RESTRICTION BOUNDS FOR THE FREE RESOLVENT AND
RESONANCES IN LOSSY SCATTERING

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Abstract. We establish high energy $L^2$ estimates for the restriction of the free Green's function to hypersurfaces in $\mathbb{R}^d$. As an application, we estimate the size of a logarithmic resonance free region for scattering by potentials of the form $V \otimes \delta_{\Gamma}$, where $\Gamma \subset \mathbb{R}^d$ is a finite union of compact subsets of embedded hypersurfaces. In odd dimensions we prove a resonance expansion for solutions to the wave equation with such a potential.

1. Introduction

Scattering by potentials is used in math and physics to study waves in many physical systems (see for example [7], [14], [22], [23] and the references therein). Examples include acoustics in concert halls and scattering of light by black holes. One case of recent interest is scattering in quantum corrals that are constructed using scanning tunneling microscopes [4], [10]. One model for this system is that of a delta function potential on the boundary of a domain $\Omega \subset \mathbb{R}^d$ (see for example [2], [4], [10]). In this paper, we study scattering by such a delta function potential on hypersurfaces $\Gamma \subset \mathbb{R}^d$.

We assume that $\Gamma \subset \mathbb{R}^d$ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces; that is, it is a union of compact subsets of graphs of $C^{1,1}$ functions. The Bunimovich stadium is an example of a domain in two dimensions which has boundary that is $C^{1,1}$, but not $C^2$. We let $\delta_{\Gamma}$ denote the surface measure on $\Gamma$, considered as a distribution on $\mathbb{R}^d$, and take $V$ to be a bounded, self-adjoint operator on $L^2(\Gamma)$. For $u \in H^{1}_{\text{loc}}(\mathbb{R}^d)$, we then define $(V \otimes \delta_{\Gamma})u := (Vu|\Gamma)\delta_{\Gamma}$.

Resonances are defined as poles of the meromorphic continuation from $\text{Im} \lambda \gg 0$ of the resolvent

$$R_V(\lambda) = (-\Delta_{V,\Gamma} - \lambda^2)^{-1},$$

where $-\Delta_{V,\Gamma}$ is the unbounded self-adjoint operator

$$-\Delta_{V,\Gamma} := -\Delta + V \otimes \delta_{\Gamma}$$

(See Section 2.1 for the formal definition of $-\Delta_{V,\Gamma}$). If the dimension $d$ is odd, $R_V(\lambda)$ admits a meromorphic continuation to the entire complex plane, and to the logarithmic covering space of $\mathbb{C} \setminus \{0\}$ if $d$ is even (see Section 7).

The imaginary part of a resonance gives the decay rate of the associated resonant states. Thus, resonances close to the real axis give information about long term behavior of waves. In particular, since the seminal work of Lax-Phillips [14] and Vainberg [21], resonance free regions near the real axis have been used to understand decay of waves.
In this paper, we demonstrate the existence of a resonance free region for delta function potentials on a very general class of $\Gamma$.

**Theorem 1.** Let $\Gamma \subset \mathbb{R}^d$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, and suppose $V$ is a self-adjoint operator on $L^2(\Gamma)$.

Then for all $\epsilon > 0$ there exists $R > 0$ such that, if $\lambda$ is a resonance for $-\Delta_{V;\Gamma}$, then

\[ \text{Im} \lambda \leq -\left(\frac{1}{2}d_\Gamma^{-1} - \epsilon\right) \log(|\text{Re} \lambda|) \quad \text{if} \quad |\text{Re} \lambda| \geq R, \]

where $d_\Gamma$ is the diameter of the convex hull of $\Gamma$. If $\Gamma$ can be written as a finite union of strictly convex $C^{2,1}$ hypersurfaces, then we can replace $\frac{1}{2}$ by $\frac{2}{3}$ in (1.1).

**Remarks:**

- These bounds on the size of the resonance free region are not generally optimal, for example in the case that $\Gamma = \partial B(0,1) \subset \mathbb{R}^2$. In [12], the first author uses a microlocal analysis of the transmission problem (1.7) to obtain sharp bounds in the case that $\Gamma = \partial \Omega$ is $C^\infty$ with $\Omega$ strictly convex.
- In the smooth, strictly convex case, scattering in other types of transmission problems was considered in [9] and [15].

Let $R_0(\lambda)$ be the analytic continuation of the outgoing free resolvent $(-\Delta - \lambda^2)^{-1}$, defined initially for $\text{Im} \lambda > 0$. Theorem 1 follows from bounds on an operator related to the free resolvent. In particular, we study the restriction of $R_0(\lambda)$ to hypersurfaces $\Gamma \subset \mathbb{R}^d$. Let $\gamma$ denote restriction to $\Gamma$, and $\gamma^*$ the inclusion map $f \mapsto f \delta_\Gamma$. Let $G(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ be obtained by restricting the kernel $G_0(\lambda,x,y)$ of $R_0(\lambda)$ to $\Gamma$,

\[ G(\lambda) := \gamma R_0(\lambda) \gamma^*. \]

Theorem 1 will follow as a consequence of the following theorem,

**Theorem 2.** Let $\Gamma \subset \mathbb{R}^d$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces. Then $G(\lambda)$ is a compact operator on $L^2(\Gamma)$, and

\[ \|G(\lambda)\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C \langle \lambda \rangle^{-\frac{1}{2}} \log(\langle \lambda \rangle) e^{d_\Gamma(\text{Im} \lambda)-}, \]

where $d_\Gamma$ is the diameter of the convex hull of $\Gamma$. Moreover, if $\Gamma$ is a finite union of compact subsets of strictly convex $C^{2,1}$ hypersurfaces, then

\[ \|G(\lambda)\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C \langle \lambda \rangle^{-\frac{2}{3}} \log(\langle \lambda \rangle) e^{d_\Gamma(\text{Im} \lambda)-}. \]

Here we set $\langle \lambda \rangle = (2 + |\lambda|^2)^{\frac{1}{2}}$, and $(\text{Im} \lambda)_- = \max(0, -\text{Im} \lambda)$. Compactness follows easily by Rellich’s embedding theorem, or the bounds on $G_0(\lambda,x,y)$ in Section 2.2. The powers $\frac{1}{2}$ and $\frac{2}{3}$ in (1.2) and (1.3), respectively, are in general optimal. This follows from the fact that the corresponding estimates for the restriction of eigenfunctions in Section 4 are the best possible. However, it is likely that the factor of $\log(\langle \lambda \rangle)$ is not needed. In Section 3 we prove estimate (1.2) in dimension two without it. Also, for the flat case in general dimensions, the estimate (1.2) holds without it. We also expect that estimate (1.3) holds for $C^{1,1}$ strictly convex hypersurfaces, but do not pursue that here.
In the case that \( \text{Im} \lambda \geq |\lambda|^\frac{3}{4} \), respectively \( \text{Im} \lambda \geq |\lambda|^\frac{5}{8} \), the above bounds can be improved upon.

**Theorem 3.** Let \( \Gamma \subset \mathbb{R}^d \) be a finite union of compact subsets of embedded \( C^{1,1} \) hypersurfaces. Then for \( \text{Im} \lambda > 0 \),

\[
\|G(\lambda)\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C (\text{Im} \lambda)^{-1}.
\]

We next use the results above to analyze the long term behavior of waves scattered by the potential \( V \otimes \delta_\Gamma \). **Theorem 1** implies in particular that there are only a finite number of resonances in the set \( \text{Im} \lambda > -A \), for any \( A < \infty \). We give a resonance expansion for the wave equation

\[
(\partial_t^2 - \Delta + V \otimes \delta_\Gamma) u = 0, \quad u(0, x) = 0, \quad \partial_t u(0, x) = g \in L^2_{\text{comp}},
\]

with wave propagator \( U(t) \) defined using the functional calculus for \( -\Delta_{V, \Gamma} \). Let \( m_R(\lambda) \) be the multiplicity of the pole of \( R_V(\lambda) \) at \( \lambda \), that is the dimension of the set of resonant states with resonance \( \lambda \), and let \( D_N \) be the domain of \( (-\Delta_{V, \Gamma})^N \).

**Theorem 4.** Let \( d > 0 \) be odd, and assume that \( \Gamma \subset \mathbb{R}^d \) is a finite union of compact subsets of embedded \( C^{1,1} \) hypersurfaces, and that \( V \) is a self-adjoint operator on \( L^2(\Gamma) \).

Let \( 0 > -\mu_1^2 > \cdots > -\mu_M^2 \) and \( 0 < \nu_1^2 < \cdots < \nu_M^2 \) be the nonzero eigenvalues of \( -\Delta_{V, \Gamma} \), and \( \{\lambda_j\} \) the resonances with \( \text{Im} \lambda < 0 \). Then for any \( A > 0 \) and \( g \in L^2_{\text{comp}} \), the solution \( U(t)g \) to \((1.4)\) admits an expansion

\[
U(t)g = \sum_{j=1}^M \sum_{k=1}^N \frac{m_R(\lambda_j)}{m_R(\lambda_j) - \lambda} e^{it\lambda_j} \int_0^t e^{-i\lambda_j s} \frac{1}{\eta_2} \mathcal{P}(\lambda_j, g) + E_A(t)g,
\]

where \( \Pi_{\mu_k} \) and \( \Pi_{\nu_k} \) respectively denote the projections onto the \( -\mu_k^2 \) and \( \nu_k^2 \) eigenspaces. The maps \( \mathcal{P}(\lambda_j, g) \) are bounded from \( L^2_{\text{comp}} \to D_{\text{loc}} \), and \( \mathcal{P}_0 \) is a symmetric map to the 0-resonances.

