Uncertainty Relations for Underdamped Langevin Dynamics

Tan Van Vu* and Yoshihiko Hasegawa†
Department of Information and Communication Engineering,
Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo 113-8656, Japan
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A universal trade-off between the precision of arbitrary currents and the dissipation cost, known as the thermodynamic uncertainty relation, has been investigated for various Markovian systems. We study here the thermodynamic uncertainty relation for underdamped Langevin dynamics. By employing information inequalities, we prove for such systems that the relative fluctuation of arbitrary currents at a steady state is constrained by both the total entropy production and the average of dynamical activity. Our numerical simulations reveal that conventional relation is violated for a specific system, implying that the total entropy production is not only the quantity that constrains the fluctuation of currents in underdamped systems. We find that the dynamical activity plays important roles in the bound, which is different to the case of overdamped dynamics. Moreover, we demonstrate that our approach can be used to derive an uncertainty relation for model systems of active matter. We illustrate our results with two systems: a single-well potential system and periodically driven Brownian particle model, and numerically verify the inequalities.

Introduction.—Fluctuations in systems which are far from thermal equilibrium have been actively studied in the last two decades. One of the central results is the fluctuation-dissipation theorems [1], which connect the response of an observable to an external perturbation with the correlation function of the system in equilibrium. A number of generalizations that hold even for out-of-equilibrium systems have also been established [2–4]. Along with that, by virtue of substantial progress in stochastic thermodynamics, universal properties of the probability distribution of thermodynamic quantities (e.g., work, heat, and entropy production), the so-called fluctuation theorems [5–7], have been derived. These principles are fundamentally important and enable us to investigate the physical properties of various systems practically.

In recent years, the thermodynamic uncertainty relation (TUR), which provides a universal bound for the relative fluctuation of arbitrary currents at the steady state in a variety of stochastic processes [8–16], is an important discovery in statistical physics. The TUR quantitively states that for arbitrary current Θ in a nonequilibrium steady state, the inequality \( \text{Var}[\Theta]/\langle\Theta\rangle^2 \geq 2k_B/(\Sigma) \) holds for finite observation time. Here, \( k_B \) is the Boltzmann constant, \( \langle\Theta\rangle \) and \( \text{Var}[\Theta] \) are the mean and variance of the current, respectively, and \( \Sigma \) is the average of the total entropy production. Qualitatively, the TUR indicates that smaller current fluctuation requires a higher thermodynamic cost in the entire system. Such precision-cost trade-off relations have been analogously reported in other contexts [17, 18]. The TURs have been developed intensively for many systems [19–24], and adapted to a range of specific problems [25–31]. Recent studies have revealed a deep connection between the TUR and information inequalities in statistical inference theory [16, 32]. That is conventional and multidimensional TURs can be derived from Cramér-Rao inequality (CRI), which is a relation between the precision of an estimator and the Fisher information.

In this Letter, we study the TURs for underdamped Langevin dynamics, wherein damping does not suppress inertial effects. By applying the CRI into systems, we derive analogous TURs for both scalar and vectorial currents, which hold for finite observation time at the steady state. Specifically, we prove that the relative fluctuation of arbitrary currents is bounded from below by a quantity involving the total entropy production and the average of dynamical activity. We run numerical simulations in a system of periodically driven Brownian particle and find that the conventional TUR is violated for several parameter settings. The violation indicates that the fluctuation of currents is not only constrained by the total entropy production but also by other measures, which are suggested to be the dynamical activity as in the derived bound. The dynamical activity is a kinetic aspect of systems and plays central roles in characterizing dynamics of systems [33–38]. In equilibrium, the fluctuation of currents is bounded only by the dynamical activity because the total entropy production vanishes. In addition, we show that a tighter TUR can be obtained for a specific class of currents. We also extend our approach and derive a TUR for active matter systems, which have been paid a lot of interests recently [39–42]. Our results are empirically verified by numerical simulations.

