A weakly universal cellular automaton in the heptagrid with three states

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Abstract In this paper, we construct a cellular automaton on the heptagrid which is planar, weakly universal and which have three states only. This result improves the best result which was with four states.

Keywords cellular automata, universality, tilings, hyperbolic geometry.

1 Introduction

In this paper, we construct a weakly universal cellular automaton on the heptagrid, see Theorem 3 at the end of the paper. Two papers, [9, 5] already constructed such a cellular automaton, the first one with 6 states, the second one with 4 states. In this paper, the cellular automaton we construct has three states only. It uses the same principle of simulating a register machine through a railway circuit, but the implementation takes advantage of new ingredients introduced by the author in his quest to lower down the number of states, see [7]. It also introduce new constructions which significantly improve the scenario used in the previous papers. The reader is referred to [3, 4, 7] for an introduction to hyperbolic geometry turned to the implementation of cellular automata in this context. A short introduction can also be found in [2]. However, it is not required to be an expert in hyperbolic geometry in order to read this paper.

We refer the reader to [6] for a discussion about weak universality. Here, we just remember the main characteristics of our cellular automaton with respect to universality. The simulation is performed by a locomotive running on a railway circuit, using the basic elements introduced in [11]. The simulation is a planar one in this meaning that the trajectory of the locomotive involves infinitely many cycles when the computation does not halt. Now, the initial configuration is infinite but, outside a finite domain, it consists of two periodic structures. This is why the automaton is called weakly universal. In the paper, we remember the basic model we use in Section 2 and we stress on the new features in Section 3 were we thoroughly describe the implementation of the model in the heptagrid. In Section 2.1 we remind the reader with the general setting of the heptarid, the tiling \( \{7, 3\} \) of the hyperbolic plane. Section 4 gives the rule of the cellular
automaton, which allows us to prove Theoreme 3.

2 Basic features

The railway circuit, mentioned in the introduction consists of tracks, crossings and switches, and a single locomotive runs over the circuit. The tracks are pieces of straight line or arcs of a circle. In the hyperbolic context, we shall replace these features by assuming that the tracks travel either on verticals or horizontalss and we shall make it clear a bit later what we call by these words.

The crossing is an intersection of two tracks, and the locomotive which arrives at an intersection by following a track goes on by the track which naturally continues the track through which it arrived. Again, later we shall make it clear what this natural continuation is. Below, Figure 1 illustrates the switches and Figure 2 illustrates the use of the switches in order to implement a memory element which exactly contains one bit of information.

The three kinds of switches are the fixed switch, the flip-flop and the memory switch. In order to understand how the switches work, notice that in all cases, three tracks abut the same point, the centre of the switch. On one side switch, there is one track, say a, and on the other side, there are two tracks, call them b and c. When the locomotive arrive through a, we say that it is an active crossing of the switch. When it arrives either through b or c, we say that it is a passive crossing.

In the fixed switch, in an active passage, the locomotive is sent either always to b or always to c, we say that the selected track is always b or it is always c. In the passive crossing, the switch does nothing, the locomotive leaves the switch through a. In the flip-flop, passive crossings are prohibited: the circuit must be managed in such a way that a passive crossing never occurs at any flip-flop. During an active passage, the selected track is changed just after the passage of the locomotive: if it was b, c before the crossing, it becomes c, b respectively after it. In the memory switch, both active and passive crossings are allowed. The selected tracks also may change and the change is dictated by the following rule: after the first crossing, only in case it is active, the selected track is always defined as the track taken by the locomotive during its last passive crossing of the switch. The selected track at a given switch defines its position.

The current configuration of the circuit is the position of all the switches of the circuit. Note that it may be coded in a finite word, even if the circuit is infinite, as at each time, only finitely many switches have been visited by the locomotive.
Figure 2 illustrates how a flip-flop and a memory switch can be coupled in order to make a one bit memory element.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The basic element of the circuit. Second row: reading the element, 0, then 1. Third row: first two elements: writing 0; last two elements: writing 1.}
\end{figure}

2.1 In the heptagrid

As mentioned in the introduction, the first weakly universal cellular automaton on the heptagrid was done by the author and a co-author, see [9]. The paper implements the solution sketchily mentioned in Subsection 2 with 6 states. In the next paper about a weakly universal cellular automaton on the heptagrid, see [5], the same model is implemented with 4 states. In [9] the cells of the tracks where the locomotive runs have a specific colour and the locomotive is implemented with two cells, the front and the rear. The front is green and the rear is red while the tracks are blue. Later, in [5], the tracks are changed: the cells where the locomotive runs are also in the quiescent state when the locomotive is far away. The tracks are delimited by milestones which are regularly put along the track. The milestones are most often blue. The locomotive is still implemented with two cells, the front being blue, as the milestones, and the rear being red. As in many of the indicated papers, the centre of the switch is signalized by the neighbouring of the centre.

2.1.1 Former implementations

Figure [4] illustrates the implementation of the crossing and of the switches performed in [9], showing in particular, the feature at which we just pointed. Fig-
Figure 3 shows a global view of how the tree structure of the tiling can be used to implement a basic element in the heptagrid. We have the following results.

**Theorem 1** (Margenstern, Song), cf. [9] – There is a planar cellular automaton on the heptagrid with 6-states which is weakly universal and rotation invariant.

**Theorem 2** (Margenstern) cf. [5] – There is a planar cellular automaton on the heptagrid with 4-states which is weakly universal and rotation invariant.

