DETECTING PLANET PAIRS IN MEAN MOTION RESONANCES VIA THE ASTROMETRY METHOD

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Received 2015 November 4; revised 2016 April 24; accepted 2016 April 27; published 2016 July 1

ABSTRACT

*Gaia* is leading us into a new era with a high astrometry precision of $\sim 10\,\mu$as. Under such precision, astrometry can play an important role in detecting and characterizing exoplanets. In particular, we can identify planet pairs in mean motion resonances (MMRs), which constrain the formation and evolution of planetary systems. In accordance with observations, we consider two-Jupiter or two-super-Earth systems in 1:2, 2:3, and 3:4 MMRs. Our simulations show that the false alarm probabilities (FAPs) of a third planet are extremely small, while the two real planets can be fitted well with a signal-to-noise ratio ($S/N > 3$). The probability of reconstructing a resonant system is related to the eccentricities and the resonance intensity. Generally, when the $S/N \geq 10$, if the eccentricities of both planets are larger than 0.01 and the resonance is quite strong, the probability of reconstructing the planet pair in MMRs is $\geq 80\%$. Jupiter pairs in MMRs are reconstructed more easily than super-Earth pairs with similar $S/N$ when we consider dynamical stability. FAPs are also calculated when we detect planet pairs in or near MMRs. The FAPs for 1:2 MMRs are the largest, i.e., FAP $> 15\%$ when $S/N \leq 10$. Extrapolating from the *Kepler* planet pairs near MMRs and assuming a $S/N \sim 3$, we discover and reconstruct a few tens of Jupiter pairs and hundreds of super-Earth pairs in 2:3 and 1:2 MMRs within 30 pc. We also compare the differences between even and uneven data cadence and find that planets are better measured with more uniform phase coverage.

*Key words:* astrometry – methods: data analysis – planetary systems

1. INTRODUCTION

Up to 2016 April 11, 1642 exoplanets had been detected, 1038 of them in multiple-planet systems\(^1\), and about 41\% of planet host stars have more than one planetary companion. Due to the high precision of the *Kepler* mission, many planets in multiple-planet systems have been confirmed using transit timing variation (TTV; Fabrycky et al. 2012; Ford et al. 2012; Steffen et al. 2012; Xie 2013, 2014). However, this method is limited when two planets are very close to the resonance center, because the period of TTV is quite long and difficult to determine well. Yang et al. (2013) show a TTV signal with a period of $\sim 1500$ days based on the *Kepler* data from as long as 1350 days. This was the longest period TTV signal to have been confirmed planets near mean motion resonances (MMRs) at the time. Most of the *Kepler* adjacent planet pairs are near or in MMRs, in particular 2:3 and 1:2 MMRs (Lissauer et al. 2011; Fabrycky et al. 2014; Ghilea 2014; Goldreich & Schlichting 2014). Observations of multi-planetary systems are very important for studying the mechanisms of dynamical interaction between planets and gas disks. Many studies hint that planets end up in MMRs after migration in the disk (Lee & Peale 2002; Papaloizou 2003; Kley et al. 2004). Additionally, planet–planet scattering may also be a major contributor to the population of resonant planets (Raymond et al. 2008). However, very few planet systems have been confirmed to be in MMRs so far because of the limitations of observations. Planets detected by transit alone lack information of the planetary masses, while planets detected by radial velocity alone yield $m \sin i$. Only a few planets have been detected by both transit and radial velocity methods. In addition, some of the orbital elements are degenerate and time series of planetary mean longitudes are usually not available for extrasolar systems, which make it difficult to know whether they are in MMRs or not. Two exceptions are the HD 82943 and HD 45364 systems. Planets detected using radial velocity around the HD 82943 and HD 45364 systems have been confirmed to be in 1:2 and 2:3 MMRs through dynamical stability analysis (Lee et al. 2006; Correia et al. 2009), i.e., the systems are stable only if the planets are in MMRs. However, some systems are still not confirmed to be in or near MMRs even though they are very close to the resonance center; for example, the period ratio of EPIC 201505350 b and c as displayed in $K2$ data is 1.503514, among the closest systems to a 2:3 commensurability detected so far (Armstrong et al. 2015).

In the past, astrometric measurements with mas precisions such as that of *HIPPARCOS* (Perryman et al. 1997) have not allowed the detection of exoplanets. A star with a Jupiter at 1 au located at 30 pc has a periodic astrometric signature of about $30\,\mu$as and is barely detected with 1 mas precision. However, with the improvement of techniques, many studies have shown that astrometric observations with $\mu$as-level precision are possible, for example using *Gaia*, launched in 2013 (Lattanzi et al. 2000, 2002, 2005; Sozzetti et al. 2001; Sozzetti 2010), and the planned *STEP* satellite (Chen 2014). *Gaia* can achieve a single-measurement astrometric error of a few tens of $\mu$as (Sahlmann et al. 2015), which is sufficient to detect a Jupiter at 1 au around a solar-like star within 30 pc. *STEP* is designed to have a single-measurement astrometric precision of 1 $\mu$as, and has the potential to detect habitable super-Earths around solar-like stars at 30 pc. Astrometry can provide more information about such planets, including the six orbital elements and the mass of each planet, which are essential to decide whether the planet pairs are in MMRs or not.

This paper is arranged as follows. In Section 2, we describe the astrometry method for detecting exoplanets and the simulation set-up. In Section 3, we investigate the limits of the signal-to-noise ratio ($S/N$) in detecting planet pairs and analyze the fitting results of the planets in our simulations. The

\(^1\) exoplanets.org
resonance-reconstruction probabilities of planet pairs in 1:2, 2:3, and 3:4 MMRs are shown in Section 4. In Section 5, we calculate the false alarm probabilities (FAPs) of a detected planet system in or near MMRs. In Section 6, we estimate how many planet pairs in MMRs can be detected and reconstructed in 30 pc. The differences between even and uneven data cadences are present in Section 7. Finally, we summarize our results and discuss how to better reconstruct planet pairs in MMRs in Section 8.

2. DETECTING EXOPLANETS USING ASTROMETRY AND THE SIMULATION SET-UP

2.1. Set-up of Planet Pairs in MMRs

The mass and radius ratios of planets observed to be in or near MMRs are shown in Figure 1. Generally, we consider planet systems containing two planets with equal mass in 1:2, 2:3, and 3:4 MMRs in this paper. We simulate planet systems with two Jupiters and two super-Earths separately. The masses of both super-Earths are set as 10 Earth masses. Hereafter, super-Earth means a planet with 10 Earth masses.

Planets in MMRs can be produced by migration and randomly perturbing the orbital elements of the planets near MMRs. We simplify the migration model by adding a slow inward semimajor axis migration to the outer planet, thus the outer orbit will approach the center of MMRs. For example, given a number of planet pairs near \((j - 1):j\) MMR \((j = 2, 3, 4)\), we add a typical migration with a timescale of \(5 \times 10^5\) yr to allow the planet systems to evolve into MMRs. We halt the migration while the planet pairs are in \((j - 1):j\) MMR \((j = 2, 3, 4)\), thus we can obtain samples of planet pairs in MMRs. With different migration times, we can obtain different eccentricities of the planet pairs in MMRs. There is a positive correlation between \(e_1\) and \(e_2\) for resonant planet pairs produced in this way. In addition, after planets migrate in the disk, they are usually locked in MMRs which are very stable (Lee et al. 2009), i.e., the resonance intensities of these systems are strong. However, if there are more planets in the disk, after the disk disappears the resonance will be disturbed and may be not as strong as when migration was halted. To complete our samples with different resonance intensities, we also produce resonant planet systems by randomly perturbing the orbital elements of the planets near \((j - 1):j\) MMR \((j = 2, 3, 4)\). We choose planet pairs with initial \(\Delta = (j - 1)P_2/(jP_1) - 1 < 0.02\) \((j = 2, 3, 4)\), \(P_1\) and \(P_2\) are the periods of the inner and outer planet, respectively, initial eccentricities are randomly distributed from 0 and 0.4, initial inclinations are randomly distributed from 0° to 5°, and other the orbital elements, \(\Omega\), \(\omega\), and \(M\), are randomly distributed from 0° to 360°. \(\Delta\) is a measure of closeness to resonance. For 2:3 and 1:2 MMRs, we have two groups of resonant planet systems produced by the two methods mentioned above. However, for the 3:4 MMRs of two Jupiters, we only have resonant systems from the random method because planets are scattered before they migrate to be captured in 3:4 MMRs.

In this paper, all the inner planets are located at 0.8−1.1 au randomly from their host stars at 30 pc and all the planet systems are in MMRs for at least \(2 \times 10^4\) yr. There are only two resonance angles \(\phi_1\) and \(\phi_2\) for \((j - 1):j\) MMR, i.e., \(\phi_1 = jx_2 - (j - 1)x_1 - \pi\) and \(\phi_2 = jx_2 - (j - 1)x_1 + \pi\), \(x_1 = M_1 + \pi\) is the mean longitude, \(M_1\) is the mean anomaly, and \(\pi = \Omega + \omega\) \((i = 1, 2)\). The subscripts 1 and 2 of the orbital elements represent the inner and outer planet, respectively. In general, a planet pair is considered to be in an MMR as long as one resonance angle is in libration (Murray & Dermott 1999; Raymond et al. 2008). To obtain refined samples of planets in MMRs, we only choose planet systems with libration amplitudes of both \(\phi_1\) and \(\phi_2\) less than 300° in \(2 \times 10^4\) yr. The systems with only one resonance angle in libration are not included in our samples. All planets in MMRs in our samples are nearly face on with inclinations between 0° and 10°. The mutual inclinations of planet pairs are less than 5°. The eccentricity distributions are shown in Figure 2.

2.2. Simulation of Astrometric Data

Detecting exoplanets by astrometry with \(\mu\)as precision has become possible since the launch of Gaia. Similarly to the radial velocity method, astrometry measures projected movements of the host star around the barycenter of the system. By measuring the movement of the star, we can acquire planetary orbits and masses. The astrometric measurements in \(x\) and \(y\) \((x\) and \(y\) represent the projected movement in the R.A. and decl. directions, respectively\) at time \(t\) relative to the reference frame of background stars are modeled with (Black & Scargle 1982)

\[
x = x_0 + \mu_x (t - t_0) - P_x \pi + X + \text{Err}_x, \tag{1}
\]

\[
y = y_0 + \mu_y (t - t_0) - P_y \pi + Y + \text{Err}_y. \tag{2}
\]

In Equations (1) and (2), \(x_0\) and \(y_0\) are the coordinate offsets, \(\mu_x\) and \(\mu_y\) are the proper motions of the star, \(P_x\) and \(P_y\) are the parallax parameters which will be provided in the observation, and \(\pi\) is the annual parallax of the star. \(x_0, y_0, \mu_x, \mu_y, \pi\) are taken as stellar parameters. \(X\) and \(Y\) are the displacements in the star’s position due to its planetary companion(s), and \(\text{Err}_x\) and \(\text{Err}_y\) are single-measurement astrometric errors. In this paper, when we simulate astrometry data, we fix \(\mu_x = 50\) mas yr\(^{-1}\) and \(\mu_y = -30\) mas yr\(^{-1}\). All the planet systems are set to be 30 pc away from us. A planet with mass \(m_p\) and semimajor axis

\[
\text{count}
\]

\[
\text{count}
\]
a_p will lead to an astrometric signature of

\[ S = 3 \left( \frac{m_p}{10 m_{\text{Earth}}} \right) \left( \frac{a_p}{1 \text{ au}} \right) \left( \frac{m_*}{m_{\text{Sun}}} \right)^{-1} \left( \frac{d}{10 \text{ pc}} \right)^{-1} \mu \text{as} \quad (3) \]

on the star with a distance of d; m_* is the stellar mass. We adopt a simple Gaussian measurement error model, i.e., Err_x and Err_y follow a Gaussian distribution with standard deviation \( \sigma_m \) in our simulations. The S/N is defined as \( S/\sigma_m \), which is similar with the definition in Casertano et al. (2008). Note that the S/N defined here is for a single measurement. Equations (1) and (2) can be complicated if one includes aberration of starlight and perspective acceleration, so we assume that these effects have been perfectly removed from measurements. \( P_x \) and \( P_y \) can be provided given the orbit of the satellite; here we use a 1 yr circular orbit to simplify the parallax model. After generating planet pairs in MMRs in Section 2.1, we use a RKF7(8) (Fehlberg 1968) N-body code which includes the full-Newtonian interaction between the planets to simulate astrometric data and sample every 0.1 yr; each simulation consists of a time series of coordinate measurements according to Equations (1) and (2) with the nominal mission lifetime set as 5 yr. Thus we have a set of 50 points \( \{x(t_i), y(t_i)\}, i = 1, 2, \ldots, 50 \) each representing a measurement at observation time \( t_i \).

