Approximately universal optimality over several dynamic and non-dynamic cooperative diversity schemes for wireless networks

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Abstract—In this work we explicitly provide the first ever optimal, with respect to the Zheng-Tse diversity multiplexing gain (D-MG) tradeoff, cooperative diversity schemes for wireless relay networks. The schemes are based on variants of perfect space-time codes and are optimal for any number of users and all statistically symmetric (and in some cases, asymmetric) fading distributions.

We deduce that, with respect to the D-MG tradeoff, channel knowledge at the intermediate relays and infinite delay are unnecessary. We also show that the non-dynamic selection decode and forward strategy, the non-dynamic amplify and forward, the non-dynamic receive and forward, the dynamic amplify and forward and the dynamic receive and forward cooperative diversity strategies allow for exactly the same D-MG optimal performance.

I. INTRODUCTION

The emerging need for reliable communications of large quantities of data, at high rates, between small and independent users with small power supplies, no antenna arrays and minimal computational capabilities, brought to the fore cooperative-diversity in wireless networks, where distributed users relay messages for one another, in order to combat the fading and additive noise that hinders their joint communication, hence improving the overall quality of service. Cooperation can be achieved by relating each user with a segment of a point-to-point communication scheme and essentially having the intermediate relays manipulate the signal from the information source in a way that the received signal at the final destination relates to that of a point to point transmission with multiple transmit antennas. Network outage analysis provides for the fundamental limits of the network’s performance.

A. Structure of the paper

In Section I we will describe the cooperative-diversity setup and the related notation that will be used in the rest of the paper, will briefly introduce the existing cooperative diversity strategies, state the existing performance bounds and quickly go over the basic aspects of the D-MG tradeoff.

In Section II Theorem I we will present the expression for optimal D-MG performance for the non-dynamic selection-decode-and-forward strategy (ND-SDAF [3], [4]) and will provide the exact coding methodology that achieves this optimality. In proving the scheme’s optimality, we will use the existence of sets of approximately universal [7] codes which hold common elements, and which maintain approximate universality even when their structure is altered. We will also generalize with respect to network topology and fading statistics.

In Section III we will present a variant of the non-dynamic linear-processing relay network [1]. Unlike with ND-SDAF, this non-dynamic receive-and-forward (ND-RAF) scheme requires for coding distribution across the relays. We will then proceed to analyze the second stage equivalent (‘two-product’) channel and prove its partial D-MG equivalence to the Rayleigh fading channel. Using the existence of sets of approximately universal codes with joint elements, we will present in Theorem 2 the optimal performance for the ND-AAF and ND-RAF scheme and the exact coding methodology that achieves this performance. As a guideline for other potential constructions, we will introduce one by one all the necessary conditions for achieving optimality. A practical scheme, more suited for a network with a large number of users and which provides for a plethora of practical advantages, will also presented. Finally, a base station network setup will be presented together with some new variants of perfect codes that have the potential to render cooperation fruitful at the lowest possible SNR.

In Theorem 4 of Section IV we present D-MG optimal schemes for the dynamic amplify-and-forward and the dynamic receive-and-forward strategies (D-AAF, D-RAF [5]). Optimality holds for any number of users, any set of channel statistics and any statistical asymmetry.

Section V offers a comparison of the cooperative diversity schemes, and through Theorem 4 it is concluded that all the above strategies have high-SNR outage regions of the same volume and thus achieve the same D-MG optimal performance. We point out that given statistical symmetry, the ND-RAF scheme offers this same optimal performance at a reduced delay, reduced decoding complexity, reduced signalling complexity, minimal computational efficiency and the highest ease for network deployment. Section VI presents some simulations. The rest of the sections are appendices.

We begin we a general description of the network.

B. Describing the network

In [3], the authors describe the case where a set

\[ \mathcal{R} = \{ R_1, R_2, \ldots, R_n, R_{n+1} \} \]
of \( n + 1 \) different terminals/relays, cooperate in their effort to communicate over \( n + 1 \) different orthogonal frequencies \( \mathcal{F} = \{ \nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1} \} \). A certain relay \( R_i \), wanting to communicate with relay \( d(R_i) \), broadcasts its original information over frequency \( \nu_i \). Depending on the availability of each intermediate relay, the set

\[
D(R_i) \subset \{ R \setminus \{ R_i \cup d(R_i) \} \} \tag{1}
\]
is then the set of all intermediate relays that cooperate with \( R_i \). Consequently, each relay \( R_j \in D(R_i) \) transmits a possibly modified version of the received signal over frequency \( \nu_j \). By the end of the transmission, \( d(R_i) \) has received the information from \( R_i \) over frequency \( \nu_i \), in a form of a superposition of faded versions of signals originating from \( R_i \) and from \( D(R_i) \).

We will focus on the case where communication takes place in the presence of additive receiver noise, and in the presence of spatially independent quasi-static fading. Furthermore, we will assume complete knowledge of the fading channel at the receiver of the final destination, and depending on the cooperative diversity strategy, we will assume complete knowledge or absolutely no knowledge of the channel at the receivers of the intermediate relays. Finally, the half-duplex condition is imposed, due to practical considerations such as the large ratio between the transmission and reception powers at the relay antennas ([3], [4], [5]).

a) Instance of a network: From [3] we see that without loss of generality we can analyze the overall network performance just by focusing on a snapshot of the network, as shown in Figure 1 where \( S \) is now the information source, \( D \) the destination, \( R_2, \ldots, R_n \) are the intermediate relays, \( g_i \) is the fading coefficient between \( S \) and intermediate relay \( R_i \), \( h_i \) is the fading from \( R_i \) to \( D \) and \( h_i \) is the fading coefficient from \( S \) to \( D \). We consider \( h_i, g_i \) to be independently distributed circularly symmetric \( \mathbb{C} \mathbb{N}(0, 1) \) random variables, remaining constant throughout the transmission. Vectors \( \nu \) and \( w \) contain the elements \( \nu_{i,j} \) and \( w_j \) corresponding to the additive receiver noise respectively affecting \( R_i \) and \( D \) at time \( t = j \). Unless we state otherwise, we ask that all \( \nu_{i,j} \) and \( w_j \) be independently distributed \( \mathbb{C} \mathbb{N}(0, 1) \) random variables. SNR will represent the ratio of the signal power to the variance of the noise at the receiver of \( D \).

C. Existing cooperative diversity strategies

We proceed to briefly introduce the above mentioned cooperative diversity strategies.

1) Non-dynamic selection-decode-and-forward: In [4], Laneman, Tse and Wornell, described the non-dynamic selection-decode-and-forward cooperation strategy for the three-node network which asks for \( R_2 \) to cooperate, by first decoding and then re-encoding, if and only if it is not in outage with respect to the information source \( S \). The strategy requires full channel knowledge at the receivers of relay \( R_2 \) and destination \( D \). Focusing on the case where each node has a single transmit-receive antenna and where the fading scalars are from the Rayleigh distribution, the scheme has \( R_2 \) decode if and only if

\[
R_{\text{source}} < \log_2 (1 + \text{SNR}|g_2|^2) \tag{2}
\]

where \( R_{\text{source}} \) is the information rate of the transmission of \( S \), measured in bits per channel use (bpcu). Again specific to the case of Rayleigh fading and of having a single transmit-receive antenna per node operating in the half-duplex environment, it was shown in [4] that given a source-to-destination information rate \( R \), measured in bits per network channel use (bpcnu), and given a multiplexing gain [6]

\[
r = \frac{R}{\log_2(\text{SNR})},
\]

then the high-SNR probability of network outage, and thus the optimal D-MG performance, is given by:

\[
d_{\text{ND-SDAFout}}(r) := \frac{\log P_{\text{outage}}(r)}{\log \text{SNR}} = 2(1 - 2r), \quad r_{\text{max}} = 1/2
\]

and that this optimal performance will be achieved by some random Gaussian codes of infinite length. This result is readily generalized to a network of \( n + 1 \) users where the cooperation strategy for intermediate relay \( R_j \) is dictated by

\[
R_j \in D(S) \Leftrightarrow R < \log_2 (1 + \text{SNR}|g_j|^2), \quad j = 2, \ldots, n. \tag{3}
\]

Furthermore, given the implicit knowledge of the multiplexing gain at the nodes\(^1\), and conditioned on infinite time duration, the same approach translates to an optimal performance of

\[
d_{\text{ND-SDAF-opt}}(r) = (n - 1) \left( 1 - \frac{2n - 1 - r}{n - 1} \right)^+ + (1 - r) \tag{4}
\]

where \( \alpha^+ := \max(0, \alpha) \). The same performance was predicted in [4] for the non-dynamic amplify and forward scheme.

Letting \( r_{\text{coop}} \) describe the multiplexing gain corresponding to the set of \( (R, \text{SNR}) \) pairs where the D-MG performance of the cooperative scheme stops being equal to the D-MG performance in the non-cooperative case, we note that given a rate \( R \), cooperation essentially applies only for SNR values greater than

\[
\text{SNR} \geq 2 r_{\text{coop}} = \frac{2n - 1}{n - 1} R.
\]

As in [6], we have \( \approx, \gtrsim, \lesssim \) and \( \leq \) denoting asymptotic exponential equality and inequalities respectively.

\(^1\)Knowledge of the code/rate and of the power constraint, together with the existing assumption of unit variance additive noise, jointly imply knowledge of the multiplexing gain.
2) Non-dynamic receive-and-forward: In a variant of the two-stage wireless relay network model proposed by Jing and Hassibi in [1], [2], we will consider the case where the \( n - 1 \) intermediate relays are only allowed to perform linear-processing (time-averaging based on space-time codes) on the received signal. Knowledge of the channel \( (h_i,g_i) \) is given only to the receiver of the final destination \( D \).

In [1], [2], the authors also present a bound on the network’s pairwise error probability (PEP), which in the high-SNR regime is a function only of SNR and of the minimum eigenvalue of any difference of any two code matrices of the corresponding distributed space-time code which performs the linear-processing (see Appendix II-C). This result, in conjunction with the finite duration random coding proposed in [1], guarantees only for full-diversity but does not guarantee any bound on the code eigenvalues. As we discuss in Appendix II-C, this implies that with finite delay random coding, the maximum achievable multiplexing gain can get arbitrarily small

\[
d_{\text{ND-RAF}}(r) \geq (n-1)(1-k r)^+ + (1-r), \quad k \gg 1
\]

thus potentially requiring for asymptotically high SNR, in order for cooperation to apply.

A better lower bound on the optimal D-MG performance of the network is obtained when the same PEP result of [1], is applied on codes that are approximately universal and thus D-MG optimal over any channel. In the same Appendix II-C we show that given such codes, expanding the existing PEP bound towards D-MG, guarantees a lower-bound on the network’s D-MG performance of:

\[
d_{\text{ND-RAF}}(r) \geq (n-1) \left(1 - \frac{4n - 1}{n-1} r \right)^+ + (1-r).
\]

This implies a maximum diversity \( d(0) = n \) and cooperation after

\[
\text{SNR}_{\min} \geq 2^{\frac{n}{n-1} R}.
\]

3) Dynamic amplify-and-forward and dynamic receive-and-forward: In this D-AAF strategy, first introduced in [5], the original source \( S \) transmits at each time instance and each intermediate relay takes turns in transmitting an amplified version of a previously received signal, and do so only at even time indexes. Again in [5], it is concluded that for \( n+1 \) users, given infinite time duration, the optimal tradeoff is given as

\[
d_{\text{D-AAF}}(r) = (n-1)(1-2r)^+ + (1-r).
\]

From the analysis in [5], we can see that the same optimal D-MG result holds when the intermediate relays do not use channel knowledge.

Having briefly introduced the cooperative strategies, we conclude the introduction with a quick exposition of the basic tools to be used in the network analysis and encoding.

