The Global Resilience of Hamiltonicity in $G(n, p)$

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Abstract
Denote by $r_g(G, \mathcal{H})$ the global resilience of a graph $G$ with respect to Hamiltonicity. That is, $r_g(G, \mathcal{H})$ is the minimal $r$ for which there exists a subgraph $H \subseteq G$ with $r$ edges, such that $G \setminus H$ is not Hamiltonian. We show that if $p$ is above the Hamiltonicity threshold and $G \sim G(n, p)$ then, with high probability, (We say that a sequence of events $(A_n)_{n=1}^{\infty}$ occurs with high probability if $\lim_{n \to \infty} P(A_n) = 1$.) $r_g(G, \mathcal{H}) = \delta(G) - 1$. This is easily extended to the full interval: for every $p(n) \in [0, 1]$, if $G \sim G(n, p)$ then, with high probability, $r_g(G, \mathcal{H}) = \max\{0, \delta(G) - 1\}$.

Keywords Random graphs · Hamilton cycles

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1 Introduction
Let $\mathcal{P}$ be a monotone increasing graph property. For a graph $G$, we define the global resilience of $G$ with respect to $\mathcal{P}$, denoted $r_g(G, \mathcal{P})$, as follows.

$$r_g(G, \mathcal{P}) := \min\{m \in \mathbb{N} \mid \exists H \subseteq G : e(H) = m, G \setminus H \text{ is not in } \mathcal{P}\}.$$ 

That is, $r_g(G, \mathcal{P})$ is the minimal number of edge removals from $G$ such that the resulting graph does not satisfy $\mathcal{P}$. This notion serves as a measure of how “strongly” $G$ satisfies the property $\mathcal{P}$, by its distance from the closest graph outside of $\mathcal{P}$. It is by no means a new notion, and many long standing results in extremal graph theory can be expressed by it. For example, the extremal number $\text{ex}(n, G)$ can be expressed as $\binom{n}{2} - r_g(K_n, \mathcal{P}_G)$, where $\mathcal{P}_G$ denotes the property of containing a copy of $G$ as
a subgraph. Turán’s theorem can now be stated as: for every \( n \geq r \geq 3 \) integers, \( r_g(K_n, \mathcal{P}_K) = (1 + o(1)) \cdot \frac{n^2}{2(r-1)} \).

We denote by \( \mathcal{H} \) the property of Hamiltonicity. It is trivial to see that, for every graph \( G \) with \( \delta(G) \geq 1 \), one has \( r_g(G, \mathcal{H}) \leq \delta(G) - 1 \). Indeed, one can ensure that \( G \setminus H \) contains no Hamilton cycle by having \( H \) contain all the edges incident in \( G \) to some vertex \( v \in V(G) \) but one. By choosing \( v \) to be a vertex with minimum degree, this trivial bound is achieved.

In this paper we show that this trivial upper bound is, in fact, the typical exact value of \( r_g(G, \mathcal{H}) \) when \( G \) is a random graph, by proving the following theorem.\(^1\)

**Theorem 1** Let \( G \sim G(n, p) \), where \( p = p(n) \) satisfies \( np - \log n - \log \log n \to \infty \). Then with high probability \( r_g(G, \mathcal{H}) = \delta(G) - 1 \).

Ore [11] proved that every \( n \)-vertex graph with at least \( \left( \frac{n-1}{2} \right) + 2 \) edges is Hamiltonian. Restated in terms of the global resilience, this implies that \( r_g(K_n, \mathcal{H}) = n - 2 \), and thus the theorem holds for \( p = 1 \).

To cover the complete range of \( p \), the following corollary is easily derived from Theorem 1.

**Corollary 1** Let \( p = p(n) \in [0, 1] \) be monotone in \( n \), and \( G \sim G(n, p) \). Then with high probability \( r_g(G, \mathcal{H}) = \max\{0, \delta(G) - 1\} \).

