Stochastic gradient descent for linear least squares problems with partially observed data

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Abstract

We propose a novel stochastic gradient descent method for solving linear least squares problems with partially observed data. Our method uses submatrices indexed by a randomly selected pair of row and column index sets to update the iterate at each step. Theoretical convergence guarantees in the mean square sense are provided. Numerical experiments are reported to demonstrate the theoretical findings.

Keywords. linear least squares problem, partially observed data, stochastic gradient descent

AMS subject classifications: 65F10, 65F20, 68W20

1 Introduction

In recent years, stochastic iterative methods for solving large-scale linear equations or linear least squares problems have been greatly developed due to low memory footprints, such as randomized Kaczmarz [22], randomized coordinate descent [13], and their extensions, e.g., [25, 17, 10, 21, 18, 15, 23, 8, 7, 5]. However, the performance of these methods are often limited [16] when solving the problems with partially observed data, which can arise due to lost of data, errors in data recording, or cost of data acquisition, etc.

In this paper we consider solving the linear least squares problem

\[ \min_{x \in \mathbb{R}^n} \|Ax - b\|_2, \] (1)

where \( A \in \mathbb{R}^{m \times n} (m \geq n) \) has full column rank and \( \| \cdot \|_2 \) denotes the Euclidean norm. The least squares solution of this problem is \( A^\dagger b \), where \( A^\dagger \) is the Moore-Penrose generalized inverse [3]. Sometimes the matrix \( A \) and the vector \( b \) are partially observed, i.e., only partial entries of \( A \) and \( b \) are available. As a concrete example, suppose \( A \) is the score matrix of \( m \) users for \( n \) services, and \( b \) contains the \( m \) weighted total scores from each user for these services. Each user may not give scores for all the \( n \) services, or may not give a weighted total score for these \( n \) services, but the survey company wants to know the weight of each service in the weighted total score. That is to say, we need to solve the linear least squares problem (1) with only partial entries of \( A \) and \( b \) are available.

Let \( \{\delta_{ij}\}_{i=1,j=1}^{m,n} \) denote independent and identically distributed (i.i.d.) Bernoulli random variables satisfying

\[ \delta_{ij} = \begin{cases} 
1, & \text{with probability } p, \\
0, & \text{with probability } 1 - p,
\end{cases} \]

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and \( \{\delta_i\}_{i=1}^m \) denote i.i.d. Bernoulli random variables satisfying
\[
\delta_i = \begin{cases} 
1, & \text{with probability } q, \\
0, & \text{with probability } 1 - q.
\end{cases}
\]

If we use \( \delta_{ij} = 1 \) or \( \delta_i = 1 \) to indicate the availability of an element in \( A \) or \( b \), and \( \delta_{ij} = 0 \) or \( \delta_i = 0 \) to indicate a missing entry, then the partially observed data are
\[
\hat{A} = [\delta_{ij} A_{ij}]_{i=1,j=1}^{m,n} \quad \text{and} \quad \hat{b} = [\delta_i b_i]_{i=1}^m.
\]

So the linear least squares problem with partially observed data is:

\[
\text{Given } \hat{A}, \hat{b}, \text{ find the unique least squares solution } A^\dagger b = \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|_2. \tag{3}
\]

Solving the problems with partially observed data by standard methods, such as data imputation \cite{9}, matrix completion \cite{4,11,12,20}, and maximum likelihood estimation \cite{6,14}, can be wasteful, produces biases, or is impractical for large-scale problems. Recently, Ma and Needell \cite{10} proposed a stochastic gradient descent (SGD) method for the linear least squares problem \cite{1} with partially observed \( A \) as given in \cite{2} and fully observed \( b \). Their method uses a randomly selected row of \( A \) to construct a stochastic gradient at each step. In this paper, we consider the more general case, i.e., both \( A \) and \( b \) are partially observed.

**Main contributions.** We propose a novel stochastic gradient descent method for solving the linear least squares problem \cite{1} with partially observed data \cite{2} and prove its convergence theory. At each step, the new method uses submatrices indexed by a randomly selected pair of row and column index sets to design a stochastic gradient. Numerical examples are reported to illustrate the convergence of the new method.

**Organization of this paper.** In the rest of this section, we give some notation. In Section 2 we describe the proposed stochastic gradient descent method and prove its convergence theory. In Section 3 we report the numerical results. Finally, we present brief concluding remarks in Section 4.

