HARDY TYPE INEQUALITIES FOR $\Delta_\lambda$-LAPLACIANS

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Abstract. We derive Hardy type inequalities for a large class of sub-elliptic operators that belong to the class of $\Delta_\lambda$-Laplacians and find explicit values for the constants involved. Our results generalize previous inequalities obtained for Grushin type operators

$$\Delta_x + |x|^{2\alpha} \Delta_y, \quad (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad \alpha \geq 0,$$

which were proved to be sharp.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain, where $N \geq 3$. The $N$-dimensional version of the classical Hardy inequality states that there exists a constant $c > 0$ such that

$$c \int_{\Omega} \frac{|u(x)|^2}{|x|^2} \, dx \leq \int_{\Omega} |\nabla u(x)|^2 \, dx,$$

for all $u \in H_0^1(\Omega)$. If the origin $\{0\}$ belongs to the set $\Omega$, the optimal constant is $c = \left(\frac{N-2}{2}\right)^2$, but not attained in $H_0^1(\Omega)$. Hardy originally proved this inequality in 1920 for the one-dimensional case.

Hardy inequalities are an important tool in the analysis of linear and non-linear PDEs (see, e.g., [6],[4],[16]), and over the years the classical Hardy inequality has been improved and extended in many directions. Our aim is to derive Hardy type inequalities for a class of degenerate elliptic operators extending the previous results by D’Ambrosio in [3]. He obtained a family of Hardy type inequalities for the Grushin type operator

$$\Delta_x + |x|^{2\alpha} \Delta_y, \quad (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $\alpha$ is a real positive constant. The class of operators we consider contains Grushin type operators and, e.g., operators of the form

$$\Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where $\alpha, \beta$ and $\gamma$ are real positive constants. Hardy type inequalities have been derived for other classes of degenerate elliptic operators such as the Kohn Laplacian on the Heisenberg group. For a wide bibliography regarding Hardy type inequalities in the sub-elliptic setting we refer to [1].

The proof of our inequalities is based on the approach introduced by Mitidieri in [15] for the classical Laplacian. Our results coincide for the particular case of Grushin type operators with the inequalities D’Ambrosio obtained in [3], where he proved that the inequalities are sharp. We derive explicit values for the constants in the inequalities, but are currently not able to show its optimality in the general case.

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The outline of our paper is as follows: We first introduce the class of operators we consider and formulate several examples. In Section 3 we explain our approach to derive Hardy type inequalities and give a motivation for the weights appearing in our inequalities. The main results are stated and proved in Section 4. In the last section we illustrate the relation between the fundamental solution and Hardy inequalities and comment on the difficulties we encounter proving the optimality of the constant in our inequalities.

2. $\Delta_\lambda$-Laplacians

Here and in the sequel, we use the following notations. We split $\mathbb{R}^N$ into
$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k},$$
and write
$$x = (x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^N, \quad x^{(i)} = (x^{(i)}_1, \ldots, x^{(i)}_{N_i}), \quad i = 1, \ldots, k.$$
The degenerate elliptic operators we consider are of the form
$$\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \cdots + \lambda_k^2 \Delta_{x^{(k)}},$$
where the functions $\lambda_i : \mathbb{R}^{N_i} \to \mathbb{R}$ are pairwise different and $\Delta_{x^{(i)}}$ denotes the classical Laplacian in $\mathbb{R}^{N_i}$. We denote by $|x|$ the euclidean norm of $x \in \mathbb{R}^m$, $m \in \mathbb{N}$, and assume the functions $\lambda_i$ are of the form
$$\lambda_1(x) = 1,$$
$$\lambda_2(x) = |x^{(1)}|^{\alpha_{21}},$$
$$\lambda_3(x) = |x^{(1)}|^{\alpha_{31}} |x^{(2)}|^{\alpha_{32}},$$
$$\vdots$$
$$\lambda_k(x) = |x^{(1)}|^{\alpha_{k1}} |x^{(2)}|^{\alpha_{k2}} \cdots |x^{(k-1)}|^{\alpha_{kk-1}}, \quad x \in \mathbb{R}^N,$$
where $\alpha_{ij} > 0$ for $i = 2, \ldots, k; j = 1, \ldots, i - 1$. Setting $\alpha_{ij} = 0$ for $j \geq i$ we can write
$$\lambda_i(x) = \prod_{j=1}^k |x^{(j)}|^{\alpha_{ij}}, \quad i = 1, \ldots, k. \quad (1)$$
This implies that there exists a group of dilations $(\delta_r)_{r>0}$,
$$\delta_r : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x^{(1)}, \ldots, x^{(k)}) = (r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)}),$$
where $1 = \sigma_1 \leq \sigma_i$ such that $\lambda_i$ is $\delta_r$-homogeneous of degree $\sigma_i - 1$, i.e.,
$$\lambda_i(\delta_r(x)) = r^{\sigma_i-1} \lambda_i(x), \quad \forall x \in \mathbb{R}^N, \ r > 0, \ i = 1, \ldots, k,$$
and the operator $\Delta_\lambda$ is $\delta_r$-homogeneous of degree two, i.e.,
$$\Delta_\lambda(u(\delta_r(x))) = r^2(\Delta_\lambda u)(\delta_r(x)) \quad \forall u \in C^\infty(\mathbb{R}^N).$$
We denote by $Q$ the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $(\delta_r)_{r>0}$, i.e.,
$$Q := \sigma_1 N_1 + \cdots + \sigma_k N_k.$$
$Q$ will play the same role as the dimension $N$ for the classical Laplacian in our Hardy type inequalities.
For functions $\lambda_i$ of the form (1) we find

\[
\sigma_1 = 1, \\
\sigma_2 = 1 + \sigma_1 \alpha_{21}, \\
\sigma_3 = 1 + \sigma_1 \alpha_{31} + \sigma_2 \alpha_{32}, \\
\vdots \\
\sigma_k = 1 + \sigma_1 \alpha_{k1} + \sigma_2 \alpha_{k2} + \cdots + \sigma_{k-1} \alpha_{kk-1}.
\]

If the functions $\lambda_i$ are smooth, i.e., if the exponents $\alpha_{ji}$ are integers, the operator $\Delta_\lambda$ belongs to the general class of operators studied by Hörmander in [11] and it is hypoelliptic (see Remark 1.3, [12]). The simplest example is the operator

\[
\partial^2_{x_1} + |x_1|^{2\alpha} \partial^2_{x_2}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad \alpha \in \mathbb{N},
\]

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, 2$, that Grushin studied in [10]. He provided a complete characterization of the hypoellipticity for such operators when lower terms with complex coefficients are added. For real $\alpha > 0$ the operator is commonly called of Grushin-type.

Operators $\Delta_\lambda$ with functions $\lambda_i$ of the form (1) belong to the class of $\Delta_\lambda$-Laplacians. Franchi and Lanconelli introduced operators of $\Delta_\lambda$-Laplacian type in 1982 and studied their properties in a series of papers. In [8] they defined a metric associated to these operators that plays the same role as the euclidian metric for the standard Laplacian. Using this metric in [9] and [12] they extended the classical De Giorgi theorem and obtained Sobolev type embedding theorems for such operators. Recently, adding the assumption that the operators are homogeneous of degree two, they were named $\Delta_\lambda$-Laplacians by Kogoj and Lanconelli in [12], where existence, non-existence and regularity results for solutions of the semilinear $\Delta_\lambda$-Laplace equation were analyzed. The global well-posedness and longtime behavior of solutions of semilinear degenerate parabolic equations involving $\Delta_\lambda$-Laplacians were studied in [13]. We finally remark that the $\Delta_\lambda$-Laplacians belong to the more general class of $X$-elliptic operators, that were introduced in [14].

