Analytic factorization of Lie group representations

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Abstract

For every moderate growth representation \((\pi, E)\) of a real Lie group \(G\) on a Fréchet space, we prove a factorization theorem of Dixmier–Malliavin type for the space of analytic vectors \(E^\omega\). There exists a natural algebra of superexponentially decreasing analytic functions \(\mathcal{A}(G)\), such that \(E^\omega = \Pi(\mathcal{A}(G)) E^\omega\). As a corollary we obtain that \(E^\omega\) coincides with the space of analytic vectors for the Laplace–Beltrami operator on \(G\).

1 Introduction

Consider a category \(\mathcal{C}\) of modules over a nonunital algebra \(\mathcal{A}\). We say that \(\mathcal{C}\) has the factorization property if for all \(\mathcal{M} \in \mathcal{C}\),

\[
\mathcal{M} = \mathcal{A} \cdot \mathcal{M} := \text{span} \{ a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M} \}.
\]

In particular, if \(\mathcal{A} \in \mathcal{C}\) this implies \(\mathcal{A} = \mathcal{A} \cdot \mathcal{A}\).

Let \((\pi, E)\) be a representation of a real Lie group \(G\) on a Fréchet space \(E\). Then the corresponding space of smooth vectors \(E^\infty\) is again a Fréchet space. The representation \((\pi, E)\) induces a continuous action \(\Pi\) of the algebra \(C^\infty_c(G)\) of test functions on \(E\) given by

\[
\Pi(f)v = \int_G f(g)\pi(g)v \, dg \quad (f \in C^\infty_c(G), v \in E),
\]

which restricts to a continuous action on \(E^\infty\). Hence the smooth vectors associated to such representations are a \(C^\infty_c(G)\)–module, and a result by Dixmier and Malliavin states that this category has the factorization property.

In this article we prove an analogous result for the category of analytic vectors.

For simplicity, we outline our approach for a Banach representation \((\pi, E)\). In this case, the space \(E^\omega\) of analytic vectors is endowed with a natural inductive limit topology, and gives rise to a representation \((\pi, E^\omega)\). To define an appropriate algebra acting on \(E^\omega\), we fix a left–invariant Riemannian metric on \(G\) and let \(d\) be the associated distance function. The continuous functions \(\mathcal{R}(G)\) on \(G\) which decay faster than \(e^{-nd(g,1)}\) for all \(n \in \mathbb{N}\) form a \(G \times G\)–module under the left–right regular representation. We define \(\mathcal{A}(G)\) to be the space of analytic vectors of this action. Both \(\mathcal{R}(G)\) and \(\mathcal{A}(G)\) form an algebra under convolution, and the action \(\Pi\) of \(C^\infty_c(G)\) extends to give \(E^\omega\) the structure of an \(\mathcal{A}(G)\)–module.
In this setting, our main theorem says that the category of analytic vectors for Banach representations of $G$ has the factorization property. More generally, we obtain a result for $F$-representations:

**Theorem 1.1.** Let $G$ be a real Lie group and $(\pi, E)$ an $F$-representation of $G$. Then

$$\mathcal{A}(G) = \mathcal{A}(G) \ast \mathcal{A}(G)$$

and

$$E^\omega = \Pi(\mathcal{A}(G)) E^\omega = \Pi(\mathcal{A}(G)) E.$$ 

Let us remark that the special case of bounded Banach representations of $(\mathbb{R}, +)$ has been proved by one of the authors in [7].

As a corollary of Theorem 1.1 we obtain that a vector is analytic if and only if it is analytic for the Laplace–Beltrami operator, which generalizes a result of Goodman [5] for unitary representations. In particular, the theorem extends Nelson’s result that $\Pi(\mathcal{A}(G)) E^\omega$ is dense in $E^\omega$ [8]. Gårding had obtained an analogous theorem for the smooth vectors [4]. However, while Nelson’s proof is based on approximate units constructed from the fundamental solution $g_t \in \mathcal{A}(G)$ of the heat equation on $G$ by letting $t \to 0^+$, our strategy relies on some more sophisticated functions of the Laplacian.

To prove Theorem 1.1 we first consider the case $G = (\mathbb{R}, +)$. Here the proof is based on the key identity

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1,$$

for the entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z} \text{erf}(z)$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$ on the complex plane $\mathbb{C}$. We consider this as an identity for the symbols of the Fourier multiplication operators $\alpha_\varepsilon(i\partial)$, $\beta_\varepsilon(i\partial)$ and $\cosh(i\varepsilon \partial)$. The functions $\alpha_\varepsilon$ and $\beta_\varepsilon$ are easily seen to belong to the Fourier image of $\mathcal{A}(\mathbb{R})$, so that $\alpha_\varepsilon(i\partial)$ and $\beta_\varepsilon(i\partial)$ are given by convolution with some $\kappa_\alpha^{\varepsilon}, \kappa_\beta^{\varepsilon} \in \mathcal{A}(\mathbb{R})$. For every $v \in E^\omega$ and sufficiently small $\varepsilon > 0$, we may also apply $\cosh(i\varepsilon \partial)$ to the orbit map $\gamma_v(g) = \pi(g)v$ and conclude that

$$(\cosh(i\varepsilon \partial) \gamma_v)^* \kappa_\alpha^{\varepsilon} + \gamma_v * \kappa_\beta^{\varepsilon} = \gamma_v.$$ 

The theorem follows by evaluating in $0$.