**Notation.** For any random variables \( \xi \) and \( \zeta \), we use \( \mathbb{E}[\xi] \) and \( \mathbb{E} [\xi | \zeta] \) to denote the expectation of \( \xi \) and the conditional expectation of \( \xi \) given \( \zeta \), respectively. For an integer \( m \geq 1 \), let \( [m] := \{1, 2, 3, \ldots, m\} \). For any vector \( b \in \mathbb{R}^m \), we use \( b_i, b^T \) and \( \|b\|_2 \) to denote, the \( i \)th entry, the transpose and the Euclidean norm of \( b \), respectively. We use \( I \) to denote the identity matrix whose order is clear from the context. For any matrix \( A \in \mathbb{R}^{m \times n} \), we use \( A_{i,j}, A_{i,:}, A_{:,j}, A^T, A^\dagger, \|A\|_F, \text{range}(A), \) and \( \sigma_{\min}(A) \) to denote the \((i,j)\) entry, the \( i \)th row, the \( j \)th column, the transpose, the Moore-Penrose pseudoinverse, the Frobenius norm, the column space, and the smallest nonzero singular values of \( A \), respectively. For index sets \( I \subseteq [m] \) and \( J \subseteq [n] \), let \( A_{I,:}, A_{:,J}, \) and \( A_{I,J} \) denote the row submatrix indexed by \( I \), the column submatrix indexed by \( J \), and the submatrix that lies in the rows indexed by \( I \) and the columns indexed by \( J \), respectively. Similarly, we use \( b_{IJ} \) to denote the column vector consisting of elements of \( b \) indexed by \( I \). Given a square matrix \( M \), we denote a matrix containing only the diagonal of \( M \) as \( \text{diag}(M) \). We use \( B \preceq A \) to denote that \( A - B \) is positive semidefinite.

## 2 Stochastic gradient descent for partially observed data

In \cite{8} we proposed a doubly stochastic block Gauss-Seidel (DSBGS) algorithm for solving a consistent linear system \( Ax = b \). By varying the row partition parameter and the column partition parameter of DSBGS, we recover the randomized Kaczmarz algorithm \cite{22}, the randomized coordinate descent algorithm \cite{13}, and the doubly stochastic Gauss-Seidel algorithm \cite{19}. Next
we provide a slightly different variant of DSBGS, which will be used to derive our stochastic gradient descent method for solving the problem (3).

Let \( \{I_1, I_2, \ldots, I_s\} \) denote a partition of \([m]\) such that, for \( i, j = 1, 2, \ldots, s \) and \( i \neq j \),

\[
I_i \neq \emptyset, \quad I_i \cap I_j = \emptyset, \quad \bigcup_{i=1}^{s} I_i = [m].
\]

Let \( \{J_1, J_2, \ldots, J_t\} \) denote a partition of \([n]\) such that, for \( i, j = 1, 2, \ldots, t \) and \( i \neq j \),

\[
J_i \neq \emptyset, \quad J_i \cap J_j = \emptyset, \quad \bigcup_{i=1}^{t} J_i = [n].
\]

Let

\[
P = \{I_1, I_2, \ldots, I_s\} \times \{J_1, J_2, \ldots, J_t\}.
\]

To solve the linear least squares problem (1), one approach is to minimize the function

\[
f(x) := \frac{1}{2st} \|Ax - b\|_2^2.
\]

If a pair of index sets \((I, J)\) is randomly selected with probability \(\frac{1}{st}\), then we obtain

\[
\begin{align*}
\mathbb{E} \left[ I_{i, J_j} (A_{I_i, J_j})^T (A_{I_i, x} - b_I) \right] & = \frac{1}{st} \sum_{i=1}^{s} \sum_{j=1}^{t} I_{i, J_j} (A_{I_i, J_j})^T (A_{I_i, x} - b_I) \\
& = \frac{1}{st} \sum_{i=1}^{s} \sum_{j=1}^{t} I_{i, J_j} (I_{i, J_j})^T A^T I_{i, J} (I_{i, J})^T (Ax - b) \\
& = \frac{1}{st} A^T (Ax - b) \\
& = \nabla f(x).
\end{align*}
\]

This yields a stochastic gradient descent method (see Algorithm 1) for the linear least squares problem (1). Note that Algorithm 1 is a slightly different variant of DSBGS [8], which randomly selects the pair \((I, J)\) with probability \(\|A_{I, J}\|_F^2/\|A\|_F^2\).

