MEDIAN GEOMETRY FOR SPACES WITH MEASURED WALLS AND FOR GROUPS

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Abstract. We show that uniform lattices of isometries of products of real hyperbolic spaces act properly discontinuously and cocompactly on a median space. For lattices in products of at least two factors, this is the strongest degree of compatibility possible with the median geometry. Our theorem is also relevant for potential Rips-type theorems for median spaces. The result follows from an analysis of a quasification of median geometry that provides a geometric characterization of spaces at finite Hausdorff distance from a median space. We explain how the case of complex hyperbolic metric spaces is different, and that such spaces cannot be at finite Hausdorff distance from a median space.

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1. Introduction.

1.a. Median spaces and spaces with walls. A median space is a metric space $(X,d)$ such that every triple of points $x_1, x_2, x_3 \in X$ admits a unique median point, i.e. a point $m \in X$ satisfying

$$d(x_i,m) + d(m,x_j) = d(x_i,x_j), \forall i,j \in \{1,2,3\}, i \neq j.$$ 

The map $X \times X \times X \to X, (x_1,x_2,x_3) \mapsto m$, endows $X$ with a ternary algebra structure, called median algebra. These were studied in [dV93, BH83, Isb80, Sho54a, Sho54b].
More geometric studies of median spaces were started in [Rol98, Nic], and more recently by Bowditch in [Bo14, Bo16, Bo20].

Examples of median metric spaces are trees, $\mathbb{R}^n$ with the $\ell^1$ metric for any $n \geq 1$, and CAT(0) cube complexes on which the Euclidean metric on cubes is replaced by the $\ell^1$ metric. According to Chepoi and Gerasimov [Che00, Ger97, Ger98] the class of 1-skeleta of CAT(0) cube complexes coincides with the class of median graphs (i.e. simplicial graphs whose 0-skeleton with the combinatorial distance is median). General median spaces can be thought of as non-discrete versions of 0-skeleton of CAT(0) cube complexes, in the same way in which real trees are non-discrete generalizations of simplicial trees. Indeed, according to Bowditch [Bo14, Bo16], the metric of a complete connected finite rank median metric space has a bi-Lipschitz equivariant deformation that is CAT(0).

The interest of the median geometry comes, among other things, from the relevance of median graphs in graph theory and computer science [BC08], and in optimization theory (see [MMR, Wil08] and references therein). Moreover, two important properties of infinite groups, Kazhdan’s Property (T) and a-T-menability, can be reformulated in terms of actions on median spaces [CDH10]. The $\delta$-hyperbolicity of metric spaces, characterized by the fact that all geodesic triangles are $\delta$-thin, is connected with both median and Banach space geometry (e.g. G. Yu [Yu] showed that every hyperbolic group acts properly on an $\ell^p$-space, for $p$ large enough). In the present paper, we investigate a common generalization called $\delta$-median geometry, for some $\delta \geq 0$, defined by the existence, for every three points $x, y, z$, of a set composed of points that are $\delta$-between $x$ and $y$, $y$ and $z$, $x$ and $z$ respectively, and of uniformly bounded diameter (see Definition 2.11). This natural notion encompasses both median geometry and $\delta$-hyperbolicity, it is closely related to the coarse median geometry [NWZ], it includes mapping class groups [Pe] and it is stable under direct products and relative hyperbolicity (see Proposition 2.10).

Another structure that appears naturally in the study of actions on Banach spaces is that of space with measured walls (see Definition 3.1). We proved in [CDH10] that, in some sense, the category of spaces with measured walls is equivalent to the one of subspaces of pseudo-metric median spaces (see Section 2), by constructing, for any given wall space $X$, a median space $\mathcal{M}(X)$ containing it. We have also shown that, without loss of generality, one can always assume that the half-spaces of a wall space are convex.

We show that a $\delta$-median structure on a wall space $X$ is in fact equivalent to being at finite distance from the associated median space $\mathcal{M}(X)$.

**Theorem 1.1.** Let $(X, \mathcal{W}, \mu)$ be a space with measured walls, $\mu$-locally finite (Definition 4.7), and such that all its half-spaces are quasi-convex (Definition 4.7). The following are equivalent:

1. the metric space $(X, \text{dist})$ associated to the wall structure is $\delta$-median;
2. the medianization $\mathcal{M}(X)$ is at finite Hausdorff distance from $X$;

Finite rank seems to be the optimal condition for the existence of such a deformation (see Section 2 for the notion of rank).
(3) \( X \) admits an isometric embedding \( \varphi \) into a median space \( \mathcal{M} \) such that \( \mathcal{M} \) is within finite Hausdorff distance from \( \varphi(X) \).

A more precise and expanded formulation of the above can be found in Theorem 1.10. Theorem 4.10 combined with Proposition 5.1 imply the following.

**Corollary 1.2.** The real hyperbolic space \( \mathbb{H}^n \) embeds isometrically and \( \text{Isom}(\mathbb{H}^n) \)-equivariantly into a locally compact median space at finite Hausdorff distance from the embedded \( \mathbb{H}^n \).

In particular, the full isometry group of the real hyperbolic space \( \mathbb{H}^n \), as well as all its uniform lattices, act properly and cocompactly on the locally compact median space associated to the usual measured walls structure on \( \mathbb{H}^n \).

1.b. Cubulable and medianizable groups. The previous results bring to light the existence, in the class of finitely generated groups, of several degrees of compatibility with median geometry, starting with cubulable, and proceeding in a decreasing order of strength, as follows.

**Definition 1.3.** A group is said to be

- **cubulable** if it acts properly discontinuously cocompactly on a CAT(0) cube complex;
- **strongly medianizable** if it acts properly discontinuously cocompactly on a median space of finite rank;
- **medianizable** if it acts properly discontinuously cocompactly on a median space.

Note that if a finitely generated group is cubulable, the CAT(0) cube complex on which it acts has finite dimension, and if it is medianizable then the median space on which it acts is locally compact.

The difference between strongly medianizable and cubulable is unclear, at this point it even seems possible that the two properties are equivalent. Some evidence comes from the fact that key properties known for cubulable groups (Tits alternative, superrigidity) have been proven for strongly medianizable groups as well \cite{Fio1, Fio2, Fio3}. Note that a Rips theorem for median spaces of finite rank as discussed in Section 1.d would give a weaker result than the equivalence between cubulable and strongly medianizable, if the connection between stabilizers is similar to the one established for actions on trees, as in \cite{BF95, GLP, Sel97, Gui05, Bes}.

On the other hand, the distinction between medianizable and strongly medianizable groups is clear: there are interesting examples of groups that are in the former class but not in the latter. Indeed, Corollary 1.2 implies the following.

**Corollary 1.4.** Uniform lattices in a product \( SO(n_1, 1) \times \cdots \times SO(n_k, 1) \), with \( k \geq 1 \), are medianizable.

For \( k \geq 2 \), irreducible uniform lattices in products \( SO(n_1, 1) \times \cdots \times SO(n_k, 1) \) are not cubulable, due to results of Chatterji-Fernos-Iozzi \cite{CFI}. Moreover, by work of Fioravanti
they are not strongly medianizable either. Thus, Corollary 1.4 is the best one can hope for, in terms of median geometry, for these lattices. It is unknown if these same lattices can act properly on an infinite dimensional CAT(0) cube complex.

Even in the case of one factor \((k = 1)\), Corollary 1.4 may turn out to be significant. It is not known if all arithmetic uniform lattices in \(SO(n, 1)\), with \(n\) odd and larger than 3, are cubulable. It is for instance the case for the uniform lattices described in [VS, LM] and [Kap, §6]. The general consensus seems to be that these lattices are cubulable, except for the construction in dimension 7 [Ber], especially for the lattices constructed using octonions instead of quaternions [Kap, Theorem 6.7]. For the congruence subgroups of these latter lattices, it is proved in [BC] that the first Betti number is always zero. Thus, to cubulate these lattices one would have to find finite index subgroups other than congruence subgroups, and it is not known if such subgroups exist.

1.c. Other rank one symmetric spaces. Quaternionic hyperbolic spaces and the Cayley hyperbolic plane over the field of octonions cannot be equipped with measured walls structures, due to the fact that their groups of isometries have Property (T). As far as the complex hyperbolic spaces are concerned, we explained in [CDH10] how, using results of Faraut and Harzallah [FH74], they can also be equipped with structures of spaces with measured walls (the wall metric being in this case \(\text{dist}^2\), where \(\text{dist}\) is the hyperbolic metric), hence their groups of isometries also act properly by isometries on a median space. However, the action is no longer cobounded.

**Corollary 1.5.**

1. The space \((\mathbb{H}^n_C, \text{dist})\) cannot be isometrically embedded into a median space. In particular the hyperbolic distance \(\text{dist}\) cannot be a wall metric.

2. For any \(\alpha \in [1/2, 1)\), whenever \(\text{dist}^\alpha\) is a wall metric, \((\mathbb{H}^n_C, \text{dist}^\alpha)\) cannot be at bounded Hausdorff distance from a median space.

1.d. Possible further applications and open questions.

**Infinite dimensional real hyperbolic spaces.** These spaces can be defined either as quotients of stabilizers of quadratic forms of signature 1 on infinite dimensional real vector spaces [Gro §6.A.III], or by an infinite dimensional hyperboloid model [BIM05, MPI]. They have a natural structure of measured walls, inducing a metric that equals the hyperbolic metric (see the discussion in Example 3.8 and references therein for details).

Some important groups have interesting actions on these spaces, such as the groups of automorphisms of (products of) regular trees and their lattices (including the Burger-Mozes examples) [BIM05] and the groups of birational transformations of complex Kähler surfaces [Ca11].

Our arguments work only partially for such spaces (see discussion in the end of Section 5). It nevertheless seems legitimate to ask the following.

**Question 1.6.** Is the infinite dimensional hyperbolic space at finite Hausdorff distance from the median space associated to it?
Rips-type theorems for median spaces. Our results are also relevant in the setting of potential extensions of Rips-type theorems to actions on median spaces [CDH10 Question 1.11]. The existing Rips-type theorems provide conditions under which a non-trivial (i.e. without global fixed point), minimal action of a group $G$ on a real tree $T \neq \mathbb{R}$, yields a non-trivial cocompact action on a simplicial tree, with stabilizers of edges virtually cyclic extensions of stabilizers of arcs in $T$ (see [BF95, GLP, Sel97, Gui05, Bes] and references therein).

