IMPROVED INEQUALITIES FOR THE NUMERICAL RADIUS VIA CARTESIAN DECOMPOSITION

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Abstract. We develop various lower bounds for the numerical radius $w(A)$ of a bounded linear operator $A$ defined on a complex Hilbert space, which improve the existing inequality $w^2(A) \geq \frac{1}{4} \|A^*A + AA^*\|$. In particular, for $r \geq 1$, we show that
\[
\frac{1}{4} \|A^*A + AA^*\| \leq \frac{1}{2} \left( \frac{1}{2} \|\Re(A) + \Im(A)\|^2 + \frac{1}{2} \|\Re(A) - \Im(A)\|^2 \right)^{\frac{1}{2}} \leq w^2(A),
\]
where $\Re(A)$ and $\Im(A)$ are the real and imaginary parts of $A$, respectively. Furthermore, we obtain upper bounds for $w^2(A)$ refining the well-known upper bound $w^2(A) \leq \frac{4}{2} \left( w(A^2) + \|A\|^2 \right)$. Separate complete characterizations for $w(A) = \frac{4}{\sqrt{2}}$ and $w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|}$ are also given.

1. Introduction

The purpose of the present article is to obtain improvements of the existing well-known upper and lower bounds for the numerical radius of bounded linear operators acting on Hilbert spaces in terms of their real and imaginary parts. This is in a continuation of the study done in recent article [4]. Let us first introduce some notations and terminologies.

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ induced by the inner product. Let $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with the identity $I$. Let $A \in \mathbb{B}(\mathcal{H})$. We denote by $|A| = (A^*A)^{\frac{1}{2}}$ the positive square root of $A$, and $\Re(A) = \frac{1}{2}(A + A^*)$ and $\Im(A) = \frac{1}{2i}(A - A^*)$, respectively, stand for the real and imaginary parts of $A$. The numerical range of $A$, denoted as $W(A)$, is defined by $W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$. We denote by $\|A\|$, $c(A)$, and $w(A)$ the operator norm, the Crawford number, and the numerical radius of $A$, respectively. Recall that
\[
c(A) = \inf \{ |\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \}
\]

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and
\[
\text{w}(A) = \sup \{ |\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \}.
\]
It is well known that the numerical radius \(w(\cdot)\) defines a norm on \(\mathbb{B}(\mathcal{H})\) and is equivalent to the operator norm \(\| \cdot \|\). In fact, the following double inequality holds:
\[
\frac{1}{2} \|A\| \leq \text{w}(A) \leq \|A\|.
\] (1.1)
The inequalities in (1.1) are sharp. The first inequality becomes equality if \(A^2 = 0\), and the second one turns into equality if \(A\) is normal. Over the years, many mathematicians have obtained various refinements of (1.1), we refer the reader to [1, 2, 9, 12, 13, 14] and references therein. In particular, Kittaneh [10] improved the inequalities in (1.1) by establishing that
\[
\frac{1}{4} \|A^*A + AA^*\| \leq \text{w}^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|.
\] (1.2)
In this paper, we obtain several refinements of the first inequality in (1.2), in terms of \(\|\Re(A) + \Im(A)\|\) and \(\|\Re(A) - \Im(A)\|\). Furthermore, we obtain upper bounds for the numerical radius of bounded linear operators improving the existing inequality \(\text{w}^2(A) \leq \frac{1}{2}(\text{w}(A^2) + \|A\|^2)\) obtained by Dragomir [8, Th. 1].

2. Main Results

We start our work with the following observation that for every \(A \in \mathbb{B}(\mathcal{H})\),
\[
\frac{1}{4} \|A^*A + AA^*\| = \frac{1}{4} \norm{(\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2}.
\] (2.1)
First by using the identity (2.1), we obtain the following improvement of the first inequality in (1.2).

**Theorem 2.1.** If \(A \in \mathbb{B}(\mathcal{H})\), then
\[
\frac{1}{4} \|A^*A + AA^*\|
\leq \frac{1}{4} \|\Re(A) + \Im(A)\|^2 + \frac{1}{4} \|\Re(A) - \Im(A)\|^2
\leq \frac{1}{4} \|\Re(A) + \Im(A)\|^2 + \frac{1}{4} \|\Re(A) - \Im(A)\|^2 + \frac{1}{4} c^2(\Re(A) + \Im(A)) + \frac{1}{4} c^2(\Re(A) - \Im(A))
\leq \text{w}^2(A).
\]

**Proof.** It follows from (2.1) that
\[
\frac{1}{4} \| A^* A + AA^* \| = \frac{1}{4} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \| \\
\leq \frac{1}{4} \| \Re(A) + \Im(A) \|^2 + \frac{1}{4} \| \Re(A) - \Im(A) \|^2.
\]

This is the first inequality, and the second follows trivially.

