Are There Four-Dimensional Small Black Rings?

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In \(d > 4\) dimensions, one can argue for the existence of small black rings using a scaling argument. We apply the same scaling argument to the \(d = 4\) case and demonstrate that it fails to say anything about the existence of \(d = 4\) small black rings, because stringy corrections get out of control. General relativity theorems say that there does not exist a black hole with toroidal topology for \(d = 4\), but we interpret this as saying that, for \(d = 4\) small black rings, stringy corrections are crucial which invalidate the assumptions those theorems are based on.
1 Introduction and Conclusion

Black holes have always been fascinating objects which deepen our understanding not only of general relativity but also of gauge theory and string theory. The discovery of the five-dimensional black ring [1] demonstrated that the uniqueness theorems of general relativity can take very different forms in different dimensions. On the other hand, recent technologies [3–12] to incorporate higher-derivative corrections predicted by string theory have enabled us to show the existence of small black holes, which cannot exist in general relativity only with a two-derivative action.

The existence of small black holes was predicted by Sen more than a decade ago by using a scaling argument [13]. In [14], a similar scaling argument was applied to argue for the existence of supersymmetric small black rings in $d \geq 5$ dimensions, which goes as follows. First, one constructs the solutions corresponding to “small black rings”, at the level of SUGRA. These solutions do not have finite horizons and actually have naked singularities. However, in the region not too close to the singularity, where curvature is much smaller than the string and Planck scales, the SUGRA solution must be trustable. Next, in this region, near the singularity, one finds the scaling region where i) moduli dependence drops out of all the fields, namely metric, $B$-field and dilaton, and ii) charge dependence enters in the solution only in a certain combination, and iii) the curvature is much smaller than the string and Planck scales so the solution is trustable. Property i) is expected due to the attractor mechanism [15–21]. If one goes further near the singularity, then the curvature becomes large and the condition iii) breaks down; stringy corrections are important. But because of the properties i) and ii), the higher derivative corrections can change the SUGRA solution only in a specific combination, and as a result, assuming that the higher derivative corrections create an event horizon, one can predict the charge dependence of the horizon geometry. Then, using the Bekenstein-Hawking-Wald relation, one can predict the entropy formula of the small black ring, which indeed reproduces the correct charge dependence of entropy predicted from the microscopic arguments, up to a numerical constant factor.

In this article we apply the same argument to the small black ring in asymptotically flat $d = 4$ spacetime to see whether it goes through in this case or not. Microscopically, this system is a fundamental heterotic string with momentum charge, winding charge, and angular momentum and, as far as the microscopic entropy counting is concerned, there is no difference

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1 For recent progress in the studies of the phase structure of higher dimensional black holes and rings (“blackfolds”), see, e.g., [2].

2 By scaling the charges appropriately, one can make the dilaton as small as one wants, using the properties i) and ii). Therefore, $g_s$ corrections are suppressed and only $\alpha'$ corrections are important.
between $d = 4$ and $d > 4$ cases. Macroscopically, on the other hand, although the argument is very similar to the higher dimensional cases, we will show that there is one crucial difference: in the region where all the conditions i)–iii) were satisfied for $d > 4$, there is one condition that fails. Namely, in this region, although i) and iii) still hold, ii) is not satisfied for $d = 4$. If we go further near the singularity, the condition ii) can be satisfied, but now the condition iii) does not hold. Therefore, for $d > 4$, there is no region where all i)–iii) are satisfied.

This suggests the following: for $d > 4$, the properties of small black rings can essentially be understood at the level of SUGRA, even before we take into account the higher derivative corrections. The small black ring spacetime is determined in the scaling region, and stringy higher derivative corrections only modifies the geometry further near the singularities to create horizons. On the other hand, for $d = 4$, there is no scaling region accessible by SUGRA, and SUGRA fails to say anything about the existence of small black rings. This seems to suggest that, as far as we start from SUGRA and take into account higher derivative correction perturbatively, it is impossible to construct small black rings in $d = 4$.

