IMPLICIT EQUATIONS INVOLVING THE \( p \)-LAPLACIAN OPERATOR

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Abstract. In this work we study the existence of solutions \( u \in W^{1,p}_0(\Omega) \) to the implicit elliptic problem
\[
    f(x,u,\nabla u, \Delta_p u) = 0 \quad \text{in } \Omega,
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 2 \), with smooth boundary \( \partial \Omega \), \( 1 < p < +\infty \), and \( f: \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \).

We choose the particular case when the function \( f \) can be expressed in the form
\[
    f(x,z,w,y) = \phi(x,z,w) - \psi(y),
\]
where the function \( \psi \) depends only on the \( p \)-Laplacian \( \Delta_p u \). We also show some applications of our results.

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1. INTRODUCTION AND MAIN RESULTS

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N, N \geq 2 \), with smooth boundary \( \partial \Omega \), let \( 1 < p < +\infty \) and let \( f: \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \). In this paper, we shall consider the following implicit elliptic problem
\[
    u \in W^{1,p}_0(\Omega), \quad f(x,u,\nabla u, \Delta_p u) = 0 \quad \text{in } \Omega,
\]
where \( \Delta_p \) denotes the \( p \)-Laplace operator, namely
\[
    \Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) \quad \forall u \in W^{1,p}(\Omega).
\]

We focus on the particular case when the function \( f \) can be expressed in the form
\[
    f(x,z,w,y) = \varphi(x,z,w) - \psi(y),
\]
where \( \varphi \) is a real-valued function defined on \( \Omega \times \mathbb{R} \times \mathbb{R}^N \), and \( \psi \) is a real-valued function defined on \( Y \), where \( Y \) is a nonempty interval of \( \mathbb{R} \) (which will be specified later). We require that \( \psi \) depends only on the \( p \)-Laplacian \( \Delta_p u \). We further distinguish among the case where \( \varphi \) is a Carathéodory function depends on \( x, u, \) and \( \nabla u \), and the case where \( \varphi \) is allowed to be highly discontinuous in each variable, for which the dependance on the gradient is not permitted.

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In both cases we first reduce problem (1.1) to an elliptic differential inclusion, but methods used are different, depending on the regularity of the function $\varphi$ and on the structure of the problem.

More precisely, in the first case we make use of a result in [14] to obtain the inclusion
\[(1.2)\quad -\Delta_p u \in F(x, u, \nabla u),\]
where $F$ is a lower semicontinuous selection of the multifunction
\[(x, z, w) \mapsto \{y \in Y : \varphi(x, z, w) - \psi(y) = 0\}.\]
A function $u \in W^{1,p}_0(\Omega)$ is called a (weak) solution to (1.2) provided there exists $v \in L^p(\Omega)$, $p'$ being the conjugate exponent of $p$, such that $v(x) \in F(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$ and
\[\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w dx = \int_\Omega vw dx \quad \forall w \in W^{1,p}_0(\Omega).\]

We start with the general case when $Y$ coincides with the whole space $\mathbb{R}$, and after we deduce, as a particular case, the existence result when $Y$ is a closed interval of $\mathbb{R}$.

The main tool to obtain existence of solutions to (1.2) is the result below, [12, Theorem 3.1], which deals with the existence of solutions for elliptic differential inclusions with lower semicontinuous right-hand side and is based on a selection theorem for decomposable-valued multifunctions (see [1] and [8]).

**Theorem 1.1.** Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^\mathbb{R}$ be a closed-valued multifunction. Suppose that:

- (h1) $F$ is $L(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable;
- (h2) for almost every $x \in \Omega$, the multifunction $(z, w) \mapsto F(x, z, w)$ turns out to be lower semicontinuous;
- (h3) there exist $a \in L^{p'}(\Omega, \mathbb{R}^+)$, $b, c \geq 0$, with $\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{p'}} < 1$, complying with
  \[\inf_{y \in F(x, z, w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1}\]
  in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

Then, (1.2) has a solution $u \in W^{1,p}_0(\Omega)$.

Here, $\lambda_{1,p}$ is the first eigenvalue of the $p$-Laplacian in the space $W^{1,p}_0(\Omega)$.
The following is our main result, which extends [8, Theorem 3.2] to the case $p \neq 2$.

**Theorem 1.2.** Let $\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function and let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that:

- (i) for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the set $\{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0\}$ has empty interior;
- (ii) for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the function $y \mapsto \varphi(x, z, w) - \psi(y)$ changes sign;
- (iii) there exist $a \in L^{p'}(\Omega, \mathbb{R}^+)$, $b, c \geq 0$, with $\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{p'}} < 1$, such that
  \[\sup\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + b|z|^{p-1} + c|w|^{p-1},\]
  for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. 

Then, there exists \( u \in W^{1,p}_0(\Omega) \) such that
\[
\psi(-\Delta_p u) = \varphi(x, u, \nabla u) \quad \text{in} \quad \Omega.
\]

When \( \varphi \) is discontinuous we essentially follow [11, Theorem 3.1] to construct an appropriate upper semicontinuous multifunction \( F \) related with \( \psi^{-1} \) and \( \varphi \), and then we solve the elliptic differential inclusion \(-\Delta_p u \in F(x, u)\) using the following [9, Theorem 2.2]

**Theorem 1.3.** Let \( U \) be a nonempty set, \( \Phi: U \to W^{1,p}_0(\Omega), \Psi: U \to L^p(\Omega) \) two operators and \( F: \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \) a convex closed-valued multifunction. Suppose that the following conditions hold true:

1. \( \Psi \) is bijective and for any \( v_h \to v \) in \( L^p(\Omega) \) there is a subsequence of \( \{\Phi(\Psi^{-1}(v_h))\} \)
2. \( F(\cdot, z) \) is measurable for all \( z \in \mathbb{R} \);
3. \( F(x, \cdot) \) has a closed graph for almost every \( x \in \Omega \);
4. There exists \( r > 0 \) such that the function
   \[
   \rho(x) := \sup_{\|z\| \leq g(r)} d(0, F(x, z)), \quad x \in \Omega,
   \]

\( \rho(x) \) belongs to \( L^p(\Omega) \) and \( \|\rho\|_{L^p} \leq r \).

Then, the problem \( \Psi(u) \in F(x, \Phi(u)) \) has at least one solution \( u \in U \) satisfying
\[
|\Psi(u)(x)| \leq \rho(x) \quad \text{for almost every} \quad x \in \Omega.
\]

Extending [11, Theorem 3.1] to the case \( p \neq 2 \), we obtain the following result. We denote by \( \pi_0 \) and \( \pi_1 \) the projections of \( \Omega \times \mathbb{R} \) on \( \Omega \) and \( \mathbb{R} \), respectively.

**Theorem 1.4.** Let \( \mathcal{F} = \{A \subseteq \Omega \times \mathbb{R} : A \text{ is measurable and there exists } i \in \{0, 1\} \text{ such that } m(\pi_i(A)) = 0\}, (\alpha, \beta) \subseteq \mathbb{R} \) be an interval which does not contain 0, \( \psi \) a continuous real-valued function defined on \( (\alpha, \beta) \), \( \varphi \) a real-valued function defined on \( \Omega \times \mathbb{R} \), and \( p > N \). Suppose that

1. \( \varphi \) is \( \mathcal{L}(\Omega \times \mathbb{R}) \)-measurable and essentially bounded;
2. the set \( D_{\varphi} = \{(x, z) \in \Omega \times \mathbb{R} : \varphi \text{ is discontinuous at } (x, z)\} \) belongs to \( \mathcal{F} \);
3. \( \varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r)) \in \mathcal{F} \) for every \( r \in \psi((\alpha, \beta)) \);
4. \( \varphi(S \setminus D_{\varphi}) \subseteq \psi((\alpha, \beta)) \).