For instance, Bestvina and Feighn prove in [BF95, Theorem 9.5] that if $G$ is finitely presented, and the action $G \curvearrowright T$ is stable (a condition satisfied by proper actions) then the required action on a simplicial tree can always be produced (with virtually cyclic stabilizers of edges, if the initial action was proper).

The interest of a Rips-type theorem for median spaces is that it would relate the negation of property (T) to actions on CAT(0) cube complexes, and it would provide conditions under which a-T-menability implies weak amenability with Cowling-Haagerup constant 1 [CDH10, §1.3].

The results in this paper show that one cannot expect, for actions on median spaces, a theorem similar to the one of Bestvina-Feighn: uniform irreducible lattices in products $SO(n_1,1) \times \cdots \times SO(n_k,1)$ are finitely presented, they act properly discontinuously, minimally and with compact quotient on median spaces, but cannot act non-trivially cocompactly with amenable stabilizers on a CAT(0) cube complex [CFI]. Still, the following question may still have a positive answer

**Question 1.7.** It is possible to obtain Rips-type theorems for stable (e.g. proper) actions of finitely presented groups on median spaces of finite rank?

This is consistent with the case of real trees, which are median spaces of rank one. For such Rips theorems, the most appropriate condition corresponding to the condition $T \neq \mathbb{R}$ for trees seems to be “median space with no global fixed point at infinity under the full isometry group, and which is not within bounded Hausdorff distance from a space $\mathbb{R}^n$ with the $\ell^1$ norm”. This follows from the fact that a finite rank median metric space has a bi-Lipschitz equivariant metric that is CAT(0), and from the results in [CM15].

1.e. Plan of the paper. In Section 2 we recall the definition of (pseudo-)median space, and define and discuss other relevant notions, such as that of $\delta$-tripodal and of geodesically $\delta$-tripodal spaces. Section 3 recalls the construction of a median space associated to a space with walls (this is done in more detail in CDH10 [Fio1, Fio2]). Section 4 is devoted to the proof of Theorem 4.10. In Section 5 we prove that the medianization of the real hyperbolic space is locally compact. Section 6 discusses the complex hyperbolic case and Corollary 1.5.

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2. Tripodal and median spaces.

Recall that a pseudo-metric space $(X, \text{pdist})$ is a space such that pdist satisfies all the properties of a distance, except for “$\text{pdist}(x, y) = 0 \Rightarrow x = y$”. Its separation is the metric space $\overline{X}$ obtained as a quotient by the equivalence relation $\text{pdist}(x, y) = 0$.

**Definition 2.1.** A point $a$ in the pseudo-metric space $X$ is said to be between two other points $x, y$ in $X$ if

$$\text{pdist}(x, a) + \text{pdist}(a, y) = \text{pdist}(x, y).$$

The interval $I(x, y)$ of endpoints $x$ and $y$ is the set of points that are between $x$ and $y$. A subset $A$ in $X$ is called convex if every point $a$ that is between two points $x, y$ in $A$ is at distance zero from a point in $A$. When $X$ is a metric space, this is equivalent to the fact that $I(x, y) \subseteq A$ for every two points $x, y$ in $A$.

A median pseudo-metric space is a space for which, given any triple of points $x, y, z$, the set $I(x, y) \cap I(y, z) \cap I(x, z)$ is non-empty and has diameter zero. Any point in the latter set is called median point for the triple $x, y, z$. When pdist is a metric, this coincides with the notion of median metric space recalled in the introduction.

The rank of a median metric space $X$ is the supremum over the set of integers $k$ such that $X$ contains an isometric copy of the set of vertices $\{-a, a\}^k$ of the cube of edge length $2a$ endowed with the induced $\ell_1$-metric, for some $a > 0$. Equivalently, it is the supremum over the set of integers $k$ so that there exist $k$ pairwise transverse convex walls.

The rank of a median pseudo-metric space $(X, \text{pdist})$ equals the rank of its separation. Given $R > 0$, the open $R$-neighbourhood of a subset $A$, i.e. $\{x \in X : \text{pdist}(x, A) < R\}$, is denoted by $\mathcal{N}_R(A)$. In particular, if $A = \{a\}$ then $\mathcal{N}_R(A) = B(a, R)$ is the open $R$-ball centered at $a$. We use the notation $\overline{\mathcal{N}}_R(A)$ and $\overline{B}(a, R)$ to designate the corresponding closed neighborhood and closed ball defined by non-strict inequalities.

A point $a$ is said to be $\delta$-between two other points $x, y$ if

$$\text{pdist}(x, a) + \text{pdist}(a, y) \leq \text{pdist}(x, y) + \delta.$$
When $\delta = 0$, $a$ is between $x$ and $y$ and it belongs to the interval $I(x, y)$.

We denote by $I_\delta(x, y)$ the set of points that are $\delta$-between $x$ and $y$.

**Remark 2.2.** One can easily see that
\[ N_R(I(x, y)) \subseteq I_{2R}(x, y). \]

The reverse inclusion is true in median metric spaces, but false for subspaces of median spaces, even with $N_R(I(x, y))$ replaced by $N_D(I(x, y))$, for some uniform $D = D(R)$. For instance, consider a simplicial tree $T$, the median space $M = T \times [0, 1]$, and the subspace $X$ of $M$ obtained by removing relative interiors of rectangles $[x_n, y_n] \times [0, 1]$ and, in the copy $T \times \{1\}$, the relative interiors of geodesics $[x_n, y_n] \times \{1\}$, with $[x_n, y_n]$ pairwise disjoint geodesics in $T$ of respective length $n$. In the space $X$ with the metric induced from $M$, $I((x_n, 1), (y_n, 1)) = \{(x_n, 1), (y_n, 1)\}$, while $I_2((x_n, 1), (y_n, 1))$ contains the entire geodesic $[x_n, y_n] \times \{0\}$.

Another example would be to consider $\mathbb{R}^2$ with the $L^1$ norm, remove the interior of pairwise disjoint translates of squares $[0, n] \times [0, n]$, $n \in \mathbb{N}$, and replace them with the other faces of a height one parallelepiped over that square, with the interior of the bottom square removed.

**Definition 2.3.** Given a subset $A$ and a point $x$ in $X$, an $\varepsilon$-projection $p$ of $x$ on $A$ is a point $p \in A$ such that $p \text{dist}(x, p) < p \text{dist}(x, A) + \varepsilon$.

As explained in the introduction, one of the aims of this paper is to investigate groups that are not cubulable, but have geometric properties that are close to cubulable, such as (strongly) medianizable (see Definition 1.3). In relation to this investigation, three natural generalizations of both median spaces and Gromov hyperbolic spaces, introduced below, play an important part. The first two of these (Definition 2.4) require, loosely speaking, that the set of ‘quasi-centers’ of any triple of points is non-empty, respectively that some of these quasi-centers come from a thin geodesic triangle. The third one (Definition 2.11) requires that moreover the set of ‘quasi-centers’ of any triple of points has uniformly bounded diameter.

**Definition 2.4.** [\(\delta\)-tripodal spaces] A (pseudo-)metric space $(X, \text{pdist})$ is called $\delta$-tripodal if, given any three points $x, y, z$,
\[ I_{2\delta}(x, y) \cap I_{2\delta}(y, z) \cap I_{2\delta}(z, x) \neq \emptyset. \]

**Examples 2.5.**
1. In metric graph theory, 0-tripodal graphs are called modular graphs. Thus, all the examples of modular graphs are also examples of 0-tripodal spaces.
2. Pseudo-modular graphs (in particular Helly graphs) are 1-tripodal [CCHO, §2.2.2.2].
3. The above, by work of Huang and Osajda [HO21], implies that certain Artin groups and Garside groups (e.g. braid groups) are tripodal.
4. Quasi-median graphs of finite cubical dimension $d$ are $d$-tripodal by work of Genevois [Gen17, Proposition 2.84].
**Definition 2.6.** [geodesically δ-tripodal (pseudo-)metric spaces] We say that \(X\) is *geodesically δ-tripodal* if given any three points \(x, y, z\), there exists a \(δ\)-*thin* geodesic triangle with vertices \(x, y, z\), namely a geodesic triangle such that each of its sides is contained in the \(δ\)-neighborhood of the union of the other two sides.

**Remark 2.7.** Geodesically \(δ\)-tripodal implies tripodal, but the converse implication is not true in general. Both examples preceding Definition 2.3 are examples of spaces that are 1-tripodal, but not geodesically 1-tripodal. These examples display the key difference between the two notions: in a \(δ\)-tripodal space in general we may not have that \(I_δ(x, y)\) is a subset of \(N_D(I(x, y))\) for all \(x, y \in X\), for some uniform constant \(D\); in a geodesically \(δ\)-tripodal space, on the other hand, for any \(μ \geq 0\) there exists \(D(μ, δ) ≥ 0\) such that \(I_μ(x, y) \subseteq N_D(I(x, y))\). Indeed, take \(t \in I_μ(x, y)\) and consider a \(δ\)-thin triangle with vertices \(x, y, t\). Take \(p \in I(x, y)\), at distance less than \(δ\) from both \(p_x \in I(x, t)\) and \(p_y \in I(y, t)\). Then \(pdist(x, p_x) ≤ pdist(x, p) + δ\) and \(pdist(y, p_y) ≤ pdist(y, p_y) + δ\) and from \(t \in I_μ(x, y)\) we get that

\[
pdist(x, p_x) + pdist(p_x, t) + pdist(t, p_y) + pdist(p_y, y) ≤ pdist(x, p) + pdist(p, y) + μ
\]

hence \(pdist(p_x, t) + pdist(t, p_y) ≤ 2δ + μ\) and so \(2pdist(t, p_y) − 2δ ≤ 2δ + μ\) since \(pdist(p_x, t) ≤ pdist(p_y, t) + 2δ\). Then \(pdist(t, p) ≤ pdist(t, p_y) + δ ≤ D(μ, δ)\), so that \(t \in N_D(I(x, y))\).

