NATURAL BOUNDARY CONDITIONS IN GEOMETRIC CALCULUS OF VARIATIONS

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Abstract. In this paper we obtain natural boundary conditions for a large class of variational problems with free boundary values. In comparison with the already existing examples, our framework displays complete freedom concerning the topology of $Y$—the manifold of dependent and independent variables underlying a given problem—as well as the order of its Lagrangian. Our result follows from the natural behavior, under boundary–friendly transformations, of an operator, similar to the Euler map, constructed in the context of relative horizontal forms on jet bundles (or Grassmann fibrations) over $Y$. Explicit examples of natural boundary conditions are obtained when $Y$ is an $(n+1)$–dimensional domain in $\mathbb{R}^{n+1}$, and the Lagrangian is first–order (in particular, the hypersurface area).

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2010 Mathematics Subject Classification. Primary 58A99, 49Q99, 35R35; Secondary 14M15, 58A20, 58A10.

Key words and phrases. Global Analysis, Calculus of Variations, Free Boundary Problems, Jet Spaces, Flags.

The authors are thankful to the referee for carefully reading the manuscript, and to the organizers of the 17th summer school in Global Analysis and its Application, held in Levoča, August 17–22, for providing a stimulating environment to their research. The first author is also thankful to the Grant Agency of the Czech Republic (GA ČR) for financial support under the project P201/12/G028. The second author is grateful to the Ph.D. program of Salerno University for financial support.
**Introduction**

Let $Y$ be a smooth (real) manifold of dimension $n+1$, with nonempty boundary $\partial Y$.

**Definition 0.1.** An $n$–dimensional submanifold $L \subseteq Y$ such that

1. $L$ is connected, compact and oriented;
2. $\partial L = L \cap \partial Y$;
3. $L$ is nowhere tangent to $\partial Y$,

is called admissible; the totality of such submanifolds is denoted by $\mathcal{A}_Y$.

Introduce a local coordinate system $(x, u)$ on $Y$, where $x := (x^1, \ldots, x^n)$. Let $\lambda = Ld^nx$ be an $r$th order Lagrangian, i.e., let $\mathcal{L} = \mathcal{L}(x, u, u_I)$ and $I$ denote a multi–index of length $\leq r$. Suppose that, in such coordinates, an element $L \in \mathcal{A}_Y$ is the graph of a function $u = u(x)$, defined on a connected and bounded domain $\Omega \subseteq \mathbb{R}^n$: then the integral

$$S_\lambda[L] := \int_\Omega \mathcal{L} \left( x, u(x), \frac{\partial |I| u}{\partial x^I} \right) d^n x$$

makes sense; it can also be given a coordinate–free formulation.

**Definition 0.2.** The variational problem with free boundary values determined by the Lagrangian $\lambda$ on $Y$ consists of finding the elements of $\mathcal{A}_Y$ which are critical for $S_\lambda$.

Indeed, if properly understood in a geometric framework, $S_\lambda$ is a real–valued function on $\mathcal{A}_Y$; the choice of the denomination is justified by (0.1): if $L$ is allowed to vary within the class $\mathcal{A}_Y$, then the function $u$ describing $L$ is “free” to take any boundary value, as long as $u$ maps $\partial \Omega$ into $\partial Y$.

The main theoretical question addressed in this paper is the following: *do the solutions to a variational problem with free boundary values fulfill some extra equation(s) besides the Euler–Lagrange equations?* A positive answer has already been given in [12, 10, 11], but without detailed proofs: Section 4 is devoted to review this result by adding the missing details.

Sections 3–4 deal with technical aspects of flag fibrations and relative $C$–spectral sequences, respectively: the reader not interested in theoretical considerations may skip them, and jump to Corollary 4.15, which summarizes their results. Section 1 explains the key used to obtain the main result (Section 5), namely the natural behavior of the relative Euler map, under boundary–friendly transformations. As certainly know all who work in geometric variational calculus and cohomological theory of nonlinear PDEs, the Euler map is but a small feature of a general theory (comprising, e.g., conservation laws, Helmholtz conditions, hamiltonian structures, recursion operators, etc.), which possesses a natural relative analog: we added Sections 3–4 just to give a glimpse of it.
The applicative purpose of this paper is to present explicit examples of natural boundary conditions. In the rather pedagogical Section 2, we review the classical analytic solution, given by van Brunt in a recent (2006) book [4], to one of the simplest examples of variational problems with free boundary values. More involved examples are suggested by real–life circumstances, as, e.g., the problem of finding the equilibrium of a soap film freely sliding along the inner wall of an arbitrarily–shaped pipe, discussed in Subsection 5.2.

The geometric point of view is the backbone of this paper: besides allowing a transparent formulation of the main problem, it provides a key tool to obtain a solution. Analytic formulation (0.1) will be used whenever it is necessary to perform actual computations, as well as a source of valuable insights. For example, the Euler–Lagrange equations

\[ \frac{dL}{du} = 0, \]

where \( \frac{dL}{du} \) is the Euler–Lagrange derivative of \( L \), are obtained by a well–known manipulation of (0.1), under the assumption that the variations of \( u \) have a compact support in \( \Omega \): hence, a solution to the main problem should be a stronger condition than the Euler–Lagrange equations themselves. This clue was confirmed by the discovery of the relative Euler map [12], reviewed in Section 4.

1. Generalities on geometric calculus of variations

The Euler map \( E \) appears, in one form or another, in all geometric frameworks for Variational Calculus that are based on the language of differential forms on jet spaces (called here Grassmann fibrations following the recent paper [13], of which we also adopt the notation). Building the Grassmann fibration \( G^*_nY \) over \( Y \) is just a coordinate–free way to add new coordinates \( u_I \), with \( |I| \leq r \), to the manifold \( Y \), in such a way that \( \lambda \) can be considered as an \( n \)–form on \( G^*_nY \). In this new perspective, (0.1) can be rewritten without mentioning the local expression of \( L \): indeed, since \( G^*_nL \equiv L \), being \( \dim L = n \), the canonical inclusion \( \iota_L : L \subseteq Y \) is lifted to an immersion \( j^rL := G^*_n\iota_L : L \rightarrow G^*_nY \), which allows to pull any Lagrangian back to \( L \). In other words, (0.1) reads

\[ S_\lambda : L \in \mathcal{A}_Y \mapsto \int_L j^rL^*\lambda \in \mathbb{R}. \]

Passing from (0.1) to (1.1) is far from being a mere aesthetic exercise. It deploys powerful tools to attack the main problem: essentially, the possibility of using transformations which mix dependent and independent variables (\( x \) and \( u \), respectively, in the above coordinate system). In Subsection 5.1 we show how a suitable change of coordinates can help avoid the lengthy computations.

\[ ^1 \text{In Vinogradov and his school's approach, } G^*_nY \text{ is denoted by } J^*(Y,n), \text{ see [14].} \]
proposed in Section 2, and how to obtain some useful formulae which, to the authors' opinion, would be very hard (though not impossible) to discover relying on pure analytic methods.

The power of transformation methods descends from the natural character of the Euler map: in the principal geometric frameworks for Variational Calculus (Krupka’s variational sequences [8, 9], Anderson’s variational bicomplex [1], and Vinogradov’s C–spectral sequence [15]) the Euler map connects two spaces, say \( L(Y) \) and \( K(Y) \), containing, respectively, the Lagrangians and the Euler–Lagrange expressions for the variational problems on \( Y \). We shall not go into the details, since a lot of excellent literature has been written on the subject; nonetheless, we stress that the natural character of the association \( Y \mapsto G^n_r Y \), where \( r \leq \infty \), i.e., the canonical way to lift transformations of \( Y \) to the Grassmann fibrations, makes the associations \( Y \mapsto \Omega(G^n_r Y, \partial G^n_r Y) \) natural as well.

Indeed, \( L(Y) \) and \( K(Y) \) are usually defined as quotients of sub–complexes (or sub–sequences) of the de Rham complex of finite (or infinite–order) Grassmann fibrations, and as such they inherit the pull–back from differential forms. In other words, any diffeomorphism \( F: Y \rightarrow Y \), determines a commutative diagram

\[
\begin{array}{ccc}
L(Y) & \xrightarrow{E_Y} & K(Y) \\
F^* & & F^* \\
\downarrow & & \downarrow \\
L(Y) & \xrightarrow{E_Y} & K(Y)
\end{array}
\]

If \( F \) is a wisely–chosen change of coordinates, then “the long way” from \( L(Y) \) to \( K(Y) \), i.e., \( F^* \circ E_Y \circ (F^*)^{-1} \) may be more convenient concerning computations. But this is just a category–theoretic restatement of the well–known transformation rule for the Euler–Lagrange equations, which was already known to E. Cartan: the purpose of this paper is to extend it to the class of variational problems with free boundary values, where the Euler–Lagrange equations are sided by the so–called natural boundary conditions, or, equivalently, they are replaced by the relative Euler–Lagrange equations.

