GLOBAL WELL-POSEDNESS AND EXPONENTIAL DECAY FOR 3D NONHOMOGENEOUS MAGNETO-MICROPOLAR FLUID EQUATIONS WITH VACUUM

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Abstract. We consider an initial boundary value problem of three-dimensional (3D) nonhomogeneous magneto-micropolar fluid equations in a bounded simply connected smooth domain with homogeneous Dirichlet boundary conditions for the velocity and micro-rotational velocity and Navier-slip boundary condition for the magnetic field. We prove the global existence and exponential decay of strong solutions provided that some smallness condition holds true. Note that although the system degenerates near vacuum, there is no need to require compatibility conditions for the initial data via time weighted techniques.

1. Introduction. The magnetohydrodynamic model is often regarded as a reasonable description of the dynamics of a plasma. But it cannot describe fluids with microstructure, such complex fluids may be of different shape, may shrink and expand, change their shape, and moreover, they may rotate, independently of the rotation and movement of the fluid. Therefore, it is necessary to refine the fluid models. In the 1970s, Ahmadi and Shahinpoor [1] proposed a magneto-micropolar fluid model, which extends the valid domain of magnetohydrodynamic equations and accounts for microrotation effect. Here we study nonhomogeneous magneto-micropolar fluid equations (see [19]) in a bounded simply connected smooth domain \( \Omega \subseteq \mathbb{R}^3 \):

\[
\begin{align*}
\rho_t + \nabla (\rho \mathbf{u}) & = 0, \\
(\rho \mathbf{u})_t + \nabla (\rho \mathbf{u} \otimes \mathbf{u}) - (\mu_1 + \xi) \Delta \mathbf{u} + \nabla P & = 2\xi \text{curl} \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{b}, \\
(\rho \mathbf{w})_t + \nabla (\rho \mathbf{u} \otimes \mathbf{w}) + 4\xi \mathbf{w} - \mu_2 \Delta \mathbf{w} - \lambda \nabla \text{div} \mathbf{w} & = 2\xi \text{curl} \mathbf{u}, \\
\mathbf{b}_t - \nu \Delta \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} & = 0, \\
\text{div} \mathbf{u} & = \text{div} \mathbf{b} = 0,
\end{align*}
\]

where \( \rho, \mathbf{u}, \mathbf{w}, \mathbf{b}, \) and \( P \) denote the density, velocity, micro-rotational velocity, magnetic field, and pressure of the fluid, respectively. The positive constants \( \mu_1, \xi, \mu_2, \)

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and \(\lambda\) stand for viscosity coefficients of the fluid, while \(\nu > 0\) is the magnetic diffusive coefficient. The system (1.1) is supplemented with the initial condition

\[
(r, pu, rw, b)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 w_0, b_0)(x), \quad x \in \Omega,
\]

and the boundary condition

\[
(u, w, b \cdot n, \nabla b \times n)(x, t) = (0, 0, 0, 0), \quad x \in \partial\Omega, \quad t \geq 0.
\]

Here \(n\) denotes the outer normal to the boundary \(\partial\Omega\).

If the motion occurs in the absence of magnetic field (that is, \(b = 0\)), (1.1) reduces to the nonhomogeneous micropolar fluid equations. When the initial density is strictly away from vacuum (i.e., \(\rho_0\) is strictly positive), the authors [4] proved some existence and uniqueness results for strong solutions. Meanwhile, Braz e Silva et al. [6] investigated global existence and uniqueness of solutions for 3D Cauchy problem through a Lagrangian approach. On the other hand, for the initial density allowing vacuum states, Łukaszewicz [18] (see also [19, Chapter 3]) obtained short-time existence of weak solutions provided that the initial functions \(u_0\) and \(w_0\) are in \(H^1_0\) and that the initial density \(\rho_0\) is uniformly bounded and satisfies \(\|\rho_0\|_{L^3} < \infty\), while Braz e Silva and Santos [12] established the global existence of weak solutions. In [10], under smallness assumptions on the initial data, weak solutions with improved regularity were obtained. At the same time, imposing a compatibility condition on the initial data, Zhang and Zhu [26] showed the global existence of strong solution with nonnegative density in \(\mathbb{R}^3\) under some smallness condition. Later on, Ye [24] improved their result by removing the compatibility condition and furthermore obtained exponential decay of strong solution (see also [23] for the case of bounded domains). There are other interesting studies on the nonhomogeneous micropolar fluid equations, such as the vanishing viscosity problem [7, 11], error estimates for spectral semi-Galerkin approximations [13], the local existence of semi-strong solutions [8], and strong solutions in thin domains [9].

Let’s turn our attention to the system (1.1). For the initial density allowing vacuum states, imposing a compatibility condition on the initial data, Zhang and Zhu [25] showed the global existence of strong solution to the three-dimensional initial boundary value problem provided that some smallness condition holds. Very recently, Tang and Sun [22] established the local existence of strong solutions to the problem (1.1)–(1.3) without using compatibility conditions. However, the global existence of strong solutions to the problem (1.1)–(1.3) with vacuum is still unknown.

In the present paper we shall investigate nonhomogeneous magneto-micropolar fluid equations with Dirichlet boundary conditions for the velocity and the micro-rotational velocity and Navier-slip boundary condition for the magnetic field in a bounded simply connected smooth domain. For such domain and such boundary conditions, we shall establish the mechanism of blowup for the velocity which provides potential structure of possible singularities of strong solutions similar to that of [23]. Then we shall combine this blow-up criterion with the method suggested in [23] to get global existence result and large time behavior of solutions for the smallness initial data. It is to be noted that the main novelty of the present paper consists in the absence of the positive lower bound for the initial density as well as the absence of compatibility conditions for the initial data. It is worth noticing that the present results also partially imply improvements in the case of density-dependent viscosity [25].

Before stating our main result, we first explain the notations and conventions used throughout this paper. For \(1 \leq p \leq \infty\) and integer \(k \geq 0\), the standard
Assume that the initial data is somewhat surprising since the be in force. Then there exists a Let the conditions in Theorem 1.1. The local existence of a unique strong solution with initial data as has been established in

\[ L^p = L^p(\Omega), \ W^{k,p} = W^{k,p}(\Omega), \ H^k = H^{k,2}(\Omega), \]
\[ H_0^1 = \{ u \in H^1 | u = 0 \text{ on } \partial\Omega \}, \ H_0^{1,\sigma} = \{ u \in H_0^1 | \text{div } u = 0 \text{ in } \partial\Omega \}, \]
\[ H = \{ v \in L^2(\Omega) : \text{div } v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \Omega \}, \]
\[ G = \{ v \in H^1(\Omega) : \text{curl } v \times n = 0 \text{ on } \partial\Omega \}, \]
\[ V = G \cap H. \]

Our main results read as follows.

**Theorem 1.1.** Assume that the initial data \((\rho_0 \geq 0, u_0, w_0, b_0)\) satisfies

\[ \rho_0 \in W^{1,q}(\Omega), \ u_0 \in H_0^{1,\sigma}(\Omega), \ w_0 \in H_0^3(\Omega), \ b_0 \in V, \ q \in (3,6]. \] (1.4)

Let \((\rho \geq 0, u, w, b)\) be a unique strong solution to the problem (1.1)–(1.3). If \(T^* < \infty\) is the maximal time of existence for that solution, then we have

\[ \lim_{T \to T^*} \| u \|_{L^r(0,T;L^r)} = \infty, \] (1.5)

where \(r \) and \(s \) satisfy

\[ \frac{2}{s} + \frac{3}{r} = 1, \ 3 < r \leq \infty. \] (1.6)

**Remark 1.1.** The local existence of a unique strong solution with initial data as in Theorem 1.1 has been established in [22]. Hence, the maximal time \(T^* \) is well-defined.

**Remark 1.2.** The conclusion in Theorem 1.1 is somewhat surprising since the criterion (1.6) is independent of the micro-rotational velocity and magnetic field and similar to that of nonhomogeneous Navier-Stokes equations [16].

Based on Theorem 1.1, we can establish the global existence of strong solutions to (1.1)–(1.3) under some smallness condition.

**Theorem 1.2.** Let the conditions in Theorem 1.1 be in force. Then there exists a small positive constant \(\varepsilon_0\) depending only on \(\| \rho_0 \|_{L^\infty}, \ \Omega, \ \mu_1, \ \xi, \ \mu_2, \ \lambda, \ \) and \(\nu\) such that if

\[ \left\{ \begin{array}{l}
\| \sqrt{\rho_0} u_0 \|_{L^2}^2 + \| \sqrt{\rho_0} w_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^2 \leq \varepsilon_0, \\
\left( \| \sqrt{\rho_0} u_0 \|_{L^2}^2 + \| \sqrt{\rho_0} w_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^2 \right)^2 \left( \| \nabla u_0 \|_{L^2}^2 + \| \nabla w_0 \|_{L^2}^2 + \| \text{curl } b_0 \|_{L^2}^2 \right) \leq \varepsilon_0,
\end{array} \right. \] (1.7)

the problem (1.1)–(1.3) has a unique global strong solution \((\rho \geq 0, u, w, b)\) such that for any \(0 < \tau < \infty \) and \(2 \leq r < q, \)

\[ \rho_1 \in L^\infty(0, \infty; L^r), \ u, w, b \in L^\infty(0, \infty; H^1) \cap L^\infty(\tau, \infty; H^2) \cap L^2(\tau, \infty; H^3), \]
\[ \nabla P \in L^\infty(\tau, \infty; L^2) \cap L^2(\tau, \infty; H^1), \]
\[ e\tau^{1/2} \nabla u, \ e\tau^{3/2} \nabla w, \ e\tau^{1/2} \nabla b \in L^2(0, \infty; L^2), \]
\[ \sqrt{\tau} \sqrt{\rho_0}, \sqrt{\tau} \sqrt{\rho_0} w, \sqrt{\tau} b \in L^\infty(0, \infty; L^2), \]
\[ \sqrt{\tau} \nabla u, \sqrt{\tau} \nabla w, \sqrt{\tau} \nabla b \in L^2(0, \infty; L^2), \]

where \(\sigma\) is a positive constant depending only on \(\Omega, \mu_1, \mu_2, \nu, \) and \(\| \rho_0 \|_{L^\infty}. \) Moreover, there exists a positive constant \(C\) depending only on \(\Omega, \mu_1, \xi, \mu_2, \lambda, \nu, q, \) and the
initial data such that for $t \geq 1$,
\begin{equation}
\begin{aligned}
\|w(t)\|_L^2 + &\|\nabla P(t)\|_H^2 + \|u(t)\|_H^2 + \|\nabla w(t)\|_H^2 + \|b(t)\|_H^2 \\
\leq &\ C e^{-\sigma t},
\end{aligned}
\end{equation}
\tag{1.9}

Remark 1.3. It should be noted that our smallness assumption (1.7) is independent of any norms of the initial data except $\|\rho_0\|_{L^\infty}$, which is different from that in [25]. Moreover, for the initial data $(\rho_0, u_0, w_0, b_0)$ satisfying (1.4), it follows from (1.7) that the problem (1.1)–(1.3) has a unique global strong solution when the initial total energy $\|\sqrt{\rho_0}u_0\|_L^2 + \|\sqrt{\rho_0}w_0\|_L^2 + \|b_0\|_L^2$ is sufficiently small.