2.3. Orbital Parameter Fitting Procedure

In general, interaction between planets can be ignored because it barely affects the motion of stars (Sozzetti et al. 2001). Most of the multiple-planet systems discovered by radial velocity techniques can be well modeled by planets on independent Keplerian orbits (Casertano et al. 2008), such as the 55 Cancri system with five planets around the primary star (Fischer et al. 2008). When a star hosts two planetary companions, we also assume that the astrometric signal of the host star is the superposition of two strictly non-interacting Keplerian orbits. Ignoring the interaction between planets, \( X \) and \( Y \) are expressed as (Catanzarite 2010)

\[ X = \sum_{i=1}^{N} (\cos E_i - e_i) A_i + \sqrt{1 - e_i^2} (\sin E_i) F_i, \quad (4) \]

\[ Y = \sum_{i=1}^{N} (\cos E_i - e_i) B_i + \sqrt{1 - e_i^2} (\sin E_i) G_i. \quad (5) \]

In Equations (4) and (5), \( E_i \) is the eccentric anomaly, \( e_i \) is the eccentricity of the planets, \( A_i, F_i, B_i, G_i \) are Thiele–Innes constants, which encode the amplitudes and orientations of the orbits such as the inclinations of planets \( I_i \), arguments of pericenter \( \omega_i \), and longitudes of ascending nodes \( \Omega_i \), \( i = 1, \ldots, N \). \( N \) is the number of planets.
We use a hierarchical scheme to fit the orbits of the planets. The details of orbit reconstruction are described in Catanzarite (2010), so here we briefly introduce the actual process:

Step 1: Ignore the planetary influence on the star and invert Equations (1) and (2) by linear least squares to calculate \( x_0, y_0, \mu_1, \mu_2, \) and \( \pi \); then we have the initial values of the stellar parameters for the next step.

Step 2: Remove the coordinate offsets, proper motion, and parallax from the data, and then analyze the residuals with the periodogram (Scargle 1982) to see if there is a significant period \( (P_t) \) which exists in both the \( x \) - and \( y \) -directions. If there is one, we obtain an initial guess of the period of the most significant planet. We identify a certain orbit when the FAP of the corresponding period is less than 1%. As we have two-dimensional time-series astrometric data, we calculate the joint periodogram defined in Catanzarite et al. (2006) as the sum of the Lomb–Scargle periodogram power from each dimension. The calculation of FAP can be found in Scargle (1982) and Horne & Baliunas (1986).

Step 3: We randomly choose the initial value of eccentricity \( e_1 \) and the moment that the planet passes its perihelion \( t_{01} \) of the planet. The stellar parameters \( x_0, y_0, \mu_1, \mu_2, \pi, \) and the period of the planet \( P_1, \) with initial values obtained in Step 1 and 2, are also fitted. Equations (1) and (2) are easily inverted by linear least squares to yield the four Thiele–Innes constants \( A_1, F_1, B_1, \) and \( G_1 \), \( X \) and \( Y \) can be calculated and we have fitted \( x \) and \( y \). We adopt the Markov chain Monte Carlo (MCMC) algorithm in our fitting procedure. After the MCMC chains converge, we have more precise stellar parameters \( P_1, e_1, \) and \( t_{01} \) for the first planet. \( I_1, \omega_1, \Omega_1, \) and \( (a_1 m_1)/m_1 \) can be calculated according to the Thiele–Innes constants.

Step 4: The projected motion of the star due to the first planet is then removed from the astrometric data; we again use the periodogram to search for significant peaks in the residuals. If there is one with a FAP smaller than 1%, then it provides an initial guess for the period of the second planet \( (P_2) \). The data are then fitted with a two-planet reflex motion model.

Step 5: Continue Steps 2–4 until no significant signal appears in the periodogram.

For each two-planet system, there are \( 5 + 2 \times 7 = 19 \) parameters to be fitted. However, to save computing time in the MCMC algorithm in Steps 3 and 4, we adopt linear equations in our fitting and only 11 parameters are fitted in the MCMC algorithm, i.e., \( \mu_1, \mu_2, \pi, x_0, y_0, P_1, P_2, t_{01}, t_{02}, e_1, \) and \( e_2 \). The other parameters of the planets can be derived from \( P_1, P_2, \) and the eight Thiele–Innes constants. If we know the mass of the star in advance, we can also obtain the semimajor axis and the mass of the each planet. Currently, with the development of spectrometry and astroseismology, the mass of the star can be measured with a precision of 10% (Creevey et al. 2007; Epstein et al. 2014). In addition, the semimajor axes of the planets are obtained through the relation between the mass of the host star and the orbital period in our fitting procedure: \( a_1 \) and \( a_2 \) are proportional to \( m_1^{1/3} \), and the masses of the planets \( m_1 \) and \( m_2 \) are proportional to \( m_2^{2/3} \). The derivation can be found in (Catanzarite 2010) and we briefly illustrate it here. The astrometric signature of planet \( i \) on the host star is \( S_i \). With the parallax we calculated, we have an estimation of the semimajor axis of the stellar reflex motion \( a_{\text{ref}} = S_i / \pi \) \((i = 1, 2)\). The center of mass equation gives the planets’ masses \( m_i a_i = m_1 \mu_{1i} \) \((i = 1, 2)\). Together with Kepler’s third law \( a_i^3 = m_i P_i^2 \), we can determine the ratios of \( a_1/a_2 \) and \( m_1/m_2 \).

Therefore, the precision of the stellar mass will not affect the characteristics of the MMRs because \( a_1/a_2, m_1/m_2 \), and other orbital elements are independent of the stellar mass. As we adopt the linear function in Equations (4) and (5), we cannot distinguish the solution of parameters \( \omega_1 \) and \( \Omega_1 \) from \( \omega_1 + 180^\circ, \Omega_1 + 180^\circ \) \((i = 1, 2)\) without the information of position variation in the direction of our line of sight. However, if the orbits of the planets are face on, the two solutions of the parameters will not influence the resonance configuration. In this paper, to simplify the problem, we assume that all the center stars have solar mass. We run each MCMC with \( 3 \times 10^5 \) iterations and statistics are derived on the last \( 1 \times 10^5 \) elements. We choose the best-fit parameters as the median of the posterior distribution. More details about the MCMC procedure can be found in the Appendix.

3. DETECTING PLANETS WITH DIFFERENT S/N

To investigate the detection of planet with different S/N, we simulate single-planet systems with different masses and semimajor axes around a solar-like star at 30 pc. We adopt the detection criterion of a planet as mentioned in Section 2, i.e., the FAP of of the corresponding period is less than 1%. The left and right panels in Figure 3 show our ability to detect and characterize the planet with observational errors of \( \sigma_m = 0.3 \mu \text{as} \) and \( 10 \mu \text{as} \). Our simulations show that planets with \( S/N > 3 \) and periods from \( \sim 0.2 \text{yr} \) (two times the data cadence) to 5 yr (the whole observation time) can be detected reliably and consistently, which is similar to that discussed in Casertano et al. (2008). Planets with \( S/N > 1 \) can be detected but with poor determination of mass. In the worst case, planets with \( S/N < 1 \) are barely measured. The requirements for \( S/N \) in the astrometry method are similar to those in the radial velocity method. In the radial velocity method, Cumming (2004) shows that the detection of periodic signals requires \( N \approx 20 – 30 \) with a single \( S/N \) of \( K/\sigma \approx 2 – 4 \), where \( K \) is the signal amplitude in radial velocity and the detection of signals \( <1\sigma \) requires \( N \geq 50 \). Plavchan et al. (2015) also shows that with 50 observations, planets with \( S/N \geq 2 \) can be detected. The lower limits of the period are due to the Nyquist sampling theorem, while the upper limits of the period are constrained by the orbital phase coverage of the planets.

According to Equation (3), a star at 30 pc with a Jupiter at 1.0 au has a periodic astrometric signature of about 30 \( \mu \text{as} \), while a star at 30 pc with a super-Earth at 1.0 au has a periodic astrometric signature of about 1 \( \mu \text{as} \). As the interactions between the planets are ignored in the fitting procedure, the Jupiter pairs with observational errors of \( \sigma_m = 10 \mu \text{as} \) and the super-Earth pairs with observational errors of \( \sigma_m = 0.3 \mu \text{as} \) are located near the line \( S/N = 3 \), which indicates that they can be detected and characterized well. The fitting results for two-planet systems in the following sections are good, i.e., the reduced chi-square value \( \chi^2_{\text{red}} \) is distributed between 0.9 and 1.3 for >80% of cases. However, we may find a third “detection” in systems without observational error. The third detection is deduced by the Keplerian model we use. The Keplerian orbit differs from the full-Newtonian orbit and, although the difference is quite small, the periodic residuals are likely to result in a third detection with a very small mass. By adding observational errors, false detection no longer occur.

The fitting errors of orbital elements are shown in Tables 1 and 2. Among the six orbital elements and the planetary mass, the semimajor axes are better determined than other orbital
Figure 3. The fitting errors $\log_{10}(\sigma_i^2 + \epsilon_i^2)/2$ of planets with observational errors $\sigma_i = 0.3$ µas (left) and 10 µas (right). We simulate a large number of single-planet systems with different planetary masses and semimajor axes to check if we can detect and characterize them using the astrometry method. All the central stars have 1 solar mass and they are 30 pc away from us. The astrometry data are generated with an even cadence of 0.1 yr. The green line at $a = 0.341$ au represents a period of 0.2 yr, which is the minimum period that can be found with a sample cadence of 0.1 yr. The magenta line at $a = 2.924$ au represents a period of 5 yr. The blue line represents planet systems with $S/N = 1$ and the dark line represents planet systems with $S/N = 3$. The region between the green and magenta lines with $S/N > 3$ can be detected and characterized well. The blue regions represent planets with small relative fitting errors, the blank regions represent planets that failed to be detected.

parameters. The relative fitting errors of mass are smaller than 0.06 when $S/N \geq 10$. When the $S/N$ reaches 3, they can be as large as 0.13. Eccentricities can be well determined when there are no observational errors. However, when $S/N \sim 3$, the absolute fitting errors of eccentricities largely increase, in particular for planet pairs in 1:2 MMRs. Other absolute fitting errors of orbital elements such as $I_i$, $\omega_i + \Omega_i$, and $M_i (i = 1, 2)$ are also very sensitive to observational errors. Here we compare the difference between the fitted $\omega_i + \Omega_i$ and true $\omega_i + \Omega_i$ ($i = 1, 2$) because the Keplerian model we use yields two orbital solutions $\omega_i, \Omega_i$ and $\omega_i + 180^\circ, \Omega_i + 180^\circ$, which were mentioned in Section 2.3. Note that as there is degeneracy between $\omega_i$ and $M_i$ when the eccentricities are very small, the absolute fitting errors of $\omega_i$ and $M_i$ decrease with the increase of eccentricities. Compared with 2:3 and 3:4 MMRs, planet pairs in 1:2 MMRs have larger relative fitting errors of masses and absolute fitting errors for other orbital parameters. The 1:2 period ratio makes it difficult to fit the orbital parameters of both planets as well as those of the 2:3 and 3:4 MMRs because of the harmonic. The average orbital parameter fitting errors of super-Earth pairs are similar to those of Jupiter pairs when they have similar $S/N$. The small relative fitting errors of planetary masses and semimajor axes guarantee the successful detection and characterization of planet systems in our simulations.

