Characterisation of orthogonal perfect fluid cosmological spacetimes

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Abstract

We consider the general orthogonal metric separable in space and time variables in comoving coordinates. We then characterise perfect fluid models admitted by such a metric. It turns out that the homogeneous models can only be either FLRW or Bianchi I while the inhomogeneous ones can only admit $G_2$ (two mutually as well as hypersurface orthogonal spacelike Killing vectors) isometry. The latter can possess singularities of various kinds or none. The non-singular family is however unique and cylindrically symmetric.

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1. Introduction

It is to the credit of general relativity (GR) that the study of the Universe as a whole has become one of the most active areas of scientific research and it goes by the name relativistic cosmology. The models of the Universe we consider are given by the exact solutions of Einstein’s field equations for gravitation. The matter content of the Universe is taken as perfect fluid. The generally accepted model is the Friedman-Lemaitre-Robertson-Walker (FLRW) model which describes a homogeneous, isotropic and expanding Universe. Howsoever successful this model be, its homogeneity and isotropy are very special properties which could by no means be considered as generic enough for the Universe. It would hence be important to find cosmological solutions of Einstein’s equations without imposing the conditions of homogeneity and isotropy.

The first step in this direction came in the form of Bianchi models that are homogeneous but anisotropic. Inhomogeneity was the main concern of what is known as $G_2$ cosmologies [1-3]. By $G_2$ we mean a spacetime that admits a two parameter orthogonally transitive group of isometries, that is, there exist two spacelike Killing vectors that are mutually as well as hypersurface orthogonal. In proper coordinates the metric depends upon only one space variable. Inhomogeneous models have been considered by some authors [4-7] of which the one due to Senovilla [6] is the most remarkable for being singularity free and yet having an acceptable physical behaviour. This was a startling result for it was generally believed on the strength of the powerful singularity theorems [8] that the occurrence of singularity is inescapable in GR so long as reasonable physical
conditions are satisfied. Here was apparently a counterexample to the theorems. Subsequent to the discovery it was found that the theorems became inapplicable because the solution in question did not obey one of the assumptions; the existence of causal compact trapped surfaces. Not only were all physical and and geometrical parameters are finite and regular for whole of the spacetime, the metric was shown to be geodesically complete [9] exhibiting the absence of a singularity of any kind.

All prior attempts to manage the big-bang singularity or to construct a non-singular cosmological model involved either unphysical behaviour for matter, like \( p < 0 \), or quantum effects and new fields or modification of GR [10,11]. Senovilla’s [6] was the first exact solution of Einstein’s equations free of any kind of singularity and possessing all physically acceptable properties. Then the question arose, was it an isolated solution or did there exist a family of non-singular models? Ruiz and Senovilla [12] considered the general \( G_2 \) metric separable in space and time in comoving coordinates and identified a large family of non-singular spacetimes with cylindrical symmetry. Is cylindrical symmetry necessary for a non-singular cosmological model? This is a pertinent question to be addressed next.

In this paper we consider a general orthogonal metric separable in space and time in comoving coordinates and examine, in all generality, the permissible fluid models. We show that the already identified non-singular family [12] is unique. It turns out that the requirement of perfect fluid imposes \( G_3, G_6 \) and \( G_2 \) isometrics on the spacetime; the first two are homogeneous Bianchi I and FLRW models while the last one alone can sustain inhomogeneity. It turns out that inhomogeneous models could be with or without singularity. However, the non-singular family is
unique and is cylindrically symmetric. We can thus characterise all the fluid models described by an orthogonal and separable metric [13].

In Sec 2 we set up the field equations for a perfect fluid for an orthogonal metric and in Sec 3 we prove a theorem characterising perfect fluid cosmological models and establishing uniqueness of the non-singular family. We conclude with a discussion.

2. Field Equations

We consider the orthogonal metric,

\[ ds^2 = Ddt^2 - Adx^2 - Bdx^2 - Cdx^3 \]  

(2.1)

with velocity field given by \( u = \sqrt{D}dt \). In comoving coordinates we assume the metric to be separable, ie \( A = A(x_\alpha)A(t) \) etc. The separability can invariantly be characterised by (i) \( \theta,_{\alpha} = \theta \dot{u}_\alpha \) and (ii) \( \sigma/\theta \) being constant over the 3-hypersurface. This can easily be verified from the following expressions for the kinematic parameters

\[ \theta = \frac{1}{2\sqrt{D}} \left( \frac{A_0}{A} + \frac{B_0}{B} + \frac{C_0}{C} \right), \]  

