On Solving Floating Point SSSP Using an Integer Priority Queue

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Abstract. We address the single source shortest path planning problem (SSSP) in the case of floating point edge weights. We show how any integer based Dijkstra solution that relies on a monotone integer priority queue to create a full ordering over path lengths in order to solve integer SSSP can be used as an oracle to solve floating point SSSP with positive edge weights (floating point P-SSSP). Floating point P-SSSP is of particular interest to the robotics community. This immediately yields a handful of faster runtimes for floating point P-SSSP; for example, $O(m + n \log \log \frac{C}{\delta})$, where $C$ is the largest weight and $\delta$ is the minimum edge weight in the graph. It also ensures that many future advances for integer SSSP will be transferable to floating point P-SSSP. Our work relies on a result known to [Dinic, 1978] and [Tsitsiklis, 1995]; that, under the right conditions, SSSP can be solved using a partial ordering of nodes, despite the fact that full orderings are typically used in practice. Thus, priority queues that do not extract keys in a monotonically nondecreasing order can be used to solve SSSP under a special set of conditions that always hold for floating point P-SSSP. In particular, any node that has a shortest-path-length (key value) within $\delta$ of the queue’s minimum key can be extracted (instead of the node with the minimum key) from the priority queue and Dijkstra’s algorithm will still run correctly. Our contribution is to show how any monotonic integer priority queue can be transformed into a suitable $\delta$-nonmonotonic floating point priority queue in order to produce the necessary partial ordering for floating point values. Monotonic integer keys for floating point values are obtained by dividing the floating point key values by $\delta$ (as was done by [Dinic, 1978]) and then converting the result to an integer. The loss of precision this division causes is the reason the resulting floating point queue is $\delta$-nonmonotonic instead of monotonic, but does not break the correctness of Dijkstra’s algorithm. We also prove that the floating point version of non-negative SSSP (which allows $\delta = 0$ in addition to $\delta > 0$) cannot be solved without creating a full ordering of nodes (and thus requires a fully monotonic priority heap); and so our method cannot be extended to work on non-negative floating point SSSP, in general.

1 Introduction

Finding the shortest path through a graph $G = (E, V)$ defined by sets of nodes $V$ and edges $E$ is a classic problem. The variation known as the “single source
shortest-path planning problem” (or SSSP) is concerned with finding all of the shortest-paths from a particular node \( s \in V \) to all other nodes \( v \in V \), where each edge \( (u, v) = \varepsilon \in E \) is associated with a length \( ||\varepsilon|| \). The first algorithm that solves SSSP was presented by Dijkstra in the 1950s using an algorithm that runs in time \( O(m + n^2) \) for the case of nonnegative integer edge lengths \cite{Dijkstra1959}. Over the years, the SSSP problem has been studied extensively by computer scientists, who have primarily focus on the classic integer version of the problem. The robotics community, on the other hand, is more interested in floating point solutions to SSSP. In robotics, graphs are embedded in the environment and edge lengths represent distances along physical trajectories that a robot traverses to move between different locations. Movement is subject to the practical geometric constraints of the environment (topology and obstacles) as well as the dynamics and kinematics of the robotic system. Thus, finding suitable trajectories (and their edge lengths) often involves solving a set of differential equations or other two-point boundary value problem. In short, the most natural numerical representation for robotics SSSP problems is not integers.

The main contribution of this paper is to show how a large class of solutions for integer SSSP can easily be modified to solve floating point SSSP with strictly positive edge weights (floating point P-SSSP). Our work is applicable to both theoretical performance bounds, as well as the practical implementation of SSSP algorithms. We note that our method breaks if edges are allowed to have length 0. This is usually not a concern for robotics applications, where an edge of length 0 implies that the two nodes it connects represent the same location in the environment — without loss of generality these can be combined into a single node and the 0 length edge removed from the graph.

The vast majority of previous work on SSSP has involved graphs with integer edge weights (related work is surveyed in Section \ref{related-work}). Sophisticated priority queue data structures have reduced the runtime of integer SSSP algorithms to \( O(m + n \log n) \) for the restrictive comparison computational model and even further for the RAM computational model that most “real” computers use e.g., \( O(m + n \log \log n) \). See Table \ref{table:results} for additional results.

The discrepancy between the state-of-the-art for integer vs. floating point SSSP can be seen in Table \ref{table:results}. That said, notable floating point results include \cite{Thorup2000a} and \cite{Thorup2004}. \cite{Thorup2000a} proves that, in the case of undirected graphs (i.e., USSSP), an algorithm that solves integer USSSP can be used as an oracle to solve floating point USSSP. \cite{Thorup2004} leverages the fact that, assuming the IEEE standard for floating point numbers and integers is used, value orderings are preserved if we interpret the bit representations of floating point numbers directly as integers. The latter trick enables many (but

\footnote{1}{In many cases the graph used for robotics is embedded within the configuration space of the robot (the space created as the product space over the degrees of freedom of the system), of which the physical environment is a lower dimensional projection.}

\footnote{2}{Or configuration space.}

\footnote{3}{At least in the robotics domain, where the graph is just a tool for reasoning about environmental connectivity with respect to the movement constraints of the robot.}
Table 1: Best theoretical runtimes prior to our work (with references)

|                | Directed (SSSP) | Undirected (USSSP) |
|----------------|-----------------|--------------------|
| Integer worst case |                 |                    |
| $O(m + n \log \log n)$ |                 | $O(m + n)$ Thorup, 2004 |
| $O(m + n \log C)$ | Thorup, 2004    |                    |
| $O(m + n (\log C \log \log C)^{1/3})$ | Raman, 1997    |                    |
| $O(n + m \log \log m)$ | Thorup, 1996    |                    |
| $O(n + m \log \log C)$ | Raman, 1997    |                    |
| $O(m + n (B + C/B))$ s.t. $B < C + 1$ | Cherkassky et al., 1996 |            |
| $O(m\Delta + n(\Delta + C/\Delta))$ s.t. $0 < \Delta < C$ | Cherkassky et al., 1996 | |
| Floating Point worst case |                 |                    |
| $O(m + n \log \log n)$ | Thorup, 2004    | $O(m + n)$ Thorup, 2000b |
| $O(m + n \frac{\log \frac{n}{\log \log n}}{\log \log n})$ | Thorup, 2004    | $O(m + n)$ Thorup, 2000a |
| $O(m + n (\log \frac{C}{\Delta})^{1/3+\epsilon})$ | Cherkassky et al., 1999 + Tsitsiklis, 1995 |            |
| $O(m\sqrt{\log \log n})$ | Thorup, 2004    |                    |
| $O(m + n \log \log n)$ | Thorup, 2000b   |                    |

Results with (*) assume strictly positive edges, $(P\text{-SSSP}), 0 < \delta \leq ||\epsilon|| \leq C < \infty, \forall \epsilon \in E$. Other results hold for both positive and nonnegative edges (N-SSSP), $0 \leq ||\epsilon|| \leq C < \infty$.

not all) results from integer SSSP to be extended to floating point SSSP, assuming IEEE standards are followed. In particular, it enables the direct application of the $O(m + n \log \log n)$ integer SSSP algorithm presented in [Thorup, 2004] to be used to solve floating point SSSP.