4. THE PROBABILITY OF RECONSTRUCTING PLANET PAIRS IN MMRs

After fitting the orbital parameters of the planets, we check the stabilities of the planet systems. If the fitted orbital parameters deviate far from the true ones, the fitted planet systems will be unstable, particularly the Jupiter pairs. We use $\beta_1$ and $\beta_2$ to indicate the probability of the fitted resonance angles $\phi_1$ and $\phi_2$ in libration. To obtain the probability of planet pairs in MMRs, we divide the total integral time of $2 \times 10^4$ yr into five equal parts and check if the resonance angles simulated in fitted systems are cycling every $4 \times 10^3$ yr. The probability of planet pairs in MMRs is defined as the fraction of time with a librating resonance angle. We use $\beta$, the larger of $\beta_1$ or $\beta_2$, to represent the probability of reconstructing a planet pair in an MMR.

4.1. The Stabilities and Probabilities in MMRs of the Fitted Planet Systems

For a two-planet system, the separation of the planets should be at least $3.5 R_H$ according to Gladman (1993) if the planets are Hill stable. $R_H$ is the Hill radius of a planet. For a Jupiter at 1 au, $3.5 R_H$ is about 0.242 au, so the outer planet should be outside 1.242 au with a period ratio $P_1/P_2$ smaller than 0.72. The Hill stability indicates that Jupiter pairs near 1:2 and 2:3 MMRs are likely to be stable, while those near 3:4 MMRs are always unstable unless they are exactly in 3:4 MMRs. This analysis corresponds to our simulations that Jupiter pairs with $P_1/P_2 \sim 3/4$ are stable only if they are in 3:4 MMRs. The 3.5 $R_H$ for a super-Earth at 1 au is 0.075 au, so the outer planet should be outside 1.075 au with a period ratio $P_1/P_2 < 1.114$, which indicates that super-Earth pairs near 2:3, 1:2, and 3:4 MMRs are most likely to be Hill stable. From observations of radial velocity and Kepler data, the occurrence rate of Super-Earths is higher than that of Jupiters (Winn & Fabrycky 2015). In addition, for planet pairs near MMRs, the fraction of both planets with $5M_{\text{Earth}} \sim 20M_{\text{Earth}}$ is 23.69% and the fraction of both planets with masses $0.5M_J \sim 2M_J$ is 2.79%. So super-Earths near MMRs are more common than two Jupiters near MMRs, in particular in Keplerian planet systems.

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5 The exoplanet data used here are from exoplanets.org.
Table 1  
Average Fitting Errors for Jupiter Pairs

| Observational | $|\varepsilon_{\text{fit}} - \varepsilon_{\text{true}}|/\varepsilon_{\text{true}}$ | $|m_{\text{fit}} - m_{\text{true}}|/m_{\text{true}}$ | $|e_{\text{fit}} - e_{\text{true}}|$ | $|i_{\text{fit}} - i_{\text{true}}|$ | $|\omega_{\text{fit}} + \Omega_{\text{fit}} - \omega_{\text{true}} - \Omega_{\text{true}}|$ | $|M_{\text{fit}} - M_{\text{true}}|$ |
|---------------|-------------------------------------------------|---------------------------------|-----------------|-----------------|-------------------------------------------------|-----------------|
| 2:3           | 0                                               | $4.0 \times 10^{-4}/1.0 \times 10^{-3}$ | $3 \times 10^{-3}/3.5 \times 10^{-3}$ | $4.3 \times 10^{-2}/5.5 \times 10^{-3}$ | $1^\circ 6/1^\circ 8$ | $14^\circ 7/14^\circ 0$ | $16^\circ 9/18^\circ 6$ |
|               | 3 μas                                           | $1.2 \times 10^{-3}/1.4 \times 10^{-3}$ | 0.024/0.013     | 0.03/0.01        | $14^\circ 3/8^\circ 7$ | $28^\circ 5/15^\circ 8$ | $42^\circ 3/20^\circ 8$ |
|               | 10 μas                                          | $4.0 \times 10^{-3}/2.3 \times 10^{-3}$ | 0.09/0.05       | 0.09/0.04        | $28^\circ 3/16^\circ 9$ | $60^\circ 4/39^\circ 1$ | $85^\circ 5/49^\circ 6$ |
| 1:2           | 0                                               | $2.7 \times 10^{-4}/1.3 \times 10^{-3}$ | $7.8 \times 10^{-3}/1.2 \times 10^{-3}$ | $2.0 \times 10^{-3}/7.6 \times 10^{-3}$ | $0^\circ 6/0^\circ 7$ | $4^\circ 7/19^\circ 2$ | $35^\circ 1/25^\circ 8$ |
|               | 3 μas                                           | $9.5 \times 10^{-4}/1.5 \times 10^{-3}$ | 0.06/0.01       | 0.03/0.07        | $13^\circ 0/8^\circ 7$ | $14^\circ 1/66^\circ 1$ | $15^\circ 8/90^\circ 3$ |
|               | 10 μas                                          | $3.0 \times 10^{-3}/3.2 \times 10^{-3}$ | 0.12/0.05       | 0.10/0.13        | $26^\circ 2/18^\circ 2$ | $33^\circ 8/72^\circ 3$ | $42^\circ 3/103^\circ 2$ |
| 3:4           | 0                                               | $8.4 \times 10^{-4}/1.5 \times 10^{-3}$ | 0.01/0.01       | $3.9 \times 10^{-3}/4.5 \times 10^{-3}$ | $3^\circ 1/2^\circ 8$ | $6^\circ 1/6^\circ 9$ | $10^\circ 0/5^\circ 0$ |
|               | 3 μas                                           | $1.3 \times 10^{-3}/1.8 \times 10^{-3}$ | 0.014/0.013     | 0.019/0.018      | $11^\circ 9/9^\circ 6$ | $7^\circ 9/7^\circ 4$ | $10^\circ 4/12^\circ 0$ |
|               | 10 μas                                          | $3.6 \times 10^{-3}/3.8 \times 10^{-3}$ | 0.06/0.05       | 0.05/0.06        | $21^\circ 9/19^\circ 3$ | $18^\circ 7/16^\circ 4$ | $26^\circ 0/27^\circ 7$ |

Note. The values to the left of “/” are fitting errors of the inner planet while those to the right are fitting errors of the outer planet.
Table 2
Average Fitting Errors for Super-Earth Pairs

| Observational | \(|a_{\text{fit}} - a_{\text{true}}|/|a_{\text{true}}|\) | \(|m_{\text{fit}} - m_{\text{true}}|/m_{\text{true}}\) | \(|e_{\text{fit}} - e_{\text{true}}|\) | \(i_{\text{fit}} - i_{\text{true}}\) | \(|\omega_{\text{fit}} + \Omega_{\text{fit}} - \omega_{\text{true}} - \Omega_{\text{true}}|\) | \(|M_{\text{fit}} - M_{\text{true}}|\) |
|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2:3 0         | \(3.7 \times 10^{-5}/5.7 \times 10^{-5}\) | \(3 \times 10^{-5}/2.56 \times 10^{-4}\) | \(1.3 \times 10^{-3}/9.8 \times 10^{-4}\) | \(0.4/0.4\) | \(9^\circ 9/8^\circ 2\) | \(7^\circ 9/6^\circ 3\) |
| 0.1 \(\mu\)as| \(1.5 \times 10^{-3}/6.5 \times 10^{-4}\) | 0.028/0.015 | \(3.1 \times 10^{-2}/9.6 \times 10^{-3}\) | 15.1/7.4 | 30.9/12.8 | 47.9/19.3 |
| 0.3 \(\mu\)as| \(4.5 \times 10^{-3}/1.8 \times 10^{-3}\) | 0.09/0.05 | 0.10/0.03 | 28.1/15.5 | 57.5/26.1 | 84.5/36.9 |
| 1:2 0         | \(9.9 \times 10^{-5}/7.0 \times 10^{-5}\) | 0.027/1.2 \times 10^{-3} | \(8.7 \times 10^{-3}/0.016\) | 1.8/0.7 | 9.9/55.9 | 8.1/68.3 |
| 0.1 \(\mu\)as| \(1.2 \times 10^{-3}/1.0 \times 10^{-3}\) | 0.06/0.02 | 0.04/0.09 | 15.4/7.6 | 17.7/68.7 | 27.2/87.6 |
| 0.3 \(\mu\)as| \(0.017/3.2 \times 10^{-3}\) | 0.13/0.13 | 0.13/0.23 | 30.2/17.1 | 44.3/78.3 | 62.9/108.1 |
| 3:4 0         | \(5.7 \times 10^{-5}/7.7 \times 10^{-5}\) | \(4.2 \times 10^{-4}/3.76 \times 10^{-4}\) | \(9.6 \times 10^{-4}/9.4 \times 10^{-4}\) | \(0.5/0.4\) | \(8.5/8.6\) | \(3.6/3.8\) |
| 0.1 \(\mu\)as| \(1.5 \times 10^{-3}/1.4 \times 10^{-3}\) | 0.019/0.016 | 0.019/0.018 | 9.7/8.7 | 20.9/16.6 | 34.9/28.8 |
| 0.3 \(\mu\)as| \(4.2 \times 10^{-3}/4.2 \times 10^{-3}\) | 0.06/0.05 | 0.06/0.06 | 19.3/17.5 | 46.5/36.9 | 62.3/55.3 |

Note. The values to the left of “/” are fitting errors of the inner planet while those to the right are fitting errors of the outer planet.
We develop an $N$-body code based on the RKF7(8) (Fehlberg 1968) integrator which includes full-Newtonian interactions between the planets to check if the fitted planet systems are stable for $2 \times 10^6$ yr. The stable fractions of the fitted planet systems and the fractions of stable planet systems with $\beta > 0.5$ for the Jupiter pairs and super-Earth pairs are shown in Tables 3 and 4. As we use the Keplerian motion to model the true motion, even when not considering observational errors, the planet systems cannot be perfectly fitted and reconstructed. In addition, when the eccentricities of the planets are very small, the orbits of the planets are circular and there is a geometrical degeneracy of $\omega_i$ and $M_i$ ($i = 1, 2$) which makes it difficult to determine $\omega_i$ and $M_i$ ($i = 1, 2$) correctly. In our simulations, super-Earth pairs in 2:3 and 3:4 MMRs with $\beta < 0.5$ mostly have eccentricities smaller than $5 \times 10^{-3}$ when not considering observational errors. For 1:2 MMRs, the 1:2 period ratio makes it difficult to obtain well-fitting results because of the influence of harmonics. For the reasons above, a small fraction of planet systems are not well reconstructed.

