The article is dedicated towards the study of fractional order nonlinear differential systems with non-instantaneous impulses involving Riemann–Liouville derivatives with fixed lower limits and appropriate integral type initial conditions in Banach spaces. First, a mild solution of the system is constructed, and subsequently, its existence is proven using Banach’s fixed point theorem. Then, results of approximate controllability are established using the concept of fractional semigroup and an iterative technique. Suitable examples are given in the end supporting the methodology along with pointing out corrections in examples presented in previous articles.

**KEYWORDS**
approximate controllability, fixed point, non-instantaneous impulses, nonlinear systems, Riemann–Liouville derivatives

**MSC CLASSIFICATION**
93B05, 26A33, 34K35

1 | INTRODUCTION

Control theory, an area of mathematics and engineering that crosses disciplinary boundaries, is appreciably used in a variety of fields, including robotics, automotive, aerospace, electrical and computer engineering, as well as image processing and biomathematical modeling. Controllability is the qualitative property of piloting a dynamical system between arbitrary initial and desired final states within set time frame using effective control functions. Since 1960s, when Kalman [1] initially proposed and developed the notion of controllability, research community across the globe has regarded it as a subject of utmost importance. Barnett [2] in 1975, and Curtain and Zwart [3] in 1995 have covered control theory in-depth for finite and infinite dimensional spaces, respectively.

Numerous books and articles closely illustrate the findings of existence and controllability for different types of integer and fractional order mathematical models involving Caputo and Riemann–Liouville derivatives. (refer to earlier studies [1–32, 33–45] and references therein). Amidst them, previous studies [12, 37, 40] established exact controllability of fractional differential models by reshaping the controllability problem to the fixed point problem. Howbeit, Triggiani [38] demonstrated that exact controllability cannot be achieved for spaces of infinite dimension under the involved assumption of the operator B or C0-semigroup that is compact. Conclusively, for spaces of infinite dimension, the notion of approximate controllability is more relatable, reasonable, and practical over the notion of exact controllability.

The admiration of fractional calculus and its study in various fields of sciences is older than the past three decades. The book by Oldham and Spanier [32] is among the first works exclusively devoted to fractional calculus and its salient findings. The understanding of the inherited traits and memory of numerous scientific, physical, and technical events
are both made possible by fractional derivatives, which offers to be the illustrative tool for the same. Along with this, fractional derivatives provide better precision and accuracy over ordinary derivatives, and for that reason, they expedite their applications in process of diffusion and viscoelasticity, image and signal processing, aeronautical and electrical engineering, biomathematical models and control theory, and so on (see earlier studies [9, 14, 25, 26, 28, 35, 36]).

In fractional calculus realm, Caputo and Riemann–Liouville derivatives have persisted as the focus of interest in the scientific community. The use of Riemann–Liouville derivative is advantageous than Caputo ones as it permits the involved function to possess discontinuity at origin, hence addressing a larger space of functions. As a matter of fact, the usual initial conditions cannot be used when working with Riemann–Liouville derivatives; instead, weighted or integral conditions are taken into consideration. The physical significance of the initial conditions used with Riemann–Liouville derivatives was made apparent by Heymans and Podlubny [22] through fractional viscoelastic systems.

Lately, the drift of the research community is in the direction to analyze the impulsive evolution systems for existence of integral solution, their stability, controllability, and much more. While modeling a real-life phenomenon, extrinsic intercessions will inevitably but inexplicably cause disturbances to occur. These disturbances or abrupt changes are termed as impulses that have a significant impact on the nature and characteristics of the solution. The existing literature mostly discuss two sorts of impulses, the very first are instantaneous impulses: These take place for a very short span of time. The latter named non-instantaneous impulses: These begin abruptly and persists for a comparatively longer duration of time; for example, injecting a medical drug into the human body is a process of abrupt occurrence but persists its effect for few minutes till the human body stabilizes. The books [4, 10, 13] and articles [5–7, 18, 21, 39, 43] contribute to the study of various impulsive differential systems. Earlier works [5, 27] address the fractional differential systems involving Caputo derivative with non-instantaneous impulses. There is no such article in the literature so far addressing the analysis for approximate controllability of Riemann–Liouville fractional evolution systems having non-instantaneous impulses, and hence is one of the motivations for the present artifact. The main motivation for analyzing the fractional impulsive evolution systems governed by Riemann–Liouville derivatives is due to the fact that non-instantaneous impulses are the frequently occurring impulses in nature that suitably describes many mathematical models.

The study of this article revolves around the following system:

\[
\begin{align*}
&\frac{d\eta}{d\tau} \psi_{\tau}(\tau, t, z(t)) = \psi_{\tau}(t, z(t)), \quad t \in (\tau, p_\tau], \\
&\frac{d\eta}{d\tau} \psi_{\tau}(\tau, t, z(t)) = \psi_{\tau}(p_{\tau}, z(t)), \quad r = 1, 2, \ldots, m, \\
&\frac{d\eta}{d\tau} \psi_{\tau}(\tau, t, z(t)) = \psi_{\tau}(p_{\tau}, z(t)), \quad r = 1, 2, \ldots, m, \\
&\frac{d\eta}{d\tau} \psi_{\tau}(\tau, t, z(t)) = \psi_{\tau}(p_{\tau}, z(t)), \quad r = 1, 2, \ldots, m, \\
&\frac{d\eta}{d\tau} \psi_{\tau}(\tau, t, z(t)) = \psi_{\tau}(p_{\tau}, z(t)), \quad r = 1, 2, \ldots, m,
\end{align*}
\]

where \(\frac{d\eta}{d\tau}\) stands for the Riemann–Liouville fractional derivative of order \(\eta\) with fixed lower limit as 0. \(A : D(A) \subseteq Z \rightarrow Z\) is densely defined and generates \(C_0\)-semigroup \(T(t)(t > 0)\). For each fixed \(t\), \(z(t)\) and \(u(t)\) belong to Banach spaces \(Z\) and \(U\), respectively. \(B : L^p([0, a]; U) \rightarrow L^q([0, a]; Z)\) is a linear map. \(h\) is a function from \([0, a] \times Z\) to \(Z\). The points \(p_\tau\) and \(t_\tau\) satisfy the relation \(0 = p_0 < t_1 < p_1 < t_2 < \ldots < p_m < t_{m+1} = a\). The impulses start at points \(t_\tau\), \(r = 1, 2, \ldots, m\) and continue for the interval \((t_\tau, p_\tau]\). For \(r = 1, 2, \ldots, m\); \(\psi_r\) are the impulsive functions to be discussed later. \(\tau(t_\tau) = \tau(t_{\tau}) = \lim_{\tau \rightarrow 0^+} z(t_\tau - \Delta)\) and \(\tau(t_{\tau}^+) = \lim_{\tau \rightarrow 0^+} z(t_\tau + \Delta)\) denote the left- and right-hand limits of \(z(t)\) at \(t_\tau\), \(r = 1, 2, \ldots, m\), respectively.

The contribution and the organization of this artifact are as follows: Section 2 gives the briefing for basic results and definitions. Section 3 is dedicated to the construction of a mild solution, carefully handling the integral initial conditions at every break point. Results for the existence of solutions based on suitable assumptions are apparent in Section 4. Section 5 accords with sufficient assumptions and proves the approximate controllability of the considered impulsive system. Section 6 presents examples in support of the theory presented along with rectification in the example given in Liu and Li [29]. The article is wounded up with concluding remarks and future scope in Section 7.

2 | PRELIMINARY FACTS

This segment provides a quick referral to some fundamental concepts and definitions which are beneficial for the smooth study of the paper. Consider the Banach space
PC_{1-\eta}(0, a] ; Z) = \{ z : z \in C \left( (\cup_{r=0}^m p_r, t_{r+1}) \cup (\cup_{r=1}^m (t_r, p_r)) ; Z \right),

\begin{align*}
\zeta(p_r) &= \zeta(p_r^+) = \lim_{\Delta \to 0^+} \zeta(p_r - \Delta) < \infty, r = 1, 2, \ldots, m, \\
(t - p_r)^{1-\eta} \|z(t)\| &< \infty, \text{ for } t \in (p_r, p_{r+1}], r = 0, 1, \ldots, m, \\
\zeta(t_r) &= \lim_{\Delta \to 0^+} \zeta(t_r - \Delta) < \infty, r = 1, 2, \ldots, m \}.
\end{align*}

Introduce the norm \( \|z\|_{[0,a]} \) on \( PC_{1-\eta}(0, a] ; Z \) as \( \|z\|_{[0,a]} = \max_{r=0,1,\ldots,m} \|z\|_r \), where \( \|z\|_r = \sup_{t \in (p_r, p_{r+1}]} (t - p_r)^{1-\eta} \|z(t)\| \) for \( r = 0, 1, \ldots, m \).