D. Preliminaries on the D-MG Tradeoff and approximate universality

Let an \( n \times T \) space-time code \( \mathcal{X} \) operate at rate \( R_x = \frac{1}{T} \log_2(|\mathcal{X}|) \) bpcu, and let \( r_x \) be the multiplexing gain (normalized rate) given by

\[
r_x = \frac{R_x}{\log_2(\text{SNR})},
\]

corresponding to

\[
|\mathcal{X}| = \text{SNR}^{nT}.
\]

For large SNR, the capacity over an \( n \times n_r \) Rayleigh fading channel is given by \( C_x \approx \min\{n,n_r\} \log_2(\text{SNR}) \), implying a maximum achievable multiplexing gain of \( r_{x,\text{max}} = \min\{n,n_r\} \). The diversity gain corresponding to a given \( r_x \), is defined by

\[
d(r_x) = -\lim_{\text{SNR} \to \infty} \frac{\log(P_e)}{\log(\text{SNR})},
\]

where \( P_e \) denotes the probability of codeword error. In a recent landmark paper, Zheng and Tse [6] showed that there exists a fundamental tradeoff between diversity and multiplexing gain, referred to as the diversity-multiplexing gain (D-MG) tradeoff. For a fixed integer multiplexing gain \( r_x \), and \( T \geq n + n_r - 1 \), the maximum achievable diversity gain \( d(r_x) \) is shown to be

\[
d(r_x) = (n-r_x)(n_r-r_x).
\]
The function for non-integral values is obtained through straight-line interpolation. For $T < n + n_r - 1$ only bounds on the maximum possible $d(r_X)$ were available [6]. It was also shown in [6] that there exist random Gaussian codes with $T \geq n + n_r - 1$ that achieve the above optimal tradeoff. For such optimal codes, the probability of error coincides with the probability of outage

$$P(\text{outage}) = P(\text{error-optimal})$$

The Zheng-Tse result sparked considerable interest in meeting this new D-MG frontier and explicitly providing D-MG optimal schemes.

The problem of constructing explicit D-MG optimal ST codes over the Rayleigh-fading channel for any pair $(n, n_r)$ was settled in [9] where it was shown that cyclic-division-algebra-based space-time codes having a certain non-vanishing determinant (NVD) property are optimal with respect to the D-MG tradeoff of the Rayleigh-fading channel. The same authors used this result to establish the D-MG optimality of constructions of space-time codes found in Belfiore et al. [20] and Kiran and Rajan [22]. These prior constructions were restrictive in terms of the values of $n$ that can be accommodated. In [9], [11], a general construction of D-MG optimal codes was provided that was valid for all $n, n_r$ and all $T \geq n$.

Furthermore, as is shown in [7] (see also [8]), the above codes are approximately universal [7] and thus D-MG optimal over all slow-fading channels, independent of the channel's statistics. As shown in the proof of Theorem 4.1 of [7] (Appendix A.2), due to the bound on their smallest eigenvalue, such codes satisfy the extra property that

$$P(\text{error} | \text{no outage}) \leq e^{-\text{SNR}^+} \leq \text{SNR}^{-\infty} \leq 0. \quad (10)$$

In a slight abuse of notation, we will use the term ‘approximately universal given statistical symmetry’ to denote a D-MG optimal scheme that also satisfies (10), and does so for all channel statistics with identically distributed path fading.

This concludes the introduction and we can now proceed with analysis and encoding for the first cooperative strategy.

II. OPTIMALITY IN THE NON-DYNAMIC SELECTION-DECODE-AND-FORWARD STRATEGY

In regards to the ND-SDAF strategy, we will here improve the existing optimal performance bound in [3], generalize to a new optimality expression that holds for a larger family of channel statistics and explicitly construct the first ever coding scheme that guarantees optimality in finite and minimum time delay, finite decoding complexity and for any number of users. We finally bound the performance for a variety of network topologies and explicitly provide encoding schemes that guarantee these bounds for any set of statistics.

a) Cooperation strategy and general coding requirements: We begin by noting that in this scheme, cooperation does not provide gains unless the probability of erroneous decoding at the intermediate relays is much smaller than the probability that some relay does not select to cooperate. Necessary for D-MG optimality is a decoding strategy that relates to outage as in [3], in which case decoding at the node (during the first stage) is required to provide for

$$P(\text{error} | R_i | \text{no outage}) \ll P(\text{error} | D \text{ with cooperation})$$

which in terms of approximate universality, translates to

$$P(\text{error} | R_i | \text{no outage}) \doteq \text{SNR}^{-\infty} \leq 0. \quad (11)$$

D-MG optimal codes that guarantee that

$$P(\text{error}) \doteq P(\text{outage}) \leq \text{SNR}^{-\infty}. \quad (12)$$

can be found in the family of random Gaussian codes, which also maximize the mutual information of both the first and second stages of the transmission, and hence optimize the overall D-MG performance of an infinite duration network. The lack of eigenvalue bounds in the finite length version of these random codes does not allow for (11) to hold, and instead as shown in [6], finite length random Gaussian codes satisfy,

$$P(\text{error} | \text{no outage}) \doteq P(\text{outage}) \geq \text{SNR}^{-\infty}. \quad (13)$$

Consequently, the use of random Gaussian codes in the first stage of the network transmission (source-to-relay) immediately imposes a requirement for infinite time duration [3], [4], [5].

A. Explicit D-MG optimal encoding for the statistically symmetric ND-SDAF network

To achieve optimality, we jointly treat the first and second stage encoding methods by first constructing a $1 \times n$ approximately universal code over SISO channels and then an $n \times n$ approximately universal code over MISO channels, which has the extra two properties that it maintains its optimality even if it is truncated and that it has the optimal number of common entries with the first stage SISO code.

b) First stage transmission and the horizontally-restricted, approximately universal, perfect code: For a relay-network where all the nodes have one transmit/receive antenna operating in half-duplex over a statistically symmetric channel, the proposed coding scheme asks for the source $S$ to sequentially transmit, during time $t = 1, 2, \ldots, n$, the $n$-length vector $\bar{k} = \theta \bar{z}$ where $\bar{z}$ comes from the $1 \times n$ horizontally-restricted perfect code $X_n$,

$$X_n = \{z \in \mathbb{Z}_{\text{odd}}^{(1)} z^{(2)} \ldots z^{(n)}\} \quad (13)$$

where the $f_i$ are from a discrete information constellation $A$ such as QAM or HEX, that scales with SNR as $|A| = \text{SNR}^+$. and where $M_n$ is the unitary lattice generator matrix for perfect codes [21], [8], [10]. Finally $\theta$ is the normalization factor such that $E[|\theta z^{(1)}|^2] = \text{SNR}$. The approximate universality of $X_n$ over the first stage SISO channel and the subsequent satisfaction of (11), are established by observing that the code carries the same information and has the same non-vanishing product distance ($X_n$ consists entirely of one of the layers of...
the CDA-perfect codes) as the simple QAM scheme, which was shown in [7, Section 3] to be approximately universal over all SISO channels.

c) Second stage transmission and the residual approximate universality of CDA codes: By \( t = n \), due to the approximate universality of \( X_b \), each intermediate relay \( R_i \in D(S) \) has correctly decoded \( k \) and will participate in the second stage of cooperation which will take place if and only if \( r \leq \frac{1}{2} \) (this choice of \( r \) will become clearer later on), in which case the network encoding scheme asks from each \( R_i \in D(S) \), \( i = 2, \cdots, n \) to transmit \( k^{(i)} \) (the \( i \)th element of \( k \)) at time \( t = n + i - 1 \), thus allowing the decoder of \( D \) to see a possibly truncated version of the diagonal restricted perfect code, \( X_d \), given as:

\[
X_d = \{ \text{diag}(\mathbf{z}) = \text{diag}(f \cdot M_n), \forall f \in \mathcal{A}^n \}.
\]  

(14)

As with \( X_b \), the corresponding information alphabet \( \mathcal{A} \) is discrete and \( M_n \), as in (13), represents the orthogonal matrix that generates a lattice in the maximal field \( \mathbb{L} \) of the division algebra of the perfect codes (see equation (59)). The above choices make \( X_d \) approximately universal over any i.i.d. MISO channel, since the product-distance of any difference of diagonals in the matrices of \( X_d \),

\[
\prod_{j=0}^{n-1} \delta z(j) = \prod_{j=0}^{n-1} \sigma (\sum_{k=0}^{n-1} f_k \beta_k) = N_{L/|z_i|} (\sum_{k=0}^{n-1} f_k \beta_k) \in \mathbb{Z}[i]
\]

is an algebraic norm and thus a Gaussian integer, meaning that

\[
\prod_{j=0}^{n-1} \delta z(j) \geq 1,
\]

which together with the fact that \( |A| = \text{SNR}^r \), satisfy all the related conditions in [7, Theorem 4.1].

The above approximate universality only relates to the event where all intermediate relays \( R_i \) are in \( D(S) \). As we have seen though, it is the case that some relays might be unable to decode, in which case the equivalent-space-time code will not be the complete \( n \times n X_d \) but instead will be missing some rows and will be of dimension \((|D(S)| + 1) \times n \). For this, we now move to establish another property, necessary for the network’s approximate universality. We will name this property as ‘residual approximate universality’.

**Definition 1:** Residual approximate universality is the property of an \( n \times n \) approximately universal space-time code, which guarantees that after removing an arbitrary number, say \( k \), of rows from each of the code matrices, it is then the case that the resulting \( (n-k) \times n \) truncated code is still approximately universal over any \( (n-k) \times n_r \) channel, for all \( n_r \).

The proof that CDA-perfect codes are residually D-MG optimal (and hence residually approximately universal) can be found in the proof of [11, Theorem 4] and is based on the fact that, given any truncated \( (n-k) \times n \) codematrix \( X \), the Hermitian nature of \( XX^H \), guarantees that the magnitude of each of its \( n-k \) ordered eigenvalues, is each lower bounded by the squared magnitude of the corresponding \( n-k \) smallest eigenvalues of the original pre-truncated matrix.

A similar argument, gives that the above diagonal restricted perfect code is also residually approximately universal, over any MISO channel with i.i.d. fading. Having constructed a proper pair of residually approximately universal codes for the two stages, we can state that:

**Theorem 1:** Given statistically symmetric Rayleigh fading, the optimal half-duplex constrained D-MG performance of the non-dynamic selection-decode-and-forward wireless network with \( n+1 \) single-antenna antennas, is given by:

\[
d_{\text{ND-SDAF} \text{opt}}(r) = (n-1)(1-2r)^+ + (1-r).
\]

(15)

This performance can be achieved by utilizing a \( 1 \times n \) horizontally-restricted perfect code during the first stage, having the intermediate relays decode if and only if they are not in outage with respect to the original transmitter, and finally having them re-transmit utilizing, together with the source, a distributed \( n \times n \) diagonal-restricted perfect code.

**Proof:** See Appendix I-A.

**Fig. 4.** Optimal D-MG performance for the single-intermediate relay, ND-SDAF scheme utilizing perfect codes. Given a rate \( R \) (bps/Hz), cooperation applies for \( \text{SNR} \geq 2^{2R} \).

For results relating to different channel statistics and different channel topologies, we refer the reader to Appendix I-B.

**III. OPTIMALITY IN THE NON-DYNAMIC RECEIVE-AND-FORWARD STRATEGY**

In this network, we initially consider the case where each node has a single receive-transmit antenna operating in half-duplex. The exact network introduced in Section I-C.2 deviates from the relay setup in [1, 2], as it requires one less intermediate relay since the destination \( D \) does not discard the direct transmission of \( S \). Unlike in the case of the ND-RAF strategy, the lack of information extraction at the relays, introduces the need that the related space-time code be completely distributable. Code distribution at the relays is discussed in Appendix IV-A.

Interest in this ‘linear-processing’ (ND-RAF) relay network is generated by the scheme’s two main advantages. The first advantage is that the intermediate relays do not require channel information and the second advantage is that the intermediate relays do not perform time and energy consuming decoding. Potential power violation issues arising from utilizing the above properties are addressed in Sections V and VI.
We proceed to further describe the relay network model and to analyze the equivalent second stage ‘two-product’ channel. The main result will be presented in Theorem 2.

