$G$–Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty

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Chapter 1

Introduction

It is well-known in classical probability theory that a expectation $\mathbb{E}$ satisfies the following relation for random variables $X$ and $Y$:

$$\mathbb{E}[aX + Y + c] = a\mathbb{E}[X] + \mathbb{E}[Y] + c, \quad \forall a, c \in \mathbb{R}. $$

A sublinear expectation $\hat{\mathbb{E}}$ satisfies the following weaker condition:

$$\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y], \quad \hat{\mathbb{E}}[|a|X + c] = |a|\hat{\mathbb{E}}[X] + c.$$ 

This $\hat{\mathbb{E}}$ keeps the monotonicity property: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

The notion of sublinear expectations is proved to be a basic tool in volatility uncertainty which is crucial in superhedging, superpricing (See [3] and [4]) and measures of risk in finance which caused a great attention in finance since the pioneer work of Artzner, Delbaen, Eber and Heath [1] and [2]. This is also the start point of a new theory stochastic calculus which gives us a new insight to characterize and calculate varies kinds of financial risk.

In this note we will introduce a crucial notion of $G$-normal distributions corresponding to the well-known normal distributions in classical probability theory. This $G$-normal distribution will bring us to a new framework of stochastic calculus of Itô’s type through the corresponding $G$-Brownian motion. We will also present analytical calculations and some new statistical methods with application to risk analysis in finance under volatility uncertainty.

Our basic point of view is: sublinear expectation theory is very like its special situation of linear expectation in the classical probability theory. Under a sublinear expectation (or even more general nonlinear expectation)
space we still can introduce the notion of distributions, of random variables, as well as the notions of joint distributions, marginal distributions, etc. We still denoted $X \sim Y$ if $X$ and $Y$ are identically distributed. We still have the notion of independence. A particularly interesting phenomenon in sublinear situations is that a random variable $Y$ is independent to $X$ does not automatically implies that $X$ is independent to $Y$.

We will prove two important theorems in the framework of sublinear expectation theory: The law of large number and the central limit theorem. A very interesting result of our new central limit theorem under a sublinear expectation is that a sequence of zero-mean independent and identically distributed random will converge in law to a ‘$G$-normal distribution’ $\mathcal{N}(0, [\sigma^2, \sigma^2])$. Briefly speaking a random variable $X$ in a sublinear expectation space is said to be $\mathcal{N}(0, [\sigma^2, \sigma^2])$-distributed if for any $Y$ independent and identically distributed w.r.t. $X$ and for any real function $a$ and $b$ we have $aX + bY \sim \sqrt{a^2 + b^2}X$. Here $\sigma^2 = \hat{E}[X^2]$ and $\sigma^2 = -\hat{E}[-X^2]$. In a special case $\sigma^2 = \sigma^2$, this $G$-normal distribution becomes a classical normal distribution $\mathcal{N}(0, \sigma^2)$.

We will define a sublinear expectation on the space of continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^d$ which will be an analogue of Wiener’s law, by which a $G$-Brownian motion is formulated. Briefly speaking a $G$-Brownian motion $(B_t)_{t \geq 0}$ is a continuous process with independent and stationary increments under a given sublinear expectation $\hat{E}[$].

$G$-Brownian motion has a very rich and interesting new structure which non trivially generalizes the classical one. We can establish the related stochastic calculus, especially $G$–Itô’s integrals (see [35, 1942]) and the related quadratic variation process $\langle B \rangle$. A very interesting new phenomenon of our $G$-Brownian motion is that its quadratic process $\langle B \rangle$ also has independent and stationary increments. The corresponding $G$–Itô’s formula is obtained. We then introduce the notion of $G$–martingales and the related Jensen inequality for a new type of “$G$–convex” functions. We have also established the existence and uniqueness of solution to stochastic differential equation under our stochastic calculus by the same Picard iterations as in the classical situation. Books on stochastic calculus e.g., [14], [32], [34], [36], [41], [47], [62], [63], [69] are recommended for understanding the present results and some further possible developments of this new stochastic calculus.

A sublinear expectation can be regarded as a coherent risk measure. This, together with the related conditional expectations $\hat{E}[\cdot | \mathcal{H}_t]_{t \geq 0}$ makes a dy-
namic risk measure: $G$–risk measure.

The other motivation of our $G$–expectation is the notion of (nonlinear) $g$–expectations introduced in [50], [51]. Here $g$ is the generating function of a backward stochastic differential equation (BSDE) on a given probability space $(\Omega, \mathcal{F}, P)$. The natural definition of the conditional $g$–expectations with respect to the past induces rich properties of nonlinear $g$–martingale theory (see among others, [4], [7], [8], [9], [15], [16], [10], [11], [38], [39], [52], [56], [57], [59]). Recently $g$–expectations are also studied as dynamic risk measures: $g$–risk measure (cf. [64], [5], [21]). Fully nonlinear super-hedging is also a possible application (cf. [3], [44] and [12] where new BSDE approach was introduced).

The notion of $g$–expectation is defined on a given probability space. In [55] (see also [54]). As compared with the framework of $g$–expectations, the theory of $G$–expectation is more intrinsic, a meaning similar to “intrinsic geometry” in the sense that it cannot be based on a given (linear) probability space. Since the classical Brownian expectation as well as many other linear and nonlinear expectations are dominated by our $G$–Expectation, our theory also provides a flexible theoretical framework.

The whole results of this paper are based on the very basic knowledge of Banach space and the parabolic partial differential equation (3.3). When this $G$-heat equation (3.3) is linear, our $G$-Brownian motion becomes the classical Brownian motion. This paper still provides an analytic shortcut to reach the sophisticated Itô’s calculus.

A very basic knowledge of Banach space and probability theory are necessary. Stochastic analysis, i.e., the theory of stochastic processes will be very helpful but not necessary. We also use some basic knowledge on the smooth solutions of parabolic partial differential equations. Good background in statistics will be very helpful. This note was written for several series of lectures: In the 2nd Workshop “Stochastic Equations and Related Topic” Jena, July 23–29, 2006; In graduate Courses of Yantai Summer School in Finance, Yantai University, July 06–21, 2007; as well as in Graduate Courses of Wuhan Summer School, July 24–26, 2007. Also in mini-course of Institute of Applied Mathematics, AMSS, April 16-18 2007 and a mini-course in Fudan University, May 2007; In graduate courses of CSFI, Osaka University, May 15–June 13, 2007. The hospitalities and encouragements of the above institutions and the enthusiasm of the audiences are the main engine to realize this lecture notes.

I thank for many comments and suggestions given during those courses,
especially to Li Juan and Hu Mingshang. References are given at the end. Historical remarks are still under preparation. This lecture note are mainly based on my recent research papers:

Peng, S. (2006) $G$–Expectation, $G$–Brownian Motion and Related Stochastic Calculus of Itô’s type, (pdf-file available in arXiv:math.PR/0601035v2, to appear in Proceedings of the 2005 Abel Symposium, Springer.)

Peng, S. (2006) Multi-dimensional $G$–Brownian motion and related stochastic calculus under $G$–expectation, Preprint, (pdf-file available in arXiv:math.PR/0601699 v2).

Peng, S. (2007) Law of large numbers and central limit theorem under non-linear expectations, in arXiv:math.PR/0702358v1 13 Feb 2007

which were stimulated by:

Peng, S. (2004) Filtration Consistent Nonlinear Expectations and Evaluations of Contingent Claims, Acta Mathematicae Applicatae Sinica, English Series 20(2), 1–24.

Peng, S. (2005) Nonlinear expectations and nonlinear Markov chains, Chin. Ann. Math. 26B(2), 159–184.
Chapter 2
Risk Measures and Sublinear Expectations

2.1 How to measure risk of financial positions

Let $\Omega$ be a set the set of scenarios. We are given a linear subspace $\mathcal{X}$ of real valued and bounded functions on $\Omega$. $\mathcal{X}$ is the collection of all possible risk positions in a financial market. We assume that all constants are in $\mathcal{X}$ and that $X \in \mathcal{X}$ implies $|X| \in \mathcal{X}$. For each $X \in \mathcal{X}$ we denote $X^* = \sup_{\omega \in \Omega} X(\omega)$ and $\|X\| = |X|^*$.

$\|\cdot\|$ is a Banach norm on $\mathcal{X}$. In this section $\mathcal{X}$ is assumed to be a Banach space, i.e., if a sequence $\{X_i\}_{i=1}^\infty$ of $\mathcal{X}$ converges uniformly to some function $X$ on $\Omega$, then $X \in \mathcal{X}$.

Remark 2.1.1 If $S \in \mathcal{X}$, then for each constant $c$, $S \lor c$, $S \land c$ are all in $\mathcal{X}$. One typical example in finance is that $S$ is the tomorrow’s price of a stock. In this case any European call or put options of forms

$$ (S - k)^+, \ (k - S)^+ $$

are in $\mathcal{X}$.

Problem 2.1.2 Prove that if $S \in \mathcal{X}$ then $\varphi(S) \in \mathcal{X}$ for each continuous function $\varphi$ on $\mathbb{R}$. 
Example 2.1.3 Let \((\Omega, \mathcal{F})\) be a measurable space and let \(L^\infty(\Omega, \mathcal{F})\) be the space of all bounded \(\mathcal{F}\)-measurable random variables. \(L^\infty(\Omega, \mathcal{F})\) is considered as a space of risk positions in a financial market. \(L^\infty(\Omega, \mathcal{F})\) is a Banach space under the norm \(\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|\).

Example 2.1.4 If \(\Omega\) is a metric space, then we can consider \(\mathcal{X} = C_b(\Omega)\), the set of all bounded and continuous functions on \(\Omega\).

2.1.1 Coherent measures of risk

A risk supervisor is responsible for taking a rule to tell traders, stocks companies, banks or other institutions under his supervision, which kind of risk positions is unacceptable and thus a minimum amount of risk capitals should be deposited to make the positions to be acceptable. The collection of acceptable positions is defined by:

\[ \mathcal{A} = \{X \in \mathcal{X}, \ X \text{ is acceptable}\} \]

This set has the following economically meaningful properties:

Definition 2.1.5 (Coherent acceptable set)

(i) Monotonicity:

\[ X \in \mathcal{A}, \ Y \geq X \implies Y \in \mathcal{A} \]

(ii) \(0 \in \mathcal{A}\) but \(-1 \notin \mathcal{A}\).

(iii) Positively homogeneity

\[ X \in \mathcal{A} \implies \lambda X \in \mathcal{A}, \ \forall \lambda \geq 0. \]

(iv) Convexity:

\[ X, Y \in \mathcal{A} \Rightarrow \alpha X + (1 - \alpha)Y \in \mathcal{A}, \ \alpha \in [0, 1] \]

Remark 2.1.6 (iii) and (iv) imply

(v) Sublinearity:

\[ X, Y \in \mathcal{A} \Rightarrow \mu X + \nu Y \in \mathcal{A}, \ \forall \mu, \nu \geq 0. \]
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Remark 2.1.7 If we remove the condition of the positive homogeneity, then $A$ is called a convex acceptable set. In this course we mainly study the coherent case. Once the rule of the acceptable set is fixed, the minimum requirement of risk deposit is then automatically determined.

Definition 2.1.8 (Risk measure related to a given acceptable set $A$) The functional $\rho(\cdot)$ defined by

$$\rho(X) = \rho_A(X) := \inf\{m \in \mathbb{R} : m + X \in A\}, \quad X \in \mathcal{X}$$

is called the risk measure related to $A$.

It is easy to see that

$$\rho(X + \rho(X)) = 0.$$ 

Proposition 2.1.9 $\rho(\cdot)$ is a coherent risk measure, namely

(a) Monotonicity: $X \geq Y$ implies $\rho(X) \leq \rho(Y)$;
(b) Constant preservation: $\rho(1) = -\rho(-1) = -1$;
(c) Sublinearity: For each $X,Y \in \mathcal{X}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
(d) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, $\forall \lambda \geq 0$.

Proof. (a), (b) are obvious. To prove (c) we first have $\rho(\mu X) = \mu \rho(X)$. In fact the case $\mu = 0$ is trivial; when $\mu > 0$,

$$\rho(\mu X) = \inf\{m \in \mathbb{R} : m + \mu X \in A\} = \mu \inf\{n \in \mathbb{R} : n + X \in A\} = \mu \rho(X).$$

Now, for each positive $\mu$ and $\nu$,

$$\rho(\mu X + \nu Y) = \inf\{m \in \mathbb{R} : m + (\mu X + \nu Y) \in A\} \leq \inf\{m \in \mathbb{R} : m + \mu X \in A\} + \inf\{n \in \mathbb{R} : n + \nu Y \in A\} \leq \rho(\mu X) + \rho(\nu Y).$$
Now let a \( \rho(\cdot) \) be a functional satisfying (a)–(d). Then we can inversely define
\[
\mathcal{A}_\rho = \{ X \in \mathcal{X} : \rho(X) \leq 0 \}.
\]
It is easy to prove that \( \mathcal{A}_\rho \) satisfies (i)–(iv).

### 2.1.2 Sublinear expectation of risk loss

From now we will denote \( X \) to be a loss position, namely \( -X \) is the corresponding financial position. Related to a coherent risk measure \( \rho \), we evaluate the risk loss \( X \) by:
\[
\hat{E}[X] := \rho(-X), \quad X \in \mathcal{X}.
\]
This functional satisfies the following properties:

(a) **Monotonicity:**
\[
X \geq Y \implies \hat{E}[X] \geq \hat{E}[Y].
\]

(b) **Constant preserving**
\[
\hat{E}[c] = c, \quad \forall c \in \mathbb{R}.
\]

(c) **Sub-additivity:** For each \( X, Y \in \mathcal{X} \),
\[
\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y].
\]

(d) **Positive homogeneity:**
\[
\hat{E}[\lambda X] = \lambda \hat{E}[X], \quad \forall \lambda \geq 0.
\]

A real valued functional \( \hat{E}[\cdot] \) defined on \( \mathcal{X} \) satisfying (a)–(d) will be called a sublinear expectation and will be systematically studied in this lecture.

**Remark 2.1.10** (c)+(d) is called sublinearity. This sublinearity implies:

(c) **Convexity:**
\[
\hat{E}[\alpha X + (1 - \alpha)Y] \leq \alpha \hat{E}[X] + (1 - \alpha)\hat{E}[Y], \quad \forall \alpha \in [0, 1];
\]
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Remark 2.1.11 \((b) + (d)\) implies \( (f) \) Cash translatability:
\[
\hat{E}[X + c] = \hat{E}[X] + c.
\]
Indeed, we have
\[
\hat{E}[X] + c = \hat{E}[X] - \hat{E}[-c] \\
\leq \hat{E}[X + c] \leq \hat{E}[X] + \hat{E}[c] = \hat{E}[X] + c.
\]

Remark 2.1.12 \((c) + (d) \iff (d) + \) the following Convexity:
\[
\hat{E}[\alpha X + (1 - \alpha)Y] \leq \alpha \hat{E}[X] + (1 - \alpha) \hat{E}[Y], \ \forall \alpha \in [0, 1].
\]

Remark 2.1.13 We recall the notion of the above expectations satisfying \((c)-(d)\) was systematically introduced by Artzner, Delbaen, Eber and Heath [1], [2], in the case where \(\Omega\) is a finite set, and by Delbaen [20] in general situation with the notation of risk measure: \(\rho(X) = \hat{E}[-X]\). See also in Huber [33] for even early study of this notion \(\hat{E}\) (called upper expectation \(\hat{E}^\ast\) in Ch.10 of [33]) in a finite set \(\Omega\). See Rosazza Gianin [64] or Peng [53], El Karoui & Barrieu [5], [24] for dynamic risk measures using \(g\)-expectations. Super-hedging and super pricing (see [25] and [26]) are also closely related to this formulation.

2.1.3 Examples of sublinear expectations

Let \(\hat{E}^1\) and \(\hat{E}^2\) be two nonlinear expectations defined on \((\Omega, \mathcal{F})\). \(\hat{E}^1\) is said to be dominated by \(\hat{E}^2\) if
\[
\hat{E}^1[X] - \hat{E}^1[Y] \leq \hat{E}^2[X - Y], \ \forall X, Y \in \mathcal{F}.
\]
The strongest sublinear expectation on \(\mathcal{F}\) is
\[
\hat{E}^\infty[X] := X^* = \sup_{\omega \in \Omega} X(\omega).
\]
Namely, all other sublinear expectations are dominated by \(\hat{E}^\infty[\cdot]\). From \(c)\) a sublinear expectation is dominated by itself.

Notation 2.1.14 Let \(\mathcal{P}_f\) be the collection of all finitely additive probability measures on \((\Omega, \mathcal{F})\).
Example 2.1.15 (A linear expectation) We consider \( L^∞_0(\Omega, F) \) the collection of risk positions with finite values. It is a subspace of \( \mathcal{X} \) consisting of risk positions \( X \) of the form
\[
X(\omega) = \sum_{i=1}^{N} x_i 1_{A_i}(\omega), \quad x_i \in \mathbb{R}, \quad A_i \in \mathcal{F}, \quad i = 1, \cdots, N. \tag{2.1}
\]
It is easy to check that, under the norm \( \| \cdot \|_∞ \), \( L^∞_0(\Omega, F) \) is dense on \( L^∞(\Omega, F) \).

For a fixed \( Q \in \mathcal{P}_f \) and \( X \in L^∞_0(\Omega, F) \) we define
\[
E_Q[X] = E_Q\left[ \sum_{i=1}^{N} x_i 1_{A_i}(\omega) \right] := \sum_{i=1}^{N} x_i Q(A_i) = \int_{\Omega} X(\omega) Q(d\omega)
\]

\( E_Q : L^∞_0(\Omega, F) \to \mathbb{R} \) is a linear functional. It is easy to check that \( E_Q \) satisfies (a)-(b). It is also continuous under \( \| X \|_∞ \).

\[
|E_Q[X]| \leq \sup_{\omega \in \Omega} |X(\omega)| = \| X \|_∞.
\]

Since \( L^∞_0 \) is dense in \( L^∞ \) we then can extend \( E_Q \) from \( L^∞_0 \) to a linear continuous functional on \( L^∞(\Omega, F) \).

Proposition 2.1.16 The functional \( E_Q[\cdot] : \mathcal{X} \to \mathbb{R} \) satisfies (a) and (b). Inversely each linear functional \( \eta(\cdot) : \mathcal{X} \to \mathbb{R} \) satisfying (a) and (b) induces a finitely additive probability measure via \( Q_\eta(A) = \eta(1_A), \ A \in \mathcal{F} \). The corresponding expectation is \( \eta \) itself:
\[
\eta(X) = \int_{\Omega} X(\omega) Q_\eta(d\omega).
\]

Notation 2.1.17 Let \( Q \in \mathcal{P}_f \) be given and let \( X : \Omega \to \mathbb{R} \) be a \( \mathcal{F} \)-measurable function such that \( |X(\omega)| < \infty \) for each \( \omega \). The distribution of \( X \) on \( (\Omega, \mathcal{F}, Q) \) is defined as a linear functional
\[
F_X[\varphi] = E_Q[\varphi(X)] : \varphi \in L^∞(\mathbb{R}, \mathcal{B}(\mathbb{R})) \to \mathbb{R}.
\]

\( F_X[\cdot] \) is a linear functional satisfying (a) and (b). Thus it induces a finitely additive probability measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) via \( F_X(B) := F_X(1_B), \ B \in \mathcal{B}(\mathbb{R}) \).

We have
\[
F_X[\varphi] = \int_{\mathbb{R}} \varphi(x) F_X(dx).
\]
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Remark 2.1.18 Usually, people call the probability measure $F_X(\cdot)$ to be the distribution of $X$ under $Q$. But we will see that in sublinear or more general situation the functional version is necessary.

Definition 2.1.19 Let $\mathbb{E}$ be a linear finitely additive probability defined on a measurable space $(\Omega, \mathcal{F})$. We say that a random variable $Y$ is independent to $X$ if, for each $\varphi \in L^\infty(\mathbb{R}^2)$

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}].$$

2.1.4 Representation of a sublinear expectation

Let a risk supervisor take a linear expectation $\mathbb{E}[\cdot]$ to be his risk measure. This means that he takes the corresponding finitely additive probability induced by $\mathbb{E}[\cdot]$ as his probability measure. But in many cases, he cannot decide precisely which probability he should take. He has a set of finitely additive probabilities $Q$. The size of $Q$ characterizes his model-uncertainty. A robust risk measure under such model uncertainty is:

$$\hat{\mathbb{E}}^Q[X] = \sup_{Q \in \mathcal{Q}} E_Q[X] : \mathcal{X} \mapsto \mathbb{R}.$$

It is easy to prove that $\hat{\mathbb{E}}^Q[\cdot]$ is a sublinear expectation.

Theorem 2.1.20 A sublinear expectation $\hat{\mathbb{E}}[\cdot]$ has the following representation: there exists a subset $\mathcal{Q} \subset \mathcal{P}_f$, such that

$$\hat{\mathbb{E}}[X] = \max_{Q \in \mathcal{Q}} E_Q[X], \quad \forall X \in \mathcal{X}.$$

Proof. It suffices to prove that, for each $X_0 \in \mathcal{X}$, there exists a $Q_{X_0} \in \mathcal{P}_f$ such that $E_{Q_{X_0}}[X] \leq \hat{\mathbb{E}}[X]$, for all $X$, and such that $Q_{X_0}(X_0) = \hat{\mathbb{E}}[X_0]$. We only consider the case $\hat{\mathbb{E}}[X_0] = 1$ (otherwise we may consider $\hat{X}_0 = X_0 - \hat{\mathbb{E}}[X_0] + 1$).

Let $U_1 := \{X : \hat{\mathbb{E}}[X] < 1\}$. Since $U_1$ is an open and convex set, by the well-known separation theorem for convex sets, there exists a continuous linear functional $\eta$ on $\mathcal{X}$ such that $\eta(X) < \eta(X_0)$, for all $X \in U_1$. Since $0 \in U_1$ we have particularly $\eta(X_0) > 0$, and we then can normalize $\eta$ so that $\eta(X_0) = 1 = \hat{\mathbb{E}}[X_0]$. Thus $\eta$ satisfies:

$$\eta(X) < 1, \quad \forall X \in \mathcal{X}, \quad \text{s.t.} \quad \hat{\mathbb{E}}[X] < 1.$$
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By the positive homogeneity of $\hat{E}$, for each $c > 0$,

$$\eta(X) < c, \quad \forall X \in \mathcal{X}, \text{ s.t. } \hat{E}[X] < c.$$ 

This implies that

$$\hat{E}[X] \geq \eta(X), \text{ for all } X \text{ such that } \hat{E}[X] > 0. \tag{2.2}$$

We now prove that $\eta$ is monotone in the sense of (a). For a given $Y \geq 0$ and $\lambda > 0$, since $-\lambda Y \in U_1$, we have

$$\eta(-\lambda Y) < \eta(X_0), \quad \text{thus } \eta(Y) > -\lambda^{-1}\eta(X_0) = -\lambda^{-1}, \quad \forall \lambda > 0.$$ 

From which it follows that $\eta(Y) \geq 0$ and thus (a) holds true. From (2.2) we have $\eta(1) \leq \hat{E}[1] = 1$. We now prove that $\eta(1) = 1$. Indeed, for each $c > 1$, we have

$$\hat{E}[2X_0 - c] = 2\hat{E}[X_0] - c = 2 - c < 1,$$

hence $2X_0 - c \in U_1$ and

$$\eta(2X_0 - c) = 2 - c\eta(1) < 1 \quad \text{or } \eta(1) > \frac{1}{c}, \quad \forall c > 1;$$

hence $\eta(1) = 1$. This, together with (2.2), yields

$$\hat{E}[X] \geq \eta(X), \text{ for all } X \in \mathcal{X}.$$ 

We thus proved the desired result.
Chapter 3

LLN and Central Limit Theorem

3.1 Preliminary

The law of normal size numbers (LLN) and central limit theorem (CLT) are long and widely been known as two fundamental results in the theory of probability and statistics. A striking consequence of CLT is that accumulated independent and identically distributed random variables tends to a normal distributed random variable whatever is the original distribution. It is a very useful tool in finance since many typical financial positions are accumulations of a large number of small and independent risk positions. But CLT only holds in cases of model certainty. In this section we are interested in CLT with variance-uncertainty. We will prove that the accumulated risk positions can converge ‘in law’ to what we call $G$-normal distribution, which is a distribution under sublinear expectation. In a special case where the variance-uncertainty becomes zero, the $G$-normal distribution becomes the classical normal distribution. Technically we introduce a new method to prove a CLT under a sublinear expectation space.

In the following two chapters we will consider the following type of spaces of sublinear expectations: Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^n)$ where $C_{l,\text{Lip}}(\mathbb{R}^n)$ denotes the
linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on $\varphi$.

$\mathcal{H}$ is considered as a space of “random variables”.

**Remark 3.1.1** In particular, if $X, Y \in \mathcal{H}$, then $|X|, X^m \in \mathcal{H}$ are in $\mathcal{H}$. More generally $\varphi(X)\psi(Y)$ is still in $\mathcal{H}$ if $\varphi, \psi \in C_{L\text{ip}}(\mathbb{R})$.

Here we use $C_{L\text{ip}}(\mathbb{R}^n)$ in our framework only for some convenience of techniques. In fact our essential requirement is that $\mathcal{H}$ contains all constants and, moreover, $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. In general $C_{L\text{ip}}(\mathbb{R}^n)$ can be replaced by the following spaces of functions defined on $\mathbb{R}^n$.

- $L^\infty(\mathbb{R}^n)$: the space bounded Borel-measurable functions;
- $C_b(\mathbb{R}^n)$: the space of bounded and continuous functions;
- $C^{k}_b(\mathbb{R}^n)$: the space of bounded and $k$-time continuously differentiable functions with bounded derivatives of all orders less than or equal to $k$;
- $C_{\text{uni}}(\mathbb{R}^n)$: the space of bounded and uniformly continuous functions;
- $C_{b,L\text{ip}}(\mathbb{R}^n)$: the space of bounded and Lipschitz continuous functions;
- $L^0(\mathbb{R}^n)$: the space of Borel measurable functions.

**Definition 3.1.2** Sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) **Monotonicity:** If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.
(b) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$.
(c) **Sub-additivity:** $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$.
(d) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \quad \forall \lambda \geq 0$.

(In many situation (c) is also called property of self–domination). The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$).
Example 3.1.3 In a game we select at random a ball from a box containing $W$ white, $B$ black and $Y$ yellow balls. The owner of the box, who is the banker of the game, does not tell us the exact numbers of $W$, $B$ and $Y$. He or she only informs us $W + B + Y = 100$ and $W = B \in \left[20, 25\right]$. Let $\xi$ be a random variable

$$
\xi = \begin{cases}
1 & \text{if we get a white ball;} \\
0 & \text{if we get a yellow ball;} \\
-1 & \text{if we get a black ball.}
\end{cases}
$$

Problem: How to measure a loss $X = \varphi(\xi)$, for a given function $\varphi$ on $\mathbb{R}$. We know that the distribution of $\xi$ is

$$
\begin{pmatrix}
-1 & 0 & 1 \\
\frac{p}{2} & 1 - p & \frac{p}{2}
\end{pmatrix}
$$

with uncertainty: $p \in [\sigma^2, \sigma^2] = [0.4, 0.5]$.

Thus the robust expectation of $X = \varphi(\xi)$ is:

$$
\hat{E}[\varphi(\xi)] := \sup_{P \in \mathcal{P}} E_P[\varphi(\xi)]
= \sup_{p \in [\sigma^2, \sigma^2]} \left[\frac{p}{2}[\varphi(1) + \varphi(-1)] + (1 - p)\varphi(0)\right].
$$

$\xi$ has distribution uncertainty.

Example 3.1.4 A more general situation is that the banker of a game can choose among a set of distribution $\{F(\theta, \mathcal{A})\}_{\mathcal{A} \in \mathcal{B}(\mathbb{R}), \theta \in \Theta}$ of a random variable $\xi$. In this situation the robust expectation of a risk position $\varphi(\xi)$ for some $\varphi \in C_{Lip}(\mathbb{R})$ is:

$$
\hat{E}[\varphi(\xi)] := \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(x) F(\theta, dx).
$$

3.2 Distributions and independence

We now consider the notion of the distributions of random variables under sublinear expectations. Let $X = (X_1, \ldots, X_n)$ be a given $n$-dimensional random vector on a sublinear expectation space $(\Omega_1, \mathcal{H}_1, \hat{E})$. We define a functional on $C_{Lip}(\mathbb{R}^n)$ by

$$
\hat{F}_X[\varphi] := \hat{E}[\varphi(X)]: \varphi \in C_{Lip}(\mathbb{R}^n) \mapsto (-\infty, \infty).
$$

The triple $(\mathbb{R}^n, C_{Lip}(\mathbb{R}^n), \hat{F}_X[\cdot])$ forms a sublinear expectation space. $\hat{F}_X$ is called the distribution of $X$. 
Definition 3.2.1 Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if
\[ \hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l,\text{Lip}}( \mathbb{R}^n). \]
It is clear that $X_1 \sim X_2$ if and only if their distributions coincide.

Remark 3.2.2 If the distribution $\hat{F}_X$ of $X \in \mathcal{H}$ is not a linear expectation, then $X$ is said to have distributional uncertainty. The distribution of $X$ has the following four typical parameters:
\[ \mu := \hat{E}[X], \quad \mu := -\hat{E}[-X], \quad \sigma^2 := \hat{E}[X^2], \quad \sigma^2 := -\hat{E}[-X^2]. \]
The subsets $[\mu, \overline{\mu}]$ and $[\sigma^2, \overline{\sigma^2}]$ characterize the mean-uncertainty and the variance-uncertainty of $X$. The problem of mean uncertainty have been studied in [Chen-Epstein] using the notion of $g$-expectations. In this lecture we are mainly concentrated on the situation of variance-uncertainty and thus set $\overline{\mu} = \mu$.

The following simple property is very useful in our sublinear analysis.

Proposition 3.2.3 Let $X,Y \in \mathcal{H}$ be such that $\hat{E}[Y] = -\hat{E}[-Y]$, i.e. $Y$ has no mean uncertainty. Then we have
\[ \hat{E}[X + Y] = \hat{E}[X] + \hat{E}[Y]. \]
In particular, if $\hat{E}[Y] = \hat{E}[-Y] = 0$, then $\hat{E}[X + Y] = \hat{E}[X]$.

Proof. It is simply because we have $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$ and
\[ \hat{E}[X + Y] \geq \hat{E}[X] - \hat{E}[-Y] = \hat{E}[X] + \hat{E}[Y]. \]

The following notion of independence plays a key role:

Definition 3.2.4 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}[]$ if for each test function $\varphi \in C_{l,\text{Lip}}( \mathbb{R}^m \times \mathbb{R}^n)$ we have
\[ \hat{E}[\varphi(X,Y)] = \hat{E}[\hat{E}[\varphi(x,Y)]_{x=X}]. \]
A random variable $Y \in \mathcal{H}$ is said to be weakly independent to $X \in \mathcal{H}^\otimes m$ if the above test functions $\varphi$ are only taken from the following class:

$$
\varphi(x, y) = \psi_0(x) + \psi_1(x)y + \psi_2(x)y^2, \quad \psi_i \in C_{Lip}(\mathbb{R}^m \times \mathbb{R}), \quad i = 0, 1, 2.
$$

**Remark 3.2.5** In the case of linear expectation, this notion of independence is just the classical one. It is important to note that under sublinear expectations the condition “$Y$ is independent to $X$” does not imply automatically that “$X$ is independent to $Y$”.

