Invertibility of the 3-core of Erdős Rényi Graphs with Growing Degree

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Abstract

Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of an Erdős Rényi graph $G(n, d/n)$ for $d = \omega(1)$ and $d \leq 3 \log(n)$. We show that as $n$ goes to infinity, with probability that goes to 1, the adjacency matrix of the 3-core of $G(n, d/n)$ is invertible.

1 Introduction

Our main result is the following theorem. Big-O notation should be interpreted with respect to $n$ going to infinity. If $A$ is the adjacency matrix of a graph $G$, then we say the 3-core of $A$ to mean the adjacency matrix of the 3-core of $G$.

**Theorem 1.** Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a random Erdős Rényi graph $G(n, d/n)$. If $d = \omega(1)$ and $d \leq 3 \log(n)$, the with probability $1 - o(1)$, the 3-core of $A$ is full rank.

**Remark 1.** One can show from our proof that this theorem holds more generally for the $k$-core of of $A$ for any fixed integer $k \geq 3$. This requires a slight modification to the proofs of Lemma 11 and Claim 7.

This provides partial progress on the following conjecture from Van Vu [6].

**Conjecture 1.** There exists a constant $d_0$, and a constant $k$ such that the $k$-core of $G(n, \frac{d}{n})$ is full rank for $d \geq d_0$.

We outline our strategy below. Let $A(3)$ denote the 3-core of an adjacency matrix $A$.

In Section 2, we state some concentration and anti-concentration lemmas. In Section 3-5, we use union bounds to prove some that with high probability, certain “sparse” dependency structures (which involve not too many rows) do not exist in $A(3)$. Combining the results in these sections with a lower bound on the size of the 3-core shows that for some constant $C$,

$$\Pr[\exists x \in \mathbb{R}^n : A(3)x = 0, \text{supp}(x) \leq n/C] = o(1).$$

(1)

The proofs in Sections 3-5 are based off of prior work of the author with DeMichele and Moreira [3], which classifies the types of linear dependencies that occur in sparse symmetric random matrices. In Section 6, we show that with high probability, $A(3)$ has no kernel vectors with large support:

$$\Pr[\exists x \in \mathbb{R}^n : A(3)x = 0, \text{supp}(x) \geq n/C] = o(1).$$

(2)

The ideas in this section are based off of ideas in the works of Ferber, Kwan, and Sauermann [4] and Costello, Tau, and Vu [1]. More closely, our proof follows the proofs of Lemmas 19-22 in [3] (also based on [4] and [1]), which show that

$$\Pr[\exists x \in \mathbb{R}^n : Ax = 0, \text{supp}(x) \geq n/C] = o(1).$$

(3)

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Our argument is slightly more complex than that of [3] because we consider $A(3)$ and not $A$.

### 1.1 Notation

For a matrix $B \in \mathbb{R}^{n \times m}$ and a sets $S \subset [n]$ and $T \subset [m]$, let $B_S$ denote the matrix $B$ restricted to rows in $S$, and let $B[T]$ denote the matrix $B$ restricted to columns in $T$.

## 2 Useful Lemmas and Definitions

**Lemma 1** (Tail Bound on Binomial). If $t > np$, then

$$\Pr[\text{Bin}(n, p) \geq t] \leq \binom{n}{t} p^t \left(\frac{1 - p}{t}\right)^t.$$  \hspace{1cm} (4)

This implies that if $t \geq 2np$,

$$\Pr[\text{Bin}(n, p) \geq t] \leq 2 \binom{n}{t} p^t \leq 2 \left(\frac{enp}{t}\right)^t.$$  \hspace{1cm} (5)

**Proof.**

\[
\Pr[\text{Bin}(n, p) \geq t] \leq \sum_{i=t}^{n} \binom{n}{i} p^i \\
\leq \sum_{i=t}^{n} \binom{n}{t} p^i \prod_{j=t+1}^{i} \left(\frac{n - j + 1}{j}\right) \\
\leq \sum_{i=t}^{n} \binom{n}{t} p^i \prod_{j=t+1}^{i} \left(\frac{n}{t}\right) \\
= \sum_{i=t}^{n} \binom{n}{t} p^i \left(\frac{pm}{t}\right)^{i-t} \\
\leq \binom{n}{t} p^t \sum_{j=0}^{\infty} \left(\frac{pm}{t}\right)^j \\
= \binom{n}{t} p^t \frac{1}{1 - \frac{pm}{t}}.
\]

We will use the following two anticoncentration lemmas.

The first lemma is due to Costello and Vu [2], and is proved with these constants in Lemma 3 of [3].

**Lemma 2** (Sparse Littlewood Offord). For any integer $m$ and $q \in (0, 1/2)$ such that $qm \geq 9$, for all $x \in \mathbb{R}^m$ with full support,

$$\Pr \left[ \sum_{i=1}^{n} x_i b_i = 0 \right] \leq \frac{1}{\sqrt{qm}},$$

where $b_i \sim \text{Ber}(q)$.

The next lemma is a quadratic version of the Littlewood-Offord theorem, due to Costello, Tau, and Vu [1]. We state a version for sparse random vectors proved by Costello and Vu in [2].

**Lemma 3** (c.f. [2] Lemma 8.4). Let $M \in \mathbb{R}^{n \times n}$ be a deterministic matrix with a least $m$ non-zero entries in each of $m$ distinct columns of $M$. Let $z \in \mathbb{R}^n$ be the random vector with i.i.d. Bernoulli entries with parameter $p \leq 1/2$. Then for any fixed $c$,

$$\Pr (z^T M z = c) = O \left( \frac{1}{(mp)^{1/4}} \right).$$
We prove Theorem 1 by ruling out the existence of all kernel vectors. We break this down based on the size of support of the kernel vector. We use the following definition and claim.

**Definition 1.** A $k$-minimal dependency is a set $k$ linearly dependent rows, where all strict subsets of these rows are linearly independent.

**Claim 1.** Any $k$-minimal dependency has no columns with exactly one non-zero entry among the $k$ rows.

**Proof.** Since the coefficients of the linear dependency must all be non-zero, if there any column had a single non-zero entry, it would not be orthogonal to the dependency coefficient vector. \( \square \)

### 3 Small Case

**Lemma 4** (Small Case). Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a random Erdős Rényi graph $G(n, d/n)$. Then there exists a constant $d_0$ such that for all $d_0 < d < 3 \log(n)$, with probability $1 - O(\frac{1}{\sqrt{n}})$, for all $k < \frac{n}{d}$, $A$ does not contain a set of $k$ rows for $k \geq 1$, which when restricted to some subset of columns, forms a $k$-minimal dependency with at least $3k - 1$ total non-zero entries.

