On the Burgers dynamical system with an external force and its Koopman decomposition

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Abstract

We prove that the Burgers flow with a steady external forcing has a unique steady state which is a sink. Although this flow cannot be linearized through Cole-Hopf transforms, we prove that it has a convergent Koopman Modes decomposition. This gives an asymptotic formula for solutions of the Burgers equation with an external force. Time dependence and the coefficients of the decomposition are proved to be eigenvalues and eigenfunctionals of the Koopman operator. Convergence of the Koopman decomposition is proved for orbits close to the sink.

The analysis of Burgers dynamical system relies on the properties of a nonlinear heat flow, that shows invariant sets with complete orbits, and invariant sets where blow-up in finite time do occur. This behaviour helps understanding some instabilities in numeric computing for fluids.

1 introduction

Koopman flows were introduced by B.O.Koopman [1] to analyse hamiltonian finite dimensional flows. An important community in fluid dynamics makes use of the Koopman decomposition to analyse data originated from fluids

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computing. The Koopman decomposition for the field $u$ of velocities of a fluid is a representation of $u$ of the following form:

$$u(t, x) = \sum_{\nu} e^{-\lambda_{\nu} t} \varphi_{\nu}(u_0) a_{\nu}(x)$$

where $\nu$ belongs to some set of indices and $u_0$ is the initial condition. The $\lambda_{\nu}$'s are called the Koopman eigenvalues, the $a_{\nu}(x)$'s are called the Koopman modes.

In the important literature on Koopman flows (N.J.Kutz - S.L.Brunton - B.W.Brunton - J.L.Proctor [3], I.Mezic [5], C.W.Rowley - S.T.Dawson [6], P.J.Schmid [7],..) proofs are only given for finite dimensional systems of ODEs, but data analysis is performed for Navier-Stokes simulations. The only attempt for a PDE is, up to our knowledge, J.Page and R.R.Kerswell’s [4], where a procedure is given for the Burgers’ equation with no external forcing. Their procedure could not be completed because they encountered a ‘degeneracy’ due to multiplicity of the eigenvalues of the Koopman operator. This degeneracy is overcome in M.Balabane-M.Mendez-S.Najem [2].

This paper is devoted to the dynamical system defined by the Burgers equation on $[0, 1]$, with an external forcing $2 \partial_x V$:

$$\partial_t u + u \partial_x u = \partial^2_{xx} u + 2 \partial_x V \quad (1)$$

$$u(t, 0) = u(t, 1) = 0 \quad u(0, x) = u_0(x)$$

We prove that the dynamical system associated with this PDE has a unique sink, attracting all orbits. Notice that the function $V$ in (1) is defined up to an additive constant, so one can assume $V > 0$.

Let $\Phi_B^t(u_0) = u(t, \cdot)$. A Koopman decomposition for $\Phi_B^t$ is proved. We prove that the time dependences in the decomposition are given by the eigenvalues of the Koopman operator $K_B^t$, and the coefficients are values at the Cauchy data of the eigenfunctionals of the Koopman operator, which is defined (Koopman []) by:

$$(K_B^t \phi)(u_0) = \phi(\Phi_B^t(u_0))$$

where $\phi$ is any continuous functional on $L^2([0, 1])$, called observable in the DMD community.

The proofs are done through the Cole-Hopf transforms, although in the presence of the forcing this transform do not intertwine Burgers equation with
a linear equation, but with a nonlinear and nonlocal heat equation, namely:

\[ \partial_t \tilde{v} = (\partial^2_{xx} - V)\tilde{v} + (\int_0^t V(x)\tilde{v}(t, x)dx)\tilde{v} \]  

(2)

\[ \partial_x \tilde{v}(t, 0) = \partial_x \tilde{v}(t, 1) = 0 \quad \tilde{v}(0, x) = \tilde{v}_0(x) \]

Properties of the flow associated with (2) are proven, and its Koopman decomposition is derived.

The behaviour of this flow gives hints for instabilities in fluid computations.

To prove properties of this flow one relies on precise estimates for the following linear heat flow:

\[ \partial_t v = \partial^2_{xx} v - V(x)v \quad \partial_x v(t, 0) = \partial_x v(., 1) = 0 \quad v(0, x) = v_0(x) \in \mathcal{P}_+ \]  

(3)

In Section 2 the Cole-Hopf transforms are given. In section 3 links between the flows are proved. In section 4 the properties of the dynamical systems defined by heat equations are proved: in subsection 4.1 are overviewed the needed basic properties of the linear heat flow (3); in subsection 4.2 are stated the properties of the nonlinear heat flow defined by (2). Section 5 is devoted to Burgers dynamical system (1) and its Koopman decomposition. Straightforward proofs are given in the core of the text, and lengthy proofs are given in section 6.

2 the Cole-Hopf transforms

The Cole-Hopf transforms are given by:

\[ u := C(v) = -2\frac{\partial_x v}{v} \quad \text{and} \quad v := H(u) = \frac{e^{-\frac{1}{2} \int_0^1 u(s)ds}}{\int_0^1 e^{-\frac{1}{2} \int_0^1 u(s)ds}dx} \]  

(4)

\( H \) is defined on \( L^2([0, 1]) \). \( H(u) \) is strictly positive on \([0, 1]\), square integrable as well as its first weak derivative on \([0, 1]\), and fulfills \( \int_0^1 H(u)(s)ds = 1 \). The set of functions with these properties is denoted by \( \mathcal{P}_1 \).

\( C \) is defined for strictly positive functions on \([0, 1]\), square integrable as well as their first weak derivative (hence continuous). The set of functions fulfilling these properties is denoted by \( \mathcal{P}_+ \).
One has, for all $u \in L^2([0,1]), C(H(u)) = u$. If $v \in \mathcal{P}_1$ then $H(C(v)) = v$. It follows that $H$ is a bijection from $L^2([0,1])$ onto $\mathcal{P}_1$, with inverse $C$. Both maps are differentiable. A useful remark is that for any (spatially) constant $\delta$ one has

$$C(\delta v) = C(v)$$

Conversely if $v_1$ and $v_2$ belong to $\mathcal{P}_+$ and fulfill $C(v_1) = C(v_2)$ then $g(v_1)v_2(x) = g(v_2)v_1(x)$ where $g(v) = \int_0^1 v(s)ds$. Hence $\frac{v_1}{v_2}$ do not depend on the space variable $x$.

## 3 linking the flows

### 3.1 Intertwining $\Phi^t_B$ and $\Phi^t_N$

Let $\Phi^t_B$ denote the flow associated with Burgers equation (1), meaning that $u(t,.) = \Phi^t_B(u_0)$ solve Burgers equation (1) with Cauchy data $u_0 \in L^2([0,1])$. $\Phi^t_B$ is a flow in $L^2([0,1])$. Let $\tilde{v}_0 = H(u_0)$ and $\tilde{v}(t,.) = H(u(t,.)$, so that $u(t,.) = C(\tilde{v}(t,.))$. Then

$$\forall t \geq 0, \ \tilde{v}(t,.) \in \mathcal{P}_1 \ \tilde{v}(0,.) = \tilde{v}_0, \ \partial_x \tilde{v}(t,0) = \partial_x \tilde{v}(t,1) = 0, \ \int_0^1 \tilde{v}(t,\xi)d\xi = 1$$

In order to get a PDE for $\tilde{v}$ one notices that:

$$\partial_t u(t,.) = -2\partial_x \frac{\partial_x \tilde{v}(t,.)}{\tilde{v}(t,.)} = -2\partial_x \frac{\partial_t \tilde{v}(t,.)}{\tilde{v}(t,.)}$$

so

$$\partial_t u + u \partial_x u = -2\partial_x \left[ \frac{\partial_t \tilde{v}}{\tilde{v}} - \left( \frac{\partial_x \tilde{v}}{\tilde{v}} \right)^2 \right] = -2\partial_x \left[ \frac{\partial_t \tilde{v}}{\tilde{v}} - \frac{\partial^2 \tilde{v}}{\tilde{v}} + \partial_x \left( \frac{\partial_x \tilde{v}}{\tilde{v}} \right) \right]$$

On the other hand

$$\partial_x (\partial_x u + 2V) = -2\partial_x (\partial_x \left( \frac{\partial_x \tilde{v}}{\tilde{v}} \right) - V)$$

Equation (1) is therefore equivalent to the existence of $n(t)$ such that:

$$\partial_t \tilde{v} = \partial^2_{xx} \tilde{v} - V\tilde{v} + n(t)$$
Integrating this equality gives:

\[ n(t) = \int_0^1 V(x) \tilde{v}(t, s) \, ds \]

and proves that \( \tilde{v} \) solves the nonlinear heat equation (2).
The converse is true because the computation is performed through equivalences.
This proves that if \( \Phi_t^N \) denotes the flow associated with equation (2) then:

\[ H \circ \Phi_t^B = \Phi_t^N \circ H \quad \text{and} \quad C \circ \Phi_t^N = \Phi_t^B \circ C \]  

(6)

In sections 4.2.1 and 4.2.2 \( \Phi_t^N \) is proved to be a flow in \( P_1 \).

### 3.2 linking \( \Phi_t^N \) and \( \Phi_t^H \)

Let \( \Phi_t^H \) denote the flow defined by equation (4). It is a flow on \( P_+ \), as proved in section 4.1.2. Let \( v_0(x) = \tilde{v}_0(x) \in P_1 \). Let \( v(t,.) = \Phi_t^H(v_0) \) and \( \tilde{v}(t,.) = \Phi_t^N(\tilde{v}_0) \) for \( t \in [0, T_*(\tilde{v}_0)] \), the time span of \( \Phi_t^N(\tilde{v}_0) \).

Let \( g(t) = \int_0^1 v(t, \xi) \, d\xi \) so \( g'(t) = -\int_0^1 V(\xi) v(t, \xi) \, d\xi \).

Let \( f(t, x) = g(t) \tilde{v}(t, x) - v(t, x) \).

