Exact Operator Solution of $A_2$-Toda Field Theory

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Abstract

Quantum $A_2$-Toda field theory in two dimensions is investigated based on the method of quantizing canonical free field. Toda exponential operators associated with the fundamental weights are constructed to the fourth order in the cosmological constant. This leads us to a conjecture for the exact operator solution.
It is well-known that Toda field theories in two dimensions admit exact classical solutions \[1\]. Recently, they have attracted renewed interests as conformal field theories incorporating $W$-symmetry \[2, 3, 4\]. The simplest Toda theory is the Liouville theory, for which extensive canonical approaches have been developed so far \[5, 6, 7\]. This includes the exact operator solution \[8, 9, 10, 11\]. Though similar development have been achieved also for Toda theories \[12, 13, 3\], complete operator solution seems to be still lacking.

In this note we investigate exact operator solution of $A_2$-Toda field theory by the method of quantizing canonical free field developed for Liouville theory \[5, 6, 7\]. We shall closely follow the arguments due to Otto and Weigt \[8\] in constructing the operator solution for Liouville theory. It should be noted, however, that the extension of their method to Toda theories is not so obvious since the latter contain more than one screening charges which are not mutually commuting in general. Furthermore, Toda exponential operators can be parametrized by an arbitrary vector in the root space. These brings about an operator ordering problem in expanding the Toda exponentials in terms of the screening charges.

Such complications may be resolved by considering special operators associated with the fundamental weights. We find that only commuting screening charges appear in the expansions of these operators. Their quantum expressions can be determined order by order in the cosmological constant from the requirement of locality \[4, 8\]. General exponential operators as well as the Toda field itself can be obtained from them. Conformality and locality alone determine the exact solution up to a constant related to the arbitrariness of the cosmological constant. This ambiguity is resolved by imposing the field equation. The canonicity of the transformation from the Toda field to the free field at the quantum level can be established directly by examining the canonical commutation relations.

Let us consider $A_2$-Toda field theory described by the classical action

\[
S = \frac{1}{\gamma^2} \int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\sigma \left( \frac{1}{2} \partial_\alpha \varphi \cdot \partial^\alpha \varphi - \mu^2 \sum_{a=1,2} e^{\alpha^a \cdot \varphi} \right),
\]

where $\varphi$ is a two-component field and $\alpha^a$ ($a = 1, 2$) stand for the simple roots normalized to $(\alpha^a)^2 = 2$. The coupling constant $\gamma$ may be fixed by the conformal invariance in the presence of matter couplings. We assume that $\varphi$ is subject to periodic boundary condition $\varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma)$. It
is well-known that the theory admits exact classical solution. We write it in the following form

\[ e^{\lambda^a \varphi} = \frac{e^{\lambda^a \psi}}{1 + \frac{\mu^2}{4} A_a B_a + \left(\frac{\mu^2}{4}\right)^2 A_a \circ A_b B_a \circ B_a}, \tag{2} \]

where \( \lambda^a \) (\( a = 1, 2 \)) are the fundamental weights satisfying \( \lambda^a \cdot \alpha^b = \delta^{ab} \) and \( \psi(x) = \psi_+(x^+) + \psi_-(x^-) \) is the canonical free field with the normal mode expansion

\[ \psi_\pm(x^\pm) = \frac{\gamma}{2} Q + \frac{\gamma}{4\pi} P x^\pm + \frac{i\gamma}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} a_n^{(\pm)} e^{-inx^\pm}. \tag{3} \]

