown that the saddle point can induce a non-trivial renormalization, independently of the fluctuations.

In order to make our point clearer consider first a lattice model with topological defects,

$$Z = \prod_x \int d\phi(x) e^{-\frac{1}{\hbar} S[\phi(x)]}.$$  \hspace{1cm} (1)

The topological defects of the size $\ell a$ ($a$ is the lattice spacing) minimize the action and the tree level $\mathcal{O}(\hbar^0)$ contributions to the saddle point approximation can be written as a

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grand canonical partition function of the topological defects \cite{2}. Suppose that we perform in dimension $d$ a real-space blocking, $a \to na$ with $n > \ell$,

$$e^{-\frac{1}{\hbar} S[\tilde{\phi}(\tilde{x})]} = \prod_{\tilde{x}} \int d\phi(\tilde{x}) \prod_{\tilde{x}} \delta \left( \tilde{\phi}(\tilde{x}) - \frac{1}{n^d} \sum_{x \in \tilde{x}} \phi(x) \right) e^{-\frac{1}{\hbar} S[\phi(x)]}. \quad (2)$$

The fraction of the topological defects which are on the surface of the blocks is suppressed by $\ell/n$. Most of the topological defects are inside of a block and they can be thought as the saddle point configurations in the elimination of the variables of the given block. As long as the topological defects are present the blocking has a tree-level, non-fluctuating contribution. Naturally the inhomogeneous saddle points break external symmetries which provide plenty of zero modes. The integration over them captures some effects of the fluctuations. The difference between the tree-level contribution and the genuine fluctuations might seem to be reduced to a technical issue at this point. But the coupling constant dependence is always more complicated and singular at the tree-level than it is in the loop contributions. Thus the scaling laws extracted from the tree- or the loop-level terms might be radically different \cite{2,3}.

We discuss in this work the tree-level renormalization effects of an $N$-component scalar field model defined by the action $S_\Lambda[\vec{\phi}]$. It is useful to introduce a constrained partition function given by

$$Z_\Phi = \int D[\vec{\phi}] \delta \left( \frac{1}{V_d} \int d^d x \vec{\phi}(x) \right) e^{-\frac{1}{\hbar} S_\Lambda[\vec{\phi}]}, \quad (3)$$

where $V_d$ is the space-time volume. The effective potential of the model \cite{1} is

$$V_{\text{eff}}(\Phi) = -\frac{\hbar}{V_d} \ln Z_\Phi. \quad (4)$$

The regulator of the path integral, the cutoff $\Lambda$ in the momentum space, is not written explicitly. We select the parameters of the action in such a manner that the $O(N)$ symmetry is spontaneously broken in the vacuum. As $\Phi$ is chosen to be smaller than the amplitude of the field in the vacuum, $\Phi < |\langle \vec{\phi}(x) \rangle|$, one expects metastable or unstable behavior for certain modes. We shall find non-trivial saddle points in this case during the blocking. The saddle point expansion actually offers a systematical treatment of the metastable phase or the spinodal instabilities. The resulting effective action for the unstable phase is found to be flat. It is remarkable that this result goes beyond the tree-approximation and remains valid to every order of $\hbar$. Domain walls are reproduced as the saddle points in the spinodal phase separation and we obtain the Maxwell-construction for the free energy, $V_{\text{eff}}(\Phi)$. A similar problem has already been discussed for $N = 1$ by keeping track of simple saddle points numerically in ref. \cite{4}. The present paper generalises the results presented there and relies on analytical methods, based on continuity assumptions inferred from the numerical study.

The saddle point approximation is used in two different manners in the paper. This is because we use the differential form of the renormalization group equations and we have two small parameters at our disposal. One is $\hbar$ to organize the loop-expansion and the other is the fraction of the modes to be eliminated in a blocking step, $\epsilon = (1 - \ell/n)^d$ in the
example mentioned above. The usual saddle point approximation is recovered when the limit \( \hbar \to 0 \) is taken at finite \( \epsilon \). This is sufficient to provide a well defined renormalization group flow. But the minimum of the action becomes highly degenerate when the infinitesimal form of the renormalization group method is employed, i.e. the limit \( \epsilon \to 0 \) is made before \( \hbar \to 0 \). The disadvantage of the degeneracy of the effective action is that the saddle point approximation is in general spoiled. This is because the minimum at the saddle point is shallow, the curvature of the action being \( \mathcal{O}(\epsilon) \). But the determination of the effective action remains reliable since \( \epsilon \) not only makes the action flat but enters as a suppression factor for the loop corrections, as well. Such a double role of \( \epsilon \) is behind an unusual feature of the saddle point expansion: the appearance of fluctuations in the order \( \mathcal{O}(\hbar^0) \). The advantage of the degeneracy is an enormous simplification of the functional integration which is demonstrated by the mean-field computation of the correlation function. The fluctuations arising in \( \mathcal{O}(\hbar^0) \) have no restoring force to the equilibrium position, a characteristic feature of the mixed phase of the first order phase transitions.

The organization of the paper is the following. Section II. introduces the infinitesimal renormalization step which provides a more powerful version of the renormalization group equation. A simple and important property of the renormalized action in the unstable region is presented in Section III for a scalar model. The renormalized trajectory is given in Section IV and Section V contains a brief analysis of the correlation function obtained on the tree-level. Finally, Section VI is for the summary.

