Non-formal deformation quantization and star-exponential of the Poincaré Group

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To the memory of Boris Fedosov

Abstract. We recall the construction of non-formal deformation quantization of the Poincaré Group $ISO(1, 1)$ on its coadjoint orbit and exhibit the associated non-formal star-exponentials.

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1. Introduction

Quite generally, a theory -either physical or mathematical- often consists in the kinematical data of an algebra $A$ together with some dynamical data encoded by some specific, say “action”, functional $\mathcal{S}$ on $A$. In most cases, the algebra is associative because it represents the “observables” in the theory, i.e. operators whose spectrality is associated with measurements. Also, the understanding of the theory passes through the determination of critical points of the action functional $\mathcal{S}$, or sometimes equivalently, by the determination of first integrals: elements of $A$ that will be preserved by the dynamics. Disposing of a sufficient numbers of such first integrals yields a satisfactory understanding of the system. In particular, the consideration of symmetries appears as a necessity - rather than a simplifying hypothesis.

Now, assume that one disposes of enough such symmetries in order to entirely determine the dynamics. In that case, the system $(A, \mathcal{S})$ could be called “integrable” or even, quite abusively, “free”. Of course, once a “free” system is understood, one wants to pass to a perturbation or “singularization” of it, for instance by implementing some type of “interactions”. This is rather clear within the physical context. Within the mathematical context, such a singularization could for example correspond to implementing a foliation. However, once perturbed, the problem remains the same: determining symmetries. One may naïvely hope that the symmetries of the unperturbed
“free” system would remain symmetries of its perturbation. It is not the case: perturbing generally implies symmetry breaking.

One idea, due essentially to Drinfel’d, is to define the perturbation process through the data of the symmetries themselves \([1]\) in the framework of deformation quantization \([2\ 3]\). This allows, even in the case of a symmetry breaking, to control the “perturbed symmetries”. More specifically, let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\) whose enveloping (Hopf) algebra is denoted by \(\mathcal{U}(\mathfrak{g})\). Consider the category of \(\mathcal{U}(\mathfrak{g})\)-module algebras i.e. associative algebras that admit an (infinitesimal) action of \(G\). A (formal) Drinfel’d twist based \([4]\) that ensures that for every \(\mathcal{U}(\mathfrak{g})\)-module algebra \((A, \mu_A)\) the formula \(\mu_A^F := \mu_A \circ F\) defines an associative algebra structure \(A_F\) on the space \(A[[\hbar]]\). Disposing of a Drinfel’d twist then allows to deform the above mentioned category. However, as expected, the deformed objects are no longer \(\mathcal{U}(\mathfrak{g})\)-module algebras. But, the data of the twist allows to define a Hopf deformation of the enveloping algebra: keeping the multiplication unchanged, one deforms the co-product \(\Delta\) of \(\mathcal{U}(\mathfrak{g})\) by conjugating under \(F\). This yields a new co-multiplication \(\Delta_F\) that together with the undeformed multiplication underly a structure of Hopf algebra \(\mathcal{U}(\mathfrak{g})_F\) on \(\mathcal{U}(\mathfrak{g})[[\hbar]]\). The latter so called non-standard quantum group \(\mathcal{U}(\mathfrak{g})_F\) now acts on every deformed algebra \(A_F\).

At the non-formal level, the notion of Drinfel’d twist based on \(\mathcal{U}(\mathfrak{g})\) corresponds to the one of universal deformation formula for the actions \(\alpha\) of \(G\) on associative algebras \(A\) of a specified topological type such as Fréchet- or C*-algebras. This -roughly- consists in the data of a two-point kernel \(K_\theta \in C^\infty(G \times G)\) \((\theta \in \mathbb{R}_0)\) satisfying specific properties that guarantee a meaning to integral expressions of the form \(a \star^A_\theta b := \int_{G \times G} K_\theta(x, y) \mu_A((\alpha_x a \otimes \alpha_y b) \text{ d}x \text{ d}y\) with \(a, b \in A\). Once well-defined, one also requires associativity of the product \(\star^A_\theta\) as well as the semi-classical limit condition: \(\lim_{\theta \to 0} \star^A_\theta = \mu_A\) in some precise topological context. This has been performed for abelian Lie groups in \([5]\) and for abelian supergroups in \([6\ 7]\).

