Characterization of Separatrices in Holomorphic Dynamical Systems

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Abstract—Multiple time scales in dynamical systems lead to a bundling of trajectories onto slow invariant manifolds (SIMs). Although they are absent in two-dimensional holomorphic dynamical systems, a bundling of orbits is often observed as well. They bundle onto special trajectories called separatrices. We apply numerical methods for the approximation of SIMs to holomorphic flows and show how a separatrix between two regions of periodic orbits can be characterized topologically. Complex time reveals a new perspective on holomorphic dynamical systems.

I. INTRODUCTION

Separatrices are structures similar to slow manifolds that characterize the flow of a holomorphic function. Their identification has become important in the context of the Riemann Hypothesis (RH) (see [1]–[3]). The challenging task for a possible proof of the RH is to characterize separatrices for the 1D-holomorphic flow of the Riemann zeta function [4].

One-dimensional holomorphic dynamical systems can be interpreted as dynamical systems in $\mathbb{R}^2$ with a special structure given by the Cauchy-Riemann equations. If a holomorphic function $f : \mathbb{C} \to \mathbb{C}$, $f : (x + iy) \mapsto u(x, y) + iv(x, y)$ is regarded as two-dimensional function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $(x, y)^T$ to $(u(x, y), v(x, y))^T$, then the Cauchy-Riemann equations yield

\[
\hat{f}'(x, y) = \begin{pmatrix}
    u_x & -v_x \\
    v_x & u_x
\end{pmatrix}.
\]

(1)

II. SEPARATRIX AND THE COSH FLOW

In this holomorphic context separatrices can be defined as follows.

Definition 1 (Broughan, [1]): A trajectory $\gamma$ is a positive (negative) separatrix if for some $z \in \gamma$ the maximum interval of existence of the path commencing at $z$ and proceeding in positive (negative) time is finite. A trajectory $\gamma$ is a separatrix if it is a positive or negative separatrix.

With this definition Broughan [1] argues that separatrices are boundary components of special regions. These regions are the union of all trajectories with the qualitatively same phase space behavior, e.g., periodic orbits around the same center.

In Figure 1 trajectories of

\[
\dot{z} = \cosh(z - 0.5)
\]

(2)

bundle near the separatrices which are the lines $\Im z = k\pi$ for every $k \in \mathbb{Z}$. This phase space behavior of trajectories is similar to those of slow-fast systems. The difference is here that separatrices are in general not slow (invariant) manifolds. Indeed, there is no spectral gap because of the Cauchy-Riemann equations.

III. APPLICATION OF SIM METHODS

Due to the common bundling behavior, we apply slow manifold approximation methods like the zero-derivative-principle (ZDP) [5] and the following boundary value problem of Lebiedz and Unger [6] to holomorphic dynamical systems:

\[
\min_{z(\cdot)} \left\| \dot{z}(t_0) \right\|_{L^2}^2
\]

s.t. \[
\dot{z} = f(z), \quad t \in [t_0, t_1]
\]

(3a) \hspace{2cm} (3b)

(3c)

The latter calculates pointwise a separatrix candidate for fixed real part of $z$ at time $t_1$, which corresponds to the reaction progress variable in the calculation of slow manifolds. The numerical results of this boundary value problem for several values of $x^*$ are plotted in Figure 1. They all lie exactly on the separatrix. The same holds for the ZDP of order 1 applied to the imaginary part as variable,
In case of system (2) it is clear that the indices of all periodic solution orbits around the same center are constant. Thus it changes only at the the curves $\Im z = k\cdot\pi$ for integers $k$, which are separatrices.

V. COMPLEX TIME

In Equation (6) $f$ is a complex-valued function $f$ with complex argument. It seems natural to consider the solution $z(t)$ of (6) as a function over a complex variable, i.e., complex time. If the real time is replaced by $t = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$, then the solution of

$$\frac{d}{dt} z(t) = f(z(t))$$

assuming $f$ to be complex differentiable w.r.t. time, satisfies

$$\frac{\partial}{\partial \tau_1} z(t) = f(z(t)) \quad \text{and} \quad \frac{\partial}{\partial \tau_2} z(t) = if(z(t)).$$

$\tau_1$ is called real time and $\tau_2$ imaginary time. Since multiplication by $i$ in $\mathbb{C}$ can be interpreted as rotation by $\frac{\pi}{2}$ in $\mathbb{R}^2$, real time trajectories and imaginary time trajectories intersect orthogonally. Imaginary time allows to traverse separatrices of the real time flow and vice versa.

The holomorphic derivative $f'(z_0)$ of a simple zero $z_0$ of $f$ determines the type of the equilibrium. If $f'(z_0)$ is real resp. imaginary, then it is a node resp. center. Otherwise, it is a focus. A node of the real time flow becomes a center of imaginary time flow and vice versa. Thus, switching the sign of $f$ and proceeding to imaginary time provide ways of “manipulating” the stability and attractivity of the dynamical system in the neighborhood of a simple zero.

Separatrices of the real time flow can be approximated by maximizing the curvature of the pure imaginary time trajectories. For each (imaginary time) trajectory this yields a point close to the separatrix.

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