A locally quadratic Glimm functional and sharp convergence rate of the Glimm scheme for nonlinear hyperbolic systems

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Abstract

Consider the Cauchy problem for a strictly hyperbolic, \( N \times N \) quasilinear system in one space dimension

\[
\begin{align*}
  u_t + A(u)u_x &= 0, \\
  u(0, x) &= \bar{u}(x),
\end{align*}
\]

(1)

where \( u \mapsto A(u) \) is a smooth matrix-valued map, and the initial data \( \bar{u} \) is assumed to have small total variation. We investigate the rate of convergence of approximate solutions of (1) constructed by the Glimm scheme, under the assumption that, letting \( \lambda_k(u), r_k(u) \) denote the \( k \)-th eigenvalue and a corresponding eigenvector of \( A(u) \), respectively, for each \( k \)-th characteristic family the linearly degenerate manifold

\[
\mathcal{M}_k \equiv \{ u \in \Omega : \nabla \lambda_k(u) \cdot r_k(u) = 0 \}
\]

is either the whole space, or it is empty, or it consists of a finite number of smooth, \( N-1 \)-dimensional, connected, manifolds that are transversal to the characteristic vector field \( r_k \). We introduce a Glimm type functional which is the sum of the cubic interaction potential defined in [6], and of a quadratic term that takes into account interactions of waves of the same family with strength smaller than some fixed threshold parameter. Relying on an adapted wave tracing method, and on the decrease amount of such a functional, we obtain the same type of error estimates valid for Glimm approximate solutions of hyperbolic systems satisfying the classical Lax assumptions of genuine nonlinearity or linear degeneracy of the characteristic families.

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1 Introduction

Consider the Cauchy problem for a general system of hyperbolic conservation laws in one space dimension

\[ u_t + F(u)_x = 0, \quad u(0, x) = \pi(x). \quad (1.1) \]

Here the vector \( u = u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) represents the conserved quantities, while the components of the vector valued function \( F(u) = (F_1(u), \ldots, F_N(u)) \) are the corresponding fluxes. We assume that the flux function \( F \) is a smooth map defined on a domain \( \Omega \subseteq \mathbb{R}^N \), and that the system (1.1) is strictly hyperbolic, i.e. that the Jacobian matrix \( A(u) = DF(u) \) has \( N \) real distinct eigenvalues

\[ \lambda_1(u) < \cdots < \lambda_N(u) \quad \forall \ u. \quad (1.3) \]

Denote with \( r_1(u), \ldots, r_N(u) \) a corresponding basis of right eigenvectors. Hyperbolic equations in conservation form physically arise in several contexts. A primary example of such systems is provided by the Euler equations of non-viscous gases, see [12].

It is well known that, because of the nonlinear dependence of the characteristic speeds \( \lambda_k(u) \) on the state variable \( u \), classical solutions to (1.1) can develop discontinuities (shock wave) in finite time, no matter of the regularity of the initial data. Therefore, in order to construct solutions globally defined in time, one must consider weak solutions interpreting the equation (1.1) in a distributional sense. Moreover, for sake of uniqueness, an entropy criterion for admissibility is usually added to rule out non-physical discontinuities. In [20] T.P. Liu proposed the following admissibility criterion valid for weak solutions to general systems of conservation laws, that generalizes the classical stability condition introduced by Lax [18].
**Definition 1.1** A shock discontinuity of the $k$-th family $(u^L, u^R)$, traveling with speed $\sigma_k[u^L, u^R]$, is Liu admissible if, for any state $u$ lying on the Hugoniot curve $S_k[u^L]$ between $u^L$ and $u^R$, the shock speed $\sigma_k[u^L, u]$ of the discontinuity $(u^L, u)$ satisfies

$$\sigma_k[u^L, u] \geq \sigma_k[u^L, u^R].$$

The existence of global weak admissible solutions to (1.1)-(1.2) with small total variation was first established in the celebrated paper of Glhmm [14] under the additional assumption that each characteristic field $r_k$ be either linearly degenerate (LD), so that

$$\nabla \lambda_k(u) \cdot r_k(u) = 0 \quad \forall u,$$

or else genuinely nonlinear (GNL) i.e.

$$\nabla \lambda_k(u) \cdot r_k(u) \neq 0 \quad \forall u.$$  

A random choice method, the Glimm scheme, was introduced in [14] to construct approximate solutions of the general Cauchy problem (1.1)-(1.2) by piecing together solutions of several Riemann problems, i.e. Cauchy problems whose initial data are piecewise constant with a single jump at the origin

$$u(0, x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x > 0. \end{cases}$$

Using a nonlinear functional introduced by Glimm, that measures the nonlinear coupling of waves in the solution, one can establish a-priori bounds on the total variation of a family of approximate solutions. These uniform estimates then yield the convergence of a sequence of approximate solutions to the weak admissible solution of (1.1)-(1.2). The existence theory for the Cauchy problem (1.1)-(1.2) based on a Glimm scheme was extended by Liu [22], Liu and Yang [23], and by Iguchi and LeFloch [17] to the case of systems with non genuinely nonlinear (NGNL) characteristic families whose flux function satisfy the more general assumption:

(H) The vector valued function $F$ is $C^3$, and for each $k \in \{1, \ldots, N\}$-th characteristic family the linearly degenerate manifold

$$\mathcal{M}_k = \{ u \in \Omega : \nabla \lambda_k(u) \cdot r_k(u) = 0 \}$$

is either empty (GNL characteristic field), or it is the whole space (LD characteristic field), or it consists of a finite number $\leq M$ of smooth, $N-1$-dimensional, connected, manifolds, and there holds

$$\nabla(\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) \neq 0 \quad \forall u \in \mathcal{M}_k.$$  

Aim of the present paper is to provide a sharp convergence rate for approximate solutions obtained by the Glimm scheme valid for strictly hyperbolic systems of
conservation laws satisfying the assumption (H). We recall that in the Glimm scheme, one works with a fix grid in the $t$-$x$ plane, with mesh sizes $\Delta t, \Delta x$. An approximate solution $u^\varepsilon$ of (1.1)-(1.2) is then constructed as follows. By possibly performing a linear change of coordinates in the $t$-$x$ plane, we may assume that the characteristic speeds $\lambda_k(u), 1 \leq k \leq N$, take values in the interval $[0, 1]$, for all $u \in \Omega$. Then, choose $\Delta t = \Delta x = \varepsilon$, and let $\{\theta_i\}_{i \in \mathbb{N}} \subset [0, 1]$ be an equidistributed sequence of numbers, which thus satisfies the condition

$$\lim_{n \to \infty} \left| \lambda - \frac{1}{n} \sum_{\ell=0}^{n-1} \chi_{[0,\lambda]}(\theta_\ell) \right| = 0 \quad \forall \lambda \in [0, 1],$$

where $\chi_{[0,\lambda]}$ denotes the characteristic function of the interval $[0, \lambda]$. On the initial strip $0 \leq t < \varepsilon$, $u^\varepsilon$ is defined as the exact solution of (1.1), with starting condition

$$u^\varepsilon(0, x) = u^\varepsilon(j \varepsilon + \theta_0 \varepsilon) \quad \forall x \in [j \varepsilon, (j+1)\varepsilon[.$$

Next, assuming that $u^\varepsilon$ has been constructed for $t \in [0, i\varepsilon[$, on the strip $i\varepsilon \leq t < (i+1)\varepsilon$, $u^\varepsilon$ is defined as the exact solution of (1.1), with starting condition

$$u^\varepsilon(i\varepsilon, x) = u^\varepsilon(i \varepsilon - (j + \theta_i) \varepsilon) \quad \forall x \in [j \varepsilon, (j+1)\varepsilon[.$$

Relying on uniform a-priori bounds on the total variation, we thus define inductively the approximate solution $u^\varepsilon(t, \cdot)$ for all $t \geq 0$.

One can repeat this construction with the same values $\theta_i$ for each time interval $[i\varepsilon, (i+1)\varepsilon[,$ and letting the mesh size $\varepsilon$ tend to zero. Hence, we obtain a sequence of approximate solutions which converge, by compactness, to some limit function $u$ that is shown to be a weak admissible solution of (1.1)-(1.2) (cfr. [19]). In order to derive an accurate estimate of the convergence rate of the approximate solutions, it was introduced in [10] an equidistributed sequence $\{\theta_i\}_{i \in \mathbb{N}} \subset [0, 1]$ enjoying the following property. For any given $0 \leq m < n$, define the discrepancy of the set $\{\theta_m, \ldots, \theta_{n-1}\}$ as

$$D_{m,n} \doteq \sup_{\lambda \in [0,1]} \left| \lambda - \frac{1}{n-m} \sum_{m \leq \ell < n} \chi_{[0,\lambda]}(\theta_\ell) \right|. \quad (1.11)$$

Then, there holds

$$D_{m,n} \leq O(1) \cdot \frac{1 + \log(n-m)}{n-m} \quad \forall n > m \geq 1. \quad (1.12)$$

Here, and throughout the paper, $O(1)$ denotes a uniformly bounded quantity, while we will use the Landau symbol $o(1)$ to indicate a quantity that approaches zero as $\varepsilon \to 0$. Relying on the existence of a Lipschitz continuous semigroup of solutions generated by (1.1), compatible with the solutions of the Riemann problems, it was proved in [10] that, for systems with GNL or LD characteristic fields, the $L^1$
convergence rate of the Glimm approximate solutions constructed in connection with a sequence enjoying the property (1.12) is \( o(1) \cdot \sqrt{\varepsilon |\log\varepsilon|} \). In the case of general systems satisfying the assumption (H), it was derived in [16] an estimate of the \( L^1 \) norm of the error in the Glimm approximate solutions of the order \( o(1) \cdot \sqrt{\varepsilon |\log\varepsilon|} \).

In the present paper, we improve this result by establishing the same convergence rate of the approximate solutions generated by the Glimm scheme for systems satisfying the assumption (H) as in the case of systems with GNL or LD characteristic fields. Namely, our result is the following.

**Theorem 1.2** Let \( F \) be a \( C^3 \) map from a domain \( \Omega \subseteq \mathbb{R}^N \) into \( \mathbb{R}^N \) satisfying the assumption (H), and assume that the system (1.1) is strictly hyperbolic. Given an initial datum \( u \) with small total variation, let \( u(t, \cdot) \) be the unique Liu admissible solution of (1.1)-(1.2). Let \( \{\theta_k\}_{k \in \mathbb{N}} \subset [0, 1] \) be a sequence satisfying (1.12) and construct the corresponding Glimm approximate solution \( u_\varepsilon \) of (1.1)-(1.2) with mesh sizes \( \Delta x = \Delta t = \varepsilon \). Then, for every \( T \geq 0 \) there holds

\[
\lim_{\varepsilon \to 0} \frac{\| u_\varepsilon(T, \cdot) - u(T, \cdot) \|_{L^1}}{\sqrt{\varepsilon |\log\varepsilon|}},
\]

(1.13)

and the limit is uniform w.r.t. \( u \) as long as \( \text{Tot.Var.}(\Omega) \) remains uniformly small.

Our result applies more generally to strictly hyperbolic \( N \times N \) quasilinear systems

\[
u_t + A(u) u_x = 0,
\]

(1.14)

not necessarily in conservation form, where \( A \) is a \( C^2 \) matrix valued map defined from a domain \( \Omega \subseteq \mathbb{R}^N \) into \( M^{N \times N}(\mathbb{R}) \), whose eigenvalues \( \lambda_k, k \in \{1, \ldots, N\} \), satisfy the assumption stated in (H). Indeed, one may alternatively assume that \( A : \Omega \to M^{N \times N}(\mathbb{R}) \) is a \( C^{1,1} \) map, and that for each NGNL \( k \in \{1, \ldots, N\} \)-th characteristic family the linearly degenerate manifold \( M_k \) consists of a finite number of connected manifolds \( M_{k,h} \), that are either \( N-1 \)-dimensional as in (H) or \( N \)-dimensional with a similar condition to (1.9) (cfr. Remark 3.6 in § 3 and Remark 6.1 in § 6).

In fact, the fundamental paper of Bianchini and Bressan [7] shows that, for any \( C^{1,1} \) map \( A : \Omega \to M^{N \times N}(\mathbb{R}) \) with strictly hyperbolic values, (1.14) generates a unique (up to the domain) Lipschitz continuous semigroup \( \{S_t : t \geq 0\} \) of vanishing viscosity solutions obtained as the (unique) limits of solutions to the (artificial) viscous parabolic approximation

\[
u_t + A(u) u_x = \mu u_{xx},
\]

(1.16)μ

when the viscosity coefficient \( \mu \to 0 \). The trajectories of such a semigroup starting from piecewise constant initial data locally coincide with the “admissible” solution of each Riemann problem determined by the jumps in the initial data. Moreover, any limit of Glimm approximations coincides with the corresponding trajectory of the semigroup generated by (1.14). In particular, in the conservative case where \( A(u) = DF(u) \) every vanishing viscosity solution of the Cauchy
problem (1.14)-(1.2) provides a weak solution of (1.1)-(1.2) satisfying the Liu admissibility conditions (1.4).

