Entanglement Generation by Communication using Phase-Squeezed Light with Photon Loss

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In order to implement fault-tolerant quantum computation, entanglement generation with low error probability and high success probability is required. We have proposed the use of squeezed coherent light as a probe to generate entanglement between two atoms by communication, and shown that the error probability is reduced well below the threshold of fault-tolerant quantum computation [Phys. Rev. A. 88, 022313 (2013)]. Here, we investigate the effect of photon loss mainly due to finite coupling efficiency to the cavity. The error probability with the photon loss is calculated by the beam-splitter model for homodyne measurement on probe light. Optimum condition on the amplitude of probe light to minimize the error probability is examined. It is shown that the phase-squeezed probe light yields lower error probability than coherent-light probe. A fault-tolerant quantum computation algorithm can be implemented under 0.59 dB loss by concatenating five-qubit error correction code.

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I. INTRODUCTION

Quantum entanglement is essential for implementation of any quantum computation schemes [1–4]. In 2001, one-way quantum computation [2, 5] has been proposed to work out the complexities of quantum gate circuits. Although the one-way quantum computation requires a large scale cluster state before the computation, it can be implemented with measurements on a quantum bit (qubit) and one-qubit unitary transformations according to the measurement outcomes. It is experimentally reported that the generation of a cluster state (where 10^5 atoms are considered to be entangled) by collisions of the atoms in an optical lattice [6]. However, the quantum computation is difficult using this cluster state, because the distance between the atoms is too small to select a single atom for the one-qubit operation. Therefore, keeping enough distance between entangled atoms is important to implement the one-way quantum computing.

Entanglement generation by communication (which uses an electromagnetic field, i.e., a quantum bus or qubus, a more detailed description will be shown in Sec. II) has been proposed [7–10] as an efficient entangler that can be used for the creating the cluster states [11–14]. Although this scheme can entangle distant atoms, the estimated error probability of entanglement generation is too large for any fault-tolerant quantum computation schemes. In order to reduce the error probability, we have proposed the use of squeezed coherent light instead of coherent light as a qubus [15] and shown that fault-tolerant one-way quantum computing [16] can be implemented using phase-squeezed light. In this proposal, we have assumed that the transmission loss for squeezed light is negligible, because the distances between quantum memories are small. However, photon loss may also arise from the coupling to cavities and the measurement of the squeezed coherent light. The travelling wave light is expected to experience a loss at the interface to typical Fabry–Pérot cavities [17]. Moreover, efficiency of the homodyne measurement is smaller than unity because of finite quantum efficiencies of photo detectors and mode mismatch between signal and local oscillator [18]. The phase-squeezed light may be collapsed by these losses, and tend to be coherent light.

In this report, we estimate the error probability of entanglement generation using squeezed coherent light under photon loss. The effect of photon loss on the entanglement generation with squeezed light in transmission has been studied for qubus in the context of quantum repeaters [19]. In ref. [19], the use of squeezed light and the displacement operation is predicted to improve the fidelity and the success probability of dispersive interaction to 0.89 and 40%, respectively, from the values 0.77 and 36% for coherent light, for a node interval of 10 km. Meanwhile, entanglement generation for quantum computers requires different characterization from that for quantum repeaters. Fidelity of entanglement between two atoms is important for quantum repeaters, since only the availability of the entanglement purification matters [20]. On the other hand, the error probability in the gates is important to implement the fault-tolerant quantum computation. When the success probability of the quantum gate is provided, the error probability of the quantum gate should be below the “threshold” values for fault-tolerance [21–24]. For example, the error probability no greater than 4 × 10^{-4} is required on the success probability 50% in Fujii and Tokumaga (FT) method [19]. Note that the fidelity denotes the distance between the ideal Bell state and generated entangled state [2], whereas the error probability denotes the probability of the unheralded error in the entanglement generation (in

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qubits, this error originates from the overlaps between nonorthogonal probe light states [8]. In addition, most fault-tolerant quantum computing architectures [25–28] require the success probability at least 1/2, while the quantum repeaters allow a smaller success probability. In fact, Praxmeyer and van Loock [19] employed homodyne measurements and window functions for quantum states discrimination to obtain the final state with a high fidelity to the ideal Bell state, which reduced the success probability to less than 1/2. We calculate the error probability with only homodyne measurements to obtain the success probability of 1/2.

