THE HIGHER-ALGEBRAIC SKELETON OF THE SUPERSTRING
– A CASE STUDY

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Abstract. A novel Lie-superalgebraic description of the superstring in the super-Minkowskian background is extracted from the Cartan–Eilenberg super-1-gerbe geometrising the higher gauge field (the Green–Schwarz super-3-cocycle) that couples to the supercharge carried by the superstring. The description assumes the form of a hierarchy of Lie superalgebras integrable to a hierarchy of Lie supergroups and provides a manifestly supersymmetric model of a family of supermanifolds defining a trivialisation of the super-1-gerbe over the embedded superstring worldsheet. The trivialisation, obtained in a purely topological formulation of the superstring dynamics dual to the standard Nambu–Goto-type one, conforms with the gerbe-theoretic representation of extended sources of higher gauge fields known from previous studies of the σ-model of the bosonic string.

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1. Introduction

The idea to probe spacetime geometry with the dynamics of distributions of charged matter has been around for a long time, cp Refs. [FI29b, FI29a, Wey29, Foc29a, Foc29b, Dir31], yielding an enhancement of the simple model of a metric spacetime \((M, g)\) accessible to a neutral pointlike particle. The enhancement incorporates the ‘higher’ geometry of the gauge fields \(H \in Z^p_{dR}(M), p \in \mathbb{N}\) (and their nonabelian counterparts) coupling to the respective charges, first neatly packaged by Lubkin [Lub63], Trautman [Tra70] et al. in the structure of fibre bundles, and later generalised as bundle \((p-)\)gerbes and related objects by Murray et al. [Mur96, MS00, Ste04], cp, in particular, Ref. [Gaj97]. The latter form a descent hierarchy of geometrisations of integral classes in (a suitable refinement of) the de Rham cohomology of \(M\) whose local sections provide us with the (suitably extended) Deligne–Beilinson cohomological data of the gauge fields, employed in the construction of simple models of charge dynamics and their geometric quantisation already by Alvarez [Alv85] and Gawêdski [Gaw88].

In the presence of Killing vector fields \(K_A \in \Gamma(TM), A \in 1, K\) of the background metric \(g\) whose flows preserve the action functional defining the charge dynamics, the enhancement may take the form of a deformation or an extension of the Lie algebra of these vector fields. Notable instantiations of the former include the algebra \([P_\mu, P_\nu] = 2qH_{\mu\nu}, \mu, \nu \in 0, 3\) of lifts of translations \(\{P_\mu\}_{\mu=0}^3\) to the space of states of a pointlike particle of charge \(q\) in a constant electromagnetic field \(H = H_{\mu\nu} dx^\mu \wedge dx^\nu \in \Omega^2(Mink(3,1))\), and the Poisson germ of the Drinfeld–Jimbo quantum-group structure in the (chiral) Wess–Zumino–Witten model on \(SU(2)\) (with the Cartan 3-form as the gauge field) in Ref. [Gaw91, Sec. 4]. The latter are amply exemplified by the infinite sequence of

\footnote{For a gentle introduction to the general theory, cp Ref. [Joh02]. An overview was given by Murray in Ref. [Mur10].}
extensions Maxwell, \( n \in \mathbb{N}^* \) of the Poincaré algebra in Refs. \[BG10\], \[GK17\] as algebraic structures encoding the rich dynamics of a (possibly backreacting) multipole distribution of charges in an external electromagnetic field (generalising the pioneering constructions: the kinematical algebras of Ref. \[BCR74\] and the Maxwell algebra of Ref. \[Sch72\]) and by the Free (super-)Differential-Algebra (FsDA) extensions of the super-Poincaré algebra considered in Ref. \[CdAI18\] in the context of superstring theory in Minkowskian spacetime that build upon the earlier constructions of Refs. \[Gre89\], \[Sie94\], \[BS95\]. The supersymmetric extensions, of central relevance to us in what follows, seem to be encompassed by the structure of the Free Lie super-Algebra (FLsA) laid out in the recent study \[GKP19\]. The common source of the enhancement in the examples listed is an interplay between the geometry (topology) of the distribution of charge consistent with its dynamics and the intrinsic cohomology of the gauge field \( H \) (tied intimately with the aforementioned higher geometry), the latter being typically assumed to satisfy the strong invariance condition \( K_A : H \in P_{gt}^M(\mathbb{M}) \) (ensuring quasi-invariance of the lagrangian density) in the case of charge distributions localised on closed submanifolds of \( \mathbb{M} \), \( cp \), in particular, Refs. \[LAGT86\], \[GSW10\], Cor. 2.2 and \[Sus18\], Sec. 3]. The two are jointly encoded by the (pre)symplectic form of the lagrangian model of dynamics, as given by the first-order formalism of Refs. \[Gaw72\], \[Kij73\], \[Kij74\], \[SS76\], \[Szc70\], \[KT79\], and so a question arises how to isolate information on (the geometry of) a particular classical solution of the dynamics given an enhancement of a reference (neutral) Killing algebra (note that the enhancement captures symmetries of the entire space of classical solutions). This is the general problem that we tackle in the present case study in the framework of supersymmetric dynamics of super-\( p \)-branes, to be investigated using methods of higher (super)geometry that we review systematically below.

The algebraic mechanisms from the previous paragraph have been encountered and employed extensively as model-building tools in the setting of (super)field theory with non-linearly realised symmetry \[CWZ99\], \[CCWZ99\], \[SS99\], \[SS97\], and supersymmetry \[VA72\], \[VA73\], \[IK75\], \[LR79\], \[ZS82\], \[IK83\], \[SW83\], \[EMW83\], \[BW84\] (originally contemplated by Schwinger \[Sch60\] and Wigner \[We68\] in the context of effective field theory with chiral symmetries) in which the fibre of the covariant configuration bundle (or the ‘field space’) carries the structure of a homogeneous space \( G/H \) of a (super)symmetry group \( G \) relative to its distinguished closed subgroup \( H \) with the tangent Lie algebra \( \mathfrak{h} \cong \text{Lie}(H) \) defining a reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \) of the tangent Lie (super)algebra \( \mathfrak{g} \) of \( G \), i.e., such that \( [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t} \). Here, the dynamics is modelled in terms of \( H \)-basic tensors on \( G \) taken from the tensor algebra of the linear space of (\( G \)-)left-invariant (LI) (super)-1-forms on the (super)manifold \( G \) and pulled back to the field space \( G/H \) along local sections of the principal \( H \)-bundle \( G \rightarrow \text{e}_{G/H} \rightarrow G/H \) (\( \pi_{G/H} \) is the quotient map) whose potential non-triviality was accounted for in \[Sus19\], Sec. 5 and whose local sections particularly favoured by physical considerations were analysed at length in the \( \mathbb{Z}/2\mathbb{Z} \)-graded setting in \[Sus20\], Sec. 2]. An in-depth study of the mechanism of spontaneous (super)symmetry breakdown by a classical solution to the dynamics in this setting, in conjunction with a clever application of the so-called Inverse Higgs Effect originally discovered by Ivanov and Ogievetsky \[IO73\], have provided us with a reinterpretation of some standard action functionals modelling charge dynamics as Goldstone fields conjugate to distinguished central charges defining extensions of geometric (i.e., neutral) (super)symmetry algebras \[GGT90\], and – crucially from the vantage point adopted in the present paper – have led to a purely topological reformulation of the Nambu–Goto-type ‘metric’ components of action functionals for structureless (as in Ref. \[Dir62\], \( cp \) also Ref. \[HAT89\]) extended distributions of charged matter in, \( i.e., \) Refs. \[Wes00\], \[GW06a\], \[GW06b\], \[McA10\], along the lines of the original idea of Hughes and Polchinski \[HP86\] developed by Gauntlett, Itoh and Townsend in Ref. \[IT99\].

The specific choice, referred to in the last paragraph, of a constructive paradigm of study of charge dynamics with the help of the Cartan calculus on a Lie (super)group paves the way for a systematic application of the techniques of Free (\( \mathbb{Z}/2\mathbb{Z} \)-graded) Differential Algebras (FDA) of Refs. \[DF82\], \[KN83\], \[CF8+83\]. In the context of interest, these are specialised to an augmentation of the canonical FDA \( \mathcal{L}(G) \) of \( LI \) (super-)1-forms on the Lie (super)group \( G \) by the (super-)\((p+1)\)-form potential of the relevant gauge field. For these, a finite ladder of integrable (super)central extensions \( 0 \rightarrow \mathfrak{a}_N \rightarrow \mathfrak{a}_{N+1} \rightarrow \mathfrak{g}_N \rightarrow \mathfrak{g} \) (altogether combining into a generically non-(super)central extension \( Y_{\mathfrak{g}} \cong \mathfrak{g}_N \rightarrow \mathfrak{g}_N \rightarrow \mathfrak{g} \) of the original Lie (super)algebra \( \mathfrak{g} \)) is sought that yields a resolution of the gauge field in the Cartan–Eilenberg (CaE) cohomology \( \text{CaE}^{p+2}(\mathfrak{g}) \cong H_{dt}^{p+2}(\mathfrak{g}) \cong \mathfrak{g}^{p+2} \) of the Lie (super)group \( \mathfrak{g} \) which integrates \( Y_{\mathfrak{g}} \). Each rung of the ladder is determined by a (super-)2-cocycle in the decomposition of the pullback of the gauge field in terms of elements of \( \mathcal{L}(\mathfrak{g}_N) \) (for \( \mathfrak{g}_N \) the Lie (super)group of

\[This\ is,\ arguably,\ most\ convincingly\ illustrated\ in\ the\ ‘top-down’\ treatment\ of\ charge\ dynamics\ in\ Refs.\ \[BG10\], \[GK17\].]
\( g_n \) in conformity with the standard classification of (super)central extensions of a Lie (super)algebra \( g_n \) by a (super)commutative Lie (super)algebra \( a_n \) by elements of the group \( H^2(g_n, a_n) \) in the \( a_n \)-valued cohomology of \( g_n \), cp Ref. [1679]; while it is not clear a priori that a finite resolution \( Y \) of this kind exists \( \{ \text{cp the Theorem in Ref. [\text{NNS}3, \text{Sec.} 7]\} \}, \) an algorithm devised by de Azcárraga and collaborators in Ref. [CaAIPB00], which essentially boils down to a systematic reconstruction of the results of the FLAs of [GKP19], does produce the desired result in the very special setting of the Green–Schwarz-type (GS) super-\( \sigma \)-models of super-\( p \)-brane dynamics in super-Minkowskian (super)geometry, \( g \) \( s \text{Mink}(d, 1|N_{D_d,1}) \cong \mathbb{R}^{d,1|N} \) \( \{ \text{Cas67, BSS1, AL682, GS834, GS841, AETW87} \} \) (here, \( N \in \mathbb{N}^* \) is the number of supercharges in a Majorana–spinor representation of \( \text{Cliff}(\mathbb{R}^{d,1}) \) of dimension \( D_{d,1} \) that generate an \( N \)-extended supersymmetry) that we work with in the present paper and on which, consequently, we focus henceforth. In fact, it was recently argued by Grasso and McArthur in Refs. [GM18a, GM18b] that these results are essentially unique \( \{ \text{when viewed as solutions to a cohomological problem in CaE}^{p+2}(g \text{Mink}(d, 1|N_{D_d,1})) \} \) (their argument exploits the assumed triviality of the de Rham cohomology of the extension).

In order to be able to interpret the extension \( Y \rightarrow G \) as a partial geometrisation of the GS super-\( (p+2) \)-cocycle \( \mathcal{H} \in \mathbb{Z}^{p+2}_{\text{dir}}(G^G) \) in the (standard) sense of Murray, one should put the \( \text{CaE} \) cohomology of \( G \) on the same footing as the underlying de Rham cohomology. That this makes sense is suggested by an old argument due to Crane and Given [RC85, Rab87] that essentially explains the discrepancy between \( \text{CaE}^n(\text{sMink}(d, 1|N_{D_d,1})) \) (for \( n = 1 \)) and \( H^0_{\text{dir}}(g \text{Mink}(d, 1|N_{D_d,1})) = 0 \) as coming, via the standard duality, from the homology of an orbifold \( g \text{sMink}(d, 1|N_{D_d,1})/\Gamma_{\text{KKR}} \) of \( g \text{sMink}(d, 1|N_{D_d,1}) \) relative to a discrete subgroup \( \Gamma_{\text{KKR}} \subset g \text{sMink}(d, 1|N_{D_d,1}) \) that had been encountered previously by Kostelecký and Rabin in their study of supersymmetric field theory on the lattice [KR88]. The orbifold has the topological structure of a fibration over its body \( g \text{Mink}(d, 1) \) with compact \( \text{Graf} \text{f} \text{mann-odd fibers} [\text{RC85}]. \) The argument led Rabin to postulate that the GS super-\( \sigma \)-model with the supertarget \( g \text{sMink}(d, 1|N_{D_d,1}) \) can be interpreted as describing propagation of loop-like distributions of supercharge within \( g \text{sMink}(d, 1|N_{D_d,1})/\Gamma_{\text{KKR}} \). But then, by a standard argument (cp, e.g., Refs. [DHVW85, DHVW86] and, in particular, [Sus12, Sec. 8.3] and Ref. [Sus13] in which the idea of a worldvolume orbifold was formalised with reference to the universal gauge principle derived in Refs. [GSWI10, GSWI13], one has to incorporate the \( \Gamma_{\text{KKR}} \)-twisted sector in the superfield theory on the cover \( g \text{sMink}(d, 1|N_{D_d,1}) \), and, indeed, this yields, e.g., a \( \text{Graf} \text{f} \text{mann-odd wrapping anomaly in the canonical picture of [Sus18a, Sec. 4.2]} \) that reproduces the Green extension of the \( g \text{sMink}(d, 1|N_{D_d,1}) \) superalgebra resolving the GS super-3-cocycle. While vital for internal consistency of our treatment, the last result shows quite explicitly how an enhancement of a neutral Killing algebra in the presence of a distribution of charged matter \( \text{combines} \) information on the cohomology of the gauge field and the topology of the distribution. We may now be more specific in defining our goal: We wish to extract a clearcut signature of the localisation of a classical superstring (and more generally super-\( p \)-brane) trajectory in the supertarget from the superalgebraic description of the gauge field that couples to it. To this end, we first need to complete the geometrisation of that field and recall from the extensive study of analogous geometrisations in the non-\( Z/2Z \) graded setting the higher-geometric representation of extended objects to which the gauge field of the \( \sigma \)-model couples – the \( D \)-branes [Pol93], with the

\begin{itemize}
  \item [\text{3}]It is certainly more natural to associate with the representative \( H \) of a class in \( \text{CaE}^{p+2}(G) \) a slim Lie \( (p+1) \)-\( \text{(super)algebra of Baez, Crans and Huerta} [B04, BH11, H11], itself a special example of the more general structure (an \( L_\omega \)-(super)algebra) encountered in the study of string field theory [Sta92, LS93]. However, to the best of the Author’s knowledge, there do not exist, up to date, any \( \text{explicit} \) constructions of the corresponding integrated structures (the so-called \( L_\omega \)-(super)groups) for the known super-\( p \)-branes with \( p > 0 \) that would be amenable to direct analysis, which precludes the discussion of a number of concrete issues of physical relevance, cp below.
  
  \item [\text{4}]Exploration of curved supergeometries was pioneered in Refs. [B98, B99, B10, B11, B12, B13].
  
  \item [\text{5}]Ataining a similar goal for physically relevant \textit{curved} supergeometries with a nontrivial topology, such as, e.g., the homogeneous spaces: \( SU(1,1,2)\text{Z}(SO(1,1) \times SO(2)) \equiv sA_{\text{SS}2} \times S^5 \) (viewed as the supertarget of the \( \text{Zhou superstring} [\text{Zhou91}], \) \( SU(1,1,2) \times SU(1,1,2) \text{Z}/(SO(1,1) \times SO(3)) \equiv sA_{\text{SS}3} \times S^5 \) (for the Park–Rey superstring [PR92]); and \( SU(2,2,4)/SO(4,1) \times SO(5) \equiv sA_{\text{SS}3} \times S^5 \) (for the Motsch–Teitly stem [MT98]), seems to call for an augmentation of the original organising principle. One natural positivity, suggested by the prime role of the asymptotic correspondence between the curved dynamics (its supergeometric data and the lagrangian model) and its flat-superspace counterpart in the construction of the former, cp Ref. [MT98], is the requirement that the \( \text{Inom} \text{ – Wigner contraction underlying the asymptotic flattening should lift to the full-fledged geometrisation of the relevant gauge superfield. The principle was laid out in Ref. [Sus18a] (with several no-go results for the Motsch–Teitly stem in its current formulation), elaborated and successfully realised for the \( \text{Zhou superparticle in sA}_{\text{SS}2} \times S^5 \) in Ref. [Sus18a], and formalised concisely in Ref. [Sus21]. It is expected to work out in conjunction with the \( S \)-expansion scheme put forward in Ref. [HS06] and later generalised in Refs. [dAI20, RS06]. This expectation is currently being investigated.
\end{itemize}
coupling encoded in the effective Dirac–Born–Infeld (DBI) lagrangian density [FT85, ACNY87, Lei89].

The train of reasoning restated above has laid the foundation for the geometrisation programme, initiated by the Author in Ref. [Sus17], elaborated in Refs. [Sus19, Sus18a, Sus18b, Sus20] and recently reviewed in Ref. [Sus21], which sets out to associate with the physically relevant CaE gauge-field super-(p + 2)-cocycles that determine the known GS-type super-σ-models on homogeneous spaces G/H of supersymmetry Lie supergroups G, as well as with the attendant supersymmetric defects [FSW08, RS09, Sus11a, Sus11b], concrete higher-geometric objects of the type conceived by Murray et al. in the non-\( \mathbb{Z}/2 \mathbb{Z} \)-graded setting (and recently reconsidered in the \( \mathbb{Z}/2 \mathbb{Z} \)-graded setting by Huerta [Hue20]), and to lift all essential geometric properties of the underlying superfield theories (such as, e.g., their \( \kappa \)-symmetry) and constitutive relations between them (such as, e.g., the fundamental asymptotic relation between the super-p-brane models with the super-AdS\(_ m \times S^ n \) targets and their super-Minkowskian counterparts) to those higher (super)geometries and the associated higher categories. The rationale for the goal thus delineated is an early observation, due to Gawędzki [Gaw88], that the existence of the said lifts is to be viewed as a condition of quantum–mechanical consistency of the \[ \sigma \]-model. Thus, the existence of the said lifts is to be viewed as a condition of quantum–mechanical consistency of the structures, properties and relations lifted. In the gerbe-theoretic picture, the D-branes of string theory are represented by trivialisations of the 1-gerbe of the gauge field of the \( \sigma \)-model over submanifolds of the target space \( M \). [Gaw90, FW90, CJM02, GR02, Saw05] – these are described by certain vector bundles whose connection acquires the interpretation of the (gerbe-twisted) gauge field of the DBI theory.

The basic geometric substrate of the principle of descent that lies at the core of Murray’s geometrisation of the class \([H] \in R^{p+2}(M, Z)\) of a gauge field \( H \) (assumed integral) [Mur96, Mur10] is a surjective submersion \( \pi M \xrightarrow{\pi M} M \) whose total space supports a smooth primitive \( B \in \Omega^ k + 1(Y M) \) for the pullback of \( H \), i.e., such that \( \pi^* M H = dB \). The \((p + 1)\)-form \( B \) can then be viewed as the trivial \( p \)-gerbe \( T H^ (p) \) of curvature \( dB \) and curving \( B \) over \( Y M \). The \( p \)-gerbe \( G^{(p)} \) for \([H]\) is subsequently erected over the nerve of the small category \( [\pi^* M, M \xrightarrow{\pi^* M} Y M] \), defined by the \((\pi^* M)\)-fibred square \( Y^2 M \equiv Y M \times_M Y M \) of the surjective submersion (cp App. A), as a family \( G \equiv \{ G^{(p-k)} \}_{k=1}^{p+1} \) of \((p-k)\)-gerbes (the \( l \)-gerbes with \( l \in \{0, -1\} \) being identified with principal \( \mathbb{C}^* \)-bundles with a compatible connection \((l = 0) \) and connection-preserving isomorphisms between them \((l = -1) \), respectively) over the respective fibred powers \( Y^{k+1} M \equiv Y^k M \times_M Y M \), the members of \( G \) being subject to various coherence constraints. Accordingly, and in keeping with the underlying (super)field-theoretic paradigm in which the \((\text{super})\text{field theory over } G/H\text{ is modelled over } G\), the point of departure of the programme advocated above is the epimorphism of Lie supergroups \( Y G \xrightarrow{\pi Y G} G \) (alongside the LI primitive for the pullback of the LI gauge field \( H \) along \( \pi Y G \)) returned by the integrable-extension algorithm described in the previous paragraph. From this point onwards, one simply turns the crank of Murray’s machine of descent and, recursively, that of de Azcárraga’s extension procedure, insisting that all extensions are consistently ad\( _n \)-equivariant in the latter (a condition essentially built into the FDA techniques employed in the procedure in the guise of the so-called minimal subalgebra [Sus17], cp Ref. [KN83, Sec. 6]), and – in the former – that all secondary surjective submersions that arise in the process are Lie-supergroup epimorphisms, and that all (connection-preserving) isomorphisms of principal \( \mathbb{C}^* \)-bundles that mark the penultimate stage of the construction and of its sub-constructions are Lie-supergroup isomorphisms, so that, by the end of the long day, we obtain a ‘bundle \( p \)-gerbe object in the category of Lie supergroups’. The ensuing Cartan–Eilenberg super-\( p \)-gerbe \( G^{(p)} \) still has to be descended to the relevant homogeneous space \( G/H \). As demonstrated by Gawędzki, Waldorf and the Author in Refs. [GSW10, GSW13, Sus12, Sus11a, Sus13], this requires that \( G^{(p)} \) carry a descendant \( H \)-equivariant structure. One of the crucial features of the advocated geometrisation scheme is that such a structure is inscribed in the very definition of the super-\( p \)-gerbe, which lends weight to the claim to naturalness of the scheme in the \((\text{super})\text{field-theoretic context under consideration. To date, the results of the programme include an explicit construction } [\text{Sus17}] \text{ of the CaE super-}\( p \)-gerbes over } s\text{Mink}(d, 1[D_{d,1}]) \text{ for the GS super-}\( p \)-branes with } p \in \{0, 1, 2\}, \text{ an extensive study } [\text{Sus18a}] \text{ of their equivariance properties inspired by the analogy with the purely} \text{Graßmann-even WZW } \sigma\text{-model } [\text{HMS85}], \text{ and an explicit construction } [\text{Sus18b}] \text{ of a super-0-gerbe over } s(\text{AdS}_2 \times S^2) \text{ for the Zhou superparticle that provides a constructive application of the principle of Inönü–Wigner contractibility proposed in Ref. [Sus18a] (cp the footnote on p.3). The CaE super-}\( p \)-gerbes (for a large class of known
super-\(p\)-brane species) were also shown \cite{Sus19,Sus20} to carry a canonical and canonically (linearised-)supersymmetric linearised \(\kappa\)-symmetry-equivariant structure, in conformity with an interpretation of \(\kappa\)-symmetry, worked out \textit{ibid.}, purely in terms of the target supergravity. In fact, it is from the latter interpretation that provides a solution to the problem posed in the present Introduction, and so we conclude the section with a recapitulation of the physical idea behind it.

The objective of the present study is to extract a supersymmetric \textit{target-space} higher-geometric description of the fundamental dynamical object of the GS super-\(\sigma\)-model with the supertarget \(G/H\), \textit{i.e.}, of the (closed) super-\(p\)-brane trajectory, from the CaE super-\(p\)-gerbe over \(G\) associated with the GS super-(\(p+2\))-cocycle that determines its Wess–Zumino term – all that in the much tractable model setting: for the superstring (\(p=1\)) in the superspace \(s\text{Mink}(d,1|D_{d,1}) \equiv \text{SO}(d,1|D_{d,1})/\text{Spin}(d,1)\). In the light of the hitherto discussion, the task boils down to identifying a (higher-)superalgebraic object related to the Lie superalgebra \(\mathfrak{g}\) of the supersymmetry supergroup \(G \equiv \text{SO}(d,1|D_{d,1})\) that captures a classical (\textit{i.e.}, critical) embedding of the superstring worldsheet in the supertarget and, in particular, the (supersymmetry) curvatures that survives such localisation. That the well-posedness of this task is non-obvious is best illustrated by the discussion of a local (tangential) Graßmann-odd supersymmetry of the GS super-\(\sigma\)-model in the standard NG formulation, aka \(\kappa\)-symmetry, discovered by de Azcárraga and Lukierski (for the superparticle) in Ref. \cite{dAL83} and subsequently rediscovered and elaborated by Siegel (for the superstring) in Refs. \cite{Sie83,Sie84}, whose existence is tied with the mechanism of restitition of equibalance of the internal degrees of freedom of both Graßmann parities in the vacua of the GS superfield theory through a removal of fermionic Goldstone modes, consistent with the structure of the supersymmetry Lie superalgebra of the theory: The symmetry couples the metric and topological terms in the action functional (\textit{i.e.}, they are not invariant \textit{separately}), and that only for a finely tuned relative normalisation of the two. It also bracket-generates a (super)algebra whose on-shell closure requires incorporation of generators of diffeomorphisms of the worldsheet \cite{McA00}. A path to \textit{target-space} geometrisation of \(\kappa\)-symmetry and field equations of the GS super-\(\sigma\)-model on \(G/H\), and so also towards a meaningful formulation of the problem of interest, was paved in Refs. \cite{Sus19,Sus20} where a duality – first noted in Ref. \cite{HP86}, later elaborated substantially in Ref. \cite{GIT90} and employed in a rederivation of a variety of (super-\(\sigma\)-) models of charge dynamics in Refs. \cite{McA10,She00,PKW01,PKW06} – was formalised, geometrised and exploited that exists between the original NG formulation of the GS super-\(\sigma\)-model and a \textit{purely topological} (super)field theory, termed the Hughes–Polchinski (HP) formulation of the GS super-\(\sigma\)-model by the Author, with the superfield space \(G/H_{\text{vac}}\) associated to another reductive decomposition \(\mathfrak{g} = (\mathfrak{t} \oplus \mathfrak{h}) \oplus h_{\text{vac}}\), \(\mathfrak{h} \oplus h_{\text{vac}} = \mathfrak{h}\) with \(h_{\text{vac}} = \text{Lie}(H_{\text{vac}})\) that encodes the spontaneous breakdown \(H \subset H_{\text{vac}}\) of the ‘invisible’ gauge symmetry \(H\) of the superfield theory. The field space of the new formulation contains additional degrees of freedom, to wit, the bosonic Goldstone fields transverse to (the body of) the vacuum of the GS super-\(\sigma\)-model and modelled on the vector space \(\mathfrak{d} \cong h/h_{\text{vac}}\). Its topologicality is reflected by the replacement of the original metric term of the NG by the pullback of an \(H_{\text{vac}}\)–basic LI super-(\(p+1\))-form on \(G\), itself (a distinguished scalar multiple of) a volume form \(\text{Vol}(t_{\text{vac}}^{(0)})\) on a fixed algebraic model \(t_{\text{vac}}^{(0)} \subset t^{(0)}\) of the body of the vacuum, to the worldvolume of the super-\(p\)-brane along suitably \(\mathfrak{d}\)-augmented sections of the NG formulation. On the higher-geometric side, this means that the dual HP dynamics is entirely determined by a \((H_{\text{vac}}\text{-equivariant})\) CaE super-\(p\)-gerbe – the tensor product of the original super-\(p\)-gerbe for the GS super-(\(p+2\))-cocycle with the trivial one with the curving given by the volume form on \(t_{\text{vac}}^{(0)}\) that we shall call, after Ref. \cite{Sus19}, the \textbf{extended Hughes–Polchinski super-\(p\)-gerbe over} \(G\) and denote as \((\lambda^{\text{HP}}_p \in \mathbb{R}^\times\) is the scalar mentioned earlier)

\[
\mathcal{G}^{(p)} \equiv \mathcal{G}^{(p)} \otimes T^{(p)}_{\lambda^{\text{HP}}_p \text{Vol}(t_{\text{vac}}^{(0)})}.
\]

The removal of the extra Goldstone modes through the Inverse Higgs Effect of Ref. \cite{O73} puts us back in the original NG formulation. It is realised by imposition of a subset of superfield equations of the HP formulation that can be interpreted as geometric constraints on the tangents of the fields of the model restricting the latter to a superdistribution in the tangent sheaf \(\mathcal{T}G\) of the target supermanifold \(G\), dubbed the HP/NG correspondence superdistribution and denoted as \(\text{Corr}(\mathfrak{g}_B^{(\text{HP})}_p)\). That all superfield equations geometrise in an analogous manner, as do the gauge-fixing conditions for the ‘invisible’ local-symmetry group \(H_{\text{vac}}\) altogether giving rise to what was named the HP vacuum superdistribution and denoted as \(\text{Vac}(\mathfrak{g}^{(\text{HP})}_p)\) in Ref. \cite{Sus20}, is a structural feature of the dual HP formulation that turns out to be instrumental in resolving the above-posed problem of extraction of super-\(p\)-brane data from \(\mathcal{G}^{(p)}\) in a manner that we outline below in the closing paragraph of the Introduction.
The (classical) vacuum of the GS super-$\sigma$-model in the HP formulation emerges as a sub-supermanifold within $G$ defined as the intersection of the HP local sections of the principal $H_{\text{vac}}$-bundle $G \to G/H_{\text{vac}}$ used in the definition of the lagrangean superfield of the theory with an integral supermanifold of $\text{Vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$. The existence of a foliation of the sections by such integral leaves calls for involutivity of $\text{Vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ that was examined in Ref. [Sus21]. There is yet another superdistribution whose regular behaviour is of essence for the consistency of the entire framework, namely, the limit of the weak derived flag (in the sense of Tanaka [Tan70]) of the projection to $\text{Vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ of the linear span of the set of generators of an enhanced (right) gauge supersymmetry that arises upon restriction of the superfield of the super-$\sigma$-model to $\text{Corr}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$. The projection removes the obvious Graßmann-even component modelled on $\mathfrak{d}$ (reflecting the enhancement $h_{\text{vac}} \supset \mathfrak{h}$ of the ‘invisible’ gauge-symmetry algebra that accompanies the transition between the two formulations) and leaves us with a superdistribution $\kappa(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ that contains a generic Graßmann-odd component – the latter is the target space-geometric realisation of the $\kappa$-symmetry of the GS super-$\sigma$-model in the topological formulation, whence the name $\kappa$-symmetry superdistribution given to it in Ref. [Sus20]. The regularity alluded to above simply means that the limit should stay within $\text{Vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$, so that it can be given the interpretation of a gauge supersymmetry of the vacuum, engendered by $\kappa(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$. This happens iff the vacuum superdistribution is involutive, in which case $\kappa(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ is readily seen to bracket-generate $\text{Vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ – the vacuum supermanifold becomes a single orbit of the gauge-symmetry supergroup obtained through integration of the Lie superalgebra $\text{vac}(\mathfrak{g}^{(\text{HP})}_{p,\lambda_1^1})$ modelling the limit. The last fact, taken in conjunction with the higher-geometric interpretation and implementation of gauge symmetries worked out in Refs. [GSW11, GSW13, Sus12, Sus11b, Sus13], leads us to the following hypothesis of Refs. [Sus20, Sus21]:

Upon restriction to the vacuum, the extended HP super-$p$-gerbe $\mathcal{G}^{(p)}$ trivialises flatly as $\mathcal{G}^{(p)} \cong \mathcal{I}_0$.

In the present paper, we prove the hypothesis for $G/H = \text{sISO}(d,1)/\text{D}(d,1)$ and $p = 1$ with the relevant super-3-cocycle of the GS super-$\sigma$-model for the superstring (whose physical content and supersymmetry, both global and local ($\kappa$-symmetry), in the dual HP formulation is reviewed for later reference in Secs. 2 and 3, respectively) as in Ref. [dAT89]. We do that by first lifting the CaE super-1-gerbe of Ref. [Sus17, Sec. 5.2], erected directly over the Lie supergroup $\text{sMink}(d,1)/\text{D}(d,1)$, to the mother Lie supergroup $\text{sISO}(d,1)/\text{D}(d,1)$ in Sect. 2 (Theorem 1), and subsequently deriving the trivialisation 1-isomorphism in Sect. 5.3 (Theorem 2) – this is readily seen by rewriting the trivialisation (symbolically, and for the value $\lambda_1^1 = 2$ determined in Sec. 3) as:

$$\mathcal{G}^{(1)}|_{\text{vacuum}} \cong \mathcal{I}^{-2\text{Vol}(\text{vacuum})} \text{vacuum},$$

$\text{cp}$ Eq. (1.1). The solution has an essential weakness: In consequence of the lack of an obvious Lie-supergroup structure on the vacuum, the 1-isomorphism can only be and is a non-supersymmetric one, and so it exists outside the framework systematically constructed in the first part of the paper (and introduced in the original papers). We strengthen the ‘raw’ result stated in Theorem 2 in the last part of Sec. 5.2 by passing to the tangent sheaf of the higher-geometric object that represents the 1-isomorphism and extracting a hierarchy of Lie superalgebras [5,6] associated with the various supermanifold components of that object and interrelated analogously but by Lie-superalgebra homomorphisms (Theorem 3) – this is the structure encoding the supersymmetric supergeometry of the vacuum that we have been after, and it seems appropriate to call it the sLieAlg-skeleton of the vacuum. We conclude our study with one further step in which we integrate the sLieAlg-skeleton to a hierarchy of Lie supergroups (Theorem 5), whereby the sLieGrp-model of the vacuum (5.10) arises.

Theorems 2–5 constitute the main results of the present study, consistent with the higher-geometric representations of physical objects charged under the gauge field geometrised, and form a solid basis for further investigation of geometrisations of gauge fields of the GS super-$\sigma$-models for super-$p$-branes on homogeneous spaces of Lie supergroups that we intend to take up in the future.\footnote{The concept is very closely related to that of the FLSsA of Ref. [GKP10].}
2. The geometrodynamics of the HP superstring

In this opening section, we recall the definition of the supersymmetric field theory of interest, modelling the propagation in Minkowskian spacetime of a loop-like distribution of charges of both Graßmann parities in equibalance. The definition calls for a supermanifold with the Minkowskian body and a supersymmetric de Rham 3-cocycle field that couples to the supercharge current engendered by the propagating loop. In our presentation, we emphasise, purposefully, the underlying Lie-supergroup structure and the associated tangential Lie-superalgebra structure.

