Semi-classical twists for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ boundary $r-$matrices of Cremmer-Gervais type

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Abstract. We obtain explicit formulas for the semi-classical twists deforming the coalgebraic structure of $U(\mathfrak{sl}_3)$ and $U(\mathfrak{sl}_4)$. In rank 2 and 3 the corresponding universal $R-$matrices quantize the boundary $r-$matrices of Cremmer-Gervais type defining Lie Frobenius structures on the maximal parabolic subalgebras in $\mathfrak{sl}_n$.

Keywords: Generalized Jordanian $r-$matrices, Cremmer-Gervais quantization, semi-classical twists

Mathematics Subject Classifications(2005): 16W30, 17B37, 81R50

1. Introduction

An interesting subclass of the boundary $r-$matrices is given by the generalized Jordanian $r-$matrices of Cremmer-Gervais type which are the boundary points of $SL_n$ adjoint action orbits containing skew-symmetric Cremmer-Gervais $r-$matrices [1, 4]. Explicitly, these boundary points in the $\mathfrak{sl}_n$ Cartan-Weyl basis are the following

$$r_p = \sum_{p=1}^{n-1} D_p \wedge E_{p,p+1} + \sum_{i<j} \sum_{m=1}^{j-i-1} E_{i,j-m+1} \wedge E_{j,i+m},$$

(1)

where

$$D_p = \frac{n-p}{n} (E_{11} + E_{22} + \cdots + E_{pp}) - \frac{p}{n} (E_{p+1,p+1} + E_{p+2,p+2} + \cdots + E_{nn}).$$

Each $r_p$ defines the structure of Lie Frobenius algebra (a Lie algebra with a nondegenerate 2-coboundary [4]) on the maximal parabolic subalgebra $p \subset \mathfrak{sl}_n$ generated by the Cartan subalgebra and all the simple root generators excluding $E_{n,n-1}$. In this letter we construct the twists and the universal $R_p-$matrices if $n = 3, 4$. The concrete $R-$matrices $R^V$ arise when one restricts $R_p = F_{p21} F_{p-1}$ to a particular $\mathfrak{sl}_n$ representation $V$. It was observed [1] that if $V_n$ is the linear space...
of polynomials of the degree \( \leq n \) then \( R^V_n \) are related to rational degeneration of the Ueno-Shibukawa operators and one obtains \( R^V_n \) explicitly. Thus this letter gives an answer to the next question what the universal \( R \)-matrices are in two first cases of physical interest, the genuine Cremmer-Gervais case \( n = 3 \) and quantization of the complexified conformal algebra \( o(4,2)_C \approx \mathfrak{sl}_4 \). The relations defining \( U \)

\[
U = \begin{pmatrix} F & 0 \\ 0 & R \end{pmatrix}
\]

explicitly. Thus this letter gives an answer to the next question what the genuine Cremmer-Gervais case \( n = 3 \) and quantization of the complexified conformal algebra \( o(4,2)_C \approx \mathfrak{sl}_4 \). The paths of constructing the parabolic twists, as we name \( F \) following [10], are almost parallel in both cases and below we describe them uniformly. We consider \( U^F_\mathfrak{sl}(\mathfrak{sl}_n-1) \), the Drinfeld-Jimbo quantization \( U_\mathfrak{sl}(\mathfrak{sl}_n-1) \) deformed by the abelian twist \( K \), and propose a method of constructing the affine twists allowing nontrivial specialization in the limit \( q \to 1 \). The construction we follow is based on factorization of a chosen singular trivial twist \( F_{n-1} := (W_{n-1} \otimes W_{n-1}) \Delta_K(W_{n-1}) \), with \( W_{n-1} \in U^F_\mathfrak{sl}(\mathfrak{sl}_n-1) \), into singular \( \Phi^{-1} \) \( W_{n-1} \) and nonsingular \( F_{n-1} \) parts such that \( F_{n-1} = \Phi^{-1} W_{n-1} \). Among all the factorizations we find the one such that \( F_{n-1} \) is a twist equivalent to an affine version of the Cremmer-Gervais twist or its analog \([5, 7, 9]\). The final step in our approach is rational degeneration of \( F_{n-1} \) which we denote \( F_{n-1} \) and construction of a homomorphism \( \tau_{n-1} \) such that \( (\tau_{n-1} \otimes \tau_{n-1})(F_{n-1}) \) is a twist on \( U_{\mathfrak{sl}}(\mathfrak{sl}_n) \), where \( \Psi_n \) turns out to be the semi-classical twist found in [8]. The final twist on \( U(\mathfrak{sl}_n) \) is obtained as the composition \( (\tau_{n-1} \otimes \tau_{n-1})(F_{n-1}) \cdot \Psi_n \).

2. Quantum affine twists for \( U_q(\mathfrak{sl}_2) \) and \( U_q(\mathfrak{sl}_3) \)

Fix the central charge \( c = 1 \) in the defining relations of \( U_q(\mathfrak{sl}_n) \). Let \( A = \left( \frac{2(a_i | a_j)}{(a_i | a_i)} \right)_{i,j=1}^n \) be the Cartan matrix of \( \mathfrak{sl}_n \) and \( \{\alpha_i\}_{i=0,\ldots,n-1} \) be the set of all simple roots of \( \mathfrak{sl}_n \) with the symmetric scalar product \((\cdot,\cdot)\) on it. With these assumptions the relations defining \( U_q(\mathfrak{sl}_n) \) over the field of rational functions \( \mathbb{Q}(q) \) are the following

\[
\begin{align*}
q^{h_{\alpha_0}}q^{h_{\alpha_1}} \cdots q^{h_{\alpha_{n-1}}} &= 1, \\
q^{h_{\alpha_i}}q^{h_{\alpha_j}} &= q^{h_{\alpha_i}}q^{h_{\alpha_j}} \\
q^{h_{\alpha_i}}e_{\pm\alpha_j}q^{-h_{\alpha_i}} &= q^{\pm(a_i | a_j)}e_{\pm\alpha_j}, \\
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij} \frac{q^{h_{\alpha_i}} - q^{-h_{\alpha_i}}}{q - q^{-1}} \\
e_{\pm\alpha_i}e_{\pm\alpha_j} &= e_{\pm\alpha_j}e_{\pm\alpha_i} \quad \text{if} \quad |i - j| \not\equiv 1 \pmod{n} \\
[e_{\pm\alpha_i}, [e_{\pm\alpha_i}, e_{\pm\alpha_j}]q] &= 0 \quad \text{if} \quad |i - j| \equiv 1 \pmod{n}, n > 2 \\
[e_{\pm\alpha_i}, [e_{\pm\alpha_i}, [e_{\pm\alpha_i}, e_{\pm\alpha_j}]q^2]] &= 0 \quad \text{if} \quad |i - j| \equiv 1 \pmod{n}, n = 2
\end{align*}
\]
and the $q-$commutator is defined as usual