From a high-level point-of-view, our work extends the applicability of integer-to-floating-point oracles to a greater subset of SSSP. In particular, it enables any solution for integer SSSP that constructs a full ordering (i.e., by extracting values from an integer priority queue in a monotonically nondecreasing sequence) to solve floating point P-SSSP. The floating point solution happens via a partial ordering on path lengths that can be found by extracting floating point values in a special sequence that may not be monotonically nondecreasing.

The trick that we use is fundamentally different than interpreting the bits of floating point numbers directly as integers, and thus applies to a different subset of SSSP algorithms than previous work. Many (and arguably most) integer SSSP
|                        | Directed (SSSP) | Undirected (USSSP) |
|------------------------|----------------|-------------------|
| Integer worst case     |                |                   |
| \(O(m + n \log \log n)\) | Thorup, 2004   | \(O(m + n)\)     |
| \(O(m + n \log C)\)    | Thorup, 2004   |                   |
| \(O(m + n (\log C \log \log C)^{1/3})\) | Raman, 1997   |                   |
| \(O(n + m \log \log m)\) | Thorup, 1996   |                   |
| \(O(n + m \log \log C)\) | Raman, 1997   |                   |
| \(O(m + n(B + C/B))\) s.t. \(B < C + 1\) | Cherkassky et al., 1996 |                   |
| \(O(m\Delta + n(\Delta + C/\Delta))\) s.t. \(0 < \Delta < C\) | Cherkassky et al., 1996 |                   |
|                        | Thorup, 2004   |                   |
|                        | Thorup, 2004   |                   |
|                        | Thorup, 2000b  |                   |
| Floating Point worst case |                |                   |
| \(*\) new result \(O(m + n \log \log C)\) | Thorup, 2000b  |                   |
| \(*\) new result \(O(m + n (\log \log C)^{1/3})\) | Thorup, 2000b  |                   |
| \(*\) new result \(O(n + m \log \log m)\) | Thorup, 2000b  |                   |
| \(*\) new result \(O(n + m \log \log C)\) | Thorup, 2000b  |                   |
| \(*\) new result \(O(m + n(B + C/B))\) s.t. \(B < C + 1\) | Thorup, 2000b  |                   |
| \(*\) new result \(O(m\Delta + n(\Delta + C/\Delta))\) s.t. \(0 < \Delta < C\) | Thorup, 2000b  |                   |
|                        | Thorup, 2004   |                   |
|                        | Thorup, 2004   |                   |
|                        | Thorup, 2000b  |                   |

Results with (*) assume strictly positive edges, (P-SSSP), \(0 < \delta \leq \|\epsilon\| \leq C < \infty\), \(\forall \epsilon \in E\). Other results hold for both positive and nonnegative edges (N-SSSP), \(0 \leq \|\epsilon\| \leq C < \infty\). Yellow indicates new results enabled by our work.

Solutions construct a full ordering based on path-length. Leveraging this previous work, we immediately get a variety of faster theoretical runtimes for floating point P-SSSP. For example, combining our work with [Thorup, 2004] yields an algorithm for floating point P-SSSP that runs in time \(O(m + n \log \log C)\), where \(\delta\) is the minimum edge length in the graph. This is particularly of note given the comment in [Thorup, 2004] that “for floating point numbers we [they] do not get bounds in terms of the maximal weight,” i.e., \(C\). Other new results enabled by our work appear highlighted in Table 2. Our works also guarantees that any new faster results for integer SSSP that come from the discovery of a new monotonic
integer priority queue will have corresponding counterparts for floating point P-SSSP.

1.1 High Level Description

Our method relies on a result known to [Dinic, 1978] and [Tsitsiklis, 1995]: that, under the right conditions, SSSP can be solved using a partial ordering of nodes, despite the fact that full orderings are typically used in practice. In particular, any node that has a shortest-path-length (key value) within $\delta$ of the queue’s minimum key can be extracted (instead of the node with the minimum key) from the priority queue and Dijkstra’s algorithm will still run correctly. We show how an integer priority queue that extracts values in a monotonically nondecreasing order can easily be converted into a floating point priority queue that is suitable for solving floating point P-SSSP. This is accomplished by calculating integer keys for floating point values by dividing the latter by $\delta$ and then converting with truncation to an integer representation. The resulting floating point priority is no longer monotonic, but rather $\delta$-nonmonotonic (see Section 2.3 for a discussion on the differences between $\delta$-nonmonotonic and nonmonotonic). While [Dinic, 1978] uses a similar idea to improve integer bucket-based priority queues for solving integer SSSP, we are the first to employ this mechanism to leverage integer priority queues for solving floating point SSSP.

Algorithmically this means that we store two keys with each node instead of one. The actual floating point key $D(v)$ that reflects the current best guess of node $v$’s shortest path length, as well as a companion integer key $\hat{D}(v) = \lfloor \frac{D(v)}{\delta} \rfloor$. We maintain the position of node $v$ within the integer queue using $\hat{D}(v)$ instead of $D(v)$. Whenever $D(v)$ is decreased we recalculate $\hat{D}(v)$ from the new $D(v)$ and then update $v$’s position in the integer queue using $\hat{D}(v)$. The differences between standard Dijkstra’s algorithm and a version using our modification to solve floating point P-SSSP can be seen by comparing Algorithms 1 and 2.

Although there is loss of precision within the integer priority queue (due to conversion from floating point to integer with truncation) any resulting node reordering is upper-bounded by $\delta$ with respect to $D(v)$. Thus, as long as $\delta > 0$, any reordering of nodes that results due to this precision loss is still sufficient to solve floating point SSSP with Dijkstra’s algorithm (i.e., the resulting $\delta$-nonmonotonic extraction order from the resulting floating point queue creates a partial ordering sufficient for Dijkstra’s algorithm to solve SSSP). The rest of this paper is dedicated to formalizing these results and discussing them in greater detail.

2 Preliminaries

A graph $G$ (either directed or underacted) is defined by its edge set $E$ and vertex set $V$. We assume that both $m = |E|$ and $n = |V|$ are finite. Each edge $\varepsilon_{ij} = (v_i, v_j)$ between two vertices $v_i$ and $v_j$ (or from $v_i$ to $v_j$ if the edge is directed) is
Algorithm 1: Dijkstra($G, s$) standard implementation

Input: A graph $G = (E, V)$ of node set $V$ and edge set $E$, a start node $s \in V$, and a priority queue $H$.
Output: Shortest path lengths $d(v)$ and parent pointers $p(v)$ with respect to the shortest path-tree $S$ for all $v \in V$.