The proof of the error bound (1.13) follows the same strategy adopted in [10], relying on the careful analysis of the structure of the solution for systems satisfying the assumption (H), developed by T.P. Liu and T. Yang in [22, 23]. Indeed, to estimate the distance between a Lipschitz continuous (in time) approximate solutions $w$ of (1.14) and the corresponding exact solution one would like to use the error bound [9]

$$
\|w(T) - S_tw(0)\|_{L^1} \leq L \int_0^T \lim_{h \to 0+} \frac{\|w(t + h) - S_hw(t)\|_{L^1}}{h} dt,
$$

(1.17)

where $L$ denotes a Lipschitz constant of the semigroup $S$ generated by (1.1). However, for approximate solutions constructed by the Glimm scheme, a direct application of this formula is of little help because of the additional errors introduced by the restarting procedures at times $t_i = i\varepsilon$. For this reason, following the wave tracing analysis in [23], it is useful to partition the elementary waves present in the approximate solution, say in a time interval $[\tau_1, \tau_2]$, into virtual waves that can be either traced back from $\tau_2$ to $\tau_1$, or are canceled or generated by interactions occurring in $[\tau_1, \tau_2]$. Thanks to the simplified wave pattern associated to this partition, one can construct a front tracking approximation having the same initial and terminal values as the Glimm approximation, and thus establish (1.13) relying on (1.17).

As one would expect, the presence of elementary waves with various composite wave patterns for systems satisfying the assumption (H), requires a careful analysis of the errors introduced by this wave-partition algorithm. As customary, the change of wave-size and wave-speeds when an interaction takes place is controlled by a Glimm functional that measures the potential interaction of waves in the solution.

For general strictly hyperbolic systems (1.14) satisfying the assumption (H), several nonlinear functionals were introduced in [22, 23, 17, 6], consisting of a standard Glimm quadratic functional, for the interaction of waves of different families, and of a cubic functional measuring the potential interaction between waves of the same family. This cubic part of the functional is defined in terms of the strengths of any pair of waves of the same family and of the absolute value of the angle between them [6] (or of the positive part of the angle between two waves [22, 23]). Such functionals work perfectly to establish uniform a-priori bounds on the total variation of the solution, but are not effective to control the quadratic order error produced by the change of wave speeds for interactions of waves of the same family, of arbitrarily small sizes.

On the other hand, in the case of systems whose characteristic families admit a single, connected, $N - 1$-dimensional degenerate manifold (1.8), it was introduced in [3] a decreasing potential interaction functional which is of second order w.r.t. the total variation (measuring the potential interaction between any pair of waves
as proportional to the product of their strengths, no matter if they belong to the same family or not).

In the present paper, in connection with a fixed threshold parameter $\delta_0 > 0$, we define a Glimm type functional $Q = Q_q + c Q$, for a suitable constant $c > 0$, which is the sum of a quadratic term $Q_q$ and of the cubic interaction potential $Q$ defined in [6]. Here, in presence of interactions between waves of the same families and strength smaller than $\delta_0$, $Q_q$ behaves as the interaction functional introduced in [3], while the decrease of $Q$ controls the possible increase of $Q_q$ at interactions involving waves of the same family and strength larger than $\delta_0$. Employing this functional we can produce a simplified wave partition pattern whose errors are controlled by the total decrease of the Glimm functional in the time interval taken in consideration, and thus yield the error estimate $(1.13)$.

**Note added.** During the completion of the present paper, we have had knowledge of a contemporary different proof of the same convergence rate $(1.13)$ provided by J. Hua, Z. Jiang and T. Yang [15], for Glimm approximations of a system $(1.14)$ satisfying the assumption (H). Their proof is obtained by using an adapted form of the functional introduced in [22, 23], that takes care of the errors in the wave-speeds at interactions between waves of the same family.

### 2 Preliminaries

Let $A$ be a smooth matrix-valued map defined on a domain $\Omega \subset \mathbb{R}^N$, with values in the set of $N \times N$ matrices. Assume that each $A(u)$ is strictly hyperbolic and denote by $\{\lambda_1(u), \ldots, \lambda_N(u)\} \subset [0, 1]$ its eigenvalues. Since we will consider only solutions with small total variation that take values in a neighborhood of a compact set $K \subset \Omega$, it is not restrictive to assume that $\Omega$ is bounded and that there exist constants $\hat{\lambda}_0 < \cdots < \hat{\lambda}_N$ such that

$$\hat{\lambda}_{k-1} < \lambda_k(u) < \hat{\lambda}_k, \quad \forall \ u, \ k = 1, \ldots, N. \quad (2.1)$$

One can choose bases of right and left eigenvectors $r_k(u), l_k(u), (k = 1, \ldots, N)$, associated to $\lambda_k(u)$, normalized so that

$$|r_k(u)| = 1, \quad \langle l_h(u), r_k(u) \rangle = \begin{cases} 1 & \text{if } k = h, \\ 0 & \text{if } k \neq h, \end{cases} \quad \forall \ u. \quad (2.2)$$

By the strict hyperbolicity of the system, in the conservative case $(1.1)$ (where $A(u) = DF(u)$), for every fixed $u_0 \in \Omega$ and for each $k \in \{1, \ldots, N\}$-th characteristic family one can construct in a neighborhood of $u_0$ a one-parameter smooth curve $S_k[u_0]$ passing through $u_0$ (called the $k$-th Hugoniot curve issuing from $u_0$), whose points $u \in S_k[u_0]$ satisfy the Rankine Hugoniot equation $F(u) - F(u_0) = \sigma(u - u_0)$ for some scalar $\sigma = \sigma_k[u_0, u]$. The curve $S_k[u_0]$ is tangent at $u_0$ to the right eigenvector $r_k(u_0)$ of $A(u_0)$ associated to $\lambda_k(u_0)$, and we say that $(u^L, u^R)$ is a shock discontinuity of the $k$-th family with speed $\sigma_k[u^L, u^R]$ if $u^R \in S_k[u^L]$. 7
We describe here the general method introduced in [7, 5] to construct the self-similar solution of a Riemann problem for a strictly hyperbolic quasilinear system \((1.14)\). As customary, the basic step consists in constructing the elementary curve of the \(k\)-th family \((k = 1, \ldots, N)\) for every given left state \(u^L\), which is a one parameter curve of right states \(s \mapsto T_k[u^L](s)\) with the property that the Riemann problem having initial data \((u^L, u^R)\), \(u^R = T_k[u^L](s)\), admits a vanishing viscosity solution consisting only of elementary waves of the \(k\)-th characteristic family. Such a curve is constructed by looking at the fixed point of a suitable contractive transformation associated to a smooth manifold of viscous traveling profiles for the parabolic system with unit viscosity \((1.16)\).

Given a fixed state \(u_0 \in \Omega\), and an index \(k \in \{1, \ldots, N\}\), in connection with the \(N + 2\)-dimensional smooth manifold of bounded traveling profiles of \((1.16)\) with speed close to \(\lambda_k(u_0)\), one can define on a neighborhood of \((u_0, 0, \lambda_k(u_0)) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}\) suitable smooth vector functions \((u, v_k, \sigma) \mapsto \tilde{r}_k(u, v_k, \sigma)\) that satisfy
\[
\tilde{r}_k(u_0, 0, \sigma) = r_k(u_0) \quad \forall \sigma, \quad (2.3)
\]
and are normalized so that
\[
\langle l_k(u_0), \tilde{r}_k(u, v_k, \sigma) \rangle = 1 \quad \forall u, v_k, \sigma. \quad (2.4)
\]
The vector valued map \(\tilde{r}_k(u, v_k, \sigma)\) is called the \(k\)-th generalized eigenvector of the matrix \(A(u)\), associated to the generalized eigenvalue
\[
\tilde{\lambda}_k(u, v_k, \sigma) \doteq \langle l_k(u_0), A(u) \tilde{r}_k(u, v_k, \sigma) \rangle,
\]
that satisfies the identity
\[
\tilde{\lambda}_k(u_0, v_k, \sigma) = \lambda_k(u_0) \quad \forall v_k, \sigma. \quad (2.5)
\]
Next, given a left state \(u^L\) in a neighborhood of \(u_0\) and \(0 < s << 1\), consider the integral system
\[
\begin{align*}
\begin{cases}
u(\tau) = u^L + \int_0^\tau \tilde{r}_k(u(\xi), v_k(\xi), \sigma(\xi)) \, d\xi, \\
v_k(\tau) = \bar{F}_k(\tau; u, v_k, \sigma) - \text{conv}_{[0,s]} \bar{F}_k(\tau; u, v_k, \sigma), \quad 0 \leq \tau \leq s, \\
\sigma(\tau) = \frac{d}{d\tau} \text{conv}_{[0,s]} \bar{F}_k(\tau; u, v_k, \sigma),
\end{cases}
\end{align*}
\]
where \(\tau \mapsto \bar{f}_k(\tau) \doteq \bar{F}_k(\tau; u, v_k, \sigma)\) is the “reduced flux function” associated to \((1.16)\) defined, by
\[
\bar{f}_k(\tau) \doteq \int_0^\tau \tilde{\lambda}_k(u(\xi), v_k(\xi), \sigma(\xi)) \, d\xi,
\]
\[
(2.7)
\]
and we let \( \text{conv}_{[0,s]} \tilde{f}_k(\tau) \) denote the lower convex envelope of \( \tilde{f}_k \) on \([0,s] \), i.e.

\[
\text{conv}_{[0,s]} \tilde{f}_k(\tau) = \inf \left\{ \theta \tilde{f}_k(y) + (1 - \theta) \tilde{f}_k(z) : \right. \\
\left. \quad \theta \in [0,1], \ y, z \in [0,s], \ \tau = \theta y + (1 - \theta) z \right\}. 
\tag{2.8}
\]

Relying on (2.3), (2.5) it is shown in [7, 5] that, for \( s \) sufficiently small, the transformation defined by the right-hand side of (2.6) maps a domain of continuous curves \( \tau \mapsto (u(\tau), v_k(\tau), \sigma(\tau)) \) into itself, and is a contraction w.r.t. a suitable weighted norm. Hence, for every \( u^L \) in a neighborhood \( \mathcal{U}_0 \) of \( u_0 \), the transformation defined by (2.6) admits a unique fixed point

\[
\tau \mapsto \left( u(\tau; u^L, s), v_k(\tau; u^L, s), \sigma(\tau; u^L, s) \right), \quad \tau \in [0, s],
\]

which provides a Lipschitz continuous solution to the integral system (2.6). The elementary curve of right states of the \( k \)-th family issuing from \( u^L \) is then defined as the terminal value at \( \tau = s \) of the \( u \)-component of the solution to the integral system (2.6), i.e., by setting

\[
T_k[u^L](s) \equiv u(s; u^L, s). \tag{2.9}
\]

Sometimes, the value (2.9) of the elementary curve issuing from \( u^L \) will be equivalently written \( T_k(s)[u^L] \). In the following it will be convenient to adopt the notations

\[
\sigma_k[u^L](s, \tau) \equiv \sigma(\tau; u^L, s)
\]

\[
\tilde{F}_k[u^L](s, \tau) \equiv \tilde{F}_k(\tau; u(\cdot; u^L, s), v_k(\cdot; u^L, s), \sigma(\cdot; u^L, s)) \quad \forall \ \tau \in [0, s],
\]

(2.10)

for the \( \sigma \)-component of the solution to (2.6), and for the reduced flux evaluated in connection with such a solution.

For negative values \( s < 0, |s| << 1 \), one replaces in (2.6) the lower convex envelope of \( \tilde{F}_k \) on the interval \([0,s]\) with its upper concave envelope on \([s,0]\) (defined in analogous way as (2.8)), and then constructs the curve \( T_k[u^L] \) and the map \( \sigma_k[u^L] \) exactly in the same way as above looking at the solution of the integral system (2.6) on the interval \([s,0]\). The elementary curve \( T_k[u^L] \) and the wave-speed map \( \sigma_k[u^L] \) constructed in this way enjoy the properties stated in the following theorem, where we let \( \mathcal{C}_I([a,b]) \) denote the set of continuous and increasing scalar functions defined on an interval \([a,b] \), and we set \( \mathcal{C}_I([a,b]) = \mathcal{C}_D([b,a]) \) in the case \( a > b \), letting \( \mathcal{C}_D([b,a]) \) denote the set of continuous and decreasing scalar functions defined on \([b,a] \).

**Theorem 2.1** ([7, 5]) Let \( A \) be a smooth, matrix valued map defined from a domain \( \Omega \subset \mathbb{R}^N \) into \( \mathbb{M}^{N \times N}(\mathbb{R}) \), and assume that the matrices \( A(u) \) are strictly hyperbolic. Then, for every \( u \in \Omega \), there exist \( N \) Lipschitz continuous curves
s \to T_k[u](s) \in \Omega \text{ satisfying } \lim_{s \to 0} \frac{d}{ds} T_k[u](s) = r_k(u), \text{ together with } N \text{ continuous functions } s \to \sigma_k[u](s, \cdot) \in C^1((0,s]) \ (k = 1, \ldots, N), \text{ defined on a neighborhood of zero, so that the following holds. Whenever } u^L \in \Omega, \ u^R = T_k[u^L](s), \text{ for some } s, \text{ letting } I = \{ \tau \in [0,s] : \sigma_k[u^L](s, \tau) \neq \sigma_k[u^L](s, \tau') \text{ for all } \tau' \neq \tau \}, \text{ the piecewise continuous function}

\begin{align*}
   u(t, x) &= \begin{cases} 
   u^L & \text{if } x/t < \sigma_k[u^L](s, 0), \\
   T_k[u^L](\tau) & \text{if } x/t = \sigma_k[u^L](s, \tau) \text{ for some } \tau \in I, \\
   u^R & \text{if } x/t > \sigma_k[u^L](s, s),
   \end{cases}
\end{align*}

(2.11)

provides the unique vanishing viscosity solution (determined by the parabolic approximation (1.16)) of the Riemann problem (1.14), (1.7).

