One head is better than two: a polynomial restriction for propositional definite Horn forgetting

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Abstract

Logical forgetting is NP-complete even in the simple case of propositional Horn formulae. An algorithm previously introduced is improved by changing the input formula before running it. This enlarges the restriction that makes the algorithm polynomial and decreases its running time in other cases. The size of the resulting formula decreases consequently.

1 Introduction

Logical forgetting is used in many logics such as propositional logic [Boo54, Moi07, Del17], answer set programming [WZZZ14, GKL16], description logics [KWW09, EIS+06], first-order logic [LR93, ZZ11], modal logics [vDHLM09], logics about actions [EF07, RHPT14], defeasible logic [AEW12] and belief revision [NCL07]. It is NP-complete even in one of the simplest cases: propositional definite Horn [Lib20a].

A way to forget variables from a definite Horn formula is to recursively replace them [Lib20a]. Forgetting from general Horn formulae can be done by turning the formula definite Horn before forgetting and adding some clauses afterwards [Lib20a]. Therefore, while this article concentrates on definite Horn formulae, the results apply to the general Horn case. In particular, it shows how efficiency increases by modifying the input formula before running the replacement algorithm. This enlarges the restriction that makes the algorithm polynomial and decreases its running time in other cases.

The replacing algorithm [Lib20a] forgets $a$ by replacing it with $bc$ if the formula contains $bc \rightarrow a$; this replacement turns $ad \rightarrow e$ into $bcd \rightarrow e$.

If a definite Horn formula contains two clauses with head $a$, both their bodies are possible replacements. A single clause $ad \rightarrow e$ becomes two. If $d$ is also to be forgotten, it is replaced as well. Again, two clauses with head $d$ make each clause two. The total is four.

Multiple clauses with the same head may produce a result that is exponentially large. Their absence guarantees polynomiality.

This is a polynomial restriction: each variable is the head of at most one clause. Otherwise? Forget is semantical: it produces a formula with the same consequence on a given alphabet [Del17, Lib20a]. It is the same as all formulae equivalent to the given one. Among them, one may be single-head even if the given one was not.
For example, \( F = \{a \rightarrow b, b \rightarrow c, a \rightarrow c\} \) is not single-head because \( c \) is the head of two clauses. Formulae with variables in multiple heads may extort exponential time from the replacing algorithm. Yet, this particular formula is equivalent to \( F = \{a \rightarrow b, b \rightarrow c\} \), which is single-head.

\[
\begin{array}{c}
  a \\
  \downarrow \\
  b \\
  \downarrow \\
  c
\end{array}
\]

Forgetting from two equivalent formulae is the same. Except that it is only takes polynomial time on single-head formulae like the second. An algorithm may first convert the given formula into a single-head formula if possible, and then perform forgetting. Equivalence guarantees that the result is the same. The single-head form guarantees that the running time is polynomial. Correctness and efficiency, what else? The cost of conversion: only if polynomial, the overall method of converting and then forgetting is polynomial-time.

This article is about efficiently turning a formula into single-head form if possible.

Such a translation is the first step of an improved algorithm for forgetting. Let \( B \) be a single-head formula equivalent to \( A \) if any, otherwise \( A \) itself. Either way, these formulae are equivalent. Since forgetting is independent on the syntax, it is the same on \( A \) and \( B \). What changes is the running time: the original algorithm may take exponential time on formulae that are single-head equivalent but not single-head; the modified algorithm takes polynomial time on them, if computing the single-head form takes polynomial time.

If that takes polynomial time.

Whether it does is an open question.

This article attempts to solve it: given a formula, find an equivalent formula that does not contain two clauses with the same head, if any.

Just following the formulation of this question and looping over the equivalent formulae does not work because of their number. A reformulation would help, something about the formula itself and its clauses. Something similar exists for similar questions. For example, whether a formula is equivalent to a Horn formula can be established by checking a certain condition over every pair of its models. This is the aim of Section 2: find conditions that express single-head equivalence but do not explicitly refer to equivalent formulae. The problem seems easy at a first glance. It is not. Loops of clauses complicates it. Two conditions are easily shown necessary to single-head equivalence, but a cyclic formula disproves their sufficiency.

Cyclic formulae proved problematic from the very beginning. Even the very first attempt at reformulating single-head equivalence fails only on formulae that contain loops. This motivates a further analysis of cyclicity in Horn clauses. The concept has been studied by Hammer and Kogan [HK95] in the context of formula minimization. Not surprisingly, this is another problem on equivalent formulae: find an equivalent formula that contains a minimal number of clauses or literal occurrences. Not surprisingly, it turns out easy on acyclic formulae. Section 3 investigates the effect of acyclicity on single-head equivalence. First, acyclicity can be defined syntactically and semantically. The first relates single-head equivalence with irredundancy. The second makes single-head equivalence the same as the first condition that was attempted at expressing it — without success in general. Seman-
tical acyclicity is a subcase of inequivalence, which is later proved a restriction that makes forgetting computable in polynomial time.

This is made possible by an algorithm that runs in polynomial time and may produce the single-head version of a formula. This algorithm is based on ordering the bodies of the clauses entailed by the formula. If the formula is single-head, the only clause of a given head has a minimal body according to both this ordering and set containment. This is not yet polynomial since minimal bodies may exponentially many. The solution is to only search for a single minimal body for each head. When the formula is inequivalent, this can be done efficiently. This is the polynomial algorithm described in Section 4 along with its optimization and implementation. It always runs in polynomial time and is always correct, but is complete only on inequivalent case. In general, it may fail to produce the single-head version of a formula even if one exists. Another article [Lib20b] presents an algorithm that is always able to find the single-head form of a formula if any, but does not always run in polynomial time.

In summary, Section 2 shows that converting a formula in single-head form if possible is not as easy as it may look; rather the contrary: even just establishing the existence of a single-head equivalent formula is complicated. The culprits are the cycles of clauses, as the problem becomes easy when they are not present; Section 3 analyzes the case of acyclic formulae. Section 4 introduces an order between sets of variables and uses it in a sound but incomplete algorithm for turning a formula in single-head form; the algorithm is implemented in the Python language [VRD11].

2 The insidious single-head equivalence

The example in the introduction makes converting a formula in single-head form look easy. It is easy when the formula is simple like $F = \{a \rightarrow b, b \rightarrow c, a \rightarrow c\}$. All it takes is to remove some clauses: since $c$ is the head of two clauses, one must be deleted; $F \setminus \{b \rightarrow c\}$ is not equivalent to $F$, but $F \setminus \{a \rightarrow c\}$ is. Problem solved: the latter formula is single-head and equivalent to $F$. This section shows that the translation is far from being that easy in general.

As a matter of fact, even checking for the existence of a single-head equivalent formula without producing it is complicated. In theory, it could be done on the set of models. Indeed, two equivalent formulae either have a single-head equivalent formula both, or they both do not. A similar concept is that of Horn equivalence: the existence of a Horn formula equivalent to a given one; it is the case if every two models of the formula intersect to another model of the same formula. Unfortunately, a similarly simple formulation of single-head equivalence does not seem to exist. Whether it does not exist, in addition to does not seem to exist, is left as an open problem by this article. The rest of this section gives some hints suggesting it does not. These are just hints. What is certain at this point is only that it is an insidious problem.

2.1 Where the traps are

The problem of single-head equivalence is insidious because it looks like it has a simple semantical formulation, but the following list shows a number of traps in the intuitive ways
it appears to be expressible.

1. If a formula is single-head, it contains at most one clause $A \rightarrow x$ for each variable $x$. If $F$ is equivalent to a single-head formula $F'$, this formula $F'$ may only contain a single clause $A \rightarrow x$ with $x$ in the head. All other clauses $B \rightarrow x$ of $F$ are consequences of $A \rightarrow x$. Formally, if $B \rightarrow x \in F$ then $F \models B \rightarrow A$.

![Diagram](image)

The first attempt at formalizing single-head equivalence is that every variable $x$ has a set of variables $A$ such that $F \models B \rightarrow x$ implies $F \models B \rightarrow A$ if $x \not\in B$. This is the case for every single-head formula by setting $A$ to the body of the only clause $A \rightarrow x$ with $x$ in the head. It is also the case for every single-head equivalent formula because the condition is semantical: it does not depend on the syntax of the formula.

**Condition 1** For each variable $x$, there exists a set of variables $A$ such that $x \not\in A$ and $F \models B \rightarrow x$ implies $F \models B \rightarrow A$ if $x \not\in B$.

This condition is necessary for $F$ being single-head equivalent, as formally proved in Lemma 2 below.

2. Every formula that is equivalent to a single-head formula satisfies Condition 1 but not the other way around. Condition 1 is not sufficient to ensure equivalence to a single-head formula. The following is a counterexample, also in the inloop.py testing file of singlehead.py program.

$$F = \{a \rightarrow b, b \rightarrow c, c \rightarrow b\}$$

![Diagram](image)

This formula is not equivalent to any single-head formula. The proof is by contradiction: a single-head definite Horn formula $F'$ is assumed equivalent to $F$. Since $F' \models b \rightarrow c$, Lemma 1 tells that $F'$ contains a clause $A \rightarrow c$ such that $F' \models \{b\} \rightarrow A$. The only variables $b$ implies are itself and $c$. Therefore, either $A$ is $\{b\}$ or is $\{c\}$. The second is ruled out as $A \rightarrow c$ would be a tautology. As a result, $F'$ contains $b \rightarrow c$. For
the same reason, it also contains \( c \rightarrow b \). Since \( F' \) is by assumption single-head, it does not contain any other clause with \( b \) or \( c \) in the head. The only other clauses it may contain have head \( a \). These are \( b \rightarrow a, c \rightarrow a \) and \( bc \rightarrow a \). None of them is entailed by \( F \). As a result, \( F' = \{ b \rightarrow c, c \rightarrow b \} \), which is not equivalent to \( F \).

In spite of \( F \) being equivalent to no single-head formula, it satisfies Condition 1 with the set \( \{ c \} \) for \( b \), the set \( \{ b \} \) for \( c \) and the set \( \emptyset \) for \( a \). For example, \( \{ c \} \) is a valid set for \( b \) because \( F \models a \rightarrow b \) is a consequence of \( F \models a \rightarrow \{ c \} \) and \( F \models \{ c \} \rightarrow b \).

The problem is that \( F \models a \rightarrow \{ c \} \) is only true because of \( a \rightarrow b \in F \): the clause \( a \rightarrow b \) is supposed to be redundant as a consequence of \( a \rightarrow \{ c \} \) and \( \{ c \} \rightarrow b \), but the first clause \( a \rightarrow \{ c \} \) only holds because of \( a \rightarrow b \).

3. The previous point proves that Condition 1 is not sufficient to single-head equivalence. It correctly states that \( B \rightarrow x \) is a consequence of \( B \rightarrow A \) and \( A \rightarrow x \), but neglects the case where \( B \rightarrow A \) only holds as a consequence of \( B \rightarrow x \). Lemma 1 allows cutting this loop: it shows that \( F \upharpoonright = B \rightarrow x \) is the same as \( F_x \upharpoonright = B \rightarrow A' \) where \( A' \rightarrow x \) is a clause of \( F \) and \( F_x \upharpoonright \) is the set of clauses of \( F \) that do not contain \( x \). Because of the single-heads, \( A' \) is the same as \( A \).

\[
\forall x \exists A . \quad F \models B \rightarrow x \quad \Rightarrow \quad F_x \models B \rightarrow A, \quad F \models A \rightarrow x
\]

This condition ensures that \( B \rightarrow A \) is not itself a consequence of \( A \rightarrow x \), since \( F_x \) does not contain this clause.

Unfortunately, \( F_x \) is a syntactic construction: it is obtained by removing certain clauses from a formula. The condition holds for all single-head formulae but not all their equivalent formulae. The following is a counterexample.

\[
F = \{ a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b \}
\]

Removing all clauses containing \( b \) results in an empty set \( F^b = \emptyset \), which entails neither \( a \rightarrow \{ c \} \) nor \( c \rightarrow \{ a \} \); therefore, neither \( \{ c \} \) nor \( \{ a \} \) are valid sets for \( b \). Removing only the clauses with \( b \) in the head gives the same outcome: \( \{ b \rightarrow a, b \rightarrow c \} \) entails neither \( a \rightarrow \{ c \} \) nor \( c \rightarrow \{ a \} \).

Yet, an equivalent single-head formula exists:

\[
F' = \{ a \rightarrow b, b \rightarrow c, c \rightarrow a \}
\]
This shows that syntactic nature of $F^x$ has practical effects, as its usage to recognize single-head equivalence may be incorrect.

4. That the counterexample involves equivalence is not an incident. Equivalence is the root of all problems of Condition [1]. This condition is incorrect only when $B \rightarrow A$ is true only because of $B \rightarrow x$. Otherwise, $B \rightarrow x$ is a consequence of $B \rightarrow A$, which is true because of other clauses, and of $A \rightarrow x$; this means that $B \rightarrow x$ can be removed and $A \rightarrow x$ left as the only clause having $x$ in the head.

The problem is when $B \rightarrow x$ is necessary to $B \rightarrow A$. This is the case when $B$ implies $x$ and some other variables $C$ that together imply $A$. Formally, $F \models B \rightarrow C \cup \{x\}$ and $F \models C \cup \{x\} \rightarrow A$. Since $F \models A \rightarrow x$, the equivalence $F \models C \cup \{x\} \equiv C \cup A$ holds. This proves that the problematic cases involve equivalences.

5. Equivalences do not always forbid single-head equivalence. For example, \{\(a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b\)\} implies the equivalence of $a$, $b$ and $c$, yet it is equivalent to the single-head formula \{\(a \rightarrow b, b \rightarrow c, c \rightarrow a\)\}. Equivalences are realized by loops of clauses, like in this case.

What single-head equivalence forbids is that variables outside the loop entail variables in the loop. This was exactly what happened in the counterexample $F = \{a \rightarrow b, b \rightarrow c, c \rightarrow b\}$, where $b$ and $c$ are equivalent but $a$ tried to entail one of them without being equivalent to them.

The converse is not a problem: a variable in a loop may entail a variable outside it.

\{\(a \rightarrow b, b \rightarrow a, a \rightarrow c\)\}
The second condition for single-head equivalence is something like: if some sets of variables $A$ and $B$ are equivalent and are entailed by another set $C$, then $C$ has to be equivalent with them. This is however too restrictive as a condition, as the next point shows.

6. The second condition as stated above works when equivalences comprise only single variables, but two equivalent sets of variables may have a common part. As an example, the following formula contain two equivalent sets $B$ and $C$, yet the non-equivalent set $A$ entails them.

$$F = \{ a \rightarrow b, \ bc \rightarrow d, \ bd \rightarrow c \}$$

This formula implies the equivalence of $B = \{ b, c \}$ and $C = \{ b, d \}$. The set $A = \{ a, c \}$ implies them without being equivalent to them. Yet, the formula is single-head.

What forbids a variable in an equivalence to be entailed by a non-equivalent set is that it already needs to be entailed by another equivalent set. However, it needs not if the variable is in all equivalent sets. This is the case here, as both the equivalent sets $B$ and $C$ contains $b$. Since $b$ does not need to be entailed by one of the equivalent sets because it belongs to both, it is free to be entailed by the outside variable $a$.

7. The second condition for single-head equivalence is that a set of variables $A$ may entail a set $B$ that is equivalent to some other set $C$, but only if $A$ is also equivalent to $B$ and $C$, at least regarding the variables that are not in all sets equivalent to $B$ and $C$. The definitions of equivalent sets and their common variables is necessary to formalize this condition:
\[
\text{EQUISET}(A, F) = \{ B \mid F \models A \equiv B \}
\]
\[
\text{EQUIALL}(A, F) = \{ x \mid \forall C \in \text{EQUISET}(A, F) . x \in C \}
\]

These two concepts allows formalizing the second condition to common equivalence.

**Condition 2** If \( F \models A \rightarrow B \) then there exists \( C \in \text{EQUISET}(B, F) \) such that \( C \setminus \text{EQUIALL}(B, F) \subseteq A \).

This condition does not depend on the syntax of the formula since the clause it mentions only occurs in an entailment, \( \text{EQUIALL}() \) is based on \( \text{EQUISET}() \) and \( \text{EQUISET}() \) again only mentions clauses in an entailment. As a result, if \( F \equiv F' \), the condition holds for \( F \) if and only if it holds for \( F' \).

Unfortunately, even this condition is only necessary to single-head equivalence. It is not sufficient.

### 2.2 Necessary conditions

Conditions [1] and [2] are proved necessary to single-head equivalence. Every single-head formula satisfy them. Since they are based on entailment and not on clause membership, they are independent on the syntax. If they hold for a formula, they hold for every equivalent one. Therefore, they are not only satisfied by single-head formulae; they are satisfied by every formula that is equivalent to a single-head one.

The proof requires a lemma proved in a previous article [Lib20a].

**Lemma 1** If \( F \) is a definite Horn formula, the following three conditions are equivalent, where \( P' \rightarrow x \) is not a tautology (\( x \notin P' \)).

1. \( F \models P' \rightarrow x \);
2. \( F^x \cup P' \models P \) where \( P \rightarrow x \in F \);
3. \( F \cup P' \models P \) where \( P \rightarrow x \in F \).

The necessity of Condition [1] is proved by the following lemma.

**Lemma 2** If \( F \) is single-head, then for each variable \( x \) there exists a set of variables \( A \) such that \( x \notin A \) and \( F \models B \rightarrow x \) implies \( F \models B \rightarrow A \) if \( x \notin B \).

**Proof.** If \( x \) is the head of no clause in \( F \) then \( F \models B \rightarrow x \) never holds. The condition is vacuously satisfied by \( A = \emptyset \).

Otherwise, \( x \) is the head of a clause \( A \rightarrow x \in F \). By assumption, clauses are not tautologic: \( x \notin A \). By Lemma 1 if \( F \models B \rightarrow x \) and \( x \notin B \) then there exists a set of variables \( C \) such that \( F \models B \rightarrow C \) and \( C \rightarrow x \in F \). Since \( F \) is single-head, \( C \rightarrow x \) is \( A \rightarrow x \). Therefore, \( F \models B \rightarrow A \). \( \square \)
The proof involves \( A \rightarrow x \in F \) but the statement of the lemma is semantical. It only involves entailments of clauses from formulae. As a result, if the statement holds for a formula it also holds for all equivalent ones. If a formula is equivalent to a single-head one, Condition 1 satisfies the latter and therefore also the former by equivalence. The same goes for Condition 2.

**Lemma 3** If \( F \) is a single-head formula, then \( F \models A \rightarrow B \) implies \( \exists C \in \text{equiset}(B, F) \). \( C \setminus \text{equiall}(B, F) \subseteq A \).

**Proof.** The core of the proof is an induction property: the following Property 2 holds for a subset of \( F \) if it holds for all smaller subsets. Induction on the size of the subset proves it for \( F \) itself.

\[
F' \models A \rightarrow d \quad F' \subseteq F \\
d \in D \in \text{equiset}(B, F) \\
\Rightarrow \exists E \cdot F \models A \rightarrow E, \; F \models E \rightarrow d, \; F \models D \rightarrow E, \; E \subseteq A \cup \text{equiall}(B, F) \quad (2)
\]

The induction step assumes that Property 2 holds when \( F' \) is replaced by every subset of \( F \) smaller than \( F' \); it requires proving Property 2 for \( F' \).

If \( d \) is in either \( A \) or \( \text{equiall}(B, F) \), Property 2 holds for \( F' \) with \( E = \{d\} \). Indeed, \( F' \models A \rightarrow d \) and \( F' \subseteq F \) imply \( F \models A \rightarrow E \); the condition \( F \models E \rightarrow d \) holds because \( d \in E \); the condition \( F \models D \rightarrow E \) holds because \( E = \{d\} \subseteq D \); finally, \( E \subseteq A \cup \text{equiall}(B, F) \) holds by assumption.

The other case is that \( d \) is neither in \( A \) nor in \( \text{equiall}(B, F) \). Since \( d \) is not in \( \text{equiall}(B, F) \), some \( D' \in \text{equiset}(B, F) \) does not include it. Since \( D' \) is in \( \text{equiset}(B, F) \), it implies all literals of \( D \), including \( d \). In formulae, \( F \models D' \rightarrow d \). Lemma 1 applies to both this implication and the assumption \( F' \models A \rightarrow d \). Since \( F \) is single-head and \( F' \) is a subset of its, the clause in \( F \) and \( F' \) is the same: \( D'' \rightarrow d \in F' \) with \( F^d \models D' \rightarrow D'' \) and \( F^d \models A \rightarrow D'' \).

Since \( D \) and \( D' \) are in \( \text{equiset}(B, F) \), both \( F \models B \rightarrow D \) and \( F \models B \rightarrow D' \) hold. Because of \( F \models D' \rightarrow D'' \), it holds \( F \models B \rightarrow D \setminus \{d\} \cup D'' \). Since \( F \models D'' \rightarrow d \), it holds \( F \models D \setminus \{d\} \cup D'' \rightarrow D \). Since \( F \models D \rightarrow B \) because of \( D \in \text{equiset}(B, F) \), it follows \( F \models D \setminus \{d\} \cup D'' \rightarrow B \). The conclusion is \( D \setminus \{d\} \cup D'' \in \text{equiset}(B, F) \).

