THE BIPARTITE BRILL–GORDAN LOCUS AND ANGULAR MOMENTUM

ABDELMALEK ABDESSELAM AND JAYDEEP CHIPALKATTI

Abstract. Given integers \( n, d, e \) with \( 1 \leq e < \frac{d}{2} \), let \( X \subseteq \mathbb{P}^{(d+n)-1} \) denote the locus of degree \( d \) hypersurfaces in \( \mathbb{P}^n \) which are supported on two hyperplanes with multiplicities \( d - e \) and \( e \). Thus \( X \) is the Brill-Gordan locus associated to the partition \( (d - e, e) \). The main result of the paper is an exact determination of the Castelnuovo regularity of the ideal of \( X \). Moreover we show that \( X \) is \( r \)-normal for \( r \geq 3 \).

In the case of binary forms (i.e., for \( n = 1 \)) we give an invariant theoretic description of the ideal generators, and furthermore exhibit a set of two covariants which define this locus set-theoretically.

In addition to the standard cohomological tools in algebraic geometry, the proof crucially relies on the nonvanishing of certain 3\( j \)-symbols from the quantum theory of angular momentum.

AMS subject classification (2000): 14F17, 20G05, 22E70, 33C20.

Keywords: angular momentum, Castelnuovo regularity, concomitants, Clebsch-Gordan coefficients, Schur modules, transvectants.

1. Introduction

This paper is a sequel to \( \mathbb{H} \), to which we refer the reader for a detailed introduction to the problem considered here. However it may be read by itself without substantial loss of continuity.

1.1. The base field will be \( \mathbb{C} \). Let \( V \) denote an \((n + 1)\)-dimensional complex vector space, with \( W = V^* = \text{span} \{ x_0, x_1, \ldots, x_n \} \). The degree \( d \) homogeneous forms in the \( x_i \) (distinguished up to scalars) are parametrized by

\[
\mathbb{P}^N = \mathbb{P}^{(d+n)-1} = \mathbb{P} S_d W = \text{Proj} \, R,
\]
where $R$ is the symmetric algebra $\bigoplus_{r \geq 0} S_r(S_d V)$. Now let $e$ be an integer such that $1 \leq e \leq \frac{d}{2}$, and define the 2n-dimensional subvariety

$$X^{(d-e,e)} = \{ [F] \in \mathbb{P}^N : F = L_1^{d-e} L_2^e \text{ for some linear forms } L_1, L_2 \}.$$ 

We will merely write $X$ for $X^{(d-e,e)}$ if no confusion is likely. In the language of [1, §1], this is the Brill-Gordan locus associated to the partition $(d - e, e)$. In the 1890s, Brill and Gordan considered the problem of finding defining equations for the following variety

$$\{ [F] \in \mathbb{P}^N : F = \prod_{i=1}^{d} L_i \text{ for some linear forms } L_i \},$$

this serves as the motivation behind this nomenclature.

Now assume $e < \frac{d}{2}$ (the case $e = \frac{d}{2}$ was treated in [1]), and consider the graded ideal $I_X \subseteq R$. The main result of this paper is the following:

**Theorem 1.1.** The Castelnuovo regularity of the ideal $I_X$ is equal to

$$m_0 = \lceil \max \{ 4, n + 2 + \frac{1}{d} - \frac{n}{2}, 2n + 1 - \frac{n}{e} \} \rceil.$$  \hfill (1)

A fortiori, the ideal is generated by forms of degree at most $m_0$.

During the course of the proof the following result emerges naturally.

**Proposition 1.2.** For $r \geq 3$, the variety $X$ is $r$-normal, i.e., the morphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(r)) \rightarrow H^0(\mathbb{P}^N, \mathcal{O}_X(r))$$

is surjective.

The imbedding $X \subseteq \mathbb{P}^N$ is stable for the natural action of the group $SL(V)$; this fact is essentially used throughout the paper.

1.2. **Binary Forms.** A particularly interesting case is that of binary forms (i.e., $n = 1$), when we get $m_0 = 4$. Together with [1, Theorem 1.4], this completely proves the following result which was first conjectured in [7].

**Theorem 1.3.** With notation as above, the Castelnuovo regularity of $I_X$ is equal to 3 if $2e = d$, and 4 otherwise.

Let $m_0$ stand for either 3 or 4. An irreducible $SL_2$-submodule

$$S_q \subseteq (I_X)_{m_0} \subseteq S_{m_0}(S_d),$$
corresponds to a covariant of binary $d$-ics of degree $m_0$ and order $q$ which identically vanishes on $X$. For the case $d = 2e$, we had explicitly described all such covariants in \[1, \S 7\] as linear combinations of compound transvectants. In Section 9 below we outline such a description for the case $d \neq 2e$. However, in this case the expressions are not as explicit as before, to the extent that they involve Clebsch-Gordan coefficients. In general, no ‘closed formulae’ are known for the latter.

Section 8 is independent of the rest of the paper. There we construct two covariants $D$ and $C_e$ which define the locus $X^{(d-e,e)}$ set-theoretically, i.e., for a binary $d$-ic $F$,

$$F \in X^{(d-e,e)} \iff C_e(F) = D(F) = 0.$$ 

The constructions are elementary, in fact they involve little beyond the Hessian and the Wronskian determinants.

1.3. We begin proving the main theorem in Section 3. Let $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^N}$ denote the ideal sheaf of $X$, then the statement to be established is

$$H^q(\mathbb{P}^N, \mathcal{I}_X(m_0 - q)) = 0 \text{ for } q \geq 1.$$ 

First we determine the ‘conductor sheaf’ supported on the singular locus of $X$, and then calculate its cohomology using the Borel-Weil-Bott theorem; this gives the required vanishing for $q \geq 2$. The case $q = 1$ occupies the bulk of the paper. We reduce it to a problem about transvectants of binary forms, and then settle the latter using some explicit combinatorial calculations in Sections 5 through 7.

As mentioned earlier, Sections 8 and 9 are devoted to binary forms. In Section 10 we treat the variety $X^{(3,2)}$ for ternary quintics. Using some machine calculations (in Macaulay-2), we express the ideal generators of $X$ as concomitants of ternary quintics.

The basic representation theory of $SL(V)$ may be found in \[11\]. All the terminology from algebraic geometry agrees with \[16\]. As for the classical invariant theory and the symbolic method, \[13, 15\] will serve as our standard references. Note however that the interpretation of the symbolic method which we will use is the one briefly given in \[1, \S 1.7\]. For the benefit of the reader with no prior familiarity with this somewhat controversial tool, a more detailed explanation would be appropriate here; it is provided in the following section.
2. THE CLASSICAL SYMBOLIC METHOD

The symbolic notation was introduced by Aronhold in [3], and most prominently developed by the German school of invariant theory led by Clebsch and Gordan [8, 14]. It is in fact a powerful reformulation of Cayley’s theory of hyperdeterminants [6]. Most presentations of this method go as follows.

2.1. Let

\[ A(x_0, x_1) = \sum_{i=0}^{m} \binom{m}{i} \alpha_i x_0^{m-i} x_1^i \]

be a generic binary form of degree \( m \). Write \( A \) ‘symbolically’ as

\[ A(x_0, x_1) = (a_0 x_0 + a_1 x_1)^m, \]

i.e., one ‘postulates’ that

\[ \alpha_i = a_0^{m-i} a_1^i \quad \text{for} \quad 0 \leq i \leq m. \]  
(2)

However, this would introduce unwanted relations such as \( \alpha_0 \alpha_2 = \alpha_1^2 \). In order to prevent these, the prescription is to use different symbols for each individual factor \( \alpha_i \) in a product like \( \alpha_0 \alpha_2 \). One therefore introduces additional letters and writes

\[ A(x_0, x_1) = (a_0 x_0 + a_1 x_1)^m = (b_0 x_0 + b_1 x_1)^m = \ldots, \]

and then the translation between monomials in the coefficients of \( A \) and those in the symbolical letters becomes

\[ \alpha_0 \alpha_2 = a_0^m b_0^{m-2} b_1^2, \quad \alpha_1 \alpha_2 = a_0^{m-1} a_1 b_0^{m-1} b_1 c_0^{m-2} c_1^2, \text{ etc.} \]

Needless to say, this explanation is far from satisfactory and could understandably seem, on a first encounter, closer to witchcraft than mathematics. In the recent mathematical literature confronting this issue, one can trace essentially two different attitudes towards the symbolic method. The first one is simply to ignore it altogether and do without it completely; however this entails throwing away a very valuable tool and comes at a cost: missing some of the gems of classical geometry which are given an appropriate display, for instance in [18]. The second is the compromise expressed in [loc. cit. pp. 290-291], where the use of this method is advocated regardless of rigor as a quick way to guess polynomial identities involving invariants; while the task of checking these identities is left for other methods, for instance the help of a computer.
This is somewhat analogous to the situation with the recent cross-fertilization between algebraic geometry and theoretical physics (see e.g. [17]). In a few but important instances, mathematical statements heuristically derived by physicists using functional integral methods were later established as theorems, either by using previously existing tools of algebraic geometry, or by devising new ones in order to bypass path integrals. In the latter situation, there is indeed the genuine difficulty of making functional integration rigorous, which is the business of constructive field theory (see e.g. [12]). However, in the case of the classical symbolic method of invariant theory, with only basic multivariate calculus as a prerequisite, we will show that there is no difficulty at all.

2.2. The simple trick is to use the easily checked identity

\[
A(x) = \frac{1}{m!} A(\frac{\partial}{\partial a_0}, \frac{\partial}{\partial a_1}) a_x^m
\]

where \( a_x = a_0 x_0 + a_1 x_1 \). By convention, the differential operators apply to whatever is on the right, and this equation, as well as the ones that follow, are to be understood verbatim rather than ‘interpreted symbolically’. We will use the notation \( \int_A da \) for the differential operator

\[
\frac{1}{m!} A(\frac{\partial}{\partial a_0}, \frac{\partial}{\partial a_1}),
\]

so that the previous equation becomes

\[
A(x) = \int_A da a_x^m .
\]

This choice of notation can be justified by the following reasons:

1. In practice, manipulating symbolic letters in the same way as dummy variables of integration is enough to guard against computational blunders.

2. An alternate way to put the symbolic method on a rigorous footing, given by Littlewood [20, pp. 326–327], precisely uses the fact that any form can be written as the sum of sufficiently many powers of linear forms. The idea is similar in spirit to the use of one’s favourite integral representation, such as the Fourier transform, for doing analytic calculations.

