Research Article

Higher-order surface FEM for incompressible Navier-Stokes flows on manifolds

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Summary
Stationary and instationary Stokes and Navier-Stokes flows are considered on two-dimensional manifolds, ie, on curved surfaces in three dimensions. The higher-order surface FEM is used for the approximation of the geometry, velocities, pressure, and Lagrange multiplier to enforce tangential velocities. Individual element orders are employed for these various fields. Streamline-upwind stabilization is employed for flows at high Reynolds numbers. Applications are presented, which extend classical benchmark test cases from flat domains to general manifolds. Highly accurate solutions are obtained, and higher-order convergence rates are confirmed.

KEYWORDS
higher-order FEM, manifold, Navier-Stokes, Stokes, surface FEM, surface PDEs

1 | INTRODUCTION

The solution of boundary value problems on curved surfaces has many practical applications in mathematics, physics, and engineering. For example, there are transport processes on interfaces, eg, in foams, biomembranes and bubble surfaces,1-3 or structure-related phenomena such as in membranes and shells.4,5 Herein, Stokes and incompressible Navier-Stokes flows on curved two-dimensional (2D) manifolds are considered. The governing equations for flows on moving surfaces were discussed in the works of Bothe and Prüss6 and Jankuhn et al7 based on fundamental surface continuum mechanics and conservation laws, and in the work of Koba et al,8 an energetic approach was presented. Earlier works in a similar context may be traced back to other works.9-12 For an excellent overview, the reader is referred to the work of Jankuhn et al.7 The aforementioned references often focus on mathematical properties such as the existence and uniqueness of the solutions or stability analyses. Applications are often two-phase flows where the fluid field in the bulk and on the moving interface are coupled. However, it is also worthwhile to consider the situation for fixed manifolds, eg, related to meteorology and oceanography where the flows take place on (part of) a sphere. Special geometries such as hyperbolic planes and spheres are discussed in other works.13-15

Herein, the focus is on the approximation of stationary and instationary (Navier-)Stokes flows on fixed manifolds based on the surface FEM as outlined in other works.16-18 The governing equations resemble the three-dimensional (3D) (Navier-)Stokes equations where the classical gradient and divergence operators are replaced by their tangential counterparts derived from tangential differential calculus.7 The equations are formulated in the classical stress-divergence form,
contrasted to the approach in the work of Nitschke et al. An additional constraint is required to enforce that the velocities remain in the tangent space of the manifold; it is labeled “tangential velocity constraint.” The models are first given in strong form and are then transformed to the weak form to enable a numerical solution based on the surface FEM. Finite element spaces of different orders are employed for the approximation of the geometry and of the involved physical fields, ie, the velocities, pressure, and the Lagrange multiplier field required to enforce the tangential velocity constraint. It is found that the balance of these element orders is critical for the accuracy and conditioning of the system of equations. In particular, the well-known Babuška-Brezzi condition applies as both the incompressibility constraint and the tangential velocity constraint are enforced using Lagrange multipliers. For the case of the instationary Navier-Stokes equations, the Crank-Nicolson time-stepping scheme is employed for the semidiscrete system of equations resulting from using the surface FEM in space. Surface FEM based on linear elements was used in the recent work by Reuther and Voigt, where the penalty method was employed to enforce tangential velocities and a projection method rather than a monolithic approach was suggested to solve for the different physical fields. Alternatives for the surface FEM are the TraceFEM and CutFEM, where the basis functions are generated from a background mesh in the bulk surrounding the manifold of interest.

Using the FEM for the Navier-Stokes flows at large Reynolds numbers requires stabilization. Herein, the streamline-upwind Petrov-Galerkin (SUPG) approach is used. Alternatively, other variants such as the Galerkin least-squares stabilization and variational multiscale approaches may also be employed. Stabilization for advection-diffusion applications on manifolds were considered in the work of Olshanskii and Reusken. The numerical results show that higher-order convergence rates are achieved provided that the finite element spaces are properly chosen. In addition, the conditioning of the system of equations depends on the element orders employed for the approximation of the individual physical fields. The presented results are based on well-known benchmark test cases in two dimensions such as driven cavity flows and cylinder flows with vortex shedding, which, herein, are extended to curved surfaces. Due to the higher-order elements, the results are highly accurate and may serve as future benchmarks in the context of (Navier-)Stokes flows on manifolds. Most test cases are carried out on parametrized surfaces; however, also the situation of flows on zero isosurfaces is covered herein.

To the best of our knowledge, this is the first time where (i) general higher-order surface FEM is used for the (in)stationary (Navier-)Stokes equations on manifolds including stabilization, (ii) numerical convergence studies are presented confirming higher-order convergence rates, and (iii) benchmark test cases are proposed and solutions presented. Furthermore, the notation employed is closely related to the typical engineering literature and aims to provide a bridge from the mathematical to the engineering community.

The paper is organized as follows. In Section 2, some requirements and properties of surfaces are described and tangential differential operators are defined based on the works of Dziuk and Elliot and Delfour and Zolésio. Section 3 covers the governing equations for (i) Stokes flow, (ii) stationary, and (iii) instationary Navier-Stokes flows on 2D manifolds. They are given in strong form, weak form, and discretized weak form according to the surface FEM. Numerical results are presented in Section 4. Convergence studies are performed for a test case for which an analytic solution is available, and it is shown that higher-order convergence rates can be achieved. For the other test cases where no analytic solutions are available, it is confirmed that in the flat 2D case, well-known reference solutions are reproduced. Various meshes with different orders and resolutions have been employed to obtain highly accurate results on curved surfaces.

Finally, a summary and outlook are given in Section 5.

2 | PRELIMINARIES

2.1 | Surfaces

The task is to solve a boundary value problem on an arbitrary surface $\Gamma$ in three dimensions. Let the surface be fixed in space over time, possibly curved, sufficiently smooth, orientable, connected (so that there is only one surface), and feature a finite area. There is a unit normal vector $n_\Gamma \in \mathbb{R}^3$ on $\Gamma$. The surface may be compact, ie, without a boundary, $\partial \Gamma = \emptyset$, see Figures 1A and 1B for examples. Otherwise, it may be bounded by $\partial \Gamma$, as shown in Figures 1C and 1D. Then, associated with $\partial \Gamma$, there is a tangential vector $t_{\partial \Gamma}$ pointing in direction of $\partial \Gamma$ and a conormal vector $n_{\partial \Gamma} = n_\Gamma \times t_{\partial \Gamma}$ pointing “outwards” and being normal to $\partial \Gamma$ and tangent to $\Gamma$. The surface may be given in parametrized form or implied, eg, based on the level-set method; both situations are considered herein. For the equivalence of these two cases and more mathematical details, see, eg, the work of Dziuk and Elliott.18
2.2 Surface operators

2.2.1 The tangential projector

On the manifold \( \Gamma \), the tangential projector \( P(x) \in \mathbb{R}^{3 \times 3} \) is defined by the normal vector as
\[
P(x) = I - n_{\Gamma}(x) \otimes n_{\Gamma}(x).
\]
Some important properties are (i) \( P \cdot n_{\Gamma} = 0 \), (ii) \( P = P^T \), and (iii) \( P \cdot P = P \).

2.2.2 Surface gradient of scalar quantities

The tangential gradient operator \( \nabla_{\Gamma} \) of a differentiable scalar function \( u : \Gamma \to \mathbb{R} \) on the manifold is given by
\[
\nabla_{\Gamma} u(x) = P(x) \cdot \nabla \tilde{u}(x), \quad x \in \Gamma,
\]
where \( \nabla \) is the standard gradient operator and \( \tilde{u} \) is a smooth extension of \( u \) in a neighborhood \( U \) of the manifold \( \Gamma \). Of course, \( \tilde{u} \) may also be some given function (rather than an arbitrary extension) in global coordinates, i.e., \( \tilde{u}(x) : \mathbb{R}^3 \to \mathbb{R} \). For the case of parametrized surfaces defined by the map \( x(r) : \mathbb{R}^2 \to \mathbb{R}^3 \) and a given scalar function \( u(r) : \mathbb{R}^2 \to \mathbb{R} \), the tangential gradient may be determined without explicitly computing an extension \( \tilde{u} \) using
\[
\nabla_{\Gamma} u(x(r)) = J(r) \cdot G^{-1}(r) \cdot \nabla_r u(r),
\]
with \( J = \frac{\partial x}{\partial r} \) being the \((3 \times 2)\)-Jacobi matrix and \( G = J^T \cdot J \) being the metric tensor (first fundamental form). Equation (2) shall be used later in the context of the FEM to determine tangential gradients of shape functions. It is noteworthy that \( \nabla_{\Gamma} u \) is in the tangent space of \( \Gamma \) and, thus, \( P \cdot \nabla_{\Gamma} u = \nabla_{\Gamma} u \) and \( \nabla_{\Gamma} u \cdot n_{\Gamma} = 0 \). The components of the tangential gradient are denoted by
\[
\nabla_{\Gamma} u(x) = (\partial_{x}^{\Gamma} u, \partial_{y}^{\Gamma} u, \partial_{z}^{\Gamma} u)^T,
\]
representing the first-order partial derivatives on \( \Gamma \). Second-order partial derivatives may be denoted by
\[
\text{He}_{ij} (u(x)) = \partial_{x_i x_j}^{\Gamma} u(x) = \partial_{x_i}^{\Gamma} \left( \partial_{x_j}^{\Gamma} u(x) \right),
\]
where \( \text{He}_{ij} (u(x)) \) is the tangential Hessian matrix. In the context of manifolds, this matrix is not symmetric,\(^35\) i.e., for mixed second derivatives \( \partial_{x_i x_j}^{\Gamma} u \neq \partial_{x_j x_i}^{\Gamma} u \) for \( i \neq j \).