Every variable \( d'' \in D'' \) is in an element \( D \setminus \{d\} \cup D'' \) of \( \text{equiset}(B, F) \). Proved above is \( F^d \models A \rightarrow D'' \), which implies \( F^d \models A \rightarrow d'' \). Since \( F^d \subseteq F \), this formula \( F^d \) is a subset of \( F \) smaller than \( F \). By the induction assumption, every subset \( F^d \) of \( F \) smaller than \( F' \) and variable \( d'' \in D'' \) satisfy Property 2. By induction, since its premise holds also its conclusion holds: \( \exists E'' \cdot F \models A \rightarrow E'', \; F \models E'' \rightarrow d'', \; F \models D \rightarrow E'', \; E'' \subseteq A \cup \text{equiall}(B, F) \).

The claim is that some set \( E \) satisfies this conclusion for \( d \). This is the case for the set \( E \) that is the union of the sets \( E'' \) for all \( d'' \in D \). The conditions \( F \models A \rightarrow E'', \; F \models D \rightarrow E'' \) and \( E'' \subseteq A \cup \text{equiall}(B, F) \) all extends from individual sets to their union. The only condition that requires some proof is \( F \models E \rightarrow d \); since \( F \models E'' \rightarrow d'' \) for every \( d'' \in D'' \) and \( E \) is the union of the sets \( E'' \), it follows \( F \models E \rightarrow D'' \). Since \( D'' \rightarrow d \in F \), the conclusion follows.
This almost proves Property 2 by induction: it is true for \( F' \subseteq F \) if it is true all subsets of \( F \) smaller than \( F' \). The missing bit is the base case: \( F' = \emptyset \). The premise of Property 2 includes \( F' \models A \rightarrow d \), which implies \( d \in A \) because \( F' \) is empty. Its conclusion holds for \( E = \{ d \} \).

Induction proves Property 2 for every \( F' \subseteq F \). In particular, it proves it for \( F' = F \).

\[
\begin{align*}
F &\models A \rightarrow d \\
&\text{and } d \in D \in \text{EQUISET}(B, F) \\
&\Rightarrow \exists E . \ F \models A \rightarrow E, \ F \models E \rightarrow d, \ F \models D \rightarrow E, \ E \subseteq A \cup \text{EQUIALL}(B, F)
\end{align*}
\]

Let \( D \) be a set such that \( F \models A \rightarrow d \) and \( d \in D \in \text{EQUISET}(B, F) \). The premises \( F \models A \rightarrow d \) and \( d \in D \in \text{EQUISET}(B, F) \) of the property hold for all elements \( d \in D \). As a result, for each \( d \in D \) the conditions \( F \models A \rightarrow E, \ F \models E \rightarrow d, \ F \models D \rightarrow E \) and \( E \subseteq A \cup \text{EQUIALL}(B, F) \) all holds for some set \( E \).

One part of the definition of \( D \in \text{EQUISET}(B, F) \) is \( F \models B \rightarrow D \). Since \( F \models D \rightarrow E \), it follows \( F \models B \rightarrow D \setminus \{ d \} \cup E \). The other part of the definition of \( D \in \text{EQUISET}(B, F) \) is \( F \models D \rightarrow B \). Since \( F \models E \rightarrow d \), it follows \( F \models D \setminus \{ d \} \cup E \rightarrow B \). The conclusion is \( D \setminus \{ d \} \cup E \in \text{EQUISET}(B, F) \).

In summary, if \( F \models A \rightarrow D \) and \( D \in \text{EQUISET}(B, F) \), replacing an arbitrary variable \( d \) in \( D \) but not in \( A \cup \text{EQUIALL}(B, F) \) with its set \( E \subseteq A \cup \text{EQUIALL}(B, F) \) results in another equiset \( D \setminus \{ d \} \cup E \) entailed by \( A \). In other words, \( D \setminus \{ d \} \cup E \) has the same properties of \( D \) they are both equisets entailed by \( A \). At the same time, \( D \setminus \{ d \} \cup E \) contains one less variable that is not in \( A \cup \text{EQUIALL}(B, F) \). This replacement can be iterated, decreasing the number of variables not in \( A \cup \text{EQUIALL}(B, F) \) until it reaches zero. Induction on this number proves that \( F \models A \rightarrow D \) and \( D \in \text{EQUISET}(B, F) \) imply \( F \models A \rightarrow C \) with \( C \in \text{EQUISET}(B, F) \) and \( C \subseteq A \cup \text{EQUIALL}(B, F) \).

This is the claim of the lemma when \( D = B \).

2.3 Insufficient conditions

The two conditions are proved insufficient to ensure single-head equivalence. The following formula satisfies both, but is equivalent to no single-head formula.

\[
F = \{ ab \rightarrow x, bx \rightarrow c, ac \rightarrow d, d \rightarrow x \}
\]

As expected, this formula hinges around a loop. Yet, it is not a simple loop where each variable entails another. The heads of the clauses in the loop are still the individual variables \( x, c, d, x \), but the first step of the loop requires \( a \) and the second \( b \). This means that both \( a \) and \( b \) are required to close the loop.
The formula is in the `conditiontwo.py` test file of the `singlehead.py` program. The following lemma proves that it satisfies the first condition.

**Lemma 4**  Formula $\Box$ satisfies Condition $\blacksquare$.

**Proof.** All variables but $x$ are already the head of a single clause; therefore, Condition $\blacksquare$ holds with $A$ equal to the body of that clause thanks to Lemma $\blacksquare$.

It also holds for $x$ with $A = \{d\}$. These sets are $\{a, b\}$, $\{a, c\}$ and their supersets. Both $F \models ab \rightarrow d$ and $F \models ac \rightarrow d$ hold, which imply the same for all supersets of $\{a, b\}$ and $\{a, c\}$. $\Box$

Condition $\blacksquare$ is fooled by $F \models ab \rightarrow d$, which however only holds thanks to the chain of implications $ab \rightarrow x$, $bx \rightarrow c$ and $ac \rightarrow d$, which require $ab \rightarrow x$. The plan was to only retain $d \rightarrow x$ and to obtain $ab \rightarrow x$ as a consequence of $F \models ab \rightarrow d$ and $d \rightarrow x$, but the first premise requires a second clause with $x$ as the head: $ab \rightarrow x$.

Proving that $F$ satisfies Condition $\blacksquare$ is more complicated because it requires considering sets of variables rather than single variables. A concept introduced below, that of inequivalence, helps in that. This is why the formal proof is delayed after some other results, in Lemma $\blacksquare$.

While $F$ satisfies both Condition $\blacksquare$ and $\blacksquare$, it is not single-head equivalent. This is proved by Lemma $\blacksquare$ which is later in the article because it requires some other results.

### 2.4 Equivalence

Another necessary condition to single-head equivalence relates the premises of the clauses that entail a variable and the same premises in the single-head equivalent formula.

**Lemma 5** If $F$ is equivalent to a single-head formula $F'$ that contains the clause $P \rightarrow x$, then $F$ contains $P' \rightarrow x$ with $F \models P \equiv P'$.

**Proof.** Since $F'$ contains $P \rightarrow x$ it also entails it: $F' \models P \rightarrow x$. By equivalence, also $F$ entails $P \rightarrow x$. Lemma $\blacksquare$ implies the existence of a set of variables $P'$ such that $x \notin P'$, $F \models P \rightarrow P'$ and $P' \rightarrow x \in F$. The latter condition implies $F \models P' \rightarrow x$. By equivalence, $F' \models P' \rightarrow x$. Again, Lemma $\blacksquare$ implies that $F' \models P' \rightarrow P''$ for some $P'' \rightarrow x \in F'$. Since $F'$ is single-head and contains $P \rightarrow x$, this is only possible if $P'' = P$. As a result, $F' \models P' \rightarrow P''$ is the same as $F' \models P' \rightarrow P$. Since $F \models P \rightarrow P'$ and the two formulae are equivalent, $F \models P \equiv P'$ is proved. $\Box$
2.5 Redundancy

Sometimes, a formula can be made single-head just by removing some redundant clauses. For example, \{a \rightarrow b, b \rightarrow c, a \rightarrow c\} is not single-head, but removing \(a \rightarrow c\) makes it so while preserving its semantics. Removing a redundant clause turns the formula into a single-head equivalent one.

Is this always the case?

If so, a formula could be made single-head by nondeterministically removing redundant clauses until the formula either becomes irredundant or single-head.

The following theorem proves that this is not always possible. Sometimes the single-head equivalent formula is a subset of the given formula, sometimes it is not.

**Theorem 1** Some formulae are equivalent to single-head formulae, but no equivalent subset of them is single-head.

**Proof.** Such a formula is \{a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b\}, in the `twoequiv.py` test file of the `singlehead.py` program. Its three variables are equivalent. The same can be achieved by a cycle of clauses, like in the single-head formula \{a \rightarrow b, b \rightarrow c, c \rightarrow a\}.

All clauses of the formula are irredundant. As a result, the formula is equivalent to none of its proper subsets, only to itself. Since it is not single-head, the claim is proved.

Turning a formula into its single-head form cannot always done just by removing clauses. Sometimes it requires adding new ones as well. The proof of the theorem shows such a case: the formula does not contain \(c \rightarrow a\), but its equivalent single-head formula does.

If a formula does not contain cycles of clauses, removing redundant clauses always leads to a single-head equivalent formula if any. This is proved in the next section, where cyclicity is formally defined.

3 Cyclicity

All examples that show that single-head equivalence is complicated contain cycles of variables: a variable implies another which implies yet another which implies the first, for example. When a formula is acyclic, Condition 1 is not only necessary to single-head equivalence but also sufficient.

3.1 Syntactical and semantical cyclicity

Cycles are made of variables implying others, like \(a \rightarrow b\). Other preconditions do not matter, so \(acd \rightarrow b\) is the same as \(a \rightarrow b\) from this point of view. More precisely, is the same as \(a \rightarrow b, c \rightarrow b\) and \(d \rightarrow b\).

Simple implications like these resemble the edges in a direct graph: variables are nodes, an implication \(a \rightarrow b\) is an edge from \(a\) to \(b\). Except that cycles in graphs are a well-defined and studied concept. To simplify notation and increase similarity, a directed graph is considered just a set of edges, each written with an arrow between two nodes, like \(a \rightarrow b\). The usual notation \(G = (N, E)\) where \(N\) is the set of nodes and \(E\) of edges only differs in allowing for isolated nodes, which do not matter to single-head equivalence. Using only the set of edges,
and further writing each as \( \text{node} \to \text{node} \) makes the graph very close to the formula it comes from.

A formula defines two graphs:

**syntactic graph:** \( Y(F) = \{y \to x \mid \exists P. P \cup \{y\} \to x \in F\} \); an edge from \( y \) to \( x \) comes from a clause with \( y \) in the body and \( x \) in the head; the syntactic graph is the graph associated to the Horn formula according to Hammer and Kogan [HK95].

**semantic graph:** \( E(F) = \{y \to x \mid \exists P. F \models P \cup \{y\} \to x \text{ and } F \not\models P \to x\} \); the edge from \( y \) to \( x \) means that \( y \) participates in entailing \( x \); participates means that it entails \( x \) with something else but that something else alone does not.

The first definition is purely syntactic: it only involves the clauses of the formula and what they contain; it does not even mention models or implications. The second is purely semantical: it only involves entailments, and does not mention any clause of the formula or its variables; it could as well be defined from a set of models instead of a formula.

The difference glows on redundant clauses. The first definition makes an edge from \( a \) to \( b \) out of a clause \( acd \to b \) even when another clause \( cd \to b \) is present. The second does not because \( a \) is redundant in implying \( b \). These two clauses are already a proof they differ. Quod erat demostrandum, they are not the same. Yet, they became so when infusing some semantics in the first. The semantical graph of a formula is identical to the syntactical graph of the set of its prime implicates.

The job of joining the gap between the two definitions is done by primality, which removes redundant clauses like \( acd \to b \) and consequently bars the edge from \( a \) to \( b \). This is proved by the following theorem, where \( PI(F) \) is the set of prime implicates of a formula \( F \).

**Lemma 6** If \( F \) is a formula that coincides with the set of its prime implicates \( PI(F) \), then \( Y(F) \) coincides with \( E(F) \).

**Proof.** The first part of the claim is that \( y \to x \in Y(F) \) implies \( y \to x \in E(F) \). By definition, \( y \to x \in Y(F) \) means \( P \cup \{y\} \to x \in F \) for some set of variables \( P \). As a result, \( F \models P \cup \{y\} \to x \). This is the first point in the definition of \( y \to x \in E(F) \). The second is \( F \not\models P \to x \). Its contrary \( F \models P \to x \) means that \( F \) implies a subset of \( P \cup \{y\} \to x \), which is therefore not a prime implicate. This contradicts the assumption that \( F \) only contains its prime implicates.

The second part of the claim is that \( y \to x \in E(F) \) implies \( y \to x \in Y(F) \). By definition, \( y \to x \in E(F) \) means that for some set of variables \( P \) the clause \( P \cup \{y\} \to x \) is entailed by \( F \) while \( P \to x \) is not. The first fact implies that \( F \) contains a prime implicate contained in \( P \cup \{y\} \to x \). Such a prime implicate contains \( x \) because otherwise \( F \) would not be a definite Horn formula. It also contains \( y \), because otherwise \( F \) would imply \( P \to x \). Therefore, this prime implicate has the form \( P' \cup \{y\} \to x \). Since \( F \) contains all its prime implicates, it also contains \( P' \cup \{y\} \to x \). This means that \( y \to x \in Y(F) \). \( \square \)

The goal is to show some sort of equality between syntactic and semantic cyclicity. This lemma proves part of that: if the semantic graph of \( F \) does not contain cycles, neither the syntactic graph of \( PI(F) \) does. Semantic acyclic implies acyclicity of one syntactic form.

Only one, not all of them. For example, \( \{a \to b, b \to c\} \) is semantically acyclic, yet its equivalent formula \( \{a \to b, b \to c, ac \to b\} \) is syntactically cyclic. Most formulae can
be made cyclic by adding some redundant clauses like $ac \rightarrow b$. The resulting formula is equivalent but cyclic.

When looking for a syntactic counterpart of semantic cyclicity, better looking for something else. The exact opposite makes sense: if a formula is semantically cyclic, all equivalent formulae are syntactically cyclic, and the other way around.

This is actually the case for arbitrary paths, not just cycles. If the semantic graph of a formula contains a path from $x$ to $y$, the same happens for the syntactic graph of every equivalent formula, and vice versa. Paths are formalized by the transitive closure of graphs.

**Definition 1** The transitive closure $G^*$ of a graph $G$ is the graph that contains an edge $y \rightarrow x$ if and only if $G$ has a path from $y$ to $x$.

The transitive closure $Y^*(F)$ of the syntactic graph is very close to the semantic graph: $E(F) \subseteq Y^*(F)$. The transitive closure of the syntactic graph contains the semantic graph. A very simple intuitive argument tells why: the syntactic graph is based on clauses; a path in it is a sequence of clauses; a sequence of clauses creates an implication, which is the root of the semantic graph.

The formal proof of this containment is based on the syntactic graph being monotone with respect to the formula, which the semantic graph is not by Lemma 10.

**Lemma 7** If $F \subseteq F'$ then $Y(F') \subseteq Y(F)$.

**Proof.** The definition of $y \rightarrow x \in Y(F)$ is that $F$ contains a clause $P \rightarrow x$ such that $y \in P$. Since $F' \subseteq F'$, this clause $P \rightarrow x$ is also in $F'$. The definition of $y \rightarrow x \in Y(F')$ is met because $y \in P$.

Monotonicity allows proving $E(F) \subseteq Y^*(F)$.

**Lemma 8** If $y \rightarrow x \in E(F)$, then $Y(F)$ contains a path from $y$ to $x$.

**Proof.** If $y = x$ the claim holds because a zero-length path between every node and itself always exists in every graph.

A consequence is that the claim holds when the formula only comprises one variable, because $x$ and $y$ coincide in this case. This is the base case of an induction over the number of variables of the formula.

The inductive case assumes $y \rightarrow x \in E(F)$ and that the claim holds for every formula smaller than $F$; the conclusion to prove is $y \rightarrow x \in Y^*(F)$.

The assumption $y \rightarrow x \in E(F)$ is defined as $F \models P \cup \{ y \} \rightarrow x$ and $F \not\models P \rightarrow x$ for some set of variables $P$.

The claim is already proved when $x = y$. The rest of the proof is for the case $x \neq y$. Since $F \not\models P \rightarrow x$, the variable $x$ is not in $P$. It is not $y$ either; therefore, $x \notin P \cup \{ y \}$.

Since $F \models P \cup \{ y \} \rightarrow x$ and $x \notin P \cup \{ y \}$, Lemma 11 tells that $F$ contains a clause $P' \rightarrow x$ such that $F^x \models P \cup \{ y \} \rightarrow P'$.

If $y \in P'$ the claim is proved because $P' \rightarrow x \in F$ and $y \in P'$ define $y \rightarrow x \in Y(F)$, which implies $y \rightarrow x \in Y^*(F)$.

The other case is $y \notin P'$. Since $P' \rightarrow x \in F$, if $F^x \models P \rightarrow P'$ then $F \models P \rightarrow x$, which is false. As a result, $F^x \not\models P \rightarrow P'$: for at least a variable $p \in P'$, it holds $F^x \not\models P \rightarrow p$. The
entailment \( F^x \models P \cup \{y\} \to p \) instead holds because \( F^x \models P \cup \{y\} \to P' \) and \( p \in P' \). This proves that \( y \to p \in E(F^x) \).

Summarizing: \( y \to x \in E(F) \) implies \( y \to p \in E(F^x) \) and \( P' \rightarrow x \in F \) with \( p \in P' \). The latter implies \( p \rightarrow x \in Y(F) \). The former implies the existence of a path from \( y \) to \( p \) in \( Y(F^x) \) by induction, since \( F^x \) has one variable less than \( F \). Since \( F^x \subset F \), the same path is in \( Y(F) \) by Lemma 7.

Since \( Y(F) \) contains a path from \( y \) to \( p \) and the edge \( p \rightarrow x \), it also contains the path from \( y \) to \( x \).

\[ \square \]

The following results are about cycles. Assuming that formulae do not contain tautologies, the syntactic graphs never contain loops, cycles comprising only one node. The semantical graph contain most of them instead, since \( F \models x \rightarrow x \) always holds and \( F \models \emptyset \rightarrow x \) holds only if \( F \models x \). Cycles in the syntactical and semantical graph match only when loops are excluded.

From this point on, only nontrivial cycles are considered, cycles containing at least two nodes. Syntactically or semantically, acyclic means the absence of nontrivial cycle.

**Lemma 9** If \( E(F) \) contains a nontrivial cycle, so does \( Y(F) \).

**Proof.** Nontrivial cycles contain at least two variables. Such a cycle in \( E(F) \) comprises a set of all-different variables \( x_1, \ldots, x_m \) such that \( m \geq 2 \) and \( E(F) \) contains \( x_i \rightarrow x_{i+1} \) for every \( i = 1, \ldots, m-1 \) and \( x_m \rightarrow x_1 \). By Lemma 8, the syntactic graph \( Y(F) \) contains a path from each \( x_i \) to \( x_{i+1} \) and from \( x_m \) to \( x_1 \). This is a path of at least 2 variables starting from \( x_1 \) and ending in \( x_1 \), a nontrivial cycle.

Cyclicity in the semantic graph imposes cyclicity in the syntactic graph. Not the other way around, as shown above. What imposes cyclicity in the semantic graph is cyclicity in the syntactic graph of all equivalent formulae.

**Theorem 2** For every formula \( F \), the graph \( E(F) \) contains nontrivial cycles if and only if \( Y(F') \) contains cycles for every \( F' \equiv F \).

**Proof.** The first part of the proof shows that if \( E(F) \) contains a nontrivial cycle and \( F' \) is equivalent to \( F \), then \( Y(F') \) contains a nontrivial cycle. Since the semantic graph is defined semantically, it is the same for equivalent formulae. In the present case, \( E(F') = E(F) \). Since \( E(F) \) contains a nontrivial cycle, do does \( E(F') \). And so does \( Y(F') \) thanks to Lemma 9.

The second part of the proof shows the converse: if \( E(F) \) does not contain any nontrivial cycle, then \( Y(F') \) does not either for some formula \( F' \) equivalent to \( F \). Such a formula \( F' \) is the set of the prime implicates \( PI(F) \) of \( F \). Since \( PI(F) \) is equivalent to \( F \), it has the same semantic graph: \( E(F) = E(PI(F)) \). Since \( PI(F) \) coincides with its set of prime implicates, its semantic graph \( E(PI(F)) \) coincides with its syntactic graph \( Y(PI(F)) \) by Lemma 6. Therefore, \( E(F) = Y(PI(F)) \). Since \( E(F) \) does not contain nontrivial cycles, \( Y(PI(F)) \) does not either.

\[ \square \]

Hammer and Kogan [HK95] introduced the syntactic graph of a formula, with its consequent definition of cyclicity. They also defined its semantical counterpart: a Boolean Horn function is acyclic if a Horn formula realizing it is acyclic. Since Boolean functions are realized by equivalent formulae, this is the same as defining acyclicity as equivalence with a
syntactically acyclic formula. Lemma 8 proves that this definition is the same as semantical acyclicity as defined in this article, in terms of implications from minimal premises $(F \models P \cup \{y\} \rightarrow x \text{ and } F \not\models P \rightarrow x)$.

The similarity is further emphasized by how close Lemma 6 and Lemma 8 relate to Lemma 4.1 by Hammer and Kogan [HK95]: the syntactic graph of a formula has a path between each variable of the body of a prime implicate and its head. They also relate to Theorem 4.3 by Boros et al. [BCK98]: if two formulae are prime and equivalent, their reachability between variables is the same.