Furthermore, we should consider the decoherence of atoms induced by the photon loss. In beam splitter model, known as a simple approach for photon loss, transmitted and reflected (lost) photons provide “which-path” information. Since atoms and probe light are entangled, atomic decoherence is induced by tracing over the lost photons mode [7, 9]. The atomic decoherence results in the phase error in the entangled state of the atoms [7, 9]. Total error probability, which thus consists of the error originated from the homodyne measurement and the phase flip error induced by the atomic decoherence, may be an obstacle to implement the fault-tolerant quantum computing. In order to solve this problem, we introduce three qubit and five qubit quantum error correction code (QECC) for the phase flips.

The report is structured as follows. We will briefly review the entanglement generation by communication using squeezed coherent light in Sec. II. Then, we will show a homodyne measurement scheme for three-state discrimination and calculate the error probability with photon loss of the squeezed coherent light in Sec. III. Finally, we will examine the feasibility of the implementation of fault-tolerant quantum computation conclude the report in Sec. IV.

II. ENTANGLEMENT GENERATION BY PHASE-SQUEEZED LIGHT AND HOMODYNE MEASUREMENT

In this section, we will briefly review entanglement generation between two atoms by squeezed light [11]. Figure 1 (a) shows process of the entanglement generation with qubits. The process consists of sequential atom-probe interaction and measurement. Two atoms have lower energy states |0⟩ and |1⟩ and an excited state |e⟩, where only the transition |1⟩ −→ |e⟩ is allowed [29, 31] as shown in Fig. 1 (b). We assume dispersive atom-photon interaction [8, 9, 31] by a large detuning ∆ and a not too large mean photon number ŉ in the cavity mode. Then, the atom-photon interaction provides a unitary operator that conditionally rotates the phase of the photon as

\[ \hat{U} = \hat{I} |0⟩⟨0| + \hat{R}(\theta) |1⟩⟨1|, \]

where θ = χt, χ = q^2/∆, g is a coupling constant, and \( \hat{R}(\theta) = e^{i\theta \hat{n}} \). The probe light is the squeezed coherent state |ξ, α⟩ where ξ = r0e^{iθ} is the squeezing parameter, described with an amplitude r0 and a phase φ. In addition, we fixed the mean photon number ŉ between the coherent state |α⟩ and the squeezed coherent state |ξ, β⟩ for a fair comparison by adjusting the amplitude to be β from the original value α as β = \( \sqrt{|α|^2 - \sinh^2(r)} \). We use the phase φ = π, since it is the optimal squeezing direction for quadrature fluctuations in a homodyne measurement [13]. For φ = π, phase fluctuations of quadrature is squeezed, thus it is called as a phase-squeezed light |r, β⟩ where r = −r0.

The interaction between the light and atom A creates an entanglement between the atomic states and the light states as \( \frac{1}{\sqrt{2}}(|0⟩|r, β⟩ + |1⟩|e^{i2θ}, βe^{iθ}⟩) \) from the initial product state \( \frac{1}{\sqrt{2}}(|0⟩ + |1⟩) ⊗ |r, β⟩ \) by the unitary operation Eq. (1). Then, the interaction between the light and atom B yields the final state

\[ |ψ_2⟩ = \frac{1}{\sqrt{2}} |0⟩ |0⟩ |r, β⟩ + \frac{1}{\sqrt{2}} \left( |0⟩ |1⟩ + |1⟩ |0⟩ \right) |e^{i2θ}, βe^{iθ}⟩ \]

