Non-Hermitian Many-Body Localization

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(Dated: November 29, 2018)

Many-body localization is shown to suppress imaginary parts of complex eigenenergies for general non-Hermitian Hamiltonians having time-reversal symmetry. We demonstrate that a real-complex transition occurs upon many-body localization and profoundly affects the dynamical stability of non-Hermitian interacting systems with asymmetric hopping that respect time-reversal symmetry. Moreover, the real-complex transition is shown to be absent in non-Hermitian many-body systems with gain and/or loss that breaks time-reversal symmetry, even though the many-body localization transition still persists.

Introduction. The reality of eigenenergies of a Hamiltonian is closely linked to the dynamical stability. While Hermiticity guarantees the reality of the eigenspectrum, certain classes of non-Hermitian Hamiltonians are known to have real eigenenergies. In particular, a real-complex transition of eigenenergies of non-Hermitian systems has attracted growing interest motivated by their experimental realization [1–19]. An important class that exhibits such a transition features parity-time (PT) symmetry [20, 21] or its generalization (pseudo-Hermiticity [22]). While previous studies focused on single-particle models [23–30], the real-complex transition has recently been discussed in many-body interacting systems [31–35].

Another important class that exhibits a real-complex transition is non-Hermitian systems with disorder and time-reversal symmetry. Hatano and Nelson [36–39] investigated a real-complex transition for a single-particle model with asymmetric hopping in a disordered potential, and found that real eigenenergies become complex when the Anderson localization of the corresponding eigenstates is destroyed due to strong non-Hermiticity. However, such a non-Hermitian localization transition has been discussed mostly for noninteracting models [40–46]. It is highly nontrivial whether a real-complex transition due to localization and time-reversal symmetry still persists in non-Hermitian many-body systems. In Ref. [37], an interaction is considered to arise from hard-core repulsion of vortices. This corresponds to the \(U = 0\) case in our model in Eq. (1) and can be treated as a single-particle problem. In Ref. [47], a non-Hermitian Bose-Hubbard model with disorder (Eq. (S-1) in Appendix I [48]) was investigated by the Hartree-Fock approximation. However, their results are not conclusive because the notion of many-body localization (MBL) [49–65] was not well established at that time. Note that this problem is also relevant to the depinning transition of vortices in type-II superconductors [36–38].

In this Letter, we show that localization suppresses imaginary parts of many-body eigenenergies for non-Hermitian interacting Hamiltonians having time-reversal symmetry and can induce a real-complex transition. By investigating disordered interacting particles with asymmetric hopping, we find a real-complex transition at which almost all eigenenergies become real as disorder is increased (Fig. 1(a) and (b)). In the phase with a real spectrum, energy absorption/emission disappears despite non-Hermiticity, demonstrating that quantum states become dynamically stable. Remarkably, this real-complex transition is solely determined from the statistics of many-body eigenenergies in the thermodynamic limit, and thus distinct from the conventional PT-symmetry breaking. We show that the transition is induced by a
non-Hermitian extension of MBL and characterized by a dramatic change in the level-spacing statistics and the entanglement entropy. We also demonstrate that real-complex transitions are absent in non-Hermitian systems with gain and/or loss that break time-reversal symmetry, though non-Hermitian MBL still survives (see Fig. 1(c)). The results are summarized in Fig. 1(d).

Suppression of complex eigenenergies due to localization. We first show that localization suppresses imaginary parts of many-body eigenenergies for generic non-Hermitian Hamiltonians having time-reversal symmetry. Let us decompose the Hamiltonian into the unperturbed part and the non-Hermitian perturbation as $\hat{H} = \hat{H}_0 + \hat{V}_{\text{NH}}$ ($\hat{H}_0$ is either Hermitian or non-Hermitian). We assume that $\hat{H}_0$ and $\hat{V}_{\text{NH}}$ can be expressed as sums of local operators. We consider a set of real eigenenergies $\{E_a\}$ of $\hat{H}_0$ and denote the corresponding right (left) eigenstates as $|E_a^R\rangle$ ($|E_a^L\rangle$), which satisfy $\langle E_a^L | \hat{E} | E_a^R \rangle = \delta_{ab}$ [66].

To see why time-reversal symmetry is crucial for the reality of eigenenergies, we consider the energy shift due to a non-Hermitian perturbation to first order. This shift, which is given by $\langle E_a^L | \hat{V}_{\text{NH}} | E_a^R \rangle$, is in general complex (Fig. 1(c)) but becomes real when time-reversal symmetry $\hat{T}$ is imposed, i.e., $[\hat{H}_0, \hat{T}] = [\hat{V}_{\text{NH}}, \hat{T}] = 0$ [67].

On the other hand, even in the presence of time-reversal symmetry, eigenenergies can coalesce and acquire imaginary parts [20] for sufficiently large non-Hermiticity $\hat{V}_{\text{NH}}$. The coalescence occurs due to mixing of eigenstates, which signals breakdown of the above perturbation theory at higher order. As detailed later, while the mixing of two adjacent excited eigenstates occurs for delocalized eigenstates of $\hat{H}_0$ [68], it does not for MBL eigenstates. In fact, for delocalized phases, many complex eigenenergies appear via coalescence of excited eigenstates [69] (Fig. 1(a)). By contrast, such coalescence is suppressed when the system enters the MBL phase (Fig. 1(b)).