$$[C, D]_q = CD - qDC, \quad [C, D] = [C, D]_1.$$  

$U_q(\hat{\mathfrak{sl}}_n)$ is a Hopf algebra. The comultiplication is uniquely defined by fixing its value on $q-$Chevalley generators

$$\Delta(e_{\alpha_i}) = q^{-h_{\alpha_i}} \otimes e_{\alpha_i} + e_{\alpha_i} \otimes 1, \quad \Delta(e_{-\alpha_i}) = e_{-\alpha_i} \otimes q^{h_{\alpha_i}} + 1 \otimes e_{-\alpha_i}$$

$$\Delta(q^{h_{\alpha_i}}) = q^{h_{\alpha_i}} \otimes q^{h_{\alpha_i}}.$$  

It is convenient to use the following notation

$$(x)_{(1)} := x \otimes 1, \quad (x)_{(2)} := 1 \otimes x$$

and its modified version

$$(x)_{(1)} := (\text{id} \otimes \text{pr}_{K_{n-1}}) \Delta(x), \quad (x)_{(2)} := (\text{pr}_{K_{n-1}} \otimes \text{id}) \Delta(x)$$

$$(x)_{(3)} := ((\text{id} - \text{pr}_{K_{n-1}}) \otimes (\text{id} - \text{pr}_{K_{n-1}})) \Delta(x)$$

where $\text{pr}_{K_{n-1}}$ the $U_q(\hat{\mathfrak{sl}}_n)$ projector to $K_{n-1} := \mathbb{Q}(q)\{q^{\pm h_{\alpha_i}}\}_{i=1,\ldots,n-1}.$

2.1. An affine twist for $U_q(\hat{\mathfrak{sl}}_2)$

Introduce a topological Hopf algebra $D^{(2)}[[\zeta]]$ as a completion in the formal series topology of the following subalgebra

$$D^{(2)} := \left\{ \sum_{l_1,l_2,l_3 \geq 0} c_{l_1,l_2,l_3}(e_{-\alpha_i})^{l_1} q^{\pm l_2 h_{\alpha_i}} (e_{\delta-\alpha_i})^{l_3} | c_{l_1,l_2,l_3} \in \mathbb{Q}(q) \right\} \subset U_q(\hat{\mathfrak{sl}}_2)$$

and consider a trivial twist $F_2$:

$$F_2 = (W_2 \otimes W_2) \Delta(W_2^{-1}),$$

$$W_2 = \exp_q^2\left(\frac{\zeta}{1 - q^2} e_{\delta - \alpha_i}\right) \exp_q^{-2}\left(-\frac{\zeta^2}{1 - q^2} q^{-h_{\alpha_i}} e_{-\alpha_i}\right) \in D^{(2)}[[\zeta]];$$

$$\exp_q^2(z) := \sum_{k \geq 0} \frac{z^k}{(k)q^2}, \quad (k)q^2! = \frac{1 - q^2}{1 - q^2} \cdots \frac{1 - q^{2n}}{1 - q^2}$$

where $\delta = \alpha_0 + \alpha.$ Along with the nonsingular $q-$exponent we make use of its singular version $e_q^2(z) := \exp_q^2\left(\frac{1}{1 - q^2} z\right)$ and of several results of $q-$calculus [3, 6] concerning its properties:

- If $[x, y]_{q^2} = 0$ then

$$e_{q^2}(x + y) = e_{q^2}(y) \cdot e_{q^2}(x), \quad (e_{q^2}(x))^{-1} = e_{q^{-2}}(q^{-2} x).$$ (2)
Proposition 1. 

Let us focus on the following product

\[ (1 - u)^{(v)} = e_q^2(u)(e_q^2(uq^{-2v}))^{-1} \]  

where \((-v + l)q^2 = (q^{-2v+2l} - 1)/(q^2 - 1).\)

where \((-v + l)q^2 = (q^{-2v+2l} - 1)/(q^2 - 1).\)

\[ e_q^2(u) \cdot e_q^2(v) = e_q^2(v) \cdot e_q^2(\frac{1}{1 - q^2}[u, v]) \cdot e_q^2(u). \] (3)

The Heine’s formula holds

\[ (1 - u)^{(v)} = e_q^2(u)(e_q^2(uq^{-2v}))^{-1} \]  

where \((-v + l)q^2 = (q^{-2v+2l} - 1)/(q^2 - 1).\)

The final step in the proof is application of the Heine’s formula that leads to an explicit form of \(F_2-aff\).
that satisfies the Drinfeld equation according to [7].

**Remark.** By Proposition 1 $F_\mathfrak{p}^{\text{aff}}$ is equivalent to $\Phi_2$ and the latter is the Cremmer-Gervais twist as it is seen from [9]. In Proposition 2 we give a proof for the case $U_q(\mathfrak{sl}_3)$ and indicate what are the simplifications one needs to take into consideration to adopt it for $U_q(\mathfrak{sl}_2)$.

### 2.2. An affine twist for $U_q(\mathfrak{sl}_3)$

Let us deform the coalgebraic structure of $U_q(\mathfrak{sl}_3)$ by the following abelian twist

\[ \mathcal{K} = q^{\frac{2}{3}h_\alpha \otimes h_\alpha + \frac{2}{9}h_\alpha \otimes h_\beta + \frac{2}{9}h_\beta \otimes h_\alpha + \frac{2}{9}h_\beta \otimes h_\beta} \]

the convenience of this choice will be justified by Proposition 2 (see also [5])

\[ \Delta_K(\hat{e}_{-\beta}) = q^{h_\alpha + \beta} \otimes \hat{e}_{-\beta} + \hat{e}_{-\beta} \otimes 1, \quad \Delta_K(\hat{e}_{-\alpha}) = q^{-h_\beta} \otimes \hat{e}_{-\alpha} + \hat{e}_{-\alpha} \otimes 1 \]

\[ \Delta_K(\hat{e}_\alpha) = \hat{e}_\alpha \otimes q^{h_\alpha + \beta} + 1 \otimes \hat{e}_\alpha \]

\[ \Delta_K(\hat{e}_{\delta - \alpha - \beta}) = \hat{e}_{\delta - \alpha - \beta} \otimes q^{-h_\beta} + 1 \otimes \hat{e}_{\delta - \alpha - \beta} \]

