Abstract—Wigner-Smith (WS) time delay concepts have been used extensively in quantum mechanics to characterize delays experienced by particles interacting with a potential well. This paper formally extends WS time delay theory to Maxwell's equations and explores its potential applications in electromagnetics. The WS time delay matrix relates a lossless and reciprocal systems scattering matrix to its frequency derivative and allows for the construction of modes that experience well-defined group delays when interacting with the system. The matrix entries for guiding, scattering, and radiating systems are energy-like overlap integrals of the electric and/or magnetic fields that arise upon excitation of the system via its ports. The WS time delay matrix has numerous applications in electromagnetics, including the characterization of group delays in multiport systems, the description of electromagnetic fields in terms of elementary scattering processes, and the characterization of frequency sensitivities of fields and multiport antenna impedance matrices.

Index Terms—Wigner Smith Time Delays, Group Delays in Multiport Systems, Frequency Derivatives of Scattering and Impedance Matrices

I. INTRODUCTION

In 1960, Felix Smith published a seminal paper Lifetime Matrix in Collision Theory, a description of procedures to characterize the time delays experienced by particles during quantum mechanical interactions [1]. Starting from the Schrödinger equation, Smith showed that the matrix

$$Q = js^1 \frac{\partial S}{\partial \omega}$$  \hspace{1cm} (1)

where $S$ is a potential wells scattering matrix and $\omega$ denotes angular frequency, fully characterizes the particles’ average time of residence in the system. Over the past 60 years, the Wigner-Smith (WS) time delay matrix $Q$ (as it has come to be known) has found many applications in quantum mechanics, including the study of particle tunneling through potential barriers [2]–[3], the characterization of photoionization and photoemission time delays [4]–[5], and the analysis of decaying quantum systems [6]. For an excellent review of the field, see [7].

References to Smith’s paper in the electromagnetics literature have been few and far in between, however. An exception is [8], where group delays of fields interacting with a two-port waveguide were characterized in terms of their WS dwell times. In optics and photonics, WS time delay concepts have been used to describe wave propagation in multimode fibers [9], to optimize light storage in highly scattering environments [10], to shape the flow of light in disordered media [11], and to characterize optical fields passing through complex cavities [12]–[13]. Another notable line of work involves the statistical characterization of chaotic fields in large enclosures and reverberation chambers by exploiting connections between WS time delays and random matrix theory [14]–[17].

This paper outlines a WS time delay theory for electromagnetics. Its contributions are threefold.

- First, it reviews WS theory from a systems perspective, using equation (1) to elucidate $Q$’s central role in characterizing group delays in lossless and reciprocal electromagnetic systems. It also describes the so-called WS modes that arise upon diagonalization of $Q$ and experience well-defined group delays when interacting with a system.

- Second, it introduces closed-form expressions for the entries of the electromagnetic WS time delay matrix for guiding, scattering, and radiating systems. Indeed, $Q$ defining equation notwithstanding, its computation may proceed without knowledge of $\frac{\partial S}{\partial \omega}$. For guiding systems (e.g. closed multiport waveguide networks) excited by Transverse Electromagnetic (TEM) waves, the elements of the WS time delay matrix (1) can be expressed in terms of volume integrals of energy-like densities involving the electric and magnetic fields that arise upon excitation of the systems ports. For guiding systems with non-TEM excitations, scattering systems (e.g. perfect electrically conducting surfaces excited by impinging waves) or radiating systems (e.g. antennas and arrays thereof), additional correction terms and renormalization procedures are called for to obtain (1).

- Third, it elucidates some important characteristics of WS modes and demonstrates the potential use of (1) in the broadband characterization of antenna systems. Specifically, it shows that WS modes naturally untangle resonant, corner/edge, and ballistic scattering phenomena as they are characterized by different dwell times within a system. It also demonstrates that knowledge of $Q$ and $S$ allows for the computation of $\frac{\partial S}{\partial \omega}$, which in turn can be used to assess the frequency dependence of impedance matrices of multiport systems. The theory and methods presented in this paper therefore can be viewed as multiport extensions of procedures for characterizing the bandwidth, quality factor, and stored energy of single-port antennas, see [18]–[23].

The three topics above are detailed in Secs. II–IV below. Conclusions and avenues for future research are provided in
The outgoing pulse’s group velocity and group delay are $\frac{\partial \omega}{\partial \beta(\omega)}$ and $Q = \gamma'(\omega)$ respectively.

**B. Group Delay in a Multi-Port System**

The above scenario is easily generalized to the linear, time-invariant, lossless, and reciprocal $M$-port system in Fig. 1 where all lines are assumed identical. Assume that line $p$ supports the time-harmonic incoming signal

$$\tilde{E}_p^i(w, \omega) = e^{j\beta(\omega)w}.$$  

On line $1 \leq m \leq M$ the system generates the outgoing signal

$$\tilde{E}_p^o(w, m, \omega) = S_{mp} e^{-j\beta(\omega)w} = S_{mp} (\tilde{E}_m^i(w, \omega))^\ast. $$

The $M \times M$ scattering matrix $S$ is unitary and symmetric, i.e.

$$S^\dagger S = I_M \quad (9a)$$

$$S = S^T \quad (9b)$$

where $I_M$ is the $M \times M$ identity matrix.

Next, assume that port $p$ is excited by the incoming narrowband pulse

$$\tilde{E}_p^i(w, t) \approx \tilde{E}_i(w, t).$$

Using $S_{mp}(\omega) = |S_{mp}(\omega)| e^{-j\gamma_{mp}(\omega)}$, the outgoing pulse on line $m$ is

$$\tilde{E}_p^o(w, m, t) = \text{Re} \left[ \int_{\omega_o - \Delta \omega}^{\omega_o + \Delta \omega} A(\omega - \omega_o) S_{mp}(\omega) e^{j(\omega t - \beta(\omega)w)} d\omega \right] \equiv |S_{mp}(\omega)| \tilde{A} \left( t - \beta'(\omega_o)w - \gamma_{mp}(\omega_o) \right) \cos (\omega_o t - \beta(\omega_o)w - \gamma_{mp}(\omega_o)) + |S_{mp}(\omega)| \left| \frac{\partial \tilde{A}}{\partial t} \right| \sin (\omega_o t - \beta(\omega_o)w - \gamma_{mp}(\omega_o)).$$

Given the narrowband and smooth nature of $\tilde{A}(t)$, the second term in (11) can be neglected. It follows that the outgoing signal’s group delay on line $m$ due to an incoming signal on line $p$ is

$$q_{mp} = \gamma'_{mp}(\omega)$$

$$= \frac{1}{S_{mp}(\omega)} S_{mp}'(\omega)$$

evaluated for $\omega = \omega_o$. To generalize (6) to multiport systems, Smith put forward the time delay matrix defined in (1). To interpret $Q$, he introduced the weighted average time delay experienced by $\tilde{E}_p^i(w, t)$ as it makes its way from input port $p$ to all output ports:

$$\langle q_{mp} \rangle \equiv \sum_{m=1}^{M} |S_{mp}(\omega)|^2 q_{mp}. \quad (13)$$

The weighting coefficient $|S_{mp}(\omega)|^2$ is the fraction of the power carried by $\tilde{E}_p^i(w, t)$ transferred to port $m$. Inserting (12) into (13) yields

$$\langle q_{mp} \rangle \equiv j \sum_{m=1}^{M} \left| S_{mp}(\omega) \right|^2 \frac{1}{S_{mp}(\omega)} S_{mp}'(\omega)$$

Fig. 1: A generic $M$-port linear, time-invariant, lossless, and reciprocal system.
diagonal elements have no direct physical interpretation but are experienced by wave packets entering the system. Therefore extremize the amount of time that a pulse dwells in the system. Specifically, the first (last) incoming WS mode represents a combination of excitations that results in an outgoing pulse experiencing the smallest (largest) possible time delay while interacting with the system.

Next, let \( \hat{W} \equiv W(\omega_i) \) for a fixed frequency \( \omega_i \), and define \( \hat{Q}(\omega) \) and \( \hat{S}(\omega) \) as

\[
\hat{Q}(\omega) = \hat{W}^\dagger Q(\omega) \hat{W} \quad (20a)
\]

\[
\hat{S}(\omega) = \hat{W}^T S(\omega) \hat{W}. \quad (20b)
\]

Note that \( \hat{Q}(\omega) = \hat{Q}(\omega) \) and \( \hat{S}(\omega) = I_M \) when \( \omega = \omega_o \).

