MEASURES OF NONCOMPACTNESS IN $\bar{N}(p,q)$ SUMMABLE SEQUENCE SPACES

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Abstract. In this paper, we first define the $\bar{N}(p,q)$ summable sequence spaces and obtain some basic results related to these spaces. The necessary and sufficient conditions for an infinite matrix $A$ to map these spaces into the spaces $c$, $c_0$, and $\ell_\infty$ is obtained and Hausdorff measure of noncompactness is then used to obtain the necessary and sufficient conditions for the compactness of linear operators defined on these spaces.

1. Introduction

We write $\omega$ for the set of all complex sequences $x = (x_k)_{k=0}^\infty$ and $\phi$, $c$, $c_0$ and $\ell_\infty$ for the sets of all finite sequences, convergent sequences, sequences convergent to zero, and bounded sequences respectively. By $e$ we denote the sequence of 1's, $e = (1, 1, 1, \ldots)$ and by $e^{(n)}$ the sequence with 1 as only nonzero term at the $n$th place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Further by $cs$ and $\ell_1$ we denote the convergent and absolutely convergent series respectively. If $x = (x_k)_{k=0}^\infty \in \omega$ then $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$ denotes the $m$–th section of $x$.

A sequence space $X$ is a linear subspace of $\omega$, such a space is called a BK space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ ($n = 0, 1, 2, \ldots$) where

$$P_n(x) = x_n, \quad x = (x_k)_{k=0}^\infty \in X.$$ 

The BK space $X$ is said to have AK if every $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$ [15, Definition 1.18]. The spaces $c_0$, $c$ and $\ell_\infty$ are BK spaces with respect to the norm

$$\|x\|_\infty = \sup_k \{|x_k| : k \in \mathbb{N}\}.$$ 

The $\beta$–dual of a subset $X$ of $\omega$ is defined by

$$X^\beta = \{a \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$ 

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If $X$ and $Y$ are Banach Spaces, then by $\mathcal{B}(X,Y)$ we denote the set of all bounded (continuous) linear operators $L : X \rightarrow Y$, which is itself a Banach space with the operator norm $\|L\| = \sup_{x} \{\|L(x)\| : \|x\| = 1\}$ for all $L \in \mathcal{B}(X,Y)$. The linear operator $L : X \rightarrow Y$ is said to be compact if its domain is all of $X$ and for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a subsequence which converges in $Y$. The operator $L \in \mathcal{B}(X,Y)$ is said to be of finite rank if $\dim R(L) < \infty$, where $R(L)$ denotes the range space of $L$. A finite rank operator is clearly compact [6, Chapter 2].

In this paper, we first define $\tilde{N}(p,q)$ summable sequence spaces as the matrix domains $X_T$ of arbitrary triangle $\tilde{N}_p^q$ and obtain some basic results related to these spaces. We then find out the necessary and sufficient condition for matrix transformations to map these spaces into $c_0$, $c$ and $\ell_\infty$. Finally we characterize the classes of compact matrix operators from these spaces into $c_0$, $c$ and $\ell_\infty$.

2. Matrix Domains

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers, we write $A_n$ for the sequence in the $n$th row of $A$, $A_n = (a_{nk})_{k=0}^\infty$. The $A$-transform of any $x = (x_k) \in \omega$ is given by $Ax = (A_n(x))_{n=0}^\infty$, where

$$A_n(x) = \sum_{k=0}^\infty a_{nk} x_k \quad n \in \mathbb{N}$$

the series on right must converge for each $n \in \mathbb{N}$.

If $X$ and $Y$ are subsets of $\omega$, we denote by $(X,Y)$, the class of all infinite matrices that map $X$ into $Y$. So $A \in (X,Y)$ if and only if $A_n \in X^X$, $n = 0, 1, 2, \ldots$ and $Ax \in Y$ for all $x \in X$. The matrix domain of an infinite matrix $A$ in $X$ is defined by

$$X_A = \{x \in \omega : Ax \in X\}$$

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has been studied by several authors see [4, 7–11]

For any two sequences $x$ and $y$ in $\omega$ the product $xy$ is given by $xy = (x_ky_k)_{k=0}^\infty$ and for any subset $X$ of $\omega$

$$y^{-1} \ast X = \{a \in \omega : ay \in X\}$$

We denote by $\mathcal{U}$ the set of all sequences $u = (u_k)_{k=0}^\infty$ such that $u_k \neq 0 \forall k = 0, 1, 2, \ldots$ and for any $u \in \mathcal{U}$, $\frac{1}{u} = \left(\frac{1}{u_k}\right)_{k=0}^\infty$.

