JULIA THEORY FOR SLICE REGULAR FUNCTIONS

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Abstract. The theory of slice regular functions is nowadays widely studied and has found elegant applications to functional calculus for quaternionic linear operators and Schur analysis. However, much less is known about their boundary behaviors. In this paper, we initiate the study of Julia theory for slice regular functions. More precisely, we establish the quaternionic versions of the Julia lemma, the Julia-Carathéodory theorem, the boundary Schwarz lemma, the Hopf lemma, and the Burns-Krantz rigidity theorem for slice regular self-mappings of the open unit ball $B \subset \mathbb{H}$ and of the right half-space $\mathbb{H}^+$. We provide an explicit example to show that in quaternionic Hopf lemma the slice derivative of a slice regular function $f$ at the boundary fixed point may not be real, in contrast to the complex version. Our result implies that the commonly believed version of the Hopf lemma turns out to be totally wrong. This new quaternionic version of the Hopf lemma also improves Osserman estimate even in the complex setting and is essentially the first significant theorem belonging to the theory of quaternions itself other than the theory of complex analysis, since the result involves the Lie brackets which reflects the non-commutative feature of quaternions.

1. Introduction

The celebrated Julia lemma [32] and the Julia-Carathéodory theorem [10, 11] for holomorphic self-mappings of the open unit disc $D \subset \mathbb{C}$ and of the right half complex plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ play an important role in the theory of hyperbolic geometry, complex dynamical systems, and composition operators; see, e.g., [12, 11, 17, 11]. These two theorems together with the boundary Schwarz lemma [31] are the most powerful tools in the theory of iterating holomorphic self-mappings, their fixed points and boundary behaviors; see, e.g., [11, 12, 17]. There are two canonical different approaches to the study of these two theorems. The usual one is the function-theoretic approach which has a strongly geometric character and depends ultimately on the Schwarz lemma and involves an asymptotic version of the Schwarz lemma, known as the Julia lemma. D. Sarason initiated the study of the Julia-Carathéodory theorem via a Hilbert space approach, which puts insight from a different perspective. In that treatment, the Julia lemma emerges as a consequence of the Cauchy-Schwarz inequality; see [39, 40] for more details.

There are many extensions for these results to higher dimensions for holomorphic functions. The Julia-Carathéodory theorem for holomorphic self-mappings on the open unit ball $B_n \subset \mathbb{C}^n$ are studied by M. Hervé [30] and by W. Rudin [38], and for holomorphic self-mappings of strongly (pseudo)convex domains and other domains...
in $\mathbb{C}^n$ by other authors, notably by M. Abate; see [1], [7], and also [3] for the most recent and complete survey on this subject; see also [6] for the bidisk version and [2] for the polydisc version. Recently a variant of the Julia-Caratheodory theorem for infinitesimal generators has been investigated in [9]. However, there are no analogous results for other classes of functions, such as regular functions in the sense of Cauchy-Fueter and slice regular functions. A great challenge may arise from extensions to the setting of quaternions due to non-commutativity.

The purpose of the present article is to generalize the Julia lemma and the Julia-Caratheodory theorem as well as the boundary Schwarz lemma to the setting of quaternions for slice regular functions of one quaternionic variable.

The theory of slice regular functions is initiated recently by Gentili and Struppa [26, 27]. It is significantly different from that of regular functions in the sense of Cauchy-Fueter and has elegant applications to the functional calculus for non-commutative operators [14] and to Schur analysis [5]. For the detailed up-to-date theory, we refer to the monographs [25, 14]. The theory of slice regular functions also centers around the non-elliptic differential operator with nonconstant coefficients, given by

$$|\text{Im}(q)|^2 \frac{\partial}{\partial x_0} + \text{Im}(q) \sum_{j=1}^3 x_j \frac{\partial}{\partial x_j},$$

where $\text{Im}(q)$ is the imaginary part of the quaternion $q = x_0 + \text{Im}(q) \in \mathbb{H}$; see [16] for more details. Furthermore, the notion of slice regularity was also extended to functions of an octonionic variable [28] and to the setting of Clifford algebras [15, 13] as well as to the setting of alternative real algebras [22].

The geometric theory of slice regular functions of one quaternionic variable has been already developed such as the Bloch-Landau theorem [19], the Bohr theorem [20], and the Landau-Toeplitz theorem [23]. Recently, the authors established the growth and distortion theorems for slice regular extensions of normalized univalent holomorphic functions using the method of finite average [36], and set up the Borel-Caratheodory theorems for slice regular functions using the method of finite average [35]. In the present article, we continue to investigate the geometric properties of slice regular functions and our starting point is the counterpart of Schwarz-Pick theorem in the setting of quaternions, which is first established in [8]; see [4] for an alternative proof.

Our main results in this article are the quaternionic versions of the Julia lemma, the Julia-Caratheodory theorem, the Hopf lemma, the Burns-Krantz rigidity theorem as well as the boundary Schwarz lemma for slice regular self-mappings. Although some results of the present paper coincide in form with those in the complex setting, they cannot be obtained directly from the original complex results using several properties of slice regular functions, except Theorem 3 and Corollary 5.

Before presenting our main results, we first recall some notations. Let $\mathbb{B}$ denote the open unit ball in the quaternions $\mathbb{H}$. For any $k > 0$ and any point $p \in \partial \mathbb{B}$, we denote

$$\mathcal{S}(p, k) = \{ q \in \mathbb{B} : |p - q|^2 < k(1 - |q|^2) \}.$$  

It is known that $\mathcal{S}(p, k)$ is an open ball internally tangent to the unit sphere $\partial \mathbb{B}$ with center $\frac{p}{1+k}$ and radius $\frac{k}{1+k}$. The boundary sphere of this ball is called an orisphere. These orispheres are crucial in the geometry theory concerning boundary behaviors.
of slice regular self-mappings of the open unit ball $\mathbb{B}$ as shown in the following quaternionic counterpart of Julia’s lemma.

**Theorem 1. (Julia)** Let $f$ be a slice regular self-mapping of the open unit ball $\mathbb{B}$ and let $\xi \in \partial \mathbb{B}$. Suppose that there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$ converging to $\xi$ as $n$ tends to $\infty$, such that the limits

$$\alpha := \lim_{n \to \infty} \frac{1 - |f(q_n)|}{1 - |q_n|}$$

and

$$\eta := \lim_{n \to \infty} f(q_n)$$

exist (finitely). Then $\alpha > 0$ and the inequality

$$\text{Re}\left( (1 - f(q)\overline{\eta})^{-*} * (1 + f(q)\overline{\eta}) \right) \geq \frac{1}{\alpha} \text{Re}\left( (1 - q\overline{\xi})^{-*} * (1 + q\overline{\xi}) \right)$$

holds throughout the open unit ball $\mathbb{B}$ and is strict except for regular Möbius transformations of $\mathbb{B}$.

In particular, the inequality (1) is equivalent to

$$\frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha \frac{|1 - q|^2}{1 - |q|^2},$$

whenever $\xi = 1$. In other words,

$$f(S(1,k)) \subseteq S(\eta, \alpha k), \quad \forall k > 0.$$

Inequality (1) is called Julia’s inequality in view of (2). It results in the quaternionic version of the Julia-Carathéodory theorem.

**Theorem 2. (Julia-Carathéodory)** Let $f$ be a slice regular self-mapping of the open unit ball $\mathbb{B}$. Then the following conditions are equivalent:

(i) The lower limit

$$\alpha := \liminf_{q \to 1} \frac{1 - |f(q)|}{1 - |q|}$$

is finite, where the limit is taken as $q$ approaches 1 unrestrictedly in $\mathbb{B}$;

(ii) $f$ has a non-tangential limit, say $\eta$, at the point 1, and the regular difference quotient

$$(q - 1)^{-*} * (f(q) - \eta)$$

has a non-tangential limit, say $f'(1)$, at the point 1;

(iii) The slice derivative $f'$ has a non-tangential limit, say $f'(1)$, at the point 1.

Moreover, under the above conditions we have

(a) $\alpha > 0$ in (i);

(b) the slice derivatives $f'(1)$ in (ii) and (iii) are the same;

(c) $f'(1) = \alpha \eta$;

(d) the quotient $\frac{1 - |f(q)|}{1 - |q|}$ has the non-tangential limit $\alpha$ at the point 1.
It is worth remarking here that \( f'(1) \) is closely related to \( \alpha \) when \( \alpha \) is finite. However, when \( \alpha = \infty \), \( f'(1) \) can be any quaternion \( \beta \) as demonstrated by the regular polynomial \( f(q) = q^n \beta /n \) with \( n > |\beta| \). Incidentally, although Theorem 1 and Theorem 2 coincide in form with those in the complex setting, they can not be obtained directly from the original complex results using several properties of slice regular functions.

The quaternionic right half-space version of the Julia-Carathéodory theorem can also be established and its proof depends ultimately on the right half-space version of the Schwarz-Pick theorem (see Sect. 4). As a direct consequence, we obtain the Burns-Krantz rigidity theorem for slice regular functions with values in the closed right half-space:

\[
\mathbb{H}^+ = \{ q \in \mathbb{H} : \text{Re}(q) \geq 0 \};
\]

see also [33, 29].

**Theorem 3. (Burns-Krantz)** If \( f : \mathbb{B} \rightarrow \mathbb{H}^+ \) is slice regular and

\[
f(q) = o(|q + 1|), \quad q \rightarrow -1,
\]

then \( f \equiv 0 \).

