Differential systems stability analysis based on matrix multiplicative criteria

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Abstract. This article proposes a method of stability analysis in the sense of Liapunov for systems of ordinary differential equations. The method is based on matrix multiplicative stability criteria obtained from transformations of numerical integration difference schemes. Criteria are formed in the form of necessary and sufficient conditions. A distinctive feature of the criteria is that they do not employ the methods of the qualitative theory of differential equations. Specifically, for the case of linear systems, the evaluation of characteristic numbers and characteristic indicators is not necessary. When analyzing the stability of nonlinear systems, the construction of Liapunov functions is not required. Matrix multiplicative form of the criteria allows computerization of stability analysis. The software implementation of the criteria is performed in real time and, based on its results, allows to make an unambiguous conclusion about the nature of stability of the system under study.

1. Introduction
Stability analysis in the sense of Liapunov of systems of ordinary differential equations (ODEs) is required in various fields of science and technology, in technological processes management, in modeling the control of robots, in the construction of high-rise buildings and aviation structures, and in solving the optimization problems in economics and management. Traditionally, stability analysis relies on qualitative theory methods and the theory of automatic control. The use of computer technology in scientific and technological research requires automation of stability analysis. It has to be performed directly in the process of computer control of objects to improve the quality of control. Therefore, the development of computer tools for analyzing the stability of differential systems is a topical research question. The approach based on the computer implementation of numerical stability analysis has its own history [1], continuation [2] and development [3]. Widely used methods, if not applied to particular types of ODEs, employ various techniques to calculate Liapunov functions [3-6]. The approach proposed in this work is based on the difference (in the general case, approximate) solution of the system, combined with stability analysis.

Consider the Cauchy problem for a nonlinear ODE system

\[ \frac{dY}{dt} = F(t, Y), \ Y(t_0) = Y_0. \] (1)

The existence of \( \delta_0 > 0 \) is assumed, when all conditions of existence and uniqueness are satisfied for an unperturbed solution on half-line \( [t_0, \infty) \) and for each of its perturbations with an initial
neighborhood vector \( \| \tilde{Y}_0 - Y_0 \| \leq \delta_0 \). It is also assumed that in the region \( R : \{ t_0 \leq t < \infty ; Y(t), \forall \tilde{Y}(t) : \| \tilde{Y}_0 - Y_0 \| \leq \delta_0 \} \), the function \( F(t, Y) \) is total, continuous and continuously differentiable on \( t \). The components of this function satisfy the following inequality:

\[
| f_k(t, Y) - f_k(t, \tilde{Y}) | \leq L | y_k - \tilde{y}_k | , \quad L = \text{const} , \quad \forall t \in [t_0, \infty) , \quad \forall (t, Y), \quad (t, \tilde{Y}) \in R , \quad \forall k = 1, \ldots, n .
\]

It is required to investigate the stability of an unperturbed solution to problem (1). Finding stability [7] is simplified in the adopted restrictions: solution \( Y = Y(t) \) is stable if \( \forall \varepsilon > 0 \) there is a \( \Delta , \quad 0 < \Delta \leq \delta_0 \), so that \( \| \tilde{Y}_0 - Y_0 \| \leq \Delta \) leads to \( \| \tilde{Y}(t) - Y(t) \| \leq \varepsilon , \quad \forall t \in [t_0, \infty) \). A solution is asymptotically stable if it is stable and there is a \( \Delta_0 , \quad 0 < \Delta_0 \leq \Delta_0 \), so that \( \| \tilde{Y}_0 - Y_0 \| \leq \Delta_0 \) leads to \( \tilde{Y}(t) - Y(t) \to 0 \) under \( t \to \infty \).

Under these conditions, a stability analysis method is developed in the Liapunov sense for ODE systems based on multiplicative matrix transformations of difference schemes of numerical integration. These transformations should result in the stability criteria that allow computer implementation [8, 9].

2. Methodology

According to [10], the use of \( X(t) = Y(t) - Z(t) \), where \( Z(t) \) is an unperturbed solution, system (1) is reduced to a system of differential equations:

\[
\frac{dX}{dt} = V(t, X) , \quad X(t_0) = X_0 ,
\]

for which \( V(t, 0) = 0 \).

