PRENILPOTENT PAIRS IN $E_{10}$

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Abstract. Tits has defined Kac–Moody groups for all root systems, over all commutative rings. A central concept is the idea of a prenilpotent pair of (real) roots. In particular, writing down his definition explicitly requires knowing all the Weyl-group orbits of such pairs. We show that for the hyperbolic root system $E_{10}$ there are so many orbits that any attempt at direct enumeration is futile. Namely, the number of orbits of prenilpotent pairs having inner product $k$ grows at least as fast as (constant)$k^7$ as $k \to \infty$.

Our purpose is to motivate alternate approaches to Tits’ groups.

Kac–Moody groups generalize reductive Lie groups to include the infinite dimensional case. Various authors have defined them in many ways, the most comprehensive approach being due to Tits [13]. Given a generalized Cartan matrix $A$, he defined a functor $\tilde{G}_A$ assigning a group to each commutative ring $R$. The main result of [13] is that any functor from commutative rings to groups, having some properties that are reasonable to expect of a “Kac–Moody group”, must agree with $\tilde{G}_A$ over every field. (See [13, Theorems 1 and 1’].)

(Tits defined a group functor $\tilde{G}_D$ for every root datum $D$. For $\tilde{G}_A$ we use the root datum with generalized Cartan matrix $A$, which is “simply connected in the strong sense” [13, p. 551]. The difference between a root datum and its generalized Cartan matrix plays no role in this paper.)

Tits defined $\tilde{G}_A(R)$ by a complicated implicitly described presentation. The key relations are his generalizations of the Chevalley relations. He begins with the free product $*_{\alpha}(U_\alpha \cong R)$, where $\alpha$ varies over all real roots. He imposes relations of the form

$$[X_\alpha(t), X_\beta(u)] = \prod_\gamma X_\gamma(v_\gamma)$$

whenever $\alpha, \beta \in \Phi$ form a prenilpotent pair. Here $U_\alpha = \{X_\alpha(t) : t \in R\}$ and similarly for the other roots, the $\gamma$’s parameterizing the product are

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the real roots in $N\alpha + N\beta$ other than $\alpha$ and $\beta$, and the parameters $v_\gamma$ depend on various choices (and anyway are unimportant in this paper).

The definition of prenilpotency is that some element of the Weyl group $W$ sends both $\alpha, \beta$ to positive roots, and some other element sends both to negative roots. When this holds, Prop. 1 of [13] and its proof show how to work out the Chevalley relation of $\alpha, \beta$, at least in principle. It is akin to working out structure constants of Kac–Moody algebras. (In fact Héé has worked out all the possible types of the relations in closed form [7].) So the essence of writing down Tits’ presentation is to list all the prenilpotent pairs. It would even be enough to find one representative of each $W$-orbit of prenilpotent pairs. Our main result, theorem 1 below, is that this is impossible in practice for the $E_{10}$ root system. The argument suggests that the same holds for all irreducible root systems of rank $> 3$ (except the spherical and affine ones; see section 2).

This negative result is balanced by the fact that in many interesting cases, including $E_{10}$, most of the prenilpotent pairs can be ignored because their Chevalley relations follow from those of other prenilpotent pairs. There are two approaches to this. The first is due to Abramenko and Mühlherr [1][11][5], and applies to Kac–Moody groups associated to 2-spherical Dynkin diagrams, over fields, with some exceptions over $F_2$ and $F_3$. The second approach is due to the author, partly in joint work with Carbone [2][3][4]. It works over general rings, but requires stronger hypotheses on the diagram. Both approaches apply to affine diagrams (of rank $\geq 3$) and all simply laced hyperbolic diagrams, including $E_{10}$. In both cases the result is that $\tilde{G}_A(R)$ is the direct limit of the family of groups $\tilde{G}_B(R)$, where $B$ varies over the 1- and 2-node subdiagrams of $A$. So one may neglect Tits’ Chevalley-style relations for prenilpotent pairs that don’t appear in the root systems of the various $B$’s. In the author’s approach, one even obtains an explicit presentation (often finite) given in terms of the Dynkin diagram, for example $\tilde{G}_{E_{10}}(R)$ and $\tilde{G}_{E_{10}}(Z)$ in theorem 1 and corollary 2 of [4].