Unlike in the work of Dixmier and Malliavin, the rigid nature of analytic functions requires a global geometric approach in the general case. The idea is to refine the functional calculus of Cheeger, Gromov and Taylor [2] for the Laplace-Beltrami operator in the special case of a Lie group. Using this tool, the general proof then closely mirrors the argument for $(\mathbb{R}, +)$.

The article concludes by showing in Section 6 how our strategy may be adapted to solve some related factorization problems.

\[1\] Some basic properties of these functions and the Gaussian error function erf are collected in the appendix.
2 Basic Notions of Representations

For a Hausdorff, locally convex and sequentially complete topological vector space $E$ we denote by $GL(E)$ the associated group of isomorphisms. Let $G$ be a Lie group. By a representation $(\pi, E)$ of $G$ we understand a group homomorphism $\pi : G \to GL(E)$ such that the resulting action

$$G \times E \to E, \quad (g, v) \mapsto \pi(g)v,$$

is continuous. For a vector $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v,$$

the corresponding continuous orbit map.

If $E$ is a Banach space, then $(\pi, E)$ is called a Banach representation.

Remark 2.1. Let $(\pi, E)$ be a Banach representation. The uniform boundedness principle implies that the function

$$w_\pi : G \to \mathbb{R}_+, \quad g \mapsto \|\pi(g)\|,$$

is a weight, i.e. a locally bounded submultiplicative positive function on $G$.

A representation $(\pi, E)$ is called an $F$-representation if

- $E$ is a Fréchet space.
- There exists a countable family of seminorms $(p_n)_{n \in \mathbb{N}}$ which define the topology of $E$ such that for every $n \in \mathbb{N}$ the action $G \times (E, p_n) \to (E, p_n)$ is continuous. Here $(E, p_n)$ stands for the vector space $E$ endowed with the topology induced from $p_n$.

Remark 2.2. (a) Every Banach representation is an $F$-representation.

(b) Let $(\pi, E)$ be a Banach representation and $\{X_n : n \in \mathbb{N}\}$ a basis of the universal enveloping algebra $U_g$ of the Lie algebra of $G$. Define a topology on the space of smooth vectors $E^\infty$ by the seminorms $p_n(v) = \|d\pi(X_n)v\|$. Then the representation $(\pi, E^\infty)$ induced by $\pi$ on this subspace is an $F$-representation (cf. [1]).

(c) Endow $E = C(G)$ with the topology of compact convergence. Then $E$ is a Fréchet space and $G$ acts continuously on $E$ via right displacements in the argument. The corresponding representation $(\pi, E)$, however, is not an $F$-representation.

2.1 Analytic vectors

If $M$ is a complex manifold and $E$ is a topological vector space, then we denote by $O(M, E)$ the space of $E$-valued holomorphic maps. We remark that $O(M, E)$ is a topological vector space with regard to the compact-open topology.

Let us denote by $g$ the Lie algebra of $G$ and by $g_C$ its complexification. We assume that $G \subset G_C$ where $G_C$ is a Lie group with Lie algebra $g_C$. Let us stress
that this assumption is superfluous but simplifies notation and exposition. We
denote by $U_C$ the set of open neighborhoods of $1 \in G_C$.

If $(\pi, E)$ is a representation, then we call a vector $v \in E$ analytic if the orbit
map $\gamma_v : G \to E$ extends to a holomorphic map to some $GU$ for $U \in U_C$. The
space of all analytic vectors is denoted by $E^{\omega}$. We note the natural embedding

$$E^{\omega} \to \lim_{U \to \{1\}} \mathcal{O}(GU, E), \quad v \mapsto \gamma_v,$$

and topologize $E^{\omega}$ accordingly.

### 3 Algebras of superexponentially decaying functions

We wish to exhibit natural algebras of functions acting on $F$-representations.
For that let us fix a left invariant Riemannian metric $g$ on $G$. The corresponding
Riemannian measure $dg$ is a left invariant Haar measure on $G$. We denote by
distance function associated to $g$ (i.e. the infimum of the lengths of
all paths connecting $g$ and $h$) and set

$$d(g) := d(g, 1) \quad (g \in G).$$

Here are two key properties of $d(g)$, see [4]:

**Lemma 3.1.** If $w : G \to \mathbb{R}_+$ is locally bounded and submultiplicative (i.e.
$w(gh) \leq w(g)w(h)$), then there exist $c, C > 0$ such that

$$w(g) \leq C e^{cd(g)} \quad (g \in G).$$

**Lemma 3.2.** There exists $c > 0$ such that for all $C > c$,

$$\int e^{-Cd(g)} \, dg < \infty.$$

We introduce the space of superexponentially decaying continuous functions
on $G$ by

$$\mathcal{R}(G) := \left\{ \varphi \in C(G) \mid \forall n \in \mathbb{N} : \sup_{g \in G} |\varphi(g)| \, e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{R}(G)$ is a Fréchet space which is independent of the particular
choice of the metric $g$. A simple computation shows that $\mathcal{R}(G)$ becomes a
Fréchet algebra under convolution

$$\varphi * \psi(g) = \int_G \varphi(x) \, \psi(x^{-1}g) \, dx \quad (\varphi, \psi \in \mathcal{R}(G), g \in G).$$

We remark that the left-right regular representation $L \otimes R$ of $G \times G$ on $\mathcal{R}(G)$
is an $F$-representation.

If $(\pi, E)$ is an $F$-representation, then Lemma 3.1 and Remark 2.1 imply that

$$\Pi(\varphi)v := \int_G \varphi(g) \, \pi(g)v \, dg \quad (\varphi \in \mathcal{R}(G), v \in E)$$
defines an absolutely convergent integral. Hence the prescription

$$\mathcal{R}(G) \times E \to E, \quad (\varphi, v) \mapsto \Pi(\varphi)v,$$
defines a continuous algebra action of $R(G)$ (here continuous refers to the continuity of the bilinear map $R(G) \times E \to E$).

Our concern is now with the analytic vectors of $(L \otimes R, R(G))$. We set $A(G) := R(G)^\omega$ and record that

$$A(G) := \lim_{U \to \{1\}} R(G)_U,$$

where

$$R(G)_U = \left\{ \varphi \in O(UGU) \mid \forall Q \subseteq U \forall n \in \mathbb{N} : \sup_{q \in G} \sup_{q_1, q_2 \in Q} |\varphi(q_1 g q_2)| e^{\text{nd}(g)} < \infty \right\}.$$ 

It is clear that $A(G)$ is a subalgebra of $R(G)$ and that

$$\Pi(A(G)) E \subset E^\omega$$

whenever $(\pi, E)$ is an $F$-representation.

4 Some geometric analysis on Lie groups

Let us denote by $V(G)$ the space of left-invariant vector fields on $G$. It is common to identify $g$ with $V(G)$ where $X \in g$ corresponds to the vector field $\tilde{X}$ given by

$$(\tilde{X} f)(g) = \frac{d}{dt} \bigg|_{t=0} f(g \exp(tX)) \quad (g \in G, f \in C^\infty(G)).$$

We note that the adjoint of $\tilde{X}$ on the Hilbert space $L^2(G)$ is given by

$$\tilde{X}^* = -\tilde{X} - \text{tr}(\text{ad} X).$$

Note that $\tilde{X}^* = -\tilde{X}$ in case $g$ is unimodular. Let us fix an orthonormal basis $X_1, \ldots, X_n$ of $g$ with respect to $g$. Then the Laplace–Beltrami operator $\Delta = d^*d$ associated to $g$ is given explicitly by

$$\Delta = \sum_{j=1}^n (\tilde{X}_j - \text{tr}(\text{ad} X_j)) \tilde{X}_j.$$

As $(G, g)$ is complete, $\Delta$ is essentially selfadjoint. We denote by

$$\sqrt{\Delta} = \int \lambda \ dP(\lambda)$$

the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define

$$f(\sqrt{\Delta}) = \int f(\lambda) \ dP(\lambda)$$

as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain

$$D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid \int |f(\lambda)|^2 \ d(P(\lambda) \varphi, \varphi) < \infty \right\}.$$
Let $c, \vartheta > 0$. We are going to apply the above calculus to functions in the space

$$
F_{c, \vartheta} = \left\{ \varphi \in \mathcal{O}(C) \mid \forall N \in \mathbb{N} : \sup_{z \in W_{N, \vartheta}} |\varphi(z)| e^{c|z|} < \infty \right\},
$$

$$
W_{N, \vartheta} = \{ z \in \mathbb{C} \mid |\text{Im} z| < N \} \cup \{ z \in \mathbb{C} \mid |\text{Im} z| < \vartheta |\text{Re} z| \}.
$$

The resulting operators are bounded on $L^2(G)$ and given by a symmetric and left invariant integral kernel $K_f \in C^\infty(G \times G)$. Hence there exists a convolution kernel $\kappa_f \in C^\infty(G)$ with $\kappa_f(x) = \kappa_f(x^{-1})$ such that $K_f(x, y) = \kappa_f(x^{-1}y)$, and for all $x \in G$:

$$
f(\sqrt{\Delta}) \varphi = \int_G K_f(x, y) \varphi(y) dy = \int_G \kappa_f(y^{-1}x) \varphi(y) dy = (\varphi * \kappa_f)(x).
$$

A theorem by Cheeger, Gromov and Taylor [2] describes the global behavior:

**Theorem 4.1.** Let $c, \vartheta > 0$ and $f \in F_{c, \vartheta}$ even. Then $\kappa_f \in \mathcal{R}(G)$.

We are going to need an analytic variant of their result.

