SPHERICALLY SYMMETRIC SOLUTIONS OF THE MULTI-DIMENSIONAL, COMPRESSIBLE, ISENTROPIC EULER EQUATIONS

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ABSTRACT. In this note, we prove that the solutions obtained to the spherically symmetric Euler equations in the recent works [2, 3] are weak solutions of the multi-dimensional compressible Euler equations. This follows from new uniform estimates made on the artificial viscosity approximations up to the origin, removing previous restrictions on the admissible test functions and ruling out formation of an artificial boundary layer at the origin. The uniform estimates may be of independent interest as concerns the possible rate of blow-up of the density and velocity at the origin for spherically symmetric flows.

1. Introduction and Main Result

The spherically symmetric, isentropic Euler equations have been a subject of active interest since at least the 1940s. In several pioneering works (cf. [4, 5]), certain special solutions were analysed, giving evidence of the possibility of finite-time blow-up of the density and velocity at the origin for spherically symmetric solutions (see also [7] for the full Euler system). However, the general question of existence of spherically symmetric solutions of the compressible, isentropic Euler equations for arbitrary spherically symmetric initial data remained open (but see [1] for the case excluding the origin).

The compressible, isentropic Euler equations in \( \mathbb{R}^n \) are

\[
\begin{aligned}
\partial_t \rho + \text{div}_x (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u} + \nabla_x p) &= 0,
\end{aligned}
\]

(1.1)

where \( \rho : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \) is the density of a given fluid (and hence \( \rho \geq 0 \)), \( \mathbf{u} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is its velocity, and the scalar function \( p(\rho) \geq 0 \) is the pressure. We write \( \mathbb{R}_+ = (0, \infty) \) throughout. In this work, we will consider the pressures given by the equation of state of a polytropic gas, that is \( p(\rho) = \kappa \rho^\gamma \) for some \( \gamma \in (1, \infty) \) and \( \kappa > 0 \). By appropriate scaling, we assume without loss of generality that \( \kappa = (\gamma - 1)^2/4\gamma \).

We consider the Cauchy problem for (1.1) by imposing initial data

\[
(\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0).
\]

(1.2)

We recall that a pair \((\rho, \mathbf{u})\) is said to be of finite energy for the Euler equations if

\[
E_*(\rho, \mathbf{u}) := \int_{\mathbb{R}^n} \left( \frac{1}{2} |\mathbf{u}|^2 + \frac{\kappa \rho^\gamma}{\gamma - 1} \right) \, dx < \infty.
\]

**Definition 1.1.** Let \((\rho_0, \mathbf{u}_0) \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n+1})\) be of finite energy, \(\rho_0 \geq 0\). We say a pair of functions \((\rho, \mathbf{u}) \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^{n+1})\) with \(\rho \geq 0\) is a finite-energy weak solution of (1.1)–(1.2) if \(E_*(\rho, \mathbf{u})(t) < \infty\) for almost every \(t > 0\) and, for all \(\varphi \in C_0^\infty(0, \infty) \times \mathbb{R}^n\),

\[
\int_0^\infty \int_{\mathbb{R}^n} (\rho \varphi_t + \rho \mathbf{u} \cdot \nabla_x \varphi) \, dx \, dt + \int_{\mathbb{R}^n} \rho_0(0) \varphi(0, x) \, dx = 0,
\]

and, for all \(\varphi \in C_0^\infty((0, \infty) \times \mathbb{R}^n; \mathbb{R}^n),\)

\[
\int_0^\infty \int_{\mathbb{R}^n} (\rho \mathbf{u} \cdot \varphi_t + (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p(\rho) \text{div}_x \varphi) \, dx \, dt + \int_{\mathbb{R}^n} \rho_0(0) \mathbf{u}_0(0) \cdot \varphi(0, x) \, dx = 0.
\]
For spherically symmetric motion, there exist scalar functions \( \rho(t, r) \) and \( u(t, r) \), where \( r = |x| \), such that
\[
\rho(t, x) = \rho(t, r), \quad u(t, x) = u(t, r) \frac{x}{|x|}.
\] (1.3)

Then, defining the momentum \( m = \rho u \), the Euler equations (1.1) take the form
\[
\begin{cases}
(r^{n-1}\rho)_t + (r^{n-1}m)_r = 0, & (t, r) \in \mathbb{R}_+ \times \mathbb{R}_+, \\
(r^{n-1}m)_t + (r^{n-1} \frac{m^2}{\rho})_r + r^{n-1}p(\rho)_r = 0, & (t, r) \in \mathbb{R}_+ \times \mathbb{R}_+.
\end{cases}
\] (1.4)

**Definition 1.2.** Let \((\rho_0, m_0) \in L^1_{loc}(\mathbb{R}_+^2 ; \mathbb{R}^2)\) with \(\rho_0 \geq 0\) and \(m_0 = \rho_0u_0\) be of finite-energy, i.e.
\[
E[\rho_0, m_0] := \int_0^\infty \left( \frac{1}{2} \frac{m_0^2}{\rho_0} + \frac{\kappa \rho_0^n}{\gamma - 1} \right) r^{n-1} \, dr = E_0 < \infty.
\]
Then a pair of functions \((\rho, m) \in L^1_{loc}(\mathbb{R}_+^2 ; \mathbb{R}^2)\) with \( \rho \geq 0 \) is a finite-energy weak solution of the spherically symmetric Euler equations (1.4) with initial data \((\rho_0, m_0)\) if \(E[\rho, m](t) < \infty\) for almost every \( t > 0 \) and, for all \( \varphi \in C_c^\infty([0, \infty)^2) \),
\[
\int_0^\infty \int_0^\infty (\rho \varphi_t + m \varphi_r) r^{n-1} \, dr \, dt + \int_0^\infty \rho_0(r) \varphi(0, r) r^{n-1} \, dr = 0,
\] (1.5)
and, for all \( \varphi \in C_c^\infty([0, \infty)^2) \) such that \( \varphi(t, 0) = 0 \) for all \( t \geq 0 \),
\[
\int_0^\infty \int_0^\infty \left( m \varphi_t + \frac{m^2}{\rho} \varphi_r + p(\rho)(\varphi_r + \frac{n-1}{r} \varphi) \right) r^{n-1} \, dr \, dt + \int_0^\infty m_0(r) \varphi(0, r) r^{n-1} \, dr = 0.
\] (1.6)

One can easily see that the formulations of Definition 1.1 and Definition 1.2 are equivalent via (1.3) (see e.g. [6, Theorem 5.7]). The main result of this note is then

**Theorem 1.3.** Suppose \( p(\rho) = \kappa \rho^n, \gamma > 1 \). Let \((\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^n; \mathbb{R}^{n+1}), \rho_0 \geq 0\), be spherically symmetric data of finite energy. Then there exists a spherically symmetric finite-energy weak solution \((\rho, u)\) of the Euler equations (1.1)–(1.2) in the sense of Definition 1.1.