Now we prove the third inequality. Let \( x \in \mathcal{H} \) with \( \| x \| = 1 \). Then from the Cartesian decomposition of \( A \), we get

\[
| \langle Ax, x \rangle |^2 = \langle \Re(A)x, x \rangle^2 + \langle \Im(A)x, x \rangle^2 \\
= \frac{1}{2} (\langle \Re(A)x, x \rangle + \langle \Im(A)x, x \rangle)^2 + \frac{1}{2} (\langle \Re(A)x, x \rangle - \langle \Im(A)x, x \rangle)^2 \\
= \frac{1}{2} (\langle \Re(A) + \Im(A) \rangle x, x)^2 + \frac{1}{2} (\langle \Re(A) - \Im(A) \rangle x, x)^2.
\]

Therefore, we have the following two inequalities:

\[
\frac{1}{2} c^2 (\Re(A) + \Im(A)) + \frac{1}{2} \| \Re(A) - \Im(A) \|^2 \leq w^2(A) 
\tag{2.2}
\]

and

\[
\frac{1}{2} c^2 (\Re(A) - \Im(A)) + \frac{1}{2} \| \Re(A) + \Im(A) \|^2 \leq w^2(A). 
\tag{2.3}
\]

It follows from (2.2) and (2.3) that

\[
\frac{1}{4} \| \Re(A) + \Im(A) \|^2 + \frac{1}{4} \| \Re(A) - \Im(A) \|^2 + \frac{1}{4} c^2 (\Re(A) + \Im(A)) + \frac{1}{4} c^2 (\Re(A) - \Im(A)) \leq w^2(A).
\]

\[\square\]

Clearly, Theorem 2.1 refines the first inequality in (1.2). Now, the following corollary is trivially inferred from Theorem 2.1.

**Corollary 2.2.** If \( A \in \mathbb{B}(\mathcal{H}) \), then

\[
\frac{1}{4} \| A^* A + AA^* \| + \frac{1}{4} c^2 (\Re(A) + \Im(A)) + \frac{1}{4} c^2 (\Re(A) - \Im(A)) \leq w^2(A).
\]

Also, the following result easily derived from (2.2) and (2.3).

**Corollary 2.3.** If \( A \in \mathbb{B}(\mathcal{H}) \), then \( w^2(A) \geq \max \{ \beta_1, \beta_2 \} \), where

\[
\beta_1 = \frac{1}{2} c^2 (\Re(A) + \Im(A)) + \frac{1}{2} \| \Re(A) - \Im(A) \|^2,
\]

\[
\beta_2 = \frac{1}{2} c^2 (\Re(A) - \Im(A)) + \frac{1}{2} \| \Re(A) + \Im(A) \|^2.
\]
Remark 2.4. (i) We have

$$\max \{ \beta_1, \beta_2 \}$$

$$= \frac{1}{2} \left\{ \frac{c^2(\Re(A) + \Im(A))^2 + c^2(\Re(A) - \Im(A))^2 + \| \Re(A) + \Im(A) \|^2}{2} \right\}$$

$$+ \frac{1}{2} \left\{ \frac{\| \Re(A) + \Im(A) \|- \Re(A) + \Im(A) \|^2}{2} \right\}$$

$$\geq \frac{1}{4} \left\{ c^2(\Re(A) + \Im(A)) + c^2(\Re(A) - \Im(A)) \right\} + \frac{1}{4} \| \Re(A) - \Im(A) \|^2 + \| \Re(A) + \Im(A) \|^2$$

$$+ \frac{1}{4} \left\{ \| \Re(A) + \Im(A) \| - \Re(A) - \Im(A) \|^2 + c^2(\Re(A) - \Im(A)) - c^2(\Re(A) + \Im(A)) \right\}$$

$$= \frac{1}{4} \| A^* A + AA^* \| + \frac{1}{4} c^2(\Re(A) + \Im(A)) + \frac{1}{4} c^2(\Re(A) - \Im(A))$$

$$+ \frac{1}{4} \left\{ \| \Re(A) + \Im(A) \|^2 - \| \Re(A) - \Im(A) \|^2 + c^2(\Re(A) - \Im(A)) - c^2(\Re(A) + \Im(A)) \right\}.$$