II. INFINITESIMAL RENORMALIZATION

We introduce the infinitesimal renormalization group in this Section. The traditional Kadanoff-Wilson blocking transformations in the Fourier space is the lowering of the cut-off \( k \to k - \delta k \),

\[
e^{-\frac{\hbar}{\pi}S_{k-\delta k}[\phi]} = \int D[\psi] e^{-\frac{\hbar}{\pi}S_{k}[\phi+\psi]},
\]

where \( \phi \) has Fourier components for \( |p| \leq k - \delta k \) and \( \tilde{\psi} \) for \( k - \delta k < |p| \leq k \). This blocking is infinitesimal if the number of modes eliminated is small compared to the number of the left over modes in the system,

\[
\epsilon = \frac{\delta k}{k} << 1.
\]

The one-loop approximation to the blocking (5) yields the exact renormalization group equations as \( \epsilon \to 0 \) [4]. The exactness comes from the fact that the functional integral (5) becomes Gaussian when \( \epsilon \to 0 \) because the higher-loop contribution are supressed by the volume of the momentum integration region,

\[
e^{-\frac{\hbar}{\pi}S_{k-\delta k}[\phi]} = e^{-\frac{\hbar}{\pi}S_{k}[\phi+\psi_{k}]} \int D[\psi] e^{-\frac{\hbar}{\pi} \sum \partial^{2}S_{k}[\phi+\psi_{k}](\tilde{\psi}-\psi_{k})^{2}} (1 + \mathcal{O}(\epsilon)).
\]

Here \( \tilde{\psi}_{k} \) denotes the saddle point which satisfies

\[
\frac{\delta S_{k}[\phi+\psi_{k}]}{\delta \psi} = 0
\]

(8)
and each eigenvalue of the inverse propagator \( \delta^2 S[\vec{\phi} + \vec{\psi}_k]/\delta \vec{\psi} \delta \vec{\psi} \) is positive. The appearance of the product \( \hbar \epsilon \) in the higher order corrections indicates the presence of the new small parameter \( \epsilon \) which renders the one-loop evolution equation exact. By performing the integration we find

\[
e^{-\frac{1}{\hbar} S_{k - \delta k}[\vec{\phi}] - S_k[\vec{\phi} - \vec{\psi}_k]} = e^{-\frac{1}{\hbar} S_k[\vec{\phi} + \vec{\psi}_k]} \left( I_k[\vec{\phi}] \right)^{N_d/2} (1 + O(\hbar \epsilon))
\]

where \( I_k[\vec{\phi}] \) is an integral over one Fourier component and \( N_d/2 \) is the number of modes to eliminate in the shell \( k - \delta k \leq |p| \leq k \). The factor \( 1/2 \) comes from the reality condition \( \vec{\phi}(-p) = \vec{\phi}^*(p) \) and

\[
N_d = \Omega_d \frac{k^{d-1}}{(2\pi)^d} V_d \delta k,
\]

where \( \Omega_d \) is the solid angle in \( d \) dimensions. The evolution in the cutoff is given by the functional finite difference equation

\[
\frac{1}{\delta k} \left( S_k[\vec{\phi} + \vec{\psi}_k] - S_k[\vec{\phi}] \right) = -\frac{\hbar}{2\delta k} Tr \ln \frac{\delta^2 S_k[\vec{\phi} + \vec{\psi}_k]}{\delta \vec{\psi} \delta \vec{\psi}} (1 + O(\hbar \epsilon))
\]

One should bear in mind that the infrared field \( \vec{\phi} \) is kept constant in computing this functional derivative. Assuming that the saddle point is trivial, \( \vec{\psi}_k = 0 \), we obtain

\[
\partial_k S_k[\vec{\phi}] = -\lim_{\delta k \to 0} \frac{\hbar}{2\delta k} Tr \ln \frac{\delta^2 S_k[\vec{\phi}]}{\delta \vec{\psi} \delta \vec{\psi}}.
\]

The limit \( \delta k \to 0 \) is safe because the trace is always \( O(\delta k) \). This is not necessarily so in the general case (11) where the saddle point \( \vec{\psi}_k \) is supposed to have a finite limit as \( \delta k \to 0 \).

The tree-level renormalization group equation, c.f. (37) below, can be obtained for any sufficiently smooth action functional. But in order to convert the complicated functional equation into a system of coupled differential equations for the renormalized coupling constants one needs an ansatz, a certain functional form for the action. This is usually given by the gradient expansion, by assuming that all important terms of the action are local. We shall consider the lowest order contributions of this expansion which are compatible with rotational invariance in Euclidean \( d \)-dimensional space,

\[
S_k[\vec{\phi}] = \int d^d x \left[ \frac{1}{2} Z_k(\vec{\phi})(\partial_\mu \vec{\phi})^2 + U_k(\vec{\phi}) \right]
\]

where the functions \( Z_k(\vec{\phi}) \) and \( U_k(\vec{\phi}) \) are \( O(N) \) invariant and depend on \( \Phi = |\vec{\phi}| \) only. The running coupling constants are defined by the expansion

\[
Z_k(\Phi) = \sum_{n=0}^{\infty} \frac{z_n}{n!} \Phi^n, \quad U_k(\Phi) = \sum_{n=0}^{\infty} \frac{g_n}{n!} \Phi^n,
\]

The qualitative features of the renormalization flow can be seen in a simpler approximation where one retains the local potential only,
The corresponding Wegner-Houghton equation is

\[ k \partial_k U_k(\Phi) = -\hbar \frac{\Omega_d k^d}{2(2\pi)^d} \ln \left( \left( k^2 + \partial_\Phi^2 U_k(\Phi) \right) \left( k^2 + \frac{1}{\Phi} \partial_\Phi U_k(\Phi) \right)^{N-1} \right) \]

where \( \Omega_d \) is the solid angle in dimension \( d \). Notice that the arguments of the logarithm is the inverse curvature of the action at the minimum and each of them is supposed to be positive, being proportional to the restoring force acting on the fluctuations around the vacuum. The solution \( U_0(\Phi) \) corresponding to the initial condition \( U(\Phi) = U_\Lambda(\Phi) \) imposed at \( k = \Lambda \) gives the effective potential, the energy density of the vacuum of the constrained model (3).