After that, the probe light is measured. Although we have estimated the error probability in the minimum error discrimination [15], it may be difficult to implement for three squeezed coherent states. In order to consider more feasible situation, we here consider the homodyne measurement on the probe light. The entangled state of the atoms |0⟩ |1⟩ + |1⟩ |0⟩ is formed by post-selecting the rotated state |e^{i2θ}, βe^{iθ}⟩. A homodyne measurement projects the quantum state onto a projection axis. Now, since we want to discriminate the |e^{i2θ}, βe^{iθ}⟩ state from the other two states in Eq. (2), the projection axis should be taken to the (p + θ)-axis for the optimal phase measurement. For this situation, the probability density distributions as shown in Fig. 2 (b) can be obtained corresponding to measurement outcome of probe light [15]. The error in entanglement generation in the homodyne measurement occurs due to overlaps of the probability density distributions representing the measurement outcomes.

III. ERROR PROBABILITY IN HOMODYNE MEASUREMENT WITH LOSS

In this section, we will examine the effects of the loss on the error probability of the entanglement generation. As mentioned in the previous section, we will calculate the overlaps of the probability density distributions for the homodyne detection of the phase-squeezed light [32].

Photon loss can be described by a beam splitter model [15] characterized by transmittance η (0 < η < 1) as shown in Fig.3. The loss is described by 1 − η. In our scheme, although photon loss may be induced in transmission, cavity coupling, and measurement of the probe
light, these losses can be represented by a single beam splitter, whose reflection corresponds to the total photon loss, since the beam splitter operation and the phase rotation operation are commute \[33\]. Practically, since the phase shift angle may be decreased by photon loss, the displacement operation before the second cavity or adjustment of atom-light interaction time in the second cavity is required. In this report, we assume the phase shifts of two cavities are same by adjustment of the interaction time.

The probability density distributions when phase-squeezed states are projected onto the \(x_\lambda\)-axis (where \(\lambda\) is the projective angle) is \[34\]

\[
P_D(\theta) \equiv |\langle x_\lambda|re^{i\varnothing}, \beta e^{i\theta}\rangle|^2 = (2\pi \Delta x_\lambda^2)^{-1/2} \exp \left\{ -\frac{(x_\lambda - \langle \hat{x}_\lambda \rangle)^2}{2\Delta x_\lambda^2} \right\}, \tag{3}
\]

where \(x_\lambda = \frac{1}{\sqrt{2}} \left\{ \alpha \exp(-i\lambda) + \hat{a}^\dagger \exp(i\lambda) \right\} (\hat{a}^\dagger \) and \(\hat{a}\) are the creation and the annihilation operator, respectively) is a quadrature operator, \(|x_\lambda\rangle\) is the normalized eigenstate of \(\hat{x}_\lambda\), \(\hat{x}_\lambda\) is the eigenvalue satisfying the eigenvalue equation \(\hat{x}_\lambda |x_\lambda\rangle = x_\lambda |x_\lambda\rangle\) is the expectation value of \(\hat{x}_\lambda\) \[32\]

\[
\langle \hat{x}_\lambda \rangle = \sqrt{\frac{\eta}{2}} (\alpha, \xi | \hat{x}_\lambda | \alpha, \xi)
\]

\[
= \frac{1}{\sqrt{2}} \{ \alpha \exp(-i\lambda) + \alpha^* \exp(i\lambda) \} \exp(i\theta), \tag{4}
\]

and \(\Delta x_\lambda^2\) is the variance of \(\hat{x}_\lambda\) \[32\]

\[
\Delta x_\lambda^2 \equiv \langle \hat{x}_\lambda^2 \rangle - \langle \hat{x}_\lambda \rangle^2
\]

\[
= \frac{\eta}{2} \left\{ \exp(2r) \sin^2 \left( \lambda - \frac{\varnothing}{2} + \theta \right) + \exp(-2r) \cos^2 \left( \lambda - \frac{\varnothing}{2} + \theta \right) \right\} + \frac{1 - \eta}{2}. \tag{5}
\]

Here, \(\langle \hat{x}_\lambda^2 \rangle\) is the expectation value of \(\hat{x}_\lambda^2\). The photon loss decreases the amplitude parameter in Eq. (3) and reduces the squeezing effect in the variance given by Eq. (4). In addition, we introduced the phase shift \(\theta\) from the projective angle \(\lambda\) to Eqs. (3)-(5) to calculate the overlaps.

