Quasi-theories and their equivariant orthogonal spectra

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Abstract. In this paper we construct orthogonal $G$–spectra up to a weak equivalence for the quasi-theory $QE^*_nG(\_)$ corresponding to certain cohomology theories $E$. The construction of the orthogonal $G$–spectrum for quasi-elliptic cohomology can be applied to the constructions for quasi-theories.

1. Introduction

In [6] we construct a functor $Q$ from the category of orthogonal ring spectra to the category of $I_G$–FSP. If $E$ is a global cohomology theory, $Q(E)$ weakly represents the cohomology theory

\[
QE^*_nG(X) := \prod_{\sigma \in G^{tor}} E^*_\Lambda(\sigma)(X^\sigma) = \left( \prod_{\sigma \in G^{tor}} E^*_\Lambda(\sigma)(X^\sigma) \right)^G.
\]

The image of global $K$–spectrum is a $I_G$–FSP representing quasi-elliptic cohomology up to a weak equivalence.

Quasi-elliptic cohomology is a variant of elliptic cohomology theories, which is the generalized elliptic cohomology theory associated to the Tate curve $\text{Tate}(q)$ over $\text{Spec}\mathbb{Z}(\!(q)\!)$ [Section 2.6, [1]]. Quasi-elliptic cohomology is defined over $\text{Spec}\mathbb{Z}[q^\pm]$. Inverting $q$ allows us to define a sufficiently non-naive equivariant cohomology theory and to interpret some constructions more easily. Its relation with Tate K-theory is

\[
QE^*\text{Ell}_G(X) \otimes_{\mathbb{Z}(\!(q)\!)} \mathbb{Z}(\!(q)\!) = (K^*_{\text{Tate}})_G(X)
\]

Motivated by quasi-elliptic cohomology, we construct quasi-theories $QE^*_nG(\_)$ in [9]. Quasi-elliptic cohomology, the theories $QE^*_nG(\_)$ defined in [11] and the generalized quasi-elliptic cohomology in Example [2.4] are all special cases of quasi-theories.

In this paper we show that the idea of constructing the functor $Q$ can be applied to construct a family of functors $Q_n$ from the category of orthogonal ring spectra to the category of $I_G$–FSP. Especially, the functor $Q_1$ is $Q$. In other words, we construct a $I_G$–FSP representing $QE^*_nG(\_)$ up to weak equivalence for each positive integer $n$ and each compact Lie group $G$.

In this paper we show the construction of functors $Q_n$. The idea is analogous to the construction of $Q$ in [6]. For the readers’ convenience, we still include all
the details in this paper. In Section 2 we recall the definition and examples of quasi-theories. In Section 3 we recall a category of orthogonal $G$–spectra introduced in [9]. In Section 4 we construct a space $QE_{G,n,m}$ representing the $m$–th $G$–equivariant quasi-theory $QE_{n,G}^m(\ast)$ up to a weak equivalence. In Section 5 we construct a $I_G$–FSP representing $QE_{n,G}^r(\ast)$ up to weak equivalence for certain cohomology theories $E$ and construct the functors $Q_n$. In the appendix, we construct some faithful group representations needed in the construction of the $I_G$–FSP.

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2. The Quasi-theory $QE_{n,G}(\ast)$

In this section we recall the quasi-theories. The main reference for that is [9]. Let $G$ be a compact Lie group and $n$ denote a positive integer. Let $G^\text{tors}_{\text{conj}}$ denote a set of representatives of $G$–conjugacy classes in the set $G^\text{tors}$ of torsion elements in $G$. Let $G^n$ denote set

\[
\{\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) | \sigma_i \in G^\text{tors}_{\text{conj}}, [\sigma_i, \sigma_j] \text{ is the identity element in } G\}.
\]

Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in G^n$. Define

\[
(2.1) \quad C_G(\sigma) := \bigcap_{i=1}^n C_G(\sigma_i);
\]

\[
(2.2) \quad \Lambda_G(\sigma) := C_G(\sigma) \times \mathbb{R}^n / \langle (\sigma_1, -e_1), (\sigma_2, -e_2), \cdots, (\sigma_n, -e_n) \rangle.
\]

where $C_G(\sigma_i)$ is the centralizer of each $\sigma_i$ in $G$ and $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{R}^n$. Let $q : T \to U(1)$ denote the representation $t \mapsto e^{2\pi it}$. Let $q_i = 1 \otimes \cdots \otimes q \otimes \cdots \otimes 1 : T^n \to U(1)$ denote the tensor product with $q$ at the $i$–th position and trivial representations at other position. The representation ring

\[R(T^n) \cong R(T)^\otimes n = \mathbb{Z}[q_1^\pm, \cdots, q_n^\pm].\]

We have the exact sequence

\[
(2.3) \quad 1 \to C_G(\sigma) \to \Lambda_G(\sigma) \xrightarrow{\pi} T^n \to 0
\]

where the first map is $g \mapsto [g, 0]$ and the second map is $\pi([g, t_1, \cdots, t_n]) = (e^{2\pi it_1}, \cdots, e^{2\pi it_n})$. Then the map $\pi^* : R(T^n) \to R\Lambda_G(\sigma)$ equips the representation ring $R\Lambda_G(\sigma)$ the structure as an $R(T^n)$–module.

This is Lemma 3.1 [9] presenting the relation between $R\Lambda_G(\sigma)$ and $R\Lambda_G(\sigma)$.

Lemma 2.1. $\pi^* : R(T^n) \to R\Lambda_G(\sigma)$ exhibits $R\Lambda_G(\sigma)$ as a free $R(T^n)$–module.

There is an $R(T^n)$–basis of $R\Lambda_G(\sigma)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \to V_\lambda|_{C_G(\sigma)}$ to $C_G(\sigma)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(\sigma)$.

Definition 2.2. For equivariant cohomology theories $\{E_H^r\}_H$ and any $G$–space $X$, the corresponding quasi-theory $QE_{n,G}^r(X)$ is defined to be

\[
\prod_{\sigma \in G^n} E_{\Lambda_G(\sigma)}(X^\sigma).
\]
Example 2.3 (Motivating example: Tate K-theory and quasi-elliptic cohomology). Tate K-theory is the generalized elliptic cohomology associated to the Tate curve. The elliptic cohomology theories form a sheaf of cohomology theories over the moduli stack of elliptic curves $\mathcal{M}_{ell}$. Tate K-theory over $\text{Spec} \mathbb{Z}((q))$ is obtained when we restrict it to a punctured completed neighborhood of the cusp at $\infty$, i.e. the Tate curve $\text{Tate}(q)$ over $\text{Spec} \mathbb{Z}((q))$ [Section 2.6, [1]]. The divisible group associated to Tate K-theory is $\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}$. The relation between Tate K-theory and string theory is better understood than most known elliptic cohomology theories.

In addition, Tate K-theory has the closest ties to Witten’s original insight that the elliptic cohomology of a space $X$ is related to the $\mathbb{T}$-equivariant K-theory of the free loop space $LX = C^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on $LX$ by rotating loops. Ganter gave a careful interpretation in Section 2, [5] of this statement that the definition of $G$-equivariant Tate K-theory for finite groups $G$ is modelled on the loop space of a global quotient orbifold.

Other than the theory over $\text{Spec} \mathbb{Z}((q))$, we can define variants of Tate K-theory over $\text{Spec} \mathbb{Z}[q]$ and $\text{Spec} \mathbb{Z}[q^\pm]$ respectively. The theory over $\text{Spec} \mathbb{Z}[q^\pm]$ is of especial interest. Inverting $q$ allows us to define a sufficiently non-naive equivariant cohomology theory and to interpret some constructions more easily in terms of extensions of groups over the circle. The resulting cohomology theory is called quasi-elliptic cohomology [12][7][8]. Its relation with Tate K-theory is

$$(2.4) \quad Q\text{Ell}_{G}^*(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K_{\text{tate}}^*)_G(X)$$

which also reflects the geometric nature of the Tate curve. $Q\text{Ell}_{G}^*(\text{pt})$ has a direct interpretation in terms of the Katz-Mazur group scheme $T$ [Section 8.7, [10]]. The idea of quasi-elliptic cohomology is motivated by Ganter’s construction of Tate K-theory [3]. It is not an elliptic cohomology but a more robust and algebraically simpler treatment of Tate K-theory. This new theory can be interpreted in a neat form by equivariant K-theories. Some formulations in it can be generalized to equivariant cohomology theories other than Tate K-theory.