The fluctuations are stable in the \( O(N) \) symmetrical phase,

\[ k^2 + \partial_\Phi^2 U_k(\Phi) > 0, \text{ and } k^2 \Phi + \partial_\Phi U_k(\Phi) > 0. \]  

(17)

If either of these inequalities is violated then the infinitesimal fluctuations around the given \( \Phi \) become unstable. This is the case with spontaneously broken symmetries. In fact, the arguments of the logarithm are decreasing monotonically as the cutoff \( k \) is lowered but they stay positive in the symmetrical phase. The symmetry broken phase is characterised by having a finite cutoff value \( k = k_{cr} \), where (17) is violated. The loop-expansion is clearly inapplicable for \( k \leq k_{cr} \). Since the Goldstone modes are the lightest excitations it is always the second inequality in (17) which indicates this instability.

Let us introduce \( \Phi(k) \), the curve on the plane \((\Phi,k)\) where (17) is first violated as \( k \) is decreased. The function \( \Phi(k) \) is decreasing and the fluctuations around the constrained vacuum of (3) with momentum \( p < k_{cr} \) are unstable if \( \Phi < \Phi_p \). This is the spinodal unstable phase where fluctuations with infinitesimal amplitude are already unstable. The Wegner-Houghton equation (12) does not apply any more in this regime because one has to take into account the non-trivial saddle points as in (11). The equation (12) should in principle be renounced for \( k \leq k_{cr} \). This is because the configurations whose action receives contributions from the spinodal unstable regions are not treated in a reliable manner. But the strategy of the loop expansion offers the remedy by assuming that the amplitude of the fluctuations is infinitesimal. Thus one may use the simple equation (12) even for \( k < k_{cr} \) when \( \Phi \gg \Phi(k) \) because the fluctuations which are treated in an unreliable manner are strongly suppressed.

We encounter an important simplification by noting that the loop expansion is applicable in the stable region, \( \Phi \gg \Phi(k) \). In fact, we can use the simplest tree-level approximation in the stable region to obtain an evaluation of \( \Phi(k) \). The \( k \) dependence is simply ignored in \( \mathcal{O}(\hbar^0) \), \( U_k(\Phi) = U_\Lambda(\Phi) \). The condition determining the curve \( \Phi(k) \) is then

\[ k^2 \Phi(k) + \partial_\Phi U_\Lambda(\Phi(k)) = 0. \]

(18)

The approach of the instability with \( N = 1 \) has been examined numerically in the local potential approximation \((Z_k = 1)\) [7]. The spinodal region was considered, as well, where nontrivial saddle points appear in the functional integration of the UV modes [4]. We give below an analytic derivation of the renormalized trajectory in the spinodal instability region for a \( N \)-component field in the approximation (13).
III. DEGENERACY OF THE ACTION

The characteristic feature of the effective action generated by the saddle points is a high level of degeneracy [4]. To understand its implication we first note that the usual saddle point approximation to the effective action or potential where all degrees of freedom are treated in the loop-expansion is based on an extended saddle point at zero momentum or a gas of localized solutions which include all momenta. The fluctuations around the saddle point are orthogonal to the saddle point configuration and are suppressed by $\bar{\hbar}$. But the saddle points of the infinitesimal blocking step appear differently. Suppose that $\delta k$ is so small that we can neglect the dependence of the Fourier transform $\tilde{\phi}(p_\mu)$ of the field variable on the length $|p|$ between $k - \delta k$ and $k$. We may then look at (11) as a path integration for a system on the $d-1$ dimensional sphere, $S_{d-1}$. The $d-1$ dimensional action, $S^{(d-1)}_k[\phi]$ is labeled by the parameter $k$. It will be shown that the continuous dependence of the action in $k$ makes $S^{(d-1)}_{k-\delta k}[\phi]$ degenerate up to corrections $O(\delta k)$. This weak variation of the action is enough to support the loop-expansion but can not suppress the fluctuations which are non-orthogonal to the saddle point for finite but small $\delta k$. This complication does not influence the value of the running coupling constants, the renormalization group flow. When the successive elimination process of the renormalization group method is used to carry out the path integral for some observables in the limit $\delta k \to 0$ then the following, different strategy is employed: We simplify the path integration enormously by taking into account that the effective action for each momentum shell is degenerate in the unstable region. This degeneracy is reminescent of the Maxwell construction for the coexisting phase of the first order phase transitions. The key element of both strategies, the $O(\delta k)$ degeneracy of the action is the subject of this Section.