In \([\text{5}]\), such universal deformation formulae have been constructed for every Piatetskii-Shapiro normal \(J\)-group \(B\). For example, the class of normal \(J\)-groups (strictly) contains all Iwasawa factors of Hermitean type non-compact simple Lie groups. A universal deformation formula in particular yields a left-invariant associative function algebra on the group \(B\). It is therefore natural to ask for a comparison with the usual group convolution algebra. Following ideas mainly due to Fronsdal in the context of the \(\star\)-representation program (representation theory of Lie group in the framework of formal \(\star\)-products), one may expect that (a non-formal version of) the notion of star-exponential \([2\ 9]\) plays a crucial role in this comparison. Moreover, such a star-exponential can give access to the spectrum of operators \([10\ 11]\) determining possible measurements of a system.
In this paper, we recall the construction of the non-formal deformation quantization (see [12]) and exhibit its star-exponential for the basic case of the Poincaré Group $ISO(1,1)$. Such a low-dimensional case illustrate the general method developed for normal $J$-group $\mathbb{B}$ (which are Kahlerian), in $\mathfrak{S}$ for star-products and in $\mathfrak{S}$ for star-exponentials. However, the Poincaré group is solvable but of course not Kahlerian, so this paper shows also that the method introduced in $\mathfrak{S}$ [8, 13] can be extended to some solvable but non-Kahlerian Lie groups.

2. Geometry of the Poincaré Group

We recall here some features concerning the geometry of the Poincaré group $G = ISO(1,1) = SO(1,1) \times \mathbb{R}^2$ and of its coadjoint orbits. First, it is diffeomorphic to $\mathbb{R}^3$, so let us choose a global coordinate system $\{(a, \ell, m)\}$ of it. Its group law can be read as

$$(a, \ell, m) \cdot (a', \ell', m') = (a + a', e^{-2a' \ell + \ell'} e^{2a' m + m'})$$

Its neutral element is $(0, 0, 0)$ and the inverse is given by: $(a, \ell, m)^{-1} = (-a, -e^{2a} \ell, -e^{-2a} m)$. By writing $(a, \ell, m) = \exp(aH) \exp(\ell E) \exp(mF)$, we can determine its Lie algebra $\mathfrak{g}$:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = 0.$$

Let us have a look to the coadjoint orbit of $G$. After a short calculation, one can find that

$$\text{Ad}_g^*(\alpha H^* + \beta E^* + \gamma F^*) = (\alpha + 2\beta \ell - 2\gamma m)H^* + \beta e^{-2a} E^* + \gamma e^{2a} F^*$$

if $g = (a, \ell, m) \in G$ and \{H*, E*, F*\} is the basis of $\mathfrak{g}^*$ dual of \{H, E, F\}. A generic orbit of $G$ is therefore a hyperbolic cylinder. We will study in particular the orbits associated to the forms $k(E^* - F^*)$ with $k \in \mathbb{R}^+_*$, which will be denoted by $\mathbb{M}_k$ or simply by $\mathbb{M}$. The Poincaré quotient $\mathbb{M}_k$ is globally diffeomorphic to $\mathbb{R}^2$, so we choose the following coordinate system:

$$(a, \ell) := \text{Ad}_{(a,\ell,0)}^* k(E^* - F^*) = k(2\ell H^* + e^{-2a} E^* - e^{2a} F^*). \quad (2.1)$$

$\mathbb{M}$ is a $G$-homogeneous space for the coadjoint action:

$$(a, \ell, m) \cdot (a', \ell') := \text{Ad}_{(a,\ell,m)}^* k(2\ell' H^* + e^{-2a'} E^* - e^{2a'} F^*) = (a + a', \ell' + e^{-2a'} \ell + e^{2a'} m). \quad (2.2)$$

Remark 2.1. Note that the affine group $\mathbb{S}$ (connected component of the identity of “ax+b”) is the subgroup of $G$ generated by $H$ and $E$, i.e. by simply considering the two first coordinates $(a, \ell)$ of $G$. Actually the identification (2.1) yields a diffeomorphism between $\mathbb{S}$ and $\mathbb{M}$ which is $\mathbb{S}$-equivariant with respect to the left action of $\mathbb{S}$ and its action on $\mathbb{M}$ by (2.2) as a subgroup of $G$. This identification $\mathbb{S} \simeq \mathbb{M}$ is useful to construct star-products.
The fundamental fields of the action (2.2), defined by
\[ X^*_{(a, \ell)} f = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX) \cdot (a, \ell)), \]
for \( X \in \mathfrak{g}, (a, \ell) \in \mathbb{M}, f \in C^\infty(\mathbb{M}) \), are given by
\[ H^*_{(a, \ell)} = -\partial_a, \quad E^*_{(a, \ell)} = -e^{-2a} \partial_\ell, \quad F^*_{(a, \ell)} = -e^{2a} \partial_\ell. \]
This permits to compute the Kostant-Kirillov-Souriau symplectic form of \( \mathbb{M} \),
\[ \omega_{(a, \ell)} = 2k da \wedge d\ell. \] (2.3)