Quasi-elliptic cohomology $Q\text{Ell}_{G}^*(-)$ is exactly the quasi-theory $QK_{1,G}^*(-)$ in Definition 2.2.

Example 2.4 (Generalized Tate K-theory and generalized quasi-elliptic cohomology). In Section 2 [5] Ganter gave an interpretation of $G$-equivariant Tate K-theory for finite groups $G$ by the loop space of a global quotient orbifold. Apply the loop construction $n$ times, we can get the $n$–th generalized Tate K-theory. The divisible group associated to it is $\mathbb{G}_m \oplus (\mathbb{Q}/\mathbb{Z})^n$.

With quasi-theories, we can get a neat expression of it. Consider the quasi-theory

$$QK_{n,G}^*(X) = \prod_{\sigma \in G_n^2} K_{\Lambda_{G}(\sigma)}^*(X^\sigma).$$

$QK_{n,G}^*(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q))^\otimes n$ is isomorphic to the $n$–th generalized Tate K-theory.

3. A new category of orthogonal $G$–spectra

It is difficult to construct a concrete representing spectrum for elliptic cohomology. In Section 4 [6] we formulate a new category of spectra with larger class of weak equivalence than that in [11]. In Section 6 [6] we construct an orthogonal $G$–spectrum for any compact Lie group $G$ representing $Q\text{Ell}_{G}^*(\text{-})$ in this new category of orthogonal $G$–spectra.
First we recall the category of orthogonal $G$–spectra in $\mathcal{G}$ and the category $GwS$ that we will work in. The weak equivalence of interest is the $\pi_{*}\mathcal{G}$–isomorphism.

**Definition 3.1.** For subgroups $H$ of $G$ and integers $q$, define the homotopy groups $\pi_{q}^{H}(X)$ of a $G$–pre spectrum $X$ by
\begin{equation}
\pi_{q}^{H}(X) = \operatorname{colim}_{V} \pi_{V}^{H}(\Omega^{V}X(V)) \text{ if } q \geq 0,
\end{equation}
where $V$ runs over the indexing $G$–spaces in the chosen universe, and
\begin{equation}
\pi_{-q}^{H}(X) = \operatorname{colim}_{V \supset R} \pi_{V}^{H}(\Omega^{V-R}X(V)) \text{ if } q > 0.
\end{equation}
A map $f : X \rightarrow Y$ of $G$–prespectra is a $\pi_{*}\mathcal{G}$–isomorphism if it induces isomorphisms on all homotopy groups.

A map of orthogonal $G$–spectra is a $\pi_{*}\mathcal{G}$–isomorphism if its underlying map of $G$–prespectra is a $\pi_{*}\mathcal{G}$–isomorphism.

**Definition 3.2.** The category $GwS$ is the homotopy category of the category of orthogonal $G$–spectra with the weak equivalence defined by
\begin{equation}
X \sim Y \text{ if } \pi_{0}^{H}(X(V)) = \pi_{0}^{H}(Y(V)),
\end{equation}
for each faithful $G$–representation $V$ and any closed subgroup $H$ of $G$.

An orthogonal $G$–spectrum $X$ in $GwS$ is said to represent a theory $H_{G}$ if we have a natural map
\begin{equation}
\pi_{0}^{H}(X(V)) = H_{V}(G/H),
\end{equation}
for each faithful $G$–representation $V$ and any closed subgroup $H$ of $G$.

**Lemma 3.3.** If a map $f : X \rightarrow Y$ of orthogonal $G$–spectra induces isomorphisms on the homotopy groups, i.e.
\begin{equation}
f : \pi_{q}^{H}(X(V)) \cong \pi_{q}^{H}(Y(V))
\end{equation}
for each faithful $G$–representation $V$, and any closed subgroup $H$ of $G$, then $f$ is a $\pi_{*}\mathcal{G}$–isomorphism.

We will work in the homotopy category of the category of orthogonal $G$–spectra with the weak equivalence defined in (3.3). This homotopy category $Gw\mathcal{T}$ is smaller than the homotopy category of orthogonal $G$–spectra that we usually talked about, where the weak equivalence involved is the $\pi_{*}\mathcal{G}$–isomorphism. However, it seems the information that each object contains is enough to define an equivariant cohomology theory.

The homotopical adjunction below is a way to describe the relation between $G$–equivariant homotopy theory and those equivariant homotopy theory for its closed subgroups. It is introduced in Definition 4.4 [6].

**Definition 3.4 (homotopical adjunction).** Let $H$ and $G$ be two compact Lie groups. Let
\begin{equation}
L : GT \rightarrow HT \text{ and } R : HT \rightarrow GT
\end{equation}
be two functors between the category of $G$–spaces and that of $H$–spaces. A left-to-right homotopical adjunction is a natural map
\begin{equation}
\operatorname{Map}_{H}(LX, Y) \rightarrow \operatorname{Map}_{G}(X, RY),
\end{equation}
which is a weak equivalence of spaces when $X$ is a $G$–CW complex.
Analogously, a right-to-left homotopical adjunction is a natural map

\[(3.8) \quad \text{Map}_G(X, RY) \to \text{Map}_H(LX, Y)\]

which is a weak equivalence of spaces when \(X\) is a \(G\)-CW complex.

\(L\) is called a homotopical left adjoint and \(R\) a homotopical right adjoint.

4. Equivariant spectra

In this section, we construct a space \(QE_{G,n,m}\) representing the \(m\)-th \(G\)-equivariant quasi-theory \(QE_{m,G}\)(\(\cdot\)) up to a weak equivalence.

Let \(G\) be a compact Lie group and \(σ \in G^n\). Let \(Γ\) denote the subgroup \(\langle σ_1, \ldots, σ_n \rangle\) of \(G\). Let

\[S_{G,σ} := \text{Map}_Γ(G, *_{K}E(Γ/K))\]

where * denotes the join, \(K\) goes over all the maximal subgroups of \(Γ\) and \(E(Γ/K)\) is the universal space of the abelian group \(Γ/K\).

**Lemma 4.1.** For any closed subgroup \(H \leq G\), \(S_{G,σ}\) satisfies

\[(4.1) \quad S^H_{G,σ} \simeq \begin{cases} \text{pt}, & \text{if for any } b \in G, \ b^{-1}Γb \not\leq H; \\ \emptyset, & \text{if there exists } b \in G \text{ such that } b^{-1}Γb \leq H. \end{cases}\]

**Proof.**

\[(4.2) \quad S^H_{G,σ} = \text{Map}_Γ(G/H, *_{K}E(Γ/K)).\]

If there exists an \(b \in G\) such that \(b^{-1}Γb \leq H\), it is equivalent to that there exists points in \(G/H\) that can be fixed by \(Γ\). But there are no points in \( *_{K}E(Γ/K) \) that can be fixed by the whole group \(Γ\). So there is no \(Γ\)-equivariant map from \(G/H\) to \( *_{K}E(Γ/K) \). In this case \(S^H_{G,σ}\) is empty.

If for any \(b \in G\), \(b^{-1}Γb \not\leq H\), it is equivalent to say that there are no points in \(G/H\) that can be fixed by \(Γ\). Any proper subgroup \(L\) of \(Γ\) is contained in some maximal subgroup of \(Γ\). \( ( *_{K}E(Γ/K))_L \) is the join of several contractible spaces \(E(Γ/K)^L \). Thus, it is contractible. So all the homotopy groups \(π_n(( *_{K}E(Γ/K))_L) \) are trivial. For any \(n \geq 1\) and any \(L\)-equivariant map

\[f : (G/H)^n \to *_{K}E(Γ/K)\]

from the \(n\)-skeleton of \(G/H\), the obstruction cocycle is zero.

Then by equivariant obstruction theory, \(f\) can be extended to the \((n+1)\)-cells of \(G/H\), and any two extensions \(f\) and \(f'\) are \(Γ\)-homotopic.

