Differential topology interacts with isoparametric foliation

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Abstract In this note, we discuss the interactions between differential topology and isoparametric foliation, with some recent progress surveyed, some open problems proposed as well as some examples and suggesting approaches provided.

1 Introduction

As is well known, two of the main topics that differential topology concerns are smooth structures on manifolds and smooth mappings between manifolds. Since the surprising discovery of exotic spheres by Milnor \cite{Milnor1956} in 1956, existence and non-existence of exotic smooth structures have induced worldwide attentions and highly intensive studies. Recall that an exotic sphere is a closed smooth manifold which is homeomorphic but not diffeomorphic to the unit sphere $S^n$. In 1963, Kervaire and Milnor \cite{Kervaire1963} gave a detailed investigation into the group $\Theta_n$ of h-cobordism classes of oriented homotopy $n$-spheres. Here a homotopy $n$-sphere $\Sigma^n$ is a closed smooth manifold which has the homotopy type of $S^n$, and $S^n$ is always homeomorphic to $S^n$ (cf. \cite{Milnor1956}). It is well known that $\Theta_n$ is isomorphic to $I_n$, the group of oriented twist $n$-spheres. What is more, according to Cerf \cite{Cerf1958}, $\Theta_n$ is isomorphic to the mapping class group $\pi_0\text{Diff}^+(S^{n-1})$ by $\pi_0\text{Diff}^+(S^{n-1}) \to \Theta_n = I_n, [\phi] \mapsto \Sigma_\phi := D^n \cup_\phi D^n$. Note that $\Sigma_\phi$ depends only on the isotopy class of $\phi \in \text{Diff}^+(S^{n-1})$. Motivated by $\Theta_n$, two inertia groups $I_0(M)$ and $I_1(M)$ are defined for a closed oriented manifold $M$.

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In fact, \( I_0(M) \) is related to the study of exotic smooth structures on \( M \), and
\( I_1(M) \) contributes to the group \( \Gamma(M) \) of \( \Gamma \)-pseudo-isotopy classes of diffeomorphisms
of \( M \).

On the one hand, from the local viewpoint of Riemannian geometry, one of
the central problems is to determine the classes of manifolds with special curva-
ture properties, for instance, manifolds with positive/nonnegative sectional, Ricci
or scalar curvature. Therefore, the curvature properties of manifolds with exotic
smooth structures, especially exotic spheres, are very interesting. We refer to [21]
and [24] for more details and the progress of this subject.

On the other hand, from the global viewpoint, it is rather fascinating to study
singular Riemannian foliations on manifolds, especially isoparametric foliations,
which are geometric generalizations of manifolds with isometry group actions. It
was E. Cartan who firstly gave a through study of isoparametric foliations on the
unit spheres in 1930’s. Up to now, the study of isoparametric foliations has become
a highly influential field in differential geometry. We recommend [41], [3], [6], [16],
[39], [38] and [35] for a systematic and complete survey of isoparametric foliations
and their applications.

This note is organized as follows. In Section 2, we will recall the basic notations.
In Section 3, we will discuss the classification of 4-manifolds with singular Rie-
mannian foliations. In Section 4, we will pay attention to isoparametric foliations
(even singular Riemannian foliations with codimension 1) on homotopy spheres. In
Section 5, some applications to exotic smooth structures will be concerned with.

2 Singular Riemannian foliation and isoparametric foliation

In this section, we will recall the basic notations and results on singular Rie-
mannian foliations (cf. [29, 43, 19, 2, 42]). Let \( M \) be a complete Riemannian manifold. A
transnormal system \( \mathcal{F} \) is a decomposition of \( M \) into connected injectively immersed
submanifolds, called leaves, such that each geodesic emanating perpendicularly to
one leaf remains perpendicular to the leaves at all its points. A singular Riemannian
foliation (SRF) is a transnormal system \( \mathcal{F} \) which is also a singular foliation, i.e.,
such that there are smooth vector fields \( X_i \) on \( M \) that span the tangent spaces
\( T_pL_p \) to the leaf \( L_p \) through each point \( p \in M \). A leaf of maximal dimension is called a
regular leaf, and its codimension is defined to be the codimension of \( \mathcal{F} \). Leaves of
lower dimensions are called singular leaves. In particular, a singular Riemannian fo-
liation all of whose leaves have the same dimension is called a (regular) Riemannian
foliation.