Before embarking the general argument we remark that the problem of finding the saddle points of the model with $N > 1$ can be reduced to the case $N = 1$ if the action is monotonically increasing with $(\partial_\mu \tilde{\phi})^2$, like in (13). To see this let us write

$$\bar{\phi}(x) + \bar{\psi}_k(x) = \eta_k(x) \mathcal{R}(x) \bar{e}$$

where $\mathcal{R}(x)$ is a $SO(N)$ matrix and $\bar{e}$ the unit vector giving the orientation of $\bar{\phi}$ in the internal space. The saddle point is a local minimum of the action whose dependence in $\mathcal{R}$ is coming by $\partial_\mu \mathcal{R}$. Since

$$(\partial_\mu \bar{\phi} + \partial_\mu \bar{\psi}_k)^2 = \partial_\mu \eta_k \partial_\mu \eta_k + \eta_k^2 \partial_\mu \mathcal{R} \bar{e} \partial_\mu \mathcal{R} \bar{e},$$

the minimum is reached by a homogeneous $\mathcal{R}$, in which case $\bar{\psi}_k$ must be parallel to $\bar{e}$. Thus the saddle point of the model $N > 1$ reduces to the case $N = 1$, which we consider below for the sake of simplicity.

The proof of the degeneracy of the effective action is based on the following assumptions:

1. the continuity of $S_k[\phi]$ as the functions of $k$ for a fixed configuration $\phi$ in the unstable region,

2. the infinite differentiability of $S_k[\phi]$ with respect to $\phi$ for any fixed value of $k$ in the unstable region,
3. the continuity of $S_k[\phi]$ and $\delta S_k[\phi]/\delta \phi(x)$ on the spinodal line, $\phi(x) = \Phi(k)$.

The saddle point configuration is a solution of the non-linear equation of motion. We have to perform the Gaussian approximation in (7) for each saddle point and sum up the contributions. Let us denote the renormalized action resulting from the saddle point $\psi_{k,\alpha}$ by $S_{k-\delta k,\alpha}[\phi]$, where $\alpha$ stands for the zero modes of the saddle point. The tree-level renormalization arises because the saddle point depends on the background field and (9) leads to read the action as a functional of the Fourier components:

$$S_{k-\delta k,\alpha}[\phi] = S_k[\phi + \psi_{k,\alpha}[\phi]] + \mathcal{O}(\hbar\epsilon).$$

In order to find the evolution of the functional derivatives of the action one has to keep in mind that the independent variables are the Fourier components $\tilde{\phi}$ and $\tilde{\psi}_k$. Thus one has to read the action as a functional of the Fourier components:

$$S_k[\phi + \psi_{k,\alpha}[\phi]] = S_k\left[\tilde{\phi}, \tilde{\psi}_{k,\alpha}[\phi]\right].$$

We shall suppress the index $\alpha$ in the expressions below. The condition (21) then implies

$$\frac{\delta S_{k-\delta k}}{\delta \tilde{\phi}(p)}[\phi] = \frac{\delta S_k}{\delta \tilde{\phi}(p)}[\phi + \psi_k] + \int \frac{d^d q}{(2\pi)^d} \frac{D S_k}{D \tilde{\psi}(q)}[\phi + \psi_k] \frac{\delta \tilde{\psi}_k(q)}{\delta \tilde{\phi}(p)} + \mathcal{O}(\hbar\epsilon).$$

Since $\psi_k$ is the saddle point,

$$\frac{\delta S_k}{\delta \tilde{\psi}(q)}[\phi + \psi_k] = 0,$$

we have

$$\frac{\delta S_{k-\delta k}}{\delta \tilde{\phi}(p)}[\phi] = \frac{\delta S_k}{\delta \tilde{\phi}(p)}[\phi + \psi_k] + \mathcal{O}(\hbar\epsilon) \quad \text{for} \quad |p| \leq k - \delta k.\quad (25)$$

The higher derivatives are obtained by taking the successive functional derivatives of (5) with respect to the Fourier components of $\tilde{\phi}$. The second derivative reads as

$$\left[\frac{\delta^2 S_{k-\delta k}}{\delta \tilde{\phi}(q)\delta \tilde{\phi}(p)}[\phi] - \frac{\delta S_k}{\delta \tilde{\phi}(q)} \frac{\delta S_{k-\delta k}}{\delta \tilde{\phi}(p)}[\phi]\right] e^{-\frac{i}{\hbar} S_{k-\delta k}[\phi]}$$

$$\int \mathcal{D}[\psi] \left[\frac{\delta^2 S_k}{\delta \tilde{\phi}(q)\delta \tilde{\phi}(p)}[\phi + \psi] - \frac{\delta S_k}{\delta \tilde{\phi}(q)} \frac{\delta S_{k-\delta k}}{\delta \tilde{\phi}(p)}[\phi + \psi]\right] e^{-\frac{i}{\hbar} S_k[\phi + \psi]}$$

$$= \left[\frac{\delta^2 S_k}{\delta \tilde{\phi}(q)\delta \tilde{\phi}(p)}[\phi + \psi_k] - \frac{\delta S_k}{\delta \tilde{\phi}(q)} \frac{\delta S_{k-\delta k}}{\delta \tilde{\phi}(p)}[\phi + \psi_k]\right] e^{-\frac{i}{\hbar} S_k[\phi + \psi_k]} \left(1 + \mathcal{O}(\hbar\epsilon)\right).$$