3. Last but not least, it takes less room on the page.
2.3. Now consider two binary forms $A(x), B(x)$ of respective degrees $m, n,$ and let $k$ be an integer such that $0 \leq k \leq \min\{m, n\}$. By definition, the $k$-th transvectant of $A$ and $B$ is the degree $m + n - 2k$ form given by

\[
(A, B)_k(x) = \frac{(m-k)! (n-k)!}{m! n!} \{\Omega_{xy}^k A(x) B(y)\} \big|_{y=x}
\]

where

\[
\Omega_{xy} = \frac{\partial^2}{\partial x_0 \partial y_1} - \frac{\partial^2}{\partial x_1 \partial y_0}
\]

is Cayley’s Omega operator, and $y = (y_0, y_1)$ is an extra set of variables. Alternately one can also expand $\Omega_{xy}$ by the binomial theorem, and write the equally useful formula

\[
(A, B)_k(x) = \frac{(m-k)! (n-k)!}{m! n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k A}{\partial x_0^i \partial x_1^{k-i}} \frac{\partial^k B}{\partial x_0^i \partial x_1^{k-i}}.
\]

Now we have

\[
(A, B)_k(x) = \frac{(m-k)! (n-k)!}{m! n!} \{\Omega_{xy}^k \left( \int_A da \ a_x^m \right) \left( \int_B db \ b_y^n \right) \} \big|_{y=x}
\]

\[
= \frac{(m-k)! (n-k)!}{m! n!} \int_A da \ \int_B db \ \left[ \{\Omega_{xy}^k a_x^m b_y^n\} \big|_{y=x} \right];
\]

because the differential operators $\int_A da, \int_B db$ commute with $\Omega_{xy}$ and substitution of $x$ into $y$, for the mere reason that the variables $a, b, x, y$ are distinct. Now an elementary calculation using Leibnitz’s rule shows that

\[
\{\Omega_{xy}^k a_x^m b_y^n\} \big|_{y=x} = \frac{m! n!}{(m-k)! (n-k)!} (a b)^k a_x^{m-k} b_y^{n-k}
\]

where the so-called symbolic bracket $(a b)$ is a compact notation for $a_0 b_1 - a_1 b_0$. Therefore

\[
(A, B)_k = \int_A da \ \int_B db \ I(a, b, x),
\]

where the ‘integrand’ $I(a, b, x)$, i.e., the classical symbolical expression for the transvectant $(A, B)_k$, is equal to $(a b)^k a_x^{m-k} b_y^{n-k}$. The particularly nice final expression justifies the combinatorial normalization factor in the original definition (3). A modern physicist might say that the classical mathematicians had the consummate wisdom of normalizing ‘sums over Wick contractions’ as probabilistic averages. One should
bear in mind that as long as the forms $A, B$ are generic, it is preferable to do the calculations on the ‘integrand’ and refrain from actually performing the ‘integral’. However, this no longer applies as soon as one substitutes a composite algebraic expression for one of the forms, for instance a decomposition into linear factors involving the roots (as in \[1\] p. 18), or the transvectant of two other forms etc.

Here is not the place for a comprehensive review (as yet unwritten) of the classical symbolic method, and its relation to the calculus of Feynman diagrams (see [1]) as well as the quantum theory of angular momentum (see [4, 5]). Nevertheless, the above should suffice in order to enable the reader to check the calculations in Sections 5 through 7 with all the necessary mathematical rigor.

3. The conductor sheaf

Let us pick up the thread from the beginning of section 1. We have an $n$-dimensional smooth subvariety

$$Z = \{ [F] \in \mathbb{P}^N : F = L^d \text{ for some } L \in W \} \subseteq X,$$

which is the $d$-fold Veronese imbedding of $\mathbb{P}^n = \mathbb{P}W$ into $\mathbb{P}^N$. There is a proper birational morphism

$$\mathbb{P}W \times \mathbb{P}W \xrightarrow{f} \mathbb{P}S_d W, \quad (L_1, L_2) \mapsto L_1^{d-e} L_2^e$$

with image $X$. It is an isomorphism over $X \setminus Z$, hence we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow f_* \mathcal{O}_{\mathbb{P}W \times \mathbb{P}W} \rightarrow \mathcal{Q} \rightarrow 0,$$

where the conductor sheaf $\mathcal{Q}$ (so called because $f$ is the normalization of $X$) has support $Z$. Let $\delta : \mathbb{P}W \rightarrow \mathbb{P}W \times \mathbb{P}W$ denote the diagonal imbedding, and $g = f \circ \delta$.

\[ \begin{array}{c}
\mathbb{P}W \\
\downarrow \delta \\
\mathbb{P}W \times \mathbb{P}W \\
\downarrow f \\
\mathbb{P}S_d W \\
\end{array} \]

**Proposition 3.1.** The pullback $g^* \mathcal{Q}$ is isomorphic to $\Omega^1_{\mathbb{P}W}$ (the cotangent sheaf of $\mathbb{P}W$).
Proof. Let $J$ denote the ideal sheaf of $\text{image}(\delta)$, then we have an exact sequence
\[
0 \rightarrow J \rightarrow \mathcal{O}_{P^W \times P^W} \rightarrow \delta_* \mathcal{O}_{P^W} \rightarrow 0.
\]

Claim 1: The inclusion $J \subseteq \mathcal{O}_{P^n \times P^n}$ factors through the natural map (see [16, p. 110])
\[
f^*f_* \mathcal{O}_{P^n \times P^n} \rightarrow \mathcal{O}_{P^n \times P^n}.
\]
Proof. Since $f$ is an affine morphism, the claim is local on $P^n \times P^n$. Hence, restricting to affine open sets, we may write
\[
f : \text{Spec } B \rightarrow \text{Spec } A, \quad \delta : \text{Spec } B/J \rightarrow \text{Spec } B.
\]
Then $f^*f_* \mathcal{O}_{P^n \times P^n}$ is locally represented by the $B$-module $B \otimes_A B_A$ (where $B_A$ denotes $B$ considered as an $A$-module). The $B$-module map
\[
J \rightarrow B \otimes_A B_A, \quad x \rightarrow x \otimes 1
\]
is the required factorization, which proves Claim 1. Composing with
\[
f^*f_* \mathcal{O}_{P^n \times P^n} \rightarrow f^* \mathcal{Q},
\]
we get a map $J \xrightarrow{q} f^* \mathcal{Q}$.

Claim 2: $q$ is surjective.
Proof. It will suffice to show that the composite $f^*f_* J \rightarrow J \xrightarrow{q} f^* \mathcal{Q}$ is surjective. We have a commutative ladder
\[
\begin{array}{cccccc}
0 & \rightarrow & \ker 1 & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_Z & \rightarrow & 0 \\
0 & \rightarrow & f_* \mathcal{J} & \rightarrow & f_* \mathcal{O}_{P^W \times P^W} & \rightarrow & g_* \mathcal{O}_{P^W} & \rightarrow & 0 \\
& & 2 & \rightarrow & 1 & \rightarrow & 4 & & \\
& & 3 & \rightarrow & & & & &
\end{array}
\]
(Since $f$ is a finite morphism, $f_*$ is exact.) Since 4 is an isomorphism, $\ker 2 = \text{coker } 3$, giving a surjection $f_* \mathcal{J} \rightarrow Q$. Since $f^*$ is right exact, $f^*f_* \mathcal{J} \rightarrow f^* \mathcal{Q}$ is also surjective. But then $q$ itself must be surjective, which is Claim 2.

Now apply $\delta^*$ to $q$, then we get a surjection $\delta^* \mathcal{J} \rightarrow g^* \mathcal{Q}$. By definition $\delta^* \mathcal{J} = \Omega^1_{P^W}$, hence we have an extension
\[
0 \rightarrow \mathcal{A} \rightarrow \Omega^1_{P^n} \rightarrow g^* \mathcal{Q} \rightarrow 0,
\]
for some $\mathcal{O}_{P^n}$-module $\mathcal{A}$.

Let $r$ denote an integer. Tensor (8) by $\mathcal{O}_{P^n}(r)$, and pass to the long exact sequence in cohomology. Assume $r \gg 0$, so that $H^1(P^n, \mathcal{A}(r)) =$
By the Borel-Weil-Bott theorem (see [23, p. 687]) $H^0(\mathbb{P}^n, \Omega^1(r))$ is an irreducible $SL(V)$-module. It surjects onto $H^0(\mathbb{P}^n, g^*\mathcal{Q}(r))$, then Schur’s lemma implies that the kernel of this surjection is zero. Thus $H^0(\mathbb{P}^n, \mathcal{A}(r)) = 0$ for $r \gg 0$, which forces $\mathcal{A} = 0$. The proposition is proved.

Lemma 3.2. Let $r \in \mathbb{Z}$. Then the group $H^q(\mathbb{P}^N, \mathcal{Q}(r))$ is nonzero for at most one value of $q$. Specifically, the only such cases are the following:

- $H^0 = S_{(rd-1,1)} V$ for $rd \geq 2$,
- $H^1 = \mathbb{C}$ for $r = 0$,
- $H^n = S_{(1-rd-n,1,\ldots,1,0)} W$ for $rd \leq -n$.

Here $S_\lambda(-)$ is the Schur functor associated to the partition $\lambda$ (see [11, Ch. 6]).

Proof. We have an isomorphism $g^* \mathcal{Q}(r) = \Omega^1_{\mathbb{P}W} \otimes \mathcal{O}_{\mathbb{P}W}(rd)$, and then the result follows from the Borel-Weil-Bott theorem.

Lemma 3.3. For $q \geq 1$, the group

$H^q(\mathbb{P}^N, f_*\mathcal{O}_{\mathbb{P}W \times \mathbb{P}W}(r))$

is nonzero iff $q = 2n$ and $r < -\frac{n}{e}$.

Proof. From the Leray spectral sequence and the Künneth formula,

$H^q(\mathbb{P}^N, f_*\mathcal{O}_{\mathbb{P}W \times \mathbb{P}W}(r)) = \bigoplus_{i+j=q} H^i(\mathbb{P}W, \mathcal{O}_{\mathbb{P}n}(rd-re)) \otimes H^j(\mathbb{P}W, \mathcal{O}_{\mathbb{P}n}(re))$.

The summand $H^i \otimes H^j$ is nonzero, iff $i = j = n$ and the twist in each factor is $< -n$ (see [16, Ch. III.5]). Since $e < d - e$, the claim follows.

4. The regularity of $X$

Now we come to the proof of the Theorem. Henceforth we always assume that $m, q$ are positive integers in the range

$m \geq 0, \quad 1 \leq q \leq N.$

(9)

Define the predicate

$\mathfrak{P}(m, q) : H^q(\mathbb{P}^N, \mathcal{I}_X(m - q)) = 0.$
Consider the following three conditions on $m$:

1. $m \geq 4$,
2. $d(m - n - 2) \geq 1 - n$,
3. $m - 2n - 1 \geq -\frac{n}{e}$.

We will reformulate the main theorem as follows:

**Theorem 4.1.** Let $m$ be a fixed positive integer. Then $\mathcal{P}(m, q)$ is true iff C1–C3 are satisfied.

The smallest integer $m_0$ satisfying C1–C3 is the one defined by formula (11) from the introduction.

We will use a spectral sequence argument which will prove the theorem for all $q > 2$. This part merely amounts to checking that C1–C3 annihilate the $E_1$ terms in the correct positions, which necessarily makes somewhat tedious reading. The cases $q = 1, 2$ will follow from Propositions 4.2 and 4.3 below, and the proof of the latter proposition will be completed only in Section 7.

**Proof.** We splice the exact sequence (7) with

$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$,

and get a complex

$C^\bullet : 0 \to C^0 \to C^1 \to C^2 \to 0$,

where $C^0 = \mathcal{O}_{\mathbb{P}^N}, C^1 = f_* \mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^n}, C^2 = \Omega^1_\mathcal{I}$. By construction, $H^0(C^\bullet) = \mathcal{I}_X$, and $H^a(C^\bullet) = 0$ for $a = 1, 2$. We have a spectral sequence

$E^a_{b} = H^a(C^b(m - q)), \quad \delta_r^{a,b} = E^a_{r,b} \to E^{a+r,b-r+1}$

$E^a_{b} \Rightarrow H^{a+b}(\mathcal{I}_X(m - q))$;

in the range $0 \leq a \leq 2, 0 \leq b \leq N$. We will refer to this spectral sequence as $\Sigma_{m-q}$.

