Unambiguous state discrimination in quantum cryptography with weak coherent states

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The use of linearly independent signal states in realistic implementations of quantum key distribution (QKD) enables an eavesdropper to perform unambiguous state discrimination. We explore quantitatively the limits for secure QKD imposed by this fact taking into account that the receiver can monitor some extent the photon number statistics of the signals even with today’s standard detection schemes. We compare our attack to the beamsplitting attack and show that security against beamsplitting attack does not necessarily imply security against the attack considered here.

I. INTRODUCTION

Quantum key distribution (QKD) is a technique to provide two parties with a secure, secret and shared key. Such a key is the necessary ingredient to the only provably secure way to communicate with guaranteed privacy, the one-time pad or Vernam cipher. The first complete protocol was given by Bennett and Brassard (BB84) following first ideas by Wiesner. It uses the fact that any channel which transmits two non-orthogonal states perfectly, automatically makes eavesdropping on this channel detectable.

We consider the BB84 protocol in a typical quantum optical implementation. Ideally, Alice sends a sequence of single photons which are at random polarized in one of the following four states: right or left circular polarized, or vertically or horizontally polarized. Bob chooses at random between two polarization analyzers, one distinguishing the circular polarized states, and the other distinguishing the linear polarized states. Following a public discussion about the basis of the sent signals and the measurement apparatus applied to them, sender and receiver can obtain a shared key made up from those signals where the measurement device gives deterministic results. This is the sifted key. Proofs of security of this scheme against the most general attack, even in the presence of noise, have been obtained.

In this article we follow another goal: we would like to illuminate to which extend already very simple attacks can render QKD impossible once realistic imperfections like lossy lines and non-ideal signal states are taken into account. The difficulties implied, for example, by the use of weak coherent states in combination with lossy lines has been pointed out earlier and this subject has been illuminated in depth in, where bounds on coverable distances are given. Positive security proofs for sufficiently short distances and taking into account the realistic signals are given for individual attacks and coherent attacks. The eavesdropping attacks which crack the secrecy of the key for set-ups exceeding this secure distances are still quite complicated. The eavesdropper needs to perform a QND measurement on the total photon number of the signal, then he has to split a photon off the occurring multi-photon signals, store that photon, and then, finally, measure it after the public discussion.

In this paper we are looking into much simpler eavesdropping strategies which make use of the opportunities arising from lossy lines and non-ideal signals. Such an attack has been proposed by Bennett and Yuen. It uses the fact that Eve can, with finite probability, discriminate the four signal states unambiguously. Whenever such a discrimination is performed successfully, the eavesdropper knows immediately which of the four signal states was sent and can send this information via a classical channel to Bob’s detector, in front of which she places a state preparation machine to prepare the identified state. This way this state does not experience the losses of the actual quantum channel without that Eve has to invest into a perfect quantum channel.

The investigation of this scenario refines Bennett’s and Yuen’s analysis since it takes into account that, to a certain extend, the photon statistics of the signals arriving at Bob’s detectors can be monitored. The results illuminate the restrictions placed on implementations of QKD on lines with strong losses. Thereby we can show that the currently widely used conditional security standard of security against beamsplitting attacks is incomplete. Especially, contrary to common belief, the use of unambiguous state discrimination can be a more efficient eavesdropping strategy than the beamsplitting attack, even for dim coherent states.

The paper is organized as follows. In Sec. we recapitulate the principles of unambiguous state discrimination. These are applied in Sec. to the signal states in the BB84 protocol with dim coherent signal states. In Sec. we introduce an eavesdropping attack based on unambiguous state discrimination (USD) attack and analyze it in detail, taking the photon number distribution of the signals arriving at Bob’s detectors into account. In Sec. we discuss the relation between the beam splitting...
attack and the USD attack. Sec. VI concludes the article with a short summary.

II. UNAMBIGUOUS DISCRIMINATION OF SIGNAL STATES

Unambiguous state discrimination is possible whenever the \( N \) states in question are linearly independent. The problem can be described by a measurement which can give the results “state 0”, “state 1”, . . . “state \( N-1 \)”, and the result “don’t know”. The constraint is that the measurement results should never wrongly identify a state, and the goal is to keep the fraction of “don’t know” results as low as possible. This problem has been investigated by Ivanovic [13] for the case of two equal probable non-orthogonal states. Peres [14] solved this problem in a formulation with Probability Operator Measures (POM). Later Jaeger and Shimony [17] extended the solution to arbitrary a priori probabilities. Peres’s solution has been generalized to an arbitrary number of equally probable non-orthogonal states which are generated from each other by a symmetry operator by Chefles and Barnett [18]. Their result can be summarized as follows: the symmetry allows to write the input states in the form

\[
|\Psi_j\rangle = \sum_{k=0}^{N-1} c_j \exp \left( 2\pi i \frac{kj}{N} \right) |\phi_j\rangle,
\]

