Notes on enriched categories with colimits of some class

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Abstract

This work is on-going. Results presented here are issued mainly from unpublished notes of the first author and contain those in the notes http://arXiv.org/abs/math.CT/0309209 and http://arXiv.org/abs/math.CT/0403164 of the second author. The latter is also the only responsible for the typos, spelling mistakes and other errors that could occur in the present paper! We intend to add further sections to this work in the near future.

Given a class $\Phi$ of weights, we study the following classes: $\Phi^+$ of $\Phi$-flat weights which are the $\psi$ for which $\psi$-colimits commute in the base $\mathcal{V}$ with limits with weight in $\Phi$; and $\Phi^-$ dually defined - of weights $\psi$ for which $\psi$-limits commute in the base $\mathcal{V}$ with colimits with weight in $\Phi$. We show that both these classes are saturated (i.e. closed with the terminology of [AK88]). We prove that for the class $\mathcal{P}$ of all weights $\mathcal{P}^+ = \mathcal{P}^-$. For any small $B$, we defined an enriched adjunction $\mathcal{B} \mathcal{V} \leftrightarrow \mathcal{B} \mathcal{V}^{op}$ and show how it restricts to an equivalence $(\mathcal{P}^-(B^{op}))^{op} \simeq \mathcal{P}^-(B)$ between subcategories of small projectives.

1 Introduction

The present notes had their beginning in the analysis of the results obtained by Borceux, Quintero and Rosicky in their article [BQR98]. These authors were concerned with extending to the enriched case the notion of accessibility and its properties, described for ordinary categories in the book [MP89] of Makkai and Paré and [AR94] of Adamek and Rosicky. They were led to discuss categories - now meaning $\mathcal{V}$-categories - with finite limits (in a suitable sense), or more generally with $\alpha$-small ones, or with filtered colimits (in a suitable sense), or more generally with $\alpha$-filtered colimits and again to discuss the connexions between these classes of categories. When we looked in detail at their work, we observed that many of the properties they discussed hold in fact for categories having colimits of any given class $\Phi$, while other holds when $\Phi$ is the class of colimits commuting in the base category $\mathcal{V}$ with the limits of some class $\Psi$ - such properties as finiteness or filteredness being irrelevant to the general results. Approaching in this abstract way, not generalisations of accessibility as such, but the study of categories with colimits (or limits) of some class, brings considerable simplifications. A recent work by Adamek, Borceux, Lack and Rosicky [ABLR02] also treats a general notion of accessibility parameterised by pair of families of ordinary limits and colimits that commute in $Set$. Rather than families of ordinary -or conical limits- we consider indexed or weighted limits and colimits. As a result of this generalisation, we will avoid some technical complications even in the case when the base $\mathcal{V}$ is $Set$. Although our positive results are quite few in number, their value may be judged by their extra light they cast on the results in [BQR98] or [ABLR02].

We begin by reviewing and completing some known facts in the first sections 2, 3. Section 4 contains a slight extension of Kelly's cocompletion theorem. Section 5 treats generally the limit/colimit commutation in the base $\mathcal{V}$ and contains a study of families of the form $\Phi^+$ of $\Phi$-flat weights - i.e. weights whose colimits commute with $\Phi$-weighted limits and families of the form $\Phi^-$.
of weights whose limits commute with $\Phi$-weighted colimits. Section 4 focusses on the family $P^-$ for $P$ the class of all weights. $P^-$ being the class of small projective, is shown to be also the class of $P$-flat weights. More results regarding the Cauchy-completion and the Isbell adjunction in the enriched context are then presented.

2 Revision of the terminology

The necessary background knowledge about enriched categories is largely contained in [Kel82], augmented by Kelly’s “Amiens” article [Kel82-2], and by the Albert-Kelly article [AK88].

We use “category”, “functor”, and “natural transformation” for “$V$-category”, “$V$-functor”, and “$V$-natural transformation”, except when more precision is needed. As usual the ordinary category $V_0$ is supposed locally small, complete and cocomplete. $V$-CAT is the 2-category of $V$-categories whereas $V$-Cat is that of small $V$-categories. Set is the category of small sets, $\text{Cat} = Set$-Cat is the 2-category of small categories, $\text{CAT} = Set$-CAT the 2-category of locally small categories.

A weight is a functor $\phi : K \to V$ with domain $K$ small; weights were called indexing-types in [Kel82] and in [Kel82-2]. Recall that the $\phi$-weighted limit $\{\phi, T\}$ of a functor $T : K \to A$ is defined representably by

$$2.1 \ A(a, \{\phi, T\}) \cong [K,V](\phi, A(a,T-)),$$

while the $\phi$-weighted colimit $\phi*S$ for $S : K^{op} \to A$ is defined dually by

$$2.2 \ A(\phi*S,a) \cong [K,V](\phi, A(S-,a)),$$

so that $\phi*S$ is equally the $\phi$-weighted limit of $S^{op} : K \to A^{op}$. Of course the limit $\{\phi, T\}$ consists not just of the object $\{\phi, T\}$ but also of the representation 2.1 or equally of the corresponding counit $\mu : \phi \to A(\{\phi, T\}, T-)$; it is by abus de langage that we usually mention only $\{\phi, T\}$. When $V = Set$, we refine the classical (or “conical”) limit of $T : K \to A$ and the classical colimit of $S : K \to A$ as

$$2.3 \ \text{lim} \ T = \{\Delta 1, T\} \ \text{and colim} \ S = \Delta 1 * S$$

where $\Delta 1 : K \to Set$ is the constant functor at the one point set 1. Recall that the weighted limits and colimits can be calculated using the classical ones when $V = Set$: for then the presheaf $\phi : K \to Set$ gives the discrete op-fibration $\hat{d} : el(\phi) \to K$ where $el(\phi)$ is the category of elements of $\phi$, and now

$$2.4 \ \{\phi, T\} = \text{lim} \{ el(\phi) \xrightarrow{\hat{d}} K \xrightarrow{T} A \},$$

$$2.5 \ \phi*S = \text{colim} \{ el(\phi) \xrightarrow{\hat{d}^{op}} K^{op} \xrightarrow{S} A \}$$

Also a functor $F : B \to C$ is said to preserve the limit $\{\phi, T\}$ as in 2.1 when $F(\{\phi, T\})$ is the limit of $FT$ weighted by $\phi$ with counit

$$\phi \xrightarrow{\mu} A(\{\phi, T\}, T-) \xrightarrow{F} B(F(\phi, T), FT-)$$

and $F$ is said to preserve the colimit $\phi*S$ as in 2.2 when $F^{op}$ preserves $\{\phi, S^{op}\}$.

We spoke above of a “class $\Phi$ of colimits” or a “class $\Psi$ of limits”; but this is loose and rather dangerous language - the only thing that one can sensibly speak of is a class $\Phi$ of weights. Then a category $A$ admits $\Phi$-limits or is $\Phi$-complete, if $A$ admits the limit $\{\phi, T\}$ for each weight $\phi : K \to V$
of $\Phi$ and each $T : K \to A$, while $A$ admits $\Phi$-colimits, or is $\Phi$-cocomplete when $A$ admits the colimit $\phi \ast S$ for each $\phi : K \to V$ in $\Phi$ and for each $S : K^{op} \to A$ (and thus when $A^{op}$ is $\Phi$-complete).

Of course a functor $A \to B$ between $\Phi$-complete categories is said to be $\Phi$-continuous when it preserves all $\Phi$-limits and one defines $\Phi$-cocontinuous dually. We write $\Phi\text{-Conts}$ for the 2-category of $\Phi$-complete categories, $\Phi$-continuous functors, and all natural transformations - which is a (non full) sub-2-category of $\mathcal{V}\text{-CAT}$; and similarly $\Phi\text{-Cocts}$ for the 2-category of $\Phi$-cocomplete categories, $\Phi$-cocontinuous functors, and all natural transformations.

To give a class $\Phi$ of weights is to give for each small $K$, those $\phi \in \Phi$ with domain $K^{op}$; let us use as in [AKSS] the notation

$$\Phi[K] = \{ \phi \in \Phi \mid \text{dom}(\phi) = K^{op} \},$$

so that

$$\Phi = \Sigma_{K \small} \Phi[K].$$

In future, we look at $\Phi[K]$ as a full subcategory of the functor category $[K^{op}, \mathcal{V}]$. The smallest class of weights is the empty class 0, and $0\text{-Cocts}$ is just $\mathcal{V}\text{-CAT}$. The largest class of weights consists of all weights with small domains, and we denote this class by $P$; the 2-category $P\text{-Cocts}$ is just the 2-category $\text{Cocts}$ of complete categories and continuous functors, and similarly $P\text{-Cocts} = \text{Cocts}$.

There may well be different classes $\Phi$ and $\Psi$ for which the sub-2-categories $\Phi\text{-Conts}$ and $\Psi\text{-Cocts}$ of $\mathcal{V}\text{-CAT}$ coincide; which is equally to say that $\Phi\text{-Cocts}$ and $\Psi\text{-Cocts}$ coincide. When $\mathcal{V} = \text{Set}$, for instance, $\text{Cocts} = P\text{-Cocts}$ coincides with $\Phi\text{-Conts}$ where $\Phi$ consists of the weights for products and for equalisers. We define the saturation $\Phi^*$ of a class $\Phi$ of weights as follows: the weight $\psi$ belongs to $\Phi^*$ when every $\Phi$-complete category is also $\psi$-complete and every $\Phi$-continuous functor is also $\psi$-continuous. Actually we change here the name given in [AKSS] where $\Phi^*$ was called the closure of $\Phi$ but we wish to avoid overusing this terminology. Clearly then, we have

$$\Phi\text{-Conts} = \Psi\text{-Conts} \iff \Phi\text{-Cocts} = \Psi\text{-Cocts} \iff \Phi^* = \Psi^*.$$ 

When $\mathcal{V} = \text{Set}$, we can of course consider $\Phi\text{-Conts}$ where $\Phi$ consists of the $\Delta 1 : K \to \text{Set}$ for all $K$ in some small class $D$ of small categories; and we might write $\Delta\text{-Conts}$ for this 2-category $\Phi\text{-Conts}$ of $D$-complete categories, $D$-continuous functors, and all natural transformations. We underline the fact that however when $\mathcal{V} = \text{Set}$, the $\Phi\text{-Conts}$ generally do NOT occur as $\Delta\text{-Conts}$ as above. A simple example of this situation is in [AKSS].