Given the conditions (9), Lemmata 3.2 and 3.3 imply that all the entries in $E_1$ away from the points

$(a, b) = (0, 0), (1, 0), (2, 0), (1, 2n), (2, 1), (2, n)$

are zero. This forces $E_2 = E_\infty$. To check the truth of $\mathcal{P}(m, q)$, we look at the terms $E^a_{b}$ in $\Sigma_{m,q}$ which are on the line $a + b = q$. Thus, for

$q \notin \{1, 2, 3, n + 2, 2n + 1\}$, \hspace{1cm} (10)

all the terms on this line are zero, and hence $\mathcal{P}(m, q)$ is true.
Firstly assume \( n > 1 \), then the numbers in (10) are all distinct. Now \( \mathfrak{P}(m, n + 2) \) holds iff \( E_{1,0}^{2,0} = 0 \equiv C_2 \) by Lemma 3.2. Similarly, \( \mathfrak{P}(m, 2n + 1) \) \( \iff \) \( E_{1,0}^{1,2n} = 0 \iff C_3 \) by Lemma 3.3. Now \( C_3 \) implies \( m \geq 3 \), and then \( \mathfrak{P}(m, 3) \) \( \iff \) \( E_{1,0}^{2,1} = 0 \). We have shown that \( \mathfrak{P}(m, q) \) is true for \( q \geq 3 \) iff \( C_1–C_3 \) hold. Hence it is enough to show that \( \mathfrak{P}(m, 2), \mathfrak{P}(m, 1) \) hold for \( m \geq 4 \). These claims will follow from Propositions 4.2 and 4.3 respectively.

Now assume \( n = 1 \), then the list in (10) is \( \{1, 2, 3\} \). Assume \( q = 3 \), and consider the entries \( E_{1,0}^{a,b} \) for \( (a, b) = (0, 3), (1, 2), (2, 1) \). The first is always zero (since \( m \geq 0 \)), and the rest are zero iff \( m \geq 4 \). Thus \( C_1 \) holds (which entails \( C_2, C_3 \)) iff \( \mathfrak{P}(m, 3) \) holds. This leaves us with \( q = 1, 2 \), and again we are done by Propositions 4.2 and 4.3.

**Proposition 4.2.** Let \( r \geq 1 \). Then the morphism
\[
\alpha_r : H^0(f_* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(r)) \to H^0(\mathcal{Q}(r))
\]
is surjective.

Since \( \alpha_{m-2} \) is the morphism \( \delta_{1,0}^{1,0} \) of \( \Sigma_{m,2} \), its surjectivity implies that \( E_2^{2,0} = 0 \), i.e., \( \mathfrak{P}(m, 2) \) holds for \( m \geq 3 \).

**Proof.** At the level of representations, the morphism is
\[
\alpha_r : S_{r(d-e)} \otimes S_{re} \to S_{(rd-1,1)}.
\]
The target of \( \alpha_r \) is an irreducible \( SL(V) \)-module, hence \( \alpha_r \) is either surjective or zero by Schur’s lemma. For \( r \geq 1 \), the sheaf \( \mathcal{Q}(r) = \Omega_{\mathbb{P}^n}(rd) \) is generated by global sections, hence the latter is impossible. This shows that \( \alpha_r \) is surjective.

Finally, consider the spectral sequence \( \Sigma_{m,1} \) with \( m \geq 4 \). The truth of \( \mathfrak{P}(m, 1) \) will follow if we can show that \( \delta_{1,0}^{1,0} \) surjects onto the kernel of \( \delta_{1,0}^{2,0} \). This is the content of the following proposition:

**Proposition 4.3.** Let \( r \geq 3 \). Then the morphism
\[
\beta_r : H^0(\mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(\mathcal{O}_X(r))
\]
is surjective.

**Proof.** For ease of reference, let us define the set
\[
\mathcal{A}_r = \{ p : 0 \leq p \leq re, \ p \neq 1 \}. \tag{11}
\]
Now the target of $\beta_r$ is

$$H^0(O_X(r)) = \ker \alpha_r = \bigoplus_{p \in A_r} S_{(rd-p,p)}.$$  \hfill (12)

(This follows from the Littlewood-Richardson rule, see \[11, Appendix A\].) Let $\pi_p$ denote the projection onto the $p$-th summand. Then, $\pi_p \circ \beta_r$ is equal to the composite

$$S_r(S_d) \xrightarrow{1} S_r(S_{d-e} \otimes S_e) \xrightarrow{2} S_r(S_{d-e}) \otimes S_r(S_e) \xrightarrow{3} S_r(d-e) \otimes S_{re} \xrightarrow{\pi_p} S_{(rd-p,p)}.$$  \hfill (13)

The map $1$ is given by applying $S_r(\cdot)$ to the coproduct map, $2$ comes from the ‘Cauchy decomposition’ (see \[2\]), and $3$ is the ‘multiplication’ map. We will show that for $p \in A_r$, the map $\pi_p \circ \beta_r$ is not identically zero, and hence surjective. Since the cokernel of $\beta_r$ is a direct summand of the target of $\beta_r$, this will prove that the cokernel is zero. By the argument of \[11, p. 11\], it is enough to show this when $\dim V = 2$. We defer the proof to the next section. \hfill □

As a corollary to the proposition, we get a formula for the character of the degree $r$ part of $I_X$.

**Corollary 4.4.** For $r \geq 3$, we have an equality

$$[(I_X)_r] = [S_r(S_d)] - \sum_{p \in A_r} [S_{(rd-p,p)}],$$

where $[-]$ denotes the formal character of an $SL(V)$-representation. \hfill □

For $r = 2$, we have the formula $[(I_X)_2] = \sum_{e < p \leq \frac{d}{2}} [S_{(2d-2p,2p)}]$. Hence the ideal has quadratic generators except when $d$ is odd and $e = \frac{d-1}{2}$.

### 5. Transvectants

We resume the proof of Proposition 4.3 under the hypothesis $\dim V = 2$. The argument is by induction on $r$, and the passage from $r$ to $r+1$ uses the symbolic calculus on binary forms. The notations will be consistent with those of Section 2.

Given binary forms $A, B$ of degrees $a, b$ in variables $x = (x_0, x_1)$, their $k$-th transvectant $(A, B)_k$ is defined by formula (3). It is the
image of $A \otimes B$ via the projection
$$S_a \otimes S_b \longrightarrow S_{a+b-2k}.$$ 

5.1. Now define the predicate
$$\Theta(r, p) : \pi_p \circ \beta_r \neq 0. \quad (14)$$

We want to show that $\Theta(r, p)$ holds for all $r \geq 3, p \in A_r$. Consider the commutative diagram
\[
\begin{array}{ccc}
S_r(S_d) \otimes S_d & \longrightarrow & S_{rd-p,p} \otimes S_d \\
\downarrow & & \downarrow u_r^{(p,p')} \\
S_{r+1}(S_d) & \longrightarrow & S_{rd+d-p',p'}
\end{array}
\]

where the horizontal maps are $(\pi_p \circ \beta_r) \otimes \text{id}$, and $\pi_{p'} \circ \beta_{r+1}$ respectively, and the vertical map $u_r^{(p,p')}$ is the composite
$$S_{rd-p,p} \otimes S_d \longrightarrow H^0(\mathcal{O}_X(r)) \otimes S_d \longrightarrow H^0(\mathcal{O}_X(r+1)) \longrightarrow S_{rd+d-p',p'}.$$

Now consider the following two statements:

I. For any $p' \in A_3$, there exists an even integer $p \in A_2$ such that $u_2^{(p,p')} \neq 0$.

II. Assume $r \geq 3$. Then for any $p' \in A_{r+1}$, there exists a $p \in A_r$ such that $u_r^{(p,p')} \neq 0$.

We claim that (I) and (II) imply $\Theta(r, p)$ for all $r \geq 3, p \in A_r$. By [1, Proposition 6.2], $\Theta(2, p)$ holds for all even $p \in A_2$. Now assume the result for $r$, and let $p' \in A_{r+1}$. Let $p$ be an integer whose existence is guaranteed by either (I) or (II), depending on whether $r$ equals or exceeds 2. By hypothesis $\pi_p \circ \beta_r$ is surjective, hence the composite
$$u_r^{(p,p')} \circ \{(\pi_p \circ \beta_r) \otimes \text{id}\}$$

is nonzero; this forces $\pi_{p'} \circ \beta_{r+1} \neq 0$.

5.2. It remains to prove (I) and (II). The map $u_r^{(p,p')}$ is defined as the composite
\[
\begin{align*}
S_{rd-2p} \otimes S_d & \overset{1}{\longrightarrow} (S_{rd-e} \otimes S_{re}) \otimes (S_{d-e} \otimes S_e) \overset{2}{\longrightarrow} \\
S_{(r+1)(d-e)} \otimes S_{(r+1)e} & \overset{3}{\longrightarrow} S_{(r+1)d-2p'},
\end{align*}
\]

where 1 is the tensor product of two coproduct maps, 2 is obtained by regrouping, and 3 is the projection.
Now let $A, B$ denote binary forms of degrees $rd - 2p, d$ respectively. We will now follow the component maps in (15), and get a step-by-step procedure for calculating the image $u_r^{(p,p')}(A \otimes B)$. Introduce new variables $y = (y_0, y_1)$, and let

$$\Lambda = \sum_{i=0}^{r(d-e)} \binom{r(d-e)}{i} l_i x_0^{r(d-e)-i} x_1^i, \quad M = \sum_{j=0}^{re} \binom{re}{j} m_j x_0^{re-j} x_1^j,$$

denote generic binary forms of degrees $r(d-e), re$. (That is to say, the $l, m$ are thought of as independent indeterminates.)

- Let $T_1 = (\Lambda, M)_p$, and $T_2 = (A, T_1)_{rd-2p}$. Then $T_2$ does not involve $x_0, x_1$.
- Obtain $T_3$ by making the substitutions

$$l_i = x_1^{r(d-e)-i}(-x_0)^i, \quad m_j = y_1^{re-j}(-y_0)^j,$$

in $T_2$.
- Let

$$T_4 = (y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1})^e B,$$

usually called a partial polarization of $B$. By construction, $T_3$ and $T_4$ have respective bidegrees $(rd - re, re)$ and $(d - e, e)$ in the sets $x, y$.
- Let $T_5 = T_3 T_4$, and $T_6 = \Omega_{xy}^{p'} T_5$.
- Finally $u_r^{(p,p')}(A \otimes B)$ is obtained by substituting $x_0, x_1$ for $y_0, y_1$ in $T_6$.

