INTEGRATION OVER A GENERIC ALGEBRA

R. Casalbuoni
Dipartimento di Fisica, Universita’ di Firenze
I.N.F.N., Sezione di Firenze

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In this paper we consider the problem of quantizing theories defined over configuration spaces described by non-commuting parameters. If one tries to do that by generalizing the path-integral formalism, the first problem one has to deal with is the definition of integral over these generalized configuration spaces. This is the problem we state and solve in the present work, by constructing an explicit algorithm for the integration over a general algebra. Many examples are discussed in order to illustrate our construction.
1 Introduction

The very idea of supersymmetry leads to the possibility of extending ordinary classical mechanics to more general cases in which ordinary configuration variables live together with Grassmann variables. More recently the idea of extending classical mechanics to more general situations has been further emphasized with the introduction of quantum groups, non-commutative geometry, etc. In order to quantize these general theories, one can try two ways: i) the canonical formalism, ii) the path-integral quantization. In refs. [1, 2] classical theories involving Grassmann variables were quantized by using the canonical formalism. But in this case, also the second possibility can be easily realized by using the Berezin’s rule for integrating over a Grassmann algebra [3]. It would be desirable to have a way to perform the quantization of theories defined in a general algebraic setting. In this paper we will make a first step toward this construction, that is we will give general rules allowing the possibility of integrating over a given algebra. Given these rules, the next step would be the definition of the path-integral. In order to define the integration rules we will need some guiding principle. So let us start by reviewing how the integration over Grassmann variables come about. The standard argument for the Berezin’s rule is translational invariance. In fact, this guarantees the validity of the quantum action principle. However, this requirement seems to be too technical and we would rather prefer to rely on some more physical argument, as the one which is automatically satisfied by the path integral representation of an amplitude, that is the combination law for probability amplitudes. This is a simple consequence of the factorization properties of the functional measure and of the additivity of the action. In turn, these properties follow in a direct way from the very construction of the path integral starting from the ordinary quantum mechanics. We recall that the construction consists in the computation of the matrix element $\langle q_f, t_f | q_i, t_i \rangle \ (t_i < t_f)$ by inserting the completeness relation

$$\int dq \ |q, t\rangle \langle q, t| = 1 \quad (1.1)$$

inside the matrix element at the intermediate times $t_a \ (t_i < t_a < t_f, \ a = 1, \cdots, N)$, and taking the limit $N \to \infty$ (for sake of simplicity we consider here the quantum mechanical case of a single degree of freedom). The relevant information leading to the composition law is nothing but the complete-
ness relation (1.1). Therefore we will assume the completeness as the basic principle to use in order to define the integration rules over a generic algebra. In this paper we will limit our task to the construction of the integration rules, and we will not do any attempt to construct the functional integral in the general case. The extension of the relation (1.1) to a configuration space different from the usual one is far from being trivial. However, we can use an approach that has been largely used in the study of non-commutative geometry [4] and of quantum groups [5]. The approach starts from the observation that in the normal case one can reconstruct a space from the algebra of its functions. Giving this fact, one lifts all the necessary properties in the function space and avoids to work on the space itself. In this way one is able to deal with cases in which no concrete realization of the space itself exists.

We will see in Section 2 how to extend the relation (1.1) to the algebra of functions. In Section 3 we will generalize the considerations of Section 2 to the case of an arbitrary algebra. In Section 4 we will discuss numerous examples of our procedure. The approach to the integration on the Grassmann algebra, starting from the requirement of completeness was discussed long ago by Martin [6].

2 The algebra of functions

Let us consider a quantum dynamical system and an operator having a complete set of eigenfunctions. For instance one can consider a one-dimensional free particle. The hamiltonian eigenfunctions are

\[ \psi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(-ikx) \] (2.1)

Or we can consider the orbital angular momentum, in which case the eigenfunctions are the spherical harmonics \( Y_{\ell m}(\Omega) \). In general the eigenfunctions satisfy orthogonality relations

\[ \int \psi_n^*(x)\psi_m(x) \, dx = \delta_{nm} \] (2.2)

(we will not distinguish here between discrete and continuum spectrum). However \( \psi_n(x) \) is nothing but the representative in the \( \langle x \rangle \) basis of the eigenkets \( |n\rangle \) of the hamiltonian

\[ \psi_n(x) = \langle x | n \rangle \] (2.3)
Therefore the eq. (2.2) reads

\[ \int \langle n|x \rangle \langle x|m \rangle \, dx = \delta_{nm} \tag{2.4} \]

which is equivalent to say that the \(|x\rangle\) states form a complete set and that \(|n\rangle\) and \(|m\rangle\) are orthogonal. But this means that we can implement the completeness in the \(|x\rangle\) space by means of the orthogonality relation obeyed by the eigenfunctions defined over this space. Another important observation is that the orthonormal functions define an algebra. In fact we can expand the product of two eigenfunctions in terms of the eigenfunctions themselves

\[ \psi_m(x)\psi_n(x) = \sum_p c_{nmp}\psi_p(x) \tag{2.5} \]

with

\[ c_{nmp} = \int \psi_n(x)\psi_m(x)\psi^*_p(x) \, dx \tag{2.6} \]

For instance, in the case of the free particle

\[ c_{kk'k''} = \frac{1}{\sqrt{2\pi}} \delta(k + k' - k'') \tag{2.7} \]

In the case of the angular momentum one has the product formula \[7\]

\[ Y_{\ell_1}^{m_1}(\Omega)Y_{\ell_2}^{m_2}(\Omega) = \sum_{L=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{M=-L}^{+L} \left[ \frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi(2L + 1)} \right] \times \langle \ell_1\ell_200|L0\rangle \langle \ell_1\ell_2m_1m_2|LM \rangle Y_L^M(\Omega) \tag{2.8} \]

where \(\langle j_1j_1m_1m_2|JM \rangle\) are the Clebsch-Gordan coefficients. A set of eigenfunctions can then be considered as a basis of the algebra (2.3), with structure constants given by (2.6). Any function can be expanded in terms of the complete set \(\{\psi_n(x)\}\), and therefore it will be convenient, for the future, to introduce a generalized Fock space \(\mathcal{F}\) build up in terms of the eigenfunctions

\[ |\psi\rangle = \begin{pmatrix} \psi_0(x) \\ \psi_1(x) \\ \vdots \\ \psi_n(x) \\ \vdots \end{pmatrix} \tag{2.9} \]
A function $f(x)$ such that

$$f(x) = \sum_n a_n \psi_n(x)$$  \hspace{1cm} (2.10)

can be represented as

$$f(x) = \langle a | \psi \rangle$$  \hspace{1cm} (2.11)

where

$$\langle a | = (a_0, a_1, \cdots, a_n, \cdots)$$  \hspace{1cm} (2.12)