**Algorithm 1:** SGD for the linear least squares problem (1)

Initialize \(x^0 \in \mathbb{R}^n\)

for \(k = 1, 2, \ldots, \) do

Pick \((I, J)\) \(\in P\) with probability \(\frac{1}{st}\)

Pick \(\alpha_k > 0\)

Set \(x^k = x^{k-1} - \alpha_k I_{i, J} (A_{I, J})^T (A_{I, x}^{k-1} - b_I)\)

Directly applying Algorithm 1 to the partially observed data (2), we obtain the update:

\[
x^k = x^{k-1} - \alpha_k I_{i, J} (A_{I, J})^T (A_{I, x}^{k-1} - b_I).
\]

Note that (see detailed calculations in the proof of Lemma 1)

\[
\mathbb{E} \left[ I_{i, J} (A_{I, J})^T (A_{I, x}^{k-1} - b_I) | x^{k-1} \right] = \frac{p^2}{st} A^T A x^{k-1} + \frac{D - p^2}{st} \text{diag}(A^T A) x^{k-1} - \frac{pq}{st} A^T b
\]

\(\neq \nabla f(x^{k-1}).\)
As a result, the iteration \[ \text{(4)} \] does not move in the negative gradient direction. Instead of using 
\[ I_{i,j}(A_{i,j})^T(A_{i,j}x^{k-1} - \hat{b}_i), \]
if we use 
\[ g(x^{k-1}) = I_{i,j}(\hat{A}_{i,j})^T \left( \frac{\hat{A}_{i,j}x^{k-1}}{p^2} - \frac{\hat{b}_i}{pq} \right) - \frac{1}{p^2} \text{diag} \left( I_{i,j}(\hat{A}_{i,j})^T \hat{A}_{i,,:} \right) x^{k-1}, \]
then we have (see Lemma \[ \text{(1)} \])
\[ \mathbb{E} \left[ g(x^{k-1}) \mid x^{k-1} \right] = \frac{1}{st} A^T(Ax^{k-1} - b) = \nabla f(x^{k-1}). \]

This yields a stochastic gradient descent method (see Algorithm 2) for solving the linear least squares problem \[ \text{(1)} \] with partially observed data \[ \text{(2)} \].

**Algorithm 2:** SGD for the linear least squares problem with partially observed data \[ \text{(2)} \]

1. Initialize \( x^0 \in \mathbb{R}^n \)
2. For \( k = 1, 2, \ldots, \) do
   1. Pick \( (I, J) \in \mathcal{P} \) with probability \( \frac{1}{st} \)
   2. Set \( g(x^{k-1}) = I_{i,j}(\hat{A}_{i,j})^T \left( \frac{\hat{A}_{i,j}x^{k-1}}{p^2} - \frac{\hat{b}_i}{pq} \right) - \frac{1}{p^2} \text{diag} \left( I_{i,j}(\hat{A}_{i,j})^T \hat{A}_{i,,:} \right) x^{k-1} \)
   3. Pick \( \alpha_k > 0 \)
   4. Set \( x^k = x^{k-1} - \alpha_k g(x^{k-1}) \)

When \( p = q = 1 \), Algorithm 2 is the same as Algorithm 1. By varying the row partition parameter \( s \) and the column partition parameter \( t \), we obtain the following special cases.

(i) \( s = m, t = n \)
\[ x^k = x^{k-1} - \alpha_k \hat{A}_{i,j} \left( \frac{\hat{A}_{i,j}x^{k-1}}{p^2} - \frac{\hat{b}_i}{pq} - \frac{1}{p^2} \text{diag} \left( \hat{A}_{i,j}^T \hat{A}_{i,:} \right) x^{k-1} \right) I_{i,j}. \]

(ii) \( s = m, t = 1 \)
\[ x^k = x^{k-1} - \alpha_k \left( \hat{A}_{i,:}^T \left( \frac{\hat{A}_{i,:}x^{k-1}}{p^2} - \frac{\hat{b}_i}{pq} \right) - \frac{1}{p^2} \text{diag} \left( \hat{A}_{i,:}^T \hat{A}_{i,:} \right) x^{k-1} \right). \]

(iii) \( s = 1, t = n \)
\[ x^k = x^{k-1} - \alpha_k \left( \hat{A}_{i,:}^T \left( \frac{\hat{A}x^{k-1}}{p^2} - \frac{\hat{b}}{pq} - \frac{1}{p^2} \text{diag} \left( \hat{A}_{i,:}^T \hat{A}_{i,:} \right) x^{k-1} \right) I_{i,:}. \]

(iv) \( s = 1, t = 1 \)
\[ x^k = x^{k-1} - \alpha_k \left( \hat{A}^T \left( \frac{\hat{A}x^{k-1}}{p^2} - \frac{\hat{b}}{pq} \right) - \frac{1}{p^2} \text{diag} \left( \hat{A}^T \hat{A} \right) x^{k-1} \right). \]

We remark that at each step the cases (i) and (ii) only use the data \( \hat{A}_{i,:} \) and \( \hat{b}_i \) to update the iterate. This is particularly appropriate for those problems with extremely large \( m \times n \) matrix \( A \) where it is not possible to load more than one row of \( A \) due to memory constraints.
2.1 Convergence analysis

First, we will prove two useful properties of the update function \( g(x) \) defined by (6). Lemma 1 shows that \( g(x) \) is a stochastic gradient of the objective function \( \frac{1}{2st} \|Ax - b\|^2_2 \). Lemma 2 provides a uniformly upper bound of the expected norm of \( g(x) \).

