Three routes to jet collimation by the Balbus-Hawley magnetorotational instability

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ABSTRACT
Three completely different lines of work have recently led to the conclusion that the magnetorotational instability (MRI) may create a hoop-stress that collimates jets. One argument is based upon consideration that magnetohydrodynamic turbulence, in general, and turbulence driven by the MRI, in particular, is more nearly viscoelastic than it is viscous. Another argument is based upon the dispersion relation for the MRI in the context of 1D simulations of core collapse. Yet a third argument rests in the results of direct numerical magnetohydrodynamic simulations of collapsars and thick accretion flows. I elaborate on my previous work regarding the first argument above and I briefly discuss how these three sets of results are all related. I also discuss the different roles played by the magnetic tension and the magnetic pressure within the context of this work. I point out that this leads to consideration of the normal stress difference between the hoop stress and the radial stress, in preference to a focus on just the hoop stress itself. Additionally, I argue that simulations of thick accretion flows and collapsars are not self-consistent if they include a phenomenological model for an MRI-induced viscous stress but disregard these other MRI-induced stress components. I comment briefly on the RHESSI observation of polarization in the gamma-ray burst GRB0212206. I argue that this polarization is consistent with a tangled field, and does not require a large-scale organized field. Finally I suggest that the role of magnetic fields in creating jets, as described here, should be understood not to work within the confines of magnetocentrifugal models of jets, but rather as an alternative to them.

Key words: MHD – turbulence – accretion, accretion discs – polarization – stars: winds, outflows – galaxies: jets

1 INTRODUCTION

It has recently been suggested by a number of authors that a magnetic field self-consistently generated by the magnetorotational instability (MRI) can collimate jets. This has been pointed out, independently, following three completely separate lines of work. Two are based in simple analytical arguments, and a third rests in the interpretation of numerical simulation.

These arguments have been placed in a variety of contexts, including supernovae and collapsars, active galactic nuclei, and protostellar jets. Certain aspects of this previous work are admittedly contextually specific. None the less I argue that MRI-generated hoop-stresses may be a nearly universal aspect of astrophysical jet creation and collimation in accretion or collapse scenarios. I explore this possibility below, where I highlight similarities in these three lines of reasoning. However, I focus primarily below on expounding on my previous work, as this has not appeared to date in peer-reviewed form.

In Williams (2001) (hereafter W01) I pointed out that, in contrast with purely hydrodynamic turbulence, magnetohydrodynamic (MHD) turbulence in an azimuthally shearing environment creates a hoop-stress,\textsuperscript{1} in analogy to the hoop-stress in viscoelastic media undergoing azimuthal shear. I estimated the magnitude of this stress in a thick accretion disk (or flow) using simple dimensional arguments, and I suggested that this hoop-stress could collimate jets. Although I discussed stresses in the general terms of MHD

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turbulence drawing its energy from azimuthal shear, I specifically pointed out in W01 (viz. p. 2, paragraph 1) that, as should be expected in an accretion scenario, this includes the Balbus-Hawley instability, i.e. the MRI. (It is not clear what other MHD instabilities exist that may draw their energy from the differential shear in the context of accretion flows.) This specific point regarding the role of the MRI was later emphasized in Williams (2003) (hereafter W03). The context of W01 was a generic, thick, steady accretion disk or flow, and in W03 I discussed protostellar jets, although most of the latter argument is broadly applicable to other jet phenomena.

It was suggested by Akiyama et al. (2003), see also Akiyama & Wheeler (2003), that a toroidal field generated by the MRI could contribute to jet creation in core-collapse supernovae. This argument was based upon the dispersion relation for the MRI. The MRI is a weak-field instability in the sense that the MRI is unstable so long as the magnetic field is less than some critical field strength. In addition, the growth rate of the MRI is comparable to the local angular velocity $\Omega$, which enables several e-foldings of the field strength in the course of core-collapse. It was thus hypothesized that the MRI would amplify a small seed field until it grew to equal the critical field strength, and it was argued that the resultant magnetic field would contribute to jet creation. This tentative conclusion was bolstered by 1D core-collapse simulations in which the MRI was treated heuristically through its dispersion relation.

Collimation by an MRI-generated field has also been argued to be happening in several recent direct numerical simulations (DNS) of both transient and quasi-steady accretion flows. This is particularly intriguing because DNS simulations, by definition, make no assumptions about the MRI or the ensuing turbulence. The first simulations in 3D seemed to show collimation in the form of a funnel surrounded by high magnetic pressure regions (Hawley et al. 2001, Hawley & Balbus 2002). This is quite different from the collimation by hoop-stress that I have argued. Meridional plane (2.5D) simulations of accretion flows in a variety of contexts seem to show collimation by stresses that the authors argue are generated by the MRI (Kudoh et al. 2002, Proga & Begelman 2003, Proga et al. 2003). Further numerical simulation and study should help clarify the role of the MRI in this collimation.

2 COLLIMATION BY TURBULENCE

2.1 A preliminary zeroth route

Tables for the components of the Reynolds stress and turbulent Maxwell stress tensors in shearing-sheet simulations of the MRI have been provided by Brandenburg et al. (1995), Hawley et al. (1995) and Hawley et al. (1996) (hereafter BNST95, HGB95 and HGB96). The stresses are due largely to a toroidal magnetic field, and the streamwise stress, corresponding to a toroidal hoop stress, is positive (indicating tension) and larger than the (off-diagonal) viscous stress; in contrast, the radial normal stress is negative (indicating pressure). Tables for the force due to a stress tensor (i.e. its divergence) can be found in a variety of standard reference sources: A positive hoop-stress $W_{\theta\theta}$ and a negative radial stress $W_{RR}$ result in a cylindrically radially-inwards (collimating) force density $F_{R}^{(2)} = -(W_{\theta\theta} - W_{RR})/R$. This leads immediately to a zeroth conclusion that the MRI can create a collimating force, and that it can do so through a turbulent, tangled field, by direct inspection of the results of BNST95 and HGB96. The primary serious potential objection to this is that the magnitude of the collimating force may not be significant, since the MRI saturates at a rather high plasma $\beta_p$ in these simulations. There remain several additional questions, such as how magnetic pressure affects this collimation. In particular note that the radial stress $W_{RR}$ is due largely to magnetic pressure, and the gradient term $\partial_\theta W_{RR}$ in the divergence of the stress $[F^{(1)}]$, infra in a real accretion flow can not be determined from local shearing-sheet simulations. The remainder of this section discusses these and other observations and questions regarding the MRI in greater detail, with the help of some simple turbulence modeling considerations, following and extending the discussion in W01 and W03.

2.2 Some simple turbulence modeling considerations

2.2.1 Introduction and notation

By viscosity, unless otherwise noted, I mean shear viscosity and not bulk viscosity. For the purposes of this paper, by viscous stress I mean that part of a stress tensor, turbulent or otherwise, which may be written in the explicit tensor form

$$w_{\text{visc}} = \nu \left[ \nabla v + (\nabla v)^T - \frac{2}{3} (\nabla \cdot v) I \right],$$

where $v$ is the velocity and $I$ is the identity. The only cases considered here are rectilinear shear, such as $\hat{v} \propto \hat{y} x$ in Cartesian coordinates, and azimuthal shear such as in an accretion disk with $R \Omega = \omega R$. In these two cases, the viscous stress tensor above reduces to a single, off-diagonal component in the respective coordinate systems, namely $w_{xy}$ and $w_{\theta\phi}$. It is assumed that this stress component is due entirely to viscosity, so that the value of the kinematic viscosity $\nu$, be it a molecular or a turbulent viscosity, is given by $\nu = w/\gamma$ where $\gamma$ is the shear rate and $w$ is the appropriate stress component above, and care is taken so that the sign of $\nu$ is consistent with the tensor equation (1) above.

My approach here as elsewhere is to ignore the mean field entirely, as counterpoint to analyses that ignore the turbulent field entirely. Thus unless noted otherwise I assume throughout the remainder of Section 2 that the mean field $\langle \mathbf{B} \rangle$ is identically zero, and the magnetic field exerts a dynamical influence entirely through a tangled field $\mathbf{B}'$. I denote the averaged Faraday tension as $4 \pi M_{ij} \approx \langle B_i B_j' \rangle$, and the turbulent Maxwell stress $\mathbf{M}$ is given by $M_{ij} = M_{ij} - \langle \rho v_i v_j' \rangle$, and the full turbulent stress tensor $\mathbf{W}$ is given by $W_{ij} = -R_{ij} + M_{ij}$.