**Example 3.2.6** We consider a case where $X, Y \in \mathcal{H}$ are identically distributed and $\hat{E}[X] = \hat{E}[-X] = 0$ but $\sigma^2 = \hat{E}[X^2] > \underline{\sigma}^2 = -\hat{E}[-X^2]$. We also assume that $\hat{E}[|X|] = \hat{E}[X^+ + X^-] > 0$, thus $\hat{E}[X^+] = \frac{1}{2}\hat{E}[|X| + X] > 0$. In the case where $Y$ is independent to $X$, we have

$$
\hat{E}[XY^2] = \hat{E}[X^+\sigma^2 - X^-\underline{\sigma}^2] = (\sigma^2 - \underline{\sigma}^2)\hat{E}[X^+] > 0.
$$

But if $X$ is independent to $Y$ we have

$$
\hat{E}[XY^2] = 0.
$$

The independence property of two random vectors $X, Y$ involves only the joint distribution of $(X, Y)$. The following result tell us how to construct random vectors with given sublinear distributions and with joint independence.

**Proposition 3.2.7** Let $X_i$ be $n_i$-dimensional random vectors respectively in sublinear expectation spaces $(\Omega_i, \mathcal{H}_i, \hat{E}_i)$, $i = 1, \cdots, N$. We can construct random vectors $Y_1, \cdots, Y_N$ in a new sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ such that $Y_i \sim X_i$ and such that $Y_{i+1}$ is independent to $(Y_1, \cdots, Y_i)$, for each $i$.

**Proof.** We first consider the case $N = 2$. We set: $\Omega = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, i.e., $\omega = (x_1, x_2) \in \Omega$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$; a space of random variables $\mathcal{H} = \{X(\omega) = \varphi(\omega), \varphi \in C_{Lip}(\Omega)\}$ and a functional $\hat{E}$ on $(\Omega, \mathcal{H})$ defined by

$$
\hat{E}[\varphi(Y)] = \hat{E}_1[\varphi(X_1)], \quad \text{where} \quad \varphi(x_1) := \hat{E}_2[\varphi(x_1, X_2)], \quad x_1 \in \mathbb{R}^{n_1},
$$

$$
\forall Y(\omega) = \varphi(\omega), \quad \varphi \in C_{Lip}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).
$$
It is easy to check that \( \hat{E} \) forms a sublinear expectation on \((\Omega, \mathcal{H})\). Now let us consider two random vectors in \( \mathcal{H} \):

\[
Y_1(\omega) = x_1, \quad Y_2(\omega) = x_2, \quad \omega = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.
\]

It is easy to check that \( Y_1, Y_2 \) meet our requirement: \( Y_2 \) is independent to \( Y_1 \) under \( \hat{E} \), and \( Y_1 \sim X_1, Y_2 \sim X_2 \). The case of \( N > 2 \) can be proved by repeating the above procedure.

**Example 3.2.8** We consider a situation where two random variables \( X \) and \( Y \) in \( \mathcal{H} \) are identically distributed and their common distribution is

\[
\hat{F}_X[\varphi] = \hat{F}_Y[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(y) F(\theta, dy), \quad \varphi \in C_{l,Lip}(\mathbb{R}).
\]

where, for each \( \theta \in \Theta \), \( \{F(\theta, A)\}_{A \in \mathcal{B}(\mathbb{R})} \) is a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). In this case “\( Y \) is independent to \( X \)” means that the joint distribution of \( X \) and \( Y \) is:

\[
\hat{F}_{X,Y}[\psi] = \sup_{\theta_1 \in \Theta} \int_{\mathbb{R}} \left[ \sup_{\theta_2 \in \Theta} \int_{\mathbb{R}} \psi(x, y) F(\theta_2, dy) \right] F(\theta_1, dx),
\]

\( \psi \in C_{l,Lip}(\mathbb{R}^2) \).

**Remark 3.2.9** The situation “\( Y \) is independent to \( X \)” often appears when \( Y \) occurs after \( X \), thus a very robust expectation should take the information of \( X \) into account.

**Definition 3.2.10** A sequence of random variables \( \{\eta_i\}_{i=1}^{\infty} \) in \( \mathcal{H} \) is said to converge in distribution under \( \hat{E} \) if for each bounded and \( \varphi \in C_{b,Lip}(\mathbb{R}) \),

\[
\left\{ \hat{E}[\varphi(\eta_i)] \right\}_{i=1}^{\infty}
\]

converges.

### 3.3 G-normal distributions

#### 3.3.1 G-normal distribution

A fundamentally important distribution in sublinear expectation theory is \( \mathcal{N}(0; [\sigma^2, \sigma^2]) \)-distributed random variable \( X \) under \( \hat{E}[\cdot] \):
### 3.3. G-NORMAL DISTRIBUTIONS

**Definition 3.3.1** *(G-normal distribution)* In a sublinear expectation space \((\Omega, \mathbb{E}, \mathcal{H})\), a random variable \(X \in \mathcal{H}\) with
\[
\sigma^2 = \mathbb{E}[X^2], \quad \sigma^2 = -\mathbb{E}[-X^2]
\]
is said to be \(N(0; [\sigma^2, \sigma^2])\)-distributed, denoted by \(X \sim N(0; [\sigma^2, \sigma^2])\), if for each \(Y \in \mathcal{H}\) which is independent to \(X\) such that \(Y \sim X\) we have
\[
aX + bY \sim \sqrt{a^2 + b^2}X, \quad \forall a, b \geq 0.
\] (3.2)

Proposition 3.2.7 tells us how to construct the above \(Y\) based on the distribution of \(X\).

**Remark 3.3.2** By the above definition, we have
\[
\sqrt{2} \mathbb{E}[X] = \mathbb{E}[X + Y] = 2 \mathbb{E}[X] \quad \text{and} \quad \sqrt{2} \mathbb{E}[-X] = \mathbb{E}[-X - Y] = 2 \mathbb{E}[-X]
\]
it follows that
\[
\mathbb{E}[X] = \mathbb{E}[-X] = 0,
\]
Namely an \(N(0; [\sigma^2, \sigma^2])\)-distributed random variable \(X\) has no mean uncertainty.

**Remark 3.3.3** If \(X\) is independent to \(Y\) and \(X \sim Y\), such that (3.2) satisfies, then \(-X\) is independent to \(-Y\), \(-X \sim -Y\). We also have \(a(-X) + b(-Y) \sim \sqrt{a^2 + b^2}(-X), \ a, b \geq 0\). Thus
\[
X \sim N(0; [\sigma^2, \sigma^2]) \quad \text{iff} \quad -X \sim N(0; [\sigma^2, \sigma^2]).
\]

The following proposition and corollary show that \(N(0; [\sigma^2, \sigma^2])\) is a uniquely defined sublinear distribution on \((\mathbb{R}, C_{\text{Lip}}(\mathbb{R}))\). We will show that an \(N(0; [\sigma^2, \sigma^2])\) distribution is characterized, or generated, by the following parabolic PDE defined on \([0, \infty) \times \mathbb{R}\):
\[
\partial_t u - G(\partial_{xx}^2 u) = 0,
\] (3.3)
with Cauchy condition \(u|_{t=0} = \varphi\), where \(G\), called the generating function of (3.3), is the following sublinear real function parameterized by \(\sigma\) and \(\sigma^2\):
\[
G(\alpha) := \frac{1}{2} \mathbb{E}[X^2 \alpha] = \frac{1}{2}(\sigma^2 \alpha^+ - \sigma^2 \alpha^-), \quad \alpha \in \mathbb{R}.
\]
Here we denote \(\alpha^+ := \max\{0, \alpha\}\) and \(\alpha^- := (-\alpha)^+\). (3.3) is called generating heat equation, or \(G\)-heat equation of the sublinear distribution \(N(0; [\sigma^2, \sigma^2])\). We also call this sublinear distribution \(G\)-normal distribution.
Remark 3.3.4 We will use the notion of viscosity solutions to the generating heat equation (3.3). This notion was introduced by Crandall and Lions. For the existence and uniqueness of solutions and related very rich references we refer to Crandall, Ishii and Lions [17]. We note that, in the situation where $\sigma^2 > 0$, the viscosity solution \((3.3)\) becomes a classical $C^{1+\frac{\alpha}{2}}$-solution (see [42], [43] and the recent works of [6] and [67]). Readers can understand \((3.3)\) in the classical meaning.

Definition 3.3.5 A real-valued continuous function $u \in C([0,T] \times \mathbb{R})$ is called a viscosity subsolution (respectively, supersolution) of \((3.3)\) if, for each function $\psi \in C^3_b((0,\infty) \times \mathbb{R})$ and for each minimum (respectively, maximum) point $(t,x) \in (0,\infty) \times \mathbb{R}$ of $\psi - u$, we have
\[ \partial_t \psi - G(\partial_{xx}^2 \psi) \leq 0 \quad \text{(respectively,} \quad \geq 0). \]
$u$ is called a viscosity solution of \((3.3)\) if it is both super and subsolution.

Proposition 3.3.6 Let $X$ be an $\mathcal{N}(0; [\sigma^2, \sigma^2])$ distributed random variable. For each $\varphi \in C_{l,\text{Lip}}(\mathbb{R})$ we define a function
\[ u(t,x) := \hat{E}[\varphi(x + \sqrt{t}X)], \quad (t,x) \in [0,\infty) \times \mathbb{R}. \]
Then we have
\[ u(t+s,x) = \hat{E}[u(t,x + \sqrt{s}X)], \quad s \geq 0. \quad (3.4) \]
We also have the estimates: For each $T > 0$ there exist constants $C, k > 0$ such that, for all $t, s \in [0, T]$ and $x, y \in \mathbb{R},$
\[ |u(t,x) - u(t,y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad (3.5) \]
and
\[ |u(t,x) - u(t+s,x)| \leq C(1 + |x|^k)|s|^{1/2}. \quad (3.6) \]
Moreover, $u$ is the unique viscosity solution, continuous in the sense of \((3.3)\) and \((3.6)\), of the generating PDE \((3.3)\).

Proof. Since
\[ u(t,x) - u(t,y) = \hat{E}[\varphi(x + \sqrt{t}X)] - \hat{E}[\varphi(y + \sqrt{t}X)] \]
\[ \leq \hat{E}[\varphi(x + \sqrt{t}X) - \varphi(y + \sqrt{t}X)] \]
\[ \leq \hat{E}[C_1(1 + |X|^k + |y|^k)|x - y|] \]
\[ \leq C(1 + |x|^k + |y|^k)|x - y|. \]
We then have (3.5). Let \( Y \) be independent to \( X \) such that \( X \sim Y \). Since \( X \) is \( \mathcal{N}(0; \sigma^2, \sigma^2) \)-distributed, then
\[
u(t + s, x) = \hat{E}[\varphi(x + \sqrt{t + s}X)] = \hat{E}[\varphi(x + \sqrt{s}X + \sqrt{t}Y)] = \hat{E}[\hat{E}[\varphi(x + \sqrt{s}z + \sqrt{t}Y)|z=x]] = \hat{E}[\hat{E}[\varphi(x + \sqrt{s}z + \sqrt{t}Y)|z=x] = \hat{E}[\hat{E}[\nu(t, x + \sqrt{s}X)].
\]
We thus obtain (3.4). From this and (3.5) it follows that
\[
u(t + s, x) - \nu(t, x) = \hat{E}[\nu(t, x + \sqrt{s}X) - \nu(t, x)] \leq \hat{E}[C_1(1 + |x|^k + |X|^k)|s|^{1/2}|X|].
\]
Thus we obtain (3.6). Now, for a fixed \((t, x) \in (0, \infty) \times \mathbb{R}\), let \( \psi \in C^{1,3}_b([0, \infty) \times \mathbb{R}) \) be such that \( \psi \geq \nu \) and \( \psi(t, x) = \nu(t, x) \). By (3.4) it follows that, for \( \delta \in (0, t) \)
\[
0 \leq \hat{E}[\psi(t - \delta, x + \sqrt{\delta}X) - \psi(t, x)] = -\partial_t \psi(t, x) \delta + \hat{E}[\partial_x \psi(t, x)\sqrt{\delta}X + \frac{1}{2}\partial_{xx}^2 \psi(t, x)\delta X^2] + \bar{C}\delta^{3/2}
= -\partial_t \psi(t, x) \delta + \hat{E}[\frac{1}{2}\partial_{xx}^2 \psi(t, x)\delta X^2] + \bar{C}\delta^{3/2}
= -\partial_t \psi(t, x) \delta + \delta \bar{G}(\partial_{xx}^2 \psi(t, x)) + \bar{C}\delta^{3/2}.
\]
From which it is easy to check that
\[
[\partial_t \psi - \bar{G} (\partial_{xx}^2 \psi)](t, x) \leq 0.
\]
It follows that \( \nu \) is a viscosity supersolution of (3.3). Similarly we can prove that \( \nu \) is a viscosity subsolution of (3.3). □

**Corollary 3.3.7** If both \( X \) and \( X \) are \( \mathcal{N}(0; \sigma^2, \sigma^2) \)-distributed. Then \( X \sim X \). In particular, \( X \sim -X \).

**Proof.** For each \( \varphi \in C_{Lip}(\mathbb{R}) \) we set
\[
u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)], \quad \bar{\nu}(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)], \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
CHAPTER 3. LLN AND CENTRAL LIMIT THEOREM

By the above Proposition, both $u$ and $\bar{u}$ are viscosity solutions of the G-heat equation (3.3) with Cauchy condition $u|_{t=0} = \bar{u}|_{t=0} = \varphi$. It follows from the uniqueness of the viscosity solution that $u \equiv \bar{u}$. In particular

$$\hat{E}[\varphi(X)] = \hat{E}[\varphi(\bar{X})].$$

Thus $X \sim \bar{X}$. ■

**Corollary 3.3.8** In the case where $\sigma^2 = \sigma^2 > 0$, $\mathcal{N}(0; [\sigma^2, \sigma^2])$ is just the classical normal distribution $\mathcal{N}(0; \sigma^2)$.

**Proof.** In fact the solution of the generating PDE (3.3) becomes a classical heat equation

$$\partial_t u = \frac{\sigma^2}{2} \partial^2_{xx} u, \quad u|_{t=0} = \varphi$$

where the solution is

$$u(t, x) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{(x-y)^2}{2\sigma^2 t}\right) dy$$

thus, for each $\varphi$,

$$\hat{E}[\varphi(X)] = u(1, 0) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{y^2}{2\sigma^2}\right) dy.$$

In two typical situations the calculation of $\hat{E}[\varphi(X)]$ is very easy:

- (i) For each convex $\varphi$, we have

$$\hat{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

Indeed, for each fixed $t \geq 0$, it is easy to check that the function $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$ is convex:

$$u(t, \alpha x + (1 - \alpha)y) = \hat{E}[\varphi(\alpha x + (1 - \alpha)y + \sqrt{t}X)]$$

$$\leq \alpha \hat{E}[\varphi(x + \sqrt{t}X)] + (1 - \alpha) \hat{E}[\varphi(x + \sqrt{t}X)]$$

$$= \alpha u(t, x) + (1 - \alpha) u(t, y)$$

It follows that $(\partial^2_{xx} u)^- \equiv 0$ and thus the G-heat equation (3.3) becomes

$$\partial_t u = \frac{\sigma^2}{2} \partial^2_{xx} u, \quad u|_{t=0} = \varphi.$$
(ii) But for each concave $\varphi$, we have,

$$\hat{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

In particular,

$$\hat{E}[X] = \hat{E}[-X] = 0, \quad \hat{E}[X^2] = \sigma^2, \quad -\hat{E}[-X^2] = \sigma^2$$

and

$$\hat{E}[X^4] = 6\sigma^4, \quad -\hat{E}[-X^4] = 6\sigma^4$$

3.3.2 Construction of $G$-normal distributed random variables

To construct an $\mathcal{N}(0; [\sigma^2, \sigma^2])$-distributed random variable $\xi$, let $u = u^\varphi$ be the unique viscosity solution of the $G$-heat equation (3.3) with $u^\varphi|_{t=0} = \varphi$. Then we take $\tilde{\Omega} = \mathbb{R}^2$, $\tilde{\mathcal{H}} = C_{\text{Lip}}(\mathbb{R}^2)$, $\omega = (x, y) \in \mathbb{R}^2$. The corresponding sublinear expectation $\tilde{E}[\cdot]$ is defined by, for each $X(\omega) = \psi(\omega)$ such that $\psi \in C_{\text{Lip}}(\mathbb{R}^2)$,

$$\tilde{E}[X] = u^\tilde{\psi}(1, 0), \text{ where we set } \tilde{\psi}(x) := u^{\psi(x, \cdot)}(1, 0).$$

We now consider two random variables $\xi(\omega) = x$, $\eta(\omega) = y$. It is clear that the

$$\tilde{E}[\varphi(\xi)] = \tilde{E}[\varphi(\eta)] = u^\varphi(1, 0), \quad \forall \varphi \in C_{\text{Lip}}(\mathbb{R}).$$

By the definition $\eta$ is independent to $\xi$ and $\xi \sim \eta$ under $\tilde{E}$. To prove that the distribution of $\xi$ and $\eta$ satisfies condition (3.2), it suffices to observe that, for each $\varphi \in C_{\text{Lip}}(\mathbb{R})$ and for each fixed $\lambda > 0$, $\tilde{x} \in \mathbb{R}$, since the function $v$ defined by $v(t, x) := u^\varphi(\lambda t, \tilde{x} + \sqrt{\lambda} x)$ solves exactly the same $G$-heat equation (3.3) but with Cauchy condition $v|_{t=0} = \varphi(\tilde{x} + \sqrt{\lambda} \cdot)$, we then have

$$\tilde{E}[\varphi(\tilde{x} + \sqrt{\lambda} \xi)] = v(1, \tilde{x}) = u^{\varphi(\sqrt{\lambda} \cdot)}(1, \tilde{x}) = u^{\varphi}(\lambda, \tilde{x}), \quad \tilde{x} \in \mathbb{R}.$$

Thus, for each $t > 0$ and $s > 0$,

$$\tilde{E}[\varphi(\sqrt{t} \xi + \sqrt{s} \eta)] = \tilde{E}\left[\tilde{E}[\varphi(\sqrt{t} x + \sqrt{s} \eta)]|_{x=\xi}\right] = u^{\varphi(s, \cdot)}(t, 0) = u^{\varphi}(t + s, 0) = \tilde{E}[\varphi(\sqrt{t + s} \xi)].$$
Namely $\sqrt{t}\xi + \sqrt{s}\eta \sim \sqrt{t+s}\xi$. Thus $\xi$ and $\eta$ are both $\mathcal{N}(0; [\sigma^2, \sigma^2])$ distributed.

We need to check that the functional $\tilde{E}[-\cdot]: C_{l,Lip}(\mathbb{R}) \mapsto \mathbb{R}$ forms a sub-linear expectation, i.e., (a)-(d) of Definition 3.1.2 are satisfied. Indeed, (a) is simply the consequence of comparison theorem, or the maximum principle of viscosity solution (see Appendix). It is also easy to check that, when $\varphi \equiv c$, then the unique solution of (3.3) is also $u \equiv c$; hence (b) holds true. (d) also holds since $u_{\lambda\varphi} = \lambda u\varphi$, $\lambda \geq 0$. The sub-additivity (c) will be proved in Appendix.

### 3.4 Central Limit Theorem

Our main result is:

**Theorem 3.4.1 (Central Limit Theorem)** Let a sequence $\{X_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ be identically distributed with each others. We also assume that, each $X_{n+1}$ is independent (or weakly independent) to $(X_1, \cdots, X_n)$ for $n = 1, 2, \cdots$. We assume furthermore that

$$
\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0, \quad \mathbb{E}[X_1^2] = \sigma^2, \quad -\mathbb{E}[-X_1^2] = \sigma^2,
$$

for some fixed numbers $0 < \sigma \leq \bar{\sigma} < \infty$. Then the sequence $\{S_n/\sqrt{n}\}_{n=1}^{\infty}$, of the sum $S_n = X_1 + \cdots + X_n$, converges in law to $\mathcal{N}(0; [\sigma^2, \bar{\sigma}^2])$:

$$
\lim_{n \to \infty} \tilde{E}[\varphi(S_n/\sqrt{n})] = \tilde{E}[\varphi(\xi)], \quad \forall \varphi \in \text{lip}_b(\mathbb{R}),
$$

(3.7)

where $\xi \sim \mathcal{N}(0; [\sigma^2, \bar{\sigma}^2])$.

**Proof.** For a small but fixed $h > 0$, let $V$ be the unique viscosity solution of

$$
\partial_t V + G(\partial_{xx}^2 V) = 0, \quad (t, x) \in [0, 1+h] \times \mathbb{R}, \quad V|_{t=1+h} = \varphi.
$$

(3.8)

We have, according to the definition of $G$-normal distribution

$$
V(t, x) = \tilde{E}[\varphi(x + \sqrt{1+h-t}\xi)].
$$

Particularly,

$$
V(h, 0) = \tilde{E}[\varphi(\xi)], \quad V(1+h, x) = \varphi(x).
$$

(3.9)
Since $a$ is a uniformly parabolic PDE and $G$ is a convex function, thus, by the interior regularity of $V$ (see Wang [67], Theorem 4.13) we have

$$\|V\|_{C^{1+\alpha,2+\alpha}[0,1]\times\mathbb{R}} < \infty, \text{ for some } \alpha \in (0, 1).$$

We set $\delta = \frac{1}{n}$ and $S_0 = 0$. Then

$$V(1, \sqrt{\delta}S_n) - V(0, 0) = \sum_{i=0}^{n-1} \{ V((i+1)\delta, \sqrt{\delta}S_{i+1}) - V(i\delta, \sqrt{\delta}S_{i}) \}$$

$$= \sum_{i=0}^{n-1} \left\{ [V((i+1)\delta, \sqrt{\delta}S_{i+1}) - V(i\delta, \sqrt{\delta}S_{i+1})] + [V(i\delta, \sqrt{\delta}S_{i+1}) - V(i\delta, \sqrt{\delta}S_{i})] \right\}$$

$$= \sum_{i=0}^{n-1} \{ I_i^i + J_i^i \}$$

with, by Taylor’s expansion,

$$J_i^i = \partial_t V(i\delta, \sqrt{\delta}S_{i})\delta + \frac{1}{2} \partial_{xx}^2 V(i\delta, \sqrt{\delta}S_{i})X_{i+1}^2\delta + \partial_x V(i\delta, \sqrt{\delta}S_{i})X_{i+1}\sqrt{\delta}$$

$$I_i^i = \int_0^1 \left[ \partial_t V((i + \beta)\delta, \sqrt{\delta}S_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}S_{i+1}) \right] d\beta \delta$$

$$+ [\partial_t V(i\delta, \sqrt{\delta}S_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}S_{i})]\delta$$

$$+ \int_0^1 \int_0^1 \left[ \partial_{xx}^2 V(i\delta, \sqrt{\delta}S_{i} + \gamma \beta X_{i+1}\sqrt{\delta}) - \partial_{xx}^2 V(i\delta, \sqrt{\delta}S_{i}) \right] \gamma d\beta d\gamma X_{i+1}^2 \delta.$$ 

Thus

$$\hat{\mathbb{E}}\left[ \sum_{i=0}^{n-1} J_i^i \right] - \hat{\mathbb{E}}\left[ - \sum_{i=0}^{n-1} I_i^i \right] \leq \hat{\mathbb{E}}\left[ V(1, \sqrt{\delta}S_n) \right] - V(0, 0) \leq \hat{\mathbb{E}}\left[ \sum_{i=0}^{n-1} J_i^i \right] + \hat{\mathbb{E}}\left[ \sum_{i=0}^{n-1} I_i^i \right] \quad (3.10)$$

We now prove that $\hat{\mathbb{E}}\left[ \sum_{i=0}^{n-1} J_i^i \right] = 0$. Indeed, the 3rd term of $J_i^i$ has mean-certainty:

$$\hat{\mathbb{E}}[\partial_x V(i\delta, \sqrt{\delta}S_{i})X_{i+1}\sqrt{\delta}] = \hat{\mathbb{E}}[-\partial_x V(i\delta, \sqrt{\delta}S_{i})X_{i+1}\sqrt{\delta}] = 0.$$ 

For the second term, we have

$$\hat{\mathbb{E}}\left[ \frac{1}{2} \partial_{xx}^2 V(i\delta, \sqrt{\delta}S_{i})X_{i+1}^2 \delta \right] = \hat{\mathbb{E}}[G(\partial_{xx}^2 V(i\delta, \sqrt{\delta}S_{i}))\delta].$$
We then combine the above two equalities with \( \partial_t V + G(\partial_{xx}^2 V) = 0 \) as well as the independence of \( X_{i+1} \) to \( (X_1, \cdots, X_i) \), it follows that
\[
\hat{E}[\sum_{i=0}^{n-1} J^i_\delta] = \hat{E}[\sum_{i=0}^{n-2} J^i_\delta] = \cdots = 0.
\]
Thus (3.10) can be rewritten as
\[
-\hat{E}[\sum_{i=0}^{n-1} I^i_\delta] \leq \hat{E}[V(1, \sqrt{\delta} S_n)] - V(0, 0) \leq \hat{E}[\sum_{i=0}^{n-1} I^i_\delta].
\]
But since both \( \partial_t V \) and \( \partial_{xx}^2 V \) are uniformly \( \alpha \)-holder continuous in \( x \) and \( \frac{\alpha}{2} \)-holder continuous in \( t \) on \([0, 1] \times \mathbb{R} \), we then have \( |I^i_\delta| \leq C_\delta^{1+\alpha/2}(1 + |X_1| + |X_{i+1}|^{2+\alpha}) \). It follows that
\[
\hat{E}[|I^i_\delta|] \leq C \delta^{1+\alpha/2}(1 + \hat{E}[|X_1|^\alpha] + \hat{E}[|X_1|^{2+\alpha})].
\]
Thus
\[
-C(\frac{1}{n})^{\alpha/2}(1 + \hat{E}[|X_1|^\alpha + |X_1|^{2+\alpha}) \leq \hat{E}[V(1, \sqrt{\delta} S_n)] - V(0, 0)
\leq C(\frac{1}{n})^{\alpha/2}(1 + \hat{E}[|X_1|^\alpha + |X_1|^{2+\alpha})].
\]
As \( n \to \infty \) we have
\[
\lim_{n \to \infty} \hat{E}[V(1, \sqrt{\delta} S_n)] = V(0, 0).
\] (3.11)
On the other hand, for each \( t, t' \in [0, 1 + h] \) and \( x \in \mathbb{R} \),
\[
|V(t, x) - V(t', x)| = |\hat{E}[\varphi(x + \sqrt{1 + h - t} \xi) - \varphi(\sqrt{1 + h - t'} \xi)]|
\leq k_\varphi|\sqrt{1 + h - t} - \sqrt{1 + h - t'}| \times \hat{E}[|\xi|]
\leq C\sqrt{|t - t'|},
\]
where \( k_\varphi \) denotes the Lipschitz constant of \( \varphi \). Thus \( |V(0, 0) - V(h, 0)| \leq C\sqrt{h} \) and, by (3.9),
\[
|\hat{E}[V(1, \sqrt{\delta} S_n)] - \hat{E}[\varphi(\sqrt{\delta} S_n)]|
= |\hat{E}[V(1, \sqrt{\delta} S_n)] - \hat{E}[V(1 + h, \sqrt{\delta} S_n)]| \leq C\sqrt{h}.
\]
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It follows from (3.11) and (3.9) that

$$\limsup_{n \to \infty} |\tilde{E}[\varphi(S_n/\sqrt{n})] - \tilde{E}[\varphi(\xi)]| \leq 2C\sqrt{h}.$$  

Since $h$ can be arbitrarily small we thus have

$$\lim_{n \to \infty} \tilde{E}[\varphi(S_n/\sqrt{n})] = \tilde{E}[\varphi(\xi)].$$

\[ \square \]

Corollary 3.4.2 The convergence (3.7) holds for the case where $\varphi$ is a bounded and uniformly continuous function.

**Proof.** We can find a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in $lip_b(\mathbb{R})$ such that $\varphi_k \to \varphi$ uniformly on $\mathbb{R}$. By

$$|\tilde{E}[\varphi(S_n/\sqrt{n})] - \tilde{E}[\varphi(\xi)]| \leq |\tilde{E}[\varphi(S_n/\sqrt{n})] - \tilde{E}[\varphi_k(S_n/\sqrt{n})]|$$

$$+ |\tilde{E}[\varphi(\xi)] - \tilde{E}[\varphi_k(\xi)]| + |\tilde{E}[\varphi_k(S_n/\sqrt{n})] - \tilde{E}[\varphi_k(\xi)]|.$$  

We can easily check that (3.7) holds. \[ \square \]
CHAPTER 3. LLN AND CENTRAL LIMIT THEOREM
Chapter 4

G-Brownian Motion: 1-Dimensional Case

4.1 1-dimensional G-Brownian motion

In this lecture I will introduce the notion of G-Brownian motion related to the G-normal distribution in a space of a sublinear expectation. We first give the definition of the G-Brownian motion.

Definition 4.1.1 A process \( \{B_t(\omega)\}_{t \geq 0} \) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called a \(G\)-Brownian motion if for each \( n \in \mathbb{N} \) and \( 0 \leq t_1, \ldots, t_n < \infty \), \( B_{t_1}, \ldots, B_{t_n} \in \mathcal{H} \) and the following properties are satisfied:

(i) \( B_0(\omega) = 0 \);

(ii) For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \) is \( \mathcal{N}(0; [\sigma^2 s, \sigma^2 s]) \)-distributed and is independent to \( (B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \).

Remark 4.1.2 The letter \( G \) indicates that the process \( B \) is characterized by its ‘generating function’ \( G \) defined by

\[
G(\alpha) := \hat{E}[\alpha B_t^2], \quad \alpha \in \mathbb{R}.
\]

We can prove that, for each \( \lambda > 0 \), \( (\lambda^{-\frac{1}{2}} B_{\lambda t})_{t \geq 0} \) is also a \( G \)-Brownian motion. For each \( t_0 > 0 \), \( (B_{t+t_0} - B_{t_0})_{t \geq 0} \) is also a \( G \)-Brownian motion. This is the scaling property of \( G \)-Brownian motion, which is the same as that of the usual Brownian motion.

In this course we assume without loss of generality that \( \sigma = 1 \) and \( \overline{\sigma} = \sigma \leq 1 \).
Theorem 4.1.3 Let \((\tilde{B}_t)_{t \geq 0}\) be a process defined in a sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\) such that
(i) \(\tilde{B}_0 = 0\);
(ii) For each \(t, s \geq 0\), the difference \(\tilde{B}_{t+s} - \tilde{B}_t\) and \(\tilde{B}_s\) are identically distributed and independent to \((\tilde{B}_t, \tilde{B}_{t_2}, \ldots, \tilde{B}_{t_n})\), for each \(n \in \mathbb{N}\) and \(0 \leq t_1 \leq \cdots \leq t_n \leq t\).
(iii) \(\tilde{B}_0 = 0\), \(\mathbb{E}[\tilde{B}_t] = \mathbb{E}[-\tilde{B}_t] = 0\) and \(\lim_{t \downarrow 0} \mathbb{E}[(\tilde{B}_t)^2] t^{-1} = 0\).
Then \(\tilde{B}\) is a \(\mathcal{G}_{\sigma^2}\)-Brownian motion with \(\sigma^2 = \mathbb{E}[\tilde{B}_1^2]\) and \(\sigma^2 = -\mathbb{E}[-\tilde{B}_1^2]\).

