Analytic regularity for Navier-Stokes-Korteweg model on pseudo-measure spaces

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ABSTRACT
The purpose of this work is to study the existence and analytic smoothing effect for the compressible Navier-Stokes system with quantum pressure in pseudo-measure spaces. This system has been considered by B. Haspot and an analytic smoothing effect for a Korteweg type system was considered by F. Charve, R. Danchin and J. Xu, both of them in Besov spaces. Here we give a better lower bound of the radius of analyticity near zero. This work is an opportunity to deepen the study of partial differential equations in pseudo-measure spaces by introducing a new functional setting to deal with non-linear terms. The pseudo-measure spaces are well-adapted to obtain a point-wise control of solutions, with a study of turbulence as perspective.

CONTENTS
1 Introduction 1
1.1 Compressible Navier-Stokes system with quantum pressure 3
1.2 Pseudo-measure spaces 3
1.3 Critical space 4
1.4 Radius of analyticity 4
1.5 Main results 5
2 The linearized system 6
2.1 Parabolic estimate for the linearized system 6
3 Global existence 6
3.1 Nonlinear estimates 6
3.2 Global existence theorem 9
4 Analyticity for global solution 11
5 Estimate near 0 13
5.1 Kato type theorem for local existence with supercritical data 13
5.2 Estimate of the radius of analyticity 15
A Characterization of analyticity with Fourier transform 18
References 20

1 Introduction
We are interested by the analytic smoothing properties of the Navier-Stokes-Korteweg system which describe a two-phase compressible and viscous fluids, of density $\rho$ and velocity field $u$. It is generally assumed that the phases are separated by a hypersurface and that the jump in the pressure across the interface is proportional to the curvature. Here we deal with a diffuse interface (DI) model that describes fluids when the change of phase corresponds to a fast but regular transition zone for the density and velocity. This type of models differs from the so-called sharp interface (SI) model when, the interface

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between phases corresponds to a discontinuity in the state space. The basic ideas of the DI model considering here, is to add to the classical compressible fluids equation a capillary term, that penalizes high variations of the density. The full derivation of the corresponding equation, that we shall name the compressible Navier-Stokes-Korteweg system is due to J. E. Dunn and J. Serrin (see \cite{10}).

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mathcal{A}u + \nabla \Pi = \text{div}(\mathcal{K}), \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{aligned}
\]

(1)

where \( \Pi := P(\rho) \) is the pressure function, \( \mathcal{A}u := \text{div}(2\mu(\rho)D_S(u)) + \nabla(\nu(\rho) \text{ div } u) \) is the diffusion operator, \( D_S(u) := \frac{1}{2}(\nabla u + \nabla u) \) is the symmetric gradient and the capillarity tensor is given by

\[
\mathcal{K} := \rho \text{ div}(\kappa(\rho) \nabla \rho)I_{\mathbb{R}^d} + \frac{1}{2}(\kappa(\rho) - \rho \kappa'(\rho))|\nabla \rho|^2 I_{\mathbb{R}^d} - \kappa(\rho) \nabla \rho \otimes \nabla \rho.
\]

This system is due to J. E. Dunn and J. Serrin in \cite{10}. The density-dependent capillarity function \( \kappa \) is assumed to be positive. Note that for smooth enough density \( \rho \) and capillarity function \( \kappa \), we have

\[
\text{div} \mathcal{K} = \rho \nabla \left( \kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho)|\nabla \rho|^2 \right).
\]

The coefficients \( \nu = \nu(\rho) \) and \( \mu = \mu(\rho) \) designate the bulk and shear viscosity, respectively, and are assumed to satisfy in the neighborhood of some reference constant density \( \bar{\rho} > 0 \) the conditions

\[
\mu > 0 \quad \text{and} \quad \nu + \mu > 0.
\]

We shall assume that the functions \( \lambda, \mu, \kappa \) and \( P \) are real analytic in a neighborhood of \( \bar{\rho} \). To simplify, we set \( \bar{\rho} = 1 \). Introducing \( a = \rho - 1 \) and denoting by \( \bar{\mu} = \mu(1), \bar{\nu} = \nu(1), \bar{\kappa} = \kappa(1), \bar{\alpha} = P'(1) \), the system (1) reads

\[
\begin{aligned}
\partial_t a + \text{div}(u) &= \bar{f}, \\
\partial_t u - \bar{\mathcal{A}}u + \bar{\alpha} \nabla a - \bar{\kappa} \Delta a &= \bar{g},
\end{aligned}
\]

(2)

where \( \bar{\mathcal{A}}u = 2\bar{\mu} \text{ div}(D_S(u)) + \bar{\nu} \nabla \text{ div } u, \bar{f} = - \text{ div}(au), \bar{g} = \sum_{i=1}^4 \bar{\tilde{g}}_i \) with

\[
\begin{aligned}
\bar{\tilde{g}}_1 &= -u \cdot \nabla u, \\
\bar{\tilde{g}}_2 &= (1 + a)^{-1} \bar{\mathcal{A}}u - \bar{\mathcal{A}}u, \\
\bar{\tilde{g}}_3 &= -(1 + a)^{-1} \nabla P(1 + a) + \bar{\alpha} \nabla a, \\
\bar{\tilde{g}}_4 &= \nabla \left( (\kappa(1 + a) - \bar{\kappa}) \Delta a + \frac{1}{2} \kappa'(1 + a)|\nabla a|^2 \right).
\end{aligned}
\]

The system (2) is a hyperbolic/parabolic coupled system, which is common for compressible Navier-Stokes type systems. In contrast with the linearized equation of the classical compressible Navier-Stokes system, it was remarked by F. Charve, R. Danchin, and J. Xu that for the linear part of (2), with external forces, both of the density and velocity are smoothed out instantaneously (see lemma 2.1.1). In 2018, authors showed in \cite{6} a Gevrey analyticity smoothed effect for all the unknowns of the compressible Navier-Stokes-Korteweg system, in Besov spaces, this is the first related result for a model of compressible fluids. In this paper, we aim to establish this smoothing effect and to estimate the radius of analyticity of the solution, in the pseudo-measure spaces.
for a particular case presented in the following subsection. Using the method used by J. Y. Chemin, I. Gallagher, and P. Zhang in [7] for semi-linear parabolic systems, we give a better estimate on the radius of analyticity near 0, the advantage to work in the pseudo-measure spaces is that we obtained point-wise time-frequency estimate of the decay of the solution, with studying the turbulence as perspective. In the following subsection, we describe a special case of the compressible Navier-Stokes-Korteweg system, so-called the incompressible Navier-Stokes system with quantum pressure, that will be discussed in this paper.

1.1 Compressible Navier-Stokes system with quantum pressure

In this note, we consider a special case, which is the so-called compressible Navier-Stokes system with quantum pressure considered by B. Haspot [12], where

\[(\mu(\rho), \nu(\rho), \kappa(\rho)) = (\mu \rho, \nu \rho, \kappa / \rho), \quad P(\rho) := \alpha \rho,\]

and \(\mu > 0, \mu + \nu > 0, \kappa > 0, \alpha > 0\) are constants.

Introducing \(\rho = \bar{\rho} e^a\), the system reads

\[
\begin{cases}
\partial_t a + \text{div}(u) = f(u, a), \\
\partial_t u - \mu \Delta u - (\mu + \nu) \nabla \text{div}(u) + \alpha \nabla a - \kappa \Delta a = g(u, a),
\end{cases}
\]

where \(g := \sum_{j=1}^3 g_i\) and

\[
\begin{align*}
f(u, a) &:= -u \cdot \nabla a, \\
g_1(u, u) &:= -u \cdot \nabla u, \\
g_2(u, a) &:= \mu \nabla a \cdot \nabla u + (\mu + \nu) \nabla a \cdot Du, \\
g_3(a, a) &:= \frac{\kappa}{2} \nabla (\nabla a \cdot \nabla a).
\end{align*}
\]

We consider the initial value condition

\[(a, u)|_{t=0} = (a_0, u_0).\]  

1.2 Pseudo-measure spaces

Let us begin by specifying some notations.

**Notation 1.2.1.** Throughout the paper, \(f \lesssim_{a_1, \ldots, a_k} g\) means that there exists a positive constant \(C\), which depends on the parameters \(a_1, \ldots, a_k\) such that \(f \leq Cg\). We denote by \(\hat{f}\) the Fourier transform with respect to the space variable of the function \(f \in C([0, T]; S'(\mathbb{R}^d))\).