The operator \( E_A(t) : L^2_{\text{comp}} \to L^2_{\text{loc}} \) has the following property: for any \( \chi \in C_0^\infty(\mathbb{R}^d) \) equal to 1 on a neighborhood of \( \Gamma \), and any \( N \geq 0 \), there exists \( T_{A, \chi, N} < \infty \) so that

\[
\|\chi E_A(t)\chi\|_{L^2 \to D_N} \leq C_{A, \chi, N} \epsilon^{-At}, \quad t > T_{A, \chi, N}.
\]

Under the assumption that \( \Gamma = \partial \Omega \) for a bounded open domain \( \Omega \subset \mathbb{R}^d \), and that \( V \) and \( \partial \Omega \) satisfy higher regularity assumptions, we obtain estimates for \( \chi E_A(t)\chi g \) in the spaces

\[
\mathcal{E}_N := H^1(\mathbb{R}^d) \cap (H^N(\Omega) \oplus H^N(\mathbb{R}^d \setminus \bar{\Omega})), \quad N \geq 1.
\]

If \( \partial \Omega \) is of \( C^{1,1} \) regularity, and \( V \) is bounded on \( H^s(\partial \Omega) \), then we show \( \mathcal{D}_1 = \mathcal{D} \subset \mathcal{E}_2 \), and convergence in \( \mathcal{E}_2 \) follows from Theorem 4. For smooth boundaries we show the following.

**Theorem 5.** Suppose that \( \Gamma = \partial \Omega \) is \( C^\infty \) and that \( V \) is bounded on \( H^s(\partial \Omega) \) for all \( s \). Then the operator \( E_A(t) \) defined in \((1.5)\) has the following property: for any \( \chi \in C_0^\infty(\mathbb{R}^d) \) equal to 1 on a neighborhood of \( \bar{\Omega} \), and integer \( N \geq 1 \), there exists \( T_{A, \chi, N} < \infty \) so that

\[
\|\chi E_A(t)\chi\|_{L^2 \to \mathcal{E}_N} \leq C_{A, \chi, N} \epsilon^{-At}, \quad t > T_{A, \chi, N}.
\]
In addition to describing resonances as poles of the meromorphic continuation of the resolvent, we will give a more concrete description of resonances in Sections 6 and 7. We show that $\lambda$ is a resonance of the system if and only if there is a nontrivial $\lambda$-outgoing solution $u \in D_{\text{loc}}$ to the equation
\[(1.6) \quad (-\Delta - \lambda^2 + V \otimes \delta_\Gamma)u = 0,\]
where we define
\[D_{\text{loc}} = \{ u : \chi u \in D \text{ whenever } \chi \in C^\infty_c(\mathbb{R}^d) \text{ and } \chi = 1 \text{ on a neighborhood of } \Gamma \}. \]
Here we say that $u$ is $\lambda$-outgoing if for some $R < \infty$, and some compactly supported distribution $g$, we can write
\[u(x) = (R_0(\lambda)g)(x) \quad \text{for } |x| \geq R.\]
Moreover, if we assume that $\Gamma = \partial \Omega$ for a $C^{1,1}$ domain $\Omega$, and that $V : H^{\frac{d}{2}}(\partial \Omega) \to H^{\frac{d}{2}}(\partial \Omega)$, we show this is equivalent to solving the following transmission problem with $u \in H^1_{\text{loc}}(\mathbb{R}^d)$, and with $u|_\Omega = u_1 \in H^2(\Omega)$, $u|_{\mathbb{R}^d \setminus \Omega} = u_2 \in H^2_{\text{loc}}(\mathbb{R}^d \setminus \Omega)$,
\[(1.7) \quad \begin{cases} (-\Delta - \lambda^2)u_1 = 0 & \text{in } \Omega \\ (-\Delta - \lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u_1 = u_2 & \text{on } \partial \Omega \\ \partial_\nu u_1 + \partial_\nu' u_2 + Vu_1 = 0 & \text{on } \partial \Omega \\ u_2 \lambda\text{-outgoing} \end{cases}\]
Here, $\partial_\nu$ and $\partial_\nu'$ are respectively the interior and exterior normal derivatives of $u$ at $\partial \Omega$.

The outline of this paper is as follows. In Section 2 we present the definition of $-\Delta_{V,\Omega}$ and its domain, as well as some preliminary bounds on the outgoing Green’s function $G_0(\lambda, x, y)$. In Section 3 we give a simple proof of Theorem 2 for $d = 2$. In Section 4 we establish Theorem 2 for $\text{Im } \lambda \geq 0$ in all dimensions, deriving the estimates from restriction estimates for eigenfunctions of the Laplacian. We include a proof of the desired restriction estimate for hypersurfaces of regularity $C^{1,1}$, since the result appears new, and also provide the proof of Theorem 3. In Section 5 we complete the proof of Theorem 2 for $\text{Im } \lambda < 0$ using the Phragmén-Lindelöf theorem. In Section 6 we demonstrate the meromorphic continuation of $R_V(\lambda)$, give the proof of Theorem 4, and relate resonances to solvability of an equation on $\Gamma$, and for $\Gamma = \partial \Omega$ to solvability of (1.7). In Section 7 we give more detailed structure of the meromorphic continuation of $R_V(\lambda)$. We establish mapping bounds for compact cutoffs of $R_V(\lambda)$, and use these to prove Theorems 4 and 5 by a contour integration argument. In Section 8 we prove a needed transmission property estimate for boundaries of regularity $C^{1,1}$.

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2. Preliminaries

2.1. Determination of $-Δ_{V,Γ}$ and its domain. We define the operator $-Δ_{V,Γ}$ using the symmetric, densely defined quadratic form

$$Q_{V,Γ}(u, w) := \langle \nabla u, \nabla w \rangle_{L^2(\mathbb{R}^d)} + \langle V u, w \rangle_{L^2(Γ)}$$

with domain $H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$.

For $Γ$ a finite union of compact subsets of $C^{1,1}$ hypersurfaces (indeed Lipschitz hypersurfaces suffice), we can bound

$$\|u\|_{L^2(Γ)} \leq C \|u\|_{L^2} \|u\|_{H^1}^{\frac{1}{2}} \leq C \epsilon \|u\|_{H^1} + C \epsilon^{-1} \|u\|_{L^2}.$$ 

It follows that there exist $c, C > 0$ such that

$$|Q_{V,Γ}(u, w)| \leq \|u\|_{H^1} \|w\|_{H^1} \quad \text{and} \quad c \|u\|_{H^1}^{2} \leq Q_{V,Γ}(u, u) + C \|u\|_{L^2}^{2}.$$ 

By Reed-Simon [16, Theorem VIII.15], $Q_{V,Γ}(u, w)$ is determined by a unique self-adjoint operator $-Δ_{V,Γ}$, with domain $D$ consisting of $u \in H^1$ for which $Q_{V,Γ}(u, u) \leq C \|u\|_{L^2}^{2}$.

For $u \in D$, by the Riesz representation theorem, we have $Q_{V,Γ}(u, w) = \langle f, w \rangle$ for some $f \in L^2(\mathbb{R}^d)$, and taking $w \in C_{c}^{\infty}(\mathbb{R}^d)$ shows that in the sense of distributions

$$-Δ u + (V u|Γ)\deltaΓ = f.$$ 

Conversely, if $u \in H^1(\mathbb{R}^d)$ and (2.1) holds for some $f \in L^2(\mathbb{R}^d)$, then by density of $C_{c}^{\infty} \subset H^1$ we have $Q_{V,Γ}(u, w) = \langle f, w \rangle$ for $w \in H^1(\mathbb{R}^d)$, hence $u \in D$, and $-Δ_{V,Γ}u$ is given by the left hand side of (2.1). We thus can write (up to a constant of proportionality)

$$\|u\|_D = \|u\|_{H^1} + \|Δ_{V,Γ}u\|_{L^2},$$

where finiteness of the second term carries the assumption that $Δ_{V,Γ}u \in L^2$.

The domain $D_N \subset D$ is defined for $N \geq 1$ by the condition $Δ_{V,Γ}u \in D_{N-1}$, and we will recursively define (consistent up to constants with the definition using the functional calculus)

$$\|u\|_{D_N} = \|u\|_{H^1} + \|Δ_{V,Γ}u\|_{D_{N-1}}, \quad N \geq 1.$$ 

Suppose that $χ \in C_{c}^{\infty}(\mathbb{R}^d \setminus Γ)$ and that $u \in H^1$ solves (2.1). Then,

$$Δ(χu) = χ f + 2∇χ \cdot \nabla u + (Δ χ)u \in L^2(\mathbb{R}^d).$$

Hence,

$$\|χu\|_{H^2} \leq C_{χ} \|u\|_D.$$ 

That is, $D \subset H^1(\mathbb{R}^d) \cap H^2_{\text{loc}}(\mathbb{R}^d \setminus Γ)$, with continuous inclusion. Similar arguments show that

$$D_N \subset H^1(\mathbb{R}^d) \cap H^2_{\text{loc}}(\mathbb{R}^d \setminus Γ), \quad N \geq 1.$$ 

The behavior of $u$ near $Γ$ may be more singular. For general $V$ acting on $L^2(Γ)$, from (2.1) and the fact that $(Vu|Γ)\deltaΓ \in H^{-\frac{1}{2},-\epsilon}(\mathbb{R}^d)$ for all $\epsilon > 0$, we conclude that $u \in H^{\frac{5}{2}-\epsilon}(\mathbb{R}^d)$. However, under additional assumptions on $V$ and $Γ$ we can give a full description of $D$ near $Γ$.

For the purposes of the remainder of this section we assume that $Γ = \partialΩ$ for some bounded open domain $Ω \subset \mathbb{R}^d$, and that $\partialΩ$ is a $C^{1,1}$ hypersurface; that is, locally $\partialΩ$ can be written as the
graph of a $C^{1,1}$ function. We assume also that $V : H^{\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$. Then since $u \in H^{1}(\mathbb{R}^{d})$, $Vu|_{\partial \Omega} \in H^{\frac{1}{2}}(\partial \Omega)$. Hence, Lemma 8.2 combined with (2.1) shows that

$$\mathcal{D} \subset \mathcal{E}_{\mathbb{C}} = H^{1}(\mathbb{R}^{d}) \cap (H^{2}(\Omega) \oplus H^{2}(\mathbb{R}^{d} \setminus \overline{\Omega})),$$

with continuous inclusion. We remark that $H^{2}(\Omega)$ and $H^{2}(\mathbb{R}^{d} \setminus \overline{\Omega})$ can be identified as restrictions of $H^{2}(\mathbb{R}^{d})$ functions; see [8] and [17, Theorem VI.5]. Thus, if $u \in \mathcal{D}$ both $u$ and its first derivatives have well defined traces on $\partial \Omega$, respectively of regularity $H^{\frac{1}{2}}(\partial \Omega)$ and $H^{\frac{1}{2}}(\partial \Omega)$.

