Error estimates for a certain class of elliptic optimal control problems

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Abstract

In this paper, error estimates are presented for a certain class of optimal control problems with elliptic PDE-constraints. It is assumed that in the cost functional the state is measured in terms of the energy norm generated by the state equation. The functional a posteriori error estimates developed by Repin in late 90’s are applied to estimate the cost function value from both sides without requiring the exact solution of the state equation. Moreover, a lower bound for the minimal cost functional value is derived. A meaningful error quantity coinciding with the gap between the cost functional values of an arbitrary admissible control and the optimal control is introduced. This error quantity can be estimated from both sides using the estimates for the cost functional value. The theoretical results are confirmed by numerical tests.

1 Introduction

This paper presents two-sided estimates for the value of the cost functional (assuming that the state equation can not be solved exactly) and shows how they can be used to generate estimates for a certain error quantity (cf. (3.13) and Theorem 3.4). In the case of unconstrained control, some estimates and numerical tests have been in presented in [4]. In [16], the case of “box constraints” is treated. Here, these results are extended considerably for constraints of more general type, a new error quantity is introduced, and the results are confirmed by numerical tests.

In section 2 definitions and standard results related to optimal control problems with elliptic state equation are recalled. Cost functionals are assumed to be of a certain type, where the state is measured in terms of the energy norm generated by the state equation. This is a special case of the general theory which can be found, e.g., from monographs [8, 17].

In section 3 the functional a posteriori error estimates (see monographs [13, 19, 10] and references therein) for the state equation are applied to generate two-sided bounds for the value of the cost functional. The strong connections between the estimates and the principal relations generating the optimal control problem are underlined. Theorem 3.4 (generalization of [16] Ch. 9, Th. 9.14 for the case of constrained control) is the analog of the Mikhlin identity (cf. Theorem 3.4) for the optimal control problem. It introduces a well motivated
error quantity and shows how the estimates for the cost function value can be used to generate two-sided bounds.

Some examples of optimal control problem of the type described in Sect. 2 are discussed in Sect. 4.1. Numerical tests in Sect. 4.3 depict how the estimates can be combined with an arbitrary (conforming) numerical method.

2 Elliptic optimal control problem

2.1 Definitions

Let $W$, $H$, and $U$ be Hilbert spaces. Their inner products and norms are denoted by subscripts, e.g., $(\cdot, \cdot)_W$ and $\| \cdot \|_W$. Moreover, $V \subset W$ is a Hilbert space generated by the inner product $(q, z)_V := (q, z)_W + (\Lambda q, \Lambda z)_H$, where $\Lambda : V \to H$ is a linear, bounded operator. The injection from $V$ to $W$ is continuous and $V$ is dense in $W$. Operator $\Lambda$ satisfies a Friedrichs type inequality

$$\| q \|_W \leq c \| \Lambda q \|_H, \quad \forall q \in V_0,$$

where a subspace $V_0 \subset V$ is closed. Assume $V_0 \subset V \subset V_0 \subset W \subset V_0^*$, where $V_0^*$ is the dual space of $V_0$.

Define linear bounded operators $B : U \to V_0^*$, $A : H \to H$, $N : U \to U$, where $A$ and $N$ are symmetric and positive definite,

$$c \| q \|_H^2 \leq (A q, q)_H \leq \bar{c} \| q \|_H^2, \quad \forall q \in H$$

and

$$k \| v \|_U^2 \leq (N v, v)_U \leq \bar{k} \| v \|_U^2, \quad \forall v \in U,$$

where $c$ and $\bar{c}$ ($k$ and $\bar{k}$) are positive constants. Thus, they generate inner products

$$(q, z)_A := (A q, z)_H, \quad (q, z)_{A-1} := (A^{-1} q, z)_H, \quad (v, w)_N := (N v, w)_U,$$

and the respective norms

$$\| q \|_A := \sqrt{(A q, q)_H}, \quad \| q \|_{A^{-1}} := \sqrt{(A^{-1} q, q)_H}, \quad \| v \|_N := \sqrt{(N v, v)_U}.$$

The adjoint operators $\Lambda^* : H \to V_0^*$ and $B^* : V_0 \to U^*$ are defined by the relations

$$\langle \Lambda^* z, q \rangle_{V_0} = (z, \Lambda q)_H, \quad \forall z \in H, \ q \in V_0$$

and

$$\langle B v, q \rangle_{V_0} = \langle v, B^* q \rangle_U, \quad \forall v \in U, \ q \in V_0,$$

where $\langle \cdot, \cdot \rangle_{V_0}$ denotes the pairing of $V_0$ and its dual space $V_0^*$. By the Riesz representation theorem, there exists an isomorphism (denoted, e.g., by $I_U : U \to U^*$) from any Hilbert space onto the corresponding dual space. The adjoint operator defines a subspace

$$Q := \{ q \in H | \Lambda^* q \in W \} \subset H.$$

The norm to $V_0^*$ is

$$\| \ell \| := \sup_{q \in V_0, q \neq 0} \frac{|\langle \ell, q \rangle_{V_0}|}{\| \Lambda q \|_A}.$$
Consider a bilinear form \( a : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R} \),
\[
a(q, z) := (A\Lambda q, \Lambda z)_{\mathcal{H}}.
\]
It is \( \mathcal{V}_0 \)-elliptic and continuous and generates an energy norm \( \| q \| := \sqrt{a(q, q)} \) in \( \mathcal{V}_0 \).