\[ \Delta_K(\hat{e}_{\delta - \beta}) = \hat{e}_{\delta - \beta} \otimes q^{-h_\alpha} + 1 \otimes \hat{e}_{\delta - \beta} + (1 - q^2) \hat{e}_\delta \otimes \hat{e}_{\delta - \alpha - \beta} q^{h_\alpha + \beta} \]

\[ \Delta_K(\hat{e}_{-\alpha - \beta}) = q^{-h_\alpha} \otimes \hat{e}_{-\alpha - \beta} + \hat{e}_{-\alpha - \beta} \otimes 1 + (1 - q^{-2}) \hat{e}_{-\beta} q^{-h_\beta} \otimes \hat{e}_{-\alpha} \]

where $\delta = \alpha_0 + \alpha + \beta$

\[ h_\alpha^\perp = \frac{2}{3}h_\alpha + \frac{4}{3}h_\beta, \quad h_\beta^\perp = \frac{4}{3}h_\alpha + \frac{2}{3}h_\beta, \quad h_{\alpha + \beta}^\perp = h_\beta^\perp - h_\alpha^\perp \]

\[ \hat{e}_{-\alpha} = q^{\frac{1}{2} h_\alpha^\perp} e_{-\alpha}, \quad \hat{e}_\alpha = e_\alpha q^{\frac{1}{2} h_\alpha^\perp}, \quad \hat{e}_{\delta - \alpha - \beta} = q^{-\frac{1}{2} h_\alpha^\perp} e_{\delta - \alpha - \beta} \]

\[ \hat{e}_{\delta - \beta} = \hat{e}_\alpha e_{\delta - \alpha - \beta} - q^2 \hat{e}_{\delta - \alpha - \beta} e_\alpha, \quad \hat{e}_{-\alpha - \beta} = \hat{e}_{-\beta} \hat{e}_{-\alpha} - q^{-2} e_{-\alpha} \hat{e}_{-\beta}. \]

**Remark.** $\mathcal{K}$ preserves the composite root generators in the sense that

\[ \hat{e}_{\delta - \alpha} = e_\alpha e_{\delta - \alpha - \beta} - q e_{\delta - \alpha - \beta} e_\alpha, \quad \hat{e}_{-\alpha - \beta} = e_{-\beta} e_{-\alpha} - q^{-1} e_{-\alpha} e_{-\beta}. \]

Denote by $K_2'$ the minimal algebra containing $K_2 \cup \{ \hat{e}_\alpha \}$ over $\mathbb{Q}(q)$. Introduce a subalgebra $D^{[3]} \subset U_q(\mathfrak{sl}_3)$ generated by all finite linear combinations with coefficients in $\mathbb{Q}(q)$:

\[ \sum_{l_1, \ldots, l_5 \geq 0} a_{l_1, \ldots, l_5} (\hat{e}_{-\beta})^{l_1} (\hat{e}_{-\alpha - \beta})^{l_2} (\hat{e}_{-\alpha})^{l_3} K_2'(\hat{e}_{\delta - \beta})^{l_4} (\hat{e}_{\delta - \alpha - \beta})^{l_5}. \]
If we complete $D^{(3)}$ up to $D^{(3)}[[\zeta]]$ and choose the following element

$$W_3 = e_q^2(\zeta \cdot q^{h_{\alpha+\beta}/2} \cdot \hat{\delta}_{-\beta}) e_q^2(\cdot - q^{2/3} \cdot q^{h_{\delta-\alpha-\beta}/2} (e_q^2(-\zeta^2 \cdot \hat{\epsilon}_{-\beta}) e_q^2(-\zeta \cdot \hat{\epsilon}_{-\alpha}) e_q^2(-\zeta^2 \cdot \hat{\epsilon}_{-\beta}))^{-1}
$$

defining the trivial twist

$$F_3 = (W_3 \otimes W_3) \Delta_K(W_3^{-1})$$

then similarly to Proposition 1 we can formulate

**Proposition 2.**

$$F_3 = \Phi_{W_3}^{-1} \cdot F_3^{\text{aff}}$$

where $F_3^{\text{aff}} \in (K_3 \otimes D^{(3)})[[\zeta]]$ is a twist and $\Phi_{W_3} = \text{Ad}(W_3 \otimes W_3)(\Lambda_1 \Lambda_2 \Lambda_3)$

$$\Lambda_1 = \exp_q^2\left(-\frac{q^{-1} \zeta^4}{1 - q^2} \cdot \hat{\delta}_{-\alpha-\beta} \otimes \hat{\delta}_{-\alpha-\beta}\right) \cdot \exp_q^2\left(-\frac{q^{-1} \zeta^3}{1 - q^2} \cdot \hat{\delta}_{-\alpha-\beta} \otimes \hat{\delta}_{-\alpha-\beta}\right)$$

$$\Lambda_2 = \exp_q^2\left(\frac{q^4}{1 - q^2} \cdot \hat{\epsilon}_{-\beta} \otimes \hat{\epsilon}_{-\alpha-\beta}\right) \cdot \hat{\epsilon}_{-\beta} = [\hat{e}_\alpha, \hat{e}_{-\alpha-\beta}]$$

$$\Lambda_3 = \exp_q^2(-\zeta(q - q^{-1}) \cdot \hat{e}_\alpha \otimes \hat{\epsilon}_{-\beta}).$$

**Proof.** A general idea of factorization of $F_3$ is to move factors from $\Delta_K(W_3^{-1})$ containing identity in the second tensor factor to the left in order to form $\Phi_{W_3}^{-1}$. Using explicit form of the coproducts and the commutation relations $[\hat{e}_\alpha, \hat{e}_{-\beta}] = [\hat{e}_{-\alpha-\beta}, \hat{e}_{-\beta}] = 0$, we expand $\Delta_K(W_3^{-1})$ into the product of $q$-exponents using $(2)$. According to the strategy of factorization we follow, first we flip the following two $q$-exponents from $\Delta_K(W_3^{-1})$ by introducing the factor $\Lambda_3$:

$$e_q^2(-\zeta^2(\hat{\epsilon}_{-\beta})_{<2>}) \cdot e_q^2(-\zeta(\hat{\epsilon}_{-\alpha})_{<1>}) =$$

$$\text{Ad}(\Lambda_3)^{-1}\left(e_q^2(-\zeta(\hat{\epsilon}_{-\alpha})_{<1>}) \cdot e_q^2(-\zeta^2 q^{-h_{\beta}/2} \otimes \hat{\epsilon}_{-\beta})\right).$$