Figure 3 illustrates the configuration of the railway circuit implementing a toy program of a register machine. All the constructions performed in previous papers and in this one too produce a configuration whose guidelines obey those illustrated by Figure 3.

Now, figures 2 and 5 show that the implementation of the basic element involves two cycles. As a register consists of infinitely many copies of a unit, again look at Figure 3 and as a unit contains two basic elements, we can see that a non halting computation entails that the locomotive runs over infinitely many cycles, so that the computation is planar.

Figure 4 illustrates the idle configuration of the crossing and the switches of the circuit for the cellular automaton constructed in the proof of Theorem 1. By idle configuration, we mean a configuration where the locomotive is not present.
3 The scenario of our simulation

Most of the cellular automata in hyperbolic spaces I constructed, myself or with a co-author, apply the same model of computation. This general scenario is described in detail in [4, 7]. Here, we simply give the guidelines in order to introduce the changes which are specific to this implementation.

3.0.2 The new scenario

In this paper, we take benefit of various improvements which I brought in the construction of weakly universal cellular automaton constructed in other contexts: in the hyperbolic 3D-space and in the tiling \{13,3\} of the hyperbolic plane, that latter automaton having two states only, see [7] for details. But here we need something more, as soon explained.

Our implementation follows the same general simulation as the one described in Subsection 2. In particular, Figure 5 is still meaningful in this new setting.
However, here, new features are introduced.

The first change is that the tracks are one-way. In some sense this is closer to what we can see for railways in real life, in particular for high-speed ones. This change entails a big change in the switches and in the crossings. There is no change for the flip-flop which was already a one-way structure from the very beginning as passive crossings are ruled out for this kind of switches. For fixed switches it introduces a very small change: for a passive crossing, we keep the structure and for the active one, as the selected track is the same, it is enough to continue the active way without branching at the centre of the switch, see Figure 6. In the same picture, we can see that the situation is different for the memory switch. This time, as there are two possible crossings of the switch and as the selected track may change, we have two one-way switches: an active one and a passive one. At first glance, the active switch looks like a flip-flop and the passive switch looks like a one-way fixed one. However, due to the working of the memory switch, we could say that the active memory switch is passive while the passive memory switch is active. Indeed, during an active crossing, the selected track is not changed contrary to what happens in the case of the flip-flop. Now, during a passive crossing, the switch looks at which track is crossed: the selected or the non-selected one. If it is the non-selected one, then the selection is changed and this change is also transferred to the active switch. Accordingly, there is a connection between the active and the passive one-way memory switches.
Now, if we wish to significantly reduce the number of states, we also have to change the locomotive: it is now reduced to a single cell which, in principle is made possible by the fact that the tracks are one-way. This is a change with respect to the weakly universal cellular automaton with 4 states, see [5]. But this is not enough: we also have to change the crossings. Contrary to what happens in the 3D-space where crossings can be replaced by bridges, which makes the situation significantly easier, crossings cannot be avoided in the plane.

In [7] we indicate a solution which allowed me to build a weakly universal cellular automaton in the hyperbolic plane with 2 states only. However, this was not performed in the pentagrid nor in the heptagrid, but in the tiling \{13, 3\}. This solution was already implemented in the pentagrid, which allowed me to reduce the number of states from 9 down to 5. Here, we improve the result for four states down to 3 states. However, the implementation of the new crossing is somehow tuned from what is done in [7] and in [8]: The pattern at the branching of [7, 8] is changed: a doubling structure is put on the track arriving to the crossing.

First, we look at a crossing of two one-way tracks. The main idea is that we organize the crossing in view of a round-about: an interference of road traffic in our railway circuit. Figure 7 illustrates this structure which we call the simple round-about. We may notice that the locomotive arriving either from A or B in Figure 7 has to turn right at the second pattern it meets on its way. Note that the locomotive arrives at a simple round-about after crossing a rhomb pattern where the locomotive is doubled: at this point a second locomotive is appended to the arriving one. At the first branching it meets, one locomotive is deleted. At the second one, the single locomotive is sent on the right track.
Figure 7: The new crossing: the one-way tracks from A and B intersect. We have a three-quarters round-about. The small disc at f represents a fixed switch. Discs 1, 2 and 3 represent the pattern which dispatches the motion of the locomotive on the appropriate way. Patterns 1 and 3 are needed as explained in the description of the scenario.

Figure 8: The new crossing: four possible one-way track. Assembling them allows to perform a two-way crossing. The notations are those of Figure 7. A₁, B₁ go opposite to A, B, respectively.

Figure 8 shows us how to assemble four one-way simple round-abouts in order to perform a true crossing for two intersection two-ways tracks. The structure of Figure 8 is called a full round-about.

But this change is not enough. We have to device a new implementation for both the flip-flop and the memory switch to which we now turn.

This new scenario for the implementation of these switches is a consequence of the reduction of the number of states to 3 ones. The idea is to put in a common structure the different working of the active part of the memory switch.
and the flip-flop. In fact, we have to dissociate the dispatching of the locomotive from the control of the selected track.

For this purpose, we introduce two new structures: the fork and the killer. The fork looks like a switch to which the locomotive arrives from the track \( a \). But, after crossing the central cell, two locomotives leave the fork: one through the track \( b \), the other through the track \( c \). Now, the role of the killer is to kill one of these locomotives as a single locomotive leaves a switch. We simply take advantage that the selection may happen a bit further from the switch: this allows us to relax the crossing of the fork leaving two locomotives. Now, the killer has two positions: an active one and a passive one. When the killer is active, if a locomotive arrives at the killer, it is destroyed while it crosses its structure. When the killer is passive, it let the locomotive leave its structure. We can see that it is enough to construct the killer in such a way that a signal arriving to it changes its status from active to passive and from passive to active.