Now we begin the $E_{10}$-specific material. Although some properties of $E_{10}$ simplify the analysis, notably its unimodularity, similar ideas should apply to other hyperbolic root systems (see section 2). The $E_{10}$ Dynkin diagram is

![Dynkin diagram for E10]

Its corresponding generalized Cartan matrix is symmetric, so it may be regarded as an inner product matrix on the root lattice $\Lambda$. The Weyl group $W$ acts on $\Lambda$ by isometries. We will never refer to imaginary
roots, so we follow Tits \[\] in using “root” to mean “real root”, i.e., “$W$-image of a simple root”. Now we can state our main result:

**Theorem 1.** Let $N(k)$ be the number of $W$-orbits of prenilpotent pairs of roots in the $E_{10}$ root system with inner product $k$. Then for some positive constant $C$ we have $N(k) \geq Ck^7$.

The constant is made effective in the proof. The theorem says nothing if $k \leq 0$, but this case is uninteresting because there are no prenilpotent pairs with $k < -1$, by lemma 3 below. Finally, although polynomial growth is generally considered “tame” in algorithmic settings, the proof shows that the problem of enumerating the prenilpotent pairs contains an infinite sequence of successively more difficult and less interesting problems in the classification theory of positive-definite quadratic forms. Such enumerations, for example “all lattices of dimension 8 and determinant $N$”, become uninteresting quite quickly. Thus our description of the direct enumeration of prenilpotent pairs as “futile”.

1. **Proof**

In this section we prove theorem \[\] by converting it into a lattice-theoretic problem. First we need to describe the roots and prenilpotent pairs entirely in terms of the root lattice $\Lambda$.

**Lemma 2.** The roots of the $E_{10}$ root system are exactly the norm 2 vectors of $\Lambda$.

**Proof.** The simple roots have norm 2 because every generalized Cartan matrix has 2’s along its diagonal. The other roots are their $W$-images and therefore have norm 2 also. Now suppose a lattice vector $r$ has norm 2. The reflection in $r$, namely $R : x \mapsto x - (x \cdot r)r$, preserves $\Lambda$ because $x \cdot r \in \mathbb{Z}$ for all lattice vectors $x$. Also, $\Lambda$ has signature $(9, 1)$, so the negative-norm vectors in $\Lambda \otimes \mathbb{R}$ fall into two components. Since $r^2 > 0$, $R$ preserves each component. Vinberg \[\] showed that $W$ is the full group of lattice isometries that preserve each component, so $R \in W$. Since every reflection in $W$ is conjugate to a simple reflection, $r$ is $W$-equivalent to a simple root. So $r$ is a root.

**Lemma 3 (\[\] Lemma 6).** Two roots in the $E_{10}$ root system form a prenilpotent pair if and only if their inner product is $\geq -1$. \[\]

At this point the proof of theorem \[\] becomes entirely number-theoretic, relying on the theory of integer quadratic forms to study certain sublattices of $\Lambda$. In the rest of this section, “$E_{10}$” will be an alternate notation for $\Lambda$. We fix $k \geq -1$ and consider prenilpotent pairs with
inner product $k$. We write $L$ for the integer span of such a prenilpotent pair; its inner product matrix is $\begin{pmatrix} 2 & k \\ k & 2 \end{pmatrix}$. The next lemma follows immediately from the previous two.