**Theorem 4.2.** Under the assumptions of the previous theorem: $\kappa_f \in \mathcal{A}(G)$.

**Proof.** We only have to establish local regularity, as the decay at infinity is already contained in [2].

The Fourier inversion formula allows to express $\kappa_f$ as an integral of the wave kernel:

$$
\kappa_f(\cdot) = K_f(\cdot, 1) = f(\sqrt{\Delta}) \delta_1 = \int_\mathbb{R} \hat{f}(\lambda) \cos(\lambda \sqrt{\Delta}) \delta_1 d\lambda.
$$

As we would like to employ $\|\cos(\lambda \sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$, we cut off a fundamental solution of $\Delta^k$ to write

$$
\delta_1 = \Delta^k \varphi + \psi
$$

for a fixed $k > \frac{1}{4} \dim(G)$ and some compactly supported $\varphi, \psi \in L^2$. Hence,

$$
\Delta^l \kappa_f(\cdot) = \int_\mathbb{R} \hat{f}^{(2k+2l)}(\lambda) \cos(\lambda \sqrt{\Delta}) \varphi d\lambda + \int_\mathbb{R} \hat{f}^{(2l)}(\lambda) \cos(\lambda \sqrt{\Delta}) \psi d\lambda.
$$

In the appendix we show the following inequality for all $n \in \mathbb{N}$ and some constants $C_n, R > 0$

$$
|\hat{f}^{(l)}(\lambda)| \leq C_n l! R^l e^{-n|\lambda|}.
$$

Using $\|\cos(\lambda \sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$ and the Sobolev inequality, we obtain

$$
|\Delta^l \kappa_f(\cdot)| \leq C_1 (2l)! S^{2l}
$$

for some $S > 0$. A classical result by Goodman [10] now implies the right analyticity of $\kappa_f$, while left analyticity follows from $\kappa_f(x) = \kappa_f(x^{-1})$. Browder’s theorem (Theorem 3.3.3 in [6]) then implies joint analyticity.  

\[\square\]
4.1 Regularized distance function

In the last part of this section we are going to discuss a holomorphic regularization of the distance function. Later on this will be used to construct certain holomorphic replacements for cut-off functions.

Consider the time–1 heat kernel $\kappa := \kappa_{e^{-\lambda^2}}$ and define $\tilde{d}$ on $G$ by

$$ \tilde{d}(g) := e^{-\Delta}d(g) = \int_G \kappa(x^{-1}g) \, d(x) \, dx. $$

**Lemma 4.3.** There exist $U \in \mathcal{U}_C$ and a constant $C_U > 0$ such that $\tilde{d} \in \mathcal{O}(GU)$ and for all $g \in G$ and all $u \in U$

$$ |\tilde{d}(gu) - d(g)| \leq C_U. $$

**Proof.** According to Theorem 4.2 the heat kernel $\kappa$ admits an analytic continuation to a superexponentially decreasing function on $GU$ for some bounded $U \in \mathcal{U}_C$. This allows to extend $\tilde{d}$ to $GU$. To prove the inequality, we consider the integral

$$ \tilde{\kappa}(y) = \int_G \kappa(x^{-1}y) \, dx $$

as a holomorphic function of $y \in GU$. By the left invariance of the Haar measure and the normalization of the heat kernel, $\tilde{\kappa} = 1$ on $G$, and hence on $GU$. Recall the triangle inequality on $G$: $|d(x) - d(g)| \leq d(x^{-1}g)$. This implies the uniform bound

$$ |\tilde{d}(gu) - d(g)| = \left| \int_G \kappa(x^{-1}gu) \, (d(x) - d(g)) \, dx \right| 
\leq \int_G |\kappa(x^{-1}gu)| \, d(x^{-1}g) \, dx 
\leq \sup_{v \in U} \int_G |\kappa(x^{-1}v)| \, d(x^{-1}) \, dx. $$

5 Proof of the Factorization Theorem

Let $(\pi, E)$ be a representation of $G$ on a sequentially complete locally convex Hausdorff space and consider the Laplacian as an element

$$ \Delta = \sum_{j=1}^{n} (-X_j - \text{tr}(\text{ad} X_j)) \, X_j $$

of the universal enveloping algebra of $g$. A vector $v \in E$ will be called $\Delta$-analytic, if there exists $\varepsilon > 0$ such that for all continuous seminorms $p$ on $E$ one has

$$ \sum_{j=0}^{\infty} \frac{\varepsilon^j}{(2j)!} p(\Delta^j v) < \infty. $$
Lemma 5.1. Let $E$ be a sequentially complete locally convex Hausdorff space and $\varphi \in O(U, E)$ for some $U \in \mathcal{U}_C$. Then there exists $R = R(U) > 0$ such that for all continuous semi-norms $p$ on $E$ there exists a constant $C_p$ such that

$$p\left(\left(\overline{X}_{i_1} \cdots \overline{X}_{i_k} \varphi\right)(1)\right) \leq C_p \, k! \, R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$.