**Theorem 1.** a) Let $X$ be a BK space with basis $(\alpha^{(k)})_{k=0}^\infty$, $u \in \mathcal{U}$ and $\beta^{(k)} = (1/u)\alpha^{(k)}$, $k = 0, 1, \ldots$. Then $(\beta^{(k)})_{k=0}^\infty$ is a basis of $Y = u^{-1} \ast X$. 

b) Let \((p_k)_{k=0}^\infty\) be a positive sequence, \(u \in \mathfrak{U}\) a sequence such that
\[|u_0| \leq |u_1| \leq \cdots \quad \text{and} \quad |u_n| \to \infty \quad (n \to \infty)\]
and \(T\) a triangle with
\[t_{nk} = \begin{cases} \frac{p_{n-k}}{u_n} & 0 \leq k \leq n \\ 0 & k > n \end{cases} \quad n = 0, 1, 2, \ldots\]
Then \((c_0)_T\) has AK.

c) Let \(T\) be an arbitrary triangle and \(B = |T|\). Then \((c_0)_B\) has AK if and only if \(\lim_{n \to \infty} t_{nk} = 0\) for all \(k = 0, 1, 2, \ldots\).

Proof. a) (cf. [1, Theorem 2])
b) \((c_0)_T\) is a BK space by Theorem 4.3.12 in [19], the norm \(\|x\|_{(c_0)_T}\) on it is defined as
\[\|x\|_{(c_0)_T} = \sup_n \left| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k}x_k \right|\]
Since \(|u_n| \to \infty \quad (n \to \infty)\) gives \(\phi \subset (c_0)_T\).
Let \(\epsilon > 0\) and \(x \in (c_0)_T\) then there exists integer \(N > 0\), such that \(|T_{n}(x)| < \frac{\epsilon}{2}\)
for all \(n \geq N\).
Let \(m > N\) then
\[\|x - x^{[m]}\|_{(c_0)_T} = \sup_{n \geq m+1} \left| \frac{1}{u_n} \sum_{k=m+1}^{n} p_{n-k}x_k \right| \quad (2.1)\]
Now
\[T_n(x) = \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k}x_k\]
\[T_m(x) = \frac{1}{u_n} \sum_{k=0}^{m} p_{n-k}x_k\]
\[T_n(x) + T_m(x) = \frac{1}{u_n} \left[ 2(p_n x_0 + \cdots + p_{n-m} x_m) + \sum_{k=m+1}^{n} p_{n-k}x_k \right]\]
Then by (2.1) we have
\[\|x - x^{[m]}\|_{(c_0)_T} \leq \sup_{n \geq m+1} \left( |T_n(x)| + |T_m(x)| \right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}\]
\[\leq \epsilon\]
Hence \( x = \sum_{k=0}^{\infty} x_k \beta(k) \).
This representation is obviously unique.
c) Same as done in (cf. [1, Theorem 2]) \( \square \)

3. \( \tilde{N}(p, q) \) Summable Sequence Spaces

Let \((p_k)_{k=0}^{\infty}, (q_k)_{k=0}^{\infty}\) be positive sequences in \( \mathbb{U} \) and \((R_n)_{n=0}^{\infty}\) the sequence with \( R_n = \sum_{j=0}^{n} p_n - j q_j \). The \( \tilde{N}(p, q) \) transform of the sequence \( (x_k)_{k=0}^{\infty} \) is the sequence \( (t_n)_{n=0}^{\infty} \) defined as

\[
t_n = \frac{1}{R_n} \sum_{j=0}^{n} p_{n-j} q_j x_j
\]

The matrix \( \tilde{N}_p^q \) for this transformation is

\[
(\tilde{N}_p^q)_{nk} = \begin{cases} 
\frac{p_{n-k} q_k}{R_n} & 0 \leq k \leq n \\
0 & k > n
\end{cases}
\] (3.1)

We define the spaces \( (\tilde{N}_p^q)_0, (\tilde{N}_p^q) \) and \( (\tilde{N}_p^q)_{\infty} \) that are \( \tilde{N}(p, q) \) summable to zero, summable and bounded respectively as

\[
(\tilde{N}_p^q)_0 = (c_0)_{\tilde{N}_p^q} = \left\{ x \in \omega : \tilde{N}_p^q x = \left( \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in c_0 \right\}
\]

\[
(\tilde{N}_p^q) = (c)_{\tilde{N}_p^q} = \left\{ x \in \omega : \tilde{N}_p^q x = \left( \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in c \right\}
\]

\[
(\tilde{N}_p^q)_{\infty} = (\ell_{\infty})_{\tilde{N}_p^q} = \left\{ x \in \omega : \tilde{N}_p^q x = \left( \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in \ell_{\infty} \right\}
\]

For any sequence \( x = (x_k)_{k=0}^{\infty} \), define \( \tau = \tau(x) \) as the sequence with \( n \)th term given by

\[
\tau_n = (\tilde{N}_p^q)_n(x) = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \quad (n = 0, 1, 2, \ldots)
\] (3.2)

This sequence \( \tau \) is called as weighted means of \( x \).

