THE NORM OF TIME-FREQUENCY LOCALIZATION OPERATORS

FABIO NICOLA AND PAOLO TILLI

Abstract. Time-frequency localization operators (with Gaussian window) \( L_F : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \), where \( F \) is a weight in \( \mathbb{R}^{2d} \), were introduced in signal processing by I. Daubechies in 1988, inaugurating a new, geometric, phase-space perspective. Sharp upper bounds for the norm (and the singular values) of such operators turn out to be a challenging issue with deep applications in signal recovery, quantum physics and the study of uncertainty principles.

In this note we provide optimal upper bounds for the operator norm \( \| L_F \|_{L^2 \to L^2} \), assuming \( F \in L^p(\mathbb{R}^{2d}) \), \( 1 < p < \infty \) or \( F \in L^p(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}) \), \( 1 \leq p < \infty \). It turns out that two regimes arise, depending on whether the quantity \( \| F \|_{L^p} / \| F \|_{L^\infty} \) is less or greater than a certain critical value. In the first regime the extremal weights \( F \), for which equality occurs in the estimates, are certain Gaussians, whereas in the second regime they are proved to be truncated Gaussians, degenerating in a multiple of a characteristic function of a ball for \( p = 1 \). This phase transition through truncated Gaussians appears to be a new phenomenon in time-frequency concentration problems.

1. Introduction and discussion of the results

The uncertainty principle is an ubiquitous theme in mathematics and still represents an endless source of challenging and inspirational problems. The literature in this connection is enormous. Here we only recall, for a deep introduction to the topic, the classical contributions [16, 23, 26] and the recent account [52]. The principle states, roughly speaking, that a function and its Fourier transform cannot both be too concentrated, or even that a time-frequency distribution cannot be too concentrated in the time-frequency space. To better develop the latter point of view, we recall the definition of the time-frequency distribution known as short-time, or windowed, Fourier transform in harmonic analysis and signal processing [21, 29, 44, 45], and coherent state transform in mathematical physics [40].

Let \( \varphi \) be the “Gaussian window”

\[
\varphi(x) = 2^{d/4} e^{-\pi |x|^2}, \quad x \in \mathbb{R}^d,
\]

2010 Mathematics Subject Classification. 42B10, 49Q20, 49R05, 81S30, 94A12.

Key words and phrases. Short-time Fourier transform, localization operator, Toeplitz operator, uncertainty principle.
normalized in such way that $\|\varphi\|_{L^2} = 1$. The short-time Fourier transform with Gaussian window of a function $f \in L^2(\mathbb{R}^d)$, is defined as

$$Vf(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \omega} f(y) \varphi(x - y) dy, \quad x, \omega \in \mathbb{R}^d.$$ 

It turns out that $V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry, so that the function $|Vf(x, \omega)|^2$ can be interpreted as the time-frequency energy density of $f$. Consequently, the integral $\int_\Omega |Vf(x, \omega)|^2 dx d\omega$, over a measurable subset $\Omega \subset \mathbb{R}^{2d}$, will be the fraction of its energy trapped in $\Omega$ (see e.g. [1, 5, 6, 25, 46, 47, 50]). Sharp upper bounds for this quantity were recently proved in [47]; the relevant estimate, that we state here in dimension $d = 1$ for simplicity, reads

$$\int_\Omega |Vf(x, \omega)|^2 dx d\omega \leq (1 - e^{-|\Omega|}) \|f\|_{L^2}^2,$$

(see Theorem 3.1 below for the complete statement), where $|\Omega|$ stands for the measure of $\Omega$, that here is supposed finite. Since the constant on the right-hand side is strictly less than one, this result can be regarded as a manifestation of the uncertainty principle. Upper bounds in the same spirit, for certain Cantor-type rotationally invariant subsets $\Omega \subset \mathbb{R}^2$ have recently been obtained in [3, 32, 33, 34] in connection with the fractal uncertainty principle. We also address to [11, 22, 28, 30, 48] for other forms of the uncertainty principle involving the short-time Fourier transform.

While the localization in the time-frequency plane by means of the characteristic function of a subset $\Omega$, as in (1.2), is quite natural, following a practice dating back at least to [16] and fully promoted in [17], one can similarly localize using other weight functions $F(x, \omega) \geq 0$ (satisfying possibly other constraints such as $F(x, \omega) \leq 1$, to avoid an amplification) and ask for similar estimates for the weighted energy

$$\langle L_F f, f \rangle = \int_{\mathbb{R}^{2d}} F(x, \omega) |Vf(x, \omega)|^2 dx d\omega.$$ 

Here we introduced the so-called time-frequency localization operator $L_F$ associated with the weight $F$, defined as

(1.3) $L_F = V^* F V$,

or weakly as $\langle L_F f, g \rangle_{L^2(\mathbb{R}^d)} = \langle F, Vf \overline{Vg} \rangle_{L^2(\mathbb{R}^{2d})}$. One can easily see that $L_F$ is indeed a bounded operator in $L^2(\mathbb{R}^d)$ for, e.g., any complex valued $F \in L^p(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$ (see e.g. [53]). This class of operators were introduced by I. Daubechies [17] as a joint time-frequency version of the celebrated Landau-Pollack-Slepian operator [36, 37, 51] from signal theory, and by F. A. Berezin [5], motivated by the quantization problem in quantum mechanics. Since then, they have been object of intensive studies, especially regarding boundedness, compactness, Schatten properties and asymptotics of the eigenvalues, even for more general weights and function spaces; see the classical references [9, 13, 19, 53], the more recent contributions [2, 4, 7, 11, 13, 24, 42, 43] and the references therein.
In this note we confine ourselves to the basic case of weights in Lebesgue spaces, but on the other hand we compute exactly the quantity
\[
\sup_{\|F\|_{L^\infty} \leq A, \|F\|_{L^p} \leq B} \|L_F\|_{L^2 \to L^2},
\]
as a function of \(A, B\), where \(1 \leq p < \infty\), and \(A, B > 0\). If \(p > 1\) we also consider the case \(A = +\infty\), i.e. we do not impose the \(L^\infty\) constraint (if \(B = +\infty\) the above supremum is trivially equal to \(A\), since \(L_F\) is the identity operator when \(F \equiv 1\)).

For simplicity, we state the bound in dimension \(d = 1\) and we address the reader to Section 4 for the extension to the multidimensional case.

**Theorem 1.1.** Let \(1 \leq p < \infty\), \(0 < A \leq +\infty\), \(0 < B < +\infty\); if \(p = 1\) we require \(A < +\infty\).

(a) If \((B/A)^p \leq (p-1)/p\) (in particular if \(p > 1\) and \(A = +\infty\)) we have
\[
\sup_{\|F\|_{L^\infty} \leq A, \|F\|_{L^p} \leq B} \|L_F\|_{L^2 \to L^2} = \left(\frac{p-1}{p}\right)^{(p-1)/p} B,
\]
and the supremum is attained at \(F\) if and only if
\[
F(z) = ce^{-\frac{1}{p-1}|z-z_0|^2}, \quad z \in \mathbb{R}^2
\]
for some \(c \in \mathbb{C} \setminus \{0\}\) with \(|c| = ((p-1)/p)^{-1/p} B\), and \(z_0 \in \mathbb{R}^2\).