**Lemma 4.** $N(k)$ equals the number of orbits of isometric embeddings $L \to E_{10}$, under the group $\mathbb{Z}/2 \times W$, where $\mathbb{Z}/2$ acts on $L$ by swapping its basis vectors and $W$ acts on $E_{10}$ in the obvious way. In particular, $N(k)$ is at least as large as the number of $O(E_{10})$-orbits of isometric copies of $L$ in $E_{10}$. \hfill \Box$

There is a general method called gluing for studying the embeddings of one lattice into another. In the current situation, one first studies the possibilities for the saturation $L^{\text{sat}} := (L \otimes \mathbb{Q}) \cap E_{10}$. (In the proof below we will restrict to the case that $L$ is already saturated.) Then, assuming $\det L \neq 0$, one studies the possibilities for $L^\perp$. In this step we take advantage of the fact that $E_{10}$ is unimodular: among other things, it implies that $L^{\text{sat}}$ and $L^\perp$ have the same determinant. For each candidate $K$ for $L^\perp$, one then considers the possible ways to glue $K$ to $L^{\text{sat}}$ in a manner that yields $E_{10}$. Gluing means finding a copy of $E_{10}$ between $K \oplus L^{\text{sat}}$ and $K^\ast \oplus (L^{\text{sat}})^\ast$, in which $K$ and $L^{\text{sat}}$ are saturated. (Asterisks indicates dual lattices.) This step boils down to analyzing the actions of $O(K)$ and $O(L)$ on the discriminant groups of $K$ and $L$ (which are finite).

Here is a review of the necessary definitions and background. A lattice $K$ means a free abelian group equipped with a $\mathbb{Z}$-valued symmetric bilinear pairing. The norm of a vector means its inner product with itself. The determinant of the inner product matrix of $K$, with respect to a basis, is independent of that basis, and is called the determinant $\det K$ of $K$. We will encounter only nondegenerate lattices, meaning those of nonzero determinant, so we assume nondegeneracy henceforth. The dual $K^\ast$ of $K$ means the set of vectors in $K \otimes \mathbb{Q}$ having integer inner product with all elements of $K$. The discriminant group $\Delta(K)$ means $K^\ast/K$, a group of order $\det K$. The inner products of elements of $\Delta(K)$ are well-defined elements of $\mathbb{Q}/\mathbb{Z}$. $K$ is called even if all vectors have even norm; all lattices we will meet have this property. If $K$ is even then the norms of elements of $\Delta(K)$ are well-defined elements of $\mathbb{Q}/2\mathbb{Z}$.

The formulation of the theory of integer quadratic forms best suited for explicit computation is due to Conway and Sloane [8, ch. 15][10]. So we follow their conventions, including the unusual one of writing $-1$ for the infinite place of $\mathbb{Q}$ and defining $\mathbb{Z}_{-1}$ and $\mathbb{Q}_{-1}$ to be $\mathbb{R}$. For any place $p$ of $\mathbb{Q}$ we write $K_p$ for the $p$-adic lattice $K \otimes \mathbb{Z}_p$. Conway and Sloane gave an elaborate notational system for isometry classes
of \( p \)-adic lattices, for example the symbols appearing in lemma [3.1(ii)] below. We refer to [8, §7 of ch. 15] for their meaning, mentioning only that each superscript consists of a sign and a nonnegative number, considered as separate objects, rather than a signed number. The sign is usually suppressed when it is +. For \( p = 2 \) there are also subscripts, which can be integers modulo 8 or the formal symbol \( \Pi \). The Sylow \( p \)-subgroup of \( \Delta(K) \), with its norm form, is the same as the discriminant group of \( K_p \).

Two lattices \( K, K' \) are said to lie in the same genus if \( K \sim K' \) for all places \( p \). In the positive definite case, the mass of a genus means \( \sum_K 1/|O(K)| \), where \( K \) varies over the isometry classes in it. This definition makes sense because a genus contains only finitely many isometry classes. It is useful because the Smith–Minkowski–Siegel mass formula computes the mass without having to first enumerate the isometry classes, and the mass is a lower bound for the number of isometry classes.