Roughly speaking, the “relative” version of the Euler map arises because of the boundary \( \partial Y \).

**Definition 1.1.** By abuse of notation\(^2\) we shall put

\[ \partial G^n_r Y := (\rho^{r,0})^{-1}(\partial Y). \]

Indeed, the canonical inclusion \( \iota: \partial G^n_r Y \subseteq G^n_r Y \) determines a differential algebra epimorphism \( \iota^*: \Omega(G^n_r Y) \rightarrow \Omega(\partial G^n_r Y) \) whose kernel ker \( \iota^* := \Omega(G^n_r Y, \partial G^n_r Y) \) is, by definition\(^3\) the ideal of relative differential forms on \( G^n_r Y \).

\(^2\)\( \partial G^n_r Y \) is more like a prolongation, or lift, to \( G^n_r Y \), of the boundary \( \partial Y \).

\(^3\)Such a construction is common in Differential Topology (see, e.g., [3]).
Much as $L(Y)$, $K(Y)$, and $E_Y$ are constructed out of (classes of) differential forms on $G^n_r Y$ and natural morphism connecting them, their “relative counterparts”, denoted by $L(Y, \partial Y)$, $K(Y, \partial Y)$, and $E^\text{rel}_Y$, respectively, are built out of relative differential forms on $G^n_r Y$. Details of this construction, carried out in the context of $\mathcal{C}$-spectral sequences (meaning, in particular, $r = \infty$), can be found in [12, 10].

Section 4 explains why the relative Euler–Lagrange equations (1.3) $E^\text{rel}_Y (\lambda) = 0$ represent a solution to the main problem. More precisely, since $K(Y, \partial Y)$ identifies with the direct sum $K(Y) \oplus K(\partial Y)$, the single equation (1.3) captures two equations simultaneously, viz., the Euler–Lagrange equations (1.4) $E_Y(\lambda) = 0$, which involve $n$ independent variables, and the natural boundary conditions (1.5) $E^\partial_Y(\lambda) = 0$, where the number of independent variables involved is $n - 1$. Besides providing a common environment for such heterogeneous equations, the formalism of flag fibrations, introduced by the first author in [11], and reviewed in Section 3, allows to write down (1.5) in a workable way.

The natural character of relative Euler map follows automatically from its very definition: in other words, the “relative” version of diagram (1.2), paraphrased by Lemma 1.1 below, needs not to be proved.

**Lemma 1.1.** Let $F$ be boundary-friendly, i.e., $F(\partial Y) \subseteq \partial Y$. Then

$$E^\text{rel}_Y = F^* \circ E^\text{rel}_Y \circ (F^*)^{-1}.$$  

Lemma 1.1 together with Corollary 4.2.1 will be employed in the last Section 5 to obtain new examples of natural boundary conditions.

2. A MOTIVATING EXAMPLE

Let $n = 1$, $Y = [a, b] \times \mathbb{R}$, and $\lambda = L(x, u, u')dx$: in this case, functions can be identified with their graphs, and $\mathcal{A} := C^\infty([a, b])$ as a subset of $\mathcal{A}_Y$. Hence, up to a (noncritical) restriction of $\mathcal{A}_Y$ to $\mathcal{A}$, the boundary problem with free boundary values determined by $\lambda$ on $Y$, entails finding the functions $u$ such that

$$\lim_{\|\hat{u} - u\| \to 0} \frac{S[\hat{u}] - S[u]}{\|\hat{u} - u\|} = 0.$$  

---

4This paper is based on the talk “A geometrical framework for Lagrangian theories which involve $n$ and $n - 1$ independent variables simultaneously” delivered by the first author on August 24, 2012, within the conference “Variations on a Theme”, dedicated to D. Krupka’s seventieth birthday.

5The norm can be either the $L^\infty$ or the $H^1$ norm on $C^2([a, b])$.  

In Chapter 7 of van Brunt’s book [4], the above problem is modified by allowing \( \tilde{u} \) to be defined on a different interval than \( u \). To fit this new setting, \( \mathcal{A} \) must give up its linear structure and norm, namely \( \mathcal{A} := \bigcup_{\xi \in \mathbb{R}} C^\infty([\tilde{x}_0, \tilde{x}_1]) \); according, \( Y := \mathbb{R}^2 \). Despite this, \( \mathcal{A} \) keeps a rather obvious metric structure \( d_{\mathcal{A}} \). Moreover, with two real numbers \( x_0 \) and \( x_1 \) and a suitable function \( \xi \), one can construct a variation \( \tilde{u} \) of \( u \), whose \( d_{\mathcal{A}} \)-distance from \( u \) is controlled by a parameter \( \epsilon > 0 \).

First, use \( x_0 \) and \( x_1 \) to define a new interval \([\tilde{x}_0, \tilde{x}_1]\), where

\[
\tilde{x}_k = x_k + \epsilon X_k, \quad k = 0, 1,
\]

and suppose, without loss of generality, that \( x_0 = \min\{x_0, \tilde{x}_0\} \) and \( \tilde{x}_1 = \max\{x_1, \tilde{x}_1\} \). Then, use \( \xi \in C^\infty([\tilde{x}_0, \tilde{x}_1]) \) to construct the variation

\[
\tilde{u} := u^* + \epsilon \xi,
\]

of \( u \), where \( u^* \) is the 2nd order polynomial extension\[^6\] of \( u \) to the interval \([x_0, \tilde{x}_1]\), i.e.,

\[
\begin{align*}
u^*(x) &= \begin{cases} u, & x \in [x_0, x_1], \\ u(x_1) + (x - x_1)u'(x_1) + \frac{(x-x_1)^2}{2}u''(x_1), & x \in (x_1, \tilde{x}_1].
\end{cases}
\end{align*}
\]

Now

\[
d_{\mathcal{A}}(u, \tilde{u}) := \|[u - \tilde{u}]| + |(x_0, u_0) - (\tilde{x}_0, \tilde{u}_0)| + |(x_1, u_1) - (\tilde{x}_1, \tilde{u}_1)|
\]

is a well-defined distance on \( \mathcal{A} \), which allows to adapt \((2.1)\) to the case when the domain of definition of \( u \) can be altered: the norm \( \|\tilde{u} - u\| \) has to be replaced by the distance \( d(u, \tilde{u}) \). Take the variation \( u \) \((2.2)\), and compute

\[
d_{\mathcal{A}}(u, \tilde{u}) \leq \|[u - \tilde{u}]| + \epsilon X_0 + |u_0 - \tilde{u}_0| + \epsilon X_1 + |u_1 - \tilde{u}_1| \leq \epsilon(3\xi + X_0 + X_1).
\]

Inequality \((2.3)\) shows that

\[
d_{\mathcal{A}}(u, \tilde{u}) = o(\epsilon).
\]

In order to estimate the numerator in \((2.1)\), compute\[^7\]

\[
S_\lambda[\tilde{u}] - S_\lambda[u] = \int_{x_0}^{\tilde{x}_1} \mathcal{L}[\tilde{u}]dx - \int_{x_0}^{x_1} \mathcal{L}[u]dx
= \int_{x_0 + \epsilon X_0}^{x_1 + \epsilon X_1} \mathcal{L}[\tilde{u}]dx - \int_{x_0}^{x_1} \mathcal{L}[u]dx
= \int_{x_0}^{x_1} (\mathcal{L}[\tilde{u}] - \mathcal{L}[u])dx + \int_{x_0 + \epsilon X_0}^{x_1 + \epsilon X_1} \mathcal{L}[\tilde{u}]dx - \int_{x_0}^{x_0 + \epsilon X_0} \mathcal{L}[\tilde{u}]dx.
\]

Equality \((2.5)\) shows that, with respect to the “fixed domain case” \((2.1)\), the variation of \( S_\lambda \) in \( u \) has two additional contributions due to the variations of the

---

\[^6\] To reduce the load of notations, we retain the same symbol \( u \) for the extension \( u^* \) of \( u \).

\[^7\] It is convenient to write \( \mathcal{L}[u] \) instead of \( \mathcal{L}(x, u(x), u'(x)) \).
endpoints of the domain of \( u \). The main advantage of the geometric approach presented in Subsection 5.1 later on, is that such a distinction between the variations of \( u \) and the variation of its domain, simply disappear. For the time being, (2.5) can be just rewritten in a more suggestive form

\[
S_{\lambda}[\hat{u}] - S_{\lambda}[u] = \epsilon \frac{\delta S_{\lambda}}{\delta \xi}[u] + \epsilon \frac{\delta S_{\lambda}}{\delta X_1}[u] - \epsilon \frac{\delta S_{\lambda}}{\delta X_0}[u]
\]

where

\[
\epsilon \frac{\delta S_{\lambda}}{\delta \xi}[u] = \int_{x_0}^{x_1} (L[\hat{u}] - L[u])dx
\]

\[
= \epsilon \left\{ \xi \frac{\partial L}{\partial u'}[u] \right|_{x_0}^{x_1} + \int_{x_0}^{x_1} \xi \left( \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} \right)[u]dx \right\} + o(\epsilon^2),
\]

and

\[
\epsilon \frac{\delta S_{\lambda}}{\delta X_k}[u] = \int_{x_k}^{x_k+\epsilon x_k} L[\hat{u}]dx
\]

\[
= \epsilon X_k L(x_k, u(x_k), u'(x_k)) + o(\epsilon^2), \quad k = 0, 1,
\]

where we used the fact that

\[
\frac{d}{dt} \bigg|_{t=0} \int_{x_k}^{x_k+\epsilon x_k} L[u]dx = L(x, u(x_k), u'(x_k))
\]

and \( L(x, \hat{u}(x_k), \hat{u}'(x_k)) - L(x, u(x_k), u'(x_k)) = o(\epsilon^2) \), for \( k = 0, 1 \).