Disordered model with asymmetric hopping. With the above general discussion in mind, we consider hard-core bosons on a one-dimensional lattice whose Hamiltonian has asymptotic hopping terms:

$$\hat{H} = \sum_{i=1}^{L} \left[ -J(e^{-g} \hat{b}_{i+1}^{\dagger} \hat{b}_{i} + e^{g} \hat{b}_{i}^{\dagger} \hat{b}_{i+1}) + U \hat{n}_{i} \hat{n}_{i+1} + h_{i} \hat{n}_{i} \right].$$

Here, $\hat{n}_{i} = \hat{b}_{i}^{\dagger} \hat{b}_{i}$ is the particle-number operator at site $i$ with the annihilation operator $\hat{b}_{i}$ of a hard-core boson, $g$ controls the non-Hermiticity, and $h_{i}$ is randomly chosen from $[-h, h]$. We assume the periodic boundary condition ($\hat{b}_{L+1} = \hat{b}_{1}$) and that $J, g, h_{i}$ and $U$ are real. This model has time-reversal symmetry whose action is complex conjugation. We consider a subspace where the total particle number is $M$. For $U = 0$ and $M = 1$, this model reduces to the Hatano-Nelson model [36]. For $g = 0$, this model can be mapped to the Hermitian XXZ model with a random magnetic field, where all the eigenstates become many-body localized at $h > h_{\text{MBL}}^{\text{MBL}} = h_{\text{c}}^{\text{MBL}}(g = 0) \sim 7$ [50, 61, 68] for $J = 1$ and $U = 2$. In the following, we assume $J = 1$, $U = 2$, and $L = 2M$ (half filling). We also consider a Bose-Hubbard model with asymmetric hopping [47] in Appendix I in Supplemental Material [48]. These non-Hermitian bosonic models can be realized experimentally in ultracold atomic systems, where strong disorder has been realized [63] and asymmetric hopping can also be implemented with a collective one-body loss [70].

Real-complex transition. We show the eigenenergies of the Hamiltonian in Eq. (1) for different values of $h$ with $g = 0.1$ in Fig. 2(a). The spectrum is symmetric with respect to the real axis due to time-reversal symmetry. The fraction of eigenenergies with nonzero imaginary parts decreases as $h$ increases.

As a quantitative indicator of the reality of eigenenergies, we define $f_{\text{im}} = \frac{D_{\text{im}}}{D}$, where $D_{\text{im}}$ is the number of eigenenergies with nonzero imaginary parts, $D$ is the total number of eigenenergies (i.e., the dimension of the Hilbert space), and the overline denotes the disorder average [71]. This quantity measures the fraction of eigenenergies with nonzero imaginary parts (see Appendix II for another indicator that measures the maximum imaginary part among all eigenenergies [48]).
Figure 2(b) shows the $h$-dependence of $f_{1m}$ for different values of $L$ with $g = 0.1$ (see Appendix III for larger $g$ [48]). As the system size increases, $f_{1m}$ increases for $h \lesssim h_c^R \approx 8$ and decreases for $h \gtrsim h_c^R$. This indicates a real-complex phase transition at the level of many-body eigenenergies at $h = h_c^R$ in the thermodynamic limit ($L \to \infty$): almost all eigenenergies have nonzero imaginary parts for $h < h_c^R$, whereas almost all eigenenergies are real for $h > h_c^R$. Note that this real-complex transition is a novel phase transition defined in terms of the statistics of the spectrum (i.e., the average over disorder and eigenstates) in the thermodynamic limit. This is in contrast to the conventional PT transition that can occur in small systems, where the PT transition is identified to be the point at which eigenstates coalesce without an average over many eigenstates. We also examine $g_{\text{SR}}^R(h)$ as a transition point by fixing $h$ and changing $g$ defined in the thermodynamic limit (see Appendix IV [48]).

Our results show that many-body eigenenergies suddenly become almost real with increasing disorder even for highly excited states. This change significantly affects the dynamical stability of the system (see Appendix V for details [48]). Figure 2(c) shows the dynamics of the real part of energy $E^R(t) = \text{Re}[(\psi(t)|\hat{H}|\psi(t))]$. While the energy should be conserved for any $h$ in the Hermitian Hamiltonian, it is sensitive to small non-Hermiticity for $h \lesssim h_c^R$. The change in energy is due to the delocalized eigenstates with nonzero imaginary parts and signals the dynamical instability. On the other hand, the system is dynamically stable for $h \gtrsim h_c^R$ where the energy is conserved except for negligibly small oscillations since almost all eigenenergies are real. Moreover, the real-complex transition is relevant for the dynamics of other quantities such as entanglement entropy and the local particle density [48].

Non-Hermitian many-body localization. We next show that the system undergoes a delocalization-localization transition due to disorder. To characterize the non-Hermitian MBLS, we first discuss the nearest-level-spacing distribution of (unfolded) eigenenergies [72] on the complex plane for sufficiently small $h$ and on the real axis for large $h$ in Fig. 3(a) [73]. For weak disorder, we find that the distribution is a Ginibre distribution $P^G_{\text{Gin}}(s) = c p(s)$ which describes the ensemble for non-Hermitian Gaussian random matrices [72, 74]. Here $p(s) = \lim_{N \to \infty} \prod_{n=1}^{N-1} e_{n}(s)^{N-2} e^{-s} \sum_{a=1}^{N-1} \sum_{m=1}^{a-1} \frac{e_{m}(s)}{n! n_{a}(s)}$ with $e_{n}(s) = \sum_{m=0}^{s^{2}} \frac{s^{m}}{m!}$ and $c = \int_{0}^{\infty} ds p(s) = 1.1429 \cdots$ [43, 72, 75]. Our result demonstrates a non-Hermitian generalization of the conjecture [50, 72, 76] that level-spacing statistics of Hermitian delocalized systems obey the Wigner-Dyson distribution. This Ginibre-type phase is a novel delocalized phase that is expected to appear for generic non-Hermitian interacting systems. Note that while the Ginibre distribution has been investigated in single-particle dissipative chaotic models [43, 77], our results are the first demonstration for quantum many-body systems. For strong $h$, the level-spacing distribution becomes Poissonian on the real axis $P^R_{\text{Po}}(s) = e^{-s}$, which is similar to the Hermitian MBL [50].