Thus,

$$w^2(A) \geq \frac{1}{4} \| A^* A + AA^* \| + \frac{1}{4} c^2(\Re(A) + \Im(A)) + \frac{1}{4} c^2(\Re(A) - \Im(A))$$

$$+ \frac{1}{4} \left\{ \| \Re(A) + \Im(A) \|^2 - \| \Re(A) - \Im(A) \|^2 + c^2(\Re(A) - \Im(A)) - c^2(\Re(A) + \Im(A)) \right\}.$$

(ii) Also, we remark that Corollary 2.3 is stronger than the recently obtained inequality in [3, Th. 2.3].

To prove the next refinement of the first inequality in (1.2), we need the following lemma, which can be found in [4, Th. 2.17].

**Lemma 2.5.** Let $A, D \in \mathbb{B}(\mathcal{H})$. Then

$$\| A + D \|^2 \leq \| A \|^2 + \| D \|^2 + \frac{1}{2} \| A^* A + D^* D \| + w(A^* D)$$

and

$$\| A + D \|^2 \leq \| A \|^2 + \| D \|^2 + \frac{1}{2} \| AA^* + DD^* \| + w(AD^*).$$

**Theorem 2.6.** If $A \in \mathbb{B}(\mathcal{H})$, then

$$\frac{1}{4} \| A^* A + AA^* \|$$

$$\leq \frac{1}{4} \left\{ \frac{3}{2} \| \Re(A) + \Im(A) \|^4 + \frac{3}{2} \| \Re(A) - \Im(A) \|^4 + \| \Re(A) + \Im(A) \|^2 \| \Re(A) - \Im(A) \|^2 \right\}^{\frac{1}{2}}$$

$$\leq w^2(A).$$
Proof. It follows from (2.1) that
\[
\frac{1}{16} \| A^* A + AA^* \|^2 \\
= \frac{1}{16} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \|^2 \\
\leq \frac{1}{16} \left\{ \| \Re(A) + \Im(A) \|^4 + \| \Re(A) - \Im(A) \|^4 + \frac{1}{2} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \| \right\} \\
\quad + \frac{1}{16} w((\Re(A) + \Im(A))^2(\Re(A) - \Im(A))^2),
\] (using Lemma 2.5)
\[
\leq \frac{1}{16} \left\{ \| \Re(A) + \Im(A) \|^4 + \| \Re(A) - \Im(A) \|^4 + \frac{1}{2} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \| \right\} \\
\quad + \frac{1}{16} \| \Re(A) + \Im(A) \|^2 \| \Re(A) - \Im(A) \|^2 \\
= \frac{1}{16} \left\{ \frac{3}{2} \| \Re(A) + \Im(A) \|^4 + \frac{3}{2} \| \Re(A) - \Im(A) \|^4 + \| \Re(A) + \Im(A) \|^2 \| \Re(A) - \Im(A) \|^2 \right\} \\
\leq w^4(A),
\]
where the last inequality is deduced from (2.4) and (2.5). \qed

Now, we state a lemma.

Lemma 2.7. [2, Th. 2.2] Let $A, D \in \mathbb{B}(\mathcal{H})$. Then
\[
\| A + D \|^2 \leq 2 \max \{ \| A^* A + D^* D \|, \| AA^* + DD^* \| \}.
\]

Based on the above lemma, we obtain the following refinement of the first inequality in (1.2).

Theorem 2.8. If $A \in \mathbb{B}(\mathcal{H})$, then
\[
\frac{1}{4} \| A^* A + AA^* \| \leq \frac{1}{2\sqrt{2}} \left( \| \Re(A) + \Im(A) \|^4 + \| \Re(A) - \Im(A) \|^4 \right)^{\frac{1}{2}} \leq w^2(A).
\]

Proof. It follows from (2.1) that
\[
\frac{1}{4} \| A^* A + AA^* \| &= \frac{1}{4} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \| \\
&\leq \frac{1}{2\sqrt{2}} \left( \| \Re(A) + \Im(A) \|^4 + \| \Re(A) - \Im(A) \|^4 \right)^{\frac{1}{2}}, \quad \text{(by Lemma 2.7)} \\
&\leq \frac{1}{2\sqrt{2}} \left( \| \Re(A) + \Im(A) \|^4 + \| \Re(A) - \Im(A) \|^4 \right)^{\frac{1}{2}} \\
&\leq w^2(A),
\]
where we deduce the last inequality from (2.4) and (2.5). \qed
We observe here that the convexity of the function $f(t) = t^2$ ensures that the first inequality in Theorem 2.8 is better than the first inequality in Theorem 2.1. Also, we observe that the second inequality in Theorem 2.8 is better than the second inequality in Theorem 2.6.

