TWISTED $K$-HOMOLOGY THEORY, TWISTED $Ext$-THEORY

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Abstract. These are notes on twisted $K$-homology theory and twisted $Ext$-theory from the $C^*$-algebra viewpoint, part of a series of talks on “$C^*$-algebras, noncommutative geometry and $K$-theory”, primarily for physicists.

Index of notation

• $\mathcal{K}$ is the algebra of compact operators on a (separable, infinite dimensional) Hilbert space $\mathcal{H}$.
• $Aut(\mathcal{K})$ is the group of automorphisms of $\mathcal{K}$.
• $PU = PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ is the group of projective unitary automorphisms of the Hilbert space $\mathcal{H}$. We will often identify $PU$ with $Aut(\mathcal{K})$ using the canonical isomorphism between these groups.
• $\mathcal{B}$ will often denote the algebra $C_0(X, \mathcal{E}_H)$ of sections, vanishing at infinity, of the unique locally trivial bundle $\mathcal{E}_H$ over $X$ with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ whose Dixmier-Douady invariant (see Introduction), $\delta(\mathcal{E}_H) = [H] \in H^3(X, \mathbb{Z})$.
• $\mathcal{A}$ will often denote an algebra obtained as an extension of $\mathcal{B}$ by $\mathcal{K}$.
• $P_H$ is the unique principal $Aut(\mathcal{K})$-bundle over $X$ whose Dixmier-Douady invariant, $\delta(P_H) = [H] \in H^3(X, \mathbb{Z})$.
• $\mathcal{F}$ denotes the space of all Fredholm operators on a Hilbert space $\mathcal{H}$.
• $\mathcal{Q} = \mathcal{Q}(\mathcal{H})$ denotes the Calkin algebra, that is $\mathcal{Q} = B(\mathcal{H})/\mathcal{K}$ where $B(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$.

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1. Introduction

Long ago, Dixmier and Douady [DD] observed that the algebra bundles with fibre $K$ over a locally compact space $X$ were classified by $[H] \in H^3(X, \mathbb{Z})$ (because $\text{Aut}(K) \cong PU$ and $\pi_j(PU) = 0$ for $j \neq 2$ but $\pi_2(PU) \cong \mathbb{Z}$).

Let $P_H$ be a principal bundle over $X$ with fibre $PU$ and $\delta(P_H) = [H] \in H^3(X, \mathbb{Z})$. (Transgression etc). Let $\mathcal{E}_H$ be the bundle $P_H \times_G K$ and $\mathcal{F}_H$ the bundle $P_H \times_G F$, where $G = PU = \text{Aut}(K)$, also acts on $K$ by conjugation. Let $C_0(X, \mathcal{E}_H)$ be the continuous sections of $\mathcal{E}_H$ vanishing at infinity. Rosenberg [Ros] defined twisted $K$-theory (section 2) and showed that $\mathcal{F}_H$ is the classifying space for twisted $K^0$, an extension of the well known theorem for $H = 0$, i.e., $\mathcal{F}$ is the classifying space for ordinary $K^0$.

In recent times, twisted $K$-theory has entered string/M-theory when the $H$-field of the Neveu-Schwarz sector is turned on. See Bouwknegt-Mathai [BM], Witten [Wi], for the linking of the physics with the $C^*$-algebras, and the references there to twisted $K$-theory. There has been considerable speculation about the role for $K$-homology (even for $C^*$-algebras) in $D$-brane theory. See Harvey-Moore [HaMo] and the references there.

Once one is in the twisted category, one asks whether $K$-homology, $Ext$-theory, and Fredholm modules can similarly be twisted and with the usual relations between them. This is indeed the case, and is really part of Kasparov’s $KK$-theory, a prediction made to the second author by R.G. Douglas.