For system (2), the following expressions are valid:

1) The derivative of functions \( v_k(t, X(t)), \quad v_k(t, \tilde{X}(t)), \quad k = 1, \ldots, n \) is bounded on the interval \([t_0, T]\):

\[
| v'_k(t, X(t)) | \leq c_1 , \quad | v'_k(t, \tilde{X}(t)) | \leq c_1 , \quad c_1 = \text{const} \quad \forall t \in [t_0, T] .
\]

2) The following inequality is satisfied: \( | v_k(t, X) - v_k(t, \tilde{X}) | \leq L_0 | x_k - \tilde{x}_k | , \quad L_0 = \text{const} , \quad \forall t \in [t_0, \infty) , \quad \forall k = 1, \ldots, n \).

Perform the following transformation of system (2):

\[
\frac{dx_k}{dt} = \frac{v_k(t, x_1, \ldots, x_n)}{x_k} x_k , \quad k = 1, \ldots, n ,
\]

or, in matrix form

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{pmatrix}
= \begin{pmatrix}
v_1(t, x_1, \ldots, x_n) & 0 & 0 & \ldots & 0 \\
0 & v_2(t, x_1, \ldots, x_n) & 0 & \ldots & 0 \\
0 & 0 & v_3(t, x_1, \ldots, x_n) & \ldots & 0 \\
0 & 0 & 0 & \ldots & v_n(t, x_1, \ldots, x_n)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} .
\]
For the zero solution of system (3), the perturbation value can be represented in the form of the Euler method with the remainder term at each step

$$\tilde{X}_{i+1} = (E + h A(t_i, \tilde{X}_i)) \tilde{X}_i + \tilde{Q}_i, \quad i = 0, 1, \ldots, $$

(4)

where $A(t_i, \tilde{X}_i) = \begin{pmatrix} v_1(t_i, \tilde{x}_1, \ldots, \tilde{x}_n) & 0 & 0 & \ldots & 0 \\ 0 & v_2(t_i, \tilde{x}_1, \ldots, \tilde{x}_n) & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & v_n(t_i, \tilde{x}_1, \ldots, \tilde{x}_n) \end{pmatrix}, \quad \|\tilde{Q}_i\| \leq c_i h^2.$$

The remainder term of the Euler method at an step is the difference between the exact solution and its approximation by the Euler method.

For any $t = \text{const}, \quad t \in [t_0, \infty), \quad h$ and the variable index $i$ are always assumed to be bound by:

$$t = t_{i+1}, \quad h = \frac{t_{i+1} - t_0}{i + 1}, \quad i = 0, 1, \ldots, $$

(5)

Transforming recurrently (4), we obtain the expression for the perturbation value through the perturbation of the initial data

$$\tilde{X}_{i+1} = \prod_{\ell=0}^{i} (E + h A(t_{i-\ell}, \tilde{X}_{i-\ell})) \tilde{X}_0 + S_i, $$

(6)

where

$$S_i = \sum_{k=1}^{i-k} \prod_{\ell=0}^{i-k} (E + h A(t_{i-\ell}, \tilde{X}_{i-\ell})) \tilde{Q}_{k-1} + \tilde{Q}_i. $$

(7)

**Lemma 1.** In the conditions under consideration $\|S_i\| = O(h)$.  

**Proof.** It follows from (7), that

$$\|S_i\| \leq \sum_{k=1}^{i-k} \prod_{\ell=0}^{i-k} (E + h A(t_{i-\ell}, \tilde{X}_{i-\ell})) \|\tilde{Q}_{k-1}\| + \|\tilde{Q}_i\|.$$  

(8)

On the basis of assertion 2) for the zero solution of (2), the inequality $\left|\frac{v_k(t, \tilde{X})}{\tilde{x}_k}\right| \leq L_0, \quad \forall k = 1, \ldots, n$ is satisfied; therefore $\|A(t, \tilde{X})\| \leq L_0, \quad \forall t \in [t_0, \infty).$

Therefore, inequality (8) has the form: $\|S_i\| \leq c_i h^2 \sum_{k=1}^{i-k} (1 + h L_0)^{i-k+1} \quad \text{or} \quad \|S_i\| \leq c_i h^2 \sum_{\ell=0}^{i} (1 + h L_0)^{i-\ell+1}$.

