EQUATIONS FOR HLM GENERALIZED FIELDS

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A general Lie algebra of operators of coordinates, momenta, and Lorentz group generators is investigated. In the limiting case, it becomes the Lie algebra of operators of the canonical quantum theory. Structure constants of the general Lie algebra depend on the known constants $c$ and $h$, as well as additional constants with dimensions of action ($H$), length ($L$), and mass ($M$). It is presented some realizations of general Lie algebra operators and equations for HLM generalized fields forming particular representation spaces of the HLM algebra.

Key words: Lie algebra, structure constants, algebra representation, fundamental physical constants, Poincaré group, canonical quantum theory.

1. INTRODUCTION

At present any theory of fundamental interactions of quantum particles is invariant with regard to Poincaré group transformations. However, for instance a realization of a Poincaré group representation on quark field is impossible in the strict sense because of non observation of quarks as free particles. One can consider only an approximation when a quark field is a field with a definite mass and a definite spin. So generalization of the group of spacetime symmetries of the canonical quantum theory that contains the Poincaré group may be essential.

Research in this direction was carried out in the context of theories dependent on new fundamental physical constants, in addition to the speed of light $c$ and the Planck constant $h$. Beginning with the work of Snyder [1], a theory with a fundamental length [2, 3] was developed with the purpose
of a removal of ultraviolet divergences in the canonical quantum electrodynamics. These divergences originate from a multiplication of quantities in neighboring spacetime points, therefore Heisenberg assumed that coordinates can be noncommutative [4]. In the work of Snyder the de Sitter and anti de Sitter groups were derived as generalizations of the Poincaré group. These groups were used for development of a theory in a quantized spacetime with the additional constant with dimensions of length. But it was observed in the Yang’s paper [5], that the interchangeability among coordinates and momenta, proposed by Born [6, 7], was lost in the Snyder’s approach. Thus, it was necessary to replace the Poincaré group with the semi-direct product of the Lorentz group and the Heisenberg group and to generalize this semi-direct product. Then the interchangeability was restored and two new constants with dimensions of length and mass appeared in the theory [5, 8]. More general theories, depending on constants with dimensions of length (L), mass (M), and action (H), were considered in [9, 10, 11, 12, 13, 14, 15].

In this paper the general \( HLM \) algebra is considered and some representations of the \( HLM \) algebra of operators dependent on the additional constants \( H, L, \) and \( M \) are presented. Equations for scalar and spinor \( HLM \) generalized fields are given. In Sec. II, we describe the general Lie algebra dependent on \( H, L, \) and \( M \), which generalizes the semi-direct sum of the Lorentz algebra and the Heisenberg algebra. In Sec. III, we present the equation for a scalar \( HLM \) generalized field and some nontrivial representations of the \( HLM \) algebra. In Sec. IV, we outline the case of \( LM \) generalized fields, which can be used for description of particles participated in an interaction conserving the \( P, C, \) and \( T \) parities, for instance, in the strong interaction. In Sec. V, we present the equations for spinor \( LM \) generalized fields with the conservation and the non-conservation of the \( P \) parity. Finally, in Sec. VI, we draw our conclusions and suggest future steps for development of the considered approach.

2. LIE ALGEBRAS OF GENERALIZED SYMMETRIES IN QUANTUM PHASE SPACE

Let us begin with consideration of groups which generalize the semi-direct product of the Lorentz group and the Heisenberg group in the eight-dimensional phase space. We restrict our consideration to the determination of a group structure near the identity element of a group, so we can investigate admissible Lie algebras of generalized groups.