For different values of \( k \in \mathbb{R}_+ \), the \( (\mathbb{M}_k, \omega) \) are symplectomorphic, so we set \( k = 1 \) in the following. Since the action of \( G \) on its coadjoint orbit \( \mathbb{M} \) is strongly hamiltonian, there exists a Lie algebra homomorphism \( \lambda : \mathfrak{g} \to C^\infty(\mathbb{M}) \) (for the Poisson bracket on \( \mathbb{M} \) associated to \( \omega \)), called the moment map and given by
\[ \lambda_H = 2\ell, \quad \lambda_E = e^{-2a}, \quad \lambda_F = -e^{2a}. \] (2.4)

**Proposition 2.2.** The exponential of the group \( G \) is given by
\[ e^{tX} = \left( \alpha t, \frac{\beta}{\alpha} e^{-\alpha t} \sinh(\alpha t), \frac{\gamma}{\alpha} e^{\alpha t} \sinh(\alpha t) \right) \]
for \( X = \alpha H + \beta E + \gamma F \).

*Proof.* It is a direct calculation using the semigroup property \( e^{(t+s)X} = e^{sX} e^{tX} \) and by deriving by \( s \). \( \square \)

For \( g = (a, \ell, m) \in G \), we can obtain straightforwardly the logarithm by inverting the above equation:
\[ \log(a, \ell, m) = aH + \frac{ae^a \ell}{\sinh(a)} E + \frac{ae^{-a} m}{\sinh(a)} F \] (2.5)

and the BCH expression:
\[ \text{BCH}(X_1, X_2) = \log(e^{X_1} e^{X_2}) = (\alpha_1 + \alpha_2)H + \frac{(\alpha_1 + \alpha_2)}{\sinh(\alpha_1 + \alpha_2)} \left( \frac{\beta_1}{\alpha_1} e^{-\alpha_2} \sinh(\alpha_1) + \frac{\beta_2}{\alpha_2} e^{\alpha_1} \sinh(\alpha_2) \right) E + \frac{(\alpha_1 + \alpha_2)}{\sinh(\alpha_1 + \alpha_2)} \left( \frac{\gamma_1}{\alpha_1} e^{\alpha_2} \sinh(\alpha_1) + \frac{\gamma_2}{\alpha_2} e^{-\alpha_1} \sinh(\alpha_2) \right) F \] (2.6)
for \( X_i = \alpha_i H + \beta_i E + \gamma_i F \in \mathfrak{g} \).
3. Deformation quantization

3.1. Star-products

Due to the identification \( \mathbb{M} \cong \mathbb{R}^2 \), we can endow the space of Schwartz functions \( \mathcal{S}(\mathbb{R}^2) \) with the Moyal product associated to the constant KKS symplectic form (2.3):

\[
(f \star_\theta^0 h)(a, \ell) = \frac{4}{(\pi \theta)^2} \int da_i d\ell_i \ f(a_1 + a, \ell_1 + \ell)h(a_2 + a, \ell_2 + \ell)e^{-\frac{\theta}{2}(a_1 \ell_2 - a_2 \ell_1)}
\]

for \( f, h \in \mathcal{S}(\mathbb{R}^2) \). It turns out that this associative star-product is \textit{covariant} for the moment map (2.4), formally in the deformation parameter \( \theta \) for \( f, h \in \mathcal{O}_\mathcal{M}(\mathbb{R}) \) is \textit{covariant}.

\[
\forall X, Y \in \mathfrak{g} : \ [\lambda_X, \lambda_Y]_{\star_\theta} = -i \theta \lambda_{[X,Y]}.
\]

Nonetheless, it is not \( G \)-\textit{invariant}, one does not have:

\[
\forall g \in G : \ g^*(f \star_\theta^0 h) = (g^* f) \star_\theta^0 (g^* h)
\]

in general, where \( g^* \) means the pullback of the action (2.2) of \( G \) on \( \mathbb{M} \): \( g^* f := f(g \cdot) \). In the following, we exhibit intertwining operators \( T_\theta \) (see [14]) in order to construct invariant star-products on \( \mathbb{M} \), i.e. satisfying (3.3).