A. Distributed space-time codes and the equivalent channel

1) Relay scheme: During the first stage, the source’s single antenna sequentially transmits a vector

\[ \mathbf{z} = \theta \mathbf{z}(1) \mathbf{z}(2) \ldots \mathbf{z}(n) \]

of \( n \) signals, where \( \mathbf{z} \) is a codeword from a \( 1 \times n \) coding scheme. Each intermediate relay \( R_t, i = 2, 3, \ldots, n \) then receives the \( n \)-length vector

\[ \mathbf{x}_i = \theta \mathbf{g}_i \mathbf{z} + \mathbf{w}_i, \]

independently performs linear-processing (time-averaging based on space-time codes) on \( \mathbf{x}_i \) and transmits

\[ \mathbf{z}_i = \mathbf{r}_i \mathbf{A}_i \]

where each \( n \times n \) matrix \( \mathbf{A}_i \) is unitary.

The signal at the receiver of the final destination \( D \) is then of the form

\[ \mathbf{y} = \sum_{i=1}^{n} h_i \mathbf{z}_i + \mathbf{w} \]

Clarifications regarding \( h_1 \) can be found in Appendix II-A.

2) Equivalent channel model: The authors in [1], [2] utilize a finite-length, random, distributed space-time code by having the linear-processing at the intermediate relays be performed by randomly chosen unitary linear-dispersion matrices [18], where each such matrix uniquely defines a codematrix row. The rationale behind this becomes clearer after rewriting [19] as:

\[ \mathbf{y} = \sum_{i=1}^{n} h_i (\theta \mathbf{g}_i \mathbf{z} + \mathbf{w}_i) \mathbf{A}_i + \mathbf{w} \]

\[ = \theta \sum_{i=1}^{n} h_i \mathbf{g}_i \mathbf{z} \mathbf{A}_i + \sum_{i=1}^{n} h_i \mathbf{w}_i \mathbf{A}_i + \mathbf{w} \]

which easily transforms to the familiar point-to-point channel model

\[ \mathbf{y} = \theta H \mathbf{X} + \mathbf{W} \]

where

\[ H = \begin{bmatrix} g_1 h_1 & g_2 h_2 & \cdots & g_n h_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} \]

\[ \mathbf{W} = \sum_{i=1}^{n} h_i \mathbf{w}_i \mathbf{A}_i + \mathbf{w} \]

Based on the results in Appendix II-A and for the sake of simplicity, we name the above channel as the ‘two-product channel’ by which we will exactly mean:

Definition 2: The ‘two-product channel’ is the \( n \times 1 \) MISO channel, modelled as in [21], with the channel matrix \( H \) as in [21] representing fading coefficients which are products of two i.i.d. \( \mathbb{C} \mathbb{N}(0, \text{SNR}^0) \) random variables. Furthermore, the effective additive noise consists of spatially and temporally white \( \mathbb{C} \mathbb{N}(0, 1) \) random variables.

B. Outage probability of the two-product channel

Due to the absence of decoding at the intermediate relays, the rate-reliability limitations are mainly due to the relays-to-destination stage of the network transmission. As a result, the optimal D-MG tradeoff of the network is a function of the half-duplex effect and of the optimal D-MG tradeoff of the second stage two-product channel.

The general method for establishing the D-MG limits of the two-product channel, relates to utilizing the existing eigenvalue bounds of the approximately universal CDA codes, finding the codes’ ‘error contribution region’ (channel region in which perfect codes decode erroneously) and equating this region to the outage region of the channel. Establishing the eigenvalue statistics of the two-product channel then provides for the volume of the outage region and for the optimality limits. With respect to these limits, we note that:

Remark 1: Since knowledge of \( g_i \) at relay \( R_i \) essentially reduces the two-product channel to the Rayleigh fading channel, it is the case that the two-product channel’s optimal D-MG tradeoff cannot be any better than that of the Rayleigh fading channel.

We proceed with the probability of outage for the two-product channel.

Proposition 1: The optimal diversity-multiplexing gain tradeoff of the \( n \times 1 \) two-product channel is given by

\[ d_{\text{eq}}(r_\lambda) = n(1 - r_\lambda) \]

Proof: From Lemma 5 (Appendix II-B), we see that for

\[ \lambda_n = HH^\dagger = \sum_{i=1}^{n} ||h_i||^2 ||g_i||^2 := \text{SNR}^{-\mu} \]

being the only non-zero eigenvalue of the two-product channel, then the corresponding outage region, which at high SNR equals the error contribution region of the perfect codes, is given by:

\[ B = \{ \mu \geq 1 - r_\lambda \}. \]

By definition of approximate universality, the probability of error outside \( B \) can be considered to be arbitrarily small, allowing us to limit our attention only to the channels with \( \lambda_n \leq \text{SNR}^0 \). As a result, knowledge of the pdf of the channel outside this region is unnecessary, with the only condition that \( f_{\lambda_n}(\lambda_n) < \infty, \forall \lambda_n > \text{SNR}^0 \). We will see later in Appendix II-F that this condition is met. Immediately from Lemma 5 in Appendix II-B, we see that for \( h_i, g_i, i = 1, 2, \ldots, n \) being \( \mathbb{C} \mathbb{N}(0, \text{SNR}^0) \) random variables, then the probability density function of \( \lambda_n \) is upper bounded as

\[ f_\lambda(\lambda) \leq \lambda^{n-1} \]

and for \( \lambda_n = \text{SNR}^{-\mu} \)

\[ f_\mu(\mu) \leq \text{SNR}^{-\mu n}. \]

We here note that in [6], [9], the pdf of \( \mu \) for the \( n \times 1 \) Rayleigh fading channel, is given by

\[ f_\mu(\mu) \leq \text{SNR}^{-\mu n} e^{\text{SNR}^{-\mu}} \]

\( \mu \) is the only non-zero eigenvalue of the two-product channel, which at high SNR equals the error contribution region of the perfect codes, is given by:

\[ B = \{ \mu \geq 1 - r_\chi \}. \]

This is immediate by first considering that for the Rayleigh fading case \( r_\chi, \max = 1 \), and then by considering Remark 1.
which reduces to the two-product pdf expression of \( f_\mu(\mu) \leq \text{SNR}^{-\mu n} \) for all \( \mu > 1 - r_\chi > 0 \), that is for all \( \mu \in B \). We then proceed as in [9], where \( B \) and \( f_\mu(\mu) \) completely defined the probability of codeword error as

\[
P_e \leq \int_{\mu \in B} f_\mu(\mu)d\mu
\]

since the double exponential nature of the probability of error, given a channel realization, acts as a binary indicator function in and out of \( B = \{ \mu > 1 - r_\chi \} \). As a result

\[
P_e \leq \int_{\mu \in B} \text{SNR}^{-\mu n}d\mu
\]

and using Varadhan’s Lemma [29] or the dominant term approach of Appendix II of [9], we get that

\[
P_e \leq \max_{\mu \in B}\{ \text{SNR}^{-\mu n} \}.
\]

(26)

For a given multiplexing gain, the maximizing eigenvalue is then \( \mu = 1 - r_\chi \), which implies that

\[
P_e \leq \text{SNR}^{-n(1-r_\chi)}
\]

This lower bounds the optimal D-MG tradeoff in the \( n \times 1 \), equivalent two-product channel as

\[
d_{eq}(r_\chi) \geq n(1-r_\chi).
\]

Remark 1 completes the proof.

We are now in position to give an explicit description of a D-MG optimal coding method for the ND-RAF network where the statistically symmetric Rayleigh fading channel is known only at the receiver of the final destination \( D \), and where each node has a single transmit-receive antenna operating under the half-duplex constraint. The rate and the power constraint are known at all nodes.

**Theorem 2:** (D-MG optimality in the ND-AAF and ND-RAF): The optimal half-duplex constrained D-MG performance of the ND-RAF network with \( n + 1 \) single-antenna users, is given by:

\[
d_{\text{ND-RAFopt}}(r) = (n-1)(1-2r)^+ + (1-r).
\]

(27)

The performance is achieved by utilizing an approximately universal \( 1 \times n \) horizontally-restricted perfect code during the first stage \( t = 1, 2, \ldots, n \). If \( r \geq \frac{1}{2} \) then the relays do not forward the message and the source begins with the new message at \( t = n+1 \). If \( r < \frac{1}{2} \) then each intermediate relay \( R_i \), \( i = 2, \ldots, n \) forwards at time \( t = i+n-1 \) what it received at time \( t = i \). Decoding at the final destination uses the received signals at either time slots \( t = 1, 2, \ldots, n \) \((r \geq \frac{1}{2})\), or at time slots \( t = 1, n, n+1, \ldots, 2n-1 \) \((r < \frac{1}{2})\).

**Proof:** For \( r > \frac{1}{2} \), the result is immediate by observing that the equivalent channel is a SISO channel and the equivalent code is the approximately universal horizontally-restricted perfect code [13]. For \( r \leq \frac{1}{2} \), the equivalent channel is the two-product channel, and the equivalent space-time code is the \( n \times n \) diagonal-restricted perfect code [14] that is D-MG optimal over all statistically symmetric channels. Consequently the proof is immediate through Proposition 1 and from the fact that transmission takes place during \( 2n - 1 \) time slots, shown in the proof of Theorem 1 to be the minimum allowed.

We now proceed to provide the general coding requirements for optimality in the ND-AAF and ND-RAF schemes, and to give bounds on the performance of networks that utilize some other existing coding methods.

**C. General coding optimality conditions in the ND-RAF relay network**

A closer look at the structure of standard perfect codes carrying \( n^2 \) information elements, reveals that approximate universality is a necessary but not a sufficient condition for network optimality, and that these standard-perfect codes do not provide for ND-RAF optimality as they fail to address the extra restrictions introduced by the half-duplex constraint. These extra constraints, exposed here in order to provide guidelines for future encoding methods, are a direct consequence of the fact that one cannot extract information at the intermediate relays, and can manipulate the received signals only with linear transformations.

1. **Condition 1:** The distributed code must have at least one row that is the ordered subset of the vector transmitted by the source: This will allow for the destination’s decoder to utilize the signal of \( S \).

2. **Condition 2:** The \( n \times T \) equivalent code should map exactly \( T \) information symbols from a discrete alphabet: Specific to the minimum delay case, we see in Lemma 7 of Appendix II-H that an \( n \times T \) space-time code, carrying \( mT \) information elements from a discrete constellation and operating (in the point-to-point sense) at multiplexing gain \( \frac{m}{m+1} \). On the other end of the spectrum, Proposition 8 tells us that the same code operating in the two-product channel with \( n \)-transmit antennas \((m \leq 1)\), will achieve maximum multiplexing gain of \( r_\chi,\text{max} \leq m \). Consequently, optimal performance requires \( m = 1 \).

Standard perfect codes, with \( m = n \), fail this condition and provide for

\[
d_{\text{network-full}}(r) = (n-1)\left(1 - \frac{n^2 + n - 1}{n-1}r\right)^+ + (1-r)
\]
guaranteeing for commencement of cooperation, given some rate $R$, only after
\[
\text{SNR}_{\text{coop-full}} \gtrsim \frac{2^{2R}}{mnR}
\]

making it so that each new user entry renders the network less cooperative.

c) Condition 3: Code does not require complex-conjucacy: The fact that complex-conjucacy is not a matrix operation, forces the source to send twice as many real information elements, thus doubling the duration of the first stage.

For the single intermediate relay case, the Alamouti code, which can be shown to be optimal over the two-product channel, only provides for
\[
d_{\text{network-alam}}(r) = (1 - 5r)^* + (1 - r)
\]
guaranteeing for commencement of cooperation only after
\[
\text{SNR}_{\text{coop-alam}} \gtrsim 2^{2R}
\]
in contrast to the optimal case, given by the diagonal-restriction perfect code of $\text{SNR}_{\text{coop-opt}} \gtrsim 2^{2R}$.