As a related note, a semantically acyclic formula may contain a syntactically cyclic subset. An example is $F = \{a \rightarrow b, a \rightarrow c, ac \rightarrow b, ab \rightarrow c\}$. This formula is semantically acyclic, as it is equivalent to its subset $\{a \rightarrow b, a \rightarrow c\}$. Removing these two clauses makes the other, previously redundant two matter: $ab \rightarrow c$ and $ac \rightarrow b$. This is a cycle, from $b$ to $c$ and back.

![Diagram](https://via.placeholder.com/150)

**3.2 Single-head equivalence of acyclic formulae**

All attempts at finding a necessary and sufficient condition to single-head equivalence failed, but they all failed because of cyclic counterexamples — formulae that satisfy a candidate condition but were not equivalent to any single-head formulae. This is why cyclicity is in this article: because it looks like the root of all complications. If so, disallowing it should give a simple necessary and sufficient condition to single-head equivalence. The following lemma proves that to be the case.

**Lemma 10** Every semantically acyclic formula that satisfies Condition [1] is equivalent to a single-head formula.

**Proof.** Let $F'$ be a formula such that $E(F')$ is nontrivially acyclic and satisfies Condition [1]. By Corollary 2, there exists a formula $F$ such that $F \equiv F'$ and $Y(F)$ is acyclic.

The claim is proved by showing that all clauses of $F$ with head $x$ are redundant but one. If $F$ does not contain any clause of head $x$, the claim is trivially true. Otherwise, let $B \rightarrow x \in F$. Since $F'$ satisfies Condition [1], a set of variables $A$ satisfies $x \notin A$, $F' \models B \rightarrow A$ and $F' \models A \rightarrow x$. Since $F \equiv F'$, the latter two conditions imply $F \models B \rightarrow A$ and $F \models A \rightarrow x$.

Lemma [1] applies because of $x \notin A$ and $F \models A \rightarrow x$, and implies that $F$ contains a clause $A' \rightarrow x$ such that $F \models A \rightarrow A'$. This entailment and $F \models B \rightarrow A$ imply $F \models B \rightarrow A'$ by transitivity.

If $B \rightarrow x$ is different from $A' \rightarrow x$, it is proved redundant: $F \setminus \{B \rightarrow x\} \models B \rightarrow x$. 

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Since $A' \rightarrow x$ is in $F$ but is not the same as $B \rightarrow x$, it is in $F\setminus\{B \rightarrow x\}$. Therefore, $A' \rightarrow x$ is entailed by $F\setminus\{B \rightarrow x\}$. Let $A''$ be a minimal subset of $A'$ such that $F\setminus\{B \rightarrow x\} \models A'' \rightarrow x$.

Because of minimality, $F\setminus\{B \rightarrow x\} \not\models (A''\setminus\{a\}) \rightarrow x$ holds for every $a \in A''$. At the same time, $F\setminus\{B \rightarrow x\}$ entails $(A''\setminus\{a\}) \rightarrow x$ because this clause is the same as $A'' \rightarrow x$. These two conditions define $a \rightarrow x \in E(F\setminus\{B \rightarrow x\})$ for every $a \in A''$.

The claim $\not\models F\setminus\{B \rightarrow x\} \models B \rightarrow x$ is proved by contradiction. Its contrary is $\models F\setminus\{B \rightarrow x\} \not\models B \rightarrow x$. It implies $\models F\setminus\{B \rightarrow x\} \not\models B \rightarrow A''$, as otherwise $B$ would imply $x$ via $A''$ since $F\setminus\{B \rightarrow x\} \models A'' \rightarrow x$. As a result, there exists $a \in A'' \setminus B$ such that $\not\models F\setminus\{B \rightarrow x\} \models B \rightarrow a$. This is one part of the condition $x \rightarrow a \in E(F\setminus\{B \rightarrow x\})$, which would prove the formula cyclic. The other part is $\models F\setminus\{B \rightarrow x\} \models (B \cup \{x\}) \rightarrow a$, which is now proved. Since $x \models B \rightarrow x$ holds in propositional logic, it holds $\models F\setminus\{B \rightarrow x\} \cup \{x\} \models B \rightarrow x$, which means that $\models F\setminus\{B \rightarrow x\} \cup \{x\}$ implies the only clause of $F$ it does not contain, and consequently it implies $F$. Since $F$ implies $B \rightarrow A'$, it also implies $B \rightarrow a$ because $a$ is in $A''$, which is a subset of $A'$. The condition $\models F\setminus\{B \rightarrow x\} \cup \{x\} \models B \rightarrow a$ can be rewritten as $\models F\setminus\{B \rightarrow x\} \models (B \cup \{x\}) \rightarrow a$. This is the second part of $x \rightarrow a \in E(F\setminus\{B \rightarrow x\})$.

The above two paragraphs prove $a \rightarrow x \in E(F\setminus\{B \rightarrow x\})$ for every $a \in A''$ and $x \rightarrow a \in E(F\setminus\{B \rightarrow x\})$ some $a \in A''$. Since $x \not\in A$ and $a \in A''$ with $A'' \subseteq A$, it holds $x \neq a$. Therefore, $E(F\setminus\{B \rightarrow x\})$ contains a nontrivial cycle. By Lemma 9, $Y(F\setminus\{B \rightarrow x\})$ contains a nontrivial cycle. Since syntactic cyclicity is monotonic by Lemma 7 also $Y(F)$ is cyclic. This is contrary to the acyclicity of $Y(F)$. The assumption that led to this contradiction is $\not\models F\setminus\{B \rightarrow x\} \models B \rightarrow x$, which is therefore false.

The conclusion $\models F\setminus\{B \rightarrow x\} \models B \rightarrow x$ implies that $\models F\setminus\{B \rightarrow x\}$ is equivalent to $F$. This proves that a clause $B \rightarrow x$ with $B \neq A''$ is redundant in $F$. Since $\models F\setminus\{B \rightarrow x\}$ is equivalent to $F$, it has the same properties. Therefore, every other clause $C \rightarrow x \in F\setminus\{B \rightarrow x\}$ with $C \neq A''$ is redundant in it. This procedure can be iterated and shows that all clauses with head $x$ but one can be removed one at a time.

The converse is always the case as shown by Lemma 2. The conclusion is that Condition 1 is the same as single-head equivalence on semantically acyclic formulae.

**Corollary 1** A semantically acyclic formula is single-head equivalent if and only if it satisfies Condition 1.

### 3.3 Cyclicality and equivalence

Simple cycles of clauses like $a \rightarrow b$, $b \rightarrow c$ and $c \rightarrow a$ are easy to detect semantically: some variable (like $a$) is equivalent to another (for example, $c$) thanks to the given clauses. In general, formula $F$ is cyclic if $F \models x \equiv y$ for some variables $x$ and $y$. Unfortunately, this is only the case for cycles of binary clauses.

A cycle may also contain clauses with more than one variable in their body. For example, $F = \{ab \rightarrow c, ab \rightarrow d, cd \rightarrow a, cd \rightarrow b\}$ require extending the condition from single variables to sets: $F \models \{a, b\} \equiv \{c, d\}$. In general, the condition is $F \models A \equiv B$, where $A$ and $B$ are different set of variables. Unfortunately, this condition is also satisfied by $F = \{a \rightarrow b\}$, which does not contain cycles, with $A = \{a\}$ and $B = \{a, b\}$.

The two sets of variables being different is not enough. If $A$ is contained in $B$ then $F \models A \equiv B$ is the same as $F \models A \rightarrow B$: this is a one-way implication, not a cycle. The
other direction $F \models B \rightarrow A$ is a consequence of monotonicity ($B \models A$), not of the clauses of $F$.

A further refinement is to define the converse of cyclicity as “if $F \models A \equiv B$” then either $A \subseteq B$ or $B \subseteq A$. This way, if $A$ and $B$ are made equivalent by $F$, then either one contains the other. The intent is to allow equivalences only if one direction is due to monotonicity of entailment. A counterexample fails this attempt: $F = \{ a \rightarrow b, a \rightarrow c \}$ makes $\{a, b\}$ equivalent to $\{a, c\}$ in spite of the absence of cycles. A variable implies other two, making two sets equivalent in spite of none being contained in the other. While $\{a\}$ is contained in both $\{a, b\}$ and $\{a, c\}$, transitivity makes $\{a, b\}$ equivalent to $\{a, c\}$ but breaks containment. In general, two sets may be equivalent to each other while not being contained one in the other; however, they must both be equivalent to a common subset of them for the formula to be acyclic.

**Condition 3** A formula $F$ is inequivalent if $F \models A \equiv B$ for two sets of variables $A$ and $B$ implies $F \models A \equiv A \cap B$.

Every acyclic formula is inequivalent, but not the other way around. Inequivalence fails at expressing acyclicity; yet, a following section shows an efficient algorithm for single-head equivalence of inequivalent formulae. Inequivalence is broader than acyclicity (includes more formulae) but still maintains polynomiality of single-head equivalence. This is why inequivalence is further analyzed in spite of not expressing acyclicity.

**Lemma 11** Every semantically acyclic formula is inequivalent, that is, satisfies Condition 3.

**Proof.** The claim is proved in the opposite direction: if a formula is not inequivalent, its semantic graph contains a nontrivial cycle.

The assumption is that $F \models A \equiv B$ holds but $F \models A \equiv A \cap B$ does not for some sets of variables $A$ and $B$.

If $F \models B \equiv A \cap B$ then $F \models A \equiv A \cap B$ because of $F \models A \equiv B$. This is contrary to assumption. Therefore, $F \not\models B \equiv A \cap B$.

The set $A$ may be redundant: $F \models A \equiv A \{a\}$ may be the case for some $a \in A$. If so, $A \{a\}$ has the same semantical properties of $A$ because of equivalence: $F \models A \{a\} \equiv B$ and $F \models A \{a\} \equiv A \cap B$. The latter implies $F \not\models A \{a\} \equiv (A \{a\}) \cap B$. This is proved by contradiction: its converse $F \models A \{a\} \equiv (A \{a\}) \cap B$ includes $F \models (A \{a\}) \cap B \rightarrow A \{a\}$, which implies $F \models (A \{a\}) \cap B \rightarrow A$ because $A \{a\}$ is equivalent to $A$. By monotonicity of entailment, $F \models A \cap B \rightarrow A$. The other direction $F \models A \rightarrow A \cap B$ of this entailment holds because $A \cap B \subseteq A$. The result $F \models A \cap B \equiv A$ contradicts the assumption, proving $F \not\models A \{a\} \equiv (A \{a\}) \cap B$. The conclusion is that if $A$ is equivalent to $A \{a\}$, the assumptions of the claim also hold for $A \{a\}$. Iteratively, all redundant elements of $A$ can be removed without affecting the assumptions. By symmetry, the same holds for $B$. If the assumptions of the lemma hold for $A$ and $B$, they also hold for some irredundant subsets of them. Both $A$ and $B$ can be assumed irredundant.

If $A \subseteq B$ then $A \cap B = A$, which contradicts $F \not\models A \equiv A \cap B$. As a result, $A$ contains some elements that are not in $B$. By symmetry, the same holds for $B$. The sets of these elements are denoted $A' = A \backslash B$ and $B' = B \backslash B$.

Let $a \in A'$. The assumption $F \models A \equiv B$ implies $F \models A \rightarrow B$. This entailment can be rewritten as $F \models (A \{a\} \cup \{a\}) \rightarrow B$. This is the first part of the definition of an edge from
a to every element \( b \in B \) in the semantic graph of \( F \). The second part is \( F \not\models (A\setminus\{a\}) \rightarrow b \).

It may not the case for all \( b \in B \) but it is for some. Otherwise, \( F \models (A\setminus\{a\}) \rightarrow b \) for all \( b \in B \) would imply \( F \models (A\setminus\{a\}) \rightarrow B \). Since \( F \models B \equiv A \), this implies \( F \models (A\setminus\{a\}) \rightarrow A \), while \( A \) is assumed irredundant. As a result, \( F \not\models (A\setminus\{a\}) \rightarrow b \) holds for at least a variable \( b \in B \). This variable \( b \) is not the same as \( a \) because \( a \in A' = A \setminus B \) while \( b \in B \). Also, \( b \not\in A\setminus\{a\} \) because otherwise \( (A\setminus\{a\}) \rightarrow b \) would be a tautology. Since \( b \in B \) but \( b \not\in A \), it follows \( b \in B \setminus A = B' \). All of this proves that for every \( a \in A' \) the semantical graph of \( F \) contains at least an edge from \( a \) to an element of \( B' \).

By symmetry, the same applies to every element of \( B' \). Following these edges, an element of \( A' \) leads to an element of \( B' \), which leads to an element of \( A' \) and so on. Since the number of variables is finite, at some point this path leads to a previous variable, forming a cycle. \( \square \)

It would be nice if the converse of this lemma also holds, giving a necessary and sufficient condition to cyclicity. Unfortunately, this is not the case, as Formula 3 shows.

\[
F = \{ab \rightarrow x, bx \rightarrow c, ac \rightarrow d, d \rightarrow x\}
\]

This formula \( F \) is inequivalent and cyclic as proved by the next lemma. Cyclicity is evident when \( F \) is show graphically, but has to be proved since semantic cyclicity does not follow from syntactic cyclicity.

![Graph of the formula](image)

**Lemma 12** Formula \( F = \{ab \rightarrow x, bx \rightarrow c, ac \rightarrow d, d \rightarrow x\} \) is inequivalent (Condition 3) and semantically cyclic.

Proof. The cycle in \( F \) is \( x \rightarrow c, c \rightarrow d, d \rightarrow x \in E(F) \). The first edge \( x \rightarrow c \) is in \( E(F) \) because \( F \) contains \( xb \rightarrow c \) but does not entail \( b \rightarrow c \). The second edge \( c \rightarrow d \) is in \( E(F) \) because \( F \) contains \( ac \rightarrow d \) but does not entail \( a \rightarrow d \). The third edge \( d \rightarrow x \) is in \( E(F) \) because \( F \) contains \( d \rightarrow x \) but does not entail \( x \).

This formula contains a cycle, which implies some sort of equivalence. Why is \( F \) inequivalent, then? For example, a cycle comprising \( x \) and \( d \) means that \( x \) entails \( d \) with other variables and \( d \) entails \( x \) with other variables. If the set of all these other variables is \( O \), then \( O \cup \{x\} \) and \( F \) imply \( O \cup \{d\} \) and vice versa: \( F \models (O \cup \{x\}) \equiv (O \cup \{d\}) \). Differing but equivalent sets of variables suggest that inequivalence does not hold, but are not enough: they should also not be equivalent to their intersection. In this case, \( O \) comprises \( a \) and \( b \), which alone imply all other variables. Therefore, \( F \) also implies \( O \equiv (O \cup \{x\}) \equiv (O \cup \{d\}) \), making inequivalence true. The same applies to every other pair of variables in the cycle. The cycle does not contradict inequivalence because it requires some variables that imply all variables in the cycle.

Inequivalence is formally proved for every pair of sets of variables \( A \) and \( B \): if \( F \) entails \( A \equiv B \) then it also entails \( A \equiv A \cap B \).
If \( A \) contains both \( a \) and \( b \), it entails all variables. If \( B \) does not contain both \( a \) and \( b \), it is not equivalent to \( A \) since \( a \) and \( b \) are not heads of any clause. Therefore, both \( A \) and \( B \) contain both \( a \) and \( b \). Their intersection contains \( \{a, b\} \), and is therefore equivalent to \( A \) because it entails all variables like \( A \) does. This proves inequivalence for all pairs of sets where one of them contains both \( a \) and \( b \).

The remaining sets may contain \( a \) or \( b \) but not both. Since \( a \) is not entailed by any other variables, if \( a \in A \) then \( a \in B \), since otherwise \( F \models B \rightarrow A \) does not hold. The same holds in the other direction, and also for \( b \) for the same reason. In other words, the remaining cases are: both \( A \) and \( B \) contain \( a \) but not \( b \), they both contain \( b \) but not \( a \), and they contain neither.

In the first case, since neither \( A \) nor \( B \) contain \( a \) and no clause has \( a \) in the head, the clauses \( ab \rightarrow x \) and \( ac \rightarrow d \) are not relevant. Therefore, inequivalence in \( F \) is the same as inequivalence in \( F \setminus \{ab \rightarrow x, ac \rightarrow d\} = \{bx \rightarrow c, d \rightarrow x\} \). This formula is syntactically acyclic. Therefore, it is semantically acyclic by Lemma 9 and consequently inequivalent by Lemma 11.

The same applies to the second case, where the clauses not containing \( b \) as preconditions are \( \{ac \rightarrow d, d \rightarrow x\} \), and are again acyclic and therefore inequivalent. In the third case the only clause left is \( \{d \rightarrow x\} \), and again acyclicity implies inequivalence. This shows that \( F \) is inequivalent.

Inequivalence is not the same as semantic acyclicity. Yet, a following section shows it useful in extending the range of tractability of single-head equivalence, from semantically acyclic to inequivalent formulae.

A further use of inequivalence is to answer a question from a previous section: are Condition 1 and 2 sufficient to single-head equivalence? Lemma 12 shows that Formula 3 is inequivalent; the next lemma shows that inequivalence implies Condition 2.

**Lemma 13** Every inequivalent formula satisfies Condition 2.

**Proof.** The condition to be proved is that \( F \models A \rightarrow B \) implies the existence of a set \( C \in \text{equiset}(B, F) \) such that \( C \setminus \text{equiall}(B, F) \subseteq A \), where:

\[
\text{equiset}(A, F) = \{B \mid F \models A \equiv B\} \\
\text{equiall}(A, F) = \{x \mid \forall C \in \text{equiset}(A, F). x \in C\}
\]

Since \( \text{equiall}(B, F) \) contains the variables that are in all sets in \( \text{equiset}(B, F) \), it can be equivalently defined as \( \text{equiall}(B, F) = \bigcap \text{equiset}(B, F) \).

This set \( \bigcap \text{equiset}(B, F) \) is proved equivalent to \( B \) if \( F \) is inequivalent. By definition, \( B' \in \text{equiset}(B, F) \) if and only if \( F \models B \equiv B' \). By inequivalence, \( F \models B \equiv B \cap B' \). Let \( B'' \) be another element of \( \text{equiset}(B, F) \). Since \( F \models B \equiv B'' \), transitivity implies \( F \models B \cap B' \equiv B'' \). By inequivalence again, \( F \models B \cap B' \equiv B \cap B' \cap B'' \), and then by transitivity \( F \models B \equiv B \cap B' \cap B'' \). This procedure can be iterated over all elements of \( \text{equiset}(B, F) \), proving that \( F \) makes their intersection equivalent to \( B \), that is, \( F \models B \equiv \bigcap \text{equiset}(B, F) \). This entailment can be rewritten \( F \models B \equiv \text{equiall}(B, F) \), which proves \( \text{equiall}(B, F) \in \text{equiset}(B, F) \).

The required set \( C \) is \( C = \text{equiall}(B, F) \), since it is in \( \text{equiset}(B, F) \) and \( C \setminus \text{equiall}(B, F) = \emptyset \subseteq A \).
Lemma 4 proves that Formula 3 satisfies Condition 1. Lemma 12 shows that it is inequivalent, and all inequivalent formulae satisfy Condition 2 by Lemma 13. Although this formula satisfies both conditions, it is not single-head equivalent as proved by Lemma 27.

**Corollary 2** Some inequivalent formulae satisfy both Condition 1 and Condition 2 but are not single-head equivalent.

### 3.4 Acyclicity and Redundancy

Theorem 1 proves that removing clauses may not make a single-head equivalent formula single-head. It proves that some formulae are equivalent to a single-head formula but none of their equivalent subsets is single-head. Its proof uses a cyclic formula. Does it hold on acyclic formulae as well? For semantically acyclic formula, yes. A counterexample shows a formula that is irredundant and not single-head but equivalent to a single-head formula.

**Lemma 14** Some semantically acyclic irredundant formulae are not single-head, but they are equivalent to a single-head formula.

**Proof.** A formula that meets the statement of the lemma is \( F = \{ bx \rightarrow a, b \rightarrow x, a \rightarrow x \} \). None of its clauses is entailed by the others.

In spite of the syntactic cycle made of \( x \rightarrow a \) and \( a \rightarrow x \), it is semantically acyclic because the first edge \( x \rightarrow a \) is only due to \( bx \rightarrow a \), but the formula entails \( b \rightarrow a \).

![Diagram](image)

The variable \( x \) is the head of two clauses of \( F \). An equivalent formula is \( F' = \{ b \rightarrow a, a \rightarrow x \} \), since \( bx \rightarrow a \) and \( b \rightarrow x \) resolve into \( b \rightarrow a \), which subsumes \( bx \rightarrow a \).

![Diagram](image)

This formula is single-head and equivalent to \( F \).

An immediate consequence of this lemma is the specialization of Theorem 1 to semantically acyclic formulae. It shows a semantically acyclic formula that is not equivalent to any of its proper subsets. Its only equivalent subset is itself. It is not single-head but is equivalent to a single-head formula.

**Corollary 3** Some semantically acyclic formulae are equivalent to single-head formulae, but no equivalent subset of them is single-head.
The formula in the proof of the lemma is semantically acyclic but syntactically cyclic. Irredundancy is a syntactical property: no clause of the formula is entailed by the others; it is not maintained when switching from a formula to an equivalent one. Being syntactic, irredundancy matches syntactical acyclicity more than semantical acyclicity. The property that is disproved for general and semantically acyclic formula holds for syntactically acyclic formulae: they are single-head equivalent if and only if they have a single-head subset.