In particular, there exist functions \( \rho(t, r) \) and \( u(t, r) \) such that
\[
\rho(t, x) = \rho(t, r), \quad u(t, x) = u(t, r) \frac{x}{|x|},
\] (1.7)
where \((\rho(t, r), m(t, r))\) with \( m = \rho u \) is a finite-energy weak solution of the spherically symmetric Euler equations (1.4) in the sense of Definition 1.2.

In [2], Chen-Perelpeitisa solved system (1.4) for weak solutions with a restricted weak formulation for \( \gamma \in (1, 3] \) via a vanishing artificial viscosity method, using the following approximate equations for viscosity \( \varepsilon > 0 \) on a truncated domain, \((t, r) \in (0, T) \times (a(\varepsilon), b(\varepsilon))\),
\[
\begin{cases}
(r^{n-1}\rho^\varepsilon)_t + (r^{n-1}m^\varepsilon)_r = \varepsilon (r^{n-1}\rho^\varepsilon)_r, \\
(r^{n-1}m^\varepsilon)_t + (r^{n-1} \frac{m^2}{\rho})_r + r^{n-1}p_\delta(\rho^\varepsilon)_r = \varepsilon (r^{n-1}m^\varepsilon)_r - \varepsilon \frac{n-1}{r} (r^{n-1}m^\varepsilon)_r,
\end{cases}
\]
(1.8)
with smooth approximate initial data
\[
(\rho^\varepsilon, m^\varepsilon)|_{r=a} = (\rho_0^\varepsilon, m_0^\varepsilon) (1.9)
\]
and mixed Dirichlet/Neumann boundary conditions
\[
(\rho^\varepsilon, m^\varepsilon)|_{r=a} = (\hat{\rho}(\varepsilon), 0), \quad (\rho^\varepsilon, m^\varepsilon)|_{r=b} = (\hat{\rho}(\varepsilon), 0),
\] (1.10)
with \( \hat{\rho}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), where \( p_\delta(\rho) = p(\rho) + \delta \rho^2 \) and \( \delta \to 0 \) as \( \varepsilon \to 0 \). Here \( a(\varepsilon) \in (0, 1) \), \( b(\varepsilon) \in (1, \infty) \) for each \( \varepsilon > 0 \) and, as \( \varepsilon \to 0 \), \( a(\varepsilon) \to 0 \), \( b(\varepsilon) \to \infty \). Subsequently, Chen and the author showed in [3] how the construction could be extended to cover the full range \( \gamma \in (1, \infty) \).

In the results of [2, 3], the weak formulation satisfied by the obtained solution \((\rho, m)\) of (1.4) required restrictions on the space of admissible test functions. In particular, in [2, 3], it is required that for both equations in (1.4) the test function \( \varphi \in C_c^\infty([0, \infty)^2) \) additionally satisfies
\( \varphi_r(t, 0) = 0 \) for all \( t \) (as well as the correct condition \( \varphi(t, 0) = 0 \) for the test function in the momentum equation). Such an assumption restricts the admissible test functions in the weak formulation of (1.1), and hence it is unclear whether the obtained solutions are indeed weak solutions of (1.1) in the proper sense of Definition 1.1.

In this note, we demonstrate that the solutions do indeed satisfy the correct weak formulation by proving uniform estimates on the approximate solutions up to the origin, \( r = 0 \), allowing for the passage to the limit with general test functions and the proof of Theorem 1.3.\(^1\) Before stating these new uniform estimates, we first recall from [2, 3] the main energy estimate.

**Proposition 1.4.** Let

\[
E_0 := \sup_{\varepsilon} \int_a^b \left( \frac{1}{2} \rho_0^\varepsilon (u_0^\varepsilon)^2 + \overline{h_\delta}(\rho_0^\varepsilon, \bar{\rho}) \right) r^{n-1} \, dr < \infty,
\]

where \( \overline{h_\delta}(\rho, \bar{\rho}) = h_\delta(\rho) - h_\delta(\bar{\rho}) - h'_\delta(\bar{\rho})(\rho - \bar{\rho}) \) and \( h_\delta(\rho) = \frac{\delta^\gamma}{r^\gamma} + \delta \rho^2 \).

Then, for each \( \varepsilon > 0 \) and any \( T > 0 \), there exists a unique, smooth solution \((\rho^\varepsilon, m^\varepsilon)\) to (1.8)–(1.10) satisfying also

\[
\sup_{t \in [0,T]} \int_a^b \left( \frac{1}{2} \rho^\varepsilon (u^\varepsilon)^2 + \overline{h_\delta}(\rho^\varepsilon, \bar{\rho}) \right) r^{n-1} \, dr \quad + \quad \varepsilon \int_0^T \int_a^b \left( h_\delta'(\bar{\rho}^\varepsilon) |\rho^\varepsilon|^2 + \rho^\varepsilon |u^\varepsilon|^2 + (n-1) \frac{\rho^\varepsilon (u^\varepsilon)^2}{r^2} \right) r^{n-1} \, dr \, dt \leq E_0.
\]

To make our uniform estimates, we suppose there exists \( M > 0 \), independent of \( \varepsilon \), such that

\[
\delta |\log(a)| \left( 1 + \frac{b^n}{\varepsilon} \right) + \rho^0 |\log(a)| + \bar{\rho} b^n + \frac{\sqrt{\varepsilon}}{a} \leq M.
\]

This can always be ensured by careful selection of \( \delta, \bar{\rho}, b, a \) depending on \( \varepsilon > 0 \).

The main new uniform estimate that we prove is a higher integrability estimate for both density and velocity. We write \( \theta = \frac{\gamma - 1}{2} \), so that \( \theta > 0 \) for all \( \gamma > 1 \).

**Lemma 1.5.** Suppose \((\rho^\varepsilon, m^\varepsilon)\) with inf \( \rho^\varepsilon > 0 \) is a smooth solution of (1.8)–(1.10) and that \( \varepsilon, \delta, a, b, \bar{\rho} \) satisfy assumption (1.12). Let \( \omega \in C_\infty([0,\infty)) \) be a test function such that \( \omega(r) = 1 \) for \( r \in [0,1] \) and \( \omega(r) \geq 0 \). Then there exists a constant \( M > 0 \), independent of \( \varepsilon \), such that

\[
\int_0^T \int_a^b \left( \rho^\varepsilon |u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta} \right) \omega(r) r^{n-1} \, dr \, dt \leq M.
\]

This estimate gives us the equi-integrability of the flux term \( \left( \rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon) \right) r^{n-1} \) in system (1.8) all the way up to the origin, \( r = 0 \), and hence allows for the passage to the limit.

The other uniform estimates that we require concern the spatial derivative of \( \rho^\varepsilon \) near the origin, appropriately weighted with the viscosity. These are stated in Lemmas 3.1–3.2 below and are designed to prove the convergence of the viscous terms to zero as \( \varepsilon \to 0 \).

The structure of this note is as follows. First, in §2, we prove Lemma 1.5 using a carefully constructed entropy function and precise estimates around \( r = 0 \). Next, in §3, we give the statements and proofs of Lemmas 3.1–3.2 concerning the spatial derivative of the density. Finally, in §4, we conclude the proof of Theorem 1.3.