The point of departure of our discussion is the \((d+1)\)-dimensional Minkowski space (for some \(d \in \mathbb{N}^*\))
\[
\mathbb{R}^{d,1} \equiv \left( \mathbb{R}^{d+1}, \eta \right), \quad \eta = \eta_{ab} E^a \otimes E^b, \quad (\eta_{ab}) = \text{diag} (-1, 1, 1, \ldots, 1)
\]
with its structure of an abelian Lie group determined by the binary operation (written in terms of the global cartesian coordinates \(\{x^a\}_{a=0}^{d}d\))
\[
\mathbb{W} : \mathbb{R}^{d,1} \times \mathbb{R}^{d,1} \rightarrow \mathbb{R}^{d,1} : \left( (x_1^a), (x_2^b) \right) \mapsto \left( x_1^a + x_2^a \right),
\]
with the commutative Lie algebra
\[
\mathfrak{min}(d,1) = \bigoplus_{a=0}^{d} \langle P_a \rangle, \quad [P_a, P_b] = 0
\]
and the associated left-invariant (LI) Maurer–Cartan form
\[
E \equiv E^a \otimes P_a, \quad E^a(x) = dx^a.
\]
To the above, there corresponds the Clifford algebra
\[
\text{Cliff} \left( \mathbb{R}^{d,1} \right) = \{ \Gamma_a \mid a \in \overline{0,d} \}, \quad \{\Gamma_a, \Gamma_b\} = 2\eta_{ab} 1
\]
that contains the spin group \(\text{Spin}(d,1)\), the universal cover of the connected component \(\text{SO}(d,1)\) of the identity element of the Lorentz group \(\text{SO}(d,1) \equiv \text{SO}(\mathbb{R}^{d,1})\) of \(\mathbb{R}^{d,1}\),
\[
1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin}(d,1) \rightarrow \pi_{\text{Spin}} \rightarrow \text{SO}(d,1) \rightarrow 1.
\]
We pick up a vector space
\[
S_{d,1} \equiv \mathbb{R}^{\times d}_{d,1}
\]
that carries a Majorana-spinor realisation
\[
S : \text{Spin}(d,1) \rightarrow \text{End}(S_{d,1})
\]
of \(\text{Spin}(d,1)\), assuming the following identities to be satisfied in this realisation: the Fierz identities
\[
(\text{2.1}) \quad \Gamma_{\alpha(\beta} \Gamma_{\gamma\delta)} = 0, \quad \alpha, \beta, \gamma, \delta \in \overline{1,D_{d,1}}, \quad \Gamma^a \equiv \eta^{-ab} \Gamma_b,
\]
and – for the corresponding charge-conjugation operator \(C \in \text{End}(S_{d,1})\) – the symmetry relations
\[
C^T = -C, \quad (C \Gamma_a)^T = C \Gamma_a \equiv \Gamma_a
\]
which, in particular, rule out the possibilities \(d \in \{5, 6, 7\}\).

Given these, we consider the associated super-Poincaré (super)group, that is the supermanifold \(\text{sISO}(d,1|D_{d,1}) = \left( \text{ISO}(d,1) \equiv \mathbb{R}^{d+1} \times L \text{Spin}(d,1), \mathcal{O}_{\text{sISO}(d,1|D_{d,1})} \equiv C^\infty (\cdot, \mathbb{R}) \circ \text{pr}_1 \otimes C^\infty (\cdot, \mathbb{R}) \circ \text{pr}_2 \otimes \wedge^* S_{d,1} \right)\)
with the crossed product in the definition of its body \(\tilde{\text{SO}}(d,1)\) determined by the vector realisation
\[
L \quad : \quad \text{Spin}(d,1) \quad \longrightarrow \quad \text{End}(\mathbb{R}^{d,1})
\]
\[
\pi_{\text{Spin}} \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
on Spin(\(d,1\)) (their ensemble being identified with a point in the spin group by a mild abuse of the notation) as

\[
m(\(\theta_1^a, x_1^a, \phi_1^{bc}\), \(\theta_2^a, x_2^a, \phi_2^{bc}\)) = (\theta_1^a + S(\phi_1)^{\alpha\beta}_\gamma \theta_2^{\alpha\beta} x_2^a + L(\phi_1)^a_b x_2^b - \frac{1}{2} \theta_1^a \Gamma^{\alpha\beta} S(\phi_1)_{\theta_2} (\phi_1 \ast \phi_2)^{ab})\),
\]

where

\[
\theta_1^a \Gamma^{a\alpha} S(\phi_1)_{\theta_2} = \theta_1^a C_{\alpha\beta\gamma} \Gamma^{a\beta\gamma} S(\phi_1)_{\theta_2} \theta_2^\beta
\]

and where \(\ast\) represents the standard binary operation on the spin group. In this picture, it is straightforward to write out coordinate expressions for the basis LI vector fields on the Lie supergroup \(\text{sISO}(d,1|D_{d,1})\),

\[
Q_\alpha(\theta, x, \phi) = S(\phi)^\alpha_\beta \left( \frac{\partial}{\partial \phi^\beta} + \frac{1}{2} \theta^\gamma \Gamma^{\alpha\beta\gamma} \frac{\partial}{\partial x^\gamma} \right) \equiv S(\phi)^\alpha_\beta Q_\beta(\theta, x),
\]

\[
P_\alpha(\theta, x, \phi) = L(\phi)^b_a \frac{\partial}{\partial x^b} \equiv L(\phi)^b_a P_b(\theta, x),
\]

\[
J_{\alpha\beta}(\theta, x, \phi) = \frac{d}{dt}\big|_{t=0} \phi^\alpha - t \phi_{\alpha\beta}, \quad (\phi_{\alpha\beta})^{cd} = \delta^{c}_{\alpha} \delta^{d}_{\beta} - \delta^{c}_{\beta} \delta^{d}_{\alpha},
\]

spanning the tangent sheaf \(T\text{sISO}(d,1|D_{d,1})\) of \(\text{sISO}(d,1|D_{d,1})\). These obey the superalgebra

\[
\{Q_\alpha, Q_\beta\} = \Gamma_{\alpha\beta}^{\alpha\beta} P_\alpha, \quad \{P_\alpha, P_\beta\} = 0, \quad \{Q_\alpha, P_\beta\} = 0,
\]

\[
[J_{\alpha\beta}, Q_\alpha] = \frac{1}{2} (Q \Gamma_{\alpha\beta})_\alpha = \frac{1}{2} \Gamma_{\alpha\beta}^{\gamma} Q_\gamma, \quad [J_{\alpha\beta}, P_\gamma] = \eta_{\gamma\alpha} P_\beta - \eta_{\gamma\beta} P_\alpha,
\]

expressed in terms of the antisymmetric products

\[
\Gamma_{\alpha\beta} = \frac{1}{2} [\Gamma_\alpha, \Gamma_\beta]
\]

and called the super-Poincaré (super)algebra and denoted as (\(D = \dim \text{sISO}(d,1|D_{d,1}) - 1\))

\[
\text{sISO}(d,1|D_{d,1}) = \bigoplus_{a=1}^{D_{d,1}} (Q_\alpha) \oplus \bigoplus_{a=0}^{D_{d,1}} (P_a) \oplus \bigoplus_{a=0}^{D_{d,1}} (J_{\alpha\beta}) \equiv \bigoplus_{A=0}^{D_{d,1}} (t_A).
\]

When referring symbolically to its (supercommutation) structure equations, we shall write (for homogeneous generators \(t_A, A \in 0, D\) of the respective Grassmann parities \(|t_A| = |A|\))

\[
[t_A, t_B] = f_{ABC}^D t_C = (-1)^{|A||B|+1} [t_B, t_A].
\]

The latter superalgebra is the key ingredient in the alternative (and equivalent) definition of the Lie supergroup \(\text{sISO}(d,1|D_{d,1})\) à la Kostant that identifies the supergroup with the super-Harish–Chandra pair

\[
\text{sISO}(d,1|D_{d,1}) \equiv (\text{ISO}(d,1), \text{sISO}(d,1|D_{d,1})),
\]

with the body Lie group \(\tilde{\text{ISO}}(d,1)\) realised on the Grassmann-odd component

\[
\text{sISO}(d,1|D_{d,1})^{(1)} \equiv \bigoplus_{\alpha=1}^{D_{d,1}} (Q_\alpha)
\]

of the Lie superalgebra \(\text{sISO}(d,1|D_{d,1})\) as

\[
\rho : \mathbb{R}^{d+1} \times \tilde{\text{Spin}}(d,1) \longrightarrow \text{End}(\text{sISO}(d,1|D_{d,1})^{(1)}) : (x, \phi) \mapsto S(\phi)^T = \rho(x, \phi).
\]

The cotangent sheaf \(T^*\text{sISO}(d,1|D_{d,1})\) of \(\text{sISO}(d,1|D_{d,1})\), dual to \(T\text{sISO}(d,1|D_{d,1})\) (as a \(\mathcal{C}_{\text{IsISO}(d,1|D_{d,1})}\)-module), is globally generated by the dual of the vector fields \(Q_\alpha, P_a, J_{\alpha\beta}\), i.e., the LI super-1-forms with the coordinate presentation

\[
q^a(\theta, x, \phi) = S(\phi)^{-1\alpha}_{\beta} \theta^\beta = S(\phi)^{-1\alpha}_{\beta} q^\beta(\theta, x),
\]

\[
p^a(\theta, x, \phi) = L(\phi)^{-1\alpha}_{b} (dx^b + \frac{1}{2} \theta^\beta \Gamma^{\alpha\beta} \theta^\gamma \partial_x) \equiv L(\phi)^{-1\alpha}_{b} p^b(\theta, x),
\]

\[
j^{ab}(\theta, x, \phi) = L(\phi)^{-1\alpha}_{c} dL(\phi)^c_d \eta^{-1\alpha}_{db}.
\]

Through the ensuing super-Maurer–Cartan equations

\[
dq^a = -\frac{1}{2} j^{ab} \wedge (\Gamma_{ab} q)^a, \quad dp^a = \frac{1}{2} q \wedge \Gamma^{a\alpha} q - \eta_{ac} j^{ab} \wedge p^c, \quad dj^{ab} = -\eta_{cd} j^{ac} \wedge j^{bd},
\]

(2.3)
they generate the Cartan–Eilenberg cochain complex of \( \text{sISO}(d, 1|D_{d,1}) \),

\[
(\Omega^* (\text{sISO}(d, 1|D_{d,1})))_{\text{sISO}(d, 1|D_{d,1})} \equiv \{ q^\alpha, p^a, \bar{\beta}^\beta \mid (\alpha, a) \in \overline{1, D_{d,1}} \times \overline{0, d}, \ b < c \in \overline{0, d} \}, d^* \equiv d .
\]

Its cohomology,

\[
H^*_\text{dR}(\text{sISO}(d, 1|D_{d,1}), \mathbb{R})_{\text{sISO}(d, 1|D_{d,1})} \equiv \text{CaE}^*(\text{sISO}(d, 1|D_{d,1})) ,
\]

the supersymmetric refinement of the de Rham cohomology of \( \text{sISO}(d, 1|D_{d,1}) \), is termed the Cartan–Eilenberg cohomology of \( \text{sISO}(d, 1|D_{d,1}) \). By the \( \mathbb{Z}/2\mathbb{Z} \)-graded version of the classic Lie-algebraic result, it is isomorphic with the Chevalley–Eilenberg cohomology of the Lie superalgebra \( \text{siso}(d, 1|D_{d,1}) \) with values in the trivial \( \text{siso}(d, 1|D_{d,1}) \)-module \( \mathbb{R} \),

\[
\text{CaE}^*(\text{sISO}(d, 1|D_{d,1})) \cong H^*(\text{siso}(d, 1|D_{d,1}), \mathbb{R}) ,
\]

a fact of prime significance for the geometrisation of physically relevant Cartan–Eilenberg (super-)cocyles discussed in Sec. \( \bullet \). Among nontrivial classes in \( \text{CaE}^*(\text{sISO}(d, 1|D_{d,1})) \), we find that of the the Green–Schwarz super-3-cocycle

(2.4) \[
\chi^{(3)} = q \wedge \overline{T}_a q \wedge p^a
\]

whose closedness follows directly from the Fierz identities \([2, 1]\).

Consider, next, the two homogeneous spaces of the super-Poincaré group associated with the respective reductive decompositions of its tangent Lie superalgebra:

\[
\text{siso}(d, 1|D_{d,1}) = \text{smint}(d, 1|D_{d,1}) \oplus \text{spin}(d, 1) ,
\]

\[
[\text{spin}(d, 1), \text{smint}(d, 1|D_{d,1})] \subset \text{smint}(d, 1|D_{d,1}) ,
\]

with

\[
\text{smint}(d, 1|D_{d,1}) = \bigoplus_{\alpha = 1}^{D_{d,1}} \{ Q_\alpha \} \oplus \bigoplus_{a = 0}^{d} \{ P_a \} ,
\]

\[
\text{spin}(d, 1) = \bigoplus_{a < b = 0} \{ J_{ab} \} ,
\]

and

\[
\text{siso}(d, 1|D_{d,1}) = (\text{smint}(d, 1|D_{d,1}) \oplus \mathfrak{d}) \oplus \text{spin}(d, 1)_{\text{vac}} ,
\]

\[
[\text{spin}(d, 1)_{\text{vac}}, \text{smint}(d, 1|D_{d,1}) \oplus \mathfrak{d}] \subset \text{smint}(d, 1|D_{d,1}) \oplus \mathfrak{d} ,
\]

with

\[
\mathfrak{d} = \bigoplus_{(\underline{a}, \underline{b}) \in \overline{0, 1} \times \overline{0, d}} \{ J_{\underline{a} \underline{b}} \} ,
\]

\[
\text{spin}(d, 1)_{\text{vac}} = \{ J_{01} \} \oplus \bigoplus_{a < b \in \overline{2, d}} \{ J_{ab} \} \equiv \text{spin}(1, 1) \oplus \text{spin}(d - 1) ,
\]

coming with the supervector-space projections

\[
p \equiv \text{pr}_1 : \text{smint}(d, 1|D_{d,1}) \oplus \text{spin}(d, 1) \longrightarrow \text{smint}(d, 1|D_{d,1}) ,
\]

\[
p_{\text{vac}} \equiv \text{pr}_1 : (\text{smint}(d, 1|D_{d,1}) \oplus \mathfrak{d}) \oplus \text{spin}(d, 1)_{\text{vac}} \longrightarrow \text{smint}(d, 1|D_{d,1}) \oplus \mathfrak{d} .
\]

The former is the super-Minkowski space

\[
\text{sMink}(d, 1|D_{d,1}) = \text{sISO}(d, 1|D_{d,1})/\text{Spin}(d, 1) ,
\]

and the latter is

\[
\text{sISO}(d, 1|D_{d,1})/\text{Spin}(d, 1)_{\text{vac}} \equiv \text{sMink}(d, 1|D_{d,1}) \times \text{Spin}(d, 1)/\text{Spin}(d, 1)_{\text{vac}} ,
\]

where

\[
\text{Spin}(d, 1)_{\text{vac}} \equiv \text{Spin}(1, 1) \times \text{Spin}(d - 1) .
\]

They are bases of the respective principal (super-)bundles

\[
\text{Spin}(d, 1) \longrightarrow \text{sISO}(d, 1|D_{d,1}) \equiv \text{sMink}(d, 1|D_{d,1}) \times_{L,S} \text{Spin}(d, 1)
\]

(2.5) \[
\text{sMink}(d, 1|D_{d,1})
\]
Indeed, the LI super-1-forms \( \theta \) right and its tangent Lie superalgebra becomes the
but the latter action does not descend to the homogeneous space.

\[ \pi_{\text{vac}} : \text{sISO}(d,1) \rightarrow \text{sISO}(d,1)/\text{Spin}(d,1), \]

on which the Lie supergroup acts (from the left) as

\[ [\ell]^K : \text{sISO}(d,1) \times \text{sISO}(d,1)/K \rightarrow \text{sISO}(d,1)/K, \]

in such a manner that

\[ [\ell]^K \circ (\text{id}_{\text{sISO}(d,1)/K} \times \pi_K) = \pi_K \circ \ell, \quad \ell \equiv m, \]

where \( K \in \{\text{Spin}(d,1), \text{Spin}(d,1)_{\text{vac}} \} \) and \( (\pi_{\text{Spin}(d,1)}, \pi_{\text{Spin}(d,1)_{\text{vac}}}) = (\pi, \pi_{\text{vac}}) \). This is a simple example of the general situation described at length in Ref. [Kos77] and, more recently, in Ref. [FLV07]. The Lie supergroup \( \text{sISO}(d,1) \) acquires the interpretation of the supersymmetry group in this context, and its tangent Lie superalgebra becomes the supersymmetry algebra. It acts on itself also from the right,

\[ \varphi \equiv m : \text{sISO}(d,1) \times \text{sISO}(d,1) \rightarrow \text{sISO}(d,1)/K \]

but the latter action does not descend to the homogeneous space.

The existence of the surjective submersions (in \( \text{sMan} \) \( \pi_K \)) can also be used to descend LI tensors from the mother Lie supergroup \( \text{sISO}(d,1)/K \) to the homogeneous space \( \text{sISO}(d,1)/K \) along local trivialising sections, an observation behind the long-established model-building technique of the so-called nonlinear realisations of (super)symmetry of Refs. [Sch67, Wei68, CWZ69, CWZ70, SS69a, SS69b, SS77, VA72, VA73, K73, LR73, UZ82, K82b, SW83, FMW83, BW84, Wes00, GK06]. In the case of a covariant tensor \( T \) (of rank \( n \)), for the descent to yield a globally defined object on the homogeneous space, \( T \) must be (right-)K-basic. Denote the tangent Lie algebra of the structure group \( K \) as \( \mathfrak{t} \in \{\text{spin}(d,1), \text{spin}(d,1)_{\text{vac}}\} \), with the understanding that \( \mathfrak{t} \subset \mathfrak{m} \oplus \mathfrak{t} = \text{siso}(d,1)_{d,1}) \) (where \( \mathfrak{m} \in \{\text{smink}(d,1)_{d,1}) \), smink(d,1)_{d,1}) = \mathfrak{m} \) is the formerly indicated direct-sum complement of \( \mathfrak{t} \), such that \( [\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m} \), and consider the induced elementwise realisation of the body Lie group \( \text{ISO}(d,1) \) on \( \text{sISO}(d,1)/d,1) \) by automorphisms in the category \( \text{sMan} \) of supermanifolds, given by

\[ |\varphi|_g : \text{ISO}(d,1) \rightarrow \text{Aut}_{\text{sMan}}(\text{sISO}(d,1)/d,1)) : g \mapsto \varphi \circ (\text{id}_{\text{sISO}(d,1)/d,1}) \times \tilde{g} \equiv |\varphi|_g, \]

where

\[ \tilde{g} \in \text{Hom}_{\text{sMan}}(R^{0,0}, \text{sISO}(d,1)/d,1)), \]

is the topological point in \( \text{sISO}(d,1)/d,1) \) corresponding to \( g \in \text{ISO}(d,1) \), whence

\[ |\varphi|_g : \text{sISO}(d,1)/d,1)) \equiv \text{sISO}(d,1)/d,1) \times R^{0,0} \xrightarrow{\text{id}_{\text{sISO}(d,1)/d,1}) \times \tilde{g}} \text{sISO}(d,1)/d,1) \times \text{sISO}(d,1)/d,1), \]

as desired. With these in hand, we can make the concept of K-basiness precise. Thus, a covariant tensor \( T \) on \( \text{sISO}(d,1)_{d,1}) \) is (right-)K-basic if it is \( \mathfrak{t} \)-horizontal,

\[ \forall_{(X_1,X_2,...,X_n)\in\text{siso}(d,1)/d,1)} : \left( \exists_{\mathfrak{t}^{\perp}} : X_i \in \mathfrak{t} \quad \implies \quad T(X_1,X_2,...,X_n) = 0 \right), \]

and (right-)K-invariant,

\[ \forall_{(k,X)\in\mathfrak{k} \times \mathfrak{t}} : \left( |\varphi|_k^*T = T \quad \land \quad \mathcal{L}_XT = 0 \right). \]

It is now easy to see that the distinguished super-3-cocycle (2.4) is \( \text{Spin}(d,1) \)-basic, and so also \( \text{Spin}(d,1)_{\text{vac}} \)-basic, as is the degenerate metric tensor

\[ \tilde{\eta} = \eta^{ab} p^a \otimes p^b. \]

Indeed, the LI super-1-forms \( \theta_L \), \( \zeta \in 0, \dim m - 1 \) dual to the basis LI vector fields from \( m \) are \( \mathfrak{t} \)-horizontal by definition and transform linearly as

\[ |\varphi|_k^* \theta_L = \rho(k)^c \zeta \theta_L^c. \]
in consequence of the assumed reductivity of the decomposition $\mathfrak{siso}(d,1|D_{d,1}) = \mathfrak{m} \oplus \mathfrak{k}$. Hence, it suffices to take a linear combination of their tensor products,

$$ T = \lambda_{\zeta_1,\zeta_2,\ldots,\zeta_n} \theta_{\zeta_1}^1 \otimes \theta_{\zeta_2}^2 \otimes \cdots \otimes \theta_{\zeta_n}^n, $$

with components $\lambda_{\zeta_1,\zeta_2,\ldots,\zeta_n}$ of a constant $\rho(\mathbb{K})$-invariant tensor as coefficients,

$$ \forall k \in \mathbb{K} : \lambda_{\zeta_1,\zeta_2,\ldots,\zeta_n} \rho(k)_{\zeta_1}^1 \rho(k)_{\zeta_2}^2 \cdots \rho(k)_{\zeta_n}^n = \lambda_{\zeta_1,\zeta_2,\ldots,\zeta_n}, $$

to obtain a $\mathbb{K}$-basic tensor $T$. The $\rho(\text{Spin}(d,1))$-invariance of the Minkowski metric is obvious, and that of $\mathbb{T}_{\alpha \beta \gamma}$ follows straightforwardly from the elementary properties of the generators of the Clifford algebra and of the charge-conjugation operator:

$$ (2.7) \quad S(\phi)^{-1} \Gamma^\alpha S(\phi) = L(\phi)^\alpha_\beta \Gamma^\beta, \quad C^{-1} S(\phi)^T C = S(\phi)^{-1}. $$

An example of a $\text{Spin}(d,1)_{\text{vac}}$-basic tensor that is not $\text{Spin}(d,1)$-basic is provided by the volume super-2-form on the subspace

$$ t_{(0)}^{(\text{vac})} = (P_0, P_1), $$

that is $(\epsilon_{ab})$ is the totally antisymmetric tensor with $\epsilon_{01} = 1$.

$$ \text{Vol}(t_{(0)}^{(\text{vac})}) = p^0 \wedge p^1 \equiv \frac{1}{2} \epsilon_{ab} p^a \wedge p^b. $$

Its (right-)Spin$(d,1)_{\text{vac}}$-invariance is ensured by the fact that $\rho(\text{Spin}(d,1)_{\text{vac}})$ restrict to $t_{(0)}^{(\text{vac})}$ as unimodular automorphisms,

$$ \forall (\phi_1, \phi_2) \in \text{Spin}(1,1) \times \text{Spin}(d-1) : \text{det} \left( \rho(\phi_1) t_{(0)}^{(\text{vac})} \right) \equiv \text{det} \left( \rho(\phi_2) t_{(0)}^{(\text{vac})} \right) \equiv \text{det} L(\phi_1) = 1. $$

Hence, in particular, there exist: a super-3-form $H_{(3)}$ and a symmetric $(2,0)$-tensor $H^a$ on sMink$(d,1|D_{d,1})$, as well as a super-3-form $H_{(3)}$ a super-2-form $\nu$ and a symmetric $(2,0)$-tensor $\nu^a$ on sISO$(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}}$ such that

$$ \chi_{(3)} = \pi^* H_{(3)}, \quad \eta_{(2)} = \pi^* \nu_{(2)}. $$

Upon putting together the explicit coordinate presentations of the various LI super-1-forms involved and identities (2.7), we readily derive

$$ H_{(3)} = q^\alpha \mathbb{T}_a q^a \wedge p^b, \quad \eta_{(2)} = \eta_{ab} p^a \otimes p^b. $$

Inspection of the group law (2.2) reveals that the homogeneous space sMink$(d,1|D_{d,1}) \subset \text{ISO}(d,1|D_{d,1})$ is, in fact, a Lie sub-supergroup with the binary operation

$$ m : \text{sMink}(d,1|D_{d,1}) \times \text{sMink}(d,1|D_{d,1}) \rightarrow \text{sMink}(d,1|D_{d,1}) $$

admitting the coordinate presentation

$$ m(\theta^a_1, x^a_1, \theta^a_2, x^a_2) = \left( \theta^a_1 + \theta^a_2, x^a_1 + x^a_2 - \frac{1}{2} \theta_1 \Gamma^a \theta_2 \right), $$

with the corresponding basis LI vector fields

$$ Q_\alpha(\theta, x) = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} \theta^\beta C_\beta^\gamma \Gamma^\gamma \frac{\partial}{\partial x^\alpha}, \quad P_\alpha(\theta, x) = \frac{\partial}{\partial x^\alpha}, $$

spanning the super-minkowskian Lie superalgebra

$$ \text{smin}(d,1|D_{d,1}) = \bigoplus_{\alpha = 1}^{D_{d,1}} \left( Q_\alpha \right) \oplus \bigoplus_{\alpha = 0}^{D_{d,1}} \left( P_\alpha \right) $$

with the structure equations

$$ \{ Q_\alpha, Q_\beta \} = \Gamma_{\alpha \beta} \cdot P_\gamma, \quad [P_\alpha, P_\beta] = 0, \quad [Q_\alpha, P_\beta] = 0. $$

Clearly, the super-1-forms $q^\alpha$ and $p^a$ are their (respective) duals, and the descended Green–Schwarz super-3-cocycle $\mathbb{H}$ defines a nontrivial class in

$$ \text{ChE}^*(\text{sMink}(d,1|D_{d,1})) \cong H^*(\text{smin}(d,1|D_{d,1}), \mathbb{R}). $$

As the de Rham cohomology of sMink$(d,1|D_{d,1})$ is trivial,

$$ H^*_{\text{dR}}(\text{sMink}(d,1|D_{d,1})) = H^*_{\text{dR}}(\text{Mink}(d,1)) = 0, $$

$$ H^*_{\text{dR}}(\text{sMink}(d,1|D_{d,1})) = H^*_{\text{dR}}(\text{Mink}(d,1)) = 0, $$
the Green–Schwarz super-3-cocycle admits a global primitive, albeit only a quasi-invariant one that can be chosen in the explicit form
\[
B_{(2)} (\theta, x) = \theta \Gamma_\theta g(\theta, x) \wedge p^2 (\theta, x).
\]

We shall write
\[
\beta_{(2)} := \pi_* B_{(2)}.
\]

The hitherto discussion provides us with all the ingredients of the two formulations of the Green–Schwarz (GS) super-σ-model of the superstring in sMink(\(d, 1[D_{d,1}]\)). The first of these is the Nambu–Goto (NG) formulation in which we have the theory of (inner-Hom) functorial embeddings
\[
\xi \in [\Sigma, \text{sMink}(\d, 1[D_{d,1}])] \cong \text{Hom}_{\text{Man}}(\Sigma \times -, \text{sMink}(\d, 1[D_{d,1}]))
\]
of a closed orientable two-dimensional manifold \(\Sigma\) (the worldsheet) in \(\text{sMink}(\d, 1[D_{d,1}])\) determined by the principle of least action applied to the Dirac–Feynman (DF) amplitude (with an obvious interpretation of the codomain)
\[
A_{\text{DF}}^{(\text{NG})} : [\Sigma, \text{sMink}(\d, 1[D_{d,1}])] \rightarrow U(1) : \xi \mapsto \exp \left[ i \int_\Sigma \sqrt{\text{det}(\xi^* \eta)} + \int_\Sigma \xi^* B_{(2)} \right],
\]
in which \(\mu_1 \in \mathbb{R}^+\) is a parameter whose numerical value is fixed by the correspondence with the other formulation stated below, \(\text{cp}\) Refs. [GS84], [GS84a]. The amplitude is to be evaluated on the Grassmann-odd hyperplanes \(\mathbb{R}^{|N|}, N \in \mathbb{N}^+\), whereby an \(N^+\)-indexed family of supersymmetric two-dimensional field theories is obtained, \(\text{cp}\) Ref. [Fre99]. The other one is the Hughes–Polchinski (HP) formulation, first postulated in [HP86], developed in Refs. [GIT90], applied in Refs. [McA00], elaborated in [Sus19, Sus20], in which we deal with the theory of (inner-Hom) functorial embeddings
\[
\tilde{\xi} \in [\Sigma, \text{sISO}(\d, 1[D_{d,1}])/\text{Spin}(\d, 1)_{\text{vac}}] \cong \text{Hom}_{\text{Man}}(\Sigma \times -, \text{sISO}(\d, 1[D_{d,1}])/\text{Spin}(\d, 1)_{\text{vac}})
\]
of the same worldsheet \(\Sigma\) in \(\text{sISO}(\d, 1[D_{d,1}])\)/\(\text{Spin}(\d, 1)_{\text{vac}}\) determined by the principle of least action applied to the DF amplitude
\[
A_{\text{DF}}^{(\text{HP})} : [\Sigma, \text{sISO}(\d, 1[D_{d,1}])]/\text{Spin}(\d, 1)_{\text{vac}}] \rightarrow U(1) : \tilde{\xi} \mapsto \exp \left[ i \int_\Sigma \tilde{\xi}^* (\lambda_1 v_{(2)} + p_1 B_{(2)}) \right],
\]
in which
\[
\lambda_1 \equiv \text{pr}_1 : \text{sMink}(\d, 1[D_{d,1}]) \times (\text{Spin}(\d, 1)/\text{Spin}(\d, 1)_{\text{vac}}) \rightarrow \text{sMink}(\d, 1[D_{d,1}])
\]
and \(\lambda_1 \in \mathbb{R}^+\) is a parameter whose numerical value we establish through a symmetry analysis in Sec. \[4.\] Here, it is presupposed that \(v_{(2)}\) is nondegenerate on classical field configurations (termed the vacua of the theory). In other words, we model the body of the vacuum on \(t^{(0)}_{\text{vac}}\). In virtue of Ref. [Sus19, Prop. 5.3] \((\text{cp}\) also Ref. [Sus20, Thm. 3.4] for a more general result), the two formulations become equivalent upon partial reduction of the latter one through imposition of a subset of its Euler–Lagrange equations. This is quite nontrivial as the super-σ-model in the HP formulation is purely topological, unlike its NG counterpart. A careful analysis of the equivalence between the two formulations sets the stage for all our subsequent higher-geometric considerations, and so we shall now spend some time studying the relevant details in the spirit of Ref. [Sus20]. In so doing, we shall refer to the two pairs:
\[
\mathfrak{s}2^{(\text{NG})} = (\text{sMink}(\d, 1[D_{d,1}]), \tilde{\eta}, \chi_{(3)}),
\]
and
\[
\mathfrak{s}2^{(\text{HP})} = (\text{sISO}(\d, 1[D_{d,1}])/\text{Spin}(\d, 1)_{\text{vac}}, \lambda_1 \text{dVol}(t^{(0)}_{\text{vac}}) + \chi_{(3)} \equiv \chi^{(\lambda_1)})
\]
as the NG superbackground and HP superbackground, respectively.

The key to understanding the equivalence lies in a patchwise smooth realisation of the two homogeneous spaces: \(\text{sISO}(\d, 1[D_{d,1}])/\text{Spin}(\d, 1)_{\text{vac}}\) and \(\text{sISO}(\d, 1[D_{d,1}])/\text{Spin}(\d, 1)\) within the mother Lie supergroup \(\text{sISO}(\d, 1[D_{d,1}])\) by means of judiciously chosen local sections of the respective principal bundles \[2.6\] and \[2.7\], along the lines of Refs. [FLV07] and [Sus20]. Let \(O_0 \in \text{Spin}(\d, 1)_{\text{vac}}\) be an open subset of \(\text{Spin}(\d, 1)/\text{Spin}(\d, 1)_{\text{vac}}\) that supports local (normal) coordinates \(\phi_{(2)}^{\text{sp}}\) : \(O_0 \xrightarrow{\phi_{(2)}^{\text{sp}}} V_0 \subset \mathfrak{d}^{(2)}\),
centred on the unital coset $e \text{Spin}(d,1)_{\text{vac}}$ (i.e., $e \phi^g_e(e \text{Spin}(d,1)_{\text{vac}}) = 0$) and a local section of the principal $\text{Spin}(d,1)_{\text{vac}}$-bundle $\text{Spin}(d,1) \to \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$ (and so also its local trivialisation), and let $\{h_i\}_{i \in I} \subset \text{Spin}(d,1), \ I \ni 0$ (with $h_0 = e$) be such that the translates $O_i = [\ell^i]_{h_i}(O_0)$, written in terms of the induced action

$$[\ell] : \text{Spin}(d,1) \times \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}} \to \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$$

of the Lie group $\text{Spin}(d,1)$ on the homogeneous space $\text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$, satisfying

$$[\ell] \circ (\text{id}_{\text{Spin}(d,1)} \times \pi_{\text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}}) = \pi_{\text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}} \circ \ell, \quad l \equiv *,$$

cover $\text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$.

The sets $U_i := \text{Mink}(d,1) \times O_i, \ i \in I$ compose an open cover of the base $\text{Mink}(d,1) \times \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$ over which the principal $\text{Spin}(d,1)_{\text{vac}}$-bundle $\text{ISO}(d,1) \to \text{Mink}(d,1) \times \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}}$ trivialises. That base is the body of the base $\text{sISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}}$ of the principal $\text{Spin}(d,1)_{\text{vac}}$-bundle $[2,3]$, and it is over the $U_i$ that we define local sections of the latter. These we take in the (shifted-)exponential form

$$\sigma^\text{vac}_i = [\ell]_{h_i} \circ e^{\theta^a \circ \text{pr}_1 \circ \sigma_{O_A} \cdot e^{x^a \circ \text{pr}_1 \circ \sigma_{F_A} \cdot e^{(\phi^g_e)^{a \ell \circ \tilde{g}}}} \circ [\ell^i]_{h^{-1}} : \mathcal{U}^\text{vac}_i \equiv (U_i, \mathcal{O}_{\text{sISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}}}) \to \text{sISO}(d,1|D_{d,1}),$$

(2.8)

defined in terms of the left action

$$[\ell] : \text{ISO}(d,1) \to \text{Aut}_{\text{sMan}}(\text{sISO}(d,1|D_{d,1})) : g \mapsto m \circ (\tilde{g} \times \text{id}_{\text{sISO}(d,1|D_{d,1})}) \equiv [\ell]_g$$

and the corresponding induced action

$$[\ell] : \text{ISO}(d,1) \to \text{Aut}_{\text{sMan}}(\text{sISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}})$$

$$: g \mapsto [\ell] \circ (\tilde{g} \times \text{id}_{\text{sISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}}}) \equiv [\ell]_g.$$}

They give rise to the **H Hughes–Polchinski section**

$$\Sigma^{\text{HP}} := \bigsqcup_{i \in I} \mathcal{V}_i, \quad \mathcal{V}_i := \sigma^\text{vac}_i(\mathcal{U}^\text{vac}_i).$$

Upon choosing a tessellation $\Delta_{\Sigma}$ of $\Sigma$ (composed of plaquettes that make up a set $\mathcal{T}_2 \subset \Delta_{\Sigma}$, edges and vertices) subordinate to $\{\mathcal{U}^\text{vac}_i\}_{i \in I}$,

$$\forall_{\tau \in \mathcal{T}_2} \exists_{i, \ell} : \tilde{\xi}(\tau) \subset U_{i\ell},$$

we may, next, rewrite the DF amplitude of the above GS super-$\sigma$-model in the HP formulation as

$$A^{\text{DF}}_{\text{HP}}[\tilde{\xi}] = \exp \left[ \frac{i}{\hbar} \sum_{\tau \in \mathcal{T}_2} \int_{\tau} \tilde{\xi}^a \sigma^\text{vac}_i \lambda_1 \text{Vol}(t^0_{\text{vac}}(\xi)) + \beta \right].$$

With the degrees of freedom of the super-$\sigma$-model (super)field thus separated into the super-minkowskian sector $(\theta^a, x^a)$ and the spin-group sector $(\phi^b_\tau)$ (we use the subscript to mark the local coordinates on $O_i$), the equivalence is completely straightforward to state: Upon expressing the non-dynamical Goldstone spin-group fields $(\phi^b_\tau)$ in $A^{\text{DF}}_{\text{HP}}$ in terms of the remaining degrees of freedom $(\theta^a, x^a)$ as dictated by the Euler–Lagrangian equations of the GS super-$\sigma$-model in the HP formulation obtained by varying the DF amplitude for $\tilde{\xi}^a$ in the direction of the spin-group fields $(\phi^b_\tau)$,

$$\tilde{\xi}^{\alpha \beta} \sigma^\text{vac}_i \rho^\beta = 0, \quad \alpha \in [2, d],$$

(2.9)

we recover the DF amplitude of the NG formulation for $\mu_1 \circ \tilde{\xi}$, written in terms of the global coordinates $(\theta^a, x^a)$ on $\text{sMink}(d,1|D_{d,1})$ for a value $\mu_1, \equiv \mu_1(\lambda_1)$ of the parameter $\mu_1$ determined uniquely by $\lambda_1$ – this is the so-called **Inverse Higgs Effect** of Ref. [4073]. It permits us to restrict our subsequent discussion to the HP formulation, with the understanding that conclusions pertinent to the standard

\[\text{Remark:} \] The sheaf-theoretic meaning of the $\sigma^\text{vac}_i$ was given in Ref. [Sus20, Sec. 2].
NG formulation of the super-σ-model can be drawn only upon restricting the tangents of the field configurations $\sigma_{i,vac}$ to the NG/HP correspondence superdistribution

$$\text{Corr}_{\text{HP}}(\mathfrak{g} \mathcal{B}_{1,\lambda_1}) \subset T\Sigma_{\text{HP}}, \quad \text{Corr}_{\text{HP}}(\mathfrak{g} \mathcal{B}_{1,\lambda_1})|_{\mathcal{V}_i} = \bigcap_{n=2}^d \ker p^2 \cap T\mathcal{V}_i.$$  

Advantages of this approach shall become clear along the way.

