Quantum noise of electron-phonon heat current

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Abstract We analyze heat current fluctuations between electrons and phonons in a metal. In equilibrium we recover the standard result consistent with the fluctuation-dissipation theorem. Here we show that heat current noise at finite frequencies, remains non-vanishing down to zero temperature. We briefly discuss the impact of electron-phonon heat current fluctuations on calorimetry, in particular in the regime of single microwave-photon detection.

Keywords Electron-phonon coupling · Heat current · Quantum fluctuations

1 Introduction

At low temperatures, phonons provide the heat bath to which electrons couple weakly in a mesoscopic electron circuit. Due to fast electron-electron relaxation [1], electrons typically obey Fermi-Dirac distribution even under non-equilibrium conditions with a well-defined temperature $T_e$ that can differ from the temperature $T_p$ of the phonons. For clean metals in three-dimensional structures, the relaxation rate of electrons to the temperature of phonons scales as $T_e^{-3}$ [2,3,4]. This result is based on the electron-phonon coupling arising from standard deformation potential model [5]. For an arbitrary temperature difference (as long as the model is valid) the average heat current between electrons and phonons scales as $T_e^5 - T_p^5$ in the same model, and this result is widely observed in experiments at sub-kelvin temperatures. Detection of single...
quanta of radiation by calorimetric means has become popular over the past years [6,7]. In this context, not only the average heat current but also its fluctuations are important. They set the fundamental bound of minimum detectable energy of the radiation quantum. Here we re-visit the standard results of heat current fluctuations under equilibrium and non-equilibrium conditions, which influence, e.g., single-photon detection in the microwave regime [8,9]. As the main result we present electron-phonon heat current noise at finite frequencies, and realize that it is non-vanishing even in the zero temperature limit. This result has interesting implications in terms of the fluctuation-dissipation theorem for heat in the quantum regime, the topic discussed over the past few years in the context of electron transmission in tunnel contacts and through general scatterers [10].

2 Description of the system and heat current operators

A sketch of the system is schematically shown in Fig. 1. The normal metal is thermally coupled to the local phonon bath with thermal conductance $G_{ep}$ at temperature $T_p$. The total Hamiltonian describing the system and the environment is given by

$$H = H_e + H_p + H_{ep},$$

where $H_e, H_p$ are the Hamiltonians of the electrons and phonons, respectively, and $H_{ep}$ is the coupling between them. The unperturbed Hamiltonian $H_0 = H_e + H_p$ can be written as

$$H_0 = \sum_k \epsilon_k a_k^\dagger a_k + \sum_q \hbar \omega_q c_q^\dagger c_q,$$

where the first part describes electron states with energy $\epsilon_k$, momentum $k$, and $a_k^\dagger$ and $a_k$ are the corresponding creation and annihilation operators. With analogous notation, the second part shows the Hamiltonian of phonons with
eigenenergies $\hbar \omega_q$, wavevector $q$, and bosonic creation and annihilation operators $c_q^\dagger$ and $c_q$. The coupling term as a perturbation of the system has the following form in a metal \[ H_{\text{ep}} = \gamma \sum_{k,q} \omega_q^{1/2} (a_k^\dagger a_{k-q} c_q + a_{k-q}^\dagger a_k c_q^\dagger). \] (3)

Here, the magnitude of $\gamma$ depends on the material properties of the system. The operator of heat flux from the electron system to phonons due to ep coupling is \[ \dot{H}_p = i \frac{\hbar}{\gamma} [H_{\text{ep}}, H_p] = i\gamma \sum_{k,q} \omega_q^{3/2} (a_k^\dagger a_{k-q} c_q - a_{k-q}^\dagger a_k c_q^\dagger), \] (4)

where we used the commutation relations for the bosonic operators, $[c_q, c_q^\dagger] = c_q$ and $[c_q^\dagger, c_q] = -c_q^\dagger$. Similarly we find the operator for heat flux to electron system \[ \dot{H}_e = -i\gamma \sum_{k,q} \omega_q^{1/2} (\epsilon_k - \epsilon_{k-q})(a_k^\dagger a_{k-q} c_q - a_{k-q}^\dagger a_k c_q^\dagger). \] (5)

3 Fluctuations of heat current

In order to find the heat current fluctuations, we evaluate the correlator $\langle \mathbb{J}(t) \mathbb{J}(0) \rangle$, where $\mathbb{J} \equiv \frac{i}{\hbar} (\dot{H}_e - \dot{H}_p)$ is the symmetric heat current operator between electron and phonon baths. Using Eqs. (4) and (5) yields \[ \mathbb{J} = -\frac{i\gamma}{2} \sum_{k,q} \omega_q^{1/2} (\hbar \omega_q + \epsilon_k - \epsilon_{k-q})(a_k^\dagger a_{k-q} c_q - a_{k-q}^\dagger a_k c_q^\dagger). \] (6)