So in this case \(S^H_{G,σ}\) is contractible. \(\square\)

**Theorem 4.2.** A homotopical right adjoint of the functor \(L_σ : GT \to C_G(σ)T, \ X \mapsto X^σ\) from the category of \(G\)-spaces to that of \(C_G(σ)\)-spaces is

\[(4.3) \quad R_σ : C_G(σ)T \to GT, \ Y \mapsto \text{Map}_{C_G(σ)}(G, Y * S_{C_G(σ),σ}).\]

**Proof.** Let \(H\) be any closed subgroup of \(G\).

First we show given a \(C_G(σ)\)-equivariant map \(f : (G/H)^n \to Y\), it extends uniquely up to \(C_G(σ)\)-homotopy to a \(C_G(σ)\)-equivariant map

\[\tilde{f} : G/H \to Y * S_{C_G(σ),σ}.\]
$f$ can be viewed as a map $(G/H)^\sigma \to Y \ast S_{C_G(\sigma), \sigma}$ by composing with the inclusion of one end of the join

$$Y \to Y \ast S_{C_G(\sigma), \sigma}, \ y \mapsto (1y, 0).$$

If $bH \in (G/H)^\sigma$, define $\tilde{f}(bH) := f(bH)$.

If $bH$ is not in $(G/H)^\sigma$, its stabilizer group does not contain $\Gamma$. By Lemma 4.1 for any subgroup $L$ of its stabilizer group, $S_{C_G(\sigma), \sigma}$ is contractible. So $(Y \ast S_{C_G(\sigma), \sigma})_{G/H} = (Y^L \ast S_{C_G(\sigma), \sigma})_{G/H}$ is contractible. In other words, if $L$ occurs as the isotropy subgroup of a point outside $(G/H)^\sigma$, $\pi_n((Y \ast S_{C_G(\sigma), \sigma})_{G/H})$ is trivial. By equivariant obstruction theory, $f$ can extend to a $C_G(\sigma)$-equivariant map $\tilde{f} : G/H \to Y \ast S_{C_G(\sigma), \sigma}$, and any two extensions are $C_G(\sigma)$-homotopy equivalent.

In addition, $S_{C_G(\sigma), \sigma}$ is contained in the end $Y$ of the join. Thus, $\text{Map}_{C_G(\sigma)}((G/H)^\sigma, Y)$ is weak equivalent to $\text{Map}_{C_G(\sigma)}(G/H, Y \ast S_{C_G(\sigma), \sigma})$.

Moreover, we have the equivalence by adjunction

$$\text{Map}_{C_G(\sigma)}(G/H, \text{Map}_{C_G(\sigma)}(Y \ast S_{C_G(\sigma), \sigma})) \cong \text{Map}_{C_G(\sigma)}(G/H, Y \ast S_{C_G(\sigma), \sigma}).$$

So we get

$$R_\sigma Y^\sigma = \text{Map}_{C_G(\sigma)}(G/H, R_\sigma Y) \cong \text{Map}_{C_G(\sigma)}((G/H)^\sigma, Y).$$

Let $X$ be of the homotopy type of a $G$-CW complex. Let $X^k$ denote the $k$-skeleton of $X$. Consider the functors $\text{Map}_{C_G(\sigma)}(\sigma, R_\sigma Y)$ and $\text{Map}_{C_G(\sigma)}((-)^\sigma), Y)$

from $G\mathcal{T}$ to $\mathcal{T}$. Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\text{Map}_{C_G(\sigma)}(\sigma, R_\sigma Y)$ to $\text{Map}_{C_G(\sigma)}((-)^\sigma), Y)$ by sending a $G$-map $F : X \to R_\sigma Y$ to the composition

$$X^\sigma \xrightarrow{F^\sigma} (R_\sigma Y)^\sigma \to Y^\sigma \subseteq Y$$

with the second map $f \to f(e)$. Note that for any $f \in (R_\sigma Y)^\sigma$, $i = 1, \cdots, n,

\begin{align*}
(\sigma_i \cdot f)(e) &= f(\sigma_i e) = f(\sigma_i) = \sigma_i \cdot f(e) \equiv (Y \ast S_{C_G(\sigma), \sigma})^\sigma = Y^\sigma
\end{align*}

and the second map is well-defined. It gives weak equivalence on orbits, as shown in (4.3). Thus, $R_\sigma$ is a homotopical right adjoint of $L_\sigma$.

The subgroup $\{(1, t) \in \Delta_G(\sigma) | t \in \mathbb{R}^n\}$ of $\Delta_G(\sigma)$ is isomorphic to $\mathbb{R}^n$. We use the same symbol $\mathbb{R}^n$ to denote it.

**THEOREM 4.3.** Let $Y$ be a $\Delta_G(\sigma)$-space. Consider the functor $L_\sigma : G\mathcal{T} \to \Delta_G(\sigma)\mathcal{T}, X \mapsto X^\sigma$ where $\Delta_G(\sigma)$ acts on $X^\sigma$ by $[g, t] \cdot x = gx$. The functor $R_\sigma : \Delta_G(\sigma)\mathcal{T} \to G\mathcal{T}$ with

$$R_\sigma Y = \text{Map}_{C_G(\sigma)}(G, Y^{\mathbb{R}^n} \ast S_{C_G(\sigma), \sigma})$$

is a homotopical right adjoint of $L_\sigma$.

**PROOF.** Let $X$ be a $G$-space. Let $H$ be any closed subgroup of $G$. For any $G$-space $X$, $\mathbb{R}^n$ acts trivially on $X^\sigma$, thus, the image of any $\Delta_G(\sigma)$-equivariant map $X^\sigma \to Y$ is in $Y^{\mathbb{R}^n}$. So we have $\text{Map}_{\Delta_G(\sigma)}(X^\sigma, Y) = \text{Map}_{C_G(\sigma)}(X^\sigma, Y^{\mathbb{R}^n})$.

First we show $f : (G/H)^\sigma \to Y^{\mathbb{R}^n}$ extends uniquely up to $C_G(\sigma)$-homotopy to a $C_G(\sigma)$-equivariant map $\tilde{f} : G/H \to Y^{\mathbb{R}^n} \ast S_{C_G(\sigma), \sigma}$. $f$ can be viewed as a
map $(G/H)^\sigma \to Y^{R^n} S_{C_G(\sigma),\sigma}$ by composing with the inclusion as the end of the join
\[ Y^{R^n} \to Y^{R^n} S_{C_G(\sigma),\sigma}, \quad y \mapsto (1y, 0). \]

The rest of the proof is analogous to that of Theorem 4.3. \qed

Theorem 4.3 implies Theorem 4.4 directly.

**Theorem 4.4.** For any compact Lie group $G$ and any integer $n$ and $m$, let $E_{G,n,m}$ denote the space representing the $m$–th $G$–equivariant $E_n$–theory. Then the theory $Q E_{n,G}$ is weakly represented by the space
\[ Q E_{G,n,m} := \prod_{\sigma \in G^2} \mathcal{R}_\sigma(E_{\Lambda G(\sigma),n,m}) \]
in the sense of (4.3)
\[ \pi_0(Q E_{G,n,m}) = Q E_{n,G}(S^0). \]
where $\mathcal{R}_\sigma(E_{\Lambda G(\sigma),n,m})$ is the space
\[ \text{Map}_{C_G(\sigma)}(G, E_{\Lambda G(\sigma),n,m} \wr S_{C_G(\sigma),\sigma}). \]

5. **Orthogonal $G$–spectrum of $Q E_{n,G}$**

In this section, we consider equivariant cohomology theories $E^n_G$ that have the same key features as equivariant complex $K$-theories. More explicitly,
- The theories $\{E^n_G\}_G$ have the change-of-group isomorphism, i.e., for any closed subgroup $H$ of $G$ and $H$–space $X$, the change-of-group map $\rho^G_H : E^n_G(G \times H X) \to E^n_H(X)$ defined by $E^n_G(G \times H X) \xrightarrow{\phi^*} E^n_H(G \times H X) \xrightarrow{i^*} E^n_H(X)$ is an isomorphism where $\phi^*$ is the restriction map and $i : X \to G \times H X$ is the $H$–equivariant map defined by $i(x) = [e, x]$.
- There exists an orthogonal spectrum $E$ such that for any compact Lie group $G$ and "large" real $G$–representation $V$ and a compact $G$–space $B$ we have a bijection $E^n_G(B) \to [B_+, E(V)]^G$. And $(E_G, \eta^E, \mu^E)$ is the underlying orthogonal $G$–spectrum of $E$.
- Let $G$ be a compact Lie group and $V$ an orthogonal $G$–representation. For every ample $G$–representation $W$, the adjoint structure map $\tilde{\sigma}^E_{V,W} : E(V) \to \text{Map}(S^W, E(V \oplus W))$ is a $G$–weak equivalence.