We begin by define pseudo-measure spaces on the whole space \(\mathbb{R}^d\). For all \(r \geq 0\), we define the pseudo-measure space of order \(r\) by setting

\[
PM^r(\mathbb{R}^d) := \left\{ g \in S'(\mathbb{R}^d) \mid \hat{g} \in L^1_{loc}(\mathbb{R}^d) \text{ and } \|g\|_{PM^r} := \sup_{\xi \in \mathbb{R}^d} \{ |\xi|^r |\hat{g}(\xi)| \} < +\infty \right\}.
\]

The pseudo-measure spaces were firstly used for fluids mechanic systems by Y. Le Jan and A.A.S. Sznitman in [13] for the incompressible Navier-Stokes system, for existence results. After, the analytic regularity was studying by P. G. Lemarié-Rieusset in [14] and W. Deng, M. Paicu and P. Zhang in [9] for the global mild solution of incompressible...
Navier-Stokes system. The introduction of pseudo-measure spaces is motivated by \cite{1}, related to the theory of turbulences (see also \cite{5} and \cite{2}). These spaces are particular case of homogeneous Besov spaces construct over the shift-invariant Banach space of distributions. Here, the so-called shift-invariant Banach space of distributions is the case of homogeneous Besov spaces related to the theory of turbulences (see also \cite{5} and \cite{2}). These spaces are particular Navier-Stokes system. The introduction of pseudo-measure spaces is motivated by \cite{4}, is finite. We also define the space \(K\) if for all positive real numbers \(\lambda\), the associated norm is invariant under the transformation

\[(a, u) \mapsto (a\lambda, u\lambda),\]

up to a constant independent of \(\lambda\). This suggests to choose initial data \((a_0, u_0)\) in the space whose norm is invariant for all positive real number \(\lambda\) by \((a_0, u_0) \mapsto (a_0(\lambda), \lambda u_0(\lambda))\).

If we deal with a pseudo-measure space, a natural candidate is the space \(PM^d\) of all \(\Omega\) such that \(\text{rad}(u) \leq 0\) for all positive real number \(\lambda\).

Let us first recall the notion of scaling for the system (3) (see \cite{8} or \cite{12}). If \((a, u)\) solves (3), then so does \((a\lambda, u\lambda)\), where

\[a_\lambda := a(\lambda^2, \cdot) \quad \text{and} \quad u_\lambda := \lambda u(\lambda^2, \cdot),\]

and \(\lambda \in \mathbb{R}^d\). This observation leads to the notion of critical spaces. We say that a functional space is a critical space for (3) if for all positive real numbers \(\lambda\), the associated space whose norm is invariant under the transformation

\[(a, u) \mapsto (a\lambda, u\lambda),\]

for all \(\lambda\) such that \(\text{rad}(u) \leq 0\) for all positive real number \(\lambda\).

We observe that the space \(K_{c,d}^\infty \times K_{c,d}^{d-1}\) verifies the invariance by scaling. The Kato spaces is useful to establish Kato types theorems (see \cite{1} and \cite{7}), such as theorem 5.1.1. We use this Kato spaces to establish global existence and regularity results.

\textbf{Definition 1.3.1.} Let \(p, r\) and \(T\) be positive real numbers. We define the Kato space \(K_T^{p,r}\) as the space of \(u \in C_c([0, T]; PM^{r+\frac{1}{2}})\) such that the quantity

\[\|u\|_{K_T^{p,r}} := \sup_{t \in [0, T]} \{ t^{\frac{1}{p}} \|u(t)\|_{PM^{r+\frac{1}{2}}} \},\]

is finite. We also define the space \(K_\infty^{p,r}\) of \(u \in C_c([0, +\infty]; PM^{r+\frac{1}{2}})\) such that

\[\|u\|_{K_\infty^{p,r}} := \sup_{t \in [0, +\infty]} \{ t^{\frac{1}{p}} \|u(t)\|_{PM^{r+\frac{1}{2}}} \},\]

is finite.

We observe that the space \(K_{c,d}^\infty \times K_{c,d}^{d-1}\) verifies the invariance by scaling. The Kato spaces is useful to establish Kato types theorems (see \cite{1} and \cite{7}), such as theorem 5.1.1. We use this Kato spaces to establish global existence and regularity results.

\section{Radius of analyticity}

If \(\Omega\) is an open subset of \(\mathbb{C}^d\), we denote by \(\mathcal{H}(\Omega)\) the set of holomorphic functions over \(\Omega\). Let \(r < d\). If \(u \in PM^r(\mathbb{R}^d)\), we define the radius of analyticity of \(u\) by setting

\[\text{rad}(u) := \sup \{ \sigma > 0 \mid e^{\sigma |D|} u \in PM^r(\mathbb{R}^d) \}.\]

If \(u = (u_1, u_2, \ldots, u_d) \in (PM^r(\mathbb{R}^d))^d\) is a vector field, we define this radius by setting

\[\text{rad}(u) := \min_{k \in [1, d]} \{ \text{rad}(u_k) \}.\]

For every \(\sigma > 0\), we denote by \(S_\sigma\) the open connected set of all \(z \in \mathbb{C}^d\) such that \(|\text{Im}(z)| < \sigma\). The following proposition justifies the denomination "radius of analyticity".
Proposition 1.4.1. Let \( r < d \) and \( \sigma > 0 \). Let \( u \) be in \( PM^{r}(\mathbb{R}^{d}) \). If \( e^{\sigma|D|}u \in PM^{r}(\mathbb{R}^{d}) \), then \( u \) extends to an unique holomorphic function \( U \) in \( \mathcal{H}(S_{\sigma}) \).

This proposition means that we can express \( u \in PM^{r}(\mathbb{R}^{d}) \), whose Fourier transform have an exponential decay, as the trace on \( \mathbb{R}^{d} \) of a function which is holomorphic on some strip \( S_{\sigma} \).

1.5 Main results

We recall that \( d \geq 2 \). Let’s assume that \( p > 2 \) is such that \( d - 3 + \frac{4}{p} > 0 \). This condition ensures that nonlinear terms are well defined. We introduce the space \( X_{T} \) of \((a, u)\in(PM^{d-1} \times PM^{d}) \times PM^{d-1} \), that we equip with the norm defined by

\[
\|(a, u)\|_{X_{T}} := \max\{\|a\|_{K^{p,d-1}}, \|a\|_{K^{p,d}}, \|u\|_{K^{p,d-1}}\}.
\]

Using the language of mild solutions of the Navier-Stokes-Korteweg system, as in [11], we prove the global existence and regularity of the solution to (3) which we state as follows (summing up theorem 3.2.1 and theorem 4.0.1).

Theorem 1.5.1. Given an initial data \((a_{0}, u_{0})\) in \((PM^{d-1} \times PM^{d}) \times PM^{d-1} \). If \( \|(a_{0}, |D|a_{0}, u_{0})\|_{PM^{d-1}} \) is small enough, then the Cauchy problem (3)–(5) has a global solution \((a, u)\) in the space \( X_{\infty} \) which space analytic at any positive time. Moreover, for any time \( t > 0 \), we have

\[
\text{rad}(a(t), u(t)) \geq c_{0}\sqrt{t},
\]

for some positive constant \( c_{0} \) which depends only on \( \nu, \mu, \kappa \) and \( \alpha \).

The first observation is that the lower bound of the radius of analyticity is similar to [11] in the case of Besov spaces. Moreover this regularity result holds for critical initial data.

In section 5, we investigate the instantaneous analytic smoothing effect of the system (3). The following theorem sums up two main results of section 5.

Theorem 1.5.2. Let \( \delta \) be in \([0, \frac{2}{p}]\). Let \((a_{0}, u_{0})\) be in \((PM^{d-1+\delta} \cap PM^{d+\delta}) \times PM^{d-1+\delta} \) an initial data. There exist a positive time \( T \) and an unique solution \((a, u)\) in \( X_{T} \) to the Cauchy problem (3)–(5). Moreover, if \( \delta = \frac{2}{p} \), we have

\[
\liminf_{t \to 0^{+}} \frac{\text{rad}(a(t), u(t))}{\sqrt{t\ln(C^{1}t)}} \geq C_{2},
\]

for some positive constants \( C_{1} \) and \( C_{2} \).