To write the orthogonality relation in terms of this new formalism it is convenient to realize the complex conjugation as a linear operation on $\mathcal{F}$. In fact, $\psi_n^*(x)$ itself can be expanded in terms of $\psi_n(x)$

$$\psi_n^*(x) = \sum_n \psi_m(x) C_{mn}$$  \hspace{1cm} (2.13)

or

$$|\psi^*\rangle = C^T |\psi\rangle$$  \hspace{1cm} (2.14)

Defining a bra in $\mathcal{F}$ as the transposed of the ket $|\psi\rangle$

$$\langle \psi | = (\psi_0(x), \psi_1(x), \cdots(x), \psi_n(x), \cdots)$$  \hspace{1cm} (2.15)

the orthogonality relation becomes

$$\int |\psi\rangle \langle \psi^* | \ dx = \int |\psi\rangle \langle \psi | C \ dx = 1$$  \hspace{1cm} (2.16)

Notice that by taking the complex conjugate of eq. (2.14), we get

$$CC^* = 1$$  \hspace{1cm} (2.17)

The relation (2.16) makes reference only to the elements of the algebra of functions that we have organized in the space $\mathcal{F}$, and it is the key element in order to define the integration rules on the algebra. In fact, we can now use the algebra product to reduce the expression (2.16) to a linear form

$$\delta_{nm} = \sum_\ell \int \psi_n(x) \psi_\ell(x) C_{\ell m} \ dx = \sum_{\ell,p} c_{n\ell p} C_{\ell m} \int \psi_p(x) \ dx$$  \hspace{1cm} (2.18)
If the set of equations
\[
\sum_p A_{nmp} \int \psi_p(x) \, dx = \delta_{nm}, \quad A_{nmp} = \sum_{\ell} c_{n\ell p} C_{\ell m}
\] (2.19)
has a solution for \( \int \psi_p(x) \, dx \), then we are able to define the integration over all the algebra, by linearity. We will show in the following that indeed a solution exists for many interesting cases. For instance a solution always exists, if the constant function is in the set \( \{ \psi_p(x) \} \). However we will not try here to define the conditions under which the equations are satisfied. Let us just show what we get for the free particle. The matrix \( C \) is easily obtained by noticing that
\[
\left( \frac{1}{\sqrt{2\pi}} \exp(-ikx) \right)^* = \frac{1}{\sqrt{2\pi}} \exp(i k' x) = \int dk' \delta(k + k') \frac{1}{\sqrt{2\pi}} \exp(-ik'x) \] (2.20)
and therefore
\[
C_{kk'} = \delta(k + k') \] (2.21)
It follows
\[
A_{kk'k''} = \int dq \, \delta(k' + q) \frac{1}{\sqrt{2\pi}} \delta(q + k - k'') = \frac{1}{\sqrt{2\pi}} \delta(k - k' - k'') \] (2.22)
from which
\[
\delta(k - k') = \int dk'' \int A_{kk'k''} \psi_{k''}(x) \, dx = \int \frac{1}{2\pi} \exp(-i(k - k')x) \, dx \] (2.23)
This example is almost trivial, but it shows how, given the structure constants of the algebra, the property of the exponential of being the Fourier transform of the delta-function follows automatically from the formalism. In fact, what we have really done it has been to define the integration rules by using only the algebraic properties of the exponential. As a result, our integration rules require that the integral of an exponential is a delta-function. One can perform similar steps in the case of the spherical harmonics, where the \( C \) matrix is given by
\[
C_{(\ell,m), (\ell',m')} = (-1)^m \delta_{\ell,\ell'} \delta_{m,-m'} \] (2.24)
and then using the constant function \( Y_0^0 = 1/\sqrt{4\pi} \), in the completeness relation.

The procedure we have outlined here is the one that we will generalize in the next Section to arbitrary algebras. Before doing that we will consider the possibility of a further generalization. In the usual path-integral formalism sometimes one makes use of the coherent states instead of the position operator eigenstates. In this case the basis in which one considers the wave functions is a basis of eigenfunctions of a non-hermitian operator

\[ \psi(z) = \langle \psi | z \rangle \]  

with

\[ a|z\rangle = |z\rangle z \]  

The wave functions of this type close an algebra, as \( \langle z^* | \psi \rangle \) do. But this time the two types of eigenfunctions are not connected by any linear operation. In fact, the completeness relation is defined on the direct product of the two algebras

\[ \int \frac{dz^* dz}{2\pi i} \exp(-z^* z) \langle z^* | z \rangle = 1 \]  

Therefore, in similar situations, we will not define the integration over the original algebra, but rather on the algebra obtained by the tensor product of the algebra times a copy. The copy corresponds to the complex conjugated functions of the previous example.

### 3 Eigenvalues and eigenvectors for a generic algebra

Let us start with a generic algebra \( \mathcal{A} \) with \( n + 1 \) elements \( x_i \), with \( i = 0, 1, \cdots, n \). In the following we will consider also the case \( n \to \infty \). We assume the multiplication rules

\[ x_i x_j = f_{ijk} x_k \]  

with the usual convention of sum over the repeated indices. The structure constants \( f_{ijk} \) define uniquely the algebraic structure. Consider for instance the case of an abelian algebra. In this case

\[ x_i x_j = x_j x_i \longrightarrow f_{ijk} = f_{jik} \]
The associativity condition reads

\[ x_i(x_j x_k) = (x_i x_j) x_k \]  \hspace{1cm} (3.3)

leading to the condition

\[ f_{ilm} f_{jkl} = f_{ijl} f_{ikm} \]  \hspace{1cm} (3.4)

We will make use of this equation in the following. An algebra being a vector space, the most general function on the algebra (that is a mapping \( \mathcal{A} \to \mathcal{A} \)) is a linear one:

\[ f(a, x) = \sum_{i=0}^{n} a^i x_i \]  \hspace{1cm} (3.5)

Of course, this relation defines a mapping between the \( n + 1 \) dimensional row-vectors and the functions on the algebra, that is a mapping between \( \mathbb{C}^{n+1} \) and \( \mathcal{A} \).