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\(^{\text{1}}\)Since the writing of this note, G.-Q. Chen has informed me in a private correspondence that he and Y. Wang have an alternative proof of the full weak formulation, however without the higher integrability estimate up to the origin.
Throughout this section and §3, we suppose that \((\rho, m)\), \(m = \rho u\), is a smooth solution of (1.8)–(1.10) such that \(\inf_{(a,b)} \rho(t,r) \geq c_{\varepsilon}(t) > 0\). For simplicity of presentation, we omit the superscript \(\varepsilon\) from functions in this section. In order to prove the higher integrability estimate of Lemma 1.5 near the origin, we begin by recalling the weak entropy pair \((\tilde{\eta}, \tilde{q})\) constructed in [8, Section II] by the formulae

\[\tilde{\eta}(\rho, \rho u) = \int_R \frac{1}{2} s|s|\rho^{2\theta} - (u - s)^2 \frac{M}{t} ds,\]

\[\tilde{q}(\rho, \rho u) = \int_R \frac{1}{2} s|s|(\theta s + (1 - \theta)u)\rho^{2\theta} - (u - s)^2 \frac{M}{t} ds.\]

We define a modified entropy pair

\[\tilde{\eta}(\rho, m) = \tilde{\eta}(\rho, \rho) - \nabla \tilde{\eta}(\rho, 0) \cdot (\rho - \rho, m) \geq 0,\]

\[\tilde{q}(\rho, m) = \tilde{q}(\rho, \rho) - \nabla \tilde{\eta}(\rho, 0) \cdot \left(m, \frac{m^2}{\rho} + p(\rho)\right).\]

As in [8, 2, 3], for a constant \(M > 0\) depending only on \(\gamma \in (1, \infty)\), we have the estimates:

\[\tilde{q}(\rho, m) \geq \frac{1}{M} (\rho|u|^3 + \rho^{\gamma + \theta}) - M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[-\hat{q} + m(\tilde{r} - \tilde{u} \tilde{m}) \leq 0,\]

\[|\tilde{\eta} - (\tilde{\eta} + \tilde{u} \tilde{m})| \leq M(\rho|u|^2 + \rho^\gamma),\]

\[|\tilde{q} - (\tilde{q} + \tilde{u} \tilde{m})| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

and, considering \(\tilde{\eta} = \tilde{u} \tilde{m}\) and \(\tilde{m} = \tilde{u} \tilde{m}\) as functions of \(\rho\) and \(u\),

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho^\gamma),\]

\[|\tilde{\eta} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho^\gamma),\]

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{\eta} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{\eta} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{\eta} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{\eta} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

\[|\tilde{q} + \tilde{u} \tilde{m}| \leq M(\rho|u|^2 + \rho + \rho^\gamma),\]

Moreover, we recall from [2, Lemma 3.4] that there exists a constant \(M > 0\), depending only on \(\gamma > 1\), such that for any \((\rho, m) \in \mathbb{R}^2\) and \(\xi \in \mathbb{R}^2\),

\[|\xi \nabla^2 \eta(\rho, m)\xi^\top| \leq M|\xi \nabla^2 \eta(\rho, m)\xi^\top|\]

We also require estimates on the growth of certain \(L^p\) norms of the density close to the origin when weighted appropriately.

**Lemma 2.1.** There exists \(M(\gamma) > 0\), independent of \(\varepsilon\), such that for \(l \in \{0, \ldots, n - 1\}\), \(T > 0\),

\[\sup_{t \in [0,T]} \int_r \rho(t, y)^{2\gamma} dy \leq M(\rho + \rho^\gamma)\]

As the proof is similar to that of [2, Lemma 3.1], we omit it here. Finally, we recall the following lemma from [2].

**Lemma 2.2** ([2, Lemma 3.2]). There exists a constant \(M = M(T) > 0\), independent of \(\varepsilon\), such that, for any \(r \in (a, b)\),

\[\int_0^T \int_r \rho(t,y)^{3} dy dt \leq M(1 + \frac{b^n}{\varepsilon}).\]

**Proof of Lemma 1.5.** We multiply the first equation in (1.8) by \(\tilde{\eta} r^{n-1}\) and the second equation by \(\tilde{m} r^{n-1}\) and sum to obtain

\[\epsilon r^{n-1}\tilde{\eta} + (r^{n-1}\tilde{q}) + (n - 1)r^{n-2}(\tilde{q} + m\tilde{\eta}) + \frac{m^2}{\rho}\tilde{m} + \tilde{m} (\tilde{q} + (n - 1)m\tilde{\eta}) - r^{n-1}(\delta\rho^2)\tilde{m}.\]
We integrate this over the region \((0, T) \times (r, b)\) to find
\[
\int_0^T \tilde{q}(t, r)r^{n-1} dt = \int_r^b (\tilde{q}(T, y) - \tilde{q}(0, y))y^{n-1} dy + \int_0^T \tilde{q}(\rho, 0)b^{n-1} dt
\]
\[
+ (n - 1) \int_0^T \int_r^b \left(-\tilde{q} + m\tilde{\eta}_0 + \frac{m^2}{\rho} \tilde{\eta}_m\right)y^{n-2} dy dt
\]
\[
+ (n - 1) \int_0^T \int_r^b \tilde{\eta}_m(\rho, 0)p(\rho)y^{n-2} dy dt
\]
\[
+ \varepsilon \int_0^T \int_r^b \left((\rho y y + \frac{n - 1}{y} \rho y) \tilde{\eta}_0 + (m y + \frac{n - 1}{y} m) \tilde{\eta}_m\right)y^{n-1} dy dt
\]
\[
+ \delta \int_0^T \int_r^b (\rho^2) \tilde{\eta}my^{n-1} dy dt.
\]

Using the upper bound of \(|\tilde{\eta}(\rho, m)| \leq \rho|u|^2 + \frac{\rho^2}{\gamma - 1} \rho^\gamma\), the identity \(\tilde{q}(\tilde{\rho}, 0) = M_0\tilde{\rho}^{\gamma + \theta}\) for some constant \(M_0 > 0\) and the non-positivity of \(-\tilde{q} + m\tilde{\eta}_0 + \frac{m^2}{\rho} \tilde{\eta}_m\) from (2.1), we obtain
\[
\int_0^T \tilde{q}(t, r)r^{n-1} dt \leq ME_0 + M_0 T \tilde{\rho}^{\gamma + \theta}b^{n-1} + (n - 1) \int_0^T \int_r^b \tilde{\eta}_m(\rho, 0)p(\rho)y^{n-2} dy dt
\]
\[
- \varepsilon \int_0^T \int_r^b \left((\rho y y + \frac{n - 1}{y} \rho y) \tilde{\eta}_0 + (m y + \frac{n - 1}{y} m) \tilde{\eta}_m\right)y^{n-1} dy dt
\]
\[
- \delta \int_0^T \int_r^b \rho^2 \tilde{\eta}my^{n-1} dy dt - \varepsilon \int_0^T \int_a^b \left((\rho y y + \frac{n - 1}{y} \rho y) \tilde{\eta}_0 + (m y + \frac{n - 1}{y} m) \tilde{\eta}_m\right)y^{n-1} dy dt
\]
by integrating by parts in the final term of (2.6) and using the boundary condition \(\tilde{\eta}_m(\tilde{\rho}, 0) = 0\).