We conclude the present field-theoretic introduction by deriving the Euler–Lagrange equations of the super-σ-model in the HP formulation. To this end, we write the variation $\mathcal{V}_r \in \{\tau, T\sigma_{i,vac}(\mathcal{U}_{\sigma_{i,vac}}^{\text{ac}})\}$ of the composite embedding $X_\tau \equiv \sigma_{i,vac} \circ \xi_\tau$ as

$$\mathcal{V}_r = \delta \theta^\alpha Q_\alpha(X_\tau) + \delta x^\alpha P_\alpha(X_\tau) + \delta \phi^\beta_{\tau_r} J_{\phi_{\tau_r}}(X_\tau) + \Delta^2_1 J_{\phi_{\tau_r}}(X_\tau),$$

in which the last term

$$\Delta^2_1 J_{\phi_{\tau_r}}(X_\tau) = \Delta^0_1 J_{01}(X_\tau) + \Delta^2_1 J_{00}(X_\tau)$$

represents spin$(d, 1)_{\text{vac}}$-vertical corrections that render $\mathcal{V}_r$ tangent to the local section $\sigma_{i,vac}(\mathcal{U}_{\sigma_{i,vac}}^{\text{ac}})$, cp Ref. [Sus20, Prop. 3.6], and calculate, with the help of the super-Maurer–Cartan equations (2.3),

$$-i h \mathcal{V}_r \cdot \delta \log A_{\text{DF}}^{(\text{HP})}(\xi) = \sum_{r \in \mathcal{R}_2} \int_{\mathcal{V}_r} \mathcal{V}_r \cdot \tilde{\xi}_r \sigma_{i,vac} \cdot \tilde{\lambda}(\lambda),$$

so that for $(\delta \theta^\alpha, \delta x^\alpha) = 0$, we obtain

$$-i h \mathcal{V}_r \cdot \delta \log A_{\text{DF}}^{(\text{HP})}(\xi) = - \sum_{r \in \mathcal{R}_2} \int_{\mathcal{V}_r} \theta^\alpha \delta \mathcal{V}_r \cdot \tilde{\xi}_r \sigma_{i,vac} \cdot (p^2 \wedge p^2).$$

At this stage, we invoke the assumption of nondegeneracy of the volume form $\text{Vol}(\mathcal{V}_{\text{vac}})$ in the vacuum (which can be viewed as a condition of its partial localisation), implying that the tangent sheaf of the latter in $\text{ISO}(d, 1)$ is (locally) spanned on the vector fields

$$\mathcal{W}_{\alpha, \beta} = \Delta^2_{\alpha, \beta} Q_\beta, \quad \alpha \in \mathcal{U}_{\text{vac}}, \quad \mathcal{W}_{\alpha, \beta} = p^2|_{\mathcal{V}_r} + \Delta^2_{\alpha, \beta} Q_\beta + \Delta^2_{\alpha, \beta} J_{\phi_{\tau_r}} + \Delta^2_{\alpha, \beta} J_{\phi_{\tau_r}} \quad \beta \in \{0, 1\},$$

written in terms of certain (even) sections $\Delta^2_{\alpha, \beta}$, $\Delta^2_{\alpha, \beta}$, and $\Delta^2_{\alpha, \beta}$ of the structure sheaf of $\mathcal{V}_r$ to be determined below, with the spin$(d, 1)_{\text{vac}}$-vertical component correcting the one along $\mathfrak{g}$ in such a way that the sum is in $T\mathcal{V}_i$, cp Ref. [Sus20, Prop. 3.6]. Taking the above into account, we obtain the formerly stated Eq. (2.3), or

$$\Delta^2_{\alpha, \beta} = 0, \quad (\alpha, \beta) \in \{0, 1\} \times \frac{2d}{2d},$$

which we write as

$$p^2 = 0, \quad \alpha \in \frac{2d}{2d},\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (2.10)$$

henceforth. Imposition of the latter leaves us with the reduced expression for the variation

$$-i h \mathcal{V}_r \cdot \delta \log A_{\text{DF}}^{(\text{HP})}(\xi) = 2\theta^\alpha \sum_{r \in \mathcal{R}_2} \int_{\mathcal{V}_r} \mathcal{V}_r \cdot \tilde{\xi}_r \sigma_{i,vac} \cdot \left(q \wedge \Gamma_a \left(\frac{1-d-\lambda_1}{2} r_{\alpha}^1 \right) q \wedge p^2\right),$$

from which we readily extract the remaining Euler–Lagrange equations (written in the above notation)

$$P_{\lambda_1} q = 0, \quad P_{\lambda_1} = \left(\Gamma_{\alpha} - \frac{1-d-\lambda_1}{2} r_{\alpha}^1 \right)$$

by setting $\delta x^\alpha = 0$. If (and only if) $\lambda_1 \in \{-2, 2, 2\}$, then the operator $P_{\lambda_1}$ is a projector,

$$P_{\lambda_2}^2 = P_{\lambda_2}$$

of rank

$$\text{rk} P_{\lambda_2} = \frac{D_{d,1}}{2},$$

with the additional properties

$$P_{\lambda_2} \Gamma_{\alpha} = \Gamma_{\alpha} \left(1 - P_{\lambda_2}\right), \quad P_{\lambda_2} \Gamma_{\alpha} = \Gamma_{\alpha} P_{\lambda_2}, \quad C^{-1} P_{\lambda_2} C = 1_{D_{d,1}} - P_{\lambda_2}$$

which leads to the emergence of a Lie sub-superalgebra

$$\{{\langle Q P_{\lambda_2}\rangle}_\alpha, {\langle Q P_{\lambda_2}\rangle}_\beta\} = {\langle \Gamma^\alpha P_{\lambda_2}\rangle}_\alpha P_{\lambda_2}, \quad [P_{\lambda_2}, P_{\lambda_2}] = 0, \quad \left[{\langle Q P_{\lambda_2}\rangle}_\alpha, P_{\lambda_2}\right] = 0,$$
\[ [J_{01}, J_{01}] = 0, \quad [J_{01}, J_{\alpha\beta}] = 0, \quad [J_{\alpha\beta}, J_{\gamma\delta}] = \delta_{\alpha\gamma} J_{\beta\delta} - \delta_{\alpha\delta} J_{\beta\gamma} + \delta_{\beta\gamma} J_{\alpha\delta} - \delta_{\beta\delta} J_{\alpha\gamma}, \]
\[ [J_{01}, (QP_{s2})_\alpha] = \frac{i}{\mathfrak{f}} ((QP_{s2}) \Gamma_{01})_\alpha, \quad [J_{\alpha\beta}, (QP_{s2})_\alpha] = \frac{1}{\mathfrak{f}} ((QP_{s2}) \Gamma_{\alpha\beta})_\alpha, \]
\[ [J_{01}, P_a] = \eta_{a1} P_0 - \eta_{a0} P_1, \quad [J_{\alpha\beta}, P_a] = 0 \]
within \( \text{siso}(d, 1|D_{d, 1}) \). The above emphasises the indispensability of the projector \( P_{s2} \) for the consistency of the field theory under consideration, and so fixes the absolute value of \( \lambda_1 \), leaving us only the immaterial choice of its sign, which we declare to be \( + \), with
\[ \mathcal{P}^{(1)} := P_{s2} \]
and
\[ \bar{\chi}^{(2)} = 2 \eta_{ab} q \wedge \bar{\Gamma}^a (1_{D_{d, 1}} - \mathcal{P}^{(1)}) q \wedge \bar{p}^b + 2 c_{ab} \bar{d}_{c\bar{c}} \bar{p}^c \wedge \bar{p}^\bar{c} \wedge \bar{J}^{bc} q \wedge \bar{\Gamma}_{\bar{a}} q \wedge \bar{p}^\bar{a}. \]

Indeed, the projector enforces a reduction of the Grassmann-odd degrees of freedom necessary for the restoration of balance between them and their Grassmann-even counterparts in the vacuum, the latter being subject to the (even) Inverse Higgs Constraints (2.10). The Constraints are transmitted unto the Grassmann-odd sector via the anticommutator of supercharges in the supersymmetry superalgebra \( \text{siso}(d, 1|D_{d, 1}) \), and so for the sake of a residual supersymmetry in the vacuum, we need a subspace in the odd component \( \text{siso}(d, 1|D_{d, 1})^{(1)} \subset \text{siso}(d, 1|D_{d, 1}) \) which the superbracket maps to the surviving Grassmann-even supersymmetries \( P_0 \) and \( P_1 \). The ratio
\[ \text{BPS}(\mathfrak{B}_1^{(1)}) \equiv \frac{\mathcal{P}^{(1)}}{D_{d, 1}} \equiv \frac{1}{\mathfrak{f}} \]
goes under the name of the BPS fraction of the vacuum.

Upon fixing a basis \( \{ \bar{Q}_\alpha \}_{\alpha = 1, 2} \) in \( \text{im} \mathcal{P}^{(1)} \),
\[ \text{im} \mathcal{P}^{(1)} = \left\{ Q_\beta \mathcal{P}^{(1)}_{\alpha} \mid \alpha \in 1, D_{d, 1} \right\} = \bigoplus_{\alpha = 1}^{D_{d, 1}} \left\{ \bar{Q}_\alpha \right\}, \]
we may write down (local) generators of the vacuum superdistribution
\[ \text{Vac}(\mathfrak{B}_1^{(1)}) \subset \text{Corr}_{\text{HP}}(\mathfrak{B}_1^{(1)}) \subset T_{\Sigma \text{HP}} \]
within the tangent sheaf \( T_{\Sigma \text{HP}} \) of the HP section \( \Sigma \text{HP} \) that is determined by the Euler–Lagrange equations of the super-\( \sigma \)-model,
\[ W_{\alpha i} = \bar{Q}_\alpha |_{V_i}, \quad \alpha \in 1, D_{d, 1}, \quad W_{\alpha i} = P_{\alpha i} + \Delta_{\alpha i} J_{\epsilon} + \Delta_{\alpha i}^S J_S, \quad a \in \{0, 1\}. \]

In the light of the physical interpretation of the superdistribution, it is natural to demand involutivity of the latter, so that it defines – in virtue of the Frobenius Theorem, Ref. [CCFT] Thm. 6.2.1 – a foliation of the HP supertarget by embedded sub-supermanifolds, identified as the vacua of the supersymmetric field theory under consideration. Transform the matrices \( \bar{\Gamma}_a \) and \( \bar{\Gamma}_{\bar{a}} \), commuting with \( \mathcal{P}^{(1)} \), into a basis of the Majorana-spinor module adapted to the decomposition of the dual module \( \text{smint}(d, 1|D_{d, 1})^{(1)} \) into \( \text{im} \mathcal{P}^{(1)} \) and its direct-sum complement,
\[ \left( 1_{D_{d, 1}} - \mathcal{P}^{(1)} \right)^T = \bigoplus_{\bar{a} = \frac{D_{d, 1}}{2} + 1}^{D_{d, 1}} \left( \bar{Q}_{\bar{a}} \right), \]
whereupon they become block-diagonal, and denote the ensuing vacuum blocks as
\[ \bar{\Gamma}_a |_{\text{im} \mathcal{P}^{(1)}} =: (\gamma_{ab})_{a \neq 1, \frac{D_{d, 1}}{2}}, \quad \text{det} \gamma_{ab} \neq 0, \quad \gamma_{11} = -\gamma_{\bar{a} \bar{b}}. \]
\[ \Gamma_{01} |_{\text{im} \mathcal{P}^{(1)}} = -1_{D_{d, 1}} \quad \Gamma_{\alpha \beta} |_{\text{im} \mathcal{P}^{(1)}} := (\gamma_{\alpha \beta})_{\alpha \neq 1, \frac{D_{d, 1}}{2}} \equiv \gamma_{\alpha \beta}. \]

Our discussion of involutivity of \( \text{Vac}(\mathfrak{B}_1^{(1)}) \) begins with the inspection of the anticommutators
\[ \{ W_{\alpha i}, W_{\beta i} \} = \gamma_{\alpha \beta} (P_0 - P_1) |_{V_i} \equiv \gamma_{\alpha \beta} (W_{0i} - W_{1i}) + \gamma_{\alpha \beta} (\Delta_{1i} - \Delta_{0i}) J_{ab} \]
from which we read off the constraints
\[ \Delta_{ab} = \Delta_{ab}^0 \equiv \Delta_{ab}, \quad a < b \in 0, d, \]
implying
\[ W_{\alpha i} = P_{\alpha i} \mid_{\mathcal{V}_i} + \Delta_i^{\beta} J_{\beta} + \Delta_i^{S} J_S. \]
Next, we compute the commutators
\[ [W_{0 i}, W_{1 i}] = \Delta_i^{01} (W_{0 i} - W_{1 i}) - (\Delta_i^{0c} + \Delta_i^{1c}) P_\beta + (P_0 - P_1) \cdot J_{\alpha b} \],
and infer the constraints
\[ \Delta_i^{\beta} = -\Delta_i^{\overline{\beta}} \Delta_i^{c}, \quad a, b \in \mathbb{Z}, d, \quad (P_0 - P_1) \cdot J_{\alpha b} = 0, \quad a < b \in \overline{0, d}. \]
Thus,
\[ W_{\alpha i} = P_{\alpha i} \mid_{\mathcal{V}_i} + \Delta_i^{\overline{0}} (J_{0 b} - J_{1 b}) + \Delta_i^{S} J_S, \]
with
\[ (P_0 - P_1) \cdot d\Delta_i^{\overline{0}} = (P_0 - P_1) \cdot d\Delta_i^{S}. \]
Finally, we examine the commutators
\[ [W_{\alpha i}, W_{\alpha i}] = -\frac{1}{2} \Delta_i^{01} W_{\alpha i} + \frac{1}{2} \Delta_i^{0c} \gamma_{b c} \Delta_i^{\overline{b}} W_{\alpha i} - \tilde{Q}_{\alpha i} \cdot d\Delta_i^{bc} J_{b c}, \]
whereby we arrive at the constraints
\[ \tilde{Q}_{\alpha i} \cdot d\Delta_i^{\overline{b}} = 0, \quad a < b \in \overline{0, d}. \]
We conclude that an involutive vacuum superdistribution (of the type assumed) is spanned on fields
\[ W_{\alpha i} = \tilde{Q}_{\alpha i} \mid_{\mathcal{V}_i}, \quad \alpha \in \mathbb{Z}, \overline{0, d}, \quad W_{\alpha i} = P_{\alpha i} \mid_{\mathcal{V}_i} + \Delta_i^{\overline{0}} (J_{0 b} - J_{1 b}) + \Delta_i^{S} J_S, \]
with
\[ (P_0 - P_1) \cdot d\Delta_i^{\overline{0}} = (P_0 - P_1) \cdot d\Delta_i^{S}, \quad \tilde{Q}_{\alpha i} \cdot d\Delta_i^{\overline{0}} = 0 = \tilde{Q}_{\alpha i} \cdot d\Delta_i^{S}. \]
Note that for the special choice
\[ \Delta_i^{\overline{0}} \equiv 0 \quad (\implies \Delta_i^{S} \equiv 0) \]
the vacuum superdistribution is modelled on the super-minkowskian component of the Lie sub-superalgebra
\[ (2.11) \quad \text{vac}(s\mathfrak{g}_{1,2}^{(\text{HP})}) = \bigoplus_{\alpha = 1}^{\frac{1}{2} d + 1} \langle \tilde{Q}_{\alpha} \rangle \oplus \langle P_0, P_1 \rangle \oplus \text{spin}(d, 1)_{\text{vac}} \]
(the hidden gauge-symmetry algebra \( \text{spin}(d, 1)_{\text{vac}} \) is realised trivially) with the superbrackets
\[ \{ \tilde{Q}_{\alpha i}, \tilde{Q}_{\beta i} \} = \tau_{\alpha \beta} P_{\alpha} \equiv \tau_{\overline{\alpha \beta}} (P_0 - P_1), \quad [P_0, P_1] = 0, \quad [\tilde{Q}_{\alpha i}, P_{\beta}] = 0, \]
\[ [J_{0 i}, J_{0 i}] = 0, \quad [J_{0 i}, J_{\overline{b} i}] = 0, \quad [J_{\overline{a} i}, J_{\overline{b} i}] = \delta_{\overline{a} \overline{b}}, J_{\overline{c} i} - \delta_{\overline{a} \overline{c}} J_{\overline{b} i} + \delta_{\overline{c} \overline{b}} J_{\overline{a} i} - \delta_{\overline{c} \overline{a}} J_{\overline{b} i}, \]
\[ [J_{0 i}, \tilde{Q}_{\alpha i}] = -\frac{1}{2} \tilde{Q}_{\alpha i}, \quad [J_{\overline{b} i}, \tilde{Q}_{\alpha i}] = \frac{1}{2} (\tilde{Q}_{\overline{b} i})_{\alpha}, \]
\[ [J_{\overline{b} i}, J_{\overline{a} i}] = \delta_{\overline{a} \overline{b}}, P_0 + \delta_{\overline{a} \overline{b}}, P_0 + \delta_{\overline{a} \overline{b}}, P_1, \quad [J_{\overline{b} i}, P_{\alpha}] = 0 \]
that we call the vacuum algebra of the superstring in \( s\text{Mink}(d, 1|D_{d, 1}) \). In the next section, we shall interpret the departure from the simple model \( \text{vac}(s\mathfrak{g}_{1,2}^{(\text{HP})}) \) in the structure of the vacuum superdistribution in terms of gauge symmetries of the field theory. In the meantime, we mark the presence of the above sheaf parameters \( \Delta_i^{\overline{0}} \) and \( \Delta_i^{S} \) (with restrictions \( \Delta_i^{\overline{0}} \mid_{\mathcal{V}_i} = \Delta_i^{\overline{0}}, \text{ and } \Delta_i^{S} \mid_{\mathcal{V}_i} = \Delta_i^{S} \), respectively) in our notation as
\[ \text{Vac}_{(\Delta)}(s\mathfrak{g}_{1,2}^{(\text{HP})}) = \bigoplus_{\alpha = 1}^{\frac{1}{2} d + 1} \{ W_{\alpha} \equiv \tilde{Q}_{\alpha} \mid_{\mathcal{V}_i} \} \oplus \bigoplus_{\alpha = 1}^{\frac{1}{2} d + 1} \{ W_{\alpha} \equiv P_{\alpha} \mid_{\mathcal{V}_i} + \Delta_i^{\overline{0}} (J_{0 b} - J_{1 b}) + \Delta_i^{S} J_S \}. \]
The disjoint union of integral leaves \( D_{i, v_i} \subset \mathcal{V}_i \) (indices \( v_i \) from an index set \( \mathcal{Y}_i \) enumerate the different leaves within \( \mathcal{V}_i \)) of the involutive vacuum superdistribution shall be denoted as
\[ (2.12) \quad \Sigma_{\text{vac}}^{(\text{HP})} = \bigcup_{i \in I} \bigcup_{v_i \in \mathcal{Y}_i} D_{i, v_i}, \]
and termed the Hughes–Polchinski vacuum foliation. It is embedded in the HP section, which we write as
\[ \iota_{\text{vac}} : \Sigma_{\text{vac}}^{(\text{HP})} \rightarrow \Sigma_{\text{HP}}^{(\text{HP})}. \]
and projects to the physical vacuum foliation 
\[ \pi_{\text{\tiny phys vac}} \mathcal{X}^\nu \equiv \bigcup_{i \in I} \bigcup_{\nu \in \Gamma_i} \pi_{\text{\tiny vac}}(\mathcal{D}_i, \nu) \].

An alternative interpretation of the vacuum superdistribution and the vacuum algebra shall be given in the next section.

3. Supersymmetries of the super-σ-model

The principle of supersymmetry lies at the core of the construction of the super-σ-model. Its field-theoretic implementation has its peculiarities that we review below upon identifying the various species of supersymmetry present.

3.1. Global supersymmetry. The GS super-σ-model in either formulation has a built-in global supersymmetry realised by the respective induced actions \([\ell]^\text{K} \) under which the integrand of the (metric) volume term is manifestly invariant (being defined in terms of the left invariant super-1-forms \( \psi \)), whereas that of the topological term is quasi-invariant, i.e., invariant up to a total exterior derivative, as demonstrated explicitly (in the \( \mathcal{S} \)-point picture) in
\[ \int_{\mathcal{S}^1} \mathbb{B}(\theta, x) = \mathbb{B}(\theta, x) + d(\mathcal{S}(\psi)^{-1} \varepsilon \Gamma_a \theta (dx^a + \frac{1}{6} \theta \Gamma_{a} \theta d\theta)), \]

whence the said invariance of the DF amplitude for \( \Sigma \) closed. The global supersymmetry of the field theory under consideration is reflected in the existence of a \( \mathfrak{s}n(1 \mid D_{d1}) \)-index family of Noether hamiltonians \( \{ h_X \}_{X \in \mathfrak{s}n(1 \mid D_{d1})} \) on its space of states. These we derive in the NG formulation in which a state is represented by the Cauchy data \( \Psi = ((\theta, x) |_{\mathcal{S}^1} \equiv \xi |_{\mathcal{S}^1}, P) \) (a configuration \( \xi |_{\mathcal{S}^1} \in [\mathcal{S}^1, \mathcal{S}(\mathbb{M}ink(d, 1|D_{d1}))] \) and its LI kinetic momentum \( P \) of a vacuum localised on an equitemporal slice of the spacetime \( \Sigma \) which we take to be (modelled on) \( \mathcal{S}^1 \subset \Sigma \). The presymplectic form of the super-σ-model in this formulation reads
\[ \Omega_\sigma[\Psi] = \delta \theta[\Psi] + \int_{\mathcal{S}^1} \mathbb{e}^\star \xi \mathbb{H}, \]

where
\[ \delta \theta[\Psi] = \int_{\mathcal{S}^1} \mathbb{V}(\mathcal{S}^1) \mathcal{P}_a \xi \mathbb{H}^a \]
is the canonical (kinetic)-action 1-form on the space(s) of states \( \mathcal{T}^* \mathcal{S}^1, \mathbb{M}ink(d, 1|D_{d1}) \), and
\[ \mathbb{e} : \mathcal{S}^1 \times [\mathcal{S}^1, \mathcal{S}(\mathbb{M}ink(d, 1|D_{d1}))] \rightarrow \mathbb{M}ink(d, 1|D_{d1}) \]
is the evaluation mapping. The presymplectic form defines a Poisson superbracket on the space of hamiltonians on the space of states, i.e., on those sections \( h \) of the structure sheaf of the latter that satisfy the relation
\[ \delta h = -\mathcal{V}_h \mathcal{A}_\sigma \]
for some vector field \( \mathcal{V}_h \), termed hamiltonian – the Poisson superbracket of two such sections: \( h_1 \) and \( h_2 \) with the respective hamiltonian vector fields \( \mathcal{V}_{h_1} \) and \( \mathcal{V}_{h_2} \) is given by
\[ [h_1, h_2]_{\Omega_\sigma} = \mathcal{V}_{h_2} \mathcal{A}_\sigma - \mathcal{V}_{h_1} \mathcal{A}_\sigma. \]

In particular, upon contracting \( \Omega_\sigma \) with the covariant\(^8\) lift
\[ \mathcal{K}_X[\Psi] \equiv \int_{\mathcal{S}^1} \mathbb{V}(\mathcal{S}^1) \mathcal{K}_X(\xi) + \mathcal{A}_X[\Psi] \]
to \( \mathcal{T}^* \mathcal{S}^1, \mathbb{M}ink(d, 1|D_{d1}) \) of the suitably spin(\( d, 1 \))-vertically corrected right-invariant (RI) vector field \( \mathcal{K}_X \in \Gamma(\mathcal{T}\mathbb{M}ink(d, 1|D_{d1})) \) on \( \mathbb{M}ink(d, 1|D_{d1}) \equiv \mathbb{M}ink(d, 1|D_{d1}) \times \{1\} \subset \text{spin}(d, 1|D_{d1}) \), expressed in terms of the sections \( \Xi_X \mathcal{S}^1 \) of \( \mathcal{O}_{\text{spin}(d, 1|D_{d1})}(\mathbb{M}ink(d, 1|D_{d1})) \) of Ref. [Sus20] Prop. 5.1, we obtain the corresponding hamiltonian
\[ \mathcal{K}_X \mathcal{A}_\sigma = -\delta h_X. \]

\(^8\)Here, covariance is determined by the super-1-form \( \vartheta \) and expressed by the requirement: \( \mathcal{L}_{\mathcal{K}_X} \vartheta = 0. \)
The relevant basis RI vector fields are
\[ R_{Q_a}(\theta, x, 0) = \frac{i}{\theta} \frac{\partial}{\partial \theta} - \frac{1}{2} \theta^\beta C_{\beta \gamma} \Gamma^\alpha_\beta \frac{\partial}{\partial x^\alpha}, \]
\[ R_{P_a}(\theta, x, 0) = \frac{\partial}{\partial x^a}, \]
\[ R_{J_a}(\theta, x, 0) = x^c (\eta_{cb} \frac{\partial}{\partial x^b} - \eta_{ca} \frac{\partial}{\partial x^a}) + \frac{1}{2} \Gamma^a_{\alpha \beta} \theta^\beta \frac{\partial}{\partial \theta^\alpha} + \frac{\theta}{2 \theta |t = 0 t \phi_{ab} = 0} \]
and give rise to the hamiltonians
\[ X^A h_A \equiv h_X = \int_{\mathcal{S}^1} (\text{Vol}(\mathcal{S}^1) P_a \xi^* (R_X \rightharpoonup \theta_a^0) + \text{ev}^* \xi^* \kappa_X), \]
with
\[ \kappa_{Q_a}(\theta, x) = \pi_{a \beta} \dot{\theta}^\beta \left( 2d x^a - \frac{1}{2} \theta \Gamma^a \theta \right), \]
\[ \kappa_{P_a}(\theta, x) = -\theta \Gamma^a \theta, \]
\[ \kappa_{J_a}(\theta, x) = -x^c \theta \Gamma_{ab} \Gamma_c \theta \theta. \]
These furnish a realisation of a centrally extended supersymmetry Lie superalgebra \( \mathfrak{iso}(d,1|D_{d,1}) \) within the Poisson superalgebra of observables on the space of states of the super-\( \sigma \)-model,
\[ [h_B, h_A]_{\Omega_{\mathcal{P}}} = f_{AB} \mathcal{C} h_C + \mathcal{W}_{AB} \]
(here, the \( f_{AB} \mathcal{C} \) are the structure constants of \( s\text{Mink}(d,1|D_{d,1}) \)), with the components of the wrapping anomaly,
\[ \mathcal{W}_{AB} = -\int_{\mathcal{S}^1} \text{ev}^* \xi^* (R_{t_A} \rightharpoonup d \kappa_{t_B} + f_{AB} \mathcal{C} \kappa_{t_C}) = \int_{\mathcal{S}^1} \text{ev}^* w_{AB}, \]
given by
\[ w_{a \beta} = d(2\Gamma_{a \alpha \beta} x^a), \quad w_{a b} = 0, \quad w_{a a} = d(-2\Gamma_{a \alpha \beta} \theta^\beta), \]
\[ w_{a b c d} = d(-\frac{1}{2} x^c \theta \Gamma_{a b} \Gamma_c \theta), \quad w_{a b c} = d(\frac{1}{2} \theta \Gamma_{a b c} \theta), \]
\[ w_{a b c d} = d(-2x^c (\eta_{ca} \Gamma_b - \eta_{cb} \Gamma_a)_{\alpha \beta} \theta^\beta - \frac{1}{6} \theta \Gamma_{a b c} \theta \Gamma_{a \alpha \beta} \theta^\beta), \quad \mathcal{W}_{AB} \equiv 0. \]

Its refinement, to be considered in Sect. [sus18a], is the first step towards geometrisation of the cohomological content of the super-\( \sigma \)-model.

3.2. Local supersymmetry of the vacuum. The GS super-\( \sigma \)-model with the physical supertarget \( s\text{ISO}(d,1|D_{d,1})/K \) realised within \( s\text{ISO}(d,1|D_{d,1}) \) by means of the local sections has an implicit local symmetry modelled on the right action of \( K \). In particular, in the NG formulation, we have the large hidden gauge group \( \text{Spin}(d,1) \). Therefore, we anticipate an enhancement of the (tangential) local symmetry in the HP formulation according to the scheme
\[ \text{spin}(d,1)_{\text{vac}} \rightarrow \text{spin}(d,1) \]
onlyy restriction to the correspondence superdistribution \( \text{Corr}_{\mathcal{P}}(s\mathfrak{B}_{1,2}^{(\mathcal{P})}) \). Inspection of the expression
\[ \sum_{(3)} l_{\text{Corr}_{\mathcal{P}}(s\mathfrak{B}_{1,2}^{(\mathcal{P})})} = 2\eta_{ab} q \wedge \Gamma^a (1_{D_{d,1}} - \sqrt{p}) q \wedge p \]
immediately corroborates our expectation: The vector fields
\[ \mathcal{T}_{\mathfrak{a}} \in \Gamma(\mathcal{T}\mathfrak{V}_{\mathcal{P}}), \quad \mathcal{T}_{\mathfrak{a}} |_{\mathcal{V}_i} = J_{\mathfrak{a}} |_{\mathcal{V}_i} + T_{\mathfrak{a} \mathfrak{b}} |_{\mathcal{V}_i}, \quad (a, b) \in \{0, 1\} \times \overline{2, d}, \]
with correcting sections \( T_{\mathfrak{a} \mathfrak{b}} \in \mathcal{O}_{\mathcal{P}}(\mathcal{V}_i) \) uniquely determined by the condition \( \mathcal{T}_{\mathfrak{a} \mathfrak{b}} \in \Gamma(\mathcal{V}_i) \), cp Ref. [sus20, Prop. 3.6], satisfy
\[ \forall (a, b) \in (0, 1) \times \overline{2, d} : \mathcal{T}_{\mathfrak{a}} \mathcal{T}_{\mathfrak{b}} \sum_{(3)} l_{\text{Corr}_{\mathcal{P}}(s\mathfrak{B}_{1,2}^{(\mathcal{P})})} = 0. \]
This enhancement does not carry any physically nontrivial information as it merely reflects the residual redundancy of our realisation of the physical supertarget within the mother Lie supergroup. Accordingly, we are inclined to fix the hidden gauge by augmenting the set of Euler–Lagrange equations derived formerly with

\[ j^{\alpha \ell} \approx 0, \quad (\alpha, \ell) \in \{0, 1\} \times \mathbb{Z}_d, \]

so that altogether we arrive at the conjunction of constraints

\[ (1_{D,1} - P^{(1)}) q \approx 0, \quad p^{\ell} \approx 0, \quad \bar{a} \in \mathbb{Z}_d, \quad j^{\varphi} \approx 0, \quad (\bar{a}, \varphi) \in \{0, 1\} \times \mathbb{Z}_d \]

as the definition of the (hidden) gauge-fixed vacuum superdistribution

\[
\text{Vac}_{\text{gff}}(\mathfrak{sB}^{(\text{HP})}_{1,2}) \equiv \text{Vac}_{(\Delta = 0)}(\mathfrak{sB}^{(\text{HP})}_{1,2}) = \bigoplus_{\alpha = 1}^{D_{d,1}} (W_\alpha \equiv \tilde{Q}_\alpha |^{\Sigma_{\text{HP}}}) \oplus \bigoplus_{\alpha \in \{0, 1\}} (W_\alpha \equiv P_\alpha |^{\Sigma_{\text{HP}}}).
\]

Clearly, the hidden gauge-symmetry (sub)distribution \(\text{spin}(d,1)_{\text{vac}}\) is a symmetry of the above vacuum superdistribution,

\[
[\text{spin}(d,1)_{\text{vac}}, \text{Vac}_{\text{gff}}(\mathfrak{sB}^{(\text{HP})}_{1,2})] \subset \text{Vac}_{\text{gff}}(\mathfrak{sB}^{(\text{HP})}_{1,2}),
\]

and so the vacuum foliation descends to the supertarget \(\text{sISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}}\).

