Solutions of the bound state Faddeev-Yakubovsky equations in three dimensions by using NN and 3N potential models

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Abstract

A recently developed three-dimensional approach (without partial-wave decomposition) is considered to investigate solutions of Faddeev-Yakubovsky integral equations in momentum space for three- and four-body bound states, with the inclusion of three-body forces. In the calculations of the binding energies, spin-dependent nucleon-nucleon (NN) potential models (named, S\(_3\), MT-I/III, YS-type and P\(_5.5\)GL) are considered along with the scalar two-meson exchange three-body potential. Good agreement of the presently reported results with the ones obtained by other techniques are obtained, demonstrating the advantage of an approach in which the formalism is much more simplified and easy to manage for direct computation.

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I. INTRODUCTION

In recent years, calculations of three- and four-body bound and scattering states based on the Faddeev-Yakubovsky (FY) scheme are performed in a novel three-dimensional (3D) approach, which avoids truncation problems and the necessity of complicated recoupling algebra that accompanies partial-wave (PW) based calculations [1]-[9]. Instead, in the 3D approach, the equations and amplitudes are formulated directly as functions of momentum vector variables. This is a straightforward procedure quite convenient for obtaining final observables such as the total energy. For a PW observable, one can easily project the final state onto the specific required partial-wave channel.

For three-nucleon (3N) and four-nucleon (4N) bound states, the FY equations with two- and three-nucleon interactions have been recently formulated in a realistic 3D approach [10]. The formalism, according to the number of spin-isospin states that one takes into account, leads to finite number of coupled three dimensional integral equations to be solved. It has been shown that considering the continuous angle variables instead of the discrete angular momentum quantum numbers in evaluation of the transition and permutation operators, coordinate transformations as well as the three-nucleon forces (3NFs) lead to less complicated expressions in comparison with the PW representation. However, it should be mentioned that with respect to the PW representation, the present formalism with the smaller number of equations leads to higher dimensionality of integral equations. In other words the price for the smaller number of equations in 3D representation is the higher dimensionality of the integral equations. It should be clear that by switching off the spin-isospin quantum numbers, one can easily reach the bosonic type of three dimensional FY integral equations which are solved in Refs. [11]-[14].

In view of the above, we can observe that one real advantage in using a non-PW approach in comparison with PW-based methods relies in a simplified computational algorithm, which is straightforward obtained from the original equations. For interacting systems with two and three particles the procedure was already shown to be quite reliable and easy to be implemented. The advantage of the 3D approach is more evident in the formulation of 4N interacting systems, where it completely avoids the extremely complicated algebra of coupling of spin-angular momentum quantum numbers. However, it is clear that such advantage of the 3D approach, when dealing with the formalism and the corresponding computation,
comes at the expense of possible numerical precision when considering more than two Jacobi momentum vector variables. In such a case, by working with the non-PW approach, after the momentum variable discretization one may have to deal with matrices larger than the ones that occur in case of PW-based calculations, making the latter procedure preferable.

By considering previous numerical comparisons between 3D and PW-based results, we should note the perfect agreement between the obtained full wave function of three-nucleon system, as well as the corresponding momentum distribution functions \[11\]. In view of these results, in case of a four-nucleon interacting system, the numerical accuracy obtained by the 3D approach is expected to be about the same as the accuracy verified in PW-based calculations. This agreement should show up in the analysis of the corresponding observables, which is partially done in the present approach by considering bound-state solutions of three- and four-nucleon systems with 3NFs.

The 3D approach has been shown to be efficient in solving the Faddeev equations for the 3N scattering calculations, especially at intermediate and higher energies \[15\]. Also, the recent proton-deuteron elastic and breakup calculations show that the 3D approach has the potential to provide a more rigorous treatment of Coulomb effects \[16\].

