L-R-smash product for (quasi) Hopf algebras

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Abstract

We introduce a more general version of the so-called L-R-smash product and study its relations with other kinds of crossed products (two-sided smash and crossed product and diagonal crossed product). We also give an interpretation of the L-R-smash product in terms of an L-R-twisting datum.

Introduction

The L-R-smash product was introduced and studied in a series of papers [1], [2], [3], [4], with motivation and examples coming from the theory of deformation quantization. It is defined as follows: if $H$ is a cocommutative bialgebra and $A$ is an $H$-bimodule algebra, the L-R-smash product $A \natural H$ is an associative algebra structure defined on $A \otimes H$ by the multiplication rule

$$(\varphi \natural h)(\varphi' \natural h') = (\varphi \cdot h_1')(h_1 \cdot \varphi') \natural h_2h_2', \quad \forall \varphi, \varphi' \in A, h, h' \in H.$$ 

If the right $H$-action is trivial, $A \natural H$ coincides with the ordinary smash product $A \# H$.

Our first remark is that, if we replace the above multiplication by

$$(\varphi \natural h)(\varphi' \natural h') = (\varphi \cdot h_2')(h_1 \cdot \varphi') \natural h_2h_1', \quad \forall \varphi, \varphi' \in A, h, h' \in H,$$

then this multiplication is associative without the assumption that $H$ is cocommutative; this more general object will also be denoted by $A \natural H$ and called L-R-smash product.

Actually, we introduce a much more general construction: we define an L-R-smash product $A \natural \mathbb{A}$ (an associative algebra), where $A$ is a bimodule algebra and $\mathbb{A}$ is a bicomodule algebra over a quasi-bialgebra $H$. Our motivation for defining the L-R-smash product in this generality stems from its relations with some constructions from [12] and [17]. Indeed, let $H$, $A$, $\mathbb{A}$ be as above.

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and also \( A \) a left \( H \)-module algebra, \( B \) a right \( H \)-module algebra and \( \mathfrak{B} \) a left \( H \)-comodule algebra; then \( A \otimes B \) becomes an \( H \)-bimodule algebra and \( \mathfrak{A} \otimes \mathfrak{B} \) an \( H \)-bicomodule algebra, and we prove that we have algebra isomorphisms \( (A \otimes B) \triangleright \triangleleft A \simeq A \triangleright A \triangleleft B \) and \( A \triangleleft (\mathfrak{A} \otimes \mathfrak{B}) \simeq A \triangleright A \triangleleft \mathfrak{B} \), where \( A \triangleright A \triangleleft B \) and \( A \triangleleft A \triangleright B \) are the two-sided smash and crossed products introduced in [7], [12]. These results combined with the ones in [7], [12], [13] suggest, and we are able to prove, that if \( H \) is moreover a quasi-Hopf algebra then \( A \triangleright \triangleright A \triangleleft \triangleleft B \triangleleft \triangleleft \triangleleft \triangleleft \triangleleft \) is isomorphic to the generalized diagonal crossed product \( A \triangleright \triangleright A \triangleleft \triangleleft B \triangleleft \triangleleft \triangleleft \) constructed in [7] (based on [12]). As a consequence of this, we obtain that over a cocommutative Hopf algebra \( H \), an \( L \)-\( R \)-smash product \( A \triangleright H \) is actually isomorphic to an ordinary smash product \( A \# H \), where the left \( H \)-action on \( A \) is now given by \( h \to \varphi = h_1 \cdot \varphi \cdot S(h_2) \).

Note that the multiplication of a diagonal crossed product (such as the quantum double of a Hopf or quasi-Hopf algebra) involves the antipode, while the one of an \( L \)-\( R \)-smash product does not. Hence, the \( L \)-\( R \)-smash product is not only a generalization of the ordinary smash product, but it can be regarded also as a substitute of the diagonal crossed product for bialgebras without antipode.

More can be said about the \( L \)-\( R \)-smash product over bialgebras and Hopf algebras. For instance, we provide a Maschke-type theorem for \( L \)-\( R \)-smash products, we give an interpretation (inspired by [11]) of the \( L \)-\( R \)-smash product in terms of what we call an \( L \)-\( R \)-twisting datum, and we find the counterpart, for twisted products, of the isomorphism between the \( L \)-\( R \)-smash product and the generalized diagonal crossed product.

For completeness and possible further use, we also introduce the dual version of the \( L \)-\( R \)-smash product (over bialgebras), called \( L \)-\( R \)-smash coproduct (generalizing Molnar’s smash coproduct).

1 Preliminaries

In this section we recall some definitions and results and fix notation that will be used throughout the paper.

We work over a commutative field \( k \). All algebras, linear spaces etc. will be over \( k \); unadorned \( \otimes \) means \( \otimes_k \). Following Drinfeld [9], a quasi-bialgebra is a fourtuple \((H, \Delta, \varepsilon, \Phi)\), where \( H \) is an associative algebra with unit, \( \Phi \) is an invertible element in \( H \otimes H \otimes H \otimes H \), and \( \Delta : H \to H \otimes H \) and \( \varepsilon : H \to k \) are algebra homomorphisms satisfying the identities

\[
(id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1},
\]

\[
(id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h,
\]

for all \( h \in H \), and \( \Phi \) has to be a normalized 3-cocycle, in the sense that

\[
(1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi),
\]

\[
(id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1.
\]

The identities [1.2], [1.3] and [1.4] also imply that

\[
(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1.
\]
The map $\Delta$ is called the coproduct or the comultiplication, $\varepsilon$ the counit and $\Phi$ the reassociator. We use the version of Sweedler’s sigma notation: $\Delta(h) = h_1 \otimes h_2$, but since $\Delta$ is only quasi-coassociative we adopt the further convention

$$(\Delta \otimes id)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = h_{1} \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of $\Phi$ by capital letters, and those of $\Phi^{-1}$ by small letters, namely

$$\begin{align*}
\Phi &= X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = Y^1 \otimes Y^2 \otimes Y^3 = \cdots \\
\Phi^{-1} &= x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = y^1 \otimes y^2 \otimes y^3 = \cdots
\end{align*}$$

The quasi-bialgebra $H$ is called a quasi-Hopf algebra if there exists a quasi-Hopf algebra $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)ah_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta,$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2\beta S(x^3) = 1. \quad (1.6)$$

The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling $\alpha$ and $\beta$, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$.

Next we recall that the definition of a quasi-bialgebra or quasi-Hopf algebra is "twist covariant" in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation or twist if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If $H$ is a quasi-bialgebra or a quasi-Hopf algebra and $F = F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = G^1 \otimes G^2$, then we can define a new quasi-bialgebra (respectively quasi-Hopf algebra) $H_F$ by keeping the multiplication, unit, counit (and antipode in the case of a quasi-Hopf algebra) of $H$ and replacing the comultiplication, reassociator and the elements $\alpha$ and $\beta$ by

$$\begin{align*}
\Delta_F(h) &= F\Delta(h)F^{-1}, \quad (1.8) \\
\Phi_F &= (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \quad (1.9) \\
\alpha_F &= S(G^1)\alpha G^2, \quad \beta_F = F^1\beta S(F^2), \quad (1.10)
\end{align*}$$

It is known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following: there exists a gauge transformation $f \in H \otimes H$ such that

$$f \Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\cop}(h)), \quad \text{for all} \ h \in H. \quad (1.11)$$

The element $f$ may be computed explicitly. First set

$$A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \quad (1.12)$$

$$B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1), \quad (1.13)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^4)\alpha A^4 \quad \text{and} \quad \delta = B^1\beta S(B^4) \otimes B^2\beta S(B^3). \quad (1.14)$$

Then $f$ and $f^{-1}$ are given by the formulae

$$f = (S \otimes S)(\Delta^{\cop}(x^1))\gamma \Delta(x^2\beta S(x^3)), \quad (1.15)$$

$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\cop}(x^3)). \quad (1.16)$$
Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If $U, V, W$ are left (right) $H$-modules, define $a_{U,V,W}, a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)),$$

$$a_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi^{-1}.$$  

The category $\mathcal{H}$ of left (right) $H$-modules becomes a monoidal category (cf. [14], [16] for the terminology) with tensor product $\otimes$ given via $\Delta$, associativity constraints $a_{U,V,W}$ (as Heyneman and Sweedler did for Hopf algebras), see [6].

We say that a $k$-vector space $A$ is a left $H$-module algebra if it is an algebra in the monoidal category $\mathcal{H}$, that is $A$ has a multiplication and a usual unit $1_A$ satisfying the following conditions:

$$aa' = (X^1 \cdot a)[(X^2 \cdot a')[(X^3 \cdot a'')] \cdot \Phi^{-1}],$$

$$h \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a'),$$

$$h \cdot 1_A = \varepsilon(h)1_A,$$  

for all $a, a', a'' \in A$ and $h \in H$, where $h \otimes a \rightarrow h \cdot a$ is the left $H$-module structure of $A$. Following [6] we define the smash product $A \# H$ as follows: as vector space $A \# H$ is $A \otimes H$ (elements $a \otimes h$ will be written $a \# h$) with multiplication given by

$$(a \# h)(a' \# h') = (x^1 \cdot a)(x^2 \cdot a') \# x^3 h_1 h'',$$

for all $a, a' \in A$, $h, h' \in H$. This $A \# H$ is an associative algebra with unit $1_A \# 1_H$ and it is defined by a universal property (as Heyneman and Sweedler did for Hopf algebras), see [6].

For further use we need also the notion of right $H$-module algebra. Let $H$ be a quasi-bialgebra. We say that a $k$-vector space $B$ is a right $H$-module algebra if $B$ is an algebra in the monoidal category $\mathcal{M}_H$, i.e. $B$ has a multiplication and a usual unit $1_B$ satisfying the following conditions:

$$(b b') b'' = (b \cdot x^1)[(b' \cdot x^2)(b'' \cdot x^3)],$$

$$b b' \cdot h = (b \cdot h_1)(b' \cdot h_2),$$

$$1_B \cdot h = \varepsilon(h)1_B,$$  

for all $b, b', b'' \in B$ and $h \in H$, where $b \otimes h \rightarrow b \cdot h$ is the right $H$-module structure of $B$.