\[ \langle a \equiv (a^0, a^1, \cdots, a^n) \leftrightarrow f(x_i) \]  \hspace{1cm} (3.6)

By proceeding as in Section 2 we introduce the space \( \mathcal{F} \) of vectors build up in terms of the generators of the algebra

\[ |x \rangle = \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad |x \rangle \in \mathcal{F} \]  \hspace{1cm} (3.7)

The mapping (3.6) becomes

\[ \langle a \equiv \langle a|x \rangle \]  \hspace{1cm} (3.8)

The action of a linear operator on \( \mathcal{F} \) is induced by its action on \( \mathbb{C}^{n+1} \)

\[ \langle a'| = \langle a|O \leftrightarrow \langle a'|x \rangle = \langle a|O|x \rangle = \langle a|x' \rangle \]  \hspace{1cm} (3.9)

where

\[ |x' \rangle = O|x \rangle \]  \hspace{1cm} (3.10)

In order to be able to generalize properly the discussion of Section 2 it will be of fundamental importance to look for linear operators having the vectors...
$|x\rangle$ as eigenvectors and the algebra elements $x_i$ as eigenvalues. As we shall see this notion is strictly related to the mathematical concept of **right and left multiplication algebras** associated to a given algebra. The linear operators we are looking for are defined by the relation

$$X_i |x\rangle = |x\rangle x_i$$

that is

$$(X_i)_{jk} x_k = x_j x_i = f_{jik} x_k$$

or

$$(X_i)_{jk} = f_{jik}$$

To relate this notion to the right multiplication algebra, let us consider the right multiplication of an arbitrary element of the algebra by a generator

$$\langle a | x \rangle_i = f(a, x) x_i = \sum_j a^j x_j x_i = \sum_j a^j f_{jik} x_k$$

$$= \sum_j a^i (X_i)_{jk} x_k = f(a X_i, x) = \langle a X_i | x \rangle$$

$$= \langle a | (X_i) | x \rangle$$

from which the (3.11) follows. Therefore the matrix $X_i$ corresponds to the linear transformation induced on the algebra by the right multiplication by the element $x_i$. In a complete analogous way we can consider column vectors $|b\rangle$, and define a function on the algebra as

$$g(x, b) = \langle \tilde{x} | b \rangle = (x_0, x_1, \ldots, x_n) \begin{pmatrix} b^0 \\ b^1 \\ \vdots \end{pmatrix} = \sum_i x_i b^i$$

Now let us consider the left multiplication

$$x_i g(x, b) = x_i \langle \tilde{x} | b \rangle = \sum_j f_{ijk} x_k b^j$$

Defining

$$(\Pi_i)_{kj} = f_{ijk}$$
we get
\[ x_i g(x, b) = \sum_j x_k (\Pi_i)_{kj} b^j = g(x, \Pi_i b) = \langle \tilde{x} | \Pi_i | b \rangle \]  
(3.18)
therefore
\[ \langle \tilde{x} | \Pi_i = x_i \langle \tilde{x} | \]  
(3.19)
The two matrices $X_i$ and $\Pi_i$ corresponding to right and left multiplication are generally different:
\[ (X_i)_{jk} = f_{jik}, \quad (\Pi_i)_{jk} = f_{ikj} \]  
(3.20)
In terms of the matrices $X_i$ and $\Pi_i$ one can characterize different algebras. For instance, consider the abelian case. It follows from eq. (3.2)
\[ X_i = \Pi_i^T \]  
(3.21)
If the algebra is associative, then from (3.4) the following three relations can be shown to be equivalent:
\[ X_i X_j = f_{ijk} X_k, \quad \Pi_i \Pi_j = f_{ijk} \Pi_k, \quad [X_i, \Pi_i^T] = 0 \]  
(3.22)
The first two say that $X_i$ and $\Pi_i$ are linear representations of the algebra. The third that the right and left multiplication commute for associative algebras.

Recalling the discussion in Section 2 we would like first consider the case of a basis originating from hermitian operators. Notice that the generators $x_i$ play here the role of generalized dynamical variables. It is then natural to look for the case in which the operators $X_i$ admit both eigenkets and eigenbras. This will be the case if
\[ \Pi_i = C X_i C^{-1} \]  
(3.23)
that is $\Pi_i$ and $X_i$ are connected by a non-singular $C$ matrix. This matrix is exactly the analogue of the matrix $C$ defined in eq. (2.14). From (3.19), we get
\[ \langle \tilde{x} | C X_i C^{-1} = x_i \langle \tilde{x} | \]  
(3.24)
By putting
\[ \langle x | = \langle \tilde{x} | C \]  
(3.25)
we have
\[ \langle x | X_i = x_i \langle x | \]  
(3.26)
In this case, the equations (3.11) and (3.26) show that $X_i$ is the analogue of an hermitian operator. We will define now the integration over the algebra by requiring that

$$\int_{(x)} |x\rangle\langle x| = 1 \quad (3.27)$$

where 1 the identity matrix on the $(n + 1) \times (n + 1)$dimensional linear space of the linear mappings on the algebra. In more explicit terms we get

$$\int_{(x)} x_i (x_k C_{kj}) = \delta_{ij} \quad (3.28)$$

or

$$\int_{(x)} x_i x_j = (C^{-1})_{ij} \quad (3.29)$$

as well as

$$\int_{(x)} f_{ijk} x_k = (C^{-1})_{ij} \quad (3.30)$$

If we can invert this relation in terms of $\int_{(x)} x_i$, we can say to have defined the integration over the algebra, because we can extend the operation by linearity. In particular, if $\mathcal{A}$ is an algebra with identity, let us say $x_0 = 1$, then, by using (3.29), we get

$$\int_{(x)} x_i = (C^{-1})_{0i} \quad (3.31)$$

and it is always possible to define the integral.