Then:

\[ \partial_t f - (\partial_{xx}^2 - V) f - \tilde{v}(t, x) (\int_0^1 V(\xi) f(t, \xi) \, d\xi) = 0 \]

and

\[ \partial_x f(t, 0) = \partial_x f(t, 1) = 0 \]

This implies, for any \( t \in [0, T] \) with \( T < T_*(\tilde{v}_0) \):

\[ \frac{d}{dt} \| f \|_{L^2}^2 \leq \int_0^1 V(\xi) f(t, \xi) \, d\xi \int_0^1 \tilde{v}(t, \xi) f(t, \xi) \, d\xi \]

\[ \leq \| V \|_{L^2} \sup_{[0, T]} \| \tilde{v} \|_{L^2} \| f \|_{L^2}^2 \]

But \( f(0, x) = 0 \) so Gromwall’s lemma gives \( f = 0 \) and:

\[ g\tilde{v} = v \]  

(7)

Because \( \Phi_t^H \) is a flow on \( P_+ \) as recalled in section 4.1, \( g(t) > 0 \), so equation (7) translates to:

\[ \Phi_t^N = \frac{1}{g(t)} \Phi_t^H \]  

(8)
Using (6), and invariance (5) of $C$ under a scaling, this gives for $t \in [0, T^*(\tilde{v}_0)]$:

$$C \circ \Phi^t_{\mathcal{N}} = C \circ \Phi^t_{\mathcal{H}} = \Phi^t_{\mathcal{B}} \circ C \quad (9)$$

Global existence of the flow of $\Phi^t_{\mathcal{N}}$ on $\mathcal{P}_1$ is proven in 4.2.3, hence $T^*(\tilde{v}_0) = \infty$.

4 on the linear and nonlinear heat flows

4.1 on the linear heat flow

Assume $V$ is a regular, strictly positive function on $[0, 1]$. Consider the linear heat equation (3) on $[0, 1]$ with $V$ for potential:

$$\partial_t v = \partial^2_{xx} v - V(x)v \quad \partial_x v(t, 0) = \partial_x v(t, 1) = 0 \quad v(0, x) = v_0(x) \in \mathcal{P}_+ \quad (10)$$

4.1.1 basic estimates

Because $v_0$ is in $H^1$ one has the estimate:

$$\forall t \geq 0 \quad \|v(t, \cdot)\|_{H^1} \leq C \|v_0\|_{H^1}$$

Regularising effect of the heat equation enables to extend this flow to Cauchy data in $L^2([0, 1])$ and one has:

$$\forall t > 0 \quad \|v(t, \cdot)\|_{H^1} \leq \frac{C_1}{\sqrt{t}} \|v_0\|_{L^2}$$

Notice that the regularising effect extends to all derivatives, so the solution with Cauchy data in $L^2([0, 1])$ lays in all Sobolev spaces and one has the following estimates:

$$\forall t > 0 \quad \|v(t, \cdot)\|_{H^s} \leq \frac{C_s}{\sqrt{t}^{s-1}} \|v_0\|_{L^2}$$

4.1.2 on positivity

By Kato’s lemma, if $v_0 > 0$ then $v(t, x) > 0$ for all $t > 0$, so the associated flow acts on $\mathcal{P}_+$, where it is denoted by by $\Phi^t_{\mathcal{H}}$:

$$\forall t \geq 0, \forall v_0 \in \mathcal{P}_+, \quad \Phi^t_{\mathcal{H}}(v_0) = v(t, \cdot) > 0 \quad (11)$$
We need below an enhancement of this property, namely that:

$$\forall t \geq 0, \forall x \in [0,1], \forall v_0 \in \mathcal{P}_+, \quad v(t,x) \geq e^{-t \sup_{[0,1]} v_0(x)}$$

(12)

It can be proved as follows: the heat equation with $V = 0$ and Neumann boundary conditions is invariant by adding a constant to the state variable. Hence positivity preservation gives $e^{t \partial^2_{xx}} v_0 \geq \inf_{[0,1]} v_0(x)$. The Trotter-Kato formula gives

$$\Phi^t_H(v_0)(t,x) = \lim_{N \to \infty} \left( e^{t \frac{1}{N} \partial^2_{xx} e^{-t \frac{1}{N} V(x)}} \right)^N v_0(x)$$

hence the lower bound.

4.1.3 on eigenvalues

The operator $A = -\partial^2_{xx} + V(x)$ is self-adjoint from $H^1([0,1])$ to its dual space, and has a bounded inverse. Hence by Rellich argument its spectrum is an increasing sequence of positive eigenvalues $(\mu_n)_{n \in \mathbb{N}}$ associated with a complete orthonormal set in $L^2([0,1])$ of eigenfunctions $(e_n(x))_{n \in \mathbb{N}}$:

$$\forall n \in \mathbb{N}, \quad \partial^2_{xx} e_n(x) - V(x)e_n(x) = -\mu_n e_n(x) \quad \partial_x e_n(0) = \partial_x e_n(1) = 0$$

(13)

Let:

$$\forall n \in \mathbb{N}, \forall v_0 \in H^1([0,1]) \quad c_n(v_0) = \int_0^1 e_n(x)v_0(x)dx$$

(14)

then

$$v(t,x) = \sum_{n=0}^{\infty} e^{-\mu_n t} c_n(v_0) e_n(x)$$

(15)

Because $\mu_0$ is a variational extremum, $e_0(x)$ do not vanish in $]0,1[$. One can assume $e_0(x) > 0$ on $]0,1[$. An orthogonality argument shows that it is the only eigenfunction with constant sign. One can even prove, comparing the Neumann minimisation with the mixed Dirichlet-Neumann minimisation that $e_0(x) > 0$ on $[0,1]$. So

$$m_0 = \inf_{[0,1]} e_0(x) > 0.$$ 

(16)
Note that this implies:

\[ \forall v_0 \in \mathcal{P}_+ \quad c_0(v_0) = m_0 \int_0^1 v_0(x) dx > 0 \quad (17) \]

Note also that due to the variational caracterisation of \( e_0 \), multiplicity of \( \mu_0 \) is equal to one, therefore \( \mu_0 < \mu_1 \).

### 4.1.4 on the spatial mean

The spatial mean of \( v(t, x) \) solving (3) is defined in [3.2] by:

\[ g(t) = \int_0^1 v(t, \xi) d\xi \quad (18) \]

Because \( v_0 \in \mathcal{P}_+ \) positivity of \( v \) implies

\[ g'(t) = -\int_0^1 V(x)v(t, x)dx \geq -(\sup V)g(t) \]

so

\[ g(t) \geq e^{-(\sup V)t} \int_0^1 v_0(x) dx > 0 \quad (19) \]

### 4.1.5 two asymptotic estimates

The first estimate, proved in [6.1] is about the mean \( g(t) \). It is needed in [4.2.8] for the Koopman decomposition of solutions of the nonlinear heat equation (2). It states that for all \( t \geq 0 \), and all \( v_0 \in \mathcal{P}_+ \), if \( v \) solves equation (3), then:

\[ |e^{\mu_0 t}g(t) - c_0(v_0)| \int_0^1 e_0(\xi)d\xi| \leq e^{-(\mu_1 - \mu_0)t}\|v_0 - c_0(v_0)e_0(x)\|_{L^2}\|1 - c_0(1)e_0(x)\|_{L^2} \quad (20) \]

The second estimate, proved in [6.2] is needed for the Koopman decomposition of the Burgers flow [1] given in section [5.3]. It states that:

\[ \forall t > 0 \quad \sup_{x \in [0,1]} |e^{\mu_0 t}v(t, x) - c_0(v_0)e_0(x)| \leq \|v_0 - c_0(v_0)e_0\|_{L^2}h_1(t) \quad (21) \]
where $h_1(t)$ do not depend on $v_0$ (it is given explicitly in section 5.3). It is a decreasing function on $]0, \infty]$ with limit zero at infinity. It is given explicitly in section 6.2.

Note that the two estimates are invariant by a multiplication of $v_0$ by a positive constant.

4.1.6 on the Koopman flow

Let $\mathcal{F}_+$. be the set of continuous maps from $\mathcal{P}_+$ to $\mathbb{R}$. The Koopman flow $\mathcal{K}_t^\mathcal{H}$ associated with $\Phi_t^\mathcal{H}$ acts on $\mathcal{F}_+$ through:

$$\forall \phi \in \mathcal{F}_+, \forall t \geq 0, \forall v_0 \in \mathcal{P}_+, \quad (\mathcal{K}_t^\mathcal{H}(\phi))(v_0) = \phi(\Phi_t^\mathcal{H}(v_0))$$

(22)

For any $n \in \mathbb{N}$ the functional $c_n$ is an eigen-functional of the Koopman flow $\mathcal{K}_t^\mathcal{H}$, with eignevalue $e^{-\mu_n t}$, because for any $v_0 \in \mathcal{P}_+$:

$$(\mathcal{K}_t^\mathcal{H}(c_n))(v_0) = c_n(\Phi_t^\mathcal{H}(v_0)) = c_n\left(\sum_{k \in \mathbb{N}} e^{-\mu_k t} e_k(v_0) e_k(x)\right) = e^{-\mu_n t} c_n(v_0)$$

(23)

Let $\nu = (q_0, q_1, \ldots, q_m)$ with $q_i \in \mathbb{N}$. Let $\lambda_\nu = \sum_{i=0}^m \mu_q$. The multiplicative property of Koopman operators implies:

$$\mathcal{K}_t^\mathcal{H}\sigma_\nu = e^{-t\lambda_\nu} \sigma_\nu \quad \text{for} \quad \sigma_\nu := \prod_{k=0}^m c_{q_k}$$

(24)

so $e^{-t\lambda_\nu}$ is an eigenvalue of $\mathcal{K}_t^\mathcal{H}$ for the eigen-functional $\sigma_\nu$.