In the next theorem, we obtain another improvement of (1.2). First we note that (2.2) and (2.3) imply the following two inequalities, respectively:

$$\frac{1}{2} \|\Re(A) - \Im(A)\|^2 \leq w^2(A)$$

and

$$\frac{1}{2} \|\Re(A) + \Im(A)\|^2 \leq w^2(A).$$

Now, by employing the convexity property of the function $f(t) = t^r$, $r \geq 1$, in the first inequality in Theorem 2.1 and using inequalities (2.4) and (2.5), we get the following inequality.

**Theorem 2.9.** If $A \in \mathbb{B}(\mathcal{H})$, then for $r \geq 1$,

$$\frac{1}{4} \|A^*A + AA^*\| \leq \frac{1}{2} \left( \frac{1}{2} \|\Re(A) + \Im(A)\|^{2r} + \frac{1}{2} \|\Re(A) - \Im(A)\|^{2r} \right)^{\frac{1}{r}} \leq w^2(A).$$

**Remark 2.10.** Clearly, Theorem 2.9 is a generalization of Theorem 2.8. We would like to remark that the second inequality in Theorem 2.9 gives more refinement as $r$ increases.

To prove our next result, we need the following lemma, which can be found in [7].

**Lemma 2.11.** Let $A, D \in \mathbb{B}(\mathcal{H})$ be positive. Then

$$\|A + D\| \leq \max \{\|A\|, \|D\|\} + \|AD\|^{\frac{1}{2}}.$$

**Theorem 2.12.** If $A \in \mathbb{B}(\mathcal{H})$, then

$$\frac{1}{4} \|A^*A + AA^*\|$$

$$\leq \frac{1}{4} \left[ \max \{\|\Re(A) + \Im(A)\|^2, \|\Re(A) - \Im(A)\|^2\} + \|\Re(A) + \Im(A)\| \|\Re(A) - \Im(A)\| \right]$$

$$\leq w^2(A).$$
Proof. Equality (2.1) and Lemma 2.11 ensure that
\[ \frac{1}{4} \| A^* A + AA^* \| \]
\[ = \frac{1}{4} \| (\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2 \| \]
\[ \leq \frac{1}{4} \left[ \max \left\{ \| \Re(A) + \Im(A) \|^2, \| \Re(A) - \Im(A) \|^2 \right\} + \| (\Re(A) + \Im(A))^2(\Re(A) - \Im(A))^2 \| \right]^{\frac{1}{2}} \]
\[ \leq \frac{1}{4} \left[ \max \left\{ \| \Re(A) + \Im(A) \|^2, \| \Re(A) - \Im(A) \|^2 \right\} + \| (\Re(A) + \Im(A)) \| \| (\Re(A) - \Im(A)) \| \right] \]
\[ \leq w^2(A), \]
in which we employ (2.4) and (2.5). \qed

We now concentrate our study on the equality of the first inequality in (1.2).

Corollary 2.13. Let \( A \in \mathbb{B}(\mathcal{H}) \). If \( w^2(A) = \frac{1}{4} \| A^* A + AA^* \| \), then the following assertions hold:

(i) There exists a sequence \( \{ x_n \} \) in \( \mathcal{H} \) with \( \| x_n \| = 1 \) such that
\[ \lim_{n \to \infty} |\langle \Re(A)x_n, x_n \rangle| = \lim_{n \to \infty} |\langle \Im(A)x_n, x_n \rangle|. \]

(ii) \( \| \Re(A) + \Im(A) \|^2 = \| \Re(A) - \Im(A) \|^2 = \frac{1}{2} \| A^* A + AA^* \|. \)

Proof. Let \( w^2(A) = \frac{1}{4} \| A^* A + AA^* \|. \) It follows from Theorem 2.1 that \( c(\Re(A) + \Im(A)) = c(\Re(A) - \Im(A)) = 0. \) This implies that there exist sequences \( \{ y_n \} \) and \( \{ z_n \} \) in \( \mathcal{H} \) with \( \| y_n \| = \| z_n \| = 1 \) such that \( \lim_{n \to \infty} \langle (\Re(A) + \Im(A))y_n, y_n \rangle = 0 \) and \( \lim_{n \to \infty} \langle (\Re(A) - \Im(A))z_n, z_n \rangle = 0. \) Thus (i) holds.