**Remark 2.1.** \( PC_{1-\eta}(0, a] ; Z \) is a dense subset of \( L^q([0,a] ; Z) \) if \( q < \frac{1}{1-\eta} \).

Throughout this article, it is considered that \( M = \sup_{t \in [0,a]} \|T(t)\| < \infty \) and \( r = \max_{r=0,1,\ldots,m} (t_{r+1} - p_r) \).

Some definitions related to fractional integrals and derivatives are as follows:

**Definition 2.1.** [25] The fractional Riemann–Liouville integral of order \( \eta \in (0, 1) \) is defined as follows:

\[
t_\eta I_\eta^n z(t) = \frac{1}{\Gamma(\eta)} \int_{t_0}^t (t - r)^{\eta-1} z(r) dr, \quad z \in PC_{1-\eta}(0, a] ; Z),
\]

where \( \Gamma \) denotes the gamma function.

**Definition 2.2.** [25] The fractional Riemann–Liouville derivative of order \( \eta \in (0, 1) \) is defined by the following expression:

\[
t_\eta D_\eta^n z(t) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_{t_0}^t (t - r)^{-\eta} z(r) dr, \quad z \in PC_{1-\eta}(0, a] ; Z).
\]

**Definition 2.3.** A function of the complex variable \( w \) defined by

\[
E_\eta(w) = \sum_{i=0}^\infty \frac{w^i}{\Gamma(\eta i + 1)}
\]

is known as the Mittag–Leffler function in one parameter.

**Lemma 2.1.** [25, 36] The underneath holds true:

(i) If \( f \in L^q[t_0, a] \), then for any point \( t \in (t_0, a] \)

\[
t_\eta D_\eta^n (t_\eta I_\eta^n f(t)) = f(t), \quad \eta \in (0, 1).
\]

(ii) If \( f \in L^q[t_0, a] \) and \( t_\eta I_\eta^{1-\eta} f(t) \in L^q[t_0, a] \), then for any point \( t \in (t_0, a] \)

\[
t_\eta I_\eta^n (t_\eta D_\eta^n f(t)) = f(t) - \frac{(t_\eta I_\eta^{1-\eta} f(t))_{t=t_0}}{\Gamma(\eta)} (t - t_0)^{\eta-1}, \quad \eta \in (0, 1).
\]

**Proposition 2.1.** [36] The underneath holds true:

(i) For \( \alpha > 0, 0 < \eta < 1 \),

\[
t_\eta I_\eta^n (t - t_0)^{\eta-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \eta)} (t - t_0)^{\alpha+\eta-1},
\]

\[
t_\eta D_\eta^n (t - t_0)^{\eta-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \eta)} (t - t_0)^{\alpha-\eta-1}.
\]
Thus, for $0 < \eta < 1$,
\[
\varphi_t^p(t - t_0)^{-\eta} = \Gamma(1 - \eta), \\
\varphi_t^p(t - t_0)^{p-1} = 0.
\]

**Lemma 2.2.** [45] The operator $T_\eta(t)$ possesses the underneath properties:

(i) For every fixed $t \geq 0$, operator $T_\eta(t)$ is linear and bounded, such that for any $z \in \mathbb{Z}$,
\[
\|T_\eta(t)z\| \leq \frac{M}{\Gamma(\eta)}\|z\|, 
\]
where $M$ is a constant such that $\|T(t)\| \leq M$ for $t \in [0, r]$.

(ii) $T_\eta(t)(t \geq 0)$ is strongly continuous.

3 | CONSTRUCTION OF MILD SOLUTION

In order to construct the mild solution $z(t)$ for system (1), we proceed with the discussion below inspired by Agarwal et al. [6]:

**Case 1:** For $t \in (0, t_1]$,
\[
z(t) = \varphi_t^p(t, \varphi(t_1^*)) \\
= \psi_1(t, \varphi(t_1^*)) \\
= \psi_1 \left( t, \varphi^p(t, \varphi(t_1^*)) + \int_0^t (t - s)^{p-1} \varphi^p(t, \varphi(t_1^*)) ds \right).
\]

**Case 2:** For $t \in (t_1, p_1]$,
\[
z(t) = \psi_1(t, \varphi(t_1^*)) \\
= \psi_1 \left( t, \varphi^p(t, \varphi(t_1^*)) + \int_0^t (t - s)^{p-1} \varphi^p(t, \varphi(t_1^*)) ds \right).
\]

**Case 3:** For $t \in (p_1, t_2]$,
\[
\varphi^p_{t_2} z(t) = \frac{1}{\Gamma(1 - \eta)} \frac{d}{dt} \int_0^t (t - s)^{1-\eta-1} \varphi(s) ds \\
= \frac{d}{dt} \int_0^{t_1} (t - s)^{\eta} \varphi(s) ds + \frac{1}{\Gamma(1 - \eta)} \frac{d}{dt} \int_{t_1}^{p_1} (t - s)^{\eta} \psi_1(\varphi(t_1^*), \varphi(t_1^*)) ds + \frac{1}{\Gamma(1 - \eta)} \frac{d}{dt} \int_{p_1}^t (t - s)^{-\eta} \varphi(s) ds \\
= \frac{-\eta}{\Gamma(1 - \eta)} \int_0^{t_1} \frac{\varphi(s)}{(t - s)^{1+\eta}} ds + \frac{\eta}{\Gamma(1 - \eta)} \int_{t_1}^{p_1} \psi_1(\varphi(t_1^*), \varphi(t_1^*)) ds + p_1 \varphi^p_{t_2} z(t) \\
= - \psi_1(t, \varphi(t_1^*)) + p_1 \varphi^p_{t_2} z(t).
\]

Thus,
\[
p_1 \varphi^p_{t_2} z(t) = A\varphi(t) + B\varphi(t) + h(t, \varphi(t)) + \psi_1(t, \varphi(t))
\]
\[
p_1 \varphi^{p-\eta}_{t_2} z(t)|_{t=p_1} = \psi_1(p_1, \varphi(t_1^*))
\]

Hence, for $t \in (p_1, t_2]$,
\[
z(t) = (t - p_1)^{p-1} \varphi^p(t, \varphi(t_1^*)) + \int_{p_1}^t (t - s)^{p-1} \varphi^p(t, \varphi(t_1^*)) ds.
\]

Continuing this process for each $r = 2, 3, \ldots, m$ and taking lower limit of Riemann–Liouville derivative as $p_r$ for each interval $(p_r, t_{r+1}]$, we define the mild solution as follows.
Definition 3.1. A function $z \in PC_{1-\eta}([0,a];Z)$ is called a mild solution of system (1) if it satisfies the following integral equation:

$$
Z(t) = \begin{cases} 
(t - p_r)^{\eta-1}T_\eta(t - p_r)\psi_r(p_r, z(t^-)) + \int_{p_r}^t (t - s)^{\eta-1}T_\eta(t - s) [Bu(s)] \, ds, & \text{for } t \in (p_r, p_{r+1}], \ r = 0, 1, \ldots, m, \\
\psi_r(t, z(t^-)), & \text{for } t \in (t_r, p_r], \ r = 1, 2, \ldots, m,
\end{cases}
$$

(2)

where

$$
\psi_0 = z_0, \quad \phi_0 = 0, \quad \phi_r(t, z(t)) = \frac{\eta}{\Gamma(1 - \eta)} \sum_{k=0}^{r-1} \left( \int_{p_k}^{p_{k+1}} \frac{z(s)}{(t - s)^{1+\eta}} \, ds + \int_{p_k}^{p_{k+1}} \psi_{k+1}(s, z(t^-_{k+1})) \, ds \right),
$$

(3)

for $r = 1, 2, \ldots, m$; and

$$
T_\eta(t) = \eta \int_0^\infty \theta \xi_\eta(\theta)T(t^\theta)d\theta,
$$

$$
\xi_\eta(\theta) = \frac{1}{\eta} \theta^{-1 - \frac{1}{\eta}} \sigma_{\eta(\theta^{-1/\eta})},
$$

(4)

\[\sigma_{\eta(\theta)} = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\eta - 1} \frac{\Gamma(n\eta + 1)}{n!} \sin(n\pi\eta), \theta \in (0, \infty),\]

and $\xi_\eta$ is a probability density function defined on $(0, \infty)$, that is, $\xi_\eta(\theta) \geq 0$ and $\int_0^\infty \xi_\eta(\theta)d\theta = 1$.