Theorem 2. The spaces \( (\tilde{N}_p^q)_0, (\tilde{N}_p^q) \) and \( (\tilde{N}_p^q)_{\infty} \) are BK spaces with respect to the norm \( \| \cdot \|_{\tilde{N}_p^q} \) given by

\[
\| x \|_{\tilde{N}_p^q} = \sup_n \left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right|
\]
If \( R_n \to \infty \ (n \to \infty) \), then \( (\tilde{N}_p^q)_0 \) has AK, and every sequence \( x = (x_k)_{k=0}^{\infty} \in (\tilde{N}_p^q) \) has unique representation

\[
x = le + \sum_{k=0}^{\infty} (x_k - l) e^{(k)}
\]  

(3.3)

where \( l \in \mathbb{C} \) is such that \( x - le \in (\tilde{N}_p^q)_0 \)

**Proof.** The sets \( (\tilde{N}_p^q)_0 \), \( (\tilde{N}_p^q) \) and \( (\tilde{N}_p^q)_{\ell_\infty} \) are BK spaces ([19] Theorem 4.3.12), Let us consider the matrix \( T = (t_{nk}) \) defined by

\[
t_{nk} = \begin{cases} \frac{p_{n-k}}{R_n} & 0 \leq k \leq n \\ 0 & k > n \end{cases} \quad n = 0, 1, 2, \ldots
\]

Then \( (\tilde{N}_p^q)_0 \) has AK by Theorem 1.

Now if \( x \in (\tilde{N}_p^q) \), then there exists a \( l \in \mathbb{C} \) such that \( x - le \in (\tilde{N}_p^q)_0 \)

Now \( \tau(e) = (\tau_n)_{n=0}^{\infty} \) where

\[
\tau_n = (\tilde{N}_p^q)_n(e) = \frac{1}{R_n} \sum_{k=0}^{\infty} p_{n-k} q_k e_k \quad (n = 0, 1, 2, \ldots)
\]

\[
= \frac{1}{R_n} \sum_{k=0}^{\infty} p_{n-k} q_k \quad \text{As } e_k = 1 \forall (k = 0, 1, 2, \ldots)
\]

\[
= 1
\]

Therefore \( \tau(e) = e \) which implies the uniqueness of \( l \).

Therefore (3.3) follows from the fact that \( (\tilde{N}_p^q)_{\ell_\infty} \) has AK.

Now \( \tilde{N}_p^q \) is a triangle, it has a unique inverse and the inverse is also a triangle [12]. Take \( H_0^{(p)} = \frac{1}{p_0} \) and

\[
H_n^{(p)} = \frac{1}{p_{n+1}} \begin{bmatrix}
p_1 & p_0 & 0 & 0 & \cdots & 0 \\
p_2 & p_1 & p_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_0 \\
p_n & p_{n-1} & p_{n-2} & \cdots & p_1
\end{bmatrix}
\]  

(3.4)

Then the inverse of matrix defined in (3.1) is the matrix \( S = (s_{nk})_{n,k=0}^{\infty} \) which is defined as see [16] in

\[
s_{nk} = \begin{cases} (-1)^{n-k} \frac{H_{n-k}^{(p)}}{q_n} R_k & 0 \leq k \leq n \\ 0 & k > n \end{cases}
\]  

(3.5)
3.1. \( \beta \) dual of \( N(p,q) \) Sequence Spaces. In order to find the \( \beta \) dual we need the following results

**Lemma 1.** [18] If \( A = (a_{nk})_{n,k=0}^{\infty} \), then \( A \in (c_0,c) \) if and only if

\[
\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \tag{3.6}
\]

\[
\lim_{n \to \infty} a_{nk} - \alpha_k = 0 \quad \text{for every } k. \tag{3.7}
\]

**Lemma 2.** [5] If \( A = (a_{nk})_{n,k=0}^{\infty} \), then \( A \in (c,c) \) if and only if conditions (3.6), (3.7) holds and