We next discuss the half-chain entanglement entropy for the right eigenstates $|E^R_{\alpha}\rangle$ [78]: $S_{\alpha} = \text{Tr}[(\rho^{L})] = \langle E^R_{\alpha}\rangle - \langle E^R_{\alpha}\rangle^{2}$, where $|E^R_{\alpha}\rangle$ is normalized to unity: $\langle E^R_{\alpha}|E^R_{\alpha}\rangle = 1$. In Fig. 3(b), we show the $L$-dependence of $S_{\alpha}/L$ averaged over the eigenstates around the middle of the spectrum [79]. This figure shows that the entanglement entropy exhibits a crossover from the volume to area law as we enter the many-body localized regime, as is similar to the Hermitian case [61, 80]. This shows that delocalized and MBL phases can be distinguished by the entanglement entropy even in non-Hermitian systems.

As discussed above, the stability of eigenstates of $\hat{H}_0$ under local perturbations $\hat{V}_{\text{NH}}$ is important for the suppression of complex eigenenergies. We consider $\mathcal{G}(L) = \langle \hat{V}_{\text{NH}} | \hat{V}_{\text{NH}} \rangle$.
\[ \frac{\langle \mathcal{E}^L_{a+1} \mid \mathcal{V}_{NH} \mid \mathcal{E}^R_{a} \rangle}{\mathcal{E}^L_{a+1} - \mathcal{E}^R_{a}} \] as a measure of the stability of the eigenstates (see Appendix VI [48]). Here we only consider \( \mathcal{E}^a_i = \mathcal{E}_a + \langle \mathcal{E}^a_i \mid \mathcal{V}_{NH} \mid \mathcal{E}^b_i \rangle \) that stays real and the labels of eigenstates (eigenenergies) \( a \)'s are taken such that \( \mathcal{E}^1_i \leq \mathcal{E}^2_i \leq \ldots \). This is a generalization to the Hermitian counterpart (i.e., \( \hat{H}_0 \) and perturbations are both Hermitian) introduced in Ref. [68], where \( \frac{\partial \mathcal{G}}{\partial L} > 0 \) for the delocalized phase due to the Srednicki ansatz on the eigenstate thermalization hypothesis [81–85] and \( \frac{\partial \mathcal{G}}{\partial L} < 0 \) for the localized phase due to quasilocal conserved quantities [55, 56, 64, 65].

In Fig. 3(c), we show the \( h \)-dependence of \( \mathcal{G}(L) \) for the non-Hermitian setting. We find \( \mathcal{G}(L) \sim \alpha L (\alpha > 0) \) for the delocalized phase and \( \mathcal{G}(L) \sim -\beta L (\beta > 0) \) for the localized phase, which is similar to the Hermitian case. From the point at which \( \mathcal{G}(L) \) is independent of \( L \), we can estimate that the non-Hermitian MBL transition occurs at \( h^\text{MBL}_c \approx 7 \pm 1 \), which is close to the real-complex transition point \( h^R \) (the small deviation is attributed to finite-size effects).

We note that the absence of coalescence of nearby eigenstates due to the non-Hermitian MBL does not necessarily lead to the complete suppression of complex eigenenergies since non-adjacent eigenstates can mix and form complex-conjugate pairs. However, for our model in Eq. (1), we can show that the coalescence process is suppressed even for non-adjacent eigenstates and hence the entirely real spectra are realized, as detailed in Appendix VII [48].

**Disordered model with gain and loss.** Finally, we show that the non-Hermitian MBL occurs even without time-reversal symmetry, whereas the real-complex transition does not. We focus on the model with gain and loss, which is experimentally feasible [18, 19]:

\[
\hat{H} = \sum_{i=1}^{L} \left[ -J(\hat{b}_{i+1}^\dagger \hat{b}_i + \text{h.c.}) + U\hat{n}_i\hat{n}_{i+1} + (h_i - i\gamma(-1)^i)\hat{n}_i \right]
\]

(2)

This model does not have time-reversal symmetry in the presence of non-Hermiticity and disorder. Figure 4(a) shows the spectrum of this Hamiltonian with \( \gamma = 0.1 \) for a single disorder realization. The eigenenergies have nonzero imaginary parts irrespective of \( h \). We confirm \( f_{\text{Im}} = 1 \) for any \( h \) and \( L \) (data not shown).

While the real-complex transition is absent, the delocalization-localization transition exists in this model. Figure 4(b) shows nearest-level-spacing distributions on the complex plane for different \( h \). For \( h \lesssim h^\text{MBL}_c \approx 5 \), the distribution is a Ginibre distribution \( P^\text{Gin}_C(s) \); for \( h \gtrsim h^\text{MBL}_c \), it is a Poisson distribution \( P^\text{Poi}_C(s) \). This change in the level statistics occurs due to the non-Hermitian MBL transition. The entanglement entropy of the eigenstates also characterizes the non-Hermitian MBL transition; it shows the volume law for \( h \lesssim h^\text{MBL}_c \approx 5 \) and the area law for \( h \gtrsim h^\text{MBL}_c \) (see Fig. 4(c)).

**Conclusion and Outlook.** We have shown that non-Hermitian MBL suppresses complex eigenenergies for generic non-Hermitian interacting Hamiltonians having time-reversal symmetry. We have demonstrated that a real-complex transition occurs upon MBL and profoundly affects the dynamical stability of interacting systems with asymmetric hopping. We have also demonstrated that real-complex transitions are absent in systems with gain and/or loss that break time-reversal symmetry, though the non-Hermitian MBL still persists.

The real-complex transition found in our work is a conceptually new type of phase transition, which never occurs in isolated systems, few-body systems, or clean systems. It is an interesting future problem to investigate other properties of the non-Hermitian MBL such as critical phenomena, where all of the non-Hermiticity, disorder, and interaction come into play. Our work is also relevant to the generalization of the notion of quantum chaos [76, 77, 86] to open interacting systems described by non-Hermitian Hamiltonians, as indicated by its Ginibre-type level statistics. Furthermore, the properties of non-Hermitian MBL, e.g., the area law of entanglement entropy, provide a new approach to understanding the critical problem of the interaction effect in the depinning transition in type-II superconductors [36]. A detailed investigation of this problem merits further study.
We are grateful to Zongping Gong, Yuto Ashida, Masaya Nakagawa, and Naomichi Hatano for fruitful discussions. This work was supported by KAKENHI Grant No. JP18H01145 and a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science” (KAKENHI Grant No. JP15H05855) from the Japan Society for the Promotion of Science. R. H. was supported by the Japan Society for the Promotion of Science through Program for Leading Graduate Schools (ALPS) and JSPS fellowship (JSPS KAKENHI Grant No. JP17J03189). K. K. was supported by the JSPS through Program for Leading Graduate Schools (ALPS).