**Examples 2.8.**

1. Geodesic Gromov hyperbolic spaces are (geodesically) \(δ\)-tripodal;
2. (geodesic) median spaces are (geodesically) 0-tripodal;
3. if \((X, d_X)\) and \((Y, d_Y)\) are (geodesically) \(δ\) and \(μ\)-tripodal respectively, then \((X \times Y, d_X + d_Y)\) is (geodesically) \((δ + μ)\)-tripodal. In particular, products of hyperbolic spaces are \(δ\)-tripodal.
4. Euclidean spaces (and Hilbert spaces) cannot be \(δ\)-tripodal for any \(δ < ∞\) unless of dimension one.

**Definition 2.9.** A finitely generated group \(G\) is (geodesically) \(δ\)-tripodal if there exists a metric \(dist\) on it, quasi-isometric to the word metrics, such that \((G, dist)\) is (geodesically) \(δ\)-tripodal.

Being (geodesically) \(δ\)-tripodal behaves well under relative hyperbolicity, as the following stability result shows.

**Proposition 2.10.** Let \(G\) be a finitely generated group hyperbolic relative to a finite family of finitely generated subgroups \(H_1, ..., H_n\). Assume that for every \(i \in \{1, ..., n\}\) the subgroup \(H_i\) is (geodesically) \(δ\)-tripodal. Then the group \(G\) is (geodesically) \(μ\)-tripodal for some \(μ ≥ 0\).

In particular, any lattice in \(SO(n, 1)\) is \(δ\)-tripodal. However, non-uniform lattices in \(SU(3, 1)\) are not \(δ\)-tripodal: a group that is \(δ\)-tripodal must have sub-cubic Dehn function [Eld], and these lattices have cubic Dehn functions. Note that M. Elder’s proof in [Eld] is purely metric, thus even though the result is formulated for word metrics and
Dehn functions, it is also true for metrics quasi-isometric to word metrics and for metric generalisations of Dehn functions, as defined in [Gro, 5.F].

Proof of Proposition 2.10. Assume that for every $i \in \{1, \ldots, n\}$ the subgroup $H_i$ admits a metric dist$_i$ quasi-isometric to a word metric, with respect to which $H_i$ is $\delta$-tripodal. The goal is to construct on the group $G$ a metric quasi-isometric to a word metric, with respect to which $G$ is $\mu$-tripodal for some $\mu \geq 0$. One can start with a word metric on $G$ with respect to a generating set containing generating sets for all parabolic groups $H_i$, and replace the metric induced on each coset $gH_i$ by the quasi-isometric metric provided by the assumption. A standard argument shows that the metric on $G$ thus modified is quasi-isometric to the initial word metric, and that $G$ endowed with this metric is still hyperbolic relative to $H_1, \ldots, H_n$.

According to [DS05b, Dr09], this relative hyperbolicity implies that there is a constant $\delta$ such that for any triangle $\Delta$, consisting of a triple of points $x, y, z$ in $G$ and (discrete) geodesics between those points, there exists a coset $gH_i$ such that if we denote by $x_1, x_2, y_1, y_2, z_1, z_2 \in gH$ the entrance and exit points of the geodesics of $\Delta$ from this coset, we have that $d(x_1, x_2), d(y_1, y_2), d(z_1, z_2) \leq \delta$, proving that $x, y, z$ are vertices of a $\delta$-thin triangle.

Definition 2.11. [$\delta$–median metric spaces] We say that a (pseudo-)metric space $(X, \text{pdist})$ is (geodesically) $\delta$–median if it is (geodesically) $\delta$-tripodal and, moreover, given any three points $x, y, z$, the non-empty set $I^{2\delta}(x, y) \cap I^{2\delta}(y, z) \cap I^{2\delta}(z, x)$ has diameter at most $D = D(\delta)$. With the convention that $D(0)$ must be 0, a 0–median space is a median space. Clearly $\delta$–median implies coarse median in the sense of Bowditch [Bo13]. According to [NWZ, Section 3], coarse median spaces are close to being $\delta$–median for some $\delta \geq 0$. In particular, they are $\delta$–median if $I^{2\delta}(x, y)$ is a subset of $N_{R(\delta)}(I(x, y))$, for every $x, y$, where $R(\delta)$ is a constant depending only on $\delta$.

The property of $\delta$–median is strictly stronger than that of $\delta$-tripodal: the building associated to $Sp(4, \mathbb{Q}_p)$ admits a metric that is 0-tripodal. Indeed, this metric corresponds to a consistent choice of a pair of orthogonal roots in each flat. This is possible because it has two pairs of roots of different lengths. For such a metric, intervals grow exponentially and medians are not unique. The metric cannot be median because $Sp(4, \mathbb{Q}_p)$ has property (T). A similar argument works for $Sp(2n, \mathbb{Q}_p)$ and $SL_4(\mathbb{Q}_p)$. They cannot be coarse median either [Hac], hence not $\delta$-median either.

It is straightforward to check that products of hyperbolic spaces are $\delta$-median. There has been a lot of lively research recently around the question of the existence of an (equivariant) embedding of a group into a finite product of hyperbolic spaces. This started with the work of Bestvina, Bromberg, Fujiwara [BBF15], and later Bestvina, Bromberg, Fujiwara and Sisto [BBFS], who introduced a set of axioms allowing to construct such an embedding, and applied it, for instance, to mapping class groups. Hagen and Petyt later proved that, at least in the case of mapping class groups, this embedding endows the
group with a structure of $\delta$-median space [HP21]. After that, Bestvina, Bromberg and Fujiwara showed that in certain cases (e.g. for mapping class groups) the construction can be adapted to yield an equivariant quasi-isometric embedding into a finite product of quasi-trees. This construction has been proven to induce a structure of $\delta$-median space on mapping class groups by Petyt [Pe].

3. THE MEDIAN SPACE ASSOCIATED TO A SPACE WITH MEASURED WALLS.

From [HP98], we recall that a wall of a set $X$ is a partition $X = h \sqcup h^c$ (where $h$ is possibly empty or the whole $X$). A collection $\mathcal{H}$ of subsets of $X$ is called a collection of half-spaces if for every $h \in \mathcal{H}$ the complementary subset $h^c$ is also in $\mathcal{H}$. We call collection of walls on $X$ the collection $\mathcal{W}_\mathcal{H}$ of pairs $w = \{h, h^c\}$ with $h \in \mathcal{H}$. For a wall $w = \{h, h^c\}$ we call $h$ and $h^c$ the two half-spaces bounding $w$. We say that a wall $w = \{h, h^c\}$ separates two disjoint subsets $A, B$ in $X$ if $A \subseteq h$ and $B \subseteq h^c$ or vice-versa and denote by $\mathcal{W}(A|B)$ the set of walls separating $A$ and $B$. In particular $\mathcal{W}(A|\emptyset)$ is the set of walls $w = \{h, h^c\}$ such that $A \subseteq h$ or $A \subseteq h^c$; hence $\mathcal{W}(\emptyset|\emptyset) = \mathcal{W}$. We use the notation $\mathcal{W}(x|y)$ to designate $\mathcal{W}(\{x\}|\{y\})$.

**Definition 3.1** (Space with measured walls [CMV04]). A space with measured walls is a set $X$, with $\mathcal{W}$ a collection of walls, $\mathcal{B}$ a $\sigma$-algebra of subsets in $\mathcal{W}$ and $\mu$ a measure on $\mathcal{B}$, such that for every two points $x, y \in X$ the set of separating walls $\mathcal{W}(x|y)$ is in $\mathcal{B}$ and it has finite measure. We denote by $\text{pdist}_\mu$ the pseudo-metric on $X$ defined by

$$\text{pdist}_\mu(x, y) = \mu(\mathcal{W}(x|y)),$$

and we call it the wall pseudo-metric.

**Remark 3.2.** Consider the set $\mathcal{H}$ of half-spaces determined by $\mathcal{W}$, and the natural projection map $p : \mathcal{H} \to \mathcal{W}$, $h \mapsto \{h, h^c\}$. The pre-images of the sets in $\mathcal{B}$ define a $\sigma$-algebra on $\mathcal{H}$, which we denote by $\mathcal{B}^\mathcal{H}$; hence on $\mathcal{H}$ can be defined a pull-back measure that we also denote by $\mu$. This allows us to work either in $\mathcal{H}$ or in $\mathcal{W}$.

**Remark 3.3.** The half-spaces are convex with respect to $\text{pdist}_\mu$. Indeed if $a$ and $b$ are two points in a half-space $h$ and $x$ is such that $\text{pdist}_\mu(a, x) + \text{pdist}_\mu(x, b) = \text{pdist}_\mu(a, b)$, one cannot assume that $\text{pdist}_\mu(x, h) = \varepsilon > 0$. Otherwise

$$\text{pdist}_\mu(a, b) = \mu(\mathcal{W}(a|x,b)) + \mu(\mathcal{W}(a|x)) \leq \mu(\mathcal{W}(a|x)) + \mu(\mathcal{W}|h))$$

+\mu(\mathcal{W}(a|x)) + \mu(\mathcal{W}(h|x)) \leq \mu(\mathcal{W}(b|x)) + \mu(\mathcal{W}(a|x))$$

whence $\text{pdist}_\mu(a, x) + \text{pdist}_\mu(x, b) - \text{pdist}_\mu(a, b) \geq 2\varepsilon > 0$.

For the definition of homomorphisms of spaces with measured walls we use a slightly modified terminology, in accordance with the one in [Fio1, Fio2].

**Definition 3.4.** Let $(X, \mathcal{W}, \mu)$ and $(X', \mathcal{W}', \mu')$ be two spaces with measured walls. A map $\phi : X \to X'$ is a homomorphism of spaces with measured walls if:
• for any \( w' = \{ h', h'c \} \in W' \) we have \( \{ \phi^{-1}(h'), \phi^{-1}(h'c) \} \in W \); this latter wall we denote by \( \phi^*(w') \);
• the map \( \phi^*: W' \to W \) is measurable and \( (\phi^*)_* \mu' = \mu \).