In this section we construct a $\mathcal{L}_G$–FSP $(Q E_n(G, \cdot), \eta E^n, \mu E^n)$ representing the theory $Q E^n_{n,G}(\cdot)$ in the category $G w S$ defined in Definition 3.2.

5.1. **The construction of $Q E_n(G, \cdot)$**

5.1.1. **The construction of $S(G,V)_\sigma$.** In this section, for each $\sigma \in G_n$, we construct an orthogonal version $S(G,V)_\sigma := \text{Sym}(V) \setminus \text{Sym}(V)^\sigma$ of the space $S_{G,\sigma}$. It is the space classified by the condition 3.1 which is also the condition classifying $S_{G,\sigma}$.

Let $V$ be a real $G$–representation. Let $\text{Sym}^n(V)$ denote the $n$–th symmetric power $V^{\otimes n}$, which has an evident $G \wr \Sigma_n$–action on it. Let
\[ \text{Sym}(V) := \bigoplus_{n \geq 0} \text{Sym}^n(V). \]

If $V$ is an ample $G$–representation, $\text{Sym}(V)$ is a faithful $H$–representation, thus, a complete $H$–universe.
The complex conjugation on $H$ subgroup $G$ of $C$ is isomorphic to $\eta$. Then $(\sigma \vee \tau)$ represents the theory $E_{\Lambda G(\sigma)}^V(\cdot)$. So we have 

$$E_{\Lambda G(\sigma)}^V(\cdot)$$

is isomorphic to 

$$[X^\sigma, \text{Map}(S(V)^{\otimes}, E((V)^{\otimes} \oplus V^\sigma))]_{\Lambda G(\sigma)}.$$
making the unit, associativity and centrality of unit diagram commute. And $\eta_\sigma(G,V)$ is $C_G(\sigma)-$equiariant and $\mu^\sigma_{(\sigma,\tau)}(\langle G, V \rangle, (H, W))$ is $C_{G \times H}(\sigma, \tau)-$equiariant.

(ii) Let $\Delta G$ denote the diagonal map $G \to G \times G$, $g \mapsto (g, g)$. Let $\tilde{\sigma}_\sigma(G, V, W) : F_\sigma(G, V) \to \text{Map}(SW^G, F_\sigma(G, V \oplus W))$ denote the map

$$x \mapsto \langle w \mapsto (\Delta G \circ \mu^\sigma_{(\sigma,\tau)}(\langle G, V \rangle, (G, W))) \rangle \langle x, \eta_\sigma(G, W)(w) \rangle \rangle.$$

Then $\tilde{\sigma}_\sigma(G, V, W)$ is a $\Lambda G(\sigma)-$weak equivalence when $V$ is an ample $G-$representation.

(iii) If $(E, \eta^E, \mu^E)$ is commutative, we have

$$\mu^\sigma_{(\sigma,\tau)}((G, V), (H, W))(x \wedge y) = \mu^\sigma_{(\sigma,\tau)}((H, W), (G, V))(y \wedge x)$$

for any $x \in F_\sigma(G, V)$ and $y \in F_\sigma(H, W)$.

The proof is straightforward and left to the readers.

5.1.3. The construction of $Q\mathbb{E}_n(G, V)$. Recall in Theorem 4.4 we construct a $G-$space $Q\mathbb{E}_{G,n,m}$ representing $Q\mathbb{E}^m_{n,G}(-)$. In this section we go a step further.

Apply Theorem 4.3 we get the conclusion below.

**Proposition 5.3.** Let $V$ be a faithful orthogonal $G-$representation. Let $B'_n(G, V)$ denote the space

$$\prod_{\sigma \in G^+_n} \text{Map}_{C_G(\sigma)}(G, F_\sigma(G, V) \ast S(G, V)_\sigma).$$

$Q\mathbb{E}^V_{n,G}(-)$ is weakly represented by $B'_n(G, V)$ in the sense $\pi_0(B'_n(G, V)) = Q\mathbb{E}^V_{n,G}(S^0)$.

The proof of Proposition 5.3 is analogous to that of Theorem 4.4 step by step. Below is the main theorem in Section 5.1. We will use formal linear combination

$$t_1a + t_2b \text{ with } 0 \leq t_1, t_2 \leq 1, t_1 + t_2 = 1$$

to denote points in join.

**Proposition 5.4.** Let $Q\mathbb{E}_{n,\sigma}(G, V)$ denote

$$\{t_1a + t_2b \in F_\sigma(G, V) \ast S(G, V)_\sigma \mid \|b\| \leq t_2 \}/\{t_1c_0 + t_2b\}.$$

It is the quotient space of a closed subspace of the joint $F_\sigma(G, V) \ast S(G, V)_\sigma$ with all the points of the form $t_1c_0 + t_2b$ collapsed to one point, which we pick as the basepoint of $Q\mathbb{E}_{n,\sigma}(G, V)$, where $c_0$ is the basepoint of $F_\sigma(G, V)$. $Q\mathbb{E}_{n,\sigma}(G, V)$ has the evident $C_G(\sigma)-$action. And it is $C_G(\sigma)-$weak equivalent to $F_\sigma(G, V) \ast S(G, V)_\sigma$. As a result, $\prod_{\sigma \in G^+_n} \text{Map}_{C_G(\sigma)}(G, Q\mathbb{E}_{n,\sigma}(G, V))$ is $G-$weak equivalent to

$$\prod_{\sigma \in G^+_n} \text{Map}_{C_G(\sigma)}(G, F_\sigma(G, V) \ast S(G, V)_\sigma).$$

So when $V$ is a faithful $G-$representation,

$$Q\mathbb{E}_n(G, V) := \prod_{\sigma \in G^+_n} \text{Map}_{C_G(\sigma)}(G, Q\mathbb{E}_{n,\sigma}(G, V))$$

weakly represents $Q\mathbb{E}^V_{n,G}(-)$ in the sense $\pi_0(Q\mathbb{E}_n(G, V)) \equiv Q\mathbb{E}^V_{n,G}(S^0)$.

**Proof.** First we show $F_\sigma(G, V) \ast S(G, V)_\sigma$ is $C_G(\sigma)-$homotopy equivalent to

$$Q\mathbb{E}^\prime_{n,\sigma}(G, V) := \{t_1a + t_2b \in F_\sigma(G, V) \ast S(G, V)_\sigma \mid \|b\| \leq t_2\}.$$ Note that $b \in S(G, V)_\sigma$ is never zero. Let $j : Q\mathbb{E}^\prime_{n,\sigma}(G, V) \longrightarrow F_\sigma(G, V) \ast S(G, V)_\sigma$ be the inclusion. Let $p : F_\sigma(G, V) \ast S(G, V)_\sigma \longrightarrow Q\mathbb{E}^\prime_{n,\sigma}(G, V)$ be the $C_G(\sigma)-$map sending $t_1a + t_2b$ to $t_1a + t_2\min(\|b\|, t_2)\min(\|b\|, t_2)\min(\|b\|, t_2)$. Both $j$ and $p$ are both
continuous and $C_G(\sigma)$–equivariant. $p \circ j$ is the identity map of $QE_{n,\sigma}^*(G, V)$. We can define a $C_G(\sigma)$–homotopy

$$H : (F_\sigma(G, V) \ast S(G, V)_\sigma) \times I \longrightarrow F_\sigma(G, V) \ast S(G, V)_\sigma$$

from the identity map on $F_\sigma(G, V) \ast S(G, V)_\sigma$ to $j \circ p$ by shrinking. For any $t_1a + t_2b \in F_\sigma(G, V) \ast S(G, V)_\sigma$, Define

\begin{equation}
H(t_1a + t_2b, t) := t_1a + t_2((1 - t)b + t \frac{\min\{||b||, t_2\}}{||b||}b).
\end{equation}

Then we show $QE_{n,\sigma}^*(G, V)$ is $G$–weak equivalent to $QE_{n,\sigma}(G, V)$. Let $q : QE_{n,\sigma}^*(G, V) \longrightarrow QE_{n,\sigma}(G, V)$ be the quotient map. Let $H$ be a closed subgroup of $C_G(\sigma)$.