**Proof** Consider separately three ranges of \( p \): sub-critical, critical and super-critical.

The global resilience in the sub-critical and critical ranges is determined by the classical result of Ajtai, Komlós, Szemerédi [1], Bollobás [4], which states that in the random graph process, the hitting time of Hamiltonicity is equal with high probability to the hitting time of \( \delta(G) \geq 2 \), which is found at time \( \frac{1}{2} \cdot n \cdot (\log n + \log \log n) + O(1) \), with high probability.

**Sub-critical:** if \( np - \log n - \log \log n \to -\infty \), then with high probability \( \delta(G) < 2 \) and \( G \) is not Hamiltonian, and therefore \( r_g(G, \mathcal{H}) = 0 = \max\{0, \delta(G) - 1\} \).

**Critical:** if \( np - \log n - \log \log n \to c \), then, by the above result, with high probability either \( \delta(G) < 2 \), or \( \delta(G) = 2 \) and \( G \) is Hamiltonian. In the first case indeed \( r_g(G, \mathcal{H}) = 0 = \max\{0, \delta(G) - 1\} \). In the second case, because \( G \) is Hamiltonian, \( 1 \leq r_g(G, \mathcal{H}) \leq \delta(G) - 1 \), as this is true for every Hamiltonian graph. But in this case \( \max\{0, \delta(G) - 1\} = 1 \) and therefore indeed \( r_g(G, \mathcal{H}) = \max\{0, \delta(G) - 1\} \).

**Super-critical:** the case \( np - \log n - \log \log n \to \infty \) is covered by Theorem 1, since in the super-critical regime, with high probability, \( \max\{0, \delta(G) - 1\} = \delta(G) - 1 \).

### 1.1 Related Work

The local resilience of a property is a similar notion of resilience, that, with respect to Hamiltonicity in random graphs, has been more thoroughly studied. Denote by \( r_l(G, \mathcal{P}) \) the local resilience of \( G \) with respect to \( \mathcal{P} \), defined as the minimal value of \( m \) such that there is a graph \( H \) with \( \Delta(H) \leq m \), and \( G \setminus H \) does not satisfy \( \mathcal{P} \). Sudakov and Vu [14] showed that there is \( C \geq 0 \) such that, for every \( \varepsilon, \delta \geq 0 \), if

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\(^1\) Here and later the logarithms have natural base.
The following graph theoretic notation is used. For a graph $G=(V,E)$ and two disjoint vertex subsets $U,W \subseteq V$, we let $E_G(U,W)$ denote the set of edges of $G$ adjacent to exactly one vertex from $U$ and one vertex from $V$, and let $e_G(U,W) = |E_G(U,W)|$. Similarly, $E_G(U)$ denotes the set of edges spanned by a subset $U$ of $V$, and $e_G(U)$ stands for $|E_G(U)|$, and $E_G(v)$ denotes $E_G(\{v\}, V \setminus \{v\})$. The (external) neighbourhood of a vertex subset $U$, denoted by $N_G(U)$, is the set of vertices in $V \setminus U$ adjacent to a vertex of $U$, and for a vertex $v \in V$ we set $N_G(v) = N_G(\{v\})$. The degree of a vertex $v \in V$, denoted by $d_G(v)$, is its number of incident edges.

While using the above notation we occasionally omit $G$ if the identity of the graph $G$ is clear from the context.

We suppress the rounding notation occasionally to simplify the presentation.