Now we turn to the specific problem of enumerating the embeddings \( L \rightarrow E_{10} \). In fact we will only estimate the number of saturated embeddings, meaning those with \( L^{\text{sat}} = L \). We take \( k \geq 3 \) to make \( L \) indefinite, avoiding the special cases \( k = -1, 0, 1, 2 \). We factor \( d := -\det L = k^2 - 4 > 0 \) as \( 2^e_23^e_35^e_5 \cdots \) and write \( f_p \) for the non-\( p \)-part \( d/p^{e_p} \) of \( d \).

**Lemma 5.** With \( L, d, e_p \) and \( f_p \) as above,

(i) \( e_2 \) is 0, 2 or \( \geq 5 \).

(ii) There exists a genus of 8-dimensional positive-definite lattices \( K \) of determinant \( d \), such that

\[
K_2 \cong \begin{cases} \II^{-8} & \text{if } e_2 = 0 \\ 1^{16} \cdot 2^{(f_2)/2} & \text{if } e_2 = 2 \\ 1^6 \cdot 2^{(f_2)/2} (2^{e_2-1})^{(f_2)/2} & \text{if } e_2 \geq 5 \end{cases}
\]

\[
K_p \cong \begin{cases} 1^{(f_p)/8} & \text{if } p > 2 \text{ and } e_p = 0 \\ 1^{(2)/7} (p^{f_p})^{(2f_2)/f_2} & \text{if } p > 2 \text{ and } e_p > 0 \end{cases}
\]

(iii) Suppose \( K \) lies in this genus. Then there are at least

\[
\frac{2 \text{ number of odd primes dividing } d}{4 |O(K)|}
\]

\( O(E_{10}) \)-orbits on the saturated sublattices of \( E_{10} \) that are isometric to \( L \) and have orthogonal complement isometric to \( K \).
(iv) The mass of this genus is

\[
\frac{d^{7/2} \zeta_d(4)}{30240\pi^4 \cdot 2 \text{ number of odd primes dividing } d} \cdot \begin{cases} 
\frac{1}{230} & \text{if } e_2 = 0 \\
\frac{1}{512} & \text{if } e_2 = 2 \\
\frac{1}{1024} & \text{if } e_2 \geq 5 
\end{cases}
\]

As mentioned above, we are using the Conway–Sloane notation for \(p\)-adic lattices. The expressions \((\cdot \overline{\cdot})\) are Kronecker symbols. These are the Legendre symbols when \(p\) is odd, and \((\frac{f}{2})\) is defined as + or − according to whether \(f^2 \equiv \pm 1 \text{ or } \pm 3 \mod 8\). The \(L\)-function \(\zeta\) in (1) is defined as in [10, §7] by

\[
\zeta_d(s) = \prod_{\text{primes } p \mid 2d} \left(1 - \left(\frac{d}{p}\right) \frac{1}{p^s}\right)^{-1} = \sum_{m \geq 1 \atop (m, 2d) = 1} \left(\frac{d}{m}\right) m^{-s}.
\]

Here \((\frac{d}{m})\) is again a Kronecker symbol. Its evaluation can be reduced to the case of prime \(m\) by using the multiplicativity of the symbol in its upper and lower terms separately.

**Proof.** (i) If \(k\) is odd then so is \(d = k^2 - 4\), so \(e_2 = 0\). If \(k\) is divisible by 4 then \(d\) is divisible by 4 but not 8, so \(e_2 = 2\). If \(k\) is twice an odd number then \(d = 4(\text{odd}^2 - 1)\) and the second factor is divisible by 8.