If the group $\Gamma$ is in $H$, since $S(G, V)_\sigma^H$ is empty, so $QE_{n,\sigma}(G, V)^H$ is the end $F_\sigma(G, V)$ and can be identified with $F_\sigma(G, V)^H$. In this case $q^H$ is the identity map.

If $\Gamma$ is not in $H$, $QE_{n,\sigma}^*(G, V)^H$ is contractible. The cone $\{c_0\} \ast S(G, V)_\sigma^H$ is contractible, so $q(\{c_0\} \ast S(G, V)_\sigma^H) = q(\{c_0\} \ast S(G, V)_\sigma^H)$ is contractible. Note that the subspace of all the points of the form $t_1c_0 + t_2b$ for any $t_1$ and $b$ is $q(\{c_0\} \ast S(G, V)_\sigma^H)$. Therefore, $QE_{n,\sigma}(G, V)^H = QE_{n,\sigma}^*(G, V)^H / q(\{c_0\} \ast S(G, V)_\sigma^H)$ is contractible.

Therefore, $QE_{n,\sigma}^*(G, V)$ is $G$–weak equivalent to $F_\sigma(G, V) \ast S(G, V)_\sigma$. \hfill \Box

**Proposition 5.5.** Let $\sigma \in G^\circ_n$. Let $Y$ be a based $\Lambda_G(\sigma)$–space. Let $\bar{Y_\sigma}$ denote the $C_G(\sigma)$–space

$$\{t_1a + t_2b \in Y^{R^n} \ast S(G, V)_\sigma ||b|| \leq t_2\}/\{t_1y_0 + t_2b\}.$$ 

It is the quotien space of a closed subspace of $Y^{R^n} \ast S(G, V)_\sigma$ with all the points of the form $t_1y_0 + t_2b$ collapsed to one point, i.e. the basepoint of $\bar{Y_\sigma}$, where $y_0$ is the basepoint of $Y$. $\bar{Y_\sigma}$ is $C_G(\sigma)$–weak equivalent to $Y^{R^n} \ast S(G, V)_\sigma$. As a result, the functor $R_\sigma : C_G(\sigma)T \longrightarrow GT$ with $R_\sigma Y = \operatorname{Map}_{C_G(\sigma)}(G, \bar{Y_\sigma})$ is a homotopical right adjoint of $L : GT \longrightarrow C_G(\sigma)T$, $X \mapsto X^\circ$.

The proof is analogous to that of Theorem 4.3 and Proposition 5.4.

**Remark 5.6.** We can consider $QE_{n,\sigma}(G, V)$ as a quotient space of a subspace of $F_\sigma(G, V) \times \operatorname{Sym}(V) \times I$

\begin{equation}
\{(a, b, t) \in F_\sigma(G, V) \times \operatorname{Sym}(V) \times I ||b|| \leq t; \text{ and } b \in S(G, V)_\sigma \text{ if } t \neq 0\}
\end{equation}

by identifying points $(a, b, 1)$ with $(a', b, 1)$, and collapsing all the points $(c_0, b, t)$ for any $b$ and $t$. In other words, the end $F_\sigma(G, V)$ in the join $F_\sigma(G, V) \ast S(G, V)_\sigma$ is identified with the points of the form $(a, 0, 0)$ in (5.6).

**Proposition 5.7.** For each $\sigma \in G^n_n$,

$$QE_{n,\sigma} : \mathcal{I}_G \longrightarrow C_G(\sigma)T, \ (G, V) \mapsto QE_{n,\sigma}(G, V)$$

is a well-defined functor. As a result,

$$QE_n : \mathcal{I}_G \longrightarrow GT, \ (G, V) \mapsto \prod_{\sigma \in G^n_n} \operatorname{Map}_{C_G(\sigma)}(G, QE_{n,\sigma}(G, V))$$

is a well-defined functor.
PROOF. Let $V$ and $W$ be $G$–representations and $f : V \to W$ a linear isometric isomorphism. Then $f$ induces a $C_G(\sigma)$–homeomorphism $F_\sigma(f)$ from $F_\sigma(G, V)$ to $F_\sigma(G, W)$ and a $C_G(\sigma)$–homeomorphism $S_\sigma(f)$ from $S(G, V)_\sigma$ to $S(G, W)_\sigma$. We have the well-defined map

$$QE_{n,\sigma}(f) : QE_{n,\sigma}(G, V) \to QE_{n,\sigma}(G, W)$$

sending a point represented by $t_1a + t_2b$ in the join to that represented by $t_1 F_\sigma(f)(a) + t_2 S_\sigma(f)(b)$. And $QE_n(f) : QE_n(G, V) \to QE_n(G, W)$ is defined by

$$\prod_{\sigma \in G^+_\sigma} \alpha_\sigma \mapsto \prod_{\sigma \in G^+_\sigma} QE_{n,\sigma}(f) \circ \alpha_\sigma.$$

It is straightforward to check that all the axioms hold. □

5.2. Construction of $\eta^{QE_n}$ and $\mu^{QE_n}$. In this section we construct a unit map $\eta^{QE_n}$ and a multiplication $\mu^{QE_n}$ so that we get a commutative $\mathcal{I}_G$–FSP representing the $QE_n$–theory in $GwS$.

Let $G$ and $H$ be compact Lie groups, $V$ an orthogonal $G$–representation and $W$ an orthogonal $H$–representation. Let $\sigma \in G^+_\sigma$. We use $x_\sigma$ to denote the basepoint of $QE_{n,G}(G, V)$, which is defined in Proposition 5.4. For each $v \in S^V$, there are $v_1 \in S^{V_{\sigma}}$ and $v_2 \in S^{V_{\sigma^+}}$ such that $v = v_1 \wedge v_2$. Let $\eta^{QE_n}_{\sigma}(G, V) : S^V \to QE_{n,G}(G, V)$ be the map

$$\eta^{QE_n}_{\sigma}(G, V)(v) := \begin{cases} (1 - \|v_2\|) \eta_{\sigma}(G, V)(v_1) + \|v_2\| v_2, & \text{if } \|v_2\| \leq 1; \\ x_\sigma, & \text{if } \|v_2\| \geq 1. \end{cases}$$

LEMMA 5.8. The map $\eta^{QE_n}_{\sigma}(G, V)$ defined in (5.10) is well-defined, continuous and $C_G(\sigma)$–equivariant.

REMARK 5.9. For any $\sigma \in G^+_\sigma$, it’s straightforward to check the diagram below commutes.

$$\begin{array}{ccc}
S^V & \xrightarrow{\eta^{QE_n}_{\sigma}(G, V)} & F_\sigma(G, V) \\
\downarrow & & \downarrow \\
S^V & \xrightarrow{\eta^{QE_n}(G, V)} & QE_{n,\sigma}(G, V)
\end{array}$$

where both vertical maps are inclusions. By Lemma 5.8 the map

$$\eta^{QE_n}(G, V) : S^V \to \prod_{\sigma \in G^+_\sigma} \text{Map}_{C_G(\sigma)}(G, QE_{n,\sigma}(G, V)), v \mapsto \prod_{\sigma \in G^+_\sigma} (\alpha \mapsto \eta_{\sigma}^{QE_n}(G, V)(\alpha \cdot v)),$$

is well-defined and continuous. Moreover, $\eta^{QE_n} : S \to QE_n$ with $QE_n(G, V)$ defined in (5.4) is well-defined.