Remark 1.4. Compared with [25], on the one hand, there is no need to impose the additional compatibility conditions on the initial data via time weighted estimates. On the other hand, the regularity assumptions for the initial data in Theorem 1.2 are weaker than those in [25]. Furthermore, we obtain exponential decay rates rather than algebraic decay.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1. Finally, we give the proof of Theorem 1.2 in Section 4.

2. Preliminaries. In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth simply connected domain. Then the following inequality holds true for all $1 < p < +\infty$, with a constant $C$ depending only on $p$ and $\Omega$:
\begin{equation}
\|f\|_{W^{1,p}} \leq C \|\nabla f\|_{L^p},
\end{equation}
\tag{2.1}
for each divergence-free vector field $f \in W^{1,p}(\Omega)$ such that either the boundary condition $(f \cdot n)|_{\partial \Omega} = 0$ or $(f \times n)|_{\partial \Omega} = 0$ is satisfied.

Proof. See [3, Lemma 9].

Next, the following divergence theorem (see [14, Theorem 2.11, p. 34]) is useful in Section 3.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. If $\phi \in H^1(\Omega)$ and $v \in L^2(\Omega)$ such that $\text{curl} v \in L^2(\Omega)$, then we have the Green’s formula:
\begin{equation}
\int_{\Omega} \text{curl} v \cdot \phi dx = \int_{\Omega} v \cdot \text{curl} \phi dx + \int_{\partial \Omega} (v \times n) \cdot \phi d\sigma.
\end{equation}

Next, we give some regularity properties for the following Stokes problem
\begin{equation}
\begin{cases}
-\mu \Delta u + \nabla P = F, & x \in \Omega, \\
\text{div} u = 0, & x \in \Omega, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\end{equation}
\tag{2.2}

Lemma 2.3. Let $k \geq 0$ be an integer and let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. For $F \in \mathcal{W}^{k,p}(\Omega)$ with $1 < p < +\infty$, there exists a unique solution $(u, P) \in \mathcal{W}^{k+2,p}(\Omega) \times \mathcal{W}^{k+1,p}(\Omega)$ to the problem (2.2) such that
\begin{equation}
\|u\|_{\mathcal{W}^{k+2,p}} + \|P\|_{\mathcal{W}^{k+1,p}} \leq C \|F\|_{\mathcal{W}^{k,p}}
\end{equation}
for some constant $C$ depending only on $\Omega, k, p,$ and $\mu$.

Proof. See [5, Theorem IV.6.6].
Finally, we give $L^2$ theory for the following vector Laplace problem
\begin{align}
-\Delta v &= f, \quad x \in \Omega,
\nu \cdot v = 0, \quad \text{curl } v \cdot n = 0, \quad x \in \partial \Omega.
\end{align}

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth simply connected domain and suppose that $f \in L^2(\Omega)$. Then the problem (2.3) has a unique solution $v \in H^2 \cap V$ such that
\[ \|v\|_{H^2} \leq C\|f\|_{L^2} \]
for some constant $C$ depending only on $\Omega$.

**Proof.** See [5, Theorems IV.9.5 and IV.9.6]. \hfill \square

3. **Proof of Theorem 1.1.** In this section, we will prove Theorem 1.1 by contradiction. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|u\|_{L^{6}(0,T;L^{6})}$ is bounded independent of any $T \in (0,T^{*})$. The a priori estimates are then sufficient for us to apply the local existence result repeatedly to extend a local solution beyond the maximal time of existence $T^{*}$, consequently, contradicting the maximality of $T^{*}$. In what follows, we sometimes use $C(f)$ to emphasize the dependence on $f$. Moreover, we write
\[ \int_{\Omega} \cdot dx = \int_{\Omega} \cdot dx. \]

Let $(\rho, u, w, b)$ be a strong solution described in Theorem 1.1. Suppose that (1.5) were false, that is, there exists a constant $M_0 > 0$ such that
\[ \lim_{T \to T^{*}} \|u\|_{L^{6}(0,T;L^{6})} \leq M_0 < \infty. \]  

**Lemma 3.1.** It holds that
\[ \sup_{0 \leq t \leq T} \|\rho\|_{L^{\infty}} \leq \|\rho_0\|_{L^{\infty}}, \]  
and
\[ \sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \]
\[ + \int_{0}^{T} \left( \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \right) dt \]
\[ \leq \|\rho_0 u_0\|_{L^2}^2 + \|\sqrt{\rho_0} w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \]  

Moreover, there exists a constant $C$ depending only on $\Omega, \mu_1, \mu_2, \nu, \|\rho_0\|_{L^{\infty}}, \|\sqrt{\rho_0} u_0\|_{L^2}, \|\sqrt{\rho_0} w_0\|_{L^2},$ and $\|b_0\|_{L^2}$ such that
\[ \sup_{0 \leq t \leq T} \left[ e^{\sigma t} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \right] \]
\[ + \int_{0}^{T} e^{\sigma t} \left( \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\text{curl } b\|_{L^2}^2 \right) dt \leq C \]
for some positive constant $\sigma$ depending only on $\Omega, \mu_1, \mu_2, \nu,$ and $\|\rho_0\|_{L^{\infty}}$.

**Proof.** 1. The desired (3.2) follows from (1.1) and $\div u = 0$. Moreover, from (1.1) and $\rho_0 \geq 0$, we have $\rho(x,t) \geq 0$. 


We get after using integration by parts, Lemma 2.2, and (1.3) that
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 \right) + \left( \mu_1 + \xi \right) \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + 4\xi \|w\|_{L^2}^2 + \lambda \|\text{div} w\|_{L^2}^2 \\
= 4\xi \int \text{curl } u \cdot w \, dx + \int b \cdot \nabla b \cdot u \, dx. \tag{3.5}
\]

Multiplying (1.1)_4 by b and applying the vector identity
\[-\Delta b = \text{curl(curl b)} - \nabla \text{div } b = \text{curl(curl b)},\]
we get after using integration by parts, Lemma 2.2, and (1.3) that
\[
\frac{d}{dt} \|b\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 + \int b \cdot \nabla b \cdot u \, dx = 0.
\]
This combined with (3.5) gives
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \left( \mu_1 + \xi \right) \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + 4\xi \|w\|_{L^2}^2 \\
= 4\xi \int \text{curl } u \cdot w \, dx \leq \xi \|\nabla u\|_{L^2}^2 + 4\xi \|w\|_{L^2}^2,
\]
which implies that
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \\
+ \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \leq 0. \tag{3.6}
\]

Integrating (3.6) over [0, T] leads to (3.3).

3. From (3.2), Sobolev’s inequality, (2.1), and (1.3), we have
\[
\|\sqrt{\rho} u\|_{L^2}^2 \leq \|\rho\|_{L^\infty} \|u\|_{L^2}^2 \leq C(\Omega, \|\rho_0\|_{L^\infty}) \|\nabla u\|_{L^2}^2,
\]
\[
\|\sqrt{\rho} w\|_{L^2}^2 \leq \|\rho\|_{L^\infty} \|w\|_{L^2}^2 \leq C(\Omega, \|\rho_0\|_{L^\infty}) \|\nabla w\|_{L^2}^2,
\]
\[
\|b\|_{L^2}^2 \leq C(\Omega) \|b\|_{H^1}^2 \leq C(\Omega) \|\text{curl } b\|_{L^2}^2.
\]

These along with (3.6) lead to
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + 2\sigma \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \leq 0
\]

for some \(\sigma\) depending only on \(\Omega, \|\rho_0\|_{L^\infty}, \mu_1, \mu_2, \) and \(\nu\). This yields immediately that, for any \(t \geq 0\),
\[
\|\sqrt{\rho} u(\cdot, t)\|_{L^2}^2 + \|\sqrt{\rho} w(\cdot, t)\|_{L^2}^2 + \|b(\cdot, t)\|_{L^2}^2 \\
\leq (\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\sqrt{\rho_0} w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2) e^{-2\sigma t}. \tag{3.7}
\]

Then we multiply (3.6) by \(e^{\alpha t}\) and use (3.7) to get that
\[
\frac{d}{dt} \left[ e^{\alpha t} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \right] \\
+ e^{\alpha t} \left( \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \right) \\
\leq \sigma e^{\alpha t} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) \\
\leq \sigma \left( \|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\sqrt{\rho_0} w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \right) e^{-\alpha t}. \tag{3.8}
\]
Integrating (3.8) in time over $[0, T]$ yields that
\[
\int_0^T e^{\sigma t} \left( \| \nabla u \|^2_{L^2} + \| \nabla w \|^2_{L^2} + \| \text{curl} b \|^2_{L^2} \right) dt \\
\leq C(\Omega, \mu_1, \mu_2, \nu, \| \rho_0 \|_{L^\infty}, \| \sqrt{\rho_0} u_0 \|^2_{L^2}, \| \sqrt{\rho_0} w_0 \|^2_{L^2}, \| b_0 \|^2_{L^2}),
\]
which together with (3.7) gives (3.4) and completes the proof of Lemma 3.1.

The following lemma concerns the higher integrability of the magnetic field, which plays a crucial role in the proof of Lemma 3.3.

