1. INTRODUCTION

The classical soft graviton theorem [1] describes the spectrum of low frequency gravitational radiation emitted during a classical scattering process, in terms of the trajectories and spin angular momenta of ingoing and outgoing objects, including hard radiation. This has been proved to subleading order in the expansion in powers of the soft frequency by taking the classical limit of the quantum soft graviton theorem. In this paper we give a direct proof of this result by analyzing the classical equations of motion of a generic theory of gravity coupled to interacting matter in space-time dimensions larger than four.

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The strategy we follow will be to take all but the linearized terms in the Einstein’s equation to the right-hand side and regard the right-hand side as the total energy-momentum tensor of the system. This includes the energy-momentum tensor of the gravitational field as defined in [17]. We can then “solve” the equations by taking the convolution of the flat space retarded Green’s function with the energy-momentum tensor. The domain of integration is then divided into two parts: a large but finite spatial volume around the region where the scattering takes place, and the region outside this volume. In the outer region we approximate the energy-momentum tensor by that of free particles corresponding to the asymptotic incoming and outgoing particles and radiation. In the inner region the energy-momentum tensor, including the non-linear terms in the Einstein’s equation, are complicated, but we determine the low frequency gravitational radiation from this region simply by using local conservation laws. As we show below, the sum of the two contributions is independent of the precise division of the space-time regions we choose, and is given solely by the asymptotic trajectories and spin angular momenta of the incoming and the outgoing particles.

We now summarize our main results. In the following we shall refer to the incoming and outgoing objects involved in the scattering as particles, even though we do not assume that they are structureless objects—even black holes, stars and bound binary systems will be counted as particles. This is justified by the fact that while describing the coupling of a gravitational field of wavelength much larger then the characteristic size of the objects, we can approximate the stress tensor for any finite size gravitating object by the stress tensor of a point particle with (generically) infinitely many multipole moments. We also regard the energy-momentum tensor of finite frequency gravitational and electromagnetic radiation, produced during the scattering, as that of a flux of massless particles, although we shall give a separate treatment of the radiation contribution to the soft factor in Sec. III. We denote the asymptotic trajectory of the $a$th particle by

$$r_{(a)} = c_{(a)} + V_{(a)}\sigma_a,$$  \hspace{1cm} (1.1)

where $c_{(a)}$ and $V_{(a)}$ are constant $D$-dimensional vectors and $\sigma_a$ is an appropriately normalized affine parameter. We also denote by $p_{(a)} \propto V_{(a)}$ the momentum of the $a$th particle, and by $\Sigma_{(a)\alpha\beta}$ the spin angular momentum carried by the $a$th particle, both counted with + sign if ingoing and – sign
if outgoing. Operationally, $c_{(a)}$, $V_{(a)}$, $P_{(a)}$, and $\Sigma_{(a)}$ can be defined through the energy-momentum tensor carried by the particle as given in (2.11). If $h_{\mu\nu} = (g_{\mu\nu} - \eta_{\mu\nu})/2$ denotes metric fluctuation, then we define

$$ e_{\alpha\beta}(x) \equiv \tilde{h}_{\alpha\beta}(x) - \frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} h_{\mu\nu}(x), \tag{1.2} $$

$$ e_{\alpha\beta}(\omega, \tilde{x}) \equiv \int dx_0 e^{i\omega x_0} e_{\alpha\beta}(x) $$

$$ = e^{i\omega|\tilde{x}|} \int du e^{i\omega u} e_{\alpha\beta}(x), \quad u \equiv x^0 - |\tilde{x}|, \tag{1.3} $$

$$ k \equiv -\omega \left( \frac{1}{|\tilde{x}|}, \right), $$

$$ N_\alpha \equiv \frac{1}{2} \left( \frac{1}{2\pi|\tilde{x}|} \right)^{(D-2)/2} \omega^{(D-4)/2}. \quad \tag{1.4} $$

We show that for large $|\tilde{x}|$, the small $\omega$ expansion of $\tilde{e}_{\alpha\beta}$ is given by (up to gauge transformations):

$$ \tilde{e}_{\alpha\beta}(\omega, \tilde{x}) = N_\alpha e^{i\omega|\tilde{x}|} \left[ \sum_a \frac{p_{(a)} \cdot p_{(a)\beta}}{k \cdot p_{(a)}} \left( \text{tr} \frac{1}{k} \eta_{\alpha\beta} J_{(a)\mu\nu} + O(\omega) \right) + \frac{1}{|\tilde{x}|^{D-2}} \frac{1}{|\tilde{x}|^{(D-3)}} \right], \tag{1.5} $$

where

$$ J_{(a)\mu\nu} = \{c_{(a)\gamma} p_{(a)\mu} - c_{(a)\mu} p_{(a)\gamma} + \Sigma_{(a)\mu\nu} \}. \tag{1.6} $$

denotes the total angular momentum carried by the $a$th particle, with the first two terms giving the orbital contribution and the last term giving the spin contribution. In (1.5) we have set $8\pi G = 1$. Here, and in the rest of the paper, all indices are raised and lowered by the Minkowski metric and all scalar products are also defined using the Minkowski metric. As indicated in the last line of (1.5), the order of the error is larger of $|\tilde{x}|^{-(D/2)}$ and $|\tilde{x}|^{-(D-3)}$. An important feature of (1.5) is that the result does not depend on any details of the actual scattering process or the nature of the interactions involved during the scattering. The leading term in (1.5), associated with the first term inside the square bracket, agrees with the results obtained in [18,19].

If a significant amount of momentum and/or angular momentum is carried away by the outgoing scalar, electromagnetic and/or gravitational radiation, then the sum over $a$ in (1.5) also includes the contribution due to radiation. An explicit form of this contribution may be written as:

$$ -N_\alpha e^{i\omega|\tilde{x}|} \left[ \int \tilde{h}_{\alpha\beta} \frac{A_{\alpha\beta}}{k \cdot A} - i \int \tilde{h}_{\alpha\gamma} \frac{k^\gamma A_{\alpha\beta}}{k \cdot A} \right]. \tag{1.7} $$

where $\tilde{h}_{\alpha\beta}$ denotes angular integration, and

$$ A^\alpha(\hat{n}^\prime) = \lim_{r \to \infty} \frac{1}{r^D-2} \int d^D r \hat{n}_{1}^\alpha T_{\alpha}^0(\hat{x}), $$

$$ B^\alpha(\hat{n}^\prime) = \lim_{r \to \infty} \frac{1}{r^D-2} \int d^D r \hat{n}_{1}^\alpha \left( x^0 T_{\alpha}^0 - x^\alpha T_0^0(\hat{x}) \right), \tag{1.8} $$

$T_{\mu\nu}$ is the contribution to the symmetric energy-momentum tensor due to massless fields. Note that there is no ambiguity regarding the definition of $T_{\mu\nu}$ for the gravitational field—it is what we get by taking the nonlinear terms in the Einstein’s equation to the right-hand side [17]. Explicit form of $T_{\mu\nu}$ for massless scalar, vector and gravitational field to the required order has been given in Appendix B. Physically $A^\alpha(\hat{n}^\prime)$ and $B^\alpha(\hat{n}^\prime)$ represent respectively the total flux of outgoing momentum and angular momentum [20] of radiation along the direction $\hat{n}^\prime$. The overall minus sign in (1.7) reflects that in (1.5) the momenta and angular momenta are counted as positive if ingoing, whereas $A^\alpha(\hat{n}^\prime)d\hat{n}^\prime$ and $B^\alpha(\hat{n}^\prime)d\hat{n}^\prime$ represent outgoing flux.

With (1.7) present on the right-hand side of (1.5), both sides of the equation contain the gravitational field $\tilde{e}_{\alpha\beta}$. However one can easily check that the contribution to (1.7) from $\tilde{e}_{\alpha\beta}$ of frequency of order $\omega$ or less is suppressed by higher powers of $\omega$ and therefore does not produce any order $\omega^0$ terms. Therefore (1.5) with (1.7) included can be regarded as an equation that determines the low frequency component of the gravitational radiation in terms of its finite frequency component and other asymptotic data. Our result is similar in spirit to the memory effect in four dimensions (see [21] for a review) where the memory term is determined in terms of finite frequency gravitational radiation and other asymptotic data.

With some work, this approach to deriving the classical soft theorem may be extended to one higher order in expansion in the soft frequency $\omega$, by including the next order terms in the expansion (2.11) of the energy-momentum tensor of matter in the far region. However the corresponding coefficients of expansion will not be universal—they will depend on the detailed properties of the incoming and the outgoing objects. Based on the quantum results of [10], we expect the corrections to $\tilde{e}_{\alpha\beta}$ at the next order to be of the form:

$$ -\frac{1}{2} N_\alpha e^{i\omega|\tilde{x}|} \sum_a \frac{1}{k \cdot p_{(a)}} k^\beta J_{(a)\mu\nu} + B_{(a)\alpha\beta}, \tag{1.9} $$

where $B_{(a)\alpha\beta}$ is some tensor that depends on the properties of the $a$th external state, but does not depend on the details of
the scattering process. \(^2\) \(B_{(\alpha)\beta\gamma\delta}\) is antisymmetric under \(\alpha \leftrightarrow \gamma\) and also under \(\beta \leftrightarrow \delta\), and symmetric under \((\alpha\gamma) \leftrightarrow (\beta\delta)\). This approach will break down at the next order due to the ambiguity described in (2.29) in determining the contribution to \(\tilde{e}_{\alpha\beta}\) from the near region. This is in agreement with the corresponding results in quantum soft graviton theorem described in [10].