In the next section, we give a more detailed analysis which leads to the conclusion above.

In the following section, we discuss our result in connection with the known no-go theorems of four-dimensional black rings.

## 2 Small black rings in four dimensions

### 2.1 $d$ dimensions

Consider heterotic string in $\mathbb{R}_t \times \mathbb{R}^{d-1} \times S^1 \times T^{9-d}$ with $4 \leq d \leq 9$ with $\mathbb{R}_t \times \mathbb{R}^{d-1}$ denoting $d$-dimensional Minkowski space. Let the radius of $S^1$ be $R_d$, and the volume of $T^{9-d}$ be $(2\pi)^{9-d} \alpha'(9-d)/2$. Our objects of interest are the BPS elementary string excitations in this theory carrying $n$ units of momentum and $-w$ units of winding charge along the circle $S^1$ and angular momentum $J$ in a two dimensional plane of $\mathbb{R}^{d-1}$. The fundamental string looks like a ring in the $d$ dimensional spacetime $\mathbb{R}_t \times \mathbb{R}^{d-1}$ and becomes a small black ring [22, 23] for $d = 5$ and is expected to be so for $d \geq 6$. These small black rings carry charges $n, w, J$ as well as charge $Q$ which corresponds to the winding number along the ring direction.

The metric for a small black ring for general $d$ is, at the two derivative level [14,24],

\[
d s^2_{str,d+1} = f_f^{-1}[-(dt - A_i dx^i)^2 + (dx^d - A_d dx^i)^2 + (f_p - 1)(dt - dx^d)^2] + dx_{d-1}^2
\]

\[e^{2\phi_{d+1}} = g^2 f_f^{-1}, \quad B_{td} = -(f_f^{-1} - 1), \quad B_{ti} = -B_{di} = f_f^{-1} A_i, \quad (2.1)\]
where $i = 1, 2, \ldots, d - 1$, and

\[
\begin{align*}
  f_f &= 1 + \frac{Q_f}{R^{d-3}} \left( \frac{x - y}{-2y} \right)^{(d-3)/2} 2F_1 \left( \frac{d - 3}{4}, \frac{d - 1}{4}; 1; -\frac{1}{y^2} \right), \\
  f_p &= 1 + \frac{Q_p}{R^{d-3}} \left( \frac{x - y}{-2y} \right)^{(d-3)/2} 2F_1 \left( \frac{d - 3}{4}, \frac{d - 1}{4}; 1; -\frac{1}{y^2} \right), \\
  A_i dx^i &= -\frac{d - 3}{2} \frac{q}{R^{d-5}} \frac{(y^2 - 1)(x - y)^{(d-5)/2}}{(-2y)^{(d-1)/2}} 2F_1 \left( \frac{d - 1}{4}, \frac{d + 1}{4}; 2; 1 - \frac{1}{y^2} \right) d\psi.
\end{align*}
\]

The $(x, y)$ coordinate system is defined by (see Appendix A):

\[
\begin{align*}
  dx_{d-1}^2 &= \frac{R^2}{(x - y)^2} \left[ \frac{dy^2}{y^2 - 1} + (y^2 - 1)d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2)d\Omega_{d-4}^2 \right].
\end{align*}
\]

The range of the coordinates is $-1 \leq x \leq 1$, $y \leq -1$, $0 \leq \psi \leq 2\pi$.

The various quantities are written in terms of microscopic quantities as

\[
\begin{align*}
  q &= \frac{16\pi G_d}{(d-3)\Omega_{d-2}\alpha'} Q, \quad R^2 = \alpha' \frac{J}{Q}, \quad Q_f &= \frac{16\pi G_d R_d}{(d-3)\Omega_{d-2}\alpha'} w, \quad Q_p = \frac{16\pi G_d}{(d-3)\Omega_{d-2}R_d} n.
\end{align*}
\]

Here $\Omega_D$ is the area of unit $D$-sphere and $G_d$ is the $d$-dimensional Newton constant obtained by regarding the $S^1$ direction as compact,

\[
16\pi G_d = \frac{16\pi G_{d+1}}{2\pi R_d} = \frac{(2\pi)^{d-3}g^2\alpha'/(d-1)/2}{R_d}.
\]

$n, w, J, Q$ are integers.