Formally,\(^{(7)}\)

\[
\Pr \left[ \exists x \in \mathbb{R}^n, T \subset [n] : x^T A|_T = 0 \land |A_{\text{supp}(x)}[T]|_1 \geq 3|\text{supp}(x)| - 1 \land \text{rank}(A_{\text{supp}(x)}[T]) = |\text{supp}(x)| - 1 \right] = o(1)
\]

The same result applies if $A$ has its first column removed.

**Proof.** We introduce some notation to prove this. Let $S \subset [n]$ be a set of size $k$, and let $A_S$ be the submatrix containing only the rows indexed by values in $S$. Let $E_{\text{Sym}}$ be the set of entries in $A_S$ whose columns are indexed by values in $S$. Let $E_{\text{SymAD}}$ be subset of entries in $E_{\text{Sym}}$ that are above the diagonal of $A$, and hence mutually independent. Let $E_{\text{Asym}}$ be the set of entries who columns are not in $S$, and finally, let $E = E_{\text{Asym}} \cup E_{\text{SymAD}}$ be the full set of mutually independent entries that determine the rows in $S$ (See Figure 1).

Formally:

\[
E_{\text{Sym}} := \{(i, j) : i, j \in S\}.
E_{\text{SymAD}} := \{(i, j) : i, j \in S, i < j\}.
E_{\text{Asym}} := \{(i, j) : i \in S, j \notin S\}.
E := E_{\text{SymAD}} \cup E_{\text{Asym}}.
\]
For each of these four sets for $T \subset [n]$, let $E_{Sym}[T]$, $E_{Sym,AD}[T]$, $E_{Asym}[T]$ and $E[T]$ denote the respective set additionally restricted to entries with $j \in T$.

We will couple the process of putting non-zero entries in $A_S$ with a random walk that counts the number of times a non-zero entry is inserted in $E_{Asym}$ in a column that already contains a non-zero entry or into $E_{Sym,AD}$.

We condition on $R$, the number of nonzero entries in $E$. Conditioned on $R = r$, the process of choosing $A_S$ is equivalent to randomly choosing $r$ locations in $E$ for these non-zero entries without replacement. (Note that this is true even if we are considering a matrix with the first column removed, even though $E_{Sym,AD}$ may include some entries which are not repeated below the diagonal, if $1 \in S$). Let $(X_i)_{i \in [R]}$ be the random walk that increases by 1 if the $i$th random location chosen is in $E_{Sym,AD}$ or if the $i$th random location is in $E_{Asym}$ and is not the first non-zero entry in its column. Otherwise, put $X_i = X_{i-1}$.

The following claim says that $X_R$ must be large for the structure described in the lemma to occur.

**Claim 2.** Suppose there is some set of at columns $T$ such that there are at least $3k - 1$ non-zeros in $A_S[T]$. Then $X_R \geq \lceil \frac{3k-1}{2} \rceil$.

**Proof.** Let $L$ be the number of non-zeros entries in $E_{Asym}[T]$, and let $M$ be the number of non-zero entries in $E_{Sym,AD}[T]$. By assumption, $2M + L \geq 3k - 1$. Let $Y$ be the number of non-zero entries in $E_{Asym}[T]$ which are not the first non-zero in their column, so $Y \leq X_R - M$. Now the number of columns in $A_S[T \setminus S]$ which have exactly one non-zero entry is at least $(L - Y) - Y$, since there are exactly $L - Y$ non-empty columns in $A_S[T]$, and at most $Y$ of those columns have more than one non-zero entry. Now

$$L - 2Y \geq L - 2X_R + 2M \geq 3k - 1 - 2X_R.$$

Hence if $X_R < \lceil \frac{3k-1}{2} \rceil$, then some column in $A_S[T]$ has exactly one non-zero entry, which rules out $A_S[T]$ being a $k$-minimal dependency.

It remains to show that $X_R$ is small enough with high enough probability to take a union bound over all $\binom{n}{k}$ sets $S$. We will break up this probability as follows. Let

$$Q := \left\lceil \frac{3k-1}{2} \right\rceil,$$

and define $R_{max} := \max (\frac{1}{4 \epsilon} n^* k^{1-\epsilon}, 2dk)$, where $\epsilon = \frac{1}{12}$.

Then

$$\Pr[\text{Structure in the lemma}] \leq \Pr[R > R_{max}] + \Pr[X_R \leq Q | R < R_{max}].$$

The following claim bounds the second term in the equation above.

**Claim 3.** For any $k < \frac{\epsilon}{\alpha}$, for $d$ a large enough constant,

$$\Pr[X_R \geq Q | R < R_{max}] \leq 2 \left( \frac{k}{3n} \right)^{k + \frac{1}{2}}. \quad (10)$$

**Proof.** We couple $X_i$ with a random walk that $(Y_i)_{i \in R}$ which increases by 1 with probability $\frac{k + R}{n}$ and otherwise stays constant. Observe that $Y_i$ stochastically dominates $X_i$, because there are at most $k + R$ columns in $E$ in which placing a non-zero entry will increase $X_i$. Then conditioned on $R = r$,

$$\Pr[X_r \geq Q] \leq \Pr \left[ \text{Bin} \left( r, \frac{k + R}{n} \right) \geq Q \right].$$

We will bound the tail of this binomial using Lemma [1]. We first check that the hypothesis of the lemma is satisfied, namely that $r \frac{k + R}{n} < 1$ for $r < R_{max}$:

If $r < R_{max} = \max (\frac{1}{4 \epsilon} n^* k^{1-\epsilon}, 3dk)$ and $k \leq \frac{n}{\alpha}$, then for $i \geq Q \geq k$,
\[
\frac{r(r + k)}{nQ} \leq \max \left( \left( \frac{(\frac{1}{4} \epsilon^r k^{1-r})^2 + (\frac{1}{4} \epsilon^r k^{1-r}) k}{kn} \right), 3d(3d + 1)k^2 \right) \\
= \max \left( \frac{1}{16e^2} \left( \frac{k}{n} \right)^{1-2e} + \frac{1}{4e} \left( \frac{k}{n} \right)^{1-e}, 3d(3d + 1)k \right) \\
\leq \max \left( \frac{1}{3e} \left( \frac{k}{n} \right)^{1-2e}, 10d^2 k \right) .
\] (12)

Since this value clearly at most \( \frac{1}{2} \), the conditions of Lemma 1 are satisfied. Hence conditioned on \( R < R_{\text{max}} \),

\[
\Pr \left[ X_R \geq Q \right] \leq 2 \left( \frac{eR(k + R)}{Qn} \right)^Q .
\] (13)

Note that the structure in the lemma cannot occur for \( k = 1 \). Plugging in \( Q = \left\lceil \frac{3k - 1}{2} \right\rceil \) and \( k \geq 2 \), conditioned on \( R < R_{\text{max}} \), we can bound

\[
\Pr \left[ X_R \geq \left\lceil \frac{3k - 1}{2} \right\rceil \right] \leq 2 \left( \frac{eR(k + R)}{Qn} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \\
\leq 2 \left( e \max \left( \frac{1}{3e} \left( \frac{k}{n} \right)^{1-2e}, 10d^2 k \right) \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \\
= 2 \max \left( \left( \frac{k}{3n} \right)^{k+\frac{1}{2}}, \left( \frac{10c_2d^2k}{n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \right) \\
\leq 2 \max \left( \left( \frac{k}{3n} \right)^{k+1}, \left( \frac{10c_2d^2k}{n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \right)
\] (14)

Here the first inequality is Equation 13, the second inequality plugs in the result of Equation 12, the third inequality uses Jenson’s inequality, and the final inequality uses the fact that \((1-2e)^{\left\lceil \frac{3k - 1}{2} \right\rceil} = \frac{5}{6} \left\lceil \frac{3k - 1}{2} \right\rceil \geq k + \frac{1}{2} \) for integers \( k \geq 2 \).