For $w \in H^{1}(\mathbb{R}^{d})$ and $u \in H^{1}(\mathbb{R}^{d}) \cap (H^{2}(\Omega) \oplus H^{2}(\mathbb{R}^{d} \setminus \overline{\Omega}))$, it follows from Green’s identities that

$$Q_{V,\partial \Omega}(u, w) = \langle -\Delta u, w \rangle_{\Omega} + \langle -\Delta u, w \rangle_{\mathbb{R}^{d} \setminus \overline{\Omega}} + \langle \partial_{\nu} u + \partial_{\nu}^{\prime} u + V u, w \rangle_{\partial \Omega},$$

where $\partial_{\nu}$ and $\partial_{\nu}^{\prime}$ denote the exterior normal derivatives from $\Omega$ and $\mathbb{R}^{d} \setminus \overline{\Omega}$. Thus, in the case that $V : H^{\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$, we can completely characterize the domain $\mathcal{D}$ of the self-adjoint operator $-\Delta_{V,\partial \Omega}$ as

$$\mathcal{D} = \{ u \in H^{1}(\mathbb{R}^{d}) \cap (H^{2}(\Omega) \oplus H^{2}(\mathbb{R}^{d} \setminus \overline{\Omega})) \text{ such that } \partial_{\nu} u + \partial_{\nu}^{\prime} u + V u = 0 \},$$

in which case $\Delta_{V,\partial \Omega} u = \Delta u|_{\partial \Omega} + \Delta u|_{\mathbb{R}^{d} \setminus \overline{\Omega}}$.

2.2. Bounds on Green’s function. We conclude this section by reviewing bounds on the convolution kernel $G_{0}(\lambda, x, y)$ associated to the operator $R_{0}(\lambda)$. It can be written in terms of the Hankel functions of the first kind,

$$G_{0}(\lambda, x, y) = C_{d} \lambda^{d-2} \left( |x - y| \right)^{-\frac{d-2}{2}} H^{(1)}_{\frac{d-1}{2}}(\lambda |x - y|),$$

for some constant $C_{d}$. If $d$ is odd, this can be written as a finite expansion

$$G_{0}(\lambda, x, y) = \lambda^{d-2} e^{i\lambda|x - y|} \sum_{j=0}^{d-2} \frac{c_{d,j}}{(\lambda |x - y|)^{j+2}}.$$

For $x \neq y$ this form extends to $\lambda \in \mathbb{C}$, and defines the analytic extension of $R_{0}(\lambda)$. In particular, we have the upper bounds

$$|G_{0}(\lambda, x, y)| \lesssim \begin{cases} |x - y|^{2-d}, & |x - y| \leq |\lambda|^{-1}, \\ e^{-\text{Im} \lambda |x - y|} |\lambda|^{\frac{d-1}{2}} |x - y|^{\frac{d-2}{2}}, & |x - y| \geq |\lambda|^{-1}. \end{cases}$$

For $d$ even, and $d \neq 2$, the bounds (2.3) hold as well as for the analytic extension to $-\pi \leq \text{arg} \lambda \leq 2\pi$. For $-\pi < \text{arg} \lambda < 2\pi$ this follows by the asymptotics of $H^{(1)}_{\frac{d}{2}}(z)$; see for example [11 (9.2.3)]. To see that it extends to the closed region, we use the relation (valid in all dimensions)

$$G_{0}(e^{i\pi} \lambda, x, y) - G_{0}(\lambda, x, y) = \frac{i}{(2\pi)^{d-1}} \int_{\mathbb{S}^{d-1}} e^{i\lambda(x - y, \omega)} d\omega = C_{d} \lambda^{d-2} \left( |x - y| \right)^{-\frac{d-2}{2}} J_{\frac{d-1}{2}}(\lambda |x - y|),$$

where $d\omega$ is surface measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^{d}$, and $e^{i\pi}$ indicates analytic continuation through positive angle $\pi$. The bounds (2.3) then follow from the asymptotics of $J_{n}(z)$ and the
bounds for $\text{Im } \lambda \geq 0$. We also note as a consequence of the above that, for $\lambda \in \mathbb{R} \setminus \{0\}$, and any sheet of the continuation in even dimensions,

$$G_0(e^{i\pi \lambda} x, y) - G_0(\lambda, x, y) = 2\pi i (\text{sgn } \lambda)^d |\lambda|^{-\frac{1}{2d-1}} (x - y),$$

where $\delta_{d-1}$ denotes surface measure on the sphere $|\xi| = |\lambda|$ in $\mathbb{R}^d$.

In the case that $d = 2$, in (2.3) one need replace $|x - y|^{2-d}$ by $-\ln |x - y|$ in the bounds for $|x - y| \leq |\lambda|^{-1}$. However, for our purposes we use only the following global bound in case $d = 2$,

$$|G_0(\lambda, x, y)| \lesssim e^{-\text{Im } \lambda |x-y|} |\lambda|^{-\frac{1}{2}} |x-y|^{-\frac{1}{2}}, \quad d = 2, \quad -\pi \leq \arg \lambda \leq 2\pi.$$

Finally, we observe that since $G_0(\lambda, x, y)$ is smooth away from the diagonal, with the singularity at $x = y$ integrable over $\Gamma$, it follows that $G(\lambda)$ is a compact operator on $L^2(\Gamma)$ for every $\lambda$. This also follows from (7.7).

3. Estimates for $d = 2$

In this section we give an elementary proof of estimate (1.2) of Theorem 2 for $d = 2$. Indeed, we can prove the following stronger result,

**Theorem 6.** Under the conditions of Theorem 2 the following holds, for $-\pi \leq \arg \lambda \leq 2\pi$,

$$\|G(\lambda)f\|_{L^2(\Gamma)} \leq \begin{cases} C \langle \lambda \rangle^{-\frac{1}{2}} (\text{Im } \lambda)^{-\frac{1}{2}} \|f\|_{L^2(\Gamma)}, & \text{Im } \lambda \geq 0, \\ C \langle \lambda \rangle^{-\frac{1}{2}} e^{-d \text{Im } \lambda} \|f\|_{L^2(\Gamma)}, & \text{Im } \lambda \leq 0. \end{cases}$$

**Proof.** We use the kernel bounds, for $x, y$ in a bounded set,

$$|G_0(\lambda, x, y)| \leq C e^{-\text{Im } \lambda |x-y|} \langle \lambda \rangle^{-\frac{1}{2}} |x-y|^{-\frac{1}{2}}.$$

By the Schur test and symmetry of the kernel, the operator norm is bounded by the following

$$\sup_x \int_{\Gamma} |G_0(\lambda, x, y)| d\sigma(y).$$

First consider $\text{Im } \lambda \leq 0$. Then $e^{-\text{Im } \lambda |x-y|} \leq e^{-d \text{Im } \lambda} \text{Im } \lambda$ for $x, y \in \Gamma$, and since $\Gamma$ is a finite union of subsets of $C^{1,1}$ hypersurfaces the desired bound follows from the following, which holds if $\Gamma$ is a bounded subset of the graph of a Lipschitz function,

$$\sup_x \int_{\Gamma} |x-y|^{-\frac{1}{2}} d\sigma(y) \leq C.$$

For $\text{Im } \lambda \geq 0$, we use instead the bound

$$\sup_x \int_{\Gamma} e^{-\text{Im } \lambda |x-y|} |x-y|^{-\frac{1}{2}} d\sigma(y) \leq C \langle \text{Im } \lambda \rangle^{-\frac{1}{2}}.$$
4. Resolvent Bounds in the Upper Half Plane

In this section, we prove Theorems 2 and 3 for Im $\lambda > 0$.

We assume that $\Gamma$ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, with induced surface measure. For $f \in L^2(\Gamma)$ we use $f\delta_\Gamma = \gamma^* f$ to denote the induced compactly supported distribution.

For Im $\lambda > 0$ let $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ be the operator with Fourier multiplier $(|\xi|^2 - \lambda^2)^{-1}$. For the proof of both Theorem 2 and Theorem 3 we will estimate

\begin{equation}
Q_\lambda(f, g) := \int R_0(\lambda)(f\delta_\Gamma)\overline{g\delta_\Gamma}.
\end{equation}

For Im $\lambda > 0$, the right hand side (4.1) agrees with the distributional pairing of $R_0(\lambda)(f\delta_\Gamma) \in H^{\frac{1}{2} - \epsilon}$ with $g\delta_\Gamma \in H^{-\frac{1}{2} - \epsilon}$, and hence by the Plancherel theorem

\begin{equation}
Q_\lambda(f, g) = \int \frac{\hat{f}\delta_\Gamma(\xi)}{|\xi|^2 - \lambda^2} \overline{\hat{g}\delta_\Gamma(\xi)}
\end{equation}

For $|\lambda| \leq 2$, the uniform bounds

$$|Q_\lambda(f, g)| \leq C \|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)}, \quad |\lambda| \leq 2,$$

follow easily from $f\delta_\Gamma, g\delta_\Gamma \in H^{-\frac{1}{2} - \epsilon}(\mathbb{R}^d)$, so we focus on $|\lambda| \geq 2$.

We start by showing that resolvent bounds for $\lambda$ in the upper half plane can be deduced from restriction bounds for $\hat{f}\delta_\Gamma$. Indeed, the following equivalence holds with $\delta_\Gamma$ replaced by any regular measure supported on a compact set.