The conventional commutation relations among quantum theory observables in the Minkowski space [16] have the form

\[
[x_i, x_j] = [p_i, p_j] = 0, \quad [p_i, x_j] = i\hbar g_{ij}I,
\]
\[[p_i, I] = [x_i, I] = [F_{ij}, I] = 0,\]

\[[F_{ij}, F_{kl}] = i\hbar(g_{jk}F_{il} - g_{ik}F_{jl} + g_{il}F_{jk} - g_{jl}F_{ik}),\]  \hspace{1cm} (1)

\[[F_{ij}, p_k] = i\hbar(g_{jk}p_i - g_{ik}p_j),\]

\[[F_{ij}, x_k] = i\hbar(g_{jk}x_i - g_{ik}x_j),\]

Relations (1) are written in the system of units, where the speed of light \(c = 1\), \(x_i\) are coordinate operators, \(p_i\) are momentum operators and \(I\) is identity operator, while \(F_{ij}\) are generators of the Lorentz group; \(i, j, k, l = 0, 1, 2, 3\).

One can define spacetime points in the Minkowski space in the language of Lie algebra representation theory as eigenvalues of operators \(x_i\), realized in a certain representation of the algebra (1). For example, representation vectors in a \(x\)–representation are as follows \(\psi_{\alpha\beta}(x)\), where \(x = \{x_0, x_1, x_2, x_3\}\) are eigenvalues of coordinate operators \(x_i\), \(\alpha, \beta\) are discrete indexes, which are subjected by operators \(S_{ij} = F_{ij} - x_ip_j + p_ix_j\). Operators \(S_{ij}\) are spin operators in some finite-dimensional representation of the Lorentz group.

Let us consider possible generalizations of the Lie algebra (1) under condition of the relativistic invariance of the theory. Thus, we suppose the fulfillment of the following conditions when we look for suitable infinitesimal groups \(G\), or their algebras \(g\) \([9, 10]\).

1. The generalized algebra should be a Lie algebra.
2. The dimensionality of the generalized algebra and the physical dimensions of operators contained in should be the same as in the algebra (1).
3. The generalized algebra should contain the Lorentz algebra as its subalgebra, and commutation relations of the Lorentz subalgebra with other generators should be the same as in the initial algebra.

The procedure of generalization of the algebra (1) described above may be named as the relativistic or the Lorentz-invariant deformation of the algebra (1) because the property of the Lorentz symmetry is conserved as the fundamental law of nature. However in some cases the canonical Poincaré invariance can be violated. The violation can occur as in commutation relations, as in realization of the commutation relations. For instance, the assumption that a generalized field put in correspondence to a particle must transform according to a definite irreducible representation of the Poincaré group can be superfluous.

The algebra with the generators \(F_{ij}, p_i, x_i,\) and \(I\), that is the maximal generalization of the algebra (1) under the conditions written above, has the following form:

\[[F_{ij}, F_{kl}] = \varphi(g_{jk}F_{il} - g_{ik}F_{jl} + g_{il}F_{jk} - g_{jl}F_{ik}).\]
\[ [p_i, x_j] = A g_{ij} I + B F_{ij} + C \epsilon_{ijkl} F^{kl}, \]
\[ [p_i, p_j] = a F_{ij} + b \epsilon_{ijkl} F^{kl}, \]
\[ [x_i, x_j] = c F_{ij} + d \epsilon_{ijkl} F^{kl}, \]
\[ [p_i, I] = \alpha x_i + \beta p_i, \quad (2) \]
\[ [x_i, I] = \gamma x_i + \delta p_i, \]
\[ [F_{ij}, x_k] = h (g_{jk} x_i - g_{ik} x_j), \]
\[ [F_{ij}, p_k] = f (g_{jk} p_i - g_{ik} p_j), \]
\[ [F_{ij}, I] = 0, \]
where \( \epsilon_{ijkl} \) is the Levi-Civita symbol.