Note that it is commonplace to refer to the funicular

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2 When discussing collimation here and in my previous work, it should be clear that I am not discussing far-downstream collimation, such as in traditional magnetocentrifugal mechanisms. Rather, I mean collimation in the jet-formation region, including the equatorial plane of accretion.
turbulent stress $M_{ij}$ as the Maxwell stress. This is strictly speaking incorrect; one must not neglect the contribution of the magnetic pressure term to the turbulent stress tensor. Inclusion of the turbulent magnetic pressure term does not affect the viscous term nor the normal stress difference, but it is none the less important in a full consideration of the dynamics of how MHD turbulence collimates jets.

Also for future reference, distinguish here between the angular velocity or Coriolis rate $\Omega$, the shear rate $2 \Omega = -R \partial_\Omega \Omega$ (here $A$ is the first galactic Oort constant), and the vorticity $(1/R) \partial_\theta (R^2 \Omega) = 2(\Omega - A)$. In the case $\Omega \propto R^{-q}$, these may be written $\Omega$, $q \Omega$, and $(2-q) \Omega$ respectively.

### 2.2.2 Failure of $\alpha$-prescription and existence of normal stress difference

The original way (Shakura & Sunyaev 1973) to write the Shakura-Sunyaev viscosity prescription, under the assumption that $v_R \ll v_\phi$, is that the viscous component of the turbulent stress tensor is proportional to the density times the sound speed squared, $W_{RR} = -\alpha_{\text{BS}} \rho c_s^2$. This represents only one of the six independent components of the turbulent stress. Through vertical integration, for a quasi-2D ($R-\theta$) disk, these six reduce to three, namely the aforementioned viscous stress and the on-diagonal stresses $W_{RR}$ and $W_{\theta\theta}$. These on-diagonal components of the turbulent stress are usually ignored. If anything, it is typically assumed that they consist either simply of the so-called turbulent pressure, $W_{RR} = W_{\theta\theta} = -P_{\text{turb}}$, or are due to a turbulent bulk viscosity. Either assumption predicts that the on-diagonal stresses are equal to one another. Particularly in the case of MHD turbulence, this may be highly inaccurate.

On-diagonal stress may be dynamically significant even when the turbulent pressure and bulk viscosity are not, because of normal stress differences. It is therefore useful to distinguish between what I call the primitive $\alpha$–viscosity prescription above where only the viscous stress $W_{RR}$ is modeled, and the extended $\alpha$–viscosity prescription in which the full turbulent stress tensor is modeled as a purely viscous stress, plus perhaps a pressure term.

The shearing-sheet simulations of the MRI mentioned above clearly show that: (i) the stress is dominated by the turbulent Maxwell stress, rather than the Reynolds stress, and (ii) as pointed out above, the cross-stream stress and the streamwise stress are of opposite sign, and both are actually larger in magnitude than the viscous stress, creating a significant normal stress difference, in gross contradiction of the extended $\alpha$–viscosity prescription. These two facts are connected (Williams 2004b).

It should be clear by examining BNST95, HGB95, HGB96, as well as Matsumoto & Tajima (1995), that the magnetic field that is being produced by the MRI is being dragged by bulk shear of the fluid, as ideal MHD tells us it should, and it is statistically aligning itself with the direction of mean shear. It is precisely by such dragging of field lines that the Maxwell stress in a turbulent medium produces a quasi-viscous off-diagonal stress, but why should the dragging of field lines stop there? Let us imagine, as I did in W01, a process where a turbulent or tangled field is constantly being created, distorted by shear, and ultimately dissipated, as appears to be the case in these shearing-sheet simulations (see fig. 2 in Williams (2004b)). Note that in equilibrium, when the turbulence is saturated, this sequence is a logical sequence, not a chronological sequence.

In fact, compare the magnetic stress $M_{ij}$ from eq. (4) of W01 to the normalized stress $M_{ij}$ of eq. (24) of BNST95, keeping in mind the switch $x \equiv y$ in going between the two papers, as well as the change of sign of $M_{xy}$ according to the differing orientation of the shear. A least-squares minimization of the difference between each of the four nonzero members of the six independent stress components taken in turn gives a fit of the normalized $M_{ij}$ of W01 to the $M_{ij}$ results of BNST95, written in the coordinate system of the latter, which corresponds to pure azimuthal shear in a disk with ordering $(R, \theta, z)$, of

$$
(M_{ij})_{W01}^{\text{fit}} = \begin{pmatrix}
0.02 & -0.14 & 0 \\
-0.14 & 0.96 & 0 \\
0 & 0 & 0.02
\end{pmatrix} \quad (2a)
$$

$$
(M_{ij})_{\text{BNST95}}^{\text{norm}} = \begin{pmatrix}
0.03 & -0.09 & 0 \\
-0.09 & 0.91 & 0 \\
0 & 0 & 0.06
\end{pmatrix} \quad (2b)
$$

for a value of the normalized relaxation time $\gamma_S = 6.9$ in the notation of W01.

In the case of the MRI the sequence of creation, distortion, and dissipation mentioned above is part of a larger feedback loop where the shear-aligned magnetic field is unstable to the MRI (locally, through the combined action of the shear operator and the Coriolis operator), creating more turbulence and completing the feedback loop:

$$
(M_{\theta\theta}+)M_{RR} \quad \text{shear} \quad M_{\theta\theta}(-M_{RR}) \quad \text{shear} \quad \text{turbulence}
$$

In principle, the feedback loop may be closed by other sources of turbulence, such as convective or inertial instabilities, or by MHD instabilities other than the MRI; it is in the nature of turbulence to stretch and contort material lines. However, note that passive turbulent dynamo action (such as magnetocvection) appears to be much less powerful than the active turbulent dynamo action of the MRI (see HGB96 regarding this point).

Indeed, let it be clear that the process I have described here and previously (W01, W03, Williams (2004a) Williams (2004b)) depends upon dynamo action in the sense of a turbulent dynamo as described by HGB96. Misunderstanding on my part regarding the definition of the word dynamo, a word which appears often to be taken to be synonymous with processes that can create self-sustaining large-scale ordered fields, led me to state otherwise in Williams (2004a), which was incorrect. On the other hand, the process does not depend upon a mean-field dynamo.

In fact, a large-scale magnetic field will complicate this picture, as will other symmetry-breaking terms such as $\nabla P \times \hat{\Omega}$. None of these are present in HGB96, nor are they present in the analysis here. A mean magnetic field is present in HGB95; the orientation of this field seems to affect the effective relaxation time for the turbulence, as I note in W03; more significantly, the presence of this field affects the saturation of the MRI. Ignoring the mean field and other

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symmetry-breaking terms for the present purposes however, I find that simple viscoelastic models of the stress produce a stress tensor \( W \) that I write here in terms of the thermodynamic pressure \( P \) for a disk in which \( v_R \ll v_\theta \) as (cf. Williams (2005) infra eq. 8)

\[
W/P = -\alpha_0 I - \alpha_1 \hat{R} \partial + \alpha_2 [ \partial \hat{R} - (1/3) I ]
\]

where the set \( \{ \alpha_1 \} \) consists of dimensionless constants, and \( \hat{R} \) and \( \partial \) are unit vectors. That is, the stress is the sum of a turbulent pressure equal to and parameterized by \( \alpha_0 \), a turbulent viscosity parameterized by \( \alpha_1 \approx \alpha_0 \) (given that \( P \approx \rho c_s^2 \)), and a streamwise normal stress difference\(^3\)\(^4\) due to the turbulent elasticity, parameterized by \( \alpha_2 \). The form chosen for this final term simply makes it traceless in 3D. For comparison with Williams (2005), \( \alpha_0 = P_{\text{turb}}/P \), \( \alpha_1 = \mu_{\text{turb}} (-\hat{R} \partial \Omega)/P \), and \( \alpha_2 = 2\xi_{\text{turb}} (\hat{R} \partial \Omega)^2 / P \). For future reference observe that \( \beta_p^{-1} = (3/2) \alpha_0 (M_{\text{kk}}/W_{\text{kk}}) = 3\alpha_0 (M_{\text{kk}}/W_{\text{kk}}) \approx (1.5 \text{ to } 2.3) \alpha_0 \approx \alpha_0 \). The lower number (1.5) is derived from HGB96 and the higher number (2.3) from BNST95.