For high values of $n$, orthogonal designs are expected to carry on the average $m \approx \frac{1}{2}$ information symbols per channel use and be of dimension $n \times n^2$. Analysis similar as that presented in the proof of Lemma 2 in Appendix II-D (also see [23]), gives that the D-MG performance in the second stage is given by $d(r) = n(1 - \frac{1}{m}) \approx n(1 - 2r)$. The first stage duration needs to be of length $2n^2m \approx n^2$ since conjugation cannot be described as a matrix transformation, providing for
\[
d_{\text{network-ortho}}(r) \approx (n - 1)(1 - 7r)^* + (1 - r),
\]
guaranteeing for commencement of cooperation only after
\[
\text{SNR}_{\text{coop-ortho}} \gtrsim 2^{2R}.
\]

D. Coding method for a network with a large number of users - the integral restriction perfect code

We here present the ‘integral-restriction perfect code’ which manages to exhibit excellent performance especially as the number of users increases. Given a discrete information set $\mathcal{A}$, the code is given by:
\[
\mathcal{X}_r = \{X = \sum_{k=0}^{n-1} f_k \Gamma^k, \forall f_k \in \mathcal{A}\}
\]

where $\Gamma$ from [60] provides the linear-dispersion matrices $A_i = \Gamma^{m-1}$, $i = 1, \cdots, n$. In addition to satisfying the necessary conditions of:
1) raw data in one row
2) one discrete information symbol per channel use
3) no complex conjugacy

it also manages to have:

4) Constellation complexity remains constant as the number of users increases: Due to the special nature of $\Gamma$ (see [8], [10]) and after a small modification, the transmitted signals essentially belong in a normalized QAM-HEX signalling set, independent of the number of users.

5) Fast encoding at the intermediate relays: Due to the sparse and discrete nature of the $A_i$’s, the scheme only requires one multiplication with a small Gaussian integer, per channel use.

6) Network topology translates to reduction of the sphere-decoding complexity at the destination: The existence of only one receive antenna at the destination translates into a sphere-decoding complexity reduction at the final destination receiver, from $O(n^2)$ to $O(n)$.

7) Allows for optimal rate, ease of construction and minimum delay, for any $n$.

E. Communication with a base-station: reducing the half-duplex effect

We will see that an increase in the number of antennas at the destination (base-station) does not only increase the diversity gain, given cooperation, but it also reduces the SNR required for cooperation to commence. This is done at the cost of extra decoding complexity at the final destination but bares no cost for the relays.

In more detail, we recall a practical relay-network scenario given in [4] that talks of several relays with one transmit/receive antenna, cooperating in their task to communicate with a single base-station. It is logical to assume that the centrality of such a base-station will allow it to utilize multiple receive antennas. This can correspond to a wireless telephony setup where each mobile user utilizes the surrounding users to increase the reliability of the transmission to the base-station.

It should be noted that the fact that the relays can only perform linear processing, prohibits having multiple antennas at the source since one cannot linearly process matrices in a meaningful way due to the additive nature of the received signal at the intermediate relays. Furthermore the intermediate relays can only have one receive-transmit antenna due to the lack of source-to-relay channel information.

The following theorem explains how, given some network rate $R$, having multiple receive antennas at the base station can allow for a relay network with single-antenna intermediate relays to potentially reduce the SNR required for cooperation to apply.

Proposition 2: Consider a base-station centered, ND-RAF relay network of $n$ users cooperating through an $n \times n$ space-time code whose D-MG performance over the equivalent (second-stage) channel is $d_{eq}(r)$. Let the code map on the average $mn$ information elements. If the base station utilizes $m$ receive antennas, $1 < m \leq n$, then cooperation is beneficial for multiplexing gains that are smaller than the multiplexing gain at the intersection of curves $d_{eq}(r(m + 1))$ and $d(r) = m(1 - r)$.

Proof: Since the information constellation satisfies $|\mathcal{A}| \leq \frac{\text{SNR}^m}{m}$ and since the entire network transmission has duration $mn + n$ time slots, then under forced cooperation we have that
\[
r = \frac{R_{\text{network}}}{\log_2(\text{SNR})} = \frac{\frac{mn}{m(n + n)}}{\frac{\log_2(\frac{\text{SNR}^m}{m})}{\log_2(\text{SNR})}} = \frac{r_X}{m + 1}.
\]
Example 1: For $n = 4$ users cooperating to communicate with a base-station, doubling the number of antennas at the base station from $m = 1$ to $m = 2$ will allow for the necessary SNR, for cooperation to apply, to be reduced from $\text{SNR} \geq 2^{2R}$ to $\text{SNR} \geq 2^{2R}$.

The above bound hints towards utilizing $n \times n$ codes that map $mn$ information elements and maintain sufficiently good eigenvalue bounds for increasing spectral efficiency. For this we turn again to the general family of CDA/perfect codes and consider the ‘$m$-layered $n \times n$ perfect code’ $X_m$, given by:

$$X_m = \left\{ X = \sum_{j=0}^{m-1} \Gamma^j \left( \text{diag}(f) \cdot G \right) \right\}$$

which maps the $mn$ information symbols from $\{f_0, f_1, \cdots, f_{m-1}\}$. The codes have not been proven to be approximately universal.

We conclude that one could accept an increase in decoding complexity and equipment, both only at the base station, in order to save power at the intermediate relays and to increase the SNR range in which cooperation is meaningful.

We now move to the dynamic receive-and-forward cooperative diversity strategy.

IV. OPTIMALITY IN THE DYNAMIC RECEIVE-AND-FORWARD STRATEGY

A. Describing the scheme’s model

For completeness, we will here reproduce the description and outage analysis of the dynamic amplify-and-forward network, first presented in [5] for infinite time duration, and we will then proceed to explicitly achieve this optimality in finite time duration.

According to this strategy, the original source $S$ transmits at each time instance, and each intermediate relay takes turns in transmitting an amplified version of a previously received signal. During a $2(n-1)$-length frame, all intermediate relays have contributed, and the frame is repeated infinite times. The set of equations that describe each frame, as given in [5], is

$$y_t = h_1 x_t + w_t, \quad t \text{ - odd}$$

$$y_t = b_i h_i (g_i x_t + v_{i,1}) + h_1 x_t + w_t, \quad i = \frac{n}{2} + 1, t \text{ - even}$$

or equivalently

$$y_t = h_1 x_t + w_t, \quad t \text{ - odd}$$

$$y_t = b_i h_i g_i x_t + h_1 x_t + b_i h_i v_{i,1} + w_t, \quad i = \frac{n}{2} + 1, t \text{ - even}$$

where $b_i$ is the amplification factor at relay $R_i$. For D-RAF, we set $b_i = 1$, which, given a minimum and usually very small SNR, will always result in reduced average power consumption and thus no power violation occurs at the cooperating relays.

Proposition 3: [5] Given infinite time duration, the optimal D-MG performance of the D-AAF and D-RAF schemes with $n - 1$ intermediate relays, is given by:

$$d_{D-RAF} = (n-1)(1-2r)^+ + (1-r)$$

Proof: For completeness, the proof in [5] is briefly reproduced in Appendix III-A.

B. Optimal explicit construction for the dynamic receive-and-forward network

We will here present an explicit construction, based on vectorized perfect codes, which achieves the optimal D-MG performance of the dynamic-receive-and-forward scheme and does so by using $2(n-1)$ frames, each of duration $2(n-1)$.

Theorem 3: We consider the scheme where the source transmits continuously from a set $x_1, x_2, \cdots, x_{4(n-1)^2}$ coming from a column-by-column vectorization of a $2(n-1) \times 2(n-1)$ D-MG optimal CDA-space-time code, and where each intermediate relay $R_i$, $i = 2, \cdots, n$ forwards at time $t = 2(i-1) + 2(n-1)k$, $k = 0, 1, \cdots, 2(n-1) - 1$ what it received at time $t = 1 + 2(n-1)k$. This $n - 1$ intermediate relay D-RAF scheme achieves the optimal D-MG performance, as described in [5], of

$$d_{D-RAF}(r) = (n-1)(1-2r)^+ + (1-r)$$

Proof: See Appendix III-B.

The above construction is optimal for any number $n$ and for any statistical asymmetry. We here note that in [12], a CDA-based D-RAF scheme is proposed that, given statistical symmetry, achieves the optimal tradeoff with delay $T = 4(n-1)$.

1) The emerging need for high-dimensional perfect codes: We have seen that in non-dynamic cooperative schemes, operating below some SNR threshold $\text{SNR}_{coop} = 2^{-T/n}$ will prompt the network to instruct the relays to not cooperate. If one insists on relaying even if $\text{SNR} < \text{SNR}_{coop}$, then through CDA-based D-RAF, the same non-cooperative D-MG performance is expected (at higher decoding and signalling complexity). At low rates and high multiplexing gains, the corresponding SNR is bound to be small, rendering D-MG inaccurate and bringing us closer to outage capacity which relates to low SNR and which accentuates the role of maximization of mutual information. For a fixed rate, as the SNR increases, approximate universality becomes more accurate and will guarantee for the fidelity of the results in Theorems II and III. This high-low SNR duality emphasizes the need for approximately universal codes that are also information lossless, offer Gaussian-like signalling and tend to maximize mutual information, i.e. perfect space-time codes. This dual requirement, the fact that in D-RAF the dimensionality of the corresponding codes grows fast with the number of users, jointly offer a substantial reason of existence for high-dimensional perfect codes [8], [10].

V. COMPARING THE PRESENTED COOPERATIVE DIVERSITY STRATEGIES

A direct consequence of Theorems II and III is that:

Theorem 4: The non-dynamic selection-decode-and-forward, the non-dynamic receive-and-forward, the dynamic amplify-and-forward, the dynamic amplify-and-forward and the dynamic receive-and-forward cooperative diversity strategies, provide for the same D-MG optimal performance. For all the above cooperative strategies, a network of $n + 1$ users, each having knowledge of the rate
and power constraint, and each having a single transmit-receive antenna operating in half-duplex over Rayleigh fading, has a high-SNR probability of outage (optimal D-MG performance) $P_{out} := \text{SNR}^{-d_{opt}(r)}$, where:

$$d_{opt}(r) = (n - 1)(1 - 2r)^+ + (1 - r).$$

Optimality is achieved in minimum delay.

![Fig. 6. D-MG performance of the 3-node network for the above cooperative diversity strategies (ND-SDAF, ND-AAF, ND-RAF, D-AAF, D-RAF).](image)

**VI. SIMULATIONS**

In Figure 7 we present a comparison of the performance of all mentioned cooperative diversity schemes. The ND-AAF scheme, in which the amplification factor is based on the channel and set to equate the power consumption over the different frequencies, constantly performs worse than all other mentioned cooperation schemes of equal dimension.

The plots assume that the users are asked to cooperate at any SNR. For the D-RAF, the amplification factor $b_2$ was set to $b_2 = 1$, with the condition that the transmission over the second frequency (second row) did not violate the power constraint. This is always the case for the SNR range of comparison. The SNR recorded is the SNR representing the total power used, given that $b_2 = 1$. The same is valid for the ND-RAF case. The code used in the D-RAF case was the vectorized $2 \times 2$ perfect code operating over 4-QAM ($M^2 = 4$).