The commutation relations (2) contain fourteen arbitrary pure imaginary parameters. Taking into account the Jacobi identities and the physical dimensions of operators involved in the commutation relations (2), ten parameters must be excluded. Then the following algebra, which is a maximal deformation of the algebra (1) under the conditions 1-3, had been obtained [9, 10]:

\[ [F_{ij}, F_{kl}] = i f (g_{jk} F_{il} - g_{ik} F_{jl} + g_{il} F_{jk} - g_{jl} F_{ik}), \]
\[ [p_i, x_j] = i f (g_{ij} I + \frac{F_{ij}}{H}), \]
\[ [p_i, p_j] = \frac{i f}{L^2} F_{ij}, \]
\[ [x_i, x_j] = \frac{i f}{M^2} F_{ij}, \]
\[ [p_i, I] = i f \left( \frac{x_i}{L^2} - \frac{p_i}{H} \right), \]
\[ [x_i, I] = i f \left( \frac{x_i}{H} - \frac{p_i}{M^2} \right), \]
\[ [F_{ij}, p_k] = i f (g_{jk} p_i - g_{ik} p_j), \]
\[ [F_{ij}, x_k] = i f (g_{jk} x_i - g_{ik} x_j), \]
\[ [F_{ij}, I] = 0. \]

The commutation relations of the algebra (3) depend on four dimensional parameters: \( L \) with the dimension of length, \( M \) with the dimension of mass, \( H \) and \( f \) with the dimension of action (\( M \) and \( L \) take real values as well as pure imaginary ones, \( c = 1 \) in the system of units being used).
In the limiting case, when $M$, $L$, and $H$ become infinitely large, the commutation relations (3) go over into the commutation relations of the canonical algebra (1) providing $f = \hbar$. A more complicated case is also possible, when $f$ is some function of the three parameters $L$, $M$, and $H$. The function $f(L, M, H)$ must tend to $\hbar$ for an agreement with the conventional commutation relations at $L \to \infty$, $M \to \infty$, and $H \to \infty$.

In other limiting cases, when $f = \hbar$, but $M$, $L$, and $H$ have different values, the commutation relations for observables of theories can be obtained, which earlier considered in the papers of Snyder [1], Yang [5], Golfand [2], Kadyshevsky [3], and Leznov [8]:

a) $H \to \infty$, $L \to \infty$ - the relativistic quantum theory with noncommutative coordinates;

b) $H \to \infty$, $M \to \infty$ - the relativistic quantum theory with noncommutative momenta;

c) $H \to \infty$ - the relativistic quantum theory with noncommutative coordinates and momenta.

The system of the commutation relations (3) specifies some class of Lie algebras which consist of semisimple algebras as well as general-type algebras. After calculation of the Killing-Cartan form, the condition of semisimplicity for the algebras (3) may be written as

$$f^2(M^2L^2 - H^2) / H^2M^2L^2 \neq 0. \quad (4)$$

In order to define all semisimple algebras of the type (3), it is convenient to perform the following transformation of generators $p_i$, $x_i$, and $I$:

$$F_{15} = Bx_i + Dp_i, \quad F_{16} = Ex_i + Gp_i, \quad F_{56} = AI, \quad (5)$$

then, under condition (4), one may obtain the commutation relations for the algebras of pseudo-orthogonal groups $O(3, 3)$, $O(2, 4)$, and $O(1, 5)$. These algebras correspond to specific values of the parameters $M^2$, $L^2$, and $H^2$ which are shown below in Table 1.

**Table 1.**

The real simple Lie algebras which correspond to different values of parameters $L^2$, $M^2$, and $H^2$.

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1. We keep the former notations for the operators $p_i$, $x_i$, and $I$, however now the $p_i$, $x_i$, and $I$ symbols mean the operators of the generalized momenta, coordinates and the generalized 'identity' operator, accordingly.
For

\[ H^2 = M^2 L^2, \quad M^2 > 0, \quad L^2 > 0 \] (6)

the \( o(1, 5) \) algebra degenerates into a semi-direct product of the \( o(1, 4) \) algebra and the algebra of 5-dimensional translations, while for

\[ H^2 = M^2 L^2, \quad M^2 < 0, \quad L^2 < 0 \] (7)

the \( o(3, 3) \) algebra degenerates into a semi-direct product of the \( o(2, 3) \) algebra and the algebra of 5-dimensional translations. The special importance have the following values of parameters: \( L^2 = 0 \), or \( M^2 = 0 \), or \( H^2 = L^2 M^2 \), because passing through these values alters a type of a considered simple algebra.