where the states \( |\phi_j\rangle \) represent some set of orthonormal states. Note that the states \( |\tilde{\Psi}_j\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp \left( 2\pi i \frac{jk}{N} \right) |\phi_j\rangle \) form another orthonormal set. It turns out that the optimal strategy for unambiguous state discrimination consists of two steps. In the first step a filter operation is performed such that the output states are either the orthonormal states \( |\tilde{\Psi}_j\rangle \) or some linear dependent states. This step can be described by a complete positive map with the two Kraus operators. They are defined with the help of the minimum coefficient \( c_{\min} = \min_j |c_j| \) as

\[
A_{\text{yes}} = \sum_{j=0}^{N-1} \frac{c_{\min}}{c_j} |\phi_j\rangle \langle \phi_j|,
\]

\[
A_{\text{no}} = \sum_{j=0}^{N-1} \sqrt{1 - \frac{c_{\min}^2}{|c_j|^2}} |\phi_j\rangle \langle \phi_j|.
\]

The conditional state in the event of successful filtering is now given as

\[
|\Psi^{(\text{yes})}_k\rangle = \sqrt{N} c_{\min} |\tilde{\Psi}_k\rangle.
\]

In a second step, we can perform a von Neumann projection measurement on this state to identify unambiguously the state \( k \) via the orthonormal state \( |\tilde{\Psi}_k\rangle \). The probability of this successful identification is given by

\[
P_D = N \min_j |c_j|^2.
\]

For the case of two equal probable non-orthogonal polarizations which can be expressed in terms of two creation operators \( b_1^\dagger \) and \( b_2^\dagger \) (corresponding, e.g., to two linear orthogonal polarizations) as

\[
a_0^\dagger = \frac{1}{\sqrt{2}} \left( b_1^\dagger + b_2^\dagger \right),
\]

\[
a_1^\dagger = \frac{1}{\sqrt{2}} \left( b_1^\dagger + i b_2^\dagger \right),
\]

\[
a_2^\dagger = \frac{1}{\sqrt{2}} \left( b_1^\dagger - b_2^\dagger \right),
\]

\[
a_3^\dagger = \frac{1}{\sqrt{2}} \left( b_1^\dagger - i b_2^\dagger \right).
\]

In terms of these two modes the signal state become therefore

\[
|\Psi_0\rangle = \frac{\alpha}{\sqrt{2}} \left| \frac{\alpha}{\sqrt{2}} \right\rangle,
\]

\[
|\Psi_1\rangle = \frac{\alpha}{\sqrt{2}} \left| \frac{i \alpha}{\sqrt{2}} \right\rangle,
\]

\[
|\Psi_2\rangle = \frac{\alpha}{\sqrt{2}} \left| \frac{-\alpha}{\sqrt{2}} \right\rangle,
\]

\[
|\Psi_3\rangle = \frac{\alpha}{\sqrt{2}} \left| \frac{-i \alpha}{\sqrt{2}} \right\rangle.
\]

We can calculate the values of the \( c_j \) in terms of the overlaps of the four states according to the formula [18]

\[
|c_j| = \frac{1}{N^2} \sum_{k,\ell} \exp \left[ -2\pi i \frac{j(k - \ell)}{N} \right] \langle \Psi_k | \Psi_\ell \rangle
\]

and find as a function of the expected photon number \( \mu = |\alpha|^2 \)

\[
|c_0| = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\cosh \frac{\mu}{2} + \cos \frac{\mu}{2}},
\]

\[
|c_1| = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\cosh \frac{\mu}{2} - \cos \frac{\mu}{2}},
\]

\[
|c_2| = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\cosh \frac{\mu}{2} + \cos \frac{\mu}{2}},
\]

\[
|c_3| = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\cosh \frac{\mu}{2} - \cos \frac{\mu}{2}}.
\]
The states are not the correct description of the situation. It is for unambiguous state discrimination.

Here the state $|c_i\rangle$ from where we can read off $P_D$ as the minimum.

$$|c_1\rangle = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\sinh \frac{\mu}{2} + \sin \frac{\mu}{2}},$$  \hspace{1cm} (15)$$

$$|c_2\rangle = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\cosh \frac{\mu}{2} - \cos \frac{\mu}{2}},$$  \hspace{1cm} (16)$$

$$|c_3\rangle = \frac{1}{\sqrt{2}} e^{-\frac{\mu}{2}} \sqrt{\sinh \frac{\mu}{2} - \sin \frac{\mu}{2}}.$$  \hspace{1cm} (17)$$

The minimum of these four functions depends on the value of $\mu$. The four functions $4|c_i|^2$ are plotted in Fig. 1 from where we can read off $P_D$ as the minimum.