2 Preliminaries

The following graph theoretic notation is used. For a graph $G=(V,E)$ and two disjoint vertex subsets $U,W \subseteq V$, we let $E_G(U,W)$ denote the set of edges of $G$ adjacent to exactly one vertex from $U$ and one vertex from $V$, and let $e_G(U,W) = |E_G(U,W)|$. Similarly, $E_G(U)$ denotes the set of edges spanned by a subset $U$ of $V$, and $e_G(U)$ stands for $|E_G(U)|$, and $E_G(v)$ denotes $E_G(\{v\}, V \setminus \{v\})$. The (external) neighbourhood of a vertex subset $U$, denoted by $N_G(U)$, is the set of vertices in $V \setminus U$ adjacent to a vertex of $U$, and for a vertex $v \in V$ we set $N_G(v) = N_G(\{v\})$. The degree of a vertex $v \in V$, denoted by $d_G(v)$, is its number of incident edges.

While using the above notation we occasionally omit $G$ if the identity of the graph $G$ is clear from the context.

We suppress the rounding notation occasionally to simplify the presentation.

2.1 Auxiliary Results

Definition 1 Let $\alpha > 0$ and $k$ a positive integer. A graph $G$ is a $(k, \alpha)$-expander if $|N_G(U)| \geq \alpha |U|$ for every vertex subset $U \subseteq V(G)$, $|U| \leq k$. 

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Definition 2 Let $G$ be a graph. A non-edge $\{u, v\} \in E(G)$ is called a booster if the graph $G'$ with edge set $E(G') = E(G) \cup \{\{u, v\}\}$ is either Hamiltonian or has a path longer than a longest path of $G$.

Lemma 1 (Pósa 1976 [12]) Let $G$ be a connected non–Hamiltonian graph, and assume that $G$ is a $(k, 2)$-expander. Then $G$ has at least $\frac{(k+1)^2}{2}$ boosters.

Definition 3 A graph $G$ is Hamilton-connected if for every two vertices $u, v \in V(G)$, $G$ contains a Hamilton path with $u, v$ as its two endpoints.

Theorem 2 (Chvátal–Erdős Theorem [5]) Let $G = (V, E)$ be a graph such that $\alpha(G) < \kappa(G)$. Then $G$ is Hamilton-connected.

2.2 Useful Inequalities

Lemma 2 [see e.g. [7] Lemma 21.1] Let $1 \leq l \leq k \leq n$ be integers. Then the following inequalities hold:

(i) $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$;
(ii) $\binom{n-1}{k-1} \leq e^{-\frac{l}{n}}$.

Lemma 3 [Corollary of Lemma 2] Let $1 \leq k \leq n$ be integers, $0 < p < 1$, and let $X \sim Bin(n, p)$. Then the following inequalities hold:

(i) $\mathbb{P}(X \geq k) \leq \left(\frac{enp}{k}\right)^k$;
(ii) $\mathbb{P}(X = k) \leq \left(\frac{enp}{k(1-p)}\right)^k \cdot e^{-np}$.

If, additionally, $k \leq np$, then

(iii) $\mathbb{P}(X \leq k) \leq (k + 1) \cdot \left(\frac{enp}{k(1-p)}\right)^k \cdot e^{-np}$.

3 Proof of Theorem 1

In this section we present a proof of Theorem 1. We prove it separately for two different ranges of the probability $p$, as the typical properties of the random graph in these two regimes is fairly different. In Sect. 3.1 we prove the claim in the sparse regime $p \leq \frac{n}{\log n}$, and in Sect. 3.2 we prove it the dense regime $n^{-0.4} \leq p \leq 1$.

3.1 Sparse Case

Theorem 3 Let $G \sim G(n, p)$ where $np - \log n - \log \log n \to \infty$ and $p \leq n^{-0.4}$. Then with high probability $r_{G}(G, \mathcal{H}) = \delta(G) - 1$.

Proof Let $d_0 = 0.001np$ and $\text{SMALL}(G) = \{v \in V(G) \mid d_G(v) < d_0\}$.
Lemma 4 With high probability $G$ has the following properties.