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$$\langle \xi^L_a \vert V_{NH} |\xi^R_a\rangle = (\langle \xi^L_a \vert \hat{V}_{NH} |\xi^R_a\rangle)^* = (|\xi^L_a\rangle \langle \xi^L_a | V_{NH} \xi^R_a \rangle)^* = (|\xi^L_a\rangle \langle \xi^L_a | V_{NH} |\xi^R_a\rangle)^*.$$

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Supplemental Material for “Non-Hermitian Many-Body Localization”

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(Dated: November 29, 2018)
I. REAL-COMPLEX TRANSITION FOR THE NON-HERMITIAN BOSE-HUBBARD MODEL

While we investigate the model with hard-core bosons in the main text, we here show that the real-complex phase transition of many-body eigenenergies occurs in the disordered Bose-Hubbard model [1] with asymmetric hopping. The Hamiltonian can be written as

\[ \hat{H}_{BH} = -J \sum_{i=1}^{L} \left( e^{-g} \hat{a}_{i+1}^\dagger \hat{a}_i + e^{g} \hat{a}_{i}^\dagger \hat{a}_{i+1} \right) + \frac{U}{2} \sum_{i=1}^{L} \hat{n}_i' (\hat{n}_i' - 1) + \sum_{i=1}^{L} h_i \hat{n}_i', \]  

(S-1)

where \( \hat{n}_i' = \hat{a}_i^\dagger \hat{a}_i \) is the particle-number operator at site \( i \) with \( \hat{a}_i \) being the annihilation operator of a boson at site \( i \). Here the prime ('') is used to distinguish \( \hat{n}_i' \) from the particle-number operator \( \hat{n}_i \) of hard-core bosons in the main text. We again consider \( J = 1, \ U = 2, \) and the half-filling case with the periodic boundary condition.

Figure S-1 shows the fraction \( f_{\text{Im}} \) of complex eigenenergies (see the main text) as a function of the disorder strength \( h \). The real-complex transition occurs at a critical disorder strength \( h_{Rc} \), as in the case of hard-core bosons discussed in the main text. Note that the numerically achievable size of the system is smaller for the Bose-Hubbard model than the case of hard-core bosons.

II. MAXIMUM VALUES OF IMAGINARY PARTS

While we discuss the fraction \( f_{\text{Im}} \) of complex eigenenergies in the main text, we can also consider the maximum value \( \Delta_{\text{Im}} \) of imaginary parts of eigenenergies as yet another indicator of the real-complex transition, which is defined as

\[ \Delta_{\text{Im}} = \max_{\alpha} |\text{Im}[E_{\alpha}]|, \]  

(S-2)

where \( E_{\alpha} \) is an eigenenergy of \( \hat{H} \). Note that \( \max_{\alpha} |\text{Im}[E_{\alpha}]| \) governs the dynamical instability of non-Hermitian systems: since the imaginary part of eigenenergies describes the rate of amplification or attenuation of that mode, the system is shown to be stable for \( t \ll |\max_{\alpha} |\text{Im}[E_{\alpha}]||^{-1} \) (see also Fig. 2(c) in the main text and Appendix V in this Supplemental Material). Note that this real-complex transition point can, in general, be different from the transition point of \( f_{\text{Im}} \) and that of the non-Hermitian many-body localization (MBL).
FIG. S-1. Real-complex transition of the non-Hermitian Bose-Hubbard model described by Eq. (S-1) as a function of the disorder strength $h$ for $g = 0.1$ (left) and 1 (right). As the system size $L$ increases, the fraction $f_{\text{Im}}$ of complex eigenenergies increases for $h \lesssim h_{R}^c$ and decreases for $h \gtrsim h_{R}^c$, where the critical disorder strength is $h_{R}^c \simeq 8$ for $g = 0.1$ and $h_{R}^c \simeq 15$ for $g = 1$. Data show the average values of $f_{\text{Im}}$ over $N_s$ samples, where $N_s = 10000$ for $L = 6, 8, 10$ and $N_s = 100$ for $L = 12$.

In Fig. S-2, we show the $h$-dependence of $\Delta_{\text{Im}}$ for different values of $L$ with asymmetric hopping (see Eq. (1) in the main text) or with gain and loss (see Eq. (2) in the main text). For the former case, $\Delta_{\text{Im}}$ slowly increases and saturates for $h \lesssim 8$ with increasing $L$, whereas it rapidly decreases for $h \gtrsim 8$. This indicates the presence of a real-complex transition at $h \simeq 8$, consistent with the case obtained for $f_{\text{Im}}$. On the other hand, for the latter case, $\Delta_{\text{Im}}$ monotonically decreases with increasing $L$ for all $h$, which indicates the absence of the real-complex transition.

III. REAL-COMPLEX TRANSITION AND NON-HERMITIAN MBL FOR STRONG NON-HERMITICITY

The real-complex transition also occurs for strong non-Hermiticity e.g., $g = 1$, in the model in Eq. (1) in the main text. Figure S-3(a) shows the $h$-dependence of $f_{\text{Im}}$ for different values of $L$ with $g = 1$. As the system size increases, $f_{\text{Im}}$ grows for $h \lesssim h_{R}^c \simeq 15$ and decreases for $h \gtrsim h_{R}^c$. Note that this transition point is far from that of the Hermitian case ($g = 0$) and depends on all of non-Hermiticity $g$, interaction $U$, and disorder $h$.