Remark 2.2 If the system (1.14) is in conservation form, i.e. in the case where \( A(u) = DF(u) \) for some smooth flux function \( F \), and if the characteristic fields satisfy the assumption (H), the general solution of the Riemann problem provided by (2.11) is a composed wave of the k-th family made of a finite number of contact discontinuities (which satisfy the Liu admissibility condition of Definition 1.1) adjacent to rarefaction waves. Namely, the regions where the \( v_k \)-component of the solution to (2.9) vanishes correspond to rarefaction waves if the \( \sigma \)-component is strictly increasing and to contact discontinuities if the \( \sigma \)-component is constant, while the regions where the \( v_k \)-component of the solution to (2.9) is different from zero correspond to contact discontinuities or to compressive shocks. In particular, whenever the solution of a Riemann problem with initial data \( u^L, u^R = T_k[u^L](s) \) contains a Liu admissible shock joining, say, two states \( T_k[u^L](s'), T_k[u^L](s''), \) \( s', s'' \in [0,s] \), one has \( \sigma_k[u^L](s, s') = \sigma_k[u^L](s, \tau) \) for all \( \tau \in [s', s''] \), and \( \sigma_k[u^L](s, s') \) provides the shock speed of the discontinuity \( (T_k[u^L](s'), T_k[u^L](s'')) \). Clearly, in a non conservative setting, “admissibility” for a jump means precisely that the jump corresponds to a traveling profile for the parabolic approximation with identity viscosity matrix (1.16).

Once we have constructed the elementary curves \( T_k \) for each \( k \)-th characteristic family, the vanishing viscosity solution of a general Riemann problem for (1.1) is then obtained by a standard procedure observing that the composite mapping

\( (s_1, \ldots, s_N) \mapsto T_N(s_N) \circ \cdots \circ T_1(s_1)[u^L] = u^R, \)

(2.12)
is one-to-one from a neighborhood of the origin in \( \mathbb{R}^N \) onto a neighborhood of \( u^L \). This is a consequence of the fact that the curves \( T_k[u] \) are tangent to \( r_k(u) \) at zero (cfr. Theorem 2.1), and then follows by applying a version of the implicit function theorem valid for Lipschitz continuous maps. Therefore, we can uniquely determine intermediate states \( u^L = \omega_0, \omega_1, \ldots, \omega_N = u^R \), and wave sizes \( s_1, \ldots, s_N \), such that there holds

\( \omega_k = T_k[\omega_{k-1}](s_k) \quad k = 1, \ldots, N, \)

(2.13)
provided that the left and right states $u^L, u^R$ are sufficiently close to each other. Each Riemann problem with initial data

$$\overline{u}_k(x) = \begin{cases} 
\omega_{k-1} & \text{if } x < 0, \\
\omega_k & \text{if } x > 0,
\end{cases} \quad (2.14)$$

admits a vanishing viscosity solution of total size $s_k$, containing a sequence of rarefactions and Liu admissible discontinuities of the $k$-th family. Then, because of the uniform strict hyperbolicity assumption (2.1), the general solution of the Riemann Problem with initial data $(u^L, u^R)$ is obtained by piecing together the vanishing viscosity solutions of the elementary Riemann problems (1.1) (2.14).

Throughout the paper, with a slight abuse of notation, we shall often call $s$ a wave of (total) size $s$, and, if $u^R = T_k[u^L](s)$, we will say that $(u^L, u^R)$ is a wave of size $s$ of the $k$-th characteristic family.

A fundamental ingredient in order to get a convergence rate for the Glimm scheme is the wave tracing procedure, which was first introduced by T.P. Liu in his celebrated paper [19] for systems with genuinely nonlinear or linearly degenerate fields, and lately extended to systems fulfilling assumption (H) [22, 23]. In this spirit, we introduce the following notion of partition of a $k$-wave $(u^L, u^R)$, defined in terms of the elementary curves $T_k$ at (2.9).

**Definition 2.3** Given a pair of states $u^L, u^R$, with $u^R = T_k[u^L](s)$ for some $s > 0$, we say that a set $\{y^1, \ldots, y^\ell\}$ is a partition of the $k$-th wave $(u^L, u^R)$ at time $i\varepsilon$, if the followings holds.

1. There exist scalars $s^h > 0, h = 1, \ldots, l$, such that, setting $\tau^h = \sum_{p=1}^h s^p$, $w^h = T_k[u^L](\tau^h)$, there holds

$$y^h = w^h - w^{h-1} \quad \forall \ h.$$  

The quantity $s^h$ is called the size of the elementary wave $y^h$.

2. Letting $\sigma = \sigma_k[u^L](s, \cdot)$ be the map in (2.10), there holds

$$\sigma(s^h) - \sigma(s^h-1) \leq \varepsilon \quad \forall \ h.$$

Moreover, we require that $\theta_{i+1} \notin \sigma(\tau^{h-1}), \sigma(\tau^h)$, for all $h$ (so to avoid further partitions of $y^h$ at $t = (i+1)\varepsilon$).

The definition is entirely similar in the case $u^R = T_k[u^L](s)$, with $s < 0$. In connection with a partition $\{y^1, \ldots, y^\ell\}$ of $(u^L, u^R)$, we define the corresponding speed of the elementary wave $y^h$ as

$$\lambda^h_k = \frac{1}{s^h} \int_{\tau^{h-1}}^{\tau^h} \sigma(\tau) \, d\tau \quad \forall \ h. \quad (2.15)$$
3 The case of a single linearly degenerate manifold

In this section we will establish the basic estimates on the change in size and speeds of the elementary waves of an approximate solution provided by the Glimm scheme, under the following simplified assumption for the hyperbolic system (1.1) (or for the quasilinear system (1.14)).

(H1) For each $k \in \{1, \ldots, N\}$-th characteristic family the linearly degenerate manifold $\mathcal{M}_k$ at (1.8) is either empty (GNL), or it is the whole space (LD), or it consists of a single smooth, $N-1$-dimensional, connected, manifold and there holds (1.9) (NGNL).

The general solution of a Riemann problem for a system satisfying the assumption (H1) consists of rarefaction waves, compressive shock and composed waves made of a single one-side contact discontinuity adjacent to a rarefaction wave. For such systems, we may consider the same type of quadratic interaction potential introduced in [3] for approximate solutions constructed by a front tracking algorithm, which in the case of solutions $u^\epsilon$ generated by a Glimm scheme can be defined by setting

$$Q_1(t) \doteq 2 \sum_{k_\alpha = k_\beta} \sum_{s_\alpha, s_\beta > 0} |s_\alpha s_\beta| + 2 \sum_{\alpha} |s^r_\alpha s^s_\alpha| + \sum_{\alpha} |s^r_\alpha|^2 +$$

$$+ c_0 \left[ \sum_{k_\alpha = k_\beta} \sum_{s_\alpha, s_\beta < 0} \sum_{x_\alpha(t) > x_\beta(t)} |s_\alpha s_\beta| \right],$$

where $c_0 > 2$ is a suitable large constant to be defined later, $s_\alpha$ denotes the size of a wave of the $k_\alpha$-th family of $u^\epsilon(t)$ located at $x_\alpha(t)$, while $s^r_\alpha$, $s^s_\alpha$ are, respectively, the (possibly zero) rarefaction and shock components of a wave $s_\alpha$. The presence of the factor 2 in the first two summands guarantees the invariance of $Q_1$ when two portions of rarefaction fans of the same family, emanating from two consecutive mesh-points, are joined together for the effect of sampling, since otherwise the quantity $Q_1$ would increase for the presence of the square of the rarefaction components. As customary, we shall define the total strength of waves in $u^\epsilon(t)$ as

$$V(t) \doteq \sum_{\alpha} |s_\alpha|.$$ (3.2)

To fix the ideas, assume that the second derivative of $\lambda_k$ in (1.9) is negative, i.e. that

$$\nabla(\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) < 0 \quad \forall u \in \mathcal{M}_k.$$ (3.3)
In order to control the nonlinear coupling of waves of the same family and with the same sign of two Riemann solutions for systems satisfying the assumption (H1), as in [3] we introduce the following definition of quantity of interaction.

**Definition 3.1** Consider two nearby waves of sizes $s', s''$ with the same sign and belonging to the the same $k$-th characteristic family, with $s'$ located at the left of $s''$. Let $u', u''$ be the left state of $s', s''$, respectively, and assume that there exist waves $s_i, k < i \leq N$, of the $i$-th family, $s_j''$, $1 \leq j < k$, of the $j$-th family, so that $u'' = (\bigcirc_{j=1}^{k-1} T_j(s_j'')) \circ (\bigcirc_{i=k}^{N} T_i(s_i'))[u']$. Then, we define the quantity of interaction between $s'$ and $s''$ as

$$I_1(s', s'') \doteq \left( |(s' + s'')^r - s'^r| + |s'^s| |s'^r| \right), \quad (3.4)$$

where $s' + s''$ must be interpreted as the size of a $k$-wave having left state $u^r \doteq \bigcirc_{j=1}^{k-1} T_j(s_j'')[u']$, while $s^r, s^s$ denote, respectively, the (possibly zero) rarefaction and shock components of a wave $s$.

**Remark 3.2** In the case where $s', s''$ are both rarefactions the quantity of interaction $I_1$ in (3.4) vanishes, while $I_1(s', s'') = |s' s''|$ whenever $s', s''$ are both shock waves.

By standard arguments (e.g., see [12] Section 9.6, Section 13.4) one can obtain as in [3] the basic estimates on the change in values of the total strength of waves $V(t)$ and of the interaction potential $Q_1(t)$, across the grid-times $\varepsilon_i$, for an approximate solution $u^*$ constructed by the Glimm scheme. Namely, defining for every pair of waves of the same family $s', s''$ the amount of cancellation $\mathcal{C}(s', s'')$ as

$$\mathcal{C}(s', s'') \doteq \begin{cases} \min\{||s'|, |s''|\} & \text{if } s's'' < 0, \\ 0 & \text{otherwise}, \end{cases} \quad (3.5)$$

the following generalization of [3] Lemma 2.1, Lemma 5.1 hold.

**Lemma 3.3** Under the assumption (H1), let $s_1', \ldots, s_N'$ and $s_1'', \ldots, s_N''$ be, respectively, the sizes of the waves in the solution of two adjacent Riemann problems $(u^L, u^M)$ and $(u^M, u^K)$, $s_i'$ and $s_i''$ belonging to the $i$-th characteristic family. Call $s_1, \ldots, s_N$ the sizes of the waves in the solution of the Riemann problem $(u^L, u^K)$, $s_i$ belonging to the $i$-th characteristic family. Then, there holds

$$\sum_{k=1}^{N} |s_k - s_k' - s_k''| = O(1) \cdot \left[ \sum_{1 \leq i \leq j \leq N} \left| s_i' s_j'' \right| + \sum_{i=1}^{N} \sum_{s_i' s_i'' < 0} \left| s_i' s_i'' \right| + \sum_{i=1}^{N} \sum_{s_i' s_i'' > 0} I_1(s_i', s_i'') \right]. \quad (3.6)$$

Moreover, for any $k \in \{1, \ldots, N\}$-th NGNL characteristic family, the following
estimates on the rarefaction components of the outgoing waves hold.

\[ |s_k^r - (s_k^r + s_k^u)^r| = O(1) \cdot \left[ \sum_{1 \leq i,j \leq N} |s_i^l s_j^r| + I_1(s_k^r, s_k^u) \right] \]

if \( s_k^r s_k^u > 0 \), \hspace{1cm} (3.7)

\[ |s_k^r - (s_k^r + s_k^u)| = O(1) \cdot \left[ \sum_{1 \leq i,j \leq N} |s_i^l s_j^r| + \min\{|s_i^l|, |s_j^r|\} \right] \]

if \( s_k^r s_k^u < 0 \), \hspace{1cm} (3.8)

where, in (3.7) \( s_k^r + s_k^u \) represents the size of a \( k \)-wave having left state \( u^r = \left( \bigcirc_{j=1}^{k-1} T_j(s_j^u) \right) \odot \left( \bigcirc_{l=1}^{k-1} T_l(s_l^u) \right)[u^L] \).