The main interest of this theorem is the amelioration of the improvement radius of analyticity near 0, proposed by F. Charve, R. Danchin and J. Xu in [11]. This result adapts to our framework the new method of J.-Y. Chemin, I. Gallagher and P. Zhang in [7] to estimate the radius of analyticity near 0 of the solution to semi-linear parabolic system. Note compared with theorem 1.5.1, that this theorem contains a local in time existence and uniqueness result for supercritical initial data and holds for arbitrary large initial data. Additionally, we remark that the constant \( C_{1} \) and the existence time interval, depend on the norm of the initial data (see theorem 5.2.3 and theorem 5.1.1).
2 The linearized system

2.1 Parabolic estimate for the linearized system

In this section we investigate the linearized system around \((u, a) = (0, 0)\). This system reads

\[
\begin{aligned}
\partial_t a + \text{div}(u) &= F, \\
\partial_t u - \mu \triangle u - (\mu + \nu)\nabla \text{div}(u) + \alpha \nabla a - \kappa \triangle a &= G,
\end{aligned}
\]  

(6)

where \(F\) and \(G\) are external forces assumed to be, smooth enough. For all \(\xi \in \mathbb{R}^d\), we define \((d + 1) \times (d + 1)\)-matrix

\[
A(\xi) := \begin{pmatrix}
0 & i\xi_1 & \cdots & \cdots & i\xi_d \\
\mu|\xi|^2 + (\mu + \nu)\xi_1^2 & \cdots & \cdots & (\mu + \nu)\xi_1\xi_d \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
i(\alpha\xi_d + \kappa\xi_d|\xi|^2) & (\mu + \nu)\xi_d\xi_1 & \cdots & \mu|\xi|^2 + (\mu + \nu)\xi_d^2
\end{pmatrix}
\]

The matrix-valued symbol \(A\) is the symbol of the space derivative operator of the linearized system. For all \(t \geq 0\), we define

\[
W(t) := e^{tA(D)}.
\]

The family of Fourier multipliers \((W(t))_{t \geq 0}\) is the semi-group of the linearized system. The key point of our study of the Navier-Stokes-Korteweg system with a quantum pressure, is a point-wise estimate of the semi-group \((W(t))_{t \geq 0}\) (that can be found in [6] and [8]). More precisely, we observe that the linear part of system [3] has a parabolic behavior.

Lemma 2.1.1. There exists a positive constant \(c_0\), depending only on \((\kappa, \mu)\), such that the following inequality holds for all \(\xi \in \mathbb{R}^d\) and \(t \geq 0\):

\[
|\langle \hat{a}, |\xi|\hat{a}, \hat{u} \rangle|(t, \xi) \lesssim e^{-c_0t|\xi|^2} |\langle \hat{a}, |\xi|\hat{a}, \hat{u} \rangle|(0, \xi) + \int_0^t e^{-c_0|\xi|^2(t-\tau)} |\langle \hat{F}, |\xi|\hat{F}, \hat{G} \rangle|(\tau, \xi)d\tau.
\]  

(7)

This lemma gives a "parabolic decay" of Fourier modes, in order to obtain the analytic regularisation. This "transfer of parabolicity" is a remarkable property of Korteweg type models for compressible fluids.

3 Global existence

3.1 Nonlinear estimates

In this section we will establish some bilinear estimate, which will be used to control the nonlinear terms of system [3]. We begin by an elementary lemma where we investigate a convolution inequality.

Lemma 3.1.1. Let \(d \geq 2\). Let \(\alpha\) and \(\beta\) be two real numbers such that \(\alpha < d, \beta < d\) and \(\alpha + \beta > d\). Then, for all \(\xi \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} \frac{1}{|\xi - \eta|^\alpha} \frac{1}{|\eta|^\beta} d\eta \lesssim_{\alpha, \beta, d} \frac{1}{|\xi|^{\alpha + \beta - d}}.
\]  

(8)
This estimate will be useful when we consider the product of two functions in pseudo-measure spaces.

**Proof.** If \( \xi = 0 \), it is classical that the left-hand side of (8) is infinite as the right-hand side this inequality. Supposed that \( \xi \neq 0 \). We set

\[
\int_{\mathbb{R}^d} \frac{1}{|\xi - \eta|^\alpha} \frac{1}{|\eta|^\beta} d\eta = \int_{B(\xi,|\xi|/2)} \frac{1}{|\xi - \eta|^\alpha} \frac{1}{|\eta|^\beta} d\eta + \int_{B(0,|\xi|/2)} \frac{1}{|\xi - \eta|^\alpha} \frac{1}{|\eta|^\beta} d\eta + \int_{\mathbb{R}^d \setminus B(0,3|\xi|/2)} \frac{1}{|\xi - \eta|^\alpha} \frac{1}{|\eta|^\beta} d\eta =: I_1 + I_2 + I_3.
\]

where \( X_\xi := B(0,|\xi|/2) \cup B(\xi,|\xi|/2) \). We only need to estimate \( I_1, I_2 \) and \( I_3 \). For the first one, let use begin by remarking that, if \( |\xi - \eta| \leq \frac{|\xi|}{2} \), then, using the inverse triangular inequality, we have \( |\eta| \geq \frac{|\xi|}{2} \). Therefore, we get

\[
I_1 \leq \int_{B(\xi,|\xi|/2)} \frac{1}{|\xi - \eta|^\alpha} d\eta \left( \frac{2}{|\xi|} \right)^\beta.
\]

We aim to estimate the first factor to the right hand side of (8). Using the change of variables \( \zeta \mapsto \zeta + \xi \), we get

\[
\int_{B(\xi,|\xi|/2)} \frac{1}{|\zeta|^\alpha} d\zeta = \int_{B(0,|\xi|/2)} \frac{1}{|\zeta|^\alpha} d\zeta.
\]

Considering the hypothesis \( \alpha < d \), we get, using polar coordinates

\[
\int_{B(0,|\xi|/2)} \frac{1}{|\zeta|^\alpha} d\zeta \lesssim_{\alpha,d} \frac{1}{|\xi|^{\alpha-d}}.
\]

Using (10) to estimate the first factor of the right of (9), we obtain

\[
I_1 \lesssim_{\alpha,d} \frac{1}{|\xi|^{\alpha+d}}.
\]

Observing that \( |\eta| \leq \frac{|\xi|}{2} \) implies \( |\xi - \eta| \geq \frac{|\xi|}{2} \) and taking into account that \( \beta < d \), from the inverse triangular inequality and using polar coordinates, we get as the same way

\[
I_2 \lesssim_{\alpha,d} \int_{B(0,|\xi|/2)} \frac{1}{|\eta|^\beta} d\eta \left( \frac{2}{|\xi|} \right)^\beta \lesssim \frac{1}{|\xi|^{\alpha+\beta-d}}.
\]

We decompose the last term, namely \( I_3 \), in two part

\[
I_3 = \int_{B(0,3|\xi|/2) \setminus X_\xi} \frac{1}{|\xi - \eta|^\alpha |\eta|^\beta} d\eta + \int_{\mathbb{R}^d \setminus B(0,3|\xi|/2)} \frac{1}{|\xi - \eta|^\alpha |\eta|^\beta} d\eta.
\]

For the first one, we have

\[
\int_{B(0,3|\xi|/2) \setminus X_\xi} \frac{1}{|\xi - \eta|^\alpha |\eta|^\beta} d\eta \lesssim_{\alpha,d} \frac{1}{|\xi|^{\alpha+\beta}} \int_{B(0,|\xi|/2)} d\eta \lesssim_{\alpha,d} \frac{1}{|\xi|^{\alpha+\beta-d}}.
\]

Observing that, if \( \eta \in \mathbb{R}^d \setminus X_\xi \), then \( |\xi - \eta| \geq |\eta| \) and using polar coordinates and the hypothesis \( \alpha + \beta > d \), we obtain

\[
\int_{\mathbb{R}^d \setminus B(0,3|\xi|/2)} \frac{1}{|\ksi - \eta|^\alpha |\eta|^\beta} d\eta \lesssim_{\alpha,d} \int_{\mathbb{R}^d \setminus B(0,3|\xi|/2)} \frac{1}{|\eta|^\alpha} d\eta \lesssim_{\alpha,d} \frac{1}{|\xi|^{\alpha+\beta-d}},
\]

that concludes the proof. \( \square \)
The lemma above give a point-wise estimate for the decay rate of the convolution, which is the base for considering products in the pseudo-measure spaces. As a consequence of lemma 3.1.1, we get following bilinear estimates.