The light-cone coordinates are defined by \( x^\pm = \tau \pm \sigma \), and we have introduced

\[
\begin{align*}
A_a(x^+) &= C_{\alpha^a} \int_0^{2\pi} dy^+ \mathcal{E}_{\alpha^a}(x^+ - y^+) e^{\alpha^a \cdot \psi_+(y^+)} , \\
B_a(x^-) &= C_{\alpha^a} \int_0^{2\pi} dy^- \mathcal{E}_{\alpha^a}(x^- - y^-) e^{\alpha^a \cdot \psi_-(y^-)} , \\
A_a \circ A_b(x^+) &= C_{\alpha^a + \alpha^b} C_{\alpha^a} \int_0^{2\pi} dy^+ dy^+ \mathcal{E}_{\alpha^a + \alpha^b}(x^+ - y^+) \mathcal{E}_{\alpha^a}(y^+ - z^+) e^{\alpha^a \cdot \psi_+(y^+)+\alpha^b \cdot \psi_+(z^+)} , \\
B_a \circ B_b(x^-) &= C_{\alpha^a + \alpha^b} C_{\alpha^a} \int_0^{2\pi} dy^- dy^- \mathcal{E}_{\alpha^a + \alpha^b}(x^- - y^-) \mathcal{E}_{\alpha^a}(y^- - z^-) e^{\alpha^a \cdot \psi_-(y^-)+\alpha^b \cdot \psi_-(z^-)}, \tag{4}
\end{align*}
\]

where \( C_\beta \equiv \sinh \frac{\gamma}{4} \beta \cdot P \) and \( \mathcal{E}_\beta(x) \equiv \exp \frac{\gamma}{4} \beta \cdot P \epsilon(x) \) with \( \epsilon(x) \) being the stair-step function defined by \( \epsilon(x) = \text{sign}(x) \) for \(-2\pi < x < 2\pi\) and \( \epsilon(x+2\pi) = \epsilon(x)+2 \). Furthermore, the convention \( \hat{1}(\hat{2}) = 2(1) \) has been employed for the indices. These functions satisfy the periodicity \( A_a(x^+ + 2\pi) = e^{2\pi \alpha^a \cdot P} A_a(x^+) \), \( A_a \circ A_b(x^+ + 2\pi) = e^{2\pi (\alpha^a + \alpha^b) \cdot P} A_a \circ A_b(x^+) \), etc. One can easily realize that \( \mathcal{E} \) is a generalization of the classical solution for Liouville theory. In the \( A_2 \)-Toda theory we have four screening charges \( A_a B_a \) and \( A_a \circ A_b B_a \circ B_a \) (\( a = 1, 2 \)), while there is only one in the Liouville case.

The key property that enable us to solve the theory exactly is that \( \mathcal{E} \) defines a canonical mapping from the interacting Toda field to the free field. By extending the analysis of ref. \[8, 10\] for Liouville theory, one can show the fundamental Poisson brackets for \( \varphi \) and the conjugate momentum \( \pi_\varphi = \frac{1}{\gamma^2} \dot{\varphi} \) by assuming the Poisson brackets for the canonical free fields

\[
\{\psi_k(\tau, \sigma), \dot{\psi}_l(\tau, \sigma')\} = \gamma^2 \delta_{kl} \delta(\sigma - \sigma') , \quad \{\psi_k(\tau, \sigma), \dot{\psi}_l(\tau, \sigma')\} = \{\dot{\psi}_k(\tau, \sigma), \dot{\psi}_l(\tau, \sigma')\} = 0, \tag{5}
\]

where the indices \( k, l = 1, 2 \) stand for the components of the free field \( \psi \). Furthermore, the theory possesses extended conformal invariance, whose generators reduce to the free field expressions after the substitution of the classical solution obtained from \( \mathcal{E} \). In particular, the stress tensor generates the pseudo-conformal symmetry for the canonical free fields exactly the same manner.
as for the interacting fields. The virtue in writing the classical solution in the form (2) is that the conformal property becomes manifest. Since both the interacting and the free classical vertex functions possess the same transformation properties under the conformal symmetry, the four screening charges $A_a B_a$ and $A_a \circ A_a B_a \circ B_a$ ($a = 1, 2$) must be of vanishing conformal dimensions. This can be verified directly from (4).