**Lemma 3.4** In the same setting of Lemma 3.3, provided that the total strength of waves is sufficiently small, there exists some constant \( c_0 > 0 \) (in (3.1)) so that there holds

\[ \Delta V \leq - \sum_{1 \leq i \leq N} C(s_i^l, s_i^u) + O(1) \cdot \left[ \sum_{1 \leq i,j \leq N} |s_i^l s_j^r| + \sum_{1 \leq i \leq N} I_1(s_i^l, s_i^u) \right] , \hspace{1cm} (3.9) \]

\[ \Delta Q_1 \leq - \frac{1}{2} \left[ \sum_{1 \leq i,j \leq N} |s_i^l s_j^r| + \sum_{1 \leq i \leq N} |s_i^l s_i^u| + \sum_{1 \leq i \leq N} I_1(s_i^l, s_i^u) \right]. \hspace{1cm} (3.10) \]

Here, as customary, we use the notations \( \Delta V = V^+ - V^- \), \( \Delta Q_1 = Q_1^+ - Q_1^- \), where \( V^- \), \( Q_1^- \) and \( V^+, Q_1^+ \) denote, respectively, the values of \( V, Q_1 \) related to the incoming waves \( s_1^1, \ldots, s_N^1, s_1^u, \ldots, s_N^u \), and to the outgoing waves \( s_1^1, \ldots, s_N^1 \).

Relying on Lemma 3.4 one deduces that there exists some constant \( C_1 > 0 \), independent of \( \varepsilon \), so that if \( V(t), Q_1(t) \) denote the total strength of waves and the interaction potential of an approximate solution \( u'(t) \) constructed by the Glimm scheme, the functional

\[ t \mapsto \mathcal{T}_1(t) = V(t) + C_1 Q_1(t) \hspace{1cm} (3.11) \]

is non increasing at any time, provided that the total initial strength \( V(0) \) is sufficiently small. Moreover, for any given \( 0 \leq m < n \), the total amount of wave interaction and cancellation taking place in the time interval \( [m \varepsilon, n \varepsilon] \) is bounded by \( O(1) \cdot (\mathcal{T}_1(m \varepsilon) - \mathcal{T}_1(n \varepsilon)) \). Denote \( \Delta \mathcal{Y}_1^{m,n} = \mathcal{Y}_1(n \varepsilon) - \mathcal{Y}_1(m \varepsilon) \) the variation of \( \mathcal{Y}_1 \) on \( [m \varepsilon, n \varepsilon] \).
A basic ingredient of the strategy followed in [10] to establish a convergence rate of the Glimm scheme is the wave tracing algorithm introduced in [22] for GNL or LD systems, and then extended in [23] to NGNL systems, which consists in partitioning the outgoing waves issuing from every mesh point \((i\varepsilon, j\varepsilon)\) in two type of waves: primary waves (i.e. waves that can be traced back from the time \(t = i\varepsilon\) to a previous time \(t = m\varepsilon < i\varepsilon\)), and secondary waves (i.e. waves that are generated by interactions occurring in the time interval \([m\varepsilon, i\varepsilon]\), or that are canceled before a later time \(t = n\varepsilon > i\varepsilon\)). The total strength of secondary waves produced in a given time interval \([m\varepsilon, n\varepsilon]\) is bounded by the total amount of interaction and cancellation occurring within \([m\varepsilon, n\varepsilon]\).

The key step of this procedure is to show that the variation of a Glimm functional provides a bound for the change in strength and for the product of strength times the variation in speeds of the primary waves. The main novelty of the analysis performed here consists in implementing a wave tracing algorithm for a NGNL system satisfying the assumption (H1) in which such bounds are obtained relying on a Glimm functional with a quadratic potential interaction, differently from the Glimm functional with a cubic potential interaction used in [23]. Namely, recalling the Definition 2.3 of a wave partition, we have the following result.

**Proposition 3.5** Under the assumption (H1), given a Glimm approximate solution and any fixed \(0 \leq m < n\), there exists a partition of elementary wave sizes and speeds \(\{y^h_k(i, j), \lambda^h_k(i, j)\}\), \(k = 1, \ldots, N\), \(i = m, m + 1, \ldots, n\), \(j \in \mathbb{Z}\), so that the following hold.

1. For every \(i, j, k\), \(\{y^h_k(i, j)\}_{0 < h \leq \ell_k(i, j)}\) is a partition of the wave of the \(k\)-th family issuing from \((i\varepsilon, j\varepsilon)\), and \(\{\lambda^h_k(i, j)\}_{0 < h \leq \ell_k(i, j)}\) are the corresponding speeds, according with Definition 2.3.

2. For every \(i, j, k\), \(\{y^h_k(i, j), \lambda^h_k(i, j)\}_{0 < h \leq \ell_k(i, j)}\) is a disjoint union of the two sets

\[
\{\tilde{y}^h_k(i, j), \tilde{\lambda}^h_k(i, j)\}, \quad \{\tilde{\tilde{y}}^h_k(i, j), \tilde{\lambda}^h_k(i, j)\},
\]

with the following properties:

\[(a)\]

\[
\sum_{j, k, h} \|\tilde{y}^h_k(i, j)\| = O(1) \cdot |\Delta Y^{m,n}| \quad \forall \ m \leq i \leq n; \quad (3.12)
\]

\[(b)\] for every fixed \(i, k, h\), there is a one-to-one correspondence between

\[
\{\tilde{y}^h_k(m, j), \tilde{\lambda}^h_k(m, j) : j \in \mathbb{Z}\} \quad \text{and} \quad \{\tilde{\tilde{y}}^h_k(i, j), \tilde{\lambda}^h_k(i, j) : j \in \mathbb{Z}\}:
\]

\[
\{\tilde{y}^h_k(m, j), \tilde{\lambda}^h_k(m, j)\} \leftrightarrow \{\tilde{\tilde{y}}^h_k(i, \ell(i, j, k, h)), \tilde{\lambda}^h_k(i, \ell(i, j, k, h))\} \quad (3.13)
\]
such that the sizes $\tilde{s}_k^h$ and the speeds $\tilde{\lambda}_k^h$ of the corresponding waves satisfy

$$
\sum_{j,k,h} \left( \max_{m \leq i \leq n} |\tilde{s}_k^h(m,j) - \tilde{s}_k^h(i,\ell_{i,j,k,h})| \right) = O(1) \cdot |\Delta T_{1}^{m,n}|, \quad (3.14)
$$

$$
\sum_{j,k,h} \left( |\tilde{s}_k^h(m,j)| \cdot \max_{m \leq i \leq n} \left| \tilde{\lambda}_k^h(m,j) - \tilde{\lambda}_k^h(i,\ell_{i,j,k,h}) \right| \right) = O(1) \cdot |\Delta T_{1}^{m,n}|.
$$

(3.15)

**Proof.** The desired partition for an approximate solution $u^\varepsilon$ will be constructed proceeding by induction on the time steps $i \varepsilon$, $m \leq i \leq n$. Assuming that a partition of elementary waves fulfilling properties 1-2 is given for all times $m \varepsilon \leq t < i \varepsilon$, we wish to produce a partition of the outgoing waves generated by the interactions occurring at $t = i \varepsilon$, so to preserve the properties 1-2. Observe first that the existence of such a partition is already guaranteed by the analysis in 23 if all interactions take place between waves of different family or of the same family with opposite sign, since for systems satisfying the assumption (H1) the change in strength and the product of strength times the variation in speeds of the primary waves is controlled by the variation of a Glimm functional with quadratic interaction potential as the part in brackets of (3.1).

Therefore, it will be sufficient to consider an interaction between two waves issuing from two consecutive mesh points $((i-1)\varepsilon, (j-1)\varepsilon)$ and $((i-1)\varepsilon, j\varepsilon)$, say $s_k^i$, $s_k^0$, belonging to a $k$-th NGNL characteristic family, and having the same sign. For the sake of simplicity, assume that $s_k^i, s_k^0 > 0$. Let $s_p$ ($p = 1, \ldots, N$) be the outgoing wave of the $p$-th family issuing from $(i\varepsilon, j\varepsilon)$, and let

$$
\left\{ y_{k}^{h}, \lambda_{k}^{h} \right\}_{0 \leq h \leq \ell^*}, \quad \left\{ y_{k}^{n}, \lambda_{k}^{n} \right\}_{0 \leq n \leq \ell'^*}, \quad (3.16)
$$

be the partitions of $s_k^i$ and $s_k^0$ enjoying the properties 1-2 on $[m \varepsilon, (i-1)\varepsilon]$, with sizes

$$
\left\{ s_k^{h} \right\}_{0 \leq h \leq \ell^*}, \quad \left\{ s_k^{n} \right\}_{0 \leq n \leq \ell'^*}. \quad (3.17)
$$

For every $p \neq k$-th wave $s_p$, we may choose a partition $\{y_{p}^{h}\}_{0 \leq h \leq \ell_p}$ as in Definition 2.3 with corresponding speeds $\{\lambda_{p}^{h}\}_{0 \leq h \leq \ell_p}$. Then, if we label all the subwaves $y_p^h$ as secondary waves $\tilde{y}_p^h$, the bound (3.12) (for $i,j,p$) is certainly satisfied thanks to the interaction estimates (3.5). Instead, for the $k$-th wave $s_k$, possibly considering a refinement of the partition of $s_k^i$ (or of $s_k^0$) we may assume that either $s_k^i + s_k^0 \leq s_k$, or $s_k^0 \leq s_k$ (in the case $s_k^i \geq s_k$), and let

$$
\ell^* = \max\{h \leq \ell^* : \sum_{q=1}^{h} s_k^q \leq s_k\}, \quad \ell'^* = \max\{h \leq \ell'^* : s_k^i + \sum_{q=1}^{h} s_k^q \leq s_k\}.
$$

Then, we define a partition of $s_k$ by means of its sizes, setting

$$
s_k^h = \begin{cases} s_k^h & \text{if } h = 1, \ldots, \ell^*, \\ s_k^{n-h} - \ell' & \text{if } \ell^* = \ell' \text{ and } h = \ell^* + 1, \ldots, \ell' + \ell'^* 
\end{cases} \quad (3.18)
$$

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(possibly refining the partitions (3.19) so to satisfy property 2 of Definition 2.3, and choosing a partition of $s_k - (s'_k + s''_k)$ as in Definition 2.3 in the case $s_k > s'_k + s''_k$. The subwaves $s^h_k$ in (3.18) inherit the same classification in primary and secondary waves of the corresponding subwaves $s'^h_k$ or $s''_k$, while all the possible subwaves of $s_k - (s'_k + s''_k)$ are labelled as secondary waves. Clearly, the bound (3.12) is again satisfied because of the interaction estimates (3.6), while the one-to-one correspondence at (3.13) and the bound (3.14) are verified by construction and by the inductive assumption. Hence, in order to conclude the proof, it remains to establish only the estimate (3.15).

By the assumption (H1), and because the incoming waves $s'_k, s''_k$ have the same sign, at most one of them can possibly be a composed wave, say $s'_k$, while $s''_k$ will be a shock. Denote as $(s'_k)^r$, $(s'_k)^s$ the rarefaction and shock component of $s'_k$, respectively. For sake of simplicity, assume that $s_k > s'_k$, i.e. that $\mathcal{T}' = \ell'$. The outgoing wave $s_k$ is either a shock or a composed wave. In the first case its Rankine-Hugoniot speed $\lambda_k$ coincides with the speeds $\lambda'_k$ of all subwaves $s'^h_k$ defined according with Definition 2.3, since for a shock wave the integrand function in (2.15) results to be a constant (cfr. Remark 2.2). Hence, letting $(\lambda'_k)^r, \lambda'_k$ denote the speeds of the shock component of $s'_k$ and of $s''_k$, respectively, by a direct computation one finds

$$
|\lambda'_k - \lambda''_k| \leq |\lambda_k - (\lambda'_k)^r| = O(1) \cdot s''_k \quad \forall h = 1, \ldots, \ell',
$$

$$
|\lambda'_k - \lambda''_k| = |\lambda_k - \lambda''_k| = O(1) \cdot s'_k \quad \forall h = \ell' + 1, \ldots, \ell' + \ell''.
$$

(3.19)

In turn, (3.19) implies

$$
\sum_{h=1}^{\ell'} s'^h_k |\lambda'_k - \lambda''_k| + \sum_{h=\ell'+1}^{\ell'+\ell''} s''_k |\lambda'_k - \lambda''_k| = O(1) \cdot s'_k s''_k,
$$

(3.20)

which, relying on the inductive assumption, yields (3.15) since in this case, by the estimate (3.7), and because the rarefaction component of $s_k$ is zero, there holds $s'_k s''_k = O(1) \cdot I_1(s'_k, s''_k)$.