Recall from [12] the notion of comodule algebra over a quasi-bialgebra.

**Definition 1.1** Let $H$ be a quasi-bialgebra. A unital associative algebra $\mathfrak{A}$ is called a right $H$-comodule algebra if there exist an algebra morphism $\rho : \mathfrak{A} \rightarrow \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that:

$$\Phi_\rho(\rho \otimes id)(\rho(a)) = (id \otimes \Delta)(\rho(a))\Phi_\rho, \quad \forall a \in \mathfrak{A},$$

$$(1_\mathfrak{A} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho),$$

$$(id \otimes \varepsilon) \circ \rho = id,$$

$$(id \otimes \varepsilon \otimes id)(\Phi_\rho) = (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_\mathfrak{A} \otimes 1_H.$$  

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Similarly, a unital associative algebra \( \mathcal{B} \) is called a left \( H \)-comodule algebra if there exist an algebra morphism \( \lambda : \mathcal{B} \to H \otimes \mathcal{B} \) and an invertible element \( \Phi_\lambda \in H \otimes H \otimes \mathcal{B} \) such that the following relations hold:

\[
(id \otimes \lambda)(\lambda(b))\Phi_\lambda = \Phi_\lambda(\Delta \otimes \lambda)(\lambda(b)), \quad \forall \ b \in \mathcal{B},
\]

\[
(1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes \lambda)(\Phi_\lambda)(\Phi \otimes 1_{\mathcal{B}}) = (id \otimes \Delta \otimes \lambda)(\Phi_\lambda)(\Delta \otimes \lambda \otimes \lambda)(\Phi_\lambda),
\]

\[
(\varepsilon \otimes id) \circ \lambda = id,
\]

\[
(id \otimes \varepsilon \otimes id)(\Phi_\lambda) = (\varepsilon \otimes \varepsilon \otimes id)(\Phi_\lambda) = \Phi_1 = 1_H \otimes 1_{\mathcal{B}}.
\]

When \( H \) is a quasi-bialgebra, particular examples of left and right \( H \)-comodule algebras are given by \( \mathfrak{A} = \mathcal{B} = H \) and \( \rho = \lambda = \Delta \), \( \Phi_\rho = \Phi_\lambda = \Phi \).

For a right \( H \)-comodule algebra \( \mathfrak{A} = H \otimes \mathcal{B} \), we will denote

\[
\rho(a) = a_{<0>} \otimes a_{<1>}, \quad (\rho \otimes id)(\rho(a)) = a_{<0,0>} \otimes a_{<0,1>} \otimes a_{<1>},
\]

for any \( a \in \mathfrak{A} \). Similarly, for a left \( H \)-comodule algebra \( \mathfrak{B} = \mathcal{B} \otimes H \), if \( b \in \mathcal{B} \) then we will denote

\[
\lambda(b) = b_{[-1]} \otimes b_{[0]}, \quad (id \otimes \lambda)(\lambda(b)) = b_{[-1]} \otimes b_{[0,-1]} \otimes b_{[0,0]},
\]

In analogy with the notation for the reassociator \( \Phi \) of \( H \), we will write

\[
\Phi_\rho = X_\rho^1 \otimes X_\rho^2 \otimes X_\rho^3 = \cdots
\]

and similarly for the element \( \Phi_\lambda \) of a left \( H \)-comodule algebra \( \mathcal{B} \).

If \( \mathfrak{A} \) is a right \( H \)-comodule algebra then we define the elements \( \tilde{\rho}_\rho, \tilde{\rho}_\rho \in \mathfrak{A} \otimes H \) as follows:

\[
\tilde{\rho}_\rho \tilde{\rho}_\rho = \tilde{\rho}_\rho \tilde{\rho}_\rho^2 = \tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3), \quad \tilde{\rho}_\rho = \tilde{y}_\rho^1 \otimes \tilde{y}_\rho^2 \otimes \tilde{y}_\rho^3 = \cdots
\]

By [12] Lemma 9.1, we have the following relations, for all \( a \in \mathfrak{A} \):

\[
\rho(a_{<0>}) = 1_{\mathfrak{A}} \otimes S(a_{<1>}), \quad \rho(\tilde{\rho}_\rho^1) = 1_{\mathfrak{A}} \otimes 1_H,
\]

\[
\rho(\tilde{\rho}_\rho^2) = \tilde{\rho}_\rho^1 \otimes \tilde{\rho}_\rho^2 = \tilde{x}_\rho^1 \otimes S^{-1}(\tilde{\rho}_\rho^3) \tilde{x}_\rho^2.
\]

By [5] it follows that \( A \bowtie \mathcal{B} \) is an associative algebra with unit \( 1_A \bowtie 1_{\mathcal{B}} \). If we take \( \mathcal{B} = H \) then \( A \bowtie H \) is just the smash product \( A \# H \). For this reason the algebra \( A \bowtie \mathcal{B} \) is called the generalized smash product of \( A \) and \( \mathcal{B} \).

The following definition was introduced in [12] under the name "quasi-commuting pair of \( H \)-coactions".

**Definition 1.2** Let \( H \) be a quasi-bialgebra. By an \( H \)-bicomodule algebra \( \mathfrak{A} \) we mean a quintuple \( (\lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda, \rho}) \), where \( \lambda \) and \( \rho \) are left and right \( H \)-coactions on \( \mathfrak{A} \), respectively, and where \( \Phi_\lambda \in H \otimes H \otimes \mathfrak{A} \), \( \Phi_\rho \in \mathfrak{A} \otimes H \otimes H \) and \( \Phi_{\lambda, \rho} \in H \otimes \mathfrak{A} \otimes H \) are invertible elements, such that:
- $(A, \lambda, \Phi_\lambda)$ is a left $H$-comodule algebra;
- $(A, \rho, \Phi_\rho)$ is a right $H$-comodule algebra;
- the following compatibility relations hold:

\[
\Phi_{\lambda, \rho}(\lambda \otimes \text{id})(\rho(u)) = (\text{id} \otimes \rho)(\lambda(u))\Phi_{\lambda, \rho}, \quad \forall \ u \in A, \tag{1.38}
\]

\[
(1_H \otimes \Phi_{\lambda, \rho})(\text{id} \otimes \lambda \otimes \text{id})(\Phi_{\lambda, \rho} \otimes 1_H) = (\text{id} \otimes \text{id} \otimes \rho)(\Phi_{\lambda, \rho})(\Delta \otimes \text{id} \otimes \text{id})(\Phi_{\lambda, \rho}), \tag{1.39}
\]

\[
(1_H \otimes \Phi_\rho)(\text{id} \otimes \rho \otimes \text{id})(\Phi_{\lambda, \rho} \otimes 1_H) = (\text{id} \otimes \text{id} \otimes \Delta)(\Phi_{\lambda, \rho})(\lambda \otimes \text{id} \otimes \text{id})(\Phi_{\rho}). \tag{1.40}
\]

As pointed out in [12], if $A$ is a bicomodule algebra then, in addition, we have that

\[
(id_H \otimes \text{id}_A \otimes \varepsilon)(\Phi_{\lambda, \rho}) = 1_H \otimes 1_A, \quad (\varepsilon \otimes \text{id}_A \otimes id_H)(\Phi_{\lambda, \rho}) = 1_A \otimes 1_H. \tag{1.41}
\]

A first example of a bicomodule algebra is $A = H$, $\lambda = \rho = \Delta$ and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda, \rho} = \Phi$. Related to the left and right comodule algebra structures of $A$ we keep notation as above. For simplicity we denote

\[
\Phi_{\lambda, \rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \Theta^1 \otimes \Theta^2 \otimes \Theta^3, \\
\Phi_{\lambda, \rho}^{-1} = \hat{\Theta}^1 \otimes \hat{\Theta}^2 \otimes \hat{\Theta}^3 = \hat{\Theta}^1 \otimes \hat{\Theta}^2 \otimes \hat{\Theta}^3.
\]

Let us denote by $H \mathcal{M}_H$ the category of $H$-bimodules; it is also a monoidal category, the associativity constraints being given by $a'_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$,

\[
a'_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)) \cdot \Phi^{-1}, \tag{1.42}
\]

for any $U, V, W \in H \mathcal{M}_H$ and $u \in U$, $v \in V$ and $w \in W$. Therefore, we may define algebras in the category of $H$-bimodules. Such an algebra will be called an $H$-bimodule algebra. More exactly, a $k$-vector space $A$ is an $H$-bimodule algebra if $A$ is an $H$-bimodule (denote the actions by $h \cdot \varphi$ and $\varphi \cdot h$, for $h \in H$ and $\varphi \in A$) which has a multiplication and a usual unit $1_A$ such that the following relations hold:

\[
(\varphi \psi)\xi = (X^1 \cdot \varphi \cdot x^1)[(X^2 \cdot \psi \cdot x^2)(X^3 \cdot \xi \cdot x^3)], \quad \forall \ \varphi, \psi, \xi \in A, \tag{1.43}
\]

\[
h \cdot (\varphi \psi) = (h_1 \cdot \varphi)(h_2 \cdot \psi), \quad (\varphi \psi) \cdot h = (\varphi \cdot h_1)(\psi \cdot h_2), \quad \forall \ \varphi, \psi \in A, h \in H, \tag{1.44}
\]

\[
h \cdot 1_A = \varepsilon(h)1_A, \quad 1_A \cdot h = \varepsilon(h)1_A, \quad \forall \ h \in H. \tag{1.45}
\]

Let $H$ be a quasi-bialgebra. Then $H^*$, the linear dual of $H$, is an $H$-bimodule via the $H$-actions

\[
<h \mapsto \varphi, h' > = \varphi(h'h), \quad < \varphi \mapsto h, h' > = \varphi(hh'), \quad \forall \ \varphi \in H^*, h, h' \in H. \tag{1.46}
\]

The convolution $< \varphi \psi, h > = \sum \varphi(h_1)\psi(h_2)$, $\varphi, \psi \in H^*$, $h \in H$, is a multiplication on $H^*$; it is not in general associative, but with this multiplication $H^*$ becomes an $H$-bimodule algebra.