Substituting (20a)–(20b) into (1) yields

\[
\hat{Q} = j\hat{S}^\dagger S'. \quad (21)
\]

Exciting the system with the time-harmonic incoming WS mode

\[
E_{WS,q}^i(w,\omega_o) \equiv \sum_{p=1}^{M} \hat{W}_{pq} E_p^i(w,\omega_o) \quad (22)
\]

results in the outgoing WS mode

\[
E_{WS,q}^o(w,\omega_o) = E_{WS,q}^i(w,\omega_o)^*. \quad (23)
\]

A narrowband outgoing pulse built from WS mode \( q \) therefore exhibits group delay \( \hat{Q}_{qq} \) w.r.t. its incoming counterpart, uniformly across all lines.

Finally, it is noted that if \( S(\omega_o) \) and \( Q(\omega_o) \) are known, (1) and (21) yield

\[
S(\omega_o + \delta \omega) \cong S(\omega_o) - j \delta \omega S(\omega_o)Q(\omega_o) \quad (24)
\]

\[
\hat{S}(\omega_o + \delta \omega) \cong I_M - j \delta \omega \hat{Q}(\omega_o) \cong e^{-j\delta \omega \hat{Q}(\omega_o)}. \]

These estimates of \( S(\omega_o + \delta \omega) \) and \( \hat{S}(\omega_o + \delta \omega) \) in turn can be used to approximate the system’s response at \( \omega_o + \delta \omega \).

For example, consider the incoming signal

\[
F_q^i(w,\omega_o + \delta \omega) \equiv \sum_{p=1}^{M} \hat{W}_{pq} E_p^i(w,\omega_o + \delta \omega) \quad (25)
\]

obtained by evolving the frequency of the \( E_p^i(w,\omega_o) \) in the WS modes in (23), while keeping their combination constants fixed. The outgoing signal \( F_q^o(w,\omega_o + \delta \omega) \) generated in response to this excitation is

\[
F_q^o(w,\omega_o + \delta \omega) = \sum_{p=1}^{M} \hat{W}_{pq} E_p^o(w,\omega_o + \delta \omega)
\]

\[
\cong e^{-j\delta \omega \hat{Q}_{qq}(\omega_o)} (F_q^i(w,\omega_o + \delta \omega))^*, \quad (26)
\]

showing that, to first order, WS modes do not couple when changing the frequency; they simply acquire an extra phase delay \( \delta \omega \hat{Q}_{qq}(\omega_o) \).

While the discussion so far assumed all lines were identical, almost all methods and conclusions presented above continue to hold true when this condition is violated. Even when the lines support waves traveling at different speeds, WS modes still describe wave packets that simultaneously exit the system, even though they disperse after that (example: non TEM waveguides).
III. WS THEORY: ELECTROMAGNETIC PERSPECTIVE

The electromagnetic WS time delay matrix \( Q(\omega) \) can be evaluated from knowledge of the fields at frequency \( \omega \) that exist throughout the system for all possible port excitations. This section presents expressions for the entries of the WS time delay matrix \( Q(\omega) \) for guiding (g), scattering (s), and radiating (r) systems. For guiding systems fed by TEM waveguides, \( Q(\omega) \)'s entries are expressed as energy-like overlap integrals of the system's electric and/or magnetic fields; correction factors involving the system's scattering matrix and waveguide impedances are used when the system is fed through non-TEM ports. For scattering and radiating systems, a renormalization procedure that extracts the system's far-fields is introduced.

A. Guiding Systems

1) Setup: Consider a lossless microwave network with perfect electrically conducting (PEC) walls that is terminated by homogeneous waveguides of uniform cross section. Let \( \Omega \) and \( d\Omega \) denote the networks volume and physical port surfaces, respectively. Next, consider the (global) curvilinear coordinate system \((u, v, w)\) shown in Fig. 2. On \( d\Omega \), \( w = 0 \) and \((u, v, w)\) is locally Cartesian. Let \( \hat{\omega} \) denote the outward pointing normal to \( d\Omega \). The physical port surfaces are assumed far removed from waveguide discontinuities, so fields there can be expressed in terms of propagating modes. Each physical port supports one or more propagating modes; let \( M_g \) denote the total number of propagating modes in all physical ports.

Assume that the network is excited by an incoming unit-power field with transverse electric and magnetic components

\[
\mathbf{E}_{p,i}(r, \omega) = n_p(\omega) e^{j\beta_p(\omega) w} \mathbf{X}_p(u, v) \quad (27a)
\]

\[
\mathbf{H}_{p,i}(r, \omega) = \frac{Z_p(\omega)}{\mu_p(\omega)} e^{j\beta_p(\omega) w} (-\hat{\omega} \times \mathbf{X}_p(u, v)) \cdot (27b)
\]

Here \( r = (u, v, w) \) and \( 1 \leq p \leq M_g \) denotes the index of a TE, TM, or TEM propagating mode. The mode's transverse profile \( \mathbf{X}_p(u, v) \) is supported on \( d\Omega_p \subset d\Omega \). Note that \( d\Omega_p = d\Omega \), when mode \( p \) and \( p \) share the same physical port. In \((27a)-(27b)\), \( Z_p(\omega) = \sqrt{\mu_p(\omega)} \), and \( \beta_p(\omega) \) are the \( p \)-th mode's impedance, power normalization factor, and propagation constant. A procedure to obtain \( \mathbf{X}_p(u, v) \), \( n_p(\omega) \), \( Z_p(\omega) \), and \( \beta_p(\omega) \) for arbitrarily shaped waveguides is outlined in Appendix A. These modes are automatically orthogonal and normalized to satisfy

\[
\int_{d\Omega} \mathbf{X}_p(u, v) \cdot \mathbf{X}_q^*(u, v) \, du \, dv = \delta_{pq} . \quad (28)
\]

The \( \mathbf{X}_q(u, v) \) are purely real, i.e. \( \mathbf{X}_q^*(u, v) = \mathbf{X}_q(u, v) \).

Next, let \( \mathbf{E}_p(r, \omega) \) and \( \mathbf{H}_p(r, \omega) \) denote the field throughout \( \Omega \) due to excitation \((27a)-(27b)\), assuming all ports are matched. Near \( d\Omega \), these fields' transverse (to \( \hat{\omega} \)) components can be expressed as

\[
\mathbf{E}_p,\parallel(r, \omega) = \mathbf{E}_p,\parallel(r, \omega) + \mathbf{E}_p,\perp(r, \omega) \quad (29a)
\]

\[
\mathbf{H}_p,\parallel(r, \omega) = \mathbf{H}_p,\parallel(r, \omega) + \mathbf{H}_p,\perp(r, \omega) , \quad (29b)
\]

where the outgoing transverse fields are

\[
\mathbf{E}_{p,\parallel}(r, \omega) = \sum_{m=1}^{M_g} \mathbf{S}_{mp}(\omega)n_m(\omega)e^{-j\beta_m(\omega)w} \mathbf{X}_m(u, v) \quad (30a)
\]

\[
\mathbf{H}_{p,\parallel}(r, \omega) = -\sum_{m=1}^{M_g} \mathbf{S}_{mp}(\omega)n_m(\omega)e^{-j\beta_m(\omega)w} \cdot (-\hat{\omega} \times \mathbf{X}_m(u, v)) \quad (30b)
\]

The above construct guarantees that \( \mathbf{S} \) is unitary. From here onwards, the \( \omega \) dependence of quantities is suppressed.