More generally, syntactically acyclic formulae do not present the main trap in single-head equivalence, the possibility that a clause \(B \rightarrow x\) seems to follow from \(A \rightarrow x \in F\) via \(F \models B \rightarrow A\) but it does not because \(F \models B \rightarrow A\) requires \(B \rightarrow x\) itself. The following lemma excludes such a dependency in certain conditions.

**Lemma 15.** If \(F \models A \rightarrow x\) and \(y \rightarrow x \not\in E(F \setminus \{B \rightarrow y\})\) then \(F \setminus \{B \rightarrow y\} \models A \rightarrow x\).

*Proof.* The starting point is the trivial entailment \(y \models B \rightarrow y\). A consequence of it is \(F \setminus \{B \rightarrow y\} \cup \{y\} \models F \setminus \{B \rightarrow y\} \cup \{B \rightarrow y\}\). The entailed formula is \(F\), which entails \(A \rightarrow x\) by assumption. Transitivity implies \(F \setminus \{B \rightarrow y\} \models A \rightarrow x\), which is the same as \(F \setminus \{B \rightarrow y\} \models (A \cup \{y\}) \rightarrow x\).

This entailment with \(F \setminus \{B \rightarrow y\} \not\models A \rightarrow x\) implies \(y \rightarrow x \in E(F \setminus \{B \rightarrow y\})\). Since this consequence is false, the first premise is true; therefore, the second premise is false: \(F \setminus \{B \rightarrow y\} \not\models A \rightarrow x\). This is the claim. \(\square\)

This lemma involves a semantic graph, flashing the vision of properties of semantically acyclic formulae. It indeed proves that certain entailments from \(F\) carry over to \(F \setminus \{B \rightarrow x\}\), but its premise is on the semantical graph of \(F \setminus \{B \rightarrow x\}\), not of \(F\).

At a first sight, the premise \(y \rightarrow x \not\in E(F \setminus \{B \rightarrow x\})\) of the lemma looks like a consequence of \(y \rightarrow x \in E(F)\). It is not.

This would be the case if the semantic graph were monotonic: a subformula has a subset of the edges of the superformula. This is disproved by the following lemma.

**Lemma 16.** There exists two formulae \(F\) and \(F'\) such that \(F \subseteq F'\) but \(E(F) \not\subseteq E(F')\).

*Proof.* The two formulae are \(F = \{ay \rightarrow x\}\) and \(F' = \{ay \rightarrow x, a \rightarrow x\}\). The former has the edge \(y \rightarrow x \in E(F)\) since it entails \(ay \rightarrow x\) but not \(a \rightarrow x\). The second does not have that edge because it entails \(a \rightarrow x\). \(\square\)

A similar counterexample fails Lemma 15 if its premise \(y \rightarrow x \not\in E(F \setminus \{B \rightarrow y\})\) is replaced by \(y \rightarrow x \not\in E(F)\). The formula is \(F = \{ay \rightarrow x, a \rightarrow b, b \rightarrow y\}\); the removed clause is \(B \rightarrow x = b \rightarrow y\). The premises of the modified lemma are satisfied: \(F \models a \rightarrow x\) and \(y \rightarrow x \not\in E(F)\); the second holds in spite of \(ay \rightarrow x \in F\) because of \(F \models a \rightarrow x\). Yet, its conclusion \(F \setminus \{b \rightarrow y\} \models a \rightarrow x\) is false.

For its intended usage the premise \(y \rightarrow x \not\in E(F)\) should hold on the whole formula, not the formula without the clause. The counterexample shows that the lemma does not extend this way. It does not because the semantic graph is not monotonic with respect to the formula as proved by Lemma 16. The syntactic graph is, as proved by Lemma 16. This allows extending Lemma 15 to \(y \rightarrow x \not\in Y^*(F)\).

**Lemma 17.** If \(F \models A \rightarrow x\) and \(y \rightarrow x \not\in Y^*(F)\) then \(F \setminus \{B \rightarrow y\} \models A \rightarrow x\).
Lemma 7, removing clauses may only remove edges. This implies \( y \to B \) is an argument in the paragraph above proves that if \( y \to \) is redundant, which is the claim. Otherwise, \( x \) is one of the two clauses of \( B \) never require \( B \to x \) to entail \( B \to A \) if they entail both \( B \to x \) and \( A \to x \). It kills all the subtlety of single-head equivalence.

A consequence is that irredundancy makes single-head the same as single-head equivalence for syntactically acyclic formula.

**Theorem 3** If an irredundant syntactically acyclic formula is single-head equivalent, it is single-head.

**Proof.** The proof assumes that \( F \) is syntactically acyclic, not single-head but equivalent to a single-head formula \( F' \); the conclusion is that \( F \) is redundant.

Let \( x \) be a variable that is the head of two clauses of \( F \). Since \( F' \) is single-head, it contains either a single clause \( A \to x \) with head \( x \) or none.

The latter case is analyzed first: \( F' \) does not contain any clause with head \( x \). Let \( B \to x \) be one of the two clauses of \( F \) with head \( x \). If \( x \in B \) then \( B \to x \) is a tautology and \( F \) is redundant, which is the claim. Otherwise, \( x \notin B \). Since \( F' \) is equivalent to \( F \), it entails \( B \to x \). Lemma \( \llbracket \) proves \( F' \equiv F' \cap \{ A \to x \} \). This formula is equivalent to \( F \) and does not contain clauses with \( x \) in the head. The argument in the paragraph above proves that if \( F \) is equivalent to a formula that has no clause with head \( x \) then \( F \) is redundant.

The remaining case is that \( F' \) contains a single clause \( A \to x \) with head \( x \).

If \( F' \) also entails a clause \( A' \to x \) with \( A' \subseteq A \) then \( F' \equiv F' \cap \{ A \to x \} \cup \{ A' \to x \} \) since \( A' \to x \equiv A \to x \) always holds and \( F' \equiv A' \to x \) holds by assumption. Since the head of these two clauses is the same, the second formula is also single-head. Iteratively, this argument proves that if \( F \) is single-head equivalent it is also equivalent to a single-head formula whose clauses \( A \to x \) are minimal: this formula does not entail any clause \( A' \to x \) with \( A' \subset A \).

Membership implies entailment: \( A \to x \in F' \) implies \( F' \mid A \to x \). Since \( F' \equiv F \) to \( F \), they entail the same clauses, including \( A \to x \). Since \( F' \) does not entail any clause \( A' \to x \) with \( A' \subset A \), the same holds for \( F \). As a result, \( F \) entails \( A \to x \) but not \( A' \mid \{ a \} \to x \) if \( a \in A \). This is the definition of \( a \to x \in E(F) \). It implies \( a \to x \in Y^*(F) \) by Lemma \( \llbracket \). This holds for every \( a \in A \).

Since \( F \) is equivalent to \( F' \), it entails \( A \to x \). Since the case \( x \in A \) is already excluded, Lemma \( \llbracket \) proves that \( F \) contains a clause \( B \to x \) such that \( F' \mid A \to B \).

By assumption, \( F \) contains two clauses of head \( x \). One is \( B \to x \). Let \( C \to x \) be the other. If \( x \in C \) then this clause is a tautology and \( F \) is redundant, which is the claim. Otherwise, \( x \notin C \). Since \( F' \) is equivalent to \( F \) it entails \( C \to x \). By Lemma \( \llbracket \) \( F' \) contains a clause \( D \to x \) such that \( F' \mid C \to D \). Since \( F' \) is single-head and contains \( A \to x \), this

\[ 23 \]
clause $D \rightarrow x$ is $A \rightarrow x$. The conclusion is $F' \models C \rightarrow A$. Since $F$ is equivalent to $F'$, it entails $C \rightarrow A$.

The entailment $F \models C \rightarrow A$ is the same as $F \models C \rightarrow a$ for every $a \in A$. By Lemma 17 if $F \setminus \{C \rightarrow x\} \not\models C \rightarrow a$ then $x \rightarrow a \in Y^*(F)$. With the edge $a \rightarrow x$ already proved in $Y^*(F)$, this forms a cycle in $Y^*(F)$. Since $F$ is syntactically acyclic, the assumption $F \setminus \{C \rightarrow x\} \not\models C \rightarrow a$ is false. The contrary is true: $F \setminus \{C \rightarrow x\} \models C \rightarrow a$. This being the case for every $a \in A$, the conclusion is $F \setminus \{C \rightarrow x\} \models C \rightarrow A$.

The following are proved so far: $F \setminus \{C \rightarrow x\} \models C \rightarrow A$, $F^x \models A \rightarrow B$ and $B \rightarrow x \in F$. Since $C \rightarrow x$ contains $x$ it is not in $F^x$. Therefore, $F^x \subseteq F \setminus \{C \rightarrow x\}$. A consequence is $F \setminus \{C \rightarrow x\} \models A \rightarrow B$. Since $C \rightarrow x$ is a clause different from $B \rightarrow x$, the condition $B \rightarrow x \in F$ implies $B \rightarrow x \in F \setminus \{C \rightarrow x\}$.

The three conclusions $F \setminus \{C \rightarrow x\} \models C \rightarrow A$, $F \setminus \{C \rightarrow x\} \models A \rightarrow B$ and $B \rightarrow x \in F \setminus \{C \rightarrow x\}$ imply $F \setminus \{C \rightarrow x\} \models C \rightarrow x$. Since $C \rightarrow x$ is a clause of $F$ by assumption, $F$ is redundant.

This theorem shows that single-head equivalent is the same as single-head for syntactically acyclic and irredundant formulae. It provides a simple algorithm for checking single-head equivalence on syntactically acyclic formulae: their redundant clauses are removed one by one in any order and the result is checked for the presence of clauses with the same head.

Does something similar exist for semantically acyclic formulae?

Irredundancy is syntactic, like syntactic acyclicity. Semantic acyclicity is semantic. What is the semantic version of irredundancy?

Irredundancy is syntactic minimality. A formula is irredundant if it has no equivalent proper subset; it has no smaller equivalent formula made only of its clauses. It is syntactic because of the latter point: “made only of its clauses”. When the specific clauses of a formula matter, syntax is in.

If irredundancy is syntactic minimality, what is semantic minimality? When the individual clauses of the formula do not matter, only size remains. Minimal is minimal by number of literal occurrences. Which is the same as number of clauses as Hammer and Kogan [HK95] proved for semantically acyclic formula.

If this analogy works, minimality matches semantical acyclicity in the same way irredundancy matches syntactic acyclicity. A semantical acyclic and minimal formula is single-head equivalent if and only if it is single-head. That this analogy works is what the next two lemmas prove.

The first lemma shows that semantic acyclicity implies syntactic acyclicity for minimal formulae. The converse is always the case.

**Lemma 18** If a definite Horn formula is equivalent to no formula with less literal occurrences and is semantically acyclic then it is also syntactically acyclic.

**Proof.** Let $F$ be a minimal formula. The first step of the proof is that it only contains prime implicates. First, it only contains some of its implicates because it entails every clause it contains. Contrary to the claim, it is assumed to entail a proper subset $C'$ of a clause $C$ it contains. Since subclauses entail superclauses, $C'$ entails $C$. As a result, $F \setminus \{C\} \cup \{C'\}$ entails $F \setminus \{C\} \cup \{C\} = F$. The converse is also the case since by assumption $F$ entails $C'$, the only clause of $F \setminus \{C\} \cup \{C'\}$ it does not contain. This makes $F$ equivalent to $F \setminus \{C\} \cup \{C'\}$, which is smaller than $F$ because it contains $C' \subset C$ in place of $C$. This
contradicts the assumption that $F$ is minimal. The conclusion is that $F$ only contains some of its prime implicates.

This conclusion can be written $F \subseteq P_I(F)$ where $P_I(F)$ is the set of the prime implicates of $F$. This implies $P_I(F) \models F$ by monotonicity. The converse is also the case since $P_I(F)$ is only made of clauses of $F$ by definition. Entailment in both direction is equivalence: $F \equiv P_I(F)$.

If $F$ is semantically acyclic, also $P_I(F)$ is semantically acyclic because they are equivalent and semantic acyclicity does not depend on the syntax.

By Lemma 6 since $P_I(F)$ coincides with its set of prime implicates and is semantically acyclic it is also syntactically acyclic. Since $F$ is a subset of $P_I(F)$, the same containment holds on their syntactic graphs by Lemma 7. $Y(F) \subseteq Y(P_I(F))$. If $Y(F)$ contains a cycle, then $P_I(F)$ contains the same cycle. This is not possible because $P_I(F)$ is syntactically acyclic. Therefore, $F$ is syntactically acyclic.

Why this lemma? Proving that semantic acyclicity implies syntactic acyclicity, it makes Theorem 3 applicable: a syntactic acyclic irredundant formula is single-head if it is single-head equivalent.

**Lemma 19** If a definite Horn formula is equivalent to no formula with less literal occurrences, is semantically acyclic and single-head equivalent, then it is single-head.

**Proof.** By Lemma 18 a formula that is minimal and semantically acyclic is also syntactically acyclic. Since it is minimal it is also irredundant. Being irredundant, syntactically acyclic and single-head equivalent, it is single-head by Theorem 3.

The converse is always the case since every single-head formula is trivially single-head equivalent. On minimal semantically acyclic formulae, single-head equivalence is the same as being single-head. Except that checking for being single-head requires only a simple scan of the formula for clauses with the same head.

The algorithm for single-head equivalence of a semantically acyclic formula is similar to that for syntactically acyclic formulae. Minimality takes the place of irredundancy: first the formula is made minimal, then the presence of clauses of the same head is checked.

Hammer and Kogan [HK95] provided a quadratic algorithm for finding the minimal formula equivalent to a given one. Checking for clauses with the same head is quadratic as well, making single-head equivalence checking quadratic in running time.

## 4 The order and the algorithm

### 4.1 Why bothering with single heads?

After all this talking about single-head equivalence, about necessary conditions to single-head equivalence, about the pitfalls of sufficient conditions to single-head equivalence, about cyclicity and single-head equivalence, about redundancy and minimality and single-head equivalence, after all of this it may look like the question is only to establish single-head equivalence — tell whether a formula is equivalent to a single-head formula. Maybe this is interesting because it tells in advance whether the forgetting algorithm takes polynomial time.
This is not the main question. Single-head equivalence does not even make the forgetting algorithm polynomial. That requires formulae that are single-head, not single-head equivalent.

The goal is not to establish single-head equivalence, but to find a single-head equivalent formula, if any.

If any, the replacing algorithm takes polynomial time on it even if it does not on the original formula. The translation from single-head equivalent to single-head reduces the running time from exponential to polynomial. It is an efficiency improvement of the forgetting algorithm, not just an efficiency estimation.

This section gives an actual method for turning a formula into a single-head equivalent one. It is incomplete: it always produces a single-head formula which may or may not be equivalent to the given one. If it is, the outcome is not only a proof of single-head equivalence, but also the required single-head equivalent formula.

The forgetting algorithm only takes polynomial time on it even if it takes exponential time on the original one. This happens for example on the following formula.

\[
\{a \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow e, e \rightarrow f, a \rightarrow c, b \rightarrow d, c \rightarrow e, d \rightarrow f\}
\]

A pictorial representation shows what is wrong with F: it is a chain of clauses surrounded by useless jumps. These redundant clauses are all entailed by transitivity, and can be removed without affecting the semantics of the formula. Yet, they are processed by the forgetting algorithm as if they were essential.

The algorithm replaces each variable to forget with the body of a clauses having it as its head. If the variable is in the head of two clauses, each of them provides a way to forget it. The algorithm nondeterministically tries both.

It forgets e by replacing it in e \(\rightarrow\) f. Since e is the head of c \(\rightarrow\) e and d \(\rightarrow\) e, it replaces it with c in a nondeterministic branch and with d in another.

If d is also to be forgotten, it branches again to replace it with either c or b. To forget c it branches once again in each of the four deterministic branches, making them eight. This behavior can be observed on the first formula of the singlehead.py test file of the forget-fork.py program, which implements the replacing algorithm for forgetting \cite{Lib20a}. The version of this formula with \(n\) variables requires \(O(2^n)\) branches.

This is not the case for the following formula, which is single-head and equivalent to the previous as shown by the chain.py test file of the singlehead.py program.

\[
\{a \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow e, e \rightarrow f\}
\]
This formula is a chain of clauses. As shown by the second formula of the `singlehead.py` test file of the `forget-fork.py` program, the forgetting algorithm starts with $e \rightarrow f$ and turns it into $d \rightarrow f$, then $c \rightarrow f$, $b \rightarrow f$ and finally into $a \rightarrow f$, the expected result. It does not branch. It takes linear time.

A variant of the algorithm that adds all possible replacements instead of branching on each would solve the problem, but the solution is worse than the problem since it requires exponential space in general while the nondeterministic algorithm runs in polynomial space.

The algorithm $SHMIN(F)$ shown at the end of this section turns $F$ into $F'$. Success is not guaranteed: while $SHMIN(F)$ is always single-head, it is not always equivalent to $F$. If it is, the replacing algorithm can be run on $SHMIN(F)$ instead of $F$, reducing time from exponential to polynomial. Otherwise, it is run on $F$ itself; the unsuccessful call to $SHMIN(F)$ only adds polynomial time to the bare algorithm. Also in this case, $SHMIN(F)$ can be added clauses to make it equivalent to $F$, and this may reduce the running time of the forgetting algorithm.

### 4.2 The induced order

The single-head form (if any) of a formula can be found by ordering the sets of literals.

If $F$ is single-head equivalent, it is equivalent to a single-head formula $F'$. For each variable $x$, this formula $F'$ may only contain a single clause $A \rightarrow x$ with $x$ as its head. If $F$ contains another clause $B \rightarrow x$, then $F$ implies it. By equivalence, also $F'$ implies it: $F' \models B \rightarrow x$. Since formulae are assumed not to contain tautologies, Lemma 1 implies $B' \rightarrow x \in F'$ with $F \models B \rightarrow B'$. Since $F'$ is single-head, $B' \rightarrow x$ is the same as $A \rightarrow x$. Therefore, $F' \models B \rightarrow A$. By equivalence, $F \models B \rightarrow A$.

In summary, if $F$ contains a clause $B \rightarrow x$, then $F$ implies $B \rightarrow A$ where $A \rightarrow x$ is the clause of head $x$ in $F'$.

The goal is to build $F'$ from $F$. To generate its clauses, like $A \rightarrow x$, the above property helps. It tells that $A$ is a set of literals such that $B \rightarrow x \in F$ implies $F \models B \rightarrow A$. This restricts the range of possible sets $A$ to the ones at the end of chains of implications from all other sets $B$ such that $B \rightarrow x \in F$.

```
B
```

```
A
```

$x$

Shortening $F \models B \rightarrow A$ to $A \leq_F B$ clarifies the aim: for each variable $x$, find a set of variables $A$ that is minimal according to $\leq_F$.

What complicates the search is that the order may not be total. The counterexample is $F = \{a \rightarrow b, c \rightarrow d, bd \rightarrow e\}$. 
The two incomparable sets are \{a, d\} and \{c, b\}. They are incomparable because none implies the other; for example, \( F \not\models ad \rightarrow cb \) because \( F \not\models ad \rightarrow c \). Still, both sets entail \( e \). Graphically, they form a sort of “cross” in the diagram, so that each set contains a literal that remains on the left of the other set, while \( e \) is on the right of both.

The figure also suggests a property they possess: they “converge” before their union is “over”. They have a common consequence that is part of their union and still implies \( e \).

That they have a common consequence is obvious: \( e \). The point is that they also have a common consequence that only contains their elements and still entails \( e \); the set \( \{b, d\} \). The following lemma proves that this is not a coincidence.

**Lemma 20** If \( F \) is a single-head formula such that \( F \models A \rightarrow x \) and \( F \models B \rightarrow x \) there exists \( C \subseteq A \cup B \) such that \( F^x \models A \rightarrow C \), \( F^x \models B \rightarrow C \) and \( F \models C \rightarrow x \).

**Proof.** The proof is by induction on the size of \( F \). If \( F \) is empty the preconditions \( F \models A \rightarrow x \) and \( F \models B \rightarrow x \) imply \( x \in A \) and \( x \in B \). The claim holds with \( C = \{x\} \).

The induction case assumes \( F \models A \rightarrow x \) and \( F \models B \rightarrow x \) and postulates that the lemma holds for every formula smaller than \( F \). If \( x \in A \) or \( x \in B \) the claim holds with \( C = \{x\} \). The rest of the proof assumes \( x \notin A \) and \( x \notin B \).

Applying Lemma 1 to \( F \models A \rightarrow x \) and \( F \models B \rightarrow x \) proves \( F^x \models A \rightarrow P \) and \( F^x \models B \rightarrow P' \) with \( P \rightarrow x, P' \rightarrow x \in F \). Since \( F \) is single-head, \( P' \) is equal to \( P \). As a result, \( P \) satisfies most of the requirements of this lemma: \( F^x \models A \rightarrow P \), \( F^x \models B \rightarrow P \) and \( F \models P \rightarrow x \). It may not meet \( P \subseteq A \cup B \).

The rest of the proof shows how to distill a set \( C \) that meets all conditions of the claim from \( P \). Let \( y \) be an element of \( P \backslash (A \cup B) \). Since it is in \( P \), both \( F^x \models A \rightarrow y \) and \( F^x \models B \rightarrow y \) hold. Since \( F \) is single-head, \( F^x \) is also single-head because it is a subset of it. It is a proper subset because it does not contain \( P \rightarrow x \). The claim of the lemma applies by induction: some set \( C_y \) satisfies \( C_y \subseteq A \cup B \), \( F^{xy} \models A \rightarrow C_y \), \( F^{xy} \models B \rightarrow C_y \), and \( F^{xy} \models C_y \rightarrow y \). Since \( F^{xy} \) is a subset of \( F^x \), the latter three imply \( F^x \models A \rightarrow C_y \), \( F^x \models B \rightarrow C_y \), and \( F^x \models C_y \rightarrow y \).