Note that the model for the MRI turbulent stresses given by Ogilvie (2003) produces a different form for the equilibrium turbulent stress, because of the explicit inclusion of the Coriolis effect through a Coriolis operator in his eq. (23). This Coriolis operator rotates the Reynolds stress about the vector \( \Omega \) in the local corotating reference frame, and it is most likely a crucial ingredient in a complete local dynamical model for MRI driven-turbulence. However, this difference does not materially affect the conclusions here: The existence of significant normal stress differences is an unavoidable feature of any realistic turbulence model for the MRI in most, if not all, contexts, and this gross feature is present in the model of Ogilvie (2003) as well. Let me be the first to point out that there are three variables to model (the three stress components), and I have done so using three parameters, so eq. (4) not predict anything by itself, although note this prescription works quite well for describing the full six-component stress tensor in 3D as well (W03). The reason this is a meaningful expression is that \( \alpha_2 \gtrsim \alpha_1 \), and that it makes physical sense to separate the normal stress into an isotropic and an anisotropic part.

2.2.3 Magnitude of the normal stress difference and the effective Weissenberg-Deborah number

The ratio of the normal stress difference to the viscous stress \( \alpha_2/\alpha_1 \) in eq. (4) is roughly proportional to the ratio of the effective stress relaxation time \( s \) to the shear time scale \( (2A)^{-1} \), with the exact constant of proportionality being a model-dependent quantity. For example, the Maxwell model of Ogilvie (2001), which I write here as

\[
s(\hat{D}, W) + W = \nu [\nabla v + \nabla v^T] + \left( v_t - \frac{2}{3} v \right) (\nabla \cdot v) I
\]

predicts \( (W_{\theta\theta} - W_{RR})/W_{RR} = 2s(2A) \). (Note that \( \hat{D} \) is the tensor generalization of the vector advection operator for the magnetic field \( B \), see Ogilvie (2001), W01, et seq.) As well, the simple \( \alpha-\delta \) model (model B) of W03, namely

\[
s(\hat{D} , M) + M = \alpha I
\]

predicts \( (W_{\theta\theta} - W_{RR})/W_{RR} = s(2A) \), under the approximation that

\[
W_{\theta\theta} - W_{RR} \approx M_{\theta\theta} - M_{RR} = M_{\theta\theta} - M_{RR}.
\]

The \( \alpha-\delta \) model was not explicitly given in W01, although the construction in W01 gives a stress of exactly the same form, in the case of steady shear that is appropriate here. The approximation in eq. (7) follows from the assumption that the turbulent Maxwell stress normal difference is much larger than the Reynolds stress normal difference, as a result of either the quicker relaxation to isotropy of the Reynolds stress compared with the turbulent Maxwell stress, or (more significantly) the assumption that the turbulent magnetic energy is greater than the turbulent kinetic energy. This second assumption is important to the model of Ogilvie (2001) as well; otherwise the use of the magnetic tensor advection operator for the advection of the full turbulent stress would be unjustified.

Independent of the question of the value of the viscosity in a disk, then, is the question of the value of the ratio of the normal stress difference to the viscous stress. Henceforth in this paper, I define the effective Weissenberg-Deborah\(^5\) number \( \tilde{W} \) to be equal to the ratio \( (W_{\theta\theta} - W_{RR})/|W_{RR}| \). In the notation of eq. (4) above and of Williams (2004b), for reference, \( \tilde{W} = \alpha_2/\alpha_1 = 2(\zeta/\mu)(2A) \). It is also useful to define \( \pi_0 \equiv \alpha_0/\alpha_1 \). The relative magnitude of the members of the set \( \{ \alpha_1 \} \) are then determined by \( \pi_0 \) and \( \tilde{W} \).

Basic dimensional arguments suggest that \( \tilde{W} \) is of the order of unity: take the limit of an inviscid, perfectly conducting fluid, with zero mean field. Under the assumption that the Coriolis parameter \( \Omega \) and shear rate are comparable, the only local background dimensionful quantities available to construct a time scale are \( \Omega^{-1} \) and, given a length scale \( L \) (such as a scale height), a sound-crossing time \( L/c_s \). For a thin disk these two are also both comparable, and this suggest that, barring some large or small dimensionless factor (such as the magnetic Prandtl number), the effective relaxation time is comparable to the shear time scale (Ogilvie 2001, W01, Ogilvie 2003, W03, Williams 2004b). One should then expect the streamwise normal stress difference to be comparable to the viscous stress, in gross contrast with the extended \( \alpha \)-viscosity prescription, and this is what is seen. On the other hand, note that almost the exact same line of reasoning has been used, historically, to argue that the original Shakura-Sunyaev \( \alpha_{\text{SS}} \approx 1 \), whereas the actual value of this parameter is typically orders of magnitude smaller. For reference, note that the two time scales above start out equal in the simulations of HGB95: at the end of the simulations, \( c_s/L \approx 5|\Omega| \) for the vertical field fiducial run and \( c_s/L \approx 4|\Omega|/3 \) for the azimuthal field fiducial run.

\(^3\) Note that rheologists often distinguish between the Weissenberg number and the Deborah number; in principle it is possible to have one large and the other small, see Section 4.2 of Phan-Thien (2002). Here I make no such distinction.
A much stronger argument can be made by appealing to the action of the MRI as a feedback loop, as in diagram (3). This feedback loop can only operate if the relaxation time of the process is long enough to create a substantial difference between the streamwise stress $M_{RR}$ and the cross-stream stress $M_{RR}$; otherwise the feedback loop will not close. This suggests that we may be arbitrarily larger than unity but it is hard to see how it could be smaller than unity.

Even more significantly, an analysis of BNST95, HGB95, and HGB96 shows that the $\alpha_2$ is not just comparable to but in fact larger than both $\alpha_0$ and $\alpha_1$. Some analysis of HGB95 is found in W03. For HGB96, the full stress tensor (taking into account the often neglected contribution of the magnetic pressure to the turbulent Maxwell stress, supra) from a naive average of their R1, R2, R3, R4, R6, and R7, is $(W_{RR}, W_{\theta\theta}, W_{zR}) = (-0.038, 0.012, -0.028, -0.015)$. The remaining two stress components, $W_{\theta R}$ and $W_{zR}$, are not provided, although symmetry suggests they are zero. I find $\alpha_0$, $\alpha_1$, and $\alpha_2$ to be 0.018, 0.015, and 0.046, respectively; this gives $W = 2.98$ and $\pi_0 = 1.19$. The model eq. (4) is also an excellent fit to the full stress tensor provided in BNST95 as well. The full normalized stress tensor (taken from their eq. (24), which provides all six components of the separate Reynolds stress and the Faraday tension contribution to the turbulent Maxwell stress $M_{ij}$, again, correcting for the neglected contribution of the magnetic pressure to the turbulent Maxwell stress, and assuming a ratio of magnetic energy to kinetic energy of 6.67 taken from averaging over all runs given in their table 1, and normalized so that $\text{Tr}(W) = -1$, is

\[
(W_{ij})_{\text{norm}} = \begin{pmatrix}
-0.780 & -1.59 & 0 \\
-1.59 & 0.487 & 0 \\
0 & 0 & -0.707
\end{pmatrix}
\] (8)

under the coordinate ordering ($R, \theta, z$) for the columns and rows. This gives a value $W = \alpha_2/\alpha_1 = 7.7$, larger than the value of $W$ obtained from HGB96. On the other hand, it is possible that the difference in $W$ is simply reflective of the differing aspect ratio of these simulations. As well, of course, future higher fidelity simulations may produce even different results.

Significantly, the variation in the value of $W$ obtained in HGB96 compared with BNST95, i.e., roughly 8 versus roughly 3, is much less than the variation between these two studies of the quoted value of $\alpha_{SS}$, which differ by a factor of almost 10. This observation is in line with the observations of HGB95 [see their discussion in paragraph 1 of p. 749, and their eq. (16) and eq. (20)] that, according to them, the traditional $\alpha$ parameterization ‘does not seem as appropriate here’ because the viscous stress correlates more strongly with the magnetic pressure than with the gas pressure. Since the field, and hence the magnetic pressure, is dominated by the streamwise toroidal field, their observations combined with the observations in the paragraph above suggest that the ratio of the stress components, and in particular, I would argue, the ratios $\alpha_0 : \alpha_1 : \alpha_2$, are a more robust result of shearing-sheet simulations than the absolute value of the viscous stress as parameterized by the traditional $\alpha_{SS}$.