**Lemma 1.** For any fixed \( x \in \mathbb{R}^n \), let

\[
g(x) = I_{:,J}(\hat{A}_{I,:})^T \left( \frac{\hat{A}_{I,:}x}{p^2} - \frac{\hat{b}_I}{pq} \right) - \frac{1-p}{p^2} \text{diag} \left( I_{:,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right)x.
\]

(6)

We have

\[
E[g(x)] = \frac{1}{st} A^T (Ax - b).
\]

**Proof.** Let \( E_\delta[\cdot] \) denote the expectation with respect to the random binary mask (i.e., \( \delta_{i,j} \) and \( \delta_i \), in total \( 2^{m(n+1)} \) possibilities) and \( E_{(I,J)}[\cdot] \) denote the expectation with respect to the random selection (\( st \) possibilities) of the pair of index sets. Then the full expected value \( E[\cdot] \) satisfies

\[
E[\cdot] = E_\delta E_{(I,J)}[\cdot].
\]

We recall that all \( \delta_{i,j} \) and \( \delta_i \) are independent. By straightforward calculations, we have

\[
E \left[ I_{:,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right] = E_\delta E_{(I,J)} \left[ I_{:,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right]
= E_\delta E_{(I,J)} \left[ I_{:,J}(I_{:,J})^T \hat{A}_{I,:} I_{:,J}(I_{:,J})^T \hat{A}_{I,J} \right]
= \frac{1}{st} E_\delta \left[ \hat{A}^T \hat{A} \right]
= \frac{1}{st} E_\delta \begin{bmatrix}
(\hat{A}_{:,1})^T \hat{A}_{:,1} & (\hat{A}_{:,1})^T \hat{A}_{:,2} & \cdots & (\hat{A}_{:,1})^T \hat{A}_{:,n} \\
(\hat{A}_{:,2})^T \hat{A}_{:,1} & (\hat{A}_{:,2})^T \hat{A}_{:,2} & \cdots & (\hat{A}_{:,2})^T \hat{A}_{:,n} \\
\vdots & \vdots & & \vdots \\
(\hat{A}_{:,n})^T \hat{A}_{:,1} & (\hat{A}_{:,n})^T \hat{A}_{:,2} & \cdots & (\hat{A}_{:,n})^T \hat{A}_{:,n}
\end{bmatrix}
= \frac{1}{st} \begin{bmatrix}
p(\hat{A}_{:,1})^T \hat{A}_{:,1} & p^2(\hat{A}_{:,1})^T \hat{A}_{:,2} & \cdots & p^2(\hat{A}_{:,1})^T \hat{A}_{:,n} \\
p^2(\hat{A}_{:,2})^T \hat{A}_{:,1} & p(\hat{A}_{:,2})^T \hat{A}_{:,2} & \cdots & p^2(\hat{A}_{:,2})^T \hat{A}_{:,n} \\
\vdots & \vdots & & \vdots \\
p^2(\hat{A}_{:,n})^T \hat{A}_{:,1} & p^2(\hat{A}_{:,n})^T \hat{A}_{:,2} & \cdots & p(\hat{A}_{:,n})^T \hat{A}_{:,n}
\end{bmatrix}
= \frac{p^2}{st} A^T A + \frac{p - p^2}{st} \text{diag}(A^T A).
\]

(7)

Similarly, we have

\[
E \left[ I_{:,J}(\hat{A}_{I,J})^T \hat{b}_I \right] = E_\delta E_{(I,J)} \left[ I_{:,J}(\hat{A}_{I,J})^T \hat{b}_I \right]
= E_\delta E_{(I,J)} \left[ I_{:,J}(I_{:,J})^T \hat{A}_{I,J} I_{:,J}(I_{:,J})^T \hat{b} \right]
= \frac{1}{st} E_\delta \left[ \hat{A}^T \hat{b} \right]
= \frac{1}{st} E_\delta \left[ \hat{A}^T \right] E_\delta \left[ \hat{b} \right]
= \frac{pq}{st} A^T b.
\]

(8)
Using (7), we have
\[
\mathbb{E} \left[ \text{diag} \left( \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T \hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \right) \right] = \text{diag} \left( \mathbb{E} \left[ \left( \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T \hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \right) \right] \right) = \frac{p}{st} \text{diag} (\mathbf{A}^T \mathbf{A}).
\] (9)