The latter is seen from the relation

$$\zeta^2 q^{h_{\alpha+\beta}/2} \cdot \hat{\epsilon}_{-\beta} + \zeta \hat{\epsilon}_{-\alpha} \otimes 1 = \text{Ad}(\Lambda_3)^{-1}\left(\zeta \cdot \hat{\epsilon}_{-\alpha} \otimes 1 + \zeta^2 q^{-h_{\beta}/2} \otimes \hat{\epsilon}_{-\beta}\right)$$

if one notices

$$[\hat{e}_\alpha, \hat{e}_{-\alpha}] = \frac{q^{h_{\alpha+\beta}/2} - q^{-h_{\beta}/2}}{q - q^{-1}}$$

and applies $(2)$. Then we move $\Lambda_3$ to the right of $\Delta_K(W_3^{-1})$ by applying $(3)$ and the commutation relations

$$[\hat{e}_{-\alpha}, \hat{e}_\alpha] = 0, \ [\hat{e}_{-\beta}, \hat{e}_\alpha] = 0, \ [\hat{e}_{-\beta}, \hat{e}_{-\alpha-\beta}] = 0$$
until we arrive at the following form of \( F_3 \)

\[
(W_3 \otimes W_3) \Lambda_3^{-1}(e_{q^2}(-\zeta^2 \hat{e}_{-\beta}) \cdot e_{q^2}(-\zeta \hat{e}_{-\alpha}) \cdot e_{q^2}(-\zeta^2 \hat{e}_{-\beta}))_{<1>} \cdot \\
e_{q^2}(-\zeta^2 q^{-h_{\beta}^+} \otimes \hat{e}_{-\beta}) \cdot e_{q^2}(-\zeta (\hat{e}_{-\alpha})_{<2>}) \cdot \exp_{q^2}(-q^{-1}\zeta^2 \hat{e}_{\alpha} q^{-h_{\beta}^+} \otimes \hat{e}_{-\alpha-\beta}) \cdot \\
e_{q^{-2}}(-\frac{q^{-1}\zeta^2}{1-q^2} (q^{h_{\beta}^+} \hat{e}_{\delta-\alpha-\beta})_{<1>}) \cdot \Lambda_3 \cdot e_{q^{-2}}(q^{-2}\zeta (q^{h_{\alpha}^+} \hat{e}_{\delta-\beta})_{<1>}) \cdot \\
e_{q^{-2}}(-\zeta^2 (\hat{e}_{-\beta})_{<2>}) \cdot e_{q^{-2}}(-\frac{q^{-1}\zeta^2}{1-q^2} (q^{h_{\beta}^+} \hat{e}_{\delta-\alpha-\beta})_{<2>}) \cdot \\
e_{q^{-2}}(q^{-2}\zeta (q^{h_{\alpha}^+} \hat{e}_{\delta-\beta})_{<2>}).
\]

Next step is the appearance of \( \Lambda_2^{-1} \) from the relation

\[
\exp_{q^2}(-q^{-1}\zeta^2 \hat{e}_{\alpha} q^{-h_{\beta}^+} \otimes \hat{e}_{-\alpha-\beta}) \cdot e_{q^{-2}}(-\frac{q^{-1}\zeta^2}{1-q^2} (q^{h_{\beta}^+} \hat{e}_{\delta-\alpha-\beta})_{<1>}) = \\
\Lambda_2^{-1} \cdot e_{q^{-2}}(-\frac{q^{-1}\zeta^2}{1-q^2} (q^{h_{\beta}^+} \hat{e}_{\delta-\alpha-\beta})_{<1>}) \cdot \exp_{q^2}(-q^{-1}\zeta^2 \hat{e}_{\alpha} q^{-h_{\beta}^+} \otimes \hat{e}_{-\alpha-\beta})
\]

holding by \( \hat{e}_{\delta-\alpha-\beta}, \hat{e}_{\delta-\beta}' \) \( q^{-2} = [\hat{e}_{\alpha}, \hat{e}_{\delta-\beta}'] q^2 = 0 \) and by (3). We move \( \Lambda_2^{-1} \) to the left of \( F_3 \) using (3) and the necessary for it relations

\[
[\hat{e}_{-\alpha}, \hat{e}_{-\alpha-\beta}] = [\hat{e}_{-\beta}, \hat{e}_{-\alpha-\beta}] = 0, \quad [\hat{e}_{\delta-\beta}', \hat{e}_{-\alpha}] = -q \hat{e}_{\delta-\alpha-\beta} q^{-h_{\beta}^+}
\]

and form \( \Phi_{W_3}^{-1} \) which leads to appearance of \( \Lambda_1^{-1} \) and results in the following form of \( F_3 \)

\[
F_3 =
\]

\[
\Phi_{W_3}^{-1}(1 \otimes W_3) \cdot e_{q^2}(-\zeta^2 q^{-h_{\beta}^+} \otimes \hat{e}_{-\beta}) \cdot e_{q^2}(-\zeta (\hat{e}_{-\alpha})_{<2>}) \cdot \\
\exp_{q^2}(-q^{-1}\zeta^2 \hat{e}_{\alpha} q^{-h_{\beta}^+} \otimes \hat{e}_{-\alpha-\beta}) \cdot \Lambda_3 \cdot e_{q^2}(-\zeta^2 (\hat{e}_{-\beta})_{<2>}) \cdot \\
e_{q^{-2}}(-\frac{q^{-1}\zeta^2}{1-q^2} (q^{h_{\beta}^+} \hat{e}_{\delta-\alpha-\beta})_{<2>}) \cdot e_{q^{-2}}(q^{-2}\zeta (q^{h_{\alpha}^+} \hat{e}_{\delta-\beta})_{<3>}) \cdot \\
e_{q^{-2}}(q^{-2}\zeta (q^{h_{\alpha}^+} \hat{e}_{\delta-\beta})_{<2>}).
\]

The last steps will be to note that

\[
[\left( e_{q^{-2}}(-q^{-2}\zeta \hat{e}_{-\alpha}) \cdot e_{q^{-2}}(-q^{-2}\zeta^2 \hat{e}_{-\beta}) \right)_{(2)}, \\
e_{q^2}(-\zeta^2 q^{-h_{\beta}^+} \otimes \hat{e}_{-\beta}) \cdot e_{q^2}(-\zeta (\hat{e}_{-\alpha})_{<2>})] = 0
\]
and rewrite \( F_3 \) in the following form with the help of (3):

\[
\Phi_{W_3}^{-1} \cdot \left( e_{q^2}(\zeta q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) \cdot e_{q^{-2}}(-q^{-2} \zeta^2 \hat{e}_{\hat{\beta}}) \right) (2) \cdot \\
e_{q^2}(-\zeta^2 q^{-h_2^+} \hat{e}_{\hat{\beta}}) \cdot e_{q^{-2}}(q^{-2} \zeta q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) .
\]