Then, there exists \( u \in W^{1,p}_0(\Omega) \) such that
\[
\psi(-\Delta_p u) = \varphi(x, u) \quad \text{in} \quad \Omega.
\]

### 1.1. Structure of the paper.
In Section 2 we will introduce the functional analytic setting we will use throughout the work. Section 3 is devoted to the case \( \varphi(x, \cdot, \cdot) \) continuous. Here we will distinguish some cases, which depend on the growth conditions on \( \varphi \) or on the choice of the set \( Y \). We also will give some examples where these situations apply. In Section 4 we will consider the discontinuous framework.
2. Preliminaries

Let $X$ be a topological space and let $V \subseteq X$. We denote by $\text{int}(V)$ the interior of $V$ and by $\overline{V}$ the closure of $V$. The symbol $\mathcal{B}(X)$ is used to denote the Borel $\sigma$-algebra of $X$.

If $(X,d)$ is a metric space, for every $x \in X, r \geq 0$ and every nonempty set $V \subseteq X$, we define

$$B(x,r) = \{ z \in X : d(x,z) \leq r \} \quad \text{and} \quad d(x,V) = \inf_{z \in V} d(x,z).$$

Let $X$ and $Z$ be two nonempty sets. A multifunction $\Phi$ from $X$ into $Z$ (symbolically $\Phi : X \rightarrow 2^Z$) is a function from $X$ into the family of all subsets of $Z$. A function $\varphi : X \rightarrow Z$ is said to be a selection of $\Phi$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$. For every set $W \subseteq Z$ we define $\Phi^-(W) = \{ x \in X : \Phi(x) \cap W \neq \emptyset \}$.

When $(X,\mathcal{A})$ is a measurable space, $Z$ is a topological space and for every open set $W \subseteq Z$ we have $\Phi^-(W) \in \mathcal{A}$, we say that the multifunction $\Phi$ is measurable. If $X$ and $Z$ are two topological spaces and, for every open (resp. closed) set $W \subseteq Z$, the set $\Phi^-(W)$ is open (resp. closed) in $X$, we say that $\Phi$ is lower semicontinuous (resp. upper semicontinuous). When $(Z,\delta)$ is a metric space, the multifunction $\Phi$ is lower semicontinuous if and only if, for every $z \in Z$, the real-valued function $x \mapsto \delta(z,\Phi(x)), x \in X$, is upper semicontinuous (see [15, Theorem 1.1]). If, moreover, $X$ is first countable, then the multifunction $\Phi$ is lower semicontinuous if and only if, for every $x \in X$, every sequence $\{x_k\}$ in $X$ converging to $x$ and every $z \in \Phi(x)$, there exists a sequence $\{z_k\}$ in $Z$ converging to $z$ and such that $z_k \in \Phi(x_k)$, for all $k \in \mathbb{N}$ (see [6, Theorem 7.1.7]).

A general result on the lower semicontinuity of a multifunction is the following [14, Theorem 1.1]

**Theorem 2.1.** Let $C, D$ be two topological spaces, with $D$ connected and locally connected, and $f$ be a real-valued function defined on $C \times D$. For all $x \in C$ we set

$$V(x) := \{ y \in D : f(x,y) = 0 \},$$

$$M(x) := \{ y \in D : y \text{ is a local extremum point for } f(x,\cdot) \},$$

$$Q(x) := V(x) \setminus M(x).$$

Suppose that

(a) for all $x \in C, f(x,\cdot)$ is continuous, and $0 \in \text{int}(f(x,D));$

(b) for all $x \in C$ and for all $A$ open subset of $D$, there exists $\bar{y} \in A$ such that $f(x,\bar{y}) \neq 0$;

(c) the set $\{(y',y'') \in D \times D : \{ x \in C : f(x,y') < 0 < f(x,y'') \} \text{ is open} \}$ is dense in $D \times D$.

Then, the multifunction $Q$ is lower semicontinuous, with nonempty closed values.

From now on, $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 2$, with a smooth boundary $\partial\Omega$, the symbol $\mathcal{L}(\Omega)$ (respectively, $m(\Omega)$) denotes the Lebesgue $\sigma$-algebra (respectively, measure) of $\Omega$, while $W^{1,p}_0(\Omega)$ stands for the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$. On $W^{1,p}_0(\Omega)$
we introduce the norm
\[ \|u\| := \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{1/p}, \quad u \in W^{1,p}_0(\Omega). \]

Let \( p^* \) be the critical exponent for the Sobolev embedding \( W^{1,p}_0(\Omega) \subseteq L^r(\Omega) \). Recall that
\[ p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{otherwise} \end{cases} \]

If \( p \neq N \), then to each \( r \in [1, p^*] \) there corresponds a constant \( c_{rp} > 0 \) satisfying
\[ \|u\|_{L^r(\Omega)} \leq c_{rp} \|u\|, \quad \forall u \in W^{1,p}_0(\Omega), \]
whereas, when \( p = N \), for every \( r \in [1, +\infty) \) we have
\[ \|u\|_{L^r(\Omega)} \leq c_{rN} \|u\|, \quad \forall u \in W^{1,N}_0(\Omega). \]

Finally, the embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega) \) is compact, provided \( 1 \leq r < p^* \). When \( p > N \), we get \( W^{1,p}_0(\Omega) \subseteq L^\infty(\Omega) \) and
\[ (2.1) \quad \|u\|_{\infty} \leq a \|u\|, \quad u \in W^{1,p}_0(\Omega), \]
for suitable \( a > 0 \); see [2, Ch. IX].

Given \( p \in ]1, +\infty[ \), the symbol \( p' \) will denote the conjugate exponent of \( p \) while \( W^{-1,p'}(\Omega) \) stands for the dual space of \( W^{1,p}(\Omega) \). Through [2, Theorem 6.4], we see that \( L^{p'}(\Omega) \) compactly embeds in \( W^{-1,p'}(\Omega) \). So, there exists \( b > 0 \) satisfying
\[ (2.2) \quad \|v\|_{W^{-1,p'}(\Omega)} \leq b \|v\|_{L^{p'}(\Omega)}, \quad \forall v \in L^{p'}(\Omega). \]

Let \( A_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega) \) be the nonlinear operator stemming from the negative \( p \)-Laplacian, i.e.,
\[ (2.3) \quad \langle A_p(u), v \rangle := \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx, \quad u, v \in W^{1,p}_0(\Omega), \]
and let \( \lambda_{1,p} \) be its first eigenvalue in \( W^{1,p}_0(\Omega) \). The following facts are well known (see, e.g., [13], Appendix A):

1. \( A_p \) is bijective and uniformly continuous on bounded sets;
2. the inverse operator \( A_p^{-1} \) is \( (W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)) \)-continuous;
3. \( \|A_p(u)\|_{W^{-1,p'}(\Omega)} = \|u\|_{W^{1,p}_0(\Omega)}^{p-1} \) in \( W^{1,p}_0(\Omega) \);
4. \( \|u\|_{L^{p'}(\Omega)} \leq \frac{1}{\lambda_{1,p}} \|u\|^p, \) for all \( u \in W^{1,p}_0(\Omega) \).

3. THE CASE WHEN \( \varphi \) IS A CARATHÉODORY FUNCTION

This section deals with the existence of solutions to the equation
\[ (3.1) \quad \psi(-\Delta_p u) = \varphi(x, u, \nabla u). \]

We first consider the case \( Y = \mathbb{R} \). Throughout the section, \( p \in ]1, +\infty[ \) and the following assumptions will be posited:

1. for every \((x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\), the set \( \{ y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0 \} \) has empty interior;
(ii) for all \((x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\), the function \(y \mapsto \varphi(x, z, w) - \psi(y)\) changes sign.

