WEIGHTED ESTIMATES FOR ONE SIDED MARTINGALE TRANSFORMS

WEI CHEN, RUI HAN, AND MICHAEL T. LACEY

Abstract. Let $T f = \sum_I \varepsilon_I (f, h_I^+) h_I^-$, Here, $|\varepsilon_I| = 1$, and $h_I$ is the Haar function defined on dyadic interval $J$. We show that, for instance,

$$\|T\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A^+}^2.$$ 

Above, we use the one sided $A_2$ characteristic for the weight $w$. This is an instance of a one sided $A_2$ conjecture. Our proof of this fact is difficult, as the very quick known proofs of the $A_2$ theorem do not seem to apply in the one sided setting.

1. One-sided martingale transform

We prove new sharp one sided weight inequalities for certain kinds of one sided martingale transforms. One sided weights are variants of the usual $A_p$ weights on the real line, for which the following supremum is finite:

$$[w]_{A^+_p} = \sup_I w(I^-)^{-p^{-1}} \left[ \frac{\sigma(I^+)}{|I^+|} \right]^{p^{-1}},$$

where $w$ is a non-negative function, locally integrable, and $\sigma = w^{-\frac{1}{p-1}}$ is also locally integrable. The set $I = (a, a+2\delta)$ is an interval, with left and right halves $I^- = (a, a+\delta)$, and $I^+ = (a+\delta, a+2\delta)$ respectively. There are variants of this notion in higher dimension which we will not directly discuss.

For the endpoint cases of $p = 1, \infty$, we use the one sided maximal function

$$M_+ f = \sup_I \frac{1_{I^-}}{|I^+|} \int_{I^+} |f| \, dx.$$ 

Define $M_-$ similarly. We set

$$[w]_{A^+_1} = \left\| \frac{M_- w}{w} \right\|_\infty, \quad [w]_{A^+_\infty} = \sup_I w(I)^{-1} \int_I M_-(w 1_I) \, dx.$$ 

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The definitions of $A_p^-$ follow similarly, and it is important to note the duality between these expressions. In particular, $[w]_{A_p^+} = [\sigma]_{A_p^-}^{p-1}$.

The qualitative aspects of the $A_p^+$ theory closely matches that of the usual $A_p$ theory. And, in the latter, the study of sharp constants, which has been under very rapid development in the $A_p$ theory. But some of the corresponding results for $A_p^+$ seem much harder to establish. The maximal function estimates were established by Sawyer [16], and we have this result, matching known results for the usual maximal function.

**Theorem 1.1.** [14, Thm1.5 and Thm 1.9] These inequalities hold:

\[ \|M_+\|_{L^p(w) \to L^{p,\infty}(w)} \lesssim [w]_{A_p^+}^{1/p}, \quad 1 < p < \infty, \]

\[ \|M_+\|_{L^p(w) \to L^p(w)} \lesssim ([w]_{A_p^+} [\sigma]_{A_p^-})^{1/p}, \quad 1 < p < \infty. \]

We remark that we have

\[ [\sigma]_{A_p^-} \leq [\sigma]_{A_p^+} = [w]_{A_p^+}^{p-1}. \]

The main contribution of this paper is to establish some sharp $A_p^+$ estimates for (maximal truncations of) one sided martingale transforms. The latter are defined by

\[ T f = \sum_{\text{dyadic } I} \varepsilon_I \langle f, h_{I^+} \rangle h_{I^-}, \]

\[ T_\delta f = \sup_{\delta > 0} \left| \sum_{\text{dyadic } I, |I| > \delta} \varepsilon_I \langle f, h_{I^+} \rangle h_{I^-} \right|, \]

where $\varepsilon_I$ takes values in $\{-1, 1\}$ and $h$ is the Haar function defined by

\[ h_I(x) = \frac{1_{I^+}(x) - 1_{I^-}(x)}{\sqrt{|I|}}. \]

Our main result is the sharp weak $L^p$ inequality, for all $1 < p < \infty$, and the sharp inequality on $L^2$.