2.2.3 Surface gradient of vector quantities

Next, operators for vector quantities \( u(x) : \Gamma \to \mathbb{R}^3 \) are considered. The “directional gradient” of \( u \) is the tensor of tangential derivatives and defined as
\[
\nabla_{\Gamma}^{\text{dir}} u(x) = \mathbf{P} \begin{bmatrix} u(x) \\ v(x) \\ w(x) \end{bmatrix} = \begin{bmatrix} \partial_{x}^{\Gamma} u & \partial_{y}^{\Gamma} u & \partial_{z}^{\Gamma} u \\ \partial_{x}^{\Gamma} v & \partial_{y}^{\Gamma} v & \partial_{z}^{\Gamma} v \\ \partial_{x}^{\Gamma} w & \partial_{y}^{\Gamma} w & \partial_{z}^{\Gamma} w \end{bmatrix} = \nabla \tilde{u} \cdot \mathbf{P}.
\]
In contrast, the covariant derivatives are
\[
\nabla_{\Gamma}^{\text{cov}} u(x) = \mathbf{P} \cdot \nabla_{\Gamma}^{\text{dir}} u(x) = \mathbf{P} \cdot \nabla \tilde{u} \cdot \mathbf{P}.
\]
One has to carefully distinguish these two different gradient operators. It is noted that \( \nabla_\Gamma^{\text{cov}} \mathbf{u} \) appears frequently in the modeling of physical phenomena on manifolds, ie, in the governing equations. On the other hand, \( \nabla_\Gamma \mathbf{u} \) is often used in straightforward extensions of identities such as product rules and divergence theorems. For example, we have for a scalar function \( f(\mathbf{x}) \) and vector functions \( \mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \)

\[
\nabla_\Gamma \mathbf{u} \cdot \nabla_\Gamma f = \nabla_\Gamma f \otimes \mathbf{u} + f \cdot \nabla_\Gamma \mathbf{u}, \\
\nabla_\Gamma \mathbf{u} \cdot \nabla_\Gamma \mathbf{v} = -\mathbf{u}^T \cdot \nabla_\Gamma \mathbf{v} + \nabla_\Gamma (\mathbf{u} \cdot \mathbf{v}).
\]

However, the relations are less straightforward for the covariant counterparts \( \nabla_\Gamma^{\text{cov}} (f \cdot \mathbf{u}) \) and \( \mathbf{v}^T \cdot \nabla_\Gamma^{\text{cov}} \mathbf{u} \), respectively. Later on, in the context of FEM implementations, it proves useful to transform covariant derivatives systematically to directional ones. This allows the computation of directional derivatives of FE (finite element) shape functions with respect to \( \mathbf{x} \in \mathbb{R}^3 \) independent of the integration of the weak form of the governing equations.

### 2.2.4 Divergence operators and divergence theorem

The divergence of a vector function \( \mathbf{u}(\mathbf{x}) : \Gamma \rightarrow \mathbb{R}^3 \) is given as

\[
\text{div}_\Gamma \mathbf{u}(\mathbf{x}) = \text{tr} \left( \nabla_\Gamma \mathbf{u} \right) = \text{tr} \left( \nabla_\Gamma^{\text{cov}} \mathbf{u} \right) = \nabla_\Gamma \cdot \mathbf{u}.
\]

For a tensor function \( \mathbf{A}(\mathbf{x}) : \Gamma \rightarrow \mathbb{R}^{3 \times 3} \), there holds

\[
\text{div}_\Gamma \mathbf{A}(\mathbf{x}) = \begin{bmatrix}
\text{div}_\Gamma (A_{11}, A_{12}, A_{13}) \\
\text{div}_\Gamma (A_{21}, A_{22}, A_{23}) \\
\text{div}_\Gamma (A_{31}, A_{32}, A_{33})
\end{bmatrix} = \nabla_\Gamma \cdot \mathbf{A}.
\]

It may be shown that \( \text{div}_\Gamma \mathbf{P} = -\mathbf{x} \cdot \mathbf{n}_\Gamma \) with \( \mathbf{x} = \text{tr}(\mathbf{H}) \) being the mean curvature and \( \mathbf{H} = \nabla_\Gamma^{\text{cov}} \mathbf{n}_\Gamma \) being the second fundamental form.

The following divergence theorem on manifolds is later needed for deriving the weak forms\(^{35,36}\):

\[
\int_\Gamma \mathbf{u} \cdot \text{div}_\Gamma \mathbf{A} \text{dA} = \int_\Gamma \nabla_\Gamma \mathbf{u} \cdot \mathbf{A} \text{dA} + \int_\Gamma \mathbf{x} \cdot \mathbf{u} \cdot \mathbf{A} \cdot \mathbf{n}_\Gamma \text{dA} + \int_\Gamma \mathbf{u} \cdot \mathbf{A} \cdot \mathbf{n}_\text{dir} \text{ds},
\]

\[(3)\]

where \( \nabla_\Gamma \mathbf{u} : \mathbf{A} = \nabla_\Gamma (\mathbf{u} \cdot \mathbf{A}^T) \). For tangential tensor functions with \( \mathbf{A} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P} \), the term involving the curvature \( \mathbf{x} \) vanishes because then \( \mathbf{A} \cdot \mathbf{n}_\Gamma = 0 \). In this case, we also have \( \nabla_\Gamma \mathbf{u} : \mathbf{A} = \nabla_\Gamma^{\text{cov}} \mathbf{u} : \mathbf{A} \).

### 3 GOVERNING EQUATIONS

In the following, we consider (i) stationary Stokes flow, (ii) stationary Navier-Stokes flow, and (iii) instationary Navier-Stokes flow on fixed manifolds. The governing equations are first given in strong and weak forms. The surface FEM is then applied for the discretization of the weak forms. As aforementioned, these models are also considered, eg, in other works\(^6-^8\) among others.

#### 3.1 Flow models in strong form

##### 3.1.1 Stationary Stokes flow

Starting point is stationary Stokes flow on a manifold. Let \( \mathbf{u}(\mathbf{x}) \in C^2(\Gamma) \) be the 3D velocity field on the surface \( \Gamma \), \( p(\mathbf{x}) \in C^1(\Gamma) \) a pressure field, and \( \mathbf{f}(\mathbf{x}) \) a tangential body force, eg, with unit \( \frac{N}{m^2} \). The governing field equations (in stress-divergence form\(^{37}\)) to be fulfilled \( \forall \mathbf{x} \in \Gamma \) are

\[
-P \cdot \text{div}_\Gamma \sigma(\mathbf{u}, p) = \mathbf{f}_\Gamma, \\
\text{div}_\Gamma \mathbf{u} = 0, \\
\mathbf{u} \cdot \mathbf{n}_\Gamma = 0.
\]

\[(4), (5), (6)\]
Stationary Navier-Stokes flow

Equation (4) expands to three momentum equations, Equation (5) is the incompressibility constraint and Equation (6) represents the tangential velocity constraint that restricts the velocities to the tangent space of $\Gamma$. Two different strain tensors are introduced

$$
\varepsilon^\text{dir}(\mathbf{u}) = \frac{1}{2} \cdot (\nabla^\text{dir}_\Gamma \mathbf{u} + (\nabla^\text{dir}_\Gamma \mathbf{u})^T),
$$  
(7)

$$
\varepsilon^\text{cov}(\mathbf{u}) = \frac{1}{2} \cdot (\nabla^\text{cov}_\Gamma \mathbf{u} + (\nabla^\text{cov}_\Gamma \mathbf{u})^T),
$$  
(8)

which are related to each other as $\varepsilon^\text{cov}(\mathbf{u}) = \mathbf{P} \cdot \varepsilon^\text{dir}(\mathbf{u}) \cdot \mathbf{P}$. The stress tensor is then defined as

$$
\sigma(\mathbf{u}, \rho) = -\rho \cdot \mathbf{P} + 2\mu \cdot \varepsilon^\text{cov}(\mathbf{u}),
$$

where $\mu \in \mathbb{R}^+$ is the (constant) dynamic viscosity. It is easily shown that

$$
-\mathbf{P} \cdot \text{div}_\Gamma \sigma(\mathbf{u}, \rho) = \nabla^\Gamma \rho - 2\mu \cdot \text{div}_\Gamma \varepsilon^\text{cov}(\mathbf{u}).
$$