**Lemma 3.1.2.** Let \( a \) and \( b \) two homogeneous Fourier multipliers of degree 1. Let \( \delta > 0 \) and let \( p > 2 \). Then, there exists a positive constant \( C_\beta \), that depends on \( \delta, p, d \) and \( b \) such that, for every \( u \) and \( v \) in \( K_{P}^{p,d-1} \), we have

\[
\| \int_{0}^{t} e^{\delta(t-s)} \beta(u,v)ds \|_{K_{P}^{p,d-1}} \leq C_\beta \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}},
\]

where \( \beta(u,v) := b(D)(u \cdot v) \). If \( p \) additionally satisfies \( d - 3 + \frac{2}{p} > 0 \), then there exists a positive constant \( C_\alpha \), that depends of \( \delta, p, d \) and \( a \) such that, for every \( u \) and \( v \) in \( K_{P}^{p,d-1} \), we have

\[
\| \int_{0}^{t} e^{\delta(t-s)} \alpha(u,v)ds \|_{K_{P}^{p,d-1}} \leq C_\alpha \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}},
\]

where \( \alpha(u,v) := u \cdot a(D)v \).

**Proof.** By applying the lemma 3.1.1 with \( \alpha = d - 1 + \frac{2}{p} \) and \( \beta = d - 2 + \frac{2}{p} \)

\[
\left| \int_{0}^{t} e^{\delta(t-s)\xi^2|\alpha(u,v)(s,\xi)ds} \right| \lesssim \int_{0}^{t} e^{\delta(t-s)\xi^2|\alpha(u,v)(s,\xi)ds},
\]

\[
\lesssim \int_{\mathbb{R}^{d}} \int_{0}^{t} e^{\delta(t-s)\xi^2|\alpha(u,v)(s,\xi)ds} ds d\eta,
\]

\[
\lesssim \left( \int_{\mathbb{R}^{d}} |\xi - \eta|^{d-1 + \frac{2}{p}} |\eta|^{d-2 + \frac{2}{p}} \right) \int_{0}^{t} e^{\delta(t-s)\xi^2} ds
\]

\[
\times \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}},
\]

\[
\lesssim \frac{1}{|\xi|^{d-3 + \frac{2}{p}}} \left( \int_{0}^{t} e^{\delta(t-s)\xi^2} ds \right) \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}}.
\]

Using the lemma 3.1.1 with \( \alpha = \beta = d - 1 + \frac{2}{p} \), we obtain by the same approach

\[
\left| \int_{0}^{t} e^{\delta(t-s)\xi^2|\beta(u,v)(s,\xi)ds} \right| \lesssim \int_{0}^{t} e^{\delta(t-s)\xi^2|\beta(u,v)(s,\xi)ds},
\]

\[
\lesssim |\xi| \int_{\mathbb{R}^{d}} \int_{0}^{t} e^{\delta(t-s)\xi^2|\beta(u,v)(s,\xi)ds} ds d\eta,
\]

\[
\lesssim \left( |\xi| \int_{\mathbb{R}^{d}} |\xi - \eta|^{d-1 + \frac{2}{p}} |\eta|^{d-1 + \frac{2}{p}} \right) \int_{0}^{t} e^{\delta(t-s)\xi^2} ds
\]

\[
\times \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}},
\]

\[
\lesssim \frac{1}{|\xi|^{d-3 + \frac{2}{p}}} \left( \int_{0}^{t} e^{\delta(t-s)\xi^2} ds \right) \| u \|_{K_{P}^{p,d-1}} \| v \|_{K_{P}^{p,d-1}}.
\]

Finally, using that the function \( y \in \mathbb{R}_+ \rightarrow e^{-\delta y} y^{1-\frac{2}{p}} \) is bounded and the change of variable \( \sigma \rightarrow t\sigma \) to make appear (taking into account the hypothesis \( p > 2 \)) the beta
function, we have
\[
\frac{1}{|\xi|^{d-3+\frac{4}{p}}} \left( \int_0^t e^{-\delta(t-s)|\xi|^2} \frac{ds}{s^{\frac{2}{p}}} \right) = \frac{1}{|\xi|^{d-1+\frac{4}{p}}} \left( \int_0^t \frac{e^{-c_0(t-s)|\xi|^2}}{\delta^{1-\frac{1}{p}}(t-s)^{1-\frac{1}{p}} s^{\frac{2}{p}}} ds \right),
\]
\[
\lesssim \frac{t^{1-\frac{1}{p}}}{\delta^{1-\frac{1}{p}}|\xi|^{d-1+\frac{4}{p}}}. \]
This concludes the proof of the lemma.

For the remainder of this paper, we supposed that \(d \geq 2\) and \(p > 2\) is such that \(d - 3 + \frac{4}{p} > 0\).

### 3.2 Global existence theorem

In this section we study the global existence of the solution to system (3) for critical initial data. The main result of this section is the following theorem.

**Theorem 3.2.1.** There exists \(\rho > 0\) and \(R > 0\) such that, for every \((a_0, u_0)\) in \((PM^{d-1} \cap PM^d) \times PM^{d-1}\) satisfying
\[
\| (a_0, |D|a_0, u_0) \|_{PM^{d-1}} \leq \rho,
\]
there exist an unique solution \((a, u)\) in \(X_\infty\) of the Cauchy problem (3), such that
\[
\|(a, u)\|_{X_\infty} \leq R.
\]

The proof is based on the Banach fixed-point theorem.

**Proof.** First observe that for all \(v \in PM^{d-1}\), since the function \(y \in \mathbb{R}_+ \mapsto e^{-c_0 y} y^{\frac{1}{p}}\) is bounded by 1 (because \(\frac{1}{p} < 1\), we have
\[
e^{-c_0 t|\xi|^2} t^{\frac{1}{p}} |\hat{\varphi}(\xi)||\xi|^{d-1+\frac{4}{p}} = \frac{e^{-c_0 t|\xi|^2} (c_0 t|\xi|^2)^{\frac{1}{p}} |\hat{\varphi}(\xi)||\xi|^{d-1}}{c_0^p} \leq \frac{1}{c_0^p} \|v\|_{PM^{d-1}},
\]
hence
\[
\|e^{c_0 t \Delta} v\|_{K_{\infty}^{p,d-1}} \leq \frac{1}{c_0^p} \|v\|_{PM^{d-1}}. \tag{13}
\]
It follows from lemma (2.1) and (13) that, for all \((a_0, u_0) \in (PM^{d-1} \cap PM^d) \times PM^{d-1}\), we have
\[
W(\cdot)(a_0, u_0) \in K_{\infty}^{p,d-1}
\]
and
\[
\|W(t)(a_0, u_0)\|_{X_\infty} \leq C_\rho \|(a_0, |D|a_0, u_0)\|_{PM^{d-1}}. \tag{14}
\]
where $\hat{C}$ is a positive constant, that depends only on $c_0$, $p$ and $d$. Combining lemma 2.1.1 and the lemma 3.1.2, we get the following estimates: for every $a$ and $b$ in $K_{p,d-1}^T \cap K_{p,d}^T$ and for all $u$ and $v$ in $K_{p,d-1}^T$,

\[
\| \int_0^t W(t-s)\hat{f}(u,a)(s)ds \|_{K_{p,d-1}^T} \leq C_f \| u \|_{K_{p,d-1}^T} \| a \|_{K_{p,d-1}^T},
\]

\[
\| \int_0^t W(t-s)|D\hat{f}(u,a)(s)ds \|_{K_{p,d-1}^T} \leq \hat{C}_f \| u \|_{K_{p,d-1}^T} \| a \|_{K_{p,d}},
\]

\[
\| \int_0^t W(t-s)g_1(u,v)(s)ds \|_{K_{p,d-1}^T} \leq C_{g_1} \| u \|_{K_{p,d-1}^T} \| v \|_{K_{p,d-1}^T},
\]

\[
\| \int_0^t W(t-s)g_2(u,a)(s)ds \|_{K_{p,d-1}^T} \leq C_{g_2} \| u \|_{K_{p,d-1}^T} \| a \|_{K_{p,d}},
\]

\[
\| \int_0^t W(t-s)g_3(a,b)(s)ds \|_{K_{p,d-1}^T} \leq C_{g_3} \| a \|_{K_{p,d}} \| b \|_{K_{p,d}},
\]

where positive constants $C_f$, $\hat{C}_f$, $C_{g_1}$, $C_{g_2}$ and $C_{g_3}$ only depends of $d$, $p$ and $c_0$. For any positive real numbers $R$, we denote by $B(0,R)$ the ball of center $0$ and radius $R$ in $X_T$.