### 2.3 The form of the stress in a thick accretion flow

#### 2.3.1 Steps towards a model

In this section and the subsequent two sections, I describe basic steps towards a turbulence-based model of jets. A more complete description of such a model will be given in a later paper. The stresses and forces to be considered here are the turbulent stress and the forces due to gravity, the pressure gradient, and the centrifugal force. The centrifugal force is the result of the $\theta \theta$ component of the inertial stress tensor; the remaining terms in this stress ($RR$, $Rz$ and $zz$) that appear in the meridional equation of motion create a ram pressure which may create forces in the meridional plane that are of comparable magnitude to the other forces described here and which may also aid in collimation. However, for the sake of simplicity here, these forces are ignored.

#### 2.3.2 The fundamental assumption

From the values given above for the set of numbers $\{\alpha_i\}$, in a Keplerian thin disk the resultant collimating force is even less than the radial thermal pressure gradient, and thus negligible. Were this not the case, the thin-disk approximation would fail to be self-consistent.

Of much more interest is the case of a geometrically thick disk or flow. It is not yet clear how turbulence saturates in this context. A global accretion flow is different from a local shearing-sheet simulation in many respects. Among other things, the restrictive assumptions (quasi-periodic shearing-box boundary conditions with no handedness) necessary to perform the shearing-sheet simulations of the MRI may affect the plasma $\beta_p$ at which the MRI saturates. In fact, as I discuss below, preliminary global simulations show the MRI to be more powerful (by which I mean $\beta_p$ is smaller) than the local shearing-sheet simulations do.

My approach here, as previously, is to fix the value of We $(\pi_0)$ for the reasons described above, but to let the overall magnitude of the viscous stress be given by a free parameter. Ignorance of the behavior of turbulence in a thick disk (or flow) is thus subsumed entirely into the viscosity prescription, and it is assumed that the other stresses (streamwise normal stress difference and turbulent pressure) occur in proportion to the viscous stress. Previously (W01) I fixed We to be of order unity. If instead we take $We \approx few$ as above, then so much the better.

It remains to be seen what is the absolute value of the viscous stress $W_{R\theta}$ and how this compares with other dimensionally similar quantities. In the next few sections I provide an unfortunate but necessary diversion into the various prescriptions for the viscosity. Largely, the variations in ways of writing the viscosity are irrelevant to the gross conclusions.

#### 2.3.3 The form of the viscosity in W01 and W03

The specific scenario discussed in W01 and W03 was a flow driven by the viscous spin-down torque on a central object completely embedded in a thick accretion flow (I did not use the word flow but rather disk, but let it be clear that this in no way should be taken to imply equilibrium as in the classic studies of thick equilibrium tori). In W01 I chose to
write \( \nu \) in the form of a proportionality constant \((a)\) times a length scale times a rate. The rate I chose was the shear rate at the fiducial radius of the central accretor \( R_s \), so that \( \nu \propto (2A)_s = \rho \Omega \). The length scale was somewhat idiosyncratically written in W01 as \( \sqrt{\beta R_s} \), and \( \beta \) was a free parameter (not the plasma beta) allowed to vary to take into account the uncertainty as to the correct length scale, and thus viscosity. (To avoid confusion with the plasma beta, henceforth label \( \beta \) and thus viscosity. (To avoid confusion with the plasma \( \beta \).)

The length scale \( \sqrt{\beta R_s} \) was somewhat too appear in the form of the product \( \alpha \beta \) this second parameter \( \beta \) is redundant in the notation of W01. Regrettably, this redundancy obfuscated the primary thrust of the paper.) In particular, I allowed the length scale (here let me call it \( L \)) to vary between \( R_s \) as a lower bound and the radius of the thick disk as an upper bound, which I denoted \( r_m R_s \) (note the omitted minus sign in W01 and W03):

\[
\nu = \alpha_{\nu \nu 1} \beta_{\nu \nu 1} R_s^2 \left( \frac{\partial \Omega}{\partial \ln R} \right) ,
\]

\[
= \alpha_{\nu \nu 1} L^2 \left( \frac{\partial \Omega}{\partial \ln R} \right) ,
\]

\[
= \alpha_{\nu \nu 1} q L^2 \Omega .
\]

The length scale \( \sqrt{\beta_{\nu \nu 1} R_s} \), or \( L \), was assumed for simplicity to be a globally defined quantity that does not vary with position, and so the viscosity above should be understood to be constant throughout the flow, for the sake of argument.

The reason for the use of the parameter \( \beta_{\nu \nu 1} \) was simply that it is sometimes taken to be implicit, when writing the viscosity in the form \( \nu \propto \ell \omega \), that the length scale \( \ell \) and the rate \( \omega \) are in some dynamical sense characteristic of the turbulence. I wished to emphasize that, for a central object completely embedded in an accretion disk that is potentially much larger than the central object, this length scale could be much larger than the radius of the central accretor, but no larger than some effective characteristic length scale of the thick accretion region.

This suggests, if \( \alpha_{\nu \nu 1} \) in eq. (9) is of the order unity, that the product \( \alpha_{\nu \nu 1} \beta_{\nu \nu 1} \) in principle may be much larger than unity. Note that this does not necessarily imply anything about the value of \( \alpha_{SS} \), which may still be of the order of or much less than unity, see below.

### 2.3.4 Shear versus vorticity versus Coriolis parameter

The form I adopted above posits the viscosity being proportional to a shear. Abramowicz et al. (1996) performed shearing-sheet simulations of the MRI with variable effective \( q \) and found that the viscosity is proportional to the ratio of shear to vorticity, \( \Omega/2A \), or equivalently, \( \nu \propto q/(2-q) c_s H \). This is an important consideration to take into account for a model of the MRI for future work on thick disks and flows. However in the rough work presented here this only introduces a factor of order unity which is negligible relative to the other uncertainties and approximations in this work.

### 2.3.5 Global versus local viscosity prescriptions

Historically the turbulent viscosity has been variously parameterized in terms of purely local quantities, purely global quantities, and mixtures of the two. For example, the Shakura-Sunyaev viscosity parameterization in its original form, supra, is a local definition, whereas the viscosity parameterization of Lynden-Bell & Pringle (1974), namely \( \nu \propto R \Omega / R_s^2 \), is in global form. Although Lynden-Bell & Pringle (1974) do not define an alpha, it is convenient to do so using their formulation, so let me here define \( \nu = \alpha_{\mu \mu 1} R \Omega / R_s^2 \). From this one may define an effective Reynolds number \( Re = 1/\alpha_{\mu \mu 1} \).

Popular mixed forms for writing the viscosity prescription include \( \nu = \alpha H^2 \Omega \) and \( \nu = \alpha c_s H \), where \( H \) is some measure of the disk thickness. These definitions are more or less equivalent to the original Shakura-Sunyaev prescription, in the case of a thin disk. This redundancy disappears in the case of a thick disk or flow; however, it is not clear what is the natural generalization of \( H \) and \( \Omega \) in a thick accretion flow, because there are are several nearly equal length scales and rates that are habitually conflated in thin-disk theory into these two quantities. For example, \( H \) in thin disk theory is neither strictly local nor is it global, it lives somewhere in between. It is variously defined to be a representative pressure scale height, a density scale height, the second moment of the vertical density distribution, or the half-thickness of the optical depth \( \tau = 2/3 \) surface. It is not sufficient to generalize it to a local quantity by defining it to be a pressure scale height, say, because strictly speaking the local pressure scale height in a thin disk is infinite at the midplane. Similarly, \( \Omega \) in thin disk theory is vaguely synonymous with the orbital frequency, the Coriolis parameter, the Keplerian orbital frequency, the shear rate, and the vorticity; none of these quantities are necessarily nearly equal one another in a thick disk. This opens up the possibility that \( \alpha \) may be anomalously larger or smaller than the accepted value, depending upon what one means by \( \alpha \), and what part of a three-dimensional accretion flow one is examining.

The choice in W01 was to write the viscosity in a purely global form, similar to the Lynden-Bell–Pringle formulation above. In particular, \( \alpha_{\mu \mu 1} = \rho_a \alpha_{\nu \nu 1} \beta_{\nu \nu 1} \). Henceforth, I dispense with \( \alpha_{\nu \nu 1} \) and \( \beta_{\nu \nu 1} \) in preference for \( \alpha_{\mu \mu 1} \).

The values of \( \alpha_{SS} \) and \( \alpha_{\mu \mu 1} \) are connected in a non-trivial way. Under the approximation that \( P \approx \rho c_s^2 \) and \( 2A \approx \Omega \), one finds

\[
\frac{\alpha_{\mu \mu 1}}{\alpha_{SS}} \approx \frac{c_s^2 / \Omega}{(v_{th})^2/\Omega} .
\]

where \( v_{th} = R \Omega \). Near the surface of the central accretor, then, \( \alpha_{\mu \mu 1} / \alpha_{SS} \approx c_s^2 / v_{th}^2 \). For \( R \gg R_s \), one may have \( \alpha_{\mu \mu 1} \gg \alpha_{SS} \) if \( c_s^2 \gg v_{th}^2 \).