2) Maxwell’s equations: Consider two sets of electromagnetic fields inside \( \Omega \): \( \{ \mathbf{E}_p(r), \mathbf{H}_p(r) \} \) and \( \{ \mathbf{E}_q(r), \mathbf{H}_q(r) \} \). The frequency derivative of Maxwell’s equations for \( \{ \mathbf{E}_p(r), \mathbf{H}_p(r) \} \) reads

\[
\nabla \times \mathbf{H}_p'(r) = j\varepsilon \mathbf{E}_p(r) + j\omega\varepsilon\mathbf{E}_p'(r) \quad (31a)
\]

\[
\nabla \times \mathbf{E}_p'(r) = -j\mu \mathbf{H}_p(r) - j\omega\mu\mathbf{H}_p'(r) , \quad (31b)
\]

Adding the dot-product of \((32a)\) and \(\frac{1}{2}\mathbf{E}_p'(r)\) to the dot-product of \((31a)\) and \(\frac{1}{2}\mathbf{E}_q'(r)\) yields

\[
\frac{1}{2} \mathbf{E}_p'(r) \cdot \nabla \times \mathbf{H}_q^*/(r) + \frac{1}{2} \mathbf{E}_q'(r) \cdot \nabla \times \mathbf{H}_p^*(r) = \frac{1}{2} j \varepsilon \mathbf{E}_q^*(r) \cdot \mathbf{E}_p(r) . \quad (33)
\]

Similarly, adding the dot-product of \((32b)\) and \(\frac{1}{2}\mathbf{H}_p'(r)\) to the dot-product of \((31b)\) and \(\frac{1}{2}\mathbf{H}_q'(r)\) results in

\[
\frac{1}{2} \mathbf{H}_p'(r) \cdot \nabla \times \mathbf{E}_q^*/(r) + \frac{1}{2} \mathbf{H}_q'(r) \cdot \nabla \times \mathbf{E}_p^*/(r) = \frac{1}{2} j \mu \mathbf{H}_q^*(r) \cdot \mathbf{H}_p(r) . \quad (34)
\]

Subtracting \((34)\) from \((33)\) yields

\[
\frac{1}{2} \nabla \cdot [\mathbf{E}_p(r) \times \mathbf{H}_q^*/(r) + \mathbf{E}_q(r) \times \mathbf{H}_p^*/(r)] = \frac{1}{2} j \varepsilon \mathbf{E}_q^*(r) \cdot \mathbf{E}_p(r) + \frac{1}{2} j \mu \mathbf{H}_q^*/(r) \cdot \mathbf{H}_p(r) . \quad (35)
\]
Finally, integrating the left and right hand sides (LHS and RHS) of \((35)\) over \(\Omega\), applying the divergence theorem, and enforcing the boundary conditions on the network’s PEC walls yields
\[
\frac{j}{2} \int_{d\Omega} \nabla \cdot \left[ \mathbf{E}_{q,\perp}(r) \times \mathbf{H}'_{p,\perp}(r) + \mathbf{E}'_{p,\perp}(r) \times \mathbf{H}_{q,\perp}(r) \right] dS
\]
\[
\quad = \frac{1}{2} \int_{\Omega} \nabla \cdot \mathbf{E}_p(r) + \mu \nabla \cdot \mathbf{H}_p(r) \cdot dV . \tag{36}
\]

3) WS Relationship: The evaluation of the LHS of \((36)\) requires expressions for \(\mathbf{E}'_{p,\perp}(r), \mathbf{H}'_{p,\perp}(r), \mathbf{E}_{q,\perp}(r), \) and \(\mathbf{H}_{q,\perp}(r)\) on \(d\Omega\). Substituting \((27a)\)–\((27b)\) and \((30a)\)–\((30b)\) into \((29a)\)–\((29b)\), and differentiating w.r.t frequency, yields
\[
\mathbf{E}'_{p,\perp}(r) = (n_p e^{j\beta_p w})' \mathbf{X}_p(u,v) \tag{37a}
\]
\[
+ \sum_{m=1}^{M_q} S_{mp} n_m e^{-j\beta_m w} \mathbf{X}_m(u,v) + \sum_{m=1}^{M_q} S_{mp} (n_m e^{-j\beta_m w})' \mathbf{X}_m(u,v) \tag{37b}
\]
\[
\mathbf{H}'_{p,\perp}(r) = \left( \frac{n_p e^{j\beta_p w}}{Z_p} \right)' \left[ -\nabla \times \mathbf{X}_p(u,v) \right] \tag{37b}
\]
\[
- \sum_{m=1}^{M_q} S'_{mp} \frac{n_m}{Z_m} e^{-j\beta_m w} \left[ -\nabla \times \mathbf{X}_m(u,v) \right] \tag{37b}
\]
\[
- \sum_{m=1}^{M_q} S_{mp} \frac{n_m}{Z_m} e^{-j\beta_m w} \left[ -\nabla \times \mathbf{X}_m(u,v) \right] . \tag{37b}
\]

Likewise, substituting \((27a)\)–\((27b)\) and \((30a)\)–\((30b)\) into \((29a)\)–\((29b)\) with \(p \rightarrow q\), and complex conjugating the result yields
\[
\mathbf{E}_{q,\perp}(r) = n_q e^{-j\beta_q w} \mathbf{X}_q^*(u,v) \tag{38a}
\]
\[
+ \sum_{m=1}^{M_q} S_{mq} n_m e^{j\beta_m w} \mathbf{X}_m^*(u,v) \quad \tag{38a}
\]
\[
\mathbf{H}_{q,\perp}(r) = \left( \frac{n_q e^{-j\beta_q w}}{Z_q} \right) \left( -\nabla \times \mathbf{X}_q^*(u,v) \right) \tag{38a}
\]
\[
- \sum_{m=1}^{M_q} S'_{mq} \frac{n_m}{Z_m} e^{j\beta_m w} \left( -\nabla \times \mathbf{X}_m^*(u,v) \right) \tag{38a}
\]
\[
- \sum_{m=1}^{M_q} S_{mq} \frac{n_m}{Z_m} e^{j\beta_m w} \left( -\nabla \times \mathbf{X}_m^*(u,v) \right) . \tag{38a}
\]

Substituting \((38a)\)–\((38b)\) and \((37a)\)–\((37b)\) into the LHS of \((36)\), and manipulating the equation produces
\[
\frac{j}{2} \int_{d\Omega} \nabla \cdot \left[ \mathbf{E}_{p,\perp}(r) \times \mathbf{H}'_{q,\perp}(r) + \mathbf{E}'_{q,\perp}(r) \times \mathbf{H}_{p,\perp}(r) \right] dS
\]
\[
\quad = j \sum_{m=1}^{M_q} S_{mq} S'_{mp} - \frac{j}{2} S_{pq} \left( \frac{1}{Z_p} \right)' [n_p]^2 \tag{39}
\]
\[
+ \frac{j}{2} S_{qp} \left( \frac{1}{Z_q} \right)' [n_q]^2 . \tag{39}
\]

Therefore, the surface integral on the LHS of \((36)\) can be computed given knowledge of scattering matrix and its frequency derivative. Next, denote the RHS of \((36)\) as
\[
\bar{Q}_{qp} = \frac{1}{2} \int_{\Omega} \left[ \nabla \cdot \mathbf{E}_p(r) + \mu \nabla \cdot \mathbf{H}_p(r) \right] dV . \tag{40}
\]

Fig. 3: Scattering system excited through free-space port defined on sphere of radius \(R\).

Substituting \((39)\) and \((40)\) into \((36)\) yields
\[
\bar{Q}_{qp} + \frac{j}{2} S_{qp} \left( \frac{1}{Z_p} \right)' [n_p]^2 - \frac{j}{2} S_{qp} \left( \frac{1}{Z_q} \right)' [n_q]^2
\]
\[
\quad = j \sum_{m=1}^{M_q} S_{pq} S'_{mp} . \tag{41}
\]

In matrix form, \((41)\) reads
\[
\bar{Q} = j S' S
\]
\[
\text{where}
\]
\[
\bar{Q} = \bar{Q} - \frac{j}{2} D S + \frac{j}{2} S' D \tag{42}
\]
\[
is the WS time delay matrix for guiding systems, and \(D\) is a diagonal matrix with \((p,p)\)-th entry
\[
\text{D}_{pp} = \left( \frac{1}{Z_p} \right)' [n_p]^2 . \tag{43}
\]

Equations \((42)\)–\((44)\) show that the entries of the WS time delay matrix for guiding systems are energy-like overlap integrals of the electric and magnetic fields that exist in \(\Omega\) upon excitation of the systems’ ports (diagonal elements are energies). Note that if all propagating modes are TEM, then \(Q^{\alpha} = Q\) because \(Z_p\) becomes frequency independent. Knowledge of the fields at frequency \(\omega\) throughout \(\Omega\) for all possible port excitations therefore allows for the computation of both \(S\) and \(Q\), and, via \((42)\), \(S\)’s frequency derivative.