(P1) $\delta(G) \geq 2$ and $\Delta(G) \leq 5np$;
(P2) $|\text{SMALL}(G)| \leq n^{0.1}$;
(P3) $G$ does not contain a path of length at most 4 with both its endpoints in $\text{SMALL}(G)$;
(P4) every $U \subseteq V(G)$ with $\frac{1}{2}d_0 \leq |U| \leq \frac{5n}{\sqrt{np}}$ spans at most $\frac{1}{15}d_0|U|$ edges;
(P5) if $U, W \subseteq V(G)$ are disjoint and $|U| = |W| = \frac{n}{\sqrt{np}}$ then $e(U, W) \geq n/2$.

Proof For each of the given properties, we bound the probability that $G \sim G(n, p)$ fails to uphold it.

(P1). Since $p$ is above the Hamiltonicity threshold, which is equal to the threshold of the property $\delta(G) \geq 2$, the first part is obvious. For the second part, by the union bound and Lemma 3(i) we get

$$\mathbb{P} (\Delta(G) \geq 5np) \leq n \cdot \mathbb{P} (\text{Bin}(n-1, p) \geq 5np) \leq n \cdot \left( \frac{e(n-1)p}{5np} \right)^{5np} \leq n^{-2},$$

where the last inequality is due to the fact that $5 \cdot \log(e/5) < -3$.

(P2). The probability that $|\text{SMALL}(G)| \geq n^{0.1}$ is at most the probability that there is a set of size $s := n^{0.1}$ with less than $d_0 \cdot s$ outgoing edges. Therefore by Lemma 3(ii)

$$\mathbb{P} \left( |\text{SMALL}(G)| \geq n^{0.1} \right) \leq \binom{n}{s} \cdot \mathbb{P} (\text{Bin}(s(n-s), p) < d_0 \cdot s) \leq \binom{n}{s} \cdot d_0^s \cdot \mathbb{P} (\text{Bin}(s(n-s), p) = d_0 \cdot s) \leq \left( \frac{en}{s} \right)^s \cdot d_0^s \cdot \left( \frac{es(n-s)p}{d_0s(1-p)} \right)^{d_0s} \cdot e^{-s(n-s)p} \leq n^{0.9s} \cdot d_0^s \cdot 3000^{d_0s} \cdot e^{-0.95snp} \leq \exp(s \cdot \log n \cdot (o(1) + 0.9 + 0.001 \cdot \log 3000 - 0.95)) = o(1).$$

(P3). Given $u, v \in V(G)$ and a path $P$ of length $\ell$ between them, the probability that $u, v \in \text{SMALL}(G)$ and $P \subseteq G$ is at most the probability that $P \subseteq G$ and $\{u, v\}$ has less than $2d_0$ outgoing edges that are not part of $P$, which by Lemma 3(ii) is at most

$$p^\ell \cdot 2d_0 \cdot \left( \frac{2enp}{2d_0(1-p)} \right)^{2d_0} \cdot e^{-2(n-3)p} \leq p^\ell \cdot e^{-1.9np}.$$
By the union bound, the probability that there is a path $P \subseteq G$ of length at most 4 with both endpoints in SMALL($G$) is at most

$$\sum_{\ell=1}^{4} \binom{n}{\ell+1} p^{\ell} \cdot e^{-1.9np} = o(1).$$

(P4). By Lemma 2(i) and Lemma 3(i), the probability that there is a set $U \subseteq V(G)$ of size $k \geq \frac{1}{2}d_0$ that contradicts (P4) is at most

$$\binom{n}{k} \cdot \mathbb{P}
\left(\text{Bin}\left(\binom{k}{2}, p\right) \geq \frac{1}{15}d_0k\right) \leq \frac{en}{k} \cdot \left(\frac{15ek^2p}{2d_0k}\right)^{d_0k/15} \leq (np)^{-0.03d_0k},$$

where the last inequality is due to the fact that $k \leq \frac{5n}{\sqrt{np}}$, and therefore

$$\frac{15ek^2p}{2d_0k} \leq \frac{37500}{\sqrt{np}} = (np)^{-0.5+o(1)}.$$ 

Therefore, the probability that $G$ does not have (P4) is at most

$$\sum_{k=d_0/2}^{5n/\sqrt{np}} (np)^{-0.03d_0k} = (1 + o(1))(np)^{-0.01d_0^2} = o(1).$$