Next, we construct the multiplication map $\mu^{QE_n}$. First we define a map

$$\mu^{QE_n}_{(\sigma, \tau)} : (G, V) \wedge QE_{n,\tau}(H, W) \to QE_{n,(\sigma, \tau)}(G \times H, V \oplus W)$$

\begin{itemize}
  \item $\mu^{QE_n}_{(\sigma, \tau)}((g, v)) : QE_{n,\sigma}(G, V) \wedge QE_{n,\tau}(H, W) \to QE_{n,(\sigma, \tau)}(G \times H, V \oplus W)$
\end{itemize}
by sending a point \([t_1a_1 + t_2b_1] \wedge [u_1a_2 + u_2b_2]\) to (5.9)
\[
\begin{align*}
&\left((1 - \sqrt{t_1^2 + u_1^2})\mu^{G,E}_{(\sigma,\tau)}((G, V), (H, W))(a_1 \wedge a_2) \text{ if } t_1^2 + u_1^2 \leq 1 \text{ and } t_2u_2 \neq 0; \\
&+ \sqrt{t_1^2 + u_1^2}(b_1 + b_2)), \\
&(1 - t_2)\mu^{G,E}_{(\sigma,\tau)}((G, V), (H, W))(a_1 \wedge a_2) + t_2b_1, \\
&(1 - u_2)\mu^{G,E}_{(\sigma,\tau)}((G, V), (H, W))(a_1 \wedge a_2) + u_2b_2, \\
&[1\mu^{G,E}_{(\sigma,\tau)}((G, V), (H, W))(a_1 \wedge a_2) + 0], \\
&x_{\sigma,\tau},
\end{align*}
\]
where \(x_{\sigma,\tau}\) is the basepoint of \(QE_{n,(\sigma,\tau)}(G \times H, V \oplus W)\).

Lemma 5.10. The map \(\mu^{QE}_{(\sigma,\tau)}((G, V), (H, W))\) defined in (5.13) is well-defined and continuous.

The basepoint of \(QE_{n}(G, V)\) is the product of the basepoint of each \(\text{Map}_{CG}(G, QE_{n,\sigma}(G, V))\), i.e. the product of the constant map to the basepoint of each \(QE_{n,\sigma}(G, V)\). We can define the multiplication \(\mu^{QE}_{(G, V), (H, W)} : QE_{n}(G, V) \wedge QE_{n}(H, W) \rightarrow QE_{n}(G \times H, V \oplus W)\) by
\[
(\prod_{\sigma \in G^n} \alpha_{\sigma}) \wedge (\prod_{\tau \in H^n} \beta_{\tau}) \mapsto \prod_{\sigma \in G^n} (\alpha_{\sigma}(\sigma')) \mu^{QE}_{(\sigma,\tau)}((G, V), (H, W))(\alpha_{\sigma}(\sigma') \wedge \beta_{\tau}(\tau')).
\]

Theorem 5.11. \(QE_{n} : I_{G} \rightarrow GT\) together with the unit map \(\eta^{QE}_{n}\) defined in (5.17) and the multiplication \(\mu^{QE}_{(G, V), (H, W)}\) gives a commutative \(I_{G} - FSP\) that weakly represents \(QE_{n,G}(-)\).

Remark 5.12. We apply a conclusion from Chapter 3, Section 1, in [13]. A \(G\)-spectrum \(Y\) is isomorphic to an orthogonal \(G\)-spectrum of the form \(X(G)\) for some orthogonal spectrum \(X\) if and only if for every trivial \(G\)-representation \(V\) the \(G\)-action on \(Y(V)\) is trivial. \(QE_{n}(G, V)\) is not trivial when \(V\) is trivial. So it cannot arise from an orthogonal spectrum.

Proposition 5.13. Let \(G\) be any compact Lie group. Let \(V\) be an ample orthogonal \(G\)-representation and \(W\) an orthogonal \(G\)-representation. Let \(\sigma^{QE}_{G,V,W} : S^{W} \wedge QE_{n}(G, V) \rightarrow QE_{n}(G, V \oplus W)\) denote the structure map of \(QE_{n}\) defined by the unit map \(\eta^{QE}_{n}(G, V)\). Let \(\sigma^{QE}_{G,V,W}\) denote the right adjoint of \(\sigma^{QE}_{G,V,W}\). Then \(\sigma^{QE}_{G,V,W} : QE_{n}(G, V) \rightarrow \text{Map}(S^{W}, QE_{n}(G, V \oplus W))\) is a \(G\)-weak equivalence.

Let \(G\) and \(H\) be compact Lie groups, \(V\) an orthogonal \(G\)-representation and \(W\) an orthogonal \(H\)-representation. We use \(x_{\sigma}\) to denote the basepoint of \(QE_{n}(G, V)\), which is defined in Proposition 5.4. Let \(\sigma \in G_{n}^{\ast}\). For each \(v \in S^{V}\), there are \(v_{1} \in S^{V_{\sigma}^{+}}\) and \(v_{2} \in S^{(V_{\sigma})^{+}}\) such that \(v = v_{1} \wedge v_{2}\). Let \(\eta^{QE}_{n}(G, V) : S^{V} \rightarrow QE_{n,\sigma}(G, V)\) be the map
\[
(5.10) \quad \eta^{QE}_{n}(G, V)(v) := \begin{cases} 
(1 - \|v_{2}\|)\eta_{\sigma}(G, V)(v_{1}) + \|v_{2}\|v_{2}, & \text{if } \|v_{2}\| \leq 1; \\
x_{\sigma}, & \text{if } \|v_{2}\| \geq 1.
\end{cases}
\]
The map \(\eta^{QE}_{n}(G, V)\) defined in (5.10) is well-defined, continuous and \(C_{G}(\sigma)-\)equivariant.
Remark 5.14. For any \( \sigma \in G_n \), it’s straightforward to check the diagram below commutes.

\[
\begin{array}{ccc}
S^\sigma & \xrightarrow{\eta_\sigma(G,V)} & F_\sigma(G,V) \\
\downarrow & & \downarrow \\
S^V & \xrightarrow{\eta_{QE_n(G,V)}} & QE_n,\sigma(G,V)
\end{array}
\]

where both vertical maps are inclusions. By Lemma 5.8 the map (5.11)

\[ \eta_{QE_n}(G,V) : S^V \rightarrow \prod_{\sigma \in G_n^o} \text{Map}_{C_{G,o}}(G, QE_n,\sigma(G,V)), \quad v \mapsto \prod_{\sigma \in G_n^o} (\alpha \mapsto \eta_\sigma^{QE_n}(G,V)(\alpha \cdot v)), \]

is well-defined and continuous. Moreover, \( \eta_{QE_n} : S \rightarrow QE_n \) with \( QE_n(G,V) \) defined in [5.3] is well-defined.

Next, we construct the multiplication map \( \mu_{QE_n} \). First we define a map

\[ \mu_{\sigma,\tau}^{QE_n}((G,V),(H,W)) : QE_n,\sigma(G,V) \wedge QE_n,\tau(H,W) \rightarrow QE_n,(\sigma,\tau)(G \times H, V \oplus W) \]

by sending a point \([t_1a_1 + t_2b_1, u_1a_2 + u_2b_2] \) to

(5.12)

\[
\begin{cases}
(1 - \sqrt{t_1^2 + u_1^2}) \mu_{\sigma,\tau}^F((G,V),(H,W))(a_1 \wedge a_2), & \text{if } t_1^2 + u_1^2 \leq 1 \text{ and } t_2u_2 \neq 0; \\
\frac{1}{2}(1 - t_2) \mu_{\sigma,\tau}^F((G,V),(H,W))(a_1 \wedge a_2) + t_2b_1, & \text{if } u_2 = 0 \text{ and } 0 < t_2 < 1; \\
\frac{1}{2}(1 - u_2) \mu_{\sigma,\tau}^F((G,V),(H,W))(a_1 \wedge a_2) + u_2b_2, & \text{if } t_2 = 0 \text{ and } 0 < u_2 < 1; \\
1 \mu_{\sigma,\tau}^F((G,V),(H,W))(a_1 \wedge a_2) + 0, & \text{if } u_2 = 0 \text{ and } t_2 = 0; \\
\end{cases}
\]

where \( x_{\sigma,\tau} \) is the basepoint of \( QE_n,(\sigma,\tau)(G \times H, V \oplus W) \). The map \( \mu_{\sigma,\tau}^{QE_n}((G,V),(H,W)) \) defined in (5.12) is well-defined and continuous.

