GEOMETRIC INVARIANCE OF THE SEMI-CLASSICAL CALCULUS
ON NILPOTENT GRADED LIE GROUPS

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Abstract. In this paper, we consider the semi-classical setting constructed on nilpotent graded Lie groups by means of representation theory. Our aim is to analyze the effects of the pull-back by diffeomorphisms on pseudodifferential operators. We restrict to diffeomorphisms that preserve the filtration and prove that they are uniformly Pansu differentiable. We show that the pull-back of a semi-classical pseudodifferential operator by such a diffeomorphism has a semi-classical symbol that is expressed at leading order in terms of the Pansu differential. Finally, we interpret the geometric meaning of this invariance in the setting of filtered manifolds.

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1. Introduction

Semi-classical analysis started to develop in the 70s and has proved a flexible tools for the analysis of PDEs [4]. It crucially relies on a microlocal viewpoint and the use of semi-classical pseudodifferential calculus. Such an approach has been recently developed on nilpotent graded Lie groups with the ambition of taking into account the non-commutativity of the group by using representation theory and the associated Fourier theory [2, 9, 6]. The symbols of the pseudodifferential operators are then fields of operators on the product $G \times \hat{G}$ of the group $G$ and its dual space $\hat{G}$. Like in the Euclidean case [16], this semi-classical pseudodifferential calculus enjoys a non-commutative symbolic calculus which describes taking the adjoints and the composition as asymptotic sums of symbols [9, 6]. These tools prove efficient for studying the propagation of oscillations or concentration effects [7] of Schrödinger equations, or related questions in PDEs such as the existence of observability inequalities [8].

In the Euclidean case, invariance to leading order by change of variables is an important property of the semiclassical pseudodifferential calculus. This invariance allows for the identification of the symbol as a function on the cotangent space, and is therefore the foundation of the semiclassical calculus on manifolds [16]. Here we investigate this property for symbols on graded Lie groups: what happens to the semi-classical pseudodifferential operators introduced in [6] when conjugated by the change of variables induced by a local smooth diffeomorphism that preserves the filtration of the group.

It turns out that a smooth function $\Phi$ between two graded Lie groups $G$ and $H$ that preserves the filtration of the groups is uniformly Pansu differentiable (see Theorem 1.5 below). After the publication of the present paper to Journal of Geometric Analysis, it was pointed out to us that this analysis had been performed in greater generality on filtered manifolds in [3, Section 7]. However, our analysis here is done from first principles on graded nilpotent Lie groups. Indeed, part of the article consists in revisiting the concept of Pansu differentiability in the context of graded nilpotent Lie groups and its link with being filtration preserving. This study allows us to define one-to-one maps between the phase spaces $G \times \hat{G}$ and $H \times \hat{H}$ associated with any smooth local diffeomorphisms $\Phi$ between two nilpotent groups $G$ and $H$ (see Theorem 1.8 below). We investigate the geometric interpretation of these results in the setting of filtered manifolds in the last Section 4.

1.1. Graded groups. A graded group $G$ is a connected simply connected nilpotent Lie group whose (finite dimensional, real) Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-gradation into linear subspaces, i.e.

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} \mathfrak{g}_j \quad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \quad 1 \leq i \leq j.$$ 

With this way of describing $\mathfrak{g}$, all but a finite number of subspaces $\mathfrak{g}_j$ are trivial. We denote by $j = n_G$ the smallest integer such that all the subspaces $\mathfrak{g}_j$, $j > n_G$, are trivial. If the first stratum $\mathfrak{g}_1$ generates the whole Lie algebra, then $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$ for all $j \in \mathbb{N}_0$ and $n_G$ is the step of the group; the group $G$ is then said to be stratified, and also (after a choice of basis or inner product for $\mathfrak{g}_1$) Carnot.

The product law on $G$ is derived from the exponential map $\exp_G : \mathfrak{g} \to G$ and the Dynkin formula for the Baker-Campbell-Hausdorff formula (see [9, Theorem 1.3.2])

$$(1.1) \quad X \star Y := \ln_G (\exp_G(X) \exp_G(Y)) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} \sum_{r,s \in \mathbb{N}_0^\ell \atop r_1+s_1 > 0} c_{r,s} \text{ad}^{r_1} X \text{ad}^{s_1} Y \ldots \text{ad}^{r_{\ell-1}} X \text{ad}^{s_{\ell-1}} Y(Y).$$
where the coefficients $c_{r,s}$ are known:

$$c_{r,s}^{-1} = \left( \sum_{j=1}^{\ell} (r_j + s_j) \right) \Pi_{i=1}^{\ell} r_i! s_i!$$

Above, the sum over $\ell$ is finite in the nilpotent case. In particular, the term for which $s_\ell > 0$ or $r_\ell = 0$ and $r_\ell > 1$ is zero, while the term $\text{ad} \text{Xad}^{-1} Y(Y)$ for $s_\ell = 0$, $r_\ell = 1$ is understood as $X$. Here $\ln_G$ denotes the inverse map to $\exp_G$; we may drop the subscript $G$ for $\exp$ and $\ln$ when the context is clear.

The exponential mapping is a global diffeomorphism from $g$ onto $G$. Once a basis $X_1, \ldots, X_n$ for $g$ has been chosen, we may identify the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with the points $x = \exp(x_1 X_1 + \cdots + x_n X_n)$ in $G$. It allows us to define the (topological vector) spaces $C^\infty(G)$ and $S(G)$ of smooth and Schwartz functions on $G$ identified with $\mathbb{R}^n$; note that the resulting spaces are intrinsically defined as spaces of functions on $G$ and do not depend on a choice of basis.

The exponential map induces a Haar measure $dx$ on $G$ which is invariant under left and right translations and defines Lebesgue spaces on $G$, together with a (non-commutative) convolution for functions $f_1, f_2 \in S(G)$ or in $L^2(G)$,

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1} x) dy, \quad x \in G.$$

We now construct a basis adapted to the gradation. Set $n_j = \dim g_j$ for $1 \leq j \leq n_G$. We choose a basis $\{X_1, \ldots, X_{n_1}\}$ of $g_1$ (this basis is possibly reduced to $\emptyset$), then $\{X_{n_1+1}, \ldots, X_{n_1+n_2}\}$ a basis of $g_2$ (possibly $\{0\}$) and so on. Such a basis $B = (X_1, \ldots, X_n)$ of $g$ is said to be adapted to the gradation; here $n = \dim g = n_1 + \cdots + n_G$.

The Lie algebra $g$ is a homogeneous Lie algebra equipped with the family of dilations $\{\delta_r, r > 0\}$, $\delta_r : g \to g$, defined by $\delta_r X = r^\ell X$ for every $X \in g_\ell$, $\ell \in \mathbb{N}$ [10]. We re-write the set of integers $\ell \in \mathbb{N}$ such that $g_\ell \neq \{0\}$ into the increasing sequence of positive integers $v_1, \ldots, v_n$ counted with multiplicity, the multiplicity of $g_{v_i}$ being its dimension. In this way, the integers $v_1, \ldots, v_n$ become the weights of the dilations and we have $\delta_r X_j = r^{v_j} X_j$, $j = 1, \ldots, n$, on the chosen basis of $g$. The associated group dilations are defined by

$$\delta_r(x) = r x := (r^{v_1} x_1, r^{v_2} x_2, \ldots, r^{v_n} x_n), \quad x = (x_1, \ldots, x_n) \in G, \quad r > 0.$$

In a canonical way, this leads to the notions of homogeneity for functions and operators. It also motivates the definition of quasi-norms on $G$ as continuous functions $|\cdot| : G \to [0, +\infty)$ homogeneous of degree 1 on $G$ which vanishes only at 0. This often replaces the Euclidean norm in the analysis on homogeneous Lie groups. Any quasi-norm $|\cdot|$ on $G$ satisfies a triangle inequality up to a constant:

$$\exists C \geq 1, \quad \forall x, y \in G, \quad |xy| \leq C(|x| + |y|).$$

Any two homogeneous quasi-norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent in the sense that

$$\exists C > 0, \quad \forall x \in G, \quad C^{-1} |x|_2 \leq |x|_1 \leq C |x|_2.$$

For example, the Haar measure is $Q$-homogeneous where

$$Q := \sum_{\ell \in \mathbb{N}} \ell n_\ell = v_1 + \cdots + v_n$$

is called the homogeneous dimension of $G$.

1.2. Diffeomorphisms between graded groups.
1.2.1. Filtration preserving maps. In this paragraph, we explain the notion of a (smooth or at least \( C^1 \)) map preserving the natural filtrations between two graded nilpotent groups \( G \) and \( H \). We first need to set some notation. For each \( x \in G \), we denote the differential of the left-translation \( L_x : G \to G \) by \( x \in G \) at the identity by \( \tau_x : G \to T_x G \). For \( y \in H \), we denote by \( \tau_y : H \to T_y H \) the analogue map on \( H \); we may omit the superscript \( G \) or \( H \) if the context is clear. These maps allow us to view derivatives between Lie groups as acting on the Lie algebras of the groups: if \( \Phi \) is a smooth map from an open set \( U \) of \( G \) to \( H \), then we define for each \( x \in U \)
\[
\partial_x \Phi := \left( \tau_{\Phi(x)}^H \right)^{-1} \circ D_x \Phi \circ \tau_x^G.
\]
By definition, the following diagram commutes:
\[
\begin{array}{ccc}
D_x \Phi & \rightarrow & T_{\Phi(x)} H \\
\tau_x^G & \uparrow & \tau_{\Phi(x)}^H \\
\mathfrak{g} & \rightarrow & \mathfrak{h} \\
\partial_x \Phi & \\
\end{array}
\]
Note that an equivalent definition for \( \partial_x \Phi : \mathfrak{g} \to \mathfrak{h} \) is given by
\[
\partial_x \Phi(V) = \lim_{t \to 0} t^{-1} \ln_H \left( \Phi(x)^{-1} \Phi(x \exp_G(tV)) \right), \quad V \in \mathfrak{g}.
\]
In this article, we will use the shorthand:
\[
\Phi_x(y) := \Phi(x)^{-1} \Phi(xy),
\]
and for instance (1.2) may be rewritten as
\[
\partial_x \Phi(V) = \lim_{t \to 0} t^{-1} \ln_H (\Phi_x(\exp_G(tV))) = \partial_{t=0}\Phi_x(\exp_G(tV)), \quad V \in \mathfrak{g}.
\]

**Definition 1.1.** Let \( \Phi \) be a smooth function from an open set \( U \) of \( G \) to \( H \). We say that \( \Phi \) preserves the filtration at \( x \in U \) when we have
\[
\partial_x \Phi(\mathfrak{g}_j) \subset \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_j, \quad j = 1, 2, \ldots.
\]

1.2.2. Pansu differentiability. For smooth diffeomorphisms (or at least sufficiently differentiable), the property of preserving the filtration is related to the notion of Pansu differentiability, which we define following [11]:

**Definition 1.2.** Let \( \Phi \) be a map from an open set \( U \) of \( G \) into \( H \). The function \( \Phi \) is Pansu differentiable at the point \( x \in U \) when for any \( z \in G \), the following limit exists:
\[
\lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-1} \left( \Phi(x)^{-1} \Phi(x \delta_{\varepsilon} z) \right) = \lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-1} \left( \Phi_x(\delta_{\varepsilon} z) \right).
\]
The limit is denoted by \( \text{PD}_x \Phi(z) \); the map \( z \to \text{PD}_x \Phi(z) \) is called the Pansu derivative of \( \Phi \) at \( x \).

When defining Pansu differentiability on an open set, we add a further hypothesis of uniformity:

**Definition 1.3.** Let \( \Phi \) be a map from an open set \( U \) of \( G \) into \( H \). The function \( \Phi \) is uniformly Pansu differentiable on \( U \) when
\[
(1) \ \Phi \text{ is Pansu differentiable at every point } x \in U,
(2) \text{ the limit in (1.4) holds locally uniformly on } U \times G.
\]
Part (2) of this definition means that for every point \((x, z) \in U \times G\), there exists a compact neighborhood of \((x, z)\) in \(U \times G\) where the limit in (1.4) holds uniformly. Note that this implies that the limit in (1.4) holds uniformly on any compact subset of \(U \times G\). A similar assumption is made in Section 4 of [15] with a subtle difference: the limit in (1.4) is supposed to be locally uniform only in \(z \in G\); however, the Pansu derivative at every point is required to be an automorphism of the group \(G\). We will see in Section 2 that condition (2) in Definition 1.3 implies that the Pansu derivative is a group morphism, and an automorphism when \(\Phi\) is a diffeomorphism.

**Example 1.4.** Let \(\Phi : \mathbb{H} \to \mathbb{R}^3; (p, q, t) \mapsto (p, q, t)\) be the identity map from the Heisenberg group to \((\mathbb{R}^3, +)\). Equip both groups with the gradation corresponding to the non-isotropic dilations \(\delta_\varepsilon(p, q, t) = (\varepsilon p, \varepsilon q, \varepsilon^2 t)\). One computes, for \(x = (p_1, q_1, t_1)\) and \(y = (p_2, q_2, t_2)\),
\[
\delta_\varepsilon^{-1}(\Phi(x)^{-1}\Phi(x\delta_\varepsilon y)) = \delta_\varepsilon^{-1}(x\delta_\varepsilon y - x) = (p_2, q_2, t_2 + \frac{1}{2\varepsilon}(p_1 q_2 - p_2 q_1))
\]
which converges as \(\varepsilon \to 0\) if and only if \(p_1 q_2 = p_2 q_1\). In particular, the identity map between \(\mathbb{H}\) and \((\mathbb{R}^3, +)\) is Pansu differentiable at the origin but not in a neighborhood of the origin.