Model.—We consider a general underdamped system of \( N \) particles, wherein the particle \( i \) is in contact with a heat reservoir in equilibrium at temperature \( T_i \). The dynamics of the system is described by a set of coupled Langevin equations as follows:

\[
\dot{r}_i = v_i, \quad m_i \dot{v}_i = -\gamma_i v_i + F_i(r) + \xi_i, \quad (1)
\]

where \( r = [r_1, \ldots, r_N]^\top \) and \( v = [v_1, \ldots, v_N]^\top \) denote the positions and velocities, respectively, \( m_i \) and \( \gamma_i \) are
the mass and damping coefficient of particle $i$, respectively, $F_i(r) = -\partial_r U(r) + f_i(r)$ is total acting force, and $\xi_i$‘s are zero-mean white Gaussian noises with covariances $\langle \xi_i(t)\xi_j(t') \rangle = 2T_i \gamma_i \delta(t-t')$. Throughout the paper, the Boltzmann constant is set $k_B = 1$. The time evolution of phase-space probability distribution function $P(r, v, t)$ can be described by Kramers equation as

$$\partial_t P(r, v, t) = \sum_{i=1}^{N} \left[ -\partial_r J_r - \partial_v J_v \right],$$

(2)

where $J_r(r, v, t) = v_i P(r, v, t)$, and $J_v(r, v, t) = 1/m_i \left[ -\gamma_i v_i + F_i(r) - T_i \gamma_i / m_i \partial_r \right] P(r, v, t)$. Hereinafter, we focus exclusively on the steady state, in which the system has stationary distribution $P^{ss}(r, v)$ and probability currents $J^{rs}_v(r, v)$ and $J^{ss}_v(r, v)$. Let $\Gamma \equiv \{ r(t), v(t) \}_{t=0}^{T}$ denote a phase-space trajectory that starts at the point $(r_0, v_0)$ and has length $T$. Along the trajectory $\Gamma$, the change in total entropy production can be defined as $\Delta S_{tot}[\Gamma] = \ln \left( P(\Gamma)/P(\Gamma') \right)$, which is a comparison of the probabilities of forward path $\Gamma$ and its time-reversed counterpart $\Gamma^\dagger \equiv \{ r(T-t), -v(T-t) \}$ [43–45]. The rate of entropy production is then calculated as [46]

$$\sigma \equiv \langle \delta_{tot}(t) \rangle = \int dr dv \sum_{i=1}^{N} \frac{m_i^2 J^{v}_{v_i}(r, v, t)^2}{T_i \gamma_i} \frac{P(r, v, t)}{P(r, v, t)},$$

(3)

where $J^{v}_{v_i}(r, v, t)$ is the irreversible current and given by $J^{v}_{v_i}(r, v, t) = -1/m_i \left[ \gamma_i v_i + T_i \gamma_i / m_i \partial_v \right] P(r, v, t)$. For arbitrary trajectory $\Gamma$, we consider a generalized observable-type current

$$\Theta[\Gamma] = \int_{0}^{T} dt \Lambda(A(r(t)))^T \circ \dot{r}(t),$$

(4)

where $\Lambda \in \mathbb{R}^{N \times 1}$ is the projection function, and $\circ$ denotes the Stratonovich product. Our aim is to derive a TUR relative to current $\Theta[\Gamma]$. In the steady state, the average of the current becomes $\langle \Theta[\Gamma] \rangle = \int dr dv \Lambda(A(r))^T P^{ss}(r, v)$.

Let us consider an auxiliary dynamics

$$\dot{v}_i = v_i, \quad m_i \dot{v}_i = H_{\theta,i}(r, v) + \xi_i,$$

(5)

where $H_{\theta,i}$ is the total force and $\theta$ is a parameter. For an arbitrary function $f[\Gamma]$, let $(f[\Gamma])_\theta \equiv \int D\Gamma f[\Gamma] P_\theta[\Gamma]$ and $\text{Var}_{\theta}[f[\Gamma]] \equiv \langle (f[\Gamma] - (f[\Gamma])_\theta)^2 \rangle_\theta$. Here, $P_\theta[\Gamma]$ is the probability of observing the trajectory $\Gamma$ generated by the auxiliary dynamics and can be expressed by path integral as [46, 47]

$$P[\Gamma] = N P_0(r_0, v_0) \prod_{i=1}^{N} \exp \left( -A_\theta[\Gamma] \right),$$