But that is not all. Indeed, the very mechanism responsible for the restitution of supersymmetry in the vacuum gives rise to an extra and physically nontrivial local supersymmetry on restriction to \(\text{Corr}_{\text{HP}}(\mathfrak{sB}^{(\text{HP})}_{1,2})\), to wit, tangential Graßmann-odd translations along \(\text{Im}P^{(1)T}\),

\[
\forall \frac{\partial_{\alpha \ell}}{} : W_\alpha \cup_{(3)} \text{Corr}_{\text{HP}}(\mathfrak{sB}^{(\text{HP})}_{1,2}) = 0.
\]

Furthermore, we readily establish

\[
(W_0 + W_1) \cup_{(3)} \text{Corr}_{\text{HP}}(\mathfrak{sB}^{(\text{HP})}_{1,2}) = 0,
\]

and so, altogether, we obtain the enhanced gauge-symmetry superdistribution

\[
\mathcal{G}S_1 \equiv \mathcal{G}S(\mathfrak{sB}^{(\text{HP})}_{1,2}) = \bigoplus_{\alpha = 1}^{D_{d,1}} (W_\alpha \oplus (W_0 + W_1) \oplus \bigoplus_{(\alpha, \ell) \in \{0, 1\}} (T_{\alpha \ell}) \subset \text{Corr}_{\text{HP}}(\mathfrak{sB}^{(\text{HP})}_{1,2}),
\]

modelled on the sub-space

\[
\mathfrak{g}S_1 \equiv \mathfrak{g}S(\mathfrak{sB}^{(\text{HP})}_{1,2}) = \text{Im}P^{(1)T} \oplus (P_0 + P_1) \oplus \text{spin}(d,1),
\]

with the component along \(\text{spin}(d,1)_{\text{vac}}\) realised trivially. Actually, in order to be able to interpret \(\mathcal{G}S_1\) as a proper local-symmetry structure of the theory, we should demand that the limit of its weak derived flag, as introduced in Ref. [sus20, Def. 4.9], stays in the correspondence superdistribution. This is, clearly, not the case for \(\mathcal{G}S_1\), and so we are led to extract from it a sub-superdistribution that satisfies this extra condition – in this manner, we arrive at the \(\kappa\)-symmetry superdistribution

\[
\kappa(\mathfrak{sB}^{(\text{HP})}_{1,2}) = \bigoplus_{\alpha = 1}^{D_{d,1}} (W_\alpha \oplus (W_0 + W_1).
\]

The latter immediately reveals its peculiarity: The limit of its weak derived flag,

\[
\kappa(\mathfrak{sB}^{(\text{HP})}_{1,2})^{\infty} = \bigoplus_{\alpha = 1}^{D_{d,1}} (W_\alpha \oplus (W_0) \oplus (W_1)
\]

is contained not only in \(\text{Corr}_{\text{HP}}(\mathfrak{sB}^{(\text{HP})}_{1,2})\), but in the vacuum superdistribution, or, more accurately,

\[
\kappa(\mathfrak{sB}^{(\text{HP})}_{1,2})^{\infty} = \text{Vac}_{\text{gff}}(\mathfrak{sB}^{(\text{HP})}_{1,2})
\]

i.e., the \(\kappa\)-symmetry superdistribution is superbracket-generating for the tangent sheaf of the gauged-fixed HP vacuum foliation \(\tilde{\Sigma}_{\text{vac}}\) its flows enveloping the integral leaves of the latter. Hence, it makes sense to think of \(\kappa(\mathfrak{sB}^{(\text{HP})}_{1,2})^{\infty}\) as the gauge (super)symmetry of the vacuum. The generating nature of the Graßmann-odd component

\[
\kappa(\mathfrak{sB}^{(\text{HP})}_{1,2})^{(1)} = \bigoplus_{\alpha = 1}^{D_{d,1}} (W_\alpha \subset \kappa(\mathfrak{sB}^{(\text{HP})}_{1,2})
\]
won it its name – the square root of (the chiral half of) the vacuum – in Ref. [Sus20]. It is the HP counterpart of the odd gauge symmetry of the super-σ-model in the NG formulation, known under the name of $\kappa$-symmetry, that was originally found and studied by de Azcarraga and Lukierski in Refs. [ALS82, ALS83] in the setting of the super-σ-model of superparticle dynamics, and subsequently rediscovered and elaborated by Siegel in Ref. [Sie83] and, in the two-dimensional setting, in Ref. [Sie84].

In the present context, the Lie superalgebra $\mathfrak{vac}(\mathfrak{so}_{1,2}^{\text{HP}})$ acquires the interpretation of the gauge-symmetry algebra of the vacuum, confirmed trivially by its inclusion in the kernel of the suitably restricted presymplectic form

$$\Omega_{\text{HP}} = \sum_{\tau \in \mathbb{Z}} \int_{S^1 \times \tau} \text{ev}^* \sigma_{\text{vac}}^* \xi$$

of the GS super-σ-model in the HP formulation (in which the space of states is parametrised by configurations $\xi_{[3]}$). The crucial feature of the gauge supersymmetry modelled by $\mathfrak{vac}(\mathfrak{so}_{1,2}^{\text{HP}})$ is its target space-geometric nature, to be contrasted with the mixed target-space/worldsheet and hence somewhat obscure nature of its NG predecessor, $\mathfrak{cp}$ target space-geometric nature, to be contrasted with the mixed target-space/worldsheet.

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$$\Omega_{\text{HP}} = \sum_{\tau \in \mathbb{Z}} \int_{S^1 \times \tau} \text{ev}^* \sigma_{\text{vac}}^* \xi$$

of the GS super-σ-model in the HP formulation (in which the space of states is parametrised by configurations $\xi_{[3]}$). The crucial feature of the gauge supersymmetry modelled by $\mathfrak{vac}(\mathfrak{so}_{1,2}^{\text{HP}})$ is its target space-geometric nature, to be contrasted with the mixed target-space/worldsheet and hence somewhat obscure nature of its NG predecessor, $\mathfrak{cp}$ target space-geometric nature, to be contrasted with the mixed target-space/worldsheet.
of the Cartan–Eilenberg cohomology of the supertarget $\text{sMink}(d,1|D_{d,1})$, as stated – after Rabin and Crane, cp Refs. [RC85, Rab87] – in Ref. [Sus17, Sec. 3], is a topologisation of the said cohomology as the dual of the singular homology of an orbifold $\text{sMink}(d,1|D_{d,1})/\Gamma_{KR}$ of the super-Minkowski space by the natural action of the Kostelecký–Rabin discrete supersymmetry group $\Gamma_{KR}$ of Ref. [KR84], generated by integer Graßmann-odd translations (in the $S$-point picture, and for a suitable choice of the Majorana representation of $\text{Cliff}(\mathbb{R}^{d,1})$). Accordingly, the GS super-$\sigma$-model ought to be interpreted as an effective description of standard loop dynamics in $\text{sMink}(d,1|D_{d,1})/\Gamma_{KR}$. Prior to investigating the consequences of the latter idea, we pause to decode its meaning and present a concrete realisation, in a semi-heuristic approach in which we gloss over any (e.g., topological) subtleties encountered along the way. Thus, we change the hitherto (Kostant’s) perspective on supermanifolds and present $\text{sMink}(d,1|D_{d,1})$ – upon invoking [Bat80, Def.-Cor.] (cp also Refs. [Jaw71, Bat79]) – as (a direct limit $N \to \infty$ of) a nested family, indexed by $N \ni N$, of DeWitt’s skeletons given essentially by (‘soul’) vector bundles

$$\text{Skel}_N(\text{sMink}(d,1|D_{d,1})) \equiv \bigoplus_{a=0}^d \big( \mathbb{R} \otimes \bigoplus_{k=1}^{E(\frac{N-1}{2})} \bigwedge^{2k} \mathbb{R}^{|N|} \big) \oplus \bigoplus_{a=0}^{D_{d,1}} \bigoplus_{l=0}^{2l+1} \bigwedge_{a=0}^{D_{d,1}} \mathbb{R}^{|N|} \to \bigoplus_{a=0}^d \mathbb{R} \equiv \mathbb{R}^{d+1}$$

of rank $2^{N-1}(d+1+D_{d,1})-d-1$ over the body $\mathbb{R}^{d+1} \equiv \text{Mink}(d,1)$, cp Ref. [DeW84, Sec. 2.1]. Practically speaking, this amounts to realising the global coordinate generators ($\theta^a, x^a$) of the structure sheaf $\mathcal{O}_{\text{sMink}(d,1|D_{d,1})}$ in the $N$th skeleton as (functional) linear combinations of elements of a basis $\{1\} \cup \{e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_m}\}_{1 \leq i_1 < \ldots < i_m \leq |N|, \, m \in \mathbb{N}}$ of the corresponding exterior algebra $\bigwedge \mathbb{R}^{|N|}$ as

$$\theta^a(N) = \sum_{l=0}^{E(\frac{N-1}{2})} \theta_{i_1i_2\ldots i_{2l+1}}^a e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{2l+1}}, \quad x^a(N) = x_0^a + \sum_{k=1}^{2l+1} x_{n_1n_2\ldots n_l}^a e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{2l+1}}$$

This presentation seems naturally compatible with Freed’s identification of the super-$\sigma$-model mapping space $[\Sigma, \text{sMink}(d,1|D_{d,1})]$ as the appropriate inner-Hom functor – indeed, we may think of the morphisms from $[\Sigma, \text{sMink}(d,1|D_{d,1})](\mathbb{R}^{d|N})$ as probing the $N$th skeleton. Now, the Rabin–Crane argument at level $N$ refers to the subgroup $\Gamma_{KR}^{(N)} \subset \text{Skel}_N(\text{sMink}(d,1|D_{d,1}))$ of the $N$th skeleton (with respect to super-minkowskian multiplication, realised in terms of the exterior product) generated\footnote{In a suitable Majorana representation of the Clifford algebra with integer-valued matrices of the generators, cp Ref. [KR84].} by odd vectors

$$\nu^a(N) = \sum_{l=0}^{E(\frac{N-3}{2})} n_{i_1i_2\ldots i_{2l+1}}^a e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{2l+1}}, \quad a \in 1, D_{d,1}$$

with all (pure-soul) coefficients in $\mathbb{Z} \ni n_{i_1i_2\ldots i_{2l+1}}^a$.

An example of the orbifold

$$\text{Skel}_N(\text{sMink}(d,1|D_{d,1}))/\Gamma_{KR}^{(N)}$$

whose homology is readily seen to encode the CaE cohomology of $\text{sMink}(d,1|D_{d,1})$ (owing to the polynomial character of the binary operation of this Lie supergroup in the $S$-point picture) was explicitly constructed in Ref. [RC85, App.], and the crucial observation of a generic nature is that it has compact odd (and even) fibre directions. The nested character of the of DeWitt’s skeletal presentation implies that the latter observation carries over to the direct limit, and so in view of our earlier remark on the interpretation of Freed’s prescription in the present context, it becomes clear that we should allow for monodromies of the embedding fields of both parities (i.e., the so-called twisted sector) in the super-$\sigma$-model with the supertarget $\text{sMink}(d,1|D_{d,1})$ when modelling the field theory with the Rabin–Crane orbifold $\text{sMink}(d,1|D_{d,1})/\Gamma_{KR}$ as the supertarget. Taking into account the top line in Eq. (8.1), we are led to consider a supercentral wrapping-charge extension

$$0 \to \mathbb{R}^{d|D_{d,1}} \to \overline{\text{smint}}(d,1|D_{d,1}) \to \text{smint}(d,1|D_{d,1}) \to 0$$

of the original supersymmetry algebra $\text{smint}(d,1|D_{d,1})$ with the supervector space structure

$$\overline{\text{smint}}(d,1|D_{d,1}) = \bigoplus_{\alpha=1}^{D_{d,1}} \big( \overline{\mathbb{Q}_\alpha} \big) \oplus \bigoplus_{\alpha=0}^{d} \big( \overline{\mathbb{P}_\alpha} \big) \oplus \bigoplus_{\beta=1}^{D_{d,1}} \big( \overline{\mathbb{Z}_\beta} \big) \oplus \bigoplus_{b=1}^{d} \big( \overline{\mathbb{R}_b} \big) \equiv \text{smint}(d,1|D_{d,1}) \oplus \mathbb{R}^{d|D_{d,1}}$$
and with the Lie-superalgebra structure determined by the relations
\[
\{\bar{Q}_\alpha, \bar{Q}_\beta\} = \Gamma^a_{\alpha\beta} (\bar{P}_a + \eta_{ab} \bar{R}_b), \quad [\bar{P}_a, \bar{P}_b] = 0, \quad [\bar{Q}_\alpha, \bar{P}_b] = 0,
\]
\[
[\bar{Q}_\alpha, \bar{R}_b] = 0, \quad [\bar{P}_a, \bar{R}_b] = 0, \quad \{\bar{Q}_\alpha, Z^\beta\} = 0, \quad [\bar{P}_a, Z^\beta] = 0, \quad [\bar{R}_a, Z^\alpha] = 0.
\]

We note, parenthetically, that considerations similar to ours were employed in Ref. [LAGT89] in a
derivation of central topological charges associated with the (even) wrapping states of the super-p-brane. It is to be emphasised, though, that neither the Rabin–Crane argument, nor the monodromy in the Graßmann-odd directions and the attendant subtlety of the twisted sector were considered in that
work.

The Graßmann-even (d-vector) component of the extension is trivial\(^{10}\) — it can be removed by the
simple redefinition
\[
\bar{P}_a \longrightarrow \bar{P}_a + \eta_{ab} \bar{R}_b
\]
that leaves us with the irreducible Graßmann-odd extension
\[(4.1) \quad 0 \longrightarrow R^{0|D_{d,1}} \longrightarrow \text{Ysminf}(d, 1|D_{d,1}) \xrightarrow{\pi_{\text{Ysminf}(d, 1|D_{d,1})}} \text{sminf}(d, 1|D_{d,1}) \longrightarrow 0
\]
with the supervector-space structure
\[
\text{Ysminf}(d, 1|D_{d,1}) = \bigoplus_{\alpha = 0}^{D_{d,1}} (\text{Y}Q_\alpha) \oplus \bigoplus_{\alpha = 0}^{D_{d,1}} (\text{YP}_\alpha) \oplus \bigoplus_{\beta = 1}^{D_{d,1}} (Z^\beta) \equiv \text{sminf}(d, 1|D_{d,1}) \oplus \mathbb{R}^{0|D_{d,1}}
\]
and the Lie superbracket
\[
\{\text{Y}Q_\alpha, \text{Y}Q_\beta\} = \Gamma^a_{\alpha\beta} \text{YP}_a, \quad [\text{YP}_a, \text{YP}_b] = 0, \quad [\text{Y}Q_\alpha, \text{YP}_\beta] = \Gamma^a_{\alpha\beta} Z^\beta,
\]
\[
\{\text{Y}Q_\alpha, Z^\beta\} = 0, \quad [\text{YP}_a, Z^\beta] = 0, \quad \{\text{Z}^\alpha, Z^\beta\} = 0,
\]
a long with a decoupled abelian algebra \(\mathbb{R}^{x_d}\),
\[
\text{sminf}(d, 1|D_{d,1}) \equiv \text{Ysminf}(d, 1|D_{d,1}) \oplus \mathbb{R}^{x_d}.
\]
The odd extension, which is none other than the Green superalgebra of Ref. [Gre89], has an attractive
cohomological feature, to wit, the pullback of the nontrivial 3-cocycle \(H^3(\mathbb{Z}/2\mathbb{Z})\) to \(\text{Ysminf}(d, 1|D_{d,1})\)
trivialises. Indeed, denote the super-1-forms dual to the new generators \(Z^\alpha\) as \(\text{Z}_\alpha\) to obtain\(^{11}\)
\[
\pi^\vee_{\text{Ysminf}(d, 1|D_{d,1})} H^3(\mathbb{Z}/2\mathbb{Z}) = \delta_{2\alpha} \wedge \pi^\vee_{\text{sminf}(d, 1|D_{d,1})} d^{2\alpha} = \delta(2\alpha) \wedge \pi^\vee_{\text{sminf}(d, 1|D_{d,1})} d^{2\alpha}.
\]

This is a manifestation of a \(\mathbb{Z}/2\mathbb{Z}\)-graded variant of the classic Lie-algebraic phenomenon: For any Lie
(super)algebra \(\mathfrak{g}\), classes in the second group \(H^2(\mathbb{Z}/2\mathbb{Z})\) of the Chevalley–Eilenberg cohomology of \(\mathfrak{g}\)
with values in a trivial (super)module \(\mathfrak{a}\) are in a one-to-one correspondence with equivalence classes of (super)central extensions
\[
0 \longrightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0
\]
of \(\mathfrak{g}\) by \(\mathfrak{a}\), cp Ref. [Sus77]. Now, the supervector space
\[
\Omega^1(\text{sMink}(d, 1|D_{d,1}))^\text{\text{sMink}(d, 1|D_{d,1})} = \bigoplus_{\alpha = 1}^{D_{d,1}} (\text{Q}^\alpha) \oplus \bigoplus_{\alpha = 0}^{D_{d,1}} (\text{P}^\alpha)
\]
carries a natural structure of a \(\text{sminf}(d, 1|D_{d,1})\)-module determined by the action
\[
\mathcal{L} : \text{sminf}(d, 1|D_{d,1}) \times \Omega^1(\text{sMink}(d, 1|D_{d,1}))^\text{\text{sMink}(d, 1|D_{d,1})} \longrightarrow \Omega^1(\text{sMink}(d, 1|D_{d,1}))^\text{\text{sMink}(d, 1|D_{d,1})}
\]
\[
(\text{X}, \omega) \longrightarrow \mathcal{L}_\text{X}\omega,
\]
\(^{10}\)This need not be so on the level of the associated Lie supergroup, cp Ref. [CdAIPB00, Sec. 2.3.1], but we shall not pursue this point in what follows.

\(^{11}\)In the present paper, the super-forms appear in a (seemingly) double rôle: as sections of the sheaf of superdifferential
forms on the Lie supergroup (regarded as a supermanifold) and as (super-)forms on its tangent Lie superalgebra. We use the same symbol(s) for both rôles, which, however, should not lead to confusion as it is always clear from the context which rôle is currently under consideration (in particular, we reserve the symbol \(\tilde{\delta}\) for the coboundary operator of the Lie-superalgebra cohomology).
and its Grassmann-odd subspace

\[ \left( \Omega^1 \left( s\text{Mink}(d, 1|D_{d,1}) \right) \right)^{s\text{Mink}(d, 1|D_{d,1})} \equiv \bigoplus_{\alpha=1}^{D_{d,1}} \left( \mathcal{Y}^\alpha \right) \]

is a trivial submodule. Accordingly, the GS super-3-cocycle

\[(4.2) \quad \mathcal{H}^{(3)} \equiv \left( \mathcal{Y}^\alpha \wedge \mathfrak{T}_{a \alpha \beta} \mathcal{Y}^\beta \right) \wedge \mathcal{Y}^\alpha \]

acquires the interpretation of a nontrivial \( \left( \Omega^1 \left( s\text{Mink}(d, 1|D_{d,1}) \right) \right)^{s\text{Mink}(d, 1|D_{d,1})} \)-valued super-2-cocycle on \( s\text{mink}(d, 1|D_{d,1}) \), and as such it gives rise to the supercentral extension \([4]\). The idea of trivialising the CaE super-\((p + 2)\)-cocycles that codefine the Green–Schwarz-type super-\(\sigma\)-models of the (half-BPS) super-minkovskian super-\(p\)-branes through the above Lie-superalgebraic mechanism (necessarily stepwise for \( p > 1 \)) was invented by de Azcúrraga et al. in Ref. \([CdAIPB00]\). Its adaptation to, interpretation and elaboration in the higher-geometric context under consideration constitutes the foundation of the geometrisation programme advanced by the Author in a series of papers \([Sus17, Sus19, Sus18a, Sus18b, Sus20, Sus21]\) that we turn to next.

The Lie-superalgebra extension \([1,1]\) integrates to a supercentral Lie-supergroup extension

\[ 1 \longrightarrow \mathbb{R}^{0|D_{d,1}} \longrightarrow \text{YsMink}(d, 1|D_{d,1}) \longrightarrow \text{Mink}(d, 1|D_{d,1}) \rightarrow 1 \]

with the supermanifold structure

\[ \text{YsMink}(d, 1|D_{d,1}) = \text{Mink}(d, 1|D_{d,1}) \times \mathbb{R}^{0|D_{d,1}} \equiv \left( \theta^\alpha, x^a, \xi_\beta \right) \]

and the Lie-supergroup structure determined by the binary operation

\[ \text{YM} : \ \text{Mink}(d, 1|D_{d,1}) \times \text{Mink}(d, 1|D_{d,1}) \rightarrow \text{Mink}(d, 1|D_{d,1}) \]

with the coordinate presentation

\[ \text{YM} \left( \left( \theta^\alpha, x^a, \xi_\beta \right), \left( \theta^\alpha, x^a, \xi_\beta \right) \right) = \left( \theta^\alpha, x^a, \xi_\beta \right) \]

ensuring the desired left-invariance of the super-1-form

\[ \omega_\alpha(\theta, x, \xi) = d\xi_\alpha - \mathcal{T}_{a \alpha \beta} \theta^\beta \left( dx^a + \frac{1}{2} \theta^\gamma \mathcal{T}_{\gamma a} d\theta \right). \]

The corresponding basis LI vector fields are

\[ \mathcal{Y}Q_\alpha(\theta, x, \xi) = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} \mathcal{T}_{a \alpha \beta} \theta^\beta \frac{\partial}{\partial x^a} + \frac{1}{4} \mathcal{T}_{a \alpha \beta} \theta^\beta \mathcal{T}_{\gamma a} \theta^\gamma \frac{\partial}{\partial \xi_\alpha}; \]

\[ \mathcal{YP}_\alpha(\theta, x, \xi) = \frac{\partial}{\partial x^a} + \frac{1}{2} \mathcal{T}_{a \alpha \beta} \theta^\beta \frac{\partial}{\partial \xi_\alpha}; \]

\[ \mathcal{Z}^\alpha(\theta, x, \xi) = \frac{\partial}{\partial \xi_\alpha}. \]

The idea of Ref. \([Sus17]\) was to take the epimorphism \([13]\)

\[ \pi_{\text{YsMink}(d, 1|D_{d,1})} = \text{pr}_1 : \ \text{YsMink}(d, 1|D_{d,1}) \equiv \text{Mink}(d, 1|D_{d,1}) \times \mathbb{R}^{0|D_{d,1}} \rightarrow \text{Mink}(d, 1|D_{d,1}) \]

in the category \( \text{sLieGrp} \) of Lie supergroups, together with the LI primitive

\[ \mathcal{Y}B^{(2)} \equiv \mathcal{Y}^\alpha \wedge \left( \pi_{\text{YsMink}(d, 1|D_{d,1})} \mathcal{Y}^\alpha \right) \]

of the CaE super-3-cocycle \( \mathcal{H}^{(3)} \) on its total space \( \text{YsMink}(d, 1|D_{d,1}) \),

\[ d\mathcal{Y}B = \pi_{\text{YsMink}(d, 1|D_{d,1})} \mathcal{H}^{(3)}; \]

as the point of departure (i.e., the surjective submersion and the curving, respectively) of the standard geometrisation procedure due to Murray, and subsequently bring the procedure to completion within \( \text{sLieGrp} \).

Thus, as the next step, we consider the fibred-product \([12]\) Lie supergroup

\[ \mathcal{Y}^{[2]}_{\text{sMink}(d, 1|D_{d,1})} = \text{Mink}(d, 1|D_{d,1}) \times_{\text{sMink}(d, 1|D_{d,1})} \text{YsMink}(d, 1|D_{d,1}), \]

\[ := \text{Mink}(d, 1|D_{d,1}) \times_{\text{YsMink}(d, 1|D_{d,1})} \text{YsMink}(d, 1|D_{d,1}) \]

\[ \text{Note that the cartesian product is not that in } \text{sLieGrp}. \]

\[ \text{Our convention on fibred products in the category } \text{sMan} \text{ is given in App. } \text{A}. \]
with the binary operation inherited from the cartesian product $\text{YsMink}(d,1|D_{d,1}) \times \text{YsMink}(d,1|D_{d,1})$ of Lie supergroups through restriction. It admits global coordinates
\[ Y^{[2]}_{\text{sMink}}(d,1|D_{d,1}) \ni \left((\theta, x, \xi_1), (\theta, x, \xi_2)\right). \]

On its tangent Lie superalgebra
\[ Y^{[2]}_{\text{sminf}}(d,1|D_{d,1}) \equiv Y_{\text{sminf}}(d,1|D_{d,1}) \oplus Y_{\text{sminf}}(d,1|D_{d,1}) \]
\[ = \bigoplus_{a=1}^{D_{d,1}} \{(Y_{Q_a}, Y_{Q_a})\} \oplus \bigoplus_{a=0}^{d} \{(Y_{P_a}, Y_{P_a})\} \oplus \bigoplus_{\beta=1}^{D_{d,1}} \{(Z^{\beta}, 0)\} \oplus \bigoplus_{\gamma=1}^{D_{d,1}} \{(0, Z^{\gamma})\}, \]
edowed with (the restriction of) the standard direct-sum superbracket, to be denoted as $[\cdot, \cdot]_\oplus$, we find the nontrivial super-2-cocycle
\[ F^{(2)} = (p r_{2}^\alpha - p r_{1}^\alpha) Y B, \]
with the coordinate presentation
\[ F^{(2)}((\theta, x, \xi_1), (\theta, x, \xi_2)) = d \theta^\alpha \wedge d (\xi_2 - \xi_1). \]

In virtue of the formerly invoked correspondence, the super-2-cocycle determines a central extension
\[ 0 \rightarrow \mathbb{R} \rightarrow I \xrightarrow{\pi_1} Y^{[2]}_{\text{sminf}}(d,1|D_{d,1}) \rightarrow 0 \]
with the supervector-space structure
\[ I \ni Y^{[2]}_{\text{sminf}}(d,1|D_{d,1}) \oplus \mathbb{R} \ni (X, r) \]
with respect to which
\[ \pi_1 \equiv p r_1 : Y^{[2]}_{\text{sminf}}(d,1|D_{d,1}) \oplus \mathbb{R} \rightarrow Y^{[2]}_{\text{sminf}}(d,1|D_{d,1}), \]
and with the Lie superbracket
\[ \left((X_1, r_1), (X_2, r_2)\right) F^{(2)} = \left([X_1, X_2]_\oplus, X_2 \uplus X_1 \uplus F^{(2)}\right), \]
\[ cp \text{ Ref. [Sus17], App. C}. \]
Thus, we have
\[ I = \bigoplus_{a=1}^{D_{d,1}} \{(L_{Q_a}), L_{P_a}\} \oplus \bigoplus_{a=0}^{D_{d,1}} \{(L_{Q^\alpha}), L_{P^\alpha}\} \oplus \bigoplus_{\beta=1}^{D_{d,1}} \{(L_{Z^\beta(1)}), L_{Z^\beta(2)}\} \oplus \bigoplus_{\gamma=1}^{D_{d,1}} \{(L_{Z^\gamma(1)}), L_{Z^\gamma(2)}\} \oplus \{(Z, Z)\} \]
with the structure equations (in which we drop the subscript $^{(2)}$ on the superbrackets)
\[ \{L_{Q^\alpha}, L_{Q^\beta}\} = \Gamma^\alpha_{\alpha \beta} L_{P^\beta}, \quad \{L_{P^\alpha}, L_{P^\beta}\} = 0, \quad \{L_{Q^\alpha}, L_{P^\beta}\} = \Gamma^\alpha_{\alpha \beta} (L_{Z^\beta(1)} + L_{Z^\beta(2)}), \]
\[ \{-L_{Q^\alpha}, L_{Z^\beta(1)}\} = \delta^\beta_{\alpha} Z = \{L_{Q^\alpha}, L_{Z^\beta(2)}\}, \quad \{L_{P^\alpha}, L_{Z^\beta(1)}\} = 0, \quad \{L_{Z^\alpha(1)}, L_{Z^\beta(1)}\} = 0, \]
\[ \{L_{P^\alpha}, L_{Z^\beta(2)}\} = 0, \quad \{L_{Z^\alpha(2)}, L_{Z^\beta(2)}\} = 0, \quad \{Z, Z\} = 0. \]

Upon denoting the super-1-form dual to the central generator $Z$ as $\zeta$, we readily establish the identity
\[ \delta\zeta = \pi_1 F^{(2)}. \]

As before, the Lie-superalgebra extension integrates to a central Lie-supergroup extension – this time, we obtain
\[ 1 \rightarrow \mathbb{C}^\times \rightarrow L \rightarrow Y^{[2]}_{\text{sMink}}(d,1|D_{d,1}) \rightarrow 1 \]
with the supermanifold structure
\[ L = Y^{[2]}_{\text{sMink}}(d,1|D_{d,1}) \times \mathbb{C}^\times \ni \left((\theta^\alpha, x^a, \xi_1^\beta), (\theta^\alpha, x^a, \xi_2^\beta), \zeta\right) \]
and the Lie-supergroup structure determined by the binary operation
\[ L m : L \times L \rightarrow L \]
with the coordinate presentation
\[ L m((\theta_1, x_1, \xi_1, (\theta_2, x_2, \xi_2)), (\theta_1, x_1, \xi_1, (\theta_2, x_2, \xi_2))) \]
\[ = \left(Y m((\theta_1, x_1, \xi_1)), Y m((\theta_2, x_2, \xi_2)), e^{i \theta_1^\alpha (\xi_2^\beta - \xi_1^\beta)} \cdot z_1 \cdot z_2\right) \]
such that the super-1-form
\[ \zeta((\theta, x, \xi_1), (\theta, x, \xi_2), z) = \frac{1}{2} \xi_1 + \theta^\alpha d(\xi_{2\alpha} - \xi_{1\alpha}) = \frac{1}{2} \xi_1 + a((\theta, x, \xi_1), (\theta, x, \xi_2)) \]
is LI. The extension has the structure of a (trivial) principal \( \mathbb{C}^\times \)-bundle
\[ \pi_{\mathcal{L}} \equiv \text{pr}_1 : \mathcal{L} \equiv \mathcal{Y}^{[2]}_{s\text{Mink}}(d, 1|D_{d,1}) \times \mathbb{C}^\times \rightarrow \mathcal{Y}^{[2]}_{s\text{Mink}}(d, 1|D_{d,1}) \]
with an obvious fibrewise action of the structure group \( \mathbb{C}^\times \) and with the LI principal connection super-1-form
\[ A_{(1)} = \frac{1}{2} \zeta \]
of curvature \( F \),
\[ dA_{(1)} = \pi_{\mathcal{L}}^* F. \]

Finally, we consider the fibred-product Lie supergroup
\[ \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \equiv \text{YsMink}(d, 1|D_{d,1}) \times_{\text{Mink}(d, 1|D_{d,1})} \text{YsMink}(d, 1|D_{d,1}) \times_{\text{Mink}(d, 1|D_{d,1})} \text{YsMink}(d, 1|D_{d,1}) \]
(defined analogously to the fibred square \( \mathcal{Y}^{[2]}_{s\text{Mink}}(d, 1|D_{d,1}) \)) and, over it, the pullback bundles
\[ \pi_{\text{pr}_{i,j}}^* \mathcal{L} \equiv \text{pr}_{i,j}^* \mathcal{L} \equiv \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \times_{\text{Mink}(d, 1|D_{d,1})} \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \times_{\text{Mink}(d, 1|D_{d,1})} \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \]
endowed with the Lie-supergroup structure obtained, through restriction, from the product one on
\[ \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \times \mathcal{L} \rightarrow \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \times \mathcal{L} \] 
On the fibred cube \( \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}) \), we have coordinates
\[ ((\theta, x, \xi_1), (\theta, x, \xi_2), (\theta, x, \xi_3)) \equiv (y_1, y_2, y_3), \]
and so for the pullback bundles, we obtain coordinates
\[ \text{pr}_{i,j}^* \mathcal{L} \ni ((y_1, y_2, y_3), (y_i, y_j, z)) \equiv (y_{1,2,3}, (y_{i,j}, z)). \]
In these, the induced binary operation
\[ \mathcal{L}_{i,j,m} : \text{pr}_{i,j}^* \mathcal{L} \times \text{pr}_{i,j}^* \mathcal{L} \rightarrow \text{pr}_{i,j}^* \mathcal{L} \]
reads
\[ \mathcal{L}_{i,j,m}(((y_{1,2,3}, (y_{i,j}, z_1)), (y_{1,2,3}, (y_{i,j}, z_2))) = \left((\text{YM}(y_1, y_1^2), \text{YM}(y_2, y_2^2), \text{YM}(y_3, y_3^2)), \mathcal{L}(y_{i,j}, z_1), (y_{i,j}, z_2)) \right). \]
Out of the first two pullback bundles, \( \text{pr}_{1,2}^* \mathcal{L} \) and \( \text{pr}_{2,3}^* \mathcal{L} \), we form the tensor-product principal \( \mathbb{C}^\times \)-bundle
\[ [\pi_{\mathcal{L}} \circ \text{pr}_1] : \text{pr}_{1,2}^* \mathcal{L} \otimes \text{pr}_{2,3}^* \mathcal{L} \rightarrow \mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1}), \]
defined as the associated bundle
\[ \text{pr}_{1,2}^* \mathcal{L} \otimes \text{pr}_{2,3}^* \mathcal{L} = (\text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L}) / \mathbb{C}^\times \],
with the projection to the base (written out in coordinates)
\[ [\pi_{\mathcal{L}} \circ \text{pr}_1](y_{1,2,3}, (y_{1,2,3}, (y_{2,3}, z))) \equiv y_{1,2,3}. \]
Here, we are quotienting out the ‘diagonal’ action
\[ \lambda : \mathbb{C}^\times \times (\text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L}) \rightarrow \text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L} \]
with the coordinate presentation
\[ \lambda(z, (y_{1,2,3}, (y_{1,2,3}, z_1)), (y_{1,2,3}, (y_{2,3}, z_2))) = \left((y_{1,2,3}, (y_{1,2,3}, z_2)), (y_{1,2,3}, (y_{2,3}, z_2)), (y_{1,2,3}, (y_{1,2,3}, z_2)) \right). \]
The tensor-product bundle inherits a natural Lie-supergroup structure from (the restricted product one on) \( \text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L} \),
\[ [\mathcal{L}_{1,2,3,m} : \text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L}] \rightarrow \text{pr}_{1,2}^* \mathcal{L} \times_{\text{pr}_1^* \mathcal{L}} \text{pr}_{2,3}^* \mathcal{L}, \]
with the coordinate presentation
\[ [\mathcal{L}_{1,2,3,m}((y_{1,2,3}, (y_{1,2,3}, (y_{1,2,3}, (y_{1,2,3}, z_1)))), (y_{1,2,3}, (y_{1,2,3}, (y_{1,2,3}, (y_{1,2,3}, z_2))))] \]
\[ ^{14}\text{Formally, we perform the quotienting in the body and subsequently take the sub-sheaf composed of } \lambda \text{-invariant sections in the structure sheaf of } \text{pr}_{1,2}^* \mathcal{L} \times_{\mathcal{Y}^{[3]}_{s\text{Mink}}(d, 1|D_{d,1})} \text{pr}_{2,3}^* \mathcal{L} \text{ as the structure sheaf of the quotient supermanifold.} \]
At this stage, it suffices to compare the base components of the principal connection super-1-forms on \( pr_{1,2}^* \mathcal{L} \otimes pr_{2,3}^* \mathcal{L} \) and \( pr_{1,3}^* \mathcal{L} \),
\[
(pr_{1,2}^* a + pr_{2,3}^* a)(y_{1,2,3}) = pr_{1,3}^* a(y_{1,2,3}),
\]
to infer the existence of a connection-preserving isomorphism of principal \( \mathbb{C}^\infty \)-bundles
\[
\mu_\mathcal{L} : pr_{1,2}^* \mathcal{L} \otimes pr_{2,3}^* \mathcal{L} \xrightarrow{\cong} pr_{1,3}^* \mathcal{L}
\]
with the coordinate presentation
\[
\mu_\mathcal{L} ((y_{1,2,3}, (y_{1,2,1})) \oplus (y_{1,2,3}, (y_{2,3, z})) = (y_{1,2,3}, (y_{1,3, z})).
\]
We reserve the suggestive (symbolic) notation
\[
\mu_\mathcal{L} \equiv 1
\]
for an isomorphism of the above trivial form. The isomorphism satisfies the coherence (groupoid) identity
\[
pr_{1,3,4}^* \mu_\mathcal{L} \circ (pr_{1,2,3}^* \mu_\mathcal{L} \otimes id_{pr_{1,4}^* \mathcal{L}}) = pr_{1,2,4}^* \mu_\mathcal{L} \circ (id_{pr_{1,2}^* \mathcal{L}} \otimes pr_{2,3,4}^* \mu_\mathcal{L})
\]
over the quadruple fibre product \( Y[1]_{\text{sMink}}(d, 1|D_{d,1}) \), the latter being equipped with the canonical projections \( pr_{i,j,k} : Y[1]_{\text{sMink}}(d, 1|D_{d,1}) \rightarrow Y[3]_{\text{sMink}}(d, 1|D_{d,1}) \), \( (i, j, k) \in \{(1,2,3), (1,3,4), (2,3,4), (1,2,4)\} \) and \( pr_{m,n} : Y[1]_{\text{sMink}}(d, 1|D_{d,1}) \rightarrow Y[2]_{\text{sMink}}(d, 1|D_{d,1}) \), \( (m, n) \in \{(1,2), (3,4)\} \) (defined in an obvious manner). Clearly, \( \mu_\mathcal{L} \) is also a Lie-supergroup isomorphism,
\[
\mu_\mathcal{L} \circ [\mathcal{L}_{1,2,3,4}] = \mathcal{L}_{1,3,4} \circ (\mu_\mathcal{L} \times \mu_\mathcal{L}).
\]
The 1-gerbe
\[
G_{1,3,4}^{(1)} := (Y_{\text{sMink}}(d, 1|D_{d,1}), \pi_{Y_{\text{sMink}}(d, 1|D_{d,1})}, YB, \mathcal{L}, \pi_\mathcal{L}, A, \mu_\mathcal{L})
\]
was named the Green–Schwarz super-1-gerbe over \( s\text{Mink}(d, 1|D_{d,1}) \) in Ref. [Sus19]. It is an example of a Cartan–Eilenberg super-1-gerbe (over the Lie supergroup \( s\text{Mink}(d, 1|D_{d,1}) \)), that is a distinguished 1-gerbe in the category of Lie supergroups. Its existence and equivariance properties, the latter to be discussed at length in Sec. [Sus19], are markers of a quantum-mechanical consistency of the GS super-\( \sigma \)-model of the superstring in \( s\text{Mink}(d, 1|D_{d,1}) \). As it stands, the super-1-gerbe is naturally associated with the NG formulation that can be phrased in terms of the differential-geometric data of the homogeneous space \( s\text{Mink}(d, 1|D_{d,1}) \) of the mother supersymmetry group \( s\text{ISO}(d, 1|D_{d,1}) \) exclusively. Instrumental in its construction is the ‘accidental’ Lie-supergroup structure on this particular homogeneous space. In the case of a generic homogeneous space \( G/H \) of a supersymmetry Lie supergroup \( G \) associated with a Lie subgroup \( H \subset |G| \) of its body \( |G| \), the geometrisation scheme exemplified above, making use of the relations between the Cartan–Eilenberg cohomology of the Lie supergroup and the Chevalley–Eilenberg cohomology of its tangential Lie superalgebra and the interpretation of the distinguished second cohomology group of the latter, has to start on \( G \) and only in the end descend to \( G/H \). The cohomology to be geometrised under such circumstances is \( \text{CaE}^*(G) \) further restricted to H-basic super-forms, and – in the light of the findings of Refs. [GSW10, Sus11B, Sus12, GSW13, Sus13] – the CaE super-1-gerbe that we seek to erect over \( G \) has to carry a descendable \( H \)-equivariant structure.\(^{15}\)

\(^{15}\)This notion shall be recalled and illustrated in Sec. [Sus19].
'gerbification' of the super-minkowskian GS super-σ-model in the supergeometrically largely tractable HP formulation.