**Theorem 3.1.** Let \(\varphi: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\) be a Carathéodory function and let \(\psi: \mathbb{R} \to \mathbb{R}\) be continuous. Suppose that (i)-(ii) hold true and, moreover,

(iii) there exist \(a \in L^p(\Omega, \mathbb{R}_+^+), b, c \geq 0\), with \(\frac{1}{p} + \frac{1}{\lambda_{1,p}} < 1\), such that

\[
\sup\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + b|z|^{p-1} + c|w|^{p-1},
\]

for all \((x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\).

Then, there exists a solution \(u \in W^{1,p}_0(\Omega)\) to equation (3.1).

**Proof.** Fix any \(x \in \Omega\). We want to apply Theorem 2.1. Choose \(C = \mathbb{R} \times \mathbb{R}^N\), \(D = \mathbb{R}\), \(f(z, w, y) = \varphi(x, z, w) - \psi(y)\), and for every \((z, w) \in \mathbb{R} \times \mathbb{R}^N\) set

\[
F(x, z, w) := \{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0, \quad y \text{ is not a local extremum point of } \psi(\cdot)\}.
\]

Hypothesis (ii) directly yields (a). Moreover, in our context, (b) is equivalent to say that, for all \((z, w) \in \mathbb{R} \times \mathbb{R}^N\), the set \(U := \{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) \neq 0\}\) is dense in \(\mathbb{R}\). Since, by (i), the set \(\mathbb{R} \setminus U\) has empty interior, it follows that \(U\) is dense in \(\mathbb{R}\), as desired.

Let us next analyze the set

\[
\{(y', y'') \in \mathbb{R} \times \mathbb{R} : \{(z, w) \in \mathbb{R} \times \mathbb{R}^N : \varphi(x, z, w) - \psi(y') < 0 \\
< \varphi(x, z, w) - \psi(y'')\} \text{ is open}\}.
\]

If one can find \(y', y'' \in \mathbb{R}\) such that

\[
\varphi(x, z, w) - \psi(y') < 0 < \varphi(x, z, w) - \psi(y''),
\]

then \(\varphi(x, z, w) \in ]\psi(y''), \psi(y')[\). So, the set

\[
\{(z, w) \in \mathbb{R} \times \mathbb{R}^N : \varphi(x, z, w) - \psi(y') < 0 < \varphi(x, z, w) - \psi(y'')\},
\]

turns out to be open, because \(\varphi(x, \cdot, \cdot)\) is continuous. Otherwise it is empty. So, the set (3.2) is the whole space \(\mathbb{R} \times \mathbb{R}\), and (c) follows.

Therefore, thanks to Theorem 2.1, the multifunction \(F(x, \cdot, \cdot)\) is lower semicontinuous, with nonempty closed values.

Moreover, for all \(y', y'' \in \mathbb{R}\) we have

\[
\{(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \varphi(x, z, w) - \psi(y') < 0 < \varphi(x, z, w) - \psi(y'')\} = \\
\{(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \varphi(x, z, w) \in ]\psi(y''), \psi(y')[\} \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N),
\]

cf. [4, Lemma III.14]. Therefore, condition (iii) of [8, Theorem 3.2], with \(\Lambda^* = \mathbb{R} \times \mathbb{R}\), is satisfied. Arguing as in that theorem we see that, if \(A \subseteq \mathbb{R}\) is open, then

\[
F^{-}(A) = \bigcup_{(y', y'') \in A \times A} \{(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \varphi(x, z, w) - \psi(y') < 0 < \varphi(x, z, w) - \psi(y'')\}.
\]

This actually implies that \(F^{-}(A) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)\), i.e. \(F\) is measurable.
Finally, fix any \( y \in F(x, z, w) \). In other words, \( y \in \psi^{-1}(\varphi(x, z, w)) \), therefore hypothesis (iii) implies that

\[
\inf_{y \in F(x, z, w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N.
\]

So all the hypotheses of Theorem 1.1 are fulfilled, and we get a solution \( u \in W_0^{1,p}(\Omega) \) to equation (1.2). Taking into account the definition of \( F \), we have \( \psi(-\Delta_p u) = \varphi(x, u, \nabla u) \), that is the thesis.

\[\Box\]

**Remark 3.2.** A very simple situation when hypothesis (iii) occurs is the following.
Suppose that \( \varphi(\Omega \times \mathbb{R} \times \mathbb{R}^N) \subseteq [\alpha, \beta] \) and \( \psi \) is such that \( \psi^{-1}(B) \) is bounded, for every \( B \) bounded subset of \( \mathbb{R} \). If \( (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \), we get \( \varphi(x, z, w) \in [\alpha, \beta] \), and so \( \psi^{-1}(\varphi(x, z, w)) \subseteq \psi^{-1}([\alpha, \beta]) \). Then, if we choose \( a \in L^p(\Omega, \mathbb{R}_0^+) \) such that \( a(x) > \sup\{|y| : y \in \psi^{-1}([\alpha, \beta])\} \) for all \( x \in \Omega \), we have

\[
|\psi^{-1}(\varphi(x, z, w))| < a(x) \leq a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N,
\]

that is hypothesis (iii).

As an application of the previous result, we consider the following example.

**Example 3.3.** Let \( g \in L^2(\Omega) \) and \( \gamma \in ]0, 1[ \). Then, for every \( \lambda \neq 0, \mu \in \mathbb{R} \), there exists a solution \( u \in W_0^{1,2}(\Omega) \) to the equation

\[
-\Delta u = g(x) + \mu(|u| + |\nabla u|)^\gamma + \lambda \sin(-\Delta u).
\]

**Proof.** Fix \( \lambda, \mu \in \mathbb{R} \). For every \( (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \) and every \( y \in \mathbb{R} \), set

\[
\varphi(x, z, w) := g(x) + \mu(|z| + |w|)^\gamma, \quad \psi(y) := y - \lambda \sin y.
\]

Since \( \lim_{y \to \pm\infty}(y - \lambda \sin y) = \pm\infty \), the function \( y \mapsto \varphi(x, z, w) - \psi(y) \) surely changes sign. Moreover, since it vanishes only at points of \( \mathbb{R} \) and not in intervals, the set

\[
\{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0\}
\]

has empty interior in \( \mathbb{R} \). Hence, hypotheses (i) and (ii) are fulfilled.

Fix now \( (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \). In order to verify hypothesis (iii), we want to find \( b, c \geq 0 \), with \( \frac{b}{\lambda_{1,2}} + \frac{\gamma}{\lambda_{1,2}^2} \leq 1 \), and \( a \in L^2(\Omega, \mathbb{R}_0^+) \) such that

\[
\max\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + b|z| + c|w|.
\]

Notice that we can consider the maximum in (3.4) instead of the supremum, since the set \( \psi^{-1}(\varphi(x, z, w)) \) is compact. Of course, (3.4) is equivalent to prove that \( |y| < a(x) + b|z| + c|w| \), for every \( y \) solution of the equation

\[
\psi(y) = \varphi(x, z, w).
\]

Thanks to Young's inequality with exponents \( 1/\gamma \) and \( 1/(1 - \gamma) \), we have

\[
|\varphi(x, z, w)| = |g(x) + \mu(|z| + |w|)^\gamma| \leq |g(x)| + |\mu||z|\gamma + |\mu||w|\gamma
\]

\[
\leq |g(x)| + \varepsilon|z| + \varepsilon|w| + C_{\gamma, \varepsilon, \mu}
\]

\[
\leq \tilde{g}(x) + \varepsilon|z| + \varepsilon|w|,
\]

where \( \varepsilon \) and \( \tilde{g}(x) \) are non-negative constants.
where \( \tilde{g}(x) := |g(x)| + C_{\gamma, \epsilon, \mu} \) for every \( x \in \Omega \). Then, if \( \tilde{g} \) is a solution to (3.5), from (3.6) it follows that

\[
|\psi(\tilde{g})| = |\varphi(x, z, w)| \leq \tilde{g}(x) + \epsilon|z| + \epsilon|w|.
\]