Also, from (i) of Remark 2.4, we have \( \| \Re(A) + \Im(A) \|^2 = \| \Re(A) - \Im(A) \|^2. \) In addition, we conclude from Theorem 2.9 that \( \| \Re(A) + \Im(A) \|^2 = \frac{1}{2} \| A^* A + AA^* \|, \) which yields (ii). \qed

Considering the matrix \( A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \), we conclude that the converse of Corollary 2.13, is not true, in general.

Remark 2.14. Considering the following two examples, we observe that the bounds obtained in Theorems 2.6 and 2.12 (also, Theorems 2.8 and 2.12) are not comparable, in general.

(i) Let \( A = \begin{pmatrix} 2 + 2i & 0 \\ 0 & 0 \end{pmatrix} \). Then \( \Re(A) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \Im(A) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \). Clearly,
∥ℜ(A) + ℑ(A)∥ = 4 and ∥ℜ(A) − ℑ(A)∥ = 0. By simple calculations, we have
\[
\frac{1}{2\sqrt{2}} \left( \|ℜ(A) + ℑ(A)\|^4 + \|ℜ(A) − ℑ(A)\|^4 \right)^{\frac{1}{2}} = 4\sqrt{2} \approx 5.65685424949,
\]
\[
\frac{1}{4} \left\{ \frac{3}{2}\|ℜ(A) + ℑ(A)\|^4 + \frac{3}{2}\|ℜ(A) − ℑ(A)\|^4 + \|ℜ(A) + ℑ(A)\|^2\|ℜ(A) − ℑ(A)\|^2 \right\}^{\frac{1}{2}}
= 2\sqrt{6} \approx 4.89897948557,
\]
\[
\frac{1}{4} \left[ \max \left\{ \|ℜ(A) + ℑ(A)\|^2, \|ℜ(A) − ℑ(A)\|^2 \right\} + \|ℜ(A) + ℑ(A)\|\|ℜ(A) − ℑ(A)\| \right] = 4.
\]
(ii) Let \(A = \begin{pmatrix} 3 + 2i & 0 \\ 0 & 4i \end{pmatrix} \). Then \(ℜ(A) = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \) and \(ℑ(A) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \). Therefore,
\[
\|ℜ(A) + ℑ(A)\| = 5 \text{ and } \|ℜ(A) − ℑ(A)\| = 4. \text{ By simple calculations, we get}
\]
\[
\frac{1}{2\sqrt{2}} \left( \|ℜ(A) + ℑ(A)\|^4 + \|ℜ(A) − ℑ(A)\|^4 \right)^{\frac{1}{2}} = \frac{1}{2\sqrt{2}} \sqrt{881} \approx 10.4940459309,
\]
\[
\frac{1}{4} \left\{ \frac{3}{2}\|ℜ(A) + ℑ(A)\|^4 + \frac{3}{2}\|ℜ(A) − ℑ(A)\|^4 + \|ℜ(A) + ℑ(A)\|^2\|ℜ(A) − ℑ(A)\|^2 \right\}^{\frac{1}{2}}
= \frac{1}{4} \sqrt{\frac{17215}{10}} \approx 10.37274071,
\]
\[
\frac{1}{4} \left[ \max \left\{ \|ℜ(A) + ℑ(A)\|^2, \|ℜ(A) − ℑ(A)\|^2 \right\} + \|ℜ(A) + ℑ(A)\|\|ℜ(A) − ℑ(A)\| \right]
= \frac{45}{4} = 11.25.
\]
Now, we obtain an upper bound for the numerical radius of bounded linear operators.
The following inequality is known as the Buzano inequality.