\[
e_{q^2}(\zeta q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) \cdot e_{q^2}(-\frac{q^2}{1-q^2} q^{h_2^+} \hat{e}_{\hat{\delta} - \alpha - \beta}) (2) \cdot e_{q^2}(-\zeta (\hat{e}_{\hat{\alpha}})_2) \cdot e_{q^{-2}}(-q^{-2} \zeta (\hat{e}_{\hat{\beta}}) (2)) \cdot \\
e_{q^{-2}}(q^{-2} \zeta^2 q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) .
\]

where we have inserted

\[
e_{q^2}(q^{-2} \zeta^2 q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) \cdot e_{q^2}(\zeta^2 q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) = 1 \] (10)

and \( \cdots \) means third, forth and fifth lines in (9). The final transformation is to move the second \( q \)-exponent in (10) to the right of \( F_3 \) so that to apply the Heine’s formula by the same trick as we did for \( F_2 \). \( F_3 = \)

\[
\Phi_{W_3}^{-1} \cdot (1 - \zeta (q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) (2) + \zeta^2 q^{-(h_2^+)(1)} (\hat{e}_{\hat{\beta}}) (2) q^{2} (\hat{e}_{\hat{\alpha} - \beta} (2)) q^{-2} \cdot (1 + q^{-2} \zeta (\hat{e}_{\hat{\alpha}}) (2)) q^{-2} (\hat{e}_{\hat{\beta}} (1)) .
\]

\[
\exp_q^2(-q^{-2} \zeta^2 \hat{e}_\alpha q^{-h_2^+} \hat{e}_{\hat{\alpha} - \beta}) \cdot \Lambda_3 \cdot e_{q^{-2}}(q^{-2} \zeta (q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) \cdot \Lambda_3 .
\]

\[
(1 - q^{-2} \zeta (q^{h_2^+} \hat{e}_{\hat{\delta} - \beta}) < 2) + q^{-2} \zeta^2 (\hat{e}_{\hat{\beta}} (2)) q^{-2} (\hat{e}_{\hat{\beta}} (1)) .
\]

Once we have factored \( F_3 \) we can give a simple proof of the Drinfeld equation

\[
(F_{3}^{aff})_{12}(\Delta \otimes \text{id})(F_{3}^{aff}) = (F_{3}^{aff})_{23}(\text{id} \otimes \Delta)(F_{3}^{aff}) .
\]

From the factorization we know that

\[
(F_{3}^{aff}) = (W_3 \otimes W_3) \Phi_3 \Delta_K(W_3^{-1}) \] (11)

where \( \Phi_3 := \Lambda_1 \Lambda_2 \Lambda_3 \). Let us consider the Drinfeld associator

\[
\text{Assoc}(F_{3}^{aff}) \equiv (F_{3}^{aff})_{23}(\text{id} \otimes \Delta)(F_{3}^{aff})((F_{3}^{aff})_{12}(\Delta \otimes \text{id})(F_{3}^{aff}))^{-1}
\]

then

\[
\text{Assoc}(F_{3}^{aff}) \in (K_2' \otimes \mathcal{D}^{(3)} \otimes \mathcal{D}^{(3)})(\mathcal{C}) .
\]

On the other hand by (11) we have

\[
\text{Assoc}(F_{3}^{aff}) = (W_3 \otimes W_3 \otimes W_3) \text{Assoc}(\Phi_3)(W_3^{-1} \otimes W_3^{-1} \otimes W_3^{-1}) \] (14)

Denote by \( \text{pr}_{K_2'} \) the projection of \( \mathcal{D}^{(3)}(\mathcal{C}) \) to \( K_2'[[\mathcal{C}]] \). Thus by (13)

\[
\text{Assoc}(F_{3}^{aff}) = (\text{pr}_{K_2'} \otimes \text{id} \otimes \text{id})(\text{Assoc}(F_{3}^{aff}))
\]
and using explicit form of $\Phi_3$ and $W_3$ we deduce that $\text{Assoc}(F_3^{\text{aff}})$ is equal to

$$(\text{pr}_{K_2} \otimes \text{id} \otimes \text{id}) \left( W_3^{\otimes 3} (\Phi_3)_{23} (\text{id} \otimes \Delta) (\Lambda_3) ((\Lambda_3)_{12} (\Lambda_1)_{23} (\Lambda_2)_{23} (\Delta \otimes \text{id}) (\Lambda_3))^{-1} W_3^{-\otimes 3} \right)$$

and the latter is $1 \otimes 1 \otimes 1$.

**Remark.** In the case of $F_2^{\text{aff}}$ the proof of the Drinfeld equation is similar and one needs to use projection to $K_1$ and follow the same lines as we did for $F_3^{\text{aff}}$.

### 3. Rational degeneration of $F_n-1$, $n = 3, 4$

#### 3.1. Rational degeneration of $F_2^{\text{aff}}$

Introduce $\mathcal{D}_A^{(2)}[[\zeta]]$ as a completion in the formal series topology of the following Hopf subalgebra in $U_q(\hat{sl}_2)$

$$\mathcal{D}_A^{(2)} := \left\{ \sum_{l_1,l_2,l_3 \geq 0} c_{l_1,l_2,l_3} (e_{-\alpha})^{l_1} \left( \frac{q^{\pm h_\alpha} - 1}{q - 1} \right)^{l_2} (e_{\delta - \alpha})^{l_3} | c_{l_1,l_2,l_3} \in A \right\}$$

where $A := \mathbb{Q}[q, q^{-1}]$ be the ring $\mathbb{Q}[q, q^{-1}]$ localized at $(q - 1)$ (the subring of rational functions nonsingular at $q = 1$). Then consider the subalgebra $\mathcal{F}_A^{(2)}[\zeta]$ generated over $A[\zeta]$ by the following elements

$$H_{\pm \alpha} = \frac{q^{\pm h_\alpha} - 1}{q - 1}, \quad f_0 = (q^{-2} - 1) q^{-h_\alpha} e_{-\alpha}, \quad f_1 = e_{\delta - \alpha} + \zeta q^{-h_\alpha} e_{-\alpha}$$

where $q^{-h_\alpha} = 1 + (q - 1) \left( \frac{q^{-h_\alpha} - 1}{q - 1} \right)$.