These notes are a short exposition of $K$-homology, $Ext$-theory, and Fredholm modules in the twisted category from the $C^*$-algebra viewpoint. They are meant primarily for physicists. One can also develop a twisted theory for any generalized cohomology theory. How to do so was explained to us by M.J. Hopkins.
2. Twisted $K$-theory and noncommutative geometry

Let $X$ be a locally compact, Hausdorff space with a countable basis of open sets, for example a smooth manifold. Let $[H] \in H^3(X, \mathbb{Z})$. Then the \textit{twisted $K$-theory} was defined by Rosenberg as

\begin{equation}
K^j(X, [H]) = K_j(C_0(X, \mathcal{E}_H)) \quad j = 0, 1,
\end{equation}

where $\mathcal{E}_H$ is the unique locally trivial bundle over $X$ with fibre $\mathcal{K}$ and structure group $\text{Aut}(\mathcal{K})$ whose Dixmier-Douady invariant, $\delta(\mathcal{E}_H) = [H]$, and $K_\bullet(C_0(X, \mathcal{E}_H))$ denotes the topological $K$-theory of the $C^*$-algebra of continuous sections of $\mathcal{E}_H$ that vanish at infinity. See [Black] or [Singer] for the definition of the topological $K$-theory of $C^*$-algebras. Notice that when $H = 0$, then $\mathcal{E}_H = X \times \mathcal{K}$; therefore $C_0(X, \mathcal{E}_H) = C_0(X) \otimes \mathcal{K}$ and by Morita invariance of $K$-theory (cf. [Black] or [Singer]), the twisted $K$-theory of $X$ coincides with the standard $K$-theory of $X$ in this case. Elements of $K^0(X, [H])$ are called (virtual) gauge-bundles in the physics literature, but we will call these twisted bundles in these notes.

In [Ros], it is shown that when $X$ is compact one has

\begin{equation}
\begin{align*}
K^0(X, [H]) &= \left[P_H, \mathcal{F}\right]^{PU} \\
K^1(X, [H]) &= \left[P_H, U(\mathcal{K}^+)\right]^{\text{Aut}(\mathcal{K})}
\end{align*}
\end{equation}

where $U(\mathcal{K}^+)$ is the group of unitaries in the unitalization of $\mathcal{K}$,

\[ U(\mathcal{K}^+) = \{ u \in U(\mathcal{H}) : u - 1 \in \mathcal{K}\}. \]

3. The twisted $Ext$ group and twisted $K$-homology

In this section we will give a brief review of the twisted $Ext$ group, which can be considered as a specialization of the general $Ext$-theory [PPV], [Kas].
Consider noncommutative \( C^* \) algebras \( \mathcal{A} \) which fit into the short exact sequence:

\[
0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \xrightarrow{\beta} \mathcal{B} \rightarrow 0
\] (3.1)

where \( \mathcal{B} = C_0(X, \mathcal{E}_{[H]}) \), for some fixed space \( X \) and NS field \( H \). In [PPV], [Kas] extensions of the form (3.1) for general nuclear \( C^* \)-algebras \( \mathcal{B} \) were investigated. We shall restrict ourselves to the special case when \( \mathcal{B} = C_0(X, \mathcal{E}_{[H]}) \). To any such extension one can associate the Busby invariant, which is a homomorphism

\[
\tau : C_0(X, \mathcal{E}_{[H]}) \rightarrow Q(H)
\] (3.2)

defined as follows. For any \( s \in C_0(X, \mathcal{E}_{[H]}) \) we choose an operator \( T_s \in \mathcal{A} \) such that \( \beta(T_s) = s \), and define \( \tau \) by: \( \tau(s) = \pi(T_s) \) where \( \pi : B(H) \rightarrow Q(H) \) is the projection. \( \tau \) is a homomorphism because \( T_s T_{s_2} - T_{s_1 s_2} \) is a compact operator. Conversely, given a homomorphism \( \tau : C_0(X, \mathcal{E}_{[H]}) \rightarrow Q(H) \) one can form an extension (3.1) as follows,

\[
0 \rightarrow \mathcal{K} \rightarrow \mathcal{A}' \rightarrow C_0(X, \mathcal{E}_{[H]}) \rightarrow 0
\] (3.3)

where the algebra \( \mathcal{A}' \) is defined as

\[
\mathcal{A}' = \{(A, f) : \pi(A) = \tau(f)\} \subset B(H) \oplus C_0(X, \mathcal{E}_{[H]}).
\] (3.4)

Moreover, (3.1) is unitarily equivalent to (3.3) in the sense that we now describe.