A generalization of (1.5), including the contribution from massless particles, exists in four space-time dimensions as well [24,25], but due to the existence of long range electromagnetic and gravitational forces, the actual formula takes a different form (see Eqs. (2.2) and (2.6) of [25] for the general formula). In particular the subleading terms now have contribution proportional to \(\ln \omega\). As in the case of \(D > 4\), the four dimensional formula has been obtained by taking the classical limit of quantum soft graviton theorem. This has also been verified in explicit examples [26–28]. We expect that a direct classical derivation of this formula should be possible along the lines discussed in the paper, but the analysis will have to be more complicated due to the reasons described below Eq. (2.16).

The rest of the paper is organized as follows. In Sec. II we prove the classical soft theorem (1.5), assuming that the contribution due to the radiation can be treated as a flux of massless particles. In Sec. III we derive the radiation contribution (1.7) explicitly by solving the soft radiation sourced by the energy-momentum tensor of massless fields. The two appendices provide some technical results on the asymptotic growth of massless fields that is used in computing the contribution to \(\tilde{e}_{\alpha\beta}\) due to radiation.

II. CLASSICAL SOFT THEOREM

We consider the situation in which a set of objects enter a given region \(\mathcal{R}\) in space, interact among themselves, and then disperse. Our goal will be to compute the spectrum of low frequency gravitational waves emitted during this process. Decomposing the metric \(g_{\mu\nu}\) as \(\eta_{\mu\nu} + 2h_{\mu\nu}\), we express the Einstein’s equation as

\[
\partial_{\rho} \partial^{\rho} h_{\mu\nu} - \partial^{\rho} \partial_{\mu} h_{\rho\nu} - \partial^{\rho} \partial_{\nu} h_{\rho\mu} + \partial_{\rho} \partial_{\mu} h_{\rho\nu} = \left\{ T_{\mu\nu} - \frac{2}{D-2} \eta_{\mu\nu} T_{\rho\rho} \right\},
\]

where we have set \(8\pi G = 1\). \(T_{\mu\nu}\) contains contribution from the matter energy-momentum tensor as well as all the nonlinear terms in the Einstein’s equation. Bianchi identity ensures that \(T_{\alpha\beta}\) satisfies the conservation law (see e.g., [17]):

\[
\partial^{\rho} T_{\alpha\beta} = 0.
\]

Choosing de Donder gauge,

\[
\partial^{\rho} h_{\mu\rho} - \frac{1}{2} \partial_{\mu} h_{\rho\rho} = 0,
\]

and defining \(e_{\mu\nu}\) through the equation

\[
e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h_{\rho\rho},
\]

a “solution” to (2.1) may be written as

\[
e_{\alpha\beta}(x) = - \int d^{D}x' G_{\epsilon}(x, x') T_{\alpha\beta}(x'),
\]

where \(G_{\epsilon}\) denotes the flat space retarded Green’s function

\[
G_{\epsilon}(x, x') = \int \frac{d^{D}k}{(2\pi)^{D-1}} e^{ik\cdot(x-x')} \frac{1}{(\epsilon + i\omega)^{2} - \epsilon_{2}},
\]

We should note however that (2.5) should be regarded as an identity involving \(e_{\alpha\beta}\) instead of a solution, since the right-hand side of the equation also involves \(e_{\mu\nu}\) through \(T_{\alpha\beta}\).

If we define

\[
\tilde{e}_{\alpha\beta}(\omega, \vec{x}) \equiv \int dx^{0} e^{i\omega x^{0}} e_{\alpha\beta}(x^{0}, \vec{x}),
\]

then using (2.5), (2.6) we get

\[
\tilde{e}_{\alpha\beta}(\omega, \vec{x}) = - \int d^{D}x' \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik\cdot x'} e_{\epsilon}(\epsilon, \vec{x}-\vec{x}')
\]

\[
\times \frac{1}{(\omega + i\epsilon)^{2} - \epsilon_{2}} T_{\alpha\beta}(x').
\]

We now decompose \(\tilde{e}\) into its component \(\tilde{e}_{\parallel}\) along \(\vec{x} - \vec{x}'\) and \(\tilde{e}_{\perp}\) transverse to \(\vec{x} - \vec{x}'\). We can evaluate the integration over \(\tilde{e}_{\parallel}\) by closing the integration contour in the upper half plane and picking up the residue from the pole at

\[
\epsilon_{\parallel} = \sqrt{(\omega + i\epsilon)^{2} - \epsilon_{2}}.
\]

After this, for large \(|\vec{x}|\), the integration over \(\tilde{e}_{\perp}\) can be done using saddle point method, with the saddle point occurring at \(\tilde{e}_{\perp} = 0\). The result takes the simple form [1]:

\[
e_{\alpha\beta}(\omega, \vec{x}) \approx i\mathcal{N} e^{i\omega |\vec{x}|} \int d^{D}x' e^{i\vec{k} \cdot \vec{x}'} T_{\alpha\beta}(x'),
\]

where

\[
\mathcal{N} = \left( \frac{2\pi}{2\pi i |\vec{x}|} \right)^{(D-2)/2} \frac{1}{2\omega}, \quad k = -\omega \left( \frac{1}{\omega} \right).
\]
\( \approx \) denotes equality up to terms containing higher powers of \( 1/|\vec{x}| \). Since \( T_{a\beta}(x') \) receives contribution from the incoming and outgoing particle trajectories which extend to infinity, we need to regulate the integration over \( x' \) by including appropriate damping factors. For the outgoing trajectories we include a damping factor of \( e^{-\epsilon|x'|} \) for some small positive number \( \epsilon \), while for the incoming trajectories we include a damping factor of \( e^{\epsilon|x'|} \). This corresponds to replacing \( \omega = -k^0 \) by \( \omega + ic \) for outgoing trajectories and by \( \omega - ic \) for the incoming trajectories.

Let us now suppose that we have a classical scattering process in which the interaction takes place mainly around the origin of the spatial coordinates \( \vec{r} \). We shall evaluate (2.9) by dividing the domain of integration over the spatial coordinates \( \vec{r} \) into two parts: \( |\vec{r}| > L \) and \( |\vec{r}| \leq L \), and call the corresponding contributions \( \tilde{e}_{a\beta}^{1} \) and \( \tilde{e}_{a\beta}^{2} \) respectively. Here \( L \) is a large but finite number so that the interaction takes place mainly in the region \( |\vec{r}| \leq L \). We do not need to assume anything about the kind of interactions that take place in this region, except that they must be consistent with the conservation laws. Outside this region we only need to take into account the long range gravitational and electromagnetic fields, and even these can be treated perturbatively. We shall assume that all initial and final particles move with finite nonzero velocity so that in the far past and far future the region \( |\vec{r}| \leq L \) is nearly empty. This can always be achieved by choosing an appropriate Lorentz frame. However, since our final formula will be written in a Lorentz covariant form, it will also be valid in the frame where some of the initial or final state particles are at rest.

In the region \( |\vec{r}| \leq L \), we do not make any assumption about \( T_{\mu\nu} \) except its conservation laws. We shall see that this is sufficient to extract the relevant contribution to the integral from this region. On the other hand in the interval \( |\vec{r}| > L \), \( T_{\mu\nu} \) will be taken to be the energy-momentum tensor of free particles whose quantum numbers are the same as those of the incoming and the outgoing particles. We do not however assume that the particles are structureless: the effect of the internal structure of the particle is encoded in the fact that the energy-momentum tensor of the particles is allowed to have derivatives of delta functions localized on the trajectory besides the leading term proportional to the delta function [22,23,29–34]. In this case the Fourier transform \( \tilde{T}_{\mu\nu}(k) \) of the energy-momentum tensor will have an expansion in powers of \( k \), with the expansion coefficients encoding the internal structure of the particle. As long as we consider wavelengths large compared to the sizes of the particles, this expansion will be valid even for big objects like neutron stars, black holes or binary systems. For our analysis we shall only need the first two coefficients in the expansion, which are determined in terms of the momentum and spin of the particle. The precise expression will be given shortly.

We begin our analysis in the \( |\vec{r}| > L \) region. In this region we have, to first order in the expansion of derivatives of delta function, see, e.g., [35]:

\[
T_{a\beta}(x') = \sum_{a} \int \sigma_{a} d[V_{(a)\alpha}\gamma_{(a)\beta}\delta^{(D)}(x' - r_{(a)}(\sigma_{a})) + V_{(a)}(\Sigma_{a})_{\alpha\beta}\delta^{(D)}(x' - r_{(a)}(\sigma_{a}))],
\]

where the sum over \( a \) runs over all the incoming and outgoing particles, \( r_{(a)}(\sigma_{a}) \) denotes the trajectory of the \( a \)th particle with \( \sigma_{a} \) labeling an appropriately normalized affine parameter along the trajectory up to a sign, \( p_{(a)} \) and \( \Sigma_{(a)} \) are respectively the \( D \)-momentum and the spin angular momentum of the \( a \)th particle, both counted with positive sign for ingoing particles and negative sign for outgoing particles, and \( V_{(a)} = dr_{(a)}(\sigma_{a}) \). Since in our notation the ingoing momenta are positive, we take \( \sigma_{a} \) to increase from \( -\tau_{a} \) at the outer end to \( 0 \) on the surface \( |\vec{r}| = L \). In this notation \( V_{(a)} = K_{(a)}p_{(a)} \) for some positive constant \( K_{(a)} \) and the trajectory begins at some cutoff point \( R_{(a)} \equiv r_{(a)}(0) \) at the outer end and ends at a point \( c_{(a)}(\alpha) \equiv r_{(a)}(\alpha) \) on the surface \( |\vec{r}| = L \). This has been illustrated in Fig. 1. Therefore by definition \( |\vec{c}_{(a)}| = L \). \( (a\ldots\beta) \) denotes symmetrization, with the convention:

\[
A_{(a}\alpha\beta) = \frac{1}{2}(A_{a\alpha\beta} + B_{a\alpha\beta}).
\]

In the sum over \( a \) in (2.11), we include the effect of outgoing radiation (gravitational or electromagnetic) by...
regarding them as a flux of massless particles.\(^4\) Therefore the sum over \(a\) includes an angular integration over outgoing finite frequency radiation. In the special case where the total energy carried away by radiation is small compared to the energies of the massive objects involved in the scattering, the contribution due to radiation in the sum in (2.11) can be ignored.