Now define real numbers \( U_0, U_1 \) by \( \epsilon U_k = \hat{u}(\hat{x}_k) - u(x_k) \), for \( k = 0, 1 \), and compute

\[
\epsilon U_k = (u + \epsilon \xi)(\hat{x}_k) - u(x_0) = u(x_k) + \epsilon X_k u'(x_k) + \epsilon \xi(x_k) + o(\epsilon^2) - u(x_k)
\]

\[
= \epsilon \left( X_k u'(x_k) + \xi(x_k) + o(\epsilon) \right).
\]

This shows that

(2.6) \( \xi(x_k) = U_k - X_k u'(x_k) + o(\epsilon), \quad k = 0, 1 \).

In view of (2.6), (2.5) reads now

\[
S_{\lambda}[\hat{u}] - S_{\lambda}[u] = \left\{ \xi \frac{\partial L}{\partial u'}[u] \right|_{x_0}^{x_1} + \int_{x_0}^{x_1} \xi \left( \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} \right)[u]dx
\]

\[
+ \sum_{k=0}^{1} (-1)^k \left[ -X_k \left( L - u' \frac{\partial L}{\partial u'} \right)(x_k, u(x_k), u'(x_k))
\right.
\]

\[
- U_k \frac{\partial L}{\partial u}(x_k, u(x_k), u'(x_k)) + \xi(x_k) \frac{\partial L}{\partial u'}(x_k, u(x_k), u'(x_k)) \bigg]\}
\]

\[
+ o(\epsilon^2).
\]
Equivalently,
\[ S_\lambda[\hat{u}] - S_\lambda[u] = \epsilon \left\{ \int_{x_0}^{x_1} \xi \frac{\delta L}{\delta u} \, dx + \sum_{k=0,1} (-1)^k \right. \]
\[ \left. \int_{x_k} x \xi \delta \frac{\delta L}{\delta u} \, dx + U_k \frac{\partial L}{\partial u'} (x_k, u(x_k), u'(x_k)) \right\} + o(\epsilon^2). \]

Plugging the last expression into (2.1), and taking into account (2.4), we finally see that \( u \) is a critical point for \( S_\lambda \) if the above term in \( \epsilon \) vanishes for all variations of \( u \), i.e., for all possible choices of \( \xi, X_0 \) and \( X_1 \). In particular, \( u \) must satisfy the (2nd order) Euler–Lagrange equations,
\[ \frac{\delta L}{\delta u} (x, u(x), u'(x), u''(x)) = 0, \]
on its domain of definition, plus a (1st order) natural boundary condition at the endpoints,
\[ \sum_{k=0,1} (-1)^k \left[ (U_k - u'X_k) \frac{\partial L}{\partial u'} + X_k L \right] (x_k, u(x_k), u'(x_k)) = 0. \]

Formula (2.7) is used in van Brunt’s book to prove Theorem 2.1 below, which answers the main question for one of the simplest (though nontrivial) examples of a variational problem with free boundary values.

**Theorem 2.1** (Transversality conditions). Let \( Y \subset \mathbb{R}^2 \) be a closed and connected smooth domain, such that \( \partial Y \) is the disjoint union of two curves \( \gamma_0 \) and \( \gamma_1 \), and \( \mathbb{R}^2 \setminus Y \) is disconnected, and \( \lambda \) be a 1st order Lagrangian. If an element \( u \in \mathcal{A}_Y \) is a solution of the variational problem with free boundary values determined by \( \lambda \), then

(1) \( u \) obeys the Euler–Lagrange equations on its domain of definition \([x_0, x_1]\);

(2) \( u \) fulfills the following transversality conditions
\[ \left[ \frac{dy_{\gamma_k}}{d\sigma} \right]_{\sigma=0} \frac{\partial L}{\partial u'} - \left[ \frac{dx_{\gamma_k}}{d\sigma} \right]_{\sigma=0} \left( u' \frac{\partial L}{\partial u'} - L \right) (x_k, u(x_k), u'(x_k)) = 0, \quad k = 0, 1, \]
where \( \gamma_k(\sigma) = (x_{\gamma_k}(\sigma), y_{\gamma_k}(\sigma)) \) and \( \gamma_k(0) = (x_k, u(x_k)) \), \( k = 0, 1 \).

**Proof.** See [1], Chapter 7. \( \square \)

In this Section we observed the lack of robustness of the functional–analytic approach: the slightest change of settings destroyed the norm on the class of admissible functions, and a (in many respects, unnatural) distance appeared in its place, which worked well only after some lengthy tricks.

\[8\text{Note that } \mathcal{A}_Y \text{ is made precisely by all curves lying in } Y, \text{ such that one endpoint belongs to } \gamma_0 \text{ and the other one to } \gamma_1, \text{ without being tangent to any of them.}\]
3. Flag fibrations

The main motivation for flag fibrations is that equations (1.4) and (1.5) involve $n$ and $n-1$ independent variables, respectively: merging them into a unique equation requires a new formalism where the number of independent variables can take (at least) two values: $n$ and $n-1$. Recall the fundamental embedding

$$G^r_n Y \subseteq G^1_{n-1} (G^r_{n-1} Y).$$

It allows to regard a point $\theta \in G^r_n Y$ as an $n$–dimensional tangent plane to $G^r_{n-1} Y$, and an element of the fibered product

$$P := G^r_n Y \times_{G^r_{n-1} Y} G^1_{n-1} (G^r_{n-1} Y)$$

as a pair consisting of an $n$–dimensional and $(n-1)$–dimensional tangent plane to $G^r_{n-1} Y$ (at the same point). Define

$$F^r_{n,n-1} Y := \{ (\theta_n, \theta_{n-1}) \in P \mid \theta_n \supset \theta_{n-1} \}.$$

In many respects, the theory of flag fibrations parallels that of Grassmann fibrations; it is useful to review here some of its characteristic features.

**Theorem 3.1.** Let $r > 0$. Then the following results hold:

- $F^r_{n,n-1} Y$ is a smooth manifold, called the ($r$th order) flag fibration of $Y$ (of signature $(n,n-1)$): it is fibered over the base $Y$, as well as all lower–order flag fibrations, i.e., the manifolds $F^s_{n,n-1} Y$, with $s < r$.

- The flag fibration $F^r_{n,n-1} Y$ is naturally fibered over the corresponding (i.e., with the same order $r$ and the same number of independent variables $n$) Grassmann fibration $G^r_n Y$.

- The image of $F^r_{n,n-1} Y$ under the canonical projection over $G^1_{n-1} (G^r_{n-1} Y)$ is a smooth submanifold, naturally understood as 1st order nonlinear partial differential equation on $G^r_{n-1} Y$ in $n-1$ independent variables: the equation of involutive $(n-1)$–planes of $G^r_{n-1} Y$.

- The equation of involutive $(n-1)$–planes of $G^r_{n} Y$ projects naturally over $F^r_{n,n-1} Y$.

- The infinite–order flag fibration $F^\infty_{n,n-1} Y$, obtained as the inverse limit of finite–order flag fibrations, identifies with the equation of involutive $(n-1)$–planes of $G^\infty_{n-1} Y$, and, hence, it can be considered as an equation on $G^\infty_{n} Y$.

- The infinite prolongation $F^\infty_{n,n-1} Y (\infty)$ of the 1st order differential equation $F^\infty_{n,n-1} Y$, understood as a pro–finite leaf space is naturally interpreted as a space of infinite–order Cauchy data.

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9 Called integral element by Bryant&Griffiths [5], or $R$–plane by Vinogradov and his school [14, 2].

10 See [7] for a definition of infinitely prolonged equations.

11 In the sense of Vinogradov’s “Secondary Calculus”: see, for instance, the introduction of [15].
• The infinitely-prolonged equation \((F_{n,n-1}^\infty Y)_{(\infty)}\) fits into a double filtration picture:

\[
\begin{array}{ccc}
F_{n,n-1}^\infty Y_{(\infty)} & \xleftarrow{p} & G_n^\infty Y \\
\xrightarrow{n} & & \xrightarrow{\downarrow} G_{n-1}^\infty Y.
\end{array}
\]

mimicking the similar diagram in the (linear) theory of flag manifolds.