On the other hand, by the definition of \( \psi \), we have

\[
|\psi(\tilde{g})| = |\tilde{g} - \lambda \sin \tilde{g}| \geq |\tilde{g}| - |\lambda|,
\]

which implies that

\[
|\tilde{g}| \leq |\psi(\tilde{g})| + |\lambda| \leq \tilde{g}(x) + |\lambda| + \epsilon|z| + \epsilon|w| < \tilde{g}(x) + \epsilon|z| + \epsilon|w|,
\]

where \( \tilde{g}(x) := \tilde{g}(x) + 2|\lambda| \), for every \( x \in \Omega \). Observe that \( \tilde{g} \in L^2(\Omega, \mathbb{R}^+) \). Then, if we choose \( \epsilon \) in such a way that

\[
\frac{\epsilon}{\lambda_{1,2}} + \frac{\epsilon}{\lambda_{1,2}^{3/2}} < 1,
\]

we have hypothesis (iii) with \( a := \tilde{g} \) and \( b := c := \epsilon \). Thanks to Theorem 3.1, there exists a solution \( u \in W_0^{1,2}(\Omega) \) to equation (3.3).

\[\square\]

In the following example the function \( \psi \) exhibits a behavior very different from the previous one.

**Example 3.4.** Let \( p \in [2, +\infty[, f \in L^p(\Omega) \) and \( \gamma \in ]0, p - 1[. \) Then, for every \( \mu \in \mathbb{R} \) and \( \lambda \in \mathbb{R}^+ \), there exists a solution \( u \in W_0^{1,p}(\Omega) \) to the equation

\[
-\Delta_p u = f(x) + \mu(|u| + |\nabla u|)^\gamma - \lambda e^{-\Delta_p u}.
\]

**Proof.** Fix \( \mu \in \mathbb{R} \) and \( \lambda \in \mathbb{R}^+ \). As before, for every \( (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \) and \( y \in \mathbb{R} \), we set

\[
\varphi(x, z, w) := f(x) + \mu(|z| + |w|)^\gamma, \quad \psi(y) := y + \lambda e^y.
\]

Since \( \lim_{y \to \pm \infty}(y + \lambda e^y) = \pm \infty \), one immediately gets that (i) and (ii) are fulfilled. In order to verify hypothesis (iii), we argue as in Example 3.3. First of all, applying Young’s inequality with exponents \( \frac{1}{\gamma}, \frac{1}{p-1-\gamma} > 1 \), we have

\[
|\varphi(x, z, w)| = |f(x) + \mu(|z| + |w|)^\gamma| \leq |f(x)| + 2^{p-1}(|\mu||z|^\gamma + |\mu||w|^\gamma) \\
\leq |f(x)| + \epsilon|z|^{p-1} + \epsilon|w|^{p-1} + C_{\gamma, \epsilon, \mu} \\
= \tilde{f}(x) + \epsilon|z|^{p-1} + \epsilon|w|^{p-1},
\]

where \( \tilde{f}(x) := |f(x)| + C_{\gamma, \epsilon, \mu} \) for every \( x \in \Omega \). Let now \( \tilde{g} \) be a solution to the equation \( \varphi(x, z, w) - \psi(y) = 0 \). Then, from the previous inequality, we have

\[
|\psi(\tilde{g})| = |\varphi(x, z, w)| \leq \tilde{f}(x) + \epsilon|z|^{p-1} + \epsilon|w|^{p-1}.
\]

On the other hand, for every \( y \in \mathbb{R} \), and in particular for \( \tilde{g} \), we have

\[
|\psi(\tilde{g})| = |\tilde{g} + \lambda e^y| \geq |\tilde{g}| - |\xi|,
\]

where \( \tilde{g}(x) := |g(x)| + C_{\gamma, \epsilon, \mu} \) for every \( x \in \Omega \). Then, if \( \tilde{g} \) is a solution to (3.5), from (3.6) it follows that

\[
|\psi(\tilde{g})| = |\varphi(x, z, w)| \leq \tilde{g}(x) + \epsilon|z| + \epsilon|w|.
\]

On the other hand, by the definition of \( \psi \), we have

\[
|\psi(\tilde{g})| = |\tilde{g} - \lambda \sin \tilde{g}| \geq |\tilde{g}| - |\lambda|,
\]

which implies that

\[
|\tilde{g}| \leq |\psi(\tilde{g})| + |\lambda| \leq \tilde{g}(x) + |\lambda| + \epsilon|z| + \epsilon|w| < \tilde{g}(x) + \epsilon|z| + \epsilon|w|,
\]

where \( \tilde{g}(x) := \tilde{g}(x) + 2|\lambda| \), for every \( x \in \Omega \). Observe that \( \tilde{g} \in L^2(\Omega, \mathbb{R}^+) \). Then, if we choose \( \epsilon \) in such a way that

\[
\frac{\epsilon}{\lambda_{1,2}} + \frac{\epsilon}{\lambda_{1,2}^{3/2}} < 1,
\]

we have hypothesis (iii) with \( a := \tilde{g} \) and \( b := c := \epsilon \). Thanks to Theorem 3.1, there exists a solution \( u \in W_0^{1,2}(\Omega) \) to equation (3.3).

\[\square\]
\(\xi\) being the only solution to the equation \(y + \lambda e^y = 0\). Indeed, fix \(y \in \mathbb{R}\). If \(y \geq \xi\), we have
\[
|y + \lambda e^y| = |y + \lambda e^y - \xi - \lambda e^\xi| = |y - \xi + \lambda (e^y - e^\xi)| \\
\geq |y - \xi| \geq |y| - |\xi|,
\]
which is (3.8). Suppose now that \(y < \xi\), then
\[
|y + \lambda e^y| = |y - \xi + \lambda (e^y - e^\xi)| = |\xi - y + \lambda (e^\xi - e^y)| \\
\geq |\xi - y| \geq |y| - |\xi|,
\]
which again gives (3.8). This implies that
\[
|\bar{y}| \leq |\psi(\bar{y})| + |\xi| \leq \bar{f}(x) + \varepsilon |z|^{p-1} + \varepsilon |w|^{p-1} + |\xi| \\
< \bar{f}(x) + \varepsilon |z|^{p-1} + \varepsilon |w|^{p-1},
\]
\[
\bar{f}(x) := \bar{f}(x) + 2|\xi|, \text{ for every } x \in \Omega \text{ (note that one cannot have } \xi = 0\).
\]
Observe that \(\bar{f} \in L^p(\Omega, \mathbb{R}_0^+)\). Then, if we choose \(\varepsilon\) in such a way that
\[
\frac{\varepsilon}{\lambda_{1,p}} + \frac{\varepsilon}{\lambda_{1,p}^1} < 1,
\]
we have hypothesis (iii) with \(a := \bar{f}\) and \(b := c := \varepsilon\). Thanks to Theorem 3.1, there exists a solution \(u \in W^{1,p}_0(\Omega)\) to equation (3.7). \(\square\)

To state the next result, we set \(\delta_\Omega := \text{diam}(\Omega)\) and denote by \(\hat{C}\) the constant given by [12, Proposition 3.3].

**Theorem 3.5.** Let \(\varphi\) and \(\psi\) as in Theorem 3.1. Suppose that hypotheses (i)-(ii) hold true and, moreover,
\[
(iii)' \text{ there exist } a \in L^q(\Omega, \mathbb{R}_0^+), \ q > N, \ g: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \text{ nondecreasing with respect to each variable separately, such that}
\]
\[
\sup\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + g(|z|, |w|),
\]
for all \((x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\);
\[
(iv) \text{ there exists } R > 0 \text{ such that}
\]
\[
\|a\|_{L^q(\Omega)} + m(\Omega)^{1/q}g(\delta_\Omega \hat{C}R^{1/(p-1)}, \hat{C}R^{1/(p-1)}) \leq R.
\]

Then, equation (3.1) has a solution \(u \in W^{1,p}_0(\Omega)\).