In the case of continuum problems, as for example when obtaining scattering observables, where partial-wave summation can be problematic, the 3D approach is expected to be particularly more efficient than a method using PW decomposition. Clearly, intrinsic limitations of the PW-based calculations are not only due to the complexity of deriving the necessary equations, but also due to the limitations in computer resources requiring very large number of angular momentum states in order to achieve convergence for the scattering observables. By increasing the energy the number of PW channels strongly proliferates and consequently leads to more numerical difficulties with respect to accuracy and storage requirements. However, as shown in Ref. \[15\], relativistic three-body scattering calculations at energies up to 1 GeV laboratory kinetic energy has been done successfully by using direct vector variable calculations, avoiding PW decomposition. Since the 3D approach does not use partial wave decomposition, carrying all the PW channels automatically, the same numerical effort is spent in observable calculations at higher or lower energies. Essentially, the 3D technique is not only shown to be a viable alternative to the well-established PW-based calculations at low-energy regions, but also it appears to be a necessary approach at higher energies where the PW approach is no longer feasible.
One should also note that channel independent observables, such as the total differential cross section, can be obtained using the 3D formalism and consequently be compared to experimental data. Since experimental data are not always available, one needs to extract from this 3D approach a channel-dependent observable, as the $NN$ phase shifts. To this aim, one can easily project the obtained final state onto the specific PW channel, as it was done by Fachruddin, leading to very accurate results in excellent agreement with established PW results [17].

Before concluding this introduction, it is useful to mention a recent alternative 3D representation for 3N bound states where the spin-isospin couplings are not explicitly carried out [18]. The novelty of this formalism is the evaluation of $NN$ $t$-matrices, the 3NFs, and the Faddeev components as products of scalar functions with scalar products of spin operators and momentum vectors. The spin operators have been removed and the final formalism leads to scalar functions of only momentum vectors.

In the present paper, our purpose is to calculate FY bound-state solutions using nucleon-nucleon potential models with three-nucleon forces, following the non-PW 3D approach as shown in Ref. [10]. We report results obtained for three- and four-nucleon binding energies by employing spin-isospin dependent NN potential models along with a scalar two-meson exchange 3NF. The main goal of the present work is to demonstrate advantages of the 3D approach in few-body systems by testing the 3D representation of the FY integral equations with several potential models not previously considered in 3D approach studies.

The current paper is organized as follows: In section II we briefly review the coupled three-dimensional FY integral equations for the $4N$ bound state. In section III we present our numerical results for three- and four-nucleon binding energies and compare them to the results obtained from other techniques. Finally, we have our summary with an outlook in section IV.
II. A BRIEF REVIEW OF FY EQUATIONS IN THREE-DIMENSIONS

In the FY formalism, the bound state of four nucleons in the presence of 3NFs is described by the following coupled equations [7]:

\[
\begin{align*}
|\psi_1\rangle &= G_0 t P \left( (1 - P_{34})|\psi_1\rangle + |\psi_2\rangle \right) + (1 + G_0 t) G_0 V_{123}^{(3)} |\Psi\rangle,
|\psi_2\rangle &= G_0 t \tilde{P} \left( (1 - P_{34})|\psi_1\rangle + |\psi_2\rangle \right),
\end{align*}
\]

(1)

where the Yakubovsky components \(|\psi_1\rangle\) and \(|\psi_2\rangle\) stand for “3+1” (K-type or 123,4) and “2+2” (H-type or 12,34) partitions of the four nucleons, respectively. \(G_0\) is the free \(4N\) propagator, the operator \(t\) is the \(NN\) transition matrix, and \(P, P_{34} \) and \(\tilde{P}\) are permutation operators. The quantity \(V_{123}^{(3)}\) defines a part of the 3NF in the cluster \((123)\), which is symmetric under the exchange of particles 1 and 2. As shown in Fig. [1] for non-PW momentum space representation of the coupled Yakubovsky components, i.e. Eq. (1), one needs two different sets of basis states:

\[
|u \alpha\rangle \equiv |u_1 u_2 u_3 \alpha_{1234}^S \alpha_{1234}^T \rangle \equiv |u_1 u_2 u_3 \left( (s_{12} \frac{1}{2})s_{13} \frac{1}{2} \right) S M_S \left( (t_{12} \frac{1}{2})t_{13} \frac{1}{2} \right) T M_T \rangle,
|v \beta\rangle \equiv |v_1 v_2 v_3 \beta_{1234}^S \beta_{1234}^T \rangle \equiv |v_1 v_2 v_3 \left( s_{12} s_{34} \right) S M_S \left( t_{12} t_{34} \right) T M_T \rangle,
\]

(2)

where these basis states are complete in the \(4N\) Hilbert space:

\[
\sum_{\xi}^A |A \xi\rangle \langle A \xi| = 1, \quad \sum_{\xi}^A \equiv \sum_{\xi} \int D^3 A \equiv \sum_{\xi} \int d^3 A_1 \int d^3 A_2 \int d^3 A_3,
\]

(3)

where \(A\) indicates each one of the \(u\) and \(v\) vector sets and \(\xi\) indicates \(\alpha\) and \(\beta\) quantum number sets. Representation of the coupled equations (1) in the introduced basis states, Eq. (2), leads to two sets of coupled integral equations (4):

\[
\begin{align*}
\langle u \alpha | \psi_1 \rangle &= \sum_{\alpha'}^u \sum_{\alpha''}^{u''} \langle u \alpha | G_0 t | u' \alpha' \rangle \langle u' \alpha' | P | u'' \alpha'' \rangle \\
&\quad \times \left( \sum_{\alpha'''}^{u'''} \langle u''' \alpha''' | 1 - P_{34} | u'' \alpha'' \rangle \langle u'' \alpha'' | \psi_1 \rangle + \sum_{\beta'}^{v'} \langle u' \alpha' | v' \beta' \rangle \langle v' \beta' | \psi_2 \rangle \right) \\
&\quad + \sum_{\alpha'}^u \sum_{\alpha''}^{u''} \langle u \alpha | (1 + G_0 t) G_0 | u' \alpha' \rangle \langle u' \alpha' | V_{123}^{(3)} | u'' \alpha'' \rangle \langle u'' \alpha'' | \psi_1 \rangle,
\end{align*}
\]

(4)