Proof. See [11]. □

Theorem 3.2 ([11], Theorem 9.1). Let \(L\) (resp., \(\Sigma\)) be a leaf (i.e., maximal integral submanifold with respect to the infinite-order contact distribution) of \(G_n^\infty Y\) (resp., \(G_{n-1}^\infty Y\)). Then the following identifications

\[
\begin{align*}
p^{-1}(L) &= G_{n-1}^\infty (L) \\
n^{-1}(\Sigma) &= J^\infty(N_\Sigma)
\end{align*}
\]

hold, where \(N_\Sigma\) is a pro-finite vector bundle called the infinite-order normal bundle.

Moreover, \(p\) and \(n\) are transverse one to another, in the sense that \(p^{-1}(L)\) (resp., \(n^{-1}(\Sigma)\)) maps non degenerately onto \(G_{n-1}^\infty Y\) (resp., \(G_n^\infty Y\)).

Equality (3.3) is the less straightforward of the two, and plays a prominent role in the description of the relative Euler map, which will be introduced in the next section.

4. Relative Euler operator and natural boundary conditions

In order to clarify the relationship between relative cohomology and variational problems with free boundary values, recall Definition 1.1 and suppose that \(\lambda = d\lambda_0\), where \(\lambda_0 \in \Omega^{n-1}(G_n^\infty Y)\) is such that

\[
\lambda_0 |_{\partial G_n^\infty Y} = 0.
\]

Then (1.1) reads

\[
S_\lambda[L] = \int_L j^* L^* \lambda = \int_L j^* L^* d\lambda_0 = \int_{\partial L} j^* L^* \lambda_0 |_{\partial L} = 0,
\]

since \(\partial L \subset \partial Y\) according to Definition 0.1. Indeed, \(j^* L\) maps \(\partial L\) into \(\partial G_n^\infty Y\) and, hence, the fact that \(\lambda_0\) vanishes on the latter implies that its pull-back \(j^* L^* \lambda_0\) vanishes on the former.

Lemma 4.1. The action \(S_\lambda\) on \(A_Y\) is determined by the equivalence class of \(\lambda\) modulo the subspace \(d\Omega^{n-1}(G_n^\infty Y, \partial G_n^\infty Y)\) of \(\Omega^n(G_n^\infty Y)\).

Proof. A paraphrase of (4.1). □
In order to simplify further analysis, we shall work, from now on, in the context of infinite Grassmann fibrations and $\mathcal{C}$–spectral sequences; in particular, a Lagrangian will be a horizontal $n$–form on $G^\infty_n Y$,

$$\lambda \in \Omega^n_h(G^\infty_n Y),$$

where $\Omega_h(G^\infty_n Y)$ is the quotient differential algebra of $\Omega(G^\infty_n Y)$ with respect to the ideal of contact forms. An expert in bicomplexes or $\mathcal{C}$–spectral sequences would say that the next corollary is the “horizontalization” of Lemma 4.1 above.

**Corollary 4.0.1.** The action $S_{\lambda}$ on $A_Y$ is determined by the relative horizontal cohomology class

$$[\lambda]_{\text{rel}} \in H^n_h(G^\infty_n Y, \partial G^\infty_n Y) := \frac{\Omega^n_h(G^\infty_n Y)}{d_h \Omega^{n-1}_h(G^\infty_n Y, \partial G^\infty_n Y)},$$

where $d_h$ is the horizontal differential.

Corollary 4.0.1 says precisely that $H^n_h(G^\infty_n Y, \partial G^\infty_n Y)$ is the space $L(Y, \partial Y)$ mentioned in Section 1. The space $K(Y, \partial Y)$ can be obtained in a similar way, using relative forms, contact ideal, and cohomology: we shall rather use an approach based on total differential operators and Spencer cohomology, as in [12]. In the same cohomological framework it will also appear the relative Euler map $E^\text{rel}_Y$, which allows to obtain the equation (1.3) out of the Lagrangian $\lambda$.

The aim of this section is to prove that (1.3) is indeed equivalent to the pair of equations (1.4)–(1.5) and, furthermore, that either the single equation (1.3), or the two coupled equations (1.4)–(1.5), provide a (nontrivial) answer to the main question stated in the Introduction. The first result can be found in [12], but its proof, which is a consequence of Theorem 3.2, was provided later in [11], and it is a consequence of the following structural result, which dictates strong restrictions on the topology of the fibration $\partial G^\infty_n Y \to G^\infty_n(Y)$.

**Corollary 4.0.2** ([12], Theorem 2). Consider $G^\infty_{n-1}(\partial Y)$ as a submanifold of $G^\infty_n(Y)$ via the embedding $\partial Y \subseteq Y$. Then

$$\partial G^\infty_n Y = J^\infty_h(N_{G^\infty_{n-1}(\partial Y)})$$

where $N$ is the infinite–order normal bundle (see Theorem 3.2), and $J^\infty_h$ means “horizontal jet bundle.”

It follows from Corollary 4.0.2 that the relative (i.e., constructed with relative forms) $\mathcal{C}$–spectral sequence of $\partial G^\infty_n Y$ is particularly simple (i.e., one–line\(^1\)) in turn, this implies that $K(Y, \partial Y)$ splits into the sum $K(Y) \oplus K(\partial Y)$ (the proof

\(^1\)Roughly speaking, the analog of jet bundle where derivatives are replaced by total derivatives. See, e.g., [11] [10] for more details.

\(^2\)See [11] for the meaning of “one–line.”
can be found in [10]). Hence, equation (1.3) splits into two equations: (1.4) and (1.5).

The second result has been stated in [11] without proof, which is provided by Lemma 4.2 below.

**Remark 1.** Proposition 4.1 below contains a general theoretical result concerning relative $C$–spectral sequences, so that there is no need to restrict ourselves to the case of one independent variable: in other words, we let $Y$ to be of dimension $n + m$, where $m$ is arbitrary, i.e., locally, to be fibered over an $n$–dimensional manifold $X$ with $m$–dimensional fiber (when needed, such a fibration is called $\pi$). Here we recall some terminology.

$\text{VSym}(Y)$ is the module of vertical symmetries (denoted by $\alpha$ in [2, 7]) of the infinite–order contact distribution on $G^\infty_n Y$, and $\mathcal{D}$ is the sub–algebra of differential operators generated by total derivatives (the $C$–differential operators, according to [2, 7]). Suppose now we work in a local chart (in particular, $\partial Y = \{x_n = 0\}$ and $\pi$ is trivial): in this case, $D^{(j)}_I$ denotes the projection on the $j^{th}$ component of the free $C^\infty(G^\infty_n Y)$–module $C^\infty(G^\infty_n Y)^m = \text{VSym}(Y)$, and $D^{(j)}_I \overset{\text{def}}{=} D_I \circ D^{(j)}_0$ for all $j = 1, \ldots, m$, and $I$ multi–index of length $n$, i.e., $I \in \mathbb{N}_0^n$. Moreover, $\mathcal{D}(C^\infty(G^n_n Y), \Omega^n_0(G^n_n Y))$ identifies with $\mathcal{D}(C^\infty(G^n_n Y), C^\infty(C^n_n Y))$ by means of the horizontal volume form $d^n x$, and $\mathcal{D}(C^\infty(\partial G^n_n Y), \Omega^n_0(\partial G^n_n Y))$ with $\mathcal{D}(C^\infty(\partial G^n_n Y), C^\infty(\partial G^n_n Y))$ by means of the horizontal volume form $d^n \omega$ on $\partial G^n_n Y$. Accordingly, the formally adjoint modules (see [16]) $\text{VSym}^\dagger(Y)$ and $\text{VSym}^\dagger(\partial G^n_n Y)$ are identified with the dual module of $\text{VSym}(Y)$ and $\text{VSym}(\partial G^n_n Y)$, which are still free, with bases $\{D^{(j)}_I\}_{j=1,\ldots,m}$ and $\{D^{(j,\alpha)}_I\}_{j=1,\ldots,m, \alpha \in \mathbb{N}_0^n}$, respectively.

Recall that $D_I$ is the composition of total derivatives $D^{(j_1)}_{i_1} \circ D^{(j_2)}_{i_2} \circ \cdots \circ D^{(j_m)}_{i_m}$, with $I = (i_1, \ldots, i_m)$, and, by our own convention, the difference between the multi–index $I$ and an integer $\alpha \leq i_n$ is the multi–index $I - \alpha := (i_1, \ldots, i_{n-1}, i_n - \alpha)$.