Two extensions (3.1) are unitarily equivalent if there is a unitary operator \( U \) on \( H \) such that the Busby invariants are related by \( \tau_2(s) = \pi(U) \tau_1(s) \pi(U)^* \). Let \( \text{Ext}(X, H) \) denote the set of unitary equivalence classes of extensions of \( C_0(X, \mathcal{E}_{[H]}) \) by \( \mathcal{K} \). A direct sum operation on \( \text{Ext}(X, [H]) \) can then be defined by taking the extension corresponding to the Busby invariant

\[
\tau_1 \oplus \tau_2 : C_0(X, \mathcal{E}_{[H]}) \rightarrow Q(H) \oplus Q(H) \rightarrow Q(H \oplus H) \cong Q(H).
\] (3.5)

Then (3.5) defines a semigroup operation on \( \text{Ext}(X, H) \). Trivial extensions are those for which the Busby invariant lifts to \( B(H) \). Equivalently, they are extensions
such that the sequence (3.1) splits. Define the twisted Ext group $\text{Ext}(X,[H])$ as being the quotient of $\text{Ext}(X,H)$ by the trivial extensions. It is shown in [PPV], [Kas] that every extension has an inverse up to the addition of a trivial extension, so that $\text{Ext}(X,[H])$ is an abelian group. It is clear that $\text{Ext}(X,[H])$ depends only on the cohomology class of $[H]$, since $C_0(X,\mathcal{E}[H])$ and $C_0(X,\mathcal{E}[H'])$ are isomorphic whenever $[H'] = [H]$.

There is a pairing of $\text{Ext}(X,[H])$ and $K^1(X,[H])$ defined as follows. (See [BM], [DK], [Ka], [Ros], [Wi] for a discussion of twisted $K$-theory, $K^\bullet(X,[H])$). An extension of the form (3.1) is determined by its Busby invariant $\tau: C_0(X,\mathcal{E}[H]) \to \mathbb{Q}(H)$, which induces homomorphisms $\tilde{\tau}: M_n(C_0(X,\mathcal{E}[H])^+) \to M_n(\mathbb{Q}(H)) \cong \mathbb{Q}(H)$ for each $n \in \mathbb{N}$. If $u$ is a unitary in $M_n(C_0(X,\mathcal{E}[H])^+)$, define the pairing

$$\text{Ext}(X,[H]) \times K^1(X,[H]) \to \mathbb{Z}$$

$$(\tau, u) \to \text{Index}(\tilde{\tau}(u)).$$

In particular, each element $\tau \in \text{Ext}(X,[H])$ defines a homomorphism $\tau_*: K^1(X,[H]) \to \mathbb{Z}$. If $\tau_* = 0$, then the six term exact sequence in K-theory (cf. section 4) corresponding to the extension (3.1) reduces to the short exact sequence

$$0 \to \mathbb{Z} = K_0(\mathcal{K}) \to K_0(\mathcal{A}) \to K^0(X,[H]) \to 0$$

and therefore defines an element of $Ext^1_{\mathbb{Z}}(K^0(X,[H]),\mathbb{Z})$ in homological algebra. It can be shown [RS] that the converse is also true, that is, one has the universal coefficient theorem:

$$0 \to Ext^1_{\mathbb{Z}}(K^0(X,[H]),\mathbb{Z}) \to Ext(X,[H]) \to \text{Hom}(K^1(X,[H]),\mathbb{Z}) \to 0.$$  