The following property is crucial for our analysis and will be proved in Section 2 below. It states that for smooth functions, the notion of uniform Pansu differentiability on an open set coincides with preserving filtration above this set:

**Theorem 1.5.** Let \(\Phi\) be a smooth function from an open set \(U\) of \(G\) to \(H\). The map \(\Phi\) is uniformly Pansu differentiable on \(U\) if and only if \(\Phi\) preserves the filtration at every point \(x \in U\). Besides, for such a map \(\Phi\), the Pansu derivative yields a smooth function \((x, z) \mapsto \text{PD}_x \Phi(z)\) on \(U \times G\).

This result has been proved in [15] in the case of Carnot groups. The filtration preserving maps then are contact maps. The Heisenberg group is a prime example of Warhust’s result [15]. One contribution of this paper is to extend the result to the case of graded groups with a proof ‘from first principle’. After the publication of the present paper to Journal of Geometric Analysis, it was pointed out to us that this analysis had been performed in greater generality on filtered manifolds in [3, Section 7].

Our proof of Theorem 1.5 will also yields the following properties:

**Corollary 1.6.** We continue with the notation and setting of Theorem 1.5. Let \(\Phi\) be a smooth function from an open set \(U\) of \(G\) which is uniformly Pansu differentiable on \(U\).

- For each \(x \in U\), \(f \circ \text{PD}_x \Phi \in C^\infty(G)\) (resp. \(S(G)\)) if \(f \in C^\infty(H)\) (resp. \(S(H)\)). The map \(f \mapsto (x \mapsto f \circ \text{PD}_x \Phi)\) yields continuous morphisms of topological vector spaces from \(C^\infty(H)\) to \(C^\infty(U, C^\infty(G))\) and also from \(S(H)\) to \(C^\infty(U, S(G))\).
- If \(\Phi\) is a smooth diffeomorphism from \(U\) onto its image, then \(G\) and \(H\) are isomorphic.

1.3. The dual set and the semi-classical pseudodifferential calculus.

1.3.1. The dual set. Recall that a representation \((\mathcal{H}_\pi, \pi)\) of \(G\) is a pair consisting of a Hilbert space \(\mathcal{H}_\pi\) and a group morphism \(\pi\) from \(G\) to the set of unitary transforms on \(\mathcal{H}_\pi\). In this paper, the representations will always be assumed (unitary) strongly continuous, and their Hilbert spaces separable. A representation is said to be irreducible if the only closed subspaces of \(\mathcal{H}_\pi\) that are stable under \(\pi\) are \(\{0\}\) and \(\mathcal{H}_\pi\) itself. Two representations \(\pi_1\) and \(\pi_2\) are equivalent if there exists a unitary transform \(U\) called an intertwining map that sends \(\mathcal{H}_{\pi_1}\) on \(\mathcal{H}_{\pi_2}\) with
\[
\pi_1 = U^{-1} \circ \pi_2 \circ U.
\]
The dual set \(\hat{G}\) is obtained by taking the quotient of the set of irreducible representations by this equivalence relation. We may still denote by \(\pi\) the elements of \(\hat{G}\) and we keep in mind that different
representations of the class are equivalent through intertwining operators. The following properties are straightforward:

**Lemma 1.7 (Pullback of Irreps).** Let θ : G → H be a continuous group homomorphism, and (π, ℋπ) be a representation of H. Then

- π ◦ θ is a representation of G,
- π ↦ π ◦ θ preserves unitarity and the unitary equivalence class of π,
- π ↦ π ◦ θ preserves irreducibility if θ is surjective.

Hence if θ is surjective, we have a map

\[ \theta^* : \hat{H} \longrightarrow \hat{G} \]
\[ [\pi] \longmapsto [\pi \circ \theta]. \]

In particular, any automorphism of G induces an automorphism of \( \hat{G} \), and hence any subgroup of Aut(G) acts on \( \hat{G} \). For instance, the dilations \( \delta_r, r > 0 \), on a graded Lie group G provide an action of \( \mathbb{R}^+ \) on \( \hat{G} \) via

\[ \delta_r \pi(x) = \pi(rx), \quad x \in G, \quad \pi \in \hat{G}, \quad r > 0. \]

1.3.2. The Fourier transform. The Fourier transform of an integrable function \( f \in L^1(G) \) at a representation \( \pi \) of G is the operator acting on \( \mathcal{H}_\pi \) via

\[ \mathcal{F}_G(f)(\pi) = \int_G f(z) (\pi(z))^* dz. \]

Note that \( \mathcal{F}_G(f)(\pi) \in \mathcal{L}(\mathcal{H}_\pi) \); in this paper, we denote by \( \mathcal{L}(\mathcal{H}) \) the Banach space of bounded operator on the Hilbert space \( \mathcal{H} \). Note also that if \( \pi_1, \pi_2 \) are two equivalent representations of G with \( \pi_1 = U^{-1} \circ \pi_2 \circ U \) for some intertwining operator \( U \), then

\[ \mathcal{F}_G(f)(\pi_1) = U^{-1} \circ \mathcal{F}_G(f)(\pi_2) \circ U. \]

Hence, this defines the measurable field of operators \( \{ \mathcal{F}_G(f)(\pi), \pi \in \hat{G} \} \) modulo equivalence. Here, the unitary dual \( \hat{G} \) is equipped with its natural Borel structure, and the equivalence comes from quotienting the set of irreducible representations of G together with understanding the resulting fields of operators modulo intertwiners.

The Plancherel measure is the unique positive Borel measure \( \mu \) on \( \hat{G} \) such that for any \( f \in \mathcal{C}_c(G) \), we have the Plancherel formula

\[ \int_G |f(x)|^2 dx = \int_{\hat{G}} \| \mathcal{F}_G(f)(\pi) \|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi). \]

Here \( \| \cdot \|_{HS(\mathcal{H}_\pi)} \) denotes the Hilbert-Schmidt norm on \( \mathcal{H}_\pi \). This implies that the group Fourier transform extends unitarily from \( L^1(G) \cap L^2(G) \) to \( L^2(\hat{G}) \) onto \( L^2(\hat{G}) := \int_{\hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* d\mu(\pi) \) which we identify with the space of \( \mu \)-square integrable fields on \( \hat{G} \).

The Plancherel formula also yields an inversion formula for any \( f \in \mathcal{S}(G) \) and \( x \in G \):

\[ f(x) = \int_{\hat{G}} \text{Tr}_{\mathcal{H}_\pi} \left( \pi(x) \mathcal{F}_G(f) \right) d\mu(\pi), \]

where \( \text{Tr}_{\mathcal{H}_\pi} \) denotes the trace of operators in \( \mathcal{L}(\mathcal{H}_\pi) \), the set of bounded linear operators on \( \mathcal{H}_\pi \). This formula makes sense since, for \( f \in \mathcal{S}(G) \), the operators \( \mathcal{F}_G(f) \) are trace-class and \( \int_{\hat{G}} | \text{Tr}_{\mathcal{H}_\pi} \mathcal{F}_G(f) | d\mu(\pi) \) is finite.
1.3.3. Semiclassical analysis. For an open set $U \subset G$, we denote by $\mathcal{A}_0(U \times \hat{G})$ the space of symbols $\sigma = \{\sigma(x, \pi) : (x, \pi) \in U \times \hat{G}\}$ of the form

$$\sigma(x, \pi) = \mathcal{F}_G \kappa_x(\pi) = \int_G \kappa_x(y)(\pi(x))^* dy,$$

where $\kappa \in C_0^\infty(U, S(G))$; that is, the map $x \mapsto \kappa_x$ is compactly supported on $U$ and valued in the set of Schwartz class functions $S(G)$ with Schwartz norms depending smoothly on $x \in U$. We call $x \mapsto \kappa_x$ the convolution kernel of $\sigma$.

As the Fourier transform is injective, it yields a one-to-one correspondence between $\mathcal{A}_0(U \times \hat{G})$ and $C_0^\infty(U, S(G))$. In this way, $\mathcal{A}_0(U \times \hat{G})$ inherits from $C_0^\infty(U, S(G))$ the structures of topological vector space and smooth compactly supported section of the Schwartz bundle over $U$. These structures are easier to grasp than being a set of measurable fields of operators on $U \times \hat{G}$ modulo intertwiners as described in Section 1.3.2. The results of this paper develop a deeper geometric interpretation of the space of symbols, here $\mathcal{A}_0(U \times \hat{G})$, in terms of densities; this will be discussed in Section 4.

With the symbol $\sigma \in \mathcal{A}_0(U \times \hat{G})$, we associate the (family of) semi-classical pseudodifferential operators $\text{Op}_\varepsilon(\sigma)$, $\varepsilon \in (0, 1]$, defined via

$$\text{Op}_\varepsilon(\sigma)f(x) = \int_{\pi \in \hat{G}} \text{Tr}_{H_\pi}(\pi(x)\sigma(x, \delta_\varepsilon \pi)\mathcal{F}_G f(\pi)) d\mu(\pi), \ f \in S(G), \ x \in U,$$

or equivalently, in terms of the convolution kernel $\kappa_x = \mathcal{F}_G^{-1}\sigma(x, \cdot)$,

$$\text{Op}_\varepsilon(\sigma)f(x) = f \ast \kappa_x^{(\varepsilon)}(x), \ f \in S(G), \ x \in U,$$

where $\kappa_x^{(\varepsilon)}$ is the following rescaling of the convolution kernel:

$$\kappa_x^{(\varepsilon)}(y) := \varepsilon^{-Q} \kappa_x(\delta_\varepsilon^{-1}y).$$

This second expression for $\text{Op}_\varepsilon(\sigma)$ explains the choice of vocabulary for the convolution kernel of a symbol.

The rescaled convolution kernel is different from the integral kernel of $\text{Op}_\varepsilon(\sigma)$, which is given by:

$$(x, y) \mapsto \varepsilon^{-Q} \kappa_x(\delta_\varepsilon^{-1}(y^{-1}x)).$$

We emphasize the convolution kernel over the integral kernel, although the latter is used in the pioneer work [12] on pseudodifferential theory on filtered manifolds (see also [13, 14]). The reason is that our approach here is different and aims to be symbolic. It is inspired from semi-classical analysis in Euclidean setting and based on the fact that the symbols obtained via the Fourier transforms of convolution kernels proved a flexible tool for dealing with applications (see [8, 7]).

Our semi-classical pseudodifferential theory may be used to analyse the oscillations of families $(u^\varepsilon)_{\varepsilon > 0}$ that are bounded in $L^2(G)$. One considers the functionals $\ell_\varepsilon$ defined on $\mathcal{A}_0(U \times \hat{G})$ by

$$\ell_\varepsilon(\sigma) = \langle \text{Op}_\varepsilon(\sigma)u^\varepsilon, u^\varepsilon \rangle_{L^2(G)}, \ \sigma \in \mathcal{A}_0(U \times \hat{G}).$$

The limit points of $\ell_\varepsilon$ as $\varepsilon$ goes to 0 have some structures. When the family $(u^\varepsilon)_{\varepsilon > 0}$ is $L^2$-normalized and after possibly further extraction of subsequences, the limit points of $\ell_\varepsilon$ define states of the $C^*$-algebra $\mathcal{A}(U \times \hat{G})$ obtained by completion of $\mathcal{A}_0(U \times \hat{G})$ for the norm

$$\|\sigma\|_{L^\infty(U \times \hat{G})} := \sup_{(x, \pi) \in U \times \hat{G}} \|\sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

For describing the structure of these limit points, we consider the set of pairs $(\gamma, \Gamma)$ where $\gamma$ is a positive Radon measure on $U \times \hat{G}$ and

$$\Gamma = \{\Gamma(x, \pi) \in \mathcal{L}(\mathcal{H}_\pi) : (x, \pi) \in U \times \hat{G}\}.$$
We want to study the case of $I^\kappa_\varepsilon$ defined on $U$ by Theorem 1.5.

Given the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in $(0, +\infty)$ with $\varepsilon_k \to 0$ as $k \to +\infty$ and a pair $\Gamma \in \mathcal{M}_+^+(U \times \hat{G})$ such that

$$\forall \varepsilon \in \mathcal{A}_0(U \times \hat{G}), \quad \text{Op}_\varepsilon(\sigma)u^{\varepsilon_k} \in L^2(G)$$

for $\varepsilon$-almost every $(x, \pi) \in U \times \hat{G}$. The equivalence class of $(\gamma, \Gamma)$ is denoted by $\Gamma \delta \gamma$, it is called a positive vector-valued measure. We denote by $M_{\text{ov}}^+(U \times \hat{G})$ the set of these equivalence classes. The positive continuous linear functionals of the $C^\ast$-algebra $\mathcal{A}(U \times \hat{G})$ is naturally identified with $M_{\text{ov}}^+(U \times \hat{G})$.

If $(u^\varepsilon)_{\varepsilon > 0}$ is a bounded family of $L^2(U)$, there exist a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in $(0, +\infty)$ with $\varepsilon_k \to 0$ and a pair $\Gamma \delta \gamma \in M_{\text{ov}}^+(U \times \hat{G})$ such that we have

$$\forall \varepsilon \in \mathcal{A}_0(U \times \hat{G}), \quad \text{Op}_\varepsilon(\sigma)u^{\varepsilon_k} \in L^2(G) \quad \text{as} \quad k \to +\infty$$

The positive vector-valued measure $\Gamma \delta \gamma$ is called a semi-classical measure of the family $(u^\varepsilon)$ for the sequence $\varepsilon_k$.

Our aim is to analyze how these notions can be transferred from (an open subset of) a graded group $G$ to another one $H$ via a smooth local diffeomorphism that preserves the filtration of the group.