![Figure 9](image_url)

**Figure 9** The flip-flop. Note the forks \( C \), \( A \) and \( S \) and the killers \( L \) and \( R \). Note the intersection close to \( R \) which need a simple round about only.

Figure 9 shows us how we can assemble three forks and two killers in order to get a flip-flop. Indeed, the locomotive arrives at \( C \) as indicated by the arrow below the fork. Then, two locomotive leave the fork. The left-hand side locomotive \( \ell \) arrives to the killer \( L \) which, in the figure, is assumed to be passive. Accordingly, \( \ell \) goes on its way on the circuit. The right-hand side locomotive \( r \) arrives to a new fork, \( A \). Then, two locomotives leave \( A \). Call \( r \) the one which goes to the left and \( s \) the one which goes to the right. We can see that \( r \) arrives to the killer \( R \) which is in the other configuration than the killer \( L \). Accordingly, \( R \) is active and it kills the locomotive \( r \). This means that the locomotive is now \( \ell \). Let us look at what happens with \( s \). This locomotive is sent to a new fork \( S \). From there, two locomotives leave the fork, \( s_\ell \) to the left, \( s_r \) to the right. We can see that \( s_\ell \), \( s_r \) arrives to \( L \), \( R \) respectively. Accordingly, both killers change their configuration: \( L \) becomes active and \( R \) becomes passive. This is exactly what a flip-flop has to do.

Figure 10 represents the assembling of forks and killers in order to get an
active memory switch. Here, we can see that the locomotive arriving to $C$ triggers the sending of two locomotives $\ell$ and $r$, to left and to right respectively. We can see that $\ell$ crosses a passive killer so that it goes on its way and will become the continuation of the locomotive which arrived to $C$. On the other hand, $r$ arrives to $R$ which is active, so that $r$ is cancelled at this moment. Now, the killers do not change their configuration. The configuration is changed only when a locomotive arrives to $S$, sent from the passive memory switch, in order to change the selection of the switch.

![Figure 10](image)

**Figure 10** The active memory switch. Note the forks $C$ and $S$ and the killers $L$ and $R$.

**Table 1** Scheme of execution of the flip-flop and the active memory switch.

|   |   |   | $\ell$ | $r$ |   |   |   |   |
|---|---|---|-------|-----|---|---|---|---|
| 0 | $\bullet$ | $b$ | $r$ | 0 | $\bullet$ | $b$ | $r$ |
| 1 | $\bullet$ | $b$ | $r$ | 1 | $\bullet$ | $b$ | $r$ |
| 2 | $\bullet$ | $b$ | $r$ | 2 | $\bullet$ | $b$ | $r$ |
| 3 | $\bullet$ | $b$ | $r$ | 3 | $\bullet$ | $b$ | $r$ |
| 4 | $\neg b$ | $\neg r$ | $b$ | 4 | $\neg b$ | $\neg r$ | $b$ | $r$ |
| $n+1$ | $\neg b$ | $\neg r$ | $b$ | $n+1$ | $\neg b$ | $\neg r$ | $b$ | $r$ |

Table 1 summarizes the working we above indicated. Note that in each sub-table, the leftmost column indicates a time. It should be stressed that between two such times, the number of steps of the cellular automaton may be very different. The only requirement is that before the visit of the next site by the locomotive, switch or crossing, all changes involved by the current site have been completed. In particular, when $S$ triggers two locomotive in order to change
the status of both $L$ and $R$, there is no need that the change happens at the same step of the cellular automaton. It is enough that when the locomotive arrives to the next site, the change has been performed. Note that we also can consider that the changes at a site are completed when there is again exactly one locomotive on the circuit. While performing the crossing of a flip-flop, there can be up to three locomotives running on the circuit at the same instant. When the flip-flop has changed the selection of the tracks, there remains exactly one locomotive.

Now that we have seen the scenario to implement the circuit, we have to precisely look at how to implement it with three states only. We turn now to this question.

### 3.0.3 Implementing the new scenario

As already announced, we need to use three states only. The first state is the quiescent state which we denote by $\mathcal{W}$. Remember it is defined by the following rule: if a cell $c$ and all its neighbours are in the state $\mathcal{W}$, then at the next top of the clock, the cell $c$ remains in the state $\mathcal{W}$. We shall also call $\mathcal{W}$ the blank and we also shall say that a cell in $\mathcal{W}$ is white. The other states are $\mathcal{B}$ and $\mathcal{R}$. The cells in these states are said to be blue and red, respectively. The state $\mathcal{B}$ is mainly used for the milestones which delimit the tracks. The state $\mathcal{R}$ is the basic state of the locomotive, remember that now it only consists of one cell. In several implementations, the marking of the structure also makes use of the state $\mathcal{R}$.

Checking the new scenario requires to first study the implementation of the tracks. We have to define verticals and horizontal.

#### Vertical and horizontal tracks

The motion is basically constructed by motions around a cell, the **pivot**, with two kinds of links from one pivot to another in a way which is slightly different from what we did in the grid $\{13, 3\}$. Figure 11 shows the elements of the track and, for each of them, how it is crossed by the locomotive.

To better understand the figure, number the sides of the central cell as follows. Number the sides increasingly while counter-clockwise turning around the cell. The milestones occur on the sides 1, 3 and 6 in the first row, 1 being the side shared by the leftmost milestone.