**Lemma 3.2.** Under the condition (3.1), there exists a positive constant $C$ depending only on $M_0$, $\Omega$, $\nu$, and the initial data such that
\[
\sup_{0 \leq t \leq T} \| b \|^4_{L^4} + \int_0^T \| b \| \| \nabla b \|^2_{L^2} dt \leq C.
\]

**Proof.** Multiplying (1.1)$_4$ by $4|b|^2 b$ and integrating the resulting equation over $\Omega$, we get
\[
\frac{d}{dt} \int |b|^4 dx - 4\nu \int \Delta b \cdot |b|^2 b dx = 4 \int (-u \cdot \nabla b + b \cdot \nabla u) \cdot |b|^2 b dx.
\]
From [2], we have
\[
-4\nu \int \Delta b \cdot |b|^2 b dx
\]
\[
= 2\nu \int |\nabla b|^2 |b|^2 dx + 2\nu \int |b|^2 \nabla |b|^2 dx - 4\nu \int_{\partial \Omega} |b|^2 (b \cdot \nabla) b \cdot ndS.
\]
Noting that $b$ satisfies $(b \cdot n)|_{\partial \Omega} = 0$, we have the identity
\[
(b \cdot \nabla) b \cdot n = -(b \cdot \nabla) n \cdot b \quad \text{on} \ \partial \Omega,
\]
which combined with (3.12) indicates that
\[
-4\nu \int \Delta b \cdot |b|^2 b dx
\]
\[
= 2\nu \int |\nabla b|^2 |b|^2 dx + 2\nu \int |b|^2 \nabla |b|^2 dx + 4\nu \int_{\partial \Omega} |b|^2 (b \cdot \nabla) n \cdot b dS.
\]
Direct calculation yields
\[
-4 \int u \cdot \nabla b \cdot |b|^2 b dx = -4 \int u^i \partial_i b^j b^k b^l b^j dx
\]
\[
= -4 \int \text{div}(|b|^4 u) dx + 4 \int \text{div} u |b|^4 dx + 4 \int u^i \partial_i b^j b^k b^l b^j dx
\]
\[
+ 4 \int u^i \partial_i b^k b^j b^l b^j dx + 4 \int u^i \partial_i b^j b^k b^l b^j dx,
\]
which along with $\text{div} u = 0$, divergence theorem, and $(u \cdot n)|_{\partial \Omega} = 0$ implies that
\[
-4 \int (u \cdot \nabla) b \cdot |b|^2 b dx = - \int \text{div}(|b|^4 u) dx + \int \text{div} u |b|^4 dx
\]
\[
= - \int_{\partial \Omega} (u \cdot n)|b|^4 dS = 0.
\]
Direct calculation shows that
\[
4 \int b \cdot \nabla u \cdot |b|^2 b dx = 4 \int b^i \partial_i u^j b^k b^l b^i dx
\]
\[
= 4 \int \text{div}(|b|^2 (u \cdot b)) b dx - 4 \int \text{div} b |b|^2 (u \cdot b) dx - 4 \int b^i \partial_i b^j u^j b^k b^l dx
\]
\[
- 4 \int b^i \partial_i b^k u^j b^l b^i dx - 4 \int b^i b^j \partial_i u^j b^k b^l dx,
\]
which together with divergence theorem, \((b \cdot n)|_{\partial \Omega} = 0, \text{ div } b = 0, \) and Hölder's inequality that for \(r \) and \(s \) satisfying (1.6),
\[
4 \int b \cdot \nabla u \cdot |b|^2 b dx \leq C \int |b|^3 |\nabla b| |u| dx
\]
\[
\leq \nu \int |b|^2 |\nabla b|^2 dx + C(\nu) \int |u|^2 |b|^4 dx
\]
\[
\leq \nu \int |b|^2 |\nabla b|^2 dx + C(\nu) \|u\|_{L^2}^2 \||b|^2\|_{L^2}^{\frac{2}{r-1}} \||b|^\frac{r}{s}\|_{L^s}
\]
\[
\leq \nu \int |b|^2 |\nabla b|^2 dx + C(\nu) \|u\|_{L^r}^s \||b|^s\|_{L^4}. \quad (3.15)
\]
Substituting (3.13)–(3.15) into (3.11), we infer from the trace inequality (see [20, Corollary]), Sobolev’s inequality, and (3.7) that
\[
\frac{d}{dt} \|b\|_{L^4}^4 + \nu \|\nabla |b|^2\|_{L^2}^2 + \nu \|\nabla b\|_{L^4}^4
\]
\[
\leq -4\nu \int_{\partial \Omega} |b|^2 (b \cdot \nabla) n \cdot bdS + C\|u\|_{L^r}^s \||b|^4\|_{L^4}
\]
\[
\leq C\|b\|_{L^2}^2 \||b|^2\|_{L^2} + C\|u\|_{L^r}^s \||b|^4\|_{L^4}
\]
\[
\leq C\|b\|_{L^2}^2 \||\nabla |b|^2\|_{L^2} + C\|u\|_{L^r}^s \||b|^4\|_{L^4}
\]
\[
\leq \nu \frac{1}{2} \||\nabla |b|^2\|_{L^2}^2 + C\|b\|_{L^4}^4 + C\|u\|_{L^r}^s \||b|^4\|_{L^4}
\]
\[
\leq \nu \frac{1}{2} \||\nabla |b|^2\|_{L^2}^2 + C\|b\|_{L^4}^4 + C\|u\|_{L^r}^s \||b|^4\|_{L^4}
\]
\[
\leq \nu \|\nabla |b|^2\|_{L^2}^2 + C\|b\|_{L^4}^4 + C\|u\|_{L^r}^s \||b|^4\|_{L^4},
\]
where \(\sigma\) is the same as in (3.7). Thus, one has
\[
\frac{d}{dt} \|b\|_{L^4}^4 + \nu \|\nabla b\|_{L^2}^2 \leq Ce^{-4\sigma t} + C\|u\|_{L^r}^s \||b|^4\|_{L^4}, \quad (3.16)
\]
which together with Gronwall’s inequality and (3.1) yields the desired (3.10).

\[\Box\]

Lemma 3.3. Under the condition (3.1), then there exists a positive constant \(C\) depending only on \(M_0, \Omega, \mu_1, \mu_2, \xi, \lambda, \nu, \) and the initial data such that, for
i \in \{0, 1, 2\},
\sup_{0 \leq t \leq T} \left[ t^i \left( \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\text{curl} b\|_{L^2}^2 \right) \right]
+ \int_0^T t^i \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} \omega_x\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\omega_t\|_{L^2}^2 \right) dt \leq C. \quad (3.17)

Proof. 1. Multiplying (1.1)_2 by \(u_t\) and (1.1)_3 by \(b_t\), we then obtain after integrating by parts that

\[ \frac{1}{2} \frac{d}{dt} \int ((\mu_1 + \xi)|\nabla u|^2 + \nu|\text{curl} b|^2) \, dx + \int (\rho|u_t|^2 + |b_t|^2) \, dx \]

leads to

\[ = 2\xi \int \nabla \times w \cdot u_t \, dx - \int \rho u \cdot \nabla u \cdot u_t \, dx + \int b \cdot \nabla b \cdot u_t \, dx + \int (b \cdot \nabla u - u \cdot \nabla b) \cdot b_t \, dx \]

which combined with (3.18) leads to

\[ \frac{d}{dt} \int \left( (\mu_2 + \xi)|\nabla u|^2 + \lambda(\text{div} w)^2 + 4\xi|w|^2 \right) \, dx + \int \rho|w_t|^2 \, dx \]

leads to

\[ = 2\xi \int \text{curl} u \cdot w_t \, dx - \int \rho u \cdot \nabla w \cdot w_t \, dx \]

and integrating by parts lead to

\[ \leq 2\xi \int \text{curl} u \cdot w_t \, dx + \frac{1}{2} \int \rho|w_t|^2 \, dx + \frac{1}{2} \int \rho|u|^2|\nabla w|^2 \, dx, \]

which combined with (3.18) leads to

\[ \frac{d}{dt} \int \left( (\mu_1 + \xi)|\nabla u|^2 + \nu|\text{curl} b|^2 + 2b \cdot \nabla u \cdot b + \mu_2|\nabla w|^2 + \lambda(\text{div} w)^2 \right. \]

we thus infer from Lemma 2.4 and (3.2) that

\[ \|u_t\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \]

\[ \leq C(\Omega, \mu_1, \xi, \|\rho_0\|_{L^\infty}) \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} \nabla u\|_{L^2}^2 + \|\text{curl} w\|_{L^2}^2 + \|b \cdot \nabla b\|_{L^2}^2 \right) \]

Recall that (u, P) satisfies the Stokes problem

we thus infer from Lemma 2.4 and (3.2) that

\[ \|u_t\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \]

\[ \leq C(\Omega, \mu_1, \xi, \|\rho_0\|_{L^\infty}) \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} \nabla u\|_{L^2}^2 + \|\text{curl} w\|_{L^2}^2 + \|b \cdot \nabla b\|_{L^2}^2 \right) \]

\[ \leq C(\Omega, \mu_1, \xi, \|\rho_0\|_{L^\infty}) \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} \nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|b \cdot \nabla b\|_{L^2}^2 \right). \]
It follows from (1.1)4, (1.3), and Lemma 2.4 that

\[ \|b\|_{H^2}^2 \leq C(\Omega, \nu) \left( \|b_t\|_{L^2}^2 + \|u\|_{H^1}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \right). \]  

(3.22)

Employing \( L^2 \)-theory of elliptic equations (see [15, Chapter 9]), we deduce from (1.1)3 that

\[ \|\nabla w\|_{H^1}^2, \]

(3.23)

\[ \leq C(\Omega, \mu_2, \xi, \lambda) \left( \|\rho w_t\|_{L^2}^2 + \|\rho u \cdot \nabla w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) \]

\[ \leq C(\Omega, \mu_2, \xi, \lambda, \|\rho_0\|_{L^\infty}) \left( \|\sqrt{\rho} w_t\|_{L^2}^2 + \|\sqrt{\rho} u \cdot \nabla w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right). \]

Combining (3.21)–(3.23) together, we have

\[ \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 \]

\[ \leq K \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 + |b_t|_{L^2}^2 \right) + C \left( \|\sqrt{\rho} u \cdot \nabla u\|_{L^2}^2 + \|\sqrt{\rho} u |\nabla w|\|_{L^2}^2 \right) \]

\[ + C \left( \|b|\nabla b\|_{L^2}^2 + \|u|\nabla b\|_{L^2}^2 + \|b|\nabla u\|_{L^2}^2 \right) + C \left( \|u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) \]  

(3.24)

for some positive constant \( K \) depending only on \( \Omega, \mu_1, \mu_2, \xi, \lambda, \nu, \) and \( \|\rho_0\|_{L^\infty} \).