5.3. A translation of this construction into the classical symbolic calculus, according to Section 2, amounts to the following algebraic calculations with differential operators, to be understood verbatim. Write

$$A(x) = \int_A da \ a_x^{rd-2p}, \quad B(x) = \int_B db \ b_x^d,$$
$$\Lambda(x) = \int_\Lambda d\lambda \ \lambda_x^{r(d-e)}, \quad M(x) = \int_M d\mu \ \mu_x^{re}.$$

Now, from the discussion in Section 2

$$T_1 = \int_\Lambda d\lambda \ \int_M d\mu (\lambda \mu)^p \ \lambda_x^{r(d-e)-p} \ \mu_x^{re-p},$$
and therefore

\[ T_2 = \frac{1}{(r^d - 2p)!^2} \left\{ \Omega^{r^d - 2p}_{xy} A(x) T_1(y) \right\} |_{y^1 = x^1}, \]

\[ = \frac{1}{(r^d - 2p)!^2} \left\{ \Omega^{r^d - 2p}_{xy} \int_A \int_A d\lambda \int_M d\mu \ (\lambda \mu)^p \ a^r_{x^d - 2p} \lambda_{y^d - e}^{r^d - p} \mu_{y^d - e}^{r^d - p} \right\} |_{y^1 = x^1}, \]

\[ = \frac{1}{(r^d - 2p)!^2} \int_A \int_A d\lambda \int_M d\mu \ (\lambda \mu)^p \ \{ J(x, y) \} |_{y^1 = x^1}, \]

where

\[ J(x, y) = \Omega^{r^d - 2p}_{xy} a^r_{x^d - 2p} \lambda_{y^d - e}^{r^d - p} \mu_{y^d - e}^{r^d - p} \]

\[ = (r^d - 2p)! \left( a_0 \frac{\partial}{\partial y_1} - a_1 \frac{\partial}{\partial y_0} \right)^{r^d - 2p} \lambda_{y^d - e}^{r^d - p} \mu_{y^d - e}^{r^d - p} \]

\[ = (r^d - 2p)!^2 (a \lambda)^{r^d - e - p} (a \mu)^{r^d - p}, \]

which (as expected) does not involve \( x \) or \( y \). Therefore

\[ T_2 = \int_A \int_A \int_M d\lambda \int_M d\mu \ (\lambda \mu)^p \ (a \lambda)^{r^d - e - p} (a \mu)^{r^d - p}. \]

Now the substitution \( l_i = x_1^{r^d - e - i} (-x_0)^i \) implies that the differential operator \( \int_A d\lambda \) can be rewritten as

\[ \int_A d\lambda = \frac{1}{[r(d - e)]!} \sum_{i=0}^{r(d - e)} \binom{r(d - e)}{i} x_1^{r(d - e) - i} (-x_0)^i \left( \frac{\partial^{r(d - e)}}{\partial \lambda_0^{r(d - e) - i}} \right) \partial \lambda_1^i. \]

Likewise, from the substitution \( m_j = y_1^{r^d - j} (-y_0)^j \), we get

\[ \int_M d\mu = \frac{1}{(re)!} \left( y_1 \frac{\partial}{\partial \mu_0} - y_0 \frac{\partial}{\partial \mu_1} \right)^{re}. \]

Now an easy calculation gives

\[ \int_M d\mu \ (\lambda \mu)^p (a \mu)^{r^d - p} \]

\[ = \frac{1}{(re)!} \left( y_1 \frac{\partial}{\partial \mu_0} - y_0 \frac{\partial}{\partial \mu_1} \right)^{re} (\lambda_0 \mu_1 - \lambda_1 \mu_0)^p (a_0 \mu_1 - a_1 \mu_0)^{r^d - p} \]

\[ = (-y_1 \lambda_1 - y_0 \lambda_0)^p (-y_1 a_1 - y_0 a_0)^{r^d - p} = (-1)^{re} \lambda_{y^d}^{r^d - p} \ a_{y^d}^{r^d - p}. \]
As a result,
\[
\int_{\Lambda} d\lambda \int_{\mathcal{M}} d\mu (\lambda \mu)^{p}(a \lambda)^{r(d-e)-p}(a \mu)^{re-p}
\]
\[
= (-1)^{re} a_y^{re-p} \left( x_1 \frac{\partial}{\partial \lambda_0} - x_0 \frac{\partial}{\partial \lambda_1} \right)^{r(d-e)} (a_0 \lambda_1 - a_1 \lambda_0)^{(d-e)-p} (\lambda_0 y_0 + \lambda_1 y_1)^p
\]
\[
= (-1)^{re} a_y^{re-p} (-x_1 a_1 - x_0 a_0)^{r(d-e)-p} (x_1 y_0 - x_0 y_1)^p,
\]
i.e.,
\[
T_3 = \int_A (-1)^{rd} a_y^{re-p} a_x^{r(d-e)-p} (xy)^p.
\]
Now
\[
T_4 = \left( y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1} \right)^e \int_B db \ b_x^d
\]
\[
= \int_B db \left( y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1} \right)^e (b_0 x_0 + b_1 x_1)^d
\]
\[
= \frac{d!}{(d-e)!} \int_B db \ b_x^{d-e} b_y^e ;
\]
from which one obtains
\[
T_6 = \frac{(-1)^{rd}d!}{(d-e)!} \left\{ \Omega'_{xy} \int_A da \int_B db (xy)^p a_x^{r(d-e)-p} b_x^{d-e} b_y^e \right\} 
\]
\[
\bigg|_{y=x},
\]
or,
\[
u_r^{(p,p')} (A \otimes B) = \frac{(-1)^{rd}d!}{(d-e)!} \int_A da \int_B db \mathcal{E}(r; p, p'),
\]
where the ‘integrand’
\[
\mathcal{E}(r; p, p') = \left\{ \Omega'_{xy} (xy)^p a_x^{r(d-e)-p} b_x^{d-e} a_y^{re-p} b_y^e \right\} 
\]
\[
\bigg|_{y=x}
\]
is an ordinary polynomial in the variables \(a_0, a_1, b_0, b_1, x_0, x_1\).

Note that if \(\mathcal{E}(r; p, p')\) vanishes identically, so does \(u_r^{(p,p')} (A \otimes B)\) for any forms \(A\) and \(B\); since differentiating zero gives zero. Conversely, if the map \(u_r^{(p,p')}\) vanishes, then by applying it to forms \(A, B\) which \textit{truly} are powers of generic linear forms, it would follow that \(\mathcal{E}(r; p, p')\) itself must vanish identically. As a result, statements (I) and (II) from Section 5.1 respectively translate into the following:

**Proposition 5.1.**  
(1) For any \(p' \in \mathcal{A}_3\), there exists an even integer \(p \in \mathcal{A}_2\) such that \(\mathcal{E}(2; p, p')\) is not identically zero.

(2) Assume \(r \geq 3\). Then for any \(p' \in \mathcal{A}_{r+1}\), there exists an integer \(p \in \mathcal{A}_r\) such that \(\mathcal{E}(r; p, p')\) is not identically zero.
The proof will be given in the next two sections. We will calculate the quantity $E(r; p, p')$ explicitly and show that its nonvanishing is equivalent to that of a numerical combinatorial sum (later denoted $S$). Finally, we break up the hypotheses into several subcases, and verify that one can always choose $p$ such that $S \neq 0$.

Broadly speaking, what one has to show is that a functorially defined construction in multilinear algebra gives a nontrivial result. A thematically similar idea involving Laguerre polynomials appears in [25, p. 57ff].

6. Transvectants of monomials and the quantum theory of angular momentum

Let

$$L_1(x) = a_0 x_0 + a_1 x_1, \quad L_2(x) = b_0 x_0 + b_1 x_1,$$

be two generic binary linear forms. Let $\alpha_1, \alpha_2, \beta_1, \beta_2,$ and $k$ be nonnegative integers such that $k \leq \min\{\alpha_1 + \alpha_2, \beta_1 + \beta_2\}$. The object of this section is to give a formula for the transvectant

$$T = (L_1^{\alpha_1} L_2^{\alpha_2}, L_1^{\beta_1} L_2^{\beta_2})_k$$

which will be used later, and also to clarify its connection with the quantum theory of angular momentum (see [4]) which was alluded to in [1].

Since $T$ is a joint covariant of $L_1(x) = a_x$ and $L_2(x) = b_x$, it is a linear combination of bracket monomials $(a b)^i a_x^{i_2} b_x^{i_3}$. By a simple degree count, we have

$$i_1 + i_2 = \alpha_1 + \beta_1, \quad i_1 + i_3 = \alpha_2 + \beta_2,$$
$$i_2 + i_3 = \alpha_1 + \beta_1 + \alpha_2 + \beta_2 - 2k.$$

This implies $i_1 = k$, $i_2 = \alpha_1 + \beta_1 - k$, $i_3 = \alpha_2 + \beta_2 - k$. Therefore

$$T = N \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] (a b)^k a_x^{\alpha_1 + \beta_1 - k} b_x^{\alpha_2 + \beta_2 - k}, \quad (16)$$
where $\mathcal{N} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right]$ is a purely numerical quantity. Now specialize to $a_0 = 1$, $a_1 = 0$, $b_0 = 0$, and $b_1 = 1$, when the right hand side becomes

$$\mathcal{N} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] x_0^{\alpha_1 + \beta_1 - k} x_1^{\alpha_2 + \beta_2 - k}.$$

By definition,

$$T = \frac{(\alpha_1 + \alpha_2 - k)! (\beta_1 + \beta_2 - k)!}{(\alpha_1 + \alpha_2)! (\beta_1 + \beta_2)!} \left( \frac{\partial^2}{\partial x_0 \partial y_1} - \frac{\partial^2}{\partial x_1 \partial y_0} \right)^k x_0^{\alpha_1} x_1^{\alpha_2} y_0^{\beta_1} y_1^{\beta_2} \bigg|_{y=x}$$

$$= \frac{(\alpha_1 + \alpha_2 - k)! (\beta_1 + \beta_2 - k)!}{(\alpha_1 + \alpha_2)! (\beta_1 + \beta_2)!} \times \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \left( \frac{\partial^2}{\partial x_0 \partial y_1} \right)^i \left( \frac{\partial^2}{\partial x_1 \partial y_0} \right)^{k-i} x_0^{\alpha_1} x_1^{\alpha_2} y_0^{\beta_1} y_1^{\beta_2} \bigg|_{y=x}.$$ 

This implies that

$$\mathcal{N} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] = \mathcal{S} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] \times \frac{(-1)^k (\alpha_1 + \alpha_2 - k)! (\beta_1 + \beta_2 - k)! k! \alpha_1 \alpha_2 \beta_1 \beta_2}{(\alpha_1 + \alpha_2)! (\beta_1 + \beta_2)!}$$

where, by definition

$$\mathcal{S} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] = \sum_{i=\max\{0,k-\alpha_2,k-\beta_1\}}^{\min\{k,\alpha_1,\beta_2\}} (-1)^i \frac{1}{i!(k-i)! (\alpha_1 - i)! (\beta_2 - i)! (\alpha_2 - k + i)! (\beta_1 - k + i)!}$$

(17)

6.1. We now connect this to the prevalent formalism in physics. From the original data $\alpha_1, \alpha_2, \beta_1, \beta_2, k$, we define numbers $j_1, j_2, j, m_1, m_2, m$ via the relations

$$j_1 = \frac{1}{2} (\alpha_1 + \alpha_2), \quad j_2 = \frac{1}{2} (\beta_1 + \beta_2), \quad j = \frac{1}{2} (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - k$$

$$m_1 = \frac{1}{2} (\alpha_2 - \alpha_1), \quad m_2 = \frac{1}{2} (\beta_2 - \beta_1), \quad m = \frac{1}{2} (\alpha_2 - \alpha_1 + \beta_2 - \beta_1).$$
Now the so-called vector-coupling or Clebsch-Gordan coefficients are the quantities
\[ C_{j_1,j_2,i}^{j,m_1,m_2,m} = \frac{(2j_1 + 1)(j_1 + j_2 - j)!(j_1 + j - j_2)!(j_2 + j - j_1)!}{(j_1 + j_2 + j + 1)!} \]
\[ \times [(j_1 - m_1)!(j_1 + m_1)!(j_2 - m_2)!(j_2 + m_2)!(j - m)!(j + m)!]^\frac{1}{2} \times \sum \frac{(-1)^i}{z}, \]
where \( z \) stands for
\[ i!(j_1 + j_2 - j - i)!(j_1 - m_1 - i)!(j_2 + m_2 - i)!(j - j_2 + m_1 + i)!(j - j_1 - m_2 + i)!, \]
and the last summation is quantified over
\[ \min\{j_1 + j_2 - j, j_1 - m_1, j_2 + m_2\} \leq i \leq \max\{0, j_2 - j - m_1, j_1 - j + m_2\}. \]

Now, using our original data,
\[ C_{j_1,j_2,i}^{j,m_1,m_2,m} = \mathcal{N} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] \]
\[ \times (-1)^k \sqrt{\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 2k + 1} (\alpha_1 + \alpha_2)! (\beta_1 + \beta_2)! \]
\[ \times \left[ \frac{(\alpha_1 + \beta_1 - k)! (\alpha_2 + \beta_2 - k)!}{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - k + 1)! (\alpha_1 + \alpha_2 - k)! (\beta_1 + \beta_2 - k)! k! \alpha_1! \alpha_2! \beta_1! \beta_2!} \right]^\frac{1}{2}. \]

Physicists also use related quantities called Wigner’s 3j-symbols given by
\[ \left( \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{array} \right) = (-1)^{j_1-j_2+m} (2j + 1)^{-\frac{1}{2}} C_{m_1,m_2,m}^{j_1,j_2,j}. \]

6.2. We will record another calculation which will be useful in the next section. Introduce auxiliary variables \( u_0, u_1, v_0, v_1, \) and let \( \{a \partial_u\} \) stand for \( a_0 \frac{\partial}{\partial u_0} + a_1 \frac{\partial}{\partial u_1} \) etc. Define
\[ U \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] (a, b, x) \]
\[ = \{a \partial_u\}^{\alpha_1} \{b \partial_u\}^{\alpha_2} \{a \partial_v\}^{\beta_1} \{b \partial_v\}^{\beta_2} (u v)^k u_x^{\alpha_1 + \alpha_2 - k} v_x^{\beta_1 + \beta_2 - k}. \]

The following result will be needed.
Lemma 6.1.

\[
\mathcal{U} \left[ \begin{array}{c} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{array} \right] (a, b, x) = (\alpha_1 + \alpha_2)!(\beta_1 + \beta_2)! \ T
\]

where \( T \) denotes the monomial transvectant \((a_1^{\alpha_1} b_2^{\alpha_2}, a_1^{\beta_1} b_2^{\beta_2})_k\).