The summation of the geometric progression on the right side implies $\|S_i\| \leq \tilde{c}_i h ((1 + h L_0)^{i+1} - 1),$

where $\tilde{c}_i = \frac{c_i}{L_0}$.

Taking into account (5), it follows that $\|S_i\| \leq \tilde{c}_i h ((1 + h L_0)^{\frac{t_{i+1}-t_0}{h}} - 1).$
Moreover, the inequality $\|S_i\| \leq \tilde{c}_i h(e^{L_i(t_{i+1}-t_i)} - 1)$ is also valid. For any $t = t_{i+1}$, $t = \text{const}$, $t \in [t_0, \infty)$, and with variable $i$ the relation $\tilde{c}_i (e^{L_i(t-t_{i})} - 1) h \to 0$ is satisfied if $h \to 0$. It follows that $\|S_i\| = O(h)$.

Under the conditions of Lemma 1,

$$\lim_{h \to 0} S_i = \tilde{0}.$$  \hfill (9)

The passage to the limit in equality (6) for any $t$ in (5) entails

$$\bar{X}(t) = \lim_{h \to 0} \lim_{i \to \infty} \prod_{\ell=0}^{i}(E + h A(t_{i-\ell}, \bar{X}_{i-\ell})) \bar{X}_0 + \lim_{h \to 0} S_i.$$  \hfill (10)

$h$ going to zero equals to $i$ going to infinity. From this, with regard to (9), for any $t \in [t_0, \infty)$:

$$\bar{X}(t) = \lim_{i \to \infty} \lim_{h \to 0} \prod_{\ell=0}^{i}(E + h A(t_{i-\ell}, \bar{X}_{i-\ell})) \bar{X}_0.$$  \hfill (11)

Theoretically, the step $h$ in (10) depends on $i$: $h(i) = \frac{t - i}{i + 1}$, $i = 0, 1, \ldots$. In the right part of (10) the infinite product does not quite correspond to the meaning of this concept, defined in mathematical analysis. The known definition includes a sequence of partial products, the change of which determines the addition of new cofactors. The infinite product in (10) is constructed as a sequence of partial products, each of which differs from the previous one not only by adding a new cofactor, but also by changing (decreasing in inverse proportion to the number of cofactors) the parameter $h$ in each of the cofactors of the new partial product.

The meaning of equality (10) is that for an arbitrary value $t$ the perturbation is equal to the infinite matrix product multiplied by the perturbation of the initial data. Thus, the magnitude of the perturbation under any $t$ is proportional to the infinite matrix product. This automatically follows the stability criterion of solutions of nonlinear ODE systems presented in theorem 1.

**Theorem 1.** Let all the conditions under consideration be satisfied. In order for the solution of problem (2) to be stable, it is necessary and sufficient to perform the inequality

$$\lim_{i \to \infty} \lim_{h \to 0} \prod_{\ell=0}^{i}(E + h A(t_{i-\ell}, \bar{X}_{i-\ell})) \leq \tilde{c}_i = \text{const} \ \forall t \in [t_0, \infty).$$  \hfill (11)

The solution is asymptotically stable if and only if (11) is satisfied; in addition, the relation is satisfied

$$\lim_{i \to \infty, h \to 0} \lim_{i \to \infty} \prod_{\ell=0}^{i}(E + h A(t_{i-\ell}, \bar{X}_{i-\ell})) \to 0.$$  \hfill (12)

The significance of the criteria (11), (12) is that they allow determining the nature of stability, asymptotic stability or instability of a nonlinear ODE system without presenting the solution in analytical form, directly from the values of difference approximations. Matrix multiplicative form of expressions under the sign of the limit allows programming the calculation of expressions as a cycle by the number of factors. This creates the possibility of computer analysis of stability without recourse to analytical methods of qualitative theory of differential equations.