**Proposition 4.1 (On the structure of $K(Y, \partial Y)$).** Let $Y$ be as in Remark 4. Then

\begin{equation}
K(Y, \partial Y) = \frac{\mathcal{D}(\text{VSym}(Y), \Omega^n_0(G^n_n Y))}{\delta(\mathcal{D}(\text{VSym}(Y), C^\infty(G^n_n Y)) \otimes \Omega^n_0(\partial G^n_n Y))},
\end{equation}

where $\delta$ is the Spencer differential. Moreover, the cohomology class of the cocycle $\square = a^I D^{(j)}_I \in \mathcal{D}(\text{VSym}(Y), \Omega^n_0(G^n_n Y))$ is identified with the pair $(E(\square), E^\partial(\square)) \in \text{VSym}^\dagger(Y) \oplus \text{VSym}^\dagger(\partial G^n_n Y)$, where $E(\square) = (-1)^{|I|} D_I(a^I) D^{(j)}_I$, and

\begin{equation}
E^\partial(\square) = \sum_{I \in \mathbb{N}^m} (-1)^{|I|-i_n} \sum_{j=1}^m \sum_{\alpha < i_n} (-1)^\alpha D_{I-i_n}(D^{(j)}_n(a^I))_{|\partial G^n_n Y}) D^{(j,i_n-\alpha-1)}_I.
\end{equation}
Proof. By the definition of relative $C$–spectral sequences \[12\], the space $K(Y, \partial Y)$ is the $n$th cohomology space of the subcomplex

$$D(VSym(Y), C^\infty(G_n^\infty Y)) \otimes \Omega_b(G_n^\infty Y, \partial G_n^\infty Y)$$

of $D(VSym(Y), \Omega_b(G_n^\infty Y))$. Expression \[4.2\] is a consequence of the fact that $\Omega_b^n(G_n^\infty Y, \partial G_n^\infty Y)$ equals $\Omega_b^n(G_n^\infty Y)$. In other words, $K(Y, \partial Y)$ has the same $n$–cycles as $K(Y)$, but fewer $n$–coboundaries, which explains why the $n$th cohomology of the subcomplex turns out to be quite larger than the cohomology of the entire complex: in turn, this explains the appearance of natural boundary conditions. For the sake of simplicity, we shall skip the index $j$.

We now prove that the relative Spencer cohomology of $\square$ is identified with \[4.3\]. To this end, observe that the elements $d_i^n d_0^{n-1} x$, for $i = 1, \ldots, n-1$, together with $x_n d_n^{n-1} x$, form a basis for $\Omega_b^n(G_n^\infty Y, \partial G_n^\infty Y)$. Accordingly, elements of $D(C^\infty(G_n^\infty Y), \Omega_b^{n-1}(C^\infty(G_n^\infty Y), \partial G_n^\infty Y))$ can be obtained by summing up the elements $\square \otimes d_i^n d_0^{n-1} x$ and $\square \otimes x_n d_n^{n-1} x$, with $\square \in D(C^\infty(G_n^\infty Y), C^\infty(G_n^\infty Y))$ for $i = 1, \ldots, n$.

We compute the Spencer differential $\delta(\square \otimes d_i^n d_0^{n-1} x) = (-1)^{i-1} (D_1 \circ \square) \otimes d^n x$, and observe that

$$(\delta(\square \otimes x_n d_0^{n-1} x))(f) = d_h(\square^n(f)) x_n d_0^{n-1} x$$

$$= d_h(\square^n(f)) \wedge (x_n d_0^{n-1} x) + \square^n(f) d(x_n d_0^{n-1} x)$$

$$= D_1 a (\square^n(f)) dx_n \wedge x_n d_0^{n-1} x + \square^n(f) dx_n \wedge d_0^{n-1} x$$

$$= (-1)^{n-1} (x_n D_1 a \circ \square^n + \square^n)(f) d^n x,$$

for arbitrary $f \in C^\infty(G_n^\infty Y)$, whence

$$\delta(\square^n \otimes x_n d_0^{n-1} x) = (-1)^{n-1} ((x_n D_1 a + 1) \circ \square^n) \otimes d^n x$$

$$= (-1)^{n-1} (D_1 a \circ x_n \circ \square^n) \otimes d^n x,$$

since $1 = [D_1 a, x_n]$.

It follows that $a \circ D_1 a \circ \square$ is cohomologous to $-D_1 a \circ \square$ for all $i = 1, \ldots, n-1$, whereas $a \circ D_1 a \circ \square = (-D_1 a + D_1 a \circ a) \circ \square$ is not generally cohomologous to $-D_1 a \circ \square$, since $D_1 a \circ a \circ \square$ is not a coboundary, unless $a$ factors through $x_n$. 

\[ end{proof} \]
From the well-known formula of elementary calculus is cohomologous to the operator
\[ \square = a^I D_I \in \mathcal{D}(C^\infty(G_n^\infty Y), \Omega_n^0), \text{ with } I \in \mathbb{N}_0^n. \] Such an operator \( \square \) is cohomologous to the operator

\[
\square' = (-1)^{|I|-i_n} D_{I-i_n}(a^I) \circ D_{x_n}^{i_n}
\]

\[
= (-1)^{|I|-i_n} (-D_{x_n}(D_{I-i_n}(a^I)) + D_{x_n} \circ D_{I-i_n}(a^I)) \circ D_{x_n}^{i_n-1}
\]

\[
= (-1)^{|I|-i_n} D_{I-i_n-1}(a^I) \circ D_{x_n}^{i_n-1} + (-1)^{|I|-i_n} D_{x_n} \circ D_{I-i_n}(a^I) \circ D_{x_n}^{i_n-1}
\]

\[
= (-1)^{|I|-i_n} D_I(a^I) + (-1)^{|I|-i_n} D_{x_n} \circ D_{I-i_n}(a^I) \circ D_{x_n}^{i_n-1}
\]

\[
+ (-1)^{|I|-i_n} D_{x_n} \circ D_{I-i_n}(a^I) \circ D_{x_n}^{i_n-2} + \ldots,
\]

i.e., to

\[
\square' = (-1)^{|I|} D_I(a^I) + \sum_{0<\alpha \leq i_n} (-1)^{|I|-\alpha} D_{x_n} \circ D_{I-\alpha}(a^I) \circ D_{x_n}^{\alpha-1},
\]

which turns into a coboundary if and only if the function \( D_I(a^I) \) is zero and all the functions \( D_{I-\alpha}(a^I) \) factor through \( x_n \), i.e., they vanish on \( \partial G_n^\infty Y \).

This means that the cohomology class \( [\square] \) is uniquely determined by \( (-1)^{|I|} D_I(a^I) \in C^\infty(G_n^\infty Y) \), i.e., by \( E(\square) \) and by the set of functions \( (-1)^{|I|-\alpha} D_{I-\alpha}(a^I) \mid_{\partial G_n^\infty Y} \in C^\infty(\partial G_n^\infty Y) \), with \( \alpha = 1, \ldots, i_n \). Notice that the latter ones can be rewritten as \( D_{I-i_n}(D_{x_n}^{\alpha}(a^I) \mid_{\partial G_n^\infty Y}) \), with \( \alpha = 0, \ldots, i_n - 1 \), because the multi-index \( I-i_n \) belongs actually to \( \mathbb{N}_0^{n-1} \) and the first \( n-1 \) total derivatives are tangent to \( \partial G_n^\infty Y \): hence the last set of functions represent the coordinates of the element \( E(\square) \), according to (4.3).

**Proposition 4.2** (On natural boundary conditions). Let \( L \) be locally given by the graph of a function \( u = u(x) \), defined on a compact and connected subset \( \Omega \subset \mathbb{R}^n \), such that \( \partial \Omega \) has equation \( x^n = 0 \). If \( L \) is critical for \( S_\lambda \), then the following equations hold on \( \partial \Omega \):

\[
\sum_{|I| \leq r, \ i_n > \alpha} (-1)^{|I|-\alpha-1} D_I^{i_n-1} \frac{\partial^{|I|-\alpha-1}}{\partial x_n^{i_n-\alpha-I}} \left( \frac{\partial L}{\partial u_I} \right) |_{\partial \Omega} = 0, \quad \alpha = 0, \ldots, r - 1.
\]

**Proof.** Let \( r \) be the order of \( \lambda \) and \( I = (i_1, \ldots, i_n) \). Put

\[
\mathcal{L}^{i_1 \cdots i_n} := \frac{\partial L}{\partial u_I} = \frac{\partial L}{\partial u_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}},
\]

From the well-known formula of elementary calculus

\[
(4.4) \quad f g^{(i)} = \left( \sum_{s=0}^{i-1} f^{(s)} g^{(i-s-1)} \right)' + (-1)^i f^{(i)} g
\]
it follows that

\[
(4.5) \quad \int_{\partial \Omega} \sum_{i_1 + \cdots + i_n \leq r} L_{i_1 i_2 \cdots i_n} \eta_{x_1} x_2 \cdots x_n d^n x
\]

\[
(4.6) = \int_{\partial \Omega} \sum_{i_1 + \cdots + i_n \leq r} \left\{ \frac{d}{dx_1} \sum_{s_1=0}^{i_1-1} (-1)^{s_1} \frac{d^{s_1} L_{i_1 i_2 \cdots i_n}}{dx_1^{s_1}} \eta_{x_1} x_2 \cdots x_n \right\} d^n x
\]

\[
(4.7) + \left\{ (-1)^{i_1} \frac{d^{i_1} L_{i_1 i_2 \cdots i_n}}{dx_1^{i_1}} \eta_{x_2 x_3 \cdots x_n} \right\} d^n x
\]