An interesting remark is that the Koopman decomposition of the flow $\Phi_t^\mathcal{H}$ is given by formula (15) where the only eigen-functionals showing are those whose index is of length one. This is due to the linearity of the flow $\Phi_t^\mathcal{H}$ and the linearity of the functional $\delta_x$ that maps any function in $H^1$ to its value at location $x$ (note that on $H^1$ these are continuous functionals).

4.2 a nonlinear heat flow

Cole-Hopf transforms intertwin Burgers equation with the following nonlinear heat equation on $[0, 1]$:

$$\partial_t \tilde{v} = \partial^2_{xx} \tilde{v} - V(x)\tilde{v} + \left(\int_0^1 V(\xi)\tilde{v}(t, \xi)d\xi\right)\tilde{v}$$

(25)
\[ \partial_x \tilde{v}(t, 0) = \partial_x \tilde{v}(t, 1) = 0 \quad \tilde{v}(0, x) = \tilde{v}_0(x) \in \mathcal{P}_+ \]

Local existence of the solution is granted in \( L^2([0, 1]) \). Let \([0, T_*(\tilde{v}_0)]\) be the maximal time span.

**4.2.1 on positivity**

The flow \( \Phi^t_N \) preserves positivity for \( t \in [0, T_*(\tilde{v}_0)] \). This can be checked using Duhamel formula and property (11) as follows, translating formula (25) to:

\[
\tilde{v}(t, x) = e^{t(\partial_x^2 - V)} \tilde{v}_0 + \int_0^t e^{(t-\tau)(\partial_x^2 - V)} \left( \int_0^1 V(\xi) \tilde{v}(\tau, \xi) d\xi \right) \tilde{v}(\tau, x) d\tau
\]

As long as \( \tilde{v} \) is positive, one has \( \int_0^1 V(\xi) \tilde{v}(t, \xi) d\xi \geq 0 \), hence applying section 4.1.2, \( e^{(t-\tau)(\partial_x^2 - V)} \left( \int_0^1 V(\xi) \tilde{v}(\tau, \xi) d\xi \right) \tilde{v}(\tau, x) \geq 0 \), hence

\[ \tilde{v}(t, x) \geq e^{t(\partial_x^2 - V)} \tilde{v}_0 \geq e^{-t} \sup_{[0,1]} \tilde{v}_0(x) \quad (26) \]

therefore \( \tilde{v} \) is trapped above \( e^{-t} \sup_{[0,1]} \tilde{v}_0(x) \) as long as it is positive, so:

\[
\forall t \in [0, T_*[, \forall x \in [0, 1] \quad \tilde{v}(t, x) > 0 \quad (27)
\]

Duhamel formula (26) shows moreover that the solution belongs to \( H^1([0, 1]) \) for all \( t \in [0, T_*(\tilde{v}_0)] \).

Positivity preservation and regularity imply that \( \mathcal{P}_+ \) is invariant by the flow defined by equation (25).

**4.2.2 on the spatial mean**

Let \( \tilde{v}_0 \in \mathcal{P}_+ \). Let \( \tilde{g}_0 = \int_0^1 \tilde{v}_0(\xi) d\xi \). Let \( \tilde{g}(t) = \int_0^1 \tilde{v}(t, \xi) d\xi \) where \( \tilde{v}(t, x) \) solves (2) on its maximal time span \( t \in [0, T_*(\tilde{v}_0)] \). One has

\[
\frac{d}{dt}(\tilde{g}(t) - 1) = \left( \int_0^1 V(\xi) \tilde{v}(t, \xi) d\xi \right) (\tilde{g}(t) - 1) \quad (28)
\]

This implies for \( t \in [0, T_*(\tilde{v}_0)] \):

\[
\tilde{g}(t) > 1 \quad \text{if} \quad \tilde{g}_0 > 1
\]

\[
\tilde{g}(t) = 1 \quad \text{if} \quad \tilde{g}_0 = 1 \quad (29)
\]
\[
\tilde{g}_0 e^{-t \sup V} \leq \tilde{g}(t) < 1 \quad \text{if} \quad 0 \leq \tilde{g}_0 < 1
\]

Note that formula (29) and positivity of \( \tilde{v} \) proven in 4.2.1 show through formula (28) that \( \tilde{g} \) is decreasing if \( \tilde{g}_0 < 1 \), and increasing if \( \tilde{g}_0 > 1 \).

Formula (27) shows that positivity is preserved by the flow defined by equation (2). This added to formula (29) shows that \( \mathcal{P}_1 \) is invariant by this flow.

We denote it by \( \Phi^t_N \) as acting in \( \mathcal{P}_1 \):

\[
\forall t \geq 0, \forall \tilde{v}_0 \in \mathcal{P}_1, \quad \Phi^t_N(\tilde{v}_0) = \tilde{v}(t,.) \in \mathcal{P}_1 \quad (30)
\]

### 4.2.3 on global existence

Let \( 0 \leq \tilde{g}_0 \leq 1 \). For \( t \in [0, T_*(\tilde{v}_0)] \) integration by parts gives:

\[
\frac{d}{dt} \frac{1}{2} \| \tilde{v} \|^2_{L^2} \leq \left( \int_0^1 V(\xi) \tilde{g}(t,\xi) d\xi \right) \| \tilde{v} \|^2_{L^2}
\]

Positivity proven in 4.2.1 gives:

\[
\frac{d}{dt} \frac{1}{2} \| \tilde{v} \|^2_{L^2} \leq (\sup_{[0,1]} V) \tilde{g}(t) \| \tilde{v} \|^2_{L^2}
\]

Formula (29) implies:

\[
\frac{d}{dt} \frac{1}{2} \| \tilde{v} \|^2_{L^2} \leq (\sup_{[0,1]} V) \| \tilde{v} \|^2_{L^2}
\]

This implies global existence of \( \tilde{v} \) in \( L^2([0,1]) \) if \( 0 \leq \tilde{g}_0 \leq 1 \), and the estimate:

\[
\| \tilde{v} \|_{L^2} \leq e^{t \sup V} \| \tilde{v}_0 \|_{L^2} \quad (31)
\]

Duhamel formula in 4.2.1 grants global existence in \( H^1([0,1]) \).

### 4.2.4 the case \( \tilde{g}_0 = 1 \): global existence and regularity

If \( \tilde{g}_0 = 1 \) an important link between equation (2) and equation (3) is the following: let \( v \) solve (3) with initial condition in \( \mathcal{P}_1 \). Because of formula
the following function \( \tilde{w} = \frac{\tilde{v}}{g(t)} \) is well-defined, fulfills \( \tilde{w}_0 = \tilde{w}(0, x) \in P_1 \), and:
\[
\partial_t \tilde{w} = \frac{\partial_v v}{g(t)} - \frac{\partial_g g}{g^2} v = \frac{1}{g(t)} (\partial^2_{xx} - V)v + \left( \int_0^1 V \; v \right) v = (\partial^2_{xx} - V) \tilde{w} + \left( \int_0^1 V \tilde{w} \right) \tilde{w} 
\]
so \( \tilde{w} \) solves equation (2). This implies
\[
\tilde{v}_0 \in P_1 \implies \tilde{v} = \frac{v}{g} \quad \text{with} \quad v_0 = \tilde{v}_0
\]
Note that this formula gives another proof of global existence of the solution of (2), because \( v(t, \cdot) \) exists globally for all \( t \geq 0 \), and \( g(t) \) is strictly positive for all \( t \geq 0 \), by formula (19).
This formula has also the following important consequence: for \( \tilde{v}_0 \in P_1 \) the solution \( \tilde{v} \) is infinitely differentiable, due to the regularity of the solution of the linear heat equation as stated in 4.1.1.

4.2.5 on blow-up

If \( \tilde{g}_0 > 1 \) the solution of equation (2) blows-up in finite time. This follows the blow-up of \( \tilde{g} \), which is a consequence of the following estimate:
\[
\frac{d}{dt} \tilde{g}(t) = (\int_0^1 V(\xi) \tilde{v}(t, \xi) d\xi) (\tilde{g}(t) - 1) \geq (\inf_{[0,1]} V) (\tilde{g}(t) - 1) \tilde{g}(t)
\]
A complete description of the blow-up of \( \tilde{v} \) can be given through the change of state variable:
\[
\tilde{z} = \frac{\tilde{v}}{\tilde{g}}
\]
where \( \tilde{v} \) solves (2) with \( \tilde{v}(0, x) \in P_+ \). \( \tilde{z} \) fulfills:
\[
\partial_t \tilde{z} = \frac{\partial\tilde{v}}{\tilde{g}} - \frac{\partial\tilde{g}}{\tilde{g}^2} \tilde{v} = (\partial^2_{xx} - V) \tilde{z} + \left( \int_0^1 V \tilde{z} \right) \tilde{z}
\]
Therefor \( \tilde{z} \) solves equation (2) and \( \tilde{z}(0, x) \in P_1 \), hence \( \tilde{z} \) exists for all \( t \geq 0 \) by 4.2.3. Equality \( \hat{g} \tilde{z} = \hat{v} \) implies that \( \tilde{v} \) blows-up at all spatial locations at the same blow-up time: the blow-up time of \( \hat{g} \).
The behaviour of solutions of (2) on \( P_+ \) can help understand some numerical instabilities of fluids computation, computing Burgers solution being, by a change of the state variable, computing solutions of (2) on \( P_1 \).
4.2.6 on steady states

If \( \tilde{s}(x) \) is a steady state solution of the nonlinear heat equation \((2)\) in \( P_+ \), it solves:

\[
\partial^2_{xx} \tilde{s} - V(x)\tilde{s} + \left( \int_0^1 V(\xi)\tilde{s}(\xi)d\xi \right)\tilde{s} = 0
\]  \( (33) \)

with Neumann boundary conditions. Equation \( (33) \) states that \( \tilde{s} \) is an eigenfunction of \( \partial^2_{xx} - V(x) \) that belongs to \( P_+ \). Hence \( \tilde{s}(x) = Ce_0(x) \) for some \( C \in \mathbb{R}_+ \) as proved in 4.1.3. Computing \( C \) from \( (33) \) shows uniqueness of the steady state in \( P_+ \), namely:

\[
\tilde{f}_0(x) = \frac{\mu_0}{\int_0^1 V(\xi)e_0(\xi)d\xi} e_0(x)
\]  \( (34) \)