Note that some transitions to the infinite values of the constants \( M, L, \) and \( H \) do not remove a considered algebra from the class of simple algebras. Let us give the examples of such transitions: \( A_\alpha \to \infty \), where \( A_\alpha \) belongs to the set \( \{M^2, L^2, H^2, (M^2, L^2)\} \). But the transitions \( B_\alpha \to \infty \) remove, where \( B_\alpha \in \{(M^2, H^2), (L^2, H^2), (M^2, L^2, H^2)\} \) or \( M^2 L^2 \to H^2 \).

Three algebras of the pseudoorthogonal groups \( O(3, 3), O(2, 4), \) and \( O(1, 5) \) are obtained by this means. Note that third condition of this section can be released and be replaced with a condition implies that the generalized algebra contains the Lorentz subalgebra solely. As this takes place, the commutation relations of the the Lorentz subalgebra generators with other ones can differ from the canonical relations. In this case the pseudo-unitary group \( SU(1, 3) \) enters into the set of generalized symmetry groups [17].

3. SOME REPRESENTATIONS OF HLM ALGEBRAS.
EQUATION FOR A GENERALIZED SCALAR FIELD

Irreducible representations of the algebras [3] are determined with the help of eigenvalues of Casimir operators. For the real simple algebras shown
in Table 1, the Casimir operators have the known forms in terms of the generators $F_{ij}$, $i,j = 0, 1, ..., 5$ of the pseudoorthogonal groups in six-dimensional spaces [18, 19]:

$$C_1 = \epsilon_{ijklmn} F^{ij} F^{kl} F^{mn}, \quad C_2 = F_{ij} F^{ij}, \quad C_3 = (\epsilon_{ijklmn} F^{kl} F^{mn})^2$$ (8)

The Casimir operators should be expressed through the operators $p_i$, $x_i$, $F_{ij}$, $i,j = 0, ..., 3$, and $I$ for using in physical applications.

Note that the general algebra (3) with two constants with the dimension of action, namely $f$ and $H$, is non invariant in regard to $T$ and $C$ transformations [9, 10]. In the framework of the orthodox theory the Planck constant is the single odd constant with respect to $T$ transformation (the speed of light enters in the commutation relations in the second power). So the transformation to conjugate or transposed operators leads to the recovery of the $T$ invariance, while it is impossible in the generalized theory with two constants $f$ and $H$. Along the same lines one may obtain $C$ non-invariance of the system (3), since the quantities with dimension of mass change their signs after replacement of particles with antiparticles. The $CT$ transformation does not change $f$ and $H$, so the system (3) is invariant under $CT$ and $P$ transformations.

At $M \to \infty$, $L \to \infty$ commutativity of coordinates among themselves, as well as momenta, restore. However, the commutation relations among $I$, $p_i$, $x_j$, and $x_j$, $p_i$ keep non-trivial. In this case it is not difficult to find a six dimensional representation of the algebra (3) with the help of the simplest matrices $e^i_j$, which contain unity at the intersection of the row $i$ and the column $j$. Accomplish these ends it is needed to form the matrices $M^i_j = -e^i_j + e^j_i$ for $i,j = 1, ..., 4$, $i < j$, and also for $i = 0$, $j = 5$, as well as the matrices $N^i_j = e^i_j + e^j_i$ for $i = 0$, $j = 1, ..., 4$, and also for $j = 5$, $i = 1, ..., 4$. Then linear combinations with real coefficients of the matrices $M^i_j$ and $N^i_j$ realize the real six dimensional representation of the initial algebra.