(b) If \((B/A)^p \geq (p-1)/p\) we have
\[
\sup_{\|F\|_{L^\infty} \leq A, \|F\|_{L^p} \leq B} \|L_F\|_{L^2 \to L^2} = A \left(1 - \frac{1}{p} e^{-(B/A)^p + (p-1)/p}\right).
\]
If \(p > 1\) the supremum in (1.6) is attained at \(F\) if and only if
\[
F(z) = c \min\{Ce^{-\frac{1}{p-1}|z-z_0|^2}, 1\}, \quad z \in \mathbb{R}^2
\]
where \(C = e^{(B/A)^p/(p-1)-1/p}\), for some \(c \in \mathbb{C} \setminus \{0\}\), with \(|c| = A\), and \(z_0 \in \mathbb{R}^2\).

If \(p = 1\) the supremum in (1.6) if attained at \(F\) if and only if \(F = c \chi_B\) for some \(c \in \mathbb{C} \setminus \{0\}\), \(|c| = A\), where \(B \subset \mathbb{R}^2\) is a ball of measure \(|B| = B/A\).

In the above cases where the supremum in attained at some \(F \neq 0\), we have
\[
\|L_F\|_{L^2 \to L^2} = |\langle L_F f, g \rangle| \text{ for some } f, g \in L^2(\mathbb{R}), \|f\|_{L^2} = \|g\|_{L^2} = 1, \text{ if and only if for some } c, c' \in \mathbb{C}, |c| = |c'| = 1 \text{ and some } (x_0, \omega_0) \in \mathbb{R}^2 \text{ we have}
\]
\[
f(x) = c' g(x) = ce^{2\pi i x \omega_0} \varphi(x - x_0), \quad x \in \mathbb{R},
\]
where \(\varphi\) is the Gaussian in (1.1) (with \(d = 1\)).

The above result has some nontrivial immediate consequences.
Corollary 1.2.

(a) Let \( 1 < p < \infty \). For every \( F \in L^p(\mathbb{R}^2) \) we have

\[
\| L_F \|_{L^2 \rightarrow L^2} \leq \left( \frac{p - 1}{p} \right)^{(p-1)/p} \| F \|_{L^p},
\]

and the equality occurs if and only if \( F \) has the form in (1.3) for some \( c \in \mathbb{C}, z_0 \in \mathbb{R}^2 \).

(b) Let \( 1 \leq p < \infty \). For every \( F \in L^\infty(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \setminus \{0\} \), satisfying \((\| F \|_{L^p}/\| F \|_{L^\infty})^p \geq (p - 1)/p\), we have

\[
\| L_F \|_{L^2 \rightarrow L^2} \leq \| F \|_{L^\infty} \left( 1 - \frac{1}{p} e^{-\left(\| F \|_{L^p}/\| F \|_{L^\infty}\right)^p + (p-1)/p} \right).
\]

If \( p > 1 \) the equality occurs in (1.10) if and only if \( F \) has the form in (1.7) for some \( c \in \mathbb{C} \setminus \{0\}, C \geq 1, z_0 \in \mathbb{R}^2 \).

If \( p = 1 \) the equality occurs in (1.10) if and only if \( F = c \chi_B \) for some \( c \in \mathbb{C} \setminus \{0\}, \) where \( B \subset \mathbb{R}^2 \) is a ball.

In the above cases of equality, if \( F \neq 0 \), we have \( \| L_F \|_{L^2 \rightarrow L^2} = |\langle L_F f, g \rangle| \) for some \( f, g \in L^2(\mathbb{R}), \| f \|_{L^2} = \| g \|_{L^2} = 1 \), if and only if for some \( c, c' \in \mathbb{C}, |c| = |c'| = 1 \) and some \((x_0, \omega_0) \in \mathbb{R}^2, (1.8) \) holds true.

Some remarks are in order. The estimate (1.9) already appeared in [10, Lemma 4], and is easily seen to be equivalent, by duality, to Lieb’s uncertainty inequality for the short-time Fourier transform (with Gaussian window) [38] (see also [13, 39]), which in dimension \( d = 1 \) reads as follows: for \( 2 \leq p < \infty \),

\[
\| \mathcal{V} f \|_{L^p(\mathbb{R}^2)} \leq (2/p)^{1/p} \| f \|_{L^2(\mathbb{R})}.
\]

However, our proof does not rely on (1.11) and therefore provides an alternative indirect proof of (1.10) (we address to [38] for applications of this estimate in quantum physics).

If \( p = 1 \), (1.10) reduces to the estimate

\[
\| L_F \|_{L^2 \rightarrow L^2} \leq \| F \|_{L^\infty} \left( 1 - e^{-\| F \|_{L^1}/\| F \|_{L^\infty}} \right),
\]

which was already proved in [27] under the additional assumption that \( F \) is real valued (so that \( L_F \) is self-adjoint) and spherically symmetric, exploiting the fact that in that case \( L_F \) diagonalizes in the Hermite bases. Also, when \( F \) is the characteristic function of a measurable subset \( \Omega \) of finite measure the latter estimate reduces to (1.2).

The appearance in (1.7) of truncated Gaussians, as extremal functions of a time-frequency concentration problem, seems an unprecedented fact in the literature, as far as we know. Also, the presence of two regimes according to a critical value for \( \| F \|_{L^p}/\| F \|_{L^\infty} \) arises naturally in the variational perspective of Theorem 1.1. Indeed, by a slicing argument and using the concentration estimates (1.1) on the superlevel sets of \( |F| \), we are led to an
optimization problem for the distribution function $u(t)$ of $|F|$, which will be solved first, in Section 2. The condition $(B/A)^p \geq (p-1)/p$ in Theorem 2.1 (b) then corresponds to the case when the optimal $u$ is strictly positive in $(0, A)$. Incidentally, this pattern seems to go to the very heart of the problem and is flexible enough to encompass more general function spaces, where the norm $\|F\|_{L^p}$ is replaced by $\int_{\mathbb{R}^2} G(|F(x, \omega)|) \, dx \, d\omega$, $G$ being a nonnegative convex function satisfying some natural conditions. Here we chose to confine ourselves to the basic case of $L^p$ spaces for the sake of concreteness.

We notice that the above estimates could be rephrased as uncertainty principles. As an illustration, suppose that for some $\varepsilon \in (0, 1)$, $f \in L^2(\mathbb{R}) \setminus \{0\}$ and $F \in L^\infty(\mathbb{R}^2)$, with $0 \leq F \leq 1$, we have

$$\int_{\mathbb{R}^2} F(x, \omega) |Vf(x, \omega)|^2 \, dx \, d\omega \geq (1 - \varepsilon) \|f\|^2_{L^2}.$$  

Then it follows from (1.12) that $\|F\|_{L^1} \geq \log(1/\varepsilon)$, which generalizes [47, Theorem 1.2], where the case $F = \chi_\Omega$ was considered.