2 L-R-smash product over quasi-bialgebras and quasi-Hopf algebras

We introduce the general version of the L-R-smash product as follows.
Proposition 2.1 Let $H$ be a quasi-bialgebra, $A$ an $H$-bimodule algebra and $A$ an $H$-biocomodule algebra. Define on $A \otimes A$ the product
\[
(\varphi \otimes u)(\psi \otimes u') = (\tilde{x}_1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{x}_2 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3) \in \tilde{x}_1 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3
\]
(2.1)
for $\varphi, \psi \in A$ and $u, u' \in A$, where $\Phi^{-1} = \tilde{x}_1^2 \otimes \tilde{x}_2^3 \otimes \tilde{x}_3^3$, $\Phi^{-1} = \tilde{x}_1^3 \otimes \tilde{x}_2^3 \otimes \tilde{x}_3^3$, and we write $\varphi \otimes u$ in place of $\varphi \otimes u$ to distinguish the new algebraic structure. Then this product defines on $A \otimes A$ a structure of associative algebra with unit $1_A \otimes 1_A$, denoted by $A \otimes A$ and called the L-R-smash product.

Proof. For $\varphi, \psi, \xi \in A$ and $u, u', u'' \in A$ we compute:
\[
[(\varphi \otimes u)(\psi \otimes u')](\xi \otimes u'')
\]
1.24 1.40
\[
= ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) ) ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) )
\]
1.43 1.24 1.28
\[
= ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) ) ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) )
\]
1.38
\[
= ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) ) ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) )
\]
1.25 1.39
\[
= ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) ) ( (\tilde{y}_3^1 \cdot \varphi \cdot \theta^3 u'_{<1>}, \tilde{y}_2^3 u_{[0]} \theta^2 u'_{<0>}, \tilde{y}_2^3) )
\]
2.1
\[
= ( (\varphi \otimes u)(\tilde{x}_1 \cdot \psi \cdot \theta^3 u'_{<1>}, \tilde{x}_2 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3) ) ( (\varphi \otimes u)(\tilde{x}_1 \cdot \psi \cdot \theta^3 u'_{<1>}, \tilde{x}_2 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3) )
\]
2.1
\[
= ( (\varphi \otimes u)(\tilde{x}_1 \cdot \psi \cdot \theta^3 u'_{<1>}, \tilde{x}_2 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3) ) ( (\varphi \otimes u)(\tilde{x}_1 \cdot \psi \cdot \theta^3 u'_{<1>}, \tilde{x}_2 u_{[0]} \theta^2 u'_{<0>}, \tilde{x}_3^3) )
\]
hence the multiplication is associative. It is easy to check that $1_A \otimes 1_A$ is the unit. \qed
Remark 2.2 It is easy to see that, in $A \triangleright A$, we have $(1 \triangleright u)(1 \triangleright u') = 1 \triangleright uu'$ for all $u, u' \in A$, hence the map $A \to A \triangleright A$, $u \mapsto 1 \triangleright u$, is an algebra map, and $(\varphi \triangleright 1)(1 \triangleright u) = \varphi \cdot u_{<1>} \triangleright u_{<0>}$. 

Examples 2.3 1) Let $A$ be a left $H$-module algebra. Then $A$ becomes an $H$-bimodule algebra, with right $H$-action given via $\varepsilon$. In this case the multiplication of $A \triangleright A$ becomes

$$(a \triangleright u)(a' \triangleright u') = (\bar{x}^1 \cdot a)(\bar{x}^2 u_{[-1]} \cdot \psi) \triangleright \bar{x}^3 u_{[0]} u',$$

for all $a, a' \in A$ and $u, u' \in A$, hence in this case $A \triangleright A$ coincides with the generalized smash product $A \triangleright \triangleleft A$.

2) As we have already mentioned, $H$ itself is an $H$-bimodule algebra. So, in this case, the multiplication of $A \triangleright H$ specializes to

$$(\varphi \triangleright h)(\psi \triangleright h') = (x^1 \cdot \varphi \cdot t^3 h^2 y^2)(x^2 h^1 t^1 \cdot \psi \cdot y^3) \triangleright x^3 h^2 t^2 h^1 y^1,$$

for all $\varphi, \psi \in A$ and $h, h' \in H$. If the right $H$-module structure of $A$ is trivial, then $A \triangleright H$ coincides with the smash product $A \# H$.

3) Let $H$ be an ordinary bialgebra, $A$ an $H$-bimodule algebra and $\triangleright$ an $H$-bicomodule algebra in the usual (Hopf) sense. In this case the multiplication of $A \triangleright H$ becomes:

$$(\varphi \triangleright u)(\psi \triangleright u') = (\varphi \cdot u_{<1>})(u_{[-1]} \cdot \psi) \triangleright u_{[0]} u'_{<0>},$$

for all $\varphi, \psi \in A$ and $u, u' \in A$. If moreover $A = H$, the multiplication of $A \triangleright H$ is

$$(\varphi \triangleright h)(\psi \triangleright h') = (\varphi \cdot h^2_1)(h^1_2 \cdot \psi) \triangleright h_2 h^1,$$  

for all $\varphi, \psi \in A, h, h' \in H$. 

In case $H$ is cocommutative, this product can be written as

$$(\varphi \triangleright h)(\psi \triangleright h') = (\varphi \cdot h^2_1)(h^1_2 \cdot \psi) \triangleright h_2 h^1,$$

(2.5)

and this was the original L-R-smash product defined in [1], [2], [3], [4].

Recall from [7] the so-called two-sided generalized smash product, defined as follows. Let $H$ be a quasi-bialgebra, $A$ a left $H$-module algebra, $B$ a right $H$-module algebra and $\triangleright$ an $H$-bicomodule algebra. If we define on $A \otimes A \otimes B$ a multiplication, by

$$(a \triangleright \triangleright u)\triangleright (a' \triangleright \triangleright u')$$

$$= (\bar{x}^1 \cdot a)(\bar{x}^2 u_{[-1]}^1 \theta^1 \cdot a') \triangleright \bar{x}^3 u_{[0]}^1 \theta^2 u_{<0>}^1 \triangleright \bar{x}^3 \rho_1 \triangleright (b \cdot \theta^3 u_{<1>}^1 \triangleright \bar{x}^3 \rho_2)$$

(2.6)

for all $a, a' \in A, u, u' \in A$ and $b, b' \in B$ (we write $a \triangleright \triangleright u \triangleright \triangleright b$ for $a \otimes u \otimes b$), and we denote this structure by $A \triangleright \triangleright A \triangleright \triangleright B$, then it is an associative algebra with unit $1_A \triangleright \triangleright 1_A \triangleright \triangleright 1_B$.

Note that, given $A, B$ as above, $A \otimes B$ becomes an $H$-bimodule algebra, with $H$-actions

$$h \cdot (a \otimes b) \cdot h' = h \cdot a \otimes b \cdot h', \quad \forall a \in A, h, h' \in H, b \in B.$$ 

(2.7)

Proposition 2.4 If $H, A, B, \triangleright$ are as above, then we have an algebra isomorphism

$$\phi : (A \otimes B) \triangleright \triangleright \triangleright A \simeq A \triangleright \triangleright \triangleright B,$$

$$\phi((a \otimes b) \triangleright \triangleright u) = a \triangleright \triangleright u \triangleright \triangleright b, \quad \forall a \in A, b \in B, u \in A.$$
Proof. We compute:
\[
\phi([((a \otimes b) \triangleright u) ([a' \otimes b') \triangleright u'])
\]
\[
= \phi((\vec{x}_\lambda^1 \cdot (a \otimes b) \cdot \theta^3 u_{<1>_{\rho}} \cdot \vec{x}_\rho^2) (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot (a' \otimes b') \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= \phi((\vec{x}_\lambda^1 \cdot a \otimes b \cdot \theta^3 u_{<1>_{\rho}} \cdot \vec{x}_\rho^2) (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot a' \otimes b' \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= \phi(((\vec{x}_\lambda^1 \cdot a)(\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot a') \otimes (b \cdot \theta^3 u_{<1>_{\rho}} \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= (\vec{x}_\lambda^1 \cdot a)(\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot a') \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= (a \triangleright <u \triangleright b)(a' \triangleright <u' \triangleright b')
\]
\[
= \phi((a \otimes b) \triangleright u)(a' \otimes b') \triangleright u',
\]
and the proof is finished. 

Recall from [7] the so-called generalized two-sided crossed product, defined as follows: if \(H\) is a quasi-bialgebra, \(\mathfrak{A}\) a right \(H\)-comodule algebra, \(\mathfrak{B}\) a left \(H\)-comodule algebra and \(\mathfrak{A}\) an \(H\)-bimodule algebra, define on \(\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{B}\) a multiplication by the formula
\[
(a \triangleright <\varphi \triangleright b)(a' \triangleright <\varphi' \triangleright b') = a \cdot_b (\varphi \cdot_b a'_{<1>} \triangleright \varphi') \cdot_b b',
\]
for all \(a, a', b, b' \in \mathfrak{A}, b, b' \in \mathfrak{B}\) and \(\varphi, \varphi' \in \mathfrak{A}\), where we write \(a \triangleright \varphi \triangleright b\) for \(a \otimes \varphi \otimes b\). Then this multiplication yields an associative algebra structure with unit \(1_\mathfrak{A} \triangleright >1_\mathfrak{A} \triangleright >1_\mathfrak{B}\), denoted by \(\mathfrak{A} \triangleright >H \triangleright > \mathfrak{B}\). For \(H\) finite dimensional and \(\mathfrak{A} = H^*\) we recover the two-sided crossed product \(\mathfrak{A} \triangleright >H^* \triangleright > \mathfrak{B}\).