B. Scattering Systems

1) Setup: Consider a lossless scatterer that resides in free space and is circumscribed by a sphere of radius \(a\) centered at the origin (Fig. 3). Let \(\Omega\) and \(d\Omega\) denote the volume and surface of a concentric sphere of radius \(R \gg a\), respectively. On \(d\Omega\), fields interacting with the scatterer can be described in terms of \(M_s = O((ka)^2)\) propagating TEM modes \([26, 27]\). Assume that the scatterer is excited by an incoming unit

\[
\text{Fig. 3: Scattering system excited through free-space port defined on sphere of radius \(R\).}
\]
power field with (naturally transverse) electric and magnetic components
\[
E^{'p,\|}_{p}(r,\omega) = \frac{n e^{j k(\omega) r}}{r} \mathbf{X}_p(\theta, \phi) \quad (45a)
\]
\[
H^{'p,\|}_{p}(r,\omega) = \frac{n e^{j k(\omega) r}}{Z} \left( -\hat{r} \times \mathbf{X}_p(\theta, \phi) \right) . \quad (45b)
\]

Here \( r = (r, \theta, \phi) \), \( \hat{r} \) is the radial unit vector, and \( 1 \leq p \leq M_s \) denotes the index of a mode with transverse mode-profile \( \mathbf{X}_p(\theta, \phi) \). In (45a)–(45b), \( Z = \sqrt{\mu_o/\varepsilon_o} \), \( n = \sqrt{Z} \), and \( k(\omega) = \omega/\sqrt{\mu_o/\varepsilon_o} \) are the mode-independent impedance, power normalization factor, and wave number; \( \mu_o \) and \( \varepsilon_o \) are the free-space permeability and permittivity. The mode profiles are assumed orthonormal, i.e.
\[
\int_0^{2\pi} \int_0^\pi \mathbf{X}_p(\theta, \phi) \cdot \mathbf{X}_q^*(\theta, \phi) \sin \theta d\theta d\phi = \delta_{pq} . \quad (46)
\]

There exists many possible choices for \( \mathbf{X}_p(\theta, \phi) \). A specific realization in terms of vector spherical harmonics is outlined in Appendix B. Note that (45a)–(45b) should not be confused with the “incident field” in scattering computations as the latter carries no net energy across \( d\Omega \).

Next, let \( E_p(r) \) and \( H_p(r) \) denote the fields throughout \( \Omega \) due to excitation (45a)–(45b). Near \( d\Omega \), these fields can be expressed as
\[
E_{p,\perp}(r,\omega) = E^{'p,\|}_{p}(r,\omega) + E^{'p,\perp}_{p}(r,\omega) \quad (47a)
\]
\[
H_{p,\perp}(r,\omega) = H^{'p,\|}_{p}(r,\omega) + H^{'p,\perp}_{p}(r,\omega) , \quad (47b)
\]

where the outgoing (automatically transverse) fields near \( d\Omega \) are
\[
E^{'p,\perp}_{p}(r,\omega) = \sum_{m=1}^{M_s} S_{mp}(\omega) \frac{e^{-j k(\omega) r}}{r} \mathbf{X}_m^*(\theta, \phi) \quad (48a)
\]
\[
H^{'p,\perp}_{p}(r,\omega) = -\sum_{m=1}^{M_s} S_{mp}(\omega) \frac{n e^{-j k(\omega) r}}{Z} \left( -\hat{r} \times \mathbf{X}_m^*(\theta, \phi) \right) . \quad (48b)
\]

The above construct guarantees that \( S \) is independent of \( R \) and that (9a) holds, i.e. that \( S \) is unitary.

2) WS Relationship: To derive the WS relationship for scattering systems, once again consider two sets of fields: \( \{E_p(r), H_p(r)\} \) and \( \{E_q(r), H_q(r)\} \). The above derivation (31a)–(36) for guiding systems continues to hold true with \( \hat{u} \rightarrow \hat{r} \). The evaluation of the LHS of (36) requires expressions for \( E^{'p,\|}_{p}(r), H^{'p,\|}_{p}(r), E^{'q,\|}_{q}(r), \) and \( H^{'q,\|}_{q}(r) \) on \( d\Omega \). Substituting (45a)–(45b) and (48a)–(48b) into (47a)–(47b), and differentiating w.r.t. frequency yields
\[
E^{'p,\|}_{p}(r) = n \left( \frac{e^{j k(\omega) r}}{r} \right)' \mathbf{X}_p(\theta, \phi) + \sum_{m=1}^{M_s} S^*_{mp} \frac{e^{-j k(\omega) r}}{r} \mathbf{X}_m^*(\theta, \phi) \quad (49a)
\]
\[
+ \sum_{m=1}^{M_s} S_{mp}(\omega) \left( \frac{e^{-j k(\omega) r}}{r} \right)' \mathbf{X}_m^*(\theta, \phi) \quad (49a)
\]
\[
H^{'p,\|}_{p}(r) = \frac{n}{Z} \left( \frac{e^{j k(\omega) r}}{r} \right)' \left( -\hat{r} \times \mathbf{X}_p(\theta, \phi) \right) - \sum_{m=1}^{M_s} S^*_{mp} \frac{n e^{-j k(\omega) r}}{Z} \left( -\hat{r} \times \mathbf{X}_m^*(\theta, \phi) \right) . \quad (49b)
\]

Likewise, substituting (45a)–(45b) and (48a)–(48b) into (47a)–(47b) with \( p \rightarrow q \), and complex conjugating the result yields
\[
E^{'q,\|}_{q}(r) = n \left( \frac{e^{-j k(\omega) r}}{r} \right)' \mathbf{X}_q^*(\theta, \phi) + \sum_{m=1}^{M_s} S^*_{mp} \frac{e^{j k(\omega) r}}{r} \mathbf{X}_m^*(\theta, \phi) . \quad (50a)
\]
\[
H^{'q,\|}_{q}(r) = -\sum_{m=1}^{M_s} S^*_{mp} \frac{n e^{j k(\omega) r}}{Z} \left( -\hat{r} \times \mathbf{X}_m^*(\theta, \phi) \right) . \quad (50b)
\]

Substituting (49a)–(49b) and (50a)–(50b) into the LHS of (36) with \( \hat{u} \rightarrow \hat{r} \), and simplifying the yields
\[
\tilde{Q}_{qp} = j \sum_{m=1}^{M_s} S^*_{qm} S^*_{mp} + 2 \mu_o \varepsilon_o R \delta_{qp} , \quad (51)
\]

where \( \tilde{Q}_{qp} \) is still given by (10). To arrive at a WS relationship that is independent of \( R \), consider the quantity \( Q^\ast_{qp,\infty} \) obtained by replacing in (41) \( \{E_p(r), H_p(r)\} \) and \( \{E_q(r), H_q(r)\} \) by \( \{E_p,\perp(r), H_p,\perp(r)\} \) and \( \{E_q,\perp(r), H_q,\perp(r)\} \), i.e.
\[
\tilde{Q}^\ast_{qp,\infty} = \frac{1}{2} \int_\Omega \left[ \varepsilon^\ast_{\perp}(r) \cdot E,\perp(r) + \mu^\ast_{\perp}(r) \cdot H,\perp(r) \right] dV . \quad (52)
\]

The quantities in the integrand in (52) are not the transverse to \( \tilde{f} \) components of \( \{E_q,\perp(r), H_q,\perp(r)\} \). Instead, they are the quantities in (47a)–(47b) evaluated for arbitrary \( r \in \Omega \). Substituting (45a)–(45b) and (48a)–(48b) into (47a)–(47b), and then evaluating (52) yields
\[
\tilde{Q}^\ast_{qp,\infty} = 2 \mu_o \varepsilon_o R \tilde{Q}_{qp} . \quad (53)
\]