\[
\Phi : X_\infty \to X_\infty \quad (a, u) \mapsto W(\cdot)(a_0, u_0) + \int_0^\cdot W(\cdot-s)(f(u,a)(s),g(u,a)(s))ds,
\]

where $(a_0, u_0) \in (PM^{d-1} \cap PM^d) \times K^{d-1}$ are the initial data. The goal is to prove the existence of a fixed point for $\Phi$. We assume that

\[
\|(a_0, |D|a_0, u_0)\|_{PM^{d-1}} < \rho,
\]

for $\rho > 0$, small enough, which we will be fixed later. We begin by proving that for some radius $R > 0$, small enough, the ball $B(0,R)$ is stable by $\Phi$. If $(a, u)$ is in $B(0,R)$, then, we deduce from (14), (15), (16), (17), (18), (19) and (20) that

\[
\|\Phi(a, u)\|_{X_\infty} \leq 5K_\Phi R^2.
\]

where $K_\Phi := \max\{C_f, \hat{C}_f, C_{g_1}, C_{g_2}, C_{g_3}\}$. Now, we assume that $R$ satisfies

\[
5K_\Phi R < \frac{1}{2},
\]

and we set

\[
\rho := \frac{R}{2C}.
\]

For $R > 0$ satisfying (22) and for this choice of $\rho$, using (21) and (14), we get for all $(a, u)$ in $B(0,R)$

\[
\|\Phi(a, u)\|_{X_\infty} < R,
\]

which means that $B(0,R)$ is stable by $\Phi$. Let $(a, u)$ and $(b, v)$ be in $X_\infty$. We have

\[
f(u, a) - f(v, b) = f(u, a - b) + f(u - v, b),
\]

\[
g_1(u, u) - g_1(v, v) = g_1(u - v, u) + g_1(v, u - v),
\]

\[
g_2(u, a) - g_2(v, b) = g_2(u, a - b) + g_2(u - v, b),
\]

\[
g_3(a, a) - g_3(b, b) = g_3(a - b, a) + g_3(b, a - b).
\]

Thus, applying (15), (16), (17), (18) and (19), we deduce, from the triangular inequality, that
\[ \|\Phi(a, u) - \Phi(b, v)\|_{X^\infty} \leq K_\Phi (\|a - u\|_{X^\infty} (\|b\|_{X^\infty} + \|a\|_{X^\infty} + \|v\|_{X^\infty}) + 2\|Db\|_{X^\infty}) + \|a - b\|_{X^\infty} \|u\|_{X^\infty} + \|D(a - b)\|_{X^\infty} (2\|u\|_{X^\infty} + \|D(a)\|_{X^\infty} + \|Db\|_{X^\infty})) \leq 8K_\Phi (\|a, u\| - (b, v))\|_{X^\infty} (\|a, u\|_{X^\infty} + \|(b, v)\|_{X^\infty}). \]

Now, if we take \((a, u)\) and \((b, v)\) in the ball \(B(0, R)\), from previous inequalities, it follows that
\[ \|\Phi(a, u) - \Phi(b, v)\|_{X^\infty} \leq 16K_\Phi R\|(a, u) - (b, v)\|_{X^\infty}. \]

However, from (22), we get
\[ 16RK_\Phi < 1. \tag{24} \]

Now, assume that \(R\) satisfies (22). Since, (22) implies (22), then, for \(\rho\) given by (22), \(\Phi\) is a strict contractive map of \(B(0, R)\) into itself. We conclude with Banach fixed-point theorem (see \(3\), theorem 5.7). \(\square\)

As a by-product of the proof of this theorem, we obtain the following result.

**Corollary 3.2.2.** Let \(T > 0\). Let \((a_0, u_0)\) be in \((PM^{d-1} \cap PM^d) \times PM^{d-1}\). There exists two positive constants \(C_1\) and \(C_2\) that depends only of \(\mu, \nu, p\) and \(d\) such that, for all solutions \((a, u)\) of \(3\) in \(X_T\), we have
\[ \|(a, u)\|_{X_T} \leq C_1\|(a_0, |D|a_0, u_0)\|_{PM^{d-1}} + C_2\|(a, u)\|^2_{X_T}. \]

### 4 Analyticity for global solution

The purpose of this section is to prove the analyticity of some global solution constructed in the previous section. Furthermore, we give an lower bound of the radius of analyticity. This result holds for small enough critical initial data. We investigate later the case of supercritical data.

**Theorem 4.0.1.** Let \(\rho\) and \(R\) as in theorem 3.2.1. For every \((a_0, u_0)\) in the space \((PM^{d-1} \cap PM^d) \times PM^{d-1}\) such that
\[ \|(a_0, |D|a_0, u_0)\|_{PM^{d-1}} \leq \frac{\rho}{2c_0\sqrt{T}}, \]
the solution of the Cauchy problem \(3\) is analytic in space for every time \(t > 0\) with a radius of analyticity bounded below by \(c_0\sqrt{T}\).

The proof of the global existence for analytic solutions follows the main scheme than the proof of global existence. The difference is the choice of the functional space where we look for the fixed point. The idea is to consider a weighted norm of the form \(e^{\delta\sqrt{T}|D|}\), where \(\delta\sqrt{T}\) gives a radius of analyticity for the solution at any positive time \(t\). This method is well-known (see \(6\) for this system). We begin by a version of lemma 3.1.2 with analytic norm. We recall that \(d \geq 2\) and \(p > 2\) is such that \(d - 3 + \frac{3}{p} > 0\).

**Lemma 4.0.2.** Let \(T\) be in \([0, +\infty]\). Let \(a\) and \(b\) two homogeneous Fourier multipliers of degree 1. Then for every \(u\) and \(v\) in the Kato space \(K_T^{p, d-1}\), we have
\[ \| \int_0^t e^{\alpha(t-s)} \Delta A_t(u, v)(s)ds \|_{K_T^{p, d-1}} \leq 2^{1 - \frac{4}{p}} e^{2c_0C_\alpha d} C_\alpha e^{c_0\sqrt{T}|D|} |u|_{K_T^{p, d-1}} e^{c_0\sqrt{T}|D|} |v|_{K_T^{p, d-1}}, \tag{25} \]
(26) where, for all \( t > 0 \), \( A_t(u, v) := e^{c_0 \sqrt{t}|D|}(u \cdot a(D)v) \) and \( B_t(u, v) := e^{c_0 \sqrt{t}|D|}(b(D)(u \cdot v)). \)

Proof. We adapt the proof of the lemma \[3.1.2\] The additional key point that we need here is the inequality

\[
e^{-\frac{c_0}{2} (t-s) |\xi|^2} e^{-c_0 \sqrt{t} |\xi - \eta|} e^{-c_0 \sqrt{t}|\eta|} \leq e^{2c_0} e^{-c_0 \sqrt{t} |\xi|}.
\]

From the inverse triangular inequality, it follows that \(-\sqrt{s}|\xi - \eta| - \sqrt{s} |\eta| \leq -\sqrt{s}|\xi|\). Hence, for establish (27), it is enough to prove that

\[
I := (\sqrt{t} - \sqrt{s}) |\xi|(1 - (\sqrt{t} + \sqrt{s})|\xi|^2) \leq 2.
\]

If \( \sqrt{t}|\xi| \geq 2 \), we deduce that \( I \leq 0 \leq 2 \) and if \( \sqrt{t}|\xi| < 2 \), then \( I \leq \sqrt{t}|\xi| \leq 2 \). Then we obtain the expected upper bound for \( I \), that concludes the proof of the lemma. \( \square \)

In lemma \[4.0.2\] constants \( C_0 \) and \( C_0 \) are the same as in lemma \[3.1.2\]. We can also notice the presence of a factor \( 2^{1 - \frac{1}{p}} \) unlike the non analytic version. To deal with the analytic setting, we introduce the analytic space \( Y_T \) of \((a, u) \in (K_{p,d-1} \cap K_{p,d}) \times K_{p,d-1}, \) that we equip with the norm defined by

\[
\| (a, u) \|_{Y_T} := \max\{\| e^{c_0 \sqrt{t}|D|} a \|_{K_{p,d-1}}, \| e^{c_0 \sqrt{t}|D|} u \|_{K_{p,d}} \} + \| e^{c_0 \sqrt{t}|D|} u \|_{K_{p,d-1}}.
\]