The basepoint of \( QE_n(G,V) \) is the product of the basepoint of each factor \( \text{Map}_{C_{G,o}}(G, QE_n,\sigma(G,V)), \) i.e. the product of the constant map to the basepoint of each \( QE_n,\sigma(G,V). \)

We can define the multiplication \( \mu_{QE_n}((G,V),(H,W)) : QE_n(G,V) \wedge QE_n(H,W) \rightarrow QE_n(G \times H, V \oplus W) \) by

\[
\left( \prod_{\sigma \in G_n^o} \alpha_\sigma \right) \wedge \left( \prod_{\tau \in H_n^o} \beta_\tau \right) \mapsto \prod_{\sigma \in G_n^o} \left( \sigma', \tau' \mapsto \mu_{(\sigma,\tau)}^{QE_n}((G,V),(H,W))(\alpha_\sigma(\sigma') \wedge \beta_\tau(\tau')) \right).
\]

Theorem 5.15. \( QE_n(G, -) : I_G \rightarrow GT \) together with the unit map \( \eta_{QE_n} \) defined in (5.11) and the multiplication \( \mu_{QE_n}((G, -),(G, -)) \) gives a commutative \( I_G-FSP \) that weakly represents \( QE^{G*}_n,(-) \).

The proof of Theorem 5.15 is analogous to that of Theorem 6.12 [6].

Remark 5.16. We apply a conclusion from Chapter 3, Section 1, in [13]. A \( G \)-spectrum \( Y \) is isomorphic to an orthogonal \( G \)-spectrum of the form \( X(G) \) for some orthogonal spectrum \( X \) if and only if for every trivial \( G \)-representation \( V \) the \( G \)-action on \( Y(V) \) is trivial. \( QE_n(V) \) is not trivial when \( V \) is trivial. So it cannot arise from an orthogonal spectrum.
In addition, we have the conclusion below.

**Proposition 5.17.** Let $G$ be any compact Lie group. Let $V$ be an ample orthogonal $G$–representation and $W$ an orthogonal $G$–representation. Let $σ_{G,V,W}^{QE_E} : S^W \smash \rightarrow QE_n(G, V) \rightarrow QE_n(G, V \oplus W)$ denote the structure map of $QE_n$ defined by the unit map $η^{QE_n}(G, V)$. Let $σ_{G,V,W}^{QE_n}$ denote the right adjoint of $σ_{G,V,W}^{QE_E}$. Then $σ_{G,V,W}^{QE_E} : QE_n(G, V) \rightarrow Map(S^W, QE_n(G, V \oplus W))$ is a $G$–weak equivalence.

The proof is analogous to that of Proposition 6.14 [6].

At last, we get the main conclusion of Section B.11.1

**Theorem 5.18.** For each positive integer $n$ and each compact Lie group $G$, there is a well-defined functor $QG_n$ from the category of orthogonal ring spectra to the category of $\mathcal{I}_G$–FSP sending $E$ to $(QE_n(G, -), η^{QE_n}, μ^{QE_n})$ that weakly represents the quasi-theory $QE_n^*(G)$.

**Appendix A.** Faithful representation of $Λ_G(σ)$

We discuss complex and real $Λ_G(σ)$–representations in Section A.1 and A.2 respectively.

**A.1. Preliminaries: faithful representations of $Λ_G(σ)$**. In this section, we construct a faithful $Λ_G(σ)$–representation from a faithful $G$–representation.

Let $G$ be a compact Lie group and $σ \in G^n$. Let $l_i$ denote the order of $σ_i$. Let $ρ$ denote a complex $G$–representation with underlying space $V$. Let $i : C_G(σ) \hookrightarrow G$ denote the inclusion. Let $\{λ\}$ denote all the irreducible complex representations of $C_G(σ)$. As said in [4], we have the decomposition of a representation into its isotypic components $i^*V \cong \bigoplus_λ V_λ$ where $V_λ$ denotes the sum of all subspaces of $V$ isomorphic to $λ$. Each $V_λ = Hom_{C_G(σ)}(λ, V) \otimes_C λ$ is unique as a subspace. Note that each $σ_i$ acts on each $V_λ$ as a diagonal matrix.

Each $V_λ$ can be equipped with a $Λ_G(σ)$–action. Each $λ(σ_i)$ is of the form $e^{2πιm_λl_i}I$ with $0 < m_λ ≤ l_i$ and $I$ the identity matrix. As shown in Lemma 4.1, we have the well-defined complex $Λ_G(σ)$–representations

\[(V_λ)_σ := V_λ \otimes_C (q^{m_λ l_i} \otimes \cdots \otimes q^{m_λ l_i})\]
and
\[(A.1) (V)_σ := \bigoplus_λ (V_λ)_σ.\]

**Proposition A.1.** Let $V$ be a faithful $G$–representation. Let $σ \in G^n$.

(i) $(V)_σ \oplus (V)_σ \otimes_C q^{-1}$ is a faithful $Λ_G(σ)$–representation.

(ii) $(V)_σ \oplus V^σ$ is a faithful $Λ_G(σ)$–representation.

**Proof.** (i) Let $[a, t] ∈ Λ_G(σ)$ be an element acting trivially on $V_σ$. Consider the subrepresentations $(V_λ)_σ$ and $(V_λ)_σ \otimes_C q^{-1}$ of $(V)_σ \oplus (V)_σ \otimes_C q^{-1}$ respectively. Let $v$ be an element in the underlying vector space $V_λ$. On $(V_λ)_σ$, $[a, t] \cdot v = e^{2πιt(\frac{m_λ}{l_i} + \frac{m_λ}{l_i})}a \cdot v = v$; and on $(V_λ)_σ \otimes_C q^{-1}$, $[a, t] \cdot v = e^{2πιt(\frac{m_λ}{l_i} + \frac{m_λ}{l_i})}a \cdot v = v$. So we get $e^{2πιt} \cdot v = v$. Thus, $t = 0$. $C_G(σ)$ acts faithfully on $V$, so it acts faithfully on $(V)_σ \oplus (V)_σ \otimes_C q^{-1}$. Since $[a, 0] \cdot w = w$, for any $w \in (V)_σ \oplus (V)_σ \otimes_C q^{-1}$, so $a = e$.

Thus, $(V)_σ \oplus (V)_σ \otimes_C q^{-1}$ is a faithful $Λ_G(σ)$–representation.
(ii) Note that $V^\sigma$ with the trivial $\mathbb{R}$-action is the representation $(V^\sigma)_\sigma \otimes_C q^{-1}$.

The representation $(V)_\sigma \oplus V^\sigma$ contains a subrepresentation $(V^\sigma)_\sigma \oplus (V^\sigma)_\sigma \otimes_C q^{-1}$, which is a faithful $\Lambda_G(\sigma)$-representation by Proposition A.1 (i). So $(V)_\sigma \oplus V^\sigma$ is faithful.

**Lemma A.2.** For any $\sigma \in G^\circ_0$, $(-)_\sigma$ defined in (A.1) is a functor from the category of $G$-spaces to the category of $\Lambda_G(\sigma)$-spaces. Moreover, $(-)_\sigma \oplus (-)_\sigma \otimes_C q^{-1}$ and $(-)_\sigma \oplus (-)^\sigma$ in Proposition A.1 are also well-defined functors from the category of $G$-spaces to the category of $\Lambda_G(\sigma)$-spaces.

**Proof.** Let $f : V \to W$ be a $G$-equivariant map. Then $f$ is $\Lambda_G(\sigma)$-equivariant for each $\sigma \in G^\circ_0$. For each irreducible complex $C_G(\sigma)$-representation $\lambda$, $f : V_\lambda \to W_\lambda$ is $C_G(\sigma)$-equivariant. And $f_\sigma : (V_\lambda)_\sigma \to (W_\lambda)_\sigma$, $v \mapsto f(v)$ with the same underlying spaces is well-defined and is $\Lambda_G(\sigma)$-equivariant. It is straightforward to check if we have two $G$-equivariant maps $f : V \to W$ and $g : U \to V$, then $(f \circ g)_\sigma = f_\sigma \circ g_\sigma$. So $(-)_\sigma$ gives a well-defined functor from the category of $G$-representations to the category of $\Lambda_G(\sigma)$-representation.