This justifies the definition of the twisted $K$-homology as being

$$K_1(X,[H]) = Ext(X,[H])$$

$$K_0(X,[H]) = Ext(X \times \mathbb{R}, p_1^*[H])$$

where $p_1: X \times \mathbb{R} \to X$ denotes projection onto the first factor.
One deduces from the definition (3.9) that the universal coefficient exact sequence (3.8) can be rewritten as

\[ 0 \to \text{Ext}^1_\mathbb{Z}(K^{*+1}(X, [H]), \mathbb{Z}) \to K^*_0(X, [H]) \to \text{Hom}(K^*(X, [H]), \mathbb{Z}) \to 0. \]

(3.10)

4. Properties of the twisted \( Ext \) groups and twisted \( K \)-homology

1) **Bott Periodicity:** Let \([H] \in H^3(X, \mathbb{Z})\). Then one has the Bott periodicity theorem for the \( C^* \)-algebra \( C_0(X, \mathcal{E}_H) \),

\[ \text{Ext}(X \times \mathbb{R}^2, p_1^*[H]) \cong \text{Ext}(X, [H]) \]

(4.1)

For details on Bott periodicity for \( C^* \)-algebras, cf. [Black] or [Singer].

This shows that if we define \( K_j(X, [H]) = \text{Ext}(X \times \mathbb{R}^j, p_1^*[H]) \) for \( j \in \mathbb{N} \), then there are at most only two distinct groups in this list, \( K_1(X, [H]) \) and \( K_0(X, [H]) \).

2) **Six term exact sequence:** Given a short exact sequence

\[ 0 \to C_0(X, \mathcal{E}_{[H]}) \to B \to J \to 0 \]

there is a six term exact sequence of twisted \( Ext \) groups, which is obtained using Bott periodicity,

\[
\begin{array}{cccccc}
K^0(J) & \longrightarrow & K^0(B) & \longrightarrow & K_0(X, [H]) \\
\delta \uparrow & & \delta \downarrow & & \\
K_1(X, [H]) & \leftarrow_{F^*} & K^1(B) & \leftarrow_{\Delta} & K^1(J)
\end{array}
\]

This enables us to compute the twisted \( Ext \) groups at least in some examples. Consider the case when \( X = S^3 \) and \([H] \) is the class of the volume form on \( S^3 \) and \( N \) is a positive integer. Then we will compute \( K_\bullet(S^3, N[H]) \). Note that this can also be done using the universal coefficient theorem.

We consider the open cover of \( S^3 \) given by the upper and lower hemispheres, \( \{ \mathcal{U}_1, \mathcal{U}_2 \} \), where \( \mathcal{U}_1 \cap \mathcal{U}_2 = S^2 \). Then representatives of \( K^1(S^3, N[H]) \) are pairs of
maps \((f_1, f_2)\), where
\[
f_i : U_i \to U(K^+)
\]
such that on the overlap \(U_1 \cap U_2 = S^2\), one has
\[
(4.2) \quad f_1 = p_{N[H]} f_2.
\]
where \(p_{N[H]}\) denotes the transition functions of the bundle \(K\)-algebra bundle \(E_{N[H]}\) with Dixmier-Douady invariant \(N[H] \in H^3(S^3, \mathbb{Z})\). Now the \(C^*\)-algebra of continuous sections of the bundle \(E_{N[H]}\), \(C(X, E_{N[H]})\) can be represented by pairs of continuous functions \((h_1, h_2)\), where
\[
h_i : U_i \to K \quad i = 1, 2
\]
and satisfying on the overlap \(U_1 \cap U_2 = S^2\)
\[
h_1 = p_{N[H]} h_2.
\]
Therefore there is a short exact sequence
\[
(4.3) \quad 0 \to C(S^3, E_{N[H]}) \xrightarrow{F} C(U_1) \otimes K \oplus C(U_2) \otimes K \xrightarrow{G} C(S^2) \otimes K \to 0
\]
where
\[
F(h_1, h_2) = h_1 \oplus h_2, \quad G(q_1 \oplus q_2) = q_1|_{S^2} - p_{N[H]}(q_2|_{S^2}).
\]
The six term exact sequence in \(Ext\)-theory associated to the short exact sequence \((4.3)\) is,
\[
\begin{array}{cccc}
K_0(S^2) & \xrightarrow{G^*} & K_0(U_1) \oplus K_0(U_2) & \xrightarrow{F^*} & K_0(S^3, N[H]) \\
\delta & & \delta & & \\
K_1(S^3, N[H]) & \xleftarrow{E^*} & K_1(U_1) \oplus K_1(U_2) & \xleftarrow{G^*} & K_1(S^2)
\end{array}
\]
Since \(0 = K_1(U_1) \oplus K_1(U_2) = K_1(S^2)\), this six term exact sequence collapses into the exact sequence
\[
(4.4) \quad 0 \to K_1(S^3, N[H]) \xrightarrow{\delta} K_0(S^2) \xrightarrow{G^*} K_0(U_1) \oplus K_0(U_2) \xrightarrow{F^*} K_0(S^3, N[H]) \to 0
\]
On analyzing the map $G^*$ explicitly, we see that if $N \neq 0$, then $\text{Ker}(G^*) = 0 = K_1(S^3, N[H])$ and that $\text{Coker}(G^*) = \mathbb{Z}_N \cong K_0(S^3, N[H])$. See [Ros] and [BM] for related computations.