We say that \( \phi \) is a **monomorphism of spaces with measured walls** if \( \phi^* \) is surjective, and that \( \phi \) is a **coarsely surjective monomorphism** if moreover there exists \( D > 0 \) such that \( X' \) is contained in the closed \( D \)-neighbourhood of \( \phi(X) \), and for every half-space \( h \) of \( X \) there exists a half-space \( h' \) of \( X' \) contained in the closed \( D \)-neighbourhood of \( \phi(h) \). Both neighbourhoods are considered with respect to the wall pseudo-metric \( \text{pdist}_{\mu'} \).

Note that being a monomorphism does not necessarily imply injectivity. On the other hand, being a homomorphism already does imply that the map is an isometric embedding with respect to the wall pseudo-distances.

**Example 3.5.** The plane \( \mathbb{R}^2 \) with the Euclidean distance can be isometrically embedded in a median space, via the map
\[
\begin{align*}
\mathbb{R}^2 & \to L^1([0, 2\pi]) \\
\begin{pmatrix} x \\ y \end{pmatrix} & \mapsto \frac{1}{4} (x \sin t + y \cos t).
\end{align*}
\]

**Example 3.6.** As mentioned in the introduction, a median metric space is endowed with a structure of convex measured walls, such that the wall pseudo-metric \( \text{pdist} \) coincides with the median metric, and isometries are automorphisms of the space with measured walls [CDH10 Section 5].

Thus, every \( L^1 \)-space has a measured walls space structure, and every \( L^p \)-space, with \( p \in (1, 2) \) (in particular, every Hilbert space), has a measured walls space structure, since they all embed isometrically in an \( L^1 \)-space [WW75].

**Example 3.7** (Finite dimensional real hyperbolic space). Define the half-spaces of the real hyperbolic space \( \mathbb{H}^n \) to be closed or open geometric half-spaces, so that the boundary of half-spaces is an isometric copy of \( \mathbb{H}^{n-1} \) (a geometric hyperplane of \( \mathbb{H}^n \)). Note that the associated set of walls \( W_{\mathbb{H}^n} \) is identified with the homogeneous space \( SO(n,1)/SO(n-1,1) \). Since the stabilizer of a hyperplane is unimodular, there is a \( SO(n,1) \)-invariant borelian measure \( \mu_{\mathbb{H}^n} \) on the set of walls. The set of walls separating two points has a compact closure, therefore it has finite measure and thus \( (\mathbb{H}^n, W_{\mathbb{H}^n}, \mu_{\mathbb{H}^n}) \) is a space with measured walls. By Crofton’s formula [CMV04 Proposition 3], up to multiplying the measure \( \mu_{\mathbb{H}^n} \) by some positive constant the wall pseudo-metric on \( \mathbb{H}^n \) coincides with the usual hyperbolic distance.

**Example 3.8** (Infinite dimensional real hyperbolic space). The infinite dimensional real hyperbolic space can be described in several ways. For instance, given the Hilbert space \( \ell^2 \) and \( O(1,\infty) \) the space of bounded operators preserving the form \( -x_1^2 + \sum_{i=2}^{\infty} x_i^2 \), then \( \mathbb{H}^\infty \) is the infinite dimensional Riemannian symmetric space of constant negative curvature \( O(1,\infty)/O(1) \times O(\infty) \), where \( O(\infty) \) represents the orthogonal unitary operators that keep the first coordinate \( x_1 \) fixed [Gro §6.A.III].
Alternatively, given a Hilbert space \( H \) and a 1-dimensional subspace \( L \) in it, one can define a bilinear symmetric form determined by the quadratic form

\[
B(v, v) = \|v_L\|^2 - \|v_{L^\perp}\|^2,
\]

where \( v_L \) and \( v_{L^\perp} \) are the orthogonal projections of \( v \) on \( L \), respectively on its orthogonal \( L^\perp \). For a fixed non-zero vector \( e \) in \( L \) one can define the hyperboloid model of the infinite dimensional hyperbolic space as

\[
\mathbb{H}^\infty = \{ v \in H \mid B(v, v) = 1, B(v, e) > 0 \},
\]

with distance defined by \( \cosh(\text{dist}_{\mathbb{H}^\infty}(x, y)) = B(x, y) \), turning it into a complete metric space with constant curvature \(-1\) \([\text{MP14}]\). Note that there is a generalization of this construction, \( \mathbb{H}^\alpha \), of Hilbert dimension \( \alpha \), for every cardinal \( \alpha \geq 2 \) \([\text{BIM05}]\). For simplicity, here we restrict to \( \alpha = \aleph_0 \).

Finally, \( \mathbb{H}^\infty \) can also be defined as the direct limit of the sequence of metric spaces \( (\mathbb{H}^n)_{n \in \mathbb{N}} \).

The structure of space with measured walls on \( \mathbb{H}^\infty \) can be defined either as the structure induced by its embedding into a Hilbert space, in the hyperboloid model, or as a direct limit of the structures of spaces with measured walls on the finite dimensional spaces \( \mathbb{H}^n \). The Crofton formula (i.e. the equality between the wall pseudo-metric and the usual hyperbolic metric) is satisfied on \( \mathbb{H}^\infty \) because it is satisfied on each \( \mathbb{H}^n \).

We now recall how a space with measured walls naturally embeds in a median space. This is done in detail in \([\text{CDH10}] \), \([\text{Fio1}] \), \([\text{Fio2}] \) and we just give the outline here.

**Definition 3.9.** A section \( s \) for the projection \( p : \mathcal{H} \to \mathcal{W} \) is called admissible if its image contains, together with a half-space \( h \), all the half-spaces \( h' \) containing \( h \). We denote by \( \mathcal{M}(X) \) the set of admissible sections.

**Remark 3.10.** We identify an admissible section \( s \) with its image \( \sigma = s(\mathcal{W}) \); with this identification, an admissible section becomes a collection of half-spaces, \( \sigma \), such that:

- for every wall \( w = \{ h, h^c \} \) either \( h \) or \( h^c \) is in \( \sigma \), but never both;
- if \( h \subset h' \) and \( h \in \sigma \) then \( h' \in \sigma \).

Such a collection of half-spaces is commonly called ultrafilter in the literature on median spaces.

For any \( x \in X \) we denote by \( \sigma_x \) the image of the section of \( p \) associating to each wall \( \{ h, h^c \} \) the half-space containing \( x \). That is, \( \sigma_x \) is the set of half-spaces \( h \in \mathcal{H} \) such that \( x \in h \). It is straightforward to see that this is an admissible section. Notice that \( p(\sigma_x \Delta \sigma_y) = \mathcal{W}(x|y) \).

Let now \( x_0 \) denote some base point in \( X \). We define

\[
\mathcal{B}^\mathcal{H} := \{ A \subseteq \mathcal{H} \text{ such that } A \Delta \sigma_{x_0} \in \mathcal{B} \text{ and } \mu(A \Delta \sigma_{x_0}) < +\infty \}.
\]

The identity \( A \Delta \sigma_{x_1} = (A \Delta \sigma_{x_0}) \Delta (\sigma_{x_0} \Delta \sigma_{x_1}) \) and the fact that \( \sigma_{x_0} \Delta \sigma_{x_1} \) is measurable with finite measure shows that in fact the median pseudo-metric space \( \mathcal{B}^\mathcal{H} \).
is independent of the chosen base point $x_0$. In particular, $\sigma_x \in B^H$ for any $x \in X$. Endowed with the pseudo-metric $\text{pdist}_\mu(A, B) = \mu(A \cup B)$, the set $B^H$ becomes a median pseudo-metric space. The map

$$\chi_{x_0} : B^H \to S^1(H, \mu), \quad \chi_{x_0}(A) = \chi_{A \cup \sigma x_0}$$

is an isometric embedding of $B^H$ into the median subspace $S^1(H, \mu) \subset L^1(H, \mu)$, where $S^1(H, \mu) = \{\chi_B, B \text{ is measurable and } \mu(B) < +\infty\}$. Notice that for $x, y \in X$ we have $\text{pdist}_\mu(x, y) = \mu(\sigma_x \cup \sigma_y)$, thus $x \mapsto \sigma_x$ is an isometric embedding of $X$ into $(B^H, \text{pdist}_\mu)$.

**Definition 3.11.** The median space associated to $X$ is the set

$$\mathcal{M}(X) := \overline{\mathcal{M}(X)} \cap B^H.$$

Since each admissible section $\sigma_x$ belongs to $\mathcal{M}(X)$, it follows that $X$ isometrically embeds in $\mathcal{M}(X)$. We will denote by $\iota : X \to \mathcal{M}(X)$ this isometric embedding. Given two elements $\tau, \tau' \in \mathcal{M}(X)$, we denote

$$W(\tau | \tau') = \{w = \{h, h^c\} \in W | h \in \tau, h^c \in \tau'\} = p(\tau \cup \tau')$$

The following result emphasizes that the construction of $\mathcal{M}(X)$ is the right one.

**Proposition 3.12** (Proposition 3.14, [CDH10]). The space $\mathcal{M}(X)$ is a median subspace of $B^H$. Let $\langle X', W' \rangle$ be another space with measured walls. Any monomorphism of spaces with measured walls $\phi : X \to X'$ induces an isometric embedding $\mathcal{M}(X) \to \mathcal{M}(X')$. In particular the group of automorphisms of $(X, W)$ acts by isometries on $\mathcal{M}(X)$.

**Remark 3.13.** To all intents and purposes, the space $\mathcal{M}(X)$ can be replaced by $\mathcal{M}_0(X)$, the metric completion of the median closure of $X$ in $\mathcal{M}(X)$. The space $\mathcal{M}_0(X)$ may in general be different from $\mathcal{M}(X)$, but is known to be equal to it when the space $\mathcal{M}_0(X)$ is locally convex [Fi02].

All the results in [CDH10] and in this paper that are formulated for $\mathcal{M}(X)$ also hold for $\mathcal{M}_0(X)$. Moreover $\mathcal{M}_0(X)$ has the advantage of being a complete geodesic metric median space when $X$ is connected. Indeed, according to Bowditch [Bo16], a complete median space is geodesic if and only if it is connected. The median map is 1–Lipschitz, therefore the image of $X$ by it in $\mathcal{M}(X)$ is connected, and as the median completion of $X$ in $\mathcal{M}(X)$ equals the increasing union of (connected) sets obtained by iterative applications of the median map to $X$, it is itself connected. The space $\mathcal{M}_0(X)$ is the median completion of this latter median completion, hence it is itself connected.