**Lemma 2.15.** ([6]) Let \(x, y, e \in \mathcal{H} \) with \(\|e\| = 1\). Then
\[
|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} (\|x\|\|y\| + \|x\|\|y\|).
\]

**Theorem 2.16.** Let \(A \in \mathcal{B}(\mathcal{H})\). Then
\[
w^2(A) \leq \frac{1}{2} \left[ \|A\|^2 \left( \min_{t \in [0,1]} \|tA^*A + (1-t)AA^*\| \right) + w^2(A^2) + w(A^2)\|A^*A + AA^*\| \right]^{\frac{1}{2}}.
\]
Proof. Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Then

\[
| \langle Ax, x \rangle |^2
= | \langle Ax, x \rangle \langle A^*x, x \rangle |
\leq \frac{1}{2} \left( | \langle Ax, A^*x \rangle | + \| Ax \||A^*x\| \right) \quad \text{(using Lemma 2.15)}
\]

\[
= \frac{1}{2} \left\{ |\langle Ax, A^*x \rangle|^2 + \| Ax \|^2 \| A^*x \|^2 + 2 |\langle Ax, A^*x \rangle| \| Ax \||A^*x\| \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ |\langle A^2x, x \rangle|^2 + \langle A^*Ax, x \rangle \langle AA^*x, x \rangle + |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \left\{ |\langle A^2x, x \rangle|^2 + \langle A^*Ax, x \rangle \langle AA^*x, x \rangle + |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
+ |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ |\langle A^2x, x \rangle|^2 + \langle (tA^*A + (1 - t)AA^*)x, x \rangle \langle ((1 - t)A^*A + tAA^*)x, x \rangle \right. \]

\[
+ |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}} \quad \text{(by the McCarthy inequality)}
\]

\[
\leq \frac{1}{2} \left\{ |\langle A^2x, x \rangle|^2 + \|tA^*A + (1 - t)AA^*\| \| (1 - t)A^*A + tAA^* \|ight.
\]

\[
+ |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ |\langle A^2x, x \rangle|^2 + \|tA^*A + (1 - t)AA^*\| \| A \|^2 \right. \]

\[
+ |\langle A^2x, x \rangle| \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ w^2(A^2) + \|tA^*A + (1 - t)AA^*\| \| A \|^2 \right. \]

\[
+ \langle A^2x, x \rangle \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ w^2(A^2) + \|tA^*A + (1 - t)AA^*\| \| A \|^2 + \langle A^2x, x \rangle \langle (AA^* + A^*A)x, x \rangle \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left\{ w^2(A^2) + \|tA^*A + (1 - t)AA^*\| \| A \|^2 + w(A^2) \| AA^* + A^*A \| \right\}^{\frac{1}{2}}
\].

Taking supremum over \( \|x\| = 1 \), we get

\[
w^2(A) \leq \frac{1}{2} \left[ \|A\|^2 (\|tA^*A + (1 - t)AA^*\|) + w^2(A^2) + w(A^2) \| AA^* + AA^* \| \right]^{\frac{1}{2}}.
\]

This holds for all \( t \in [0, 1] \), so considering minimum over \( t \in [0, 1] \), we have

\[
w^2(A) \leq \frac{1}{2} \left[ \|A\|^2 \left( \min_{t \in [0, 1]} \|tA^*A + (1 - t)AA^*\| \right) + w^2(A^2) + w(A^2) \| AA^* + AA^* \| \right]^{\frac{1}{2}},
\]

as required. \( \Box \)
Remark 2.17. (i) Dragomir [8, Th. 1] proved that for $A \in \mathbb{B}(\mathcal{H})$, 

$$
w^2(A) \leq \frac{1}{2} \left( \| A \|^2 + w(A^2) \right). \tag{2.6}
$$

We would like to remark that the inequality in Theorem 2.16 is sharper than that in Dragomir’s result [8, Th. 1].

(ii) Now, we consider $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then, by simple calculations, we arrive at

$$
\frac{1}{2} \left[ \| A \|^2 \left( \min_{t \in [0,1]} \| tA^*A + (1-t)AA^* \| \right) + w^2(A^2) + w(A^2)\| A^*A + AA^* \| \right]^{\frac{1}{2}} = \frac{3}{4}
$$

and

$$
\frac{1}{2} \| A^*A + AA^* \| = 1.
$$

Again, considering another example $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$, we have

$$
\frac{1}{2} \left[ \| A \|^2 \left( \min_{t \in [0,1]} \| tA^*A + (1-t)AA^* \| \right) + w^2(A^2) + w(A^2)\| A^*A + AA^* \| \right]^{\frac{1}{2}} = \sqrt{5}
$$

and

$$
\frac{1}{2} \| A^*A + AA^* \| = 2.
$$

Thus, we conclude that the second inequality in (1.2) and our obtained inequality in Theorem 2.16 are not comparable, in general.