### 2.2 Optimal control problem

The state equation is
\[
a(y(v), q) = \langle \ell + Bv, q \rangle_{\mathcal{V}_0}, \quad \forall q \in \mathcal{V}_0,
\]
where \( \ell \in \mathcal{V}_0^*, \ v \in \mathcal{U}_{ad} \subset \mathcal{U} \) is the control, and \( y(v) \in \mathcal{V}_0 \) is the corresponding state. Let \( \mathcal{U}_{ad} \subset \mathcal{U} \) be a non-empty, convex, and closed set. The cost functional \( J : \mathcal{U} \rightarrow \mathbb{R} \) is
\[
J(v) := \| y(v) - y^d \|_{\mathcal{A}}^2 + \| v - u^d \|_{\mathcal{N}}^2,
\]
where \( u^d \in \mathcal{U} \) and \( y^d \in \mathcal{V}_0 \). The optimal control problem is to find \( u \in \mathcal{U}_{ad} \), such that
\[
J(u) \leq J(v), \quad \forall v \in \mathcal{U}_{ad}.
\]
Under earlier assumptions, \( J \) is \( \mathcal{U} \)-elliptic, coercive, and lower semi-continuous. Thus, the solution of the optimal control problem exists and is unique (see, e.g., [8, Chap. II, Th. 1.2]).

**Remark 2.1.** Cost functional of type
\[
J_2(v) := \| \Lambda y(v) - \sigma^d \|_{\mathcal{A}}^2 + \| v - u^d \|_{\mathcal{N}}^2,
\]
can be shifted using a projection: Find \( y^d \in \mathcal{V}_0 \) such that
\[
(\mathcal{A}(\Lambda y^d - \sigma^d), \Lambda q)_{\mathcal{W}} = 0, \quad \forall q \in \mathcal{V}_0.
\]
Then, \( J(v) = J_2(v) - \| \Lambda y^d - \sigma^d \|_{\mathcal{A}}^2 \)

The derivative of \( J \) at \( v \) is
\[
\langle J'(v), w \rangle_{\mathcal{U}} = \lim_{t \to 0^+} \frac{1}{t} \langle (J(v + tw) - J(v)), w \rangle_{\mathcal{U}} = 2(\mathcal{B}w, y(v) - y^d)_{\mathcal{V}_0} + (v - u^d, w)_{\mathcal{N}} = 2(\mathcal{I}_0^{-1}\mathcal{B}^*(y(v) - y^d) + \mathcal{N}(v - u^d), w)_{\mathcal{U}}.
\]
The necessary conditions for the optimal control problem (2.5) are (2.3) and
\[
\langle J'(v), v - u \rangle_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}
\]
(see, e.g., [8] Ch. I, Th. 1.3], [17] Le. 2.21], i.e.,
\[
(\mathcal{I}_0^{-1}\mathcal{B}^*(y(u) - y^d) + \mathcal{N}(u - u^d), v - u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.
\]
Note that for the cost functional of type (2.4), there is no need to define an adjoint state to present the necessary conditions (compare [8] Chap. II, Th. 1.14).

The following proposition (dating back to [12], see, e.g., [3] Chap. I, Pr. 2.2 or [2] Chap. 7, Pr. 7.4) allows to write (2.8) in a different form.
Proposition 2.1. Including the earlier assumptions, let $x \in U$. Then, the following conditions are equivalent,

(i) \((u - x, v - u)_{\mathcal{N}} \geq 0, \forall v \in U_{\text{ad}}\),

(ii) \(|x - u|_{\mathcal{N}} = \inf_{v \in U_{\text{ad}}} |x - v|_{\mathcal{N}},\)

(iii) \(u = \Pi_{\text{ad}}^N x, \) where \(\Pi_{\text{ad}}^N : U \to U_{\text{ad}} \) is a projection.

Proof. Assume (i). The identity
\[
|x - v|_{\mathcal{N}}^2 - |x - u|_{\mathcal{N}}^2 = |u - v|_{\mathcal{N}}^2 + 2(u - x, v - u)_{\mathcal{N}} \geq 0
\]
leads at
\[
|x - u|_{\mathcal{N}} \leq |v - x|_{\mathcal{N}} \quad \text{for arbitrary } v \in U_{\text{ad}}, \quad \text{i.e., (ii)}.
\]
Assume (ii). Let \(v \in U_{\text{ad}}\) be arbitrary and \(t \in (0, 1)\), then by the convexity of \(U_{\text{ad}}\)
\[
|x - u|_{\mathcal{N}}^2 \leq |x - (1 - t)u + tv|_{\mathcal{N}}^2 = ||(x - u) + t(u - v)||_{\mathcal{N}}^2.
\]
Expanding the right side leads at
\[
2t(x - u, u - v)_{\mathcal{N}} \leq t^2 ||(u - v)||_{\mathcal{N}}^2, \quad \text{tending } t \text{ to zero yields (i)}.
\]
Conditions (ii) and (iii) equal by definition.

Remark 2.2. Typical choice is \(\mathcal{N} = \alpha \text{Id}\), where \(\alpha > 0\) and \(\text{Id}\) denotes the identity mapping. Then (2.9) becomes
\[
u = \Pi_{\text{ad}}^N (u^d - N^{-1}I_u^{-1}B^*(y(u) - y^d)).
\]

Remark 2.3. If \(U_{\text{ad}} = U\), then \(\Pi_{\text{ad}}^N = \text{Id}\) and (2.9) reduces to
\[
u = u^d - N^{-1}I_u^{-1}B^*(y(u) - y^d).
\]

Substituting (2.10) to (2.3) yields a following linear problem: Find \(y(u) \in V_0\) satisfying
\[
a(y(u), z) + \langle BN^{-1}I_u^{-1}B^*y(u), z \rangle_{V_0} = \langle \ell + Bu^d, z \rangle_{V_0} + \langle BN^{-1}I_u^{-1}B^*y^d, z \rangle_{V_0}, \quad \forall z \in V_0.
\]

3 Estimates

3.1 Estimates for the state equation

The solution \(y(v) \in V_0\) of (2.3) minimizes a quadratic energy functional (see, e.g., [8, Chapter I, Theorem 1.2 and Remark 1.5]), i.e.,
\[
E(y(v)) \leq E(q) := \|q\|^2 - 2(\ell + Bv, q)_{V_0}, \quad \forall q \in V_0.
\]