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} a_{nk} \quad \text{exists for all } k. \tag{3.8}
\]

**Lemma 3.** [5] If \( A = (a_{nk})_{n,k=0}^{\infty} \), then \( A \in (\ell_\infty,c) \) if and only if condition (3.7) holds and

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n \to \infty} a_{nk} \right| \tag{3.9}
\]

**Theorem 3.** Let \((p_k)_{k=0}^{\infty}, (q_k)_{k=0}^{\infty}\) be positive sequences, \(R_n = \sum_{j=0}^{n} p_{n-j}q_j\) and \(a = (a_k) \in \omega \), we define a matrix \(C = (c_{nk})_{n,k=0}^{\infty}\) as,

\[
c_{nk} = \begin{cases} 
R_k \left[ \sum_{j=k}^{n} (-1)^{j-k} \left( \frac{p_{j-k}}{q_j} \right) a_j \right] & 0 \leq k \leq n \\
0 & k > n \end{cases} \tag{3.10}
\]

and consider the sets

\[
c_1 = \left\{ a \in \omega : \sup_n \sum_k |c_{nk}| < \infty \right\} \quad ; \quad c_2 = \left\{ a \in \omega : \lim_{n \to \infty} c_{nk} \text{ exists for each } k \in \mathbb{N} \right\}
\]

\[
c_3 = \left\{ a \in \omega : \lim_{n \to \infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n \to \infty} c_{nk} \right| \right\} \quad ; \quad c_4 = \left\{ a \in \omega : \lim_{n \to \infty} \sum_k c_{nk} \text{ exists } \right\}
\]

Then \( [(\mathring{N}_p^q)_0]^\beta = c_1 \cap c_2, \quad [(\mathring{N}_p^q)_\infty]^\beta = c_1 \cap c_2 \cap c_4 \) and \( [(\mathring{N}_p^q)_\infty]^\beta = c_2 \cap c_3 \).

**Proof.** We prove the result for \( [(\mathring{N}_p^q)_0]^\beta \).

Let \( x \in (\mathring{N}_p^q)_0 \) then there exists a \( y \) such that \( y = \mathring{N}_p^q x \).
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Hence

\[
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left( \bar{N}_p^q \right)^{-1} y_k \\
= \sum_{k=0}^{n} a_k \left[ \sum_{j=0}^{k} (-1)^{k-j} R_j \left( \frac{H_{j-k}^{(p)}}{q_j} \right) y_j \right] \\
= \sum_{k=0}^{n} R_k \left[ \sum_{j=k}^{n} (-1)^{j-k} \left( \frac{H_{j-k}^{(p)}}{q_j} a_j \right) \right] y_k \\
= (Cy)_n
\]

So \( ax = (a_n x_n) \in cs \) whenever \( x \in \left( \bar{N}_p^q \right)_0 \) if and only if \( Cy \in cs \) whenever \( y \in c_0 \).

Using Lemma 1 we get \( \left( \left( \bar{N}_p^q \right)_0 \right)^\beta = c_1 \cap c_2 \).

Similarly using Lemma 2 and Lemma 3 the \( \beta \) dual of \( \left( \bar{N}_p^q \right) \) and \( \left( \bar{N}_p^q \right)_\infty \) can be found same way we can show the other two results as well. \( \square \)

Let \( X \subset \omega \) be a normed space and \( a \in \omega \). Then we write

\[
\|a\|^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : \|x\| = 1 \right\}
\]

provided the term on the right side exists and is finite, which is the case whenever \( X \) is a BK space and \( a \in X^\beta \) [19, Theorem 7.2.9].

**Theorem 4.** For \( \left( \left( \bar{N}_p^q \right)_0 \right)^\beta \), \( \left( \left( \bar{N}_p^q \right) \right)^\beta \) and \( \left( \left( \bar{N}_p^q \right)_\infty \right)^\beta \) the norm \( \| \cdot \|^* \) is defined as

\[
\|a\|^* = \sup_n \left\{ \sum_{k=0}^{n} R_k \left| \sum_{j=k}^{n} (-1)^{j-k} \left( \frac{H_{j-k}^{(p)}}{q_j} a_j \right) \right| \right\}
\]

**Proof.** If \( x^{[n]} \) denotes the \( n \)th section of the sequence \( x \in \left( \bar{N}_p^q \right)_0 \) then using (3.2) we have

\[
\tau_k^{[n]} = \tau_k(x^{[n]}) = \frac{1}{R_k} \sum_{j=0}^{k} p_{n-j} q_j x_j^{[n]}
\]