Then, applying again (4.4) to the term (4.7), we obtain

\[
(4.8) \quad \frac{d}{dx_2} \left( \sum_{s_2=0}^{i_2-1} (-1)^{s_2} \frac{d^{s_2} L_{i_1 i_2 \cdots i_n}}{dx_2^{s_2}} \right) \eta_{x_2 x_3 \cdots x_n}
\]

\[
\frac{d}{dx_2} \left( \sum_{s_2=0}^{i_2-1} (-1)^{s_2} \frac{d^{s_2} L_{i_1 i_2 \cdots i_n}}{dx_2^{s_2}} \right) \eta_{x_2 x_3 \cdots x_n}
\]

Again, by (4.4), we develop term (4.9):

\[
(4.10) \quad (4.5) = (4.6) + (4.8) + \cdots
\]

\[
+ \int_{\partial \Omega} \sum_{i_1 + \cdots + i_n \leq r} \left\{ \frac{d}{dx_2} \left( \sum_{s_2=0}^{i_2-1} (-1)^{s_2} \frac{d^{s_2} L_{i_1 i_2 \cdots i_n}}{dx_2^{s_2}} \right) \right\} d^n x
\]

\[
+ \left\{ (-1)^{i_1} \frac{d^{i_1} L_{i_1 i_2 \cdots i_n}}{dx_2^{i_1}} \right\} d^n x
\]

Since \( \partial \Omega = \{ x_n = 0 \} \), all terms appearing on line (4.10) disappear, being of the form

\[
\int_{\partial \Omega} \frac{d}{dx_1} (F) d^n x = \int_{\partial \Omega} d(F d^{n-1} x) = \int_{\partial \Omega} F d^{n-1} x|_{\partial \Omega} = 0, \quad \forall i = 1, 2, \ldots, n - 1.
\]

On the other hand, (4.12) is the Euler–Lagrange; it remains just (4.11), i.e.,

\[
(4.13) \quad \int_{\partial \Omega} \sum_{i_1 + \cdots + i_n \leq r} \left\{ (-1)^{|I|+1} \frac{d^{i_1 \cdots i_n} L}{dx^{|I|+1}} \right\} d^n x.
\]
Since (4.13) has to vanish for all variations $\eta$, all equations (4.16) must be satisfied.

**Lemma 4.2.** Let $L \in A_Y$ be a solution to the variational problem with free boundary values determined by $\lambda$ on $Y$. Then equation (1.4) holds on $L$ and equation (1.5) holds on $\partial L$.

Before providing a proof, it is convenient to cast a bridge between the approach based on total differential operators to the space $K(Y)$, sketched in Remark 1, and a perhaps more familiar one, based on “1–contact, $n$–horizontal” $(n + 1)$–forms, or forms “of type (1, n)”. Namely, (1.4) can be written down as

\[(4.14)\quad E_Y(\lambda) = \frac{\delta L}{\delta u} \omega \wedge d^n x,\]

where $\omega$ is the zero–order contact form, and $\omega \wedge d^n x$ plays the role of the generator $D_0$ (see in Remark 1 of the module $K(Y)$). Equation (4.14) clarifies the above sentence “(1.4) holds on $L$”: it means that (4.14), pulled back to $L$ via $j^\infty L$, vanishes.

Similarly, the results contained in Corollary 4.0.2 and Proposition 4.1 give a solid basis to the sentence “holds on $\partial L$”, since

\[(4.15)\quad E_{\partial Y}(\lambda) = E_{\partial Y,\alpha}(\lambda) \omega^\alpha \wedge d_{n-1} x,\]

where now the $\omega^\alpha$'s are the zero–order contact forms on $J^\infty_n (N_{G^\infty_n} (\partial Y))$, and $\omega^\alpha \wedge d_{n-1} x$ plays the role of the generators $D_{0j}^{\alpha}$, where $j = 1$ (see in Remark 1), of the module $K(\partial G^\infty_n Y)$.

According, $E_{\partial Y,\alpha}(\lambda)$ can be obtained as the coefficient of $D_{0j}^{\alpha}$, in (4.3), where $j = 1$, and $a^j = \frac{\delta L}{\delta a^j}$. Again, the sentence “(1.5) holds on $\partial L$” means that (4.15), pulled back to $\partial L$ via $j^\infty \partial L$, vanishes (see also Theorem 11.1 in [11]).

**Proof.** The first fact is obvious: if $L$ a solution of a variational problem with free boundary values, then $L \cap \partial L$ is a solution of the Euler–Lagrange equation determined by the same Lagrangian $\lambda$, i.e., equation (1.4) holds.

We stress that, in order to prove the second fact, it is necessary to have the result on the structure of equation (1.5) provided by Corollary 4.0.2. Namely, (1.5) is localizable, in the sense that its left–hand side belong to a module of sections, and hence it vanishes locally if and only if it vanishes globally. Then, we can choose a coordinate system $(x, u)$ such that $L$ is the graph of a function $u = u(x)$ on $\Omega$ and $\partial Y = \{x_n = 0\}$. Since $L$ is critical, all equations (4.16) must hold true on $\partial \Omega$; on the other hand, the above discussions showed that equations (4.16) are nothing but $E_{\partial Y,\alpha} = 0$: hence, (4.15) vanishes, i.e., (1.5) must be valid on $\partial L$.
For readers not interested in theoretical details, we collect the main result of the last two sections into a convenient (though redundant) Corollary.

**Corollary 4.2.1** (A solution of the main problem). Let $L \in A_Y$ be a solution to the variational problem with free boundary values determined by $\lambda$ on $Y$. Then, the natural boundary conditions $E^\lambda_Y(\lambda) = 0$ are satisfied on $\partial L$. In local coordinates, $E^\lambda_Y(\lambda) = E^\lambda_{Y,\alpha}(\lambda)\omega^\alpha \wedge d_{n-1}x$, where:

1. the $\omega^\alpha$’s are the zero–order contact forms on the infinite jet of a suitable pro–finite vector bundle over $\partial Y$ which arises in the theory of flag fibrations over $Y$;
2. if $\alpha$ is less than the order of the Lagrangian $\lambda$, and $L$ is locally the graph of a function $u = u(x)$ on $\Omega \subseteq \mathbb{R}^n$, the component $E^\lambda_{Y,\alpha}(\lambda)$ is given by

$$E^\lambda_{Y,\alpha}(\lambda) = \sum_{|I| \leq r, \ i_n > \alpha} (-1)^{|I|-\alpha-1} \frac{d^{|I|-\alpha}}{dx^n_{i_n}} \left( \frac{d^{\alpha-1}}{dx^n_{i_n} - \alpha} \left( \frac{\partial L}{\partial u^I} \right) \bigg| \partial \Omega \right) .$$

5. **Applications**

Together, Lemma 1.1 and Corollary 4.2.1 provide a powerful tool for writing down concrete examples of natural boundary conditions. Computations presented in this section will be simplified by some “tricks” based on multi–linear algebra and total differentials (Remarks 2 and 3 below).

**Remark 2** (Top differential forms). A brute–force attempt to change variables in a multi–dimensional integral may lead to a meaningless formula

$$\int f(x)d^n x = \int f(x(t)) \frac{d^n x}{dt} d^n t .$$

Nonetheless, since the $C^\infty(\mathbb{R}^n)$–module $\Omega^n(\mathbb{R}^n)$ is freely generated by $d^n t$, any $n$–form can be identified with a function. In particular, $d^n t$ identifies with 1, and $d^n x$ with the Jacobian of the change of variables $x = x(t)$, thus recovering the meaning of $\frac{d^n x}{dt}$. From now on, all $n$–forms will be identified with functions: hence, an expression like $\Xi(\omega)$, where $\Xi$ is a vector field and $\omega$ an $n$–form, is not the Lie derivative of $\omega$, but the function $\Xi(f)$, where $f$ is uniquely defined by $\omega = fd^n t$.

**Remark 3** (Total differentials). Formula (5.1) can be adapted to variational integrals, just by replacing differentials by total differentials, namely

$$\int f(x, u(x), u_1(x), \ldots, u_n(x)) d^n x = \int f(x(t, y), u(t, y), u_1(t, y, y_1, \ldots, y_n), \ldots, u_n(t, y, y_1, \ldots, y_n)) \frac{d^n x}{dt} \frac{d^n t}{dt} .$$
where now $\overline{d^n x} = \overline{dx^1} \wedge \cdots \wedge \overline{dx^n}$ (i.e., the operator $d$ used in Section 4). Recall that

$$\overline{d}x = D_t (x) dt,$$

where $D_t = \partial_t + y_i \partial_{u^i}$ is the total derivative operator with respect to $t$. In this context, horizontal $n$–forms on $G^1_n Y$, i.e., the space with coordinates $(t, y, y_1, \ldots, y_n)$, are identified with functions on the same space, via the horizontal volume form $\overline{d^n} t$. Accordingly, $\overline{d^n} x$ is the “total Jacobian” associated with the change of variables $(x, u) = (x(t, y), u(t, y))$.