Proof. There exists a small neighborhood of 0 in $g$ in which the mapping

$$\Phi : g \to E, \ X \mapsto \varphi(\exp(X)),$$

is analytic. Let $X = t_1 X_1 + \cdots + t_n X_n$. Because $E$ is sequentially complete, $\Phi$ can be written for small $X$ and $t$ as

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = k} \left(\overline{X}_{\alpha_1} \cdots \overline{X}_{\alpha_k} \varphi\right)(1) \, t^\alpha.$$  

As this series is absolutely summable, there exists a $R > 0$ such that for every continuous semi-norm $p$ on $E$ there is a constant $C_p$ with

$$p\left(\left(\overline{X}_{i_1} \cdots \overline{X}_{i_k} \varphi\right)(1)\right) \leq C_p \, k! \, R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$.

As a consequence we obtain:

Lemma 5.2. Let $(\pi, E)$ be a representation of $G$ on some sequentially complete locally convex Hausdorff space $E$. Then analytic vectors are $\Delta$-analytic.

In Corollary 5.6 we will see that the converse holds for $F$-representations.

Let $(\pi, E)$ be an $F$-representation of $G$. Then for each $n \in \mathbb{N}$ there exists $c_n, \ell_n > 0$ such that

$$\|\pi(g)\|_n \leq \ell_n \cdot e^{c_n d(g)} \quad (g \in G),$$

where

$$\|\pi(g)\|_n := \sup_{p_n(v) \leq 1 \atop v \in E} p_n(\pi(g)v).$$

For $U \in \mathcal{U}_C$ and $n \in \mathbb{N}$ we set

$$\mathcal{F}_{U,n} = \left\{ \varphi \in O(GU, E) \mid \forall Q \subset U \forall \varepsilon > 0 : \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq)) \, e^{-\left(c_n + \varepsilon\right) d(g)} < \infty \right\}.$$  

We are also going to need the subspace of superexponentially decaying functions in $\bigcap_n \mathcal{F}_{U,n}$:

$$\mathcal{R}(GU, E) = \left\{ \varphi \in O(GU, E) \mid \forall Q \subset U \forall n, N \in \mathbb{N} : \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq)) \, e^{N d(g)} < \infty \right\}.$$  

We record:
Lemma 5.3. If $\kappa \in \mathcal{A}(G)_V$, then right convolution with $\kappa$ is a bounded operator from $\mathcal{F}_U,n$ to $\mathcal{F}_V,n$ for all $n \in \mathbb{N}$.

We denote by $C_\varepsilon$ the power series expansion $\sum_{j=0}^{\infty} \varepsilon^{2j} (2j)! \Delta_j$ of $\cosh(\varepsilon \sqrt{\Delta})$.

Note the following consequence of Lemma 5.1:

Lemma 5.4. Let $U,V \in \mathcal{U}_C$ such that $V \subset U$. Then there exists $\varepsilon > 0$ such that $C_\varepsilon$ is a bounded operator from $\mathcal{F}_U,n$ to $\mathcal{F}_V,n$ for all $n \in \mathbb{N}$.

As in the Appendix, consider the functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z} \operatorname{erf}(z)$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$, which belong to the space $\mathcal{F}_{2\varepsilon,\vartheta}$. We would like to substitute $\sqrt{\Delta}$ into our key identity \( (A.3) \)

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1$$

and replace the hyperbolic cosine by its Taylor expansion.

Lemma 5.5. Let $U \in \mathcal{U}_C$. Then there exist $\varepsilon > 0$ and $V \subset U$ such that for any $\varphi \in \mathcal{F}_U,n$, $n \in \mathbb{N}$,

$$(C_\varepsilon \varphi) \ast \kappa_\alpha^\varepsilon + \varphi \ast \kappa_\beta^\varepsilon = \varphi$$

holds as functions on $GV$.

Proof. Note that $\kappa_\alpha^\varepsilon, \kappa_\beta^\varepsilon \in \mathcal{A}(G)$ according to Theorem 4.2. We first consider the case $E = C$ and $\varphi \in L^2(G)$. With $|\alpha_\varepsilon(z) \cosh(\varepsilon z)|$ being bounded, $\cosh(\varepsilon \sqrt{\Delta})$ maps its domain into the domain of $\alpha_\varepsilon(\sqrt{\Delta})$, and the rules of the functional calculus ensure

$$\varphi - \beta_\varepsilon(\sqrt{\Delta}) \varphi = (\alpha_\varepsilon(\cdot) \cosh(\varepsilon \cdot))(\varphi) = (\cosh(\varepsilon \sqrt{\Delta}) \varphi) \ast \kappa_\alpha^\varepsilon$$