We also observe that we could regard the above results as optimal bounds for the norm of Toeplitz operators in the Bargmann-Fock space of entire functions [54], the “translation” between the two frameworks being standard (cf. [27]). Also, this general scheme should give sharp bounds for the weighted $L^q$ integral $\int_{\mathbb{R}^2} F(x, \omega) |Vf(x, \omega)|^q \, dx \, d\omega$, $2 \leq q < \infty$. Interestingly, similar results should hold for wavelet localization operators [19, 20], the estimate analogous to (1.2) being recently proved in [49]; see also [12, 35, 41] for other deep applications of this circle of ideas in the hyperbolic setting. We will investigate these issues in a sequel, in preparation.

2. A PRELIMINARY VARIATIONAL PROBLEM

In this section we address a variational problem which arises naturally in the study of the norm of localization operators.

Let $1 \leq p < \infty$, $0 < A \leq +\infty$ and $0 < B < +\infty$ be fixed. If $p = 1$, we require that $A < +\infty$.

Consider the maximization problem

(2.1) \[ \sup_{u \in C} I(u) \]

where $C = C_{A,B}$ denotes the set of decreasing functions $u : (0, A) \to [0, +\infty)$, satisfying the constraint

(2.2) \[ p \int_0^A t^{p-1} u(t) \, dt \leq B^p. \]
The functional $I(u)$ is defined as
\begin{equation}
I(u) := \int_0^A \left(1 - e^{-u(t)}\right) dt.
\end{equation}

**Theorem 2.1.** There exists a $u \in \mathcal{C}$ which attains the supremum in (2.1).
Moreover, the maximizing function is unique, and is given by
\[ u(t) = \max\{-\log(\lambda t^{p-1}), 0\} \quad t \in (0, A), \]
where $\lambda > 0$ is a constant that depends on $p$, $A$ and $B$. Moreover, for this $u$, there is equality in the constraint (2.2).

We first show the existence of an extremal function.

**Proposition 2.2.** The supremum in (2.1) is attained by at least one function $u \in \mathcal{C}$. Moreover, every extremal function $u$ achieves equality in the constraint (2.2).

**Proof.** First of all we observe that, for every $u \in \mathcal{C}$ and every $t \in (0, A)$,
\[ t^p u(t) \leq p \int_0^t \tau^{p-1} u(\tau) d\tau \leq B^p, \]
so that
\begin{equation}
(2.4) \quad u(t) \leq \frac{B^p}{t^p} \quad \forall t \in (0, A), \quad \forall u \in \mathcal{C}
\end{equation}
(when $p > 1$ and $A = +\infty$ this implies that $I(u) < \infty$ for every $u \in \mathcal{C}$, while the finiteness of $I(u)$ is trivial when $A < +\infty$, regardless of $p$).

Now let $u_n \in \mathcal{C}$ be a maximizing sequence for problem (2.1). Then, by the previous bound, $u_n(t) \leq B^p/t^p$ for every $n$ and $t > 0$. By Helly’s selection theorem we can extract a subsequence, still denoted by $u_n$ in the following, pointwise converging on to a decreasing function $u : (0, A) \to [0, +\infty)$. Now, by the Fatou lemma, we have
\[ \lim_{n \to \infty} I(u_n) = I(u), \]
which gives the first part of the statement.

Suppose now that the supremum in (2.1) is attained at $u \in \mathcal{C}$. If (2.2) holds with the strict inequality then, for some $\sigma > 1$, the function $\sigma u$ still verifies (2.2), whereas $I(\sigma u) > I(u)$, because $u$ is not 0 a.e. (recall $B > 0$), and the function $1 - e^{-s}$ in (2.3) is strictly increasing. This contradicts that $u$ is an extremal function. \qed
We now show that we can remove the monotonicity assumption in the definition of the class $C$, while retaining the same extremal functions for the corresponding maximization problem.

Namely, let $C' = C'_{A,B}$ be the set of measurable functions $u : (0, A) \to [0, +\infty)$, satisfying the constraint (2.2), and consider the maximization problem

$$\sup_{u \in C'} I(u).$$

Proposition 2.3. We have

$$\sup_{u \in C} I(u) = \sup_{u \in C'} I(u).$$

As a consequence, if the supremum on the left-hand side is attained at some $u \in C$, also the supremum on the right-hand side is attained at the same $u$.

Proof. Let $u \in C'$. Since it satisfies (2.2), all its superlevel sets $\{u > s\}$, $s > 0$, have finite measure and we can consequently consider its decreasing rearrangement $u^* : (0, A) \to [0, +\infty)$ (cf. [31, Section 10.12]). Since the function $t^{p-1}$ is increasing, we have

$$p \int_0^A t^{p-1} u^*(t) \, dt \leq p \int_0^A t^{p-1} u(t) \, dt \leq B^p,$$

so that $u^* \in C$.

On the other hand, $u$ and $u^*$ are equi-measurable, so that

$$I(u) = I(u^*).$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We already known from Proposition 2.2 that the supremum in (2.1) is attained at some $u \in C$ and that the equality holds in (2.2) for such $u$. Moreover, by Proposition 2.3 such $u$ is also en extremal function for the problem (2.5).

Let $0 < M \leq A$ be such that $u(t) > 0$ for $0 < t < M$ and $u(t) = 0$ for $t > M$ (so that $M = A$ if $u > 0$ on $(0, A)$). Let $0 < a, b < M$ and $\eta \in L^\infty(0, +\infty)$ be a function supported in $[a, b]$ and satisfying

$$\int_a^b t^{p-1} \eta(t) \, dt = 0.$$  

Then for $\varepsilon \in \mathbb{R}$ small enough (depending on $\eta$), $u + \varepsilon \eta \in C'$ and the function $I(u + \varepsilon \eta)$ has a maximum at $\varepsilon = 0$. Hence

$$0 = \frac{d}{d\varepsilon} I(u + \varepsilon \eta)|_{\varepsilon = 0} = \int_a^b e^{-u(t)} \eta(t) \, dt,$$

where we applied the dominated convergence theorem in the second equality. By a simple approximation argument the same holds true for every
\( \eta \in L^2([a, b]) \) satisfying (2.7). Hence, the restriction to \([a, b]\) of the function \( e^{-u(t)} \) is orthogonal, in \( L^2([a, b]) \), to every such \( \eta \), and therefore

\begin{equation}
(2.7) \quad e^{-u(t)} = \lambda t^{p-1}
\end{equation}

for a.e. \( t \in [a, b] \), for some constant \( \lambda \) (possibly depending on \( a \) and \( b \)). But \( a \) and \( b \) are arbitrary, so that (2.7) holds, for some constant \( \lambda > 0 \), for a.e. \( t \in (0, M) \), for every \( t \in (0, b) \), being \( u(t) \) decreasing.