The specialization $\mathcal{F}_{A,q=1}^{(2)}[\zeta] = \mathcal{F}_A^{(2)}[\zeta] / (q - 1)$ is a Hopf algebra with the following structure

$$[H_\alpha, f_1] = -2f_1, \quad [H_\alpha, f_0] = -2f_0, \quad f_1 f_0 - f_0 f_1 = -\zeta f_0^2 \quad (15)$$

$$\Delta(f_1) = f_1 \otimes 1 + 1 \otimes f_1 + H_\alpha \otimes f_0, \quad (16)$$

and $H_{-\alpha} = -H_{-\alpha}, f_0$ are primitive. The relations (15) define $\mathcal{F}_{A,q=1}^{(2)}[\zeta]$ as an associative algebra with the basis $\{ f_0^k H_\alpha^l f_0^m \}_{k,l,m \geq 0}$ over $\mathbb{Q}[\zeta]$ obtained by specialization $q = 1$ from its quantum version. Comparing the semi-classical basis with its quantum analog $\{ (f_0)^k (H_\alpha)^l f_0^m \}_{k,l,m \geq 0}$
we deduce that (15) are the only relations defining $\mathcal{F}_{\mathcal{A}, q=1}[\zeta]$. Completing $\mathcal{F}_{\mathcal{A}, q=1}[\zeta]$ up to a topological Hopf algebra $\mathcal{F}_{\mathcal{A}, q=1}[\zeta] := (\mathcal{F}_{\mathcal{A}, q=1}[\zeta])[\zeta]$ we see that $F_2^{\text{aff}}$ given by

$$F_2^{\text{aff}} = (1 \otimes 1 - \zeta \otimes f_1 - \zeta^2 (h_\alpha/2)_q \otimes f_0) (-\frac{1}{2} h_\alpha \otimes 1)$$

can be specialized to a twist of $\mathcal{F}_{\mathcal{A}, q=1}[\zeta]$ where it is given by the formula of [7]

$$\overline{F}_2^{\text{aff}} = (1 \otimes 1 - \zeta \otimes \overline{f}_1 - \zeta^2 (H_\alpha/2) \otimes \overline{f}_0) - \frac{1}{2} H_\alpha \otimes 1. \quad (17)$$

3.2. Rational degeneration of $F_3^{\text{aff}}$

Let

$$\mathcal{D}_A^{(3)} := \sum_{l_1, \ldots, l_5 \geq 0} c_{l_1, \ldots, l_5} (\hat{e}_-\beta)^{l_1} (\hat{e}_{-\alpha-\beta})^{l_2} (\hat{e}_-\alpha)^{l_3} K_2''(\hat{e}_{\delta-\beta})^{l_4} (\hat{e}_{\delta-\alpha-\beta})^{l_5}$$

where $c_{l_1, \ldots, l_5} \in A$. $K_2''$ is an algebra generated by $H_{\pm, \alpha} = \frac{q^{\pm h_\beta}}{q-1} - 1$, $H_{\pm, \beta} = \frac{q^{\pm h_\beta}}{q-1}$, and $\hat{e}_\alpha$ over $A$. Complete $\mathcal{D}_A^{(3)}$ up to $\mathcal{D}_A^{(3)}[[\zeta]]$ and consider the Hopf subalgebra

$$\mathcal{F}_A^{(3)}[\zeta] := \{ \sum_{l_1, \ldots, l_5 \geq 0} c_{l_1, \ldots, l_5} (f_2)^{l_1} (f_0)^{l_2} (\hat{e}_{-\alpha})^{l_3} K_2''(f_3)^{l_4} (f_1)^{l_5} | c_{l_1, \ldots, l_5} \in A[\zeta] \}$$

where

$$f_0 = (q - q^{-1}) \hat{e}_{-\alpha-\beta}, \quad f_1 = q^{h_\alpha} \hat{e}_{\delta-\alpha-\beta} + q^{-1} \zeta \hat{e}_{-\alpha-\beta}$$
$$f_2 = (1 - q^{-2}) \hat{e}_{-\beta}, \quad f_3 = q^{h_\alpha} \hat{e}_{\delta-\beta} - \zeta \hat{e}_{-\beta}.$$

**Proposition 3.** $F_2^{\text{aff}}$ restricts to a twist of $\mathcal{F}_A^{(3)}[[\zeta]] := (\mathcal{F}_A^{(3)}[\zeta])[\zeta]$. 

*Proof.* Note that by (2) we have the following identity

$$\exp_{q^2}(-q^{-1} \zeta^2 \hat{e}_\alpha q^{h_\beta} \otimes \hat{e}_{-\alpha-\beta}) = \exp_{q^2}(\zeta^2 \hat{e}_\alpha \frac{H_{\alpha}^+ - H_{\beta}^-}{1+q} \otimes f_0) \exp_{q^{-2}}(-q^{-1} \zeta^2 q^{h_\alpha} \hat{e}_\alpha \otimes \hat{e}_{-\alpha-\beta})$$
which allows to bring $F^{\text{aff}}_3$ to the following form
\[
(1 \otimes 1 - \zeta 1 \otimes f_3 - \zeta^2 (h_1^+ / 2) q_2 \otimes f_2)_{q^2}^{(-\frac{1}{2} h_3^+ \otimes 1)}.
\]
\[
(1 \otimes 1 + q^{-2} \zeta 1 \otimes \bar{e}_\alpha)_{q_2}^{(-\frac{1}{2} h_3^+ \otimes 1)} \cdot \exp_{q^2}(\zeta^2 \bar{e}_\alpha \frac{H_\alpha^+ - H_\beta^+}{1 + q} \otimes f_0).
\]
\[
\exp_{q^2}(-\zeta q h_\alpha^+ \epsilon_\alpha \otimes f_1) \cdot \exp_{q^2}(-q \zeta \bar{e}_\alpha \otimes f_2).
\]
\[
(1 \otimes 1 - q^{-2} \zeta q h_\alpha^+ \otimes f_3 - \zeta^2 (h_\alpha^+ / 2) q_2 \otimes f_2)_{q^{-2}}^{(\frac{1}{2} h_3^+ \otimes 1)}.
\]
correctly defined on $\mathcal{F}_A^{(3)}[[\zeta]]$.