3) **When $H$ is torsion:** In this case, there is an argument in [Gr], [Wi] which shows that the free part of $K^0(X, [H])$ is isomorphic to the free part of $K^0(X)$. Therefore $\text{Hom}(K^0(X, [H]), \mathbb{Z}) \cong \text{Hom}(K^0(X), \mathbb{Z})$, and by the universal coefficient theorem, we see that the free part of the twisted $K$-homology $K_0(X, [H])$ is isomorphic to the free part of $K_0(X)$ in this case, and in particular, $K_0(X, [H]) \otimes \mathbb{Q} \cong K_0(X) \otimes \mathbb{Q}$.

4) **The fundamental class when $H = 0$:** A noteworthy case, to be reviewed later in these talks, is the extension

$$0 \to \mathcal{K} \to \mathcal{A} \to C(S^*X) \to 0$$

where $\mathcal{A}$ is the closure in the norm topology of the algebra of singular integral operators (pseudodifferential operators of order zero) and $S^*X$ is the sphere bundle of the cotangent bundle of a smooth manifold $X$. The extension (4.5) does not split, and is called the fundamental class in $\text{Ext}(S^*X)$, cf. [Kas], [BD].

5. **Description of twisted $K$-homology in terms of Fredholm operators**

We recall here the definition of $KK^1(\mathcal{B}, \mathbb{C})$, where $\mathcal{B} = C_0(X, \mathcal{E}_{[H]})$ and $\mathbb{C}$ is the algebra of complex numbers, a very special case of Kasparov’s $KK(\mathcal{B}, \mathcal{D})$ theory for general $C^*$-algebras $\mathcal{D}$. This will provide us with a Fredholm module picture for twisted $\text{Ext}$-theory. A Fredholm module is a triple $(\mathcal{H}, \phi, F)$, where,

- $\mathcal{H}$ is a separable Hilbert space;
- $\phi : C_0(X, \mathcal{E}_{[H]}) \to B(\mathcal{H})$ is a $*$-homomorphism;
• $F$ is self-adjoint and satisfies: $(F^2 - 1)\phi(a) \in \mathcal{K}$, and $[F, \phi(a)] \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Let $E_1(\mathcal{B})$ denote the set of all Fredholm modules over $\mathcal{B}$. Let $D_1(\mathcal{B})$ denote the subset of Fredholm modules satisfying $(F^2 - 1)\phi(a) = 0 = [F, \phi(a)]$. They are called degenerate Fredholm modules.

The direct sum of two Fredholm modules is again a Fredholm module. Moreover, the direct sum of degenerate Fredholm modules is again a degenerate Fredholm module. Two Fredholm modules $(\mathcal{H}_i, \phi_i, F_i)$, $i = 0, 1$ are said to be unitarily equivalent if there is a unitary in $B(\mathcal{H}_0, \mathcal{H}_1)$ intertwining the $\phi_i$ and the $F_i$.