Complex representations with dimensions equal to four and eight can be formed by the similar procedure [20]. These representations are constructed with the help of the well-known Pauli $\sigma$ matrices ($\sigma_0$ is the identity matrix). For instance, generators of the eight-dimensional representation are bilinear combinations of Clifford algebra elements:

$$\Gamma_0 = \sigma_2 \otimes \sigma_3 \otimes \sigma_0, \quad \Gamma_1 = i\sigma_2 \otimes \sigma_2 \otimes \sigma_1,$$

$$\Gamma_2 = i\sigma_2 \otimes \sigma_0 \otimes \sigma_2, \quad \Gamma_3 = i\sigma_2 \otimes \sigma_2 \otimes \sigma_3,$$

$$\Gamma_4 = -i\sigma_2 \otimes \sigma_1 \otimes \sigma_0, \quad \Gamma_5 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0,$$ (9)
Now, the bilinear combinations \( F_{ab} = i f[\Gamma_a, \Gamma_b]/4 \) and the inverse to (5) transformation, when \( F_{ij}, p_i, x_i, \) and \( I \) are written via \( F_{ab} \), give a representation of the considered algebra.

Let us give the infinite dimensional representation of the generators \( F_{ij}, p_i, x_i, \) and \( I \) with the help of differential operators in the \( \xi \)-representation (in the formulae below \( a \) is a free parameter). This representation is the most similar to a representation of the canonical quantum theory and is dependent on the constant \( H \).

\[
p_i = i\hbar \frac{\partial}{\partial \xi_i}, \quad I = i\hbar (a + \frac{\xi^m}{H} \frac{\partial}{\partial \xi^m})
\]

\[
F_{ij} = i\hbar (\xi_i \frac{\partial}{\partial \xi^j} - \xi_j \frac{\partial}{\partial \xi^i}),
\]

\[
x_i = i\hbar (a\xi_i + \frac{\xi^m}{H} \frac{\partial}{\partial \xi^m} - \frac{\xi^2}{2H} \frac{\partial}{\partial \xi^i})
\]

We can write an equation for a one-component field \( \Phi(\xi) \) (a \( HLM \) generalized scalar field) with the help of the quadratic Casimir operator \( C_2 \), that is represented through the physical operators \( p_i, x_i, F_{ij}, i,j = 0,...,3, \) and \( I \). Then taking into account the relations (5), we obtain

\[
\left( \sum_{i<j} F_{ij} F^{ij} \left( \frac{1}{M^2 L^2} - \frac{1}{H^2} \right) + I^2 + \frac{x_i p^i + p_i x^i}{H} - \frac{x_i x^i}{L^2} - \frac{p_i p^i}{M^2} \right) \Phi(\xi) = 0 \tag{11}
\]

The equation (11) is analogous with the Klein-Gordon-Fock equation of the canonical quantum theory. The explicit form of the Casimir operator \( C_2 \), i.e. the left-hand side of the equation (11), was obtained in Ref. [21] at \( H = \infty \) and arbitrary values of \( L \) and \( M \). In a three-dimensional space the explicit form of the quadratic Casimir operator \( C_2 \) was found in Ref. [22] and was afterward used in Ref. [23].

4. GROUPS OF \( LM \) GENERALIZED SYMMETRIES FOR QUANTUM PARTICLES

One may choose groups with specified properties taking into account the properties of quantum particles and their symmetries. For instance, commutation relations (3) at \( H = \infty \) are suited for symmetries of quantum particles, which conserved the \( P, C, \) and \( T \) invariance. This has been noted in Sec. 3 and this case may be applied to strong interaction particles such as quarks.
Then we have the following commutation relations for $LM$ algebras in the system of units with $\hbar = 1$.

\[
\begin{align*}
[F_{ij}, F_{kl}] &= i(g_{jk}F_{il} - g_{ik}F_{jl} + g_{il}F_{jk} - g_{jl}F_{ik}), \\
[F_{ij}, p_k] &= i(g_{jk}p_i - g_{ik}p_j), \\
[F_{ij}, x_k] &= i(g_{jk}x_i - g_{ik}x_j), \\
[p_i, x_j] &= ig_{ij}I, \\
[F_{ij}, I] &= 0,
\end{align*}
\]

\[ (p_i, p_j) = (i/L^2)F_{ij}, \]
\[ (x_i, x_j) = (i/M^2)F_{ij}, \]
\[ (p_i, I) = (i/L^2)x_i, \]
\[ (x_i, I) = (-i/l/M^2)p_i, \]