The classical boundary Schwarz lemma claims that

\[
\xi f(\xi)f'(\xi) \geq 1.
\]

for any holomorphic function \( f \) on \( \mathbb{D} \cup \{ \xi \} \) satisfying \( f(\mathbb{D}) \subseteq \mathbb{D} \), \( f(0) = 0 \) and \( f(\xi) \in \partial\mathbb{D} \) for some point \( \xi \in \partial\mathbb{D} \). However, its quaternionic variant has an additional item in terms of Lie brackets reflecting the non-commutativity of quaternions.

For a given element \( \xi \in \mathbb{H} \), we denote by \([\xi]\) the associated 2-sphere:

\[
[\xi] = \{ q\xi q^{-1} : q \in \mathbb{H} \setminus \{0\} \}.
\]

Recall that two quaternions belong to the same sphere if and only if they have the same modulus and the same real part.

**Theorem 4. (Schwarz)** Let \( \xi \in \partial\mathbb{B} \) and \( f \) be a slice regular function on \( \mathbb{B} \cup [\xi] \) such that \( f(\mathbb{D}) \subseteq \mathbb{B} \) and \( f(\xi) \in \partial\mathbb{B} \). Then:

(i) It holds the following sharp estimate

\[
\xi \left( f(\xi)f'(\xi) + [\xi, f(\xi)R_\xi R_\xi f(\xi)] \right) \geq \frac{2}{S + \frac{1 - |f(0)|^2}{|f(\xi) - f(0)|^2}},
\]

where

\[
S := \text{Re} \left( f'(0)(f(\xi) - f(0))^{-1}\xi(1 - f(0)\overline{f(\xi)})^{-1} \right),
\]

and

\[
R_\xi f(q) := (q - \xi)^{-*} \ast (f(q) - f(\xi)).
\]

Moreover, equality in inequality 5 holds if and only if \( f \) is of the form

\[
f(q) = \left( 1 - q(1 - qa)\overline{\eta} \right)^{-*} \ast (q\overline{\eta} - a)\overline{v} \ast \left( f(0) - q(1 - qa)\overline{\eta} \right)^{-*} \ast (q\overline{\eta} - a)\overline{v},
\]

where

\[
a \in [-1, 1), \quad v = (f(0) - f(\xi))^{-1}\xi(1 - f(\xi)\overline{f(0)}) \in \partial\mathbb{B},
\]
Let Corollary 1.

(ii) If further

\[ f^{(k)}(0) = 0, \quad \forall \ k = 0, 1, \ldots, n - 1 \]

for some \( n \in \mathbb{N} \), then

\[
\mathcal{E} \left( f(\xi)f'(\xi) + [\xi, f(\xi)R_{\xi}R_{\xi}f(\xi)] \right) \geq n + \frac{2}{\mathcal{T} + \frac{1 - |f^{(n)}(0)/n!|^2}{|f(\xi) - \xi f^{(n)}(0)/n!|^2}},
\]

where

\[
\mathcal{T} := \text{Re} \left( \frac{f^{(n+1)}(0)}{(n+1)!} \left( \xi^{-n}f(\xi) - f^{(n)}(0)/n! \right) \xi \left( 1 - f^{(n)}(0)\xi^{-n}f(\xi)/n! \right)^{-1} \right).
\]

Moreover, equality holds for the last inequality if and only if \( f \) is of the form

\[ f(q) = q^n \left( 1 - q(1 - qa\bar{\eta})^{-\ast} + (q\bar{\eta} - a)(f^{(n)}(0)v/n!)^{-\ast} \right)^{\ast} \left( f^{(n)}(0)/n! - q(1 - qa\bar{\eta})^{-\ast}(q\bar{\eta} - a)v \right), \]

where

\[ a \in [-1, 1], \quad v = \left( \xi^n f^{(n)}(0)/n! - f(\xi) \right)^{-1} \xi \left( f^{(n)}(0)/n! \right) \in \partial\mathbb{B}, \]

and

\[ \eta = (\xi^n - f(\xi)f^{(n)}(0)/n!)^{-1} \xi(\xi^n - f(\xi)f^{(n)}(0)/n!) \in \partial\mathbb{B}. \]

In particular,

\[
\mathcal{E} \left( f(\xi)f'(\xi) + [\xi, f(\xi)R_{\xi}R_{\xi}f(\xi)] \right) > n
\]

unless \( f(q) = q^nu \) for some \( u \in \partial\mathbb{B} \).

**Corollary 1.** Let \( \xi \in \partial\mathbb{B} \) and \( f \) be a slice regular function on \( \mathbb{B} \cup [\xi] \) such that \( f(\mathbb{B}) \subseteq \mathbb{B} \) and \( f(\xi) \in \partial\mathbb{B} \). Then:

(i) It holds the following sharp estimate

\[
\mathcal{E} \left( f(\xi)f'(\xi) + [\xi, f(\xi)R_{\xi}R_{\xi}f(\xi)] \right) \geq \frac{2|f(\xi) - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|^2}.
\]

Moreover, equality holds for the last inequality if and only if \( f \) is of the form

\[ f(q) = \left( 1 - q(1 - qa\bar{\eta})^{-\ast} + (q\bar{\eta} - a)f^{(n)}(0)v \right)^{\ast} \left( f^{(n)}(0)/n! - q(1 - qa\bar{\eta})^{-\ast}(q\bar{\eta} - a)v \right), \]

where

\[ a \in [-1, 0], \quad v = (f(0) - f(\xi))^{-1} \xi(1 - f(\xi)f^{(0)}) \in \partial\mathbb{B}, \]

and

\[ \eta = (1 - f(\xi)f^{(0)})^{-1} \xi(1 - f(\xi)f^{(0)}) \in \partial\mathbb{B}. \]
Corollary 2. Let $\xi \in \partial \mathbb{B}$ and $f$ be a slice regular function on $\mathbb{B} \cup \{\xi\}$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$, $f(0) = 0$ and $f(\xi) = \xi$. Then

$$f'(\xi) - [\xi, R_\xi f(\xi)] \geq \frac{2}{1 + \text{Re} f'(0)}.$$ 

Moreover, equality holds for the last inequality if and only if $f$ is of the form

$$f(q) = q^n (1 - q(1 - qa)\xi^{-n} * (\xi^{-n} - a)\xi)^{-1} \xi \left( (\xi^n - f(\xi)\xi^n(0)/n!)/n! \right) \in \partial \mathbb{B},$$

and

$$\eta = (\xi^n - f(\xi)\xi^n(0)/n!)^{-1} \xi \left( (\xi^n - f(\xi)\xi^n(0)/n!) \right) \in \partial \mathbb{B}.$$ 

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Moreover, equality holds for the last inequality if and only if $f$ is of the form

$$f(q) = q^n (1 - q(1 - qa)\xi^{-n} * (\xi^{-n} - a)\xi)^{-1} \xi \left( (\xi^n - f(\xi)\xi^n(0)/n!)/n! \right) \in \partial \mathbb{B},$$

and

$$\eta = (\xi^n - f(\xi)\xi^n(0)/n!)^{-1} \xi \left( (\xi^n - f(\xi)\xi^n(0)/n!) \right) \in \partial \mathbb{B}.$$ 

It is worth remarking here that the Lie brackets in the preceding corollary do not vanish and $f'(\xi)$ may not be a real number, in general; see Example 2 in Section 3 for more details. This means that in the setting of quaternions the commonly believed fact that

$$f'(\xi) > 1$$

may fail; see [24 Theorem 9.24]. However, the same line of the proof of Theorem 4 implies simultaneously the following theorem.

Theorem 5. Let $\xi \in \partial \mathbb{B}$ and $f$ be a slice regular function on $\mathbb{B} \cup \{\xi\}$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$, $f(0) = 0$ and $f(\xi) \in \partial \mathbb{B}$. Then

$$|f'(\xi)| \geq \frac{2}{1 + \text{Re} f'(0)}.$$ 

Moreover, equality holds for the last inequality if and only if $f$ is of the form

$$f(q) = q^n (1 - qa\xi)^{-n} * (q - a\xi)\xi$$

for some constant $a \in [-1, 1]$.

As a direct consequence of Theorem 4 we have the following special case where $\xi = 1$, which is also called Hopf’s lemma. Incidentally, it can be proved alternatively in virtue of Theorem 2.

Corollary 3. (Hopf) Let $f$ be a slice regular function on $\mathbb{B} \cup \{1\}$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$ and $f(1) = 1$. Then the following statements hold true:
Our result becomes can assume that \( \xi \) to regular function on \( D \) open unit disk in Theorem 4 is a new result. More precisely, for any holomorphic function which implies established by Osserman in \([34]\):

These two estimate improve the following estimate (also called Osserman inequality) for some constant \( a \in [-1, 1] \).

If further for some \( n \in \mathbb{N} \), then

Moreover, equality holds for the last inequality if and only if

for some constant \( a \in [-1, 1] \).