Using Eqs. (3)-(5) and assuming the direction of the projection axis as \(\lambda = (\pi + \theta)/2\), we obtain the probability density distributions of the measurement outcomes on the superposition state described by Eq. (2) as shown in Fig. 4. The overlap of probability density distributions are increased by the photon loss from those calculated for loss-less depicted in Fig. 2 (b).

The error probability is obtained rigorously by subtracting the overlap between the probability density distributions of \(|r, \beta\rangle\) and \(|re^{i\varnothing}, \beta e^{i\theta}\rangle\) from the sum of the overlaps between \(|r, \beta\rangle\) and \(|re^{i\varnothing}, \beta e^{i\theta}\rangle\) and the overlaps between \(|re^{i\varnothing}, \beta e^{i\theta}\rangle\) and \(|re^{i\varnothing}, \beta e^{i\theta}\rangle\). The overlap between the arbitrary probability density distributions \(P_D(\theta_1)\) and \(P_D(\theta_2)\) can be considered as the sum of the area from the mid-point (the cross-over point) \(\sqrt{\eta} \xi_{12}\) between \(P_D(\theta_1)\) and \(P_D(\theta_2)\) to positive infinity and the

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**FIG. 1.** (Color online) (a) The process of entanglement generation between quantum memories by communication. Light interacts with two atoms, and the phase rotation by the interaction is measured after the interaction. (b) The energy level scheme of a Λ-configured three-level atom. The phase of light rotates only when the atom is in the state \(|1\rangle\).

**FIG. 2.** (Color online) (a) A schematic representation of the phase-squeezed light in the phase space after interacting with atoms A and B (at time at \(\tau_2\) in Fig. 1(a)), described by ellipses. The atomic entanglement state \(|0\rangle \langle 1| + |1\rangle \langle 0|\) is formed by post-selecting the rotated coherent state \(|re^{i\varnothing}, \beta e^{i\theta}\rangle\). (b) Probability density distributions for the outcomes of the quadrature measurement of the probe beam corresponding to the quantum states in Fig. 2 (a) projected on the \((p + \theta)\)-axis, which is the optimal projection angle to obtain the lowest error probability in the homodyne measurement.
area from negative infinity to the mid-point, thus it can be described as:

\[
P_{E1}(\theta_1, \theta_2) = \begin{cases} 
    c_A \int_{\sqrt{\beta} \zeta_{12}}^{\infty} P_D(\theta_1) d\lambda + c_B \int_{\infty}^{\sqrt{\beta} \zeta_{12}} P_D(\theta_2) d\lambda + c_B \int_{-\infty}^{\sqrt{\beta} \zeta_{12}} P_D(\theta_1) d\lambda, & \theta_1 = 0 \text{ and } \theta_2 = \theta \ (i.e., \ overlap \ between \ probability \ distributions \ P_D(\theta_1) \ and \ P_D(\theta_2)), \text{ factors } c_A \ and \ c_B \ are \ non-zero. \\
    \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda \right) \left( \frac{\theta_2 - \theta_1}{2} \leq \lambda \leq \frac{\theta_2 - \theta_1}{2} \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
    \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda \right) \left( \frac{\theta_1 - \theta}{2} \leq \lambda \leq \frac{\theta_3 - \theta}{2} \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
    2P_{E1}(\theta_1, \theta_2) - \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda \right), & \theta_1 = \theta_2 = \theta \\
    2P_{E1}(\theta_1, \theta_2) - \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda \right) \left( \frac{\theta_1 - \theta}{2} \leq \lambda \leq \frac{\theta_3 - \theta}{2} \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
    2P_{E1}(\theta_1, \theta_2) - \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
\end{cases}
\]

where \( \zeta_{13} = \frac{\cos \lambda \cos \theta_1 + \cos \lambda \cos \theta_2 + \sin \lambda \sin \theta_1 + \sin \lambda \sin \theta_2}{2\sqrt{2}} \) (7)