We now show that, if \( M < A \), the function \( u \) is continuous at \( t = M \) as well, i.e. \( \lim_{t \to M^-} u(t) = 0. \) We argue similarly, by considering variations \( \eta \) of the form

\[
\eta(t) = -1 \text{ for } t \in (M - \delta, M), \quad \eta(t) = c \text{ for } t \in (M, M + \delta), \quad \eta(t) = 0 \text{ elsewhere, where } \delta > 0 \text{ is small enough and } c = c(\delta) \text{ is adjusted so that}
\]

\[
\int_{M-\delta}^{M+\delta} t^{p-1} \eta(t) \, dt = 0;
\]

hence

\begin{equation}
(2.8) \quad c(\delta) = \frac{M^p - (M - \delta)^p}{(M + \delta)^p - M^p} \to 1 \quad \text{as } \delta \to 0^+.
\end{equation}

Assume, by contradiction, that \( \lim_{t \to M^-} u(t) > 0. \) Then if \( \varepsilon > 0 \) is small enough, \( u + \varepsilon \eta \in \mathcal{C}' \), so that

\[
0 \geq \lim_{\varepsilon \to 0^+} \frac{I(u + \varepsilon \eta) - I(u)}{\varepsilon} = \int_{M-\delta}^{M+\delta} e^{-u(t)} \eta(t) \, dt.
\]

Multiplying by \( \delta \) and letting \( \delta \to 0^+ \), using (2.8), we obtain

\[
- \lim_{t \to M^-} e^{-u(t)} + 1 \leq 0,
\]

namely \( \lim_{t \to M^-} u(t) \leq 0 \), which gives the desired contradiction.

Summing up, we have

\[
u(t) = \max\{-\log(\lambda t^{p-1}), 0\} \quad 0 < t < A.
\]

The uniqueness of the extremal function is clear, because the constant \( \lambda > 0 \) is uniquely determined by the condition that \( u \) satisfies (2.2) with the sign of equality, being the function

\[
\lambda \rightarrow p \int_0^A t^{p-1} \max\{-\log(\lambda t^{p-1}), 0\} \, dt
\]

strictly decreasing on \((0, +\infty)\). This concludes the proof.

\text{Remark 2.4.} In Theorem 2.1 when \( p = 1 \), the case \( A = +\infty \) was excluded. Indeed, in that case there is not an extremal function: for \( u \in \mathcal{C} \) we have, by (2.2) (with \( p = 1, A = +\infty \)),

\[
\int_0^{+\infty} (1 - e^{-u(t)}) \, dt < \int_0^{+\infty} u(t) \, dt = B,
\]

where \( B > 0 \).
whereas if we consider the sequence of functions $u_n$ on $(0, +\infty)$ given by $u_n(t) = B/n$ for $t \in (0, n)$, and $u_n(t) = 0$ elsewhere, we have

$$\int_0^{+\infty} (1 - e^{-u_n(t)}) \, dt = n(1 - e^{-B/n}) \to B \quad \text{as } n \to \infty.$$ 

Also, observe that in the case $p = 1$ (hence $A < +\infty$), the unique extremal function in Theorem 2.1 reduces to the constant function $u(t) = B/A$, $0 < t < A$. This is also a consequence of Jensen inequality, being the function $1 - e^{-s}$ strictly concave: we have

$$\frac{1}{A} \int_0^A (1 - e^{-u(t)}) \, dt \leq 1 - e^{-\frac{1}{A} \int_0^A u(u) \, dt},$$

with equality if and only if $u$ is constant.

Remark 2.5. For future reference we observe that if $A < +\infty$ the extremal function $u(t) = u_\lambda(t)$ in Theorem 2.1 is strictly positive on $(0, A)$, i.e. $\lambda A^{p-1} \leq 1$, if and only if $(B/A)^p \geq (p-1)/p$.

Indeed, let $\lambda A^{p-1} \leq 1$. Then

$$B^p = p \int_0^A t^{p-1} u(t) \, dt = -A^p \left( \log \lambda + (p-1) \log A - \frac{p-1}{p} \right) \geq \frac{p-1}{p} A^p,$$

because $\log \lambda + (p-1) \log A \leq 0$, so that $(B/A)^p \geq (p-1)/p$. We observe that in this case

$$\lambda = A^{-(p-1)} e^{-(B/A)^p + (p-1)/p}.$$ 

Instead, if $M := \lambda^{-1/(p-1)} < A$ (so that $M \in (0, A)$ is the first zero of $u(t)$),

$$B^p = p \int_0^A t^{p-1} u(t) \, dt = -M^p \left( \log \lambda + (p-1) \log M - \frac{p-1}{p} \right) = \frac{p-1}{p} M^p,$$

so that

$$(B/A)^p < (B/M)^p = \frac{p-1}{p}.$$ 

Also, we notice that when $(B/A)^p \leq (p-1)/p$ (in particular for $p > 1$ and $A = +\infty$), if $u$ denotes again the corresponding extremal function, we have

$$I(u) = \left( \frac{p-1}{p} \right)^{(p-1)/p} B$$

whereas if $(B/A)^p \geq (p-1)/p$ it turns out

$$I(u) = A \left( 1 - \frac{1}{p} e^{-(B/A)^p + (p-1)/p} \right).$$
Incidentally, it is clear from these results that, if $p > 1$ and $(B/A)^p \geq (p - 1)/p$,
\[
A \left(1 - \frac{1}{p} e^{-(B/A)^p + (p-1)/p}\right) \leq \left(\frac{p-1}{p}\right)^{(p-1)/p} B,
\]
which can also be checked directly; indeed
\[
x^{-1} \left(1 - \frac{1}{p} e^{-x + (p-1)/p}\right)^p \leq \left(\frac{p-1}{p}\right)^{(p-1)}
\]
for $x \geq (p-1)/p$, being the function on the left-hand side strictly decreasing for $x > 0$.

3. Proof of the main result in dimension 1 (Theorem 1.1)

This section is devoted to the proof of Theorem 1.1. To this end, we recall from [47] the following result.

**Theorem 3.1.** [47, Theorem 1.1] For every $f \in L^2(\mathbb{R})$, $\|f\|_{L^2} = 1$, and every measurable subset $\Omega \subset \mathbb{R}^2$ of measure $|\Omega| < \infty$, we have
\[
\int_{\mathbb{R}^2} |\mathcal{V}f(x, \omega)|^2 \, dx \, d\omega \leq 1 - e^{-|\Omega|}.
\]
The equality occurs if and only if
\[
(3.1) \quad f(x) = c e^{2\pi i \omega \cdot \varphi(x-x_0)}, \quad x \in \mathbb{R},
\]
where $\varphi$ is the Gaussian in (1.1) (with $d = 1$), for some $c \in \mathbb{C}$, $|c| = 1$, and $\Omega$ is equivalent (up to a set of measure 0) to a ball centered at $(x_0, \omega_0)$.