The practical computer implementation of the criteria will be performed with a constant step at a fixed interval. This is done for two reasons: first, the difference in the simulation results is not detected...
experimentally when changing the step in the set of factors, and secondly, the constancy of the step fundamentally reduces the simulation time [8, 11].

It is required to verify the reliability of the stability analysis using the presented criteria in a numerical experiment. Reliability is estimated by comparing it with the known estimation on the basis of the computer analysis described in [12, 13].

For comparison, the following criterion will be used: in the conditions under consideration, for the stability of the solution of problem (1), the existence of such \( \Delta_1 \), \( 0 < \Delta_1 \leq \delta_0 \) is necessary and sufficient, that for all the solutions \( Y = Y(t) \), \( Y(t_0) = \tilde{Y}_0 \) under the constraint \( 0 < \| \tilde{Y}_0 - Y_0 \| \leq \Delta_1 \) the following inequality is satisfied

\[
\left| \frac{\tilde{y}_k(t) - y_k(t)}{\tilde{y}_k(t_0) - y_k(t_0)} \right| \leq \tilde{c}_2, \quad \tilde{c}_2 = \text{const}, \quad \forall t \in [t_0, \infty), \quad k = 1, \ldots, n.
\] (13)

For asymptotic stability under the same conditions, it is necessary and sufficient for (13) to hold and such \( \Delta_2 \leq \Delta_1 \) to exist that inequality \( 0 < \| \tilde{Y}_0 - Y_0 \| \leq \Delta_2 \) entails

\[
\lim_{t \to \infty} \left| \frac{\tilde{y}_k(t) - y_k(t)}{\tilde{y}_k(t_0) - y_k(t_0)} \right| = 0, \quad k = 1, \ldots, n.
\] (14)

With the computer implementation of criteria (13), (14), the exact solution and perturbation are replaced by approximate values, which are based on the difference method.

3. **Modification of stability criteria for the case of a linear ODE system**

The Cauchy problem for a linear homogeneous ODE system is considered

\[
\frac{dY}{dt} = A(t)Y, \quad Y(t_0) = Y_0,
\] (15)

where \( Y \) is a vector-function whose coordinates are the desired functions \( y_1, y_2, \ldots, y_n \) from the independent variable \( t \); \( A(t) \) is a coefficient matrix, \( n \times n \). It is assumed that all functions \( a_{ij}(t) \) \( i, j = 1, \ldots, n \) are defined, continuous, and continuously differentiable on a segment \( [t_0, T] \), with any choice of \( T = \text{const}, \quad T \in [t_0, \infty) \); \( Y(t_0) = Y_0 \) is the given initial vector. Everywhere below the canonical norm of the matrix is used \( \| A \| = \max_{1 \leq i \leq n} \sum_{k=1}^{n} |a_{ik}| \) and the agreed norm of the vector \( \| Y \| = \max_{1 \leq k \leq n} |y_k| \).

It is assumed that for (15) all conditions of existence and uniqueness of the solution are satisfied \( Y = Y(t), \quad Y(t_0) = Y_0 \), in \( [t_0, \infty) \); the same conditions are assumed to be satisfied for each element of the set of perturbed solutions \( \tilde{Y} = \tilde{Y}(t) \), corresponding to the perturbed initial vector \( \tilde{Y}(t_0) = \tilde{Y}_0 \), satisfying the inequality \( \| \tilde{Y}_0 - Y_0 \| \leq \delta, \quad \delta > 0 \).

The following statements are valid in the assumptions under consideration [8, 9].