We now turn to quantum theory. We impose the standard commutation relations on the normal modes

$$ [Q_k, P_l] = i\delta_{kl}, \quad [a_k^{(+)}, a_l^{(+)}] = [a_k^{(-)}, a_l^{(-)}] = n\delta_{kl}\delta_{n+m,0}. \tag{6} $$

Then the procedure to achieve exact operator solution for the interacting fields goes as follows: We first construct the exponential operator $e^{\nu \lambda_a \varphi}$ with $\nu$ an arbitrary parameter by assuming it to be a product of a free field vertex operator and a power series of the screening charges with arbitrary coefficients depending on the zero-mode momenta $P$. These coefficients will be determined up to a constant order by order in the cosmological constant $\mu^2$ by the requirement of locality. The Toda field will be obtained as the derivatives of $e^{\nu \lambda_a \varphi}$ with respect to $\nu$ at $\nu = 0$. We then impose the field equations at the first order in $\mu^2$. This determine the unknown constant. We finally establish the full equations of motion and the canonical commutation relations. In the present work the analysis is carried out to the fourth order in $\mu^2$ for the general exponential operator and to some yet higher orders for special cases with $\nu$ being negative integers. This leads us to a conjecture of the exact solution.

To write down the Toda exponential we must find appropriate quantum definition for the screening charges. Since they must be $2\pi$ periodic in $\sigma$ and be of vanishing conformal weight, we define

$$ Y_a(x) = \int_0^{2\pi} dy^+ dy^- : E_{\alpha^a} (x^+ - y^+) \hat{E}_{\alpha^a} (x^- - y^-) V_{\eta \alpha^a} (y) : , $$

$$ Y_{\bar{a}a}(x) = \int_0^{2\pi} dy^+ dy^- \int_0^{2\pi} dz^+ dz^- : E_{\alpha^a} (x^+ - y^+) \hat{E}_{\alpha^a} (x^- - y^-) V_{\eta \alpha^a} (y) : $$

$$ \times : E_{\alpha^a} (x^+ - z^+) \hat{E}_{\alpha^a} (y^+ - z^+) \hat{E}_{\alpha^a} (y^- - z^-) V_{\eta \alpha^a} (z) : , \tag{7} $$

where $V_{\beta}(x) \equiv e^{\beta \psi(x)} :$ is the free field vertex operator and $: :$ stands for free field normal ordering for oscillators and symmetric ordering for zero-mode operators, i.e., $e^{\beta Q} f(P) := e^{\frac{1}{2}\beta Q} f(P) e^{\frac{1}{2}\beta Q}$. We have rescaled $\psi$ and $P$ appearing in $E_{\alpha^a}(x)$ by a parameter $\eta$. It is fixed by requiring that the
of these variables the screening charges (7) can be cast into the following form

\[
\mathcal{Y}_a(x) = \int_0^{2\pi} dy^+ dy^- q^{(\omega^a+1)\varepsilon(x,y)} V_{\eta\alpha^a}(y), \\
\mathcal{Y}_{a\bar{a}}(x) = \int_0^{2\pi} dy^+ dy^- \int_0^{2\pi} dz^+ dz^- q^{(\omega^a+\omega^{\bar{a}}+1)\varepsilon(x,y)+\omega^{\bar{a}}\varepsilon(y,z)} V_{\eta\alpha^a}(y)V_{\eta\alpha^{\bar{a}}}(z),
\]
where we have introduced \(\varepsilon(x, y) \equiv \epsilon(x^+ - y^+) + \epsilon(x^- - y^-)\).

The screening charges (7) are hermitian for the standard assignment of hermiticity for the normal modes. More importantly, one can verify the mutual commutativity of \(V_{\eta\lambda^a}(x), \mathcal{Y}_a(x)\) and \(\mathcal{Y}_{a\bar{a}}(x)\). We thus expand the Toda exponential as

\[
e^{\nu\lambda^a \cdot \varphi(x)} = V_{\eta\lambda^a}(x) \sum_{n,m=0}^{\infty} \left( \frac{\mu^2}{4} \right)^{n+2m} C_{nm}^{a}(\omega; \nu) \mathcal{Y}_a^m(x) \mathcal{Y}_{a\bar{a}}^m(x),
\]
where the coefficients \(C_{nm}^{a}\) may depend on the zero-mode momenta without affecting the conformality and we assume \(C_{00}^{a}(\omega, \nu) = 1\). Note that there arises no ordering ambiguity due to the commutativity of operators mentioned above. This greatly simplifies the following analysis.