(6)

where $A_\theta[\Gamma] \equiv \int_0^T dt \left\{ \frac{1}{4T_i \gamma_i} (m_i \dot{v}_i - H_{\theta,i}(r, v))^2 \right\}$ is an Onsager-Machlup action functional and $N$ is a term that does not depend on $\theta$. The integral in Eq. (6) is interpreted as the continuum limit of an Itô sum with pre-point discretization. Note that writing the crossing term $\int dt \dot{v}_i H_{\theta,i}(r, v)$ in Stratonovich integral (mid-point discretization) results in a different form of path integral. However, it can be shown that both pre- and mid-point discretization schemes reduce to the same path-integral representation in the case of additive noise [46]. Since $(\Theta[\Gamma])_\theta$ is a function of $\theta$, i.e., $(\Theta[\Gamma])_\theta = \psi(\theta)$, we can regard $\Theta[\Gamma]$ as an estimator of $\psi(\theta)$. It has been proved that the precision of this estimator is bounded from below by Fisher information as [16]

$$\frac{\text{Var}_{\theta}[\Theta]}{(\partial \psi(\theta))_\theta^2} \geq \frac{1}{\Sigma},$$

(7)

where $\Sigma = \int \text{Var}_{\theta}[f[\Gamma]]_\theta$, i.e., $(\partial \psi(\theta))_\theta = \langle \delta_{tot}(t) \rangle$ is the Fisher information and is calculated via path integral as

$$\Sigma = \int \text{Var}_{\theta}[f[\Gamma]]_\theta = \frac{1}{2} \left\langle \int_0^T dt \sum_{i=1}^{N} (\partial_r H_{\theta,i})^2 \right\rangle_\theta.$$  

(8)

To derive a bound on current fluctuation, we use the virtual perturbation technique [48] and consider the auxiliary dynamics with the following modified drift terms

$$H_{\theta,i}(r, v) = -\gamma_i v_i + \frac{1}{(1+\theta)^2} F_i(r)$$

$$+ \frac{1}{m_i} (1 - \frac{1}{1+\theta}) \partial_v P^{ss}(r, v)/(1+\theta)$$

(9)

This auxiliary dynamics has the steady-state distribution $P^{ss}_{\theta}(r, v) = P^{ss}(r, v/(1+\theta))/(1+\theta)^N$. The current of auxiliary dynamics is related to that of the original dynamics as $(\Theta[\Gamma])_\theta = \langle 1+\theta \rangle (\Theta[\Gamma])$. By letting $\theta = 0$ in Eq. (7), we obtain the following inequality:

$$\frac{\text{Var}[\Theta]}{(\Theta)^2} \geq \frac{2}{\Sigma},$$

(10)

where $\Sigma = \int (9\sigma + 4\bar{Y}) + \Omega$ and

$$\bar{Y} = \sum_{i=1}^{N} \left( \frac{1}{T_i \gamma_i} (F_i(r))^2 - 3\frac{\gamma_i}{T_i} (v_i^2) + \frac{2\gamma_i}{m_i} \right),$$

(11)

$$\Omega = 2 \left\langle \left( \sum_{i=1}^{N} v_i \partial_v P^{ss}(r, v)/P^{ss}(r, v) \right)^2 \right\rangle - 2N^2.$$  

(12)

The inequality in Eq. (10) is the main result of our paper, which holds for arbitrary time scale. As can be seen, the lower bound is not equal to the total entropy production as in conventional TUR of overdamped systems, wherein $\Sigma = \int \sigma$. In addition to the total entropy production, the bound $\Sigma$ contains $\bar{Y}$ which involves the moments of forces and velocities, and a boundary term $\Omega$ which is always nonnegative as $\Omega = 2 \langle (\partial_r \ln P_0(r_0, v_0))^2 \rangle_{\theta=0}$. In the large time limit, i.e., $T \to \infty$, the boundary term can
be neglected; thus, our result reduces to a bound that has
been derived for one-dimensional system by using large
deformation theory in Ref. [49].

Now, let us interpret the physical meaning of the term Υ. The action functional in path integral with mid-point
discretization scheme is [50]

\[-\mathcal{A}^m_\mu[\Gamma] = - \int_0^T dt \left\{ \frac{1}{4T_i \gamma_i} \left( m_i \dot{v}_i + \gamma_i v_i - F_i \right)^2 - \frac{\gamma_i}{2m_i} \right\}. \tag{13}\]