1. $H \equiv \emptyset$;
2. for all $v \in V$ do
3. \[ D(v) = \infty \] ; /\* $S = \emptyset$ /\*
4. $p(v) = \text{NULL}$ ;
5. $D(s) = 0$ ;
6. $v = s$ ;
7. while $v \neq \text{NULL}$ do
8. \[ d(v) = D(v) ; \] /\* $S = S \cup \{v\}$ /\*
9. for all $u$ s.t. $(v, u) \in E$ do
10. if $d(v) + \|v, u\| < D(u)$ then
11. \[ D(u) = d(v) + \|v, u\| ; \]
12. $p(u) = v$ ;
13. \[ \text{updateValue}(H, u, D(u)) ; \]
14. $v = \text{extractMin}(H)$ ;

assumed to have a predefined length (or edge-length or cost) $\|\varepsilon_{ij}\| = \|(v_i, v_j)\|$ such that $0 \leq \|\varepsilon_{ij}\| \leq \infty$ for all $\varepsilon_{ij} \in E$.

A path $P(v_i, v_j)$ is an ordered sequence of edges $\varepsilon_1, \ldots, \varepsilon_\ell$ such that $\varepsilon_1 = (v_i, v_1)$ and $\varepsilon_k = (v_{k-1}, v_k)$ for all $k \in \{2, \ldots, \ell - 1\}$ and $\varepsilon_\ell = (v_{\ell-1}, v_j)$ and where $\{\varepsilon_k \in P(v_i, v_j)\} \subset V$.

The shortest path $P^*(v_i, v_j)$ is the shortest possible path from $v_i$ to $v_j$. Formally,

$$P^*(v_i, v_j) \equiv \arg \min_{P(v_i, v_j)} \sum_{\varepsilon \in P(v_i, v_j)} \|\varepsilon\|$$

We are interested in paths from a particular “start-node” $s$ to nodes $v$, and define $d(v)$ to be the length of the shortest possible path from $s$ to $v$.

$$d(v) \equiv \min_{P(s, v)} \sum_{\varepsilon \in P(s, v)} \|\varepsilon\|$$

Dijkstra’s algorithm works by incrementally building a “shortest-path-tree” $S$ outward from $s$, one node at a time. Dijkstra’s algorithm appears in Algorithm 1. Each node that is not yet part of the growing $S$ refines a “best-guess” $D(v)$ of its actual shortest-path-length $d(v)$, with the restriction that $d(v) \leq D(v)$. Dijkstra’s algorithm guarantees/requires that $d(v) = D(v)$ for the node in $V \setminus S$ with minimum $D(v)$. In modern versions of the algorithm, a min-priority-heap $H$ (or related data structure) is used to keep track of $D(v)$ values.

The min-priority-heap $H$ is initialized to empty, best-guesses $D(v)$ are initialized to $\infty$, parent pointers $p(v)$ with respect to the shortest path tree $S$ are
### Algorithm 2: Dijkstra\( (G, s) \)

**Input:** A graph \( G = (E, V) \) of node set \( V \) and edge set \( E \), a start node \( s \in V \), and a monotone integer priority queue \( H \).

**Output:** Shortest path lengths \( d(v) \) and parent pointers \( p(v) \) with respect to the shortest path-tree \( S \) for all \( v \in V \).

1. \( H = \emptyset \);
2. for all \( v \in V \) do
   3. \( D(v) = \infty \);
   4. \( p(v) = \text{NULL} \);
   5. \( \delta = \infty \);
3. for all \( (v, u) \in E \) do
   4. if \( \| (v, u) \| < \delta \) then
      5. \( \delta = \| (v, u) \| \);
6. \( D(s) = 0 \);
7. \( v = s \);
8. while \( v \neq \text{NULL} \) do
   9. \( d(v) = D(v) \);
   10. for all \( u \) s.t. \( (v, u) \in E \) do
      11. if \( d(v) + \| (v, u) \| < D(u) \) then
          12. \( D(u) = d(v) + \| (v, u) \| \);
          13. \( \hat{D}(v) = \lfloor D(v) \rfloor \) \( \text{/* \lfloor \cdot \rfloor \text{ converts to integer with truncation} */ \) ;
          14. \( p(u) = v \);
          15. \( \text{updateValue}(H, u, \hat{D}(v)) \);
      16. \( v = \text{extractMin}(H) \);

initialized to NULL, and the start node \( s \) is given an actual distance of 0 from itself, lines 1-5, respectively.

Each iteration involves “processing” the node \( v \in V \setminus S \) with minimum \( D(v) \) lines 3-13 (where \( v \) is removed, for all practical purposes, from future consideration during the algorithm’s execution). Such a node \( v \) is extracted from the heap on line 11 (in the first iteration we know to use \( s \), line 5). Next, each neighbor \( u \) of \( v \) checks if \( d(v) + \| (v, u) \| < D(u) \), that is, if the distance from \( s \) through the shortest-path-tree to \( v \) plus the distance from \( v \) to \( u \) through edge \( (v, u) \in E \) is less than \( u \)'s current best-guess. If so, then \( u \) updates its best-guess and parent pointer to reflect the better path via \( v \), lines 10-13. All neighbors \( u \) of \( v \) perform the update \( D(u) = \min(D(u), d(v) + \| (v, u) \|) \). The heap is adjusted to account for changing \( D(u) \) on line 13.

Our modification that enables any integer based solution (i.e., a monotone integer priority queue) to be used for a floating point problem appears in Algo-
It relies on knowledge of $\delta$, the length of shortest edge in $E$.

$$\delta = \min_{\varepsilon \in E} \|\varepsilon\|$$

If $\delta$ is not known a priori then it can be obtained by scanning all edges in the graph in time $O(m)$, lines 3-5. We store two keys with each node instead of one. The actual floating point key $D(v)$ that reflects the current best guess of node $v$’s shortest path length, as well as a companion integer key $\hat{D}(v) = \lfloor D(v) / \delta \rfloor$, line 16, that is used to update the position of $v$ within the integer priority queue.

A number of the previous (integer-based) algorithms that are compatible with our method have runtime dependent on the maximum weight $C$ and $w$, the word size of the computer being used (currently 32 or 64 in most computers). The effects of $w$ and $C$ on runtime are buried in the heap operations, and dependent on the particular algorithm being used. If a value of $C$ is explicitly needed by a particular priority queue then it can be found by scanning edges at the beginning of the algorithm, similarly to $\delta$.

### 2.1 Problem Variants

#### SSSP

The single source shortest path planning problem is defined:

Given $G = (V, E)$ and a particular node $s \in V$, then for all $v \in V$, find the shortest path $P^*(s, v)$.

The problem is considered solved once we have produced a data structure containing both:

1. The shortest-path lengths $d(v)$ for all $v \in V$ from $s$.
2. The shortest path tree that can be used to extract the shortest path from $s$ to any $v$ (at least for any $v$ such that $d(v) < \infty$).

The latter can be accomplished by storing the parent of each node with respect to $S$, allowing each shortest path to be extracted by following back pointers in the fashion of gradient descent from $v$ to $s$ and then reversing the result.