A singular Riemannian foliation \((M, \mathcal{F})\) of codimension 1 is called an isopara-
metric foliation if all the regular leaves have constant mean curvature. Each regular
leaf of an isoparametric foliation is called an isoparametric hypersurface, and the
singular leaves are called focal submanifolds. Moreover, if each regular leaf has
constant principal curvatures, then \( \mathcal{F} \) is called a totally isoparametric foliation (cf.
[18]). According to [43], there is essentially a correspondence between transnormal
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(resp. isoparametric) functions on $M$ and transnormal systems (resp. isoparametric foliations) of codimension 1 on $M$.

Next to the basic notations, we are in a position to provide some interesting examples of SRF.

**Example 2.1**

(1). Homogeneous SRF:

- Let $G$ be a Lie group that acts on a Riemannian manifold $M$ by isometries, and $\mathcal{F}$ be the set of orbits of $G$. Then $\mathcal{F}$ is a singular Riemannian foliation on $M$, called homogeneous SRF. If the codimension of $\mathcal{F}$ is equal to 1, i.e., the isometric action is a cohomogeneity one action, then $\mathcal{F}$ is a homogeneous isoparametric foliation. And homogeneous isoparametric foliations must be totally isoparametric foliations.

(2). Isoparametric hypersurfaces in unit spheres ($\mathbb{S}^n$):

Due to É. Cartan, isoparametric hypersurfaces in real space forms have constant principal curvatures, and the spherical case is more complicated. Let $g$ be the number of distinct principal curvatures. For isoparametric hypersurfaces in $\mathbb{S}^n$, $g$ must be $1, 2, 3, 4$ or $6$. If $g=1, 2, 3$ or $6$, isoparametric hypersurfaces are all homogeneous (cf. [28]). For the case $g=4$, isoparametric hypersurfaces must be OF-FKM type or homogeneous except for one case (cf. [3, 2]). Consequently, there exist infinitely many nonhomogeneous, but totally isoparametric foliations on unit spheres.

(3). Isoparametric (Equifocal) hypersurfaces in compact symmetric spaces:

In the complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric, every totally isoparametric foliation must be homogeneous (cf. [18]). Based on the classification result for $\mathbb{S}^n$, [29] obtained the classification of isoparametric foliations on $\mathbb{C}P^n$ except for one case. So far, the problem for classification of isoparametric foliations on compact symmetric spaces is far from being touched, except for the cases $\mathbb{S}^n$ and $\mathbb{C}P^n$.

(4). SRF and Regular Riemannian foliation on (homotopy) spheres:

Homogeneous SRF and nonhomogeneous SRF are abundant on unit spheres (cf. [40, 36]). Compared to SRF, regular Riemannian foliations are rather rare on unit spheres, even on homotopy spheres. They occur only when the dimension of the leaves is $1, 3$ or $7$ (cf. [26]). However, a topological classification is still unknown so far.

Motivated by the examples (2-3) above, it is natural to pose the following

**Question 2.2** Is every totally isoparametric foliation on a compact symmetric space other than $\mathbb{S}^n$ a homogeneous isoparametric foliation?

Now, we turn to the topological side of SRF with codimension 1. In this note, by a foliated diffeomorphism between two foliated manifolds, we mean a diffeomorphism maps leaves to leaves, and such foliations are called equivalent. And two equivalent foliated manifolds are denoted by $(M, \mathcal{F}) \cong (M', \mathcal{F'})$. Note that the foliated diffeomorphism here needs not to be an isometry.

According to [29], a codimension 1 SRF $\mathcal{F}$ on a closed simply connected manifold $N$ has exactly two closed singular leaves $M_{\pm}$, and $N$ has a decomposition of two disk bundles $E_\pm$ of the normal vector bundles $\xi_\pm$ over $M_{\pm}$ of rank $m_\pm > 1$, i.e.,
$N \cong E_\phi := E_+ \cup_\phi E_-$ with the gluing diffeomorphism $\phi : \partial E_+ \to \partial E_-$ (cf. [14]). The codimension 1 SFR $\mathcal{F}$ on $N$ induces a codimension 1 SRF $\mathcal{F}_\phi$ on $E_\phi$. The leaves of $\mathcal{F}_\phi$ are just the concentric tubes around the zero sections in $E_\pm$. Hence, the equivalence class of $(N, \mathcal{F})$ can be represented by $(E_\phi, \mathcal{F}_\phi)$.

