A Classification of Hyperfocused 12-Arcs

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Abstract
A $k$-arc in $PG(2, q)$ is a set of $k$ points no three of which are collinear. A hyperfocused $k$-arc is a $k$-arc in which the $\binom{k}{2}$ secants meet some external line in exactly $k$ points. Hyperfocused $k$-arcs can be viewed as 1-factorizations of the complete graph $K_k$ that embed in $PG(2, q)$. We study the 526,915,620 1-factorizations of $K_{12}$, determine which are embeddable in $PG(2, q)$, and classify hyperfocused 12-arcs. Specifically we show that if a 12-arc $K$ is a hyperfocused arc in $PG(2, q)$ then $q = 2^{5k}$ and $K$ is a subset of a hyperconic including the nucleus.

Keywords Hyperfocused arcs · 1-Factorizations · Projective plane · Secret sharing scheme
1 Introduction and Definitions

This article concerns special types of $k$-arcs, called hyperfocused arcs, in classical finite projective planes $PG(2, q)$. For more on finite projective planes, see [9] and the references therein. A $k$-arc in $PG(2, q)$ is a set $K$ of $k$ points no three of which are collinear. It is well known that $k \leq q + 1$ if $q$ is odd and $k \leq q + 2$ if $q$ is even. A $(q + 1)$-arc is called an oval, and a $(q + 2)$-arc is called a hyperoval. Based on the result of Cherowitzo and Holder discussed below, we will restrict our attention to the case when $q$ is even, that is, $q = 2^h$. Recall that a conic is a set of points satisfying an irreducible homogeneous quadratic equation. The nucleus of an oval is the point where all of its tangent lines intersect. A hyperconic is a hyperoval consisting of a conic together with its nucleus.

Lines in the plane are called external, tangent, or secant depending on whether the line meets the arc in 0, 1, or 2 points, respectively. Let $\ell$ be a line external to a $k$-arc $K$. We say $K$ is hyperfocused on $\ell$ if the $\binom{k}{2}$ secants of $K$ meet $\ell$ in exactly $k - 1$ points. The line $\ell$ is called the focus line, and the set of $k - 1$ points in which the secants meet $\ell$ is called the focus set. We call an arc $K$ hyperfocused provided that it is hyperfocused on some external line. It is clear that $k$ must be even for a hyperfocused $k$-arc to exist because secant lines must partition the arc. Since a 2-arc is trivially hyperfocused on any line, we say a hyperfocused $k$-arc is non-trivial if $k \geq 4$.

The study of hyperfocused arcs in classical planes was initially motivated by the construction of certain secret sharing schemes (see [8, 11] and [2] for details), though it has proven to be an interesting geometry problem in its own right (see [6] for example). Following from a result of Bichara and Korchmáros [1], Cherowitzo and Holder [2] showed that if a non-trivial hyperfocused arc exists in $PG(2, q)$ then $q$ is even. Further, it was shown that if a hyperfocused $k$-arc is not a hyperoval or a hyperoval minus two points, then $k \leq \frac{q}{2}$.

Hyperfocused $k$-arcs were classified in all classical planes $PG(2, q)$ for $k = 4, 6, 8$ by Drake and Keating in [4], where they use the setting of Desarguesian nets, and independently by Cherowitzo and Holder in [2] where they also classified hyperfocused 10-arcs. Cherowitzo and Holder exploited the relationship between the 1-factorizations of $K_k$ and hyperfocused $k$-arcs in order to complete their classifications. Hyperfocused 12-arcs were classified in $PG(2, 32)$ by Faina et al. in [5], where they also showed the non-existence of hyperfocused 14-arcs in $PG(2, 32)$.