As preparation for (ii) and (iii), we give the \(p\)-adic invariants of \(L\). Its determinant and signature are \(-d\) and 0, and

\[
L_2 \cong \begin{cases} 
\frac{1}{2^2} & \text{if } e_2 = 0 \\
2 \left(\frac{-2}{1-f_2}\right)^2 & \text{if } e_2 = 2 \\
2 \left(2^{e_2-1} \left(\frac{-f_2}{1-f_2}\right)^1\right) & \text{if } e_2 \geq 5
\end{cases}
\]

\[
L_p \cong \begin{cases} 
1 \left(\frac{-p}{2}\right)^2 & \text{if } p > 2 \text{ and } e_p = 0 \\
1 \left(\frac{p}{2}\right)^1 \left(\frac{-p}{p^r}\right)^1 & \text{if } p > 2 \text{ and } e_p > 0
\end{cases}
\]

These can be worked out explicitly using the methods of [8, ch. 15]. (It helps to observe that if \(k\) is even then \(L \cong \langle 2 \rangle \oplus \langle -d/2 \rangle\). And while \(L\) does not satisfy this if \(k\) is odd, \(L_p\) does if \(p\) is also odd.) Defining \(L^\text{neg}\) as \(L\) with all inner products negated, its local forms \(L_{p^k-1}^\text{neg}\) are as follows. If \(e_p = 0\) then \(L_{p^k}^\text{neg}\) is isometric to \(L_p\), while if \(e_p > 0\) then its Conway–Sloane symbol is got from that of \(L_p\) by negating subscripts (if \(p = 2\)) or multiplying each superscript by \(\left(\frac{-1}{p}\right)\) (if \(p > 2\)).

(ii) Following [8, §7.7 of ch. 15], there exists a \(Z\)-lattice \(K\) of determinant \(d\), having specified local forms \(K_{p^k-1, 2, 3, \ldots}\), if and only if both the
following hold. First, $\det K_p = d \cdot (\mathbb{Q}_p^\times)^2$ for all places $p$, and second, the oddity formula holds:

$$\text{signature}(K_{-1}) + \sum_{p \geq 3} p\text{-excess}(K_p) \equiv \text{oddity}(K_2) \pmod{8}.$$  

The determinant condition is easy to check directly. The oddity formula can be verified as follows. We constructed the local forms of $K$ as $K_2 = 1_6 \oplus L_2^\neg$ and $K_p = 1_{\frac{d}{p}} \oplus L_p^\neg$ for $p > 2$. We observe that $K_{-1}$ and $L_{-1}$ have the same signature mod 8, that the 2-adic lattice $1_6$ has oddity 0, and that for $p > 2$ the $p$-adic lattice $1_{\frac{d}{p}}$ has $p$-excess equal to 0. Since $L^\neg$ exists, its local forms $L_p^\neg$ satisfy the oddity formula. Since the corresponding formula for the $K_p$ has the same terms, it also holds. So $K$ exists.

In the language of [8, §3 of ch. 4], this is the question of how one may glue $L$ to $K$ to obtain $E_{10}$. Here are the details. Since $L$ is an even lattice, the norms of elements of $\Delta(L)$ are well-defined modulo 2. By the description of $K$ in the proof of (ii) $\Delta(K)$ is isometric to $\Delta(L)$ with all norms negated. So there are totally isotropic subgroups $G$ (for “graph”) of $\Delta(L) \oplus \Delta(K)$ which project isomorphically to each summand. The preimage of $G$ in $L^* \oplus K^*$ is integral and even (by $G$’s isotropy) and unimodular (since it contains $L \oplus K$ of index $d$ and $\det(L \oplus K) = -d^2$). By Theorem 5 of [12, §V.2], there is only one even unimodular lattice of signature $(9, 1)$, namely $E_{10}$. So we have an embedding $L \to E_{10}$ with orthogonal complement $K$. This embedding is saturated because $G \cap (\Delta(L) \times \{0\}) = \{0\}$.