Suppose there exists a boundary $\partial \Gamma$ of the manifold that consists of two nonoverlapping parts, the Dirichlet boundary, ie, $\partial \Gamma_D$, and the Neumann boundary, ie, $\partial \Gamma_N$. The corresponding boundary conditions are given as

$$
\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) \text{ on } \partial \Gamma_D,
$$

$$
\sigma(\mathbf{x}) \cdot \mathbf{n}_{\partial \Gamma}(\mathbf{x}) = \bar{\mathbf{i}}(\mathbf{x}) \text{ on } \partial \Gamma_N,
$$

where the prescribed velocities $\bar{\mathbf{u}}$ and tractions $\bar{\mathbf{i}}$ are in the tangent space of $\Gamma$, ie, $\bar{\mathbf{u}} \cdot \mathbf{n}_\Gamma = \bar{\mathbf{i}} \cdot \mathbf{n}_\Gamma = 0$.

Note that, in general, there are no explicit boundary conditions needed for the pressure $p$. In cases where no Neumann boundary is present, ie, $\partial \Gamma_N = \emptyset$ and $\partial \Gamma_D = \partial \Gamma$, the pressure is defined up to a constant.\textsuperscript{37,38} This includes compact manifolds where $\partial \Gamma = \emptyset$. In such situations, the pressure may be prescribed at a given point on $\Gamma$ or it is imposed by a constraint in the form of $\int_{\Gamma} \rho \, dA = 0$.

### Vorticity on manifolds.

The vorticity $\omega$ is a physical quantity frequently computed in flow problems. In the context of manifolds, we shall define

$$
\omega = \nabla^\text{cov}_\Gamma \times \mathbf{u}.
$$  
(10)

Note that $\omega$ is colinear to the normal vector $\mathbf{n}_\Gamma$; hence, $\mathbf{P} \cdot \omega = 0$. Therefore, it is useful to determine the signed magnitude of $\omega$, ie, the scalar function

$$
\omega^*(\mathbf{x}) = \omega \cdot \mathbf{n}_\Gamma = \pm ||\omega||, \quad \forall \mathbf{x} \in \Gamma.
$$  
(11)

This scalar quantity may also be obtained using directional derivatives, ie, $\omega^* = (\varepsilon^\text{dir}_\Gamma \times \mathbf{u}) \cdot \mathbf{n}_\Gamma$.

#### 3.1.2 Stationary Navier-Stokes flow

For stationary Navier-Stokes flow, a nonlinear advection term is added to Equation (4), resulting into

$$
\rho \cdot (\mathbf{u} \cdot \nabla^\text{cov}_\Gamma) \mathbf{u} - \mathbf{P} \cdot \text{div}_\Gamma \sigma(\mathbf{x}) = f_t(\mathbf{x}),
$$  
(12)

where $\rho \in \mathbb{R}^+$ is the (constant) fluid density with unit $\text{kg}/\text{m}^2$ and $(\mathbf{u} \cdot \nabla^\text{cov}_\Gamma) \mathbf{u} := (\nabla^\text{cov}_\Gamma \mathbf{u}) \cdot \mathbf{u}$. It is quite common to express the body force in the form $f_t(\mathbf{x}) = \rho \cdot g_x(x)$, where $g_x$ may consider gravity as $g_x = \mathbf{P} \cdot [0, 0, -9.81]^T \text{ m/s}^2$ for instance. The remaining Equations (5) and (6) and the boundary conditions (9) remain unchanged. The solution of the nonlinear governing equations can be obtained iteratively based on the Newton-Raphson method or other fixed-point iterations such as Picard iterations. Because the advection operator is not self-adjoint, well-known stability issues may arise for large Reynolds numbers in a numerical context.

#### 3.1.3 Instationary Navier-Stokes flow

For instationary Navier-Stokes flow, the momentum equation (4) changes to

$$
\rho \cdot (\partial_t \mathbf{u}(\mathbf{x}, t) + (\mathbf{u} \cdot \nabla^\text{cov}_\Gamma) \mathbf{u} - g_x(x, t)) - \mathbf{P} \cdot \text{div}_\Gamma \sigma(\mathbf{x}, t) = 0.
$$  
(13)

The functions representing the physical fields live in space (on $\Gamma$) and time, ie, in the time interval $\tau = [0, T]$. Therefore, Equations (5), (6), and (13) have to be solved in the space-time domain $\Gamma \times \tau$. Herein, we restrict ourselves to spatially fixed manifolds $\Gamma$. 
The boundary conditions (9) also extend in time dimension; hence, there are prescribed velocities $\tilde{u}(x, t)$ along $\partial \Gamma_D \times t$ and tractions $\tilde{t}(x, t)$ along $\partial \Gamma_N \times t$. Furthermore, an initial condition is needed

$$u(x, 0) = u_0(x), \text{ with } \text{div}_\Gamma u_0 = 0 \text{ and } u_0 \cdot n_\Gamma = 0, \quad \forall x \in \Gamma \text{ at } t = 0. \tag{14}$$

### 3.2 Flow models in weak form

The following trial and test function spaces are introduced:

$$S_u = \{u \in H^1(\Gamma)^3, u = \tilde{u} \text{ on } \partial \Gamma_D\}, \tag{15}$$

$$V_u = \{w_u \in H^1(\Gamma)^3, w_u = 0 \text{ on } \partial \Gamma_D\}, \tag{16}$$

$$S_p = \mathcal{L}^2(\Gamma), \tag{17}$$

$$S\lambda = \mathcal{V}_\lambda = \mathcal{L}^2(\Gamma). \tag{18}$$

As mentioned previously, if no Neumann boundary exists, i.e., $\partial \Gamma_N = \emptyset$, the pressure is defined up to a constant and one may replace $S_p$ by

$$S_p^0 = \left\{p \in \mathcal{L}^2(\Gamma), \quad \int_\Gamma p \, dA = 0 \right\}. \tag{19}$$

#### 3.2.1 Stationary Stokes flow

The weak form of the Stokes problem becomes the following. Given viscosity $\mu \in \mathbb{R}^+$, body force $f(x)$ in $\Gamma$, and traction $\tilde{t}(x)$ on $\partial \Gamma_N$, find the velocity field $u(x) \in S_u$, pressure field $p(x) \in S_p$, and Lagrange multiplier field $\lambda(x) \in S\lambda$ such that for all test functions $(w_u, w_p, w_\lambda) \in V_u \times V_p \times V_\lambda$, there holds in $\Gamma$

$$\int_\Gamma \nabla_{\Gamma} \cdot w_u \cdot \sigma(u, p) \, dA + \int_\Gamma \lambda \cdot (w_u \cdot n_\Gamma) \, dA = \int_\Gamma w_u \cdot f \, dA + \int_\Gamma w_u \cdot \tilde{t} \, ds, \tag{20}$$

$$\int_\Gamma w_p \cdot \text{div}_\Gamma u \, dA = 0, \tag{21}$$

$$\int_\Gamma w_\lambda \cdot (u \cdot n_\Gamma) \, dA = 0. \tag{22}$$

In order to obtain Equation (20), the divergence theorem (3) was applied to $-\int_\Gamma w_u \cdot \text{div}_\Gamma \sigma \, dA$ where the curvature term vanishes due to $\sigma \cdot n_\Gamma = 0$. Using the definition of the stress tensor, we get

$$\int_\Gamma \nabla_{\Gamma} \cdot w_u \cdot (p \cdot P) \, dA = \int_\Gamma \nabla_{\Gamma} \cdot (p \cdot P) \, dA + 2\mu \cdot \int_\Gamma \nabla_{\Gamma} \cdot \varepsilon^\text{cov}(u) \, dA. \tag{23}$$