Factors \( c_A \) and \( c_B \) are determined with the factors of \( |\alpha, \beta\rangle, |\alpha e^{i\theta}, \beta e^{i\theta}\rangle \) and \( |\alpha e^{i\theta}, \beta e^{i\theta}\rangle \) in Eq. (2). The factors are selected according to the photon states to be considered in estimating the overlap. For example of

\[
P_E = \left \{ \begin{array}{ll}
2P_{E1}(\theta_1, \theta_2) - \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda \right) \left( \frac{\theta_2 - \theta_1}{2} \leq \lambda \leq \frac{\theta_2 - \theta_1}{2} \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
2P_{E1}(\theta_1, \theta_2) - \frac{1}{4} \left( \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_1) d\lambda - \int_{\sqrt{\beta} \zeta_{13}}^{\infty} P_D(\theta_2) d\lambda \right), & \theta_1 = 0 \text{ and } \theta_2 = \theta \\
\end{array} \right.
\]

where \( \zeta_{13} \) is obtained by the replacement \( \theta_2 \to \theta_3 \) in Eq. (6). Note that, when overlaps \( P_{E1}(\theta_1, \theta_2) \) and \( P_{E1}(\theta_2, \theta_3) \) are symmetric to the \( \theta_2 \)-axis and the projection angle is set to \( \lambda = (\pi + \theta_2)/2 \) (i.e., quantum states of light in \( |\psi_2\rangle \) is projected on to \( \theta_2 \)-axis) as shown in Fig. 2 (b), \( \theta_2 = \theta_3 \) is satisfied, and thus the two overlaps are the same. In our proposal, we assumed the same phase shift angle for both atoms in Fig. 1 (a). Therefore, we can assume \( P_{E1}(\theta_1, \theta_2) + P_{E1}(\theta_2, \theta_3) = 2P_{E1}(\theta_1, \theta_2) \) in Eq. (8).

Here, we examine the effect of photon loss on the error probability with Eq. (8). Fig. 5 (a) plots the error probability as a function of the transmittance \( \eta \) for various values of squeezing amplitude \( r \). In the loss less case (\( \eta = 1 \)), \( \bar{n} = 10^4 \) and \( \theta = 0.01 \), we obtain the error probability of entanglement generation with the homodyne measurement to be \( P_E = 0.23 \) with the squeezing amplitude \( r = 0 \) (for coherent states) and \( P_E = 6.52 \times 10^{-6} \) with the experimentally reported maximum value of the squeezing parameter \( r = 1.5 \). As photon loss increases (i.e., smaller transmittance \( \eta \)), the error probabilities increase and converges to 0.5 even for large squeezing amplitudes. This is because the overlaps between the three phase-squeezed states shown in Fig. 4 depend on not only coherent amplitude but the variance \( \Delta x^2_\lambda \) given by Eq. (5). The “squeezed” variance is collapsed to the variance of coherent light \( \Delta x^2_\lambda \) = 1/2 due to the invasion of the vacuum mode by the photon loss. Actually, the variance \( \Delta x^2_\lambda \) for phase (i.e. \( p \)-quadrature; \( \Delta x^2_\lambda + \pi/2 \)) for phase-squeezed states increases as \( \eta \) decreases, and reaches the value for the coherent state at \( \eta = 0 \), as shown in Fig. 5 (b). Nevertheless, the phase-squeezed state provides better error probability than the coherent state for phase measurement as long as \( \eta > 0 \).

\[\text{IV. DISCUSSION AND CONCLUSION}\]