The first element is the **standard** one, on the first row of the figure. The second element is the **turn**, on the second row of the figure. There the locomotive goes from side 7 to side 2, exactly the reverse motion of the previous one and the milestones are on sides 1, 3, 4 and 6. The last element is the **junction**, on the third row of Figure 11. There, the locomotive goes from side 5 to side 2, with milestones on sides 1, 3, 4, 6 and 7. In the first row of Figure 11 we can say that the pivot is the blue cell sharing side 1. As will be seen in Section 4, we can append rules so that the same motion is possible for each of these elements with two contiguous red cells instead of a single one.

In Figure 12 we make use of the standard element and of the turn in order to perform tracks travelling along a branch of a Fibonacci tree: we call this track
a **vertical**. As can be seen on the figure, the standard elements are turning around a pivot, running along a half of its sides and passing from one pivot to the other thanks to the turn. The turn is absolutely needed in order to obtain tracks which can go from any tile to any other one.

![Figure 11](image1.png)

**Figure 11** The elements of the track. For each of them, the figures show how it is crossed by the locomotive. From top to bottom: the standard element, the turn and the junction.

![Figure 12](image2.png)

**Figure 12** A vertical motion for the locomotive. First two rows: a single locomotive; last two rows: two contiguous locomotives travelling together.

In Figure 13 we can see the implementation of what we call a **horizontal**: it is a track running along a level of a Fibonacci tree. The pivots are on such a level and the track itself, where the locomotive runs, consists of the sons of the nodes which are on the level of the pivots. To go from the sons of a node to the sons of the next node, we precisely need the junction. As can be seen on the figure, the same motion can be managed with two contiguous locomotives.
travelling on the same path.

As seven Fibonacci trees whose root are displayed around a fixed tile constitute a circular path of any radius, combining this with verticals as also show in Figure 13 we can see that we indeed can construct a path from a tile to another: we construct a shortest path whose tiles play the role of pivots and we go from one pivot to the next one through a turn or through a junction, depending on the configuration.

![Figure 13](image)

A horizontal with a locomotive, first four rows and then, two contiguous locomotives.

Remark that as horizontal tracks run on three consecutive levels, a two-way portion of the tracks require at least seven consecutive levels as we need at least one level to separate the tracks run on each direction. Similarly, a two-way section of vertical lines must be separated by several nodes on the same level at the level where the distance between the supporting line is minimal: this guarantees that the lines are not secant. These constraints require much space for the implementation, but in the hyperbolic plane, we are never short of space.

Implementing the crossings

As indicated in Subsubsection 3.0.3 the tracks are organized according to what is depicted in Figures 7 and 8. From the latter figure, it is enough to
focus on the implementation of Figure 7. From the implementation of the tracks, we only have to look at the implementation of the patterns symbolically denoted as 1, 2, 3, f and by a rhomb in the figure. As f is a fixed switch whose implementation is indicated a bit further, we simply implement 1 and the rhomb as 2 and 3 are strict copies of 1. Figure 15 illustrates this implementation and the behaviour of the locomotive when it crosses pattern 1. We can see that the difference strongly depends on whether a single locomotive or two of them arrive to the pattern.

From Figure 7, the locomotive always arrives to the pattern from the same cell, namely 9(5). Then the locomotive goes to the cell 1(6) if it is alone. If two contiguous locomotives arrive, then a single one arrives to the cell 0 and then goes on its way on the round-about. First, Figure 14 illustrates the working of the rhombic pattern which duplicates a locomotive arriving to the pattern.

**Figure 14** The rhombic pattern of the crossings. A single locomotive arrives at the pattern, two contiguous ones leave the pattern.
Figure 15 The pattern of the crossings. Notice the difference of behaviour depending on the number of locomotives. When a single locomotive is detected, the locomotives goes to the track. When two contiguous locomotives are detected, one of them is cancelled and the other goes on its way on the round-about.

Next, Figure 15 illustrates the pattern which is common to 1, 2 and 3 in Figure 7. We can see the difference of behaviour depending on the number of contiguous locomotives which arrive at the pattern. If a single locomotive arrives at the pattern, then the locomotive is sent to the track which leaves the round-about at this branching point. This is what is illustrated by the first two rows of Figure 15. If two contiguous locomotives arrive at the pattern, then one locomotive is destroyed before it is seen by the first cell of the track which leaves the round-about in the pattern. Then, the second locomotive is sent to the way which continues the round-about. This is illustrated by the last two rows of Figure 15.

Implementation of a fixed switch

Figure 16 illustrates the working of the passive crossing of a fixed switch.

The upper half of Figure 16 illustrates the case of a single locomotive. In the first two rows of the figure, the locomotives arrives to the switch from the left. In the last two rows, it arrives from the right.

Now, the lower half of Figure 16 illustrates the case of two contiguous locomotives. Again, the first two rows of the figure illustrate the case when the locomotives arrive to the switch from the left. The last two rows illustrate the case when these locomotives arrive to the right.

With one-way tracks, we need a single kind of passive fixed switch. Indeed, there is no need of an active fixed switch, as the locomotive never goes in the non-selected direction. The active switch is reduced to the track which goes in
the selected direction, as illustrated by Figure 5.

We remain with the more complicated switches: the flip-flop and the memory switch. Remember from Sub-section 3.0.2 that we split the memory switch into two sub-switches, the active memory switch and the passive one, due to the introduction of one way tracks for the locomotive. In the same sub-section, we have seen that the flip-flop and the active memory switch can be implemented by using the same structures, the fork and the killer: we only have to dispatch them in different way in order to implement the two kinds of switches. Now, we could probably use the fork to implement the passive memory switch, but it would require another structure, let us call it the sensor. As we succeeded to implement a sensor with what is indeed another fork in a single structure, we start with the passive memory switch.
The fixed switch: passive crossing by the green locomotive and then the red one.