We will discuss now the transformation properties of the integration measure with respect to an automorphism of the algebra. In particular, we will restrict our analysis to the case of a simple algebra (that is an algebra having as ideals only the algebra itself and the null element). Let us consider an invertible linear transformation on the basis of the algebra leaving invariant the multiplication rules (that is an automorphism)

$$x'_i = S_{ij} x_j \quad (3.32)$$

with

$$x'_i x'_j = f_{ijk} x'_k \quad (3.33)$$

This implies the following conditions for the transformation $S$

$$S_{il} S_{jm} f_{imp} = f_{ijk} S_{kp} \quad (3.34)$$
This relation can be written in a more convenient form in terms of the matrices $X_i$ and $\Pi_i$

\[ SX_i S^{-1} = (S^{-1})_{ij} X_j, \quad S^T \Pi_i S^T = (S^{-1})_{ij} \Pi_j \] (3.35)

In the case we are considering here $X_i$ and $\Pi_i$ are related by the $C$ matrix (see eq. (3.23), and therefore we get

\[ (C^{-1} S^T C) X_i (C^{-1} S^T C) = SX_i S^{-1} \] (3.36)

For a simple algebra, one can show that the enveloping algebra of the right and left multiplications forms an irreducible set of linear operators [8], and therefore by the Shur’s lemma we obtain

\[ C^{-1} S^T C = k S^{-1} \] (3.37)

where $k$ is a constant. It follows

\[ \langle x | \rightarrow \langle \tilde{x} | S^T C = k \langle \tilde{x} | CS^{-1} = k \langle x | S^{-1} \] (3.38)

Now we require

\[ \int_{(x')} |x'\rangle \langle x'| = \int_{(x)} |x\rangle \langle x| \] (3.39)

which is satisfied by taking

\[ \int_{(x')} = \frac{1}{k} \int_{(x)} \] (3.40)

In fact

\[ \int_{(x')} |x'\rangle \langle x'| = \frac{1}{k} \int_{(x)} S|x\rangle k \langle x | S^{-1} = 1 \] (3.41)

Let us consider now the case in which the automorphism $S$ can be exponentiated in the form

\[ S = \exp(\alpha D) \] (3.42)

Then $D$ is a derivation of the algebra, as it follows from

\[ (x_i x_j)' = x_i' x_j' \rightarrow \exp(\alpha D)(x_i x_j) = (\exp(\alpha D)x_i)(\exp(\alpha D)x_j) \] (3.43)

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by taking \( \alpha \) infinitesimal. If it happens that for this particular automorphism \( S \), one has \( k = 1 \), the integration measure is invariant (see eq. (3.40)). Then, the integral satisfies
\[
\int_{(x)} D(f(x)) = 0 \tag{3.44}
\]
for any function \( f(x) \) on the algebra. On the contrary, a derivation always defines an automorphism of the algebra by exponentiation. So, if the corresponding \( k \) is equal to one, the equation (3.44) is always valid.

Of course it may happen that the \( C \) matrix does not exist. This would correspond to the case of non-hermitian operators discussed in Section 2. So we look for a copy \( \mathcal{A}^* \) of the algebra. By calling \( x^* \) the elements of \( \mathcal{A}^* \), the corresponding generators will satisfy
\[
x^*_i x^*_j = f_{ijk} x^*_k \tag{3.45}
\]
It follows
\[
\langle \tilde{x}^* | \Pi_i = x^*_i \langle \tilde{x}^* \rangle \tag{3.46}
\]
Then, we define the integration rules on the tensor product of \( \mathcal{A} \) and \( \mathcal{A}^* \) in such a way that the completeness relation holds
\[
\int_{(x,x^*)} |x\rangle \langle \tilde{x}^*| = 1 \tag{3.47}
\]
or
\[
\int_{(x,x^*)} x^*_i x^*_j = \delta_{ij} \tag{3.48}
\]
This second type of integration is invariant under orthogonal transformation or unitary transformations, according to the way in which the \( * \) operation acts on the transformation matrix \( S \). If \( * \) acts on complex numbers as the ordinary conjugation, then we have invariance under unitary transformations, otherwise if \( * \) leaves complex numbers invariant, then the invariance is under orthogonal transformations. Notice that the invariance property does not depend on \( S \) being an automorphism of the original algebra or not.

The two cases considered here are not mutually exclusive. In fact, there are situations that can be analyzed from both points of view.

We want also to emphasize that this approach does not pretend to be complete and that we are not going to give any theorem about the classification of the algebras with respect to the integration. What we are giving is
rather a set of rules that one can try to apply in order to define an integration
over an algebra. As argued before, there are algebras that do not admit the
integration as we have defined in (3.29) or in (3.48). Consider, for instance,
a simple Lie algebra. In this case we have the relation

\[ f_{ijk} = f_{jki} \] (3.49)

which implies

\[ X_i = \Pi_i \] (3.50)

or \( C = 1 \). Then the eq. (3.29) requires

\[ \delta_{ij} = \int_{(x)} x_i x_j = \int_{(x)} f_{ijk} x_k \] (3.51)

which cannot be satisfied due to the antisymmetry of the structure constants.
Therefore, we can say that, according to our integration rules, there are
algebras with a complete set of states and algebras which are not complete.

4 Examples

In this Section we will discuss several examples of both types of integration.

4.1 The bosonic case

We will start trying to reproduce the integration rules in the bosonic case.
It is convenient to work in the coherent state basis. The coherent states are
defined by the relation

\[ a|z\rangle = |z\rangle z \] (4.1)

where \( a \) is the annihilation operator, \([a, a^\dagger] = 1\). The representative of a state
at fixed occupation number in the coherent state basis is

\[ \langle n|z \rangle = \frac{z^n}{\sqrt{n!}} \] (4.2)

So we will consider as elements of the algebra the quantities

\[ x_i = \frac{z^i}{\sqrt{i!}}, \quad i = 0, 1, \cdots, \infty \] (4.3)
The states in $\mathcal{F}$ are therefore

\[
\begin{pmatrix}
1 \\
z \\
z^2/\sqrt{2!} \\
\vdots \\
\end{pmatrix}
\] (4.4)

The algebra is defined by the multiplication rules

\[x_ix_j = \frac{z^{i+j}}{\sqrt{i!j!}} = x_{i+j} \sqrt{\frac{(i+j)!}{i!j!}}\] (4.5)

from which

\[f_{ijk} = \delta_{i+j,k} \sqrt{\frac{k!}{i!j!}}\] (4.6)

It follows

\[(X_i)_{jk} = \delta_{i+j,k} \sqrt{\frac{k!}{i!j!}}\] (4.7)

and

\[(\Pi_i)_{jk} = \delta_{i+k,j} \sqrt{\frac{j!}{i!k!}}\] (4.8)