Next, assume that the outgoing wave $s_k$ is made of a rarefaction component $(s_k)^r < (s'_k)^r$ and of a shock component $(s_k)^s$. Then, possibly considering a refinement of the partition of $s_k$ (and hence of the partition of $s'_k$), there will be some index $\ell'' < \ell'$ so that $\sum_{h=1}^{\ell''} s_k = (s_k)^r$ and $\sum_{h=\ell''+1}^{\ell'+\ell''} s_k = (s_k)^s$. Notice that $(s_k)^s$ can be seen as a shock wave generated by an interaction between a composed wave with rarefaction component of size $(s'_k)^r - (s_k)^r$ and shock component $(s'_k)^s$, and of the shock wave $s''_k$, for which we can apply the above estimates on the variation of wave speeds. Hence, the wave speeds $\lambda'_k$ of $s'^h_k$ defined as in (2.15)
satisfy

\[
|\lambda^h_k - \lambda^{n_h}_k| = \begin{cases} 
O(1) & \forall \ h = 1, \ldots, \ell', \\
\sigma' & \forall \ h = h', \ldots, \ell', 
\end{cases}
\]

\[
|\lambda^h_k - \lambda^{n_h}_k| = O(1) \cdot \left( (s'_k)^r - (sk)^r + (s'_k)^s \right) \quad \forall \ h = \ell' + 1, \ldots, \ell' + \ell''. \tag{3.21}
\]

This implies

\[
\sum_{h=1}^{\ell'} s'_k |\lambda^h_k - \lambda^{n_h}_k| + \sum_{h=\ell'+1} s'_k |\lambda^h_k - \lambda^{n_h}_k| = O(1) \cdot \left( (s'_k)^r - (sk)^r + (s'_k)^s \right) s''_k = O(1) \cdot I_1(s'_k, s''_k). \tag{3.22}
\]

which in turn, relying on the inductive assumption, yields again (3.16) since in this case, by the estimate (3.7) and because \(s''_k = (s''_k)^s\), one has \(\left[ (s'_k)^r - (sk)^r + (s'_k)^s \right] s''_k = O(1) \cdot I_1(s'_k, s''_k)\). This completes the proof of the proposition. \(\square\)

Proposition 3.5 provides for NGNL systems satisfying the assumption (H1) the same type of result that was established in [10, Proposition 2] for systems with GNL or LD characteristic families. In order to obtain the desired convergence rate (1.13) one can now simply repeat the proofs of [10, Propositions 3-4] and of the final estimates in [10, § 6], which all rely only on the conclusion of [10, Proposition 2] and thus remain valid within our more general framework of NGNL systems. We will give a brief description of them in Section 6.

**Remark 3.6** The conclusion of Theorem 1.2 established so far for smooth systems satisfying the assumption (H1), remains valid if we assume that the flux function \(F\) is \(C^{2,1}\) and that, for each \(k\)-th characteristic family not fulfilling (H1), the linearly degenerate manifold \(M_k\) in (1.3) is a \(C^{1,1}\) \(N\)-dimensional, connected manifold, \(F\) is \(C^3\) on \(\Omega \setminus M_k\), the vector field \(r_k\) is transversal to the boundary of \(M_k\), and letting \(\partial^+ M_k, \partial^- M_k\) denote the connected components of the boundary of \(M_k\), where \(r_k\) points towards \(\Omega \setminus M_k\) and \(M_k\), respectively, there holds

\[
\nabla^+ (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) < 0 \quad \forall u \in \partial^+ M_k, \tag{3.23}
\]

\[
\nabla^- (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) < 0 \quad \forall u \in \partial^- M_k
\]

\((\nabla^\pm (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) = \lim_{h \to 0^\pm} \frac{\nabla \lambda_k \cdot r_k(u + hr_k(u)) - \nabla \lambda_k \cdot r_k(u)}{h}\) denoting the one-side second derivatives of \(\lambda_k\). Indeed, the only difference in the structure of the elementary waves of a NGNL \(k\)-th family satisfying such assumptions instead of (H1) comes from the possible presence of two-sided contact discontinuities. In fact, under the above assumptions, the general solution of a Riemann problem of the \(k\)-th family will be either a rarefaction wave, or a shock wave (which can be either a compressive shock or a contact discontinuity), or a composed wave made of a rarefaction wave adjacent to one (one-sided or two-sided) contact discontinuity or
several (two-sided) contact discontinuities. Then, we may consider the interaction potential $Q_{1}$ in (3.1), where the shock component of a composed wave $s_\alpha$ containing several contact discontinuities $s_{\alpha,1}, \ldots, s_{\alpha,l}$ is $s_\alpha^s \doteq \sum_{p=1}^{l} s_{\alpha,p}^s$, and for every such wave we add the term $2 \sum_{p \neq q} |s_{\alpha,p}^s s_{\alpha,q}^s|$. One can easily verify that employing this definition of $Q_{1}$ and the same definition of quantity of interaction $I_{1}$ in (3.4), the estimates stated in Lemma 3.3 continue to hold, provided that

$$\inf \left\{ s > 0 : R_k[u](s) \in \partial^+ M_k, \, u \in \partial^- M_k \right\} > 0 \quad (3.24)$$

($R_k[u](s)$ denoting the integral curve of $r_k$), which is certainly true up to a possible slight restriction of the domain $\Omega$. Relying on Lemma 3.3, one then deduces Lemma 3.4 and thus can establish the key Proposition 3.5 with the same arguments as above.

4 A new interaction potential

We turn now our attention to an approximate solution $u^\varepsilon$ constructed by the Glimm scheme for an hyperbolic system (1.1) (or for the quasilinear system (1.14)) that satisfy the assumption (H) stated in the Introduction. We recall that for such systems the general solution of a Riemann problem contains composed waves made of several contact discontinuities adjacent to rarefaction waves (instead of just a single contact discontinuity adjacent to a rarefaction wave as for the systems treated in § 3). We will say that a wave $s$ of this type, belonging to the $k$-th characteristic family, crosses all connected components of $M_k$ that are transversal to the $k$-th elementary curve $T_k$ issuing from the left state of $s$ and terminating on the right state of $s$. Notice that, for each $k$-th NGNL family, and for every connected component $M_{k,h}$ of $M_k$, the first derivative $\nabla \lambda_k \cdot r_k$ has opposite signs on the connected components of $\Omega \setminus M_{k,h}$ adjacent to $M_{k,h}$, and as a consequence the second derivative $\nabla (\nabla \lambda_k \cdot r_k) \cdot r_k$ has opposite signs on any pair of consecutive components $M_{k,h}, M_{k,h+1}$. Thus, by continuity we may assume that there exists some constant $\delta_0 > 0$ so that

$$\nabla (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) \neq 0 \quad \forall \ u \ \text{s.t.} \ d(u, M_k) \leq 6\delta_0, \quad (4.1)$$

where $d(u, M_k) \doteq \inf_{w \in M_k} |u - w|$ denotes the distance of a state $u$ from $M_k$.

Remark 4.1 Condition (4.1) implies that every wave $s$ of a $k$-th NGNL family with strength $|s| \leq 3\delta_0$ crosses at most one connected component of $M_k$. Moreover, if an interaction takes place between two waves of the $k$-th characteristic family with strength $\leq \delta_0$, then, by the interaction estimates in [6, Theorem 3.7], the outgoing wave of the $k$-th family crosses as well at most one connected component of $M_k$.

By Remark 4.1 as far as the waves of the NGNL families involved in an interaction have all strength smaller than $\delta_0$, we can establish the same kind of estimates
of Proposition 3.5 employing the quadratic interaction potential in (3.1) even for systems satisfying the more general assumption (H). On the other hand, observe that if we consider an interaction between two shock waves of a $k$-th NGNL family, say $s', s''$, with speeds $\lambda', \lambda''$, respectively, and we assume that $s', s''$ have the same sign, then, letting $\lambda$ denote the shock speed of the outgoing wave of the $k$-th family, by the interaction estimates in [6, Theorem 3.7] there holds

$$[s\Delta \lambda] \doteq |s'| |\lambda - \lambda'| + |s''| |\lambda - \lambda''| = O(1) \cdot \frac{|s's''| |\lambda' - \lambda''|}{|s' + s''|}. \tag{4.2}$$

Notice that $|s's''| |\lambda' - \lambda''|$ has precisely the same order of the quantity of which it decreases the interaction potential $Q$ introduced in [6] whenever interactions of this type take place. Therefore, if we assume that at least one of the incoming waves of the $k$-th family has strength $\geq \delta_0$, we deduce from (4.2) that $[s\Delta \lambda] = O(1) \cdot |\Delta Q|/\delta_0$. Hence, for such interactions one may derive the same kind of estimates on the products of the wave strengths times the variation of the wave speeds of Proposition 3.5 employing the cubic interaction potential $Q$ defined in [6].

In view of the above observations, we shall introduce now a functional $Q$ that is the sum of a quadratic and of a cubic interaction potential. The latter is the interaction potential for waves of the same family and with the same sign defined in [6], valid for general strictly hyperbolic systems (1.14), which takes the form

$$Q(t) = \sum_{k_a = k_{\beta}, s_a, s_{\beta} > 0} \left[ \int_0^{s_a} \int_0^{s_{\beta}} \sigma_\alpha(\tau) - \sigma_\beta(\tau') \ d\tau' d\tau \right]. \tag{4.3}$$

The summation here extends to all pair of waves $s_\alpha, s_\beta$ of the $k_a \in \{1, \ldots, N\}$ family with the same sign (including $s_\alpha = s_\beta$), of the approximate solution $u^e(t)$, and $\sigma_\alpha = \sigma_{k_a}[\omega_\alpha](s_\alpha, \cdot)$ denotes the map in (2.10), where $\omega_\alpha$ is the left state of $s_\alpha$. Such a functional controls the nonlinear coupling of waves of the same family with the same sign.

The quadratic part $Q_q$ of the functional $Q$ enjoys two basic properties:

1. it decreases whenever it takes place an interactions between “small” waves of the same family, i.e. waves whose strength is smaller than $\delta_0$, and the amount of decreasing satisfies the same type of estimate (3.10) obtained for systems with a single linearly degenerate manifold;

2. the possible increase of $Q_q$ caused by interactions involving “large” waves of the same family, i.e. waves of strength larger than $\delta_0$, is controlled by the decrease of $Q$.

Thus, for general hyperbolic systems (1.14) satisfying the assumption (H), we shall consider a potential interaction of the form

$$Q(t) \doteq Q_q(t) + c Q(t), \tag{4.4}$$
where \( c > 2 \) is a suitable constant to be specified later.

Towards the definition of \( Q_q \), let us first introduce some further notations. Given a composed wave \( s \) of a \( k \)-th NGNL family, let \( \{ s^h \}_{h=1}^l \) be its decomposition in rarefaction and shock components, and write \( h \in \mathcal{R} \) (respectively \( h \in \mathcal{S} \)) if \( s^h \) is a rarefaction (respectively a shock) wave. Thus, letting \( w^{h-1}, w^h \) denote the left and right states of each wave \( s^h \), one has \( w^h = T_k[w^{h-1}](s^h) \). Next, for every given shock \( s^h, h \in \mathcal{S} \), we define a convex-concave sub decomposition \( \{ s^{h,p} \}_{p=1}^{q_h} \), as follows. Assuming for the sake of simplicity that \( s^h > 0 \), let \( 0 = \tau_0 < \tau_1 < \cdots < \tau^{q_h} = s^h \) be a partition of \( s^h \) determined by the inflection points of the reduced flux \( \tau \mapsto \tilde{F}^h_k(\tau) \doteq F_k[w^{h-1}](s^h, \tau) \) in \((2,10)\), and set \( s^{h,p} \doteq \tau^p - \tau^{p-1} \).

We will write \( p \in \prec \) (respectively \( p \in \succ \)) if \( \tilde{F}^h_k \) is convex (respectively concave) on \([\tau^{p-1}, \tau^p]\), and we will call \( s^{h,p} \) a convex (respectively concave) component of \( s^h \) if \( p \in \prec \) (respectively \( p \in \succ \)). Then, considering the affine map

\[
\varphi(s) = \begin{cases} 
1 & \text{if } |s| \geq 2\delta_0, \\
(|s| - \delta_0)/\delta_0 & \text{if } \delta_0 \leq |s| < 2\delta_0, \\
0 & \text{if } |s| \leq \delta_0, 
\end{cases}
\]

(4.5)

we define the intrinsic interaction potential of \( s^h, h \in \mathcal{S} \), as

\[
q(s^h) = \varphi(s^h) \cdot \left[ 2 \sum_{p \neq q} |s^{h,p}s^{h,q}| + \sum_{p \in \prec} |s^{h,p}|^2 \right],
\]

(4.6)

where the first summand runs over all indexes \( p, q \in \mathcal{R} \), \( p \neq q \), and \( q(s^h) \) is understood to be zero if \( s^h \) has zero convex component. Notice that, by definition \((4.3)\), for shocks \( s^h \) with non zero convex component, \( q(s^h) \) can possibly be zero only when \( h = 1 \) or \( h = l \), i.e. when \( s^h \) is the first or the last component of \( s \). In fact, all other shock components of \( s \) are two-sided contact discontinuities which necessarily must cross at least two connected components of \( M_k \), and hence their strengths are certainly larger than \( 2\delta_0 \) because of \((4.1)\).

Now, defining the inner interaction potential of a composed wave \( s \) as

\[
Q^I(s) = 2 \sum_{h \in \mathcal{S}, \kappa \in \mathcal{R}} |s^h s^\kappa| + \sum_{h \in \mathcal{S}} q(s^h) + \sum_{\kappa \in \mathcal{R}} |s^\kappa|^2,
\]

(4.7)

we can finally provide the definition of the quadratic interaction potential enchanting properties 1-2 by setting

\[
Q_q(t) \doteq 2 \sum_{k_\alpha = k_\beta} \left| s_{\alpha, \beta} \right| + \sum_{\alpha} Q^I(s_\alpha) + c \left[ \sum_{k_\alpha = k_\beta} + \sum_{k_\alpha < k_\beta} \sum_{x_\alpha(t) > x_\beta(t)} \left| s_{\alpha, \beta} \right| \right],
\]

(4.8)

where, as usual, \( x_\alpha(t) \) denotes the position of the wave \( s_\alpha \), and \( k_\alpha \) its characteristic family while \( c \) is the same constant that appears in \((4.4)\). Here, the second
summation runs over all composed waves \(s_\alpha\) present in \(w'(t)\). Notice that \(Q_\alpha\) differs from the interaction potential \(Q_1\) defined in §3 only for the presence of the inner interaction potential \(Q^I\) of the composed waves that replaces the corresponding terms of the second and third summands in (4.1). On the other hand, whenever \(|s_\alpha| \leq \delta_0\), we clearly have \(Q^I(s_\alpha) = |s_\alpha^h s_\alpha^r| + |s_\alpha^q|^2\), \(h \in S, \kappa \in R\), and thus one recovers the same expression present in \(Q_1\).