The strategy of [14] was to find a “scaling region” in spacetime where

i) all the moduli dependence drops out from the solution (metric, dilaton, and $B$-field), and

ii) charge dependence shows up only in specific combinations in the solution, and

iii) the curvature is small enough for the metric obtained from the two-derivative action to be trustable.

If one goes closer to the ring singularity, the condition iii) above breaks down and the solution obtained by two-derivative action is no longer trustable. In this strongly coupled region, higher derivative correction will modify the two-derivative solution, but it must do so in a specific way determined by the specific charge combinations appearing in the metric in the weakly curved region. This allows one to exactly determine the charge dependence of the higher-derivative corrected metric, up to some unknown functions. That is enough to show
that the entropy derived from the Wald entropy formula agrees with the microscopic entropy up to a factor.

The $d > 4$ case was discussed in [14], and it was shown that the all conditions i)–iii) can indeed be met, if we go close enough to the ring (i.e., we take $|y|$ to be large enough), but not too close (i.e., we keep $|y|$ to be not too large) so that the curvature is small. We will see that for a $d = 4$ small black ring, if we go close enough to the ring so that i) and ii) are satisfied, the last condition iii) is no longer satisfied.

### 2.2 Four dimensions

Let us focus on the $d = 4$ case. In this case there is no $dΩ_{d-4}$ term in (2.3), and by defining $x = \cos φ$,

$$
dx_3 = \frac{R^2}{(x - y)^2} \left[ \frac{dy^2}{y^2 - 1} + (y^2 - 1)dψ^2 + dφ^2 \right]. \quad (2.7)$$

The harmonic functions are

$$f_f = 1 + \frac{Q_f}{\pi R} \sqrt{\frac{x - y}{-2y}} 2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{y^2}\right) = 1 + \frac{2Q_f}{\pi R} \sqrt{\frac{x - y}{-2y}} \frac{K(\frac{2z}{z - 1})}{\sqrt{1 - z}},$$

$$f_p = 1 + \frac{Q_p}{\pi R} \left[ \log |y| + 3 \log 2 + \mathcal{O}\left(\frac{1}{|y|}, \frac{\log |y|}{|y|}\right) \right],$$

$$A = -\frac{qR}{2} \frac{(y^2 - 1)}{(x - y)^{1/2}(-2y)^{3/2}} 2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; 1 - \frac{1}{y^2}\right),$$

$$= -\frac{qR}{\pi} \sqrt{\frac{-2y}{x - y}} K(\frac{2z}{z - 1}) + (z - 1)E(\frac{2z}{z - 1}),$$

$$= -\frac{4qR}{\pi} \left[ \log |y| + 3 \log 2 - 2 + \mathcal{O}\left(\frac{1}{|y|}, \frac{\log |y|}{|y|}\right) \right], \quad (2.8)$$

where $z \equiv \sqrt{1 - 1/y^2}$, and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kinds, respectively:

$$K(m) = \int_0^{\pi/2} \frac{dθ}{\sqrt{1 - m \sin^2 θ}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 θ} \ dθ. \quad (2.9)$$

For large enough $|y|$ satisfying

$$\log |y| \gg \frac{R}{Q_f}, \frac{R}{Q_p}, 1, \quad (2.10)$$

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we can approximate these harmonic functions as
\[ f_f \simeq \frac{Q_f}{\pi R} \log |y|, \quad f_p \simeq \frac{Q_p}{\pi R} \log |y|, \quad A_\psi \simeq -\frac{qR}{\pi} \log |y|. \] (2.11)