\[
\begin{align*}
\langle v \beta | \psi_2 \rangle &= \sum_{\beta'}^{v'} \sum_{\beta''}^{v''} \langle v \beta | G_0 t | v' \beta' \rangle \langle v' \beta' | \tilde{P} | v'' \beta'' \rangle \\
&\quad \times \left( \sum_{\alpha'}^u \sum_{\alpha''}^{u''} \langle v'' \beta'' | u' \alpha' \rangle \langle u' \alpha' | 1 + P_{34} | u'' \alpha'' \rangle \langle u'' \alpha'' | \psi_1 \rangle + \langle v'' \beta'' | \psi_2 \rangle \right).
\end{align*}
\]
To evaluate the above coupled integral equations, one needs to evaluate the matrix elements of two-body $t$-matrices, permutation operators, as well as the coordinate transformations. These have been evaluated in detail in Ref. [10]. It is useful to mention that one needs the free 4N basis states $|A\gamma\rangle$, where the spin-isospin parts $\gamma$ are given as: $|\gamma\rangle \equiv |\gamma_s \gamma_T\rangle \equiv |m_{s_1} m_{s_2} m_{s_3} m_{s_4} m_{t_1} m_{t_2} m_{t_3} m_{t_4}\rangle$. In changing the 4N basis states, i.e. $|\alpha\rangle$ and $|\beta\rangle$, to the free 4N basis states $|\gamma\rangle$, one needs to calculate the usual Clebsch-Gordan coefficients $\langle \gamma|\alpha\rangle = g_{\gamma\alpha} \equiv g_{\gamma\alpha}^S g_{\gamma\alpha}^T$ and $\langle \gamma|\beta\rangle = g_{\gamma\beta} \equiv g_{\gamma\beta}^S g_{\gamma\beta}^T$ (see Ref. [10]). After the mentioned operators and coordinate transformations are carried out, the coupled Yakubovsky equations can be obtained explicitly:
\[
\langle \textbf{u} \alpha | \psi_1 \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \times \int d^3 u_2 \sum_{\gamma', \gamma''} g_{\gamma' \gamma''} \delta_{m''_s m''_s} \delta_{m'_t m'_t} \delta_{m'_{s_3} m'_{t_3}} \delta_{m''_{t_3} m''_{t_3}} \times \langle \textbf{u}_1 m''_{s_1} m''_{s_2} m''_{t_1} m''_{t_2} | t(\epsilon) | -\frac{1}{2} \textbf{u}_2 - \textbf{u}_2' m'_{s_2} m'_{s_3} m'_{t_2} m'_{t_3} \rangle_a \\
\times \left\{ \sum_{\alpha''} g_{\gamma' \alpha''} \langle \textbf{u}_2 + \frac{1}{2} \textbf{u}_2' \textbf{u}_2' \textbf{u}_3 \alpha'' | \psi_1 \rangle - \sum_{\alpha''} g_{\gamma'_{1243} \alpha''} \langle \textbf{u}_2 + \frac{1}{2} \textbf{u}_2' \frac{1}{3} \textbf{u}_2' + \frac{8}{9} \textbf{u}_3 - \frac{1}{3} \textbf{u}_3 \alpha'' | \psi_1 \rangle + \sum_{\beta'} g_{\gamma' \beta'} \langle \textbf{u}_2 + \frac{1}{2} \textbf{u}_2' - \frac{2}{3} \textbf{u}_3 \frac{1}{2} \textbf{u}_2' - \frac{2}{3} \textbf{u}_3 \beta' | \psi_2 \rangle \right\} + \left\{ \langle \textbf{u} \alpha | V_{123}^{(3)} | \Psi \rangle \times \frac{1}{2} \sum_{\gamma', \gamma'', \alpha''} g_{\alpha \gamma' \gamma''} \sum_{\alpha''} g_{\gamma' \alpha''} \frac{\delta_{m''_{s_3} m''_{s_2}} \delta_{m''_{s_2} m''_{t_2}} \delta_{m''_{s_1} m''_{t_1}} \delta_{m''_{t_3} m''_{t_3}}}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \times \langle \textbf{u}_1 m'_{s_1} m'_{s_2} m'_{t_2} m'_{t_2} | t(\epsilon) | \textbf{u}_1' m''_{s_1} m''_{s_2} m''_{t_1} m''_{t_1} \rangle \langle \textbf{u}_1' \textbf{u}_2 \textbf{u}_3 \alpha'' | V_{123}^{(3)} | \Psi \rangle \right\} \right. \\
\langle \textbf{v} \beta | \psi_2 \rangle = \frac{1}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{3m}} \times \int d^3 v_3' \sum_{\gamma', \gamma''} g_{\beta \gamma''} \delta_{m''_{s_3} m''_{s_1}} \delta_{m''_{s_2} m''_{t_2}} \delta_{m''_{s_1} m''_{t_1}} \delta_{m''_{t_3} m''_{t_3}} \times \langle \textbf{v}_1 m''_{s_1} m''_{s_2} m''_{t_1} m''_{t_1} | t(\epsilon^*) | \textbf{v}_3' m'_{s_3} m'_{s_4} m'_{t_3} m'_{t_4} \rangle \times \left\{ \sum_{\alpha'} g_{\gamma' \alpha'} \langle \textbf{v}_3' \frac{2}{3} \textbf{v}_2 + \frac{2}{3} \frac{1}{2} \textbf{v}_2 - \textbf{v}_3' \alpha' | \psi_1 \rangle - \sum_{\alpha'} g_{\gamma'_{1243} \alpha'} \langle \textbf{v}_3' \frac{2}{3} \textbf{v}_2 - \frac{2}{3} \frac{1}{2} \textbf{v}_2 + \textbf{v}_3' \alpha' | \psi_1 \rangle + \sum_{\beta'} g_{\gamma' \beta'} \langle \textbf{v}_3' \textbf{v}_2 \textbf{v}_3' \beta' | \psi_2 \rangle \right\},
\right.
\]
\]