Note that, given \(\mathfrak{A}, \mathfrak{B}\) as above, \(\mathfrak{A} \otimes \mathfrak{B}\) becomes an \(H\)-bicomodule algebra, with the following structure:
\[
\rho(a \otimes b) = (a_{<0>} \otimes b) \otimes a_{<1>}, \quad \lambda(a \otimes b) = b_{[-1]} \otimes (a \otimes b_{[0]}),
\]
\[
\Phi_\rho = (\vec{x}_\lambda^1 \otimes \vec{x}_\lambda^2) \otimes (\vec{x}_\lambda^3 \otimes \vec{x}_\lambda^1), \quad \Phi_{\lambda, \rho} = 1_H \otimes (1_\mathfrak{A} \otimes 1_\mathfrak{B}) \otimes 1_H,
\]
for all \(a \in \mathfrak{A}\) and \(b \in \mathfrak{B}\), see [12].

**Proposition 2.5** If \(H, \mathfrak{A}, \mathfrak{B}\) are as above, then we have an algebra isomorphism
\[
\tau : \mathfrak{A} \triangleright > (\mathfrak{A} \otimes \mathfrak{B}) \simeq \mathfrak{A} \triangleright > \mathfrak{A} \triangleright > \mathfrak{B},
\]
\[
\tau(\varphi \triangleright > (a \otimes b)) = a \triangleright > \varphi \triangleright b, \quad \forall \varphi \in \mathfrak{A}, a \in \mathfrak{A}, b \in \mathfrak{B}.
\]

**Proof.** We compute:
\[
\tau((\varphi \triangleright > (a \otimes b))(\varphi' \triangleright > (a' \otimes b')))
\]
\[
= \tau((\vec{x}_\lambda^1 \cdot \varphi \cdot (a' \otimes b')_{<1>} \cdot \vec{x}_\rho^2) (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot a' \otimes b' \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= \tau((\vec{x}_\lambda^1 \cdot \varphi \cdot a'_{<1>} \vec{x}_\rho^2 (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= \tau((\vec{x}_\lambda^1 \cdot \varphi \cdot a'_{<1>} \vec{x}_\rho^2 (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= \tau((\vec{x}_\lambda^1 \cdot \varphi \cdot a'_{<1>} \vec{x}_\rho^2 (\vec{x}_\lambda^2 u_{[-1]}^\theta \cdot \vec{x}_\rho^3)) \triangleright \vec{x}_\rho^3 u_{[0]}^\theta \cdot u_{<0>_{\rho}} \cdot \vec{x}_\rho^1
\]
\[
= (a \triangleright > \varphi \triangleright b)(a' \triangleright > \varphi' \triangleright b')
\]
\[
= \tau(\varphi \triangleright > (a \otimes b)) \tau(\varphi' \triangleright > (a' \otimes b'))
\]
and the proof is finished. □
Assume that $H$ is a quasi-Hopf algebra, $\mathcal{A}$ is an $H$-bimodule algebra and $\mathcal{A}$ an $H$-bicomodule algebra. Recall from [7] the so-called generalized diagonal crossed product (which for $\mathcal{A} = H^*$ gives the diagonal crossed product from [12]). Namely, define the element

$$\Omega = (\tilde{X}_\rho|_{[-1]}|_1 \tilde{X}_\rho|_{[-1]}|_2 \tilde{X}_\rho^2) \otimes (\tilde{X}_\rho|_{[0]} \tilde{X}_\rho^2) \otimes S^{-1}(f_1 \tilde{X}_\rho^3) \otimes S^{-1}(f_2 \tilde{X}_\rho^3)$$

in $H^\otimes 2 \otimes \mathcal{A} \otimes H^\otimes 2$, where $\Phi_\rho = \tilde{X}_\rho \otimes \tilde{X}_\rho \otimes S^{-1}(f \tilde{X})$, $\Phi^{-1}_\rho = \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^2$, $\Phi^{-1}_\rho = \theta_1 \otimes \theta_2 \otimes \theta_3$ and $f = f^1 \otimes f^2$ is the twist defined in (1.15). Then define a multiplication on $\mathcal{A} \otimes \mathcal{A}$, by:

$$(\varphi \triangleright u)(\psi \triangleright u') = (\Omega \cdot \varphi \cdot \Omega^5)((2u_{<0}_1) \cdot \psi \cdot S^{-1}(u_{<1}_1) \Omega^4) \triangleright \Omega^3 u_{<0}_0 u' \cdot \Omega^5$$

for all $\varphi, \psi \in \mathcal{A}$ and $u, u' \in \mathcal{A}$. Then this multiplication defines an associative algebra structure with unit $1_B \triangleright 1_B$, which will be denoted by $\mathcal{A} \bowtie \mathcal{A}$.

It was proved in [7] that, for an $H$-bimodule algebra of type $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is a left $H$-module algebra and $\mathcal{B}$ is a right $H$-module algebra, we have $(\mathcal{A} \otimes \mathcal{B}) \bowtie \mathcal{A} \approx \mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{A}) \bowtie \mathcal{A}$. Also, it was proved in [7] that, for an $H$-bicomodule algebra of type $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is a right $H$-comodule algebra and $\mathcal{B}$ is a left $H$-comodule algebra, we have $\mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{B}) \approx \mathcal{A} \bowtie H \bowtie \mathcal{A} \bowtie \mathcal{B}$, hence, by Proposition 2.5 we obtain $\mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{B}) \approx \mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{B})$. This raises the natural question whether we actually have $\mathcal{A} \bowtie \mathcal{A} \approx \mathcal{A} \bowtie \mathcal{A}$ for any $H$-bimodule algebra $\mathcal{A}$ and any $H$-bicomodule algebra $\mathcal{A}$. We will see that this is the case, and for the proof we need first to recall some formulae from [7]. Namely, if we denote by $\tilde{\theta}_\rho^1 \bowtie \tilde{\theta}_\rho^2$ another copy of $\tilde{q}_p$, then we have:

$$\Theta^1 \Theta^2 \tilde{q}_p \tilde{q}_p^1 (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<0} \otimes \tilde{\theta}_\rho^2 \tilde{\theta}_\rho^1 (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<1} \otimes (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<2} = \tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2 \tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2 (\Theta^1 \Theta^2)_{<0} \otimes (\Theta^1 \Theta^2)_{<1} \otimes (\Theta^1 \Theta^2)_{<2}$$

$$\Theta^1 \Theta^2 \tilde{q}_p^1 \tilde{q}_p^2 (\Theta^1 \Theta^2)_{<0} \otimes S^{-1}(\Theta^1 \Theta^2)_{<1} \otimes S^{-1}(\Theta^1 \Theta^2)_{<2}$$

Then we have:

$$\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2 \tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2 (\Theta^1 \Theta^2)_{<0} \otimes S^{-1}(\Theta^1 \Theta^2)_{<1} \otimes S^{-1}(\Theta^1 \Theta^2)_{<2} = (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<0} \otimes (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<1} \otimes (\tilde{\theta}_\rho^1 \tilde{\theta}_\rho^2)_{<2}$$

Theorem 2.6 Let $H$ be a quasi-Hopf algebra, $\mathcal{A}$ an $H$-bimodule algebra and $\mathcal{A}$ an $H$-bicomodule algebra. Then the map

$$\nu : \mathcal{A} \bowtie \mathcal{A} \rightarrow \mathcal{A} \bowtie \mathcal{A}$$

for all $\varphi \in \mathcal{A}$ and $u \in \mathcal{A}$, is an algebra isomorphism, with inverse

$$\nu^{-1} : \mathcal{A} \bowtie \mathcal{A} \rightarrow \mathcal{A} \bowtie \mathcal{A}$$

and this isomorphism is compatible with the isomorphisms $(\mathcal{A} \otimes \mathcal{B}) \bowtie \mathcal{A} \approx \mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{B}) \bowtie \mathcal{B}$ and $\mathcal{A} \bowtie (\mathcal{A} \otimes \mathcal{B}) \approx \mathcal{A} \bowtie \mathcal{A} \bowtie \mathcal{B}$. 

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Proof. First we establish that $\nu$ is an algebra map. We compute:

\[
\nu((\varphi \bowtie u)(\psi \bowtie u')) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, u' <1>)
\]

\[
= (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, u' <1>)
\]

\[
\nu((\varphi \bowtie u)(\psi \bowtie u')) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, u' <1>)
\]

as needed. The fact that $\nu(1 \bowtie 1) = 1$ is trivial.

We prove now that $\nu$ and $\nu^{-1}$ are inverses. Indeed, we have:

\[
\nu^{-1}(\varphi \bowtie u) = \Theta^1 \varphi \cdot \varphi \cdot S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

\[
= (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

\[
\nu^{-1}(\varphi \bowtie u) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

\[
\nu^{-1}(\varphi \bowtie u) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

\[
\nu^{-1}(\varphi \bowtie u) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

\[
\nu^{-1}(\varphi \bowtie u) = (\Theta^1 \varphi \cdot \varphi \cdot \Omega^3 S^{-1}(\Theta^3) q^2_0 \Theta^2_0 \Theta^3_0 <1, u <0, 0 <1, (p^1_0) <1>)
\]

Examples 2.7 1) If $A$ is a left $H$-module algebra regarded as an $H$-bimodule algebra with trivial right $H$-action, then $A \bowtie A$ and $A \bowtie \omega A$ both coincide with $A \bowtie A$, and the isomorphism
\( \nu \) is just the identity.

