Abstract

I present an example in which the individuals’ preferences are strict orderings, and under the majority rule, a transitive social ordering can be obtained and thus a non-empty choice set can also be obtained. However, the individuals’ preferences in that example do not satisfy any conditions (restrictions) of which at least one is required by Inada (1969) for social preference transitivity under the majority rule. Moreover, the considered individuals’ preferences satisfy none of the conditions of value restriction (VR), extremal restriction (ER) or limited agreement (LA), some of which is required by Sen and Pattanaik (1969) for the existence of a non-empty social choice set. Therefore, the example is an exception to a number of theorems of social preference transitivity and social choice set existence under the majority rule. This observation indicates that the collection of the conditions listed by Inada (1969) is not as complete as might be supposed. This is also the case for the collection of conditions VR, ER and LA considered by Sen and Pattanaik (1969). This observation is a challenge to some necessary conditions in the current social choice theory. In addition to seeking new conditions, one possible way to deal with this challenge may be, from a theoretical prospective, to represent the identified conditions (such as the VR, ER and LA) in terms of a common mathematical tool, and then, people may find more.

Keywords: social choice theory, majority rule, social preference transitivity, social choice set, necessary condition

1 Introduction

Arrow’s Impossibility Theorem (Arrow, 1950,1951) has aroused much interest of researchers in social choice theory in seeking restrictions imposed on the individuals’ preferences so that the majority decision rule can produce a transitive social ordering or at least, a non-empty social choice set. So far there have been several conditions that are identified as necessary and/or sufficient for social preference transitivity and/or for the existence of none-empty social choice set (Arrow,1951; Black,1958; Inada,1964,1969; Ward, 1965; Sen, 1966; Sen & Pattanaik, 1969; Duggan,2016; and many others).

In this paper, I am mainly concerned with those conditions considered by Inada (1969) when it is for the social transitivity, and those considered by Sen and Pattanaik (1969) when it is for non-empty social choice set. The reason why I discuss the conditions with reference to these two published papers lies in that, Inada (1969) claimed that the collection of the listed conditions for social preference transitivity is complete, and that, Sen and Pattanaik (1969) presented several necessary and sufficient conditions for the existence of non-empty social choice set. I will demonstrate by an example, that even though all the considered conditions are violated, the majority rule may still yield a transitive social ordering and
a non-empty social choice set. This observation challenges the necessity (rather than sufficiency) aspects of some theorems in the literature.

2 Preliminary

In this section, I introduce some preliminaries related to the discussion.

2.1 Assumption and notation

The individuals preferences are assumed to be weak orderings over a finite set of alternatives. I further assume that the number of alternatives is more than 2 and that the individuals number is more than 1.

As usual, for two alternatives \(x\) and \(y\), individual \(i\)'s preference "\(x\) is at least as good as \(y\)" is denoted by \(xR_{i}y\), "\(x\) is preferred to \(y\)" is denoted by \(xP_{i}y\), and "\(x\) is indifferent to \(y\)" is denoted by \(xI_{i}y\). Similarly, \(R\), \(P\) and \(I\) are used to denote the social preferences.

Let \(N(xRy)\), \(N(xPy)\) and \(N(xIy)\) denote the numbers of individuals regarding \(xR_{i}y\), \(xP_{i}y\) and \(xI_{i}y\), respectively. The simple majority decision rule means:

- \(xRy\) if and only if \(N(xRy) \geq N(yRx)\),
- \(xPy\) if and only if \(N(xRy) > N(yRx)\),
- \(xIy\) if and only if \(N(xRy) = N(yRx)\).

For a triple of alternatives \(x, y, z\), the social transitivity means:

- If \(xRy\) and \(yRz\) then \(xRz\). Particularly,
  - If \(xPy\) and \(yPz\) then \(xPz\),
  - If \(xPy\) and \(yIz\) then \(xPz\),
  - If \(xIy\) and \(yPz\) then \(xPz\),
  - If \(xIy\) and \(yIz\) then \(xIz\).

A choice set of alternative set \(X\) is defined by a subset \(C(X)\) of \(X\) such that every element in this subset is socially at least as good as every element in \(X\) (Sen & Pattanaik,1969).