The reverse (i.e., sink) search that involves finding all paths to $s$ (instead of from $s$) can be solved using basically the same algorithm except that the roles played by in- and out- neighbors are swapped and the extracted path is not reversed.

For the sake of clarity and brevity in the rest of our presentation, we now formally define abbreviations for a number of variations on SSSP. Variations are related to the range of allowable edge weights (e.g., whether or not zero-length edges allowed), and if edges are directed or undirected.

**N-SSSP** “Nonnegative SSSP” refers to the special case of SSSP such that edges have nonnegative weights, $0 \leq \|\varepsilon\|$ for all $\varepsilon \in E$.

**P-SSSP** “Positive SSSP” refers to the special case of SSSP such that edges have positive weights, $0 < \delta \leq \|\varepsilon\|$ for all $\varepsilon \in E$. 

BN-SSSP  “Bounded Nonnegative SSSP” refers to the special case of N-SSSP such that edge weights are bounded above, $0 \leq \|\varepsilon\| \leq C < \infty$ for all $\varepsilon \in E$.

BP-SSSP  “Bounded Positive SSSP” refers to the special case of P-SSSP such that edge weights are bounded above, $0 < \|\varepsilon\| \leq C < \infty$ for all $\varepsilon \in E$.

USSSP  The single source shortest path planning problem for undirected graphs is defined:

Given $G = (V, E)$ such that for all $(u, v) \in E$ there exists $(v, u) \in E$ such that $\| (u, v) \| = \| (v, u) \|$, and a particular node $s \in V$, then for all $v \in V$, find the shortest path $P^*(s, v)$.

The prefixes N, P, BN, and BP can also be used with USSSP, e.g., P-USSSP is “Positive USSSP.”

Floating point vs. integer  We explicitly differentiate between variants of problems that use either integer or floating point edge weights. For example, integer SSSP is the subset of SSSP that uses integer weights; floating point BP-SSSP is the subset of BP-SSSP that uses floating point weights; etc.

2.2 Orderings

Dijkstra’s original algorithm is provably correct (see [Dijkstra, 1959]), based on guarantees that the next node $v$ processed at any step has the following properties:

1. $v \in V \setminus S$.
2. Either $v = s$ or $v$ has some neighbor $u$ such that $u \in S$.
3. $D(v) \leq D(v')$, for all nodes $v' \in V \setminus S$.

As has often been remarked, it works by finding a full ordering on $d(v)$ for all $v \in V$. The priority heap data structure enforces (3) and determines this ordering.

However, a partial ordering based on $\delta$ is sufficient to solve P-SSSP; this is formally proven in Section 3 and has previously been shown by [Tsitsiklis, 1995] and, according to [Thorup, 1999], even earlier in Russian by [Dinic, 1978]. We make extensive use of the latter partial ordering in the current paper.

Let $v_1, \ldots, v_n$ denote the order in which Dijkstra’s algorithm processes nodes.

FO (Full Ordering)  A full ordering of nodes based on path-length level-sets is a sequence $v_1, \ldots, v_n$ such that:

$$i < j \Rightarrow d(v_i) \leq d(v_j).$$

in other words, the fact that the $j$-th node is extracted after the $i$-th node implies that the shortest path from the $j$-th node is at least as long as the one from the $i$-th node.
\(\delta\text{-PO (Partial Ordering Based on} \ \delta\text{)}\) A partial ordering of nodes based on \(\delta\) values is a sequence \(v_1, \ldots, v_n\) such that:

\[i < j \Rightarrow d(v_i) < d(v_j) + \delta\]

where \(\delta\) is the minimum edge weight in the graph. Note that \(\delta\text{-PO}\) is similar to FO except that nodes \(v\) and \(u\) can be swapped if \(d(v)\) and \(d(u)\) are within \(\delta\). As we will show shortly, \(\delta\text{-PO}\) is sufficient for solving P-SSSP.

### 2.3 Monotonicity

A *monotone priority queue* is, unfortunately, given a variety of definitions in the literature. According to [Raman, 1996] “A monotone priority queue has the property that the value of the minimum key stored in the priority queue is a non-decreasing function of time.” [Cherkassky et al., 1999] alternatively state that “In monotone priority queues the extracted keys form a monotone, non-decreasing sequence.”

While all heaps that meet the second definition necessarily meet the first; the reverse is not true, in general. The difference between these two definitions is particularly important for our work. The queues we use to create \(\delta\text{-PO}\) have the property that the minimum value stored in the priority queue is monotonically non-decreasing but do not, in general, extract nodes from the priority queue in a monotonically non-decreasing order. Moreover, they do not even guarantee that the node with the minimum value is extracted; but instead guarantee only that the node extracted from the heap has a key that is within \(\delta\) of the minimum. We will use the term “\(\delta\)-nonmonotonic” to describe the floating point heap we use/require, in order to highlight this important property.

We note that the class of \(\delta\)-nonmonotonic priority queues are a relaxed version of “monotone priority queues” (where we use quotation marks to denote that either the first or second definition may be used). Thus our special \(\delta\)-nonmonotonic priority queues can be assumed to require no more runtime or space than “monotone priority queues” and may even require less. The special (useful) case of \(\delta\)-nonmonotonicity should not be confused with standard nonmonotonicity. A nonmonotonic priority queue always extracts the minimum value and makes no assumptions on the order, partial or otherwise, that key values are added to and/or extracted from the queue (since less assumptions can be made on the values stored in the queue, “nonmonotonic priority queues” require more runtime and space than “monotone priority queues”; this is the opposite of our special case \(\delta\)-nonmonotonicity).

### 2.4 Computational Model

We assume the existence of an algorithmic solution to integer N-SSSP that creates a FO using a monotone integer priority queue. As such, we inherit whatever computational model is assumed by this underlying integer-based algorithm. In general, basic arithmetic and related operations on \(w\)-length words are assumed
to require $O(1)$ time; where $w = O(\log C)$ so that extra computations cannot be “hidden” by performing them in parallel on unnecessarily ample numerical representations. Previous methods with runtimes dependent on $C$ assume that $w \geq \log C$ (i.e., so that it is possible to efficiently represent integers and floating point numbers in memory), and thus $w = \Theta(\log C)$.

We additionally assume that the computational model supports floating point numbers and that floating-point-to-integer conversion (with truncation) is supported in $O(1)$ time. Similar to other floating point algorithms, we also assume cumulative errors due to fixed floating point precision are tolerated within the definition of “correct solution.” This is commonly handled by representing floating point numbers at twice the normal precision during mathematical operations within the CPU and then rounding to the nearest represented floating point number for RAM storage.

3 Sufficiency of $\delta$-PO Partial Ordering for Solving P-SSSP

We now prove that Dijkstra’s algorithm will correctly solve (integer and floating point) P-SSSP if nodes are processed according to $\delta$-PO, the partial ordering based on $\delta$ as defined in Section 2.2. We note that [Tsitsiklis, 1995] presents an alternative proof of the same idea.