(P5). By the union bound, the probability that there are $U, W \subseteq V(G)$ of size $\frac{n}{\sqrt{np}}$ with less than $\frac{1}{2}n$ edges between them is at most

$$\left(\binom{n}{\frac{n}{\sqrt{np}}} \cdot \mathbb{P} \left(\text{Bin}\left(\frac{n}{p}, p\right) = \frac{1}{2}n\right) \leq n \cdot \left(e^2np \right)^{\frac{n}{\sqrt{np}}} \cdot \left(\frac{2en}{(1-p)n}\right)^{\frac{1}{2n}} \cdot e^{-n} = o(1).$$

Lemma 5 With high probability, for every subgraph $H \subseteq G$ with $e(H) = \delta(G) - 2$, the graph $G \setminus H$ contains a subgraph $\Gamma_0$ that is an $(\frac{n}{4}, 2)$-expander with at most $d_0n$ edges.

Proof We prove this by showing that if $G$ satisfies properties (P1)–(P5) then, for every $H \subseteq G$ with $\delta(G) - 2$ edges, $G \setminus H$ contains a subgraph $\Gamma_0$ as required. To this end we consider a random subgraph of $G \setminus H$ with at most $d_0n$ edges and show that it is an $(\frac{n}{4}, 2)$-expander with positive probability, which implies existence.

Construct a random subgraph of $G \setminus H$ as follows. For every $v \in V(G)$ set $E_v$ to be $E_{G \setminus H}(v)$ in the case $d_{G \setminus H}(v) \leq d_0$, and otherwise set $E_v$ to be a subset of $E_{G \setminus H}(v)$ of size $d_0$, chosen uniformly at random and independently of all other choices. The random subgraph $\Gamma$ is the $G \setminus H$-subgraph with edge set $\bigcup_{v \in V(G)} E_v$. Observe that
the minimum degree of a graph $\Gamma$ drawn this way is at least $\min\{\delta(G \setminus H), d_0\} \geq 2$, that $d_G(v) = d_{G \setminus H}(v)$ for every $v \in \text{SMALL}(G)$, and that $e(\Gamma) \leq d_0n$.

We bound from above the probability that $\Gamma$ contains a subset $U$ with at most $n/4$ vertices with less than $2|U|$ neighbours. Let $|U| = k \leq \frac{n}{4}$ and denote $U_1 = U \cap \text{SMALL}(G)$, $U_2 = U \setminus U_1$ and $k_1, k_2$ the sizes of $U_1, U_2$ respectively. Observe that (P3) implies that (i) every vertex in $U_2$ has at most one neighbour in $U_1 \cup N_G(U_1)$, and therefore $|N_\Gamma(U_2) \cap (U_1 \cup N_\Gamma(U_1))| \leq k_2$; and (ii) distinct vertices in $\text{SMALL}(G)$ have non-intersecting neighbourhoods, and therefore $|N_\Gamma(U_1)| \geq \delta(\Gamma) \cdot k_1 \geq 2k_1$.

First we show that if $k_2 \leq \frac{n}{\sqrt{np}}$ then $|N_\Gamma(U)| \geq 2|U|$ with probability 1. We separate the proof into different cases according to the value of $k_2$.

1. $k_2 = 1$. If $k_1 = 0$ then $U$ is a singleton, and $N_\Gamma(U)$ contains at least two vertices since $\delta(\Gamma) \geq 2$.