There is a distinction which is often important, but which is not covered by the notation above. We spoke of $\mathcal{V}\text{-CAT}$ as a 2-category, the category $\mathcal{V}\text{-CAT}(A,B)$ having as its objects the $\mathcal{V}$-functors $T : A \to B$ and as its arrows, the $\mathcal{V}$-natural transformations $\alpha : T \to S : A \to B$. When $A$ is small, however, we also have the $\mathcal{V}$-category $[A,B]$, whose underlying category $[A,B]_0$ is $\mathcal{V}\text{-CAT}(A,B)$. Thus small $\mathcal{V}$-categories form not only a 2-category $\mathcal{V}\text{-Cat}$, but also a $\mathcal{V}$-category (as $\mathcal{V}\text{-Cat}$ is a monoidal closed 2-category and as any closed category, it is enriched over itself). Again, even when $A$ is not small, we can (as in [Ke82] section 3.12) see $[A,B]$ as a $\mathcal{V}'$-category where $\mathcal{V}'$ is an extension of $\mathcal{V}$ to a $\text{Set}'$-category (with $\text{Set}'$ a universe containing $\text{Obj} (\text{Set})$ and $\text{Obj} (\mathcal{B})$ for all categories $\mathcal{B}$), such that $\mathcal{V} \to \mathcal{V}'$ preserves all limits that exist in $\mathcal{V}$ and all $\text{Set}'$-small colimits that exist in $\mathcal{V}$. So we take the view that $[A,B]$ always exists as a $\mathcal{V}'$-category, which is fact a $\mathcal{V}$-category when $A$ is small. Thus, even for large $A$, $A$ is $\text{Set}'$-small and we may see $\mathcal{V}\text{-CAT}(A,B)$ as the ordinary category underlying the $\mathcal{V}'$-category $[A,B]$, so that $\mathcal{V}$-categories form not only the 2-category $\mathcal{V}\text{-CAT}$ cut also a $\mathcal{V}'$-$\mathcal{V}$-CAT-category denoted $\mathcal{V}'\text{-CAT}$. This has a full subcategory given by the $\Phi$-complete categories; and if $A$ and $B$ are $\Phi$-complete, the $\Phi$-continuous functors $A \to B$ determine a full subcategory $\Phi\text{-Conts}[A,B]$ of the $\mathcal{V}'$-category $[A,B]$. Note the use of square brackets in the name of the $\mathcal{V}'$-category $\Phi\text{-Conts}[A,B]$ (which is a
\( \mathcal{V} \)-category for small \( \mathcal{A} \); its underlying category is our earlier \( \Phi \text{-}\text{Conts}(\mathcal{A}, \mathcal{B}) \) with round brackets. Thus the 2-category \( \Phi \text{-}\text{Conts} \) really underlies a \( \mathcal{V}'\text{-}\text{CAT} \)-category that we might note \( \Phi \text{-}\text{Conts} \) with

2.9 \( \Phi \text{-}\text{Conts}(\mathcal{A}, \mathcal{B}) = \Phi \text{-}\text{Conts}[\mathcal{A}, \mathcal{B}] \)

(Note that we suppress the mention of \( \mathcal{V} \) in \( \Phi \text{-}\text{Conts} \) or \( \Phi \text{-}\text{Conts} \); one cannot know the class \( \Phi \) without knowing \( \mathcal{V} \). There may be a problem when \( \Phi = \mathcal{P} \), so that \( \Phi \text{-}\text{Conts} \) is just \( \text{Conts} \); one can write \( \mathcal{V} \text{-}\text{Conts} \) if necessary.) Of course the definition of \( \Phi \text{-}\text{Conts} \) makes it clear that \( \Psi \) can equally be written as

2.10 \( \Phi \text{-}\text{Conts} = \Psi \text{-}\text{Conts} \leftrightarrow \Phi \text{-}\text{Cocts} = \Psi \text{-}\text{Cocts} \Rightarrow \Phi^* = \Psi^* \).

3 Free cocompletions of categories and saturated classes of weights

Another piece of background knowledge that we need to recall concerns the “left bi-adjoint” to the forgetful 2-functor \( U_\Phi : \Phi \text{-}\text{Cocts} \rightarrow \mathcal{V}\text{-}\text{CAT} \), or rather to \( U_\Phi : \Phi \text{-}\text{Cocts} \rightarrow \mathcal{V}'\text{-}\text{CAT} \). (Note that it is convenient to deal with colimits rather than limits.)

In the language of \[\text{Kel82}\], a presheaf \( F : \mathcal{A}^{\text{op}} \rightarrow \mathcal{V} \) is said to be accessible if it is the left Kan extension of its restriction to some small full subcategory of \( \mathcal{A}^{\text{op}} \). Even though \( \mathcal{A}^{\text{op}}, \mathcal{V} \) is when \( \mathcal{A} \) is not small, only a \( \mathcal{V}' \)-category, its full subcategory \( \mathcal{P} \mathcal{A} \) having as objects the accessible presheaves is a \( \mathcal{V} \)-category, as follows from \[\text{Kel82}\] 4.41; of course \( \mathcal{P} \mathcal{A} \) coincides with \( \mathcal{A}^{\text{op}}, \mathcal{V} \) when \( \mathcal{A} \) is small. Every representable \( \mathcal{A}(\cdot, a) \) is accessible, so that the Yoneda embedding \( Y : \mathcal{A} \rightarrow \mathcal{A}^{\text{op}}, \mathcal{V} \) takes its values in \( \mathcal{P} \mathcal{A} \). Remember that the functor category \( \mathcal{A}^{\text{op}}, \mathcal{V} \) admits all small colimits, these being formed pointwise in \( \mathcal{V} \). Each object of \( \mathcal{P} \mathcal{A} \), being a left Kan extension of the given kind, is a small colimit of representables. Recall from \[\text{Kel82}\] p.154 that given a full subcategory \( \mathcal{A} \) of \( \mathcal{B} \) and a family \( \Phi \) of weights, the closure of \( \mathcal{A} \) in \( \mathcal{B} \) under \( \Phi \)-colimits is the smallest full replete subcategory of \( \mathcal{B} \) containing \( \mathcal{A} \) and closed under the formation \( \Phi \)-colimits in \( \mathcal{B} \). Since it is shown in \[\text{Kel82}\] Proposition 5.34 that \( \mathcal{P} \mathcal{A} \) is closed in \( \mathcal{A}^{\text{op}}, \mathcal{V} \) under small colimits, we see that \( \mathcal{P} \mathcal{A} \) is precisely the closure of \( \mathcal{A} \) in \( \mathcal{A}^{\text{op}}, \mathcal{V} \) under small colimits (when \( \mathcal{A} \) is embedded in \( \mathcal{A}^{\text{op}}, \mathcal{V} \) by the Yoneda embedding \( Y \)). For any class \( \Phi \) of weights, and any category \( \mathcal{A} \), write \( \Phi(\mathcal{A}) \) for the closure of \( \mathcal{A} \) in \( \mathcal{P} \mathcal{A} \) (or equally in \( \mathcal{A}^{\text{op}}, \mathcal{V} \)) under \( \Phi \)-colimits, with \( Z : \mathcal{A} \rightarrow \Phi(\mathcal{A}) \) and \( W : \Phi(\mathcal{A}) \rightarrow \mathcal{A}^{\text{op}}, \mathcal{V} \) for the full inclusions, so that \( Y : \mathcal{A} \rightarrow \mathcal{A}^{\text{op}}, \mathcal{V} \) is the composite \( WZ \). We now reproduce (the main point of) \[\text{Kel82}\] Theorem 5.35. The proof below is a little more direct than that given there, which referred back to earlier results. The result itself must be older still, at least for certain classes \( \Phi \).

Proposition 3.1 For any \( \Phi \)-cocomplete category \( \mathcal{B} \) there is an equivalence of \( \mathcal{V}' \)-categories

3.1 \([Z, 1] : \Phi \text{-}\text{Cocts}[\Phi(\mathcal{A}), \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{B}]\)

with an equivalence inverse given by \( \text{Lan}_Z \). It follows that \( \Phi \text{-}\text{Cocts}[\Phi(\mathcal{A}), \mathcal{B}] \) is like \([\mathcal{A}, \mathcal{B}]\), a \( \mathcal{V} \)-category when \( \mathcal{A} \) is small. Thus \( \Phi(\cdot) \) provides a left bi-adjoint, not only to the forgetful \( \mathcal{V}'\text{-}\text{CAT} \)-functor \( \Phi \text{-}\text{Cocts} \rightarrow \mathcal{V}\text{-}\text{CAT} \), but even to the forgetful \( \mathcal{V}'\text{-}\text{CAT} \)-functor \( \Phi \text{-}\text{Cocts} \rightarrow \mathcal{V}'\text{-}\text{CAT} \).