Subtracting (53) from both sides of (51) yields
\[
Q_{qp} = j \sum_{m=1}^{M_s} S^*_{qm} S^*_{mp} , \quad (54)
\]

where \( Q^\ast_{qp} = \tilde{Q}_{qp} - \tilde{Q}^\ast_{qp,\infty} \). In matrix form (54) reads
\[
Q^\ast_{qp} = j S^\dagger S^\ast \quad (55)
\]

with
\[
Q_{qp} = \frac{1}{2} \sum_{R^3} \left[ E^\dagger_q(r) \cdot H_p(r) - E^\dagger_{\perp,q}(r) \cdot E_{\perp,p}(r) \right] dV \quad (56)
\]
\[
+ \frac{1}{2} \mu_o \sum_{R^3} \left[ H^\dagger_q(r) \cdot H_p(r) - H^\dagger_{\perp,q}(r) \cdot H_{\perp,p}(r) \right] dV ,
\]

where the domain of integration was changed from \( \Omega \) to \( R^3 \) because the bracketed integrands converge rapidly. The above renormalization procedure is different from Smith’s (11), who used an averaging scheme to render all integrals convergent. Instead, the procedure resembles that used in (21, 23) for expressing the energy stored in antenna fields. Just like for}
WS time delay matrix for scattering systems are energy-like overlap integrals of fields that exist throughout Ω for all possible port excitations. The renormalization procedure in (56) extracts the time delay caused by the scattering process from the total time waves naturally dwell within Ω (which tends to infinity as $R \to \infty$). Evaluation of $\mathbf{S}$ and $\mathbf{Q}$ at frequency $\omega$ once again permits the computation of $\mathbf{S}$’s frequency derivative via (55).

C. Radiating Systems

1) Setup: Radiating systems are hybrids of the guiding and scattering systems considered in Secs. III-A and III-B. Consider a radiating system composed of lossless antennas that are fed by PEC waveguides of uniform cross section (Fig. 4). The antennas and their feeds reside in free space and are circumscribed by a sphere of radius $a$. Let $\Omega$ denote the volume of a concentric sphere of radius $R = a$, and let $d\Omega$ denote the union of the concentric sphere’s surface and the waveguides’ physical ports. On $d\Omega$, fields interacting with the system can be characterized in terms of $M_r = M_g + M_s$ propagating modes.

Assume that the system is excited by an incoming unit-power field with transverse electric and magnetic components $\mathbf{E}^I_p(r, \omega)$ and $\mathbf{H}^I_p(r, \omega)$ near $d\Omega$. If $p \leq M_g$, then $\mathbf{E}^I_p(r, \omega)$ and $\mathbf{H}^I_p(r, \omega)$ are given by (27a)-(27b). Likewise, if $p > M_g$, then $\mathbf{E}^I_p(r, \omega)$ and $\mathbf{H}^I_p(r, \omega)$ are given by (45a)-(45b).

Let $\mathbf{E}_p(r, \omega)$ and $\mathbf{H}_p(r, \omega)$ denote the electric and magnetic fields throughout $\Omega$ due to $\mathbf{E}^I_p(r, \omega)$ and $\mathbf{H}^I_p(r, \omega)$. Near $d\Omega$, the total transverse electric and magnetic fields $\mathbf{E}_p(r, \omega)$ and $\mathbf{H}_p(r, \omega)$ are given by (29a)-(29b) with the transverse outgoing fields given by

$$\mathbf{E}_p(r, \omega) = \sum_{m=1}^{M_g} \mathbf{S}_{mp}(\omega) n_m(\omega) e^{-j\beta_m(\omega)w} \mathbf{X}_m(u, v)$$

$$+ \sum_{m=M_g+1}^{M_r} \mathbf{S}_{mp}(\omega) e^{-j(k\omega)\hat{r}} \mathbf{X}_m^*(\theta, \phi)$$

(57a)

$$\mathbf{H}_p(r, \omega) = -\sum_{m=1}^{M_g} \mathbf{S}_{mp}(\omega) \frac{n_m(\omega)}{Z_m(\omega)} e^{-j\beta_m(\omega)w}$$

$$\cdot \left( -\hat{w} \times \mathbf{X}_m(u, v) \right)$$

$$- \sum_{m=M_g+1}^{M_r} \mathbf{S}_{mp}(\omega) \frac{n_m(\omega) e^{-j(k\omega)\hat{r}}}{Z_m}$$

$$\cdot \left( -\hat{r} \times \mathbf{X}_m^*(\theta, \phi) \right)$$

(57b)

All modal quantities in (57a)-(57b) were defined in Sec. III-A and Sec. III-B.

2) The WS Relationship: The WS relationship can be derived by following the same procedure as in Secs. III-A and III-B. First, consider two sets of fields: $\{\mathbf{E}_p(r), \mathbf{H}_p(r)\}$ and $\{\mathbf{E}_q(r), \mathbf{H}_q(r)\}$. The derivation in (31a)-(36) continues to hold true with $\hat{w} \to \hat{r}$ if $r$ is on the spherical surface of radius $R$. Expressions for $\mathbf{E}^I_p(r), \mathbf{H}^I_p(r), \mathbf{E}^I_q(r),$ and $\mathbf{H}^I_q(r)$ on $d\Omega$ can be derived using (29a)-(29b), (27a)-(27b), (45a)-(45b), and (57a)-(57b). Substituting these expressions into (36) and simplifying the result yields

$$\mathbf{Q}_{qp} = J \sum_{m=1}^{M_r} \mathbf{S}^\dagger_{qm} \mathbf{S}_{mp} \left( \frac{1}{Z_p} \right)^\dagger \left( n_p \right)^2 \delta_p \leq M_s$$

$$+ \frac{1}{2} \sum_{m=1}^{M_g} \mathbf{S}^\dagger_{qm} \left( \frac{1}{Z_q} \right)^\dagger \left( n_q \right)^2 \delta_q \leq M_s$$

$$+ R \sqrt{\mu_0 \varepsilon_0} \sum_{m=M_g+1}^{M_r} \mathbf{S}_{mp} \mathbf{S}^\dagger_{qm}$$

(58)

where $\mathbf{Q}_{qp}$ is still given by (40), and $\delta_f = 1$ if $f$ is true and is 0 otherwise. Note that the second and third terms on the RHS of (58) are due to non-TEM waveguide modes and resemble the second and third terms on the LHS of (41). The final term on the RHS of (58) is proportional to $R$ and resembles the second term on the RHS of (51). As in Sec. III-B a WS relationship that is independent of $R$ is obtained by introducing $\tilde{\mathbf{Q}}_{qp, \infty}$, which is still given by (52). Substituting $\mathbf{E}_p(r)$ and $\mathbf{H}_p(r)$ into (52) and evaluating the resulting integral yields

$$\tilde{\mathbf{Q}}_{qp, \infty} = R \sqrt{\mu_0 \varepsilon_0} \left( \delta_q \delta_q \geq M_s \right)$$

$$+ \sum_{m=M_g+1}^{M_r} \mathbf{S}_{mp} \mathbf{S}^\dagger_{qm}$$

(59)

which is identical to (53) for the case of $M_g = 0$. Using (59) and (58) results in

$$\tilde{\mathbf{Q}}_{qp} = \frac{1}{2} \sum_{m=1}^{M_g} \mathbf{S}^\dagger_{qm} \left( \frac{1}{Z_p} \right)^\dagger \left( n_p \right)^2 \delta_p \leq M_s$$

$$+ \frac{1}{2} \sum_{m=1}^{M_g} \mathbf{S}^\dagger_{qm} \left( \frac{1}{Z_q} \right)^\dagger \left( n_q \right)^2 \delta_q \leq M_s$$

$$- \sum_{m=M_g+1}^{M_r} \mathbf{S}_{mp} \mathbf{S}^\dagger_{qm}$$

(60)

where $\tilde{\mathbf{Q}}_{qp}$ is given by the same as expression as $\mathbf{Q}_{qp}$ in (56). In matrix form, (60) reads

$$\mathbf{Q}^r = j \mathbf{S}^\dagger \mathbf{S}'$$

(61)
where  
\[ Q^e = \tilde{Q}^e + \frac{j}{2} \begin{bmatrix} S_{gg} & S_{gs}^\dagger \end{bmatrix} \begin{bmatrix} D_g & 0 \\ 0 & 0 \end{bmatrix} \]

and  
\[ Q^h = \tilde{Q}^h - \frac{j}{2} \begin{bmatrix} D_g & 0 \\ 0 & 0 \end{bmatrix} \]

with the \( M_g \times M_g \) diagonal matrix \( D_g \) still given by (44). In (62), the scattering matrix \( S \) was decomposed into four blocks that separate the waveguide and free-space ports. Equations (61) and (62) show that the WS time delay matrix for radiating systems can be computed from knowledge of the fields throughout \( \Omega \) due to all possible port excitations. Once again, knowledge of \( S \) and \( Q \) at frequency \( \omega \) permits the computation of \( S' \), which in turn can be used to compute the frequency derivative of the antenna impedance matrix (see Sec. III-E below).