Combining (7), (8) and (9) yields
\[
\mathbb{E} [\mathbf{g}(\mathbf{x})] = \frac{1}{st} \mathbf{A}^T (\mathbf{A} \mathbf{x} - \mathbf{b}).
\]

This completes the proof. \(\square\)

**Lemma 2.** For any fixed \(\mathbf{x} \in \mathbb{R}^n\), let \(\mathbf{g}(\mathbf{x})\) be given as in (4). We have
\[
\mathbb{E} \left[ \|\mathbf{g}(\mathbf{x})\|_2^2 \right] \leq \frac{2}{stp^2} \|\mathbf{A}\|_F^2 \|\mathbf{A} \mathbf{x} - \mathbf{b}\|_2^2 + \frac{2(1 - q)}{stp^2q} \|\mathbf{A}\|_F^2 \|\mathbf{b}\|_2^2 + \frac{2(1 - p)}{stp^3} \|\mathbf{A}^T \mathbf{A}\|_2 \|\mathbf{x}\|_2^2 + \frac{2(1 - p)^2}{stp^3} \|\mathbf{A}^T \mathbf{A}\|_2 \|\mathbf{x}\|_2^2.
\]

**Proof.** By straightforward calculations, we have
\[
\|\mathbf{g}(\mathbf{x})\|_2^2 = \left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \mathbf{x} - \frac{\hat{\mathbf{b}}_T}{pq}) \right\|^2_2 - \frac{1 - p}{p^2} \text{diag} \left( \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T \hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \right) \mathbf{x}^2_2\]
\[
\leq \left( \left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\frac{\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \mathbf{x}}{p^2} - \frac{\hat{\mathbf{b}}_T}{pq}) \right\|_2^2 + \left\| \frac{1 - p}{p^2} \text{diag} \left( \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T \hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \right) \mathbf{x} \right\|_2^2 \right)^2
\leq 2 \left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\frac{\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \mathbf{x}}{p^2} - \frac{\hat{\mathbf{b}}_T}{pq}) \right\|_2^2 + \frac{2(1 - p)^2}{p^2} \|\mathbf{A}^T \mathbf{A}\|_2 \|\mathbf{x}\|_2^2.
\] (10)

and
\[
\left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\frac{\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}} \mathbf{x}}{p^2} - \frac{\hat{\mathbf{b}}_T}{pq}) \right\|_2^2 = \left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\frac{\mathbf{A} \mathbf{x}}{p^2} - \frac{\mathbf{b}}{pq}) \right\|_2^2 \leq \left\| \mathbf{I}_{\mathcal{I},\mathcal{J}} (\hat{\mathbf{A}}_{\mathcal{I},\mathcal{J}})^T (\mathbf{I}_{\mathcal{I},\mathcal{J}})^T (\frac{\mathbf{A} \mathbf{x}}{p^2} - \frac{\mathbf{b}}{pq}) \right\|_2^2
\leq \|\mathbf{A}^T \mathbf{A}\|_F^2 \left\| \frac{\mathbf{A} \mathbf{x}}{p^2} - \frac{\mathbf{b}}{pq} \right\|_2^2.
\] (11)