The following relations are easily derived:

$$\nabla_{\Gamma} \cdot w_u = (p \cdot P) + 2\mu \cdot \varepsilon^\text{cov}(u). \tag{24}$$

It is readily verified that solutions of the strong form also fulfill the weak form in Equations (19)-(21). This is obvious for Equations (20) and (21) due to Equations (5) and (6), respectively. For the momentum equations, it is noted that (19) is fulfilled for $-\text{div}_\Gamma \sigma(u, p) + \lambda \cdot n_\Gamma = f$. Restricting this to the tangential space by multiplication with the projector $P$ yields the strong form of the momentum equations (4) because $P \cdot n_\Gamma = 0$. It is thus also seen that the Lagrange multiplier field $\lambda$ may be physically interpreted as a force in normal direction.
3.2.2 | Stationary Navier-Stokes flow

The weak form of the stationary Navier-Stokes equations is similar to the aforementioned Stokes problem; however, Equation (19) is replaced by

\[
\rho \cdot \oint_{\Gamma} \left( \mathbf{w}_u \cdot \left( \mathbf{u} \cdot \nabla_{\Gamma}^{\text{cov}} \right) \right) \mathbf{u} \, dA + \oint_{\Gamma} \nabla_{\Gamma}^{\text{dir}} \mathbf{w}_u : \sigma(\mathbf{u}, p) \, dA + \oint_{\Gamma} \lambda \cdot (\mathbf{w}_u \cdot \mathbf{n}_f) \, dA = \oint_{\Gamma} \mathbf{w}_u \cdot \hat{\mathbf{f}} \, dA + \oint_{\partial \Gamma_N} \mathbf{w}_u \cdot \hat{\mathbf{n}} \, ds,
\]

where the added advection term is readily identified.

3.2.3 | Instationary Navier-Stokes flow

The weak form of the instationary Navier-Stokes problem is as follows. Given density \( \rho \in \mathbb{R}^+ \), viscosity \( \mu \in \mathbb{R}^+ \), body force \( \mathbf{f}(\mathbf{x}, t) \) in \( \Gamma \times \tau \), traction \( \mathbf{t}(\mathbf{x}, t) \) on \( \partial \Gamma_N \times \tau \), and initial condition \( u_0(\mathbf{x}) \) on \( \Gamma \) at \( t = 0 \), according to (14), find the velocity field \( \mathbf{u}(\mathbf{x}, t) \in L_2(\tau; S_u) \), pressure field \( p(\mathbf{x}, t) \in L_2(\tau; S_p) \), and Lagrange multiplier field \( \lambda(\mathbf{x}, t) \in L_2(\tau; S_\lambda) \) such that for all test functions \( (\mathbf{w}_u, w_p, w_\lambda) \in \mathcal{V}_u \times \mathcal{V}_p \times \mathcal{V}_\lambda \), there holds in \( \Gamma \times \tau \)

\[
\rho \cdot \oint_{\Gamma} \left( \mathbf{w}_u \cdot \left( \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\Gamma}^{\text{cov}}) \mathbf{u} - \mathbf{g} \right) \right) \, dA + \oint_{\Gamma} \nabla_{\Gamma}^{\text{dir}} \mathbf{w}_u : \sigma(\mathbf{u}, p) \, dA + \oint_{\Gamma} \lambda \cdot (\mathbf{w}_u \cdot \mathbf{n}_f) \, dA = \oint_{\Gamma} \mathbf{w}_u \cdot \hat{\mathbf{f}} \, dA,
\]

\[
\oint_{\Gamma} w_p \cdot \text{div}_\Gamma \mathbf{u} \, dA = 0,
\]

\[
\oint_{\Gamma} w_\lambda \cdot (\mathbf{u} \cdot \mathbf{n}_f) \, dA = 0.
\]

3.3 | Surface FEM for flows on manifolds

3.3.1 | Surface meshes

Assume that a suitable surface mesh composed by higher-order triangular or quadrilateral Lagrange elements of order \( q \) may be generated with desired element sizes and all nodes on \( \Gamma \). Well-known necessary requirements of meshes such as the shape regularity of the elements and bounds on inner angles are fulfilled. The shape of each (physical) element in the mesh results from a map of the corresponding reference element with \( n_q \) nodes

\[
x(\mathbf{r}) = \begin{bmatrix} x(r, s) \\ y(r, s) \\ z(r, s) \end{bmatrix} = \sum_{i=1}^{n_q} N_i^q(\mathbf{r}) \mathbf{x}_i.
\]

\( N_i^q(\mathbf{r}) \) are classical Lagrangian shape functions of order \( q \) in reference coordinates \( \mathbf{r} \in \mathbb{R}^2 \) and \( \mathbf{x}_i \in \Gamma \) are the nodal coordinates. The resulting mesh is an approximation \( \Gamma_q^h \subset C^0 \) of the exact surface \( \Gamma \). Clearly, \( \Gamma_q^h \) is defined parametrically through the map (26) even if the original \( \Gamma \) was implicitly given, eg, by the zero isosurface of a level-set function. See other works\(^{39-41} \) for the automatic generation of higher-order meshes on zero isosurfaces. The discrete unit normal vector is

\[
n_i^h = \frac{\partial_r \mathbf{x} \times \partial_s \mathbf{x}}{\| \partial_r \mathbf{x} \times \partial_s \mathbf{x} \|}
\]

and is not smooth across element edges due to the \( C^0 \)-continuity of the surface mesh. The discrete tangent and conormal vectors \( \mathbf{t}_{\text{dir}}^h \) and \( \mathbf{n}_{\text{dir}}^h \) are easily obtained along the element edges on the boundary of \( \Gamma_q^h \). The definitions of the surface operators from Section 2.2 readily extend to the case of a discrete manifold \( \Gamma_q^h \) and are not repeated here.

3.3.2 | Surface FEM

We use higher-order surface FEM as detailed, eg, in the works of Demlow\(^{26} \) and Dziuk and Elliott\(^{48} \) for the discretization of the weak forms from Section 3.2. Finite element spaces of different orders are involved. As aforementioned, suitable surface meshes of order \( q \) may be generated defining approximations \( \Gamma_q^h \in C^0 \) of \( \Gamma \). Let there be a “geometry mesh” of order \( q = k_{\text{geom}} \) with the sole purpose to approximate the geometry of the manifold \( \Gamma_q^h = \Gamma_{k_{\text{geom}}}^h \) and define the element maps (26). In particular, this mesh is not used to imply a finite element space for the approximation of the weak forms.
Next, a finite element space of order \( k \) is generated on \( \Gamma^h \) for which it is assumed that there is a second mesh of order \( k \). The two meshes feature the same element types and number of elements with identical coordinates at the corners; however, the total number of nodes differs due to the individual orders. It is emphasized that the coordinates of the nodes in the \( k \)th order mesh are, in fact, never needed and it is only the connectivity, which is required to set up the finite element space.

Associated to triangular or quadrilateral elements in the \( k \)th order mesh, there is a fixed set of local basis functions \( \{ N^k_i(r) \} \) defined in a reference element with \( i = 1, \ldots, n_i \) and \( n_i \) being the number of nodes per element. Classical Lagrange basis functions with \( N^k_i(r) = \delta_{ij} \) are used herein. Based on the map (26), which is completely determined by the geometry mesh, one may generate \( \{ N^k_i(x(r)) \} \) for all \( x \in \Gamma^h \) and tangential derivatives \( \nabla_{\Gamma} N^k_i(x(r)) \) are determined based on Equation (2). This is only an isoparametric map when \( k = k_{\text{geom}} \). Summing up the element contributions for nodes belonging to several elements, this generates a set of global \( C^0 \)-continuous basis functions \( \{ M^k_i(x(r)) \} \) in \( \Gamma^h \) with \( i = 1, \ldots, n^k_{\text{nodes}} \) and \( n^k_{\text{nodes}} \) being the number of nodes of the \( k \)th order surface mesh. Note that to generate the nodal basis \( \{ M^k_i(x(r)) \} \), only the coordinates of the geometry mesh are needed; however, not from the \( k \)th order mesh. A general finite element space of order \( k \) is now defined by

\[
Q^h_k = \left\{ u^h \in C_0\left( \Gamma^h_{k_{\text{geom}}} \right) : u^h = \sum_{i=1}^{n^k_{\text{nodes}}} M^k_i(x(r)) \cdot u_i, \ u_i \in \mathbb{R} \right\} \subset H^1\left( \Gamma^h_{k_{\text{geom}}} \right).
\]