**Lemma 4.1.** Suppose that for some $\alpha \in (0, 1)$ the following estimate holds for $r > 0$,

\begin{equation}
\int \left| \frac{\hat{f}\delta_\Gamma(\xi)}{|\xi| - r} \right|^2 \delta(|\xi| - r) \leq C |r|^\alpha \|f\|_{L^2(\Gamma)}^2.
\end{equation}

Then, for $\lambda$ in the upper half plane with $|\lambda| \geq 2$,

$$|Q_\lambda(f, g)| \leq C |\lambda|^{\alpha - 1} \log |\lambda| \|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)},$$

where $Q_\lambda$ is as in (4.1).

**Proof.** Consider first the integral in (4.2) over $|\xi| - |\lambda| \geq 1$. Since $||\xi|^2 - \lambda^2| \geq ||\xi|^2 - |\lambda|^2|$, by the Schwartz inequality and (4.3) this piece of the integral is bounded by

$$\|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)} \int_{|\xi| - |\lambda| \geq 1} (|r|^\alpha |r^2 - |\lambda|^2|^{-1} dr \leq C |\lambda|^{\alpha - 1} \log |\lambda| \|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)}.$$

Next, if Im $\lambda \geq 1$, then $||\xi|^2 - \lambda^2| \geq |\lambda|$, and by (4.3)

$$\left| \int_{|\xi| - |\lambda| \leq 1} \frac{\hat{f}\delta_\Gamma(\xi)}{|\xi|^2 - \lambda^2} \overline{\hat{g}\delta_\Gamma(\xi)} d\xi \right| \leq C |\lambda|^{\alpha - 1} \|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)}.$$
Thus, we may restrict our attention to \(0 \leq \text{Im} \lambda \leq 1\) and \(||\xi| - |\lambda|| \leq 1\). For this piece we use that (4.3) implies
\[
(4.4) \quad \int |\nabla_\xi \hat{f}\delta_r(\xi)|^2 \delta(|\xi| - r) \leq C (r^\alpha \|f\|_{L^2(\Gamma)}^2),
\]
due to the compact support of \(f\delta_r\).

We consider \(\text{Re} \lambda \geq 0\), the other case following similarly, and write
\[
\frac{1}{r^2 - \lambda^2} = \frac{1}{|\xi| + \lambda} \frac{\xi}{|\xi|} \cdot \nabla_\xi \log(|\xi| - \lambda),
\]
where the logarithm is well defined since \(\text{Im}(|\xi| - \lambda) < 0\). Let \(\chi(r) = 1\) for \(|r| \leq 1\) and vanish for \(|r| \geq \frac{3}{2}\). We then use integration by parts, together with (4.3) and (4.4) to bound
\[
\left| \int \chi(|\xi| - |\lambda|) \frac{1}{|\xi| + \lambda} \hat{f}\delta_r(\xi) \overline{g}\delta_r(\xi) \frac{\xi}{|\xi|} \nabla_\xi \log(|\xi| - \lambda) \, d\xi \right| \leq C |\lambda|^\alpha \|f\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)}.
\]

To conclude the proof of Theorem 2, we need to show that (4.3) holds with \(\alpha = \frac{1}{2}\) when \(\Gamma\) is a compact subset of a \(C^{1,1}\) hypersurface, and with \(\alpha = \frac{1}{3}\) for a compact subset of a strictly convex \(C^{2,1}\) hypersurface. Since we work locally we assume that \(\Gamma\) is given by the graph \(x_n = F(x')\), where by an extension argument we assume that \(F\) is a \(C^{1,1}\) function (respectively \(C^{2,1}\) function) defined on \(\mathbb{R}^n\), and we replace surface measure on \(\Gamma\) by \(dx'\). By scaling we may assume that \(|\nabla F| \leq \frac{1}{2r}\). We may also assume that \(F(0) = 0\).

Let \(r \geq 1\), and let \(\delta_{S^{d-1}} = \delta(|\xi| - r)\) be surface measure on the sphere \(S^{d-1}\) of radius \(r\). Assume that \(g(\xi)\) is a function belonging to \(L^2(S^{d-1})\), and define
\[
T g(x) = \int e^{i\langle x, \xi \rangle} g(\xi) \delta(|\xi| - r).
\]
Let \(\chi(x') \in C^\infty_c(\mathbb{R}^{d-1})\) be supported in the unit ball. By duality, (4.3) with \(\alpha = \frac{1}{2}\) is equivalent to the following estimate
\[
(4.5) \quad \left( \int |(Tg)(x', F(x'))|^2 \chi(x') \, dx' \right)^{\frac{1}{2}} \leq C r^{\frac{\alpha}{2}} \|g\|_{L^2(S^{d-1})},
\]
and for \(\alpha = \frac{1}{3}\) is equivalent to the same estimate with \(r^{\frac{\alpha}{2}}\) replaced by \(r^{\frac{2}{3}}\).

The estimate (4.5) is known as a restriction estimate for eigenfunctions of the Laplacian. \(L^p\) generalizations in the setting of a smooth Riemannian manifold, with restriction to a smooth submanifold, were studied by Burq, Gérard and Tzvetkov in [6]. The \(L^2\) estimates, again in the smooth setting, were noted by Tataru [19] as being a corollary of an estimate of Greenleaf and Seeger [13]. These estimates were generalized to the setting of restriction to smooth submanifolds in Riemannian manifolds with metrics of \(C^{1,1}\) regularity by Blair [3]. In making a change of coordinates to flatten a submanifold the resulting metric has one lower order of regularity, thus the estimates of [3] do not apply directly to \(C^{1,1}\) submanifolds, and so we include here the proof of the \(L^2\) estimate on \(C^{1,1}\) hypersurfaces of Euclidean space. The estimate for strictly convex \(C^{2,1}\)
hypotheses does follow from [3], so we consider here just the case of a general $C^{1,1}$ hypersurface and $\alpha = \frac{1}{2}$.

We derive (4.5) from the following square function estimate for solutions to the wave equation.

**Lemma 4.2.** Suppose that $f \in L^2(\mathbb{R}^d)$ and $\hat{f}(\xi)$ is supported in the region $\frac{3}{4}r \leq |\xi| \leq \frac{3}{2}r$. Then

$$\left( \int_0^1 \left\| \left( \cos(t\sqrt{-\Delta})f \right)(x', F(x')) \right\|^4_{L^2(\mathbb{R}^{d-1}, dx')} dt \right)^{\frac{1}{4}} \leq C r^\frac{5}{4} \| f \|_{L^2(\mathbb{R}^d)}.$$

The reduction of (4.5) to Lemma 4.2 is attained by letting $f = \psi Tg$, where $\psi \in C_0^\infty(\mathbb{R}^d)$ equals 1 on the ball of radius 3. Then $\cos(t\sqrt{-\Delta})f = \cos(tr)Tg$ for $|x| < 2$ and $|t| < 1$. On the other hand, $\hat{f} = \hat{\psi} * (g \delta_{\mathbb{S}^{d-1}_r})$ is rapidly decreasing away from the sphere $|\xi| = r$, so the difference between $\hat{f}$ and its truncation to $\frac{3}{2}r \leq |\xi| \leq \frac{3}{2}r$ is easily handled. Also, a simple calculation shows that, uniformly over $r$,

$$\| \hat{\psi} * (g \delta_{\mathbb{S}^{d-1}_r}) \|_{L^2(\mathbb{R}^d)} \leq C \| g \|_{L^2(\mathbb{S}^{d-1}_r)}.$$

**Proof of Lemma 4.2.** Given a function $F_r$ such that $\sup_{x'} |F_r(x') - F(x')| \leq r^{-1}$, then (4.6) holds if we can show that

$$\left( \int_0^1 \left\| \left( \cos(t\sqrt{-\Delta})f \right)(x', F_r(x')) \right\|^4_{L^2(\mathbb{R}^{d-1}, dx')} dt \right)^{\frac{1}{4}} \leq C r^\frac{5}{4} \| f \|_{L^2(\mathbb{R}^d)}.$$

This follows from the fact that (4.7), together with the frequency localization of $f$, implies the gradient bound, uniformly over $s$,

$$\left( \int_0^1 \left\| \partial_s \left( \cos(t\sqrt{-\Delta})f \right)(x', F_r(x') + s) \right\|^4_{L^2(\mathbb{R}^{d-1}, dx')} dt \right)^{\frac{1}{4}} \leq C r^\frac{5}{4} \| f \|_{L^2(\mathbb{R}^d)}.$$

We will take $F_r$ to be a mollification of the $C^{1,1}$ function $F$ on the $r^{-\frac{3}{2}}$ spatial scale. Precisely, let $F_r = \phi_{r^{1/2}} * F$, where $\phi_{r^{1/2}} = r^{-\frac{1}{2}} \phi(\frac{r}{2} x)$, with $\phi$ a Schwartz function of integral 1. Then

$$\sup_x |F_r(x) - F(x)| \leq C r^{-1}, \quad \sup_x |\nabla F_r(x) - \nabla F(x)| \leq C r^{-\frac{3}{2}},$$

and $F_r$ is a smooth function with derivative bounds

$$\sup_x |\partial_x^\alpha F_r(x)| \leq C r^{\frac{|\alpha| - 2}{2}}, \quad |\alpha| \geq 2.$$  

In establishing (4.7) we may replace $\cos(t\sqrt{-\Delta})$ by $\exp(it\sqrt{-\Delta})$. We then use a $TT^*$ argument to reduce to proving mapping properties for an operator on $\Gamma_r \times [0, 1]$. Precisely, let $K_r(t-s, x-y) = \rho(r^{-1}D) \exp(i(t-s)\sqrt{-\Delta})$, $D := -i\partial$, and

$$\rho(r^{-1}D) \exp(i(t-s)\sqrt{-\Delta}),$$
where \( \rho \) is a smooth function supported in the region \( \frac{1}{2} < |\xi| < 2 \). It then suffices to show that

\[
\left\| \int_0^1 K_r(t - s, (x' - y', F_r(x') - F_r(y'))) f(s, y') dy' ds \right\|_{L^1([0,1], L^2(\mathbb{R}^{d-1}))} \leq C r^{\frac{1}{2}} \|f\|_{L^{1/3}(\mathbb{R}, L^2(\mathbb{R}^{d-1}))}.
\]

By the Hardy-Littlewood-Sobolev inequality

\[
\|t^{-\sigma} * f\|_{L^q(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})}
\]

where \( \frac{1}{q} + 1 = \frac{\sigma}{d} + \frac{1}{p} \).