1.4. **Main result.** Let us consider two graded groups, $G$ and $H$, and a smooth diffeomorphism, $\Phi$, from an open set $U$ of $G$ to $H$ that is filtration preserving and thus uniformly Pansu differentiable by Theorem 1.5.

We denote by $J_\Phi$ the Jacobian of $\Phi$ with respect to Haar measures chosen on $G$ and $H$, and we consider the operator $\mathcal{U}_\Phi$ which associates to a function $f$ defined on $\Phi(U) \subset H$ the function

$$\mathcal{U}_\Phi(f) := J_\Phi^{1/2} f \circ \Phi$$

defined on $U \subset G$. The map $\mathcal{U}_\Phi$ is a unitary operator from $L^2(\Phi(U))$ onto $L^2(U)$.

Conjugation by $\mathcal{U}_\Phi$ pulls back any $S \in \mathcal{L}(L^2(\Phi(U)))$ to an operator

$$\mathcal{U}_\Phi \circ S \circ \mathcal{U}_\Phi^{-1} \in \mathcal{L}(L^2(U)).$$

We want to study the case of $S = \text{Op}_\varepsilon(\sigma)$, $\sigma \in \mathcal{A}_0(\Phi(U) \times \hat{H})$.

The map $\Phi$ also induces a transformation $\mathcal{I}_\Phi$ between convolution kernels. Indeed, for a function $\kappa : x \mapsto \kappa_x$ in $C^\infty_c(\Phi(U), S(H))$, we associate the function $\mathcal{I}_\Phi \kappa$ defined via

$$\mathcal{I}_\Phi \kappa(z) = J_\Phi(x) \kappa_{\Phi(x)}(\text{PD}_x \Phi(z)), \quad \forall (x, z) \in U \times G.$$  

By Corollary 1.6 $\mathcal{I}_\Phi \kappa$ is in $C^\infty_c(U, S(G))$ and $\mathcal{I}_\Phi$ is an isomorphism of topological vector spaces from $C^\infty_c(\Phi(U), S(H))$ onto $C^\infty_c(U, S(G))$. The geometric meaning of this map is discussed in Section 4.

By Lemma 1.7 the map

$$\mathcal{G}_\Phi : \begin{cases} U \times \hat{G} & \to \Phi(U) \times \hat{H} \\ (x, \pi) & \mapsto (\Phi(x), \pi \circ (\text{PD}_x \Phi)^{-1}) \end{cases}$$
is well-defined. Moreover, it induces an isomorphism of topological vector spaces
\[(\hat{G}\Phi)^*: \mathcal{A}_0(\Phi(U) \times \hat{H}) \to \mathcal{A}_0(U \times \hat{G}),\]
defined in the following way: for \(\sigma \in \mathcal{A}_0(\Phi(U) \times \hat{H})\), the symbol \((\hat{G}\Phi)^*\sigma\) given by
\[(1.11) \quad (\hat{G}\Phi)^*\sigma(x, \pi) := \sigma(\Phi(x), \pi \circ PD_x\Phi^{-1}), \quad (x, \pi) \in U \times \hat{G},\]
is in \(\mathcal{A}_0(U \times \hat{G})\). Indeed, its convolution kernel is \(I_{\Phi\kappa}\) when \(\kappa: x \mapsto \kappa_x\) denotes the convolution kernel of \(\sigma\).

**Theorem 1.8.** Let \(G\) and \(H\) be two graded groups, and \(\Phi\) a smooth diffeomorphism from an open set \(U\) of \(G\) to \(H\). Assume that \(\Phi\) is filtration preserving on \(U\) (and thus uniformly Pansu differentiable on \(U\)). Let \(\sigma \in \mathcal{A}_0(\Phi(U) \times \hat{H})\), then in \(\mathcal{L}(L^2(U))\),
\[U_{\Phi} \circ \text{Op}_\varepsilon(\sigma) \circ U_{\Phi}^{-1} = \text{Op}_\varepsilon((\hat{G}\Phi)^*\sigma) + O(\varepsilon).\]

Note that when we assume that the smooth diffeomorphism \(\Phi\) is uniformly Pansu differentiable on \(U\), this implies that the graded groups \(G\) and \(H\) are isomorphic (see Corollary 1.6).

Theorem 1.8 is proved in Section 3 below. It crucially relies on the Pansu differentiability of filtration preserving maps (see Section 2). It also has straightforward consequences on semi-classical measures that we now present. We associate with \(\Phi\) a map from \(\mathcal{M}^+_{ov}(U \times \hat{G})\) into \(\mathcal{M}^+_{ov}(\Phi(U) \times \hat{H})\) which maps \(\Gamma d\gamma \in \mathcal{M}^+_{ov}(U \times \hat{G})\) on \(\Gamma^{\Phi} d\gamma^{\Phi} \in \mathcal{M}^+_{ov}(\Phi(U) \times \hat{H})\) defined by
\[
\int_{\Phi(U) \times \hat{H}} \text{Tr}_{\mathcal{H}_{\varepsilon'}}((\sigma(x', \pi') \Gamma^{\Phi}(x', \pi')) \, d\gamma^{\Phi}(x', \pi') = \int_{U \times \hat{G}} \text{Tr}_{\mathcal{H}_{\varepsilon}}(((\hat{G}\Phi)^*\sigma)(x, \pi) \Gamma(x, \pi)) \, d\gamma(x, \pi).
\]
Then, the result of Theorem 1.8 implies the next corollary.

**Corollary 1.9.** Let \(G\) and \(H\) be two graded groups, and let \(\Phi\) a smooth diffeomorphism from an open set \(U\) of \(G\) to \(H\). Assume that \(\Phi\) is filtration preserving on \(U\) and thus uniformly Pansu differentiable on \(U\). Let \((f^\varepsilon)_{\varepsilon>0}\) be a bounded family in \(L^2(U)\) and \(\Gamma d\gamma\) be a semi-classical measure of \((f^\varepsilon)_{\varepsilon>0}\) for the sequence \(\varepsilon_k\). Then, \(\Gamma^{\Phi} d\gamma^{\Phi}\) is a semi-classical measure of the family \((U^*_{\Phi} f^\varepsilon)_{\varepsilon>0}\) for the sequence \(\varepsilon_k\).

The proof of Theorem 1.8 is developed in Section 3 and heavily uses the results of Section 2 via Theorem 1.5 and its consequences in Corollary 1.6. Similar results should hold in the microlocal case where no specific scale \(\varepsilon\) is chosen, as developed in [5]. Finally, we develop the geometric interpretation of the convolution kernel in Section 4.

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2. Filtration preserving diffeomorphisms

Our aim in this section is to prove the equivalence between the uniform Pansu differentiability on an open set and the preservation of the filtration above this set, see Theorem 1.5. We will start with introducing some notations and concepts in order to give a precise meaning to these properties.

2.1. Pansu differentiability. In this section, we recall basic properties of the Pansu derivative:

**Lemma 2.1.** Let \(\Phi\) be a map from an open set \(U\) of a graded Lie group \(G\) to another graded Lie group \(H\).

1. Let \(x \in U\). If \(\Phi\) is Pansu differentiable at the point \(x\), then the Pansu derivative at \(x\) is a 1-homogeneous map \(z \mapsto PD_x\Phi(z)\) from \(G\) to \(H\) satisfying \(PD_x\Phi(0_G) = 0_H\). Moreover, the function \(\Phi\) is continuous at \(x\).
(2) If \( \Phi \) is uniformly Pansu differentiable on \( U \), then the Pansu derivative at every point \( x \in U \) is a group morphism from \( G \) to \( H \):

\[
\forall z_1, z_2 \in G, \quad \PD_x \Phi(z_1) \circ \PD_x \Phi(z_2) = \PD_x \Phi(z_1z_2).
\]

**Proof.** (1) This comes from the definition and the observation that for \( r > 0 \),

\[
\delta_{r^{-1}} \left( \Phi(x)^{-1} \Phi(x\delta_r z) \right) = \delta_r \left[ \delta_{r^{-1}} \left( \Phi(x)^{-1} \Phi(x\delta_r z) \right) \right].
\]

Passing to the limit as \( r \) goes to 0, we obtain

\[
\PD_x \Phi(\delta_r z) = \delta_r [\PD_x \Phi(z)].
\]

Then, fixing a homogeneous quasi-norm on \( G \) and using that \( z = \delta_{|z|} \) with \( |z| = 1 \), one writes

\[
\Phi(x)^{-1} \Phi(xz) = \delta_{|z|} \left( \delta_{|z|}^{-1} \Phi(x)^{-1} \Phi(x\delta_{|z|} z) \right) = \delta_{|z|} (\PD_x z + O(|z|))
\]

Therefore, \( \Phi(x)^{-1} \Phi(xz) \rightarrow 0 \) as \( |z| \rightarrow 0 \), whence the continuity of \( x \mapsto \Phi(x) \).

(2) We write

\[
\delta_{\varepsilon^{-1}} \left( \Phi(x)^{-1} \Phi(x\delta_{\varepsilon}(z_1 z_2)) \right) = \delta_{\varepsilon^{-1}} \left( \Phi(x)^{-1} \Phi(x\delta_{\varepsilon}(z_1)) \right) \delta_{\varepsilon^{-1}} \left( \Phi(x\delta_{\varepsilon} z_1)^{-1} \Phi(x\delta_{\varepsilon}(z_1) \delta_{\varepsilon}(z_2)) \right)
\]

and use the uniformity of the convergence. \( \Box \)

The hypothesis of uniformity for the Pansu differentiability is needed to show that the Pansu derivative is a group morphism (see Part (2) above) but also for the following composition property:

**Lemma 2.2.** Let \( F, G, H \) be three graded Lie groups, let \( \Phi \) be a map from an open set \( U \) of \( G \) to \( H \), and let \( \Psi \) be a map from an open set \( U' \) of \( F \) to \( G \). We assume that \( \Psi(U') \subset U \). If \( \Phi \) and \( \Psi \) are uniformly Pansu differentiable on \( U \) and \( U' \) respectively, then their composition \( \Phi \circ \Psi \) is uniformly Pansu differentiable on \( U' \) with

\[
\forall w \in U', \ z \in F \quad \PD_w (\Phi \circ \Psi)(z) = \PD_{\Psi(w)} \Phi(\PD_w \Psi(z)).
\]

**Proof.** Write

\[
\tilde{z}_\varepsilon := \delta_{\varepsilon^{-1}} \left( \Psi(w)^{-1} \Psi(w\delta_{\varepsilon} z) \right),
\]

for \((w, z)\) ranging in a compact subset of \( U' \times F \) and \( \varepsilon \in (0, 1] \). Observe that

\[
\delta_{\varepsilon^{-1}} \left( \Phi \circ \Psi(w)^{-1} \Phi \circ \Psi(w\delta_{\varepsilon} z) \right) = \delta_{\varepsilon^{-1}} \left( \Phi(\Psi(w))^{-1} \Phi(\Psi(w)\delta_{\varepsilon} z) \right).
\]

By continuity of \( \Phi \), and uniform Pansu differentiability of \( \Psi \), the pair \((\Psi(w), \tilde{z}_\varepsilon)\) ranges in a compact subset of \( \Psi(U') \times G \) on which the map

\[
(w', z') \mapsto \delta_{\varepsilon^{-1}} \left( \Phi(w')^{-1} \Phi(w'\delta_{\varepsilon} z') \right)
\]

converges uniformly as \( \varepsilon \rightarrow 0 \) (by uniform Pansu differentiability of \( \Phi \)). Since \( \lim_{\varepsilon \rightarrow 0} \tilde{z}_\varepsilon = \PD_w \Psi(z) \), the limit of (2.1) as \( \varepsilon \rightarrow 0 \) is \( \PD_{\Psi(w)} \Phi(\PD_w \Psi(z)) \). \( \Box \)

**Remark 2.3.** Let us consider a map \( \Phi \) from the open set \( U \) of a graded group \( G \) to a graded group \( H \) such that \( \Phi \) is a bijection from \( U \) onto its image \( \Phi(U) \) which is open. If \( \Phi \) and its inverse \( \Phi^{-1} \) are uniformly Pansu differentiable on \( U \) and \( \Phi(U) \) respectively, then we may apply Lemmata 2.1 and 2.2 and use \( \PD_x \Id = \Id \) to obtain

\[
\forall x \in U \quad (\PD_x \Phi)^{-1} = \PD_{\Phi(x)} (\Phi^{-1}).
\]

In particular, the groups \( G \) and \( H \) are isomorphic.
2.2. Filtration preserving smooth maps. In this section, we study how to characterize smooth maps that are filtration preserving.

A matrix-valued viewpoint will be helpful for a deeper understanding of Definition 1.1. For a smooth function $\Phi$ from an open set $U$ of $G$ to $H$, we denote by $M_\Phi(x)$ the matrix of $\mathcal{d}_x\Phi$ for the bases

$$B = (X_1, X_2, \ldots, X_{\dim G}) \text{ and } C = (Y_1, Y_2, \ldots, Y_{\dim H})$$

of $\mathfrak{g} = \oplus_{j=1}^{\infty}\mathfrak{g}_j$ and $\mathfrak{h} = \oplus_{j=1}^{\infty}\mathfrak{h}_j$ adapted to the respective gradations (see Section 1.1). This matrix can be written by blocks $M_{\Phi,i,j}(x)$ associated with the gradation. As the map $\mathcal{d}_x\Phi$ is linear, in order to identify the blocks $M_{\Phi,i,j}(x)$, it is enough to let $V$ vary in $\mathfrak{g}_j$ and calculate the projection on $\mathfrak{h}_j$ of $\mathcal{d}_x\Phi(V)$. For this, we denote by $\text{pr}_{\mathfrak{h},j}$ the projection onto $\mathfrak{h}_j$ along $\oplus_{j' \neq j}\mathfrak{h}_{j'}$. We may allow ourselves to remove the subscript $\mathfrak{h}$ (and write $\text{pr}_j$ instead of $\text{pr}_{\mathfrak{h},j}$) when the context is clear.