Definition 3.2. Let $z(t, u)$ be a mild solution of system (1) at time $t$ corresponding to a control $u(\cdot) \in L^q([0,a];U)$. The set $K_o(h) = \{z(a, u) \in Z; u(\cdot) \in L^q([0,a];U)\}$ is called the reachable set for final time $a$. If $K_o(h)$ becomes dense in $Z$, system (1) is approximately controllable on $[0,a]$.

4  |  EXISTENCE OF MILD SOLUTION

This segment establishes the existence and uniqueness of mild solution for system (1) utilizing the Banach fixed point theorem. The results are based on the below mentioned assumptions:

(H0) $q \in (\frac{1}{\eta}, \frac{1}{1-\eta})$.

(H1) A constant $\kappa > 0$ exists in a way satisfying

$$
\|h(t, z) - h(t, y)\|_Z \leq \kappa \|z - y\|_Z, \forall z, y \in Z.
$$

(H2) A function $\zeta(\cdot)$ in $L^q([0,a];\mathbb{R}^+)$, $q > \frac{1}{\eta}$, and a constant $d > 0$ exists such that

$$
\|h(t, z)\|_Z \leq \zeta(t) + d(t - p_r)^{1-\eta}\|z\|_Z \text{ for a.e. } t \in (p_r, p_{r+1}] ; r = 0, 1, \ldots, m \text{ and } \forall z \in Z.
$$

(H3) For $r = 1, 2, \ldots, m$, the impulsive functions $\psi_r$ defined from $[t_r, p_r] \times Z$ to $Z$ are continuous, and there exist constants $b_r \in (0, 1)$, such that

$$
\|\psi_r(t, z) - \psi_r(t, y)\|_Z \leq b_r \|z - y\|_Z.
$$
(H4) The constant \( \nu < 1 \), where

\[
\nu = \max_{r=1,2,\ldots,m} \left\{ \frac{Mb_r}{\Gamma(\eta)(t_r - p_{r-1})^{1-\eta}} + \frac{M \kappa \Gamma(\eta)}{\Gamma(2\eta)} (t_{r+1} - p_r)^\eta \right. 
\left. + 
\frac{M}{\Gamma(1+\eta)\Gamma(1-\eta)} \left( \sum_{k=0}^{r-1} \frac{(t_{k+1} - p_k)^\eta}{(t_r - p_{k+1})^{1-\eta}} + \sum_{k=0}^{r-2} \frac{b_{k+1}(t_{r+1} - p_r)}{(t_{k+1} - p_k)^{1-\eta}(p_r - p_{k+1})^\eta} + \frac{M b_r(t_{r+1} - p_r)^{1-\eta}}{(t_r - p_{r-1})^{1-\eta}} \right) \right\}.
\]

**Theorem 4.1.** The nonlinear system (1) admits a unique mild solution in \( PC_{1-\eta}([0,a];Z) \) for each control \( u(\cdot) \in L^q([0,a];U) \) under the assumptions (H0) – (H4).

**Proof.** Consider the operator \( G \) as

\[
(Gz)(t) = \begin{cases}
(t-p_r)^{\eta-1} T_\eta(t-p_r) \psi_r(p_r,z(t_r^-)) + \int_{p_r}^{t} (t-s)^{\eta-1} T_\eta(t-s) [Bu(s) + h(s,z(s)) + \phi_r(s,z(s))] ds, & \text{for } t \in (p_r,t_{r+1}], \ r = 0, 1, \ldots, m \\
\psi_r(t, Gz(t_r^-)), & \text{for } t \in (t_r,p_r], \ r = 1, \ldots, m.
\end{cases}
\]  

(5)

First, it is evident that \( G \) maps \( PC_{1-\eta}([0,a];Z) \) into itself. It is now required to prove \( G \) as a contraction operator on \( PC_{1-\eta}([0,a];Z) \). Let \( z, y \in PC_{1-\eta}([0,a];Z) \), then

**Case 1:** For \( t \in (p_r,t_{r+1}], \ r = 0, 1, 2, \ldots, m, \)

\[
(t-p_r)^{\eta-1} \| (Gz)(t) - (Gy)(t) \| \leq \| T_\eta(t-p_r) [\psi_r(p_r,z(t_r^-)) - \psi_r(p_r,y(t_r^-))] \| + (t-p_r)^{\eta-1} \int_{p_r}^{t} (t-s)^{\eta-1} \| T_\eta(t-s) [(h(s,z(s)) - h(s,y(s)))] ds \\
+ (t-p_r)^{\eta-1} \int_{p_r}^{t} (t-s)^{\eta-1} \| T_\eta(t-s) [\phi_r(s,z(s)) - \phi_r(s,y(s))] \| ds \\
\leq \frac{M}{\Gamma(\eta)} \| [\psi_r(p_r,z(t_r^-)) - \psi_r(p_r,y(t_r^-))] \| z + \frac{M \kappa (t-p_r)^{1-\eta}}{\Gamma(\eta)} \int_{p_r}^{t} (t-s)^{\eta-1} \| z(s) - y(s) \| z ds \\
+ \frac{M (t-p_r)^{1-\eta}}{\Gamma(\eta)} \int_{p_r}^{t} (t-s)^{\eta-1} \| \phi_r(s,z(s)) - \phi_r(s,y(s)) \| z ds.
\]  

(6)

Solving \( (t-p_r)^{\eta-1} \int_{p_r}^{t} (t-s)^{\eta-1} \| \phi_r(s,z(s)) - \phi_r(s,y(s)) \| ds \) separately for \( r = 1, 2, \ldots, m \), we proceed with

\[
\phi_r(s,z(s)) = \frac{\eta}{\Gamma(1-\eta)} \sum_{k=0}^{r-1} \left( \int_{p_k}^{p_{k+1}} \frac{z(s_1)}{(s-s_1)^{1+\eta}} ds_1 + \int_{p_k}^{t_{k+1}} \psi_{k+1}(s_1, z(t_{k+1}^-)) \frac{\psi_{k+1}(s_1, z(t_{k+1}^-))}{(s-s_1)^{1+\eta}} ds_1 \right), \text{ for } r = 1, 2, \ldots, m,
\]

and \( \phi_0 = 0. \)

So, basing our following evaluations for \( t \in (p_r,t_{r+1}], \ r = 1, \ldots, m, \)

\[
\int_{p_k}^{t_{k+1}} \| z(s_1) - y(s_1) \| ds_1 = \int_{p_k}^{t_{k+1}} \frac{(s_1 - p_k)^{\eta-1} \| z(s_1) - y(s_1) \| ds_1}{(s_1 - p_k)^{1-\eta}(s - s_1)^{\eta+1}} \leq \frac{(t_{k+1} - p_k)^{\eta}}{\eta (s - t_{k+1})^{\eta+2}} \| z - y \|_{k}.
\]  

(7)
Using (7) and \( t - t_{k+1} > t - p_r, k = 0, 1, \ldots, r - 1 \), we obtain

\[
(t - p_r)^{-\eta} \int_{t_r}^{t} \int_{s_k}^{s} \left( \frac{\|z(s_1) - y(s_1)\|}{(s - s_1)^{r+1}} \right) ds_1 ds \\
\leq \frac{(t - p_r)^{-\eta} (t_{k+1} - p_k)^{\eta} \|z - y\|_k}{\eta} \\
= (t - p_r)^{-\eta} (t_{k+1} - p_k)^{\eta} \frac{\|z - y\|_k}{\eta (p_r - t_{k+1})^r (t - t_{k+1})} \\
\leq \frac{\|z - y\|_k}{\eta} \left( \frac{t_{k+1} - p_k}{p_r - t_{k+1}} \right)^{\eta}.
\]