Now let \(\omega \in C^\infty_c([0, \infty))\) be as in the statement of the lemma, so that \(\omega(r) = 1\) for \(r \in [0, 1]\) and \(\omega(r) \geq 0\). We multiply (2.7) by \(\omega(r)\), apply the lower bound of (2.1) for \(\tilde{q}\), and integrate in \(r\) from \(a\) to \(b\) to see
\[
\int_0^T \int_a^b (\rho|u|^3 + \rho^{\gamma + \theta})\omega(r)r^{n-1} dr dt \leq M(E_0 + \tilde{\rho}^{\gamma + \theta}b^{n-1} + I),
\]
where
\[
I = (n - 1) \int_0^T \int_a^b \int_r^b \tilde{\eta}_m(\rho, 0)p(\rho)y^{n-2}\omega(r) dy dr dt
\]
\[
- \varepsilon \int_0^T \int_a^b \int_r^b \left((\rho y y + \frac{n - 1}{y} \rho y) \tilde{\eta}_0 + (m y + \frac{n - 1}{y} m) \tilde{\eta}_m\right)y^{n-1}\omega(r) dy dr dt
\]
\[
- \delta \int_0^T \int_a^b \int_r^b \rho^2 \tilde{\eta}my^{n-1}\omega(r) dr dt
\]
\[
= I_1 + \cdots + I_5.
\]

We treat \(I_1\) first, recalling Lemma 2.1 and (2.1) to bound
\[
|I_1| \leq M \int_0^T \int_a^b \int_r^b \tilde{\rho}^\theta \rho^{\gamma} y^{n-2}\omega(r) dy dr dt
\]
\[
\leq MT \tilde{\rho}^\theta \int_a^b (r^{-1} + \tilde{\rho}^{b^{n-1}})\omega(r) dr
\]
\[
\leq MT \tilde{\rho}^\theta (|\log(a)| + \tilde{\rho}^{b^{n-1}}).
\]
We consider next $I_2$, using integration by parts to re-write the inner integral as:

$$I_2 = \varepsilon \int_0^T \int_a^b \int_r^b \left( \rho u(\tilde{\eta}_r) \eta + m_b(\tilde{\eta}_m) \right) y^{n-1} \omega(r) \, dy \, dr \, dt$$

$$+ \varepsilon(n-1) \int_0^T \int_a^b \int_r^b m\tilde{\eta}_m y^{n-3} \omega(r) \, dy \, dr \, dt + \varepsilon \int_0^T \int_a^b \tilde{\eta}_r(t, r) r^{n-1} \omega(r) \, dr \, dt$$

$$= I_2^1 + I_2^2 + I_2^3.$$

The first term is easily seen to be bounded from (2.4) and the main energy estimate, Proposition 1.4, giving a contribution of $|I_2^1| \leq ME_0$.

For the second term, we use (2.1) to bound

$$|m\tilde{\eta}_m| \leq M \left( \rho u^2 + \rho^2 + \rho \rho^2 \right) \leq M \left( \eta^\ast(\rho, m) + \rho \tilde{\rho} \gamma^{-1} \right).$$

Thus we find, noting $\rho \tilde{\rho} \gamma^{-1} \leq \tilde{\gamma} \gamma$ by Young’s inequality,

$$|I_2^2| \leq M \varepsilon \int_0^T \int_a^b \int_r^b \left( \eta^\ast(\rho, m) + \rho \tilde{\rho} \gamma^{-1} \right) y^{n-3} \omega(r) \, dy \, dr \, dt$$

$$\leq M \varepsilon \int_0^T \int_a^b \int_r^b \left( \frac{1}{n} \eta^\ast(\rho, m) y^{n-1} + \frac{1}{r} \tilde{\rho} \gamma^{-2} \right) \omega(r) \, dy \, dr \, dt$$

$$\leq M \varepsilon \left( \frac{1}{n} + \tilde{\rho} \gamma^{-2} \right).$$

Next, we treat the final term, $I_2^3$, by integrating by parts and using $\omega(a) = 1$ to find

$$I_2^3 = -\varepsilon \int_0^T \int_a^b \tilde{\eta}_r(t, r) (\omega_r(r) + \frac{n-1}{r} \omega(r)) r^{n-1} \, dr \, dt - \varepsilon \int_0^T \tilde{\eta}(t, a) a^{n-1} \, dt.$$

Using (2.1), we easily bound the first term by

$$\left| \varepsilon \int_0^T \int_a^b \tilde{\eta}_r(t, r) (\omega_r(r) + \frac{n-1}{r} \omega(r)) r^{n-1} \, dr \, dt \right| \leq M \frac{\varepsilon}{a}.$$

For the second term, we again apply (2.1) and the boundary condition $a(t, a) = 0$ to note that $|\tilde{\eta}(t, a)| \leq M(\tilde{\rho} \gamma + 1)$. The contribution from the constant is clearly bounded, so we focus on the $\tilde{\rho} \gamma(t, a)$ term. From the fundamental theorem of calculus and Lemma 2.1, we obtain

$$\varepsilon \int_0^T \tilde{\rho} \gamma(t, a) a^{n-1} \, dt = -\varepsilon \int_0^T \int_a^b (\tilde{\rho} \gamma^{-1}) \, dr \, dt + \varepsilon \int_0^T \tilde{\rho} \gamma^{-1} \, dt$$

$$= -\varepsilon \int_0^T \int_a^b (\gamma \tilde{\rho} \gamma^{-1} \rho \gamma^{-1} + (n-1) \rho \gamma^{-1} \gamma^{-1} ) \, dr \, dt + \varepsilon T \tilde{\rho} \gamma^{-1}$$

$$\leq M \varepsilon \int_0^T \int_a^b (\gamma^{-1} \rho \gamma^{-1} + \gamma^{-1}) \, dr \, dt + M \tilde{\rho} \gamma^{-1}$$

$$\leq M \left( 1 + \frac{\varepsilon}{a} + \tilde{\rho} \gamma^{-1} \right),$$

where we have applied the Cauchy-Young inequality and main energy estimate, Proposition 1.4. Thus, combining (2.10)–(2.12) we have the bound

$$|I_2| \leq M \left( \frac{\varepsilon}{a} + \tilde{\rho} \gamma^{-1} + 1 \right).$$

To bound $I_3$, we recall (2.3) and Lemma 2.2 to see

$$|I_3| \leq M \delta \int_0^T \int_a^b \int_r^b \rho^2 (\rho^{-1} |\rho y| + |u_y|) y^{n-1} \omega(r) \, dy \, dr \, dt$$

$$\leq M \delta \int_0^T \int_a^b \int_r^b (\rho^{-2} |\rho y|^2 + \rho |u_y|^2 + \tilde{\rho}^5) y^{n-1} \omega(r) \, dy \, dr \, dt$$

$$\leq M \left( 1 + \delta \frac{b^n}{\varepsilon} \right),$$
where we have used the main energy estimate and \( \delta \leq \varepsilon \) to control the derivative terms.