Let us illustrate this point with the following equivalences:

**Lemma 2.4.** Let $\Phi$ be a smooth function from an open set $U$ of $G$ to $H$, and let $x \in U$.

The following are equivalent:

(i) $\Phi$ preserves the filtration at $x$,

(ii) the matrix $M_\Phi(x)$ defined above is block-upper-diagonal in the sense that all the blocks $M_{\Phi,i,j}(x)$, $i > j$, strictly below the diagonal are 0,

(iii) we have for any $i > j$

$$\forall V \in \mathfrak{g}_j, \quad \text{pr}_{\mathfrak{h},i}(\mathcal{d}_x\Phi(V)) = 0.$$

We obtain easily the following implication between preserving the filtration and uniform Pansu differentiability:

**Lemma 2.5.** Let $\Phi$ be a smooth function from an open set $U$ of $G$ to $H$, and let $x \in U$. If $\Phi$ is Pansu differentiable at $x$, then $\Phi$ preserves the filtration at $x$.

**Proof.** Let $V \in \mathfrak{g}_j$. We have by taking $t = \varepsilon^j$ in (1.2),

$$\text{pr}_{\mathfrak{h},i}(\mathcal{d}_x\Phi(V)) = \lim_{\varepsilon \to 0} \text{pr}_{\mathfrak{h},i}(\varepsilon^{-j} \ln_H(\Phi_x(\exp_G(\varepsilon^j V))))$$

while the properties of dilations yield

$$\text{pr}_{\mathfrak{h},i}(\varepsilon^{-j} \ln_H(\Phi_x(\exp_G(\varepsilon^j V)))) = \varepsilon^{i-j} \text{pr}_{\mathfrak{h},i} \circ \ln_H(\delta_{\varepsilon^{-1}}\Phi_x(\exp_G(\delta_{\varepsilon} V))).$$

As $\Phi$ is Pansu differentiable at $x$, the argument inside $\text{pr}_{\mathfrak{h},i} \circ \ln_H$ above has a limit. Hence, if $i > j$, we have $\text{pr}_{\mathfrak{h},i}(\mathcal{d}_x\Phi(V)) = 0$ and we conclude with Lemma 2.4.

In the next sections, we will analyze the reverse implication to the one in Lemma 2.5, using this matrix-valued point of view; this will give Theorem 1.5.

2.3. Characterization of Pansu differentiability for smooth maps. If $\Phi$ is Pansu differentiable at $x \in U$, we set

$$\mathfrak{p} \mathcal{d}_x\Phi := \ln_H \circ \text{PD}_x\Phi \circ \exp_G,$$

so that we have the following diagram:

$$\begin{array}{ccc}
G & \xrightarrow{\text{PD}_x\Phi} & H \\
\exp_G & \uparrow & \exp_H \\
\mathfrak{g} & \xrightarrow{\mathfrak{p} \mathcal{d}_x\Phi} & \mathfrak{h}
\end{array}$$

This defines the map $\mathfrak{p} \mathcal{d}_x\Phi : \mathfrak{g} \to \mathfrak{h}$. An equivalent definition for this map is given by

$$(2.2) \quad \mathfrak{p} \mathcal{d}_x\Phi(V) = \lim_{t \to 0} \delta_{t^{-1}} \ln_H(\Phi(x)^{-1}\Phi(x\exp_G(\delta t V))) = \lim_{t \to 0} \delta_{t^{-1}} \ln_H(\Phi_x(\exp_G(\delta t V))).$$
for \( V \in \mathfrak{g} \), having used the shorthand (1.3). Clearly, \( \Phi \) is Pansu differentiable at \( x \in U \) if and only if the limit in (2.2) exists for all \( V \in \mathfrak{g} \), and it is uniformly Pansu differentiable on \( U \) if and only if these limits hold locally uniformly on \( U \times \mathfrak{g} \).

The map \( p \cdot \partial_x \Phi \) may not be linear in general but it will be under mild hypotheses. Indeed, when \( z \mapsto \text{PD}_x \Phi(z) \) is a continuous group morphism, for instance when \( \Phi \) is uniformly Pansu differentiable on an open neighbourhood of \( x \), then \( p \cdot \partial_x \Phi \) is linear with

\[
\text{PD}_x \circ \exp_G(V) = (\exp_H \circ p \cdot \partial_x \Phi)(V) \quad \text{and} \quad p \cdot \partial_x \Phi(V) = \partial_{t=0} \text{PD}_x \circ \exp_G(tV), \quad V \in \mathfrak{g}.
\]

As above, we can adopt a matrix-valued point of view and define \( \text{PM}_\Phi(x) \) to be the matrix whose columns are the images of \( \mathcal{B} \) by \( p \cdot \partial_x \Phi \), that is, the vectors \( p \cdot \partial_x \Phi(X_j), \, j = 1, \ldots, \dim G \), expressed in the basis \( \mathcal{C} \). The (rectangular) matrix \( \text{PM}_\Phi(x) \) is of the same size as \( M_\Phi(x) \). It makes sense to look at \( \text{PM}_\Phi(x) \) and its blocks \( \text{PM}_{\Phi,i,j}(x) \) associated with the gradation, or rather to the quantities \( \text{pr}_{h,i}(p \cdot \partial_x \Phi(V)), \, V \in \mathfrak{g}_j \). As a consequence of the definitions of the objects involved and of homogeneous properties, for each \( V \in \mathfrak{g}_j \) with \( i, j = 1, 2, \ldots \),

\[
(2.3) \quad \lim_{\varepsilon \to 0} \varepsilon^{-i} \text{pr}_{h,i} \circ \ln_H(\Phi_x(\exp_G(\varepsilon^j V))) = \text{pr}_{h,i}(p \cdot \partial_x \Phi(V)).
\]

Actually, very few of the quantities \( \text{pr}_{h,i}(p \cdot \partial_x \Phi(V)), \, V \in \mathfrak{g}_j \), are non-zero:

**Lemma 2.6.** Let \( \Phi \) be a smooth function from an open set \( U \) of \( G \) to \( H \), and let \( x \in U \). We always have (regardless of whether \( \Phi \) is uniformly Pansu differentiable or preserves the filtration) for \( V \in \mathfrak{g}_j \)

\[
\lim_{\varepsilon \to 0} \varepsilon^{-i} \text{pr}_{h,i} \circ \ln_H(\Phi_x(\exp_G(\varepsilon^j V))) = \begin{cases} 0 & \text{when } j > i, \\ \text{pr}_{h,i}(\partial_x \Phi(V)) & \text{when } j = i. \end{cases}
\]

**Proof of Lemma 2.6.** For each \( x \in U \) and \( V \in \mathfrak{g}_j \), we have

\[
\varepsilon^{-i} \text{pr}_{h,i} \circ \ln_H(\Phi_x(\exp_G(\varepsilon^j V))) = \varepsilon^{-i} \text{pr}_{h,i} \varepsilon^{-j} \ln_H(\Phi_x(\exp_G(\varepsilon^j V)));
\]

and the last argument of \( \text{pr}_{h,i} \) tends to \( \partial_x \Phi(V) \) as \( \varepsilon \) goes to 0. \( \square \)

**Remark 2.7.** (1) By Lemma 2.6 and equation (2.3), if \( \Phi \) is Pansu differentiable at \( x \) then for each \( V \in \mathfrak{g}_j \) with \( i, j = 1, 2, \ldots \),

\[
\text{pr}_{h,i}(p \cdot \partial_x \Phi(V)) = \lim_{\varepsilon \to 0} \varepsilon^{-i} \text{pr}_{h,i} \circ \ln_H(\Phi_x(\exp_G(\varepsilon^j V))) = \begin{cases} 0 & \text{when } j > i, \\ \text{pr}_{h,i}(\partial_x \Phi(V)) & \text{when } j = i. \end{cases}
\]

(2) Let us give a matrix interpretation of Part (1). The matrix \( \text{PM}_\Phi(x) \) defined above is block-lower-diagonal in the sense that all the blocks \( \text{PM}_{\Phi,i,j}(x) \), \( i < j \), strictly above the diagonal are 0. Furthermore, the diagonal blocks coincide with those of \( M_\Phi(x) \). We will see later that one can say more (see Theorem 2.9). This matrix interpretation does not depend on the bases \( \mathcal{B} \) and \( \mathcal{C} \) chosen to describe the matrix as long as they are adapted to the gradations.

The existence of the limits on the left-hand side of (2.3) turns out to also be a sufficient condition for a smooth map to be Pansu differentiable, and thus gives a characterization of smooth Pansu differentiable maps (or at least for \( C^k \)-maps for some \( k \) large enough).

**Proposition 2.8.** Let \( \Phi \) be a smooth function from an open set \( U \) of \( G \) to \( H \). The function \( \Phi \) is uniformly Pansu differentiable on \( U \) if and only if for each \( x \in U \), \( V \in \mathfrak{g}_j \), with \( i, j = 1, 2, \ldots \), the limit of

\[
(2.4) \quad \varepsilon^{-i} \text{pr}_{h,i} \circ \ln_H(\Phi_x(\exp_G(\varepsilon^j V)));
\]

exists as \( \varepsilon \) goes to 0 and, all these limits hold locally uniformly on \( U \times \mathfrak{g}_j \) for each \( i = 1, 2, \ldots \).
Proof of Proposition 2.8. We drop the indices of the groups and Lie algebras in the notation for the logarithmic and exponential maps and the projections.

In view of Remark 2.7 (1), it remains to show the reverse implication. Hence, we assume that the limits in (2.4) exist and we aim at proving that for any \( x \in U, \ V \in \mathfrak{g} \), \( i = 1, 2, \ldots \), the limit of \( \text{pr}_i \circ \ln_H \delta_{x^{-1}} (\Phi_x(\exp_G(\delta_x V))) \) exists and that this holds locally uniformly with respect to \( (x, V) \) in \( U \times \mathfrak{g} \). We will prove this recursively on \( i = 1, 2, \ldots \) First, we need to set some conventions.

Let us recall that the map \( \Theta : \mathbb{R}^{\dim G} \rightarrow \mathbb{R}^{\dim G} \) given by the following exponential coordinates of the second kind on \( \mathfrak{g} \)

\[
\Theta(V) = (V_1, \ldots, V_{n_G}) \in \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n_G} \quad \text{where} \quad \exp(V) = \exp(V_1) \cdots \exp(V_{n_G}),
\]
is a global diffeomorphism of \( \mathbb{R}^{\dim G} \), and that

\[
\Theta(\delta_x V) = (\varepsilon V_1, \ldots, \varepsilon^n V_{n_G}).
\]

We will use the equality

\[
\Phi_x(\exp V) = \Phi_x(\exp V_1) \Phi_x(\exp V_2) \Phi_x(\exp V_3) \cdots \Phi_x(\exp V_n),
\]
which yields

\[
\Phi_x(\exp \delta_x V) = \Phi_x(\exp (\varepsilon V_1)) \Phi_x(\exp (\varepsilon V_2)) \cdots \Phi_x(\exp (\varepsilon V_n)),
\]

\[
(2.5) \quad \Phi_x(\exp \delta_x V) = \Phi_x(\exp (\varepsilon V_1)) \Phi_x(\exp (\varepsilon V_2)) \cdots \Phi_x(\exp (\varepsilon V_n)).
\]

We will use the star product \([3, 1]\) from the Dynkin formula and its properties coming from the gradation property for \( H \) via the properties we now describe. Here \( m \geq 2 \) is an integer, which will be equal to \( n_G \) below. Consider the star product of \( m \) elements \( W_1, \ldots, W_m \in \mathfrak{h} \) projected onto \( \mathfrak{h}_1 \) and along \( \oplus j \neq 1 \mathfrak{h}_{j'} \) along \( \oplus j > 1 \mathfrak{h}_{j'} \), with the convention \( \text{pr}_{r < 1} := 0 \). We have:

\[
P_i(W_1, \ldots, W_m) := \text{pr}_{r}(\text{pr}_{r < 1} (W_1 \star \cdots \star W_m) - (\text{pr}_{r}(W_1) + \cdots + \text{pr}_{r}(W_m))).
\]
is polynomial, valued in \( \mathfrak{h}_1 \) and homogeneous in the sense that

\[
P_i(\delta_\varepsilon W_1, \ldots, \delta_\varepsilon W_m) = \delta_\varepsilon P_i(W_1, \ldots, W_m) = \delta_\varepsilon P_i(W_1, \ldots, W_m), \quad r > 0, \ W_1, \ldots, W_m \in \mathfrak{h}.
\]

Furthermore, \( P_i \) depends only on the projections of the vectors onto \( \mathfrak{h}_{j'}, j' < i \). In order to express this technically, for each \( j \in \mathbb{N} \), we denote by \( \text{pr}_{r < j} = \text{pr}_1 + \cdots + \text{pr}_{j-1} \) the projection onto \( \oplus j' < j \mathfrak{h}_{j'} \) along \( \oplus j > j \mathfrak{h}_{j'} \), with the convention \( \text{pr}_{r < 1} := 0 \). We have:

\[
P_i(W_1, \ldots, W_m) = P_i(\text{pr}_{r < 1} W_1, \ldots, \text{pr}_{r < 1} W_m).
\]

Let us come back to (2.5) and set \( m = n_G \). We use a recursive argument and start with \( i = 1 \); in this case, \( P_1 = P_{1,n_G} \) (defined above) is identically zero. Therefore, composing (2.5) with \( \varepsilon^{-1} \text{pr}_1 \circ \ln \) yields the expression

\[
\varepsilon^{-1} \text{pr}_1 \circ \ln (\Phi_x(\delta_\varepsilon \exp V)) = \varepsilon^{-1} \text{pr}_1 \circ \ln \Phi_x(\exp (\varepsilon V_1)) + \varepsilon^{-1} \text{pr}_1 \circ \ln \Phi_x(\exp (\varepsilon V_2)) + \cdots
\]

whose limit exists locally uniformly by assumption, see (2.4) for \( i = 1 \).