Recall that “processing” $v_i$ is the act of removing $v_i$ from the heap and updating its neighbors $v \in \{v_j \mid (v_i, v_j) \in E\}$ with respect to any path-length decreases that can be achieved via edges from $v_i$.

**Lemma 1.** Assuming $\delta$-PO exists, and nodes are processed according to $\delta$-PO with Dijkstra’s algorithm, and $d(v_i) = D(v_i)$ when $v_i$ is processed for $i = 1, \ldots, j - 1$, then $d(v_j) = D(v_j)$.

**Proof.** (by contradiction) Assume instead $d(v_j) < D(v_j)$. Therefore, there must exist some $v_k$ and edge $\varepsilon_{kj}$ such that $d(v_k) + \|\varepsilon_{kj}\| = d(v_j) < D(v_j)$. This has two ramifications, first:

$$d(v_k) + \delta \leq d(v_j)$$

because $\delta \leq \|\varepsilon_{kj}\|$ by definition; and second:

$$j < k$$

(otherwise, if $k < j$, then we would have already set $D(v_j) = d(v_j) = d(v_k) + \|\varepsilon_{kj}\|$).

But by the definition of $\delta$-PO the fact that $j < k$ implies:

$$d(v_k) + \delta > d(v_j)$$

yielding the necessary contradiction. \qed

Now we are able to prove the correctness of Dijkstra’s algorithm using $\delta$-PO by showing that all nodes $v$ are processed from the $\delta$-PO heap only if $d(v) = D(v)$, i.e., the shortest path from $v$ to $s$ has already been computed.
Theorem 1. Assuming a $\delta$-PO exists, and nodes are processed according to $\delta$-PO, then Dijkstra’s algorithm correctly sets $d(v) = D(v)$ for all $v \in V$.

Proof. The proof is by induction on $j$. The inductive step relies on Lemma 1 for $j = n, \ldots, 2$. The base case for $j = 1$ is given by the fact that $v_1 = s$ and $d(s) = D(s) = 0$ before the algorithm starts. $\Box$

4 Any Heap that Constructs an FO for Integer N-SSSP is an Oracle for $\delta$-PO for Floating Point P-SSSP

We now show how any monotone integer heap $H$ that produces a FO when given integer weights for integer P-SSSP can be used as an oracle to create a $\delta$-nonmonotonic floating point heap that produces a $\delta$-PO for floating point P-SSSP. Recall that (for floating point SSSP) the floating point key for a node $v$ is given by $D(v)$. We define a companion “integer key” $\hat{D}(v) = \lfloor \frac{D(v)}{\delta} \rfloor$, and maintain the position $v$ within $H$ using $\hat{D}(v)$ instead of $D(v)$. In other words, whenever $D(v)$ is decreased we also recalculate $\hat{D}(v)$ and then update $v$ in $H$ using $\hat{D}(v)$.

By definition of the conversion with truncation operator $\lfloor \cdot \rfloor$ and the fact that $\delta > 0$ we are guaranteed that $D(v) - \delta \hat{D}(v) < \delta$ for all nodes at any point during Dijkstra’s execution. Assuming that $i < j$, then at the instant $v_j$ is processed there are two cases:

$$\hat{D}(v_j) - \hat{D}(v_i) = 0$$

and

$$\hat{D}(v_j) - \hat{D}(v_i) \geq 1.$$

In the first case, $\hat{D}(v_j) - \hat{D}(v_i) = 0$ implies that $|D(v_i) - D(v_j)| < \delta$ and so either $D(v_i) < D(v_j) + \delta$, in the case that $D(v_i) > D(v_j)$, or $D(v_i) \leq D(v_j)$ otherwise — and so $D(v_i) < D(v_j) + \delta$ trivially. In the second case $\hat{D}(v_j) - \hat{D}(v_i) \geq 1$ implies that $D(v_i) < D(v_j)$ and so $D(v_i) < D(v_j) + \delta$ trivially. This discussion yields the following theorem:

Theorem 2. Assuming a heap $H$ exists that creates a FO when used by Dijkstra’s algorithm on integer P-SSSP, then that heap can be used to create a $\delta$-PO partial ordering for floating point P-SSSP.

Proof. By induction, it is easy to see (from the above discussion) that nodes are processed such that $i < j \Rightarrow d(v_i) < d(v_j) + \delta$. Thus, nodes are processed in a $\delta$-PO. Combining this result with Theorem 1 completes the proof. $\Box$

In the worst case, all nodes always require a unique integer key. Thus, worst-case runtime for the resulting float P-SSSP algorithm is identical to that for the underlying integer N-SSSP algorithm with respect to $n$ and $m$. That said, it is easy to imagine adding additional machinery to some preexisting heaps so that all nodes with the same integer key $\hat{D}(v_i)$ are inserted into a single bucket, and the heap sorts buckets in place of individual nodes — an idea that may reduce
the expected running time in some case, although we do not pursue it further in the current paper.

A notable difference between using an integer solution as an oracle for floating point data vs. using that same integer solution on raw integer data is the fact that we calculate integer keys by dividing floating point values by $C$. Thus, when used in conjunction with an integer priority queue solution that normally has a runtime dependent on $C$ (e.g., bucketing based queues), the $C$ used in the raw integer N-SSSP solution must be replaced by $C/\delta$ in the resulting floating point P-SSSP algorithm. This leads to the following corollary:

**Corollary 1.** Assuming an integer P-SSSP solution is used as an oracle to solve a floating point P-SSSP, the latter will have similar runtime vs. the former with respect to $n$ and $m$, while $C$ will be replaced by $C/\delta$.

Integer N-SSSP trivially includes all of P-SSSP, since N-SSSP is a super-set of P-SSSP. This leads to two additional corollaries.

**Corollary 2.** Assuming a heap $H$ exists that creates a FO when used by Dijkstra’s algorithm on integer N-SSSP, then that heap can be used to create a $\delta$-PO partial ordering for floating point N-SSSP.

**Corollary 3.** Assuming an integer N-SSSP solution is used as an oracle to solve a floating point P-SSSP, the latter will have similar runtime vs. the former with respect to $n$ and $m$, while $C$ will be replaced by $C/\delta$.

## 5 Faster Runtimes For Floating Point P-SSSP

We now discuss new runtimes for floating point P-SSSP that are enabled by Corollary 1. The work by [Thorup, 2004] solves integer N-SSSP in time $O(m + n \log \log C)$, combining their result with corollary gives:

**Corollary 4.** Floating point P-SSSP can be solved in time $O(m + n \log \log \frac{C}{\delta})$.

**Corollary 5.** Floating point P-SSSP can be solved in expected time $O(m + n \log \log \frac{C}{\delta})$.

Similarly, [Raman, 1997] proves that a combination of [Ahuja et al., 1990] and [Cherkassky et al., 1999] can solve integer N-SSSP in time $O(m + n (\log C \log \log C)^{1/3})$ so:

**Corollary 6.** Floating point P-SSSP can be solved in time $O(m + n (\log \frac{C}{\delta} \log \log \frac{C}{\delta})^{1/3})$.