As was done for \( d > 4 \) in \cite{14}, let us consider the following scaling of charges:
\[ J \gg Q \gg 1, \quad n \sim w, \quad nw \sim JQ, \quad 1 - \frac{JQ}{nw} \sim 1. \] (2.12)

Then, using the relations (2.4), (2.5), we see that the condition (2.10) becomes\(^3\)
\[ \log |y| \gg \frac{1}{g^2 Q} \cdot \frac{R^2}{g^2 Q \alpha'}, 1. \] (2.14)

This can certainly be met if we take \( Q \) to be large.

The string frame, four-dimensional curvature goes for large \( |y| \) as
\[ R \sim \left( \frac{|y| \log |y|}{R} \right)^2. \] (2.15)

Therefore, for the metric (2.1), which was derived using the two derivative action, to be trustable, we need
\[ |y| \log |y| \ll \frac{R}{\sqrt{\alpha'}}. \] (2.16)

Because \( |y| \gg 1 \) from (2.14), note that (2.16) implies
\[ |y| \ll \frac{R}{\sqrt{\alpha'}.} \] (2.17)

Using (2.11), we see that for large \( |y| \) satisfying (2.14), the metric (2.1) takes the following form:
\[ ds_{str, 5}^2 = \frac{Q_p}{Q_f} (dx^4 - dt)^2 + \frac{2\pi R}{Q_f \log |y|} dt (dx^4 - dt) + 2 \frac{qR^2}{Q_f} d\psi (dx^4 - dt) \]
\[ + R^2 \frac{dy^2}{y^4} + R^2 d\psi^2 + \frac{R^2}{y^2} d\phi^2, \] (2.18)
\[ e^{2\Phi_5} = \frac{\pi R g^2}{Q_f \log |y|}. \] (2.19)

\(^3\)For comparison, the corresponding relations for \( d > 4 \) are
\[ |y|^{d-4} \gg \left( \frac{R}{\sqrt{\alpha'}} \right)^{d-4} \frac{1}{g^2 Q} \left( \frac{R}{\sqrt{\alpha'}} \right)^{d-4} \frac{R^2}{g^2 Q \alpha'}, 1. \] (2.13)
\[ B = -\frac{\pi R}{Q_f \log |y|} dt \wedge (dx^4 - dt) - \frac{R^2 q}{Q_f} (dt - dx^4) \wedge d\psi + \text{const.} \quad (2.20) \]

Or, in terms of microscopic numbers (Eq. (2.4)), the metric is

\[ ds^2_{\text{str,5}} = \frac{n\alpha'}{wR_4^2} (dx^4 - dt)^2 + \frac{4\pi}{g^2 w} \sqrt{\frac{J}{Q} \log |y|} dt (dx^4 - dt) + 2 \frac{J\alpha'}{wR_4} d\psi (dx^4 - dt) \]
\[ + R^2 \frac{dy^2}{y^4} + R^2 d\psi^2 + \frac{R^2}{y^2} d\phi^2, \]

\[ e^{2\Phi_5} = \frac{2\pi}{w} \sqrt{\frac{J}{Q} \log |y|}, \]

\[ B = -\frac{2\pi}{wg^2} \sqrt{\frac{J}{Q} \log |y|} dt \wedge (dx^4 - dt) - \frac{J\alpha'}{wR_4} (dt - dx^4) \wedge d\psi + \text{const.} \]

Let us introduce the coordinates analogous to the ones introduced in [14]:

\[ \chi \equiv \sqrt{\frac{J}{Q}} \psi + \sqrt{\frac{JQ}{1}} \frac{1}{w} R_4 (x^4 - t), \quad \sigma \equiv \frac{1}{R_4} \sqrt{n} \frac{JQ}{w^2} (x^4 - t), \]

\[ \tau \equiv \frac{2\pi RR_4}{\alpha'^{3/2} \sqrt{nw - JQg^2} t}, \quad \rho \equiv -\frac{R}{\sqrt{\alpha' y}}. \]