The functional \( \mathcal{A}^m_\mu \) can be decomposed into two contributions: time-antisymmetric (S) and time-symmetric (\( K_i, K_i^* \)) components [51] as

\[-\mathcal{A}^m_\mu[\Gamma] = S_i[\Gamma] + K_i[\Gamma] + K_i^*[\Gamma], \]

where

\[S_i[\Gamma] = - \int_0^T dt \frac{1}{2T_i} \left( m_i \dot{v}_i - F_i \right) \circ v_i, \tag{14}\]

\[K_i[\Gamma] = \int_0^T dt \left\{ \frac{1}{4T_i \gamma_i} \left( 2m_i \dot{v}_i \circ F_i - \gamma_i^2 v_i^2 - F_i^2 \right) + \frac{\gamma_i}{2m_i} \right\}, \tag{15}\]

\[K_i^*[\Gamma] = - \int_0^T dt \frac{m_i^2 \dot{v}_i^2}{2T_i \gamma_i}. \tag{16}\]

The time-antisymmetric part is corresponding to the integrated entropy flux from the system into the reservoir and is thermodynamically consistent with the definition of medium entropy production \( \Delta s_m \equiv \sum_{i=1}^N \frac{1}{T_i} \int_0^T dt \left[ \gamma_i v_i - \xi_i \right] \circ v_i. \) The kinetic terms \( K_i^* \) relates to the mean square velocity of the particle and may quantitatively amount of activity. However, \( K_i \) should be interpreted as a part of the functional measure and determines the functional space over which to integrate [52]. Specifically, \( K_i^* \) collects paths regular enough, those for which \( d\dot{v}_i^2/dt \) remains finite when \( dt \to 0 \). The time-symmetric term \( K_i \) is identified as the dynamical activity, which has been introduced in the literature [29, 53–57]. In discrete-state Markov jump processes, the dynamical activity characterizes the time scale of the system and serves as an essential term in speed-limit inequality [38] and kinetic uncertainty relation [58]. Taking the average of \( K_i \), we explicitly obtain

\[\aver{K_i} = \frac{T}{4} \left( \frac{1}{T_i \gamma_i} \aver{F_i(r)^2} - 3\frac{\gamma_i}{T_i} \aver{\dot{v}_i^2} + 4\frac{\gamma_i}{m_i} \right). \tag{17}\]

It can be easily confirmed that \( \Upsilon = 4/\aver{T} \sum_{i=1}^N \aver{K_i}. \) Therefore, the term \( \Upsilon \) in the lower bound of TUR is exactly the average of dynamical activity. This implies that the fluctuation of arbitrary currents in underdamped systems is not only constrained by total entropy production, but also by its mean dynamical activity. The role of dynamical activity in the TUR will be discussed later.

Next we show that a tighter TUR can be achieved for a certain class of currents. Let us consider a current of general form \( \Theta[\Gamma] = \int_0^T dt \mathbf{A}(r, v)^\top \circ \dot{r} \), where the projection function \( \mathbf{A}(r, v) \) involves the velocities variable. Suppose that \( \mathbf{A}(r, v) \) has the property that \( \mathbf{A}(r, (1 + \theta) v) = (1 + \theta)^d \mathbf{A}(r, v) \) for all \( \theta \in \mathbb{R} \) and some nonnegative integer \( d \). For example, \( \mathbf{A}(r, v) \) can be a homogeneous polynomial of \( v \), i.e., all nonzero terms have the same degree \( d \). From the relation \( \theta \partial_\theta\Theta_{\theta=0} = (d + 1)\Theta \), we have \( \partial_\theta\Theta_{\theta=0} = (d + 1)\Theta \). Consequently, we obtain a tighter TUR for such current as

\[\frac{\Var[\Theta]}{\aver{\Theta}^2} \geq \frac{2(d+1)^2}{\Sigma}. \tag{18}\]

When the project function contains only \( r \), i.e., \( d = 0 \), Eq. (18) reduces to Eq. (10).

For the equilibrium systems (i.e., the external force \( f_i(r) = 0 \) and the temperature \( T_i = T \) for all \( i = 1, \ldots, N \)), the steady-state distribution is of a Maxwell-Boltzmann type

\[P^\text{eq}(r, v) = C \exp \left\{ \frac{-1}{T} \frac{1}{2} \sum_{i=1}^N m_i v_i^2 + U(r) \right\}, \tag{19}\]

where \( C \) is the normalizing constant. It can be observed that in the equilibrium overdamped systems, the average of current and total entropy production always vanish; thus, the conventional TUR becomes a trivial inequality. However, for underdamped dynamics, the current does not always vanish even in equilibrium state, for instance, \( \mathbf{A}(r, v) = v \). This implies that the conventional TUR does not hold for equilibrium underdamped systems as the total entropy production is zero. On the other hand, the dynamical activity is always positive, i.e., \( \aver{K_i} = T/4 \left( \aver{F_i(r)^2} / (T \gamma_i) + \gamma_i / m_i \right) > 0 \). Therefore, the fluctuation of currents in equilibrium state is bounded only by the average of dynamical activity.