There are of course higher order terms in the expansion (2.11) containing more derivatives of \(\delta^{(D)}(x' - r_a(\sigma_a))\), but they will not contribute to the soft theorem to subleading order. This can be seen as follows. First we see that when we substitute (2.11) into (2.9), the \(\delta^{(D)}(x' - r_a(\sigma_a))\) term in the first term gets localized at \(r_a(\sigma_a)\), but the \(\sigma_a\) integration produces a linearly divergent term from the large \(\sigma_a\) region in the \(k \to 0\) limit. This divergence is regulated by the \(\epsilon^1:\(k^p\)) term in (2.9), producing an inverse power of \(k\). This gives the leading term. Since the second term in (2.11) contains a derivative of \(\delta^{(D)}(x' - r_a(\sigma_a))\), we can first integrate by parts, bringing down a factor of \(k \cdot V(\sigma_a)\) and then repeat the previous argument. As a result this term is of order unity and begins contributing at the subleading order. Terms involving higher derivatives of \(\delta^{(D)}(x' - r_a(\sigma_a))\) will bring down more powers of \(k\). Therefore they will not contribute at the subleading order.

Using (2.11), the contribution \(\tilde{e}_{aq}(x)\) is given by:

\[
\tilde{e}_{aq}^1(x) = iNe^{ia|x|} \int d^Dx' e^{ik\cdot x'} T_{aq}(x') \\
= iNe^{ia|x|} \sum_a \int_{-\infty}^{0} d\sigma_a e^{ikr_a(\sigma_a)} \\
\times \{ V(a)\delta^{(D)}(x' - r_a(\sigma_a)) \} \\
+ iNe^{ia|x|} \sum_a e^{ik\cdot c_a(\sigma_a)} \frac{1}{\tilde{n}(\sigma_a) \cdot V(a) \delta(\tau_a)} \\
\times \{ \tilde{V}(a)\delta^{(D)}(x' - r_a(\sigma_a)) \},
\]

where we define,

\[
\tilde{n}(\sigma_a) = (0, \tilde{\sigma}(\sigma_a)/|\tilde{\sigma}(\sigma_a)|).
\]

The term in the penultimate line of (2.13) is a boundary term at \(|\tilde{x}'| = L\) that arises from having to integrate by parts the term involving \(\delta^{(D)}(x' - r_a(\sigma_a))\) in (2.11). This can be seen as follows. First we represent the boundary term as

\[
- iNe^{ia|x|} \sum_a e^{ik\cdot c_a(\sigma_a)} \int d^Dx' d\sigma_a \delta(|\tilde{x}'| - L) \\
\times \{ V(a)\delta^{(D)}(x' - r_a(\sigma_a)) \}.
\]

We now carry out the integration over \(x'\) using the \(\delta^{(D)}(x' - r_a(\sigma_a))\) factor. This replaces \(\delta(|\tilde{x}'| - L)\) by \(\delta(|\tilde{x}'| - L) - \delta(-|\tilde{x}'| - L)\). We then carry out the \(\sigma_a\) integration using this delta function, which has support at \(\sigma_a = 0\) since \(|\tilde{x}'| = |\tilde{\sigma}(\sigma_a)| = L\). This generates a factor of \(-1/\tilde{n}(\sigma_a) \cdot V(a)\), with the minus sign reflecting the fact that \(V(a)\) is counted as positive if ingoing.

There are similar terms from the outer boundary where \(r_a(\sigma_a)\) takes value \(K(\sigma_a)\), but the \(ic\) prescription described below (2.10) dampens these terms in the limit \(R_\infty^0 \to \infty\) for outgoing particles and \(R_\infty^\alpha \to -\infty\) for incoming particles. Using the trajectory equation \(r_a(\sigma_a) = c(a) + V(a)\sigma_a\), we can carry out the integration over \(\sigma_a\) in (2.13). If we now use the relation \(\tilde{V}(a) = K(a)c(a)\) for some positive constant \(K(a)\), we can express (2.13) as:

\[
\tilde{e}_{aq}^1(x) = N e^{ia|x|} \sum_a e^{ik\cdot c_a(\sigma_a)} \\
\times \left[ \frac{1}{k \cdot \tilde{n}(\sigma_a) \cdot V(a) \delta(\tau_a)} \{ P(a)\delta^{(D)}(x' - r_a(\sigma_a)) \} \right].
\]

In arriving at this expression we have dropped all terms proportional to \(e^{i\omega K(a)\tau_a}\) by appropriate addition of \(\pm ic\) terms to \(\omega = -k^0\).

Let us estimate the error we made in the above calculation by taking \(T_{\mu\nu}\) to be the energy-momentum tensor produced by free particles. Since we are computing the result to subleading order \(\epsilon^0\), we shall estimate the error to this order. The first error stems from the fact that the particles are not free, but are under the influence of each other’s long range gravitational (and possibly electromagnetic) fields. These forces fall off as \(1/r^D\) when the distance between the particles is of order \(r\)—with all distances being measured with respect to the flat metric. Integrating this once we see that the correction to \(P(a)\) (and hence also \(\tilde{V}(a)\)) fall off as \(1/r^D\). Therefore the integral of the error over part of the trajectory from \(r\) to \(\infty\) will fall off as \(1/r^{D-4}\). Since \(r \geq L\) in the integration region for evaluating \(\tilde{e}_{aq}^1\), the net error in the computation of \(\tilde{e}_{aq}^1\) is bounded by \(1/L^{D-4}\). For \(D > 4\), this error vanishes in the large \(L\) limit.

There may also be contributions to \(\tilde{e}_{aq}^1\) from \(T_{\alpha\beta}\) stored in the long range fields (gravitational and electromagnetic). We shall show in Sec. III that this contribution can be included in the sum over \(a\) in (2.16) by regarding the radiative field contribution as a sum over the flux of massless particles. In the final expression, the additional contribution to \(\tilde{e}_{aq}^1\) due to radiation will be given by (1.7).

We now turn to the contribution \(\tilde{e}_{aq}^2\) to (2.9) from the \(|\tilde{x}'| = L\) region. We have:

\[
\tilde{e}_{aq}^2(x) = i Ne^{ia|x|} \int d^Dx' e^{ik\cdot x'} T_{aq}(x'),
\]
This gives:

\[ \int_{|\vec{x}'| < L} d^Dx' \partial_{\alpha a} (\epsilon^{ik'x'})^T \rho (x') \].

Using integration by parts and the conservation law (2.2), this can be expressed as

\[ \int_{|\vec{x}'| = L} d^Dx' (\omega^{ik} - L) \ldots \]

We can evaluate the right-hand side of (2.19) by noting that on the boundaries $|\vec{x}'| = L$ the energy-momentum tensor may be approximated by those of the free particles entering and leaving the region $|\vec{x}'| = L$. We now use (2.11) to express (2.19) as,

\[ \int_{|\vec{x}'| = L} d^Dx' (\omega^{ik} - L) \ldots \]

Expanding the $e^{ikc(a)}$ factor in powers of $k \cdot c(a)$ and using the momentum conservation law

\[ \sum_a P(a) = 0, \]

we can express (2.23) as

\[ ik^2 \tilde{\rho}_{al} (k) = -iN \epsilon^{iaol} \sum_a \left( ik \cdot c(a)P(a)_\beta - \frac{i}{2} k^l \Sigma(a)_{\beta\gamma} \right) \]

\[ + \frac{i}{2} k^l \frac{P(a)_\beta}{\Sigma(a)_{\beta\gamma}} \Sigma(a)_{\gamma\rho} \tilde{\rho}_{al}^\rho + \frac{i}{2} k \cdot P(a) \Sigma(a)_{\beta\gamma} \tilde{\rho}_{al}^\rho + O(\alpha^2). \]

Since in the definition (2.17) of $\tilde{\rho}^\rho_{al}$ the integration over $\vec{x}'$ is confined to a finite region (which also effectively makes the integration over $x^0$ bounded since by assumption the region $|\vec{x}'| = L$ becomes empty for large $|x^0|$), $\tilde{\rho}^\rho_{al}$ is an analytic function of $k^\mu$ near $k = 0$ and should admit a Taylor series expansion in $k^\rho$. We propose the following solution for $\tilde{\rho}^\rho_{al}(k)$:

\[ \tilde{\rho}^\rho_{al} = -iN \epsilon^{iaol} \sum_a \left( c(a)aP(a)_\beta - \frac{1}{2} \Sigma(a)_{\beta\alpha} \right) \]

\[ + \frac{1}{2} \frac{P(a)_\beta}{\Sigma(a)_{\beta\alpha}} \Sigma(a)_{\alpha\rho} \tilde{\rho}^\rho_{al} + \frac{1}{2} \frac{P(a)_\alpha}{\Sigma(a)_{\alpha\rho}} \Sigma(a)_{\beta\rho} \tilde{\rho}^\rho_{al} + O(k). \]

It satisfies (2.25) up to terms of order $\alpha$. We also need to check that this is symmetric under exchange of $\alpha$ and $\beta$. For this we note that angular momentum conservation implies:
\[ \sum_a (c_{(a)\alpha} P_{(a)\beta} - c_{(a)\beta} P_{(a)\alpha} + \Sigma_{(a)\alpha\beta}) = 0. \]  

(2.27)

Adding (2.27) multiplied by \(iN e^{i\omega|\vec{k}|}/2\) to (2.26), we get:

\[ \hat{\varepsilon}_{a\beta} = -iN e^{i\omega|\vec{k}|} \sum_a \left\{ c_{(a)(a)\beta} + \frac{1}{2} \frac{P_{(a)\beta}}{\hat{R}_{(a)}} \Sigma_{(a)\alpha\beta} \hat{R}^\rho_{(a)} + \frac{1}{2} \frac{P_{(a)\alpha}}{\hat{R}_{(a)}} \Sigma_{(a)\beta\alpha} \hat{R}^\rho_{(a)} \right\} + O(k), \]

(2.28)

which is manifestly symmetric under \(\alpha \leftrightarrow \beta\).