The benefit for measuring \(y(v) - y^d\) in the \(\| \cdot \|\) -norm in (2.4) (instead of, e.g., \(\| \cdot \|_{W}-\text{norm}\)) is due to the following results (Theorem 3.1 is due to [11] and generalized in [10]).
Theorem 3.1. Let $y(v)$ be the solution of (3.1) and $z \in V_0$ be arbitrary, then
\[
\| y(v) - z \|^2 = E(z) - E(y(v)).
\] (3.2)

Proof. By (2.3),
\[
\| y(v) - z \|^2 = \| y(v) \|^2 - 2a(y(v), z) + \| z \|^2
- 2(\ell(y(v), y(v)) + (\ell + Bv, y(v)))_{V_0}
= - \| y(v) \|^2 + 2(\ell + Bv, z)_{V_0} + \| z \|^2 - 2(\ell + Bv, y(v))_{V_0}
= E(z) - E(y(v)).
\]

\[\square\]

Theorem 3.2. Let $y(v)$ be the solution of (3.1) and $z \in V_0$ be arbitrary, then
\[
\sup_{q \in V_0} \mathcal{M}^2(z, q, v) = \| y(v) - z \|^2\inf_{\beta > 0} \mathcal{M}^2(z, \tau, \beta, v),
\] (3.3)

where
\[
\mathcal{M}^2(z, q, v) := E(z) - E(q)
\] and
\[
\mathcal{M}^2(z, \tau, \beta, v) := (1 + \beta)\| \tau - A\Lambda z \|^2_{A^{-1}} + \frac{1 + \beta}{\beta} |A^*\tau + Bv + \ell|^2.
\] (3.4)

Proof. $\mathcal{M}^2$ is obtained directly from (3.1) and (3.2). For $\overline{\mathcal{M}}^2$, see, e.g., [13 Chap. 6, (6.2.3)], [16 Chap. 7, (7.1.19)]. Upper bounds of more general type have been presented already in [14] [15].

\[\square\]

Remark 3.1. It is easy to confirm that the supremum over $\mathcal{M}^2$ is obtained at $q = y(v)$ and the infimum over $\overline{\mathcal{M}}^2$ is attained at $\tau = A\Lambda y(v)$ and $\beta \to 0$.

3.2 Estimates for the cost functional

Applying Theorem 3.2 to the first term of (2.4), leads to two-sided bounds for $J(v)$. These bounds are guaranteed, have no gap, and do not depend on $y(v)$, i.e., they do not require the solution of the state equation.

Theorem 3.3. For any $v \in U$,
\[
\sup_{q \in V_0} J(v, q) = J(v) = \inf_{\tau \in V} \overline{J}(v, \tau, \beta),
\] (3.5)

where
\[
J(v, q) := \mathcal{M}^2(y^d, q, v) + \| v - u^d \|^2_X
\] (3.6)

and
\[
\overline{J}(v, \tau, \beta) := \mathcal{M}^2(y^d, \tau, \beta, v) + \| v - u^d \|^2_X.
\] (3.7)
Theorem 3.3 can be used to estimate \( J(u) \). By \( (2.5) \) and \( (3.5) \),
\[
\inf_{v \in U_{ad}} J(v, q) \leq J(u) \leq J(v, \tau, \beta), \quad \forall q \in V_0, \ v \in U, \ \tau \in H, \ \beta > 0,
\]
where all inequalities hold as equalities if \( v = u, \ q = y(u), \ \tau = A\Lambda y(u), \) and \( \beta \to 0 \). In view of \( (3.5) \), it is very important that the minimizer of \( \mathcal{J}(v, q) \) over \( v \in U_{ad} \) can be explicitly computed. Computation of the minimizers of \( \mathcal{J} \) require further assumptions of the structure of the problem (cf. Propositions 4.1 and 4.2).

**Proposition 3.1.** For all \( v \in U_{ad} \) and \( q \in V_0 \),
\[
\mathcal{J}(\hat{v}(q), q) = \inf_{v \in U_{ad}} J(v, q),
\]
\[
\mathcal{J}(v, \hat{q}(v)) = \sup_{q \in V_0} J(v, q),
\]
where \( \hat{q}(v) = y(v) \) (from \( (2.3) \)) and
\[
\hat{v}(q) := \Pi_{ad}^N (u^d + N^{-1}B^*(y^d - q)).
\]

**Proof.** The condition \( \hat{q}(v) = y(v) \) follows directly from Remark 3.1.

By \( (3.1), \ (3.3), \) and \( (3.6), \ \mathcal{J} \) has the following form
\[
\mathcal{J}(v, q) = \| y^d \|^2 - 2\langle \ell, y^d \rangle - \| q \|^2 + 2\langle \ell, q \rangle + 2(Bv, q - y^d)v_n + \| v - u^d \|_N^2
\]
\[
= \| v \|_N^2 - 2\langle v, u^d \rangle_N - 2(Bv, y^d - q)v_n - \| q \|^2 + 2\langle \ell, q \rangle + 2(Bv, q)v_n + \text{const}.
\]

Clearly, it is quadratic w.r.t \( v \) and the minimizer \( \hat{v} \in U_{ad} \) is identified by the following variational inequality (see, e.g., [K, Chap. 1, Th. 1.2]):
\[
(\hat{v}, v - \hat{v})_N \geq (v - \hat{v}, u^d)_N + \langle B(v - \hat{v}), y^d - q \rangle_{V_n}, \quad \forall v \in U_{ad}.
\]

Reorganizing and \( (2.2) \) yields
\[
(\hat{v} - u^d + N^{-1}B^*(q - y^d), v - \hat{v})_N \geq 0, \quad \forall v \in U_{ad},
\]
and Proposition 2.1 leads at \( (3.10) \).