Strictly speaking, the operator product \(V_{\eta\lambda^a} \mathcal{Y}_a \mathcal{Y}_{a\bar{a}}\) becomes ill-defined on the physically interesting region of \(\nu, g\) for sufficiently large \(n, m\). We may, however, consider it as an analytic continuation from the region where it is well-defined.

To determine the coefficients \(C_{nm}^{a}\), we require the locality conditions \([e^{\nu\lambda^a \cdot \varphi(\tau, \sigma)} \cdot e^{\nu\lambda^{\bar{a}} \cdot \varphi(\tau, \sigma')} - 1] = 0\) for \(\sigma \neq \sigma'\). These lead to the following constraints

\[
\sum_{n+r+2(m+s)=J \atop n+s+m+J=K} \left[ C_{nm}^{a}(\omega + \kappa \lambda^a; \nu) C_{rs}^{a}(\omega + (\kappa + \nu) \lambda^a + (n + m) \alpha^a + m \alpha^{\bar{a}}; \nu) I_{nm;rs}^{a} (\nu, \kappa; x, x') + C_{rs}^{a}(\omega + \nu \lambda^a; \nu) C_{nm}^{a}(\omega + (\kappa + \nu) \lambda^a + (r + s) \alpha^a + s \alpha^{\bar{a}}; \nu) I_{rs;nm}^{a} (\nu, \kappa; x', x) \right] = 0,
\]

\[
\sum_{n+r+2(m+s)=J \atop n+s+m+J=K} \left[ C_{nm}^{a}(\omega + \kappa \lambda^a; \nu) C_{rs}^{a}(\omega + \kappa \lambda^a + \nu \lambda^{\bar{a}} + (n + m) \alpha^a + m \alpha^{\bar{a}}; \nu) I_{nm;rs}^{a} (\nu, \kappa; x, x') - C_{rs}^{a}(\omega + \nu \lambda^{\bar{a}}; \nu) C_{nm}^{a}(\omega + \kappa \lambda^a + \nu \lambda^{\bar{a}} + (r + s) \alpha^a + s \alpha^{\bar{a}}; \nu) I_{rs;nm}^{a} (\nu, \kappa; x', x) \right] = 0.
\]
with \( x^\pm = \tau \pm \sigma \) and \( x'^\pm = \tau \pm \sigma' \). We have introduced

\[
I_{nm;rs}^a(\kappa, \nu; x, x') \equiv V_{\kappa\eta\lambda^a}(x)\mathcal{V}_a^n(x)\mathcal{V}_a^m(x)V_{\nu\eta\lambda^b}(x')\mathcal{V}_b^r(x')\mathcal{V}_a^s(x') .
\] (13)

The sum should be taken over nonnegative integers \( n, m, r, s \) for given \( J \), the order of \( \mu^2 \), and \( K \), the number of \( V_{\kappa\eta\lambda^a} \) contained in the operator products, with \( J \geq K \). Due to the symmetry \( a \leftrightarrow \bar{a} \) we have only to consider the case \( a = 1 \).

At the lowest order, the locality conditions for \( J = K = 0 \) are trivially satisfied since the free field vertex operators commute at equal time. To illustrate how (11) and (12) work, we consider the case \( a = 10 \). The constraint (11) can be cast into the following form

\[
\int_0^{2\pi} dy^+ dy^- \left[ q^{(\omega^a + \nu \lambda^a + \omega'; y) + \nu \epsilon(x,y)} C_{10}^a(\varpi + \nu \lambda^a; \nu) + q^{(\omega^a + \nu \lambda^a + \nu \epsilon(x',y)} C_{10}^a(\varpi + \nu \lambda^a; \nu) \right.
\]

\[
- q^{(\nu \epsilon(x,y))} C_{10}^a(\varpi; \nu) - q^{(\nu \epsilon(x,y,\nu))} C_{10}^a(\varpi; \nu) \left. \right] \times V_{\kappa\eta\lambda^a}(x)V_{\nu\eta\lambda^b}(x')V_{\eta\lambda^a}(y) = 0 .
\] (14)