To conclude this section we recall some of the examples in our previous paper [13].

**Example 1.** Let $\alpha$ be a real positive constant and $k = 2$. We consider the Grushin-type operator

\[
\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}},
\]

where $\lambda = (\lambda_1, \lambda_2)$, with $\lambda_1(x) = 1$ and $\lambda_2(x) = |x^{(1)}|^\alpha$, $x \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Our group of dilations is

\[
\delta_r \left( x^{(1)}, x^{(2)} \right) = \left( r x^{(1)}, r^{\alpha+1} x^{(2)} \right),
\]

and the homogenous dimension with respect to $(\delta_r)_{r>0}$ is $Q = N_1 + N_2(\alpha + 1)$. More generally, for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$ with real constants $\alpha_i > 0$, $i = 1, \ldots, k-1$, we consider

\[
\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1} \Delta_{x^{(2)}} + \cdots + |x^{(1)}|^{2\alpha_{k-1}} \Delta_{x^{(k)}}.
\]
The group of dilations is given by
\[ \delta_r \left( x^{(1)}, \ldots, x^{(k)} \right) = \left( r x^{(1)}, r^{1+\alpha_1} x^{(2)}, \ldots, r^{1+\alpha_{k-1}} x^{(k)} \right), \]
and the homogeneous dimension is \( Q = N + \alpha_1 N_2 + \alpha_2 N_3 + \cdots + \alpha_{k-1} N_k. \)

**Example 2.** For a given multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{k-1}) \) with real constants \( \alpha_i > 0, \ i = 1, \ldots, k-1, \) we define
\[ \Delta_{\lambda} = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1} \Delta_{x^{(2)}} + |x^{(2)}|^{2\alpha_2} \Delta_{x^{(3)}} + \cdots + |x^{(k-1)}|^{2\alpha_{k-1}} \Delta_{x^{(k)}}. \]
Then, in our notation \( \lambda = (\lambda_1, \ldots, \lambda_k) \) with
\[ \lambda_1(x) = 1, \ \lambda_i(x) = |x^{(i-1)}|^{\alpha_{i-1}}, \ i = 2, \ldots, k, \ x \in \mathbb{R}^N_1 \times \cdots \times \mathbb{R}^N_k, \]
and the group of dilations is given by
\[ \delta_r \left( x^{(1)}, \ldots, x^{(k)} \right) = \left( r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)} \right) \]
with \( \sigma_1 = 1 \) and \( \sigma_i = \alpha_{i-1} \sigma_{i-1} + 1 \) for \( i = 2, \ldots, k. \) In particular, if \( \alpha_1 = \ldots = \alpha_{k-1} = \alpha, \) the dilations become
\[ \delta_r \left( x^{(1)}, \ldots, x^{(k)} \right) = \left( r x^{(1)}, r^{\alpha+1} x^{(2)}, \ldots, r^{\alpha k-1 + \cdots + 1} x^{(k)} \right). \]

**Example 3.** Let \( \alpha, \beta \) and \( \gamma \) be positive real constants. For the operator
\[ \Delta_{\lambda} = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}} + |x^{(2)}|^{2\beta} |x^{(2)}|^{2\gamma} \Delta_{x^{(3)}}, \]
where \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) with
\[ \lambda_1(x) = 1, \ \lambda_2(x) = |x^{(1)}|^{\alpha}, \ \lambda_3(x) = |x^{(1)}|^{\beta} |x^{(2)}|^{\gamma}, \ x \in \mathbb{R}^N_1 \times \mathbb{R}^N_2 \times \mathbb{R}^N_3, \]
we find the group of dilations
\[ \delta_r \left( x^{(1)}, x^{(2)}, x^{(3)} \right) = \left( r x^{(1)}, r^{\alpha+1} x^{(2)}, r^{\beta+(\alpha+1)\gamma+1} x^{(3)} \right). \]

3. **How we approach Hardy-type inequalities**

Our Hardy type inequalities are based on the following approach indicated by Mitidieri in [15].

Let \( \Omega \subset \mathbb{R}^N, \ N \geq 3, \) be an open subset and \( p > 1. \) We assume \( u \in C_0^1(\Omega), \) and the vector field \( h \in C^1(\Omega; \mathbb{R}^N) \) satisfies \( \text{div} h > 0. \) The divergence theorem implies
\[ \int_{\Omega} |u(x)|^p \text{div} h(x) \, dx = -p \int_{\Omega} |u(x)|^{p-2} u(x) \nabla u(x) \cdot h(x) \, dx, \]
where \( \cdot \) denotes the inner product in \( \mathbb{R}^N. \) Taking the absolute value and using H"older’s inequality we obtain
\[ \int_{\Omega} |u(x)|^p \text{div} h(x) \, dx = -p \int_{\Omega} |u(x)|^{p-2} |u(x) \nabla u(x) \cdot h(x)| \, dx \]
\[ \leq p \left( \int_{\Omega} |u(x)|^p \text{div} h(x) \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} \frac{|h(x)|^p}{(\text{div} h(x))^{p-1}} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}, \]
and it follows that
\[ \int_{\Omega} |u(x)|^p \text{div} h(x) \, dx \leq p^p \int_{\Omega} \frac{|h(x)|^p}{(\text{div} h(x))^{p-1}} |\nabla u(x)|^p \, dx. \quad (2) \]
If we choose the vector field
\[ h_\varepsilon(x) := \frac{x}{(|x|^2 + \varepsilon)^{\frac{N}{2}}}, \]
where \( \varepsilon > 0 \), then
\[ \text{div} h_\varepsilon(x) = \frac{N - p |x|^2}{(|x|^2 + \varepsilon)^{\frac{N}{2}}}, \quad |h_\varepsilon(x)| = \frac{|x|}{(|x|^2 + \varepsilon)^{\frac{N}{2}}}. \]

Assuming that \( N > p \) we have \( \text{div} h_\varepsilon > 0 \), and from inequality \( (2) \) we obtain
\[ \frac{1}{p^p} \int_{\Omega} \left( N - p \frac{|x|^2}{|x|^2 + \varepsilon} \right) \frac{|u(x)|^p}{(|x|^2 + \varepsilon)^{\frac{N}{2}}} dx \leq \int_{\Omega} \left( N - p \frac{|x|^2}{|x|^2 + \varepsilon} \right)^{-(p-1)} \frac{|x|^p}{(|x|^2 + \varepsilon)^{\frac{N}{2}}} |\nabla u(x)|^p dx. \]

Taking the limit \( \varepsilon \) tends to zero, the classical Hardy inequality follows from the dominated convergence theorem,
\[ \left( \frac{N - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^p} dx \leq \int_{\Omega} |\nabla u(x)|^p dx, \]
and by a density argument it is satisfied for all functions \( u \in H^1_0(\Omega) \). If the origin \( \{0\} \) belongs to the domain \( \Omega \), the constant \( \frac{N - p}{p} \) is optimal, but not attained in \( H^1_0(\Omega) \).