Based on this, the following discrete trial and test function spaces are defined:

\[
S^h_u = \left\{ u^h \in [Q^h_k]^3 : u^h = \tilde{u}^h \text{ on } \partial\Gamma^h_D \right\},
\]

\[
V^h_u = \left\{ \mathbf{w}^h_u \in [Q^h_k]^3 : \mathbf{w}^h_u = \mathbf{0} \text{ on } \partial\Gamma^h_D \right\},
\]

\[
S^h_p = V^h_p = Q^h_{k_p},
\]

\[
S^h_j = V^h_j = Q^h_{k_j}.
\]

Although shape functions for the pressure and the Lagrange multiplier for enforcing the tangential velocity constraint may be discontinuous, we restrict ourselves to classical \( C^0 \)-continuous approximations. Note that individual orders \( k_u, k_p, \) and \( k_j \) are associated to the approximations of velocities \( u^h \), pressure \( p^h \), and Lagrange multiplier field \( \lambda^h \), respectively. Analogous to the continuous case, one may impose that the functions in \( S^h_p \) have to fulfill \( \int_{\Gamma} p^h \, dA = 0 \) if no Neumann boundary is present.

### 3.3.3 Stationary Stokes flow

The discrete weak form of the Stokes problem reads as follows. Given viscosity \( \mu \in \mathbb{R}^+ \), body force \( f^h(x) \) in \( \Gamma^h \), and traction \( t^h(x) \) on \( \partial\Gamma^h_N \), find the velocity field \( \mathbf{u}^h(x) \in S^h_u \), pressure field \( p^h(x) \in S^h_p \), and Lagrange multiplier field \( \lambda^h(x) \in S^h_j \) such that for all test functions \( (\mathbf{w}_u^h, w_p^h, w_j^h) \in V^h_u \times V^h_p \times V^h_j \), there holds in \( \Gamma^h \)

\[
\int_{\Gamma} \nabla_{\Gamma} \cdot \mathbf{w}_u^h \cdot \mathbf{u}^h \, dA + \int_{\Gamma} \lambda^h \cdot (\mathbf{w}_u^h \cdot \mathbf{n}_u^h) \, dA = \int_{\Gamma} \mathbf{w}_u^h \cdot f^h \, dA + \int_{\partial \Gamma_N} \mathbf{w}_u^h \cdot \mathbf{t}^h \, ds,
\]

\[
\int_{\Gamma} w_p^h \cdot \text{div}_{\Gamma} \mathbf{u}^h \, dA = 0,
\]

\[
\int_{\Gamma} w_j^h \cdot (\mathbf{u}^h \cdot \mathbf{n}_u^h) \, dA = 0.
\]

The usual element assembly yields a linear system of equations in the form

\[
\begin{bmatrix}
K & G & L \\
G^T & 0 & 0 \\
L^T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
p \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f \\
0 \\
0
\end{bmatrix},
\]
with \([\mathbf{u}, \mathbf{b}, \lambda]^T = [\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}, \lambda]^T\) being the sought nodal values of the velocity components, pressure, and Lagrange multiplier. For the implementation, it is interesting to compare the system (34) with the system obtained for a classical 3D Stokes problem

\[
\begin{bmatrix}
K_{3D} & G_{3D} & 0 \\
G_{3D}^T & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
\mathbf{p} \\
\lambda \\
\end{bmatrix}
= \begin{bmatrix}
f \\
0 \\
0 \\
\end{bmatrix}.
\]  

(35)

Assume a function, which generates \(K_{3D}\) and \(G_{3D}\) based on 3D FE shape functions (including classical partial derivatives with respect to \(x\)), evaluated at given integration points in 3D. The same function may be used for generating \(K\) and \(G\) provided that (i) the integration points are restricted to \(Γ^h\) with proper weights; (ii) the classical partial derivatives in \(V\) are replaced by the tangential derivatives as in \(V_{Γ}^{dir}\), and (iii) the contribution to \(K\) at the current integration point, ie, \(K(x_i)\), is projected as \(K(x_i) = P(x_i) \cdot K_{3D}(x_i) \cdot P(x_i)\), which is due to Equation (22). The same shall later hold for the advection matrix \(C(\mathbf{u})\) in the Navier-Stokes equations.

As expected in the context of the Lagrange multiplier method, the matrix in Equation (34) has a saddle-point structure and is typical for a mixed FEM. The well-known Babuška-Brezzi condition\(^{20-22}\) must be fulfilled to obtain useful solutions for flows at high Reynolds numbers.\(^{37,38}\) In particular, the SUPG method is used for the stabilization. Different definitions of the stabilization parameter \(r_{SUPG}\) are found\(^{30,44,45}\) and

\[
r_{SUPG} = \left[ \left( \frac{2}{\Delta t} \right)^2 + \left( \frac{2 \| \mathbf{u_c} \|}{h_c} \right)^2 + \left( \frac{4 \mu}{h_c^2} \right)^2 \right]^{-1/2}
\]

is used herein with element-averaged velocity \(\mathbf{u_c}\), element length \(h_c\), and \(\Delta t \to \infty\) for the stationary case. When stabilization is not necessary because no oscillations occur, \(r_{SUPG} = 0\). Note that, in the stabilization term, second-order derivatives appear (only in the element interiors). The definition of tangential second-order derivatives is given, eg, in the work of Delfour and Zolésio.\(^{35}\)

Element assembly results in a nonlinear system of equations of the form

\[
\begin{bmatrix}
K^* + C(\mathbf{u}) & G^* & L \\
G^T & 0 & 0 \\
L^T & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
\mathbf{p} \\
\lambda \\
\end{bmatrix}
= \begin{bmatrix}
f \\
0 \\
0 \\
\end{bmatrix}.
\]  

(36)
which is no longer symmetric (partly) due to the advection matrix \( C(u) \). The distinguishing feature of \( K^* \) and \( G^* \) (compared with \( K \) and \( G \) of the Stokes problem) are the added SUPG-stabilization terms. The issues related to mixed FEMs and the Babuška-Brezzi condition remain relevant.

### 3.3.5 Instationary Navier-Stokes flow

The discrete weak form of the instationary Navier-Stokes problem is the following. Given density \( \rho \in \mathbb{R}^+ \), viscosity \( \mu \in \mathbb{R}^+ \), body force \( \rho \cdot g^h(x, t) \) in \( \Gamma^h \times \tau \), traction \( \mathbf{t}^h(x, t) \) on \( \partial \Gamma^h \times \tau \), and initial condition \( u^h_0(x) \) on \( \Gamma^h \) at \( t = 0 \) according to (14), find the velocity field \( u^h(x, t) \in L_2(\tau; S^h) \), pressure field \( p^h(x, t) \in L_2(\tau; \mathcal{S}^h) \), and Lagrange multiplier field \( \lambda^h(x, t) \in L_2(\tau; S^h) \) such that for all test functions \( (w^h_u, w^h_p, w^h_\lambda) \in \mathcal{V}^h_u \times \mathcal{V}^h_p \times \mathcal{V}^h_\lambda \), there holds in \( \Gamma^h \times \tau \)

\[
\begin{align*}
\rho \int_{\Gamma} u^h \cdot (\partial_t u^h + (u^h \cdot \nabla^\text{cov}) u^h - g^h) \, dA + & \int_{\Gamma} \nabla^\text{dir} w^h_u : \sigma(u^h, p^h) \, dA + \int_{\Gamma} \lambda^h \cdot (w^h_u \cdot n^h_t) \, dA \\
- \int_{\partial \Gamma_N} w^h_u \cdot \mathbf{t}^h \, ds + & \int_{\Gamma} w^h_p \cdot \text{div}_\Gamma u^h \, dA + \int_{\Gamma} w^h_\lambda \cdot (u^h \cdot n^h_t) \, dA \\
+ \sum_{e=1}^{n_e} & \int_{\Gamma_e} r_{\text{SUPG}} \left( (u^h \cdot \nabla^\text{cov} w^h_u) \cdot \left[ \rho \cdot (\partial_t u^h + (u^h \cdot \nabla^\text{cov}) u^h - g^h) - \text{div}_\Gamma \sigma(u^h, p^h) \right) \right] = 0.
\end{align*}
\]

This yields a system of nonlinear semidiscrete equations for \( t \in \tau \)

\[
M \cdot \dot{u}(t) + (K^* + C(u)) \cdot u(t) + G^* \cdot p(t) + L \cdot \lambda(t) = f(t),
\]

\[
G^T \cdot u(t) = 0,
\]

\[
L^T \cdot u(t) = 0.
\]

with initial condition \( u(0) \). This system may be advanced in time by using finite difference schemes, and the Crank-Nicolson method is employed herein.