Integrating equation \( (13) \) gives

\[
\int_0^1 V(\xi)e_0(\xi)d\xi = \mu_0 \int_0^1 e_0(\xi)d\xi,
\]

so:

\[
\tilde{f}_0(x) = \frac{e_0(x)}{\int_0^1 e_0(\xi)d\xi}
\]  \( (35) \)

This proves that \( \tilde{f}_0 \in P_1 \), so the flow on \( P_1 \) has a unique steady state. It is a sink: this can be checked using the formula \( (8) \) and asymptotics \( (20) \) and \( (21) \) with \( t \to \infty \) as follows:

\[
\lim_{t \to \infty} \tilde{v}(t, x) = \lim_{t \to \infty} \frac{e^{\mu_0 t} V(t, x)}{e^{\mu_0 t} g(t)} = \frac{c_0(v_0)e_0(x)}{c_0(v_0) \int_0^1 e_0(\xi)d\xi} = \tilde{f}_0(x)
\]

4.2.7 on spectral elements of the Koopman operator

In this section are computed spectral elements of the Koopman flow \( K^t_N \) associated with the flow \( \Phi^t_N \). These elements are the building blocks of the Koopman decomposition of \( \Phi^t_N \) performed in the next section.

To give a precise definition of the Koopman operator \( K^t_N \), one needs the set \( F_1 \) of observables on \( P_1 \): it is the set of maps from \( P_1 \) to \( \mathbb{R} \) which are continuous for the \( H^1 \) norm given by:

\[
\|\bar{v}_0\|_{H^1}^2 = \|\bar{v}_0\|_{L^2}^2 + \|\partial_x \bar{v}_0\|_{L^2}^2
\]

The Koopman operator \( K^t_N \) maps \( \phi \in F_1 \) to \( K^t_N(\phi) \in F_1 \) by the formula:

\[
\forall \bar{v}_0 \in P_1 \quad (K^t_N(\phi))(\bar{v}_0) = \phi(\Phi^t_N(\bar{v}_0))
\]  \( (36) \)
This definition is consistent because $P_1$ is invariant under $\Phi^t_N$ due to (27), (29) and (31).

To compute spectral elements of $K^t_N$ one notices that all functionals $c_n$ given by (14) belong to $F_1$. Moreover strict positivity of $c_0$ on $P_1$ proved in (17) shows that $\frac{1}{c_0}$ is an observable on $P_1$. Therefore

$$\forall q \in \mathbb{N}, \quad \frac{c_q}{c_0} \in F_1$$

(37)

Formulas (8) and (23) give, for $q \geq 0$:

$$\forall \tilde{v}_0 \in P_1 \quad K^t_N(c_q)(\tilde{v}_0) = c_q(\Phi^t_N(\tilde{v}_0)) = \frac{1}{g(t)} c_q(\Phi^t_N(\tilde{v}_0)) = \frac{1}{g(t)} e^{-\mu q t} c_q(\tilde{v}_0)$$

Therefore, the multiplicative property of $K^t_N$ gives:

$$\forall q \geq 0, \quad K^t_N \left( \frac{c_q}{c_0} \right) = e^{-(\mu q - \mu_0) t} \frac{c_q}{c_0}$$

(38)

Applying once more the multiplicative property gives eigenvalues and eigenobservables of $K^t_N$, namely that $e^{-\lambda_\nu t}$ is an eigenvalue of $K^t_N$, with eigenfunctional $\psi_\nu$, where:

$$\nu = (q_0, q_1, ..., q_m) \in \mathbb{N} \times (\mathbb{N}\{0\})^m \quad \text{for} \quad m \in \mathbb{N}$$

(39)

$$\lambda_\nu = \sum_{i=0}^k (\mu_{q_i} - \mu_0) \quad \text{and} \quad \psi_\nu = \prod_{i=0}^m \frac{c_{q_i}}{c_0}$$

(40)

4.2.8 on Koopman decomposition

Let $\tilde{v}_0 \in P_1$. In order to motivate the Koopman decomposition for $\Phi^t_N(\tilde{v}_0)$, it is worth considering formulas (32) and (15), written using (16) as:

$$\tilde{v}(t, x) = \Phi^t_N(\tilde{v}_0) = \frac{1}{\int_0^1 e_0(\xi) d\xi} \sum_{n=0}^\infty e^{-(\mu_n - \mu_0) t} \frac{c_n(\tilde{v}_0)}{c_0(\tilde{v}_0)} e_n(x)$$

(41)

where

$$\tilde{k}(t) = \sum_{q=1}^\infty e^{-(\mu_q - \mu_0) t} \frac{c_q(\tilde{v}_0)}{c_0(\tilde{v}_0)} p_q \quad \text{with} \quad p_q = \frac{\int_0^1 e_q(\xi) d\xi}{\int_0^1 e_0(\xi) d\xi}$$
Formula (41) is not a Koopman decomposition for $\Phi^t_N(\tilde{v}_0)$ because the time evolution is not given by exponentials. But all quantities involved are spectral elements of the Koopman operator, computed in the above section.

In order to derive a Koopman decomposition for $\Phi^t_N(\tilde{v}_0)$ one must expand

$$ (1 + \tilde{k})^{-1} = \sum_{m=0}^{\infty} (-1)^m \tilde{k}^m. $$

For this end one must have $|\tilde{k}| < 1$. Formula (20) gives:

$$ |\tilde{k}(t)| \leq \frac{e^{-(\mu_1 - \mu_0)t}}{c_0(\tilde{v}_0) \int_0^1 e_0(\xi) d\xi} \| \tilde{v}_0 - c_0(\tilde{v}_0) e_0(\cdot) \|_{L^2} 1 - c_0(1) e_0(\cdot) \|_{L^2} $$

Let:

$$ \tau_N(\tilde{v}_0) = \frac{1}{\mu_1 - \mu_0} \log(\frac{\| \tilde{v}_0 - c_0(\tilde{v}_0) e_0(\cdot) \|_{L^2} 1 - c_0(1) e_0(\cdot) \|_{L^2}}{c_0(\tilde{v}_0) \int_0^1 e_0(\xi) d\xi}) $$

Section 6.3 gives the procedure by which, for $t \geq 0$ fulfilling $t > \tau_N(\tilde{v}_0)$, formula (41) can be written as:

$$ \tilde{v}(t, x) = \sum_{q_0=0}^{\infty} \sum_{m=0}^{\infty} \sum_{(q_1, \ldots, q_m) \in (\mathbb{N} \setminus \{0\})^m} e^{-\lambda_{\nu} t} \psi_{\nu}(\tilde{v}_0) b_\nu(x) $$

(42)

with

$$ \nu = (q_0, \ldots, q_m) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})^m \quad b_\nu(x) = (-1)^m \prod_{i=1}^m p_{q_i} e_{q_0}(x) \quad \text{for} \quad m \neq 0 $$

and, for $m = 0$, the last sum in (42) reduced to one element, namely:

$$ e^{-\mu_{q_0} t} \frac{c_{q_0}(\tilde{v}_0)}{c_0(\tilde{v}_0) \int_0^1 e_0(\xi) d\xi} $$

This is a Koopman decomposition for $\Phi^t_N(\tilde{v}_0)$. The Koopman modes are the functions $b_\nu(x)$. For each mode the time behaviour is given by the eigenvalue $e^{-\lambda_{\nu} t}$ of the Koopman operator. The coefficient is given by the value of the associated eigen-functional $\psi_{\nu}$ at the Cauchy data.

It is interesting to notice that each Koopman mode corresponds to (infinitely) many eigenvalues.

The only nondecreasing component in this decomposition is for $(q_0, m) = (0, 0)$. It corresponds, as expected, to the sink $\tilde{f}_0$. 

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It is important to point out that the Koopman decomposition is valid for all $t \geq 0$ (i.e. $\tau_N(\tilde{v}_0) < 0$) if the Cauchy data $\tilde{v}_0$ belongs to a neighbourhood of the sink $\tilde{f}_0$, namely:

$$\Omega_N = \{ \tilde{v}_0 \in P_1 ; \frac{\|\tilde{v}_0 - c_0(\tilde{v}_0)e_0\|_{L^2}^2}{c_0(\tilde{v}_0) \int_0^1 e_0(\xi)d\xi} < 1 \}$$  \hspace{1cm} (43)

Notice that although the spectral elements, building bricks of the Koopman decomposition, are defined on all $P_1$, the decomposition exists only locally on $P_1$.