In fact this yields an embedding for every one of the $|O(\Delta(L))|$ many possibilities for $G$. This orthogonal group has order at least 2 to the power of the number of odd primes dividing $d$, since for each odd prime $p|d$ we may negate just the $p$-part of $\Delta(L)$. Furthermore, the inclusions $L \to E_{10}$ associated to two such subgroups $G, G'$ are isometric if and only if $G$ and $G'$ are equivalent under the action of $O(L) \times O(K)$ on $\Delta(L) \oplus \Delta(K)$. The desired result (iii) now follows from the claim: $O(L)$’s action on $\Delta(L)$ factors through a group of order $\leq 4$. To prove the claim, let $r, r'$ be a pair of simple roots for the subgroup of $O(L)$ generated by reflections in norm 2 roots. Then $O(L)$ is generated by negation, the reflections in $r, r'$, and an isometry exchanging $r, r'$ (if one exists). In particular, the subgroup generated by the two reflections has index $\leq 4$. Then one checks directly that reflection in a norm 2 root acts trivially on $\Delta(L)$.

The mass $m(K)$ can be worked out by following the intricate but explicit procedure in [10]. Here are the details, using the language

\[ \text{signature}(K_{-1}) + \sum_{p \geq 3} p\text{-excess}(K_p) \equiv \text{oddity}(K_2) \pmod{8}. \]
introduced there. First, following [10, §7],
\[
m(K) = \text{std}(K) \prod_{\text{primes } p|2d} \frac{m_p(K)}{\text{std}_p(K)}
\]
where \text{std}(K) is given in [10, table 3] as \(\zeta_D(4)/30240\pi^4\), the definition of \(D\) is \((-1)^{d_4}d = d\), and for \(p|2d\) we have
\[
\text{std}_p(K) := \frac{1}{2(1 - p^{-2})(1 - p^{-4})(1 - p^{-6})}.
\]

For computing \(m_p(K)\) we refer to [10, §4–5]. We treat the case of odd \(p|d\) first. The Jordan constituents \(1(\frac{2}{1})^7\) and \((p^{e_p/p0})^3\) have species 7 and 1 respectively. So their diagonal factors are \(M_p(7) = \text{std}_p(K)\) and \(M_p(1) = 1/2\). Since there are only two Jordan constituents, there is a single cross-term, namely \((p^{e_p/p0})^3\)\(^{\frac{3}{2}}\)\(^{-1}\) = \(p^{7e_p/2}\). By definition, \(m_p(K)\) is the product of the diagonal factors and this cross-term. This gives \(m_p(K)/\text{std}_p(K) = \frac{1}{2}p^{7e_p/2}\).

The calculation of \(m_2(K)\) is similar but more intricate, and we must treat all three possibilities for \(K_2\). First suppose \(e_2 = 0\), so \(K_2 \cong 1_{\frac{8}{2}}\). The single Jordan constituent is free with type II, dimension 8 and sign \(-\), so it has species \(8-,\) hence diagonal term
\[
M_2(8-) = \frac{1}{2(1 - 2^{-2})(1 - 2^{-4})(1 - 2^{-6})(1 + 2^{-4})} = \frac{16}{15} \text{std}_2(K).
\]
There are no type I constituents (hence no bound love forms) and no cross-terms. Since type II constituents account for 8 dimensions, \(m_2(K)\) picks up a factor \(2^{-8}\). So \(m_2(K)/\text{std}_2(K) = \frac{16}{15}2^{-8} = \frac{1}{216}2^{7e_2/2}\).