This approach can be generalized to deduce Hardy type inequalities for degenerate elliptic operators. For the operators \( \Delta_\lambda \) with functions \( \lambda_i \) of the form \( (1) \) and a function \( u \) of class \( C^1(\Omega) \) we define
\[ \nabla_\lambda u := (\lambda_1 \nabla_{x(1)} u, \ldots, \lambda_k \nabla_{x(k)} u), \quad \lambda_i \nabla_{x(i)} := (\lambda_i \partial_{x_1}, \ldots, \lambda_i \partial_{x_{N_j}}), \quad i = 1, \ldots, k. \]

We will obtain a wide family of Hardy type inequalities, that include as particular cases inequalities of the form
\[ \left( \frac{Q - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^\lambda} dx \leq \int_{\Omega} \psi(x) |\nabla_\lambda u(x)|^p dx, \]  \hfill (3)
\[ \left( \frac{Q - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^\lambda} dx \leq \int_{\Omega} |\nabla_\lambda u(x)|^p dx, \]  \hfill (4)
where \( Q \) is the homogeneous dimension, and \( \varphi \) and \( \psi \) are suitable weight functions. Moreover, \( [\cdot]_\lambda \) is a homogeneous norm that replaces the euclidean norm in the classical Hardy inequality.

We introduce the following notation. For a vector field \( \varphi \) of class \( C^1(\Omega; \mathbb{R}^N) \) we define
\[ \text{div}_\lambda \varphi := \sum_{i=1}^k \lambda_i \text{div}_{x(i)} \varphi, \quad \text{div}_{x(i)} \varphi := \sum_{j=1}^N \partial_{x(j)} \varphi, \]
and for \( \varepsilon > 0 \) we set
\[ \lambda^\varepsilon := (\lambda_1^\varepsilon, \ldots, \lambda_k^\varepsilon), \quad \lambda_i^\varepsilon(x) := \prod_{j=1}^k \left( |x(j)|^2 + \varepsilon \right)^{\alpha_{ij} \varepsilon^{\frac{1}{2}}}, \quad i = 1, \ldots, k. \]

The following lemma follows from the divergence theorem and can be shown similarly as inequality \( \text{(2)} \). See also Theorem 3.5 in \( [3] \) for the particular case of Grushin-type operators.
**Lemma 1.** Let \( \varepsilon \geq 0 \) and \( h \in C^1(\Omega; \mathbb{R}^N) \) be such that \( \text{div}_\lambda h > 0 \). Then, for every \( p > 1 \) and \( u \in C^1_0(\Omega) \) we have

\[
\int_\Omega |u(x)|^p \text{div}_\lambda h(x) \, dx \leq p \int_\Omega \frac{|h(x)|^p}{(\text{div}_\lambda h(x))^{p-1}} |\nabla u|^p \, dx.
\]

**Proof.** We define

\[
\sigma^\varepsilon := \begin{pmatrix}
I_1 & 0 & \cdots & 0 \\
0 & \lambda_2^\varepsilon I_2 & \vdots & \\
& \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_k^\varepsilon I_k
\end{pmatrix},
\]

where \( I_i \) denotes the identity matrix in \( \mathbb{R}^{N_i} \), \( i = 1, \ldots, k \). The divergence theorem implies

\[
0 = \int_{\partial \Omega} |u|^p h \cdot \sigma^\varepsilon \nu \, d\zeta = \int_\Omega \text{div}_\lambda ((|u|^p h) \sigma^\varepsilon) \, dx = \int_\Omega p |u|^{p-2} u \nabla x^\lambda \cdot h \, dx + \int_\Omega |u|^p \text{div}_\lambda h \, dx,
\]

where \( \nu \) denotes the outward unit normal at \( \zeta \in \partial \Omega \). Applying Hölder’s inequality we obtain

\[
\int_\Omega |u|^p \text{div}_\lambda h \, dx = -\int_\Omega p |u|^{p-2} u \nabla x^\lambda \cdot h \, dx \leq \int_\Omega p |u|^{p-1} |\nabla x^\lambda u||h| \, dx
\]

\[
\leq p \left( \int_\Omega |u|^p \text{div}_\lambda h \, dx \right)^{\frac{p-1}{p}} \left( \int_\Omega \frac{|h|^p}{(\text{div}_\lambda h)^{p-1}} |\nabla x^\lambda u|^p \, dx \right)^{\frac{1}{p}},
\]

and the statement of the lemma follows. \( \square \)

To illustrate our approach we first consider Hardy type inequalities of the form (3), i.e.,

\[
\left( \frac{Q-p}{p} \right)^p \int_\Omega \frac{|u(x)|^p}{||x||^p_\lambda} \, dx \leq \int_\Omega \psi(x) |\nabla x^\lambda u|^p \, dx,
\]

with a certain weight function \( \psi \) and homogeneous norm \( [\cdot]_\lambda \).

Motivated by Lemma (1) we look for a function \( h \) satisfying

\[
\text{div}_\lambda h(x) = \frac{Q-p}{||x||^p_\lambda}.
\]

If we choose

\[
h(x) = \frac{1}{||x||^p_\lambda} \begin{pmatrix}
\sigma_{1,x^{(1)}} \\
\vdots \\
\sigma_{k,x^{(k)}}
\end{pmatrix}
\]

we obtain

\[
\text{div}_\lambda h(x) = \frac{Q}{||x||^p_\lambda} - p \frac{1}{|x|^{p-1}_\lambda} \sum_{i=1}^k \sigma_i x^{(i)} \cdot \nabla x^{(i)}(||x||_\lambda).
\]

Consequently, the homogeneous norm \( [\cdot]_\lambda \) should fulfill the relation

\[
\sum_{i=1}^k \sigma_i x^{(i)} \cdot \nabla x^{(i)}(||x||_\lambda) = ||x||_\lambda.
\]

(5)

On the other hand, computing the absolute value of \( h \) we obtain

\[
|h(x)|^2 = \frac{1}{||x||^2_\lambda} \prod_{i=1}^k \frac{1}{\lambda_i(x)} \left( \prod_{j \neq 1} \lambda_j(x)^2 |x^{(1)}|^2 + \cdots + \prod_{j \neq k} \lambda_j(x)^2 |x^{(k)}|^2 \right).
\]
which motivates to consider the homogeneous norm

$$[[x]]_{\lambda} = \left( \prod_{j \neq 1} \lambda_j (x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j \neq k} \lambda_j (x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1 + \sum_{i=1}^{k} (\sigma_i - 1))}}. \quad (6)$$

The exponent is determined by requiring $[[\cdot]]_{\lambda}$ to be $\delta_r$-homogeneous of degree one. Since the functions $\lambda_i$ are of the form (1), the relation (5) is satisfied.

4. Hardy Inequalities for $\Delta_{\lambda}$-Laplacians

4.1. Our homogeneous norms. As in Section 2 we consider $\Delta_{\lambda}$-Laplacians

$$\Delta_{\lambda} = \lambda_1^2 \Delta_{x^{(1)}} + \cdots + \lambda_k^2 \Delta_{x^{(k)}}$$

with functions $\lambda_i$ of the form (1),

$$\lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{|\alpha_{ij}|}, \quad i = 1, \ldots, k,$$

which are $\delta_r$-homogeneous of degree $\sigma_i - 1$ with respect to a group of dilations

$$\delta_r(x) = (r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)}), \quad x \in \mathbb{R}^N, \ r > 0.$$

Using our previous notations follow the relations

$$\sum_{j=1}^{k} \alpha_{ij} \sigma_j = \sigma_i - 1, \quad \lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{|\alpha_{ij}|}, \quad \prod_{i=1}^{k} \lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{\sum_{i=1}^{k} \alpha_{ij}}.$$

**Definition 2.** As in (6) we define the homogenous norm $[[\cdot]]_{\lambda}$ associated to the $\Delta_{\lambda}$-Laplacian by

$$[[x]]_{\lambda} := \left( \prod_{i \neq 1} \lambda_i (x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} \lambda_i (x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1 + \sum_{i=1}^{k} (\sigma_i - 1))}}, \quad x \in \mathbb{R}^N.$$

Under our hypothesis the homogeneous norm can be written as

$$[[x]]_{\lambda} = \left( \prod_{j=1}^{k} |x^{(j)}|^{\sum_{i \neq 1}^{k} 2\alpha_{ij} \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j=1}^{k} |x^{(j)}|^{\sum_{i \neq k}^{k} 2\alpha_{ij} \sigma_k^2 |x^{(k)}|^2} \right)^{\frac{1}{2(1 + \sum_{i=1}^{k} (\sigma_i - 1))}}.$$

We compute our homogeneous norm $[[\cdot]]_{\lambda}$ for some of the operators in our previous examples.