As well known, consistent decision and rational choice are fundamental in social choice theory. For a set of individual preferences (a preference profile), however, the majority decision rule may not yield a transitive social preference ordering or a non-empty social choice set (Arrow,1950,1951; Sen & Pattanaik,1969). In the following two subsections, I will list some conditions related to social preference ordering and social choice set.

2.2 Conditions listed by Inada

Inada (1969) listed the conditions which he proved that the collection is complete, that is, the conditions he listed cover all possible conditions for social transitivity under the majority rule. For the convenience of readers, I reproduce Inada's description as follows.

Inada's Description (Inada,1969): For the simple majority decision rule to yield a transitive social preference, the individuals’ preferences over any alternative triple \((x, y, z)\) should satisfy at least one of the following patterns:

(i) dichotomous preferences (number of voters is free): \(xPyPz\) do not appear as possible individual preferences;
(ii) echoic preferences (number of voters is free): if an individual has preference \( xPyPz \), then \( zPx \) is not allowed for anyone else;

(iii) antagonistic preferences (number of voters is free): if an individual has preference \( xPyPz \), then the preference of anyone else should be \( zPyPx \) or \( xIz \);

(iv) value-restricted preferences (number of voters is odd): there is one alternative and one value ("best," "worst," or "medium") such that the alternative never has that value in any concerned individual’s preference;
   (a) single-peaked preferences,
   (b) single-caved preferences,
   (c) two-group-separated preferences,

(v) taboo preferences (number of voters is odd): \( xIyIz \) does not appear, and \( x \) is "best" or \( y \) is "worst" for any individual orderings.

As sufficiency conditions, I remark that, the single-peaked restriction was first proposed by Black (1948, 1958) and later formally by Arrow (1951), the value restriction was proposed by Sen (1966), and the other restrictions in the above list were proposed by Inada (1964, 1969).

2.3 Conditions dealt with by Sen and Pattanaik

Sen and Pattanaik (1969) considered the following conditions:

(i) value restriction (VR): as aforementioned;

(ii) extremal restriction (ER): for an ordered triple \( (x, y, z) \) of alternatives, \( \exists i : x > y \land y > z \rightarrow \forall j : z > x \rightarrow z > y \land y > x \);

(iii) limited agreement (LA): in a triple of distinct alternatives there is an ordered pair \( (x, y) \) such that \( \forall i : xRiy \);

where ER combines three patterns, i.e., dichotomous preferences, echoic preferences and antagonistic preferences; and LA is a weaker version of the pattern of taboo preferences.

Sen and Pattanaik (1969) applied VR, ER and LA to several theorems regarding necessary and/or sufficient conditions for social preference transitivity and non-empty social choice set existence. To avoid possible confusion, I reproduce their descriptions in the following even with their original reference numbers remained.

**Theorem V (Sen and Pattanaik (1969))** The necessary and sufficient condition for a set of individual orderings to be in the domain of the majority-decision-SDF is that every triple of alternatives must satisfy at least one of the conditions, VR, ER, and LA.

**Theorem VIII (Sen and Pattanaik (1969))** The necessary and sufficient condition for a set of individual orderings to be in the domain of the majority-decision SDF is that every triple must satisfy value restriction.

**Theorem XI (Sen and Pattanaik (1969))** The necessary and sufficient condition for a set of individual orderings to be in the domain of majority-decision-SWF is that every triple of alternatives must satisfy extremal restriction.

**Theorem XII (Sen and Pattanaik (1969))** A necessary condition for a set of individual strict orderings to be in the domain of a majority-decision-SWF is that every triple of alternatives must satisfy value restriction, but it is not a sufficient condition.

The terms of SWF and SDF are abbreviations for "social welfare function" and "social decision function", respectively.
3 Example, observation and discussion

3.1 Example

I consider the following example.

**Example 1** Suppose that 5 voters provide their preference orderings over an alternative triple \{x, y, z\} as follows.