As in the case of $g = 0.1$ explained in the main text, the non-Hermitian MBL transition
FIG. S-2. Maximum values of imaginary parts of eigenenergies for (a) the model with asymmetric hopping ($g = 0.1$) described by Eq. (1) in the main text and (b) the model with gain and loss ($\gamma = 0.1$) described by Eq. (2) in the main text. In (a), $\Delta_{\text{Im}}$ is nearly independent of the system size $L$ for $h \lesssim 8$ and rapidly decreases for $h \gtrsim 8$ with increasing $L$. The results for $L = 16$ are shown only for $h \leq 14$ because we have few eigenenergies with nonzero imaginary parts for larger $h$. In (b), $\Delta_{\text{Im}}$ increases for any $h$ with increasing the system size, which means the absence of the real-complex transition. We use $N_s = 10000$ for $L = 6, 8, 10, 12$, $N_s = 1000$ for $L = 14$, and $N_s = 10$ for $L = 16$ for both (a) and (b).

leads to the real-complex transition described above for $g = 1$. Figure S-3(b) shows the nearest-level-spacing distribution of (unfolded) eigenenergies [2] on the complex plane for sufficiently small $h$ ($h = 8$) and that on the real axis for large $h$ ($h = 20$). For weak disorder, we find that the distribution is a Ginibre distribution $P_{\text{Gin}}(s)$ (see the main text for the definition). On the other hand, for large $h$, the level-spacing distribution becomes a Poisson distribution on the real axis $P_{\text{Po}}(s) = e^{-s}$. This is similar to the case for $g = 0.1$.

We next discuss the half-chain entanglement entropy $S$ for the right eigenstates (see the main text for the definition). In Fig. S-3(c), we show the $L$-dependence of the ratio $S/L$ averaged over the eigenstates around the middle of the real part of the spectrum for different $h$. We find that the entanglement entropy decreases as we enter the localized regime, and that delocalized and MBL phases can be distinguished also for $g = 1$.

In Fig. S-3(d), we show the $h$-dependence of $G(L)$ for the perturbation $\hat{V}_{\text{NH}} = \hat{b}_i^{\dagger} \hat{b}_{i+1}$ (see the main text for the definition). We see $G(L) \sim \alpha L$ ($\alpha > 0$) in the delocalized phase and $G(L) \sim -\beta L$ ($\beta > 0$) in the localized phase. We can identify the non-Hermitian MBL
FIG. S-3. Real-complex transition and non-Hermitian MBL indicators in the model described in Eq. (1) in the main text for strong non-Hermiticity with $g = 1$. (a) Dependence of $f_{\text{Im}}$ on $h$ for different $L$ with $g = 1$. As the system size increases, $f_{\text{Im}}$ increases for $h \lesssim h_c^R \simeq 15$ and decreases for $h \gtrsim h_c^R$. (b) Nearest-level-spacing distribution of (unfolded) eigenenergies on the complex plane for $h = 8$ and that on the real axis for $h = 20$. For $h = 8$, the distribution is a Ginibre distribution $P_{\text{Gin}}^C(s)$ (see the main text for the definition) rather than a Poisson distribution $P_{\text{Pois}}^C(s) = \frac{\pi s}{2} e^{-\frac{\pi}{4} s^2}$ on the complex plane. For $h = 20$, the level-spacing distribution becomes a Poisson distribution on the real axis $P_{\text{Pois}}^R(s) = e^{-s}$ rather than the Wigner-Dyson distribution $P_{\text{WD}}^R(s) = \frac{\pi s}{2} e^{-\frac{\pi}{4} s^2}$. Statistics are taken for eigenstates in the middle of the spectrum (we take eigenstates whose energies lie within $\pm 10\%$ from the middle of the real and imaginary parts of the eigenspectrum). (c) Half-chain entanglement entropy $S/L$ obtained by averaging $S_\alpha/L$ over disorder and eigenstates in the middle of the spectrum (we take eigenstates whose energies lie within $\pm 2\%$ from the middle of the real part of the eigenspectrum). For weak $h$, $S/L$ is almost constant, indicating the volume law of the entanglement entropy in the delocalized state. For stronger $h$, the eigenstates are localized and $S/L$ decreases with increasing $L$. We use $N_s = 100$ for $L = 6, 8, 10, 12, 14$ and $N_s = 10$ for $L = 16$. (d) Stability $\mathcal{G}$ of eigenstates with real eigenenergies for varying disorder strength $h$ against the perturbation $\hat{V}_{\text{NH}} = \hat{b}_i^\dagger \hat{b}_{i+1}$ (see the main text for the definition). With increasing $h$, $\mathcal{G}$ changes from $\sim \alpha L$ to $\sim -\beta L$ ($\alpha, \beta > 0$), where the transition occurs at $h_{cMBL}^R \simeq 14 \pm 1$. We use $N_s = 1000$ for $L = 6, 8, 10, 12$ and $N_s = 100$ for $L = 14$. 

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transition point to be the one at which $G(L)$ is independent of $L$, giving $h_{c}^{MBL} \simeq 14 \pm 1$, which is close to $h_{c}^{R}$ for $g = 1$ (the small deviation is attributed to finite-size effects).

IV. REAL-COMPLEX TRANSITION WITH VARYING NON-HERMITICITY $g$

Here by fixing $h$ and varying $g$ for the model in Eq. (1) in the main text, we investigate the transition point $g_{c}^{R}$ of the real-complex transition. We first show the $g$-dependences of $f_{im}$ and $\Delta_{im}$ for $h = 2 < h_{0c}^{MBL}$ in Fig. S-4(a). Both $f_{im}$ and $\Delta_{im}$ increase for all $g$ with increasing the system size. The figure also implies that this behavior holds for infinitesimal $g$, which means $g_{c}^{R} = 0$ for $h = 2$.

Next, we show the $g$-dependences of $f_{im}$ and $\Delta_{im}$ for $h = 18 > h_{0c}^{MBL}$ in Fig. S-4(b). Both $f_{im}$ and $\Delta_{im}$ decrease for $g \lesssim 2$ and increase for $g \gtrsim 2$ with increasing the size of the system. This means that the real-complex transition occurs at $g_{c}^{R} \simeq 2$ for $h = 18$ in the thermodynamic limit. In general, we have $g_{c}^{R} > 0$ only for $h > h_{0c}^{MBL}$ in this model.