- For Grushin-type operators

$$\Delta_{\lambda} = \Delta_x + |x|^{2\alpha} \Delta_y, \quad (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where the constant $\alpha$ is non-negative, the definition leads to the same distance from the origin that D’Ambrosio considered in [3],

$$[[x, y]]_{\lambda} = \left( |x|^{2(1+\alpha)} + (1 + \alpha)^2 |y|^2 \right)^{\frac{1}{2(1+\alpha)}}.$$
• For operators of the form
  \[ \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} \Delta_z, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}, \]
  with non-negative constants \( \alpha \) and \( \beta \), we obtain
  \[ [[(x, y, z)]_\lambda] = \left( |x|^{2(1+\alpha+\beta)} + (1 + \alpha)^2 |x|^{2\beta} |y|^2 + (1 + \beta)^2 |x|^{2\alpha} |z|^2 \right)^{\frac{1}{2(1+\alpha+\beta)}}. \]

• For \( \Delta_\lambda \)-Laplacians of the form
  \[ \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^2 \Delta_z, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}, \]
  where the constants \( \alpha, \beta \) and \( \gamma \) are non-negative, we get
  \[ [[(x, y, z)]_\lambda] = \left( |y|^{2\gamma} |x|^{2(1+\alpha+\beta)} + (1 + \alpha)^2 |x|^{2\beta} |y|^2 |x|^{2\alpha} |z|^2 \right)^{\frac{1}{2(1+\alpha+\beta+\gamma)}}, \]
  where \( \mu = \beta + (1 + \alpha) \gamma. \)

**Proposition 3.** Our homogeneous norm \([[]]_\lambda \) satisfies the following properties:

1. It is \( \delta_\cdot \)-homogeneous of degree one, i.e.,
   \[ [[\delta_r(x)]_\lambda] = r [[x]]_\lambda. \]

2. It fulfills the relation
   \[ \sum_{i=1}^{k} \sigma_i (x^{(i)} \cdot \nabla x^{(i)}) [[x]]_\lambda = [[x]]_\lambda. \]

**Proof.** (1) Let \( x \in \mathbb{R}^N \). The homogeneity of the functions \( \lambda_i \) implies

\[
[[\delta_r(x)]_\lambda] = \left( \prod_{i \neq 1} (\lambda_i(\delta_r(x)))^2 \sigma_1^2 |r \sigma_1 x^{(1)}|^2 + \cdots + \prod_{i \neq k} (\lambda_i(\delta_r(x)))^2 \sigma_k^2 |r \sigma_k x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}}
\]

\[
= \left( \prod_{i \neq 1} r^{2\sigma_1} r^{2(\sigma_i - 1)} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} r^{2\sigma_k} r^{2(\sigma_i - 1)} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}}
\]

\[
= \left( r^{2+\sum_{i=1}^{2(\sigma_i - 1)} \prod_{i \neq 1} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}} = r [[x]]_\lambda.
\]

(2) We observe

\[
x^{(i)} \cdot \nabla x^{(i)} [[x]]_\lambda
\]

\[
= \frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))} \left( \prod_{i \neq 1} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{-\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}}
\]

\[
\left( 2 \sum_{j \neq 1} \alpha_{jr} \prod_{i \neq 1} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + 2 \sum_{j \neq k} \alpha_{jr} \prod_{i \neq k} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 + 2 \prod_{i \neq l} (\lambda_i(x))^2 \sigma_l^2 |x^{(l)}|^2 \right).
\]
4.2. Main results. We denote by $\dot{W}^1_{\lambda,p}(\Omega)$ the closure of $C^1_0(\Omega)$ with respect to the norm
\[
\|u\|_{\dot{W}^1_{\lambda,p}(\Omega)} := \left( \int_{\Omega} |\nabla_\lambda u(x)|^p \, dx \right)^{\frac{1}{p}},
\]
and for $\psi \in L^1_{loc}(\Omega)$ such that $\psi > 0$ a.e. in $\Omega$ we define the space $\dot{W}^1_{\lambda,p}(\Omega,\psi)$ as the closure of $C^1_0(\Omega)$ with respect to the norm
\[
\|u\|_{\dot{W}^1_{\lambda,p}(\Omega,\psi)} := \left( \int_{\Omega} |\nabla_\lambda u(x)|^p \psi(x) \, dx \right)^{\frac{1}{p}}.
\]

**Theorem 4.** Let $p > 1$ and $\mu_1, \ldots, \mu_k$, $s \in \mathbb{R}$ be such that $Q > s - \sum_{i=1}^k \sigma_i \mu_i$. Then, for every $u \in \dot{W}^1_{\lambda,p}(\Omega,\psi)$, we have
\[
\left( \frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{p} \right)^p \int_{\Omega} \prod_{l=1}^k \frac{|x^{(l)}|^{\mu_i}}{\|x\|^p_{\lambda}} |u(x)|^p \, dx \leq \int_{\Omega} \psi(x) |\nabla_\lambda u(x)|^p \, dx,
\]
where $\psi(x) = \frac{[[x]]^{p(1+\sum_{i=1}^k (\sigma_i - 1)) - s}}{\prod_{l=1}^k |x^{(l)}|^{p\sum_{j=1}^k \alpha_{ji} - \mu_i}}$.

In particular, for $s = p$ and $\mu_1 = \cdots = \mu_k = 0$ we get
\[
\left( \frac{Q - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{\|x\|^p_{\lambda}} \, dx \leq \int_{\Omega} \frac{[[x]]^{\sum_{i=1}^k (\sigma_i - 1)}}{\prod_{l=1}^k \lambda_i(x)^p} |\nabla_\lambda u(x)|^p \, dx,
\]
and choosing $s = p(1 + \sum_{i=1}^k (\sigma_i - 1))$ and $\mu_i = p \sum_{j=1}^k \alpha_{ji}$ we obtain
\[
\left( \frac{Q - p}{p} \right)^p \int_{\Omega} \frac{\prod_{l=1}^k \lambda_i(x)^p}{\|x\|^p_{\lambda}^{p(1+\sum_{i=1}^k (\sigma_i - 1))}} |u(x)|^p \, dx \leq \int_{\Omega} |\nabla_\lambda u(x)|^p \, dx.
\]