**Proof of Theorem 1.1** (a) Suppose that $\|F\|_{L^\infty} \leq A$, $\|F\|_{L^p} \leq B$. Since $p < \infty$ we know (see e.g. [53]) that $L_F$ is a compact operator on $L^2(\mathbb{R})$. Hence there exist $f, g \in L^2(\mathbb{R})$, $\|f\|_{L^2} = \|g\|_{L^2} = 1$, such that $\|L_F\|_{L^2 \to L^2} = \langle L_F f, g \rangle$. By the Cauchy-Schwarz inequality we have
\[
(3.2) \quad \|L_F\|_{L^2 \to L^2} = \langle L_F f, g \rangle = \int_{\mathbb{R}^2} F(x, \omega) \mathcal{V}f(x, \omega) \overline{\mathcal{V}g(x, \omega)} \, dx \, d\omega
\]
\[
\leq \left( \int_{\mathbb{R}^2} |F(x, \omega)|^2 \mathcal{V}f(x, \omega) \overline{\mathcal{V}g(x, \omega)} \, dx \, d\omega \right)^{1/2} \left( \int_{\mathbb{R}^2} |F(x, \omega)|^2 \overline{\mathcal{V}g(x, \omega)} \, dx \, d\omega \right)^{1/2}.
\]
We now apply the “layer cake” formula (cf. [40, Page 26])
\[
|F(z)| = \int_0^{+\infty} \chi_{\{|F|>t\}}(z) \, dt, \quad z \in \mathbb{R}^2
\]
and Theorem 3.1 to estimate
\[
\int_{\mathbb{R}^2} |F(x, \omega)|^2 \mathcal{V}f(x, \omega) \overline{\mathcal{V}g(x, \omega)} \, dx \, d\omega = \int_{0}^{+\infty} \int_{|F|>t} |\mathcal{V}f(x, \omega)|^2 \, dx \, d\omega \, dt
\]
\[
\leq \int_0^{+\infty} (1 - e^{-u(t)}) \, dt,
\]
where
\[
u(t) = |\{|F|>t\}|
\]
is the distribution function of $|F|$.

From Theorem 2.1 and Remark 2.5 we deduce that
\[ \int_0^{+\infty} (1 - e^{-u(t)}) dt \leq \left( \frac{p-1}{p} \right)^{(p-1)/p} B. \]

Since the same estimate holds for $\int_{\mathbb{R}^2} |F(x,\omega)||\nabla g(x,\omega)|^2 \, dx \, d\omega$, we obtain (1.4).

Concerning the characterization of the extremal functions, observe that if the supremum in (1.4) is attained at $F$, then there exists equality in the above chain of inequalities. In particular by Theorem 2.1 the distribution function $u(t)$ is given by
\begin{equation}
(3.3) \quad u(t) = \max\{-\log(\lambda t^{p-1}), 0\} \quad 0 < t < A,
\end{equation}
with $\lambda = ((p-1)/p)^{(p-1)/p} B^{(p-1)}$ (cf. Remark 2.5). Moreover
\[ \int_0^{+\infty} \int_{|F|>t} |\nabla f(x,\omega)|^2 \, dx \, d\omega \, dt = \int_0^{+\infty} (1 - e^{-u(t)}) \, dt, \]
which together with Theorem 3.1 implies that
\[ \int_{|F|>t} |\nabla f(x,\omega)|^2 \, dx \, d\omega = 1 - e^{-u(t)} \]
for a.e. $t > 0$. Since the right-hand side is a continuous functions of $t$, and the left-hand side is decreasing, this equality holds for every $t > 0$, indeed. By Theorem 3.1 this implies that $f$ has the form in (3.1) for some $(x_0,\omega_0) =: z_0 \in \mathbb{R}^2$, $|c| = 1$ and the sets $\{|F| > t\}$ are balls of center $z_0$. The same happens with $f$ replaced by $g$, necessarily for the same $z_0$.

Summing up, $f$ and $g$ have the form in (3.1) possibly for different constants $c$, and $|F|$ is spherically symmetric around $z_0$, hence, by (3.3),
\[ |F(z)| = c e^{-\frac{1}{p-1}|z-z_0|^2}, \quad \text{a.e. } z \in \mathbb{R}^2, \]
with $c = \lambda^{-1/(p-1)} = ((p-1)/p)^{-1/p} B$.

Coming back to (3.2) we see that
\[ c' \int_{\mathbb{R}^2} F(x,\omega)||\nabla f(x,\omega)||^2 \, dx \, d\omega = \int_{\mathbb{R}^2} |F(x,\omega)||\nabla f(x,\omega)||^2 \, dx \, d\omega \]
for some $c' \in \mathbb{C}$, $|c'| = 1$, and therefore
\[ c' F(z)||\nabla f(z)||^2 = |F(z)||\nabla f(z)||^2, \quad \text{a.e. } z \in \mathbb{R}^2. \]
Since $|\nabla f(z)||^2 > 0$ (indeed, $|\nabla f(z)||^2$ is easily seen to be a Gaussian function) we have $c' F = |F|$, which gives the desired expression (1.5).

(b) The proof follows the same pattern as above, and therefore we just point out the differences. Assuming $\|F\|_{L^\infty} \leq A$, $\|F\|_{L^p} \leq B$ and arguing
as above, from Theorem 2.1 and Remark 2.5 since now \((B/A)^p \geq (p-1)/p\), we obtain
\[
\int_0^{+\infty} (1 - e^{-u(t)}) \, dt \leq A \left( 1 - \frac{1}{p} e^{-(B/A)^p + (p-1)/p} \right),
\]
which gives at once the desired estimate (1.3).

As far as the extremal functions is concerned, by Theorem 2.1 and Remark 2.5 the optimal distribution function \(u(t) = |\{ |F| > t \}|\) is given now by
\[
u(t) = -\log(\lambda t^{p-1}), \quad 0 < t < A,
\]
with \(\lambda = A^{-1} e^{-(B/A)^p + (p-1)/p}\), whereas for \(p = 1\)
\[
|F(z)| = A \chi_B
\]
where \(B\) is a ball of measure \(B/A\). Then we conclude as above.

The last sentence in the statement is also clear from the above proof. □

**Remark 3.2.** The proof of Theorem 3.1 shows that, if \((B/A)^p \geq (p-1)/p\), the maximization problem
\[
\sup \|F\|_{L^\infty} \leq A, \|F\|_{L^p} \leq B, \|L_F\|_{L^2 \to L^2}
\]
is in fact equivalent to
\[
\sup \|L_F\|_{L^2 \to L^2}.
\]
It easy to see that also if \((B/A)^p < (p-1)/p\) we have
\[
(3.4) \sup_{\|F\|_{L^\infty} = A, \|F\|_{L^p} = B} \|L_F\|_{L^2 \to L^2} = \sup_{\|F\|_{L^\infty} \leq A, \|F\|_{L^p} \leq B} \|L_F\|_{L^2 \to L^2} = \left( \frac{p-1}{p} \right)^{(p-1)/p} B.
\]
Indeed, consider the extremal function \(F\) in (1.5), which satisfies \(\|F\|_{L^p} = B\) and \(\left( \|F\|_{L^p} / \|F\|_{L^\infty} \right)^p = (p-1)/p\). If \((B/A)^p < (p-1)/p\), i.e., \(A > \|F\|_{L^\infty}\), and \(A < +\infty\), adding to \(F\) a suitable multiple of a characteristic function, and normalizing in \(L^p\), produces a function \(F'\), with \(\|F'\|_{L^p} = B\), \(\|F'\|_{L^\infty} = A\) and \(\|F - F'\|_{L^p}\) arbitrarily small. Then \(\|L_{F'}\|_{L^2 \to L^2}\) will be arbitrarily close to \(\|L_F\|_{L^2 \to L^2}\) by (1.9). A similar argument holds for \(A = +\infty\). This gives (3.4).