We can also argue that the solution (2.26) is unique. To see this we assume the contrary, that there is another solution. Then the difference \(d_{a\beta}\) between the two solutions will be analytic function of \(k^\mu\) near \(k = 0\) and will satisfy the constraint \(k^\mu d_{a\beta} = 0\). It is easy to check that there is no function \(d_{a\beta}(k)\) that satisfies this requirement, is analytic at \(k = 0\) and does not vanish at \(k = 0\). The first term in the power series expansion in \(k_\mu\) that satisfies this constraint is proportional to

\[ k^2 \eta_{a\beta} - k_\alpha k_\beta. \]

Adding (2.16) and (2.28), and expanding in powers of \(k\), we get:

\[ \hat{\varepsilon}_{a\beta} = \hat{\varepsilon}^1_{a\beta} + \hat{\varepsilon}^2_{a\beta} = N e^{i\omega|\vec{k}|} \left[ \sum_a \frac{P_{(a)a\beta}}{k \cdot P_{(a)}} - \sum_a \frac{1}{2} \frac{P_{(a)a}}{k^2} \Sigma_{(a)a\gamma} J_{(a)\gamma\beta} \right] + O(k), \]

(3.20)

where

\[ J_{(a)\gamma\beta} = \{ c_{(a)\alpha} P_{(a)\beta} - c_{(a)\beta} P_{(a)\alpha} + \Sigma_{(a)\alpha\beta} \}. \]

(3.21)

This is the classical soft graviton theorem to subleading order. We emphasis that the sum over \(a\) in (3.20) includes integration over the flux of gravitational (and electromagnetic if any) radiation, with \(J_{(a)\alpha\gamma}\) representing the flux of angular momentum carried by the radiation. Explicit expression for these contributions has been given in (1.7). In Sec. III we shall derive (1.7) by directly working with massless fields instead of regarding them as flux of massless particles.

We conclude this section by exploring the possibility of extending the analysis to higher orders in the frequency \(\omega\) of the soft graviton:

1. In order to extend our computation of \(\hat{\varepsilon}^1_{a\beta}\) to higher order in \(\omega\), we need to keep terms in the expression (2.11) involving higher number of derivatives of the delta function. However the coefficients of these terms are not expected to be universal. Instead they will depend on the internal structures of the objects involved in the scattering. Nevertheless, these contributions will still be independent of the details of the scattering process, being sensitive only to the properties of the incoming and the outgoing objects. For Kerr black holes in four dimensions, some of the coefficients of higher derivatives of the delta function have been computed in [22,23].

2. Presence of the term (2.29) in the expression for \(\hat{\varepsilon}^2_{a\beta}\) begin affecting the soft radiation at the sub-subleading order. These contributions are expected to depend on the details of the scattering process and not just on the properties of the incoming and the outgoing objects. Therefore our approach cannot unambiguously determine the low frequency gravitational radiation in terms of the properties of the incoming and outgoing objects beyond the sub-subleading order and further details of the theory are required to determine the metric.

We have described in (1.9) the expected correction to \(\hat{\varepsilon}_{a\beta}\) at the sub-subleading order in the expansion in powers of \(\omega\). We hasten to add however that this expectation is based on the quantum soft graviton theorem derived in [10], and we have not derived (1.9) from a classical analysis.

### III. SOFT RADIATION FROM FIELDS

Our goal in this section will be to compute soft radiation sourced by fields. We shall use (2.9) for this computation, by dividing the integration region into the \(|\vec{x}| > L\) and \(|\vec{x}| \leq L\) parts as in Sec. II. We shall divide the analysis into two parts. In the first part we shall derive the analog of (2.11) for radiation. In the second part we shall use this result to compute soft radiation from the radiative stress tensor.

#### A. General form of the stress tensor of radiation

We begin by introducing some notations. Let us first define:

\[ t' = x^0, \quad r' = |\vec{x}|, \quad \hat{n}' = \vec{x}/r', \quad n' = (1, \hat{n}'), \]

\[ \hat{n}' = (0, \hat{n}'), \quad u' = t' - r', \]

(3.1)

where in (3.1), \(n'\) and \(\hat{n}'\) are to be regarded as contravariant vectors. We shall denote by \(\partial_{n}\) the derivative \(\partial/r \partial x^{n}\). We also define the transverse derivative \(\partial_{\perp}\) as follows. If \(\tilde{\delta}\hat{n}'\) denotes an infinitesimal vector orthogonal to \(\hat{n}'\), so that \(\hat{n}' + \tilde{\delta}\hat{n}'\) is still a unit vector to first order, then we define \(\partial_{\perp} S\) for any function \(S'(\hat{n}')\) via the relation:

\[ S(\hat{n}' + \tilde{\delta}\hat{n}') = S(\hat{n}') + \delta\hat{n}' \cdot \partial_{\perp} S, \quad \hat{n}' \cdot \partial_{\perp} S = 0. \]

(3.2)

It is easy to verify the following identities,
\[
\partial_\mu r' = \tilde{n}^{\mu}, \quad \partial_\mu u' = -n^{\mu}, \\
\partial_\mu \tilde{n}^i = -r^{-1}(n^\mu \tilde{n}^i - \delta^i_{\mu}) = -r^{-1} \eta^i_{\mu}, \\
\partial _\perp n^\mu_\perp = \eta^\perp_{\mu} = \partial _\perp \tilde{n}_\perp^\mu, \quad (3.3)
\]

where
\[
\eta^\perp_{\mu} \equiv \eta_{\mu \nu} + n^\mu n^\nu - n^\mu \tilde{n}^\nu - n^\nu \tilde{n}^\mu,
\]
is the projection operator into the space transverse to \(n'\) and \(\tilde{n}'\). This gives, for any function \(f(r', u, \tilde{n}')\):
\[
\partial_\mu f = \tilde{n}_\perp^\mu \partial_\perp f - n^\mu \partial_\nu f + \frac{1}{r} \eta^\perp_{\mu} \partial_\perp f. \quad (3.5)
\]

We shall now derive the analog of (2.11)—the asymptotic form of the stress tensor \(T_{\mu \nu}\) associated with massless fields, including gravitational and electromagnetic field. As in Sec. II, the stress tensor of the gravitational field will be defined to be whatever appears on the right hand side of the Einstein’s equation if the left-hand side contains only the linear terms. We claim that the relevant part of the stress tensor produced by radiation has the following expansion to order \(r'^{-D-1}\):
\[
T_{\mu \nu}(x') = \frac{1}{r^{D-1}} R(\tilde{n}', u') n^\mu n^\nu - \frac{1}{r^{D-1}} \{n^\mu R_{\perp \nu}(\tilde{n}', u') + n^\nu R_{\perp \mu}(\tilde{n}', u')\} \\
- \frac{1}{r^{D-1}} n^\mu n^\nu \tilde{\partial}_\perp \tilde{R}_{\perp}(\tilde{n}', u') \\
+ \frac{1}{r^{D-1}} \partial_\nu N_{\mu \nu}(\tilde{n}', u'), \quad (3.6)
\]

where \(R_{\perp}(\tilde{n}', u')\) is a vector with only transverse components:
\[
R_{\perp} = (0, \tilde{R}_{\perp}), \quad \tilde{n}' \tilde{R}_{\perp}(\tilde{n}', u') = 0. \quad (3.7)
\]

Furthermore, we show below that for large \(|u'|\), \(R\) and \(R_{\perp}\) fall off at least as fast as \(|u'|^{-(D-2)}\) and \(|u'|^{-(D-3)}\) respectively. \(N_{\mu \nu}(\tilde{n}', u')\) is a general tensor that vanishes as \(u' \to \pm \infty\).

The justification for (3.6) can be given as follows. We begin with the leading term \(R n^\mu n^\nu/r^{D-1}\). Any component \(f_\alpha\) of a massless field, irrespective of its spin, has a leading behaviour of the form \(f_\alpha(\tilde{n}', u')/r^{D-2}/2\) for large \(r\). Furthermore, as reviewed in Appendix A, \(f_\alpha(\tilde{n}', u')\) falls off at large \(|u'|\) [18,19,37]. Using (3.5) we now see that the leading term in \(\partial_\mu f_\alpha\) is given by \(-n^\mu \partial_\nu f_\alpha(\tilde{n}', u')/r^{D-2}/2\). Since the relevant term in the stress tensor in the asymptotic region comes from the square of the first derivative of the field, we see that in order to get a term of the form \(1/r^{D-3}\) in the stress tensor we must take the term \(n^\mu n^\nu \partial_\nu f_\alpha f_\beta/f_\alpha f_\beta/r^{D-3}\) and appropriately contract the indices. Since \(n^\mu\) cannot contract with itself or the transverse indices, by taking the leading order term in the fields to carry only transverse polarization—which is possible for \(D > 4\)—we can ensure that \(n^\mu\) and \(n^\nu\) remain uncontracted and only the transverse indices are contracted with each other. This shows that the \(1/r^{D-3}\) term in \(T_{\mu \nu}\) must be proportional to \(n^\mu n^\nu\), i.e., it takes the form given in the first term in (3.6). Using (3.3), (3.5) one can show that this term satisfies the energy-momentum conservation law \(\partial_\mu T_{\mu \nu} = 0\) by itself.