where \( a\langle |t(\epsilon)\rangle \rangle_a \) and \( a\langle |t(\epsilon^*)\rangle \rangle_a \) are anti-symmetrized NN \( t \)-matrices. This spin-isospin 3D formalism can be simplified to the bosonic case by switching off the spin-isospin quantum numbers (see Refs. [13, 14]).
III. NUMERICAL RESULTS FOR THE THREE- AND FOUR-NUCLEON BINDING ENERGIES

In this section, we present numerical results for the three- and four-nucleon binding energies. The details of the numerical algorithm for solving the coupled three-dimensional integral equations can be found in Refs. [10, 13, 14].

A. Results for NN Potential Models

In order to check our proposed 3D formulation for the three- and four-nucleon bound states, we apply the formalism to the following spin-dependent NN potential models: $S_3$ [19], $YS$ [20], MT-I/III [21] and $P_{5,5\text{GL}}$ [22]. We are aware that realistic NN potentials have already been used even for nuclei with $A > 4$, but the main goal of the present work is the test of the 3D representation of the FY equations for more realistic potentials that we have been used before in such 4B calculations. The parameters of the above potentials are given in Table I.

Our results will be compared to several techniques: the VAR [23] and HH [24]-[27] methods, several types of approximations for the subsystem kernels of the four-body problem by operators of finite rank (SKFR) [28]-[32], the integrodifferential equation approaches SIDE [33] and IDEA [34], the CRC [35], the DFY [1, 36], the FY (PW) [3], and last, but not least, 2DI [20]. Our results for the triton and $\alpha$-particle binding energies are shown in tables II-V in comparison to the results of other techniques. Table II collects the binding energies for the $S_3$ potential, Table III for the YS type potentials, Table IV for the MT-I/III potential, and Table V for the $P_{5,5\text{GL}}$ potential.

As shown in Table II, our result for the $\alpha$-particle binding energy for the spin-dependent (spin-averaged) $S_3$ potential with value $-28.8 \ (-25.7)$ MeV is in good agreement with results of HHE, SIDE, DFY techniques and especially with FY result in PW decomposition. Also, our result for the triton binding energy with values $-8.20$ and $-6.41$ MeV, corresponding to spin-dependent and -averaged versions of this potential, are in excellent agreement with FY results in PW decomposition. It should be pointed that the results with spin-averaged version of the potentials differ from previous results where the original version of the potentials was used. The difference between obtained results of original and averaged versions of
TABLE I. List of parameters of the NN potentials used in this work. Each potential contains two parts, $V_0$ and $V_1$ where the indices 0 and 1 denote the spin of the $2N$ subsystem. Each part is written as a sum of a few terms; each is expressed as $V_{s_i} f(\mu_{s_i}, r(p, p'))$, where $f(\mu_{s_i}, r) = \exp(-\mu_{s_i} r^2)$ for Gauss-type potential, $f(\mu_{s_i}, r) = \exp(-\mu_{s_i} r)/r$ for Yukawa-type potential and $f(\mu_{s_i}, p, p') = \frac{\xi_i^2}{m_i} \frac{(p p')^{2i-2}}{(p+i+\mu_{s_i})! (p'_i+i+\mu_{s_i})!}$ for separable potentials. The potential strengths $V_{s_i}$ are in MeV for $S_3$, in $fm^{-3}$ for YS and $P_{5.5GL}$ and dimensionless for MT-I/III. The range parameters, exchanged masses for MTI/III, $\mu_{s_i}$ are in $fm^{-2}$ for $S_3$ and in $fm^{-1}$ for others. For separable potentials $\xi_1 = 1.0000$ and $\xi_2 = 2.9499$.