While these algorithms/corollaries cover a large chunk of the problem space defined by $n \times m \times C$, there are a variety of additional integer N-SSSP and P-SSSP algorithms that provide the best theoretical runtime for less frequently addressed regions of that space. We now briefly summarize these. Leveraging integer-based result from [Thorup, 1996]:

**Corollary 7.** Floating point P-SSSP can be solved in time $O(m + n \log \log m)$. 

and [Raman, 1997]:

**Corollary 8.** Floating point P-SSSP can be solved in time $O(n + m \log \log \frac{C}{\delta})$.

and [Cherkassky et al., 1996] for both:

**Corollary 9.** Floating point P-SSSP can be solved in time $O(m + n(B + \frac{C}{B\delta}))$ for user defined parameter $B < C + 1$

and:

**Corollary 10.** Floating point P-SSSP can be solved in time $O(m\Delta + n(\Delta + \frac{C}{\Delta\delta}))$, for user defined $\Delta$.

Corollary 10 is the final corollary regarding runtime presented in our current paper.

### 6 Correct Dijkstra’s on N-SSSP Requires a Full Ordering

It would be nice if our method could somehow be extended to handle the case $\delta = 0$ in addition to $\delta > 0$; however, we now prove this is impossible. We first show that $\delta$-PO is insufficient to correctly solve N-SSSP, and then proceed to prove that correctly solving N-SSSP requires a full ordering.

#### 6.1 Insufficiency of $\delta$-PO to Solve N-SSSP using Dijkstra’s Algorithm

We have defined $\delta$-PO such that it only applies to P-SSSP. Even if we attempt to accommodate N-SSSP by letting $\delta = 0$ then $\delta$-PO degenerates into $i < j \Rightarrow d(v_i) < d(v_j)$ where the latter inequality is strict. Thus, $\delta$-PO can only exist if $\|\varepsilon_{ij}\| > 0$ for all $\varepsilon_{ij} \in E$ — otherwise the existence of some $\|\varepsilon_{ij}\| = 0$ would guarantee that for some $i$ and $j$ it is the case that $i < j$ despite the fact that $d(v_i) = d(v_j)$, which would violate $\delta$-PO. This contradiction leads to the following lemma:

**Lemma 2.** $\delta$-PO is insufficient to solve N-SSSP.

#### 6.2 A Necessary Condition for Correct Dijkstra’s on N-SSSP

Lemma 2 suggests that allowing zero-length edges fundamentally changes the nature of the problem. However, it falls short of disproving that all best-path-length-based partial orderings (that are not also full orderings) are similarly doomed with respect to N-SSSP. In pursuit of the latter, we now prove a condition required for a correct Dijkstra’s algorithm and then define another ordering that is related to $\delta$-PO but slightly more general.

We begin by proving a necessary condition for any ordering that correctly solves N-SSSP. This particular condition is useful as an analytical tool simply because we can show that it is necessary (Theorem 3) and we can also show
that it implies a full ordering is required for floating point N-SSSP in the worst case (Theorem 4). In particular, the following condition guarantees that if the best path to \( v_c \) involves a zero-length sub-path immediately before reaching \( v_c \), and also there exists a worse path to \( v_c \) via some other node \( v_b \), then all of the nodes involved in the zero-length subpath must be processed before \( v_c \) in order to correctly solve N-SSSP.

**Condition A.** If for all \( P^*(s, v_c) \) it is the case that there exists some \( v_a \) and \( P^*(v_a, v_c) \subset P^*(s, v_c) \) such that \( \|P^*(v_a, v_c)\| = 0 \), then:

\[
\begin{aligned}
\text{if } & d(v_c) = d(v_a) < d(v_b) \\
\text{then } & i \leq c \text{ for all } i \text{ s.t. } v_i \in P^*(v_a, v_c) \quad (4)
\end{aligned}
\]

where we technically only require that \( P^*(v_a, v_c) \) is a zero-length subset of any optimal path to \( v_c \) such that \( v_a \neq v_c \) and \( d(v_c) = d(v_a) \). In other words, we can break ties between two or more optimal paths arbitrarily.

**Lemma 3.** Condition A is necessary to correctly solve N-SSSP using Dijkstra’s algorithm.

**Proof.** (by contradiction) Assume that \( c < i \). Then by construction it is possible to create a graph satisfying Condition A such that nodes are processed in an order such that \( b < c < i \), and consequently \( D(v_c) = \|P^*(s, v_b)\| + \|\varepsilon_{bc}\| \geq d(v_c) \) at the instant \( v_c \) is processed.

6.3 Another Ordering

Let AO be defined as a sequence \( v_1, \ldots, v_n \) such that both:

- Condition A is satisfied
- \( i < j \Rightarrow d(v_i) \leq d(v_j) + \epsilon \)

where \( \epsilon \) is a constant that we are allowed to choose per each problem instance.

The definition of AO allows \( d(v_i) = d(v_j) \) and so it can handle zero-length edges even when \( \epsilon = 0 \) (it handles them trivially when \( \epsilon > 0 \)).

6.4 For All \( \epsilon \geq 0 \), AO Induces a Full Ordering for N-SSSP in the Worst Case

We now prove that, for all \( \epsilon \geq 0 \), AO induces a full ordering for N-SSSP in the worst case. We begin by examining AO when \( \epsilon > 0 \).

Consider Figure 1, the dotted edges have zero length while the solid edges have length defined by their height span in the figure (width is used for illustrative purposes only). By construction, for all \( i \) and \( j \) such that \( 2 \leq i < n \) and \( 2 < j \leq n \) and \( i < j \) edge lengths are defined such that \( 0 < \|\varepsilon_{1i}\| < \|\varepsilon_{1j}\| \) and \( \|\varepsilon_{1j}\| - \|\varepsilon_{1i}\| < \epsilon \) and \( \|\varepsilon_{ij}\| = 0 \) Therefore, \( d(v_k) = d(v_2) \) for all \( k \) such that \( 1 < k \leq n \).
Lemma 4. Condition A induces a full ordering in the worst case for any $\epsilon > 0$

Proof. (by construction) For each $k$ such that $2 < k < n$ there exists $P^*(v_{k-1}, v_k) \subset P^*(s, v_k)$ such that $\|P^*(v_{k-1}, v_1)\| = 0$ and $d(v_k) = d(v_2) < d(v_n)$ and $\|P^*(v_n, v_k)\| = 0$ and $\|P^*(s, v_n)\| + \|\epsilon_{nk}\| = dv_n > d(v_k)$. Thus, Condition A requires that $k - 1 \leq k < n$. Trivially we know that $s = v_1$ and so $1 < k$, thus there is only one possible extraction order that will enable Dijkstra’s algorithm to yield a correct solution to this N-SSSP: $1, 2, \ldots, n$ — and this is a full ordering. Moreover, this can be accomplished for any $\epsilon > 0$.