D. Alternative Expressions for \( Q \)

In the previous sections, the entries of the WS time delay matrix for guiding, scattering, and radiating systems were expressed in terms of integrals of both electric and magnetic fields over \( \Omega \). By manipulating Maxwell’s equations (31a)–(31b) and their frequency derivatives (32a)–(32b), the following alternatives to (36) may be derived:

\[ \int_{\Omega} \tilde{w} \cdot \left[ \nabla \times \mathcal{E}^e_p(r) \times \nabla \times \mathcal{E}^e_q(r) \right] dS = \frac{2k^2}{\omega} \int_{\Omega} \mathcal{E}^e_p(r) \cdot \mathcal{E}^e_q(r) dV \]  

(63a)

\[ \int_{\Omega} \tilde{w} \cdot \left[ \nabla \times \mathcal{H}^e_p(r) \times \nabla \times \mathcal{H}^e_q(r) \right] dS = \frac{2k^2}{\omega} \int_{\Omega} \mathcal{H}^e_p(r) \cdot \mathcal{H}^e_q(r) dV \]  

(63b)

The above expressions only require integration of electric or magnetic fields.

Using (63a) instead of (36) to derive the WS relationship for guiding system results in  
\[ Q^{g,e} = jS^\dagger S' \]  

(64)

where  
\[ Q^{g,e} = \tilde{Q}^e - \frac{j}{2} \left( D_g + \frac{1}{\omega} I_{M_g} \right) S \]

\[ + \frac{j}{2} S^\dagger \left( D_g + \frac{1}{\omega} I_{M_g} \right) \]

(65)

and  
\[ \tilde{Q}^h = \varepsilon \int_{\Omega} \mathcal{E}^e_p(r) \cdot \mathcal{E}^e_q(r) dV \]  

(66)

Likewise, using (63b) to derive the WS relationship for guiding systems yields  
\[ Q^{g,h} = jS^\dagger S' \]  

(67)

where  
\[ Q^{g,h} = \tilde{Q}^h - \frac{j}{2} \left( D_g - \frac{1}{\omega} I_{M_g} \right) S \]

Note that (41) can be retrieved by adding (65) and (68).

Similar WS relations involving \( Q^e \) and \( Q^h \) can be derived for scattering and radiating systems starting from (65) and (68).

Expressions for \( Q \) involving only electric or magnetic fields are useful in many computational electromagnetics settings that model only one field type.

E. Impedance Formulation

WS relationship (1) can be used to obtain the frequency derivative of a system’s impedance matrix.

Recall that the \( M \times M \) scattering matrix relates the amplitudes of incoming and outgoing waves \( a \) and \( b \) as  
\[ b = Sa \]  

(70)

The impedance matrix, on the other hand, relates ports voltages and currents \( v \) and \( i \)  
\[ v = Z i \]

(71)

where  
\[ v = N (a + b) \]  

(72a)

\[ i = NY (a - b) \]  

(72b)

Here, \( N \) and \( Y \) are diagonal matrices whose \((p, p)\)-th entries are mode power normalization constants \( n_p \) (or \( n \)) and admittances \( Z_p^{-1} \) (or \( Z^{-1} \)), respectively.

Defining \( \tilde{Z} = N^{-1} Z N Y \), it follows from (70)–(72b) that  
\[ \tilde{Z} = (I_M + S) (I_M - S)^{-1} \]  

(73)

Alternatively, \( S \) may be written in terms of \( \tilde{Z} \) as  
\[ S = \left( \tilde{Z} + I_M \right)^{-1} \left( \tilde{Z} - I_M \right) \]  

(74)

Taking the frequency derivative of (74) by applying the chain rule yields  
\[ \tilde{Z}' = \left( \tilde{Z} + I_M \right) S' (I_M - S)^{-1} \]  

(75)

Finally, using the WS relationship (1) in (75) yields the frequency derivative of the impedance matrix in terms of the WS time delay matrix \( Q \) and the scattering matrix \( S \)  
\[ \tilde{Z}' = -j \left( \tilde{Z} + I_M \right) S Q (I_M - S)^{-1} \]  

(76)

or  
\[ Z' = -j N \left( \tilde{Z} + I_M \right)^{-1} S Q (I_M - S)^{-1} Y^{-1} N^{-1} \]  

(77)

For radiating systems, the frequency derivative of the antenna impedance matrix is easily extracted from (77).
IV. ILLUSTRATIVE EXAMPLES

This section applies WS methods to the characterization of fields with well-defined time delays in guiding and scattering systems. It also demonstrates the use of equations (61) and (77) to compute the frequency derivative of antenna impedance matrices. The examples in this sections are merely illustrative in nature. While WS methods have applications in many branches of electromagnetics, their treatment is beyond the scope of this paper.

A. Guiding Systems

1) A PEC-Terminated Rectangular Waveguide: This first didactic example verifies the WS relationship (42) for an air-filled rectangular waveguide with dimensions $a \times b$ and length $l$ that is terminated in a short. Since both $S$ and $S'$ are diagonal, so is $Q$. Assume the waveguide is excited by a unit-power $TM_{mn}$ mode. The total electric and magnetic fields inside the waveguide are

\[
\mathbf{E}_p(u,v,w) = \mathbf{E}_{p,\parallel}(u,v,w) + \mathbf{E}_{p,w}(u,v,w) \hat{\mathbf{w}} \tag{78a}
\]

\[
\mathbf{H}_p(u,v,w) = \mathbf{H}_{p,\parallel}(u,v,w), \tag{78b}
\]

where $\mathbf{E}_{p,1}(u,v,w)$ and $\mathbf{H}_{p,1}(u,v,w)$ are given by (29a) and (29b) with mode profile

\[
\mathbf{X}_p(u,v) = \frac{1}{\sqrt{k^2 ab/4}} \left[ \sin \left( \frac{n\pi u}{a} \right) \cos \left( \frac{m\pi v}{b} \right) \hat{u} + \sin \left( \frac{m\pi u}{a} \right) \cos \left( \frac{n\pi v}{b} \right) \hat{v} \right]. \tag{79}
\]

wave impedance $Z_p = \beta_p \eta / k$, propagation constant $\beta_p = \sqrt{k^2 - k_c^2}$, cutoff wave number $k_c = \sqrt{(m\pi/a)^2 + (n\pi/b)^2}$, and a diagonal scattering matrix $S$ with $(p,p)$-th entry

\[
S_{pp} = -e^{-2j\beta_p l}. \tag{80}
\]

In (78a), the longitudinal component of $\mathbf{E}_p(r)$ reads

\[
\mathbf{E}_{p,w}(u,v,w) = \frac{j \eta k_c}{\beta_p} \frac{2}{\sqrt{ab}} \sin \left( \frac{m\pi u}{a} \right) \sin \left( \frac{n\pi v}{b} \right) \left[ e^{j\beta_p w} - S_{pp} e^{-j\beta_p w} \right]. \tag{81}
\]

Substituting (78a)-(78b) into (40) and evaluating the resulting integral yields

\[
Q_{pp}^g = \frac{2l}{\cos \theta_p} \frac{\sqrt{\mu_0 \varepsilon_0}}{1 + \omega^2 \tan^2 \theta_p \sin(2\beta_p l)} \tag{82}
\]

where $\cos \theta_p = \beta_p / k$. The last two terms on the LHS of (41) are easily shown to equal the negative of the second term on the RHS of (82), yielding

\[
Q_{pp}^g = \frac{2l}{\cos \theta_p} \frac{\sqrt{\mu_0 \varepsilon_0}}{\cos \theta_p}. \tag{83}
\]

Using (83) and the fact that

\[
S'_{pp} = -j \frac{2l}{\cos \theta_p} \frac{\sqrt{\mu_0 \varepsilon_0}}{S_{pp}}, \tag{84}
\]

which follows from (80), it is easily verified that the WS relationship (42) holds true, i.e. that $Q_{pp}^g = j S_{pp} S'_{pp}$. Not surprisingly, the time delay $Q_{pp}^g$ in (83) equals the time needed for light to travel the length of the zig-zag ray path from the port to the short and back, traditionally associated with the $TM_{mn}$ mode. The above analysis can be repeated for $TE_{mn}$ modes with identical results.