**Theorem 1.2.** The following inequalities are uniform over all one sided martingale transforms.

\[ \|T\|_{L^p(w) \to L^{p,\infty}(w)} \lesssim [w]_{A_p^+}^{1/p} [w]_{A_p^-}^{1/p'}, \quad 1 < p < \infty, \]

\[ \|T\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_p^+}^{1/2} \max\{[\sigma]_{A_p^-}, [w]_{A_p^-}^{1/2}\}^{1/2}. \]

This is a very modest assertion, as compared to the advanced state of the $A_p$ theory. We will address what we think it true, and some of the unexpected complications we encountered, in the concluding section of this paper.
Concerning the proof, it is noteworthy that the proof of the maximal function estimates (1.2) are relatively simple. One uses the weak-type result, together with a sharp reverse Hölder estimate \[14\], Thm 1.8. Marcinkiewicz interpolation finishes the proof. (This type of argument was first identified by Buckley \[3\].)

No such argument can work for the martingale transforms, since the \(L^\infty\) endpoint estimate is too large. Indeed, most of the strategies that are so successful for the usual \(A_p\) weights do not seem to generalize to the one sided setting. We adapt methods from \[11, 13\] in order to complete the proof. Namely, we establish a general version of a distributional inequality, which is the key to plus/minus characteristic of the \(A_p\) condition. This is combined with testing conditions, and a corona construction. There are new complications in carrying out this program, and some serious obstructions to proving what one would expect to be true.

The theory of \(A_p^+\) weights was started by Sawyer \[16\]. He showed the natural analog of Muckenhoupt’s famous theorem. The \(A_p^+\) condition is characterized by those positive a.e. weights \(w\) for which \(M_+\) is bounded on \(L^p(w)\). Amir, Foranzi and Martín-Reyes \[1\] established several natural analogs of the \(A_p^+\) theory for singular integrals. The recent paper of Chill and Król \[5\] gives an elegant development of the \(A_p^+\) theory, especially with an eye towards applications in parabolic PDEs. While there is a sizable literature on \(A_p^+\) weights, even in the singular integral setting, the results we could find were qualitative in nature. As far as we are aware, the question of sharp constants has not been addressed before for singular integral like objects in the \(A_p^+\) setting.

2. Background

It is a useful remark that the one weight inequality \(|\|Tf\|_{L^p(w)} \lesssim |\|f\|_{L^p(\sigma)}|\) is equivalent to the two weight inequality
\[
|\|T(\sigma_1 f)\|_{L^p(w)} \lesssim |\|f\|_{L^p(\sigma)}, \quad \sigma = w^{1-p'}.
\]

The latter inequality is a two weight inequality, and is convenient as it dualizes correctly, namely the inequality above is equivalent to \(|\|T^*(w\phi)\|_{L^{p'}(\sigma)} \lesssim |\|\phi\|_{L^{p'}(w)}|\).

There is a characterizations of the two weight inequality (2.1) in the case of \(p = 2\), due to Nazarov, Treil and Volberg \[15\]. It completely solves the question for dyadic operators. For the sake of clarity, we recall it, in the context of our one sided martingale transforms.

**Theorem 2.1.** \[15\] Let \((w, \sigma)\) be a pair of weights, and \(T\) a one sided martingale transform. We have \(|\|T(\sigma)\|_{L^2(\sigma) \to L^2(w)} \lesssim \mathcal{F}\), where \(\mathcal{F}\) is the best constant in the inequalities below, uniformly over intervals \(I\).

\[
|\|1_I T(\sigma 1_I)\|_{L^2(w)} \leq \mathcal{F} \sigma(I)^{1/2},
\]

\[
|\|1_I T^*(w 1_I)\|_{L^2(\sigma)} \leq \mathcal{F} w(I)^{1/2}.
\]
The condition \((2.2)\) is referred to as a testing condition. It is a fundamental reduction in complexity in proving the \(L^2\) bound.

A corresponding theorem in the \(L^p\) setting is necessarily more complicated. A true characterization has not yet been found, but must be of a vector valued nature, see \([17]\). Indeed, the main result of this last paper is powerful, but are stated with an assumption about 'quadratic \(A_p\) condition', \([17, (3.2)]\), which is ill suited to the one sided case.

We do not need a full characterization, and if one restricts attention to dyadic Calderón-Zygmund operators, very sharp conditions can be given. We will use this corollary to \([11, \text{Thm 4.3}]\). It is a two weight inequality for maximal truncations of one sided martingale transforms.