Proof of theorem \[4.0.1\] First, we remark that for every \( \xi \in \mathbb{R}^d \) and positive time \( t \), we have

\[
e^{-c_0 \sqrt{t} |\xi|^2} \frac{1}{t} e^{c_0 \sqrt{t} |\xi|} = \left( \frac{2}{c_0} \right)^{\frac{1}{p}} \times \left( e^{-\frac{c_0}{2} t |\xi|^2} \left( \frac{c_0}{2} t |\xi|^2 \right)^{\frac{1}{p}} \right) \times \left( e^{c_0 \sqrt{t} |\xi|} e^{-\frac{c_0}{2} t |\xi|^2} \right).
\]

Since \( \frac{1}{p} < 1 \), the second factor of the right-hand side of the previous identity is lower than 1. Using the inequality (27), the third factor to the right-hand of (28) is increased by \( e^{2c_0} \). Hence, for all \( v \in PM_{d-1} \), we have

\[
e^{-c_0 \sqrt{t} |\xi|^2} \frac{1}{t} e^{c_0 \sqrt{t} |\xi|} |\tilde{\rho}(\xi)| |\xi|^{d-1+\frac{2}{p}} \leq \frac{2^p e^{2c_0}}{c_0^{\frac{1}{p}}} \| v \|_{PM_{d-1}}.
\]

We suppose that the initial data \((a_0, u_0) \in PM_{d-1} \cap PM^{d}) \times PM^{d-1} \) satisfy

\[
\|(a_0, |D|a_0, u_0)\|_{PM_{d-1}} < \tilde{\rho}, \quad (a, u) \rightarrow W(\cdot)(a_0, u_0) + \int_0^T W(\cdot - s)(f(u, a)(s), g(u, a)(s))ds,
\]

for some positive real number \( \tilde{\rho} \), small enough. Using the lemma \[4.0.2\] we deduce by the same way of the proof of global existence, that for a radius \( \bar{R} := \frac{\tilde{R}}{2^{1 - \frac{1}{p} e^{2c_0}}} \) and for \( \bar{\rho} := \frac{\rho}{2^{1 - \frac{1}{p} e^{2c_0}}} \). The map

\[
\Psi : Y_\infty \rightarrow Y_\infty \quad (a, u) \rightarrow W(\cdot)(a_0, u_0) + \int_0^T W(\cdot - s)(f(u, a)(s), g(u, a)(s))ds,
\]

have a unique fixed-point \((a, u)\) in the ball of center 0 and radius \( \bar{R} \) in \( Y_\infty \).
Furthermore, if the initial data \((a_0, u_0)\) satisfy \([29]\), using \(\tilde{\rho} < \rho\), we deduce the existence of a global solution, constructed as the unique fixed point of \(\Phi\) in the ball \(B(0, R)\) of \(X_\infty\). Moreover the fixed-point of \(\Psi\) found previously, namely \((a, u)\), is in the ball \(B(0, R)\) of the space \(X_\infty\) and is the unique fixed-point of \(\Phi\) in this ball. Indeed, it is enough to observe that

\[
\|\Phi(a, u)\|_{X_\infty} \leq \|\Psi(a, u)\|_{Y_\infty} \leq \tilde{R} < R,
\]

keeping in mind that \(\tilde{\rho} < \rho\). In particular, we conclude that, if the initial data \((a_0, u_0)\) satisfy \([29]\), the solution of theorem 3.2.1 is analytic and, at any positive time \(t\), its radius of analyticity is bounded below by \(c_0\sqrt{t}\).

5 Estimate near 0

Now, we turn our attention to the case of supercritical initial data. First we give a local in time existence and uniqueness theorem (in the subsection 5.1), so-called Kato type theorem. This result holds for supercritical initial data, which we will pick arbitrarily in time existence and uniqueness theorem (in the subsection 5.1), so-called Kato type theorem. This result holds for supercritical initial data, which we will pick arbitrarily that does not depend on the initial data \((a, u)\) of the solution during the existence time. Furthermore, we don’t need any smallness assumptions on the initial data, but the existence interval gets smaller as the norm of the initial data gets bigger.

**Proof.** Let \((a_0, u_0)\) be in \((PM^{d-1+\delta} \cap PM^{d+\delta}) \times PM^{d-1+\delta}\) and \(T\) a positive time will be chosen later. Let \(v \in PM^{d-1+\delta}\). We deduce from the lemma 2.1.1 that

\[
\|W(t)v\|_{K_p^{d-1}} \lesssim \|e^{\triangle t}v\|_{K_p^{d-1}}.
\]  

(30)

Furthermore, since the function \(y \in \mathbb{R}_+ \mapsto e^{-ca_0y}\frac{1}{\sqrt{\xi}}\) is bounded, for every \(t \in [0, T]\) and \(\xi \in \mathbb{R}^d\) we have

\[
e^{-ca_0t|\xi|^d}e^{\frac{1}{2}T\|v\|_{PM^{d-1+\delta}}},
\]

\[1\text{We recall that the fixed-point of the Banach fixed-point, like in }[3]\text{ theorem 5.7, is unique.}
hence
\[ \| e^{\alpha t} \Delta v \|_{P_{M}^{d-1+\frac{2}{q}}} \lesssim c_0, \quad T^\frac{2}{q} \| v \|_{P_{M}^{d-1+\delta}}. \]

From (30) and the estimate above, we conclude that there exist a positive constant \( C_\delta \), such that
\[ \| W(t)(a_0, u_0) \|_{X_T} \leq C_\delta T^\frac{2}{q} \| (a_0, |D|a_0, u_0) \|_{P_{M}^{d-1+\delta}}. \] (31)

Now, we consider the map
\[ \Phi : X_T \to X_T \]
\[ (a, u) \mapsto W(\cdot)(a_0, u_0) + \int_0^T W(\cdot, s)(f(u, a)(s), g(u, a)(s)) ds. \]

As in the proof of the global existence theorem, for some positive radius \( R \) we have
\[ \| \Phi(a, u) - W(\cdot)(a_0, u_0) \|_{X_T} \leq 5K_\Phi R^2, \] (32)
\[ \| \Phi(a, u) - \Phi(b, v) \|_{X_T} \leq 16K_\Phi R \| (a, u) - (b, v) \|_{X_T}. \] (33)

Assume that the radius \( R \) and the time \( T \) satisfy
\[ 16RK_\Phi < 1 \quad \text{and} \quad T = c_\delta \| (a_0, |D|a_0, u_0) \|_{P_{M}^{d-1+\delta}}^{-\frac{2}{q}} \]

where \( c_\delta := \left( \frac{R}{2C_\delta} \right)^\frac{2}{q} \). Then, using (31), (32) and (33), we observe that \( \Phi \) is a contraction of the ball \( B(0, R) \) of \( X_T \) into itself. We conclude the proof by applying the Banach fixed-point theorem.

Theorem 5.1.2. Let \( \delta \in [0, \frac{2}{p}] \). For any initial data \((a_0, u_0) \in (P_M^{d-1+\delta} \cap P_M^{d+\delta}) \times P_M^{d-1+\delta} \), there exists a positive real number \( T \) such that the Cauchy problem (3) has unique solution \((a, u)\) in the space \( X_T \), such that for all \( t \in [0, T] \), \((a(t), u(t))\) is real analytic with
\[ \operatorname{rad}((a(t), u(t))) \geq c_0 \sqrt{t}. \]

Moreover, there exists a positive constant \( d_\delta \), that does not depend on the initial data \((a_0, u_0)\), such that \( T \geq d_\delta \| (a_0, |D|a_0, u_0) \|_{P_{M}^{d-1+\delta}}^{-\frac{2}{q}} \).