We now turn to the subleading terms in (3.6). We begin by writing down the most general expression for the order \(1/r^{D-1}\) term in \(T_{\mu \nu}\):
\[
\frac{1}{r^{D-1}} \{A n^\mu n^\nu + (n^\mu B_{\perp \nu} + n^\nu B_{\perp \mu}) + C (n^\mu \tilde{n}_\perp^\nu + n^\nu \tilde{n}_\perp^\mu) \\
+ F \tilde{n}_\perp^\mu + (\tilde{n}_\perp^\mu G_{\perp \nu} + \tilde{n}_\perp^\nu G_{\perp \mu}) + H_{\perp \mu \nu}\}, \quad (3.8)
\]

where \(A\) and \(C\) are scalars, \(B_{\perp \mu}\) are transverse vectors and \(H_{\perp \mu \nu}\) is a transverse symmetric tensor, all the quantities being functions of \(\tilde{n}'\) and \(u'\). We now demand \(\partial_\nu T_{\mu \nu} = 0\) and use (3.3). We get, at order \(1/r^{D-1}\):
\[
\partial_\nu C = 0, \quad \partial_\nu F = 0, \quad \partial_\nu G_{\perp \nu} = 0. \quad (3.9)
\]

Since for fixed \(r'\) the stress tensor must vanish for \(u' \to -\infty\), this gives:
\[
C = 0, \quad F = 0, \quad G_{\perp \nu} = 0. \quad (3.10)
\]

Vanishing of the term proportional to \(1/r^{D}\) in \(\partial_\mu T_{\mu \nu}\) give
\[
A - \partial_\perp B_{\perp} = 0, \quad \partial_\perp H_{\perp \perp}^{\mu \nu} = 0. \quad (3.11)
\]

The right-hand sides of these equations could actually have terms proportional to \(u'^4\)-derivatives of higher order coefficients of expansion, but since such terms can be absorbed into a redefinition of \(N_{\alpha \beta}\) in (3.6), we ignore them. This brings the result almost to the desired form (3.6) with the identification \(B_{\perp \nu} = -R_{\perp \nu}\), except that we need to show that the transverse tensor \(H_{\perp \mu \nu}\) has the form of the term proportional to \(\partial_\nu N_{\alpha \beta}\) in (3.6). This can be seen as follows:

(1) In the specific coordinate system that we are using, we have chosen the polarization tensors of the fields \(\phi_\alpha\) to be transverse (and also traceless for the radiative part of the metric). Therefore, neither the \(u'\) nor the \(r'\) derivative can contract with an index of the polarization tensor. One way to get rid of the free

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5This result is well known (see, e.g., [36]), but we include the argument for completeness.
u index from \( \partial_u \) is to pick the other derivative to be \( \partial_x \) and contract \( \partial_x \) with \( \partial_x \)—we see from (3.5) that this is possible since \( n' n' = 1 \). This leaves behind the indices from the polarization tensors, which could supply the indices of the transverse tensor \( H_{\mu \nu} \) (and, for the gravitational field, the left-over transverse indices can contract with each other). Since \( \partial_x \phi_0 \propto f_a(\hat{\mathbf{r}}, u') r^{-D/2} \) and \( \partial_x \phi_a \propto \partial_x f_a(\hat{\mathbf{r}}, u') r^{-D/2}/2 \), such contributions to \( H_{\mu \nu} \) will have the structure \( r^{-(D-1)} \partial_x f_a(\hat{\mathbf{r}}, u') \) with the polarizations appropriately contracted. It is easy to see that these terms are total derivatives in \( u' \) and therefore can be absorbed into the term proportional to \( \partial_x N_a \phi_0 \) in (3.6). For example for gauge fields we shall have \( H_{\mu \nu} \propto (f_i \partial_x f_j + f_j \partial_x f_i) = \partial_x (f_i f_j) \) and for the gravitational field we have \( H_{\mu \nu} \propto (f_i \partial_x f_{j'} + f_{j'} \partial_x f_i) = \partial_x (f_i f_j) \).

(2) For the gravitational field we need to also consider a possible contribution to \( T_{ij} \) proportional to \( \partial_x h_{\mu x} \partial_x h_{\nu i} \), where \( i, j \) are transverse directions and the two \( u \) indices coming from \( \partial_u \) are contracted with the two \( r \) indices of \( h_{\mu x} \). Even if by a choice of gauge we take the order \( r^{-(D-2)/2} \) term in \( h_{\mu x} \), to have only transverse components, \( h_{rr} \) could have a term of order \( r^{-D/2} \). Therefore, \( \partial_x h_{\mu x} \partial_x h_{\nu i} \) could give a contribution to \( T_{ij} \) of order \( r^{-(D-1)} \). However, as shown in Eq. (B35) in Appendix B, equations of motion forces the leading term in \( h_{rr} \) to be of order \( r^{-(D+2)/2} \). Therefore we cannot get a contribution of order \( r^{-(D-1)} \) to \( T_{ij} \) with \( i, j \) transverse from the \( \partial_x h_{\mu x} \partial_x h_{\nu i} \) term. This establishes (3.6).

In Appendix B we shall verify (3.6) explicitly for massless scalar, vector and tensor fields, where we also express (3.6) in Bondi coordinates. From this analysis it will also be clear that \( R \) and \( R' \) contain two powers of \( f_a(\hat{\mathbf{r}}, u') \), with \( R \) having two \( u' \) derivative acting on these factors and \( R' \) having one \( u' \) derivative acting on one of the factors. On the other hand, it follows from the analysis of [18,19,37]—rederived in Appendix A—that \( f_a(\hat{\mathbf{r}}, u') \) falls off at least as fast as \( |u'|^{-(D-4)/2} \) for large \( u' \). Therefore \( R \) falls off at least as fast as \( |u'|^{-(D-2)} \) and \( R' \) falls off at least as fast as \( |u'|^{-(D-3)} \) for large \( |u'| \).

Before concluding this section we shall discuss a subtle point. In odd dimensions, the expansion of a massless field \( \phi_a \) in inverse powers of \( r' \) also contains integer powers of the form \( r^{-(D-3)} \). In five dimensions this could upset the expansion (3.6), by producing a term of order \( r^{-(D-3)} \) from the product of the leading term in \( \phi_a \) of order \( r^{-(3/2)} \) and the subleading term of order \( r^{-(2)} \). This is larger than the subleading term of order \( r^{-4} \) given in (3.6). It was shown however in [38] that the order \( r^{-(D-3)} \) term in the expansion of \( \phi_a \) is \( u' \)-independent, and therefore when we try to construct the stress tensor from the field, we must necessarily act either a radial derivative or a transverse derivative on this component.\(^6\) This produces an extra power of \( 1/r' \), making the contribution to the stress tensor of order \( r'^{-9/2} \). This is smaller than the subleading term in (3.6) which falls off as \( r'^{-4} \).

**B. Computation of \( \tilde{e}_{a\phi} \)**

We shall now substitute (3.6) into (2.9) to compute \( \tilde{e}_{a\phi} \). As usual we divide the integration range into two parts, \( r' > L \) and \( r' \leq L \), to compute the contributions to \( \tilde{e}_{a}^{1} \phi \) and \( \tilde{e}_{a}^{2} \). We begin with the contribution to \( \tilde{e}_{a}^{1} \phi \), given by

\[
iN e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \int_{r' > L} d^{D}x' e^{i k \cdot x'} T_{R\phi}(x').
\] (3.12)

First we note that the contribution from the last term in (3.6) proportional to \( \partial_x N_a \phi_0 \) may be analyzed by integration by parts. There are no boundary terms since for fixed \( r' \), integration over \( u' \equiv t' - r' \) runs from \(-\infty \) to \( \infty \) and the integrand falls off at the two ends. Acting on the \( e^{i k \cdot x'} \) factor at fixed \( r', \hat{\mathbf{r}}, \partial_{u'} \), the \( \partial_{u'} \) term brings down a factor proportional to \( \omega = |\mathbf{k}| \). The integrand multiplying it falls off as \( d r'/r' \) for large \( r' \), and the \( e^{i k \cdot x'} \) factor renders the integral finite, with at most a \( \ln \omega \) divergence for small \( \omega \). Since \( \ln \omega \) terms are sub-subleading in the soft expansion, we can ignore this term in our computation.

Similarly one can show that higher order terms in the expansion of \( T_{R\phi} \), beyond those given in (3.6), do not contribute to (3.12) to subleading order. For this let us consider a term in \( T_{R\phi} \) of order \( r'^{-9(D-1)} \) for any positive number \( a \). We substitute this into (3.12) and evaluate it in the \( k \to 0 \) limit. First let us assume that the integrand falls off sufficiently fast for large \( |u'| \) so that the \( u' \) integral gives a finite result. Then the integration over the spatial coordinates is proportional to \( \int_{r' > L} d^{D-1}r'/r'^{(D-1+\alpha)} \). This goes as \( L^{-\alpha} \) and is therefore suppressed in the large \( L \) limit. Exceptions to this are terms coming from products of Coulomb components of the fields, which remain \( u' \) independent [38] over a time scale of order \( r' \). After this period the sources producing the Coulomb field will move away to a distance farther than \( r' \) and the field will begin to decrease. Therefore for the contribution to the stress tensor from the product of these terms, the \( u' \) integration can give terms of order \( r' \). However since the Coulomb component appears at order \( r'^{-(D-3)} \) and its derivatives are of order \( r'^{-(D-2)} \), its contribution to the stress tensor will be of order

\(^6\)Since in our set up the sources of the gravitational field travel with finite velocity at late time, we expect the Coulomb part to be not \( u' \) independent, but it should fall off for \( |u'| \gg r' \) since the sources move away to a distance further than \( r' \) for \( |u'| \gg r' \). Nevertheless the important point is that the \( u' \) derivative is of the same order as \( r' \) derivative and therefore does not produce any contribution to \( \partial_x \phi_0 \) of order \( r'^{-2} \).
Therefore even if the $u'$ integral produces a factor of $r'$, the spatial integral will be of the form
\[ \int_{r' > L} d^{D-1}r'/r^{(2D-3)} \sim L^{-(D-4)} \] For $D > 4$, this is suppressed for large $L$.