We consider \( \mathcal{P}_\theta \) an invertible multiplier on \( \mathbb{R} \): \( \mathcal{P}_\theta \in \mathcal{O}^\times_M(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}), \ \forall h \in \mathcal{S}(\mathbb{R}) \ f h \in \mathcal{S}(\mathbb{R}) \ \text{and} \ f^{-1} h \in \mathcal{S}(\mathbb{R}) \} \), and \( \phi_\theta \) defined by:

\[
\phi_\theta(a, \ell) = (a, \frac{2}{\theta} \sinh(\frac{\theta \ell}{2})), \quad \phi_\theta^{-1}(a, \ell) = (a, \frac{2}{\theta} \arcsinh(\frac{\theta \ell}{2})).
\]

We define the operator \( T_\theta = \mathcal{P}_\theta(0) \mathcal{F}^{-1} \circ \phi_\theta^{-1} \circ \mathcal{P}_\theta^{-1} \circ \mathcal{F} \), from \( \mathcal{S}(\mathbb{R}^2) \) to \( \mathcal{S}'(\mathbb{R}^2) \), where \( \mathcal{P}_\theta^{-1} \) acts by multiplication by \( \mathcal{P}_\theta(e^{-1}) \) and the partial Fourier transformation is given by:

\[
\mathcal{F} f(a, \xi) = \hat{f}(a, \xi) := \int d\ell \ e^{-i \xi \ell} f(a, \ell).
\]

The normalization is chosen so that \( T_\theta 1 = 1 \). On its image, \( T_\theta \) is invertible. The explicit expressions are:

\[
T_\theta f(a, \ell) = \frac{\mathcal{P}_\theta(0)}{2\pi} \int dt d\xi \ \cosh(\frac{\theta t}{2}) \mathcal{P}_\theta(t)^{-1} e^{\frac{\theta}{2} \sinh(\frac{\theta \ell}{2}) - i \xi t} f(a, \xi)
\]

\[
T_\theta^{-1} f(a, \ell) = \frac{1}{2\pi \mathcal{P}_\theta(0)} \int dt d\xi \ \mathcal{P}_\theta(t) e^{-\frac{\theta}{2} \sinh(\frac{\theta \ell}{2}) \xi + it \ell} f(a, \xi)
\]

In [14], it has been shown that this intertwining operator yields an associative product on \( T_\theta(\mathcal{S}(\mathbb{R}^2)) \): \( f \star_{\theta, T}^0 h := T_\theta((T_\theta^{-1} f) \star_\theta^0 (T_\theta^{-1} h)) \) which is \( G \)-invariant. Its explicit expression is: \( \forall f, h \in T_\theta(\mathcal{S}(\mathbb{R}^2)) \),

\[
(f \star_{\theta, T}^0 h)(a, \ell) = \frac{4}{(\pi \theta)^2} \int da_i d\ell_i \cosh(2(a_1 - a_2)) \frac{\mathcal{P}_\theta(\frac{4}{\theta}(a_1 - a)) \mathcal{P}_\theta(\frac{4}{\theta}(a - a_2))}{\mathcal{P}_\theta(\frac{4}{\theta}(a_1 - a_2))} \frac{\mathcal{P}_\theta(\frac{4}{\theta}(a_1 - a_2)) \mathcal{P}_\theta(\frac{4}{\theta}(a - a_2))}{\mathcal{P}_\theta(\frac{4}{\theta}(a_1 - a_2)) \mathcal{P}_\theta(0)}
\]

\[
e^{\frac{2\theta}{\theta}(\sinh(2(a_1 - a_2)) \ell + \sinh(2(a_2 - a)) \ell_1 + \sinh(2(a - a_1)) \ell_2)} f(a_1, \ell_1) h(a_2, \ell_2).
\]

3.2. Star-exponential of the Poincaré Group

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3.2. Schwartz space

In [8], a Schwartz space adapted to $\mathbb{M}$ has been introduced, which is different from the usual one $\mathcal{S}(\mathbb{R}^2)$ in the global chart $\{(a, \ell)\}$ ([2,1]).