Now

\[
\left( \frac{10ed^2k}{n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} = \left( \frac{k}{3n} \right)^{k+1} \left( \frac{k(30ed^2)^3}{3n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \left( \frac{30ed^2}{1} \right)^{1+1k_{\text{odd}}} \\
= \left( \frac{k}{3n} \right)^{k+1} \left( \frac{k(30ed^2)^3}{3n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \left( \frac{30ed^2}{1} \right)^{1+1k_{\text{odd}}} 
\] (15)

Note that the function

\[
\left( \frac{k(30ed^2)^3}{3n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \left( \frac{30ed^2}{1} \right)^{1+1k_{\text{odd}}}
\]

is convex in \( k \) (when the parity of \( k \) is fixed), so its maximum is achieved at either the minimum or maximum value of \( k \), up to parity.

It is easy to check that when \( 2 \leq k < \frac{n}{2}d \), \( \frac{3n}{2(30ed^2)^3} \leq \frac{3n}{2(30ed^2)^3} \) (where the last inequality holds for \( d \) large enough), we have

\[
\left( \frac{k(30ed^2)^3}{3n} \right)^{\left\lceil \frac{3k - 1}{2} \right\rceil} \left( \frac{30ed^2}{1} \right)^{1+1k_{\text{odd}}} < 1.
\] (16)
Hence for all $k$ in this range, we have

\[
\left( \frac{10ed^2k}{n} \right)^{\left\lceil \frac{2k-1}{2} \right\rceil} \leq \left( \frac{k}{3n} \right)^{k+\frac{1}{4}}.
\]  

(17)

Plugging this in to Equation 14, we have

\[
\Pr \left[ X_R \geq \left\lceil \frac{3k - 1}{2} \right\rceil \right] \leq 2 \left( \frac{k}{3n} \right)^{k+\frac{1}{4}}.
\]  

(18)

This yields the claim.

Then next claim bounds the probability that $R \geq R_{\text{max}}$.

**Claim 4.**

\[
\Pr[R \geq R_{\text{max}}] \leq 2 \left( \frac{k}{en} \right)^k e^{-2 \log(n)}
\]  

(19)

**Proof.** Recall that $R$ is the number of non-zero entries in the $E_{\text{Asym}}$ and $E_{\text{SymAD}}$. Now $|E_{\text{Asym}} \cup E_{\text{SymAD}}| \leq kn$, and each entry in $E_{\text{Asym}} \cup E_{\text{SymAD}}$ is non-zero with probability $d/n$.

Hence

\[
\Pr[R \geq R_{\text{max}}] \leq \Pr \left[ \text{Bin} \left( \frac{kn}{2}, \frac{d}{n} \right) \geq R_{\text{max}} \right] \leq 2 \left( \frac{dke}{R_{\text{max}}} \right)^{R_{\text{max}}},
\]  

(20)

where the binomial bound comes from Lemma 1.

Recall that $R_{\text{max}} = \max \left( \frac{k}{4e} n^{1/\epsilon}, 3dk \right)$. Let $\gamma := \log_2(n/k)$ such that $R_{\text{max}} = k \max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)$.

Then

\[
\Pr[R > R_{\text{max}}] \leq 2 \left( \frac{de}{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} \right)^{k \max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)}
\]  

\[
= 2 \left( \frac{k}{en} \right)^k \left( \frac{de}{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} \right)^{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} \left( \frac{ne}{k} \right)^k
\]  

\[
= 2 \left( \frac{k}{en} \right)^k \left( \frac{de}{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} \right)^{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} n^{1-\gamma}
\]  

(21)

If $n^{\gamma \epsilon} < (4de^2)^2$, then

\[
\left( \frac{de}{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} \right)^{\max \left( \frac{n^{\gamma \epsilon}}{4e}, 3d \right)} n^{1-\gamma} \leq \left( \frac{e}{3} \right)^{3d (4de^2)^2 / \epsilon} e^{n^{1-\gamma} (3d \log(\frac{e}{3}) + \frac{4}{\epsilon} \log(4de^2) + 1)} \leq e^{-n^{1/2}},
\]  

(22)

where the last inequality follows from taking $d$ a large enough constant and noticing $\gamma \leq \frac{2 \log(4e^2 d)}{\epsilon \log(n)} \leq \frac{1}{2}$ for large enough $n$.

If $n^{\gamma \epsilon} \geq (4de^2)^2$, then

\[
\]
\[
\left( \frac{de}{\max \left( \frac{d \gamma}{4e}, 3d \right)} \right)^{\max \left( \frac{d \gamma}{4e}, 3d \right)} n^{1-\gamma} e \leq \left( \frac{4de}{\gamma} \right)^{\frac{d \gamma}{4e}} n^{2\gamma} \leq \left( \frac{1}{n^{\gamma}} \right)^{\frac{d \gamma}{4e}} n^{1-\gamma} \]  

(23)

Taking a logarithm, we have
\[
\log \left( \frac{1}{n^{\gamma}} \right)^{\frac{d \gamma}{4e}} = n^{1-\gamma} \left( 2\gamma \log(n) - \frac{\gamma e}{8e} n^{\gamma} \log(n) \right)
= n^{1-\gamma} \gamma \log(n) \left( 2 - \frac{en^{\gamma}}{8e} \right)
\leq -2n^{1-\gamma} \gamma \log(n),
\]

where in the last line we used the bound \( n^{\gamma} > (ed)^2 \) to show \( \frac{en^{\gamma}}{8e} \geq \frac{e(4d^2)^2}{8e} \geq 4 \) for \( d \) a large enough constant.

Now \( n^{\gamma} > (4de^2)^2 \) implies \( \gamma > \frac{2 \log(4de^2)}{\epsilon \log(n)} \). The function \( n^{1-\gamma} \gamma \log(n) \) achieves its minimum at one endpoint of the interval \( \gamma \in \left[ \frac{2 \log(d)}{\epsilon \log(n)}, 1 \right) \) (this can be checked by observing that the log of the function is concave in \( \gamma \)).