Having now discussed the variations in writing the viscosity, I now return to the physics of collimation. Following W01, for concreteness, assume that in the thick disk, \( \Omega \sim R^{-3} \), and in particular \( \Omega \) may differ greatly from the Keplerian value \( \Omega_K \).

### 2.4 The ordering of forces and speeds

The force density that arise from taking the divergence of the stress tensor may be written as the sum of three forces, see the three terms on the right hand side of eq. (21) below, respectively called \( F^{(3)} \), \( F^{(2)} \), and \( F^{(1)} \). Here in this paper I do not consider \( F^{(0)} \). The normal stress differences create a cylindrically radially inwards specific force \( f_R^{(2)} = F^{(2)} / \rho = \rho c_s^2 \)
\(- (\text{We}) W_{\theta \theta} / R = - (\text{We}) \nu / \partial R \Omega\) (c.f. W01, eq. 6). In the Shakura-Sunyaev local form, this may be written \(f_R^{(2)} = -(\text{We}) \alpha_{SS} c_s^2 / R\), and in the Lynden-Bell-Pringle global form, \(f_R^{(2)} = -(\text{We}) \alpha_{\text{IBP}} R^2 \Omega / (\Omega / R)\). For \(\text{We} \simeq 1\) and for \(\text{Re} \leq 1\) (i.e. \(\alpha_{\text{IBP}} \gtrsim 1\)) as assumed in W01, the collimating force due to the MHD turbulence is clearly significant.

Consider here the relative strength of various specific forces. For simplicity here, assume that our test fluid element is close to the equatorial plane, \(z / R \ll 1\). The set of forces I wish to consider here consists of the force due to the first normal stress difference \(f_R^{(2)}\), the centrifugal force \(f_c = \Omega^2 R\), and the force due to gravity \(f_g = \Omega R^2\). The force due to the turbulent pressure \(f_{\text{tew}}^{(2)}\) (which is essentially the same as the force due to the magnetic pressure because \(M_{\theta \theta} \gg R_{\theta \theta}\)), and the force resulting from pressure \(f_P = - (\partial R P) / \rho\) will be discussed in the subsequent section below.

Before proceeding with the evaluation of forces, however, it is handy to describe the collimation stress in terms of a wave speed, just as Akiyama et al. (2003) make use of the Alfvén speed to describe the strength of the magnetic field. A tangled field produced by a turbulent conducting medium supports Alfvén-like waves much like a viscoelastic medium supports elastic waves (Gruzinov & Diamond (1996), Schekochihin et al. (2002), Williams (2004b)). The speed of these transverse elastic waves \(v_{\text{tew}}\) is dependent upon direction, for an anisotropic stress tensor. The wave speed due to the toroidal component of the tangled, turbulent field is similar to the expression for the Alfvén wave speed, \(v_{\text{tew}}^2(\hat{\theta}) = (B_\theta B_\theta) / 4\pi \rho\). Then, for the toroidal mode, 

\[
\frac{v_{\text{tew}}^2(\hat{\theta})}{c_s^2} = \frac{M_{\theta \theta}}{\rho} \tag{11}
\]

Note that 

\[
M_{\theta \theta} = \left(2\alpha_0 + \frac{2}{3} \alpha_2\right) P = \left(2\pi_0 + \frac{2}{3} \text{We}\right) \alpha_{SS} \rho c_s^2 \tag{12}
\]

\[
\simeq (\text{We}) \alpha_{SS} \rho c_s^2
\]

and therefore for the toroidal mode 

\[
v_{\text{tew}}^2(\hat{\theta}) \simeq (\text{We}) \alpha_{SS} c_s^2. \tag{13}
\]

Typically the shearing-sheet simulations show \((\text{We}) \alpha_{SS}\) is small, on the order of \(1/(f_{\text{ew}})\), e.g. 1/20 for HGR96. It is conceivable that high-fidelity global simulations will produce markedly different results; perhaps it is possible that \((\text{We}) \alpha_{SS}\) becomes larger than unity. Incidentally, note that the principle axes of the stress tensor \(M_{ij}\) do not coincide with the coordinate axes, so an initially prograde or retrograde azimuthal wave will be refracted radially inwards or outwards respectively. Henceforth let it be understood in this paper that by \(v_{\text{tew}}\) I mean the speed of the azimuthally traveling wave \(v_{\text{tew}}(\hat{\theta})\). Also note that \(v_{\text{tew}}^2\) is equal to the energy per unit mass stored in the azimuthal part of the turbulent magnetic field. The relative magnitude of the various forces can be stated in terms of the relative magnitude of various speeds, or the relative specific energy densities as well, as given below.

The condition for the turbulent collimating force \(f_R^{(2)}\) to dominate (i.e. have a larger magnitude than) the centrifugal force is 

\[
\frac{v_{\text{tew}}^2}{c_s^2} < (\text{We}) \alpha_{SS}, \tag{14}
\]

or, equivalently, 

\[
\left(\frac{R}{R_s}\right)^{2-q} < (\text{We}) \alpha_{\text{IBP}}. \tag{15}
\]

Even more succinctly, eq. (14) may be restated with the help of the wave speed expression above, so the condition becomes 

\[
v_{\theta}^2 < v_{\text{tew}}^2. \tag{16}
\]

That is, the radial force due to the turbulent normal stress difference dominates the centrifugal force when the azimuthal flow velocity becomes sub-(quasi)-Alfvénic, or equivalently, when the energy in the shear-aligned magnetic field is greater than the kinetic energy of rotation. Interestingly, this critical field strength is the same as the MRI saturation field given in eq. (8) of Akiyama et al. (2003).

The condition for the turbulent collimating force \(f_R^{(2)}\) to dominate gravity, in terms of the Keplerian orbital speed \(v_K\), is 

\[
\frac{v_K^2}{c_s^2} < (\text{We}) \alpha_{SS}, \tag{17}
\]

or, equivalently, 

\[
\left(\frac{R}{R_s}\right)^{q-1} < (\text{We}) \alpha_{\text{IBP}} \left[\frac{\Omega_s}{(\Omega R)^2}\right]^2 \tag{18}
\]

Again, more succinctly, eq. (17) may be restated with the help of the wave speed expression above, so the condition becomes 

\[
\frac{GM_s}{R} = v_K^2 < v_{\text{tew}}^2. \tag{19}
\]

That is, the radial force due to the turbulent normal stress difference is stronger than gravity (but pushes in the same direction!) when the energy per unit mass stored in the turbulent azimuthal field exceeds the gravitational binding energy per unit mass.

Of course, there is nothing magical at all in eq. (16) and eq. (19). The question is whether the turbulence is ever so strong that these conditions are ever approached. Roughly, for collimation by turbulence to work, it appears likely that we must have that \(v_{\text{tew}}^2\) is a substantial fraction of \(v_{\theta}^2\), and preferably 

\[
v_{\theta}^2 < v_{\text{tew}}^2, \tag{16}
\]

and then if we assume \((\text{We}) \alpha_{SS} < 1\) this implies 

\[
v_{\theta}^2 < v_{\text{tew}} \approx c_s^2. \tag{20}
\]

The end result, perhaps not surprisingly, is that the turbulent elastic radial force \(f_R^{(2)}\) is unequivocally important when the rotation and the Keplerian orbital speed become sub-(quasi)-Alfvénic as in eq. (16), and furthermore subsonic as well if we assume as is usual that \(c_s^2 > v_{\theta}^2\). These are not conditions that are ordinarily met in a thin disk, where radial pressure support becomes significant. Note that the Shakura-Sunyaev formulation implies that turbulent pressure and hence magnetic pressure occurs...
in proportion to the thermal pressure. Even if this is not precisely true in reality, it suggests that if we are to discuss radial thermal pressure support, then we should as well address radial magnetic pressure support.

2.5 The role of magnetic and turbulent pressure

This brings me now to address one final potential objection to collimation by turbulence-induced magnetic hoop-stresses, namely that magnetic pressure might push out and counteract the inward pull of the hoop-stress.

This was explored in W01. As I describe in further detail here, the existence of magnetic pressure may actually be beneficial to the jet-producing scenario I have described. It does, however, lead to a focus on the normal stress differences, rather than the hoop-stresses themselves.