**Proof.** As before, fix \(x \in \Omega\), and for all \((z, w) \in \mathbb{R} \times \mathbb{R}^N\), define
\[
F(x, z, w) := \{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0,
\]
y is not a local extremum point of \(\psi(\cdot)\}.

Reasoning like in the previous theorem, the multifunction \(F\) actually takes nonempty closed values, is lower semicontinuous w.r.t. \((z, w)\) and \(L(\Omega) \otimes B(\mathbb{R} \times \mathbb{R}^N)\)-measurable.

Fix now \(y \in F(x, z, w)\). Since, in other words, \(y \in \psi^{-1}(\varphi(x, z, w))\), hypothesis (iii)’ implies that
\[
\inf_{y \in F(x, z, w)} |y| < a(x) + g(|z|, |w|) \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.
\]
Taking into account (iv), we see that all the hypotheses of [12, Theorem 3.4] are fulfilled. Therefore, there exists \( u \in W_0^{1,p}(\Omega) \) such that \(-\Delta u \in F(x,u,\nabla u)\). Exploiting the definition of \( F \), this means that \( u \) is a solution to equation (3.1).

\[ \square \]

As an application of the previous result, we consider the following example, which has been inspired by [5, Corollary 1]. Observe that, unlike [5], here we consider a function \( \varphi \) which is not necessarily continuous w.r.t. the variable \( x \), but only in a suitable \( L^q(\Omega) \).

Moreover, here we deal with partial differential equations.

**Example 3.6.** Let \( h \in L^q(\Omega) \), with \( q > N \). Then, for every \( k \neq 0 \) and any sufficiently small \( \|h\|_q \) there exists a solution \( u \in W_0^{1,2}(\Omega) \) to the equation

\[ -\Delta u = h(x) + u^3 + |\nabla u|^2 + k \sin(-\Delta u). \]

**Proof.** Fix \( k \in \mathbb{R} \) and for all \((x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\) and all \( y \in \mathbb{R} \) define

\[ \varphi(x,z,w) := h(x) + z^3 + |w|^2, \quad \psi(y) := y - k \sin y. \]

Reasoning like in Example 3.3, we have that hypotheses (i)-(ii) are fulfilled.

In order to verify hypothesis (iii)', let \( g(|z|,|w|) := |z|^3 + |w|^2 \), for all \((z,w) \in \mathbb{R} \times \mathbb{R}^N\).

Of course \( g: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is nondecreasing w.r.t. each variable, separately. Let \( \tilde{y} \) be a solution to the equation \( \psi(y) = \varphi(x,z,w) \). It follows that

\[ |\psi(\tilde{y})| = |\tilde{y} - k \sin \tilde{y}| = |\varphi(x,z,w)| \]

\[ \leq |h(x)| + |z|^3 + |w|^2 = |h(x)| + g(|z|,|w|). \]

On the other hand, we always have

\[ |\psi(\tilde{y})| = |\tilde{y} - k \sin \tilde{y}| \geq |\tilde{y}| - |k|, \]

which implies that

\[ |\tilde{y}| \leq |\psi(\tilde{y})| + |k| \leq |h(x)| + g(|z|,|w|) + |k| \]

\[ < \tilde{h}(x) + g(|z|,|w|), \]

where \( \tilde{h}(x) := |h(x)| + 2|k| \), for every \( x \in \Omega \). Of course, \( \tilde{h} \in L^q(\Omega, \mathbb{R}_0^+) \). Hence we get hypothesis (iii)'.

Moreover, in order to verify hypothesis (iv), we have to check if there exists \( R > 0 \) such that

\[ \|\tilde{h}\|_{L^q(\Omega)} + m(\Omega)^{1/q}g(\delta_{\Omega}\dot{\psi}R,\dot{\psi}R) \leq R, \]

that is

\[ \|\tilde{h}\|_{L^q(\Omega)} + m(\Omega)^{1/q}\delta_{\Omega}^3\dot{\psi}^3R^2 + m(\Omega)^{1/q}\dot{\psi}^2R^2 \leq R. \]

(3.9)

If \( 0 < R << 1 \), choosing \( \tilde{h} \) in such a way that \( \|\tilde{h}\|_{L^q(\Omega)} < \frac{R}{2} \), we have that (3.9) is immediately satisfied, since the terms containing \( R^2 \) and \( R^3 \) are negligible with respect to \( R \). So all the hypotheses of Theorem 3.5 are fulfilled, and we get the thesis.

\[ \square \]

The next result provides solutions to equation (3.1) when the function \( \psi \) is of the form \( y \mapsto y - h(y) \), with \( h \) continuous and bounded. Note that here we have to require a specific growth condition on \( \varphi \).
Theorem 3.7. Let \( \varphi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) be a Carathéodory function and let \( h \in L^\infty(\mathbb{R}) \) be continuous. Suppose that (i)-(ii) hold true and, moreover,

(iii)' there exist \( f \in L^p(\Omega, \mathbb{R}_0^+) \), with \( f(x) \geq \|h\|_\infty \) for all \( x \in \Omega, \mu > 0, \gamma \in [0, p-1] \) such that

\[
\sup_{(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N} |\varphi(x, z, w)| < f(x) + \mu(|z| + |w|)^\gamma.
\]

Then, there exists a solution \( u \in W^{1,p}_0(\Omega) \) to the equation

\[
- \Delta_p u - h(-\Delta_p u) = \varphi(x, u, \nabla u).
\]

Proof. Fix \( x \in \Omega \) and define, for all \( (z, w) \in \mathbb{R} \times \mathbb{R}^N \),

\[
F(x, z, w) := \{y \in \mathbb{R} : \varphi(x, z, w) - (y - h(y)) = 0, \ y \text{ is not a local extremum point of } y \mapsto y - h(y)\}.
\]

Reasoning as in the above proofs ensures that \( F \) is lower semicontinuous w.r.t. \((z, w)\), with nonempty closed values, and \( \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N) \)-measurable.

Fix \((x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \). If \( y \in F(x, z, w) \), then it solves the equation \( \varphi(x, z, w) = y - h(y) \). Two cases occur. First, \( \gamma \in [1, p-1] \). Applying Young’s inequality with exponents \( \frac{p-1}{\gamma}, \frac{p-1}{p-1-\gamma} > 1 \), we have

\[
|y| = |y - h(y) + h(y)| \leq |y - h(y)| + |h(y)| \leq |\varphi(x, z, w)| + \|h\|_\infty
\]

\[
< f(x) + \mu(|z| + |w|)^\gamma + \|h\|_\infty
\]

\[
\leq 2f(x) + 2^{\gamma-1}\mu(|z| + |w|)^\gamma
\]

\[
\leq 2f(x) + 2^{\gamma-1}\mu(\varepsilon|z|^{p-1} + \varepsilon|w|^{p-1} + K_{\varepsilon})
\]

\[
\leq 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu(\varepsilon|z|^{p-1} + |w|^{p-1}),
\]

that is \( |y| < 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu(\varepsilon|z|^{p-1} + |w|^{p-1}) \), where \( C_\varepsilon := 2^{\gamma-1}\mu K_{\varepsilon} \). Hence

\[
\inf_{y \in F(x, z, w)} |y| < 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu(\varepsilon|z|^{p-1} + |w|^{p-1}).
\]

If we choose \( \varepsilon \) in such a way that

\[
\frac{2^{\gamma-1}\mu\varepsilon}{\lambda_{1,p}} + \frac{2^{\gamma-1}\mu\varepsilon}{\lambda_{1,p}^1} < 1,
\]

hypothesis (h3) of Theorem 1.1 is fulfilled, with \( a := 2f + C_\varepsilon \in L^p(\Omega, \mathbb{R}_0^+) \) and \( b := c := 2^{\gamma-1}\mu\varepsilon \).