Implementation of a passive memory switch

Both idle configurations are illustrated by Figure 17: the left- and the right-hand side switch.

The stable configurations of the passive memory switch: the side of the selected track is that of the picture.
The working of this switch is illustrated by Figure 18. The upper half of the figure illustrates the working of the left-hand side switch. In the first two lines, the locomotive arrives from the left so that nothing happens. In the next two lines, the locomotive arrives from the right-hand side: this time the selection is changed to the right-hand side one.

In the lower half, the figure illustrates the working of the right-hand side switch. In the first two lines, the locomotive arrives from the right so that nothing is changed. In the last two lines it arrives from the left: accordingly, the selection is changed to the left-hand side one.
Implementation of the flip-flop and of the active memory switch

According to Figures 10 and 9 what we have to implement is the fork and the killer.

First, let us have a look at Figure 19. It illustrates the working of the killer. In the upper half, we have the situation of a locomotive arriving to the killer. We say that the killer is blue, red depending on whether its central cell in Figure 19 is blue or red.

![Figure 19](image)

**Figure 19** The control, structure used both by the flip-flop and by the active memory switch. When killer is blue, the locomotive crosses the killer and goes on its way on the track. When the killer is red, the locomotive is destroyed. The lower half of the figure: when the signal arrives at the killer, it changes its colour.

On the upper half of the figure, we can see that if the killer is red, a locomo-
tive arriving there cannot pass the killer. But when it is blue, the locomotive can pass the killer and so it goes on its way on the track. The lower part of the figure shows us how a signal arriving to the killer changes its colour. We can see that the change from red to blue takes 5 steps while the change from blue to red takes 3 steps only. As will be seen in Section 4, in the change from red to blue, we have to take into accounts rules for other motions which apply also in this case.

Figure 20 illustrates the fork. We can see that a locomotive arriving at the fork is duplicated and each new locomotive goes on a different track.

![Figure 20](image)

**Figure 20** The fork, the other structure used both by the flip-flop and by the active memory switch.

In Sub-section 3.0.2 we have seen how the killer and the fork are combined in order to obtain the working of a flip-flop or that of an active memory switch.

It remains to remark that the signal which arrives to both killers come from the switch itself in the case of the flip-flop and that it comes from another switch in the case of the active memory switch. In that latter case, the signal comes from the passive memory switch: when the locomotive arrives from the non-selected track, Figure 18 shows that a signal is emitted from the switch: see the lines 3 and 7 of the figure. The track which arrives to $S$ in Figure 11 comes from a passive memory switch.

Figure 21 illustrates the connection between the passive memory switch, the blue disk with a purple circle on the right-hand side of the figure, and the active memory switch, the orange square, with a black border, on the left-hand side of the picture. We may imagine that inside the orange square we have a copy of Figure 10 considering that the orange arrow which arrives to the square from above arrives to the fork $S$ inside the square. Also, the tracks which leave the killers in the copy of Figure 10 are the tracks which leave the upper corners of the orange square in Figure 21. With this figure, Figures 10 and 19 we can check that the working of the active memory switch is conformal to what we have described in Sub-section 3.0.2.

Here, we may again note that up to three locomotives may occur at the same time on the circuit. Say that the time of execution of an elementary action in our circuit coincide with the arrival of the locomotive at the central cell of either a rhombic pattern, a passive memory switch, or a fork $C$. Then,
as already mentioned, we may construct the circuit in such a way that when the locomotive starts an elementary action, it is the single locomotive of the circuit. With this condition, it is clear that the just considered implementation is correct with respect to the scenario described in Sub-section 3.0.2.

![Figure 21](image)

**Figure 21** The connection from the passive memory switch to the active one. The active switch is represented by the big orange square.

4 The rules

In order to prove the existence of the cellular automaton whose working was described in the previous sections, we have to implement its rules. Here, we display all of them, but we list them in several groups according to the presentation of the previous implementation. We present the rules according to the same order as we presented the implementation in Subsection 3.0.3.

The rules are rotation invariant. This means that if we perform a circular permutation on the neighbours of a cell, this does not change the new state. Accordingly, in the tables we give, the rules are rotationally invariant: a given rule \( \rho \) has four over rules with the same current state and the same new state, the states of the neighbours being a circular permutation of those indicated by \( \rho \). The rules have been checked by a computer program written by the author. The program insures that the rules are rotation invariant and that there is not conflict between them. Also, the program constructed the figures used in Section 3 so that it allowed me to check that the rules make the cellular automaton to perform what it is expected to do.
4.0.4 Rules for the tracks

First, we have the rules for the motion of the locomotive on the tracks which are displayed in Table 2. The rules are presented in the same order as established while checking them during the construction by a computer program.

As indicated in the caption of the table, the first two columns of rules are used for the vertical tracks and the last two columns contain additional rules needed by the horizontal tracks. Of course, as the standard element is present in both kinds of tracks, several rules used for the vertical tracks are also used for the horizontal ones. Some rules are conservative: they do not change the state of the cell. In particular, these rules apply to those of the milestones which remain unchanged while the locomotive passes close to them. However, some of them may change their state for one or two steps of the computation. Then we apply the other rules which are not conservative. Many rules are motion rules: either they apply to the locomotive itself or two of them when two of them are contiguous. We also consider as motion rules, rules which change the state of a milestone in order to contribute to the motion. This is the case, in particular, for the rules which apply to those signals which may change, according to the kind of switch to which they are associated. Note that in many cases a motion rule is not conservative, but in a few cases, a motion rule is also conservative: in particular, when the locomotive leaves a cell $c$, the rules which apply to $c$ just after the locomotive left it is conservative.