We here examine the impact of the photon loss on the implementation of fault-tolerant quantum computation in terms of the error probability derived in Eq. (8). To this end, we should consider the fact that the photon loss induces phase flip error, though the atomic decoherence in addition to the measurement induced error described in Sec. III. As mentioned in Sec. I, photon loss induces atomic decoherence, since atoms and probe light are entangled. The atomic decoherence is determined by the number of lost photons \( |\alpha|^2 (1 - \eta) \) of probe light and the coupling strength between atom and probe light \( (1 - \cos \theta) \). Thus the decoherence can be written as \( e^{-\gamma} \), where \( \gamma = |\alpha|^2 (1 - \eta)(1 - \cos \theta) \). A critical problem for the fault-tolerant computing is that the atomic decoherence produces a phase flip error. Actually, from this decoherence, final state \( |\psi_{i2}\rangle \) can be rewritten...
as mixed state \[|\psi\rangle_2\]

\[
(P_S^A P_S^B + P_F^A P_F^B) |\psi_2\rangle + P_F^A P_S^B |\psi_2^{\text{flip}A}\rangle |\psi_2^{\text{flip}A}\rangle + P_S^A P_F^B |\psi_2^{\text{flip}B}\rangle |\psi_2^{\text{flip}B}\rangle
\]

where \(P_S^A = (1 + \gamma_A)/2\) and \(P_S^B = (1 + \gamma_B)/2\) are no-phase flip error probabilities for atom A and B, \(P_F^A = 1 - P_S^A\) and \(P_F^B = 1 - P_S^B\) are phase flip error probabilities for atom A and B, respectively. Moreover, \(|\psi_2^{\text{flip}A}\rangle\) and \(|\psi_2^{\text{flip}B}\rangle\) are the phase flipped states of \(|\psi_2\rangle\) for atom A and B, respectively. These states can be written as

\[
|\psi_2^{\text{flip}A}\rangle = \frac{1}{2} (|0\rangle - |1\rangle) |r, \beta\rangle
\]

\[
+ \frac{1}{\sqrt{2}} \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) |r e^{i2\theta}, \beta e^{i\theta}\rangle
\]

\[
- \frac{1}{2} |1\rangle |r e^{i4\theta}, \beta e^{i2\theta}\rangle
\]

Finaly, combining the error probability of the state discrimination, the total error probability is obtained as

\[
E_{\text{tot}} = (P_S^A P_S^B + P_F^A P_F^B) P_E + (P_F^A P_S^B + P_S^A P_F^B) (1 - P_E).
\]

As mentioned above, since the atomic decoherence, i.e., the phase flip error is determined by the number of lost photons and coupling strength, the total error probability can be minimized by optimizing the coherent amplitude \(\alpha\) and/or the phase shift angle \(\theta\). Note that, photon loss effects for atoms A and B are different, because the different numbers of the photons interact with the atoms. This difference induces the unequal atomic decoherence on atoms A and B. Therefore, we should consider the phase error probabilities in atoms A and B separately. For atom A, the decoherence \(\gamma_A = |\alpha|^2 (1 - \eta_A \eta_B) (1 - \cos \theta)\) arises from the loss from the first cavity to the detectors in the homodyne measurement. Here, \(\eta_A\) and \(\eta_B\) are transmissions from the atom A to the atom B, and from the atom
the mean photon number is \( \bar{n} \) function of set the coherent amplitude of the phase-squeezed light \( r \) also depict the threshold of the FT method by the lower error probability is below 1 can be implemented with realistic resources, when the dashed line) as shown in Fig. 6. The Knill’s method [25] can be implemented with realistic resources, when the error probability is below 1 \( \times 10^{-2} \) (shown by the line in Fig. 6). Even for a small photon loss, this value of the error probability \( E_{\text{tot}} \) is hard to realize, and the implementation of the Knill’s method is difficult even if strong phase-squeezed light is used. We here introduce the concatenation of a few-qubit QECC [24] to correct the phase flip error. Such error correction has been implemented in solid-state qubits such as diamond spin systems [38]. We estimate the probabilities of the successful error correction on the logical errors corresponding to the atoms A and B are

\[
\begin{align*}
P_{\text{F3cor}}^A & = (P_S^A)^3 + 3(P_S^A)^2 P_F^A, \\
P_{\text{S3cor}}^A & = 1 - P_{\text{F3cor}}^A, \\