Proof. Let \((a_0, u_0)\) be in \((P_M^{d-1+\delta} \cap P_M^{d+\delta}) \cap P_M^{d-1+\delta} \) and \( T \) a positive time that will be chosen later. Now, we pick \( v \) in \( P_M^{d-1+\delta} \). The lemma (2.1.1) provides
\[ \| e^{\alpha t} D |W(t)v| \|_{K^{p,d-1}} \lesssim \| e^{\alpha t} e^{\alpha v \sqrt{|D|}} |v| \|_{K^{p,d-1}}. \] (34)

Moreover, for all \( t \in [0, T] \), we have
\[ e^{-\alpha t |\xi|^2} T^\frac{2}{q} e^{\alpha v \sqrt{|D|}} |\hat{\varphi}(\xi)| |\xi|^{d-1+\frac{2}{p}} = t^\frac{2}{q} \left( e^{-\frac{\alpha}{2} |\xi|^2 |t| |\xi|^{\frac{p}{2} - \frac{1}{p}}} \right) \times \left( |\hat{\varphi}(\xi)| |\xi|^{d-1+\delta} \right) \times \left( e^{-\frac{\alpha}{2} |\xi|^2} e^{\alpha v \sqrt{|D|}} \right) \leq T^\frac{2}{q} \left( \frac{c_0}{2} \right)^{\frac{p}{2} - \frac{1}{p}} e^{2c_0} \| c \|_{P^{d-1+\delta}}. \]

Combining the last estimate and (34), we get
\[ \| e^{c_0 \sqrt{|D|}} |W(t)v| \|_{K^{p,d-1}} \lesssim T^\frac{2}{q} \| v \|_{P_M^{d-1+\delta}}. \]
Hence, there exists a positive constant $D_\delta$ such that
\[
\|W(t)(a_0, u_0)\|_{Y_T} \leq D_\delta T^\delta \|e^{\sqrt{\delta} |D|}(a_0, |D|a_0, u_0)\|_{P_M^{d-1+\delta}}. \tag{35}
\]

Now we consider the map
\[
\Phi : Y_T \rightarrow Y_T
\]
\[
(a, u) \mapsto W(\cdot)(a_0, u_0) + \int_0^t W(\cdot - s)(f(u, a)(s), g(u, a)(s))ds.
\]

Let $R$ be a positive radius that will be chosen later. From the lemma 4.0.2, we deduce that there exists a positive constant $K_\phi$ such that, for all $(a, u)$ and $(b, v)$ in the ball $B(0, R)$ of $Y_T$, we have
\[
\|\Phi(a, u) - W(\cdot)(a_0, u_0)\|_{Y_T} \leq 5K_\phi R^2
\]
and
\[
\|\Phi(a, u) - \Phi(b, v)\|_{Y_T} \leq 16K_\phi R\|a - (b, v)\|_{Y_T}.
\]

For $R > 0$ and $T$ such that
\[
16RK_\phi < 1 \quad \text{and} \quad T = d_\delta\|a_0, |D|a_0, u_0\|^{\frac{1}{1+\delta}},
\]
where $\delta := \left(\frac{R}{2D_\delta}\right)^\frac{1}{\delta}$, as in the proof of theorem 5.1.1 from the Banach fixed-point theorem we conclude that there exists a unique fixed-point $(a, u) \in Y_T$ of $\Phi$ which solve (35-38). Since $(a, u) \in Y_T$, for all $t$ in $[0, T]$, we get $\text{rad}((a(t), u(t))) \leq c_0\sqrt{T}$, which completes the proof.

\[\square\]

5.2 Estimate of the radius of analyticity

In this section we show an improvement of the estimate of the radius of analyticity near 0. We will adapt the method used by J.-Y. Chemin, I. Gallagher and P. Zhang in [7] to our context, in order to obtain an estimate of the radius of analyticity when the initial data is in the space $(P M_p^{p,d-1+\frac{\alpha}{2}} \cap P M_{p,d}^{p,d+\frac{\alpha}{2}} \times P M_p^{p,d-1+\frac{\beta}{2}}$.

For $f \in L^1_{loc}([0, T]; S'(\mathbb{R}^d))$, we set
\[
\widehat{f}(t) := e^{-\frac{\lambda^2}{4(1-\varepsilon)c_0}|t|} + \frac{\lambda}{\sqrt{T}} |D|f(t) \quad (t \in [0, T]) \tag{36}
\]

For all $T > 0$ and for each $t \in [0, T]$ and $\varepsilon > 0$, we define the Fourier multiplier
\[
\theta(t, D, \varepsilon) := -\frac{\lambda^2}{4(1-\varepsilon)c_0} + \frac{\lambda}{\sqrt{T}} |D|.
\]

In order to study the radius of analyticity, we begin to establish some nonlinear estimate in the new analytic norm provided by $e^{\theta(\cdot, D, \varepsilon)}$.

**Lemma 5.2.1.** Let $\varepsilon > 0$ and $T > 0$. For $\alpha, \beta, u$ and $v$ as in lemma 5.1.2, we have the following inequalities
\[
\|\int_0^t e^{\alpha_0(t-s)} \Delta u_v(u, v)(s)ds\|_{K_T^{p,d-1}} \leq \varepsilon\cdot c_0 e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}} \|u\|_{K_T^{p,d-1}} \|\theta\|_{K_T^{p,d-1}}, \tag{37}
\]
\[
\|\int_0^t e^{\alpha_0(t-s)} \Delta \beta_v(u, v)(s)ds\|_{K_T^{p,d-1}} \leq \varepsilon\cdot c_0 e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}} \|u\|_{K_T^{p,d-1}} \|\theta\|_{K_T^{p,d-1}}, \tag{38}
\]
where $\alpha_v := e^{\theta(t, D, \varepsilon)}\alpha$ and $\beta_v := e^{\theta(t, D, \varepsilon)}\beta$. 

15
Proof. First, we recall some properties of symbols of $\theta(t, D, \varepsilon)$. For every $t$ and $s$ in $[0, T]$ and for all $\xi$ and $\eta$ in $\mathbb{R}^d$, we have

\[
\begin{align*}
\theta(t, \xi, \varepsilon) &= \theta(t - s, \xi, \varepsilon) + \theta(s, \xi, \varepsilon), \\
\theta(t, \xi, \varepsilon) - c_0 t |\xi|^2 &= -\frac{\lambda^2}{4(1 - \varepsilon)c_0} t + \lambda \frac{t}{\sqrt{T}} |\xi| - c_0 t |\xi|^2 \leq \varepsilon c_0 t |\xi|^2, \\
\theta(t, \xi, \varepsilon) &\leq \theta(t, \xi - \eta, \varepsilon) + \theta(t, \eta, \varepsilon) + \frac{\lambda^2}{4(1 - \varepsilon)c_0} t.
\end{align*}
\] (39) (40) (41)

Let $\xi \in \mathbb{R}^d$. Using (39) and (41), for all $t \in [0, T]$, we get

\[
\begin{align*}
\left| \int_0^t e^{-c_0(t-s)}|\xi|^2 (\bar{u}(u, v)(s, \xi)ds \right| \leq \left| \int_0^t e^{-c_0(t-s)}|\xi|^2 + \theta(t-s, \xi, \varepsilon) \left(e^{\theta(s, \xi, \varepsilon)\alpha(u, v)(s, \xi)} - 1 \right) ds \right|,
\end{align*}
\]

Therefore, we deduce from (41) that, for all $s \in [0, T]$, we have

\[
\begin{align*}
e^{\theta(s, \xi, \varepsilon)\alpha(u, v)(s, \xi)} &\leq \int e^{\theta(s, \xi, \varepsilon)\alpha(u, v)(s, \xi)} |\bar{u}(s, \xi - \eta)| |\eta| |\bar{v}(s, \eta)| d\eta, \\
&\leq \int e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} |\bar{u}(s, \xi - \eta)| |\eta| |\bar{v}(s, \eta)| d\eta, \\
&\leq e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} \int |\bar{u}(s, \xi - \eta)| |\eta| |\bar{v}(s, \eta)| d\eta, \\
&\leq e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} \int \frac{d\eta}{|\xi - \eta|^{d-1} + \frac{\lambda^2}{s} |\eta|^{d-1} + \frac{\lambda^2}{p} s} \|u\|_{K_T^{p, d-1}} \|v\|_{K_T^{p, d-1}}.
\end{align*}
\]

Using the lemma 3.1.1, we obtain

\[
e^{\theta(s, \xi, \varepsilon)\alpha(u, v)(s, \xi)} \leq e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} \left( \frac{\lambda^2}{s} \right)^{d-1} \frac{\lambda^2}{p} \|u\|_{K_T^{p, d-1}} \|v\|_{K_T^{p, d-1}}.
\]

Similarly to the end of the proof of the lemma 5.1.2, we deduce from (42), that

\[
\begin{align*}
\int_0^t e^{-c_0(t-s)}|\xi|^2 (\bar{u}(u, v)(s, \xi)ds \leq e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} \left( \frac{\lambda^2}{s} \right)^{d-1} \frac{\lambda^2}{p} \|u\|_{K_T^{p, d-1}} \|v\|_{K_T^{p, d-1}}.
\end{align*}
\]

The first inequality follows. The proof of the second inequality is similar, with some modifications in the same manner as the proof of (41).

As in the proof of theorem 3.2.1 using lemma 5.2.1, we can prove the following estimate.