The other steps of the recursion are proved in the following manner. At a general recursive step \( i = 2, 3, \ldots \), we have

\[
\text{pr}_i \circ \ln \Phi_x(\exp V) = Q_{x,i}(V) + \text{pr}_i \circ \ln \Phi_x(\exp V_1) + \text{pr}_i \circ \ln \Phi_x(\exp V_2) + \cdots,
\]

where

\[
Q_{x,i}(V) := P_i(\ln \Phi_x(\exp V_1), \ln \Phi_x(\exp V_2), \ldots),
\]

having used the polynomial \( P_i = P_{i,n_G} \) defined above. The properties of \( P_i \) imply

\[
\varepsilon^{-1} Q_{x,i}(\delta_\varepsilon V) = P_i(\delta_\varepsilon^{-1} \circ \text{pr}_{r < i} \circ \ln \Phi_x(\delta_\varepsilon V_1), \delta_\varepsilon^{-1} \circ \text{pr}_{r < i} \circ \ln \Phi_x(\delta_\varepsilon V_2), \ldots).
\]
Applying the recursive assumption to each term involving $\text{pr}_{<i}$, the limit of $\varepsilon^{-i}Q_{x,i}(\delta_{\varepsilon}V)$ holds locally uniform as $\varepsilon$ goes to 0. Therefore, in the expression
\[
\varepsilon^{-i}\text{pr}_{i} \circ \ln \Phi(x)(\exp(\delta_{\varepsilon}V)) = \varepsilon^{-i}Q_{x,i}(\delta_{\varepsilon}V) + \varepsilon^{-i}\text{pr}_{i} \circ \ln \Phi(x)(\exp(\delta_{\varepsilon}V_{1})) + \varepsilon^{-i}\text{pr}_{i} \circ \ln \Phi(x,\exp(\delta_{\varepsilon}V_{1}))(\exp(\delta_{\varepsilon}V_{2})) + \ldots,
\]
the first term on the right-hand side has a locally uniform limit and the other ones too by the hypotheses in [2.4]. This shows the $i^{th}$ step and terminates the proof.

2.4. A refinement on Theorem [1.5] This section is devoted to the statement of the theorem below which implies Theorem [1.5] and Corollary [1.6] but is more technical. It will be shown in Section [2.5].

We shall use the following notation: with a subspace $v$ of $g$ and an open subset $U$ of $G$, we associate the open set
\[
\Omega_{v,U} = \{(x, V,\varepsilon) \in U \times v \times (0, +\infty) : x \exp_{G}(\delta_{\varepsilon}V) \in U\}
\]
and the set
\[
\mathcal{R}_{v,U} = \{(x, V,\varepsilon) \in U \times v \times [0, +\infty) : x \exp_{G}(\delta_{\varepsilon}V) \in U\}.
\]

**Theorem 2.9.** Let $\Phi$ be a smooth function from an open set $U$ of the graded Lie group $G$ to the graded Lie group $H$.

(i) The map $\Phi$ is uniformly Pansu differentiable on $U$ if and only if $\Phi$ preserves the filtration at every point $x \in U$.

(ii) Besides, for such a map $\Phi$:

1. The Pansu derivative yields a smooth function $(x, z) \mapsto \text{PD}_{x}\Phi(z)$ on $U \times g$ and the map $x \mapsto \text{PD}_{x}\Phi$ is smooth from $U$ to $\text{Hom}(G, H)$.

2. For any $x \in U$, $V \in g_{j}$,

\[
\text{pr}_{h,i}(\text{p-}\Phi(x))(V) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\text{pr}_{h,i} \circ \ln_{H}(\Phi(x,\exp_{G}(\varepsilon V))) = \begin{cases} 0 & \text{when } j \neq i, \\ \text{pr}_{h,i}(\Phi(V)) & \text{when } j = i. \end{cases}
\]

Consequently, the Jacobian $J_{\Phi}(x)$ equals the Jacobian of the map $z \mapsto \text{PD}_{z}(x)$.

3. For any $(x, V) \in U \times g$ and any $\varepsilon > 0$ small enough:

\[
\delta_{\varepsilon}^{-1}(\Phi(x)^{-1}\Phi(x,\exp_{G}(\delta_{\varepsilon}V))) = \exp_{H}(\text{p-}\Phi(x) + \varepsilon\rho(x, V,\varepsilon)),
\]

where the function $\rho$ is valued in $h$, continuous on $\mathcal{R}_{g,U}$ and smooth on the open subset $\Omega_{g,U}$.

Theorem [1.5] follows from Part (i) while the second part of Corollary [1.6] follows from Part (i) and Remark [2.3]. The first part of Corollary [1.6] is a consequence of Theorem [2.9] (ii) (1).

Before entering into the proof Theorem [2.9] let us give a technical corollary that will be useful in Section [3].

**Corollary 2.10.** Here we fix a homogeneous a quasi-norm $| \cdot |_{G}$ on $G$ and we keep the same notation for the function on the underlying Lie algebra $g$, and similarly for $H$. We denote by $B_{r}$ the corresponding closed balls about 0 of radius $r > 0$.

We continue with the assumptions of Theorem [2.9]. We fix a compact subset $K$ of $U$. Let $r_{0} > 0$ be so that $K B_{r_{0}} \subset U$. Then there exists a constant $C > 0$ such that for any $(x, V,\varepsilon) \in K \times g \times (0, 1]$ satisfying $\varepsilon|V|_{G} \leq r_{0}$, we have:

\[
|\rho(x, V,\varepsilon)|_{H} \leq C(1 + |V|_{G}) \sup_{x \in K, |W|_{G} \leq 1, \varepsilon \in [0, r_{0}]} |\rho(x, W,\varepsilon)|_{H}.
\]
Proof of Corollary 2.10. We set $V = \delta_{|G} V_0$ and we compute

$$\delta_{x_1} \ln \Phi_x(\exp_G \delta_{x} V) = \delta_{|G} \left( \delta_{x_1} \ln \Phi_x(\exp_G \delta_{x_1} V_0) \right)$$

$$= \delta_{|G} (p - \partial_x \Phi(V_0) + \varepsilon |G \rho(x, V_0, \varepsilon |G))$$

$$= p - \partial_x \Phi(V) + \delta_{|G} (\varepsilon |G \rho(x, V_0, \varepsilon |G)).$$

We deduce

$$\rho(x, V, \varepsilon) = \delta_{|G} (|G \rho(x, V_0, \varepsilon |G)).$$

We conclude with the equivalence between homogeneous quasi-norms together with the observation that if $| \cdot |_G$ denotes the homogeneous quasi-norm given by

$$|(x_1, \ldots, x_{\dim G})|_G = |x_1|^{1/\nu_1} + \ldots + |x_{\dim G}|^{1/\nu_{\dim G}}$$

then for any $t \in \mathbb{R}$ and $W \in \mathfrak{g}$, $|tW|_G \leq (1 + |t|)|W|_G$.

\[\square\]

2.5. Proof of Theorem 2.9

2.5.1. A technical lemma. The proof of Theorem 2.9 relies on the following property which is of interest on its own, all the more that it holds for any Lie group, not necessarily nilpotent.

Lemma 2.11. Let $\Phi$ be a smooth function from an open set $U$ of $G$ to $H$.

1. For every $x \in U$ and $V \in \mathfrak{g}$, we have for $t$ in a small neighbourhood of 0,

$$\partial_t \ln_H \Phi_x(\exp_G (tV)) = \sum_{p=0}^{\infty} \overline{c}_p \text{ad}^p (\ln_H \Phi_x(\exp_G (tV))) \left( \partial^{x_{\exp_G (tV)}} \Phi(V) \right),$$

where the coefficients $\overline{c}_p \in \mathbb{R}$ comes from the Dynkin formula in [1.1]. In particular $\overline{c}_0 = 1$.

2. We have

$$\ln_H \Phi_x(\exp_G (tV))|_{t=0} = 0,$$

$$\partial_{t=0} \ln_H \Phi_x(\exp_G (tV)) = \partial_x \Phi(V),$$

$$\partial_{t=0}^2 \ln_H \Phi_x(\exp_G (tV)) = \partial_{t=0} \partial_{\exp_G (tV)} \Phi(V),$$

and more generally

$$\partial_{t=0}^{k+1} \ln_H \Phi_x(\exp_G (tV)) = \partial_{t=0}^k \partial_{\exp_G (tV)} \Phi(V) + \sum_{p, k_2 \in \mathbb{N}} \frac{k_2}{k} \overline{c}_p p! \text{ad}^p (\partial_x \Phi(V)) \partial_{t=0}^{k_2} \partial_{\exp_G (tV)} \Phi(V).$$

Proof. We observe that

$$\partial_t \ln_H \Phi_x(\exp_G (tV)) = \partial_{u=0} \ln_H \left( \Phi_x(\exp_G (tV)) \Phi_{x_{\exp_G (tV)}}(\exp_G (uV)) \right) = \partial_{u=0} X \ast Y_u,$$

where the star product was defined in the introduction (see [1.1]) and

$$X := \ln_H \Phi_x(\exp_G (tV)) \text{ and } Y_u := \ln_H \Phi_{x_{\exp_G (tV)}}(\exp_G (uV)).$$

We now apply Dynkin’s formula in [1.1] and notice that since $Y_0 = 0$, only two kinds of terms appear. Firstly, there are those with $\ell \geq 1$, $s_\ell = 1$, $s_1 = \cdots = s_{\ell-1} = 0$, $r_1 > 0$, ..., $r_{\ell-1} > 0$, $r_\ell \geq 0$. Secondly there are the ones with $\ell > 1$, $s_\ell = 0$, $s_{\ell-1} = 1$, $s_1 = \cdots = s_{\ell-2} = 0$, $r_1 = 1$, $r_{\ell-1} \geq 0$, $r_1, \cdots, r_{\ell-2} > 0$. Therefore, the (finite) sum becomes:

$$\partial_{u=0} X \ast Y_u = \sum_{p=0}^{\infty} \overline{c}_p \text{ad}^p (\partial_{u=0} Y_u).$$

Using $\partial_{u=0} Y_u = \partial_{\exp_G (tV)} \Phi(V)$ proves Part 1. Besides, the term $p = 0$ comes from $n = 1$, $r_1 = 0$ and $s_1 = 1$, whence $\overline{c}_0 = c_{0,1} = 1$. This shows Part (1).
The first two relations in Part (2) come readily from the definitions of the objects involved. This together with the consequence of Part (1)

\[ \partial_{t}^{k+1} \ln H \circ \Phi_{x}(\exp_{G}(tV)) = \sum_{p=0}^{\infty} \bar{c}_{p} \sum_{k_{1}+k_{2}+\ldots=k} \binom{k_{2}}{k} \partial_{t_{1}=0}^{k_{1}} \partial_{t_{2}=0}^{k_{2}} \partial_{t_{3}=0} \ldots \partial_{t_{p}=0} \text{ad}^{p} \left( \ln H \circ \Phi_{x}(\exp_{G}(t_{1}V)) \right) \left( d_{\exp_{G}(t_{2}V)} \Phi(V) \right). \]

implies the rest of Part (2).

\[ \square \]

2.5.2. Proof of Theorem 2.9 Let us start with the equivalence (i).

**Proof of Part (i) in Theorem 2.9** By Lemma 2.4, it suffices to prove the reverse implication: we assume that \( \Phi \) preserves the filtration at every \( x \in U \) and we want to show that it is uniformly Pansu differentiable on \( U \).

Let \( V \in g_{j} \). By Proposition 2.8, it suffices to show that the limit in (2.4) exists and holds locally uniformly. Here, we will show the existence of the limits as the local uniformity will be a natural consequence of the existence. We only need considering \( j < i \) by Lemma 2.6.