PROOF: The left Kan extension \( \text{Lan}_Z G \), if it exists, is given by \( \text{Lan}_Z G(F) = \tilde{Z} F * G \) where \( \tilde{Z}(F) = \Phi(\mathcal{A})(Z, F) \). However \( \Phi(\mathcal{A})(Z, F) = [\mathcal{A}^{\text{op}}, \mathcal{V}](Z - F, WF) = [\mathcal{A}^{\text{op}}, \mathcal{V}](Y - F, WF) \), which by Yoneda is isomorphic to \( WF \). Consider the full subcategory of \( \Phi(\mathcal{A}) \) given by those \( F \) for which \( WF * G \) does exist; it contains the representatives by Yoneda; it is also closed under \( \Phi \)-colimits according to \[\text{Kel82}\] (3.22) since these colimits exist in \( \mathcal{B} \); so it is all of \( \Phi(\mathcal{A}) \). What is more: \( \text{Lan}_Z G = W * G : \Phi(\mathcal{A}) \rightarrow \mathcal{B} \) preserves \( \Phi \)-colimits as \( W \) preserves \( \Phi \)-colimits and \( - * G \) is cocontinuous (straightforward from \[\text{Kel82}\] (3.22) again). So one does have indeed a functor \( \text{Lan}_Z : [\mathcal{A}, \mathcal{B}] \rightarrow \Phi \text{-}\text{Cocts}[\Phi(\mathcal{A}), \mathcal{B}] \), while one has trivially the restriction functor
[Z, 1] : \Phi\text{-Cocts}(\Phi(A), B) \to [A, B]. The canonical \( G \to Lan_Z(G)Z \) is invertible for all \( G \) since \( Z \) is fully faithful by \[\text{Kc82}\] (4.22). So that it remains to consider the canonical \( \alpha : Lan_Z(SZ) \to S \) for a \( \Phi \)-cocontinuous \( S \) : \( \Phi(A) \to B \). The \( F \)-component of \( \alpha \) for \( F \in \Phi(A) \) is the canonical \( \alpha_F : WF \circ SZ \to SF \); and clearly the collection of those \( F \) for which \( \alpha_F \) is invertible contains the representables and is closed under \( \Phi \)-colimits, therefore it is the totality of \( \Phi(A) \).

\[\text{Proposition 3.4}\]

An explicit description of the saturation \( \Phi^* \) of a class \( \Phi \) of weights was given by Albert and Kelly in \[\text{AK88}\], in the following terms:

\[\Phi^* \text{ of the class } \Phi \text{ if and only if the object } \psi \text{ of } \Phi(K) = [K^{op}, V] \text{ lies in the full subcategory } \Phi(K) \text{ of } [K^{op}, V].\]

There is another useful way of putting this. When \( K \) is small, both \( \Phi[K] \) and \( \Phi(K) \) makes sense for any class \( \Phi \); and in fact we have

3.2 \( \Phi[K] \subseteq \Phi(K), \)

since for \( \phi : K^{op} \to V \) the Yoneda isomorphism

3.3 \( \phi \cong \phi * Y \)

exhibits \( \phi \) as an object of \( \Phi(K) \) when \( \phi \in \Phi \). We can write Proposition \[\text{3.4}\] as

3.4 \( \Phi^*[K] = \Phi(K) \)

so that \( \Phi \) is a saturated class precisely when

3.5 \( \Phi[K] = \Phi(K) \)

for each small \( K \). In other words the class \( \Phi \) is saturated precisely when, for each small \( K \), \( \Phi(K) \) contains the representables \( K(-, k) \) and is closed in \( [K^{op}, V] \) under \( \Phi \)-colimits.

Note that the saturation \( 0^* \) of the empty class 0 consists precisely in the representables - now in the sense that \( 0^*[K] = 0^*(K) \) consists exactly in the functors isomorphic to some \( K(-, k) : K^{op} \to V \).

It follows of course from the definitions of \( \Phi(A) \) and of \( \Phi^* \) that

3.6 \( \Phi^*(A) = \Phi(A) \)
for any $A$, small or not. We cannot write $\text{KSp}$ when $K$ is replaced by a non-small $A$, since then $\Phi[A]$ has no meanings; but a partial replacement for it is provided by the following which was Proposition 7.4 in [AK88].

**Proposition 3.5** If the presheaf $F : A^{\text{op}} \to V$ lies in $\Phi(A)$ for some closed class $\Phi$ then $F$ is a $\Phi$-colimit of representables; that is $F \cong \phi \ast ZS$ for some $\phi : K^{\text{op}} \to V$ in $\Phi$ and some $S : K \to A$.

It may be useful to understand extreme special cases of one’s notation: we mentioned above the empty class 0 of weights, with $0[K] = 0$ and with $0(K) = 0^\ast(K) = 0^\ast[K]$ consisting of the representables. Another extreme case involves the empty $V$-category 0 with no objects. Of course $P0 = [0^{\text{op}},V]$ is the terminal category 1; its unique object is the unique functor $! : 0^{\text{op}} \to V$ and 1(!!) is the terminal object 1 of $V$. (This differs in general from the unit $V$-category $I$, with one object $\ast$ but with $I(\ast,\ast)=I$, the unit for $\otimes$.) So for any class $\Phi$, we have $\Phi[0] = 0$ if $!:0^{\text{op}} \to V$ is not in $\Phi$, and $\Phi[0] = 1$ otherwise. Now $\Phi(0)$ is the closure of 0 in $P0$ under $\Phi$-colimits, and any diagram $T : K \to 0$ has $K = 0$, so that $\Phi(0)$ is 0 if $! \notin \Phi$ and otherwise $\Phi(0)$ contains $! \ast Y = !$, giving $\Phi(0) = 1$. So in fact $\Phi(0) = \Phi[0]$, being 0 or 1. Both are possible for a closed $\Phi$ for $P0 = 1$, while the Albert-Kelly theorem (Proposition 3.4) gives 0(!) is the terminal object 1 of $V$; then $\Phi(0) = 0[0] = 0^\ast[0] = 0$. Both are possible for a closed $\Phi$ for $P0 = 1$, while the Albert-Kelly theorem (Proposition 3.4) gives 0(0) = 0[0] = 0[0] = 0.

Before ending this section, we recall a result characterising $\Phi$-cocomplete categories, along with a short proof. This was Proposition 4.5 in [AK88].

**Proposition 3.6** For any class $\Phi$ of weights, a category $A$ admits $\Phi$-colimits if and only if the fully faithful embedding $Z : A \to \Phi(A)$ admits a left adjoint; that is, if and only if the full subcategory $A$ given by the representables is reflective in $\Phi(A)$.

**PROOF:** If $A$ is reflective, it admits $\Phi$-colimits because $\Phi(A)$ does so. Suppose conversely that $A$ admits $\Phi$-colimits, and write $B$ for the full subcategory of $[A^{\text{op}},V]$ given by those objects admitting a reflection into $A$; then $B$ contains $A$ and $B$ is closed in $[A^{\text{op}},V]$ under $\Phi$-colimits since $A$ admits these; so that $B$ contains $\Phi(A)$, as desired. \hfill $\blacksquare$

## 4 The embedding $A \to \Phi(A)$

We recall from Proposition 5.62 of [Kel82] a result characterising categories of the form $\Phi(A)$ - or more precisely functors of the form $Z : A \to \Phi(A)$ (which we also write when necessary as $Z_A$). At the same time, we give a direct proof; for the proof in [Kel82] refers back to earlier results in that book.

We begin with a piece of notation: for a category $A$ and a class $\Phi$ of weights, we write $A_\Phi$ for the full subcategory of $A$ generated by those $a \in A$ for which the representables $A(a,-) : A^{\text{op}} \to V$ preserve all $\Phi$-colimits (That is all $\Phi$-colimits that exist in $A$). There is no agreed name for $A_\Phi$; the objects of $A_\Phi$ are usually called finitely presentable when the $\Phi$-colimits are the classical filtered colimits; while when $\Phi$ is the class $P$ of all weights the objects of $A_\Phi$ were called small projective in [Kel82], but have also been called atoms by some authors. Therefore we propose to name the objects of $A_\Phi$ $\Phi$-atoms. When $A$ admits $\Phi$-colimits and hence $\Phi^+$-colimits, it follows from the definition of $A_\Phi$ that

4.1 $A_\Phi^+ = A_\Phi$.

The following is the characterisation result of Proposition 5.62 [Kel82] with a slightly expanded form of its statement.

**Proposition 4.1** In order that $G : A \to B$ be equivalent to the free $\Phi$-cocompletion $Z : A \to \Phi(A)$ of $A$ for a class $\Phi$ of weights, the following conditions are necessary and sufficient:

- (i) $G$ is fully faithful (allowing us to treat $A$ henceforth as a full subcategory of $B$);
• (ii) $\mathcal{B}$ is $\Phi$-cocomplete;

• (iii) the closure of $\mathcal{A}$ in $\mathcal{B}$ under $\Phi$-colimits is $\mathcal{B}$ itself;

• (iv) $\mathcal{A}$ is contained in the full subcategory $\mathcal{B}_\Phi$ of $\Phi$-atoms of $\mathcal{B}$.

When these conditions hold, the functor $\tilde{G} : \mathcal{B} \to [A^{op}, \mathcal{V}]$ defined (in the usual notation of [Kel82]) by $\tilde{G} b = B(G\cdot b)$ is fully faithful and takes its image in the full subcategory $\Phi(A)$ of $[A^{op}, \mathcal{V}]$ thereby defining a fully faithful $K : \mathcal{B} \to \Phi(A)$; and this $K$ is in fact an equivalence - inverse being given by $\text{Lan}_Z(G) : \Phi(A) \to \mathcal{B}$, which by Proposition 3.1 is the unique $\Phi$-cocontinuous extension of $G$ to $\Phi(A)$.