In Table 3 we can see that when $\sigma_\mu \lesssim 10$ mas, more than 90% of the fitted Jupiter pairs in 2:3 and 1:2 MMRs are stable. For Jupiter pairs in 3:4 MMRs, even without observational errors, only half of the fitted planet systems are stable. Although the fitting errors of the Jupiter pairs in 3:4 MMRs are similar to those in 2:3 MMRs, it is more difficult for planets to be locked in 3:4 MMRs than in 2:3 MMRs, therefore the stable fractions of Jupiter pairs in 3:4 MMRs are much lower than those in 2:3 MMRs. When only considering the MMR-reconstruction probabilities in stable fitted systems, more than 80% of Jupiter pairs in 3:4 MMRs can be reconstructed with $\beta > 0.5$. We check the long-time stabilities of a few systems in 3:4 MMRs with $\beta < 0.5$ and find that all these systems are unstable in 0.5 Myr. Consequently, with a longer stability checking time, Jupiter pairs with low probabilities in 3:4 MMRs can be excluded from the stable samples, thus the MMR-reconstruction probabilities in stable fitted systems could also approach 100%.

For super-Earth pairs, the results in Table 4 show that the stable fractions of fitted planet systems are generally much larger than those of the Jupiter pairs. As we have mentioned above, if the two Jupiters are not in an MMR, they are likely to be unstable according to Hill stability. In observations, planets with Jupiter mass observed to be near MMRs are usually confirmed to be in MMRs according to their dynamical stability (Lee et al. 2006; Correia et al. 2009). In this paper, we did not perform such a study. So, when considering the MMR-reconstruction probabilities in stable fitted systems, the fraction of super-Earth pairs with $\beta > 0.5$ is smaller than that of the Jupiter pairs. Considering the similar fitting errors of orbital parameters with similar S/N (Tables 1 and 2), and the fact that resonance width increases with planetary mass (Deck et al. 2013), stable Jupiter pairs are more likely to stay in

### Table 3

| Observational Error | Even Cadence | Fraction of Stable Systems | Fraction of Stable Planet Systems with $\beta > 0.5$ |
|---------------------|--------------|---------------------------|---------------------------------------------------|
| 2:3                 | 0            | 95%                       | 95% $\pm$ 1%                                     |
| (1319)$^b$          | 3 $\mu$as    | 95%                       | 87% $\pm$ 1%                                     |
| 10 $\mu$as          |              | 90%                       | 58% $\pm$ 1%                                     |
| 1:2                 | 0            | 99%                       | 98% $\pm$ 1%                                     |
| (1370)$^c$          | 3 $\mu$as    | 99%                       | 71% $\pm$ 3%                                     |
| 10 $\mu$as          |              | 91%                       | 58% $\pm$ 3%                                     |
| 3:4                 | 0            | 49%                       | 99% $\pm$ 1%                                     |
| (926)$^d$           | 3 $\mu$as    | 48%                       | 98% $\pm$ 1%                                     |
| 10 $\mu$as          |              | 42%                       | 85% $\pm$ 1%                                     |

Notes. The uncertainties are calculated as the difference between the fraction of $\beta > 0.5$ from all stable systems and the fraction of $\beta < 0.5$ from N/2 stable systems. $N$ is the sample number of each MMR shown in the parenthesis. N/2 samples are chosen randomly to guarantee that the N/2 samples have similar distributions of eccentricities and $\Delta$ to the whole samples.

$^a$ The MMR-reconstruction probability.

$^b$ Sample number of Jupiter pairs in 2:3 MMRs.

$^c$ Sample number of Jupiter pairs in 1:2 MMRs.

$^d$ Sample number of Jupiter pairs in 3:4 MMRs.

### Table 4

| Observational Error | Even Cadence | Fraction of Stable Systems | Fraction of Stable Planet Systems with $\beta > 0.5$ |
|---------------------|--------------|---------------------------|---------------------------------------------------|
| 2:3                 | 0            | 99%                       | 93% $\pm$ 1%                                     |
| (812)$^a$           | 0.1 $\mu$as  | 99%                       | 76% $\pm$ 2%                                     |
| 0.3 $\mu$as         |              | 99%                       | 42% $\pm$ 1%                                     |
| 1:2                 | 0            | 100%                      | 91% $\pm$ 1%                                     |
| (562)$^b$           | 0.1 $\mu$as  | 100%                      | 79% $\pm$ 1%                                     |
| 0.3 $\mu$as         |              | 100%                      | 49% $\pm$ 2%                                     |
| 3:4                 | 0            | 99%                       | 92% $\pm$ 1%                                     |
| (895)$^c$           | 0.1 $\mu$as  | 99%                       | 77% $\pm$ 2%                                     |
| 0.3 $\mu$as         |              | 98%                       | 40% $\pm$ 1%                                     |

Notes. The uncertainties are calculated similarly to those in Table 3.

$^a$ Sample number of super-Earth pairs in 2:3 MMRs.

$^b$ Sample number of super-Earth pairs in 1:2 MMRs.

$^c$ Sample number of super-Earth pairs in 3:4 MMRs.
MMRs in our reconstruction. The fraction of fitted planet systems with $b > 0.5$ is larger than 70% when $S/N > 10$. When $S/N = 3$, the fractions largely drop to 40%–60%. We will investigate the relation between the MMR-reconstruction probability in stable fitted systems and $\Delta$, eccentricity, and resonance intensity in the following sections.

4.2. MMR-reconstruction with Different $\Delta$

The distributions of $\Delta$ for the Jupiter pairs and super-Earth pairs in our samples are shown in Figures 4 and 5. We can see that $\Delta$ concentrate around small values. For Jupiter pairs, $\Delta \sim 10^{-3}$, while for super-Earth pairs, $\Delta \sim 10^{-4}$. As the resonance width increases with the mass of the planet pairs (Deck et al. 2013), the values of $\Delta$ for Jupiters are much larger than for super-Earths. Planet pairs in 2:3 MMRs have a much wider distribution of $\Delta$ than those in 1:2 and 3:4 MMRs. Because, in our simulations, planet pairs with large $\Delta$ are generated by migration, Jupiter pairs in 3:4 MMRs tend to have small $\Delta$ due to a lack of samples from migration. As we used a simplified migration model, planet pairs with small eccentricities usually have large $\Delta$. In Figure 2 we can see that planet pairs from migration in 2:3 MMRs have more samples with small eccentricities than planet pairs in 1:2 and 3:4 MMRs, so the $\Delta$ distribution is broader for 2:3 MMR pairs than for the 1:2 and 3:4 MMR pairs. In this section we will check the relation between $\Delta$ and the MMR-reconstruction probabilities.

Although the relative fitting errors of the semimajor axes of the planets are very small (Tables 1 and 2), the absolute fitting errors of $\Delta$ can be large. As $\Delta$ is calculated according to average periods of the planet pairs in $2 \times 10^4$ yr, a small variation of the initial semimajor axis will lead to large difference in $\Delta$. We calculate $\Delta_{\text{fit}}$ according to the average fitted periods of the planets in $2 \times 10^4$ yr and find that the average differences between $\Delta$ and $\Delta_{\text{fit}}$ are around $2 \times 10^{-4}$, without observational errors, for both Jupiter and super-Earth pairs. When there are observational errors, the average differences between $\Delta$ and $\Delta_{\text{fit}}$ reach $10^{-3}$.

Figure 5. Distribution of the normalized distance $\Delta$ from the resonant center of the super-Earth pairs. The top, middle, and bottom panels are samples of the 2:3, 1:2, and 3:4 MMRs. Samples with large $\Delta$ are not shown here.

Figure 6. Relations between the MMR-reconstruction probabilities of the Jupiter pairs in MMRs and $\Delta$. The top, middle, and bottom panels are the results of the 2:3, 1:2, and 3:4 MMRs. The red, blue, and magenta lines show the results with observational errors of $0.0 \mu\text{as}$, $0.1 \mu\text{as}$, and $0.3 \mu\text{as}$, respectively.

Figure 7. Relations between the MMR-reconstruction probabilities of the super-Earth pairs in MMRs and $\Delta$. The top, middle, bottom panels are the results of the 2:3, 1:2, and 3:4 MMRs. The red, blue, and magenta lines show the results with observational errors of $0.0 \mu\text{as}$, $0.1 \mu\text{as}$, and $0.3 \mu\text{as}$, respectively.

To check the correlations between $\Delta$ and the MMR-reconstruction probability, we sort the samples in each MMR with increasing $\Delta$ and divide them into ten parts with the same number of samples. $\overline{\beta}(\Delta)$ is defined as the average value of $\beta$. 

$\overline{\beta}(\Delta)$
for planets in each part. Figures 6 and 7 show $\beta(\Delta)$ at different $\Delta$. We do not show the tenth part with the largest $\Delta$ for extremely large variations. If observations are carried out without any errors, we can reconstruct nearly all the systems in MMRs independent of $\Delta$.

For Jupiter pairs in 2:3 and 1:2 MMRs, there is a decrease of $\beta(\Delta)$ with the increase of $\Delta$. With the increase of $\Delta$, the resonance becomes fragile and small variations of $\Delta$ may destroy the resonance, therefore the MMR-reconstruction probability decreases. If $\sigma_m = 3\,\mu$as, the MMR-reconstruction probabilities are larger than 60%, with a slight decrease of $\Delta$. When observational errors are large enough, e.g., $\sigma_m = 10\,\mu$as, $\beta(\Delta)$ are less than 80% for both the 2:3 and 1:2 MMRs with large dispersions. $\beta(\Delta)$ of the 3:4 MMRs are mostly constrained by stability and still >80%.

For super-Earth pairs, similarly to Jupiter pairs, $\beta(\Delta)$ should decrease with the increase of $\Delta$. However, there is no obvious negative correlation. This is because of the large absolute fitting errors of $\Delta$ for super-Earth pairs. As we mentioned in the previous paragraph, the average differences between $\Delta$ and $\Delta_{fit}$ reach $10^{-3}$, which is much larger than the distribution range of $\Delta$ for super-Earth pairs, but smaller than that for Jupiter pairs. So there is no positive correlation between $\Delta$ and $\Delta_{fit}$ for super-Earth, while $\Delta_{fit}$ increases with $\Delta$ for Jupiter pairs, i.e., the negative correlation between $\Delta$ and $\beta(\Delta)$ is hidden by the large fitting errors of $\Delta$ for super-Earths. $\beta(\Delta)$ are smaller than those of the Jupiter pairs with similar S/N. As we calculate $\beta(\Delta)$ in the stable systems, Jupiter pairs that remain stable are more likely to be in MMRs because of Hill stability. When $\sigma_m = 0.1\,\mu$as, $\beta(\Delta)$ for 2:3 MMRs drops from 80% to 60% with the increase of $\Delta$. For 1:2 and 3:4 MMRs, $\beta(\Delta)$ is about 80%. When $\sigma_m = 0.3\,\mu$as, $\beta(\Delta)$ is smaller than 60%.

The large dispersions make the relation between $\beta(\Delta)$ and $\Delta$ a little obscure, which also indicates that some other factors can influence the MMR-reconstruction probabilities, such as the eccentricity and resonance intensity. We will analyze the relations between MMR-reconstruction, and eccentricity and resonance intensity in the following subsections.

4.3. MMR-reconstruction with Different Eccentricities

In addition to $\Delta$, eccentricities also have an important influence on the probability of reconstructing a planet pair in an MMR. We divide the systems in each MMR into four parts according to the eccentricities of the planets: I: $e_1 > 0.1$, $e_2 > 0.1$; II: $e_1 < 0.1$, $e_2 > 0.1$; III: $e_1 > 0.1$, $e_2 < 0.1$; and IV: $e_1 < 0.1$, $e_2 < 0.1$. We calculate the average values of $\beta$ (hereafter $\beta(e)$) in each part with different observational errors for the Jupiter pairs and super-Earth pairs, as shown in Figures 8 and 9. To better illustrate the relation between MMR-reconstruction probability and eccentricity, we only calculate $\beta(e)$ in eccentricity bins with a number of planet pairs larger than 20. The uncertainties due to Poisson statistics are shown as error bars displayed on the eccentricity bins. Different colors represent different observational errors. Obviously, $\beta(e)$ with large error decreases in all cases.