Thus
\[
-2n^{1-\gamma} \gamma \log(n) \leq \max \left( -2n^{1-\gamma} \frac{2 \log(4de^2)}{\epsilon}, -2 \log(n) \right) \leq -2 \log(n),
\]

so if \( n^{\gamma} \geq (4de^2)^2 \), we have
\[
\left( \frac{de}{\max \left( \frac{d \gamma}{4e}, 3d \right)} \right)^{\max \left( \frac{d \gamma}{4e}, 3d \right)} n^{1-\gamma} e \leq e^{-2 \log(n)}.
\]

(26)

Combining the two cases (Equations 22 and 26) into Equation 21 we have
\[
\Pr[R > R_{\max}] \leq 2 \left( \frac{k}{en} \right)^k e^{-2 \log(n)}
\]

(27)

as desired.

We finish the proof of the main lemma by union bounding over all \( k \), and all set \( S \) of \( k \) rows.

Plugging in the result of Claim 4, we have
\[
\sum_{k=1}^{n/d^7} \left( \begin{array}{c} n \\ k \end{array} \right) \Pr[R > R_{\max}] \leq \sum_{k=1}^{n/d^7} \left( \begin{array}{c} n \\ k \end{array} \right) k \left( \frac{k}{en} \right)^k e^{-2 \log(n)} \leq \sum_{k=1}^{n/d^7} \frac{4}{n^2} \leq \frac{1}{n}.
\]

(28)

Hence with probability at least \( \frac{1}{n} \), for all sets of \( k < n/d^7 \) rows, there less than \( R_{\max} = \max(3dk, n^{1/12}k^{11/12}) \) non-zeros among the rows. Assuming \( R < R_{\max} \), we plug in the result of Claim 4 to obtain:
\[ \sum_{k=1}^{n/d^7} \binom{n}{k} \Pr[X_R \geq Q \land R < R_{max}] \leq \sum_{k=1}^{n/d^7} \binom{n}{k} 4 \left( \frac{k}{3n} \right)^{k+\frac{1}{2}} \leq 4 \sum_{k=1}^{n/d^7} \binom{ne}{k} \left( \frac{k}{3n} \right)^{k+\frac{1}{2}} \leq 4 \sum_{k=1}^{n/d^7} \left( \frac{e}{3} \right)^k \left( \frac{k}{3n} \right)^{1/3} = O \left( \frac{1}{\sqrt{n}} \right). \] (29)

This union bound completes the proof of the first part of the lemma which states that with probability \(1 - O(\frac{1}{\sqrt{n}})\), the structure described in the lemma does not exist in \(A\).

\[ \square \]

4 Medium Case

**Lemma 5** (Medium Case). Let \(A \in \mathbb{R}^{n \times n}\) be the adjacency matrix of a random Erdős Rényi graph \(G(n, d/n)\). There exists a constant \(d_0\) such that for all \(d > d_0\),

\[ \Pr \left[ \exists x \in \mathbb{R}^n, T \subseteq [n] : x^T A[T] = 0, \frac{n}{d^7} \leq |\text{supp}(x)| < \frac{2n}{d^7}, |T| \geq \left( 1 - \frac{1}{d^7} \right) n \right] = e^{-\Theta(n)}. \] (30)

The same result applies if \(A\) has its first column removed.

**Proof.** We union bound over all \(k\), and all \(\binom{n}{k}\) sets \(S\) of \(k\) rows which may be the support of \(x\). We will lower bound the number of length \(k\) columns in \(A_S\) which have exactly one non-zero entry. We consider only the \(n - k\) mutually independent columns. Let \(X_i\) be the event that column \(i\) has one non-zero entry. Then

\[ \Pr[X_i] = k \frac{d}{n} \left( 1 - \frac{d}{n} \right)^{k-1}. \] (31)

Let \(c := \frac{n}{d^k}\), such that \(\frac{1}{2} \leq c < d^6\). Now because \(1 - x \geq e^{-\sqrt{x}}\), we have

\[ \left( 1 - \frac{d}{n} \right)^{k-1} \geq \left( 1 - \frac{d}{n} \right)^k \geq \left( 1 - \frac{d}{n} \right)^{\frac{n}{2c}} \geq e^{-\frac{1}{\sqrt{2c}}}, \]

so

\[ \Pr[X_i] \geq \frac{e^{-\frac{1}{\sqrt{2c}}}}{c} \geq \frac{1}{2c}, \] (32)

where we have bounded \(\frac{d}{n} < \frac{1}{2}\) and used the fact that \(c \geq 1/2\).

If \(x^T A[T] = 0\) for \(\text{supp}(x) = S\), then necessarily \(\sum X_i < \frac{n}{d^7}\), because no column in \(A_S[T]\) can have exactly one non-zero entry.

By a Chernoff bound, for \(d\) at least a large enough constant,

\[ \Pr \left[ \sum X_i < \frac{n}{d^7} \right] \leq \Pr \left[ \text{Bin} \left( n - k, \frac{1}{2c} \right) < \frac{n}{d^7} \right] \leq \Pr \left[ \text{Bin} \left( n, \frac{1}{2}, \frac{1}{2c} \right) < \frac{n}{d^7} \right] \leq e^{-n \left( \frac{1}{2c} \right)^2 \frac{n}{d^7}} \leq e^{-\frac{n}{d^7}}, \] (33)

where we have used in the final inequality the fact that \(c \leq d^6\).

Union bounding over all \(\binom{n}{k}\) choices of \(S\), and all choices of \(k\) the probability of having the structure in the lemma is at most
\[
\sum_{k=\frac{n}{d}}^{\frac{2n}{d}} \binom{n}{k} e^{-\frac{ne}{2d}} \leq n \max_{c \in \left[\frac{1}{d} \cdot d\right]} \left( \frac{ne}{2d} \right) \frac{n}{e} e^{-\frac{ne}{2d}} \\
= n \max_{c \in \left[\frac{1}{d} \cdot d\right]} (ecd) \frac{n}{e} e^{-\frac{ne}{2d}} \\
= \max_{c \in \left[\frac{1}{d} \cdot d\right]} e^{n\left(\frac{\log(n)}{2d} + \frac{\log(ed)}{ed} - \frac{1}{2d}\right)}.
\]

Since \( c \) grows at most polynomially in \( d \), for \( d \) a large enough constant, the exponent becomes negative, so the probability that the structure in the lemma exists in \( A \) is \( e^{-\Theta(n)} \).