![Figure 1](attachment:image.png)

FIG. 1. The fourfold weight $4|c_i|^2$ of the four canonical states $|\phi_i\rangle$ as a function of the mean photon number $\mu$. The lower bound of these four curves gives the optimum probability for unambiguous state discrimination.

It turns out, however, that for realistic sources these states are not the correct description of the situation. The reason is that Eve does not have a phase reference, that means that for a given polarization she does not see the coherent state $|n\rangle$ but the phase averaged density matrix $\frac{1}{\pi} \int_0^{2\pi} |e^{i\phi}\alpha\rangle \langle e^{i\phi}\alpha|\, d\phi$. This results in signal states which are mixtures of Fock states with a Poissonian photon number distribution described by the density matrix

$$\rho = e^{-\mu} \sum_n \frac{\mu^n}{n!} |n\rangle \langle n|.$$  \hspace{1cm} (18)$$

Here the state $|n\rangle$ denotes the Fock state with $n$ photons in one of the four BB84 polarization states. The optimal strategy to discriminate between the four possible density matrices can be logically decomposed into a QND measurement on the total photon number in the modes $b_1$ and $b_2$ together and a following measurement which unambiguously discriminates between the four resulting conditional states for each total photon number. The justification for this is that the total photon number via the QND measurement “comes free”, since the execution of this measurement does not change the signal states. However, given the resulting information, we know the optimal strategy on the conditional states according to [18]. Therefore we find that the total probability of unambiguous state discrimination $P_D$ is given in terms of the respective probabilities for each photon number subspace $P_D^{(n)}$ as

$$P_D = \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} P_D^{(n)}. $$  \hspace{1cm} (19)$$

The conditional states resulting from the QND measurement and corresponding to $n$ photons in total satisfy again the symmetry condition which allows to apply the results by Chefles and Barnett. We find for the four coefficients (as a function of the photon number $n > 0$) the expressions

$$|c_0\rangle = \sqrt{\frac{1}{4} + 2^{-1+n/2}} \cos \left(\frac{\pi}{4}n\right),$$  \hspace{1cm} (20)$$

$$|c_1\rangle = \sqrt{\frac{1}{4} + 2^{-1+n/2}} \sin \left(\frac{\pi}{4}n\right),$$  \hspace{1cm} (21)$$

$$|c_2\rangle = \sqrt{\frac{1}{4} - 2^{-1+n/2}} \cos \left(\frac{\pi}{4}n\right),$$  \hspace{1cm} (22)$$

$$|c_3\rangle = \sqrt{\frac{1}{4} - 2^{-1+n/2}} \sin \left(\frac{\pi}{4}n\right).$$  \hspace{1cm} (23)$$

Therefore the maximum probability of unambiguous state discrimination for fixed value of $n$ is given by

$$P_D^{(n)} = \begin{cases} 0 & n \leq 2 \\ 1 - 2^{1-n/2} & n \text{ even} \\ 1 - 2^{-1-n/2} & n \text{ odd.} \end{cases} $$  \hspace{1cm} (24)$$

It is possible to sum up the contributions from different photon numbers from the Poissonian distribution and we obtain the expression

$$P_D = \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} P_D^{(n)} $$

$$= 1 - e^{-\mu} \left(\sqrt{2} \sinh \frac{\mu}{\sqrt{2}} + 2 \cosh \frac{\mu}{\sqrt{2}} - 1\right). $$  \hspace{1cm} (25)$$

This result is compared to the result for coherent states in Fig. 2. As expected, the probability for unambiguous state identification is lower for the mixture of Fock-states than for the coherent states. An expansion in terms of the photon number $\mu$ gives $P_D = \frac{1}{12} \mu^3 + O(\mu^4)$ for both situations. For lower than third order the signal states are not linearly independent, so that no unambiguous state discrimination is possible. Note that an actual implementation does not necessarily need to follow the decomposition into a QND and another measurement. We just need to implement one generalized measurement. Actually, Bennett et al. [13] and Yuen [9] gave a simple beamsplitter setup which obtains a discrimination probability of $P_D = \frac{1}{12} \mu^3 + O(\mu^4)$. 


IV. UNAMBIGUOUS STATE DISCRIMINATION AS EAVESDROPPING STRATEGY

We now consider the realistic situation that Alice uses the phase-averaged coherent states as signal states which are described by a Poissonian photon number distribution with mean photon number \( \mu \). In this scenario we fix our eavesdropping strategy, to which we refer as the unambiguous state discrimination attack (USD attack) as follows: The unambiguous state discrimination allows Eve to identify a fraction of the signals without error. For this fraction, she can prepare a corresponding state close to Bob’s detectors such that no errors appear for these signals. Whenever the identification does not succeed, she sends the vacuum signal to Bob to avoid errors, which therefore will not be relevant in the considered scenario.