Therefore we need to evaluate (3.12) with $T_{\mu\nu}$ given by (3.6), ignoring the term proportional to $\partial_\mu N_{\mu\nu}$. For this we first make a change of variables:
\[ \vec{x} \rightarrow \vec{x} + \vec{R}_L(\vec{n}', \vec{u}')/R(\vec{n}', \vec{u}'). \] (3.13)

This induces the transformations:
\[ r' \rightarrow r', \quad u' \rightarrow u', \quad \vec{n}' \rightarrow \vec{n}' + \vec{R}_L(\vec{n}', \vec{u}'), \quad d^Dx' \rightarrow d^Dx' \left\{ 1 + \frac{1}{r'} \partial_{\vec{n}} \left( \frac{\vec{R}_L(\vec{n}', \vec{u}')}{R(\vec{n}', \vec{u}')} \right) \right\}, \] (3.14)

where we have ignored terms that are suppressed by two powers of $r'$. Using this in (3.12) we get
\[ \tilde{e}^1_{\alpha\beta} = -\mathcal{N} e^{i\omega|x|} \int_{r' > L} d^Dx' e^{ikn'x'} + i\omega n'x' \parallel R_L/R \parallel \frac{1}{r'^2} \left\{ R(\vec{n}', \vec{u}')n'_\alpha n'_\beta + \mathcal{O} \left( \frac{1}{r'^2} \right) \right\} \right\}. \] (3.15)

In particular the order $1/r^{(D-1)}$ terms in (3.6) get cancelled (except for the $\partial_\mu N_{\mu\nu}$ term, which we have argued does not contribute to \( \tilde{e}^1_{\alpha\beta} \) to subleading order). We now express $e^{ikn'x'}$ as $(ik \cdot n')^{-1} \partial_{\vec{n}}(e^{ik\cdot\vec{n}'x'})$, and then integrate by parts over $r'$, arriving at
\[ \tilde{e}^1_{\alpha\beta} = -\mathcal{N} e^{i\omega|x|} \int d\vec{n}' \int d' u' \frac{1}{k \cdot n'} e^{ikn'L + i\omega n'L} + i\omega R_L/R \parallel \frac{1}{r'^2} \left\{ R(\vec{n}', \vec{u}')n'_\alpha n'_\beta + \mathcal{O} \left( \frac{1}{r'^2} \right) \right\} \right\}. \] (3.16)

where $d\vec{n}'$ denotes integration over the angular variables and we have used:
\[ d^Dx' = d\vec{n}' du' r'^{D-2} dr'. \] (3.17)

The first term on the right-hand side of (3.16) represents the boundary contribution from $r' = L$. As usual, we have ignored boundary terms at infinity. The term in the second line has integrand of order $1/(r')^3$, and even when we expand the exponential factor to order $k^\alpha$ to pick the subleading term, the integrand will be of order $1/(r')^2$. Therefore the integral goes as $1/L$ and can be ignored. This gives, to subleading order in the expansion in powers of $\omega \equiv |k^0|$, \( \tilde{e}^1_{\alpha\beta} \) is:
\[ \tilde{e}^1_{\alpha\beta} = -\mathcal{N} e^{i\omega|x|} \int d\vec{n}' \int d'u' \frac{1}{k \cdot n'} \left\{ 1 + i\omega \cdot n'L \right\} + i\omega u' + ik \cdot R_L/R \parallel R(\vec{n}', \vec{u}')n'_\alpha n'_\beta \right\}. \] (3.18)

We now turn to the computation of
\[ \tilde{e}^2_{\alpha\beta}(x) = i\mathcal{N} e^{i\omega|x|} \int_{r' \leq L} d^Dx' e^{ik\cdotx'} T_{\mu\nu}(x'). \] (3.19)

The calculation will follow the same steps as the ones described below (2.17). We have
\[ k_\alpha e^{2\alpha\beta}(x) = \mathcal{N} e^{i\omega|x|} \int_{r' \leq L} d^Dx' \left\{ \partial_\beta e^{ik\cdotx'} \parallel T_{\mu\nu}(x') \right\} \] (3.20)

where in the second step we have carried out an integration by parts, picking up the boundary term at $r' = R$ and using the conservation law \( \partial_\alpha T_{\mu\nu}(x') = 0 \). Since the total incoming momentum flux is equal to the total outgoing momentum flux, we have\[ -(r')^{D-2} \int d\vec{n}' \int d'u' \hat{n}_\alpha' T_{\mu\nu}^{\beta}(x') \parallel r' = L = 0. \] (3.21)

Using this we can express (3.20) as
\[ k_\alpha e^{2\alpha\beta}(x) = i\mathcal{N} e^{i\omega|x|} (r')^{D-2} \int d\vec{n}' \int d'u' \hat{n}_\alpha' k \cdot x' T_{\mu\nu}^{\beta}(x') \parallel r' = L + \mathcal{O}(\omega^2). \] (3.22)

We can take the solution to (3.22) to be
\[ e^{2\alpha\beta}(x) = i\mathcal{N} e^{i\omega|x|} (r')^{D-2} \int d\vec{n}' \int d'u' \hat{n}_\alpha' x' T_{\mu\nu}^{\beta}(x') \parallel r' = L + \mathcal{O}(k). \] (3.23)

This does not look symmetric under $\alpha \leftrightarrow \beta$, but using the conservation of total angular momentum (see footnote 7):

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Footnote 7: actually it is the sum of the stress tensor of the radiation and matter that is conserved. So we really need to combine (3.20) with (2.18) and set to zero the total contribution to $\partial_\alpha T_{\mu\nu}$. Similarly neither (2.24) nor (3.21) is true individually, but their sum is true, and one should analyze the contribution to $\tilde{e}^2_{\alpha\beta}$ from matter and radiation together. A similar remark holds for the angular momentum conservation laws (2.27) and (3.24).
From the analysis in the penultimate paragraph of Sec. III A we know that $R$ falls off as $|u'|^{-(D-2)}$ and $R_L$ fall off as $|u'|^{-(D-3)}$ for large $u'$. Using these results in (3.27), (3.28) we see that the integrand in (3.27) falls off at least as fast as $|u'|^{-(D-3)}$ for large $|u'|$. Therefore its integral over $u'$ yields finite result for $D > 4$. This observation is particularly relevant for odd $D$ since the retarded Green’s function $G_r(x, x')$ has support for $x$ lying inside the future light-cone of $x'$, instead of on the future light-cone of $x'$.

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**APPENDIX A: RADIATIVE FIELDS AT LARGE RETARDED TIME**

In this appendix we shall study the asymptotic fall-off of massless fields in the scattering process at large retarded time. We shall assume that we have chosen a gauge such that the field equation of a massless field $\phi_a$ takes the form:

$$\partial^\mu \partial_\mu \phi_a = -j_a$$

(A1)

for some source term $j_a$. In this case the solution is given by:

$$\phi_a = - \int d^D x' G_r(x, x') j_a(x'),$$

(A2)

where $G_r$ is the retarded Green’s function. It was shown in [1] (and reviewed in Sec. II) that for large $|\vec{x}|$, (A2) takes the form:

$$\phi_a(t, \vec{x}) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} \tilde{\phi}_a(\omega, \vec{x}),$$

(A3)

with,

$$\tilde{\phi}_a(\omega, \vec{x}) = \frac{i}{2\omega} e^{i\omega |\vec{x}|} \left(\frac{D-2}{2\pi i |\vec{x}|}\right)^{(D-2)/2} \int d^D x' e^{i\omega (\vec{r} - \vec{n} \cdot \vec{x}')} j_a(x'),$$

(A4)

Now we know from the analysis of Sec. II that for small $\omega$:

$$\int d^D x' e^{i\omega (\vec{r} - \vec{n} \cdot \vec{x}')} j_a(x') \approx \frac{A(\hat{n})}{\omega} + \mathcal{O}(1),$$

(A5)

for some function $A(\hat{n})$. Since small $\omega$ behaviour of $\tilde{\phi}_a(\omega, \vec{x})$ controls the large time behavior of $\phi_a(t, \vec{x})$, we get, from (A2)–(A4),
\( \phi_a(t, \vec{x}) \approx \frac{i}{4\pi} \left( \frac{1}{2\pi|\vec{x}|} \right)^{(D-2)/2} \tilde{A}(\hat{t}) u^{-(D-4)/2} \),
\( u \equiv t - |\vec{x}|. \)

In even dimensions \( D > 4 \) the integral gives \( \delta(u) \) or its derivatives [18,37], and therefore the expression is localized around \( u = 0 \). In odd dimensions, changing integration variable from \( \omega \) to \( y \equiv ou \), we get:
\( \phi_a(t, \vec{x}) \approx \frac{i}{4\pi} \left( \frac{1}{2\pi|\vec{x}|} \right)^{(D-2)/2} \tilde{A}(\hat{t}) u^{-(D-4)/2} \)
\( \times \int dy e^{-iy(D-6)/2}. \)  

This shows that for \( D > 4 \), \( \phi_a(t, \vec{x}) \) falls off as \( u^{-(D-4)/2} \) for large \( u \). This agrees with the results of [19] and is one of the results used in our analysis in Sec. III for computing the contribution to \( \tilde{c}_{ab} \) due to stress tensor of massless fields.

**APPENDIX B: STRESS TENSOR OF RADIATION AT LARGE DISTANCE**

In this appendix we shall verify the general form (3.6) of the stress tensor associated with massless fields by explicitly constructing the stress tensor of massless scalar, vector and tensor fields. The asymptotic form of various massless fields that we shall use for this computation can be found in [38,39], but we also review their derivation. To simplify notation, we shall drop the subscript \( R \) from \( T_{R\mu\nu} \) and drop the primes from the coordinate labels used in (3.6).