### 4 NUMERICAL RESULTS

The following error measures are computed in the convergence studies. When analytic (exact) velocity and pressure fields, ie, \( u_{\text{ex}} \) and \( p_{\text{ex}} \), are known, the velocity error is determined by

\[
\varepsilon_u = \sum_{i=1}^{3} \sqrt{ \int_{\Gamma} (u_i^h(x) - u_{\text{ex}}(x))^2 \, dA }.
\]

and the pressure error calculated as

\[
\varepsilon_p = \sqrt{ \int_{\Gamma} (p^h(x) - p_{\text{ex}}(x))^2 \, dA }.
\]

When analytic solutions are not available, it is useful to evaluate the error of the FE approximations in the strong form of the momentum or continuity equations, integrated over all element interiors. For the example of stationary Stokes flow, the corresponding residual errors are defined as

\[
\varepsilon_{\text{mom}} = \sqrt{ \int_{\Gamma} \left( \mathbf{P} \cdot \text{div}_\Gamma \sigma(u^h, p^h) + f^h \right)^2 \, dA }.
\]

and

\[
\varepsilon_{\text{cont}} = \sqrt{ \int_{\Gamma} \left( \text{div}_\Gamma u^h \right)^2 \, dA }.
\]

This can be easily extended to the case of Navier-Stokes flows where the advection term is added to the integrand in (40). Moreover, the error in the tangential velocity constraint from Equation (6) may be computed in a similar manner. The
evaluation of the error $\varepsilon_{\text{mom}}$ involves second-order derivatives, and convergence can only be expected for higher-order elements and sufficiently smooth solutions.

### 4.1 Stokes flow on an axisymmetric surface

A test case is developed for which analytic solutions are available. An axisymmetric surface with height $L = 5$ and radius $r(z) = 1 + \frac{1}{5} \cdot \sin(1 + 3 \cdot z)$, $z \in [0, L]$, is generated as illustrated in Figure 2A. Let $r_0 = r(0)$ and $r'_0 = \frac{dr(0)}{dz}$. In parametrized form, one may also define $\Gamma$ based on the map $x(a) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$x(a) = \begin{bmatrix} \cos a \cdot r(b) \\ \sin a \cdot r(b) \\ b \end{bmatrix}$$

with $a \in [0, 2\pi]$, $b \in [0, 5]$.

The lower boundary at $z = 0$ is the Dirichlet boundary $\Gamma_D$, where the inflow in conormal direction of the manifold is prescribed as

$$\hat{u}(x) = \frac{u^*}{\|u^*\|} \quad \text{and} \quad u^* = \begin{bmatrix} r'_0 \cdot \cos \theta \\ r'_0 \cdot \sin \theta \\ 1 \end{bmatrix},$$

with angle $\theta$ given by $\tan \theta = \gamma / \rho$. The upper boundary at $z = L$ is the outflow boundary where zero tractions are applied as Neumann boundary conditions. The density and viscosity are set to $\rho = 1$ and $\mu = 0.01$, respectively.

The mass flow on the lower boundary is

$$Q_0 = \int_{\Gamma_D} \hat{u}(x) \cdot n_{\Gamma_D} \, ds = 2\pi \cdot r_0$$

and due to mass conservation, the mass flow along the height follows as $Q(z) = 2\pi \cdot r'_0 / \rho$. As the flow field is expected to be axisymmetric for this test case and the tangential velocity constraint applies, one may compute the velocity components as

$$\begin{bmatrix} u_{\text{ex}}(x) \\ v_{\text{ex}}(x) \\ w_{\text{ex}}(x) \end{bmatrix} = \frac{r_0}{r \cdot \sqrt{1 + \left(\frac{dr}{dz}\right)^2}} \cdot \begin{bmatrix} \frac{dr}{dz} \cdot \gamma / r \\ \frac{dr}{dz} \cdot \gamma / r \\ 1 \end{bmatrix}.$$
See Figure 3 for a graphical representation. It is noted that the mass flow $Q(z)$, velocity magnitude $\|u\|$, and the vertical velocity component $w$ are only functions of $z$, ie, they do not vary in $x$- and $y$-directions.

Finite element approximations are carried out on various meshes composed by triangular or quadrilateral Lagrange elements of different orders. For the convergence studies, meshes with $n_z = \{4, 6, 10, 14, 20, 30, 40, 60\}$ elements over the height are chosen; the number of elements in circumferential direction is $n_\theta = \text{round}\left(\frac{2\pi r_0}{L} \cdot n_z\right)$. The meshes are perturbed, as illustrated in Figures 2B to 2D, to avoid perfectly axisymmetric meshes which, otherwise, could have improved the convergence rates for this special case.

The individual element orders used for the convergence studies are indicated by a four-tuple $\{k_{\text{geom}}, k_u, k_p, k_\lambda\}$. To be precise, this tuple summarizes the employed orders for the geometry, $k_{\text{geom}}$, the velocities, $k_u$, the pressure, $k_p$, and the
Lagrange multiplier for enforcing the tangential velocity constraint, $k$. For each tuple, meshes with different resolutions (given by $n_z$ and $n_θ$) are considered and errors calculated, each time resulting in one curve in the convergence plots as indicated in the legends.

Systematic studies of different combinations of element orders showed that equal-order approximations for the velocity and pressure, i.e., $k_p = k_u$, do not converge satisfactorily (or at all), which is well known from the standard context of the incompressible Navier-Stokes equations in 2D and 3D due to the Babuška-Brezzi condition. For the studies outlined in this paper, we shall choose $k_p = k_u - 1$, which is a popular choice for FEM approximations of classical incompressible flows and known as Taylor-Hood elements.\(^{46}\)

For the first study, we use $2 \leq k_u \leq 5$, $k_p = k_u - 1$, and $k_λ = k_u - 1$ which, later on, becomes the recommended standard setting. Convergence plots for $ε_u$ and $ε_p$ are given in Figure 4. The thick solid lines are for $k_{geom} = k_u + 1$. It is noteworthy that, for quadrilateral elements, setting $k_{geom} = k_u$ leads to almost identical results as seen from the thin dashed lines in Figures 4C and 4D. This does not necessarily hold for triangular elements; see Figures 4A and 4B where the convergence may drop by one order when setting $k_{geom} = k_u$ rather than $k_{geom} = k_u + 1$. This is later confirmed for the errors $ε_{mom}$ and $ε_{cont}$ in Figure 6. Therefore, we recommend to choose the geometry one order higher than $k_u$, which is done in the

![Figure 5](Colour figure can be viewed at wileyonlinelibrary.com)

**Figure 5** Influence of the order of the Lagrange multiplier field for enforcing the tangential velocity constraint: (A) convergence results in $ε_u$ and (B) conditioning $κ$ for the axisymmetric test case. [Colour figure can be viewed at wileyonlinelibrary.com]

![Figure 6](Colour figure can be viewed at wileyonlinelibrary.com)

**Figure 6** Convergence results in $ε_{mom}$ and $ε_{cont}$ for the axisymmetric test case. (A) and (B) for triangular elements, (C) and (D) for quadrilateral elements. The legends decode the orders $\{k_{geom}, k_u, k_p, k_λ\}$ of the meshes. [Colour figure can be viewed at wileyonlinelibrary.com]
It is important to note in Figure 4 that the convergence rates in the pressure are optimal, i.e., \( m_p = k_p + 1 \); however, in the velocities one-order suboptimal, \( m_u = k_u \). We have traced this back to the influence of the order \( k_j \) of the Lagrange multiplier field. This is demonstrated in Figure 5 where Figure 5A shows the error \( \epsilon_u \) and Figure 5B the condition number \( \kappa \) of the corresponding system of equations (obtained with MATLAB’s \texttt{condest} function). As aforementioned, \( k_{\text{geom}} = k_u + 1 \) and \( k_p = k_u - 1 \). Figure 5 shows that setting \( k_j = 1 \) yields convergence rates \( m_u = 2 \), independent of the other orders (black lines). Setting \( k_j = k_u \) yields optimal convergence rates \( m_u = k_u + 1 \) for the velocities (red lines); however, there is a dramatic influence on the conditioning, which scales with \( \kappa \sim O(h^{-6}) \) in this case rather than with \( O(h^{-2}) \) for all choices where \( k_j < k_u \). Therefore, we set \( k_j = k_u - 1 \) in the following and accept the suboptimal convergence in the velocities.