According to (21) and (25) we arrive at

$$\frac{\delta^2 S_{k-\delta k}}{\delta \tilde{\phi}(q)\delta \tilde{\phi}(p)}[\phi] = \frac{\delta^2 S_k}{\delta \tilde{\phi}(q)\delta \tilde{\phi}(p)}[\phi + \psi_k] + \mathcal{O}(\hbar\epsilon).$$

The further derivatives can be found by induction,
\[ \frac{\delta^n S_{k-\delta k}}{\delta \phi(p_1) \ldots \delta \phi(p_n)}[\phi] = \frac{\delta^n S_k}{\delta \phi(p_1) \ldots \delta \phi(p_n)}[\phi + \psi_k] + \mathcal{O}(\hbar \epsilon), \]

for \(|p_i| \leq k - \delta k\). Let us now take the derivative of (23) with respect to \(\tilde{\phi}(q)\):

\[ \frac{\delta^2 S_{k-\delta k}}{\delta \phi(q) \delta \phi(p)}[\phi] = \frac{\delta^2 S_k}{\delta \phi(q) \delta \phi(p)}[\phi + \psi_k] + \int \frac{d^d q'}{(2\pi)^d} \frac{\delta^2 S_k}{\delta \psi(q') \delta \phi(p)}[\phi + \psi_k] \frac{\delta \tilde{\psi}_k(q')}{\delta \phi(q)} + \mathcal{O}(\hbar \epsilon) \quad (29) \]

which, according to (27), leads to, in the limit \(\epsilon \to 0\),

\[ \int \frac{d^d q'}{(2\pi)^d} \frac{\delta^2 S_k}{\delta \psi(q') \delta \phi(p)}[\phi + \psi_k] \frac{\delta \tilde{\psi}_k(q')}{\delta \phi(q)} = 0. \quad (30) \]

The derivative of the saddle point is supposed to be nonvanishing even as \(\epsilon \to 0\). Since the background field \(\phi\) can be chosen arbitrarily we conclude that

\[ \frac{\delta^2 S_k}{\delta \tilde{\psi}(q') \delta \phi(p)}[\phi + \psi_k] = 0 \quad \text{for} \quad k - \delta k < \vert q \vert \leq k. \quad (31) \]

But if we take the derivative of (24) with respect to \(\tilde{\phi}(p)\), we find

\[ \frac{\delta^2 S_k}{\delta \tilde{\psi}(q) \delta \phi(p)}[\phi + \psi_k] + \int \frac{d^d q'}{(2\pi)^d} \frac{\delta^2 S_k}{\delta \psi(q') \delta \tilde{\psi}(q')}[\phi + \psi_k] \frac{\delta \tilde{\psi}_k(q')}{\delta \phi(p)} = 0. \quad (32) \]

Because of the non vanishing of the derivative of the saddle point when \(\epsilon \to 0\), (31) and (32) imply that

\[ \frac{\delta^2 S_k}{\delta \tilde{\psi}(q) \delta \tilde{\psi}(q')}[\phi + \psi_k] = 0 \quad \text{for} \quad k - \delta k < \vert p \vert, \vert q \vert \leq k. \quad (33) \]

One can easily find by induction, that for any \(n \geq 1\)

\[ \frac{\delta^n S_k}{\delta \psi(p_1) \ldots \delta \psi(p_n)}[\phi + \psi_k] = 0 \quad \text{for} \quad k - \delta k < \vert p_1 \vert, \ldots, \vert p_n \vert \leq k. \quad (34) \]

Therefore if \(\psi\) has non vanishing Fourier components for \(|p|\) between \(k\) and \(k - \delta k\) and \(\phi\) is the background field (with non vanishing Fourier components for \(|p| \leq k - \delta k\)), we can write,

\[ S_k[\phi + \psi] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int_{k_j} (\tilde{\psi}(k_j) - \tilde{\psi}(k_j)) \frac{\delta^n S_k}{\delta \psi(k_1) \ldots \delta \psi(k_n)}[\phi + \psi_k] = S_k[\phi + \psi_k] \quad (35) \]

What we found is that \(S_k\) does not depend on the Fourier components \(|p| = k\) in the unstable region. This result is independent of the choice of the saddle point \(\psi_k\). The tree-level renormalization makes the action flat in the modes to be eliminated.

Note that the continuity of \(S_k[\phi]\) in \(k\) for a fixed \(\phi\) was needed in our argument. In fact, the conclusion about the flatness is reached for \(S_k[\phi]\), the initial condition of the infinitesimal blocking step. This initial condition must be close to the result of the blocking, \(S_{k-\delta k}[\phi]\)
according to the continuity of the renormalization group flow in $k$. Thus our assumptions 1 and 2 yield the flatness of the action within the unstable region.

It is important to keep in mind that this flatness holds up to corrections $O(\delta k)$ so the eigenvalues of the propagator in computing the functional integral (7) are $O(h\delta k^{-1})$. Taking into account that the integration volume $O(\delta k)$, the one-loop correction to the tree-level result is $O(h)$, as usual. Thus there is no arbitrary small parameter, like $\epsilon$, available in separating the tree- and the (one)loop-level contributions to the renormalization group equation. In other words, the simplification offered by the smallness of the loop integration volume suppresses the higher-loop contributions only and leaves both the tree and the one-loop contributions important.