The idea that we wish to pursue now consists in lifting the supercentral extension $\text{Ysmint}(d,1|D_{d,1})$ of the super-minkowskian Lie superalgebra equivariantly to an extension of the full supersymmetry Lie superalgebra $\text{siso}(d,1|D_{d,1})$. Taking a closer look at the lift of the GS super-3-cocycle (4.3),

$$\chi \equiv (p^a \wedge \Gamma_{a\alpha\beta} q^\beta) \wedge q^\alpha,$$

and the precise relation between the lifted LI super-1-forms $q^\alpha$ in it and the $\bar{q}^\alpha$ that we previously identified as the basis of the (trivial) $\text{smint}(d,1|D_{d,1})$-module defining the extension,

$$(4.3) \quad q^\alpha (\theta, x, \phi) = S(\phi)^{-1} \alpha \beta \bar{q}^\beta (\theta, x),$$

we are readily led to postulate the lift

$$\text{Ysiso}(d,1|D_{d,1}) = (\bigoplus_{\alpha=1}^{D_{d,1}} (\text{YQ}_\alpha) \oplus \bigoplus_{\alpha=0}^{D_{d,1}} (\text{YP}_\alpha) \oplus \bigoplus_{\beta=1}^{D_{d,1}} (\text{Z}_\beta)) \oplus \bigoplus_{b=0}^{D_{d,1}} (\text{YJ}_{bc}) \equiv \text{Ysiso}(d,1|D_{d,1}) \oplus \text{spin}(d,1)$$

in the form of a $\text{spin}(d,1)$-module Lie superalgebra with superbrackets

$$\{\text{YQ}_\alpha, \text{YQ}_\beta\} = \Gamma_{a\beta} \text{YP}_a, \quad \{\text{YP}_\alpha, \text{YP}_\beta\} = 0, \quad \{\text{YQ}_\alpha, \text{YP}_\beta\} = \Gamma_{a\alpha\beta} \text{Z}_\beta,$$

$$\{\text{YQ}_\alpha, \text{Z}_\beta\} = 0, \quad \{\text{YP}_\alpha, \text{Z}_\beta\} = 0, \quad \{\text{Z}_\alpha, \text{Z}_\beta\} = 0,$$

$$\{\text{YJ}_{ab}, \text{YQ}_\alpha\} = \frac{1}{2} \Gamma_{ab \alpha} \text{YQ}_\beta, \quad \{\text{YJ}_{ab}, \text{YP}_\alpha\} = \eta_{bc} \text{YP}_a - \eta_{ac} \text{YP}_b, \quad \{\text{YJ}_{ab}, \text{Z}_\alpha\} = -\frac{1}{2} \Gamma_{ab \alpha} \text{Z}_\beta,$$

that extends the analogous structure on the equivariant lift $\text{siso}(d,1|D_{d,1})$ of $\text{smint}(d,1|D_{d,1})$. Thus, the extra generators $\text{Z}_\alpha$ transform under $\text{spin}(d,1)$ as spinors, in conformity with Eq. (4.3). The only components of the ensuing super-Jacobiator that are not trivially null (e.g., because of being identical with their un-extended counterparts) read

$$\text{sJac}(\text{YQ}_\alpha, \text{YQ}_\beta, \text{YQ}_\gamma) = 3! \Gamma_{a\alpha\beta} \text{YQ}_\gamma,$$

$$\text{sJac}(\text{YQ}_\alpha, \text{YP}_\beta, \text{YJ}_{bc}) = \left( (\Gamma_{a\alpha\beta} (\text{YQ}_\gamma)_{(\alpha\beta)} - \eta_{ab} \Gamma_{c\alpha\beta} \eta_{ac} \Gamma_{b\alpha\beta}) \right) \text{Z}_\gamma,$$

$$\text{sJac}(\text{Z}_\alpha, \text{YJ}_{ab}, \text{YJ}_{cd}) = \frac{1}{4} \left( \Gamma_{ab \alpha} \Gamma_{cd} - \Gamma_{ad \alpha} \Gamma_{bc} - 2 \eta_{ad} \Gamma_{bc} - 2 \eta_{ac} \Gamma_{bd} + 2 \eta_{bd} \Gamma_{ac} \right) \text{Z}_\gamma.$$

The first of these vanishes in virtue of the Fierz identities (2.1). Next, we compute

$$\left( \Gamma_{a \alpha \beta} \Gamma_{bc} \right)^T = \Gamma_{cb \alpha} \Gamma_{a \beta},$$

and use it to perform the reduction

$$\Gamma_{a \alpha \beta} \Gamma_{bc} + (\Gamma_{a \alpha \beta} \Gamma_{bc})^T = -2 \eta_{ab} \Gamma_{c \alpha \beta} + 2 \eta_{ac} \Gamma_{b \alpha \beta} = 0,$$

which implies that the second component of the super-Jacobiator is zero. Finally, the third component can be rewritten as

$$\text{sJac}(\text{Z}_\alpha, \text{YJ}_{ab}, \text{YJ}_{cd}) = -q^\alpha \left( \text{sJac}(\text{YQ}_\beta, \text{YJ}_{ab}, \text{YJ}_{cd}) \right) \text{Z}_\beta = 0,$$

which concludes the proof of existence of a Lie-superalgebra structure on $\text{Ysiso}(d,1|D_{d,1})$ with the superbracket postulated above. Denote as

$$\text{π}_{\text{Ysiso}(d,1|D_{d,1})} : \text{Ysiso}(d,1|D_{d,1}) \longrightarrow \text{siso}(d,1|D_{d,1})$$

the Lie-superalgebra epimorphism obtained by linearly extending the assignment

$$\text{π}_{\text{Ysiso}(d,1|D_{d,1})} : (\text{YQ}_\alpha, \text{YP}_\alpha, \text{Z}_\alpha, \text{YJ}_{bc}) \longmapsto (\text{Q}_\alpha, \text{P}_\alpha, 0, \text{J}_{bc}),$$

and let the duals of the generators $Z^\alpha$ be $z_\alpha$. We then obtain the desired identity

$$\delta(z_\alpha \wedge \text{π}_{\text{Ysiso}(d,1|D_{d,1})} \text{YQ}_\alpha) = \text{π}_{\text{Ysiso}(d,1|D_{d,1})} \chi.$$

Next, we readily enhance the Lie superalgebra $\text{Ysiso}(d,1|D_{d,1})$ to a super-Harish–Chandra pair (i.e., to a Lie supergroup)

$$\text{YsISO}(d,1|D_{d,1}) = (\text{ISO}(d,1), \text{Ysiso}(d,1|D_{d,1}))$$
Thus, trivially satisfying the identity

\[ \text{On the level of the underlying supervector spaces, we have the corresponding linear maps} \]

and the corresponding relation

\[ \text{with the body Lie group } \tilde{\text{ISO}}(d,1) \text{ realised on the Grassmann-odd component} \]

\[
\text{of } Y_{\text{iso}}(d,1|D_{d,1}) \text{ as } \\
Y_{\rho} : \mathbb{R}^{\times d+1} \rtimes \text{Spin}(d,1) \to \text{End}(Y_{\text{iso}}(d,1|D_{d,1})) : (x,\phi) \mapsto \theta(\phi)^T \otimes S(\phi)^{-1} \equiv Y_{\rho}(x,\phi). \]

Thus, \( \pi_{Y_{\text{iso}}(d,1|D_{d,1})} \) integrates to a Lie-supergroup epimorphism

\[ \pi_{Y_{\text{iso}}(d,1|D_{d,1})} = \text{pr}_1 \times \text{id}_{\text{Spin}(d,1)} : Y_{\text{iso}}(d,1|D_{d,1}) \equiv Y_{\text{Mink}}(d,1|D_{d,1}) \rtimes_{L,S,S^-} \text{Spin}(d,1) \to \text{sISO}(d,1|D_{d,1}), \]

and we have a coordinate description of the Lie supergroup

\[ Y_{\text{Mink}}(d,1|D_{d,1}) \rtimes_{L,S,S^-} \text{Spin}(d,1) \ni (\theta^\alpha, x^\alpha, \xi, \phi^b \phi^c) \]

in which the binary operation

\[ Y_{\text{Mink}} : Y_{\text{iso}}(d,1|D_{d,1}) \times Y_{\text{iso}}(d,1|D_{d,1}) \to Y_{\text{iso}}(d,1|D_{d,1}) \]

takes the form

\[ Y_{\text{Mink}} \left( \left( \theta_1^\alpha, x_1^\alpha, \xi_1, \phi_1^b \phi_1^c \right), \left( \theta_2^\alpha, x_2^\alpha, \xi_2, \phi_2^b \phi_2^c \right) \right) = \left( \theta_1^\alpha + S(\phi_1)^\alpha \phi_2^b x_2^b + L(\phi_1)^\alpha_{\beta} x_2^\beta, \xi_1 + \xi_2, \phi_2 \right) \theta_2. \]

\[ \xi_1 \circ \xi_2 \cap S(\phi_1)^{-1} \theta_1 \circ \xi_2 \gamma L(\phi_1)^{\beta}_c x_2^c - \frac{1}{6} \theta_1 \Gamma^\gamma \left( 2 \theta_1^\gamma \theta_2^\delta \delta^\beta \right). \]

The LI super-1-forms \( z_\alpha \) admit the explicit coordinate presentation

\[ z_\alpha(\theta, x, \xi, \phi) = \tilde{z}_\beta(\theta, x, \xi) S(\phi)^\beta_{\alpha}, \]

and we arrive at the anticipated identity

\[ \pi_{Y_{\text{Mink}}(d,1|D_{d,1})} \equiv d(z_\alpha \wedge \pi_{Y_{\text{iso}}(d,1|D_{d,1})})^\alpha, \]

whence also the choice of the curving

\[ Y_{\beta} = z_\alpha \wedge \pi_{Y_{\text{Mink}}(d,1|D_{d,1})} Y_{\alpha} \]

of the CaE super-1-gerbe over sISO(1,1|D_{d,1}) under reconstruction. The canonical projection \( \pi \) of \( \text{Diag.}(2,5) \) lifts to the extensions as the supermanifold morphism

\[ Y_{\pi} \equiv \text{pr}_1 : Y_{\text{Mink}}(d,1|D_{d,1}) \equiv Y_{\text{Mink}}(d,1|D_{d,1}) \times \text{Spin}(d,1) \to Y_{\text{Mink}}(d,1|D_{d,1}) \]

with the property

\[ \pi \circ \pi_{Y_{\text{Mink}}(d,1|D_{d,1})} = \pi_{Y_{\text{Mink}}(d,1|D_{d,1})} \circ Y_{\pi}, \]

and we establish the descent relation

\[ Y_{\beta} \equiv Y_{\pi} Y_{\beta}. \]

On the level of the underlying supervector spaces, we have the corresponding linear maps

\[ Y_{\rho} \equiv \text{pr}_1 : Y_{\text{Mink}}(d,1|D_{d,1}) \oplus \text{spin}(d,1) \to Y_{\text{Mink}}(d,1|D_{d,1}) \]

trivially satisfying the identity

\[ \pi_{Y_{\text{Mink}}(d,1|D_{d,1})} \circ Y_{\rho} = p \circ \pi_{Y_{\text{Mink}}(d,1|D_{d,1})}, \]

and the corresponding relation

\[ Y_{\beta} \equiv Y_{\rho} Y_{\beta}. \]

between super-2-forms on the respective Lie superalgebras.

From this point onwards, the construction proceeds along the same lines as for sMink(1,1|D_{d,1}). Thus, we take the fibred-square Lie super group

\[ Y^{[2]}_{\text{ISO}}(d,1|D_{d,1}) \equiv Y_{\text{ISO}}(d,1|D_{d,1}) \times_{\text{sISO}(d,1|D_{d,1})} Y_{\text{ISO}}(d,1|D_{d,1}) \]

\[ \equiv Y^{[2]}_{\text{sMink}}(d,1|D_{d,1}) \rtimes_{L,S,S^-} \text{Spin}(d,1), \]

endowed with the canonical projection

\[ Y^{[2]}_{\pi} \equiv Y_{\pi} \times Y_{\pi} : Y_{\text{ISO}}(d,1|D_{d,1}) \times_{\text{sISO}(d,1|D_{d,1})} Y_{\text{ISO}}(d,1|D_{d,1}) \]
and its tangent Lie superalgebra

\[ Y^{[2]}_{\text{s}ISO}(d, 1|D_{d,1}) \equiv Y_{\text{sISO}}(d, 1|D_{d,1}) \oplus_{\text{sISO}(d, 1|D_{d,1})} Y_{\text{sISO}}(d, 1|D_{d,1}) \]

\[ = \bigoplus_{a=1}^{D_{d,1}} \left( \left( Y_{Q_a}, Y_{Q_a} \right) \right) \oplus \bigoplus_{a=0}^{d} \left( \left( Y_{P_a}, Y_{P_a} \right) \right) \oplus \bigoplus_{\beta=1}^{D_{d,1}} \left( \left( Z_\beta, 0 \right) \right) \oplus \bigoplus_{\gamma=1}^{D_{d,1}} \left( \left( 0, Z_\gamma \right) \right) \oplus \bigoplus_{b,c=0}^{d} \left( \left( Y_{J_{bc}}, Y_{J_{bc}} \right) \right) \]

\[ \cong Y^{[2]}_{\text{s}smft}(d, 1|D_{d,1}) \oplus \text{spin}(d, 1), \]

coming with the supervector-space projection

\[ Y^{[2]}_{p} \equiv Y_{p} \oplus Y_{p} \equiv \pi_{1} : Y^{[2]}_{\text{s}smft}(d, 1|D_{d,1}) \oplus \text{spin}(d, 1) \rightarrow Y^{[2]}_{\text{s}smft}(d, 1|D_{d,1}). \]

The nontrivial super-2-cocycle

\[ \mathcal{F} = \left( \pi_{2}^{*} - \pi_{1}^{*} \right) Y_{p} \equiv Y^{[2]}_{p} \quad \mathcal{F}^{(2)} \]

don \( Y^{[2]}_{\text{s}ISO}(d, 1|D_{d,1}) \) engenders a central extension

\[ (4.4) \quad 0 \rightarrow \mathbb{R} \rightarrow \tilde{\mathcal{I}} \rightarrow Y^{[2]}_{\text{s}ISO}(d, 1|D_{d,1}) \rightarrow 0 \]

with the supervector-space structure

\[ \tilde{\mathcal{I}} = \left( \bigoplus_{a=1}^{D_{d,1}} \left( \mathcal{F} Q_{a} \right) \oplus \bigoplus_{a=0}^{d} \left( \mathcal{F} P_{a} \right) \oplus \bigoplus_{\beta=1}^{D_{d,1}} \left( \mathcal{F} Z_{\beta}^{(1)} \right) \oplus \bigoplus_{\gamma=1}^{D_{d,1}} \left( \mathcal{F} Z_{\gamma}^{(2)} \right) \right) \oplus \left( \mathcal{Z} \right) \oplus \bigoplus_{b,c=0}^{d} \left( \mathcal{F} J_{bc} \right) \cong \mathcal{I} \oplus \text{spin}(d, 1), \]

with respect to which

\[ \pi_{1} = \pi_{1} \oplus \text{id}_{\text{spin}(d, 1)} : \mathcal{I} \oplus \text{spin}(d, 1) \rightarrow Y^{[2]}_{\text{s}ISO}(d, 1|D_{d,1}), \]

and with the Lie superbracket

\[ \left( \mathcal{F} Q_{a}, \mathcal{F} Q_{\beta} \right) = \Gamma^{a}_{\alpha \beta} \mathcal{F} P_{a}, \quad \left[ \mathcal{F} P_{a}, \mathcal{F} P_{b} \right] = 0 , \quad \left[ \mathcal{F} Q_{a}, \mathcal{F} P_{a} \right] = \Gamma^{a}_{\alpha \beta} \left( \mathcal{F} Z_{\beta}^{(1)} + \mathcal{F} Z_{\beta}^{(2)} \right), \]

\[ -\left( \mathcal{F} Q_{a}, \mathcal{F} Z_{\beta}^{(1)} \right) = \delta^{a}_{\beta} \mathcal{Z} = \left\{ \mathcal{F} Q_{a}, \mathcal{F} Z_{\beta}^{(1)} \right\}, \quad \left[ \mathcal{F} P_{a}, \mathcal{F} Z_{\alpha}^{(m)} \right] = 0 , \quad \left\{ \mathcal{F} P_{a}, \mathcal{F} Z_{\beta}^{(m)} \right\} = 0, \]

\[ \left[ \mathcal{F} Q_{a}, \mathcal{Z} \right] = 0 , \quad \left[ \mathcal{F} P_{a}, \mathcal{Z} \right] = 0 , \quad \left[ \mathcal{F} Z_{\alpha}^{(m)}, \mathcal{Z} \right] = 0 , \quad \left[ \mathcal{Z}, \mathcal{Z} \right] = 0, \]

\[ \left[ \mathcal{F} J_{ab}, \mathcal{F} Q_{a} \right] = \frac{1}{2} \Gamma^{a}_{\beta \alpha} \mathcal{F} Q_{\beta}, \quad \left[ \mathcal{F} J_{ab}, \mathcal{F} P_{a} \right] = \eta_{bc} \mathcal{F} P_{a} - \eta_{ac} \mathcal{F} P_{b}, \]

\[ \left[ \mathcal{F} J_{ab}, \mathcal{F} Z_{\alpha}^{(m)} \right] = -\frac{1}{2} \Gamma^{a}_{\beta \alpha} \mathcal{F} Z_{\beta}^{(m)}, \quad \left[ \mathcal{F} J_{ab}, \mathcal{Z} \right] = 0, \]

\[ \left[ \mathcal{F} J_{ab}, \mathcal{F} J_{cd} \right] = \eta_{ad} \mathcal{F} J_{bc} - \eta_{ac} \mathcal{F} J_{bd} + \eta_{bc} \mathcal{F} J_{ad} - \eta_{bd} \mathcal{F} J_{ac}. \]

With \( \zeta \) denoting the super-1-form dual to \( \mathcal{Z} \) and

\[ \mathcal{F} p \equiv \pi_{1} : 1 \oplus \text{spin}(d, 1) \rightarrow \mathcal{I}, \]

we obtain, similarly as before,

\[ \delta_{\zeta} = \pi_{1}^{*} \mathcal{F} \equiv \mathcal{F}^{*} \pi_{1}^{*} \mathcal{F}. \]

The above Lie-superalgebra extension integrates to a central Lie-supergroup extension

\[ 1 \rightarrow \mathbb{C}^{\times} \rightarrow \mathcal{F} \rightarrow \mathcal{F} \rightarrow Y^{[2]}_{\text{sISO}}(d, 1|D_{d,1}) \rightarrow 1 \]

with the supermanifold structure

\[ \mathcal{F} = Y^{[2]}_{\text{sISO}}(d, 1|D_{d,1}) \times \mathbb{C}^{\times} \cong \mathcal{F} \times \text{Spin}(d, 1) \]

for which

\[ \pi_{\mathcal{F}} \equiv \pi_{\mathcal{F}} \times \text{id}_{\text{Spin}(d, 1)} : \mathcal{F} \rightarrow Y^{[2]}_{\text{s}Mink}(d, 1|D_{d,1}) \times \text{Spin}(d, 1) \equiv Y^{[2]}_{\text{sISO}}(d, 1|D_{d,1}), \]

and with the Lie-supergroup structure determined by the binary operation

\[ \mathcal{F} m : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F} \]

with the coordinate presentation

\[ \mathcal{F} m\left( ((\theta_{1}, x_{1}, \xi_{1,1}, \phi_{1}), (\theta_{1}, x_{1}, \xi_{1,2}, \phi_{1}), z_{1}), ((\theta_{2}, x_{2}, \xi_{2,1}, \phi_{2}), (\theta_{2}, x_{2}, \xi_{2,2}, \phi_{2}), z_{2}) \right) \]

\[ \rightarrow \text{Y} \text{Mink}(d, 1|D_{d,1}) \times_{\text{Y} \text{Mink}(d, 1|D_{d,1})} \text{Y} \text{Mink}(d, 1|D_{d,1}), \]
\[ \left( \begin{array}{c} Ym(\theta_1, x_1, \xi_{1,1}, \phi_1), (\theta_2, x_2, \xi_{2,1}, \phi_2) \\ Ym(\theta_1, x_1, \xi_{1,2}, \phi_1), (\theta_2, x_2, \xi_{2,2}, \phi_2) \end{array} \right), e^{i\beta} (\xi_2, -\xi_2, \phi) \cdot S(\phi) = z_1 \cdot z_2. \]

Its form ensures left-invariance of the super-1-form

\[ \zeta((\theta, x, \xi_1, \phi), (\theta, x, \xi_2, \phi), z) = \frac{d\zeta}{z} + \theta^\alpha d(\xi_2 - \xi_1) \equiv \frac{d\zeta}{z} + \gamma^a(\theta, x, \xi, \phi), (\theta, x, \xi_2, \phi). \]

Writing

\[ \tilde{\mathcal{F}} \equiv \text{pr}_1 : \mathcal{L} \times \text{Spin}(d, 1) \rightarrow \mathcal{L}, \]

with

\[ \pi_\mathcal{L} \circ \tilde{\mathcal{F}} = \pi \circ \pi_\mathcal{L}, \]

we obtain

\[ \zeta = \tilde{\mathcal{F}}^* \zeta. \]

Once again, we end up with the structure of a (trivial) principal \( C^* \)-bundle

\[ \pi_{\tilde{\mathcal{F}}} : \tilde{\mathcal{F}} \rightarrow \gamma^a(\mathcal{D}_d, 1) \]

with the LI principal connection super-1-form

\[ A_{(1)}^{\mathcal{F}} = \zeta \]

of curvature \( \mathcal{F} \),

\[ dA_{(1)}^{\mathcal{F}} = \pi^* \mathcal{F}. \]

A reasoning fully analogous to the one presented in the case of \( \mathcal{L} \) leads to the trivial groupoid structure

\[ \mu_{\tilde{\mathcal{F}}} = 1 : \text{pr}^*_{1,2} \tilde{\mathcal{F}} \otimes \text{pr}^*_{2,3} \tilde{\mathcal{F}} \rightarrow \text{pr}^*_{1,3} \tilde{\mathcal{F}} \]

that isomorphically maps the Lie-supergroup structure on its domain to the one on its codomain. Also the latter admits a Lie-superalgebraic description, which we state hereunder for later reference. Its reconstruction starts with the self-explanatory definition of the pullback Lie superalgebras

\[ \text{pr}^*_{i,j} \equiv \gamma^a(\mathcal{D}_d, 1) \]

with the respective bases

\[ \begin{align*}
\text{pr}^*_{1,2} & = \bigoplus_{\alpha=1}^{D_d} \left( \left( YQ_\alpha, YO_\alpha, YO_\alpha, \tilde{\mathcal{F}}Q_\alpha \equiv \tilde{\mathcal{F}}Q_\alpha^{(1,2)} \right) \oplus \bigoplus_{\alpha=0}^{d} \left( \left( YP_\alpha, YP_\alpha, YP_\alpha, \tilde{\mathcal{F}}P_\alpha \equiv \tilde{\mathcal{F}}P_\alpha^{(1,2)} \right) \right. \\
& \bigoplus_{\beta=1}^{D_d} \left( \left( Z^\beta, 0, 0, \tilde{\mathcal{F}}Z_{(1)}^\beta \equiv \tilde{\mathcal{F}}Z_{(1)}^{(1,2)} \right) \oplus \bigoplus_{\gamma=1}^{D_d} \left( \left( 0, Z^\gamma, 0, \tilde{\mathcal{F}}Z_{(2)}^\gamma \equiv \tilde{\mathcal{F}}Z_{(2)}^{(1,2)} \right) \right. \\
& \bigoplus_{\beta=1}^{D_d} \left( \left( 0, 0, Z^\beta, 0 \right) \equiv \tilde{\mathcal{F}}Z_{(3)}^{(1,2)} \right) \oplus \left( \left( 0, 0, 0, Z \right) \equiv \mathcal{Z}^{(1,2)} \right) \\
& \bigoplus_{\beta=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(1,2)} \right) \right. \\
& \bigoplus_{b=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(1,2)} \right) \right) \\
& \bigoplus_{\gamma=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(1,2)} \right) \right) \\
& \bigoplus_{\gamma=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(1,2)} \right) \right)
\end{align*} \]

\[ \begin{align*}
\text{pr}^*_{2,3} & = \bigoplus_{\alpha=1}^{D_d} \left( \left( YQ_\alpha, YO_\alpha, YO_\alpha, \tilde{\mathcal{F}}Q_\alpha \equiv \tilde{\mathcal{F}}Q_\alpha^{(2,3)} \right) \oplus \bigoplus_{\alpha=0}^{d} \left( \left( YP_\alpha, YP_\alpha, YP_\alpha, \tilde{\mathcal{F}}P_\alpha \equiv \tilde{\mathcal{F}}P_\alpha^{(2,3)} \right) \right. \\
& \bigoplus_{\beta=1}^{D_d} \left( \left( Z^\beta, 0, 0, \tilde{\mathcal{F}}Z_{(1)}^\beta \equiv \tilde{\mathcal{F}}Z_{(1)}^{(2,3)} \right) \oplus \bigoplus_{\gamma=1}^{D_d} \left( \left( 0, Z^\gamma, 0, \tilde{\mathcal{F}}Z_{(2)}^\gamma \equiv \tilde{\mathcal{F}}Z_{(2)}^{(2,3)} \right) \right. \\
& \bigoplus_{\beta=1}^{D_d} \left( \left( 0, 0, Z^\beta, 0 \right) \equiv \tilde{\mathcal{F}}Z_{(3)}^{(2,3)} \right) \oplus \left( \left( 0, 0, 0, Z \right) \equiv \mathcal{Z}^{(2,3)} \right) \\
& \bigoplus_{\beta=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(2,3)} \right) \right. \\
& \bigoplus_{b=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(2,3)} \right) \right) \\
& \bigoplus_{\gamma=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(2,3)} \right) \right) \\
& \bigoplus_{\gamma=1}^{d} \left( \left( YJ_{bc}, YJ_{bc}, YJ_{bc}, \tilde{\mathcal{F}}J_{bc} \equiv \tilde{\mathcal{F}}J_{bc}^{(2,3)} \right) \right)
\end{align*} \]

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\[ \bigoplus_{b < c = 0} \left\{ \left( (Y_{J_{bc}}, Y_{J_{bc}}, Y_{J_{bc}}), \mathcal{F} J_{bc} \right) \right\}, \]

\[ \text{pr}^*_{1,3} \mathcal{I} = \bigoplus_{a=1}^d \left( \left( (Y Q_a, Y Q_a, Y Q_a), \mathcal{F} Q_a \right) \right) \oplus \bigoplus_{a=0}^d \left( \left( (Y P_a, Y P_a, Y P_a), \mathcal{F} P_a \right) \right) \]

\[ \bigoplus_{\beta=1}^{D_{d,1}} \left\{ \left( (Z, 0, 0), \mathcal{F} Z \right) \right\} \oplus \bigoplus_{d=1}^{D_{d,1}} \left\{ \left( (0, Z, \gamma), \mathcal{F} Z \right) \right\} \]

\[ \bigoplus_{d=1}^d \left\{ \left( (Y_{J_{bc}}, Y_{J_{bc}}, Y_{J_{bc}}), \mathcal{F} J_{bc} \right) \right\} \]

\[ \text{pr}^*_{1,2} \mathcal{I} \oplus \text{pr}^*_{2,3} \mathcal{I} = \left( \text{pr}^*_{1,2} \mathcal{I}_\mathbb{R} \oplus \text{pr}^*_{2,3} \mathcal{I}_\mathbb{R} \right) / \sim, \]

by identifying the generators \( (Z^{(1,2)}, 0) \sim_R (0, Z^{(2,3)}) \),

so that

\[ \text{pr}^*_{1,2} \mathcal{I} \otimes \text{pr}^*_{2,3} \mathcal{I} = \bigoplus_{a=1}^{D_{d,1}} \left( \left( \mathcal{F} Q_{\alpha}^{(1,2,3)}, \mathcal{F} Q_{\beta}^{(1,2,3)} \right) \right) \oplus \bigoplus_{a=0}^d \left( \left( \mathcal{F} P_{\alpha}^{(1,2,3)}, \mathcal{F} P_{\beta}^{(1,2,3)} \right) \right) \]

\[ \bigoplus_{\beta=1}^{D_{d,1}} \left( \left( \mathcal{F} Z_{(1,2)}^{(1,2,3)}, \mathcal{F} Z_{(3)}^{(1,2,3)} \right) \right) \oplus \bigoplus_{d=1}^{D_{d,1}} \left\{ \left( (Z^{(1,2)}, 0), \right) \right\} \]

\[ \bigoplus_{d=1}^d \left\{ \left( (\mathcal{F} J_{bc}, \mathcal{F} J_{bc}), \mathcal{F} J_{bc} \right) \right\}, \]

with the superbracket (defined as the projection of the restricted direct-sum superbracket on \( \text{pr}^*_{1,2} \mathcal{I} \otimes \mathcal{F} \mathfrak{s}(d, 1|D_{d,1}) \)) \( \text{pr}^*_{2,3} \mathcal{I} \), computed for arbitrary representatives of the equivalence classes of arguments, back to the quotient)

\[ \{ \mathcal{F} Q_{\alpha}^{(1,2,3)}, \mathcal{F} Z_{(1,2,3)} \} = \Gamma_{\alpha,\beta} \mathcal{F} P_{\alpha}^{(1,2,3)}, \]

\[ [ \mathcal{F} Q_{\alpha}^{(1,2,3)}, \mathcal{F} P_{\alpha}^{(1,2,3)} ] = 0, \]

\[ [ \mathcal{F} Z_{(1,2,3)}, \mathcal{F} Q_{\alpha}^{(1,2,3)} ] = 0, \]

Comparing the above with the superbracket of \( \text{pr}^*_{2,3} \mathcal{I} \), we infer the existence of a Lie-superalgebra isomorphism

\[ \mu_{\mathcal{I}} : \text{pr}^*_{1,2} \mathcal{I} \otimes \text{pr}^*_{2,3} \mathcal{I} \xrightarrow{2} \text{pr}^*_{1,3} \mathcal{I} \]

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given by the unique linear extension of the assignment
\[
\left( \mathcal{Z}^{(1,2;2,3)}_a, \mathcal{F}_a^{(1,2;2,3)}, \mathcal{Z}'^{(1,2;2,3)}_a, \mathcal{Z}^{(3,2)}_a, \mathcal{Z}^{(1,2;2,3)}_a, \mathcal{Z}'^{(1,2;2,3)}_a, \mathcal{F}_{bc}^{(1,2;2,3)} \right)
\]
\[
\rightarrow \left( \mathcal{Z}^{(1,3)}_a, \mathcal{F}_a^{(1,3)}, \mathcal{Z}'^{(1,3)}_a, \mathcal{Z}^{(3,2)}_a, \mathcal{Z}^{(1,3)}_a, \mathcal{Z}'^{(1,3)}_a, \mathcal{F}_{bc}^{(1,3)} \right).
\]
This is the Lie-superalgebraic counterpart of the groupoid structure \( \mu_{\mathcal{Z}} \), its triviality being reflected in the identity
\[
\mu_{\mathcal{Z}}(\mathcal{Z}^{1,2;2,3}) = \mathcal{Z}^{1,3},
\]
which we encode in the same notation:
\[
\mu_{\mathcal{Z}} \equiv 1
\]
as the one used for the trivial \( \mu_{\mathcal{Z}} \).

By the end of the long day, we conclude that

**Theorem 1.** The GS super-1-gerbe over \( s\text{Min}(d,1|D_{d,1}) \) canonically lifts \((\text{Spin}(d,1))-\text{equivariantly}\) to a CaE super-1-gerbe over \( s\text{ISO}(d,1|D_{d,1}) \).

