Bounds on the 2-rainbow domination number of graphs

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Abstract

A 2-rainbow domination function of a graph $G$ is a function $f$ that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$, such that for any $v \in V(G)$, $f(v) = \emptyset$ implies $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$. The 2-rainbow domination number $\gamma_r^2(G)$ of a graph $G$ is the minimum $w(f) = \Sigma_{v \in V} |f(v)|$ over all such functions $f$. Let $G$ be a connected graph of order $|V(G)| = n \geq 3$. We prove that $\gamma_r^2(G) \leq 3n/4$ and we characterize the graphs achieving equality. We also prove a lower bound for 2-rainbow domination number of a tree using its domination number. Some other lower and upper bounds of $\gamma_r^2(G)$ in terms of diameter are also given.

Keywords: domination number, 2-rainbow domination number, Cartesian product

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1 Introduction

We follow the notation of [1] in this paper. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. $P_k$ and $C_k$ denote a path and a cycle of order $k$, respectively. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. The diameter of $G$ is the maximum distance between vertices of $G$, denoted by $\text{diam}(G)$. A penultimate vertex is any neighbor of a vertex with degree one (the vertex of degree one is also called a leaf in a tree), and a pendent edge is an edge incident with a vertex of degree one. A star is a tree isomorphic to a bipartite graph $K_{1,k}$ for $k \geq 1$. A double-star $DS_{r,s}$ is a tree with diameter 3 in which there are exactly two penultimate vertices with degrees $r + 1$ and $s + 1$, respectively. A set $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A thorough study of domination concepts appears in [8]. For a pair of graphs $G$ and $H$, the Cartesian product $G \square H$ of $G$ and $H$ is the graph with vertex set

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Let $f$ be a function that assigns to each vertex a set of colors chosen from the set $\{1, \cdots, k\}$; that is, $f : V(G) \rightarrow \mathcal{P}\{1, \cdots, k\}$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u \in N(v)} f(u) = \{1, \cdots, k\}$. Then $f$ is called a $k$-rainbow dominating function ($k$RDF) of $G$. The weight, $w(f)$, of a function $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a $k$-rainbow dominating function is called the $k$-rainbow domination number of $G$, which we denote by $\gamma_{rk}(G)$. We say that a function $f$ is a $\gamma_{rk}(G)$-function if it is a $k$RDF and $w(f) = \gamma_{rk}(G)$. The concept of rainbow domination was introduced in [4], and used in obtaining some bounds on the paired-domination number of Cartesian products of graphs, see also [3]. A more ambitious motivation for the introduction of this invariant was inspired by the following famous open problem [10]:

**Vizing’s Conjecture.** For any graphs $G$ and $H$, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

In the language of domination of Cartesian products, Hartnell and Rall [7] obtained a couple of observations about rainbow domination, for instance, $\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G)$. Rainbow domination of a graph $G$ coincides with the ordinary domination of the Cartesian product of $G$ with the complete graph, in particular $\gamma_2(G) = \gamma(G \Box K_2)$ for any graph $G$ [4]. Notably a lower bound for the 2-rainbow domination number of a graph expressed in terms of its ordinary domination could bring a new approach to the much desired proof of Vizing’s conjecture. In particular, Brešar, Henning and Rall [4] proposed the following problem:

**Problem 1.** (Brešar, Henning and Rall [4]). For any graphs $G$ and $H$, $\gamma_2(G \Box H) \geq \gamma(G)\gamma(H)$.

A Roman domination function of a graph $G$ is a function $g : V \rightarrow \{0, 1, 2\}$ such that every vertex with 0 has a neighbor with 2. The Roman domination number $\gamma_R(G)$ is the minimum of $g(V(G)) = \sum_{v \in V(G)} g(v)$ over all such functions. In [12], the authors showed the following result:

**Theorem 1.** (Wu and Xing [12]) Let $G$ be a graph. Then $\gamma(G) \leq \gamma_2(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

In [11], Wu showed the following weaker form of Problem [4] by Theorem [11]:

**Theorem 2.** (Wu [11]) For any graphs $G$ and $H$, $\gamma_R(G \Box H) \geq \gamma(G)\gamma(H)$.

In fact, Both Problem [4] and Theorem [2] are improvements of the result given by Clark and Suen [2]:

**Theorem 3.** (Clark and Suen [2]) For any graphs $G$ and $H$, $2\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

Nevertheless the concept of rainbow domination seems to be of independent interest as well and it attracted several authors who provided structural and algorithmic results on this invariant [5] [6] [9] [13]. In particular, it was shown that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is NP-complete even when restricted to bipartite graphs or chordal graphs [5]. Also a few exact values and bounds for the 2-rainbow domination number were given for some special classes of graphs, including generalized Petersen graphs [5] [13].
For a graph $G$, let $f: V(G) \rightarrow \mathcal{P}\{1, 2\}$ be a 2RDF of $G$ and $(V_0, V_1^1, V_1^2, V_2)$ be the ordered partition of $V(G)$ induced by $f$, where $V_0 = \{v \in V(G) \mid f(v) = \emptyset\}$, $V_1^1 = \{v \in V(G) \mid f(v) = \{1\}\}$, $V_1^2 = \{v \in V(G) \mid f(v) = \{2\}\}$ and $V_2 = \{v \in V(G) \mid f(v) = \{1, 2\}\}$. Note that there exists a 1-1 correspondence between the functions $f: V(G) \rightarrow \mathcal{P}\{1, 2\}$ and the ordered partitions $(V_0, V_1^1, V_1^2, V_2)$ of $V(G)$. Thus we will write $f = (V_0, V_1^1, V_1^2, V_2)$ for simplicity.

In this paper we present some general bounds on the 2-rainbow domination number of a graph that are expressed in terms of the order and domination number of a graph. More specifically, we show that $\gamma_{r2}(G) \leq 3|V(G)|/4$ and we characterize the graphs achieving equality. We also prove a lower bound for the 2-rainbow domination number of a tree using its domination number. The latter lower bound goes in the direction of the original goal, mentioned above, to obtain a new approach for establishing Vizing’s conjecture. Some other lower and upper bounds of $\gamma_{r2}(G)$ in terms of diameter are also given.

## 2 Main results

Our aim in this section is to determine some bounds on the 2-rainbow domination number of graphs.

### 2.1 Upper bounds

We first recall a few definitions. A subdivision of an edge $uv$ is obtained by removing edge $uv$, adding a new vertex $w$, and adding edges $uw$ and $vw$. Let $t \geq 2$. A spider (wounded spider) is the graph formed by subdividing some edges (at most $t - 1$ edges) of a star $K_{1,t}$. The unique center of $K_{1,t}$ is also called the center of the spider. Only one vertex of the spider $P_4$ can be called the center.

**Proposition 1.** Let $G$ be a spider of order $|V(G)| = n \geq 3$, then $\gamma_{r2}(G) \leq 3n/4$. Moreover, the equality only holds for a path of order four.

**Proof.** Let $u$ be the center of $G$. Suppose $u$ has $x$ penultimate neighbors and $y$ non-penultimate neighbors. Then $n = 2x + y + 1$.

If $x \geq 3$ or $y \geq 2$, we set

$$f(v) = \begin{cases} 
{1, 2} & v = u, \\
{1} \text{ or } {2} & v \text{ is at distance two to } u, \\
\emptyset & \text{otherwise.}
\end{cases}$$

If $x = 2$ and $y \leq 1$, we set

$$f(v) = \begin{cases} 
{1} & v = u, \\
{2} & v \text{ is a leaf,} \\
\emptyset & \text{otherwise.}
\end{cases}$$

In both cases, $\gamma_{r2}(G) \leq w(f) < 3n/4$.

If $x = y = 1$, then $G$ is a path of order four. Clearly, $\gamma_{r2}(G) = 3 = 3n/4$. ■
Theorem 4. Let $T$ be a tree of order $n \geq 3$, then $\gamma_{r2}(T) \leq 3n/4$.

Proof. We use induction on $n$. The base step handles trees with few vertices or small diameter. If $\text{diam}(T) = 2$, then $T$ has a dominating vertex, and $\gamma_{r2}(T) \leq 2$. This beats $n \geq 3$. If $\text{diam}(T) = 3$, then $T$ has a dominating set of size two, which yields $\gamma_{r2}(T) \leq 4$. This handles the desired bound for such trees with at least six vertices. When $n = 4$ or $n = 5$, then $T$ is a spider and the theorem holds by Proposition 1. Moreover, if $T$ is a path of order four, then it achieves this bound.