**Proof.** Using a graphical notation for symmetrizers as in [9] would make the truth of the lemma visually obvious. Alternatively, one can do the following. Write each factor in (18) as a sum over indices with values in \( \{0, 1\} \):

\[
\{ a \partial_u \} = \sum_i a_i \partial_{u_i}, \quad (u v) = \sum_{i,j} u_i \epsilon_{ij} v_j,
\]

\[
u x = \sum_i u_i x_i \quad \text{etc.},
\]

where \( \epsilon = (\epsilon_{ij}) \) is the antisymmetric \( 2 \times 2 \) matrix with \( \epsilon_{01} = 1 \). One has to use disjoint sets of indices for each individual factor in (18), i.e., a total of \( 2p \) indices, with \( p = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \). Now expand \( U \) completely, which gives an expression of the form

\[
\mathcal{U} = \sum_{I,J} A_{I,J} \frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_p}} z_{j_1} \cdots z_{j_p}
\]

where \( I = (i_1, \ldots, i_p) \) and \( J = (j_1, \ldots, j_p) \) are collections of indices running from 1 to 4, by definition

\[
z = (z_1, z_2, z_3, z_4) = (u_0, u_1, v_0, v_1),
\]

and finally \( A_{I,J} \) are coefficients depending on \( a, b, x \), the detailed expression of which we spare the reader. Now

\[
\frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_p}} z_{j_1} \cdots z_{j_p} = \sum_{\sigma} \prod_{\nu=1}^{p} \delta_{i_{\nu}j_{\sigma(\nu)}}
\]

where \( \sigma \) denotes a permutation of the set \( \{1, \ldots, p\} \). Since

\[
\frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_p}} z_{j_1} \cdots z_{j_p} = \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_p}} z_{i_1} \cdots z_{i_p},
\]

and one can exchange the role of the dummy summation indices \( I \) and \( J \), we get

\[
\mathcal{U} = \sum_{I,J} A_{J,I} \frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_p}} z_{j_1} \cdots z_{j_p}.
\]
Now undo the previous expansion of $U$ to find
\[
U \begin{bmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{bmatrix} (a, b, x) = (\partial_u)_{x_1}^{\alpha_1+k} (\partial_v)_{x_2}^{\beta_1+k} W(u, v),
\]
where $(\partial_u)_{x_1} = x_0 \frac{\partial}{\partial u_0} + x_1 \frac{\partial}{\partial u_1}$ etc., and
\[
W(u, v) = \left( \frac{\partial^2}{\partial u \partial v} - \frac{\partial^2}{\partial u_1 \partial v_0} \right)^k a_u^\alpha b_u^\beta a_v^\beta b_v^\beta.
\]
Since $W(u, v)$ is homogeneous in $u, v$ of respective degrees $\alpha_1 + \alpha_2 - k$ and $\beta_1 + \beta_2 - k$,
\[
U \begin{bmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{bmatrix} (a, b, x) = (\alpha_1 + \alpha_2 - k)! (\beta_1 + \beta_2 - k)! W(u, v)|_{u, v = x},
\]
and the lemma follows. \hfill \square

**Corollary 6.2.**
\[
U \begin{bmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{bmatrix} (a, b, x) = N \begin{bmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ k \end{bmatrix} (\alpha_1 + \alpha_2)! (\beta_1 + \beta_2)! \times (a b)^k a_x^{\alpha_1+\beta_1-k} b_x^{\alpha_2+\beta_2-k}.
\]

**7. Proof of Proposition 5.1**

In this section we will prove the remaining proposition, and hence complete the proof of the main theorem.

Let us write
\[
\mathcal{M} = \Omega_{x y}^{p'} (x y)^p a_x^{r(e-d)-p} b_x^{d-e} a_y^{r-e+p} b_y^e,
\]
then
\[
\mathcal{E} = \mathcal{E}(r; p, p') = \mathcal{M}|_{y=x}
\]
is the expression to be calculated. Introduce pairs of variables $u_0, u_1, v_0, v_1$ as in Section 6.2. In the notation introduced there,
\[
\{a \partial_u\}^{r(d-e)-p} \{b \partial_u\}^{d-e} u_x^{(r+1)(d-e)-p} = ((r+1)(d-e)-p)! a_x^{r(d-e)-p} b_x^{d-e},
\]
and similarly
\[
\{a \partial_v\}^{r-e-p} \{b \partial_v\}^{e} v_y^{(r+1)e-p} = ((r+1)e-p)! a_y^{r-e+p} b_y^e.
\]
Hence
\[ M = \frac{1}{[(r+1)e-p]![(r+1)(d-e)-p]!} \Omega_{xy}^{p'} \]
\[ (xy)^p \{ a \partial_a \}^{r(d-e)-p} \{ b \partial_a \}^{d-e} \{ a \partial_e \}^{re-p} \{ b \partial_e \}^{e} u_x^{(r+1)(d-e)-p} v_y^{(r+1)e-p} \].

We can commute those partial differential operators which act on disjoint sets of variables, this gives
\[ M = \frac{\{ a \partial_a \}^{r(d-e)-p} \{ b \partial_a \}^{d-e} \{ a \partial_e \}^{re-p} \{ b \partial_e \}^{e}}{[(r+1)e-p]![(r+1)(d-e)-p]!} \times \mathcal{P}, \]
where
\[ \mathcal{P} = \Omega_{xy}^{p'} \{(xy)^p u_x^{(r+1)(d-e)-p} v_y^{(r+1)e-p}\}. \]

7.1. Now let \( m = (r+1)(d-e)-p, n = (r+1)e-p \), so that \( n < m \). (Of course, this \( n \) is entirely unrelated to the one from Section 1.1. The latter plays no role in this calculation.) By the Clebsch-Gordan series (see [15, Ch. IV]),
\[ u_x^m v_y^n = \sum_{j=0}^{n} \frac{(m)}{(j)} \frac{(n)}{(j)} \frac{(m+n-j+1)}{(j+1)} (xy)^j (u_x^m, v_y^n)_j^{y_n-j}, \]
where, by definition,
\[ (u_x^m, v_y^n)_j^{y_n-j} = \frac{(m-j)!}{(m+n-2j)!} \{ y \partial_x \}^{n-j} (u v)^j u_x^{m-j} v_y^{n-j}. \]

Now introduce new variables \( w_0, w_1 \), and rewrite the last expression as
\[ (u_x^m, v_y^n)_j^{y_n-j} = \frac{(u v)^j}{(m+n-j+1)} \{ u \partial_w \}^{m-j} \{ v \partial_w \}^{n-j} u_x^{m-j} w_y^{n-j}. \]

Then \( \mathcal{P} \) can be written as
\[ \Omega_{xy}^{p'} (xy)^p u_x^m v_y^n = \Omega_{xy}^{p'} \left[ (xy)^p \sum_{j=0}^{n} \frac{(m)}{(j)} \frac{(n)}{(j)} \frac{(m+n-j+1)}{(j+1)} (xy)^j (u_x^m, v_y^n)_j^{y_n-j} \right] \]
\[ = \Omega_{xy}^{p'} \left[ (xy)^p \sum_{j=0}^{n} \mathcal{U}_j \right] \]
where
\[ \mathcal{U}_j = \frac{(m)}{(j)} \frac{(n)}{(j)} \frac{(m+n-j+1)}{(j+1)} \frac{(x y)^j}{(m+n-2j)!} (u v)^j \{ u \partial_w \}^{m-j} \{ v \partial_w \}^{n-j} u_x^{m-j} w_y^{n-j}. \]
Again the point is that the differential operators can be commuted! Hence
\[ P = \sum_{j=0}^{n} \frac{m!}{j!(m-j)!} \frac{n!}{(n-j)!} \frac{j!(m+n-2j+1)!}{(m+n-j+1)!} \times \frac{1}{(m+n-2j)!} \]
\[ \times (u v)^j \{ u \partial_w \}^{m-j} \{ v \partial_w \}^{n-j} \times Q, \]
where
\[ Q = \Omega_{xy}^{p+j}(x^m y^n)^{w_x m-j w_y n-j} \]
The last expression occurs frequently in classical invariant theory. It is calculated, for instance, in [13, §3.2.6]. If \( p' > p + j \) then \( Q \) is zero, and if \( p' \leq p + j \) then it equals
\[ \frac{(p+j)!}{(p+j-p')!} \times \frac{(m+n+p-j+1)!}{(m+n+p-p'-j+1)!} \]
\[ \times (u v)^j \{ u \partial_w \}^{m-j} \{ v \partial_w \}^{n-j} \times P \]
\[ (x^m y^n)^{w_x m-j w_y n-j}. \]
As a result,
\[ P = \sum_{j} \mathbb{1}_{\{ v \leq j \leq n \}} \frac{m!}{j!(m-j)!} \frac{n!}{(n-j)!} \frac{j!(m+n-2j+1)!}{(m+n-j+1)!} \times \frac{1}{(m+n-2j)!} \]
\[ \times \frac{(p+j)!}{(p+j-p')!} \times \frac{(m+n+p-j+1)!}{(m+n+p-p'-j+1)!} \]
\[ \times (u v)^j \{ u \partial_w \}^{m-j} \{ v \partial_w \}^{n-j} \times P \]
\[ (x^m y^n)^{w_x m-j w_y n-j}. \]
Here \( \mathbb{1}_{\{ \} } \) denotes the characteristic function of that set.