(2.2)

\[ \sigma^2 = \frac{1}{36D} \left[ \left( \frac{B_0}{B} + \frac{C_0}{C} - \frac{2A_0}{A} \right)^2 + \left( \frac{C_0}{C} + \frac{A_0}{A} - \frac{2B_0}{B} \right)^2 + \left( \frac{A_0}{A} + \frac{B_0}{B} - \frac{2C_0}{C} \right)^2 \right] \]  

(2.3)

and

\[ \dot{u}_\alpha = -\frac{D_\alpha}{2D}, \]  

(2.4)
where $A_0 = \partial A/\partial t$ and $D_\alpha = \partial D/\partial x^\alpha$.

We note a general result arising out of the following two relations [14],

\[
\theta,\alpha = \frac{3}{2} \left[ (\sigma^i + \omega^i)_;i - (\sigma_{\alpha i} + w_{\alpha i})\dot{u}^i \right] \tag{2.5}
\]

\[
= \theta \dot{u}_\alpha + \frac{1}{\sqrt{g_{00}}} \left( \ln \sqrt{|g/g_{00}|} \right), \tag{2.6}
\]

where $\theta, \sigma, \omega, \dot{u}_\alpha$ are the kinematic parameters; expansion, shear, rotation and acceleration, $\dot{u}_\alpha = u_{\alpha;i}u^i$. Note that we have assumed $u_i = \sqrt{g_{00}}\delta_i^0$.

We infer from the above relations:

**Lemma:** In the absence of shear and vorticity, the expansion of fluid is constant over the 3-space orthogonal to the fluid congruence and, further, the acceleration also vanishes when the quantity $g/g_{00}$ is a separable function of space and time in comoving coordinates.

**Corollary:** For the vorticity free spacetime with separability (as is the case for the metric (2.1)), acceleration can be non-zero only if shear is non-zero.

According to the Raychaudhuri equation [15], in the absence of vorticity acceleration is necessary for halting the collapse to avoid the singularity which in our case can only exist if shear is non-zero. Thus non-singular solutions represented by the metric (2.1) will always have to be both inhomogeneous and anisotropic.

Einstein’s field equations are

\[
R_{ik} - \frac{1}{2}Rg_{ik} = -8\pi T_{ik}, \tag{2.7}
\]
where for a perfect fluid

\[ T_{ik} = (\rho + p)u_i u_k - pg_{ik} \]  \hspace{1cm} (2.8)

The explicit expressions for \( T_{ik} \) [16] look quite formidable and rather intimidating. Fortunately, we have discovered an underlying order in them that allows us to write the rest of them from a given two (one each of diagonal and off diagonal) by prescribing the appropriate permutation rules. We begin with

\[ -32\pi AT_0^1 = -2 \left( \frac{B_0}{B} + \frac{C_0}{C} \right) + \frac{A_0}{A} \left( \frac{B_1}{B} + \frac{C_1}{C} \right) + \frac{B_0}{B} \left( -\frac{B_1}{B} + \frac{D_1}{D} \right) + \frac{C_0}{C} \left( -\frac{C_1}{C} + \frac{D_1}{D} \right) \] \hspace{1cm} (2.9)

\[ -32\pi T_1^1 = \frac{1}{A} \left[ \frac{B_1 C_1}{BC} + \frac{D_1}{D} \left( \frac{B_1}{B} + \frac{C_1}{C} \right) \right] 
+ \frac{1}{B} \left[ 2 \left( \frac{C_2}{C} + \frac{D_2}{D} \right) + \frac{C_2}{C} \left( -\frac{B_2}{B} + \frac{C_2}{C} \right) + \frac{D_2}{D} \left( -\frac{B_2}{B} + \frac{C_2}{C} + \frac{D_2}{D} \right) \right] 
+ \frac{1}{C} \left[ 2 \left( \frac{B_3}{B} + \frac{D_3}{D} \right) + \frac{B_3}{B} \left( \frac{B_3}{B} - \frac{C_3}{C} \right) \right] 
+ \frac{D_3}{D} \left( -\frac{B_3}{B} + \frac{C_3}{C} \right) + \frac{D_3}{D} \left( \frac{B_3}{B} - \frac{C_3}{C} + \frac{D_3}{D} \right) \right] 
+ \frac{1}{D} \left[ -2 \left( \frac{B_0}{B} + \frac{C_0}{C} \right) + \frac{B_0}{B} \left( \frac{B_0}{B} + \frac{C_0}{C} - \frac{D_0}{D} \right) - \frac{C_0}{C} \left( \frac{C_0}{C} - \frac{D_0}{D} \right) \right] \] \hspace{1cm} (2.10)

where a subscript denotes partial differentiation and here the assumption of separability is not effected.