in $L^2(G)$ for all $\varphi \in D(\cosh(\varepsilon \sqrt{\Delta}))$. For such $\varphi$, the partial sums of $C_\varepsilon \varphi$ converge to $\cosh(\varepsilon \sqrt{\Delta}) \varphi$ in $L^2(G)$, and hence almost everywhere. Indeed,

$$\left\| \cosh(\varepsilon \sqrt{\Delta}) \varphi - \sum_{j=0}^{N} \frac{\varepsilon^{2j}}{(2j)!} \Delta_j^j \varphi \right\|^2_{L^2(G)}$$

$$= \int \left( dP(\lambda) \left( \cosh(\varepsilon \sqrt{\Delta}) \varphi - \sum_{j=0}^{N} \frac{\varepsilon^{2j}}{(2j)!} \Delta_j^j \varphi \right) , \cosh(\varepsilon \sqrt{\Delta}) \varphi - \sum_{k=0}^{N} \frac{\varepsilon^{2k}}{(2k)!} \Delta_k^k \varphi \right)$$

$$= \int \left( \cosh(\varepsilon \lambda) - \sum_{k=0}^{N} \frac{(\varepsilon \lambda)^{2k}}{(2k)!} \right)^2 \langle dP(\lambda) \varphi, \varphi \rangle$$

$$= \sum_{j,k=N+1}^{\infty} \int \frac{(\varepsilon \lambda)^{2j}}{(2j)!} \frac{(\varepsilon \lambda)^{2k}}{(2k)!} \langle dP(\lambda) \varphi, \varphi \rangle ,$$

and the right hand side tends to 0 for $N \to \infty$, because

$$\sum_{j,k=0}^{\infty} \int \frac{(\varepsilon \lambda)^{2j}}{(2j)!} \frac{(\varepsilon \lambda)^{2k}}{(2k)!} \langle dP(\lambda) \varphi, \varphi \rangle = \int \cosh(\varepsilon \lambda)^2 \langle dP(\lambda) \varphi, \varphi \rangle < \infty .$$
In particular, given \( \varphi \in \mathcal{R}(GU, E) \) and \( \lambda \in E' \), we obtain \( C_\varepsilon \lambda(\varphi) = \cosh(\varepsilon \sqrt{\Delta}) \lambda(\varphi) \) almost everywhere and

\[
C_\varepsilon(\lambda(\varphi)) * \kappa_\alpha^\varepsilon + \lambda(\varphi) * \kappa_\beta^\varepsilon = \lambda(\varphi)
\]
as analytic functions on \( G \) for sufficiently small \( \varepsilon > 0 \).

Since the above identity holds for all \( \lambda \in E' \), we obtain

\[
C_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi
\]
on any connected domain \( GV, 1 \in V \subset U \), on which the left hand side is holomorphic.

Recall the regularized distance function \( \tilde{d}(g) = e^{-\Delta d(g)} \) from Lemma 4.3 and set \( \chi_\delta(g) := e^{-\delta \tilde{d}(g)} (\delta > 0) \). Given \( \varphi \in \mathcal{F}_{U,n} \), \( \chi_\delta \varphi \in \mathcal{R}(GU, E) \) and

\[
C_\varepsilon(\chi_\delta \varphi) * \kappa_\alpha^\varepsilon + (\chi_\delta \varphi) * \kappa_\beta^\varepsilon = \chi_\delta \varphi.
\]
The limit \( \chi_\delta \varphi \to \varphi \) in \( \mathcal{F}_{U,n} \) as \( \delta \to 0 \) is easily verified. From Lemma 5.3 we also get \( (\chi_\delta \varphi) * \kappa_\beta^\varepsilon \to \varphi * \kappa_\beta^\varepsilon \) as \( \delta \to 0 \). Finally Lemma 5.3 and Lemma 5.4 imply

\[
C_\varepsilon(\chi_\delta \varphi) * \kappa_\alpha^\varepsilon \to C_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon \quad (\delta \to 0).
\]
The assertion follows.

Proof of Theorem 1.1. Given \( v \in E^\omega \), the orbit map \( \gamma_v \) belongs to \( \bigcap_n \mathcal{F}_{U,n} \) for some \( U \in \mathcal{U}_C \). Applying Lemma 5.5 to the orbit map and evaluating at \( 1 \) we obtain the desired factorization

\[
v = \gamma_v(1) = \Pi(\kappa_\alpha^\varepsilon)(C_\varepsilon(\gamma_v)(1)) + \Pi(\kappa_\beta^\varepsilon)(\gamma_v(1)).
\]

Note the following generalization of a theorem by Goodman for unitary representations \[5, 10\].

Corollary 5.6. Let \( (\pi, E) \) be an \( F \)-representation. Then every \( \Delta \)-analytic vector is analytic.

Remark 5.7. a) A further consequence of our Theorem 1.1 is a simple proof of the fact that the space of analytic vectors for a Banach representation is complete.

b) We can also substitute \( \sqrt{\Delta} \) into Dixmiér’s and Malliavin’s presentation of the constant function 1 on the real line \[3\]. This invariant refinement of their argument shows that the smooth vectors for a Fréchet representation are precisely the vectors in the domain of \( \Delta^k \) for all \( k \in \mathbb{N} \).