1) Matrix \( A(t) \) is limited on the segment \( [t_0, T] \) on the norm:

\[
\| A(t) \| \leq c, \quad c = \text{const} \quad \forall T \in [t_0, \infty).
\]
2) The derivative of functions \( f_k(t, Y(t)), f_k(t, \tilde{Y}(t)) \)
\( f_k(t_i, Y_i) = a_{k1}(t_i) y_{1(i)} + ... + a_{kn}(t_i) y_{n(i)}, k = 1, ..., n \) is limited on the segment \([t_0, T]\):
\[
\left| f_k'(t, Y(t)) \right| \leq c_2, \quad \left| f_k'(t, \tilde{Y}(t)) \right| \leq c_2, \quad c_2 = \text{const} \quad \forall t \in [t_0, T].
\]

Euler’s method of approximate solution of system (15) has the form \( Y_{i+1} = (E + h A(t_i))Y_i \). For the perturbed solution of system (15) the relation is valid \( \tilde{Y}_{i+1} = (E + h A(t_i))\tilde{Y}_i \). Variables \( t, h, i \), like previously, are assumed to be connected by (5).

For the difference between the perturbed and unperturbed solution we obtain an exact equality
\[
\tilde{Y}_{i+1} - Y_{i+1} = (E + h A(t_i)) (\tilde{Y}_i - Y_i) + Q_{E,i},
\]
where \( \|Q_{E,i}\| \leq c_i h^2 \) [8, 9].

Recurrent transformation of (16) entails
\[
\tilde{Y}_{i+1} - Y_{i+1} = \prod_{j=0}^{i} (E + h A(t_{i-j})) (\tilde{Y}_0 - Y_0) + P_i,
\]
where
\[
P_i = \sum_{k=1}^{i} \prod_{j=0}^{i-k} (E + h A(t_{i-j})) Q_{E,k-1} + Q_{E,i}.
\]

**Lemma 2.** In the considered conditions, the relation \( \|P_i\| = O(h) \) takes place [8].

**Theorem 2.** Let all considered conditions for system (15) be satisfied. In order for the solution of problem (15) to be stable, it is necessary and sufficient to fulfill the inequality
\[
\lim_{t \to \infty} \left\| \prod_{j=0}^{i} (E + h A(t_{i-j})) \right\| \leq c_3 = \text{const} \quad \forall t \in [t_0, \infty).
\]

The solution is asymptotically stable if and only if (18) is satisfied and, moreover, the following relation is satisfied
\[
\lim_{t \to \infty} \lim_{i \to \infty} \left\| \prod_{j=0}^{i} (E + h A(t_{i-j})) \right\| \to 0.
\]

The significance of these criteria is that they, along with (11) and (12) create prerequisites for computer analysis of the stability of linear ODE systems. In contrast to the nonlinear case, the construction of the criteria does not require the calculation of the approximate solution of the system.

If matrix \( A \) is constant, the criteria notation is simplified [8]. The stability criterion (18) will take the form
\[
\lim_{i \to \infty} B^{i+1} \leq \bar{c}_i = \text{const} \quad \forall t \in [t_0, \infty),
\]
where \( B = E + h A \).

The asymptotic stability criterion (19) transforms into the following relation:
\[
\lim_{t \to \infty} \lim_{i \to \infty} B^{i+1} \to 0.
\]

In this particular case, the proposed criteria differ in that they do not require information about the characteristic polynomial of the matrix and its roots.
4. Numerical and program experiment

The Lorentz system is investigated for stability

\[
\begin{align*}
\frac{dy_1}{dt} &= \sigma (y_2 - y_1), \\
\frac{dy_2}{dt} &= -y_1y_3 + ry_1 - y_2, \\
\frac{dy_3}{dt} &= y_1y_2 - by_3.
\end{align*}
\] (20)

Initially, the stability analysis is performed with initial conditions \( y_{10} = y_{20} = \sqrt{b(r-1)} \), \( y_{30} = r - 1 \) with parameter values \( \sigma = 10.1 \), \( r = 24.74 \), \( b = 2.666667 \).