Further calculations give the expectation
\[
\mathbb{E} \left[ \frac{1}{stp^2} \|\mathbf{A} \mathbf{x} - \mathbf{b}\|_2^2 + \frac{1 - q}{stp^2q} \|\mathbf{A}\|_F^2 \|\mathbf{b}\|_2^2 + \frac{1 - p}{stp^3} \|\mathbf{A}\|_F^2 \|\mathbf{x}\|_2^2 \right] = \frac{1}{st} \mathbb{E}_{\mathcal{I},\mathcal{J}} \left[ \frac{1}{p^2} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} + \frac{1 - p}{p^3} \text{diag}(\mathbf{A}^T \mathbf{A}) \mathbf{x} - \frac{2}{p^2q} \mathbf{x}^T \mathbf{A}^T \mathbf{b} + \frac{1 - p}{p^2q} \mathbf{b}^T \mathbf{b} \right]
\leq \frac{1}{st} \mathbb{E}_{\mathcal{I},\mathcal{J}} \left[ \frac{1}{p^2} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} + \frac{1 - p}{p^3} \text{diag}(\mathbf{A}^T \mathbf{A}) \mathbf{x} - \frac{2}{p^2q} \mathbf{x}^T \mathbf{A}^T \mathbf{b} + \frac{1 - p}{p^2q} \mathbf{b}^T \mathbf{b} \right].
\] (12)
It follows from
\[
\text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) = \text{diag} \left( I_{n,J}(I_{n,J})^T \hat{A}^T I_{n,J}(I_{n,J})^T \hat{A} \right) \\
= \sum_{j=1}^{n} I_{n,J}(I_{n,J})^T \left( I_{n,J}(I_{n,J})^T \hat{A}^T I_{n,J}(I_{n,J})^T \hat{A} \right) I_{n,J}(I_{n,J})^T \\
= \sum_{j \in J} I_{n,J}(I_{n,J})^T \hat{A}^T I_{n,J}(I_{n,J})^T \hat{A} I_{n,J}(I_{n,J})^T \\
= \sum_{j \in J} I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,J} \left( I_{n,J} \right)^T \\
= I_{n,J} \text{diag} \left( (\hat{A}_{I,J})^T \hat{A}_{I,J} \right) \left( I_{n,J} \right)^T
\]
that
\[
0 \leq \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) \leq \text{diag} \left( \hat{A}^T \hat{A} \right) \leq \text{diag} \left( A^T A \right).
\]
This yields
\[
\begin{align*}
\left\| \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) \right\|_F^2 & = x^T \left( \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) \right)^2 x \\
& \leq x^T \text{diag} \left( A^T A \right) \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) x.
\end{align*}
\]
Then by (9) and (13), we have
\[
\begin{align*}
E \left[ \left\| \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) \right\|_F^2 \right] & \leq E \left[ x^T \text{diag} \left( A^T A \right) \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) x \right] \\
& \leq \frac{P}{st} x^T \left( \text{diag} \left( A^T A \right) \right)^2 x \\
& = \frac{P}{st} \left\| \text{diag} \left( A^T A \right) x \right\|_2^2.
\end{align*}
\]
Combining (10), (11), (12), and (14) yields
\[
\begin{align*}
E \left[ \left\| g(x) \right\|_2^2 \right] & \leq 2E \left[ \left\| A_{I,J} \right\|_F^2 \left\| \left( \hat{A} x - \hat{b} \right) \right\|_2^2 \right] \\
& + \frac{2(1-p)^2}{stp^2} E \left[ \left\| \text{diag} \left( I_{n,J}(\hat{A}_{I,J})^T \hat{A}_{I,:} \right) \right\|_2^2 \right] \\
& \leq \frac{2}{stp^2} \left\| A \right\|_F^2 \left\| A x - b \right\|_2^2 + \frac{2(1-q)}{stp^2} q \left\| A \right\|_F^2 \left\| b \right\|_2^2 \\
& + \frac{2(1-p)}{stp^2} \left\| A \right\|_F^2 x^T \text{diag}(A^T A) x \\
& + \frac{2(1-p)^2}{stp^2} \left\| \text{diag}(A^T A) x \right\|_2^2.
\end{align*}
\]
This completes the proof. □

Next, we give the main result of this paper, which shows the convergence behavior of Algorithm 2 with a constant step size.

**Theorem 3.** Let \( x^k \) denote the \( k \)th iterate of Algorithm 2 applied to the linear least squares problem (1) with partially observed data (2). For a constant step size \( 0 < \alpha < \frac{\sigma_{\text{min}}^2(A)}{st\rho} \) (i.e.,
all $\alpha_k = \alpha$, it holds
\[
\mathbb{E} \left[ \|x^k - A^\dagger b\|_2^2 \right] \leq \left( 1 - \frac{2\alpha \sigma_{\min}^2(A)}{\sigma_{\min}(A) - \alpha \sigma} \right)^k \|x^0 - A^\dagger b\|_2^2 + \frac{\alpha C}{\sigma_{\min}(A) - \alpha \sigma}.
\]
where
\[
\rho = \mathbb{E} \|B^TB\|_2, \quad B = \frac{1}{p^2} I_{n,\mathcal{J},\mathcal{I}}^T (\tilde{A}_{\mathcal{I},\mathcal{I}}) \mathbb{E} \tilde{A}_{\mathcal{I},\mathcal{I}} - \frac{1 - \rho}{p^2} \text{diag} \left( I_{n,\mathcal{J},\mathcal{I}}^T \tilde{A}_{\mathcal{I},\mathcal{I}} \right),
\]
and
\[
C = \frac{2}{p^2} \|A\|^2 \|AA^\dagger b - b\|_2^2 + \frac{2(1 - \rho)}{p^2 q} \|A\|^2 \|b\|_2^2 + \frac{2(1 - \rho)^2}{p^4} \|\text{diag}(A^T A) A^\dagger b\|_2^2.
\]