Let \( D = P \backslash (A \cup B) \) and \( C = \bigcup_{y \in D} C_y \). Combining the above conditions for all \( y \in D \) results in \( C \subseteq A \cup B \), \( F^x \models A \rightarrow C \), \( F^x \models B \rightarrow C \), and \( F^x \models C \rightarrow D \). The first condition \( C \subseteq A \cup B \) implies \( P \backslash D \cup C \subseteq A \cup B \) since \( D \) comprises all elements of \( P \) that are not in \( A \cup B \). The second condition \( F^x \models A \rightarrow C \), combines with \( F^x \models A \rightarrow P \) to produce \( F^x \models A \rightarrow (P \backslash D \cup C) \). By symmetry, the third condition produces \( F^x \models B \rightarrow (P \backslash D \cup C) \). The fourth condition \( F^x \models C \rightarrow D \) implies \( F \models (P \backslash D \cup C) \rightarrow (P \backslash D \cup D) \), which can be rewritten as \( F \models (P \backslash D \cup C) \rightarrow x \). Combined with \( F \models P \rightarrow x \), it gives \( F \models (P \backslash D \cup C) \rightarrow x \). The claim is proved. \( \square \)
If both $A$ and $B$ entail $x$ in a single-head formula, both $A$ and $B$ entail the body of the only clause having $x$ as its head. What is not obvious is that this convergence of entailments can be tracked back earlier than that, to a set only comprising variables of $A$ and $B$.

The lemma holds on single-head formulae, but its intended usage is on formulae that are equivalent to single-head ones. The construction $F^x$ does not survive equivalence: $F^x$ is not equivalent to $F^x'$ even if $F'$ is equivalent to $F$. Still, $F^x \models A \rightarrow C$ implies $F \models A \rightarrow C$, which implies $F' \models A \rightarrow C$. The lemma with $F$ in place of $F^x$ holds on single-head equivalent formulae.

Lemma 21 If $F$ is a single-head equivalent formula such that $F \models A \rightarrow x$ and $F \models B \rightarrow x$, there exists $C \subseteq A \cup B$ such that $F \models A \rightarrow C$, $F \models B \rightarrow C$ and $F \models C \rightarrow x$.

Proof. Since $F$ is single-head equivalent, a single-head formula $F'$ with $F \equiv F'$ exists. Lemma 20 applies to $F'$. Everything entailed by $F'^x$ is also entailed by $F'$, and also by $F$ by equivalence. Replacing $F'^x$ by $F$ in the statement of Lemma 20 results in the claim. □

This lemma implies that $\leq_F$ is a downwards directed order if $F$ is single-head equivalent: every pair of elements it compares has a lower bound. However, it proves more than this: a lower bound is contained in the union of the pair.

The lemma can be seen as yet another necessary condition to single-head equivalence, but again is not sufficient. A counterexample is $\{a \rightarrow b, b \rightarrow c, c \rightarrow b\}$, which is not single-head equivalent but satisfies it. This formula is already defined as Formula [1] and proved not single-head equivalent. Yet, it satisfies the claim of the lemma. This is proved for each variable that is the head of a clause, $b$ and $c$. For example, $c$ is entailed by $\{a\}$, $\{b\}$ and $\{a, b\}$. The union of one of these three sets with itself is itself; the union of one with another contains $b$, which implies $c$. The same holds for $b$ by symmetry.

4.3 The formula of minimal bodies

The order between sets of literals suggests how to find a single-head equivalent formula, if any: for every variable $x$, if more than one set of literals imply it, only the minimal one is taken. For example, if $F \models A \rightarrow x$, $F \models B \rightarrow x$ and $F \models C \rightarrow x$, the three sets $A$, $B$ and $C$ are compared according to $\leq_F$. If $C$ is less than $A$ and $B$, then $F \models A \rightarrow C$ and $F \models B \rightarrow C$ hold. Maybe $C \rightarrow x$ is sufficient, since the other two clauses $A \rightarrow x$ and $B \rightarrow x$ are consequences of it and $A \rightarrow C$ and $B \rightarrow C$. Why “maybe”? At this point the traps of single-head equivalence should be clear: $A \rightarrow C$ may be a consequence of $A \rightarrow x$. Or not.

The method works in the other way around. Just because $C$ is minimal does not mean that it is the body of the clause with $x$ in the head in the single-head equivalent formula. But if $C$ is that body, it is minimal. If only one minimal element exists, the clause of the single-head form is found. If the formula is inequivalent (Condition 3), uniqueness is guaranteed. The minimal bodies make the single-head equivalent formula.

$$MIN(F) = \{A \rightarrow x \mid F \models A \rightarrow x \text{ and } \not\exists B. x \notin B, \ F \models B \rightarrow x \text{ and } B <_F A \text{ or } B \subset A\}$$

An inequivalent formula $F$ is single-head equivalent if and only if it is equivalent to $MIN(F)$. A mechanism to establish single-head equivalence is to build $MIN(F)$ and check whether it is single-head and equivalent to $F$. This not only proves single-head equivalence, but also produces the single-head equivalent formula: $MIN(F)$. 29
This procedure is correct but not complete. If it tells that the formula is single-head equivalent, it is. Otherwise, it may still be. An example is the following formula, which is single-head equivalent as proved by the `singlehead.py` program on the `incomplete.py` test file but \( \text{MIN}(F) \) is equal to \( F \), which is not single-head.

\[
F = \{a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow a\}
\]  

The following lemmas tell how \( \text{MIN}(F) \) relates to \( F \) and its single-head equivalence. Their proofs sometimes involve sets that are strictly contained one in the other, like \( B \subset A \). A caveat is that \( B \subset A \) implies \( B \leq_F A \) but not \( B <_F A \). Even if \( B \) is strictly contained into \( A \), they may still be equivalent: \( B \equiv_F A \). As an example, \( F |\equiv \{a, b\} \equiv \{b\} \) if \( F \) is \( \{b \rightarrow a\} \).

A first obvious property is that if \( \text{MIN}(F) \) is single-head and equivalent to \( F \), then \( F \) is single-head equivalent. This is almost the definition of single-head equivalence, only restricted to \( \text{MIN}(F) \). It makes a formal lemma only for being referenced from the following proofs.

**Lemma 22** If \( \text{MIN}(F) \) is single-head and equivalent to \( F \), then \( F \) is single-head equivalent.

**Proof.** By definition, \( F \) is single-head equivalent if and only if it is equivalent to a single-head formula. Such a formula is \( \text{MIN}(F) \).

What is less obvious is that the converse holds in case of inequivalence (Condition \( 3 \)): if \( F \) is inequivalent and single-head equivalent, then \( \text{MIN}(F) \) is single-head and equivalent to it. Inequivalence makes \( \text{MIN}(F) \) the single-head version of \( F \), if any. This claim can be broken in two: \( F \) is equivalent to \( \text{MIN}(F) \), and \( \text{MIN}(F) \) is single-head. Both claims require \( F \) to be inequivalent and single-head equivalent.

Some preliminary results are about single-head formulae.

**Lemma 23** If \( F \) is single-head and contains \( A \rightarrow x \), it does not entail any non-tautologic clause \( B \rightarrow x \) with \( B <_F A \).

**Proof.** By contradiction, \( F \models B \rightarrow x \) is assumed for some \( B <_F A \) with \( x \notin B \). The comparison \( B <_F A \) is defined as \( F \models A \rightarrow B \) and \( F \not\models B \rightarrow A \). By Lemma \( 1 \), \( F \models B \rightarrow x \) and \( x \notin B \) imply the existence of a clause \( C \rightarrow x \in F \) such that \( F \models B \rightarrow C \). If \( C \) were the same as \( A \), it would contradict \( F \not\models B \rightarrow A \). As a result, \( C \) is different from \( A \). Therefore, \( F \) contains two different clauses \( A \rightarrow x \) and \( C \rightarrow x \) with the same head, contradicting the assumption that it is single-head.

Single-head formulae are always equivalent to their formulae or minimal bodies.

**Lemma 24** If \( F \) is single-head, then \( F \equiv \text{MIN}(F) \).
Proof. Since $MIN(F)$ only contains clauses entailed by $F$, it is entailed by $F$. The claim requires proving the converse: if $F$ is single-head, it is entailed by $MIN(F)$.

This is the same as showing $MIN(F) \models A \rightarrow x$ for every $A \rightarrow x \in F$. It is proved by contradiction, assuming $MIN(F) \not\models A \rightarrow x$ for some clause $A \rightarrow x \in F$.

Since $A \rightarrow x$ is not entailed by $MIN(F)$ is not in $MIN(F)$ either. Since it is entailed by $F$, the definition of $MIN(F)$ implies the existence of a non-tautologic clause $B \rightarrow x$ such that $F \models B \rightarrow x$ and either $B <_F A$ or $B \subset A$. These two possibilities can be split differently: the first is $B <_F A$, the second is $B \not<_F A$ and $B \subset A$. The first makes $F \models B \rightarrow x$ and $x \not\in B$ contradict Lemma 23.

The second case is $B \not<_F A$ and $B \subset A$. The containment $B \subset A$ makes $A \rightarrow B$ a tautology. The consequence $F \models A \rightarrow B$ defines $B \leq_F A$. With $B \not<_F A$, it implies $B \equiv_F A$. Let $C$ be a minimal subset of $A$ that is equivalent to it: none of its proper subsets is equivalent to $A$. Such a minimal subset exists because the comparison is by the subset ordering. If $C \rightarrow x \in MIN(F)$ then $MIN(F) \models C \rightarrow x$, which implies $MIN(F) \models A \rightarrow x$ since $C \subset A$. A consequence of $C \rightarrow x \not\in MIN(F)$ is $F \models D \rightarrow x$ with $D <_F C$ or $D \not<_F C$ and $D \subset C$ for some non-tautologic clause $D \rightarrow x$.

If $D <_F C$ then $D <_F A$ because of $C \equiv_F A$; this makes $F \models D \rightarrow x$ and $x \not\in D$ contradict Lemma 23. The other case is $D \not<_F C$ and $D \subset C$. The second condition is the start of the chain of consequences $\models C \rightarrow D, F \models C \rightarrow D$ and $D \leq_F C$. With $D \not<_F C$, the latter implies $D \equiv_F C$. This makes $D \subset C$ contradict the minimality of $C$ among the subsets of $A$ that are equivalent to it.

The converse of this lemma requires $F$ to be inequivalent Condition 3.

Lemma 25 If $F$ is single-head and inequivalent (Condition 3) then $MIN(F)$ is single-head.

Proof. The proof is by contradiction: $A \rightarrow x$ and $B \rightarrow x$ are assumed to both belong to $MIN(F)$, with $A \not\equiv B$; this condition is shown to contradict the assumptions.

By definition, all clauses of $MIN(F)$ are entailed by $F$, including $A \rightarrow x$ and $B \rightarrow x$. By Lemma 21 $F$ also entails a clause $C \rightarrow x$ with $C \subset A \cup B$ such that both $F \models A \rightarrow C$ and $F \models B \rightarrow C$ hold. These two entailments define $C \leq_F A$ and $C \leq_F B$.

Since $F \models C \rightarrow x$, if either $C <_F A$ or $C \subset A$ were true, then $A \rightarrow x \not\in MIN(F)$. As a result, both $C <_F A$ and $C \subset A$ are false. By definition, $C <_F A$ is $C \leq_F A$ and $A \not<_F C$; since $C <_F A$ is false, either $C \leq_F A$ is false or $A \leq_F C$ is true. But the first is true. Therefore, the second is true: $A \leq_F C$. With $C \leq_F A$, it proves $A \equiv_F C$. By symmetry, $B \equiv_F C$.

This proves $A \equiv_F B$. Condition 3 implies $A \equiv_F A \cap B$. This equivalence and $F \models A \rightarrow x$ imply $F \models A \cap B \rightarrow x$.

Since $A \cap B \subset A$, two cases are possible: either this containment is strict or it is an equality. The first case, $A \cap B \subset A$, contradicts the assumption $A \rightarrow x \in MIN(F)$ because $F \models A \cap B \rightarrow x$. In the second case, $A \cap B = A$, it holds $A \subset B$; since $A$ and $B$ are different, this containment is strict: $A \subset B$; this contradicts the assumption $B \rightarrow x \in MIN(F)$ because $F \models A \rightarrow x$.

Combining the latter two lemmas tells that if $F$ is inequivalent, checking whether $MIN(F)$ is single-head and equivalent to $F$ is a way to verify the single-head equivalence of $F$. 

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Lemma 26  If $F$ is inequivalent, it is single-head equivalent if and only if $MIN(F)$ is single-head and equivalent to $F$.

Proof. If $MIN(F)$ is single-head and equivalent to $F$ then $F$ is single-head equivalent by Lemma 22. In the other direction, if $F$ is single-head equivalent then it is equivalent to a single-head formula $F'$ by definition. By Lemma 24, $MIN(F)$ is equivalent to $F'$, and to $F$ by transitivity. Since inequivalence is a semantical property, $F'$ is also inequivalent. By Lemma 25, $MIN(F')$ is single-head. Since the formula of minimal bodies is defined semantically, $MIN(F')$ is the same as $MIN(F)$ and is therefore single-head. 

What happens if $F$ is not inequivalent? The previous counterexample $F = \{ a \to b, b \to a, b \to c, c \to a \}$ shows that the lemma does not extend: while $F$ is equivalent to the single-head formula $\{ a \to b, b \to c, c \to a \}$, the formula $MIN(F)$ is not single-head. Indeed, $\{ a \} \equiv_F \{ b \}$, which implies that both $a \to c$ and $b \to c$ are in $MIN(F)$ because they are entailed by $F$ and no set is less than $\{ a \}$ or $\{ b \}$ according to $\leq_F$.

A consequence of this lemma is that a formula used in a previous counterexample is not single-head equivalent: $F = \{ ab \to x, bx \to c, ac \to d, d \to x \}$. The proof was delayed to this point, where Lemma 26 makes it easy.

Lemma 27  Formula $\mathcal{F}$ is not single-head equivalent.

Proof. Lemma 12 proved Formula $\mathcal{F}$ inequivalent. Lemma 26 proves that an inequivalent formula $F$ is single-head equivalent if and only if $MIN(F)$ is single-head and equivalent to it. The claim is proved by showing that the model $M = \{ a, b \}$ that sets $a$ and $b$ to true and all other variables to false satisfies $MIN(F)$ but not $F$.

It does not satisfy $F$ because it falsifies $ab \to x \in F$. That $M$ satisfies $MIN(F)$ is proved by contradiction, by assuming it does not. This means that it falsifies a clause of $MIN(F)$. Since $M$ assigns true to $a$ and $b$ and false to the other variables, this clause may only have a subset of $\{ a, b \}$ in its body and a variable among $c$, $d$ and $x$ in the head. The body is not empty because $MIN(F)$ only contains clauses entailed by $F$, which does not entail any clause with an empty body. The body does not comprise $a$ only because otherwise $MIN(F)$ contains a clause $a \to h$ with $h \in \{ c, d, x \}$ while $F$ entails no such clause. For the same reason, the body does not comprise $b$ only.

Therefore, the clause of $MIN(F)$ is $ab \to h$ with $h \in \{ c, d, x \}$. For each of the three possible heads, a set $D$ that satisfies $F \models D \to h$, $h \notin D$ and $D <_F \{ a, b \}$ is shown; this proves $ab \to h \notin MIN(F)$, contrary to assumption. Making $<_F$ explicit, what is proved is $F \models D \to h$, $h \notin D$, $F \models \{ a, b \} \to D$ and $F \not\models D \to \{ a, b \}$.

$h = c$ ; the required set $D$ is $\{ b, x \}$; the clause $bx \to c$ is in $F$ and is therefore entailed by it; $F \models ab \to bx$ holds because $\{ a, b \}$ entails all variables; $F \not\models bx \to ab$ holds because no clause of $F$ has head $a$;

$h = d$ ; the required set $D$ is $\{ a, c \}$; the clause $ac \to d$ is in $F$ and is therefore entailed by it; $F \models ab \to ac$ holds because $\{ a, b \}$ entails all variables; $F \not\models ac \to ab$ holds because no clause of $F$ has head $b$;

$h = x$ ; the required set $D$ is $\{ d \}$; the clause $d \to x$ is in $F$ and is therefore entailed by it; $F \models ab \to d$ holds because $\{ a, b \}$ entails all variables; $F \not\models d \to ab$ holds because no clause of $F$ has head $a$.

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In all three cases, a contradiction is reached from the assumption that \( M \) is not a model of \( \text{MIN}(F) \). Therefore, it is. It is not a model of \( F \), which implies \( F \not\equiv \text{MIN}(F) \). Since \( F \) is inequivalent, it is not single-head equivalent by Lemma 26.

\[ \square \]

### 4.4 Computing a formula of minimal bodies

If \( \text{MIN}(F) \) is single-head and equivalent to \( F \), forgetting can be computed in polynomial time because it can be done on \( \text{MIN}(F) \). Inequivalence is not necessary. Forgetting on \( \text{MIN}(F) \) instead of \( F \) can be done even if \( F \) is not inequivalent, if it is equivalent to \( \text{MIN}(F) \) and \( \text{MIN}(F) \) is single-head.

Inequivalence enters into play when \( \text{MIN}(F) \) is either not single-head or not equivalent to \( F \). Neither implies that \( F \) is not single-head equivalent in general. They only do if \( F \) is also inequivalent. This information is algorithmically useful because it ends the quest for a single-head equivalent formula. Forgetting is done on \( F \).

Checking inequivalence is not necessary. Checking whether \( \text{MIN}(F) \) is equivalent to \( F \) is easy since both formulae are Horn. Computing \( \text{MIN}(F) \) is the problem. This formula may be large. An alternative is to find only a clause \( C \rightarrow x \in \text{MIN}(F) \) for each variable \( x \), if any. Such a formula is the same as \( \text{MIN}(F) \) if \( \text{MIN}(F) \) is single-head, which is the only case where \( \text{MIN}(F) \) is useful anyway.

This is the theme of this section: find a formula \( \text{SHMIN}(F) \) that contains a clause \( C \rightarrow x \in \text{MIN}(F) \) for each \( x \), if any. Such a formula is single-head by construction, and equivalent to \( \text{MIN}(F) \) if \( \text{MIN}(F) \) is single-head. If \( \text{MIN}(F) \) is also equivalent to \( F \), then \( \text{SHMIN}(F) \) is a single-head version of \( F \).

The starting point is a clause \( A \rightarrow x \in F \) for each variable \( x \). If no other clause \( B \rightarrow x \) entailed by \( F \) is such that either \( B <_F A \) or \( B \subset A \), then \( A \rightarrow x \) is in \( \text{MIN}(F) \); it is a valid choice for the clause of \( x \) in \( \text{SHMIN}(F) \). Otherwise, \( B \rightarrow x \) is entailed by \( F \) where \( B \) is either less than or contained in \( A \). The same argument applies to \( B \rightarrow x \) in place of \( A \rightarrow x \). Other clauses \( B' \rightarrow x \) with the same properties are irrelevant because the goal is to find a single minimal body for the head \( x \), not all of them.

A subset \( B \subset A \) can be found by looping over all variables \( a \in A \) and testing whether \( F \) implies \( A \setminus \{a\} \rightarrow x \). If so, \( A \rightarrow x \) is replaced by \( A \setminus \{a\} \rightarrow x \) and the search continues from there. The procedure stops when \( A \) is subset-minimal: no \( B \subset A \) satisfies \( F \models B \rightarrow x \).

This state is insufficient for \( A \rightarrow x \in \text{MIN}(F) \) because \( B <_F A \) and \( F \models B \rightarrow x \) may still hold for some set \( B \) that is not a subset of \( A \). The problem is that exponentially many sets of variables \( B \) are to be checked. The set \( \text{BCN}(A,F) \) is used to reduce this number.

\[
\text{BCN}(A,F) = \{ x \mid F \models A \rightarrow x \}
\]

This is the set of variables entailed by \( A \) according to \( F \) \cite{Lib20a}. It is the base of a condition that ensures that a set less than another exists.

**Lemma 28** If \( F \models \text{BCN}(A,F) \setminus \{a,x\} \rightarrow x \) and \( F \not\models \text{BCN}(A,F) \setminus \{a,x\} \rightarrow a \) with \( a \in A \), then \( B = \text{BCN}(A,F) \setminus \{a,x\} \) satisfies \( F \models B \rightarrow x \) and \( B <_F A \).

**Proof.** The claim \( F \models B \rightarrow x \) coincides with the assumption \( F \models \text{BCN}(A,F) \setminus \{a,x\} \rightarrow x \). The condition \( B <_F A \) is defined as \( F \models A \rightarrow B \) and \( F \not\models B \rightarrow A \). The first holds because
of $B \subset BCN(A, F)$, which means that $F |= A \rightarrow b$ holds for every $b \in B$; the second holds because the assumption $F \not|= BCN(A, F) \{a, x\} \rightarrow a$ is the same as $F \not|= B \rightarrow a$, where $a$ is an element of $A$.