2) If \( \mathbb{A} = H \), the maps \( \nu : A \bowtie H \to A \bowtie H \) and \( \nu^{-1} : A \bowtie H \to A \bowtie H \) are given by

\[
\begin{align*}
\nu(\varphi \bowtie h) &= X^1 \cdot \varphi \cdot S^{-1}(X^3)q_R^2 X_2^2 h_2 \bowtie q_1^2 X_1^2 h_1, \\
\nu^{-1}(\varphi \bowtie h) &= x^1 \cdot \varphi \cdot S^{-1}(x^3 h_2 p_R^2) \bowtie x^2 h_1 p_R^1,
\end{align*}
\]

for all \( \varphi \in A \) and \( h \in H \), where \( q_R = q_1^1 \otimes q_2^2 = Y^1 \otimes S^{-1}(\alpha Y^3)Y^2 \) and \( p_R = p_1^1 \otimes p_2^2 = y^1 \otimes y^2 \beta(S(y^3)) \).

3) Let \( H \) be a Hopf algebra with bijective antipode, \( \mathbb{A} \) an \( H \)-bimodule algebra and \( \mathbb{A} \) an \( H \)-bicorepresentation in the usual (Hopf) sense. Then the maps \( \nu : A \bowtie \mathbb{A} \to A \bowtie \mathbb{A} \) and \( \nu^{-1} : A \bowtie \mathbb{A} \to A \bowtie \mathbb{A} \) become:

\[
\begin{align*}
\nu(\varphi \bowtie u) &= \varphi \cdot u_{<1>} \bowtie u_{<0>}, \\
\nu^{-1}(\varphi \bowtie u) &= \varphi \cdot S^{-1}(u_{<1>}) \bowtie u_{<0>},
\end{align*}
\]

for all \( \varphi \in A \) and \( u \in \mathbb{A} \); if moreover \( \mathbb{A} = H \), they become

\[
\begin{align*}
\nu(\varphi \bowtie h) &= \varphi \cdot h_2 \bowtie h_1, \\
\nu^{-1}(\varphi \bowtie h) &= \varphi \cdot S^{-1}(h_2) \bowtie h_1,
\end{align*}
\]

for all \( \varphi \in A \) and \( h \in H \).

Let now \( H \) be a finite dimensional quasi-Hopf algebra. Recall that the quantum double \( D(H) \) (generalizing the usual Drinfeld double of a Hopf algebra) was first introduced by Majid in [15] by an implicit Tannaka-Krein reconstruction procedure, and more explicit descriptions were obtained afterwards by Hausser and Nill in [12], [13]. According to one of these descriptions, the algebra structure of \( D(H) \) is just the diagonal crossed product \( H^* \bowtie H \). By transferring the whole structure of \( D(H) \) via the map \( \nu \), we can thus obtain a new realization of \( D(H) \), having the L-R-smash product \( H^* \bowtie H \) for the algebra structure.

We study the invariance under twisting of the L-R-smash product and first recall some facts from [12], [7].

Let \( H \) be a quasi-bialgebra, \( \mathbb{A} \) an \( H \)-bimodule algebra and \( F \in H \otimes H \) a gauge transformation. If we introduce on \( \mathbb{A} \) another multiplication, by \( \varphi \circ \varphi' = (G^1 \cdot \varphi \cdot F^1)(G^2 \cdot \varphi' \cdot F^2) \) for all \( \varphi, \varphi' \in \mathbb{A} \), where \( F^{-1} = G^1 \otimes G^2 \), and denote this structure by \( F \mathbb{A} \), then \( F \mathbb{A} \) is an \( H \)-comodule algebra, with the same unit and \( H \)-actions as \( \mathbb{A} \).

Suppose that we have a left \( H \)-comodule algebra \( \mathfrak{B} \); then on the algebra structure of \( \mathfrak{B} \) one can introduce a left \( H_F \)-comodule algebra structure (denoted by \( \mathfrak{B}^{F^{-1}} \) in what follows) putting \( \lambda^{F^{-1}} = \lambda \) and \( \Phi^{F^{-1}} = \Phi(F^{-1} \otimes 1_{\mathfrak{B}}) \). Similarly, if \( \mathfrak{A} \) is a right \( H \)-comodule algebra, one can introduce on the algebra structure of \( \mathfrak{A} \) a right \( H \)-comodule algebra structure (denoted by \( F \mathfrak{A} \) in what follows) putting \( F \rho = \rho \) and \( F \Phi = (1_{\mathfrak{A}} \otimes F) \Phi \). One may check that if \( \mathbb{A} \) is an \( H \)-comodule algebra, the left and right \( H_F \)-comodule algebras \( \mathbb{A}^{F^{-1}} \) respectively \( F \mathbb{A} \) actually define the structure of an \( H_F \)-comodule algebra on \( \mathbb{A} \), denoted by \( F \mathbb{A} \), which has the same \( \Phi_{\lambda, \rho} \) as \( \mathbb{A} \).

**Proposition 2.8** With notation as above, we have an algebra isomorphism

\[
\mathbb{A} \bowtie \mathbb{A} \cong F \mathbb{A} \bowtie F \mathbb{A}^{F^{-1}},
\]

given by the trivial identification.
Proof. Let $F^1 \otimes F^2$ and $G^1 \otimes G^2$ be two more copies of $F$ and $F^{-1}$ respectively. We compute the multiplication in $F_A F^{-1} \cong F_A F^{-1}$:

$$(\varphi \triangleleft u)(\psi \triangleleft u')$$

$$= (F^1 \bar{x}_1^3 \cdot \varphi \triangleleft \theta^3 u'_{<1>} \cdot \bar{x}_2^2 G^1) \circ (F^2 \bar{x}_1^3 u_{[-1]} \theta^2 \cdot \psi \triangleleft \bar{x}_3^2 G^2) \triangleleft \bar{x}_1^3 u_{[0]} \theta^2 u'_{<0>} \bar{x}_1^1$$

$$= (G^1 F^1 \bar{x}_1^3 \cdot \varphi \triangleleft \bar{x}_2^2 G^1) \circ (G^2 F^2 \bar{x}_1^3 u_{[-1]} \theta^2 \cdot \psi \triangleleft \bar{x}_3^2 G^2 F^2) \triangleleft \bar{x}_1^3 u_{[0]} \theta^2 u'_{<0>} \bar{x}_1^1$$

$$= (\bar{x}_1^3 \cdot \varphi \triangleleft \theta^3 u'_{<1>} \cdot \bar{x}_2^2) (\bar{x}_1^3 u_{[-1]} \theta^1 \cdot \psi \triangleleft \bar{x}_3^2) \triangleleft \bar{x}_1^3 u_{[0]} \theta^2 u'_{<0} \bar{x}_1^1,$$

which is the multiplication of $A \triangleleft A$.

3 More properties of the L-R-smash product over bialgebras and Hopf algebras

The first property we want to emphasize is a direct consequence of Theorem 2.6.

Proposition 3.1 Let $H$ be a cocommutative Hopf algebra and $A$ an $H$-bimodule algebra. Then the L-R-smash product $A \triangleright H$ is isomorphic to an ordinary smash product $A \# H$, where the left $H$-action on $A$ is now given by $h \to \varphi = h_1 \cdot \varphi \cdot S(h_2)$, for all $h \in H$ and $\varphi \in A$.

Proof. Since $H$ is cocommutative (hence in particular $S^2 = id$), the generalized diagonal crossed product $A \rtimes H$ (which is isomorphic to $A \triangleright H$) has multiplication:

$$(\varphi \rtimes h)(\varphi' \rtimes h') = \varphi(h_1 \cdot \varphi' \cdot S(h_3)) \rtimes h_2 h'$$

$$= \varphi(h_1 \cdot \varphi' \cdot S(h_2)) \rtimes h_2 h'$$

$$= \varphi(h_1 \to \varphi') \rtimes h_2 h',$$

which is just the multiplication of $A \# H$.

The following provides us with a Maschke-type theorem for L-R-smash products, the proof is similar to the one for crossed products, see [18], Theorem 7.4.2.

Theorem 3.2 Let $H$ be a finite dimensional Hopf algebra such that $H^*$ is semisimple and $H^*$ is unimodular, and let $A$ be an $H$-bimodule algebra. Then:

(i) If $V \in A \triangleright H^\mathcal{M}$ and $W \subseteq V$ is a submodule which has a complement in $A \mathcal{M}$, then $W$ has also a complement in $A \triangleright H^\mathcal{M}$.

(ii) If $A$ is semisimple Artinian, then so is $A \triangleright H$.

Proof. Obviously (ii) follows from (i), so we prove (i) now. Let $t \in H$ be an integral with $\varepsilon(t) = 1$. By [19], formula (15), we have

$$S^{-1}(t_3) g^{-1} t_1 \otimes t_2 = 1 \otimes t,$$

where $g$ is the distinguished group-like element of $H$; our hypothesis that $H^*$ is unimodular implies $g = 1$, hence we also get

$$S^{-1}(t_4) t_1 \otimes t_2 \otimes t_3 = 1 \otimes t_1 \otimes t_2.$$

Let now $\pi : V \to W$ be an $A$-projection. We construct the averaging function as in [18], namely:

$$\tilde{\pi}(v) = (1 \triangleright S(t_1)) \cdot \pi((1 \triangleright t_2) \cdot v), \quad \forall v \in V.$$
Before proving that the generalized smash product \( A \rhd H \) is \( A \)-linear, note the following formulae:

\[
(1 \rhd h)(\varphi \rhd 1) = (h_1 \cdot \varphi \cdot S^{-1}(h_3) \rhd 1)(1 \rhd h_2), \tag{3.2}
\]
\[
\varphi \rhd h = (\varphi \cdot S^{-1}(h_2) \rhd 1)(1 \rhd h_1), \tag{3.3}
\]

for all \( \varphi \in A \) and \( h \in H \), which follow by direct computation using the formula for the multiplication in \( A \rhd H \). Let now \( \varphi \in A \) and \( v \in V \); we compute:

\[
\tilde{\pi}((\varphi \rhd 1) \cdot v) = (1 \rhd S(t_1)) \cdot \pi((1 \rhd t_2)(\varphi \rhd 1) \cdot v)
\]
\[
= (1 \rhd S(t_1))(t_2 \cdot S^{-1}(t_4) \rhd 1)(1 \rhd t_3) \cdot v)
\]
\[
= (S(t_1)t_2 \cdot \varphi \cdot S^{-1}(t_4) \rhd S(t_1)2 \cdot \pi((1 \rhd t_3) \cdot v)
\]
\[
= (\varphi \cdot S^{-1}(t_3) \rhd S(t_1)) \cdot \pi((1 \rhd t_2) \cdot v)
\]
\[
= (\varphi \cdot S^{-1}(t_3)S^{-1}(t_1) \rhd 1)(1 \rhd S(t_1)) \cdot \pi((1 \rhd t_2) \cdot v)
\]
\[
= (\varphi \cdot S^{-1}(t_4)t_1 \rhd 1)(1 \rhd S(t_2)) \cdot \pi((1 \rhd t_3) \cdot v)
\]

hence \( \tilde{\pi} \) is \( A \)-linear. The rest of the proof is identical to the one in [19], Theorem 7.4.2. \( \square \)