In particular we get

\[(X_1)_{jk} = \sqrt{k} \delta_{j+1,k}, \quad (\Pi_1)_{jk} = \sqrt{k+1} \delta_{j-1,k}\] (4.9)

Therefore $X_1$ and $\Pi_1$ are nothing but the representative, in the occupation number basis, of the annihilation and creation operators respectively. It follows that the $C$ matrix cannot exist, because $[X_1, \Pi_1] = 1$, and a unitary transformation cannot change this commutation relation into $[\Pi_1, X_1] = -1$. For an explicit proof consider, for instance, $X_1$

\[
X_1 = \begin{pmatrix}
0 & \sqrt{1} & 0 & 0 & \cdots \\
0 & 0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & 0 & \sqrt{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\] (4.10)
If the matrix \( C \) would exist it would be possible to find states \( \langle z | \) such that

\[
\langle z | X_1 = z \langle z |
\]

(4.11)

with

\[
\langle z | = (f_0(z), f_1(z), \cdots)
\]

(4.12)

This would mean

\[
\langle z | X_1 = (0, f_0(z), \sqrt{2} f_1(z), \cdots) = (zf_0(z), zf_1(z), zf_2(z), \cdots)
\]

(4.13)

which implies

\[
f_0(z) = f_1(z) = f_2(z) = \cdots = 0
\]

(4.14)

Now, having shown that no \( C \) matrix exists, we will consider the complex conjugated algebra with generators constructed in terms of \( z^* \), where \( z^* \) is the complex conjugate of \( z \). Then the equation

\[
\langle \tilde{z}^* | \Pi_i = z_i^* \langle \tilde{z}^* |
\]

(4.15)

is satisfied by

\[
\langle \tilde{z}^* | = (1, \frac{z^*}{\sqrt{1!}}, \frac{z^{*2}}{\sqrt{2!}}, \cdots)
\]

(4.16)

and the integration rules give

\[
\int_{(z,z^*)} \frac{z^i z^j}{\sqrt{i! j!}} = \delta_{i,j}
\]

(4.17)

We see that our integration rules are equivalent to the gaussian integration

\[
\int_{(z,z^*)} = \int \frac{dz^* dz}{2\pi i} \exp(-|z|^2)
\]

(4.18)

Another interesting example is again the algebra of multiplication of the complex numbers but now defined also for negative integer powers

\[
z^n z^m = z^{n+m}, \quad -\infty \leq n, m \leq +\infty
\]

(4.19)

with \( z \) restricted to the unit circle

\[
z^* = z^{-1}
\]

(4.20)
Defining the vectors in $\mathcal{F}$ as

$$|z\rangle = \begin{pmatrix} . \\
z^{-i} \\
. \\
1 \\
z \\
. \\
z^{i} \\
. \\
\end{pmatrix}$$  \hspace{1cm} (4.21)

the $X_i$ and $\Pi_i$ matrices are given by

$$(X_i)_{ij} = \delta_{i+j,k}, \quad (\Pi_i)_{ij} = \delta_{i+k,j}$$  \hspace{1cm} (4.22)

and now we can construct a $C$ matrix connecting these two set of matrices. This is easier seen by looking for a bra which is eigenvector of $X_i$

$$\langle z|X_i = z^i\langle z|$$  \hspace{1cm} (4.23)

In components, by putting

$$\langle z| = (\cdots, f_{-i}(z), \cdots, f_{0}(z), \cdots, f_{i}(z), \cdots)$$  \hspace{1cm} (4.24)

we get

$$f_{j}(z)(X_i)_{jk} = f_{j}(z)\delta_{i+j,k} = f_{k-i}(z) = z^i f_{k}(z)$$  \hspace{1cm} (4.25)

This equation has the solution

$$f_{i}(z) = z^{-i}$$  \hspace{1cm} (4.26)

therefore

$$\langle z| = (\cdots, z^{i}, \cdots, 1, \cdots, z^{-i}, \cdots)$$  \hspace{1cm} (4.27)

The matrix $C$ is given by

$$(C)_{ij} = (C^{-1})_{ij} = \delta_{i,-j}$$  \hspace{1cm} (4.28)

or, more explicitly by

$$C = \begin{pmatrix} . & . & . & . \\
. & 0 & 0 & 1 \\
. & 0 & 1 & 0 \\
. & 1 & 0 & 0 \\
. & . & . & . \\
\end{pmatrix}$$  \hspace{1cm} (4.29)
In fact

\[(C \Pi_i C^{-1})_{lp} = \delta_{l,-m} \delta_{i+n,m} \delta_{n,-p} = \delta_{i-p,-l} = \delta_{i+l,p} = (X_i)_{lp}\] (4.30)

Notice that the $C$ matrix is nothing but the representation in $\mathcal{F}$ of the complex conjugation ($z \rightarrow z^* = z^{-1}$). The completeness relation reads now

\[\int_{(z)} z^i z^{-j} = \delta_{ij}\] (4.31)

from which

\[\int_{(z)} z^k = \delta_{k0}\] (4.32)

Our algebraic definition of integral can be interpreted as an integral along a circle $C$ around the origin. In fact we have

\[\int_{(z)} = \frac{1}{2\pi i} \int_C \frac{dz}{z}\] (4.33)

### 4.2 The $q$-oscillator

A generalization of the bosonic oscillator is the $q$-bosonic oscillator [9]. We will use the definition given in [10]

\[bb - q\bar{b}b = 1\] (4.34)

with $q$ real and positive. We assume as elements of the algebra $\mathcal{A}$, the quantities

\[x_i = \frac{z^i}{\sqrt{i_q!}}\] (4.35)

where $z$ is a complex number,

\[i_q = \frac{q^i - 1}{q - 1}\] (4.36)

and

\[i_q! = i_q(i - 1)_q \cdots 1\] (4.37)

The structure constants are

\[f_{ijk} = \delta_{i+j,k} \sqrt{\frac{k_q!}{i_q!j_q!}}\] (4.38)
and therefore
\[
(X_i)_{jk} = \delta_{i+j,k} \sqrt{\frac{k!}{i_j! j_k!}}, \quad (\Pi_i)_{jk} = \delta_{i+k,j} \sqrt{\frac{j!}{i_k! k_i!}} \quad (4.39)
\]

In particular
\[
(X_1)_{jk} = \delta_{j+1,k} \sqrt{k_q}, \quad (\Pi_1)_{jk} = \delta_{j-1,k} \sqrt{(k+1)_q} \quad (4.40)
\]