This lemma gives a method for finding sets $B$ such that $B <_F A$ and $F |= B \rightarrow x$. Similar to the loop over the sets $A \{a\}$, it allows looping over all elements $a$ of $A$ instead of all sets of variables. If the checks $F |= BCN(A, F) \{a, x\} \rightarrow x$ and $F \not|= BCN(A, F) \{a, x\} \rightarrow a$ succeed, $BCN(A, F) \{a, x\}$ replaces $A$ because it is less than $A$ according to $<_F$. The algorithm $SHMIN(F)$ uses this kind of loop.

Algorithm 1 $SHMIN(formula F)$

1. $R = \emptyset$

2. for each $A \rightarrow x \in F$

   (a) if $R$ contains a clause $B \rightarrow x$
       then continue

   (b) $G = true$

   (c) while $G$

       i. $G = false$

       ii. for each $a \in A$

           A. $B = BCN(A, F) \{a, x\}$

           B. if $F |= B \rightarrow x$ and $F \not|= B \rightarrow a$
               then $A = B$, $G = true$, break

   (d) $G = true$

   (e) while $G$

       i. $G = false$

       ii. for each $a \in A$

           A. $B = A \{a\}$

           B. if $F |= B \rightarrow x$
               then $A = B$, $G = true$, break

   (f) $R = R \cup \{A \rightarrow x\}$

3. return $R$

The two loops in Step 2c and Step 2e are separated and in this order because of how minimality according to $<_F$ and $\subset$ interact. A subset of a body that is minimal according to $<_F$ and $\subset$ is still minimal. Instead, a $\subset$-minimal body may still be greater than another that is not $\subset$-minimal. First minimizing according to $<_F$ and then to $\subset$ ensures that a second minimization according to $<_F$ is not required.
Each iteration of the two loops checks a linear number of entailments, which are polynomial-time because $F$ is Horn. The question is the number of iterations: polynomial, exponential or infinite? Since the loops terminate if $G$ is false and the only instruction that sets it to true is with $A = B$, each iteration either is the last or replaces a set $A$ with another set $B$ that is strictly lower than it or strictly contained in it. Infinite chains are impossible. Proving the iteration polynomial many is the job of the next lemma.

**Lemma 29** The computation of $\text{SHMIN}(F)$ requires polynomial time.

**Proof.** Each iteration of the two loops of Algorithm 1 takes polynomial time because it checks some entailments from Horn clauses, which take polynomial time. The claim is a consequence of the number of iterations being polynomial.

Both loops terminate if $G$ is false, and the only instructions that set it to true are executed only after replacing $A$ with $B$. Apart from the last iteration of each loop, all others replace $A$ with $B$.

The iterations of the first loop (Step 2c) replace $A$ with $B = BCN(A, F) \{a, x\}$ only if $B <_F A$, as proved by Lemma 28. A consequence of $B <_F A$ which is now proved is $BCN(B, F) \subset BCN(A, F)$.

This claim is proved in two parts: first, $b \in BCN(B, F)$ implies $b \in BCN(A, F)$; second, $BCN(B, F) = BCN(A, F)$ is a contradiction.

The definition of $B <_F A$ is $F|\{a, x\} = A \rightarrow B$ and $F \not|\{\not\} = B \rightarrow A$. The definition of $b \in BCN(B, F)$ is $F|\{a, x\} = B \rightarrow b$. Transitivity implies $F|\{a, x\} = A \rightarrow b$, which defines $b \in BCN(A, F)$. This proves $BNC(B, F) \subset BNC(A, F)$.

This containment is proved strict by implying a contradiction from its converse: $BNC(B, F) = BCN(A, F)$. Since $A \rightarrow a$ is a tautology if $a \in A$, it is valid and therefore entailed by $F$. This fact $F|\{a, x\} = A \rightarrow a$ defines $a \in BCN(A, F)$. This is the case for every $a \in A$, which implies $A \subseteq BNC(A, F)$. Since $BNC(A, F)$ is equal to $BNC(B, F)$ by assumption, $A \subseteq BNC(B, F)$ follows. This is defined as $F|\{a, x\} = B \rightarrow a$ for every $a \in A$, or $F|\{a, x\} = B \rightarrow A$. Contradiction is reached since $F \not|\{a, x\} = B \rightarrow A$ is part of the definition of $B <_F A$.

The conclusion is that if $B$ replaces $A$ in an iteration of the first loop (Step 2c), then $BCN(B, F) \subset BCN(A, F)$. The set $BCN(A, F)$ strictly decreases at every iteration. Since this set contains at most all variables of $F$, the iterations cannot be more than the variables. The number of iterations is linear.

A similar but simpler argument applies to the second loop (Step 2e): the set $A$ decreases at every step since it may only be replaced by $A \{a\}$ for some $a \in A$. Only a linear number of iterations are possible.

Algorithm 1 takes polynomial time. It is efficient, but is still to be proved useful. Another step in that direction is proving that it always produces a single-head formula.

**Lemma 30** $\text{SHMIN}(F)$ is a single-head formula.

**Proof.** The claim is proved by contradiction, assuming instead that Algorithm 1 returns two clauses with the same head. The return value is $R$. This set is only changed the end of each iteration of the main loop at Step 2. The head $x$ of the second clause that is added to $R$ did not change during the iteration of the main loop. It is the same in Step 2c. Since this is the second clause of head $x$ that is added to $R$, the check succeeds, cutting the loop short and
preventing the second clause to be added to $R$. This contradicts the assumption that both clauses are eventually in $R$.

Algorithm 1 takes polynomial time and produces a single-head formula $SHMIN(F)$. If this set is equivalent to $F$, then $F$ is single-head equivalent by definition.

The converse does not hold in general, but only if $F$ is inequivalent: if $F$ is both single-head and inequivalent, then $F \equiv SHMIN(F)$. This shows that $SHMIN(F)$ is not just a candidate for being the single-head form of $F$ if any, but a good candidate since it is the single-head form of $F$ when $F$ is inequivalent. This is proved in the next section.

When $F$ is not inequivalent, the algorithm may end with a body that is not minimal according to $<_F$. Lemma 28 only gives a sufficient condition (if – then), not a necessary and sufficient one (if and only if). Even after exhausting all sets $B = BCN(A, F) \{a, x\}$, the minimality of $A$ is not guaranteed. Another set $B <_F A$ may still exist.

This drawback is not fatal to the intended usage of the final clause: it is collected in a set $SHMIN(F)$ that is checked for equivalence with $F$. If the system fails to find $B <_F A$, the result will not be equivalent to $F$, and $F$ is not replaced by $SHMIN(F)$ for computing forgetting. Efficiency is harmed, not correctness.

Lemma 28 does not exclude that some sets less than $A$ are missed when checking only the sets $BCN(A, F) \{a, x\}$ with $a \in A$ instead of all sets of literals, but does not prove they may exist either. The following lemma proves they may do. Additionally, it shows that they may do even when no proper subset of $A$ entails $x$ and $F$ is single-head.

**Lemma 31** There exists a single-head formula $F$, a set of variables $A$ and a variable $x$ such that both $F \models A \{a\} \rightarrow x$ and $F \not\models BCN(A, F) \{a, x\} \rightarrow a$ are false for every $a \in A$, but there exists a set of variables $B$ such that $F \models B \rightarrow x$ and $B <_F A$.

**Proof.** The formula $F$ and the set of variables $A$ are as follows.

$$F = \{ab \rightarrow d, ad \rightarrow b, bd \rightarrow a, d \rightarrow x\}$$
$$A = \{a, b\}$$

Every single element of $BCN(A, F) \{x\} = \{a, b, d\}$ is entailed by the other two, making these subsets equivalent to $A$. Yet, the two elements $a$ and $b$ cannot be recovered once both removed: $\{d\}$ is strictly less than $A$ while still satisfying $F \models d \rightarrow x$.

The claim is proved for $a$; it holds for $b$ by symmetry. The first requirement is the falsity of $F \models A \{a\} \rightarrow x$; it holds because $A \{a\} = \{b\}$, and $b$ alone does not imply $x$. The second is the falsity of $F \not\models BCN(A, F) \{a, x\} \rightarrow a$, which is the same as the truth of $F \models BCN(A, F) \{a, x\} \rightarrow a$; it holds because $BCN(A, F) \{a, x\} = \{a, b, d, x\} \{a, x\} = \{b, d\}$; this set implies $a$ thanks to $bd \rightarrow a \in F$. 36
The set $B$ of the statement of the lemma is $\{d\}$. It satisfies $F \models B \rightarrow x$ because $B \rightarrow x$ is $d \rightarrow x$, which is in $F$. The other requirement $B <_F A$ is defined as $F \models A \rightarrow B$ and $F \not\models B \rightarrow A$. The first holds because $A \rightarrow B$ is $ab \rightarrow d$, which is in $B$. The second holds because $B \rightarrow A$ is $d \rightarrow a$ and $d \rightarrow b$, and $F$ is consistent with the model that assigns true to $d$ and $x$ and false to $a$ and $b$.

This lemma proves that no set $B = A \setminus \{a\}$ or $B = BCN(A, F) \setminus \{x,a\}$ may replace $A$, yet a set $B$ strictly smaller than $A$ satisfies $F \models B \rightarrow x$. If Algorithm 1 starts from the clause $ab \rightarrow x$ when analyzing the formula $F \cup \{ab \rightarrow x\}$, it outputs that clause because no strictly lower set or proper subset is found by looping over a single variable in $A$. Since $F$ is single-head and entails $ab \rightarrow x$, the formula $F \cup \{ab \rightarrow x\}$ is single-head equivalent. Yet, $SHMIN(F)$ contains the wrong clause $ab \rightarrow x$.

### 4.5 Inequivalence and single-head equivalence

Lemma 32 shows that $F \models B \rightarrow x$ and $B <_F A$ do not imply the existence of a variable $a \in A$ such that $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$ and $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$. In short, the converse of Lemma 28 does not always hold. Removing single variables from $BCN(A, F) \setminus \{x\}$ does not always provide a body less than $A$ even if one exists.

It does when the formula is inequivalent: a lower body is always found this way, if one exists. A minimal body eventually results. Algorithm 1 always finds a single-head equivalent formula if one exists.

**Lemma 32** If $F$ is inequivalent (Condition 3) and entails $A \rightarrow x$ with $x \not\in A$, it entails a clause $B \rightarrow x$ with $x \not\in B$ and $B <_F A$ if and only if $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$ and $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$ both hold for some $a \in A$.

**Proof.** Lemma 28 proves that $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$ and $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$ if and only if $BCN(A, F) \setminus \{a, x\}$ satisfies $F \models B \rightarrow x$ and $B <_F A$.

The rest of the proof is for the converse: $F \models B \rightarrow x$, $x \not\in B$ and $B <_F A$ imply $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$ and $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$ for some $a \in A$ if $F$ is inequivalent.

Let $A'$ be the set of variables of $BCN(A, F)$ that are not entailed by $B$:

$$A' = BCN(A, F) \setminus BCN(B, F) = \{a \in BCN(A, F) \mid F \not\models B \rightarrow a\}$$

The first part $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$ of the claim is proved for all elements $a \in A'$, the second $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$ for at least an element $a \in A' \cap A$.

The assumption $B <_F A$ includes $F \models A \rightarrow B$, which implies $B \setminus \{x\}$ thanks to the assumption $x \not\in B$. Since $B$ does not entail any element of $A'$ by construction, it does not contain any. This makes the containment further strengthen to $B \subseteq BCN(A, F) \setminus A' \setminus \{x\}$, which is needed below. Since $F$ entails $B \rightarrow x$, it also entails its superclause $BCN(A, F) \setminus \{a, x\} \rightarrow x$ for every $a \in A'$.

The other part of the claim is proved by showing that $F \models BCN(A, F) \setminus \{a, x\} \rightarrow a$ for all $a \in A'$ contradicts the assumption $B <_F A$. A consequence is $F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a$ for at least a variable $a \in A'$; this variable is then proved to belong to $A$.

By construction, $F$ entails $A \rightarrow BCN(A, F)$; this implies $F \models A \rightarrow BCN(A, F) \setminus \{a, x\}$ for every variable $a$. The converse implication $F \models BCN(A, F) \setminus \{a, x\} \rightarrow A$ holds for all
a ∈ A' because x does not belong to A by assumption and \( BCN(A, F) \setminus \{a, x\} \) by assumption implies a, the only element of A it does not contain. Implication in both directions is equivalence: \( F \models BCN(A, F) \setminus \{a, x\} \equiv A \). This holds for every \( a \in A' \): all sets \( BCN(A, F) \setminus \{a, x\} \) for \( a \in A' \) are equivalent to A. Therefore, they are also equivalent to each other. By Condition 3 they are equivalent to their intersection \( BCN(A, F) \setminus A' \setminus \{x\} \), which is therefore equivalent to A.

By construction, \( A' \) comprises all elements \( BCN(A, F) \) that \( F \cup B \) does not imply; as a result, \( F \cup B \) implies the others: \( F \models B \rightarrow BCN(A, F) \setminus A' \). An immediate consequence is \( F \models B \rightarrow BCN(A, F) \setminus A' \setminus \{x\} \). The converse implication is a consequence of \( B \subseteq BCN(A, F) \setminus A' \setminus \{x\} \), proved above. This proves the equivalence \( F \models B \equiv BCN(A, F) \setminus A' \setminus \{x\} \). Since \( BCN(A, F) \setminus A' \setminus \{x\} \) is equivalent to A, so is B. This contradicts the assumption \( B <_F A \).

This contradiction disproves the assumption \( F \models BCN(A, F) \setminus \{a, x\} \rightarrow a \) for all \( a \in A' \). Therefore, it proves \( F \not\models BCN(A, F) \setminus \{a, x\} \rightarrow a \) for at least an element \( a \in A' \). If \( a \not\in A \) then \( A \subseteq BCN(A, F) \setminus \{a, x\} \) since \( x \not\in A \). This implies \( F \not\models A \rightarrow a \), which is a contradiction because the definition of \( a \in A' \) includes \( a \in BCN(A, F) \), which is \( F \models A \rightarrow a \) by definition. Therefore, \( a \in A \).

This lemma proves that the mechanism of checking one variable at time when searching for a lower body is correct in the inequivalent case. Algorithm 1 switches from a body to a lower body if any, returning a body only when minimal.

**Lemma 33** If \( F \) is inequivalent then \( SHMIN(F) \subseteq MIN(F) \).

**Proof.** Let \( A \rightarrow x \) be a clause in \( SHMIN(F) \). It is proved to be in \( MIN(F) \). The claim requires \( F \models A \rightarrow x \) and that \( F \models B \rightarrow x \) does not hold if \( B <_F A \) or \( B \subset A \).

The first part of the claim is an invariant in Algorithm 1: the clause \( A \rightarrow x \) is initially a clause of \( F \), and \( B \) replaces \( A \) only if \( F \models B \rightarrow x \).

The second part of the claim takes most of the proof.

Algorithm 1 returns a clause \( A \rightarrow x \) only when its second loop at Step 2c ends. This is only the case when \( G \) remains false during an entire iteration, which happens only when \( F \models A \setminus \{a\} \rightarrow x \) holds for no variable \( a \in A \). If \( F \models B \rightarrow x \) holds with \( B \subset A \) then \( A \setminus B \) contains at least a variable \( b \) because containment is strict. Since \( B \) does not contain \( b \), it is contained in \( A \setminus \{b\} \). A consequence of \( F \models B \rightarrow x \) is \( F \models A \setminus \{b\} \rightarrow x \), which contradicts the assumption with \( a = b \). This proves that if \( A \rightarrow x \) is in \( SHMIN(F) \) then \( F \models B \rightarrow x \) does not hold if \( B \subset A \).

This is the first part of the definition of \( A \rightarrow x \in MIN(F) \). The second is that \( F \models B \rightarrow x \) does not hold if \( B <_F A \).

Let \( A' \) be the value of \( A \) at the end of the first loop of Algorithm 1 at Step 2c. The loop ends only when \( G \) remains false during an entire iteration, which is only the case when \( BCN(A, F) \setminus \{a, x\} \models x \) and \( BCN(A, F) \setminus \{a, x\} \not\models a \) do not hold at the same time for any \( a \in A \). Lemma 32 proves that \( B <_F A \) and \( F \models B \rightarrow x \) with \( x \not\in B \) implies the opposite of that. As a result, \( B <_F A' \) holds for no non-tautologic clause \( B \rightarrow x \) entailed by \( F \).

This is not the claim yet, because the set \( A' \) that is proved minimal is the value of \( A \) at the end of the first loop of the algorithm, not its final value. Minimality extends to that thanks to the invariant \( A \equiv_F A' \) of the second loop of Algorithm 1 at Step 2c.
This invariant is proved by induction on the number of iterations. The claim \( A' \equiv_F A \) is proved at the beginning of the second loop of the algorithm (base case) and is then assumed at the beginning of an iteration of the loop and proved at the end (induction case).

The base case is the start of the second loop. Since \( A' \) is the value of \( A \) at the end of the first loop of Algorithm \([1]\) it is the same as the value of \( A \) at the beginning of the second. The claim holds because \( A = A' \) implies \( A \equiv_F A' \).

The induction case is about an arbitrary iteration of the loop. The inductive assumption is \( A \equiv_F A' \) when the iteration starts, the inductive claim is the same at the end. The value of \( A \) changes only if \( F \models A \{a\} \rightarrow x \) with \( a \in A \). Since \( A \rightarrow A \{a\} \) is a tautology, \( F \) implies it. This defines \( A \{a\} \leq_F A \). The inductive assumption \( A \equiv_F A' \) implies \( A \{a\} \leq_F A' \). Less than or equal to are two possibilities: less than, or equal to. In this case, \( A \{a\} \leq_F A' \) is either \( A \{a\} <_F A' \) or \( A \{a\} \equiv_F A' \). The first is not possible because it contradicts the previously proved property that \( B <_F A' \) holds for no non-tautologic clause \( B \rightarrow x \) entailed by \( F \). The only actual possibility is the second: \( A \{a\} \equiv_F A' \). Since \( A \{a\} \) replaces \( A \), the inductive claim follows: the next value of \( A \) is equivalent to \( A' \).

Since \( B <_F A' \) holds for no nontautologic clause \( B \rightarrow x \) entailed by \( F \) and \( A' \) is equivalent to all following values of \( A \), this property holds for the final value of \( A \). This is the second part of the definition of \( A \rightarrow x \) being in \( \text{MIN}(F) \).

Having proved that \( \text{SHMIN}(F) \) only contains clauses of \( \text{MIN}(F) \), remains to prove the converse: it contains all of them. This claim requires an additional assumption: \( F \) is not only inequivalent but also single-head equivalent. The second is necessary because \( \text{MIN}(F) \) is not single-head otherwise, while \( \text{SHMIN}(F) \) always is.

Intuitively, \( \text{SHMIN}(F) \) starts from a clause of \( F \) and produces a clause entailed by it with a minimal body. Instead, \( \text{MIN}(F) \) contains all entailed clauses with a minimal body. The difference is one versus all, which disappears in the single-head equivalent case. With a caveat: \( \text{SHMIN}(F) \) starts with a clause in \( F \) while \( \text{MIN}(F) \) is purely semantical. This gap is filled by the next lemma.

**Lemma 34** If \( \text{MIN}(F) \) contains a clause with head \( x \) then \( F \) contains a clause with head \( x \).

**Proof.** The proof is by contradiction: let \( A \rightarrow x \) be a clause of \( \text{MIN}(F) \) such that \( F \) contains no clause with head \( x \). Since \( \text{MIN}(F) \) only contains clauses entailed by \( F \), this particular clause \( A \rightarrow x \) is entailed by \( F \).

Let \( M \) be the model that sets all variables to true but \( x \). This model satisfies every clause \( B \rightarrow y \) of \( F \) because \( y \) is by assumption different from \( x \), and is therefore assigned true by \( M \). The clause \( A \rightarrow x \) is instead falsified by \( M \) because all variables in \( A \) are different from \( x \) and therefore assigned to true while \( x \) is assigned to false.

This lemma shows that all clauses in \( \text{MIN}(F) \) have a head that is also a head in \( F \), and therefore a head in \( \text{SHMIN}(F) \). In the single-head equivalent case, this was the only missing bit holding \( \text{SHMIN}(F) \) away from \( \text{MIN}(F) \).

**Lemma 35** If \( F \) is inequivalent and single-head equivalent, then \( \text{SHMIN}(F) = \text{MIN}(F) \).

**Proof.** Lemma \([3]\) proves \( \text{SHMIN}(F) \subseteq \text{MIN}(F) \) since \( F \) is inequivalent. The claim follows from \( \text{MIN}(F) \subseteq \text{SHMIN}(F) \), which is proved if \( F \) is single-head equivalent.
Lemma 34 proves that $A \rightarrow x \in \text{MIN}(F)$ implies that $F$ contains some clauses with the same head. In other words, $F$ contains one or more clauses of head $x$. The iterations of the main loop of Algorithm 1 at Step 2 are performed on all clauses of $F$, including them. Since $x$ is never changed during an iteration, it is the head of the clause added to $R$ if any. The first iteration where the clause has head $x$ is such that: $R$ does not contain any clause of head $x$ because this is the first; the iteration is therefore not cut short at Step 2a and a clause is added to $R$, a clause of head $x$. Let $B \rightarrow x \in \text{SHMIN}(F)$ be such a clause. Since $\text{SHMIN}(F) \subseteq \text{MIN}(F)$ by Lemma 33, $B \rightarrow x$ is in $\text{MIN}(F)$. By Lemma 24, $\text{MIN}(F)$ is single-head. Since it contains $A \rightarrow x$, this is the only clause of head $x$ it contains. Therefore, $B$ is equal to $A$, proving $A \rightarrow x \in \text{SHMIN}(F)$.