We need to study this strategy under realistic constraints. An important constraint is that the transmittance of the quantum channel connecting both parties is given by the transmission efficiency \( \eta_L \). We consider a detection setup where Bob monitors each polarization mode in the chosen basis by one detector. These detectors have a finite detection efficiency \( \eta_B \), in which we include any additional loss on Bob’s side, e.g., from a polarizing beamsplitter. The detectors are modeled as “yes/no” detectors, which either fire, or they do not fire; they cannot distinguish the number of impinging photons.

It is clear that once Eve identified a signal she is interested to produce a signal in the corresponding polarization such that Bob will detect it despite his inefficient detectors. One strategy is to send a stronger signal than the original one in the correct polarization. This will work as long as Bob measures in the polarization basis which includes the signal polarization (sifted key), but it will lead to an increased coincidence rate of clicks in both of Bob’s detectors otherwise. Our analysis extends previous analysis to include the additional constraint put on the eavesdropping strategy by the fact that Bob observes not only the rate of clicks of one or the other detector, but also the rate of events when both detectors fire, each monitoring one of the orthogonal polarization modes. The latter event will be observed ideally only when Alice and Bob use different bases, independently of the presence of absence of an eavesdropper. Eve’s aim is to reproduce these two observables with the minimum number of non-vacuum signals to make efficient use of the successfully identified signals.

In the absence of Eve, whenever Alice and Bob use the same polarization basis Bob’s expects to find at most one detector clicks; the probability of a click is

\[
P_1 = 1 - \exp(-\eta_L \eta_B \mu),
\]

as follows from the Poissonian photon-number statistics of coherent states.

Whenever Alice and Bob use different bases a double-click may occur; its probability is

\[
P_2 = \left[1 - \exp\left(-\frac{\eta_L \eta_B \mu}{2}\right)\right]^2.
\]

What happens in the presence of Eve depends on the signals Eve sends for the successfully detected Alice’s signals. It is clear that Eve can avoid the occurrence of double clicks when Alice and Bob measure in the same basis, since she unambiguously determined the signal. Therefore it is not useful to monitor the double click rate when Alice and Bob use the same basis.

Note that we do not need to include detector dark count rates or errors due to misalignment into account. The reason for that is that we will investigate the limit when the USD attack gives complete information to Eve while it reproduces the expected probabilities \( \bar{P}_1 \) and \( \bar{P}_2 \). The values of these probabilities in the absence of an eavesdropper and the reproduced values resulting from a successful USD attack will be affected in the same way by the error mechanisms of dark counts and misalignment etc., so that the resulting real observed rates will still be indistinguishable.

A. Eve sends \( n \)-photon states

Let us suppose now, that whenever Eve succeeds in the unambiguous state discrimination she sends a number state (with correct polarization) containing \( N \) photons to Bob. If she fails she simply sends no photon.

If Alice and Bob use the same basis, at most one of two Bob’s detectors will click. The probability of this event is given by

\[
P_1^{(N)} = P_D \left[ 1 - \binom{m}{0} \eta_B^0 (1 - \eta_B)^N \right] = P_D \left[ 1 - (1 - \eta_B)^N \right].
\]
This is the probability that one detector clicks if a state \( |N\rangle \) comes, multiplied by the probability that Eve succeeds in USD (and sends \( |N\rangle \)).

If Alice and Bob use different bases, we can think of the photons as being equally and independently distributed to both Bob’s detectors. The probability to find photons as being equally and independently distributed we obtain finally for the double-click probability

\[
\Pi_{k\ell} = P_D \left( \frac{1}{2} \sum_{m=k}^{N} \frac{N}{m} \left[ \binom{m}{k} \eta_B^k (1 - \eta_B)^{m-k} \right] \right) \times \left[ (N - m\ell) \eta_B^{\ell} (1 - \eta_B)^{N-m-\ell} \right],
\]

where the summation limits stem from obvious constraints \( m \geq k, N - m \geq \ell \). Thus the probability of double click in Bob’s “yes-no” detectors when Eve is active and while Alice and Bob use different polarization bases reads

\[
P^{(N)}_2 = P_D \left[ 1 - \sum \Pi_{0\ell} - \sum \Pi_{k0} + \Pi_{00} \right],
\]

(note that \( \Pi_{00} \) would be subtracted two times). Because of the symmetry of the configuration, obviously, \( \sum_{\ell=0}^N \Pi_{0\ell} = \sum_{k=0}^N \Pi_{k0} \). With the expressions

\[
\Pi_{00} = P_D (1 - \eta_B)^N,
\]

\[
\sum_{k=0}^N \Pi_{k0} = P_D \sum_{m=0}^N \left( \frac{N}{m} \right) 2^{-N} \sum_{k=0}^m \binom{m}{k} \eta_B^k (1 - \eta_B)^{N-k}
\]

\[
= P_D \left( 1 - \frac{\eta_B}{2} \right)^N
\]

we obtain finally for the double-click probability

\[
P^{(N)}_2 = P_D \left[ 1 - 2 \left( 1 - \frac{\eta_B}{2} \right)^N + (1 - \eta_B)^N \right]. \quad \text{(29)}
\]