To bound \( I_4 \), we apply the bound \( |\tilde{\eta}_m| \leq M(|u| + \rho^\theta) \) and the Cauchy-Young inequality to show

\[
I_4 \leq M\delta \int_0^T \int_a^b \int_r^b \rho^2 |\tilde{\eta}_m| y^{n-2}\omega(r) \, dy \, dr \, dt \\
\leq M\delta \int_0^T \int_a^b \int_r^b (\rho^2 |u| + \rho^{2+\theta}) y^{n-2}\omega(r) \, dy \, dr \, dt \\
\leq M\delta \int_0^T \int_a^b \int_r^b (\rho|u|^2 + \rho^3 + \rho^{2+\theta}) y^{n-2}\omega(r) \, dy \, dr \, dt.
\]

In the case that \( \theta \leq 1 \) (i.e. \( \gamma \leq 3 \)), we then estimate further using Lemma 2.2,

\[
|I_4| \leq M\delta \int_0^T \int_a^b \frac{1}{r} \int_r^b (\rho|u|^2 + \rho^3 + 1) y^{n-1}\omega(r) \, dy \, dr \, dt \\
\leq M\delta |\log(a)| (1 + \frac{b^n}{\varepsilon}).
\]

On the other hand, if \( \theta > 1 \) then \( \gamma > 3 \) and \( \gamma > 2 + \theta \), so we use the Cauchy-Young inequality to bound

\[
|I_4| \leq M\delta \int_0^T \int_a^b \frac{1}{r} \int_r^b (\rho|u|^2 + \rho^3 + 1) y^{n-1}\omega(r) \, dy \, dr \, dt \\
\leq M\delta |\log(a)| (1 + (\gamma^1 + 1)b^n).
\]

Finally, \( I_5 \) is treated analogously to \( I_4 \), giving a bound of

\[
|I_5| \leq \delta M \left( 1 + \frac{b^n}{\varepsilon} \right).
\]

By (1.12), all of the above bounds for the terms \( I_1, \ldots, I_5 \) become uniform with respect to \( \varepsilon \), hence we conclude from (2.8) (and the obvious estimate \( (\bar{\rho}^\gamma + 1) \leq \varepsilon^{-1} \)) that

\[
\int_0^T \int_a^b (\rho|u|^3 + \rho^{\gamma+\theta}) \omega(r) r^{n-1} \, dr \, dt \leq M \left( \bar{\rho}^\theta |\log(a)| + \bar{\rho}^\gamma b^{n-1} + \frac{\varepsilon}{a} + \delta |\log(a)| (1 + \frac{b^n}{\varepsilon}) + 1 \right) \leq M,
\]

and so we conclude the proof of the lemma. \( \square \)

3. Viscous terms

We begin this section with the two main estimates we need to demonstrate convergence to zero of the viscous terms in the weak formulation of the approximate equations, system (1.8).

**Lemma 3.1.** Let \( \omega = \omega(r) \in C_0^\infty(\mathbb{R}) \) be a test function such that \( \omega(r) = 1 \) for \( r \in [0, 1] \), \( \omega(r) = 0 \) for \( r \geq 2 \). Then for any \( \Delta \in (0, \frac{1}{2}) \), there exists a constant \( M > 0 \), independent of \( \Delta \) and \( \varepsilon \), such that

\[
\varepsilon \int_0^T \int_a^b \rho_r^2 \chi_{\{\rho < \Delta\}} \omega(r)^2 r^{n-1} \, dr \, dt \leq M \left( \sqrt{\varepsilon}(1 + \Delta^{4-\gamma}) + \frac{\Delta}{a} + \frac{\Delta^{3/2}}{\sqrt{\varepsilon}} \right). \quad (3.1)
\]

**Lemma 3.2.** Let \( \omega = \omega(r) \in C_0^\infty(\mathbb{R}) \) be a test function such that \( \omega(r) = 1 \) for \( r \in [0, 1] \), \( \omega(r) = 0 \) for \( r \geq 2 \). Then for any \( \Delta \in (0, \frac{1}{2}) \), there exists a constant \( M > 0 \), independent of \( \Delta \) and \( \varepsilon \), such that

\[
\varepsilon \int_0^T \int_a^b \rho_r^2 \rho \chi_{\{\rho < \Delta\}} \omega(r)^2 r^{n-1} \, dr \, dt \leq M \left( |\log \Delta| + \frac{\sqrt{\Delta}}{\sqrt{\varepsilon}} + \frac{\sqrt{\Delta}}{a} + \sqrt{\varepsilon}|\log \Delta|(1 + \Delta^{2-\gamma}) \right). \quad (3.2)
\]

The proofs of these two lemmas are motivated by the following observation. Let \( \varphi = \varphi(\rho) \) be a twice differentiable function, \( \omega = \omega(r) \in C_0^\infty(\mathbb{R}) \), and multiply the first equation in (1.8) by
\( \varphi'(\rho)\omega(r)^2 \). A simple calculation yields
\[
(r^{n-1}\varphi\omega^2)_t + (r^{n-1}\varphi\omega^2)_r + r^{n-1}(\rho\varphi' - \varphi)(u_r + \frac{n-1}{r}u)\omega^2 - 2r^{n-1}\varphi u\omega_r = \varepsilon(r^{n-1}\varphi'\rho_r\omega^2) - \varepsilon r^{n-1}\varphi''\rho_r^2\omega^2 - 2\varepsilon r^{n-1}\varphi\omega_r.
\]

(3.3)

Thus, for any such \( \varphi \),
\[
\varepsilon \int_0^T \int_a^b \varphi''\rho_r^2\omega^2 r^{n-1} \, dr \, dt = \left. -\varepsilon \int_a^b \varphi\omega^2 r^{n-1} \, dr \right|_0^T + \int_0^T \int_a^b (\varphi - \rho\varphi')u_r\omega^2 r^{n-1} \, dr \, dt + 2\int_0^T \int_a^b \varphi u\omega_r r^{n-1} \, dr \, dt + \frac{n-1}{r} \int_0^T \int_a^b (\varphi - \rho\varphi')u\omega^2 r^{n-1} \, dr \, dt - 2\varepsilon \int_0^T \int_a^b \varphi\omega_r r^{n-1} \, dr \, dt,
\]

(3.4)

where we have used the boundary conditions \( \rho_r = u = 0 \) at \( a \) and the compact support of \( \omega \).