3. Adding (3.24) multiplied by \( \frac{1}{2K} \) to (3.19), we get from Hölder’s inequality, Sobolev’s inequality, (3.10), and Young’s inequality that

\[ A'(t) + \frac{1}{2} \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 + |b_t|_{L^2}^2 \right) + \frac{1}{2K} \left( \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 \right) \]

\[ \leq C \int \left( \rho |u|^2 |\nabla u|^2 + \rho |u| |\nabla w|^2 + |u|^2 |\nabla b|^2 + |b|^2 |\nabla u|^2 + |b|^2 |\nabla b|^2 \right) dx \]

\[ + C \left( \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) \]

\[ \leq C \|\rho\|_{L^\infty} \|u\|_{L^2} \|\nabla u\|_{L^2}^2 + C \|\rho\|_{L^\infty} \|u\|_{L^2} \|\nabla w\|_{L^2}^2 + C \|u\|_{L^2} \|\nabla b\|_{L^2}^2 \]

\[ + C \|b\|_{L^2} \|\nabla b\|_{L^2} + C \|b\|_{L^2} \|\nabla u\|_{L^2} + C \left( \|u\|_{L^2} + \|\nabla w\|_{L^2} \right) \]

\[ \leq C \|u\|_{L^2} \|\nabla u\|_{L^2}^2 + C \|u\|_{L^2} \|\nabla w\|_{L^2}^2 \]

\[ + C \|u\|_{L^2} \|\nabla b\|_{L^2}^2 \]

\[ + C \|\nabla u\|_{L^2} \|\nabla b\|_{L^2} + C \|\nabla b\|_{L^2} \|\nabla b\|_{L^2} + C \left( \|u\|_{L^2} + \|\nabla w\|_{L^2} \right) \]

\[ \leq \frac{1}{4K} \left( \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 \right) \]

\[ + C \|u\|_{L^2} \left( \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \right) \]

\[ + C \left( \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \right), \]

(3.25)

where \( r \) and \( s \) satisfy (1.6). This along with Lemma 2.1 implies that

\[ A'(t) + \frac{1}{2} \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 + |b_t|_{L^2}^2 \right) \]

\[ + \frac{1}{4K} \left( \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 \right) \]

\[ \leq C \|u\|_{L^2} \left( \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \right) \]

\[ + C \left( \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \right). \]  

(3.26)
Here
\begin{equation}
A(t) := \int ((\mu_1 + \xi) |\nabla u|^2 + \mu_2 |\nabla w|^2 + \lambda (\text{div } w)^2 + 4\xi |w|^2 - 4\xi \text{ curl } u \cdot w) \, dx \\
+ \int (\nabla \cdot b)^2 + 2b \cdot \nabla u \cdot b) \, dx
\end{equation}

satisfies
\begin{equation}
\frac{\mu_1}{2} \|\nabla u\|^2_{L^2} + \mu_2 \|\nabla w\|^2_{L^2} + \nu \|\text{ curl } b\|^2_{L^2} - L \|b\|^4_{L^2} \\
\leq A(t) \leq \frac{3\mu_1}{2} \|\nabla u\|^2_{L^2} + (\mu_2 + \lambda) \|\nabla w\|^2_{L^2} + \nu \|\text{ curl } b\|^2_{L^2} + L \|b\|^4_{L^2} \tag{3.27}
\end{equation}
due to
\begin{equation}
| -4\xi \int \text{ curl } u \cdot w \, dx| \leq \xi \|\nabla u\|^2_{L^2} + 4\xi \|w\|^2_{L^2}, \tag{3.28}
\end{equation}
and
\begin{equation}
\int 2b \cdot \nabla u \cdot b \, dx \leq 2\|\nabla u\|_{L^2} \|b\|^2_{L^2} \leq C(\Omega) \|\nabla u\|_{L^2} \|b\|^2_{L^2} \\
\leq \frac{\mu_1}{2} \|\nabla u\|^2_{L^2} + L \|b\|^4_{L^2}
\end{equation}

for some positive constant $L$ depending only on $\Omega$ and $\mu_1$. Consequently, adding (3.16) multiplied by $L + 1$ to (3.26) gives rise to
\begin{equation}
B'(t) + \frac{1}{2} \left( \|\sqrt{\rho} u\|^2_{L^2} + \|\sqrt{\rho} w\|^2_{L^2} + \|b\|^2_{L^2} \right) \\
+ \frac{1}{4K} \left( \|u\|^2_{H^2} + \|b\|^2_{H^2} + \|\nabla w\|^2_{H^1} \right) \\
\leq C \left[ \|u\|^2_{L^2} \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{ curl } b\|^2_{L^2} + \|b\|_{L^2}^4 \right] \\
+ C \left[ \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{ curl } b\|^2_{L^2} + \|b\|_{L^2}^4 \right] + Ce^{-2\alpha t}, \tag{3.29}
\end{equation}
where
\begin{equation}
B(t) := A(t) + (L + 1) \|b\|^4_{L^2}
\end{equation}
satisfies
\begin{equation}
\frac{\mu_1}{2} \|\nabla u\|^2_{L^2} + \mu_2 \|\nabla w\|^2_{L^2} + \nu \|\text{ curl } b\|^2_{L^2} + \|b\|^4_{L^2} \\
\leq B(t) \leq C \left( \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{ curl } b\|^2_{L^2} + \|b\|^4_{L^2} \right), \tag{3.30}
\end{equation}
due to (3.27). Thus, we obtain from (3.29), Gronwall’s inequality, (3.1), (3.3), and (3.30) that
\begin{equation}
\sup_{0 \leq t \leq T} \left( \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{ curl } b\|^2_{L^2} \right) \\
+ \int_0^T \left( \|\sqrt{\rho} u\|^2_{L^2} + \|\sqrt{\rho} w\|^2_{L^2} + \|b\|^2_{L^2} + \|u\|^2_{H^2} + \|b\|^2_{H^2} + \|\nabla w\|^2_{H^1} \right) \, dt \leq C. \tag{3.31}
\end{equation}
4. Multiplying (3.29) by $t$ together with (3.30) yields

$$ \frac{d}{dt}(tB(t)) + \frac{1}{2} \left( t \| \sqrt{\rho} u_t \|_{L^2}^2 + t \| \sqrt{\rho} w_t \|_{L^2}^2 + t \| b_t \|_{L^2}^2 \right) $$

$$ + \frac{1}{4K} \left( t \| u \|_{H^2}^2 + t \| b \|_{H^2}^2 + t \| \nabla w \|_{H^1}^2 \right) $$

$$ \leq C \| u \|^2_{L^r}(t \| \nabla u \|_{L^2}^2 + t \| \nabla w \|_{L^2}^2 + t \| \text{curl } b \|_{L^2}^2 + t \| b \|_{L^4}^4) $$

$$ + Ct \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) $$

$$ + Ct e^{-4\sigma t} + C \| \nabla u \|_{L^2}^2 + C \| \nabla w \|_{L^2}^2 + C \| \text{curl } b \|_{L^2}^2 + C \| b \|_{L^4}^4 $$

$$ \leq C \| u \|^2_{L^r}(t \| \nabla u \|_{L^2}^2 + t \| \nabla w \|_{L^2}^2 + t \| \text{curl } b \|_{L^2}^2 + t \| b \|_{L^4}^4) $$

$$ + Ct \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) $$

$$ + Ct e^{-4\sigma t} + C \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) + C \| b \|_{L^2} \| b \|^3_{L^4} $$

$$ \leq C \| u \|^2_{L^r}(t \| \nabla u \|_{L^2}^2 + t \| \nabla w \|_{L^2}^2 + t \| \text{curl } b \|_{L^2}^2 + t \| b \|_{L^4}^4) $$

$$ + Ct \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) $$

$$ + Ct e^{-4\sigma t} + C \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) + C \| b \|_{L^2} \| b \|^3_{L^4} \quad (3.32) $$

From (3.4), we have

$$ \int_0^T t \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) dt $$

$$ = \int_0^T (te^{-\sigma t}) \cdot e^{\sigma t} \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) dt $$

$$ \leq \sup_{0 \leq t \leq T} (te^{-\sigma t}) \int_0^T e^{\sigma t} \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) dt \leq C. \quad (3.33) $$

From (3.3) and (3.31), we find that

$$ \int_0^T \| \text{curl } b \|_{L^2}^2 dt \leq \left( \sup_{0 \leq t \leq T} \| \text{curl } b \|_{L^2} \right) \int_0^T \| \text{curl } b \|_{L^2}^2 dt \leq C. \quad (3.34) $$

Thus, we obtain from (3.32), Gronwall’s inequality, (3.30), (3.1), (3.33), (3.3), and (3.34) that

$$ \sup_{0 \leq t \leq T} \left[ t \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 + \| \text{curl } b \|_{L^2}^2 \right) \right] $$

$$ + \int_0^T t \left( \| \sqrt{\rho} u_t \|_{L^2}^2 + \| \sqrt{\rho} w_t \|_{L^2}^2 + \| b_t \|_{L^2}^2 + \| u \|_{H^2}^2 + \| b \|_{H^2}^2 + \| \nabla w \|_{H^1}^2 \right) dt \leq C. \quad (3.35) $$

For $i = 2$, we can derive the similar result and omit the details for simplicity. \qed

