Interpolation of an analytic family of operators on variable exponent Morrey spaces

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Abstract. In this paper we show the validity of Stein’s interpolation theorem on variable exponent Morrey spaces.

1. Introduction

The Stein interpolation theorem, where the interpolation is given with regards to an analytic family of operators, is an essential tool pervading modern Fourier analysis. For example, the first non-trivial progress on spherical summation of multiple Fourier series was obtained with the usage of this theorem, see [7] for more details. Stein’s interpolation theorem is given in the framework of Lebesgue spaces and we were not able to find such an interpolation theorem for Morrey spaces. It is interesting to note that the Riesz-Thorin interpolation theorem when the domain space is a Morrey type space does not hold for appropriate counter examples see [18]. Hence, the proved Stein type result will deal only when the target space are Morrey type spaces but the domain is a Lebesgue type space. For interpolation type results on Morrey-Campanato spaces, we refer to [9, 17, 28] and references therein.

In 1938 C. Morrey [19] studied Morrey spaces for the first time in connection to its applications in partial differential equations. Until recently, a rapid growth has been seen in the study of Morrey type spaces because of its applications in major fields of engineering and sciences (see e.g. [8]). For a comprehensive study of Morrey spaces we refer to [2, 22, 21]. Function spaces with non-standard growth has seen a major focus in recent times (see e.g. [14, 15]) because of its wide range of applications e.g. in the area of image processing [1, 27], the study of thermorheological fluids [4] and modeling of electrorheological fluids [23].

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Let $X$ and $Y$ be two quasi-metric measure spaces (QMMSs). In this manuscript, a version of Stein’s interpolation theorem is proved in the framework when the target space is a variable exponent Morrey space $L^{q(x),l(x)}(Y)$ and the domain space is the variable exponent Lebesgue space $L^{p(\cdot)}(X)$. It is worth mentioning that these results are new even for the constant case.

Throughout the paper, constants (often different constants in the same series of inequalities) will mainly be denoted by $c$ or $C$; by the symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}$. We denote by $p_0(x)$ the function $p(x)/c_0$, $1 < p(x) < \infty$; the relation $a \approx b$ means that there are positive constants $c_1$ and $c_2$ such that $c_1 a \leq b \leq c_2 a$.

2. Preliminaries

Let $X$ be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be quasi-metric if the following conditions are satisfied:

(a) $d(x, y) = 0$ for all $x \in X$.
(b) $d(x, y) > 0$ for all $x, y \in X$ and $x \neq y$.
(c) There is a constant $c_0 > 0$ such that $d(x, y) = c_0 d(y, x)$ for all $x, y \in X$.
(d) There is a constant $c_1 > 0$ such that $d(x, y) \leq c_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

Let $\mu$ be a complete measure such that the set of all compactly supported continuous functions are dense in $L^1(X)$. We refer the triplet $(X, d, \mu)$ as quasi-metric measure spaces (QMMS), where $d$ is a quasi-metric.

Let $d_X = \text{diam}(X) = \sup\{d(x, y) : x, y \in X\}$. Let us denote by $B(x, r) = \{y \in X : d(x, y) < r\}$ a ball of radius $r > 0$ and centered at $x$. Throughout this paper, it will be assumed that $0 < \mu(B(x, r)) < \infty$ for every $r > 0$ and $x \in X$. It is evident that the assumption that all balls have finite measure together with the condition $d_X < \infty$ imply $\mu(X) < \infty$.

Variable exponent spaces. Let $\Omega$ be a $\mu$-measurable set in $(X, \mu)$ with positive measure. We denote:

$$p^-(\Omega) := \inf_{\Omega} p, \quad p^+(\Omega) := \sup_{\Omega} p$$

for a $\mu$-measurable function $p$ on $\Omega$. Suppose that $1 \leq p^-(\Omega) \leq p^+(\Omega) < \infty$. We say that a $\mu$-measurable function $f$ on $\Omega$ belongs to $L^{p(\cdot)}(\Omega)$ (or to $L^{p(x)}(\Omega)$) if

$$S_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} d\mu(x) < \infty.$$
It is a Banach space with respect to the norm (see e.g. [11, 16, 24, 25])
\[
\|f\|_{L^p(\Omega)} = \inf \left\{ \eta > 0 : S_{p(\cdot),\Omega}\left(\frac{f}{\eta}\right) \leq 1 \right\}.
\]

For the following propositions we refer to [16, 24, 25].