Proof. We need only to prove that \(\tilde{B}_t\) is \(\mathcal{N}(0; [\sigma^2 t, \sigma^2 t])\)-distributed. We first prove that
\[
\mathbb{E}[\tilde{B}_1^2] = \sigma^2 t \quad \text{and} \quad -\mathbb{E}[-\tilde{B}_1^2] = \sigma^2 t.
\]
We set \(b(t) := \mathbb{E}[\tilde{B}_1^2]\). Then \(b(0) = 0\) and \(\lim_{t \downarrow 0} b(t) \leq \mathbb{E}[(\tilde{B}_t)^2] t^{-1} \rightarrow 0\). Since for each \(t, s \geq 0\),
\[
b(t + s) = \mathbb{E}[(\tilde{B}_{t+s})^2] = \mathbb{E}[(\tilde{B}_{t+s} - \tilde{B}_s + \tilde{B}_s)^2] = \mathbb{E}[(\tilde{B}_{t+s} - \tilde{B}_s)^2 + \tilde{B}_s^2]
= b(t) + b(s).
\]
Thus \(b(t)\) is linear and uniformly continuous in \(t\); hence there exists a constant \(\sigma \geq 0\) such that \(\mathbb{E}[\tilde{B}_1^2] = \sigma^2 t\). Similarly, there exists a constant \(\sigma \in [0, \sigma]\) such that \(-\mathbb{E}[-\tilde{B}_1^2] = \sigma^2 t\). We have \(\sigma^2 = \mathbb{E}[\tilde{B}_1^2] \geq -\mathbb{E}[-\tilde{B}_1^2] = \sigma^2\).

We now prove that \(\tilde{B}_t\) is \(\mathcal{N}(0; [\sigma^2 t, \sigma^2 t])\)-distributed. We just need to prove that, for each fixed \(\varphi \in C_{l.Lip}(\mathbb{R})\), the function
\[
u(t, x) := \mathbb{E}[\varphi(x + \tilde{B}_t)], \quad (t, x) \in [0, \infty) \times \mathbb{R}
\]
is the viscosity solution of the following \(G_{\sigma^2 t}\)-heat equation
\[
\partial_t u - G(\partial_{xx}^2 u) = 0, \quad \text{for } t > 0; \quad u|_{t=0} = \varphi.
\]
with \(G(a) = G_{\sigma^2 t}(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-)\). Thus for each \(\mathcal{N}(0; [\sigma^2 t, \sigma^2 t])\)-distributed \(\xi\), we have
\[
\mathbb{E}[\varphi(\tilde{B}_t)] = \mathbb{E}[\varphi(\sqrt{t} \xi)], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}).
\]
Thus \(\tilde{B}_t \sim \sqrt{t} \xi \sim \mathcal{N}(0; [\sigma^2 t, \sigma^2 t])\).

We first prove that \(u\) is locally Lipschitz in \(x\) and locally \(1/2\)-Hölder in \(t\). In fact, for each fixed \(t\), \(u(t, \cdot) \in C_{l.Lip}(\mathbb{R})\) since
\[
|\mathbb{E}[\varphi(x + \tilde{B}_t)] - \mathbb{E}[\varphi(y + \tilde{B}_t)]| \leq \mathbb{E}[|\varphi(x + \tilde{B}_t) - \varphi(y + \tilde{B}_t)|] \leq |\mathbb{E}[C(1 + |x|^m + |y|^m + |\tilde{B}_t|^m)|x - y|] \leq C_1(1 + |x|^m + |y|^m)|x - y|.
\]
For each $\delta \in [0, t]$, since $\tilde{B}_t - \tilde{B}_\delta$ is independent to $\tilde{B}_\delta$, we also have
\begin{align*}
u(t, x) &= E[\varphi(x + \tilde{B}_\delta + (\tilde{B}_t - \tilde{B}_\delta))] \\
&= E[E[\varphi(y + (\tilde{B}_t - \tilde{B}_\delta))]_{y=x+\tilde{B}_\delta}],
\end{align*}

hence
\begin{equation}
u(t, x) = E[u(t - \delta, x + \tilde{B}_\delta)]. \quad (4.2)
\end{equation}

Thus
\begin{align*}
|u(t, x) - u(t - \delta, x)| &= |E[u(t - \delta, x + \tilde{B}_\delta) - u(t - \delta, x)]| \\
&\leq E[|u(t - \delta, x + \tilde{B}_\delta) - u(t - \delta, x)|] \\
&\leq E[C(1 + |x|^m + |\tilde{B}_\delta|^m)|\tilde{B}_\delta|] \\
&\leq C_1(1 + |x|^m)\sigma^\frac{1}{2}.
\end{align*}

To prove that $u$ is a viscosity solution of (4.1), we fix a time-space $(t, x) \in (0, \infty) \times \mathbb{R}$ and let $v \in C^{2,3}_b([0, \infty) \times \mathbb{R})$ be such that $v \geq u$ and $v(t, x) = u(t, x)$. From (4.2) we have
\begin{equation}
v(t, x) = E[u(t - \delta, x + \tilde{B}_\delta)] \leq E[v(t - \delta, x + \tilde{B}_\delta)]
\end{equation}

Therefore by Taylor’s expansion,
\begin{align*}
0 &\leq E[v(t - \delta, x + \tilde{B}_\delta) - v(t, x)] \\
&= E[v(t - \delta, x + \tilde{B}_\delta) - v(t, x + \tilde{B}_\delta) + (v(t, x + \tilde{B}_\delta) - v(t, x))] \\
&= E[-\partial_t v(t, x)\delta + \partial_x v(t, x)\tilde{B}_\delta + \frac{1}{2}\partial^2_{xx} v(t, x)\tilde{B}_\delta^2 + I_\delta] \\
&\leq -\partial_t v(t, x)\delta + E[\frac{1}{2}\partial^2_{xx} v(t, x)\tilde{B}_\delta^2] - E[I_\delta] \\
&= -\partial_t v(t, x)\delta + G(\partial^2_{xx} (t, x))\delta - E[I_\delta].
\end{align*}

where
\begin{align*}
I_\delta &= \int_0^1 -[\partial_t v(t - \beta\delta, x + \tilde{B}_\delta) - \partial_t v(t, x)]\delta d\beta \\
&\quad + \int_0^1 \int_0^1 [\partial^2_{xx} v(t, x + \alpha\tilde{B}_\delta) - \partial^2_{xx} v(t, x)]\alpha d\beta d\alpha \tilde{B}_\delta^2
\end{align*}
With the assumption (iii) we can check that \( \lim_{\delta \to 0} \mathbb{E}[|I_\delta|]\delta^{-1} = 0 \); from which we get \( \partial_t v(t, x) - G(\partial^2_{xx}(t, x)) \leq 0 \); hence \( u \) is a viscosity supersolution of (4.1). We can analogously prove that \( u \) is a viscosity subsolution. But by the definition of \( G \)-normal distribution 

\[ \tilde{B}_t \sim \sqrt{\xi} \sim \mathcal{N}(0; [\sigma^2 t, \sigma^2 t]). \]

Thus \( (\tilde{B}_t)_{t \geq 0} \) is a \( G \)-Brownian motion.

**4.1.1 Existence of \( G \)-Brownian motion**

In the rest of this course, we denote by \( \Omega = C_0(\mathbb{R}^+) \) the space of all \( \mathbb{R} \)-valued continuous paths \( (\omega_t)_{t \in \mathbb{R}^+} \) with \( \omega_0 = 0 \), equipped with the distance

\[ \rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t \in [0,i]} |\omega^1_t - \omega^2_t|) \wedge 1]. \]

For each fixed \( T \geq 0 \), we consider the following space of random variables:

\[ (\mathcal{H}^0_T) = L^0_{ip}(\mathcal{F}_T) := \{ X(\omega) = \varphi(\omega_{t_1}, \cdots, \omega_{t_m}), \forall m \geq 1, \] \[ t_1, \cdots, t_m \in [0, T], \forall \varphi \in C_{l,Lip}(\mathbb{R}^m) \}. \]

It is clear that \( \mathcal{H}^0_T \subseteq L^0_{ip}(\mathcal{F}_T) \), for \( t \leq T \). We also denote

\[ (\mathcal{H}^0) = L^0_{ip}(\mathcal{F}) := \bigcup_{n=1}^{\infty} L^0_{ip}(\mathcal{F}_n). \]

**Remark 4.1.4** Obviously \( C_{l,Lip}(\mathbb{R}^m) \) and then \( L^0_{ip}(\mathcal{F}_T) \), \( L^0_{ip}(\mathcal{F}) \) are vector lattices. Moreover, since \( \varphi, \psi \in C_{l,Lip}(\mathbb{R}^m) \) implies \( \varphi \cdot \psi \in C_{l,Lip}(\mathbb{R}^m) \) thus \( X, Y \in L^0_{ip}(\mathcal{F}_T) \) implies \( X \cdot Y \in L^0_{ip}(\mathcal{F}_T) \).

We will consider the canonical space and set \( B_t(\omega) = \omega_t, t \in [0, \infty), \) for \( \omega \in \Omega \).

For each fixed \( T \in [0, \infty) \), we set

\[ L_{ip}(\mathcal{F}_T) := \{ \varphi(B_{t_1}, \cdots, B_{t_n}) : 0 \leq t_1, \cdots, t_n \leq T, \ \varphi \in C_{l,Lip}(\mathbb{R}^n), \ n \in \mathbb{N} \}. \]

In particular, for each \( t \in [0, \infty) \), \( B_t \in L_{ip}(\mathcal{F}_t) \). We are given a sublinear function \( G(a) = G_{\sigma,1}(a) = \frac{1}{2}(a^+ - \sigma^2 a^-), a \in \mathbb{R} \). Let \( \xi \) be a \( G \)-normal
4.1. 1-DIMENSIONAL G-BROWNIAN MOTION

distributed, or \(\mathcal{N}(0; [\sigma^2, 1])\)-distributed, random variable in a sublinear expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})\).

We now introduce a sublinear expectation \(\hat{\mathbb{E}}\) defined on \(\mathcal{H}_T^0 = L_{ip}^0(\mathcal{F}_T)\), as well as on \(\mathcal{H}^0 = L_{ip}^0(\mathcal{F})\), via the following procedure: For each \(X \in \mathcal{H}_T^0\) with

\[
X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})
\]

for some \(\varphi \in C_l, \text{Lip}(\mathbb{R}^m)\) and \(0 = t_0 < t_1 < \cdots < t_m < \infty\), we set

\[
\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] = \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],
\]

where \((\xi_1, \ldots, \xi_m)\) is an \(m\)-dimensional \(G\)-normal distributed random vector in a sublinear expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})\) such that \(\xi_i \sim \mathcal{N}(0; [\sigma^2, 1])\) and such that \(\xi_{i+1}\) is independent of \((\xi_1, \ldots, \xi_i)\) for each \(i = 1, \ldots, m - 1\).

The related conditional expectation of \(X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})\) under \(\mathcal{H}_{t_j}\) is defined by

\[
\hat{\mathbb{E}}[X|\mathcal{H}_{t_j}] = \hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_j}] = \psi(B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}})
\]

where

\[
\psi(x_1, \cdots, x_j) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)]
\]

It is easy to check that \(\hat{\mathbb{E}}[\cdot]\) consistently defines a sublinear expectation on as well as on \(L_{ip}^0(\mathcal{F})\) satisfying (a)–(d) of Definition 3.1.2.

**Definition 4.1.5** The expectation \(\hat{\mathbb{E}}[\cdot]: L_{ip}^0(\mathcal{F}) \mapsto \mathbb{R}\) defined through the above procedure is called \(G\)-expectation. The corresponding canonical process \((B_t)_{t \geq 0}\) in the sublinear expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})\) is called a \(G\)-Brownian motion.

**Proposition 4.1.6** We list the properties of \(\hat{\mathbb{E}}[\cdot|\mathcal{H}_t]\) that hold for each \(X, Y \in \mathcal{H}^0 = L_{ip}^0(\mathcal{F})\):

(a') If \(X \geq Y\), then \(\hat{\mathbb{E}}[X|\mathcal{H}_t] \geq \hat{\mathbb{E}}[Y|\mathcal{H}_t]\).

(b') \(\hat{\mathbb{E}}[\eta|\mathcal{H}_t] = \eta\), for each \(t \in [0, \infty)\) and \(\eta \in \mathcal{H}^0_t\).

(c') \(\hat{\mathbb{E}}[X|\mathcal{H}_t] - \hat{\mathbb{E}}[Y|\mathcal{H}_t] \leq \hat{\mathbb{E}}[X - Y|\mathcal{H}_t]\).
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(d') \( \hat{E}[\eta X|\mathcal{H}_t] = \eta^+ \hat{E}[X|\mathcal{H}_t] + \eta^- \hat{E}[-X|\mathcal{H}_t], \) for each \( \eta \in \mathcal{H}_t^0. \)

We also have

\[
\hat{E}[\hat{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \hat{E}[X|\mathcal{H}_{t\wedge s}], \quad \text{in particular} \quad \hat{E}[\hat{E}[X|\mathcal{H}_t]] = \hat{E}[X].
\]

For each \( X \in L^0_{ip}(\mathcal{F}_T), \) \( \hat{E}[X|\mathcal{H}_t] = \hat{E}[X], \) where \( L^0_{ip}(\mathcal{F}_T) = \mathcal{H}_T \) is the linear space of random variables of the form

\[
\varphi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_{m+1}} - B_{t_m}),
\]

\( m = 1, 2, \ldots, \varphi \in C_{t,Lip}(\mathbb{R}^m), t_1, \ldots, t_m, t_{m+1}, \in [t, \infty). \)

**Remark 4.1.7** (b') and (c') imply:

\[
\hat{E}[X + \eta|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \eta.
\]

Moreover, if \( Y \in L^0_{ip}(\mathcal{F}) \) satisfies \( \hat{E}[Y|\mathcal{H}_t] = -\hat{E}[-Y|\mathcal{H}_t] \) then

\[
\hat{E}[X + Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t].
\]

**Example 4.1.8** For each \( s < t, \) we have \( \hat{E}[B_t - B_s|\mathcal{H}_s] = 0 \) and, for \( n = 1, 2, \ldots, \)

\[
\hat{E}[|B_t - B_s|^n|\mathcal{H}_s] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} |x|^n \exp\left(-\frac{x^2}{2(t-s)}\right)dx.
\]

But we have

\[
\hat{E}[-|B_t - B_s|^n|\mathcal{H}_s] = \hat{E}[|-B_{t-s}|^n] = -\sigma^n \hat{E}[|B_{t-s}|^n].
\]

Exactly as in classical cases, we have

\[
\hat{E}[(B_t - B_s)^2|\mathcal{H}_s] = t - s, \quad \hat{E}[(B_t - B_s)^4|\mathcal{H}_s] = 3(t - s)^2,
\]

\[
\hat{E}[(B_t - B_s)^6|\mathcal{H}_s] = 15(t - s)^5, \quad \hat{E}[(B_t - B_s)^8|\mathcal{H}_s] = 105(t - s)^4,
\]

\[
\hat{E}[|B_t - B_s|^2|\mathcal{H}_s] = \frac{\sqrt{2(t-s)}}{\sqrt{\pi}}, \quad \hat{E}[|B_t - B_s|^3|\mathcal{H}_s] = \frac{2\sqrt{2}(t-s)^{3/2}}{\sqrt{\pi}},
\]

\[
\hat{E}[|B_t - B_s|^5|\mathcal{H}_s] = 8\frac{\sqrt{2}(t-s)^{5/2}}{\sqrt{\pi}}.
\]
Definition 4.1.9 A process \((M_t)_{t \geq 0}\) is called a \(G\)-martingale (respectively, \(G\)-supermartingale; \(G\)-submartingale) if for each \(t \in [0, \infty)\), \(M_t \in L^0_{\text{lip}}(\Omega)\) and for each \(s \in [0, t]\), we have
\[
\mathbb{E}[M_t|\mathcal{H}_s] = M_s, \quad \text{(respectively,} \quad \leq M_s; \quad \geq M_s).\]

Example 4.1.10 \((B_t)_{t \geq 0}\) and \((-B_t)_{t \geq 0}\) are \(G\)-Martingale. \((B^2_t)_{t \geq 0}\) is a \(G\)-submartingale since
\[
\hat{\mathbb{E}}[B^2_t|\mathcal{H}_s] = \hat{\mathbb{E}}[(B_t - B_s)^2 + B^2_s + 2B_s(B_t - B_s)|\mathcal{H}_s]
= \hat{\mathbb{E}}[(B_t - B_s)^2] + B^2_s = t - s + B^2_s \geq B^2_s.
\]
4.1.2 Complete spaces of sublinear expectation

We briefly recall the notion of nonlinear expectations introduced in [55]. Following Daniell (see Daniell 1918 [18]) in his famous Daniell’s integration, we begin with a vector lattice.

Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a vector lattice of real functions defined on \( \Omega \) containing 1, namely, \( \mathcal{H} \) is a linear space such that \( 1 \in \mathcal{H} \) and that \( X \in \mathcal{H} \) implies \( |X| \in \mathcal{H} \). \( \mathcal{H} \) is a space of random variables. We assume the functions on \( \mathcal{H} \) are all bounded. Notice that

\[
    a \wedge b = \min\{a, b\} = \frac{1}{2}(a + b - |a - b|), \quad a \vee b = -[(-a) \wedge (-b)].
\]

Thus \( X, Y \in \mathcal{H} \) implies that \( X \wedge Y, X \vee Y, X^+ = X \vee 0 \) and \( X^- = (-X)^+ \) are all in \( \mathcal{H} \).

In this course we are mainly concerned with space \( \mathcal{H}^0 = L^0_{lip}(\mathcal{F}) \). It satisfies

\[
    X_1, \ldots, X_n \in \mathcal{H} \implies \varphi(X_1, \ldots, X_n) \in \mathcal{H}, \quad \forall \varphi \in C_{lip}(\mathbb{R}^n).
\]

**Remark 4.1.11** For each fixed \( p \geq 1 \), we observe that \( \mathcal{H}^0_p = \{ X \in \mathcal{H}, \hat{E}[|X|^p] = 0 \} \) is a linear subspace of \( \mathcal{H} \). To take \( \mathcal{H}^0_p \) as our null space, we introduce the quotient space \( \mathcal{H}/\mathcal{H}^0_p \). Observe that, for every \( \{X\} \in \mathcal{H}/\mathcal{H}^0_p \) with a representation \( X \in \mathcal{H} \), we can define an expectation \( \hat{E}[\{X\}] := \hat{E}[X] \) which still satisfies (a)–(d) of Definition 3.1.2. We set \( \|\cdot\|_p \) forms a Banach norm in \( \mathcal{H}/\mathcal{H}^0_p \).

**Lemma 4.1.12** For \( r > 0 \) and \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
    |a + b|^r \leq \max\{1, 2^{r-1}\}(|a|^r + |b|^r), \quad \forall a, b \in \mathbb{R}; (4.4)
\]

\[
    |ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \quad (4.5)
\]

**Proposition 4.1.13** For each \( X, Y \in L^0_{lip}(\mathcal{F}) \), we have

\[
    \hat{E}[|X + Y|^r] \leq C_r(\hat{E}[|X|^r] + \hat{E}[|Y|^r]), \quad (4.6)
\]

\[
    \hat{E}[|XY|] \leq \hat{E}[|X|^p]^{1/p} \cdot \hat{E}[|Y|^q]^{1/q}, \quad (4.7)
\]

\[
    \hat{E}[|X + Y|^p]^{1/p} \leq \hat{E}[|X|^p]^{1/p} + \hat{E}[|Y|^p]^{1/p}. \quad (4.8)
\]

In particular, for \( 1 \leq p < p' \), we have \( \hat{E}[|X|^p]^{1/p} \leq \hat{E}[|X|^{p'}]^{1/p'} \).
Proof. (4.6) follows from (4.4). We set
\[ \xi = \frac{X}{\mathbb{E}[|X|^p]^{1/p}}, \quad \eta = \frac{Y}{\mathbb{E}[|Y|^q]^{1/q}}. \]
By (4.5) we have
\[ \hat{\mathbb{E}}[|\xi\eta|] \leq \hat{\mathbb{E}}[|\xi|^p/p + |\eta|^q/q] \leq \hat{\mathbb{E}}[|\xi|^p/p] + \hat{\mathbb{E}}[|\eta|^q/q] = \frac{1}{p} + \frac{1}{q} = 1. \]
Thus (4.7) follows. We now prove (4.8):
\[ \hat{\mathbb{E}}[|X + Y|^{p}] = \hat{\mathbb{E}}[|X + Y| \cdot |X + Y|^{p-1}] \]
\[ \leq \hat{\mathbb{E}}[|X| \cdot |X + Y|^{p-1}] + \hat{\mathbb{E}}[|Y| \cdot |X + Y|^{p-1}] \]
\[ \leq \hat{\mathbb{E}}[|X|^p]^{1/p} \cdot \hat{\mathbb{E}}[|X + Y|^{(p-1)q}]^{1/q} \]
\[ + \hat{\mathbb{E}}[|Y|^p]^{1/p} \cdot \hat{\mathbb{E}}[|X + Y|^{(p-1)q}]^{1/q}. \]
We observe that \((p-1)q = p\). Thus we have (4.8).
For each \(p, q > 0\) with \(\frac{1}{p} + \frac{1}{q} = 1\) we have
\[ \|XY\| = \hat{\mathbb{E}}[|XY|] \leq \|X\|_p \|X\|_q. \]
With this we have \(\|X\|_p \leq \|X\|_{p'}\) if \(p \leq p'\).

Remark 4.1.14 It is easy to check that \(\mathcal{H}/\mathcal{H}_0^p\) is a normed space under \(\|\cdot\|_p\).
We then extend \(\mathcal{H}/\mathcal{H}_0^p\) to its completion \(\hat{\mathcal{H}}_p\) under this norm. (\(\hat{\mathcal{H}}_p, \|\cdot\|_p\))
is a Banach space. The sublinear expectation \(\hat{\mathbb{E}}[\cdot]\) can be also continuously extended from \(\mathcal{H}/\mathcal{H}_0\) to \(\hat{\mathcal{H}}_p\), which satisfies (a)-(d).

For any \(X \in \mathcal{H}\), the mappings
\[ X^+(\omega) : \mathcal{H} \mapsto \mathcal{H} \quad \text{and} \quad X^- (\omega) : \mathcal{H} \mapsto \mathcal{H} \]
satisfy
\[ |X^+ - Y^+| \leq |X - Y| \quad \text{and} \quad |X^- - Y^-| = |(-X)^+ - (-Y)^+| \leq |X - Y|. \]
Thus they are both contraction mappings under \(\|\cdot\|_p\) and can be continuously extended to the Banach space (\(\hat{\mathcal{H}}_p, \|\cdot\|_p\)).
We define the partial order "\(\geq\)" in this Banach space.
Definition 4.1.15 An element X in \((\hat{\mathcal{H}}, \|\cdot\|)\) is said to be nonnegative, or \(X \geq 0\), if \(X = X^+\). We also denote by \(X \geq Y\), or \(Y \leq X\), if \(X - Y \geq 0\).

It is easy to check that \(X \geq Y\) and \(Y \geq X\) implies \(X = Y\) in \((\hat{\mathcal{H}}_p, \|\cdot\|_p)\).

The sublinear expectation \(\hat{\mathbb{E}}[\cdot]\) can be continuously extended to \((\hat{\mathcal{H}}_p, \|\cdot\|_p)\) on which (a)–(d) still hold.

4.1.3 \(G\)-Brownian motion in a complete sublinear expectation space

We can check that, for each \(p > 0\) and for each \(X \in L^{0}_{ip}(\mathcal{F})\) with the form \(X(\omega) = \varphi(B_{t_1}, \cdots, B_{t_n})\), for some \(\varphi \in C_{l,Lip}(\mathbb{R}^m)\),

\[
\hat{\mathbb{E}}[|X|] = 0 \iff \hat{\mathbb{E}}[|X|^p] = 0 \iff \varphi(x) \equiv 0, \forall x \in \mathbb{R}^m.
\]

For each \(p \geq 1\), \(\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}}\), \(X \in L^{0}_{ip}(\mathcal{F}_T)\) (respectively, \(L^{0}_{ip}(\mathcal{F})\)) forms a norm and that \(L^{0}_{ip}(\mathcal{F}_T)\) (respectively, \(L^{0}_{ip}(\mathcal{F})\)), can be continuously extended to a Banach space, denoted by

\[
\mathcal{H}_T = L^{p}_{G}(\mathcal{F}_T) \quad \text{(respectively, } \mathcal{H} = L^{p}_{G}(\mathcal{F}))\).
\]

For each \(0 \leq t \leq T < \infty\) we have \(L^{0}_{G}(\mathcal{F}_t) \subseteq L^{0}_{G}(\mathcal{F}_T) \subseteq L^{p}_{G}(\mathcal{F})\). It is easy to check that, in \(L^{0}_{G}(\mathcal{F}_T)\) (respectively, \(L^{0}_{G}(\mathcal{F})\)), \(\hat{\mathbb{E}}[\cdot]\) still satisfies (a)–(d) in Definition 3.1.2.

We now consider the conditional expectation introduced in (4.3). For each fixed \(t = t_j \leq T\), the conditional expectation \(\hat{\mathbb{E}}[\cdot | \mathcal{H}_t] : L^{0}_{ip}(\mathcal{F}_T) \mapsto \mathcal{H}_t^{0}\) is a continuous mapping under \(\|\cdot\|\) since \(\hat{\mathbb{E}}[\hat{\mathbb{E}}[X | \mathcal{H}_t]] = \hat{\mathbb{E}}[X], X \in L^{0}_{ip}(\mathcal{F}_T)\) and

\[
\hat{\mathbb{E}}[\hat{\mathbb{E}}[X | \mathcal{H}_t] - \hat{\mathbb{E}}[Y | \mathcal{H}_t]] \leq \hat{\mathbb{E}}[X - Y],
\]

\[
\left\|\hat{\mathbb{E}}[X | \mathcal{H}_t] - \hat{\mathbb{E}}[Y | \mathcal{H}_t]\right\| \leq \|X - Y\|.
\]

It follows that \(\hat{\mathbb{E}}[\cdot | \mathcal{H}_t]\) can be also extended as a continuous mapping \(L^{p}_{G}(\mathcal{F}_T) \mapsto L^{p}_{G}(\mathcal{F}_t)\). If the above \(T\) is not fixed, then we can obtain \(\hat{\mathbb{E}}[\cdot | \mathcal{H}_t] : L^{1}_{G}(\mathcal{F}) \mapsto L^{p}_{G}(\mathcal{F}_t)\).

Proposition 4.1.16 The properties of Proposition 4.1.6 of \(\hat{\mathbb{E}}[\cdot | \mathcal{H}_t]\) still hold for \(X, Y \in L^{0}_{G}(\mathcal{F})\):
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(a') If \( X \geq Y \), then \( \hat{E}[X|\mathcal{H}_t] \geq \hat{E}[Y|\mathcal{H}_t] \).

(b') \( \hat{E}[\eta|\mathcal{H}_t] = \eta \), for each \( t \in [0, \infty) \) and \( \eta \in L^1_G(\mathcal{F}_t) \).

(c') \( \hat{E}[X|\mathcal{H}_t] - \hat{E}[Y|\mathcal{H}_t] \leq \hat{E}[X-Y|\mathcal{H}_t] \).

(d') \( \hat{E}[\eta X|\mathcal{H}_t] = \eta^+ \hat{E}[X|\mathcal{H}_t] + \eta^- \hat{E}[-X|\mathcal{H}_t] \), for each bounded \( \eta \in L^1_G(\mathcal{F}_t) \).

(e') \( \hat{E}[\hat{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \hat{E}[X|\mathcal{H}_{t\wedge s}] \), in particular, \( \hat{E}[\hat{E}[X|\mathcal{H}_t]] = \hat{E}[X] \).

(f') For each \( X \in L^1_G(\mathcal{F}_t) \) we have \( \hat{E}[X|\mathcal{H}_t] = \hat{E}[X] \).

**Definition 4.1.17** An \( X \in L^1_G(\mathcal{F}) \) is said to be independent of \( \mathcal{F}_t \) under the \( G \)-expectation \( \hat{E} \) for some given \( t \in [0, \infty) \), if for each real function \( \Phi \) suitably defined on \( \mathbb{R} \) such that \( \Phi(X) \in L^1_G(\mathcal{F}) \) we have

\[
\hat{E}[\Phi(X)|\mathcal{H}_t] = \hat{E}[\Phi(X)] .
\]

**Remark 4.1.18** It is clear that all elements in \( L^1_G(\mathcal{F}) \) are independent of \( \mathcal{F}_0 \). Just like the classical situation, the increments of \( G \)-Brownian motion \( (B_{t+s} - B_s)_{t \geq 0} \) is independent of \( \mathcal{F}_s \). In fact it is a new \( G \)-Brownian motion since, just like the classical situation, the increments of \( B \) are identically distributed.

**Example 4.1.19** For each \( n \in \mathbb{N} \), \( 0 \leq t < \infty \) and \( X \in L^1_G(\mathcal{F}_t) \), since

\[
\hat{E}[B_{T-t}^{2n-1}] = \hat{E}[-B_{T-t}^{2n-1}] ,
\]

we have, by (f') of Proposition 4.1.16,

\[
\hat{E}[X(B_T - B_t)^{2n-1}] = \hat{E}[X^+ \hat{E}[(B_T - B_t)^{2n-1}|\mathcal{H}_t] + X^- \hat{E}[-(B_T - B_t)^{2n-1}|\mathcal{H}_t]]
\]

\[
= \hat{E}[|X| \cdot \hat{E}[B_{T-t}^{2n-1}]] ,
\]

\[
\hat{E}[X(B_T - B_t)|\mathcal{H}_t] = \hat{E}[-X(B_T - B_t)|\mathcal{H}_t] = 0 .
\]

We also have

\[
\hat{E}[X(B_T - B_t)^2|\mathcal{H}_t] = X^+(T-t) - \sigma^2 X^-(T-t) .
\]

**Remark 4.1.20** It is clear that we can define an expectation \( \mathbb{E}[] \) on \( L^0_{ip}(\mathcal{F}) \) in the same way as in Definition 4.1.1 with the standard normal distribution \( \mathcal{F} = \mathcal{N}(0, 1) \) in place of \( \mathcal{F}_\xi = \mathcal{N}(0; [\sigma, 1]) \) on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). Since \( \mathcal{F} \) is dominated by \( \mathcal{F}_\xi \) in the sense \( \mathbb{E}[\varphi] - \mathbb{E}[\psi] \leq \mathcal{F}_\xi[\varphi - \psi] \), then \( \mathbb{E}[] \) can be continuously
extended to $L^1_G(\mathcal{F})$. $E[\cdot]$ is a linear expectation under which $(B_t)_{t \geq 0}$ behaves as a Brownian motion. We have

$$E[X] \leq \hat{E}[X], \quad \forall X \in L^1_G(\mathcal{F}).$$

(4.9)

In particular, $\hat{E}[-B_{T-t}^{2n-1}] = \hat{E}[B_{T-t}^{2n-1}] \geq E[-B_{T-t}^{2n-1}] = 0$. Such kind of extension under a domination relation was discussed in details in [55].

The following property is very useful.

**Proposition 4.1.21** Let $X, Y \in L^1_G(\mathcal{F})$ be such that $\hat{E}[Y|\mathcal{H}_t] = -\hat{E}[-Y|\mathcal{H}_t]$, then we have

$$\hat{E}[X + Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t].$$

In particular, if $t = 0$ and $\hat{E}[Y|\mathcal{H}_0] = \hat{E}[Y] = \hat{E}[-Y] = 0$, then $\hat{E}[X + Y] = \hat{E}[X]$. 