Remark 4.2 Consider a shock wave \(s\) with strength \(|s| \leq \delta_0\) that crosses a connected component \(\mathbb{M}_{k,h}\) of \(\mathbb{M}_k\). According with the above definitions \(s\) is decomposed in a convex and a concave component \(s^-\), \(s^-\). Relying on [1, Propositions 2.1-2.2], we deduce that, choosing \(\delta_0\) sufficiently small, there holds

\[
|s^-| \leq c_1 |s^-|,
\]

for some constant \(0 < c_1 < 1\). Such a bound will be useful in the study of the variation of the intrinsic interaction potential \(q(s)\) in presence of interactions. Notice that the above estimate holds even in the case, instead of \(s\), we assume that there is some even index \(p\) so that the following weaker condition is satisfied:

\[
D^j_{r_k} \lambda_k(u) = 0 \quad \forall \; j < p, \quad D^p_{r_k} \lambda_k(u) \neq 0 \quad \forall \; u \in \mathbb{M}_{k,h},
\]

where \(D^j_{r_k} \lambda_k(u)\) denotes the \(j\)-th derivative of \(\lambda_k\) along \(r_k\), inductively defined by setting \(D^1_{r_k} \lambda_k(u) = \nabla \lambda_k(u) \cdot r_k(u)\), and \(D^j_{r_k} \lambda_k(u) = D^{j-1}_{r_k} \lambda_k(u) \cdot r_k(u)\) for all \(j > 1\).

Towards an analysis of the interaction potential above introduced, we first define a quadratic quantity of interaction as in Section §3 for waves of the same family and with the same sign, to measure the decrease of the quadratic functional \(Q_q\) in (1.8) when waves of this type with strength \(\leq \delta_0\) are involved in an interaction.

Definition 4.3 Consider two nearby waves of sizes \(s', s''\) with the same sign and belonging to the the same \(k\)-th characteristic family, with \(s'\) located at the left of \(s''\). Assume that \(|s'|, |s''| \leq \delta_0\) and, with the same notations of Definition §3, suppose that the state \(u^p\) belongs to the connected component of \(\Omega \setminus \mathbb{M}_k\) lying between two consecutive manifolds \(\mathbb{M}_{k,h-1}, \mathbb{M}_{k,h}\) of \(\mathbb{M}_k\). Then, in the case \(s', s'' > 0\), we define the quantity of interaction between \(s'\) and \(s''\) as

\[
I(s', s'') = \begin{cases} 
|\{(s' + s'')^r - s'^r\} + |s''s'| |s''s'| \ 
& \text{if} \quad \nabla (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u)_{u \in \mathbb{M}_{k,h}} < 0, \\
|\{(s' + s'')^r - s''r\} + |s'rs'\|s''s'| \ 
& \text{if} \quad \nabla (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u)_{u \in \mathbb{M}_{k,h}} > 0.
\end{cases}
\]

An entirely similar definition is given in the case \(s', s'' < 0\). For notational convenience we also set \(I(s', s'') = 0\) for every pair of waves \(s', s''\) of the same family that have opposite sign.

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Next, following [6, Definition 3.5], we introduce a definition of quantity of interaction for a general strictly hyperbolic system (1.14), which measures the decrease of the cubic functional $Q$ in (4.3) when waves of the same family and with the same sign interact together.

**Definition 4.4** Consider two nearby waves of sizes $s', s''$ with the same sign and belonging to the same $k$-th characteristic family, with left states $u', u''$, respectively. Let $\tilde{F}' = \tilde{F}_k[u'](s', \cdot)$ and $\tilde{F}'' = \tilde{F}_k[u''](s'', \cdot)$ be the reduced flux with starting point $u', u''$, evaluated along the solution of (2.6) on the interval $[0, s']$, and $[0, s'']$, respectively (cfr. def. (2.10)). Then, assuming that $s' \geq 0$, we say that the amount of interaction $J(s', s'')$ between $s'$ and $s''$ is the quantity

$$
J(s', s'') \doteq \int_{0}^{s'} \left[ \text{conv}_{[0, s']} \tilde{F}'(\xi) - \text{conv}_{[0, s' + s'']} \tilde{F}' \cup \tilde{F}''(\xi) \right] d\xi 
+ \int_{s'}^{s' + s''} \left| \tilde{F}'(s') + \text{conv}_{[0, s'']} \tilde{F}''(\xi - s') - \text{conv}_{[0, s' + s'']} \tilde{F}' \cup \tilde{F}''(\xi) \right| d\xi,
$$

(4.12)

where $\tilde{F}' \cup \tilde{F}''$ is the function defined on $[0, s' + s'']$ as

$$
\tilde{F}' \cup \tilde{F}''(s) = \begin{cases} 
\tilde{F}'(s) & \text{if } s \in [0, s'], \\
\tilde{F}''(s') & \text{if } s \in [s', s' + s''].
\end{cases}
$$

(4.13)

Here, $\text{conv}_{[a,b]}f$, $\text{conc}_{[a,b]}f$ denote the lower convex envelope and the upper concave envelope of $f$ on $[a,b]$, defined as in (2.13). In the case where $s' < 0$, one replaces in (4.12) the lower convex envelope with the upper concave one, and vice-versa. As in Definition 4.3, for notational convenience we also set $J(s', s'') \doteq 0$ for every pair of waves $s', s''$ of the same family that have opposite sign.

**Remark 4.5** Notice that by the Lipschitz continuity of the derivative $(u, s) \mapsto D_s\tilde{F}_k[u](s, \cdot)$ of the reduced flux (2.10) (cfr. [23]), it follows $J(s', s'') = O(1) \cdot |s's''|$. Moreover, by Remark 2.2, one can easily verify that, in the conservative case, if $s', s''$ are both shocks of the $k$-th family that have the same sign, then the amount of interaction in (4.12) takes the form

$$
J(s', s'') = \left| s's'' \right| \left| \sigma_k[u^k, u^M] - \sigma_k[u^M, u^R] \right|,
$$

i.e. it is precisely the product of the strength of the waves times the difference of their Rankine Hugoniot speeds.

Relying on the results in [6] Section 3 and on Lemma 3.3, we will show now that the interaction potential $Q$ defined by (4.3), (4.4), (4.5), is decreasing at every interaction, and that the variation of the total strength of waves $V$ in an approximate solution $u^\varepsilon$ is controlled by $|\Delta Q|$. 

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Lemma 4.6 Under the assumption (H), in the same setting of Lemmas 3.3-3.4, there exists some constant $c > 0$ (in (4.13), (4.8)), so that there holds

$$\Delta V \leq - \sum_{1 \leq i \leq N} C(s'_i, s''_i) + O(1) \cdot \left[ \sum_{1 \leq i, j \leq N} |s'_i s''_j| + \sum_{1 \leq i \leq N} \mathcal{J}(s'_i, s''_i) \right], \quad (4.14)$$

$$\Delta Q \leq - \frac{1}{2} \left[ \sum_{1 \leq i, j \leq N} |s'_i s''_j| + \sum_{1 \leq i \leq N} |s'_i s''_i| + \sum_{1 \leq i \leq N} I(s'_i, s''_i) + \sum_{1 \leq i \leq N} \mathcal{J}(s'_i, s''_i) \right] + O(1) \cdot V^- \cdot \sum_{1 \leq i \leq N} C(s'_i, s''_i). \quad (4.15)$$

Proof. A proof of the estimate (4.14) can be found in [6], thus we will focus our attention on (4.15). For the sake of simplicity, we shall consider only the case in which the two adjacent Riemann problems are solved by a single wave, say $s'$ and $s''$, $s'$ on the left of $s''$. We distinguish three cases, depending on the strengths of $s'$ and $s''$ and on their characteristic families.

1. $s'$ and $s''$ are waves of the $k'$ and $k'' < k'$ characteristic families.

To fix the ideas, let $s', s'' > 0$. Observe that for every NGNL $k$-family, since condition (1.2) implies that the characteristic vector field $r_k$ is transversal to $\mathcal{M}_k$, by construction it follows that the $k$-elementary curves $T_k$ are transversal to each manifold $\mathcal{M}_{k, h}$. As a consequence of this property one can easily verify that, letting $u', u''$ be the left states of $s', s''$, and denoting $u'^+, u''^+$ the left states of the outgoing waves $s_{k'}$ and $s_{k''}$ of the $k'$ and $k''$ characteristic families, there holds

$$\Sigma' = \left[ \sum_{h \in S} \sum_{p \in \omega} s'^{h,p} - \sum_{h \in S} \sum_{p \in \omega} s'^{h,p} \right] + \left[ \sum_{\kappa \in \mathcal{R}} s'^{\kappa} - \sum_{\kappa \in \mathcal{R}} s'^{\kappa} \right] = O(1) \cdot \left\| T_{k'} [u'] - T_{k'} [u'^+] \right\|_{L^\infty} \quad (4.16)$$

$$\Sigma'' = \left[ \sum_{h \in S} \sum_{p \in \omega} s''^{h,p} - \sum_{h \in S} \sum_{p \in \omega} s''^{h,p} \right] + \left[ \sum_{\kappa \in \mathcal{R}} s''^{\kappa} - \sum_{\kappa \in \mathcal{R}} s''^{\kappa} \right] = O(1) \cdot \left\| T_{k''} [u''] - T_{k''} [u''^+] \right\|_{L^\infty}.$$

Here the $L^\infty$ norm in the first and second equality is referred to the intervals $[0, \min \{s', s_{k'}\}]$ and $[0, \min \{s'', s_{k''}\}]$, respectively. Then, since the
interaction estimates in [6, Section 3] imply
\[
\left\| T_k'[u'] - T_k'[u]\right\|_{L^\infty} = \mathcal{O}(1) \cdot s'',
\]
\[
\left\| T_k''[u'''] - T_k''[u''']\right\|_{L^\infty} = \mathcal{O}(1) \cdot s',
\]  
(4.17)

relying on (4.16)-(4.17), we deduce the following bounds on the variation of the inner interaction potential at (4.7)
\[
\Delta Q^I(s') = \mathcal{O}(1) \cdot \Sigma' \cdot s = \mathcal{O}(1) \cdot s's'',
\]
\[
\Delta Q^I(s'') = \mathcal{O}(1) \cdot \Sigma'' \cdot s'' = \mathcal{O}(1) \cdot s's''.
\]  
(4.18)

Hence, using (4.14), (4.18), one obtains
\[
\Delta Q \leq -\frac{1}{2} I(s', s'') + \mathcal{O}(1) \cdot J(s', s''),
\]  
(4.19)

from which we derive (4.15), choosing \( c > 0 \) sufficiently large in (4.8).

2. \( s' \) and \( s'' \) are both \( k \)-waves and \( s's'' < 0 \).

By definition of \( Q^I \), and with the same analysis in the previous point, one deduces that in this case the inner interaction potential of the outgoing \( k \)-wave \( s \) satisfies \( Q^I(s) \leq \min\{Q^I(s'), Q^I(s'')\} + \mathcal{O}(1) \cdot |s's''| \). Hence \( \Delta Q^I = \mathcal{O}(1) \cdot |s's''| \), and thus we obtain the same estimate in (4.19). On the other hand, relying on [6, Proposition 4.1] we derive
\[
\Delta Q \leq \mathcal{O}(1) \cdot V^- \cdot C(s', s''),
\]  
(4.20)

which, together with (4.19), yields (4.15), choosing \( c > 0 \) sufficiently large in (4.8).

3. \( s' \) and \( s'' \) are both \( k \)-waves and \( s's'' > 0 \).

To fix the ideas, let \( s', s'' > 0 \), and call \( s \) the outgoing \( k \)-wave. We shall distinguish a number of cases, depending on the strengths of \( s', s'' \).

(a) \( \max\{s', s''\} \leq \delta_0/2 \).

In this case, by definitions (4.5)-(4.8) one has \( q(s') = q(s'') = q(s' + s'') = 0 \), and \( \Delta Q_q \leq \Delta q + \Delta Q_1 + \mathcal{O}(1) \cdot |\Delta V| \), where \( \Delta Q_1 \) denotes the variation of the interaction potential \( Q_1 \) in (3.11) (related to the waves involved in the interaction). Hence, relying on (4.14), and applying (3.10), we deduce \( \Delta q = \mathcal{O}(1) \cdot |\Delta V| = \mathcal{O}(1) \cdot J(s', s'') \) and
\[
\Delta Q_q \leq -\frac{1}{2} I(s', s'') + \mathcal{O}(1) \cdot J(s', s'').
\]  
(4.21)
On the other hand, due to [3, Proposition 4.1], we get
\[ \Delta Q \leq -\frac{1}{2} J(s', s''), \]  
(4.22)
which yields (4.15) choosing \( c > 0 \) sufficiently large in (4.4).