The successive cyclic permutations \( A \rightarrow B \rightarrow C \rightarrow A \) and \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) will give \( T_0^2, T_0^3 \) from \( T_0^1 \); \( T_3^2, T_3^3 \) from \( T_2^1 \); and \( T_2^2, T_3^3 \) from \( T_1^1 \). To write \( T_2^1 \) from \( T_0^1 \), let \( 0 \rightarrow i2 \) (i.e. \( A_0 \rightarrow iA_2, T_0^1 \rightarrow iT_2^1 \)) and \( B \rightarrow C \rightarrow D \rightarrow B \) while \( T_0^0 \) follows.

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from $T^1_i$ for $2 \rightarrow 3 \rightarrow 1 \rightarrow i0 \rightarrow -2(T^1_i \rightarrow T^0_0)$ and $A \rightarrow D \rightarrow B \rightarrow C \rightarrow A$. Thus we can write all ten $T^k_i$, given the two, one each of diagonal and off diagonal.

3. Characterisation and Uniqueness

The conditions implied by the perfect fluid character of the source are: $T_{\alpha0} = 0$, $T_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $T^1_1 = T^2_2 = T^3_3$. We shall a priori assume no isometries of any kind except the separability of the metric (2.1) in comoving coordinates. The implementation of the fluid conditions will lead to $G_3$ (homogeneity) and $G_6$ (both homogeneity and isotropy) symmetries for homogeneous models and $G_2$ (only admitting two spacelike Killing vectors) symmetry for inhomogeneous models with or without singularity. Then all the fluid models described by the metric (2.1) are characterised and uniqueness of the already identified family of non-singular models is demonstrated. We prove the following theorem [13].

Theorem : The separable metric (2.1) can only represent the following kinds of perfect fluid cosmological models:

(a) if homogeneous, then Bianchi I and FLRW models,

(b) if inhomogeneous, then models with or without singularity.

Further the non-singular family as already identified [12] is unique.

Proof : Let us first of all write the three representative equations; $T_{10} = 0, T_{12} = 0$ and $T^1_1 = T^2_2$,

\[
\frac{A_0}{A} \left( \frac{B_1}{B} + \frac{C_1}{C} \right) + \frac{B_0}{B} \left( - \frac{B_1}{B} + \frac{D_1}{D} \right) + \frac{C_0}{C} \left( - \frac{C_1}{C} + \frac{D_1}{D} \right) = 0, \quad (3.1)
\]
\[-2\left(\frac{C_1}{C} + \frac{D_1}{D}\right)_2 + \frac{B_1}{B}\left(\frac{C_2}{C} + \frac{D_2}{D}\right) + \frac{C_1}{C}\left(-\frac{C_2}{C} + \frac{A_2}{A}\right) + \frac{D_1}{D}\left(-\frac{D_2}{D} + \frac{A_2}{A}\right) = 0, \quad (3.2)\]

\[
\begin{align*}
\frac{1}{A} & \left[ -2\left(\frac{C_1}{C} + \frac{D_1}{D}\right)_1 + \frac{C_1}{C}\left(\frac{A_1}{A} + \frac{B_1}{B} - \frac{C_1}{C}\right) + \frac{D_1}{D}\left(\frac{A_1}{A} + \frac{B_1}{B} - \frac{D_1}{D}\right) \right] \\
+ \frac{1}{B} & \left[ 2\left(\frac{C_2}{C} + \frac{D_2}{D}\right)_2 - \frac{C_2}{C}\left(\frac{A_2}{A} + \frac{B_2}{B} - \frac{C_2}{C}\right) + \frac{D_2}{D}\left(\frac{A_2}{A} + \frac{B_2}{B} - \frac{D_2}{D}\right) \right] \\
+ \frac{1}{C} & \left[ 2\left(\frac{B_3}{B} + \frac{A_3}{A}\right)_3 - \frac{A_3}{A}\left(\frac{A_3}{A} - \frac{C_3}{C} + \frac{D_3}{D}\right) + \frac{B_3}{B}\left(\frac{B_3}{B} - \frac{C_3}{C} + \frac{D_3}{D}\right) \right] \\
+ \frac{1}{D} & \left[ 2\left(\frac{A_0}{A} - \frac{B_0}{B}\right)_4 - \frac{A_0}{A}\left(\frac{A_0}{A} - \frac{C_0}{C} + \frac{D_0}{D}\right) - \frac{B_0}{B}\left(\frac{B_0}{B} - \frac{C_0}{C} + \frac{D_0}{D}\right) \right] = 0.
\end{align*}
\]