| Potential | Type | i  | $V_{0i}$ | $\mu_{0i}$ | $V_{1i}$ | $\mu_{1i}$ |
|-----------|------|----|----------|------------|----------|------------|
| $S_3$     | Gauss | 1  | 1000.0   | 3.00       | 1000.0   | 3.00       |
|           |       | 2  | -326.7   | 1.05       | -166.0   | 0.80       |
|           |       | 3  | 43.0     | 0.60       | 23.0     | 0.40       |
| YS        | Separable | | | | |
| YS-I      |       | 1  | -0.1490  | 1.165      | -0.4160  | 1.450      |
| YS-II     |       | 1  | -0.1430  | 1.150      | -0.3815  | 1.406      |
| YS-III    |       | 1  | -0.1323  | 1.130      | -0.3815  | 1.406      |
| YS-IV     |       | 1  | -0.1323  | 1.130      | -0.3628  | 1.406      |
| MT-I/III  | Yukawa | 1  | 7.39     | 3.110      | 7.39     | 3.110      |
|           |       | 2  | -2.64    | -1.555     | -3.22    | -1.555     |
| $P_{5.5GL}$ | Separable | 1  | 0.13230  | 1.130      | -0.18752 | 1.2766     |
|           |       | 2  | -0.18752 | 1.7610     |           |             |

the potentials is to be expected and it is quite natural.

The calculated triton and $\alpha$-particle binding energies for separable, spin-dependent Yamaguchi type potentials with different methods are listed in Table III. Our results for the $\alpha$-particle (triton) binding energy for YS I, II, III and IV with values $-45.9$ ($-11.05$),
TABLE II. Triton and α-particle binding energies for $S_3$ potential in MeV.

| Method       | $E_t$ | $E_\alpha$ |
|--------------|-------|------------|
| VAR [23]     | -26.47|            |
| HHE [25]     | -26.01|            |
| SIDE [33]    | -8.20 | -27.93     |
| CRC [35]     | -28.74|            |
| DFY [1]      | -28.79|            |
| FY(PW) [3]   | -8.20 | -28.80     |
| FY(3D)       | -8.20 | -28.8      |
| SIDE$^{\text{av}}$ [33] | -25.38|            |
| DFY$^{\text{av}}$ [1]   | -25.50|            |
| HHE$^{\text{av}}$ [24]  | -25.97|            |
| DFY$^{\text{av}}$ [36]  | -25.68|            |
| FY(PW)$^{\text{av}}$ [3] | -6.41 | -25.69     |
| FY(3D)$^{\text{av}}$    | -6.41 | -25.7      |
| Exp.         | -8.48 | -28.30     |

$-44.4 (-10.70), -42.4 (-10.13), -37.8 (-8.47)$ MeV, are in excellent agreement with the 2DI results.

As demonstrated in Table IV, the calculation of the α-particle binding energy by using the spin-dependent and spin-averaged version of MT-I/III potential in the FY(PW) scheme converges to values of $-30.29$ and $-28.83$ MeV, while the triton binding energy converges to values $-8.54$ and $-7.55$ MeV, correspondingly. As shown in this table our calculations for spin-dependent version of this potential yields the values $-8.54$ and $-30.3$ MeV for triton and α-particle binding energies correspondingly, which are in good agreement with the FY (PW) results. Also, our results for the triton and α-particle binding energies with the spin-averaged version of this potential with values $-7.57$ and $-28.8$ MeV are also in excellent agreement with the corresponding FY (PW) results.