We can see that eccentricities are essential for variation of $\beta(e)$ in different parts. For planet pairs in 2:3 and 1:2 MMRs, $\beta(e)$ in part I are larger than $\beta(e)$ in part IV. The increase of $\beta(e)$ from part IV to part I is obvious for Jupiter pairs in 2:3 and 1:2 MMRs. Stability constraints are quite strong in Jupiter pairs in 3:4 MMRs, few planet pairs remain in parts III and IV, and $\beta(e)$ in part I are larger than those in part II. For super-Earth pairs in 3:4 MMRs, the increase of $\beta(e)$ from part IV to part I is not obvious, this is because the average amplitudes of resonance angles are not well-distributed from part I to part IV, the influence of which on $\beta$ will be discussed in the following section. However, $\beta(e)$ in part IV is still the smallest in 3:4.
MMRs. The positive correlation between $\bar{f}(e)$ and eccentricities indicates that eccentricities are important for MMR-reconstruction, because the precisions of $\omega_i$ and $M_i$ are very sensitive to the precision of $e_i$ ($i = 1, 2$). Simulations show that when eccentricities are smaller than 0.01, $\omega_{i,\text{fit}} + M_{i,\text{fit}}$ may obviously deviate from the true value. Even if $\omega_{i,\text{fit}} + M_{i,\text{fit}}$ ($i = 1, 2$) equals to true value, it is difficult to decide both $\omega_i$ and $M_i$ ($i = 1, 2$) accurately when $e_i < 0.01$ ($i = 1, 2$). The geometrical degeneracy of $\omega$ and $M_i$ ($i = 1, 2$) makes us reconstruct planet pairs with small eccentricities in MMRs ambiguously. Large $e_i$ can avoid this degeneracy and result in more accurate $\omega_i$ and $M_i$. In addition, the resonance widths increase with the eccentricities of the planet (Deck et al. 2013). With similar absolute fitting errors of eccentricities, planet pairs with large eccentricities are more likely to remain in MMRs. Therefore, $\bar{f}(e)$ increases with $e_i$ ($i = 1, 2$).

### 4.4. MMR-reconstruction with Different Resonance Intensities

To check how well we reconstruct the MMRs, we compare the average amplitudes of $\phi_1$ and $\phi_2$ in fitted systems (hereafter $A_{\phi,\text{fit}}$) with those in real systems (hereafter $A_{\phi}$) for each type of MMR at $\beta_i > 0.5$ ($i = 1, 2$). With similar absolute fitting errors of eccentricities, planet pairs with large eccentricities are more likely to remain in MMRs. Therefore, $\bar{f}(e)$ increases with $e_i$ ($i = 1, 2$).

Figure 10. The residuals of the average amplitudes of resonance angles $\phi_1$ and $\phi_2$ in fitted systems ($A_{\phi,\text{fit}}$) with those in real systems ($A_{\phi}$) of the Jupiter pairs. The top histograms are the distributions of the average amplitudes of resonance angles $\phi_1$ and $\phi_2$, i.e., $A_{\phi_1}$ and $A_{\phi_2}$. The top two, middle two, and bottom two scatter diagrams in each column show the residuals of $A_{\phi_1}$ and $A_{\phi_2}$ with observational errors of 0, 3, and 10, respectively. The red crosses represent Jupiter pairs with $\beta_i > 0.5$ and the blue crosses represent Jupiter pairs with $\beta_i < 0.5$ ($i = 1, 2$). The dotted lines represent $A_{\phi_i} = 30^\circ$ ($i = 1, 2$). The values on the left side of the dotted lines represent the blue fractions of Jupiter pairs with $A_{\phi_i} < 30^\circ$ while the values on the right side represent the blue fractions of Jupiter pairs with $A_{\phi_i} > 30^\circ$ ($i = 1, 2$). $\mu$ and $\sigma$ are the mean value and standard deviation of the residuals of $A_{\phi_1}$ and $A_{\phi_2}$ with $\beta_i > 0.5$ ($i = 1, 2$). The left, middle, and right panels show the residuals of the 2:3, 1:2, and 3:4 MMRs, respectively.

In Figures 10 and 11, the red crosses represent planet pairs with $\beta_i > 0.5$ while the blue ones represent those with $\beta_i < 0.5$ ($i = 1, 2$). We adopt the Gaussian distribution to fit the residuals with $\beta_i > 0.5$ ($i = 1, 2$) and obtain a mean value $\mu$ and standard deviation $\sigma$ for each type of MMR at different observational errors. As shown in Figures 10 and 11, we find that with the increase of observational errors, both the mean values and standard deviations become larger and larger, indicating that fitted resonance angles deviate more and more from the true values. In addition, the samples with $\beta_i < 0.5$ ($i = 1, 2$) have residuals far away from 0, indicating that some of the blue crosses only deviate from 0 because in systems with $\beta_i < 0.5$ the resonance angles $\phi_i$ only librate in $2 \times 10^4 \cdot \beta_i$ yr, so the average amplitudes should be larger than those with $\beta_i > 0.5$ ($i = 1, 2$).

The blue crosses mostly come from real systems with large $A_{\phi}$ ($i = 1, 2$), which means weak MMRs. To show the distribution of blue crosses clearly, we plot a dotted line in Figures 10 and 11, which represents $A_{\phi_i} = 30^\circ$ ($i = 1, 2$), to divide the total samples into two categories. The values on the left and right sides of the dotted line, respectively, represent the fractions of the blue crosses in the two categories with 2:3 MMR $m_1 = m_2 = 10^{-3}$, 1:2 MMR $m_1 = m_2 = 10^{-3}$, and 3:4 MMR $m_1 = m_2 = 10^{-3}$.
We find that the blue fractions on the right side are generally larger than those on the left side for Jupiter pairs in 2:3 and 1:2 MMRs. For Jupiter pairs in 3:4 MMRs, the blue fractions on each side are similar because the stability will exclude part of the systems with large $f_A$. For super-Earth pairs, there are many more systems with $e < 0.01$ than for systems containing Jupiter pairs, because planets with larger masses more easily excite their eccentricities in our sample simulations. According to Section 4.3, small eccentricities lead to a small MMR-reconstruction probability, thus the blue fraction with large $A_0$ is not always larger than the blue fraction with small $A_0$. To exclude the non-uniform distribution of systems with small eccentricities in the two categories with different $A_0$, we choose samples of the super-Earth pairs with $e_1 > 0.01$ and $e_2 > 0.01$ to recalculate the blue fractions of the two categories, which are shown as blue values in Figure 11. We find that the blue fractions for $A_{0i} > 30^\circ$ are generally larger than for $A_{0i} < 30^\circ$ in 2:3, 1:2 MMRs.

Beyond that, there are obvious differences between the blue fractions of $f_{A1}$ and $f_{A2}$ for Jupiter pairs in 2:3 and 1:2 MMRs. Compared to $f_{A1}$, $f_{A2}$ concentrates upon a smaller value for 2:3 MMRs. Naturally, with the same level of observational errors and similar deviations from the original values of $\phi_1$ and $\phi_2$, $f_{A2}$ is more likely to remain in libration than $\phi_1$, therefore the planet pairs with $\beta_2 < 0.5$ are fewer than those with $\beta_1 < 0.5$. In contrast, for 1:2 MMRs, $A_{01}$ concentrates upon a much smaller value than $A_{02}$, so $\phi_1$ is much easier to reconstruct in libration than $\phi_2$. The positive correlation between the blue fraction and $f_{Ai}$ ($i = 1, 2$) indicates that the stronger the intensities of the MMRs, the easier the MMRs can be reconstructed.

The analyses above indicate that the MMR-reconstruction probabilities are related to the eccentricities and resonance intensities of the planet pairs. To better compare the difference of MMR-reconstruction probabilities between Jupiter pairs and super-Earth pairs, we calculate the fraction of planet pairs with $\beta > 0.5$ among samples with appropriate eccentricities and strong intensities, i.e., the eccentricities of both planets are larger than 0.01 and the average amplitude of at least one resonance angle is smaller than 30°. The results are shown in Table 5. Except for planet pairs in 1:2 MMRs with S/N = 10, Jupiter pairs can be better reconstructed in MMRs than super-Earth pairs, especially when S/N = 3, because of Hill stability and because planet pairs with larger masses have larger resonance widths, according to Deck et al. (2013). In fact, it is quite difficult to explain all the differences between super-Earth pairs and Jupiter pairs. Although we have confined the eccentricity and resonance intensity to compare the differences
in MMR-reconstructions between Jupiter pairs and super-Earth pairs, we cannot eliminate the sample bias between Jupiter and super-Earth pairs totally. A more refined sample control should be helpful in eliminating the exception.

5. THE ANALYSIS OF THE FALSE ALARM PROBABILITY OF PLANET PAIRS IN OR NEAR MMRs

In Section 4, we have investigated the probability of reconstructing planet pairs in MMRs. Consequently, the fraction of well-reconstructed planet pairs with \( \beta > 0.5 \) in MMRs (hereafter denoted as \( P_{\beta-1} \)) can be obtained in accordance with the previous section. In fact, some systems are not in but near MMRs. We are also interested in the FAPs (hereafter denoted as \( P_{0-1} \)) of mistaking planet pairs near MMRs for planet pairs in MMRs.

To obtain the FAPs of mistaking near-MMR systems for Jupiter pairs, we simulate 1600 Jupiter pairs with different ranges of \( \mu \) and Table 6 confirms that, among planet pairs in MMRs, the FAP is different from 0.01 when \( \mu \) is different from 0.3.

To estimate the FAPs of mistaking near-MMR systems for super-Earth pairs, we simulate 1600 super-Earth pairs with \( \mu \) in MMRs in 2 systems in MMRs, we simulate 1600 Jupiter pairs with different ranges of \( \mu \) and Table 7 confirms that, among planet pairs in MMRs, the FAP is different from 0.01 when \( \mu \) is different from 0.3.

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The uncertainties are according to the uncertainties in Tables 6 and 8. As we have

\[
\mu \pm 5 \%
\]

Notes. The uncertainties are calculated similarly to those in Table 3.

| Observational Errors | Jupiter Pairs | Super-Earth Pairs |
|-----------------------|--------------|-------------------|
|                       | 0            | 3 \( \mu s \)       | 10 \( \mu s \) | 0            | 0.1 \( \mu s \) | 0.3 \( \mu s \) |
| 2:3 MMR               | \( \Delta\text{fit} < 0.01 \) | 99% ± 1% | 96% ± 1% | 62% ± 1% | \( \Delta\text{fit} < 5 \times 10^{-4} \) | 98% ± 1% | 87% ± 2% | 56% ± 1% |
| 1:2 MMR               | \( \Delta\text{fit} < 5 \times 10^{-3} \) | 99% ± 1% | 81% ± 4% | 69% ± 1% | \( \Delta\text{fit} < 1.6 \times 10^{-4} \) | 95% ± 1% | 92% ± 1% | 66% ± 5% |
| 3:4 MMR               | \( \Delta\text{fit} < 1 \times 10^{-3} \) | 100% ± 1% | 97% ± 1% | 87% ± 4% | \( \Delta\text{fit} < 5 \times 10^{-4} \) | 96% ± 1% | 86% ± 1% | 51% ± 5% |

Note. The uncertainties are calculated similarly to those in Table 3.

| Observational Errors | \( F_{0-1} \) \( a \) | \( F_{1-2} \) \( b \) |
|-----------------------|------------------------|------------------------|
|                       | 2:3 (\( \Delta\text{fit} < 0.01 \)) | 1:2 (\( \Delta\text{fit} < 5 \times 10^{-3} \)) |
| Without Error         | 5% ± 1%                | 27% ± 1%               |
| 3 \( \mu s \)         | 27% ± 1%                | 35% ± 1%               |
| 10 \( \mu s \)        | 40% ± 1%                | 42% ± 1%               |

Notes. The uncertainties are according to the uncertainties in Tables 6 and 8. As we have \( F_{0-1} = \frac{P_{0-1}}{P_{0-1} + P_{1-2}} \), the uncertainty of \( F_{0-1} \) can be estimated as

\[
P_{0-1} \cdot P_{1-2} \cdot P_{0-1} \cdot P_{1-2} - P_{0-1} \cdot P_{0-1} \cdot P_{1-2} \cdot P_{1-2}
\]

\( a \) \( F_{0-1} \) is the FAP when we detect a planet pair in an MMR. It is calculated on the basis of the possibility we mistake a planet pair near but not in an MMR for one in an MMR and the possibility we detect a true planet pair in an MMR.