In what follows, the equality condition of the derived TUR is discussed. The lower bound in Eq. (10) can be attained if and only if the equality condition in Eq. (7) is satisfied with \( \theta = 0 \). Equivalently, the following relation must hold for arbitrary trajectory \( \Gamma \)

\[\partial_\theta \ln P_\theta[\Gamma]_{\theta=0} = \mu \left( \Theta[\Gamma] - \psi(0) \right), \tag{20}\]

where \( \mu \) is an arbitrary scaling coefficient. However, it can be proved that the lower bound cannot be attained [46]. This is different with the case of overdamped systems, in which the TUR is saturated in the case of near-equilibrium systems [13] or a specific choice of the current and drift term [10].

**Multidimensional TUR.**—Generally, there exists a correlation between the currents in high-dimensional real-world systems. Simultaneously observing multiple currents is expected to obtain more statistical information of the distribution. Therefore, multidimensional TUR which includes several currents in the observable provides a tighter bound than scalar TUR [32]. Such bound can be applied to study the trade-off relation between power
and efficiency in steady-state heat engines [32, 59]. Here, we derive multidimensional TUR for underdamped systems. We consider a vectorial observable \( \Theta \in \mathbb{R}^{m \times 1} \), defined by

\[
\Theta[\Gamma] = \int_0^T dt \Lambda(r(t)) \circ \dot{r}(t),
\]  

(21)

where \( \Lambda \in \mathbb{R}^{m \times n} \) is an arbitrary matrix-valued function of \( r \). By applying the CRI for multivariate estimator, we obtain the following inequality:

\[
\text{Cov}_\theta(\Theta) \geq \partial_i(\Theta)_{\theta}^I(\theta)^{-1}\partial_i(\Theta)_{\theta}^\top,
\]  

(22)

where \( \text{Cov}_\theta(\Theta) \in \mathbb{R}^{m \times m} \) is the covariance matrix of the estimator \( \Theta \), and the matrix inequality \( X \succeq Y \) expresses that \( X - Y \) is positive semi-definite. By considering the same auxiliary dynamics as in Eq. (9), we obtain a multidimensional TUR as

\[
\text{Cov}(\Theta) \succeq \frac{2}{\Sigma}(\Theta)_{\Theta}^\top.
\]  

(23)

In the case of scalar observable, this inequality becomes the TUR in Eq. (10). From Eq. (23), one can derive various bounds for current fluctuation. For instance, by applying Eq. (23) for two-dimensional currents, i.e., \( \Theta = [\Theta_1, \Theta_2]^\top \), the condition of positive semi-definite matrix yields a tighter bound as

\[
\text{Var}[\Theta_1] \geq \frac{2}{\Sigma}(\Theta_1)^2 + \sup_{\Theta_2} \left( \frac{\text{Cov}(\Theta_1, \Theta_2) - \frac{\Sigma}{2}(\Theta_1)(\Theta_2)^\top}{\text{Var}[\Theta_2] - \frac{\Sigma}{2}(\Theta_2)^2} \right).
\]  

(24)

In addition to the variance of individual current, above inequality also involves the correlation of two currents. A remarkable point in Eq. (24) is that if there exists a current \( \Theta_1 \) that satisfies \( \text{Var}[\Theta_1] = 2/\Sigma(\Theta_1)^2 \), then \( \text{Cov}(\Theta_1, \Theta_2) = 2/\Sigma(\Theta_1)(\Theta_2) \) holds for an arbitrary non-vanishing current \( \Theta_2 \). Although the equality condition of the derived TUR cannot be attained, inequalities like Eq. (24) provide insight into the correlation of currents. In particular, for overdamped dynamics, the current \( \Theta_{\text{tot}} \) of stochastic total entropy production with a specific form of drift function satisfies the equality condition [16]; thus, \( \text{Cov}[\Theta_{\text{tot}}; \Theta] = 2(\Theta) \) for arbitrary current \( \Theta \), which characterizes a universal property of stochastic entropy production. Another direction is deriving an inequality directly from the definition of semidefinite positive matrix, \( \Xi^\top \text{Cov}(\Theta) - 2/\Sigma(\Theta)(\Theta)^\top \Xi \geq 0 \) for all \( \Xi \in \mathbb{R}^{m \times 1} \) (since if \( \Xi \) is a nonsingular positive semi-definite matrix so is \( \Xi^{-1} \)). By choosing \( \Xi = (\Theta) \), we obtain the following inequality [46]:

\[
(\Theta)^\top \text{Cov}(\Theta)^{-1}(\Theta) \leq \frac{\Sigma}{2},
\]  