**Lemma 3.4.** Under the condition (3.1), then there exists a positive constant $C$ depending only on $M_0$, $\Omega$, $\mu_1$, $\mu_2$, $\xi$, $\lambda$, $\nu$, and the initial data such that, for $i \in \{1, 2\}$,

$$ \sup_{0 \leq t \leq T} \left[ t^i \left( \| \sqrt{\rho} u_t \|_{L^2}^2 + \| \sqrt{\rho} w_t \|_{L^2}^2 + \| b_t \|_{L^2}^2 \right) \right] $$

$$ + \int_0^T t^i \left( \| \nabla u_t \|_{L^2}^2 + \| \nabla w_t \|_{L^2}^2 + \| \text{curl } b_t \|_{L^2}^2 \right) dt \leq C. \quad (3.36) $$
Proof. 1. Differentiating (1.1)\textsubscript{2} and (1.1)\textsubscript{3} with respect to $t$ and using (1.1)\textsubscript{1} give rise to

$$\begin{align*}
\rho u_{tt} + \rho u \cdot \nabla u_t - (\mu_1 + \xi)\Delta u_t + \nabla P_t \\
= \text{div}(\rho u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + 2\xi \nabla \times w_t + b_t \cdot \nabla b_t + b \cdot \nabla b_t, \\
(3.37)
\end{align*}$$

Multiplying (3.37) by $u_t$, (3.38) by $w_t$, and integrating the resulting equality by parts over $\Omega$ and summing them, we obtain that

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int (\rho |u_t|^2 + \rho |w_t|^2) dx + (\mu_1 + \xi) \int |\nabla u_t|^2 dx + 4\xi \int |w_t|^2 dx & \\
+ \mu_2 \int |\nabla w_t|^2 dx + \lambda \int (\text{div} w_t)^2 dx & \\
= \int (\text{div}(\rho u_t)(u_t + u \cdot \nabla u) \cdot u_t - \rho u_t \cdot \nabla u \cdot u_t) dx & \\
+ \int (\text{div}(\rho u_t)(w_t + u \cdot \nabla w) \cdot w_t - \rho u_t \cdot \nabla w \cdot w_t) dx & \\
+ 2\xi \int (\nabla \times w_t \cdot u_t + \nabla \times u_t \cdot w_t) dx & + \int b_t \cdot \nabla b_t \cdot u_t dx & + \int b \cdot \nabla b_t \cdot u_t dx & \\
=: I_1 + I_2 + I_3 + I_4 + I_5. & & \\
(3.39)
\end{align*}$$

After integration by parts, we derive from (3.2), Sobolev’s inequality, and (3.17) that

$$\begin{align*}
I_1 \leq C & \int \rho |u| (|u_t| \nabla u_t + |u| \nabla^2 u |u_t| + |u| \nabla u |\nabla u_t| + |u|^2 |u_t|) dx & \\
& + \int \rho |u_t|^2 |\nabla u| dx & \\
\leq C & \|u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^6} \|\nabla u_t\|_{L^2} + C \|u\|_{L^6}^2 \|\nabla^2 u\|_{L^2} \|u_t\|_{L^6} & \\
& + C \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} & \\
& + C \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^6} \|u_t\|_{L^6} + \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6}^2 & \\
\leq C & \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2} & \\
& + C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u_t\|_{H^1} \|\nabla u_t\|_{L^2} & \\
& + C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6} \|\sqrt{\rho} u_t\|_{L^6}^2 & \\
\leq C & \|\sqrt{\rho} u_t\|_{L^6} \|\nabla u_t\|_{L^2}^2 & + C \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2} & \\
& \leq \frac{H}{6} \|\nabla u_t\|_{L^2}^2 + C (\|\sqrt{\rho} u_t\|_{L^6}^2 + \|\nabla u\|_{H^1}^2). & & \\
(3.40)
\end{align*}$$

Similarly, one has

$$\begin{align*}
I_2 \leq C & \int \rho |w| (|w_t| \nabla w_t + |u| \nabla^2 w |w_t| + |u| \nabla w |\nabla w_t| + |\nabla u| \nabla w |w_t|) dx & \\
& + \int \rho |u_t| |w_t| |\nabla w| dx & \\
\leq \frac{H_1}{2} \|\nabla w_t\|_{L^2}^2 + C (\|\sqrt{\rho} u_t\|_{L^6}^2 + \|\sqrt{\rho} w_t\|_{L^6}^2 + \|\nabla w\|_{H^1}^2). & & \\
(3.41)
\end{align*}$$
Integration by parts, we obtain from Cauchy-Schwarz inequality that

$$I_3 = 4\xi \int \nabla \times \mathbf{u}_t \cdot \mathbf{w}_t \, dx \leq 4\xi \| \mathbf{w}_t \|_{L^2}^2 + \xi \| \nabla \mathbf{u}_t \|_{L^2}^2. \tag{3.42}$$

By Hölder's inequality, Sobolev's inequality, (2.1), and (3.31), we have, for $\delta > 0,$

$$|I_4| \leq \| \mathbf{b}_t \|_{L^3} \| \nabla \mathbf{b} \|_{L^2} \| \mathbf{u}_t \|_{L^6} \leq C \| \mathbf{b}_t \|_{L^2}^2 \| \mathbf{b}_t \|_{L^2}^2 \| \nabla \mathbf{u}_t \|_{L^2} \leq \frac{\mu_1}{6} \| \nabla \mathbf{u}_t \|_{L^2}^2 + C(\delta) \| \mathbf{b}_t \|_{L^2}^2 + \frac{\delta}{2} \| \text{curl} \mathbf{b}_t \|_{L^2}^2. \tag{3.43}$$

Integrating by parts together with $\text{div} \mathbf{b} = 0$ and $(\mathbf{b} \cdot \mathbf{n})|_{\partial \Omega} = 0$, we derive from Hölder's inequality, Sobolev's inequality, (2.1), and (3.31) that

$$|I_5| = \left| \int \mathbf{b} \cdot \nabla \mathbf{u}_t \cdot \mathbf{b}_t \, dx \right| \leq \| \mathbf{b} \|_{L^6} \| \nabla \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} \leq C \| \mathbf{b} \|_{H^1} \| \nabla \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} \| \mathbf{b}_t \|_{H^1} \leq \frac{\mu_1}{6} \| \nabla \mathbf{u}_t \|_{L^2}^2 + C(\delta) \| \mathbf{b}_t \|_{L^2}^2 + \frac{\delta}{2} \| \text{curl} \mathbf{b}_t \|_{L^2}^2. \tag{3.44}$$

Substituting (3.40)–(3.44) into (3.39), we obtain that

$$\frac{d}{dt} \left( \| \sqrt{\rho} \mathbf{u}_t \|_{L^2}^2 + \| \sqrt{\rho} \mathbf{w}_t \|_{L^2}^2 \right) + \mu_1 \| \nabla \mathbf{u}_t \|_{L^2}^2 + \mu_2 \| \nabla \mathbf{w}_t \|_{L^2}^2 \leq C \left( \| \sqrt{\rho} \mathbf{u}_t \|_{L^2}^2 + \| \sqrt{\rho} \mathbf{w}_t \|_{L^2}^2 \right) + C \left( \| \sqrt{\rho} \mathbf{u}_t \|_{L^2}^2 + \| \sqrt{\rho} \mathbf{w}_t \|_{L^2}^2 \right) + C \| \mathbf{b}_t \|_{L^2}^2 + \delta \| \text{curl} \mathbf{b}_t \|_{L^2}^2. \tag{3.45}$$

2. Differentiating (1.14) with respect to $t$ and multiplying the resulting equations by $\mathbf{b}_t$, we obtain after integration by parts using (3.17) and (2.1) that

$$\frac{1}{2} \frac{d}{dt} \int |\mathbf{b}_t|^2 dx + \nu \int |\text{curl} \mathbf{b}_t|^2 dx \leq C \left( \| \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} + \| \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} \right) \| \nabla \mathbf{b}_t \|_{L^2} \leq C \left( \| \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} + \| \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} \right) \| \mathbf{b}_t \|_{H^1} \| \nabla \mathbf{b}_t \|_{L^2} \leq C \left( \| \nabla \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{H^1} + \| \nabla \mathbf{u}_t \|_{L^2} \| \mathbf{b}_t \|_{L^2} \| \mathbf{b}_t \|_{H^1} \right) \| \mathbf{b}_t \|_{H^1} \leq \frac{\nu}{2} \| \text{curl} \mathbf{b}_t \|_{L^2}^2 + C \| \nabla \mathbf{u}_t \|_{L^2}^2 + C \| \mathbf{b}_t \|_{L^2}^2,$$

which implies that

$$\frac{d}{dt} \| \mathbf{b}_t \|_{L^2}^2 + \nu \| \text{curl} \mathbf{b}_t \|_{L^2}^2 \leq C_1 \| \nabla \mathbf{u}_t \|_{L^2}^2 + C \| \mathbf{b}_t \|_{L^2}^2 \tag{3.46}$$

for some positive constant $C_1$ independent of $T$. Adding (3.45) multiplied by $\frac{2C_1}{\nu}$ to (3.46) and then choosing $\delta = \frac{\mu_1 \nu}{4C_1}$, we deduce that

$$\frac{d}{dt} \left[ 2C_1 \mu_1^{-1} \left( \| \sqrt{\rho} \mathbf{u}_t \|_{L^2}^2 + \| \sqrt{\rho} \mathbf{w}_t \|_{L^2}^2 \right) + \| \mathbf{b}_t \|_{L^2}^2 \right] + C_1 \| \nabla \mathbf{u}_t \|_{L^2}^2 \leq C \left( \| \sqrt{\rho} \mathbf{u}_t \|_{L^2}^2 + \| \mathbf{b}_t \|_{L^2}^2 + \| \mathbf{u}_t \|_{L^2}^2 + \| \nabla \mathbf{w}_t \|_{H^1} \right), \tag{3.47}$$

which multiplied by $t^i$ ($i \in \{1, 2\}$) together with Gronwall's inequality and (3.17) yields (3.36). \qed
Lemma 3.5. Under the condition \((3.1)\), then there exists a positive constant \(C\) depending only on \(M_0\), \(\Omega\), \(\mu_1\), \(\mu_2\), \(\xi\), \(\lambda\), \(\nu\), and the initial data such that

\[
\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C. \tag{3.48}
\]