6 Related Problems

We conclude this article with a discussion of how our techniques can be modified to deal with a number of similar questions.

In the context of the introduction, given a nonunital algebra \( A \), a category \( C \) of \( A \)-modules is said to have the strong factorization property if for all \( M \in C \),

\[
M = \{ am \mid a \in A, m \in M \}.
\]
6.1 A Strong Factorization of Test Functions

Our methods may be applied to solve a related strong factorization problem for test functions. On $\mathbb{R}^n$ the Fourier transform allows to write a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ as the convolution $\psi * \Psi$ of two Schwartz functions, and [9] posed the natural problem whether one could demand $\psi, \Psi \in \mathcal{R}(\mathbb{R}^n)$. We are going to prove this in a more general setting.

**Theorem 6.1.** For every real Lie group $G$

$$C_c^\infty(G) \subset \{ \psi * \Psi \mid \psi, \Psi \in \mathcal{R}(G) \}.$$ 

As above, we first regularize an appropriate distance function and set

$$l(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} \log(1 + |z|).$$

**Lemma 6.2.** The function $l(z)$ is entire and approximates $\log(1 + |z|)$ in the sense that for all $N > 0, \vartheta \in (0, 1)$ there exists a constant $C_{N, \vartheta}$ such that

$$|l(z) - \log(1 + |z|)| \leq C_{N, \vartheta} (z \in \mathcal{W}_{N, \vartheta}).$$

Let $m \in \mathbb{N}$. We would like to substitute the square root of the Laplacian associated to a left invariant metric $G$ into a decomposition

$$1 = \hat{\psi}_m(z) \hat{\Psi}_m(z)$$

of the identity. In the current situation we use $\hat{\psi}_m(z) = e^{-ml(z)}$ and $\hat{\Psi}_m(z) = e^{ml(z)}$. Denote the convolution kernels of $\hat{\psi}_m(\sqrt{\Delta})$ and $\hat{\Psi}_m(\sqrt{\Delta})$ by $\psi_m$ resp. $\Psi_m$.

The ideas from the proof of Theorem 4.2 may be combined with the results of [2] to obtain:

**Lemma 6.3.** Let $\chi \in C_c^\infty(G)$ with $\chi = 1$ in a neighborhood of $1$. Then $\chi \Psi_m$ is a compactly supported distribution of order $m$ and $(1 - \chi) \Psi_m \in \mathcal{R}(G) \cap C^\infty(G)$. Given $k \in \mathbb{N}$, $\psi_m \in \mathcal{R}(G) \cap C^k(G)$ for sufficiently large $m$.

Therefore $\hat{\Psi}_m(\sqrt{\Delta})$ maps $C_c^\infty(G)$ to $\mathcal{R}(G)$. The functional calculus leads to a factorization

$$\text{Id}_{C_c^\infty(G)} = \hat{\psi}_m(\sqrt{\Delta}) \hat{\Psi}_m(\sqrt{\Delta})$$

of the identity, and in particular for any $\varphi \in C_c^\infty(G)$,

$$\varphi = (\hat{\Psi}_m(\sqrt{\Delta}) \varphi) * \psi_m \in \mathcal{R}(G) * \mathcal{R}(G).$$

6.2 Strong Factorization of $\mathcal{A}(G)$

It might be possible to strengthen Theorem 1.1 by showing that the analytic vectors have the strong factorization property.

**Conjecture 6.4.** For any $F$-representation $(\pi, E)$ of a real Lie group $G$,

$$E^\omega = \{ \Pi(\varphi)v \mid \varphi \in \mathcal{A}(G), v \in E^\omega \}.$$
We provide some evidence in support of this conjecture and verify it for Banach representations of \((\mathbb{R}, +)\) using hyperfunction techniques.

**Lemma 6.5.** *The conjecture holds for every Banach representation of \((\mathbb{R}, +)\).*

**Proof.** Let \((\pi, E)\) be a representation of \(\mathbb{R}\) on a Banach space \((E, \|\cdot\|)\). Then there exist constants \(c, C > 0\) such that \(\|\pi(x)\| \leq C e^{c|x|}\) for all \(x \in \mathbb{R}\). If \(v \in E^{\omega}\), there exists \(R > 0\) such that the orbit map \(\gamma_v\) extends holomorphically to the strip \(S_R = \{z \in \mathbb{C} | \text{Im} \ z \in (-R, R)\}\). Let

\[
\mathcal{F}_+ (\gamma_v) (z) = \int_{-\infty}^{0} \gamma_v(t)e^{-itz} \, dt, \quad \text{Im} \ z > c,
\]

\[
-\mathcal{F}_- (\gamma_v) (z) = \int_{0}^{\infty} \gamma_v(t)e^{-itz} \, dt, \quad \text{Im} \ z < -c.
\]