We see that $X_1$ and $\Pi_1$ satisfy the $q$-bosonic algebra
\[
X_1 \Pi_1 - q \Pi_1 X_1 = 1 \quad (4.41)
\]

For $q$ real and positive, no $C$ matrix exists, so, according to our rules
\[
\int_{(z,z^*)_q} \frac{z^i z^*_j}{i_q! j_q!} = \delta_{ij} \quad (4.42)
\]

This integration can be expressed in terms of the so called $q$-integral (see ref. [11]), by using the representation of $n_q!$ as a $q$-integral
\[
n_q! = \int_0^{1/(1-q)} d_q t \ e_{1/q}^{-qt} t^n \quad (4.43)
\]

where the $q$-exponential is defined by
\[
e_q^t = \sum_{n=0}^{\infty} \frac{z^n}{n_q!} \quad (4.44)
\]

and the $q$-integral through
\[
\int_0^a d_q t \ f(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \quad (4.45)
\]

Then the two integrations are related by ($z = |z| \exp(i\phi)$)
\[
\int_{(z,z^*)_q} = \int \frac{d\phi}{2\pi} \int d_q (|z|^2) \ e_{1/q}^{-q|z|^2} \quad (4.46)
\]
4.3 The fermionic case

We will discuss now the case of the Grassmann algebra $\mathcal{G}_1$, with generators $1, \theta$, such that $\theta^2 = 0$. The multiplication rules are

$$\theta^i \theta^j = \theta^{i+j}, \quad i, j, i+j = 0, 1$$

and zero otherwise (see Table 1).

|   | 1 | $\theta$ |
|---|---|---------|
| 1 | 1 | $\theta$ |
| $\theta$ | $\theta$ | 0 |

Table 1: Multiplication table for the Grassmann algebra $\mathcal{G}_1$.

From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \quad i, j, k = 0, 1$$

from which the explicit expressions for the matrices $X_i$ and $\Pi_i$ follow

$$(X_0)_{ij} = f_{i0j} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(X_1)_{ij} = f_{i1j} = \delta_{i+1,j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(\Pi_0)_{ij} = f_{0ji} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\Pi_1)_{ij} = f_{iji} = \delta_{i,j+1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Notice that $X_1$ and $\Pi_1$ are nothing but the ordinary annihilation and creation Fermi operators with respect to the vacuum state $|0\rangle = (1, 0)$. The $C$ matrix exists and it is given by

$$(C)_{ij} = \delta_{i+j,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The ket and the bra eigenvectors of $X_i$ are

$$|\theta\rangle = \begin{pmatrix} 1 \\ \theta \end{pmatrix}, \quad \langle \theta| = (\theta, 1)$$
and the completeness reads
\[ \int_{\mathcal{G}_1} |\theta\rangle \langle \theta| = \int_{\mathcal{G}_1} \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \] (4.52)
or
\[ \int_{\mathcal{G}_1} \theta^i \theta^{1-j} = \delta_{i,j} \] (4.53)
which means
\[ \int_{\mathcal{G}_1} 1 = 0, \quad \int_{\mathcal{G}_1} \theta = 1 \] (4.54)

The case of a Grassmann algebra \( \mathcal{G}_n \), which consists of \( 2^n \) elements obtained by \( n \) anticommuting generators \( \theta_1, \theta_2, \ldots, \theta_n \), the identity, 1, and by all their products, can be treated in a very similar way. In fact, this algebra can be obtained by taking a convenient tensor product of \( n \) Grassmann algebras \( \mathcal{G}_1 \), which means that the eigenvectors of the algebra of the left and right multiplications are obtained by tensor product of the eigenvectors of eq. (4.51). The integration rules extended by the tensor product give
\[ \int_{\mathcal{G}_n} \theta_1 \theta_2 \cdots \theta_n = 1 \] (4.55)
and zero for all the other cases, which is equivalent to require for each copy of \( \mathcal{G}_1 \) the equations (4.54). It is worth to mention the case of the Grassmann algebra \( \mathcal{G}_2 \) because it can be obtained by tensor product of \( \mathcal{G}_1 \times \mathcal{G}_1^* \). Then we can apply our second method of getting the integration rules and show that they lead to the same result with a convenient interpretation of the measure. The algebra \( \mathcal{G}_2 \) is generated by \( \theta_1, \theta_2 \). An involution of the algebra is given by the mapping
\[ ^* : \quad \theta_1 \leftrightarrow \theta_2 \] (4.56)
with the further rule that by taking the * of a product one has to exchange the order of the factors. It will be convenient to put \( \theta_1 = \theta, \ \theta_2 = \theta^* \). This allows us to consider \( \mathcal{G}_2 \) as \( \mathcal{G}_1 \otimes \mathcal{G}_1^* \equiv (\mathcal{G}_1, \theta^*) \). Then the ket and bra eigenvectors of left and right multiplication in \( \mathcal{G}_1 \) and \( \mathcal{G}_1^* \) respectively are given by
\[ |\theta\rangle = \begin{pmatrix} 1 \\ \theta \end{pmatrix}, \quad \langle \theta^*| = (1, \theta^*) \] (4.57)
with

$$\langle \tilde{\theta}^* | \Pi_i = \theta^* i \langle \tilde{\theta}^* |$$  (4.58)

The completeness relation reads

$$\int_{(G_1, \cdot, \cdot)} | \theta \rangle \langle \tilde{\theta}^* | = \int_{(G_1, \cdot, \cdot)} \left( \begin{array}{cc} 1 & \theta^* \\ \theta & \theta \theta^* \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$  (4.59)

This implies

$$\int_{(G_1, \cdot, \cdot)} 1 = \int_{(G_1, \cdot, \cdot)} \theta \theta^* = 1$$

$$\int_{(G_1, \cdot, \cdot)} \theta = \int_{(G_1, \cdot, \cdot)} \theta^* = 0$$  (4.60)

These relations are equivalent to the integration over $G_2$ if we do the following identification

$$\int_{(G_1, \cdot, \cdot)} = \int_{G_2} \exp(-\theta^* \theta)$$  (4.61)

Notice that the factor $\exp(-\theta^* \theta)$ plays the same role of the factor $\exp(-|z|^2)$ appearing in the gaussian measure (eq. (4.18)). In fact it has the same origin, it comes out of the norm

$$\langle \tilde{\theta}^* | \theta \rangle = 1 + \theta^* \theta = \exp(\theta^* \theta)$$  (4.62)

4.4 The case of parastatistics

We will discuss now the case of a paragrassmann algebra of order $p, G^p_1$, with generators 1, and $\theta$, such that $\theta \theta^{\dagger + 1} = 0$. The multiplication rules are defined by

$$\theta^i \theta^j = \theta^{i + j}, \quad i, j, i + j = 0, \ldots, p$$  (4.63)

and zero otherwise (see Table 2).