Then we have \[ \langle \mathbb{J}(t) \mathbb{J}(0) \rangle = \frac{\gamma^2}{4\hbar^2} \sum_{k,q} \omega_q (\hbar \omega_q + \epsilon_k - \epsilon_{k-q})^2 \left[ \langle a_k^\dagger(t) a_k(0) a_{k-q}(0) a_{k-q}^\dagger c_q(0) c_q^\dagger(0) \rangle 
+ \langle a_{k-q}^\dagger(t) a_{k-q}(0) a_k(t) a_k^\dagger c_q(0) c_q(0) \rangle \right]. \] (7)

We use the time dependence of the creation and annihilation operators, $a_k(t) = a_k e^{-i\epsilon_k t / \hbar}$ and $c_q(t) = c_q e^{-i\omega_q t}$, taking the expectation values of the products of the operators $\langle a_k^\dagger a_k \rangle = f(\epsilon_k)$ and $\langle c_q^\dagger c_q \rangle = n(\omega_q)$, where $f(\epsilon) = (1 + e^{\beta_\epsilon})^{-1}$ and $n(\omega) = (e^{\beta_\omega} - 1)^{-1}$ are Fermi and Bose distribution for electrons and phonons, respectively, with related inverse temperatures $\beta_\epsilon = (k_B T_e)^{-1}$ and $\beta_\omega = (k_B T_p)^{-1}$. Integrating for the noise power $S_\gamma(\omega) = \int dt \langle \mathbb{J}(t) \mathbb{J}(0) \rangle e^{i\omega t}$ leads to \[ S_\gamma(\omega) = \frac{\gamma^2}{4\hbar^2} \sum_{k,q} \omega_q (\hbar \omega_q + \epsilon_k - \epsilon_{k-q})^2 \left[ e^{i(\epsilon_k - \epsilon_{k-q} - \hbar \omega_q + \hbar \omega_q) t / \hbar} f(\epsilon_k)[1 - f(\epsilon_{k-q})] 
+ e^{i(\epsilon_k - \epsilon_{k-q} + \hbar \omega_q + \hbar \omega_q) t / \hbar} f(\epsilon_{k-q})[1 - f(\epsilon_k)] n(\omega_q) \right]. \] (8)
Now, we integrate over time, and replace \(\sum_q \rightarrow D(q) \int d^3q = \frac{V}{(2\pi)^3} \int_0^{+\infty} dq q^2 \int_{-1}^{+1} d(\cos \theta)\) in spherical coordinates, where \(\theta\) is the angle between \(k\) and \(q\), and \(\sum_k \rightarrow N(0) \int d\epsilon_k, N(0) (D(q) = \frac{V}{(2\pi)^3})\) denotes the density of states of electrons (phonons), here \(V\) is the volume of the system. Further, \(\epsilon_k = \frac{\hbar^2k^2}{2m}, \epsilon_{k\pm q} \simeq \epsilon_k \pm \frac{\hbar^2k_Fq \cos \theta}{m}\), where the last approximation is due to \(k \approx k_F\) and \(q \ll k_F\), where \(k_F\) is the Fermi wave vector and \(m\) is the mass of electron. Moreover \(\omega_q\) is replaced by \(c_q q\), where \(c_q\) is the speed of sound. Then Eq. (5) leads to

\[
S_3(\omega) = \frac{\gamma^2 N(0)V}{8\pi \hbar} \int_{-\infty}^{+\infty} d\epsilon_k \int_0^{+\infty} dq q^2 \int_{-1}^{+1} d(\cos \theta) \omega_q (\hbar \omega_q + \frac{\hbar^2k_F}{m}q \cos \theta)^2 \times
\]

\[
\left[ f(\epsilon_k)[1 - f(\epsilon_k - \frac{\hbar^2k_F}{m}q \cos \theta)][1 + n(\omega_q)]\delta(\frac{\hbar^2k_F}{m}q \cos \theta - \hbar \omega_q + \hbar \omega)
\right.
\]

\[
+ f(\epsilon_k - \frac{\hbar^2k_F}{m}q \cos \theta)[1 - f(\epsilon_k)]n(\omega_q)\delta(\frac{\hbar^2k_F}{m}q \cos \theta + \hbar \omega_q + \hbar \omega)\right].
\]

Collecting the angle dependent terms and integrating over \(\cos \theta\) and using notation \(\epsilon \equiv \hbar \omega_q = \hbar c_q q\), we have

\[
S_3(\omega) = \frac{\Sigma V}{96\zeta(5)k_B^2} \int_0^{+\infty} d\epsilon \epsilon^2 \left[ (2\epsilon - \hbar \omega)^2 \epsilon - \hbar \omega \right.
\]

\[
\left. + (2\epsilon + \hbar \omega)^2 \epsilon + \hbar \omega \right]
\]

\[
(10)
\]

where we have defined the electron-phonon coupling constant \([4][13]\) as \(\Sigma = \frac{12\pi^2 N(0)m(5)k_B^5}{\pi k_F^4 c^5}\) and \(\zeta(z)\) denotes the Riemann zeta function.