**Theorem 2.2.** Let \(w, \sigma\) be two weights, and \(T\) a one sided martingale transform. We have the weak-type bound

\[
\|T_{\sharp}^*(\sigma f)\|_{L^{p,\infty}(w)} \lesssim (\mathcal{M}_p^+ + \mathcal{F}_p)\|f\|_{L^p(\sigma)}
\]

where the two constants on the right are the best constants in the following inequalities, holding for all functions \(f\), and intervals \(I\).

\begin{align}
(2.3) & \quad \|M_+(\sigma f)\|_{L^{p,\infty}(w)} \leq \mathcal{M}_p^+\|f\|_{L^p(\sigma)}, \\
(2.4) & \quad \int_I T_{\sharp}^*(\sigma 1_I f) \, dw \leq \mathcal{F}_p\|f\|_{L^p(\sigma)}w(I)^{1/p'}.
\end{align}

The Theorem as formulated above is not a corollary, since the cited \([11, \text{Thm 4.3}]\) has the usual maximal function appearing in \((2.3)\). The proof however extends immediately to the version above. We will comment more of this in the final section of the paper.

The condition \((2.4)\) does not look at all like the corresponding testing condition \((2.2)\). Let us explain here why they are similar. The maximal truncation operator is not linear. But its boundedness is equivalent to the boundedness of a family of linear operators given as follows. For any measurable \(\delta : \mathbb{R} \to (0, \infty)\), let us set

\[
T_{\delta}f(x) = \sum_{I: |I| > \delta(x)} \langle f, h_{I^+} \rangle h_{I^-}(x)
\]

be a linearization of the maximal truncations. It is easy to see that \(T_{\delta}\) is bounded if and only if the family of linear operators \(T_{\delta}\) satisfy a norm bound independent of the choice of measurable \(\delta\).

Using linearity, note that the integral in \((2.4)\) can be written as

\[
\langle T_{\delta}(f 1_I \sigma), \varphi_I \cdot w \rangle \leq \mathcal{F}_p\|f\|_{L^p(\sigma)}w(I)^{1/p'}.
\]

This inequality should hold uniformly over all choices of truncation \(\delta\), and \(|\varphi_I| = 1_I\). Therefore, the condition \((2.4)\) is the same as

\[
(2.5) \quad \|1_I T_{\delta}^*(w \varphi_I)\|_{L^{p'}(\sigma)} \leq \mathcal{F}_p w(I)^{1/p'}.
\]
This is the form of (2.4) that we will use. It is a testing condition.

The appearance of $T_\delta^*$ is not so familiar. Crucially, these operators satisfy a weak $L^1$ inequality. We state here, and remark that the proof is not easy.

**Proposition 2.3.** [11, Thm. 9.3] $T_\delta^*$ maps $L^1$ to weak $L^1$, uniformly over the choice of the one sided martingale transform and the choice of the linearization, $\delta$.

Finally, we need a Lemma which states that a weak type distributional estimate is sufficient for a John-Nirenberg type estimate.

**Lemma 2.4.** ([8, Lemma 5.5], [11, Lemma 10.2]) Let $\{\phi_I : I \in \mathcal{D}\}$ be a collection of functions indexed by dyadic intervals which are supported on $I$, and constant on the grandchildren of $I$. Suppose that there is a constant $C$ so that for any dyadic interval $I_0$, and any collection $\mathcal{E}$ of dyadic subintervals $I \subset I_0$, there holds

\[
\left| \left\{ \left| \sum_{I \in \mathcal{E}} \phi_I \right| > C \right\} \right| < \frac{1}{2} |I_0|.
\]

Then, we have

\[
\left| \left\{ \left| \sum_{I \in \mathcal{E}} \phi_I \right| > (C + 1)\lambda \right\} \right| < 2^{(1-\lambda)/2} |I_0|, \quad \lambda > 1.
\]

3. The Distributional Lemma

The key component is a one sided variant of a distributional lemma discovered in [13]. Fix an interval $I_0$, and for integers $a \in \mathbb{Z}$, let $K_a$ be those intervals dyadic $I \subseteq I_0^-$ for which we have

\[
2^a < \langle \sigma \rangle_{I_0^+}^{p-1} \langle w \rangle_{I^-} \leq 2^{a+1},
\]

\[
\langle w \rangle_{I^-} \leq 2 \langle w \rangle_{I_0^-}.
\]