7.2. Now recall that
\[ M = \{ a \partial_w \}^{r(d-e)-p} \{ b \partial_w \}^{d-e} \{ a \partial_v \}^{re-p} \{ b \partial_v \}^e \times P \]
with \( m = (r+1)(d-e) - p \), and \( n = (r+1)e - p \). The quantity we are interested in is \( E = M|_{y=x} \). When we set \( y = x \) in (19), the term corresponding to \( j = p' - p \) is the only one that survives. Therefore \( E \) equals
\[ \mathbb{1}_{\{ v \leq p' \leq n \}} \times \{ a \partial_u \}^{r(d-e)-p} \{ b \partial_u \}^{d-e} \{ a \partial_v \}^{re-p} \{ b \partial_v \}^e \times \]
\[ (u v)^{p'-p} \{ u \partial_w \}^{m-p'+p} \{ v \partial_w \}^{n-p'+p} w_x^{m+n-2p'+2p} \times \]
\[ p'!(m+n+2p-p'+1)! \]
\[ (p' - p)!(m-p'+p)!(n-p'+p)!(m+n-p'+p+1)!(m+n-2p'+2p)! \]
Now
\[ \{ u \partial_w \}^{m-p'+p} \{ v \partial_w \}^{n-p'+p} w_x^{m+n-2p'+2p} = (m+n-2p'+2p)! u_x^{m-p'+p} v_x^{n-p'+p}. \]
The condition $p' - p \leq n$ is equivalent to the hypothesis $p' \leq (r + 1)e$, and can therefore be dropped. Hence

$$E = \mathbb{1}_{[p \leq p']} \times \mathcal{U} \times \frac{p'!}{(p' - p)!((r + 1)d - p' + 1)!} \times \frac{((r + 1)(d - e) - p')!((r + 1)e - p')!((r + 1)d - p' - p + 1)!}{(r + 1)(d - e) - p, d - e, re - p, e, p' - p}$$

where

$$\mathcal{U} = \mathcal{U} \left[ \begin{array}{c} r(d - e) - p, d - e \\ re - p, e \\ p' - p \end{array} \right]$$

in the notation of Section 6.2. As a result, $E$ is nonzero iff the sum

$$S = S \left[ \begin{array}{c} r(d - e) - p, d - e \\ re - p, e \\ p' - p \end{array} \right]$$

defined by Formula (17) is nonzero. Let

$$\mathcal{L} (d, e, r, p', p) = \max \{0, p' - p - (d - e), p' - re\},$$

$$\mathcal{H} (d, e, r, p', p) = \min \{p' - p, e, r(d - e) - p\},$$

henceforth written as $\mathcal{L}$ and $\mathcal{H}$ if no confusion is likely. The index of summation in the definition of $S$ runs from $\mathcal{L}$ to $\mathcal{H}$.

7.3. In [1, §6] we were able to produce closed formulae for such coefficients in analogous cases, and then to determine whether they were nonzero. This was due to the particular form of these coefficients, which allowed the use of some standard summation theorems for hypergeometric series. On the contrary, we now have five independent parameters $d, e, r, p', p$, and no such closed formulae seem to apply. It does not seem either that all the situations where there exist such summation formulae (say for terminating hypergeometric series or Wigner’s $3nj$-symbols) have been classified in the framework of Wilf-Zeilberger theory (cf. [19, 22, 27]). Moreover, the determination of the zeros of $3nj$-symbols (even in the simplest case of $3j$-symbols) is an outstanding open problem in the quantum theory of angular momentum (see [5, Ch. 5, Topic 10] or [24]). It is quite intriguing that some of these zeros have been given an explanation involving exceptional Lie groups. The issue may well be related to the article by Dixmier [10] (see also [21, Chap. 6]), where, among a bestiary of nonassociative algebras, the octonions are realised by a construction using transvectants of binary forms.
7.4. We can conclude the proof of Proposition 5.1 only because we have some freedom in the choice of \( p \). We break up the allowed values of \( r \) and \( p' \) into several cases, and by analysing which entries realize the maximum and minimum in (20), it is always possible to choose the value of \( p \) in such a way that the sum defining \( S \) has at most two terms.

In the angular momentum parlance, these correspond to ‘stretched’ 3\( j \)-symbols. The trickiest part is the case \( r = 2 \), since there we have fewer choices for \( p' \).

**Case 1 :** \( r = 2, 0 \leq p' \leq 2e \) and \( p' \) even.
Choose \( p = p' \). Therefore \( L = H = 0 \), and
\[
S = \frac{1}{e!(2(d - e) - p')!(2e - p')!(d - e)!} \neq 0.
\]

**Case 2 :** \( r = 2, 0 \leq p' \leq 2e \) and \( p' \) odd.
Choose \( p = p' - 1 \). Therefore \( L = 0, H = 1 \), and
\[
S = \frac{(-p' + 1)(d - 2e)}{e!(2(d - e) - p' + 1)!(2e - p' + 1)!(d - e)!} \neq 0.
\]

**Case 3 :** \( r = 2, 2e < p' \leq \min\{2(d - e), 3e\} \).
Choose \( p = 2e \). Therefore \( L = H = p' - 2e \), and
\[
S = \frac{(-1)^{p' - 2e}}{(p' - 2e)!(3e - p')!(2(d - e) - p')!(d - e)!} \neq 0.
\]

**Case 4 :** \( r = 2, 2(d - e) < p' \leq 3e \), and \( p' \) even.
Choose \( p = 2d - p' \). Therefore \( L = H = p' - 2e \), and
\[
S = \frac{(-1)^{p' - 2e}}{(p' - 2e)!(3e - p')!(p' - 2(d - e))!(3(d - e) - p')!} \neq 0.
\]

**Case 5 :** \( r = 2, 2(d - e) < p' < 3e \), and \( p' \) odd.
Choose \( p = 2d - p' - 1 \). Therefore \( L = p' - 2e, H = p' - 2e + 1 \), and
\[
S = \frac{(-1)^{p' - 2e}(d - 2e)(p' + 3)}{(p' - 2e + 1)!(3e - p')!(p' - 2(d - e) + 1)!(3(d - e) - p')!} \neq 0.
\]

**Case 6 :** \( r = 2, p' = 3e \) and \( p' \) odd (i.e., \( e \) is odd).
Choose \( p = 2d - p' - 1 \). Therefore \( L = H = e \), and
\[
S = \frac{-1}{(5e - 2d + 1)!(3d - 6e - 1)!} \neq 0.
\]
Case 7: \( r \geq 3, 0 \leq p' \leq re \) and \( p' \neq 1 \).
Choose \( p = p' \). Therefore \( \mathcal{L} = \mathcal{H} = 0 \), and
\[
S = \frac{1}{e!(r(d-e) - p')!(re - p')!(d-e)!} \neq 0.
\]

Case 8: \( r \geq 3 \), and \( re < p' \leq \min\{r(d-e), (r+1)e\} \).
Choose \( p = re \). Therefore \( \mathcal{L} = \mathcal{H} = p' - re \), and
\[
S = \frac{(-1)^{p'-re}}{(p'-re)!(r+1)e - p')!(r(d-e) - p')!(d-e)!} \neq 0.
\]

Case 9: \( r \geq 3 \), and \( r(d-e) < p' \leq (r+1)e \).
Choose \( p = rd - p' \). Therefore \( \mathcal{L} = \mathcal{H} = p' - re \), and
\[
S = \frac{(-1)^{p'-re}}{(p'-re)!(r+1)e - p')!(p' - r(d-e))!(r+1)(d-e) - p')!} \neq 0.
\]
The proof of Proposition 5.1 (and hence that of the main theorem) is complete. \( \square \)

8. Binary forms-I

In this section we will construct two covariants of binary \( d \)-ics which together define the locus \( X \) set-theoretically.

8.1. Let \( F \) be a binary \( d \)-ic. For nonnegative integers \( \mu, \nu \), we will use the notation
\[
F_{x_0^\mu x_1^\nu} = \frac{\partial^{\mu+\nu} F}{\partial x_0^\mu \partial x_1^\nu}.
\]

Now write \( F = \prod l_i^{\alpha_i} \), where \( l_i \) are pairwise non-proportional linear forms, and \( \alpha_i > 0 \). Assume furthermore that \( F \) is not a power of a linear form. Define \( g_F = \gcd (F_{x_0}, F_{x_1}) \). (By our assumption, both \( F_{x_i} \) are nonzero.)

Lemma 8.1. With notation as above, \( g_F = \prod l_i^{\alpha_i-1} \).

Proof. Evidently \( g = \prod l_i^{\alpha_i-1} \) divides both the \( F_{x_i} \), hence write \( F_{x_0} = gA, F_{x_1} = gB \). Divide Euler’s equation \( dF = x_0 F_{x_0} + x_1 F_{x_1} \) by \( g \), then \( d \prod l_i = x_0 A + x_1 B \). If \( A, B \) have a common linear factor, it must be one of the \( l_i \), say \( l_1 \). But
\[
A = \sum_i \alpha_i \frac{\partial l_i}{\partial x_0} (\prod_{j \neq i} l_j),
\]
so \( l_1 | A \) implies \( \frac{\partial l_1}{\partial x_0} = 0 \). The same argument on \( B \) leads to \( \frac{\partial l_1}{\partial x_1} = 0 \), so \( l_1 = 0 \). This is absurd, hence \( A, B \) can have no common factor, i.e. \( g = g_F \).

Now define
\[
Y = \bigcup_{0 \leq e \leq [\frac{d}{2}]} X^{(d-e,e)},
\]
the locus of binary \( d \)-ics with at most two distinct linear factors.

**Lemma 8.2.** For a binary \( d \)-ic \( F(x_0, x_1) \), the following are equivalent:

(i) \( F \in Y \).

(ii) The forms \( \mathcal{U} = \{ x_0 F_{x_0}, x_0 F_{x_1}, x_1 F_{x_0}, x_1 F_{x_1} \} \) are linearly dependent.

**Proof.** Assume (i), then \( F = x_0^{d-e} x_1^e \) or \( x_0^d \) after a change of variables, and (ii) is immediate. If (ii) holds, then there exist linear forms \( l, m \) (not both zero) such that \( l F_{x_0} = m F_{x_1} \). But then either \( F_{x_0}, F_{x_1} \) have a common factor of degree \( \geq d - 2 \), or one of them is zero. In the latter case \( F \) is a power of a linear form. In the former case, the previous lemma implies that \( F \) has at most two distinct linear factors. \( \square \)

Define \( \mathcal{D}(F) \) to the Wronskian of the sequence \( \mathcal{U} \), i.e.,
\[
\mathcal{D}(F) = \det \begin{vmatrix}
(x_0 F_{x_0}) x_0^3 & (x_0 F_{x_0}) x_0^2 x_1 & (x_0 F_{x_0}) x_0 x_1^2 & (x_0 F_{x_0}) x_1^3 \\
(x_0 F_{x_1}) x_0^3 & (x_0 F_{x_1}) x_0^2 x_1 & (x_0 F_{x_1}) x_0 x_1^2 & (x_0 F_{x_1}) x_1^3 \\
(x_1 F_{x_0}) x_1^3 & (x_1 F_{x_0}) x_1^2 x_1 & (x_1 F_{x_0}) x_1 x_0^2 & (x_1 F_{x_0}) x_0^3 \\
(x_1 F_{x_1}) x_1^3 & (x_1 F_{x_1}) x_1^2 x_1 & (x_1 F_{x_1}) x_1 x_0^2 & (x_1 F_{x_1}) x_0^3 \\
\end{vmatrix}
\]

(21)

It is a covariant of \( F \) of degree 4 and order \( 4d - 12 \). By the previous lemma,
\[
F \in Y \iff \mathcal{D}(F) = 0.
\]

(22)

Recall that a binary form \( A(x_0, x_1) \) is a power of a linear form, iff its Hessian
\[
\text{He} (A) = A_{x_0}^2 A_{x_1}^2 - (A_{x_0 x_1})^2
\]
is identically zero. Moreover, for such a form all covariants of degree greater than one are identically zero.
8.2. Now fix an integer \(1 \leq e \leq d/2\), and define a rational covariant
\[
A_e(F) = \frac{F^{2d-2e-2}}{\text{He}(F)^{d-e}}.
\]
Assume \(F \in Y, \text{He}(F) \neq 0\). By a change of variable we may write
\(F = x_0^{d-f} x_1^f\), for some \(1 \leq f \leq d/2\). Then up to a nonzero multiplicative factor, \(\text{He}(F) = x_0^{2d-2f-2} x_1^{2f-2}\), and by a direct substitution
\[
A_e(F) = x_0^{2f-2e} x_1^{2d-2e-2f}.
\]
Hence \(A_e(F)\) can be a power of a linear form, iff either \(f = e\) or \(e + f = d\), i.e., iff \(F \in X^{(d-e,e)}\). Hence we have proved the following:

**Proposition 8.3.** A binary \(d\)-ic \(F\) (which is not a \(d\)-th power of a linear form) lies in \(X^{(d-e,e)}\), iff \(D(F) = \text{He}(A_e(F)) = 0\).