(25)

which relates the means and covariances of currents. For uncorrelated currents \( \Theta \), i.e., \( \text{Cov}[\Theta_i, \Theta_j] = 0 \) for all \( i \neq j \), one can obtain a bound from Eq. (25) as

\[
\sum_{i=1}^M \frac{(\Theta_i)^2}{\text{Var}[\Theta_i]} \leq \frac{\Sigma}{2}.
\]  

(26)

Equations (25) and (26) have analogously appeared in Ref. [32] for overdamped Langevin dynamics. Equation (26) can be considered a generalization of Eq. (10). In what follows, we numerically verify our results with several systems.

**Examples.—**For illustration, we first consider an equilibrium system under the symmetric single-well potential \( U(r) = \alpha r^n/(2n) \) with \( n \) is a positive integer. The total force acting on the particle is \( F(r) = -\alpha r^{n-1} \). For this system, the dynamical activity \( \Upsilon \) and the boundary term \( \Omega \) can be calculated analytically, i.e.,

\[
\Upsilon = \frac{\alpha^2}{T \gamma} \left( \frac{\alpha}{2n \gamma} \right)^{-2+1/n} \frac{\Gamma(2-\frac{1}{n})}{\Gamma(\frac{1}{n})} + \gamma/m, \quad \Omega = 4,
\]  

(27)

where \( \Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \) is the gamma function. Since the total entropy production vanishes, the lower bound \( \Sigma \) can be expressed as \( \Sigma = 4T \Upsilon + 4 \). We consider current \( \Theta[\Gamma] = \int_0^T dt v^2 \), which is proportional to the accumulated kinetic energy of the particle. According to Eq. (18), the TUR corresponding to this current becomes \( \text{Var}(\Theta)/\langle \Theta \rangle^2 \geq 8/\Sigma \). To validate this inequality, we randomly sample parameters \( m, \gamma, \alpha, T \), and \( \mathcal{T} \) (parameter ranges are shown in the caption of Fig. 1), and numerically solve the Langevin equation \( \dot{X}_S \) times with time step \( \Delta t = 2 \times 10^{-4} \) for each of the selected parameter settings. The value of \( n \) is chosen from the set \{1, 2, 3\}. The case \( n = 1 \) corresponds to the harmonic oscillator which is a fundamental model in physics. For each random realizations of parameters, we calculate \( \mathcal{E}_{\text{eq}} = 8\langle \Theta \rangle^2/(\Sigma \cdot \text{Var}[\Theta]) \) which should be \( \mathcal{E}_{\text{eq}} \leq 1 \), and

![FIG. 1. Schematics of (a) single-well potential system and (b) periodically driven Brownian particle model. (c) Numerical verification of the TURs. The parameters are \( m, \gamma, \alpha, T \in [0, 10] \), and \( \mathcal{T} \in [1, 10] \). Blue circles corresponds to \( \mathcal{E}_{\text{eq}} \) in single-well potential system. Green circles and red triangles denote \( \mathcal{E}_{\text{eq}} \) (derived TUR), \( \mathcal{E}_{\text{con}} \) (conventional TUR), respectively, in driven particle model. The TURs are satisfied if \( \mathcal{E} \leq 1 \).](image-url)
plot $E_{\text{eq}}$ as a function of $\Sigma$ in Fig. 1(c) (blue circles). The dashed line corresponds to the saturated case of the TUR. It can be observed that all blue circles are located below the line, indicating that the TUR is satisfied for all parameter settings.