Next, for \( k = 0, 1, \ldots, r - 1 \) and \( r = 1, 2, \ldots, m \), we have

\[
\int_{t_{k+1}}^{p_{k+1}} \frac{\|\psi_{k+1}(s_1, z(t_{k+1})) - \psi_{k+1}(s_1, y(t_{k+1}))\|}{(s - s_1)^{r+1}} ds_1 \\
\leq \frac{b_{k+1} (t_{k+1} - p_k)^{\eta} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} ds_1 \\
\leq \frac{b_{k+1} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} \\
\leq \frac{b_{k+1} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} \\
(9)
\]

Using (9) for \( k = 0, 1, 2, \ldots, r - 2 \) and \( r = 2, 3, \ldots, m \), we obtain

\[
(t - p_r)^{-\eta} \int_{t_r}^{t} \int_{s_k}^{s} \frac{\|\psi_{k+1}(s_1, z(t_{k+1})) - \psi_{k+1}(s_1, y(t_{k+1}))\|}{(s - s_1)^{r+1}} ds_1 ds \\
\leq \frac{b_{k+1} (t_{k+1} - p_k)^{\eta} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} ds_1 ds \\
\leq \frac{b_{k+1} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} \\
\leq \frac{b_{k+1} \|z - y\|_k}{\eta (t_{k+1} - p_k)^{1-\eta} (s - s_1)^{r+1}} \\
(10)
\]

and for \( k = r - 1 \) and \( r = 1, 2, \ldots, m \), it is

\[
(t - p_r)^{-\eta} \int_{t_r}^{t} \int_{s_{r-1}}^{s} \frac{\|\psi_{r}(s_1, z(t_r)) - \psi_{r}(s_1, y(t_r))\|}{(s - s_1)^{r+1}} ds_1 ds \\
\leq \frac{(t - p_r)^{-\eta} b_r \|z - y\|_r}{\eta (t_r - p_{r-1})^{1-\eta} (s - s_1)^{r+1}} ds_1 ds \\
\leq \frac{(t - p_r)^{-\eta} b_r \|z - y\|_r}{\eta (t_r - p_{r-1})^{1-\eta} (s - s_1)^{r+1}} \\
\leq \frac{(t - p_r)^{-\eta} b_r \|z - y\|_r}{\eta (t_r - p_{r-1})^{1-\eta} (s - s_1)^{r+1}} \\
(11)
\]

Combining Equations (3), (8), (10), and (11), we obtain

\[
(t - p_r)^{-\eta} \int_{t_r}^{t} \frac{(s)^{r-1} \|\phi_r(s, z(s)) - \phi_r(s, y(s))\|}{\eta (1 - \eta)} ds \\
\leq \frac{\|z - y\|_0}{\eta (1 - \eta)} \left( \sum_{k=0}^{r-1} \frac{(t_{k+1} - p_k)}{p_r - t_{k+1}} \right)^n + \sum_{k=0}^{r-2} \frac{b_{k+1} (t_{k+1} - p_r)}{(t_{k+1} - p_k)^{1-\eta} (p_r - p_{k+1})^\eta} + \frac{b_r (t_{r+1} - p_r)}{(t_r - p_{r-1})^{1-\eta} (1 + \eta) (1 - \eta)}.
\]

(12)
Applying Equation (12) in Equation (6), we proceed as

\[
(t - p_r)^{1-\eta} \| (Gz)(t) - (Gy)(t) \|
\leq \frac{M_b}{\Gamma(\eta)} \| z - y \|_{[0,a]}^{1-\eta} + \frac{M \kappa (t - p_r)^{1-\eta}}{\Gamma(\eta)} \| z - y \| \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{\eta-1} ds
\]

where \( Banach\ space \)

\[
+ \frac{M}{\Gamma(1 + \eta) \Gamma(1 - \eta)} \left( \sum_{k=0}^{r-1} \frac{b_{k+1}(t_{k+1} - p_k)}{p_r - t_{k+1}} \right) + \frac{M b_r(t_{r+1} - p_r)^{1-\eta}}{(t_r - p_{r-1})^{1-\eta}} \| z - y \|_{[0,a]}
\]

\[
= \nu \| z - y \|_{[0,a]}. \tag{13}
\]

**Case 2:** For \( t \in (t_r, p_r), \ r = 1, 2, \ldots, m, \)

\[
(t - p_{r-1})^{1-\eta} \| (Gz)(t) - (Gy)(t) \|
\leq b_r (t - p_{r-1})^{1-\eta} \| G(z(\cdot)) - G(y(\cdot)) \| Z
\]

\[
\leq b_r \left[ \frac{M b_r}{\Gamma(\eta)} \| (t_r - p_{r-1})^{1-\eta} \| \right] + \frac{M \kappa \Gamma(\eta)}{\Gamma(2\eta)} (t_{r+1} - p_r)^{\eta}
\]

\[
+ \frac{M}{\Gamma(1 + \eta) \Gamma(1 - \eta)} \left( \sum_{k=0}^{r-1} \frac{b_{k+1}(t_{k+1} - p_k)}{p_r - t_{k+1}} \right) + \frac{M b_r(t_{r+1} - p_r)^{1-\eta}}{(t_r - p_{r-1})^{1-\eta}} \| z - y \|_{[0,a]}
\]

\[
\leq \nu \| z - y \|_{[0,a]}. \tag{14}
\]

Therefore, on combining Equations (13) and (14), we obtain

\[
\| Gz - Gy \|_{[0,a]} \leq \nu \| z - y \|_{[0,a]}. \]

Thus, \( G \) is contraction and it is evident through Banach fixed point theorem that \( G \) possess a unique fixed point \( z(\cdot) \) in \( PC_1(\eta)([0, a]; Z) \) which serves as the requisite solution of system (1).

\[\Box\]

### 5  CONTROLLABILITY RESULTS

We define the following operators:

The Nemytskii operator corresponding to the nonlinear function \( h \) for \( t \in (p_r, t_{r+1}], \ r = 0, 1, \ldots, m, \) be denoted by \( \Omega_h : C_1(\eta)([p_r, t_{r+1}]; Z) \rightarrow \mathcal{L}([p_r, t_{r+1}]; Z) \) is defined by

\[
\Omega_h(z)(t) = h(t, z(t)),
\]

where Banach space \( C_1(\eta)([p_r, t_{r+1}]; Z) = \{ z : (t - p_r)^{1-\eta}z(t) \in C([p_r, t_{r+1}]; Z) \} \). The Nemytskii operator corresponding to the function \( \phi \), be denoted by \( \Omega_{\phi} : C_1(\eta)([p_r, t_{r+1}]; Z) \rightarrow \mathcal{L}([p_r, t_{r+1}]; Z) \) is defined by

\[
\Omega_{\phi}(z)(t) = \phi_t(t, z(t)).
\]
The bounded linear operator $\mathbb{F} : L^q([p_r, t_{r+1}]; Z) \to Z$ as

$$\mathbb{F}z = \int_{p_r}^{t_{r+1}} (t_{r+1} - s)^{\eta-1} T_{\eta}(t_{r+1} - s) z(s) ds.$$ 

The following assumptions are made to prove approximate controllability of system (1).