In Table V we present the triton and α-particle binding energies for the $P_{5,5}GL$ potential calculated with the SKFR and FY methods. Our results for triton and α-particle binding
TABLE III. $\alpha$-particle binding energy for YS-type potentials in MeV. The numbers in parenthesis are corresponding to triton binding energies.

| Method          | YS-I     | YS-II    | YS-III   | YS-IV    |
|-----------------|----------|----------|----------|----------|
| FY(PW) \[3\]   | -45.87 (-11.05) |          |          |          |
| SKFR \[31\]    | -45.73   |          |          |          |
| SKFR \[29\]    | -45.59   |          |          |          |
| SKFR \[30\]    | -45.32   |          |          |          |
| 2DI \[20\]     | -45.7 (-11.05) | -44.2 (-10.71) | -42.3 (-10.13) | -37.7 (-8.48) |
| FY(3D)          | -45.9 (-11.05) | -44.4 (-10.70) | -42.4 (-10.13) | -37.8 (-8.47) |
| Exp.            |          | -28.30 (-8.48) |          |          |

TABLE IV. Triton and $\alpha$-particle binding energies for Malfliet-Tjon I/III potential in MeV.

| Method          | $E_t$    | $E_\alpha$ |
|-----------------|----------|------------|
| SKFR \[32\]    | -29.6    |            |
| SKFR \[28\]    | -30.36   |            |
| SIDE \[33\]    | -8.54    | -29.74     |
| DFY \[1\]      | -8.54    | -30.31     |
| IDEA \[34\]    | -8.86    | -30.20     |
| HH \[27\]      |          | -30.33     |
| EIHH \[26\]    | -8.72    | -30.71     |
| DFY(PW) \[1\]  |          | -30.312    |
| FY(PW) \[3\]   | -8.54    | -30.29     |
| FY(3D)          | -8.54    | -30.3      |
| FY(PW)$^{av}$ \[3\] | -7.55 | -28.83     |
| FY(3D)$^{av}$   | -7.55    | -28.8      |
| Exp.            |          | -8.48      | -28.30   |
TABLE V. Triton binding energy for $P_{3,5}GL$ potential in MeV. The numbers in parenthesis are $\alpha$-particle binding energies.

| Method       | $E_t$          |
|--------------|----------------|
| SKFR [29]    | -29.10         |
| FY(PW) [3]   | -28.87 (-8.04) |
| FY(3D)       | -28.9 (-8.04)  |
| Exp.         | -28.30 (-8.48) |

energies with values $-8.04$ and $-28.9$ MeV are in excellent agreement with the corresponding PW results. In the next section, we present our results for binding energies with the inclusion of 3NFs.

B. Results for $NN$ with $3N$ Potential Models

In our calculations with a 3NF, we use a model of the 3NF which is based on multi-meson exchanges. We study two different types of 3NFs, a purely attractive and a superposition of attractive and repulsive, which are named MT3-I and MT3-II respectively, Ref. [12]. The parameters of these 3NFs are chosen so that the correction due to these 3NFs to the triton binding energy calculated with the modified Malfliet-Tjon (MT2-II) NN potential is small, and they lead to binding energies near to the experimental triton binding energy.

As shown in Table VII our results for the $\alpha$-particle (triton) binding energies with the addition of the MT3-I and MT3-II 3NFs, while the averaged version of MT-I/III is used as the NN potential, are $-35.7 (-8.68)$ and $-34.5 (-8.45)$ MeV, respectively. Unfortunately we could not compare these results for binding energies with other calculations, but we have listed our recent results with different combination of MT-V NN potential and mentioned 3N potential models, i.e. MT3-I and MT3-II, [14]. As one can see from the comparison of our results with and without 3NFs (while MT-I/IIIave is used as NN potential model) with the previously calculated binding energies (while MT-V is used as NN potential model) the MT-I/IIIave NN potential model provide more reasonable results in comparison to MT-V for triton and $\alpha$-particle binding energies.
TABLE VI. Triton and $\alpha$-particle binding energies with and without 3NFs in MeV.