**Remark 3.2.** By \( (3.5) \) and \( (3.8) \), \( \mathcal{J}(v, \tau, \beta) \) is an upper bound of \( J(u) \) for all \( v \in U_{ad}, \ \tau \in Q, \) and \( \beta > 0 \) and \( \mathcal{J}(v, q) \) is a lower bound for \( J(v) \) for all \( q \in U_{ad} \), but it is a lower bound of \( J(u) \) only if \( v = \hat{v}(q) \) (see \( (3.10) \)).

**Remark 3.3.** Lower bound \( \mathcal{J} \) generates a saddle point formulation for the original optimal control problem \( (2.7) \). Find \( (\hat{v}, \hat{q}) \) satisfying
\[
\mathcal{J}(\hat{v}, \hat{q}) \leq \mathcal{J}(\hat{v}, q) \leq \mathcal{J}(v, q), \quad \forall v \in U_{ad}, q \in V_0.
\]

Note that \( \mathcal{J} \) is convex, lower semi-continuous, and coercive w.r.t. \( v \) and concave, upper semi-continuous, and anti-coercive w.r.t. \( q \). \( U_{ad} \) is convex, closed, and non-empty, and \( V_0 \) is convex, closed, and non-empty. Thus, the solution of \( (3.11) \) exists and is unique (see, e.g., [K, Chap. VI, Pr. 2.4]). By Remark 3.1, \( \hat{v} = u \) and \( \hat{q} = y(u) \). Moreover, \( \hat{v}(y(u)) = u \), where \( \hat{v} \) is defined in \( (3.10) \). The left and right-hand-side of \( (3.11) \) yield \( (3.1) \) and \( (2.8) \) (i.e., necessary conditions \( (2.3) \) and \( (2.7) \), respectively).
Remark 3.4. By \((3.8)\), \(J(u) \leq J(v) \leq \mathcal{J}(v, \tau, \beta)\) and it is easy to see that \(J(u) = \lim_{\beta \to 0} \mathcal{J}(u, \mathcal{A}y(u), \beta)\). Thus, the upper bound generates a minimization problem
\[
\mathcal{J}(u, \mathcal{A}y(u), 0) = \min_{v \in \mathcal{U}_{\text{ad}}} \mathcal{J}(v, \tau, \beta),
\]
where the constraint related to \((2.3)\) does not appear.

3.3 Estimates for an error quantity

The following identity can be viewed as an analog of \((3.1)\) for the optimal control problem.

**Theorem 3.4.** For any \(v \in \mathcal{U}_{\text{ad}},\)
\[
\|y(v) - y(u)\|^2 + \|v - u\|_N^2 + \langle J'(v), v - u \rangle_U = J(v) - J(u).
\]  
(3.12)

**Proof.** We have,
\[
J(v) - J(u) = \|y(v) - y(u)\|^2 + 2a(y(v) - y(u), y(u) - y^d) + \|v - u\|_N^2 + 2(v - u, v - u^d)_N.
\]
By \((2.3)\) and \((2.6)\),
\[
a(y(v) - y(u), y(u) - y^d) = \langle \mathcal{B}(v - u), y(u) - y^d \rangle_{V_0}
\]
and
\[
2a(y(v) - y(u), y(u) - y^d) + 2(v - u, v - u^d)_N = \langle J'(u), v - u \rangle_U.
\]

**Remark 3.5.** If \(\mathcal{U}_{\text{ad}} = \mathcal{U}\), then \(\langle J'(u), v \rangle_U = 0\), for all \(v \in \mathcal{U}\) and \((3.12)\) reduces to \((16)\), Ch. 9, Th. 9.14].

Equality \((3.12)\) shows that it is reasonable to include \(\langle J'(u), v - u \rangle_U\) to the applied error measure. Obviously, \(\langle J'(u), v - u \rangle_U\) is positive for any \(v \in \mathcal{U}_{\text{ad}}\) by \((2.7)\), it is convex and vanishes if \(v = u\). Thus, the error measure is
\[
\mathrm{err}^2(v) := \|y(v) - y(u)\|^2 + \|v - u\|_N^2 + \langle J'(u), v - u \rangle_U.
\]  
(3.13)

The “derivative weight” guarantees that the sensitivity of the cost functional at the optimal control is taken into account. Most importantly, \(\mathrm{err}(v)\) can be estimated from both sides by computable functionals, which do not require the knowledge of the optimal control \(u\), the respective state \(y(u)\), or the exact state \(y(v)\). Indeed, applying \((3.5)\), \((3.8)\), and \((3.9)\) to the right hand side of \((3.12)\) yields the following theorem:

**Theorem 3.5.** For any \(v \in \mathcal{U}_{\text{ad}},\)
\[
\sup_{q \in V_0} \inf_{v_2 \in \mathcal{U}_{\text{ad}}} \mathrm{err}^2(v, q, v_2, \tau, \beta) = \mathrm{err}^2(v) = \inf_{v_2 \in \mathcal{U}_{\text{ad}}} \mathrm{err}^2(v, \tau_2, \beta_2, q_2),
\]  
(3.14)

where
\[
\mathrm{err}^2(v, q, v_2, \tau, \beta) := J(v, q) - J(v_2, \tau, \beta)
\]
and
\[
\mathrm{err}^2(v, \tau_2, \beta_2, q_2) := J(v, \tau_2, \beta_2) - J(v(q_2), q_2).
\]
Remark 3.6. By Remark 3.2, 3.6, 3.7, and 3.14, the equality 3.14 is attained at
\[
\text{err}^2(v, y(v), u, \mathcal{A}y(u), 0) = \text{err}^2(v) = \text{err}^2(v, \mathcal{A}y(v), 0, y(u)).
\]

Remark 3.7. Obviously \( J(v) \) and \( \text{err}^2(v) \) are positive. However, e.g., the lower bound \( \tilde{J}(v_2) \) for \( J(u) \) may be negative if \( q_2 \) is not close enough to \( y(u) \) and \( \text{err}^2(v, q, v_2, \tau, \beta) \) may be negative value if \( v_2 \) is not “good enough” in comparison with \( v \), or the upper bound \( \tilde{J}(v_2, \tau, \beta) \) is not “sharp enough”.