**Proof.** We just need to use twice the sub-additivity of $\hat{E}[\cdot|\mathcal{H}_t]$: 

\[
\hat{E}[X + Y|\mathcal{H}_t] \geq \hat{E}[X|\mathcal{H}_t] - \hat{E}[-Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t] \geq \hat{E}[X + Y|\mathcal{H}_t].
\]

\[\blacksquare\]

**Example 4.1.22** We have

$$\hat{E}[B_t^2 - B_s^2|\mathcal{H}_s] = \hat{E}[(B_t - B_s + B_s)^2 - B_s^2|\mathcal{H}_s] = E[(B_t - B_s)^2 + 2(B_t - B_s)B_s|\mathcal{H}_s] = t - s,$$

since $2(B_t - B_s)B_s$ satisfies the condition for $Y$ in Proposition 4.1.21, and

\[
\hat{E}[(B_t^2 - B_s^2)^2|\mathcal{H}_s] = \hat{E}[(B_t - B_s + B_s)^2 - B_s^2]^2|\mathcal{H}_s] = E[(B_t - B_s)^4 + 4(B_t - B_s)^3B_s + 4(B_t - B_s)^2B_s^2|\mathcal{H}_s] \leq \hat{E}[(B_t - B_s)^4 + 4|B_t - B_s|B_s| + 4(t - s)B_s^2 = 3(t - s)^2 + 8(t - s)^{3/2}|B_s| + 4(t - s)B_s^2.
\]
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4.1.4 Itô’s integral of G–Brownian motion

Bochner’s integral

Definition 4.1.23 For \( T \in \mathbb{R}_+ \), a partition \( \pi_T \) of \([0, T]\) is a finite ordered subset \( \pi = \{t_1, \cdots, t_N\} \) such that \( 0 = t_0 < t_1 < \cdots < t_N = T \). We denote \( \mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, \cdots, N-1\} \).

We use \( \pi^N_T = \{t_0^N, t_1^N, \cdots, t_N^N\} \) to denote a sequence of partitions of \([0, T]\) such that \( \lim_{N \to \infty} \mu(\pi^N_T) = 0 \).

Let \( p \geq 1 \) be fixed. We consider the following type of simple processes: For a given partition \( \{t_0, \cdots, t_N\} = \pi_T \) of \([0, T]\), we set

\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t),
\]

where \( \xi_i \in L^p(G) \), \( i = 0, 1, 2, \cdots, N-1 \), are given. The collection of this form of processes is denoted by \( M^{p,0}_G(0, T) \).

Definition 4.1.24 For an \( \eta \in M^{p,0}_G(0, T) \) with \( \eta_t = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t) \) the related Bochner integral is

\[
\int_0^T \eta_t(\omega)dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).
\]

Remark 4.1.25 For each \( \eta \in M^{p,0}_G(0, T) \) we set

\[
\hat{E}_T[\eta] := \frac{1}{T} \int_0^T \hat{E}[\eta_t]dt = \frac{1}{T} \sum_{j=0}^{N-1} \hat{E}[^{\xi_j}(\omega))(t_{j+1} - t_j).
\]

It is easy to check that \( \hat{E}_T : M^{p,0}_G(0, T) \mapsto \mathbb{R} \) forms a sublinear expectation satisfying (a)–(d) of Definition 3.1.2. From Remark 4.1.11 we can introduce a natural norm \( \|\eta\|_{M^{p,0}_G(0, T)} = \left\{ \int_0^T \hat{E}[|\eta_t|^p]dt \right\}^{1/p} \). Under this norm \( M^{p,0}_G(0, T) \) can be continuously extended to a Banach space.
**Definition 4.1.26** For each $p \geq 1$, we will denote by $M^p_G(0,T)$ the completion of $M^{p,0}_G(0,T)$ under the norm

$$
\|\eta\|_{M^p_G(0,T)} = \left\{ \int_0^T \mathbb{E}[|\eta_t|^p]dt \right\}^{1/p}.
$$

For $\eta \in M^{p,0}_G(0,T)$, we have

$$
\mathbb{E}[\frac{1}{T} \int_0^T \eta_t(\omega)dt]^p \leq \frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|^p]dt \leq \frac{1}{T} \int_0^T \sum_{j=0}^{N-1} \mathbb{E}[|\xi_j(\omega)|^p](t_{j+1} - t_j) dt = \frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|^p]dt.
$$

We then have

**Proposition 4.1.27** The linear mapping $\int_0^T \eta_t(\omega)dt : M^{p,0}_G(0,T) \mapsto L^p_G(F_T)$ is continuous; thus it can be continuously extended to $M^p_G(0,T) \mapsto L^p_G(F_T)$. We still denote this extended mapping by $\int_0^T \eta_t(\omega)dt$, $\eta \in M^p_G(0,T)$. We have

$$
\mathbb{E}[\frac{1}{T} \int_0^T \eta_t(\omega)dt]^p \leq \frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|^p]dt, \quad \forall \eta \in M^p_G(0,T). \tag{4.10}
$$

We have $M^p_G(0,T) \supset M^q_G(0,T)$, for $p \leq q$.

**Itô’s integral of G–Brownian motion**

**Definition 4.1.28** For each $\eta \in M^{2,0}_G(0,T)$ with the form

$$
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j,t_{j+1})}(t),
$$

we define

$$
I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).
$$
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Lemma 4.1.29 The mapping $I : M_{G}^{2,0}(0,T) \rightarrow L_{G}^{2}(\mathcal{F}_{T})$ is a linear continuous mapping and thus can be continuously extended to $I : M_{G}^{2}(0,T) \rightarrow L_{G}^{2}(\mathcal{F}_{T})$: We have

\[
\hat{\mathbb{E}}\left[ \int_{0}^{T} \eta(s)dB_{s} \right] = 0, \quad (4.11)
\]

\[
\hat{\mathbb{E}}\left[ \left( \int_{0}^{T} \eta(s)dB_{s} \right)^{2} \right] \leq \int_{0}^{T} \hat{\mathbb{E}}[\eta(t)^{2}]dt. \quad (4.12)
\]

Definition 4.1.30 We define, for a fixed $\eta \in M_{G}^{2}(0,T)$, the stochastic integral

\[
\int_{0}^{T} \eta(s)dB_{s} := I(\eta).
\]

It is clear that (4.11) and (4.12) still hold for $\eta \in M_{G}^{2}(0,T)$.

Proof of Lemma 4.1.29. From Example 4.1.19 for each $j$,

\[
\hat{\mathbb{E}}[\xi_{j}(B_{t_{j+1}} - B_{t_{j}})|\mathcal{H}_{t_{j}}] = \hat{\mathbb{E}}[-\xi_{j}(B_{t_{j+1}} - B_{t_{j}})|\mathcal{H}_{t_{j}}] = 0.
\]

We have

\[
\hat{\mathbb{E}}\left[ \int_{0}^{T} \eta(s)dB_{s} \right] = \hat{\mathbb{E}}\left[ \int_{0}^{t_{N-1}} \eta(s)dB_{s} + \xi_{N-1}(B_{t_{N}} - B_{t_{N-1}}) \right]
\]

\[
= \hat{\mathbb{E}}\left[ \int_{0}^{t_{N-1}} \eta(s)dB_{s} + \hat{\mathbb{E}}[\xi_{N-1}(B_{t_{N}} - B_{t_{N-1}})|\mathcal{H}_{t_{N-1}}] \right]
\]

\[
= \hat{\mathbb{E}}\left[ \int_{0}^{t_{N-1}} \eta(s)dB_{s} \right].
\]

We then can repeat this procedure to obtain (4.11). We now prove (4.12):

\[
\hat{\mathbb{E}}\left[ \left( \int_{0}^{T} \eta(s)dB_{s} \right)^{2} \right] = \hat{\mathbb{E}}\left[ \left( \int_{0}^{t_{N-1}} \eta(s)dB_{s} + \xi_{N-1}(B_{t_{N}} - B_{t_{N-1}}) \right)^{2} \right]
\]

\[
= \hat{\mathbb{E}}\left[ \left( \int_{0}^{t_{N-1}} \eta(s)dB_{s} \right)^{2} \right] + \hat{\mathbb{E}}\left[ 2 \left( \int_{0}^{t_{N-1}} \eta(s)dB_{s} \right) \xi_{N-1}(B_{t_{N}} - B_{t_{N-1}}) + \xi_{N-1}(B_{t_{N}} - B_{t_{N-1}})^{2} \right] \hat{\mathcal{H}}_{t_{N-1}}
\]

\[
= \hat{\mathbb{E}}\left[ \left( \int_{0}^{t_{N-1}} \eta(s)dB_{s} \right)^{2} \right] + \xi_{N-1}^{2}(t_{N} - t_{N-1})].
\]
Thus \( \hat{E}[(\int_0^t \eta(s) dB_s)^2] \leq \hat{E}[(\int_0^{t_{N-1}} \eta(s) dB_s)^2] + \hat{E}[\xi_{N-1}^2] \). We then repeat this procedure to deduce
\[
\hat{E}[(\int_0^T \eta(s) dB_s)^2] \leq \sum_{j=0}^{N-1} \hat{E}[(\xi_j)^2](t_{j+1} - t_j) = \int_0^T \hat{E}[(\eta(t))^2] dt.
\]

We list some main properties of the Itô’s integral of \( G \)-Brownian motion. We denote for some \( 0 \leq s \leq t \leq T \),
\[
\int_s^t \eta(u) dB_u := \int_0^T I_{[s,t]}(u) \eta_u dB_u.
\]
We have

**Proposition 4.1.31** Let \( \eta, \theta \in M^2_G(0,T) \) and let \( 0 \leq s \leq r \leq t \leq T \). Then in \( L^1_G(F_T) \) we have
(i) \( \int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u \),
(ii) \( \int_s^t (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u \), if \( \alpha \) is bounded and in \( L^1_G(F_s) \),
(iii) \( \hat{E}[X + \int_r^T \eta_u dB_u | H_s] = \hat{E}[X], \forall X \in L^1_G(F) \).

**Quadratic variation process of \( G \)-Brownian motion**

We now study a very interesting process of the \( G \)-Brownian motion. Let \( \pi^N_t \), \( N = 1, 2, \cdots \), be a sequence of partitions of \([0, t]\). We consider
\[
B_t^2 = \sum_{j=0}^{N-1} [B_{t_{j+1}}^2 - B_{t_j}^2] = \sum_{j=0}^{N-1} 2B_{t_j}(B_{t_{j+1}} - B_{t_j}) + \sum_{j=0}^{N-1} (B_{t_{j+1}} - B_{t_j})^2.
\]
As \( \mu(\pi^N_t) \to 0 \) the first term of the right side tends to \( 2 \int_0^t B_s dB_s \). The second term must converge. We denote its limit by \( \langle B \rangle_t \), i.e.,
\[
\langle B \rangle_t = \lim_{\mu(\pi^N_t) \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}} - B_{t_j})^2 = B_t^2 - 2 \int_0^t B_s dB_s. \quad (4.13)
\]
By the above construction \( \{\langle B \rangle_t \}_{t \geq 0} \), is an increasing process with \( \langle B \rangle_0 = 0 \). We call it the \textit{quadratic variation process} of the \( G \)-Brownian motion \( B \). Clearly \( \langle B \rangle \) is an increasing process. It characterizes the part of statistic uncertainty of \( G \)-Brownian motion. It is important to keep in mind that \( \langle B \rangle \) is not a deterministic process unless the case \( \sigma = 1 \), i.e., when \( B \) is a classical Brownian motion. In fact we have

**Lemma 4.1.32** We have, for each \( 0 \leq s \leq t < \infty \)

\[
\begin{align*}
\hat{E}[\langle B \rangle_t - \langle B \rangle_s | \mathcal{H}_s] &= t - s, \\
\hat{E}[-(\langle B \rangle_t - \langle B \rangle_s)| \mathcal{H}_s] &= -\sigma^2(t - s).
\end{align*}
\]

(4.14) \hspace{1cm} (4.15)

**Proof.** By the definition of \( \langle B \rangle \) and Proposition 4.1.31-(iii),

\[
\hat{E}[\langle B \rangle_t - \langle B \rangle_s | \mathcal{H}_s] = \hat{E}[B_t^2 - B_s^2 - 2 \int_s^t B_u dB_u | \mathcal{H}_s]
\]

\[
= \hat{E}[B_t^2 - B_s^2 | \mathcal{H}_s] = t - s.
\]

The last step can be checked as in Example 4.1.22. We then have (4.14). (4.15) can be proved analogously with the consideration of \( \hat{E}[-(B_t^2 - B_s^2) | \mathcal{H}_s] = -\sigma^2(t - s) \). \( \blacksquare \)

To define the integration of a process \( \eta \in M^1_G(0, T) \) with respect to \( d \langle B \rangle \), we first define a mapping:

\[
Q_{0,T}(\eta) = \int_0^T \eta(s) d \langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j((\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M^1_G(0, T) \mapsto L^1(\mathcal{F}_T).
\]

**Lemma 4.1.33** For each \( \eta \in M^1_G(0, T) \),

\[
\hat{E}[||Q_{0,T}(\eta)||] \leq \int_0^T \hat{E}[||\eta_s||] ds.
\]

(4.16)

Thus \( Q_{0,T} : M^1_G(0, T) \mapsto L^1(\mathcal{F}_T) \) is a continuous linear mapping. Consequently, \( Q_{0,T} \) can be uniquely extended to \( L^1_T(0, T) \). We still denote this mapping by

\[
\int_0^T \eta(s) d \langle B \rangle_s = Q_{0,T}(\eta), \quad \eta \in M^1_G(0, T).
\]
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We still have

\[
\hat{\mathbb{E}}\left[ \int_0^T \eta(s)d\langle B \rangle_s \right] \leq \int_0^T \hat{\mathbb{E}}[|\eta_s|]ds, \quad \forall \eta \in M_G^1(0,T).
\]

(4.17)

Proof. From Lemma 4.1.32 (4.16) can be checked as follows:

\[
\hat{\mathbb{E}}\left[ \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) \right] \leq \sum_{j=0}^{N-1} \hat{\mathbb{E}}[|\xi_j| \cdot \hat{\mathbb{E}}[\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j} | H_{t_j}]]
\]

\[
= \sum_{j=0}^{N-1} \hat{\mathbb{E}}[|\xi_j|(t_{j+1} - t_j)]
\]

\[
= \int_0^T \hat{\mathbb{E}}[|\eta_s|]ds.
\]

A very interesting point of the quadratic variation process \( \langle B \rangle \) is, just like the \( G \)-Brownian motion \( B \) itself, the increment \( \langle B \rangle_{t+s} - \langle B \rangle_s \) is independent of \( F_s \) and identically distributed like \( \langle B \rangle_t \). In fact we have

Lemma 4.1.34 For each fixed \( s \geq 0 \), \( \langle B \rangle_{t+s} - \langle B \rangle_s \) is independent of \( F_s \). It is the quadratic variation process of the Brownian motion \( B_t^s = B_{s+t} - B_s, t \geq 0 \), i.e., \( \langle B \rangle_{s+t} - \langle B \rangle_s = \langle B^s \rangle_t \).

Proof. The independence follows directly from

\[
\langle B \rangle_{s+t} - \langle B \rangle_s = B_{t+s}^2 - 2 \int_0^{s+t} B_t dB_r - [B_s^2 - 2 \int_0^s B_r dB_r]
\]

\[
= (B_{t+s} - B_s)^2 - 2 \int_s^{s+t} (B_r - B_s)d(B_r - B_s)
\]

\[
= \langle B^s \rangle_t.
\]

Proposition 4.1.35 Let \( 0 \leq s \leq t \), \( \xi \in L_G^1(F_s) \). Then

\[
\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] = \hat{\mathbb{E}}[X + \xi(B_t - B_s)^2]
\]

\[
= \hat{\mathbb{E}}[X + \xi(\langle B \rangle_t - \langle B \rangle_s)].
\]
Proof. By (4.13) and Proposition 4.1.21 we have

\[ \hat{E}[X + \xi ((B_t^2 - B_s^2))] = \hat{E}[X + \xi ((B_t - B_s) + 2 \int_s^t B_u dB_u)] \]

\[ = \hat{E}[X + \xi ((B_t - B_s))]. \]

We also have

\[ \hat{E}[X + \xi ((B_t^2 - B_s^2))] = \hat{E}[X + \xi \{(B_t - B_s)^2 + 2(B_t - B_s)B_s\}] \]

\[ = \hat{E}[X + \xi (B_t - B_s)^2]. \]

We have the following isometry

**Proposition 4.1.36** Let \( \eta \in M^2_G(0, T) \). We have

\[ \hat{E}[\left( \int_0^T \eta(s)dB_s \right)^2] = \hat{E}\left[ \int_0^T \eta^2(s)d\langle B\rangle_s \right]. \tag{4.18} \]

Proof. We first consider \( \eta \in M^2_G(0, T) \) with the form

\[ \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t) \]

and thus \( \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}). \) From Proposition 4.1.21 we have

\[ \hat{E}[X + 2\xi_j(B_{t_{j+1}} - B_{t_j})\xi_i(B_{t_{i+1}} - B_{t_i})] = \hat{E}[X], \quad \text{for } X \in L^1_G(\mathcal{F}), \ i \neq j. \]

Thus

\[ \hat{E}[\left( \int_0^T \eta(s)dB_s \right)^2] = \hat{E}\left[ \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}) \right]^2 \hat{E}\left[ \sum_{j=0}^{N-1} \xi_j^2(B_{t_{j+1}} - B_{t_j})^2 \right]. \]

From this and Proposition 4.1.35 it follows that

\[ \hat{E}[\left( \int_0^T \eta(s)dB_s \right)^2] = \hat{E}\left[ \sum_{j=0}^{N-1} \xi_j^2(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) \right] = \hat{E}\left[ \int_0^T \eta^2(s)d\langle B \rangle_s \right]. \]

Thus (4.18) holds for \( \eta \in M^2_G(0, T) \). We can continuously extend the above equality to the case \( \eta \in M^2_G(0, T) \) and prove (4.18). \( \blacksquare \)
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The distribution of \( \langle B \rangle_t \)

The quadratic variation process \( \langle B \rangle \) of G-Brownian motion \( B \) is a very interesting process. We have seen that the G-Brownian motion \( B \) is a typical process with variance uncertainty but without mean-uncertainty. This uncertainty is concentrated in \( \langle B \rangle \). Moreover, \( \langle B \rangle \) itself is a typical process with mean-variance. This fact will be applied to measure the mean-uncertainty of risky positions.

Lemma 4.1.37 We have
\[
\mathbb{E}[(B^s)_t^2 | \mathcal{H}_s] = \mathbb{E}[(B)_t^2] \leq 10t^2. \tag{4.19}
\]

Proof.
\[
\mathbb{E}[(B)_t^2] = \mathbb{E}[(B^t)^2 - 2 \int_0^t B_u dB_u]^2]
\leq 2\mathbb{E}[(B^t)^4] + 8\mathbb{E}[(\int_0^t B_u dB_u)^2]
\leq 6t^2 + 8 \int_0^t \mathbb{E}[(B_u)^2] du
= 10t^2.
\]

Definition 4.1.38 A random variable \( \xi \) of a sublinear expectation space \( (\Omega, \tilde{\mathcal{H}}, \tilde{\mathbb{E}}) \) is said to be \( \mathcal{U}_{[\mu, \overline{\mu}]} \)-distributed for some given interval \( [\mu, \overline{\mu}] \subset \mathbb{R} \) if for each \( \varphi \in C_{1,Lip}(\mathbb{R}) \) we have
\[
\tilde{\mathbb{E}}[\varphi(\xi)] = \sup_{x \in [\mu, \overline{\mu}]} \varphi(x).
\]

Theorem 4.1.39 Let \((b_t)_{t \geq 0}\) be a process defined in \((\Omega, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\) such that
(i) \( b_0 = 0 \).
(ii) For each \( t, s \geq 0 \), the difference \( b_{t+s} - b_t \) and \( b_s \) are identically distributed and independent to \((b_{t_1}, b_{t_2}, \ldots, b_{t_n})\) for each \( n \in \mathbb{N} \) and \( 0 \leq t_1, \ldots, t_n \leq t \).
(iii) \( b_0 = 0 \) and \( \lim_{t \downarrow 0} \frac{\tilde{\mathbb{E}}[b_t^2]}{t} = 0 \).
Then \( b_t \) is \( \mathcal{U}_{[\mu, \overline{\mu}]} \)-distributed with \( \overline{\mu} = \tilde{\mathbb{E}}[b_1] \) and \( \mu = -\tilde{\mathbb{E}}[-b_1] \).
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Proof. We need only to prove that $b_t$ is $\mathcal{U}_{[\mu t, \mu]}$-distributed. We first prove that

$$\mathbb{E}[b_t] = \mu t \text{ and } -\mathbb{E}[-b_t] = \mu t.$$  

We set $\varphi(t) := \mathbb{E}[b_t]$. Then $\varphi(0) = 0$ and $\lim_{t \to 0} \varphi(t) = 0$. Since for each $t, s \geq 0$

$$\varphi(t + s) = \mathbb{E}[b_{t+s}] = \mathbb{E}[(b_{t+s} - b_s) + b_s] = \varphi(t) + \varphi(s).$$

Thus $\varphi(t)$ is linear and uniformly continuous in $t$ which means $\mathbb{E}[b_t] = \mu t$. Similarly $-\mathbb{E}[-b_t] = \mu t$.

We now prove that $b_t$ is $\mathcal{U}_{[\mu t, \mu]}$-distributed. We just need to prove that for each fixed $\varphi \in C_{\text{L.Lip}}(\mathbb{R})$, the function

$$u(t, x) := \mathbb{E}[\varphi(x + b_t)], \ (t, x) \in [0, \infty) \times \mathbb{R}$$

is the viscosity solution of the following $G_{\mu}$-drift equation

$$\partial_t u - 2G(\partial_x u) = 0, \text{ for } t > 0, \ u|_{t=0} = \varphi \quad (4.20)$$

with $G(a) = G_{\mu}(a) = \frac{1}{2}(\mu a^+ - \mu a^-)$. Since the function

$$\tilde{u}(t, x) = \max_{v \in [\mu, \mu]} \varphi(x + vt)$$

is the unique solution of the PDE $(4.20)$.

We first prove that $u$ is locally Lipschitz in $(t, x)$. In fact, for each fixed $t$, $u(t, \cdot) \in C_{\text{L.Lip}}(\mathbb{R})$ since

$$|\mathbb{E}[\varphi(x + b_t)] - \mathbb{E}[\varphi(y + b_t)]| \leq |\mathbb{E}||\varphi(x + b_t) - \varphi(y + b_t)||$$

$$\leq |\mathbb{E}[C(1 + |x|^m + |y|^m + m|b_t|^m)\|x - y||]$$

$$\leq C_1(1 + |x|^m + |y|^m)|x - y|.$$  

For each $\delta \in [0, t]$, since $b_t - b_\delta$ is independent to $b_\delta$, we also have

$$u(t, x) = \mathbb{E}[\varphi(x + b_\delta + (b_t - b_\delta))]$$

$$= \mathbb{E}[\mathbb{E}[\varphi(y + (b_t - b_\delta))]_{y=x+b_\delta}]$$
hence
\[ u(t, x) = \tilde{E}[u(t - \delta, x + b\delta)]. \] (4.21)

Thus
\[
|u(t, x) - u(t - \delta, x)| = |\tilde{E}[u(t - \delta, x + b\delta) - u(t - \delta, x)]|
\leq \tilde{E}|u(t - \delta, x + b\delta) - u(t - \delta, x)|
\leq \tilde{E}[C(1 + |x|^m + |b\delta|^m)|b\delta|]
\leq C_1(1 + |x|^m)\delta.
\]

The space point \((t, x) \in (0, \infty) \times \mathbb{R}\) and let \(v \in C^{2,2}_b([0, \infty) \times \mathbb{R})\) be such that \(v \geq u\) and \(v(t, x) = u(t, x)\). From (1.21) we have
\[
v(t, x) = \tilde{E}[u(t - \delta, x + b\delta)] \leq \tilde{E}[v(t - \delta, x + b\delta)]
\]
Therefore from Taylor’s expansion
\[
0 \leq \tilde{E}[v(t - \delta, x + b\delta) - v(t, x)]
= \tilde{E}[v(t - \delta, x + b\delta) - v(t, x + b\delta) + (v(t, x + b\delta) - v(t, x))]
= \tilde{E}[-\partial_t v(t, x)\delta + \partial_x v(t, x)b\delta + I_\delta]
\leq -\partial_t v(t, x)\delta + \tilde{E}[\partial_x v(t, x)b\delta] + \tilde{E}[I_\delta]
= -\partial_t v(t, x)\delta + 2G(\partial_x (t, x))\delta + \tilde{E}[I_\delta].
\]

where
\[
I_\delta = \int_0^1 [-\partial_t v(t - \beta\delta, x + \beta b\delta) + \partial_t v(t, x)]d\beta\delta
+ \int_0^1 [\partial_x v(t - \beta\delta, x + \beta b\delta) - \partial_x v(t, x)]d\beta b\delta.
\]

With the assumption that \(\lim_{t\downarrow 0} \tilde{E}[b_\delta^2]t^{-1} = 0\) we can check that
\[
\lim_{\delta \downarrow 0} \tilde{E}[||I_\delta||]\delta^{-1} = 0;
\]
from which we get \(\partial_t v(t, x) - 2G(\partial_x (t, x)) \leq 0\); hence \(u\) is a viscosity supersolution of (4.20). We can analogously prove that \(u\) is a viscosity subsolution. It follows that \(b_1\) is \(\mathcal{U}_{[\mu, \overline{\mu}]}\)-distributed. The proof is complete. ■
Corollary 4.1.40 For each $t \leq T < \infty$, we have
\[\mu(T - t) \leq b_T - b_t \leq \overline{\mu}(T - t), \quad \text{in } L^1_G(F).\]

**Proof.** It is a direct consequence of
\[\mathbb{E}[(b_T - b_t - (T - t))^+] = \sup_{\mu \leq \eta \leq \overline{\mu}} \mathbb{E}[(\eta - \overline{\mu}^+)(T - t)] = 0\]
and
\[\mathbb{E}[(b_T - b_t - \sigma^2(T - t))^+] = \sup_{\mu \leq \eta \leq \overline{\mu}} \mathbb{E}[(\eta - \mu^T)(T - t)] = 0.\]

**Corollary 4.1.41** We have
\[\mathbb{E}[(B_s^2)_{t} | \mathcal{H}_s] = \mathbb{E}[(B_t^2)] = t^2\]
as well as
\[\mathbb{E}[(B_s^3)_{t} | \mathcal{H}_s] = \mathbb{E}[(B_t^3)] = t^3, \quad \mathbb{E}[(B_s^4)_{t} | \mathcal{H}_s] = \mathbb{E}[(B_t^4)] = t^4.\]

**Theorem 4.1.42** For each $x \in \mathbb{R}$, $Z \in M^2_G(0, T)$ and $\eta \in M^1_G(0, T)$ the process
\[M_t = x + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \in [0, T]\]
is a martingale:
\[\mathbb{E}[M_t | \mathcal{H}_s] = M_s, \quad 0 \leq s \leq t \leq T.\]

**Proof.** Since \(\mathbb{E}[\int_s^t Z_r dB_r | \mathcal{H}_s] = \mathbb{E}[\int_s^t Z_r dB_r] | \mathcal{H}_s] = 0\) we only need to prove that
\[\tilde{M}_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds\]
is a $G$-martingale. It suffices to consider the case where $\eta$ is a simple process, i.e., $\eta_t = \sum_{k=0}^{N-1} \xi_k 1_{[t_k, t_{k+1})}(t)$. In fact we only need to consider one-step case in which we have
\[
\mathbb{E}[\tilde{M}_{t_{k+1}} - \tilde{M}_{t_k} | \mathcal{H}_{t_k}] = \mathbb{E}[\xi_k (\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}) - 2G(\xi_k)(t_{k+1} - t_k) | \mathcal{H}_{t_k}] = \\
\mathbb{E}[\xi_k (\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}) | \mathcal{H}_{t_k}] - 2G(\xi_k)(t_{k+1} - t_k) = \\
\sum_{k=0}^{N-1} \xi_k^+ \mathbb{E}[\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}] + \xi_k^- \mathbb{E}[\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}] - 2G(\xi_k)(t_{k+1} - t_k) = 0.
\]

\[\blacksquare\]
Problem 4.1.43 For each \( \xi = \varphi(B_t - B_0, \cdots, B_{t_N} - B_{t_{N-1}}) \) we have the following representation:

\[
\xi = \mathbb{E}[\xi] + \int_0^T z_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s)ds.
\]

4.2 Itô’s formula for \( G \)-Brownian motion

We have the corresponding Itô’s formula of \( \Phi(X_t) \) for a \( G \)-Itô process \( X \). For simplification we only treat the case where the function \( \Phi \) is sufficiently regular. We first consider a simple situation.

Lemma 4.2.1 Let \( \Phi \in C^2(\mathbb{R}^n) \) be bounded with bounded derivatives and \( \{\partial_{x^\mu x^\nu}^2 \Phi\}_{\mu, \nu=1}^n \) are uniformly Lipschitz. Let \( s \in [0, T] \) be fixed and let \( X = (X^1, \cdots, X^n)^T \) be an \( n \)-dimensional process on \([s, T]\) of the form

\[
X^\nu_t = X^\nu_s + \alpha^\nu(t - s) + \eta^\nu((B)_t - \langle B \rangle_s) + \beta^\nu(B_t - B_s),
\]

where, for \( \nu = 1, \cdots, n, \alpha^\nu, \eta^\nu \) and \( \beta^\nu \), are bounded elements of \( L^2_G(F_s) \) and \( X_s = (X^1_s, \cdots, X^n_s)^T \) is a given \( \mathbb{R}^n \)-vector in \( L^2_G(F_s) \). Then we have

\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu} \Phi(X_u) \beta^\nu dB_u + \int_s^t \partial_{x^\nu} \Phi(X_u) \alpha^\nu du + \int_s^t \left[D_{x^\nu} \Phi(X_u) \eta^\nu + \frac{1}{2} \partial_{x^\mu x^\nu} \Phi(X_u) \beta^\mu \beta^\nu \right] d\langle B \rangle_u.
\]

Here we use the Einstein convention, i.e., each single term with repeated indices \( \mu \) and/or \( \nu \) implies the summation.