The other conclusions can be proved in a similar way.

**Proposition A.3.** Let $H$ and $G$ be two compact Lie groups. Let $\sigma \in G^\circ_0$ and $\tau \in H^\circ_0$. Let $V$ be a $G$-representation and $W$ a $H$-representation.

(i) We have the isomorphisms of representations $(V \oplus W)_{(\sigma, \tau)} = (V_\sigma \oplus W_\tau)$ as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{T^\sigma} \Lambda_H(\tau)$-representations;

$(V \oplus W)_{(\sigma, \tau)} \oplus (V \oplus W)_{(\sigma, \tau)} \otimes_C q^{-1} = ((V)_\sigma \oplus (V)_\sigma \otimes_C q^{-1}) \oplus ((W)_\tau \oplus (W)_\tau \otimes_C q^{-1})$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{T^\sigma} \Lambda_H(\tau)$-representations;

and $(V \oplus W)_{(\sigma, \tau)} \oplus (V \oplus W)_{(\sigma, \tau)} = ((V)_\sigma \oplus V^\sigma) \oplus ((W)_\tau \oplus W^\tau)$ as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{T^\sigma} \Lambda_H(\tau)$-representations.

(ii) Let $\phi : H \to G$ be a group homomorphism. Let $\phi_\tau : \Lambda_H(\tau) \to \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from $\phi$. Then we have

$\phi_\tau^* (V)_{\phi(\tau)} = (V)_\tau$,

$\phi_\tau^* ((V)_{\phi(\tau)} \oplus (V)_{\phi(\tau)} \otimes_C q^{-1}) = (V)_\tau \oplus (V)_\tau \otimes_C q^{-1}$,

$\phi_\tau^* ((V)_{\phi(\tau)} \oplus V^{\phi(\tau)}) = (V)_\tau \oplus V^\tau$

as $\Lambda_H(\tau)$-representations.

**Proof.** (i) Let $\{\lambda_G\}$ and $\{\lambda_H\}$ denote the sets of all the irreducible $C_G(\sigma)$-representations and all the irreducible $C_H(\tau)$-representations. Then $\lambda_G$ and $\lambda_H$ are irreducible representations of $C_{G \times H}(\sigma, \tau)$ via the inclusion $C_G(\sigma) \hookrightarrow C_{G \times H}(\sigma, \tau)$ and $C_H(\tau) \hookrightarrow C_{G \times H}(\sigma, \tau)$.

The $\mathbb{R}$-representation assigned to each $C_{G \times H}(\sigma, \tau)$-irreducible representation in $V \oplus W$ is the same as that assigned to the irreducible representations of $V$ and $W$. So we have

$(V \oplus W)_{(\sigma, \tau)} = (V_\sigma \oplus W_\tau)$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{T^\sigma} \Lambda_H(\tau)$-representations.

Similarly we can prove the other two conclusions in (i).

(ii) Let $\sigma = \phi(\tau)$. If $(\phi_\tau^* V)_{\lambda_H}$ is a $C_H(\tau)$-subrepresentation of $\phi_\tau^* V_{\lambda_G}$, the $\mathbb{R}$-representation assigned to it is the same as that to $V_{\lambda_G}$. So we have $\phi_\tau^* (V)_{\phi(\tau)} = (V)_\tau$ as $\Lambda_H(\tau)$-representations.

Similarly we can prove the other two conclusions in (ii).
A.2. real \( \Lambda_G(\sigma) \)-representation. In this section we discuss real \( \Lambda_G(\sigma) \)-representation and its relation with the complex \( \Lambda_G(\sigma) \)-representations introduced in Lemma 2.1. The main reference is [2] and [4].

Let \( G \) be a compact Lie group and \( \sigma \in G^a_z \).

DEFINITION A.4. A complex representation \( \rho : G \to Aut_C(V) \) is said to be self dual if it is isomorphic to its complex dual \( \rho^* : G \to Aut_C(V^*) \) where \( V^* := Hom_C(V, \mathbb{C}) \) and \( \rho^*(g) = \rho(g^{-1})^* \).

For any compact Lie group, we use \( RO(G) \) to denote the real representation ring of \( G \). We have the real version of Lemma 2.1 below. The proof of Lemma A.5 is left to the readers.

LEMMA A.5. Let \( \sigma \in G^a_z \). Then the map \( \pi^* : RO(\mathbb{T}^n) \to RO(\Lambda_G(\sigma)) \) exhibits \( RO(\Lambda_G(\sigma)) \) as a free \( RO(\mathbb{T}^n) \)-module.

In particular there is an \( RO(\mathbb{T}) \)-basis of \( RO(\Lambda_G(\sigma)) \) given by irreducible real representations \( \{V_\lambda\} \). There is a bijection between \( \{V_\lambda\} \) and the set \( \{\lambda\} \) of irreducible real representations of \( C_G(\sigma) \). When \( \sigma \) is trivial, \( V_\lambda \) has the same underlying space \( V \) as \( \lambda \). When \( \sigma \) is nontrivial, \( V_\lambda = ((\lambda \otimes \mathbb{R} \mathbb{C}) \circ \mathbb{C} (\eta_1 \otimes \cdots \otimes \eta_n)) \oplus ((\lambda \otimes \mathbb{R} \mathbb{C}) \circ \mathbb{C} (\eta_1 \otimes \cdots \otimes \eta_n))^* \) where each \( \eta_i \) is a complex \( \mathbb{R} \)-representation such that \( (\lambda \otimes \mathbb{R} \mathbb{C})(\sigma_i) \) acts on \( V \otimes \mathbb{R} \mathbb{C} \) via the scalar multiplication by \( \eta_i(1) \). The dimension of \( V_\lambda \) is twice that of \( \lambda \).

As in (A.1), we can construct a functor \((-)^R_\sigma\) from the category of real \( G \)-representations to the category of real \( \Lambda_G(\sigma) \)-representations with
\[
(A.2) \quad (V)^R_\sigma = (V \otimes \mathbb{R} \mathbb{C})_\sigma \oplus (V \otimes \mathbb{R} \mathbb{C})^*_\sigma.
\]

PROPOSITION A.6. Let \( V \) be a faithful real \( G \)-representation. For each \( \sigma \in G^a_z \), \( (V)^R_\sigma \) is a faithful real \( \Lambda_G(\sigma) \)-representation.

PROOF. Let \([a, t] \in \Lambda_G(\sigma)\) be an element acting trivially on \((V)^R_\sigma\). Assume \( t \in [0, 1) \). Let \( v \in (V \otimes \mathbb{R} \mathbb{C})_\sigma \) and let \( v^* \) denote its correspondence in \((V \otimes \mathbb{R} \mathbb{C})^*_\sigma\). Then \([a, t](v + v^*) = (ae^{2\pi int} + ae^{-2\pi int})(v + v^*) = v + v^*\) where \( m \) is a nonzero number determined by \( \sigma \). Thus \( a \) is equal to both \( e^{2\pi int}I \) and \( e^{-2\pi int}I \). Thus \( t = 0 \) and \( a \) is trivial.

So \((V)^R_\sigma\) is a faithful real \( \Lambda_G(\sigma) \)-representation. \( \square \)

PROPOSITION A.7. Let \( H \) and \( G \) be two compact Lie groups. Let \( \sigma \in G^a_z \) and \( \tau \in H^a_z \). Let \( V \) be a real \( G \)-representation and \( W \) a real \( H \)-representation.
(i) We have the isomorphisms of representations \((V \oplus W)^R_\sigma = (V^R_\sigma \oplus W^R_\tau)\) as \( \Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times \Lambda_H(\tau) \)-representations.
(ii) Let \( \phi : H \to G \) be a group homomorphism. Let \( \phi_\tau : \Lambda_H(\tau) \to \Lambda_G(\phi(\tau)) \) denote the group homomorphism obtained from \( \phi \). Then \( \phi_\tau^*(V)^R_\phi(\tau) = (V)^R_\tau \), as \( \Lambda_H(\tau) \)-representations.

The proof is left to the readers.

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QUASI-THEORIES AND THEIR EQUIVARIANT ORTHOGONAL SPECTRA

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