4 Examples, algorithms and numerical tests

4.1 Examples

In the following examples, the domain \( \Omega \subset \mathbb{R}^d \) is open, simply connected and has a piecewise Lipschitz-continuous boundary \( \Gamma \). Spaces are \( W = L^2(\Omega) \), \( V = H^1(\Omega) \), \( H = L^2(\Omega, \mathbb{R}^d) \), and \( Q = H(\text{div}, \Omega) \). Operators are \( \Lambda = \nabla \), \( \Lambda^* = -\text{div} \), \( A = \text{Id} \), and \( N = \alpha \text{Id} \) \((\alpha > 0)\). Then \( a(q, z) := (\nabla q, \nabla z)_{L^2(\Omega, \mathbb{R}^d)} \) and \( \| w \| = \|\nabla w\|_{L^2(\Omega, \mathbb{R}^d)} \). The examples differ only by the selection of \( \mathcal{V}_0, \mathcal{U}, \mathcal{B} \), and \( \ell \).

4.1.1 Dirichlet problem, distributed control

Let \( U := L^2(\Omega), \mathcal{V}_0 := H^1_0(\Omega) \), and \( \langle \ell, w \rangle = (f, w)_{L^2(\Omega)} \), where \( f \in L^2(\Omega) \). Moreover, \( B = \text{Id} \), i.e., \( (Bv, q) = (v, q)_{L^2(\Omega)} \). The analog of (2.1) is the Friedrichs inequality
\[
\| q \|_{L^2(\Omega)} \leq c_0 \| \nabla q \|_{L^2(\Omega, \mathbb{R}^d)}, \quad \forall q \in H^1_0(\Omega).
\]

The cost functional (2.4) is
\[
J(v) := \| \nabla (y(v) - y^d) \|_{L^2(\Omega, \mathbb{R}^d)}^2 + \alpha \| v - u^d \|_{L^2(\Omega)}^2. \tag{4.1}
\]

The state equation (2.3) is
\[
(\nabla y(v), \nabla z)_{L^2(\Omega, \mathbb{R}^d)} = (f + v, z)_{L^2(\Omega)}, \quad \forall z \in H^1_0(\Omega) \tag{4.2}
\]
and it has the classical form
\[
\begin{cases}
-\Delta y(v) = f + v & \text{a.e. in } \Omega, \\
y(v) = 0 & \text{on } \Gamma.
\end{cases}
\]

The majorant (3.4) is
\[
M^2(q, \tau, \beta, v) = (1 + \beta)\| \tau - \nabla z \|_{L^2(\Omega, \mathbb{R}^d)}^2 + \frac{1 + \beta}{\beta} \varepsilon^2 \| \text{div} \tau + f + v \|_{L^2(\Omega)}^2.
\]

The counterpart of the Proposition 3.1 is below.

Proposition 4.1. For all \( v \in \mathcal{U}_d, \tau \in H(\text{div}, \Omega) \), and \( \beta > 0 \)
\[
\begin{align*}
\mathcal{J}(v(\tau, \beta), \tau, \beta) &= \inf_{\mathcal{U}_d} \mathcal{J}(v, \tau, \beta), \\
\mathcal{J}(v(\tau, \beta), \beta) &= \inf_{\tau \in H(\text{div}, \Omega)} \mathcal{J}(v(\tau, \beta), \beta), \\
\mathcal{J}(v(\tau, \beta), \tau) &= \inf_{\beta > 0} \mathcal{J}(v(\tau, \beta), \tau),
\end{align*}
\]
where
\[ \hat{v}(\tau, \beta) = \Pi_{\text{ad}} \left( \frac{\alpha \beta}{(1+\beta)^2} u^d - \text{div} \tau - \hat{f} \right), \] (4.3)
\[ \hat{v} := \hat{v}(\tau, \beta) \text{satisfies} \]
\[ \beta(\hat{v}, \xi)_{L^2(\Omega, \mathbb{R}^3)} + \frac{c_\Omega^2}{(1+\beta)^2} \langle \text{div} \tau, \text{div} \xi \rangle_{L^2(\Omega)} \]
\[ = \beta(\nabla y^d, \xi)_{L^2(\Omega, \mathbb{R}^3)} + \frac{c_\Omega^2}{1+\beta} (f + \text{div} \xi)_{L^2(\Omega)}, \quad \forall \xi \in H(\text{div}, \Omega), \quad (4.4) \]
and
\[ \hat{\beta}(v, \tau) = \frac{c_\Omega \| \text{div} \tau + f + v \|_{L^2(\Omega)}}{\| \tau - \nabla y^d \|_{L^2(\Omega, \mathbb{R}^3)}}. \] (4.5)

**Proof.** The upper bound \( \mathcal{J} \) can be rewritten as follows,
\[ \mathcal{J}(v, \tau, \beta) = (1+\beta)\|\tau - \nabla z\|_{L^2(\Omega, \mathbb{R}^3)}^2 + \frac{1+\beta}{\beta} c_\Omega^2 \| \text{div} v + f \|_{L^2(\Omega)}^2 + \alpha \| v - u^d \|_{L^2(\Omega)}^2 \]
\[ = \left( \frac{1+\beta}{\beta} c_\Omega^2 + \alpha \right) \| v \|_{L^2(\Omega)}^2 - 2 \left( \alpha u^d - \frac{1+\beta}{\beta} c_\Omega^2 (\text{div} \tau + f), v \right)_{L^2(\Omega)} + \text{const w.r.t. } v. \]
Thus, the minimizer \( \hat{v} \in U_{\text{ad}} \) satisfies
\[ \left( \frac{1+\beta}{\beta} c_\Omega^2 + \alpha \right) (\hat{v}, w - \hat{v})_{L^2(\Omega)} \geq \left( \alpha u^d - \frac{1+\beta}{\beta} c_\Omega^2 (\text{div} \tau + f), w - \hat{v} \right)_{L^2(\Omega)}, \quad \forall w \in U_{\text{ad}}. \]
Reorganizing leads at
\[ (\hat{v} - \frac{\alpha \beta}{(1+\beta)^2} u^d + \text{div} \tau + f, w - \hat{v})_{L^2(\Omega)}, \quad \forall w \in U_{\text{ad}}, \]
and Proposition 2.1 yields (4.3).