Proof. By Lemma 1 we have
\[
\mathbb{E} \left[ g(x^{k-1}) | x^{k-1} \right] = \frac{1}{s^T} A^T (Ax^{k-1} - b).
\]
By Lemma 2 we have
\[
\mathbb{E} \left[ \|g(A^\dagger b)\|_2^2 \right] \leq \frac{C}{s^T}.
\]
Straightforward calculations yield
\[
\|x^k - A^\dagger b\|_2^2 = \|x^{k-1} - A^\dagger b - \rho g(x^{k-1})\|_2^2
\]
\[
= \|x^{k-1} - A^\dagger b\|_2^2 - 2\alpha (x^{k-1} - A^\dagger b)^T g(x^{k-1}) + \alpha^2 \|g(x^{k-1})\|_2^2
\]
\[
\leq \|x^{k-1} - A^\dagger b\|_2^2 - 2\alpha (x^{k-1} - A^\dagger b)^T g(x^{k-1})
\]
\[
+ \alpha^2 \|g(x^{k-1}) - g(A^\dagger b)\|_2 + \|g(A^\dagger b)\|_2^2
\]
\[
\leq \|x^{k-1} - A^\dagger b\|_2^2 - 2\alpha (x^{k-1} - A^\dagger b)^T g(x^{k-1})
\]
\[
+ 2\alpha^2 \|g(x^{k-1}) - g(A^\dagger b)\|_2^2 + 2\alpha^2 \|g(A^\dagger b)\|_2^2,
\]
and
\[
\mathbb{E} \left[ \|g(x^{k-1}) - g(A^\dagger b)\|_2^2 | x^{k-1} \right] = \mathbb{E} \left[ (x^{k-1} - A^\dagger b)^T B^T B (x^{k-1} - A^\dagger b) | x^{k-1} \right]
\]
\[
\leq \mathbb{E} \left[ B^T B \right] \|x^{k-1} - A^\dagger b\|_2^2
\]
\[
= \rho \|x^{k-1} - A^\dagger b\|_2^2.
\]
Combining (15), (16), (17), and (18) yields
\[
\mathbb{E} \left[ \|x^k - A^\dagger b\|_2^2 | x^{k-1} \right] \leq \|x^{k-1} - A^\dagger b\|_2^2 - \frac{2\alpha}{s^T} (x^{k-1} - A^\dagger b)^T A^T (Ax^{k-1} - b)
\]
\[
+ 2\alpha^2 \rho \|x^{k-1} - A^\dagger b\|_2^2 + 2\alpha^2 \frac{C}{s^T}
\]
\[
= \|x^{k-1} - A^\dagger b\|_2^2 - \frac{2\alpha}{s^T} (x^{k-1} - A^\dagger b)^T A^T (Ax^{k-1} - A^\dagger b)
\]
\[
+ 2\alpha^2 \rho \|x^{k-1} - A^\dagger b\|_2^2 + 2\alpha^2 \frac{C}{s^T}
\]
\[
\leq \left( 1 - \frac{2\alpha \sigma_{\min}^2(A)}{\sigma_{\min}(A) - \alpha \sigma} \right)^k \|x^{k-1} - A^\dagger b\|_2^2 + 2\alpha^2 \frac{C}{s^T}.
\]
Therefore, by the law of total expectation, we have
\[
E \left[ \| x^k - A^1 b \|_2^2 \right] = E \left[ E \left[ \| x^k - A^1 b \|_2^2 | x^{k-1} \right] \right]
\leq \left( 1 - \frac{2\alpha \sigma_{\text{min}}^2(A)}{st} + 2\alpha^2 \rho \right) E \left[ \| x^{k-1} - A^1 b \|_2^2 \right] + 2\alpha^2 \frac{C}{st}
\leq \ldots
\leq \left( 1 - \frac{2\alpha \sigma_{\text{min}}^2(A)}{st} + 2\alpha^2 \rho \right)^k \| x^0 - A^1 b \|_2^2 + \frac{2\alpha^2 C}{st \sigma_{\text{min}}^2(A)} - 2\alpha^2 \rho
\leq \left( 1 - \frac{2\alpha \sigma_{\text{min}}^2(A)}{st} + 2\alpha^2 \rho \right)^\ell \| x^0 - A^1 b \|_2^2 + \frac{2\alpha^2 C}{\sigma_{\text{min}}^2(A) - \alpha \rho}
\]
This completes the proof. \qed

When \( p = q = 1 \) and \( b \in \text{range}(A) \), Theorem 3 implies that \( x^k \) in Algorithm 1 using sufficiently small positive constant \( \alpha \) converges to \( A^1 b \).