**Proposition 1 (Hölder’s inequality).** Let $\Omega$ be a $\mu$-measurable subset of $X$ and let $1 \leq p^-(\Omega) \leq p^+(\Omega) < \infty$. Then for every $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p^+(\cdot)}(\Omega)$ the following inequality holds.

\[
\left| \int_{\Omega} f(x)g(x)d\mu(x) \right| \leq \left( \frac{1}{p^-(\Omega)} + \frac{1}{(p^+(\Omega))'} \right) \|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p^+(\cdot)}(\Omega)}
\]

The following lemma has been taken from [5, p. 27].

**Lemma 1.** Let $\Omega$ be a $\mu$-measurable subset of $X$ and let $1 \leq p^-(\Omega) \leq p^+(\Omega) < \infty$. Then the following inequality holds.

\[
\|f\|_{L^{p(\cdot)}(\Omega)} \leq S_{p(\cdot),\Omega}(f) + 1,
\]

**Definition 1.** We say that a $\mu$-measurable function $p : X \to [1, \infty)$ belongs to the class $\mathcal{P}^{p,\log}_\mu(X)$ if for every $x, y \in X$ such that $\mu(B(x, d(x, y))) \leq 1/2$ the following inequality holds.

\[
|p(x) - p(y)| \leq \frac{-A}{\ln \mu(B(x, d(x, y)))}
\]

The following lemma can be found in [22, 14].

**Lemma 2.** Let $(X, d, \mu)$ be a QMMS with $\mu(X) < +\infty$ and let $p \in \mathcal{P}^{p,\log}_\mu(X)$. Then

\[
\|x_{B(x, r)}\|_{L^{p(\cdot)}} \leq C(\mu(B(x, r)))^{1/p(x)}.
\]

Morrey spaces with variable exponent $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ where $\Omega$ is an open subset of $\mathbb{R}^n$ were introduced simultaneously by Almeida et al. [3], Kokilashvili et al. [12, 13], Ohno [20] and X. Fan [6] in more or less similar manner. Let $1 \leq p(\cdot) < p^+(\Omega) < \infty$ and $0 \leq \lambda(\cdot) \leq 1$ be $\mu$-measurable functions. We say that a $\mu$-measurable function $f \in L^{p(\cdot)}(\Omega)$ belongs to $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ if

\[
I_{p(\cdot),\lambda(\cdot)}(f) = \sup_{x \in \Omega, r > 0} \left( \frac{1}{\mu(B(x, r))} \right)^{\lambda(x)} \int_{B(x, r)} |f(y)|^{p(y)}d\mu(y) < \infty.
\]
The norm on variable exponent Morrey spaces can be introduced in the following ways (see e.g. [3, 12, 13, 22]):

\[ \|f\|_1 = \inf \{ \eta > 0 : I_{p(\cdot), \lambda(\cdot)}(f/\eta) \leq 1 \}, \]

and

\[ \|f\|_2 = \sup_{x \in \Omega, r > 0} \| (\mu(B(x, r)))^{-\lambda(x)/p(x)} f \|_{L^{p(\cdot)}(B(x, r))}, \]

and

\[ \|f\|_3 = \sup_{x \in \Omega, r > 0} (\mu(B(x, r)))^{-\lambda(x)/p(x)} \|f\|_{L^{p(\cdot)}(B(x, r))}. \]

It can be checked easily by means of simple computations that \( \|f\|_1 = \|f\|_2 \). Further, if the exponent \( p \) is such that \( p \in \mathcal{P}_\mu^{\log}(X) \) (see e.g. [22]) then both the norms \( \|f\|_2 \) and \( \|f\|_1 \) are equivalent to \( \|f\|_3 \). We define the norm on variable exponent Morrey space as:

\[ \|f\|_{L^{p(\cdot), \lambda(\cdot)}(X)} = \|f\|_3. \]

It is easy to see that if the parameter \( \lambda = 0 \), then \( L^{p(\cdot)}(X) = L^{p(\cdot), 0}(X) \). When \( p(x) \equiv \text{const} \) and \( \lambda(x) \equiv \text{const} \) then \( L^{p(\cdot), \lambda(\cdot)}(X) \) is reduced to the case of classical Morrey space \( L^{p, \lambda}(X) \).