In the following sections, we use the perspective of Cherowitzo and Holder to study hyperfocused 12-arcs by examining the 1-factorizations of $K_{12}$, which were classified in [3]. We show there is a unique 1-factorization of $K_{12}$ which embeds in $PG(2, q)$, $q = 2^h$, and so there is a unique hyperfocused 12-arc, which is a subset of a hyperconic. Our method uses a computer search of the non-isomorphic 1-factorizations of $K_{12}$ along with a new necessary condition for a 1-factorization to embed as a hyperfocused arc.
2 Hyperfocused Arcs and 1-Factorizations

Let $K$ be a hyperfocused arc. If we identify the $k$ points of a hyperfocused $k$-arc with the vertices of the complete graph $K_k$ then the secant lines to the arc naturally correspond to the edges of the graph. For each point in the focus set, the secant lines through that point partition $K$ into pairs. This corresponds to a set of disjoint edges that cover the vertices of the complete graph, which is called a 1-factor. Thus, the focus set determines a 1-factorization, a set of disjoint 1-factors that cover the edges of the graph. Given a 1-factorization $\mathcal{F}$ of a complete graph, we will call that 1-factorization embeddable if there is a hyperfocused arc in some finite classical plane that determines $\mathcal{F}$.

A 1-factorization of a complete graph induces an edge coloring of that graph, where each color class is a 1-factor. For convenience we will at times refer to this induced edge coloring to state facts about the 1-factorization. We say that two subgraphs have the same coloring if there is an isomorphism between them that preserves the colors of edges.

In order to determine which 1-factorizations of $K_{12}$ are embeddable in $PG(2, q)$ we will first develop some necessary conditions that will help us eliminate 1-factorizations that cannot be embedded. To discuss them properly we recall the concept of the diagonal line of a quadrangle $ABCD$. In $PG(2, 2^h)$ the quadrangle $ABCD$ can be completed to a Fano subplane. The unique line meeting this subplane in a line of the subplane that is external to the quadrangle is known as the diagonal line. Our first necessary condition comes from Cherowitzo and Holder. We use $C_4$ to denote the cycle on four vertices and $K_4$ to denote the complete graph on four vertices; see [12] for any other graph theory terminology.

**Lemma 1 ([2])** Let $\mathcal{F}$ be the 1-factorization obtained from the focus set of a hyperfocused arc in $PG(2, 2^h)$. If there are two 1-factors in $\mathcal{F}$ with a $C_4$ in their union, then there must be a unique third 1-factor that completes the $C_4$ to a $K_4$.

**Proof** Let $\mathcal{H}$ be a hyperfocused $k$-arc in $PG(2, 2^h)$ that has focus line $\ell$. Let $P, Q$ be two points on $\ell$ associated with 1-factors in $\mathcal{F}$ with a $C_4$ in their union. We may assume that $P = AB \cap CD$ and $Q = AD \cap BC$ where $A, B, C, D \in \mathcal{H}$. The diagonal line of the quadrangle $ABCD$ contains both $P$ and $Q$ so must be $\ell$. Hence the point $R = AC \cap BD$ (which necessarily lies on $PQ$ since $q$ is even) is associated with the desired 1-factor.

We add the following necessary condition for a 1-factorization to embed in $PG(2, 2^h)$. We make significant use of Desargues’ theorem.
**Theorem 2** *(Desargues’ Theorem)* Two triangles are in perspective from a point if and only if they are in perspective from a line.

**Lemma 3** Let $F$ be the 1-factorization obtained from the focus set of a hyperfocused arc in $PG(2, 2^h)$. If there are two disjoint copies of $K_4 - e$ with the same coloring (see Fig. 2), then the remaining edges of each $K_4$ both have the same color as well.

**Proof** Let $\mathcal{H}$ be a hyperfocused arc with focus line $\ell$, and let $\mathcal{F}$ be the 1-factorization obtained from the focus set of $\mathcal{H}$. Additionally let $F_1, \ldots, F_5$ be 1-factors in $\mathcal{F}$. Assume that $A, B, C, D, E, F, G, H$ belong to $\mathcal{H}$, and that $(AB), (EF) \in F_1$, $(AC), (EG) \in F_2$, $(BC), (FG) \in F_3$, $(BD), (FH) \in F_4$, and $(CD), (GH) \in F_5$, where $(XY)$ denotes the edge between the vertices corresponding to the points on $\ell$. See Fig. 2.