As a consequence of Theorem 2.61 and Theorem 3.2, we obtain the very well-known result (see [19], Proposition 7):

**Corollary 3.3** If \( H \) is a semisimple cosemisimple Hopf algebra, then \( D(H) \) is semisimple.

Let now \( H \) be a bialgebra, \( A \) an \( H \)-bimodule algebra and \( \mathbb{A} \) an \( H \)-bicomodule algebra in the usual (Hopf) sense. Let also \( A \) be an algebra in the Yetter-Drinfeld category \( H \mathcal{YD} \), that is \( A \) is a left \( H \)-module algebra, a left \( H \)-comodule algebra (with left \( H \)-comodule structure denoted by \( a \mapsto a_{(-1)} \otimes a_{(0)} \in H \otimes A \)) and the Yetter-Drinfeld compatibility condition holds:

\[
h_1 a_{(-1)} \otimes h_2 \cdot a_{(0)} = (h_1 \cdot a)_{(-1)} h_2 \otimes (h_1 \cdot a)_{(0)}, \quad \forall h \in H, \ a \in A. \tag{3.4}
\]

Consider first the generalized smash product \( A \rhd_A A \), an associative algebra. From the condition (3.2), it follows that \( A \rhd_A A \) becomes an \( H \)-bimodule algebra, with \( H \)-actions

\[
h \cdot (\varphi \rhd_A a) = h_1 \cdot \varphi \rhd_A h_2 \cdot a,
\]
\[
(\varphi \rhd_A a) \cdot h = \varphi \cdot h \rhd_A a,
\]

for all \( h \in H, \varphi \in A \) and \( a \in A \), hence we may consider the algebra \((A \rhd_A A) \rhd \mathbb{A}\). Then, consider the generalized smash product \( A \rhd_A A \), an associative algebra. Using the condition (3.4), one can see that \( A \rhd_A A \) becomes an \( H \)-bicoremodule algebra, with \( H \)-coactions

\[
\rho : A \rhd_A A \to (A \rhd_A A) \otimes H, \quad \rho(a \rhd_A u) = (a \rhd_A u_{(0)}) \otimes u_{(1)}),
\]
\[
\lambda : A \rhd_A A \to H \otimes (A \rhd_A A), \quad \lambda(a \rhd_A u) = a_{(-1)} u_{(-1)} \otimes (a_{(0)} \rhd_A u_{(0)}),
\]

for all \( a \in A \) and \( u \in \mathbb{A} \), hence we may consider the algebra \( A \rhd_A (A \rhd_A A) \).

**Proposition 3.4** We have an algebra isomorphism \((A \rhd_A A) \rhd \mathbb{A} \equiv A \rhd_A (A \rhd_A A)\), given by the trivial identification.
Proof. The multiplication in \((A\triangleright A)\triangleright A\) is:
\[
((\varphi\triangleright a)\triangleright u)((\varphi'\triangleright a')\triangleright u')
= ((\varphi\cdot u'_{<1>})(u_{[-1]}_{-1}\cdot (\varphi'\triangleright a'))\triangleright u'_{<0>}
= (\varphi \cdot u'_{<1>})(u_{[-1]}_{-1}\cdot \varphi'\triangleright u'_{<1>1}\cdot (\varphi'\triangleright u'_{<0>})
= ((\varphi \cdot u'_{<1>})(a_{[-1]}_{-1}u_{[-1]}_{-1}\cdot (\varphi'\triangleright u'_{<0>})
\]

hence the two multiplications coincide. \(\square\)

Corollary 3.5 If \(H, A, A'\) are as above and \(A'\) is a left \(H\)-module algebra, then we have an algebra isomorphism \((A'\triangleright A)\triangleright A' \equiv (A'\triangleright A)\triangleright A\), given by the trivial identification.

4 L-R-twisting data

We start by recalling the set-up used in [11], slightly modifying terminology. Let \((A, \mu)\) be a (unital) associative algebra. Assume that there exists a bialgebra \(H\) such that \(A\) is a left \(H\)-module algebra (denote by \(\pi : H \otimes A \to A\), \(\pi(h \otimes a) = h \cdot a\) the action), a left \(H\)-comodule algebra (denote by \(\psi : A \to H \otimes A\), \(\psi(a) = a_{[-1]} \otimes a_{(0)}\) the coaction) and the following compatibility condition holds:
\[
(h \cdot a)_{(-1)} \otimes (h \cdot a)_{(0)} = a_{(-1)} \otimes h \cdot a_{(0)}, \quad \forall \ h \in H, \ a \in A.
\] (4.1)

We call the triple \((H, \pi, \psi)\) a left twisting datum for \((A, \mu)\) (in [11] it is called a very strong left twisting datum). If we define a new multiplication on \(A\), by
\[
a \star b = a_{(0)}(a_{(-1)} \cdot b), \quad \forall \ a, b \in A,
\] (4.2)

then this multiplication defines a new algebra structure on \(A\), with the same unit. The product \(\star\) is called the left twisted product.

Example 4.1 Let \(H\) be a Hopf algebra with bijective antipode \(S\), \(A\) an \(H\)-bimodule algebra with actions \(h \otimes \varphi \mapsto h \cdot \varphi\) and \(\varphi \otimes h \mapsto \varphi \cdot h\) for all \(h \in H, \varphi \in A\), and \(\Lambda\) an \(H\)-comodule algebra with coactions \(u \mapsto u_{[-1]} \otimes u_{[0]}\), \(u \mapsto u_{<0>} \otimes u_{<1>}\) for all \(u \in \Lambda\), and denote also by \(u_{[-1]} \otimes u_{[0]} \otimes u_{(1)} := u_{<0>1} \otimes u_{<0>|0} \otimes u_{<1>} = u_{[-1]} \otimes u_{[0]} \otimes u_{(1)},\). Then \(A\) becomes a left \(H \otimes H^{op}\)-module algebra with action \(\pi : H \otimes H^{op} \otimes A \to A\), \(\pi(h \otimes h' \otimes \varphi) = h \cdot \varphi \cdot h'\), and \(\Lambda\) becomes a left \(H \otimes H^{op}\)-comodule algebra (see [11]) with coaction \(\psi : \Lambda \to (H \otimes H^{op}) \otimes \Lambda\), \(u \mapsto (u_{[-1]} \otimes S^{-1}(u_{(1)})) \otimes u_{[0]}\). It is easy to check that \((H \otimes H^{op}, \pi \otimes \text{id}, (\tau \otimes \text{id})(\text{id} \otimes \psi))\),
where \( \tau \) is the usual twist, is a left twisting datum for \( A \otimes \mathbb{A} \), and the corresponding twisted product is

\[
(\varphi \otimes u) \ast (\varphi' \otimes u') = \varphi(u_{(-1)} \cdot \varphi' \cdot S^{-1}(u_{(1)})) \otimes u_{(0)} u',
\]

and this is exactly the multiplication of the generalized diagonal crossed product \( A \bowtie \mathbb{A} \).

Now we introduce the L-R-version of the above construction. Let \((A, \mu)\) be an algebra and assume there exists a bialgebra \( H \) such that:

(i) \( A \) is an \( H \)-bimodule algebra with actions denoted by \( \pi_l : H \otimes A \to A \), \( \pi_l(h \otimes a) = h \cdot a \) and \( \pi_r : A \otimes H \to A \), \( \pi_r(a \otimes h) = a \cdot h \);

(ii) \( A \) is an \( H \)-bicomodule algebra, with coactions denoted by \( \psi_l : A \to H \otimes A \), \( a \mapsto a_{[-1]} \otimes a_{[0]} \) and \( \psi_r : A \to A \otimes H \), \( a \mapsto a_{<0>} \otimes a_{<1>} \);

(iii) The following compatibility conditions hold, for all \( h \in H \) and \( a, b \in A \):

\[
\begin{align*}
(h \cdot a)_{[-1]} \otimes (h \cdot a)_{[0]} &= a_{[-1]} \otimes h \cdot a_{[0]}, \\
(h \cdot a)_{<0>} \otimes (h \cdot a)_{<1>} &= h \cdot a_{<0>} \otimes a_{<1>}, \\
(a \cdot h)_{[-1]} \otimes (a \cdot h)_{[0]} &= a_{[-1]} \otimes a_{[0]} \cdot h, \\
(a \cdot h)_{<0>} \otimes (a \cdot h)_{<1>} &= a_{<0>} \cdot h \otimes a_{<1>}.
\end{align*}
\]