**Proof of Lemma 3.1.** We define, for \( \Delta \in (0, \frac{1}{2}) \) fixed,
\[
\varphi(\rho) = \begin{cases} \frac{\rho^2}{2} + \Delta(\rho - \Delta), & \rho < \Delta, \\ \frac{\rho^2}{2}, & \rho \geq \Delta. \end{cases}
\]

(3.5)

Then we have that
\[
\varphi''(\rho) = \mathbb{1}_{\{\rho \leq \Delta\}}(\rho),
\]

\[
\varphi(\rho) - \rho\varphi'(\rho) = -\frac{1}{2} \min\{\rho^2, \Delta^2\}.
\]

Then from (3.4), we obtain
\[
\varepsilon \int_0^T \int_a^b \rho_r^2 \mathbb{1}_{\{\rho \leq \Delta\}} \omega^2 r^{n-1} \, dr \, dt = \left. -\varepsilon \int_a^b \varphi\omega^2 r^{n-1} \, dr \right|_0^T - \frac{1}{2} \int_0^T \int_a^b \min\{\rho^2, \Delta^2\} u_r\omega^2 r^{n-1} \, dr \, dt + 2\int_0^T \int_a^b \varphi u\omega_r r^{n-1} \, dr \, dt - \frac{n-1}{r} \int_0^T \int_a^b \min\{\rho^2, \Delta^2\} u\omega^2 r^{n-1} \, dr \, dt - 2\varepsilon \int_0^T \int_a^b \varphi\omega_r r^{n-1} \, dr \, dt = J_1 + \cdots + J_5.
\]

(3.6)

To bound \( J_1 \), we simply observe that \( |\varphi(\rho)| \leq \Delta\rho \) for all \( \rho > 0 \). Thus
\[
|J_1| \leq M \sup_{t \in [0,T]} \int_a^b \Delta\rho\omega^2 r^{n-1} \, dr \leq M\Delta
\]

by the main energy estimate, Proposition 1.4, where we have used the compact support of \( \omega \) and the estimate \( \rho \leq M(1 + K_\delta(\rho, \rho)) \).

The next simplest term to control is \( J_3 \), which we bound in a similar way, giving an estimate of
\[
|J_3| \leq M \int_0^T \int_a^b \Delta(\rho + \rho u^2)\omega r^{n-1} \, dr \, dt \leq M\Delta,
\]

where we again use the main energy estimate and \( M \) depends on \( \|\omega\|_{L^\infty} \).
Turning now to $J_2$, we estimate

$$|J_2| \leq \frac{1}{2\sqrt{\varepsilon}} \int_0^T \int_a^b \Delta^{3/2} \sqrt{\varepsilon} \rho u_r \omega r^{n-1} \, dr \, dt \leq \frac{M\Delta^{3/2}}{\sqrt{\varepsilon}} \left( \varepsilon \int_0^T \int_a^b \rho u_r^2 r^{n-1} \, dr \, dt \right)^{1/2} \leq \frac{M\Delta^{3/2}}{\sqrt{\varepsilon}},$$

by the main energy estimate, where $M$ also depends on $|\text{supp}\omega|$.

Next, we use that $r > a$ in the domain of integration and Proposition 1.4 to bound

$$|J_4| \leq M \int_0^T \int_a^b \frac{n-1}{r} \Delta \rho u_r \omega r^{n-1} \, dr \, dt \leq M \frac{\Delta}{a} \int_0^T \int_a^b (\rho + \rho u^2) \omega r^{n-1} \, dr \, dt \leq \frac{M\Delta}{a}. $$

We consider $J_5$ on the two regions $\{\rho < \Delta\}$ and $\{\rho \geq \Delta\}$ by writing

$$J_5 = -2\varepsilon \int_0^T \int_a^b \min\{\rho, \Delta\} \rho u_r \omega r^{n-1} \, dr \, dt$$

$$= -2\varepsilon \int_0^T \int_a^b \rho \mathbb{1}_{\{\rho < \Delta\}} \rho u_r \omega r^{n-1} \, dr \, dt - 2\varepsilon \int_0^T \int_a^b \Delta \mathbb{1}_{\{\rho \geq \Delta\}} \rho u_r \omega r^{n-1} \, dr \, dt$$

$$= J^1_5 + J^2_5.$$

Considering the second term first, we use the Cauchy-Young inequality to bound

$$|J^2_5| \leq 2\sqrt{\varepsilon} \int_0^T \int_a^b \sqrt{\varepsilon} \Delta^{2/2} r^{2\gamma-2} \rho \mathbb{1}_{\{\rho \geq \Delta\}} \omega r^{n-1} \, dr \, dt$$

$$\leq M\sqrt{\varepsilon} \int_0^T \int_a^b \varepsilon \rho^{\gamma-2} |\rho|^2 r^{n-1} \, dr \, dt + M\Delta^2 \sqrt{\varepsilon} \int_0^T \int_a^b \rho^{2-\gamma} \mathbb{1}_{\{\rho \geq \Delta\}} \omega r^{n-1} \, dr \, dt.$$

In the case that $\gamma \in (1,2)$, we make the estimate $\rho^{2-\gamma} \leq \rho^{\gamma+1}$ and apply the main energy estimate to obtain

$$|J^2_5| \leq M\varepsilon.$$

On the other hand, for $\gamma > 2$, we estimate $\rho^{2-\gamma} \leq \Delta^{2-\gamma}$ on the region $\rho \geq \Delta$ to obtain

$$|J^2_5| \leq M\sqrt{\varepsilon}(1 + \Delta^{4-\gamma}),$$

where we have used the main energy estimate to bound the first term of $J^2_5$.

Turning finally to $J^1_5$, we use the Cauchy-Young inequality to estimate

$$|J^1_5| \leq \frac{\varepsilon}{2} \int_0^T \int_a^b |\rho|^2 \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt + M\varepsilon \int_0^T \int_a^b \Delta^2 \omega^2 r^{n-1} \, dr \, dt$$

$$\leq \frac{\varepsilon}{2} \int_0^T \int_a^b |\rho|^2 \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt + M\varepsilon \Delta^2.$$

Combining this with the estimate above for $J^2_5$, we obtain

$$|J_5| \leq \frac{\varepsilon}{2} \int_0^T \int_a^b |\rho|^2 \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt + M\varepsilon \Delta^2 + M\sqrt{\varepsilon}(1 + \Delta^{4-\gamma}).$$

Thus, combining the estimates for $J_1, \ldots, J_5$ in (3.6),

$$\varepsilon \int_0^T \int_a^b \rho_0^2 \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt \leq M \left( \sqrt{\varepsilon}(1 + \Delta^{4-\gamma}) + \frac{\Delta}{a} + \frac{\Delta^{3/2}}{\sqrt{\varepsilon}} \right).$$