From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \quad i, j, k = 0, 1, \ldots, p$$  (4.64)

from which we obtain the following expressions for the matrices $X_i$ and $\Pi_i$:

$$(X_i)_{jk} = \delta_{i+j,k}, \quad (\Pi_i)_{jk} = \delta_{i+k,j}, \quad i, j, k = 0, 1, \ldots, p$$  (4.65)
In analogy with the Grassmann algebra we can construct the $C$ matrix

$$(C)_{ij} = \delta_{i+j,p} \tag{4.66}$$

In fact

$$(CX_iC^{-1})_{lj} = \delta_{i+m,p}\delta_{i+m,n}\delta_{n+q, p} = \delta_{i+p-l, p-q} = \delta_{i+q,l} = (\Pi_i)_{lj} \tag{4.67}$$

The ket and the bra eigenvectors of $X_i$ are given by

$$|\theta\rangle = \begin{pmatrix} 1 \\ \theta \\ \cdot \\ \cdot \\ \cdot \\ \theta^p \end{pmatrix}, \quad \langle \theta | = (\theta^p, \cdot, 1) \tag{4.68}$$

and the completeness reads

$$\int_{\mathcal{G}^p_1} \theta^i \theta^{p-j} = \delta_{ij} \tag{4.69}$$

which means

$$\int_{\mathcal{G}^p_1} 1 = \int_{\mathcal{G}^p_1} \theta = \int_{\mathcal{G}^p_1} \theta^{p-1} = 0 \tag{4.70}$$

$$\int_{\mathcal{G}^p_1} \theta^p = 1 \tag{4.71}$$

in agreement with the results of ref. [3] (see also [12]).
4.5 The algebra of quaternions

The quaternionic algebra is defined by the multiplication rules

\[ e_A e_B = -\delta_{AB} + \epsilon_{ABC} e_C, \quad A, B, C = 1, 2, 3 \quad (4.72) \]

where \( \epsilon_{ABC} \) is the Ricci symbol in 3 dimensions. The quaternions can be realized in terms of the Pauli matrices \( e_A = -i\sigma_A \). The automorphism group of the quaternionic algebra is \( SO(3) \), but it is more useful to work in the so-called split basis

\[ u_0 = \frac{1}{2}(1 + i e_3), \quad u_0^* = \frac{1}{2}(1 - i e_3) \]
\[ u_+ = \frac{1}{2}(e_1 + i e_2), \quad u_- = \frac{1}{2}(e_1 - i e_2) \quad (4.73) \]

In this basis the multiplication rules are given in Table 3.

|   | \( u_0 \) | \( u_0^* \) | \( u_+ \) | \( u_- \) |
|---|---|---|---|---|
| \( u_0 \) | \( u_0 \) | 0 | \( u_+ \) | 0 |
| \( u_0^* \) | 0 | \( u_0^* \) | 0 | \( u_- \) |
| \( u_+ \) | 0 | \( u_+ \) | 0 | \( -u_0 \) |
| \( u_- \) | \( u_- \) | 0 | \( -u_0^* \) | 0 |

Table 3: Multiplication table for the quaternionic algebra.

The automorphism group of the split basis is \( U(1) \), with \( u_0 \) and \( u_0^* \) invariant and \( u_+ \) and \( u_- \) with charges +1 and −1 respectively. The vectors in \( \mathcal{F} \) are

\[ |u\rangle = \begin{pmatrix} u_0 \\ u_0^* \\ u_+ \\ u_- \end{pmatrix} \quad (4.74) \]

The matrices \( X_A \) and \( \Pi_A \) satisfy the quaternionic algebra because this is an associative algebra. So \( X_+ \) and \( X_- \) satisfy the algebra of a Fermi oscillator (apart a sign). It is easy to get explicit expressions for the left and right
multiplication matrices and check that the $C$ matrix exists and that it is given by

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix} \quad (4.75)$$

Therefore

$$\langle u | = (u_0, u_0^*, -u_-, -u_+) \quad (4.76)$$

The exterior product is given by

$$|u\rangle \langle u| = \begin{pmatrix}
u_0 \\
u_0^* \\
u_+ \\
u_-
\end{pmatrix} (u_0, u_0^*, -u_-, -u_+)
= \begin{pmatrix}
u_0 & 0 & 0 & -u_+ \\
0 & u_0^* & -u_- & 0 \\
u_+ & u_0 & 0 & 0 \\
u_- & 0 & 0 & u_0^*
\end{pmatrix} \quad (4.77)$$

According to our integration rules we get

$$\int_{(u)} u_0 = \int_{(u)} u_0^* = 1, \quad \int_{(u)} u_+ = \int_{(u)} u_- = 0 \quad (4.78)$$

In terms of the original basis for the quaternions we get

$$\int_{(u)} 1 = 2, \quad \int_{(u)} e_A = 0 \quad (4.79)$$

and we see that, not unexpectedly, the integration coincides with taking the trace in the $2 \times 2$ representation of the quaternions. That is, given an arbitrary functions $f(u)$ on the quaternions we get

$$\int_{(u)} f(u) = Tr[f(u)] \quad (4.80)$$

By considering the scalar product

$$\langle u' | u \rangle = u'_0 u_0 + u_0^* u_0 - u_-^* u_+ - u_+^* u_- \quad (4.81)$$
we see that
\[ \langle u| u \rangle = 2 \] (4.82)
and
\[ \int \langle u'| u \rangle = u'_0 + u''_0 = 1 \] (4.83)
Therefore \( \langle u'| u \rangle \) behaves like a delta-function.

