A forcing extension in which no absolute $\kappa$-Borel set of density $\kappa \in (\aleph_1, c)$ condenses onto compacta

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Abstract

It is consistent that the continuum be arbitrary large, there is a forcing extension in which no absolute $\kappa$-Borel set $X$ of density $\kappa$, $\aleph_1 < \kappa < c$, condenses onto a compact metric space.

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1. Introduction

The density $d(X)$ of a topological space $X$ is the smallest cardinality of a dense subset of $X$. The cardinal function $w(X)$ is the weight of $X$, which is defined by $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \omega$. For a metrizable space $X$, we have $d(X) = w(X)$.

Recall that the family of hyper-Borel sets of $X$, denoted $HB(X)$, to be the smallest family of subsets of $X$ which contains (i) the open sets of $X$, (ii) $X \setminus B$ whenever $B \in HB(X)$, and (iii) $\bigcup B_t$ whenever each $B_t \in HB(X)$ and $\{B_t\}$ is a $\sigma$-discrete family of subsets of $X$. A set $B \subset X$ is called $\kappa$-Borel if $B$ is hyper-Borel in $X$ of class $\alpha$ for some ordinal $\alpha$ of cardinal $\leq \kappa$. The $\aleph_0$-Borel sets are the ordinary Borel sets [4].

A space $X$ is called absolute $\kappa$-Borel, if $X$ is homeomorphic to a $\kappa$-Borel subset of some complete metrizable space. Thus, absolute $\aleph_0$-Borel sets are the ordinary absolute Borel sets.

Since metrizable compact spaces have cardinality at most continuum, every metric space admitting a condensation (i.e. a bijective continuous...
mapping) onto a compactum (= compact metric space) has density at most continuum.

Let $\text{FIN}(\kappa, 2)$ be the partial order of finite partial functions from $\kappa$ to 2, i.e., Cohen forcing.

**Proposition 1.1.** (Corollary 3.13 in [1]) Suppose $M$ is a countable transitive model of $\text{ZFC} + \text{GCH}$. Let $\kappa$ be any cardinal of $M$ of uncountable cofinality which is not the successor of a cardinal of countable cofinality. Suppose that $G$ is $\text{FIN}(\kappa, 2)$-generic over $M$, then in $M[G]$ the continuum is $\kappa$ and for every uncountable $\gamma < \kappa$ if $F : \gamma^\omega \to \omega^\omega$ is continuous and onto, then there exists a $Q \in [\gamma]^\omega$ such that $F(Q^\omega) = \omega^\omega$.

2. Main result

**Theorem 2.1.** It is consistent that the continuum be arbitrary large, there is a forcing extension in which no absolute $\kappa$-Borel set $X$ of density $\kappa$, $\aleph_1 < \kappa < \aleph_2$, condenses onto a metrizable compact space.

**Proof.** Suppose $M$ is a countable transitive model of $\text{ZFC} + \text{GCH}$. Suppose that $G$ is $\text{FIN}(\kappa, 2)$-generic over $M$ (Proposition 1.1).

The proof of Theorem 3.7 in [3] uses Cohen reals, but the same idea shows that this generic extension has the property that

(*) for every family $\mathcal{F}$ of Borel subsets of $\omega^\omega$ with size $\aleph_1 < |\mathcal{F}| < \aleph_2$, if $\bigcup \mathcal{F} = \omega^\omega$ then there exists $\mathcal{F}_0 \in [\mathcal{F}]^{\omega_1}$ with $\bigcup \mathcal{F}_0 = \omega^\omega$ (see Proposition 3.14 in [1]).

Fix $\aleph_1 < \kappa < \aleph_2$. Let $X$ be an absolute $\kappa$-Borel set of density $\kappa$.

Assume that there is a condensation $g$ of $X$ onto a compact metric space $K$. Since $X$ is an absolute $\kappa$-Borel set of density $\kappa$ there is a continuous bijection $f : A \to X$ where $A$ is a closed subset of $\kappa^\omega$ (Theorem 5 in [4]) and a continuous surjection $q : \kappa^\omega \to A$ (Theorem 4 in [3]). Since $f$ is a continuous bijection and $d(X) = \kappa$, $d(A) = \kappa$.

Then we have the continuous bijection $h = g \circ f : A \to K$ from $A$ onto $K$.

Let $\sum = [\kappa]^\omega \cap M$. Note that $|\sum| < \aleph_2$ since in $M$, $|\kappa^\omega| > \kappa$ if and only if $\kappa$ has cofinality $\omega$, but in that case $|\kappa^\omega| = |\kappa^+| < \aleph_2$. Since the forcing is c.c.c. $M[G] \models \kappa^\omega = \bigcup \{Y^\omega : Y \in \sum\}$.