PROOF: The necessity of the first three conditions is clear; for that of the forth, writing as before

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{Y} & [A^{op}, \mathcal{V}] \\
\downarrow & & \downarrow \\
\Phi(A) & \xleftarrow{Z \mathcal{W}} & \mathcal{B}
\end{array}
$$

for the full inclusions, the point is that $\Phi(A)(Za, -) \cong [A^{op}, \mathcal{V}](Ya, W-)$ preserves $\Phi$-colimits since $W : \Phi(A) \to [A^{op}, \mathcal{V}]$ does so by definition of $\Phi(A)$, while $[A^{op}, \mathcal{V}](Ya, -) : [A^{op}, \mathcal{V}] \to \mathcal{V}$ preserves all small colimits, being isomorphic by Yoneda to the evaluation $E_a$.

Turning to the proof of sufficiency, to say that $\tilde{G}$ is fully faithful is just to say (see Theorem 5.1 [Kel82]) that $G : \mathcal{A} \to \mathcal{B}$ is dense; and in the present case that is so by Theorem 5.19 of [Kel82] since $\mathcal{B}$ is the closure of $\mathcal{A}$ in $\mathcal{B}$ under $\Phi$-colimits and $\mathcal{A}$ lies in $\mathcal{B}_\Phi$. The proof is so short that we may recall it: we consider the full subcategory of $\mathcal{B}$ given by those $b$ for which $G b, c : b(b, c) \to [A^{op}, \mathcal{V}](Gb, Gc)$ is invertible for all $c$, observing that it is all of $\mathcal{B}$ since it contains $\mathcal{A}$ - since $G$ is fully faithful - and is closed under $\Phi$-colimits - as $\mathcal{A} \subseteq \mathcal{B}_\Phi$.

Next, to see that each $Gb = B(G\cdot b)$ lies in the full replete subcategory $\Phi(A)$ of $[A^{op}, \mathcal{V}]$, consider the full subcategory of $\mathcal{B}$ given by those $b$ for which this is so: this contains $\mathcal{A}$ since $G \cdot Ga \cong Ya$ because $G$ is fully faithful, and it is closed in $\mathcal{B}$ under $\Phi$-colimits since $G$ preserves these by (iv) and since $\Phi(A)$ is closed under these in $[A^{op}, \mathcal{V}]$; so it is all of $\mathcal{B}$. Thus $\tilde{G}$ is indeed of the form $\tilde{G} = WK$ for some fully faithful $K : \mathcal{B} \to \Phi(A)$.

It remains to show that $K$ and $S = \text{Lan}_Z(G) : \Phi(A) \to \mathcal{B}$ are equivalence-inverses. Recall from Proposition 3.1 that $S$ is equally the essentially unique $\Phi$-cocontinuous functor with $SZ = G$. Since $\tilde{G} = WK$ and $G \cong SZ$, one has $WKSZ \cong \tilde{G} G \cong Y \cong WZ$, giving $KSZ \cong Z$ since $W$ is fully faithful and then giving $KS \cong 1$ by Proposition 3.1 since $KS$ and $1$ are $\Phi$-cocontinuous ($K$ being so because $WK = \tilde{G}$ is so). Finally $KS \cong 1$ gives $SKS \cong K$ and $SK \cong 1$ since $K$ (as we saw) is fully faithful.

This is of course of particular interest in the case of a small $\mathcal{A}$ in Proposition 3.1. If we suppose the class $\Phi$ saturated for simplicity, we may cast the result for small $\mathcal{A}$’s in the form

**Proposition 4.2** For a saturated class $\Phi$ of weights, the following properties of a category $\mathcal{B}$ are equivalent:

• (i) For some small $\mathcal{K}$, there is an equivalence $\mathcal{B} \cong \Phi(\mathcal{K})$;

• (ii) $\mathcal{B}$ is $\Phi$-cocomplete and has a small full subcategory $\mathcal{A} \subseteq \mathcal{B}_\Phi$ such that every object of $\mathcal{B}$ is a $\Phi$-colimit of a diagram in $\mathcal{A}$;

• (iii) $\mathcal{B}$ is $\Phi$-cocomplete and has a small full subcategory $\mathcal{A} \subseteq \mathcal{B}_\Phi$ such that the closure of $\mathcal{A}$ in $\mathcal{B}$ under $\Phi$-colimits is $\mathcal{B}$ itself.

Under the hypothesis (iii) - and so a fortiori under (ii) - if $G : \mathcal{A} \to \mathcal{B}$ denotes the inclusion, the functor $\tilde{G} : \mathcal{B} \to [A^{op}, \mathcal{V}]$ is fully faithful, with $\Phi(A)$ for its replete image.
Remarks 4.3

- (a) When $\Phi$ is the class $\mathcal{P}$ of all weights, we get a characterisation here of the functor category $\mathcal{P}\mathcal{K} = [\mathcal{K}^{\text{op}}, \mathcal{V}]$ for a small $\mathcal{K}$; note that it differs from the characterisation given in \cite{Kel82} Theorem 5.26 which replaces the condition that $\mathcal{B}$ be the colimit closure of $\mathcal{A}$ by the condition that $\mathcal{A}$ be strongly generating in $\mathcal{B}$; but that these conditions are very similar in strength by \cite{Kel82} Proposition 3.40.

- (b) When $\mathcal{V} = \text{Set}$, various special cases of this are well known, besides that where $\Phi = \mathcal{P}$; for instance the cases where $\Phi$-colimits are finite coproducts, or finite colimits or filtered colimits, or absolute colimits. We will return to this last example in section 6.

- (c) Theorem 2.4 of \cite{BQR98} is the special case where $\mathcal{V}$ is locally finitely presentable as a closed category in the sense of \cite{Kel82} and $\Phi$ is the saturated class of flat presheaves.

Of particular interest regarding Proposition 4.2 are the saturated families of the form $\Psi^+$ and $\Phi^-$, for arbitrary classes of weights $\Psi$ and $\Phi$. In particular we shall see that when $\Phi = \mathcal{P}$ is the family of all weights, $\Phi^+(\mathcal{A})$ is the Cauchy-completion of $\mathcal{A}$. Also the fact of considering families of weighted limits/colimits instead of mere conical limits/colimits is crucial even in the case $\mathcal{V} = \text{Set}$. In \cite{ABL02}, Adamek, Borceux, Lack and Rosicky investigated the notion of general commutation of families of limits/colimits - the limits and colimits considered are classical - i.e. defined in terms of diagrams - not weight. To obtain a result similar to Proposition 4.2 (that one may call “parameterised accessibility”), they needed to impose the technical condition of “soundness” on classes of limits. The next section should be a key argument in favour of weighted limits/colimits, that should also invalidate the wrong idea that “weighted limits/colimits in Sets are just the same as usual ones”.

5 Limits and colimits commuting in $\mathcal{V}$

The new observations to which we now turn on begin with the general theory of the commutativity in $\mathcal{V}$ of limits and colimits. For a pair of weights $\psi : \mathcal{K}^{\text{op}} \to \mathcal{V}$ and $\phi : \mathcal{L}^{\text{op}} \to \mathcal{V}$, to say that

$5.1 \phi* - : [\mathcal{L}, \mathcal{V}] \to \mathcal{V}$ preserves $\psi$-limits

is equally to say that

$5.2 \{\psi, -\} : [\mathcal{K}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$ preserves $\phi$-colimits

because each in fact asserts the invertibility, for every functor $S : \mathcal{K}^{\text{op}} \otimes \mathcal{L} \to \mathcal{V}$, of the canonical comparison morphism

$5.3 \phi^? * \{\psi-, S(-,?)\} \to \{\psi-, \phi* S(-,?)\}$.

The reader who would not find the above statement immediate, would be happy to consider the following proposition.

**Proposition 5.1** Let $F : \mathcal{A}^{\text{op}} \to \mathcal{V}$, $P : \mathcal{K} \to \mathcal{V}$ and $G : \mathcal{A} \otimes \mathcal{K} \to \mathcal{V}$ equivalent to $G' : \mathcal{A} \to [\mathcal{K}, \mathcal{V}]$ and also to $G'' : \mathcal{K} \to [\mathcal{A}, \mathcal{V}]$. Then $F* - : [\mathcal{A}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{P, G''\}$ if and only if $\{P, -\} : [\mathcal{K}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $F* G'$.

**PROOF:** Let

$P \xrightarrow{\eta} [\mathcal{A}, \mathcal{V}](\{P, G''\}, G''-) \quad \text{be the unit of } \{P, G''\}$ and

$F \xrightarrow{\lambda} [\mathcal{K}, \mathcal{V}](G'-, F* G') \quad \text{be the unit of } \{F* G'\}$.
be the unit of \(F \ast G'\). We need to show that

\[
\begin{array}{ccc}
(1) & P & \longrightarrow \{A, \mathcal{V}\}((P, G'), G'\} \xrightarrow{F \ast} \{F \ast \{P, G''\}, (F \ast G''\}]
\end{array}
\]

exhibits \(F \ast \{P, G''\}\) as \(\{P, F \ast G'\}\) if and only if

\[
\begin{array}{ccc}
(2) & F & \longrightarrow \{K, \mathcal{V}\}(F \ast G', G'\} \xrightarrow{\{P,\}} \{(P, F \ast G'), \{P, G'\}\}
\end{array}
\]

exhibits \(\{P, F \ast G'\}\) as \(F \ast \{P, G''\}\).

First note that given \(x \in \mathcal{V}\), any natural in \(k\),

\[
(1') Pk \rightarrow_k [x, (F \ast G')k]
\]

corresponds via Yoneda to a natural in \(v\), \([v, x] \rightarrow [K, \mathcal{V}](P, [v, (F \ast G')])\). Since

\[
[K, \mathcal{V}](P, [v, (F \ast G')]) \cong [v, \{P, F \ast G'\}],
\]

it corresponds also to an arrow

\[
(1'') x \rightarrow \{P, F \ast G'\}.
\]

Also that \((1')\) exhibits \(x\) as the limit \(\{P, F \ast G'\}\) is equivalent to the fact that \((1'')\) is iso. Analogously any natural in \(a\),

\[
(2') Fa \rightarrow \{(P, G''\}a, x\]
\]

corresponds to an arrow

\[
(2'') F \ast \{P, G''\} \rightarrow x,
\]

and \((2')\) exhibits \(x\) as the colimit \(F \ast \{P, G''\}\) if and only if \((2'')\) is iso.