Next, the error is observed in the strong form of the momentum and continuity equations, i.e., \( \epsilon_{\text{mom}} \) and \( \epsilon_{\text{cont}} \); see Equations (39) and (40). Results for \( k_p = k_j = k_u - 1 \) are depicted in Figure 6 for triangular and quadrilateral elements. Again, the thick lines refer to \( k_{\text{geom}} = k_u + 1 \) and the thin dashed lines to \( k_{\text{geom}} = k_u \). As aforementioned for the \( L_2 \)-errors in the velocities and pressure, this makes a difference (of one order) for triangular elements, however, not for quadrilateral elements. When using \( k_{\text{geom}} = k_u + 1 \) on the safe side, the convergence rate in \( \epsilon_{\text{mom}} \) is \( m_{\text{mom}} = k_u - 1 \) as expected due to the presence of second-order derivatives of \( u \) in the momentum equations. The expected convergence rate in \( \epsilon_{\text{cont}} \) is \( m_{\text{cont}} = k_u \) due to the presence of first-order derivatives of \( u \) in the continuity equation.

### 4.2 Driven cavity flows on manifolds

The stationary Navier-Stokes model is considered in this example. Starting point is the driven cavity for the case of a flat 2D domain as depicted in Figure 7A. This case has well-documented reference solutions for a variety of Reynolds numbers.\(^47\) There, a flow inside a quadratic domain \( \Omega_{2D} = (0, 1) \times (0, 1) \) with no-slip boundary conditions on the left, right, and lower wall develops under a shear flow of \( u = 1.0 \) and \( v = 0.0 \) applied on the upper boundary until a stationary solution is reached. The Reynolds number is computed as \( \text{Re} = \rho \cdot u \cdot L / \mu \).

Herein, the situation is extended to curved surfaces in 3D by deforming the flat 2D domain \( \Omega_{2D} \) in \( z \)-direction using functions \( z(x, y) \). In particular, two different maps A and B are used

\[
\begin{align*}
\text{map A:} & \quad z(x, y) = \alpha \cdot (-1 + 8x + 2y - 8x^2) \cdot (1 - y), \\
\text{map B:} & \quad z(x, y) = \beta \cdot (1 - y) \cdot \sin((2x - 1)x) \cdot \cos((2y - 1)y) ,
\end{align*}
\]

where \( \alpha \) and \( \beta \) scale the height in \( z \)-direction; see Figures 7B and 7C for examples. The advantage is that, for \( \alpha = 0 \) and \( \beta = 0 \), the flat situation is recovered and the reference solutions in the work of Ghia et al\(^47\) are relevant. We have confirmed that these solutions are recovered with great accuracy also for any rigid body transformation of \( \Omega_{2D} \) into three dimensions. The density is chosen as \( \rho = 1 \) and two different viscosities of \( \mu = 0.01 \) and \( \mu = 0.001 \), leading to Reynolds numbers of

![Figure 7](http://wileyonlinelibrary.com)
FIGURE 8  Velocity and pressure fields for the driven cavity test case with $\mu = 0.001$ for map A with $\alpha = 0.4$ and map B with $\beta = 0.4$. [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 9  Different meshes for the driven cavity test case in top view. [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 10  Velocity profiles for the driven cavity test case for different $\alpha$ and $\beta$. The vertical profiles show the velocity component $u(x)$ and the horizontal profiles $v(x)$. The scaling factor of the velocities is 0.5. [Colour figure can be viewed at wileyonlinelibrary.com]

Re = 100 and Re = 1000 for the flat case, respectively. Solutions for the velocity magnitude and pressure field for some example manifolds are displayed in Figure 8.

The meshes feature quadrilateral elements of different orders and are refined toward the boundaries to capture the resulting boundary layers. See Figure 9 for the meshes in $\Omega_{2D}$, which are mapped to 3D according to maps A and B from above for various scaling coefficients $\alpha$ and $\beta$. The number of elements per dimension is $n = \{10, 20, 30, 50, 70, 100\}$. For the numerical studies, $k_p = k_\lambda = k_u - 1$ and $k_{geom} = k_u + 1$ is used as recommended previously.
FIGURE 11  Velocity profiles for the driven cavity test case for different $\alpha$ and $\beta$. The horizontal and vertical profiles show the velocity component $w(x)$ scaled by the factor 1 [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 12  Velocity profiles for the driven cavity test case following Figure 10. Results of coarse meshes with $10 \times 10$ elements are compared with the high-accuracy results (with $100 \times 100$ elements with $k_u = 4$). A, Flat manifold; B, Manifold according to map $A$ with $\alpha = 1.0$ [Colour figure can be viewed at wileyonlinelibrary.com]

Just as for the reference solutions in the work of Ghia et al., the results are presented as velocity profiles along the horizontal and vertical centerlines in $\Omega_{2D}$. Figure 10 shows the profiles for the velocity component $u$ along the vertical centerline and $v$ along the horizontal centerline for the two maps with different scaling factors $\alpha$ and $\beta$, respectively. The crosses indicating the reference solution from the work of the aforementioned author are only relevant for the flat case where $\alpha = \beta = 0$. The results for the velocity component $w$ along the two centerlines are given in Figure 11. These results have the quality of benchmark solutions and have been obtained with $k_u = 4$ and 100 elements per dimensions. The convergence of other element orders and mesh resolutions toward these profiles have been confirmed, and a small selection is shown in Figure 12. Without stabilization, the typical oscillations are seen for this rather high Reynolds number for coarse meshes with low order. As no analytical solutions for the velocities and pressure are available, it is impossible to provide convergence results in $\varepsilon_u$ and $\varepsilon_p$. Furthermore, the singular pressure in the upper left and right corners lead to singularities in the derivatives of other physical fields. Thus, it cannot be expected that (optimal) convergence in $\varepsilon_{\text{mom}}$ and $\varepsilon_{\text{cont}}$ is achieved.
4.3 Flows on zero-level sets

The next test case shows the potential to solve flows on zero-level sets with the proposed models. Stationary Stokes and Navier-Stokes flows are considered. The scalar function \( \phi(x) : \mathbb{R}^3 \to \mathbb{R} \) is based on the work of Dziuk and Elliott\(^{18} \) and defined as

\[
\phi(x) = (x^2 + y^2 - 4)^2 + (x^2 + z^2 - 4)^2 + (y^2 + z^2 - 4)^2 + (x^2 - 1)^2 + (y^2 - 1)^2 + (z^2 - 1)^2 - 15.
\]

The zero isosurface of \( \phi \) implies the compact manifold of interest, ie, \( \Gamma = \{ x : \phi(x) = 0 \} \), and is depicted in Figure 13.

In the first step, meshes with linear triangular elements are generated using distmesh.\(^{48} \) A scaling parameter \( h \) may be chosen, which defines an average element length. In the second step, higher-order elements are mapped to this linear surface mesh and their element nodes are “lifted”\(^{18} \) such that they are on the manifold \( \Gamma \). Thereby, a higher-order accurate representation \( \Gamma_h \) is obtained.

As there are no boundaries present, an acceleration field in \( z \)-direction drives the flow. That is, on the right-hand side, \( g = P \cdot [0, 0, g_z]^T \), where \( g_z \) is determined by

\[
g_z(x) = \begin{cases} 
\exp(-\frac{x^2}{2\sigma_0^2}) & \text{with } \sigma_0 = 0.15 \text{ for } x < 0 \text{ and } y < 0, \\
0 & \text{else,}
\end{cases}
\]

and visualized in Figure 14A. It is virtually nonzero only for the left front “pillar” of the domain. The density is \( \rho = 1 \) and the viscosity is \( \mu = 0.05 \). For the case of stationary Navier-Stokes flow, the corresponding velocity magnitude, pressure fields, and vorticity \( \omega^* \) according to Equation (11) are seen in Figures 14B to 14D, respectively.