However, this criterion is not aesthetically satisfactory insomuch as it appeals to a rational (as opposed to a polynomial) covariant. To amend this, we will deduce a formula for the Hessian of a quotient of two forms and then apply it to \(A_e\).

8.3. The **Hessian of a quotient**. Let \(P, Q\) denote generic binary forms of degrees \(p, q \geq 0\) respectively. (By convention, 1 is the generic degree zero form.) Define
\[
z_1 = \frac{p^2 (p-1) (2p-2q-1) (p-q-1)}{2(2p-1)},
\]
\[
z_2 = \frac{q^2 (q-1) (2p-2q+1) (p-q-1)}{2(2q-1)},
\]
\[
z_3 = pq (p-q-1).
\]

**Theorem 8.4.** With notation as above, we have the following formal identity:
\[
\text{He} \left( \frac{P}{Q} \right) = \frac{J(P,Q)}{Q^4},
\]
where
\[
J(P,Q) = z_1 Q^2 (P,P)_2 + z_2 P^2 (Q,Q)_2 + z_3 (P^2, Q^2)_2.
\]

**Proof.** If either \(p\) or \(q\) is zero, then the theorem reduces to an easy calculation, hence we may assume \(p, q \geq 1\). Let \(U = \frac{P}{Q}\), first we will show that \(J = Q^4 \text{He}(U)\) is a polynomial. Let us write
\[
\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},
\]
then by quotient rule,
\[
\partial_{ij} U = \frac{\partial_{ij} P}{Q} - \frac{\partial_i P \partial_j Q + \partial_j P \partial_i Q}{Q^2} - \frac{P \partial_{ij} Q}{Q^2} + \frac{2P \partial_i Q \partial_j Q}{Q^3}
\]
(24)
Here \(e_\ast(i, j)\) are simply names for those consecutive expressions. Now
\[
\text{He} (U) = (\partial_{0,0} U) (\partial_{1,1} U) - (\partial_{0,1} U)^2
\]
is a linear combination of terms
\[
E(a, b) = e_a(0, 0) e_b(1, 1) + e_b(0, 0) e_a(1, 1) - 2e_a(0, 1) e_b(0, 1),
\]
for \(1 \leq a, b \leq 4\). The terms \(E(2, 4), E(4, 4)\) are zero, so the only term with a (possible) denominator of \(Q^5\) is \(E(3, 4)\). Let us write
\[
E(3, 4) = -\frac{2P^2}{Q^5} E'(3, 4),
\]
we will show that in fact \(Q\) divides \(E'(3, 4)\). Now we have an identity
\[
q^2 (2q - 1)(q - 1) (Q^2, Q)_2 = q^2 (q - 1)^2 Q (Q, Q)_2 + E'(3, 4); \quad (25)
\]
this follows by directly calculating the left hand side with formula (4).
So far the entire argument works if \(Q\) is any sufficiently differentiable function of \(x_0, x_1\). But now we can use the homogeneity of \(Q\) to rewrite the left hand side of (25). Since \(E'(3, 4) = 0\) for \(q = 1\), we may assume
\[
q \geq 2. \quad \text{The Gordan series} \quad \begin{pmatrix} Q & Q & Q \\ q & q & q \\ 0 & 0 & 2 \end{pmatrix} \quad \text{gives an identity}
\]
\[
(Q^2, Q)_2 = \frac{3q - 2}{2(2q - 1)} Q (Q, Q)_2.
\]
(See [15, Ch. IV] for the derivation of the series.) We have shown that \(Q\) divides \(E'(3, 4)\), hence \(J\) is a polynomial covariant.

We can continue the calculation of \(J\) from (24), but it is easier to proceed as follows. By counting degrees, we see that \(J(P, Q)\) is a joint covariant of \(P, Q\) which is quadratic in \(P, Q\) separately and has order \(2p + 2q - 4\). We claim that every such joint covariant is a linear combination of
\[
Q^2 (P, P)_2, \quad P^2 (Q, Q)_2, \quad (P^2, Q^2)_2.
\]
(26)
This amounts to counting the number of copies of the representation \(S_{2p+2q-4}\) inside \(S_2(S_p) \otimes S_2(S_q)\). A straightforward expansion shows that there are three such copies (see [26, §4.2]). It is easy to see by
specialization that the covariants in (26) are linearly independent for generic $P, Q$, so they must form a basis for this space.

Hence we may write $J$ as in (23) for some constants $z_i$. Specialize to $P = x_0^p, Q = x_1^q$, then $(P, P)_2 = (Q, Q)_2 = 0$ and $(P^2, Q^2)_2 = x_0^{2p-2} x_1^{2q-2}$. On the other hand, $Q^4 \text{He}(U) = p q (p - q - 1) x_0^{2p-2} x_1^{2q-2}$.

This forces $z_3 = p q (p - q - 1)$.

Similarly, specialize $P, Q$ to the pairs $(x_0^p, x_1^{q-1})$ and $(x_0^{p-1} x_1, x_0^q)$, and get two more linear equations involving the $z_i$. Solving these, we get the theorem. \hfill \Box

This formula has a simple but interesting corollary. If $p = q + 1$, then $\text{He}(\frac{P}{Q})$ is identically zero.

Finally write $\mathcal{C}_e(F) = J(F^{2d-2e-2}, \text{He}(F)^{d-e})$; then we can state a criterion which involves only polynomial covariants:

**Theorem 8.5.** Let $F$ be a binary $d$-ic. Then

$$F \in X^{(d-e,e)} \iff \mathcal{C}_e(F) = \mathcal{D}(F) = 0.$$ 

**8.4. A formula for $\mathcal{D}$.** One can write down a formula for the covariant $\mathcal{D}$ in terms of compound transvectants. The proofs will only be sketched. Define

$$\xi_1 = (2d - 1)(2d - 3)(2d - 5)^3,$$

$$\xi_2 = -9 (d - 3)(2d - 5)(2d - 7)(2d - 3)^2,$$

$$\xi_3 = 4 (d - 1)(d - 3)(d - 4)(2d - 9)(4d - 7).$$

**Proposition 8.6.** If $d \geq 6$, then up to a multiplicative scalar

$$\mathcal{D}(F) = \xi_1 (F^2, F^2)_6 + \xi_2 (F^2, (F, F)_2)_4 + \xi_3 (F^2, (F, F)_4)_2.$$ \hfill (27)

One may argue as follows: for $d \geq 6$, there are three copies of $S_{4d-12}$ inside $S_4(S_d)$, and a basis for this space is given by the three covariants which occur in (27). Hence $\mathcal{D}(F)$ can be written as their linear combination. To determine the actual coefficients, specialize to $F = x_0^{d-e} x_1^e, e = 2, 3$ (when $\mathcal{D}$ must vanish) and solve a system of linear equations. \hfill \Box

**Remark 8.7.** It is a priori clear that the $\xi_i$ should be rational functions in $d$ (or polynomials after clearing denominators). However, we can see no conceptual explanation of the fact that they should split into linear factors over $Q$. 

These are the formulae in low degrees:

\[ \mathcal{D}(F) = \begin{cases} 
(F^2, F^2)_6 & \text{for } d = 3, \\
7(F^2, F^2)_6 - 5(F^2, (F, F)_2)_4 & \text{for } d = 4, \\
129(F^2, F^2)_6 - 250(F^2, (F, F)_2)_4 & \text{for } d = 5.
\end{cases} \]

To prove these, notice that there are two copies of \( S_{d-12} \) in \( S_4(S_d) \) for \( d = 4, 5 \) and argue as before. For degree 3 forms, \( \mathcal{D} \) is simply the discriminant.

9. Binary forms-II

In this section we write down the covariants which correspond to the quartic generators of \( I_{X(d-e,e)} \). Given a triple of integers \( I = (i, j, k) \), define a covariant

\[ \mathcal{E}_I(F) = (((F, F)_{2i}, F)_{2j}, F)_{2k}. \]

Let \( F \) be a general point of \( X^{(d-e,e)} \), then we may write \( F = x_0^{d-e} x_1^e \) after a change of variables. Using formula (16),

\[ \mathcal{E}_I(F) = \omega_I x_0^{4(d-e)-(2i+j+k)} x_1^{4e-(2i+j+k)} \]

where \( \omega_I \) is the rational number

\[
\mathcal{N} \left[ \begin{array}{c} d-e, e \\ d-e, e \\ 2i \\
\end{array} \right] \times \mathcal{N} \left[ \begin{array}{c} 2(d-e)-2i, 2e-2i \\ d-e, e \\ j \\
\end{array} \right] \times \]

\[
\mathcal{N} \left[ \begin{array}{c} 3(d-e)-(2i+j), 3e-(2i+j) \\ d-e, e \\ k \\
\end{array} \right]
\]

Thus, as a monomial, \( \mathcal{E}_I(F) \) depends only on the sum \( 2i+j+k \). Given triples \( I = (i, j, k), I' = (i', j', k') \) such that \( 2i + j + k = 2i' + j' + k' \), define

\[ \Psi_{I,I'}(F) = \omega_I \mathcal{E}_{I'}(F) - \omega_{I'} \mathcal{E}_I(F). \]

**Proposition 9.1.** The locus \( X^{(d-e,e)} \) is scheme-theoretically generated by the coefficients of all the covariants \( \Psi_{I,I'} \).

**Proof.** The proof is in essence identical to [1, Theorem 7.2], hence we omit the details. \( \square \)
We have been unable to give ‘closed formulae’ for the $\omega_I$, indeed this is directly traceable to the difficulty that no closed expression is known for a general Clebsch-Gordan coefficient $C^{j_1,j_2,j}_{m_1,m_2,m}$.

The covariants corresponding to the quadratic generators of $I_X(d-e,e)$ are easily described, they are $\{(F,F)_{2i} : e+1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor\}$.

10. Ternary Quintics

In this section we work out the case $n = 2$, $(d-e,e) = (3,2)$, and describe the ideal generators invariant-theoretically. We have made rather heavy use of machine-computations, specifically the programs Macaulay-2 and Maple.

10.1. Define generic forms

\[
L_1 = a_0 x_0 + a_1 x_1 + a_2 x_2, \quad L_2 = b_0 x_0 + b_1 x_1 + b_2 x_2
\]

\[
F = c_0 x_0^5 + c_1 x_0^4 x_1 + \cdots + c_{20} x_2^5,
\]

where $a, b, c$ are independent indeterminates. Write $F = L_1^3 L_2^2$ and equate the coefficients of the monomials in $x_0, x_1, x_2$. This expresses each $c_i$ as a polynomial in $a_0, \ldots, b_2$, and hence defines a ring map

\[
\mathbb{C}[c_0, \ldots, c_{20}] \longrightarrow \mathbb{C}[a_0, \ldots, b_2].
\]

The kernel of this map is $I_X$. We calculated it in Macaulay-2, and found that its resolution begins with

\[
\ldots \rightarrow R(-4) \otimes M^{(4)} \oplus R(-3) \otimes M^{(3)} \rightarrow R \rightarrow R/I_X \rightarrow 0,
\]

where $M^{(3)}, M^{(4)}$ are vector spaces of dimensions 455 and 1470 respectively. Thus there are no generators in degrees $\geq 5$. Since $SL(V)$ acts on this resolution, the $M^{(i)}$ are $SL(V)$-modules.