### B. Eve sends a mixture of number states

Of course, there is no reason to restrict Eve only to the use of number states. After successful state discrimination she can send to Bob any pure state or mixture. However, from Bob’s point of view these signals are effectively mixtures of photon number states because of the nature of his detectors (they may be described by the pair of projectors: \( P_{\text{no}} = |0\rangle \langle 0| \) and \( P_{\text{yes}} = \sum_{n=1}^\infty |n\rangle \langle n| \)). Therefore it is sufficient to analyze only a mixture of photon number states in the polarization of the identified signal, so that only the photon number statistics remains to be chosen by Eve.

As already mentioned, Bob is interested only in the number of single clicks (in case that his and Alice’s bases coincide) and double clicks (if the bases differ). One can plot very illustrative diagram displaying relations between corresponding single-click and double-click probabilities (see Fig. 3).

The situation where Eve sends number states to Bob is represented by a dot for each value of the photon number \( N \). The positions of these dots have been calculated for fixed values of \( \eta_L \) and \( \mu \). Coordinates of a point corresponding to any mixture of number states can always be expressed as a linear convex combination of coordinates corresponding to individual number states. Because of the convexity of the above mentioned curve all such points must lie inside (or on the boundary) of the polygon with vertices at the points corresponding to number states (i.e. in the area highlighted in Fig. 3).

We can explicitly prove the convexity of the boundary formed by the points for fixed photon number. The points with \( x \)-coordinate \( P^{(N)}_1 \) [Eq. (28)] and \( y \)-coordinate \( P^{(N)}_2 \) [Eq. (29)] lie on a continuous curve which can be expressed with help of Eq. (28) by a real

**FIG. 3.** Diagram displaying relations between “single-click” and “double-click” probabilities. The highlighted area contains all possible combinations of Bob’s detection probabilities stemming from Eve’s activity (described in the text) for a given detection efficiency (here, particularly, \( \eta_B = 0.5 \)) and a given mean photon number in states sent by Alice (\( \mu = 4 \)). It is a region of insecure key generation. The shape of the area depends on \( \eta_L \), the scaling on \( \mu \) [through discrimination probability \( P_D(\mu) \)]. The separate dotted curve represents a set of all possible “working points” without an eavesdropper, i.e. a set of all possible pairs of expected \( P_1 \) and \( P_2 \). Any particular position of a working point depends on the values of the line transmittance (\( \eta_L \)), the detection efficiency (\( \eta_B \)), and the mean photon number (\( \mu \)). The value of \( \mu = 4 \) is chosen to make the diagram well readable. The structure is the same for lower, realistic values.
continuation of the parameter $N$ as

$$N = \frac{\ln(1 - x/P_D)}{\ln(1 - \eta_B)}.$$

Substituting into Eq. (29) we obtain the explicit equation of the curve

$$y = \left[2 - 2 \left(1 - \frac{x}{P_D}\right)^\kappa - \frac{x}{P_D}\right] P_D,$$

(30)

where

$$\kappa = \frac{\ln(1 - \eta_B/2)}{\ln(1 - \eta_B)}.$$

Calculating the second derivative of Eq. (30) with respect to $x$ and using the fact that $\eta_L$, $\eta_B$, and $x/P_D$ take values in the interval between 0 and 1, it follows that the curve given by Eq. (30) is convex. This proves that the highlighted area in Fig. 3 is indeed convex.

C. Insecure parameter regime

The convex area defined in the previous section can be called a region of insecurity. We define the working point of a setup as the one point whose coordinates are given by expected values in the absence of an eavesdropper. If this working point falls into the region of insecurity, Eve can get complete information on the key without a risk of being disclosed.

The set of all possible working points is represented by the dotted curve in the diagram. Expected single-click probability $P_1$ [Eq. (28)] represents $x$-coordinate, expected double-click probability $P_2$ [Eq. (29)] represents $y$-coordinate. From Eq. (29) the exponential can be expressed and substituted into Eq. (27). Thus the explicit equation of the working point curve reads

$$y = \left[1 - (1 - x)^{1/2}\right]^2.$$

(31)

We have to answer the question for which values of parameters $\eta_L$, $\eta_B$, and $\mu$ the working point lies in the region of insecurity.