The periodicity \((\psi, x^4) \cong (\psi + 2\pi, x^4) \cong (\psi, x^4 + 2\pi R_4)\) implies the following periodicity:

\[ (\sigma, \chi) \cong \left( \sigma, \chi + 2\pi \frac{J}{Q} \right) \cong \left( \sigma + 2\pi \sqrt{\frac{n}{w}} \frac{1}{\sqrt{\alpha'}}, \chi + 2\pi \frac{\sqrt{JQ}}{w} \right). \]

Also note that the third condition in (2.10) and the condition (2.17) become in terms of \(\rho\) as follows:

\[ 1 \ll \rho \ll \frac{R}{\sqrt{\alpha' y}}. \]

In the new coordinate system, the metric (2.21) takes the following simple form:

\[ \frac{ds^2_{\text{str,5}}}{\alpha'} = d\chi^2 + d\sigma^2 + \frac{2d\sigma d\tau}{\log(\frac{R}{\sqrt{\alpha' \rho}})} + d\rho^2 + \rho^2 d\phi^2, \]

\[ e^{2\Phi_5} = \frac{2\pi}{w} \sqrt{\frac{J}{Q} \log(\frac{R}{\sqrt{\alpha' \rho}})}, \]

\[ B = -\frac{1}{\log(\frac{R}{\sqrt{\alpha' \rho}})} d\tau \wedge d\sigma + \frac{d\sigma \wedge d\chi}{\sqrt{nw/JQ - 1}}. \]
In contrast with the $d > 4$ case, we were unable to quite eliminate the unwanted charge dependence from the metric in the region (2.26) (recall that $R/\sqrt{\alpha'} = \sqrt{J/Q}$). In order to eliminate the unwanted charge dependence from (2.26) completely, we need to be able to approximate

$$\log\left(\frac{R}{\sqrt{\alpha'} \rho}\right) = \log\left(\frac{R}{\sqrt{\alpha'}}\right) + \log \frac{1}{\rho} \approx \log \frac{1}{\rho}. \quad (2.27)$$

This would mean

$$\frac{R}{\sqrt{\alpha'}} \lesssim \frac{1}{\rho}. \quad (2.28)$$

If this inequality were to hold, the charge dependence would enter only in (the exponential of) dilation as an overall factor as one can see from (2.26). Because (the exponential of) dilation enters in the action as a overall factor at the string tree level, this charge dependence would not influence the equation of motion. However, (2.25) means that $1 \ll \frac{R}{\sqrt{\alpha'}}$, which combined with (2.28) would mean

$$\rho \ll 1, \quad \text{or} \quad \frac{R}{\sqrt{\alpha'}} \ll |y|. \quad (2.29)$$

This would contradict with (2.25) or (2.17), the condition for the curvature to be small. Therefore, in this region, the solution (2.26) obtained by two-derivative action is actually not valid. Namely, we could eliminate the unwanted charge dependence, but for that we have come too close to the ring singularity and the curvature, and thus the $\alpha'$ corrections, have gone out of control.

Instead, one could stay in the region where the curvature is not very strong and the solution (2.26) is valid. In this region, the conditions i) and iii) in (2.6) are met, but the condition ii) is not completely satisfied. But because the solution (2.26) is trustable, we can start from it and solve the $\alpha'$-corrected equations of motion toward smaller $\rho$. Then the solution must take the following form:

$$\frac{ds^2_{str,5}}{\alpha'} = g_{\alpha\beta}\left(\rho, \frac{J}{Q}\right)\, d\zeta^\alpha d\zeta^\beta + f_1\left(\rho, \frac{J}{Q}\right)\, d\phi^2 + d\rho^2,$$

$$e^{2\Phi_5} = \frac{1}{w} \sqrt{\frac{J}{Q}} f_2\left(\rho, \frac{J}{Q}\right),$$

$$\frac{B}{\alpha'} = b_{\alpha\beta}\left(\rho, \frac{J}{Q}\right)\, d\zeta^\alpha d\zeta^\beta + \frac{d\sigma \wedge d\chi}{\sqrt{n w J Q - 1}}. \quad (2.30)$$