Next, we study a Brownian particle circulating on a ring of length $L$ under a periodic potential $U(r) = \alpha L / (2 \pi n \cos (2 \pi r / L))$ with $n > 0$ is an integer. The total force acting on the particle is $F(r) = -\partial_r U(r) + f_{\text{nc}}$, where $f_{\text{nc}}$ is a constant external force that drives the particle out of equilibrium. In the steady state, the rate of total entropy production becomes $\sigma = f_{\text{nc}}(v) / T$. We validate the TUR in Eq. (10) for $\Theta[\Gamma] = \int_0^T dt \dot{r}$ with Monte Carlo simulations. This current is corresponding to the accumulated distance traveled by the particle. We set $L = 1$ and randomly choose $f_{\text{nc}} \in [0.1, 10]$ and $n \in [1, 10]$, while other parameter ranges and the simulation setting are the same as in the first example. According to Eq. (10), we have $\text{Var}[\Theta] / \langle \Theta \rangle^2 \geq 2 / \Sigma$. For each parameter setting, we numerically evaluate $E_{\text{eq}} = 2 \langle \Theta \rangle^2 / \left( (\Sigma - \Omega) \text{Var}[\Theta] \right)$ to accurately calculating $\Omega$ is difficult, and plot $E_{\text{eq}}$ as a function of $\Sigma - \Omega$ in Fig. 1(c). Since $\Omega$ is nonnegative, the validity of the TUR is empirically verified if $E_{\text{eq}} \leq 1$ holds. We note that it is possible that $E_{\text{eq}} > 1$ even when the TUR is satisfied. Simultaneously, we also calculate $E_{\text{con}} = 2 \langle \Theta \rangle^2 / (\Sigma \text{Var}[\Theta])$ which corresponds to the conventional TUR, and plot in same figure (red triangles). The case that $E_{\text{con}} > 1$ implies the violation of the conventional relation. As can be seen, all green circle points lie below the dashed line wherein the equality is attained; thus, it can be concluded that Eq. (10) holds for all realizations. On the contrary, some triangle points locate above the line, which indicates that the conventional TUR is violated for several parameter settings.

**TUR in active matter systems**—We here derive the TUR for active Ornstein-Uhlenbeck particles (AOPUs), which is a model system of active matter and has been studied in the literature [60, 61]. The system contains $N$ self-propelled particles and the dynamics of the particle $i$ evolves as

$$\dot{r}_i = -\mu \partial_r \Phi(r) + \eta_i, \quad \tau \eta_i = -\eta_i + \xi_i,$$

(28)

where $\mu$ is the mobility of particles, $\Phi(r)$ is an interaction potential, and $\eta_i$’s are Ornstein-Uhlenbeck processes with covariances $\langle \eta_i(t) \eta_j(t') \rangle = D_i \delta_{ij} \exp \left( -|t - t'| / \tau \right) / \tau$. Here, $\tau$ is the persistent time. In the limit $\tau \to 0$, $\eta_i$’s become white noise with delta-function covariances, and $r_i$’s become Markovian processes. The system can be mapped in to an underdamped dynamics by introducing velocities $v_i = \dot{r}_i$ and taking time derivative of Eq. (28) [60, 61]

$$\tau \dot{v}_i = -v_i - \mu \left( 1 + \tau \sum_{k=1}^N v_k \partial_{r_k} \right) \partial_r \Phi(r) + \xi_i.$$

(29)

By extending our approach, it is easy to obtain $\text{Var}[\Theta] / \langle \Theta \rangle^2 \geq 2 / \Sigma$ [46], where $\Sigma = \Omega + \mathcal{T} (9\sigma - 4\Psi)$. Here, $\Psi$ is given by

$$\Psi = \sum_{i=1}^N \frac{v_i^2}{D_i} \left( \sum_{j=1}^N \langle v_i v_j \partial_{r_i} F_j(r) \rangle + 2 \left( \sum_{j=1}^N v_j G_{ij}(r) \right)^2 \right) + 3D_i / \tau^2 (G_{ij}(r)),$$

(30)

and $G_{ij}(r) = -\delta_{ij} / \tau - \mu \partial_{r_i} \Phi, F_i(r) = -\mu / \tau \partial_r \Phi$. We note that the definition of total entropy production for AOPUs is not unique and depends on the chosen coarse-grained model. The definition employed in our paper is the same as in Ref. [60].

**Conclusions.**—In this Letter, we have derived TURs for arbitrary currents in underdamped Langevin dynamics by using information inequalities. Our results indicated that the fluctuation of currents is controlled not only by the total entropy production but also by the average of dynamical activity, which has been reported in other contexts [29]. Applying the statistical inference theory into stochastic thermodynamics is shown to be a promising approach in the development of thermodynamic bounds. As future works, deriving the TUR for general currents of form $\int dt \left( \langle A_1(r, v) \dot{v} + A_2(r, v) \dot{v} \rangle \right)$ or for other relevant systems requires further investigation.

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