The phrases hoop-stress, azimuthal stress and toroidal stress are all succinct, equivalent and physically intuitive. The phrase ‘positive normal stress difference between the azimuthal stress and the radial stress’ is more precise, but not succinct. One may have toroidal hoop-stresses without normal stress differences, but this is physically equivalent to a negative pressure. Collimation in such a scenario can only occur through the radial gradient term below.

The existence of a positive (tension) turbulent magnetic hoop-stress requires a normal stress difference: the on-diagonal turbulent magnetic stress can not be of the form of a negative pressure; the normal stresses can not all be positive, since their sum is negative semi-definite. Recall that the $R$-component of the divergence of a symmetrical (stress) tensor in cylindrical coordinates is given by

$$
(\nabla \cdot W)_R = \partial_R W_{RR} - \frac{W_{\theta \theta} - W_{RR}}{R} + \partial_z W_{Rz}. \quad (21)
$$

For a purely azimuthal field, spatially stochastic or otherwise, the stress $W_{RR}$ is negative and due entirely to magnetic pressure, and it appears in the force equation through its gradient. In contrast, the normal stress difference $W_{\theta \theta} - W_{RR}$ is due entirely to the magnetic Faraday term, i.e. the tension of field lines, and it appears in the force equation only through the geometrical factor $1/R$, not through its gradient. In the more general case of a tangled field which is predominantly but not entirely azimuthal, not only will $W_{\theta \theta}$ have some contributions from the magnetic pressure, but $W_{RR}$ will have some positive contributions from the tension term as well. However, the normal stress difference $W_{\theta \theta} - W_{RR}$, absent Reynolds stresses, is still due entirely to the magnetic tension term.

Now let us examine outflow collimation in engineering. Consider jets from nozzles, such as a garden hose nozzle or a rocket nozzle. These jets are confined and collimated at their base by the anisotropic elastic stress in the collimating medium (i.e. the nozzle). Similarly, the explosive gases that propel the slug of a rifle or cannon are effective at propelling the projectile because of the anisotropic elastic stress in the surrounding barrel. The metal in the barrel is radially in compression (negative stress) and azimuthally in tension (positive stress). In the case of both the steady dynamics of nozzles and the transient dynamics of guns, the collimation is thus due to a normal stress difference in the surrounding material.

3.1 Introduction

As we shall see below, the estimates for the strength of the collimation as discussed above and in W01 and W03 are consistent with the estimates for the strength of the MRI argued by Akiyama et al. (2003).

It is well known that fluid dynamical instabilities that lead to turbulence may be self-limiting, in the sense that the turbulence creates transport that, in many cases, damp out the driving terms in the instability. Thus, for example, an adverse entropy gradient in a stellar interior drives convection, which in turn reduces the adverse entropy gradient. Similarly, it was suggested by Akiyama et al. (2003) (see also Akiyama & Wheeler 2003) that, under conditions in a core-collapse supernova where the MRI is locally unstable, a small seed field should grow via the MRI until the field reaches the critical field strength at which the MRI is no longer unstable. Since the instability grows exponentially, it was argued that it dominates linear-in-time processes such as passive field wrapping. The authors found that given initial angular velocity of the supernova progenitor core of the order of a few radians per second, the combination of collapse and conservation of angular momentum on shells combine to create strong MRI-unstable differential rotation in core collapse. Further they point out that, although the ultimate field orientation is not clear, a toroidal field configuration seems likely, considering the large differential shear in core collapse.
3.2 Estimates for the field saturation

Akiyama et al. (2003) provide a number of different estimates for the saturation field of the MRI. One estimate may be obtained by setting the length scale $\ell \sim dR/d(\ln \Omega)$ equal to the characteristic mode scale. This results in a field strength $B_{\text{sat}}^2 \simeq 4\pi \rho R^2 \Omega^2$ or, equivalently, $(v_A)^2_{\text{sat}} = v_s^2$. Akiyama et al. (2003) point out that this estimate of the field strength is substantiated by the simulations of HGB96, namely $2\pi B_{\text{sim}} = B_{\text{sat}}$. In point of fact, the results of HGB96 refer to the mean square of the field components, as the mean field in these simulations is identically zero, as I discuss at length above. From the point of view of calculating the stresses and forces due to the field, the distinction is largely irrelevant. This distinction does make a large difference, however, when it is argued that the MRI-induced stresses of Akiyama et al. (2003) depend upon processes such as helicity conservation that, through inverse cascades, create a significant mean field. While such processes may indeed occur in core-collapse supernovae, I would argue that Akiyama et al. (2003) may be re-read with an understanding that the mean field may be replaced throughout their analysis with a tangled field with a mean stress, and this difference in interpretation does not change the analysis at all.

Another estimate is related to this first estimate by setting the mode length equal to the local radius, resulting in a field strength

$$B_{\text{rad}}^2 \simeq 4\pi \rho R^2 \Omega (R \gamma / \Omega)^2$$ (22)

Other estimates are found using the maximum unstable growing mode (with a vertical seed field), or by setting the wavelength of the maximum growing mode equal to the shear length.

Again, one of the issues that is naturally raised in their analysis as well as in my analysis is the question of local time scales and length scales. For example, the difference between the estimates $B_{\text{rad}}^2$ and $B_{\text{sat}}^2$ is a factor of the ratio of the shear to Coriolis parameters.

Finally, they find the ordering

$$c_s \gg r \Omega \sim v_A.$$

In particular, they find that the plasma $\beta_p$ is large, specifically the magnetic pressure is on the order of a few of the gas pressure. They note the less the argument that the resultant MHD luminosity may be significant enough to propel and collimate jets in core-collapse supernovae.

Note that the similarity in the conclusions regarding the magnitude of the hoop-stresses based on turbulence modeling and on the dispersion relation shows how the the saturation level of the MRI and the value of the effective viscosity is related to the MRI dispersion relation.

Akiyama et al. (2003) do not address the magnetic pressure in their work.

4 COMPARISON WITH SIMULATION

Let me now briefly address recent simulations in the light of the preceding discussion.

Only lately has it become possible to perform global DNS magnetohydrodynamic accretion flow simulations in three dimensions that are capable of showing the action of the MRI. These global simulations (Hawley, Balbus & Stone 2001) show, as the authors point out, that the behavior of the flow is not adequately described by a hydrodynamic $\alpha$-model. This should not be surprising because, by inspection as I discussed above and previously, neither are local simulations adequately described by a hydrodynamic $\alpha$-model.

Hawley, Balbus & Stone (2001) simulated the non-radiative magnetohydrodynamics of an initially constant angular momentum torus in three dimensions. They gave meridional contour plots of the magnetic pressure and the off-diagonal $(R - \theta)$ Maxwell stress, but as these plots are unscaled and as they do not show the other stress components, it is impossible to see if the MRI is creating hoop stresses through the tension term in the magnetic stress. The magnetic pressure appears to be anti-collimating (as I argued in W01) in the equatorial plane, and collimating above the plane. They found that after several orbits the MRI creates magnetic field with plasma $\beta_p$ on the order of roughly 5–10 close to the equatorial plane, whereas several scale heights above the equatorial plane $\beta_p$ is less than 1. Thus, even on the equatorial plane the MRI saturates to a level much higher (i.e. $\beta_p$ is smaller) than in the zero mean-field shearing-sheet simulations of HGB96, which yield a plasma $\beta_p$ of roughly 40, and the MRI saturates rather to a level comparable to the nonzero mean-field simulations of HGB95, which yield a plasma $\beta_p$ of the order of 7 for the azimuthal field runs and of the order of 2.4 for the vertical seed field (as determined from a naive average of all the runs on their tables 1 and 3, and excluding run Z3). Note that in the case of HGB95 these values for the $\beta_p$ include contributions due to the mean field and the turbulent field.

The further exploration of a very similar set of simulations in Hawley & Balbus (2002) allows us to place these simulations more firmly in the context of our remarks in the previous sections. This second set of simulations has been applied to the very hot ($\sim 10^{13}$ K) accretion flow of Sgr A*; nevertheless the authors use a very simple single-component equation of state which allows one to estimate the ratio of sound speed to orbital speed from their fig. 8, namely

$$c_s^2 \simeq (1/10)v_A^2.$$

Using $\beta_p \simeq 5$ for the saturation of the MRI on the equatorial plane, then, I have

$$v_A^2 \simeq (1/250)v_s^2,$$

and

$$v_A^2 < c_s^2 < v_s^2.$$

On the other hand, the situation changes above the equatorial plane where $\beta_p^{-1}$ is much larger, with a correspondingly larger Alfvén speed relative to sound speed. Here, $v_A^2$ is likely a much larger fraction of $v_s^2$ or $v_K^2$, but it is difficult to tell by how much.