(3.3)

*I No isometry:* We assume no isometry to begin with. From eqn. (3.1), \(T_{\alpha 0} = 0,\)

will imply

\[
\frac{D_1}{D} = n_1 C_1, \quad \frac{D_2}{D} = n_1 C_2, \quad \frac{D_3}{D} = \frac{n_1}{k_1} B_3,
\]

(3.4)

where

\[
P_1 + k_1 Q_1 = n_1, \quad B_1 = k_1 C_1
\]

(3.5)

and

\[
P_1 = \frac{C_0/C - A_0/A}{B_0/B + C_0/C}, \quad Q_1 = \frac{B_0/B - A_0/A}{B_0/B + C_0/C}
\]

(3.6)

others follow by the cyclic permutation. Here \(n_{\alpha}\) and \(k_{\alpha}\) are constants.

We integrate the exact differential

\[
d(lnD) = (lnD)_1 dx_1 + (lnD)_2 dx_2 + (lnD)_3 dx_3
\]

(3.7)
along two different paths to give

\[ D(x_\alpha) = C^{n_1}(x_\alpha), \quad A = C^{1/k_2}(x_\alpha) \quad \text{and} \quad B = C^{k_1}(x_\alpha), \quad (3.8) \]

that is, the space dependence of the metric is all but determined. It remains to find \( C(x_\alpha) \). Further for the time dependence we get

\[ k_2(1 + k_1) \frac{A_0}{A} + (1 + k_2) \frac{B_0}{B} + (1 + k_1 k_2) \frac{C_0}{C} = 0 \quad (3.9) \]

and \( n_1 = 1 + k_1 + 1/k_2 \).

It may be noted that we have so far used only the three equations \( T_{\alpha 0} = 0 \) to obtain the relations (3.8) and (3.9) which leave only \( C(x_\alpha) \) and two of \( A(t), B(t), C(t) \) to be determined. Substituting (3.8) in (3.2) and its permutants leads to \( C(x_\alpha) = \text{const.} \). Thus the metric can only represent a homogeneous Bianchi I model.

On the other hand when \( A_0/A = B_0/B = C_0/C \), the shear vanishes and so does the acceleration. The spacetime is then both homogeneous and isotropic which determine FLRW uniquely [17]. When \( A_0/A = B_0/B \neq C_0/C \), eqn. (3.1) and its permutants will imply either Bianchi I or the spacetime admits a \( G_1 \) isometry. This is the case we consider next.

\( II \quad G_1 \) isometry : Let \( \partial/\partial x_3 \) be the spacelike Killing vector and hence the metric is a function of only two space variables, \( x_1 \) and \( x_2 \).
Note that eqn. (3.3) has the form

\[ \frac{f_1}{A(t)} + \frac{f_2}{B(t)} + \frac{f_3}{C(t)} = F(t) \]  

(3.10)

where \( f_1, f_2, f_3 \) are functions of \( x_\alpha \), containing respectively derivatives with respect to \( x_1, x_2 \) and \( x_3 \). In this case \( f_3 = 0 \) and \( T_{30} \equiv 0 \). Eqn. (3.10) gives rise to two cases: (i) \( A_0/A = B_0/B \neq C_0/C \) and (ii) \( A_0/A \neq B_0/B \neq C_0/C \) and \( f_1 = \text{const.}, f_2 = \text{const.} \). That means \( \sigma \) is non-zero to give non-zero acceleration.

It could be a viable case for a non-singular model as well. In (i) we can set \( A = B \) for \( A(t) = B(t) \) is implied by \( A_0/A = B_0/B \) (a constant multiple can always be absorbed) and \( A(x_\alpha) = B(x_\alpha) \) can be done by an appropriate coordinate transformation. Eqn. (3.1) implies \( C(x_\alpha) = D^\lambda(x_\alpha) \) and eqns. (3.1) – (3.3) give three equations to determine the space dependence of the metric. The Lie group analysis of the equations (see Appendix) leads to the inference that the functional dependence can only occur in the form \( A(x_1 + x_2), A(x_1^2 + x_2^2) \) and \( A(x_1/x_2) \). The first two cases reduce to single variable dependence by suitable coordinate transformation, which we consider separately. The last case is obviously singular and could not be considered as a viable case for any kind of cosmology.