Definition 3.1. The Schwartz space of $\mathbb{M}$ is defined as

$$\mathcal{S}(\mathbb{M}) = \{f \in C^\infty(\mathbb{M}) \mid \forall \alpha = (k, p, q, n) \in \mathbb{N}^4, \|f\|_\alpha := \sup_{(a, \ell)} \left| \frac{\sinh(2a)^k}{\cosh(2a)^p} \ell^q a^p f(a, \ell) \right| < \infty \}.$$ 

The space $\mathcal{S}(\mathbb{M})$ corresponds to the usual Schwartz space in the coordinates $(r, \ell)$ with $r = \sinh(2a)$. It is stable by the action of $G$:

$$\forall f \in \mathcal{S}(\mathbb{M}), \forall g \in G : g^* f \in \mathcal{S}(\mathbb{M}),$$

due to the formulation of the action of $G$ in the coordinates $(r, \ell)$:

$$(r, \ell, m)(r', \ell') = \left( r \sqrt{1 + r'^2} + r' \sqrt{1 + r^2}, \ell' + (\sqrt{1 + r'^2} - r') \ell 
+ (\sqrt{1 + r'^2} + r') m \right).$$

Moreover, $\mathcal{S}(\mathbb{M})$ is a Fréchet nuclear space endowed with the seminorms $(\|f\|_\alpha)$. For $f, h \in \mathcal{S}(\mathbb{M})$, the product $f \ast_{\theta, p} h$ is well-defined by ([3,5]). However, it is not possible to show that it belongs to $\mathcal{S}(\mathbb{M})$ unless we consider this expression as an oscillatory integral. Let us define this concept. For $F \in \mathcal{S}(\mathbb{M}^2)$, one can show using integrations by parts that:

$$\int \int da_1da_2 \ell_1 \ell_2 e^{2\pi i (\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)} F(a_1, a_2, \ell_1, \ell_2) =$$

$$\int \int da_1da_2 \ell_1 \ell_2 e^{2\pi i (\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)} \left( \frac{1 - \frac{\sinh^2(2a_1)}{1 + \sinh^2(2a_1)}}{1 + \sinh^2(2a_2)} \right)^{k_1} \left( \frac{1 - \frac{\sinh^2(2a_2)}{1 + \sinh^2(2a_2)}}{1 + \sinh^2(2a_1)} \right)^{k_2}$$

$$\left( \frac{1 - \frac{\sinh^2(2a_2)}{1 + \ell_2^2}}{1 + \ell_1^2} \right)^{p_1} \left( \frac{1 - \frac{\sinh^2(2a_1)}{1 + \ell_1^2}}{1 + \ell_2^2} \right)^{p_2} F(a_1, a_2, \ell_1, \ell_2)$$

$$= \int \int da_1da_2 \ell_1 \ell_2 e^{2\pi i (\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)} \left( \frac{1}{1 + \sinh^2(2a_1)} \right)^{k_1}$$

$$\left( \frac{1}{1 + \sinh^2(2a_2)} \right)^{k_2} \left( \frac{1}{1 + \ell_1^2} \right)^{p_1} \left( \frac{1}{1 + \ell_2^2} \right)^{p_2} \left( \frac{DF(a_1, a_2, \ell_1, \ell_2)}{(1 + \sinh^2(2a_2))^{k_2}(1 + \ell_1^2)^{p_1}(1 + \ell_2^2)^{p_2}} \right) (3.6)$$

for any $k_i, p_i \in \mathbb{N}$, and where $D$ is a linear combination of products of bounded functions (with every derivatives bounded) in $(a_i, \ell_i)$ with powers of $\partial_{\ell_i}$ and $\frac{1}{\cosh(2a_i)} \partial_{a_i}$. The first expression of (3.6) is not defined for non-integrable functions $F$ bounded by polynomials in $r_i := \sinh(2a_i)$ and $\ell_i$. However, the last expression of (3.6) is well-defined for $k_i, p_i$ sufficiently large. Therefore it gives a sense to the first expression, now understood as an oscillatory integral, i.e. as being equal to the last expression. This definition of oscillatory integral ([5, 8]) is unique, in particular unambiguous in the powers $k_i, p_i$ because of the density of $\mathcal{S}(\mathbb{M})$ in polynomial functions in $(r, \ell)$ of a given degree. Note
that this corresponds to the usual oscillatory integral [16] in the coordinates $(r, \ell)$.