### 4.2.9 on absolute convergence

The series in formula (42) is convergent under the assumption $\tilde{v}_0 \in \Omega_N$, but it is not absolutely convergent. This means that the summation order is mandatory. In order to have convergence in the $\sup_x$ norm, enabling through Fubini theorem a summation in arbitrary order, and analytic use of the formula, through integration for instance, one needs a more restrictive assumption. To state it let:

$$C_V = \sup_{x \in [0,1]} V(x)$$ and $$h_2(t) = \sqrt{\sum_{1}^{\infty} \frac{e^{-2(\mu_q-\mu_0)t}}{\mu_q^3}}$$  \hspace{1cm} (44)

The function $h_2(t)$ is defined for all $t \geq 0$, due to formula (60) below. It is decreasing and has limit zero at infinity. For any $\tilde{v}_0 \in P_1$ let:

$$\tilde{\tau}_N(\tilde{v}_0) = \inf \{ t > 0 ; h_2(t) < \frac{c_0(\tilde{v}_0) \int_0^1 e_0(\xi)d\xi}{C_V \|V\|_{L^2} \|\tilde{v}_0 - c_0(\tilde{v}_0)e_0\|_{H^1}} \}$$  \hspace{1cm} (45)

If $t > \max(\tilde{\tau}_N(\tilde{v}_0), \tau_N(\tilde{v}_0))$ then the Koopman decomposition of $\tilde{v} = \Phi^t_{\Lambda_N} \tilde{v}_0$ is given by the following series, which is absolutely convergent in the $\sup_x$ norm:

$$\tilde{v}(t, x) = \sum_{\nu \in \mathcal{B}} e^{-\lambda_{\nu}t} \psi_{\nu}(\tilde{v}_0)b_{\nu}(x)$$  \hspace{1cm} (46)

where:

$$\mathcal{B} = \bigcup_{m=0}^{\infty} B_m$$

$$\forall m \in \mathbb{N}; \quad B_m = \{ \nu = (q_0, \ldots, q_m); q_0 \in \mathbb{N}, \text{ and } q_i \in \mathbb{N}\setminus\{0\} \} \quad \text{for } i = 1, \ldots, m$$

The proof is given in section [6.4](#).
A consequence of formula (45) is that there exists a neighbourhood $\tilde{\Omega}_N$ of the sink $\tilde{f}_0$ such that for any $\tilde{v}_0 \in \tilde{\Omega}_N$ absolute convergence occurs for all $t > 0$. $\tilde{\Omega}_N$ is the set of functions $\tilde{v}_0 \in P_1$ such that:

$$\|\tilde{v}_0 - c_0(\tilde{v}_0)e_0\|_{H^1} < \frac{c_0(\tilde{v}_0)\int_0^1 e_0(\xi)\,d\xi}{\max(C_V\|V\|_{L^2}h_2(0), \|1 - c_0(1)e_0(x)\|_{L^2})}$$  \hspace{1cm} (47)

\section{on the Burgers flow}

\subsection{on global existence and regularity}

The intertwining of the Burgers flow $\Phi^t_B$ with the nonlinear heat flow $\Phi^t_N$ through the Cole-Hopf transforms, as given by formula (6), is valid for all initial data $u_0 \in L^2([0,1])$ in the Burgers equation (11), because $H$ maps bijectively $L^2([0,1])$ onto $P_1$. Therefore the global existence of the nonlinear heat flow $\Phi^t_N$ proved in 4.2.4 implies global existence for $t \geq 0$ of the Burgers orbits $\Phi^t_B(u_0)$, for all Cauchy data $u_0 \in L^2([0,1])$.

It is worth noticing that the Cole-Hopf transform $C$ maps $(s+1)$-differentiable and strictly positive functions to $s$-differentiable functions. Therefore regularity of the flow $\Phi^t_N$ stated in 4.2.4 implies that for any given Cauchy data $u_0 \in L^2([0,1])$ the orbit $u(t, x) = \Phi^t_N(u_0)$ lays in the set of infinitely differentiable functions in the space variable, for $t > 0$. Thus the product $t u \partial_x u$ in equation (11) is meaningful.

\subsection{on steady states for the Burgers flow}

The intertwining asserted by formula (3), implies that a steady state for the Burgers flow $\Phi^t_B$ is image through $C$ of a steady state of the nonlinear heat flow $\Phi^t_N$ on $P_1$. Therefore section 4.2.6 shows that the only steady state for $\Phi^t_B$ is the sink:

$$s_0 = C(\tilde{f}_0)$$

Formula (5) gives

$$s_0(x) = C\left(\frac{e_0(x)}{\int_0^1 e_0(\xi)\,d\xi}\right) = -2 \frac{\partial_x e_0(x)}{e_0(x)}$$  \hspace{1cm} (48)

This can also be checked by a direct proof, that is given in section 6.5.
5.3 the linear Koopman flow for Burgers equation

Let $u_0 \in L^2([0,1])$. Formula (9) gives the Burgers orbit of $u_0$ as the image by $C$ of the linear heat orbit $\Phi^{t_H}_H(v_0)$ where $v_0 = H(u_0)$ because:

$$\Phi^{t_B}_B(u_0) = \Phi^{t_B}_B \circ C = C \circ \Phi^{t_H}_H(v_0) = -2 \frac{\partial_x \Phi^{t_H}_H(v_0)}{\Phi^{t_H}_H(v_0)}$$

Let $v(t,.) = \Phi^{t_H}_H(v_0)$. The above formula gives:

$$u(t,.) = \Phi^{t_B}_B(u_0) = -2 \frac{\partial_x v}{c_0(v_0)e_0 + (e^{\mu_0 t}v - c_0(v_0)e_0)}$$

Estimate (21) gives:

$$\forall t > 0 \sup_{x \in [0,1]} |e^{\mu_0 t}v(t,x) - c_0(v_0)e_0(x)| \leq \|v_0 - c_0(v_0)e_0\|_{L^2} h_1(t)$$

so for

$$\frac{\|v_0 - c_0(v_0)e_0\|_{L^2}}{c_0(v_0)e_0} h_1(t) < 1$$

one gets

$$u(t,.) = -2 \frac{e^{\mu_0 t} \partial_x v}{c_0(v_0)e_0} \frac{1}{1 + \tilde{k}_B} = -2 \frac{e^{\mu_0 t} \partial_x v}{c_0(v_0)e_0} \sum_{m=0}^{\infty} (1)^m \frac{m}{m} \tilde{k}_B$$

where

$$\tilde{k}_B = \frac{e^{\mu_0 t}v - c_0(v_0)e_0}{c_0(v_0)e_0}$$

Writing $v$ as the series given by formula (15), it is proved in 6.6, using the same procedure as for the Koopman flow of the nonlinear heat equation in 4.2.8, that:

$$u(t,x) = \sum_{q_0=0}^{\infty} \sum_{m=0}^{\infty} \sum_{(q_1,\ldots,q_m) \in (\mathbb{N}\backslash\{0\})^m} e^{-\lambda_\nu t} \varphi_{\nu}(u_0) a_{\nu}(x)$$

where

$$\forall m \in \mathbb{N}, \forall \nu = (q_0,\ldots,q_m) \in \mathcal{B}_m, \quad \lambda_\nu = \sum_{0}^{m} (\mu_{q_i} - \mu_0)$$
\[ \varphi_{\nu}(u_0) = \prod_{0}^{m} c_0(H(u_0)) \quad a_{\nu}(x) = 2(-1)^{m+\frac{j}{2}} \frac{\partial_x e_{\theta_0}(x)}{e_{\theta}(x)} \prod_{1}^{m} e_{q_j}(x) \] (51)

Formula (50) is a Koopman decomposition for the Burgers flow. It needs assumption (49) be fulfilled, that can be rewritten in order to highlight the link with neighbourhoods of the sink \(s_0\). let:

\[ \forall u_0 \in L^2([0,1]) \quad \tau_B(u_0) = h_1^{-1}(\frac{m_0^2}{\|H(u_0) - \tilde{f}_0\|_{L^2}}) \]

The main result is that for \(t > \tau_B(u_0)\) assumption (49) is fulfilled and \(\Phi^t_B(u_0)\) has the Koopman decomposition (50). The proof is given in section 6.7.

To make explicit the link with neighbourhoods of the sink \(s_0\), let:

\[ \forall \theta_0 > 0 \quad \Omega_B(\theta_0) = \{u_0 \in L^2([0,1]); \quad \|H(u_0) - \tilde{f}_0\|_{L^2} < \frac{m_0^2}{h_1(\theta_0)}\} \] (52)

These are neighbourhoods of the sink \(s_0\) of the Burgers flow, because \(H\) is continuous and the \(H^1\) norm dominates the \(L^2\) norm. For any Cauchy data \(u_0 \in \Omega_B(\theta_0)\) the orbit \(\Phi^t_B(u_0)\) has a Koopman decomposition for all \(t > \theta_0\). This is proved in section 6.7.

Notice that the only non-decreasing component in the decomposition corresponds to \((q_0, m_0) = (0, 0)\). It is the sink \(s_0(x)\), as expected.

### 5.4 on absolute convergence of formula (50)

Using formula (50) to compute the Koopman decomposition of observables such as mappings of the state variable \(u(t,.)\) through analytic functions, or observables like the energy that shows products and spatial integration, needs convergence of (50) in the \(\sup_{x \in [0,1]} \) norm.

In order to have a workable assumption that grants absolute convergence, let:

\[ h_5(t) = \sqrt{\sum_{q=1}^{\infty} e^{-2(\mu_q - \mu_0)t} (1 + \sqrt{\mu_q})^2} \]

It is a continuous decreasing function with \(h_5(0) = \infty\) and \(h_5(\infty) = 0\). For \(u_0 \in L^2\) let \(\tilde{\tau}_B\) be defined by:

\[ h_5(\tilde{\tau}_B(u_0)) = \frac{m_0^2}{\|H(u_0) - c_0(H(u_0))e_0\|_{L^2}} \]
The main result is that for $t > \tilde{\tau}_B(u_0)$ the Koopman decomposition (50) of $\Phi^t_B(u_0)$ is absolutely convergent.

The proof is given in (6.8).

An interesting interpretation of this result follows orthonormality of the family $(e_q)$ that gives:

$$\|H(u_0) - \tilde{f}_0\|^2_{L^2} = \|H(u_0) - c_0(H(u_0))e_0\|^2_{L^2} + \|c_0(H(u_0))e_0 - \tilde{f}_0\|^2_{L^2}$$

so

$$\|H(u_0) - c_0(H(u_0))e_0\|^2_{L^2} < \|H(u_0) - \tilde{f}_0\|^2_{L^2}$$

therefore if:

$$\tilde{\Omega}_B(\theta_0) = \{u_0 \in L^2([0, 1]); \|H(u_0) - \tilde{f}_0\|^2_{L^2} < \frac{m^2_0}{h_5(\theta_0)}\} \quad (53)$$

the result asserted above implies that the Koopman decomposition is absolutely convergent for $t > \theta_0$ for Cauchy data $u_0 \in \tilde{\Omega}_B(\theta_0)$.

