Harmonic analysis

A note on weighted bounds for rough singular integrals

Une note sur les bornes pondérées pour les intégrales singulières rugueuses

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1. Introduction

Consider a class of rough homogeneous singular integrals defined by

\[ T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y) \frac{\Omega(y/|y|)}{|y|^n} dy, \]

with \( \Omega \in L^\infty(S^{n-1}) \) and having zero average over the sphere.

In [7], Hytönen, Roncal and Tapiola proved that

\[ \|T_\Omega\|_{L^2(w) \to L^2(w)} \leq C_n \|\Omega\|_{L^\infty} [w]_{A_2}^2, \]

(1.1)

where \([w]_{A_2} = \sup_Q \frac{1}{|Q|} \int_Q w \int_Q w^{-1} \). Different proofs of this result, via a sparse domination, were given by Conde-Alonso, Culiuc, Di Plinio, and Ou [3], and by the author [8]. Recently, (1.1) was extended to maximal singular integrals by Di Plinio, Hytönen, and Li [4].

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1 The author was supported by ISF grant No. 447/16 and ERC Starting Grant No. 713927.

https://doi.org/10.1016/j.crma.2017.11.016

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It was conjectured in [7] that the quadratic dependence on \([w]_{A_2}\) in (1.1) can be improved to the linear one. In this note, we obtain a strengthening of (1.1), which, to a certain extent, supports this conjecture.

**Theorem 1.1.** For every \(w \in A_2\), we have

\[
\|M \circ T_\Omega\|_{L^2(w) \to L^2(w)} \leq C_n \|\Omega\|_{L^\infty} [w]_{A_2}^2,
\]

and this bound is optimal, in general.

Here \(M\) denotes the standard Hardy–Littlewood maximal operator. Notice that \(\|M\|_{L^2(w) \to L^2(w)} \leq [w]_{A_2}\), and this bound is sharp [1]. Therefore, (1.2) cannot be obtained via a simple combination of the sharp linear bound for \(M\) with (1.1). The proof of (1.2) is based essentially on the technique introduced in [8].

2. Preliminaries

Recall that a family of cubes \(\mathcal{S}\) is called sparse if there exists \(0 < \alpha < 1\) such that for every \(Q \in \mathcal{S}\), one can find a measurable set \(E_Q \subset Q\) with \(|E_Q| \geq \alpha |Q|\), and the sets \(\{E_Q\}_{Q \in \mathcal{S}}\) are pairwise disjoint.

Given a sublinear operator \(T\), define the maximal operator \(M_{p, T}\) by

\[
M_{p, T}f(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |T(f \chi_{\mathbb{R}^n \setminus 3Q})|^p dy \right)^{1/p}.
\]

Denote \(\langle f \rangle_{p, Q} = \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{1/p}\).

**Proposition 2.1.** Assume that \(T\) and \(M_{p, T}\) are of weak type \((1, 1)\) and, moreover, \(\|M_{p, T}\|_{L^1 \to L^{1, \infty}} \leq K_p\) for all \(p \geq 2\). Then

\[
\|T\|_{L^2(w) \to L^2(w)} \leq C_n \|\Omega\|_{L^\infty} + K(w)^2.
\]

**Proof.** This is just a combination of several known facts. By [8, Cor. 3.2], for every suitable \(f, g\), there exists a sparse family \(\mathcal{S}\) such that

\[
|\langle Tf, g \rangle| \leq C_n \|T\|_{L^1 \to L^{1, \infty}} + K_p \sum_{Q \in \mathcal{S}} \langle f \rangle_{1, Q} \langle g \rangle_{p, Q} |Q| \quad (p > 1).
\]

But it was shown in [3] (see the proof of Corollary A1 there) that this sparse bound implies (2.1). \(\square\)

In particular, \(T_\Omega\) with \(\Omega \in L^\infty\) satisfies the hypothesis of Proposition 2.1, namely, it was proved in [8] that

\[
\|M_{p, T_\Omega} f\|_{L^{1, \infty}} \leq C_n \|\Omega\|_{L^\infty} p \|f\|_{L^1} \quad (p \geq 1).
\]

(2.2)

3. Proof of Theorem 1.1

First, by a general extrapolation argument found in [9], the sharpness of (1.2) follows from \(\|M \circ T_\Omega\|_{L^p \to L^p} \geq \frac{c}{(p-1)^2}\) as \(p \to 1\). The latter relation holds for a subclass of \(T_\Omega\) with kernels satisfying the standard nondegeneracy assumptions. In particular, it can be easily checked for the Hilbert transform.

Turn to the proof of (1.2). By homogeneity, one can assume that \(\|\Omega\|_{L^\infty} = 1\). The proof is based on two pointwise estimates:

\[
M(T_\Omega f)(x) \lesssim M Mf(x) + M_{1, T_\Omega} f(x)
\]

(3.1)

and

\[
M_{p, (M_1, T_\Omega)} f(x) \lesssim Mf(x) + M_{p, T_\Omega} f(x) \quad (p \geq 2)
\]

(3.2)

(use the usual notation \(A \lesssim B\) if \(A \leq C_B\)).