The first part of the next Theorem shows that this concept of oscillatory integral is necessary [8] for $S(M)$ to obtain an associative algebra, while the other parts have been treated in [12].

**Theorem 3.2.** Let $P : \mathbb{R} \to C^\infty(\mathbb{R})$ be a smooth map such that $P_0 \equiv 1$, and $P_\theta(a)$ as well as its inverse are bounded by $C \sinh(2a)^k$, $k \in \mathbb{N}$, $C > 0$.

- Then, the expression (3.3), understood as an oscillatory integral, yields a $G$-invariant non-formal deformation quantization.
- In particular, $(S(M), \star_{\theta, P})$ is a Fréchet algebra.
- For $f, h \in S(M)$, the map $\theta \mapsto f \star_{\theta, P} h$ is smooth and admits a $G$-invariant formal star-product as asymptotic expansion in $\theta = 0$.
- Every $G$-invariant formal star-product on $M$ can be obtained as an expansion of a $\star_{\theta, P}$, for a certain $P$.
- For $P_\theta(a) = P_\theta(0) \sqrt{\cosh(a\ell/2)}$, one has the tracial identity: $\int f \star_{\theta, P} h = \int f \cdot h$.

We denote $\star_{\theta}$ the product $\star_{\theta, P}$ with $P_\theta(a) = \sqrt{\cosh(a\ell/2)}$, therefore satisfying the tracial identity.

**3.3. Schwartz multipliers**

Let us consider the topological dual $S'(M)$ of $S(M)$. In the coordinates $(r, \ell)$, it corresponds to tempered distributions. By denoting $\langle -, - \rangle$ the duality bracket between $S'(M)$ and $S(M)$, one can extend the product $\star_{\theta}$ (with tracial identity) as $\forall T \in S'(M)$, $\forall f, h \in S(M)$,

$\langle T \star_{\theta} f, h \rangle := \langle T, f \star_{\theta} h \rangle$ and $\langle f \star_{\theta} T, h \rangle := \langle T, h \star_{\theta} f \rangle$,

which is compatible with the case $T \in S(M)$. Then, we define [13]:

$$M_{\star_{\theta}}(M) := \{ T \in S'(M), \ f \mapsto T \star_{\theta} f, \ f \mapsto f \star_{\theta} T \text{ are continuous from } S(M) \text{ into itself} \},$$

and the product can be extended to $M_{\star_{\theta}}(M)$ by:

$\forall S, T \in M_{\star_{\theta}}(M), \forall f \in S(M) : \langle S \star_{\theta} T, f \rangle := \langle S, T \star_{\theta} f \rangle = \langle T, f \star_{\theta} S \rangle$.

We can equip $M_{\star_{\theta}}(M)$ with the topology associated to the seminorms:

$$\| T \|_{B, \alpha, L} = \sup_{f \in B} \| T \star_{\theta} f \|_{\alpha} \text{ and } \| T \|_{B, \alpha, R} = \sup_{f \in B} \| f \star_{\theta} T \|_{\alpha},$$

where $B$ is a bounded subset of $S(M)$, $\alpha \in \mathbb{N}^4$ and $\| f \|_{\alpha}$ is the Schwartz seminorm introduced in Definition 3.1. Note that $B$ can be described as a set satisfying $\forall \alpha$, $\sup_{f \in B} \| f \|_{\alpha}$ exists.

**Proposition 3.3.** $(M_{\star_{\theta}}(M), \star_{\theta})$ is an associative Hausdorff locally convex complete and nuclear algebra, with separately continuous product, called the multiplier algebra.
4. Construction of the star-exponential

4.1. Formal construction

Let us follow the method developed in [13]. We want first to find a solution to the following equation

$$\partial_t f_t(a, \ell) = \frac{i}{\theta} (\lambda_X \star_\theta f_t)(a, \ell)$$