As \( \Phi \) preserves the filtration at every \( x \in U \), by Lemma 2.4, we have in a neighborhood of \( t = 0 \)

\[ \text{pr}_{h,i}(d_{\exp_{G}(tV)} \Phi(V)) = 0. \]

Therefore, differentiating in \( t \), we obtain that in the same neighborhood of \( t = 0 \)

\[ \forall \ell = 0, 1, 2 \ldots \partial_{t=0}^{\ell} \text{pr}_{h,i}(d_{\exp_{G}(tV)} \Phi(V)) = 0. \]

More precisely, by Lemma 2.11 we also have

\[ \forall p > j \quad \text{pr}_{h,p}(d_{\exp_{G}(tV)} \Phi(V)) = 0, \]

and equation (2.6) may be generalised into

\[ \forall \ell = 0, 1, 2 \ldots \partial_{t=0}^{\ell} d_{\exp_{G}(tV)} \Phi(V) \in h_{1} \oplus \ldots \oplus h_{j}. \]

The latter relation implies that we have for any \( p \in \mathbb{N} \) and \( \ell = 1, 2, \ldots : \)

\[ \text{ad}^{p}(d_{x} \Phi(V)) \left( \partial_{t=0}^{\ell} d_{\exp_{G}(tV)} \Phi(V) \right) \in h_{1} \oplus \ldots \oplus h_{(p+1)j}. \]

The previous fact together with Lemma 2.11 (2) implies for any \( k = 2, 3, \ldots \)

\[ \partial_{t=0}^{\ell} \ln H \circ \Phi_{x}(\exp_{G}(tV)) \in h_{1} \oplus \ldots \oplus h_{(k-1)j}. \]

We now combine these observations with the Taylor expansion of \( t \mapsto \text{pr}_{h,i} \circ \ln H \circ \Phi_{x}(\exp_{G}(tV)) \) for \( t = \varepsilon^{j} \). We push the expansion at a higher order depending on how far \( j \) is from \( i \). Indeed, for \( j < i \), we can find an integer \( k \geq 1 \) such that \( \frac{i}{k+1} < j < \frac{i+1}{k} \) (with the convention that \( \frac{1}{0} = +\infty \) if \( k = 1 \)) and we push the Taylor expansion up to order \( k \):

\[ \ln H \circ \Phi_{x}(\exp_{G}(tV)) = \sum_{k'=1}^{k} \frac{1}{k'!} \varepsilon^{j k'} \text{pr}_{h,i} \circ \ln H \circ \Phi_{x}(\exp_{G}(t'V)) + O(|t|^{k+1}). \]

By the first part of Lemma 2.11 (2), we have

\[ \ln H \circ \Phi_{x}(\exp_{G}(tV))|_{t=0} = 0, \quad \partial_{t=0} \ln H \circ \Phi_{x}(\exp_{G}(tV)) = d_{x} \Phi(V) = 0 \]

by (2.6). Therefore, by (2.9), as long as \( (k-1)j < i \), we have

\[ \text{pr}_{h,i} \circ \ln H \circ \Phi_{x}(\exp_{G}(\varepsilon^{j}V)) = \sum_{k'=1}^{k} \frac{1}{k'!} \varepsilon^{j k'} \text{pr}_{h,i} \circ \ln H \circ \Phi_{x}(\exp_{G}(t'V)) + O(\varepsilon^{j(k+1)}) = O(\varepsilon^{j(k+1)}). \]

This shows the existence of the limits in (2.4) for \( j < i/(k-1) \) such that \( j > i/(k+1) \), and thus recursively for all \( j < i \). Furthermore, one checks easily that they hold locally uniformly, so Part (i) of Theorem 2.9 is proved.

\[ \square \]
The next points of Theorem 2.9 come from the preceding analysis:

Proof of Part (ii) in Theorem 2.9. Additionally to the limits in (2.4), we have obtained that for any \( V \in \mathfrak{g}_j \) and \( \text{pr}_{h,i} \in \mathfrak{h}_i \), we have

\[
\begin{align*}
&\text{for } i \neq j, \quad \varepsilon^{-1} \text{pr}_{h,i} \circ \ln_H \circ \Phi_x (\exp_G (\varepsilon^j V)) = \varepsilon r_{i,j} (x, V, \varepsilon), \\
&\text{for } i = j, \quad \varepsilon^{-1} \text{pr}_{h,i} \circ \ln_H \circ \Phi_x (\exp_G (\varepsilon^j V)) = \text{pr}_{h,i} \circ \Phi_x (V) + \varepsilon r_{i,j} (x, V, \varepsilon),
\end{align*}
\]

where the functions \( r_{i,j} \) are valued in \( \mathfrak{h}_i \), smooth on the open subset \( \Omega_{g,j,U} \) and continuous on \( \mathcal{R}_{g,j,U} \). This implies Point (2) and that \( (x, V) \mapsto \text{PD}_x \Phi (\exp_G V) \) is smooth on \( U \times \mathfrak{g}_j \) for any \( j = 1, 2, \ldots \)

Then, the smoothness of PD\(_x\) stated in Point (1) follows from the Baker-Campbell-Hausdorff formula and the fact that it is a group morphism. Finally, the existence of \( R \) in Point (3) follows for general \( V \in \mathfrak{g}_i \), using (2.5) and the Baker-Campbell-Hausdorff formula. \( \square \)

3. Invariance of the semi-classical calculus on nilpotent Lie group

This section is devoted to the proof of Theorem 1.8. We will first recall (see [6, 7]) some properties of the semi-classical calculus that will be useful in the proof.

3.1. \( L^2 \)-boundedness in the semi-classical calculus. If \( \sigma \in \mathcal{A}_0 (U \times \hat{G}) \), then

\[
(3.1) \quad \| \text{Op}_1 (\sigma) \|_{\mathcal{L}(L^2(U))} \leq \int_{G} \sup_{x \in U} |\kappa_x (z)| dz =: \| \sigma \|_{\mathcal{A}_0 (U \times \hat{G})},
\]

and the right-hand side defines the semi-norm \( : \| \cdot \|_{\mathcal{A}_0 (U \times \hat{G})} \) on \( \mathcal{A}_0 (U \times \hat{G}) \). Consequently, the operators \( \text{Op}_\varepsilon (\sigma) \) are uniformly bounded on \( L^2 (U) \) with bound:

\[
\| \text{Op}_\varepsilon (\sigma) \|_{\mathcal{L}(L^2(U))} \leq \int_{G} \sup_{x \in U} |\kappa_x^\varepsilon (z)| dz = \int_{G} \sup_{x \in U} |\kappa_x (z)| dz = \| \sigma \|_{\mathcal{A}_0 (U \times \hat{G})}.
\]

The next lemma shows that the behaviour of the integral kernel near the diagonal \( \{(x, x) : x \in U\} \) in \( U \) contains all the information about \( \text{Op}_\varepsilon (\sigma) \) at leading order.

Definition 3.1. A cut-off along the diagonal of \( U \) is a function \( \chi \in \mathcal{C}_c^\infty (U \times U) \) such that for each \( x \in U \), the map \( z \mapsto \chi (x, xz^{-1}) \) extends trivially to a smooth function on \( G \) that is identically equal to 1 on a neighborhood \( U_1 \) of 0 and vanish outside a bounded neighborhood \( U_0 \) of 0; moreover, the neighborhoods \( U_1 \) and \( U_0 \) are assumed to be independent of \( x \in U \). We may call the smallest \( U_0 \) the diagonal support of \( \chi \).

For example, if \( \chi_1 \in \mathcal{C}_c^\infty (G) \) is a function equal to 1 close to 0 and with support small enough, then \( (x, y) \mapsto \chi_1 (xy^{-1}) \) is a cut-off along the diagonal of \( U \) whose diagonal support is the support of \( \chi \).

Lemma 3.2. Let \( \sigma \in \mathcal{A}_0 (U \times \hat{G}) \) and let \( \chi \in \mathcal{C}_c^\infty (U \times U) \) be a cut-off along the diagonal as introduced in Definition 3.1. Let \( S^\varepsilon \) be the operator with integral kernel

\[
(x, y) \mapsto \kappa_x^\varepsilon (y^{-1} x) \chi (x, y).
\]

Then, for all \( N \in \mathbb{N} \), there exists a constant \( C = C_{N, \sigma, \chi} > 0 \) such that

\[
\forall \varepsilon \in (0, 1] \quad \| \text{Op}_\varepsilon (\sigma) - S^\varepsilon \|_{\mathcal{L}(L^2(U))} \leq C \varepsilon^N.
\]

Proof. One observes that for all \( N \in \mathbb{N} \), there exists a bounded function \( \theta \) defined on \( G \times G \) such that, for all \( x, z \in G \)

\[
\chi (x, xz^{-1}) = 1 + \theta (x, z) |z|_G^N
\]

\[
\square
\]
where $|:|_G$ is a homogeneous quasi-norm on $G$. Set $\widetilde{\kappa}_{x,\varepsilon}(z) = \kappa_x(z)(1 - \chi(x, x\delta_{\varepsilon}z^{-1}))$, then there exists a constant $c_0 > 0$ such that

$$\|\widetilde{\kappa}_{x,\varepsilon}\|_{A_0(U \times G)} \leq c_0\varepsilon^N \int_G \sup_{x \in U} |\kappa_x(z)| |z|^N dz.$$

This induces the estimate for the operator $S_\varepsilon$.

3.2. Proof of Theorem 1.8. Let $\sigma \in A_0(\Phi(U) \times \widehat{H})$ and denote by $\kappa : x \mapsto \kappa_x$ its convolution kernel. Set $\tilde{\sigma} := (\widehat{\Phi} \ast \sigma) \in A_0(U \times \widehat{G})$ and denote by $\tilde{\kappa} = \mathcal{I}_\Phi \kappa$ its convolution kernel. Note that the $x$-supports are transported via $\Phi$:

$$x \text{- supp } \tilde{\kappa} \subset \Phi^{-1}(x \text{- supp } \kappa) := K_0 \subset U.$$

Let us first proceed to a reduction of the problem by using Lemma 3.2. Let $\chi \in C^\infty(\Phi(U) \times \Phi(U))$ be a cut-off along the diagonal of $\Phi(U)$. Consider the function $\tilde{\chi}$ defined on $U \times U$ by

$$\tilde{\chi}(x, y) = \chi(\Phi(x), \Phi(y)).$$

Then $\tilde{\chi}$ is a cut-off along the diagonal of $U$.

With the two cut-off functions $\chi$ and $\tilde{\chi}$ in hands, by Lemma 3.2, we can restrict to proving that the operator $R_\varepsilon$ whose integral kernel is

$$(x, y) \mapsto \varepsilon^{-Q}J_{\varepsilon}(y)^{1/2}J_{\varepsilon}(x)^{1/2} \kappa_{\varepsilon}(x) \left( \delta_{\varepsilon}^{-1}(\Phi(y)^{-1}\Phi(x)) \right) \chi(\Phi(x), \Phi(y)) - \varepsilon^{-Q} \tilde{\kappa}_{x,\varepsilon}(x, y),$$

satisfies $\|R_\varepsilon\|_{L^2(U)} \leq c_0 \varepsilon$ for some constant $c_0 > 0$.

We observe that the operator $R_\varepsilon$ may be written in the form

$$R_\varepsilon f(x) = f \ast r_{x,\varepsilon}(x), \quad x \in U, \ f \in S(G),$$

where $r_{x,\varepsilon}(y) = \varepsilon^{-Q}r_{x,\varepsilon}(\delta_{\varepsilon}^{-1}y)$ and $r_{x,\varepsilon}$ is the function in $C^\infty_c(U, S(G))$ given by

$$r_{x,\varepsilon}(z) := J_{\varepsilon}(x\delta_{\varepsilon}z^{-1})^{1/2}J_{\varepsilon}(x)^{1/2} \kappa_{\varepsilon}(x) \left( \delta_{\varepsilon}^{-1}(\Phi(x\delta_{\varepsilon}z^{-1})^{-1}\Phi(x)) \right) \chi(x, x\delta_{\varepsilon}z^{-1})$$

$$- \tilde{\kappa}_{x,\varepsilon}(x, x\delta_{\varepsilon}z^{-1})
= J_{\varepsilon}(x\delta_{\varepsilon}z^{-1})^{1/2}J_{\varepsilon}(x)^{1/2} \kappa_{\varepsilon}(x) \left( \delta_{\varepsilon}^{-1}(\Phi(x\delta_{\varepsilon}z^{-1})^{-1}\Phi(x)) \right) \chi(x, x\delta_{\varepsilon}z^{-1})$$

$$- J_{\varepsilon}(x) \kappa_{\varepsilon}(x) (\text{PD}_{x,\varepsilon}(\Phi)) \tilde{\chi}(x, x\delta_{\varepsilon}z^{-1}).$$

We now aim to prove that

$$\exists c_0 > 0 \forall \varepsilon \in (0, 1] \quad I_\varepsilon := \int_G \sup_{x \in U} |r_{x,\varepsilon}(z)| \leq c_0 \varepsilon,$$

as, by (3.1), this implies $\|R_\varepsilon\|_{L^2(U)} \leq c_0 \varepsilon$.

We observe that the $x$-support of $r_{x,\varepsilon}$ is included in $K_0$. We may assume that the diagonal support of $\chi$ is as small as we need below and therefore that the diagonal support of $\tilde{\chi}$ is included in a small ball $\bar{B}_{r_0}$ of $G$ with a radius $r_0$ as small as we need; here, we have fixed a homogeneous quasi-norm $|:|_G$. We can decompose $I_\varepsilon \leq I_{1,\varepsilon} + I_{2,\varepsilon}$ with

$$I_{1,\varepsilon} := \int_G \sup_{x \in K_0} \left| J_{\varepsilon}(x\delta_{\varepsilon}z^{-1})^{1/2} - J_{\varepsilon}(x)^{1/2} \right| \left| J_{\varepsilon}(x)^{1/2} \kappa_{\varepsilon}(x) (\text{PD}_{x,\varepsilon}(\Phi)) \right| \chi(x, x\delta_{\varepsilon}z^{-1}) \, dz,$$

$$I_{2,\varepsilon} := C_2 \int_G \sup_{x \in K_0} \kappa_{\varepsilon}(x) \left( \delta_{\varepsilon}^{-1}(\Phi(x\delta_{\varepsilon}z^{-1})^{-1}\Phi(x)) \right) - \kappa_{\varepsilon}(x) \left( \text{PD}_{x,\varepsilon}(\Phi) \right) \chi(x, x\delta_{\varepsilon}z^{-1}) \, dz,$$

where

$$C_2 := \sup_{K_0} \left| J_{\varepsilon} \right|^{1/2} \sup_{K_0 \bar{B}_{r_0}} |J_{\varepsilon}|^{1/2}.$$
For $I_{1,\varepsilon}$, we use the Taylor estimates due to Folland and Stein (see [10, Section 1.41] or [9, Section 3.1.8]); we have for any $x \in K_0$ and $z \in \delta_\varepsilon^{-1} B_{r_0}$
\[
\left| J_\Phi(x \delta_\varepsilon z^{-1})^{1/2} - J_\Phi(x)^{1/2} \right| \lesssim |\delta_\varepsilon z^{-1}|_G \sup_{x' \in K_0', j=1, \ldots, \dim G} |X_j(J_\Phi^{1/2})(x')|
\]
for some compact subset $K_0'$ of $U$. Since $y \mapsto \kappa_{\Phi}(x) (PD_x \Phi(y)) \in C_c^\infty(U, S(G))$, this implies that $I_1 \leq c_1 \varepsilon$ for some constant $c_1 > 0$.