In (ii) \( A_0/A \neq B_0/B \neq C_0/C \), following the same route we get from \( T_{\alpha 0} = 0; A, B \) and \( D \) in terms of \( C(x_\alpha) \) as before. Then \( T_{12} = 0 \) determines \( C(x_\alpha) = (f(x_1) + f(x_2))^c \). Eqn. (3.10) represents two equations; \( f_1 = \text{const.}, f_2 = \text{const.} \) and two more similar equations. These will ultimately determine \( C(x_\alpha) = \text{const.} \) and again the spacetime is Bianchi I.
Thus $G_1$ symmetry does not yield a viable fluid model.

### III $G_2$ isometry:
Finally we have the spacetime general enough to sustain an inhomogeneous fluid. All fluid models are inhomogeneous and anisotropic. In view of eqns. (2.5) and (2.6), it follows that inhomogeneous spacetime has necessarily to be anisotropic. Inhomogeneous models can have singularities of different kinds or none.

Ruiz and Senovilla [12] have thoroughly analysed this case and have shown that the spacetime possesses a rich singularity structure. The metric (2.1) will have models with singularity but not always of the big-bang kind as well as models free of singularity. The latter family is shown to be unique and cylindrically symmetric. Since non-singular solutions are allowed only in this case, the identified non-singular family is unique for the general orthogonal metric (2.1).

This completes the proof of the theorem.

The most general non-singular metric [12] is given by,

$$
\begin{align*}
ds^2 &= \cosh^{1+n}(at)\cosh^{n-1}(nar)\left(dt^2 - \frac{\sinh^2(nar)}{P^2}dr^2\right) \\
&\quad - \cosh^{1+n}(at)\frac{P^2}{n^2a^2L^2\cosh \frac{n-1}{n}(nar)}d\phi^2 - \frac{\cosh^{1-n}(at)}{\cosh \frac{n-1}{n}(nar)}dz^2,
\end{align*}
$$

(3.11)

where

$$
L = K - \frac{K-1}{2n}, \quad P^2 = \cosh^2(nar) + (K-1)\cosh \frac{2n-1}{n}(nar) - K
$$

(3.12)
and $K, n, a$ are constants. The coordinates range as $-\infty < t, z < \infty$, $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$ and the metric has cylindrical symmetry.

The fluid parameters are given by

\[ 8\pi \rho = X \left[ (n - 1)(2n - 1)(n + 3)K \cosh^{-2}(nar) + (n + 1)(n - 3) \cosh^{-2}(at) \right] \tag{3.13} \]

\[ 8\pi p = X \left[ (n - 1)^2(2n - 1)K \cosh^{-2}(nar) + (n + 1)(n - 3) \cosh^{-2}(at) \right] \tag{3.14} \]

where

\[ X = \frac{a^2}{4} \cosh^{-(1+n)}(at) \cosh^{-n}(nar). \tag{3.15} \]

Both density and pressure are positive and $p \leq \rho$ for $K \geq 0$. The equation of state $\rho = 3p$ for radiation is admitted when $n = 3$. Senovilla’s model [6] further requires $K = 1$. The case $K = 0$ gives the stiff fluid equation of state, $\rho = p$ [18]. The case $K = 1$ has been considered separately and it has been shown that radial heat flux can be incorporated without disturbing the singularity–free character of the metric [18,19].

4. Discussion

The main result of the paper is that the already identified family of non-singular cosmological models is unique not only for the $G_2$ metric but also for the general orthogonal metric separable in space and time in comoving coordinates. Thus the complete set of non-singular solutions has been identified. In the process of
establishing this result we have also been able to characterise all perfect fluid models described by the metric (2.1). They are: homogeneous Bianchi I and FLRW; and inhomogeneous with or without singularity (different kinds of singularities occur [12]). We assume no isometries a priori, the perfect fluid conditions imply $G_3$ and $G_6$ symmetries for homogeneous, and $G_2$ for inhomogeneous models.

The non–singular character of fluid models singles out cylindrical symmetry. Like inhomogeneity, it is only a necessary condition but not sufficient. A kind of formal connection can be indicated between the non–singular metric (3.11) with $K = 1$ and the FLRW open model. The former can be thought of as arising out of a natural inhomogenisation of the latter [20]. The unfortunate feature of the metric (3.11) is that anisotropy does not decay with time (since $\sigma/\theta = \text{const}$), which means it can never evolve into FLRW. There may, however, occur a non–singular solution when the assumption of separability is dropped, which may isotropise to FLRW at late times. That would be a very significant result for cosmology, but the situation becomes mathematically formidable. We are currently investigating this question for spherical and cylindrical symmetry.