Otherwise, $k_1 > 0$ and in particular $\text{SMALL}(G)$ is not empty, so $\delta(G) \leq d_0$ and

$$|N_\Gamma(U)| \geq |N_\Gamma(U_1) \setminus U_2| + |N_\Gamma(U_2) \setminus (N_\Gamma(U_1) \cup U_1)|$$

$$\geq \min\{d_G(v) \mid v \in U_1\} \cdot k_1 - 1 + d_0 - (\delta(G) - 2) - 1.$$

If all vertices in $U_1$ have degree at least 3 in $\Gamma$ then this expression is indeed at least $3k_1 + 1 \geq 2|U|$. Additionally, if $\delta(G) \leq d_0 - 2$ then this expression is again at least $2k_1 + 2 = 2|U|$. The remaining case is if some vertices in $U_1$ have degree 2 in $\Gamma$ (and therefore in $G \setminus H$), and $\delta(G) = d_0 - 1$. But for this to be possible, all $d_0 - 3$ edges in $H$ must have at least one end in $U_1$, which implies that at most one edge was removed from the unique vertex in $U_2$. This results in

$$|N_\Gamma(U)| \geq |N_\Gamma(U_1) \setminus U_2| + |N_\Gamma(U_2) \setminus (N_\Gamma(U_1) \cup U_1)|$$

$$\geq \delta(\Gamma) \cdot k_1 - 1 + d_0 - 2$$

$$\geq 2k_1 + 2 = 2|U|.$$

2. $2 \leq k_2 \leq \frac{1}{10}d_0$. Since there are at least two vertices, there is a vertex $v \in U_2$ such that

$$d_{G \setminus H}(v) \geq d_G(v) - \frac{1}{2}(\delta(G) - 2) - 1 \geq \frac{1}{2}d_G(v) \geq \frac{1}{2}d_0,$$

and therefore also $d_\Gamma(v) \geq \frac{1}{2}d_0$, and $e_\Gamma(v, V(G) \setminus U_2) \geq \frac{1}{2}d_0 - k_2 \geq \frac{2}{5}d_0$. We get

$$|N_\Gamma(U)| \geq |N_\Gamma(U_1) \setminus U_2| + |N_\Gamma(U_2) \setminus (N_\Gamma(U_1) \cup U_1)|$$

$$\geq 2k_1 - k_2 + \frac{2}{5}d_0 - k_2$$

$$\geq 2k_1 + 2k_2 = 2|U|.$$

3. $\frac{1}{10}d_0 \leq k_2 \leq \frac{n}{\sqrt{np}}$. In this case $|N_\Gamma(U_2)| \geq 4k_2$. Indeed, if $|N_\Gamma(U_2)| \leq 4k_2$ then $U_2 \cup N_\Gamma(U_2)$ is contained in a set of size $5k_2$, which is between $\frac{1}{2}d_0$ and $\frac{5n}{\sqrt{np}}$, that
spans at least $\frac{1}{2}d_0k_2 - e(H) \geq \frac{1}{15}d_0 \cdot (5k_2)$ edges in $G$, a contradiction to (P4).

We get

$$|N_G(U)| \geq |N_G(U_1) \setminus U_2| + |N_G(U_2) \setminus (N_G(U_1) \cup U_1)|$$

$$\geq 2k_1 - k_2 + 4k_2 - k_2$$

$$\geq 2k_1 + 2k_2 = 2|U|.$$ 

For the remaining case $\frac{n}{\sqrt{np}} \leq k_2 \leq \frac{n}{4}$, we show that $|N_G(U)| \geq 2|U|$ with positive probability. Indeed, assume that $|N_G(U)| < 2|U|$, then $|V(G) \setminus (U \cap N_G(U))| \geq \frac{1}{2}|n|$. In particular, there are disjoint sets $U' \subseteq U$ and $W \subseteq V(G) \setminus (U \cap N_G(U))$, each of size $\frac{n}{\sqrt{np}}$, such that $e_G(U', W) = 0$. Observe that by (P5), $e_{G \setminus H}(U', W) \geq \frac{1}{2}|n| - \delta(G) \geq \frac{1}{3}|n|$. For a given pair of subsets $U', W$, the probability for this is at most