Hence translation invariance in \( t \) shows that (4.9) is a consequence of the following fixed-time estimate, for \( |t| < 1 \),

\[
\left\| \int K_r(t, (x' - y', F_r(x') - F_r(y'))) f(y') dy' \right\|_{L^2(\mathbb{R}^{d-1})} \leq C r^{\frac{5}{4}} |t|^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^{d-1})}.
\]

If \( |t| \leq r^{-1} \), then \( K_r \) satisfies

\[
|K_r(t, x - y)| \leq C_N r^d (1 + r |x - y|)^{-N},
\]

and (4.10) follows by the Schur test. To prove (4.10) for \( |t| > r^{-1} \), we decompose the convolution kernel \( K_r(t, \cdot) \) as an almost orthogonal sum of terms, each of which behaves as a normalized convolution operator. Fix \( t \in [r^{-1}, 1] \), and let \( \delta = r^{\alpha/4} t^{-\frac{1}{2}} \). Let \( \eta_j \) count the elements of the lattice of spacing \( \delta \) for which \( |\eta_j| \in [\frac{1}{2}r, 2r] \), and write

\[
\rho(r^{-1} \xi) = \sum_j Q_j(\xi),
\]

where \( Q_j \) is supported in the cube of sidelength \( \delta \) centered on \( \eta_j \), and the following bounds hold on the derivatives of \( Q_j \), uniformly over \( r, t \) and \( j \),

\[
|\partial_\xi^\alpha Q_j(\xi)| \leq C_N \delta^{-|\alpha|}.
\]

We then write \( K_r(t, x) = \sum K_j(x) \), where we suppress the dependence on \( r, t \) and \( j \), and set

\[
K_j(x) = (2\pi)^{-d} \int e^{i(x, \xi) + it|\xi|} Q_j(\xi) d\xi.
\]

The multiplier \( t|\xi| - t|\eta_j|^{-1} \eta_j, \xi \) satisfies the derivative bounds (4.11) on the support of \( Q_j \), hence we may write

\[
e^{i(x, \xi) + it|\xi|} Q_j(\xi) = e^{i(x + t|\eta_j|^{-1} \eta_j, \xi)} \tilde{Q}_j(\xi),
\]

with \( \tilde{Q}_j \) having the same support and derivative conditions as \( Q_j \). Consequently, we may write

\[
K_j(x) = \delta^d e^{i(x, \eta_j) + it|\eta_j|} \chi_0(\delta(x + t|\eta_j|^{-1} \eta_j)),
\]

where \( \chi_0 \) is a Schwartz function, with seminorm bounds independent of \( j \). We let

\[
\tilde{K}_j(x', y') = K_j(x' - y', F_r(x') - F_r(y')).
\]

It follows from the Schur test that

\[
\|\tilde{K}_j\|_{L^2 \to L^2} \leq C \delta.
\]
To handle the sum over $j$ we establish the estimate
\begin{equation}
\|\tilde{K}_j K_i^*\|_{L^2 \to L^2} + \|\tilde{K}_j^* K_i\|_{L^2 \to L^2} \leq C_N \delta^2 (1 + \delta^{-1}|\eta_i - \eta_j|)^{-N},
\end{equation}
from which the bound \eqref{eq:12} follows by the Cotlar-Stein lemma. Since $K_j$ and $K_j^*$ have similar form, we restrict attention to the first term in \eqref{4.12}.

The kernel $(K_j K_j^*)(x', z')$ has absolute value dominated by
\[
\delta^{2d} \int (1 + \delta |x + t|_{\eta_j}^{-1} \eta_j - y|)^{-N} (1 + \delta |z + t|_{\eta_i}^{-1} \eta_i - y|)^{-N} dy',
\]
where we use the notation $y = (y', F_r(y'))$, and similarly for $x$ and $z$.

Suppose that $|\langle \eta_j \rangle_n| \geq \frac{1}{4}|\eta_j|$. Then since $|F_r(x') - F_r(y')| \leq \frac{1}{10}|x' - y'|$,
\[
x' + t|\eta_j|^{-1} \eta_j - y'| + 10|F_r(x') + t|\eta_j|^{-1}(\eta_j)_n - F_r(y')| \geq 5t,
\]
and the Schur test leads to the bound
\[
\|\tilde{K}_j K_i^*\|_{L^2 \to L^2} \leq C_N \delta^2 (1 + \delta t)^{-N},
\]
which is stronger than \eqref{4.12} since $|\eta_i - \eta_j| \leq 6r$. The same estimate holds if $|\langle \eta_i \rangle_n| \geq \frac{1}{4}|\eta_i|$.

We thus assume that $|\langle \eta_j \rangle_n| \leq \frac{1}{4}|\eta_j|$, and similarly for $\eta_i$. Consider then the case where $|\langle \eta_i - \eta_j \rangle_n| \geq |\langle \eta_i - \eta_j \rangle_n'|$. Then we have
\[
|\langle \eta_j|^{-1} \eta_j - \eta_i|^{-1} \eta_i\rangle_n| \geq \frac{1}{2 + 2\sqrt{2}} |\langle \eta_j|^{-1} \eta_j - \eta_i|^{-1} \eta_i\rangle'|
\]
and since $\frac{1}{2^r} \leq |\eta_j|, |\eta_j| \leq 2r$,
\[
|\langle \eta_j|^{-1} \eta_j - \eta_i|^{-1} \eta_i\rangle_n| \geq \frac{1}{4\sqrt{2}} r^{-1}|\eta_i - \eta_j|.
\]
Then with $|\nabla F_r| \leq \frac{1}{10},$
\[
x' - z' + t|\eta_j|^{-1} \eta_j - |\eta_i|^{-1} \eta_i'\rangle' + 10|F_r(x') - F_r(z') + t|\eta_j|^{-1} \eta_j - |\eta_i|^{-1} \eta_i_n|
\geq \frac{5}{4\sqrt{2}} \delta^{-2} |\eta_j - \eta_i|,
\]
hence using just the absolute bounds on the kernels, the operator norm is seen to be bounded by $C_N \delta^2 (1 + \delta^{-1}|\eta_j - \eta_i|)^{-N}$ as desired.

We thus consider the case that $|(\eta_j - \eta_i)_n| \leq |(\eta_j - \eta_i)_n'|$. Up to a factor of modulus 1, the kernel $(K_j K_j^*)(x', z')$ can be written as
\[
\delta^{2d} \int e^{-i(y',\eta_j' - \eta_i') - iF_r(y') |\eta_j - \eta_i_n|} \chi_j(\delta(x + t|\eta_j|^{-1} \eta_j - y)) \chi_i(\delta(z + t|\eta_i|^{-1} \eta_i - y)) dy',
\]
where again $y = (y', F_r(y'))$, and similarly for $x$ and $z$. Since $|\nabla F_r(y')| \leq \frac{1}{10}$, and $|(\eta_j - \eta_i)_n| \leq |\eta_j' - \eta_i'|$, we have
\[
|\eta_j' - \eta_i' + \nabla F_r(y') (\eta_j - \eta_i)_n| \geq \frac{1}{2}|\eta_j - \eta_i|.
\]
Using the estimates (1.8), and that \( r^2 \leq \delta \), an integration by parts argument dominates the kernel \((K_j K_j^*)(x', z')\) by
\[
\delta^{2d} (1 + \delta^{-1}|\eta_j - \eta_k|)^{-N} \int (1 + \delta |x + t|\eta_j - \eta_j - y|)^{-N} (1 + \delta |z + t|\eta_k - \eta_j - y|)^{-N} dy',
\]
which leads to the desired norm bounds, concluding the proof of (4.12). \(\square\)

For the proof of Theorem 3, first consider the case that \(f = g\) and \(\Gamma\) is a graph \(x_n = F(x')\), and \(\text{Im}\lambda \geq 1\). We then have uniform bounds
\[
\sup_{\xi_n} \int |\hat{f}\delta_{\Gamma}(\xi', \xi_n)|^2 d\xi' \leq C \|f\|^2_{L^2(\Gamma)}.
\]
We use the lower bound \(||\xi|^2 - \lambda^2| \geq |\lambda|\|\text{Im}\lambda|\) to dominate
\[
\int_{|\xi_n| \leq 2|\lambda|} \frac{|\hat{f}\delta_{\Gamma}(\xi)|^2}{||\xi|^2 - \lambda^2|} d\xi \leq C \langle\text{Im}\lambda\rangle^{-1} \|f\|^2_{L^2(\Gamma)}.
\]
For \(|\xi_n| \geq 2|\lambda|\) we have \(||\xi|^2 - \lambda^2| \geq |\xi_n|^2\), hence
\[
\int_{|\xi_n| \geq 2|\lambda|} \frac{|\hat{f}\delta_{\Gamma}(\xi)|^2}{||\xi|^2 - \lambda^2|} d\xi \leq C \langle\text{Im}\lambda\rangle^{-1} \|f\|^2_{L^2(\Gamma)}.
\]
The case \(f \neq g\) and \(\Gamma\) a finite union of graphs follows by a partition of unity argument and the Schwarz inequality.

5. Resolvent Bounds in the Lower Half Plane

For \(\lambda \in \mathbb{R}\), the resolvent \(R_0(\lambda)\) is defined as the limit \(R_0(\lambda + i0)\) from \(\text{Im}\lambda > 0\). The estimates of the previous sections then give that, for \(\lambda \in \mathbb{R}\) with \(|\lambda| > 2\), and for some \(a > 0\) and \(b \in \{0, 1\}\), we have
\[
||\gamma R_0(\lambda) \gamma^*||_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C |\lambda|^{-a} (\log |\lambda|)^b.
\]
In this section we extend this to bounds for \(\text{Im}\lambda < 0\).