**Remark 3.14.** The median space $\mathcal{M}(X)$ has a structure of measured walls constructed explicitly from the one on $X$: for each $h \in \mathcal{H}$ define $h_\mathcal{M}$ to be the set of $\sigma \in \mathcal{M}(X)$ such that $h \in \sigma$. The complement of $h_\mathcal{M}$ in $\mathcal{M}(X)$ is the set of $\sigma \in \mathcal{M}(X)$ such that $h \notin \sigma$, or equivalently by the properties of admissible sections $h^c \in \sigma$. In other words $(h_\mathcal{M})^c = (h^c)_\mathcal{M}$. Thus $\{h_\mathcal{M}\}_{h \in \mathcal{H}}$ is a collection of half-spaces - which we will denote by $\mathcal{H}_\mathcal{M}$. We denote by $\mathcal{W}_\mathcal{M}$ the associated set of walls on $\mathcal{M}(X)$. Using the bijection $\mathcal{W} \to \mathcal{W}_\mathcal{M}$ induced by $h \mapsto h_\mathcal{M}$ we define on $\mathcal{W}_\mathcal{M}$ a $\sigma$-algebra $\mathcal{B}_\mathcal{M}$ and a measure $\mu_\mathcal{M}$. 
Note that \( \iota : X \to \mathcal{M}(X) \) is a monomorphism of spaces with measured walls. Note also that the distance on \( \mathcal{M}(X) \) coincides with the distance induced by the measured walls structure.

We leave it to the reader to check that the median space associated with \( \mathcal{M}(X) \) endowed with this structure of space with measured walls is \( \mathcal{M}(X) \) itself.

When speaking of a structure of measured walls on the particular median space \( \mathcal{M}(X) \), we shall henceforth always assume that it is the one described above.

4. DISTANCES TO THE ASSOCIATED MEDIAN SPACE.

In this section we investigate under what circumstances a space with measured walls is within bounded Hausdorff distance of its associated median space, or a complete median space in general. Such a measured wall space would have to satisfy metric properties similar to those of a median space, up to bounded perturbation. We briefly recall here a few more properties of median spaces, as discussed in [CDH10], ending with the property that will be central to our approach.

Let \((X, \dist)\) be first a general metric space. A gate between a point \(x\) and a subset \(Y\) of \(X\) is a point \(p \in Y\) that is between \(x\) and any point \(y \in Y\). A subset \(Y\) is gate convex if every point \(x \in X \setminus Y\) has a gate.

Every gate convex set is closed and convex. The converse is true if \(X\) is a complete median space.

In particular, in a complete median space the closure of every half-space is gate convex.

This is the property that we will quasify and use to discuss the finiteness of the Hausdorff distance of a measured wall space to its associated median space.

**Definition 4.1.** A measured wall space is said to have \((\epsilon, K)\)-gated half-spaces, where \(\epsilon\) and \(K\) are non-negative constants, if for every half-space \(h\) of \(X\), every point \(x\) outside \(h\), and every \(\epsilon\)-projection \(p\) of \(x\) on \(h\) (see Definition 2.3), the measure of the set of walls separating \(x\) from \(p\) and intersecting \(h\) is bounded by \(K\).

When we want to avoid mentioning the constants \(\epsilon\) and \(K\), we say that the measured wall space has quasi-gated half-spaces.

It turns out that indeed the above property suffices.

**Lemma 4.2.** Let \(X\) be a measured wall space with the property that all its half-spaces are \((\epsilon, K)\)-gated, for some non-negative constants \(\epsilon\) and \(K\).

Then for every \(\tau \in \mathcal{M}(X)\), and \(x \in X\), \(\epsilon\)-projection of \(\tau\) onto \(X\), each wall in \(\mathcal{W}(x|\tau)\) cuts the ball \(\bar{B}(x, 2K + \epsilon)\).

**Proof.** Take \(w = \{h, h^c\} \in \mathcal{W}(x|\tau)\), so that \(x \in h\) and \(h^c \in \tau\). Let \(p \in h^c\) be a point such that \(\pdist_{\mu}(p, x) \leq \pdist_{\mu}(h^c, x) + \epsilon\). By assumption, we have that

\[
\mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|h^c)) \leq K.
\]

We can write

\[
\pdist_{\mu}(\tau, \sigma_p) = \mu((\tau \triangle \sigma_x) \triangle (\sigma_p \triangle \sigma_x)) = \mu(\mathcal{W}(x|\tau) \setminus \mathcal{W}(x|p)) + \mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|\tau)).
\]
By the admissibility of \( \tau \), \( W(x|h^c) \subseteq W(x|\tau) \). It follows that \( W(x|p) \setminus W(x|\tau) \subseteq W(x|h^c) \setminus W(x|p) \), so that, using the assumption once again, we have \( \mu(W(x|p) \setminus W(x|\tau)) \leq K \).

On the other hand

\[
\mu(W(x|\tau) \setminus W(x|p)) = \text{pdist}_{\mu}(\tau, \sigma_x) - \mu(W(x|\tau) \cap W(x|p)) \\
\leq \text{pdist}_{\mu}(\tau, \iota(X)) + \epsilon - \mu(W(x|h^c) \cap W(x|p)).
\]

Now \( \mu(W(x|h^c) \cap W(x|p)) = \mu(W(x|p)) - \mu(W(x|p) \setminus W(x|h^c)) \), which by assumption is larger than \( \text{pdist}_{\mu}(x, p) - K \). Combining all, we get that

\[
\text{pdist}_{\mu}(\tau, \sigma_p) \leq K + \text{pdist}_{\mu}(\tau, \iota(X)) + \epsilon - \text{pdist}_{\mu}(x, p) + K,
\]

hence that \( \text{pdist}_{\mu}(x, p) \leq 2K + \epsilon \), as desired.

Thus, the quasification of the property “closed walls are gate convex” formulated above suffices to conclude that \( X \) is within bounded distance from \( M(X) \), provided that we assume that the set of walls intersecting a ball has finite measure. This is the condition that we introduce next.

**Notation 4.3.** Let \( (X, W, \mu) \) be a space with measured walls. For any subset \( Y \subseteq X \), we denote by \( W(Y) \) the set of walls separating two points of \( Y \) (we also say that these walls *cut* \( Y \)). The set \( W(Y) \) is not *a priori* measurable, unless \( Y \) is countable for instance, in which case \( W(Y) = \bigcup_{y,y' \in Y} W(y|y') \).

For an arbitrary subset \( Y \) we write \( \overline{\mu}(W(Y)) \leq K \) if for any measurable subset \( E \subseteq W(Y) \) we have that \( \mu(W(E)) \leq K \).

**Definition 4.4.** A wall space \( (X, W, \mu) \) is called *\( \mu \)-locally finite* if for every \( R > 0 \), there exists \( M = M(R) \geq 0 \) such that \( \overline{\mu}(W(B(x,R))) \leq M < +\infty \) for every \( x \in X \), where the open ball \( B(x,R) \) is with respect to the wall metric \( \text{pdist}_{\mu} \). When we want to avoid specifying the measure \( \mu \) and there is no risk of confusion, we simply say that the measured wall space is *measurably locally finite*.

**Example 4.5.**

(1) The measured wall space structures on \( \mathbb{R}^n \) endowed with either the Euclidean norm or the \( \ell^1 \)-norm are measurably locally finite, as is the structure on the real hyperbolic space \( \mathbb{H}^n \).

(2) Discrete wall spaces whose wall distance is uniformly proper are \( \mu \)-locally finite.

(3) More generally, if for each ball we have a covering

\[
W(B(x,R)) \subseteq W(x|a_1) \cup \cdots \cup W(x|a_n(R)),
\]

with \( d(x, a_i) \leq g(R) \), then the measured wall space is measurably locally finite.

Lemma 4.2 implies the following.

**Lemma 4.6.** Let \( (X, W, \mu) \) be a measured wall space that is \( \mu \)-locally finite and has the property that all its half-spaces are \( (\epsilon, K) \)-gated, where \( \epsilon \) and \( K \) are non-negative reals independent of the half-space.
Then $\mathcal{M}(X)$ is within Hausdorff distance at most $\beta(K, \epsilon) := \bar{\mu}(W(\bar{B}(x, 2K + \epsilon)))$ from $\iota(X)$. In particular $X$ is $\beta$-median.

Another consequence of the property that half-spaces are quasi-gated is a property of quasi-convexity of walls. The latter quasi-convex property is slightly different from the one introduced in [CDH10], where we defined quasi-convex sets to be sets $Y$ such that for every $a, b \in Y$, $I(a, b) \subset N_M(Y)$ for some uniform $M \geq 0$. Here, we extend the notion of quasi-convexity as follows.

**Definition 4.7.** For every $\delta \geq 0$, we call a subset $Y(\delta, M)$-quasi-convex if, for every $a, b \in Y$,

$$I(\delta)(a, b) \subset N_M(Y),$$

for some uniform $M \geq 0$.

With this terminology, we obtain the following.

**Lemma 4.8.** Let $X$ be a measured wall space such that all its half-spaces are $(\epsilon, K)$-gated, where $\epsilon$ and $K$ are non-negative reals.

Then, for every $\delta \geq 0$, all the half-spaces of $X$ are $(\delta, K + \delta)$-quasi-convex.

**Proof.** Let $\delta \geq 0$, $x, y$ two distinct points in a half-space $h$ and $a \in I(\delta)(x, y)$. Let $p$ be an $\epsilon$-projection of $a$ onto $h$. Then $W(a|p) \setminus W(a|h)$ has measure at most $K$.