As an example of illustration of motion rules, consider rules 7 up to 11. Rules 7, 8, 9 and 11 govern the motion of a single locomotive as it can easily be checked on figure 11. Rule 10 is an additional rule needed by the motion of two contiguous locomotives moving together.

|   |   |   |   |   |
|---|---|---|---|
| 1 |WBWBBBBB| 19 |WBBWBBWBB| 31 |WBBWBBWBB| 49 |WBBWBBWBB|
| 2 |WBWBBBBB| 20 |WBBWBBWBB| 32 |WBBWBBWBB| 50 |WBBWBBWBB|
| 3 |WBWBBBBB| 21 |WBBWBBWBB| 33 |WBBWBBWBB| 51 |WBBWBBWBB|
| 4 |WBWBBBBB| 22 |WBBWBBWBB| 34 |WBBWBBWBB| 52 |WBBWBBWBB|
| 5 |WBWBBBBB| 23 |WBBWBBWBB| 35 |WBBWBBWBB| 53 |WBBWBBWBB|
| 6 |WBWBBBBB| 24 |WBBWBBWBB| 36 |WBBWBBWBB| 54 |WBBWBBWBB|
| 7 |WBWBBBBB| 25 |WBBWBBWBB| 37 |WBBWBBWBB| 55 |WBBWBBWBB|
| 8 |WBWBBBBB| 26 |WBBWBBWBB| 38 |WBBWBBWBB| 56 |WBBWBBWBB|
| 9 |WBWBBBBB| 27 |WBBWBBWBB| 39 |WBBWBBWBB| 57 |WBBWBBWBB|
|10 |WBWBBBBB| 28 |WBBWBBWBB| 40 |WBBWBBWBB| 58 |WBBWBBWBB|
|11 |WBWBBBBB| 29 |WBBWBBWBB| 41 |WBBWBBWBB| 59 |WBBWBBWBB|
|12 |WBWBBBBB| 30 |WBBWBBWBB| 42 |WBBWBBWBB| 60 |WBBWBBWBB|
|13 |WBWBBBBB| 31 |WBBWBBWBB| 43 |WBBWBBWBB| 61 |WBBWBBWBB|
|14 |WBWBBBBB| 32 |WBBWBBWBB| 44 |WBBWBBWBB| 62 |WBBWBBWBB|
|15 |WBWBBBBB| 33 |WBBWBBWBB| 45 |WBBWBBWBB| 63 |WBBWBBWBB|
|16 |WBWBBBBB| 34 |WBBWBBWBB| 46 |WBBWBBWBB| 64 |WBBWBBWBB|
|17 |WBWBBBBB| 35 |WBBWBBWBB| 47 |WBBWBBWBB| 65 |WBBWBBWBB|
|18 |WBWBBBBB| 36 |WBBWBBWBB| 48 |WBBWBBWBB| 66 |WBBWBBWBB|

And so, we can see that the motion on the tracks requires 62 rules.
4.0.5 Rules for the crossing

Table 3 provides additional rules for the crossing. It can be noticed in Figures 14 and 15 that the central cell of the rhombic pattern and pattern 1 have different neighbourhoods. For the rhombic pattern it is BBWRRW and for pattern 1 it is BBBRRWR. The W-neighbours of the B-part of the ring around the central cell is the same in both patterns, but they are placed in opposite places which explains the difference of working of the patterns and also the fact that they share many rules so that it is not easy to clearly identify which rules are used in one of these patterns and not in the other.

Table 3 The rules for the crossings.

| Rule | Description |
|------|-------------|
| 63   | BBBRRWRW    |
| 64   | BBBRRWRW    |
| 65   | BBBRRWRW    |
| 66   | BBBRRWRW    |
| 67   | BBBRRWRW    |
| 68   | BBBRRWRW    |
| 69   | BBBRRWRW    |
| 70   | BBBRRWRW    |
| 71   | BBBRRWRW    |
| 72   | BBBRRWRW    |
| 73   | BBBRRWRW    |
| 74   | BBBRRWRW    |
| 75   | BBBRRWRW    |
| 76   | BBBRRWRW    |
| 77   | BBBRRWRW    |
| 78   | BBBRRWRW    |
| 79   | BBBRRWRW    |
| 80   | BBBRRWRW    |

We can see from Tables 2 and 3 that the crossings require 64 additional rules, not taking into account the many rules from Table 2 it uses too.

4.0.6 Rules for the fixed switch

These rules are displayed by Table 4.

Table 4 The rules for the fixed switch.

| Rule | Description |
|------|-------------|
| 127  | BBBRRBBW    |
| 128  | BBBRRBBW    |
| 129  | BBBRRBBW    |
| 130  | BBBRRBBW    |
| 131  | BBBRRBBW    |
| 132  | BBBRRBBW    |
| 133  | BBBRRBBW    |
| 134  | BBBRRBBW    |
| 135  | BBBRRBBW    |
| 136  | BBBRRBBW    |
| 137  | BBBRRBBW    |
| 138  | BBBRRBBW    |
| 139  | BBBRRBBW    |
| 140  | BBBRRBBW    |
| 141  | BBBRRBBW    |

23
As mentioned in the table, the switch may be crossed either by a single locomotive or by two locomotives always running contiguously.