$$\prod_{u \in U'} \mathbb{P}(e_G(u, W) = 0) \leq \prod_{u \in U'} \left(\frac{d_{G \setminus H}(u) - e_{G \setminus H}(u, W)}{d_0}\right)$$

$$\leq \prod_{u \in U'} e^{-\frac{d_0 e_{G \setminus H}(u, W)}{d_{G \setminus H}(u)}}$$

$$\leq \exp\left(-\frac{d_0}{\Delta(G)} \cdot e_{G \setminus H}(U', W)\right)$$

$$\leq \exp\left(-\frac{1}{15000}n\right),$$

where the second inequality is due to Lemma 2(ii), and in the last inequality we used the fact that $G$ has (P1), and therefore $\Delta(G) \leq 5np$. Since there are $\exp(o(n))$ pairs of subsets $U', W$ of size $\frac{n}{\sqrt{np}}$, by the union bound the probability that two subsets of this size with no edges between them in $\Gamma$ exist is of order $o(1)$. Consequently, the random subgraph $\Gamma$ is an $(\frac{n}{4}, 2)$-expander with probability $1 - o(1)$, implying that $G \setminus H$ contains a sparse expander, as claimed.

**Lemma 6** With high probability, for every subgraph $H \subseteq G$ with $e(H) = \delta(G) - 2$ and every non-Hamiltonian $(\frac{n}{4}, 2)$-expander $\Gamma \subseteq G$ with $e(\Gamma) \leq 2d_0n$, the graph $G \setminus (H \cup \Gamma)$ contains a booster with respect to $\Gamma$.

**Proof** By Lemma 1, a non-Hamiltonian $(\frac{n}{4}, 2)$-expander has at least $\frac{n^2}{32}$ boosters. For a given subgraph $H$ with $\delta(G) - 2$ edges and a given expander $\Gamma$, the probability that none of the boosters are in $G \setminus H$ is at most

$$\mathbb{P}\left(\text{Bin}\left(\frac{n^2}{32}, p\right) \leq e(H)\right) \leq \delta(G) \cdot \left(\frac{en^2p}{32 \cdot \delta(G) \cdot (1 - p)}\right)^{\delta(G)} \cdot e^{-\frac{n^2p}{32}} \leq e^{-\frac{n^2p}{33}}.$$

By the union bound, the probability that there is an expander subgraph $\Gamma \subseteq G$ with at most $2d_0n$ edges, and no boosters with respect to $\Gamma$ in $G$, is at most

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Lemma 7, an event which occurs with high probability. Then, given edges in $G$ only possible if $H$ contains an $(\frac{n}{4}, 2)$-expander subgraph $\Gamma_0$ with at most $d_0n$ edges. Then, while $\Gamma_i$ is not Hamiltonian, $G \setminus H$ contains a booster with respect to it, which we add to $\Gamma_i$ to obtain $\Gamma_{i+1}$. Repeating this for at most $n$ steps we obtain a Hamiltonian subgraph of $G \setminus H$.

### 3.2 Dense Case

Theorem 4 Let $G \sim G(n, p)$ where $n^{-0.4} \leq p \leq 1$. Then with high probability $r_{\mathcal{H}}(G, \mathcal{H}) = \delta(G) - 1$.

**Proof**

Lemma 7 With high probability $G$ has the following properties.

(Q1) $\delta(G) \geq \frac{1}{4}np$;
(Q2) if $U \subseteq V(G)$ and $|U| = \frac{1}{8}np$ then $e(U) \geq n$;
(Q3) if $U, W \subseteq V(G)$ are disjoint and $|U| = |W| = \frac{1}{8}np$ then $e(U, W) \geq n$.