In the numerical studies, \( 2 \leq k_u \leq 5, k_p = k_s = k_u - 1, \) and \( k_{\text{geom}} = k_u + 1 \) are used. As there is no analytical solution available, convergence results are only shown in \( \epsilon_{\text{mom}} \) and \( \epsilon_{\text{cont}} \) in Figure 15. Higher-order rates are clearly achieved. In order to make the solution more quantitative, the velocity profiles for \( w(x) \) in the horizontal \( xy \)-plane (at \( z = 0 \)) are shown in Figure 16. The four closed black lines represent the intersection of the plane with the vertical “pillars” of the zero isosurface. Figure 16A shows \( w(x) \) as a third dimension, and Figure 16B shows the same result where \( w(x) \) is plotted in normal direction of the plane-pillar intersections with a scaling factor of 0.4. A clear convergence to these profiles was observed when using meshes with different resolutions and orders.
4.4 | Cylinder flows

As an example for the instationary Navier-Stokes equations, the following test case is based on a channel flow around a cylinder according to the work of Schäfer and Turek. The geometry is first described in 2D, labeled $\Omega_{2D}$, and later on mapped to obtain curved surfaces in 3D. In 2D, the cylinder with a diameter of 0.1 is placed slightly unsymmetrically in $y$-direction of the channel in $[0, 2.20] \times [0, 0.41]$ (see Figure 17A). No-slip boundary conditions are applied on the upper and lower wall and on the cylinder surface. A quadratic velocity profile for $u$, with $u_{\text{max}} = 1.5$, and $v = 0$ is applied at the
inflow on the left side of the domain. At the outflow, traction-free boundary conditions are used. The density and viscosity are prescribed as $\rho = 1.0$ and $\mu = 0.001$. This results in a Reynolds number of $\text{Re} = \rho \cdot u_m \cdot L / \mu = 100$ when taking the cylinder diameter as a length scale $L$ and the average inflow velocity $u_m = 1.0$ at the inflow. At this Reynolds number, periodic flow patterns known as the Kármán vortex street are observed behind the cylinder. Reference solutions are given for the lift and drag coefficients $c_L$ and $c_D$ of the cylinder and the current implementation confirms these numbers for the flat case (ie, in 2D or when the flat 2D domain is transformed by a rigid body motion to 3D). The reference Strouhal number $St = D / (u_m T)$, with the diameter $D = 0.1$ of the cylinder, and the time $T$ for 2 periods of the curve of $c_D$ is given as $0.295 \leq St \leq 0.305$, resulting in a frequency of about $f = 3.33$ Hz.
The 2D domain is mapped to three dimensions using two different maps. Assume that the coordinates of the 2D domain \( \Omega_{2D} \), as seen in Figure 17A, are given in coordinates \((a, b)\). Map A, \( x(a) : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), is defined as

\[
\begin{align*}
    x(a) &= \cos \left( \frac{\pi \cdot a}{2.2} \right) \cdot (b + 0.35), \\
y(a) &= \sin \left( \frac{\pi \cdot a}{2.2} \right) \cdot (b + 0.35), \\
z(x(a), y(a)) &= 2 + \frac{1}{2} \sqrt{x^2 + y^2} - \sin \left( 3\sqrt{x^2 + y^2} \right).
\end{align*}
\]

For map B, we first define an intermediate mapping \( r(a) : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), applying some twist to the domain

\[
\begin{align*}
r(a) &= a, \\
s(a) &= -(1 + q(a)) \cdot (b - 0.205) \cdot \cos \left( \frac{\pi}{6}(1 - \frac{25}{11} \cdot a) \right), \\
t(a) &= -(1 + q(a)) \cdot (b - 0.205) \cdot \sin \left( \frac{\pi}{6}(1 - \frac{25}{11} \cdot a) \right),
\end{align*}
\]

with \( q(a) = -0.2/2.42 \cdot a^2 + 0.44/2.42 \cdot a. \) This is further mapped by \( x(r) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined as

\[
\begin{align*}
x(r) &= \cos \left( \frac{205}{198} \cdot \pi \cdot r \right) \cdot (s + \frac{9}{5}), \\
y(r) &= \sin \left( \frac{205}{198} \cdot \pi \cdot r \right) \cdot (s + \frac{9}{5}), \\
z(r) &= t + \frac{1}{5} \sin(3r).
\end{align*}
\]

The resulting curved manifolds according to maps A and B are visualized in Figures 17B and 17C, respectively. Note that also the inflow velocities are mapped accordingly based on the Jacobians of the respective mappings to ensure that they are in the tangent space at \( \partial \Omega \).

The initial condition on the manifolds is \( u_0(x) = 0 \). The observed time interval is \( t = [0, 6] \) and the inflow velocities are ramped by a cubic function in time

\[
R(t) = \begin{cases} 
-2 \cdot ((t^*)^3 + 3 \cdot (t^*)^2) & \text{for } t \leq t^*, \\
1 & \text{else},
\end{cases}
\]

with \( t^* = 0.96 \). That is, after \( t^* \), the full velocity profile is active at the inflow. Figures 18 and 19 show the velocity magnitude, pressure field, and vorticity \( \omega^* \) at time \( t = 6 \) for the two mappings. The expected vortex shedding can be clearly seen.

Two different meshes with 972 and 1920 elements each are used, which are refined at the no-slip boundaries to resolve the boundary layers. They are visualized for \( \Omega_{2D} \) in Figure 20 and mapped to the manifolds accordingly. We use element orders of \( k_{geom} = 4, k_u = 3, k_p = 2, \) and \( k_z = 2 \) in the numerical studies shown here. Higher orders achieved virtually indistinguishable results for the quantities shown below. It is also noted that the Crank-Nicolson method used for the time discretization is only second-order accurate. For the time discretization, \( n_{step} = \{150, 300, 600, 1200, 2400, 4800\} \) time steps are used. To make the results more quantitative, the stresses at the cylinder wall are summed up to obtain a force

![Image](image_url)

**FIGURE 19** Physical fields for the cylinder flow test case according to map B. A., Velocity magnitude \( ||u|| \); B, Pressure \( p \); C, Vorticity \( \omega^* \) [Colour figure can be viewed at wileyonlinelibrary.com]
resultant $F(t) = \|F(t)\|$ in 3D. This is the equivalent of the lift and drag coefficients for the flat 2D case. Furthermore, the pressure difference between the front and back position of the cylinder (in $\Omega_{2D}$, mapped to three dimensions) is computed, i.e., $\Delta p(t) = p_{\text{front}}(t) - p_{\text{back}}(t)$.

The results for map A are shown in Figure 21 for the different number of time steps. It can be seen that, after about 2 seconds, the expected vortex shedding is almost established. After 3 seconds, the resulting oscillations remain virtually unchanged. The time interval $[5.2, 6]$ is shown in more detail in Figures 21B and 21D for $F(t)$ and $\Delta p(t)$, respectively. The convergence with increasing number of time steps is clearly demonstrated. Figure 22 shows the results in the same style for map B; the same conclusions may be drawn. The spatial convergence is investigated in Figure 23 where it is found that the coarse and fine mesh employed here obtain very similar results for the chosen element orders. The frequency of the oscillations for map A is $f_A = 2.191$Hz and for map B is $f_B = 3.078$Hz; for the flat case, the frequency is $f = 3.33$Hz.
5 | CONCLUSIONS

The surface FEM with higher-order elements has been applied to solve Stokes and Navier-Stokes flows on (fixed) manifolds. For the governing equations, the classical gradient and divergence operators are replaced by their tangential counterparts. An additional constraint is needed to ensure that the velocities are in the tangent space of the manifold. Stabilization is required for the case of Navier-Stokes flows at large Reynolds numbers, and the standard SUPG approach is used herein.
For the discretization, the surface FEM is employed with quadrilateral or triangular elements. Element spaces of different orders are used for (i) the geometric approximation of the manifold, ie, \( k_{\text{geom}} \); (ii) the approximation of the velocity fields, ie, \( k_u \); (iii) the pressure field, ie, \( k_p \); and (iv) the Lagrange multiplier field for the enforcement of the tangential velocity constraint, ie, \( k_j \). The choice of these orders affects the properties of the resulting FEM in terms of conditioning, accuracy, and stability. Particularly useful combinations for a chosen order \( k_u \) are \( k_{\text{geom}} = k_u + 1 \) and \( k_p = k_j = k_u - 1 \). Some benchmark test cases for flows on manifolds are proposed and higher-order convergence rates are achieved. The notation used in this work is closely related to the engineering literature for the FEM in fluid mechanics. Implementational matters are outlined.

There is a large potential for future research related to this work. One may investigate different stabilization methods such as Galerkin least-squares stabilization and variational multiscale methods. Stabilization may also be useful to circumvent the Babuška-Brezzi condition and enable equal-order shape functions for the velocities and pressure. The tangential velocity constraint may be more efficiently enforced based on penalty methods or other Lagrange multiplier approaches such as the Uzawa method. We believe that flows on manifolds have a strong potential for fundamental research in mathematics, physics, and engineering.

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