5.1. A 1st order, one–dimensional example. Consider again the variational problem with free boundary values of Theorem 2.1, Section 2. Let $\mathcal{L} = \mathcal{L}(t, y, y')$ be its Lagrangian, and recall that $\partial Y$ is the disjoint union of two curves in the $(t, y)$–plane. Then, if $\gamma(\sigma) = (t_\gamma(\sigma), y_\gamma(\sigma))$ is one of them, a critical point $y = y(x)$ for $S_\lambda$ must fulfill the natural boundary condition

$$\left[ y'_\gamma(0) \frac{\partial \mathcal{L}}{\partial y'} - t'_\gamma(0) \left( y'_\gamma \frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} \right) \right] (t, y(t), y'(t)) = 0$$

where $(t, y(t)) = \gamma(0)$ (see (2.8)).

Equation (5.2) can be obtained in a transparent geometrical way, without introducing ad hoc metrics on the set $A$. Just use a change of coordinates $(t, y) \mapsto (x, u)$,

$$x = x(t, y)$$
$$u = u(t, y)$$

which “rectifies” the curve $\gamma$, i.e., such that $F_\gamma(\gamma)$ is, for instance, the $u$–axis of the $(x, u)$–plane. Then, lift $F$ to a contact transformation $(t, y, y') \mapsto (x, u, u')$ of $G^1_1 Y$, where

$$u' = \frac{u_t + y'u_y}{x_t + y'x_y}.$$

It is a simple exercise to get (5.3) (see, for instance, [2], Section 1.2); nevertheless, in view of the next generalization, we prefer to justify it, by using the total differential operator $\overline{d}$. Recall that

$$\overline{df} = D_t (f) dt,$$

for $f = f(t, y, y')$, with $D_t = \partial_t + y' \partial_{y'}$ being the total derivative operator in $t$. As announced in Remark 3, we shall identify horizontal one–forms on the $(t, y, y')$–space with functions: hence, (5.3) above can be written as

$$u' = \overline{\frac{du}{dx}} = \frac{D_t (u) dt}{D_t (x) dt}.$$
It is worth noticing that the inverse transformation $F^{-1}$ is the same as \( (5.5) \)

\[
y' = \frac{dy}{dt} = \frac{D_x(y)dx}{D_x(t)dx} = \frac{y_x + u'u_u}{t_x + u't_u},
\]

where now the total derivative operator is taken with respect to \( x \). It follows that

\[
\frac{\partial y'}{\partial u'} = \frac{\partial}{\partial u'} \left( \frac{D_x(y)}{D_x(t)} \right).
\]

Finally, since \( dt = \overline{dt} \) (see \( (5.4) \)), the Lagrangian \( \lambda \) reads

\[
\lambda = \overline{\mathcal{L}}(t, y, y') dt = \mathcal{L}(t, y, y')(D_x(t)dx)
\]

in the \( (x, u, u') \)-space, where \( D_x(t) \) plays the role of “total Jacobian” (Remark 3). In other words, \( (F^*)^{-1}(\lambda) = \mathcal{L}dx \), where \( \mathcal{L} = \mathcal{L}(x, u, u') \) is given by

\[
(5.8) \quad \mathcal{L} = (F^*)^{-1}(\mathcal{L})D_x(t) = \mathcal{L}(t(x, u), y(x, y), y'(x, u, u'))D_x(t)(x, u, u').
\]

Now everything is ready. \( (F^*)^{-1}(\lambda) \) determines a variational problem with free boundary values on \( F(Y) \) and \( \partial F(Y) = F(\partial Y) \), by construction, consists of two curves, one of which is the \( u \)-axis: by Corollary 4.2.1, on such an axis the natural boundary conditions take a particularly simple form

\[
\frac{\partial \mathcal{L}}{\partial u'}(0, u(0), u'(0)) = 0.
\]

It remains to express \( (5.9) \) in terms of the coordinates \( (t, y, y') \), i.e., to apply Lemma 1.1. We compute

\[
\frac{\partial \mathcal{L}}{\partial u'}(5.8) = \frac{\partial}{\partial u'}(\mathcal{L}D_x(t)) = \frac{\partial \mathcal{L}}{\partial u'}D_x(t) + \mathcal{L} \frac{\partial D_x(t)}{\partial u'}
\]

\[
= \frac{\partial \mathcal{L}}{\partial y'} \frac{\partial}{\partial u'} \left( \frac{D_x(y)}{D_x(t)} \right) D_x(t) + \mathcal{L} \frac{\partial D_x(t)}{\partial u'}
\]

\[
= \frac{\partial \mathcal{L}}{\partial y'} \frac{D_x(t)D_x(t) - D_x(y)\frac{\partial D_x(t)}{\partial u'}}{D_x(t)} + \mathcal{L} \frac{\partial D_x(t)}{\partial u'}
\]

\[
= \frac{\partial \mathcal{L}}{\partial y'} \left( \frac{\partial}{\partial u'}(D_x(y)) - \frac{y'}{D_x(t)} \frac{\partial D_x(t)}{\partial u'} \right) + \mathcal{L} \frac{\partial D_x(t)}{\partial u'}
\]

\[
(5.9) \quad \frac{\partial \mathcal{L}}{\partial y'} \left( \frac{\partial}{\partial u'}(D_x(y)) - \frac{y'}{D_x(t)} \frac{\partial D_x(t)}{\partial u'} \right) + \mathcal{L} \frac{\partial D_x(t)}{\partial u'}.
\]
It suffices to observe that
\begin{align}
\frac{\partial D_x}{\partial u'}(t) &= \frac{\partial (t_x + u't_u)}{\partial u'} = t_u, \tag{5.10} \\
\frac{\partial D_y}{\partial u'}(y) &= \frac{\partial (y_x + u'y_u)}{\partial u'} = y_u. \tag{5.11} 
\end{align}

Substituting (5.10) and (5.10) into (5.9), one gets
\begin{align}
\frac{\partial L}{\partial y'}(y - t'u' + t'u) + L'u = 0,
\end{align}

which, evaluated at \((0, u(0), u'(0))\), returns (5.2), since, by the choice of \(F\),
\begin{align}
t'_a(0) &= t_u(\gamma(0)), \\
y'_a(0) &= y_u(\gamma(0)).
\end{align}

5.2. A 1st order, multi–dimensional example. Now we pass to an \(n\)–dimensional example: as we shall see, the geometric methods used before generalize effortlessly to this case; an analogous generalization of the methods used in Section 2 i.e., defining a metric structure on the space of all functions defined on a compact and connected subset of \(\mathbb{R}^n\), would introduce a lot of technical difficulties, obscuring the simple solution of the problem.

**Theorem 5.1** (Generalized transversality conditions). Let \(Y\) be a closed smooth domain, with nonempty (smooth) boundary, of \(\mathbb{R}^{n+1} = (t, y)\), with \(t = (t^1, \ldots, t^n)\), and \(\lambda\) be a 1st order Lagrangian, locally given by \(L = L(t, y, y_1, \ldots, y_n)\). If \(y = y(t)\) is a critical point for \(S_\lambda\), then, for any point \(\theta \in \partial Y\), the normal vector \(\nu_\theta\) to the hypersurface \(\partial Y\) must be orthogonal to \(H(t, y, y_1(t), \ldots, y_n(t))\), where \(H\) is the \(\mathbb{R}^{n+1}\)–valued function on \(G_1 Y\) given by
\begin{align}
H := \left( \frac{\partial L}{\partial y_1}, \ldots, \frac{\partial L}{\partial y_n}, L - \frac{\partial L}{\partial y_i} \right),
\end{align}

and \(\theta = (t, y(t))\).

**Proof.** Choose a change of coordinates \((t, y) \mapsto (x, u)\),
\begin{align}
x &= x(t, y), \\
u &= u(t, y),
\end{align}

such that \(F(\partial Y)\) has equation \(x^n = 0\). In analogy with (5.6),
\begin{align}
y_i &= \frac{\partial y}{\partial u'} = \frac{\partial y}{\partial t} \wedge \frac{\partial u^{-1}}{\partial t} t = \frac{\omega_i}{\partial t}, \tag{5.12} 
\end{align}

where
\begin{align}
\omega_i := \frac{\partial t^1}{\partial t} \wedge \cdots \wedge \frac{\partial t^{-1}}{\partial t} \wedge \frac{\partial y}{\partial t} \wedge \frac{\partial t^{i+1}}{\partial t} \wedge \cdots \wedge \frac{\partial t^n}{\partial t},
\end{align}
and we use again the convention that horizontal $n$–forms are identified with functions via the (horizontal) volume form $dx^n$ introduced in Remark 3. Developing all total differentials appearing in (5.12), one recovers the familiar formula for the lifting of $F$, as it can be found, e.g. in [2], Section 1.2.