3 Numerical results

In this section, we report numerical experiments to illustrate the theoretical results. In each experiment, all data are available. Partially observed data are realized by the mask independent random variables \( \delta_{ij} \) and \( \delta_i \). This makes the error \( \| x^k - A^1 b \|_2^2 \) of Algorithm 2 computable. The initial guess \( x^0 = 0 \) and the relative error \( \| x^k - A^1 b \|_2^2 / \| A^1 b \|_2^2 \) is averaged over 10 trials. All experiments are performed using MATLAB on a laptop with 2.7-GHz Intel Core i7 processor, 16 GB memory, and Mac operating system.

In Algorithm 2, for simplicity, we use the row partition \( \{I_i\}_{i=1}^s \) with \( s = \left\lfloor \frac{m}{\ell} \right\rfloor \):
\[
I_i = \{(i-1)\ell + 1, (i-1)\ell + 2, \ldots, i\ell\}, \quad i = 1, 2, \ldots, s - 1,
I_s = \{(s-1)\ell + 1, (s-1)\ell + 2, \ldots, m\},
\]
and the column partition \( \{J_j\}_{j=1}^t \) with \( t = \left\lfloor \frac{n}{\tau} \right\rfloor \):
\[
J_j = \{(j-1)\tau + 1, (j-1)\tau + 2, \ldots, j\tau\}, \quad j = 1, 2, \ldots, t - 1,
J_t = \{(t-1)\tau + 1, (t-1)\tau + 2, \ldots, n\}.
\]

In each experiment, the matrix \( A \) is generated from a standard normal distribution:
\[
A = \text{randn}(m, n),
\]
so \( A \) is a full column rank matrix with probability one. For the case \( b \in \text{range}(A) \), we use
\[
b = A * \text{randn}(n, 1),
\]
and for the case \( b \notin \text{range}(A) \), we use
\[
b = A * \text{randn}(n, 1) + \text{null}(A^*) * \text{ones}(m - n, 1).
\]

Figure 1 shows the results of Algorithm 2 using \( \ell = 2 \), \( \tau = n \), a constant step size \( \alpha = 10^{-4} \) and varied proportions (i.e., \( p \) and \( q \)) of available data. Figure 2 shows the performance of Algorithm 2 using \( \ell = 2 \), \( \tau = n \), \( p = 0.9 \), \( q = 0.9 \), and different constant \( \alpha \). These experimental
results support the theoretical findings presented in Theorem 3. Using a constant step size, Algorithm 2 converges to some radius (proportional to $\alpha$) around the solution. The proportions (i.e., $p$ and $q$) of available data affect the convergence horizon. In particular, as $p$ and $q$ decrease the size of the convergence horizon increases. When $p = q = 1$ and $b \in \text{range}(A)$, Algorithm 2 behaves as DSBGS does on the consistent linear system $Ax = b$.

Based on these numerical experiments, we can design a step size updating strategy: (i). choose pairs $\{ (\beta_i, T_i) \}_{i=1}^{K}$ satisfying $\beta_1 > \beta_2 > \cdots > \beta_K > 0$ and $T_1 \leq T_2 \leq \cdots \leq T_K$; (ii) use step size $\beta_1$ in the first $T_1$ iterations, and use step size $\beta_2$ in the following $T_2$ iterations, and so on. The performance of Algorithm 2 using this step size updating strategy with $\beta_1 = 10^{-4}$, $\beta_2 = 10^{-4.5}$, $\beta_3 = 10^{-5}$ and $T_1 = 3 \times 10^4$, $T_2 = 4 \times 10^4$, $T_3 = 1.3 \times 10^5$, for the same data used in Figure 2 is given in Figure 3. Compared with the constant step size strategy, the new strategy significantly reduces the number of iterations.

4 Concluding remarks

We have proposed a stochastic gradient descent method for solving linear least squares problems with partially observed data. We prove that this method generates a sequence converging to some radius around the least squares solution. Numerical experiments on synthetic data illustrate the theoretical results. Finding appropriate step size selection strategies such as that used for Figure 3 and applying the resulting method on real world data should be valuable topics in the future study.

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Figure 2: The performance of Algorithm 2 using $\ell = 2$, $\tau = n$, $p = 0.9$, $q = 0.9$, and different constant $\alpha$. Here, $m = 1000$ and $n = 200$. Left: $b \in \text{range}(A)$. Right: $b \notin \text{range}(A)$.

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Figure 3: The performance of Algorithm 2 using $\ell = 2$, $\tau = n$, $p = 0.9$, $q = 0.9$, and updating $\alpha_k$. Here, $m = 1000$ and $n = 200$. Left: $b \in \text{range}(A)$. Right: $b \notin \text{range}(A)$.

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