Lemma 10.1. We have the following isomorphisms of $SL(V)$-modules:

\[
M^{(3)} = S_{(9,3)} \oplus S_{(9,0)} \oplus S_{(7,5)} \oplus S_{(7,2)} \oplus S_{(6,3)} \oplus S_{(3,3)} \oplus S_{(3,0)},
\]

\[
M^{(4)} = S_{(16,4)} \oplus S_{(14,6)} \oplus S_{(12,8)} \oplus S_{(10,10)}.
\]

Proof. Since $M^{(3)} = (I_X)_3$, the first isomorphism follows from Corollary 4.4. (Throughout this example, all the inner and outer products of Schur functions were calculated using the Maple package ‘SF’.)
The degree 4 piece of $I_X$ is a direct sum of two parts: multiples of degree 3 generators by linear forms, and the new generators $M^{(4)}$. Hence

$$[(I_X)_4] = [M^{(3)} \otimes S_5] - [N^{(4)}] + [M^{(4)}],$$

where $N^{(4)}$ denotes the module of first syzygies in degree 4. (We know practically nothing about $N^{(4)}$, but we will see that this is no obstacle.) Now $[(I_X)_4]$ can be calculated by Corollary 4.4 and $[M^{(3)} \otimes S_5]$ by the Littlewood-Richardson rule. Hence the difference $[M^{(4)}] - [N^{(4)}]$ is known, we write it as

$$[M^{(4)}] - [N^{(4)}] = \mathcal{Y} + \mathcal{Z},$$

where $\mathcal{Y}$ (resp. $\mathcal{Z}$) is a positive (resp. negative) linear combination of Schur polynomials. The actual calculation shows that

$$\mathcal{Y} = [S_{(16,4)} \oplus S_{(14,6)} \oplus S_{(12,8)} \oplus S_{(10,10)}].$$

It follows that each summand on the right must appear in $M^{(4)}$. Now the direct sum has dimension $585 + 504 + 315 + 66 = 1470 = \dim M^{(4)}$. Hence $M^{(4)}$ must in fact coincide with this sum. \hfill $\square$

10.2. By the standard formalism of [20], a submodule $S_{(a,b)} \subseteq S_r(S_5)$ corresponds to a concomitant of degree $r$, order $a - b$ and class $b$ of ternary quintics. We will illustrate how to write down such a concomitant symbolically. For instance, let $\Psi$ correspond to the inclusion $S_{(16,4)} \subseteq M^{(4)}$. Decomposing $S_4(S_5)$, we detect that it has two copies of $S_{(16,4)}$, hence ternary quintics have two independent concomitants $\Psi_1, \Psi_2$ of degree 4, order 12 and class 4. Now consider the following Young tableau of shape $(16,4)$ filled with four symbolic letters $\alpha, \beta, \gamma, \delta$, each occurring 5 times:

$$\begin{array}{cccccccccccc}
\alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \gamma & \gamma & \gamma & \gamma & \delta & \delta & \delta & \delta
\end{array}$$

Reading this tableau columnwise, we can construct the concomitant

$$\Psi_1 = (\alpha \beta u)^2 (\alpha \gamma u) (\alpha \delta u) \alpha_x \beta_x^3 \gamma_x^4 \delta_x^4.$$

We will abbreviate this as $\Psi_1 = (5, 3, 4, 4|0, 2, 1, 1)$. (This means that in the top row of the tableau $\alpha$ occurs five times, followed by $\beta$ thrice etc. Such a notation is possible because we will choose all of our tableaux to be semistandard for the order $\alpha < \beta < \gamma < \delta$.) Similarly, let $\Psi_2 = (5, 1, 5, 5|0, 4, 0, 0)$. To show that the $\Psi_i$ are linearly
independent, it is sufficient to evaluate them on any specific form, in fact \( F = x_0^5 - x_1^5 \) would do. (We checked this in Maple.) Hence \( \Psi_1, \Psi_2 \) form a basis of the space of concomitants of degree 4, order 12 and class 4.

Now write \( \Psi = \eta_1 \Psi_1 + \eta_2 \Psi_2 \), and evaluate on \( x_0^3 x_1^2 \in X^{(3,2)} \). By hypothesis \( \Psi \) must vanish identically, this gives the equation

\[
\left( \frac{57}{2500} \eta_1 + \frac{3}{50} \eta_2 \right) x_0^8 x_1^4 u_2^4 = 0.
\]

Hence \( \eta_1 : \eta_2 = 50 : -19 \), which determines \( \Psi \) (up to a scalar).

We have worked out the complete list for all the summands in (28).

In degree 3 (where we need only three symbolic letters), the concomitants are

\[
\langle 5, 4, 1|0, 1, 3|0, 0, 1 \rangle, \quad \langle 5, 3, 3|0, 2, 0|0, 0, 2 \rangle, \quad \langle 5, 3, 0|0, 2, 4|0, 0, 1 \rangle,
\langle 5, 3, 1|0, 2, 2|0, 0, 2 \rangle, \quad \langle 5, 2, 1|0, 3, 2|0, 0, 2 \rangle, \quad \langle 5, 1, 0|0, 4, 2|0, 0, 3 \rangle,
\langle 5, 1, 1|0, 4, 0|0, 0, 4 \rangle.
\]

(29)

In degree 4, they are

\[
50 \langle 5, 3, 4, 4|0, 2, 1, 1 \rangle - 19 \langle 5, 1, 5, 5|0, 4, 0, 0 \rangle,
5 \langle 5, 5, 4, 0|0, 0, 1, 5 \rangle - 8 \langle 5, 4, 0, 0|0, 1, 5, 0 \rangle,
\langle 5, 1, 0, 4 \rangle^2 + 2 \langle 5, 3, 2, 2|0, 2, 3, 3 \rangle,
\langle 5, 3, 2, 0|0, 2, 3, 5 \rangle.
\]

(30)

In conclusion we have the following result:

**Proposition 10.2.** Let \( F \) be a ternary quintic with zero scheme \( C \subseteq \mathbb{P}^2 \). Then \( C \) consists of a triple line and a double line, iff all the concomitants in (29) and (30) vanish on \( F \). □

**Acknowledgements:** The first author would like to express his gratitude to Professors David Brydges and Joel Feldman for the invitation to visit the University of British Columbia. The second author would like to thank Professor James Carrell for his invitation to visit UBC. We are indebted to Daniel Grayson and Michael Stillman (authors of Macaulay-2), John Stembridge (author of the ‘SF’ package for Maple), the digital libraries maintained by the Universities of Cornell, Göttingen and Michigan, as well as J-Stor and Project Gutenberg.
References

[1] A. Abdesselam, and J. Chipalkatti. Brill-Gordan loci, transvectants and an analogue of the Foulkes conjecture. [math.AG/0411110] Preprint, 2004.
[2] K. Akin, D. Buchsbaum, J. Weyman. Schur functors and Schur complexes. Adv. Math., vol. 44, pp. 207–278, 1982.
[3] S. Aronhold. Theorie der homogenen Functionen dritten Grades von drei Veränderlichen. J. Reine Angew. Math., vol. 55, pp. 97–191, 1858.
[4] L. C. Biedernham, and J. D. Louck. Angular momentum in quantum physics. Encyclopedia of mathematics and its applications, vol. 8, Addison-Wesley, Reading, Massachusetts, 1981.
[5] L. C. Biedernham, and J. D. Louck. The Racah-Wigner algebra in quantum mechanics. Encyclopedia of mathematics and its applications, vol. 9, Addison-Wesley, Reading, Massachusetts, 1981.
[6] A. Cayley. On linear transformations. No. 14 in vol. I of Collected Mathematical Works, Cambridge University Press, 1889.
[7] J. Chipalkatti. On equations defining coincident root loci. J. of Algebra, vol. 267, pp. 246–271, 2003.
[8] A. Clebsch. Theorie der binären algebraischen Formen. Leipzig, 1872.
[9] P. Cvitanović. Group Theory (with contributions by H. Elvang and T. Kennedy). Available as a webbook at [http://www.nbi.dk/GroupTheory](http://www.nbi.dk/GroupTheory)
[10] J. Dixmier. Certaines algèbres non associatives définies par la transvection des formes binaires. J. Reine Angew. Math., vol. 346, pp. 110–128, 1984.
[11] W. Fulton and J. Harris. Representation Theory, A First Course. Graduate Texts in Mathematics. Springer–Verlag, New York, 1991.
[12] K. Gawedzki. Lectures on conformal field theory. In Quantum fields and strings: a course for mathematicians (Princeton 1996/1997), vol. 2, pp. 727-805, P. Deligne et al. (ed.), Providence RI, Amer. Math. Soc., 1999.
[13] O. Glenn. A Treatise on the Theory of Invariants. Ginn and Co., Boston, 1915. (Available as an eBook from Project Gutenberg at [http://www.gutenberg.net](http://www.gutenberg.net))
[14] P. Gordan and G. Kershensteiner. Vorlesungen über Invariantentheorie. Leipzig, 1887.
[15] J. H. Grace and A. Young. The Algebra of Invariants, 1903. Reprinted by Chelsea Publishing Co., New York, 1965.
[16] R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics. Springer–Verlag, New York, 1977.
[17] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. Mirror symmetry. Clay Mathematics Monographs, 1. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003.
[18] B. Hunt. The geometry of some special arithmetic quotients. Lecture Notes in Mathematics, no. 1637. Springer-Verlag, Berlin, 1996.
[19] C. Krattenthaler, and S. K. Rao. Automatic generation of hypergeometric identities by the beta integral method. *J. Comput. Appl. Math.*, vol. 160, pp. 159–173, 2003.

[20] D. E. Littlewood. Invariant theory, tensors and group characters. *Phil. Trans. of the Royal Society of London, Series A, Mathematical and Physical Sciences*, vol. 239, no. 807, pp. 305–365, 1944.

[21] S. Okubo. *Introduction to octonion and other non-associative algebras in physics*. Cambridge University Press, 1995.

[22] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A = B*. A. K. Peters Ltd., Wellesley, Massachusetts, 1996.

[23] O. Porras. Rank varieties and their resolutions. *J. Algebra*, vol. 186, no. 3, pp. 677–723, 1996.

[24] J. Raynal, J. Van der Jeugt, S. K. Rao, and V. Rajeswari. On the zeros of 3j coefficients: polynomial degree versus occurrence order. *J. Phys. A*, vol. 26, no. 11, pp. 2607–2623, 1993.

[25] N. I. Shepherd-Barron. The rationality of some moduli spaces of plane curves. *Compositio Math.*, vol. 67, no. 1, pp. 51–88, 1988.

[26] B. Sturmfels. *Algorithms in Invariant Theory*. Texts and Monographs in Symbolic Computation, Springer-Verlag, Wien, 1993.

[27] H. S. Wilf, and D. Zeilberger. Rational functions certify combinatorial identities. *J. Amer. Math. Soc.*, vol. 3, pp. 147–158, 1990.
Abdelmalek Abdesselam
Department of Mathematics
University of British Columbia
1984 Mathematics Road
Vancouver, BC V6T 1Z2
Canada.
abdessel@math.ubc.ca

Jaydeep Chipalkatti
Department of Mathematics
University of Manitoba
433 Machray Hall
Winnipeg MB R3T 2N2
Canada.
chipalka@cc.umanitoba.ca

LAGA, Institut Galilée
CNRS UMR 7539
Université Paris XIII
99 Avenue J.B. Clément
F93430 Villetaneuse
France.