(b) \( \delta_0/2 < \max\{s', s''\} \leq 2\delta_0 \).

To fix the ideas, assume that \( s' \) crosses a connected component \( \mathcal{M}_{k,h} \) of \( \mathcal{M}_k \) where \( \nabla(\nabla\lambda_k \cdot r_k) \cdot r_k < 0 \). Because of (4.1), this implies that the wave \( s'' \) on the right of \( s' \) must be a shock with zero convex component and hence \( q(s'') = 0 \). For sake of simplicity, we shall treat only the case in which also \( s' \) is a shock and \( s' \leq \delta_0 \leq s/2 \), the other cases being similar or simpler since for such values of \( s' \), \( s \) there is the largest possible increase of \( q \) due to the fact that, by definitions (4.5), (4.6), one has \( q(s) = \varphi(s) = 0 \). Under these assumptions, by definitions (4.6), (4.7) we have
\[ Q^I(s') = q(s) = 2s^\sim s^\sim + (s^\sim)^2 \]
(4.23)
\[ \leq 2s^\sim - (s' - s^\sim) + O(1) \cdot J(s', s''). \]
Moreover, observe that \( s' \leq \delta_0 \leq s/2 \) implies \( s' \leq s'' + O(1) \cdot J(s', s'') \). Therefore, using (4.23), and recalling that by Remark 4.2 one has \( s^\sim < s' \), we find
\[ \Delta Q_q \leq -2s' s'' + Q^I(s) + O(1) \cdot |\Delta V| \]
\[ \leq -2s''(s' - s^\sim) + O(1) \cdot J(s', s'') \]
(4.24)
\[ \leq O(1) \cdot J(s', s''). \]

Hence, (4.24) together with (4.22), that continues to hold, yields (4.15) choosing \( c > 0 \) sufficiently large in (4.4).

(c) \( \min\{s', s''\} \leq 2\delta_0 < \max\{s', s''\} \).

To fix the ideas assume that \( s' \) is a composed wave of size \( s' \leq 2\delta_0 \), crossing a connected component of \( \mathcal{M}_k \) where \( \nabla(\nabla\lambda_k \cdot r_k) \cdot r_k < 0 \). Because of (4.1), this implies that the first component \( s'^{\sim 1} \) of the (possible composed) wave \( s'' \) on the right of \( s' \) must be a shock of size \( s'^{\sim 1} > 2\delta_0 \). For sake of simplicity we shall treat only the case in which also \( s' \) is a shock, the other cases being similar. Observe that, letting \( u', u'' \) be the left states of \( s', s'' \), calling \( u'^+ \) the left state of the outgoing \( k \)-wave \( s \), and letting \( s'^\sim, s^\sim \) denote the convex and concave components of \( s' \),

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by the same arguments at point 1 we find

\[ \Sigma = \left[ s^\sim + \sum_{h \in S} \sum_{p \in S} s^{nh:p} - \sum_{h \in S} \sum_{p \in S} s^{h:p} \right] + \left[ \sum_{\kappa \in R} s^{\kappa} - \sum_{\kappa \in R} s^{\kappa} \right] \]

\[ = \mathcal{O}(1) \cdot \left[ \| T_k[u'] - T_k[u'^+] \|_{L^\infty} + \| T_k[u'' - T_k[u'^+] (s'^+ \cdot \cdot T_k[u'^+]] \|_{L^\infty} \right], \]

where the \( L^\infty \) norm of the two terms in the equality is referred to the intervals \([0, s']\) and \([0, \min\{s'', s - s'\}]\), respectively. Then, applying the interaction estimates in [6, Section 3], we derive

\[ \Sigma = \mathcal{O}(1) \cdot J(s', s''). \]

On the other hand, observe that by definition (4.5) the above assumptions imply \( \varphi(s'^{n'}) = 1, \varphi(s^1) = \mathcal{O}(1) \cdot J(s', s'') \), since the first component \( s^1 \) of \( s \) satisfies the lower bound \( s^1 \geq s'^{n'} + \mathcal{O}(1) \cdot J(s', s'') > 2\delta_0 + \mathcal{O}(1) \cdot J(s', s'') \). Thus, relying on (4.14), (4.20), and because \( s' < s'^{n'} \), we obtain

\[ \Delta Q^I \leq 2s^\sim (s'^\sim + s'^{n'}) + (s^\sim)^2 + \mathcal{O}(1) \cdot (\Sigma + |\Delta V|) \]

\[ \leq -2s'^{n'} (s'^\sim - s'^\sim) + 2s' s'^{n'} + \mathcal{O}(1) \cdot J(s', s''), \]

which in turn, recalling that by Remark 4.2 one has \( s'^{n'} < s'^\sim \), yields

\[ \Delta Q_q \leq -2s' s'^{n'} + \Delta Q^I + \mathcal{O}(1) \cdot |\Delta V| \]

\[ \leq \mathcal{O}(1) \cdot \left( J(s', s'') + |\Delta V| \right) \]

\[ = \mathcal{O}(1) \cdot J(s', s''). \]

Hence, (4.28) together with (4.22), that continues to hold, yields (4.15) choosing \( c > 0 \) sufficiently large in (4.4).

(d) \( \min\{s', s''\} > 2\delta_0 \).

We shall treat only the case in which the last component \( s'^{n'} \) of \( s' \) and the first component \( s'^{n'} \) of \( s'' \) are both shocks of size \( > 2\delta_0 \). The other cases are simpler or reducible to one of the previous cases (a), (b), (c). Then, by definition (4.5) there holds \( \varphi(s'^{n'}) = \varphi(s'^{n'}) = 1 \). Moreover, with the same notations and with the same arguments of point (c), we
have
\[
\Sigma = \left[ \sum_{h \in \mathcal{S}} \sum_{p \in \mathcal{P}} s^{th,p} + \sum_{h \in \mathcal{S}} \sum_{p \in \mathcal{P}} s^{th,p} - \sum_{h \in \mathcal{S}} \sum_{p \in \mathcal{P}} s^{bh,p} \right] + \\
\left[ \sum_{\kappa \in \mathcal{R}} s^{\kappa} + \sum_{\kappa \in \mathcal{R}} s^{\kappa} - \sum_{\kappa \in \mathcal{R}} s^{\kappa} \right]
\]
\[
= \mathcal{O}(1) \cdot \left[ \left\| T_k [u'] - T_k [u'^+] \right\|_{L^\infty} + \left\| T_k [u'''] - T_k [u'^+] (s' + \cdot) \right\|_{L^\infty} \right],
\]
\[
= \mathcal{O}(1) \cdot J(s', s'').
\]

Thus, relying on (4.14), (4.29), we derive
\[
\Delta Q_\eta \leq \Delta Q^f + \mathcal{O}(1) \cdot |\Delta V|
\]
\[
= \mathcal{O}(1) \cdot (\Sigma + |\Delta V|)
\]
\[
= \mathcal{O}(1) \cdot J(s', s''),
\]
which, together with (4.22), that continues to hold, yields (4.15) choosing \( c > 0 \) sufficiently large in (3.4).

\[\square\]

Relying on the above result one can prove that there exists \( C > 0 \) so that, assuming \( V(0) \) sufficiently small, the Glimm functional
\[
t \mapsto \Upsilon(t) = V(t) + C Q(t)
\]
is non increasing at any time, and at every discrete time \( t = i \varepsilon \) there holds
\[
\Delta \Upsilon(i \varepsilon) \leq -\frac{1}{2} \left( \text{[amount of cancellation at } t = n \varepsilon] + |\Delta Q(i \varepsilon)| \right).
\]
Hence, for any given \( 0 \leq m < n \), the total amount of wave interaction and cancellation taking place in the time interval \([m \varepsilon, n \varepsilon]\) is bounded by \( \mathcal{O}(1) \cdot |\Upsilon^{m,n}| \), where
\[
\Delta \Upsilon^{m,n} = \Upsilon(n \varepsilon) - \Upsilon(m \varepsilon)
\]
denotes the variation of \( \Upsilon \) on \([m \varepsilon, n \varepsilon]\).

5 Wave tracing for general non genuinely nonlinear systems

We will show now how to implement a wave tracing algorithm for a NGNL system satisfying the assumption (H) so that the change in strength and the product of strength times the variation in speeds of the primary waves be bounded by the variation of the Glimm functional in (4.31). Namely, recalling the Definition 2.3 of a wave partition, we have the following result analogous to Proposition 3.5.
Proposition 5.1 Under the assumption (H), the same conclusions of Proposition 3.5 hold, with \( \Delta \Upsilon_{m,n} \) in place of \( \Delta \Upsilon_1^{m,n} \).

Proof. As in the proof of Proposition 3.5 in order to produce a partition for an approximate solution \( u^\varepsilon \) that fulfills properties 1-2, one may proceed by induction on the time steps \( i\varepsilon, m \leq i < n \). Then, assuming that such a partition is given for all times \( m\varepsilon \leq t > i\varepsilon \), our goal is to show how to define a partition of the outgoing waves generated by the interactions that take place at \( t = i\varepsilon \), preserving the properties 1-2. As observed in the proof of Proposition 3.5 it will be sufficient to focus our attention on interactions between waves of the same family and with the same sign, since whenever any other interaction occurs for a system satisfying the assumption (H), the change in strength and the product of strength times the variation in speeds is controlled by the variation of a Glimm functional with a quadratic interaction potential as the part in brackets of (4.8) (cfr. [23, Lemma 3.2 and Theorem 5.1]).

Thus, consider an interaction between two waves, say \( s'_k, s''_k \), issuing from two consecutive mesh points \((i-1)\varepsilon, (j-1)\varepsilon\) and \((i-1)\varepsilon, j\varepsilon\), belonging to a \( k \)-th NNGNL characteristic family, and having the same sign. Observe that, if \(|s'_k|, |s''_k| \leq \delta_0/2\), then relying on the estimates (4.14), (4.15) provided by Lemma 4.6, one obtains the desired partition proceeding precisely as in the proof of Proposition 3.5. Hence, we shall treat only the case where \( \max\{|s'_k|, |s''_k|\} > \delta_0/2 \). For sake of simplicity, we assume that \( s'_k, s''_k > 0 \) and that the outgoing \( k \)-wave \( s_k \) is a shock, the other cases being entirely similar. Let

\[
\{y^{n_h}_h, \lambda^{n_h}_h\}_{0 < h \leq \ell'}, \quad \{y^{n_h}_h, \lambda^{n_h}_h\}_{0 < h \leq \ell''}
\]

be the partitions of \( s'_k \) and \( s''_k \) enjoying the properties 1-2 (on the interval \([m\varepsilon, (i-1)\varepsilon])\), with sizes

\[
\{s^{n_h}_h\}_{0 < h \leq \ell'}, \quad \{s^{n_h}_h\}_{0 < h \leq \ell''}.
\]

Then, define a partition \( \{y^p_h\}_{0 < h \leq \ell_p} \) of the outgoing wave \( s_p \) \( (p = 1, \ldots, N) \) of the \( p \)-th family issuing from \((i\varepsilon, j\varepsilon)\) (with corresponding speeds \( \{\lambda^p_h\}_{0 < h \leq \ell_p} \) as in Proposition 3.5). In particular, a partition of \( s_k \) is defined by means of its sizes as

\[
s^h_k = \begin{cases} s^{n_h}_h & \text{if } h = 1, \ldots, \ell', \\ s^{n_h}_{\ell'} & \text{if } \ell' = \ell' + 1, \ldots, \ell' + \ell'' \end{cases}
\]

(with the same notations of the proof of Proposition 3.5).

Clearly, such partitions continue to satisfy the bounds (3.12), (3.14) and the one-to-one correspondence at (3.13), thanks to the estimate (4.14), and because of the inductive assumption. Therefore, in order to conclude the proof, it remains to establish only the estimate (3.15) on the wave speeds. To this end, notice that the Rankine-Hugoniot speed \( \lambda_k \) of the outgoing \( k \)-wave \( s_k \) coincides with the speeds \( \lambda^h_k \) of all subwaves \( s^h_k \) defined according with Definition 2.3, since for a shock wave
the integrand function $\sigma(\cdot)$ in (2.15) results to be a constant (cfr. Remark 2.2). Moreover, by the choice of the speeds of a partition at (2.15), one has

$$
\lambda_k^{\ell} = \frac{1}{s_k^{\ell}} \int_{\tau_k^{\ell-1}}^{\tau_k^{\ell}} \sigma'(\tau) \, d\tau, \quad \lambda_k'^{\ell} = \frac{1}{s_k'^{\ell}} \int_{\tau_k'^{\ell-1}}^{\tau_k'^{\ell}} \sigma''(\tau) \, d\tau,
$$

(5.4)

where $\tau_k^{\ell} = \sum_{p=1}^{\ell} s_k^p$, $\tau_k'^{\ell} = \sum_{p=1}^{\ell} s_k'^p$, and

$$
\sigma'(\cdot) = \sigma_k[\omega_k'(s_k', \cdot)], \quad \sigma''(\cdot) = \sigma_k[\omega_k''(s_k'', \cdot)],
$$
denote the map in (2.10) defining the speed of the rarefaction and shock components of $s_k'$ and $s_k''$, respectively ($\omega_k', \omega_k''$ being the left states of $s_k', s_k''$).