After transformation into form (2), system (20) will take the form

\[
\begin{align*}
\frac{dx_1}{dt} &= \sigma ((x_2 + z_2) - (x_1 + z_1)) - \sigma (z_2 - z_1), \\
\frac{dx_2}{dt} &= -(x_1 + z_1)(x_3 + z_3) + r (x_1 + z_1) - (x_2 + z_2) + z_1z_3 - rz_1 + z_2, \\
\frac{dx_3}{dt} &= (x_1 + z_1)(x_2 + z_2) - b(x_3 + z_3) - z_1z_2 + bzh.
\end{align*}
\] (21)

Computer analysis of system stability (21) is performed based on criteria (11) – (14). A software model of stability analysis of ODE systems based on matrix multiplicative criteria is constructed [11]. The basic program uniformly implements cycles of the criteria, while the infinite product of the left part of the criteria is approximately realized as a partial product with the help of difference schemes. The result of its work is the output of the matrix norm of the value of the partial product at the current step. From the behavior of this norm, we can conclude about the stability, instability or asymptotic stability of the considered ODE systems. Unlimited growth of the norm denotes instability; its limitation corresponds to stability; its going to zero characterizes asymptotic stability.

Based on the numerical experiment, the boundaries of numerical parameters are established, at which the reliability of the computer analysis of stability according to the proposed criteria is preserved. Specifically, it was found that the software model with a range of interval from \([0, 100]\) to \([0, 10000]\) of the difference solution of the ODE system and at the step length from \([h]=10^{-7}\) to \([h]=10^{-4}\) reliably reveals the nature of the stability of the system of this class according to the proposed criteria.

In the case of establishing such boundaries as parameters of the program model, it achieves invariance with respect to the form of the right part of the system, the step value, and the length of the integration interval. In addition, the model is invariant with respect to the difference schemes of the approximate solution of the system.

This invariance of the constructed model allows reliable modeling of the stability of arbitrary ODE systems based on arbitrary difference methods in a relatively wide range of numerical parameters.

The presented interpretations of the asymptotic behavior of the solution should be considered close to reliable. Computer modeling cannot completely formally replace the mathematical study of the nature of stability, leaving the final solution of the problem to the qualitative theory. At the same time, in practice, within the framework of a wide program and numerical experiment, the developed method of analyzing the stability of ODE systems has always led to an exhaustively reliable assessment of the nature of stability.
Stability analysis of the system (21) is carried out at the interval [0, 1000] in step $h = 10^{-5}$. Based on the following operators (Delphi), computer stability analysis is performed according to criteria (11) and (12):

$$
\begin{align*}
\text{function } f_1(t, z_1, z_2, z_3 : \text{extended}) : \text{extended}; \\
\text{begin } f_1 := \sigma \times (z_2 - z_1); \\
\text{end;}
\end{align*}
$$

$$
\begin{align*}
\text{function } f_2(t, z_1, z_2, z_3 : \text{extended}) : \text{extended}; \\
\text{begin } f_2 := -(z_1^* + z_3) + (r^* z_1) - z_2; \\
\text{end;}
\end{align*}
$$

$$
\begin{align*}
\text{function } f_3(t, z_1, z_2, z_3 : \text{extended}) : \text{extended}; \\
\text{begin } f_3 := (z_1^* z_2) - (b^* z_3); \\
\text{end;}
\end{align*}
$$

begin
\begin{align*}
t := 0; \\
k := 0; \\
z_1 := \sqrt{b \times (r - 1)}; \\
z_2 := \sqrt{b \times (r - 1)}; \\
z_3 := r - 1; \\
x_1 := 0.00001; \\
x_2 := 0.00001; \\
x_3 := 0.00001; \\
\end{align*}

if \((x_1 < 0) \text{ and } (x_2 < 0) \text{ and } (x_3 < 0)\) then begin
\begin{align*}
p_1 := (1 + (h \times f_1(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_1)); \\
p_2 := (1 + (h \times f_2(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_2)); \\
p_3 := (1 + (h \times f_3(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_3)); \\
end;
\end{align*}