Such relation holds true only when the integrand identically vanishes. After some algebraic manipulations we obtain

\[
C_{10}^a(\varpi - \nu \lambda^a; \nu) + q^{-(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi; \nu)
\]

\[
- q^{\nu \epsilon(x,y,x')} C_{10}^a(\varpi; \nu) - q^{-(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi - \nu \lambda^a; \nu) = 0 ,
\] (15)

where we have introduced \( \epsilon(x, y, z) \equiv \epsilon(x, y) + \epsilon(y, z) \). Since \( \epsilon(x, y, x') \) takes the values 0, ±2 and cannot be equal to ±4, this is equivalent to the three independent relations

\[
C_{10}^a(\varpi - \nu \lambda^a; \nu) + C_{10}^a(\varpi; \nu) - C_{10}^a(\varpi - \nu \lambda^a; \nu) = 0 ,
\]

\[
C_{10}^a(\varpi - \nu \lambda^a; \nu) + q^{2(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi; \nu) - q^{2\nu \epsilon(x,y,\nu)} C_{10}^a(\varpi - \nu \lambda^a; \nu) - q^{2(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi - \nu \lambda^a; \nu) = 0 ,
\] (16)

\[
C_{10}^a(\varpi - \nu \lambda^a; \nu) + q^{-(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi; \nu) - q^{2\nu \epsilon(x,y,\nu)} C_{10}^a(\varpi - \nu \lambda^a; \nu) - q^{2(\omega^a - \nu \lambda^a; \nu)} C_{10}^a(\varpi - \nu \lambda^a; \nu) = 0 .
\]

While the constraint (12) leads to

\[
C_{10}^a(\varpi - \nu \lambda^a; \nu) = C_{10}^a(\varpi; \nu) .
\] (17)

This implies that \( C_{10}^a(\varpi; \nu) \) is independent of \( \varpi^a \) as is expected since the lowest order analysis is essentially the same with that of Liouville theory. The solution to (16) has been argued in detail in refs. 8, 10. We simply give the result for the generic value of \( g \)

\[
C_{10}^a(\varpi; \nu) = \frac{c_0[\nu]}{[\varpi^a + 1][\varpi^a - \nu + 1]} ,
\] (18)
where we have introduced $q$-numbers $[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$. The arbitrary constant $c_0$ can be fixed by imposing the field equations at the first order in $\mu^2$, where we can replace the Toda fields in the Toda potential by $\eta \psi$. This leads to

$$c_0 = (8\pi g \sin 2\pi g)^{-1}.$$  

We generalize all this to $C_{n0}(\varpi; \kappa)$ for arbitrary $n$ since (11) for $J = K$ are equivalent to the locality conditions in the Liouville case. One can obtain these coefficients from the results of ref. [2].

At the second order in $\mu^2$, we have only to determine $C_{01}^a$. This can be achieved by considering the constraint (12) for $J = 2$ and $K = 1$. By the analysis similar to the first order case we obtain the condition

$$q^a(\varpi^a - \kappa)= (x, y_1, y_2, x') C_{01}^{a}(\varpi - \nu \lambda \bar{a}; \kappa) + q^{-(\nu a + 1)}(x, y_1, y_2, x') C_{10}^{a}(\varpi; \nu)$$

$$- q^{\varpi a}(x, y_1, y_2, x') C_{01}^{a}(\varpi; \kappa) - q^{-(\nu a - \kappa + 1)}(x, y_1, y_2, x') C_{01}^{a}(\varpi - \kappa \lambda \bar{a}; \nu)$$

$$= - C_{01}^{a}(\varpi - \nu \lambda \bar{a}; \kappa) C_{10}^{a}(\varpi + \alpha_2; \nu) + q^{-\varepsilon(x, y_1, y_2, x')}(\varpi + \alpha_2; \kappa) C_{10}^{a}(\varpi - \kappa \lambda \bar{a}; \nu),$$