### 4.0.7 Rules for the passive memory switch

Table 5 gives the rules for the passive memory switch. It concerns both versions of the switch: one when the left-hand side track is selected, the other when the right-hand side track is selected. The cells which signalize the selected and the non-selected tracks are the cells sharing side 2 and side 7 of the central cell which is a blue milestone, see Figure 18. In the first two rows of the figure, the red signal is in the cell sharing side 7.

**Table 5** The rules for the passive memory switch.

|     |     |     |     |     |
|-----|-----|-----|-----|-----|
| gauche, venue | 168 | VBRWBRWV | 186 | BRRWWRBB |
| de droite | 169 | VBRWBRWV | 187 | WRWWWBRV |
| 150 | VBBBWBRR | 170 | VBBBWBRR | 188 | RWBRRBBR |
| 151 | VBBBWBRR | 171 | VBBBWBRR | 189 | WRBRRBBR |
| 152 | VBBBWBRR | 172 | VBBBWBRR | 190 | BBBBBBRR |
| 153 | VBBBWBRR | 173 | VBBBWBRR | 191 | BBBBBBRR |
| 154 | VBBBWBRR | 174 | VBBBWBRR | 192 | BBBBBBRR |
| 155 | VBBBWBRR | 175 | VBBBWBRR | 193 | BBBBBBRR |
| 156 | VBBBWBRR | 176 | VBBBWBRR | 194 | BBBBBBRR |
| 157 | VBBBWBRR | 177 | BBBBBBRR | 195 | BBBBBBRR |
| 158 | VBBBWBRR | 178 | BBBBBBRR | 196 | BBBBBBRR |
| 159 | VBBBWBRR | 179 | BBBBBBRR | 197 | BBBBBBRR |
| 160 | VBBBWBRR | 180 | VBBBWBRR | 198 | BBBBBBRR |
| 161 | VBBBWBRR | 181 | VBBBWBRR | 199 | VBBBWBRR |
| 162 | VBBBWBRR | 182 | VBBBWBRR | 200 | VBBBWBRR |
| 163 | VBBBWBRR | 183 | VBBBWBRR | 201 | VBBBWBRR |
| 164 | VBBBWBRR | 184 | VBBBWBRR | 202 | VBBBWBRR |
| 165 | VBBBWBRR | 185 | VBBBWBRR | 203 | BBBBBBRR |
| 166 | VBBBWBRR | 186 | VBBBWBRR | 204 | BBBBBBRR |
| 167 | VBBBWBRR | 187 | VBBBWBRR | 205 | BBBBBBRR |

We note that when the locomotive, always a single one, crosses the switch through the non-selected track, the central cell turns from blue to red for just one time. This is performed by rule 150 when the locomotive comes from the right-hand side and rule 195 when it comes from the left-hand side. The shutdown of this signal is performed by rules 152 and 195 respectively. We have then rules which perform the exchange between red and blue in both neighbours of the central cell sharing side 2 and side 7: rules 189 and 205 for the cell sharing side 7, rules 176 and 204 for the cell sharing side 1. At the same time, the cell sharing side 1 turns from white to red for just this time: this triggers a new locomotive which is sent to the active switch. The rule initiating the locomotive is rule 156, and rule 157 turns back the cell to white, contributing to the motion of the new locomotive.
Table 6 Execution trace of the crossing of the passive memory switch by the locomotive through the non-selected track. The neighbours are identified by the side shared with the central cell.

|   |   |   |   |
|---|---|---|---|
| 2 | 1 | 7 | 0 |
| B | W | R | B |
| B | W | R | R |
| R | R | B | B |
| R | W | B | B |

4.0.8 Rules for the fork and for the killer

Here, the rules will be divided into two tables: Table 7 and Table 8 for the killer and for the fork respectively.

As explained in Section 8, the killer is a controlling structure. In Figure 19, the central cell is a milestone which is a signal for the locomotive. Note that a single locomotive arrives to the killer. If the signal is blue, the locomotive goes on its way and leaves the structure. If the signal is red, the locomotive is destroyed by the structure. For instance, rules 217 and 218 which apply to the central cell witness that the locomotive passes its way when the signal is blue. Rules 219, 220 and 221 apply when the signal is red and we can see on rules 219 and 221 that the locomotive disappears: rule 219 is applied just after rule 221 was applied. The 'killing' rule is rule 168, a rule already used by the passive memory switch.

Table 7 The rules for the killer.

| Rule | Colour | Colour | Colour |
|------|--------|--------|--------|
| 210  | BBBBBB | WWWWWW | VVRVRR |
| 211  | BBBBBB | WWWWWW | VVRVRR |
| 212  | BBBBBB | WWWWWW | VVRVRR |
| 213  | BBBBBB | WWWWWW | VVRVRR |
| 214  | BBBBBB | WWWWWW | VVRVRR |
| 215  | BBBBBB | WWWWWW | VVRVRR |
| 216  | BBBBBB | WWWWWW | VVRVRR |
| 217  | BBBBBB | WWWWWW | VVRVRR |
| 218  | BBBBBB | WWWWWW | VVRVRR |
| 219  | BBBBBB | WWWWWW | VVRVRR |
| 220  | BBBBBB | WWWWWW | VVRVRR |
| 221  | BBBBBB | WWWWWW | VVRVRR |
| 222  | BBBBBB | WWWWWW | VVRVRR |
| 223  | BBBBBB | WWWWWW | VVRVRR |
| 224  | BBBBBB | WWWWWW | VVRVRR |
| 225  | BBBBBB | WWWWWW | VVRVRR |
| 226  | BBBBBB | WWWWWW | VVRVRR |
| 227  | BBBBBB | WWWWWW | VVRVRR |
| 228  | BBBBBB | WWWWWW | VVRVRR |
| 229  | BBBBBB | WWWWWW | VVRVRR |
| 230  | BBBBBB | WWWWWW | VVRVRR |
| 231  | BBBBBB | WWWWWW | VVRVRR |
| 232  | BBBBBB | WWWWWW | VVRVRR |
| 233  | BBBBBB | WWWWWW | VVRVRR |
| 234  | BBBBBB | WWWWWW | VVRVRR |
| 235  | BBBBBB | WWWWWW | VVRVRR |
| 236  | BBBBBB | WWWWWW | VVRVRR |
| 237  | BBBBBB | WWWWWW | VVRVRR |
| 238  | BBBBBB | WWWWWW | VVRVRR |
| 239  | BBBBBB | WWWWWW | VVRVRR |