Now suppose \(e_2 = 2\), so \(K_2 \cong 1_{\frac{8}{2}}2_{f_2-1}^{f_2-1}\). The first constituent is bound, 6-dimensional and has type II, hence species 7, hence diagonal factor \(M_2(7) = \text{std}_2(K)\). Before analyzing the second constituent we remark that \(f_2 \equiv 3 \mod{4}\). To see this, recall from the proof of \([i]\) that \(e_2 = 2\) exactly when \(k = 2l\) with \(l\) even. From \(d = 4(l^2 - 1)\) we get \(f_2 = (l + 1)(l - 1)\), and observe that one factor on the right is 1 mod 4 while the other is 3 mod 4. Now, the octane value of \(2_{f_2-1}^{f_2-1}\) is the subscript \(f_2 - 1\), plus 0 or 4 according to whether the sign \(\left(\frac{f_2}{2}\right)\) is + or -. We have just shown that \(f_2 \equiv 3\) or 7 mod 8. Either case yields the octane value 6, hence species 1, hence diagonal factor \(M_2(1) = 1/2\). There is one bound love form, contributing a factor of \(1/2\) to \(m_2(K)\). There is one cross-term, contributing a factor \(2^{\frac{7}{6}-}\). There are no adjacent type I constituents, and 6 dimensions total of
PRENILPOTENT PAIRS IN $E_{10}$

Type II constituents, contributing a factor $2^{-6}$. So

$$m_2(K) / \text{std}_2(K) = \frac{1}{2} \cdot \frac{1}{2} \cdot 2^6 \cdot 2^{-6} = \frac{1}{512} 2^{7\epsilon_2/2}.$$ 

Finally, suppose $\epsilon_2 \geq 5$, so $K_2 \cong 1^6_{II} 2^1_{II} (2^{e_2-1})_{f_2}$. The constituent $1^6_{II}$ is bound, hence has species 7, hence diagonal factor $M_2(7) = \text{std}_2(K)$. The constituent $2^1_{II}$ is free with octane value $-1$, hence species 0+, hence diagonal factor $M_2(0+) = 1$. The last constituent $(2^{e_2-1})_{f_2}$ is free. Considering each of the four possibilities for $f_2 \mod 8$ shows that the octane value is always $\pm 1$, so this constituent also has species 0+ and diagonal factor 1. There are three bound love forms, contributing a factor $2^{-3}$ to $m_2(K)$. There is a cross-term for each pair of constituents, and the cross-factor is their product, namely

$$2^4 \cdot (2^{e_2-1})_{f_2} \cdot (2^{e_2-2})_{f_2} = \frac{1}{2} 2^{7\epsilon_2/2}$$

Finally, there are no adjacent type I constituents, and 6 dimensions total of type II constituents, contributing a factor $2^{-6}$. Multiplying all the factors together yields

$$m_2(K) / \text{std}_2(K) = 1 \cdot 1 \cdot 1 \cdot 2^{-3} \cdot \frac{1}{2} 2^{7\epsilon_2/2} \cdot 2^{-6} = \frac{1}{1024} 2^{7\epsilon_2/2}.$$

We have now computed all the ingredients in (1), and assembling them yields $\Box$

**Proof of theorem 1.** Because each lattice has at least two symmetries, the number of lattices $K$ in the genus described in lemma 5(ii) is at least twice the mass given in (iv). One can show that the largest possible order for a finite subgroup of $GL_8(\mathbb{Z})$ is $|W(E_8)|$ (see [9] and its references). Using this we may replace $|O(K)|$ by $|W(E_8)|$ in lemma 5(iii). Multiplying this modified version of the quantity in (iii) by twice the mass, we see that the number of $O(E_{10})$-orbits of saturated copies of $L$ is at least

$$\frac{2d^{7/2} \zeta_d(4)}{30240 \pi^4 \cdot 4|W(E_8)| \cdot 1024}.$$ 

By lemma 4 this is also a lower bound for $N(k)$. Next we note

$$\zeta_d(4) \geq 1 - \frac{1}{2^4} - \frac{1}{3^4} - \cdots = 2 - \sum_{n=1}^{\infty} n^{-4} = 2 - \pi^4 / 90$$

Therefore

$$N(k) \geq \frac{2(k^2 - 4)^{7/2} (2 - \pi^4 / 90)}{30240 \pi^4 \cdot 4|W(E_8)| \cdot 1024} > 2.1 \times 10^{-19} (k^2 - 4)^{7/2}.$$
This argument works for $k \geq 3$. To complete the proof we must also observe that $N(1), N(2) > 0$ by the presence of $A_2$ and $E_9$ diagrams in the $E_{10}$ diagram. 