We shall work with Bondi coordinates defined as:
\( u \equiv t - r, \quad r, \quad \theta_K \equiv \hat{h}_K. \)  

For a given vector \( A_\mu \), we get, using (3.3)
\[ A_\mu = \partial_\mu u A_u + \partial_\mu r A_r + \partial_\mu \theta^K A_K \]
\[ = -n_\mu A_u + \tilde{n}_\mu A_r + \frac{1}{r} \partial_\mu \theta^K A_K. \]  

so that we have
\[ A_r = n^\mu A_\mu, \quad A_u = (n^\mu - \tilde{n}^\mu) A_\mu. \]

In this coordinate system, the expected form of \( T_{\mu\nu} \) given in (3.6) takes the form:
\[ T_{uu} = \frac{1}{r^{D-2}} R - \frac{1}{r^{D-1}} D^K R_K, \]
\[ T_{ru} = 0, \quad T_{rr} = 0, \]
\[ \frac{1}{r} T_{uk} = \frac{1}{r^{D-3}} R_K, \quad \frac{1}{r} T_{rk} = 0, \quad \frac{1}{r} T_{KL} = 0. \]  

up to terms that are either of order \( 1/r^{D-1} \) and total derivative in \( u \) or of order \( 1/r^{D} \). \( R_K \) is related to \( R_{\perp i} \) in (3.6) via the relation,
\[ \partial_{\perp i} \theta^K R_K = R_{\perp i}. \]

The metric in this coordinate system is given by
\[ ds^2 = -2drdu - du^2 + r^2 Q_{KL} d\theta^K d\theta^L, \]  

where \( Q_{KL} \) is the metric on the unit sphere. The inverse metric has the form:
\[ g^{ru} = -1, \quad g^{rr} = 1, \quad g^{uu} = 0, \]
\[ g^{KL} = r^{-2} Q^{KL}, \quad g^{K0} = 0, \quad g^{K0} = 0. \]

The nonvanishing Christoffel symbols of the Minkowski metric in Bondi coordinate system are
\[ \Gamma^r_K = r Q_{KL}, \quad \Gamma^K_L = -r Q_{KL}, \]
\[ \Gamma^K_L = r^{-1} \delta^K_L, \quad \Gamma^K_L = \tilde{\Gamma}^K_L, \]

where \( \tilde{\Gamma}^K_L \) is the Christoffel symbol on the unit \( (D - 2) \) sphere labeled by the coordinates \( \theta^K \). We shall denote by \( D_K \) the covariant derivative on the unit sphere computed with the metric \( Q_{KL} \) and the Christoffel symbol \( \tilde{\Gamma}^K_L \), and by \( D^K \) the combination \( Q^{KL} D_L \).

Our goal will be to verify (B4) for massless fields. Let us first consider the case of a massless scalar field with stress tensor:
\[ T^\theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \phi \partial_\rho \phi. \]

In the Bondi coordinates the Laplace equation \( \square \phi = 0 \) takes the form:
\[ -2 \partial_\mu \partial^\mu \phi + \partial_\nu \phi + r^{-2} D^K D_K \phi - r^{-1} (D - 2) \partial_\nu \phi \]
\[ + r^{-1} (D - 2) \partial_\nu \phi = 0. \]

The asymptotic form of the scalar field produced during a classical scattering process has the form:
\[ \phi = \frac{1}{r^{(D-2)/2}} g(\hat{r}, u) + \frac{1}{r^{D/2}} \tilde{g}(\hat{r}, u) + O\left( \frac{1}{r^{(D+2)/2}} \right). \]  

Here \( g \) is some function that falls off for large \( |u| \) according to the results of Appendix A, and \( \tilde{g} \) will be determined shortly. In (B11) we have ignored a possible Coulomb term of order \( r^{-2} \) in \( D = 5 \), since we have argued in Sec. III that these terms do not contribute to \( T_{\mu\nu} \) to the required order.

Substituting (B11) into (B10) we find that the order \( r^{-D/2} \) term automatically vanishes and the order \( r^{-(D+2)/2} \) term gives:

\( ^{\text{In standard notation in general relativity, e.g., in } [38], g \text{ and } \tilde{g} \text{ would be denoted as } g^{(\frac{D}{2})} \text{ and } g^{(\frac{D}{2})} \text{ respectively. We shall avoid using this notation for brevity, but the translation is straightforward. The same translation can be made for the other fields introduced below, e.g., } a_\mu \text{ and } \tilde{a}_\mu \text{ will stand for } A_\mu^{(\frac{D}{2})} \text{ and } A_\mu^{(\frac{D}{2})} \text{ respectively, and } f_\mu \text{ and } \tilde{f}_\mu \text{ will stand for } h_\mu^{(\frac{D}{2})} \text{ and } h_\mu^{(\frac{D}{2})} \text{ respectively.} \)
\[ \partial_u \tilde{g}(\tilde{n}, u) = -\frac{1}{2} D^K D_K g(\tilde{n}, u) + \frac{(D - 2)(D - 4)}{8} g(\tilde{n}, u). \]  

(B12)

This gives, to order \( r^{-D/2} \),

\[ \partial_u \tilde{\phi} = \frac{1}{r^{(D-2)/2}} \partial_u g(\tilde{n}, u) - \frac{1}{r^{D/2}} D^K D_K g(\tilde{n}, u) + \frac{1}{r^{D/2}} \frac{(D - 2)(D - 4)}{8} g(\tilde{n}, u), \]

\[ \partial_r \tilde{\phi} = -\frac{D - 2}{2} \frac{1}{r^{D/2}} g(\tilde{n}, u), \quad \frac{1}{r} \partial_K \tilde{\phi} = \frac{1}{r^{D/2}} \partial_K g(\tilde{n}, u), \]  

(B13)

and therefore

\[ \partial_u \phi \partial^u \phi = -2 \partial_u \phi \partial_u \phi + \partial_r \phi \partial_r \phi + r^{-2} Q^{KL} \partial_K \phi \partial_L \phi = D_2 - \frac{2}{r^{D-1}} g(\tilde{n}, u) \partial_u g(\tilde{n}, u) + O(r^{-D}). \]  

(B14)

We now express the radiative part of the gauge field in the far region as

\[ A_\mu(x) = \frac{1}{r^{(D-2)/2}} a_\mu(\tilde{n}, u) + \frac{1}{r^{D/2}} \tilde{a}_\mu(\tilde{n}, u) + O(r^{-(D-2)/2}). \]  

(B18)

where the function \( a_\mu(\tilde{n}, u) \) falls off for large \( |u| \). As described in Sec. III, there are also Coulombic modes [38], but their contribution to the stress tensor can be ignored at this order. The Lorentz gauge condition \( \partial^K A_\mu = 0 \) gives \( \partial_\mu(n^\mu a_\mu) = 0 \). Since \( a_\mu \) falls off at large \( |u| \), we get \( n^\mu a_\mu = 0 \). We can use the residual gauge freedom \( A_\mu \rightarrow A_\mu + \partial_\mu \phi \) with \( \Box \phi = 0 \) to also set \( \tilde{n}^\mu \tilde{a}_\mu = 0 \). In the \( u, r, \theta^K \) coordinate system this gives, from (B3),

\[ A_u = \frac{1}{r^{D/2}} \tilde{a}_u, \quad A_r = \frac{1}{r^{D/2}} \tilde{a}_r, \]

\[ \frac{1}{r} A_K = \frac{1}{r^{(D-2)/2}} a_K + \frac{1}{r^{D/2}} \tilde{a}_K, \]  

(B19)

up to corrections of order \( r^{-D/2-1} \). Substituting (B19) into (B17) we get equations analogous to (B12):

\[ \partial_u \tilde{a}_u = 0, \quad \partial_u \tilde{a}_r = D^K a_K, \]

\[ \partial_u \tilde{a}_K = -\frac{1}{2} D^L D_L a_K + \frac{1}{8} (D^2 - 6D + 12) a_K. \]  

(B20)

The first equation, together with the fact that \( \tilde{a}_u \) vanishes in the far past, allows us to set \( \tilde{a}_u \) to 0.

In the \( (u, r, \theta^K) \) coordinate system, different components of the field strength \( F_{\mu \nu} \equiv (\partial_\mu A_\nu - \partial_\nu A_\mu) \) up to order \( r^{-D/2} \) are given by:

\[ F_{ur} = \frac{1}{r^{D/2}} D^K a_K, \]

\[ \frac{1}{r} F_{uk} = \frac{1}{r^{(D-2)/2}} \partial_\nu a_K \]

\[ + \frac{1}{r^{D/2}} \left\{ \frac{1}{2} D^L D_L a_K + \frac{D^2 - 6D + 12}{8} a_K \right\}, \]

\[ \frac{1}{r^2} F_{KL} = \frac{D - 4}{2} \frac{1}{r^{D/2}} a_K, \]

\[ \frac{1}{r^3} F_{KL} = \frac{1}{r^{D/2}} (D_K a_L - D_L a_K). \]  

(B21)

From this we can calculate the energy-momentum tensor:
\[ T^\mu_\nu = F^\mu_\rho F^\rho_\nu - \frac{1}{4} \eta^\mu_\nu F_{\rho\sigma} F^{\rho\sigma}, \quad (B22) \]

ignoring corrections of order \( r^{-D} \) and total derivatives in \( u \) in terms of order \( r^{-(D-1)} \). We first note that to this order

\[ F_{\mu\nu} F^{\mu\nu} = -4r^{-2} Q^{KL} F_{LK} F_{RL} = -2 \frac{D-4}{D-1} Q^{KL} a_K a_L. \quad (B23) \]