**Lemma 5.1.** Suppose that for \(\lambda \in \mathbb{R}, |\lambda| > 2\), the following holds
\[
||\gamma R_0(\lambda) \gamma^*||_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C |\lambda|^{-a} (\log |\lambda|)^b.
\]
Then for \(\text{Im}\lambda \leq 0, |\lambda| \geq 2\), and \(\arg \lambda \in [-\pi, 0] \cup [\pi, 2\pi]\) in the case that \(d\) is even,
\[
||\gamma R_0(\lambda) \gamma^*||_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C |\lambda|^{-a} (\log |\lambda|)^b e^{-d_\Gamma |\text{Im}\lambda|}
\]
where \(d_\Gamma\) is the diameter of \(\Gamma\).

**Proof.** First consider the case that \(d\) is odd. Suppose that \(||f||_{L^2(\Gamma)} = ||g||_{L^2(\Gamma)} = 1\), and consider the function
\[
F(\lambda) = e^{-id\lambda} \lambda^a (\log |\lambda|)^b Q_\lambda(f, g), \quad \text{Im}\lambda \leq 0, \quad |\lambda| \geq 2,
\]
where \(\log |\lambda|\) is defined for \(\arg \lambda \in (\frac{\pi}{2}, \frac{5\pi}{2})\). Then \(|F(\lambda)| \leq C\) for \(\lambda \in \mathbb{R} \setminus [-2, 0]\) and for \(|\lambda| = 2\). On the other hand, the resolvent kernel bounds (2.3) and the Schur test show that \(|F(\lambda)|\) has at most polynomial growth in \(\lambda\) for \(\text{Im}\lambda \leq 0\), since the kernel \(|x - x'|^{2-d}\) is integrable over a \(d-1\).
dimensional hypersurface. It follows by the Phragmén-Lindelöf theorem that $|F(\lambda)| \leq C$ in the lower half plane.

In the case that $d$ is even, we note that the bounds of the lemma hold for $R_0(\lambda)$ if $\mathrm{arg}\lambda = 2\pi$ and $|\lambda| \geq 2$. This follows since $R_0(e^{i\pi}\lambda) - R_0(\lambda)$ satisfies the same bounds as $R_0(\lambda)$ for $\mathrm{arg}\lambda = 0$, and by (2.4) we have $R_0(e^{i\pi}\lambda) - R_0(e^{i\pi}\lambda) = R_0(e^{i\pi}\lambda) - R_0(\lambda)$. We may thus apply the Phragmén-Lindelöf theorem on the sheet $\pi \leq \mathrm{arg}\lambda \leq 2\pi$. A similar argument works for $-\pi \leq \mathrm{arg}\lambda \leq 0$. 

6. APPLICATION TO RESONANCE FREE REGIONS

In this section we establish Theorem 1. First, we demonstrate the meromorphic continuation of $R_V(\lambda)$ from $\mathrm{Im}\lambda \geq 0$ to $\lambda \in \mathbb{C}$ (to the logarithmic cover in even dimensions) following arguments similar to those in the case where $V \in L^\infty_{\mathrm{comp}}$. We assume $\Gamma$ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces, and that $\rho \in C^\infty_0(\mathbb{R}^d)$ with $\rho = 1$ on a neighborhood of $\Gamma$. Let

$$K(\lambda) = (V \otimes \delta_\Gamma)R_0(\lambda).$$

The operator $K(\lambda)\rho : H^{-1}(\mathbb{R}^d) \to H^{-1}_{\mathrm{comp}}$ is compact on $H^{-1}(\mathbb{R}^d)$ by Rellich’s embedding theorem. Furthermore, $I + K(\lambda)\rho$ is invertible if $\mathrm{Im}\lambda \gg 0$. To see this, note that $g + K(\lambda)\rho g = 0$ and $g \in H^{-1}(\mathbb{R}^d)$ implies that $g = f\delta_\Gamma$ where $f \in L^2(\Gamma)$. It follows that $f + VG(\lambda)f = 0$, which implies $f = 0$ for $\mathrm{Im}\lambda \gg 0$ by Theorem 3.

Consequently $(I + K(\lambda)\rho)^{-1}$ continues to a meromorphic family of Fredholm operators on $H^{-1}(\mathbb{R}^d)$ for $\lambda \in \mathbb{C}$ (or to the logarithmic cover in even dimensions); see e.g. Proposition 7.4 of [20], Chapter 9. Since $K = \gamma^*V\gamma R_0$, we have that

$$(I + K(\lambda)\rho)^{-1}\gamma^* = \gamma^*(I + VG(\lambda))^{-1},$$

where $(I + VG(\lambda))^{-1}$ acts on $L^2(\Gamma)$. Hence,

$$(I + K(\lambda)\rho)^{-1} = I - (I + K(\lambda)\rho)^{-1}K(\lambda)\rho$$

$$= I - \gamma^*(I + VG(\lambda))^{-1}V\gamma R_0(\lambda)\rho.$$

The meromorphic extension of the resolvent $R_V(\lambda)$ for $-\Delta_V\Gamma$ then equals

$$R_V(\lambda) = R_0(\lambda) (I + K(\lambda)\rho)^{-1} (I - K(\lambda)(1 - \rho))$$

$$= \left( R_0(\lambda) - R_0(\lambda)\gamma^*(I + VG(\lambda))^{-1}V\gamma R_0(\lambda)\rho \right) (I - K(\lambda)(1 - \rho)).$$

In particular, if $g \in H^{-1}_{\mathrm{comp}}$ we can take $\rho g = g$ to obtain

$$(6.1) \quad R_V(\lambda)g = R_0(\lambda)g - R_0(\lambda)\gamma^*(I + VG(\lambda))^{-1}V\gamma R_0(\lambda)g.$$

Consequently, $R_V(\lambda) : H^{-1}_{\mathrm{comp}} \to H^1_{\mathrm{loc}}$, and its image is $\lambda$-outgoing.

The resolvent set $\Lambda$ is defined as the set of poles of $R_V(\lambda)$. Since

$$(I - K(\lambda)(1 - \rho))(I + K(\lambda)(1 - \rho)) = I,$$

the preceding arguments show that $\Lambda$ agrees with the poles of $(I + VG(\lambda))^{-1}$, which by the Fredholm property agrees with the set of $\Lambda$ for which $(I + VG(\lambda))$ has nontrivial kernel. If
\[\|G(\lambda)\|_{L^2 \to L^2} < C^{-1}\] where \(C = \|V\|_{L^2 \to L^2}\), then \(I + VG(\lambda)\) is invertible by Neumann series. By Theorem 2, when \(\text{Im } \lambda < 0\) this is the case provided that (for a different \(C\))

\[|\text{Im } \lambda| \leq d^{-1}_r (a \ln |\lambda| - \ln C - b \ln (\ln |\lambda|)) ,\]

which completes the proof of Theorem 1.

We now observe that if \(f\) solves \((I + VG(\lambda))f = 0\) with \(f \in L^2(\Gamma)\), then

\[(6.2) u = R_0(\lambda)(f\delta_\Gamma)\]

is a \(\lambda\)-outgoing solution to \(-\Delta_{V;\Gamma} u = \lambda^2 u\). Indeed, \(u\) is \(\lambda\)-outgoing by definition, \(u \in H^1_{\text{loc}}\), and \((-\Delta - \lambda^2)u = f\delta_\Gamma\). On the other hand, \((V \otimes \delta_\Gamma)u = (VG(\lambda)f)\delta_\Gamma = -f\delta_\Gamma\).

To see that all such solutions arise this way we use the following extension of the Rellich uniqueness theorem, that there are no global \(\lambda\)-outgoing solutions to \((-\Delta - \lambda^2)u = 0\). To prove this, note that for \(0 < \arg \lambda < \pi\) and \(g\) a compactly supported distribution , \(R_0(\lambda)g\) is exponentially decreasing in \(|x|\), so Green's identities yield, for \(u = R_0(\lambda)g\) and for \(R \gg 0\), that

\[u(x) = \int_{|y|=R} \left(G_0(\lambda, x, y) \partial_{\nu^*}u(y) - \partial_y G_0(\lambda, x, y)u(y)\right) d\sigma(y), \quad |x| > R.\]

By analytic continuation this holds for all \(\lambda\). If \(u\) is an entire solution then the right hand side is real-analytic in \(R\), and we may let \(R \to 0\) to deduce that \(u \equiv 0\).

Suppose that \(u \in H^1_{\text{loc}}\) is a \(\lambda\)-outgoing solution to \(-\Delta_{V;\Gamma} u = \lambda^2 u\). By the uniqueness theorem it follows that

\[(6.3) u = -R_0(\lambda)(V \otimes \delta_\Gamma)u = -\int_{\Gamma} G_0(\lambda, x, y) (Vu)(y) .\]

Hence if \(f = Vu|_\Gamma\), then \(f + VG(\lambda)f = 0\). By (6.3) the correspondence between \(u\) and \(V u|_\Gamma\) is one-to-one. It follows that the corresponding space of solutions \(u\) for any \(\lambda\) is finite dimensional, since it is in one-to-one correspondence with the kernel of a Fredholm operator.

Suppose now that \(\Gamma = \partial \Omega\) for a compact domain \(\Omega \subset \mathbb{R}^d\) with \(C^{1,1}\) boundary. Assume also that \(V : H^4(\partial \Omega) \to H^2(\partial \Omega)\). Then the analysis leading to (2.2) shows that \(u = u_1 \oplus u_2\) satisfies the transmission problem (1.7). Conversely, suppose \(u = u_1 \oplus u_2\) belongs to \(\mathcal{E}_2\) and satisfies (1.7). For \(w \in C^\infty(\mathbb{R}^d)\), Green’s identities yield

\[\int_{\mathbb{R}^d} u(-\Delta - \lambda^2)w = \int_{\partial \Omega} (\partial^\nu u + \partial_y u) w = -\int_{\partial \Omega} (Vu) w .\]

Hence \(u\) is a \(\lambda\)-outgoing \(H^1_{\text{loc}}\) distributional solution to \((-\Delta - \lambda^2)u + (V \otimes \delta_{\partial \Omega})u = 0\), and by the above \(\lambda\) is a resonance.