The following lemma gives the embedding of variable Morrey spaces into variable Lebesgue space in the case \( d_X < \infty \). Here we present the proof of this lemma for the sake of completeness.

**Lemma 3.** Let \( (X, d, \mu) \) be a QMMS. Suppose that \( 1 \leq p(\cdot) < p^+(X) < \infty \) and \( 0 \leq \lambda(\cdot) \leq 1 \). Then for every \( f \in L^{p(\cdot), \lambda(\cdot)}(X) \), \( x \in X \) and \( r > 0 \) we have

\[ \|f\|_{L^{p(\cdot)}(B(x, r))} \leq (\mu(B(x, r)))^{\lambda(x)/p(x)} \|f\|_{L^{p(\cdot), \lambda(\cdot)}(X)}. \]

Moreover, if \( \mu(X) < \infty \) then

\[ \|f\|_{L^{p(\cdot)}(X)} \leq c_{p, \lambda, \mu} \|f\|_{L^{p(\cdot), \lambda(\cdot)}(X)}. \]

**Proof.** Suppose that \( f \in L^{p(\cdot), \lambda(\cdot)}(X) \). Let \( x \in X \) and \( r > 0 \), then

\[ \|f\|_{L^{p(\cdot)}(B(x, r))} = (\mu(B(x, r)))^{\lambda(x)/p(x)} \frac{1}{(\mu(B(x, r)))^{\lambda(x)/p(x)}} \|f\|_{L^{p(\cdot)}(B(x, r))} \leq (\mu(B(x, r)))^{\lambda(x)/p(x)} \|f\|_{L^{p(\cdot), \lambda(\cdot)}(X)}. \]

Since \( p \) is bounded, hence taking supremum with respect to \( x \in X \) and \( r > 0 \) we have the following estimate
\[ \|f\|_{L^{p(\cdot)}(X)} \leq \max\{1, (\mu(X))^{(\frac{1}{p'})^+(X)}\} \|f\|_{L^{p(\cdot), \mu(\cdot)}(X)} \leq c_{p, \mu} \|f\|_{L^{p(\cdot), \mu(\cdot)}(X)}. \]

Consequently, via Hölder’s inequality, for \( f \in L^{p(\cdot), \mu(\cdot)}(X) \) and \( g \in L^{p'(\cdot)}(X) \) there is a positive constant \( c \) such that,

\[
\int_X f(y)g(y)\,d\mu(y) \leq c \|f\|_{L^{p(\cdot), \mu(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \tag{1}
\]

holds.

### 3. Interpolation of analytic family of operators in variable exponent Morrey spaces

In this section we prove the main result of this paper. We prove the Stein interpolation type theorem for analytic family of operators.

**Definition 2.** A function \( f(z) \) analytic on an open strip \( 0 < \text{Re}(z) < 1 \) and continuous and bounded on the closed strip is said to be of admissible growth if for \( a < \pi \) the following inequality

\[
\sup_{|y| \leq r} \sup_{0 \leq x \leq 1} |f(x + iy)| \leq Ce^{ar},
\]

holds, where \( C \) is a positive constant.

The next lemma is due to Hirschman and can be found in e.g. [10].

**Lemma 4 (Hirschman Lemma).** Let \( f(z) \) be analytic on an open strip \( 0 < \text{Re}(z) < 1 \) and continuous and bounded on the closed strip and of admissible growth there. Let

\[
\log|f(iy)| \leq A_0(y), \quad \log|f(1 + iy)| \leq A_1(y),
\]

then for \( 0 \leq t \leq 1 \) the following inequality

\[
\log|f(t)| \leq \left( \frac{1}{2} \right) \left[ \int_{-\infty}^{\infty} \frac{\sin(\pi t)}{\cosh(\pi y) - \cos(\pi t)} A_0(y)\,dy \right. \\
\quad + \left. \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi t)}{\cosh(\pi y) + \cos(\pi t)} A_1(y)\,dy \right],
\]

holds.