Condition (4.4) can be easily derived, since \( M^2 \) is quadratic w.r.t. \( \tau \in H(\text{div}, \Omega) \) and (4.5) results from solving a one-dimensional minimization problem.

The relation (3.10) becomes
\[ \hat{v}(q) = \Pi_{\text{ad}} (u^d + \frac{1}{\alpha} (y^d - q)), \] (4.6)
where \( \Pi_{\text{ad}} : L^2(\Omega) \rightarrow U_{\text{ad}} \) is a projection.

**Example 4.1.** If \( U_{\text{ad}} = L^2(\Omega) \), then by (2.11) \( y(u) \in H^1_0(\Omega) \) satisfies
\[ (\nabla y(u), \nabla z)_{L^2(\Omega, \mathbb{R}^3)} + \frac{1}{\alpha} (y(u), z)_{L^2(\Omega)} \]
\[ = (f + \frac{1}{\alpha} y^d + u^d, z)_{L^2(\Omega)}, \quad \forall z \in H^1_0(\Omega) \] (4.7)
and \( \Pi_{\text{ad}} = \text{Id} \) in (4.3) and (4.6).

**Example 4.2.** Let
\[ U_{\text{ad}} = \{ v \in L^2(\Omega) \mid \psi_- \leq v \leq \psi_+ \quad \text{a.e in } \Omega \}, \] (4.8)
then the projection operator \( \Pi_{\text{ad}} : L^2(\Omega) \rightarrow U_{\text{ad}} \) is
\[ \Pi_{\text{ad}} v = \min \{ \psi_+, \max \{ \psi_-, v \} \}. \]
Example 4.3. Let

\[ U_{ad} = \{v \in L^2(\Omega) \mid \|v\|_{L^2(\Omega)} \leq M\}, \]

then the projection operator \( \Pi_{ad} : L^2(\Omega) \to U_{ad} \) is

\[
\Pi_{ad} v = \begin{cases} 
\frac{M v}{\|v\|_{L^2(\Omega)}} & \text{if } \|v\|_{L^2(\Omega)} > M, \\
0 & \text{else}
\end{cases}
\]

Finally, functional a posteriori error estimates for the problem (4.7) are recalled. (see, e.g., [16, Ch. 4.2], and [10, Ch. 3.2]).

Theorem 4.1. Let \( y \) be the solution of (4.7) and \( z \in H^1_0(\Omega) \), then

\[
\|\nabla (y - z)\|_{L^2(\Omega, \mathbb{R}^d)}^2 + \frac{1}{\alpha} \|q - q\|_{L^2(\Omega)}^2 = \inf_{(\tau, \beta, \nu) \in \mathcal{M}(z, \beta, \nu)} \mathcal{M}(z, \tau, \beta, \nu),
\]

where

\[
\mathcal{M}(z, \tau, \beta, \nu) := (1 + \beta)\|\nabla z - \tau\|_{L^2(\Omega, \mathbb{R}^d)}^2 + \left(1 + \beta \right) \left( \frac{1}{\alpha} \|\nu\|_{L^2(\Omega)} \right)^2 + \beta \left( \frac{1}{\alpha} \|\mathcal{R}(z, \tau)\|_{L^2(\Omega)} \right)^2
\]

and

\[
\mathcal{R}(z, \tau) = \text{div} \tau - \frac{1}{\alpha} z + f + \frac{1}{\alpha} y^d + u^d.
\]

4.1.2 Neumann problem, boundary control

The boundary \( \Gamma \) consists of two parts \( \Gamma_N \cup \Gamma_D \), where \( \Gamma_D \) has a positive measure. By the trace theorem there exists a bounded linear mapping \( \gamma : H^1_0(\Omega) \to L^2(\Gamma_N) \),

\[
\|\gamma q\|_{L^2(\Gamma_N)} \leq c \|q\|_{H^1(\Omega)},
\]

such that \( \gamma v = v|_{\Gamma} \) for all \( v \in C^1(\bar{\Omega}) \). Let \( U := L^2(\Gamma_N) \) and

\[
\nabla_0 := V_0 := \{w \in H^1(\bar{\Omega}) \mid w \text{ has zero trace on } \Gamma_D\}.
\]

Moreover, \( \langle Bv, q \rangle = \langle v, q \rangle_{L^2(\Gamma_N)} \) and \( \langle \ell, q \rangle = (f, q)_{L^2(\Omega)} - (g, \gamma q)_{L^2(\Gamma_N)} \), where \( f \in L^2(\Omega) \) and \( g \in L^2(\Gamma_N) \).

The cost functional (2.4) is

\[
J(v) := \|\nabla (y(v) - y^d)\|_{L^2(\Omega)}^2 + \alpha \|q - u^d\|_{L^2(\Gamma_N)}^2,
\]

and the state equation (3.1) is

\[
(\nabla y(v), \nabla q)_{L^2(\Omega, \mathbb{R}^d)} = (f, q)_{L^2(\Omega)} + (g + v, \gamma q)_{L^2(\Gamma_N)}, \quad \forall q \in V_0.
\]