Suppose now \( \gamma \in [0, 1[ \). Since, for every \( a, b \geq 0 \) we have \( (a + b)^\gamma \leq a^\gamma + b^\gamma \), reasoning as before yields

\[
|y| < 2f(x) + \tilde{C}_\varepsilon + \mu\varepsilon(|z|^{p-1} + |w|^{p-1}),
\]

where \( \tilde{C}_\varepsilon := \mu K_{\varepsilon} \). If we now choose \( \varepsilon \) in such a way that

\[
\frac{\mu\varepsilon}{\lambda_{1,p}} + \frac{\mu\varepsilon}{\lambda_{1,p}^1} < 1,
\]

hypothesis (h3) of Theorem 1.1 is again fulfilled, with \( a := 2f + \tilde{C}_\varepsilon \in L^p(\Omega, \mathbb{R}_0^+) \) and \( b := c := \mu\varepsilon \).
In both cases, there exists \( u \in W_0^{1,p}(\Omega) \) such that \(-\Delta_p u \in F(x,u,\nabla u)\). Through a familiar argument, this entails that \( u \) is a solution to equation (3.10).

We conclude this section considering the case when \( Y \) is a closed interval of \( \mathbb{R} \). Observe that here no growth conditions on \( \varphi \) are required.

**Theorem 3.8.** Let \( \varphi : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a Carathéodory function and let \( \psi : [\alpha, \beta] \rightarrow \mathbb{R} \) be continuous. Suppose that:

1. for every \((x,z,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N\), the set \( \{ y \in [\alpha, \beta] : \varphi(x,z,w) - \psi(y) = 0 \} \) has empty interior;
2. for every \((x,z,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N\), the function \( y \mapsto \varphi(x,z,w) - \psi(y) \) changes sign in \([\alpha, \beta]\).

Then, there exists a solution \( u \in W_0^{1,p}(\Omega) \) to equation (3.1).

**Proof.** As before, fix \( x \in \Omega \), and for all \((z,w)\in\mathbb{R}\times\mathbb{R}^N\) define
\[
F(x,z,w) := \{ y \in [\alpha, \beta] : \varphi(x,z,w) - \psi(y) = 0, \ y \text{ is not a local extremum point of } \psi(\cdot) \}.
\]

A familiar argument ensures that \( F \) takes nonempty closed values, is lower semicontinuous w.r.t. \((z,w)\) and \( L(\Omega) \otimes B(\mathbb{R} \times \mathbb{R}^N)\)-measurable.

Fix now \( y \in F(x,z,w) \). In particular we have \(|y| \leq \max\{|\alpha|, |\beta|\} \). Then, hypothesis (h3) of Theorem 1.1 is immediately satisfied with \( a(x) := 2 \max\{|\alpha|, |\beta|\} \) for every \( x \in \Omega \) and \( b := c := 0 \). Therefore, there exists \( u \in W_0^{1,p}(\Omega) \) such that \(-\Delta_p u \in F(x,u,\nabla u)\), i.e. \( u \) is a solution to (3.1).

As application of the previous theorem, we consider two examples, which differ by the behavior of the function \( \psi \). In both cases, the condition which permits to get a solution is the boundedness of \( \varphi \).

**Example 3.9.** Let \( f \in L^\infty(\Omega), k \in \mathbb{N}, k \text{ even and such that } k\pi > \|f\|_\infty \), and let \( \psi : [-k\pi, k\pi] \rightarrow \mathbb{R} \) be defined by \( \psi(y) = y \cos(y) \). Then, there exists a solution \( u \in W_0^{1,p}(\Omega) \) to the equation
\[
(3.11) \quad \psi(-\Delta_p u) = f \quad \text{ in } \Omega.
\]

**Proof.** Observe that assumption (1) is clearly satisfied. Moreover, for every \( x \in \Omega \), we have
\[
f(x) - \psi(k\pi) = f(x) - k\pi \cos(k\pi) = f(x) - k\pi (-1)^k = f(x) - k\pi < 0
\]
and \( f(x) - \psi(-k\pi) = f(x) + k\pi \cos(-k\pi) = f(x) + k\pi > 0 \).

Therefore, hypothesis (2) is also satisfied. Thanks to Theorem 3.8, there exists at least a solution \( u \in W_0^{1,p}(\Omega) \) to equation (3.11).

Note that the interval \([\alpha, \beta]\) could be unbounded, as the following example shows.
Example 3.10. Let $p \in [1, +\infty[$, $f \in L^p(\Omega)$, $\gamma > 0$ and $\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Suppose that there exists $\lambda \in \mathbb{R}^+$ such that

\begin{equation}
\label{eq:3.12}
\sup_{(x,z,w)\in\Omega \times \mathbb{R} \times \mathbb{R}^N} |\varphi(x, z, w)| < \lambda.
\end{equation}

Then, there exists a solution $u \in W^{1,p}_0(\Omega)$ to equation

$$
\varphi(x, u, \nabla u) - \lambda e^{\Delta_p u} - \Delta_p u = 0.
$$

**Proof.** Define $\psi(y) := \lambda e^{-y} - y$ for every $y \in [0, +\infty[$. Observe that hypothesis (1) is immediately satisfied. Moreover, thanks to (3.12), for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ we have

$$
\varphi(x, z, w) - \psi(0) = \varphi(x, z, w) - \lambda < 0
$$

and

$$
\lim_{y \to +\infty} (\varphi(x, z, w) - \psi(y)) = +\infty.
$$

Therefore, hypothesis (2) holds true too, and the conclusion follows from Theorem 3.8.

\[ \square \]

4. The discontinuous framework

This section is devoted to the proof of Theorem 1.4, which we rewrite here, for the reader’s convenience. Given $(x, z) \in S := \Omega \times \mathbb{R}$, set $\pi_0(x, z) = x$ and $\pi_1(x, z) = z$. Moreover, fix $p > N$ and define

$$
\mathcal{F} = \{A \subseteq S : A \text{ is measurable and there exists } i \in \{0, 1\} \text{ such that } m(\pi_i(A)) = 0\}.
$$

**Theorem 4.1.** Let $(\alpha, \beta) \subseteq \mathbb{R}$ be such that $0 \notin (\alpha, \beta)$, $\psi : (\alpha, \beta) \to \mathbb{R}$ be continuous and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$. Suppose that the following conditions hold true:

(i) $\varphi$ is $\mathcal{L}(\Omega \times \mathbb{R})$-measurable and essentially bounded;

(ii) the set $D_{\varphi} = \{(x, z) \in S : \varphi \text{ is discontinuous at (x, z)}\}$ belongs to $\mathcal{F}$;

(iii) $\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r)) \in \mathcal{F}$ for every $r \in \psi((\alpha, \beta))$;

(iv) $\varphi(S \setminus D_{\varphi}) \subseteq \psi((\alpha, \beta))$.

Then, there exists $u \in W^{1,p}_0(\Omega)$ such that

$$
\psi(-\Delta_p u) = \varphi(x, u) \quad \text{in} \quad \Omega.
$$

**Proof.** The first part essentially follows the proof of [11, Theorem 3.1]. Thanks to assumption (i), there exists a constant $c > 0$ such that

$$
S \setminus D_{\varphi} \subseteq \{(x, z) \in S : |\varphi(x, z)| \leq c\}.
$$

Set

$$
a = \min \varphi(S \setminus D_{\varphi}) \quad \text{and} \quad b = \max \varphi(S \setminus D_{\varphi}).
$$

Hypothesis (iv) allows us to choose $y', y'' \in (\alpha, \beta)$ in such a way that $\psi(y') = a$ and $\psi(y'') = b$. Pick a continuous function $\lambda : [0, 1] \to (\alpha, \beta)$ complying with $\lambda(0) = y'$, $\lambda(1) = y''$, and define $\bar{\psi}(t) = \psi(\lambda(t))$, $t \in [0, 1]$.