### 7. Resonance Expansion for the Wave Equation

In this section we prove Theorems 4 and 5. Let \(\Lambda\) denote the set of resonances; since we work in odd dimensions \(\Lambda\) is a discrete subset of \(\mathbb{C}\). The elements of \(\Lambda\) such that \(\text{Im } \lambda > 0\) consist of \(i\mu_j\) where \(-\mu_j^2\) are the non-zero eigenvalues of \(-\Delta_{V;\Gamma}\). That there are only a finite number of
eigenvalues follows by relative compactness of $V \otimes \delta_{\Gamma}$ with respect to $-\Delta$. The resolvent near $i\mu_j$ takes the form

$$R_V(\lambda) = \frac{-\Pi_{\mu_j}}{\lambda^2 + \mu_j^2} + \text{holomorphic} = \frac{i \Pi_{\mu_j}}{2\mu_j(\lambda - i\mu_j)} + \text{holomorphic},$$

where $\Pi_{\mu_j}$ is projection onto the $-\mu_j^2$-eigenspace of $-\Delta_{V,\Gamma}$. In particular we note that

$$\text{Res}(e^{-i\lambda} R_V(\lambda), i\mu_j) = i(2\mu_j)^{-1} e^{i\mu_j} \Pi_{\mu_j}.$$  \hfill (7.1)

In dimension $d = 1$, if $0 \in \Lambda$ it is not an eigenvalue, whereas for $d \geq 5$ the corresponding solutions to (1.6) for $\lambda = 0$ must be square-integrable. For $d = 3$, if $0 \in \Lambda$ there may be square-integrable and/or non square-integrable solutions to (1.6), depending on whether the corresponding $f = Vu|_{\Gamma}$ in (6.2) has vanishing integral.

For $|\lambda| \ll 1$ and $\text{Im} \lambda > 0$, the spectral bound $\|R_V(\lambda)\|_{L^2 \to L^2} \leq C(|\lambda| \text{Im} \lambda)^{-1}$ shows that

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{i\mathcal{P}_0}{\lambda} + \text{holomorphic},$$

where by inspection $\Pi_0$ is projection onto the 0-eigenspace of $-\Delta_{V,\Gamma}$. Since $R_V^*(\lambda) = R_V(\lambda)$ for $\text{Im} \lambda > 0$, it follows that $\mathcal{P}_0$ is a symmetric map of $L^2_{\text{comp}}$ to solutions of (1.6) with $\lambda = 0$. Hence,

$$\text{Res}(e^{-i\lambda} R_V(\lambda), 0) = i\lambda \mathcal{P}_0 + i\mathcal{P}_0.$$  \hfill (7.2)

In contrast to the case of $V \in L^\infty_{\text{comp}}$, there may be resonances $\lambda \in \mathbb{R} \setminus \{0\}$. For an example in one dimension of $V$ and $\Gamma$ with such resonances consider $\Gamma = \{-\pi, 0, \pi\}$, and $V$ given by

$$(Vu|_{\Gamma})(x) = \begin{cases} u(0), & x = \pm \frac{\pi}{2} \\ \frac{\pi}{2} u + u(-\frac{\pi}{2}), & x = 0 \end{cases}$$

Then the function

$$u(x) = \begin{cases} \cos(x), & |x| \leq \frac{\pi}{2} \\ 0, & |x| \geq \frac{\pi}{2} \end{cases}$$

is a resonant state, that is an outgoing solution to (1.6), for $\lambda = \pm 1$.

In fact, all resonances in $\mathbb{R} \setminus \{0\}$ must correspond to compactly supported eigenfunctions of $-\Delta_{V,\Gamma}$. To see this, suppose that $\lambda \in \mathbb{R} \setminus \{0\}$ and let $u \in \mathcal{D}_{\text{loc}}$ solve $-\Delta_{V,\Gamma} u = \lambda^2 u$. For $R > 0$, writing

$$0 = \int_{|x| \leq R} \pi(-\Delta u + (V \otimes \delta_{\Gamma}) u - \lambda^2 u) = \int_{|x| \leq R} (|\nabla u|^2 - \lambda^2 |u|^2) + \int_{|x| = R} \pi \partial_\nu u + \int_\Gamma \pi V u$$

shows that $\text{Im} \int_{|x| = R} \pi \partial_\nu u = 0$. The proof of Proposition 1.1 and Lemma 1.2 of [20, Chapter 9] then show that $u \equiv 0$ on $|x| \geq R_0$, hence by analytic continuation $u$ vanishes on the unbounded component of $\mathbb{R}^d \setminus \Gamma$.

We note that if $\Gamma$ coincides with the boundary of the unbounded component of $\mathbb{R}^d \setminus \Gamma$ then there are no resonances $\lambda \in \mathbb{R} \setminus \{0\}$, since $0 = u|_{\Gamma}$ implies that $(V \otimes \delta_{\Gamma}) u = 0$. Hence $u$ is a compactly supported eigenfunction of $-\Delta$ on $\mathbb{R}^d$, and must vanish identically.
The resonances in $\mathbb{R} \setminus \{0\}$ form a finite set by Theorem 1 where $\lambda \in \mathbb{R} \setminus \{0\}$ is a resonance if $\lambda^2$ is an eigenvalue. The real resonances are thus symmetric about 0. We indicate them by $\pm \nu_j$, with $\nu_j > 0$. By inspection, for $\text{Im}\, \lambda > 0$ near $\pm \nu_j$ we have

$$R_V(\lambda) = \frac{-\Pi_{\nu_j}}{\lambda^2 + \nu_j^2} + \text{holomorphic} = \frac{\mp \Pi_{\nu_j}}{2\nu_j(\lambda \mp \nu_j)} + \text{holomorphic},$$

where $\Pi_{\nu_j}$ is projection onto the $\nu_j^2$ eigenspace, hence

$$(7.3) \quad \text{Res}(e^{-i\lambda} R_V(\lambda), \pm \nu_j) = \mp(2\nu_j)^{-1} e^{\mp i\nu_j} \Pi_{\nu_j}.$$

The remaining resonances form a discrete set $\{\lambda_j\} \subset \{\text{Im}\, \lambda < 0\}$, with respective multiplicity $m_R(\lambda_j)$. Since $\lambda_j \neq 0$, the Laurent expansion of $R_V(\lambda)$ about $\lambda_j$ can be written in the following form

$$R_V(\lambda) = i \sum_{k=1}^{m_R(\lambda_j)} \frac{(-\Delta_V \Omega - \lambda_j^2)^{k-1} \mathcal{P}_{\lambda_j}}{(\lambda^2 - \lambda_j^2)^k} + \text{holomorphic}.$$

Here $\mathcal{P}_{\lambda_j} : L^2_{\text{comp}} \to \mathcal{D}_{\text{loc}}$ is given by

$$\mathcal{P}_{\lambda_j} = -\frac{1}{2\pi} \int_{\lambda_j} R_V(\lambda) 2\lambda d\lambda,$$

and $(-\Delta_V \Omega - \lambda_j^2)^{m_R(\lambda_j)} \mathcal{P}_{\lambda_j} = 0$. We can thus write

$$(7.4) \quad \text{Res}(e^{-i\lambda} R_V(\lambda), \lambda_j) = i \sum_{k=0}^{m_R(\lambda_j)-1} t^k e^{-it\lambda_j} \mathcal{P}_{\lambda_j,k},$$

where $\mathcal{P}_{\lambda_j,k} : L^2_{\text{comp}} \to \mathcal{D}_{\text{loc}}$. When $k = m_R(\lambda_j) - 1$, $\mathcal{P}_{\lambda_j,k} g$ is $\lambda_j$-outgoing, as seen by writing the Laurent expansion of $R_V(\lambda)$ in terms of that for $(I + K(\lambda_1)\rho)^{-1}$. In particular, if $m_R(\lambda_j) = 1$, then $\text{Res}(e^{-i\lambda} R_V(\lambda), \lambda_j) = i(2\lambda_j)^{-1} e^{-it\lambda_j} \mathcal{P}_{\lambda_j}$, where $\mathcal{P}_{\lambda_j}$ maps $L^2_{\text{comp}}$ to $\lambda_j$-outgoing solutions of $(-\Delta_V \Omega - \lambda_j^2) u = 0$.

### 7.1. Resonant Estimates

We first establish bounds on the cutoff of $R_V(\lambda)$, for $\lambda$ in the resonance free region established in Section 6.

**Lemma 7.1.** Suppose that $\Gamma$ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces. Then for all $\epsilon > 0$ there exists $R < \infty$, so that if $\chi \in C^\infty_c(\mathbb{R}^d)$ equals 1 on a neighborhood of $\Gamma$, $\text{Re}\, \lambda > R$, and $\text{Im}\, \lambda \geq -\frac{1}{2} d_{\Gamma}^{-1} - \epsilon \log(\text{Re}\, \lambda)$, then

$$\|\chi R_V(\lambda) \chi g\|_{L^2} \leq C \langle \lambda \rangle^{-1} e^{2d_{\chi}(\text{Im}\, \lambda)} \|g\|_{L^2},$$

$$\|\chi R_V(\lambda) \chi g\|_{H^1} \leq C e^{2d_{\chi}(\text{Im}\, \lambda)} \|g\|_{L^2},$$

$$\|\chi R_V(\lambda) \chi g\|_{\mathcal{D}} \leq C \langle \lambda \rangle e^{2d_{\chi}(\text{Im}\, \lambda)} \|g\|_{L^2},$$

where $d_{\chi} = \text{diam}(\text{supp}\, \chi)$, and $R_V(\lambda)$ is the meromorphic continuation of $(-\Delta_V \Gamma - \lambda^2)^{-1}$ from $\text{Im}\, \lambda > 0$. If $\text{Im}\, \lambda \geq 1$, $|\text{Re}\, \lambda| > R$, then the estimates hold with $\chi \equiv 1$, setting $d_{\chi}(\text{Im}\, \lambda) = 0$.