It has the classical form

\[
\begin{align*}
-\Delta y(v) &= f \quad \text{a.e. in } \Omega, \\
y(v) &= 0 \quad \text{on } \Gamma_D, \\
\frac{\partial y(v)}{\partial n} &= g + v \quad \text{on } \Gamma_N.
\end{align*}
\]
Remark 4.1. with the lower bound of (2.3) is generated, e.g., by finite elements or Fourier series. The approximations from \( U \) and \( \mathcal{Q}_h \) coincide with any existing numerical scheme, which generates approximations of the optimal control (and/or state). Computation of the derived estimates requires some finite dimensional subspaces. Hereafter, assume that constants satisfy

\[
\|q\|_{L^2(\Omega)} \leq c_{\Omega,2}\|\nabla q\|_{L^2(\Omega,\mathbb{R}^2)} \quad \text{and} \quad \|q\|_{L^2(\Gamma_N)} \leq c_{\Gamma_N}\|\nabla q\|_{L^2(\Omega,\mathbb{R}^2)}, \quad \forall q \in V_0.
\]

Proposition 4.2. For all \( q \in H_0^1(\Omega) \), \( \tau \in H(\text{div}, \Omega) \), and \( \beta > 0 \)

\[
\mathcal{J}(q, \hat{\tau}, \hat{\beta}) = \inf_{\tau \in H(\text{div}, \Omega)} \mathcal{J}(q, \tau, \beta),
\]

\[
\mathcal{J}(q, \tau, \hat{\beta}) = \inf_{\beta > 0} \mathcal{J}(q, \tau, \beta),
\]

where \( \hat{\tau} \) satisfies

\[
\beta(\hat{\tau}, \xi)_{L^2(\Omega,\mathbb{R}^d)} + c_{\Omega}^2(\text{div} \hat{\tau}, \text{div} \xi)_{L^2(\Omega)} + c_{\Omega}^2(\frac{\partial \tau}{\partial n}, \frac{\partial \xi}{\partial n})_{L^2(\Gamma_N)} = \beta(\nabla q, \xi)_{L^2(\Omega,\mathbb{R}^d)} + c_{\Omega}^2(f + v, \text{div} \xi)_{L^2(\Omega)} + c_{\Omega}^2(g + v, \frac{\partial \xi}{\partial n})_{L^2(\Gamma_N)}, \quad \forall \xi \in H(\text{div}, \Omega)
\]

and

\[
\hat{\beta} = \left( c_{\Omega,2}^2\|\text{div} \tau + f\|_{L^2(\Omega)}^2 + c_{\Gamma_N}^2\|\frac{\partial \tau}{\partial n} + g + v\|_{L^2(\Gamma_N)}^2 \right)^{1/2} / \|\tau - \nabla q\|_{L^2(\Omega,\mathbb{R}^2)}.
\]

4.2 Algorithms

The results of Sect. 3 give grounds for several error estimation Algorithms. Note that the estimates in Theorems 3.3 and 3.5 are valid for any approximations from \( U_{ad} \). There is no need for Galerkin orthogonality, extra regularity, or mesh dependent data. Thus they can be combined with any existing numerical scheme, which generates approximations of the optimal control (and/or state). Computation of the derived estimates requires some finite dimensional subspaces. Hereafter, assume that \( U_{ad} \subset U \subset V_0 \subset V_{ad} \) and \( Q^h \subset Q \) are given. They can be generated, e.g., by finite elements or Fourier series. The approximate solution of (2.3) is \( y^h(v) \in V_0^h \subset V_0 \) that satisfies

\[
a(y^h(v), z) = (Bv + \ell, z)_{V_0}, \quad \forall z \in V_0^h. \quad (4.9)
\]

Remark 4.1. By Remark 3.2, the evaluation of (the approximation of) \( \mathcal{J}(v) \) by computing \( y^h(v) \) from (4.9) and \( J_h(v) := \| y^h(v) - y^d \|^2 + \| v - w^d \|_{\lambda_N}^2 \) coincides with the lower bound \( \mathcal{J}(v, y^h(v)) = \max_{y \in V_0^h} \mathcal{J}(v, y) \).

The generation of the estimates for the cost function value \( \mathcal{J}(v) \) for a given approximation \( v \in U_{ad} \) is depicted as Algorithm 1.

In order to test the presented error estimates, a projected gradient method (see, e.g., [5, 7]) is applied to generate a sequence approximations. Method consists of line searches along (anti)gradient directions, where all evaluated points are first projected to the admissible set. A projected gradient method with
Algorithm 1 Generation of bounds for the cost functional value

input: $v \in U_{ad}$ {approximation of the control}, $V_h^0$ {subspace for state}, $Q_h$ {subspace for the flux of state}, $I_{max}$ {maximum number of iterations}, $\varepsilon$ {stopping criteria}

$y^h = \argmax_{y \in V_h^0} J(v, y)$ {compute $y^h(v)$ from (4.9)}

$\hat{v}^h = \argmin_{v \in U_{ad}} J(v, y^h)$ {compute $\hat{v}(y^h)$ by (3.10)}

$\beta^0 = 1$

for $k = 1$ to $I_{max}$ do

$\tau^k = \argmin_{\tau \in \tau^h} J(v, \tau^k, \beta^{k-1})$

$\beta^k = \argmin_{\beta > 0} J(v, \tau^k, \beta)$

if $J(v, \tau^{k-1}, \beta^{k-1}) - J(v, \tau^k, \beta^k) < \varepsilon$ then break

end if

end for

$J_h(v) = J(v, y^h)$

$\overline{J}_h(v) = J(v, \tau^k, \beta^k)$

$\underline{J}_h(u) = J(\hat{v}^h, y^h)$

output: $\underline{J}_h(u)$ {lower bound for $J(v)$}, $\overline{J}_h(v)$ {upper bound for $J(v)$}, $\underline{J}_h(u)$ {lower bound for $J(u)$}, $\overline{J}_h(u)$ {upper bound for $J(u)$}

error estimates is depicted as Algorithm 2. At the beginning of every projected gradient step Algorithm 1 is used to generate approximations for the cost functional. After the execution of Algorithm 2 ($N$ iteration steps taken), cost estimates are recalled to generate two-sided estimates for $err(v)$ (i.e., the difference $J(v) - J(u)$) at each iteration step ($k = 1, \ldots, N$) as follows:

$err^2(v^k) \geq \underline{J}_h^2(v^k) - \overline{J}_h(v^N)$

$err^2(v^k) \leq \overline{J}_h(v^k) - \underline{J}_h^2(u)$

Note that the iterate of the last step ($N$’th step) is used to generate as accurate bounds as possible for $J(u)$.