(H5) There is a constant $\kappa > 0$ such that

$$\|h(t, z) - h(t, y)\|_Z \leq \kappa (t - p_r)^{1-\eta}\|z - y\|_Z \forall z, y \in Z \text{ and } t \in (p_r, p_{r+1}], r = 0, 1, \ldots, m.$$ 

(H6) For $r = 1, 2, \ldots, m$, the impulsive functions $\psi_r$ defined from $[t_r, p_r] \times Z$ to $Z$ are continuous, and there exist constants $c_r \in (0, 1]$, such that

$$\|\psi_r(t, z)\| \leq c_r\|z\| \text{ for each } t \in [t_r, p_r] \text{ and } z \in Z.$$ 

(H7) $\mu E_{\eta}(M \kappa r) < 1.$

(H8) For any $\epsilon > 0$ and $\theta'(-) \in L^q([p_r, t_{r+1}]; Z)$, there exists a $u'(-) \in L^q([0, a]; Z)$, and hence, $u'(-) \in L^q([p_r, t_{r+1}]; U)$ satisfying

$$\|\mathbb{F}\theta' - \mathbb{F}Bu'\|_Z < \epsilon$$

$$\|Bu'(-)\|_{L^q([p_r, t_{r+1}]; Z)} < \mathfrak{N} \|\theta'(-)\|_{L^q},$$

where $\mathfrak{N}$ is a constant independent of $\theta'(-) \in L^q([p_r, t_{r+1}]; Z)$ and for each $r = 1, 2, \ldots, m,$

$$\mathfrak{N} \left[ \kappa (t_{r+1} - p_r)^{\frac{1}{\eta}} + \frac{1}{\Gamma(1 - \eta)} \sum_{k=0}^{r-1} \frac{(t_{k+1} - p_k)^\eta (t_{r+1} - p_r)^{\frac{1}{\eta}}}{(p_r - t_{k+1})^{1+\eta}} + \frac{b_{k+1}(t_{r+1} - p_r)^{\frac{1-\eta}{\eta}}}{(t_{k+1} - p_k)^{1-\eta}(1 - \eta)^{\frac{1}{\eta}}}) \frac{\alpha E_{\eta}(M \kappa r)}{1 - \mu E_{\eta}(M \kappa r)} < 1. \right. \tag{15}$$

**Lemma 5.1.** Under the assumptions (H0), (H2), (H3), and (H5)–(H7), any mild solution of system (1) satisfies the following:

(i) $\|z(-, u)\|_{[0, a]} \leq \Lambda E_{\eta}(M \kappa r)$ for any $u(-) \in L^q([0, a]; U),$ 

(ii) $\|z(-, u) - y(-, v)\|_{[0, a]} \leq \frac{\alpha E_{\eta}(M \kappa r)}{1 - \mu E_{\eta}(M \kappa r)} \|Bu - Bv\|_{L^q},$

where

$$\Lambda = \max_{r=0, 1, \ldots, m} \left\{ \frac{M}{\Gamma(\eta)} \left[ \frac{c_r}{(t_r - p_{r-1})^{1-\eta}} \|z\|_{1-\eta} + \left( \frac{q - 1}{q \eta - 1} \right)^{\frac{1}{\eta}} (t_r - p_r)^{-1}\left(\|Bu\|_{L^q} + \|z\|_{L^q} \right) \right. \right.$$

$$\left. + \left( \sum_{k=0}^{r-1} \frac{(t_{k+1} - p_k)^\eta}{(p_r - t_{k+1})^{1+\eta}} + (t_{r+1} - p_r) \sum_{k=0}^{r-2} \frac{c_{k+1}}{(p_r - p_{k+1})^{1+\eta}} \right) + c_r \Gamma(\eta)(t_{r+1} - p_r)^{-1-\eta}\|z\|_{r-1} \right] \} ,$$

$$\rho = \frac{M}{\Gamma(\eta)} \left( \frac{q - 1}{q \eta - 1} \right)^{\frac{1}{\eta}} \left. \frac{1}{r_{1-\eta}} \right.$$

and

$$\mu = \max_{r=1, 2, \ldots, m} \left\{ \frac{Mb_r}{\Gamma(\eta)(t_r - p_{r-1})^{1-\eta}} + \frac{M}{\Gamma(1 + \eta)(1 + \eta)} \left( \sum_{k=0}^{r-1} \frac{(t_{k+1} - p_k)^\eta}{(p_r - t_{k+1})^{1+\eta}} + \sum_{k=0}^{r-2} \frac{b_{k+1}(t_{r+1} - p_r)}{(t_{k+1} - p_k)^{1-\eta}(p_r - p_{k+1})^{\eta}} \right) \right.$$

$$\left. + \frac{Mb_r(t_{r+1} - p_r)^{1-\eta}}{(t_r - p_{r-1})^{1-\eta}} \right\} ,$$
Proof.

(i) Let \( z \in C_{1-\eta}([0, a]; Z) \) be a mild solution of system (1) corresponding to the control \( u(\cdot) \in L^\infty([0,a]; U) \), then

\[
\begin{align*}
\mathcal{Z}(t) &= \begin{cases}
(t - p_r)^{1-\eta} T_\eta(t - p_r) \psi_r(p_r, z(t_r)) + \int_{p_r}^t (t - s)^{\eta-1} T_\eta(t - s) [B(u(s)) + h(s, z(s)) + \phi_r(s, z(s))] \, ds, & \text{for } t \in (p_r, t_{r+1}], \ r = 0, 1, \ldots, m, \\
\psi_r(t, z(t_r)), & \text{for } t \in (t_r, p_r), \ r = 1, 2, \ldots, m.
\end{cases}
\end{align*}
\]

Case 1: For \( t \in (p_r, t_{r+1}], \ r = 0, 1, 2, \ldots, m \)

\[
(t - p_r)^{1-\eta} \|z(t)\| \leq \|T_\eta(t - p_r)\| (t - p_r)^{1-\eta} \int_{p_r}^t (t - s)^{\eta-1} \left[ \|B(u(s)) + h(s, z(s)) + \phi_r(s, z(s))\| \right] \, ds
\]

\[
\leq \frac{M}{\Gamma(\zeta)} \left[ c_r \|z(t_r)\| + (t - p_r)^{1-\eta} \int_{p_r}^t (t - s)^{\eta-1} \left[ \|B(u(s)) + h(s, z(s))\| + \|\phi_r(s, z(s))\| \right] \, ds \right]
\]

\[
\leq \frac{M}{\Gamma(\zeta)} \left[ c_r \frac{(t - p_r - 1)^{1-\eta}\|z(t_r)\|}{(t - p_r - 1)^{1-\eta}} + (t - p_r)^{1-\eta} \int_{p_r}^t (t - s)^{\eta-1} \left[ \|B(u(s))\| + \|\phi_r(s, z(s))\| \right] \, ds \right]
\]

\[
\leq \frac{M}{\Gamma(\zeta)} \left[ c_r \frac{(t - p_r - 1)^{1-\eta}\|z(t_r)\|}{(t - p_r - 1)^{1-\eta}} + (t - p_r)^{1-\eta} \int_{p_r}^t (t - s)^{\eta-1} \left[ \|B(u(s))\| + \|\phi_r(s, z(s))\| \right] \, ds \right]
\]

\[
+ \frac{c_r(t + 1 - p_r)}{(p_r - p_{k+1})^\eta} \leq c_r \Gamma(\eta) (t_{r+1} - p_r)^{1-\eta} \|z\|_{r-1}.
\]

Thus,

\[
(t - p_r)^{1-\eta} \|z(t)\| \leq \Lambda + \frac{Md(t_{r+1} - p_r)^{1-\eta}}{\Gamma(\zeta)} \int_{p_r}^t (t - s)^{\eta-1} \|z(s)\| \, ds.
\]

Using generalized Gronwall’s inequality ([42]), it follows that

\[
(t - p_r)^{1-\eta} \|z(t)\| \leq \Lambda E_{\eta}(Md\tau).
\]

Case 2: For \( t \in (t_r, p_r), \ r = 1, 2, \ldots, m, \)

\[
\begin{align*}
(t - p_{r-1})^{1-\eta} \|z(t)\| &= (t - p_{r-1})^{1-\eta} \|\psi_r(t, z(t_r))\|
\leq c_r (t - p_{r-1})^{1-\eta} \|z(t_r)\|
\leq \Lambda E_{\eta}(Md\tau).
\end{align*}
\]

Clearly, on combining the results for both the cases discussed above, that is, from Equations (16) and (17), it is proved that

\[
\|z\|_{[0,a]} \leq \Lambda E_{\eta}(Md\tau).
\]

(ii) Similarly, proceeding to prove the next inequality, we have the following.