**Proof of Lemma 3.2.** We let $\Delta \in (0, \frac{1}{2})$ and define the function $\psi(\rho)$ by

$$\psi(\rho) = \begin{cases} \rho \log \rho - \rho, & \rho < \Delta, \\ \rho \log \Delta - \Delta, & \rho \geq \Delta, \end{cases}$$

so that

$$\psi(\rho) - \rho \psi'(\rho) = -\min\{\rho, \Delta\},$$

$$\psi''(\rho) = \frac{1}{\rho} \mathbb{1}_{\{\rho < \Delta\}}. \quad (3.8)$$
Then from (3.4), we obtain
\[
\varepsilon \int_0^T \int_a^b \frac{\rho^2}{\rho} \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt = - \int_0^T \int_a^b \psi \omega^2 r^{n-1} \, dr \, dt - \int_0^T \int_a^b \min \{\rho, \Delta\} u_\gamma \omega^2 r^{n-1} \, dr \, dt \\
+ 2 \int_0^T \int_a^b \psi \omega \omega_r r^{n-1} \, dr \, dt - \int_0^T \int_a^b \frac{n-1}{r} \min \{\rho, \Delta\} u_\gamma \omega^2 r^{n-1} \, dr \, dt \\
- 2\varepsilon \int_0^T \int_a^b \psi_r \omega \omega_r r^{n-1} \, dr \, dt \\
= J_1 + \cdots + J_5.
\]

As \( \rho |\log \rho| \mathbb{1}_{\{\rho < \Delta\}} \leq \Delta |\log \Delta| \) for \( \Delta \in (0, \frac{1}{2}) \), we bound \( J_1 \) by
\[
|J_1| \leq M \sup_{t \in [0,T]} \int_a^b (\Delta |\log \Delta| + \rho |\log \Delta|) \omega^2 r^{n-1} \, dr \leq M |\log \Delta|,
\]
where we have employed the main energy estimate, Proposition 1.4, and \( \rho \leq M (1 + h_0(\rho, \bar{\rho})) \).

\( J_3 \) is bounded similarly, using the estimate \( \rho u |\log \rho| \leq \rho u^2 + \rho (\log \rho)^2 \leq \rho u^2 + \Delta (\log \Delta)^2 \) for \( \rho < \Delta \), giving an estimate of \( |J_3| \leq M |\log \Delta| \), where \( M \) depends on \( ||\omega_r||_{L^\infty} \).

To control \( J_2 \), we again employ the main energy estimate and Hölder’s inequality to obtain
\[
|J_2| \leq \int_0^T \int_a^b \min \{\rho, \Delta\} |u_r| \omega^2 r^{n-1} \, dr \, dt \\
\leq M \sqrt{\Delta} \left( \int_0^T \int_a^b \rho |u_r|^2 r^{n-1} \, dr \, dt \right)^\frac{1}{2} \leq M \sqrt{\Delta} \frac{\epsilon}{\sqrt{\epsilon}}.
\]

For \( J_4 \), we use the Cauchy-Schwarz inequality to bound
\[
|J_4| \leq \int_0^T \int_a^b \frac{n-1}{r} \min \{\rho, \Delta\} |u| \omega^2 r^{n-1} \, dr \, dt \\
\leq M \sqrt{\Delta} \left( \int_0^T \int_a^b \rho |u|^2 r^{n-1} \, dr \, dt \right)^\frac{1}{2} \\
\leq M \sqrt{\Delta} \frac{\epsilon}{\sqrt{\epsilon}}.
\]

Finally, to estimate \( J_5 \), we use the Cauchy-Young inequality to bound
\[
|J_5| = 2\varepsilon \left| \int_0^T \int_a^b (\log \rho \mathbb{1}_{\{\rho < \Delta\}} + \log \Delta \mathbb{1}_{\{\rho \geq \Delta\}}) |\rho_r| \omega \omega_r r^{n-1} \, dr \, dt \right| \\
\leq \varepsilon \int_0^T \int_a^b \frac{\rho^2}{\rho} \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt + M \varepsilon \int_0^T \int_a^b (\log \rho)^2 \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt \\
+ \varepsilon |\log \Delta| \int_0^T \int_a^b |\rho_r| \mathbb{1}_{\{\rho \geq \Delta\}} \omega \omega_r r^{n-1} \, dr \, dt \\
\leq \frac{\varepsilon}{2} \int_0^T \int_a^b \frac{\rho^2}{\rho} \mathbb{1}_{\{\rho < \Delta\}} \omega^2 r^{n-1} \, dr \, dt + M \varepsilon \frac{\epsilon}{\sqrt{\epsilon}} \left( \int_0^T \int_a^b \max \{\Delta^{2-\gamma}, 1\} |\rho_r|^2 \omega(r)^2 r^{n-1} \, dr \, dt \right)^\frac{1}{2},
\]

where we have also applied the main energy estimate, Proposition 1.4.

Combining the estimates for \( J_1, \ldots, J_5 \) in (3.9), we conclude the proof. \( \square \)
4. PROOF OF THEOREM 1.3

We begin by recalling the following theorem from [2, 3].

**Theorem 4.1.** Let \((\rho_0, m_0) \in L^1_{\text{loc}}(\mathbb{R}^+)^2, \rho_0 \geq 0,\) be of finite energy,
\[
E[\rho_0, m_0] < \infty,
\]
and suppose that for \(\varepsilon > 0,\) the parameters \(\bar{\rho}(\varepsilon), \delta(\varepsilon), b(\varepsilon)\) satisfy
\[
\bar{\rho} \varepsilon b^3 + \delta b^3 \leq M,
\]
where \(M\) is independent of \(\varepsilon.\) Let \((\bar{\rho}_0, \bar{m}_0)\) with \(\inf \bar{\rho}_0 > 0\) be smooth functions on \((a(\varepsilon), b(\varepsilon))\)
such that

- \((\rho_0, m_0) \to (\rho_0, m_0)\) for almost every \(r \in \mathbb{R}^+ \) as \(\varepsilon \to 0,\) where we extend \((\rho_0, m_0)\) from \((a, b)\) to \(\mathbb{R}^+\) by zero;
- \((\rho_0, m_0)\) satisfies the boundary conditions (1.10) as well as the compatibility conditions:
  \[
  \left(\rho_0 - \bar{\rho}_0, \frac{1}{\varepsilon}(r^{n-1} m_0 - \bar{m}_0)\right)_{r=0} = 0,
  \left(\rho_0 - \bar{\rho}_0, \frac{1}{\varepsilon}(r^{n-1} m_0 - \bar{m}_0)\right)_{r=b} = 0;
  \]
- \(E[\rho_0, m_0] \to E[\bar{\rho}_0, \bar{m}_0] = 0\) as \(\varepsilon \to 0.\)

Then there exist unique classical solutions \((\rho^\varepsilon, m^\varepsilon)\) of (1.8)–(1.10) (extended by 0 to \(\mathbb{R}^2\)) which converge \((\rho^\varepsilon, m^\varepsilon) \to (\rho, m)\) almost everywhere in \(\mathbb{R}^2\) and in \(L^p_{\text{loc}}(\mathbb{R}^+)^2 \times L^q_{\text{loc}}(\mathbb{R}^+)^2\) for \(p \in [1, \gamma+1)\) and \(q \in \left[1, \frac{\gamma+1}{\gamma+3}\right).\)