It is worth remarking here that, even in the complex setting, the result obtained in Theorem 4 is a new result. More precisely, for any holomorphic function \( f \) on the open unit disk \( \mathbb{D} \cup \{1\} \) (\( \mathbb{D} \) is transitive under the action of holomorphic functions, we can assume that \( \xi = 1 \) satisfying \( f(\mathbb{D}) \subseteq \mathbb{D} \) and \( f(1) = 1 \), it can extend regularly to \( \mathbb{B} \cup \{1\} \). We denote this unique regular extension still by \( f \). Thus \( f \) is a slice regular function on \( \mathbb{B} \cup \{1\} \) such that \( f(\mathbb{B}) \subseteq \mathbb{B} \) and \( f(1) = 1 \). The assertion that \( f(\mathbb{B}) \subseteq \mathbb{B} \) follows easily from a convex combination identity in [36]. For all such \( f \), our result becomes

\[
f'(1) \geq \frac{2}{\text{Re}\left(1 - f(0)^2 + f'(0)\right)}.
\]

which implies

\[
f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.
\]

These two estimate improve the following estimate (also called Osserman inequality) established by Osserman in [34]:

\[
f'(1) \geq \frac{2(1 - |f(0)|^2)}{1 - |f(0)|^2 + |f'(0)|}.
\]

This new estimate in (9) for holomorphic self-mappings of the open unit disk \( \mathbb{D} \), with boundary regular fixed point 1, is initially proved in [21] Theorem 3 via analytic semigroup approach and Julia-Carathéodory theorem for univalent holomorphic self-mappings of \( \mathbb{D} \), which is derived by the method of extremal length. The method presented in [21] can not be used to get the extremal functions for
which equality holds in \([13]\). The proof of this paper (see the proof of Corollary \([3]\) below for details) is very elementary, and has its extra advantage to get the extremal functions.

Moreover, notice that the Julia-Carathéodory theorem in Theorem \([3]\) holds only for the real boundary points \(\xi = \pm 1 \in \partial \mathbb{B}\). As shown by Theorem \([3]\) the relation
\[
f'(\xi) = \alpha \xi f(\xi)
\]
does no longer hold in general in the setting of quaternions under the condition that
\[
\alpha := \lim \inf_{B \ni q \to \xi} \frac{1 - |f(q)|}{1 - |q|} < +\infty
\]
in contrast to the complex setting. Consequently, the general Julia-Carathéodory theorem (the case that \(\pm 1 \neq \xi \in \partial \mathbb{B}\)) will be much more delicate and requires further research. At the same time, this phenomenon reflects fully the special role of the real axis in the theory of slice regular functions.

The outline of this paper is as follows. In Section 2, we set up basic notations and give some preliminary results. In Section 3, we give the detailed proofs of main results for slice regular self-mappings of the open unit ball \(\mathbb{B}\). To this end, we shall establish the Lindelöf principle and the Lindelöf inequality. The analogous results for slice regular self-mappings of the right half-space \(\mathbb{H}^+\) are established in Section 4, of which the starting point is the right half-space version of the Schwarz-Pick theorem.

## 2. Preliminaries

We recall in this section some preliminary definitions and results on slice regular functions. To have a more complete insight on the theory, we refer the reader to the monograph \([25]\).

Let \(\mathbb{H}\) denote the non-commutative, associative, real algebra of quaternions with standard basis \(\{1, i, j, k\}\), subject to the multiplication rules
\[
i^2 = j^2 = k^2 = ijk = -1.
\]
Every element \(q = x_0 + x_1i + x_2j + x_3k\) in \(\mathbb{H}\) is composed by the real part \(\text{Re}(q) = x_0\) and the imaginary part \(\text{Im}(q) = x_1i + x_2j + x_3k\). The conjugate of \(q \in \mathbb{H}\) is then \(\bar{q} = \text{Re}(q) - \text{Im}(q)\) and its modulus is defined by \(|q|^2 = q\bar{q} = |\text{Re}(q)|^2 + |\text{Im}(q)|^2\).

We can therefore calculate the multiplicative inverse of each \(q \neq 0\) as \(q^{-1} = |q|^{-2}q\).

Every \(q \in \mathbb{H}\) can be expressed as \(q = x + yI\), where \(x, y \in \mathbb{R}\) and
\[
I = \frac{\text{Im}(q)}{|\text{Im}(q)|}
\]
if \(\text{Im} q \neq 0\), otherwise we take \(I\) arbitrarily such that \(I^2 = -1\). Then \(I\) is an element of the unit 2-sphere of purely imaginary quaternions
\[
\mathbb{S} = \{ q \in \mathbb{H} : q^2 = -1 \}.
\]

For every \(I \in \mathbb{S}\) we will denote by \(\mathbb{C}_I\) the plane \(\mathbb{R} \oplus IR\), isomorphic to \(\mathbb{C}\), and, if \(\Omega \subseteq \mathbb{H}\), by \(\Omega_I\) the intersection \(\Omega \cap \mathbb{C}_I\). Also, for \(R > 0\), we will denote the open ball centred at the origin with radius \(R\) by
\[
B(0, R) = \{ q \in \mathbb{H} : |q| < R \}.
\]

We can now recall the definition of slice regularity.
Definition 1. Let \( \Omega \) be a domain in \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is called slice regular if, for all \( I \in \mathbb{S} \), its restriction \( f_I \) to \( \Omega_I \) is holomorphic, i.e., it has continuous partial derivatives and satisfies
\[
\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0
\]
for all \( x + yI \in \Omega_I \).

As shown in [12], the natural domains of definition of slice regular functions are the so-called axially symmetric slice domains.

Definition 2. Let \( \Omega \) be a domain in \( \mathbb{H} \). \( \Omega \) is called a slice domain if \( \Omega \) intersects the real axis and \( \Omega_I \) is a domain of \( \mathbb{C}_I \) for any \( I \in \mathbb{S} \).

Moreover, if \( x + yI \in \Omega \) implies \( x + yS \subseteq \Omega \) for any \( x, y \in \mathbb{R} \) and \( I \in \mathbb{S} \), then \( \Omega \) is called an axially symmetric slice domain.

From now on, we will omit the term ‘slice’ when referring to slice regular functions and will focus mainly on regular functions on \( B(0,R) = \{ q \in \mathbb{H} : |q| < R \} \) and the right half-space \( \mathbb{H}^+ = \{ q \in \mathbb{H} : \text{Re}(q) > 0 \} \). For regular functions the natural definition of derivative is given by the following (see [26, 27]).

Definition 3. Let \( f : B = B(0,R) \to \mathbb{H} \) be a regular function. For each \( I \in \mathbb{S} \), the \( I \)-derivative of \( f \) at \( q = x + yI \) is defined by
\[
\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)
\]
on \( B_I \). The slice derivative of \( f \) is the function \( f' \) defined by \( \partial_I f \) on \( B_I \) for all \( I \in \mathbb{S} \).

The definition is well-defined because, by direct calculation, \( \partial_I f = \partial_J f \) in \( B_I \cap B_J \) for any choice of \( I, J \in \mathbb{S} \). Furthermore, notice that the operators \( \partial_I \) and \( \partial_I \) commute, and \( \partial_I f = \frac{\partial f}{\partial x} \) for regular functions. Therefore, the slice derivative of a regular function is still regular so that we can iterate the differentiation to obtain the \( n \)-th slice derivative
\[
\partial_I^n f = \frac{\partial^n f}{\partial x^n}, \quad \forall \ n \in \mathbb{N}.
\]

In what follows, for the sake of simplicity, we will denote the \( n \)-th slice derivative by \( f^{(n)} \) for every \( n \in \mathbb{N} \). Incidentally, the slice derivative \( f' \) is initially called Cullen derivative in [26, 27] and is denoted by \( \partial_C f \) due to the work of Cullen [18]. Here we follow the standard notations and terminology in the monograph [25].

In the theory of regular functions, the following splitting lemma (see [27]) relates closely slice regularity to classical holomorphy.

Lemma 1. (Splitting Lemma) Let \( f \) be a regular function on \( B = B(0,R) \). Then for any \( I \in \mathbb{S} \) and any \( J \in \mathbb{S} \) with \( J \perp I \), there exist two holomorphic functions \( F, G : B_I \to \mathbb{C}_I \) such that
\[
f_I(z) = F(z) + G(z)J, \quad \forall \ z = x + yI \in B_I.
\]

Since the regularity does not keep under point-wise product of two regular functions a new multiplication operation, called the regular product (or \( * \)-product), appears via a suitable modification of the usual one subject to noncommutative setting. The regular product plays a key role in the theory of slice regular functions. On open balls centred at the origin, the \( * \)-product of two regular functions is defined by means of their power series expansions (see, e.g., [24, 12]).
Definition 4. Let \( f, g : B = B(0, R) \to \mathbb{H} \) be two regular functions and let
\[
f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n
\]
be their series expansions. The regular product (or \(*\)-product) of \( f \) and \( g \) is the function defined by
\[
f * g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^{n} a_k b_{n-k} \right)
\]
and it is regular on \( B \).

Notice that the \(*\)-product is associative and is not, in general, commutative. Its connection with the usual pointwise product is clarified by the following result [24, 12].

Proposition 1. Let \( f \) and \( g \) be regular on \( B = B(0, R) \). Then for all \( q \in B \),
\[
f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if} \quad f(q) \neq 0; \\ 0 & \text{if} \quad f(q) = 0. \end{cases}
\]

We remark that if \( q = x + yI \) and \( f(q) \neq 0 \), then \( f(q)^{-1}qf(q) \) has the same modulus and same real part as \( q \). Therefore \( f(q)^{-1}qf(q) \) lies in the same 2-sphere \( x + yS \) as \( q \). Notice that a zero \( x_0 + y_0I \) of the function \( g \) is not necessarily a zero of \( f * g \), but some element on the same sphere \( x_0 + y_0S \) does. In particular, a real zero of \( g \) is still a zero of \( f * g \). To present a characterization of the structure of the zero set of a regular function \( f \), we need to introduce the following functions.