The semi-classical twist $F^{\text{aff}}_3$ is obtained by specializing $q = 1$:
\[
(1 \otimes 1 - \zeta 1 \otimes \bar{f}_3 - \frac{1}{2} \zeta^2 \bar{H}_\beta^+ \otimes \bar{f}_2)_{1}^{(-\frac{1}{2} h_3^+ \otimes 1)} \cdot (1 \otimes 1 + \zeta 1 \otimes e_\alpha)_{1}^{(-\frac{1}{2} h_3^+ \otimes 1)}.
\]
\[
\exp\left(\frac{1}{2} \zeta^2 \bar{e}_\alpha (H_\alpha^+ + H_\beta^+) \otimes \bar{f}_0 \right) \cdot \exp\left(-\zeta \bar{e}_\alpha \otimes \bar{f}_1 \right) \cdot \exp\left(-\zeta e_\alpha \otimes \bar{f}_2 \right) \cdot (1 \otimes 1 - \zeta 1 \otimes \bar{f}_3 - \frac{1}{2} \zeta^2 \bar{H}_\alpha^+ \otimes \bar{f}_2)_{1}^{(\frac{1}{2} h_3^+ \otimes 1)}.
\]

4. Universal quantization of $n = 3, 4$ generalized Jordanian $r$–matrices

Let us denote by $U^{\Psi_n}(\mathfrak{s}l_n)[[\zeta]]$ the completed universal enveloping algebra $U(\mathfrak{s}l_n)[[\zeta]]$ with the twisted coproduct
\[
\Delta_{\psi_n}(\cdot) = \psi_n \Delta(\cdot) \psi_n^{-1}.
\]

**Proposition 4.** There exists a twist $\Psi_n$ in $U(\mathfrak{s}l_n)[[\zeta]]$ and a homomorphism $\iota_{n-1}$ such that
\[
\iota_{n-1} : \mathcal{F}_A^{(n-1)}[[\zeta]] \rightarrow U^{\Psi_n}(\mathfrak{s}l_n)[[\zeta]].
\]

**Proof.** If $n = 3$, then choose the extended Jordanian twist [8]
\[
\Psi_3 = \exp(\zeta E_{32} \otimes E_{13} e^{-\sigma_{12}^\zeta} \cdot \exp(D_1 \otimes \sigma_{12}^\zeta)
\]
where $\sigma_{12}^\zeta = \ln(1 - \zeta E_{12})$ and $E_{ij}$ are the elements of $U(\mathfrak{s}l_n)$ corresponding to the elements of Cartan-Weyl basis of $\mathfrak{s}l_n$. In the deformed $U^{\Psi_3}(\mathfrak{s}l_3)[[\zeta]]$ we find the following elements and their coproducts
\[
\Delta_{\psi_3}(E_{23}) = E_{23} \otimes 1 + 1 \otimes E_{23} - 2 \zeta D_1 \otimes E_{13} e^{-\sigma_{12}^\zeta}
\]
and $D_1$, $E_{13}e^{-\sigma_{12}\zeta}$ with primitive coproducts. Define $\iota_2$ by its values on the generators of $\mathcal{F}_{\mathcal{A},q=1}^{(2)}[[\zeta]]$:

$$\iota_2(\overline{H}_\alpha) = -2D_1, \quad \iota_2(\overline{f}_0) = E_{13}e^{-\sigma_{12}}, \quad \iota_2(\overline{f}_1) = E_{23}$$

and extend it to $\mathcal{F}_{\mathcal{A},q=1}^{(2)}[[\zeta]]$ as a homomorphism into $U^{\Psi_4}(\mathfrak{sl}_3)[[\zeta]]$. The defining relations (15) and coproducts (16) are preserved by $\iota_2$, thus the statement holds in this case. Next, if $n = 4$ then take

$$\Psi_4 = \exp(\zeta E_{32} \otimes E_{13}e^{-\sigma_{12}\zeta} + \zeta E_{42} \otimes E_{14}e^{-\sigma_{12}\zeta}) \cdot \exp(D_1 \otimes \sigma_{12}^{-\zeta})$$

and define $\iota_3 : \mathcal{F}_{\mathcal{A},q=1}^{(3)}[[\zeta]] \to U^{\Psi_4}(\mathfrak{sl}_3)[[\zeta]]$ by the following relations

$$\begin{align*}
\iota_3(\overline{H}_\alpha) &= D_2 - 2D_3, \quad \iota_3(\overline{H}_\beta) = D_3 - 2D_2, \quad \iota_3(\overline{f}_0) = -E_{14}e^{-\sigma_{12}\zeta} \\
\iota_3(\overline{f}_1) &= E_{24}, \quad \iota_3(\overline{f}_2) = E_{13}e^{-\sigma_{12}\zeta}, \quad \iota_3(\overline{f}_3) = E_{23} \\
\iota_3(\overline{e}_\alpha) &= -E_{43}, \quad \iota_3(\overline{e}_{-\alpha}) = -E_{34}
\end{align*}$$

where $E_{24} = E_{24} - \zeta E_{34}E_{13}e^{-\sigma_{12}\zeta}$. The only nonprimitive coproducts of the generators in $\iota_3(\mathcal{F}_{\mathcal{A},q=1}^{(3)})[[\zeta]]$ are the following

$$\Delta_{\Psi_4}(E_{23}) = E_{23} \otimes 1 + 1 \otimes E_{23} + \zeta (D_3 - 2D_2) \otimes E_{13}e^{-\sigma_{12}\zeta} + \zeta E_{43} \otimes E_{14}e^{-\sigma_{12}\zeta}$$

$$\Delta_{\Psi_4}(E_{24}') = E_{24}' \otimes 1 + 1 \otimes E_{24}' - \zeta (D_2 + D_3) \otimes E_{14}e^{-\sigma_{12}\zeta} - \zeta E_{13}e^{-\sigma_{12}\zeta} \otimes E_{34}.$$ 

Let us consider the structure of $\mathcal{F}_{\mathcal{A},q=1}^{(3)}[[\zeta]]$. As a topological Hopf algebra it is the completion of $\mathcal{F}_{\mathcal{A},q=1}^{(3)}[[\zeta]] := \{ \sum_{l_1, \ldots, l_s \geq 0} c_{l_1, \ldots, l_s} (\overline{f}_2)^{l_1} (\overline{f}_0)^{l_2} (\overline{e}_{-\alpha})^{l_3} (\overline{H}_\alpha)^{l_4} (\overline{H}_\beta)^{l_5} (\overline{e}_\alpha)^{l_6} (\overline{f}_3)^{l_7} (\overline{f}_1)^{l_s} \}$.