Let $\sum' := \{Y \in \sum : Y^\omega \cap A \neq \emptyset\}$. For any $Y \in \sum'$ the continuous image $h(Y^\omega \cap A)$ (note that $Y^\omega \cap A$ is Polish because $A$ is closed) is an analytic
set (a $\Sigma^1_1$ set) and, hence the union of $\omega_1$ Borel sets in $K$ (see Ch.3, § 39, Corollary 3 in [2]).

Thus $h(Y^\omega \cap A) = \bigcup\{B(Y, \beta) : \beta < \omega_1\}$ where $B(Y, \beta)$ is a Borel subset in $K$ for each $Y \in \sum'$ and $\beta < \omega_1$.

Let $\theta = \min\{|S| : S \subseteq \{B(Y, \beta) : Y \in \sum', \beta < \omega_1\}\}$ and $\bigcup S = K$. Note that $\theta \leq |\sum'| \leq |\sum| < c$.

Claim 1. $\theta \geq \kappa$.

Assume that $\theta < \kappa$. Let $S = \{B(Y_\zeta, \beta_\zeta) : \zeta \in \theta\}$. Consider a function $\phi : \{B(Y_\zeta, \beta_\zeta) : \zeta \in \theta\} \to \sum'$ such that $\phi(B(Y_\zeta, \beta_\zeta)) = Y_\zeta \in \sum'$ where $h(Y_\zeta, Y_\zeta) \cap A)$ contains in decomposition the set $B(Y_\zeta, \beta_\zeta)$ and $\zeta \in \theta$. Since $h$ is a condensation and $K = \bigcup\{h(Y_\zeta, \zeta, A) : \zeta \in \theta\}$, $X = \bigcup\{f(Y_\zeta) : \zeta \in \theta\}$ and $A \subseteq \bigcup\{Y_\zeta, \zeta \in \theta\}$.

Let $Q = \bigcup\{Y_\zeta, \zeta \in \theta\}$ then $Q \in [\kappa]^{<\theta}$. Note that $A \subseteq \bigcup\{Y_\zeta, \zeta \in \theta\} \subseteq \bigcup\omega, \omega \subset \kappa^\omega$ and $w(Q^\omega) \leq \theta$. Since $f$ is continuous, $w(f(Q^\omega)) \leq w(Q^\omega)$. But $\kappa = w(X) = w(f(Q^\omega)) \leq w(Q^\omega) \leq \theta$ is a contradiction.

Thus, $\kappa \leq \theta \leq |\sum| < c$.

Since $K$ is Polish, there is a continuous surjection $p : \omega^\omega \to K$. Given a family $\mathcal{F} = \{p^{-1}(B(Y_\zeta, \beta_\zeta) : \zeta \in \theta\}$ of $\theta$-many Borel sets ($\aleph_1 < \theta < c$) whose union is $\omega^\omega$.

By property $(\star)$, there is a subfamily $\mathcal{F}_0 = \{F_\alpha : F_\alpha = p^{-1}(B(Y_\zeta, \beta_\zeta) : \alpha < \omega_1\}$ of size $\omega_1$ whose union is $\omega^\omega$. Then the family $\{B(Y_\zeta, \beta_\zeta) : \alpha < \omega_1\}$ of size $\omega_1$ whose union is $K$. It follows that $\theta \leq \aleph_1$, by Claim 1, it is a contradiction.

Proposition 2.2. There is a forcing extension in which for any $\kappa \in (\aleph_1, c)$ there exists a metric space $X$ of density $\kappa$ such that $X$ condenses onto $[0, 1]$ but any completion $\tilde{X}$ of $X$ is not condensed onto a metrizable compact space.

Consider the forcing extension in Theorem 2.1. Let $[0, 1] = \prod\{A_\alpha : \alpha \in \kappa\}$ be a partition such that $A_\alpha$ is dense in $[0, 1]$ for each $\alpha \in \kappa$. Let $X = \bigoplus\{A_\alpha : \alpha \in \kappa\}$. Clear that $X$ condenses onto $[0, 1]$, but any completion $\tilde{X}$ of $X$ is an absolute Borel set of density $\kappa$. Hence, by Theorem 2.1 $\tilde{X}$ is not condensed onto a metrizable compact space.

References

[1] W.R. Brian, A.W. Miller, Partitions of $2^\omega$ and completely ultrametrizable spaces, Topology and its Applications, 184 (2015), 61–71.
[2] K. Kuratowski, Topology I, Academic Press, New York, 1966.

[3] A.W. Miller, *Infinite combinatorics and definability*, Annals of Pure and Applied Logic, 41 (1989), 179–203.

[4] R.W. Hansell, *On the nonseparable theory of Borel and Souslin sets*, Bull. Am. Math. Soc., 78:2 (1972), 236–241.

[5] A.H. Stone, *Non-separable Borel sets*, Rozpr. Math., 28 (1962), 3–40.