Definition 5. Let \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) be a regular function on \( B = B(0, R) \). We define the regular conjugate of \( f \) as
\[
f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n,
\]
and the symmetrization of \( f \) as
\[
f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^{n} a_k \bar{a}_{n-k} \right).
\]
Both \( f^c \) and \( f^s \) are regular functions on \( B \).

We are now able to define the inverse element of a regular function \( f \) with respect to the \(*\)-product. Let \( \mathcal{Z}_{f^s} \) denote the zero set of the symmetrization \( f^s \) of \( f \).

Definition 6. Let \( f \) be a regular function on \( B = B(0, R) \). If \( f \) does not vanish identically, its regular reciprocal is the function defined by
\[
f^{-s}(q) := f^s(q)^{-1} f^c(q)
\]
and it is regular on \( B \setminus \mathcal{Z}_{f^s} \).

The following result shows that the regular quotient is nicely related to the pointwise quotient (see [43, 44]).
Proposition 2. Let $f$ and $g$ be regular on $B = B(0, R)$. Then for all $q \in B \setminus Z_f$, 

$$f^{-*} \ast g(q) = f(T_f(q))^{-1} g(T_f(q)),$$

where $T_f : B \setminus Z_f \to B \setminus Z_f$ is defined by $T_f(q) = f^*(q)^{-1} q f^*(q)$. Furthermore, $T_f$ and $T_{f'}$ are mutual inverses so that $T_f$ is a diffeomorphism.

Let us set

$$U(x_0 + y_0 S, R) = \{ q \in \mathbb{H} : \|(q - x_0)^2 + y_0^2\| < R^2 \}$$

for all $x_0, y_0 \in \mathbb{R}$ and all $R > 0$. The following result was proved in \[15\]; see Theorems 4.1 and 6.1 there for more details.

Theorem 6. Let $f$ be a regular function on a symmetric slice domain $\Omega$, and let $q_0 = x_0 + iy_0 \in U(x_0 + y_0 S, R) \subseteq \Omega$. Then there exists $\{A_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$ such that

$$f(q) = \sum_{n=0}^{\infty} ((q - x_0)^2 + y_0^2)^n (A_{2n} + (q - q_0)A_{2n+1})$$

for all $q \in U(x_0 + y_0 S, R)$.

As a consequence, for all $v \in \mathbb{H}$ with $|v| = 1$ the directional derivative of $f$ along $v$ can be computed at $q_0$ as

$$\frac{\partial f}{\partial v}(q_0) = \lim_{t \to 0} \frac{f(q_0 + tv) - f(q_0)}{t} = vA_1 + (q_0 v - \overline{v q_0}) A_2,$$

where

$$A_1 = R_{q_0} f(\overline{q_0}) = \partial_s f(q_0), \quad A_2 = R_{q_0} R_{q_0} f(q_0).$$

In particular, there holds that

$$f'(q_0) = R_{q_0} f(q_0) = A_1 + 2 \text{Im}(q_0) A_2.$$

3. Proofs of Main Theorems

In this section, we shall give the proofs of Theorems \[14\] except that of Theorem 3, which will be given in the next section.

Proof of Theorem \[11\]. The Schwarz-Pick Theorem shows, for all $q \in \mathbb{B}$,

$$\left| (1 - f(q) \overline{f(q)})^{-*} \ast (f(q) - f(q_n)) \right| \leq \left| (1 - q \overline{q_n})^{-*} \ast (q - \overline{q_n}) \right|,$$

which together with Proposition \[4\] implies that

$$\left| \frac{f \circ T_{1-t\overline{\overline{q_n}}}(q) - f(q_n)}{1 - f \circ T_{1-t\overline{\overline{q_n}}}(q) f(q_n)} \right| \leq \frac{|T_{1-t\overline{\overline{q_n}}}(q) - q_n|}{|1 - T_{1-t\overline{\overline{q_n}}}(q) q_n|}. $$

We square both sides and then minus by one to yield

$$\left| \frac{1 - f \circ T_{1-t\overline{\overline{q_n}}}(q) f(q_n)}{1 - \left| f \circ T_{1-t\overline{\overline{q_n}}}(q) \right|^2} \right|^2 \leq \frac{\left| 1 - T_{1-t\overline{\overline{q_n}}}(q) q_n \right|^2}{1 - |q|^2} \frac{1 - \left| f(q_n) \right|^2}{1 - |q_n|^2}. $$

Letting $n \to \infty$, we obtain that

$$\left| \frac{1 - f \circ T_{1-t\overline{\overline{q_n}}}(q) \xi}{1 - \left| f \circ T_{1-t\overline{\overline{q_n}}}(q) \right|^2} \right|^2 \leq \alpha \left| \frac{1 - T_{1-t\overline{\overline{q_n}}}(q) \xi}{1 - |q|^2} \right|^2. $$

(11)
Replacing \( q \) due to the fact that \( \| T \| \). By Proposition 2, notice that the term in the first brackets can be written as an imaginary constant, say \( \alpha > 0 \) it implies that

\[
1 - \frac{|f \circ T_1 - f(q)|^2}{|1 - f \circ T_1 - f(q)\bar{q}|^2} \geq \frac{1}{\alpha} \frac{1 - |q|^2}{|1 - T_1 - Id\xi(q)\bar{\xi}|^2}.
\]

This is equivalent to the inequality

\[
\text{Re} \left( (1 - f \circ T_1 - f(q)\bar{q})^{-1} (1 + f \circ T_1 - f(q)\bar{q}) \right) \geq \frac{1}{\alpha} \text{Re} \left( (1 - T_1 - Id\xi(q)\bar{\xi})^{-1} (1 + T_1 - Id\xi(q)\bar{\xi}) \right).
\]

That is, in terms of the regular product,

\[
\text{Re} \left( (1 - f(q)\bar{q})^{-*} (1 + f(q)\bar{q}) \right) \geq \frac{1}{\alpha} \text{Re} \left( (1 - q\bar{\xi})^{-*} (1 + q\bar{\xi}) \right).
\]

In particular, when \( \xi = 1 \), inequality (11) becomes

\[
\frac{1 - f \circ T_1 - f(q)\bar{q}}{1 - \| f \circ T_1 - f(q)\bar{q} \|^2} \leq \alpha \frac{1 - |q|^2}{1 - |q|^2}
\]

due to the fact that

\[
|1 - T_1 - Id(q)| = |1 - q|.
\]

By Proposition 2, \( T_1 - f(q) \) is a homeomorphism with inverse \( T_1 - f(q) \) since \( f(B) \subseteq B \). Replacing \( q \) by \( T_1 - f(q) \) in inequality (12) gives that

\[
\frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha \frac{|1 - T_1 - f(q)|^2}{1 - |q|^2} = \alpha \frac{|1 - q|^2}{1 - |q|^2}, \quad \forall q \in B.
\]

If equality holds for Julia's inequality (11) at some point \( q_0 \in B \), then the function

\[
(1 - f(q)\bar{q})^{-*} (1 + f(q)\bar{q}) = \frac{1}{\alpha} (1 - q\bar{\xi})^{-*} (1 + q\bar{\xi})
\]
is an imaginary constant, say \( I_{T_0} \) in virtue of the maximum principle for real part of regular functions (see Lemma 2 in [35]). A simple calculation shows that

\[
f(q) = \left( 1 + \frac{1}{\alpha} (1 - q\bar{\xi})^{-*} (1 + q\bar{\xi}) + I_{T_0} \right)^{-*} \left( \frac{1}{\alpha} (1 - q\bar{\xi})^{-*} (1 + q\bar{\xi}) + I_{T_0} - 1 \right) \eta.
\]

Notice that the term in the first brackets can be written as

\[
\frac{1}{\alpha} (1 - q\bar{\xi})^{-*} (1 + \alpha + I\alpha t_0) \ast (1 + q\bar{u}),
\]
where

\[
u = ((1 - \alpha) + I\alpha t_0) \xi ((1 + \alpha) - I\alpha t_0)^{-1} \in \partial \mathbb{B},
\]
since \( \alpha > 0 \). The other term can be treated similarly. Consequently, \( f \) can be represented as

\[
f(q) = (1 + q\bar{u})^{-*} (q + u)v,
\]
where

\[
v = ((1 + \alpha) + I\alpha t_0)^{-1} \xi ((1 + \alpha) - I\alpha t_0) \eta \in \partial \mathbb{B}.
\]

It follows that the equality in Julia's inequality can hold only for regular M"obius transformations of \( \mathbb{B} \) onto \( \mathbb{B} \), and a direct calculation shows that it does indeed hold for all such regular M"obius transformations. Now the proof is complete. \( \square \)

To prove Theorem 2 we shall need a quaternionic version of Lindel"of's principle, which follows easily from the corresponding result in the complex setting and the splitting lemma.
Lemma 2. (Lindelöf) Let $f$ be a regular function on $\mathbb{B}$ and bounded in each non-tangential approach region at 1. If for some continuous curve $\gamma \in \mathbb{B} \cap \mathbb{C}_I$ ending at 1 for some $I \in \mathbb{S}$, there exists the limit
$$
\eta = \lim_{t \to 1^-} f(\gamma(t)),
$$
then $f$ also has the non-tangential limit $\eta$ at 1.