Proof. 1. We obtain from Lemma 2.3, Sobolev’s inequality, \((3.2)\), and Young’s inequality that for \(\sigma\) being as in Lemma 3.1,

\[
\|u\|_{W^{2,4}} \leq C (\|\rho u\|_{L^4} + \|\rho u \cdot \nabla u\|_{L^4} + \|\nabla w\|_{L^4} + \|b \cdot \nabla \sigma\|_{L^4})
\]

\[
\leq C \|\rho u\|_{L^4} + C \|\rho\|_{L^\infty} \|u\|_{L^4} + C \|\nabla w\|_{L^4} \|\nabla w\|_{H^1}^\frac{2}{3} + C \|b\|_{L^\infty} \|\nabla \sigma\|_{L^4}
\]

\[
\leq C (\|\rho u\|_{L^4} + C \|u\|_{H^1}^2 + \|\nabla w\|_{H^1}^2 + \|b\|_{H^2}^2) + C \|\nabla w\|_{L^4}^2 + C t^{-\frac{4}{3}}
\]

which combined with Sobolev’s embedding theorem, \((3.31)\), and \((3.9)\) implies that

\[
\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C \int_0^T \|u\|_{W^{2,4}} \, dt \leq C \int_0^T \|\rho u\|_{L^4} \, dt + C. \tag{3.49}
\]

2. By Hölder’s inequality, Sobolev’s inequality, and \((3.2)\), we have

\[
\|\rho u\|_{L^4} \leq \|\rho\|_{L^\infty} \|\sqrt{\rho} u\|_{L^2} \|\nabla u\|_{L^2} \leq C \|\sqrt{\rho} u\|_{L^2} \|\nabla u\|_{L^2},
\]

which together with Hölder’s inequality implies for any \(0 \leq a < b < \infty\),

\[
\int_a^b \|\rho u\|_{L^4} \, dt \leq C \int_a^b t^{\frac{1}{2}} \|\sqrt{\rho} u\|_{L^2} \cdot t^{-\frac{1}{2}} \|\nabla u\|_{L^2} \, dt. \tag{3.50}
\]

As a consequence, if \(T \leq 1\), we obtain from \((3.50)\), Hölder’s inequality, and \((3.36)\) that

\[
\int_0^T \|\rho u\|_{L^4} \, dt \leq C \sup_{0 \leq t \leq T} (t \|\sqrt{\rho} u\|_{L^2})^{\frac{1}{2}} \left( \int_0^T t^{-\frac{1}{2}} \, dt \right)^{\frac{1}{2}} \left[ \int_a^b t \|\nabla u\|_{L^2} \, dt \right]^{\frac{1}{2}}
\]

\[
\leq CT^\frac{a}{2} \leq C. \tag{3.51}
\]

If \(T > 1\), one deduces from \((3.51)\), \((3.50)\), Hölder’s inequality, and \((3.36)\) that

\[
\int_0^T \|\rho u\|_{L^4} \, dt = \int_0^1 \|\rho u\|_{L^4} \, dt + \int_1^T \|\rho u\|_{L^4} \, dt
\]

\[
\leq C + C \sup_{1 \leq t \leq T} (t^2 \|\rho u\|_{L^2})^{\frac{1}{2}} \left( \int_1^T t^{-\frac{2}{3}} \, dt \right)^{\frac{1}{2}} \left[ \int_a^b t^2 \|\nabla u\|_{L^2} \, dt \right]^{\frac{1}{2}}
\]

\[
\leq C + C \left( 1 - T^{-\frac{4}{3}} \right)^{\frac{1}{2}} \leq C. \tag{3.52}
\]

Hence, we infer from \((3.51)\) and \((3.52)\) that

\[
\int_0^T \|\rho u\|_{L^4} \, dt \leq C.
\]

This combined with and \((3.49)\) leads to the desired \((3.48)\). □
Let $q$ be as in Theorem 1.1, then there exists a positive constant $C$ depending only on $M_0$, $\Omega$, $\mu_1$, $\mu_2$, $\xi$, $\lambda$, $\nu$, $q$, and the initial data such that, for $r \in [2, q)$,
\[
\sup_{0 \leq t \leq T} \left( \|\rho\|_{W^{1,q}} + \|\rho_t\|_{L^r} \right) \leq C. \tag{3.53}
\]

Proof. Taking spatial derivative $\nabla$ on the transport equation (1.1)$_1$ together with (1.1)$_4$ leads to
\[
\partial_t \nabla \rho + \mathbf{u} \cdot \nabla^2 \rho + \nabla \mathbf{u} \cdot \nabla \rho = 0.
\]
Thus standard energy methods yield that, for $q \in (3, 6]$,
\[
\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \rho\|_{L^q},
\]
which combined with Gronwall’s inequality and (3.48) gives that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^q} \leq C. \tag{3.54}
\]
Noticing the following fact
\[
\|\rho_t\|_{L^r} = \|\mathbf{u} \cdot \nabla \rho \|_{L^r} \leq \|\nabla \rho\|_{L^q} \|\mathbf{u}\|_{L^\infty \omega} \leq \|\nabla \rho\|_{L^q} \|\nabla \mathbf{u}\|_{L^2},
\]
which together with (3.54) and (3.17) yields
\[
\sup_{0 \leq t \leq T} \|\rho_t\|_{L^r} \leq C. \tag{3.55}
\]
Thus, the desired (3.53) follows from (3.2), (3.54), and (3.55).

\[\square\]

Lemma 3.7. Let $q$ be as in Theorem 1.1, then there exists a positive constant $C$ depending only on $M_0$, $\Omega$, $\mu_1$, $\mu_2$, $\xi$, $\lambda$, $\nu$, $q$, and the initial data such that
\[
\sup_{0 \leq t \leq T} \left[ t \left( \|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{P}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{H^1}^2 + \|\mathbf{b}\|_{H^2}^2 \right) \right] + \int_0^T t \left( \|\mathbf{u}\|_{H^3}^2 + \|\nabla \mathbf{P}\|_{H^1}^2 + \|\mathbf{w}\|_{H^3}^2 + \|\mathbf{b}\|_{H^3}^2 \right) \, dt \leq C. \tag{3.56}
\]

Proof. 1. We obtain from (3.21), (3.22), (3.2), Sobolev’s inequality, and (3.17) that
\[
\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{P}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \\
\leq C \left( \|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \frac{1}{2} \right) \\
+ C \left( \|\mathbf{b}\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 \right)
\]
\[
\leq C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\]
\[
\leq C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\]
\[
\leq C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\]
\[
\leq C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2 + \frac{1}{2} \left( \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{b}\|_{H^2}^2 \right) \right)
\]
\[
+ C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 \right), \tag{3.57}
\]
which together with (3.36) and (3.35) yields
\[
\sup_{0 \leq t \leq T} \left[ t \left( \|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{P}\|_{L^2}^2 + \|\mathbf{b}\|_{H^2}^2 \right) \right] \leq C. \tag{3.58}
\]
From (3.23), (3.2), Sobolev’s inequality, and (3.17), one has
\[
\|\nabla w\|_{H^1}^2 \leq C \left( \|\rho w\|_{L^2}^2 + \|\rho \mathbf{b} \cdot \nabla \mathbf{w}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\leq C \left( \|\nabla \mathbf{w}\|_{L^2}^2 + C \|\rho\|_{L^{\infty}} \|\mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\leq C \left( \|\nabla \mathbf{w}\|_{L^2}^2 + C \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\leq C \left( \|\nabla \mathbf{w}\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathbf{w}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\]
which combined with (3.36) and (3.17) yields
\[
\sup_{0 \leq t \leq T} (t \|\nabla w\|_{H^1}^2) \leq C.
\] (3.60)

2. We get from (3.20), (3.2), (3.54), Sobolev’s inequality, (3.57), and (3.59) that
\[
\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{P}\|_{H^1}^2 \leq C \left( \|\rho \mathbf{u}\|_{L^2}^2 + \|\rho \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{b} \cdot \nabla \mathbf{b}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\leq C \left( \|\nabla \mathbf{w}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\nabla \rho\|_{L^2}^2 + C \|\mathbf{b} \cdot \nabla \mathbf{b}\|_{L^2}^2 + C \|\nabla \mathbf{w}\|_{L^2}^2 \right)
\leq C \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + C \|\nabla \mathbf{b}\|_{L^2}^2 \right)
\]

which together with (3.17), (3.4), (3.36), and (3.58) implies
\[
\int_0^T t \left( \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{P}\|_{H^1}^2 \right) dt \leq C.
\] (3.61)

Similarly, one can deduce that
\[
\int_0^T t \|\mathbf{w}\|_{H^1}^2 dt \leq C, \quad \int_0^T t \|\mathbf{b}\|_{H^1}^2 dt \leq C.
\] (3.62)

Hence, the desired (3.56) follows from (3.58) and (3.60)–(3.62).

With Lemmas 3.1–3.7 at hand, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We argue by contradiction. Suppose that (1.5) were false, that is, (3.1) holds. Note that the general constant $C$ in Lemmas 3.1–3.7 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.7 are uniformly bounded for any $t < T^*$. Hence, the function
\[
(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})(x, T^*) := \lim_{t \to T^*} (\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})(x, t)
\]
satisfies the initial condition (1.4) at $t = T^*$. Therefore, taking $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})(x, T^*)$ as the initial data, one can extend the local strong solution beyond $T^*$, which contradicts the maximality of $T^*$. Thus we finish the proof of Theorem 1.1.