1. Necessary condition for insecurity

If the expected probability of single clicks satisfies $\bar{P}_1 > \bar{P}_1(N)$ for all $N$ then the working point will certainly not fall to the region of insecurity, which is clearly illustrated in Fig. 3. This leads to the necessary condition for insecurity given by $\bar{P}_1 < P_D$. To evaluate the implication for the experimental parameters, we substitute Eq. (28)

$$\eta_L\eta_B < -\ln[1 - P_D(\mu)]/\mu.$$

(32)

An analysis of this expression shows that for a fixed expected photon number $\mu$ a system cannot be cracked by an USD attack if the total transmission efficiency $\eta_L\eta_B$ is higher than a certain threshold which depends on the expected photon number $\mu$. This dependence is evaluated numerically in Fig. 4. The surprising aspect is, that the threshold does not go to 1 as $\mu$ goes to infinity. Instead we find

$$\eta_L\eta_B(\infty) = \lim_{\mu \to \infty} -\left[1 - P_D(\mu)/\mu\right] = (1 - 2^{-1/2}) \approx 0.293.$$  

(33)

This shows, that the implementation of quantum cryptography with weak coherent states cannot be cracked completely by the USD attack for all values of the expected photon number $\mu$ as long as the total transmission satisfies $\eta_L\eta_B \geq 1 - 2^{-1/2}$.

2. Sufficient condition of insecurity

In this section we will derive precise conditions determining when a working point falls into the region of insecurity. In a first step we will show that for parameters of practical applications it is sufficient to consider the scenario the working point falls below the straight line going through the origin and the vertex $N = 2$. This condition corresponds to

$$x_w \geq \frac{x_w}{y_w}.$$

(34)

The coordinates of points used in this condition are defined in Tab. IV. In the second step we can then determine whether in this scenario the working point lies inside or
the observation that we can do without any extra approximation is based on to find its analytical solution. The analytical statement which, whenever condition (36) is fulfilled, determines cially for non negligible loss. Due to the complicated dependence of the mean photon number in coherent states sent by Alice must be lower than a threshold \( \mu \) given by
\[
\mu < \mu_2 = \frac{-2}{\eta_L \eta_B} \ln \left( \frac{4 - 3 \eta_B}{4 - \eta_B} \right).
\]
We find that \( \mu_2 \in [1/\eta_L, 2 \ln 3/\eta_L] \) for any \( \eta_B \) and, especially, always \( \mu_2 \geq 1 \). As we can see, this condition is satisfied in all current experiments and does not pose a serious restriction to the validity of our analysis especially for non negligible loss.

Now let us turn our attention to the condition (37) which, whenever condition (36) is fulfilled, determines whether the working point is in the region of insecurity. It can be expressed in the following form
\[
F(\mu, \eta_L, \eta_B) := x_w \eta_B - 2 y_w (1 - \eta_B) - P_D \eta_B^2 \leq 0, \quad (37)
\]
Due to the complicated dependence of \( P_D \) on \( \mu \) we failed to find its analytical solution. The analytical statement we can do without any extra approximation is based on the observation that
\[
\frac{\partial F}{\partial \mu} \Big|_{\mu=0} = \eta_L \eta_B^2 > 0 \quad \text{and} \quad F(0, \eta_L, \eta_B) = 0.
\]
This implies that there exists always a range of values for \( \mu \) starting from \( \mu = 0 \) for which we have \( F > 0 \), i.e. the security of the key distribution cannot be cracked completely by the USD attack.

It is easy to evaluate condition (37) numerically. In Fig. 5 we give an example for the values of line transmittance \( \eta_L = 0.1 \) and detection efficiency \( \eta_B = 0.5 \) (so that \( \mu_2 \approx 13.46 \)). In this particular case, the transmission becomes insecure in about 2.07 photons. It is not completely satisfying to have to fall back to numerical methods to investigate the security against the USD attack. Fortunately, it is possible to get some analytic results in a situation which is relevant to applications.

3. Partly accessible loss in a system with large loss

The results of the previous sections illuminate to which extent Eve can achieve perfect eavesdropping by making an unambiguous state discrimination measurement followed by sending the identified signals directly to Bob’s detectors, thereby bypassing the lossy quantum channel.

However, Eve does not necessarily need to access the whole lossy quantum channel to be successful. By accessing we mean, that Eve can avoid these losses either by replacing a quantum channel by a perfect, loss-free one, or by replacing it by classical communication and state preparation. The formulas of the previous sections still apply if we collect in the quantity \( \eta_B \) all those losses on the way to Bob’s detector which are not accessible to Eve, while \( \eta_L \) denotes now only that loss that is accessible to her. It is instructive to look at the limit of high non-accessible losses (\( \eta_B \ll 1 \)). In that case we can approximate the function \( F \) of equation (37) by
\[
F \approx \eta_B^2 \left( \eta_L \mu - \frac{1}{2} \eta_L^2 \mu^2 - P_D \right).
\]
The insecurity criterion \( F \leq 0 \) in the region \( \mu < \mu_2 \) (from Eqn. (36)) then leads to the condition
\[
\eta_L \leq \frac{1}{\mu} \left( 1 - \sqrt{1 - 2P_D} \right),
\]
which is independent of \( \eta_B \). It can be approximated by

\[
\eta_L \leq \frac{2P_D}{1 - 2P_D}
\]
\[ \eta_L \leq \eta_{L\text{crit}} \approx \frac{P_D}{\mu} \approx \frac{1}{12} \mu^2. \] (40)