Now we are in a position to prove the Julia-Carathéodory theorem.

Proof of Theorem. The equivalence between (ii) and (iii) follows directly from the corresponding result in the complex setting and the splitting lemma.

The implication (ii) $\Rightarrow$ (i) follows from the inequality
$$
\frac{1 - |f(r)|}{1 - r} \leq \frac{\eta - f(r)}{1 - r}, \quad \forall \ r \in (0, 1).
$$

Now we prove the implication (i) $\Rightarrow$ (ii). Under assumption (i), there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$ converging to 1 as $n$ tends to $\infty$, such that
$$
\alpha = \lim_{n \to \infty} \frac{1 - |f(q_n)|}{1 - |q_n|}
$$
and
$$
\lim_{n \to \infty} f(q_n) = \eta
$$
for some $\eta \in \partial \mathbb{B}$. It follows from Julia’s inequality \((2)\) that
$$
\frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \frac{\alpha |1 - q|^2}{1 - |q|^2}.
$$
Fix a non-tangential approach region at 1, say the region
\begin{equation}
(13) \quad R(1, k) = \{q \in \mathbb{B} : |q - 1| < k(1 - |q|)\},
\end{equation}
where $k$ is a constant greater than one. For all $q \in R(1, k)$ we have
\begin{equation}
(14) \quad \frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha k |q - 1| \frac{(1 - |q|)}{1 - |q|^2} \leq \alpha k |q - 1|,
\end{equation}
which implies that $f(q)$ tends to $\eta$ as $q$ tends to 1 within $R(1, k)$. In other words, $f$ has a non-tangential limit $\eta$ at 1.

It remains to prove that the difference quotient
$$
(q - 1)^{-1}(f(q) - \eta)
$$
has a non-tangential limit $\alpha \eta$. To this end, notice that
$$
\frac{|\eta - f(q)|}{1 + |f(q)|} \leq \frac{|\eta - f(q)|^2}{1 - |f(q)|^2},
$$
from which and inequality \((14)\) we have
$$
|(q - 1)^{-1}(f(q) - \eta)| \leq \alpha k (1 + |f(q)|) \leq 2\alpha k,
$$
whenever $q \in R(1, k)$. Consequently, the regular function
$$
g(q) := (q - 1)^{-1}(f(q) - \eta)
$$
is bounded in each non-tangential approach region at 1. The Lindelöf’s principle in Lemma \cite{2} thus reduces the proof to the existence of the radial limit
$$
\lim_{r \to 1^-} \frac{\eta - f(r)}{1 - r} = \alpha \eta.
$$
To consider this radial limit, we observe from the definition of \( \alpha \) as the lower limit in (i) that

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} \geq \alpha.
\]

On the other hand, setting \( q = r \) in the Julia’s inequality (2) yields that

\[
\frac{1 - |f(r)|}{1 - r} \leq \frac{|\eta - f(r)|^2}{1 - |f(r)|^2} \leq \alpha \frac{(1 - r)^2}{1 - r} \frac{1 + r}{1 - r}.
\]

Since \( f \) has a non-tangential limit \( \eta \in \partial B \) at 1, it follows that

\[
\limsup_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} \leq \alpha.
\]

Consequently,

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} = \alpha.
\]

Furthermore,

\[
\left( \frac{1 - |f(r)|}{1 - r} \right)^2 \leq \left( \frac{|\eta - f(r)|}{1 - r} \right)^2 \leq \alpha \frac{1 - |f(r)|^2}{1 - r^2} = \alpha \frac{1 - |f(r)|}{1 - r} \frac{1 + |f(r)|}{1 + r},
\]

which together with (17) implies that

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} = \alpha.
\]

By (17) and (18), we have

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{|\eta - f(r)|} = 1.
\]

Since

\[
\frac{1 - |f(r)|}{|\eta - f(r)|} \leq \frac{1 - \text{Re}(f(r)\overline{\eta})}{|1 - f(r)\overline{\eta}|} \leq 1,
\]

it follows from that

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{|\eta - f(r)|} = 1.
\]

This forces that

\[
\lim_{r \to 1^-} \frac{1 - |f(r)|}{|\eta - f(r)|} = 1.
\]

Therefore,

\[
\lim_{r \to 1^-} \frac{\eta - f(r)}{1 - r} = \lim_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} \frac{1 - \text{Re}(f(r)\overline{\eta})}{|1 - f(r)\overline{\eta}|} \eta
\]

\[
= \lim_{r \to 1^-} \frac{1 - |f(r)|}{1 - r} \frac{1 - |f(r)|}{1 - r} \frac{|\eta - f(r)|}{1 - |f(r)|} \frac{1 - |f(r)|}{1 - r} \frac{1 - |f(r)|}{1 - r} \eta
\]

\[
= \alpha \eta,
\]

which completes the proof of the implication (i) \( \Rightarrow \) (ii).

Now we assume that all conditions (i)-(iii) hold. By carefully checking the above proof, we see that assertions (a)-(c) hold true. It remains to verify (d).

From assertions (i), (ii), and (c), we know that \( \alpha < \infty \) and the difference quotient

\[
(q - 1)^{-1}(f(q) - \eta)
\]
has a non-tangential limit $\alpha\eta$ at point 1. Let us set

$$g(q) = (q - 1)^{-1}(f(q)\overline{\eta} - 1) - \alpha.$$  

Then $g$ is regular on $\Bbb{B}$ and has the non-tangential limit 0 at point 1. Since

$$|f(q)|^2 = \left| 1 + (q - 1)(\alpha + g(q)) \right|^2,$$

it follows that

$$(21) \quad \frac{1 - |f(q)|^2}{1 - |q|^2} = 2\alpha \frac{\text{Re}(1 - q)}{1 - |q|^2} + 2 \frac{\text{Re}((1 - q)g(q))}{1 - |q|^2} - \frac{|1 - q|^2}{1 - |q|^2} |\alpha + g(q)|^2.$$  

Fix the non-tangential approach region $\mathcal{R}(1, k)$ as in (13) and consider the non-tangential limit as $q \to 1$ within $\mathcal{R}(1, k)$. It is clear that the second term on the right-hand side of the preceding equality approaches 0 and so does the third term. Since the equality

$$\frac{\text{Re}(1 - q)}{1 - |q|^2} = \frac{1}{2} \left( 1 + \frac{|1 - q|^2}{1 - |q|^2} \right)$$

shows that

$$\lim_{q \to 1, q \in \mathcal{R}(1, k)} \frac{\text{Re}(1 - q)}{1 - |q|^2} = \frac{1}{2},$$

from which it follows that the first term on the right-hand side of equality (21) approaches to $\alpha$ as $q \to 1$ within $\mathcal{R}(1, k)$. Therefore,

$$\lim_{q \to 1, q \in \mathcal{R}(1, k)} \frac{1 - |f(q)|^2}{1 - |q|^2} = \alpha,$$

which is equivalent to

$$\lim_{q \to 1, q \in \mathcal{R}(1, k)} \frac{1 - |f(q)|}{1 - |q|} = \alpha.$$  

Now the proof is complete. \qed

Next we come to prove the quaternionic version of Hopf’s theorem. As mentioned in the introduction part, it is just a direct consequence of Theorem 4. Here we provide an alternative and easier proof, in virtue of Theorem 2.

Proof of Corollary 3. (i) Let $f$ be as described in Corollary 3. Set

$$g(q) := (1 - f(q)f(0))^{-*} \ast (f(q) - f(0))(1 - f(0))(1 - f(0))^{-1},$$

which is a regular function on $\Bbb{B} \cup \{1\}$ such that $g(\Bbb{B}) \subseteq \Bbb{B}$, $g(0) = 0$ and $g(1) = 1$. Moreover, an easy calculation shows that

$$(22) \quad f'(1) = \frac{|1 - f(0)|^2}{1 - |f(0)|^2} g'(1),$$

and

$$(23) \quad g'(0) = \frac{f'(0)}{1 - |f(0)|^2}(1 - f(0))(1 - f(0))^{-1},$$
which is no more than one in modulus. Applying Julia-Carathéodory Theorem and Julia’s inequality \[2\] to the regular function \( h(q) := q^{-1}g(q) \) mapping \( B \) to \( \overline{B} \) yields that

\[
g'(1) = 1 + h'(1) \geq 1 + \frac{|1 - g'(0)|^2}{1 - |g'(0)|^2} = \frac{2(1 - \text{Re } g'(0))}{1 - |g'(0)|^2}.
\]

In particular,

\[
g'(1) \geq \frac{2}{1 + \text{Re } g'(0)}.
\]

Substituting equalities in \(22\) and \(23\) to \(25\) yields that

\[
f'(1) \geq \frac{2}{\text{Re } (1 - f(0)^2 + f'(0))(1 - f(0))^{-2}}.
\]

If equality holds for the last inequality, then equalities also hold for Julia’s inequality \(2\) at point \( q = 0 \) and inequality \(25\), it follows from Theorem 1 and the assumption that \( f(1) = 1 \) that

\[
g(q) = q(1 - qa)^{-*} * (q - a)
\]

for some constant \( a \in [-1, 1] \). Consequently, \( f \) is of the form

\[
f(q) = \left(1 + q(1 - qa)^{-*} * (q - a)(1 - f(0))^{-1}(1 - f(0))^{-1}(1 - f(0)) \right)^{-*}
\]

\[
* \left(f(0) + q(1 - qa)^{-*} * (q - a)(1 - f(0))^{-1}(1 - f(0)) \right)
\]

for some constant \( a \in [-1, 1] \). Therefore, the equality in inequality \(26\) can hold only for regular functions of the form \(25\), and a direct calculation shows that it does indeed hold for all such regular functions. Now the proof of (i) is complete.