**Definition 3 (Analytic Family of Operators).** Let \((X_1, d_1, \mu_1)\) and \((X_2, d_2, \mu_2)\) be QMMSs. Consider a family of linear operators \( \{T_z\}_{z \in \mathbb{C}} \). We shall call this family of linear operators to be analytic if:
For each \( z \in \mathbb{C} \), \( T_z \) maps simple functions in \((X_1, d_1, \mu_1)\) on measurable functions in \((X_2, d_2, \mu_2)\).

For \( z \in S \), \( r > 0 \) and a.e. \( y \in X_2 \), the function \( F_{y, r}(z) \) defined by
\[
F_{y, r}(z) := \int_{B(y, r)} T_z \left[ a_1^{m_1(z) + b_1(z)} Z_{A_1}(\cdot) \right](x_2) \times a_2^{m_2(z) + b_2(z)} Z_{A_2}(x_2) \, d\mu_2(x_2),
\]
exists, is continuous and bounded on the strip \( S = \{ z : 0 \leq \text{Re}(z) \leq 1 \} \) and analytic on \( \text{int}(S) \), where \( a_k \) are positive real numbers and \( m_k, b_k \) are measurable functions for \( k = 1, 2 \).

We shall call \( \{ T_z \}_{z \in \mathbb{C}} \) of admissible growth if \( F_{y, r}(z) \) is of admissible growth in the sense of Definition 2.

**Remark 1.** Although the definition of an analytic family of operators given in Definition 3 seems cumbersome at first sight, but it should be noted that in the non-variable framework this definition coincides with the definition given by Stein in [26].

We now formulate and prove the Stein interpolation theorem in the variable exponent framework.

**Theorem 1.** Let \((X, \mu)\) and \((Y, v)\) be \( \sigma \)-finite, complete QMMSs. For \( k = 0, 1 \), assume that \( 1 \leq p_k(\cdot), q_k(\cdot) < q_k^*(Y) < \infty \) and \( 0 \leq \lambda_k(\cdot) \leq 1 \). Suppose that we have an analytic family of linear operators \( T_z : L_{p_k(\cdot)}(X) \to L_{q_k(\cdot)}(Y) \) which is of admissible growth in the strip \( S := \{ z : 0 \leq \text{Re}(z) \leq 1 \} \). Further suppose that the following inequalities
\[
\| T_{it} f \|_{L_{q_1}^{\lambda_1(\cdot)}(X)} \leq M_0(t) \| f \|_{L_{p_1}^{\lambda_0(\cdot)}(X)} \quad (3)
\]
\[
\| T_{1+it} f \|_{L_{q_1}^{\lambda_1(\cdot)}(X)} \leq M_1(t) \| f \|_{L_{p_1}^{\lambda_0(\cdot)}(X)} \quad (4)
\]
hold for all simple functions \( f \). Also we assume that
\[
\log |M_k(t)| \leq C e^{l|t|} \quad l < \pi \text{ for } k = 0, 1.
\]
For \( z \in S := \{ z : 0 < \text{Re}(z) < 1 \} \), define \( p_z, q_z \) and \( \lambda_z \) by
\[
\frac{1}{p_z(x)} = \frac{1 - z}{p_0(x)} + \frac{z}{p_1(x)},
\]
\[
\frac{1}{q_z(x)} = \frac{1 - z}{q_0(x)} + \frac{z}{q_1(x)},
\]
and
\[
\frac{\lambda_z(x)}{q_z(x)} = (1 - z) \frac{\lambda_0(x)}{q_0(x)} + \frac{z \lambda_1(x)}{q_1(x)}.
\]
Then, given any $y \in A(0,1)$, the inequality
\[
\|T_0 f\|_{L_{\theta(y)}(X,Y)} \leq cM_0\|f\|_{L_{\theta(y)}(X)}
\]
holds for every $f \in L^{\theta(y)}(X)$, where
\[
\log M_0 = \left( 1 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi \theta) - \cos(\pi \theta)} \log M_0(y) \, dy \right) + \left( 1 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi \theta) + \cos(\pi \theta)} \log M_1(y) \, dy \right).
\]