Conversely, given disk bundles $E_\pm$ over $M_\pm$ with rank greater than 1 and a diffeomorphism $\phi : \partial E_+ \to \partial E_-$, and let $\mathcal{F}_\phi$ be the foliation consisting of concentric tubes on $E_\phi := E_+ \cup_\phi E_-$. By the fundamental construction theorem in [34], there exists a bundle-like metric $g_\phi$ such that $(E_\phi, \mathcal{F}_\phi)$ becomes an isoparametric foliations. It follows that isoparametric foliations on closed simply connected manifolds require no more on the topology than codimension 1 SRF. Moreover, to study classification of equivalence classes of codimension 1 SRF, one only needs to study foliations in the form $(E_\phi, \mathcal{F}_\phi)$ determined by two pairs of disk bundles $E_\pm \subset \xi_\pm$ and gluing diffeomorphisms $\phi : \partial E_+ \to \partial E_-$. 

3 Classification of 4-manifolds with SRF

In this section, we will be concerned with the singular Riemannian foliations on simply connected 4-manifolds. Geometry and topology of 4-manifolds form an extremely rich and also complicated research field. This is the lowest dimension in which exotic smooth structures arise, e.g., the noncompact 4-spaces $\mathbb{R}^4$ and compact $m\mathbb{C}P^2 \# n\mathbb{C}P^2$ for many pairs of $(m \geq 1, n \geq 2)$ (cf. [12, 1]). Up to now, it is not known whether there is a 4-manifold with only one smooth structure, even for $S^4$, the affirmative side of which is called the smooth Poincaré conjecture. Therefore, it is natural to study 4-manifolds with additional structures, especially 4-manifolds with SRF. The partial classification results are listed as the following.

**Theorem 3.1** Known classification results with SRF:

(1) Homogeneous SRF:

a) If a closed simply connected 4-manifold $N$ admits a cohomogeneity one action, then $N$ is diffeomorphic to one of the standard manifolds $S^4$, $\mathbb{C}P^2$, $S^2 \times S^2$, or $CP^2 - \mathbb{C}P^2$ ([33, 20, 27, 22])

b) If a closed simply connected 4-manifold $N$ admits a $T^2$ action (with cohomogeneity two), then $N$ is diffeomorphic to a connected sum of copies of standard $S^4$, $\pm \mathbb{C}P^2$ and $S^2 \times S^2$ ([30, 31]).

b) If a closed simply connected 4-manifold $N$ admits a $S^1$ action (with cohomogeneity three), then $N$ is diffeomorphic to a connected sum of copies of standard $S^4$, $\pm \mathbb{C}P^2$ and $S^2 \times S^2$ ([10, 77]).

(2) General SRF:

a) If a homotopy 4-sphere $\Sigma^4$ admits a codimension 1 SRF, then $\Sigma^4$ is diffeomorphic to $S^4$ ([17]).

b) If a closed simply connected 4-manifold $N$ admits a SRF $\mathcal{F}$ of codimension 3, then $\mathcal{F}$ must be a homogeneous SFR induced by some $S^1$ action ([13]).
Based on the results above, Ge and Radeschi \cite{15} completely classified closed simply connected 4-manifolds with SRF of general codimension, which generalizes all the differentiable classification results with SRF in the literature.

**Theorem 3.2 (\cite{15})** Let $N$ be a closed simply connected 4-manifold.

(a) If $N$ admits a SRF of codimension 1, then $N$ is diffeomorphic to one of the 5 standard manifolds $S^4$, $\mathbb{CP}^2$, $S^2 \times S^2$, or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.

(b) If $N$ admits a SRF of codimension greater than 1, then $N$ is diffeomorphic to a connected sum of copies of standard $S^4$, $\pm \mathbb{CP}^2$ and $S^2 \times S^2$.

Moreover, as a corollary of Theorem 3.2, they obtained a complete differentiable classification of codimension 1 SRF on closed simply connected 4-manifolds.

**Corollary 3.3 (\cite{15})** Let $N$ be a closed simply connected 4-manifold admitting a SRF $\mathcal{F}$ of codimension 1 with regular leaf $M$ and two singular leaves $M_{\pm}$. Then the foliated diffeomorphism classification of $(N, \mathcal{F})$ in terms of $(M, M_{\pm})$ is given in the following Table 1:

| $N$       | $\mathcal{F}$ | $M$       | $M_{\pm}$ | Homog | T-Isopar | Isopar |
|-----------|---------------|-----------|-----------|-------|----------|--------|
| $S^4$     | $L(1, 1)$     | $pt$, $pt$|           | Yes   | Yes      | Yes    |
|           | $L(0, 1)$     | $S^1$, $S^2$|          |       |          |        |
|           | $SO(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | $\mathbb{RP}^2$, $\mathbb{RP}^2$| |       |          |        |
| $\mathbb{CP}^2$ | $L(1, 1)$     | $pt$, $S^2$|           |       |          |        |
|           | $L(4, 1)$     | $\mathbb{RP}^2$, $S^2$| |       |          |        |
| $S^m \times S^n$ | $L(2m, 1)$, $m > 0$ | $S^m$, $S^n$| | No |          |        |
| $\mathbb{CP}^2 \# \mathbb{CP}^2$ | $L(1, 1)$     | $S^2$, $S^2$| | Yes |          |        |
|           | $L(2, 1)$     | $S^2$, $S^2$| |       |          |        |
| $\mathbb{CP}^2 \# - \mathbb{CP}^2$ | $L(2m + 1, 1)$, $m > 0$ | $S^m$, $S^2$| | Yes |          |        |
|           | $L(0, 1)$     | $S^2$, $S^2$| | No |          | Unknown |

where the column “Homog” (resp. “T-Isopar”, “Isopar”) means whether there exist a homogeneous (resp. totally isoparametric, isoparametric) representative in the foliated diffeomorphism class.

To conclude this section, the following conjecture (essentially due to K. Grove) related to non-negatively curved manifolds is stated as follows.

**Conjecture 3.4 (\cite{15})** Any closed simply connected non-negatively curved manifold admits a SRF of codimension 1, under the same metric or a different bundle-like metric.

An affirmative answer to the conjecture above will solve affirmatively

**Conjecture 3.5 (\cite{15, 20})** A closed simply connected non-negatively curved 4-manifold is diffeomorphic to one of the 5 standard manifolds $S^4$, $\mathbb{CP}^2$, $S^2 \times S^2$, or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.

Note that the Conjecture 3.5 is still open even for the homeomorphism case.
4 Isoparametric foliations on homotopy spheres

The study of codimension 1 SRF and isoparametric foliation on homotopy spheres was initiated by Ge and Tang in [17]. Based on their research, they proposed the following problem.

**Problem 4.1** ([17]) Are there always isoparametric foliations on $\Sigma^n$ ($n \neq 4$) with the same focal submanifolds as those on $S^n$?

To attack this problem, Qian and Tang in [34] obtained a fundamental construction, as mentioned in Section 2, which answered the Problem 4.1 partially. In particular, they proved that each homotopy sphere $\Sigma^n$ ($n > 4$) admits an isoparametric foliation with two points as the focal submanifolds.

In [14], the following theorem has been proved, which solved the Problem 4.1 completely.

**Theorem 4.2** ([14]) For each homotopy sphere $\Sigma^n$ ($n > 4$), there exists a 1-1 correspondence between the sets of equivalence classes (under foliated diffeomorphisms) of isoparametric foliations on $\Sigma^n$ and $S^n$.

Consequently, the classification of equivalence classes of isoparametric foliations on homotopy spheres is equivalent to the case on $S^n$. However, this classification on $S^n$ ($n > 4$) is still far from reached though almost completed for the round metric. For the special case when the focal submanifolds are two points, by usage of differential topology, [14] established

**Theorem 4.3** ([14])

1) Every homotopy sphere $\Sigma^n$ ($n \neq 4, 5$) and $S^4$ admit exactly one equivalent class of isoparametric foliations with two points as focal submanifolds.

2) Either $S^5$ admits nonequivalent isoparametric foliations with two points as focal submanifolds, or $\pi_0(\text{Diff}(S^4)) \cong \mathbb{Z}_2$ (i.e., pseudo-isotopy implies isotopy on $S^4$).

In fact, for the general case, [14] also observed some new and somehow exotic examples, and investigated a feasible scheme:

(Step 1) Classify disk bundles $E_\pm$ which decompose the sphere as

$$\mathbb{S}^n = E_\phi = E_+ \bigcup_\phi E_-.$$  

There exist many “exotic” disk bundles $\tilde{E}_\pm \cong E_\pm$. Hence there are many non-equivalent isoparametric foliations $(\tilde{E}_\phi, \tilde{\mathcal{F}}_\phi) \not\cong (E_\phi, \mathcal{F}_\phi)$:

(a) $S^m \times D^k$ admits non-trivial disk bundle structures for (cf. [8])

$$(m, k) = (7, 4), (8, 4), (9, 4), (11, 4), (11, 5), (11, 6).$$

Therefore each of $S^{11}, S^{12}, S^{13}, S^{15}, S^{16}, S^{17}$ admits non-equivalent isoparametric foliations whose focal submanifolds are all standard spheres $(S^m, S^{k-1})$.