This being the case for every clause of $\text{MIN}(F)$, it proves $\text{MIN}(F) \subseteq \text{SHMIN}(F)$. □

The final destination of this string of lemmas is the equality of $\text{MIN}(F)$ and $\text{SHMIN}(F)$ if $F$ is both inequivalent and single-head equivalent: Algorithm 1 calculates $\text{MIN}(F)$ in this case. This gives a way for checking single-head equivalence in the inequivalent case.

**Lemma 36** An inequivalent formula $F$ is single-head equivalent if and only if $F \equiv \text{SHMIN}(F)$.

**Proof.** The first direction of the lemma does not require inequivalence: $F \equiv \text{SHMIN}(F)$ implies that $F$ is single-head equivalent. Lemma 30 proves that $\text{SHMIN}(F)$ is single-head. As a result, if $F \equiv \text{SHMIN}(F)$ then $F$ is equivalent to the single-head formula $\text{SHMIN}(F)$.

The other direction of the lemma assumes $F$ inequivalent and single-head equivalent and proves $F \equiv \text{SHMIN}(F)$. When $F$ is both inequivalent and single-head equivalent, Lemma 35 proves $\text{SHMIN}(F) = \text{MIN}(F)$ and Lemma 26 proves $F \equiv \text{MIN}(F)$. Together, these two facts imply $F \equiv \text{SHMIN}(F)$.

A consequence of this lemma is that single-head equivalence can be computed in polynomial time on inequivalent formulae. The polynomial algorithm not only tells whether a single-head equivalent formula exists but produces it. That formula can be then fed to the forgetting algorithm, which runs in polynomial time because the formula is single-head.

**Theorem 4** Computing a single-head formula equivalent to a given inequivalent formula if any is polynomial-time.

**Proof.** The polynomial algorithm computes $\text{SHMIN}(F)$ and checks $F \equiv \text{SHMIN}(F)$. This establishes the single-head equivalence of $F$ by Lemma 36. If this check succeeds, then $\text{SHMIN}(F)$ is a single-head formula equivalent to $F$; that is single-head is proved by Lemma 30. Generating it takes polynomial time thanks to Lemma 29; checking equivalence also takes polynomial time because both $F$ and $\text{SHMIN}(F)$ are Horn formulae.

Even if the formula is not inequivalent, producing $\text{SHMIN}(F)$ takes polynomial time. Inequality guarantees its equality with $\text{MIN}(F)$ if $F$ is single-head equivalent.

This theorem provides a way to forget variables from a formula that is both inequivalent and single-head equivalent: first compute the single-head formula that is equivalent to the given one, which can be done in polynomial time since the formula is also inequivalent, and then forget the variables from that formula.

**Corollary 4** Forgetting variables from inequivalent and single-head equivalent formulae can be computed in polynomial time.
4.6 The real consequences

The previous section shows how to calculate $\text{SHMIN}(F)$ by a sequence of steps that check the following conditions for all $a \in A$:

- $F \models BCN(A, F) \setminus \{a, x\} \rightarrow x$,
- $F \not\models BCN(A, F) \setminus \{a\} \rightarrow a$,
- $F \models A \setminus \{a\} \rightarrow x$.

These conditions can be tested exploiting the real consequences, the variables that are inferred from $A$ with at least a derivation step. This excludes the variables of $A$ that are entailed just because they are in $A$, not because of some clauses of $F$.

$$RCN(A, F) = \{a \mid F \models BCN(A, F) \setminus \{a\} \rightarrow a\}$$

While $BCN(A, F)$ contains all variables that $A$ implies, $RCN(A, F)$ only contains its real consequences, those implied thanks to at least a clause of $F$. This is formally proved by the following lemma.

**Lemma 37** A variable $x$ is in $RCN(A, F)$ if and only if $F$ contains a clause $B \rightarrow x$ such that $F \models A \rightarrow B$.

**Proof.** The definition of $x \in RCN(A, F)$ is $F \models BCN(A, F) \setminus \{x\} \rightarrow x$.

Since $x \not\in BCN(A, F) \setminus \{x\}$, Lemma 1 implies the existence of a clause $B \rightarrow x \in F$ such that $F \models BCN(A, F) \setminus \{x\} \rightarrow B$. Since $F \models A \rightarrow BCN(A, F)$, transitivity and monotonicity tell $F \models A \rightarrow B$, the required conclusion.

The other direction assumes the existence of a clause $B \rightarrow x \in F$ with $F \models A \rightarrow B$. The latter implies $B \subseteq BCN(A, F)$. Since no clause in $F$ is tautologic by assumption, $x$ is not in $B$. Therefore, $B \subseteq BCN(A, F) \setminus \{x\}$. By monotonicity, $F \models B \rightarrow x$ implies $F \models BCN(A, F) \setminus \{x\} \rightarrow x$.

This lemma clarifies the difference between $RCN(A, F)$ and $BCN(A, F)$: both require each of their variables $x$ to be entailed from $F \cup A$, but the first also imposes this entailment to result from a clause $B \rightarrow x$, the second do not. Only when $x \in A$ this difference matters.

The next section shows an algorithm for $RCN(B, F)$. The rest of this one shows how $RCN(B, F)$ is used. The first way is to calculate $BCN(B, F)$.

**Lemma 38** For every formula $F$ and set of variables $A$, it holds $BCN(A, F) = A \cup RCN(A, F)$.

**Proof.** The claim is that $x \in BCN(A, F)$ is the same as either $x \in A$ or $x \in RCN(A, F)$.

The condition $x \in BCN(A, F)$ is defined as $F \models A \rightarrow x$. Lemma 1 tells it equivalent to $F$ containing a clause $B \rightarrow x$ such that $F \models A \rightarrow B$ if $x$ is not in $A$. Reformulated, “something exists if a condition is false” is the same as “either the condition is true or something exists”. In the present case, either $x \in A$ or $F \models A \rightarrow B$ for some $B \rightarrow x \in F$. As proved by Lemma 37, the second possibility is equivalent to $x \in RCN(A, F)$. Overall, either $x \in A$ or $x \in RCN(A, F)$. 

The three checks required to determine $SHMIN(F)$ can be expressed in terms of $RCN()$.

The check $F \models A \{a\} \rightarrow x$ is equivalent to $x \in BCN(A \{a\}, F)$, which Lemma 38 proved the same as $x \in A \{a\} \cup RCN(A \{a\}, F)$.

The check $F \models BCN(A, F) \{a, x\} \rightarrow x$ is equivalent to $x \in B \cup RCN(B, F)$ with $B = BCN(A, F) \{a, x\}$ for the same reason.

The check $F \not\models BCN(A, F) \{a, x\} \rightarrow a$ is only needed if the previous check succeeds, since Lemma 28 requires both. In so, the following lemma recasts it in terms of $RCN(A, F)$.

**Lemma 39** If $F \models BCN(A, F) \{a, x\} \rightarrow x$ holds with $a \in A$, then $F \not\models BCN(A, F) \{a, x\} \rightarrow a$ is equivalent to $a \in A \setminus RCN(A, F)$.

Proof. A consequence of $F \models BCN(A, F) \{a, x\} \rightarrow x$ is $F \models BCN(A, F) \{a, x\} \equiv BCN(A, F) \{a\}$ because the entailing set contains all elements of the entailed set but $x$ and entails $x$ by assumption. The condition $F \not\models BCN(A, F) \{a, x\} \rightarrow a$ is therefore the same as $F \not\models BCN(A, F) \{a\} \rightarrow a$, which is the exact opposite of the definition of $a \in RCN(A, F)$. Since $a$ is by assumption an element of $A$, the condition $a \not\in RCN(A, F)$ is the same as $a \in A \setminus RCN(A, F)$. \hfill \Box

Another use of $RCN(A, F)$ is to prove some sets to be strictly greater than certain subsets of them.

**Lemma 40** If $a \in A \setminus RCN(A, F)$ then $A \{a\} <_F A$.

Proof. The claim $A \{a\} <_F A$ is $F \models A \rightarrow A \{a\}$ and $F \not\models A \{a\} \rightarrow A$. The first condition holds because $A \{a\}$ is a subset of $A$.

The second condition $F \not\models A \{a\} \rightarrow A$ is proved as follows. Since $a$ is not in $RCN(A, F) = \{a \mid F \models BCN(A, F) \{a\} \rightarrow a\}$, the entailment $F \models BCN(A, F) \{a\} \rightarrow a$ does not hold. Since $A \subseteq BCN(A, F)$, by monotonicity of entailment $F \models A \{a\} \rightarrow a$ does not hold either. This implies $F \not\models A \{a\} \rightarrow A$ since $A \{a\}$ does not imply $a$, which is in $A$. \hfill \Box

### 4.7 Finding the real consequences

The following algorithm calculates the real consequence $RCN(B, F)$ of $B$ according to $F$. It employs unit propagation [CA93] but keeps the given variables $B$ separated from the generated ones $H$.

**Algorithm 2** $rcnucl(variables B, formula F)$:

1. $H = \emptyset$
2. $U = \emptyset$
3. while $H$ changes:
   
   for every $B' \rightarrow x \in F$ such that $B' \subseteq B \cup H$:
   
   i. $H = H \cup \{x\}$

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Lemma 41 The first return value of Algorithm 2 is $RCN(B, F)$.

Proof. The proof relies on Lemma 37: $x \in RCN(B, F)$ is equivalent to $F \models B \rightarrow B'$ for a clause $B' \rightarrow x \in F$; the entailment is equivalent to $B' \subseteq BCN(B, F)$. Since $x$ enters $H$ if and when $B' \subseteq B \cup H$ holds for a clause $B' \rightarrow x \in F$, the claim follows from $B \cup H$ always being a subset of $BCN(B, F)$ and eventually becoming equal to it. The first fact implies that $x$ enters $H$ only if $x \in RCN(B, F)$, the second implies that if $x \in RCN(B, F)$ then $x$ eventually enters $H$. Once containment $B \cup H \subseteq BCN(B, F)$ is proved, equality becomes equivalent to containment in the other direction: $BCN(B, F) \subseteq B \cup H$.

The first part of the claim is that $B \cup H \subseteq BCN(B, F)$ holds during the entire run of the algorithm. It is proved by induction on the number of iterations of the loop. It holds at the beginning because $B \subseteq BCN(B, F)$ holds and $H$ is initially empty. While $B$ does not change, $H$ is updated by the instruction $H = H \cup \{x\}$ when a clause $B' \rightarrow x \in F$ satisfies $B' \subseteq B \cup H$. The inductive assumption $B \cup H \subseteq BCN(B, F)$ and transitivity of containment imply $B' \subseteq BCN(B, F)$. This is defined as $F \models B \rightarrow b$ for every $b \in B'$, or $F \models B \rightarrow B'$. Together with $B' \rightarrow x \in F$, it implies $F \models B \rightarrow x$, which defines $x \in BCN(B, F)$. Therefore, $B \cup H \subseteq BCN(B, F)$ still holds after the addition of $x$ to $H$.

The second part of the claim is that $B \cup H$ eventually contains all of $BCN(B, F)$.

A part of the proof is to show that $B \cup H$ is larger when running the algorithm on a larger formula. Formally, if $x$ is in $B \cup H$ at some point when running the algorithm on $B$ and $F'$ then it is also in $B \cup H$ when running it on $B$ and $F$ if $F' \subseteq F$.

This is proved by induction on the number of iterations of the loop. Initially this number is zero, providing the base case of induction. In both runs $H$ is empty. Since $B$ is the same in both runs, also $B \cup H$ is the same. This proves the base case of induction. The inductive step assumes that $B \cup H$ on $F'$ is contained in $B \cup H$ on $F$ at the beginning of an iteration, and requires proving the same at its end. If $x$ is added to $H$ when running on $F'$, then $B' \rightarrow x \in F'$ holds for some $B' \subseteq B \cup H$; the same conditions are also true when running on $F$: the first $B' \rightarrow x \in F$ because $F' \subseteq F$, the second $B' \subseteq B \cup H$ because of the inductive assumption. Therefore, $x$ is also added when running on $F$.

This proves that $B \cup H$ monotonically increases with $F$.

The claim that $BCN(B, F) \subseteq B \cup H$ holds at some point of the algorithm is proved by induction on the size of $F$. When $F$ is empty $BCN(B, F) = B$. The claim $BCN(B, F) \subseteq B \cup H$ follows.

In the inductive case, the claim is that $BCN(B, F) \subseteq B \cup H$ holds at some point when running on $B$ and $F$; the assumption is that the same holds for every formula smaller than $F$.

The claim is the same as $x \in BCN(B, F)$ implying $x \in B \cup H$. Lemma 41 tells that $x \in BCN(B, F)$ implies either $x \in B$ or $F^x \models B \rightarrow B'$ for some clause $B' \rightarrow x \in F$. In the first case, $x \in B$ implies $x \in B \cup H$ and the claim is proved. In the second case,
\( F^x \models B \rightarrow B' \) is the same as \( b \in BCN(F^x, B) \) for every \( b \in B' \). The inductive assumption tells that \( b \in BCN(F^x, B) \) implies \( b \in B \cup H \) at some point of running the algorithm on \( B \) and \( F^x \). It has been proved that \( B \cup H \) is larger when running on a larger formula. Therefore, \( b \in B \cup H \) also holds when running the algorithm on \( B \) and \( F \). This holds for every \( b \in B' \), implying \( B' \subseteq B \cup H \). Since \( B' \rightarrow x \in F \), the algorithm adds \( x \) to \( H \), making the inductive claim \( x \in B \cup H \) true.

The algorithm collects all clauses it uses in its second return value. They could be found by a separate algorithm, but producing them while calculating \( RCN(B, F) \) only requires the extra time of adding them to a set.

They are the clauses that are relevant to proving that something is a consequence of \( F \cup B \). All others are irrelevant. Formally, this set contains a clause if and only if its precondition is entailed by \( F \cup B \).

**Lemma 42** The second return value of Algorithm 2 is

\[
UCL(B, F) = \{ B' \rightarrow x \in F \mid B' \subseteq BCN(B, F) \}.
\]

**Proof.** By Lemma 38, \( BCN(B, F) = B \cup RCN(B, F) \). The claim is therefore equivalent to \( UCL(B, F) \) being the set of clauses \( B' \rightarrow x \in F \) such that \( B' \subseteq B \cup RCN(B, F) \).

A clause \( B' \rightarrow x \) enters the set \( U \) if and only if \( B' \subseteq B \cup H \). The set \( H \) is never removed elements, and its final value is \( RCN(B, F) \) by Lemma 41. Therefore, \( H \subseteq RCN(B, F) \) holds during the entire run of the algorithm. This proves that all clauses in \( U \) satisfy \( B' \subseteq B \cup RCN(B, F) \).

In the other direction, let \( B' \rightarrow x \in F \) be a clause satisfying \( B' \subseteq B \cup RCN(B, F) \). In the final iteration, \( H = RCN(B, F) \) by Lemma 41. As a result, \( B' \subseteq B \cup H \). The clause is added to \( U \) if not already.

\( \square \)

### 4.8 Finding the real consequences, quickly

The \texttt{rcnucl()} implementation in the \texttt{singlehead.py} program uses the standard Python functions for sets: a formula is a set of clauses, each clause is a set of literals; the main loop is over the clauses of the formula, each iteration checks whether the negative variables of a clause are contained in \( B \cup H \).

From the point of view of code simplicity, such an implementation is unbeatable: apart from the initializations and the return statement, all is done in six lines of code.

Yet, its efficiency is not optimal.

Since it implements unit propagation, it benefits from its optimizations: clause indexing and unassigned variables counts [CA93].

Scanning the formula each time in search for a clause containing some variables is inefficient since a variable may be contained in just few clauses. A reverse index avoids such a wasteful scan. Also, a clause may itself be large to scan every time.

Not all clauses are actually needed, and not their entire content. Only the clauses that contain variables in \( B \) or in \( H \) in their body matter, and what matters of them is only whether their body only comprises such variables. The first point is achieved by pointers from variables to clauses; the second by storing the number of variables that are in the body of each clause but not in \( B \cup H \). Such an index comprises:
for each clause, the index holds a record comprising an integer and a variable; the
integer is initialized with the size of its body, the variable is its head;

for each variable, the index contains a list of pointers; each points to the record of a
clause containing the variable in its body.

The variables in $B$ are added to a queue, which is processed a variable at time. For each
variable, its list of pointers is scanned; for each pointer, the integer in the pointed record is
decreased; if this number reaches zero, the variable in the record is added to $H$ and to the
queue if not already in $B \cup H$; the latter check requires constant time thanks to a vector
representing $B \cup H$.

The pointers avoid the loop over all clauses by pointing directly to the clauses containing
each variable in their body. The integer avoids the scan of the clause. The variable is already
guaranteed to be in the body thanks to the pointer. Of the other variables of the body, what
matters is only whether they are all in $B \cup H$. This is the same as the number of variables
in the body but not in $B \cup H$ reaching zero.

4.9 The algorithm, improved

Algorithm 4 makes a clause of $SHMIN(F)$ from each clause $A \rightarrow x$ of $F$ by iteratively
replacing $A$ with another set $B$ such that $B <_F A$ and $F \models B \rightarrow x$; when such a set $B$ no
longer exists, it continues with sets $B \subset A$ such that $F \models B \rightarrow x$. Each phase allows for
some improvements.

• In the first phase, only the sets $B = BCN(A, F) \{a, x\}$ for every $a \in A$ are checked. For such sets, Lemma 28 proves that $F \models B \rightarrow x$ and $F \not\models B \rightarrow a$ imply $B <_F A$, which makes $B$ a valid replacement for $A$.

The first condition $F \models B \rightarrow x$ is the same as $x \in BCN(B, F)$, or $x \in B \cup RCN(B, F)$. Since $B$ is $BCN(A, F) \{a, x\}$, it does not contain $x$. Checking $x \in RCN(B, F)$ is enough.

If the first condition $F \models B \rightarrow x$ is met, the second $F \not\models B \rightarrow a$ is the same as
$a \in A \setminus RCN(A, F)$ by Lemma 39. This check is also expressed in terms of $RCN()$.

If they both hold, $B$ replaces $A$; the next step employs $RCN(A, F)$, but this set needs not to be calculated again since it the same as $RCN(B, F)$ for the set $B$ that replaced $A$.

• In the second phase, only the sets $B = A \setminus \{a\}$ for every $a \in A$ are checked. This is not
a restriction since $B \subset A$ and $F \models B \rightarrow x$ imply the same for every $B' = A \setminus \{a\}$ with
$a \in A \setminus B$.

The check $F \models B \rightarrow x$ is the same as $x \in BCN(B, F)$, or $x \in B \cup RCN(B, F)$. This
is the same as $x \in RCN(B, F)$ since $x \notin B$. Indeed, $B$ is a subset of $A$, either the
result of the first phase or one of its subsets. In turn, this is either the original clause
or a set $BCN(A, F) \{a, x\}$; the second does not contain $x$ by construction, the first
because no clause of $F$ is tautologic by assumption.

The second phase begins only when the first cannot continue. No $B$ satisfies both
$F \models B \rightarrow x$ and $B <_F A$ if $F$ is inequivalent; otherwise, completeness is not guaranteed
anyway. The second phase searches for sets $B$ such that $F \models B \rightarrow x$ and $B \subset A$. Since the first phase is over, $F \models B \rightarrow x$ implies that $B <_F A$ is not possible; $B \subset A$ implies $F \models A \rightarrow B$, which defines $B \leq_F A$. A consequence of $B \leq_F A$ and the impossibly of $B <_F A$ is $A \equiv_F B$.

Equivalence allows for a little improvement in the second phase. The base is Lemma 40 if $a \in A \setminus R C N(A, F)$ then $A \setminus \{a\} <_F A$. Since $B <_F A$ does not hold during the second phase, the sets $A \setminus \{a\}$ with $a \in A \setminus R C N(A, F)$ do not need to be checked, only the ones with $a \in A \cap R C N(A, F)$ do.

If $F$ is not inequivalent, $B$ may not be equivalent to $A$. Correctness is still guaranteed because of the final check $F \equiv S H M I N(F)$, while completeness is not anyway.

The complete algorithm with all these efficiency improvements in place follows.

```python
def shmin(f):
    s = set()
    d = set()

    for c in f:
        h = head(c)
        b = body(c)

        # only one clause for each head

        if h in d:
            continue
        d |= {h}

        # minimize according to $<_F$

        a = None
        r, u = rcnucl(b, f)
        while a != b and b - r:
            a = b
            for e in b - r:
                nb = (b | r) - {e, h}
                nr, nu = rcnucl(nb, u)
                if h in nr:
                    b = nb
                    r = nr
            u = nu
            break

        # minimize according to set containment

        a = None
```
while a != b and b & r:
    a = b
    for e in b & r:
        nb = b - {e}
        nr, nu = rcnuc1(nb, u)
        if h in nr:
            b = nb
            r = nr
            u = nu
            break
s |= {frozenset([h]) | frozenset([l for l in b])}
return s

The aim of $SHMIN(F)$ is to accelerate forgetting: if it is equivalent to $F$ forgetting can be done on it in place of $F$.

Any other single-head formula could be used in the same way, but $SHMIN(F)$ is a good choice because it is guaranteed to be equivalent to $F$ in at least one case: when $F$ is inequivalent and single-head equivalent. It is not in general, as shown by the formula $F = \{a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b\}$ in the twoequiv.py test file of the singlehead.py program. A single-head formula equivalent to $F$ is for example $\{a \rightarrow b, b \rightarrow c, c \rightarrow a\}$.