In the limiting case $L \to \infty$, $M \to \infty$ we obtain the canonical commutation relations \([11]\). The equation of the second order for $LM$ generalized fields have the following form:

\[
\left( \sum_{i<j} F_{ij}F^{ij} \left( \frac{1}{M^2L^2} \right) + I^2 - \frac{x_i x_i}{L^2} - \frac{p_ip_i}{M^2} \right) \Phi(\xi) = 0 \tag{13}
\]

**5. EQUATIONS FOR $LM$ GENERALIZED SPINOR FIELDS**

Let us consider equations for $LM$ generalized spinor fields which are invariant in regards to transformations generated by semisimple algebras \([12]\ [24] [25]\). The linear with respect to operators $p_i$, $x_j$, $F_{ij}$, and $I$ equations can be found with the help of the method considered in Ref. \([20]\). In the first place we write an equation for the generalized field $\psi(\xi)$ with four components:

\[
(\gamma_i p^i - \gamma_i \gamma_5 x^i \zeta_1 \zeta_2 \sqrt{\frac{M^2}{L^2} - \gamma_5 I \zeta_2 \sqrt{-M^2}}} - \sum_{i<j} \gamma_i \gamma_j F^{ij} \frac{\zeta_1}{\sqrt{L^2}} - n) \psi(\xi) = 0, \tag{14}
\]

where $F^{ij}$ are the Lorentz group generators, $n$ is a free number, $\gamma_i$, $i = 0, 1, 2, 3$, are the Dirac matrices, $\zeta_1 = \pm 1$, $\zeta_2 = \pm 1$. 

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Equation (14) is not invariant with respect to spatial parity $P$. We give below the $P$ invariant equation for the generalized field $\Psi(\xi)$ with eight components, which is suitable for description of quantum particles with $P$ invariant interactions.

$$
(\sigma_0 \otimes \gamma_i) P^i - \sigma_3 \otimes \gamma_5 x^i \zeta_1 \zeta_2 \sqrt{-\frac{M^2}{L^2}} - \sigma_3 \otimes \gamma_3 I \zeta_2 \sqrt{-M^2} - \sigma_0 \otimes \sum_{i<j} \gamma_i \gamma_j F^{ij} \frac{\zeta_1}{\sqrt{L^2}} - \sigma_0 \otimes n) \Psi(\xi) = 0,
$$

(15)

where $\sigma_0$ is the identity $(2 \times 2)$ matrix, $\sigma_3$ is the Pauli matrix.

The coefficients in Eq.(5) obtained in Ref. [26] are specified with the coefficients in Eqs.(14) and (15), which contain the discrete parameters $\zeta_1$ and $\zeta_2$. The explicit form of these coefficients will allow some time to study in detail solutions of Eqs. (14) and (15).

6. CONCLUSIONS

Equations (11), (14), and (15) for the HLM generalized scalar and spinor fields are represented in this paper. These equations are invariant relatively symmetries generated by the algebras with commutation relations (3) or (12), which are generalizations of canonical commutation relations (1). The special cases of finite-dimensional representations and the infinite-dimensional representation of the generalized algebras are exemplified in Sec. 3 and 5.

It is possible to consider applications of the HLM generalized fields either in a domain of super-low distances, as it was assumed by Snyder, or, for example, in a domain of confinement of color particles, because the validity of the requirement of the Poincaré symmetry for color particles is not straightforward. A procedure of a experimental determination of a color particle mass value can be cited as an example. In this case the color particle must be insulated from color interactions and be placed in a weak magnetic field. Then one can determine a particle mass value via a path of the particle with the proviso that a particle electric charge is known. However, this measurement procedure cannot be realized. Thus, of fundamental importance is a search for generalizations of the Poincaré symmetry and investigations of their properties. A future step forward in this direction is the study of peculiarities of solutions of Eqs. (11), (14), and (15) and possibilities of their use for description of quantum particles.
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