**Lemma 3.1.** For an absolute constant $c > 0$ and constant $C_p > 0$ that depends (only) on $p$, we have

\[
\sigma\left( \left| T_{\delta,K_a}^*(w\varphi_{I_0}) \right| > C_p \lambda \langle w \rangle_{I_0^-} \right) \lesssim \begin{cases} 
\lambda^{-\frac{2p'}{p'+1}} \cdot \sigma(I_0^+), & \text{for } 0 < \lambda < 1, \\
e^{-c\lambda} \cdot \sigma(I_0^+), & \text{for } \lambda \geq 1.
\end{cases}
\]

Above, for any collection $\mathcal{K}$,

\[
(T_{\delta,K}^*(f))(x) := \sum_{I \in \mathcal{K}} \varepsilon_I \langle f, h_{I^-} 1_{|I| > \delta(x)} \rangle h_{I^+}(x).
\]

The function $\varphi_I$ satisfies $|\varphi_I| = 1_I$. 
The point of this inequality is that it holds in the two weight setting, provided the $A^+_p$ product is approximately constant. Moreover, one take care to note that the set on the left is in $I_0^c$, and its $\sigma$-measure is controlled by $\sigma(I_0^+)$.  

Proof. The collection $K_a$ is further divided into collections $K^b_a$, for $b \in \mathbb{N}$, where $I \in K_a^b$ if $I \in K_a$ and in addition,  

\begin{equation}
2^{-b}\langle w \rangle_{I_0} < \langle w \rangle_{I} \leq 2^{1-b}\langle w \rangle_{I_0}.
\end{equation}

By \ref{3.2}, this is a decomposition of $K_a$.

We show that for constants $c, C > 0$, and $\lambda > 0$,  

\begin{equation}
\sigma\left(\left|T^*_\delta w \varphi_{I_0}\right| > C\lambda^{2^{-1-b}}\langle w \rangle_{I_0}\right) \lesssim 2^{b/(p-1)-c\lambda}\sigma(I_0^+).
\end{equation}

Nevertheless, this is a strong condition, and another easy subadditivity argument completes the proof of \ref{3.3}.

Indeed, for $0 < \lambda < 1$. Since \[\sum_{b=0}^{\infty} 2^{-b(1-\frac{p+1}{2p})} = 1/(1 - 2^{1-\frac{p}{2p}}) =: C_p \leq \infty,\] we have  

\begin{equation}
\begin{aligned}
\sigma\left(\left|T^*_\delta w \varphi_{I_0}\right| > 2CC_p\lambda^{2}\langle w \rangle_{I_0}\right) &\lesssim \sum_{b=0}^{\infty} \sigma\left(\left|T^*_\delta w \varphi_{I_0}\right| > C\lambda^{2^{-1-b}}\langle w \rangle_{I_0}\right) \\
&\lesssim \sigma(I_0^+) \sum_{b=0}^{\infty} 2^{b/(p-1)-c\lambda}\sigma(I_0^+) \\
&\lesssim \lambda^{1-\frac{2-p}{p+1}} \cdot \sigma(I_0^+).
\end{aligned}
\end{equation}

For $\lambda \geq 1$. Since \[\sum_{b=0}^{\infty} 2^{-b/2} < 10,\] we have  

\begin{equation}
\begin{aligned}
\sigma\left(\left|T^*_\delta w \varphi_{I_0}\right| > 20C\lambda^{2}\langle w \rangle_{I_0}\right) &\lesssim \sum_{b=0}^{\infty} \sigma\left(\left|T^*_\delta w \varphi_{I_0}\right| > C\lambda^{2^{-1-b/2}}\langle w \rangle_{I_0}\right) \\
&\lesssim e^{-c\lambda} \cdot \sigma(I_0^+) \sum_{b=0}^{\infty} 2^{b/(p-1)-c2^{b/2}} \lesssim e^{-c\lambda} \cdot \sigma(I_0^+).
\end{aligned}
\end{equation}

This proves \ref{3.3}.

In general, we know little more than the local integrability condition on $w \varphi_{I_0}$. But we have this consequence of the weak type inequality Proposition 2.3: There is a constant $C$ so that for any subinterval $J \subset I_0$ and any collection $E \subset K^b_a$ of subintervals of $J$, we have  

\begin{equation}
\left|\left\{\left|T^*_\delta E w \varphi_{I_0}\right| > C\lambda^{2^{-1-b}}\langle w \rangle_{I_0}\right\}\right| \leq \frac{1}{2}|J|.
\end{equation}

This follows from the dyadic structure, and condition \ref{3.5}, which controls the local $L^1$ norms of $w \varphi_{I_0}$.