\textbf{Proof.} Since $T_0$ is linear, we may assume that $f = 0$, otherwise the inequality holds for $f = 0$. By the homogeneity of the norm and the scaling argument we may assume that $\|f\|_{L^{\theta(y)}(X)} = 1$. Now we need to show that
\[
\|T_0 f\|_{L_{\theta(y)}(X,Y)} \leq cM_0.
\]
We will show (6) for simple functions in $X$ and since the span of simple functions is dense in $L^{\theta(y)}(X)$ we will have the estimate for all $f \in L^{\theta(y)}(X)$.

Let us assume $f$, $g$ are simple and complex valued functions defined on $X$ and $Y$ respectively by,
\[
f(x) = \sum_{j=1}^{m} a_j e^{i\varphi_j} \chi_{A_j}(x), \quad x \in X
\]
\[
g(y) = \sum_{k=1}^{n} b_k e^{i\varphi_k} \chi_{B_k}(y), \quad y \in Y
\]
where $a_j, b_k > 0$ and $\varphi_j, \varphi_k \in \mathbb{R}$, $\mu(A_j), \mu(B_k) < \infty$, and $\{A_j\}$ and $\{B_k\}$ are, respectively, pairwise disjoint. Now define,
\[
f_\varepsilon(z) = \sum_{j=1}^{m} a_j^{p_0(y)/p_j(z)} e^{i\varphi_j} \chi_{A_j}(x),
\]
\[
g_\varepsilon(z) = \sum_{k=1}^{n} b_k^{q_0(z)/q_k(y)} e^{i\varphi_k} \chi_{B_k}(y).
\]

Finally, for every $y \in Y$, $r > 0$ and $z \in \mathbb{C}$, we put
\[
F_{y,r}(z) := \int_{B(y,r)} T_0 (f_\varepsilon(z)) g_\varepsilon(z) \, dv(s).
\]
Substituting the values of $f_\varepsilon$ and $g_\varepsilon$ in the last expression we have
\[
F_{y,r}(z) = \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{B(y,r)} T_0 [a_j^{p_0(y)/p_j(z)} \chi_{A_j}(\cdot)](s) b_k^{q_0(z)/q_k(y)} \chi_{B_k}(s) \, dv(s).
\]
Hence for almost every \( y \in Y \), \( F_{y,r}(z) \) is analytic on \( \text{int}(S) \) and continuous and bounded on \( S \) and of admissible growth, since \( T_z \) is an analytic family of linear operators of admissible growth.

Since \( A_j \) are pairwise disjoint and \( a_j > 0 \), we have for \( z = it \) (\( t \in \mathbb{R} \))

\[
S_{p_0(\cdot), B(y,r)}(f_z) = \int_{B(y,r)} \left| \sum_{j=1}^{m} a_j^{p_0(x)/p_1(x)} e^{iz A_j(x)} \right|^{p_0(x)} \ d\mu(x)
\]

\[
= \int_{B(y,r)} \left\| \sum_{j=1}^{m} a_j^{p_0(x)/p_1(x)-1/p_0(x)} e^{iz A_j(x)} \right\|^{p_0(x)} \ d\mu(x)
\]

\[
= \int_{B(y,r)} \left\| \sum_{j=1}^{m} a_j^{p_0(x)} e^{iz A_j(x)} \right\|^{p_0(x)} \ d\mu(x)
\]

\[
= S_{p_0(\cdot), B(y,r)}(f)
\]

\[
\leq 1
\]

since \( \|f\|_{L^{p_0(\cdot)}(X)} \leq 1 \). Hence \( \|f_z\|_{L^{p_0(\cdot)}(B(y,r))} \leq 1 \). A similar argument shows that \( \|g_z\|_{L^{q_0(\cdot)}(B(y,r))} \leq 1 \) for \( z = it \). Now by Hölder’s inequality, Lemma 3 and (3) we have