FIG. 6. The secure parameter regime for the losses accessible to Eve for large Bob’s losses \((\eta_B \ll 1)\) is the region above the solid line \((F > 0)\). In the region with \(F \leq 0\) and \(\mu < \mu_2\) the system is insecure. In the remaining region we have \(F \leq 0\), but since \(\mu > \mu_2\), we cannot make any definitive statements about security.

Condition (39) is shown in Fig. 6 as a solid line. To make statements about security against the USD attack, we need to consider additionally condition (36), which can be approximated by \(\mu < \frac{1}{\eta_L}\) in leading order of \(\eta_B\) and is shown as a dashed line. We now can conclude that the system is secure against USD attacks in the regime of small detection efficiencies \(\eta_B\) if we are in the parameter region with \(F > 0\) and \(\mu < \mu_2\). Furthermore, the system is insecure in the region \(F \leq 0\) and \(\mu < \mu_2\). For the region with \(\mu > \mu_2\) we can only make indirect statements. One is, that if the system is secure for a pair of values \((\mu, \eta_L)\), then it must be secure for all values \((\mu, \eta'_L)\) with \(\eta'_L > \eta_L\), otherwise Eve could gain an advantage by not accessing all the loss available to her. Therefore the only region about which we cannot make a statement with the present calculation is the region with \(\mu > 4.1\) and \(\eta_L > \frac{1}{\mu}\). Here more detailed calculation would be necessary.

Note that these considerations are valid for \(\eta_B \ll 1\) and only in this limit \(\eta_B\) does no longer play any role. For higher values of \(\eta_B\) this changes.

D. Remark to the statistical nature of the problem

One should keep in mind that all of Bob’s measurements have a statistical character. Bob does not measure probabilities but finite numbers of clicks which naturally fluctuates. In practice Bob must set certain limits of a “confidential interval” of acceptable numbers of detector clicks. This effects that in some cases Bob will reject the transmission even if no eavesdropper is present. A more serious implication is that there is always some non-zero probability that Eve will not be detected even if the working point lies outside the insecurity region.

Note that Eve does not need to eavesdrop all the time—she may let pass a fraction of the signal sequence without any intervention. Her (deterministic) information on the key decreases with this strategy. But both Bob’s single- and double-click probabilities also change. The point corresponding to such an eavesdropping strategy (in the diagram as in Fig. 6) shifts along the straight line connecting “full-time” Eve’s strategy point with Alice’s and Bob’s working point. The relative shift equals the fraction of transmission during which Eve is active.

For practical purposes it would be necessary to determine the probability that Eve’s information on the key (due to the USD attack) will be smaller than a certain chosen limit, as a function of the limits of the confidential interval and of the length of the key. This represents a challenge for the further research in this field.

V. USD ATTACK VERSUS BEAMSPLITTING ATTACK

Traditionally, security against the beamsplitting attack \([14]\) has been used as a practical level of security. In the beamsplitting attack the lossy line is replaced by an ideal loss-free line complemented by a beamsplitter such that the total loss of the original line is reproduced. The eavesdropper stores any photons coming out of the free arm of the beamsplitter. Whenever the eavesdropper and the receiver obtain a photon, which is possible for multi-photon signals, Eve can measure her signal after she learns the polarization basis in the public announcement and she will learn thereby the bit value of these signals completely.

It is interesting to note that security against beamsplitting attack suggests that one can obtain a secure key even for large average photon numbers. In the absence of errors, the gain rate of secure key bits per signal bit can be approximated in a way similar to that used in \([13]\) for the optimal individual attack. This approximation is given by

\[ G_{\text{BS}} = \frac{1}{2} \left( p_{\text{exp}} - p_{\text{split}} \right), \] (41)

where the factor \(1/2\) stems from discarding signals with unequal polarization basis. Then \(p_{\text{exp}}\) is the probability that Bob receives a signal, while \(p_{\text{split}}\) is the joint probability that Eve learned the bit value of a signal and that the signal is received by Bob. To point out the basic problem of the beamsplitting attack it is sufficient to consider the case of \(\eta_B = 1\) and of coherent states. Then we find for Poissonian photon statistics and a transmission rate \(\eta\) of the system

\[ p_{\text{exp}} = 1 - \exp(-\eta\mu), \] (42)

\[ p_{\text{split}} = p_{\text{exp}} \left( 1 - \exp[-(1 - \eta)\mu] \right), \] (43)

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\[ G_{\text{BS}} = \frac{1}{2} \exp(-(1 - \eta)\mu) [1 - \exp(-\eta\mu)], \] (44)

which is always positive. Actually, the optimum is obtained for \( \mu \approx 1 \). It is clear from our analysis, however, that for large values of \( \mu \) and typical loss rates, the USD attack will render the quantum key distribution protocol completely insecure.