4. THE MULTIDIMENSIONAL CASE

Looking for an extension of Theorem 1.1 to the case of dimension \(d > 1\), and following the same pattern as in its proof, suitable estimates for
\[ \int_{\Omega} |\nabla f(x, \omega)|^2 \, dx \, d\omega \] are needed, where \( f \in L^2(\mathbb{R}^d) \) and \( \Omega \subset \mathbb{R}^{2d} \) is a measurable subset of finite measure. The relevant bound was proved in [47, Theorem 4.1] and reads

\[ (4.1) \quad \int_{\Omega} |\nabla f(x, \omega)|^2 \, dx \, d\omega \leq G(|\Omega|) \| f \|_{L^2}^2, \]

where

\[ (4.2) \quad G(s) := \frac{1}{(d-1)!} \int_0^{\pi(s/\omega_{2d})^{1/d}} \tau^{d-1} e^{-\tau} \, d\tau, \]

\( \omega_{2d} \) being the measure of the unit ball in \( \mathbb{R}^{2d} \). Moreover, the equality occurs in (4.1) if and only if

\[ f(x) = ce^{2\pi i x \cdot \omega_0} \varphi(x-x_0), \]

for some \( c \in \mathbb{C}, (x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d \), and \( \Omega \) is equivalent to a ball of center \( (x_0, \omega_0) \), where \( \varphi \) is the Gaussian in (1.1).

For future reference we recall the definition of the (lower) incomplete \( \gamma \) function as

\[ (4.3) \quad \gamma(k, s) := \int_0^s \tau^{k-1} e^{-\tau} \, d\tau \]

where \( k \geq 1 \) is an integer and \( s \geq 0 \), so that

\[ (4.4) \quad \frac{\gamma(k, s)}{(k-1)!} = 1 - e^{-s} \sum_{j=0}^{k-1} \frac{s^j}{j!} \]

and

\[ (4.5) \quad G(s) = \frac{\gamma(d, \pi(s/\omega_{2d})^{1/d})}{(d-1)!}. \]

Let \( 1 \leq p < \infty \), \( 0 < A \leq +\infty \), \( 0 < B < +\infty \), with \( A < +\infty \) if \( p = 1 \). Arguing as in the proof of Theorem 1.1 one is now faced with the problem

\[ \sup_{u \in \mathcal{C}} \tilde{I}(u) \]

where \( \mathcal{C} = \mathcal{C}_{A,B} \) is the same class of functions occurring in Section 2 (namely \( u : (0, A) \to [0, +\infty) \) is decreasing and satisfies the constraint (2.2)), and the functional \( \tilde{I} \) is defined as

\[ \tilde{I}(u) = \int_0^A G(u(t)) \, dt. \]

Again we see that \( \tilde{I}(u) \) is finite for \( u \in \mathcal{C} \), because \( G(s) \leq Cs \) for \( s > 0 \). This problem can be addressed essentially by the same argument as in Section 2. We deduce that there exists a unique extremal function \( u \in \mathcal{C} \); moreover \( u \) is continuous on \( (0, A) \) and in the interval where it is strictly positive we have

\[ G'(u(t)) = \lambda t^{p-1} \]
for some constant $\lambda > 0$. Since
\[
G'(s) = \frac{\pi^d}{d!} e^{-\pi(s/\omega_{2d})^{1/d}}
\]
we find
\[
(4.6) \quad u(t) = u_\lambda(t) = \max \left\{ \frac{\omega_{2d}}{\pi^d} (-\log(\lambda t^{p-1}))^{d}, 0 \right\} \quad 0 < t < A,
\]
for a new constant $\lambda > 0$ uniquely determined by the fact that $u_\lambda$ satisfies the constraint (2.2) with equality, i.e.
\[
(4.7) \quad p \int_0^A t^{p-1} u_\lambda(t) \, dt = B^p.
\]
Indeed, the expression on the left-hand side, for fixed $A$ as above, is strictly decreasing as a function of $\lambda > 0$. We will write
\[
(4.8) \quad \lambda = \psi(A, B)
\]
where $\psi$ in general is not an elementary function (when $p > 1$). Also, observe that, if $p > 1$ and $A < +\infty$, $u_\lambda$ is strictly positive on $(0, A)$, i.e.
\[
(4.9) \quad \sup \|F\|_{L^\infty} \leq A, \quad \|F\|_{L^p} \leq B, \quad \|L_F\|_{L^2 \to L^2} = \left( \frac{p-1}{p} \right)^{d(p-1)/p} B,
\]
and the supremum is attained at $F$ if and only if
\[
(4.10) \quad F(z) = ce^{-\frac{\pi^{1/d}}{\pi^d} |z-z_0|^2} \quad z \in \mathbb{R}^{2d}
\]
for some $c \in \mathbb{C} \setminus \{0\}$, with $|c| = (\pi^d/(d!\omega_{2d}))^{1/p}((p-1)/p)^{-d/p} B$ and $z_0 \in \mathbb{R}^{2d}$.

(b) If $(B/A)^p \geq \frac{d!\omega_{2d}}{\pi^d} \left( \frac{p-1}{p} \right)^d$,
\[
(4.11) \quad \sup \|F\|_{L^\infty} \leq A, \quad \|F\|_{L^p} \leq B, \quad \|L_F\|_{L^2 \to L^2} = \int_0^A G(u_\lambda(t)) \, dt,
\]
for some constant $\lambda > 0$. Since
\[
G'(s) = \frac{\pi^d}{d!} e^{-\pi(s/\omega_{2d})^{1/d}}
\]
we find
\[
(4.6) \quad u(t) = u_\lambda(t) = \max \left\{ \frac{\omega_{2d}}{\pi^d} (-\log(\lambda t^{p-1}))^{d}, 0 \right\} \quad 0 < t < A,
\]
for a new constant $\lambda > 0$ uniquely determined by the fact that $u_\lambda$ satisfies the constraint (2.2) with equality, i.e.
\[
(4.7) \quad p \int_0^A t^{p-1} u_\lambda(t) \, dt = B^p.
\]
Indeed, the expression on the left-hand side, for fixed $A$ as above, is strictly decreasing as a function of $\lambda > 0$. We will write
\[
(4.8) \quad \lambda = \psi(A, B)
\]
where $\psi$ in general is not an elementary function (when $p > 1$). Also, observe that, if $p > 1$ and $A < +\infty$, $u_\lambda$ is strictly positive on $(0, A)$, i.e.
\[
(4.9) \quad \sup \|F\|_{L^\infty} \leq A, \quad \|F\|_{L^p} \leq B, \quad \|L_F\|_{L^2 \to L^2} = \left( \frac{p-1}{p} \right)^{d(p-1)/p} B,
\]
and the supremum is attained at $F$ if and only if
\[
(4.10) \quad F(z) = ce^{-\frac{\pi^{1/d}}{\pi^d} |z-z_0|^2} \quad z \in \mathbb{R}^{2d}
\]
for some $c \in \mathbb{C} \setminus \{0\}$, with $|c| = (\pi^d/(d!\omega_{2d}))^{1/p}((p-1)/p)^{-d/p} B$ and $z_0 \in \mathbb{R}^{2d}$.