Turn the sufficient conditions for a John-Nirenberg estimate in Lemma 2.4. The inequality \ref{3.7} is the hypothesis \ref{2.6}. The other hypotheses of Lemma 2.4 are easy
to check. It follows that for \( \lambda > 0 \),
\[
\left\{ x : |T_{\delta,K_a}(x)(w_\varphi I_0)| > C\lambda 2^{1-b}\langle w \rangle I_0 \right\} \lesssim 2^{-c\lambda}|I_0|.
\]
(3.8) Note that for \( 0 < \lambda < 1 \), this estimate is trivial.

Our task is to convert this Lebesgue measure estimate into \( \sigma \)-measure. By combining conditions (3.1) and (3.5), we have for any \( I \in K_{a,b} \)
\[
\langle \sigma \rangle_{I^+}^{p-1} \sim 2^a\langle w \rangle_{I^+}^{-1} \sim 2^{a+b}\langle w \rangle_{I_0}^{-1} \sim 2^b\langle \sigma \rangle_{I_0}^{p-1}.
\]
That is, \( \sigma(I^+) \sim 2^{b/(p-1)}\rho|I| \), with constant \( \rho \) independent of \( I \in K_{a,b} \).

The set on the left in (3.8) is a union of children of intervals \( I^- \), for \( I \in K_{a,b} \).
Therefore, (3.6) follows. And, the Lemma is proved. \( \square \)

4. Proof of Theorem 1.2

There are two assertions, the weak \( L^p \) bound, and the \( L^2 \) bound. Both are proved by appeal to the two weight theorems, Theorem 2.2 and Theorem 2.1. By inspection, it suffices to prove the weak \( L^p \) bound for the maximal truncations. This inequality specialized to the case of \( p = 2 \), and taking duality into account, gives the full \( L^2 \) result.

From Theorem 2.2, we need to estimate the quantities \( \mathfrak{M}_p^+ \) and \( \mathfrak{F}_p \) defined in (2.3) and (2.4). The first \( \mathfrak{M}_p^+ \) is the weak type norm for the maximal function \( M_{+} \). That is given in (1.1), and is smaller than what is claimed. So we turn to the second constant, \( \mathfrak{F}_p \) which is the testing constant for the maximal truncations of one sided martingale transform. Namely, the task is to show that uniformly over all dyadic intervals \( I \), that the inequality (2.5) holds. More exactly, we need to see that
\[
\|1_I T_{\delta}^+(w_\varphi I)| |_{L^{p/(p)}} \leq [w]^{1/p}_{A^+_\infty} [w]^{1/p'}_{A^\infty} \cdot w(I)^{1/p'}.
\]
(4.1)

The martingale transform is a sum over all dyadic intervals. We can restrict the sum to those dyadic intervals \( I \) that intersect \( I_0 \). Those \( I \) that strictly contain \( I_0 \) can also be dismissed, since they contribute nothing to the left side of (2.5). The case of \( I = I_0 \) is trivial. Thus, the difficult case is \( I \nsubseteq I_0 \), and we restrict attention to this case, by assuming that the multiplier coefficients \( \epsilon_I = 0 \) unless \( I \subseteq I_0 \).

Our principal tools are the Distribution Lemma 3.1 and a corona type decomposition. The latter is needed to get to a point that we can apply Lemma 3.1. For an integer \( a \) with \( 2^a < [w]_{A^+_{\infty}} \), let
\[
K_a = \{ I \nsubseteq I_0 : 2^a < \langle \sigma \rangle_{I^+}^{p-1}\langle w \rangle_{I^-} \leq 2^{a+1} \}.
\]

Assuming that \( \epsilon_I = 0 \) unless \( I \in K_a \), we will show that
\[
\int_{I_0} T_{\delta,K_a}^+(w_\varphi I_0)^{p'} \sigma(dx) \lesssim 2^{a(p'-1)}[w]_{A^\infty} w(I_0).
\]
(4.2)
Above, we are using the notation of (3.4). Summing over \( a \) will prove (4.1).