In the $n = 2$ ND-SDAF, the code used was the $2 \times 3$ version of the diagonal restricted code, operating over 16-QAM, and in the $n = 2$ ND-RAF, the code used was the $2 \times 4$ version of the diagonal restricted code, again operating over 16-QAM. We note that this choice is not optimal but was used for reasons of network-rate uniformity. No information was allowed to be wasted at the receiver. In the $n = 8$ ND-RAF, the code used was the $8 \times 8$ integral-restriction perfect code. The decoding strategy of the first stage of the ND-SDAF network was such that the first stage horizontally-restricted code did not limit the error performance of the scheme. This was ensured by allowing the relay to decode only if

$$|g_2|^2 > \frac{\delta M^2 - 1}{\text{SNR}}$$

where $\delta$ is chosen so that the effective rate is not the maximum allowed 8/3 bpcu, but instead $7/3 = 2.33$ bpcu. This small drop in rate allowed for decoding to take place well out of outage, thus guaranteeing that the probability of error at the relay (given a horizontally-restricted perfect code and a given $\delta$) is less than the probability of error at the receiver of the destination, given that cooperation took place. The choice of $\delta$ and of the corresponding effective rates were arbitrary and somewhat heuristic. The following figure gives us an idea of the different choices and their utility. What we can conclude is that if the users have the choice to accept or reject cooperation then the preferred decoding strategy should be the one that allows for the maximum effective rate and intersects the performance of the cooperation scheme before the performance of the cooperation scheme intersects the performance of the non-cooperative case. On the other hand, if the users are forced to cooperate independent of SNR and rate, then the preferred decoding strategy presents a tradeoff between the effective rate and the SNR required for the probability of error at the intermediate relay to become smaller than the probability of error given cooperation.

**VII. CONCLUSION**

In this work we explicitly provided the first ever D-MG optimal encoding schemes for several cooperative diversity strategies for wireless relay networks. These practical perfect-code based schemes are optimal over a broad range of channel
statistics and topologies. Bringing the results together we were able to conclude that the ND-SDAF, the ND-RAF (ND-AAF) and the D-RAF (D-AAF) have exactly the same D-MG optimal tradeoff.

We proceed to recap the findings that relate to the different schemes.

- For statistically symmetric ND-SDAF, ND-AAF and ND-RAF, optimality is provided by combining the horizontally-restricted perfect code over the first network stage, and the diagonally restricted perfect code over the second stage. This optimality holds for any i.i.d. fading, any number of users and with the least possible delay of $T = 1, T = n$ or $T = 2n - 1$.

- Optimality in D-AAF and D-RAF is achieved by letting the source transmit continuously from a set $x_1, x_2, \cdots, x_{4(n-1)^2}$ coming from a column-by-column vectorization of a $2(n-1) \times 2(n-1)$ D-MG optimal CDA space-time code, and where each intermediate relay $R_i$, $i = 2, \cdots, n$ forwards at time $t = 2(i-1) + 2(n-1)k$, $k = 0, 1, \cdots, 2(n-1) - 1$ what it received at time $t = 1 + 2(n-1)k$. Optimality holds for any number of users and any statistical distribution of the fading.

- For D-MG optimality, in a finite time duration ND-SDAF network, the first stage code needs to satisfy $P(\text{err} \mid \text{no outage}) < < P(\text{outage})$, and the second stage code needs to maintain (the corresponding) D-MG optimality even when any number of its rows are deleted.

- It was noted that D-MG optimal random Gaussian codes do not satisfy the first stage condition unless they are of infinite length.

In comparing the ND-RAF, in a statistically symmetric network, with the other schemes we see that it:

- requires no channel knowledge at the intermediate relays (compared to full knowledge for the ND-SDAF)
- has minimum delay of 1 or $2n - 1$ (compared to $T = 4(n-1)$ of the least-delay D-AAF scheme [12])
- has signalling complexity of $n$ (compared to complexity of $2(n-1)$ for the D-AAF scheme)
- has sphere decoding complexity of $n$ (compared to $4(n-1)$ of the best proposed D-AAF scheme).

As a final thought, it is pointed out that in terms of D-MG, what was believed to require complete channel knowledge at the receivers of the intermediate relays, require encoding over infinite time duration and require decoding of infinite complexity, was exactly achieved by newly constructed perfect code variants, given absolutely no channel knowledge, given minimum time duration and small decoding complexity.

### Appendix I

**Results and proofs relating to the ND-SDAF scheme**

**A. Proof of Theorem 1**

Given the approximately universal $X_i$, and the residually approximately universal $X_d$, it is the case that the network’s full-duplex D-MG performance will be optimal over any channel with i.i.d fading, and will be given as,

$$P_{\text{err}} = P_{X_0}(1, 1) \left( \sum_{i=0}^{n-1} P(|D(r_0)| = i)P_{X_0}(i, 1) \right)$$

$$= P_{\text{out}}(1, 1) \left( \sum_{i=0}^{n-1} P(|D(r_0)| = i)P_{\text{out}}(i, 1) \right)$$

for the minimum delay case, where $P_{\text{out}}(k, l)$ represents the probability of outage in the $k \times l$ channel. For the Rayleigh fading MISO channel, we have that

$$P_{\text{out}}(i, 1) = (P_{\text{out}}(1, 1))^i \leq \text{SNR}^{-i(1-r)}$$

resulting in the full-duplex probability of network transmission error, for the statistically symmetric Rayleigh fading case, to be given by

$$P_{\text{err}} = P_{\text{out}}(1, 1) \left( \sum_{i=0}^{n-1} (P_{\text{out}}(1, 1))^{n-1-i}(P_{\text{out}}(1, 1))^i \right)$$

$$= (P_{\text{out}}(1, 1))^n \leq \text{SNR}^{-n(1-r)}.$$

In regards to the half-duplex constrained D-MG performance in the statistically symmetric case, we observe that the overall network transmission duration was $2n - 1$ time slots. The task is now to prove that the duration cannot be any less. For this we first turn to Proposition 3, which tells us that for a maximum (second stage - MISO) multiplexing gain of $r_{\text{max}} = 1$, we need to map $n$ discrete symbols in $X_d$. For this we need at least $n$ time slots in the first stage (SISO) since any smaller duration would violate the $r_{\text{max}} = 1$ limit placed by the first stage SISO channel. The proof is then complete by observing that we need at least one time slot for every row of $X_d$ that does not correspond to direct transmission.

**B. D-MG tradeoff for different fading distributions and different channel topologies**

**a) Optimal D-MG performance for some other fading distributions:** Due to statistical symmetry, it will be the case that, for any channel probability density function (pdf), the network’s D-MG optimality requires the same outage-based decoding strategy as, as well as the joint element property and the residual approximate universality of the utilized space-time codes. For an exact network performance expression, we also need to know the codes’ D-MG performance over the corresponding channels. Deviating from Rayleigh fading, analysis in [7, Prop 5.2] provides the optimal tradeoff curve for the class of $n \times 1$ MISO channels with i.i.d fading coefficients $\{c_i\}_{i=1}^n$, to be

$$d^*(r_x) = \alpha n(1 - r_x), \quad 0 \leq r_x \leq 1$$

where $\alpha := \lim_{t \to 0} \frac{\log \left( \frac{P(|c_i|^2 \leq t)}{\log(t)} \right)}{\log(t)}.$ We conclude that the overall network half-duplex D-MG optimal performance in such a channel is

$$d_{\text{ND-SDAFopt}}(r) = \alpha(n - 1)(1 - 2r)^+ + \alpha(1 - r).$$
1) Encoding schemes for networks with multiple receive and transmit antennas: We consider the special case where each terminal has \( m \) receive-transmit antennas, and where the fading across the different antennas has an arbitrary statistical distribution, with this distribution being the same over any path. This topology brings to the fore the ’horizontally-stacked perfect code’ and the following bound.

Proposition 4: The full-duplex D-MG performance of the ND-SDAF with \( n + 1 \) nodes, each having \( m \) transmit-receive antennas, is given by the integral-point-wise linear plot governed by:

\[
d^* (r_x) = n(m - r_x)^2, \quad r_x,_{\text{max}} = m
\]

and the half-duplex performance is lower bounded by the integral-point-wise linear plot governed by:

\[
d_{\text{ND-SDAFm}} (r) \geq n(m - 2r)^2, \quad 0 \leq r \leq r_{\text{coop}} = \frac{(m - r)^2}{r_{\text{coop}}},
\]

where \( r_{\text{coop}} \) is the intersection of integral-point-wise linear plots \( n(m - 2r)^2 \) and \( (m - r)^2 \). The bound is met by utilizing an \( m \times m(n - 1) \) horizontally-stacked perfect code (horizontal stacking of \( n - 1 \) independent matrices from an \( m \times m \) perfect code) during the first stage, and a distributed \( m(n - 1) \times m(n - 1) \) perfect code over the second stage.

Proof: See Appendix [1].

The proof of approximate universality of the horizontally-stacked CDA code is found in [11, Section V.B], and is based on the fact that the Hermitian of \( X \) guarantees that the magnitude of each of the \( m \) ordered eigenvalues of any \( X^{\dagger} \), is lower bounded by the magnitude of the corresponding eigenvalue of any of the \( X^{(k)}, \ k = 1, 2, \cdots, n - 1 \).

2) Approximately universal codes and the most general relay network setup: We here note that, given any intermediate relay decoding strategy, it is the case that the generalization of the above multi-antenna, multi-node, CDA-based scheme, will allow for an overall network full-duplex D-MG performance, that is optimal over all other point-to-point encoding schemes, even for the most general case of having

- an arbitrary number of receive antennas varying over each terminal
- an arbitrary number \( n_S \) and \( n_t \), of transmit antennas varying over the different nodes \( S \) and \( \{R_i\}_1^n \)
- arbitrary and possibly different probability distribution functions for the fading across the different antenna pairs
- an arbitrary number of users
- an arbitrary decoding strategy
- an arbitrary network transmission duration.

Specifically, for some \( T \) depending on the network setup, this generalized optimal network encoding scheme asks for an \( n_S \times T \) CDA code to be used for transmission during the first stage, and which will be part of a \((n_S + \sum_{i=2}^n n_i) \times (n_S + \sum_{i=2}^n n_i)\) CDA code that will be distributed across all relays, for transmission during the second stage.

C. Proof of D-MG bound on the multi-antenna ND-SDAF scheme (Proposition 4)

The encoding scheme first asks for the source \( S \) to map \( m^2(n - 1) \) information elements \( \{f_1\}_{i=1}^{m^2(n-1)} \), from a discrete constellation \( A_m \) of cardinality \( |A_m| = \text{SNR}^{-\theta} \), into \( n - 1 \) codematrixes \( X^{(k)} \times_{k=1}^{n-1} \), each from an approximately universal CDA space-time code \( X_m \) of cardinality \( |X_m| = \text{SNR}^{-\theta} \). Having done so, \( S \) sequentially retransmits \( \theta X^{(1)} \times_{k=2}^{n-1} X^{(n-1)} \), essentially sending a \( m \times m(n - 1) \) codematrix \( X = [X^{(1)} \times_{k=2}^{(n-1)} X^{(n-1)}] \) from the horizontally-stacked CDA code whose approximate universality was proved in [11, Section V.B]. The resulting guarantee that \( P(\epsilon) \) (no outage) \( \approx \text{SNR}^{-\infty} \), makes it such that, by \( t = (n - 1)m \), all intermediate relays in \( D(S) \) have correctly decoded \( X \) and gained full knowledge of \( \{f_1\}_{i=1}^{m^2(n-1)} \). Consequently, each relay in \( D(S) \) now re-maps the correct information into \( m(n - 1) \) elements from a discrete constellation \( A_{m(n-1)} \) of cardinality \( |A_{m(n-1)}| = \text{SNR}^{-\infty} \), and eventually re-maps these new information elements into the codematrix \( X_{m(n-1)} \) of \( (m(n - 1) \times m(n - 1) \) CDA code.

In the second stage, the \( k^{th} \) transmit antenna \( k = 1, \cdots, m \) of intermediate relay \( R_j \in D(S) \), transmits the \( ((j - 1)m + k)^{th} \) row of codematrix \( X_{m(n-1)} \). As a result, the participating relays essentially construct a \( \{D(S)\}^{+1} m \times mn \) punctured version of the \( mn \times mn \) CDA code. Due to the residual approximate universality of the CDA codes, we have that this \( D(S) m \times mn \) punctured code is approximately universal over any \( \{D(S)\} m \times nr \) channel, for all \( nr \). Furthermore we know that the optimal decoding strategy will ask for the relays to decode if and only if the corresponding \( mxm \) channel is not in outage. This, combined with the fact that the codes, utilized on both stages, have the required approximate universality and residual D-MG optimality property, allows for the proposed encoding scheme to provide for the optimal full-duplex D-MG performance over the network.