(b) If $(B/A)^p \geq \frac{d!\omega_{2d}}{\pi^d} \left( \frac{p-1}{p} \right)^d$,
\[
(4.11) \quad \sup \|F\|_{L^\infty} \leq A, \quad \|F\|_{L^p} \leq B, \quad \|L_F\|_{L^2 \to L^2} = \int_0^A G(u_\lambda(t)) \, dt,
\]
for some constant $\lambda > 0$. Since
\[
G'(s) = \frac{\pi^d}{d!} e^{-\pi(s/\omega_{2d})^{1/d}}
\]
where \( \lambda = \psi(A, B) \), cf. (1.2) and (4.8).

If \( p > 1 \) the supremum in (4.11) is attained at \( F \) if and only if

\[
F(z) = c \min \{ Ce^{-\frac{1}{d} (|z| - z_0)^2}, 1 \} \quad z \in \mathbb{R}^d
\]

with \( C = A^{-1} \psi(A, B)^{-1/(p-1)} \), for some \( c \in \mathbb{C} \setminus \{0\} \), with \(|c| = A\) and \( z_0 \in \mathbb{R}^d \).

If \( p = 1 \) (4.11) can be written more explicitly as

\[
\sup_{\|F\|_L^\infty \leq A, \|F\|_{L^1} \leq B} \|L_F\|_{L^2 \to L^2} = A \gamma(d, \pi \omega_{2d}^{-1/d} (B/A)^{1/d}) (d-1)!
\]

and the supremum is attained at \( F \) if and only if \( F = c \chi_B \) for some \( c \in \mathbb{C} \setminus \{0\} \), with \(|c| = A\), where \( B \subset \mathbb{R}^d \) is a ball of measure \(|B| = B/A\).

In the above cases where the supremum in (4.11) is attained at some \( F \neq 0 \), we have \( \|L_F\|_{L^2 \to L^2} = |\langle L_F f, g \rangle| \) for some \( f, g \in L^2(\mathbb{R}^d) \), \( \|f\|_{L^2} = \|g\|_{L^2} = 1 \), if and only if for some \( c, c' \in \mathbb{C} \), \(|c| = |c'| = 1\) and some \((x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d \) we have

\[
f(x) = c'g(x) = ce^{2\pi ix \cdot \omega} \varphi(x - x_0), \quad x \in \mathbb{R}^d,
\]

where \( \varphi \) is the Gaussian in (1.1).

The following estimates therefore follow immediately.

**Corollary 4.2.**

(a) Let \( 1 < p < \infty \). For every \( F \in L^p(\mathbb{R}^d) \) we have

\[
\|L_F\|_{L^2 \to L^2} \leq \left( \frac{p-1}{p} \right)^{d(p-1)/p} \|F\|_{L^p},
\]

and the equality occurs and if and only if \( F \) has the form in (4.11) for some \( c \in \mathbb{C} \), \( z_0 \in \mathbb{R}^d \).

(b) Let \( 1 \leq p < \infty \). For every \( F \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \setminus \{0\} \), satisfying

\[
\frac{\|F\|_{L^p}^p}{\|F\|_{L^\infty}^p} \geq \frac{d!\omega_{2d}}{\pi^d} \left( \frac{p-1}{p} \right)^d,
\]

we have

\[
\|L_F\|_{L^2 \to L^2} \leq \int_0^{\|F\|_{L^\infty}} G(u\lambda(t)) \, dt,
\]

where \( \lambda = \psi(\|F\|_{L^\infty}, \|F\|_{L^p}) \), cf. (1.2) and (4.8).

If \( p > 1 \) the equality occurs in (4.16) if and only if \( F \) has the form in (4.11) for some \( c \in \mathbb{C} \setminus \{0\} \), \( C \geq 1 \), \( z_0 \in \mathbb{R}^d \).

If \( p = 1 \) (4.11) can be written more explicitly as

\[
\|L_F\|_{L^2 \to L^2} \leq \|F\|_{L^\infty} \gamma(d, \pi \omega_{2d}^{-1/d} (\|F\|_{L^1}/\|F\|_{L^\infty})^{1/d}) (d-1)!
\]
and the equality occurs if and only if $F = c \chi_B$ for some $c \in \mathbb{C} \setminus \{0\}$, where $B \subset \mathbb{R}^{2d}$ is a ball.

In the above cases of equality, if $F \neq 0$, we have $\|L_F\|_{L^2 \to L^2} = |\langle L_F f, g \rangle|$ for some $f, g \in L^2(\mathbb{R}^d)$, $\|f\|_{L^2} = \|g\|_{L^2} = 1$, if and only if for some $c, c' \in \mathbb{C}$, $|c| = |c'| = 1$ and some $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, (4.14) holds true.

**Proof of Theorem 4.1.** The desired result can be obtained by following the same pattern as in the proof of Theorem 1.1. Hence, we point out only the needed modifications.

(a) Using (4.1) we are led to consider the optimization problem at the beginning of this section. The unique solution $u_\lambda \in C$ is given in (4.6), with the constant $\lambda$ as in (4.7). The corresponding extremal functions $F$ will be therefore the Gaussians in (4.10), and the norm $\|L_F\|_{L^2 \to L^2}$ can be computed using the formula in [17, Formula (11)] for the eigenvalues in the case of spherically symmetric weights (keeping in mind the different normalization for the Fourier transform), which gives the expression on the right-hand side of (4.9).

(b) The desired results follow again by the same argument, using (4.1) and the results at the beginning of this section.

When $p = 1$, the function $u$ in (4.6) is constant: $u(t) = B/A$. The more explicit formula (4.13) then follows from (4.11) and (4.5). □

**References**

[1] L. D. Abreu and M. Dörfler. An inverse problem for localization operators. *Inverse Problems*, 28(11):115001, 16, 2012.

[2] L. D. Abreu, K. Gröchenig, and J. L. Romero. On accumulated spectrograms. *Trans. Amer. Math. Soc.*, 368(5):3629–3649, 2016.

[3] L. D. Abreu, Z. Mouayn, and F. Voigtlaender. A fractal uncertainty principle for Bergman spaces and analytic wavelets, arXiv:2201.11705, 2022.