The resulting CaE super-1-gerbe
\[
\mathcal{G}^{(1)}_{\text{GS}} := (Ys\text{ISO}(d,1|D_{d,1}), \pi_{Ys\text{ISO}(d,1|D_{d,1})}, Y, \pi_{\mathcal{Z}}, A_{\mathcal{Z}}, \mu_{\mathcal{Z}}) \equiv \pi^* \mathcal{G}^{(1)}_{\text{GS}},
\]
a distinguished \((\text{Spin}(d,1))-\text{equivariant}\) pullback of \( \mathcal{G}^{(1)}_{\text{GS}} \), shall be called the **lifted Green–Schwarz super-1-gerbe over** \( s\text{ISO}(d,1|D_{d,1}) \). It constitutes the point of departure of a full-fledged ‘gerbification’ of the GS super-\( \sigma \)-model in the purely topological HP formulation that we shall carry out in what follows. Its first step consists in extending \( \mathcal{G}^{(1)}_{\text{GS}} \) by the trivial CaE super-1-gerbe
\[
\mathcal{I}^{(1)}_{2\text{Vol}(t_{\text{vac}}^{(0)})} := (s\text{ISO}(d,1|D_{d,1}), \text{id}_{s\text{ISO}(d,1|D_{d,1})}, 2\text{Vol}(t_{\text{vac}}^{(0)}), s\text{ISO}(d,1|D_{d,1}) \times \mathbb{C}^*, \text{pr}_1, \text{pr}_2^* \partial_{C^*}, 1)
\]
over the supersymmetry group \( s\text{ISO}(d,1|D_{d,1}) \) associated with the LI super-2-form \( 2\text{Vol}(t_{\text{vac}}^{(0)}) \) (featuring as its curving). Above, the total space \( s\text{ISO}(d,1|D_{d,1}) \times \mathbb{C}^* \times (\theta^\alpha, x^a, z) \) of the trivial principal \( \mathbb{C}^* \)-bundle
\[
\text{pr}_1 : s\text{ISO}(d,1|D_{d,1}) \times \mathbb{C}^* \longrightarrow s\text{ISO}(d,1|D_{d,1})
\]
carries the product Lie-supergroup structure and comes equipped with the trivial principal connection super-1-form \( \text{pr}_2^* \partial_{C^*} \) with the coordinate presentation
\[
\text{pr}_2^* \partial_{C^*}(\theta, x, z) \equiv \partial_{C^*}(z) := \frac{\text{id}_z}{z},
\]
manifestly LI with respect to the said Lie-supergroup structure. The tensor product
\[
\mathcal{G}^{(1)}_{\text{GS}} \otimes \mathcal{I}^{(1)}_{2\text{Vol}(t_{\text{vac}}^{(0)})} := (Ys\text{ISO}(d,1|D_{d,1}), \pi_{Ys\text{ISO}(d,1|D_{d,1})}, Y, 2Y^* \text{Vol}(t_{\text{vac}}^{(0)}), Y, \pi_{\mathcal{Z}}, A_{\mathcal{Z}}, \mu_{\mathcal{Z}}) \equiv \mathcal{G}^{(1)}_{\text{HP}}
\]
is also a CaE super-1-gerbe, to be referred to – after Ref. [Sus15, Def.6.5], but taking into account its supersymmetry established above – the **extended Hughes–Polchinski super-1-gerbe over** \( s\text{ISO}(d,1|D_{d,1}) \). The 1-gerbe over \( \Sigma_{\text{HP}} \) with restrictions \( \mathcal{G}^{(1)}_{\text{HP}}|_{\mathcal{V}_i} \) over the components \( \mathcal{V}_i \) of that supermanifold shall be denoted as
\[
\mathcal{G}^{(1)}_{\Sigma_{\text{HP}}} := \bigsqcup_{i \in I} \mathcal{G}^{(1)}_{\text{HP}}|_{\mathcal{V}_i}.
\]
In the remainder of the present paper, we investigate at great length structural properties of its **vacuum restriction**
\[
\mathcal{G}^{(1)}_{\text{vac}} := t_{\text{vac}}^* \mathcal{G}^{(1)}_{\Sigma_{\text{HP}}} \equiv \mathcal{G}^{(1)}_{\text{vac}}|_{\Sigma_{\text{HP}}} := (Y_{\Sigma_{\text{HP}}}, \pi_{Y_{\Sigma_{\text{HP}}}}^*, Y_{\text{vac}}, \pi_{Y_{\text{vac}}}, Y_{\text{vac}}, \mathcal{F}_{\text{vac}}, \pi_{\mathcal{F}_{\text{vac}}}, A_{\mathcal{F}_{\text{vac}}}, \mu_{\mathcal{F}_{\text{vac}}} \equiv \mathcal{G}^{(1)}_{\text{vac}})
\]
with view to understanding the quantum-mechanical aspect of the vacuum of the GS super-\( \sigma \)-model and of its global and local supersymmetry, as encoded by the (super-)gerbe theory of the field theory of interest.
5. The supersymmetry of the super-1-gerbes

Prequantisable symmetries of the (super-)σ-model have specific gerbe-theoretic manifestations that ensure the existence of their consistent lift to the Hilbert space of the (super)field theory. These have been known for quite some time from the extensive study of the subject carried out in the non-$\mathbb{Z}/2\mathbb{Z}$-graded geometric category. They fall into the two classes, mentioned previously, with a fundamentally different ontological status and, accordingly, a different higher-geometric implementation, to wit, the global and the local symmetries that we discuss in sequence hereunder.

5.1. Higher global supersymmetry. We begin with global symmetries that set in correspondence inequivalent field configurations. In the non-$\mathbb{Z}/2\mathbb{Z}$-graded setting, these are represented by families of 1-gerbe 1-isomorphisms indexed by elements of the symmetry group that identify the 1-gerbe $G^{(1)}$ of the σ-model as invariant under the element-wise realisation of the group, a fact established firmly in Refs. [Sus11a, Sus11b]. More specifically, given a Lie group $G$ of those isometries of the target $M$ whose action on fields of the σ-model induced from its action

$$\lambda : G \times M \rightarrow M : (g, m) \mapsto \lambda_g(m)$$
onumber

on the target preserves the DF amplitude, we demand the existence of 1-isomorphisms

$$\Phi_g : \lambda_g^* G^{(1)} \xrightarrow{\cong} G^{(1)}, \quad g \in G.$$ 

In the case of a homogeneous space $G/K$ of a supersymmetry Lie supergroup $G$ (relative to its Lie subgroup $K$), this simple scenario requires, in general, a straightforward sheaf-theoretic adaptation that separately takes into account invariance under the element-wise action

$$[[\ell]]^{K} : |G| \rightarrow \text{Aut}_{s\text{Man}}(G/K) : g \mapsto [\ell]^K \circ (\gamma \times \text{id}_{G/K})$$

of the body Lie group $|G|$, and that under the element-wise tangential action of the Lie superalgebra $\mathfrak{g}$ of the Lie supergroup $G$,

$$d\Phi_X : \mathcal{L}_X G^{(1)} \xrightarrow{\cong} \mathcal{I}_0^{(1)}, \quad X \in \mathfrak{g}.$$ 

Here, $\mathcal{L}_{X,s\text{Man}} G^{(1)}$ is a super-1-gerbe obtained from $G^{(1)}$ by Lie-differentiating local data of the latter in the direction of the fundamental vector field $X$ for the induced action $[\ell]^K$ of $G$ on $G/K$, and $\mathcal{I}_0^{(1)}$ is the flat trivial super-1-gerbe with a null curving. In the super-minkowskian setting, we may readily put the components of the structure sheaf of the supersymmetry supergroup $s\text{ISO}(d,1|D_{d,1})$ of both parities on the same footing by using the global generators $\theta^a$ of the Graßmann-odd component of that sheaf, alongside the remaining coordinates $(x^a, \phi^{bc})$, and demand the existence of 1-isomorphisms that we write, in a self-explanatory notation\footnote{That is, in the coordinate picture, in which $[\ell]^{\text{Spin}(d,1)} \equiv [\ell]^{\text{Spin}(d,1)}((\varepsilon, y, \psi))$.} as

$$\Phi((\varepsilon, y, \psi)) : [\ell]^{\text{Spin}(d,1)}_{(\varepsilon, y, \psi)} \xrightarrow{\cong} G^{(1)}_{GS}, \quad (\varepsilon, y, \psi) \in s\text{ISO}(d,1|D_{d,1}).$$

Actually, we shall go one step further and consider, instead, the corresponding 1-isomorphisms

$$\tilde{\Phi}_{(\varepsilon, y, \psi)} : \ell_{(\varepsilon, y, \psi)}^{s\text{ISO}(d,1|D_{d,1})} \xrightarrow{\cong} G^{(1)}_{GS}, \quad (\varepsilon, y, \psi) \in s\text{ISO}(d,1|D_{d,1}).$$

for its lift. In so doing, we get a chance to appreciate the structural merits of the geometrisation scheme adopted in which the implementation of the global supersymmetry is seen to essentially trivialise. Thus, we take as the surjective submersions of the pullback super-1-gerbe $\ell_{(\varepsilon, y, \psi)}^{s\text{ISO}(d,1|D_{d,1})}$ the same very one as for $G^{(1)}_{GS}$ — that this makes sense follows from the identity

$$\pi_{s\text{ISO}(d,1|D_{d,1})} \circ \ell_{(\varepsilon, y, \psi)} = \ell_{(\varepsilon, y, \psi)} \circ \pi_{(\varepsilon, y, \psi)} Y_{s\text{ISO}(d,1|D_{d,1})}, \quad \pi_{(\varepsilon, y, \psi)} Y_{s\text{ISO}(d,1|D_{d,1})} \equiv \pi_{s\text{ISO}(d,1|D_{d,1})};$$

written for $\ell_{(\varepsilon, y, \psi)} \equiv Y\ell_{(\varepsilon, y, 0, \psi)} \equiv Y\ell((\varepsilon, y, 0, \psi), \cdot)$ and ensured by the equivariance of $\pi_{s\text{ISO}(d,1|D_{d,1})}$. For this choice of the surjective submersion, we find

$$\tilde{\Phi}_{(\varepsilon, y, \psi)} Y_{(2)} = Y_{(2)}.$$
and so we infer that $\gamma \beta$ is the curving of the pullback super-1-gerbe. Continuing along these lines, we take $\tilde{\mathcal{F}}$ as the principal $\mathbb{C}^*$-bundle of the pullback super-1-gerbe, a choice legitimised by the identity

$$\pi_{\mathcal{F}} \circ \tilde{\mathcal{F}}^{[2]} \equiv \mathcal{F}^{[2]}(\varepsilon, y, 0) \equiv \mathcal{F}^{[2]}(\varepsilon, y, 1),$$

in which $\tilde{\mathcal{F}}^{[2]}(\varepsilon, y, 0, \psi, 1) \equiv \mathcal{F}^{[2]}(\varepsilon, y, 0, \psi, 1)$. The left-invariance of $\mathcal{A}_{\mathcal{F}}^{(1)}$:

$$\mathcal{A}^{(1)}_{\mathcal{F}} = \mathcal{A}_{\mathcal{F}}^{(1)},$$

now permits us to take $\mathcal{A}_{\mathcal{F}}^{(1)}$ as the principal $\mathbb{C}^*$-connection on the pullback principal $\mathbb{C}^*$-bundle. The construction is consistently completed by taking

$$\tilde{\mathcal{F}}^{[3]}(\varepsilon, y, \psi) \equiv \mu_{\mathcal{F}}$$

as the groupoid structure of the pullback super-1-gerbe (for an obvious definition of $\tilde{\mathcal{F}}^{[3]}(\varepsilon, y, \psi)$). Altogether, then, we obtain

$$\tilde{\ell}^{(1)}(\varepsilon, y, \psi) \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)} = \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)},$$

whence also

$$\tilde{\Phi}(\varepsilon, y, \psi) \equiv \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)}.$$

Given the nature of the trivial correction $\tilde{\mathcal{F}}^{(1)}_{\text{2Vol}(\varepsilon_{\text{vac}})}$, we ultimately obtain 1-isomorphisms

$$\tilde{\Phi}(\varepsilon, y, \psi) \equiv \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)} : \tilde{\ell}^{(1)}(\varepsilon, y, \psi) \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)} \rightarrow \tilde{\mathcal{F}}_{\mathcal{GS}}^{(1)}, \quad (\varepsilon, y, \psi) \in \text{so}(d, 1|D_d, 1).$$

This is the anticipated higher-geometric realisation of the global supersymmetry of the GS super-$\sigma$-model in the HP formulation.

5.2. Higher $\kappa$-symmetry & the sLieAlg-skeleton of the vacuum. Next, we pass to local symmetries that relate (gauge-)equivalent field configurations, or – according to the passive interpretation of symmetry – different (and equivalent) coordinate descriptions of a given field configuration, and signal reducibility of the set of degrees of freedom of the field theory to those charting the space of orbits of the action of the gauge group. Whenever an action $\lambda$ of a group $G$ of global symmetries of a $\sigma$-model with the target $M$ is rendered local, or gauged, the ensuing (gauged) field theory effectively describes a $\sigma$-model on the orbispace $M/G$, or actually descends to the quotient manifold if the latter exists, cp Ref. [Bun12], Sec. 8] and Ref. [GSW13], Sec. 9], which happens, e.g., when $\lambda$ is free and proper. A quantum-mechanical consistency of the descent of the $\sigma$-model to the orbispace calls for the existence of a $G$-equivariant structure on the associated 1-gerbe $\mathcal{G}^{(1)}$ (of, say, curvature $\text{curv}(\mathcal{G}^{(1)}) \in Z^3_{\text{DR}}(M)$), to arise over the nerve $N^*(\mathbb{G}M) \equiv G^* \times M$

\begin{equation}
\cdots \xrightarrow{d^{(3)}} G^* \times M \xrightarrow{d^{(2)}} G \times M \xrightarrow{d^{(1)}} M \end{equation}

of the action (Lie) groupoid $\mathbb{G}M$, i.e., a simplicial manifold with face maps (written for $x \in M$, $g, g_k \in G$, $k \in I, m \in N^*$)

$$d^{(1)}_0 (x, g) = x \equiv \text{pr}_2 (g, x), \quad d^{(1)}_1 (x) = \lambda_g (x),$$

$$d^{(m)}_0 (g_m, g_{m-1}, \ldots, g_1, x) = (g_{m-1}, g_{m-2}, \ldots, g_1, x),$$

$$d^{(m)}_m (g_m, g_{m-1}, \ldots, g_1, x) = (g_m, g_{m-1}, \ldots, g_2, \ell_{g_1} (x)),$$

$$d^{(m)}_i (g_m, g_{m-1}, \ldots, g_1, x) = (g_m, g_{m-1}, \ldots, g_{m+2-i}, g_{m+1-i}, g_{m-i}, g_{m-1-i}, \ldots, g_1, x), \quad i \in I, m \in N^*,$$

cp Ref. [GSW13]. The first component of the structure is a 1-isomorphism

$$\mathcal{Y} : d^{(1)}_0 \mathcal{G}^{(1)} \xrightarrow{\cong} d^{(1)}_0 \mathcal{G}^{(1)} \otimes \mathcal{I}^{(1)}_{\mathcal{G}M}$$

of 1-gerbes over the arrow manifold $G \times M$ of $\mathbb{G}M$, written in terms of the 2-form

$$\theta_{\mathcal{G}M} = \text{pr}_2^* \kappa_A \wedge \text{pr}_1^* \theta_A^1 - \frac{1}{2} \text{pr}_2^* (\kappa_A \cup \kappa_B) \text{pr}_1^* (\theta_B^1 \wedge \theta_B^2) \in \Omega^2 (G \times M)$$. 

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in whose definition \( \mathcal{K}_A \equiv \mathcal{K}_{\mu A} \), the \( \theta^0_1 \) are components of the \( \mathfrak{g} \)-valued Maurer–Cartan 1-form

\[
\theta_L = \theta^0_1 \otimes t_A \in \Omega^1(G) \otimes \mathfrak{g}
\]

associated with the generators \( \{t_A\}_{A \in \text{dim} \mathfrak{g}} \) of the Lie algebra \( \mathfrak{g} \), and the \( \kappa_A \) are 1-forms on \( M \) satisfying the identities

\[
\mathcal{K}_A - \text{curv}(\mathcal{G}^{(1)}) = -d\kappa_A,
\]

as required for \( G \) to be a symmetry of the \( \sigma \)-model in the first place. The second component is a 2-isomorphism

\[
\begin{array}{c}
\left( d^{(1)}_0 \circ d^{(2)}_1 \right)^* \mathcal{G}^{(1)} \\
\downarrow d^{(2)}_0 \otimes \gamma \\
\left( d^{(1)}_0 \circ d^{(2)}_1 \right)^* \mathcal{T}^{(1)} \otimes \mathcal{G}^{(1)} \\
\downarrow d^{(2)}_0 \otimes \varepsilon_L \\
\end{array}
\]

between the 1-isomorphisms over \( G^2 \times M \), satisfying, over \( G^3 \times M \), the coherence condition

\[
d^{(3)}_1 \gamma \ast (\text{id} \circ d^{(2)}_0 \otimes d^{(3)}_2) \ast \gamma = d^{(2)}_2 \gamma \ast (d^{(3)}_1 \gamma \ast \text{id} \circ d^{(2)}_0 \otimes d^{(3)}_2) \ast \gamma
\]

in which \( \ast \) and \( \bullet \) are the horizontal and vertical compositions of 1-gerbe 2-isomorphisms, respectively. Thus, altogether, a 1-gerbe with a \( G \)-equivariant structure relative to the 2-form \( \vartheta_L \) is the triple

\[
\left( \mathcal{G}^{(1)}, \mathcal{T}, \gamma \right)
\]

as defined and constrained by the conditions of coherence above. The principle of descent for \( \lambda \) such that \( M/G \) is a smooth manifold is now encoded in the equivalence

\[
\mathfrak{Grb}^\mathcal{V}(M/G) \cong \mathfrak{Grb}^\mathcal{V}(M)_\{\vartheta_L = 0\}
\]

between the bicategory \( \mathfrak{Grb}^\mathcal{V}(M/G) \) of 1-gerbes (with a connective structure) over \( M/G \) and the bicategory \( \mathfrak{Grb}^\mathcal{V}(M)_\{\vartheta_L = 0\} \) of 1-gerbes (with a connective structure) over \( M \) with a \( G \)-equivariant structure relative to \( \vartheta_L = 0 \), cp Ref. [GSW10, Thm. 5.3]. Generically, the 2-form \( \vartheta_L \) does not vanish, and then a fairly complex construction of Refs. [GSW10, Sus12, GSW13, Sus13] has to be carried out to formulate loop dynamics with the global symmetry \( G \) gauged. The construction employs a bundle \( \mathcal{P}_G \times \mathcal{M} \) associated with a principal G-bundle \( \mathcal{P}_G \) over \( \Sigma \) and endowed with the Crittenden connection induced from that on \( \mathcal{P}_G \) and with an action of the gauge group \( \Gamma(\mathcal{A}d\mathcal{P}_G) \) of global sections of the adjoint bundle \( \mathcal{A}d\mathcal{P}_G \equiv \mathcal{P}_G \times \mathcal{A}d \mathcal{G} \). It goes well beyond the classic minimal-coupling scheme. The loop dynamics descends to \( M/G \) if the latter exists as a smooth manifold, or is taken to model it otherwise. It may also happen that the 1-gerbe of the \( \sigma \)-model carries a \( G \)-equivariant structure relative to the vanishing 2-form \( \vartheta_L = 0 \), in which case the 1-gerbe, and with it the \( \sigma \)-model, directly descends to resp. models loop dynamics on the orbispace \( M/G \). This is the very special situation that we encounter below.

Bearing in mind that the existence of a gerbe-theoretic realisation of a symmetry is requisite for its quantum-mechanical consistency, we shall, now, put the \( \kappa \)-symmetry of Sec. 3.2 in the above framework. In trying to do that, though, we stumble upon a peculiarity of the symmetry that takes us out of the standard scheme. The symmetry is realised in its full and integrated form only after imposition of the Euler–Lagrange equations of the super-\( \sigma \)-model, i.e., it is a gauge symmetry of the vacuum. Therefore, when looking for a higher-geometric signature of \( \kappa \)-symmetry, we should investigate the vacuum restriction \( \mathcal{G}^{(1)}_{\text{vac}} \) of the extended HP super-1-gerbe. Taking into account the higher-geometric interpretation of gauge symmetries, we seek to establish a sISO\((d, 1|D_{d, 1})_{\text{vac}} \)-equivariant structure on the latter. The very definition of the \( \kappa \)-symmetry superdistribution (and of the limit of its weak derived flag) makes it obvious that the structure, if present, is descendable – indeed, \( \mathcal{G}^{(1)}_{\text{vac}} \) is a flat 1-gerbe.

---

\[^{17}\text{For a general supertarget } G/H_{\text{vac}} \text{ realised patchwise within } G \text{ by means of local sections of the principal } H_{\text{vac}} \text{-bundle } G \longrightarrow G/H_{\text{vac}}, \text{ there is yet another problematic peculiarity that we encounter, to wit, the symmetry seems to be well-defined (on } G \text{) only in its infinitesimal (tangential) form due to the intrinsic ambiguities of the patchwise realisation over intersections of elements of the trivialising cover of } G/H_{\text{vac}}.\]
then the correspondence \([2.2]\) in conjunction with our earlier description of the leaves of \(S^{\HP}_{\vac}\), as full orbits of \(s\ISO(d,1|D_{d,1})\) leads us to posit, as a hypothesis to be verified, the nullity of \(G^{(1)}_{\vac}\), that is the existence of a 1-isomorphism

\[
\tau : G^{(1)}_{\vac} \xrightarrow{\cong} T_0^{(1)},
\]

in which \(T_0^{(1)}\) is to be understood as (the pullback of) the unique 1-gerbe over the 0-dimensional orbispace of a leaf of the lifted vacuum foliation with respect to the action of the \(\kappa\)-symmetry group.

This hypothesis was first formulated (without a mention of the lift) in Ref. [Sus21, Rem. 7.15], cp also Ref. [Sus21, Sec. 7]. Below, we prove it directly (essentially in the de Rham cohomology) and in a more manifestly supersymmetric procedure, suggested by the track of thought delineated in Ref. [Sus21], in which we stay in the tangent sheaf of a leaf of \(S^{\HP}_{\vac}\) and exploit the Lie-superalgebra structure on its model \(\socf\).

We begin our investigation on the leaf \(D_{i,v_i} \subset S^{\HP}_{\vac} \cap V_i\) of the vacuum superdistribution \([2.12]\), embedded in the superdomain with the previously introduced local coordinates \((\theta^a, x^a, \phi^b, \phi^c)\). We have (keeping the pullbacks by \(\iota_{\vac}\) implicit to unburden the notation)

\[
\mathcal{X} \cap D_{i,v_i} = 0,
\]

and so we pass to consider the curving of the extended HP super-1-gerbe restricted to

\[
\mathcal{Y}D_{i,v_i} \equiv \text{YSISO}(d,1|D_{d,1}) \mid D_{i,v_i} \triangleright \langle (\theta^a, x^a, \xi^b, \phi^c) \rangle,
\]

whereby we find

\[
\mathcal{Y}D_{i,v_i} = \text{YSISO}(d,1|D_{d,1}) \mid D_{i,v_i} \triangleright \langle (\theta^a, x^a, \xi^b, \phi^c) \rangle,
\]

with

\[
\bar{\Delta}(\theta, x, \phi_i) = L(\phi_i)^{-1} a L(\phi_i)^{-1}_b \left( dx^a \wedge \theta \Gamma^b \wedge dx^b \wedge \theta \Gamma^a \right) + \frac{1}{2} \theta \Gamma^a \wedge \theta \Gamma^b \wedge \theta \Gamma^c
\]

However, on \(D_{i,v_i}\), where

\[
\mathcal{Y}D_{i,v_i} = \mathcal{Y}D_{i,v_i} \mid D_{i,v_i},
\]

we obtain the identity

\[
L(\phi_i)^{-1} a L(\phi_i)^{-1}_b \theta \Gamma^a \wedge \theta \Gamma^b \wedge \theta \Gamma^c = 0,
\]

and so also

\[
\bar{\Delta}(\theta, x, \phi_i) = 0,
\]

whence also (for the totally skew tensor \(\epsilon_{ab} = -\epsilon_{ba}\), with \(\epsilon_{01} = 1\))

\[
\mathcal{Y}D_{i,v_i} = \mathcal{Y}D_{i,v_i} \mid D_{i,v_i},
\]

Indeed, the identities

\[
dL(\phi_i)^{-1}_b = -j^{ac}(\theta, x, \phi_i) \eta_{c,d} L(\phi_i)^{-1}_d
\]

that obtain on \(D_{i,v_i}\) yield the desired result

\[
d(\epsilon_{ab} L(\phi_i)^{-1}_a L(\phi_i)^{-1}_b x^c x^d) - 2L(\phi_i)^{-1}_a L(\phi_i)^{-1}_b dx^a \wedge dx^b = 0.
\]

Following the standard gerbe-theoretic procedure, we erect a trivial principal \(\mathbb{C}^*\)-bundle over \(D_{i,v_i}\),

\[
\pi_{D_{i,v_i}} \equiv \pi_{1} : \mathcal{E}_{i,v_i} \equiv \mathcal{Y}D_{i,v_i} \times \mathbb{C}^* \rightarrow \mathcal{Y}D_{i,v_i},
\]

and endow it with the principal connection super-1-form \((\theta^a, x^a, \phi^c, \xi^b, z) \in \mathcal{Y}D_{i,v_i} \times \mathbb{C}^*)

\[
\mathcal{A}_{(1)}(\theta, x, \xi, \phi, z) = \frac{i d z}{z} - \theta^a d \xi^a - \epsilon_{a b} L(\phi_i)^{-1}_a L(\phi_i)^{-1}_b x^c x^d = \frac{i d z}{z} + A_{(1)}(\theta, x, \xi, \phi, z).
\]

The bundle may subsequently be pulled back to \(\mathcal{Y}D_{i,v_i} \equiv \mathcal{Y}D_{i,v_i} \times \mathcal{E}_{i,v_i} \mathcal{Y}D_{i,v_i},\) along the canonical projections \(p_n : \mathcal{Y}D_{i,v_i} \rightarrow \mathcal{Y}D_{i,v_i}, n \in \{1, 2\}\), whereupon we obtain the two (trivial) principal \(\mathbb{C}^*\)-bundles

\[
\pi_{\mathcal{Y}D_{i,v_i}} \equiv \pi_{1} : \mathcal{P}^\mathcal{E}_{i,v_i} \equiv \mathcal{Y}D_{i,v_i} \times \mathcal{E}_{i,v_i} \mathcal{Y}D_{i,v_i}, n \in \{1, 2\}
\]

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with
\[ \overline{pr}_n \equiv pr_2 : \text{pr}_n^* \mathcal{E}_{i,v_i} \to \mathcal{E}_{i,v_i} \]
and with the respective principal connection super-1-forms
\[ \overline{pr}_n^* A_{\mathcal{E}_{i,v_i}} \equiv \text{pr}_2^* A_{\mathcal{E}_{i,v_i}} . \]
Next, we tensor the second of these bundles, \( \text{pr}_2^* \mathcal{E}_{i,v_i} \), with the restriction of the principal \( \mathbb{C}^* \)-bundle \( \widetilde{\mathcal{D}} \) of the extended HP super-1-gerbe to \( \mathcal{Y}^{[2]} D_{i,v_i} \) and look for a connection-preserving principal \( \mathbb{C}^* \)-bundle isomorphism
\[ \alpha_{\mathcal{E}_{i,v_i}} : \widetilde{\mathcal{D}}|_{\mathcal{Y}^{[2]} D_{i,v_i}} \otimes \text{pr}_2^* \mathcal{E}_{i,v_i} \xrightarrow{\sim} \text{pr}_1^* \mathcal{E}_{i,v_i} . \]
Direct comparison of the base components of the respective connection super-1-forms,
\[ (\mathcal{Y}^{[2]} \pi^* a + \text{pr}_2^* A_{\mathcal{E}_{i,v_i}})((\theta, x, \xi_1, \phi_i), (\theta, x, \xi_2, \phi_i)) = \text{pr}_2^* A_{\mathcal{E}_{i,v_i}}((\theta, x, \xi_1, \phi_i), (\theta, x, \xi_2, \phi_i)) , \]
indicates that we may take the isomorphism in the trivial form, with the coordinate presentation
\[ \alpha_{\mathcal{E}_{i,v_i}}(((\theta, x, \xi_1, \phi_i), (\theta, x, \xi_2, \phi_i), 1) \otimes ((\theta, x, \xi_1, \phi_i), (\theta, x, \xi_2, \phi_i), (\theta, x, \xi_2, \phi_i, z))) = ((\theta, x, \xi_1, \phi_i), (\theta, x, \xi_2, \phi_i), (\theta, x, \xi_1, \phi_i, z)) , \]
or, symbolically,
\[ \alpha_{\mathcal{E}_{i,v_i}} = 1 . \]
This is automatically compatible with the (trivial) groupoid structure on (the fibres of) \( \widetilde{\mathcal{D}}|_{\mathcal{Y}^{[2]} D_{i,v_i}} \), and so we conclude that the quadruple
\[ \tau_{i,v_i} = (\mathcal{E}_{i,v_i}, \pi_{\mathcal{E}_{i,v_i}}, A_{\mathcal{E}_{i,v_i}}, \alpha_{\mathcal{E}_{i,v_i}}) \]
defines a trivialisation
\[ \tau_{i,v_i} : G^{(1)}_\text{vac}|_{\mathcal{D}_{i,v_i}} \xrightarrow{\sim} \mathcal{I}^{(1)}_0|_{\mathcal{D}_{i,v_i}} . \]
Combining the local trivialisations over the entire vacuum foliation gives us the sought-after global trivialisation
\[ \tau \equiv (\mathcal{E}, \pi_{\mathcal{E}}, A_{\mathcal{E}}, \alpha_{\mathcal{E}}) : G^{(1)}_\text{vac} \xrightarrow{\sim} \mathcal{I}^{(1)}_0 , \quad \tau = \bigsqcup_{i \in I} \bigsqcup_{v_i \in Y_i} \tau_{i,v_i} . \]
Of course, ultimately, we want to make statements about the extended HP super-1-gerbe descended to the physical vacuum foliation \( \Sigma^\text{HP phys vac} \). For the results of the above analysis to descend to the homogeneous space \( s\text{ISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}} \supset \Sigma^\text{HP phys vac} \), we need essentially to equip \( \tau \) with a descendable \( \text{Spin}(d,1)_{\text{vac}} \)-equivariant structure. Luckily, our construction provides us with such a structure, namely – the trivial one. Indeed, upon invoking the relation between the local sections \( \sigma^\text{vac}_j \) of Eq. (2.8) over nonempty intersections \( U_{i,j}^{\text{vac}} \equiv U_i^{\text{vac}} \cap U_j^{\text{vac}} \) of superdomains \( U_i^{\text{vac}} \) (with global coordinates \( (\theta^a, x^a, \xi_\beta, \phi^E_\beta) \) and \( U_j^{\text{vac}} \) (with global coordinates \( (\theta^a, x^a, \xi_\beta, \phi^E_\beta) \)),
\[ \sigma_j^{\text{vac}}|_{U_{i,j}^{\text{vac}}} = [p]|_{U_{i,j}^{\text{vac}}} (\sigma_i^{\text{vac}}|_{U_{i,j}^{\text{vac}}}) , \]
expressed in terms of the transition maps \( h_{ij} : O_{ij} \to \text{Spin}(d,1)_{\text{vac}} \) of the principal \( \text{Spin}(d,1)_{\text{vac}} \)-bundle \( \text{Spin}(d,1) \to \text{Spin}(d,1)/\text{Spin}(d,1)_{\text{vac}} \) (inherited by that of Eq. (2.3)), we readily verify the desired gluing property for the base components of the relevant restrictions \( A_{\mathcal{E}_{i,v_i}}^{(1)} \) and \( A_{\mathcal{E}_{j,v_j}}^{(1)} \) of the principal \( \mathbb{C}^* \)-connection \( A_{\mathcal{E}} \) on \( \mathcal{E} \),
\[ A_{j,v_j}(\theta, x, \xi, \phi_j) = A_{i,v_i}(\theta, x, \xi, \phi_i) , \]
which follows from the block-diagonal structure of the matrix \( L(h) \) for \( h \in \text{Spin}(d,1)_{\text{vac}} \) with respect to the decomposition \( \text{min}(d,1) = (P_0, P_1) \oplus (\oplus_{k=2}^d (P_k)) \). We conclude that

**Theorem 2.** The super-1-gerbe of the GS super-\( \sigma \)-model in the HP formulation descended from the extended HP super-1-gerbe to the supertarget \( s\text{ISO}(d,1|D_{d,1})/\text{Spin}(d,1)_{\text{vac}} \) trivialises upon restriction to the vacuum of the (super)field theory.
The main principle underlying the scheme of geometrisation proposed in Ref. \cite{Sus17} (and recalled in Sec. 3) is the invariance of all structures under consideration with respect to the global supersymmetry enforced through restriction of the standard constructions (of 1-gerbes, their 1- and 2-isomorphisms) due to Murray et al. to the category of Lie supergroups. There seems to be no obvious way of implementing this principle in the last construction that leads up to Theorem 2. Quite simply because there is no natural Lie-supergroup structure on the vacuum of the super-σ-model. Rather than trying to save the day, at least partially, by imposing invariance with respect to a residual global supersymmetry of the vacuum (an idea that we leave for a future study), we rectify the present situation by passing to the tangent sheaves of the various supermanifolds entering the definition of the vacuum restriction \(G^{(1)}_{\text{vac}}\) of the extended HP super-1-gerbe of the superstring and read off the Lie-superalgebraic trace of the trivialisation \(\tilde{\chi}\).

The point of departure of our analysis is the concise restatement of a faithful Lie-superalgebraic model of the composite diagram

in \(\text{sMan}\) (actually, in \(\text{sLieGrp}\)), decorated with the relevant CaE data (as well as indicators of the various supercentral extensions), in the form

\[\text{Ref. [Sus20, Prop. 5.5].}\]

\[\text{18 Cp Ref. [Sus20, Prop. 5.5].}\]
to be referred to as the \textbf{sLieAlg-skeleton} of the supermanifold diagram. Now, given the supermanifold diagram describing the trivialisation of the extended HP super-1-gerbe over the vacuum foliation,

\begin{equation}
\alpha_\epsilon \equiv 1 : \widetilde{\mathcal{D}}_{\text{vac}} \otimes \text{pr}_2^* \mathcal{E} \rightarrow \text{pr}_1^* \mathcal{E} \\
\mathbb{C}^* \rightarrow \mathcal{E}, (1) \quad \pi_\epsilon
\end{equation}

\begin{equation}
\begin{array}{ccc}
\mathcal{Y}[2]\Sigma_{\text{vac}}^{\text{HP}} & \xrightarrow{\text{pr}_1} & \mathcal{Y}\Sigma_{\text{vac}}^{\text{HP}}, \mathcal{Y}_{\text{vac}}^{(2)} \\
\text{pr}_2 & & \pi_{\text{vac}} \\
\Sigma_{\text{vac}}^{\text{HP}}, \mathcal{Y}_{\text{vac}}^{(3)} & & 1
\end{array},
\end{equation}

\begin{equation}
\sum_{\text{vac}}^{\text{HP}} \supset D_{i,v_i} \rightarrow \text{SIso}(d,1|D_{d,1}),
\end{equation}

we may enquire as to the existence of a consistent extension

\begin{equation}
0 \rightarrow \text{vac}(\mathfrak{sB}_{1/2}^{(\text{HP})}) \xrightarrow{\text{vac}} \text{SIso}(d,1|D_{d,1})
\end{equation}

of the latter, written for some sub-superspace $\mathbb{R}^{0|\Delta_{d,1}}$ of $\mathbb{R}^{0|D_{d,1}}$ (with $\Delta_{d,1} \leq D_{d,1}$) and for a Lie superalgebra $\mathcal{Y}_{\text{vac}}(\mathfrak{sB}_{1/2}^{(\text{HP})})$ to be established together with a Lie-superalgebra monomorphism $\mathcal{Y}_{\text{vac}}$ and the extension $\pi_{\text{vac}}(\mathfrak{sB}_{1/2}^{(\text{HP})})$, and such that there exists a \textbf{sLieAlg-skeleton} of Diag. [5.3] of the
in which

\[ 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Y}_{\text{vac}}(\mathfrak{sB}_{1,2}^{(\text{HP})}) \longrightarrow 0 \]

is a central extension determined by the super-2-cocycle

\[ \mathcal{Y}_{\beta_{\text{vac}}} = \mathcal{Y}_{J_{\text{vac}}} \mathcal{Y}_{\beta}^{(2)} \]

and such that the super-1-form \( \tilde{\zeta} \) on \( \mathcal{E} \) dual to the central generator given as the image of \( 1 \in \mathbb{R} \) in \( \mathcal{Y}_{\text{vac}}(\mathfrak{sB}_{1,2}^{(\text{HP})}) \) trivialises the pullback of \( \mathcal{Y}_{\beta_{\text{vac}}} \) along \( \pi_{\epsilon} \),

\[ \tilde{\delta}_{\zeta_{\epsilon}} = -\pi_{\epsilon} \mathcal{Y}_{\beta_{\text{vac}}}^{(2)} \]

and in which \( \mathcal{Y}_{J_{\text{vac}}} \) is a central extension

\[ 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{Y}_{J_{\text{vac}}} \longrightarrow \mathcal{Y}_{\text{vac}}(\mathfrak{sB}_{1,2}^{(\text{HP})}) \longrightarrow 0 \]

consistent with that of Eq. (4.4) in the sense expressed by the diagram

with the Lie-superalgebra monomorphism \( \mathcal{F}_{J_{\text{vac}}} \). A constructive positive answer to the question thus posed is laid out below.