In order to produce it, $SHMIN(F)$ would have to replace $b \rightarrow a$ with $c \rightarrow a$, which it does not because $\{b\} \equiv_F \{c\}$. Replacing a set with an equivalent one would deprive the algorithm of its termination guarantee since equivalent sets form loops. Checking for sets already analyzed is unfeasible because they may be exponentially many.

When $F$ is single-head equivalent but not inequivalent, $shmin(f)$ may not find the single-head version of $F$. Yet, it may. Formula 4 is an example: $F = \{a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow a\}$; this formula is in the incomplete.py test file of the singlehead.py. The $shmin(f)$ function sometimes generates $\{a \rightarrow b, b \rightarrow c, c \rightarrow a\}$, which is equivalent to $F$, and sometimes $\{a \rightarrow b, b \rightarrow a, b \rightarrow c\}$, which is not. It depends on the order of the clauses in the main loop. All clauses of $F$ have a minimal body since $F$ makes every literal equivalent to each other. As a result, $shmin(f)$ returns $a \rightarrow b, b \rightarrow c$ and whichever between $b \rightarrow a$ and $c \rightarrow a$ comes first in the main loop. The latter makes the output equivalent to $F$, the former does not. While depending on the order of analysis of the clauses is an undesirable algorithm behavior, this example also shows a positive feature of $shmin(f)$: it may find a single-head equivalent formula even if $MIN(F)$ is not single-head.

When $SHMIN(F)$ is not equivalent to $F$, is its calculation wasted time? Maybe not. Not completely, at least. Since $SHMIN(F)$ outputs a clause $B \rightarrow x$ only if $F \models B \rightarrow x$, these clauses are all entailed by $F$. Globally, $F \models SHMIN(F)$. Non-equivalence may only be due to $SHMIN(F) \not\models A \rightarrow x$ for some $A \rightarrow x \in F$. Such clauses can be added to $SHMIN(F)$, or the algorithm be run again to determine a minimal-body clause for each. The resulting formula is still better than the original because it turns a single-head equivalent subformula into a single-head subformula. For example, if $SHMIN(F)$ implies all clauses of $F$ but $a \rightarrow b$, then $F \setminus \{a \rightarrow b\}$ is single-head equivalent and $SHMIN(F)$ is a single-head formula equivalent to it. The replacing algorithm for forgetting performs well on formulae
like \( SHMIN(F) \cup \{a \rightarrow b\} \) that contain only two same heads. The addition of \( a \rightarrow b \) only doubles the recursive calls at most. Running time is still polynomial.

### 4.10 Disproving single-head equivalence

The algorithm for \( SHMIN(F) \) tries to produce a single-head formula that is semantically close to \( F \). When computing forgetting, all of \( SHMIN(F) \) is required, possibly with the addition of other clauses if it is not equivalent to \( F \). If the aim is instead just to check whether a single-head equivalent formula exists, generating all of \( SHMIN(F) \) is wasteful. For example, if \( F \) contains \( a \rightarrow x \) and \( b \rightarrow x \) but implies neither \( a \rightarrow b \) nor \( b \rightarrow a \), it is not single-head equivalent. Computing all of \( SHMIN(F) \) is unnecessary.

Unfortunately, such a property only concerns an individual head, and looking at one head at a time may not be sufficient. A counterexample is \( \{a \rightarrow b, b \rightarrow c, c \rightarrow b\} \), in the testing file `local.py` of the `singlehead.py` program. It is single-head but for \( b \), and is equivalent to \( \{a \rightarrow c, b \rightarrow c, c \rightarrow b\} \), which is single-head but for \( c \). For each of its variables, an equivalent formula that is single-head on that variable exists. Yet, no equivalent formula is single-head on all variables. Every property concerning the heads separately fails at recognizing that it is not single-head equivalent.

Nonetheless, a sufficient condition may be useful anyway, one that allows to sometimes cut \( SHMIN(F) \) short because no single-head formula equivalent to \( F \) exists.

A minimal modification of the algorithm is to compare all sets \( BCN(A, F)\backslash\{a, x\} \) that meet the conditions for replacing \( A \). If two of them are incomparable, they may lead to different minimal bodies. Unfortunately, this is not always the case. A counterexample is \( F = \{abd \rightarrow x, ab \rightarrow d, d \rightarrow x\} \). Removing either \( a \) or \( b \) from \( A = \{a, b, d\} \) produces a set that can replace it: \( \{a, d\} \) entails \( x \) but not \( b \); \( \{b, d\} \) entails \( x \) but not \( a \). These sets are incomparable, yet the formula is equivalent to the single-head formula \( \{ab \rightarrow d, d \rightarrow x\} \), as shown by the `alternatives.py` test file of the `singlehead.py` program.

Even sets of variables Algorithm 1 no longer replaces may be incomparable even if the formula is single-head.

The counterexample is based on the formula in the proof of Lemma 33. There, the formula \( F \) was architected to make a set \( A \) minimal by having each of its elements entailed by the others and the consequences of \( A \). Here, two copies of the same structure make two minimal sets incomparable:

\[
F = \{ab \rightarrow d, ad \rightarrow b, bd \rightarrow a, a'b' \rightarrow d', a'd' \rightarrow b', b'd' \rightarrow a', dd' \rightarrow x\}
\]
This formula is single-head. It is used as an example in the `incomparable.py` test file of the `singlehead.py` program. Since it implies \( abd' \rightarrow x \) and \( a'b'd \rightarrow x \), adding these two clauses preserves equivalence. Therefore, the resulting formula is single-head equivalent.

The same line of proof of Lemma \([31]\) shows that \( \mathcal{A} = \{a, b, d'\} \) is minimal when restricting to single-variable removal. The starting point is \( BCN(A, F) = \{a, b, d, d', x\} \). It makes \( BCN(A, F) \setminus \{a, x\} = \{b, d, d', x\} \), which entails \( a' \); the same holds for \( b \) by symmetry; for \( d' \), it holds \( BCN(A, F) \setminus \{d', x\} = \{a, b, d\} \), which does not entail \( x \). Either way, removing a single variable from \( BCN(A, F) \setminus \{x\} \) does not produce a set that replaces \( A \).

The same applies to \( \mathcal{B} = \{d, a', b'\} \) by symmetry.

These two sets \( A \) and \( B \) are incomparable: \( A \) does not imply \( a' \in B \) and \( B \) does not imply \( a \in A \).

Conclusion: the single-head equivalent formula \( F \cup \{abbd' \rightarrow x, a'b'd \rightarrow x\} \) makes two sets \( A \) and \( B \) minimal but incomparable when removing only a single variable. Yet, the formula is single-head equivalent. Minimal-yet-incomparable sets do not disprove single-head equivalence.

Still better, two minimal-yet-incomparable sets do not disprove single-head equivalence. A similar example with three, four and more sets is easy to make by replicating the core of the counterexample, the clauses around \( a, b \) and \( d \). Arbitrarily many sets may be minimal but incomparable. Still, many is not all. Lemma \([5]\) proves that not all sets implying the same variable can be incomparable in a single-head formula.

This is not obvious, since the lemma proves something seemingly unrelated: if a formula \( F \) is equivalent to the single-head formula \( F' \) that contains \( A \rightarrow x \), then \( F \) contains \( B \rightarrow x \) with \( A \equiv_F B \). What does it tell about comparability? Every nontautologic clause \( C \rightarrow x \) entailed by \( F \) is also entailed by \( F' \), and \( F' \models C \rightarrow x \) implies \( F' \models C \rightarrow A \) by Lemma \([1]\) since \( A \rightarrow x \) is the only clause of \( F' \) with head \( x \). Equivalence implies \( F \models C \rightarrow A \), which defines \( A \leq_F C \), which implies \( B \leq_F C \) because of \( A \equiv_F B \).

The body of a clause of \( F \) is less than or equal to all other bodies of nontautologic clauses entailed by \( F \). The algorithm only creates bodies of clauses entailed by \( F \). The sufficient condition that disproves single-head equivalence is: if none of the bodies of \( F \) is less than or equal to all bodies created by the algorithm, then \( F \) is not single-head equivalent. This is the case even for bodies that are not minimal. What matters is only that the clause with
that body is entailed by $F$, and all bodies generated by the algorithm are.

**Corollary 5** If $F$ entails $C \rightarrow x$ but contains no clause $B \rightarrow x$ such that $B \leq_F C$, then it is not single-head equivalent.

This condition is only sufficient to disprove single-head equivalence. In the other way around, it is yet another necessary condition. Its sufficiency is disproved by a previous example: $F = \{a \rightarrow b, b \rightarrow c, c \rightarrow b\}$, in the `inloop.py` test file of the `singlehead.py` program. It is proved not single-head equivalent in Section 2. Yet, it satisfies the condition: the entailed clauses are $a \rightarrow b$, $a \rightarrow c$, $b \rightarrow c$ and $c \rightarrow b$ and their superclauses. The first, third and fourth are in $F$. The second $a \rightarrow c$ satisfies the condition because of $\{b\} \leq_F \{a\}$ and $b \rightarrow c \in F$. The superclauses satisfy the condition because their bodies are greater than the body of the subclause.

### 4.11 Python implementation

The `singlehead.py` program implements the `shmin()` function. It can be called directly on a formula and tells whether it is single-head equivalent according to `shmin()`.

```
$ singlehead.py -f 'a->b' 'abd->c' 'b=d' 'b->c'
```

The clauses are passed each as a commandline option. Formulae like `ab->cd` or `ab=cd` are accepted in place of clauses. Variables are single characters, which bounds them to the ones accepted by the Python interpreter, currently about a million.

The program first outputs the formula in definite Horn form, with subformulae like `ab->cd` or `ab=cd` turned into clauses. For each clause of the formula, it prints the bodies replacing its because of $\leq_F$ and then because of $\subset_F$, followed by the final clause of `shmin()`. Eventually, it prints the generated formula on a single line, and whether it is equivalent to the input formula.

```
## cmdline formula ##
formula: a->b abd->c b=d b->c
b->d | | | b->d
a->b | dc | d | d->b
dba->c | db | b | b->c
d->b | [head already in shmin]
b->c | [head already in shmin]
shmin: b->d d->b b->c
shmin equivalent: False
expected result: None
```

As an example, the fourth line `a->b | dc | d | d->b` is the result of processing the clause `a->b`. Its body `a` is first replaced by `dc` because of $dc <_F a$. No further replacement is possible according to the order, this is way the separator `|` follows. The body `bc` contains `d`, which implies the head `b`. Therefore, `bc` is replaced by `d`. No subset of it implies `b`. The final clause is `d->b`, the last part of the line.
A line like this is printed for each input clause: the input clause, the bodies replacing its according to $<_F$ and then to $\subset$, and the final clause. Three pipe characters separate the two minimization phases from each other and from the input and output clauses.

An exception is $d\rightarrow b \mid [\text{head already in shmin}]$, meaning that the input clause $d\rightarrow b$ is not processed at all because a clause of the same head is already generated.

The line $\text{shmin: } b\rightarrow d \ d\rightarrow b \ b\rightarrow c$ shows the output formula. The following line tells whether it is equivalent to the input formula. A positive answer is certain: the output formula is single-head and equivalent to the given one. A negative answer is inconclusive, as the input formula may still be single-head equivalent.

The last line of the output is redundant for formulae given on the command line, but essential to automated testing. If the formula is given in a file rather than the commandline arguments, an expected result can be provided. The name of the file is provided as an argument, possibly preceded by the -t option:

```
singlehead.py -t tests/conditiontwo.py
```

As an example, the formula disproving the sufficiency of Condition 2 to single-head equivalence is in the file `conditiontwo.py`. Such a file contains one or more calls to the `analyze()` function. Its first argument is a string describing the formula, the second is the expected result of the test, the following are the clauses of the formula.

```
analyze(
  'second condition is insufficient',
  False,
  'ab->x', 'bx->c', 'ac->d', 'd->x')
```

Since the program fails to produce a single-head equivalent formula, the test is deemed passed because of the second argument `False`. The string `TEST PASSED` ends the program output. Other possible outcomes are `TEST FAILED` and `TEST INCONCLUSIVE`. The latter is the mark of incompleteness of the program: the formula is declared single-head equivalent, but the program fails to prove it.

## 5 Conclusions

If a definite Horn formula does not have two clauses with the same head, logical forgetting [Del17] is easy to compute and produces a polynomially sized output [Lib20a]. The single-head restriction is easy to tell: no two clauses have the same head. A formula that is not single-head may still be equivalent to one that is. Forgetting can be performed on that, since it is the same on equivalent formulae. The single-head equivalence concept is where the troubles begin.

Checking the clauses of a formula for duplicated heads is easy as it only requires a simple scan of the formula. Checking for an equivalent formula with this property is difficult because of the many equivalent formulae. A parallel with a classical problem is in order: checking the size of a formula is straightforward; checking whether a formula is equivalent to one of a given size is not [Cou94, CS02, UVSV06]. It took twenty years just for being precisely classified in the polynomial hierarchy [Sto76, Uma01].

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The first direction of attack is to turn the definition based on the many equivalent formulae into a condition based on the semantics of the formula alone. Equivalent formulae are either all single-head equivalent or they are all not. Given a set of models, the theory predicts that all formulae satisfied precisely by them are the same on single-head equivalence. Another parallel explains the aim: a set of propositional interpretations are the models of some Horn formula if and only if the intersection of any two of them is also one of them; the intersection is defined as the model setting a variable to true only if and only if both intersecand models do. This condition is easy to express, at least in theory: any two models, their intersection. It tells whether a set of models is Horn-expressible or not without checking the many equivalent formulae that express the set of models. Similarly, a set of models is either single-head expressible or not. The definition of single-head expressible is that some single-head formula has precisely this set of models. What would be useful is a necessary and sufficient condition to single-head expressible, simple like “two models, their intersection”.

Such a condition seems not too hard to find. If a formula is single-head equivalent, it is equivalent to a single-head formula. That formula contains at most a clause with each variable in the head. The problem is to tell its body. Still better, the problem is to tell whether a single body suffices, and all others reduce to them. Such reducibility proved slippery ground: sometimes, a clause seems redundant because it reduces to another, but the reduction requires the clause itself. An entire section of this article shows necessary conditions to single-head equivalence that all fell on sufficiency. A series of examples and counterexamples show that single-head equivalence is not obvious to express when looking at a set of models. A particularly significant example is Formula $3$:

$$
F = \{ab \rightarrow x, bx \rightarrow c, ac \rightarrow d, d \rightarrow x\}.
$$

In spite of being just a mere, single Boolean formula, it is one of the main outcomes of this article. It exemplifies the complication of clauses reducing to others: $ab \rightarrow x$ appears to be reducible to $d \rightarrow x$ thanks to $F \models ab \rightarrow d$, but this entailment requires $ab \rightarrow x$ itself. It also shows useful elsewhere in the article, but its main feature is to negate the semantical simplicity of single-head equivalence. Every necessary and sufficient condition to single-head equivalence and every algorithm for single-head equivalence must be tested on this formula. Doing so tells whether the main pitfall of single-head equivalence is ducked or not.

The two necessary conditions in Section 2 fell on it. They tell the formula single-head equivalent. It is not. They may be necessary, but they are not sufficient.

This Section 2 looks like a string of failures: a first condition is found that looks obviously the same as single-head equivalence, but is not; a three-clauses counterexample fails it; another condition patches it, but its being syntactic stops it; equivalence between sets of variables shows to be the problem, but does not forbid single-head equivalence; a second condition seems to account for equivalence between sets of variables; Formula $3$ crushes this delusion. All of this may look like someone banging their head on the wall over and over again, blind to the large door nearby. But maybe the door is not so close, and maybe not that large. The sequence of failed conditions, examples and counterexamples make it look not.

The counterexamples all share a feature: their clauses form cycles. Not by chance: forbidding cycles makes all problems disappear. Necessary and sufficient conditions become easy, an efficient algorithm for checking single-head equivalence becomes possible.

Cyclicity in Horn clauses is a well-establish concept [Ang87, Ang87, GS05]. Not surprisingly, the related problem of formula minimization is easy on acyclic formulae [HK95].
Single-head equivalence is a form of formula minimization where formulae are measured not by the total number of clauses or literal occurrences but by the maximal number of same heads, and the question is whether a formula can be minimized to measure one. Like formula minimization, checking single-head equivalence is easy on acyclic formulae.

Still better, it is easy on semantically acyclic formulae. Because a formula may be syntactically cyclic and may be semantically cyclic. The distinction is the same as that of cyclic functions and cyclic formulae by Hammer and Kogan [HK95]. Semantical matches semantical, and single-head equivalence is a semantical concept. It is the semantical version of being single-head: it depends on the models of the formula rather than its clauses—its syntax. A formula is single-head equivalent if it is equivalent to a formula that is single-head; a formula is semantically acyclic if it is equivalent to a formula whose clauses do not form cycles. The parallel is evident.

Acyclic formulae do not suffer from the subtleties of single-head equivalence. The very first sufficient condition that failed to be necessary in general succeeds on semantically acyclic formulae.

Syntactic acyclicity has its part too. It implies a property that does not hold in general, not even on all semantically acyclic formulae: if a formula is single-head equivalent, it is equivalent to a single-head subset of its. Irredundancy highlights the value of this result: a syntactically acyclic irredundant formula is single-head equivalent if and only if it is single-head. Single-head equivalence can be checked by making the formula irredundant, for example removing one by one its redundant clauses, and checking whether the result is single-head or not. The same property does not hold on semantically acyclic formulae. Not a surprise: subsets are syntactic, as well as irredundancy. The semantic cognate of irredundancy is minimization: semantically acyclic formulae can be checked for single-head equivalence by minimizing them.

Both syntactically and semantically acyclic formulae have their algorithm for checking single-head equivalence, and for finding a single-head equivalent formula if any: make the formula irredundant or minimal, and check it for duplicated heads. Nothing similar works in the general definite Horn case: an irredundant or minimal formula may have duplicate heads and still being single-head equivalent.

What helps is ordering the bodies of the clauses according to entailment. A body is greater than another if implies it. The bodies of a single-head formula are minimal. Since the order is defined semantically, they are also minimal for all equivalent formulae. Seen from the other direction, if a formula is equivalent to a single-head formula, the bodies of that are minimal according to the order, which is the same for both formulae. The problem shifts from finding a single-head formula that is equivalent to the given one to finding the minimal bodies.

Minimizing is no easy task since sets of variables are exponentially many. The implemented algorithm exploits testing the removal of a single variable at time. When reaching a minimum according to the ordering, the set of literals is further minimized according to set containment.

The algorithm proves complete on inequivalent formulae, an extension of semantically acyclic formulae. It is therefore complete also in its subcases of semantically and syntactically acyclic formulae. It is not complete in the general definite Horn case. Yet, it is superior to the methods for acyclic formulae based on making the formula irredundant or minimal. First, it works on the larger set of inequivalent formulae. Second, while it does not work
in general like them it is still useful: it produces a formula that can be completed by the addition of other clauses. Even when the formula is not single-head equivalent, it may reduce the duplicated heads, thereby speeding the forgetting algorithm.

Inequivalence means that a formula makes two sets of variables equivalent only if it makes them equivalent to their intersection. Every syntactically or semantically acyclic formula is inequivalent. Not the other way around: Formula 3 is inequivalent but cyclic. Inequivalent formulae are easy to check for single-head equivalence using the algorithm based on the order between sets of variables. Yet, inequivalence do not inherit the necessary and sufficient conditions for single-head equivalence: Formula 3 again meets them both but is not single-head equivalent.

The algorithm based on the order of the bodies always runs in polynomial time but is incomplete in general: it may fail to find a single-head equivalent formula even if one exists. An alternative algorithm is complete, but sometimes requires exponential time. Instead of searching a body for each head, it searches for the appropriate head for each body.

An algorithm is polynomial but incomplete, another is complete but not polynomial. The question is whether the problem itself can be solved in polynomial time at all. The complexity of single-head equivalence is an open question. It is in NP since it can be solved by guessing and checking: the guessed formula has at most as many clauses as variables because of the single-head restriction, and checking it for equivalence is polynomial-time because of the Horn restriction. Whether the problem is also NP-hard is still to be established.

Some tests of the algorithm suggest a probabilistic workaround. On some formulae, it succeeds. On some others, it fails. But on yet some others, it sometimes fails and sometimes succeeds. For example, the singlehead.py program run on the incomplete.py test file may not find a single-head equivalent formula only to find it the second time it is run on the same formula.

The log of execution shows why: the program minimizes the first clause for each head and discards the others. This makes it depend on the order of analysis of the clauses in the formula. A formula is a set of clauses, and standard implementations of sets do not guarantee the same order of visit of the elements in a set. The main loop of the algorithm may sometimes follow an order and sometimes another. If it fails, it may still succeed when run a second or third time on the same formula.

Or not. The implementations of sets do not ensure the same order every time, but do not guarantee they are different either, and especially they do not guarantee that all orders are eventually followed. The easiest way to get close to that is to randomly sort the clauses of the formula. With high probability, all possible orders of clauses are tested when running the algorithm many times.

Probabilities may also come to help within the algorithm itself. When it does not find a minimal body is because it only tests bodies obtained by removing a single variable. None of these may be less than the current one, while another body with two or more elements removed is. A solution is to sometimes replace the current body with one that is not less than it but just equivalent to it. Doing it always impairs termination since equivalent bodies form loops. Doing it randomly does not. This way, termination is maintained because sometimes the current body is not replaced but a minimal body is found with positive probability.
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