Let \( a \in \left( \left( \bar{N}_p^q \right)_0 \right)^\beta \), then for any non-negative integer \( n \) define the sequence \( d^{[n]} \) as

\[
d_k^{[n]} = \left\{ \begin{array}{ll}
R_k \left[ \sum_{j=k}^{n} (-1)^{j-k} \left( \frac{H_{j-k}^{(p)}}{q_j} a_j \right) \right] & \text{if } 0 \leq k \leq n \\
0 & \text{if } k > n
\end{array} \right.
\]
Let \( \|a\|_{\Pi} = \sup_{n} \|d[n]\|_1 = \sup_{n} \left( \sum_{k=0}^{\infty} |d_k^{[n]}| \right) \) where \( \Pi = \left[ \left( \tilde{N}_{p}^{q} \right)^{\beta} \right] \). Then

\[
\sum_{k=0}^{\infty} a_k x_k^{[n]} = \left| \sum_{k=0}^{n} a_k \left( \sum_{j=0}^{k} (-1)^{k-j} H_{k-j}^{(p)} \frac{R_j \tau_j^{[n]}}{q_k} \right) \right| \leq \sup_{k} \left| \tau_k^{[n]} \right| \cdot \left( \sum_{k=0}^{n} R_k \left| \sum_{j=k}^{n} (-1)^{j-k} H_{j-k}^{(p)} \frac{a_j}{q_j} \right| \right)
\]

\[
\leq \sup_{k} \left| \tau_k^{[n]} \right| \cdot \left( \sum_{k=0}^{n} R_k \left| \sum_{j=k}^{n} (-1)^{j-k} H_{j-k}^{(p)} \frac{a_j}{q_j} \right| \right)
\]

\[
= \|x[n]\|_{\tilde{N}_{p}^{q}} \|d[n]\|_1
\]

\[
= \|a\|_{\Pi} \|x[n]\|_{\tilde{N}_{p}^{q}}.
\]

Hence

\[
\|a\|^{\ast} \leq \|a\|_{\Pi}.
\]  (3.11)

To prove the converse define the sequence \( x^{(n)} \) for any arbitrary \( n \) by

\[
\tau_k \left( x^{(n)} \right) = \text{sign} \left( d_k^{[n]} \right) \quad \text{for} \quad k = 0, 1, 2, \ldots.
\]

Then

\[
\tau_k \left( x^{(n)} \right) = 0 \quad \text{for} \quad k > n \quad \text{i.e.} \quad x^{(n)} \in \left( \tilde{N}_{p}^{q} \right)_0, \quad \|x^{(n)}\|_{\tilde{N}_{p}^{q}} = \|\tau_k \left( x^{(n)} \right)\|_{\infty} \leq 1.
\]

and

\[
\left| \sum_{k=0}^{\infty} a_k x_k^{(n)} \right| = \left| \sum_{k=0}^{n} d_k^{[n]} x_k^{(n)} \right| \leq \sum_{k=0}^{n} \left| d_k^{[n]} \right| \leq \|a\|^\ast.
\]

Since \( n \) is arbitrarily choosen so

\[
\|a\|_{\Pi} \leq \|a\|^\ast.
\]  (3.12)

From (3.11) and (3.12) we get the required conclusion. \( \square \)

Some well known results that are required for proving the compactness of operators are

**Proposition 1.** (cf. [13], Theorem 7) Let \( X \) and \( Y \) be BK spaces, then \( (X, Y) \subset B(X, Y) \) that is every matrix \( A \) from \( X \) into \( Y \) defines an element \( L_{A} \) of \( B(X, Y) \) where

\[
L_{A}(x) = A(x) \quad \forall \ x \in X.
\]
Also $A \in (X, \ell_\infty)$ if and only if
$$
\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty.
$$

If $(b^{(k)})_{k=0}^\infty$ is a basis of $X$, $Y$ and $Y_1$ are FK spaces with $Y_1$ a closed subspace of $Y$, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A (b^{(k)}) \in Y_1$ for all $k = 0, 1, 2, \ldots$.

**Proposition 2.** (cf. [14], Proposition 3.4) Let $T$ be a triangle

(i) If $X$ and $Y$ are subsets of $\omega$, then $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.