Observe that the triangles $\Delta ABC$ and $\Delta EFG$ are in perspective from the line $\ell$ and so must be in perspective from a point. Call that point $V$. Now, triangles $\Delta BCD$ and $\Delta FGH$ are also in perspective from $\ell$ and so must also be in perspective from $V$. Observe that triangles $\Delta ACD$ and $\Delta EGH$ are also in perspective from $V$ and so must be in perspective from a line. Since $AC$ meets $EG$ on $\ell$ and $CD$ meets $GH$ on $\ell$ we must have that $AD$ meets $EH$ on $\ell$. Thus $(AD)$ and $(EH)$ are in the same 1-factor (hence have the same color), which completes both $K_4 - e$ to a $K_4$. \hfill \square

The following theorems from Cherowitzo and Holder, which classify all hyperfocused arcs contained in a hyperconic and contain the nucleus, as well as Pascal’s theorem, are the final pieces of information needed for us to complete our classification.

**Theorem 4** *(Pascal’s Theorem, Braikenridge-Maclaurin Theorem)* The six points of a hexagon lie on a conic if and only if the pairs of opposite sides meet in three collinear points.

**Theorem 5** *(2) A set of points $K$ with $3 < |K|$ on a hyperconic in $PG(2, q)$, $q = 2^h$ which includes the nucleus $N$ of the conic is hyperfocused on a secant line to that conic which does not meet $K$ if and only if $K \setminus \{N\}$ is projectively equivalent to a set of points $\{(a, a^2, 1) : a \in H\}$ determined by a multiplicative subgroup of $H$ of $(GF(q^2), \cdot)$ (described in theorem 3.3 of [2]).

**Theorem 6** *(2) A set of points $K$ with $3 < |K|$ on a hyperconic in $PG(2, q)$, $q = 2^h$ which includes the nucleus $N$ of the conic is hyperfocused on an exterior line to that conic if and only if $K \setminus \{N\}$ is projectively equivalent to a set of points determined by...
3 Classifying Hyperfocused 12-Arcs

In order to classify hyperfocused 12-arcs we examined the list of 1-factorizations of $K_{12}$. This list was graciously provided in usable form by Kaski and Östergård from their paper [10]. We studied each 1-factorization and checked, via computer, the necessary conditions from Lemma 1 and Lemma 3. The code can be found at [7].

When performing the computer search we break the computation down into two parts. The first is a filter of the 526, 915, 620 1-factorizations of $K_{12}$ to determine which satisfy Lemma 1. This filtering produced a list of only 253 1-factorizations. We then checked the list of 253 against the conditions of Lemma 3, which yielded a list of 2 remaining 1-factorizations.

To filter using Lemma 1, we used the following process. In each 1-factorization, we examine each 1-factor $F$ and each pair $P$ of edges from $F$. The pair $P$ contains four vertices that we use to check Lemma 1. There are two pairs of disjoint edges between these vertices that are not in $F$. Lemma 1 states that if one of the pairs of disjoint edges is contained in another 1-factor, then the other pair must also be contained in a third 1-factor. So we check if exactly one of the pairs is contained in a 1-factor, and if so, exclude the 1-factorization from our list. Otherwise the 1-factorization makes it through our filter. As noted above, there are 253 such 1-factorizations that made it through this step. As an optimization, in a given 1-factorization we check every 1-factor as described above except for one. Since two 1-factors are needed for a violation of Lemma 1, the method described above will detect the violation starting from either 1-factor.