Hence we may assume that $\text{diam}(T) \geq 4$. Given a subtree $T'$ with $n'$ vertices, where $n' \geq 3$, the induction hypothesis yields a 2RDF $f'$ of $T'$ with weight at most $3n'/4$. We find such $T'$ and add a bit more weight to obtain a 2RDF $f$ of $T$. Let $P$ be a longest path in $T$ chosen to maximize the degree of the penultimate vertex $v$ on it, and let $u$ be the non-leaf neighbor of $v$.

Case 1. $d_T(v) > 2$.

We obtain $T'$ by deleting $v$ and its leaf neighbors. Define $f$ on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = \{1, 2\}$ and $f(x) = \emptyset$ for each leaf $x$ adjacent to $v$. Since color set $\{1, 2\}$ on $v$ takes care of its neighbors, $f$ is a 2RDF for $T$. Since $\text{diam}(T) \geq 4$, we have $n' \geq 3$, and $w(f) = w(f') + 2 \leq 3n'/4 + 2 = 3(n - 3)/4 + 2 < 3n/4$.

Case 2. $d_T(v) = d_T(u) = 2$.

We obtain $T'$ by deleting $u$ and $v$ and the leaf neighbor $l$ of $v$. If $n' = 2$, then $T$ is a path of order five and has a 2RDF of weight $3 < 3n/4$. Otherwise, the induction hypothesis applies. Define $f$ on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = \{1, 2\}$ and $f(u) = f(l) = \emptyset$. Again $f$ is a 2RDF, and the computation $w(f) < 3n/4$ is the same as in Case 1.

Case 3. $d_T(v) = 2$ and $d_T(u) > 2$.

By the choice of path $P$, every penultimate neighbor of $u$ has degree 2.

Subcase 3.1. Every neighbor of $u$ is penultimate or a leaf.

Then $\text{diam}(T) = 4$ and $T$ is a spider. By Proposition 1, $\gamma_{r2}(T) < 3n/4$, since $T$ is not a path of order four.

Subcase 3.2. There exists a neighbor $t$ of $u$ which is neither penultimate nor a leaf.

Then $T - tu$ contains two components $T'$ and $T''$ such that $T''$ is a spider containing $u$. Now $|V(T')| = n' \geq 3$ and the induction hypothesis applies that $\gamma_{r2}(T') \leq 3|V(T')|/4 = 3n'/4$. By Proposition $\gamma_{r2}(T'') \leq 3|V(T'')|/4$. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + \gamma_{r2}(T'') \leq 3n/4$. $\blacksquare$

Let $L_k$ consist of the disjoint union of $k$ copies of $P_4$ plus a path through the center vertices of these copies, as illustrated in Figure 1. Let $G$ be a graph having an induced subgraph $P_4$ such that only the center of $P_4$ can be adjacent to the vertices in $G - P_4$, then every 2RDF of $G$ must have weight at least 3 on $P_4$. In $L_k$, there are $k$ disjoint $P_4$ of this form, so $\gamma_{r2}(L_k) \geq 3k = 3n/4$. Indeed, we can assemble such copies of $P_4$ in many ways, and this allows us to characterize the trees achieving equality in Theorem 4.

Theorem 5. Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{r2}(T) = 3n/4$ if and only if $V(T)$ can be partitioned into sets inducing $P_4$ such that the subgraph induced by the center vertices of these $P_4$ is connected.
Proof. We have observed that if an induced subgraph $H$ of $G$ is isomorphic to $P_4$, and its noncenter vertices have no neighbors outside $H$ in $G$, then every 2RDF of $G$ must have weight at least 3 on $V(H)$. Thus in any tree with the structure described, weight at least 3 is needed on every $P_4$ in the specified partition. To show that equality requires this structure, we examine the cases more closely in the proof of Theorem 4. The proof is by induction on $n$. In the base cases and Cases 1 and 2, we produce a 2RDF with weight less than $3n/4$ except for $P_4$. Define $u$, $T'$, $T''$, $n'$, $t$ as in the inductive part of Case 3. The equality holds only if $n' = n - 4$ and $T''$ is a $P_4$ path. Equality also requires $\gamma_{r,2}(T') = 3n'/4$, so by the induction hypothesis $T'$ has the specified form.