Narrowing to the specific cases of the Rayleigh fading channel, we first observe that for \( i = 0, 1, \cdots, n - 1 \), it is the case that the expression

\[
P(i)P(\epsilon)(im, m) \equiv (P(\epsilon)(im, m))^{n-1}P(\epsilon)(im, m)
\]

is maximized for \( i = 0 \). Consequently, using the dominant summand approach as in [6], [9], we conclude that the optimal full-duplex D-MG performance over the network, is given by

\[
\frac{P_{\text{err}}}{\text{SNR}^{-\infty}} \geq P_{\text{out}}(m, m) \left( \sum_{i=0}^{n-1} P(\epsilon)(im, m) \right)
\]

The fact that \( 2m(n - 1) \) time slots are required for the network to complete the transmission of the \( m^2(n - 1) \) information symbols, from a constellation \( A_m \) of cardinality \( |A_m| = \text{SNR}^{-\theta} \), concludes the proof.
Appendix II
Proofs relating to the ND-RAF scheme

A. The equivalent second stage ‘two-product’ channel

We first note that we can equate having the first linear dispersion matrix $A_1$ be equal to the identity matrix $I_n$, to correspond to the direct uncoded transmission between the source and final destination. Decoding with consideration of the source’s signal, results in savings of one intermediate relay. Having set $A_1 = I_n$ translates to considering the source as an intermediate relay, with $g_1 = 1$ known to the receiver and the transmitter. For uniformity we will assume that the transmitter does not know $g_1$. As we proceed, it will become apparent that this does not affect our analysis. Consequently, we will henceforth consider $g_i, h_i, \ i = 1, \ldots, n$ to be i.i.d, zero-mean, complex Gaussian random variables. As in [1], we now consider the channel equations (21) and (22), and observe that the $h_i$’s are known to the receiver, it is the case that the effective additive noise elements $W|h_i$ are spatially and temporally white, zero mean, Gaussian random variables, with variance

$$Var(W|h_i) = (1 + \sum_{i=1}^{n} |h_i|^2)I_n. \tag{30}$$

In Appendix II-B we see that for unitary $A_i$’s, in the SNR scale of interest, the $W|h_i$ can be considered, without loss of generality, to be $\mathbb{C}N(0,1)$ random variables, conditioned on considering the $g_i, h_i$ fading coefficients to be i.i.d. $\mathbb{C}N(0, SNR^0)$ random variables. This is exactly the two-product channel. \hfill \square

B. Proof of the i.i.d nature of fading

As stated in [1], [2], $W|h_i$ are spatially and temporally white $\mathbb{C}N(0, 1 + \sum_{i=1}^{n} |h_i|^2)$ random variables. The probability density function of $h = \sum_{i=1}^{n} |h_i|^2$ is

$$p(h) = \frac{h^{n-1}e^{-h}}{(n-1)!}.$$  

For $\epsilon > 0,$

$$p(h = SNR^0) = \frac{SNR^{(n-1)}e^{-SNR^0}}{(n-1)!},$$

and due to the double exponential term we have that $p(h = SNR^0) = SNR^{-\infty} = 0.$ On the other hand

$$p(h = SNR^{-\epsilon}) = \frac{SNR^{-(n-1)}e^{-SNR^{-\epsilon}}}{(n-1)!},$$

and since $e^{-SNR^{-\epsilon}} (n-1)! \rightarrow 1$ we have that

$$p(h = SNR^{-\epsilon}) = SNR^{-(n-1)}\epsilon^{(n-1)}$$

which implies that $Var(W|h_i) = SNR^0$.

Now we observe that

$$\frac{E[\|H\theta X\|_F^2]}{E[\|W|h_i\|_F^2]} = n_t SNR = \frac{n_t n_r Var(g_1 h_1)E[\|\theta X\|_F^2]}{n_r TVar(W|h_i)}$$

and thus

$$\frac{Var(g_1)Var(h_1)}{Var(W|h_i)}E[\|\theta X\|_F^2] = TSNR.$$  

Finally we note that in the SNR scale of interest, we may interchange the SNR$^0$ and 1 values and have $Var(g_1) = Var(h_1) = SNR^0$ and $Var(W|h_i) = 1$, since the substitution maintains the power-SNR requirements.

C. Existing performance bounds for the ND-RAF (linear-processing scheme) (Proposition 6)

a) The diversity result from [1]: Having established the whiteness of the additive noise, the authors in [1], [2] proceed to prove the following:

Proposition 5: (Variant of the main result in [1]) Over the effective channel described in (21), with the channel fading coefficient matrix as in (22), utilizing full-rank matrices as in (22) and with white additive noise as in (50), it is then the case that in the high SNR regime, the pairwise error probability based diversity is $n$.

In expanding the above PEP bound to consider multiplexing gains other than $r = 0$, we get:

Proposition 6: Given random coding, the existing PEP bound in Proposition 5 translates to a D-MG bound (of the entire network) of

$$d_{ND-RAF}(r) \geq (n-1)(1 - kr)^+ + (1 - r), \quad k >> 1.$$

Given coding based on approximately-universal codes, the same PEP bound translates to a D-MG bound for the entire network of

$$d_{ND-RAF}(r) \geq (n-1) \left( 1 - \frac{4n-1}{n-1} \right)^+ + (1 - r).$$

Proof: It is not difficult to see that in the high-SNR regime, the PEP diversity of the second stage of the network scheme, as given in [1], [2], is a function only of SNR and of the minimum eigenvalue of any difference of any two codematrices (at a given rate), and is upper bounded as:

$$PEP \leq SNR^{-n}(l'_n)^{-n} \tag{31}$$

where $l'_n$ is the smallest eigenvalue of the power normalized version of the code where the codematrices $X'$ are of the form

$$X' = \frac{X}{\sqrt{n}}$$

such that $E[\|X'\|_F^2] = 1$.

For the channel model given as $Y = \theta H X + W$, with $E[\|X\|_F^2] = E^2$ and $\theta$ such that $E[\|\theta X\|_F^2] = SNR$, it was shown in [9] that a non-vanishing determinant $\det(\Delta X \Delta X') \geq SNR^0$, guarantees a minimum partial eigenvalue product of

$$\prod_{i=n-j}^{n} l_i \geq (E^2)^{-(n-j)}$$

and minimum code-eigenvalue of $l_n \geq (E^2)^{-(n-1)}$.

Interestingly, in the normalized case of $E[\|X'\|_F^2] = 1$ with $l'_n = \frac{1}{\sqrt{n}}$, the minimum partial eigenvalue product becomes

$$\prod_{i=n-j}^{n} l'_i \geq (E^2)^{-n} \quad \forall j$$
which accentuates nicely the fact that code performance depends solely on its smallest eigenvalue. For when the $n \times n$ CDA code $\mathcal{X}$ maps $n^2$ information symbols from a discrete constellation $\mathcal{A}$, we have that

$$E^2 = |\mathcal{A}| = |\mathcal{X}| \frac{1}{n} = (2^{n^2}) \frac{1}{n} = (2^{n \log \mathrm{SNR}}) \frac{1}{n} = \mathrm{SNR}^\frac{1}{n}.$$

As a result $\lambda_\mu (\mathrm{SNR}^\frac{1}{n}) = \mathrm{SNR}^{-r}$ which brings us to

$$\text{PEP} \leq \text{SNR}^{nr} \mathrm{SNR}^{-r}.$$

The overall probability of error is then

$$P_e \leq |\mathcal{X}| (\text{PEP}) = \text{SNR}^{nr} \mathrm{SNR}^{-r}$$

which allows for lower-bounding the D-MG tradeoff of the two-product channel as

$$d_{\text{equiv}}(r) \geq n - 2rn$$

implying a maximum diversity $d_{\text{equiv}}(0) = n$ and maximum multiplexing gain of $r_{\text{equiv}} \geq \frac{1}{2}$. Consequently, a temporally disjoint first and second stage result in cooperation applying at the solution of $n(1 - 4r) = 1 - r$. This concludes the proof.

Given approximately universal codes, the PEP based D-MG bound can only guarantee that cooperation will apply for $r < \frac{1}{4n-1}$ which, given some rate $R$, translates to having cooperation only after $\mathrm{SNR} > 2^{\frac{n-1}{4n-1} R}$. Given finite duration random codes, cooperation might require $\mathrm{SNR} > 2^{\frac{n-1}{4n-1} R}, k > 1$.

D. Lemma on the error contribution region of perfect codes

Lemma 5: For $\lambda_n = \mathcal{H}^T = \sum_{i=1}^{n} |h_i|^2 |g_i|^2 := \mathrm{SNR}^{-\mu}$ being the only non-zero eigenvalue of the two-product channel, then the corresponding outage region which at high SNR equals the error contribution region of the perfect codes, is given by:

$$\mathcal{B} = \{ \mu \geq 1 - r \}. \quad (33)$$

Proof: Directly from the mismatched eigenvalue theorem presented in [9], we know that the minimum Euclidean distance between any two codemates after the action of the channel, is lower bounded as

$$d_E^2 = \|\theta \mathcal{H} X\|_2^2 \geq \theta^2 \lambda_n l_n \quad (34)$$

where $l_n$ corresponds to the smallest code eigenvalue of $\mathcal{H} \mathcal{A} \mathcal{D}^\dagger$ for the case of the perfect code $\mathcal{X}$, which maps $n^2$ information symbols from a set $\mathcal{A}$, we know that $\theta^2 = \mathrm{SNR}^{1-\frac{1}{n}}$, $l_n \geq \mathrm{SNR}^{-\frac{1}{n}} (n-1)$. As a result,

$$d_E^2(\mu) \geq \mathrm{SNR}^{1-\frac{1}{n}} \mathrm{SNR}^{-\frac{1}{n} (n-1)} \mathrm{SNR}^{-\mu} := \mathrm{SNR}^{-\mu}. \quad (35)$$

Observing that the pairwise error probability, given a channel, is upper bounded as

$$\text{PEP}(\mu) \leq e^{-d_E^2(\mu)}$$

allows for the entire probability of error, given a channel $\mu$, to be upper bounded as:

$$P(\text{err}|\mu) \leq |\mathcal{X}| \cdot \text{PEP}(\mu) = \mathrm{SNR}^{nr} e^{-\mathrm{SNR}^{-\mu}}.$$  

It is clear that simply because we have a double exponentially decreasing term multiplied with a polynomially increasing term. As a result, we have the error contribution region, being defined as the set $\mathcal{B}$ of eigenvalues $\mu$ such that $P(\text{err}|\mu) \neq 0$. This region requires that $c \leq 0$. Consequently, (35) gives that $c = 1 - r - \mu \leq 0$ which concludes the proof.