4. Proof of Theorem 1.2. Throughout this section, we denote

\[
C_0 := \left( \|\sqrt{\rho_0} \mathbf{u}_0\|_{L^2}^2 + \|\sqrt{\rho_0} \mathbf{w}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2 \right).
\]
Note that Lemma 3.1 also holds true due to its independence of the condition (3.1).
Lemma 4.1. Let \((\rho, u, w, b)\) be a strong solution to the problem (1.1)–(1.3) on \((0, T), \) then there exist positive constants \(C\) and \(L\) depending only on \(\|\rho_0\|_{L^\infty}, \Omega, \mu_1, \mu_2, \lambda, \nu, \) and \(\xi\) such that, for any \(t \in (0, T), \)

\[
\sup_{0 \leq s \leq t} \left( \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \right) \\
+ \frac{1}{2L} \int_0^t \left( \|u\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 + \|b\|_{H^2}^2 \right) \, ds \\
\leq \mu \left( \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|\text{curl } b_0\|_{L^2}^2 \right) \\
+ CC_0 \sup_{0 \leq s \leq t} \|\nabla u\|_{L^2}^2 + C \sqrt{C_0} \sup_{0 \leq s \leq t} \|\text{curl } b\|_{L^2}^2 \\
+ C \sqrt{C_0} \sup_{0 \leq s \leq t} \left( \|\nabla u\|_{L^2} + \|\text{curl } b\|_{L^2} \right) \\
\times \int_0^t \left( \|u\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 + \|b\|_{H^2}^2 \right) \, ds + CC_0, \tag{4.1}
\]

where \(\mu := \mu_1 + \mu_2 + \lambda + 2\xi + 8\xi d^2 + \nu\) with \(d\) being the diameter of \(\Omega.\)

**Proof.** 1. Setting

\[
C(t) := \int \left( (\mu_1 + \xi) |\nabla u|^2 + \nu |\text{curl } b|^2 + \mu_2 |\nabla w|^2 + \lambda (\text{div } w)^2 + 4\xi |w|^2 - 4\xi \text{curl } u \cdot w \right) \, dx,
\]

then we have

\[
\mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \\
\leq C(t) \leq (\mu_1 + 2\xi) \|\nabla u\|_{L^2}^2 + (\mu_2 + \lambda + 8\xi d^2) \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2, \tag{4.2}
\]

due to (3.28) and the following Poincaré inequality (see [21, (A.3), p. 266])

\[
\|f\|_{L^2}^2 \leq d^2 \|\nabla f\|_{L^2}^2, \text{ for } f \in H^1_0(\Omega),
\]

with \(d\) the diameter of \(\Omega.\) Hence, integrating (3.19) with respect to the time over \((0, t)\) together with (4.2) gives rise to

\[
\sup_{0 \leq s \leq t} \left( \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\text{curl } b\|_{L^2}^2 \right) \\
+ \int_0^t \left( \|\sqrt{\rho} u_s(s)\|_{L^2}^2 + \|\sqrt{\rho} w_s(s)\|_{L^2}^2 + \|b_s(s)\|_{L^2}^2 \right) \, ds \\
\leq \mu \left( \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|\text{curl } b_0\|_{L^2}^2 \right) + 4 \sup_{0 \leq s \leq t} \int |b|^2 |\nabla u| \, dx \\
+ C \int_0^t \left( \rho |u|^2 |\nabla u|^2 + \rho |u|^2 |\nabla w|^2 + |b|^2 |\nabla u|^2 + |u|^2 |\nabla b|^2 \right) \, dx \, ds, \tag{4.3}
\]

where \(\mu := \mu_1 + \mu_2 + \lambda + 2\xi + 8\xi d^2 + \nu.\) Let \(K\) be as in (3.24), integrating (3.24) multiplied by \(\frac{1}{2K}\) with respect to \(t\) and adding the resulting inequality to (4.3), we
derive that
\[
\begin{align*}
& \sup_{0 \leq s \leq t} (\mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + \nu \|\nabla b\|_{L^2}^2) \\
& \quad + \frac{1}{2K} \int_0^t (\|u\|^2_{H^2} + \|\nabla w\|^2_{H^1} + \|b\|^2_{H^2}) \, ds \\
& \leq \mu \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 \right) + 4 \sup_{0 \leq s \leq t} \int |b|^2 |\nabla u| \, dx \\
& \quad + K \int_0^t \left(\rho |u|^2 |\nabla u|^2 + |u|^2 |\nabla w|^2 + |b|^2 |\nabla u|^2 + |u|^2 |\nabla b|^2 + |b|^2 |\nabla b|^2\right) \, dx 
\end{align*}
\]  \tag{4.4}

for some positive constant \( K \) depending only on \( \Omega, \|\rho_0\|_{L^\infty}, \mu_1, \mu_2, \xi, \lambda, \) and \( \nu \).

2. By Hölder’s inequality, Sobolev’s inequality, (2.1), and (3.3), we have
\[
\int |b|^2 |\nabla u| \, dx \leq \|b\|^2_{L^2} \|\nabla u\|_{L^2} \leq \|b\|^2_{L^2} \|\nabla u\|^2_{L^2} \\
\leq C \|b\|^2_{L^2} \|\nabla u\|_{L^2} \leq \frac{\mu_1}{8} \|\nabla u\|^2_{L^2} + C \|b\|_{L^2} \|\nabla u\|^3_{L^2} \\
\leq \frac{\mu_1}{8} \|\nabla u\|^2_{L^2} + C \sqrt{C_0} \|\nabla u\|^3_{L^2},
\]
which yields
\[
4 \sup_{0 \leq s \leq t} \int |b|^2 |\nabla u| \, dx \leq \frac{\mu_1}{2} \sup_{0 \leq s \leq t} \|\nabla u\|^2_{L^2} + C \sqrt{C_0} \sup_{0 \leq s \leq t} \|\nabla u\|^3_{L^2}. \tag{4.5}
\]

Similarly, one has
\[
K \int \left(\rho |u|^2 |\nabla u|^2 + \rho |u|^2 |\nabla w|^2 + |b|^2 |\nabla u|^2 + |u|^2 |\nabla b|^2 + |b|^2 |\nabla b|^2\right) \, dx \\
\leq \frac{\mu_1}{2} \|\nabla u\|^2_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|^2_{L^2} + K \|\rho\|^2_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \\
+ K \|b\|_{L^2} \|b\|_{L^2} \|\nabla u\|^2_{L^2} + K \|b\|_{L^2} \|\nabla b\|^2_{L^2} \|\nabla u\|_{L^2} + K \|b\|_{L^2} \|\nabla b\|^2_{L^2} \\
\leq C \|\nabla u\|^2_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|^2_{L^2} + C \|b\|^2_{L^2} \|\nabla u\|^2_{L^2} + C \|b\|^2_{L^2} \|\nabla u\|^2_{L^2} \\
\leq C \sqrt{C_0} \|\nabla u\|^2_{L^2} \|\nabla b\|^2_{L^2} + C \|b\|_{L^2} \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{H^1} + \|b\|^2_{H^2} \\
+ C \|\nabla u\|^2_{L^2} \|\nabla b\|_{H^2},
\]
which together with (3.3) implies
\[
K \int_0^t \left(\rho |u|^2 |\nabla u|^2 + \rho |u|^2 |\nabla w|^2 + |b|^2 |\nabla u|^2 + |u|^2 |\nabla b|^2 + |b|^2 |\nabla b|^2\right) \, dx \\
\leq C \sqrt{C_0} \sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2}) \int_0^t (\|u\|^2_{H^2} + \|\nabla w\|^2_{H^1} + \|b\|^2_{H^2}) \, ds \\
+ C \sup_{0 \leq s \leq t} \|\nabla u\|^2_{L^2} \int_0^t \|\nabla b\|^2_{L^2} \, ds + \frac{1}{4K} \int_0^t \|b\|^2_{H^2} \, ds \\
\leq C \sqrt{C_0} \sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2}) \int_0^t (\|u\|^2_{H^2} + \|\nabla w\|^2_{H^1} + \|b\|^2_{H^2}) \, ds \\
+ C \sqrt{C_0} \sup_{0 \leq s \leq t} \|\nabla u\|^2_{L^2} + \frac{1}{4K} \int_0^t \|b\|^2_{H^2} \, ds. \tag{4.6}
\]

Substituting (4.5) and (4.6) into (4.4) leads to the desired (4.1). \qed
Lemma 4.2. Let \( (\rho, u, w, b) \) be a strong solution to the problem (1.1)–(1.3) on \((0, T)\) and \(\mu\) be as in Lemma 4.1. Then there exists a positive constant \(\varepsilon_0\) depending only on \(\|\rho_0\|_{L^\infty}, \Omega, \mu_1, \mu_2, \lambda, \nu,\) and \(\xi\) such that
\[
\sup_{0 \leq t \leq T} \left( \mu_1 \|\nabla u\|^2_{L^2} + \mu_2 \|\nabla w\|^2_{L^2} + \nu \|\text{curl } b\|^2_{L^2} \right)
\leq 2\mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) + C C_0, \tag{4.7}
\]
provided that
\[
\left\{ \begin{array}{l}
\|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\sqrt{\rho_0} w_0\|^2_{L^2} + \|b_0\|^2_{L^2} \leq \varepsilon_0, \\
(\|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\sqrt{\rho_0} w_0\|^2_{L^2} + \|b_0\|^2_{L^2})(\|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2}) \leq \varepsilon_0.
\end{array} \right. \tag{4.8}
\]

Proof. Define functions \(E(t)\) and \(\Phi(t)\) as follows
\[
E(t) := \sup_{0 \leq s \leq t} \left( \mu_1 \|\nabla u\|^2_{L^2} + \mu_2 \|\nabla w\|^2_{L^2} + \nu \|\text{curl } b\|^2_{L^2} \right)
+ \frac{1}{2L} \int_0^t \left( \|u\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 + \|b\|_{H^2}^2 \right) ds,
\]
\[
\Phi(t) := C_0 \sup_{0 \leq s \leq t} \left( \mu_1 \|\nabla u\|^2_{L^2} + \nu \|\text{curl } b\|^2_{L^2} \right),
\]
where \(L\) is the same as in (4.1). In view of the regularities of \(u\) and \(w\), one can obtain that both \(E(t)\) and \(\Phi(t)\) are continuous functions on \((0, T)\). By (4.1), there is a positive constant \(M\) depending only on \(\|\rho_0\|_{L^\infty}, \Omega, \mu_1, \mu_2, \lambda,\) and \(\xi\) such that
\[
E(t) \leq \mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) + M \left( \sqrt{\Phi(t)} + \Phi(t) \right) E(t) + MC_0. \tag{4.9}
\]
We set
\[
\tilde{\varepsilon}_0 := \min \left\{ \frac{1}{16M^2}, \frac{3}{32M^2} \right\},
\]
and suppose that
\[
2\mu C_0 \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) \leq \tilde{\varepsilon}_0. \tag{4.10}
\]
We claim that
\[
\Phi(t) < \min \left\{ \frac{1}{4M^2}, \frac{1}{16M^2} \right\}, \quad 0 \leq t \leq T.
\]
Otherwise, by the continuity and monotonicity of \(\Phi(t)\), there is a \(T_0 \in (0, T]\) such that
\[
\Phi(T_0) = \min \left\{ \frac{1}{4M^2}, \frac{1}{16M^2} \right\}. \tag{4.11}
\]
On account of (4.11), it follows from (4.9) that
\[
E(T_0) \leq \mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) + \frac{1}{2} E(T_0) + MC_0,
\]
and hence
\[
E(T_0) \leq 2\mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) + 2MC_0.
\]
Recalling the definition of \(E(t)\) and \(\Phi(t)\), we deduce from the above inequality and (4.10) that
\[
\Phi(T_0) \leq C_0 E(T_0) \leq C_0 \left( 2\mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl } b_0\|^2_{L^2} \right) + 2MC_0 \right)
\leq \frac{3}{2} \tilde{\varepsilon}_0 = \min \left\{ \frac{3}{32M}, \frac{3}{64M^2} \right\},
\]
provided that

\[ C_0^2 \leq \frac{\varepsilon_0}{4M}. \]

This contradicts with (4.11).