where $\varepsilon(x, y, z, w) \equiv \varepsilon(x, y) + \varepsilon(y, z) + \varepsilon(z, w)$. Since $\varepsilon(x, y_1, y_2, x')$ takes only the values 0, 1, 2, 3, 4, (21) implies five constraints. They can be solved with respect to $C_{01}^{a}(\varpi; \kappa)$. We thus obtain the second order coefficients

$$C_{20}^{a}(\varpi; \nu) = \frac{c_0^2[\nu][\nu + 1]}{[2][\varpi a + 2][\varpi a + 3][\varpi a - \nu + 1][\varpi a - \nu + 2]},$$

$$C_{01}^{a}(\varpi; \nu) = - \frac{c_0^2[\nu]}{[\varpi a + \varpi a + 1][\varpi a + \varpi a - \nu + 1][\varpi a][\varpi a + 1]}.$$  

The analysis can be carried out for higher orders. The complication of the constraints (11) and (12), however, grows rapidly with the order of $\mu^2$. We only quote here the results of the third and the fourth order cases.

Third order coefficients:

$$C_{30}^{a}(\varpi; \nu) = \frac{c_0^3[\nu][\nu + 1][\nu + 2]}{[2][3][\varpi a + 3][\varpi a + 4][\varpi a + 5][\varpi a - \nu + 1][\varpi a - \nu + 2][\varpi a - \nu + 3]},$$

$$C_{11}^{a}(\varpi; \nu) = - \frac{c_0^3[\nu][\nu + 1]}{[\varpi a + 2][\varpi a - \nu + 1][\varpi a + \varpi a + 2][\varpi a + \varpi a - \nu + 1][\varpi a - 1][\varpi a + 1]}.$$
Fourth order coefficients:

\[
C^a_{40}(\varpi; \nu) = c_0^4 \frac{[\nu]\nu + 1][\nu + 2][\nu + 3]}{[2][3][4]} \frac{1}{[\varpi^a + 4][\varpi^a + 5][\varpi^a + 6][\varpi^a + 7]}
\times \frac{1}{[\varpi^a - \nu + 1][\varpi^a - \nu + 2][\varpi^a - \nu + 3][\varpi^a - \nu + 4]},
\]

\[
C^a_{21}(\varpi; \nu) = -c_0^4 \frac{[\nu]\nu + 1}[2] \frac{1}{[\varpi^a + 3][\varpi^a + 4][\varpi^a - \nu + 1][\varpi^a - \nu + 2]}
\times \frac{1}{[\varpi^a + \varpi^a + 3][\varpi^a + \varpi^a - \nu + 1][\varpi^a - 2][\varpi^a + 1]},
\]

\[
C^a_{02}(\varpi; \nu) = c_0^4 \frac{[\nu][\nu + 1]}{[2]} \frac{1}{[\varpi^a + \varpi^a + 2][\varpi^a + \varpi^a + 3][\varpi^a + \varpi^a - \nu + 1][\varpi^a + \varpi^a - \nu + 2]}
\times \frac{1}{[\varpi^a][\varpi^a + 1]^2[\varpi^a + 2]},
\]

(23)

These results suffice to infer the general form of \(C^a_{nm}\) for arbitrary \(n, m\). We thus arrive at the conjecture for the coefficients

\[
C^a_{nm}(\varpi; \nu) = (-1)^m c_0^{n+2m} \frac{\Gamma_q[\nu + n + m]}{[n]!\Gamma_q[\nu]} \frac{\Gamma_q[\varpi^a + \varpi^a + n + m]\Gamma_q[\varpi^a + \varpi^a - \nu + 1]}{\Gamma_q[\varpi^a + \varpi^a + n + 2m]\Gamma_q[\varpi^a + \varpi^a - \nu + m + 1]}
\times \frac{\Gamma_q[\varpi^a + n + m]\Gamma_q[\varpi^a - \nu + 1]}{\Gamma_q[\varpi^a + 2n + m]\Gamma_q[\varpi^a - \nu + n + 1]\Gamma_q[\varpi^a - n + m]\Gamma_q[\varpi^a + n + 1]},
\]

(24)

where \(\Gamma_q[x]\) is the \(q\)-deformed \(\Gamma\)-function defined by \(\Gamma_q[x + 1] = [x]\Gamma_q[x]\) with \(\Gamma_q[1] = 1\), and \([n]! \equiv \Gamma_q[n + 1]\) stands for the \(q\)-factorial. With (19) and (24), (11) represents the \(q\)-deformation of the classical Toda exponential. In fact one can show that in the limit \(g \to 0\) with \(8\pi g \varpi \to \gamma P\) kept finite our operator expressions reduce to the classical solution (2). This is the central result of the present work.