We are now ready to consider $\epsilon \geq 0$.

Theorem 3. Assuming that Dijkstra’s algorithm processes nodes according to a partial ordering $i < j \Rightarrow d(v_i) \leq d(v_j) + \epsilon$, where $\epsilon \geq 0$, any partial ordering that yields a correct algorithm will induce a full ordering in the worst case.

Proof. There are two cases: $\epsilon > 0$ and $\epsilon = 0$. In the first case, we have $\epsilon > 0$ and Lemmas 3 and 4 guarantee that in the worst case we will create a full ordering. In the second case, we have $\epsilon = 0$ and so we recover the definition of a full ordering, i.e., $i < j \Rightarrow d(v_i) \leq d(v_j)$.

7 Contributions and Related Work

Our primary problem of interest is the floating point version of P-SSSP. The primary result of this paper is a general technique that allows any monotonic integer priority queue to be used to solve floating point P-SSSP, as well as understanding why this is possible and where the technique breaks down.
Our contributions regarding the theoretical runtime of Dijkstra’s algorithm can most easily be seen by comparing Tables 1 and 2. It is important to note that our new runtime results are only applicable to cases involving directed graphs with positive floating point weights, i.e., floating point P-SSSP.

For disconnected graphs it is possible that \( m < n \), and so we report runtimes in terms of both \( m \) and \( n \) (other authors have occasionally omitted terms involving only \( n \) because \( n = O(m) \) for connected graphs). A technical report that includes some of the ideas in this work appears in Otte, 2015; however, Otte, 2015 focuses on a few select applications of the general idea presented in the current paper.

7.1 Comparison Computational Model

Dijkstra’s algorithm for N-SSSP was originally presented in [Dijkstra, 1959] with a runtime of \( O(m + n^2) \) and assuming a comparison based computational model. [Williams, 1964] refined this to \( O(m \log(n) + n) \) using a more sophisticated heap. The Fibonacci heap by [Fredman and Tarjan, 1987] allows the (integer and floating point) N-SSSP to be solved in \( O(m + n \log n) \) — this is believed to be the fastest possible theoretical runtime for a comparison-based computational model.

7.2 RAM Computational Model

An alternative style of algorithm assumes a RAM computational model in order to use heaps based on some form of bucketing. [Dial, 1969] uses an approximate heap data structure to achieve runtime \( O(m + Cn) \) for integer BN-SSSP. [Ahuja et al., 1990] proves that a combination of a Radix-heap and a Fibonacci-heap will solve integer BN-SSSP in \( O(m + n \sqrt{\log C}) \). [Fredman and Willard, 1990] use AF-heaps to achieve \( O(m + n \log n \log \log n) \). Note that AF-heaps rely on Q-heaps, which assume that edge weights are integers. The ideas of [Dial, 1969] are extended by [Tsitsiklis, 1995] to floating point BP-SSSP in a RAM model with a runtime of \( O(n + m + C/\delta) \).

For integer BN-SSSP in the RAM model, [Cherkassky et al., 1996] uses a more complex variation of [Dial, 1969] to achieve time \( O(m + n(B + C/B)) \) and time \( O(m \Delta + n(\Delta + C/\Delta)) \), for user defined parameters \( B < C + 1 \) and \( \Delta \). [Thorup, 1996] achieves \( O(n + m \log \log m) \). [Raman, 1997] shows that combining results from [Ahuja et al., 1990] with those from [Cherkassky et al., 1996] yields a \( O(m + n(\log C \log \log C)^{1/3}) \) solution to integer BN-SSSP, and that [Thorup, 1996] can be extended to achieve \( O(n + m \log \log C) \). A later solution by [Cherkassky et al., 1999] achieves \( O(m + n \log C) \). Although this is slightly worse than the result by [Raman, 1997], it is of interest because the hot-queue data structure it uses can be modified to solve floating point BP-SSSP in \( O(m + n \log \frac{C}{\log \log C}) \) by incorporating the earlier ideas from [Tsitsiklis, 1995].

[Thorup, 2004] presents an integer priority queue that has constant time for the decrease key operation and is thus able to achieve a runtimes of \( O(m + \)
Using the trick of interpreting IEEE floating point numbers as integers enables the algorithm to also solve floating point N-SSSP in time $O(m + n \log \log n)$. However, as the author of [Thorup, 2004] notes, “for floating point numbers we [they] do not get bounds in terms of the maximal weight” (i.e., $C$). We note that, as proved in Corollary 4, our method can be combined with the integer result of $O(m + n \log \log n)$ from [Thorup, 2004] to yield a new algorithm with a runtime of $O(m + n \log \log C/\delta)$ for floating point BP-SSSP. This highlights one way our work differs from previous work. A handful of other improvements that are newly obtained by combining our work with previous integer results appear in Table 2.

### 7.3 Expected Times in RAM Model

With regard to expected running times over randomizations (of edge lengths) and assuming RAM and integer N-SSSP, [Fredman and Willard, 1993] gives expected time $O(m \sqrt{\log n} + n)$. More recent results by [Thorup, 1996] yield the expected time of $O(m + n (\log n)^{1/2 + \epsilon})$, while [Raman, 1996] and [Raman, 1997] give expected times $O(m + n (\log n \log \log n)^{1/2})$ and $O(m + n (\log C)^{1/4 + \epsilon})$ and $O(m + n (\log n)^{1/3 + \epsilon})$. [Cherkassky et al., 1999] achieves the slightly worse expected time $O(m + n (\log C)^{1/3 + \epsilon})$; however, we note that this hot-queues-based solution can be combined with [Tsitsiklis, 1995] to solve floating point N-SSSP in expected time $O(m + n (\log C)^{1/3 + \epsilon})$ so we include this in Table 1. [Thorup, 2000a] proves a general reduction from N-SSSP to sorting that leverages the randomized sorting algorithm in [Han and Thorup, 2002] to produce an integer N-SSSP algorithm that runs in expected time $m \sqrt{\log n}$ (this can be extended to floating point P-SSSP using the trick of interpreting IEEE floats as integers).

The more recent work of [Thorup, 2004] provides solutions in worst case time (and therefore also expected time) $O(m + n \log n)$ and $O(m + n \log \log C)$ for integer N-SSSP and BN-SSSP, respectively. As noted above [Thorup, 2004] can be applied directly to floating point N-SSSP (by assuming IEEE standard floats are interpreted as integers) with runtime $O(m + n \log n)$, but analogous floating point results in terms of $C$ could not be achieved. On the other hand, our results from the current paper can be combined with [Thorup, 2004] to yield a new algorithm with a runtime of $O(m + n \log \log C/\delta)$ for floating point BP-SSSP.