Next we show no copy of $P_4$ in $T$ such that both the two penultimate vertices on $P_4$ with degree at least three in $T$. Suppose there is a spanning subgraph $H'$ isomorphic to the graph shown in Figure 2, then we give a 2RDF $f$ for $H'$ as follows:

$$f(v) = \begin{cases} \{1, 2\} & v = x \text{ or } y, \\ \{1\} & v \notin N[x] \cup N[y], \\ \emptyset & \text{otherwise.} \end{cases}$$

By Theorem 4 $\gamma_{r,2}(T) \leq \gamma_{r,2}(H') + \gamma_{r,2}(T - H') \leq 8 + 3(n - 12)/4 < 3n/4$, a contradiction. ■

Recall that the corona $HoK_1$ of a graph $H$ is obtained by attaching one pendent edge at each vertex of $H$. Since the rainbow domination number does not increase when edges are added to a graph, we infer from Theorem 4 and 5 the following general upper bound.
Corollary 1. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{r2}(G) \leq 3n/4$. Moreover, the equality holds if and only if $G$ is $P_4$ or $C_{4 \circ K_1}$ or $V(G)$ can be partitioned into $k$ copies of $P_4$ ($k \geq 3$) and all the copies of $P_4$ can only be connected by their centers.

Proof. If $G$ has the specified form, then for each copy of $P_4$ in the partition of $V(G)$, every 2RDF of $G$ puts weight at least 3 on it.

Suppose $\gamma_{r2}(G) = 3n/4$ and $G$ is not a tree. Since adding edges can not increase the 2-rainbow domination number, every spanning tree of $G$ has the form specified in Theorem 5. If $n = 4$, then $G$ is $P_4$. If $n = 8$, then it is easy to check that the only extremal graph is $C_{4 \circ K_1}$. If $n \geq 12$, let $T$ be a spanning tree of $G$ has the form specified in Theorem 5.

The following result for the 2-rainbow domination number of paths is given by Brešar and Kraner Šumajak.

Proposition 2. ($[5]$) $\gamma_{r2}(P_n) = \lceil \frac{n}{2} \rceil + 1$.

We conclude this subsection with an upper bound in terms of diameter.

Theorem 6. For any connected graph $G$ on $n$ vertices,

$$\gamma_{r2}(G) \leq n - \left\lfloor \frac{\text{diam}(G) - 1}{2} \right\rfloor.$$ 

Furthermore, this bound is sharp.

Proof. Let $P = v_1v_2 \cdots v_{\text{diam}(G)+1}$ be a diametral path in $G$ and $f$ be a $\gamma_{r2}$-function of $P$. By Proposition 2, the weight of $f$ is $\frac{\text{diam}(G)+1}{2} + 1$. Define $g : V(G) \to \mathcal{P}\{1, 2\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = \{1\}$ for $x \in V(G) - V(P)$. Obviously $g$ is a 2RDF for $G$. Hence,

$$\gamma_{r2}(G) \leq w(f) + (n - \text{diam}(G) - 1) = n - \left\lfloor \frac{\text{diam}(G) - 1}{2} \right\rfloor.$$ 

The family of all paths of even order achieves the bound, and the proof is complete.

2.2 Lower bounds

We present a lower bound on the 2-rainbow domination number of a tree expressed in terms of its domination number, maximum degree, and the number of its leaves and penultimate vertices. Given a tree, $T$ we denote by $\ell(T)$ the number of leaves in $T$, and by $p(T)$ the number of penultimate vertices in $T$. 


Theorem 7. For any tree $T$ on at least three vertices, $\gamma_{r=2}(T) \geq \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil$, where $\Delta(T)$ denotes the maximum degree in $T$.

Proof. The proof is by induction on the order of $T$. First we handle trees with small diameter. If $diam(T) \leq 2$ then $\gamma(T) = 1$, $\gamma_{r=2}(T) = 2$, and one can easily find that the required inequality holds. Moreover, we have $\gamma_{r=2}(G) = \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil$ precisely when $T$ is isomorphic to $K_{1,r}$ for $r > 1$. If $diam(T) = 3$ then another simple analysis shows that the inequality holds, and the equality is achieved for $DK_{r,s}$ with $r \geq s \geq 4$ and $DK_{r,1}$ with $r \geq 2$.