5 Large Case

**Lemma 6** (Large Case). Let \( A \in \mathbb{R}^{n \times n} \) be the adjacency matrix of a random Erdős Rényi graph \( G(n, d/n) \). There exist constants \( d_0 \) and \( C \) such that for all \( d > d_0 \),

\[
\Pr \left( \exists x \in \mathbb{R}^n, T \subseteq [n] : x^T A[T] = 0, \frac{2n}{d} \leq |\text{supp}(x)| < \frac{n}{C}, |T| \geq \left( 1 - \frac{1}{d} \right) n \right) = e^{-\Theta(n)}.
\]  

(35)

The same result applies if \( A \) has its first column removed.

**Proof.** We union bound over all \( k \), and all \( \binom{n}{k} \) sets \( S \) of \( k \) rows which may be the support of \( x \), and over all sets \( T \) of size \( \left( 1 - \frac{1}{d} \right) n \).

Fix a set of \( k \) rows \( S \) and a subset \( T \) of columns. We will consider only the at most \( n - k - \frac{n}{d} \) mutually independent columns of \( A_S[T] \), that is, the columns of the matrix \( A_S[T \setminus S] \).

Consider the following process, where we draw these independent columns one at a time. Let \( A_i \in \mathbb{R}^k \) be the nullspace of the \( i \) columns drawn, and let \( D_i \) be the dimension of the smallest subspace of \( A_i \) which contains all vectors in \( A_i \) which have no zeros.

By Lemma 2 if \( D_i > 0 \), then we can choose an arbitrary vector \( v \) in \( A_i \) with support \( k \), and with probability at least \( 1 - \frac{1}{\sqrt{kd/n}} \), the \( (i+1) \)th column drawn is not orthogonal to \( v \). In this case \( D_{i+1} = D_i - 1 \).

If at any point \( D_i \) becomes 0, then this means there can be no dependency involving all the rows. It follows that since \( D_0 = k \), using a Chernoff bound yields

\[
\Pr[D_{|T\setminus S|} \neq 0] \leq \Pr \left[ \text{Bin}(n-k-n \cdot 1 - \frac{1}{\sqrt{kd/n}}) < k \right] \\
\leq \Pr \left[ \text{Bin} \left( \frac{n}{2} \cdot 1 - \frac{1}{\sqrt{kd/n}} \right) < k \right] \\
\leq e^{\frac{1}{2} \left( 1 - \frac{1}{\sqrt{kd/n}} \right) \left( 1 - \frac{1}{\sqrt{kd/n}} - \frac{2k}{n} \right) \leq e^{-n\epsilon},}
\]

where \( \epsilon > 0.002 \). To achieve this value of \( \epsilon \), we plugged in \( k < \frac{n}{2} \) in the last inequality.

Finally union bounding, the probability of having the structure in the lemma is at most

\[
\sum_{k=\frac{n}{d}}^{n} \binom{n}{k} \binom{n}{\frac{n}{d}} e^{-c\epsilon n} \leq n (eC)^{\frac{n}{d}} (ed)^{\frac{n}{2}} e^{-\epsilon n} \leq e^{n\left(\frac{\log(n)}{2d} + \frac{\log(ed)}{ed} - \epsilon \right)},
\]

(37)

which for constants \( C \) and \( d \) large enough, is \( e^{-\Theta(n)} \). This concludes the lemma. \( \Box \)
6 3-Core is Full Rank

We will need the following claims.

Claim 5. Let $A$ be a matrix with columns $A_i$ for $i \in [n]$. Let $H_i$ be the space spanned by the column vectors $A_1, A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n$. Let $S$ be the set of all $i$ such that $A_i \in H_i$. Then there exists some $y$ with $\text{supp}(y) = S$ such that $Ay = 0$.

Proof. We prove this by linearly combining the dependencies. Without loss of generality, let $S = [k]$ for some $k$. For all $i \in [k]$, since $A_i \in H_i$, there exists some $x^{(i)}$ such that $x^{(i)}_i \neq 0$ and $Ax^{(i)} = 0$. Observe that for each $i \in [k]$, we have $\text{supp}(x^{(i)}) \subset S$ - otherwise it would imply that some column $A_j$ for $j \notin S$ is spanned by $H_j$. Choose random coefficients $c_i$ for $i \in [k]$ from any continuous distribution and let $y = \sum_i c_i x^{(i)}$. Then with probability 1, $y$ is nonzero on all $i \in S$.

Let $e_i$ denote the $i$th standard basis vector. In the rest of this section, for a matrix $B$, we use $\text{Span}(B)$ to denote the column span of $B$.

Claim 6. With the terminology of the previous claim, if and only if $e_i \in \text{Span}(A^T)$, then $A_i \notin H_i$.

Proof. For the forward direction, let $v$ be such that $A^Tv = e_i$. The for any $w$ with $i \in \text{supp}(w)$, we have $w^TA^Tv = w_i \neq 0$, so it is impossible that $Aw = 0$.

For the converse, suppose $A_i \notin H_i$. Then there must exist some $w_i$ such that $\langle w_i, A_i \rangle \neq 0$, but $w_i \perp A_j$ for all $j \neq i$. However, this implies that $Aw_i = \langle w_i, A_i \rangle e_i$, so $e_i \in \text{Span}(A^T)$, which is a contradiction.

We restate our main theorem.

Theorem 2. Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a random Erdős Rényi graph $G(n, d/n)$. If $d = \omega(1)$ and $d \leq 3 \log(n)$, the with probability $1 - o(1)$, the 3-core of $A$ is full rank.

We will use the following notation: For an adjacency matrix $M$, let $C_k(M)$ be the set of vertices in the $k$-core of the graph with adjacency matrix $M$. Further, let $M(3)$ denote adjacency matrix of the 3-core of $M$. For a vector $v$ and a set $S$, let $v[S]$ denote the restriction of $v$ to the indices in $S$. For a matrix $M$ and set $S$, in the proof of Lemma 8, we abuse notation and let $M[S]$ denote the restriction of $M$ to the columns and rows in $S$. Thus $M(3) = M[C_3(M)]$.

The following lemma, which follows from [5], states that the 3-core of $A$ is large.

Lemma 7 (c.f. Theorem 2 in [5]). Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a random Erdős Rényi graph $G(n, d/n)$ with $d = \omega(1)$.

For any constant $k \geq 3$, with probability at least $1 - O(1/n)$,

$$|C_k(A)| \geq n \left(1 - e^{-d^{3/4}}\right).$$

Our main tool to prove Theorem 2 is the following lemma. It will allow us to rule out having large dependencies in $A(3)$.

Lemma 8. Let $A$ be as in the proposition, and and let $A^{(i)}$ denote the submatrix of $A$ given by the $i$th row and column removed. Let $A_i$ denote the $i$th column of $A$, and let $A_i'$ be this vector with the $i$th entry removed.