**Case 1:** For \( t \in (p_r, t_{r+1}], r = 0, 1, 2, \ldots, m, \)

\[
(t - p_r)^{1-\eta} \| z(t) - y(t) \| \leq \| T_\eta (t - p_r) \psi_r (p_r, z(t_r)) - \psi_r (p_r, y(t_r)) \| + (t - p_r)^{1-\eta} \left( \int_{p_r}^{t} (t - s)^{\eta-1} \| T_\eta (t - s) (B_u(s) - B_v(s)) \| \, ds \right)
+ \int_{p_r}^{t} (t - s)^{\eta-1} \| T_\eta (t - s) (h(s, z(s)) - h(s, y(s))) \| \, ds
+ \int_{p_r}^{t} (t - s)^{\eta-1} \| T_\eta (t - s) (\phi_r (s, z(s)) - \phi_r (s, y(s))) \| \, ds
\leq \frac{M}{\Gamma(\eta)} \left[ b_r \| z(t_r) - y(t_r) \| + (t - p_r)^{1-\frac{1}{\eta}} \left( \frac{q - 1}{\eta q - 1} \right) \| B_u - B_v \|_{L^q} \right.
+ \int_{p_r}^{t} (t - s)^{\eta-1} \| z(s) - y(s) \| \, ds + \int_{p_r}^{t} (t - s)^{\eta-1} \| \phi_r (s, z(s)) - \phi_r (s, y(s)) \| \, ds
\leq \frac{M}{\Gamma(\eta)} \left[ b_r \| z(t_r) - y(t_r) \| + (t - p_r)^{1-\frac{1}{\eta}} \left( \frac{q - 1}{\eta q - 1} \right) \| B_u - B_v \|_{L^q} \right.
+ \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{1-\eta} \| z(s) - y(s) \| \, ds + \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{1-\eta} \| \phi_r (s, z(s)) - \phi_r (s, y(s)) \| \, ds
\leq \frac{M}{\Gamma(\eta)} \left[ b_r \| z(t_r) - y(t_r) \| + (t - p_r)^{1-\frac{1}{\eta}} \left( \frac{q - 1}{\eta q - 1} \right) \| B_u - B_v \|_{L^q} \right.
+ \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{1-\eta} \| z(s) - y(s) \| \, ds + \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{1-\eta} \| \phi_r (s, z(s)) - \phi_r (s, y(s)) \| \, ds
\leq \mu \| z - y \|_{[0,a]} + \frac{M r^{1-\frac{1}{\eta}}}{\Gamma(\eta)} \left( \frac{q - 1}{\eta q - 1} \right) \| B_u - B_v \|_{L^q} + \frac{M r^{1-\eta}}{\Gamma(\eta)} \int_{p_r}^{t} (t - s)^{\eta-1} (s - p_r)^{1-\eta} \| z(s) - y(s) \| \, ds.
\]

On applying generalized Gronwall’s identity ([45]), it leads to

\[
(t - p_r)^{1-\eta} \| z(t) - y(t) \| \leq \left( \mu \| z - y \|_{[0,a]} + \phi \| B_u - B_v \|_{L^q} \right) \mathcal{E}_\eta (MKr).
\]

**Case 2:** For \( t \in (t_r, p_{r+1}], r = 1, 2, \ldots, m, \)

\[
(t - p_{r-1})^{1-\eta} \| z(t) - y(t) \| = (t - p_{r-1})^{1-\eta} \| \psi_r (t, z(t_r)) - \psi_r (t, y(t_r)) \|
\leq b_r (t - p_{r-1})^{1-\eta} \| z(t_r) - y(t_r) \| \leq \left( \mu \| z - y \|_{[0,a]} + \phi \| B_u - B_v \|_{L^q} \right) \mathcal{E}_\eta (MKr).
\]

On combining the results from Cases 1 and 2, that is, from Equations (19) and (20), the following is obtained

\[
\| z - y \|_{[0,a]} \leq \left( \mu \| z - y \|_{[0,a]} + \phi \| B_u - B_v \|_{L^q} \right) \mathcal{E}_\eta (MKr),
\]
In order to manifest approximate controllability of nonlinear control system (1), it is adequate to claim that

\[ \|z - y\|_{[0, a]} \leq \frac{\rho E_p(M_k \tau)}{1 - \mu E_p(M_k \tau)} \|Bu - By\|_{L^t}. \]  

(21)

This accomplishes the proof. \( \square \)

**Theorem 5.1.** The nonlinear control system (1) becomes approximately controllable, provided the assumptions (H0), (H1), and (H3)–(H5) hold true and A generates the differentiable semigroup \( T(t) \).

**Proof.** In order to manifest approximate controllability of nonlinear control system (1), it is adequate to claim that \( D(A) \subset K_0(g) \), as it is well-known that domain of \( A \), \( D(A) \) is dense in \( Z \).

First, we will prove the approximate controllability of (1) in \([0, t]\) for \( t \in (0, t_1] \). For any \( z_0 \in Z \), it is understood that \( t_1 \eta^{-1} T_\eta(t_1) z_0 \in D(A) \) because \( T(t) \) is differential semigroup (implies \( D(A) = Z \)). Now, in view of Liu and Li [29], for \( g \in D(A) \), existence of a function \( \zeta \in L^q([0, t_1]; Z) \) can be shown such that \( F \zeta = g - t_1 \eta^{-1} T_\eta(t_1) z_0 \), like \( \zeta(t) = \frac{(t_1 \eta(t_1) - t)^{q-1}}{t_1} \left( T_\eta(t_1 - t) + 2t \frac{d T_\eta(t_1 - t)}{dt} \right) (g - t_1 \eta^{-1} T_\eta(t_1) z_0), t \in (0, t_1) \).

Next step is to show the existence of a control function \( u_0^1(\cdot) \in L^q([0, t_1]; U) \) in a way that the underneath inequality holds

\[ \|g - t_1 \eta^{-1} T_\eta(t_1) z_0 - F \Omega h(z_0) - F Bu^0_1\|_Z < \epsilon, \]

where \( z_0(t) \) is a mild solution of system (1) in accord with the control \( u^0_1(t) \). With the use of Hypothesis (H8), we can say that for any given \( \epsilon > 0 \) and \( u^0_1(\cdot) \in L^q([0, t_1]; U) \), there exists a control \( u^0_2(\cdot) \in L^q([0, t_1]; U) \) satisfying

\[ \|g - t_1 \eta^{-1} T_\eta(t_1) z_0 - F \Omega h(z_1) - F Bu^0_1\|_Z < \frac{\epsilon}{2^2}, \]

where \( z_1(t) = z(t; u^0_1), t \in [0, t_1] \). Denote \( z_2(t) = z(t, u^0_2) \), again by Hypothesis (H8), there exists \( \omega^0_2 \in L^q([0, t_1]; U) \) satisfying

\[ \|F[\Omega h(z_2) - \Omega h(z_1)] - F(Bo^0_2)\|_Z < \frac{\epsilon}{2^3}, \]

and

\[ \|Bo^0_2\|_{L^t} \leq \kappa \|\Omega h(z_2) - \Omega h(z_1)\|_{L^t} \]

\[ = \kappa \left( \int_0^{t_1} \|h(t, z_2(t)) - h(t, z_1(t))\| Z dt \right)^{\frac{1}{q}} \]

\[ \leq \kappa \left( \int_0^{t_1} \|z_2(t) - z_1(t)\| Z dt \right)^{\frac{1}{q}} \]

\[ \leq \kappa \left( \int_0^{t_1} \|z_2 - z_1\| Z dt \right)^{\frac{1}{q}} \]

\[ \leq \kappa \left( \frac{\rho E_p(M_k \tau)}{1 - \mu E_p(M_k \tau)} \|Bu^0_1 - Bu^0_2\|_{L^t} \right). \]

Now, define

\[ u^0_2(t) = u^0_2(t) - \omega^0_2(t), \quad u^0_3(t) \in U, \]

then

\[ \|g - t_1 \eta^{-1} T_\eta(t_1) z_0 - F \Omega h(z_2) - F Bu^0_1\|_Z \]

\[ \leq \|g - t_1 \eta^{-1} T_\eta(t_1) z_0 - F \Omega h(z_1) - F Bu^0_1\|_Z + \|FBo^0_2 - [F \Omega h(z_2) - F \Omega h(z_1)]\|_Z \]

\[ < \left( \frac{1}{2^2} + \frac{1}{2^3} \right) \epsilon. \]

\( \square \)
By applying inductions, a sequence \{u^0_n\} in \(L^q([0, t_1]; U)\) is obtained such that

\[
\| \varphi - t_1^{n-1} T_n(t_1) z_0 - \mathcal{F} Bu^0_{n+1} \| \leq \left( \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n+1}} \right) \epsilon,
\]

where \(z_0(t) = z(t, u^0_n(t))\) and

\[
\| Bu^0_{n+1} - Bu^0_n \|_{L^q} < N \kappa t_n^{\frac{1}{2}} \frac{\omega E_n(M \kappa r)}{1 - \mu E_n(M \kappa r)} \| Bu^0_n - Bu^0_{n-1} \|_{L^q}.
\]

By (15), it is evident that the sequence \(\{Bu^0_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^q([0, t_1]; Z)\). Thus, for any \(\epsilon > 0\), a positive integer \(n_0\) can be found satisfying

\[
\| Bu^0_{n_0+1} - Bu^0_{n_0} \| < \frac{\epsilon}{2}.
\]

Therefore,

\[
\| \varphi - t_1^{n-1} T_n(t_1) z_0 - \mathcal{F} Bu^0_{n+1} \| \leq \| \varphi - t_1^{n-1} T_n(t_1) z_0 - \mathcal{F} Bu^0_n \| + \| \mathcal{F} Bu^0_{n+1} - \mathcal{F} Bu^0_n \| \leq \left( \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n+1}} \right) \epsilon + \frac{\epsilon}{2} < \epsilon.
\]

Thus, the approximate controllability of (1) is proved in the interval [0, t_1].