Define the Fourier transform \(\mathcal{F}(\gamma_v)\) of \(\gamma_v\) by

\[
\mathcal{F}(\gamma_v) (x) = \mathcal{F}_+ (\gamma_v) (x + 2ic) - \mathcal{F}_- (\gamma_v) (x - 2ic).
\]

Note that \(\|\mathcal{F}(\gamma_v) (x)\| e^{r|x|}\) is bounded for every \(r < R\). Let \(g(z) := \frac{R}{2} \text{erf}(z)\) and write \(\mathcal{F}(\gamma_v)\) as

\[
\mathcal{F}(\gamma_v) = e^{-g} e^{g} \mathcal{F}(\gamma_v)
\]

Define the inverse Fourier transform \(\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))\) for \(x \in \mathbb{R}\) by

\[
\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))(x) = \int_{\text{Im} \ t = -2c}^{\text{Im} \ t = 2c} \mathcal{F}_+ (\gamma_v) (t) e^{itx} \, dt - \int_{\text{Im} \ t = -2c}^{\text{Im} \ t = 2c} \mathcal{F}_- (\gamma_v) (t) e^{itx} \, dt.
\]

Applying the inverse Fourier transform to both sides of (1) and evaluating at 0 yields

\[
v = (2\pi)^{-1} \Pi (\mathcal{F}^{-1} (e^{-g} \mathcal{F}(\gamma_v))(0)).
\]

The assertion follows because \(\mathcal{F}^{-1} (e^{-g}) \in \mathcal{A}(\mathbb{R})\). □

Strong factorization likewise holds for Banach representations of \((\mathbb{R}^n, +)\). Using the Iwasawa decomposition we are able to deduce from this the conjecture for \(SL_2(\mathbb{R})\).

**A An Identity of Entire Functions**

Consider the following space of exponentially decaying holomorphic functions

\[
\mathcal{F}_{c, \theta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N, \theta}} |\varphi(z)| e^{c|z|} < \infty \right\},
\]

\[
\mathcal{W}_{N, \theta} = \{z \in \mathbb{C} \mid |\Re z| < N\} \cup \{z \in \mathbb{C} \mid |\Im z| < \theta |\Re z|\}.
\]

To understand the convolution kernel of a Fourier multiplication operator on \(L^2(\mathbb{R})\) with symbol in \(\mathcal{F}_{c, \theta}\), or more generally functions of \(\sqrt{\Delta}\) on a manifold as in Section 4, we need some properties of the Fourier transformed functions.
Lemma A.1. Given $f \in \mathcal{F}_{c,\vartheta}$, there exist $C, R > 0$ such that

$$\left| \hat{f}^{(k)}(z) \right| \leq C_k R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$.

Proof. Given $f \in \mathcal{F}_{c,\vartheta}$, the Fourier transform extends to a superexponentially decaying holomorphic function on $\mathcal{W}_{c,\vartheta}$. It follows from Cauchy’s integral formula that

$$\left| \hat{f}^{(k)}(z) \right| \leq C_k R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$.

Some important examples of functions in $\mathcal{F}_{c,\vartheta}$ may be constructed with the help of the Gaussian error function [11]

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} \, dt.$$ 

The error function extends to an odd entire function, and $\text{erf}(z) - 1 = O(z^{-1} e^{-z^2})$ as $z \to \infty$ in a sector $\{| \text{Im} z | < \vartheta \text{ Re } z \}$ around $\mathbb{R}_+$. 

Remark A.2. The function

$$z \text{ erf}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * |z| - \frac{1}{\sqrt{\pi}} e^{-z^2}$$

is just one convenient regularization of the absolute value $|z|$, and the basic properties we need also hold for other similarly constructed functions. For example replace the heat kernel $\frac{1}{\sqrt{\pi}} e^{-z^2}$ by a suitable analytic probability density.

For any $\varepsilon > 0$, some algebra shows that the even entire functions $\alpha_{\varepsilon}(z) = 2 e^{-\varepsilon z \text{ erf}(z)}$ and $\beta_{\varepsilon}(z) = 1 - \alpha_{\varepsilon}(z) \cosh(\varepsilon z)$ decay exponentially as $z \to \infty$ in $\mathcal{W}_{N,\vartheta}$ for any $\vartheta < 1$. Hence $\alpha_{\varepsilon}, \beta_{\varepsilon} \in \mathcal{F}_{2\varepsilon,\vartheta}$. Our later factorization hinges on a multiplicative decomposition of the constant function 1:

Lemma A.3. For all $\varepsilon > 0$, $\vartheta \in (0, 1)$, the functions $\alpha_{\varepsilon}, \beta_{\varepsilon} \in \mathcal{F}_{2\varepsilon,\vartheta}$ satisfy the identity

$$\alpha_{\varepsilon}(z) \cosh(\varepsilon z) + \beta_{\varepsilon}(z) = 1.$$ 

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[11] see e.g. [http://functions.wolfram.com](http://functions.wolfram.com)