The structure of the first extension, \( \mathcal{Y}_{\text{vac}}(\mathfrak{sB}_{1,2}^{(\text{HP})}) \), is readily read off from the commutator \([\mathcal{Y}Q_{\alpha}, \mathcal{Y}P_{\alpha}]\) of \( \mathcal{Y}_{\text{siso}(d,1|D_{d,1})} \) spanning \( \text{im}(1_{\text{vac}}^{(1)})^{T} \) within the sub-superspace \( \sum_{D_{d,1}}^{d} \text{span} \mathcal{Y}_{\alpha_{\epsilon}} \) and to the \( \mathcal{Y}P_{d,1} \) spanning the lift of \( 1_{\text{vac}}^{(6)} \), we are naturally restricted to the subspace

\[ \text{im}(1^{(1)}) \equiv \left\{ P^{(1)}_{\alpha} Z^{\alpha} \mid \alpha \in 1, D_{d,1} \right\} \subset \bigoplus_{\alpha=1}^{D_{d,1}} \langle Z^{\alpha} \rangle. \]
Denote its basis as \( \{ \hat{Z}_\alpha \} \) to postulate

\[
(5.9) \quad Y_{\text{vac}}(sB_{1,2}^{(HP)}) := \left( \bigoplus_{\alpha=1}^{D_d} (Y \hat{Q}_\alpha) \oplus (Y P_0, Y P_1) \oplus \bigoplus_{\beta=1}^{D_d} (\hat{Z}_\beta) \right) \oplus \left( (Y J_{01}) \oplus \bigoplus_{\tilde{a} \tilde{b} \epsilon \mu = 2} (Y J_{\tilde{a} \tilde{b}}) \right)
\]

with the Lie-superalgebra structure induced by the restriction of the superbracket of \( Y_{\text{iso}}(d, 1|D_d, 1) \). We confirm the self-consistency of the postulate by inspecting the brackets

\[
[Y J_{01}, (P^{(1)} Z)^\alpha] = -\frac{1}{2} \Gamma_{01}^{\alpha \beta} (P^{(1)} Z)^\beta
\]

and

\[
[Y J_{\tilde{a} \tilde{b}}, (P^{(1)} Z)^\alpha] = -\frac{1}{2} \Gamma_{\tilde{a} \tilde{b}}^{\alpha \beta} (P^{(1)} Z)^\beta.
\]

Thus, we have

\[ \Delta_{d, 1} \equiv \frac{D_d}{2} \]

and

\[
\pi_{Y_{\text{vac}}(sB_{1,2}^{(HP)})} \equiv \pi_{Y_{\text{iso}}(d, 1|D_d, 1)} |_{Y_{\text{vac}}(sB_{1,2}^{(HP)})}.
\]

On the new Lie superalgebra, we find the super-2-form

\[
\tilde{\gamma}_{\text{vac}}^{(2)} = \tilde{z}_\alpha \wedge \pi_{Y_{\text{vac}}(sB_{1,2}^{(HP)})} \tilde{q}_\beta + 2 \pi_{Y_{\text{vac}}(sB_{1,2}^{(HP)})} (p^0 \wedge p^1),
\]

written in terms of the duals \( \tilde{z}_\alpha \) of the \( \hat{Z}_\alpha \) and the duals \( \tilde{q}_\beta \) of the \( \hat{Q}_\beta \). The super-2-form satisfies the identity

\[
\delta \gamma_{\text{vac}}^{(2)} = \pi_{\gamma_{\text{vac}}} \chi \equiv 0,
\]

and so it determines a central extension \( (5.8) \) with the supervector-space structure

\[
\epsilon = \left( \bigoplus_{\alpha=1}^{D_d} (\epsilon \hat{Q}_\alpha) \oplus (\epsilon P_0, \epsilon P_1) \oplus \bigoplus_{\beta=1}^{D_d} (\epsilon \hat{Z}_\beta) \right) \oplus (\epsilon J_{01}) \oplus \bigoplus_{\tilde{a} \tilde{b} \epsilon \mu = 2} (\epsilon J_{\tilde{a} \tilde{b}})
\]

\[
\cong (Y_{\text{min}}(d, 1|D_d, 1) \oplus \mathbb{R}) \oplus \text{spin}(d, 1)_{\text{vac}}
\]

and the associated projection

\[
\pi_{\epsilon} \equiv \text{pr}_1 \oplus \text{id}_{\text{spin}(d, 1)_{\text{vac}}}: (Y_{\text{min}}(d, 1|D_d, 1) \oplus \mathbb{R}) \oplus \text{spin}(d, 1)_{\text{vac}} \rightarrow Y_{\text{min}}(d, 1|D_d, 1) \oplus \text{spin}(d, 1)_{\text{vac}},
\]

and with the superbrackets

\[
\{ \epsilon \hat{Q}_\alpha, \epsilon \hat{Q}_\beta \} = -\delta_{\alpha \beta} \epsilon P_0, \quad [\epsilon P_0, \epsilon P_1] = 2 \hat{Z}, \quad [\epsilon \hat{Q}_\alpha, \epsilon \hat{Q}_\beta] = \pi_{\alpha \beta} \epsilon \hat{Z},
\]

\[
\{ \epsilon \hat{Q}_\alpha, \epsilon \hat{Z}_\beta \} = -\delta_{\alpha \beta} \hat{Z}, \quad [\epsilon \hat{P}_0, \epsilon \hat{Z}_\alpha] = 0, \quad [\epsilon \hat{Z}_\alpha, \epsilon \hat{Z}_\beta] = 0,
\]

\[
[\epsilon J_{01}, \epsilon \hat{Q}_\alpha] = \frac{1}{2} \gamma_{01}^{\alpha \beta} \epsilon \hat{Q}_\beta, \quad [\epsilon J_{\tilde{a} \tilde{b}}, \epsilon \hat{Q}_\alpha] = \frac{1}{2} \gamma_{\tilde{a} \tilde{b}}^{\alpha \beta} \epsilon \hat{Q}_\beta,
\]

\[
[\epsilon J_{01}, \epsilon \hat{P}_0] = \delta_{1a} \epsilon P_0 + \delta_{2a} P_1, \quad [\epsilon J_{\tilde{a} \tilde{b}}, \epsilon \hat{P}_0] = 0,
\]

\[
[\epsilon J_{01}, \epsilon \hat{Z}_\beta] = -\frac{1}{2} \gamma_{01}^{a \beta} \epsilon \hat{Z}, \quad [\epsilon J_{\tilde{a} \tilde{b}}, \epsilon \hat{Z}_\alpha] = \frac{1}{2} \gamma_{\tilde{a} \tilde{b}}^{a \beta} \epsilon \hat{Z}_\alpha,
\]

\[
[\epsilon J_{01}, \epsilon \hat{Z}] = 0, \quad [\epsilon J_{\tilde{a} \tilde{b}}, \epsilon \hat{Z}] = 0.
\]

Let \( \tilde{\zeta}_\epsilon \) be the dual of \( \hat{Z} \). Clearly, it satisfies the desired relation \( (5.8) \), and so the first stage of the construction is complete.

In the next step, we form the pullback Lie superalgebra

\[
Y^{[2]}_{\text{vac}} \equiv Y^{[2]}_{\text{vac}}(sB_{1,2}^{(HP)}) \oplus \pi_{\epsilon} \hat{I}
\]
with

$$\pi_{\mathfrak{y}^{2}}|_{\tilde{\mathfrak{y}}_{\text{vac}}} \equiv \text{pr}_{1} : \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} \rightarrow \mathfrak{y}^{2}_{\text{vac}}\mathfrak{s} \mathfrak{B}_{1,2}^{(\text{HP})}, \quad \tilde{\mathfrak{y}}_{\text{vac}} \equiv \text{pr}_{2} : \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} \rightarrow \tilde{\mathfrak{y}}$$

and with the basis

$$\mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} = \bigoplus_{\alpha \in \mathfrak{y}^{(0,1)}} \left( \left( \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}}, \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}} \right) \equiv \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right) \oplus \left( \left( \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}}, \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right) \equiv \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right)$$

and the superbracket obtained from the direct-sum one on \( \mathfrak{y}^{2}_{\text{vac}}\mathfrak{s} \mathfrak{B}_{1,2}^{(\text{HP})} \oplus \tilde{\mathfrak{y}} \) through restriction. With this Lie superalgebra in hand, we may, at last, finish the construction of the \( \textsf{sLieAlg} \)-skeleton of \( \text{Diag.} \{ \tilde{\mathfrak{y}} \} \). To this end, consider the pullback Lie superalgebras

$$\text{pr}_{n}^{*} \epsilon \equiv \mathfrak{y}^{2}_{\text{vac}}\mathfrak{s} \mathfrak{B}_{1,2}^{(\text{HP})} \text{pr}_{n} \oplus \epsilon, \quad n \in \{ 1, 2 \}$$

with the respective bases

$$\text{pr}_{1}^{*} \epsilon = \bigoplus_{\alpha \in \mathfrak{y}^{(0,1)}} \left( \left( \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}}, \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right) \equiv \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right) \oplus \left( \left( \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}}, \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right) \equiv \mathfrak{y}^{2}_{\text{vac}} \mathfrak{y}^{2}_{\text{vac}} \right)$$

and superbrackets induced from the direct-sum ones, and form the ‘tensor-product’ Lie superalgebra

$$\mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} \otimes \text{pr}_{2}^{*} \epsilon \equiv \left( \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} \right) \pi_{\mathfrak{y}^{2}}|_{\tilde{\mathfrak{y}}_{\text{vac}}} \otimes \text{pr}_{1}^{*} \epsilon) / \sim_{\text{pr}},$$

based on the identification

$$\left( \mathfrak{z}_{\text{vac}}, 0 \right) \sim_{\mathbb{R}} \left( 0, \mathfrak{z}^{(2)} \right).$$

The latter is the supervector space

$$\mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}} \otimes \text{pr}_{2}^{*} \epsilon = \bigoplus_{\alpha \in \mathfrak{y}^{(0,1)}} \left( \left( \mathfrak{z}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}}, \mathfrak{z}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}} \right) \equiv \mathfrak{z}^{(2)} \right) \oplus \left( \left( \mathfrak{z}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}}, \mathfrak{z}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}} \right) \equiv \mathfrak{z}^{(2)} \right)$$

endowed with the superbracket

$$\{ \tilde{\mathfrak{y}}_{\text{vac}}, \tilde{\mathfrak{y}}_{\text{vac}} \} = \mathfrak{y}^{2}_{\text{vac}} \mathfrak{s} \mathfrak{B}_{1,2}^{(\text{HP})}, \quad [P_{0}, P_{1}] = 2Z^{\otimes}, \quad [\tilde{\mathfrak{y}}_{\text{vac}}, P_{0}^{(2)}] = \mathfrak{y}^{2}_{\text{vac}} \tilde{\mathfrak{y}}_{\text{vac}} \left( \tilde{Z}^{(1)} + \tilde{Z}^{(2)} \right),$$

$$\{ \tilde{\mathfrak{y}}_{\text{vac}}, \tilde{\mathfrak{y}}_{\text{vac}} \} = -\delta_{\mathfrak{y}^{(0,1)}} \tilde{Z}^{\otimes}, \quad [\tilde{\mathfrak{y}}_{\text{vac}}, \tilde{\mathfrak{y}}_{\text{vac}}] = 0.$$
\[
\begin{align*}
[P_\alpha, \hat{Z}_\alpha^{(m)}] &= 0, & \{\hat{Z}_\alpha^{(m)}, \hat{Z}_\beta^{(m)}\} &= 0, \\
[Q_\alpha, \hat{Z}_\alpha^s] &= 0, & [P_\alpha, \hat{Z}_\alpha^s] &= 0, & \{\hat{Z}_\alpha^{s\alpha}, \hat{Z}_\beta^s\} &= 0, & \{\hat{Z}_\alpha^s, \hat{Z}_\beta^s\} &= 0, \\
[J_{\alpha 01}, \tilde{Q}_{\alpha 01}^s] &= 1 \gamma_{01} \hat{Z}_\alpha^{s\alpha}, & [J_{\alpha \beta 01}, \tilde{Q}_{\alpha \beta 01}^s] &= 1 \gamma_{\alpha \beta} \hat{Z}_\alpha^{s\beta}, \\
[J_{\alpha 01}, P_{\alpha 01}^s] &= \delta_{1\alpha} P_{01}^s + \delta_{\beta \alpha 01} P_{01}^\beta, & [J_{\alpha \beta 01}, P_{\alpha \beta 01}^s] &= 0, \\
[J_{\alpha 01}, \hat{Z}_\alpha^{s\alpha}] &= -\frac{1}{2} \gamma_{01} \hat{Z}_\alpha^{s\beta}, & [J_{\alpha \beta 01}, \hat{Z}_\alpha^{s\alpha}] &= -\frac{1}{2} \gamma_{\alpha \beta} \hat{Z}_\alpha^{s\beta}, \\
[J_{01}, \hat{Z}_\alpha^{s\alpha}] &= 0, & [J_{01}, \hat{Z}_\alpha^s] &= 0, \\
[J_{01}, J_{\alpha \beta 01}^s] &= 0, & [J_{01}, J_{\alpha \beta 01}^s] &= \delta_{\alpha \beta} J_{01}^s - \delta_{\beta \alpha} J_{01}^\beta + \delta_{\beta \alpha} J_{01}^\beta - \delta_{\alpha \beta} J_{01}^\beta.
\end{align*}
\]

Comparison of the above with the superbracket of $\text{pr}_1^* \epsilon$ reveals the existence of the sought-after Lie-superalgebra isomorphism

\[
\alpha_\tau \equiv 1 : Y^2 \text{vac} \overset{\chi}{\otimes} \text{pr}_2^* E \xrightarrow{\pi} \text{pr}_1^* E
\]

given by the unique linear extension of the assignment

\[
(\tilde{Q}_{\alpha 01}^s, \hat{Z}_{(1)}^s, \hat{Z}_{(2)}^s, J_{\alpha 01}, J_{\alpha \beta 01}) \rightarrow (\delta \tilde{Q}_{\alpha 01}^{\beta(1)}, \delta \hat{Z}_{(1)}^{\beta(1)}, \delta \hat{Z}_{(2)}^{\beta(2)}, \delta \hat{Z}_{01}^{\beta}, \delta \hat{Z}_{01}^{\beta}, \delta \hat{Z}_{01}^{\beta}).
\]

We summarise our findings in

**Theorem 3.** The null trivialisation $\tau$ of the descended super-1-gerbe of the GS super-$\sigma$-model in the HP formulation stated in Theorem 2, admits a $\text{sLieAlg}$-skeleton.

The $\text{sLieAlg}$-skeleton determines a formal setting in which we may quite naturally address the question of existence of a $\text{sLieGrp}$-model of the vacuum, by which we mean a diagram

\[
\begin{align*}
\alpha_E &\equiv 1 : Y^2 \text{vac} \overset{\tilde{\Theta}}{\otimes} \text{pr}_2^* E \xrightarrow{\pi} \text{pr}_1^* E \\
\end{align*}
\]

We proceed with the (sub-)supermanifold description of the Lie supergroup $\text{sISO}(d, 1|D_{d, 1})$ vac $\subset \text{sISO}(d, 1|D_{d, 1})$ as the locus of the coordinate equations

\[
(1 - P^{(1)})^\alpha_\beta \theta^\beta = 0, \quad \alpha \in 1, D_{d, 1}, \quad \bar{z}^\alpha = 0, \quad \bar{a} \in 2, d,
\]

Using an adapted basis in $\text{siso}(d, 1|D_{d, 1})$ obtained through completion of the one for $\text{vac}(sB_{1, 2}^{(HP)})$ given in Eq. (2.11), we thus obtain an embedding

\[
J_{\text{vac}} : \text{sISO}(d, 1|D_{d, 1}) \text{vac} \equiv \text{sMink}(d, 1|D_{d, 1}) \text{vac} \ltimes L, S \text{ Spin}(d, 1) \text{vac} \hookrightarrow \text{sISO}(d, 1|D_{d, 1}),
\]
with \(\text{sMink}(d, 1|D_{d, 1})_{\text{vac}} \subset \text{sMink}(d, 1|D_{d, 1})\) defined by the top-line equations above. The embedding admits the explicit coordinate description

\[
J_{\text{vac}}(d, 0, 0, 0, 0, 0, 0),
\]

where the zeros correspond to the nullified coordinates on \(\ker P^{(1)}\), the \(x^\beta\) and the \(\phi^S\), respectively. The binary operation \(m_{\text{vac}}\) on the embedded Lie supergroup is inherited, through restriction, from that on \(\text{sISO}(d, 1|D_{d, 1})\),

\[
m_{\text{vac}} \equiv J_{\text{vac}}^{-1} \circ m \circ (J_{\text{vac}} \times J_{\text{vac}}) : \text{sISO}(d, 1|D_{d, 1})_{\text{vac}} \times \text{sISO}(d, 1|D_{d, 1})_{\text{vac}} \to \text{sISO}(d, 1|D_{d, 1})_{\text{vac}}
\]

and reads, in coordinates,

\[
m_{\text{vac}}((d, 0, 0, 0, 0, 0, 0), (d, 0, 0, 0, 0, 0, 0)) = (d, 0, 0, 0, 0, 0, 0),
\]

where we have used the notation \(\bar{S}(\tilde{\phi}_1)\) and \(\bar{L}(\tilde{\phi}_1)\) for the 'vacuum' blocks of the block-diagonal (in the adapted basis) matrices \(S(\tilde{\phi}_1, 0)\) and \(L(\tilde{\phi}_1, 0)\), respectively. In the next step, we take the sub-supermanifold

\[
\text{YsMink}(d, 1|D_{d, 1})_{\text{vac}} \equiv \text{sMink}(d, 1|D_{d, 1})_{\text{vac}} \times \mathbb{R}^0[D_{d, 1}],
\]

of \(\text{YsMink}(d, 1|D_{d, 1})\) with the second cartesian factor defined by the coordinate equations

\[
\xi^\alpha(1 - P^{(1)})^\alpha = 0, \quad \alpha \in 1, D_{d, 1},
\]

and use it to write the desired embedding

\[
Y_{\text{Jvac}} : \text{YsISO}(d, 1|D_{d, 1})_{\text{vac}} \equiv \text{YsMink}(d, 1|D_{d, 1})_{\text{vac}} \rtimes_{L, S, S^{-T}} \text{Spin}(d, 1)_{\text{vac}} \to \text{YsISO}(d, 1|D_{d, 1})
\]

in adapted coordinates (cp. Eq. (5.9)) as

\[
Y_{\text{Jvac}}(\tilde{\phi}^\alpha, \tilde{x}^\alpha, \tilde{\xi}^\alpha, \tilde{\phi}^S) = (\tilde{\phi}^\alpha, 0, 0, 0, 0, 0, 0).
\]

The sub-supermanifold submerges surjectively onto \(\text{sISO}(d, 1|D_{d, 1})_{\text{vac}}\),

\[
\pi_{\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}} : \text{YsISO}(d, 1|D_{d, 1})_{\text{vac}} \to \text{sISO}(d, 1|D_{d, 1})_{\text{vac}}
\]

as

\[
\pi_{\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}}(\tilde{\phi}^\alpha, \tilde{x}^\alpha, \tilde{\xi}^\alpha, \tilde{\phi}^S) = (\tilde{\phi}^\alpha, \tilde{x}^\alpha, \tilde{\phi}^S).
\]

The inherited binary operation

\[
Y_{m_{\text{vac}}} \equiv Y_{J_{\text{vac}}}^{-1} \circ Y_{m} \circ (Y_{J_{\text{vac}}} \times Y_{J_{\text{vac}}}) : \text{sISO}(d, 1|D_{d, 1})_{\text{vac}} \times \text{sISO}(d, 1|D_{d, 1})_{\text{vac}} \to \text{sISO}(d, 1|D_{d, 1})_{\text{vac}}
\]

uses the same objects \(\bar{S}\) and \(\bar{L}\) as \(m_{\text{vac}},\)

\[
Y_{m_{\text{vac}}}(\tilde{\phi}^\alpha, \tilde{x}^\alpha, \tilde{\xi}^\alpha, \tilde{\phi}^S) = (\tilde{\phi}^\alpha + \bar{S}(\tilde{\phi}_1) \tilde{\phi}^\alpha, \tilde{x}^\alpha, \tilde{\xi}^\alpha + \tilde{\phi}^S),
\]

and gives rise, as usual, to the left regular action

\[
Y_{\tilde{\ell}} \equiv Y_{m_{\text{vac}}},
\]

of the Lie supergroup \(\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}\) on itself. In the coordinate presentation of \(Y_{m_{\text{vac}}},\) we have taken into account the identity

\[
\pi_{\tilde{\alpha}} \otimes \pi_{\tilde{\alpha}} = 0
\]

and dropped the term in the coordinate expression for \(Y_{m}\) trilinear in the Graßmann-odd coordinates accordingly.

On the Lie supergroup \(\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}\), we find the standard \(Y_{\text{vac}}(s\mathbb{O}_1, 2)\)-valued LI Maurer–Cartan super-1-form

\[
\tilde{\theta}_1 = \pi_{\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}} \tilde{\phi}^\alpha \otimes YQ_{\tilde{\phi}} + \pi_{\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}} \tilde{\phi}^\alpha \otimes YP_{\tilde{\phi}} + \tilde{\xi}^\alpha \otimes \tilde{Z}^\alpha + \pi_{\text{YsISO}(d, 1|D_{d, 1})_{\text{vac}}} \tilde{\phi}^S \otimes YJ_{\tilde{\phi}}
\]

whose components along \(\text{YsMink}(d, 1|D_{d, 1})\), with the coordinate presentations (derived in a procedure analogous to the one leading to the formula for their counterparts on \(\text{sISO}(d, 1|D_{d, 1})\))

\[
\tilde{\phi}(\tilde{\phi}, \tilde{x}) = \tilde{S}(\tilde{\phi})^{-1} (\tilde{\phi} \otimes \tilde{\phi} \otimes \tilde{\phi}) = \tilde{S}(\tilde{\phi})^{-1} \tilde{\phi} \tilde{x} \tilde{\phi},
\]

\[
\tilde{x}(\tilde{\phi}, \tilde{x}) = \tilde{L}(\tilde{\phi})^{-1} (\tilde{d} \tilde{d} + \tilde{x} \tilde{d} \tilde{d}) = \tilde{L}(\tilde{\phi})^{-1} \tilde{d} \tilde{x},
\]

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\[
\tilde{\varepsilon}_{\beta}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}) = (d\tilde{\xi}_{\beta} - \tau_{\alpha\beta\gamma} \tilde{\theta}^\alpha d\tilde{x}^\gamma) \tilde{S}(\tilde{\phi})^\beta_{\alpha} = \tilde{\varepsilon}_{\beta}(\tilde{\theta}, \tilde{x}, \tilde{\xi}) \tilde{S}(\tilde{\phi})^\beta_{\alpha},
\]
enter the definition of the LI super-2-cocycle
\[
\tilde{\beta}_{(2)} = \pi_{\text{YsISO}(d,1|D_{d,1})_{\text{vac}}}^{\text{vac}} \tilde{\varphi}^\alpha \wedge \tilde{\varepsilon}_{\alpha} + 2\pi_{\text{YsISO}(d,1|D_{d,1})_{\text{vac}}}^{\text{vac}} (\tilde{\varphi}^0 \wedge \tilde{\varphi}^1)
= \pi_{\text{YsISO}(d,1|D_{d,1})_{\text{vac}}}^{\text{vac}} \text{pr}_2^* \tilde{\varphi}^\alpha \wedge \text{pr}_1^* \tilde{\varepsilon}_{\alpha} + 2\pi_{\text{YsISO}(d,1|D_{d,1})_{\text{vac}}}^{\text{vac}} \text{pr}_2^* (\tilde{\varphi}^0 \wedge \tilde{\varphi}^1),
\]
whose final (descended) form follows from the unimodularity of \(\rho_{\text{Spin}(d,1)}^{\text{vac}}\).

Following the logic of the geometrisation programme, we seek to associate with the latter a central extension
\[
(5.12) \quad 1 \longrightarrow C^* \longrightarrow E \equiv \text{YsISO}(d,1|D_{d,1})_{\text{vac}} \times C^* \xrightarrow{\pi_E \pi_{\text{pr}}^1} \text{YsISO}(d,1|D_{d,1})_{\text{vac}} \longrightarrow 1
\]
integrating the formerly obtained Lie-superalgebra extension \([5.7]\). To this end, we compute the non-LI primitive of \(\tilde{\beta}_{(2)}\),
\[
\tilde{\beta}_{(2)}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}) = d(\tilde{\varphi}^\alpha \tilde{\xi}_{\alpha} + \varepsilon_{ab} \tilde{\varphi}^a \tilde{\xi}^b),
\]
and define the principal \(C^*\)-connection super-1-form
\[
\tilde{A}_E \in \Omega^1(E)
\]
on the total space \(E \ni (\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}, \tilde{\gamma})\) of the principal \(C^*\)-bundle
\[
\pi_E : E \longrightarrow \text{YsISO}(d,1|D_{d,1})_{\text{vac}}
\]
explicitly as
\[
\tilde{A}_E(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}) = \frac{i d\tilde{\xi}}{2} - \tilde{\theta}^\alpha d\tilde{\xi}_{\alpha} - \varepsilon_{ab} \tilde{\varphi}^a d\tilde{\zeta}^b = \frac{i d\tilde{\xi}}{2} + \tilde{\Lambda}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}),
\]
determining the Lie-supergroup structure on \(E\) through imposition of the usual demand that the primitive \(-\tilde{A}_E^{(1)}\) of the pullback \(\pi_E^* \tilde{\beta}_{(2)}\) be LI. From the direct computation (carried out for \((\tilde{\varphi}^\alpha, \tilde{\varphi}^a, \tilde{\xi}_{\alpha}, \tilde{\zeta}^a, \tilde{\gamma}^a) \in \text{YsISO}(d,1|D_{d,1})_{\text{vac}}\)) of
\[
Y_{\tilde{\varepsilon}_{(\tilde{\xi}, \tilde{\varphi}, \tilde{\xi}, \tilde{\phi})}} \tilde{A}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}) = \tilde{A}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}) + d(\tilde{\xi} \tilde{S}(\tilde{\psi})^{-1} \tilde{\xi} - \varepsilon_{ab} \tilde{\varphi}^a \tilde{L}(\tilde{\psi})^{-1} \tilde{\xi}^b - \frac{1}{2} \tilde{\xi} \tilde{L}(\tilde{\psi})^{-1} \tilde{\xi})
- \frac{1}{2} \varepsilon_{ab} \tilde{S}(\tilde{\psi}) \tilde{\theta} \tilde{L}(\tilde{\psi})^{-1} \tilde{\xi},
\]
using the identities
\[
\varepsilon_{ab} \tilde{\varphi}^a \otimes \tilde{\varphi}^b = 0, \quad \varepsilon_{ab} \tilde{\varphi}^a = -\tilde{\varphi}_b,
\]
and hence leading to
\[
(Y_{\tilde{\varepsilon}_{(\tilde{\xi}, \tilde{\varphi}, \tilde{\xi}, \tilde{\phi})}} - \text{id}_{\text{YsISO}(d,1|D_{d,1})_{\text{vac}}}) \tilde{A}(\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi})
= d(\tilde{\xi} \tilde{S}(\tilde{\psi})^{-1} \tilde{\xi} - \varepsilon_{ab} \tilde{\varphi}^a \tilde{L}(\tilde{\psi})^{-1} \tilde{\xi}^b + \frac{1}{2} (\tilde{\varphi}^a + \tilde{L}(\tilde{\psi})^{-1} \tilde{\xi}) \tilde{\xi} \tilde{S}(\tilde{\psi}) \tilde{\theta}),
\]
we read off the candidate for the binary operation:
\[
\text{Em}_{\text{vac}} : E \times E \longrightarrow E
\]
in the coordinate form
\[
\text{Em}_{\text{vac}}((\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}, 1), (\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}, 1)) = (Y_{\text{Em}_{\text{vac}}})((\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}, 1), (\tilde{\theta}, \tilde{x}, \tilde{\xi}, \tilde{\phi}, \tilde{\zeta}, 1)),
\]
\[
e^{i(\tilde{\xi} \tilde{S}(\tilde{\phi})^{-1} \tilde{\xi} - \varepsilon_{ab} \tilde{\varphi}^a \tilde{L}(\tilde{\phi})^{-1} \tilde{\xi}^b + \frac{1}{2} (\tilde{\varphi}^a + \tilde{L}(\tilde{\phi})^{-1} \tilde{\xi}^b) \tilde{\xi} \tilde{S}(\tilde{\phi})) \tilde{\xi} + \tilde{\xi} \tilde{S}(\tilde{\phi}) \tilde{\theta}) \cdot (\tilde{\xi}, \tilde{\xi}).
\]
Through inspection, we readily prove
Proposition 4. The supermanifold $E$, together with the supermanifold morphism $E_{\text{vac}}$ defined above as the binary operation and the pair of supermanifold morphisms

$$E_{\text{vac}} : E \to E, \quad E_{\text{vac}} : \mathbb{R}^{0|0} \to E$$

with the coordinate presentations

$$E_{\text{vac}}(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) = \left(-\tilde{S}(\tilde{\phi})^{-1} \frac{\partial}{\partial \tilde{\phi}}, -\tilde{L}(\tilde{\phi})^{-1} \frac{\partial}{\partial \tilde{\phi}}, (\tilde{\xi} + \tilde{z} a_{\tilde{\phi}, \tilde{\phi}}(\tilde{\phi})) \tilde{S}(\tilde{\phi})^{-1} \frac{\partial}{\partial \tilde{\phi}}, e^{i\tilde{\phi} \cdot \tilde{\phi}}^{-1}\right),$$

$$E_{\text{vac}}(\bullet) = (0, 0, 0, 1)$$

as the inverse and the unit, respectively, is a Lie supergroup that centrally extends $Y_{SISO}(d,1|D_{d,1})_{\text{vac}}$ as in Eq. (6.12).

At this stage, it remains to establish the existence of the trivial Lie-supergroup isomorphism

$$\alpha_E \equiv 1 : Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \circ \text{pr}^*_2 E \to \text{pr}^*_1 E$$

over the fibred square

$$Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}} \equiv YsISO(d,1|D_{d,1})_{\text{vac}} \times sISO(d,1|D_{d,1})_{\text{vac}} \times Y_{sISO}(d,1|D_{d,1})_{\text{vac}}$$

of the surjective submersion $YsISO(d,1|D_{d,1})_{\text{vac}}$ (endowed with the Lie-supergroup structure induced from the product one on $YsISO(d,1|D_{d,1})_{\text{vac}} \times Y_{sISO}(d,1|D_{d,1})_{\text{vac}}$ through restriction) and check that it is simultaneously a connection-preserving principal $\mathbb{C}^*$-bundle isomorphism. Here, $Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}}$ is the Lie sub-supergroup of the product Lie supergroup $Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}} \times \tilde{\mathcal{F}}$ whose support is the principal $\mathbb{C}^*$-bundle

$$\pi Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \equiv \text{pr}_1 : Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \equiv Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}} \times Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \to Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}}$$

with coordinates

$$Y^{[2]} \pi^*(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) \in Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}} \times Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}}$$

and the tensor product in $Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \circ \text{pr}^*_2 E$ is defined analogously to the one on $\mathcal{F}$. Comparing the base components of the principal $\mathbb{C}^*$-connection super-1-forms of the two principal $\mathbb{C}^*$-bundles to be related by $\alpha_E \equiv 1$,

$$Y^{[2]} \pi^*(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) - \tilde{\phi}(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) = 0,$$

we conclude that the bundles are related by the connection-preserving isomorphism indicated. We merely need to check if the latter is a Lie-supergroup homomorphism. That this is, indeed, the case follows from the equality

$$e^{i\theta_{\phi}}(\tilde{\xi}, \tilde{\phi})^{-1} \theta_{\phi} \cdot \tilde{\phi}(\tilde{\phi}) = e^{i\theta_{\phi}}(\tilde{\xi}, \tilde{\phi})^{-1} \theta_{\phi} \cdot \tilde{\phi}(\tilde{\phi}) \cdot \overline{\theta}_{\phi}(\tilde{\phi}),$$

and the ‘phase’ factors in – on the one hand (the left-hand side of the equality sign) – the $\alpha_E$-image of the product of the

$$Y^{[2]} \alpha^*(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) \in Y^{[2]} sISO(d,1|D_{d,1})_{\text{vac}} \times Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}}$$

with $n \in \{1, 2\}$, and – on the other hand (the right-hand side of the equality sign) – the product of their $\alpha_E$-images

$$Y^{[2]} \alpha^*(\tilde{\phi}, \tilde{\xi}, \tilde{\zeta}) \in Y^{[2]} J_{\text{vac}}^* \tilde{\mathcal{F}} \circ \text{pr}^*_2 E.$$

Thus, altogether, we arrive at

Theorem 5. The sLieAlg-skeleton of the null trivialisation $\tau$ of the super-1-gerbe of the GS super-$\sigma$-model in the HP formulation from theorem 3. integrates to a sLieGrp-model.

Theorems 2 attest to the veracity of our expectation with regard to (the triviality of) the vacuum restriction of the extended HP super-1-gerbe over $sISO(d,1|D_{d,1})$. We conclude our study by demonstrating how this fact implies the existence of a descendent $sISO(d,1|D_{d,1})_{\text{vac}}$-equivariant structure on the restriction. In so doing, we localise our analysis over a single leaf $\mathcal{D}_{t,v_i}$ of the vacuum foliation within $\Sigma^{\text{vac}}_{\text{HP}}$, with the understanding that the mechanisms discovered in its course descend
to the physical vacuum in $\Sigma_{\text{phys vac}}^{\text{HP}}$. Moreover, we exclude the ‘hidden’ gauge-symmetry subgroup $\text{Spin}(d,1)_{\text{vac}} \subset \text{sISO}(d,1)_{\text{vac}}$ from our discussion as the latter is an artifact of the realisation of the physical supertarget $\text{sISO}(d,1)_{\text{vac}}/\text{Spin}(d,1)_{\text{vac}}$ in the mother Lie supergroup $\text{sISO}(d,1)_{D(d,1)}$, with the corresponding equivariance explicitly built into the construction of the (extended) super-1-gerbe. This leaves us with the visible $\kappa$-symmetry group

$$\kappa_{\text{vis}} \equiv \text{sMink}(d,1)_{D(d,1)} \overset{j_{\text{vis}}}{\longrightarrow} \text{sISO}(d,1)_{D(d,1)}$$

of the superstring in $\text{Mink}(d,1)_{D(d,1)}$ (embedded in an obvious manner in the mother supersymmetry group) as the Lie supergroup for which we are to establish an equivariant structure on $\vartheta_{\text{vac}}^{(1)} \lvert_{D_{i,u_i}}$.