On the other hand

$$W(a|h) \sqcup W(x|y) \subseteq W(a|x) \cup W(a|y),$$

whence

$$\mu(W(a|h)) + \pdist(\mu)(x, y) \leq \pdist(\mu)(x, a) + \pdist(\mu)(a, y) \Rightarrow \mu(W(a|h)) \leq \delta.$$

It follows that $\pdist(\mu)(a, p) \leq K + \delta$. □

To complete the characterization of measured wall spaces within finite distance of their associated median space, we have to consider a converse of the statement in Lemma 4.6. However, the property alone that $\mathcal{M}(X)$ is within bounded distance from $\iota(X)$ does not suffice to imply that half-spaces are quasi-gated. The first example in Remark 2.2 shows this, since the space $X$ is within bounded Hausdorff distance of $\mathcal{M}(X) = M$, but for the midpoints $(a_n, 0)$ of the geodesics $[x_n, y_n] \times \{0\}$ in $X$, there exist no points $p_n$ in the half-space $h = T \times [1/2, 1] \cap X$ such that $W(a_n, p_n) \setminus W(a_n, h)$ has uniformly bounded measure. Indeed, for every $p_n \in h$, the measure of $W(a_n, p_n) \setminus W(a_n, h)$ is at least $n/2 - 1$. The reason for this is that in this example half-spaces are not quasi-convex in the previous sense. Quasi-convexity is therefore a condition independent of the ‘bounded distance from $\mathcal{M}(X)$’ condition and, as it is also a necessary condition for the half-spaces to be quasi-gated, it has to be added to obtain a converse. This is the subject of the next lemma. It is to be noted that, when the quasi-convexity of the half-spaces is required, the condition ‘within bounded distance from $\mathcal{M}(X)$’ can be weakened to ‘tripodal’.

**Lemma 4.9.** Let $X$ be a measured wall space that is $\delta$-tripodal when endowed with the wall pseudo-metric, for some $\delta \geq 0$, and such that the half-spaces are $(2\delta, M)$-quasi-convex, for some $M \geq 0$ independent of the half-space.
For every half-space $h$ of $X$, every point $x$ outside $h$, and every $\epsilon$-projection $p$ of $x$ on $h$, the walls separating $x$ from $p$ and intersecting $h$ must intersect $B(p, 2\delta + M + \epsilon)$.

In particular, if $X$ is measurably locally finite, its half-spaces are $(\epsilon, \bar{\mu}(B(p, 2\delta + M + \epsilon))$-gated.

Proof. Let $x$ be a point outside a half-space $h_0$, and let $p$ be an $\epsilon$-projection of $x$ on $h_0$. Let $\mathcal{E}$ be a measurable subset of $\mathcal{W}$ contained in $\mathcal{W}(x|p) \setminus \mathcal{W}(x|h_0)$. For any wall $w = \{h, h^c\} \in \mathcal{E}$ assume the notation is such that $x \in h$ (so $p \in h^c$). Since the wall $w$ does not separate $x$ from $h_0$, the intersection $h \cap h_0$ is not empty. Take $q \in h \cap h_0$. As $X$ is $\delta$-tripodal, there exists a point $m$ that is $2\delta$-between any two of the three points $x, p, q$. In particular, since the walls are $(2\delta, M)$-quasi-convex, $m \in I_{2\delta}(x, q) \subseteq \mathcal{N}_M(h)$, and $m \in I_{2\delta}(p, q) \subseteq \mathcal{N}_M(h_0)$ so

$$\pdist(x, m) \geq \pdist(x, h_0) - M \geq \pdist(x, p) - \epsilon - M$$

hence $\pdist(x, p) - \pdist(x, m) \leq M + \epsilon$. Finally, from $m \in I_{2\delta}(x, p)$ we compute

$$\pdist(p, m) \leq \pdist(p, x) - \pdist(m, x) + 2\delta \leq 2\delta + M + \epsilon.$$

This shows that $w \in \mathcal{W}(B(p, \epsilon + M + 2\delta))$. \hfill \Box

Combining all the preceding lemmas, we obtain the following result, that immediately implies Theorem \ref{thm:main}.

**Theorem 4.10.** Let $(X, \mathcal{W}, \mu)$ be a space with measured walls that is $\mu$-locally finite (see Definition \ref{def:meas}). The following are equivalent:

1. the Hausdorff distance from $\iota(X)$ to the associated median space $\mathcal{M}(X)$ is at most $\delta$ and all the half-spaces of $X$ are $(2\delta, M)$-quasi-convex, for some $M \geq 0$ independent of the half-space;
2. there exists a median space $\mathcal{M}$ and a monomorphism $\varphi : X \to \mathcal{M}$ such that $\mathcal{M}$ is within Hausdorff distance at most $\delta$ from $\varphi(X)$, and all the half-spaces of $X$ are $(2\delta, M)$-quasi-convex;
3. there exists $\delta$ and $M$ non-negative constants such that $X$ with the wall pseudo-metric is $\delta$-tripodal, and all the half-spaces of $X$ are $(2\delta, M)$-quasi-convex;
4. the space $X$ has $(\epsilon, K)$-gated half-spaces, for some $\epsilon$ and $K$ non-negative constants.

To complete the picture, we note that for a measured wall space $X$ satisfying the equivalent assumptions of Theorem \ref{thm:main}, a connexion can be established between its half-spaces $h$ and the corresponding half-spaces $h_{\mathcal{M}}$ of $\mathcal{M}(X)$.

**Proposition 4.11.** If $(X, \mathcal{W}, \mu)$ is a measured wall space at Hausdorff distance $\eta$ from $\mathcal{M}(X)$ and with $(2\eta, M)$-quasi-convex walls, then for every half-space $h$ in $X$ the corresponding half-space in $\mathcal{M}(X)$

$$h_{\mathcal{M}} = \{\sigma \text{ admissible section } ; h \in \sigma\}.$$ 

is at Hausdorff distance at most $3\eta$ from $h$. 
In particular, with the terminology of Definition \[3.4\] in this case the inclusion \(i : X \to \mathcal{M}(X)\) is a coarsely surjective monomorphism of spaces with measured walls.

**Proof.** By definition \(h\), identified with its embedding in \(\mathcal{M}(X)\), \(\{\sigma_x ; x \in h\}\), is contained in \(h_\mathcal{M}\). For every \(\tau \in h_\mathcal{M}\), there exists \(x \in X\) such that \(\mu(\tau \triangle \sigma_x) \leq \eta\). If moreover \(x \in h\) then we are done. Assume therefore that \(x \notin h\).

There exists \(y \in h\) such that \(\text{pdist}_\mu(y, x) \leq 2\eta\), whence \(\mu(\sigma_y \triangle \sigma_x) \leq 2\eta\). We can now use

\[\tau \triangle \sigma_y = (\tau \triangle \sigma_x) \triangle (\sigma_y \triangle \sigma_x)\]

to conclude that \(\mu(\tau \triangle \sigma_y) \leq 3\eta\). \(\square\)

Note that the property of being at finite Hausdorff distance from its median space is not inherited by subsets, not even when they are geodesic. Indeed, every \(L^2\) space can be embedded isometrically into an \(L^1\) space [WW75]. A particular case of the equivalence in Theorem \[4.10\] is the following.

**Theorem 4.12.** Let \((X, W, \mu)\) be a space with measured walls that is \(\mu\)-locally finite and such that for every \(\delta > 0\) there exists \(\mu\) such that \(I_\delta(x, y) \subseteq N_\mu(I(x, y))\). The following are equivalent:

1. the Hausdorff distance from \(i(X)\) to the associated median space \(\mathcal{M}(X)\) is finite;
2. there exists a median space \(\mathcal{M}\) and a monomorphism \(\varphi : X \to \mathcal{M}\) such that \(\mathcal{M}\) is within finite Hausdorff distance from \(\varphi(X)\);
3. there exists \(\delta\) such that \(X\) with the wall pseudo-metric is \(\delta\)-tripodal;
4. the space \(X\) has \((\epsilon, K)\)-gated half-spaces, for some \(\epsilon\) and \(K\) non-negative constants.

**Corollary 4.13.** A space with measured walls that is \(\mu\)-locally finite and geodesically \(\delta\)-tripodal (e.g. geodesic Gromov hyperbolic or finite product of geodesic Gromov hyperbolic spaces) with respect to the distance \(\text{pdist}_\mu\) is at finite Hausdorff distance of its associated median space.

**Corollary 4.14.** A finite product \(X\) of finite dimensional real hyperbolic spaces is within bounded distance from its associated median space \(\mathcal{M}(X)\).

5. **Local compactness of the medianization of real hyperbolic spaces.**

In this section, we will show that the medianization of a real hyperbolic space is locally compact. To do so, we will show that balls in that medianization are **totally bounded**, namely they can be covered by finitely many balls of any given radius. In a metric space, a subset is compact if and only if it is complete and totally bounded. More precisely, the aim of this section is to show the following.

**Proposition 5.1.** Let \(n \geq 1\). For any \(R \geq 0\) and any \(\varepsilon > 0\), any ball of radius \(R\) in \(\mathcal{M}(\mathbb{H}^n)\) can be covered by a finite collection of balls of radius \(\varepsilon\).
In what follows, we fix a basepoint \( x_0 \) in \( \mathbb{H}^n \). We start with some notation. For every point \( x \neq x_0 \), we denote by \( H_x \) the unique hyperplane orthogonal to the geodesic \([x, x_0]\) and containing \( x \), and by \( W_x \) the unique wall determined by \( H_x \).

**Lemma 5.2.** For every admissible section \( \tau \in \mathcal{M}(\mathbb{H}^n) \) and non-trivial geodesic \([a, x_0]\) in \( \mathbb{H}^n \), one of the following cases occurs:

1. there exists \( \theta \in (a, x_0) \) such that \( \tau \) coincides with \( \sigma_a \) on every \( W_x \) with \( x \in (\theta, x_0] \) and \( \tau \) coincides with \( \sigma_{x_0} \) on every \( W_x \) with \( x \in [a, \theta) \);
2. \( \tau \) coincides with \( \sigma_a \) on every \( W_x \) with \( x \in (a, x_0) \);
3. \( \tau \) coincides with \( \sigma_{x_0} \) on every \( W_x \) with \( x \in (a, x_0) \).