Since this is a total derivative we can ignore its contribution to \( T_{\mu\nu} \). Therefore we get, ignoring terms of order \( r^{-D} \) and total \( u \)-derivative terms of order \( r^{-(D-1)} \):

\[ T_{uu} = r^{-2} Q^{KL} F_{LK} F_{uL} \]
\[ = \frac{1}{D-2} Q^{KL} \partial_u a_K a_L - \frac{1}{D-1} Q^{KL} \partial_u a_K D_L D_M a_L, \]
\[ T_{rr} = 0, \quad \frac{1}{r} T_{rK} = 0, \quad T_{ur} = r^{-2} Q^{KL} F_{LK} F_{uL} = 0, \]
\[ \frac{1}{r^2} T_{KL} = \frac{1}{r} g^{uu} (F_{KL} F_{uL} + F_{Ku} F_{Lu}) = 0, \]
\[ \frac{1}{r} T_{uk} = \frac{1}{r} g^{uu} F_{uu} F_{Ku} + r^{-2} Q^{LM} \frac{1}{r} F_{uu} F_{KM} \]
\[ = \frac{1}{D-1} \partial_u a_K D_L a_L \]
\[ + \frac{1}{D-1} Q^{LM} \partial_u a_L (D_K a_M - D_M a_K). \quad (B24) \]

This has the same form as (B4) if we identify:

\[ R = Q^{KL} \partial_u a_K a_L, \]
\[ R_k = \partial_u a_K D_L a_L + Q^{LM} \partial_u a_L (D_K a_M - D_M a_K), \]
\[ D^K R_K = Q^{KL} \partial_u a_K D^M D_M a_L + \partial_u \mathcal{N}. \quad (B25) \]

It is easy to check that the last two equations are consistent with each other for suitable choice of \( \mathcal{N} \).

Finally we shall analyze the stress tensor associated with asymptotic gravitational field. We shall use the de Donder gauge:

\[ \partial^\rho h_{\mu\nu} - \frac{1}{2} \partial_\mu h_{\nu}^\rho = 0. \quad (B26) \]

so that the linearized equations of motion take the form \( \Box h_{\mu\nu} = 0 \). We expand the radiative part of \( h_{\mu\nu} \) in the far region as

\[ h_{\mu\nu} = \frac{1}{r^{D-2}/2} f_{\mu\nu}(\hat{n}, u) + \frac{1}{r^{D-2}} \tilde{f}_{\mu\nu}(\hat{n}, u) \]
\[ + \mathcal{O}(r^{-(D-2)/2}). \quad (B27) \]

ignoring the Coulombic modes as usual. The gauge condition (B26) gives, at leading order,

\[ n^\rho f_{\mu\nu} - \frac{1}{2} n_\nu f_{\rho}^\rho = 0. \quad (B28) \]

In this gauge we still have residual gauge symmetry

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}. \quad \Box \xi_{\mu} = 0, \quad (B29) \]

which induces a transformation

\[ f_{\mu\nu} \rightarrow f_{\mu\nu} + n_{\mu} a_{\nu} + n_{\nu} a_{\mu}, \quad (B30) \]

for any function \( a_{\mu}(\hat{n}, u) \). By adjusting \( a_{\mu} \) we can set

\[ f_{\rho}^\rho = 0, \quad \tilde{n}^\rho f_{\rho\nu} = 0. \quad (B31) \]

In Bondi coordinates this corresponds to the following expansion of the various components of \( h_{\mu\nu} \) up to order \( r^{-(D-2)/2} \):

\[ h_{uu} = \frac{1}{r^{D/2}} \tilde{f}_{uu}, \quad h_{rr} = \frac{1}{r^{D/2}} \tilde{f}_{rr}, \]
\[ h_{ru} = \frac{1}{r^{D/2}} \tilde{f}_{ru}, \quad h_{uk} = \frac{1}{r^{D/2}} \tilde{f}_{uk}, \]
\[ \frac{1}{r} h_{kK} = \frac{1}{r^{(D-2)/2}} \tilde{f}_{kK}, \quad \frac{1}{r} h_{KL} = \frac{1}{r^{(D-2)/2}} \tilde{f}_{KL} + \frac{1}{r^{D/2}} \tilde{f}_{KL}. \quad (B32) \]

We can now write down the \( \Box h_{\mu\nu} = 0 \) equations in the Bondi coordinate system, and substitute (B32) into these equations to determine \( \tilde{f}_{\mu\nu} \)’s in terms of \( f_{\mu\nu} \)’s as in the case of scalar fields and gauge fields. Explicit form of these equations can be found in [38,39]. For the sake of brevity we shall not describe the full set of equations for \( h_{\mu\nu} \), but give one example. The \( rr \) component of the equations of motion takes the form:

\[ -2 \partial_\rho \partial_\nu h_{rr} + \partial_\nu^2 h_{rr} + r^{-2} D^K D_K h_{rr} - r^{-1} (D - 2) \partial_\mu h_{rr} \]
\[ + r^{-1} (D - 2) \partial_\nu h_{rr} - 4 r^{-2} D^K h_{Kr} + 2 r^{-4} Q^{KL} h_{KL} \]
\[ + 2 r^{-2} (D - 2) (h_{ur} - h_{rr}) = 0. \quad (B33) \]

Upon substituting (B32) into this equation we find that the order \( r^{-(D+2)/2} \) terms in the equation gives

\[ \partial_\mu \tilde{f}_{rr} = -Q^{KL} f_{KL} = 0, \quad (B34) \]

where in the last step we have used the last equation of (B32). Vanishing of \( \partial_\mu \tilde{f}_{rr} \) is an important ingredient that was used in Sec. III to show that at order \( r^{-(D-1)} \), the transverse component of the gravitational stress tensor is a total derivative in \( u \)—we shall also see this explicitly in (B39). Similar analysis with the other components of the \( \Box h_{\mu\nu} = 0 \) equation leads to the following set of equations for the \( \tilde{f}_{\mu\nu} \)’s in the Bondi coordinates:
\[ \partial_u \tilde{T}_{uu} = 0, \quad \partial_u \tilde{T}_{ur} = 0, \quad \partial_u \tilde{T}_{rr} = 0, \]
\[ \partial_u \tilde{T}_{uk} = 0, \quad \partial_u \tilde{T}_{fK} = D^f \tilde{T}_{KL}, \]
\[ \partial_u \tilde{T}_{KL} = -\frac{1}{2} D^M D_M \tilde{T}_{KL} + \frac{1}{8} (D^2 - 6D + 16) \tilde{T}_{KL}. \] (B35)

We can now use this to compute the energy-momentum tensor of gravitational radiation. In the asymptotic region we only need to take the terms quadratic in \( h_{\mu \nu} \). This is given by [17][10]:
\[ T_{\mu \nu} = -h_{\mu \nu} R^{(1)}_{\rho \sigma} + \eta_{\mu \nu} h^{\rho \sigma} R^{(1)}_{\rho \sigma} + R^{(2)}_{\rho \sigma}, \] (B36)
where \( R^{(1)}_{\rho \sigma} \) and \( R^{(2)}_{\rho \sigma} \) represent contributions to the Ricci tensor \( R_{\rho \sigma} \) linear and quadratic in \( h_{\mu \nu} \) respectively:
\[ R^{(1)}_{\rho \sigma} = \partial_\mu \partial_\nu h_{\rho \sigma} - \partial_\rho \partial_\sigma h_{\mu \nu} - \partial_\mu \partial_\sigma h_{\nu \rho} + \partial_\sigma \partial_\rho h_{\mu \nu}, \] (B37)
\[ R^{(2)}_{\rho \sigma} = -2 h^{\rho \sigma} [\partial_\mu \partial_\nu h_{\rho \sigma} - \partial_\rho \partial_\sigma h_{\mu \nu} - \partial_\mu \partial_\sigma h_{\nu \rho} + \partial_\sigma \partial_\rho h_{\mu \nu}] + 2 \partial_\rho h_{\nu \sigma} - \partial_\sigma h_{\nu \rho} + \partial_\nu h^{\rho \sigma} \partial_\mu h_{\rho \sigma} - \partial_\mu h^{\rho \sigma} \partial_\nu h_{\rho \sigma} + \partial_\sigma h^{\rho \sigma} \partial_\mu h_{\rho \sigma} - \partial_\mu h^{\rho \sigma} \partial_\sigma h_{\rho \sigma} + \partial_\nu h^{\rho \sigma} \partial_\sigma h_{\rho \sigma} - \partial_\sigma h^{\rho \sigma} \partial_\nu h_{\rho \sigma}, \] (B38)

\[ \text{In the Bondi coordinates, } T_{\mu \nu} \text{ will have the same form, except that the derivatives } \partial_\mu \text{ will have to be replaced by } D_\mu - \text{covariant derivatives computed with the metric (B6) and connection (B8), and } \eta_{\mu \nu} \text{ will have to be replaced by the form of the metric given in (B6). The calculation is straightforward, yielding the result:} \]
\[ T_{uu} = \frac{1}{10 D^2 - 2 D^f} \frac{1}{f^{KL}} \partial_u f_{KL} D_M D_M f^{K L}, \]
\[ T_{uk} = \frac{1}{10 D^2 - 2 D^f} \left( \partial_u f_{MN} D_K f^{MN} - 2 D_M f^{LM} D_M f_{LK} + 2 D_M f^{LM} \partial_u f_{LK} \right), \] (B39)

with all the other components vanishing to this order. This has the form given in (B4) with:
\[ R = \partial_u f_{KL} \partial_u f^{KL}, \]
\[ R_K = \partial_u f_{MN} D_K f^{MN} - 2 D_M f^{LM} D_M f_{LK} + 2 D_M f^{LM} \partial_u f_{LK}, \]
\[ D^K R_K = \partial_u f_{KL} D^M D_M f_{KL} + \partial_u N. \] (B40)

In particular the last two equations are compatible for suitable choice of \( N \). The expression for \( T_{uk} \) given in (B39) also agrees with the result of [40] in \( D = 4 \).

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