(ii) If $X$ and $Y$ are BK spaces and $A \in (X, Y_T)$, then
$$
\|L_A\| = \|L_B\|
$$

Using Proposition 1 and Theorem 4 we conclude the following corollary:

**Corollary 1.** Let $(p_k)_{k=0}^\infty, (q_k)_{k=0}^\infty$ be given positive sequences, and $R_n = \sum_{k=0}^n p_{n-k} q_k$ then

i) $A \in ((N^q_p)_0, \ell_\infty)$ if and only if
$$
\sup_{n,m} \left\{ \sum_{k=0}^m R_k \left| \sum_{j=k}^m (-1)^{j-k} \frac{H_j^{(p)}}{q_j} a_{nj} \right| \right\} < \infty \quad (3.13)
$$

and
$$
\frac{A_n H_n^{(p)} R}{q} \in c_0 \quad \forall \ n = 0, 1, \ldots \quad (3.14)
$$

ii) $A \in ((N_0^q), \ell_\infty)$ if and only if condition (3.13) holds and
$$
\frac{A_n H_n^{(p)} R}{q} \in c \quad \forall \ n = 0, 1, 2, \ldots \quad (3.15)
$$

iii) $A \in ((\tilde{N}_0^q)_0, \ell_\infty)$ if and only if condition (3.13) holds.

iv) $A \in ((\tilde{N}_0^q)_0, c_0)$ if and only if condition (3.13) holds and
$$
\lim_{n \to \infty} a_{nk} = 0 \quad \text{for all} \ k = 0, 1, 2, \ldots \quad (3.16)
$$

v) $A \in ((\tilde{N}_0^q)_0, c)$ if and only if condition (3.13) holds and
$$
\lim_{n \to \infty} a_{nk} = \alpha_k \quad \text{for all} \ k = 0, 1, 2, \ldots \quad (3.17)
$$

vi) $A \in ((\tilde{N}_0^q), c_0)$ if and only if conditions (3.13), (3.14) and (3.16) holds and
$$
\lim_{n \to \infty} \sum_{k=0}^\infty a_{nk} = 0 \quad \text{for all} \ k = 0, 1, 2, \ldots \quad (3.18)
$$
vii) \( A \in \left( \left( N_{p}^{q} \right)_{c} \right) \) if and only if conditions (3.13), (3.14) and (3.17) holds
and
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \quad \text{for all } k = 0, 1, 2 \ldots
\] (3.19)

From theorem 2,4 and Proposition 2 we conclude the following corollary

**Corollary 2.** Let \( X \) be a BK-space and \( (p_{k})_{k=0}^{\infty}, (q_{k})_{k=0}^{\infty} \) be positive sequences, \( R_{n} = \sum_{k=0}^{n} p_{n-k} q_{k} \) then
i) \( A \in (X, (N_{p}^{q})_{\infty}) \) if and only if
\[
\sup_{m} \left\| \frac{1}{R_{m}} \sum_{n=0}^{m} p_{m-n} q_{n} A_{n} \right\|^{*} < \infty
\] (3.20)

ii) \( A \in (X, (N_{p}^{q})_{0}) \) if and only if (3.20) holds and
\[
\lim_{m \to \infty} \left( \frac{1}{R_{m}} \sum_{n=0}^{m} p_{m-n} q_{n} A_{n} \left( c^{(k)} \right) \right) = 0 \quad \forall \, k = 0, 1, 2 \ldots
\] (3.21)

where \( (c^{(k)}) \) is a basis of \( X \).

iii) \( A \in (X, (N_{p}^{q})) \) if and only if (3.21) holds and
\[
\lim_{m \to \infty} \left( \frac{1}{R_{m}} \sum_{n=0}^{m} p_{m-n} q_{n} A_{n} \left( c^{(k)} \right) \right) = \alpha_{k} \quad \forall \, k = 0, 1, 2 \ldots
\] (3.22)

4. **HAUSDORFF MEASURE OF NONCOMPACTNESS**

Let \( S \) and \( M \) be the subsets of a metric space \( (X, d) \) and \( \epsilon > 0 \). Then \( S \) is called an \( \epsilon \)-net of \( M \) in \( X \) if for every \( x \in M \) there exists \( s \in S \) such that \( d(x, s) < \epsilon \). Further, if the set \( S \) is finite, then the \( \epsilon \)-net \( S \) of \( M \) is called **finite \( \epsilon \)-net** of \( M \). A subset of a metric space is said to be **totally bounded** if it has a finite \( \epsilon \)-net for every \( \epsilon > 0 \) [17].

If \( \mathcal{M}_{X} \) denotes the collection of all bounded subsets of metric space \( (X, d) \). If \( Q \in \mathcal{M}_{X} \) then the **Hausdorff Measure of Noncompactness** of the set \( Q \) is defined by
\[
\chi(Q) = \inf \{ \epsilon > 0 : Q \text{ has a finite } \epsilon \text{-net in } X \}
\]
The function \( \chi : \mathcal{M}_{X} \to [0, \infty) \) is called **Hausdorff Measure of Noncompactness** [2].