**Proof** An upper bound of order $o(1)$ on the probability that $G \sim G(n, p)$ fails to uphold any of the three properties follows from applying the union bound and standard bounds on binomial distributions.

The proof of Theorem 3.2 now follows the lemma. We prove that if $G$ has properties (Q1)–(Q3) then $G \setminus H$ is Hamiltonian for every such $H$ with $e(H) = \delta(G) - 2$.

Let $v_0 \in V(G)$ be a vertex with $d_{G \setminus H}(v_0) = \delta(G \setminus H)$ and denote $G' := (G \setminus H) - v_0$. Then $\delta(G') \geq \frac{1}{2}\delta(G)$. Indeed, if $\delta(G') < \frac{1}{2}\delta(G)$, then there is a vertex $v_1 \in V(G \setminus \{v_0\})$ such that both $v_0$ and $v_1$ have degree at most $\frac{1}{2}\delta(G) + \frac{1}{2}$ in $G \setminus H$, which is only possible if $H$ has at least $\delta(G) - 1$ edges, which is a contradiction to its definition.

We now claim that $k(G') > \alpha(G')$, and therefore by Theorem 2 we conclude that $G'$ is Hamilton-connected. Since $d_{G \setminus \{v_0\}}(v_0) \geq 2$ this implies that $G \setminus H$ is Hamiltonian.

Indeed, by (Q2), every vertex subset with $\frac{1}{8}np$ vertices spans at least $n - \delta(G) > 0$ edges in $G'$, and therefore $\alpha(G') < \frac{1}{8}np$.

On the other hand, let $V(G') = U \cup X \cup W$ be a partition of $V(G')$ such that $U, W$ are non-empty and $e_{G'}(U, W) = 0$. Assume without loss of generality that $|U| \leq |W|$, Then $|U| < \frac{1}{8}np$, since otherwise by (Q3) we have $e_{G'}(U, W) \geq n - \delta(G) > 0$. Additionally, by (Q1) we have $\delta(G') \geq \frac{1}{2}\delta(G) \geq \frac{1}{4}np$. Since $U \cup X$ contains all of
the neighbors of a vertex \( u \in U \) we get \( |X| \geq d_{G'}(u) - |U| \geq \frac{1}{4}np - \frac{1}{8}np = \frac{1}{8}np \). Therefore \( \kappa(G') \geq \frac{1}{8}np \).

4 Concluding Remarks

We note that the proof of Theorem 1 (and, in fact, Corollary 1) presented in this paper can be adjusted slightly to prove the following statement, where here PM denotes the property of containing a perfect matching.

**Proposition 1** Let \( p(n) \in [0, 1] \) and \( G \sim G(n, p) \), where \( n \) is even. Then with high probability \( rg(G, PM) = \delta(G) \).

For the critical and sub-critical regimes the same reasoning as in the proof of Corollary 1 can be applied, where for the critical regime we replace the probability that a graph is Hamiltonian with the result by Erdős and Rényi [6] regarding perfect matchings. Also observe that, given Theorem 1, only the sparse case of the super-critical regime requires a proof. Indeed, in the dense case, if \( G \sim G(n, p) \) then, with high probability, for every \( H \) with \( e(H) \leq \delta(G) - 1 \) the graph \( G \setminus H \) contains a Hamilton path, also implying that it contains a perfect matching.

The sparse case of the super-critical regime can be proved by applying some small adjustments to the proof of Theorem 3. Here, property (P1) in Lemma 4 should be changed to state that \( \delta(G) \geq 1 \). A slight adjustment to Lemma 5 then shows that, with high probability, \( G \setminus H \) contains a sparse \((n/4, 1)\)-expander for any \( H \) with at most \( \delta(G) - 1 \) edges. The last part of the proof is identical, as \((k, 1)\)-expanders are also known to have \((k+1)^2/2\) boosters (with respect to maximum size matchings, rather than maximum length paths. See e.g. [7], Lemma 6.3).

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