The next part of the computation is to check the surviving 1-factorizations against Lemma 3. For each 1-factorization, we examine each set of four unordered vertices from $K_{12}$ and then another set of four ordered vertices disjoint from the original four. These vertices make up the vertices of our two disjoint $K_4$s. The choosing of unordered and ordered sets of vertices accounts for the different ways the edges may correspond. Next, we counted how many pairs of corresponding edges between the two $K_4$s have the same color, and if there are exactly five pairs of corresponding edges with the same color, Lemma 3 is violated and we exclude that 1-factorization from our list. At the end of this process we were left with only 2 1-factorizations.

We now consider both of these 1-factorizations, which are given below. We identify the vertices of the complete graph $K_{12}$ with the integers $0, 1, \ldots, 11$ and the
edges as order pairs \((x, y)\) with \(x, y \in \{0, 1, \ldots, 11\}\). Further, we identify the 1-factors with letters \(A, \ldots, K\). The first 1-factorization we want to examine is:

\[
\begin{align*}
A & : [(0, 1), (2, 3), (4, 5), (6, 7), (8, 9), (10, 11)] \\
B & : [(0, 2), (1, 3), (4, 6), (5, 7), (8, 10), (9, 11)] \\
C & : [(0, 3), (1, 2), (4, 7), (5, 6), (8, 11), (9, 10)] \\
D & : [(0, 4), (1, 5), (2, 8), (3, 9), (6, 11), (7, 10)] \\
E & : [(0, 5), (1, 4), (2, 10), (3, 11), (6, 8), (7, 9)] \\
F & : [(0, 9), (1, 8), (2, 4), (3, 5), (6, 10), (7, 11)] \\
G & : [(0, 11), (1, 10), (2, 5), (3, 4), (6, 9), (7, 8)] \\
H & : [(0, 6), (1, 7), (2, 11), (3, 10), (4, 8), (5, 9)] \\
I & : [(0, 7), (1, 6), (2, 9), (3, 8), (4, 11), (5, 10)] \\
J & : [(0, 10), (1, 11), (2, 6), (3, 7), (4, 9), (5, 8)] \\
K & : [(0, 8), (1, 9), (2, 7), (3, 6), (4, 10), (5, 11)]
\end{align*}
\]

Assume that this 1-factorization embeds and without loss of generality assume that we have coordinates \(A = (1, 0, 0)\), \(B = (0, 1, 0)\), \(0 = (0, 0, 1)\), and \(3 = (1, 1, 1)\). Thus the focus line is \([0, 0, 1]^T\) and since \((03)\) meets the focus line at \(C\) we must have \(C = (1, 1, 0)\). Completing the Fano plane \(0123ABC\) yields \(1 = (1, 0, 1)\), \(2 = (0, 1, 1)\). The point \(4\) cannot be on the line \([0, 0, 1]^T\) so assume it has coordinates \((x, y, 1)\). Observe that \((45)\) passes through \(A\) and \((46)\) passes through \(B\) so we must have coordinates \(5 = (s, y, 1)\) and \(6 = (x, t, 1)\). Since \(4567ABC\) forms a Fano plane we must also have \(7 = (s, t, 1)\) with \(x + s = y + t\). Since \(D = (04) \cap (1, 5)\) we must have that \(D = (x, y, s + x + 1)\) and using the fact that the focus line is \([0, 0, 1]^T\) we get \(s = x + 1\) and consequently \(t = y + 1\). Further we have \(E = (x + 1, y, 0)\), \(F = (x, y + 1, 0)\), and \(G = (x + 1, y + 1, 0)\). Now, \((06)\) meets the focus line at \(H\), so \(H = (x, y + 1, 0)\), forcing \(H = F\), a contradiction. Thus this 1-factorization is not embeddable. We now consider the only remaining 1-factorization.
Lemma 7. If \( a \in GF(2^{5k}) \) satisfies \( a^5 = a^4 + a^3 + a + 1 \) then \( T(a) = T(1) \).
Table 1  Coordinates of the hyperfocused arc and focus set