Lemma 5.2.2. Let $\varepsilon > 0$ and let $\delta$ be in $[0, \frac{2}{p})$. Let $(a_0, u_0) \in (PM^{d-1+\delta} \cap PM^d) \times PM^{d-1+\delta}$ an initial data and $(a, u) \in X_T$ a solution of the Cauchy problem \[ \mathbb{P} \]. There exist two positive constants $C$, which depend only of $\nu$, $\mu$, $\alpha$, $\kappa$, and $C_\varepsilon$, which depend only of $\nu$, $\mu$, $\alpha$, $\kappa$, and $\varepsilon$, such that

\[
\|(a, u)\|_{X_T} \leq C \left( \|e^{c_0 t \Delta} (a_0, u_0)\|_{X_T} + C_\varepsilon e^{\frac{\lambda^2}{4(1 - \varepsilon)c_0} s} \|(a, u)\|_{X_T}^2 \right).
\] (43)
The lemma above combined with a bootstrap argument is the key point to prove the following theorem, that is the main result of this section.

**Theorem 5.2.3.** Let \((a_0, u_0) \in (PM^{d-1+\frac{2}{p}} \cap PM^{d+\frac{2}{p}}) \times PM^{d-1+\frac{2}{p}}\). If there exists a solution \((a, u) \in X_{T^*}\) of (3) for some positive times \(T^*\), then

\[
\liminf_{T \to 0^+} \frac{\text{rad}(a(t), u(t))}{\sqrt{T \ln \left( T \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}} \right)}} \geq \sqrt{\frac{T}{p}}. \tag{44}
\]

**Proof.** We use a bootstrap argument. Let \(\varepsilon > 0\). For every \(T \in [0, T^*]\), we denote by \(H(T)\) the following induction hypothesis,

\[
\| (a, u) \|_{X_T} \leq \frac{1}{C_4} \| e^{-\lambda_T^2 t} (a_0, u_0) \|_{X_T}, \tag{45}
\]

where the positive real number \(\lambda_T\) will be chosen later and

\[
D_\varepsilon := \frac{1}{C_4} 4\mu C,
\]

with

\[
\mu := \frac{1}{2} \frac{1}{2C + 4}. \tag{46}
\]

If \(H(T)\) is satisfying, we deduce from the lemma 5.2.2 that

\[
\| (a, u) \|_{X_T} \leq C \| e^{\varepsilon \cot \Delta} (a_0, u_0) \|_{X_T} + \frac{1}{4\mu} \| (a, u) \|_{X_T},
\]

that is

\[
\| (a, u) \|_{X_T} \leq \frac{4\mu C}{1 - 4\mu} \| e^{\varepsilon \cot \Delta} (a_0, u_0) \|_{X_T}. \tag{47}
\]

For all \(T \in [0, T^*]\), we have

\[
\| e^{\varepsilon \cot \Delta} (a_0, u_0) \|_{X_T} \leq D_{p, \varepsilon} T^\frac{1}{p} \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}}. \]

Let us define

\[
T_\varepsilon := \eta_\varepsilon \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}}^{-p} \tag{48},
\]

where

\[
\eta_\varepsilon := \left( \frac{D_\varepsilon}{2D_{p, \varepsilon}} \right)^\frac{1}{p}.
\]

Then, for every \(T \in [0, T_\varepsilon]\), we have

\[
2D_{p, \varepsilon} T^\frac{1}{p} \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}} \leq D_\varepsilon.
\]

Now, for all \(T \in [0, T^*]\), we define the positive real number \(\lambda_T\) by setting

\[
\lambda_T^2 := \frac{4(1 - \varepsilon)}{p} \ln \left( \frac{\eta_\varepsilon}{T \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}}} \right). \tag{49}
\]
According to (16), we have \( \frac{4\mu C}{1 - 4\mu} < 2 \). For \( T \in [0, T_\varepsilon] \), assuming \( H(T) \), we deduce from (47), that
\[
\| (a, u) \|_{X_T} \leq \frac{4\mu C}{1 - 4\mu} D_{p, \varepsilon} T^{\frac{1}{\mu}} \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}},
\]
\[
< 2D_{p, \varepsilon} T^{\frac{1}{\mu}} \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}},
\]
\[
= D_\varepsilon e^{-\frac{x_0^2}{4(1-\varepsilon)R_0}}.
\]
This in turn shows that \( H(T) \) holds for \( T \in [0, T_\varepsilon] \). Moreover, for all \( T \in [0, T_\varepsilon] \), from (48), it follows
\[
T^{\frac{1}{\mu}} \| e^{\lambda T \sqrt{T} |D|(a(T), |D|a(T), u(T))} \|_{PM^{d-1+\frac{2}{p}}} \leq D_\varepsilon.
\]
Hence, for every \( T \in [0, T_\varepsilon] \), we have
\[
R(T) \geq \left( \frac{4(1 - \varepsilon)}{p} T \right)^{\frac{1}{\mu}} \ln \left( \frac{\eta_\varepsilon}{T \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}}} \right),
\]
where \( R(T) := \text{rad}(a(t), u(t)) \). This shows that
\[
\liminf_{T \to 0^+} \frac{R(T)}{T \ln \left( T \| (a_0, |D|a_0, u_0) \|_{PM^{d-1+\frac{2}{p}}} \right)} \geq \sqrt{\frac{4(1 - \varepsilon)}{p}}, \tag{49}
\]
Since (49) holds for \( \varepsilon > 0 \) chosen arbitrarily, the theorem is proved. \( \square \)

For the case of critical initial data, the proof of theorem 1.3 (b) of [7] for the semi-linear parabolic equation cannot be adjusted to our functional framework, due to the point-wise feature of pseudo-measure spaces.

A Characterization of analyticity with Fourier transform

In this appendix we prove proposition [A.0.1]

**Proposition A.0.1.** Let \( r < d \) and \( \sigma > 0 \). Let \( u \) be in \( PM^r(\mathbb{R}^d) \). If \( e^{\sigma |D|u} \in PM^r(\mathbb{R}^d) \), then \( u \) extends to a unique holomorphic function \( U \) in \( H(S_\sigma) \).

**Proof.** Since \( \hat{u} \in L^1_{loc}(\mathbb{R}^d) \), we deduce that \( \hat{u} \) and \( |\hat{u}|^2 \) are integrable on the a neighborhood of 0 and using that \( e^{\sigma |D|u} \in PM^r(\mathbb{R}^d) \), it is easy to conclude that \( \hat{u} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Then, for almost every \( x \in \mathbb{R}^d \), we have
\[
u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{u}(\xi) d\xi.
\]
We denote by \( v(x) \) the right-hand side of this inequality. It is sufficient to prove that the function \( x \in \mathbb{R}^d \mapsto v(x) \) extends to a holomorphic function on \( S_\sigma \). If \( \sigma \in ]0, \sigma[ \), then for all \( z \in S_\sigma \), we have
\[
|e^{iz\cdot\xi} \hat{u}(\xi)| \leq e^{\text{Im}(z)||\xi||} |\hat{u}(\xi)|
\]
\[
\leq e^{\sigma||\xi||} |\hat{u}(\xi)|
\]
\[
\leq e^{-\sigma(\sigma - \delta)||\xi||} ||e^{\sigma|D|u}\|_{PM^r}.
\]
Using the hypothesis $r < d$, we deduce by a classical argument that the function $\xi \mapsto e^{iz \cdot \hat{u}(\xi)}$ is in $L^1(\mathbb{R}^d)$. This legitimate, for every $z \in S_\sigma$, the definition of the quantities

$$U(z) := \frac{1}{(2\pi)^d} \int e^{iz \cdot \hat{u}(\xi)} d\xi.$$  

From

$$|e^{iz \cdot \hat{u}(\xi)}| \leq \frac{e^{-(\sigma-\tilde{\sigma})|\xi|}}{|\xi|} \|e^{\sigma|D|}u\|_{PM^r},$$  \hspace{1cm} (51)

that holds for each $z \in S_{\tilde{\sigma}}$ and $\xi \in \mathbb{R}^d$, and observing that the right-hand side of (51) defines a $L^1(\mathbb{R}^d)$ function that does not depend on $z \in S_{\tilde{\sigma}}$, we deduce that $U \in \mathcal{H}(S_{\tilde{\sigma}})$. Since $\tilde{\sigma}$ is arbitrarily chosen in $]0, \tilde{\sigma}[$, we deduce that $U$ is holomorphic over $S_\sigma$. \hfill \Box
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