Now the result follows from the fact the arrow \((1)\) above corresponds by the bijection \((1') - (1'')\) to the same arrow as \((2)\) by \((2') - (2'')\). This is now left to the reader.

When these statements are true for every such \(S\), we say that \(\phi\)-colimits commute with \(\psi\)-limits in \(\mathcal{V}\). For classes \(\Phi\) and \(\Psi\) of weights, if [5.1] (equivalently [5.2]) holds for all \(\phi \in \Phi\) and all \(\psi \in \Psi\), we say that \(\Phi\)-colimits commute with \(\Psi\)-limits in \(\mathcal{V}\). For any class \(\Psi\) of weights we may consider the class \(\Psi^+\) of all weights \(\phi\) for which \(\phi\)-colimits commute with \(\Psi\)-limits in \(\mathcal{V}\); and for any class \(\Phi\) of weights we may consider the class \(\Phi^-\) of all weights \(\psi\) for which \(\Phi\)-colimits commute with \(\psi\)-limits in \(\mathcal{V}\). This forms of course a Galois connection (with \(\Phi \subseteq \Psi^+\) if and only if \(\Psi \subseteq \Phi^-\)). Note that \([\mathcal{L}, \mathcal{V}]\) and \(\mathcal{V}\) in [5.4] admit all (small) limits; so that by the definition above of the saturation \(\Psi^+\) of a class \(\Psi\) of weights if \(\phi \ast -\) preserves all \(\Psi\)-limits, it also preserves \(\Psi^+\)-limits. From this and a dual argument, one concludes that:

**Proposition 5.2** For any class \(\Phi\) and \(\Psi\) of weights, the class \(\Phi^-\) and \(\Psi^+\) are saturated classes of weights; so that \(\Psi^+ = \Psi^+\) and \(\Phi^- = \Phi^-\). Moreover \(\Psi^+ = \Psi^+\) and \(\Phi^- = \Phi^-\).

When \(\Psi\) consists of the weights for finite limits (in the usual case for ordinary categories, or in the sense of [Kes82], when \(\mathcal{V}\) is “locally finitely presentable as a closed category”, its has been customary to call the elements of \(\Psi^+\) the flat weights, as they are those \(\phi\) having \(\phi \ast - : [\mathcal{L}, \mathcal{V}] \rightarrow \mathcal{V}\) left exact. We might therefore for a general \(\Psi\) think of the elements of \(\Psi^+\) as \(\Psi\)-flat weights.

Recall that the limit functor \(\{\psi, -\}\) of [4.2] is just the representable functor \([\mathcal{K}^{op}, \mathcal{V}](\psi, -) : [\mathcal{K}^{op}, \mathcal{V}] \rightarrow \mathcal{V}\). Accordingly \(\psi : [\mathcal{K}^{op}, \mathcal{V}] \rightarrow \mathcal{V}\) lies in \(\Phi^-\) for a given class \(\Phi\) if and only if it lies in the subcategory \([\mathcal{K}^{op}, \mathcal{V}]_S\) of \(\Phi\)-atoms:

**5.4** \(\Phi^-[\mathcal{K}] = \Phi^-[\mathcal{K}] = [\mathcal{K}^{op}, \mathcal{V}]_S\)

9
Part of the saturation of $\Phi^-$ - namely the closedness of $\Phi^-[K]$ in $[K^{op}, V]$ under $\Phi^-$-colimits is the special case for $A = [K^{op}, V]$ of the following more general result:

**Proposition 5.3** For any class $\Phi$ of weights and any category $A$, the full subcategory $A_{\Phi}$ of $A$ is closed in $A$ under $\Phi^-$-colimits that exists in $A$.

**PROOF:** Let the colimit $\psi * S$ exist, where $\psi : K^{op} \to V$ lies in $\Phi^-$ and $S : K \to A$ takes its values in $A_{\Phi}$. Then by definition

$$A(\psi * S, a) \cong [K^{op}, V](\psi, A(S-, a)).$$

Since each $A(SK, -)$ preserves $\Phi$-colimits, and since $[K^{op}, V](\psi, -)$ preserves $\Phi$-colimits by $\ref{prop5.2}$ it follows that $A(\psi * S, -)$ preserves $\Phi$-colimits, that is to say $\psi * S \in A_{\Phi}$. $\blacksquare$

Let us add ad-hoc proof of the closeness of $\Phi[K]$ under $\Phi^+$-colimits in $[K^{op}, V]$ (though this does not bring anything more than Proposition $\ref{prop5.2}$)

**Lemma 5.4** For any $\Psi$-flat $G : A^{op} \to V$ and any functor $H : A \to [C^{op}, V]$ with values $\Psi$-flat functors, the colimit $G * H$ is again $\Psi$-flat.

**PROOF:** Consider $F : K \to V$ in $\Psi$ and $L : K \to [C, V]$. One has the successive isomorphisms:

$$(G * H) * \{F, L\} \cong G * (H * \{F, L\}) \quad (\text{Kels2}, \ (3.23) \ "\text{continuity of a colimit in its index"})$$

$$\cong G * \{\{F, H * L\}\} \quad (\text{since for all } a, Ha * - \text{ preserves } \Psi\text{-limits})$$

$$\cong \{F, G * (H * L)\} \quad (\text{since } G * \text{ preserves } \Psi\text{-limits})$$

$$\cong \{F, (G * H) * L\} \quad (\text{Kels2}, \ (3.23)).$$

The resulting isomorphism $(G * H) * \{F, L\} \cong \{F, (G * H) * L\}$ corresponds actually to the preservation of $\{F, L\}$ by $(G * H) * -$.

### 6 Cauchy completion and Isbell adjunction

This last section is devoted to the study of the particular class of weights $Q = P^-$ where $P$ stands as before for the class of all weights. That is to say that $Q$ is the class defined by:

**6.1** $\phi \in Q \Leftrightarrow \phi$-limits commute with all small colimits in $V$.

and equivalently, as we shall see, that the weights in $Q$ are the so-called small projective ones. According to Proposition $\ref{prop5.2}$, we already know that $Q$ is a saturated class. We shall establish the following alternative characterisations for $Q$. From Proposition $\ref{prop5.2}$

**6.2** $\phi : A^{op} \to V$ lies in $Q$ if and only the corresponding module $\mathcal{I} \longrightarrow A$ is left adjoint.

This last result shows that the Cauchy-completion of a small category $A$ is just $Q(A) = Q[\mathcal{A}]$. Also from Proposition $\ref{prop6.11}$

**6.3** $Q$ is the class of $P$-flat weights, i.e. $Q = P^+$.

The end of the section treats an enriched adjunction à la Isbell that restricts to an equivalence between subcategories of small projectives.

We need to recall a little terminology and few facts regarding adjoints, Kan extensions and liftings in a bicategory. Let us consider a bicategory $B$ with horizontal composition “$*$” and vertical one “$\circ$” to fix the notations. $B^{op}$ will denote the bicategory obtained from $B$ by reversing arrows, and $B^{coop}$ will be the one obtained by reversing arrows and 2-cells. We borrow from [StWa78] the following terminology. Given a diagram as below in $B$
the arrow $h$ together with 2-cell $\epsilon$ constitutes the right lifting of $g$ through $f$ if and only if pasting with $\epsilon$ at $h$ determines a bijection of 2-cells

$$B(C, B)(f * k, g) \leftrightarrow B(C, A)(k, h).$$

In this situation we shall loosely write $h = \text{rlift}(f, g)$. The right lifting is said respected by a $k : X \to C$ when the 2-cell $\epsilon * k$ exhibits $h * k$ as the right lifting of $g * k$ through $f$, it is absolute when it is respected by any such arrow. Note that a right lifting in $B$ is a right Kan extension in $B^{\text{op}}$ or a left Kan extension in $B^{\text{coop}}$. One has also a dual notion of left liftings that correspond to left Kan extensions in $B^{\text{op}}$.

The following result characterising adjoints in a bicategory goes back to early work of Lawvere and Bénabou on closed bicategories. In fact the full result is really present in the one object-closed-bicategory situation given by a monoidal closed category, and that was studied in the symmetric case by [KeLa80].

**Proposition 6.1** In ANY bicategory, the following statements are equivalent:

- (i) the arrow $s : A \to B$ has a right adjoint;
- (ii) for all $g : C \to B$, the right lifting $\text{rlift}(s, g)$ of $g$ through $s$ exists and is absolute;
- (iii) the right lifting $\text{rlift}(s, 1)$ of $1 : B \to B$ through $s$ exists and is respected by $s$.

When $s$ has a right adjoint, this one is $\text{rlift}(s, 1)$ the right lifting of $1$ through $s$.

**PROOF:** (i) $\Rightarrow$ (ii). Let $B$ denote the ambient bicategory. Let also $\eta : 1 \Rightarrow t * s : A \to A$ and $\epsilon : s * t \Rightarrow 1 : B \to B$ denote respectively the unit and counit of $s - t$. For any 1-cells $g : C \to B$ and $x : C \to A$ the “pasting with $\epsilon$”: $(\epsilon * g) \cdot (s * -) : B(C, A)(x, t * g) \to B(C, B)(s * x, g)$ admits for inverse the “pasting with $\eta$”: $(t * -) \cdot (\eta * x)$. This shows that $\epsilon * g : s * t * g \Rightarrow g$ exhibits $t * g$ as the right lifting of $g$ through $s$. This lifting is trivially absolute.