The summation over the saddle points gives finally (with the dependence on $\alpha$ properly displayed)

$$e^{-\frac{1}{\bar{h}}S_{k-\delta k}[\phi]} = \sum_{\alpha}^{N[\phi]} dX_{\alpha} F(X_{\alpha}) e^{-\frac{1}{\bar{h}}S_{k}[\phi+\psi_{k,\alpha}]} = \sum_{\alpha}^{N[\phi]} dX_{\alpha} F(X_{\alpha}), \tag{36}$$

where $X_{\alpha}$ stands for the zero modes of the saddle point $\Psi_{k}(\alpha)$, $\alpha = 1, \cdots, N[\phi]$, and $F(X_{\alpha})$ is the corresponding integration measure. Since the change of the number of the saddle points, $N[\phi]$ is necessarily discontinuous in $\phi(x)$, and the zero mode integral is always finite with inhomogeneous saddle points, the assumption 3 asserts that $N[\phi]$ is constant. The zero mode entropy can thus be ignored, giving

$$S_{k-\delta k}[\phi] = S_{k}[\phi + \psi_{k}(\alpha_{0})] + O(\bar{h}\epsilon), \tag{37}$$

where $\psi_{k}(\alpha_{0})$ is an arbitrary saddle point.

### IV. Renormalization Group Flow

We present a solution of the renormalization group equation (37) in this Section. The argument is based on the assumption 3.

The solution of the tree-level evolution equation is largely simplified by the high degree of the degeneracy, (35). This allows us to use a simple saddle point to evaluate $S_{k-\delta k}[\phi]$. We use a plane wave

$$\psi_{k}(x) = 2\rho_{k} \cos(\hat{\omega}_{\mu}(k)x_{\mu} + \theta(k)) \tag{38}$$

where the unit vector $\hat{\omega}_{\mu}(k)$ and the phase $\theta(k)$ are the collective coordinates corresponding to the zero modes. The breakdown of the $O(d)$ rotational symmetry gives rise to $\omega_{\mu}$. We tacitly assumed large but finite quantization box with periodic boundary conditions which guarantees the translational symmetry. The translation in the direction of $\hat{\omega}(k)$ is no longer symmetry, the corresponding zero mode is $\alpha(k)$.

The value of the action (13) at $\chi(x) = \Phi + 2\rho \cos(k_{\mu}x_{\mu})$ is

$$V_{d}^{-1}S_{k}[\chi] = U_{k}(\Phi) + \sum_{n=1}^{\infty} \frac{\rho^{2n}}{(n!)^{2}} \left[ nk^{2}\partial_{\phi}^{2n-2}Z_{k}(\Phi) + \partial_{\phi}^{2n}U_{k}(\Phi) \right]. \tag{39}$$
We have seen with (35) that the action is \( \rho \)-independent and thus the coefficient of \( \rho^2 \) vanishes,

\[
n k^2 \partial_\phi^{2n-2} Z_k(\Phi) + \partial_\phi^{2n} U_k(\Phi) = 0,
\]

(40)

to find

\[
Z_k(\Phi) = z(k)
\]

\[
U_k(\Phi) = -\frac{1}{2} z(k) k^2 \Phi^2 + u(k)
\]

(41)

for an even potential.

According to the assumption 3 the saddle point \( \psi_k \) is a continuous function of the background \( \Phi \) at \( \Phi = \Phi(k) \). This implies that \( U_k(\Phi), Z_k(\Phi) \) together with \( \partial_\phi U_k(\Phi) \) and \( \partial_\phi Z_k(\Phi) \) are continuous functions, as well, for \( \Phi = \Phi(k) \) because the saddle point is found by setting the first derivative of the action to zero. The continuity of \( Z_k(\Phi) \) leads to

\[
Z_k^{\text{unst}}(\Phi) = Z_k^{\text{st}}(\Phi(k))
\]

\[
U_k^{\text{unst}}(\Phi) = -\frac{1}{2} Z_k^{\text{st}}(\Phi(k)) k^2 \Phi^2 + u(k)
\]

(42)

where the quantities in the stable \( (\Phi > \Phi(k)) \) and the unstable region \( (\Phi < \Phi(k)) \) are supplied by a subscript ‘st’ and ‘unst’, respectively. \( Z_k^{\text{st}}(\Phi(k)) \) comes from the solution of the Wegner-Houghton equation in the stable region.

As a consequence of the continuity of \( \partial_\phi U_k(\Phi) \) in \( \Phi \), we find:

\[
\frac{\partial U_k^{\text{st}}}{\partial \phi}(\Phi(k)) = \frac{\partial U_k^{\text{unst}}}{\partial \phi}(\Phi(k)) = -Z_k^{\text{st}}(\Phi(k)) k^2 \Phi(k)
\]

(43)

which results the definition of the boundary \( \Phi(k) \)

\[
Z_k^{\text{st}}(\Phi(k)) k^2 \Phi(k) + \frac{\partial U_k^{\text{st}}}{\partial \phi}(\Phi(k)) = 0.
\]

(44)

Here both \( Z_k^{\text{st}}(\Phi(k)) \) and \( U_k^{\text{st}}(\Phi(k)) \) are given by the solution of the Wegner-Houghton equation in the stable region.

Note the disappearance of \( N \) from the equation. One would have thought that the Goldstone modes which go unstable earlier as the cutoff is lowered make different boundary \( \Phi(k) \) for \( N > 1 \) and \( N = 1 \). But it turned out that the instability line for \( N = 1 \) is given by (44) instead of the vanishing of the first factor in the right hand side of (16).