repeat
\begin{align*}
x_1 := x_1 + h \times f_1(t, z_1, z_2, z_3, x_1, x_2, x_3); \\
x_2 := x_2 + h \times f_2(t, z_1, z_2, z_3, x_1, x_2, x_3); \\
x_3 := x_3 + h \times f_3(t, z_1, z_2, z_3, x_1, x_2, x_3); \\
z_1 := z_1 + h \times f_1(t, z_1, z_2, z_3); \\
z_2 := z_2 + h \times f_2(t, z_1, z_2, z_3); \\
z_3 := z_3 + h \times f_3(t, z_1, z_2, z_3); \\
end;
\end{align*}

if \((x_1 < 0) \text{ and } (x_2 < 0) \text{ and } (x_3 < 0)\) then begin
\begin{align*}
p_1 := p_1 + (h \times f_1(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_1)); \\
p_2 := p_2 + (h \times f_2(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_2)); \\
p_3 := p_3 + (h \times f_3(t, z_1, z_2, z_3, x_1, x_2, x_3) / x_3)); \\
end;
\end{align*}

begin
\begin{align*}
k := k + 1; \\
t := t + h; \\
\text{if } k \geq 2000000 \text{ then begin}; \\
\text{norma} := \sqrt{\text{sqr}(p_1) + \text{sqr}(p_2) + \text{sqr}(p_3)}; \\
\text{writeln}("t = \text{t:5:0; ", \text{normal} = "\text{norma})); \\
k := 0; \\
end; \\
until t > 1000; \\
\end{align*}

With the program implementation of criteria (13) and (14), the value of the initial data perturbations is considered equal $10^{-5}$.

Subsequent operators (Delphi) implement criteria (13) and (14):

\begin{align*}
t := 0; \\
k := 0; \\
y_1 := \sqrt{b \times (r - 1)}; \\
y_2 := \sqrt{b \times (r - 1)}; \\
y_3 := r - 1; \\
\end{align*}
\text{yv1:=y1+eps1; yv2:=y2+eps2; yv3:=y3+eps3;}
\text{repeat}
\text{y_1:=y1; y_2:=y2; yv_1:=yv1; yv_2:=yv2;}
\text{y1:=y1+h*f1(t,y1,y2,y3); y2:=y2+h*f2(t,y_1,y2,y3);}
\text{y3:=y3+h*f3(t,y_1,y_2,y3); yv1:=yv1+h*f1(t,yv1,yv2,yv3); yv2:=yv2+h*f2(t,yv_1,yv2,yv3); yv3:=yv3+h*f3(t,yv_1,yv_2,yv3); k:=k+1; t:=t+h;}
\text{if k>=1000000 then begin}
\text{delta1:=(yv1-y1)/eps1; delta2:=(yv2-y2)/eps2; delta3:=(yv3-y3)/eps3; norma2:=sqrt(sqr(delta1)+sqr(delta2)+sqr(delta3)); writeln('t=',t:5:0,'   ','norma2=',norma2); k:=0; end; until t>1000;}

\begin{table}
\begin{tabular}{|c|c|c|}
\hline
\text{t} & \text{norma}_1 & \text{norma}_2 \\
\hline
100 & 1.51967175299834 & 1.51964747983185 \\
200 & 1.52667845430781 & 1.52670187348336 \\
300 & 1.76118191134798 & 1.76121501496277 \\
400 & 1.8920386766054 & 1.892097309 \\
500 & 1.76639158006125 & 1.76638239850 \\
600 & 1.55902574756429 & 1.55900554927315 \\
700 & 1.60539288275143 & 1.60542163108715 \\
800 & 1.85387046776363 & 1.85390210312243 \\
900 & 1.9579549038943 & 1.9578822793121 \\
1000 & 1.80352238881595 & 1.80348440583217 \\
\hline
\end{tabular}
\caption{Results of system stability analysis (20) with initial conditions $y_{10} = y_{20} = \sqrt{b(r-1)}$, $y_{30} = r-1$ with parameter values $\sigma=10.1$, $r=24.74$, $b=2.666667$.}
\end{table}

The second column of the table shows the norm values corresponding to stability criterion (11). The third column presents the norm values corresponding to criterion (12).

The presented norm values are limited by a constant, which in accordance with criteria (11) and (12) indicates stability.