(ii) trivially implies (iii).

(iii) $\Rightarrow$ (i).

The 1-cell

$$A \xleftarrow{s} 1 \xrightarrow{s} B \xleftarrow{s} A$$

factorises as
since $s$ respects the absolute right lifting

Now

equals

and because $\epsilon$ is a right lifting

equals

$B \leftarrow t \rightarrow A = B \rightarrow t \rightarrow A$
As adjoints $F \dashv G$ in a bicategory $\mathcal{B}$ correspond to adjoints $G \dashv F$ in $\mathcal{B}^{op}$, there are dual statements obtained by using left or right Kan extensions or left liftings. In detail

**Proposition 6.2** In any bicategory, the following assertions are equivalent:

- the arrow $s : A \to B$ has a right adjoint;
- for all $g : A \to C$, the left Kan extension $\text{lan}_s(g)$ exists and is absolute;
- the left Kan extension $\text{lan}_s(1)$ of $1 : A \to A$ along $s$ exists and is preserved by $s$.

When $s$ has a right adjoint, this one is $\text{lan}_s(1)$.

And

**Proposition 6.3** In any bicategory, the following assertions are equivalent:

- (i) the arrow $t : B \to A$ has a left adjoint;
- (ii) for all $f : C \to A$, the left lifting $\text{llift}(t, f)$ of $f$ through $t$ exists and is absolute;
- (iii) the lifting $\text{llift}(t, 1)$ of $1 : A \to A$ through $t$ exists and is respected by $t$.
- (iv) for all $g : B \to C$, the right Kan extension $\text{ran}_t(g)$ of $g$ along $t$ exists and is absolute;
- (v) the right Kan extension $\text{ran}_t(1)$ of $1 : B \to B$ along $t$ exists and is preserved by $t$.

When $t$ has a left adjoint, this one is $\text{llift}(t, 1)$ or also $\text{ran}_t(1)$.

There is one lemma that we shall use later

**Proposition 6.4** In a bicategory if an arrow $f : A \to B$ has a right adjoint $g : B \to A$ then for any $h : C \to B$, the right lifting of $h$ through $f$ exists and is given by $\text{rlift}(f,h) \cong g \ast h$.

PROOF: The counit $\epsilon : f \ast g \to 1 : B \to B$ of the adjunction $f \dashv g$ exhibits $g$ as the right lifting $\text{rlift}(f, 1)$ of $1 : B \to B$ through $f$, and this right lifting is absolute. Composing then with any $h : C \to B$ yields the right lifting $\text{rlift}(f,h)$.

We shall now consider the more specific case when the considered bicategory is closed, i.e. such a bicategory admits all right Kan extensions and all right liftings and is locally complete and cocomplete. The older terminology for this was biclosed, but the more modern terminology recognises that a bicategory may be right-closed or left-closed, or both and to be both is to be close.

In a closed bicategory $\mathcal{B}$, for any objects $A$ and any arrow $h : C \to B$, one has the natural isomorphisms in $f, g$:

6.4

$\mathcal{B}(A,B)(f,\text{ran}_g(h)) \cong \mathcal{B}(C,B)(f \ast g, h) \cong \mathcal{B}(C,A)(g,\text{rlift}(f,h))$

and thus an adjunction

6.5

$\text{rlift}(\mathcal{B}) \dashv \text{ran}_\mathcal{B} : \mathcal{B}(A,B) \to \mathcal{B}(C,A)^{op}$

Note also that according to 6.4 and 6.8

**Remark 6.5** the composite isomorphism 6.4 in the case $C = B$ and $h = 1_B : B \to B$, maps bijectively left adjoints $A \to B$ to right adjoints $B \to A$. 

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We can now turn to the specific case of the closed bicategory \( \mathcal{V} \text{-Mod} \) of modules with domains and codomains \textit{small} categories. For categories \( A \) and \( B \) not necessarily small, a module \( A \to B \) is a functor \( B^{\text{op}} \otimes A \to \mathcal{V} \) sending \((b, a)\) to \( f(b, a)\) and a morphism \( \alpha : f \to g : A \to B \) of module is a natural transformation with components \( \alpha_{b,a} : f(b, a) \to g(b, a) \). So the modules from \( A \) to \( B \) form an ordinary category \( \mathcal{V} \text{-MOD}(A, B) \) that underlines a \( \mathcal{V}' \)-category \( \mathcal{V} \text{-MOD}[A, B] \) (for a suitable \( \mathcal{V}' \) extending \( \mathcal{V} \)), which is a \( \mathcal{V} \)-category when \( A \) and \( B \) are small. Of course a module \( A \to B \) can equally be seen as a functor \( A \to [B^{\text{op}}, \mathcal{V}] \), and so (to within equivalence) as a cocontinuous functor \( P_A \to [B^{\text{op}}, \mathcal{V}] \). Therefore when \( A \) and \( B \) are small, \( \mathcal{V} \text{-MOD}(A, B) \) is equivalent to \( \text{Cocts}([A^{\text{op}}, \mathcal{V}], [B^{\text{op}}, \mathcal{V}]) \) and the cocontinuous functors here are equally the left adjoint ones - see [Kel82], Theorem 4.82). For modules \( f : A \to B \) and \( g : B \to C \) we have a composite \( gf : A \to C \) defined by

\[
(gf)(c, a) = \int^b g(c, b) \otimes f(b, a)
\]

whenever the integral exists - which surely does when \( B \) is small. This composition is associative up to coherent isomorphisms, with as identities-to-within-isomorphisms the \( I_A : A \to A \) given by the hom functor \( A(-,-) : A^{\text{op}} \otimes A \to \mathcal{V} \); so that the modules between \textit{small} categories are the arrows of a bicategory \( \mathcal{V} \text{-Mod} \). This \( \mathcal{V} \text{-Mod} \) is \textit{closed} as it admits all right liftings and all right Kan extensions as follows. Given modules \( f : A \to B \), \( g : B \to C \) and \( h : A \to C \), we consider new modules \( \lfloor f, h \rfloor : B \to C \) and \( \{ g, h \} : A \to B \) by

\[
\lfloor f, h \rfloor(c, b) = \int_a [f(b, a), h(c, a)]
\]

and

\[
\{ g, h \}(b, a) = \int_c [g(c, b), h(c, a)]
\]

whenever these integrals exist - so that \( \lfloor f, h \rfloor \) certainly exists when \( A \) is small, while \( \{ g, h \} \) certainly exists when \( C \) is small. When \( gf \) and \( \lfloor f, h \rfloor \) exist, to give a morphism \( gf \to h \) is to give a family \( g(c, b) \otimes f(b, a) \to h(c, a) \) natural (meaning \( \mathcal{V} \)-natural) in each variable; which is equally to give a family \( g(c, b) \to [f(b, a), h(c, a)] \) natural in each variable, and so as morphism \( g \to \lfloor f, h \rfloor \) of modules. Accordingly so far as the modules in question exist we have natural bijections

\[
\mathcal{V} \text{-MOD}(A, C)(gf, h) \cong \mathcal{V} \text{-MOD}(B, C)(\lfloor f, h \rfloor) \cong \mathcal{V} \text{-MOD}(A, B)(\{ g, h \}).
\]

exhibiting \( \lfloor f, h \rfloor \) as the right Kan extension of \( h \) along \( f \) and \( \{ g, h \} \) as the right lifting of \( h \) through \( g \).

Each functor \( T : A \to B \) gives rise to modules \( T_* : A \to B \) and \( T^* : B \to A \), where

\[
T_*(b, a) = B(b, Ta) \text{ and } T^*(a, b) = B(Ta, b);
\]

and a natural \( \rho : T \to S : A \to B \) gives rise to evident morphisms of modules

\[
\rho_* : T_* \to S_* \text{ and } \rho^* : S^* \to T^*.
\]
Since $\rho \mapsto \rho_*$ and $\rho \mapsto \rho^*$ are fully faithful by Yoneda, any of $\rho, \rho_*$ and $\rho^*$ determines the others. For a functor $T : \mathcal{A} \to \mathcal{B}$, and modules $g : \mathcal{B} \to \mathcal{C}$ and $h : \mathcal{D} \to \mathcal{B}$ where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are not necessarily small, substituting the values [6.10] into [6.6] [6.7] and [6.8] and using the Yoneda isomorphisms gives natural isomorphisms

6.11

\[(gT_*)(c, a) \cong g(c, Ta) \cong [T^*, g](c, a)\]

6.12

\[(T^*h)(a, d) \cong h(Ta, d) \cong \{ T_*, h \}(a, d).\]

In particular for $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{C}$, one has natural isomorphisms

\[(ST)_* \cong ST_* \text{ and } (ST)^* \cong T^*S^*;\]

so that there are locally fully faithful embeddings of bicategories

6.13

\[(-)_* : \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Mod} \text{ and } (-)^* : \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Mod}^\text{coop}.\]

We shall now discuss adjunctions in $\mathcal{V}\text{-Mod}$. Consider a functor $T : \mathcal{A} \to \mathcal{B}$ in $\mathcal{V}\text{-Cat}$ and a module $k : \mathcal{D} \to \mathcal{B}$ in $\mathcal{V}\text{-Mod}$. Taking the isomorphism [6.12] \[\{ T_*, k \} \cong T^*k\] shows that the right lifting of $k$ through $T_*$ is absolute so by [6.11] $T_*$ has right adjoint $T^*$. In any bicategory left adjoints are often called maps and it is common to denote by $f^*$ the right adjoint of $f$. This is then consistent with the above notation $T_* \dashv T^*$ above provided that we identify functor with their associated module via the fully faithful $(-)_*$ of [6.13]. However the functors are not the only maps in $\mathcal{V}\text{-Mod}$ that we shall characterise now.