For $I_{2,\varepsilon}$, we apply Theorem 2.39 and Corollary 2.10 with the usual (Euclidean) Taylor estimate to obtain:
\[
|\kappa_{\Phi}(x) (\delta_\varepsilon^{-1} (x \delta_\varepsilon z^{-1})^{-1} \Phi(x))) - \kappa_{\Phi}(x) (PD_x \Phi(z))| \\
= |\kappa_{\Phi}(x) \circ \exp_H (p - \partial_x \Phi(\ln G z) + \varepsilon \rho(x, \ln G z, \varepsilon)) - \kappa_{\Phi}(x) \circ \exp_H (p - \partial_x \Phi(\ln G z))| \\
\leq \sup_{|W| \leq \varepsilon |\rho(x, \ln G z, \varepsilon)|} |D_{p - \partial_x \Phi(\ln G z)} W \kappa_{\Phi}(x) \circ \exp_H | \varepsilon \rho(x, \ln G z, \varepsilon)| \\
\leq D_N (1 + |p - \partial_x \Phi(\ln G z)|)^{-N} \varepsilon,
\]
for any $N \in \mathbb{N}$ and some constant $D_N > 0$, as long as $x \in K_0$, $\delta_\varepsilon z^{-1} \in B_{r_0}$, and $\varepsilon \in (0, 1]$, since $y \mapsto \kappa_y \in C_c^\infty(U, S(G))$. This implies $I_2 \leq c_2 \varepsilon$ for some constant $c_2 > 0$, which concludes the proof.

4. GEOMETRIC MEANING OF THE CONVOLUTION KERNEL

As for the semi-classical calculus in the Euclidean setting, it is important to keep in mind the geometric aspects of the elements that one considers. Indeed, the objects defined using the group Fourier Transform depend on a choice of Haar measure on the graded Lie group $G$. This makes little difference in the group context, where the Haar measure is determined up to a constant, but will matter more if one wants to extend this non-commutative semi-classical approach on manifolds. In this section, we explain how to avoid choosing a normalization for the Haar measure when defining semi-classical quantization on graded Lie groups. We achieve this by giving a geometrically intrinsic definition of the convolution kernel and thus of the semiclassical symbols.

We will start with well-known geometric considerations and set out their usual conventions. The manifolds (often denoted by $M$) are assumed to be smooth. If $F = \cup_{x, y \in M} F_x$ is a fiber bundle over $M$, $\pi_F : F \to M$ will denote its canonical projection. Here, all the fiber bundles are smooth, and we denote by $\Gamma(F)$ the space of its smooth sections.

4.1. Generalities on bundles. Let $E$ be a real vector bundle over a manifold $M$ of rank $r$.

For $s \in \mathbb{R}^*$, the $s$-density bundle associated to $E$ is the set $|\Lambda|^s(E)$ of all maps $\mu_x : (E_x)^r := E_x \times \cdots \times E_x \to \mathbb{R}$, for which
\[
\mu_x(A v_1, \ldots, A v_r) = |\det A|^s \mu(v_1, \ldots, v_r); \quad A \in \text{GL}(E_x).
\]

When $E = TM$, we simply write $|\Lambda|^s M$ for the $s$-density bundle over $M$. An $s$-density $\mu$ on $M$ is a smooth section of $|\Lambda|^s M$. When $s = 1$ or $s = \frac{1}{2}$, we say that $\mu$ is a density or half-density respectively.

We say that a density $\mu$ is a positive density when $\mu_x(v_1, \ldots, v_r) > 0$ for all $x \in M, v_i \in E_x$. A nonvanishing $n$-form $\omega$ on an open set $U \subset M$ determines a positive density $|\omega|$, and identifies $s$-densities on $U$ with functions via $\mu = f |\omega|^s$. In particular, an $s$-density $\mu$ on $M$, is written
\[
\mu(y) = f(y) |dy|^s
\]
in local coordinates $(y^i)$.

The canonical $L^2$-space. Note that two half-densities determine a 1-density; indeed, given $u = f \mu^2$ and $v = g \mu^2$, we have $uv = f g \mu$. Furthermore, the integral of a 1-density is invariantly defined.
Therefore we may define the canonical Hilbert space of half-densities, $L^2(M)$, as the closure of the set of compactly supported half-densities with respect to

$$\langle u, v \rangle_{L^2(M)} := \int_M u v.$$

Upon choosing a positive density $\mu$ on $M$, we obtain a unitary map between the $L^2$ space associated to $\mu$, and the canonical Hilbert space

$$(4.1) \quad U_\mu : L^2(M, \mu) \longrightarrow L^2(M); \quad f \longmapsto f \mu^{\frac{1}{2}}.$$

**Densities and mappings.** The $s$-density bundle, $|\Lambda|^s M$, is a trivial line bundle (though not canonically so), as every smooth manifold has a global positive density. A choice of positive densities $\mu$ and $\nu$ on two manifolds $M$ and $N$ respectively allows for the definition of the Jacobian of a smooth map $\Phi : M \rightarrow N$. Indeed, there exists a positive function $J_\Phi$ on $M$, called the Jacobian of $\Phi$, such that

$$\Phi^* \nu = J_\Phi \mu.$$

More generally $\Phi^* (f \nu) = (f \circ \Phi) J_\Phi^* \mu$. If in addition, $\Phi$ is a diffeomorphism, then

$$\Phi^* \circ U_\nu = U_\mu \circ U_\Phi,$$

where $U_\Phi$ is defined as in (1.8) and $U_\mu, U_\nu$, in (4.1).

**Vertical bundles.** Consider a fiber bundle $F = \cup_{x \in M} F_x$ over a manifold $M$. With $f \in F$, we associate $x = \pi_F(f)$ and $f_x$ the corresponding point in the fiber $F_x$. The vertical space $V_f(F)$ above a point $f \in F$ is the tangent space $T_{f_x} F_x$ to the fiber $F_x$ at $f_x$. The resulting vector bundle $V(F) = \cup_{f \in F} V_f(F)$ over $F$ is called the vertical bundle of $F$; it is given by the kernel of the map $d\pi_F : TF \rightarrow TM$. Smooth sections of $|\Lambda|^1 V(F)$ are called vertical densities on $F$.

**Example 4.1** (The vertical bundle of a vector bundle). Suppose $F = E$ is a vector bundle. Then the vertical space $V_e(E) = T_{e_x} E_x$ above $e \in E$ (here $x = \pi_E(e)$) is naturally identified with $E_x$. Hence, the vertical bundle $V(E)$ is isomorphic to the pull-back bundle $(\pi_E)^*(E) = \cup_{e \in E} E_{\pi_E(e)}$ of $E$ by the isomorphism of vector bundles $\text{vert}^E : (\pi_E)^* (E) \longrightarrow V(E)$:

$$\text{vert}_e^E (V) := \left. \frac{d}{dt} \right|_{t=0} U + t V, \quad U = e_x + E_x, \quad V \in E_x \cong T_{e_x} E_x = V_e(E), \quad x = \pi_E(e).$$

This example may be generalised to the case of a group bundle:

**Example 4.2** (The vertical bundle of a Lie group bundle). Suppose $F = G = \cup_{x \in M} G_x$ is a Lie group bundle. The vertical space $V_g(G) = T_{g_x} (G_x)$ above $g \in G$ (here $x = \pi_G(g)$) is naturally isomorphic to the Lie algebra $g_x$ of the Lie group $G_x$. Therefore, the vertical bundle $V(G)$ is naturally isomorphic to the pull-back bundle $\cup_{g \in G} g_{\pi_G(g)} = (\pi_G)^*(g)$ of the corresponding Lie algebra bundle $g = \cup_{x \in M} g_x$. More precisely, the natural isomorphism of vector bundles $\text{vert}^G : (\pi_G)^*(g) \longrightarrow V(G)$ is given via

$$\text{vert}^G_g (V) := \left. \frac{d}{dt} \right|_{t=0} g_x \exp_{G_x} (tV), \quad V \in g_x \cong T_{g_x} (G_x) = V_g(G), \quad x = \pi_G(g).$$

4.2. **Filtered manifolds.** A filtered manifold is a manifold $M$ equipped with a filtration of the tangent bundle $TM$ by vector bundles

$$M \times \{ 0 \} = H^0 \subseteq H^1 \subseteq \ldots \subseteq H^{n_M} = TM \quad \text{satisfying} \quad [\Gamma(H^i) \cap \Gamma(H^j)] \subseteq \Gamma(H^{i+j});$$

here we are using the convention that $H^i = TM$ for $i > n_M$. For each $x \in G$, the quotient

$$\Theta_x M := \bigoplus_i \left( H^i_x / H^{i-1}_x \right)$$
Lemma 4.5. The next statement summarises the main results in Section 2:

The osculating maps that we may call the group and Lie algebra bundles over \( M \).

Example 4.3. Equiregular sub-Riemannian manifolds, in particular contact manifolds, are naturally filtered manifolds. Indeed, suppose \( M \) is a smooth manifold equipped with a bracket-generating distribution \( \mathcal{D} \subset T M \). Let \( \Gamma^1 = \Gamma(\mathcal{D}) \) be the set of smooth sections of \( \mathcal{D} \), and

\[
G^i := [\Gamma^1, \Gamma^{i-1}] + \Gamma^{i-1}, \quad i > 1.
\]

As \( \mathcal{D} \) is bracket generating, there exists \( i \) such that \( G^i = TM \), and we denote by \( r \) the smallest such integer \( i \). If, for every \( i = 1, \ldots r \), there exists a subbundle \( H^i \subset TM \) for which \( \Gamma^i = \Gamma(H^i) \) is the set of smooth sections on \( H^i \), then \( M \) is said to have an equiregular sub-Riemannian structure. Clearly, the \( H^i \)'s provide the structure of filtered manifold.

Remark 4.4. Sub-Riemannian (and semi-Riemannian) structures are not in general equiregular. Examples include the Martinet distribution or the Grushin plane. The latter examples also fall outside of the scope of this paper because they are not open subsets of graded Lie groups. An approach to a pseudodifferential calculus on the non-equiregular case is given in [1].

In what follows, it will be helpful to describe a smooth function \( f \) on \( G M \) by denoting the associated function on the fiber \( G_x M \) via

\[
G_x M \ni z_x \mapsto f_x(z_x), \quad \text{for each } x \in M.
\]

We will thus denote by \( S(G M) := \cup_{x \in M} S(G_x M) \) the (smooth Fréchet) bundle of fiberwise Schwartz functions on the group \( G_x M \) over \( x \in M \). The space \( \Gamma_c(S(G M)) \) of compactly supported (smooth) sections of \( S(G M) \) may be described as the space of smooth function \( f \) on \( G M \) that are compactly supported in \( x \in M \), and for which the functions \( z_x \mapsto f_x(z_x) \) are Schwartz on \( G_x M \) in a way varying smoothly with \( x \in M \).

A morphism of filtered manifolds is a smooth map \( \Phi : M \to N \) between two filtered manifolds \( M \) and \( N \) that respects the filtrations. Denoting by \( H^{M,i} \) and \( H^{N,i} \) the vector bundles giving the filtrations of \( TM \) and \( TN \) respectively, this means that \( d_x \Phi(H^{M,i}_x) \subseteq H^{N,i}_{\Phi(x)} \) holds at every \( x \in M \) and \( i = 0, 1, \ldots \). At every \( x \in M \), we may define a Lie algebra morphism \( G_x \Phi : G_x M \to G_{\Phi(x)} N \) via

\[
G_x \Phi(V \mod H^{M,i-1}_x) = d_x \Phi(V) \mod H^{N,i-1}_{\Phi(x)}, \quad V \in H^{M,i}_x, \quad i = 1, \ldots
\]

We denote by \( G_x \Phi : G_x M \to G_{\Phi(x)} N \) the corresponding group morphism. This induces \( G \Phi \) morphisms between the osculating bundles

\[
G \Phi : G M \to G N \quad \text{and} \quad G \Phi : G M \to G N
\]

that we may call the group and (resp.) Lie algebra osculating maps.

Open sets of graded Lie groups are naturally equipped with a structure of filtered manifolds. The next statement summarises the main results in Section 2.

Lemma 4.5. Let \( \Phi : U \to H \) be a smooth map from an open set \( U \) of a graded group \( G \) to a graded group \( H \). The following are equivalent:

- \( \Phi \) preserves the filtration in the sense of Definition [1.1]
- \( \Phi \) is locally uniformly Pansu differentiable on \( U \) in the sense of Definition [1.2]
- \( \Phi : U \to H \) is a morphism of filtered manifolds (as explained above).
Moreover, in this case, the Pansu derivative and the osculating map coincide at every $x \in U$:

$$G_x \Phi = P \xi_x \Phi \text{ on } G \quad \text{and} \quad \mathfrak{g}_x \Phi = p \xi_x \Phi \text{ on } \mathfrak{g}.$$ 

4.3. **Densities on filtered manifolds.** Let $M$ be a filtered manifold.