Next, stability analysis of the system (20) is performed with parameter values $\sigma=10.1$, $r=24.1$, $b=2.666667$ and with unchanged initial conditions.

\begin{table}
\begin{tabular}{|c|c|c|}
\hline
\text{t} & \text{norma}_1 & \text{norma}_2 \\
\hline
100 & 2.52270794332446E-0001 & 2.5226756026934E-0001 \\
200 & 3.63735659052882E-0002 & 3.6373311658328E-0002 \\
300 & 5.17172426019912E-0003 & 5.17172646538134E-0003 \\
400 & 7.24760612262447E-0004 & 7.24765833067368E-0004 \\
500 & 1.00165120010766E-0004 & 1.00166419672831E-0004 \\
600 & 1.3676924487112E-0005 & 1.36771977329476E-0005 \\
700 & 1.85139681068736E-0006 & 1.8514552337542E-0006 \\
800 & 2.49820934268658E-0007 & 2.49695936142241E-0007 \\
900 & 3.38598736538033E-0008 & 3.34255692705603E-0008 \\
1000 & 4.6501509006704E-0009 & 7.7503394862876E-0010 \\
\hline
\end{tabular}
\caption{Results of system stability analysis (20) with initial conditions $y_{10} = y_{20} = \sqrt{b(r-1)}$, $y_{30} = r-1$ with parameter values $\sigma=10.1$, $r=24.1$, $b=2.666667$.}
\end{table}

The monotone decrease of norm values by both criteria testifies to asymptotic stability.
Stability analysis of the system (20) is performed below with parameter values $\sigma = 10.1, \ r = 25.15, \ b = 2.66667$ with the same initial conditions.

Table 3. Results of system stability analysis (20) with initial conditions

| $y_{10} = y_{20} = \sqrt{b(r-1)}$, $y_{30} = r-1$ with parameter values $\sigma = 10.1, \ r = 25.15, \ b = 2.66667$. | $t$ | $\text{norma}_1$ | $\text{norma}_2$ |
|---|---|---|---|
| 100 | 6.37314416942599E+0000 | 6.37311804407407E+0000 |
| 200 | 2.1662565925625E+0001 | 2.16628662987714E+0001 |
| 300 | 6.86776224902555E+0001 | 6.86791253395545E+0001 |
| 400 | 2.17172947384400E+0002 | 2.17175449681681E+0002 |
| 500 | 7.58681452010085E+0002 | 7.58671242297656E+0002 |
| 600 | 2.91025306391316E+0003 | 2.91018923033866E+0003 |
| 700 | 1.1040770858193E+0004 | 1.10406419527835E+0004 |
| 800 | 3.94803677105081E+0004 | 3.94804629815105E+0004 |
| 900 | 1.45280085563410E+0005 | 1.45278781733407E+0005 |
| 1000 | 5.8589771965835E+0006 | 6.24010744318018E+0006 |

The monotone growth of norm values indicates instability.

Computer implementation of the criteria demonstrated feasibility of using this approach in practice. The analysis based on these criteria allows to determine the nature of stability, asymptotic stability or instability of ODE systems without presenting the solution in analytical form, directly from the values of different approximations of the solution and the right part of the system. The results of the numerical experiment indicate the equivalence of the considered methods of computer analysis of stability.

5. Conclusions

The paper presents a method of computer analysis of general ODE systems stability that is based on matrix multiplicative stability criteria as necessary and sufficient conditions. Computer analysis based on the criteria is performed in real time and its results lead to an unambiguous conclusion about the nature of stability of the system under study. An important feature of the proposed criteria is that for their application it is sufficient to submit (at some (initial) point) the values of the right part of the ODE system to the input of a standard cyclic procedure. After that, no analytical transformations of the right side are required; nor is it necessary to enter additional data while running the program, or interrupt it. It follows that to analyze the stability of the control system, the model of which is described by the ODE system of the considered type, it is possible to use a standard block in the form of a procedure that programmatically implements the presented criteria. Because this does not require additional data input, computer stability analysis can be performed in real time. Thus, the analysis will correspond adequately to the actual values of the system parameters.

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