Consider the following situation in $\mathcal{V}\text{-Mod}$

\[
\begin{array}{c}
\mathcal{D} \\
\downarrow h \\
\mathcal{C} \\
\downarrow g \\
\mathcal{B} \\
\downarrow f \\
\mathcal{A}
\end{array}
\]

where the bottom triangle is a right lifting. The module $h$ gives by restriction for each object $d$ of $\mathcal{D}$ a functor $H_d : \mathcal{C}^{\text{op}} \to \mathcal{V}$. Similarly $f$ gives for each $a$ in $\mathcal{A}$ a functor $F_a : \mathcal{B}^{\text{op}} \to \mathcal{V}$ and the module $g$ corresponds to a functor $G : \mathcal{B}^{\text{op}} \to [\mathcal{C}, \mathcal{V}]$. One may check straightforwardly that

6.14 the composition with $h$ respects the right lifting \{ $f, g$ \} of $g$ through $f$.

if and only if

6.15 for all $a$ in $\mathcal{A}$ and all $d$ in $\mathcal{D}$, $H_d * - : [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves the limit \{ $F_a, G$ \}.

Writing now $G'$ for the functor $\mathcal{C} \to [\mathcal{B}^{\text{op}}, \mathcal{V}]$ corresponding to $g$, one gets according to [5.23] that the statement above is also equivalent to the fact that

6.16 for all $a$ in $\mathcal{A}$ and all $d$ in $\mathcal{D}$, the colimit $H_d * G'$ is preserved by \{ $F_a, -$ \} : [\mathcal{B}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$.
At the light of these observations, we may now reformulate Proposition 6.6 for the case when the ambient bicategory is $\mathcal{V}$-$\text{Mod}$. Note that the assertion (ii) below is the direct translation by 6.14-6.16 of the fact that for any module $g : \mathcal{C} \longrightarrow \mathcal{B}$ the right lifting $\{ f, g \}$ is respected by composition with any module $\mathcal{I} \longrightarrow \mathcal{C}$.

**Proposition 6.6** Given small categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $F : \mathcal{A} \rightarrow [\mathcal{B}^{op}, \mathcal{V}]$ that corresponds to the module $f : \mathcal{I} \longrightarrow \mathcal{B}$, the following conditions on $F$ or $f$ are equivalent:

- (i) $f$ (as a module) has a right adjoint $f^*$;
- (ii) for each $a$ in $\mathcal{A}$, the representable functor $\{ Fa, - \} = [\mathcal{B}^{op}, \mathcal{V}](Fa, -) : [\mathcal{B}^{op}, \mathcal{V}] \rightarrow \mathcal{V}$ is cocontinuous;
- (iii) for each pair $a,c$ in $\mathcal{A}$, the functor $Fc * - : [\mathcal{B}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves the limit $\{ Fa,Y' \}$ of $Y' : \mathcal{B}^{op} \rightarrow [\mathcal{B}, \mathcal{V}]$ weighted by $Fa : \mathcal{B}^{op} \rightarrow \mathcal{V}$.
- (iv) for each pair $a,c$ in $\mathcal{A}$, the representable functor $[\mathcal{B}^{op}, \mathcal{V}](Fa, -) : [\mathcal{B}^{op}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves the colimit $Fc * Y$ of $Y : \mathcal{B} \rightarrow [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $Fc : \mathcal{B}^{op} \rightarrow \mathcal{V}$ giving (since $Fc * Y \cong Fc$ by Yoneda) an isomorphism $Fc * [\mathcal{B}^{op}, \mathcal{V}](Fa,Y-) \cong [\mathcal{B}^{op}, \mathcal{V}](Fa,Fc)$.

When these are satisfied, we have

**6.17**

\[ f^*(a,b) \cong [\mathcal{B}^{op}, \mathcal{V}](Fa,Yb). \]

Proposition 6.6 gives in the particular case $\mathcal{A} = \mathcal{I}$:

**Proposition 6.7** Given a small category $\mathcal{B}$ and a presheaf $F : \mathcal{B}^{op} \rightarrow \mathcal{V}$ that corresponds to the module $f : \mathcal{I} \longrightarrow \mathcal{B}$, the following conditions on $F$ or $f$ are equivalent:

- (i) $f$ (as a module) has a right adjoint $f^*$;
- (ii) the representable functor $\{ F, - \} = [\mathcal{B}^{op}, \mathcal{V}](F, -) : [\mathcal{B}^{op}, \mathcal{V}] \rightarrow \mathcal{V}$ is cocontinuous;
- (iii) the functor $F * - : [\mathcal{B}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves the limit $\{ F,Y' \}$ of $Y' : \mathcal{B}^{op} \rightarrow [\mathcal{B}, \mathcal{V}]$ weighted by $F : \mathcal{B}^{op} \rightarrow \mathcal{V}$.
- (iv) the representable functor $\{ F, - \} = [\mathcal{B}^{op}, \mathcal{V}](F, -) : [\mathcal{B}^{op}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves the colimit $F * Y$ of $Y : \mathcal{B} \rightarrow [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $F : \mathcal{B}^{op} \rightarrow \mathcal{V}$ giving (since $F * Y \cong F$ by Yoneda) an isomorphism $F * \{ F,Y- \} \cong \{ F,F \}$.

When these are satisfied, we have

**6.18**

\[ f^*(b) \cong [\mathcal{B}^{op}, \mathcal{V}](F,Yb). \]

Once again there are dual statements to 6.6 and 6.7 obtained by translating part of 6.3. In detail:

**Proposition 6.8** For any functor $G : \mathcal{A}^{op} \rightarrow [\mathcal{B}, \mathcal{V}]$ that corresponds to the module $g : \mathcal{B} \longrightarrow \mathcal{A}$, the following conditions on $G$ or $g$ are equivalent:

- (i) $g$ (as a module) has a left adjoint $g_*$;
- (ii) for each $a$, the representable functor $\{ Ga, - \} = [\mathcal{B}, \mathcal{V}](Ga, -) : [\mathcal{B}, \mathcal{V}] \rightarrow \mathcal{V}$ is cocontinuous;
• (iii) for each pair $a,c$ in $A$, the functor $Gc\ast - : [B^{op}, V] \to V$ preserves the limit $\{Ga\}$ of $Y : B \to [B^{op}, V]$ weighted by $Ga : B \to V$.

• (iv) for each pair $a,c$ in $A$, the representable functor $[B, V](Ga, -) : [B, V] \to V$ preserves the colimit $Gc \ast Y'$ where $Y' : B^{op} \to [B, V]$.

When these are satisfied, we have

6.19

$$g_*(a,b) \cong [B, V](Ga,Y'b).$$

Eventually for the case $A = I$ one has

Proposition 6.9 For any weight $G : B \to V$ that corresponds to the module $g : B \to I$, the following conditions on $G$ or $g$ are equivalent:

• (i) $g$ (as a module) has a left adjoint $g_*$;

• (ii) the representable functor $\{G, -\} = [B, V](G, -) : [B, V] \to V$ is cocontinuous;

• (iii) the functor $G \ast - : [B^{op}, V] \to V$ preserves the limit $\{G, Y\}$ of $Y : B \to [B^{op}, V]$ weighted by $G : B \to V$.

• (iv) the representable functor $\{G, -\} : [B, V] \to V$ preserves the colimit $G \ast Y'$.

When these are satisfied, we have

6.20

$$g_*(b) \cong [B, V](G,Y'b).$$

For any presheaves $F,H : B^{op} \to V$ on a small $B$, that correspond respectively to modules $f, h : I \to B : [B^{op}, V](F, H)$ is the right lifting of $h$ through $f$. Therefore as a consequence of Proposition 6.4.

Proposition 6.10 Given a module $f : I \to B$ corresponding to a presheaf $F : B^{op} \to V$, if $f$ has a right adjoint $g : B \to I$ then for any left module $h : I \to B$ corresponding to a presheaf $H : B^{op} \to V$

$$[B^{op}, V](F, H) \cong g \ast h.$$

Proposition 6.11 For any $G : B \to V$ with $B$ small, that corresponds to a module $g : B \to I$, the following assertions are equivalent:

• (i) $g$ is a right adjoint;

• (ii) $- \ast G : [B^{op}, V] \to V$ is representable;

• (iii) $- \ast G : [B^{op}, V] \to V$ is continuous.

Proof: (i) $\Rightarrow$ (ii). If $g$ has left adjoint $f : I \to V$ that corresponds to a functor $F : B^{op} \to V$ then $\{F,-\} \cong - \ast G \cong G \ast -$ by 6.10 $G \ast -$ being representable is therefore continuous.

(ii) $\Rightarrow$ (iii) well-known.

(iii) $\Rightarrow$ (i). $G \ast - = - \ast G : [B^{op}, V] \to V$ being continuous preserves in particular the limit $\{G, Y\}$ of $Y : B \to [B^{op}, V]$ weighted by $G$ and by Proposition 6.11 this equally asserts that $g$ is a right
adjoint.

In [Str83] Street characterised right adjoints modules $g : B \to I$ with $B$ small as the ones for which the corresponding presheaf $G : B \to \mathcal{V}$ is a weight of absolute colimits. A similar argument to the one used for the proof (i) $\to$ (ii) above, yields half of the result. Actually if $G$ is a weight of absolute colimits then the colimit $G * Y'$ is certainly preserved by $\{G, -\}$ and then by $\textbf{6.9}$ $g$ is right adjoint.