**Proof.** Suppose that we are neither in case (2) nor in case (3). Thus, there exists a point \( u \in (a, x_0) \) such that \( \tau \) coincides with \( \sigma_a \) on \( W_u \) and a point \( v \in (a, x_0) \) such that \( \tau \) coincides with \( \sigma_{x_0} \) on \( W_u \). Clearly for every \( x \in [u, x_0) \), \( \tau \) coincides with \( \sigma_a \) on \( W_x \), and for every \( y \in [a, v] \), \( \tau \) coincides with \( \sigma_{x_0} \) on \( W_y \). In particular \( u \) must be between \( v \) and \( x_0 \). Let \( \delta \) be the supremum of the distances to \( x_0 \) of points such as \( u \) and let \( \theta \) be the point on \([a, x_0]\) that is at distance \( \delta \) from \( x_0 \). As \( \theta \) appears as limit of a sequence of points \( u_n \) as above and \( (\theta, x_0) = \bigcup [u_n, x_0] \), the first property in (1) follows, and the second property follows according to the choice of \( \theta \) and the fact that for every \( x \in (a, x_0) \), \( \tau \) must equal either \( \sigma_a \) or \( \sigma_{x_0} \). \( \Box \)

**Notation 5.3.** We denote by \( \theta_\tau(a) \) the point \( \theta \) in case (1), the endpoint \( a \) in case (2) and the endpoint \( x_0 \) in case (3). Note that when \( a \) varies on the sphere \( S(x_0, R) \), the parameter \( \theta_\tau(a) \) defines a map

\[
\delta_\tau : S(x_0, R) \rightarrow [0, R] \\
a \mapsto \dist(x_0, \theta_\tau(a)).
\]

**Lemma 5.4.** For every admissible section \( \tau \in \mathcal{M}(\mathbb{H}^n) \), the map \( \delta_\tau \) defined above is continuous.

**Proof.** Let \( a_0 \) be a fixed point on \( S(x_0, R) \) and let \( a \) be an arbitrary point on \( S(x_0, R) \), such that the angle between \([x_0, a]\) and \([x_0, a_0]\) is at most \( \epsilon \). For every point \( y_0 \) on \([a_0, x_0]\), the point \( y \) on \([a, x_0]\) that is nearest to \( x_0 \) such that \( H_y \) does not intersect \( H_{y_0} \) satisfies

\[
\dist(x_0, y) \leq \dist(x_0, y_0) + \kappa \epsilon R,
\]

for a universal constant \( \kappa \). This implies that \( \dist(x_0, \theta_\tau(a)) \leq \dist(x_0, \theta_\tau(a_0)) + \kappa \epsilon R \). The opposite inequality is obtained by swapping \( a_0 \) and \( a \) in the previous argument. \( \Box \)

Recall that for every given subset \( A \) of \( X \) we denote by \( \mathcal{H}(A) \) the set of half-spaces \( h \in \mathcal{H} \) such that \( h \cap A \neq \emptyset \), and by \( \mathcal{W}(A) \) the set of walls \( w = \{h, h^c\} \) such that both \( h \) and \( h^c \) are in \( \mathcal{H}(A) \). These are the walls cutting \( A \).

According to Lemmas 4.1 and 4.2 given \( \tau \) an arbitrary admissible section in \( \mathcal{M}(\mathbb{H}^n) \), and \( x \in \mathbb{H}^n \) a point such that \( \pdist_{\mu, \mathcal{M}}(\tau, \sigma_x) \leq \pdist_{\mu, \mathcal{M}}(\tau, \iota(\mathbb{H}^n)) + 1 \), any wall that
separates $x$ from $\tau$ also cuts the ball $B(x, \rho)$, where $\rho$ only depends on the hyperbolicity constant and the dimension of $\mathbb{H}^n$. This implies that
\[
\tau = [\sigma_x \cap \mathcal{H}(B(x, \rho))] \cup [\tau \cap \mathcal{H}(B(x, \rho))],
\]
meaning that $\tau$ and $\sigma_x$, as admissible sections, coincide on the walls that do not intersect $B(x, \rho)$ but may differ on the walls intersecting that ball. In particular, $\dist_{\mathcal{M}(X)}(\tau, \sigma_x) \leq \Delta_\rho$, where $\Delta_\rho$ is the measure of $\mathcal{H}(B(x, \rho))$.

Since $\mathcal{M}(\mathbb{H}^n)$ is at finite Hausdorff distance from $\mathbb{H}^n$, to prove Proposition 5.1 it suffices to consider balls centered at some $\sigma_{x_0}$, for $x_0 \in \mathbb{H}^n$ fixed. Let $\tau$ be an arbitrary admissible section in this ball $B(\sigma_{x_0}, R)$ and $x$ a 1-projection of the given $\tau$ in $\mathcal{M}(\mathbb{H}^n) \subset \mathcal{M}(\mathbb{H}^n)$. The point $x$ is within distance at most $R + \Delta_\rho$ from $x_0$. In particular, for $R' = R + \Delta_\rho + \rho$ we can write
\[
\tau = [\sigma_{x_0} \cap \mathcal{H}(B(x_0, R'))] \cup [\tau \cap \mathcal{H}(B(x_0, R'))],
\]
meaning that $\tau$ coincides with $\sigma_{x_0}$ on walls that are far enough from $x_0$. We now have everything in place for the proof of Proposition 5.1.

**Proof of Proposition 5.1** To show that the ball of radius $R$ is totally bounded, it suffices to find, for every $\epsilon > 0$, a finite number of admissible sections $\tau_1, \ldots, \tau_m$ in the closed ball of radius $R$ around $\sigma_{x_0}$ such that for every admissible section $\tau$ at distance at most $R$ from $\sigma_{x_0}$, there exists some $\tau_i$ such that the symmetric difference
\[
\tau \triangle \tau_i = [\tau \cap \mathcal{H}(B(x_0, R'))] \triangle [\tau_i \cap \mathcal{H}(B(x_0, R'))] (= \text{pdist}(\tau, \tau_i))
\]
has measure at most $C\epsilon$, where $C$ is a constant depending on the hyperbolicity constant and the dimension of $\mathbb{H}^n$. To construct those admissible sections, we consider $a_1, \ldots, a_k$ points on the sphere $S(x_0, R')$ such that for every $a \in S(x_0, R')$ there exists an $a_i$ such that the angle between $[x_0, a]$ and $[x_0, a_i]$ is at most $\epsilon$. For each $i \in \{1, 2, \ldots, k\}$ we consider $b_1(i), \ldots, b_n(i)$ on $[x_0, a_i]$ ordered from $x_0$ to $a_i$, dividing it into subgeodesics of equal length $\epsilon$. This process subdivides the ball of radius $R'$ into an a grid of mesh $\epsilon$.

On the finite set of walls $W_F = \bigcup_{i=1}^{k} \bigcup_{j=1}^{n} W_{b_j(i)}$ the number of admissible sections is finite. Let $\tau_1, \ldots, \tau_m$ be admissible sections at distance at most $R$ from $\sigma_{x_0}$ such that all the possible restrictions $\tau|_{W_F}$ of an admissible section at distance at most $R$ from $\sigma_{x_0}$ are realized by some $\tau_i$. In particular, for any $\tau$ at distance at most $R$ from $\sigma_{x_0}$ there exists $\tau_i$ such that $\tau$ restricted to $W_F$ coincides with $\tau_i$.

We are left to show that the distance from $\tau$ to $\tau_i$ in $\mathcal{M}(X)$ is at most $C\epsilon$. To this end, assume that $\theta_\tau(a_r)$ is inbetween $b_j(i)$ and $b_{j+1}(i)$, possibly equal to $b_j(i)$. Then on $W_{b_{j+1}(i)}$ $\tau$ equals $\sigma_{x_0}$ and on $W_{b_j(i)}$ $\tau$ equals $\sigma_a$. Whence the same is true for $\tau_i$ and $\theta_{\tau_i}(a_r)$ is inbetween $b_j(i)$ and $b_{j+1}(i)$. It follows that the distance between $\theta_\tau(a_r)$ and $\theta_{\tau_i}(a_r)$ is at most $\epsilon$.

On an arbitrary $a \in S(x_0, R')$, the above and Lemma 5.4 imply that the distance between $\theta_\tau(a)$ and $\theta_{\tau_i}(a)$ is at most $(1 + 2\kappa R)\epsilon$. 

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Thus, for every \( a \in S(x_0, R') \), the sections \( \tau \) and \( \tau_i \) coincide on every \( W_x \) with \( x \in [a, x_0] \), except maybe on a small sub-interval of length at most \((1 + 2\kappa R)\epsilon\). An integration allows to conclude. \( \square \)

**Remark 5.5.** Proposition 5.1 can be extended to spaces with measured wall \((X, W, \mu)\) that are \( \mu \)-locally finite, \( \delta \)-tripodal and with all half-spaces \((2\delta, M)\)-quasi-convex, for some \( \delta, M \) non-negative constants, such that \( X \) endowed with the metric \( \text{dist}_\mu \) is geodesic, proper and \( \text{CAT}(0) \). For simplicity, we also assume that the space \( X \) is geodesically complete (i.e. such that every geodesic segment can be extended to a bi-infinite geodesic line, not necessarily unique). For instance, Hadamard manifolds (i.e. complete simply connected manifolds of non-positive sectional curvature) and, more generally, complete \( \text{CAT}(0) \) spaces that are homology manifolds are geodesically complete \( \text{[BH, §II.5.12]} \). This would allow to define a map like the one introduced with Notation 5.3, though very likely the argument can be adapted to work in the general case, where the map should be defined not on \( S(x_0, R) \) but on \( T(x_0, R) \), where the latter is the set of points \( a \) in \( B(x_0, R) \) such that the geodesic \([x_0, a]\) cannot be extended beyond \( a \). Due to the \( \text{CAT}(0) \) property, for every half-space \( h \) not containing a basepoint \( x_0 \), there exists a unique geodesic \([x_0, x]\) orthogonal to \( h \) with length equal to \( \text{dist}_\mu(x_0, h) \). In this more general setting, not all the points \( x \) in the space are contained in a half-space \( h \) such that \( x \) is the closest point to \( x_0 \) in \( h \), only some of them are.

Lemma 5.2 is verified in this case too, with the same comment as before, that only some points \( x \in [a, x_0] \) are contained in a half-plane orthogonal to \([x, x_0]\). Therefore, there may be a gap between the point \( u \) that is the farthest from \( x_0 \), such that \( \tau \) coincides with \( \sigma_a \) on \( W_u \) and the point \( v \) that is nearest to \( x_0 \) such that \( \tau \) coincides with \( \sigma_{x_0} \) on \( W_v \). One could choose any point between \( v \) and \( u \) as \( \theta_f(a) \) in this case.