| Potential | $E_t$  | $E_\alpha$ |
|-----------|--------|------------|
| MT-I/III $^{\text{ave}}$ | -7.55  | -28.8      |
| MT-I/III $^{\text{ave}}$+MT3-I | -8.68  | -35.7      |
| MT-I/III $^{\text{ave}}$+MT3-II | -8.45  | -34.5      |
| MT-V $^{[13]}$ | -7.74  | -31.3      |
| MT-V+MT3-I $^{[14]}$ | -8.92  | -38.8      |
| MT-V+MT3-II $^{[14]}$ | -8.70  | -37.5      |
| Exp. | -8.48  | -28.30     |

All these numbers are not meant to provide insight into the physics of three and four interacting nucleons, but have the purpose to demonstrate the high accuracy of numerical results that one can obtain by considering the present non-PW approach, in face of other existent methods. The advantages of the method relies in a simplified and straightforward formalism, which is appropriate to treat typical nuclear forces consisting of attractive and repulsive (short range) parts. The results presented indicate that the 3D approach leads to numerical results with the same accuracy of PW-based methods, whereas it leads to integral equations with much less analytical and algebraic complexity in comparison to corresponding equations formulated in PW-based methods. In a 3D case, there are only a finite number of coupled three-dimensional integral equations to be solved; whereas, in the PW case, after truncation, one has a finite number of coupled equations with kernels containing relatively complicated geometrical expressions.

IV. SUMMARY AND OUTLOOK

In summary, in the present paper we solve the FY three-dimensional integral equations for spin-dependent and spin-averaged NN potential models, i.e., $S_3$, MT I/III, YS-type and $P_{5,5}GL$ and the scalar two-meson exchange three-body interaction. These potentials provide reasonable results for binding energies in comparison to the potential models that have been used in previous works. Our results for these potential models are in good agreement with
the corresponding previous values when considering VAR, HHE, SKFR, SIDE and DFY techniques. In particular, they are matched with PW calculations in the FY scheme.

This non-PW approach, by directly working with momentum vector variables, is being revealed as an efficient good alternative to other methods to treat three- and four-nucleon bound-state calculations. Recently, following this approach, the coupled FY equations have been formulated with and without 3NFs, as a function of vector Jacobi momenta, where the formalism is given in terms of the magnitudes of the momenta and the angles between them. It has been demonstrated that the three-dimensional FY integral equations can be handled in a straightforward and numerically reliable fashion. In comparison to commonly used angular momentum decompositions, this direct approach leads to a finite number of coupled equations with kernels containing very simplified expressions.

It should be clear that this approach is more efficient for scattering problems, especially in the energy regions where the PW-based calculations have slow convergence. The formulation of 3N scattering and $^3$H photodisintegration in a realistic 3D approach has been done successfully [37]-[38] and the calculation is underway. Molecular, atomic, and nuclear or subnuclear physics are but a few examples of various fields of physics where quantum mechanical few-body problems play an important role. Since the 3D approach is general, it can be applied to any system from molecules to elementary particles. Another valuable application of this non-PW approach, are the few-body atomic bound states with realistic potentials.

We should also mention a renormalization group approach that our group has considered when solving integral equations for the nucleon-nucleon (NN) interaction [39]. In leading order, by using the one-pion-exchange potential (OPEP) plus a Dirac-delta function, is considered a non-perturbative renormalization procedure, relying on a subtracted kernel where a scaling parameter is introduced. The role of the scaling parameter is similar to the cut-off momentum parameter but with a big advantage in view of its flexibility. Since the approach is renormalization group invariant, one can arbitrarily move the reference scale without affecting the relevant physical results. An extension of this approach is being submitted for publication [40], where a recursive subtraction procedure is applied to the scattering matrix solution with next-leading-order (NLO) and next-to-next-leading-order (NNLO) two-pion exchange interactions. Also, we are considering the application of the present 3D approach for the NN interaction in the renormalization group scheme that was
used in Ref. [39].

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