2. Other hyperbolic root lattices

The details of the previous section were $E_{10}$-specific, but the same philosophy looks likely to apply to the other symmetrizable hyperbolic root systems. This suggests the same enumeration-is-impracticable conclusion in rank $> 3$. We have not worked out the details, because for us the $E_{10}$ result is enough to motivate the improvements to Tits’ presentation that we mentioned in the introduction. But it seems valuable to give an outline of how the calculations would go.

By a hyperbolic root system we mean one arising from an irreducible Dynkin diagram that is neither affine nor spherical, but whose irreducible proper subdiagrams are. There are 238 such Dynkin diagrams, of which 142 are symmetrizable; see for example [ ]. Symmetrizability is equivalent to the root lattice $\Lambda$ possessing an inner product invariant under the Weyl group $W$. This is obviously a prerequisite to applying lattice-theoretic methods. Hyperbolicity implies that $\Lambda$ has Lorentzian signature and that $W$ has finite index in $O(\Lambda)$.

The roots are the $W$-images of the simple roots, so there are only finitely many root norms. For each pair of such norms $N, N'$, we can study prenilpotent pairs $r, r'$ with norms $N, N'$. The analogue of lemma 3 is that $r, r'$ form a prenilpotent pair if and only if $k := r \cdot r'$ is larger than $-\sqrt{NN'}$. By taking $k > \sqrt{NN'}$ we may suppose the span $L$ of $r, r'$ is indefinite. We are interested in the number $N(k)$ of $W$-orbits of such prenilpotent pairs.

Next one studies the embeddings of $L$ into $\Lambda$ as in lemma 5 which of course depend on $d := -\det L \approx k^2$. One can follow the same argument to bound below the number of $O(L)$-orbits of saturated copies of $L$ in $\Lambda$. First one would have to work out which genera could occur as $L^\perp$. If there are any, then we fix one and and restrict attention to saturated copies of $L$ for which $L^\perp$ lies in that genus. Then one would work out the mass of that genus. The essential part of the mass calculations in lemma 5(iv) are the cross-terms, because they provide the $d^{7/2}$ term that yields theorem 1. The corresponding term for $\Lambda$ would be $d^{(\dim \Lambda - 3)/2}$. This suggests that the number of $O(L)$-orbits of prenilpotent pairs (with various parameters fixed as above) grows at least as fast as a multiple of $k^{\dim \Lambda - 3}$.

An obstruction to turning this into a proof is that there may be some embeddings of $L$ into $\Lambda$ that send the basis vectors to non-roots.
We expect that the finiteness of \([\text{O}(\Lambda) : W]\) means that this difficulty can be more or less ignored. The point is that each \(\text{O}(\Lambda)\)-orbit of embeddings \(L \to \Lambda\) splits into at most \([\text{O}(\Lambda) : W]\) many \(W\)-orbits. So for each \(k\), we expect that \(N(k)\) is either 0 or at least a constant times \(k^{\dim \Lambda - 3}\).

This suggests that if \(\dim \Lambda > 3\) then tabulating the prenilpotent pairs is not feasible. But the \(\dim \Lambda = 3\) case is borderline and may be amenable to direct attack. Indeed, Carbone and Murray [6] have studied one particular case with \(\dim \Lambda = 3\). What is special about the \(\dim \Lambda = 3\) case is that \(L^\perp\) is 1-dimensional, and every 1-dimensional genus has a unique member and mass \(1/2\). So the main contribution to \(N(k)\) will be some analogue of the term

\[
2^\text{number of odd primes dividing } d
\]

from lemma [7](iii).

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