Let us show first how to complete the proof using these estimates. By (2.2), \(M_{1, T_\Omega}\) is of weak type \((1, 1)\). Applying (2.2) again along with (3.2) yields \(\|M_{p, (M_1, T_\Omega)}\|_{L^1 \to L^{1, \infty}} \lesssim p\). Therefore, by Proposition 2.1,

\[
\|M_{1, T_\Omega}\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}^2.
\]

This estimate, combined with (3.1) and Buckley’s linear bound for \(M [1]\), implies (1.2).
It remains to prove (3.1) and (3.2). We start with (3.1). This estimate follows from the definition of $M_{1,T_{Ω}}$ and the standard fact that, for every cube $Q$ containing the point $x$,

$$\frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{3Q})| \lesssim MMf(x).$$

(3.3)

For the sake of completeness, we outline the proof of (3.3). Combining the weak type $(1,1)$ and the $L^2$ boundedness of $T_{Ω}$ (see [2,10]) with interpolation and Yano’s extrapolation [6, p. 43], we obtain

$$\frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{3Q})| \lesssim \|f\|_{L\log L,3Q}.$$

By Stein’s $L \log L$ result [11],

$$\|f\|_{L\log L, Q} \lesssim \frac{1}{|Q|} \int_{Q} Mf,$$

which, along with the previous estimate, implies (3.3).

Turn to the proof of (3.2). Let $R$ be an arbitrary cube containing the point $x$. Let $y \in R$ and let $Q$ be an arbitrary cube containing $y$.

Assume that $\ell_{Q} \leq \frac{1}{2}\ell_{R}$. Then $Q \subset 2R$ and $3Q \subset 3R$. Hence,

$$\frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus (3R \cup 3Q)})| \lesssim M_{2R}(T_{Ω}(f \chi_{\mathbb{R}^n \setminus 3R}))(y).$$

(3.4)

where $M_{2R}$ denotes the maximal operator restricted to the cube $2R$.

Suppose now that $\ell_{R} < 2\ell_{Q}$. Then $R \subset 5Q$ and $3R \subset 9Q$. We obtain

$$|T_{Ω}(f \chi_{15Q \setminus (3R \cup 3Q)}) \chi_{Q}(z)| \lesssim \frac{1}{|Q|} \int_{15Q} |f| \lesssim Mf(x).$$

Also,

$$\frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus 15Q})| \lesssim \frac{1}{|5Q|} \int_{5Q} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus 15Q})| \lesssim M_{1,Tn} f(x),$$

and therefore,

$$\frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus (3R \cup 3Q)})| dz \lesssim Mf(x) + M_{1,Tn} f(x).$$

This estimate, combined with (3.4), implies

$$M_{1,Tn}(f \chi_{\mathbb{R}^n \setminus 3R})(y) = \sup_{Q \ni y} \frac{1}{|Q|} \int_{Q} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus (3R \cup 3Q)})| \lesssim M_{2R}(T_{Ω}(f \chi_{\mathbb{R}^n \setminus 3R}))(y) + Mf(x) + M_{1,Tn} f(x).$$

Therefore, by the $L^p$-boundedness of $M$,

$$\left( \frac{1}{|R|} \int_{R} M_{1,Tn}(f \chi_{\mathbb{R}^n \setminus 3R})^p dy \right)^{1/p} \lesssim \left( \frac{1}{|2R|} \int_{2R} |T_{Ω}(f \chi_{\mathbb{R}^n \setminus 3R})|^p dy \right)^{1/p} + Mf(x) + M_{1,Tn} f(x).$$

Combining this estimate with

$$|T_{Ω}(f \chi_{\mathbb{R}^n \setminus 3R}) \chi_{2R}(y)| \lesssim Mf(x) + |T_{Ω}(f \chi_{\mathbb{R}^n \setminus 6R}) \chi_{2R}(y)|$$

and using also that, by Hölder’s inequality, $M_{1,Tn} f \leq M_{p,Tn} f$, we obtain

$$\left( \frac{1}{|R|} \int_{R} M_{1,Tn}(f \chi_{\mathbb{R}^n \setminus 3R})^p dy \right)^{1/p} \lesssim Mf(x) + M_{p,Tn} f(x),$$

which proves (3.2), and therefore, Theorem 1.1 is completely proved.
Remark 3.1. Define the maximal singular integral by
\[ T^*_\Omega f(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} f(x - y) \frac{\Omega(|y|)}{|y|^n} \, dy \right|. \]

As we have mentioned in the introduction, it was shown in [4] that
\[ \|T^*_\Omega\|_{L^2(\mathbb{R})} \leq C_\varepsilon \|\Omega\|_{L^\infty} \|\chi\|_{A_2}^2. \]  
(3.5)

We outline an alternative proof of this result based on Theorem 1.1. Assume that \( \|\Omega\|_{L^\infty} = 1 \). Denote
\[ T_j f(x) = \int_{2^j < |y| \leq 2^{j+1}} f(x - y) \frac{\Omega(|y|)}{|y|^n} \, dy. \]

Let \( \Phi \) be a Schwartz function supported in the unit ball and such that \( \int \Phi = 1 \). Set \( \Phi_k(x) = 2^{-kn} \Phi(2^{-k} x) \). It was proved in [5] that
\[ T^*_\Omega f(x) \lesssim Mf(x) + M(T_\Omega f)(x) + \sum_{j=0}^{\infty} A_j f(x), \]  
(3.6)

where
\[ A_j f(x) = \sup_{k \in \mathbb{Z}} |T_{j+k} f(x) - \Phi_k \ast T_{j+k} f(x)|. \]

The standard Fourier transform estimates (which again can be found in [5]) show that \( \|A_j\|_{L^2(\mathbb{R})} \lesssim 2^{-\alpha j} \) with some fixed \( 0 < \alpha < 1 \). Also, the standard convolution estimates imply \( A_j f(x) \lesssim Mf(x) \). Applying the interpolation with a change of measures along with the sharp reverse Hölder estimates (as it was done in [7]), we obtain
\[ \left\| \sum_{j=0}^{\infty} A_j f \right\|_{L^2(\mathbb{R})} \lesssim [w]_{A_2}^2 \|f\|_{L^2(\mathbb{R})}, \]

which, along with (3.6) and Theorem 1.1, proves (3.5).

Acknowledgement

I would like to thank the referee for useful comments that improved the presentation.

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