**Definition 4.1 (The Corona).** Let

\[
C_{a,1} := \{ I \in \mathcal{K}_a : I^- \text{ is maximal w.r.t. inclusion} \}.
\]

For \( I \in C_{a,1} \), let \( C_{a,2}(I) \) be the “bad” children of \( I \), defined as below

\[
C_{a,2}(I) := \{ J \in \mathcal{K}_a : J^- \subset I^- \quad \langle w \rangle_J^- > 2 \langle w \rangle_{I^-}, \quad J^- \text{ is maximal} \},
\]

and let

\[
C_{a,2} = \bigcup_{I \in C_{a,1}} C_{a,2}(I).
\]

We also define \( C_{a,t} \) for \( t \geq 3 \) inductively. Let

\[
C_a = \bigcup_{t=1}^{\infty} C_{a,t}.
\]

We refine the corona decomposition.

**Definition 4.2.** For \( J \in \mathcal{K}_a \), let \( J^s \) stand for the minimal stopping interval \( S \in C_a \) with \( J^- \subseteq S^- \).

The collections below form a partition of \( \mathcal{K}_a \).

\[
\mathcal{K}_a(S) := \{ J \in \mathcal{K}_a : J^s = S \}, \quad S \in C_a,
\]

We are now at the core of the argument. Set \( \tau_S = T^{*}_{\delta,\mathcal{K}_a(S)}(w \varphi_{I_0}) \), and

\[
X_{S,n} = 1_{\{2^{n-1} \langle w \rangle_{S^-} < |\tau_S| \leq 2^n \langle w \rangle_{S^-}\}} \tau_S, \quad n \in \mathbb{Z}.
\]

Then, estimate

\[
\|1_{I_0} T^{*}_{\delta,\mathcal{K}_a}(w \varphi_{I_0})\|_{L^{p'}(\sigma)} = \left\| \sum_{S \in C_a} \tau_S \right\|_{L^{p'}(\sigma)} \\
\leq \sum_{n=-\infty}^{\infty} \left\| \sum_{S \in C_a} X_{S,n} \right\|_{L^{p'}(\sigma)} \\
\lesssim \sum_{n=-\infty}^{\infty} \left[ \sum_{S \in C_a} \|X_{S,n}\|_{L^{p'}(\sigma)} \right]^{1/p'}.
\]

The last inequality follows from the construction of the corona: the values \( \langle w \rangle_{S^-} \) form a geometric sequence of reals.

Concerning this norms in (4.3), the case of \( n \geq n_p \) and \( n < n_p \) are different, here \( n_p \) is the least positive integer such that \( 2^{n_p} \geq C_p \). For \( n \geq n_p \), we have, again by the
construction of the corona and the distribution estimate (3.3) with \( \lambda = 2^{n-1}/C_p \geq 1, \)

\[
\sum_{S \in C_a} \| X_{S,n} \|_{L^{p'}(\sigma)}^p \lesssim 2^{np'} e^{-c2^{n-1}} \sum_{S \in C_a} \langle w \rangle_S^p \sigma(S^+)
\lesssim 2^{np'} e^{-c2^{n-1}} 2^{a(p'-1)} \sum_{S \in C_a} \langle w \rangle_S - |S^+|
\lesssim 2^{np'} e^{-c2^{n-1}} 2^{a(p'-1)} [w]_{A^\infty} \sigma(I_0).
\]

This estimate decreases rapidly in \( n \), so that it is strong enough to imply (4.2).

The case of \( n < n_p \), is very similar. We use the distribution estimate (3.3) with \( \lambda = 2^{n-1}/C_p < 1, \)

\[
\sum_{S \in C_a} \| X_{S,n} \|_{L^{p'}(\sigma)}^p \lesssim 2^{np'} 2^{-n \frac{2p'}{p+1}} \sum_{S \in C_a} \langle w \rangle_S^p \sigma(S^+)
\lesssim 2^{np/(p+1)} 2^{a(p'-1)} \sum_{S \in C_a} \langle w \rangle_S - |S^+|
\lesssim 2^{np/(p+1)} 2^{a(p'-1)} [w]_{A^\infty} \sigma(I_0).
\]

Since \( n \leq n_p \), this is again strong enough to complete the proof of (4.2). And the proof is finished. \( \square \)

5. Complements

1. It would be very natural to seek a proof of our main theorem using ideas related to sparse bounds. The latter subject was started in [12], and has been very successful. But we could not find such a bound in the current setting. That is why we returned to an earlier proof [13] of the \( A_2 \) bound, one that was explored in [8, 9, 11].