Define an equivalence relation $\sim$ on $E(\mathcal{B})$ generated by unitary equivalence, addition of degenerate elements and ‘compact perturbations’ of $(\mathcal{H}, \phi, F)$. Here a Fredholm module $(\mathcal{H}, \phi, F')$ is said to be a compact perturbation of $(\mathcal{H}, \phi, F)$ if $(F - F')\phi(a) \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Then $KK^1(\mathcal{B}, \mathbb{C})$ is the set of equivalence classes of $E_1(\mathcal{B})$ under the equivalence relation $\sim$.

Given a Fredholm module $(\mathcal{H}, \phi, F)$, we will define a $\mathcal{K}$ extension of $\mathcal{B}$ of the form (3.1) as follows. Observe that $P = 1/2F + 1/2$ is a projection modulo $\mathcal{K}$. Define the Busby map $\tau$ by $\tau(a) = \pi(P\phi(a)P)$ for all $a \in \mathcal{B}$, where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the projection. Then $\tau$ gives the desired $\mathcal{K}$ extension of $\mathcal{B}$ of the form (3.1). The Busby map corresponding to $1 - P$ is an inverse for $\tau$, and we have a well defined map

$$KK^1(\mathcal{B}, \mathbb{C}) \to Ext(X, [H]) = K_1(X, [H]).$$

It is not too hard to show that this map is an isomorphism

$$KK^1(\mathcal{B}, \mathbb{C}) \cong Ext(X, [H]) = K_1(X, [H]).$$
This gives a Fredholm module description of twisted $Ext$-theory, or equivalently of twisted $K$-homology theory.

There is also a $\mathbb{Z}_2$-graded Fredholm module description of the twisted $K$-homology group $K_0(X, [H])$, which we will now discuss. A $\mathbb{Z}_2$-graded Fredholm module is a triple $(H, \phi, F)$, where $H$ is a separable $\mathbb{Z}_2$-graded Hilbert space, $\phi : C_0(X, E_{[H]}) \to B(H)$ is a $\ast$-homomorphism which is of even degree, $F$ is an odd degree self-adjoint operator on $H$ and satisfies $(F^2 - 1)\phi(a) \in \mathcal{K}$, $[F, \phi(a)] \in \mathcal{K}$ for all $a \in \mathcal{B}$. Let $E_0(\mathcal{B})$ denote the set of all $\mathbb{Z}_2$-graded Fredholm modules over $\mathcal{B}$. Let $D_0(\mathcal{B})$ denote the subset of $\mathbb{Z}_2$-graded Fredholm modules satisfying $(F^2 - 1)\phi(a) = 0 = [F, \phi(a)]$. They are called degenerate $\mathbb{Z}_2$-graded Fredholm modules.

The direct sum of two $\mathbb{Z}_2$-graded Fredholm modules is again a $\mathbb{Z}_2$-graded Fredholm module, with respect to the total $\mathbb{Z}_2$-grading. Moreover, the direct sum of degenerate $\mathbb{Z}_2$-graded Fredholm modules is again a degenerate $\mathbb{Z}_2$-graded Fredholm module. Two $\mathbb{Z}_2$-graded Fredholm modules $(H_i, \phi_i, F_i)$, $i = 0, 1$ are said to be unitarily equivalent if there is a unitary in $B(H_0, H_1)$ intertwining the $\phi_i$ and the $F_i$.

Define an equivalence relation $\sim$ on $E_0(\mathcal{B})$ generated by unitary equivalence, addition of degenerate elements and ‘compact perturbations’ of $(H, \phi, F)$. Here a $\mathbb{Z}_2$-graded Fredholm module $(H, \phi, F')$ is said to be a compact perturbation of $(H, \phi, F)$ if $(F - F')\phi(a) \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Then $KK^0(\mathcal{B}, \mathbb{C})$ is the set of equivalence classes of $E_0(\mathcal{B})$ under the equivalence relation $\sim$. It follows from the discussion above and Bott periodicity that $KK^0(\mathcal{B}, \mathbb{C}) = KK^1(\mathcal{B} \otimes C_0(\mathbb{R}), \mathbb{C}) \cong Ext(X \times \mathbb{R}, p^1_1[H]) = K_0(X, [H])$.