We remain to look at the change of colour in the signal. Rules 236 and 243 applying to the neighbours sharing side 6 and 7 of the central cell respectively detect the locomotive coming from a fork: either from that of the passive memory switch or from that of the flip-flop. Rules 222 and 223 change the colour of the signal: from blue to red and from red to blue respectively. As mentioned in the implementation, the change to blue when the signal is red takes two more step
that the opposite change.

We arrive to the rules for the fork. Rule 267 allows the locomotive to arrive at the central cell of the fork which is also the central cell in the pictures of Figure 20. Again, rule 167 of the passive memory switch and rule 85 from the crossing allow the creation of a locomotive both of the neighbours sharing sides 1 and 7 respectively. Next, rules 277 and 281 of Table 8 allow the new locomotives to leave the cells where they were created. Later, specific motion rules allow the locomotives to leave the fork: rule 279, rule 21 attract the locomotive to a neighbour of its previous place and rules 283 and rule 22 allow it to leave that second place.

Table 8 The rules for the fork.

| Rule | State 1 | State 2 | State 3 |
|------|---------|---------|---------|
| 257  | WWRBBBWR | 272 | WRBBBWR |
| 258  | WWBRBBWR | 273 | BBWBRRW |
| 259  | WWRBBBWR | 274 | BBWBRRW |
| 260  | WRWWWWWR | 275 | WRWWWWWR |
| 261  | WRBBBWR | 276 | WRBBBWR |
| 262  | WRBBBWR | 277 | WRBBBWR |
| 263  | WRBBBWR | 278 | WRBBBWR |
| 264  | WRBBBWR | 279 | WRBBBWR |
| 265  | WRBBBWR | 280 | WRBBBWR |
| 266  | WRBBBWR | 281 | WRBBBWR |
| 267  | WRBBBWR | 282 | WRBBBWR |
| 268  | WRBBBWR | 283 | WRBBBWR |
| 269  | WRBBBWR | 284 | WRBBBWR |
| 270  | WRBBBWR | 285 | WRBBBWR |
| 271  | WRBBBWR | 286 | WRBBBWR |

With these rules and with this study illustrated by the figures, we completed the proof of the following result:

**Theorem 3** There is a rotation invariant cellular automaton on the heptagrid with 3 states which is planar and weakly universal.

References

[1] F. Herrmann, M. Margenstern, A universal cellular automaton in the hyperbolic plane, *Theoretical Computer Science*, 296, (2003), 327-364.

[2] Cellular Automata and Combinatoric Tilings in Hyperbolic Spaces, a survey, *Lecture Notes in Computer Sciences*, Editors: Crisitan Calude, Michael J. Dinneen, V. Vajnovszki, 2731, (2003), 48-72, doi: 10.1007/3-540-45066-1.

[3] M. Margenstern, *Cellular Automata in Hyperbolic Spaces*, vol. 1, *Theory*, Collection: Advances in Unconventional Computing and Cellular Automata, Editor: Andrew Adamatzky, Old City Publishing, Philadelphia, (2007), 422p.
[4] M. Margenstern, *Cellular Automata in Hyperbolic Spaces, vol. 2, Implementation and computations*, Collection: *Advances in Unconventional Computing and Cellular Automata*, Editor: Andrew Adamatzky, Old City Publishing, Philadelphia, (2008), 360p.

[5] M. Margenstern, A universal cellular automaton on the heptagrid of the hyperbolic plane with four states, *Theoretical Computer Science*, 412, (2011), 33-56

[6] M. Margenstern, About Strongly Universal Cellular Automata, *Electronic Proceedings in Theoretical Computer Science*, 128(17), (2013), 93-125.

[7] M. Margenstern, *Small Universal Cellular Automata in Hyperbolic Spaces: A Collection of Jewels*, Collection: *Emergence, Complexity and Computation*, Editors: Ivan Zelinka, Andrew Adamatzky, Guanrong Chen, Springer Verlag, (2013), 331p., doi: 10.1007/978-3-642-36663-5.

[8] M. Margenstern, A weakly universal cellular automaton in the pentagrid with five states, to be published.

[9] M. Margenstern, Y. Song, A universal cellular automaton on the ternary heptagrid, *Electronic Notes in Theoretical Computer Science*, 223, (2008), 167-185.

[10] A new universal cellular automaton on the pentagrid, *Parallel Processing Letters*, 19(2), (2009), 227-246, doi: 10.1142/S0129626409000195.

[11] I. Stewart, A Subway Named Turing, Mathematical Recreations in *Scientific American*, (1994), 90-92.