(ii) The result follows easily from (i) by considering the regular function \( h(q) := q^{-n}f(q) \) and noticing that

\[
h(0) = \frac{f^{(n)}(0)}{n!}, \quad h'(0) = \frac{f^{(n+1)}(0)}{(n+1)!}.
\]

\[\square\]

To prove the boundary Schwarz lemma (Theorem 4), we make the best use of the classical Hopf’s lemma, which can be viewed as the real version of the boundary Schwarz lemma. We remark that, unlike in the complex setting, the boundary Schwarz lemma cannot be simplified to the specific case that \( \xi = 1 \) because the theory of regular composition is unavailable.

**Proof of Theorem 4.** By assumption, the function \(|f|^2\) attains its maximum at the boundary point \( \xi \in \partial B \) so that the directional derivative of \(|f|^2\) along \( \xi \) at the point \( \xi \) satisfies that

\[
\frac{\partial |f|^2}{\partial \xi}(\xi) > 0,
\]

in virtue of the classical Hopf’s lemma. Moreover,

\[
\frac{\partial |f|^2}{\partial \tau}(\xi) = 0, \quad \forall \tau \in T_\xi(\partial B) \cong \mathbb{R}^3.
\]
Indeed, for any unit tangent vector \( \tau \in T_\xi(\partial \mathbb{B}) \), take a smooth curve \( \gamma : (-1, 1) \to \mathbb{B} \) such that
\[
\gamma(0) = \xi, \quad \gamma'(0) = \tau.
\]
By definition we have
\[
\frac{\partial |f|^2}{\partial \tau}(\xi) = \left( \frac{d}{dt} |f(\gamma(t))|^2 \right)_{t=0} = 0,
\]
since the function \( |f(\gamma(t))|^2 \) in \( t \) attains its maximum at the point \( t = 0 \).
Moreover, Theorem 6 shows that
\[
\partial f(\xi) = \bar{v}\partial_s f(\xi) + (\xi v - v\bar{\xi})R_\xi R_\xi f(\xi)
\]
for all \( v \in \mathbb{H} \) with \( |v| = 1 \). Therefore,
\[
\frac{\partial |f|^2}{\partial v}(\xi) = 2 \left( \frac{\partial f}{\partial v}(\xi), f(\xi) \right)
\]
\[
= 2 \left( \bar{v}\partial_s f(\xi) + (\xi v - v\bar{\xi})R_\xi R_\xi f(\xi), f(\xi) \right)
\]
\[
= 2 \left( v, f(\xi)\bar{\partial}_s f(\xi) + \xi f(\xi)R_\xi R_\xi f(\xi) - f(\xi)R_\xi R_\xi f(\xi) \right)
\]
\[
= 2 \left( v, f(\xi)f'(\xi) + [\bar{\xi}, f(\xi)R_\xi R_\xi f(\xi)] \right),
\]
where \( \left( p, q \right) \) denotes the standard inner product on \( \mathbb{H} \cong \mathbb{R}^4 \), i.e.,
\[
\left< p, q \right> = \text{Re}(p\bar{q}), \quad \forall p, q \in \mathbb{H}.
\]
In the last equality we have used the fact that
\[
f'(\xi) = \partial_s f(\xi) + 2 \text{Im}(\xi)R_\xi R_\xi f(\xi).
\]
Now it follows from (30) and (31) that
\[
f(\xi)f'(\xi) + [\bar{\xi}, f(\xi)R_\xi R_\xi f(\xi)] \perp T_\xi(\partial \mathbb{B})
\]
so that in view of (29) and (31) there exists a real number \( \lambda > 0 \) such that
\[
f(\xi)f'(\xi) + [\bar{\xi}, f(\xi)R_\xi R_\xi f(\xi)] = \lambda f(\xi)
\]
and
\[
\bar{\lambda} \left( f(\xi)f'(\xi) + [\bar{\xi}, f(\xi)R_\xi R_\xi f(\xi)] \right) = \lambda \frac{1}{2} \frac{\partial |f|^2}{\partial \xi}(\xi) = \frac{\partial |f|}{\partial \xi}(\xi) > 0.
\]
To obtain the desired sharp estimate in (5), we need a technical trick. Denote
\[
v = (f(0) - f(\xi))^{-1} \xi (1 - f(\xi)\bar{f}(0))^{-1},
\]
which belongs to \( \partial \mathbb{B} \), for \( f(\xi) \in \partial \mathbb{B} \) by assumption. Set
\[
g(q) := (1 - f(q)\bar{f}(0))^{-1} * (f(0) - f(q))v,
\]
which is a regular function on \( \mathbb{B} \cup \{\xi\} \) such that \( g(\mathbb{B}) \subseteq \mathbb{B} \). Moreover, it is evident that \( g(0) = 0 \) and
\[
g'(0) = -\frac{f'(0)}{1 - |f(0)|^2}v.
\]
Denote
\[
\eta = T_{1-f(0)*}\bar{f}'(\xi) \in \partial \mathbb{B},
\]
which is a boundary fixed point of $g$. Indeed, it easily follows from Proposition 2 and (34) that
\begin{equation}
    g(\eta) = (1 - f(\xi f(0)))^{-1} \xi (1 - f(\xi f(0))) = T_{1 - f(0) \ast f^c}(\xi) = \eta,
\end{equation}
and hence the regular function $g$ satisfies all the assumptions in Theorem 3.

We next claim that
\begin{equation}
    |\xi| (f(\xi) f'(\xi) + [\xi, f(\xi) R_{\xi} R_{\xi} f(\xi)]) = \frac{|f(0) - f(\xi)|^2}{1 - |f(0)|^2} \left( \eta (\eta g^c(\eta)) + \eta, R_\eta R_\eta g(\eta) \right).
\end{equation}

One can deduce this equality by direct verification, but that argument seems quite a tedious calculation and is a bit more complicated than the following one, which goes as follows. Due to equality in (32), it suffices to prove that
\begin{equation}
    \frac{\partial |f|^2}{\partial \xi}(\xi) = \frac{|f(0) - f(\xi)|^2}{1 - |f(0)|^2} \frac{\partial |g|^2}{\partial \eta}(\eta).\tag{39}
\end{equation}

First, from (34) we obtain that
\[ f(q) = (1 - g(q) v f(0))^* (f(0) - g(q) v). \]
This together with Proposition 2 implies
\begin{equation}
    f(q) = \left( 1 - g \circ T_{1 - g f(0)v}^{-1} f(0) v \right)^{-1} \left( f(0) - g \circ T_{1 - g f(0)v}^{-1} f(0) v \right),
\end{equation}
from which one easily deduces that
\[ 1 - |f(0)|^2 = \frac{(1 - |f(0)|^2)(1 - |g \circ T_{1 - g f(0)v}^{-1} f(0) v|^2)}{1 - |g \circ T_{1 - g f(0)v}^{-1} f(0) v|^2}. \]
Consequently,
\begin{equation}
    \frac{\partial |f|^2}{\partial \xi}(\xi) = \lim_{t \to 0^+} \frac{1 - |f(\xi - t\xi)|^2}{t}, \tag{41}
\end{equation}
\[ = \frac{1 - |f(0)|^2}{|f(0) - g \circ T_{1 - g f(0)v}^{-1} f(0) v|^2} \lim_{t \to 0^+} \frac{1 - |g \circ T_{1 - g f(0)v}^{-1} f(0) v|^2}{t}. \]

We next show that the limit on the right-hand-side of the preceding equality is exactly the directional derivative of $|g|^2$ along $\eta$ at the boundary point $\eta \in \partial B$. An direct calculation gives that
\[ 1 - g f(0)v = (1 - |f(0)|^2)(1 - f f(0))^{-}, \]
from which one easily obtain that
\[ T_{1 - g f(0)v} = T_{1 - f f(0)v} = T_{1 - f(0) f^c}^{-}. \]
This fact together with the notation of $\eta$ in (34) implies that
\begin{equation}
    \eta = T_{1 - f(0) f^c}^{-}(\xi) = T_{1 - g f(0)v}^{-}(\xi).\tag{42}
\end{equation}
Therefore, the curve
\[ t \mapsto \Gamma(t) := T_{1 - g f(0)v}^{-}(\xi - t\xi) \]
is a smooth curve defined on some interval \((−\varepsilon, \varepsilon)\) with some positive number \(\varepsilon\) small enough such that 
\[
\Gamma(0) = \Gamma'(0) = T_{1-\overline{f(0)}v}(\xi) = \eta \in \partial B.
\]
Consequently, 
\[
\lim_{t \to 0^+} \frac{1 - |g \circ T_{1-\overline{f(0)}v}(\xi - t\xi)|^2}{t} = \frac{\partial |g|^2}{\partial \eta}(\eta).
\]
Furthermore, it follows from \((40)\) and \((42)\) that 
\[
g \circ T_{1-\overline{f(0)}v}(\xi) = g(\eta) = \eta = (1 - f(\xi)\overline{f(0)})^{-1} \xi (f(0) - f(\xi))
\]
and hence 
\[
|f(0) - g \circ T_{1-\overline{f(0)}v}(\xi)|^2 = \frac{1 - |f(0)|^2}{|f(0) - f(\xi)|}.
\]
Now equality \((43)\) follows from \((41), (43)\) and \((44)\). This completes the proof of equality in \((48)\).