**Proof.** We deduce the inequalities from Lemma II. To this end we define
\[
[[x]]_{\varepsilon,\lambda} := \left( \sum_{j=1}^k \left( \prod_{i \neq j} \lambda_i(x)^2 |x^{(j)}|^2 + \varepsilon \right) \right)^{\frac{1}{2(1+\sum_{i=1}^k (\sigma_i - 1))}}.
\]
and consider the function
\[ h_\varepsilon(x) := \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}}}{||x||_{\varepsilon,\lambda}^s} \left( \sigma_1 x^{(1)} \lambda_1^\varepsilon(x), \cdots, \sigma_k x^{(k)} \lambda_k^\varepsilon(x) \right). \]

We obtain
\[
\text{div}_{\varepsilon} h_\varepsilon(x) = \sum_{i=1}^{k} \nabla x^{(i)} \cdot \left( \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\mu_i}}{||x||_{\varepsilon,\lambda}^s} \sigma_i x^{(i)} \right)
= \prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}} \left( \sum_{i=1}^{k} \left( N_i \sigma_i + \sigma_i \mu_i |x^{(i)}|^2 - s \frac{1}{||x||_{\varepsilon,\lambda}^s} \sigma_i x^{(i)} \cdot \nabla x^{(i)} (||x||_{\varepsilon,\lambda}) \right) \right)
= \prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}} \left( Q + \sum_{i=1}^{k} \sigma_i \mu_i |x^{(i)}|^2 + \varepsilon - s \frac{1}{||x||_{\varepsilon,\lambda}^s} \sum_{i=1}^{k} \sigma_i x^{(i)} \cdot \nabla x^{(i)} (||x||_{\varepsilon,\lambda}) \right),
\]
and using Proposition 3 we observe that
\[
\lim_{\varepsilon \to 0} \frac{1}{||x||_{\varepsilon,\lambda}^s} \sum_{i=1}^{k} \sigma_i x^{(i)} \cdot \nabla x^{(i)} (||x||_{\varepsilon,\lambda}) = 1.
\]

Since \( Q > s - \sum_{i=1}^{k} \sigma_i \mu_i \) we have \( \text{div}_{\varepsilon} h_\varepsilon > 0 \) for all sufficiently small \( \varepsilon > 0 \). Moreover, we compute
\[
|h_\varepsilon(x)| = \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}}}{||x||_{\varepsilon,\lambda}^s} \left( \sum_{i=1}^{k} \sigma_i^2 |x^{(i)}|^2 \lambda_i^\varepsilon(x)^{\frac{1}{2}} \right)
= \prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}} \left( \sum_{i=1}^{k} \Pi_{j \neq i} \lambda_j^\varepsilon(x)^2 \sigma_i^2 |x^{(i)}|^2 \right)^{\frac{1}{2}} \frac{1}{||x||_{\varepsilon,\lambda}^s} \Pi_{i=1}^{k} \lambda_i^\varepsilon(x)
= \left( \sum_{i=1}^{k} \Pi_{j \neq i} \lambda_j^\varepsilon(x)^2 \sigma_i^2 |x^{(i)}|^2 \right)^{\frac{1}{2}} \frac{1}{||x||_{\varepsilon,\lambda}^s} \frac{1}{\Pi_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{1}{2} (\sum_{j=1}^{k} \alpha_{ji} - \mu_i)}},
\]
and Lemma 1 applied to \( h_\varepsilon \) yields
\[
\frac{1}{p^p} \int_{\Omega} \eta_\varepsilon(x) \prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\mu_i}{2}} |u(x)|^p dx
\leq \int_{\Omega} \eta_\varepsilon(x)^{-(p-1)} \frac{\left( \sum_{i=1}^{k} \Pi_{j \neq i} \lambda_j^\varepsilon(x)^2 \sigma_i^2 |x^{(i)}|^2 \right)^{\frac{1}{2}} \frac{1}{||x||_{\varepsilon,\lambda}^s}}{\Pi_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{1}{2} (\sum_{j=1}^{k} \alpha_{ji} - \mu_i)}} |\nabla u(x)|^p dx,
\]
where
\[
\eta_\varepsilon(x) = \left( Q + \sum_{i=1}^{k} \sigma_i \mu_i \frac{|x^{(i)}|^2}{|x^{(i)}|^2 + \varepsilon} - s \frac{1}{||x||_{\varepsilon,\lambda}^s} \sum_{i=1}^{k} \sigma_i x^{(i)} \cdot \nabla x^{(i)} (||x||_{\varepsilon,\lambda}) \right).
\]
Since
\[
\lim_{\varepsilon \to 0} \eta_\varepsilon(x) = Q + \sum_{i=1}^{k} \sigma_i \mu_i - s,
\]
the theorem now follows from the dominated convergence theorem by taking the limit \( \varepsilon \) tends to zero. \( \square \)

We formulated a very general family of Hardy-type inequalities, the parameters allow to adjust the weights and to move them from one side of the inequality to the other. Particular choices lead to inequalities of the form (3) or (4).

**Remark 1.** For Grushin-type operators \( \Delta_{\lambda} = \Delta_x + |x|^{2\alpha} \Delta_y \), \( \alpha \geq 0 \), \( (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \), we recover the Hardy inequalities of Theorem 3.1 in [3], where it was proved that the constants are optimal. Our results show that the hypothesis \( N_1 > p - \mu \) in Theorem 3.1 in [3] can be omitted.

Alternatively, we can consider homogeneous distances from the origin \( \| \cdot \|_{\lambda} \) that satisfy the relation

\[
\sum_{j=1}^{k} \sigma_j \left( x^{(j)} \cdot \nabla x^{(j)} \right) \| x \|_{\lambda} = \| x \|_{\lambda}, \quad x \in \mathbb{R}^N,
\]

for instance,

\[
\| x \|_{\lambda} := \left( \sum_{j=1}^{k} |x^{(j)}|^2 \prod_{i \neq j} \sigma_i \right)^{1/(2 \prod_{i=1}^{k} \sigma_i)}, \quad x \in \mathbb{R}^N,
\]

or

\[
\| x \|_{\lambda} := \left( \sum_{j=1}^{k} (\sigma_j |x^{(j)}|)^2 \prod_{i \neq j} \sigma_i \right)^{1/(2 \prod_{i=1}^{k} \sigma_i)}, \quad x \in \mathbb{R}^N.
\]

**Remark 2.** For Grushin-type operators the second distance \( \| \cdot \|_{\lambda} \) coincides with our homogeneous norm \([\cdot]_{\lambda}\) and with the distance considered by D’Ambrosio in [3].

We compute the first of the homogeneous distances for our previous examples.

- For operators of the form

\[
\Delta_{\lambda} = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} \Delta_z, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},
\]

with non-negative constants \( \alpha \) and \( \beta \), we obtain

\[
\|(x, y, z)\|_{\lambda} = \left( |x|^{2(1+\alpha)(1+\beta)} + |y|^{2(1+\beta)} + |z|^{2(1+\alpha)} \right)^{1/(2(1+\alpha)(1+\beta))}.
\]

- For \( \Delta_{\lambda} \)-Laplacians of the form

\[
\Delta_{\lambda} = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},
\]

where the constants \( \alpha, \beta \) and \( \gamma \) are non-negative, we get

\[
\|(x, y, z)\|_{\lambda} = \left( |x|^{2(1+\alpha)(1+\mu)} + |y|^{2(1+\mu)} + |z|^{2(1+\alpha)} \right)^{1/(2(1+\alpha)(1+\mu))},
\]

where \( \mu = \beta + (1 + \alpha) \gamma \).