In analogy with (5.7),
\[
\frac{\partial y_i}{\partial u_n} = \frac{\partial}{\partial u_n} \left( \frac{\omega_i}{\partial t} \right), \quad i = 1, 2, \ldots, n.
\]

Finally, analogously to (5.8), we obtain $(F^*)^{-1}(\lambda) = \tilde{\mathcal{L}} d^n t$, where
\[
\tilde{\mathcal{L}} = \frac{\partial}{\partial u_n} (L d^n t)
\]
where, as explained by Remark 3, $\frac{d}{\partial x} dt = \frac{d}{\partial x} dx$ is just an unconventional way to write down the “total Jacobian” of $F$.

Again, $(F^*)^{-1}(\lambda)$ determines a variational problem with free boundary values on $F(Y)$ and $\partial F(Y) = F(\partial Y)$, by construction, is the hyperplane $x^n = 0$; by Corollary 4.2.1 on such a hyperplane the natural boundary conditions read
\[
(5.14) \quad \frac{\partial \tilde{\mathcal{L}}}{\partial u_n} (x^1, \ldots, x^{n-1}, 0, u(x^1, \ldots, x^{n-1}, 0), \ldots, u_n(x^1, \ldots, x^{n-1}, 0)) = 0.
\]

Finally, let us write down (5.14) in terms of the coordinates $(t, y, y_1, \ldots, y_n)$.

We compute
\[
\frac{\partial \tilde{\mathcal{L}}}{\partial u_n} = \frac{\partial}{\partial u_n} \left( L d^n t \right) = \frac{\partial L}{\partial u_n} d^n t + L \frac{\partial (d^n t)}{\partial u_n}
\]
\[
= \frac{\partial L}{\partial y_i} \frac{\partial}{\partial u_n} \left( \frac{\omega_i}{\partial t} \right) d^n t + L \frac{\partial (d^n t)}{\partial u_n}
\]
\[
= \frac{\partial L}{\partial y_i} \left( \frac{\partial}{\partial u_n} (\omega_i) - \omega_i \frac{\partial}{\partial u_n} (d^n t) \right) + L \frac{\partial (d^n t)}{\partial u_n}
\]
\[
= \frac{\partial L}{\partial y_i} \left( \frac{\partial}{\partial u_n} (\omega_i) - \omega_i \frac{\partial}{\partial u_n} (d^n t) \right) + L \frac{\partial (d^n t)}{\partial u_n}.
\]

We denote
\[
\nu := \left( \frac{\partial \omega_1}{\partial u_n}, \ldots, \frac{\partial \omega_n}{\partial u_n}, \frac{\partial (d^n t)}{\partial u_n} \right) \in \mathbb{R}^{n-1}.
\]

Then, (5.15) coincides with $\nu \cdot H$. 

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It remains to show that $\nu$ is indeed the normal vector to $F(\partial Y)$. To this end, it is convenient to pass to the determinantal notation for $n$-forms. Namely, 

$$
\omega_i = \begin{vmatrix}
t^1_{x1} + u_1 t^1_u & \cdots & y_{x1} + u_1 y_u & \cdots & t^n_{x1} + u_1 t^n_u \\
t^1_{x2} + u_2 t^1_u & \cdots & y_{x2} + u_2 y_u & \cdots & t^n_{x2} + u_2 t^n_u \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
t^1_{x_{n-1}} + u_{n-1} t^1_u & \cdots & y_{x_{n-1}} + u_{n-1} y_u & \cdots & t^n_{x_{n-1}} + u_{n-1} t^n_u \\
t^1_{xn} + u_n t^1_u & \cdots & y_{xn} + u_n y_u & \cdots & t^n_{xn} + u_n t^n_u 
\end{vmatrix}
$$

contains $u_n$ only in the last line: hence,

$$
\frac{\partial \omega_i}{\partial u_n} = \begin{vmatrix}
t^1_{x1} + u_1 t^1_u & \cdots & y_{x1} + u_1 y_u & \cdots & t^n_{x1} + u_1 t^n_u \\
t^1_{x2} + u_2 t^1_u & \cdots & y_{x2} + u_2 y_u & \cdots & t^n_{x2} + u_2 t^n_u \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
t^1_{x_{n-1}} + u_{n-1} t^1_u & \cdots & y_{x_{n-1}} + u_{n-1} y_u & \cdots & t^n_{x_{n-1}} + u_{n-1} t^n_u \\
t^1_{xn} + u_n t^1_u & \cdots & y_{xn} + u_n y_u & \cdots & t^n_{xn} + u_n t^n_u 
\end{vmatrix}
$$

Subtracting from the $j^{th}$ row the $n^{th}$ row multiplied by $u_j$, for all $j = 1, \ldots, n-1$, the determinant does not change, i.e.,

$$
(5.16) \quad \frac{\partial \omega_i}{\partial u_n} = \begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 
\end{vmatrix}
$$

Similarly,

$$
(5.17) \quad \frac{\partial (\partial^n t)}{\partial u_n} = \begin{vmatrix}
t^1_{x1} & \cdots & t^1_{x1} & \cdots & t^n_{x1} \\
t^1_{x2} & \cdots & t^1_{x2} & \cdots & t^n_{x2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
t^1_{x_{n-1}} & \cdots & t^1_{x_{n-1}} & \cdots & t^n_{x_{n-1}} \\
t^1_{xn} & \cdots & t^1_{xn} & \cdots & t^n_{xn} 
\end{vmatrix}
$$

Observe that (5.16) and (5.17) are the multi-dimensional analogues of (5.10) and (5.11), respectively. In other words, $\nu$ is composed of the $n \times n$ minors (with sign) of the $n \times (n + 1)$ matrix

$$(T_1, T_2, \ldots, T_{n-1}, T)^t,$$

where the $n$ vectors

$$
T_i = (t_{xi}, y_{xi}), \quad i = 1, \ldots, n-1, \\
T = (t_u, y_u),
$$

form a basis for the tangent space of $F(\partial Y) = \{x_n = 0\}$, i.e., $\nu = T_1 \times \cdots \times T_{n-1} \times T$. \qed
Notice that (5.15) is formally the same as (5.9): the synthetic language of multi–linear algebra allowed to handle all the “total Jacobian” determinants involved in the proof, without any additional difficulty as compared with the one–dimensional case.

5.3. The soap film. We generalize now a classical example that can be found in Giaquinta and Hildebrandt’s book [2] (Section 2.4). Namely, in the hypotheses of Theorem 5.1 above, suppose that \( \lambda \) is the (hypersurface) area Lagrangian, i.e., locally,
\[
L = \sqrt{1 + \sum_{i=1}^{n} y_i^2}.
\]
Then
\[
H = \frac{1}{\sqrt{1 + \sum_{i=1}^{n} y_i^2}} (y_1, y_2, \ldots, y_n, -1)
\]
is precisely the unit normal vector to the surface \( y = y(t) \). This proves the last result of this paper.

Corollary 5.1.1. Let \( Y \subseteq \mathbb{R}^{n+1} \) be a closed smooth domain with smooth nonempty boundary. If a hypersurface \( L \in A_Y \) is a solution of the variational problem with free boundary values determined by the area functional, then \( L \) must intersect orthogonally \( \partial Y \) everywhere.

In particular, Corollary 5.1.1 shows that a soap film, whose boundary is constrained to slide over the inner surface of a fixed domain (e.g., a pipe of arbitrary shape), tends toward a position of equilibrium where it forms a right angle with the walls of the container (besides, of course, possessing zero mean curvature).

Remark 4. If \( n = 2 \) and \( Y = D \times \mathbb{R}^2 \), where \( D \subseteq \mathbb{R} \) is diffeomorphic to a closed disk, then a surface \( L \) from Corollary 5.1.1 above is forced to be the graph of a constant function \( y = y(t_1, t_2) \). Indeed, since \( Y \) is a surface with zero mean curvature, its maximum is attained on \( \partial D \). Hence, there exists a point \( \theta \in \partial L \), such that \( \partial L = L \cap \partial Y \) has negative curvature. But \( L \) hits \( \partial Y \) orthogonally in \( \theta \), thus, along the normal direction to \( \partial L \), the surface \( L \) must possess positive curvature, i.e., there must exist a point \( \theta' \in L \), in a neighborhood of \( \theta \), such that the \( y \)-component of \( \theta' \) is greater than the \( y \)-component of \( \theta \), thus contradicting the fact that \( \theta \) corresponds to a maximum. It follows that \( L \) must be the graph of a constant function. It would be nice to generalize this simple observation to multi–dimensional cases.

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