The sets $\tilde{\Omega}_B(\theta_0)$ are neighbourhoods of the sink $s_0$ in $L^2([0, 1])$, by continuity of $H$ and because on $\mathcal{P}_1$ the $H^1$ norm dominates the $L^2$ norm.

### 5.5 Koopman eigen-observables for the Burgers flow

Let $\mathcal{F}$ be the set of continuous maps from $L^2([0, 1])$ to $\mathbb{R}$. The Koopman flow $\mathcal{K}^t_B$ associated with the Burgers flow $\Phi^t_B$ is the flow on $\mathcal{F}$ given by:

$$\forall \phi \in \mathcal{F}, \forall t \geq 0, \forall u_0 \in L^2([0, 1]), \quad (\mathcal{K}^t_B(\phi))(u_0) = \phi(\Phi^t_B(u_0)) \quad (54)$$

Note that for any $t \geq 0$, $\mathcal{K}^t_B$ is linear on $\mathcal{F}$ and fulfills the multiplicative property:

$$\forall t \geq 0, \forall \phi_1 \in \mathcal{F}, \forall \phi_2 \in \mathcal{F}, \quad \mathcal{K}^t_B(\phi_1\phi_2) = \mathcal{K}^t_B(\phi_1)\mathcal{K}^t_B(\phi_2) \quad (55)$$

Formula (5) intertwins Burgers flow with the nonlinear heat flow. Formulas (39) and (40) give eigen-observables for the nonlinear heat flow. This gives eigen-observables for the Burgers flow as follows:

$$\forall m \in \mathbb{N}, \forall \nu = (q_0, ..., q_m) \in (\mathbb{N} \setminus \{0\})^{m+1}, \forall u_0 \in L^2([0, 1])$$

$$(\mathcal{K}^t_B(\psi_\nu \circ H))(u_0) = (\psi_\nu \circ H)((\Phi^t_B(u_0)) = \psi_\nu((H\circ \Phi^t_B)(u_0)) = \psi_\nu((\Phi^t_{\lambda \circ H})(u_0)) =$$
This proves, using notations of formula (50), that for any \(m \in \mathbb{N}\), and any \(\nu = (q_0, \ldots, q_m) \in (\mathbb{N} \setminus \{0\})^{m+1}\):

\[
e^{-\lambda_\nu t}
\]

is an eigenvalue of \(K^t_B\) with associated eigen-observable

\[
\varphi_\nu = \prod_{i=1}^k \frac{c_{q_i} \circ H}{c_0 \circ H}
\]

To conclude this section it is worth noticing that formulas (56) and (57) show that the eigenvalues of the Koopman decomposition (50) are eigenvalues of the Koopman operator \(K^t_B\) associated with \(\Phi^t_B\), and that the coefficients in the Koopman decomposition (50) are values of eigen-functionals at the Cauchy data.

Although convergence of the Koopman decomposition (50) is a local phenomena, eigen-functionals are defined globally on \(\mathcal{F}\).

6 proofs for the estimates

6.1 proof of formula (20)

Let \(v_0 \in \mathcal{P}_+\). Let \(v(t, x)\) solve equation (3), so it is given by formula (15), and \(g(t)\) is given by:

\[
g(t) = \sum_{0}^{\infty} c_n(v_0)e^{-\mu_n t} \int_{0}^{1} e_n(\xi) d\xi
\]

so

\[
|e^{\mu_0 t} g(t) - c_0(v_0) \int_{0}^{1} e_0(\xi) d\xi| \leq e^{-(\mu_1 - \mu_0)t} \sum_{1}^{\infty} |c_n(v_0)| \int_{0}^{1} e_n(\xi) d\xi | \leq
\]

\[
e^{-(\mu_1 - \mu_0)t} \sqrt{\sum_{1}^{\infty} |c_n(v_0)|^2} \sqrt{\sum_{1}^{\infty} (\int_{0}^{1} e_n(\xi) d\xi)^2} =
\]

\[
e^{-(\mu_1 - \mu_0)t} \|v_0 - c_0(v_0)e_0(x)\|_{L^2} \|1 - c_0(1)e_0(x)\|_{L^2}
\]
6.2 proof of formula (21)

Let $v_0 \in \mathcal{P}_+$. Let $v(t, x)$ solve equation (3): it is given by formula (15). Three inequalities are needed for the proof:

First is the usual Sobolev injection inequality:

$$\forall f \in H^1, \sup_x |f(x)| \leq C_1 \|f\|_{H^1}$$  \hspace{1cm} (58)

Second is the following estimate for the $H^1$ norm of $e_n(x)$:

$$\|e_n\|_{H^1} \leq \sqrt{1 + \mu_n}$$ \hspace{1cm} (59)

proved through:

$$\frac{1}{\mu_n} \int_0^1 (\partial_x e_n)^2 \leq \frac{1}{\mu_n} \left( \int_0^1 (\partial_x e_n)^2 + \int_0^1 V e_n^2 \right) \leq$$

$$\frac{1}{-\mu_n} \int_0^1 (\partial_{xx}^2 - V) e_n(\xi) e_n(\xi) d\xi = \|e_n\|_{L^2} = 1$$

Third, because $(\partial_{xx}^2 - V)^{-1}$ is bounded from $H^{-1}$ to $H^1$, one has:

$$\sum_{n=0}^{\infty} \frac{1}{\mu_n} < \infty$$ \hspace{1cm} (60)

The proof goes as follows:

$$\sup_{x \in [0,1]} |v(x) - c_0(v_0)e^{-\mu_0 t} e_0(x)| \leq C_1 \|v(x) - c_0(v_0)e^{-\mu_0 t} e_0(x)\|_{H^1} =$$

$$C_1 \|\sum_{n=0}^{\infty} e^{-\mu_n t} c_n(v_0) e_n(x)\|_{H^1} \leq C_1 \sum_{n=0}^{\infty} e^{-\mu_n t} |c_n(v_0)| \|e_n(x)\|_{H^1} \leq$$

$$C_1 \sum_{n=0}^{\infty} e^{-\mu_n t} |c_n(v_0)| \sqrt{1 + \mu_n} \leq C_1 \sqrt{\sum_{n=0}^{\infty} |c_n(v_0)|^2} \sqrt{\sum_{n=0}^{\infty} (1 + \mu_n) e^{-2\mu_n t}} =$$

$$e^{-\mu_0 t} \|v_0 - c_0(v_0)e_0\|_{L^2} h_1(t)$$

with

$$h_1(t) = C_1 \sqrt{\sum_{n=0}^{\infty} (1 + \mu_n) e^{-2(\mu_n - \mu_0) t}}$$

The function $h_1(t)$ is defined for all $t > 0$ because of the third inequality above. It is obviously decreasing, and has limit zero at infinity.
6.3 proof of formula (42)

Let \( \tilde{v}_0 \in P_1 \). The condition \( t > \tau_{N}(\tilde{v}_0) \) implies \( |\tilde{k}(t)| < 1 \) so

\[
\frac{1}{1 + \tilde{k}(t)} = \sum_{m=0}^{\infty} (-1)^m (\tilde{k}(t))^m
\]

One expresses the product \((\tilde{k}(t))^m\) as the sum of the products of the term of rank \( q_1 \) in the series corresponding to the first factor, times the term of rank \( q_2 \) of the second factor, ..., till the term of rank \( q_m \) in the last factor, and gets:

\[
\frac{1}{1 + \tilde{k}(t)} = \sum_{m=0}^{\infty} (-1)^m \sum_{(q_1,...,q_m) \in (N^*)^m} e^{-\sum_{i=0}^{m}(\mu_{q_i}-\mu_0)t} \prod_{i=1}^{m} \frac{c_{q_i}(\tilde{v}_0)}{c_0(\tilde{v}_0)} \prod_{i=1}^{m} p_{q_i}
\]

Using this last expression one re-writes formula (41) as:

\[
\tilde{v}(t, x) = \sum_{q_0=0}^{\infty} \sum_{m=0}^{\infty} \sum_{(q_1,...,q_m) \in (N^*)^m} (-1)^m e^{-\sum_{i=0}^{m}(\mu_{q_i}-\mu_0)t} \prod_{i=1}^{m} \frac{c_{q_i}(\tilde{v}_0)}{c_0(\tilde{v}_0)} \prod_{i=1}^{m} p_{q_i} \int_0^1 e_{q_0}(\xi) d\xi
\]

This proves formula (42)

6.4 proof of formula (46)

Formula (46) is the formula (42) proved above, with a change in the order of the summation. This change is allowed if the series (46) is absolutely convergent, meaning that the sum of the \( \sup_{x \in [0,1]} |e_q(x)| \) norm of all terms is convergent. To prove absolute convergence one needs the following estimates:

1- Estimating \( \sup_{x \in [0,1]} |e_q(x)| \): for any \((x, y) \in [0,1]^2\),

\[
|e_q(x)| \leq |e_q(y)| + \int_y^x |\partial_x e_q(\xi)| d\xi \leq |e_q(y)| + \sqrt{\int_0^1 (\partial_x e_q(\xi))^2 d\xi}
\]

After integrating in \( y \), Cauchy-Schwarz inequality gives:

\[
|e_q(x)| \leq 1 + \sqrt{\int_0^1 (\partial_x e_q(\xi))^2 d\xi} \leq 1 + \sqrt{\int_0^1 \partial_x^2 e_q(\xi) e_q(\xi) d\xi} =
\]

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\[ 1 + \sqrt{\mu_q \int_0^1 e_q^2(\xi) d\xi} - \int_0^1 V(\xi) e_q^2(\xi) d\xi \leq 1 + \sqrt{\mu_q} \]

Therefore

\[ \sup_{x \in [0,1]} |e_q(x)| \leq 1 + \sqrt{\mu_q} \quad (61) \]