2. The source of many of our difficulties seem to be linked to this point. A key element of the recent developments in the \( A_p \) theory is the following fact: For any weight \( \mu \) on \( \mathbb{R} \), the maximal function

\[
M_\mu f = \sup_{I : \mu(I) > 0} 1_{I} \mu(I)^{-1} \int_I f \, d\mu
\]

is weakly bounded on \( L^1(\mu) \). This does not seem to be true for the plus versions of this maximal function.

3. The absence of this ‘universal maximal function’ reminds us of a similar obstruction in the multiparameter setting. Recent results of Barron and Pipher [2] have
shown that in fact sparse bounds do not hold in that setting. Would some version of these arguments hold in the one sided case?

4. We pose the question: What is the simplest natural condition that one can place on a weight $\mu$ so that the maximal operator

$$M^+_\mu f = \sup_{I : \mu(I^+) > 0} 1_{I^-} \mu(I^+)^{-1} \int_{I^+} f \, d\mu$$

is weakly bounded on $L^1(\mu)$? Or on some $L^p(\mu)$, for $1 \leq p < \infty$. Is it $\mu \in A^+_\infty$?

5. Our main theorem admits a straightforward extension to paraproducts, that is operators of the form

$$Tf = \sum_I \tau_I (f)_{I^+} h_{I^-},$$

where $\tau_I$ is a Carleson sequence. It likewise admits an extension to the setting where the martingale transforms are replaced with Haar shifts with complexity, in the sense of [9, 10]. In the latter case, of complexity, one wants bounds that are polynomial in complexity. There are versions of our two weight theorems Theorem 2.1 and Theorem 2.2 that account for complexity. In the interest of clarity, we have not pursued these points here.

6. One of the ways that the classical $A_p$ theory and the $A^+_p$ are similar is in the area of extrapolation. A robust theory holds in both places. A quantified version of the extrapolation of $A^+_p$ is described in Carro, Lorente and Martín-Reyes [4]. The extrapolation of strong type norms in the $A^+_p$ setting is described in [6]. That discussion is not quantitative, but there should not be a major obstacle to devising such a theory. With it, one could deduce new strong type inequalities from our inequality, (1.3).

7. Given the role of two weight inequalities Theorem 2.1 and Theorem 2.2 in our argument, it might be reasonable to investigate these theorems in the setting of one sided operators. We comment that [11, Thm 4.7] has a general strong type two weight inequality. But, like for Theorem 2.2, the discussion is not geared towards one sided operators. Moreover, the proof of that theorem is not so easy, and the sufficient conditions are substantially more involved. So, we felt that appealing to the one sided version of that result was not in the spirit of this paper.

8. The paper of Vuorinen [17] aims for a characterization of the strong type inequality for general dyadic operators. But, note that the main theorem of this paper begins with an assumption of a ‘quadratic two weight $A_p$’ condition, specified in section 3 of that paper. This condition is not of a one sided nature. So again, the Theorem might be attractive to apply in the one sided setting, but it is not completely straightforward to do so.

9. The overarching conjecture here concerns one sided Calderón-Zygmund operators, and a one sided version of the main results of [7, 9]. These are operators $T,$
bounded on $L^2$, that have a kernel representation

$$\langle Tf, g \rangle = \int K(x, y)f(y)g(x) \, dx \, dy$$

where $K(x, y)$ satisfies the standard size and smoothness conditions

$$|K(x, y)| \lesssim |x - y|^{-1}, \quad x \neq y$$

$$|\nabla^\alpha K(x, y)| \lesssim |x - y|^{-1-\alpha},$$

where $0 < \alpha \leq 1$ is fixed. But, one imposes the one sided condition: $K(x, y) = 0$ if $x < y$.

**Conjecture 5.1.** Let $T$ be a one sided Calderón-Zygmund operator. Then, there holds

$$\|T\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p^2} \max\{[\sigma]_{A_p^\infty}^{1/p}, [w]_{A_p^1}^{1/p'}\} \lesssim [w]_{A_p}^{\max\{1,(p-1)^{-1}\}}.$$  

10. One of the beautiful aspects of the proof of the $A_2$ theorem in [9] is the Hytönen Representation Theorem, which gives a representation of a Calderón-Zygmund operator as a rapidly convergent in complexity sum of dyadic shifts. We could not prove the analogous result in the one sided setting. Does a one sided Hytönen Representation hold? We see no reason why it should not hold, but could not find a proof.

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