V. DYNAMICS IN THE NON-HERMITIAN SYSTEM WITH ASYMMETRIC HOPPING

We address the nonequilibrium dynamics of the non-Hermitian model described in Eq. (1) in the main text. We assume that the dynamics of the non-Hermitian Hamiltonian is described by

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi_{0}\rangle / \| e^{-i\hat{H}t} |\psi_{0}\rangle \|, \quad (S-3)$$

where the initial state $|\psi_{0}\rangle$ is taken as a charge-density-wave state

$$|\psi_{0}\rangle = |1010 \cdots 10\rangle. \quad (S-4)$$

Figure S-5(a) shows the real part of the energy discussed in Fig. 2 in the main text:

$$E^{R}(t) = \Re \langle \psi(t) | \hat{H} | \psi(t) \rangle, \quad (S-5)$$

where $\hat{H}$ is the non-Hermitian Hamiltonian in Eq. (1) in the main text and $\cdots$ denotes the disorder average. We first note that $E^{R}(t)$ is constant due to the energy conservation for the Hermitian case ($g = 0$). On the other hand, $E^{R}(t)$ is no longer constant for weak
FIG. S-4. Dependences of \( f_{\text{Im}} \) and \( \Delta_{\text{Im}} \) on the non-Hermiticity \( g \) in the model described by Eq. (1) in the main text. (a) With weak disorder \( h = 2 < h_{\text{MBL}}^{\text{c}} \), both \( f_{\text{Im}} \) and \( \Delta_{\text{Im}} \) increase for all \( g \) with increasing the system size. (b) With strong disorder \( h = 18 > h_{\text{MBL}}^{\text{c}} \), both \( f_{\text{Im}} \) and \( \Delta_{\text{Im}} \) decrease for \( g \lesssim 2 \) and increase for \( g \gtrsim 2 \) with increasing the system size.

Disorder with \( h = 2 \) in the presence of the non-Hermitian perturbation, however small it may be (\( g = 0.1 \)). This means that the system is unstable in the delocalized phase in that the macroscopic energy is absorbed and emitted significantly with time, which is linked to the delocalized eigenstates with nonzero imaginary parts. However, for strong disorder with \( h = 14 \), the energy is kept constant except for negligibly small oscillations. This means that the system is stable in the localized phase in that there is no absorption and emission of the macroscopic energy, which is linked to the reality of almost all eigenenergies.

Next, Fig. S-5(b) shows the time evolution of the half-chain entanglement entropy

\[
S(t) := \text{Tr}_{L/2} |\langle \psi(t) | \psi(t) \rangle|.
\]

(S-6)

In the delocalized regime with weak disorder \( h = 2 \), \( S(t) \) grows similarly for both the Hermitian (\( g = 0 \)) and non-Hermitian (\( g = 0.1 \)) models for \( t \lesssim 4 \). On the other hand, in the
longer time \((t \gtrsim 10)\), while \(S(t)\) saturates in the Hermitian case, it gradually decreases in the non-Hermitian case. The decay again shows that this non-Hermitian model is unstable due to the nonzero imaginary parts of eigenenergies. In the localized regime with strong disorder \(h = 14\), \(S(t)\) exhibits a logarithmic growth similar to the Hermitian model \([3]\) for a long time. This again indicates that the non-Hermitian MBL phase is dynamically stable despite its non-Hermiticity, in contrast to the delocalized phase.

Finally, we show the dynamics of the local particle density in Fig. S-5(c):

\[
m(t) = \langle \psi(t)|n_1|\psi(t)\rangle.
\]  

(S-7)

In the delocalized regime with weak disorder \(h = 2\), \(m(t)\) saturates for \(g = 0\) and decays for a long time for \(g = 0.1\). On the other hand, in the localized regime with strong disorder \(h = 14\), \(m(t)\) behaves similarly for \(g = 0\) and \(g = 0.1\). Note that the stationary value of \(m(t)\) depends sensitively on the initial states in this case due to the presence of local conserved quantities \([4, 5]\).

VI. PERTURBATION THEORY FOR NON-HERMITIAN HAMILTONIANS

Here we justify the measure of eigenstate stability introduced in the main text,

\[
G(L) = \ln \frac{|\langle E_{L+1}|\hat{V}_{\text{NH}}|E_{L}^{R}\rangle|}{|E_{L+1}^{R} - E_{L}^{R}|},
\]  

(S-8)

where \(|E_{L}^{R}\rangle\) (\(|E_{L}^{L}\rangle\)) is a right (left) eigenstate of \(\hat{H}_0\) and \(E_{a}' = E_{a} + \langle E_{L}^{L}|\hat{V}_{\text{NH}}|E_{L}^{R}\rangle\) is a shifted eigenenergy of \(\hat{H}_0\). This measure is justified from the perturbation theory as follows. We consider the original Hamiltonian as

\[
\hat{H} = \hat{H}_0 + \hat{V}_{\text{NH}} = \sum_a (\mathcal{E}_a + \Delta E^{(0)}_a) |\mathcal{E}_a^{R}\rangle \langle \mathcal{E}_a^{L}| + \hat{V}_{\text{NH}} - \sum_a \Delta E^{(0)}_a |\mathcal{E}_a^{R}\rangle \langle \mathcal{E}_a^{L}|,
\]  

(S-9)

where \(\Delta E^{(0)}_a = \langle \mathcal{E}_a^{L}|\hat{V}_{\text{NH}}|\mathcal{E}_a^{R}\rangle\). We treat the second and third terms as a perturbation. Then, the first-order perturbation about the eigenstate leads to

\[
|\mathcal{E}_{a,\text{perturbed}}^{R}\rangle = |\mathcal{E}_a^{R}\rangle + \sum_{b(\neq a)} \frac{\langle \mathcal{E}_b^{L}|\hat{V}_{\text{NH}}|\mathcal{E}_a^{R}\rangle}{\mathcal{E}_b' - \mathcal{E}_a'} |\mathcal{E}_b^{R}\rangle.
\]  