E. Lemma on the effective pdf in the two-product channel

Lemma 6: For $h_i, g_i, i = 1, 2, \ldots, n$ being $\mathbb{C}\mathbb{N}(0, \mathrm{SNR}^0)$ random variables, then the probability density function of $\lambda_n = \sum_{i=1}^{n} |h_i|^2 |g_i|^2$ is upper bounded as

$$f_\lambda(\lambda) \leq \lambda^{n-1} \quad (36)$$

and for $\lambda_n = \mathrm{SNR}^{-\mu}$

$$f_\mu(\mu) \leq \mathrm{SNR}^{-\mu} \quad (37)$$

Proof: We first recall that for $x, y$ being two independent random variables and for $z = xy$, we have that $f_{z,x}(z, x) = \frac{1}{|x|} f_{x,y}(z, \frac{z}{x})$ and thus

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{|w|} f_{x,y}(x, \frac{z}{x}) dx$$

Now consider $h_i, g_i$ being two independent identically distributed $\mathbb{C}\mathbb{N}(0, k)$, $k = \mathrm{SNR}^0$, random variables and thus $x = |h_i|^2, y = |g_i|^2$, two i.i.d. exponential random variables with $f_z(x) = e^{-kx}$, $f_y(y) = e^{-ky}$, $f_{x,y}(x, y) = e^{-kx} e^{-ky}$ and $f_{x,y}(x, \frac{y}{x}) = e^{-kx} e^{-\frac{k}{x}}$, $x, y, z \in \mathbb{R}^+$. As a result,

$$f_z(z) = \int_{0}^{\infty} \frac{1}{w} e^{-k w} e^{-k \frac{z}{w}} dw = \int_{0}^{\infty} \frac{1}{w} e^{-k (w+\frac{z}{w})} dw.$$  

From equation 3.471.9 of [26] we see that $f_z(z) = 2 K_0(2k\sqrt{z})$ i.e.

$$f_{|h_i|^2,|g_i|^2}(z) = 2 K_0(2k\sqrt{z})$$

where $K_0(\cdot)$ corresponds to the modified Bessel function of the second kind.

From [27, Section 6.6], we observe that

$$\lim_{x \to 0} \frac{K_0(x)}{-\log(x)} \to 1.$$

Observing that for $x > 0.5$, $K_0(x) \leq \frac{\sqrt{\pi}}{\sqrt{2}} e^{-x}$ is always finite (and decreasing in a double-exponential rate), allows one to interchange $\mathrm{SNR}^0$ and 1. Using the absolute unity, we also note that as $x \to 0$ then

$$\frac{K_0(1-\epsilon)}{-\log(1-\epsilon)} \approx e^{-1}$$

We then choose a guaranteed to exist $\epsilon$ such that $\frac{K_0(1-\epsilon)}{-\log(1-\epsilon)} = k = \mathrm{SNR}^0$. It is easy to see that $1 - \epsilon$ is closer to 1 than to 0. This combined with the monotonically decreasing nature of $K_0(x)$, gives that

$$\int_{0}^{1-\epsilon} K_0(x) dx > \int_{1-\epsilon}^{1} K_0(x) dx.$$
and as a result of the dominant summand effect
\[ \int_{0}^{1} K_{0}(x)dx \approx \int_{0}^{1} K_{0}(x)dx + 0. \]
Consequently, the term \( K_{0}(x)dx \approx \int_{0}^{1} K_{0}(x)dx \) can be substituted by the smaller term \( \int_{1-x}^{1} K_{0}(x)dx \). Combined with the fact that \( K_{0}(1) = K_{0}(SNR) = SNR^{0} \) and that
\[ \frac{1}{\log(x)} > \log(x), \quad \forall x > 2, \]
it is the case that for all \( z \leq 1 \), we get
\[ \int_{0}^{z} K_{0}(\sqrt{z})dz \leq \int_{0}^{z} K_{0}(\sqrt{z})dz + k \int_{0}^{z} \log(\sqrt{z})dz = 0. \]
Recall that if \( \lambda \in B \) then \( \lambda \leq SNR^{0} \). The fact that all the summands of the channel eigenvalue \( \lambda = \sum_{i=1}^{\lambda} |g_i|^2|h_i|^2 \) are positive, suggests that if \( \lambda \in B \) then \( |g_i|^2|h_i|^2 \leq SNR^{0} \). This justifies limiting our attention to the \( |g_i|^2|h_i|^2 \), \( x \leq SNR^{0} \). As a result, for \( h_i, g_i \) being \( \mathcal{C}(N(0, SNR^{0})) \) random variables, then the cumulative distribution function of \( |h_i|^2\log|g_i|^2 \), up to \( SNR^{0} \), is upper bounded as
\[ P(|h_i|^2\log|g_i|^2 \leq z) \leq z(1 - \log(z)) \]
(38)
since
\[ F(z) = - \int_{0}^{z} \log(t)dt \leq \frac{1}{2} t \log(t)dt \]
and since \( 0 \log(0) = 0 \), we get that
\[ P(|h_i|^2\log|g_i|^2 < z) \leq z(1 - \log(z)). \]
(39)
To complete the proof of the main Lemma 6 we use the following proposition:

**Proposition 7:** Consider \( n \) independent identically distributed random variables \( a_1, \cdots, a_n \) with \( a_i > 0 \), \( \forall i \). It is then the case that for \( \lambda = \sum_{i=1}^{n} a_i \) then
\[ P(\lambda < t)^{\leq n}(a_i < t)^{\leq n}. \]

**Proof:** Sketch of proof of Proposition 7 In the \( n = 2 \) case, the \((a_1, a_2)\) pairs such that \( a_1 + a_2 < t \) are the points inside the triangle formed by the positive \( a_1\)-axis, the positive \( a_2\)-axis and the line \( a_2 = t - a_1 = f(a_1) \). As a result, for a certain \( a_2' \), it is the case that \( a_2' + a_1 < t, \forall a_1 < f^{-1}(a_2') \). The corresponding triangle is completely enclosed in the \( t \times t \) square whose lower-left corner is at the \((0,0)\) point. As a result
\[ P(a_1 + a_2 < t) \leq P(a_1 < t)P(a_2 < t) = (P(a_1 < t)^2). \]
For the 3-dimensional case, the triangle is replaced by a pyramid which is enclosed in a 3-dimensional cube with edge of length \( t \), and in the general \( n \)-dimensional case the \( n \)-dimensional pyramid is enclosed in an \( n \)-dimensional hypercube again with edge of length \( t \). This concludes the proof of Proposition 7.

Consequently
\[ F_{\lambda}(\lambda) \leq [\lambda(1 - \log(\lambda))]^{n} \]
Keeping in mind that the range of interest is \( \lambda \leq SNR^{0} \) and that \( F_{\lambda}(SNR^{0}) = SNR^{0} \), then for \( \lambda = SNR^{\delta}, SNR \to \infty, 1 > \delta > 0 \), it is the case that \( (SNR^{\delta}(1 - \log(SNR^{\delta})))^{m} = (SNR^{\delta})^{m}, m \geq 0 \). As a result, the cumulative distribution function (cdf) is expressed as a polynomial with only one term as
\[ F_{\lambda}(\lambda) \leq \lambda^{n} \]
which allows us, after differentiation, to apply the existing upper bound inequality of the cdf to the pdf, and get that
\[ f_{\lambda}(\lambda) \leq \lambda^{n-1}. \]
(40)
For \( \lambda = SNR^{-\mu}, \mu = -\log(\lambda)/\log(SNR) \), then \( f_{\mu}(\mu) = f_{\lambda}(\lambda) \) with
\[ \mu_{\lambda} = -\frac{1}{\log(SNR)} \lambda. \]
As a result,
\[ f_{\mu}(\mu) \leq SNR^{-\mu n} \log(SNR) \]
(41)
As in [6], the fact that
\[ \lim_{SNR \to \infty} \log((\log(SNR))) / \log(SNR) \to 0 \]
indicates that the \( \log(SNR) \) term does not contribute to the SNR exponent of the probability of error, and as a result
\[ f_{\mu}(\mu) \leq SNR^{-\mu n} \]
(42)
which concludes the proof of Lemma 6.

**F. Proof of the antenna limitations of the relay network**

The fact that the relays can only perform linear processing, prohibits multiple antennas at the source since one cannot linearly process matrices in a meaningful way due to the additive nature of the received signal at the intermediate relays. Furthermore the intermediate relays can only have one receive-transmit antenna due to lack of source-to-relay channel information.

**G. Relation of number of mapped symbols to \( r_{max} \)**

**Proposition 8:** Consider a full-diversity \( n \times T, T \geq n \), space-time code \( \mathcal{X} \) that maps, through linear combining, on the average \( mT \) information elements from a discrete constellation \( \mathcal{A} \). Given that the code operates in the Rayleigh fading with \( n\)-transmit and \( r \) receive antennas, with \( n + r \geq m \), or operates in the two-product channel with \( n\)-transmit antennas \((m \leq 1)\), then such a code can achieve a maximum multiplexing gain of \( r_{max} \leq m \).

**Proof:** We have that \( |\mathcal{X}| = 2^{RT} = 2^{rT\log_{2}SNR} = SNR^{T} = |\mathcal{A}|^{m} \) which implies that \( |\mathcal{A}| = SNR^{\pi} \) and since the constellation is discrete \( \mathbb{E}[\|a \in \mathcal{A}|^{2}] = |\mathcal{A}| \). The fact that each element \( X_{i,j} \) of a code matrix is a linear combination, with coefficients independent of SNR, of elements of \( \mathcal{A} \), gives us that \( \mathbb{E}[\|X_{i,j}|^{2}] = |\mathcal{A}| = SNR^{\pi} \). For \( \theta \) such that \( \mathbb{E}[\|\theta X_{i,j}|^{2}] = |\mathcal{A}| = SNR^{\pi} \) we have that \( \theta^{2} = SNR^{1-\pi} \).
Without loss of generality we can assume that there exist two codeword matrices \( X_1, X_2 \in \mathcal{X} \), with \( X_1 \) mapping the information \( mn \)-tuple \( (\alpha_1, 0, 0, \cdots, 0) \), where \( \alpha_1 = \text{SNR}^{1/2} \). As a result, the determinant and trace of the difference matrix \( \Delta X \), is a polynomial of degree less than \( n \) over \( \alpha = a_1 - a_2 = \text{SNR}^0 \), with coefficients independent of SNR, i.e.

\[
\text{det}(\Delta X \Delta X^\dagger) = Tr(\Delta X \Delta X^\dagger) = \text{SNR}^0
\]

and thus with all its eigenvalues \( l_i \equiv \text{SNR}^0 \). The corresponding pairwise error probability \( PEP(X \rightarrow X') \), in the Rayleigh fading channel, then serves as a lower bound to the codeword error probability \( P_e \), i.e.,

\[
P_e \geq PEP(X \rightarrow X') \geq \frac{1}{\prod_{j=1}^{n} [1 + \frac{g_j}{\text{SNR}}]^{n_r}} = \text{SNR}^{-n \cdot n(1 - \frac{1}{m})}
\]

\[
\Rightarrow d(r) \leq n_r n \left(1 - \frac{P}{m}\right)
\]

which proves the claim for the Rayleigh fading case. Remark \( \text{[4]} \) concludes the proof.

**H. Relation between rate reduction due to half-duplex and number of mapped symbols**

**Lemma 7:** Consider a distributed \( n \times n \) space-time code, with one of its linear-dispersion matrices equal to the identity matrix, and with D-MG performance over the two-product channel given by \( d_{eq}(r) \). Furthermore assume that the space-time code carries \( mn \) information elements from a discrete constellation. It is then the case that the D-MG tradeoff of the entire half-duplex constrained linear-processing relay network utilizing this code, is given by

\[
d_{\text{network}}(r) = d_{eq}(r(m + 1)).
\]

**Proof:** We first observe that linear-processing does not allow for the intermediate relays to extract information from received signals since the distributed code \( \mathcal{X} \) can only hold linear combinations of the received symbols. This forces the source to transmit the \( mT \) information symbols one at a time, and the intermediate relays to transmit over \( T \) time slots, without information extraction and re-encoding. This means that the lack of decoding at the intermediate relays does not translate to arbitrarily high rates. It is in fact the case that the rates are limited both by the capacity-outage corresponding to the dimensions of the second stage channel, as well as from the fact that the transmission during the second stage will have to take place over \( \frac{mT}{m} = T \) time slots.

For \( r \) corresponding to the multiplexing gain of the second stage, it is the case that \( |A| = \text{SNR}^{-T} \) and as a result the information constellation satisfies

\[
|A| \leq \text{SNR}^{-T}.
\]

Consequently, in the \( T = n \) case, for \( d(r) \) corresponding to the transmission over the \( n \) time slots of the second-stage, it is then the case that \( d(r(m + 1)) \) corresponds to the entire \( mn + n \) time-slots (first and second stage duration).