The awareness of this problem is low, and it is thought that it can be avoided by complementing the beamsplitting attack with the additional requirement to keep the average photon number low, much lower than 1, as to keep the set-up in the quantum domain. This seems rather odd, since there is no obvious justification for this requirement. More importantly, even for photon numbers \( \mu \ll 1 \) we find that for sufficiently large loss the transmission becomes insecure according to the USD attack while the analysis according to the beamsplitting attack makes us believe that we are dealing with a secure key. It seems that the USD attack is underestimated since the probability of success in the unambiguous state discrimination goes with \( \mu^3 \) since only for three or more photons the four signal states are actually linear independent. The beamsplitting attack, however, succeeds with a probability of order \( \mu^2 \), since already two photons can be split by the beamsplitter.

This seems to imply that beamsplitting is the more powerful attack. However, this is not the case since the two attacks vary in their power differently as the loss of the system increases. In the USD attack the probability to identify a signal depends only on the average photon number \( \mu \), and once this probability is high enough to generate the expected number of signals for the receiver (which depends on the amount of loss) then the transmission becomes insecure. In the beamsplitting attack, on the other hand, the total probability of identified signals \( p_{\text{split}} \) depend on \( \mu \) and on the transmission coefficient \( \eta \), and this probability goes down with increasing loss for fixed \( \mu \). And indeed, we find that \( p_{\text{exp}} > p_{\text{split}} \).

In other words, the beamsplitting attack becomes less efficient with increasing loss. This is easy to see in a simple example of a two-photon signal. The probabilities \( p(n, 2 - n) \) that \( n = 0, 1, 2 \) photons arrive at Bob’s detectors and \( n - 2 \) photons go to Eve in the beamsplitting attack are given by

\[ p(0, 2) = (1 - \eta)^2, \] (45)
\[ p(1, 1) = 2\eta(1 - \eta), \] (46)
\[ p(2, 0) = \eta^2. \] (47)

This means, that for high losses (\( \eta \ll 1 \)) most likely both photons are sent to Eve. Probability of this event is \( p(0, 2) \approx 1 - 2\eta \), while the splitting probability goes down as \( p(1, 1) \approx 2\eta \). The respective probabilities for \( n \)-photon signals are of the same order of magnitude in \( \eta \). Therefore, clearly, there is a crossover as a function of \( \eta \) where for fixed average photon number \( \eta \) the USD attack is more efficient than the beamsplitting attack.

We would like to stress again that from a technological point of view the USD attack seems to be easier to implement than the beamsplitting attack. This is based on two points. Firstly, experience indicates that a complete measurements destroying the quantum state completely, as possible by the USD attack, are easier to realize (at least in some approximation) than the realization of a quantum channel with reduced loss, as required by the beamsplitting attack. Secondly, the beamsplitting attack implies the use of quantum memory, which could store the split-off signal photons until the polarization bases for each signal are announced.

Finally, we would like to point out again that a security proof for realistic signals with Poissonian photon number distribution exists for individual attacks \( \cite{12} \) and coherent attacks \( \cite{13} \). Naturally, these security proofs include the security against the beamsplitting attack and against the USD attack.

VI. CONCLUSIONS

We have quantitatively analyzed an attack against realistic quantum crypto systems which enables an eavesdropper to gain information on the key without causing any errors in case of a lossy channel or poor detection efficiencies. It uses unambiguous discrimination of linearly independent signal states. This attack does not require the ability to store quantum states or to perform complicated quantum dynamics. Moreover, the attack does not require to substitute the lossy quantum channel by a perfect one.

We have derived a set of conditions which allow to judge whether a given system can be totally insecure under the USD attack. We have shown a secure parameter regime in terms of the total transmission efficiency and the mean photon number. In the important limit of small detection efficiencies \( \eta_B \) we have obtained an analytic result so that we can give explicitly a set of parameters (line transmittances, detector efficiencies and mean photon numbers in coherent states sent by Alice) for which the transmission is secure/insecure under the USD attack. In theory, the signal can always be chosen to be weak enough to allow secure communication. In practice, however, the detector noise places restrictions on that end \( \cite{11} \). Finally, we showed that security against beamsplitting attacks does not necessarily imply security against the USD attack. This implies that we need to search for a better conditional security criterion against attacks deemed practically with currently available technology.

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