\footnote{Because dilaton is suppressed for (2.12), we can use the heterotic string action at the string tree level. Although we do not know the string corrected action, it is expected that the dilation gets small near a fundamental string even if we included such corrections.}
where $g_{\alpha\beta}, f_1, f_2$, and $b_{\alpha\beta}$ are some unknown functions representing our ignorance of the $\alpha'$-corrected higher derivative action. $\zeta^\alpha$ stands collectively for the coordinates $\tau, \sigma$ and $\chi$. Because those unknown functions depend on $J/Q$, even if we assume that the $\alpha'$ corrections lead to a finite horizon, we cannot determine the Wald entropy of the small black ring as was possible for $d > 4$ [14]. Namely, because the condition ii) is not satisfied, we do not have enough control over the higher derivative corrections.

In summary, for $d = 4$, there does not exist a region where all the conditions i)–iii) in (2.6) are met. If we go very close to the ring (very large $|y|$ satisfying (2.29)), the charge dependence seems to disappear from the metric (2.1), but in that region the curvature has become much larger than the string scale and the metric (2.1) itself is not valid. It is possible that there does exist a region where i)–iii) are all satisfied, but it should be very near the ring where the curvature is of the order of the string scale, and we need to know all $\alpha'$ corrections in order to be able to study such a region.

### 3 Discussion

What is the relation of our result to the general relativity theorems [25] on the topology of event horizons of four-dimensional black holes? Based on topological censorship [26], a powerful theorem about the topology of four-dimensional black holes was proved in [27, 28]. Let $\mathcal{M}$ be the spacetime, and assume that this spacetime can be conformally included into a spacetime-with-boundary, $\mathcal{M}' = \mathcal{M} \cup I$. Then the topological censorship amounts to the statement that every causal curve whose initial and final endpoints belong to $I$ is endpoint-homotopic to a curve in $I$. Based on this topological censorship, it was shown in [28] that the sum of the genera of event horizons is bounded by a certain number determined by the property of $I$. In particular, when the spacetime is asymptotically flat, this theorem says that the horizon of every black hole must have genus 0, and there cannot be a black ring with toroidal topology [27, 28].

One might think that this is consistent with the result we found in this note, and that there does not exist a small black ring in four dimensions. Note however that topological censorship is under certain assumptions[5] which use notions of classical geometry based on point particles. In string theory, the fundamental object has a finite size and the validity of such classical notions cannot be taken for granted particularly when the curvature of spacetime is of the order of the string scale. Therefore for the objects such as small black holes or rings, it is logically possible that this theorem cannot be applied.

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[5] See, for example, page 8 of [28].
One related fact is that, in string theory, there is redefinition ambiguity in the metric field:

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + a \alpha' R_{\mu\nu} + b \alpha' R g_{\mu\nu} + \ldots, \]  

(3.1)

which does not change the action up to two-derivative terms. However, under (3.1), geodesic length of a curve, which is an invariant quantity in classical gravity, changes. In particular, such an effect is not negligible if the curvature of spacetime is of the order of string scale. Therefore, in such situations, arguments based on topological censorship must be reconsidered.