It is important to note that in the limit \( k \to 0 \) the potential in (12) recovers the Maxwell construction: the effective potential \( U_{\text{eff}} = U_{k=0} \) is a flat function of the field between the minima \( \pm \Phi_{k=0} \). The Maxwell construction which is a well-known feature of the vacuum of the coexisting phase has been seen numerically in [4]. The potential (12) has already been reported in the approximation \( Z = 1 \) for \( N = 3 \) with a smooth cutoff by retaining the simple plane wave saddle point [4]. But the smooth cutoff makes other saddle points than the single plane wave important. Our result, valid for any \( N \) underlines the generality of the Maxwell construction. The limit \( N \to \infty \) can be used, as well, to obtain the Maxwell construction.
But it remains an unjustified step in this scheme to ignore the massive mode. The formal argument is that the $N - 1$ Goldstone modes overweight the single massive mode. Since the saddle point appears in this latter the large $N$ scheme can not trace down the stabilisation mechanism in the spinodal phase.

Finally we mention the simple relation satisfied by the amplitude of the plane-wave saddle point,

$$\tilde{\psi}_k(p) = \rho_k(\Phi) \left[ e^{i\theta(k)\delta(k\omega(k) - p)} + e^{-i\theta(k)\delta(k\omega(k) + p)} \right]$$

$$\psi_k(x) = 2\rho_k(\Phi) \cos(k\omega(k)x + \theta(k))$$  \hspace{1cm} (45)

We found that the action is flat for the field whose magnitude is less than $\Phi(k)$. Since the action must increase if the field goes beyond this limit we have

$$\Phi(k) = \max_x \{ \Phi + \psi_k(x) \} = \Phi + 2\rho_k(\Phi)$$  \hspace{1cm} (46)

to determine the amplitude $\rho_k$. First we note that the saddle point, as well as its functional derivative with respect to the field have non-vanishing limits when $\epsilon \to 0$, as we expected when giving the arguments leading to a flat action. Then it is instructive to take the limit $k \to 0$. The saddle point becomes homogeneous with the amplitude $2\rho_0(\Phi)$, and it can be interpreted as the ”polarization” of the vacuum due to the external field $\Phi$. The lesson of (46) is that the external field plus the polarization always add up to $\Phi(0)$, the vacuum expectation value. This complete ”screening” of the external field is the result of the degeneracy of the effective action and is in agreement with the Maxwell construction.

V. CORRELATION FUNCTION

The characteristic feature of the saddle point approximation is the large number of zero modes, the high degree of degeneracy of the action. This makes the tree-level contribution to the correlation functions highly non-trivial. We demonstrate this by computing the correlation function for the constrained model (3)

$$G^\Phi_{\ell,m}(p,q) = \frac{1}{Z} \int D[\tilde{\phi}] \tilde{\phi}_\ell(p) \tilde{\phi}_m(q) e^{-\frac{1}{\hbar} \mathcal{S}_\Lambda[\phi + \Phi]}$$  \hspace{1cm} (47)

in the mean-field approximation where only one plane-wave mode is retained on a homogeneous background $\Phi$. The mean field approximation consists of keeping a single variable active in the path integral in the presence of a non-trivial background. One usually follows this strategy on a real space lattice where a site variable is integrated over in the presence of a homogeneous background field. Such a mean-field approximation is unacceptable within the spinodal unstable phase. Instead, we suggest to make this approximation in the momentum space. The living mode will be a plane wave on the background of $\Phi$. The plane wave will take large amplitude which creates an inhomogeneous ground state supporting the phase separation. The average amplitude of the plane-wave may serve as a measure of the ”readyness” of the system to create the phase separation with a given length scale.

The shallow action is always a potential threat for the loop-expansion. This problem was circumvented in the computation of the tree-level renormalization group by noting that the
number of modes contributing to the loop integrals is small, as well. This is not a sufficient argument any more to suppress the contributions of the fluctuations around the shallow minima for the expectation values, such as (47). What happens is that the fluctuations within the shallow part of the action do contribute to (47) in $\mathcal{O}(h^0)$. When the successive integration of the renormalization group procedure is used for the evaluation of the path integral in the limit $\delta k \to 0$ then we may disregard the saddle point structure and make the integration within each momentum shell, using the appropriate effective action. This integration is simplified in the unstable region where the effective action is flat. Thus the system is strongly "disordered" in terms of the Fourier transformed variables $\tilde{\phi}(p)$ within the mixed phase due to the absence of the restoring force acting on the fluctuations around the vacuum.

The $\mathcal{O}(h^0)$ contribution to (47) is be obtained by restricting the integration into the regime where the effective action is flat and therefore

$$G^\phi_{l,m}(p,q) \simeq \left[ \int D\phi[\tilde{\phi}] \right]^{-1} \int D\phi[\tilde{\phi}] \delta_l(p) \tilde{\phi}_m(q) = \delta_{l,m} \left[ \int D\phi[\tilde{\phi}] \right]^{-1} \int D\phi[\tilde{\phi}] \delta_l(p) \tilde{\phi}_l(q), \quad (48)$$

where $\int D\phi$ stands for the integration over the functional space where the action is flat. The approximation for the computation of the correlation function consists in replacing the fields by plane waves, just as the saddle points, but for which we will integrate the amplitude over the whole range where the action is flat, because of the degeneracy of the latter. We will call $\Omega_N$ the solid angle in the internal space and $\alpha$ the angle between the azimutal axis and the field $\tilde{\phi}$. Let $k_p$ be the momentum of the plane wave. We will then take