2- Estimating \( \sup_{x \in [0,1]} |b_\nu(x)| \): the boundary conditions give:

\[ \int_0^1 e_q(\xi) d\xi = -\frac{1}{\mu_q} \int_0^1 \partial_{xx} e_q(\xi) d\xi + \frac{1}{\mu_q} \int_0^1 V(\xi) e_q(\xi) d\xi = \frac{1}{\mu_q} \int_0^1 V(\xi) e_q(\xi) d\xi \]

therefore, using (61), one has for any \( \nu = (q_0, \ldots, q_m) \in B_m \):

\[ \sup_{x \in [0,1]} |b_\nu(x)| = \sup_{x \in [0,1]} |e_{q_0}(x)| \frac{\prod_{i=1}^m |\int_0^1 e_{q_i}|}{(\int_0^1 e_0)^{m+1}} \leq \frac{1 + \sqrt{\mu_{q_0}}}{(\int_0^1 e_0(\xi) d\xi)^{m+1}} \prod_{i=1}^m \frac{|\int_0^1 V(\xi) e_{q_i}(\xi) d\xi|}{\mu_{q_i}} \quad (62) \]

Moreover

\[ \sum_0^\infty (\int_0^1 V(\xi) e_q(\xi) d\xi)^2 = \|V\|_{L^2}^2 < \infty \]

3- Estimating \( \psi_\nu(\bar{v}_0) \):

Let \( \bar{v}_0 \in \mathcal{P}_+ \). It has a square integrable derivative, and fulfills Neumann conditions on the boundary. For \( q \in \mathbb{N}, q \neq 0 \), one first notice that due to orthogonality of the eigenfunctions:

\[ c_q(\bar{v}_0) = \int_0^1 \bar{v}_0(\xi) e_q(\xi) d\xi = \int_0^1 (\bar{v}_0(\xi) - c_0(\bar{v}_0)e_0(\xi)) e_q(\xi) d\xi = \]

\[ \frac{1}{\mu_q} \int_0^1 \partial_x (\bar{v}_0 - c_0(\bar{v}_0)e_0)(\xi) \partial_x e_q(\xi) d\xi + \frac{1}{\mu_q} \int_0^1 (\bar{v}_0 - c_0(\bar{v}_0)e_0)(\xi) V(\xi) e_q(\xi) d\xi \]

so

\[ |c_q(\bar{v}_0)| \leq \frac{1}{\mu_q} \|\partial_x (\bar{v}_0 - c_0(\bar{v}_0)e_0)\|_{L^2} \|\partial_x e_q\|_{L^2} + \frac{1}{\mu_q} (\sup V) \|\bar{v}_0 - c_0(\bar{v}_0)e_0\|_{L^2} \]
Integration by parts gives \( \| \partial_x e_q \|_{L^2} \leq \sqrt{\mu_q} \), so:

\[
|c_q(\tilde{v}_0)| \leq \frac{1}{\sqrt{\mu_q}} \| \partial_x (\tilde{v}_0 - c_0(\tilde{v}_0)e_0) \|_{L^2} + \frac{1}{\mu_q} (\sup V) \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{L^2}
\]

For \( C_V = \max (1, \sup V) \), the above estimate writes:

\[
|c_q(\tilde{v}_0)| \leq \frac{C_V}{\sqrt{\mu_q}} \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}
\]  \( (63) \)

Therefore, if \( \nu = (q_0, \ldots, q_m) \in \mathcal{B}_m \):

\[
|\psi_\nu(\tilde{v}_0)| \leq C^{m+1}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m+1}(\prod_{q=0}^{m} \mu_{q_i})^{-\frac{1}{2}} \text{ if } q_0 \neq 0
\]  \( (64) \)

and

\[
|\psi_\nu(\tilde{v}_0)| \leq C^{m}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m}(\prod_{q=1}^{m} \mu_{q_i})^{-\frac{1}{2}} \text{ if } q_0 = 0
\]

4- Final step - convergence in the \( \sup_{x \in [0,1]} \) norm:

The series below is a series of positive numbers. Therefore convergence (and sum) do not depend on the order in indexing, by use of Fubini’s theorem. This gives, with \( v_q = |\int_0^1 V(\xi)e_q(\xi)d\xi| \):

\[
\sum_{\mathcal{B}} e^{-\lambda \nu t} |\psi_\nu(\tilde{v}_0)| \sup_{x \in [0,1]} |b_\nu(x)| \leq
\]

\[
\sum_{m=0}^{\infty} \sum_{\nu \in \mathcal{B}, q_0 = 0} e^{-\lambda \nu t}(1 + \sqrt{\mu_0}) \frac{C^{m}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m+1}(\prod_{i=1}^{m} \mu_{q_i})^{-\frac{1}{2}}}{(c_0(\tilde{v}_0))^{m}(\int_0^1 e_0(\xi)d\xi)^{m+1}} + \]

\[
\sum_{m=0}^{\infty} \sum_{\nu \in \mathcal{B}, q_0 \neq 0} e^{-\lambda \nu t}(1 + \sqrt{\mu_0}) \frac{C^{m+1}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m+1}(\prod_{i=1}^{m} \mu_{q_i})^{-\frac{1}{2}}}{(c_0(\tilde{v}_0))^{m+1}(\int_0^1 e_0(\xi)d\xi)^{m+1}} =
\]

\[
\sum_{m=0}^{\infty} \sum_{q=1}^{\infty} (1 + \sqrt{\mu_0}) \frac{C^{m}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m}(\int_0^1 e_0(\xi)d\xi)^{m+1}}{(c_0(\tilde{v}_0))^{m}(\int_0^1 e_0(\xi)d\xi)^{m+1}} \left( \sum_{q=1}^{\infty} e^{-(\mu_q - \mu_0)t} \frac{v_q}{\mu_q} \right)^m +
\]

\[
\sum_{m=0}^{\infty} \sum_{q_0=1}^{\infty} e^{-(\mu_{q_0} - \mu_0)t}(1 + \sqrt{\mu_0}) \frac{C^{m+1}_V \| \tilde{v}_0 - c_0(\tilde{v}_0)e_0 \|_{H^1}^{m+1}(\int_0^1 e_0(\xi)d\xi)^{m+1}}{(c_0(\tilde{v}_0))^{m+1}(\int_0^1 e_0(\xi)d\xi)^{m+1}} \left( \sum_{q=1}^{\infty} e^{-(\mu_q - \mu_0)t} \frac{v_q}{\mu_q} \right)^m \leq
\]
\[ \sum_{m=0}^{\infty} \left( 1 + \sqrt{\mu_0} \right) \frac{C_V^m v_0 - c_0(v_0)e_0}{(c_0(v_0))(\int_0^1 e_0(\xi)d\xi)^{m+1}} \| V \|_{L^2}^m (h_2(t))^m + \]

\[ \sum_{m=0}^{\infty} \sum_{q_0=1}^{\infty} e^{-(\mu q_0 - \mu_0)t} \frac{1 + \sqrt{\mu q_0}}{\sqrt{\mu q_0}} \frac{C_V^{m+1} v_0 - c_0(v_0)e_0}{(c_0(v_0))(\int_0^1 e_0(\xi)d\xi)^{m+1}} \| V \|_{L^2}^m (h_2(t))^m \]

Formula (45) gives for \( t > \tau_N(\tilde{v}_0) \):

\[ \varepsilon(t) = \frac{C_V h_2(t) \| V \|_{L^2} \| v_0 - c_0(v_0)e_0 \|_{H^1}}{c_0(v_0)(\int_0^1 e_0(\xi)d\xi)} < 1 \]

so

\[ \sum_B e^{-\lambda v_\nu(\tilde{v}_0)} \sup_{x \in [0,1]} |b_\nu(x)| \leq \frac{C_0}{1 - \varepsilon(t)} \sum_{q_0=0}^{\infty} e^{-(\mu q_0 - \mu_0)t} \frac{1 + \sqrt{\mu q_0}}{\sqrt{\mu q_0}} \]

with \( C_0 = \frac{1}{\int_0^1 e_0(\xi)d\xi} \max(\max(1, \sqrt{\mu_0}), \max(1, \frac{C_v \| L^2 \| v_0 - c_0(v_0)e_0 \|_{H^1})) \))

The last series do converge for all \( t > 0 \) due to formula (60).

6.5 on formula (48): the sink for Burgers equation

Let \( u_0 \in L^2([0,1]) \). Let \( v_0 = H(u_0) \) and \( v(t,.) = \Phi^t_H(v_0) \). Formulas (6) and (8) state that

\[ \Phi^t_B(u_0) = C(\Phi^t_N(v_0)) = C(\frac{\Phi^t_H(v_0)}{g(t)}) \]

Formula (20) gives:

\[ e^{\mu_0 t} g(t) = c_0(v_0) \int_0^1 e_0(\xi)d\xi + \varepsilon_1(t) \]

where

\[ |\varepsilon_1(t)| < e^{-(\mu_1 - \mu_0)t} \| v_0 - c_0e_0(0) \|_{L^2} \| 1 - c_0(1) e_0 \|_{L^2} \]

Formula (21) gives:

\[ e^{\mu_0 t} v(t,x) = c_0(v_0)e_0(x) + \varepsilon_2(t,x) \]

where

\[ \sup_{x \in [0,1]} |\varepsilon_2(t,x)| \leq \| v_0 - c_0(v_0)e_0 \|_{L^2} h_1(t) \]
and $h_1(t)$ has limit zero at infinity.
Continuity of $C$ implies:

$$\lim_{t \to \infty} \Phi(t, x) = \frac{v(t, x)}{g(t)} = C\left(\frac{e_0(x)}{\int_0^1 e_0(\xi) d\xi}\right) = C(\tilde{f}_0) = s_0$$

6.6 proof for formula (50)

Let $u_0 \in L^2([0, 1])$ and $v_0 = H(u_0)$. Let $u(t, x)$ solve (1) and $v(t, x)$ solve (3).