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Here by “exotic” we mean non-equivalent as disk bundles but with diffeomorphic total spaces.
(b) The tangent bundles of homotopy spheres $T\Sigma^n \cong T\mathbb{S}^n$ are “exotic”. Therefore, for instance, $\mathbb{S}^{14}$ admits 15 (ignore orientation) non-equivalent isoparametric foliations whose focal submanifolds are $(\Sigma^7, \mathbb{S}^6)$ for $\Sigma^7 \in \Theta_7 \cong \mathbb{Z}_{28}$.

(Step 2) Solve the question: For $\phi : \partial E_+ \to \partial E_- (i = 0, 1)$, $E_{\phi_i} \cong E_{\phi_1}$, when

$$(E_{\phi_1}, \partial E_{\phi_1}) \cong (E_{\phi_1}, \partial E_{\phi_1})?$$

It relies heavily on the study of diffeomorphism groups, especially on the (pseudo-) isotopy theory. For instance, a nearly sufficient condition is that all diffeomorphisms of the boundaries of $E_{\phi}$ that extendable to the total spaces are isotopic to bundle isomorphisms (cf. Theorem 2.6 of [14]). Theorem 4.3 above was derived from this investigation and known results in the (pseudo-) isotopy theory.

5 Application to existence of exotic smooth structures

We start with the definitions of two inertia groups. Let $M^n$ be a closed oriented manifold. Recall $I_0(M) \subset \Theta_n$ and $I_1(M) \subset \Theta_{n+1}$ are two inertia groups of $M$ (cf. [25]).

$I_0(M)$ consists of all $\Sigma \in \Theta_n$ such that $M#\Sigma \cong M$. For $\Sigma \in \Theta_n \setminus I_0(M)$, it is evident that $M#\Sigma \not\cong M$ is homeomorphic to $M$ and hence induces an exotic oriented smooth structure on $M$. Moreover, different cosets in $\Theta_n/I_0(M)$ give distinct oriented smooth structures. Hence, there exist at least $|\Theta_n|/|I_0(M)|$ distinct oriented smooth structures on $M$.

To define $I_1(M)$, we first recall the disk theorem in [32], i.e., any orientation-preserving diffeomorphism is isotopic to one that restricts to the identity on an embedded disk. Thus, for any $\phi \in \text{Diff}^{+}(\mathbb{S}^n)$, we can assume $\phi : \mathbb{S}^n = D^n_+ \cup_{id} D^n_- \to D^n_+ \cup_{id} D^n_-$ satisfy $\phi|_{D^n_+} = id$ up to isotopy. Then $I_1(M)$ consists of all $\Sigma_{\phi} \in \Theta_{n+1}$ such that the diffeomorphism of $M$ which differs from identity only on an $n$-disk in $M$, and there coincides with $\phi$, is concordant to the identity. The coset space $\Theta_{n+1}/I_1(M)$ corresponds to a subset of $|\Theta_{n+1}|/|I_1(M)|$ elements in $\Gamma(M)$ (also in $\pi_0(\text{Diff}^{+}(M))$).

In [25], Levine established an elegant relation between two inertia groups, i.e., $I_1(M) = I_0(M \times \mathbb{S}^1)$. Inspired by study of isoparametric foliations on homotopy spheres, the following observation on inertia groups is acquired in [14].

**Theorem 5.1 ([14])** Let $M^{n-1}$ be a closed oriented embedded hypersurface in a closed oriented manifold $N^n$. Then $I_1(M^{n-1}) \subseteq I_0(N^n)$. Consequently, $I_0(M^{n-1} \times \mathbb{S}^1) \subseteq I_0(N^n)$, i.e., $M^{n-1} \times \mathbb{S}^1$ has the smallest $I_0$ among all $n$-manifolds containing $M$.

As an application of the theorem above, it follows from the fact the product of standard spheres has $I_0 = 0$ (cf. [34]) that

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3 Here by “nearly” we emphasize: it is not known that whether the diffeomorphism between $E_{\phi_0}$ and $E_{\phi_1}$ maps each of $E_{\phi}$ to itself, hence Theorem 2.6 of [14] might be not applicable.
Theorem 5.2 (14) Let $M^{n-1}$ be a closed embedded hypersurface in $\mathbb{S}^n$. Then there exist at least $|\Theta_n + k|$ distinct oriented smooth structures on $M^{n-1} \times P^k \times S^1$, where $P^k = S^{k_1} \times S^{k_2} \times \cdots \times S^{k_l}$ is a product of standard spheres of total dimension $k = \sum_{i=1}^l k_i \geq 0$.

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