For any $u, r < s, t \in [n]$,

$$A_i\left[u^r, s^t\right] = 0.$$
\[
\Pr[\exists x : A(3)x = 0, \text{supp}(x) > t] \leq \frac{n}{t - \frac{n}{d}} \max_{x \in \mathbb{R}^n : \text{supp}(x) \geq s} \Pr[x^T A_n = 0]
+ \frac{n}{t - \frac{n}{d}} \Pr[\exists x : A^{(i)}(3)x = 0, r < |\text{supp}(x)| < s]
+ \frac{dr}{t - \frac{n}{d}}
+ \frac{n}{t - \frac{n}{d}} \Pr[\exists x \neq 0 : A^{(n)}(3)x = e_1, |\text{supp}(x)| < s]
+ \frac{n}{t - \frac{n}{d}} \max_{X \in \mathcal{T}_{n-n/d}^{n}} \Pr[A_n X A_n = 0]
+ o\left(\frac{n}{t - \frac{n}{d}}\right),
\]

where \(\mathcal{T}_{p,q}^{m}\) denotes the set of matrices in \(\mathbb{R}^{m \times m}\) with some set of \(p\) columns that each have at least \(q\) non-zero entries.

**Proof.** For \(i \in [n]\), let \(H_i\) be the space spanned by the column vectors

\[
A_1[C_3(A^{(i)}) \cup i], A_2[C_3(A^{(i)}) \cup i], \ldots, A_{i-1}[C_3(A^{(i)}) \cup i], A_{i+1}[C_3(A^{(i)}) \cup i], \ldots, A_n[C_3(A^{(i)}) \cup i].
\]

Let

\[
T := \{i : C_3(A) = C_3(A^{(i)}) \cup i\}.
\]

Then \(A(3)x = 0\) for some \(x\) implies that for all

\[
i \in \text{supp}(x) \cap T,
\]

we have

\[
A_i[C_3(A^{(i)}) \cup i] = A_i[C_3(A)] \in H_i,
\]

since

\[
x_i A_i[C_3(A)] = -\sum_{j} x_j A_j[C_3(A)].
\]

We first claim that the set \(T\) is large, such that if \(\text{supp}(x)\) is large, we must have many \(i\) for which \(A_i[C_3(A^{(i)}) \cup i] \in H_i\). Let \(c_i := |C_3(A^{(i)})|\).

**Claim 7.** With probability \(1 - o(1),\)

\[
|T| \geq n \left(1 - \frac{1}{d}\right).
\]

**Proof.** Fix \(i\), and consider the probability that \(C_3(A) = C_3(A^{(i)}) \cup i\).
Notice that $C_3(A^{(i)})$ is independent of $A_i$, and hence
\[
\Pr[C_3(A) = C_3(A^{(i)}) \cup i] \geq \Pr[\supp(A_i) \geq 3 \land A_{ij} = 0 \forall j \notin C_3(A^{(i)})]
\]
\[
= \mathbb{E}_{c_i} \left[ \left( 1 - \frac{d}{n} \right)^{n-1-c_i} \Pr \left[ \text{Bin} \left( c_i, \frac{d}{n} \right) \geq 3 \right] \right]
\]
\[
\geq \mathbb{E}_{c_i} \left[ e^{-d(n-1-c_i)/n} \left( 1 - \Pr \left[ \text{Bin} \left( c_i, \frac{d}{n} \right) = 1 \right] - \Pr \left[ \text{Bin} \left( c_i, \frac{d}{n} \right) = 2 \right] \right) \right]
\]
\[
= \mathbb{E}_{c_i} \left[ e^{-d(n-1-c_i)/n} \left( 1 - \frac{c_id}{n} \left( 1 - \frac{d}{n} \right)^{c_i-1} - \frac{c_i^2d^2}{n^2} \left( 1 - \frac{d}{n} \right)^{c_i-2} \right) \right]
\]
\[
\geq \mathbb{E}_{c_i} \left[ e^{-d(n-1-c_i)/n} \left( 1 - de^{-d(c_i-1)/n} - d^2e^{-d(c_i-2)/n} \right) \right]
\]
\[
\geq \Pr \left[ c_i \geq n \left( 1 - \frac{1}{2d^3} \right) \right] e^{-\frac{1}{d^2}} (1 - e^{-d/2})
\]
\[
\leq \Pr \left[ c_i \geq n \left( 1 - \frac{1}{2d^3} \right) \right] \left( 1 - \frac{1}{d^{3/2}} \right).
\]

where the last two inequalities hold for $d$ large enough. By Lemma 7 with probability $1 - O(1/n)$, we have $c_i \geq n \left( 1 - e^{-d^{1/4}} \right)$, so we have
\[
\Pr[C_3(A) = C_3(A^{(i)}) \cup i] \geq 1 - \frac{1}{d^{3/2}} - O \left( \frac{1}{n} \right).
\]

Now by Markov’s inequality,
\[
\Pr \left[ |\{i\} \setminus T| \geq \frac{n}{d} \right] \leq \frac{d}{n} \sum_i \left( 1 - \Pr[C_3(A) = C_3(A^{(i)}) \cup i] \right)
\]
\[
\leq \frac{d}{n} \left( 1 - d^{-3/2} - O \left( \frac{1}{n} \right) \right) = o(1).
\]

Let $E_i$ denote the event that $A_i[C_3(A^{(i)}) \cup i] \in H_i$. Let $X_i$ be the indicator of this event, and let $X = \sum_i X_i$. Then by Markov’s inequality, since if $|T| \geq n \left( 1 - \frac{1}{3} \right)$, at most $n/d$ vertices can be in the 3-core but not in $T$, we have
\[
\Pr \left[ \exists x : A(3)x = 0, |\supp(x)| \geq t \right] \leq \Pr \left[ |T| < n \left( 1 - \frac{1}{d} \right) \right] + \Pr \left[ X \geq t - n/d \right]
\]
\[
\leq o(1) + \frac{\mathbb{E}[X]}{t - n/d}
\]
\[
\leq o(1) + \frac{\sum_i \Pr[E_i]}{t - n/d}
\]
\[
= o(1) + \frac{\mathbb{E}[X]}{t - n/d} \Pr[E_n].
\]

We will break down the probability $\Pr[E_n]$ into several cases, depending on the size of the support of vectors in the kernel of $A^{(n)}(3)$. Let $S \subset C_3(A^{(n)})$ be the set of all $j$ such that $e_j \in \text{Span}(A[C_3(A^{(n)})])$. By Claim 5 and Claim 6
\[
k := \max(\supp(x) : A^{(n)}(3)x = 0) = c_n - |S|.
\]

\textbf{Case 1:} $A^{(n)}(3)$ has a vector $x$ with large support, that is, $k \geq s$. 

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**Case 2:** $A^{(n)}(3)$ has a vector $x$ with medium support, that is, $r < k < s$.