Lemma 5.3 may not be satisfied, due to the fact that not every point \( x \) is contained in a half-space \( h \) such that \( x \) is the closest point to \( x_0 \) in \( h \).

Still, the proof of Proposition 5.1 can be made to work, by covering the ball \( B(x_0, R') \) with finitely many cones \( C(a_i, \epsilon) \) composed of geodesics \([x_0, a]\) at an angle at most \( \epsilon \) from \([x_0, a_i]\), where \( a_i \in S(x_0, R') \), by considering, for each cone, only the set \( W_i \) of walls orthogonal to geodesics in the cone, by taking the set \( B_i \) of points that appear as intersections of \([x_0, a_i]\) with walls in \( W_i \), that is \([x_0, a_i] \cap \tilde{h} \cap h^\epsilon \), with \( \{h, h^\epsilon\} \in W_i \), and by taking \( b_1(i), \ldots, b_n(i) \) to be the extension of an \( \epsilon \)-net of \( B_i \).

With the finite set of walls defined as \( W_F = \bigcup_{i=1}^k \bigcup_{j=1}^n W_{b_j(i)} \), where \( W_{b_j(i)} \) is the wall intersecting the corresponding geodesic in \( b_j(i) \), the previous argument works, for \( \epsilon \) small enough. We leave the details as an exercise to the reader.

Proposition 5.1 combined with Corollary 4.14 imply the following.

**Corollary 5.6.** Let \( X \) be a finite product of finite dimensional real hyperbolic spaces. Every uniform lattice in \( \text{Isom}(X) \) acts on the median space \( \mathcal{M}(X) \) properly discontinuously and with compact quotient.
Remark 5.7. It follows from results of [Fio1, Fio4] that irreducible lattices as in Corollary 5.6 cannot act properly discontinuously cocompactly on a median space of finite rank. This implies that:

- the median space $\mathcal{M}(X)$ has infinite rank;
- the action described in Corollary 5.6 is the best type of action on a median space that can be found for such lattices.

It is natural to ask in what measure the previous results extend to infinite dimensional hyperbolic spaces (see Example 3.8). A number of important groups have interesting actions on these spaces, such as the groups of automorphisms of (products of) regular trees and their lattices (including the Burger-Mozes examples) [BIM05], the groups of birational transformations of complex Kähler surfaces [Ca11]. Even the groups of isometries of finite dimensional real hyperbolic spaces have actions on $\mathbb{H}^\infty$ that are deformations of the standard ones and present a number of more interesting features [MP14].

For the space $X = \mathbb{H}^\infty$ with its measured walls structure, a few of the previous arguments work. In particular:

1. for every $\lambda > 0$, $I_\lambda(x,y)$ is contained in $N_{\lambda + \delta}(I(x,y))$, for every $x, y$, where $\delta$ only depends on the hyperbolicity constant. Therefore, all the half-spaces are quasi-convex in the most general sense.

2. the half-spaces are quasi-gated only in the following sense: given a half-space $h$, a point $x$ outside $h$ and an $\varepsilon$-projection $p$ of $x$ onto $h$, every wall in $W(x|h) \setminus W(x|h)$ intersects a ball $B(p, \delta)$, with $\delta$ depending only on the hyperbolicity constant and on $\varepsilon$ (Lemma 4.9).

However, as $X$ is not measurably locally finite, the rest of the argument fails. Taking a limit of the finite dimensional case does no work either, as the bound we obtain for the distance to the associated median space in that case depends on volumes of balls that increase with the dimension. It is nevertheless natural to ask whether the proof of the result cannot be completed by different means.

Question 5.8. Is the infinite dimensional hyperbolic space at finite Hausdorff distance from the median space associated to it?

6. The complex hyperbolic space.

In the case of complex hyperbolic spaces $\mathbb{H}_C^n$, their hyperbolic metric $\text{dist}_{\mathbb{H}_C^n}$ cannot be induced by a wall structure (see the proof of Corollary 1.5 at the end of this section). The square root of this metric however is known to come from a wall structure (possibly also larger powers $\text{dist}_{\mathbb{H}_C^n}^\alpha$ with $\alpha \in (1/2, 1)$ might, but nothing is proven in this respect). Here we explain that no metric $\text{dist}_{\mathbb{H}_C^n}^\alpha$ can be $\delta$-tripodal, hence the wall space it would be induced by cannot be within bounded Hausdorff distance from a median space. This is explained in Proposition 6.2 below, which follows from simple considerations about snowflaked metrics. For the sake of completeness, we provide the entire (easy computational) argument here.
Lemma 6.1. Let $a \geq b \geq 0$, let $0 < \alpha < 1$ and $\beta > 1$. The following inequalities hold:

1. $(a + b)^\beta \geq a^\beta + b^\beta$;
2. $a^\alpha + b^\alpha - (a + b)^\alpha \geq b^\alpha(2 - 2^\alpha) \geq 0$.

Proof. The inequalities are trivial when $b = 0$. Thus we may assume that $b > 0$, in fact without loss of generality we may assume that $b = 1$ and $a \geq 1$.

1 follows from the fact that the function $f(x) = (x + 1)^\beta - x^\beta - 1$ is increasing for $x \geq 1$ and $f(1) = 2^\beta - 2 > 0$.

2 follows from the fact that the function $g(x) = x^\alpha + 1 - (x + 1)^\alpha$ is increasing for $x \geq 1$ and $g(1) = 2 - 2^\alpha$.

Proposition 6.2 (snowflaked metric spaces). Let $(X, \text{dist})$ be a metric space and let $0 < \alpha < 1$ be such that $\text{dist}^\alpha$ is a metric.

1. For every two points $x, y$ in $X$, the interval with respect to the metric $\text{dist}^\alpha$ reduces to $\{x, y\}$.

2. Assume that $X$ contains triples of points $x_n, y_n, z_n$ such that
   $$\lim_{n \to \infty} \min \{\text{dist}(x_n, y_n), \text{dist}(x_n, z_n), \text{dist}(y_n, z_n)\} = \infty.$$ 
   Then $(X, \text{dist}^\alpha)$ cannot be $\delta$–median, for any constant $\delta > 0$.

Proof. 1 We denote the distance $\text{dist}^\alpha$ by $d$, and $\frac{1}{\alpha}$ by $\beta$. We thus have that $\text{dist} = d^\beta$.

Let $z$ be a point between two points $x, y$ in $X$, with respect to the metric $d$. Thus
$$d(x, y) = d(x, y) + d(y, z),$$
whence

$$\text{dist}(x, y) = [d(x, z) + d(z, y)]^\beta \geq \text{dist}(x, z) + \text{dist}(z, y) \geq \text{dist}(x, y).$$

The first inequality in (3) follows from Lemma 6.1 (1). Note that equality holds if and only if the two numbers is zero.

Since the first and the last terms in (3) are equal, all inequalities become equalities, in particular from the above we can deduce that either $d(x, z) = 0$ or $d(z, y) = 0$.

2 The proof is a coarse version of the proof of (1). We argue by contradiction, and assume that $(X, d)$ is $\delta$–median, for some positive constant $\delta$.

For each triple $x_n, y_n, z_n$ with
$$d_n = \min \{\text{dist}(x_n, y_n), \text{dist}(x_n, z_n), \text{dist}(y_n, z_n)\}$$
diverging to infinity, consider a $\delta$–median point $m_n$ with respect to the metric $d$. For simplicity, in what follows we drop the index $n$.

Without loss of generality we can assume that $\text{dist}(m, x) \leq \text{dist}(m, y) \leq \text{dist}(m, z)$.

We can write
$$d(x, y) + \delta \geq d(x, m) + d(m, y) \geq [\text{dist}(x, m) + \text{dist}(m, y)]^\alpha \geq \text{dist}(x, y)^\alpha = d(x, y).$$

In the second inequality of (3), we used Lemma 6.1 (2).
Using (4) and again Lemma 6.1 (2), we can write that

\[(5) \quad \delta \geq \text{dist}(x, m)^\alpha + \text{dist}(m, y)^\alpha - \left[\text{dist}(x, m) + \text{dist}(m, y)\right]^\alpha \geq (2 - 2^\alpha)\text{dist}(m, x).\]

If we repeat the arguments in (4) and (5), with \(x, y\) replaced by \(y, z\), we obtain that

\[(2 - 2^\alpha)\text{dist}(m, y) \leq \delta,\]

whence

\[\text{dist}(x, y) \leq \frac{2\delta}{2 - 2^\alpha}.\]

This contradicts the fact that \(\text{dist}(x_n, y_n) \geq d_n\), and \(d_n\) diverges to infinity. \(\square\)

**Corollary 6.3.** Let \((X, \text{dist})\) be a geodesic Gromov hyperbolic metric space with boundary at infinity containing at least three points, and let \(\alpha < 1\) be such that \(\text{dist}^\alpha\) is a metric. Then \((X, \text{dist}^\alpha)\) cannot be \(\delta\)-tripodal for any \(\delta > 0\).

In particular if the snowflaked metric \(\text{dist}^\alpha\) is induced by a measured walls structure, then no isometric embedding of \((X, \text{dist}^\alpha)\) into a median metric space \(M\) has image within bounded distance of \(M\).

Consider now the hyperbolic space \(\mathbb{H}^n_\mathbb{C}\) with \(n \geq 2\), endowed with the hyperbolic distance \(\text{dist}\). Recall that \(\mathbb{H}^n_\mathbb{C}\) admits a structure of space with measured walls such that the induced distance is \(\text{dist}^{1/2}\). Possibly the exponent \(1/2\) can be increased to some other value \(\alpha \in (1/2, 1)\). In either case, the space with measured walls thus obtained cannot be within finite distance of a median space due to Proposition 6.2. Thus we can finish the proof of Corollary 1.5 stated in the introduction.

**Proof of Corollary 1.5** Indeed, if it were the case, then \(\mathbb{H}^n_\mathbb{C}\) would admit a convex subset with convex complementary, hence a convex hypersurface of codimension one, every convexity with respect to the hyperbolic distance. The existence of a hypersurface as described is impossible.

\(\square\) is an immediate consequence of Corollary 6.3

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