If $\bar{\psi}$ is constant, then $a = b$ and, consequently, $\varphi(S \setminus D_{\varphi}) = \{a\}$. Let $u \in W^{1,p}_0(\Omega)$ be such that $-\Delta_p u = y'$. Since $\bar{\psi}(-\Delta_p u) = \psi(y') = a$, the conclusion will be achieved by showing that the set $\Omega_{\varphi} = \{x \in \Omega : (x, u(x)) \in D_{\varphi}\}$ has measure zero.
First of all, observe that an elementary computation gives
\begin{equation}
\Omega_{\varphi} \subseteq \pi_0(D_{\varphi}) \cap u^{-1}(\pi_1(D_{\varphi}))
\end{equation}
and, due to (ii), \( m(\pi_i(D_{\varphi})) = 0 \), for some \( i \in \{0, 1\} \). Suppose \( i = 0 \). From (4.1) we obtain
\[ m(\Omega_{\varphi}) \leq m(\pi_0(D_{\varphi}) \cap u^{-1}(\pi_1(D_{\varphi}))) \leq m(\pi_0(D_{\varphi})) = 0, \]
whence \( m(\Omega_{\varphi}) = 0 \). Let now \( i = 1 \). Lemma 1 in [3] ensures that \( \nabla u(x) = 0 \) a.e. in \( u^{-1}(\pi_1(D_{\varphi})) \). Thanks to [7, Theorem 1.1], we have \( y' = 0 \) on \( \{x \in \Omega : \nabla u(x) = 0\} \), and, in particular, on \( u^{-1}(\pi_1(D_{\varphi})) \) (notice that our calculation showed that \( u^{-1}(\pi_1(D_{\varphi})) \subseteq \{x \in \Omega : \nabla u(x) = 0\} \)). Since \( y' \in (\alpha, \beta) \), this is possible if and only if \( m(u^{-1}(\pi_1(D_{\varphi}))) = 0 \). From (4.1) again we get
\[ m(\Omega_{\varphi}) \leq m(\pi_0(D_{\varphi}) \cap u^{-1}(\pi_1(D_{\varphi}))) \leq m(u^{-1}(\pi_1(D_{\varphi}))), \]
which implies \( m(\Omega_{\varphi}) = 0 \).

Suppose now that \( \tilde{\psi} \) is non constant and choose \( t_1, t_2 \in [0, 1] \) fulfilling
\[ \tilde{\psi}(t_1) = \min_{t \in [0, 1]} \tilde{\psi}(t), \quad \tilde{\psi}(t_2) = \max_{t \in [0, 1]} \tilde{\psi}(t). \]
Obviously, \( t_1 \neq t_2 \) and there is no loss of generality in assuming \( t_1 < t_2 \). Let \( h : \tilde{\psi}([0, 1]) \to [0, 1] \) be defined by \( h(r) = \min(\tilde{\psi}^{-1}(r) \cap [t_1, t_2]) \), for every \( r \in \tilde{\psi}([0, 1]) \).

We claim that \( h \) is strictly increasing. Indeed, pick \( r_1, r_2 \in \tilde{\psi}([0, 1]) \), with \( r_1 < r_2 \). Then, \( h(r_1) \neq h(r_2) \) and \( t_1 < h(r_2) \). From \( \tilde{\psi}(h(r_2)) = r_2 > r_1 \), \( \tilde{\psi}(t_1) \leq r_1 \), taking into account the continuity of \( \tilde{\psi} \), we immediately infer \( h(r_1) < h(r_2) \).

Therefore, the family \( D_k \) of all discontinuity points of the function \( k : \mathbb{R} \to (\alpha, \beta) \) given by
\[ k(r) = \begin{cases} 
\lambda(h(\tilde{\psi}(t_1))) & \text{if } r \in [-\infty, \tilde{\psi}(t_1)[ \\
\lambda(h(r)) & \text{if } r \in \tilde{\psi}([0, 1]) \\
\lambda(h(\tilde{\psi}(t_2))) & \text{if } r \in ]\tilde{\psi}(t_2), +\infty[ 
\end{cases} \]
is at most countable. Owing to hypotheses (ii) and (iii), this implies that the set
\begin{equation}
D = D_{\varphi} \cup \left\{ \bigcup_{r \in D_k} [\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r))] \right\}
\end{equation}
has measure zero.

Define now \( f(x, z) := k(\varphi(x, z)) \), \( (x, z) \in S \). Of course, the function \( f : S \to \mathbb{R} \) is bounded, since \( f(S) \subseteq \lambda([0, 1]) \). Moreover, as in [11, Theorem 3.1], we see that \( f \) is continuous. Put
\[ F(x, z) = \overline{\sigma}\left( \bigcap_{\delta > 0} \bigcap_{E \in \mathcal{E}} f(B_\delta(x, z) \setminus E) \right), \]
where
\[ \mathcal{E} = \{ E \subseteq S : m(E) = 0 \} \]
and \( B_\delta(x, z) = \{(x', z') \in S : |x - x'| + |z - z'| \leq \delta \} \).
A standard argument (see, e.g. [11, Theorem 3.1]), ensures that \( F \) is upper semicontinuous, and nonempty, convex and closed-valued. Further, \( F(\cdot, z) \) is measurable for every \( z \in \mathbb{R} \), \( F(x, \cdot) \) has a closed graph for almost all \( x \in \Omega \), and

\[
F(x, z) = \{ f(x, z) \}, \quad \text{as soon as } (x, z) \in S \setminus D.
\]

Consider now the problem

\[
-\Delta_p u \in F(x, u) \text{ in } \Omega, \ u \in W^{1,p}_0(\Omega).
\]

A solution will be obtained via Theorem 1.3. So, let us verify its hypotheses. Choose \( U := A_p^{-1}(L^p(\Omega)) \), where \( A_p \) is given in (2.3), \( \Phi(u) := u \) and \( \Psi(u) := A_p(u) \) for every \( u \in U \). Observe that the operator \( A_p : U \to L^p(\Omega) \) is bijective.

Let \( v_h \rightharpoonup v \) in \( L^p(\Omega) \). Since \( \{ v_h \} \) is bounded in \( L^p(\Omega) \), and \( L^p(\Omega) \) compactly embeds in \( W^{-1,p'}(\Omega) \), there exists a subsequence, still denoted by \( \{ v_h \} \), such that \( v_h \to v \) in \( W^{-1,p'}(\Omega) \). Since, from property \((p_2)\), \( A_p^{-1} \) is strongly continuous, it follows that \( \{ A_p^{-1}(v_h) \} \) converges to \( A_p^{-1}(v) \) almost everywhere in \( \Omega \).

Let now \( g : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \) be defined by

\[
g(t) := a(bt)^{1/(p-1)} \quad \forall t \in \mathbb{R}^+_0,
\]

where the constants \( a \) and \( b \) derive from inequalities (2.1)-(2.2). Clearly, \( g \) is monotone increasing in \( \mathbb{R}^+_0 \). Moreover, taking into account property \((p_3)\), if \( u \in U \) then

\[
\|u\|_{\infty} \leq a\|u\| = a\|A_p(u)\|_{W^{-1,p'}(\Omega)}^{1/(p-1)} \leq a(b\|A_p(u)\|_{p'})^{1/(p-1)} = g(\|A_p(u)\|_{p'})..
\]

This shows \((i_1)\). Since hypotheses \((i_2)\) and \((i_3)\) are already satisfied, we have only to check \((i_4)\). Define, for every \( x \in \Omega \),

\[
\rho(x) := \sup_{|z| \leq g(r)} d(0, F(x, z)).
\]

Reasoning as in [10, Theorem 3.1], we see that \( \|\rho\|_{p'} \leq r \) once the same property holds true for the function \( x \mapsto j(x) := \sup_{|z| \leq g(r)} |f(x, z)| \).