4.3 Numerical tests

Finite dimensional subspaces are generated by the finite element method (see, e.g., [2]). In these tests, $U = L^2(\Omega)$, $V_0 = H^1_0(\Omega)$, and $Q = H(div, \Omega)$. Subspaces $DG^p_h \subset L^2(\Omega)$, $V^p_h \subset H^1_0(\Omega)$, and $RT^p \subset H(div, \Omega)$ are generated by Discontinuous Galerkin elements, Lagrange elements, and Raviart-Thomas elements, respectively. Superscripts $p$ denote the order of basis functions. All the numerical tests were performed using FEniCS (see [9] Ch. 3 for detailed descriptions of the applied elements and for additional references).

Example 4.4. Let $\Omega = (0, 1)^2$. Consider the optimal control problem generated by (4.7), (4.8), and $U_{ad}$ defined by (4.9), where $\psi_\epsilon(x_1, x_2) = -3$ and
Algorithm 2 Projected gradient method with guaranteed cost estimates

\textbf{input:} $v^0 \in U_{ad}$ \{initial approximation of the control\}, $V_h^0$ \{subspace for state\}, $Q_h^b$ \{subspace for the flux of state\}, $I_{PG}^{max}$ \{maximum number of iterations (projected gradient)\}, $\varepsilon_{PG}$ \{stopping criteria (projected gradient)\}, $I_{max}$ \{maximum number of iterations ($J$ minimization)\}, $\varepsilon$ \{stopping criteria ($J$ minimization)\}

\begin{verbatim}
for $k = 0$ to $I_{PG}^{max}$ do
\{ GenerateCostEstimates($v^k$, $V_h^0$, $Q_h^b$, $I_{max}$, $\varepsilon$) \}
\quad $d^k = 2 (B^T y^d - y(v^k))$ \{ search direction \}
\quad $s^k \arg\min_{0 \leq \lambda \leq \lambda_{max}} J_h (\Pi_{ad} (v^k + \lambda d^k))$ \{ step length (golden section method) \}
\quad $v^{k+1} = v^k + s^k d^k$ \{ update approximation \}
\quad if $\|v^k - v^{k-1}\| < \varepsilon_{PG}$ then
\quad\quad break
\end{verbatim}

\textbf{output:} \{$v^k\}_{k=1}^N$ \{sequence of approximations\}, \{$J_h(v^k)\}_{k=1}^N$ \{lower bounds for $J(v^n)$\}, \{$J_h(v^k)\}_{k=1}^N$ \{upper bounds for $J(v^n)$\}, $J_h^m(u)$ \{lower bound for $J(u)$\}

$\psi_+(x_1, x_2) = 3$. Select
\begin{align*}
y(x_1, x_2) &= \sin(k_1 \pi x_1) \sin(k_2 \pi x_2), \\
y^d(x_1, x_2) &= \sin(k_1 \pi x_1) \sin(k_2 \pi x_2) + \beta \sin(m_1 \pi x_1) \sin(m_2 \pi x_2), \\
u^d(x_1, x_2) &= 0 \\
u(x_1, x_2) &= \max \left\{ \psi_-(x_1, x_2), \min \left\{ \psi_+(x_1, x_2), \frac{\pi}{k} \sin(m_1 \pi x_1) \sin(m_2 \pi x_2) \right\} \right\} \\
f(x_1, x_2) &= \pi^2 (k_1^2 + k_2^2) \sin(k_1 \pi x_1) \sin(k_2 \pi x_2) - u(x_1, x_2),
\end{align*}

where $k_1, k_2, m_1, m_2 \in \mathbb{Z}$ and $\beta \in \mathbb{R}$.

In Example 4.3, select $k_1 = 1, k_2 = 1, m_1 = 2, m_2 = 1, \beta = 0.5$, and $\alpha = 0.05$. A mesh of 50×50 cells divided to triangular elements is being used. Consider first linear elements, i.e., $p_1 = p_2 = p_3 = 1$, the amount of corresponding global degrees of freedom are $\dim(DG_h^1) = 15000$, $\dim(V_h^1) = 2601$, and $\dim(RT_h^1) = 7600$. The bounds generated by Algorithm 2 ($I_{PG}^{max} = 10$) are depicted in Figure 1. If the order of approximation for state and flux are increased, i.e., subspaces $V_h$ and $Q_h$ are enhanced, then the accuracy of error bounds improves significantly (see Fig. 2). Here $\dim(V_h^2) = 10201$ and $\dim(RT_h^2) = 25200$.

In previous examples, $J(v)$ and $J(u)$ (and other integrals also) were computed using a uniformly refined mesh and 121 integration points in each triangle.

Obviously, the negative lower bound for the error could be rejected immediately. Sharp lower bound requires a very good approximation of the optimal control $v \approx u$ and the corresponding flux of the respective state $\tau \approx \nabla y(u)$. Then the upper bound $J(u) \leq J(v) \leq J(v, \tau, \beta)$ would be very efficient. However, ten steps of the projected gradient method does not provide a very accurate
Figure 1: Estimates for the cost function value (top) and the error quantity (bottom), where subspaces for control, state, and flux are $\text{DG}_h^1$, $V_h^1$, and $\text{RT}_h^1$, respectively.
Figure 2: Estimates for the cost function value (top) and the error quantity (bottom), where subspaces for control, state, and flux are $\text{DG}_h^1$, $V_h^2$, and $\text{RT}_h^2$, respectively.
approximation. It is a matter of further numerical tests to apply more efficient approximation methods (see, e.g., [6]) and to apply the element wise contributions of the error estimates to generate adaptive sequences of subspaces.

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