2) Notched Waveguide: Consider the waveguide shown in Fig. 5 due to the excitation described below, the structures height along $v$ is immaterial. The system is excited by $\nu$-polarized electric fields with mode profiles

\[
\mathbf{X}_m(u) = \sqrt{\frac{2}{a_{\text{port}}} \sin \left( \frac{m\pi u}{a_{\text{port}}} \right)} \hat{v}. \tag{85}
\]

and frequency $f = 30$ GHz. Physical ports 1 and 2 support 24 and 28 propagating modes, respectively. In contrast to the previous example, all modes couple; $S$ and $Q$ therefore are dense $M_g \times M_g$ matrices with $M_g = 52$. $S$ and $Q$ are computed using a third order accurate finite element method, and the WS relationship holds true to 7 digits ($|Q - j S S'| / |Q| \approx 10^{-7}$). Fig. 6 shows the electric field distribution $\mathbf{E}_{e,1}(u,w)$ throughout $\Omega$ due to excitation of physical port 1 by an incoming field with mode profile (85) with $m = 1$. Many different scattering phenomena are in play, causing the field to be devoid of obvious structure.

Next, the WS time delay matrix is diagonalized (see (16)). Fields associated with select WS modes (port excitations specified by columns of $W$ (see (16)) are shown in Fig. 7. The structured nature of these fields (relative to that in Fig. 3) is immediately apparent. Time delays ($Q$'s diagonal elements) are converted to equivalent spatial shifts measured in centimeter (100 times the eigenvalue multiplied by the free-space speed of light) (Fig. 8). Four different delay/shift regimes are observed (a-d).

a. WS modes 1-14. Fields originating in, and reflecting back to physical port 1, Consider WS mode #1 shown in Fig. 7a, this mode is characterized by a shift of 30.2 cm, or just over twice the distance from physical port 1 to wall A, representing the shortest possible path a field originating from either aperture can take before exiting the system. Other modes in this category, e.g. WS modes #3 and #5,
share similar characteristics but experience slightly larger delays/shifts than WS mode #1 as they travel at a (small) angle w.r.t. the \( w \)-axis.

b. WS modes 15-22. Fields originating in physical port 1 and exiting though physical port 2 (and vice versa). Consider WS mode #15 shown in Fig. 7d; this mode is characterized by a shift of 48.0 cm, or 0.5 cm more than the structures physical length of 47.5 cm. After all modes originating in, and reflecting back to, physical aperture 1 have been exhausted, the shortest possible time a signal can dwell in the system is by traversing the entire cavity (from physical port 1 to 2, or vice versa), avoiding contact with walls A and B. Other modes in this category, e.g. WS mode #18, share similar characteristics but experience slightly larger delays than WS mode #15 as they travel at a (small) angle w.r.t. the \( w \)-axis.

c. WS modes 23-40. Fields originating in, and reflecting back to, physical port 2. These modes are similar to WS modes 1-14, but originate in physical aperture 2 and bounce off wall B. The shift of WS mode #23 is 58.1 cm, or slightly higher than twice the distance from physical port 2 to wall B and back.

d. WS modes 41-52. Highly resonant fields, traveling at steep angles w.r.t. the \( w \)-axis. These modes originate either in physical aperture 1 or 2 and are characterized by large time delays.

**B. Scattering Systems**

1) PEC Strip: Consider the strip shown in Fig. 9a. The strip is 8 cm wide and centered about the origin (the location of the strip relative to the origin affects the WS time delays, as they are derived from scattering matrices defined on spheres/cylinders centered at the origin). The strip is illuminated by \( \text{TM}_z \) fields at \( f = 30 \text{ GHz} \). The strips dense \( S \) and \( Q \) matrices are computed using an integral equation code considering \( M_s = 100 \) excitations by cylindrical harmonics; because \( 100 \gg 2k \text{ (width of strip)}/2 \approx 50 \), these excitations adequately resolve all the systems degrees of freedom. The WS relationship (55) is found to hold true to 5 digits (\(|Q - jS^\dagger S'|/|Q| \approx 10^{-5}\)). Very much like the fields due to a waveguide port excitation in Fig. 6, all cylindrical harmonics excite both surface and edge scattering phenomena, causing each of them to experience a different delay (as defined by (12)).

Next, the WS time delay matrix is diagonalized (see (16)). Total fields (i.e. sums of incident and scattered fields) and the strips currents for select WS modes (port excitations specified by columns of \( W \) (see (16)) are shown in Figs. 9a-9f. Time delays (\( Q \)'s diagonal elements) are converted to equivalent spatial shifts just as was done for the notched waveguide in Sec. IV-A2 (Fig. 10). Two different delay/shift regimes are observed.

a. WS edge modes. The total field and current of WS mode #1 are shown in Figs. 9a and 9d; this mode is characterized
Fig. 9: Field distribution (Figs. (a)–(c)) and magnitude of current (Figs. (d)–(f)) for select WS modes of the PEC strip. Distance is measured from the left edge of the strip.

Fig. 10: Spatial shifts (delays) of WS modes for strip. Shifts are only shown for the first 10 modes as all others are zero.

by a (negative!) shift of \(-8.1\, \text{cm}\), or just over twice the distance from the strips edge to the origin. The total field and currents are concentrated near the strips edges; the center of the strip is virtually quiescent (the same to a large extent holds true for the incident and scattered field (not shown)). The total fields associated with this mode are concentrated along the strips axis; the two “beams” (coming in predominantly from the \(\pm x\)-axis) that excite the strip reflect back upon hitting the edge, causing them to travel (roughly) 8 cm less (from the edge to the origin and back) than a field that does not interact with the strip. WS mode #1 is characterized by an even current distribution on the strip. WS mode #2 (not shown) is very similar to mode #1, except that it supports odd currents and fields.

b. WS Geometric Optics (GO)-like modes. The fields and current distributions of WS modes #3 and #16 are shown in Figs. 9b–9c and 9e–9f. These modes are GO-like in nature, and consist of beam-like incident fields that avoid the strips edges while exciting quasi-periodic currents on the strip, producing beams that specularly reflect away from the strip. These modes are characterized by very small time delays/spatial shifts as the GO rays round trip time is virtually identical to that experienced by a wave that does not interact with the strip. For each WS mode hitting the strip from the top, there is one hitting it from the bottom (not shown; while in principle these modes are degenerate, the symmetry was broken here by positioning the strip a very small distance above the origin). Note that the WS GO fields extremize group time delays (see Sec. II-C), in accordance with the generalized Fermat principle for optics and high-frequency electromagnetic fields [28].