We strengthen the assumptions of Theorem 4.1 by imposing assumption (1.12), as well as the slightly stronger condition, guaranteed by appropriate choice of \(a,\)
\[
\sqrt{\varepsilon} a(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.
\]

Now we let \(\varphi \in C^\infty([0, \infty)^2),\) multiply the first equation in (1.8) by \(\varphi\) and integrate by parts on \([0, T] \times (a(\varepsilon), b(\varepsilon)),\) using the boundary conditions (1.10), to obtain
\[
\int_a^T \int_a^b \left(\rho^\varepsilon(t,r)\varphi_t(t,r) + m^\varepsilon(t,r)\varphi_r(t,r)\right) r^{n-1} dr dt + \int_a^b \rho_0(r)\varphi(0,r) r^{n-1} dr
\]
\[
= \varepsilon \int_a^T \int_a^b \rho^\varepsilon(t,r)\varphi_t(t,r) r^{n-1} dr dt.
\]

As \(\varphi\) has compact support in \([0, \infty)^2,\) we may apply the uniform bound of Lemma 1.5 and the almost everywhere convergence \((\rho^\varepsilon, m^\varepsilon) \to (\rho, m)\) to deduce that the left hand side of (4.2) converges as \(\varepsilon \to 0\) to
\[
\int_a^T \int_a^b \left(\rho(t,r)\varphi_t(t,r) + m(t,r)\varphi_r(t,r)\right) r^{n-1} dr dt + \int_a^b \rho_0(r)\varphi(0,r) r^{n-1} dr.
\]

On the other hand, by Lemma 3.1, we have
\[
\left| \varepsilon \int_a^T \int_a^b \rho^\varepsilon(t,r)\varphi_t(t,r) r^{n-1} dr dt \right| \leq M \left(\varepsilon^2 \int_a^T \int_a^b |\rho^\varepsilon|^2 |\varphi_r|^2 r^{n-1} dr dt\right)^{\frac{1}{2}} \to 0
\]
as \(\varepsilon \to 0,\) where \(M\) depends on \(\varphi_r,\) thus demonstrating (1.5).

Let now \(\varphi \in C^\infty_c([0, \infty)^2)\) be such that \(\varphi(t,0) = 0\) and take a sequence \(\{\varphi^\varepsilon\}_{\varepsilon > 0} \in C^\infty_c(\mathbb{R}^2),\)
uniformly bounded in \(W^{1, \infty}(\mathbb{R}^2),\) such that \(\varphi^\varepsilon \to \varphi\) strongly in \(W^{1,p}(\mathbb{R}^2)\) for all \(p < \infty\) and \(\varphi^\varepsilon(t,r) = 0\) for \(r \in [0, a(\varepsilon)]\) and \(t \in [0, T].\) We choose the sequence \(\varphi^\varepsilon\) such that the supports of the \(\varphi^\varepsilon\) are contained in a fixed compact set in \([0, \infty)^2.\) We multiply the second equation in
(1.10), and the uniform compact support of \( \varphi^\varepsilon \) and integrate by parts on \([0, T] \times (a(\varepsilon), b(\varepsilon))\), using the boundary conditions (1.10) and \( \varphi^\varepsilon(t, a) = 0 \), to obtain
\[
\int_0^T \int_a^b \left( m^\varepsilon \varphi_t^\varepsilon + \frac{(m^\varepsilon)^2}{\rho^\varepsilon} \varphi_r^\varepsilon + p_3(\rho^\varepsilon) (\varphi_r^\varepsilon + \frac{n-1}{r} \varphi^\varepsilon) \right) r^{n-1} \, dr \, dt \\
+ \int_a^b m_0^\varepsilon(r) \varphi^\varepsilon(0, r) r^{n-1} \, dr = \varepsilon \int_0^T \int_a^b \left( (r^{n-1} m^\varepsilon)_r \varphi_r^\varepsilon + \frac{n-1}{r} (r^{n-1} m^\varepsilon)_r \varphi^\varepsilon \right) r^{n-1} \, dr \, dt.
\]
(4.3)

Then, using again Lemma 1.5 and the uniform compact support of \( \varphi^\varepsilon \), we see the convergence
\[
\lim_{\varepsilon \to 0} \int_0^T \int_a^b \left( m^\varepsilon \varphi_t^\varepsilon + \frac{(m^\varepsilon)^2}{\rho^\varepsilon} \varphi_r^\varepsilon + p_3(\rho^\varepsilon) (\varphi_r^\varepsilon + \frac{n-1}{r} \varphi^\varepsilon) \right) r^{n-1} \, dr \, dt \\
= \int_0^T \int_a^b \left( m \varphi_t + \frac{m^2}{\rho^\varepsilon} \varphi_r + p(\rho) (\varphi_r + \frac{n-1}{r} \varphi) \right) r^{n-1} \, dr \, dt,
\]
where for the final term, \( p_3(\rho^\varepsilon) \frac{n-1}{r} \varphi^\varepsilon r^{n-1} \), we note \( |\varphi^\varepsilon(t, r)| \leq r ||\varphi^\varepsilon||_{L^\infty} \) is compact, we note by Proposition 1.4 that
\[
|\varepsilon \int_0^T \int_a^b n \frac{1}{r^2} m^\varepsilon \varphi^\varepsilon r^{n-1} \, dr \, dt| \leq \varepsilon \int_0^T \int_a^b (\rho^\varepsilon + \rho^\varepsilon |u^\varepsilon|^2) r^{n-3} \varphi^\varepsilon \, dr \, dt \leq M \varepsilon a^2 + M \varepsilon,
\]
which converges to 0 as \( \varepsilon \to 0 \) by (4.1).

For the remaining term, we apply Hölder’s inequality to bound
\[
|\varepsilon \int_0^T \int_a^b m^\varepsilon \varphi_r^\varepsilon r^{n-1} \, dr \, dt| = |\varepsilon \int_0^T \int_a^b (\rho^\varepsilon u^\varepsilon + \rho^\varepsilon u^\varepsilon) \varphi_r^\varepsilon r^{n-1} \, dr \, dt| \\
\leq (\varepsilon \int_0^T \int_a^b \rho^\varepsilon |u^\varepsilon|^2 r^{n-1} \, dr \, dt)^{\frac{1}{2}} \left( \varepsilon \int_0^T \int_a^b \rho^\varepsilon |\varphi^\varepsilon|^2 r^{n-1} \, dr \, dt \right)^{\frac{1}{2}} \\
+ (\varepsilon^2 \int_0^T \int_a^b (\rho^\varepsilon)^2 |\varphi^\varepsilon|^2 r^{n-1} \, dr \, dt)^{\frac{1}{2}} \left( \varepsilon \int_0^T \int_a^b \rho^\varepsilon (u^\varepsilon)^2 r^{n-1} \, dr \, dt \right)^{\frac{1}{2}}
\]
which converges to 0 as \( \varepsilon \to 0 \) by the main energy estimate, Proposition 1.4, and Lemmas 3.1 and 3.2, thus demonstrating (1.6) and hence concluding the proof of Theorem 1.3.

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