If \( |z| \leq g(r) \), then

\[
\int_\Omega |f(x, z)|^{p'} dx \leq \int_\Omega \|f(\cdot, z)\|_{p'}^{p'} dx = \|f(\cdot, z)\|_{\infty}^{p'} m(\Omega),
\]

whence

\[
\int_\Omega |j(x)|^{p'} dx = \int_\Omega \sup_{|z| \leq g(r)} |f(x, z)|^{p'} dx \leq \sup_{|z| \leq g(r)} \int_\Omega |f(x, z)|^{p'} dx
\]
\[
\leq \sup_{|z| \leq g(r)} \|f(\cdot, z)\|_{\infty}^{p'} m(\Omega) = \|f(\cdot, z)\|_{\infty}^{p'} m(\Omega).
\]

Choosing \( r \geq \|f(\cdot, z)\|_{\infty} m(\Omega)^{1/p'} \), we get \( j \in L^{p'}(\Omega) \) and \( \|j\|_{p'} \leq r \).

Now, thanks to Theorem 1.3, there exists \( u \in U \subseteq W^{1,p}_0(\Omega) \) such that

\[
-\Delta_p u(x) \in F(x, u(x)) \quad \text{a.e. in } \Omega
\]
and $|\Delta_p u(x)| \leq \rho(x)$, for almost every $x \in \Omega$. Define $\Omega_f = \{ x \in \Omega : (x,u(x)) \in D \}$. From (4.2), it follows that

$$\Omega_f \subseteq \{ \pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi)) \} \cup \left\{ \bigcup_{r \in D_k} \left[ \pi_0(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r)))) \right] \right\},$$

which, in particular, implies

$$m(\Omega_f) \leq m\left( \pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi)) \right) + m\left( \bigcup_{r \in D_k} \left[ \pi_0(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r)))) \right] \right) \leq m\left( \pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi)) \right) + \bigcup_{r \in D_k} m\left( \pi_0(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \text{int}(\varphi^{-1}(r)))) \right).$$

Assumption (ii) entails $m(\pi_i(D_\varphi)) = 0$ for some $i \in \{0,1\}$. Likewise, due to (iii), for each $r \in D_k$, there exists $i_r \in \{0,1\}$ such that $m(\pi_{i_r}(\varphi^{-1}(r) \cap \text{int}(\varphi^{-1}(r)))) = 0$. Therefore, reasoning as in the case when $\psi$ is constant, we obtain $m(\Omega_f) = 0$. This implies $F(x,u(x)) = \{ f(x,u(x)) \}$ and so, on account of (4.3),

$$-\Delta_p u(x) = f(x,u(x)) \quad \text{a.e. in } \Omega.$$

Therefore,

$$\psi(-\Delta_p u(x)) = \psi(f(x,u(x))) = \psi(k(\varphi(x,u(x)))) = \varphi(x,u(x)),$$

which completes the proof.

Hypothesis (iv) and the assumption $0 \notin (\alpha,\beta)$ are essential to obtain the existence of a solution for equations as in the previous theorem. Below we list two examples, apparently very similar, and such that one admits a solution while the other one doesn’t.

**Example 4.2.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

and let $\psi : [1, +\infty[ \to \mathbb{R}$ be such that $\psi(y) = y$. Consider the following equation

$$-\Delta_p u = \varphi(u). \quad (4.4)$$

Note that equation (4.4) doesn’t have any solution $u \in W^{1,p}_0(\Omega)$. Suppose on the contrary that $u$ is such a solution. Since $\varphi(u) \geq 0$, then from equation (4.4) we have $-\Delta_p u \geq 0$, and so the Strong Maximum Principle implies that $u \equiv 0$ or $u > 0$. Suppose that $u \equiv 0$, then this would imply that $-\Delta_p u \equiv 0$, which is in contrast with (4.4). Suppose now that $u > 0$. Then, taking into account the definition of $\varphi$ and equation (4.4), we have $-\Delta_p u = 0$. This fact, jointly with the boundary condition $u|_{\partial\Omega} = 0$, implies $u \equiv 0$ which is again impossible.
It is also evident from the definition of \( \varphi \) that hypothesis (iv) and \( 0 \notin (\alpha, \beta) \) cannot be verified simultaneously.

Fix now \( \lambda \in ]0, 1[ \) and consider the function \( \tilde{\varphi} : \mathbb{R} \to \mathbb{R} \) be defined by

\[
\tilde{\varphi}(z) = \begin{cases} 
1 & \text{if } z \neq 0 \\
\lambda & \text{if } z = 0.
\end{cases}
\]

In this case \( 0 \notin [1, +\infty[ \) and hypothesis (iv) is now verified, since

\[
\{1\} = \overline{\varphi(\mathbb{R} \setminus \{0\})} \subseteq \psi([1, +\infty[) = [1, +\infty[,
\]
and so we get a solution to the equation \( -\Delta_p u = \tilde{\varphi}(u) \).

**References**

[1] G. Bartuzel and A. Fryszkowski, *On the existence of solutions for inclusion \( \Delta u \in F(x, \nabla u) \)*, In Marz, editor, Proceedings of the fourth conference on numerical treatment of ordinary differential equations, volume 65 of Seminarberichte/Humboldt-Univ. zu Berlin, Sekt. Mathematik, Berlin (1984).

[2] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York (2011).

[3] G. Buttazzo, G. Dal Maso and E. De Giorgi, *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl., 74, pp. 274-282 (1983).

[4] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer-Verlag Berlin Heidelberg (1977).

[5] T. Kaczynski, *Implicit Differential Equations which are not Solvable for the Highest Derivative*, Lecture Notes in Math., 1475, pp. 218-224 (1991).

[6] E. Klein and A.C. Thompson, *Theory of Correspondences*, Wiley, New York (1984).

[7] H. Lou, *On Singular Sets of Local Solutions to p-Laplace Equations*, Chin. Ann. Math., 29B, 5, pp. 521-530 (2008).

[8] S.A. Marano, *Implicit Elliptic Differential Equations*, Set-Valued Analysis, 2, pp. 545-558 (1994).

[9] S.A. Marano, *On a Dirichlet problem with p-Laplacian and set-valued nonlinearity*, Bull. Aust. Math. Soc., 86, pp. 83-89 (2012).

[10] S.A. Marano, *Elliptic Boundary-Value Problems with Discontinuous Nonlinearities*, Set-Valued Analysis, 3, pp. 167-180 (1995).

[11] S.A. Marano, *Implicit Elliptic Boundary-Value Problems with Discontinuous Nonlinearities*, Set-Valued Analysis, 4, pp. 287-300 (1996).

[12] S.A. Marano and S.J.N. Mosconi, *Lower semi-continuous differential inclusions with p-Laplacian*, Libertas Mathematica, 33 N. 1, pp. 109-123 (2013).

[13] I. Peral, *Multiplicity of solutions for the p-Laplacian*, ICTP Lecture Notes of the Second School of Nonlinear Functional Analysis and Applications to Differential Equations, Trieste (1997).

[14] B. Ricceri, *Applications de théorèmes de semi-continuité inférieure*, C.R. Acad. Sci. Paris, Série I, 295, pp. 75-78 (1982).

[15] B. Ricceri, *On multifunctions with convex graph*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 77, pp. 64-70 (1984).