2) PEC Cavity: Consider the square cavity shown in Fig. 11a. The cavities base is 8.24 cm long and contains a hole
of width 1.6 cm. The cavity is centered about the origin and illuminated by TM$_z$ fields at $f = 30$ GHz. The cavity's dense $S$ and $Q$ matrices are computed using an integral equation code using $M_s = 120$ cylindrical harmonic excitations, which adequately resolve all of the system's degrees of freedom. The WS relationship (55) is found to hold true to 4 digits ($\left| Q - j S^\dagger S \right| / \left| Q \right| \approx 10^{-4}$). The WS time delay matrix is diagonalized (see (16)) and total fields and currents for select WS modes are shown in Figs. 11a–11h. Spatial shifts computed from $Q$'s eigenvalues are shown in (Fig. 12). Three different delay/shift regimes are observed.

a. **WS corner modes.** The total field and current of WS mode #1 are shown in Figs. 11a and 11e; just like for the strip, this mode is characterized by a negative spatial shift. The total field and currents are concentrated near the cavity's four corners; the cavity's facets and aperture are virtually quiescent. The cavity supports a total of four corner modes (inset in Fig. 12) with different current and field symmetries across the origin.

b. **WS GO-like modes.** The fields and current distributions of WS modes #5 and #15 are shown in Figs. 11b–11c and 11f–11g. These modes are GO-like in nature and consist of beam-like incident fields that avoid the cavity's corners and aperture while exciting quasi-periodic currents on, and causing specular reflections from, its facets. These modes spend less time in the system than free space fields that do not interact with the system, causing them to be characterized by negative time delays/shifts.

c. **WS cavity modes.** The total field and current of WS mode #120 are shown in Figs. 11d and 11h. This mode is excited by a beam focused on the cavity's aperture, exciting a strong quasi-resonant field in its interior. This field bounces back and forth many times between the cavity's walls, causing it to experience a large positive time delay before escaping. The cavity supports a total of four such modes, each characterized by a different aperture field distribution. All cavity modes are characterized by large currents near the aperture and virtually quiescent total fields on the cavity's exterior walls away from the aperture.

The WS modes in region d are composed of fields that do not interact with the cavity. Indeed, because $M_s = 120 > 2k\sqrt{2}($width of cavity$)/2$, many cylindrical harmonics do not reach the cavity and their WS combinations do not experience any time delay.
Consider two strips with length 15 cm and width 0.5 cm that are separated by 15 cm. Both dipoles are center-fed by a voltage source. Matrices $Q$ and $S$ are computed using an integral equation code with $M_g = 2$ TEM waveguide ports and $M_s = 198$ scattering ports (many of which are not excited). Figure 13 shows the self and mutual input impedance of the two-element dipole array in the frequency band 0.5–2.5 GHz.

The frequency derivative of the antenna impedance matrix is obtained in two ways: (i) using the WS relationship, i.e. by computing $Q$ from the field in the system for all port excitations using (62), computing $S'$ using (61), and finally using (77) to compute $Z'$, which contains a 2 × 2 block with the frequency derivatives of antenna port self and mutual impedances, and (ii) via a finite difference in frequency approximation. Fig. 14 shows the frequency derivative of the self and mutual impedances of the two-dipole system computed via finite difference method (–) and WS theory (×).

V. CONCLUSIONS AND AVENUES FOR FUTURE RESEARCH

This paper presented a WS theory for electromagnetic fields. Following a review of basic WS concepts, closed-form expressions for the entries of the WS time delay matrix involving energy-like overlap integrals of port-excited fields were presented. Furthermore, the nature of WS modes in guiding and scattering systems was elucidated, and the use of the WS time delay matrix for characterizing the frequency sensitivity of antenna impedance matrices was illustrated.

Applications of the WS time delay concepts abound in electromagnetics. The authors foresee many more uses of WS methods, including

- the design of multiport and distributed system that exhibit precise delays, including filter banks, antenna arrays, and (meta)materials;
- the systematic phenomenological classification of fields interacting with guiding, scattering, and radiating systems, e.g. in the identification of scattering centers for radar cross section analysis;
- the broadband characterization of multiport antenna systems;
- the construction of fast frequency-sweep computational methods for characterizing broadband electromagnetic phenomena.

Work on several of the above topics is in progress and will be reported in future papers.

APPENDIX A

WAVEGUIDE MODES

Fields inside waveguides can be expanded in terms of TE, TM, and TEM modes. Consider the Helmholtz equation

$$\nabla^2 \Phi_p(u, v) + k^2_{c,p} \Phi_p(u, v) = 0. \quad (86)$$
The TM mode profile is expressed as [29]

$$\mathbf{X}_{p,TM}(u,v) = -\nabla \mathbf{f}(u,v), \quad (87)$$

where $\mathbf{f}(u,v)$ is the solution to (86) subject to boundary condition

$$\Phi_p(u,v) = 0 \quad (88)$$
on the cross-section’s boundary $dS$. The TE mode profile is given by

$$\mathbf{X}_{p,TE}(u,v) = -\hat{\mathbf{w}} \times \nabla \Phi_p(u,v) \quad (89)$$

where $\Phi_p(u,v)$ is the solution to (86) subject to the boundary condition

$$\frac{\partial \Phi_p(u,v)}{\partial n} = 0 \quad (90)$$
on $dS$, where $(\partial/\partial n)$ denotes derivative w.r.t. the outward normal direction. Finally, the TEM mode profile is given by

$$\mathbf{X}_{p,TEM}(u,v) = -\nabla \Phi_p(u,v) \quad (91)$$

subject to the boundary condition

$$\frac{\partial \Phi_p(u,v)}{\partial \tau} = 0 \quad (92)$$

where $(\partial/\partial n)$ denotes derivative w.r.t. the tangential direction. In (86), $k_{c,p}$ is the cutoff wave number and $\beta_p = \sqrt{k^2 - k_{c,p}^2}$ is the propagation constant. The wave impedance is

$$Z_p = \begin{cases} \frac{k \beta_p}{\eta} & \text{if } p \text{ is a TE mode} \\ \eta & \text{if } p \text{ is a TM mode} \end{cases} \quad (93)$$

APPENDIX B

MODES FOR SCATTERING SYSTEMS

Free-space electric fields in spherical coordinate systems can be expanded in terms of vector spherical harmonics (VSH). Electric fields derived from VSH are either TE or TM (to $\mathbf{f}$) in nature. Incoming TE and TM fields can be expressed as [30], [31]

$$\mathbf{I}_{lm,TE}(r, \theta, \phi) = j^{l+1} k \eta^{1/2} h_l^{(1)}(kr) \mathbf{X}_{1lm}(\theta, \phi) \quad (94)$$

$$\mathbf{I}_{lm,TM}(r, \theta, \phi) = \frac{j k^{1/2} \partial kr h_l^{(1)}(kr)}{r} \frac{\partial}{\partial \phi} \mathbf{X}_{2lm}(\theta, \phi) + \frac{\sqrt{l(l+1)} j^{1/2} \eta h_l^{(1)}(kr)}{r} \mathbf{X}_{3lm}(\theta, \phi), \quad (95)$$

where $l = \{1, \ldots, \infty \}$ and $m = \{-l, \ldots, l\}$ are mode indices, $\eta = \sqrt{\mu_o/\varepsilon_o}$ is the free-space intrinsic impedance, and $h_l^{(1)}(z)$ is the $l$-th order spherical Hankel function of the first kind. The vector spherical harmonic $\mathbf{X}_{ilm}(\theta, \phi)$ is

$$\mathbf{X}_{1lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (r Y_{lm}(\theta, \phi)) \quad (96a)$$

$$\mathbf{X}_{2lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}(\theta, \phi) \quad (96b)$$

$$\mathbf{X}_{3lm}(\theta, \phi) = \hat{r} Y_{lm}(\hat{r}) \quad (96c)$$

where the scalar spherical harmonic $Y_{lm}(\theta, \phi)$ is [30], [31]

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^m_m(\cos \theta) e^{im\phi} \quad (97)$$

Here, $P^m_m(x)$ is the associated Legendre polynomial of degree $l$ and order $m$ [32].

As $r \to \infty$, incoming VSH electric fields [94]–[95] can be approximated as

$$\lim_{r \to \infty} \mathbf{I}_{lm,TE}(r, \theta, \phi) \approx \eta^{1/2} \frac{e^{i k r}}{r} \mathbf{X}_{1lm}(\theta, \phi) \quad (98a)$$

$$\lim_{r \to \infty} \mathbf{I}_{lm,TM}(r, \theta, \phi) \approx \eta^{1/2} \frac{e^{i k r}}{r} \mathbf{X}_{2lm}(\theta, \phi) \quad (98b)$$

where the large argument approximation of the spherical Hankel functions was used [32]. Note that $\mathbf{X}_{1lm}$ and $\mathbf{X}_{2lm}$ are transverse to $\mathbf{f}$. Equation (98a) directly follows from (98b). The outgoing vector spherical waves in (48a) are obtained by complex conjugating $\mathbf{I}_{lm,TE}(r, \theta, \phi)$ and $\mathbf{I}_{lm,TM}(r, \theta, \phi)$.

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