Proof. For each positive integer \( N \) we set \( \delta = (t - s)/N \) and take the partition

\[
\pi_N^{[s, t]} = \{t^N_0, t^N_1, \cdots, t^N_N\} = \{s, s + \delta, \cdots, s + N\delta = t\}.
\]

We have

\[
\Phi(X_t) = \Phi(X_s) + \sum_{k=0}^{N-1} \Phi(X^N_{t_{k+1}}) - \Phi(X^N_{t_k})] \\
= \Phi(X_s) + \sum_{k=0}^{N-1} \partial_{x^\mu} \Phi(X^N_{t_k}) (X^\mu_{t_{k+1}} - X^\mu_{t_k}) \\
+ \frac{1}{2} \partial_{x^\mu x^\nu} \Phi(X^N_{t_k}) (X^\mu_{t_{k+1}} - X^\mu_{t_k}) (X^\nu_{t_{k+1}} - X^\nu_{t_k}) + \eta^\nu_{N_k})]
\]

(4.24)
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where

\[ \eta^N_t = [\partial^2_{x^\mu,x^\nu} \Phi(X^N_t) + \theta_t(X^N_t - X_t^N)] - \partial^2_{x^\mu,x^\nu} \Phi(X_t^N)(X_{t_{k+1}}^N - X_{t_k}^N)(X_{t_{k+1}}^N - X_{t_k}^N) \]

with \( \theta_t \in [0, 1] \). We have

\[ \hat{E}[|\eta_t^N|] = \hat{E}[|\partial^2_{x^\mu,x^\nu} \Phi(X_t^N) + \theta_t(X_t^N - X_t^N)] - \partial^2_{x^\mu,x^\nu} \Phi(X_t^N)(X_{t_{k+1}}^N - X_{t_k}^N) \times \]

\[ (X_{t_{k+1}}^N - X_{t_k}^N) | \]

\[ \leq c \hat{E}[|X_{t_{k+1}}^N - X_{t_k}^N|^3] \leq C[\delta^3 + \delta^{3/2}], \]

where \( c \) is the Lipschitz constant of \( \{\partial^2_{x^\mu,x^\nu} \Phi\}_{\mu,\nu=1}^n \). Thus \( \sum_k \hat{E}[|\eta_k^N|] \to 0 \).

The rest terms in the summation of the right side of (4.24) are \( \xi_t^N + \zeta_t^N \) with

\[ \xi_t^N = \sum_{k=0}^{N-1} \{ \partial_{x^\mu} \Phi(X^N_t)[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^\mu(\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N}) + \beta^\mu(B_{t_{k+1}^N} - B_{t_k^N})] \]

\[ + \frac{1}{2} \partial^2_{x^\mu,x^\nu} \Phi(X^N_t)^\beta \beta^\nu(B_{t_{k+1}^N} - B_{t_k^N})(B_{t_{k+1}^N} - B_{t_k^N}) \}

and

\[ \zeta_t^N = \frac{1}{2} \sum_{k=0}^{N-1} \partial^2_{x^\mu,x^\nu} \Phi(X^N_t)[\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^\nu(\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N})] \times [\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^\nu(\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N})] \]

\[ + \beta^\nu(\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N}) + \eta^\nu(\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N})(B_{t_{k+1}^N} - B_{t_k^N}). \]

We observe that, for each \( u \in [t_k^N, t_{k+1}^N) \)

\[ \hat{E}[|\partial_{x^\mu} \Phi(X_t^N) - \sum_{k=0}^{N-1} \partial_{x^\mu} \Phi(X^N_{t_{k+1}^N}) \mathbf{1}_{[t_k^N, t_{k+1}^N)}(u)|^2] \]

\[ = \hat{E}[|\partial_{x^\mu} \Phi(X_t^N) - \partial_{x^\mu} \Phi(X_{t_k^N})|^2] \]

\[ \leq c^2 \hat{E}[|X_t^N - X_{t_k^N}|^2] \leq C[\delta + \delta^2]. \]

Thus \( \sum_{k=0}^{N-1} \partial_{x^\mu} \Phi(X^N_{t_k^N}) \mathbf{1}_{[t_k^N, t_{k+1}^N)}(\cdot) \) tends to \( \partial_{x^\mu} \Phi(X) \) in \( M^2_G(0, T) \). Similarly,

\[ \sum_{k=0}^{N-1} \partial^2_{x^\mu,x^\nu} \Phi(X_{t_k^N}) \mathbf{1}_{[t_k^N, t_{k+1}^N)}(\cdot) \rightarrow \partial^2_{x^\mu,x^\nu} \Phi(X) \) in \( M^2_G(0, T) \).
Let $N \to \infty$. From the definitions of the integrations with respect to $dt$, $dB_t$ and $d\langle B \rangle_t$ the limit of $\xi^N_t$ in $L^2_G(\mathcal{F}_t)$ is just the right hand of (4.23). By the estimates of the next remark we also have $\zeta^N_t \to 0$ in $L^1_G(\mathcal{F}_t)$. We then have proved (4.23).

**Remark 4.2.2** We have the following estimates: For $\psi^N \in M_G^1(0,T)$ such that $\psi^N_t = \sum_{k=0}^{N-1} \xi^N_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(t)$ and $\pi^N_T = \{0 \leq t_0, \ldots, t_N = T\}$ with $\lim_{N \to \infty} \mu(\pi^N_T) = 0$ and $\sum_{k=0}^{N-1} \hat{E}[|\xi^N_{t_k}|(t^N_{k+1} - t^N_k)] \leq C$ for all $N = 1,2,\ldots$, we have

$$\hat{E}[(\sum_{k=0}^{N-1} \xi^N_{t_k}(t^N_{k+1} - t^N_k)^2)] \to 0,$$

and, thanks to Lemma 4.1.34,

$$\hat{E}[(\sum_{k=0}^{N-1} \xi^N_{t_k}(t^N_{k+1} - t^N_k)^2)] \leq \sum_{k=0}^{N-1} \hat{E}[\xi^N_{t_k}] \hat{E}[(\langle B \rangle^{t^N_{k+1}}_{t_k} - \langle B \rangle^{t^N_k}_{t_k})^2|\mathcal{H}_{t_k}^N]$$

$$= \sum_{k=0}^{N-1} \hat{E}[\xi^N_{t_k}](t^N_{k+1} - t^N_k)^2 \to 0,$$

as well as

$$\hat{E}[(\sum_{k=0}^{N-1} \xi^N_{t_k}(\langle B \rangle^{t^N_{k+1}}_{t_k} - \langle B \rangle^{t^N_k}_{t_k})(B^{t^N_{k+1}}_{t_k} - B^{t^N_k}_{t_k}))]
\leq \sum_{k=0}^{N-1} \hat{E}[\xi^N_{t_k}] \hat{E}[(\langle B \rangle^{t^N_{k+1}}_{t_k} - \langle B \rangle^{t^N_k}_{t_k})|B^{t^N_{k+1}}_{t_k} - B^{t^N_k}_{t_k}]$$

$$\leq \sum_{k=0}^{N-1} \hat{E}[\xi^N_{t_k}] \hat{E}[(\langle B \rangle^{t^N_{k+1}}_{t_k} - \langle B \rangle^{t^N_k}_{t_k})^2]^{1/2} \hat{E}[|B^{t^N_{k+1}}_{t_k} - B^{t^N_k}_{t_k}|^2]^{1/2}$$

$$= \sum_{k=0}^{N-1} \hat{E}[\xi^N_{t_k}](t^N_{k+1} - t^N_k)^{3/2} \to 0.$$
We also have
\[
\hat{E}[\sum_{k=0}^{N-1} \xi_k^N ((B)_{t_{k+1}^N} - (B)_{t_k^N})(t_{k+1}^N - t_k^N)] \\
\leq \sum_{k=0}^{N-1} \hat{E}[\xi_k^N (t_{k+1}^N - t_k^N) \cdot \hat{E}[(B)_{t_{k+1}^N} - (B)_{t_k^N} | \mathcal{H}_{t_k^N}]] \\
= \sum_{k=0}^{N-1} \hat{E}[\xi_k^N (t_{k+1}^N - t_k^N)^2] \to 0
\]

and
\[
\hat{E}[\sum_{k=0}^{N-1} \xi_k^N (t_{k+1}^N - t_k^N)(B_{t_{k+1}^N} - B_{t_k^N})] \\
\leq \sum_{k=0}^{N-1} \hat{E}[\xi_k^N (t_{k+1}^N - t_k^N) \hat{E}[B_{t_{k+1}^N} - B_{t_k^N}]] \\
= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{N-1} \hat{E}[\xi_k^N (t_{k+1}^N - t_k^N)^{3/2}] \to 0.
\]

We now consider a more general form of Itô’s formula. Consider
\[
X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^\nu dB_s + \int_0^t \beta_s^\nu dB_s.
\]

**Proposition 4.2.3** Let \(\alpha^\nu, \beta^\nu, \eta^\nu, \nu = 1, \ldots, n,\) be bounded processes of \(M_G^2(0,T).\) Then for each \(t \geq 0\) and \(\Phi\) in \(L_G^2(\mathcal{F}_t)\) we have
\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_x \Phi(X_u)\beta_u^\nu dB_u + \int_s^t \partial_x \Phi(X_u)\alpha_u^\nu du \\
+ \int_s^t [\partial_x \Phi(X_u)\eta_u^\nu + \frac{1}{2} \partial_{xx} \Phi(X_u)\beta_u^\nu \beta_u^\nu] d\langle B \rangle_u.
\]

**Proof.** We first consider the case where \(\alpha, \eta\) and \(\beta\) are step processes of the form
\[
\eta_k(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{1}_{(t_k,t_{k+1})}(t).
\]
CHAPTER 4. G-BROWNIAN MOTION: 1-DIMENSIONAL CASE

From Lemma 4.2.1 it is clear that (4.25) holds true. Now let

\[
X_t^{\nu,N} = X_0^\nu + \int_0^t \alpha_s^{\nu,N} ds + \int_0^t \eta_s^{\nu,N} \langle B \rangle_s + \int_0^t \beta_s^{\nu,N} dB_s,
\]

where \(\alpha^N, \eta^N\) and \(\beta^N\) are uniformly bounded step processes that converge to \(\alpha, \eta\) and \(\beta\) in \(M^2_G(0,T)\) as \(N \to \infty\), respectively. From Lemma 4.2.1

\[
\Phi(X_t^{\nu,N}) - \Phi(X_0) = \int_s^t \partial_{x^\nu} \Phi(X_u^N) \beta_u^{\nu,N} dB_u + \int_s^t \partial_{x^\nu} \Phi(X_u^N) \alpha_u^{\nu,N} du \quad (4.26)
\]

\[
+ \int_s^t [\partial_{x^\nu} \Phi(X_u^N) \eta_u^{\nu,N} + \frac{1}{2} \partial^2_{x^\mu x^\nu} \Phi(X_u^N) \beta_u^{\mu,N} \beta_u^{\nu,N}] \langle B \rangle_u.
\]

Since

\[
\hat{E}[|X_t^{\nu,N} - X_t^\nu|^2] \leq 3 \hat{E}[|\int_0^t (\alpha_s^N - \alpha_s) ds|^2] + 3 \hat{E}[|\int_0^t (\eta_s^{\nu,N} - \eta_s^\nu) \langle B \rangle_s|^2]
\]

\[
+ 3 \hat{E}[|\int_0^t (\beta_s^{\nu,N} - \beta_s^\nu) dB_s|^2] \leq 3 \int_0^T \hat{E}[|\alpha_s^{\nu,N} - \alpha_s^\nu|^2] ds + 3 \int_0^T \hat{E}[|\eta_s^{\nu,N} - \eta_s^\nu|^2] ds
\]

\[
+ 3 \int_0^T \hat{E}[|\beta_s^{\nu,N} - \beta_s^\nu|^2] ds.
\]

Furthermore

\[
\partial_{x^\nu} \Phi(X^N) \eta^N + \partial^2_{x^\mu x^\nu} \Phi(X^N) \beta^\mu,N \beta^\nu,N \to \partial_{x^\nu} \Phi(X) \eta^\nu + \partial^2_{x^\mu x^\nu} \Phi(X) \beta^\mu \beta^\nu;
\]

\[
\partial_{x^\nu} \Phi(X^N) \alpha^N \to \partial_{x^\nu} \Phi(X) \alpha^\nu;
\]

\[
\partial_{x^\nu} \Phi(X^N) \beta^N \to \partial_{x^\nu} \Phi(X) \beta^\nu.
\]

We then can pass limit in both sides of (4.26) and get (4.25).
4.3 Stochastic differential equations

We consider the following SDE defined on $M^2_G(0, T; \mathbb{R}^n)$:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B \rangle_s + \int_0^t \sigma(X_s)dB_s, \quad t \in [0, T].$$

(4.27)

where the initial condition $X_0 \in \mathbb{R}^n$ is given and $b, h, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lipschitz functions, i.e., $|\varphi(x) - \varphi(x')| \leq K|x - x'|$, for each $x, x' \in \mathbb{R}^n$, $\varphi = b, h$ and $\sigma$, respectively. Here the horizon $[0, T]$ can be arbitrarily large.

The solution is a process $X \in M^2_G(0, T; \mathbb{R}^n)$ satisfying the above SDE. We first introduce the following mapping on a fixed interval $[0, T]$:

$$\Lambda_t(Y) := X_0 + \int_0^t b(Y_s)ds + \int_0^t h(Y_s)d\langle B \rangle_s + \int_0^t \sigma(Y_s)dB_s, \quad t \in [0, T].$$

We immediately have

**Lemma 4.3.1** For each $Y, Y' \in M^2_G(0, T; \mathbb{R}^n)$ we have the following estimate:

$$\mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \leq C \int_0^t \mathbb{E}[|Y_s - Y'_s|^2]ds, \quad t \in [0, T],$$

where $C = 3K^2$.

**Proof.** This is a direct consequence of the inequalities (4.10), (4.12) and (4.17). □

We now prove that SDE (4.27) has a unique solution. By multiplying $e^{-2Ct}$ on both sides of the above inequality and then integrating them on $[0, T]$ it follows that

$$\int_0^T \mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt \leq C \int_0^T e^{-2Ct} \int_0^t \mathbb{E}[|Y_s - Y'_s|^2]dsdt$$

$$= C \int_0^T \int_s^T e^{-2Cs} \mathbb{E}[|Y_s - Y'_s|^2]ds dt$$

$$= (2C)^{-1} C \int_0^T (e^{-2Cs} - e^{-2CT}) \mathbb{E}[|Y_s - Y'_s|^2]ds.$$
We then have
\[ \int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt \leq \frac{1}{2} \int_0^T \hat{\mathbb{E}}[|Y_t - Y'_t|^2]e^{-2Ct}dt. \]

We observe that the following two norms are equivalent in \( M^2_G(0,T; \mathbb{R}^n) \):
\[ \int_0^T \hat{\mathbb{E}}[|Y_t|^2]dt \sim \int_0^T \hat{\mathbb{E}}[|Y_t|^2]e^{-2Ct}dt. \]

From this estimate we can obtain that \( \Lambda(Y) \) is a contraction mapping. Consequently, we have

**Theorem 4.3.2** There exists a unique solution \( X \in M^2_G(0,T; \mathbb{R}^n) \) of the stochastic differential equation (4.27).

### 4.4 Backward SDE

We consider the following type of BSDE
\[ Y_t = \hat{\mathbb{E}}[\xi + \int_t^T f(s,Y_s)ds|\mathcal{H}_t], \quad t \in [0,T]. \quad (4.28) \]

where \( \xi \in L^1_G(\mathcal{F}_T; \mathbb{R}^m) \), \( f(t,y) \in M^1_G(0,T; \mathbb{R}^m) \), \( t \in [0,T] \), \( y \in \mathbb{R}^m \), are given such that
\[ |f(t,y) - f(t,y')| \leq k|y - y'|. \]

We have

**Theorem 4.4.1** There exists a unique solution \( (Y_t)_{t \in [0,T]} \in M^1(0,T) \).

\[ \Lambda_t(Y) = \hat{\mathbb{E}}[\xi + \int_t^T f(s,Y_s)ds|\mathcal{H}_t] \]
4.4. BACKWARD SDE

\[ \int_0^T \mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|]e^{\beta t}dt \leq \int_0^T \mathbb{E}[\int_t^T |f(s, Y_s) - f(s, Y'_s)| ds]e^{\beta t}dt \]
\[ \leq C \int_0^T \int_t^T \mathbb{E}[|Y_s - Y'_s|]e^{\beta t}ds dt \]
\[ = C \int_0^T \mathbb{E}[|Y_s - Y'_s|] \int_0^s e^{\beta t}dt ds \]
\[ = \frac{C}{\beta} \int_0^T \mathbb{E}[|Y_s - Y'_s|](e^{\beta s} - 1)ds \]
\[ \leq \frac{C}{\beta} \int_0^T \mathbb{E}[|Y_s - Y'_s|]e^{\beta s}ds. \]

We choose \( \beta = 2C \), then

\[ \int_0^T \mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|]e^{\beta t}dt \leq \frac{1}{2} \int_0^T \mathbb{E}[|Y_s - Y'_s|]e^{\beta s}ds. \]

**Exercise.** We define a deterministic function \( u = u(t, x) \) on \((t, x) \in [0, T] \times \mathbb{R}^d \) by

\[ dX^{x,t}_s = f(X^{t,x}_s)ds + h(X^{t,x}_s)d\langle B \rangle_s + \sigma(X^{t,x}_s)dB_s, \quad s \in [t, T], \quad X^{x,t}_t = x \]
\[ Y^{t,x}_s = \mathbb{E}[\Phi(X^{t,x}_T) + \int_s^T g(X^{t,x}_r, Y^{t,x}_r)dr | \mathcal{H}_s], \quad s \in [t, T]. \]

where \( \Phi \) and \( g \) are given \( \mathbb{R}^m \)-valued Lipschitz functions on \( \mathbb{R}^d \) and \( \mathbb{R}^d \times \mathbb{R}^m \), respectively. \( u \) is a viscosity solution of a fully nonlinear parabolic PDE. Try to find this PDE and prove this interpretation.
Chapter 5

Vector Valued $G$-Brownian Motion

5.1 Multidimensional $G$–normal distributions

For a given positive integer $n$ we will denote by $(x, y)$ the scalar product of $x, y \in \mathbb{R}^n$ and by $|x| = (x,x)^{1/2}$ the Euclidean norm of $x$. We denote by $\mathcal{S}(n)$ the space of all $d \times d$ symmetric matrices. $\mathcal{S}(n)$ is a $\frac{d(d+1)}{2}$-dimensional Euclidean space of the norm $(A,B):=\text{tr}[AB]$. We also denote $\mathcal{S}(n)^+$ subset of non-negative matrices in $\mathcal{S}(n)$.

For a given $d$-dimensional random vector $\xi = (\xi_1, \cdots, \xi_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ with zero mean uncertainty:

$$\hat{E}[(\xi_1, \cdots, \xi_d)] = 0, \quad \hat{E}[\xi] = 0.$$ 

We set

$$G(A) = G_\xi(A) := \frac{1}{2}\hat{E}[(A\xi, \xi)], \quad A \in \mathcal{S}(d).$$

This function $G(\cdot) : \mathcal{S}(n) \mapsto \mathbb{R}$ characterizes the variance-uncertainty of $\xi$. Since $\xi \in \mathcal{H}$ implies $\hat{E}[|\xi|^2] < \infty$, there exists a constant $C$ such that

$$|G(A)| \leq C|A|, \quad \forall A \in \mathcal{S}(n).$$

Proposition 5.1.1 The function $G(\cdot) : \mathcal{S}(n) \mapsto \mathbb{R}$ is a monotonic and sublinear mapping:
CHAPTER 5. VECTOR VALUED G-BROWNIAN MOTION

(a) \( A \geq B \implies G(A) \geq G(B) \);

(b) \( G(\lambda A) = \lambda^+ G(A) + \lambda^- G(-A) \);

(c) \( G(A + B) \leq G(A) + G(B) \).

Moreover, there exists a bounded, convex and closed subset \( \Gamma \subset S^+(d) \) (the non-negative elements of \( S(d) \)) such that

\[
G(A) = \frac{1}{2} \sup_{B \in \Gamma} (A, B).
\]  

(5.1)

We introduce \( d \)-dimensional \( G \)-normal distribution.

**Definition 5.1.2** In a sublinear expectation space \( (\Omega, \hat{E}, \mathcal{H}) \) a \( d \)-dimensional random vector \( \xi = (\xi_1, \cdots, \xi_d) \in \mathcal{H}^d \) is said to be \( G \)-normal distributed if for any random vector \( \zeta \) in \( \mathcal{H}^d \) independent to \( \xi \) such that \( \zeta \sim \xi \) we have

\[
a\xi + b\zeta \sim \sqrt{a^2 + b^2} \xi, \quad \forall a, b \geq 0.
\]

Here \( G(\cdot) \) is a real sublinear function defined on \( S(n) \) by

\[
G(A) := \frac{1}{2} \hat{E}[(A\xi, \xi)], \quad A \in S(n).
\]

Since \( G \) is uniquely determined by \( \Gamma \subset S(n) \) via (5.1), \( \xi \) is also called \( \mathcal{N}(0, \Gamma) \) distributed, denoted by \( \xi \sim \mathcal{N}(0, \Gamma) \).

Proposition 3.2.7 tells us how to construct the above \((\xi, \zeta)\) based on the distribution of \( \xi \).

**Remark 5.1.3** By the above definition, we have

\[
\sqrt{2}\hat{E}[\xi] = \hat{E}[\xi + \zeta] = 2\hat{E}[\xi]
\]

and

\[
\sqrt{2}\hat{E}[-\xi] = \hat{E}[-\xi - \zeta] = 2\hat{E}[-\xi]
\]

it follows that

\[
\hat{E}[\xi] = \hat{E}[-\xi] = 0,
\]

Namely \( G \)-normal distributed random variable \( \xi \) has no mean uncertainty.

**Remark 5.1.4** If \( \xi \) is independent to \( \zeta \) and \( \xi \sim \zeta \), such that (5.2) satisfies, then \( -\xi \) is independent to \( -\zeta \) and \( -\xi \sim -\zeta \). We also have \( a(-\zeta) + b(-\zeta) \sim \sqrt{a^2 + b^2}(-\zeta), \ a, b \geq 0 \). Thus

\[
\xi \sim \mathcal{N}(0; \Gamma) \quad \text{iff} \quad -\xi \sim \mathcal{N}(0; \Gamma).
\]
The following proposition and corollary show that \( \mathcal{N}(0; \Gamma) \) is a uniquely defined sublinear distribution on \((\mathbb{R}^n, C_{l.Lip}(\mathbb{R}^n))\). We will show that an \( \mathcal{N}(0; \Gamma) \) distribution is characterized, or generated, by the following parabolic PDE defined on \([0, \infty) \times \mathbb{R}^n\):

**Proposition 5.1.5** For a given \( \varphi \in C_{l.Lip}(\mathbb{R}^d) \) the function

\[
u(t,x) = \mathbb{E}[\varphi(x + \sqrt{t}\xi)], \quad (t,x) \in [0, \infty) \times \mathbb{R}^n
\]

satisfies

\[
u(t+s,x) = \mathbb{E}[\nu(t,x + \sqrt{s}\xi)]
\]

\( \nu(t,\cdot) \in C_{l.Lip}(\mathbb{R}^n) \) for each \( t \) and \( \nu \) is locally Hölder in \( t \). Moreover \( \nu \) is the unique viscosity solution of the following PDE

\[
\frac{\partial \nu}{\partial t} - G(D^2 \nu) = 0, \quad \nu|_{t=0} = \varphi,
\]

(5.2)

here \( D^2 \nu \) is the Hessian matrix of \( \nu \), i.e., \( D^2 \nu = (\partial^2_{x_ix_j} \nu)_{i,j=1}^d \).

**Remark 5.1.6** In the case where \( \xi \) has zero variance–uncertainty, i.e., there exists \( \gamma_0 \in \mathbb{S}(n)^+ \), such that

\[
G(A) = \frac{1}{2} (A, \gamma_0).
\]

The above PDE becomes a standard linear heat equation and thus, for \( G^0 = G_{\gamma_0} \), the corresponding \( G_{\gamma_0} \)-distribution is just the \( d \)-dimensional classical normal distribution with zero mean and covariance \( \gamma_0 \), i.e., \( \xi_0 \sim \mathcal{N}(0, \gamma_0) \).

In a typical case where \( \gamma_0 = I_d \in \Gamma \) we have

\[
\mathbb{E}[\varphi(\xi)] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp[-\frac{1}{2} \sum_{i=1}^d (x_i)^2] \varphi(x) dx, \quad \varphi \in C_{l.Lip}(\mathbb{R}^d).
\]

In the case where \( \gamma_0 \in \Gamma \) from comparison theorem of PDE

\[
\mathbb{F}_\xi[\varphi] \geq \mathbb{F}_{\xi_0}[\varphi], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^d).
\]

(5.3)

More generally, for each subset \( \Gamma' \subset \Gamma \) let

\[
G'(A) := \sup_{\gamma \in \Gamma'} \langle A, \gamma \rangle
\]

the corresponding \( G \)-normal distribution \( \mathcal{N}(0, \Gamma') \) is dominated by \( P^G \) in the following sense:

\[
\mathbb{F}_{G_{\Gamma'}}[\varphi] - \mathbb{F}_{G_{\Gamma'}}[\psi] \leq \mathbb{F}_{G_{\Gamma}}[\varphi - \psi], \quad \forall \varphi, \psi \in C_{l.Lip}(\mathbb{R}^d).
\]
Remark 5.1.7 In the previous chapters we have discussed 1-dimensional case which corresponds to \( d = 1 \) and \( \Gamma = [\sigma^2, \sigma^2] \subset \mathbb{R} \). In this case the nonlinear heat equation (5.2) becomes
\[
\frac{\partial u}{\partial t} - G(\partial^2_{xx}u) = 0, \quad u|_{t=0} = \varphi \in C_{Lip}(\mathbb{R}),
\]
with \( G(a) = \frac{1}{2}(\sigma^2a^+ - \sigma^2a^-) \). In the multidimensional cases we also have the following typical nonlinear heat equation:
\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{d} G_i(\partial^2_{xx,i}u) = 0, \quad u|_{t=0} = \varphi \in C_{Lip}(\mathbb{R}^d),
\]
where \( G_i(a) = \frac{1}{2}(\sigma_i^2a^+ - \sigma_i^2a^-) \) and \( 0 \leq \sigma_i \leq \sigma_i \) are given constants. This corresponds to:
\[
\Gamma = \{ \text{diag}[\gamma_1, \cdots, \gamma_d] : \gamma_i \in [\sigma_i^2, \sigma_i^2], \quad i = 1, \cdots, d \}. 
\]
x + \sqrt{t}\xi is \( \mathcal{N}(x, t\Gamma) \)-distributed.

Definition 5.1.8 We denote
\[
P_t^G(\varphi)(x) = \hat{E}[\varphi(x + \sqrt{t} \times \xi)] = u(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d. \tag{5.4}
\]
Since for each \( \varphi \in C_{Lip}(\mathbb{R}^d) \) we have the Markov-Nisio chain rule:
\[
P_{t+s}^G(\varphi)(x) = P_s^G(P_t^G(\varphi)(\cdot))(x).
\]

Remark 5.1.9 This chain rule was first established by Nisio \cite{43,46} in terms of “envelope of Markovian semi-groups”. See \cite{52} for an application to generates a sublinear expectation space.

Lemma 5.1.10 Let \( \xi \sim \mathcal{N}(x, t\Gamma) \). Then for each \( a \in \mathbb{R}^d \), the random variable \( (a, \xi) \) is \( \mathcal{N}(0, [\sigma^2, \sigma^2]) \) distributed, where
\[
\sigma^2 = \hat{E}[(a, \xi)^2], \quad \sigma^2 = -\hat{E}[-(a, \xi)^2].
\]

Proof. If \( \zeta \) is independent to \( \xi \) and \( \zeta \sim \xi \) then \( (a, \zeta) \) is also independent to \( (a, \xi) \) and \( (a, \zeta) \sim (a, \xi) \). It follows from \( \alpha \xi + \beta \zeta \sim \sqrt{\alpha^2 + \beta^2} \xi \) that
\[
\alpha(a, \xi) + \beta(a, \zeta) = (a, \alpha \xi + \beta \zeta) \sim \sqrt{\alpha^2 + \beta^2}(a, \xi).
\]
Thus \( (a, \xi) \sim \mathcal{N}(0, [\sigma^2, \sigma^2]) \). \( \blacksquare \)
5.2  $G$–Brownian motions under $G$–expectations

In the rest of this paper, we set $\Omega = C^d_0(\mathbb{R}^+)$, the space of all $\mathbb{R}^d$–valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$. $\Omega$ is the classical canonical space and $\omega = (\omega_t)_{t \geq 0}$ is the corresponding canonical process. It is well–known that in this canonical space there exists a Wiener measure $(\Omega, \mathcal{F}, P)$ under which the canonical process $B_t(\omega) = \omega_t$ is a $d$–dimensional Brownian motion.

For each fixed $T \geq 0$ we consider the following space of random variables:

$$L_{ip}^0(\mathcal{F}_T) := \{X(\omega) = \varphi(\omega_{t_1}, \ldots, \omega_{t_m}) : \forall m \geq 1, t_1, \ldots, t_m \in [0, T], \varphi \in C_{t, \text{Lip}}(\mathbb{R}^{d \times m})\}.$$  

It is clear that $\{\mathcal{H}_t^0\}_{t \geq 0}$ constitute a family of sub-lattices such that $\mathcal{H}_0^0 \subseteq L_{ip}^0(\mathcal{F}_T)$, for $t \leq T < \infty$. $L_{ip}^0(\mathcal{F}_t)$ representing the past history of $\omega$ at the time $t$. Its completion will play the same role of Brownian filtration $\mathcal{F}_t^B$ as in classical stochastic analysis. We also denote

$$L_{ip}^0(\mathcal{F}) := \bigcup_{n=1}^{\infty} L_{ip}^0(\mathcal{F}_n).$$  

Remark 5.2.1  It is clear that $C_{t, \text{Lip}}(\mathbb{R}^{d \times m})$ and then $L_{ip}^0(\mathcal{F}_T)$, $L_{ip}^0(\mathcal{F})$ are vector lattices. Moreover, since $\varphi, \psi \in C_{t, \text{Lip}}(\mathbb{R}^{d \times m})$ imply $\varphi \cdot \psi \in C_{t, \text{Lip}}(\mathbb{R}^{d \times m})$ thus $X, Y \in L_{ip}^0(\mathcal{F}_T)$ imply $X \cdot Y \in L_{ip}^0(\mathcal{F}_T)$; $X, Y \in L_{ip}^0(\mathcal{F})$ imply $X \cdot Y \in L_{ip}^0(\mathcal{F})$.

We will consider the canonical space and set $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$.

Definition 5.2.2  The $d$-dimensional process $(B_t)_{t \geq 0}$ is called a $G$–Brownian motion under a sublinear expectation $(\Omega, \mathcal{H}, \tilde{E})$ defined on $L_{ip}^0(\mathcal{F})$ if $B_0 = 0$ and

(i) For each $s, t \geq 0$ and $\psi \in C_{t, \text{Lip}}(\mathbb{R}^d)$, $B_t \sim B_{t+s} - B_s \sim \mathcal{N}(0, G)$;

(ii) For each $m = 1, 2, \ldots$, $0 = t_0 < t_1 < \cdots < t_m < \infty$ the increment $B_{t_m} - B_{t_{m-1}}$ is independent to $B_{t_1}, \ldots, B_{t_{m-1}}$.

Definition 5.2.3  The related conditional expectation of $\varphi(B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}})$ under $\mathcal{H}_{t_k}$ is defined by

$$\tilde{E}[\varphi(B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_k}] = \varphi_{m-k}(B_{t_1}, \ldots, B_{t_k}),$$ (5.5)

where $\varphi_{m-k}(x^1, \ldots, x^k) = \tilde{E}[\varphi(x^1, \ldots, x^k, B_{t_{k+1}} - B_{t_k}, \ldots, B_{t_m} - B_{t_k})]$. 