6. **Topological $K$-homology**

We now give a Baum-Douglas type description of twisted $K$-homology, called topological twisted $K$-homology. The basic objects are twisted $K$-cycles. A twisted
K-cycle on a topological space is a triple \((M, E, \phi)\), where \(M\) is a compact \(\text{Spin}^C\) manifold, \(E \to M\) is a twisted bundle on \(M\), and \(\phi : M \to X\) is a continuous map. Two twisted \(K\)-cycles \((M, E, \phi)\) and \((M', E', \phi')\) are said to be isomorphic if there is a diffeomorphism \(h : M \to M'\) such that \(h^*(E') \cong E\) and \(h^*\phi' = \phi\). Let \(\Pi(X, H)\) denote the collection of all twisted \(K\)-cycles on \(X\).

- **Bordism**: \((M_i, E_i, \phi_i) \in \Pi(X, H), i = 0, 1\) are said to be bordant if there is a triple \((W, E, \phi)\) where \(W\) is a compact \(\text{Spin}^C\) manifold with boundary \(\partial W\), \(E\) is a twisted bundle over \(W\) and \(\phi : W \to X\) is a continuous map, such that \((\partial W, E|_{\partial W}, \phi|_{\partial W})\) is isomorphic to the disjoint union \((M_0, E_0, \phi_0) \cup (-M_1, E_1, \phi_1)\). Here \(-M_1\) denotes \(M_1\) with the reversed \(\text{Spin}^C\) structure.

- **Direct sum**: Suppose that \((M, E, \phi) \in \Pi(X, H)\) and \(E = E_0 \oplus E_1\). Then \((M, E, \phi)\) is isomorphic to \((M, E_0, \phi) \cup (M, E_1, \phi)\).

- **Twisted bundle modification**: Let \((M, E, \phi) \in \Pi(X, H)\) and \(H\) be an even dimensional \(\text{Spin}^C\) vector bundle over \(M\). Let \(\hat{M} = S(H \oplus 1)\) denote the sphere bundle of \(H \oplus 1\). Then \(\hat{M}\) is canonically a \(\text{Spin}^C\) manifold. Let \(S\) denote the bundle of spinors on \(H\). Since \(H\) is even dimensional, \(S\) is \(\mathbb{Z}_2\)-graded,

\[ S = S^+ \oplus S^- \]

into bundles of \(1/2\)-spinors on \(M\). Define \(\hat{E} = \pi^*(S^+ \otimes E)\), where \(\pi : \hat{M} \to M\) is the projection. Finally, \(\hat{\phi} = \pi^*\phi\). Then \((\hat{M}, \hat{E}, \hat{\phi}) \in \Pi(X, H)\) is said to be obtained from \((M, E, \phi)\) and \(H\) by twisted bundle modification.

Let \(\sim\) denote the equivalence relation on \(\Pi(X, H)\) generated by the operations of bordism, direct sum and twisted bundle modification. Notice that \(\sim\) preserves the parity of the dimension of the twisted \(K\)-cycle. Let \(K^0_0(X, [H])\) denote the quotient \(\Pi^{even}(X, H)/\sim\), where \(\Pi^{even}(X, H)\) denotes the set of all even dimensional twisted \(K\)-cycles in \(\Pi(X, H)\), and let \(K^1_1(X, [H])\) denote the quotient \(\Pi^{odd}(X, H)/\sim\), where \(\Pi^{odd}(X, H)\) denotes the set of all odd dimensional twisted \(K\)-cycles in \(\Pi(X, H)\).
Then it is possible to show as in [BD] that $K_j^t(X, [H]) \cong K_j(X, [H])$, $j = 0, 1$, providing a topological description of twisted $K$-homology.

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