Collimation by the MRI appears to be seen in the 2.5D simulations of Kudoh et al. (2002), Proga & Begelman (2003), and Proga et al. (2003). Note that all three components of the velocity and magnetic field are included in these simulations. However, the constraint of axisymmetry in these simulations necessarily restricts the action of the MRI; in particular, this constraint eliminates the non-axisymmetric modes which act upon the azimuthal field, and which I called upon to complete the feedback loop in
the creation of turbulent azimuthal hoop-stresses in equilibrium that were the basis of my analyses. There therefore remains the question of the extent to which the collimation in Kudoh et al. (2002), Proga & Begelman (2003) and Proga et al. (2003) is a transient effect. This may be a moot point in the case of collapsars, but it is essential to the case of steady accretion.

Proga & Begelman (2003) present axisymmetric simulations of the accretion of low angular momentum gas onto a black hole. Despite the initial low angular momentum, at late stages of the accretion the Alfvén speed, sound speed, azimuthal fluid velocity and Keplerian velocity are all roughly comparable (see their figures 9 and 10). In contrast, in the collapsar simulations of Proga et al. (2003), these velocities sometimes differ by more than an order of magnitude, and again \( v_a^2 < c_s^2 < v_K^2 \). In both cases, the radial pressure gradient in the equatorial plane is anti-collimating. Also in both cases, the field is clearly dominated by the toroidal component.

Proga et al. (2003) state that it is the gradient of this toroidal field in particular that drives the outflow. Let me point out that in the meridional plane, the gradient of the toroidal field appears only through the magnetic pressure term. By rough examination of the printed results of Proga et al. (2003), it appears the toroidal field exerts a force through the curvature term that is comparable to the force due to its gradient. That is, I argue that the term \( \partial_r(B_\phi^2/r) \) in the divergence of the stress may be of approximately equal significance to the results of Proga et al. (2003) as the gradient term \( \partial_r(B_\phi^2) \) that the authors address. (As an aside, note that also of comparable importance, depending upon location in the flow, is, by inspection, the ram pressure from infall, so that it appears that the jet is in part not only magnetically confined but inertially confined as well.)

A final intriguing note regarding the simulations of Proga et al. (2003) is that the toroidal field component changes polarity with time and location. This confirms that the MRI can collimate jets through a disorganized rather than an organized (mean-field) toroidal field, as I suggested in W01 and W03. Note that the possibility of jet collimation by a tangled field has also been investigated in the context of the far-downstream jet by Li (2002) who showed that a tangled field avoids the long-mode kink instabilities that plague jets collimated by an ordered field.

Finally, let me address the recent core-collapse simulations of Thompson et al. (2005) who insert MRI-induced viscous dissipation into 1D simulations. Thompson et al. (2005) point out that Akiyama et al. (2003) overestimate the viscous time scale, and that viscous process that smooth out gradients in \( \Omega \) must be taken into consideration on the time scale case of core collapse. As a result, viscous heating may significantly alter the core collapse dynamics. The proof-of-concept work of Akiyama et al. (2003), however, demonstrates that one must take into account the streamwise hoop stress as well. My primary goal here and elsewhere has been to focus upon the hoop stress, as this had not been addressed in the literature. However, a full accounting must at the very least address all three components, as I have attempted here and elsewhere, and not just one or another component. (The physical significance of the remaining three stress components, which may come into play when one moves above or below the equatorial plane of symmetry, is not explored here.) For example, the viscous stress (and not just the normal stresses) is important to the model of W01, as this stress transports energy and angular momentum, which the other stresses do not. My analysis of the results of Thompson et al. (2005) shows that at, say, the fiducial radius of 20km, the inwards MRI-induced force in the equatorial plane, which the authors do not include in their code, is no more than half an order of magnitude less than the outwards centrifugal force that the authors do include in their code, if one takes the modest value \( We = 3 \). Admittedly, both sets of simulations are severely limited by the inherent assumption of spherical symmetry of a 1D code. These are valuable proof-of-principle studies; future progress demands multidimensional MHD codes.

5 CONCLUSION

In this paper I have discussed my previous work (W01, W03, Williams 2004a) regarding the hypothetical creation and collimation of jets by MHD turbulence and by the MRI in particular. I have added on to this work with a much more extensive and in-depth discussion of the dynamics of turbulent stress and the modeling of turbulent stresses in accretion flows. Also, I have placed this work within the context of other recent work, in particular the semianalytic work of Akiyama et al. (2003) based on the dispersion relation for the MRI, as well as the DNS simulations of Hawley et al. (2001), Hawley & Balbus (2002), Proga & Begelman (2003), and Proga et al. (2003).

I have focused in this paper on expounding on how simple MHD turbulence modeling considerations in the context of steady shear, as discussed in Williams (2004b), lead to the conclusion that MHD turbulence and MRI-driven turbulence in particular can create collimating forces. As discussed previously (e.g. W03) I also show how collimation can be seen directly in the results of local shearing-sheet simulations of the MRI, which significantly predate global MRI simulations. Turbulent collimation by quasi-viscoelastic stresses, as discussed, has a natural analog in the Weissenberg effect, whereby viscoelastic fluids climb a spinning rod. Thus, the turbulent MHD collimation of astrophysical jets as discussed has a natural analog to the prehaps more prosaic physics of the dynamics of ordinary viscoelastic fluids such as egg whites which climb spinning rods.

Indeed, this research was originally inspired by my serendipitous discovery in the literature of a laboratory viscoelastic phenomenon well-known to rheologists (Giesekus (1963); Thomas & Walters (1964); National Committee for Fluid Mechanics Films (1972)), namely the viscoelastic flows discussed in W01, which seemed to bear a remarkable similarity to jet phenomena. This lead naturally to the question of whether turbulent conductive fluids supporting a tangled magnetic field can in some ways behave like viscoelastic fluids, and thus create a collimating force by tangled fields in analogy to the collimating force due to normal stress difference in polymeric laboratory viscoelastic fluids. Fortuitously enough for this line of reasoning, simulations of the MRI in particular demonstrate that MRI-induced turbulent stresses do in fact look in many ways like the stresses in viscoelastic, as opposed to viscous, fluids.

Of course, turbulent magnetized fluids are not really vis-
The primary open question is not whether the MRI can create confining magnetic hoop-stresses, but rather whether these stresses are strong enough to be a dominant confining mechanism for jet creation. Indeed, the ordering of speeds given in eq. (20) may be difficult to achieve in steady-state accretion flow scenarios. The weaker condition that the turbulent magnetic fields be dynamically significant, but perhaps not strong enough by themselves alone to create an anisotropic outflow, can be restated that the ratio $v_{\text{ew}}^2/v_{\text{ew}}$ not be much larger than unity. Conventional wisdom would suggest that this condition is difficult to meet in accretion flows in AGN and protostellar systems. On the other hand, our understanding of the physics of these systems is far from complete. Below, I offer as a hypothesis (see hypothesis (1)) the notion that in these accretion systems, when jets are produced, $v_{\text{ew}}^2$ is a substantial fraction of, if not larger than, $v_{\text{ew}}^2$, at the jet base.

Regardless of the validity of this hypothesis, it does appear that, as I argued in W01 and W03, the MHD turbulent hoop-stresses in accretion are at least of equal magnitude to the turbulent viscous stresses. Thus, global simulations of the inner regions of accretion and jet launching that treat turbulence with a purely viscous $\alpha$ model are of questionable physical relevance, as pointed out by Hawley et al. (2001). For example, this result casts doubt on the results of, e.g., Abramowicz et al. (2002), because these authors do not include the turbulent hoop-stresses in their radial force balance equation, and as noted above, the resultant force changes the value of the Bernoulli $b$, which must be taken into consideration. Furthermore, there is a simple and computationally cost-effective way to include these hoop-stresses in 2D, by the use of a viscoelastic turbulence model rather than a viscous model.