\( b \) \( F_{1-2} \) is the FAP when we detect a planet pair near but not in an MMR. It is calculated on the basis of the possibility we mistake a planet pair in an MMR for one near an MMR, and the possibility we detect a true planet pair near but not in an MMR.

Earth pairs are smaller than 0.02. We do not analyze \( P_{0-1} \) for Jupiter pairs near the 3:4 MMRs because of their weak stabilities, i.e., \( P_{0-1} \sim 0 \) when considering the long-term stabilities of these systems. For Jupiter pairs and super-Earth pairs in each MMR, we divide the samples into three ranges according to \( \Delta\text{fit} \) or \( \Delta \), and calculate \( P_{0-1} \) in each range of \( \Delta\text{fit} \) (outside the brackets) and \( \Delta \) (in the brackets). The \( P_{0-1} \) for all samples are shown in the fourth row of each MMR in Tables 6 and 7.

The fourth row of each MMR in Tables 6 and 7 shows that there is a positive correlation between \( P_{0-1} \) and observational error when \( P_{0-1} \) is calculated via \( \Delta \). The larger the observational error is, the further the fitted orbital parameters deviate from their true values, thus the fitted planet pairs can arrive at some islands of MMRs far away from the initial position in phase space and they are probably in MMRs. Unlike the positive correlation between \( P_{0-1} \) and observational error, there is a negative correlation between \( P_{0-1} \) and \( \Delta\text{fit} \), i.e., the larger \( \Delta\text{fit} \) is, the further the planet pair is away from the MMR center, thus it is less likely to be mistaken for a planet pair in an MMR. However, \( P_{0-1} \) has no obvious correlation with \( \Delta \).

When we detect a planet pair with a period ratio near 1:2, 2:3, or 3:4 MMRs, and the simulation shows that it is in an MMR based on the fitted parameters, the detected planet pair in the MMR might be a false alarm. To calculate the FAP \( F_{0-1} \) for a detected planet system in an MMR, we need the values of both \( P_{0-1} \) and \( P_{1-2} \). If we assume the same number \( N_{p} \) of planet pairs in or near MMRs, \( N_{p} \cdot P_{0-1} \) planet pairs near MMRs will be mistaken as planet pairs in MMRs, while \( N_{p} \cdot P_{1-2} \) planet pairs in MMRs can be well reconstructed. Finally, \( F_{0-1} \) is expressed as:

\[
F_{0-1} = \frac{P_{0-1}}{P_{0-1} + P_{1-2}}.
\]

Note that the meaning of \( P_{0-1} \) is different from that of \( F_{0-1} \). From Equation (6), we can see that even if \( P_{0-1} = 1 \), \( F_{0-1} \) is greater than 0, but smaller than \( P_{0-1} \).

On the other hand, there is another FAP when we reconstruct a planet system near an MMR. Take \( P_{0-1} \) as the probability of reconstructing a system near but not in an MMR, and take \( P_{1-2} \) as the probability of mistaking an in-MMR system for a near-MMR system. Similar to the derivation of \( F_{0-1} \), the FAP for a near-MMR system \( F_{1-2} \) is expressed as:

\[
F_{1-2} = \frac{P_{1-2}}{P_{0-1} + P_{1-2}}.
\]

It is easy to obtain \( P_{0-1} = 1 - P_{1-2} \) and \( P_{0-1} = 1 - P_{0-1} \). In observations, only \( \Delta\text{fit} \) can be obtained, so it is suitable to adopt the values of \( P_{1-2} \), \( P_{0-1} \), \( P_{0-1} \), and \( P_{1-2} \) calculated via \( \Delta\text{fit} \). \( P_{0-1} \) in Table 8 are slightly larger than the values in the last column in Table 3 and 4, because they are calculated among planet pairs with \( \Delta\text{fit} \) in the same range of \( \Delta\text{fit} \) shown in the third rows in Tables 6 and 7.

Tables 9 and 10 show the final \( F_{0-1} \) and \( F_{1-2} \) of a planet system detected in or near MMRs. Generally, the larger the observational errors are, the larger \( F_{0-1} \) and \( F_{1-2} \) are. For both Jupiter and super-Earth pairs, \( F_{0-1} \) and \( F_{1-2} \) are sensitive to the observational errors. \( F_{0-1} \) of Jupiter pairs and super-Earth pairs in 1:2 MMRs are very similar, which are larger than 20% even without observational errors. With the same observational errors, \( F_{0-1} \) and \( F_{1-2} \) for planet pairs in 2:3 and 3:4 MMRs are smaller than those in 1:2 MMRs. Note that the particularly large FAP for planet pairs in 1:2 MMRs is mainly induced by
Table 10  
FAP of the Two-Super-Earth System

| Observational Error | $F_{0-1}^a$ | $F_{1-0}^b$ |
|---------------------|-------------|-------------|
|                     | 2:3 ($\Delta_{\text{fit}} < 5 \times 10^{-4}$) | 1:2 ($\Delta_{\text{fit}} < 1.6 \times 10^{-4}$) | 3:4 ($\Delta_{\text{fit}} < 5 \times 10^{-4}$) | 2:3 ($\Delta_{\text{fit}} < 5 \times 10^{-4}$) | 1:2 ($\Delta_{\text{fit}} < 1.6 \times 10^{-4}$) | 3:4 ($\Delta_{\text{fit}} < 5 \times 10^{-4}$) |
| Without Error       | 6% ± 2%     | 26% ± 1%    | 7% ± 1%      | 2% ± 1%     | 7% ± 1%      | 4% ± 1%     |
| 0.1 $\mu$as         | 21% ± 1%    | 39% ± 1%    | 25% ± 1%     | 14% ± 2%    | 16% ± 1%     | 16% ± 1%    |
| 0.3 $\mu$as         | 44% ± 1%    | 50% ± 2%    | 47% ± 1%     | 44% ± 1%    | 52% ± 3%     | 47% ± 2%    |

Notes. The uncertainties are according to uncertainties in Table 9.  
$^a$ The same with $F_{0-1}$ in Table 9.  
$^b$ The same with $F_{1-0}$ in Table 9.
the significantly large $P_{0-1}$. As we have mentioned before, the 1:2 period ratio makes it more difficult to fit planet pairs as well as planet pairs with a different period ratio. So it is more likely that planet pairs near 1:2 MMRs are mistaken for those in 1:2 MMRs. When $S/N \sim 3$, both $\mathcal{F}_{0-1}$ and $\mathcal{F}_{1-0}$ are larger than 30%, therefore, if we detect a planet system in or near MMRs with low $S/N$, the system should be checked carefully.

6. THE POTENTIAL OF DISCOVERING PLANET PAIRS IN MMRs

After calculating the MMR-reconstruction probabilities, we can estimate the number of planet pairs in MMRs ($N_{\text{MMR}}$) which can be measured by astrometry if we know the frequency of Jupiter pairs and super-Earth pairs in MMRs around nearby stars.

Based on observations before the Kepler Mission, Casertano et al. (2008) estimate the number of multiple-planet systems that Gaia can detect. In their paper, they list all the multiple-planet systems detected and calculate the fraction of the multiple-planet systems which meet the condition $S/N > 3$ with single-measurement precision set to be 8 $\mu$as. Then they extrapolate the results to the planet systems Gaia can detect and finally estimate the number of multiple-planet systems they can find. However, in this paper, it is hard to estimate $N_{\text{MMR}}$ in the same way due to a lack of samples with parallax measurements. Among the 415 multiple-planet systems detected, 76 have parallax measurements, and only 27 systems have planet pairs near MMRs. These samples are very rare and no super-Earth pairs near MMRs appear in the 27 systems, so we choose another way to estimate $N_{\text{MMR}}$.

The number of planet pairs in MMRs reconstructed by astrometry measurements can be expressed as: $N_{\text{MMR}} = N_c \times f_1 \times f_2 \times f_3 \times f_4 \times f_5$. $N_c$ is the number of target stars, here we adopt $N_c = 3 \times 10^5$ based on the fact that there are more than $3 \times 10^8$ bright stars ($V < 10$) within 30 pc (The Hipparcos and Tycho catalogs). $f_1$ is the probability that a star host planets. $f_2$ is the probability that the planets are in multiple-planet systems, $f_3$ is the probability that there are planets in MMRs in multiple-planet systems, $f_4$ is the probability that planets in MMRs with Jupiter-like or super-Earth-like masses. $f_5$ is the probability that the planets in MMRs can be reconstructed by astrometry. According to Cassan et al. (2012), each Milky Way star hosts at least one planet, i.e., $f_1$ is set as 100%. We calculate $f_2$, $f_3$, and $f_4$ based on the planets discovered so far. According to observations of the Kepler mission, $f_2 \approx 41%$. There is observational bias in the Kepler mission which tends to discover planets closer to the host star; planets further away from the host star have a smaller probability of being detected. Therefore, planet pairs in or near MMRs in observations mostly have semimajor axes $<0.5$ au. For planets detected by transit, the occurrence rates of terrestrial planets decrease from 50 to 300 day, however, occurrence rates for planets with larger periods are barely constrained (Burke et al. 2015). For planets detected by radial velocity, Cumming et al. (2008) found evidence for a sharp rise in occurrence of planets with periods $\geq 1$ yr. Winn & Fabrycky (2015) summarized the basic picture of planet probability density: giant planets have a probability density nearly constant in log $P$ between 2 and 2000 day while smaller planets (1–4 $R_\oplus$) have a probability nearly constant in log $P$ between 10 and 300 day. Here, we simply assume that the occurrences of MMRs far away from the host star (1 au) are similar to those near the host star. As few planets in MMRs have been confirmed, we set $f_5$ as the probability of near-MMR planet pairs in multiple-planet systems. This is reasonable because many studies (Lithwick & Wu 2012; Batygin & Morbidelli 2013; Xie 2014; Chatterjee & Ford 2015) hint that planet pairs in MMRs can evolve into the observed MMR offset due to several mechanisms such as tidal dissipation and planet–planetesimal disk interaction. Currently, 415 multiple-planet systems have been detected, and 135, 91, and 20 planet systems contain planet pairs near 2:3, 1:2, and 3:4 MMRs, i.e., $f_3 = 21.9\%, 32.5\%$, and 4.8% for 2:3, 1:2, and 3:4 MMRs, respectively. In addition, among the planet pairs near MMRs, the fraction of both planets with masses $5M_\oplus \sim 20M_\oplus$ is 23.69% and the fraction of both planets with masses $0.5M_\oplus \sim 2M_\oplus$ is 2.79%. We choose $f_4 = 2.79\%$ for Jupiter pairs and $f_4 = 23.69\%$ for super-Earth pairs. In our simulations, planet pairs in or near MMRs have inclinations between 0° and 10°, however, planet pairs in MMRs with inclinations of $\sim 90°$ can also be reconstructed with a certain probability. We perform simulations for a super-Earth pair in a 2:3 MMR with their inclinations increased from 0° to 90° and find that the MMR-reconstruction probability decreases if inclinations are $\geq 50°$. Here we simply assume that the MMR-reconstruction probability decreases linearly with increase of inclinations, i.e., $f_5(i) = f_5(0°)(1 - [i]/(\pi/2))([i] = [i - \pi/2, \pi/2])$. $f_5(i) \approx 0°$ is approximated by the MMR-reconstruction probability of planet pairs with nearly face-on orbits which is calculated in Section 4, i.e., the last columns in Tables 3 and 4. In addition, assuming a uniform distribution of the planets’ orbital angular momentum vector, the probability density of inclination $dP(i)/di = \sin|i|/2(i - [i - \pi/2, \pi/2])$. So we have $f_5 = f_5(0°) \int_{-\pi/2}^{\pi/2} \sin|i| /2(1 - [i]/(\pi/2)) \approx 0.36f_5(i \approx 0°)$. Although $f_5(i \approx 0°)$ is obtained by simulation of planet pairs near 1 au, planet pairs at different locations will lead to the same results with the same $S/N$, if we rescale the observational errors and data samplings consistent with the locations of the inner planet.