For the convenience of the reader we first formulated Hardy type inequalities for the particular case of our homogeneous norms \([\cdot]_{\lambda}\). The following result generalizes our previous Theorem [4].
Theorem 5. Let $p > 1$ and $\mu_1, \ldots, \mu_k, s, t \in \mathbb{R}$ be such that $Q > s + t - \sum_{i=1}^{k} \mu_i \sigma_i$. Then, for every $u \in W^{1,p}_{\lambda}(\Omega, \psi)$ we have

$$
\left(\frac{Q - s - t + \sum_{i=1}^{k} \mu_i \sigma_i}{p}\right)^{\frac{p}{p-1}} \int_{\Omega} \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{\|x\|_\lambda^{p-1} [([x])_\lambda^s]} |u(x)|^p \, dx \leq \int_{\Omega} \psi(x) |\nabla u(x)|^p \, dx,
$$

where $\psi(x) = \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i - p \sum_{j=1}^{k} \alpha_{ij}}}{\|x\|_\lambda^{p(1+\sum_{i=1}^{k} (\sigma_i - 1))}}$.

In particular, for $s = 0, \mu_i = 0$ and $t = p$ we obtain

$$
\left(\frac{Q - p}{p}\right)^{\frac{p}{p-1}} \int_{\Omega} \frac{|u(x)|^p}{\|x\|_\lambda^{p-1}} \, dx \leq \int_{\Omega} \frac{|([x])_\lambda^{p(1+\sum_{i=1}^{k} (\sigma_i - 1))} \prod_{j=1}^{k} \lambda_j(x)^p}{\|
abla u(x)\|^p} \, dx.
$$

For $t = 0$ we recover our Hardy inequalities in Theorem 4 with our homogeneous norms $[\cdot]_\lambda$.

Proof. We deduce the inequalities from Lemma 1. To this end we define the function

$$
h_\varepsilon(x) := \frac{\prod_{i=1}^{k} |x^{(i)}|^{2 + \varepsilon}}{\|x\|_\lambda^s [([x])_\lambda^s]} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1(x)} , \ldots , \frac{\sigma_k x^{(k)}}{\lambda_k(x)}\right),
$$

where $\| \cdot \|_{\varepsilon, \lambda}$ is a smooth approximation of $\| \cdot \|_{\lambda}$, e.g.,

$$
\|x\|_{\varepsilon, \lambda} = \left(\sum_{j=1}^{k} (|x^{(j)}|^2 + \varepsilon) \Pi_{i \neq j} \sigma_i\right)^{\frac{1}{2}} \frac{1}{\prod_{i=1}^{k} \sigma_i},
$$

or

$$
\|x\|_{\varepsilon, \lambda} = \left(\sum_{j=1}^{k} (|x^{(j)}|^2 + \varepsilon) \Pi_{i \neq j} \sigma_i\right)^{\frac{1}{2}} \frac{1}{\prod_{i=1}^{k} \sigma_i}.
$$

Similarly to the proof of Theorem 4 we obtain

$$
|h_\varepsilon(x)| = \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{1}{2}}}{\|x\|_\lambda^s [([x])_\lambda^s]} \left(Q + \sum_{i=1}^{k} \sigma_i |x^{(i)}|^2 \right) - s \frac{1}{\|x\|_\lambda} \sum_{i=1}^{k} \sigma^i x^{(i)} \cdot \nabla x^{(i)}([x])_\lambda
$$

$$
- \frac{1}{\|x\|_\lambda} \sum_{i=1}^{k} \sigma^i x^{(i)} \cdot \nabla x^{(i)}([x])_\lambda.
$$

We define

$$
\eta_\varepsilon(x) := Q + \sum_{i=1}^{k} \sigma_i |x^{(i)}|^2 - s \frac{1}{\|x\|_\lambda} \sum_{i=1}^{k} \sigma^i x^{(i)} \cdot \nabla x^{(i)}([x])_\lambda
$$

$$
- \frac{1}{\|x\|_\lambda} \sum_{i=1}^{k} \sigma^i x^{(i)} \cdot \nabla x^{(i)}([x])_\lambda
$$
and observe that
\[ \lim_{\varepsilon \to 0} \eta_\varepsilon(x) = Q + \sum_{i=1}^{k} \sigma_i \mu_i - s - t. \]

By assumption, \( Q > s + t - \sum_{i=1}^{k} \sigma_i \mu_i \), which implies that \( \text{div}_{\lambda} h_\varepsilon > 0 \) for all sufficiently small \( \varepsilon > 0 \). Lemma \( \square \) applied to the function \( h_\varepsilon \) leads to the inequality
\[ \frac{1}{p^p} \int_{\Omega} \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\alpha_i j_i}{2}}}{\|x\|_{p,\lambda}} |u(x)|^p \; dx \leq \int_{\Omega} \eta_\varepsilon(x)^{(p-1)} \psi_\varepsilon(x) |\nabla_{\lambda} u(x)|^p \; dx, \]

where \( \psi_\varepsilon(x) = \frac{\prod_{i=1}^{k} (|x^{(i)}|^2 + \varepsilon)^{\frac{\alpha_i - p \sum_{j=1}^{k} \alpha_i j_j}{2}}}{\|x\|_{p,\lambda}^{\sum_{j=1}^{k} \sum_{j \neq i} \lambda_j^2 (x_j^2 + \varepsilon)^{-2}}} \). By taking the limit \( \varepsilon \) tends to zero the statement of the theorem follows. \( \square \)

Finally, we formulate Hardy type inequalities without weights.

**Theorem 6.** Let \( N_1 > p > 1 \). Then, for every \( u \in W^{1,p}_\lambda(\Omega) \) we have
\[
\left( \frac{N_1 - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{\|x\|^p_{\lambda}} \; dx \leq \int_{\Omega} |\nabla_{\lambda} u(x)|^p \; dx,
\]

and compute
\[
\text{div}_{\lambda} h_\varepsilon(x) = \frac{N_1 - p \frac{|x^{(1)}|^2}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}}}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} > 0,
\]

\[
|h_\varepsilon(x)| = \frac{|x^{(1)}|}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}}.
\]

Since \( N_1 > p \) we have \( \text{div}_{\lambda} h_\varepsilon > 0 \), and Lemma \( \square \) applied to \( h_\varepsilon \) yields the inequality
\[ \frac{1}{p^p} \int_{\Omega} \left( \frac{N_1 - p \frac{|x^{(1)}|^2}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}}}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} \right) \frac{|u(x)|^p}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} \; dx \leq \int_{\Omega} \left( \frac{N_1 - p \frac{|x^{(1)}|^2}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}}}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} \right)^{-(p-1)} \frac{|x^{(1)}|^p}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} |\nabla_{\lambda} u(x)|^p \; dx. \]

The first inequality of the theorem now follows from the dominated convergence theorem by taking the limit \( \varepsilon \) tends to zero. \( \square \)
5. SOME REMARKS ON THE OPTIMALITY OF THE CONSTANT

The family of Hardy inequalities in Theorem 4 coincides for the particular case of Grushin type operators with the results D’Ambrosio obtained in [3], where he proved that the constants in the inequalities are optimal. The optimality was shown similarly to the proof in the classical case using the explicit form of the function for which the Hardy inequality becomes an equality. For more general $\Delta_\lambda$-Laplacians this function is unknown, and at present we are not able to prove that our Hardy type inequalities are sharp.

The following observations indicate the relationship between the fundamental solution for the Laplacian, the Hardy inequality and the function used to show the optimality of the constant. We will only consider the case $p = 2$ here.