It is proved in [55] that \( E[\cdot] \) consistently defines a nonlinear expectation satisfying (a)–(d) on the vector lattice \( L^0_{ip}(\mathcal{F}_T) \) as well as on \( L^0_{ip}(\mathcal{F}) \). It follows that \( E[|X|], X \in L^0_{ip}(\mathcal{F}_T) \) (respectively \( L^0_{ip}(\mathcal{F}) \)) is a norm and thus \( L^0_{ip}(\mathcal{F}_T) \) (respectively, \( L^0_{ip}(\mathcal{F}) \)) can be extended, under this norm, to a Banach space. We denote this space by

\[ \mathcal{H}_T = L^1_G(\mathcal{F}_T), \quad T \in [0, \infty), \quad \text{ (respectively, } \mathcal{H} = L^1_G(\mathcal{F})). \]

For each \( 0 \leq t \leq T < \infty \), we have \( L^1_G(\mathcal{F}_t) \subseteq L^1_G(\mathcal{F}_T) \subseteq L^1_G(\mathcal{F}) \). In \( L^1_G(\mathcal{F}_T) \) (respectively, \( L^1_G(\mathcal{F}_T) \)), \( E[\cdot] \) still satisfies (a)–(d) in Definition 3.1.2.

**Remark 5.2.4** It is suggestive to denote \( L^0_{ip}(\mathcal{F}_t) \) by \( \mathcal{H}_t^0 \), \( L^1_G(\mathcal{F}_t) \) by \( \mathcal{H}_t \) and \( L^1_G(\mathcal{F}) \) by \( \mathcal{H} \) and thus consider the conditional expectation \( E[\cdot|\mathcal{H}_t] \) as a projective mapping from \( \mathcal{H} \) to \( \mathcal{H}_t \). The notation \( L^1_G(\mathcal{F}_t) \) is due to the similarity with \( L^1(\Omega, \mathcal{F}_t, P) \) in classical stochastic analysis.

**Definition 5.2.5** The expectation \( E[\cdot]: L^1_G(\mathcal{F}) \mapsto \mathbb{R} \) introduced through the above procedure is called \textit{G–expectation}, or \textit{G–Brownian expectation}. The corresponding canonical process \( B \) is said to be a \textit{G–Brownian motion} under \( \hat{E}[\cdot] \).

For a given \( p > 1 \) we also denote \( L^p_G(\mathcal{F}) = \{ X \in L^1_G(\mathcal{F}) : |X|^p \in L^1_G(\mathcal{F}) \} \). \( L^p_G(\mathcal{F}) \) is also a Banach space under the norm \( \|X\|_p := (\hat{E}[|X|^p])^{1/p} \). We have (see Appendix)

\[ \|X + Y\|_p \leq \|X\|_p + \|Y\|_p \]

due to the similarity with \( L^1(\Omega, \mathcal{F}_t, P) \).

With this we have \( \|X\|_p \leq \|X\|_{p'} \) if \( p \leq p' \).

We now consider the conditional expectation introduced in Definition 5.2.3 (see [55]). For each fixed \( t = t_k \leq T \) the conditional expectation \( \hat{E}[\cdot|\mathcal{H}_t]: L^0_{ip}(\mathcal{F}_T) \mapsto L^0_{ip}(\mathcal{F}_t) \) is a continuous mapping under \( \|\cdot\| \). Indeed, we have \( \hat{E}[\hat{E}[X|\mathcal{H}_t]] = \hat{E}[X], X \in L^0_{ip}(\mathcal{F}_T) \) and since \( P^G_t \) is subadditive

\[ \hat{E}[X|\mathcal{H}_t] - \hat{E}[Y|\mathcal{H}_t] \leq \hat{E}[X - Y|\mathcal{H}_t] \leq \hat{E}[|X - Y||\mathcal{H}_t] \]

We thus obtain

\[ \hat{E}[\hat{E}[X|\mathcal{H}_t] - \hat{E}[Y|\mathcal{H}_t]] \leq \hat{E}[X - Y] \]
We list the properties of Proposition 5.2.6. If the above (i) it follows that \( \hat{E}[\cdot | \mathcal{H}_t] \) can be also extended as a continuous mapping
\[
\hat{E}[\cdot | \mathcal{H}_t] : L^1_G(\mathcal{F}_T) \mapsto L^1_G(\mathcal{F}_t)
\]
If the above \( T \) is not fixed then we can obtain \( \hat{E}[\cdot | \mathcal{H}_t] : L^1_G(\mathcal{F}) \mapsto L^1_G(\mathcal{F}_t) \).

**Proposition 5.2.6** We list the properties of \( \hat{E}[\cdot | \mathcal{H}_t] \), \( t \in [0,T] \) that hold in \( L^0_{ip}(\mathcal{F}_T) \) and still hold for \( X, Y \in L^1_G(\mathcal{F}_T) \):

(i) \( \hat{E}[X|\mathcal{H}_t] = X \), for \( X \in L^1_G(\mathcal{F}_t) \), \( t \leq T \).
(ii) If \( X \geq Y \) then \( \hat{E}[X|\mathcal{H}_t] \geq \hat{E}[Y|\mathcal{H}_t] \).
(iii) \( \hat{E}[X|\mathcal{H}_t] - \hat{E}[Y|\mathcal{H}_t] \leq \hat{E}[X - Y|\mathcal{H}_t] \).
(iv) \( \hat{E}[\hat{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \hat{E}[X|\mathcal{H}_{t\wedge s}], \hat{E}[\hat{E}[X|\mathcal{H}_t]] = \hat{E}[X] \).
(v) \( \hat{E}[X + \eta|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \eta, \eta \in L^1_G(\mathcal{F}_t) \).
(vi) \( \hat{E}[\eta X|\mathcal{H}_t] = \eta \hat{E}[X|\mathcal{H}_t] + \eta^{-} \hat{E}[-X|\mathcal{H}_t] \) for bounded \( \eta \in L^1_G(\mathcal{F}_t) \).
(vii) We have the following independence:
\[
\hat{E}[X|\mathcal{H}_t] = \hat{E}[X], \forall X \in L^1_G(\mathcal{F}_T), \forall T \geq 0,
\]
where \( L^1_G(\mathcal{F}_T) \) is the extension under \( \|\cdot\| \) of \( L^0_{ip}(\mathcal{F}_T) \) which consists of random variables of the form \( \varphi(B^t_{t_1}, B^t_{t_2}, \ldots, B^t_{t_m}), \varphi \in C_{l,Lip}(\mathbb{R}^m), t_1, \ldots, t_m \in [0,T], m = 1, 2, \ldots \). Here we denote
\[
B^t_s = B_{t+s} - B_t, \quad s \geq 0.
\]
(viii) The increments of \( B \) are identically distributed:
\[
\hat{E}[\varphi(B^t_{t_1}, B^t_{t_2}, \ldots, B^t_{t_m})] = \hat{E}[\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_m})].
\]

The meaning of the independence in (vii) is similar to the classical one:

**Definition 5.2.7** An \( \mathbb{R}^n \) valued random variable \( Y \in (L^1_G(\mathcal{F}))^n \) is said to be independent to \( \mathcal{H}_t \) for some given \( t \) if for each \( \varphi \in C_{l,Lip}(\mathbb{R}^n) \) we have
\[
\hat{E}[\varphi(Y)|\mathcal{H}_t] = \hat{E}[\varphi(Y)].
\]
It is seen that the above property (vii) also holds for the situation \( X \in L^1_G(\mathcal{F}_t) \) where \( L^1_G(\mathcal{F}_t) \) is the completion of the sub-lattice \( \cup_{T \geq 0} L^1_G(\mathcal{F}_T) \) under \( \| \cdot \| \).

From the above results we have

**Proposition 5.2.8** For each fixed \( t \geq 0 \) \((B^t_s)_{s \geq 0}\) is a \( G \)-Brownian motion in \( L^1_G(\mathcal{F}_t) \) under the same \( G \)-expectation \( \hat{E}[\cdot] \).

**Remark 5.2.9** We can also prove that the time scaling of \( B \), i.e., \( \hat{B} = (\sqrt{\lambda}B_{t/\lambda})_{t \geq 0} \) also constitutes a \( G \)-Brownian motion.

The following property is very useful.

**Proposition 5.2.10** Let \( X, Y \in L^1_G(\mathcal{F}_t) \) be such that \( \hat{E}[Y|\mathcal{H}_t] = -\hat{E}[-Y|\mathcal{H}_t] \), for some \( t \in [0, T] \). Then we have

\[
\hat{E}[X + Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t].
\]

In particular, if \( \hat{E}[Y|\mathcal{H}_t] = \hat{E}[-Y|\mathcal{H}_t] = 0 \) then \( \hat{E}[X + Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] \).

**Proof.** This follows from the two inequalities \( \hat{E}[X + Y|\mathcal{H}_t] \leq \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t] \) and \( \hat{E}[X + Y|\mathcal{H}_t] \geq \hat{E}[X|\mathcal{H}_t] - \hat{E}[-Y|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \hat{E}[Y|\mathcal{H}_t] \).

\[ \blacksquare \]

**Example 5.2.11** We have

\[
\hat{E}[(AB, B_t)] = \sigma_A t = 2G(A) t, \quad \forall A \in \mathbb{S}(d).
\]

More general, for each \( s \leq t \) and \( \eta = (\eta^{ij})_{i,j=1}^d \in L^2_G(\mathcal{F}_s; \mathbb{S}(d)) \)

\[
\hat{E}[(\eta B^s_t, B^s_t)|\mathcal{H}_s] = \sigma_\eta t = 2G(\eta) t, \quad s, t \geq 0.
\] (5.6)

**Definition 5.2.12** We will denote in the rest of this paper

\( B^a_t = (a, B_t), \quad \text{for each } a = (a_1, \ldots, a_d)^T \in \mathbb{R}^d. \) (5.7)

Since \( (a, B_t) \) is normal distributed \( (a, B_t) \sim \mathcal{N}(0, [\sigma_a a^T, \sigma_{-a} a^T]) \):

\[
\hat{E}[\varphi(B^a_t)] = \hat{E}[(a, (a, \sqrt{t}\xi))]
\]

Thus, according to Definition 5.2.2 for \( d \)-dimensional \( G \)-Brownian motion \( B^a \) forms a \( 1 \)-dimensional \( G_a \)-Brownian motion for which the \( G_a \)-expectation coincides with \( \hat{E}[\cdot] \).
Example 5.2.13 For each $0 \leq s - t$ we have

$$\hat{E}[\psi(B_t - B_s)|\mathcal{H}_s] = \hat{E}[\psi(B_t - B_s)].$$

If $\varphi$ is a real convex function on $\mathbb{R}$ and at least not growing too fast then

$$\hat{E}[X\varphi(B_t^a - B_s^a)|\mathcal{H}_t] = X^+\hat{E}[\varphi(B_t^a - B_s^a)|\mathcal{H}_t] + X^-\hat{E}[\varphi(B_t^a - B_s^a)|\mathcal{H}_t]$$

$$= \frac{X^+}{\sqrt{2\pi(T-t)\sigma_{aa^T}}} \int_{-\infty}^{\infty} \varphi(x) \exp\left(-\frac{x^2}{2(T-t)\sigma_{aa^T}}\right)dx$$

$$- \frac{X^-}{\sqrt{2\pi(T-t)\sigma_{aa^T}}} \int_{-\infty}^{\infty} \varphi(x) \exp\left(-\frac{x^2}{2(T-t)\sigma_{aa^T}}\right)dx.$$

In particular, for $n = 1, 2, \cdots$,

$$\hat{E}[|B_t^a - B_s^a|^n|\mathcal{H}_s] = \hat{E}[|B_{t-s}^a|^n]$$

$$= \frac{1}{\sqrt{2\pi(t-s)\sigma_{aa^T}}} \int_{-\infty}^{\infty} |x|^n \exp\left(-\frac{x^2}{2(t-s)\sigma_{aa^T}}\right)dx.$$

But we have $\hat{E}[|B_t^a - B_s^a|^n|\mathcal{H}_s] = \hat{E}[|B_{t-s}^a|^n]$ which is 0 when $\sigma_{aa^T} = 0$ and equals to

$$\frac{-1}{\sqrt{2\pi(t-s)\sigma_{aa^T}}} \int_{-\infty}^{\infty} |x|^n \exp\left(-\frac{x^2}{2(t-s)\sigma_{aa^T}}\right)dx, \text{ if } \sigma_{aa^T} < 0.$$

Exactly as in the classical cases we have $\hat{E}[B_t^a - B_s^a|\mathcal{H}_s] = 0$ and

$$\hat{E}[(B_t^a - B_s^a)^2|\mathcal{H}_s] = \sigma_{aa^T}^2(t-s), \quad \hat{E}[(B_t^a - B_s^a)^4|\mathcal{H}_s] = 3\sigma_{aa^T}^2(t-s)^2,$$

$$\hat{E}[(B_t^a - B_s^a)^6|\mathcal{H}_s] = 15\sigma_{aa^T}^3(t-s)^3, \quad \hat{E}[(B_t^a - B_s^a)^8|\mathcal{H}_s] = 105\sigma_{aa^T}^4(t-s)^4,$$

$$\hat{E}[|B_t^a - B_s^a|^2|\mathcal{H}_s] = \frac{\sqrt{2(t-s)\sigma_{aa^T}}}{\sqrt{\pi}}, \quad \hat{E}[|B_t^a - B_s^a|^3|\mathcal{H}_s] = \frac{2\sqrt{2}(t-s)\sigma_{aa^T}^{3/2}}{\sqrt{\pi}},$$

$$\hat{E}[|B_t^a - B_s^a|^5|\mathcal{H}_s] = \frac{8\sqrt{2}(t-s)\sigma_{aa^T}^{5/2}}{\sqrt{\pi}}.$$

Example 5.2.14 For each $n = 1, 2, \cdots, 0 \leq t \leq T$ and $X \in L^1_G(\mathcal{F}_t)$, we have

$$\hat{E}[X(B_t^a - B_s^a)|\mathcal{H}_t] = X^+\hat{E}[(B_t^a - B_s^a)|\mathcal{H}_t] + X^-\hat{E}[-(B_t^a - B_s^a)|\mathcal{H}_t] = 0.$$
CHAPTER 5. VECTOR VALUED G-BROWNIAN MOTION

This, together with Proposition 5.2.10, yields
\[
\hat{E}[Y + X(B^a_T - B^a_t)|\mathcal{H}_t] = \hat{E}[Y|\mathcal{H}_t], \quad Y \in L^1_G(\mathcal{F}).
\]
We also have,
\[
\hat{E}[X(B^a_T - B^a_t)^2|\mathcal{H}_t] = X^+\hat{E}[(B^a_T - B^a_t)^2|\mathcal{H}_t] + X^-\hat{E}[-(B^a_T - B^a_t)^2|\mathcal{H}_t]
\]
\[
= [X^+\sigma_{aa^\tau} + X^-\sigma_{-a^\tau a}](T-t).
\]

Remark 5.2.15 It is clear that we can define an expectation \(E^0[\cdot]\) on \(L^0_{ip}(\mathcal{F})\) in the same way as in Definition 5.2.2 with the standard normal distribution \(N(0,1)\) in place of \(N(0,\Gamma)\). If \(I_d \in \Gamma\) then it follows from (5.3) that \(N(0,1)\) is dominated by \(N(0,\Gamma)\):
\[
E^0[\varphi(B_t)] - E^0[\psi(B_t)] \leq \hat{E}[(\varphi - \psi)(B_t)].
\]
Then \(E^0[\cdot]\) can be continuously extended to \(L^1_G(\mathcal{F})\). \(E^0[\cdot]\) is a linear expectation under which \((B_t)_{t \geq 0}\) behaves as a classical Brownian motion. We have
\[
-\hat{E}[-X] \leq E^0[X] \leq \hat{E}[X], \quad -\hat{E}[-X|\mathcal{H}_t] \leq E^0[X|\mathcal{H}_t] \leq \hat{E}[X|\mathcal{H}_t]. \tag{5.8}
\]
More generally, if \(\Gamma' \subset \Gamma\) since the corresponding \(P' = P'^{G_{\Gamma'}}\) is dominated by \(P^G = P^G_{\Gamma}\), thus the corresponding expectation \(\hat{E}'\) is well-defined in \(L^1_G(\mathcal{F})\) and \(\hat{E}'\) is dominated by \(\hat{E}\):
\[
\hat{E}'[X] - \hat{E}'[Y] \leq \hat{E}[X - Y], \quad X, Y \in L^1_G(\mathcal{F}).
\]
Such kind of extension through the above type of domination relations was discussed in details in [52]. With this domination we then can introduce a large kind of time consistent linear or nonlinear expectations and the corresponding conditional expectations, not necessarily to be positive homogeneous and/or subadditive, as continuous functionals in \(L^1_G(\mathcal{F})\). See Example 5.3.14 for a further discussion.

Example 5.2.16 Since
\[
\hat{E}[2B^a_s(B^a_t - B^a_s)|\mathcal{H}_s] = \hat{E}[-2B^a_s(B^a_t - B^a_s)|\mathcal{H}_s] = 0
\]
we have
\[
\hat{E}[(B^a_t)^2 - (B^a_s)^2|\mathcal{H}_s] = \hat{E}[(B^a_t - B^a_s + B^a_s)^2 - (B^a_s)^2|\mathcal{H}_s]
\]
\[
= \hat{E}[(B^a_t - B^a_s)^2 + 2(B^a_t - B^a_s)B^a_s|\mathcal{H}_s]
\]
\[
= \sigma_{aa^\tau}(t-s)
\]
5.3. Itô’s integral of G–Brownian motion

5.3.1 Bochner’s integral

Definition 5.3.1 For $T \in \mathbb{R}_+$ a partition $\pi_T$ of $[0, T]$ is a finite ordered subset $\pi = \{t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$. 

$$
\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, \ldots, N - 1\}.
$$

We use $\pi_T^N = \{t_0^N, t_1^N, \ldots, t_N^N\}$ to denote a sequence of partitions of $[0, T]$ such that $\lim_{N \to \infty} \mu(\pi_T^N) = 0$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: For a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$ we set

$$
\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t),
$$

where $\xi_k \in L^p_G(\mathcal{F}_{t_k})$, $k = 0, 1, 2, \cdots, N - 1$ are given. The collection of these processes is denoted by $M^{p, 0}_G(0, T)$.

Definition 5.3.2 For an $\eta \in M^{p, 0}_G(0, T)$ with $\eta_t = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t)$, the related Bochner integral is

$$
\int_0^T \eta_t(\omega)dt = \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k).
$$
Remark 5.3.3 We set for each \( \eta \in M^0_G(0,T) \)
\[
\hat{E}_T[\eta] := \frac{1}{T} \int_0^T \hat{E}[\eta]dt = \frac{1}{T} \sum_{k=0}^{N-1} \hat{E}\xi_k(\omega)(t_{k+1} - t_k).
\]
It is easy to check that \( \hat{E}_T : M^1_G(0,T) \rightarrow \mathbb{R} \) forms a nonlinear expectation satisfying (a)–(d) of Definition 3.1.2. We then can introduce a natural norm
\[
||\eta||_T^1 = \hat{E}_T[|\eta|] = \frac{1}{T} \int_0^T \hat{E}[|\eta|]dt.
\]
Under this norm \( M^0_G(0,T) \) can be extended to \( M^1_G(0,T) \) which is a Banach space.

Definition 5.3.4 For each \( p \geq 1 \) we denote by \( M^p_G(0,T) \) the completion of \( M^0_G(0,T) \) under the norm
\[
(\frac{1}{T} \int_0^T ||\eta||^p dt)^{1/p} = \left( \frac{1}{T} \sum_{k=0}^{N-1} \hat{E}||\xi_k(\omega)||^p(t_{k+1} - t_k) \right)^{1/p}.
\]
We observe that,
\[
\hat{E}[\frac{1}{T} \int_0^T \eta(\omega)dt] \leq \frac{1}{T} \sum_{k=0}^{N-1} ||\xi_k(\omega)|| (t_{k+1} - t_k) = \frac{1}{T} \int_0^T \hat{E}[|\eta|]dt. \quad (5.9)
\]
We then have

Proposition 5.3.5 The linear mapping \( \int_0^T \eta(\omega)dt : M^1_G(0,T) \rightarrow L^1_G(\mathcal{F}_T) \) is continuous and thus can be continuously extended to \( M^1_G(0,T) \rightarrow L^1_G(\mathcal{F}_T) \). We still denote this extended mapping by \( \int_0^T \eta(\omega)dt, \eta \in M^1_G(0,T) \).

Since \( M^p_G(0,T) \subset M^1_G(0,T) \) for \( p \geq 1 \), this definition makes sense for \( \eta \in M^p_G(0,T) \).
5.3. ITÔ’S INTEGRAL OF G–BROWNIAN MOTION

5.3.2 Itô’s integral of G–Brownian motion

We still use $B^a_t := (a, B_t)$ as in (5.7).

**Definition 5.3.6** For each $\eta \in M^2_G(0, T)$ with the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t)$$

we define

$$I(\eta) = \int_0^T \eta(s) dB^a_s := \sum_{k=0}^{N-1} \xi_k(B^a_{t_{k+1}} - B^a_{t_k}).$$

**Lemma 5.3.7** We have, for each $\eta \in M^2_G(0, T)$,

$$\hat{E}[\int_0^T \eta(s) dB^a_s] = 0,$$  \hspace{1cm} (5.10)

$$\hat{E}[(\int_0^T \eta(s) dB^a_s)^2] \leq \sigma_{aa} \int_0^T \hat{E}[\eta^2(s)] ds.$$  \hspace{1cm} (5.11)

Consequently, the linear mapping $I : M^2_G(0, T) \mapsto L^2_G(\mathcal{F}_T)$ is continuous and thus can be extended to $I : M^2_G(0, T) \mapsto L^2_G(\mathcal{F}_T)$.

**Definition 5.3.8** We define, for a fixed $\eta \in M^2_G(0, T)$, the stochastic calculus

$$\int_0^T \eta(s) dB^a_s := I(\eta).$$

It is clear that (5.10) and (5.11) still hold for $\eta \in M^2_G(0, T)$.

**Proof of Lemma 5.3.7.** From Example 5.2.14 for each $k$

$$\hat{E}[\xi_k(B^a_{t_{k+1}} - B^a_{t_k})] = 0.$$

We have

$$\hat{E}[\int_0^T \eta(s) dB^a_s] = \hat{E}[\int_0^{t_{N-1}} \eta(s) dB^a_s + \xi_{N-1}(B^a_{t_N} - B^a_{t_{N-1}})]$$

$$= \hat{E}[\int_0^{t_{N-1}} \eta(s) dB^a_s + \hat{E}[\xi_{N-1}(B^a_{t_N} - B^a_{t_{N-1}})] | \mathcal{H}_{t_{N-1}}]$$

$$= \hat{E}[\int_0^{t_{N-1}} \eta(s) dB^a_s].$$
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We then can repeat this procedure to obtain (5.10). We now prove (5.11).

**Proposition 5.3.9**

We have

\[ \mathbb{E}[\left( \int_0^T \eta(s) dB_s^a \right)^2] = \mathbb{E}\left[ \int_0^{t_{N-1}} \eta(s) dB_s^a + \xi_{N-1}(B_{t_{N-1}}^a - B_{t_{N-1}}^a)^2 \right] \]

\[ = \mathbb{E}\left[ \int_0^{t_{N-1}} \eta(s) dB_s^a \right]^2 + \mathbb{E}[2 \int_0^{t_{N-1}} \eta(s) dB_s^a] \xi_{N-1}(B_{t_{N-1}}^a - B_{t_{N-1}}^a)^2 + \mathbb{E}[\xi_{N-1}^2 \sigma_{aa^2}(t_{N-1} - t_{N-1})]. \]

Thus \( \mathbb{E}[\left( \int_0^{t_{N}} \eta(s) dB_s^a \right)^2] \leq \mathbb{E}\left[ \int_0^{t_{N-1}} \eta(s) dB_s^a \right]^2 + \mathbb{E}[\xi_{N-1}^2 \sigma_{aa^2}(t_{N} - t_{N-1})]. \)

We then repeat this procedure to deduce

\[ \mathbb{E}[\left( \int_0^T \eta(s) dB_s^a \right)^2] = \sigma_{aa^2} \sum_{k=0}^{N-1} \mathbb{E}[\xi_k^2](t_{k+1} - t_k) = \int_0^T \mathbb{E}[(\eta(t))^2] dt. \]

We list some main properties of the Itô’s integral of \( G \)-Brownian motion. We denote for \( 0 \leq s \leq t \leq T \)

\[ \int_s^t \eta_u dB_u^a := \int_0^T I_{[s,t]}(u) \eta_u dB_u^a. \]

We have

**Proposition 5.3.9** Let \( \eta, \theta \in M^2_G(0, T) \) and \( 0 \leq s \leq r \leq t \leq T \). Then in \( L_G^1(\mathcal{F}_T) \) we have

(i) \( \int_s^r \eta_u dB_u^a = \int_s^r \eta_u dB_u^a + \int_r^T \eta_u dB_u^a \).

(ii) \( \int_s^r (\alpha \eta_u + \theta_u) dB_u^a = \alpha \int_s^r \eta_u dB_u^a + \int_s^r \theta_u dB_u^a \), if \( \alpha \) is bounded and in \( L^1_G(\mathcal{F}_s) \).

(iii) \( \mathbb{E}[X + \int_r^T \eta_u dB_u^a | \mathcal{H}_s] = \mathbb{E}[X | \mathcal{H}_s], \forall X \in L^1_G(\mathcal{F}) \).

(iv) \( \mathbb{E}[\left( \int_r^T \eta_u dB_u^a \right)^2 | \mathcal{H}_s] \leq \sigma_{aa^2} \int_r^T \mathbb{E}[|\eta_u|^2 | \mathcal{H}_s] du. \)

### 5.3.3 Quadratic variation process of \( G \)-Brownian motion

We now consider the quadratic variation process of \( G \)-Brownian motion. It concentrically reflects the characteristic of the ‘uncertainty’ part of the
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G–Brownian motion $B$. This makes a major difference from the classical Brownian motion.

Let $\pi_t^N$, $N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$. We consider

\[
(B_t^a)^2 = \sum_{k=0}^{N-1} [(B_{t_{k+1}}^a) - (B_{t_k}^a)]^2 = \sum_{k=0}^{N-1} 2B_{t_k}^a (B_{t_{k+1}}^a - B_{t_k}^a) + \sum_{k=0}^{N-1} (B_{t_{k+1}}^a - B_{t_k}^a)^2
\]

As $\mu(\pi_t^N) = \max_{0 \leq k \leq N-1} (t_{k+1}^N - t_k^N) \to 0$, the first term of the right side tends to $2 \int_0^t B_s^a dB_s^a$. The second term must converge. We denote its limit by $\langle B^a \rangle_t$, i.e.,

\[
\langle B^a \rangle_t = \lim_{\mu(\pi_t^N) \to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}}^a - B_{t_k}^a)^2 = (B_t^a)^2 - 2 \int_0^t B_s^a dB_s^a. \tag{5.12}
\]

By the above construction $\langle B^a \rangle_t$, $t \geq 0$, is an increasing process with $\langle B^a \rangle_0 = 0$. We call it the **quadratic variation process** of the G–Brownian motion $B^a$. Clearly $\langle B^a \rangle$ is an increasing process. It is also clear that for each $0 \leq s \leq t$ and each smooth real function $\psi$ such that $\psi(\langle B^a \rangle_{t-s}) \in L^1_G(\mathcal{F}_{t-s})$ we have $\mathbb{E}[\psi(\langle B^a \rangle_{t-s})] = \mathbb{E}[\psi(\langle B^a \rangle_t - \langle B^a \rangle_s)]$. We also have

\[
\langle B^a \rangle_t = \langle B^{-a} \rangle_t = \langle -B^a \rangle_t.
\]

It is important to keep in mind that $\langle B^a \rangle_t$ is not a deterministic process except in the case $\sigma_{aa} = -\sigma_{-aa}$ and thus $B^a$ becomes a classical Brownian motion. In fact we have

**Lemma 5.3.10** For each $0 \leq s \leq t < \infty$

\[
\mathbb{E}[\langle B^a \rangle_t - \langle B^a \rangle_s | \mathcal{H}_s] = \sigma_{aa}(t-s), \tag{5.13}
\]

\[
\mathbb{E}[\langle -B^a \rangle_t - \langle B^a \rangle_s | \mathcal{H}_s] = \sigma_{-aa}(t-s). \tag{5.14}
\]

**Proof.** From the definition of $\langle B^a \rangle$, Proposition 5.3.9(iii) and Example 5.2.16

\[
\mathbb{E}[\langle B^a \rangle_t - \langle B^a \rangle_s | \mathcal{H}_s] = \mathbb{E}[(B_t^a)^2 - (B_s^a)^2 - 2 \int_s^t B_u^a dB_u^a | \mathcal{H}_s] = \mathbb{E}[(B_t^a)^2 - (B_s^a)^2 | \mathcal{H}_s] = \sigma_{aa}(t-s).
\]
We then have (5.13). (5.14) can be proved analogously by using the equality
\[ \mathbb{E}[-((B^a_t)^2 - (B^a_s)^2)|\mathcal{H}_s] = \sigma_{aa}\tau(t-s). \]
An interesting new phenomenon of our $G$-Brownian motion is that its quadratic process $\langle B \rangle$ also has independent increments. In fact, we have

**Lemma 5.3.11** An increment of $\langle B^a \rangle$ is the quadratic variation of the corresponding increment of $B^a$, i.e., for each fixed $s \geq 0$

\[ \langle B^a \rangle_{t+s} - \langle B^a \rangle_s = \langle (B^a)^a \rangle_t, \]
where $B^a_t = B_{t+s} - B_s$, $t \geq 0$ and $(B^a)^a_t = (\mathbf{a}, B^a_t)$.