| Point | Coordinates       | Determined by | Point | Coordinates       | Determined by       |
|-------|-------------------|---------------|-------|-------------------|---------------------|
| 0     | (1, 0, 0)         | Nucleus       | A     | (1, 1, 0)         | (01) ∩ (23)         |
| 1     | (0, 1, 0)         | Assumed       | B     | (a, 0, 1)         | (02) ∩ (14)         |
| 2     | (0, 0, 1)         | Assumed       | C     | (0, a, 1)         | (12) ∩ [1, 1, a]^T  |
| 3     | (1, 1, 1)         | Assumed       | D     | (a + 1, 1, 1)     | (03) ∩ [1, 1, a]^T  |
| 4     | (a, a^2, 1)       | Assumed       | E     | (1, a + 1, 1)     | (13) ∩ [1, 1, a]^T  |
| 5     | (a + 1, a^2 + 1, 1)| (D1) ∩ C     | F     | (a^2 + a, a^2, 1)| (04) ∩ [1, 1, a]^T  |
| 6     | (a^2 + a, a^2, a^2 + 1)| (B3) ∩ (F2) | G     | (a, a^2, a + 1)  | (24) ∩ [1, 1, a]^T  |
| 7     | (a + 1, 1, a^2 + 1)| (D2) ∩ (E4) | H     | (a^2 + a + 1, a^2 + 1, 1)| (05) ∩ [1, 1, a]^T |
| 8     | (a^2 + a, a^4 + a^2, 1)| (F1) ∩ C   | I     | (1, a^2 + a + 1, a + 1)| (35) ∩ [1, 1, a]^T  |
| 9     | (a^2 + a + 1, a^4 + a^2 + 1, 1)| (H1) ∩ C | J     | (a^5 + 1, a^5 + a^3, a^5 + 1)| (06) ∩ (39) |
| 10    | (a^3 + a + 1, a^3 + a^2 + a + 1, a^2 + 1)| (E0) ∩ (F3) | K     | (1, a^2 + a, a^4 + a^3 + a^2 + a)| (07) ∩ (28) |
| 11    | (a^3 + a^2 + a, a^3 + a, a^2 + 1) | (C0) ∩ (H2) |       |                   |                     |
Proof Since finite fields are separable extensions of their ground field, we use the formula

\[ T(a) = [GF(q) : GF(2)] \sum_{x \in R} x, \]

where \( R \) is the set of roots of the minimal polynomial of \( a \). Since \( x^5 + x^4 + x^3 + x + 1 \) is irreducible, it must be the minimal polynomial of \( a \). The coefficient of \( x^4 \) is 1 in this polynomial, thus we have \( \sum_{x \in R} x = 1 \). Therefore, \( T(a) = [GF(q) : GF(2)] = T(1) \), as desired.

It is well known that the polynomial \( t^2 + t + a \) is irreducible if and only if \( T(a) = 1 \). It follows from Lemma 7 that this polynomial is irreducible if and only if \( T(1) = 1 \), which happens if and only if \( 5k \) is odd, and hence, \( k \), is odd. Hence the line \( [1, 1, a]^T \) is external to \( C \) when \( k \) is odd, and secant to \( C \) when \( k \) is even. Thus our classification follows from Theorems 5 and 6.

Theorem 8 A 12-arc \( K \) is a hyperfocused arc in \( PG(2, q) \) if and only if \( q = 2^{5k} \) and \( K \) is a subset of a hyperconic including the nucleus. Specifically, the focus line is secant to the conic when \( k \) is even so the non-nucleus points are projectively equivalent to a set of points determined by a subgroup of \( (GF(q)^*) \); and the focus line is exterior when \( k \) is odd, so the non-nucleus points are projectively equivalent to a set of points determined by a subgroup of \( \mathbb{Z}_{q+1} \).

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Data Availability The datasets generated during and/or analyzed during the current study are available from the authors on reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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