Finally, we estimate the probabilities of discovering and reconstructing the planet systems using the astrometry method, as shown in Table 11. As all the planet systems in our simulations are at 30 pc, the MMR-reconstruction probabilities are the inferior limits. The number of Jupiter pairs in MMRs that can be detected and reconstructed is much smaller than that of the super-Earth pairs. This is reasonable, as planet systems containing two giants are rare in observations. With an observational $S/N = 3$, we can find tens of giant planet pairs in 2:3 and 1:2 MMRs. The reconstruction of super-Earth pairs in MMRs requires higher precision to reach $S/N \sim 3$, hundreds of super-Earth pairs in 2:3 and 1:2 MMRs will be identified.

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| S/N | Jupiter Pairs | Super-Earth Pairs |
|-----|--------------|-------------------|
|     | ~10          | ~3                |
| 2:3 MMR | 24          | 16            |
|       | 128          | 147         |
| 1:2 MMR | 20          | 24            |
|       | 110          | 171         |
| 3:4 MMR | 6           | 6             |
|       | 39           | 20           |

Table 11: Number of Planet Pairs in MMRs that Can Be Detected and Reconstructed by Astrometry in 30 pc

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The exoplanet data used in this section are from exoplanets.org.
pairs in 3:4 MMRs are much fewer. With a higher S/N = 10, about 1.2 times more Jupiter pairs and 1.8 times more super-Earth pairs in MMRs can be reconstructed than from results with S/N = 3. Jupiter pairs in 3:4 MMRs are strictly constrained by stability. Even with S/N = 3, \( \sim 100\% \) of all Jupiter pairs near 3:4 MMRs can be detected by direct orbital fitting or dynamical analysis, therefore, improvement of S/N cannot enhance the number of Jupiter pairs. The discovery and reconstruction of planet pairs in MMRs is essential for planet formation and evolution theories. High precision astrometry should lead to great progress.

### 7. EVEN AND UNEVEN DATA CADENCE

The simulations above were all carried out with even data cadence. In fact, most observations have uneven data cadence because of multiple realistic observational limits. To find out the influence of uneven data cadence on planet detection using the astrometry method, we compare the differences between even and uneven data cadence in this section. Although the uneven data cadences discussed here are not realistic cadence schemes for \textit{Gaia} and \textit{STEP}, it is important to explore how large the influence is.

For single-planet systems, we simulate 100 super-Earth systems and 100 Jupiter systems which are 30 pc from us. All the planets are 0.9 au from the host star and their eccentricities are distributed from 0.01 to 0.5. All observations have a set of 50 data points. We choose eight different data cadences c1–c8 of simulated astrometry data as follows:

- **c1**: 80% data points are randomly distributed near the perigee, i.e., \(-43\degree 2 < f < 43\degree 2\), where \( f \) is the true anomaly (hereafter the same). 20% are randomly distributed near the apogee, i.e., \( 129\degree 6 < f < 216\degree \) (hereafter the same);
- **c2**: 20% are randomly distributed near the perigee, while 80% are randomly distributed near the apogee;
- **c3**: 50% are randomly distributed near the perigee, while 50% are near the apogee;
- **c4**: 50% are randomly distributed near the mid-point of the apogee and perigee, i.e., \( 43\degree 2 < f < 129\degree 6\), the others are randomly distributed on the opposite side, where \( 230\degree 4 < f < 316\degree 8\);
- **c5**: 40% are randomly distributed near the perigee, while 40% are randomly distributed near the apogee, the other 20% are randomly distributed in the left regions;
- **c6**: \( f \) of all data points is randomly distributed;
- **c7**: the times of all data points are uniformly distributed, i.e., the even data cadence adopted before this section;
- **c8**: all data points are distributed with uniform orbital phase coverage, i.e., there is one data point in each range of \( f \) with a width of 7\degree 2.

Diagrammatic sketches of the eight data cadences are shown in Figure 12.

To illustrate the non-uniformity of the data points, we divide the whole phase coverage of \( f \) into 25 parts, each with a width of 14\degree 4. Then we count the number of data points in each part and calculate the variance (\( \sigma_{\text{phase}} \)) of them. The variance represents the phase coverage of the observation, i.e., the smaller the variance is, the more complete the phase coverage is. For each kind of data cadence, the variance changes slowly with eccentricity, therefore, we calculate the average variance in each bin of eccentricities; the bin range is set to be 0.1. Set S/N ~10, i.e., \( \sigma_m = 0.1 \mu\text{as} \) for the super-Earth and \( \sigma_m = 3 \mu\text{as} \) for the Jupiter; we fit the planet
parameters with data sets c1–c8. The differences between the true and fitted astrometric signatures caused by the planets are shown in Figures 13 and 14. The residuals are expressed as 

$$\sigma^2_{\text{phase}} = \frac{\sum_{i=1}^{N} (X_{\text{fit}}(t_i) - X_{\text{true}}(t_i))^2 + (Y_{\text{fit}}(t_i) - Y_{\text{true}}(t_i))^2}{N}. \quad N = 50$$

The number of data points.

Similar characteristics for single Jupiter and super-Earth systems are obtained in our simulations, which is reasonable, because the simulations are performed with similar S/N. The left panels of Figures 13 and 14 show the variance of each data cadence at different eccentricities of Jupiters and super-Earths, respectively. The right panels show the corresponding residuals at each variance. In the left panels, $\sigma^2_{\text{phase}}$ increases from c8 to c1. The cases of c8 have zero variances, while $\sigma^2_{\text{phase}}$ for c6 and c7, which have much better phase coverage than c1–c5, is smaller than 3. $\sigma^2_{\text{phase}}$ of c1 is similar to that of c2, because $f$ of data points near the perigee and apogee is uniformly distributed as shown in the top two panels in Figure 12. The same reason can also explain the similarity of $\sigma^2_{\text{phase}}$ in c3 and c4. In the right panel of Figures 13 and 14, with the increase of average variance the residual also increases, indicating that more uniform and complete phase coverage will ensure a better orbit fitting of the planets. With the similar $\sigma^2_{\text{phase}}$ in c1 and c2, the residuals are nearly the same, i.e., the residuals are not sensitive to the samplings with more data near the perigee or apogee. The similar residuals of c3 and c4 show that there are no differences whether data sampling near the perigee/apogee or not. For even data cadence c7, when eccentricities are larger, we will have more data points near the apogee if we sample every 0.1 yr, and the variance increases with the eccentricities. Accordingly, the increase of variance leads to the increase of residuals with the eccentricities, while there are no such obvious correlations for other cadences. Empirically, if $\sigma^2_{\text{phase}} < 3$, e.g., c6–c8, the residuals are smaller than observational errors $\sigma_m$ for single-planet systems.

For single-planet systems, even with extremely uneven data cadence, all the planets are detected with precise periods, although the residuals vary greatly. When it comes to two-planet systems, things are quite different. The large fitted residuals of the first planet may contaminate the signal of the secondary planet, thus the period of the secondary planet is difficult to determine accurately. So we compare the differences between even and uneven data cadences for two-planet systems to see how large the influence is. As we adopt a Keplerian orbit for each planet, the motion of the host star will be irregular rather than a Keplerian orbit. So it is difficult to clearly choose data points near the perigee or apogee for both planets. For two-planet systems, the star moves around the common center of mass and is located in different quadrants at different times. Define $\alpha$ as the angle of the data vectors $[x(t_i), y(t_i)] (i = 1, \ldots, N)$ with the x-axis. We choose all the simulated astrometry data of super-Earth pairs in 2:3 MMRs in Section 4, and test four kinds of data samples (d1-d4) as follows:

- **d1:** Samples in regions with $45^\circ < \alpha < 90^\circ$ and $245^\circ < \alpha < 270^\circ$.
- **d2:** Samples in regions with $0^\circ < \alpha < 90^\circ$ and $180^\circ < \alpha < 270^\circ$.
- **d3:** Samples in regions with $-45^\circ < \alpha < 90^\circ$ and $135^\circ < \alpha < 270^\circ$.
- **d4:** Samples every 0.1 yr, i.e., even data samples.

Table 12 shows the results of the four kinds of sampling for super-Earth pairs in 2:3 MMRs with $\sigma_m = 0.1 \mu$as. We choose $\sigma_m = 0.1 \mu$as in order to ensure a large S/N ~10. Therefore, planets can be detected with large confidence and we can compare the influence of different sampling schemes on MMR-reconstruction in our simulations. Similarly to single-planet systems, we can calculate the variance $\sigma^2_{\text{phase},i}$ for each planet. From data cadences d1 to d4, the mean values of $\sigma^2_{\text{phase},i}$, denoted as $\sigma^2_{\text{phase},i}$ (i = 1, 2), for both planets largely drop. The reason is obvious because the larger the regions we sample in, the more uniform and complete the phase coverage will be. When sampling only in a very small region, take d1 for example, only about 27% of the results converge at $\chi^2_{\text{red}} < 1.5$.

**Figure 13.** The left panel shows the variances of the data cadences at different eccentricities. The green, blue, cyan, magenta, light gray, dark, red, and purple lines represent the data cadences c1–c8, respectively. The right panel shows the difference between the true and fitted astrometric signatures caused by a Jupiter with a standard deviation of observational error $\sigma_m = 3 \mu$as at different variances, $\sigma^2_{\text{phase}}$, and eccentricities, $e$, of the data cadences. The different colors represent the same data cadences as those in the left panel. The symbols dot, cross, asterisk, diamond, and left triangle represent the mean variance and residuals with mean eccentricities $e = 0.05, 0.15, 0.25, 0.35, \text{ and } 0.45$, respectively. The circles are the mean values of variances and residuals for each data cadence.

**Figure 14.** Similar to Figure 13 but for a super-Earth with observational error $\sigma_m = 0.1 \mu$as.
while for d3 and d4 all results can converge at small \( c < 1.5 \) as shown in the third row in Table 12. Among the results with \( c < 1.5 \), the average MMR-reconstruction probabilities \( \overline{\beta} \) also increase with the phase coverage in the fourth row. We investigate the fitting errors of eccentricities, which decrease from d1 to d4, and lead to the increase of \( \overline{\beta} \) for d1 is smaller than the others, because \( \overline{\beta} \) becomes very small if the period of one planet is determined ambiguously. In our simulations, about 10% of the fitted super-Earth pairs with small \( c < 1.5 \) have large fitting errors of the semimajor axis for the secondary planet (\( \delta_{a_1} > 0.1 \)) while the periods of both planets in d2–d4 are determined well with \( \delta_{a_2} < 0.05 \) (\( i = 1, 2 \)) as shown in the fifth row in Table 12. For planet pairs with \( c > 2 \), which occurs only in d1 and d2, most of them are characterized with false periods of the secondary planets with \( \delta_{a_1} > 0.1 \). Therefore, these planet pairs in 2:3 MMRs can barely be reconstructed.