The point of departure of our considerations is the action (super)groupoid

$$\kappa_{\text{vis}, i,u_i} \cong \kappa_{\text{vis}} \times D_{i,u_i} \overset{\lambda_{\text{vis}}, \theta_{\text{pr}}(\text{Inv}_{\text{vis}} \times \text{Id}_{i,u_i})}{\longrightarrow} D_{i,u_i} \equiv \kappa^0_{\text{vis}, i,u_i} ,$$

with the target map $\lambda_{\text{vis}}$ purposely turned into a left action, so that we may directly employ the construction (1.1) introduced previously. Over the arrow supermanifold of this category, we set up a pair of surjective submersions

$$\pi^{\text{F}}_\text{D}_{i,u_i} \equiv \text{pr}_1 : \pi^{\text{F}} \text{YD}_{i,u_i} \equiv \kappa_{\text{vis}, i,u_i}^1 \times \pi_{\text{vis}} \text{SO}(d)_{D(d,1)} \text{YD}_{i,u_i} \longrightarrow \kappa_{\text{vis}, i,u_i}^1 , \quad f \in \{ \lambda_{\text{vis}}, \text{pr}_2 \}$$

with

$$\tilde{f} \equiv \text{pr}_2 : f^{\ast} \text{YD}_{i,u_i} \longrightarrow \text{YD}_{i,u_i} ,$$

and demand the existence of a principal $C^\ast$-bundle

$$\mathcal{F} \longrightarrow \lambda_{\text{vis}}^\ast \text{YD}_{i,u_i} \text{pr}_1 \times \text{pr}_1 \pi_{\text{pr}}^2 \text{YD}_{i,u_i} \equiv \text{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i}$$

over the $\kappa_{\text{vis}, i,u_i}$-fibred product of the two, with a principal $C^\ast$-connection super-1-form

$$A_{\mathcal{F}} \in \Omega^1(\mathcal{F})$$

satisfying the identity

$$dA_{\mathcal{F}} = \pi^{\text{F}}_{\mathcal{F}} \left( \text{pr}_2^{\ast} \circ \tilde{\lambda}_{\text{pr}}^\ast - \text{pr}_1^{\ast} \circ \tilde{\lambda}_{\text{vis}}^\ast \right) \text{Y}\beta ,$$

and such that there exists, over

$$\lambda_{\text{vis}}^\ast \text{YD}_{i,u_i} \text{pr}_1 \times \text{pr}_1 \pi_{\text{pr}}^2 \text{YD}_{i,u_i} \lambda_{\text{vis}}^\ast \text{YD}_{i,u_i} \text{pr}_1 \times \text{pr}_1 \pi_{\text{pr}}^2 \text{YD}_{i,u_i} \equiv \text{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} ,$$

a connection-preserving isomorphism

$$\alpha_{\mathcal{F}} : \text{pr}_1^{\ast} \tilde{\lambda}_{\text{vis}}^\ast \mathcal{F} \otimes \text{pr}_3 \mathcal{F} \overset{\equiv}{\longrightarrow} \text{pr}_1^{\ast} \mathcal{F} \otimes \text{pr}_3 \mathcal{F}$$

of the principal $C^\ast$-bundles obtained by tensoring, in the manner discussed earlier, the pullbacks of the bundles

$$\pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \text{pr}_1 : \pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \mathcal{Y}_{\mathcal{F}^{\ast} \mathcal{F}} \text{D}_{i,u_i} \text{pr}_2 \times \pi_{\mathcal{F}^{\ast} \mathcal{F}} \mathcal{F} \longrightarrow \mathcal{Y}_{\mathcal{F}^{\ast} \mathcal{F}} \text{D}_{i,u_i} \equiv f^{\ast} \text{YD}_{i,u_i} \times \kappa_{\text{vis}, i,u_i}^1 \times \pi_{\text{pr}}^2 \text{YD}_{i,u_i}$$

given by

$$(5.13) \quad \pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \text{pr}_1 : \pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1 \times \pi_{\text{pr}}^2 \text{D}_{i,u_i} \text{pr}_1 \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1 \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1$$

(for $f = \lambda_{\text{vis}}$) and

$$\pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \text{pr}_1 : \pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1 \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1 \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1$$

(for $f = \text{pr}_2$, respectively, with the pullback bundles

$$\pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \text{pr}_1 : \pi_{\mathcal{F}^{\ast} \mathcal{F}} \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1 \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \text{pr}_1$$

Using Eq. (5.3), we readily find

$$\text{pr}_2^{\ast} \circ \tilde{\lambda}_{\text{vis}}^\ast \text{Y}\beta \equiv \text{d} \left[ \left( \text{pr}_1^{\ast} \circ \tilde{\lambda}_{\text{vis}}^\ast - \text{pr}_2^{\ast} \circ \tilde{\lambda}_{\text{vis}}^\ast \right) \text{A}_{\text{vis}, i,u_i} \right] ,$$

and so we postulate

$$\mathcal{F} \equiv \mathcal{Y}_{\lambda_{\text{vis}}} \text{D}_{i,u_i} \times C^\ast \equiv \left( \{( \tilde{\xi}, \tilde{\eta} ) \}, \{( \theta, x, \phi ) \}, \{( \theta - S(\phi) \tilde{\xi}, x - L(\phi) \tilde{\eta} + \frac{1}{2} \theta \Theta S(\phi) \tilde{\xi}, \xi_{\phi} \} \right) ,$$

$$\{( \tilde{\xi}, \tilde{\eta} ) \}, \{( \theta, x, \phi ) \}, \{( \theta, x, \xi_{\phi} \} \equiv \varphi$$

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Inspection of the base components of the relevant principal $\mathbb{C}^\times$-connection super-1-forms,
\[
a((\theta - S(\phi) \bar{\varepsilon}, x - L(\phi) \bar{y} + \frac{1}{2} \theta \Gamma S(\phi) \bar{\varepsilon}, \xi_1, \phi)) - A_{i,\nu_{1}}(\theta, x, \xi_{2}, \phi)
\]
\[
= \pi_{\mathcal{F}}a_{i,\nu_{1}}(\varphi_{1}) + a((\theta, x, \xi_{2}), (\theta, x, \xi_{1}))
\]
written for
\[
\varphi_{A} = (((\bar{\varepsilon}, \bar{y}), (\theta, x, \nu)), (\theta - S(\phi) \bar{\varepsilon}, x - L(\phi) \bar{y} + \frac{1}{2} \theta \Gamma S(\phi) \bar{\varepsilon}, \xi_{2A-1}, \phi)),
\]
reveals that we may take $\alpha_{\mathcal{F}}$ in the trivial form
\[
\alpha_{\mathcal{F}} \equiv 1,
\]
automatically compatible with $\mu_{\mathcal{F}} \equiv 1$. Thus, we obtain data
\[
\Upsilon_{\mathfrak{vis}} \equiv (\Upsilon_{\lambda = 2}D_{i,\nu_{1}}, \text{id}_{\mathcal{F}}_{\lambda = 2}D_{i,\nu_{1}}, \mathcal{F}, \pi_{\mathcal{F}}, A_{\mathcal{F}}, \alpha_{\mathcal{F}})
\]
of the 1-isomorphism
\[
\Upsilon_{\mathfrak{vis}} : \lambda_{\mathfrak{vis}}^{\ast}G_{\text{vac}}^{(1)} \xrightarrow{\cong} \text{pr}_{2}^{\ast}G_{\text{vac}}^{(1)}
\]
of the (descendable) $\kappa_{\mathfrak{vis}}$-equivariant structure sought after.

In order to complete the construction, we move to
\[
\kappa_{\mathfrak{vis},i,\nu_{1}} \equiv \kappa_{\mathfrak{vis}}^{2} \times D_{i,\nu_{1}}
\]
with its face maps
\[
d_{0}^{(2)} = \text{id}_{\mathfrak{vis},i,\nu_{1}}, \quad d_{1}^{(2)} = \lambda_{\mathfrak{vis}}^{\ast} \times D_{i,\nu_{1}}, \quad d_{2}^{(2)} = \text{id}_{\mathfrak{vis},i,\nu_{1}}
\]
to $\kappa_{\mathfrak{vis},i,\nu_{1}}$, and define, over the fibre products
\[
\pi_{2}^{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}} \equiv \text{id}_{\mathfrak{vis},i,\nu_{1}}, \quad \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \equiv (\kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}, \lambda_{\mathfrak{vis}}^{\ast}DYD_{i,\nu_{1}}) \xrightarrow{\pi_{1}^{\ast}D_{i,\nu_{1}}} \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}}
\]
the respective principal $\mathbb{C}^\times$-bundles
\[
\pi_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \equiv \text{id}_{\mathfrak{vis},i,\nu_{1}}, \quad \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \equiv \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}}
\]
We identify\(^{19}\)
\[
\iota_{1} \equiv (\text{id}_{\mathfrak{vis},i,\nu_{1}}, \text{id}_{\mathfrak{vis},i,\nu_{1}}) : \quad d_{0}^{(2)} \ast \pi_{2}^{\ast}YD_{i,\nu_{1}} \xrightarrow{\mathfrak{vis}} d_{0}^{(2)} \ast \lambda_{\mathfrak{vis}}^{\ast}YD_{i,\nu_{1}},
\]
\[
\iota_{2} \equiv (\text{id}_{\mathfrak{vis},i,\nu_{1}}, \text{id}_{\mathfrak{vis},i,\nu_{1}}) : \quad d_{0}^{(2)} \ast \lambda_{\mathfrak{vis}}^{\ast}YD_{i,\nu_{1}} \xrightarrow{\mathfrak{vis}} d_{0}^{(2)} \ast \lambda_{\mathfrak{vis}}^{\ast}YD_{i,\nu_{1}},
\]
\[
\iota_{3} \equiv (\text{id}_{\mathfrak{vis},i,\nu_{1}}, \text{id}_{\mathfrak{vis},i,\nu_{1}}) : \quad d_{0}^{(2)} \ast \pi_{2}^{\ast}YD_{i,\nu_{1}} \xrightarrow{\mathfrak{vis}} d_{0}^{(2)} \ast \pi_{2}^{\ast}YD_{i,\nu_{1}},
\]
and define (in an obvious shorthand notation)
\[
\Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \equiv \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \times \lambda_{\mathfrak{vis}}^{\ast}DYD_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}, \quad d_{0}^{(2)} \ast \lambda_{\mathfrak{vis}}^{\ast}YD_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}, \quad d_{0}^{(2)} \ast \pi_{2}^{\ast}YD_{i,\nu_{1}}
\]
\[
\xrightarrow{\mathfrak{vis}} d_{0}^{(2)} \ast \lambda_{\mathfrak{vis}}^{\ast}YD_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}, \quad d_{0}^{(2)} \ast \pi_{2}^{\ast}YD_{i,\nu_{1}},
\]
in terms of which we may write
\[
\text{pr}_{1,2}^{\ast}(\Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}) \equiv \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}, \quad \text{pr}_{1,2}^{\ast}(\Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}) \equiv \Upsilon_{\lambda_{\mathfrak{vis},2}D_{i,\nu_{1}}}^{\ast}D_{i,\nu_{1}} \times \kappa_{\mathfrak{vis},i,\nu_{1}}^{d_{2}^{(2)}}.
\]
\(^{19}\)Coordinate expressions for the identification mappings can be read off directly from Ref. [Bus19, Sec. 4.2].
and

\[
\left(\text{pr}_{1,3}(\circ \varepsilon_2 \circ \varepsilon_1)\right)^* (\mathcal{Y}^2_{\lambda_{\text{vis}},2;2} D_{i,v} \times_{pr^2} \mathcal{F}) \equiv \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v} \times_{\text{pr}^1} (\mathcal{Y}^2_{\lambda_{\text{vis}},2;1} D_{i,v} \times_{pr^2} \mathcal{F}) \xrightarrow{\text{pr}^1} \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}.
\]

Subsequently, we form the tensor product of the former two,

\[
\text{pr}^1_2 : \text{pr}^1_{1,2} \left(\mathcal{Y}^2_{\lambda_{\text{vis}},2;2} D_{i,v} \times_{pr^2} \mathcal{F}\right) \otimes \text{pr}^2_{1,3} \left(\mathcal{Y}^2_{\lambda_{\text{vis}},2;0} D_{i,v} \times_{pr^2} \mathcal{F}\right) \rightarrow \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v},
\]

and identify it as the principal $C^*$-bundle of the product 1-isomorphism $d_0^{(2)} \mathcal{Y}_{\text{vis}} \circ d_2^{(2)} \mathcal{Y}_{\text{vis}}$, to be related to the latter one, in which we recognise the principal $C^*$-bundle of $d_0^{(2,2)} \mathcal{Y}_{\text{vis}}$. The relation is readily established through comparison of the base components of the respective principal $C^*$-connection super-1-forms. For that purpose, write

\[
k_{1,2} := \left(\left(\tilde{\varepsilon}_1, \tilde{y}_1\right), \left(\tilde{\varepsilon}_2, \tilde{y}_2\right), \left(\theta, x, \phi\right)\right) \in \kappa_{\text{vis}, i,v}^2
\]

and consider

\[
\left(\left(k_{1,2}, \left(\left(\tilde{\varepsilon}_1, \tilde{y}_1\right), \left(\theta - S(\phi) \tilde{\varepsilon}_2, x - L(\phi) \tilde{y}_2 + \frac{1}{2} \theta \bar{T} S(\phi) \tilde{\varepsilon}_2, \phi\right)\right), \right), \right.
\]

\[
\left(\theta - S(\phi) \tilde{\varepsilon}_2, x - L(\phi) \tilde{y}_2 + \frac{1}{2} \theta \bar{T} S(\phi) \tilde{\varepsilon}_2, \tilde{y}_2, \phi\right), \right),
\]

\[
\left(k_{1,2}, \left(\left(\tilde{\varepsilon}_1, \tilde{y}_1\right), \left(\theta - S(\phi) \tilde{\varepsilon}_2, x - L(\phi) \tilde{y}_2 + \frac{1}{2} \theta \bar{T} S(\phi) \tilde{\varepsilon}_2, \phi\right)\right), \right), \right)
\]

\[
\left(\theta - S(\phi) \tilde{\varepsilon}_2, x - L(\phi) \tilde{y}_2 + \frac{1}{2} \theta \bar{T} S(\phi) \tilde{\varepsilon}_2, \phi, \xi_2, \phi\right), \right), \right.
\]

\[
\left(k_{1,2}, \left(\left(\tilde{\varepsilon}_1, \tilde{y}_1\right), \left(\theta, x, \phi\right), \left(\theta, x, \phi, \xi_3\right)\right)\right)
\]

\[
\equiv \left(\left(k_{1,2}, \varphi_1\right), \left(k_{1,2}, \varphi_2\right), \left(k_{1,2}, \varphi_3\right)\right) \in \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}.
\]

The connection super-1-forms now compare as

\[
\left(\text{pr}^1_{1,2} \circ \text{pr}^2_{1,3} \mathcal{Y}^2 \times_{pr^2} \mathcal{F}\right) \otimes \left(\text{pr}^2_{1,3} \mathcal{Y}^2 \times_{pr^2} \mathcal{F}\right) \xrightarrow{\text{pr}^1_{1,3}} \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}
\]

\[
= \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v} \times_{\text{pr}^1} (\mathcal{Y}^2_{\lambda_{\text{vis}},2;1} D_{i,v} \times_{pr^2} \mathcal{F}) \xrightarrow{\text{pr}^1_{1,3}} \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v},
\]

whence also the choice

\[
\beta := \text{pr}^*_{1,2} (\mathcal{Y}^2 \times_{pr^2} \mathcal{F}) \otimes \text{pr}^*_{1,3} (\mathcal{Y}^2 \times_{pr^2} \mathcal{F}) \xrightarrow{\text{pr}^1_{1,3}} \mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}
\]

\[
\text{trivially compatible with } \alpha_x \equiv 1. \text{ The ensuing 2-isomorphism}
\]

\[
\gamma_{\text{vis}} \equiv \left(\mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}, \text{id}_{\mathcal{Y}^3_{\lambda_{\text{vis}},22} D_{i,v}}, \beta\right) : d_0^{(2)} \mathcal{Y}_{\text{vis}} \circ d_2^{(2)} \mathcal{Y}_{\text{vis}} \xrightarrow{\approx} d_1^{(2)} \mathcal{Y}_{\text{vis}}
\]

of the $\kappa_{\text{vis}}$-equivariant structure under reconstruction is manifestly (and trivially) coherent. We summarise the results of our check in

**Theorem 5.** The vacuum restriction of the extended super-1-gerbe of the GS super-$\sigma$-model in the HP formulation from theorem 3. carries a canonical descendant $\kappa_{\text{vis}}$-structure.

The last result – a consequence of the trivialisation mechanism discussed earlier that we dissected above for the sake of illustration that may prove useful in geometrically more involved circumstances – completes our systematic investigation of the higher-geometric and -algebraic content of the super-$\sigma$-model of the superstring in sMink($d, 1|D_{d,1}$) in the purely topological HP formulation in which that content becomes particularly manifest and structured. It leaves us with a fairly complete picture of the (classical) vacuum of the theory and its global and local supersymmetries, alongside their very natural (super)gerbe-theoretic realisations with Lie-superalgebraic and -supergroup ‘skeletla’. We hope to return to the line of research drawn hereabove in the future.

6. CONCLUSIONS & OUTLOOK

In the present paper, we have associated with the classical vacuum of the super-$\sigma$-model for the superstring in the super-Minkowski spacetime (i.e., with the embedded superstring worldsheet) a higher Lie-superalgebraic object – the sLieAlg-skeleton of Theorem 3. – that models, in the category of Lie superalgebras, the tangent sheaf of the null trivialisation of the vacuum restriction of an extended super-1-gerbe geometrising, through an adaptation of the general scheme of Ref. [Mur96] to the super-geometric setting proposed in Ref. [Sus17], the topological action functional of the super-$\sigma$-model in the Hughes–Polchinski formulation consistently with the supersymmetries present. The geometrisation has been obtained, in Theorem 1., as an equivariant lift of the one originally constructed in Ref. [Sus17] over the physical supertarget sMink($d, 1|D_{d,1}$) to the full supersymmetry group sISO($d, 1|D_{d,1}$), and
the said trivialisation, postulated in Ref. [Sus21] and proven as Theorem 2., can be viewed as a higher-geometric manifestation of the nature of the vacuum, which is that of an integral leaf of an involutive superdistribution (over the extended supertarget $\mathfrak{sISO}(d,1|D_{d,1})$ of its tangential gauge supersymmetries, the latter being bracket-generated by the $\kappa$-symmetry superdistribution of Sec. 4.2 and modelled on the Lie sub-superalgebra $\mathfrak{vac}(\mathfrak{Z}_3^{\text{H},\text{E}})$ of the supersymmetry algebra $\mathfrak{siso}(d,1|D_{d,1})$ of the super-$\sigma$-model given in Eq. (2.11)). The $\mathfrak{sLieAlg}$-skeleton has been demonstrated to integrate to a higher-geometric object – the $\mathfrak{sLieGrp}$-model of Theorem 4. – in the category of Lie supergroups that acquires the interpretation of the higher gauge supersymmetry group of the vacuum. The statement of null trivialisation of the vacuum-restricted super-1-gerbe has been shown, in Theorem 5., to strengthen the statement of descendable equivariance of that super-1-gerbe with respect to the extended $\kappa$-symmetry group of the superstring $\mathfrak{sISO}(d,1|D_{d,1})_{\mathfrak{vac}} \subset \mathfrak{sISO}(d,1|D_{d,1})$ at the root of the $\mathfrak{sLieGrp}$-model, anticipated, already in Ref. [Sus13] and in a more structured form in Ref. [Sus20], on the basis of the interpretation of the latter supergroup as the gauge supersymmetry group of the vacuum, cp Refs. [GSW10, GSW13]. In the light of the long-established interpretation of the higher-geometric objects associated with the cohomological content of the (super-)$\sigma$-model as structures encoding, through the transgression mechanism of Refs. [Gaw88, Sus11a], the pre-quantisation of the (super)field theory under consideration, the findings of the present paper are to be understood as novel markers of quantum-mechanical coherence of the super-$\sigma$-model and, simultaneously, as strong and nontrivial evidence for the internal consistency of the gerbe-theoretic approach to Green–Schwarz-type super-$\sigma$-models advanced in Ref. [Sus17] and developed in the series of papers [Sus19, Sus18a, Sus18b, Sus20, Sus21] that followed. First and foremost, however, they provide us with a realisation of the goal set up in the Introduction, which consists in extracting a higher (super-)algebraic representation of the fundamental object of the super-field theory under consideration – the superstring (trajectory/current) – from the geometrisation of the background gauge field obtained through an extension of the supersymmetry algebra.

The geometrisation of the *supersymmetrically invariant* cohomological content of the super-minkowskian super-$\sigma$-model reviewed and elaborated in the present paper can be understood, in the spirit of Refs. [RCS5, Rab87], as standard geometrisation, à la Murray, of the cohomological content of a super-$\sigma$-model with an orbifold of the supermanifold $\mathfrak{sMink}(d,1|D_{d,1})$ with respect to a natural action of the discrete Kostelecký–Rabin group $\Gamma_{KR} \subset \mathfrak{sMink}(d,1|D_{d,1})$ as the supertarget, the orbifold having a highly nontrivial topology, also in the Grassmann-odd fibre. This remark disperses the illusion of triviality of the constructions considered which may arise as a consequence of the topological triviality of (the body of) the apparent supertarget $\mathfrak{sMink}(d,1|D_{d,1})$. It legitimises the present choice of the superstring supergeometry as the one in which the novel phenomena entailed by the $\mathbb{Z}/2\mathbb{Z}$-grading of the target geometry and the supersymmetry of the dynamics that takes place in it and captured by the discrepancy between the de Rham cohomology and its physically favoured supersymmetric refinement are most neatly and tractably separated from the standard ones known from the study of $\sigma$-models with topologically nontrivial targets. That said, it is only natural, and very well justified from the physical point of view, to look for analogons of the structures and mechanisms reported herein in superstring superbackgrounds with topologically nontrivial curved supertargets. The results for the family of superbackgrounds over the $\text{AdS}_p \times S^q$ obtained in Refs. [Sus18a, Sus18b] and Ref. [Sus20] provide a firm basis for such developments.

The key idea of the paper, which boils down to associating a particular diagram in the category of Lie superalgebras decorated with (and determined by) Chevalley–Eilenberg cohomological data and integrable to the corresponding diagram in the category of Lie supergroups, to the vacuum of the super-$\sigma$-model through a supersymmetrically invariant trivialisation of the super-1-gerbe that geometrises the relevant Cartan–Eilenberg super-3-cocycle upon restriction to the embedded (vacuum) superstring worldsheet, also admits an obvious generalisation to other species of BPS states encountered in superstring theory. Indeed, there are two independent sources of such structures, of a different physical status, that we may derive from the known super-$(p+2)$-cocycles that define Green–Schwarz-type super-$\sigma$-models for super-$p$-branes, to wit,

- trivialisations of the super-$p$-gerbes geometrising the super-$(p+2)$-cocycles over the embedded super-$p$-brane vacua in the Hughes–Polchinski formulation, with curvatures given by the volume super-$(p+1)$-forms of the vacua (as in the present paper, in which $p = 1$);
- arbitrary modules of the same super-$p$-gerbes arising over sub-supermanifolds within the respective supertargets endowed with actions of subgroups of the supersymmetry groups and defining (Dirichlet) boundary conditions in the super-$p$-brane super-$\sigma$-models.
The latter class naturally extends to that of supersymmetric super-p-gerbe bimodules associated with worldvolume defects in these super-σ-models. We shall study their higher Lie-superalgebraic incarnation at length in an upcoming paper.

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Appendix A. A Convention

Let \( \mathcal{M}_1, \mathcal{M}_2 \) and \( \mathcal{N} \) be supermanifolds, and let \( \varphi_n : \mathcal{M}_n \to \mathcal{N}, \ n \in \{1,2\} \) be supermanifold morphisms of which (at least) one is a surjective submersion. We then define the fibred product of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) over \( \mathcal{N} \) as the supermanifold

\[
\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2 \equiv \mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2
\]

embedded in the cartesian product \( \mathcal{M}_1 \times \mathcal{M}_2 \) (cp Ref. [Keß19, Sec. 2.4.9]) and described by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2 & \xrightarrow{\varphi_2} & \mathcal{M}_2 \\
\downarrow{\text{pr}_1} & & \downarrow{\text{pr}_2} \\
\mathcal{M}_1 & \xrightarrow{\varphi_1} & \mathcal{N}
\end{array}
\]

Its existence is ensured by Ref. [Keß19, Prop. 3.2.11].

References

[ACNY87] A. Abouelsaood, C.G. Callan, Jr., C.R. Nappi, and S.A. Yost, “Open strings in background gauge fields”, Nucl. Phys. B280 (1987), 599–624.

[AETW87] A. Achúcarro, J.M. Evans, P.K. Townsend, and D.L. Wiltshire, “Super p-branes”, Phys. Lett. B198 (1987), 441–446.

[AF05] G. Arutyunov and S. Frolov, “Superstrings on AdS_4 x CP^3 as a coset sigma-model”, JHEP 09 (2005), 129.

[Alv85] O. Alvarez, “Topological quantization and cohomology”, Commun. Math. Phys. 100 (1985), 279–309.

[Bat80] M. Batchelor, “The structure of supermanifolds”, Trans. Amer. Math. Soc. 253 (1979), 329–338.

[Bat80] M. Batchelor, “Two approaches to supermanifolds”, Trans. Amer. Math. Soc. 258 (1980), 257–270.

[BC97] J.C. Baez and A.S. Crans, “Higher-dimensional algebra VI: Lie 2-algebras”, Theor. Appl. Categor. 12 (2004), 492–528.

[BCR70] H. Bacry, P. Combe, and J.L. Richard, “Group-theoretical analysis of elementary particles in an external electromagnetic field. I. The relativistic particle in a constant and uniform field”, Nuovo Cim. A 67 (1970), 267–299.

[BG11] S. Bonanos and J. Gomis, “Infinite sequence of Poincaré group extensions: structure and dynamics”, J. Phys. A 43 (2010), 015201.

[BH11] J.C. Baez and J. Huerta, “Division algebras and supergeometry II”, Adv. Theor. Math. Phys. 15 (2011), no. 5, 1371–1410.

[BLN+97] I.A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin, and M. Tonin, “Quantum superstring”, Phys. Lett. B280 (1987), 599–624.

[BLN+97] I.A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin, and M. Tonin, “Covariant action for the superfive-brane of M theory”, Phys. Rev. Lett. 78 (1997), 4332–4334.

[BSS1] L. Brink and H. Schwach, “Quantum superspace”, Phys. Lett. B100 (1981), 310–312.

[BSS1] L. Brink and H. Schwach, “Quantum superspace”, Phys. Lett. B100 (1981), 310–312.

[BSS95] E. Bergshoeff and E. Sezgin, “Super p-brane theories and new spacetime superalgebras”, Phys. Lett. B354 (1995), 256–263.

[BST86] E. Bergshoeff, E. Sezgin, and P.K. Townsend, “Superstrings in D = 3, 4, 6, 10 curved superspaces”, Phys. Lett. B169 (1986), 191–196.

[BST87] E. Bergshoeff, E. Sezgin, and P. K. Townsend, “Supermembranes and eleven-dimensional supergravity”, Phys. Lett. B189 (1987), 75–78.

[BW84] J. Bagger and J. Wess, “Partial breaking of extended supersymmetry”, Phys. Lett. B138 (1984), 105–110.

[Cas76] R. Casalbuoni, “Relativity and Supersymmetries”, European Mathematical Society, 2011.

[Cvd96] C.G. Callan, Jr., S.R. Coleman, J. Wess, and B. Zumino, “Structure of phenomenological Lagrangians. II”, Phys. Rev. 177 (1969), 2247–2250.

[CFG+83] L. Castellani, P. Fre, F. Giani, K. Pilch, and P. van Nieuwenhuizen, “Gauging of d = 11 supergravity?”, Annals Phys. 215 (1983), 35–77.
M.K. Murray, M.B. Green and J.H. Schwarz, “Super-translations, superstrings and Chern–Simons forms”, Phys. Lett. B223 (1989), 157–164.

M.B. Green, “Super-translations, superstrings and Chern–Simons forms”, Phys. Lett. B223 (1989), 157–164.

M.B. Green and J.H. Schwarz, “Covariant description of superstrings”, Phys. Lett. B136 (1984), 367–370.

M.B. Green and J.H. Schwarz, “Properties of the covariant formulation of superstring theories”, Nucl. Phys. B243 (1984), 285–306.

K. Gawędzki, R.R. Suszek and K. Waldorf, “Global gauge anomalies in two-dimensional bosonic sigma models”, Comm. Math. Phys. 302 (2010), 513–580.

K. Gawędzki, R.R. Suszek and K. Waldorf, “Global gauge anomalies in two-dimensional bosonic sigma models”, Comm. Math. Phys. 302 (2010), 513–580.
“An Introduction to bundle gerbes”, pp. 237–260, Oxford University Press, 2010.

J. Polchinski, “Dirichlet Branes and Ramond–Ramond Charges”, Phys. Rev. Lett. 75 (1995), 4724–4727.

J. Park and S.-J. Rey, “Green–Schwarz superstring on AdS₃ × S³”, JHEP 01 (1999), 001.

J.M. Rabin, “Supersmalsfold cohomology and the Wess–Zumino term of the covariant superstring action”, Commun. Math. Phys. 108 (1987), 375–389.

J.M. Rabin and L. Crane, “Global properties of supermanifolds”, Commun. Math. Phys. 100 (1985), 141–160.

I. Runkel and R.R. Suszek, “Gerbe-holonomy for surfaces with defect networks”, Adv. Theor. Math. Phys. 13 (2009), 1137–1219.

“Affine su(2) fusion rules from gerbe 2-isomorphisms”, J. Geom. Phys. 61 (2011), 1527–1552.

J.S. Schwinger, “Chiral dynamics”, Phys. Lett. B24 (1967), 473–476.

R. Schrader, “The Maxwell Group and the Quantum Theory of Particles in Classical Homogeneous Electromagnetic Fields”, Fortsch. Phys. 20 (1972), 701–734.

W. Siegel, “Hidden local supersymmetry in the supersymmetric particle action”, Phys. Lett. B128 (1983), 397–399.

Light cone analysis of covariant superstrings”, Nucl. Phys. B236 (1984), 311–318.

“Randomizing the superstring”, Phys. Rev. D 50 (1994), 2799–2805.

A. Salam and J. A. Strathdee, “Nonlinear realizations. 1. The role of Goldstone bosons”, Phys. Rev. 184 (1969), 1750–1759.

J. Stasheff, “Differential graded Lie algebras, quas-hopf algebras and higher homotopy algebras”, Quantum Groups. Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall 1990 (P.P. Kulish, ed.), Lecture Notes in Mathematics, vol. 1510, Springer, 1992, pp. 120–137.

D. Stevenson, “Bundle 2-gerbes”, Proc. London Math. Soc. 88 (2004), 405–435.

D. Sullivan, “Infinitesimal computations in topology”, Publ. Math. IHÉS 47 (1977), 269–331.

R.R. Suszek, “Defects, dualities and the geometry of strings via gerbes. I. Dualities and state fusion through defects”, Hamb. Beitr. Math. Nr. 360 (2011) [arXiv preprint: 1101.1126 [hep-th]].

“Defects, dualities and the geometry of strings via gerbes II. Generalised geometries with a twist, the gauge anomaly and the gauge-symmetry defect”, Hamb. Beitr. Math. Nr. 361 (2011) [arXiv preprint: 1209.2343 [hep-th]].

“Gauge Defect Networks in Two-Dimensional CFT”, Symmetries and Groups in Contemporary Physics, Proceedings of The XXIXth International Colloquium on Group-Theoretical Methods in Physics, Chern Institute of Mathematics, August 20-26, 2012, Nankai Series in Pure, Applied Mathematics and Theoretical Physics, World Scientific, 2013, pp. 411–416.

“Equivariant Cartan–Eilenberg supergerbes for the Green-Schwarz superbranes I. The super-Minkowskian case”, arXiv preprint: 1706.05682 [hep-th].

“Equivariant Cartan–Eilenberg supergerbes for the Green-Schwarz superbranes III. The wrapping anomaly and the super-AdS₅ × S⁵ background”, arXiv preprint: 1808.04470 [hep-th].

“Equivariant prequantisation of the super-0-brane in AdS₂ × S² – a toy model for supergerbe theory on curved spaces”, arXiv preprint: 1810.00856 [hep-th].

“Equivariant Cartan–Eilenberg supergerbes II. Equivariance in the super-Minkowskian setting”, arXiv preprint: 1905.05255 [hep-th].

“The square root of the vacuum I. Equivariance for the k-symmetry superdistribution”, arXiv preprint: 2002.10012 [hep-th].

“Higher Supergeometry for the Super-σ-Model”, Rev. Roumaine Math. Pures Appl. LXVI (2021), in print.

S. Samuel and J. Wess, “A superfield formulation of the non-linear realization of supersymmetry and its coupling to supergravity”, Nucl. Phys. B221 (1983), 153–177.

W. Szczyrba, “A symplectic structure on the set of Einstein metrics. A canonical formalism for general relativity”, Commun. Math. Phys. 51 (1976), 163–182.

N. Tanaka, “On differential systems, graded Lie algebras and pseudo-groups”, J. Math. Kyoto Univ. 10 (1970), no. 1, 1–82.

A.M. Trautman, “Fibre bundles associated with space-time”, Rep. Math. Phys. 1 (1970), 29–62.

T. Uematsu and C.K. Zachos, “Structure of phenomenological lagrangians for broken supersymmetry”, Nucl. Phys. B201 (1982), 250–268.

D.V. Volkov and V.P. Akulov, “Possible universal neutrino interaction”, Pisma Zh. Eksp. Teor. Fiz. 16 (1972), 438–440.

Is the neutrino a Goldstone particle?”, Phys. Lett. 46B (1973), 109–110.

P. van Nieuwenhuizen, “Free graded differential superalgebras.”, “Group Theoretical Methods in Physics. Proceedings of the Xth International Colloquium held at Boğaziçi University, Istanbul, Turkey, August 23-28, 1982” (Serdaroğlu M. and E. İnönü, eds.), Lecture Notes in Physics, vol. 180, Springer, 1983, pp. 228–247.

Th.Th. Voronov, “Geometric Integration Theory on Supermanifolds”, Cambridge Scientific Publishers, 2014.

S. Weinberg, “Nonlinear realizations of chiral symmetry”, Phys. Rev. 166 (1968), 1568–1577.

P.C. West, “Automorphisms, nonlinear realizations and branes”, JHEP 02 (2000), 024.

H. Weyl, “Elektron und Gravitation. I.”, Z. Phys. 56 (1929), 330–352.

J.-G. Zhou, “Super 0-brane and GS superstring actions on AdS₂ × S²”, Nucl. Phys. B559 (1999), 92–102.

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