Further, we need to prove approximate controllability in [0, t] for \(t \in (p_1, t_2]\). For any \(\psi_1(p_1, z(t_1)) \in Z\), it is understood that \((t_2 - p_1)^{n-1} T_n(t_2 - p_1) \psi_1(p_1, z(t_1)) \in D(A)\) because \(T(t)\) is differential semigroup. Now, for \(\varphi \in D(A)\), existence of a function \(\zeta_1 \in L^q([p_1, t_2]; Z)\) can be shown such that \(\zeta_1 = \varphi - (t_2 - p_1)^{n-1} T_n(t_2 - p_1) \psi_1(p_1, z(t_1))\).

Next step is to show the existence of a control function \(u^1_\epsilon(\cdot) \in L^q([p_1, t_2]; U)\) in a way that the underneath inequality holds

\[
\| \varphi - (t_2 - p_1)^{n-1} T_n(t_2 - p_1) \psi_1(p_1, z(t_1)) - \mathcal{F} \Omega_h(z_1) - \mathcal{F} \Omega_\psi(z_1) - \mathcal{F} Bu^1_\epsilon \| < \epsilon,
\]

where \(z_1(t) = z(t; u^1_\epsilon), \ t \in (p_1, t_2]\). Denote \(z_2(t) = z(t; u^1_\epsilon)\), again by Hypothesis (H8), there exists \(\omega^1_\epsilon \in L^q([p_1, t_2]; U)\) satisfying

\[
\| \mathcal{F} \Omega_h(z_2) - \mathcal{F} \Omega_h(z_1) + \mathcal{F} \Omega_\psi(z_2) - \mathcal{F} \Omega_\psi(z_1) - \mathcal{F} Bu^1_\epsilon \| < \frac{\epsilon}{2^2},
\]

and

\[
\| Bu^1_\epsilon \|_{L^q} \leq \mathcal{N} \left[ \| \Omega_h(z_2) - \Omega_h(z_1) \|_{L^q} + \| \Omega_\psi(z_2) - \Omega_\psi(z_1) \| \right]
\]

\[
= \mathcal{N} \left[ \int_{p_1}^{t_2} \| h(t, z_2(t)) - h(t, z_1(t)) \|_{Z} q dt \right] + \left[ \int_{p_1}^{t_2} \| \phi_1(t, z_2(t)) - \phi_1(t, z_1(t)) \|_{Z} \right] \frac{1}{1-q} \epsilon,
\]

\[
= \mathcal{N} \left[ \tilde{k} (t_2 - p_1)^{\frac{1}{2}} \| z_2 - z_1 \|_0 + \frac{1}{(1 - \eta)^2} \left( \frac{t_1 - p_0}{p_1 - t_1 (1 + \eta)} \right) \right] \| z_2 - z_1 \|_0 + \frac{1}{(1 - \eta)^2} \left( \frac{t_1 - p_0}{p_1 - t_1 (1 + \eta)} \right) \| z_2 - z_1 \|_0
\]

\[
= \mathcal{N} \left[ \tilde{k} (t_2 - p_1)^{\frac{1}{2}} + \frac{1}{(1 - \eta)^2} \left( \frac{t_1 - p_0}{p_1 - t_1 (1 + \eta)} \right) \right] \| z_2 - z_1 \|_0.
\]
Now, define
\[ u^{1}_{3}(t) = u^{1}_{2}(t) - \omega^{1}_{2}(t), \quad u^{1}_{3}(t) \in U, \]
then
\[
\|\varphi^{*} - F\Omega_{h}(z_{2}) - F\Omega_{\phi_{1}}(z_{2}) - FBu^{1}_{3}\|_{Z} \leq \|\varphi^{*} - F\Omega_{h}(z_{1}) - F\Omega_{\phi_{1}}(z_{1}) - FBu^{1}_{3}\|
\]
\[ + \|FBu^{1}_{3} - [F\Omega_{h}(z_{2}) - F\Omega_{h}(z_{1})] - [F\Omega_{\phi_{1}}(z_{2}) - F\Omega_{\phi_{1}}(z_{1})]\|_{Z}
\]
\[ < \left( \frac{1}{2^{2}} + \frac{1}{2^{3}} + \frac{1}{2^{n+1}} \right) \varepsilon. \]

By applying inclusions, a sequence \( \{u^{1}_{n}\} \) in \( L^{q}([p, t_{2}]; U) \) is obtained such that
\[
\|\varphi^{*} - F\Omega_{h}(z_{n}) - F\Omega_{\phi_{1}}(z_{n}) - FBu^{1}_{n+1}\|_{Z} < \left( \frac{1}{2^{2}} + \frac{1}{2^{3}} + \cdots + \frac{1}{2^{n+1}} \right) \varepsilon,
\]
where \( z_{n}(t) = z(t, u^{1}_{n}(t)) \) and
\[
\|Bu^{1}_{n+1} - Bu^{1}_{n}\|_{L^{q}} < N \left[ \tilde{k}(t_{2} - p_{1}) \frac{1}{(t_{1} - p_{1})^{\eta}} \right] < \frac{\varepsilon}{2}.
\]

By (15), it is evident that the sequence \( \{Bu^{1}_{n}\}_{n \in \mathbb{N}} \) is a Cauchy sequence on \( L^{q}([p, t_{2}]; Z) \). Thus, for any \( \varepsilon > 0 \), a positive integer \( n_{0} \) can be found satisfying
\[
\|FBu^{1}_{n+1} - FBu^{1}_{n}\|_{Z} < \frac{\varepsilon}{2}.
\]
Therefore,
\[
\|\varphi^{*} - F\Omega_{h}(z_{n}) - F\Omega_{\phi_{1}}(z_{n}) - FBu^{1}_{n+1}\|_{Z} \leq \|\varphi^{*} - F\Omega_{h}(z_{n}) - F\Omega_{\phi_{1}}(z_{n}) - FBu^{1}_{n+1}\|_{Z} + \|FBu^{1}_{n+1} - FBu^{1}_{n}\|_{Z}
\]
\[ < \left( \frac{1}{2^{2}} + \frac{1}{2^{3}} + \cdots + \frac{1}{2^{n+1}} \right) \varepsilon + \frac{\varepsilon}{2} < \varepsilon.
\]
Thus, the approximate controllability of (1) is proved in the interval \([0, t_{2}]\). Similarly, repeating the process for \( r = 2, 3, \ldots, m \), we finally get
\[
\|\varphi^{*} - (a - p_{m})^{\theta - 1} T_{\theta}(a - p_{m})\psi_{1}(p_{m}, z(t_{m})) - F\Omega_{h}(z_{c}) - F\Omega_{\phi_{m}}(z_{c}) - FBu^{m}\|_{Z} < \varepsilon
\]
which establishes the approximate controllability of system (1) in \([0, a]\). \( \square \)

**Remark 5.1.** It should be noted that if the semigroup \( T(t) \) is not differentiable, the proposed concepts in the above theorem hold valid for \( z_{0}, \psi_{r}(p_{r}, z(t_{r})) \in D(A) \) instead of \( z_{0}, \psi_{r}(p_{r}, z(t_{r})) \in Z \).