\[
|F_{y,r}(it)| \leq \left| \int_{B(y,r)} T(f_z(s))g_z(s) \ dv(s) \right|
\]

\[
\leq c \|Tf_z\|_{L^{p_0(\cdot)}(B(y,r))} \|g_z\|_{L^{q_0(\cdot)}(B(y,r))}
\]

\[
\leq c \|Tf_z\|_{L^{p_0(\cdot)}(B(y,r))}
\]

\[
\leq c(v(B(y,r)))^{\gamma_0(y)/q_0(y)} \|Tf_z\|_{L^{q_0(\cdot)}(X)}
\]

\[
\leq c(v(B(y,r)))^{\gamma_0(y)/q_0(y)} M_0(t) \|f_z\|_{L^{q_0(\cdot)}(X)}
\]

\[
\leq c(v(B(y,r)))^{\gamma_0(y)/q_0(y)} M_0(t).
\]

An analogous argument with \( \text{Re}(z) = 1 \) for the exponents \( p_1 \) and \( q_1 \) yields,

\[
|F_{y,r}(1 + it)| \leq c(v(B(y,r)))^{\gamma_1(y)/q_1(y)} M_1(t).
\]
Invoking Hirschman’s Lemma we have:

\[
\log|F_{y,r}(\theta)| \leq \left( \frac{1}{2} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) - \cos(\pi \theta)} \, \log\left( (v(B(y, r)))^{x_0(y)/q_0(y)} M_{0}(s) \right) \, ds \\
+ \left( \frac{1}{2} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) + \cos(\pi \theta)} \, \log\left( (v(B(y, r)))^{x_1(y)/q_1(y)} M_1(s) \right) \, ds \\
\leq \left( \log(v(B(y, r)))^{x_0(y)/q_0(y)} \right) \left( \frac{\sin(\pi \theta)}{2} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) - \cos(\pi \theta)} \, ds \\
+ \left( \log(v(B(y, r)))^{x_1(y)/q_1(y)} \right) \left( \frac{\sin(\pi \theta)}{2} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) + \cos(\pi \theta)} \, ds + \log M_{\theta}.
\]

By making the change of variables \( e^{\pi s} = u \) in the above integrals we have

\[
\frac{1}{2} \left( \log(v(B(y, r)))^{x_0(y)/q_0(y)} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) - \cos(\pi \theta)} \, ds = 1 - \theta
\]

and

\[
\frac{1}{2} \left( \log(v(B(y, r)))^{x_1(y)/q_1(y)} \right) \int_{-\infty}^{\infty} \frac{\sin(\pi \theta)}{\cosh(\pi s) + \cos(\pi \theta)} \, ds = \theta.
\]

Hence,

\[
\log|F_{y,r}(\theta)| \leq (1 - \theta) \log(v(B(y, r)))^{x_0(y)/q_0(y)} \\
+ \theta \log(v(B(y, r)))^{x_1(y)/q_1(y)} + \log M_{\theta} \\
\leq \log(v(B(y, r)))^{(1-\theta)(x_0(y)/q_0(y)) + \theta(x_1(y)/q_1(y))} + \log M_{\theta} \\
\leq \log(v(B(y, r)))^{(1-\theta)(x_0(y)/q_0(y)) + \theta(x_1(y)/q_1(y))} + \log M_{\theta} \\
\leq \log(v(B(y, r)))^{x_0(y)/q_0(y)} + \log M_{\theta},
\]

which yields

\[
|F_{y,r}(\theta)| \leq c(v(B(y, r)))^{x_0(y)/q_0(y)} M_{\theta}.
\]

Also,

\[
\sup_{\|g\|_{L^{q_0}(B(y, r))} \leq 1} F_{y,r}(\theta) \sim \|T_{0}f\|_{L^{q_0}(B(y, r))}.
\]

Hence for almost every \( y \in Y \) and \( r > 0 \) we have,

\[
(v(B(y, r)))^{-x_0(y)/q_0(y)} \|T_{0}f\|_{L^{q_0}(B(y, r))} \leq cM_{\theta},
\]
which implies that,
\[ \| T_0 f \|_{L^{q(y)}_\infty(Y)} \leq cM_0. \]

This completes the proof.

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