The basic properties of **Hausdorff Measure of Noncompactness** can be found in ( [3], [15], [2]).

Some of those properties are
If $Q, Q_1$ and $Q_2$ are bounded subsets of a metric space $(X, d)$, then

\[ \chi(Q) = 0 \iff Q \text{ is totally bounded set}, \]
\[ \chi(Q_1) = \chi(Q_2), \]
\[ Q_1 \subset Q_2 \Rightarrow \chi(Q_1) \leq \chi(Q_2), \]
\[ \chi(Q_1 \cup Q_2) = \max \{\chi(Q_1), \chi(Q_2)\}, \]
\[ \chi(Q_1 \cap Q_2) = \min \{\chi(Q_1), \chi(Q_2)\}. \]

Further if $X$ is a normed space then Hausdorff Measure of Noncompactness $\chi$ has the following additional properties connected with the linear structure.

\[ \chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2) \]
\[ \chi(\eta Q) = |\eta| \chi(Q) \quad \eta \in \mathbb{C} \]

The most effective way of characterizing operators between Banach Spaces is by applying Hausdorff Measure of Noncompactness. If $X$ and $Y$ are Banach spaces, and $L \in \mathcal{B}(X, Y)$, then the Hausdorff Measure of Noncompactness of $L$, denoted by $\|L\|_\chi$ is defined as

\[ \|L\|_\chi = \chi(L(S_X)) \]

Where $S_X = \{x \in X : \|x\| = 1\}$ is the unit ball in $X$.

From ( [12], Corollary 1.15) we know that

\[ L \text{ is compact if and only if } \|L\|_\chi = 0 \]

Proposition 3. ([2], Theorem 6.1.1, $X = c_0$) Let $Q \in M_{c_0}$ and $P_r : c_0 \to c_0 \ (r \in \mathbb{N}$ be the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. Then, we have

\[ \chi(Q) = \lim_{r \to \infty} \left( \sup_{x \in Q} \|(I - P_r)(x)\| \right) \]

where $I$ is the identity operator on $c_0$.

Proposition 4. (cf. [2], Theorem 6.1.1) Let $X$ be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, and $Q \in M_X$ and $P_n : X \to X \ (n \in \mathbb{N}$ be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then, we have
\[ \frac{1}{a} \lim_{n \to \infty} \sup_{x \in Q} \| (I - P_n)(x) \| \leq \chi(Q) \]

\[ \leq \inf_{n} \left( \sup_{x \in Q} \| (I - P_n)(x) \| \right) \leq \lim_{n \to \infty} \sup_{x \in Q} \| (I - P_n)(x) \| \]

where \( a = \lim_{n \to \infty} \sup \| I - P_n \| \), and \( I \) is the identity operator on \( c \).

If \( X = c \) then \( a = 2 \). (see [2], p.22).

5. Compact operators on the spaces \( (\overline{N}_p^q)_0 \), \( (\overline{N}_p^q)_\infty \)

**Theorem 5.** Consider the matrix \( A \) as in Corollary 1, and for any integers \( n,s, n > s \) set

\[ \| A \|^{(s)} = \sup_{n > s} \sup_{m} \left\{ \sum_{k=0}^{m} R_k \left( \sum_{j=k}^{m} (-1)^{j-k} \frac{H_j^{(p)}}{q_j} a_{nj} \right) \right\} \]  

(5.1)

If \( X \) be either \( (\overline{N}_p^q)_0 \) or \( (\overline{N}_p^q)_\infty \) and \( A \in (X, c_0) \). Then

\[ \| L_A \|_{\chi} = \lim_{s \to \infty} \| A \|^{(s)}. \]  

(5.2)

If \( X \) be either \( (\overline{N}_p^q)_0 \) or \( (\overline{N}_p^q)_\infty \) and \( A \in (X, c) \). Then

\[ \frac{1}{2} \cdot \lim_{s \to \infty} \| A \|^{(s)} \leq \| L_A \|_{\chi} \leq \lim_{r \to \infty} \| A \|^{(s)}. \]  

(5.3)

and if \( X \) be either \( (\overline{N}_p^q)_0 \) or \( (\overline{N}_p^q)_\infty \) and \( A \in (X, \ell_\infty) \). Then

\[ 0 \leq \| L_A \|_{\chi} \leq \lim_{s \to \infty} \| A \|^{(s)}. \]  

(5.4)