**APPENDIX III**

**Proofs relating to the D-RAF scheme**

From \([5]\), we see that due to the independence of the information symbols, the power constraints are satisfied when

\[
b_i \leq \sqrt{\frac{\text{SNR}}{|g_i|^2 \text{SNR} + 1}}.
\]

We denote by \( \phi_i \) and \( \xi_i \) the exponential orders of \( h_i \) and \( g_i \), respectively, i.e.

\[
h_i \equiv \text{SNR}^{-\phi_i}, \quad g_i \equiv \text{SNR}^{-\xi_i}
\]

and as in \([5]\), we observe that

\[
p_{\phi_i}(\phi_i) = \begin{cases} \text{SNR}^{-\phi_i}, & \phi_i < 0 \\ \text{SNR}^{-\phi_i}, & \phi_i \geq 0 \end{cases}
\]

The same holds for \( \xi_i \). As a result, in accordance with \([6], [9], [5]\) and without loss of generality we will limit our attention to \( \xi_i, \phi_i \in \mathbb{R}^+ \). Given that the probability that \( \xi_i, \phi_i \in \mathbb{R}^- \) can be considered to be arbitrarily small, we may restrict the amplification factors to be \( b_i = 1 \).

**A. Proof of Proposition \[7\]**

Beginning with the single intermediate relay case, a frame \( [\mathbf{x}\mathbf{y}] \), is given by

\[
\mathbf{y} = [y_1 \ y_2] = \mathbf{z} + \mathbf{w} = [h_1 x_1 \ h_2 g_2 x_1 + h_1 x_2] + [w_1 \ h_2 v_{2,1} + w_2].
\]

For compactness we will denote SNR by \( \rho \). For

\[
G_2 = \begin{bmatrix} h_1 & 0 \\ h_2 g_2 & h_1 \end{bmatrix}
\]

denoting the equivalent channel defined by the above equations, we have \( \Sigma = \rho G_2 G_2^\dagger \) denoting the covariance matrix of the signal and fading part of the observed vector at the destination, with the expectation taken over the signal-set and the fading coefficients held fixed. Furthermore, \( \Sigma_n \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 + |h_2|^2 \end{bmatrix} \) denotes the covariance matrix of the additive noise part of the observed vector at the destination. Using the fact that the largest eigenvalue \( l_{\text{max}} \) of \( \mathbb{E}[x_i x_j^\ast] \) satisfies \( l_{\text{max}} = \rho \) and directly from \([45, 46]\), the mutual information during a single 2-length frame is given in \([5]\) to be

\[
I(\mathbf{x}; \mathbf{y}|h_1, h_2, g_2) \equiv \log_2 \det(I_2 + \rho G G^\dagger \Sigma_n^{-1}) \geq \max\{2(1 - \phi_1)^+ - \phi_2 - \xi_2\} \log_2 \rho.
\]

We now consider the outage event, that is the event where the corresponding maximum mutual information allowed by a set of fading coefficients is less than the rate of transmission \( R = r \log_2 \rho \). That is

\[
\mathcal{O} = \{ (h_1, h_2, g_2) \in \mathbb{C}^3 \mid I(\mathbf{x}; \mathbf{y}|h_1, h_2, g_2) < RT \}
\]
where $T_f$ is the duration of the frame. Furthermore, due to (46), we are only interested in

$$O^+ = \{ (\phi_1, \phi_2, \xi_2) \in \mathbb{R}^+ \mid \max \{ 2(1-\phi_1)^+, 1-\phi_2-\xi_2 \} < 2r \}.$$  

The probability of outage is given by

$$P_{out}(r) = \rho^{out}(r) = \int_{(\phi_1, \phi_2, \xi_2) \in O} P(\phi_1, \phi_2, \xi_2) d\phi_1 d\phi_2 d\xi_2$$

$$= \sup_{(\phi_1, \phi_2, \xi_2) \in O^+} \rho^{-(\phi_1+\phi_2+\xi_2)}. \quad (50)$$

The last equation comes from the dominant term approach in [28], extensively used in [6],[5],[9], or directly from Varadhan’s lemma [29]. Consequently

$$d_{out}(r) = \max\{ 2(1-\phi_1)^+, 1-\phi_2-\xi_2 \} < 2r \{ \phi_1 + \phi_2 + \xi_2 \} \quad (51)$$

which readily leads to the final expression

$$d_{D-RAF}(r) = (1-r) + (1-2r)^+.$$

Now moving to the multiple intermediate relay case, we again assume Gaussian random coding across the frame, and introduce $\Sigma_s$, $i=2,3,\cdots,n$

$$\Sigma_{s_i} = \rho G_i G_i^\dagger = \rho \begin{bmatrix} |h_1|^2 & h_1 h_{i+1}^* g_{i+1}^* \\ h_{i+1}^* g_{i+1} & |h_{i+1}|^2 + |g_{i+1}|^2 \\ \end{bmatrix}$$

and $\Sigma_{n_i}, i=2,3,\cdots,n$

$$\Sigma_{n_i} = \begin{bmatrix} \sigma_n^2 & 0 \\ 0 & |h_{i+1}|^2 \sigma_n^2 \\ \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 1 + |h_{i+1}|^2 \\ \end{bmatrix}.$$

As a result, the block diagonal covariance matrices are now

$$\Sigma_s = \begin{bmatrix} \Sigma_{s_2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & \Sigma_{s_2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2\times 2} & \cdots & 0_{2\times 2} & \Sigma_{s_n} \\ \end{bmatrix} \quad \Sigma_n = \begin{bmatrix} \Sigma_{n_2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & \Sigma_{n_2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2\times 2} & \cdots & 0_{2\times 2} & \Sigma_{n_n} \\ \end{bmatrix} \quad (52)$$

Proceeding as in the single intermediate relay case, concludes the proof for the probability of outage corresponding to the equivalent channel

$$G = \begin{bmatrix} G_2 & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & G_3 & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2\times 2} & \cdots & 0_{2\times 2} & G_n \\ \end{bmatrix} \quad (53)$$

and modify to reflect noise whitening by

$$\Sigma = \mathbb{E}\left\{ \begin{bmatrix} w_1 \\ h_2 v_{2,1} + w_2 \end{bmatrix} \begin{bmatrix} w_1^* \\ h_2^* v_{2,1}^* + w_2^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ 0 & |h_2|^2 + 1 \end{bmatrix} \quad (56)$$

$$\Sigma^{-1} = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$$

$$= \Sigma^{-1} \begin{bmatrix} h_1 & 0 \\ g_2 h_2 & h_1 \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} + \Sigma^{-1} \begin{bmatrix} w_1 \\ h_2 v_{2,1} + w_2 \end{bmatrix} \begin{bmatrix} w_3 \\ h_2 v_{2,3} + w_4 \end{bmatrix}. \quad (57)$$

For

$$\Sigma^{-1} \begin{bmatrix} w_1 \\ h_2 v_{2,1} + w_2 \end{bmatrix} \begin{bmatrix} w_3 \\ h_2 v_{2,3} + w_4 \end{bmatrix} := \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix}$$

and

$$\Sigma^{-1} \begin{bmatrix} h_1 \\ g_2 h_2 \end{bmatrix} \begin{bmatrix} 0 \\ h_1 \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ 1+|h_2|^2 & 1+|h_2|^2 \end{bmatrix}$$

we also let

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \Sigma^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ A g_2 h_2 & h_1 A \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_3 \\ u_2 \\ u_4 \end{bmatrix}$$

where $A = \frac{1}{1+|h_2|^2}$. We now define

$$H_{eff} = \begin{bmatrix} h_1 & 0 \\ A g_2 h_2 & h_1 A \end{bmatrix}$$

and proceed to find the outage probability by calculating

$$\det(I + \rho H_{eff} H_{eff}^\dagger) = \det \begin{bmatrix} 1 + \rho |h_1|^2 \\ \rho (A g_2 h_2 h_1^*) + 1 + \rho \{|A g_2 h_2|^2 + |h_1 A|^2\} \\ \end{bmatrix}$$

$$= \left( 1 + \rho |h_1|^2 \right) \left( 1 + \rho \{|A g_2 h_2|^2 + |h_1 A|^2\} \right) - \rho^2 |h_1|^2 |A g_2 h_2|^2$$

$$= 1 + \rho |h_1|^2 + A^2 \left( \rho |h_2 g_2|^2 + \rho |h_1|^2 + \rho^2 |h_1|^4 \right).$$

This gives that

$$P_{out}(r) = P_r \left\{ \log [1 + \rho |h_1|^2 + A^2 (\rho |h_2 g_2|^2 + \rho |h_1|^2 + \rho^2 |h_1|^4)] < 2r \log(\rho) \right\}$$

and then for

$$\overline{h}_1 := |h_1|^2, \quad \overline{h}_2 := |h_2|^2, \quad \overline{g}_2 := |g_2|^2,$$
the CDA code-matrix form is

\[ \sum_{i, j, k} \begin{bmatrix} \beta_k \\ \alpha_k \end{bmatrix} \begin{bmatrix} \gamma_i \\ \delta_i \end{bmatrix} = \begin{bmatrix} \beta_i \\ \alpha_i \end{bmatrix} \]

for some ‘non-norm’ element \( L \). The Galois group \( \text{Gal} \) corresponding to the left-regular representation of elements of degree \( F \) is created for an arbitrary choice of integral \( f \). From there we construct a cyclic division algebra \( D = \mathbb{L} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L} \). A space-time code \( X \) can be associated to \( D \) by selecting the set of matrices corresponding to the left-regural representation of elements of a finite subset of \( D \). For an arbitrary choice of integral basis \( \{ \beta_i \}_{i=0}^{n-1} \) for a submodule in the ring of integers \( \mathbb{O}_L \) of \( \mathbb{L} \) over \( \mathbb{F} \), the elements of the signaling set are of the form \( \ell_i = \sum_{j=0}^{n-1} f_{i,j} \beta_j \), \( \ell_i \in \mathbb{L} \), \( f_{i,j} \in \mathbb{O}_L \). As a result, the CDA code-matrix form is

\[ X = \sum_{j=0}^{n-1} \Gamma^j \left( \text{diag}(f_j, G) \right) = \sum_{u=1}^{n} f_A u \]  

(58)

with \( f_j = [f_{j,0}, f_{j,1}, \cdots, f_{j,n-1}] \) and the concatenated \( f \) now being the \( n^2 \)-length sequence of \( \mathbb{O}_L \) information elements leaving the transmitter node (after normalization). Each \( n^2 \times n \) matrix \( A_u \) (corresponding to relay \( u \)) is created by first letting \( A_{n,i} = \text{diag} [ \{ \beta_i \}, \sigma^2(\beta_i), \cdots, \sigma^{n-1}(\beta_i) ] \) for \( i = 0, 1, \cdots, n-1 \) and then recursively creating \( A_{u,i} = \Gamma^{n-u} A_{n,i} \), \( u = 1, 2, \cdots, n \) where

\[ G = \begin{bmatrix} \sigma^0(\beta_0) & \cdots & \sigma^{n-1}(\beta_0) \\
\vdots & \ddots & \vdots \\
\sigma^0(\beta_{n-1}) & \cdots & \sigma^{n-1}(\beta_{n-1}) \end{bmatrix} \]

\[ \Gamma = \begin{bmatrix} 0 & 0 & \cdots & \gamma \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \end{bmatrix} \]

\[ A_u = \begin{bmatrix} A_{u,0} \\
\vdots \\
A_{u,n-1} \end{bmatrix} \]

(60)

\[ X = \begin{bmatrix} f_A u_1 \\
\vdots \\
f_A u_n \end{bmatrix} \]

(61)

The perfect code requirement that the lattice generator matrix \( G \) and the power sharing matrices \( \Gamma \), \( j = 0, 1, \cdots, n-1 \) in (60), be unitary matrices guarantees that \( A_u^* A_u = I \), essential for the code’s information losslessness.

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