We call \((H, \pi_l, \pi_r, \psi_l, \psi_r)\) an L-R-twisting datum for \( A \). Given such a datum, we define a new multiplication on \( A \), by

\[
a \bullet b = (a_{[0]} \cdot b_{<1>})(a_{[-1]} \cdot b_{<0>}), \quad \forall \ a, b \in A,
\]

and call it the L-R-twisted product. Obviously it has the same unit \( 1 \) as \( A \). Note the following easy consequences of the axioms:

\[
\begin{align*}
h \cdot (a \bullet b) &= (h_1 \cdot a_{[0]} \cdot b_{<1>})(h_2 a_{[-1]} \cdot b_{<0>}), \\
(a \bullet b) \cdot h &= (a_{[0]} \cdot b_{<1>} h_1)(a_{[-1]} \cdot b_{<0>} \cdot h_2), \\
(a \bullet b)_{[-1]} \otimes (a \bullet b)_{[0]} &= a_{[-1]} b_{<0>_{[-1]}} \otimes (a_{[0]} \cdot b_{<1>})(a_{[-1]} \cdot b_{<0>_{[0]}}, \\
(a \bullet b)_{<0>} \otimes (a \bullet b)_{<1>} &= (a_{[0]} b_{<1>})(a_{[-1]} \cdot b_{<0>_{<0>}}) \otimes a_{<0>} b_{<0>_{<1>}}.
\end{align*}
\]

**Proposition 4.2** \((A, \bullet, 1)\) is an associative unital algebra.

**Proof.** We compute:

\[
\begin{align*}
(a \bullet b) \bullet c &= ((a \bullet b)_{[0]} \cdot c_{<1>})(a \bullet b)_{[-1]} \cdot c_{<0>} \\
&= (a_{[0]} b_{<1>} c_{<1>})(a_{[-1]} \cdot b_{<0>_{[0]} \cdot c_{<1>_{2}}})(a_{[0]} b_{<0>_{[-1]} \cdot c_{<0>}}), \\
(a \bullet (b \bullet c)) &= (a_{[0]} \cdot b_{<1>}(a_{[-1]} \cdot b_{<0>_{<0>}} \cdot c_{<1>_{2}}))(a_{[-1]} c_{<0>_{<0>}}), \\
&= (a_{[0]} b_{<1>_{<1>}} c_{<0>_{<0>}})(a_{[-1]} b_{<0>_{<0>}} \cdot c_{<1>_{<0>}})(a_{[-1]} b_{<0>_{<1>}} \cdot c_{<0>_{<0>}}),
\end{align*}
\]

and the two terms are equal because \( A \) is an \( H \)-bicomodule. \( \square \)

**Remark 4.3** An L-R-twisting datum is in particular a left twisting datum, but in general the corresponding twisted products \( \bullet \) and respectively \( \ast \) are different. On the other hand, any left twisting datum can be regarded as an L-R-twisting datum with trivial right action and coaction, and in this case the corresponding twisted products coincide.
As a particular case of an L-R-twisting datum, obtained if the left action and coaction are trivial, we obtain the following concept.

**Definition 4.4** Let \((A, \mu)\) be a (unital) associative algebra. Assume that there exists a bialgebra \(H\) such that \(A\) is a right \(H\)-module algebra (denote by \(\pi_r : A \otimes H \to A\), \(\pi_r(a \otimes h) = a \cdot h\) the action), a right \(H\)-comodule algebra (denote by \(\psi_r : A \to A \otimes H\), \(\psi_r(a) = a_{\cdot 0} \otimes a_{\cdot 1}\) the coaction) and the following compatibility condition holds, for all \(a \in A\) and \(h \in H\):

\[
(a \cdot h)_{\cdot 0} \otimes (a \cdot h)_{\cdot 1} = a_{\cdot 0} \cdot h \otimes a_{\cdot 1}.
\]

We call the triple \((H, \pi_r, \psi_r)\) a right twisting datum for \((A, \mu)\). If we define a new multiplication on \(A\), by

\[
a \circ b = (a \cdot b_{\cdot 1}) b_{\cdot 0}, \quad \forall \ a, b \in A,
\]

then this multiplication defines a new algebra structure on \(A\), with the same unit. The product \(\circ\) is called the right twisted product.

**Proposition 4.5** Let \(A\) be an algebra and \((H, \pi_l, \pi_r, \psi_l, \psi_r)\) an L-R-twisting datum for \(A\), with notation as before. Then the L-R-twisted product \(\star\) can be obtained as a left twisting followed by a right twisting and also viceversa.

**Proof.** First consider the left twisted product algebra \((A, \star)\); it is easy to see that \((H, \pi_r, \psi_r)\) is a right twisting datum for \((A, \star)\), and the corresponding right twisted product becomes:

\[
a \circ b = (a \cdot b_{\cdot 1}) \star b_{\cdot 0} = (a \cdot b_{\cdot 1})_{[0]}((a \cdot b_{\cdot 1})_{[-1]} \cdot b_{\cdot 0}) = (a_{[0]} \cdot b_{\cdot 1})_{[-1]}(a_{[-1]} \cdot b_{\cdot 0}) = a \cdot b.
\]

Similarly, one can start with the right twisted product algebra \((A, \diamond)\), for which \((H, \pi_l, \psi_l)\) is a left twisting datum, and the corresponding left twisted product coincides with the L-R-twisted product.

**Example 4.6** Let \(H\) be a bialgebra, \(A\) an \(H\)-bimodule algebra with actions \(h \otimes \varphi \mapsto h \cdot \varphi\) and \(\varphi \otimes h \mapsto \varphi \cdot h\) for all \(h \in H\), \(\varphi \in A\), and \(\mathbb{A}\) an \(H\)-bicomodule algebra with coactions \(u \mapsto u_{[-1]} \otimes u_{[0]}\), \(u \mapsto u_{\cdot 0} \otimes u_{\cdot 1}\) for all \(u \in \mathbb{A}\). Take the algebra \(A = \mathcal{A} \otimes \mathbb{A}\), which becomes an \(H\)-bimodule algebra with actions \(h \cdot (\varphi \otimes u) = h \cdot \varphi \otimes u\) and \((\varphi \otimes u) \cdot h = \varphi \cdot h \otimes u\), for all \(h \in H\), \(\varphi \in A\), \(u \in \mathbb{A}\), and an \(H\)-bicomodule algebra, with coactions \(\varphi \otimes u \mapsto u_{[-1]} \otimes (\varphi \otimes u_{[0]}),\)

\[
\varphi \otimes u \mapsto (\varphi \otimes u_{\cdot 0}) \otimes u_{\cdot 1}.
\]

Moreover, one checks that the conditions (4.13)-(4.16) are satisfied, hence we have an L-R-twisting datum for \(A \otimes \mathbb{A}\). The corresponding L-R-twisted product is:

\[
(\varphi \otimes u) \circ (\varphi' \otimes u') = ((\varphi \otimes u)_{[0]} \cdot (\varphi' \otimes u')_{\cdot 1})(\varphi \otimes u)_{[-1]} \cdot (\varphi' \otimes u')_{\cdot 0}) = ((\varphi \otimes u_{[0]}) \cdot u'_{\cdot 1})(u_{[-1]} \cdot (\varphi' \otimes u')_{\cdot 0}) = (\varphi \cdot u'_{\cdot 1})(u_{[-1]} \cdot \varphi') \otimes u_{[0]} u'_{\cdot 0},
\]

and this is exactly the multiplication of the L-R-smash product \(A \bowtie \mathbb{A}\).

If \(H\) is a Hopf algebra with bijective antipode \(S\), \(A\) is an \(H\)-bimodule algebra and \(\mathbb{A}\) an \(H\)-bicomodule algebra, we have proved in Section 2 that \(A \bowtie \mathbb{A} \simeq A \triangleright \bowtie \mathbb{A}\) as algebras. We derive now two interpretations of this result at the level of twisting data and twisted products.
Theorem 4.7  With notation as above, let $A$ be an algebra and $(H, \pi_l, \pi_r, \psi_l, \psi_r)$ an $L$-$R$-twisting datum for $A$, $H$ being a Hopf algebra with bijective antipode $S$. If we denote by $\pi$ respectively $\psi$ the left $H \otimes H^{op}$-module algebra respectively comodule algebra structures on $A$ defined as in Example 4.7, then $(H \otimes H^{op}, \pi, \psi)$ is a left twisting datum for $A$. Moreover, the corresponding twisted algebras $(A, \bullet)$ and $(A, \star)$ are isomorphic, and the isomorphism is defined by:

$$\lambda : (A, \bullet) \rightarrow (A, \star), \quad \lambda(a) = a_{<0>} \cdot S^{-1}(a_{<1>}), \quad \forall \ a \in A,$$

$$\lambda^{-1} : (A, \star) \rightarrow (A, \bullet), \quad \lambda^{-1}(a) = a_{<0>} \cdot a_{<1>}, \quad \forall \ a \in A.$$  

In particular, for $A = A \otimes \mathbb{A}$, we obtain $A \uparrow \mathbb{A} \cong A \bowtie \mathbb{A}$.  

Proof. To prove that $(H \otimes H^{op}, \pi, \psi)$ is a left twisting datum for $A$, one has to check (4.1) and this follows using (4.3)-(4.6). We only prove that $\lambda$ is an algebra isomorphism. The fact that $\lambda^{-1} = \lambda^{-1} \lambda = id$ follows easily using (4.6); obviously $\lambda(1) = 1$, hence we only have to check that $\lambda$ is multiplicative:

$$\lambda(a \cdot b) = (a \cdot b)_{<0>} \cdot S^{-1}((a \cdot b)_{<1>})$$

$$= [(a[0]_{<0>} \cdot b_{<1>})(a[-1] \cdot b_{<0> <1>})] \cdot S^{-1}(a[0]_{<1>} b_{<0> <1>})$$

$$= [a[0]_{<0>} \cdot b_{<1>} S^{-1}(b_{<0> <1>}) S^{-1}(a[0]_{<1>})]$$

$$= [a[0]_{<0>} \cdot b_{<1>} S^{-1}(a[0]_{<1>})]$$

$$= [a[0]_{<0>} \cdot S^{-1}(a[0]_{<1>})]$$

$$= (a_{<0>} \cdot S^{-1}(a_{<1>})) \cdot (a[0]_{<0>} \cdot S^{-1}(a_{<1>})$$

$$= (a_{<0>} \cdot S^{-1}(a_{<1>})) \cdot (a[0]_{<0>} \cdot S^{-1}(a_{<1>})$$

$$= (a_{<0>} \cdot S^{-1}(a_{<1>})) \cdot (a[0]_{<0>} \cdot S^{-1}(a_{<1>})$$

$$= (a_{<0>} \cdot S^{-1}(a_{<1>})) \cdot (a[0]_{<0>} \cdot S^{-1}(a_{<1>})$$

$$= (a_{<0>} \cdot S^{-1}(a_{<1>})) \cdot (a[0]_{<0>} \cdot S^{-1}(a_{<1>})$$

and the proof is finished.  