**Proof.** Let \( F = \{ x \in X : \| x \| \leq 1 \} \) if \( A \in (X, c_0) \) and \( X \) is one of the spaces \( (\overline{N}_p^q)_0 \) or \( (\overline{N}_p^q)_\infty \), then by Proposition 3

\[ \| L_A \|_{\chi} = \chi(AF) = \lim_{s \to \infty} \left[ \sup_{x \in F} \| (I - P_s)A x \| \right] \]  

(5.5)

Again using Proposition 1 and Corollary 1 we have

\[ \| A \|^{*} = \sup_{x \in F} \| (I - P_s)A x \| \]  

(5.6)

From (5.5) and (5.6) we get

\[ \| L_A \|_{\chi} = \lim_{s \to \infty} \| A \|^{(s)}. \]
Since every sequence \( x = (x_k)_{k=0}^{\infty} \in c \) has a unique representation
\[
x = le + \sum_{k=0}^{\infty} (x_k - l)e^{(k)}
\]
where \( l \in \mathbb{C} \) is such that \( x - le \in c_0 \)

We define \( P_s : c \to c \) by
\[
P_s(x) = le + \sum_{k=0}^{s} (x_k - l)e^{(k)}, \quad s = 0, 1, 2, \ldots
\]
Then \( \|I - P_s\| = 2 \) and using (5.6) and Proposition 4 we get
\[
\frac{1}{2} \cdot \lim_{s \to \infty} \|A\|^{(s)} \leq \|LA\|_\chi \leq \lim_{s \to \infty} \|A\|^{(s)}
\]

Finally we define \( P_s : \ell_\infty \to \ell_\infty \) by
\[
P_s(x) = (x_0, x_1, \ldots, x_s, 0, 0, \ldots), \quad x = (x_k) \in \ell_\infty.
\]
Clearly \( AF \subset P_s(AF) + (I - P_s)(AF) \)
So using the properties of \( \chi \) we get
\[
\chi(AF) \leq \chi[P_s(AF)] + \chi[(I - P_s)(AF)]
= \chi[(I - P_s)(AF)]
\leq \sup_{x \in F} \|(I - P_s)A(x)\|
\]
Hence by Proposition 1 and Corollary 1 we get
\[
0 \leq \|LA\|_\chi \leq \lim_{s \to \infty} \|A\|^{(s)}
\]

□

A direct corollary of the above theorem is

**Corollary 3.** Consider the matrix \( A \) as in Corollary 1, and \( X = (\bar{N}^p_q)_0 \) or \( X = (\bar{N}^p_q) \) then if \( A \in (X, c_0) \) or \( A \in (X, c) \) we have
\[
LA \text{ is compact if and only if } \lim_{s \to \infty} \|A\|^{(s)} = 0
\]
Further, for \( X = (\bar{N}^p_q)_0 \) or \( X = (\bar{N}^p_q) \) or \( X = (\bar{N}^p_q)_{\infty} \), if \( A \in (X, \ell_\infty) \) then we have
\[
LA \text{ is compact if } \lim_{s \to \infty} \|A\|^{(s)} = 0 \quad (5.7)
\]

In (5.7) it is possible for \( LA \) to be compact although \( \lim_{s \to \infty} \|A\|^{(s)} \neq 0 \), that is the condition is only sufficient condition for \( LA \) to be compact.

For example, let the matrix \( A \) be defined as \( A_n = e^{(1)} \) \( n = 0, 1, 2, \ldots \) and the positive sequences \( q_n = 3^n \) \( n = 0, 1, 2, \ldots \) and \( p_0 = 1, p_1 = 1, p_k = 0 \), \( \forall k = \)
2, 3, . . .

Then by (3.13) we have

$$\sup_{n,m} \left\{ \sum_{k=0}^{m} R_k \left| \sum_{j=k}^{m} (-1)^{j-k} \frac{H_{j-k}}{q_j} a_{nj} \right| \right\} = \sup_{m} \left( 2 - \frac{2}{3^m} \right) = 2 < \infty$$

Now by Corollary 1 we know $A \in \left( (\bar{N}_p^q)_\infty, \ell_\infty \right)$.

But

$$\|A\|^{(s)} = \sup_{n>s} \left[ 2 - \frac{2}{3^m} \right] = 2 - \frac{1}{2 \cdot 3^s} \forall s$$

Which gives $\lim_{s \to \infty} \|A\|^{(s)} = 2 \neq 0$.

Since $A(x) = x_1$ for all $x \in (\bar{N}_p^q)_\infty$, so $L_A$ is a compact operator.

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