4.3.1. **Haar system.** Let us fix a smooth Haar system $\{\mu_x\}_{x \in M}$ for $\mathbb{G}M$, that is, a choice of Haar measure $\mu_x$ for each of the group fibers $\mathbb{G}_x M$ for which

$$\forall f \in C^\infty(\mathbb{G}M), \quad x \mapsto \int_{z \in \mathbb{G}_x M} f(z) \mu_x(z) \in C^\infty(M).$$

A Haar measure $\mu_x$ on the group fiber $\mathbb{G}_x M$ naturally identifies with a left-invariant density, which we also call $\mu_x$, on $\mathbb{G}_x M$. By left-invariance, each $\mu_x$ corresponds to a positive element of the one-dimensional vector space $|\Lambda| \mathfrak{g}_x M$. Indeed, smooth Haar systems for $\mathbb{G}M$ correspond in this way to smooth positive sections of $|\Lambda| \mathfrak{g}M$, which we call **Haar densities** on $M$.

4.3.2. **Vertical densities.** As in Examples 4.1 and 4.2, the vertical space $\mathcal{V}_z(\mathbb{G}M) = T_z \mathbb{G}_z M$ above $z \in \mathbb{G}M$ with $x = \pi_{\mathbb{G}M}(z) \in M$ identifies naturally with the Lie algebra $\mathfrak{g}_x M$. Hence, the vertical bundle $\mathcal{V}(\mathbb{G}M)$ is naturally isomorphic to the pullback bundle $(\pi_{\mathbb{G}M})^*|\mathfrak{g}M|$ of the osculating Lie algebra bundle. More precisely, the natural isomorphism of vector bundles

$$\text{vert} : (\pi_{\mathbb{G}M})^*|\mathfrak{g}M| \rightarrow \mathcal{V}(\mathbb{G}M)$$

is given via:

$$\text{vert}_z(V) := \frac{d}{dt} \bigg|_{t=0} z \exp\mathbb{G}_x M(tV), \quad V \in \mathfrak{g}_x M \cong T_z \mathbb{G}_z M = \mathcal{V}_z(\mathbb{G}M), \quad x = \pi_{\mathbb{G}M}(z).$$

This isomorphism allows us to lift densities $x \mapsto \mu_x$ in $|\Lambda| \mathfrak{g}M$ to vertical densities $z \mapsto \tilde{\mu}_z$ via

$$\tilde{\mu}_z(\text{vert}_z(V_1), \ldots, \text{vert}_z(V_n)) = \mu_x(V_1, \ldots, V_n); \quad V_1, \ldots, V_n \in \mathfrak{g}_x M, \quad x = \pi_{\mathbb{G}M}(z).$$

This lift yields a map of sections: $\Gamma(|\Lambda| \mathfrak{g}M) \rightarrow \Gamma(|\Lambda| \mathcal{V}(\mathbb{G}M))$. We shall subsequently blur the distinction between a Haar density on $M$, its lift to a vertical density on $\mathbb{G}M$, and a smooth Haar system on $\mathbb{G}M$.

4.3.3. **Haar system and smooth functions on $\mathbb{G}M$.** With respect to a smooth Haar system $\{\mu_x\}_{x \in M}$ on $M$, any smooth function $\tilde{\kappa}$ on $\mathbb{G}M$ yields the vertical density

$$\kappa(z) = \tilde{\kappa}(z) \mu_z, \quad z \in \mathbb{G}M.$$

Conversely, as dim$(|\Lambda| \mathfrak{g}_M M) = 1$ at every $x \in M$, any vertical density $\kappa$ on $\mathbb{G}M$ may be written as in (4.2) for a unique smooth function $\tilde{\kappa}$ on $\mathbb{G}M$. This defines an isomorphism of topological vector spaces

$$I_\mu : C^\infty(\mathbb{G}M) \rightarrow C^\infty(\mathbb{G}M, |\Lambda| \mathcal{V}); \quad \tilde{\kappa} \mapsto \kappa = \tilde{\kappa} \mu.$$

In the next paragraph, we will use the (smooth Fréchet) bundle $\mathcal{S}(\mathbb{G}M, |\Lambda| \mathcal{V})$ of fiberwise Schwartz densities; that is, the collection $\{\kappa_x : z_x \mapsto \kappa_x(z_x)\}_{x \in M}$ of densities on the fibers $\mathbb{G}_x M$ for which the function $z_x \mapsto \kappa(z_x)$ is Schwartz.

4.3.4. **Schwartz vertical densities.** A vertical density $\kappa$ is said to be a **Schwartz vertical density** when the corresponding function $\tilde{\kappa} \in C^\infty(\mathbb{G}M)$ from (4.2) is in $\Gamma_c(\mathcal{S}(\mathbb{G}M))$. Although this definition requires the choice of a smooth Haar system $\mu = \{\mu_x\}_{x \in M}$ on $M$, the resulting property is independent of $\mu$ as the $x$-support of $x \mapsto \kappa_x$ is assumed to be compact. We denote by $\Gamma_c(\mathcal{S}(\mathbb{G}M, |\Lambda| \mathcal{V}))$ the space of vertical Schwartz densities (compactly supported over $M$). Although these spaces are independent of the choice of Haar system, fixing such a Haar system $\mu$ establishes an identification between $\Gamma_c(\mathcal{S}(\mathbb{G}M))$ and $\Gamma_c(\mathcal{S}(\mathbb{G}M, |\Lambda| \mathcal{V}))$ via the isomorphism of topological vector spaces:

$$I_\mu : \Gamma_c(\mathcal{S}(\mathbb{G}M)) \rightarrow \Gamma_c(\mathcal{S}(\mathbb{G}M, |\Lambda| \mathcal{V})); \quad \tilde{\kappa} \mapsto \kappa = \tilde{\kappa} \mu.$$
4.4. Convolution kernels and semi-classical symbols as geometric items.

4.4.1. Convolution kernels as vertical Schwartz densities. We start with the following general observation:

**Lemma 4.6.** Let $\mathbb{F} : \mathbb{G}M \to \mathbb{G}N$ be a smooth map. Assume that $\mathbb{F}_x : \mathbb{G}_x M \to \mathbb{G}_y N$ (with $y = \pi_{GN}(\mathbb{F}(z))$) is a group morphism for each $x \in M$. Then, the differential of $\mathbb{F}$ preserves the vertical bundles:

$$d\mathbb{F}(\mathcal{V}(\mathbb{G}M)) \subseteq \mathcal{V}([\mathbb{G}N]) .$$

**Proof.** For each $z \in \mathbb{G}M$ and $V \in \mathfrak{g}_x M$ with $x = \pi_{GM}(z)$, we have:

$$d_z \mathbb{F}_x (\text{vert}_z(V)) = \left. \frac{d}{dt} \right|_{t=0} \mathbb{F}_x(z, \exp_{\mathbb{G}_x M}(tV)) = \left. \frac{d}{dt} \right|_{t=0} \mathbb{F}_x(z, \exp_{\mathbb{G}_y N}(t\mathfrak{F}_x(V)))$$

$$= \text{vert}_{\mathbb{F}(z)}(\mathfrak{F}_x(V)), \quad \square$$

where $\mathfrak{F}_x(V) := \left. \frac{d}{dt} \right|_{t=0} \mathbb{F}_x(\exp_{\mathbb{G}_x M}(tV))$.

Given a map $\mathbb{F}$ as in Lemma 4.16, we may now define the pullback of a vertical density $\nu$ on $\mathbb{G}N$ to a vertical density $\mathbb{F}^*\nu$ on $\mathbb{G}M$ via

$$(\mathbb{F}^*\nu)_x(V_1, \ldots, V_n) := \nu_z(\mathbb{F}_x(V_1), \ldots, \mathbb{F}_x(V_n)) ; \quad V_1, \ldots, V_n \in T_z(\mathbb{G}_x M), x = \pi_{GM}(z).$$

If in addition $\mathbb{F}$ is a diffeomorphism (and hence an isomorphism of filtered manifolds), the pushforward is defined by $\mathbb{F}_* := (\mathbb{F}^{-1})^*$. Given a morphism of filtered manifolds $\Phi : M \to N$, we may apply the above to $\mathbb{F} = \mathbb{G}\Phi$ the osculating map, and define the pullback $(\mathbb{G}\Phi)^*\nu$ of a vertical density $\nu$ on $\mathbb{G}N$ to $\mathbb{G}M$. If $\Phi$ is in addition a diffeomorphism, the pushforward yields the following isomorphism of topological vector spaces:

$$\mathbb{G}\Phi^* : \Gamma_c(S(\mathbb{G}N, |\Lambda|V)) \to \Gamma_c(S(\mathbb{G}M, |\Lambda|\nu)) .$$

If $\mu$ and $\nu$ are vertical densities on $M$ and $N$ respectively, we define the operator

$$\mathcal{I}_\Phi : \Gamma_c(S(\mathbb{G}N)) \to \Gamma_c(S(\mathbb{G}M)) , \quad \text{via} \quad \mathbb{G}\Phi^* \circ \mathcal{I}_\nu = \mathcal{I}_\mu \circ \mathcal{I}_\Phi .$$

**Lemma 4.7.** Assume $M$ and $N$ are open subsets of graded Lie groups $G$ and $H$ (resp.). If $\Phi : M \to N$ is a diffeomorphism that preserves the filtrations of the group, then it is an isomorphism of filtered manifolds and the map $\mathcal{I}_\Phi$ agrees with $\mathbb{G}\Phi^*$. This result shows that convolution kernels have the geometric structure of vertical Schwartz densities.

**Lemma 4.7** motivates $\Gamma_c(S(\mathbb{G}M, |\Lambda|V))$ as the geometric space of convolution kernels for future study of semi-classical pseudodifferential operators on filtered manifolds. It is also interesting to notice that setting $\kappa^{(e)} := (\delta_{e^{-1}})_{x}\kappa$, we have for any smooth Haar system, by homogeneity of Haar measures on graded Lie groups,

$$((\delta_{e^{-1}})_{x}\kappa)(z) = (\delta_{e^{-1}}\kappa)_{x}(z) = (\delta_{e^{-1}}(\kappa\mu))_{x}(z) = \kappa_{x}(\delta_{e^{-1}}z)\delta_{e^{-1}}\mu_{x} = \varepsilon^{-Q}\kappa_{x}(\delta_{e^{-1}}z)\mu_{x} = : \kappa^{(e)}(z)\mu_{x},$$

where $Q$ is the (constant) homogeneous dimension of the group fibers of $\mathbb{G}M$.

**Proof of Lemma 4.7.** Let $G$ and $H$ be graded Lie groups. We may assume that both groups are modeled on $\mathbb{R}^p$, in which case the Lebesgue measure $|dz|$ is a Haar measure for both. If $U$ is an open subset of $G$, then

$$\mathcal{V}(\mathbb{G}U) \cong (U \times G) \times \mathfrak{g} \quad \text{whence} \quad |\Lambda|\mathcal{V}(\mathbb{G}U) \cong (U \times G) \times |\Lambda|\mathfrak{g} ,$$

whence

$$|\Lambda|\mathcal{V}(\mathbb{G}U) \cong (U \times G) \times |\Lambda|\mathfrak{g} ,$$

then

$$|\Lambda|\mathcal{V}(\mathbb{G}U) \cong (U \times G) \times |\Lambda|\mathfrak{g} .$$
and similarly for any open subset of $H$. With these identifications, the vertical lift of $|dz|$ on $U \subset G$ to $|A|\mathcal{N}(GU)$ is also written $|dz|$. By (4.2), any vertical density $\kappa$ on $H$ has the form

$$\kappa_x(z) = \tilde{\kappa}_x(z)|dz|.$$ 

Suppose $\Phi : U \subset G \rightarrow \Phi(U) \subset H$ is a filtration-preserving diffeomorphism from the open subset $U \subset G$ to $\Phi(U) \subset H$. By Lemma 4.5 we have $G_x(\Phi(z)) = PD_x\Phi(z)$ for any $z \in G_xU = G$, so

$$(\Phi^{\ast}\kappa)_x(z) := \tilde{\kappa}_{\Phi(x)}(G_x(\Phi(z)))\Phi^{\ast}|dz| = \tilde{\kappa}_{\Phi(x)}(PD_x\Phi(z))J_\Phi(z)|dz|.$$ 

Therefore, vertical Schwartz densities on groups transform according to (1.9).

4.4.2. Towards a set of semi-classical symbols on filtered manifolds. The fiberwise Fourier transform of $\kappa \in \Gamma_c(S(GM,|A|\mathcal{N}))$ is defined as

$$\mathcal{F}(\kappa)(x,\pi) := \int_{z \in G_xM} \kappa(z_x)\pi(z_x)^*; \quad \pi \in \hat{G}_xM.$$ 

Here $\hat{G}_xM$ is the unitary dual to the group fiber $G_xM$ (note that this does not require a choice of Haar system). Then, a natural candidate for the set of semi-classical symbols on $M$ is the image of $\Gamma_c(S(GM,|A|\mathcal{N}))$ by $\mathcal{F}$; we denote it by $A_0(\hat{G}M)$. Elements of this space are fields of operators on $\hat{G}M := \bigcup_{x \in GM} \hat{G}_xM$ (modulo intertwiners). In the case where $M$ is an open subset of a graded group $G$, $A_0(\hat{G}M)$ coincides with the set $A_0(U \times \hat{G})$ of Section 1.3.3.

Given an isomorphism $\Phi : M \rightarrow N$ of filtered manifolds, the operator defined by

$$\hat{G}\Phi : \hat{G}M \rightarrow \hat{G}N; \quad \pi_x \mapsto \pi_x \circ G\Phi(x)^{-1}.$$ 

is a generalization of (1.10). Besides, for $\kappa \in \Gamma_c(S(GN,|A|\mathcal{N}))$ and $\sigma(x,\pi) = \mathcal{F}(\kappa)(x,\pi)$ we have

$$\mathcal{F}(G\Phi^{\ast}\kappa)(x,\pi) = \hat{G}\Phi^{\ast}\sigma(x,\pi)$$

So when $M$ and $N$ are open subsets of graded Lie groups, we are left with (1.11) and the geometric frame is consistent.

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