A second question, from the point of view of turbulence modeling, is the relative magnitude of of the MRI versus other instabilities (such as convection and, possibly, inertial instabilities) in creating turbulent hoop-stresses. As has been argued previously, the MRI, as a magnetohydrodynamic instability rather than a purely hydrodynamic instability, injects energy directly into the Maxwell stress tensor rather than indirectly, such as the case in convection, where energy may be expected to be transferred to the turbulent Maxwell stress by coupling to the Reynolds stress. Thus it is quite reasonable to expect that turbulent Maxwell stresses are relatively more significant than Reynolds stresses in purely MRI-driven turbulence, and this appears to be the case. In fact in W01 I assumed that the turbulent Maxwell stresses dominate the Reynolds stresses. This is not a necessary condition for the creation of collimating hoop stresses, but it helps, because otherwise one must contrive that the relaxation time for the Maxwell stresses is much longer than the relaxation time for the Reynolds stresses, since the action of the shear operator on the former creates positive (tension) streamwise stress whereas the action of the shear operator on the latter creates negative streamwise stress. In addition, in the case of the MRI the dimensional arguments presented above and in W01 and W03 argue that the effective relaxation time is roughly equal to $\Omega$ or $A$. These arguments no longer hold when shear and Coriolis forces no longer drive the instability.

A third question is the relative importance of tangled versus organized fields in this collimation. It remains to be seen the extent to which mean-field dynamo processes may be significant in MRI-generated collimating forces. Rather, here and previously I have emphasized tangled fields. I argue collimation by MRI-induced tangled fields to be a more robust result than collimation by MRI-induced organized fields. First, I have argued above that simulations may be interpreted as showing that the creation of small-scale fields is a more reliable feature of the MRI than creation of large-scale mean fields. Second, if the MRI fails to create a small-scale tangled field, then it also fails to provide a mechanism for turbulent transport of angular momentum. This would quite clearly be a problem since it is currently accepted that turbulence driven by the MRI is most likely largely if not solely responsible for angular momentum transport in ionized disks. A third point, particularly apropos in the case of transient accretion events such as collapsars, is that the creation of a tangled field may proceed faster than the creation of a large-scale mean field, to the extent that creation of a mean field depends upon inverse cascades acting over several orders of magnitude in length-scale.

Thus, Lyutikov et al. (2003), in the context of gamma-ray burst (GRB) fireballs, argue that the prompt creation of a large-scale field is problematic. However, citing Gruzinov & Waxman (1999) who in turn cite Loeb & Perna (1998), they argue that the linear polarization of GRB021206 seen by the RHESSI satellite as described by Coburn & Boggs (2003) implies a large-scale field [as indeed Coburn & Boggs (2003) argue as well] and that, since it is difficult to create such a field quickly, the field must exist prior to the catastrophe producing the GRB: ‘Such fields cannot be generated in a hydrodynamically-dominated outflow, which is causally disconnected on large scales. Thus, the large scale magnetic fields should be present [at the beginning]’. Lyutikov et al. (2003). The authors claim that ‘[A large polarization] implies that magnetic field coherence scale is larger than the size of the visible emitting region...’. The truth of this assertion depends upon what is meant by ‘coherence scale’. In particular, it is wrong to assert that the polarization implies a large-scale field: tangledness should in no way be confused with isotropy.

As an analogy, the orientation of mesogens in the nematic phase of liquid-crystal displays (LCDs) is coherent on a macroscopic scale. This large-scale anisotropy (or anisotropy) makes the speed of light depend upon polarization, even though the size of an individual mesogen is typically smaller than an optical wavelength (Rosenblatt, private communication). Likewise, a large-scale mean dyad field $B_iB_j$ does not imply a large-scale mean vector field $B_i$. Processes that do not discriminate between $B$ and $-B$ care about the former, not the latter; this includes the linear polarized emission of cyclotron and synchrotron radiation. In the case of cyclotron and synchrotron radiation, only the presence of circular polarization discriminates between a mean field and a mean dyad (Matsumiya & Ioka (2003)). Thus, large linear synchrotron polarization is possible due to a tangled field, so long as $M_{ij}$ is strongly anisotropic (Sagiv et al. (2004)).
Again regarding tangled versus ordered fields, Thompson et al. (2005) state, regarding the work of Akiyama et al. (2003), that ‘[The argument presented by Akiyama et al. (2003)] assumes that the magnetic field generated by the MRI can form the organized large-scale fields required for collimation and jet formation.’ Strictly speaking this may be true, since Akiyama et al. (2003) refer to mean fields rather than tangled fields. But, inasmuch as Akiyama et al. (2003) may be reinterpreted as describing the magnitude of the stress due to a tangled field rather then an ordered field as described above, the estimates for the forces due to the MRI remain the same. Thus, while the authors of Akiyama et al. (2003) were perhaps unaware of it, the processes they described does not actually depend upon the creation of organized large-scale fields, and in this sense Thompson et al. (2005) are incorrect.

Fourth and finally, the role of magnetic turbulent pressure and other stresses such as the neglected inertial stresses requires further analysis. I have argued here that the turbulent magnetic pressure will be anti-collimating in jet scenarios, and that it provides a reservoir of energy to accelerate the jet, but this conclusion is more tenuous than the more robust conclusions regarding the collimating normal stress differences caused by MHD turbulence. In addition, is conceivable that neither magnetic stresses nor inertial stresses are large enough by themselves to launch and collimate jets, but that the two acting together may create jets. Discussion of inertial stresses in particular is left to future work; notwithstanding their potential dynamical significance, I hypothesize below that the MHD turbulent stresses must be dynamically significant in a broad range of jet-producing accretion flows.

These problems discussed above, as well as many others, remain to be answered completely. None the less, I have shown here that my previous estimates of the forces due to the MRI remain the same. Thus, while the authors of Akiyama et al. (2003) were perhaps unaware of it, the processes they described does not actually depend upon the creation of organized large-scale fields, and in this sense Thompson et al. (2005) are incorrect.

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These problems discussed above, as well as many others, remain to be answered completely. None the less, I have shown here that my previous estimates of the turbulent collimating force are consistent with other estimates, and in particular the three lines of evidence and reasoning I presented above all reduce to the same condition, which may be stated in two different ways: The energy in the toroidal field is a sizeable fraction the orbital kinetic energy, or equivalently, the local toroidal magnetic wave speed is a sizeable fraction of the orbital speed. The difference between the theoretical approach based on the MRI dispersion relation and the approach based on turbulence modeling is simply whether the wave in question is a purely Alfvén wave or an Alfvén-like turbulent transverse elastic wave. It is not yet clear how sizeable a field is necessary, and whether condition (20) need indeed be met anywhere in the flow for a turbulent toroidal magnetic field to be dynamically significant.

Notwithstanding these remaining questions, then, I offer as hypotheses the notions that:

(i) MHD turbulence generated by the MRI (as well as perhaps additional instabilities) creates dynamically significant toroidal hoop stresses (more properly, a toroidal normal stress difference) that contribute significantly to jet creation and collimation;

(ii) this process does not depend upon mean-field dynamo processes, and in particular much of the field energy may be in a tangled field rather than a mean field;

(iii) inasmuch as MHD turbulence (driven by the MRI in particular) is thought to be the predominant source of turbulent viscous angular momentum transport in hot ionized disks and accretion flows, the corresponding turbulent elastic collimation process described here may be a nearly universal aspect of jet formation, applicable not just to the transient dynamics of collapsars but to jets in a broad range of quasi-steady-state systems such as AGN and quasars, microquasars, and protostellar systems; this hypothesis is offered notwithstanding the predominant understanding of thick accretion flows that would argue that such azimuthal turbulent magnetic fields should be relatively weak;

(iv) that this jet creation and collimation, as described, is not offered as a process that occurs in addition to traditional magnetocentrifugal acceleration mechanisms, but rather as an alternative to them.

Blandford & Payne (1982), Shu et al. (1994) and others have argued that jets are created by material flung out following large-scale open magnetic field lines. To be quite explicit, as a hypothesis, I reject the notion that such magnetocentrifugal mechanisms are responsible for tightly collimated jets. In such scenarios, power can not be transferred to the jet without transferring angular momentum as well. Note that recent observations of certain jets suggest that the fields may have helical structure (Gabuzda, Murray & Cronin (2004)) and the jets may be rotating (Coffey et al. (2004)). However, this in itself does not demonstrate magnetocentrifugal acceleration. Indeed, it would be surprising if Nature conspired to remove all of the residual angular momentum from material before accelerating it in a jet. The presence of a helical magnetic field, rather than a purely azimuthal or vertical field, can be explained as the result of advection, and does not necessarily say what is the dynamical role of that field, and in particular it does not necessarily imply that material is in some sense being flung out along field lines. Rather it is suggested here that the predominant role of the magnetic field in the central engine of a broad array of jet sources is the creation of turbulent hoop-stresses as described, which create and collimate an outflow deep in the core of the central engine. In analogy to viscoelastic laboratory phenomena, and in contrast to magnetocentrifugal mechanisms, in such a scenario material can take part in a collimated outflow not because angular momentum is transferred to it, but because angular momentum is removed from it.

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