Let $\Omega \subset \mathbb{R}^N$ be a domain, $N \geq 3$, and $\Phi$ be the fundamental solution of $-\Delta$ on $\Omega$, i.e.,

$$-\Delta \Phi = c\delta_0, \quad \Phi > 0,$$

for some constant $c > 0$, where $\delta_0$ denotes the Dirac delta function. Moreover, let $u \in C_0^1(\Omega)$ and $v := u\Phi^{-\frac{1}{2}}$. Then, the following identities follow from integration by parts and the properties of the fundamental solution (see also [2]),

$$\int_{\Omega} |\nabla u|^2 dx = \frac{1}{4} \int_{\Omega} \frac{|\nabla \Phi|^2}{|\Phi|^2} u^2 dx + \frac{1}{2} \int_{\Omega} \nabla \Phi \nabla (v^2) dx + \int_{\Omega} |\nabla v|^2 \Phi dx$$

$$= \frac{1}{4} \int_{\Omega} \frac{|\nabla \Phi|^2}{|\Phi|^2} u^2 dx + \frac{1}{2} c v^2(0) + \int_{\Omega} |\nabla v|^2 \Phi dx$$

$$= \frac{1}{4} \int_{\Omega} \frac{|\nabla \Phi|^2}{|\Phi|^2} u^2 dx + \int_{\Omega} |\nabla v|^2 \Phi dx \geq \frac{1}{4} \int_{\Omega} \frac{|\nabla \Phi|^2}{|\Phi|^2} u^2 dx,$$  \hspace{1cm} (7)

where we used that $v(0) = u(0)\Phi(0)^{-\frac{1}{2}} = 0$. In particular, for $\Phi(x) = \frac{1}{|x|^{N-2}}$ we obtain

$$\frac{|\nabla \Phi(x)|^2}{|\Phi(x)|^2} = \frac{(N-2)^2}{|x|^2},$$

and (7) implies the classical Hardy inequality.

To show the optimality of the constant $\frac{(N-2)^2}{4}$ we consider the identity

$$\int_{\Omega} |\nabla u(x)|^2 - \frac{(N-2)^2}{4} \frac{u(x)^2}{|x|^2} dx = \int_{\Omega} \left| \nabla u(x) + \frac{N-2}{2} \frac{x}{|x|^2} u(x) \right|^2 dx.$$

A solution of the equation

$$\nabla u(x) = -\frac{N-2}{2} \frac{x}{|x|^2} u(x)$$

is the function

$$u(x) = \frac{1}{|x|^{\frac{N-2}{2}}}.$$

For this function the Hardy inequality becomes an equality, but it does not belong to $H^1(\Omega)$ if the domain $\Omega$ contains the origin $\{0\}$. Minimizing sequences are typically used to show that
the inequality is sharp, e.g., sequences of the form
\[ u_\varepsilon(x) = \frac{1}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}} - \frac{1}{(\varepsilon + 1)^{\frac{N-2}{2}}}, \quad \varepsilon > 0 \]
(see [6] or [3]).

The fundamental solution for the Grushin-type operator
\[ \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y, \quad \alpha \geq 0, \quad z = (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \]
is of the form
\[ \Phi(x,y) = \frac{c}{[[(x,y)]]_{\lambda}^{Q-2}}, \]
for some constant \( c \geq 0 \) (see [5]). The identity
\[ \int_{\Omega} |\nabla_\lambda u|^2 \, dz = \frac{1}{4} \int_{\Omega} \frac{|\nabla_\lambda \Phi|^2}{|\Phi|^2} u^2 \, dz + \int_{\Omega} |\nabla_\lambda \Phi|^2 \Phi \, dz, \]
where \( v = u \Phi^{-\frac{1}{2}} \), can be shown like the equality (7) above. In particular, we obtain the
Hardy type inequality
\[ \int_{\Omega} |\nabla_\lambda u(z)|^2 \, dz \geq \frac{1}{4} \int_{\Omega} \frac{|\nabla_\lambda \Phi(z)|^2}{|\Phi(z)|^2} u(z)^2 \, dz = \frac{(Q-2)^2}{4} \int_{\Omega} \frac{|x|^{2\alpha}}{[[(x,y)]]_{\lambda}^{2(1+\alpha)}} u(z)^2 \, dz, \]
which is a particular case of the inequalities in Theorem 4.

To show the optimality of the constant we consider the identity
\[ \int_{\Omega} |\nabla_\lambda u(z) - \varphi(z) u(z)|^2 \, dz = \int_{\Omega} |\nabla_\lambda u(z)|^2 + |u(z)|^2 (|\varphi(z)|^2 + \text{div}_\lambda \varphi(z)) \, dz \]
and observe that the function
\[ \varphi(x,y) = -\frac{Q-2}{2} \frac{|x|^{2\alpha}}{[[(x,y)]]_{\lambda}^{2(1+\alpha)}} \left( x, \frac{(1 + \alpha)y}{|x|^\alpha} \right), \]
which we applied in the proof of Theorem 4, satisfies
\[ |\varphi(x,y)|^2 + \text{div}_\lambda \varphi(x,y) = -\left( \frac{Q-2}{2} \right)^2 \frac{|x|^{2\alpha}}{[[(x,y)]]_{\lambda}^{2(1+\alpha)}}. \]

A solution of the equation
\[ \nabla_\lambda u(x,y) = -\frac{Q-2}{2} \frac{|x|^{2\alpha}}{[[(x,y)]]_{\lambda}^{2(1+\alpha)}} \left( x, \frac{(1 + \alpha)y}{|x|^\alpha} \right) u(x,y) \]
is the function
\[ u(x,y) = \frac{1}{[[(x,y)]]_{\lambda}^{Q-2}}, \]
which D’Ambrosio used in [3] to prove the optimality of the constant. It transforms the Hardy
inequality into an equality, but does not belong to the class \( \tilde{W}_{\lambda}^{1,2}(\Omega) \) if the domain \( \Omega \) contains
the origin (see [3], p.728).

We finally discuss \( \Delta_\lambda \)-Laplacians with functions \( \lambda \) of the form (11). The fundamental solution
as well as the function that transforms the Hardy inequality into an equality are unknown
in this general case. If $\Phi$ denotes the fundamental solution, $u \in C_0^1(\Omega)$ and $v = u\Phi^{-\frac{1}{2}}$, the following identity follows like the equality (7) for the classical Laplacian

$$
\int_{\Omega} |\nabla u|^2 dx = \frac{1}{4} \int_{\Omega} |\nabla \Phi|^2 u^2 dx + \int_{\Omega} |\nabla v|^2 \Phi dx \geq \frac{1}{4} \int_{\Omega} |\nabla \Phi|^2 u^2 dx.
$$

On the other hand, suitable to analyze the optimality of the constants in our family of Hardy type inequalities is the relation

$$
\int_{\Omega} \left| \frac{\varphi(x)}{\psi(x)} u(x) - \psi(x) \nabla u(x) \right|^2 dx = \int_{\Omega} \psi(x)^2 |\nabla u(x)|^2 + u(x)^2 \left( \frac{|\varphi(x)|^2}{\psi(x)^2} + \text{div}_x \varphi(x) \right) dx,
$$

which follows from integration by parts, where $\varphi : \mathbb{R} \to \mathbb{R}^N$ is a vector field and $\psi : \mathbb{R} \to \mathbb{R}$ a scalar function. Comparing with the first inequality in Theorem 4 we choose

$$
\psi(x)^2 = \frac{[x]^2 \left( 1 + \sum_{i=1}^k (\sigma_i - 1) - s \right)}{\prod_{i=1}^k [x(i)]^2 \sum_{j=1}^k \alpha_j - \mu_i},
$$

and observe that the function

$$
\varphi(x) = -\frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \prod_{i=1}^k |x(i)|^{\mu_i} \left( \frac{\sigma_1 x_1^{(1)}}{\lambda_1(x)}, \ldots, \frac{\sigma_k x_k^{(k)}}{\lambda_k(x)} \right),
$$

which we used to prove the theorem, satisfies

$$
\left( \frac{|\varphi(x)|^2}{\psi(x)^2} + \text{div}_x \varphi(x) \right) = - \left( \frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \prod_{i=1}^k |x(i)|^{\mu_i} \right)^2 \frac{\prod_{i=1}^k |x(i)|^{\mu_i}}{[x]^2 \sum_{j=1}^k \alpha_j - \mu_i}.
$$