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Appendix

An \(n\)th order system of differential equations

\[
E(x, y, y', \ldots, y^{(n)}) = 0 \quad (A.1)
\]

will have the Lie (point) symmetry \[21\]

\[
G = \xi_i(x_i, y_i) \frac{\partial}{\partial x_i} + \eta_i(x_i, y_i) \frac{\partial}{\partial y_i} \quad (A.2)
\]

iff

\[
G^{[n]}E \bigg|_{E=0} = 0, \quad (A.3)
\]

where \(G^{[n]}\) is the \(n\)th extension of \(G\) needed to take care of the \(n\)th derivatives in \((A.1)\).

We analyse the following three equations:

\[
2 \left( \frac{D_1}{D} \right)_2 - (\alpha - 2) \frac{D_1 D_2}{D^2} - \frac{D_1 A_2 + D_2 A_1}{DA} = 0, \quad (A.4)
\]

\[
2 \left( \frac{D_1}{D} \right)_1 - (\alpha - 2) \frac{D_1^2}{D^2} - \frac{2D_1 A_1}{DA} = 2 \left( \frac{D_2}{D} \right)_2 - (\alpha - 2) \frac{D_2^2}{D^2} - \frac{2D_2 A_2}{DA}, \quad (A.5)
\]

\[
\frac{D}{A} \left[ 2\lambda \left( \frac{D_1}{D} \right)_1 + \lambda(\lambda + 1) \frac{D_1^2}{D^2} - 2 \left( \frac{A_1}{A} \right)_1 - (\lambda + 1) \frac{A_1 D_1}{AD} \right]
+ \frac{D}{B} \left[ -2 \left( \frac{A_2}{A} \right)_1 - 2 \left( \frac{D_2}{D} \right)_2 + (\lambda - 1) \frac{D_2^2}{D^2} + (\lambda + 1) \frac{A_2 D_2}{AD} \right] = l, \quad (A.6)
\]

where \(\alpha = (1 + 2\lambda - \lambda^2)/(1 + \lambda)\) and \(l\) and \(\lambda\) are constants.
The standard Lie analysis [22] gives the Lie point symmetries of the above equations as

\[ G_1 = \frac{\partial}{\partial x_1} \]  \hspace{1cm} (A.7)

\[ G_2 = \frac{\partial}{\partial x_2} \]  \hspace{1cm} (A.8)

\[ G_3 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - 2A \frac{\partial}{\partial A} \]  \hspace{1cm} (A.9)

\[ G_4 = A \frac{\partial}{\partial A} + D \frac{\partial}{\partial D} \]  \hspace{1cm} (A.10).

To reduce the system of partial differential equations (A.4)–(A.6) to one of ordinary differential equations (and hence solve them) we need to decide on an appropriate independent variable. From (A.7) and (A.8) we have the possibility of independent variables

\[ u = c_1 x_1 + c_2 x_2 \]  \hspace{1cm} (A.11)

and from (A.9)

\[ u = \frac{x_2}{x_1}. \]  \hspace{1cm} (A.12)

The fourth symmetry implies that \( u = u(x_1, x_2) \).

Now, (A.11) implies

\[ A = A(x_1 + x_2) \quad D = D(x_1 + x_2) \]  \hspace{1cm} (A.13)
while (A.12) implies

\[
A = \frac{1}{x_1} A \left( \frac{x_2}{x_1} \right) \tag{A.14}
\]

with

\[
D = x_1 D \left( \frac{x_2}{x_1} \right) \tag{A.15}
\]

coming from the addition of (A.10). On the other hand, \( u = u(x_1, x_2) \) implies

\[
A = A(u), \quad D = D(u) \tag{A.16}
\]

which further implies

\[
u = f(x_1^2 + x_2^2). \tag{A.17}\]

We could take other combinations of the symmetries. The only one of relevance is (A.9) + \( k \) (A.10). This gives

\[
A = x_1^{k-2} f \left( \frac{x_2}{x_1} \right), \quad D = x_1^k \left( \frac{x_2}{x_1} \right) \tag{A.18}
\]

and is equivalent to (A.14) and (A.15) for the purposes of determining whether \( A \) and \( D \) contain singularities.
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