By virtue of the claim we showed in the above, we derive from (4.9) that

\[ E(t) \leq 2\mu \left( \|\nabla u_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\text{curl} b_0\|^2_{L^2} \right) + CC_0, \quad 0 < t < T, \]

provided that (4.8) holds true. This implies the desired (4.7) and consequently completes the proof of Lemma 4.2. \( \square \)

**Lemma 4.3.** Let (4.8) be satisfied and \( \sigma \) be as in Theorem 1.2, then for \( \zeta(T) := \min \{1, T\} \), there exists a positive constant \( C \) depending only on \( \Omega, \mu_1, \xi, \mu_2, \lambda, \nu, q, \) and the initial data such that

\[
\sup_{\zeta(T) \leq t \leq T} \left[ e^{\sigma t} \left( \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|\nabla w_t\|^2_{L^2} + \|\text{curl} b_t\|^2_{L^2} \right) \right] \leq C. \tag{4.12}
\]

**Proof.** 1. We obtain from Lemma 4.2 that

\[
\sup_{0 \leq t \leq T} \left( \|\nabla u\|^2_{L^2} + \|\text{curl} b\|^2_{L^2} \right) \leq C. \tag{4.13}
\]

Choosing \( s = 4 \) and \( r = 6 \) in (3.29) together with Sobolev’s inequality and (4.13) yields

\[
B'(t) + \frac{1}{2} \left( \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|b_t\|^2_{L^2} + \|\nabla w_t\|^2_{H^1} \right) \leq C \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{curl} b\|^2_{L^2} + \|\nabla w\|^2_{H^1} + \|\text{curl} b\|^2_{L^2}
\]

\[
+ C (\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{curl} b\|^2_{L^2}) + Ce^{-4\sigma t}
\]

which multiplied by \( e^{\sigma t} \) together with (3.30), (3.7), (2.1), and (4.13) implies

\[
\frac{d}{dt} (e^{\sigma t} B(t)) + e^{\sigma t} \left( \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|\nabla w_t\|^2_{H^1} + \|\text{curl} b_t\|^2_{L^2} \right) \leq C \|\nabla u\|^2_{L^2} e^{\sigma t} + \|\nabla w\|^2_{L^2} + \|\text{curl} b\|^2_{L^2} + \|\nabla w\|^2_{H^1} + \|\text{curl} b\|^2_{L^2}
\]

\[
+ C e^{\sigma t} (\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{curl} b\|^2_{L^2}) + Ce^{-3\sigma t}
\]

This along with Gronwall’s inequality, (3.30), (3.3), and (3.4) that

\[
\sup_{0 \leq t \leq T} \left[ e^{\sigma t} \left( \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\text{curl} b\|^2_{L^2} \right) \right] \leq C.
\]
2. We obtain from (3.47) multiplied by $e^{\sigma t}$, Gronwall’s inequality, and (4.15) that for $\zeta(T) := \min\{1, T\}$,

$$\sup_{\zeta(T) \leq t \leq T} \left[ e^{\sigma t} \left( \|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|\sqrt{\rho} w_{tt}\|_{L^2}^2 + \|b_t\|_{L^2}^2 \right) \right] \leq C, \tag{4.16}$$

which together with (3.57), (3.59), and (4.15) yields

$$\sup_{\zeta(T) \leq t \leq T} \left[ e^{\sigma t} \left( \|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2 + \|b\|_{H^2}^2 + \|\nabla w\|_{H^1}^2 \right) \right] \leq C. \tag{4.17}$$

Hence, the desired (4.12) follows from (4.16) and (4.17).

Now, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\varepsilon_0$ be the constant stated in Lemma 4.2 and suppose that the initial data $(\rho_0, u_0, w_0, b_0)$ satisfies (1.4) and

$$\left\{ \begin{array}{l}
\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\sqrt{\rho_0} w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \leq \varepsilon_0, \\
\left( \|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\sqrt{\rho_0} w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \right) \left( \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|\text{curl} b_0\|_{L^2}^2 \right) \leq \varepsilon_0.
\end{array} \right.$$  

According to [22, Theorem 1.1], there is a unique local strong solution $(\rho, u, w, b)$ to the problem (1.1)–(1.3). Let $T^*$ be the maximal existence time to the solution. We will show that $T^* = \infty$. Suppose, by contradiction, that $T^* < \infty$, then by (1.5), we deduce that for $(s, r) = (4, 6)$,

$$\int_0^{T^*} \|u\|_{L^6}^4 dt = \infty,$$

which combined with the Sobolev inequality $\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}$ leads to

$$\int_0^{T^*} \|\nabla u\|_{L^2}^4 dt = \infty. \tag{4.18}$$

By Lemma 4.2, for any $0 < T < T^*$, there holds

$$\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \leq C,$$

which implies that

$$\int_0^{T^*} \|\nabla u\|_{L^2}^4 dt \leq C^2 T^* < \infty,$$

contradicting to (4.18). This contradiction provides us that $T^* = \infty$, and thus we obtain the global strong solution. The exponential decay rate (1.9) follows from (4.12). Furthermore, (1.8) follows from $\int_0^T \|u\|_{L^6}^4 dt < \infty$, Lemmas 3.1–3.7, and (2.1). This finishes the proof of Theorem 1.2.

REFERENCES

[1] G. Ahmadi and M. Shahinpoor, Universal stability of magneto-micropolar fluid motions, *Internat. J. Engrg. Sci.*, 12 (1974), 657–663.
[2] H. Beirão da Veiga and F. Crispo, Sharp inviscid limit results under Navier type boundary conditions. An $L^p$ theory, *J. Math. Fluid Mech.*, 12 (2010), 397–411.
[3] L. C. Berselli and S. Spirito, On the vanishing viscosity limit of 3D Navier-Stokes equations under slip boundary conditions in general domains, *Commun. Math. Phys.*, 316 (2012), 171–198.
[4] J. L. Boldrini, M. A. Rojas-Medar, and E. Fernández-Cara, Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, *J. Math. Pures Appl.*, 82 (2003), 1499–1525.
[5] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, Springer, New York, 2013.
[6] P. Braz e Silva, F. W. Cruz, M. Loayza and M. A. Rojas-Medar, Global unique solvability of nonhomogeneous asymmetric fluids: A Lagrangian approach, *J. Differ. Equ.*, 269 (2020), 1319–1348.

[7] P. Braz e Silva, F. W. Cruz and M. A. Rojas-Medar, Vanishing viscosity for nonhomogeneous asymmetric fluids in $\mathbb{R}^3$: the $L^2$ case, *J. Math. Anal. Appl.*, 420 (2014), 207–221.

[8] P. Braz e Silva, F. W. Cruz and M. A. Rojas-Medar, Semi-strong and strong solutions for variable density asymmetric fluids in unbounded domains, *Math. Methods Appl. Sci.*, 40 (2017), 757–774.

[9] P. Braz e Silva, F. W. Cruz and M. A. Rojas-Medar, Global strong solutions for variable density incompressible asymmetric fluids in thin domains, *Nonlinear Anal. Real World Appl.*, 55 (2020), 103125, 14 pp.

[10] P. Braz e Silva, F. W. Cruz and M. A. Rojas-Medar, Weak solutions with improved regularity for the nonhomogeneous asymmetric fluids equations with vacuum, *J. Math. Anal. Appl.*, 473 (2019), 567–586.

[11] P. Braz e Silva, E. Fernández-Cara and M. A. Rojas-Medar, Vanishing viscosity for nonhomogeneous asymmetric fluids in $\mathbb{R}^3$, *J. Math. Anal. Appl.*, 332 (2007), 833–845.

[12] P. Braz e Silva and E. G. Santos, Global weak solutions for variable density incompressible fluids, *J. Math. Anal. Appl.*, 332 (2007), 833–845.

[13] F. W. Cruz and P. Braz e Silva, Error estimates for spectral semi-Galerkin approximations of asymmetric fluids with variable density, *J. Math. Fluid Mech.*, 21 (2019), 27 pp.

[14] V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag, 1986.

[15] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[16] H. Kim, A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations, *SIAM J. Math. Anal.*, 37 (2006), 1417–1434.

[17] P. L. Lions, *Mathematical Topics in Fluid Mechanics*, Oxford University Press, Oxford, 1996.

[18] G. Łukaszewicz, On nonstationary flows of incompressible asymmetric fluids, *Math. Methods Appl. Sci.*, 13 (1990), 219–232.

[19] G. Łukaszewicz, *Micropolar Fluids*, Birkhäuser, Boston, 1999.

[20] A. Lunardi, *Interpolation Theory, 3rd edition*, Edizioni della Normale, Pisa, 2018.

[21] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, 2008.

[22] T. Tang and J. Sun, Local well-posedness for the density-dependent incompressible magneto-micropolar system with vacuum, *Discrete Contin. Dyn. Syst. Ser. B*, 26 (2021), 6017-6026.

[23] G. Wu and X. Zhong, Global strong solution and exponential decay of 3D nonhomogeneous asymmetric fluid equations with vacuum, *Acta Math. Sci. Ser. B (Engl. Ed.)*, 41 (2021), 1428–1444.

[24] Z. Ye, Remark on exponential decay-in-time of global strong solutions to 3D inhomogeneous micropolar equations, *Discrete Contin. Dyn. Syst. Ser. B*, 24 (2019), 6725–6743.

[25] P. Zhang and M. Zhu, Global regularity of 3D nonhomogeneous incompressible magneto-micropolar system with the density-dependent viscosity, *Comput. Math. Appl.*, 76 (2018), 2304–2314.

[26] P. Zhang and M. Zhu, Global regularity of 3D nonhomogeneous incompressible micropolar fluids, *Acta Appl. Math.*, 161 (2019), 13–34.

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