### 6 | EXAMPLES

**Example 6.1.** Examine the following fractional differential problem with Riemann–Liouville derivative involving non-instantaneous impulses:

\[
\begin{align*}
0D^{\frac{3}{2}}_{t} z(t, n) &= Az(t, n) + Bu(t, n) + h(t, z(t, n)), \quad t \in \bigcup_{r=0}^{m}(p_{r}, t_{r+1}] \subset [0, 1], \quad n \in \mathbb{N}, \\
z(t, n) &= \psi_{r}(t, z(t, n)), \quad t \in \bigcup_{r=1}^{m}(t_{r}, p_{r}], \quad n \in \mathbb{N}, \\
\left. 0D^{\frac{3}{2}}_{t} z(t, x) \right|_{t=0} &= z_{0, n}, \quad n \in \mathbb{N} \\
p_{r}D^{\frac{3}{2}}_{t} z(t, n) &= \psi_{r}(p_{r}, z(t, n)), \quad n \in \mathbb{N}.
\end{align*}
\]
Let \( W = W' = l^2 \) be the space of square summable infinite sequences \( w = (w_1, w_2, \ldots, w_n, \ldots) \) with the norm 
\[ ||w|| = \sqrt{\sum_{n=1}^{\infty} |w_n|^2}. \]
The operator \( A : D(A) \subset W \rightarrow W \) is defined as 
\[ Aw = (-w_1, -\frac{1}{2}w_2, \ldots, -\frac{1}{n}w_n, \ldots), \]
where 
\[ D(A) = \left\{ w \in W | \sum_{n=1}^{\infty} | -\frac{1}{n} \langle w, e_n \rangle |^2 < \infty \right\}. \]

\( A \) can be written in the form of 
\[ Aw = \sum_{n=1}^{\infty} (-\frac{1}{n}) \langle w, e_n \rangle e_n, \ w \in D(A), \]
where \( e_n, n \in \mathbb{N} \) are the eigen vectors corresponding to the eigen values \(-\frac{1}{n}\), respectively, and \( \{e_1, e_2, \ldots, e_n, \ldots\} \) forms an orthonormal basis of \( W \). A differentiable semigroup \( T(t)(t > 0) \) in \( W \) having \( A \) as its infinitesimal generator is written as
\[ T(t)w = \sum_{n=1}^{\infty} e^{-\frac{1}{n}t} \langle w, e_n \rangle e_n, \ w \in W \text{ and } \|T(t)\| \leq 1, M = 1. \]

The map \( B : W \rightarrow W \) is defined as 
\[ Bu = \sum_{n=2}^{\infty} \langle u, e_n \rangle e_n. \]

System (22) can be written as System (1) in abstract form and thus follows approximate controllability from Theorem 2 under assumptions \( H(1)–H(8) \).

**Remark 6.1.** Approximate controllability of system (1) is proved by assuming that \( T(t) \) forms a differentiable semigroup which leads to \( D(A) = Z \), and together with closed graph theorem, it follows that \( A \) is bounded. For this reason, the example presented by Liu and Li [29] does not support the theory as the chosen \( A \) is the differential (unbounded) operator.

**Example 6.2.** Examine the below mentioned initial value problem with Riemann–Liouville derivative involving non-instantaneous impulses:
\begin{align*}
0D_t^{\frac{1}{2}} z(t, x) &= \frac{d}{dx} z(t, x) + u(t, x) + h(t, z(t, x)), \ t \in \cup_{m=0}^{\infty} (p_r, t_r + 1] \subset [0, 1], x \in [0, \pi], \\
z(t, x) &= \psi_r(t, z(t_r, x)), \ t \in [p_r, t_r], x \in [0, \pi], \\
z(0, x) &= 0 = z(t, x), \ t \in (0, 1], \\
0D_t^{\frac{1}{2}} z(t, x)|_{t=0} &= z_0(x) \in D(A), \ x \in [0, \pi], \\
p_r D_t^{\frac{1}{2}} z(t, x)|_{t=p_r} &= \psi_r(p_r, z(t_r, x)) \in D(A), \ x \in [0, \pi].
\end{align*}

(23)

Let \( W = W' = L^2([0, \pi]) \), the map \( B = I \), and the operator \( A : D(A) \subset W \rightarrow W \) defined as 
\[ Aw = w'', \]
where 
\[ D(A) = \left\{ w \in W | w, \ \frac{\partial w}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 w}{\partial x^2} \in W \text{ and } w(0) = 0 = w(\pi) \right\}. \]
Then, $A$ can be written in the form of

$$Aw = \sum_{l=1}^{\infty} (-l^2)(w, a_l)a_l, \ w \in D(A),$$

where $a_l(x) = \sqrt{\frac{2}{\pi}} \sin(lx) (l \in \mathbb{N})$ are the eigen functions corresponding to the eigen values $-l^2$, respectively, and $\{a_1, a_2, \ldots\}$ forms an orthonormal basis of $W$. A semigroup $T(t)(t > 0)$ in $W$ having $A$ as its infinitesimal generator is written as

$$T(t)w = \sum_{l=1}^{\infty} e^{-lt}(w, a_l)a_l, \ w \in W \ \text{and} \ \|T(t)\| \leq e^{-1}, M = 1.$$ 

Let us choose the nonlinear function $h$ as

$$h(t, z(t,x)) = 1 + (t - p_r)^2 + \delta(t - p_r)^{\beta}[z(t,x) + \sin z(t,x)],$$

where $t \in (p_r, p_{r+1}]$ and $r = 0, 1, \ldots, m$. $\delta$ and $\beta$ are constants and $\beta \geq 1 - \eta$.

Now,

$$\|h(t, z(t,x)) - h(t, y(t,x))\| \leq |\delta|(t - p_r)^{\beta}\|z(t,x) - y(t,x) + \sin z(t,x) - \sin y(t,x)\|$$

$$\leq |\delta|(t - p_r)^{\beta+\eta-1}(t - p_r)^{1-\eta}\|z(t,x) - y(t,x)\|
+ \left\|2\cos\left(\frac{z(t,x) + y(t,x)}{2}\right)\sin\left(\frac{z(t,x) - y(t,x)}{2}\right)\right\|
\leq 2|\delta|(t - p_r)^{1-\eta}\|z(t,x) - y(t,x)\|
\leq 2|\delta|\|z(t,x) - y(t,x)\|$$

and

$$\|h(t, z(t,x))\| \leq 1 + (t - p_r)^2 + |\delta|(t - p_r)^{\beta}\|z(t,x) + z(t,x)\|$$

$$\leq 1 + (t - p_r)^2 + 2|\delta|(t - p_r)^{\beta+\eta-1}(t - p_r)^{1-\eta}\|z(t,x)\|
\leq (1 + (t - p_r)^2) + 2|\delta|(t - p_r)^{1-\eta}\|z(t,x)\|.$$ 

It is evident that assumptions $(H1), (H2)$, and $(H5)$ are satisfied with $\kappa = \tilde{\kappa} = d = 2|\delta|$. Similarly, the assumptions $(H3)$ and $(H6)$ are satisfied by choosing suitable impulsive functions. Further, the assumptions $(H4), (H7)$, and $(H8)$ are satisfied by choosing $\delta$ to be sufficiently close to zero. Thus, approximate controllability of (23) follows from Theorem 2 without assuming $T(t)$ as differentiable semigroup.

### 7 | CONCLUDING REMARKS

The article established the results for the existence and approximate controllability of the non-instantaneous impulsive fractional differential systems involving Riemann–Liouville derivatives. Approximate controllability has been achieved with the use of Lemma 3, interval-wise Nemytskii operators, and iterative techniques for control sequence. The article rectifies the inappropriate use of unbounded operator in the example presented by Liu and Li [29] (see Remark 3). The present work opens up the potential study in the direction of various analyses towards fractional evolution systems governed by Riemann–Liouville derivatives perturbed by non-instantaneous impulses, which cover a wide range of applications. The proposed future work emerges from relaxing the Lipschitz continuity on the nonlinear operator. If the nonlinear operator $h$ is not Lipschitz, then even the existence of solution is a matter of prime investigation. Also, the present findings can be extended for the partial approximate controllability or finite approximate controllability of the considered system with general nonlocal conditions or for a delayed epidemic model with Riemann–Liouville derivatives. For some ideas, see earlier studies [8, 15, 30, 31].
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