One can calculate the average heat flux into the phonon bath \(\dot{Q}_{ep} = \langle \dot{H}_p \rangle\) by applying the Kubo formula in the interaction picture to Eq. (4) as

\[
\dot{Q}_{ep} = -\frac{i}{ \hbar} \int_0^{+\infty} dt' \langle[\hat{H}_p(t), H_{ep}(t')]\rangle.
\]

We obtain then the known result

\[
\dot{Q}_{ep} = \Sigma V(T_e^5 - T_p^5).
\]

For the noise spectrum at equal temperatures, \(T_e = T_p\), and \(\omega = 0\), we obtain

\[
S_3(0) = 10\Sigma V k_B T^6,
\]

(13)

which is again the well known classical result \([5]\). In this regime \(\dot{Q}_{ep}\) yields the thermal conductance between electrons and phonons, \(G_{ep}\), by differentiation with respect to \(T_e\) at \(T = T_e = T_p\). We have \(G_{ep} = d\dot{Q}_{ep}/dT_e = 5\Sigma V T^4\), which satisfies the fluctuation-dissipation relation

\[
S_3(0) = 2k_B T^2 G_{ep}.
\]

(14)

For zero temperature \(\beta_p, \beta_e \rightarrow \infty\), we see that only the first term in the
Fig. 2 Heat current noise $S_I(\Omega)$ normalized by its classical value $S_I(0)$, where $\Omega = \beta \hbar \omega$. Solid red line is the full expression based on Eq. (16); solid blue line shows the first order approximation $1 + \Omega^2$ while dashed black line is the second order result based on Eq. (17). Top scale indicates the temperature $T$ with a detector of cut-off frequency $f_c$ for achieving the frequency on the bottom axis. Inset: The same data as in the main frame over a wider frequency range.

The integrand of Eq. (10) is non-zero and just over the interval $0 < \epsilon < \hbar \omega$. Then,

$$S_\gamma(\omega) = \frac{\sum V}{96\zeta(5) k_B^4} \frac{(\hbar \omega)^6}{60}. \quad (15)$$

This result is in analogy with the expressions of zero-temperature noise of heat current in charge transport through a scatterer, for instance a tunnel junction [10,11,12]. In that case $S_\gamma \propto \omega^3$, which is consistent with our result in the following way. The thermal conductance in a tunnel junction scales as $\propto T$. The fluctuation dissipation theorem then yields $S_\gamma(0) \propto T^3$, to be compared to $T^6$ in the electron-phonon system (Eq. (15)). Now, replacing $k_B T$ by $\hbar \omega$ in the two cases, we get the corresponding zero temperature finite frequency noise $\propto \omega^3$ and $\propto \omega^6$, respectively.
4 Discussion of fluctuations at finite frequency

At equal temperatures and finite frequency, Eq. (10) can be written in the dimensionless form

\[
S_I(\Omega) = \frac{S_I(0)}{960 \zeta(5)} \int_0^\infty du \, u^2 \left[ \frac{(2u - \Omega)^2}{1 - e^{-u}} \frac{u - \Omega}{e^{u-\Omega} - 1} \right. \\
\left. + \frac{(2u + \Omega)^2}{e^u - 1} \frac{u + \Omega}{1 - e^{-(u+\Omega)}} \right],
\]

(16)

where \( \Omega \equiv \beta \hbar \omega \). The expansion of \( S_I(\Omega) \) in \( \Omega \) up to second order yields

\[
\frac{S_I(\Omega)}{S_I(0)} = 1 + \frac{\Omega}{2} + \left( \frac{1}{12} + \frac{7}{240 \zeta(5)} \right) \Omega^2.
\]

(17)

We present the result of Eq. (16) for the noise power in the case of equal temperatures as a function of \( \Omega \) in Fig. 2. The analytic approximations up to the first and second order in \( \Omega \) are also shown, and we see that the second order result follows the exact result up to \( \Omega \sim 1 \).

We finally comment on the observability of the finite frequency corrections. Experimental techniques to measure temperature focus traditionally into the low frequency regime. Yet RF-techniques developed for charge transport [14] and circuit QED (Quantum Electro-Dynamics) experiments [15] have recently been adapted to thermometry to address temporal evolution of temperature and noise in heat currents down to sub-\( \mu \)s time scales [16,17,18]. With these methods, noise up to \( f_c \sim 1...10 \) MHz frequencies becomes experimentally feasible. The experimentally available \( \Omega \) range (0 < \( \Omega < \Omega_{max} \)) in Fig. 2 can be obtained by setting \( \Omega_{max} = \hbar f_c / k_B T \), where \( T \) is the temperature of the experiment. We indicate \( T \) on the top axis of Fig. 2 scaled by cut-off frequency \( f_c \) of the thermometer. As an example we see that for \( f_c = 10 \) MHz, one needs to measure at \( T = 5 \) mK to achieve \( \Omega_{max} = 0.1 \). This electronic temperature is within the range of present day experiments on nanostructures [19,20]. In this case a correction of \( \sim 5\% \) in \( S_3 \) can be observed. However, at the time when the experimental observation of even the classical heat current noise remains elusive, the experiment on quantum heat current noise is still a challenge.

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