The coproducts of the generators and the commutation relations are obtained from their quantum counterparts

$$\begin{align*}
\Delta(\overline{f}_1) &= \overline{f}_1 \otimes 1 + 1 \otimes \overline{f}_1 - \zeta (\overline{H}_\alpha + \overline{H}_\beta) \otimes \overline{f}_0 + \zeta \overline{f}_2 \otimes \overline{e}_{-\alpha} \\
\Delta(\overline{f}_3) &= \overline{f}_3 \otimes 1 + 1 \otimes \overline{f}_3 + \zeta \overline{H}_\beta \otimes \overline{f}_2 + \zeta \overline{e}_\alpha \otimes \overline{f}_0,
\end{align*}$$
where we have written only the generators with nonprimitive coproducts,
\[
[\overline{e}_\alpha, \overline{e}_{-\alpha}] = \overline{H}_\alpha, \quad [\overline{e}_\alpha, \overline{f}_0] = -\overline{f}_2, \quad [\overline{e}_\alpha, \overline{f}_1] = \zeta \overline{f}_2 + \overline{f}_3 + \zeta \overline{f}_2 \overline{H}_\alpha
\]
\[
\overline{e}_\alpha, \overline{f}_2 = 0, \quad [\overline{e}_\alpha, \overline{f}_3] = 0, \quad [\overline{f}_0, \overline{e}_{-\alpha}] = 0
\]
\[
[\overline{f}_1, \overline{e}_{-\alpha}] = \zeta \overline{f}_0 \overline{e}_{-\alpha}, \quad [\overline{f}_2, \overline{e}_{-\alpha}] = \overline{f}_0, \quad [\overline{f}_3, \overline{e}_{-\alpha}] = -\zeta \overline{f}_0 - \overline{f}_1 + \zeta \overline{f}_2 \overline{e}_{-\alpha}
\]
\[
[\overline{f}_0, \overline{f}_1] = -\zeta \overline{f}_0^2, \quad [\overline{f}_0, \overline{f}_2] = 0, \quad [\overline{f}_0, \overline{f}_3] = \zeta \overline{f}_2 \overline{f}_0
\]
\[
[\overline{f}_1, \overline{f}_2] = 0, \quad [\overline{f}_1, \overline{f}_3] = \zeta \overline{f}_2 \overline{f}_1, \quad [\overline{f}_2, \overline{f}_3] = \zeta \overline{f}_2^2.
\]
To see that any other relation in \( \mathcal{F}_{\mathcal{A}, q=1}^{(3)}[[\zeta]] \) follows from the introduced ones, we consider the quantum analogues of the commutation relations as the ordering rules in \( \mathcal{F}_{\mathcal{A}}^{(3)}[[\zeta]] \) with the normal ordering \( f_2 < f_0 < \hat{e}_{-\alpha} < H_\alpha < H_\beta < \hat{e}_\alpha < f_3 < f_1 \), then by the Diamond lemma [2] any monomial \( m \in \mathcal{F}_{\mathcal{A}}^{(3)}[[\zeta]] \) can be brought to the form (18), indeed otherwise there would exist a nontrivial relation between the ordered monomials and any such relation must have zero coefficients as the normal ordering in \( \mathcal{F}_{\mathcal{A}}^{(3)}[[\zeta]] \) compatible with the ordering in \( \mathcal{D}_{\mathcal{A}}^{(3)}[[\zeta]] \): \( \hat{e}_{-\beta} < \hat{e}_{-\alpha-\beta} < \hat{e}_{-\alpha} < H_\alpha < H_\beta < \hat{e}_\alpha < \hat{e}_{\delta-\beta} < \hat{e}_{\delta-\alpha-\beta} \). Thus any monomial in \( \mathcal{F}_{\mathcal{A}, q=1}^{(3)}[[\zeta]] \) can be uniquely ordered as well and no additional relations come from \( \mathcal{F}_{\mathcal{A}}^{(3)}[[\zeta]] \). Now it is direct to check that the extension of \( \iota_3 \) to a homomorphism preserves the coproducts and relations in \( \mathcal{F}_{\mathcal{A}}^{(3)}[[\zeta]] \).

The main new result of this paper is an explicit form of \( F_p \) when \( n = 4 \) is obtained as \( F_p = (\iota_3 \otimes \iota_3)(\overline{F}_p^{3\mathbb{Z}}) \cdot \Psi_4 \) and is the following
\[
(1 \otimes 1 - \zeta \otimes E_{23} + \zeta^2 D_3 \otimes E_{13} e^{-\sigma_{12}^\zeta})^{(D_3 \otimes 1)} \cdot (1 \otimes 1 - \zeta \otimes E_{34})^{(D_4 \otimes 1)},
\]
\[
\exp(-\zeta^2 E_{43}(D_2 + D_3) \otimes E_{14} e^{-\sigma_{12}^\zeta}) \cdot \exp(\zeta E_{43} \otimes (E_{24} - \zeta E_{34} E_{13} e^{-\sigma_{12}^\zeta})).
\]
\[
\exp(\zeta E_{43} \otimes E_{13} e^{-\sigma_{12}^\zeta}) \cdot (1 \otimes 1 - \zeta \otimes E_{23} + \zeta^2 D_2 \otimes E_{13} e^{-\sigma_{12}^\zeta})^{(D_2 - D_3 \otimes 1)}.
\]
\[
\exp(\zeta E_{32} \otimes E_{13} e^{-\sigma_{12}^\zeta} + \zeta E_{42} \otimes E_{14} e^{-\sigma_{12}^\zeta}) \cdot \exp(D_1 \otimes \sigma_{12}^\zeta).
\]
Forming the universal \( R \)-matrix \( R_p = F_{p_{21}} F_p^{-1} \) we obtain in the first order in \( \zeta \) the Gerstenhaber-Giaquinto \( n = 4 \) \( r_p \)-matrix:
\[
D_1 \wedge E_{12} + E_{14} \wedge E_{42} + E_{13} \wedge E_{32} + D_2 \wedge E_{23} + E_{24} \wedge E_{43} + D_3 \wedge E_{34} + E_{13} \wedge E_{43}.
\]
Note that there is a whole family of homomorphisms \( \iota_3^{(a)} \):

\[
\iota_3^{(a)}(H_{\alpha}) = D_2 - 2D_3, \quad \iota_3^{(a)}(H_{\beta}) = D_3 - 2D_2, \quad \iota_3^{(a)}(f_0) = \frac{1}{a} E_{14} e^{-\sigma_{12}^{-\xi}}
\]

\[
\iota_3^{(a)}(f_1) = -\frac{1}{a} E_{24}, \quad \iota_3^{(a)}(f_2) = E_{13} e^{-\sigma_{12}^{-\xi}}, \quad \iota_3^{(a)}(f_3) = E_{23}
\]

\[
\iota_3^{(a)}(\bar{e}_\alpha) = a E_{45}, \quad \iota_3^{(a)}(\bar{e}_{-\alpha}) = \frac{1}{a} E_{34}
\]

and thus \((\iota_3^{(a)} \otimes \iota_3^{(a)})(F_{\text{aff}})^* \Psi_4\) for any \( a \neq 0 \) leads to quantization of

\[
D_1 \wedge E_{12} + E_{13} \wedge E_{32} + E_{14} \wedge E_{42} + D_2 \wedge E_{23} + E_{24} \wedge E_{43} + \frac{1}{a} D_3 \wedge E_{34} + a E_{13} \wedge E_{43}.
\]

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Semi-classical twists for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ boundary $r$–matrices

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