So far our analysis has been restricted to the Toda exponentials associated with the fundamental weights. We can, however, define arbitrary exponential operators \(e^{\beta \cdot \varphi}\) in terms of them as a composite operator:

\[
e^{\beta \cdot \varphi} = \left( \sum_{n,m=0}^{\infty} \left( \frac{\mu^2}{4} \right)^{n+2m} C^a_{nm}(\varpi + \alpha^a \cdot \beta \lambda^a; \alpha^a \cdot \beta) Y^m_a Y^m_{a\bar{a}} \right) V_{\alpha^a \beta}
\times \left( \sum_{r,s=0}^{\infty} \left( \frac{\mu^2}{4} \right)^{r+2s} C^a_{rs}(\varpi; \alpha^a \cdot \beta) Y^r_a Y^s_{a\bar{a}} \right),
\]

(25)

where \(\beta\) is an arbitrary vector in the \(A_2\)-root space. In particular we need such operators for \(\beta = \alpha^a\) to verify the field equations.
The Toda exponential (10) reduces to a finite polynomial in the screening charges when \( \nu \) is a negative integer as in the Liouville case. The composition rule (25) can be used to construct such operators as the \((-\nu)\)-th power of \( e^{-\lambda^a \cdot \varphi} \), which is given exactly in our analysis. Since the latter operator satisfies the locality, any composite of such operators is necessarily local. It can be shown that the coefficients \( C^a_{nm}(\varpi; \nu) \) can be obtained recursively from

\[
\sum_{n,m} \left( \frac{\mu^2}{4} \right)^{n+2m} C^a_{nm}(\varpi; \nu) Y^m_a \bar{Y}^m_{\bar{a}} = \prod_{k=0}^{-\nu-1} \left( 1 + \frac{\mu^2}{4} C^a_{10}(\varpi + (k - \nu - 1)\lambda^a; -1) Y_a + \left( \frac{\mu^2}{4} \right)^2 C^a_{01}(\varpi + (k - \nu - 1)\lambda^a; -1) Y_{\bar{a}} \right). \tag{26}
\]

This can be used to check the validity of (24). We have worked out the fifth order coefficients for \( \nu \) being some negative integers. They are consistent with (24).

To complete our construction of the exact operator solution, we must verify the field equations and the canonical commutation relations for the interacting Toda field obtained from \( \lambda^a \cdot \varphi \equiv \frac{d}{d\nu} e^{\nu \lambda^a \cdot \varphi} \bigg|_{\nu=0} \). This can be carried out in exactly the same way for the locality analysis. We have confirmed that the Toda fields satisfy the operatorial field equations and the canonical commutation relations

\[
[\varphi_j(\tau, \sigma), \dot{\varphi}_k(\tau, \sigma')] = [\eta \psi_j(\tau, \sigma), \eta \dot{\psi}_k(\tau, \sigma')] \tag{27}
\]

to the fourth order in \( \mu^2 \). Thus the operatorial mapping \( \eta \psi \to \varphi \) can be considered as canonical.

In summary we have investigated quantum \( A_2 \)-Toda field theory based on the method of quantizing canonical free fields. We have obtained exact operator field solution satisfying the fundamental requirements of quantum theory to fourth order of the cosmological constant. This lead us to the conjecture for the full order expression. Though the analysis have been restricted to \( A_2 \) case, it can be extended to other Toda theories of higher ranks. Our results also suggests that the algebraic approach developed for Liouville theory \([3, 11]\) can be extended to Toda field theories. We will argue the issue elsewhere.


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