Let \(I \in \mathbb{S}\) be such that \(\eta \in \partial B \cap C_I\) and let us split \(g_I\) as 
\[
g_I(z) = G(z) + H(z)J,
\]
where \(J \in \mathbb{S}\) and \(J \perp I\), and \(G, H\) are holomorphic self-mappings of \(B_I\). Then 
\[
|g_I(z)|^2 = |G(z)|^2 + |H(z)|^2
\]
and 
\[
g'_I = G'(z) + H'(z)J
\]
for any \(z \in B_I\). Moreover, 
\[
G(\eta) = \eta, \quad H(\eta) = 0, \quad \Re g'(0) = \Re G'(0).
\]
Now applying the classical Julia-Carathéodory theorem and Julia inequality in Julia lemma in the complex setting (see \((10)\) p. 48 and p. 51), one can easily deduce (as in the proof of Corollary \((37)\) see \((37)\) for more details) that 
\[
\frac{\partial |g|}{\partial \eta}(\eta) = \frac{\partial |G|}{\partial \eta}(\eta) \geq 1 + \frac{|1 - G'(0)|^2}{1 - |G'(0)|^2} = \frac{2(1 - \Re G'(0))}{1 - |G'(0)|^2}.
\]
In particular, 
\[
\frac{\partial |g|}{\partial \eta}(\eta) \geq \frac{2}{1 + \Re G'(0)} = \frac{2}{1 + \Re g'(0)}.
\]
Substituting \((33), (34)\) and \((47)\) into \((48)\) yields the desired sharp estimate in \((5)\).

If equality hold for inequality in \((5)\), then equalities also hold in the Julia inequality (see \((10)\) p. 51)) at point \(z = 0\) and in inequality \((17)\), it follows from the condition for equality in the Julia inequality and that for equality in inequality \((47)\) that 
\[
G(z) = z \frac{z\eta - a}{1 - a\eta z^2}, \quad \forall z \in B_I,
\]
for some constant \(a \in [-1, 1]\). Furthermore, it follows from equality in \((45)\) that 
\[
|H(z)|^2 = |g_I(z)|^2 - |G(z)|^2 \leq 1 - |G(z)|^2, \quad \forall z \in B_I,
\]
which together with \((48)\) implies that \(H \equiv 0\), and hence 
\[
g(q) = \text{ext } G(q) = q(1 - qa\eta)^{-s} * (q\eta - a).
\]
Consequently, $f$ must be of the form

$$(49) \ f(q) = (1 - q(1_a - qa) - *(q_a - a)A(1) - *(q_a - a)A),$$

where $a \in [-1, 1]$, and $v$ and $\eta$ are the same as those in (33) and (36), respectively. Therefore, the equality in inequality (31) can hold only for regular self-mappings of the form (49), and a direct calculation shows that it does indeed hold for all such regular self-mappings. This completes the proof of (i) and it remains to prove (ii).

However, (ii) follows easily from (i) by considering the regular function $h(q) := q^{-n}f(q)$ and noticing that

$$h(0) = \frac{f(n)(0)}{n!}, \quad h'(0) = \frac{f(n+1)(0)}{(n + 1)!}.$$

Moreover,

$$f(q)f'(q) + [\xi, f(q)RqRqf(q)] = n\xi + h(q)h'(q) + [\xi, h(q)RqRqh(q)]$$

as one easily verifies. Now the proof is complete. \hfill \square

**Proof of Corollary 1.** We only to prove (i). Inequality (7) follows immediately from inequality (5), and equality in (7) holds if and only if

$$S := \text{Re} \left( f'(0) (f(q) - f(0)) - 1 \xi (1 - f(0))f(0) \right)^{-1},$$

which is equivalent to $G'(0) \in [0, 1]$, i.e. $a \in [-1, 0]$. Here the function $G$ is the one in (48).

Some useful remarks are in order.

**Remark 1.** It is quite natural to ask if the quality

$$\xi \left( f(q)f'(q) + [\xi, f(q)RqRqf(q)] \right)$$

in Corollary 1 is no other than

$$\xi f(q)f'(q)$$

in the complex setting. Unfortunately, the Lie brackets

$$[\xi, f(q)RqRqf(q)]$$

in Corollary 1 do not vanish in general. Moreover, all the products of $\xi$, $f(q)$ and $f'(q)$ in any different orders may fail simultaneously to be real numbers so that the inequality

$$\xi f(q)f'(q) \geq \frac{2|f(q) - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|^2}$$

does not hold, neither all of its modified versions free of orders. These facts can be demonstrated by the following counterexample.

**Example 1.** Let $I \in S$ be fixed. Set

$$f(q) = (1 + qI/2)^{-*}*(q - I/2).$$

Then it is a slice regular M"{o}bius transformation of $B$ onto $B$. It is evident to see that

$$f(q) = (q^2 + 4)^{-1}(3q - 2(q^2 + 1)I)$$

so that it satisfies all the assumptions of Corollary 1.
Now we set \( \xi = J \), where \( J \in \mathbb{S} \) is fixed such that \( J \perp I \). An easy calculation thus shows that

\[
f(J) = J, \quad f'(J) = \frac{1}{3}(5 + 4IJ),
\]
and

\[
A_1 = \partial_s f(J) = 1, \quad A_2 = R_{-J}R_J f(J) = -\frac{1}{3}(2I + J).
\]

Therefore,

\[
\left[ J, f(J)R_{-J}R_J f(J) \right] = -\frac{1}{3}[J, J(2I + J)] = \frac{4}{3}I \neq 0.
\]

On the other hand,

\[
\mathcal{J}\left( f(J)f'(J) + \left[ J, f(J)R_{-J}R_J f(J) \right] \right) = \frac{2|f(J) - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|} = \frac{5}{3}
\]
as predicated by Corollary 1.

Now we provide an example to show that in Corollary 2 the inequality \( f'(\xi) > 1 \) may fail, or rather that \( f'(\xi) \) may not be a real number.

**Example 2.** We now construct a function \( g \) such that \( g'(J) \) is no longer a real number. To this end, we set

\[
g(q) = -qf(q)J = -q(1 + qI/2)^{-*}(q - I/2)J,
\]
where \( f \) is as described in Example 1 and \( I, J \in \mathbb{S} \) with \( J \perp I \).

It is evident that this function is a Blaschke product of order 2 so that it is regular on \( \mathbb{B} \), and satisfies \( g(\mathbb{B}) \subseteq \mathbb{B} \), \( g(0) = 0 \) and \( g(J) = -Jf(J)J = J \). This means that \( g \) satisfies all assumptions given in Corollary 2.

However, we find that \( g'(J) \) is indeed not a real number. In fact, by the Leibniz rule we have

\[
g'(q) = -(f(q) + qf'(q))J.
\]

Consequently,

\[
g'(J) = \frac{4}{3}(2 - IJ) \notin \mathbb{R}.
\]

On the other hand, a simple calculation shows that

\[
\left[ J, R_{-J}R_J g(J) \right] = -\frac{4}{3}IJ
\]
and

\[
g'(J) - \left[ J, R_{-J}R_J g(J) \right] = \frac{8}{3} > 1
\]
as predicated by Corollary 2.

**Remark 2.** Theorem 4 also provides a lower bound for the slice derivative of \( f \) at \( \xi \) in modulus. Notice that

\[
\frac{\partial f}{\partial \xi}(\xi) = \xi f'(\xi)
\]
and

\[
\frac{\partial |f|}{\partial \xi}(\xi) = \overline{\xi}\left( f(\xi)f'(\xi) + \left[ \xi, f(\xi)R_{\xi}R_{\xi} f(\xi) \right] \right)
\]
from (32), thus the obvious inequality
\[ \left| \frac{\partial f}{\partial \xi}(\xi) \right| \geq \left| \frac{\partial |f|}{\partial \xi}(\xi) \right| \]
shows that the left-hand side of each inequality in Theorem 4 can be replaced by \(|f'(\xi)|\).

Remark 3. In the proof of the desired sharp estimate in (5), we only use the weaker estimate
\[ \frac{\partial |g|}{\partial \eta}(\eta) \geq \frac{2}{1 + \text{Re} g'(0)} \]
which follows from (46). If we make the best use of estimate in (46), we will obtain more precise estimate than that in (5). The estimate will be very complicated at least in form, but it is same as that in (5) if the functions of concern are the extremal functions given in (6).

4. JULIA-CARATHÉODORY THEOREM IN $\mathbb{H}^+$

In this section, we establish the Julia-Carathéodory theorem for slice regular self-mappings of the right half-space
\[ \mathbb{H}^+ := \{ q \in \mathbb{H} : \text{Re}(q) > 0 \} \]
The proof depends ultimately on the right half-space version of the Schwarz-Pick theorem.

Theorem 7. (Schwarz-Pick) Let \( f : \mathbb{H}^+ \rightarrow \mathbb{H}^+ \) be a regular function. Then for any \( q_0 \in \mathbb{H}^+ \) we have
\[ \left| (g(q) + g(q_0))^{-*} (g(q) - g(q_0)) \right| \leq \left| (q + q_0)^{-*} (q - q_0) \right|, \quad \forall q \in \mathbb{H}^+. \]
Inequality is strict (except at \( q = q_0 \)) unless \( f \) is a regular Möbius transformation from \( \mathbb{H}^+ \) onto itself.