**Case 3:** $A^{(n)}(3)$ does not have any vectors with large or medium support vectors in its kernel, that is, $k \leq r$.

We can expand

$$
\Pr[\mathcal{E}_n] = \Pr[\mathcal{E}_n|k \geq s] \Pr[k \geq s] + \Pr[\mathcal{E}_n|r < k < s] \Pr[r < k < s] + \Pr[\mathcal{E}_n|k \leq r] \Pr[k \leq r].
$$

(45)

Define $a := A_n[C_3(A^{(n)})]$ to be the restriction of $A_n$ to the indices in the 3-core of $A^{(n)}$.

To evaluate the probability of the first case, we condition on $A^{(n)}$, and let $x$ be any vector of support at least $s$ in the kernel of $A^{(n)}(3)$. Observe that $\mathcal{E}_n$ cannot hold if $x^T a$ is non-zero. Since $a$ is independent from $x$, we have

$$
\Pr[\mathcal{E}_n|k \geq s] \Pr[k \geq s] \leq \max_{x: \text{supp}(x) \geq s} \Pr[x^T a = 0] \leq \max_{x \in \mathbb{R}^n: \text{supp}(x) \geq s} \Pr[x^T A_n = 0].
$$

(46)

Here the second inequality comes from considering $x$ over a larger domain. Combined with Equation 43, the contribution from this case yields the first term in the right hand side of Equation 38.

For the second case, we bound

$$
\Pr[\mathcal{E}_n|r < k < s] \Pr[r < k < s] \leq \Pr[r < k < s] \leq \Pr[\exists x: A^{(n)}(3)x = 0, r < |\text{supp}(x)| < s].
$$

(47)

Combined with Equation 43, the contribution from this case yields the second term in the right hand side of Equation 38.

In this third case, we will show conditions under which we can algebraically construct a vector $v$ such that $A[C_3(A^{(n)}) \cup n]v = e_n$. This will imply by Claim 6 that under those conditions, $A_n[C_3(A^{(n)}) \cup n] \notin H_n$.

To define these conditions, recall that $S \subset C_3(A^{(n)})$ is the set of all $j$ such that $e_j \in \text{Span}(A[C_3(A^{(n))})]$. For $j \in S$, let $w_j$ be any vector such that $A^{(n)}(3)w_j = e_j$. We next construct a sort of “pseudoinverse” matrix $B \in \mathbb{R}^{C_3(A^{(n)}) \times C_3(A^{(n)})}$ as follows: For $j \in S$, define $B_{jk}$ to be the $j$th entry of $w_k$. Define all other entries of $B$ to be zero.

The following claim shows a condition for $\mathcal{E}_n$ not holding.

**Claim 8.** If $\text{supp}(a) \subset S$ and $a^T B a \neq 0$, then $e_n \in \text{Span}(A[C_3(A^{(n)}) \cup n])$.

**Proof.** Let

$$w' := B a = \sum_{j \in S} a_j w_j$$

such that

$$A^{(n)}(3)w' = \sum_{j \in S} a_j e_j.$$

Hence if $\text{supp}(a) \subset S$,

$$A^{(n)}(3)w' = a.$$

In this case, define $w \in \mathbb{R}^{C_3(A^{(n)}) \cup n}$ to be the vector equal to $w'$ on the coordinates in $C_3(A^{(n)})$ and $-1$ on coordinate $n$. Then

$$A[C_3(A^{(n)}) \cup n]w)_j = 0 \quad \forall j \in C_3(A^{(n)})$$

(48)

and

$$A[C_3(A^{(n)}) \cup n]w)_n = a^T B a.$$

(49)
Evidently, if $a^T Ba \neq 0$, then
\[
A[C_3(A^{(n)}) \cup n]w = e_n,
\]
so $e_n \in \text{Span}(A[C_3(A^{(n)}) \cup n])$.

By definition, in the third case, we have $|S| \geq n - 1 - r$.
Hence
\[
\Pr[\mathcal{E}_n \land k \leq r] \leq \Pr[\text{supp}(a) \not\subseteq S \land |S| \geq c_n - r]
+ \Pr[a^T Ba = 0 \land |S| \geq c_n - r].
\]
(50)

Notice that $S$ is a function of $A^{(n)}$ and so $a$ is independent from $S$. It is easy to check that for any set $S$,
\[
\Pr[\text{supp}(a) \not\subseteq S] \leq 1 - \left(1 - \frac{d}{n}\right)^{n-|S|} \leq \frac{d(n-|S|)}{n}.
\]
As by Lemma (7), with probability $1 - o(1)$, we have $c_n \geq n \left(1 - \frac{1}{d^2}\right)$, we have
\[
\Pr[\text{supp}(a) \not\subseteq S \land |S| \geq c_n - r] \leq o(1) + \frac{d(n - n(1 - \frac{1}{d^2}) + r)}{n} = o(1) + \frac{1}{d} + \frac{dr}{n} = o(1) + \frac{dr}{n}.
\]
(51)

Combined with Equation 43, the contribution from this equation yields the third term in the right hand side of Equation 38.

We will break up the second term in Equation 50 by conditioning on whether the support of $B$ has many entries or not, and then by using the independence of $a$ from $B$:
\[
\Pr[a^T Ba = 0 \land |S| \geq c_n - r] \leq \Pr[B \not\subseteq \mathcal{T}_{c_n - r - u,s} \land |S| \geq c_n - r] + \max_{X \in \mathcal{T}_{c_n - r - u,s}} \Pr[a^T X a = 0].
\]
(52)

For the second probability on the right hand side, notice that for any positive integers $a \leq b$, if $x$ and $y$ are random vectors from some product distributions $P^{\otimes a}$ and $P^{\otimes b}$ respectively, then
\[
\max_{X \in \mathcal{T}_{a - r - u,s}} \Pr[x^T X x = 0] \leq \max_{X \in \mathcal{T}_{b - r - u,s}} \Pr[y^T X y = 0].
\]
(53)

Hence since $c_n \geq n \left(1 - \frac{1}{d^2}\right)$ with probability $1 - o(1)$, we have
\[
\max_{X \in \mathcal{T}_{c_n - r - u,s}} \Pr[a^T X a = 0] \leq o(1) + \max_{X \in \mathcal{T}_{n - u/d^2 - r - u,s}} \Pr[A^T_n X A_n = 0].
\]
(54)

To bound the first probability on the right hand side of Equation 52, observe that if $|S| \geq c_n - r$ and $B \not\subseteq \mathcal{T}_{c_n - r - u,s}$, there must exist at least $u$ different $j \in S$ such that $\text{supp}(w_j) \leq s$.
So
\[
\Pr[B \not\subseteq \mathcal{T}_{c_n - r - u,s} \land |S| \geq c_n - r] \leq \Pr[\{j : \exists x \neq 0 : A^{(n)}(3)x = e_j, |\text{supp}(x)| < s\} \geq u]
\leq \frac{n}{u} \Pr[\exists x \neq 0 : A^{(n)}(3)x = e_1, |\text{supp}(x)| < s],
\]
(55)

where the last inequality follows by Markov’s inequality.