Voter 1: \[xP_1yP_1z,\]
Voter 2: \[xP_2yP_2z,\]
Voter 3: \[xP_3yP_3z,\]
Voter 4: \[yP_4zP_4x,\]
Voter 5: \[zP_5xP_5y.\]

Applying the majority decision rule to the aggregation of the individuals’ preferences, we obtain a transitive social ordering \(xP_1yP_1z\), and thus a non-empty social choice set \(\{x\}\). This outcome can be easily verified by

(1) According to “If more than 50 per cent of the concerned electors share the same ordering, then no matter what orderings the others hold, majority decision will yield a social ordering” (Sen & Pattanaik 1969; Sen, 1970), one can know that a transitive social ordering can be obtained in this example since the first 3 (among 5 concerned voters) voters share the same ordering.

(2) From the preference profile, one can see that

\[
\begin{align*}
N(xy) &= 4, \quad N(yx) = 1, \\
N(zy) &= 4, \quad N(zy) = 1, \\
N(zx) &= 3, \quad N(zx) = 2.
\end{align*}
\]

Therefore, the social ordering is \(xP_1yP_1z\), which indicates that the social choice set is \(\{x\}\).

3.2 Observation

It is interesting to enquire: (1) whether the conditions listed by Inada (1969) are fulfilled by the individuals’ preferences in Example 1; and (2) whether the conditions considered by Sen and Pattanaik (1969) are fulfilled by the individuals’ preferences in Example 1. The answer is NOT. The readers may be surprised at this answer, so I’d like to check them exhaustively in the following.

(1) Whether the conditions listed by Inada (1969) are satisfied by the profile in Example 1?

(i) dichotomous preferences (number of voters is free): VIOLATED, since \(xP_1yP_1z\) appears as the voter 1’s preference;

(ii) echoic preferences (number of voters is free): VIOLATED, since we have both \(xP_1yP_1z\) and \(zP_3x\);

(iii) antagonistic preferences (number of voters is free): VIOLATED, since we have both \(xP_1yP_1z\) and \(zP_3xP_3y\);

(iv) value-restricted preferences (number of voters is odd): VIOLATED, since any of the three alternatives can be ranked at any position from \{1, 2, 3\} by the last three voters;

(a) single-peaked preferences,

(b) single-caved preferences,

(c) two-group-separated preferences,

(v) taboo preferences (number of voters is odd): VIOLATED, since we have \(xP_3yP_3z, yP_4zP_4x\) and \(zP_5xP_5y\).
(2) Whether the conditions considered by Sen and Pattanaik (1969) are satisfied by the profile in Example 1?

(i) value restriction (VR): VIOLATED, as shown above;

(ii) extremal restriction (ER): VIOLATED, because for one instance, we have $xP_1yP_1z$ and $zP_4x$ but we do NOT have $zP_4yP_4x$.

(iii) limited agreement (LA): VIOLATED, since we have $xP_3yP_3z$, $yP_4zP_4x$ and $zP_5xP_5y$.

3.3 Discussion

One can observe from the subsections 3.1 and 3.2 that, under the majority rule a transitive social ordering can be obtained and thus a non-empty social choice set can also be obtained each without the fulfillment of any of the conditions listed by Inada (1969) or any of the conditions considered by Sen and Pattanaik (1969). This observation is a challenge to the necessity (rather than sufficiency) aspects of Inada's description which is reproduced in subsection 2.2, and a challenge to the necessity (rather than sufficiency) aspects of Sen and Pattanaik's theorems which are reproduced in subsection 2.3.

In other words, the description of Inada as reproduced in subsection 2.2 and the theorems of Sen and Pattanaik as reproduced in subsection 2.3 deserve further consideration. To make the collection of conditions complete, conditions other than VR, ER or LA need to be collected (if already exist) or new conditions need to be identified.

4 Concluding remarks

It is demonstrated that the majority rule may yield a transitive social ordering and a non-empty social choice set without the fulfillment of any of the conditions that are supposed necessary by two published papers. Particularly, the observation indicates that the collection of the conditions listed by Inada (1969) is not as complete as might be supposed for social preference transitivity. This is also the case for the collection of conditions VR, ER and LA considered by Sen and Pattanaik (1969) for non-empty social choice set. This paper proposes a challenge to some necessary conditions in current social choice theory. In addition to seeking new conditions, one possible way to deal with this challenge may be, from a theoretical prospective, to represent the identified conditions (such as the VR, ER and LA) in terms of a common mathematical tool, and then, people may find more. A preliminary work as such can be found in Hou (2022).

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