We denote the non-tangential cone at the boundary point 0 of \( \mathbb{H}^+ \) by
\[ \mathcal{S}_\gamma = \{ q \in \mathbb{H}^+ : \text{Re}(q) > \gamma |q| \} \]
for any \( \gamma > 0 \).

Theorem 8. (Julia-Carathéodory) Let \( f : \mathbb{H}^+ \rightarrow \mathbb{H}^+ \) be a regular function and set
\[ c := \inf \left\{ \frac{\text{Re} f(q)}{\text{Re}(q)} : q \in \mathbb{H}^+ \right\} \geq 0. \]
Then the following hold true:
(i) for any \( q \in \mathbb{H}^+ \),
\[ \text{Re} f(q) \geq c \text{Re}(q); \]
(ii) for any \( \gamma > 0 \),
\[ \lim_{|q| \to \infty, q \in \mathcal{S}_\gamma} q^{-1} f(q) = \lim_{|q| \to \infty, q \in \mathcal{S}_\gamma} \frac{\text{Re} f(q)}{\text{Re}(q)} = c; \]
(iii) for any \( \gamma > 0 \),
\[ \lim_{|q| \to \infty, q \in \mathcal{S}_\gamma} f'(q) = c. \]
Proof. We put
\begin{equation}
(50) \quad g(q) := f(q) - cq, \quad \forall q \in \mathbb{H}^+,
\end{equation}
Then by definition \( \text{Re} g(q) \geq 0 \) for all \( q \in \mathbb{H}^+ \). Moreover, we may assume that
\( \text{Re} g(q) > 0 \), \( \forall q \in \mathbb{H}^+ \), in virtue of the maximum principle for real parts of regular functions, see Lemma 2 in [35]. Otherwise, \( g(q) = Ht_0 \) for some \( t_0 \in \mathbb{R} \) and some \( I \in \mathbb{S} \). Thus the results are obvious.

It follows from the Schwarz-Pick theorem that for all \( q, q_0 \in \mathbb{H}^+ \),
\[
\left| (g(q) + g(q_0))^{-1} \ast (g(q) - g(q_0)) \right| \leq \left| (q + q_0)^{-1} \ast (q - q_0) \right|,
\]
which is equivalent to
\[
\left| \frac{g \circ T_{g+g(q_0)}(q) - g(q)}{g \circ T_{g+g(q_0)}(q) + g(q_0)} \right| \leq \left| \frac{H_{Id+\overline{q_0}}(q) - q_0}{H_{Id+\overline{q_0}}(q) + q_0} \right|,
\]
or
\begin{equation}
(51) \quad \left| \frac{g(q) - g(q_0)}{g(q) + g(q_0)} \right| \leq \left| \frac{p - q_0}{p + q_0} \right| := r,
\end{equation}
where
\begin{equation}
(52) \quad p = H_{Id+\overline{q_0}} \circ T_{g+g(q_0)}(q).
\end{equation}

We set
\[
h(q) = \frac{g(q) - \text{Im} g(q_0)}{\text{Reg}(q_0)}.
\]
Then (51) becomes
\begin{equation}
(53) \quad \left| \frac{h(q) - 1}{h(q) + 1} \right| \leq r = \left| \frac{p - q_0}{p + q_0} \right|.
\end{equation}

Consequently,
\begin{equation}
(54) \quad |h(q)| \leq \frac{1 + r}{1 - r} = \frac{(1 + r)^2}{(1 - r)^2} = \left( \frac{|p + q_0| + |p - q_0|}{|p + q_0|^2 - |p - q_0|^2} \right)^2.
\end{equation}

By the definition of \( p \) in (52), we have
\[
|p| = |q|, \quad \text{Re}(p) = \text{Re}(q),
\]
so that (54) leads to
\begin{equation}
(55) \quad |h(q)| \leq \frac{(|q| + |q_0|)^2}{\text{Re}(q)\text{Re}(q_0)}.
\end{equation}

This implies that
\[
|q^{-1}g(q)| \leq \left| q^{-1}\text{Im} g(q_0) \right| + \left| q^{-1}(g(q) - \text{Im} g(q_0)) \right|
\]
\[
= \frac{|\text{Im} g(q_0)|}{|q|} + |h(q)| \frac{\text{Reg}(q_0)}{|q|}
\]
\begin{equation}
(56) \quad \leq \frac{|\text{Im} g(q_0)|}{|q|} + \left( \frac{|q| + |q_0|}{\text{Re}(q)\text{Re}(q_0)} \right)^2 \text{Reg}(q_0) \frac{|q|}{|q|}.
\end{equation}
Then for all $q \in S$, we have
\begin{equation}
|q^{-1}g(q)| \leq \frac{|\text{Im}(g(q))|}{|q|} + \frac{(|q| + |g(q)|)^2 \text{Re}(g(q))}{\gamma |q|^2 \text{Re}(q)} \tag{57}
\end{equation}

By the definition of $g$ in (50), it is evident that
\[\inf \left\{ \frac{\text{Re}(g(q))}{\text{Re}(q)} : q \in H^+ \right\} = 0.\]

For any $\epsilon > 0$, there thus exists a $q_0 \in H^+$ such that
\[\text{Re}(g(q_0)) \leq \gamma \epsilon,\]
and hence by letting $|q| \to \infty$ in (57) we obtain
\[\limsup_{|q| \to \infty} |q^{-1}g(q)| \leq \epsilon.\]

Namely,
\begin{equation}
\lim_{|q| \to \infty} q^{-1}g(q) = 0, \quad q \in S, \tag{58}
\end{equation}

since $\epsilon$ is an arbitrary positive number. At the same time, for all $q \in S$,
\[\frac{\text{Reg}(q)}{\text{Re}(q)} \leq \frac{|g(q)|}{|q|} \frac{|q|}{\text{Re}(q)} \leq \frac{1}{\gamma} |q^{-1}g(q)|,\]
which together with (58) implies that
\[\lim_{|q| \to \infty} \frac{\text{Reg}(q)}{\text{Re}(q)} = 0.\]

Now the proof of the assertions (i) and (ii) is complete.

It remains to prove (iii). For any given $\gamma \in (0, 1)$, it is easy to see that the closed ball
\[\overline{B}(q, \gamma |q|) \subset S,\]
whenever $q \in S_{2\gamma}$. For any $I \in \mathbb{S}$ and $q \in S_{2\gamma} \cap C_I$ we have
\begin{equation}
f'(q) - c = \frac{1}{2\pi I} \int_{\partial B(q, \gamma |q|) \cap C_I} ds \frac{df(s) - cs}{(s - q)^2}. \tag{59}
\end{equation}

Notice that for any $s \in \partial B(q, \gamma |q|) \cap C_I$ we have $|s - q| = \gamma |q|$ so that
\[(1 - \gamma)|q| \leq |s| \leq (1 + \gamma)|q|.
\]
Since $|f(s) - cs| = |s||s^{-1}f(s) - c|$ and
\[\partial B(q, \gamma |q|) \cap C_I \subseteq \{ s \in S : |s| \geq (1 - \gamma)|q| \}\]

it follows from (59) that
\begin{equation}
|f'(q) - c| \leq \left(1 + \frac{1}{\gamma} \right) \sup_{s \in S} \frac{|s^{-1}f(s) - c|}{|s| \geq (1 - \gamma)|q|} \tag{60}
\end{equation}
which approaches to 0 as \( q \) tends to \( \infty \) in \( S_2 \), due to assertion (ii). Consequently,

\[
\lim_{|q| \to \infty} f'(q) = c.
\]

Now the proof is complete. \( \square \)

**Remark 4.** There exists a regular function \( f : \mathbb{H}^+ \to \mathbb{H}^+ \) such that

\[
c := \inf \left\{ \frac{\text{Re} f(q)}{\text{Re}(q)} : q \in \mathbb{H}^+ \right\} = 0.
\]

A simple example is the constant function \( f(q) = 1 \).

As a simple consequence of the preceding theorem, we have the following result.

**Corollary 4.** Let \( f : \mathbb{H}^+ \to \mathbb{H}^+ \) be a regular function. Then there exists a positive number \( \beta \), finite or infinite, such that for any \( \gamma > 0 \) we have

\[
\lim_{|q| \to \infty} qf(q) = \beta.
\]

**Proof.** By assumption, \( f \) has no zeros and so does \( f^{-*} \) (see Proposition 3.9 in [25]). The result immediately follows by applying the preceding theorem to \( f^{-*} \). \( \square \)

Another consequence is the following rigidity theorem.

**Corollary 5.** Let \( f : \mathbb{H}^+ \to \mathbb{H}^+ \) be a regular function. If there exist a \( I \in \mathbb{S} \) and some \( \theta \in (-\pi/2, \pi/2) \) such that

\[
\liminf_{r \to \infty} r |f(re^{i\theta})| = 0,
\]

then \( f \equiv 0 \).

Finally, we give the proof of the Burns-Krantz theorem.

**Proof of Theorem 3.** Let \( \varphi \) be the Cayley transformation from \( \mathbb{H}^+ \) to \( \mathbb{B} \), i.e.

\[
\varphi(q) = (1 + q)^{-1}(1 - q), \quad \forall q \in \mathbb{H}^+.
\]

The result immediately follows by applying the preceding corollary to the regular function \( g = f \circ \varphi \). \( \square \)

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