The second interpretation is inspired by the following two facts. First, by Proposition 4.5 one can see that $A \uparrow \mathbb{A}$ may be written as a right twisting of the generalized smash product $A \bowtie \mathbb{A}$. Second, we have the observation in [10] that the Drinfeld double can be obtained as a two step twisting procedure, where at the first step the smash product is obtained; this can be extended to a generalized diagonal crossed product $A \bowtie \mathbb{A}$, thus obtaining $A \bowtie \mathbb{A}$ as a left twisting of $A \uparrow \mathbb{A}$. Hence, our general result looks as follows:

Theorem 4.8  Let $A$ be an algebra and $(H, \pi_l, \pi_r)$ a right twisting datum for $A$, with notation as above, where $H$ is a Hopf algebra with bijective antipode. Define $\pi_l : H^{op} \otimes A \rightarrow A$, $\pi_l(h \otimes a) = a \cdot h$.
and \( \psi : A \to H^{\text{op}} \otimes A, a \mapsto S^{-1}(a_{<1>}) \otimes a_{<0>} \). Then \( (H^{\text{op}}, \pi, \psi) \) is a left twisting datum for \( A \), and the corresponding twisted algebras \( (A, \circ) \) and \( (A, \star) \) are isomorphic, via the maps:

\[
\lambda : (A, \circ) \to (A, \star), \quad \lambda(a) = a_{<0>} \cdot S^{-1}(a_{<1>}), \quad \forall \ a \in A,
\]

\[
\lambda^{-1} : (A, \star) \to (A, \circ), \quad \lambda^{-1}(a) = a_{<0>} \cdot a_{<1>}, \quad \forall \ a \in A.
\]

In particular, for \( A = A \rhd_h \mathbb{A} \), we obtain \( A \rhd_h \mathbb{A} \simeq A \rtimes_h \mathbb{A} \).

\textbf{Proof.} Similar to the proof of Theorem 4.7. \( \square \)

We end this section with a partial answer to the following natural question. Suppose that \( A \) is an algebra and \( (H, \pi, \psi) \) is a left twisting datum for \( A \); then how far is the twisted algebra \( (A, \star) \) from being isomorphic to a generalized smash product?

\textbf{Proposition 4.9} Let \( A \) and \( (H, \pi, \psi) \) be as above. Define the algebras \( A^H = \{a \in A / h \cdot a = \varepsilon(h)a, \ \forall \ h \in H\} \) and \( A^\text{co}(H) = \{a \in A / a_{(-1)} \otimes a_{(0)} = 1 \otimes a\} \). Then \( A^H \) is a left \( H \)-comodule algebra and \( A^\text{co}(H) \) is a left \( H \)-module algebra. If moreover we have that \( ab = ba \) for all \( a \in A^\text{co}(H) \) and \( b \in A^H \), then the map \( \lambda : A^\text{co}(H) \rhd_h A^H \to (A, \star) \), \( \lambda(a \otimes b) = ab \), is an algebra map.

\textbf{Proof.} The assertions concerning \( A^H \) and \( A^\text{co}(H) \) follow easily from (4.11). Assume now that

\[
ab = ba, \quad \forall \ a \in A^\text{co}(H), \ b \in A^H. \tag{4.13}
\]

Then we compute, for all \( a, a' \in A^\text{co}(H) \) and \( b, b' \in A^H \):

\[
\lambda(a \rhd_h b) \star \lambda(a' \rhd_h b') = (ab) \star (a'b')
\]

\[
= a_{(0)}b_{(0)}(a_{(-1)}b_{(-1)} \cdot (a'b'))
\]

\[
= ab_{(0)}(b_{(-1)} \cdot (a'b')) \quad (\text{since } a \in A^\text{co}(H))
\]

\[
= ab_{(0)}(b_{(-1)} \cdot a')(b_{(-1)}b_{(1)})b'
\]

\[
= ab_{(0)}(b_{(-1)} \cdot a')b_{(1)}b' \quad (\text{since } b' \in A^H)
\]

\[
= a(b_{(-1)} \cdot a')b_{(1)}b' \quad (\text{by (4.13)})
\]

\[
= \lambda(a(b_{(-1)} \cdot a') \rhd_h b_{(1)}b')
\]

\[
= \lambda((a \rhd_h b)(a' \rhd_h b')),
\]

hence \( \lambda \) is an algebra map. \( \square \)

\section{5 L-R-smash coproduct over bialgebras}

Throughout this section, \( H \) will be a given bialgebra. We introduce the L-R-smash coproduct \( C \rhd_h H \), dualizing the L-R-smash product \( A \rhd_h H \) and generalizing Molnar’s smash coproduct.

Let \( C \) be an \( H \)-bicomodule coalgebra, that is:

\( \text{(i) } C \) is an \( H \)-bicomodule, with structures

\[
\rho : C \to C \otimes H, \quad \rho(c) = c_{<0>} \otimes c_{<1>},
\]

\[
\lambda : C \to H \otimes C, \quad \lambda(c) = c_{(-1)} \otimes c_{(0)},
\]

\[...\]
for all \( c \in C \). We record the (bi) comodule conditions:

\[
c^{(-1)} \otimes c^{(0)(-1)} \otimes c^{(0)(0)} = (c^{(-1)})_1 \otimes (c^{(-1)})_2 \otimes c^{(0)}, \tag{5.1}
\]
\[
c^{<0><0> \otimes c^{<0><1>} \otimes c^{<1>}} = c^{<0>} \otimes (c^{<1>})_1 \otimes (c^{<1>})_2, \tag{5.2}
\]
\[
c^{<0>(-1) \otimes c^{<0>(0) \otimes c^{<1>}} = (c^{(-1)}) \otimes c^{(0) <0> \otimes c^{(0)<1>}}. \tag{5.3}
\]

(ii) \( C \) is a coalgebra, with comultiplication \( \Delta_C : C \to C \otimes C \), \( \Delta_C(c) = c_1 \otimes c_2 \), and counit \( \varepsilon_C : C \to k \).

(iii) \( C \) is a left \( H \)-comodule coalgebra, that is, for all \( c \in C \):

\[
c_1^{(-1)} c_2^{(-1)} \otimes c_1^{(0)} \otimes c_2^{(0)} = c^{(-1)} \otimes (c^{(0)})_1 \otimes (c^{(0)})_2, \tag{5.4}
\]
\[
c^{(-1)} \varepsilon_C(c^{(0)}) = \varepsilon_C(c) 1_H. \tag{5.5}
\]

(iv) \( C \) is a right \( H \)-comodule coalgebra, that is, for all \( c \in C \):

\[
c_1^{<0>} \otimes c_2^{<0> \otimes c_1^{<1>}} \otimes c_2^{<1> = (c^{<0>})_1 \otimes (c^{<0>})_2 \otimes c^{<1>}, \tag{5.6}
\]
\[
\varepsilon_C(c^{<0>} c^{<1>}) = \varepsilon_C(c) 1_H. \tag{5.7}
\]

We denote \( C \otimes H \) by \( C \uparrow\downarrow H \) and elements \( c \otimes h \) by \( c \uparrow h \). Define the maps

\[
\Delta : C \uparrow\downarrow H \to (C \uparrow\downarrow H) \otimes (C \uparrow\downarrow H), \quad \varepsilon : C \uparrow\downarrow H \to k,
\]
\[
\Delta(c \uparrow h) = (c_1^{<0>} \uparrow c_2^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>}), \quad \varepsilon(c \uparrow h) = \varepsilon_C(c) \varepsilon_H(h).
\]

**Proposition 5.1** \((C \uparrow\downarrow H, \Delta, \varepsilon)\) is a coalgebra, called the \( L-R \)-smash coproduct.

**Proof.** The counit axiom is immediate, so we check coassociativity. We compute:

\[
(id \otimes \Delta)(\Delta(c \uparrow h)) = (c_1^{<0>} \uparrow c_2^{(-1)} h_1) \otimes \Delta(c_2^{<0>} \uparrow h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{(-1)} h_1) \otimes \Delta((c_2^{<0>})_1 \uparrow (c_2^{<0>})_2 \uparrow h_2 c_1^{<1>})
\]
\[
= (c_1^{<0>} \uparrow c_2^{(-1)} c_3^{(-1)} h_1) \otimes ((c_2^{<0>})_1 \uparrow (c_2^{<0>})_2 \uparrow h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{(-1)} c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>})
\]
\[
= (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>})
\]
\[
= (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>})
\]
\[
= (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>})
\]
\[
= (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>}) = (c_1^{<0>} \uparrow c_2^{<0>} \uparrow c_3^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow c_3^{(-1)} h_2 c_1^{<1>})
\]
\[
= \Delta(c_1^{<0>} \uparrow c_2^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow h_2 c_1^{<1>}) = \Delta(c_1^{<0>} \uparrow c_2^{(-1)} h_1) \otimes (c_2^{<0>} \uparrow h_2 c_1^{<1>})
\]

finishing the proof. \( \square \)
Remark 5.2 Obviously, if the right $H$-comodule structure of $C$ is trivial (i.e. $c^{<0>} \otimes c^{<1>} = c \otimes 1$ for all $c \in C$), then $C \rtimes H$ coincides with Molnar’s smash coproduct from [17].

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