$$\tilde{\phi}_l(p) = r_l(p) \left[ e^{i\theta_p \delta(p - k_p)} + e^{-i\theta_q \delta(p + k_p)} \right] \quad (49)$$

and we will summ over all the possible orientations of $k_p$. In the internal space, we have $r_l = r \cos \alpha_l$ and the correlation function is in this approximation

$$G^\phi_{l,m} = \delta_{l,m} \left[ \int D[k] D[\theta] D[\Omega_N] D\phi[r] \right]^{-1} \int D[k] D[\theta] D[\Omega_N] D\phi[r] r(p) r(q) \cos \alpha_p \cos \alpha_q \times \left[ e^{i\theta_p \delta(p - k_p)} + e^{-i\theta_q \delta(p + k_p)} \right] \left[ e^{i\theta_q \delta(q - k_q)} + e^{-i\theta_q \delta(q + k_q)} \right] \quad (50)$$

The integration over the phases $\theta$ will give a result different from 0 only if $|p| = |q|$, otherwise $\theta_p$ and $\theta_q$ are independant variables and therefore remain only the terms where we have the difference $\theta_p - \theta_q$ and one summation over the direction of $k = k_p = k_q$, which we can take as a summation over $k$ because of the Dirac distributions. The correlation function is

$$G^\phi_{l,m} = \delta_{l,m} \left[ \int D[\tilde{\phi}] \int d^4k \int d\Omega_N \int_{0}^{r_p} r^{N-1} dr \right]^{-1} \int D[\tilde{\phi}] \int d^4k \int d\Omega_N \cos^2 \alpha \int_{0}^{r_p} r^{N-1} dr \left[ \delta(p - k) \delta(q + k) + \delta(p + k) \delta(q - k) \right] \quad (51)$$

where we denoted $\int D[\tilde{\phi}]$ the integration over all Fourier components other then $\tilde{\phi}(p)$.

Let us introduce
the volume in the Fourier space where the saddle point amplitude $\rho_k(\Phi)$ is non-vanishing. Here $\kappa_{\Phi}$ satisfies $\Phi(\kappa_{\Phi}) = 0$ and $\kappa_0 = k_{cr}$. The correlation function is finally

$$G_{l,m}^\Phi(p,q) = 2\delta_{l,m} \frac{(2\pi)^d \rho_p^2}{\Gamma_d(\Phi) N + 2} \delta(p + q)$$

(53)

This expression differs from the correlation function given in ref. [4] for $N = 1$ because of the integration over the $O(h^0)$ fluctuations which yield a multiplicative constant only. Note that the integration over the phase $\theta$ restored the translational invariance.

The integration region over the modulus is given by (46):

$$\rho_p(\Phi) = \frac{1}{2}(\Phi(p) - \Phi)$$

(54)

and it extends over the plane waves for which the action is degenerate. The real-space correlation function can easily be obtained in closed form as a function of $|\vec{x} - \vec{y}| = r$ in dimension $d = 3$ for $\Phi = 0$,

$$G_{\ell,\ell}^{\Phi=0}(r) = \frac{18}{g(N + 2)(k_{cr}r)^3} \left[ \left( \frac{3}{r^2} - k_{cr}^2 \right) \sin(k_{cr}r) - \frac{3k_{cr}}{r} \cos(k_{cr}r) \right].$$

(55)

where we took the following bare potential:

$$U_{\Lambda}(\Phi) = \frac{g_2}{2} \Phi^2 + \frac{g_4}{24} \Phi^4 = -\frac{k_{cr}^2}{2} \Phi^2 + \frac{g}{24} \Phi^4$$

(56)

This function together with the ones for $\Phi = \Phi(0)/3$, $2\Phi(0)/3$, and $\Phi(0)$ is shown in Fig. 1. for $g_2 = -0.1$, $g_4 = 0.01$. The vanishing of the Fourier transform (53) for $p > k_{cr}$ induces an oscillatory behaviour of the diffraction integrals. The characteristic length scale of the large amplitude oscillations due to the domain wall structure is $\xi_{macr} = k_{cr}^{-1}$. Note the qualitative difference between the correlation function in the stable phase and (55): The spinodal instability induces a non-monotical behaviour, coming from the trigonometrical functions, instead of the exponential decrease.

**VI. SUMMARY**

It is demonstrated that the tree-level contributions to the blocking relation may induce a non-trivial renormalization. Whenever this happens the system displays instabilities and the vacuum is inhomogeneous and highly non-trivial.

The assumption of the continuity of the effective action in the cutoff within the unstable region allows us to show that the action is flat in the unstable region. This result is valid in each order of the loop expansion. The flatness gives rise the Maxwell construction for the free energy. A diffraction pattern-like correlation function was obtained in the mean-field approximation.
Our argument is based on the Euclidean field theory and as such can be relevant for the description of equilibrium states. It is important to mention that one can, in principle, repeat the steps in the Minkowski space-time. The result is a dynamical renormalization group approach to the formation of the condensate. A somehow similar program has already been followed in ref. [11] which supports the flatness of the effective potential in the spinodal unstable region.

The sharp momentum space cutoff is used in this work. It is well known that the strong dependence on the momentum at the cutoff induces non-local behavior in the real space which in turn spoils the gradient expansion [8]. Though this does not happen in the tree-level, discussed in this paper, it remains to be seen if the loop corrections can be summed up in a consistent scheme in the exterior, stable regime.
FIG. 1. The correlation function in the mean-field approximation.
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