**Proof.**

\[
\begin{align*}
\langle B^a \rangle_{t+s} - \langle B^a \rangle_s &= (B^a_{t+s})^2 - 2 \int_0^{t+s} B^a_u dB^a_u - \left((B^a_s)^2 - 2 \int_0^s B^a_u dB^a_u\right) \\
&= (B^a_{t+s} - B^a_s)^2 - 2 \int_s^{t+s} (B^a_u - B^a_s) dB^a_u \\
&= (B^a_{t+s} - B^a_s)^2 - 2 \int_0^t (B^a_{s+u} - B^a_s) d(B^a_{s+u} - B^a_s) \\
&= \langle (B^a)^a \rangle_t.
\end{align*}
\]

**Lemma 5.3.12** We have

\[ \hat{\mathbb{E}}[(\langle B^a \rangle_t)^2] = \hat{\mathbb{E}}[(\langle B^a \rangle_{t+s} - \langle B^a \rangle_s)^2|\mathcal{H}_s] = \sigma_{aa}\tau t^2, \quad s, t \geq 0. \]  
(5.15)

**Proof.** We set $\varphi(t) := \hat{\mathbb{E}}[(\langle B^a \rangle_t)^2]$.

\[
\varphi(t) = \hat{\mathbb{E}}[(\langle B^a \rangle_t)^2 - 2 \int_0^t B^a_u dB^a_u)^2] \\
\leq 2\hat{\mathbb{E}}[(\langle B^a \rangle_t)^4] + 8\hat{\mathbb{E}}[(\int_0^t B^a_u dB^a_u)^2] \\
\leq 6\sigma_{aa}\tau t^2 + 8\sigma_{aa}\tau \int_0^t \hat{\mathbb{E}}[(B^a_u)^2] du \\
= 10\sigma_{aa}\tau t^2.
\]
This also implies \( \hat{E}[(B^a)_t - (B^a)_s]^2] = \varphi(t - s) \leq 10\sigma_{aa^T}(t-s)^2 \). For each \( s \in [0, t] \)

\[
\varphi(t) = \hat{E}[(B^a)_s + (B^a)_t - (B^a)_s]^2] \\
\leq \hat{E}[(B^a)_s]^2 + \hat{E}[(B^a)_t - (B^a)_s]^2 + 2\hat{E}[(B^a)_t - (B^a)_s] \langle B^a \rangle_s] \\
= \varphi(s) + \varphi(t-s) + 2\hat{E}[\hat{E}[(B^a)_t - (B^a)_s]|\mathcal{H}_s] \langle B^a \rangle_s] \\
= \varphi(s) + \varphi(t-s) + 2\sigma_{aa^T}^2 s(t-s).
\]

We set \( \delta_N = t/N, t_k^N = kt/N = k\delta_N \) for a positive integer \( N \). From the above inequalities

\[
\varphi(t_N^k) \leq \varphi(t_{N-1}^k) + \varphi(\delta_N) + 2\sigma_{aa^T}^2 t_{N-1}^k \delta_N \\
\leq \varphi(t_{N-2}^k) + 2\varphi(\delta_N) + 2\sigma_{aa^T}^2 (t_{N-1}^k + t_{N-2}^k) \delta_N \\
\ldots .
\]

We then have

\[
\varphi(t) \leq N\varphi(\delta_N) + 2\sigma_{aa^T}^2 \sum_{k=0}^{N-1} t_k^N \delta_N \leq 10t^2\sigma_{aa^T}^2 /N + 2\sigma_{aa^T}^2 \sum_{k=0}^{N-1} t_k^N \delta_N.
\]

Let \( N \to \infty \) we have \( \varphi(t) \leq 2\sigma_{aa^T}^2 \int_0^t sds = \sigma_{aa^T}^2 t^2 \). Thus \( \hat{E}[(B^a)_t]^2 \leq \sigma_{aa^T}^2 t^2 \). This, together with \( \hat{E}[\langle B^a \rangle_t^2] \geq E_0[\langle B^a \rangle_t^2] = \sigma_{aa^T}^2 t^2 \), implies (5.13). In the last step, the classical normal distribution \( P^0_1 \), or \( N(0, \gamma \gamma^T_0) \), \( \gamma_0 \in \Gamma \), is chosen such that

\[
tr[\gamma \gamma^T_0 aa^T] = \sigma_{aa^T}^2 = \sup_{\gamma \in \Gamma} tr[\gamma \gamma^T aa^T].
\]

Similarly we have

\[
\hat{E}[(B^a)_t - (B^a)_s]^3|\mathcal{H}_s] = \sigma_{aa^T}^3(t-s)^3, \\
\hat{E}[(B^a)_t - (B^a)_s]^4|\mathcal{H}_s] = \sigma_{aa^T}^4(t-s)^4. \\
(5.16)
\]

**Proposition 5.3.13** Let \( 0 \leq s \leq t, \xi \in L^1_G(\mathcal{F}_s) \) and \( X \in L^1_G(\mathcal{F}) \). Then

\[
\hat{E}[X + \xi((B^a)^2 - (B^a)_s^2)] = \hat{E}[X + \xi((B^a)^2 - (B^a)_s^2)] \\
= \hat{E}[X + \xi((B^a)_t - (B^a)_s^2)].
\]
Proof. From (5.12) and Proposition 5.2.10 we have
\[
\hat{E}[X + \xi((B^a_t)^2 - (B^a_s)^2)] = \hat{E}[X + \xi((B^a_t - B^a_s)^2 + 2 \int_s^t B^a_u dB^a_u)]
\]
\[
= \hat{E}[X + \xi((B^a_t - B^a_s)]).
\]
We also have
\[
\hat{E}[X + \xi((B^a_t)^2 - (B^a_s)^2)] = \hat{E}[X + \xi\{(B^a_t - B^a_s)^2 + 2(B^a_t - B^a_s)B^a_s]\}
\]
\[
= \hat{E}[X + \xi(B^a_t - B^a_s)^2].
\]

Example 5.3.14 We assume that in a financial market a stock price \((S_t)_{t \geq 0}\) is observed. Let \(B_t = \log(S_t), t \geq 0\), be a 1-dimensional \(G\)-Brownian motion \((d = 1)\) with \(\Gamma = [\sigma^*, \sigma^*]\), with fixed \(\sigma^* \in [0, \frac{1}{2})\) and \(\sigma^* \in [1, \infty)\). Two traders \(a\) and \(b\) in a same bank are using their own statistic to price a contingent claim \(X = \langle B \rangle_T\) with maturity \(T\). Suppose, for example, under the probability measure \(\mathbb{P}_a\) of \(a\), \(B_t(\omega)_{t \geq 0}\) is a (classical) Brownian motion whereas under \(\mathbb{P}_b\) of \(b\), \(\frac{1}{2}B_t(\omega)_{t \geq 0}\) is a Brownian motion, here \(\mathbb{P}_a\) (respectively, \(\mathbb{P}_b\)) is a classical probability measure with its linear expectation \(E_a\) (respectively, \(E_b\)) generated by the heat equation \(\partial_t u = \frac{1}{2} \partial^2_{xx} u\) (respectively, \(\partial_t u = \frac{1}{4} \partial^2_{xx} u\)). Since \(E_a\) and \(E_b\) are both dominated by \(\hat{E}\) in the sense of (3), they can be both well-defined as a linear bounded functional in \(L^1_G(F_T)\). This framework cannot be provided by just using a classical probability space because it is known that \(\langle B \rangle_T = T, \mathbb{P}_a-a.s., \) and \(\langle B \rangle_T = \frac{T}{4}, \mathbb{P}_b-a.s.\). Thus there is no probability measure on \(\Omega\) with respect to which \(\mathbb{P}_a\) and \(\mathbb{P}_b\) are both absolutely continuous. Practically this sublinear expectation \(\hat{E}\) provides a realistic tool of dynamic risk measure for a risk supervisor of the traders \(a\) and \(b\): Given a risk position \(X \in L^1_G(F_T)\) we always have \(\hat{E}[-X | \mathcal{H}_t] \geq E_a[-X | \mathcal{H}_t] \vee E_b[-X | \mathcal{H}_t]\) for the loss \(-X\) of this position. The meaning is that the supervisor uses a more sensitive risk measure. Clearly no linear expectation can play this role. The subset \(\Gamma\) represents the uncertainty of the volatility model of a risk regulator. The larger the subset \(\Gamma\), the bigger the uncertainty, thus the stronger the corresponding \(\hat{E}\).
It is worth considering to create a hierarchic and dynamic risk control system for a bank, or a banking system, in which the Chief Risk Officer (CRO) uses...
\( \hat{E} = \hat{E}^G \) for his risk measure and the Risk Officer of the \( i \)th division of the bank uses \( \hat{E}^i = \hat{E}^{G_i} \) for his one, where

\[
G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma \gamma^T A], \quad G_i(A) = \frac{1}{2} \sup_{\gamma \in \Gamma_i} \text{tr}[\gamma \gamma^T A], \quad \Gamma_i \subset \Gamma, \quad i = 1, \ldots, I.
\]

Thus \( \hat{E}^i \) is dominated by \( \hat{E} \) for each \( i \). For a large banking system we can even consider to create \( \hat{E}^{ij} = \hat{E}^{G^{ij}} \) for its \((i,j)\)th sub-division. The reason is: In general, a risk regulator’s statistics and knowledge of a specific risk position \( X \) are less than a trader who is concretely involved in the business of the financial loss \( X \).

To define the integration of a process \( \eta \in M^1_G(0,T) \) with respect to \( d\langle B^a \rangle \) we first define a mapping:

\[
Q_{0,T}(\eta) = \int_0^T \eta(s) d\langle B^a \rangle_s := \sum_{k=0}^{N-1} \xi_k(\langle B^a \rangle_{t_{k+1}} - \langle B^a \rangle_{t_k}) : M^1_G(0,T) \mapsto L^1(\mathcal{F}_T).
\]

**Lemma 5.3.15** For each \( \eta \in M^1_G(0,T) \)

\[
\hat{E}[|Q_{0,T}(\eta)|] \leq \sigma_{aaT} \int_0^T \hat{E}[|\eta_s|] ds.
\]  

(5.17)

Thus \( Q_{0,T} : M^1_G(0,T) \mapsto L^1(\mathcal{F}_T) \) is a continuous linear mapping. Consequently, \( Q_{0,T} \) can be uniquely extended to \( M^1_G(0,T) \). We still denote this mapping by

\[
\int_0^T \eta(s) d\langle B^a \rangle_s = Q_{0,T}(\eta), \quad \eta \in M^1_G(0,T).
\]

We still have

\[
\hat{E}[| \int_0^T \eta(s) d\langle B^a \rangle_s |] \leq \sigma_{aaT} \int_0^T \hat{E}[|\eta_s|] ds, \quad \forall \eta \in M^1_G(0,T).
\]  

(5.18)

**Proof.** With the help of Lemma 5.3.10 (5.17) can be checked as follows:

\[
\hat{E}[\left| \sum_{k=0}^{N-1} \xi_k(\langle B^a \rangle_{t_{k+1}} - \langle B^a \rangle_{t_k}) \right|] \leq \sum_{k=0}^{N-1} \hat{E}[|\xi_k|] \cdot \hat{E}[|\langle B^a \rangle_{t_{k+1}} - \langle B^a \rangle_{t_k}| \mathcal{H}_k] = \sum_{k=0}^{N-1} \hat{E}[|\xi_k|] \sigma_{aa}(t_{k+1} - t_k) = \sigma_{aaT} \int_0^T \hat{E}[|\eta_s|] ds.
\]
We have the following isometry.

**Proposition 5.3.16** Let \( \eta \in M^2_G(0, T) \). We have

\[
\hat{E}[\left( \int_0^T \eta(s)dB_s^a \right)^2] = \hat{E}[\int_0^T \eta^2(s)d\langle B^a \rangle_s].
\]  

(5.19)

**Proof.** We first consider \( \eta \in M^2_{G,0}(0, T) \) with the form

\[
\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t)
\]

and thus \( \int_0^T \eta(s)dB_s^a := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}}^a - B_{t_k}^a). \) By Proposition 5.2.10 we have

\[
\hat{E}[X + 2\xi_k(B_{t_{k+1}}^a - B_{t_k}^a)\xi_l(B_{t_{l+1}}^a - B_{t_l}^a)] = \hat{E}[X], \text{ for } X \in L^1_G(F), l \neq k.
\]

Therefore

\[
\hat{E}[\left( \int_0^T \eta(s)dB_s^a \right)^2] = \hat{E}\left[ \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}}^a - B_{t_k}^a) \right]^2 = \hat{E}\left[ \sum_{k=0}^{N-1} \xi_k^2(B_{t_{k+1}}^a - B_{t_k}^a)^2 \right].
\]

This, together with Proposition 5.3.13 yields that

\[
\hat{E}[\left( \int_0^T \eta(s)dB_s^a \right)^2] = \hat{E}\left[ \sum_{k=0}^{N-1} \xi_k^2(\langle B^a \rangle_{t_{k+1}} - \langle B^a \rangle_{t_k}) \right] = \hat{E}[\int_0^T \eta^2(s)d\langle B^a \rangle_s].
\]

Thus (5.19) holds for \( \eta \in M^2_{G,0}(0, T) \). Then we can continuously extend this equality to the case \( \eta \in M^2_G(0, T) \) and obtain (5.19). ■

### 5.3.4 Mutual variation processes for \( G \)-Brownian motion

Let \( a = (a_1, \ldots, a_d)^T \) and \( \bar{a} = (\bar{a}_1, \ldots, \bar{a}_d)^T \) be two given vectors in \( \mathbb{R}^d \). We then have their quadratic variation processes \( \langle B^a \rangle \) and \( \langle B^{\bar{a}} \rangle \). We then can define their mutual variation processes by

\[
\langle B^a, B^{\bar{a}} \rangle_t := \frac{1}{4}[\langle B^a + B^{\bar{a}} \rangle_t - \langle B^a - B^{\bar{a}} \rangle_t]
\]

\[
= \frac{1}{4}[\langle B^{a+\bar{a}} \rangle_t - \langle B^{a-\bar{a}} \rangle_t].
\]
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Since \( \langle B^{a-\bar{a}} \rangle = \langle B^{\bar{a}} - a \rangle = \langle -B^{a-\bar{a}} \rangle \) we see that \( \langle B^{a}, B^{\bar{a}} \rangle_t = \langle B^{\bar{a}}, B^{a} \rangle_t \). In particular we have \( \langle B^{a}, B^{a} \rangle = \langle B^{a} \rangle \). Let \( \pi^N_t, N = 1, 2, \ldots \), be a sequence of partitions of \([0, t] \). We observe that

\[
\sum_{k=0}^{N-1} (B_{i_{k+1}^N}^{a} - B_{i_{k}^N}^{a}) (B_{i_{k+1}^N}^{\bar{a}} - B_{i_{k}^N}^{\bar{a}}) = \frac{1}{4} \sum_{k=0}^{N-1} [(B_{i_{k+1}^N}^{a+\bar{a}} - B_{i_{k}^N}^{a+\bar{a}})^2 - (B_{i_{k+1}^N}^{a-\bar{a}} - B_{i_{k}^N}^{a-\bar{a}})^2].
\]

Thus as \( \mu(\pi^N_t) \to 0 \) we have

\[
\lim_{N \to 0} \sum_{k=0}^{N-1} (B_{i_{k+1}^N}^{a} - B_{i_{k}^N}^{a}) (B_{i_{k+1}^N}^{\bar{a}} - B_{i_{k}^N}^{\bar{a}}) = \langle B^{a}, B^{\bar{a}} \rangle_t.
\]

We also have

\[
\langle B^{a}, B^{\bar{a}} \rangle_t = \frac{1}{4} [\langle B^{a+\bar{a}} \rangle_t - \langle B^{a-\bar{a}} \rangle_t]
\]

\[
= \frac{1}{4} [(B_t^{a+\bar{a}})^2 - 2 \int_0^t B_s^{a+\bar{a}} dB_s^{a+\bar{a}} - (B_t^{a-\bar{a}})^2 + 2 \int_0^t B_s^{a-\bar{a}} dB_s^{a-\bar{a}}]
\]

\[
= B_t^{a} B_t^{\bar{a}} - \int_0^t B_s^{a} dB_s^{\bar{a}} - \int_0^t B_s^{\bar{a}} dB_s^{a}.
\]

Now for each \( \eta \in M^1_G(0, T) \) we can consistently define

\[
\int_0^T \eta_s d \langle B^{a}, B^{\bar{a}} \rangle_s = \frac{1}{4} \int_0^T \eta_s d \langle B^{a+\bar{a}} \rangle_s - \frac{1}{4} \int_0^T \eta_s d \langle B^{a-\bar{a}} \rangle_s.
\]

**Lemma 5.3.17** Let \( \eta^N \in M^1_G(0, T), N = 1, 2, \ldots \), be of form

\[
\eta^N_t(\omega) = \sum_{k=0}^{N-1} \xi^N_k(\omega) I_{(i_{k}^N, i_{k+1}^N)}(t)
\]

with \( \mu(\pi^N_T) \to 0 \) and \( \eta^N \to \eta \) in \( M^1_G(0, T) \) as \( N \to \infty \). Then we have the following convergence in \( L^1_G(\mathcal{F}_T) \):

\[
\int_0^T \eta^N(s) d \langle B^{a}, B^{\bar{a}} \rangle_s := \sum_{k=0}^{N-1} \xi^N_k(B_{i_{k+1}^N}^{a} - B_{i_{k}^N}^{a}) (B_{i_{k+1}^N}^{\bar{a}} - B_{i_{k}^N}^{\bar{a}})
\]

\[
\to \int_0^T \eta(s) d \langle B^{a}, B^{\bar{a}} \rangle_s.
\]
For notational simplification we denote $B^i = B^{e_i}$, the $i$-th coordinate of the $G$–Brownian motion $B$, under a given orthonormal basis $(e_1, \cdots , e_d)$ of $\mathbb{R}^d$. We denote by

$$\langle (B^i)_t \rangle_{ij} = \langle B^i, B^j \rangle_t.$$ 

\langle B \rangle_t, \ t \geq 0, is an $\mathbb{S}(n)$-valued process. Since

\begin{align*}
\hat{E}(\langle B \rangle_t, A) &= 2G(A)t, \ A \in \mathbb{S}(n).
\end{align*}

We have

\begin{align*}
\hat{E}[\langle B \rangle_t, A] &= 2G(A)t, \ A \in \mathbb{S}(n).
\end{align*}

Now we set a function

$$v(t, X) = \hat{E}[\varphi(X + \langle B \rangle_t)], \ (t, X) \in [0, \infty) \times \mathbb{S}(n).$$

**Proposition 5.3.18** $v$ solves the following first order PDE:

$$\partial_t v = 2G(Dv), \ v|_{t=0} = \varphi,$$

where $Dv = (\partial_{x_{ij}})_{i,j=1}^d$. We also have

$$v(t, X) = \sup_{\Lambda \in \Gamma} \varphi(X + t\Lambda).$$

**Proof.** (Sketch) We have

\begin{align*}
v(t + \delta, X) &= \hat{E}[\varphi(X + \langle B \rangle_\delta + \langle B \rangle_{t+\delta} - \langle B \rangle_\delta)] \\
&= \hat{E}[v(t, X + \langle B \rangle_\delta)].
\end{align*}

The rest part of the proof is as in one dimensional case. ■

**Corollary 5.3.19** We have

$$\langle B \rangle_t \in t\Gamma := \{ t \times \gamma : \gamma \in \Gamma \}.$$

or, equivalently $d_{\Gamma}(\langle B \rangle_t) = 0$, where $d_U(x) = \inf \{|x - y| : y \in U\}$.

**Proof.** Since

$$\hat{E}[d_{\Gamma}(\langle B \rangle_t)] = \sup_{\Lambda \in \Gamma} d_{\Gamma}(t\Lambda) = 0,$$

It follows that $d_{\Gamma}(\langle B \rangle_t) = 0$ in $L^1_G(\mathcal{F})$. ■
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5.3.5 Itô’s formula for G–Brownian motion

We have the corresponding Itô’s formula of Φ(X_t) for a “G-Itô process” X. For simplification we only treat the case where the function Φ is sufficiently regular.

Lemma 5.3.20 Let Φ ∈ C^2(\mathbb{R}^n) be bounded with bounded derivatives and \{\partial^{2}_{\mu\nu} \Phi\}_{\mu,\nu=1}^{n} are uniformly Lipschitz. Let s ∈ [0,T] be fixed and let X = (X^1, ⋯, X^n)^T be an n–dimensional process on [s,T] of the form

\[ X_t^\nu = X_s^\nu + \alpha^\nu(t-s) + \eta^\nu ij(\langle B^i, B^j \rangle_t - \langle B^i, B^j \rangle_s) + \beta^\nu j (B^j_t - B^j_s), \]

where, for \nu = 1, ⋯, n, i, j = 1, ⋯, d, \alpha^\nu, \eta^\nu ij and \beta^\nu j are bounded elements of L^2_G(F_s) and X_s = (X^1_s, ⋯, X^n_s)^T is a given \mathbb{R}^n–vector in L^2_G(F_s).

Then we have

\[ \Phi(X_t) - \Phi(X_s) = \int_{s}^{t} \partial_{\nu} \Phi(X_u) \beta^\nu j dB^j_u + \int_{s}^{t} \partial_{\nu} \Phi(X_u) \alpha^\nu du \] (5.20)

\[ + \int_{s}^{t} [\partial_{\nu} \Phi(X_u) \eta^\nu ij + \frac{1}{2} \partial^{2}_{\mu\nu} \Phi(X_u) \beta^\nu i \beta^\nu j] d \langle B^i, B^j \rangle_u. \]

Here we use the Einstein convention, i.e., the above repeated indices \mu, \nu, i and j (but not k) imply the summation.

Proof. For each positive integer N we set \delta = (t - s)/N and take the partition

\[ \pi_s^N = \{t_0^N, t_1^N, ⋯, t_N^N\} = \{s, s + \delta, ⋯, s + N\delta = t\}. \]

We have

\[ \Phi(X_t) - \Phi(X_s) = \sum_{k=0}^{N-1} [\Phi(X_{t_{k+1}^N}) - \Phi(X_{t_k^N})] \]

\[ = \sum_{k=0}^{N-1} [\partial_{\mu} \Phi(X_{t_k^N})(X_{t_{k+1}^N}^\mu - X_{t_k^N}^\mu) \]

\[ + \frac{1}{2} [\partial^{2}_{\mu\nu} \Phi(X_{t_k^N})(X_{t_{k+1}^N}^\mu - X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu) + \eta_k^N]] \] (5.21)

where

\[ \eta_k^N = [\partial^{2}_{\mu\nu} \Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N}))(X_{t_{k+1}^N}^\mu - X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu) \]

\[ - \partial^{2}_{\mu\nu} \Phi(X_{t_k^N})(X_{t_{k+1}^N}^\mu - X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu) ] \]
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with \( \theta_k \in [0, 1] \). We have

\[
\mathbb{E}[|\eta_{t_k}^N|] = \mathbb{E}[|\partial_{x_{\mu,x}}^2 \Phi(X_{t_k}^N + \theta_k (X_{t_{k+1}}^N - X_{t_k}^N)) - \partial_{x_{\mu,x}}^2 \Phi(X_{t_k}^N)|] \\
\times (X_{t_{k+1}}^\mu - X_{t_k}^\mu)(X_{t_{k+1}}^\nu - X_{t_k}^\nu)] \\
\leq c \mathbb{E}[|X_{t_{k+1}}^N - X_{t_k}^N|] \leq C|\delta^3 + \delta^{3/2}|,
\]

where \( c \) is the Lipschitz constant of \( \{\partial_{x_{\mu,x}}^2 \Phi\}_{\mu, \nu = 1}^d \). In the last step we use Example 5.2.13 and (5.16). Thus \( \sum_k \mathbb{E}[|\eta_{t_k}^N|] \to 0 \). The rest terms in the summation of the right side of (5.21) are \( \xi_t^N \) with

\[
\xi_t^N = \sum_{k=0}^{N-1} \{ \partial_{x^\mu} \Phi(X_{t_k}^N)[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu i j}(\langle B^i, B^j \rangle_{t_{k+1}^N} - \langle B^i, B^j \rangle_{t_k^N})] \\
+ \beta^{\mu ij}(B_{t_{k+1}^N}^i - B_{t_k^N}^i) + \frac{1}{2} \partial_{x_{\mu,x}^\nu}^2 \Phi(X_{t_k}^N) \beta^{\mu i} \beta^{\nu j}(B_{t_{k+1}^N}^i - B_{t_k^N}^i) \}
\]

and

\[
\zeta_t^N = \frac{1}{2} \sum_{k=0}^{N-1} \partial_{x_{\mu,x}^\nu}^2 \Phi(X_{t_k}^N)[\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\rho \mu i j}(\langle B^i, B^j \rangle_{t_{k+1}^N} - \langle B^i, B^j \rangle_{t_k^N})] \\
\times [\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\rho \mu i j}(\langle B^i, B^j \rangle_{t_{k+1}^N} - \langle B^i, B^j \rangle_{t_k^N})] \\
+ \eta^{\mu i j}(\langle B^i, B^j \rangle_{t_{k+1}^N} - \langle B^i, B^j \rangle_{t_k^N})] \beta^{\nu l}(B_{t_{k+1}^N}^l - B_{t_k^N}^l).
\]

We observe that, for each \( u \in [t_k^N, t_{k+1}^N] \)

\[
\mathbb{E}[|\partial_{x^\mu} \Phi(X_u) - \sum_{k=0}^{N-1} \partial_{x^\mu} \Phi(X_{t_k}^N)I_{[t_k^N, t_{k+1}^N]}(u)|^2] \\
= \mathbb{E}[|\partial_{x^\mu} \Phi(X_u) - \partial_{x^\mu} \Phi(X_{t_k}^N)|^2] \\
\leq c^2 \mathbb{E}[|X_u - X_{t_k}^N|^2] \leq C|\delta + \delta^2|.
\]

Thus \( \sum_{k=0}^{N-1} \partial_{x^\mu} \Phi(X_{t_k}^N)I_{[t_k^N, t_{k+1}^N]}(\cdot) \) tends to \( \partial_{x^\mu} \Phi(X) \) in \( M_G^2(0, T) \). Similarly,

\[
\sum_{k=0}^{N-1} \partial_{x_{\mu,x}^\nu}^2 \Phi(X_{t_k}^N)I_{[t_k^N, t_{k+1}^N]}(\cdot) \to \partial_{x_{\mu,x}^\nu}^2 \Phi(X) \) in \( M_G^2(0, T) \).
Let $N \to \infty$. From Lemma 5.3.17 as well as the definitions of the integrations of $dt$, $dB_t$ and $d\langle B \rangle_t$ the limit of $\xi^N_t$ in $L^2_G(\mathcal{F}_t)$ is just the right hand side of (5.20). By the next Remark we also have $\xi^N_t \to 0$ in $L^2_G(\mathcal{F}_t)$. We then have proved (5.20). □

**Remark 5.3.21** In the proof of $\xi^N_t \to 0$ in $L^2_G(\mathcal{F}_t)$, we use the following estimates: for $\psi^N \in \mathcal{M}_{G}^{1,0}(0,T)$ such that $\psi^N_t = \sum_{k=0}^{N-1} \xi^N_{t_k} (t_{k+1}^N - t_k^N)(t)$, and $\pi^N_T = \{0 \leq t_0, \ldots, t_N = T\}$ with $\lim_{N \to \infty} \mu(\pi^N_T) = 0$ and $\sum_{k=0}^{N-1} \tilde{E}[|\xi^N_{t_k}|] (t_{k+1}^N - t_k^N) \leq C$, for all $N = 1, 2, \cdots$, we have $\tilde{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{k+1}^N - t_k^N)^2] \to 0$ and, for any fixed $a, \bar{a} \in \mathbb{R}^d$,

$$\tilde{E}[\sum_{k=0}^{N-1} \xi^N_k (\langle B^a \rangle_{t_{k+1}^N} - \langle B^a \rangle_{t_k^N})^2] \leq \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k| \cdot \tilde{E}[\langle (B^a)^a \rangle_{t_{k+1}^N} - \langle (B^a)^a \rangle_{t_k^N}] | \mathcal{H}_{t_k^N}]]$$

$$= \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k|] \sigma_{aa}^2 (t_{k+1}^N - t_k^N)^2 \to 0,$$

$$\tilde{E}[\sum_{k=0}^{N-1} \xi^N_k (\langle B^a \rangle_{t_{k+1}^N} - \langle B^a \rangle_{t_k^N})(t_{k+1}^N - t_k^N)]$$

$$\leq \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k|] (t_{k+1}^N - t_k^N) \cdot \tilde{E}[\langle (B^a)^a \rangle_{t_{k+1}^N} - \langle (B^a)^a \rangle_{t_k^N}] | \mathcal{H}_{t_k^N}]$$

$$= \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k|] \sigma_{aa}^2 (t_{k+1}^N - t_k^N)^2 \to 0,$$

as well as

$$\tilde{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{k+1}^N - t_k^N) (B^a_{t_{k+1}^N} - B^a_{t_k^N})] \leq \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k|] (t_{k+1}^N - t_k^N) \tilde{E}[|B^a_{t_{k+1}^N} - B^a_{t_k^N}|]$$

$$= \sqrt{\frac{2\sigma_{aa}}{\pi}} \sum_{k=0}^{N-1} \tilde{E}[|\xi^N_k|] (t_{k+1}^N - t_k^N)^{3/2} \to 0.$$
and

\[ \hat{\mathbb{E}}[\sum_{k=0}^{N-1} \xi_k^N (\langle B^a \rangle_{t_k^{N}}^N - \langle B^a \rangle_{t_k^N}) (B_{t_{k+1}^N}^\bar{a} - B_{t_k^N}^\bar{a})] \]

\[ \leq \sum_{k=0}^{N-1} \hat{\mathbb{E}}[|\xi_k^N|] \hat{\mathbb{E}}[|\langle B^a \rangle_{t_{k+1}^N}^N - \langle B^a \rangle_{t_k^N}|] \]

\[ \leq \sum_{k=0}^{N-1} \hat{\mathbb{E}}[|\xi_k^N|] \hat{\mathbb{E}}[(\langle B^a \rangle_{t_{k+1}^N}^N - \langle B^a \rangle_{t_k^N})^2]^{1/2} \hat{\mathbb{E}}[B_{t_{k+1}^N}^\bar{a} - B_{t_k^N}^\bar{a}]^{1/2} \]

\[ = \sum_{k=0}^{N-1} \hat{\mathbb{E}}[|\xi_k^N|] \sigma_1^{1/2} \sigma_\bar{a}^{1/2} (t_{k+1}^N - t_k^N)^{3/2} \rightarrow 0. \]

We now can claim our \( G \)-Itô’s formula. Consider

\[ X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu \, ds + \int_0^t \eta_s^{\nu ij} \, dB_s^i + \int_0^t \beta_s^{\nu j} \, dB_s^j \]

**Proposition 5.3.22** Let \( \alpha^\nu, \beta^{\nu ij} \) and \( \eta^{\nu ij}, \nu = 1, \ldots, n, i, j = 1, \ldots, d \) be bounded processes of \( M^2_G(0, T) \). Then for each \( t \geq 0 \) and \( \Phi \in L^2_G(\mathcal{F}_t) \) we have

\[ \Phi(X_t) - \Phi(X_0) = \int_0^t \partial_{x^\nu} \Phi(X_u) \beta_u^{\nu j} dB_u^j + \int_0^t \partial_{x^\nu} \Phi(X_u) \alpha_u^\nu du + \int_0^t \partial_{x^\nu} \Phi(X_u) \eta_u^{\nu ij} + \frac{1}{2} \partial_{x^\nu}^{2, \nu} \Phi(X_u) \beta_u^{\nu i} \beta_u^{\nu j} dB_u^i dB_u^j \]

**Proof.** We first consider the case where \( \alpha, \eta \) and \( \beta \) are step processes of the form

\[ \eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t). \]

From the above Lemma, it is clear that (5.22) holds true. Now let

\[ X_t^{\nu, N} = X_0^\nu + \int_0^t \alpha_s^{\nu, N} \, ds + \int_0^t \eta_s^{\nu ij, N} d\langle B^i, B^j \rangle_s + \int_0^t \beta_s^{\nu j, N} dB_s^j \]
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where \( \alpha^N, \eta^N \) and \( \beta^N \) are uniformly bounded step processes that converge to \( \alpha, \eta \) and \( \beta \) in \( M^2_G(0, T) \) as \( N \to \infty \). From Lemma 5.3.20

\[
\Phi(X_t^N) - \Phi(X_0) = \int_0^t \frac{\partial x}{\nu} \Phi(X_u^N) \beta_{u}^{\nu i,N} dB_u^i + \int_0^t \frac{\partial x}{\nu} \Phi(X_u^N) \alpha_{u}^{\nu,N} du \\
+ \int_0^t \left[ \frac{\partial^2 x}{\nu x} \Phi(X_u^N) \eta_{u}^{\nu i,N} + \frac{1}{2} \frac{\partial^2 x}{x^2} \Phi(X_u^N) \beta_{u}^{\mu i,N} \beta_{u}^{\nu j,N} \right] d\langle B^i, B^j \rangle_u
\]

(5.23)

Since

\[
\hat{E}[|X_t^{N,\mu} - X_t^{\mu}|^2] \\
\leq C \int_0^T \{ \hat{E}[(\alpha_s^{\mu,N} - \alpha_s^{\mu})^2] + \hat{E}[|\eta_s^{\mu,N} - \eta_s^{\mu}|^2] + \hat{E}[(\beta_s^{\mu,N} - \beta_s^{\mu})^2] \} ds
\]

We then can prove that, in \( M^2_G(0, T) \),

\[
\frac{\partial x}{\nu} \Phi(X_u^N) \eta_{u}^{\nu i,N} \to \frac{\partial x}{\nu} \Phi(X) \eta_{t}^{\nu i} \\
\frac{\partial^2 x}{x^2} \Phi(X_u^N) \beta_{u}^{\mu i,N} \beta_{u}^{\nu j,N} \to \frac{\partial^2 x}{x^2} \Phi(X) \beta_{t}^{\mu i} \beta_{t}^{\nu j} \\
\frac{\partial x}{x} \Phi(X_u^N) \alpha_{u}^{\nu,N} \to \frac{\partial x}{x} \Phi(X) \alpha_{t}^{\nu} \\
\frac{\partial x}{x} \Phi(X_u^N) \beta_{u}^{\nu j,N} \to \frac{\partial x}{x} \Phi(X) \beta_{t}^{\nu j}
\]

We then can pass limit in both sides of (5.23) to get (5.22). ■

Example 5.3.23

\[
(B_t, AB_t) = \sum_{i,j} A_{ij} B_t^i B_t^j = 2 \sum_{i,j} [A_{ij} \int_0^t B_t^i dB_t^j + A_{ij} \langle B^i, B^j \rangle_t] \\
= 2 \sum_{i,j} [A_{ij} \int_0^t B_t^i dB_t^j + \langle A, \langle B \rangle_t \rangle].
\]

Thus

\[
\hat{E}[(A, \langle B \rangle_t)] = \hat{E}[(B_t, AB_t)] = 2G(A)t.
\]
5.4 $G$–martingales, $G$–convexity and Jensen’s inequality

5.4.1 The notion of $G$–martingales

We now give the notion of $G$–martingales:

**Definition 5.4.1** A process $(M_t)_{t \geq 0}$ is called a $G$–martingale (respectively, $G$–supermartingale, $G$–submartingale) if for each $0 \leq s \leq t < \infty$, we have $M_t \in L^1_G(\mathcal{F}_t)$ and

$$\mathbb{E}[M_t | \mathcal{H}_s] = M_s, \quad (\text{resp.,} \leq M_s, \geq M_s).$$

It is clear that for a fixed $X \in L^1_G(\mathcal{F})$ $\mathbb{E}[X | \mathcal{H}_t]_{t \geq 0}$ is a $G$–martingale. In general how to characterize a $G$–martingale or a $G$–supermartingale is still a very interesting problem. But the following example gives an important characterization:

**Example 5.4.2** Let $M_0 \in \mathbb{R}$, $\varphi = (\varphi^i)_{i=1}^d \in M^2_G(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M^2_G(0,T;\mathbb{S}(d))$ be given and let

$$M_t = M_0 + \int_0^t \varphi_u^i dB_u^i + \int_0^t \eta^{ij}_u d\langle B^i, B^j \rangle_u - \int_0^t 2G(\eta_u) du, \quad t \in [0,T].$$

Then $M$ is a $G$–martingale on $[0,T]$. To consider this it suffices to prove the case $\eta \in M^2_G(0,T;\mathbb{S}(d))$, i.e.,

$$\eta_t = \sum_{k=0}^{N-1} \eta_k I_{[t_k,t_{k+1})}(t).$$

We have for $s \in [t_{N-1}, t_N]$,

$$\mathbb{E}[M_t | \mathcal{H}_s] = M_s + \mathbb{E}[\eta^{ij}_{t_{N-1}}(\langle B^i, B^j \rangle_t - \langle B^i, B^j \rangle_s) - 2G(\eta_{t_{N-1}})(t-s)|\mathcal{H}_s]$$

$$= M_s + \mathbb{E}[\eta^{ij}_{t_{N-1}}(B^i_t - B^i_s)(B^j_t - B^j_s)|\mathcal{H}_s] - 2G(\eta_{t_{N-1}})(t-s)$$

$$= M_s.$$

In the last step we apply the relation (5.6). We then can repeat this procedure, step by step backwardly, to prove the result for any $s \in [0,t_{N-1}]$.