Furthermore, in string theory we know examples where the notion of metric itself loses its sense. Consider NS5-branes in flat 10-dimensional space. Away from the NS5-branes, one can trust the SUGRA metric [29], but as one approaches the NS5-brane, dilaton becomes large and one cannot trust the solution any more; in this strongly coupled region, the metric is not a good variable to describe the physics. However, in this case we know what to do: we need to go to the S-dual picture where one has D5-branes instead of NS5-branes and the low energy physics is described by the dual metric variable. The original metric and dual metric are good variables in different regions in spacetime, and theorems about the original metric (in the region of its validity) has little to say about the dual metric (in the region of its validity). Therefore, it is conceivable that, whatever general relativity theorems say about the metric away from the singularity of the small black ring, in the strongly coupled region very near the ring a dual metric becomes more appropriate, whose horizon topology those theorems have nothing to say about.

To summarize, we showed that the scaling argument that could be used for \( d > 4 \) small black rings to derive their entropy formula is not valid for the \( d = 4 \) small black ring. This is because for \( d = 4 \) there does not exist a scaling region where all the conditions i)–iii) in (2.6) are met, which existed for \( d > 4 \). General relativity theorems say that there does not exist a black hole with toroidal topology in asymptotically flat four-dimensional spacetime, but we interpret this as saying that stringy corrections are crucial for the four-dimensional small black ring which invalidates the assumptions underlying those theorems.

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A \ x, y \ coordinates

Denote the Cartesian coordinates for $\mathbb{R}^{d-1}$ by $x_{d-1} = (x^1, x^2, x^3, \ldots, x^{d-1})$. The relation of these coordinates to the $(x, y, \psi, \Omega_{d-4})$ coordinates are as follows:

$$x^1 = u \cos \psi, \quad x^2 = u \sin \psi, \quad (x^3, x^4, \ldots, x^{d-1}) = (v \xi^1, \ldots, v \xi^{d-3}), \quad (A.1)$$

where $u, v \in [0, \infty)$, $\psi \in [0, 2\pi)$ and $(\xi^1, \ldots, \xi^{d-3})$ are coordinates of $S^{d-4}$ satisfying $(\xi^1)^2 + \cdots + (\xi^{d-3})^2 = 1$. Then the relation to the $(x, y)$ coordinate is [30]

$$u = \frac{\sqrt{y^2 - 1} R}{x - y}, \quad v = \frac{\sqrt{1 - x^2} R}{x - y}, \quad (A.2)$$

$$x = -\frac{u^2 + v^2 - R^2}{\Sigma}, \quad y = -\frac{u^2 + v^2 + R^2}{\Sigma}, \quad \Sigma = \sqrt{(u^2 + v^2 + R^2)^2 + 4R^2v^2}. \quad (A.3)$$

The flat metric for $\mathbb{R}^{d-1}$ is

$$dx_{d-1}^2 = du^2 + u^2 d\psi^2 + dv^2 + v^2 d\Omega_{d-4}^2$$

$$= \frac{R^2}{(x - y)^2} \left[ \frac{dy^2}{y^2 - 1} + (y^2 - 1) d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2) d\Omega_{d-4}^2 \right]. \quad (A.4)$$

Note that, for $d = 4$, $S^{d-4}$ is made of two points: $\xi^1 = \pm 1$, and (A.1) becomes

$$x^1 = u \cos \psi, \quad x^2 = u \sin \psi, \quad x^3 = \pm v. \quad (A.5)$$

In terms of the $(x, y)$ coordinates, the computation of harmonic functions goes e.g. as

$$f_f = 1 + \frac{Q_f}{L} \int_0^L \frac{dv}{|x - F(v)|^{d-3}}$$

$$= 1 + \frac{Q_f}{2\pi} \left( \frac{x - y}{-2R^2 y} \right)^{(d-3)/2} \int_0^{2\pi} \frac{d\theta}{(1 - \sqrt{1 - y^2 \cos \theta})^{d-3/2}}$$

$$= 1 + Q_f \left( \frac{x - y}{-2R^2 y} \right)^{(d-3)/2} \, _2F_1 \left( \frac{d - 3}{4}, \frac{d - 1}{4}; 1; 1 - \frac{1}{y^2} \right). \quad (A.6)$$

See the appendix of [14] for more details.

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