(S-10)

The absolute value of the coefficient in the second term should be sufficiently small for the perturbation series to converge. We take \(b = a + 1\) to discuss the localization transition \([6]\) and define it as a measure of the stability of eigenstates as in the main text.
FIG. S-5. (a) Time evolution of the real part of the energy $E^R(t)$ in Eq. (S-5) for different $g$. For the Hermitian case ($g = 0$), $E^R(t)$ is constant due to the energy conservation. For weak disorder with $h = 2$ and $g = 0.1$, $E^R(t)$ changes significantly with time. This means that the system is unstable in the delocalized phase. For strong disorder with $h = 14$ and $g = 0.1$, the energy is kept constant except for negligibly small oscillations. This means that the system is stable in the localized phase. (b) Time evolution of the half-chain entanglement entropy $S(t)$ defined in Eq. (S-6). In the delocalized regime with weak disorder $h = 2$, $S(t)$ exhibits similar growth in both the Hermitian ($g = 0$) and non-Hermitian ($g = 0.1$) models for $t \lesssim 4$. On the other hand, in the longer times ($t \gtrsim 10$), while $S(t)$ saturates in the Hermitian case, it gradually decreases in the non-Hermitian case. In the localized regime with strong disorder $h = 14$, $S(t)$ in non-Hermitian model exhibits a logarithmic growth similar to the Hermitian model in a long-time regime. This indicates that the non-Hermitian MBL phase is dynamically stable despite its non-Hermiticity, in contrast to the delocalized phase. (c) Time evolution of the local particle density $m(t)$ in Eq. (S-7). In the delocalized regime with weak disorder $h = 2$, $m(t)$ saturates for $g = 0$ and keeps decaying for $g = 0.1$ in a long-time regime. In the localized regime with strong disorder $h = 14$, $m(t)$ behaves similarly for $g = 0$ and $g = 0.1$ with a smaller decay. All the data show the averages over $N_s = 100$ samples for the system with $L = 12$.

Note that $\mathcal{G}(L)$ is a generalization of the stability measure for the Hermitian setting introduced in Ref. [6]. They consider the stability of eigenstates of a Hermitian $\hat{H}_0$ against a Hermitian perturbation $\hat{V}$ with its measure $\mathcal{G}_H(L) = \ln \frac{||\mathcal{E}_a+1||_{\mathcal{E}_0}}{||\mathcal{E}^0_{a+1}||_{\mathcal{E}_0}}$. Here $\mathcal{E}'_a = \mathcal{E}_a + \langle \mathcal{E}_a | \hat{V} | \mathcal{E}_0 \rangle$ and the labels of eigenstates (eigenenergies) $\mathcal{E}_a$’s are taken such that $\mathcal{E}'_1 \leq \mathcal{E}'_2 \leq \cdots$. When $\hat{H}_0$ belongs to the delocalized phase, the Srednicki ansatz on the eigenstate thermalization hypothesis [7–11] leads to $\mathcal{G}(L) \sim \frac{s}{2}L$, where $s$ is the entropy density. This means that two
adjacent eigenstates are mixed by the perturbation. On the other hand, when $\hat{H}_0$ belongs to the MBL phase, $|E_a\rangle$ and $|E_{a+1}\rangle$ are characterized by quasi-local conserved quantities [4, 5, 12, 13] and local perturbations cannot mix these two states, resulting in $G(L) \sim -\kappa L (\kappa > 0)$ [6]. As we have seen in the main text, similar behavior can be seen for the stability measure of the non-Hermitian setting $G(L)$, which is relevant for the real-complex transition.

VII. SIMILARITY TRANSFORMATION OF THE NON-HERMITIAN MANY-BODY HAMILTONIAN WITH ASYMMETRIC HOPPING

As we have noted in the main text, the absence of coalescence between adjacent eigenstates due to the non-Hermitian MBL does not necessarily lead to complete suppression of complex eigenenergies. In fact, due to statistical fluctuations of disorder, some (but rare) spatial regions are susceptible (resonant) to local perturbations [12–15], leading to the mixing of non-adjacent eigenstates and the formation of complex conjugate pairs. In such resonant regions, non-Hermitian effects should be treated non-perturbatively.

On the other hand, for our model in Eq. (1) in the main text, we can perform a similarity transformation [16–20] such that the non-Hermitian perturbation only acts on nonresonant regions. Through investigation of this transformed Hamiltonian, the mixing of eigenstates is found to be suppressed even for non-adjacent eigenstates and hence the entirely real spectrum emerges. Below we explain these points in detail.

For simplicity, we focus on the Hamiltonian $\hat{H}$ in Eq. (1) in the main text for sufficiently small $g$. We consider $\hat{H} = \hat{H}_0 + \hat{V}_{NH}$, where

$$\hat{H}_0 = \sum_{i=1}^{L} \left[ -J(\hat{b}_i^\dagger \hat{b}_{i+1} + h.c.) + U\hat{n}_i \hat{n}_{i+1} + h_i \hat{n}_i \right]$$

(S-11)
is Hermitian and

$$\hat{V}_{NH} = -J \sum_{i=1}^{L} \left[ (e^{-g} - 1)\hat{b}_i^\dagger \hat{b}_{i+1} + (e^g - 1)\hat{b}_i \hat{b}_{i+1} \right] = \sum_{i=1}^{L} \hat{v}_{i,i+1}.$$  

(S-12)

In the localized phase of $\hat{H}_0$ with large $h$, the local tunneling amplitude $|\langle \mathcal{E}_b | \hat{v}_{i,i+1} | \mathcal{E}_a \rangle|$ by the perturbation $\hat{v}_{i,i+1}$ is smaller than $|\mathcal{E}_a - \mathcal{E}_b|$ for most $i$’s, where we assume that the two eigenstates $|\mathcal{E}_a\rangle$ and $|\mathcal{E}_b\rangle$ are approximately product states and connected by the hopping terms $\hat{b}_{i+1}^\dagger \hat{b}_i$ ($\hat{b}_i^\dagger \hat{b}_{i+1}$). This is because moving particles in the localized regions is
energetically costly (i.e., $|E_a - E_b| \sim |h_{i+1} - h_i|$ is sufficiently large). Thus, the effect of $\hat{v}_{i,i+1}$ can be treated perturbatively. On the other hand, for some (but rare) $i$'s, localization becomes very weak due to the statistical fluctuation of $h_i$ and particles are relatively mobile, leading to $|\langle E_b|\hat{v}_{i,i+1}|E_a\rangle| > |E_a - E_b|$. Thus, for the original model, we cannot control the perturbation due to such resonant $i$'s.