(4.1)

for $X = \alpha H + \beta E + \gamma F \in \mathfrak{g}$, with initial condition $\lim_{t \to 0} f_t(a, \ell) = 1$. To remove the integral of this equation, we apply the partial Fourier transformation (3.4) to obtain

$$\mathcal{F}(\lambda H \star_\theta f) = \left(2i\partial_{\xi} + i\theta a \partial_a\right) \hat{f}, \quad \mathcal{F}(\lambda E \star_\theta f) = e^{-2a - \frac{\theta_{\ell}}{2}} \hat{f},$$

$$\mathcal{F}(\lambda F \star_\theta f) = -e^{2a + \frac{\theta_{\ell}}{2}} \hat{f},$$

so that the equation (4.1) can be reformulated as

$$\partial_t \hat{f}_t(a, \xi) = \frac{i}{\theta} \left[2i\alpha \partial_{\xi} + \frac{i\theta a}{2} \partial_a + \beta e^{-2a} - \gamma e^{2a} \right] \hat{f}_t(a, \xi).$$

The existence of a solution of this equation which satisfies the BCH property directly relies on the covariance (3.2) of the Moyal product. We have the explicit following result.

**Proposition 4.1.** For $X = \alpha H + \beta E + \gamma F \in \mathfrak{g}$, the expression

$$E^{\star_\theta}(t\lambda X)(a, \ell) = e^{\frac{i}{\theta} \left(2\ell\alpha t + \frac{\theta_{\ell}}{2} \sinh(\alpha t)(\beta e^{-2a} - \gamma e^{2a})\right)}$$

is a solution of the equation (4.1) with initial condition $\lim_{t \to 0} f_t(a, \ell) = 1$. Moreover, it satisfies the BCH property: $\forall X, Y \in \mathfrak{g}$,

$$E^{\star_\theta}(\lambda_{BCH(X,Y)}) = E^{\star_\theta}(\lambda_X) \star_\theta E^{\star_\theta}(\lambda_Y).$$

(4.2)

**Proof.** By performing the following change of variables $b = -\frac{a}{\alpha} - \frac{\theta_{\ell}}{2\alpha}$ and $c = -\frac{a}{\alpha} + \frac{\theta_{\ell}}{2\alpha}$, we reformulate the equation as

$$\partial_t \hat{f}_t = \partial_b \hat{f}_t + \frac{i\beta}{\theta} e^{2ab} - \frac{i\gamma}{\theta} e^{-2ab},$$

whose solution is given by $\hat{f}_t(b, c) = \exp \left(-\int_0^t \left(\frac{i\beta}{\theta} e^{2as} - \frac{i\gamma}{\theta} e^{-2as}\right) ds\right) h(b + t, c)$, where $h$ is an arbitrary function. By assuming the initial condition, we obtain the expression of $E^{\star_\theta}(tX)(a, \ell)$. The BCH property is given by direct computations from the expression of the product (3.1) and from (2.6). \qed

Finally, we push this solution by $T_\theta$:

$$E^{\star_\theta, p}(tX)(a, \ell) := T_\theta E^{\star_\theta}(tT_\theta^{-1}\lambda X)(a, \ell)$$

$$= \frac{\mathcal{P}_\theta(0) \cosh(\alpha t)}{\mathcal{P}_\theta(\frac{2\alpha t}{\theta})} e^{\frac{i}{\theta} \sinh(\alpha t)(2\ell + \frac{\theta_{\ell}}{2} e^{-2a} - \frac{\gamma}{\alpha} e^{2a}) + \frac{2\mathcal{P}_\theta(0)}{\mathcal{P}_\theta(\frac{2\alpha t}{\theta})} \alpha t}.$$

It also satisfies the BCH property (4.2).
4.2. Multiplier property

For the star-product with tracial property, we want to define the star-exponential at the non-formal level. We can use the oscillatory integral in the star-product to show \[13\]:

**Theorem 4.2.** For any \(X \in \mathfrak{g}\), the function

\[ E_{\star \theta}(tX)(a, \ell) = \sqrt{\cosh(\alpha t)} e^{\frac{\theta}{2} \sinh(\alpha t)(2\ell + \beta \alpha e^{-2\alpha} - \frac{\gamma}{2} e^{2\alpha})} \]

lies in the multiplier algebra \(\mathcal{M}_{\star \theta}(\mathbb{M})\).

As it belongs to a specific “functional space”, the function \(E_{\star \theta}\) is called the non-formal star-exponential of the group \(G\) for the star-product \(*_{\theta}\). The BCH property

\[ E_{\star \theta}(\text{BCH}(X, Y)) = E_{\star \theta}(X) *_{\theta} E_{\star \theta}(Y) \]

now makes sense in the topological space \(\mathcal{M}_{\star \theta}(\mathbb{M})\). This functional framework is useful for applications of the star-exponential discussed in the introduction.

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