In the following theorem, we present another refinement of (2.6).

Theorem 2.18. Let $A \in \mathbb{B}(\mathcal{H})$. Then

$$
w^4(A) \leq \frac{1}{4} \left[ w^2(A^2) + \frac{1}{4} \| (A^*A)^2 + (AA^*)^2 \| + \frac{1}{2}w(A^*A^2A^*) + w(A^2)\| A^*A + AA^* \| \right].
$$
**Proof.** Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \). It follows from Lemma 2.15 that
\[
\langle A^* Ax, x \rangle \langle AA^* x, x \rangle = \langle A^* Ax, x \rangle \langle, AA^* x, x \rangle \\
\leq \frac{\|A^* Ax\| \|AA^* x\| + \langle AA^* x, A^* Ax \rangle}{2} \\
\leq \frac{1}{4} (\|A^* Ax\|^2 + \|AA^* x\|^2) + \frac{1}{2} \|A^* A^2 A^* x, x \rangle \\
= \frac{1}{4} \langle((A^*)^2 + (AA^*)^2) x, x \rangle + \frac{1}{2} \|A^* A^2 A^* x, x \rangle \\
\leq \frac{1}{4} \|(A^*)^2 + (AA^*)^2\| + \frac{1}{2} w(A^* A^2 A^*).
\]
Following the proof of Theorem 2.16, we infer that
\[
|\langle Ax, x \rangle|^4 \\
\leq \frac{1}{4} \{ |\langle A^2 x, x \rangle|^2 + \langle A^* Ax, x \rangle \langle AA^* x, x \rangle + |\langle A^2 x, x \rangle| \langle((AA^* + A^*)x, x \rangle \} \\
\leq \frac{1}{4} \{ w^2(A^2) + \langle A^* Ax, x \rangle \langle AA^* x, x \rangle + w(A^2) \|AA^* + A^* A\| \} \\
\leq \frac{1}{4} \{ w^2(A^2) + \frac{1}{4} \|(A^*)^2 + (AA^*)^2\| + \frac{1}{2} w(A^* A^2 A^*) + w(A^2) \|AA^* + A^* A\| \}.
\]
Considering supremum over \( \|x\| = 1 \), we arrive at the desired inequality. \(\square\)

**Remark 2.19.** Clearly, for \( A \in \mathcal{B}(\mathcal{H}) \), we have
\[
\frac{1}{4} \left[ w^2(A^2) + \frac{1}{4} \|(A^*)^2 + (AA^*)^2\| + \frac{1}{2} w(A^* A^2 A^*) + w(A^2) \|A^* A + AA^*\| \right] \\
\leq \frac{1}{4} \left[ w^2(A^2) + \frac{1}{2} \|A\|^4 + \frac{1}{2} \|A^* A^2 A^*\| + 2w(A^2) \|A\|^2 \right] \\
\leq \frac{1}{4} \left[ w^2(A^2) + \frac{1}{2} \|A\|^4 + \frac{1}{2} \|A\|^4 + 2w(A^2) \|A\|^2 \right] \\
= \left[ \frac{\|A\|^2 + w(A^2)}{2} \right]^2.
\]
Therefore, Theorem 2.18 refines inequality (2.6).

In our next theorem, we obtain an inequality involving norm and numerical radius of a bounded linear operator. First we recall the following well-known identity from [15, p. 85]:
\[
w(A) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta} A)\|.
\] (2.7)

**Theorem 2.20.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then
\[
w^3(A) \leq \frac{1}{4} \left[ w^3(A) + \|A\| \|A^2\| + w(A) \|A^* A + AA^*\| \right].
\]
Proof. By a short calculation, we get

\[ \Re^3(A) = \frac{1}{4} \Re(A^3) + \frac{1}{8} (A^2A^* + A^{*2}A) + \frac{1}{4} (A^*A + AA^*) \Re(A). \]

Since \( \Re(A) \) is selfadjoint, we have

\[ \|\Re(A)\|^3 = \left\| \frac{1}{4} \Re(A^3) + \frac{1}{8} (A^2A^* + A^{*2}A) + \frac{1}{4} (A^*A + AA^*) \Re(A) \right\| \]
\[ \leq \frac{1}{4} \|\Re(A^3)\| + \frac{1}{8} \|A^2A^* + A^{*2}A\| + \frac{1}{4} \|A^*A + AA^*\| \|\Re(A)\| \]
\[ \leq \frac{1}{4} \|\Re(A^3)\| + \frac{1}{4} \|A^2\| \|A\| + \frac{1}{4} \|A^*A + AA^*\| \|\Re(A)\|. \]