An object $a$ of a category $A$ (especially of a cocomplete $A$) is said in [Kel82] to be small projective when the representable $A(a, -)$ preserves all small colimits. Of course the representables in $[B^{\text{op}}, \mathcal{V}]$ are among the small projectives: for $[B^{\text{op}}, \mathcal{V}](Yb, -)$ is isomorphic to the evaluation $E_b$ which preserves all colimits. Proposition 6.6 above asserts in particular that $f : A \to B$ has a right adjoint (or is a map) precisely when each $f(-, a) : B^{\text{op}} \to \mathcal{V}$ is a small projective in $[B^{\text{op}}, \mathcal{V}]$.

Note the special case of a functor $T : A \to B$ where $T(-, a) = B(-, Ta)$ and the representable $B(\cdot, Ta)$ is a small projective. There is another way of looking at the small projectives in the category $[B^{\text{op}}, \mathcal{V}]$ when $B$ is small: an object $\phi$ of $[B^{\text{op}}, \mathcal{V}]$ is a weight, and $[B^{\text{op}}, \mathcal{V}](\phi, \psi)$ is just the $\phi$-weighted limit of $\psi : B^{\text{op}} \to \mathcal{V}$. To say, therefore that $\phi$ is a small projective is equally to say that $\phi$-limits commute in $\mathcal{V}$ with all small colimits. As mentioned earlier we shall write $Q$ for the class of small projective weights. In the language of the previous section, $Q = \mathcal{P}^-$, and according to $\textbf{6.11}$ also $Q = \mathcal{P}^\perp$. For any small $B$, as $Q$ is saturated, the closure $Q(B)$ of $B$ in $[B^{\text{op}}, \mathcal{V}]$ under $Q$-colimits is $Q[B]$, as $Q$ is equally the class of weights corresponding in the sense of $\textbf{6.6}$ to left modules, $Q(B)$ is the Cauchy-completion of $B$ in the sense of [Law73].

When the category $B$ is not small, we can still consider the closure $Q(B)$ of $B$ in $[B^{\text{op}}, \mathcal{V}]$ (or equally in $\mathcal{P}(B)$) under $Q$-colimits. On the other hand, we can consider the full subcategory of $\mathcal{P}(B)$ given by its small projectives that is $\mathcal{P}(B)^p$. By Proposition 5.8 $\mathcal{P}(B)^p$ is closed in $\mathcal{P}(B)$ under $Q$-colimits. Since the representables lie in $\mathcal{P}(B)^p$, it follows that $Q(B) \subseteq \mathcal{P}(B)^p$. When $\mathcal{V} = \text{Set}$ the converse is true, so that $Q(B)$ consists even for large $B$ of the small projectives in $\mathcal{P}(B)$. To check this last point one can adapt the proof of [Kel82] to show that any accessible $F : B^{\text{op}} \to \text{Set}$ such that $[B^{\text{op}}, \text{Set}](F, -)$ preserves small colimits is a retract of representables, thus a $Q = \mathcal{P}^\perp$-weighted colimit of representables, and thus in $Q(B)$.

For $B$ small, the operations $\textbf{6.18}$ and $\textbf{6.20}$ above define actually two functors:

6.21

$[B^{\text{op}}, \mathcal{V}][-, Y] : [B^{\text{op}}, \mathcal{V}] \to [B, \mathcal{V}]^{\text{op}}$

and

6.22

$[B, \mathcal{V}][-, Y'] : [B, \mathcal{V}] \to [B^{\text{op}}, \mathcal{V}]^{\text{op}}$.

Letting $B = \mathcal{V}$-$\text{Mod}$, $A = I$ and $h : C \to B = 1_B : B \to B$ in $\textbf{6.8}$ we get that their underlying Set-functors define an adjoint pair:

6.23

$\mathcal{V}$-$\text{Cat}(B^{\text{op}}, \mathcal{V}) \to \mathcal{V}$-$\text{Cat}(B, \mathcal{V})^{\text{op}}$.

The adjunction above goes back in the case $\mathcal{V} = \text{Set}$ to Isbell. Let us call it the Isbell adjunction. Also by Remark $\textbf{6.5}$ Propositions $\textbf{6.7}$ and $\textbf{6.9}$

**Proposition 6.12** The Isbell adjunction as in $\textbf{6.23}$ restricts to an equivalence
\[ Q(B^{op})^{op} \cong Q(B) \]

where \( Q(B) \) consists of the small projectives in \([B^{op}, V]\).

We shall further study more in detail the Isbell adjunction and shows that it lifts to a \( V \)-adjunction.

Recall that for a small \( B \), the functor category \([B^{op}, V]\) is the free cocomplete category on \( B \), in the sense that, for each cocomplete \( C \), we have an adjoint equivalence of categories:

6.24

\[
\text{Cocts}([B^{op}, V], C) \xrightarrow{\text{Lan}_Y} [B, C]
\]

where \([Y, 1]\) sends the cocontinuous \( S : [B^{op}, V] \to C \) to its restriction \( SY : B \to C \), while \( \text{Lan}_Y \) sends \( G : B \to C \) to \( \text{Lan}_Y(G) = -^*G : [B^{op}, V] \to C \) which in fact has the right adjoint \( \tilde{G} : C \to [B^{op}, V] \) where

6.25

\[
\tilde{G}c = C(G -, c)
\]

as in the diagram

6.26

Recall too that there is a dual version involving \( \hat{G} : C^{op} \to [B, V] \) where

6.27

\[
\hat{G}c = C(c, G-).
\]

Now consider the case where \( G \) is the Yoneda embedding

6.28 \( y = Y^{op} : B \to [B, V]^{op} \)

whereupon 6.29 becomes

6.29

\[
[B, V]^{op} \xrightarrow{-^*y} [B^{op}, V] \]

\[
[B^{op}, V] \xrightarrow{\text{Lan}_Y} [B, C] \]

\[
[C, C] \xrightarrow{\text{Lan}_Y} [B, C]
\]

\[
[B, C] \xrightarrow{\text{Lan}_Y} [B, V]
\]

\[
[B, V] \xrightarrow{\text{Lan}_Y} [B^{op}, V]
\]

\[
[y, Y] \xrightarrow{\text{Lan}_Y} [B^{op}, V]
\]

\[
[y, Y] \xrightarrow{\text{Lan}_Y} [B^{op}, V]
\]

\[
[y, Y] \xrightarrow{\text{Lan}_Y} [B^{op}, V]
\]
Here

6.30

\[ \hat{y}(\varphi) = [\mathcal{B}, \mathcal{V}]^{op}(y-, \varphi) = [\mathcal{B}, \mathcal{V}](\varphi, Y'-) \]

so that

6.31

\[ \hat{y}(\varphi) = \psi \text{ where } \psi(b) = [\mathcal{B}, \mathcal{V}](\varphi, Y'b), \]

as in 6.22. Again, since \( \psi \ast y \) is the colimit in \([\mathcal{B}, \mathcal{V}]^{op}\) of \( y : \mathcal{B} \to [\mathcal{B}, \mathcal{V}]^{pp} \) weighted by \( \psi : \mathcal{B}^{op} \to \mathcal{V} \), it is equally the limit \( \{ \psi, Y' \} \) in \([\mathcal{B}, \mathcal{V}]\) of \( y^{op} = Y' : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}] \) weighted by \( \psi : \mathcal{B}^{op} \to \mathcal{V} \); and since limits in \([\mathcal{B}, \mathcal{V}]\) are formed pointwise from the limits in \(\mathcal{V}\), its value \( (\psi \ast y)b = \{ \psi, Y' \}b \) at \( b \in \mathcal{B} \) is given by

6.32

\[ (\psi \ast y)(b) = \{ \psi, Yb \} = [\mathcal{B}^{op}, \mathcal{V}](\psi, Yb). \]

An immediate consequence of this, using 6.27, is the observation that

6.33

\[ - \ast y = \hat{Y}^{op} : [\mathcal{B}^{op}, \mathcal{V}] \to [\mathcal{B}, \mathcal{V}]^{op} \]

which may be combined with the trivially true

6.34

\[ \hat{y} = (Y^{rop})^{\sim} : [\mathcal{B}, \mathcal{V}]^{op} \to [\mathcal{B}^{op}, \mathcal{V}] \]

in order to write the adjunction in 6.29 as

6.35

\[ \hat{Y}^{op} \dashv (Y^{rop})^{\sim} \]

Omitting the \( b \) in 6.32 gives us a form which we may compose with 6.30, namely

6.36

\[ \psi \ast y = [\mathcal{B}^{op}, \mathcal{V}](\psi, Y-) \]

so that - compare 6.31 - we have

6.37

\[ \psi \ast y = \varphi \text{ where } \varphi(b) = [\mathcal{B}^{op}, \mathcal{V}](\psi, Yb), \]

as in 6.21. Now we observe that there is an alternative analysis of this equivalence between the \( \psi \)'s in \( Q(\mathcal{B}) \) and the \( \varphi \)'s in \( Q(\mathcal{B}^{op}) \). Accordingly to the general picture in 6.26 referring to \([\mathcal{B}^{op}, \mathcal{V}]\) as the free cocompletion of \( \mathcal{B} \), when \( \psi \in Q(\mathcal{B}) \), the cocontinuous \( [\mathcal{B}^{op}, \mathcal{V}](\psi, -) : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V} \) is given by:

6.38
In more detail, $\psi$ lies in $Q(B)$ precisely when $[B^{\text{op}}, \mathcal{V}](\psi, -)$ is cocontinuous (or equally left adjoint); and then $[B^{\text{op}}, \mathcal{V}](\psi, -) = - * \varphi$ where $\varphi = [B^{\text{op}}, \mathcal{V}](\psi, Y^-) = \psi * y$. On the other hand, if we start with $\varphi : B \to \mathcal{V}$ and ask whether it is of the form $[B^{\text{op}}, \mathcal{V}](\psi, Y^-)$ for some $\psi \in Q(B)$, we observe that it so precisely when $- * \varphi : [B^{\text{op}}, \mathcal{V}] \to \mathcal{V}$ is representable.

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