4.6 The algebra of octonions

We will discuss now how to integrate over the octonionic algebra (see [13]).
This algebra (said also a Cayley algebra) is defined in terms of the multiplication table of its seven imaginary units \( e_A \)
\[ e_A e_B = -\delta_{AB} + a_{ABC} e_C, \quad A, B, C = 1, \ldots, 7 \] (4.84)
where \( a_{ABC} \) is completely antisymmetric and equal to +1 for \( (ABC) = (1, 2, 3), (2, 4, 6), (4, 3, 5), (3, 6, 7), (6, 5, 1), (5, 7, 2) \) and \( (7, 1, 4) \). The automorphism group of the algebra is \( G_2 \). We define also in this case the split basis as
\[ u_0 = \frac{1}{2}(1 + ie_7), \quad u^*_0 = \frac{1}{2}(1 - ie_7) \]
\[ u_i = \frac{1}{2}(e_i + ie_{i+3}), \quad u^*_i = \frac{1}{2}(e_i - ie_{i+3}) \] (4.85)
where \( i = 1, 2, 3 \). In this basis the multiplication rules are given in Table 4.

|   | \( u_0 \) | \( u^*_0 \) | \( u_j \) | \( u^*_j \) |
|---|---|---|---|---|
| \( u_0 \) | \( u_0 \) | 0 | \( u_j \) | 0 |
| \( u^*_0 \) | 0 | \( u^*_0 \) | 0 | \( u^*_j \) |
| \( u_i \) | 0 | \( u_i \) | \( \epsilon_{ijk} u^*_k \) | \(-\delta_{ij} u_0 \) |
| \( u^*_i \) | \( u^*_i \) | 0 | \(-\delta_{ij} u_0 \) | \( \epsilon_{ijk} u_k \) |

Table 4: Multiplication table for the octonionic algebra.

This algebra is non-associative and in the split basis it has an automorphism group \( SU(3) \). The non-associativity can be checked by taking, for instance,
\[ u_i (u_j u^*_k) = u_i (-\delta_{jk} u_0) = 0 \] (4.86)
and comparing with
\[ (u_i u_j) u_k^* = \epsilon_{ijm} u_m^* u_k^* = -\epsilon_{ijk} \epsilon_{kmn} u_n \] (4.87)

The vectors in \( \mathcal{F} \) are
\[ |u\rangle = \begin{pmatrix} u_0 \\ u_0^* \\ u_i \\ u_i^* \end{pmatrix} \] (4.88)

and one can easily evaluate the matrices \( X \) and \( \Pi \) corresponding to right and left multiplication. We will not give here the explicit expressions, but one can easily see some properties. For instance, one can evaluate the anticommutator \( [X_i, X_j^*]_+ \), by using the following relation
\[ [X_i, X_j^*]_+ |u\rangle = X_i |u\rangle u_j^* + X_j^* |u\rangle u_i = (|u\rangle u_i) u_j^* + (|u\rangle u_j^*) u_i \] (4.89)

The algebra of the anticommutators of \( X_i, X_i^* \) turns out to be the algebra of three Fermi oscillators (apart from the sign)
\[ [X_i, X_j]_+ = -\delta_{ij}, \quad [X_i, X_j]_+ = 0, \quad [X_i^*, X_j^*]_+ = 0 \] (4.90)

The matrices \( X_0 \) and \( X_0^* \) define orthogonal projectors
\[ X_0^2 = X_0, \quad (X_0^*)^2 = X_0^*, \quad X_0 X_0^* = X_0^* X_0 = 0 \] (4.91)

Further properties are
\[ X_0 + X_0^* = 1 \] (4.92)

and
\[ X_i^* = -X_i^T \] (4.93)

Similar properties hold for the left multiplication matrices. One can also show that there is a matrix \( C \) connecting left and right multiplication matrices. This is given by
\[ C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1_3 \\ 0 & 0 & -1_3 & 0 \end{pmatrix} \] (4.94)

where \( 1_3 \) is the \( 3 \times 3 \) identity matrix. It follows that the eigenbras of the matrices of type \( X \), are
\[ \langle u | = (u_0, u_0^*, -u_i^*, -u_i) \] (4.95)
For getting the integration rules we need the external product

\[
|u\rangle\langle u| = \begin{pmatrix} u_0^* \\ u_i \\ u_i^* \\ u_i^* \\ u_i^* \\ u_i^* \end{pmatrix} (u_0, u_0^*, -u_j^*, -u_j)
\]

\[
= \begin{pmatrix} u_0 & 0 & 0 & -u_j \\ 0 & u_0^* & -u_j^* & 0 \\ 0 & u_i & \delta_{ij}u_0 & -\epsilon_{ijk}u_k^* \\ u_i^* & 0 & -\epsilon_{ijk}u_k & \delta_{ij}u_0^* \end{pmatrix}
\]

(4.96)

According to our rules we get

\[
\int_{(u)} u_0 = \int_{(u)} u_0^* = 1, \quad \int_{(u)} u_i = \int_{(u)} u_i^* = 0
\]

(4.97)

Other interesting properties are

\[
\langle u|u \rangle = u_0 + u_0^* + 3u_0^* + 3u_0 = 4
\]

(4.98)

and using

\[
\langle u'|u \rangle = u_0'u_0 + u_0'u_0 - u_i'u_i - u_i'u_i^*
\]

(4.99)

we get

\[
\int_{(u)} \langle u'|u \rangle = u_0' + u_0'^* = 1
\]

(4.100)

Showing that \(\langle u'|u \rangle\) behaves like a delta-function.
5 Conclusions and outlook

In this paper we have shown how it is possible to define an integral over an arbitrary algebra. The main idea is to restate the completeness relation in the configuration space (the space spanned by the eigenkets \( |x\rangle \) of the position operator), in terms of the wave functions (the functions on the configuration space). In this way the completeness relation can be understood in algebraic terms and this has allowed us to define the integration rules over an arbitrary algebra in terms of the completeness relation itself. The physical motivation to require the completeness relation is that it ensures the composition law for probabilities, as discussed in the Introduction.

The motivations of the present work come from searching a way of quantizing a theory defined on a configuration space made up of non-commuting variables, the simplest example being the case of supersymmetry. The work presented here is only a first approach to this subject. First of all we have limited our investigation to the construction of the integration rules, but we have not tried to study under which conditions they are satisfied in a given algebra. Or, said in a different way, we have not looked for a classification of algebras with respect to the integration rules we have defined. Second, in order to build up the functional integral, a further step is necessary. One needs a different copy of the given algebra to each different time along the path-integration. This should be done by taking convenient tensor products of copies of the algebra. Given these limitations, we think, however, that the step realized in this work is a necessary one in order to solve the problem of quantizing the general theories discussed here.
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