P_{\text{F3cor}}^B & = (P_S^B)^3 + 3(P_S^B)^2 P_F^B, \\
P_{\text{S3cor}}^B & = 1 - P_{\text{F3cor}}^B.
\end{align*}
\]

Then, the total error probability can be obtained by substituting Eqs. (13) and (14) to Eq. (12). Figure 7 (a) plots the optimized (minimized) error probability for \( \theta = 0.01 \) as a function of the coherent amplitude \( \alpha \) with the amplitude of the squeezing parameter \( r = 0 \) and transmittance \( \eta = 0.9 \) (solid line), \( r = 0 \) and \( \eta = 0.99 \) (dashed line), \( r = 1.5 \) and \( \eta = 0.9 \) (dotted line), and \( r = 1.5 \) and \( \eta = 0.99 \) (dot-dashed line) as shown in Fig. 6. The Knill’s method [25] can be implemented with realistic resources, when the error probability is below 1 \( \times 10^{-2} \) (shown by the line in Fig. 6). Even for a small photon loss, this value of the error probability \( E_{\text{tot}} \) is hard to realize, and the implementation of the Knill’s method is difficult even if strong phase-squeezed light is used. We here introduce the concatenation of a few-qubit QECC [24] to correct the phase flip error. Such error correction has been implemented in solid-state qubits such as diamond spin systems [38]. We estimate the probabilities of the successful error correction on the logical errors corresponding to the atoms A and B are

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P_{\text{S3cor}}^A & = 1 - P_{\text{F3cor}}^A, \\

P_{\text{F3cor}}^B & = (P_S^B)^3 + 3(P_S^B)^2 P_F^B, \\
P_{\text{S3cor}}^B & = 1 - P_{\text{F3cor}}^B.
\end{align*}
\]

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P_{\text{S3cor}}^A & = 1 - P_{\text{F3cor}}^A, \\

P_{\text{F3cor}}^B & = (P_S^B)^3 + 3(P_S^B)^2 P_F^B, \\
P_{\text{S3cor}}^B & = 1 - P_{\text{F3cor}}^B.
\end{align*}
\]

Then, the total error probability can be obtained by substituting Eqs. (13) and (14) to Eq. (12). Figure 7 (a) plots the optimized (minimized) error probability for \( \theta = 0.01 \) as a function of the coherent amplitude \( \alpha \) with the amplitude of the squeezing parameter \( r = 0 \) and transmittance \( \eta = 0.9 \) (solid line), \( r = 0 \) and \( \eta = 0.99 \) (dashed line), \( r = 1.5 \) (dashed line), and (b) plots the conditions of the optimization of the error probability as a function of \( \eta \). Note that, in phase-squeezed light, since the mean photon number is \( \bar{n} = |\alpha|^2 + \sinh r \) [31], we set the coherent amplitude of the phase-squeezed light to \( \beta = \sqrt{|\alpha|^2 - \sinh r} \) in order to make a fair comparison with coherent light. Therefore, the solid line and the dashed line in Fig. 7 (b) correspond to amplitudes \( \alpha \) and \( \beta \), respectively. As mentioned above, the upper line at \( 1 \times 10^{-5} \) denotes the threshold of the Knill’s method. We also depict the threshold of the FT method by the lower line at \( 4 \times 10^{-4} \) in Fig. 7(a). For the error probability below this threshold, FT method can be also implemented. The FT method requires the transmittance \( \eta \geq 0.978 \) (loss of 0.10 dB) and \( \eta \geq 0.999 \) (loss of 0.004 dB) for the squeezing parameters \( r = 1.5 \) and \( r = 0 \) (coherent light), respectively. On the other hand, the Knill’s method requires the transmittance \( \eta \geq 0.913 \) (loss of 0.40 dB) and \( \eta \geq 0.984 \) (loss of 0.07 dB) for the squeezing parameters \( r = 1.5 \) and \( r = 0 \) (coherent light), respectively. We can further improve the loss tolerance by increasing the code length of the QECC. For five-qubit code, the logical error
and the no-error probabilities for atom A and B are
\[ P_{F_{5\text{cor}}}^A = (P_{S_{5\text{cor}}}^A)^5 + 5(P_{S_{5\text{cor}}}^A)^4P_F^A + 10(P_{S_{5\text{cor}}}^A)^3(P_F^A)^2, \]
\[ P_{S_{5\text{cor}}}^A = 1 - P_{F_{5\text{cor}}}^A. \]  
(15)