Plugging this and Equation 54 into Equation 52 yields
\[
\Pr[a^T Ba = 0 \land |S| \geq c_n - r] \leq \frac{n}{u} \Pr[\exists x \neq 0 : A^{(n)}(3)x = e_1, |\text{supp}(x)| < s]
+ \max_{X \in \mathcal{T}_{n - u/d^2 - r - u,s}} \Pr[A^T_n X A_n = 0] + o(1).
\]
(56)
Combining this with \[51\] and \[50\] yields

\[
\Pr[\mathcal{E}_n \land k \leq r] \leq o(1) + \frac{dr}{n} + \max_{x \in \mathbb{R}^{n-1}} \Pr \left[ A_n^T X A_n = 0 \right] + \frac{n}{u} \Pr \left[ \exists x \neq 0 : A(n)^{(3)} x = e_1, |\text{supp}(x)| < s \right].
\]

Plugging this and Equations \[46\] and \[47\] into Equation \[45\] and finally Equation \[43\] yields the lemma. \(\square\)

We are now ready to prove our main theorem.

**Proof of Theorem 2.** The proof follows by instantiating Lemma 8 with the following values: \(t = \frac{n}{C}\), \(s = \frac{n}{C}\), \(r = \frac{n}{d \log(d)}\), \(u = \frac{n}{2}\). Here \(C\) is the constant from Lemma 6.

With the following values, it is immediate from the sparse Littlewood-Offord (Lemma 2) and quadratic Littlewood-Offord theorems (Lemma 3) that the first and last terms in Lemma 8 respectively are \(o(1)\). The third term is at most \(O(1/\log(d))\) which is also \(o(1)\). The two lemmas stated immediately after this proof (Lemmas 9 and 10) will show that the second and fourth terms are \(o(1)\).

Assuming these two lemmas, it follows that with probability \(1 - o(1)\), there are no dependencies of at least \(n/C\) rows \(A(3)\). To rule out any smaller dependencies in \(A(3)\), we use Lemmas 4, 5, and 6.

If there were a \(k\)-minimal dependency in \(A(3)\), then it would mean that the restriction of these \(k\) rows in \(A\) to the set \(T = C_3(A)\) would:

1. Contain \(\geq 3k\) non-zero entries, because \(A(3)\) is a 3-core; and
2. Be a \(k\)-minimal dependency.

For \(k \leq \frac{n}{d}\), by Lemma 4, no such structure exists in \(A\) with probability \(1 - o(1)\).

If \(\frac{n}{d} < k < \frac{n}{C}\), then applying Lemmas 5 and 6 with \(T = C_3(A)\) means that, if \(|C_3(A)| \geq n(1-d^7)\), with probability at least \(1 - e^{-\Theta(n)}\), there is no dependency of these sizes in \(A(3)\). By Lemma 7, \(|C_3(A)| \geq n(1-d^7)\) with probability \(1 - o(1)\).

This rules out all dependencies in \(A(3)\) with probably \(1 - o(1)\), proving the theorem. \(\square\)

It remains to prove lemmas 9 and 10.

**Lemma 9.** Let \(A \in \mathbb{R}^{n \times n}\) be the adjacency matrix of a random Erdős Rényi graph \(G(n, d/n)\). Let \(d = \omega(1)\) and let \(C\) be the constant in Lemma 6.

\[
\Pr \left[ \exists x : A(3) x = 0, \frac{n}{d \log(d)} < |\text{supp}(x)| < \frac{n}{C} \right] = o(1).
\]

**Proof.** This is immediate from the medium and large case Lemmas 5 and 6 since the 3-core has size at least \(n \left(1 - \frac{1}{d^7}\right)\) with probability \(1 - o(1)\). Hence if there was a dependency of \(k\) rows \(S\) in the 3-core, it would mean there would have to be a set \(T = C_3(A)\) of size at least \(n \left(1 - \frac{1}{d^7}\right)\) such that for \(\frac{n}{d \log(d)} < |S| < \frac{n}{C}\), the restriction of the \(A_S[T]\) forms a minimal dependency. \(\square\)

**Lemma 10.** Let \(A \in \mathbb{R}^{n \times n}\) be the adjacency matrix of a random Erdős Rényi graph \(G(n, d/n)\). Let \(d = \omega(1)\) and let \(C\) be the constant in Lemma 6. Then

\[
\Pr \left[ \exists x : A(3) x = e_1, |\text{supp}(x)| < \frac{n}{C} \right] = o(1)
\]

We reduce can reduce Lemma 10 to Lemma 11 which we should be better suited to prove with our medium/small-case techniques.

**Lemma 11.** Let \(A \in \mathbb{R}^{n \times n}\) be the adjacency matrix of a random Erdős Rényi graph \(G(n, d/n)\). Let \(d = \omega(1)\) and let \(C\) be the constant in Lemma 6.

Let \(A'\) be \(A(3)\) with the first column removed. Then

\[
\Pr[\exists x : x^T A' = 0, |\text{supp}(x)| \leq \frac{n}{C}] = o(1).
\]
Proof of Lemma 10 assuming Lemma 11. If such a vector $x$ satisfied $A(3)x = e_1$, then necessarily $A'x' = 0$, where $x'$ is $x$ with the first (zero) coordinate removed, and $A'$ is $A(3)$ with the first row removed. This occurs with probability at most $o(1)$ by Lemma 11.

Proof of Lemma 11. By Lemmas 5 and 6 with probability $1-o(1)$, there are no vectors $x$ such that $x^T A' = 0$ where $\frac{n}{d7} \leq |\text{supp}(x)| < n/C$.

Now, suppose there was some set $S$ of size $k < \frac{n}{d7}$ such that $A'_S$ is a $k$-minimal dependency. This would mean that in $A$, there exists a subset of rows $U$ such there are at least $3k$ non-zero entries in $A_S[U]$ and at most 1 column with a single non-zero entry. Indeed, letting $U = C_A(A)$, then there must be at least $3k$ non-zero entries among $A_U[S]$. Further, since $A'_S$ is a $k$-minimal dependency, then there can be no columns in $A_S[U]$ with a single 1 except for possibly the first column, which does not appear in $A'$.

If $1 \in U$ and $A_S[\{1\}]$ has exactly one non-zero entry, then let $T := U \setminus 1$. Otherwise, let $T := U$. It follows that $A_S[T]$:

- has at least $3k - 1$ nonzero entries and
- has no columns with a single non-zero entry.

By the proof of Lemma 4 with probability $1-o(1)$, no such structure exists in $A$.

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