It is proved in 5.3 that under the assumption (49) one has

$$u(t, .) = -2 e^{\mu_0 t} \partial_x v \sum_{m=0}^{\infty} (-1)^m \left(\frac{e^{\mu_0 t} - c_0(v_0)e_0}{c_0(v_0)e_0}\right)^m$$

By formula (15):

$$v(t, x) = \sum_{q=0}^{\infty} e^{-\mu_q t} c_q(v_0) e_q(x)$$

so

$$u(t, x) = \sum_{q_0=0}^{\infty} \sum_{m=0}^{\infty} 2(-1)^{m+1} e^{-(\mu_{q_0} - \mu_0)t} \frac{c_{q_0}(v_0)}{c_0(v_0)} \frac{\partial_x c_{q_0}(v_0)}{e_0(x)} \left(\sum_{q=1}^{\infty} e^{-(\mu_q - \mu_0)t} e_q(v_0) e_q(x)\right)^m$$

Writing the power $m$ of the series as a sum of products of the term of rank $q_1$ from the first factor, term of rank $q_2$ from the second factor,..., till term of rank $q_m$ from the $m$-th factor, one gets:

$$u(t, x) =$$

$$\sum_{q_0=0}^{\infty} \sum_{m=0}^{\infty} 2(-1)^{m+1} e^{-\sum_{i=0}^{m}(\mu_{q_i} - \mu_0)} \frac{\prod_{i=0}^{m} c_{q_i}(v_0)}{c_0(v_0)} \frac{\partial_x c_{q_0}(v_0)}{e_0(x)} \left(\sum_{q=1}^{\infty} e^{-(\mu_q - \mu_0)t} e_q(v_0) e_q(x)\right)^m$$

This is formula (50).

6.7 on formula (52)

Let $u_0 \in L^2([0, 1])$ and $v_0 = H(u_0) \in P_1$. Orthonormality of the sequence $(e_q)$ gives:

$$\|v_0 - c_0(v_0)e_0\|_{L^2}^2 = \|v_0 - \tilde{f}_0\|_{L^2}^2 - \|c_0(v_0)e_0 - \tilde{f}_0\|_{L^2}^2$$
so the condition (49) is fulfilled if

\[ \|v_0 - \tilde{f}_0\|_{L^2} < \frac{m_0 c_0(v_0)}{h_1(t)} \]

but

\[ c_0(v_0) = \int_0^1 v_0(\xi) e_0(\xi) d\xi \geq m_0 \int_0^1 v_0(\xi) d\xi = m_0 \]

so the condition (49) is fulfilled if

\[ h_1(t) < \frac{m_0^2}{\|v_0 - \tilde{f}_0\|_{L^2}} \]

Because \( h_1(t) \) is a decreasing continuous function with \( h_1(0) = \infty \) and \( h_1(\infty) = 0 \) this proves that for any \( u_0 \in L^2([0, 1]) \), if

\[ t > \tau_B(u_0) = h_1^{-1}(\frac{m_0^2}{\|H(u_0) - \tilde{f}_0\|_{L^2}}) \]

then the decomposition (50) holds.

Continuity of the map \( H \) from \( L^2([0, 1]) \) to \( P_1 \) equipped with the \( L^2 \) norm shows that the map \( \tau_B \) is continuous from \( L^2([0, 1]) \) to \( \mathbb{R}_+ \).

6.8 on absolute convergence of formula (50)

Let \( u_0 \in L^2(0, 1) \). This section is devoted to prove absolute convergence in the \( \sup_{[0,1]} \) norm of the following series, i.e. of the Koopman decomposition of \( \Phi^t_B \):

\[ u(t, x) = \sum_B e^{-\lambda_\nu t} \varphi_\nu(u_0) a_\nu(x) \]

that means convergence of

\[ \sum_B e^{-\lambda_\nu t} |\varphi_\nu(u_0)| \sup_{x \in [0,1]} |a_\nu(x)| \] (66)

where for \( \nu = (q_0, \ldots, q_m) \in B \):

\[ \lambda_\nu = \sum_0^m (\mu_{q_i} - \mu_0) \]

\[ \varphi_\nu(u_0) = \prod_0^m \frac{c_{q_i}(H(u_0))}{c_0(H(u_0))} \]

\[ a_\nu(x) = 2(-1)^{m+1} \frac{\partial_x e_{q_0}(x)}{e_0(x)} \prod_1^m \frac{e_{q_i}(x)}{e_0(x)} \] (67)
All estimates needed are already proven, but one: an estimate with respect to $\mu_q$ of $\sup_{[0,1]} |\partial_x e_q(x)|$: because $\partial_x e_q(x)$ fulfills Dirichlet boundary condition one has:

$$\sup_{[0,1]} |\partial_x e_q(x)|^2 \leq \int_0^1 (\partial_{xx} e_q(\xi))^2 d\xi = \int_0^1 (V(\xi) - \mu_q)^2 e_q^2(\xi) d\xi \leq \sup_{[0,1]} |V(x) - \mu_q|^2$$

so

$$\sup_{[0,1]} |\partial_x e_q(x)| \leq C_V (1 + \mu_q)$$  \hspace{1cm} (68)

1- estimate for $\sup_{[0,1]} |a_{\nu}(x)|$:

Formulas (61) and (68) give for $\nu = (q_0, \ldots, q_m)$:

$$\sup_{x \in [0,1]} |a_{\nu}(x)| \leq \frac{2C_V}{m_0} (1 + \mu_{q_0}) \prod_{i=1}^m \frac{1 + \sqrt{\mu_{q_i}}}{m_0}$$

2- estimate for $|\varphi_{\nu}(u_0)|$:

Orthonormalisation of $e_q$ gives for $q \neq 0$:

$$c_q(H(u_0)) = \int_0^1 H(u_0)(\xi)e_q(\xi) d\xi = \int_0^1 (H(u_0)(\xi) - c_0(H(u_0))e_0(\xi))e_q(\xi) d\xi = c_q(H(u_0)) - c_0(H(u_0))e_0 := c'_q(u_0)$$

This, with formula (65) gives

$$|\varphi_{\nu}(u_0)| \leq \frac{\prod_{i=0}^m |c'_q|}{m_0^{m+1}} \text{ if } q_0 \neq 0$$

$$|\varphi_{\nu}(u_0)| \leq \frac{|c_0(H(u_0))| \prod_{i=1}^m |c'_q|}{m_0^{m+1}} \text{ if } q_0 = 0$$

Therefore the generic term in formula (66) can be estimated by:

$$\frac{2C_V}{2^{(m+1)}} e^{-\lambda_{\nu}t}(1 + \mu_{q_0}) \prod_{i=0}^m |c'_q| \prod_{i=1}^m (1 + \sqrt{\mu_{q_i}})$$

with $c'_q(u_0) = c_0(H(u_0))$.

Because it is a positive series, the sum do not depend on the summation order. Summing first in $(q_1, \ldots, q_m)$ one gets:

$$\sum_B e^{-\lambda_{\nu}t} |\varphi_{\nu}(u_0)| \sup_{x \in [0,1]} |a_{\nu}(x)| \leq$$

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If the following assumption is fulfilled:

$$h_3(t, u_0) = \sum_{q_0=1}^{\infty} \frac{e^{-(\mu_{q_0} - \mu_0)t}}{m_0^2} \sum_{m=0}^{\infty} \frac{e^{-(\mu_q - \mu_0)t}}{m^2} (1 + \sqrt{\mu_q}|c'_q|)^m$$

one gets

$$\sum_{g} e^{-\lambda_{g}|\varphi_{\nu}(u_0)|} \sup_{\alpha \in [0,1]} |a_{\nu}(x)| \leq \frac{2C_V}{m_0^2(1 - h_3(t, u_0))} \sum_{q_0=0}^{\infty} |c'_{q_0}|(1 + \mu_{q_0})e^{-(\mu_{q_0} - \mu_0)t}$$

$$\leq \frac{2C_V h_4(t)}{m_0^2(1 - h_3(t, u_0))} \sqrt{\sum_{q_0=0}^{\infty} (1 + \mu_{q_0})^2 e^{-2(\mu_{q_0} - \mu_0)t}} \sqrt{\sum_{q_0=0}^{\infty} |c'_{q_0}|^2}$$

with

$$h_4(t) = \sqrt{\sum_{q_0=0}^{\infty} (1 + \mu_q)^2 e^{-2(\mu_{q_0} - \mu_0)t}}$$

The function $h_4(t)$ is a continuous function for $t > 0$ following (60). The full series is convergent because:

$$\sum_{q_0=0}^{\infty} |c'_{q_0}|^2 = \|H(u_0) - c_0(H(u_0))e_0\|_{L^2}^2 < \infty$$

This completes the proof, and gives absolute convergence, provided assumption (69) is fulfilled.

In order to fit assumption (69) in a topological setting, one notices that:

$$m_0^2 h_3(t, u_0) \leq \sqrt{\sum_{q=1}^{\infty} e^{-2(\mu_q - \mu_0)t}(1 + \sqrt{\mu_q})^2} \sum_{q=1}^{\infty} |c'_q|^2 = h_5(t)\|H(u_0) - c_0(H(u_0))e_0\|_{L^2}$$
with
\[
  h_5(t) = \sqrt{\sum_{q=1}^{\infty} e^{-2(\mu_q-\mu_0)t}(1 + \sqrt{\mu_q})^2}
\]

\( h_5(t) \) is a continuous function for \( t > 0 \) because of (60). Therefore the assumption (69) is fulfilled for \( t > \tilde{\tau}_B(u_0) \) with \( \tilde{\tau}_B(u_0) \) defined by:

\[
  h_5(\tilde{\tau}_B(u_0)) = \frac{m_0^2}{\|H(u_0) - c_0(H(u_0))e_0\|_{L^2}}
\]

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