**Remark 5.4.3** It is worth mentioning that for a $G$–martingale, in general, $-M$ is not a $G$–martingale. But in the above example when $\eta \equiv 0$ then $-M$ is still a $G$–martingale. This makes an essential difference of the $dB$ part and the $d\langle B \rangle$ part of a $G$–martingale.
5.4.2 \textit{G–convexity and Jensen’s inequality for G–expectation}

A very interesting question is whether the well–known Jensen’s inequality still holds for \textit{G–expectation}. In the framework of \textit{g–expectation} this problem was investigated in [4] in which a counterexample is given to indicate that, even for a linear function which is obviously convex, Jensen’s inequality for \textit{g–expectation} generally does not hold. Stimulated by this example [10] proved that Jensen’s inequality holds for any convex function under a \textit{g–expectation} if and only if the corresponding generating function \textit{g} = \textit{g}(t, z) is superhomogeneous in \textit{z}. Here we will discuss this problem from a quite different point of view. We will define a new notion of convexity:

\textbf{Definition 5.4.4} A \textit{C^2–function} \textit{h : R} \rightarrow \textit{R} is called \textit{G–convex} if the following condition holds for each \((y, z, A) \in \textit{R} \times \textit{R}^d \times \textit{S}(d)\):

\[ G(h'(y)A + h''(y)zz^T) - h'(y)G(A) \geq 0, \]  

(5.24)

where \(h'\) and \(h''\) denote the first and the second derivatives of \(h\), respectively.

It is clear that in the special situation where \(G(D^2u) = \frac{1}{2} \Delta u\) a \textit{G–convex function} becomes is a convex function in the classical sense.

\textbf{Lemma 5.4.5} The following two conditions are equivalent:

\textbf{(i)} The function \(h\) is \textit{G–convex}.

\textbf{(ii)} The following Jensen inequality holds: For each \(T \geq 0\),

\[ \hat{E}[h(\varphi(B_T))] \geq h(\hat{E}[\varphi(B_T)]), \]  

(5.25)

for each \textit{C^2–function} \(\varphi\) such that \(h(\varphi(B_T))\) and \(\varphi(B_T)\) \textit{\in L^1_{G}(\mathcal{F}_T)}\).

\textbf{Proof.} (i) \(\Rightarrow\) (ii): From the definition \(u(t, x) := P_t^G[\varphi](x) = \hat{E}[\varphi(x + B_t)]\) solves the nonlinear heat equation (5.2). Here we only consider the case where \(u\) is a \textit{C^{1,2}}-function. Otherwise we can use the language of viscosity solution. By simple calculation we have

\[ \partial_t h(u(t, x)) = h'(u)\partial_t u = h'(u(t, x))G(D^2u(t, x)), \]

or

\[ \partial_t h(u(t, x)) - G(D^2h(u(t, x)))) - f(t, x) = 0, \quad h(u(0, x)) = h(\varphi(x)), \]
where we denote
\[ f(t, x) = h'(u(t, x))G(D^2u(t, x)) - G(D^2h(u(t, x))). \]

Since \( h \) is \( G \)-convex it follows that \( f \leq 0 \) and thus \( h(u) \) is a \( G \)-subsolution. It follows from the maximum principle that \( h(P^G_t(\varphi)(x)) \leq P^G_t(h(\varphi))(x) \). In particular \((5.25)\) holds. Thus we have (ii).

(ii) \( \implies \) (i): For a fixed \((y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)\) we set \( \varphi(x) := y + \langle x, z \rangle + \frac{1}{2} \langle Ax, x \rangle \). From the definition of \( P^G_t \) we have \( \partial_t(P^G_t(\varphi)(x))|_{t=0} = G(D^2\varphi)(x) \).

With (ii) we have
\[ h(P^G_t(\varphi)(x)) \leq P^G_t(h(\varphi))(x). \]

Thus, for \( t > 0 \),
\[ \frac{1}{t}[h(P^G_t(\varphi)(x)) - h(\varphi(x))] \leq \frac{1}{t}[P^G_t(h(\varphi))(x) - h(\varphi(x))] \]

We then let \( t \) tend to 0:
\[ h'(\varphi(x))G(D^2\varphi(x)) \leq G(D^2_{xx}h(\varphi(x))). \]

Since \( D_x\varphi(x) = z + Ax \) and \( D^2_{xx}\varphi(x) = A \) we then set \( x = 0 \) and obtain \((5.24)\). 

**Proposition 5.4.6** The following two conditions are equivalent:
(i) the function \( h \) is \( G \)-convex.
(ii) The following Jensen inequality holds:
\[ \hat{E}[h(X)|\mathcal{H}_t] \geq h(\hat{E}[X|\mathcal{H}_t]), \quad t \geq 0, \quad (5.26) \]
for each \( X \in L^1_G(\mathcal{F}) \) such that \( h(X) \in L^1_G(\mathcal{F}) \).

**Proof.** The part (ii) \( \implies \) (i) is already provided by the above lemma. We can also apply this lemma to prove \((5.26)\) for the case \( X \in L^0_{ip}(\mathcal{F}) \) of the form \( X = \varphi(B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}) \) by using the procedure of the definition of \( \hat{E}[] \) and \( \hat{E}[]|\mathcal{H}_t \) given in Definitions \( 5.2.2 \) and \( 5.2.3 \), respectively. We then can extend this Jensen’s inequality, under the norm \( ||\cdot|| = \hat{E}[||\cdot||] \) to the general situation. 

\[ \blacksquare \]
5.5. STOCHASTIC DIFFERENTIAL EQUATIONS

Remark 5.4.7 The above notion of \( G \)-convexity can be also applied to the case where the nonlinear heat equation (5.2) has a more general form:

\[
\frac{\partial u}{\partial t} - G(u, \nabla u, D^2 u) = 0, \quad u(0, x) = \psi(x)
\]

(see Examples 4.3, 4.4 and 4.5 in [55]). In this case a \( C^2 \)-function \( h : \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( G \)-convex if the following condition holds for each \((y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d)\):

\[
G(y, h'(y)z, h''(y)zz^T) - h'(y)G(y, z, A) \geq 0.
\]

We don’t need the subadditivity and/or positive homogeneity of \( G(y, z, A) \).

A particularly interesting situation is the case of \( g \)-expectation for a given generating function \( g = g(y, z), \ (y, z) \in \mathbb{R} \times \mathbb{R}^d \), in this case we have the following \( g \)-convexity:

\[
\frac{1}{2} h''(y)|z|^2 + g(h(y), h'(y)z) - h'(y)g(y, z) \geq 0.
\]

(5.28)

This situation is systematically studied in Jia and Peng [37].

Example 5.4.8 Let \( h \) be a \( G \)-convex function and \( X \in L^1_G(F) \) such that \( h(X) \in L^1_G(F) \). Then \( Y_t = h(\hat{\mathbb{E}}(X|\mathcal{H}_t)), \ t \geq 0, \) is a \( G \)-submartingale: For each \( s \leq t, \)

\[
\hat{\mathbb{E}}[Y_t|\mathcal{H}_s] = \hat{\mathbb{E}}[h(\hat{\mathbb{E}}(X|\mathcal{F}_t)|\mathcal{F}_s)] \geq h(\hat{\mathbb{E}}(X|\mathcal{F}_s)) = Y_s.
\]

5.5 Stochastic differential equations

We consider the following SDE driven by \( G \)-Brownian motion.

\[
X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h_{ij}(X_s)d\langle B^i, B^j \rangle_s + \int_0^t \sigma_j(X_s)dB^j_s, \quad t \in [0, T],
\]

(5.29)

where the initial condition \( X_0 \in \mathbb{R}^n \) is given and

\[
b, h_{ij}, \sigma_j : \mathbb{R}^n \mapsto \mathbb{R}^n
\]

are given Lipschitz functions, i.e., \(|\varphi(x) - \varphi(x')| \leq K|x - x'|\), for each \( x, x' \in \mathbb{R}^n, \ \varphi = b, \eta_{ij} \) and \( \sigma_j \), respectively. Here the horizon \([0, T]\) can be
arbitrarily large. The solution is a process \( X \in M^2_G(0, T; \mathbb{R}^n) \) satisfying the above SDE. We first introduce the following mapping on a fixed interval \([0, T]\):

\[
\Lambda_t(Y) := Y \in M^2_G(0, T; \mathbb{R}^n) \mapsto M^2_G(0, T; \mathbb{R}^n)
\]

by setting \( \Lambda \equiv X_t, t \in [0, T] \), with

\[
\Lambda_t(Y) = X_0 + X_0 + \int_0^t b(Y_s)ds + \int_0^t h_{ij}(Y_s)d\langle B^i, B^j \rangle_s + \int_0^t \sigma_j(Y_s)dB^j_s.
\]

We immediately have

**Lemma 5.5.1** For each \( Y, Y' \in M^2_G(0, T; \mathbb{R}^n) \) we have the following estimate:

\[
\hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \leq C \int_0^t \hat{\mathbb{E}}[|Y_s - Y'_s|^2]ds, \ t \in [0, T],
\]

where the constant \( C \) depends only on \( K, \Gamma \) and the dimension \( n \).

**Proof.** This is a direct consequence of the inequalities (5.9), (5.11) and (5.18). \( \blacksquare \)

We now prove that SDE (5.29) has a unique solution. By multiplying \( e^{-2Ct} \) on both sides of the above inequality and then integrating them on \([0, T]\), it follows that

\[
\int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt \leq C \int_0^T e^{-2Ct} \int_0^t \hat{\mathbb{E}}[|Y_s - Y'_s|^2]dsdt
\]

\[
= C \int_0^T \int_s^T e^{-2Ct}dt \hat{\mathbb{E}}[|Y_s - Y'_s|^2]ds
\]

\[
= (2C)^{-1}C \int_0^T (e^{-2Cs} - e^{-2CT})\hat{\mathbb{E}}[|Y_s - Y'_s|^2]ds.
\]

We then have

\[
\int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt \leq \frac{1}{2} \int_0^T \hat{\mathbb{E}}[|Y_t - Y'_t|^2]e^{-2Ct}dt.
\]

We observe that the following two norms are equivalent in \( M^2_G(0, T; \mathbb{R}^n) \)

\[
\int_0^T \hat{\mathbb{E}}[|Y_t|^2]dt \sim \int_0^T \hat{\mathbb{E}}[|Y_t'|^2]e^{-2Ct}dt.
\]

From this estimate we can obtain that \( \Lambda(Y) \) is a contraction mapping. Consequently, we have
Theorem 5.5.2 There exists a unique solution \( X \in M^2_G(0, T; \mathbb{R}^n) \) of the stochastic differential equation (5.29).
CHAPTER 5. VECTOR VALUED G-BROWNIAN MOTION
Chapter 6

Other Topics and Applications

6.1 Nonlinear Feynman-Kac formula

Consider SDE:

\[ dX_t^{t,x} = b(X_t^{t,x})dt + h(X_t^{t,x})dB_t, \quad X_t^{t,x} = x. \]

\[ Y_s^{t,x} = \mathbb{E}[\Phi(X_T^{t,x}) + \int_s^T g(X_r^{t,x}, Y_r^{t,x})dr + \int_s^T f(X_r^{t,x}, Y_r^{t,x})d\langle B \rangle_r | \mathcal{H}_s]. \]

It is clear that \( u(t, x) := Y_t^{t,x} \in \mathcal{H}_t \), thus it is a deterministic function of \((t, x)\). \( u(t, x) \) solves

\[
\begin{align*}
\partial_t u + \sup \{(b(x) + h(x)\gamma, \nabla u) + \frac{1}{2} (\sigma(x)\gamma \sigma^T(x), D^2 u) & + g(x, u) + f(x, u)\gamma \} = 0, \\
u|_{t=T} &= \Phi.
\end{align*}
\]

Example 6.1.1 Let \( B = (B^1, B^2) \) be a 2-dimensional G-Browian motion with

\[ G(A) = G_1(a_{11}) + G_2(a_{22}) \]

\[ G_1(a) = \frac{1}{2} (\sigma^2 a^+ - \sigma^2 a^-) \]

In this case by Itô’s formula.
\[ \begin{align*}
\frac{dX_s}{s} &= \mu X_s ds + \nu X_s dB_s^1 + \sigma X_s dB_s^2, \quad X_t = x.
\end{align*} \]

\[ u(t, x) = \hat{E}[\varphi(X_{t,x}^t)] = \hat{E}[\varphi(X_{T}^t)|\mathcal{F}_t]. \]

We have

\[ u(t, x) = \hat{E}[u(t + \delta, X_{t+\delta}^t)]. \]

From which it is easy to prove that

\[ \partial_t u + \sup_{\gamma \in [\sigma^2_1, \sigma^2_2]} (\mu + v\gamma) x \partial_x u + \frac{x^2}{2} \sup_{\gamma \in [\sigma^2_1, \sigma^2_2]} [\partial^2_{xx} u] = 0. \]

### 6.2 Markov-Nisio process

**Definition 6.2.1** A \( n \)-dimensional process \((X_t)_{t \geq 0}\) on \((\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \geq 0}, \hat{E})\) is called a Markov-Nisio process if for each \( 0 \leq s \leq t \) and each \( \varphi \in C^b_b(\mathbb{R}^n) \) there exists a \( \psi \in C^b_b(\mathbb{R}^n) \) such that

\[ \hat{E}[\varphi(X_t)|\mathcal{H}_s] = \psi(X_s). \]

**Remark 6.2.2** When \( \hat{E} \) is a linear expectation then this notion describes a classical Markovian process related to Markovian group. Nisio’s semigroup (see Nisio [45], [46]) extended Markovian group to sublinear cases to describe the value function of an optimal control system.

### 6.2.1 Martingale problem

For a given function \( F(x, p, A) : \mathbb{R}^d \times \mathbb{R}^d \times S(d) \rightarrow \mathbb{R} \) we call a sublinear expectation space solves the martingale problem related to \( F \) if there exists a time consistent nonlinear expectation space \((\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \geq 0}, \hat{E})\) such that, for each \( \varphi \in C^b_\infty(\mathbb{R}^n) \) we have

\[ M_t := \varphi(X_t) - \varphi(X_0) - \int_0^t F(X_s, D\varphi(X_s), D^2\varphi(X_s)) ds, \quad t \geq 0 \]

is a \( \hat{E} \)-martingale. (see P. [54], [55]).
6.3 Pathwise properties of $G$-Brownian motion

It is proved that
\[ \hat{\mathbb{E}}[X] = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[X] \]
where $\mathcal{P}$ is a family of probabilities on $(\Omega, \mathcal{B}(\Omega))$, $\Omega = C_0(0, \infty; \mathbb{R}^d)$. We set
\[ \hat{c}(A) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[1_A] \]
\(\hat{c}(\cdot)\) is a Choquet capacity. We have

**Theorem 6.3.1 (Denis-Hu-Peng)** There exists a continuous version of $G$-Brownian motion: We can find a pathwise process $(\tilde{B}_t(\omega))_{t \geq 0}$ on some $\tilde{\Omega} \subset \Omega$, with $\hat{c}(\tilde{\Omega}^c) = 0$, such that, for each $\omega \in \tilde{\Omega}$, $\tilde{B}_t(\omega) \in C_0(0, \infty)$ and $\hat{\mathbb{E}}[|B_t - \tilde{B}_t|] = 0$, $\forall t \in [0, \infty)$.
Chapter 7

Appendix

We will use the following known result in viscosity solution theory (see Theorem 8.3 of Crandall Ishii and Lions [17]).

**Theorem 7.0.2** Let $u_i \in USC((0, T) \times Q_i)$ for $i = 1, \ldots, k$ where $Q_i$ is a locally compact subset of $\mathbb{R}^{N_i}$. Let $\varphi$ be defined on an open neighborhood of $(0, T) \times Q_1 \times \cdots \times Q_k$ and such that $(t, x_1, \cdots, x_k)$ is once continuously differentiable in $t$ and twice continuously differentiable in $(x_1, \cdots, x_k) \in Q_1 \times \cdots \times Q_k$. Suppose that $\hat{t} \in (0, T)$, $\hat{x}_i \in Q_i$ for $i = 1, \cdots, k$ and

$$w(t, x_1, \cdots, x_k) := u_1(t, x_1) + \cdots + u_k(t, x_k) - \varphi(t, x_1, \cdots, x_k) \leq w(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k)$$

for $t \in (0, T)$ and $x_i \in Q_i$. Assume, moreover that there is an $r > 0$ such that for every $M > 0$ there is a $C$ such that for $i = 1, \cdots, k$

$$b_i \leq C \quad \text{whenever} \quad (b_i, q_i, X_i) \in P^2 + u_i(t, x_i),$$

$$|x_i - \hat{x}_i| + |t - \hat{t}| \leq r \quad \text{and} \quad |u_i(t, x_i)| + |q_i| + \|X_i\| \leq M. \quad \text{(7.1)}$$

Then for each $\varepsilon > 0$, there are $X_i \in \mathbb{S}(N_i)$ such that

(i) $(b_i, D_x \varphi(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k), X_i) \in P^{2+} u_i(\hat{t}, \hat{x}_i), \quad i = 1, \cdots, k$;

(ii) 

$$-(\frac{1}{\varepsilon} + \|A\|) \leq \begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{bmatrix} \leq A + \varepsilon A^2$$

(iii) $b_1 + \cdots + b_k = \varphi_t(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k)$

where $A = D^2 \varphi(\hat{x}) \in \mathbb{S}^{kN}$.
Observe that the above conditions (7.1) will be guaranteed by having \( u_i \) be subsolutions of parabolic equations given in the following theorem.

**Theorem 7.0.3** (Domination Theorem) Let \( m \)-order polynomial growth functions \( u_i \in USC([0, T] \times \mathbb{R}^N) \) be subsolutions of

\[
\partial_t u - G_i(D^2 u) = 0, \quad i = 1, \ldots, k, \tag{7.2}
\]

on \((0, T) \times \mathbb{R}^N\). We assume that \( \{G_i\}_{i=1}^k \) satisfies the following domination condition:

\[
\sum_{i=1}^k G(X_i) \leq 0, \quad \text{for all } X_i \in \mathcal{S}(N), \text{ such that } \sum_{i=1}^k X_i \leq 0, \tag{7.3}
\]

and \( G_1 \) is monotonic, i.e., \( G_1(X) \geq G_1(Y) \) if \( X \geq Y \). Then the following domination holds: If the initial condition satisfies \( u_1(0, x) + \cdots + u_k(0, x) \leq 0 \) for each \( x \in \mathbb{R}^N \) then we have

\[
u_1(t, x) + \cdots + u_k(t, x) \leq 0, \quad \forall (t, x) \in (0, T) \times \mathbb{R}^N.
\]

**Proof of Theorem 7.0.3** We first observe that for \( \bar{\delta} > 0 \), since \( D^2|x|^{2m} \geq 0 \), the function defined by \( \tilde{u}_1 := u_1 - \bar{\delta}/(T - t) - \bar{\delta}|x|^{2m} \) is also a subsolution of (7.2) with \( i = 1 \), and satisfies with a strictly inequality:

\[
\partial_t \tilde{u}_1 - G_1(D^2 \tilde{u}_1) = \partial_t u_1 - G_1(D^2 u_1 + \bar{\delta}D^2|x|^{2m}) - \frac{\bar{\delta}}{(T - t)^2} \leq \partial_t u_1 - G_1(D^2 u_1) - \frac{\bar{\delta}}{(T - t)^2} \leq -\frac{\bar{\delta}}{(T - t)^2}.
\]

Since \( u_1 + u_2 + \cdots + u_k \leq 0 \) follows from \( \tilde{u}_1 + u_2 + \cdots + u_k \leq 0 \) in the limit \( \bar{\delta} \downarrow 0 \), it suffices to prove the theorem under the additional assumptions

\[
\partial_t u_1 - G_1(D^2 u_1) \leq -c, \quad c := \bar{\delta}/T^2 \quad \text{and} \quad \lim_{t \to T, |x| \to \infty} u_1(t, x) = -\infty \quad \text{uniformly in } [0, T) \times \mathbb{R}^N.
\]

To prove our result, we assume to the contrary that \( u_1(s, z) + \cdots + u_k(s, z) = \bar{\delta} > 0 \) for some \((s, z) \in (0, T) \times \mathbb{R}^N\) and \( \bar{\delta} > 0 \). We will apply Theorem 7.0.2 for \( x = (x_1, \cdots, x_k), x_i \in \mathbb{R}^N \) and

\[
w(t, x) := \sum_{i=1}^k u_i(t, x_i), \quad \varphi_\alpha(x) := \frac{\alpha}{2} \left( \sum_{i=1}^{k-1} |x_{i+1} - x_i|^2 + |x_k - x_1|^2 \right).
\]
Since for each large $\alpha > 0$ the maximum of $w - \varphi_\alpha$ achieved at some $(t^\alpha, x^\alpha)$ uniformly inside a compact subset of $[0, T) \times \mathbb{R}^{k \times N}$. Set

$$M_\alpha = \sum_{i=1}^{k} u_i(t^\alpha, x_i^\alpha) - \varphi_\alpha(t^\alpha, x^\alpha).$$

It is clear that $M_\alpha \geq \delta$. We can also check that (see [17] Lemma 3.1)

$$\begin{cases}
(i) \lim_{\alpha \to \infty} \varphi_\alpha(t^\alpha, x^\alpha) = 0. \\
(ii) \lim_{\alpha \to \infty} M_\alpha = \lim_{\alpha \to \infty} u_1(t^\alpha, x_1^\alpha) + \cdots + u_k(t^\alpha, x_k^\alpha) = \sup_{t, x}[u_1(\hat{t}, \hat{x}) + \cdots + u_k(\hat{t}, \hat{x})].
\end{cases} \tag{7.4}$$

where $(\hat{t}, \hat{x})$ is a limit point of $(t^\alpha, x^\alpha)$.

If $t^\alpha = 0$, we have

$$0 < \delta \leq M_\alpha \leq \sum_{i=1}^{k} u_i(0, x_i^\alpha) - \varphi_\alpha(t^\alpha, x^\alpha).$$

But by $\sum_{i=1}^{k} u_i(0, x_i^\alpha) \to \sum_{i=1}^{k} u_i(0, \hat{x}_i) \leq 0$. So $t^\alpha$ must be strictly positive for large $\alpha$. It follows from Theorem 7.0.2 that, for each $\varepsilon > 0$ there exists $X_i \in \mathbb{S}(N)$ such that

$$(b_i^\alpha, D_{x_i^\alpha} \varphi(t^\alpha, x^\alpha), X_i) \in \bar{J}^{2;+}_{Q_i} u_i(t^\alpha, x_i^\alpha) \text{ for } i = 1, \cdots, k.$$ \(\text{and such that } \sum_{i=1}^{k} b_i^\alpha = 0,

$$\begin{pmatrix}
X_1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & X_{k-1} & 0 \\
0 & \cdots & 0 & X_k
\end{pmatrix} \leq A + \varepsilon A^2 \tag{7.5}$$

where $A = D^2 \varphi_\alpha(x^\alpha) \in \mathbb{S}^{3N}$ is explicitly given by

$$A = \alpha J_{3N} + \alpha I_{3N}, \quad J_{3N} = \begin{pmatrix}
I_N & \cdots & -I_N & -I_N \\
\vdots & \ddots & \vdots & \vdots \\
-I_N & \cdots & I_N & -I_N \\
-I_N & \cdots & -I_N & I_N
\end{pmatrix}.$$
The second inequality of (7.5) implies \( \sum_{i=1}^k X_i \leq 0 \). Setting
\[
p_1 = D_{x_1} \varphi_\alpha(x_\alpha) = \alpha(2x_1^\alpha - x_3^\alpha - x_2^\alpha),
\]
\[
p_k = D_{x_k} \varphi_\alpha(x_\alpha) = \alpha(2x_k^\alpha - x_{k-1}^\alpha - x_1^\alpha),
\]
We have \( \sum_{i=1}^k p_i = 0 \), \( \sum_{i=1}^k b_i = 0 \) and \((b_i, p_i, X_i) \in J_0^2 \cup u_1(t^\alpha, x_\alpha)\). Thus
\[
b_1 - G_1(X_1) \leq -c,
b_i - G_i(X_i) \leq 0, \quad i = 2, 3, \cdots, k.
\]
This, together with the domination condition (7.3) of \( G_1 \), implies
\[-c = -\sum_{i=1}^k b_i - c \geq -\sum_{i=1}^k G_i(X_i) \geq 0.\]
This induces a contradiction. The proof is complete.

**Corollary 7.0.4** (Comparison Theorem) Let \( G_1, G_2 : \mathbb{S}(N) \mapsto \mathbb{R} \) be given functions bounded by continuous functions on \( \mathbb{S}(N) \) and let \( G_1 \) be monotone and
\[G_1(X) \geq G_2(X), \quad \forall X \in \mathbb{S}(N).\]
Let \( u_2 \in LSC((0, T) \times \mathbb{R}^N) \) be a viscosity supersolution of \( \partial_t u - G_1(D^2 u) = 0 \) and \( u_2 \in USC((0, T) \times \mathbb{R}^N) \) be a viscosity subsolution of \( \partial_t u - G_2(D^2 u) = 0 \), such that \( u_1(0, x) \geq u_2(0, x) \), for all \( x \in \mathbb{R}^N \). Then we have \( u_1(t, x) \leq u_2(t, x) \) for all \((t, x) \in [0, \infty) \times \mathbb{R}^N\).

**Proof.** It suffices to observe that \(-u_1 \in USC((0, T) \times \mathbb{R}^N)\) is viscosity subsolutions of \( \partial_t u - G_*(D^2 u) = 0 \), with \( G_*(X) := -G_1(-X) \). We also have, for each \( X_1 + X_2 \leq 0 \),
\[
G_*(X_1) + G_2(X_2) = G_2(X_2) - G_1(-X_1)
\leq G_1(X_2) - G_1(-X_1)
\leq G_1(X_1 + X_2) \leq 0.
\]
We thus can apply the above domination theorem to get \(-u_1 + u_2 \leq 0\).
Corollary 7.0.5 (Domination Theorem) Let $G_i : S(N) \mapsto \mathbb{R}$, $i = 0, 1$, be two given mappings bounded by some continuous functions on $S(N)$ and let $u_i \in LSC((0, T) \times \mathbb{R}^N)$ be viscosity supersolutions of $\partial_t u - G_i(D^2 u) = 0$ respectively for $i = 0, 1$ and let $u_2$ be a viscosity subsolution of $\partial_t u - G_1(D^2 u) = 0$. We assume that $G_0$ is monotone and that $G_0$ dominates $G_1$ in the following sense:

$$G_1(X) - G_1(Y) \leq G_0(X - Y), \quad \forall X, Y \in S(N).$$

Then the following domination holds: If

$$u_2(0, x) - u_1(0, x) \leq u_0(0, x), \quad \forall x \in \mathbb{R}^N,$n

then $u_2(t, x) - u_1(t, x) \leq u_0(t, x)$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^N$.

**Proof.** It suffices to observe that $-u_i \in USC((0, T) \times \mathbb{R}^N)$ are viscosity subsolutions of $\partial_t u - G_i^*(D^2 u) = 0$, with $G_i^*(X) := -G_i(-X)$. We also have, for each $X_1 + X_2 + X_3 \leq 0$,

$$G_0(X_1) + G_1(X_2) + G^1(X_3) = G_1(X_3) - G_1(-X_2) - G_0(-X_1)$$

$$\leq G_0(X_3 + X_2) - G_0(-X_1) \leq 0.$$

We thus can apply the above domination theorem to get $u_2(t, x) - u_1(t, x) \leq u_0(t, x).$
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