[4] L. D. Abreu, J. M. Pereira, and J. L. Romero. Sharp rates of convergence for accumulated spectrograms. *Inverse Problems*, 33(11):115008, 12, 2017.

[5] L. D. Abreu and M. Speckbacher. Deterministic guarantees for $L^1$-reconstruction: A large sieve approach with geometric flexibility. *IEEE Proceedings SampTA*, 2019.

[6] L. D. Abreu and M. Speckbacher. Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces. *Bull. Sci. Math.*, 171:Paper No. 103032, 25, 2021.

[7] D. Bayer and K. Gröchenig. Time-frequency localization operators and a Berezin transform. *Integral Equations Operator Theory*, 82(1):95–117, 2015.

[8] F. A. Berezin. Wick and anti-Wick symbols of operators. *Mat. Sb. (N.S.),* 86(128):578–610, 1971.

[9] F. A. Berezin and M. A. Shubin. *The Schrödinger equation*, volume 66 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 1983 Russian edition by Yu. Rajabov, D. A.

\[1\] Alternatively, for such $F$ the eigenfunctions of $L_F$ are the Hermite functions [17], and the operator $L_F$, when applied to a tensor product type function, splits as expected. Hence one can compute $\|L_F\|_{L^2 \to L^2}$ from the one dimensional case in Theorem 1.1.
Leîtes and N. A. Sakharova and revised by Shubin. With contributions by G. L. Litvinov and Leîtes.

[10] P. Boggiatto, E. Carypis, and A. Oliaro. Two aspects of the Donoho-Stark uncertainty principle. *J. Math. Anal. Appl.*, 434(2):1489–1503, 2016.

[11] A. Bonami and B. Demange. A survey on uncertainty principles related to quadratic forms. *Collect. Math.*, (Vol. Extra):1–36, 2006.

[12] O. F. Brevig, J. Ortega-Cerdà, K. Seip, and J. Zhao. Contractive inequalities for Hardy spaces. *Funct. Approx. Comment. Math.*, 59(1):41–56, 2018.

[13] E. A. Carlen. Some integral identities and inequalities for entire functions and their application to the coherent state transform. *J. Funct. Anal.*, 97(1):231–249, 1991.

[14] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[15] E. Cordero and F. Nicola. Sharp continuity results for the short-time Fourier transform and for localization operators. *Monatsh. Math.*, 162(3):251–276, 2011.

[16] M. G. Cowling and J. F. Price. Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality. *SIAM J. Math. Anal.*, 15(1):151–165, 1984.

[17] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.

[18] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.

[19] C. Fernández and A. Galbis. Compactness of time-frequency localization operators on $L^2(\mathbb{R}^d)$. *J. Funct. Anal.*, 233(2):335–350, 2006.

[20] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1952. 2d ed.

[21] K. Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.

[22] K. Gröchenig. Uncertainty principles for time-frequency representations. In *Advances in Gabor analysis*, Appl. Numer. Harmon. Anal., pages 11–30. Birkhäuser Boston, Boston, MA, 2003.

[23] K. Gröchenig. Uncertainty principles for time-frequency representations. In *Advances in Gabor analysis*, Appl. Numer. Harmon. Anal., pages 11–30. Birkhäuser Boston, Boston, MA, 2003.

[24] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):Paper No. 15, 13, 2022.

[25] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[26] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[27] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[28] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[29] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[30] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[31] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[32] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[33] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[34] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[35] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[36] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.

[37] Y. V. Galperin and K. Gröchenig. Uncertainty principles as embeddings of modulation spaces. *J. Fourier Anal. Appl.*, 16(1):181–202, 2012.
[33] H. Knutsen. Daubechies’ time-frequency localization operator on Cantor type sets. J. Funct. Anal., 282(9):Paper No. 109412, 28, 2022.

[34] H. Knutsen. A fractal uncertainty principle for the short-time Fourier transform and Gabor multipliers, arXiv:2204.03068, 2022.

[35] A. Kulikov. Functionals with extrema at reproducing kernels, arXiv:2203.12349, 2022.

[36] H. J. Landau. An overview of time and frequency limiting. In Fourier techniques and applications (Kensington, 1983), pages 201–220. Plenum, New York, 1985.

[37] H. J. Landau and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. II. Bell System Tech. J., 40:65–84, 1961.

[38] E. H. Lieb. Proof of an entropy conjecture of Wehrl. Comm. Math. Phys., 62(1):35–41, 1978.

[39] E. H. Lieb. Integral bounds for radar ambiguity functions and Wigner distributions. J. Math. Phys., 31(3):594–599, 1990.

[40] E. H. Lieb and M. Loss. Analysis, volume 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2001.

[41] E. H. Lieb and J. P. Solovej. Wehrl-type coherent state entropy inequalities for SU(1, 1) and its AX + B subgroup. In Partial differential equations, spectral theory, and mathematical physics—the Ari Laptev anniversary volume, EMS Ser. Congr. Rep., pages 301–314. EMS Press, Berlin, 2021.

[42] F. Luef and E. Skrettingland. Convolutions for localization operators. J. Math. Pures Appl. (9), 118:288–316, 2018.

[43] F. Luef and E. Skrettingland. On accumulated Cohen’s class distributions and mixed-state localization operators. Constr. Approx., 52(1):31–64, 2020.

[44] S. Mallat. A wavelet tour of signal processing. The sparse way. Elsevier/Academic Press, Amsterdam, third edition, 2009. With contributions from Gabriel Peyré.

[45] C. Muscalu and W. Schlag. Classical and multilinear harmonic analysis. Vol. I, volume 137 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2013.

[46] F. Nicola, J. L. Romero, and S. I. Trapasso. On the existence of optimizers for time-frequency concentration problems, arXiv:2112.09675, 2021.

[47] F. Nicola and P. Tilli. The Faber-Krahn inequality for the short-time Fourier transform. Invent. Math., 2022, open access: https://doi.org/10.1007/s00222-022-01119-8.

[48] F. Nicola and S. I. Trapasso. A note on the HRT conjecture and a new uncertainty principle for the short-time Fourier transform. J. Fourier Anal. Appl., 26(4):Paper No. 68, 7, 2020.

[49] J. Ramos and P. Tilli. A Faber-Krahn inequality for wavelet transforms, arXiv:2205.07908, 2022.

[50] K. Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. SIAM J. Math. Anal., 22(3):856–876, 1991.

[51] D. Slepian. Some comments on Fourier analysis, uncertainty and modeling. SIAM Rev., 25(3):379–393, 1983.

[52] A. Wigderson and Y. Wigderson. The uncertainty principle: variations on a theme. Bull. Amer. Math. Soc. (N.S.), 58(2):225–261, 2021.

[53] M. W. Wong. Wavelet transforms and localization operators, volume 136 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2002.

[54] K. Zhu. Analysis on Fock spaces, volume 263 of Graduate Texts in Mathematics. Springer, New York, 2012.
Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

Email address: fabio.nicola@polito.it

Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

Email address: paolo.tilli@polito.it