Fortunately, we can show that the Hamiltonian in Eq. (1) in the main text can be transformed into a matrix whose non-Hermitian perturbation only acts on the non-resonant regions. To see this, we consider $\hat{V}_i = e^{g\theta_i}\hat{n}_i = 1 + (e^{g\theta_i} - 1)\hat{n}_i$ (similar transformations called the imaginary gauge transformation are used in Refs. [16–18]). Then

$$\hat{V}_i\hat{b}_i\hat{V}_i^{-1} = \hat{b}_i[1 + (e^{g\theta_i} - 1)\hat{n}_i] = \hat{b}_i + (e^{g\theta_i} - 1)(1 - 2\hat{n}_i)\hat{b}_i = e^{g\theta_i}\hat{b}_i,$$

where we have used $\{\hat{b}_i, \hat{b}_i^\dagger\} = 1$ and $\hat{b}_i^2 = 0$. Similarly,

$$\hat{V}_i\hat{b}_i^\dagger\hat{V}_i^{-1} = [1 + (e^{g\theta_i} - 1)\hat{n}_i]\hat{b}_i^\dagger = \hat{b}_i^\dagger + (e^{g\theta_i} - 1)\hat{b}_i^\dagger(1 - 2\hat{n}_i) = e^{g\theta_i}\hat{b}_i^\dagger.$$

Thus we obtain

$$\hat{V}_{i+1}\hat{V}_i\hat{b}_i\hat{V}_{i+1}^{-1}\hat{V}_i^{-1} = e^{g(\theta_{i+1} - \theta_i)}\hat{b}_i\hat{V}_{i+1}^{-1} = e^{g(\theta_{i+1} - \theta_i)}\hat{b}_i^\dagger\hat{b}_i^\dagger,$$

$$\hat{V}_{i+1}\hat{V}_i\hat{b}_i^\dagger\hat{V}_{i+1}^{-1}\hat{V}_i^{-1} = e^{-g(\theta_{i+1} - \theta_i)}\hat{b}_i^\dagger\hat{b}_i^\dagger,$$

$$\hat{V}_i\hat{n}_i\hat{V}_i^{-1} = \hat{n}_i,$$

$$\hat{V}_{i+1}\hat{V}_i\hat{n}_i\hat{V}_{i+1}^{-1}\hat{V}_i^{-1} = \hat{n}_{i+1}\hat{n}_i.$$

Now, we consider a similarity transformation $\hat{H}' = \hat{\nu}\hat{H}\hat{\nu}^{-1}$, where $\hat{\nu} = \bigotimes_{i=1}^L \hat{V}_i$. Note that $\hat{H}'$ and $\hat{H}$ have the same eigenenergies. In fact, the eigenstate $|E^{R\prime}_\alpha\rangle$ of $\hat{H}'$ with the eigenenergy $E^{R\prime}_\alpha$ is obtained from $|E^{R\prime}_\alpha\rangle$ of $\hat{H}'$ with the same eigenenergies as $|E^{R}_\alpha\rangle = \hat{\nu}^{-1}|E^{R\prime}_\alpha\rangle$, since,

$$\hat{H}|E^{R}_\alpha\rangle = \hat{\nu}\hat{H}\hat{\nu}^{-1}|E^{R\prime}_\alpha\rangle = \hat{\nu}^{-1}\hat{H}'|E^{R\prime}_\alpha\rangle = E\hat{\nu}^{-1}|E^{R\prime}_\alpha\rangle = E\alpha|E^{R\prime}_\alpha\rangle.$$

(S-19)

Then, we can investigate the energy spectrum of $\hat{H}'$ instead of $\hat{H}$.
By choosing $\theta_i$’s appropriately, we can obtain $\hat{H}'$ for which the non-Hermitian perturbations only act on non-resonant regions. To see this, we consider a simplified situation, where sites from 1 to $x$ may have resonant regions and other sites are non-resonant. We can assume that $x$ is much smaller than $L$ because resonant regions are rare in the localized phase [21]. If we choose

$$
\theta_i = i \quad (1 \leq i \leq x + 1),
\theta_i = -i + 2x + 2 \quad (x + 2 \leq i \leq 2x + 2),
\theta_i = 0 \quad (2x + 3 \leq i \leq L),
$$

then

$$
\hat{H}' = \hat{V} \hat{H} \hat{V}^{-1}
= - J \sum_{i=1}^{L} (e^{-g_{z_i} \hat{b}_{i+1}^\dagger \hat{b}_i} + e^{g_{z_i} \hat{b}_i^\dagger \hat{b}_{i+1}}) + \sum_{i=1}^{L} U \hat{n}_i \hat{n}_{i+1} + \sum_{i=1}^{L} h_i \hat{n}_i
= \hat{H}_0 + \hat{V}'_{\text{NH}},
$$

where

$$
\begin{align*}
    z_i &= 0 \quad (1 \leq i \leq x, i = L), \\
    z_i &= 2 \quad (x + 2 \leq i \leq 2x + 1), \\
    z_i &= 1 \quad (2x + 3 \leq i \leq L - 1).
\end{align*}
$$

Since $\hat{V}'_{\text{NH}}$ only acts on the non-resonant regions, it does not mix the eigenstates, leading to further suppression of complex eigenenergies that are, in general, non-adjacent.

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