### 7.4 Undirected Graphs (BN-USSSP)

For the case of undirected edges with bounded positive integer weights (integer BN-USSSP) [Thorup, 1999] presents a method that runs in $O(m + n)$, and then extends this to floating point BN-USSSP in [Thorup, 2000a]. In [Thorup, 2000a] it is noted that an alternative solution with significantly less overhead is also possible in $O(\log(\delta) + \alpha(m,n,m+n))$, where $\alpha(m,n)$ is the inverse Ackermann function using $m$ and $n$. [Wei and Tanaka, 2013] [Wei and Tanaka, 2014] present
modifications to [Thorup, 2000a] that achieve better practical performance by removing the necessity of an “unvisited node structure” but do not change theoretical runtime bounds (or the constant factor).

7.5 Other Special Cases

The special case of (integer or floating point) P-SSSP involving \( z \) distinct weights can be solved in time \( O(m + n) \) if \( zn \leq 2m \) and \( O(m \log \frac{m}{m} + n) \), otherwise, using [Orlin et al., 2010]. Integer or floating point N-SSSP on planar graphs can be solved in \( O(n \sqrt{n}) \) with a method by [Federickson, 1987]. Note that \( m = O(n) \) for planar graphs.

7.6 Partial Orderings

The main conceptual difference between [Thorup, 1999, Thorup, 2000a] and our work is in the details of the partial orderings that are used. These differences cause both (A) theoretical ramifications regarding the subsets of SSSP for which a particular is applicable, and (B) practical differences affecting ease of implementation and performance. The particular partial ordering we investigate \( \delta \)-PO is grounded in the notion of shortest-path-length. Our method is designed for floating point P-SSSP, while [Thorup, 1999, Thorup, 2000a] is for integer and floating point BN-USSSP. The floating point method in [Thorup, 2000a] uses the integer algorithm in [Thorup, 1999] as an oracle; our work extends the basic oracle concept to a greater subset of SSSP using a completely different technique.

The observation that P-SSSP can be solved using \( \delta \)-PO was first observed by [Dinic, 1978], who used the trick of dividing by minimum edge weight to improve the use of bucket based integer priority queues for solving integer SSSP. [Tsitsiklis, 1995] proves the result for floating point P-SSSP and uses it in a variation of [Dial, 1969] to solve floating point BP-SSSP; although, without explicitly considering the runtime of the resulting algorithm with respect to \( C \) or \( \delta \). Our work can be also be considered an application of [Dinic, 1978] and a generalization of [Tsitsiklis, 1995]. We show that any monotone integer priority queue that creates a PO for integer N-SSSP (or P-SSSP) can be used as an oracle to create a \( \delta \)-PO for floating point P-SSSP, and thereby solve floating point P-SSSP. Finally, we prove that \( \delta \)-PO is insufficient to solve floating point N-SSSP and that any related ordering that correctly solves floating point P-SSSP must create a FO in the worst-case.

8 Remarks

We now discuss a few points that we find interesting.

8.1 Integer N-SSSP vs. Floating Point N-SSSP

Even though we have shown that heaps designed to solve integer N-SSSP can immediately be used to solve floating point P-SSSP in general, we fall short of
achieving a similarly general oracle method that can solve floating point N-SSSP. The reason for this discrepancy is intimately related to zero-length edges and enforcement of Condition A. It is impossible to make the granularity of the partial ordering small enough (i.e., without creating full ordering) to guarantee that processing nodes based on \( \hat{D}(v) = D(v)/\delta \) instead of \( D(v) \) yields a correct algorithm. Indeed, this result extends to any method relying on \( \hat{D}(v) = D(v)/x \), regardless of how \( x \) is chosen. There will always be problems when \( x = 0 \).

More interesting is the result from Lemma 3 which proves that Condition A is necessary for any ordering that correctly solves N-SSSP. Note that this does not conflict with previous results for integer N-SSSP case — it is trivial to show that full orderings guarantee Condition A, and all state-of-the-art integer N-SSSP algorithms based on shortest-path-length partial orders use full orderings. Indeed, the main ramification of Lemma 3 appears to be that it is impossible for any version of Dijkstra’s algorithm to solve general-case floating point N-SSSP by using a (non-full) partial ordering based on shortest-path length. Thus, future work on the general floating point N-SSSP problem must either: (1) use an alternative method that in which orderings consider more than just best-path-length (as [Thorup, 1999] has done for the USSSP case), or (2) attempt to create faster full-order inducing heaps that use floating point keys directly.

8.2 Floats, Ints and the IEEE Standard

It has often been remarked, e.g., by [Thorup, 2000a], that given the IEEE standards for floating point and integer representations it is often possible to use floating point numbers directly in some integer heaps by reinterpreting their binary values directly as integers. This is obviously true for comparison based algorithms, but is even true for some of the bucket-based structures such as Radix heaps.

In contrast, the idea presented in the current paper is representation agnostic. This is theoretically important because there are a variety of ways to represent real numbers for computational purposes other than IEEE standard, for example, unnormalized versions of binary and decimal floating point numbers and more exotic alternatives such as: continued fractions [Vuillemin, 1990], logarithmic [Swartzlander and Alexopoulos, 1975], semi-logarithmic [Muller et al., 1998], level-index [Clenshaw and Olver, 1984], floating-slash [Matula and Kornerup, 1985], and others [Vuillemin, 1994, Muller et al., 2009]. Our results hold for all binary representations of positive real numbers such that an \( O(1) \) integer conversion operation exists, and an integer-based heap exists that can solve N-SSSP or P-SSSP for the resulting integer representation.

9 Summary and Conclusions

We provide a new proof that using Dijkstra’s algorithm directly with partial ordering based on shortest-edge-length and best-path-length is sufficient to solve
the floating point positive edge weight case of the single source shortest path planning problem (SSSP) — we note this result has been previously obtained by [Dinic, 1978] and [Tsitsiklis, 1995] using alternative analysis.

Based on this result, we present a simple yet general method that enables a large class of results from the nonnegative (and positive) integer case of SSSP to be extended to the positive floating point case of SSSP. In particular, any integer priority queue that solves integer SSSP by creating a full ordering (by extracting nodes in a monotonically nondecreasing sequence of best-path-length key values) can be used as an oracle to solve the floating point SSSP with positive edge weights (by creating a sufficient partial ordering). This immediately yields a handful of faster theoretical runtimes for the positive floating point case of SSSP — both worst case and expected times and for various relationships between \( n, m, \) and \( C \). The idea is easy to implement in practice, i.e., in conjunction with many practical heaps that solve integer SSSP. Moreover, it guarantees that many future advances for the integer case, both theoretical and practical, will immediately yield better results for the positive floating point case of SSSP.

Finally, we prove that Dijkstra’s Algorithm must use a full ordering to solve the nonnegative floating point of SSSP in the worst case (and thus our results cannot be extended to nonnegative floating point SSSP). The latter is due to complications inherent with using zero-length edges, and suggests that future work for the floating point case of SSSP must either involve a departure from Dijkstra’s Algorithm or the discovery of heaps that are able to create full orderings over floating point numbers more quickly.

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