Consequently, the Hardy type inequality in Theorem 4 is an equality if $u$ is a solution of the equation

$$
\nabla \varphi(x) = \frac{\varphi(x)}{\psi(x)^2} u(x),
$$

i.e.,

$$
\nabla_{x(i)} u(x) = -\frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \frac{\prod_{j \neq i} \lambda_j(x)^2}{[x]^2 \sum_{j=1}^k (\sigma_j - 1)} \sigma_i x_i^{(i)} u(x), \quad i = 1, \ldots, k, \quad (9)
$$

which we are unable to solve in the general case.

We consider this inequality (9) for our previous examples and $s = 2$, $\mu_1 = \cdots = \mu_k = 0$.

- For operators of the form

  $$
  \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} \Delta_z, \quad \alpha, \beta \geq 0, \quad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},
  $$

  our homogeneous norm is

  $$
  \|[(x, y, z)]_\lambda = \left( |x|^{2(1+\alpha+\beta)} + (1 + \alpha)^2 |x|^2 |y|^2 + (1 + \beta)^2 |x|^2 |z|^2 \right)^{\frac{1}{2(1+\alpha+\beta)}},
  $$

  where $\lambda$ is the scalar function.
and we obtain the system of equations
\[
\nabla_x \ln u(x, y, z) = -\frac{Q-2}{2} \frac{|x|^{2(\alpha+\beta)}x}{|x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^2 + (1+\beta)^2|x|^{2\alpha}|z|^2},
\]
\[
\nabla_y \ln u(x, y, z) = -\frac{Q-2}{2} \frac{|y|^{2\gamma}y}{|y|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^2 + (1+\beta)^2|x|^{2\alpha}|z|^2},
\]
\[
\nabla_z \ln u(x, y, z) = -\frac{Q-2}{2} \frac{|z|^{2\alpha}z}{|z|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^2 + (1+\beta)^2|x|^{2\alpha}|z|^2}.
\]

For the operators
\[
\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z, \quad \alpha, \beta, \gamma \geq 0, \ (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},
\]
our homogeneous norm was defined as
\[
[[\lambda]] = \left( |y|^{2\gamma} |x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\beta)^2|x|^{2\alpha}|z|^2 \right)^{\frac{1}{2(1+\alpha+\beta)}}.
\]
where \( \lambda = \beta + (1+\alpha)\gamma \), and we have to solve the system
\[
\nabla_x \ln u(x, y, z) = -\frac{Q-2}{2} \frac{|x|^{2(\alpha+\beta)}y^{2\gamma}x}{|y|^{2\gamma} |x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\mu)^2|x|^{2\alpha}|z|^2},
\]
\[
\nabla_y \ln u(x, y, z) = -\frac{Q-2}{2} \frac{|y|^{2\gamma}y^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\mu)^2|x|^{2\alpha}|z|^2}{|y|^{2\gamma} |x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\mu)^2|x|^{2\alpha}|z|^2},
\]
\[
\nabla_z \ln u(x, y, z) = -\frac{Q-2}{2} \frac{(1+\mu)|x|^{2\alpha}z}{|y|^{2\gamma} |x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\mu)^2|x|^{2\alpha}|z|^2}.
\]

Using the identity \( \nabla \) for the inequalities involving homogeneous distances \( \| \cdot \|_\lambda \) in Theorem \( \mathbb{S} \) we find
\[
\psi(x)^2 = \frac{1}{\|x\|_\lambda^2} \sum_{j=1}^k \left( \frac{\sigma_j |x(j)|}{\lambda_j(x)} \right)^2.
\]
Moreover, the function
\[
\varphi(x) = -\frac{Q-2}{2} \frac{1}{\|x\|_\lambda^2} \left( \frac{\sigma_1 x^{(1)}}{\lambda_1(x)} , \ldots , \frac{\sigma_k x^{(k)}}{\lambda_k(x)} \right)
\]
satisfies
\[
\frac{|\varphi(x)|^2}{\psi(x)^2} + \text{div}_\lambda \varphi(x) = \left( \frac{Q-2}{2} \frac{1}{\|x\|_\lambda^2} \right)^2.
\]
This leads to the equation
\[
\nabla_\lambda u(x) = \frac{\varphi(x)}{\psi(x)^2} u(x),
\]
i.e.,
\[
\nabla_{\lambda}^{(i)} \ln u(x) = \frac{1}{u^{(i)}(x)} \nabla^{(i)} u(x) = -\frac{Q-2}{2} \frac{\prod_{j \neq i} \lambda_j(x)^2}{|[x]|_{\lambda}^{2(1+\sum_{j=1}^k (\sigma_j-1))}} \sigma_i x^{(i)},
\]
which coincides with our previous problem \( \mathbb{S} \).
Remark 3. Finally, we remark that with respect to the group of dilations
\[ \delta_r(x) = (r^{\sigma_1}x^{(1)}, \ldots, r^{\sigma_k}x^{(k)}) , \]
the \( \lambda \)-gradient is \( \delta_r \)-homogeneous of degree 1,
\[ \nabla_\lambda(u(\delta_r(x))) = r(\nabla_\lambda u)(\delta_r(x)) , \]
and the function
\[ \eta(x) = \frac{\varphi(x)}{\psi(x)^2} \]
is \( \delta_r \)-homogeneous of degree \(-1\),
\begin{align*}
\eta(\delta_r(x)) &= \frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \prod_{j=1}^k \lambda_j(\delta_r(x))^2 \frac{\left( \sigma_1(\delta_r x^{(1)}) \right) \cdots \left( \sigma_k(\delta_r x^{(k)}) \right)}{\left( \lambda_1(\delta_r x^{(1)}) \right) \cdots \left( \lambda_k(\delta_r x^{(k)}) \right)} , \\
&= \frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \prod_{j=1}^k \frac{\lambda_j(x)^2}{r^{2(1+\sum_{i=1}^k (\sigma_i - 1))} [[x]]^{2(1+\sum_{i=1}^k (\sigma_i - 1))}} \left( \frac{\sigma_1(x)^{(1)}}{r^{\sigma_1-1} \lambda_1(x)} \right) \cdots \left( \frac{\sigma_k(x)^{(k)}}{r^{\sigma_k-1} \lambda_k(x)} \right) \\
&= r^{-1} \eta(x) .
\end{align*}
Consequently, we deduce from equality (11) the relation
\[ \frac{\nabla_\lambda(u(\delta_r(x)))}{u(\delta_r(x))} = \frac{r(\nabla_\lambda u)(\delta_r(x))}{u(\delta_r(x))} = r \eta(\delta_r(x)) = \eta(x) = \frac{\nabla_\lambda u(x)}{u(x)} .
\]

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