Let $T$ be a tree. By the above we may assume that $diam(T) \geq 4$. Let $P$ be a diametral path with the leaf $w$ as one of its ends. Suppose $v$ is the neighbor of $w$ and $u$ is the neighbor of $v$ that is not a leaf (hence $u$ also lies on $P$). Let $L$ denote the vertex set containing $v$ and all leaves adjacent to $v$ and $F(u)$ be all the possible color sets among all $\gamma_{r=2}$-function of $T - L$. Then $\gamma(T - L) \leq \gamma(T) \leq \gamma(T - L) + 1$, $\Delta(T) - 1 \leq \Delta(T - L) \leq \Delta(T)$ and $p(T - L) \leq p(T) \leq p(T - L) + 1$.

Case 1. $d_T(v) = 2$ and $F(u) = \{\{1\}, \{2\}, \{1, 2\}\}$.

In this case $\gamma_{r=2}(T) = \gamma_{r=2}(T - L) + 1$. By induction hypothesis $\gamma_{r=2}(T - L) \geq \gamma(T - L) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil$. We finally get

$$\gamma_{r=2}(T) = \gamma_{r=2}(T - L) + 1$$

$$\geq \gamma(T - L) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 1$$

$$\geq \gamma(T) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil$$

$$\geq \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil.$$}

Since if $p(T - L) = p(T)$, then $\ell(T - L) = \ell(T)$. Otherwise $p(T - L) = p(T) - 1$ and $\ell(T - L) = \ell(T) - 1$. The last inequality is obtained.

Case 2. $d_T(v) \geq 3$ or $d_T(v) = 2$ and $F(u) = \emptyset$.

In this case $\gamma_{r=2}(T) = \gamma_{r=2}(T - L) + 2$. Then we get

$$\gamma_{r=2}(T) = \gamma_{r=2}(T - L) + 2$$

$$\geq \gamma(T - L) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 2$$

$$\geq \gamma(T) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 1$$

$$\geq \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil.$$}

In the last inequality we use that the excess of leaves in $T$ with respect to $T - L$ does not go beyond $\Delta(T)$.

In the above proof we mentioned several examples of trees with diameter at most $3$ that achieve the bound in Theorem 7. We pose a characterization of all these extremal graphs as an open problem.
Next we give a lower bound of the 2-rainbow domination number of an arbitrary graph in terms of its diameter.

**Theorem 8.** For any connected graph $G$, $\gamma_{r2}(G) \geq \lceil \frac{2\text{diam}(G)+2}{5} \rceil$.

**Proof.** Let $f = (V_0, V_1^1, V_1^2, V_2)$ be a 2RDF of $G$. Consider an arbitrary path of length $\text{diam}(G)$. This diametral path includes at most two edges from the induced subgraph $\langle N[v] \rangle_G$ for each vertex $v \in V_1^1 \cup V_1^2 \cup V_2$. Furthermore, if vertex $v \in V_0$, then it is adjacent to a vertex with color set $\{1,2\}$, or adjacent to two different vertices with color set $\{1\}$ and $\{2\}$, respectively. Hence excluding the edges mentioned above, the diametral path includes at most $m$ in $\{|V_1^1|, |V_1^2| \} + |V_2| - 1$ other edges joining the neighborhoods of the vertices of $V_1^1 \cup V_1^2 \cup V_2$. Therefore

$$\text{diam}(G) \leq 2(|V_1^1| + |V_1^2| + |V_2|) + \min\{|V_1^1|, |V_1^2|\} + |V_2| - 1$$

$$\leq 2(|V_1^1| + |V_1^2| + |V_2|) + (|V_1^1| + |V_1^2|)/2 + |V_2| - 1$$

$$= 5/2(|V_1^1| + |V_1^2| + 2|V_2|) - 2|V_2| - 1$$

$$\leq 5/2\gamma_{r2}(G) - 1.$$ 

Then the desired result follows. 

Clearly, the bound of Theorem 8 is sharp, e.g. for $G$ isomorphic to $P_3$ or $C_4$.

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