Then, relying on the interaction estimates in [6, Section 3], with the same type of arguments used in the proof of [6, Lemma 3.9] one obtains the following estimate on the wave speeds, similar to the one provided by [23, Theorem 3.1]:

$$
\lambda_k \cdot (s_k' + s_k'') = \int_0^{s_k'+s_k''} \sigma(\tau) \, d\tau + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'')
$$

$$
= \int_0^{s_k'} \sigma'(\tau) \, d\tau + \int_0^{s_k''} \sigma''(\tau) \, d\tau + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'') + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'')
$$

(5.5)

Thus, since by the monotonicity property of $\sigma'(\cdot)$ and $\sigma''(\cdot)$, we have

$$
\lambda_k^{\ell} - \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'') \leq \lambda_k \leq \lambda_k^{\ell} + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'') \quad \forall \ h,
$$

using (5.8) we derive

$$
|\lambda_k^{\ell} - \lambda_k| = \lambda_k^{\ell} - \lambda_k + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'')
$$

$$
= \frac{1}{s_k^{\ell} + s_k'} \left[ \sum_{p=1}^{\ell'} s_k^p (\lambda_k^{\ell} - \lambda_k^p) + \sum_{p=1}^{\ell''} s_k'^p (\lambda_k^{\ell} - \lambda_k'^p) \right] + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'')
$$

(5.6)

which, in turn, yields

$$
\sum_{h=1}^{\ell'} s_k^h |\lambda_k^h - \lambda_k| = \frac{1}{s_k^{\ell} + s_k'} \left[ \sum_{h=1}^{\ell'} s_k^h s_k^p (\lambda_k^{\ell} - \lambda_k^p) + \sum_{h=1}^{\ell''} s_k'^h s_k'^p (\lambda_k^{\ell} - \lambda_k'^p) \right] + \mathcal{O}(1) \cdot \mathcal{J}(s_k', s_k'') \cdot \frac{s_k'}{s_k^{\ell} + s_k'}.
$$
Notice that the terms of the first double sum on the right hand side of (5.6) are antisymmetric in \((h, p)\), and hence the first summand vanishes. Moreover, recalling (5.4), we have
\[
\ell' \sum_{h=1}^{\ell'} s_k^h s_k^{p'} (\lambda_k^h - \lambda_k^p) =
\]
\[
= \ell' \sum_{p=1}^{\ell'} s_k^{p'} \sum_{h=1}^{\ell'} \int_{k^{-h-1}}^{s_k^h} \sigma'(\tau) \, d\tau - \ell' \sum_{h=1}^{\ell'} s_k^h \sum_{p=1}^{\ell'} \int_{k^{-h-1}}^{s_k^{p'}} \sigma''(\eta) \, d\eta
\]
\[= s_k^{p'} \int_0^{s_k^h} \sigma'(\tau) \, d\tau - s_k^h \int_0^{s_k^{p'}} \sigma''(\eta) \, d\eta
\]
\[= \int_0^{s_k^h} \int_0^{s_k^{p'}} \left[ \sigma'(\tau) - \sigma''(\eta) \right] \, d\eta \, d\tau .
\]  
(5.7)

On the other hand, observe that the term in (4.3) corresponding to the outgoing shock wave \(s_k\) vanishes (being the map \(\sigma\) constant), and hence one clearly has
\[
\int_0^{s_k^h} \int_0^{s_k^{p'}} \left| \sigma'(\tau) - \sigma''(\eta) \right| \, d\eta \, d\tau = \mathcal{O}(1) \cdot |\Delta Q(i\varepsilon)| .
\]  
(5.8)

Thus, since the assumption \(\max\{s_k^l, s_k^{p'}\} > \delta_0/2\) implies \(s_k^l + s_k^{p'} > \delta_0/2\), from (5.6)-(5.8) it follows
\[
\sum_{h=1}^{\ell'} s_k^h |\lambda_k^h - \lambda_k| \leq \frac{2}{\delta_0} \int_0^{s_k^h} \int_0^{s_k^{p'}} \left| \sigma'(\tau) - \sigma''(\eta) \right| \, d\eta \, d\tau + \mathcal{O}(1) \cdot \mathcal{J}(s_k^l, s_k^{p'})
\]
\[\leq \mathcal{O}(1) \cdot |\Delta Q(i\varepsilon)| .
\]  
(5.9)

An entirely similar estimate can be derived for the components of the partition of \(s_k^{p'}\), so that there holds
\[
\sum_{h=1}^{\ell'} s_k^{p'} |\lambda_k^{p'} - \lambda_k| = \mathcal{O}(1) \cdot |\Delta Q(i\varepsilon)| .
\]  
(5.10)

Therefore, relying on the inductive assumption, from (5.9)-(5.10) we recover the desired estimate (3.15), which completes the proof of the proposition.  

\[\square\]

6 Conclusion

Here we briefly describe how to get the proof of Theorem 1.2 following the ideas contained in [10] and relying on the results established in the previous section.
Step 1.

We use the partition of waves of an approximate solution $u^\varepsilon$ into

primary waves $\{\tilde{y}^h_k, \tilde{\lambda}_k^h\}$,

secondary waves $\{\tilde{y}^h_k, \tilde{\lambda}_k^h\}$,

provided by Proposition 5.1 to construct a piecewise constant approximation $w = w(t, x)$ of $u^\varepsilon(t, x)$ in a time interval $[m\varepsilon, n\varepsilon]$ that enjoys the following properties (see [10, Section 4]).

1. The wave fronts in $w$ are of two kinds, primary and secondary.

2. Each primary front originates at $t = m\varepsilon$ and ends at $t = n\varepsilon$;

3. There is a one-to-one correspondence between primary fronts and primary waves $\{\tilde{y}^h_k\}$. In particular, the primary front corresponding to $\tilde{y}^h_k(m, j)$ has constant size $\tilde{s}^h_k(m, j)$ and, in view of Proposition 5.1, joins with a segment the points $(m\varepsilon, j\varepsilon)$ and $(n\varepsilon, \ell(n,j,k,h)\varepsilon)$ of the $(t, x)$ plane.

4. The left and right states of the primary front corresponding to $\tilde{y}^h_k(m, j)$, say $u^h,L_k(m, j)$, $u^h,R_k(m, j)$, are always related by

$$u^h,R_k(m, j) = T_k[u^h,L_k(m, j)](\tilde{s}^h_k(m, j)).$$

Moreover, there holds

$$w(m\varepsilon) = u^\varepsilon(m\varepsilon).$$

5. Let $u^L_j(t)$, and $u^R_j(t)$ be the left and right state of a secondary front $x_j(t)$ of $w$ at time $t \in [m\varepsilon, n\varepsilon]$. Then, letting $CW$ denote the set of all pairs of crossing primary waves in $u^\varepsilon$ (i.e. all pairs of waves $\tilde{y}^h_k(m, j)$, $\tilde{y}^h_k'(m, j')$ for which $j < j'$, $k > k'$ and $\ell(n,j,k,h) \geq \ell(n,j',k',h)$), there holds

$$\sum_\beta |u^R_\beta(t) - u^L_\beta(t)| = O(1) \cdot \left[ \sum_{j,k,h} \left| \tilde{s}^h_k(m, j) \right| + \sum_{CW} \left| \tilde{s}^h_k(m, j) \tilde{s}^h_k'(m, j') \right| \right]$$

$$= O(1) \cdot |\Delta \Upsilon^{m,n}|,$$

where the summand on the left hand side runs over all secondary fronts in $w(t)$, while the second summand on the right hand side runs over all pairs of crossing primary waves in $u^\varepsilon$.

6. All secondary fronts travel with speed 2, strictly larger than all characteristic speeds.
Step 2.

Using the same arguments of [10, Section 5], relying on (1.12), (1.17), (3.14), (3.15), one can prove that

\[ \| S_{(n-m)\varepsilon} w(m\varepsilon) - w(n\varepsilon) \|_{L_1} = \]
\[ = O(1) \cdot \left| \Delta \Upsilon^{m,n} \right| + \frac{1 + \log(n - m)}{n - m} + \varepsilon \right) (n - m)\varepsilon, \quad (6.1) \]

\[ \| u^\varepsilon(n\varepsilon) - w(n\varepsilon) \|_{L_1} = O(1) \cdot |\Delta \Upsilon^{m,n}| \cdot (n - m)\varepsilon, \]

where \( S_{(n-m)\varepsilon} w(m\varepsilon) \) is the semigroup trajectory related to (1.1) with initial datum \( w(m\varepsilon) = u^\varepsilon(m\varepsilon) \) evaluated at time \( t = (n - m)\varepsilon \).

Step 3.

Now, as in [10, Section 6], let \( T = \overline{m\varepsilon} + \varepsilon' \), for some \( \overline{m} \in \mathbb{N} \), \( 0 \leq \varepsilon' < \varepsilon \), and fix a positive constant \( \rho > 2\varepsilon \). Then, we define inductively integers \( 0 = m_0 < m_1 < \ldots < m_\kappa = \overline{m} \) in this way.

1. if \( \Upsilon(m_i\varepsilon) - \Upsilon((m_i + 1)\varepsilon) \leq \rho \), let \( m_{i+1} \) be the largest integer less or equal to \( \overline{m} \) such that \( (m_{i+1} - m_i)\varepsilon \leq \rho \) and \( \Upsilon(m_i\varepsilon) - \Upsilon(m_{i+1}\varepsilon) \leq \rho \);

2. if \( \Upsilon(m_i\varepsilon) - \Upsilon((m_i + 1)\varepsilon) > \rho \), set \( m_{i+1} = m_i + 1 \).

On every interval \( [m_i\varepsilon, m_{i+1}\varepsilon] \) where 1. holds, we construct a piecewise constant approximation of \( u^\varepsilon \) according to Step 1, and using (6.1) we derive

\[ \| u^\varepsilon(m_{i+1}\varepsilon) - S_{(m_{i+1} - m_i)\varepsilon} u^\varepsilon(m_i\varepsilon) \|_{L_1} = O(1) \cdot \left| \Delta \Upsilon^{m_i,m_{i+1}} \right| + \frac{1 + \log(m_{i+1} - m_i)}{m_{i+1} - m_i} + \varepsilon \right) (m_{i+1} - m_i)\varepsilon. \quad (6.2) \]

On the other hand, on each interval \( [m_i\varepsilon, m_{i+1}\varepsilon] \) where 2. is verified, by the Lipschitz continuity of \( u^\varepsilon \) and applying (1.17) we find

\[ \| u^\varepsilon(m_{i+1}\varepsilon) - S_{(m_{i+1} - m_i)\varepsilon} u^\varepsilon(m_i\varepsilon) \|_{L_1} = O(1) \cdot \varepsilon. \quad (6.3) \]

Hence, observing that the cardinality of both classes of intervals 1.-2. is bounded by \( O(1) \cdot \rho^{-1} \), from (6.2) - (6.3) we finally deduce

\[ \| u^\varepsilon(T) - S_T \overline{u} \|_{L_1} = O(1) \cdot \left[ \rho + \frac{\varepsilon}{\rho} \log \frac{\rho}{\varepsilon} + \varepsilon \left( 1 + \frac{1}{\rho} \right) \right], \]

which yields (1.13) choosing \( \rho \approx \sqrt{\varepsilon} \cdot \log \left| \log \varepsilon \right| \).
Remark 6.1 By the same observations of Remark 3.6, one deduces that the conclusion of Theorem 1.2 remains valid if we assume that the flux function $F$ is $C^{2,1}$ and that, for each $k$-th NGNL characteristic family, the linearly degenerate manifold $M_k$ in (1.8) is the union of a finite number of connected manifolds $M_{k,h}$, that are either $N-1$-dimensional as in (H), or $N$-dimensional and in this case the following conditions hold. The flux function $F$ is $C^3$ on $\Omega \setminus M_k$, the vector field $r_k$ is transversal to the boundary of $M_{k,h}$, and letting $\partial^+ M_{k,h}$, $\partial^- M_{k,h}$ denote the connected components of the boundary of $M_{k,h}$ where $r_k$ points towards $\Omega \setminus M_{k,h}$ and $M_{k,h}$, respectively, the one-sided second derivatives of $\lambda_k$ (see Remark 3.6) satisfy

$$\nabla^+ (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) < 0 \quad \forall u \in \partial^+ M_k,$$

$$\nabla^- (\nabla \lambda_k \cdot r_k)(u) \cdot r_k(u) < 0 \quad \forall u \in \partial^- M_k,$$

or the opposite inequalities. Indeed, if we again add the term $2 \sum_{p \neq q} |s^s_{\alpha,p} s^s_{\alpha,q}|$ in the interaction potential $Q_q$ defined by (4.7)–(4.8), for every wave $s_\alpha$ containing several contact discontinuities $s^s_{\alpha,1}, \ldots, s^s_{\alpha,l}$, the estimates stated in Lemma 4.6 continue to hold, provided that

$$\inf \left\{ s > 0 : R_k[u](s) \in \partial^+ M_{k,h}, u \in \partial^- M_{k,h} \right\} > 0,$$

which is certainly true up to a possible slight restriction of the domain $\Omega$. Relying on Lemma 4.6, one then establishes Proposition 5.1 with the same arguments of Section 5 and thus conclude as above.

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