\[ P_{F_{5\text{cor}}}^B = (P_{S_{5\text{cor}}}^B)^5 + 5(P_{S_{5\text{cor}}}^B)^4P_F^B + 10(P_{S_{5\text{cor}}}^B)^3(P_F^B)^2, \]
\[ P_{S_{5\text{cor}}}^B = 1 - P_{F_{5\text{cor}}}^B. \]  
(16)

By using five-qubit QECC, the Knill’s method and the FT method can be implemented for \( \eta \geq 0.873 \) (loss of 0.59 dB) and \( \eta \geq 0.953 \) (loss of 0.21 dB), respectively. Although this QECC for phase flip error is simple, it still increases the needs on the atomic resources. Nonetheless, since this increase is polynomial, three and five (or more) qubit QECC are feasible.

Here, we estimate the experimentally feasible values of the total loss. The homodyne measurement loss is estimated to be 0.07 using Si photodiodes at 860 nm [59]. Smaller coupling loss, compared with conventional Fabry–Pérot cavities, is predicted to be less than 0.05 using microtroidal resonators, which are developed to achieve strong coupling in an atom-cavity system [11]. Although the predicted coupling loss has not been realized yet, the progress in fabrication technology will achieve the value in the near future. Therefore, the microtroidal resonator is a strong candidate for the atom-cavity system with the dispersive interaction. The combination of the Si photodiodes and the microtroidal resonators will provide the total loss as small as 0.1–0.2 (0.46–0.97 dB). This value of the total loss implies that the Knill’s method can be implemented using phase-squeezed state with the five-qubit QECC.

In summary, we have calculated the error probability of entanglement generation by the phase-squeezed light to implement fault-tolerant quantum computing methods. When coherent light is used for qubus, implementation of any fault-tolerant quantum computation schemes is difficult even if the loss less case, since the error probability exceeds 0.01. By contrast, when phase-squeezed light is used for qubus, fault-tolerant quantum computation can be implemented up to 0.40 dB and 0.10 dB loss (with three-qubit error correction code) and up to 0.59 dB and 0.21 dB loss (with five-qubit error correction code) for Knill’s method and FT method, respectively. These results suggest that scalable quantum computing can be performed using the near-future technology. The most remarkable fact found in this study is that squeezed light works better than coherent light under large photon loss. It suggests that the phase-squeezed light should be used for signal discrimination by phase measurement.

Finally, we list issues for future study. First, we mentioned that the photon loss does not affect phase shift operation in principle in Sec. II. However, in practical systems such as qubus scheme, the decrease of the photon number and squeezing amplitude may affect the phase shift angle at the second cavity. Therefore, we should calculate the error probability including the difference of photon numbers, or including the displacement operation before the second cavity. Second, although we have calculated the error probability on the homodyne measurement, the minimum error discrimination provides the optimal error probability for this qubus scheme as shown in the previous report [15]. Therefore, the minimum error probability with photon loss should be derived. Third, the realization of the minimum error measurement for three-phase-squeezed states itself is a challenging research task. Fourth, we assumed that phase-squeezed light with large coherent amplitudes to show that the qubus entangler using phase-squeezed light works with a low error probability. In principle, such phase-squeezed light can be obtained by displacing the vacuum squeezed state. However, displacement from a squeezed vacuum to a largely phase-squeezed light pulse with a coherent amplitude of \( \beta \gg 1 \) has not been reported experimentally yet. The realization of the phase-squeezed light pulse with large coherent and squeezing amplitudes is thus a research task in the future [42].

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