Now let \( \theta \in \mathbb{R} \). Replacing \( A \) with \( e^{i\theta}A \) in the last inequality yields that

\[ \|\Re(e^{i\theta}A)\|^3 \leq \frac{1}{4} \|\Re(e^{3i\theta}A^3)\| + \frac{1}{4} \|A^2\| \|A\| + \frac{1}{4} \|A^*A + AA^*\| \|\Re(e^{i\theta}A)\|. \]

Taking supremum over all \( \theta \in \mathbb{R} \) and using identity (2.7), we derive that

\[ w^3(A) \leq \frac{1}{4} \left[ w(A^3) + \|A\| \|A^2\| + w(A) \|A^*A + AA^*\| \right], \]

as desired.

**Remark 2.21.** Let \( A \in \mathbb{B}(\mathcal{H}) \) with \( A \neq 0 \) and \( A^2 = 0 \). It follows from Theorem 2.20 that

\[ w(A) \leq \frac{1}{2} \sqrt{\|A^*A + AA^*\|}. \]

This inequality combined with the first inequality in (1.2) ensures that

\[ w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|}. \]

It should be mentioned that the reverse part is not true, in general. To see this, consider the matrix \( A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Then one can easily verify that \( w(A) = \frac{1}{\sqrt{2}} = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \), but \( A^2 \neq 0 \).

At the end of the article, we give separate complete characterizations for \( w(A) = \frac{1}{2} \|A\| \) and \( w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \). First we need the following lemma. Its proof can be found in [5, Th. 2.14].

**Lemma 2.22.** Let \( A \in \mathbb{B}(\mathcal{H}) \). Then

(i) \( w(A) = \frac{1}{2} \|A\| \) if and only if \( \overline{W(A)} \) is a circular disk with center at the origin and radius \( \frac{1}{2} \|A\| \);
(ii) \( w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \) if and only if \( \overline{W(A)} \) is a circular disk with center at the origin and radius \( \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \).

**Theorem 2.23.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then

(i) \( w(A) = \frac{1}{2}\|A\| \) if and only if \( w(A + \lambda I) = \frac{1}{2}\|A\| + |\lambda| \) for all \( \lambda \in \mathbb{C} \);

(ii) \( w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \) if and only if \( w(A + \lambda I) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} + |\lambda| \) for all \( \lambda \in \mathbb{C} \).

**Proof.** (i) The sufficient part is trivial, so we only prove the necessary part. Let \( w(A) = \frac{1}{2}\|A\| \). Clearly, \( \overline{W(A + \lambda I)} = \overline{W(A)} + \lambda \) for all \( \lambda \in \mathbb{C} \). Therefore, it follows from Lemma 2.22 that \( \overline{W(A + \lambda I)} \) is a circular disk with center at \( \lambda \) and radius \( \frac{1}{2}\|A\| \). This implies that \( w(A + \lambda I) = \frac{1}{2}\|A\| + |\lambda| \).

(ii) The proof follows as in (i). \( \square \)

**Remark 2.24.** Let \( A \in \mathcal{B}(\mathcal{H}) \). We would like to remark that if \( \overline{W(A)} \) is a circular disk with center at the origin, then \( w(A + \lambda I) = w(A) + |\lambda| \) for all \( \lambda \in \mathbb{C} \). Hence, it follows from [5, Lemma 2.13] that if \( \|\Re(e^{i\theta}A)\| = k \) (a constant) for all \( \theta \in \mathbb{R} \), then \( w(A + \lambda I) = w(A) + |\lambda| \) for all \( \lambda \in \mathbb{C} \). This shows that if \( \|\Re(e^{i\theta}A)\| = k \) (a constant) for all \( \theta \in \mathbb{R} \), then \( w(A + \lambda I) \geq w(A) \) for all \( \lambda \in \mathbb{C} \), that is, \( A \) is Birkhoff–James numerical radius orthogonal to \( I \), (for the details of Birkhoff–James numerical radius orthogonality, we refer to [11, 16]). Finally, we remark that if either \( w(A) = \frac{1}{2}\|A\| \) or \( w(A) = \frac{1}{2} \sqrt{\|A^*A + AA^*\|} \), then \( A \) is Birkhoff–James numerical radius orthogonal to \( I \).

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