The mean values of \( \beta \) are all < 0.08 for planet pairs with \( c > 2 \). We define the average variance of the two-planet system as \( \sigma^2_{\text{phase}} = (\sigma^2_{\text{phase,1}} + \sigma^2_{\text{phase,2}})/2 \). Consistent with single-planet systems, if \( \sigma^2_{\text{phase}} < 3 \), e.g., d2–d4, the MMR-

| Data Cadence | d1 | d2 | d3 | d4 |
|--------------|----|----|----|----|
| \( \sigma^2_{\text{phase,1}}/\sigma^2_{\text{phase,2}} \) | 3.92/5.54 | 2.36/3.18 | 2.21/2.49 | 1.15/0.80 |
| Fraction of \( \chi^2 < 1.5 \) | 27.12% | 98.37% | 100% | 100% |
| \( \overline{\beta} \) of \( \chi^2 < 1.5 \) | 0.63 | 0.72 | 0.75 | 0.77 |
| Fraction of \( \delta_{a_1} > 0.1 \) and \( \delta_{a_2} < 0.05 \) when \( \chi^2 < 1.5 \) | 9.52% | 0 | 0 | 0 |
| Fraction of \( \chi^2 > 2 \) | 60.26% | 1.40% | 0 | 0 |
| \( \overline{\beta} \) of \( \chi^2 > 2 \) | 0.01 | 0.08 | ... | ... |
| Fraction of \( \delta_{a_1} > 0.1 \) and \( \delta_{a_2} < 0.05 \) when \( \chi^2 > 2 \) | 93.36% | 66.67% | ... | ... |

Note. d1–d4 represents the four kinds of data cadence for super-Earth pairs in 2:3 MMRs in Section 7. \( \sigma^2_{\text{phase,i}} (i = 1, 2) \) is the mean value of \( \sigma^2_{\text{phase}} \) for each planet and \( \overline{\beta} \) is the mean MMR-reconstruction probability for all super-Earth pairs in 2:3 MMRs. The fraction of \( \delta_{a_1} > 0.1 \) and \( \delta_{a_2} < 0.05 \) when \( \chi^2 < 1.5 \) (or \( \chi^2 > 2 \)) represent the fraction of planet pairs with relative fitting errors of the semimajor axes \( \delta_{a_1} > 0.1 \) and \( \delta_{a_2} < 0.05 \) among fitted planet pairs with \( \chi^2 < 1.5 \) (or \( \chi^2 > 2 \)).

Figure 15. The trace plot of the iteration number against the value of the parameters at each iteration for a Jupiter pair in a 3:4 MMR with \( \sigma_n = 10 \) μas. The dark lines are the true values of each parameter of the Jupiter pair.
reconstructed probabilities are much better than \( d_1 \) with \( \sigma_{\text{phase}} > 4 \).

The comparison between even and uneven data cadence indicates that it is important to have more uniformly distributed data points in astrometry measurements. Although even data cadence is difficult to achieve in real observations considering the limitations of observational windows, we can obtain good MMR-reconstruction probabilities if the data sampling has a small variance \( \sigma_{\text{phase}} < 3 \) according to our results. Choosing an even data cadence would be suitable for most cases except systems with very eccentric planet pairs in MMRs, which are very rare.

8. DISCUSSION AND CONCLUSION

Astrometry is an ancient technique used to discover asteroids and planets in the solar system. With improvements to the technique, the astrometry method can be extended to discovering exoplanets around nearby stars and to obtain more information about the orbit of planets. Using these orbital elements and the mass of the star, we can reconstruct planet systems in MMRs. In Section 2, we introduce the astrometry methods for detecting planets and the fitting procedure of planetary parameters. Based on observations about planet pairs near MMRs (Figure 1), we consider planet pairs with equal masses, i.e., Jupiter pairs and super-Earth pairs. We also present how to simulate samples of planetary systems in 1:2, 2:3, and 3:4 MMRs via migration and random methods. The distributions of eccentricities and \( \Delta \) for each MMR of the Jupiter pairs and super-Earth pairs are shown in Figures 2, 4, and 5. In Section 3, we show that planets with \( S/N > 3 \) can be detected reliably (Figure 3) in our simulations. As we use the Keplerian orbit to model the true orbit, the difference may lead to false detections of third planets when there is no observational error, however, they can be ignored in our MMR-reconstruction because of their small masses and large separations with the detected planets.

In Section 4, we show the probabilities of reconstructing Jupiter pairs and super-Earth pairs in 1:2, 2:3, and 3:4 MMRs. The main conclusions are listed as follows:

1. The fitting errors of planet pairs are sensitive to observational errors according to Tables 1 and 2. The fitting errors lead to an obvious decrease of the MMR-reconstruction probabilities \( \beta \) with the decrease of \( S/N \) as shown in Tables 3 and 4.

2. With the increase of \( \Delta \), there is a decrease in MMR-reconstruction probability \( \beta \) for Jupiter pairs in 2:3 and 1:2 MMRs in Figure 6, which is not obvious for super-Earth pairs in Figure 7.

3. There is a positive correlation between the MMR-reconstruction probability and the eccentricity of the planets.
for both Jupiter and super-Earth pairs in Figures 8 and 9. Planet pairs with $e > 0.11$ and $e > 0.12$ are better reconstructed than those with $e < 0.11$ and $e < 0.12$, because large eccentricity can avoid the degeneracy between $\omega$ and $M$, and the resonance width increases with eccentricity (Deck et al. 2013).

4. MMR-reconstruction probabilities are larger for planet pairs with strong resonance intensities with $A_{\psi i} < 30^i (i = 1, 2)$ illustrated in Figures 10 and 11.

5. With similar S/N, the MMR-reconstruction probabilities of Jupiter pairs are larger than those of super-Earth pairs when considering stability, as shown in Table 5.

In Section 5, we calculate the FAPs for reconstructing a planet system in or near MMRs. Our main conclusions are:

1. $P_{0-1}$ as the probability of mistaking a near-MMR system for a resonant system has a positive correlation with observational error, however, it decreases with the increase of $\Delta_{\text{fit}}$. The results are presented in Table 6 and 7.

2. The FAPs for planets reconstructed to be in MMRs $\mathcal{F}_{0-1}$ are the largest for planet pairs in 1:2 MMRs. It is difficult to produce a stable Jupiter pair near 3:4 MMRs, thus $\mathcal{F}_{0-1} \sim 0$. Both $\mathcal{F}_{0-1}$ and $\mathcal{F}_{1-0}$ are sensitive to observational errors. As shown in Tables 9 and 10, when S/N $\sim 3$, both $\mathcal{F}_{0-1}$ and $\mathcal{F}_{1-0}$ are larger than 30%, so planets with small S/N detected to be in MMRs should be checked carefully.

In Section 6, we estimate the number of planet systems to be discovered in MMRs via astrometry, as shown in Table 11. There are about $3 \times 10^7$ stars with $V < 10$ within 30 pc from the Sun. After assuming the occurrence of planet pairs in MMRs, we estimate that with S/N = 3, tens of planet pairs with Jupiter masses in 2:3 and 1:2 MMRs can potentially be reconstructed, and hundreds of super-Earth pairs in 2:3 and 1:2 MMRs can be detected; planet pairs in 3:4 MMRs are very few because of their rareness based on observations.

In Section 7, we compare the difference between even and uneven data cadences. Extremely uneven data cadences with $s > 4\pi$ lead to large fitting errors in single-planet systems, while data cadences with good phase coverage with $s_{\text{phase}} < 3$ have good fitting results (see Figures 13 and 14). Although it is difficult to obtain even data cadences in real observations, it is important to have enough data points to guarantee good phase coverage. Using a defined parameter $s_{\text{phase}}$ in two-planet systems, the MMR-reconstruction probabilities with $s_{\text{phase}} < 3$ are similar to those for even data cadences (see Table 12).
Currently, the precision of the Gaia program is about a few tens of μas, which can help us find planets of Jupiter mass. If it can reach a precision of about 10 μas, the probability of reconstructing a Jupiter pair in 2:3 and 1:2 MMRs is >50% at least (see Table 3). If a Jupiter pair with such S/N is reconstructed in an MMR, it should be checked very carefully because of the large FAP ~40%. The target precision of the STEP program for bright stars is about 1 μas. For a super-Earth 1 au from the host star and 30 pc from us, S/N ~1, which makes it very difficult to identify the super-Earth. However, if the planets are 10 pc from us, the S/N ~3, which will ensure a probability of 40% with FAP ~40% for 2:3 and 1:2 MMRs. We expect higher precision astrometry (~0.1 μas) in the future, thus we will have the opportunity to detect planets with masses even smaller than Earth, and the probability of reconstructing super-Earth pairs in MMRs will be improved to as large as 75% (Table 4). All planet systems in our simulations are at 30 pc, with similar observational errors. We can reconstruct planet pairs in MMRs with larger probability and smaller FAP if they are closer to us.

In this paper, we adopt a mission lifetime of 5 yr, comparable to Gaia and STEP. Thus it is appropriate to detect planet systems around 1 au from the host star. The data cadence and the time allocated for observations will influence our planet detection via astrometry. A shorter data cadence helps to detect planets closer to the host star and a longer mission lifetime enables planets with longer period to be detected. Only 50 data points are used in this paper; more data could improve the fitting precision and thus perhaps lead to a larger MMR-reconstruction probability. In addition, recent studies (Giuppone et al. 2009, 2012) have shown the potential of detecting and characterizing planet pairs in MMRs using radial velocity data. In combination with high precision radial velocity data, we can improve the precision of the eccentricities which can help to determine ω + Ω accurately. Consequently, planet pairs in MMRs can be reconstructed with larger probabilities and smaller FAPs. Additionally, although fitting errors for planets in 1:2 MMRs are larger than those in 2:3 and 3:4 MMRs, it is difficult to conclude whether planet pairs in 1:2 MMRs are more difficult to reconstruct than the other two MMRs according to results of our simulations, because many factors influence the MMR-reconstruction probabilities, e.g., different MMRs have different resonance widths and resonance structures in the e1 – e2 phase diagram, it is difficult to obtain a large number of samples with exactly the same distribution of Δ, and the same eccentricities and amplitudes of resonance angles for different MMRs. The same reason applies to comparisons...
between super-Earth pairs and Jupiter pairs, and it is also difficult to figure out the significant differences between them.

We only simulate planet pairs with equal masses in the first order MMRs in this paper. Other planet systems in MMRs with different masses such as a Jupiter and a super-Earth can also be reconstructed with proper observational precision. Planet pairs in high order MMRs such as 1:3 and 3:5 are not considered here. As these MMRs are much weaker and have a narrower convergence width than the MMRs discussed in this paper, they need higher precision to be reconstructed.

We thank the anonymous referee for providing helpful comments. This research is supported by the Key Development Program of Basic Research of China (No. 2013CB834900), the National Natural Science Foundations of China (Nos. 11503009 and 11333002), Strategic Priority Research Program The Emergence of Cosmological Structures of the Chinese Academy of Sciences (Grant No. XDB09000000), the Natural Science Foundation for the Youth of Jiangsu Province (No. BK20130547), 985 Project of Ministration of Education and Superiority Discipline Construction Project of Jiangsu Province, “Search for Terrestrial Exo-Planets,” and the Strategic Priority Research Program on Space Science Chinese Academy of Sciences (Grant No. XDA04060900).