**Remark:** The region in which this estimate is valid can be improved by replacing $\frac{1}{2}$ by $\frac{2}{3}$ if the components of $\Gamma$ are subsets of strictly convex $C^{2,1}$ hypersurfaces.
Consequently, and note by (7.8) and (7.9) that the bounds of Lemma 7.1 except for the ones on $\|g\|_{H^s}$, $s \leq t \leq s + 2$.

In addition, when $\text{Im} \lambda \geq 1$ these estimates hold globally, that is with $\chi \equiv 1$.

This in turn leads to the following restriction estimates

$$\|\gamma R_0(\lambda)\chi g\|_{L^2(\Gamma)} \leq C \langle \lambda \rangle^{-s-\frac{1}{2}d_x(\text{Im} \lambda)^{-}} \|g\|_{H^s}, \quad -\frac{3}{2} < s < \frac{1}{2},$$

and

$$\|\nabla R_0(\lambda)\chi g\|_{L^2(\Gamma)} \leq C \langle \lambda \rangle^{-s+\frac{1}{2}d_x(\text{Im} \lambda)^{-}} \|g\|_{H^s}, \quad -\frac{3}{2} < s < \frac{3}{2}.$$  

(7.5)

To prove (7.5) we use the following interpolation bound separately on each component of $\Gamma$,

$$\|g\|_{L^2(\Gamma)} \leq C_{t,t'} \|g\|_{H^t} \|g\|_{H^{t'}}^\theta \|g\|_{H^s}^{1-\theta}, \quad 0 \leq t < \frac{1}{2} < t', \quad \theta(t-\frac{1}{2})+(1-\theta)(t'-\frac{1}{2}) = 0.$$  

(7.5)

By duality we have the following extension estimate,

$$\|\chi R_0(\lambda)\gamma f\|_{H^\frac{1}{2}(\Gamma)} \leq C \langle \lambda \rangle^{s-\frac{1}{2}d_x(\text{Im} \lambda)^{-}} \|f\|_{L^2(\Gamma)}, \quad -\frac{1}{2} < s < \frac{3}{2}.$$  

(7.6)

By restriction, note that (7.6) implies

$$\|G(\lambda)f\|_{H^1(\Omega)} \leq C \langle \lambda \rangle^\frac{1}{2} e^{(d_2+d_1)\text{Im} \lambda} \|f\|_{L^2(\Gamma)}, \quad \epsilon > 0,$$

where the norm on the left is the sum of the $H^\frac{1}{2}$ norms on the distinct $C^{1,1}$ components of $\Gamma$.

Now fix $g \in L^2(\mathbb{R}^d)$, set $u = R_V(\lambda)\gamma g$. Then by (7.5) we have $u = R_0(\lambda)\chi g - w$, where $w = R_0(\lambda)\gamma(I + V G(\lambda))^{-1} V \gamma R_0(\lambda)\chi g$.

By Theorem 2 for $|\text{Re} \lambda|$ large enough and $\text{Im} \lambda \geq -(\frac{1}{2}d_1-\epsilon)\log(|\text{Re} \lambda|)$, the operator $I + V G(\lambda)$ is invertible on $L^2(\Gamma)$, and we have

$$\|(I + V G(\lambda))^{-1}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C, \quad \|V G(\lambda)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} < 1.$$  

Thus, for $\frac{3}{2} < s < \frac{1}{2}$,

$$\|(I + V G(\lambda))^{-1} V \gamma R_0(\lambda)\chi g\|_{L^2(\Gamma)} \leq C \langle \lambda \rangle^{-s-\frac{1}{2}d_x(\text{Im} \lambda)^{-}} \|g\|_{H^s}.$$

Then (7.6) gives the following, for $\frac{3}{2} < s < \frac{1}{2}$, and with global bounds if $\text{Im} \lambda \geq 1$,

$$\|\chi w\|_{L^2} \leq C \langle \lambda \rangle^{-s-1} e^{2d_2(\text{Im} \lambda)^{-}} \|g\|_{H^s},$$

(7.8)

$$\|\chi w\|_{H^1} \leq C \langle \lambda \rangle^{-s-2} e^{2d_2(\text{Im} \lambda)^{-}} \|g\|_{H^s}.$$  

(7.9)

By the $L^2 \rightarrow H^1$ bounds for $\chi R_0(\lambda)\chi$ the same holds for $s = 0$ with $w$ replaced by $u$, which yields the bounds of Lemma 7.1 except for the ones on $\|\chi u\|_D$.

To obtain bounds on $\|\chi u\|_D$, we write

$$\Delta(\chi u) = -\chi g + 2(\nabla \chi) \cdot \nabla u + (\Delta \chi) u - \lambda^2 \chi u + (V \otimes \delta_\Gamma)u,$$

and note by (7.8) and (7.9) that

$$\|\nabla \chi \cdot \nabla u\|_{L^2} + \|(\Delta \chi) u\|_{L^2} + \|\lambda^2 \chi u\|_{L^2} \leq C \langle \lambda \rangle e^{2d_2(\text{Im} \lambda)^{-}} \|g\|_{L^2}.$$  

Consequently,

$$\|\Delta_{V,G}(\chi u)\|_{L^2} \leq C \langle \lambda \rangle e^{2d_2(\text{Im} \lambda)^{-}} \|g\|_{L^2}.$$  

Proof. We recall the Sobolev estimates for the cutoff of the free resolvent, see e.g. [23, Chapter 3]
yielding the desired bound on $\|\chi u\|_D$. \qed

7.2. Proof of Theorem 4. We prove here the case $N = 1$ of Theorem 4; the case $N \geq 2$ will be handled following the proof of Theorem 5. We follow the treatment in [18], and suppose that $g \in H^s$ for some $0 < s < \frac{d}{2}$ and proceed by density in $L^2$. As above write

$$R_V(\lambda)\chi g = w + R_0(\lambda)\chi g.$$ 

Choose $\alpha \geq 1$ so that $\mu_j < \alpha$ for all $j$, where $-\mu_j$ are the negative eigenvalues of $-\Delta_{V,\Gamma}$. By the spectral theorem we can write

$$U(t)\chi g = \frac{1}{2\pi} \int_{-\infty+\frac{i\alpha}{2}}^{\infty+\frac{i\alpha}{2}} e^{-it\lambda} R_V(\lambda)\chi g \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty+\frac{i\alpha}{2}}^{\infty+\frac{i\alpha}{2}} e^{-it\lambda} (w + R_0(\lambda)\chi g) \, d\lambda. \tag{7.10}$$

The integral is norm convergent in $L^2(\mathbb{R}^d)$, by (7.8) and the norm convergence of the free resolvent integral. After localizing by $\chi$ on the left, for $t$ sufficiently large we seek to deform the contour $\mathbb{R} + i\alpha$ to

$$\Sigma_A = \{ \lambda \in \mathbb{C} : \text{Im} \lambda = -A - c\log(2 + |\text{Re} \lambda|) \}$$

where we choose $c < \frac{1}{9}d^{-1}$, and assume $A$ is such that there are no resonances on $\Sigma_A$. We will show that the integral over $\Sigma_A$ is norm convergent for $g \in H^s$ if $s > 0$, so to justify the contour change we need to show that for $t$ sufficiently large the integrals over

$$\gamma_{\pm R}(v) = \{ \pm R + iv : -(A + c\log(2 + R)) \leq v \leq \alpha \}, \quad \gamma_{R,\infty} = \{ x + i\alpha : |x| \geq R \}$$

tend to 0 as $R \to \infty$. Note that for $R$ large enough, Theorem 4 shows that there are no resonances between $\mathbb{R} + i\alpha$ and $\Sigma_A$ with $|\text{Re} \lambda| \geq R$, and hence none on $\gamma_{\pm R}$.

We introduce the following notation,

$$E_\gamma(t) f = \frac{1}{2\pi} \int_{\gamma} e^{-it\lambda} R_V(\lambda)f \, d\lambda.$$ 

Then for $t > 2d\chi$, and $R$ large enough,

$$\|\chi E_{\gamma_{\pm R}}(t)\chi g\|_{L^2} \leq C e^{at} (R^{-1})(A + A + c\log(2 + R)) \|g\|_{L^2} \to 0 \quad \text{as} \quad R \to \infty.$$ 

The norm convergence of (7.10) shows that $\|\chi E_{\gamma_{R,\infty}}(t)\chi g\|_{L^2} \to 0$ as $R \to \infty$. We then assume $c(t-2d\chi) \geq 3$ and calculate

$$\|\chi E_{\Sigma_A}(t)\chi g\|_D \leq C_A \chi e^{-A(t-2d\chi)} \int_{-\infty}^\infty e^{-3\log(2+|R|)} (A + |R|) \, dR \leq C_A \chi e^{-At} \|g\|_{L^2}.$$ 

In particular the integral is norm convergent, and the contour deformation is allowed.

Thus, if we let $\Omega_A$ denote the collection of poles of $R_V(\lambda)$ in the set $\text{Im} \lambda > -A - c\log(2 + |\text{Re} \lambda|)$, then

$$\chi U(t)\chi g = \chi E_{\Sigma_A}(t)\chi g - i\chi \sum_{z \in \Omega_A} \text{Res}(e^{-it\lambda} R_V(\lambda), z) \chi g,$$

and by density this holds for $g \in L^2(\mathbb{R}^d)$. Observe that if $g \in L^2_{\text{comp}}$ then we can take $\chi = 1$ on the support of $g$, and drop the cutoff $\chi$ to write a global equality in $L^2_{\text{loc}}$. To have estimates on
the remainder in $\mathcal{D}$, though, requires cutting off by $\chi$ and taking $t > 2d_\chi + C$, consistent with the propagation of singularities. The expressions (7.1), (7.2), (7.3), and (7.4) now complete the proof of Theorem 4 for $N = 1$, where we observe that the terms from poles in $\Omega_A$ with $\text{Im} \lambda \leq -A$ can be absorbed into $E_A(t)$.

7.3. Higher order estimates for smooth domains. We start with the following lemma, where we now assume that $\Gamma = \partial \Omega$ is $C^\infty$, and that $V : H^s(\partial \Omega) \to H^s(\partial \Omega)$ for all $s \geq 0$. Recall that we set $\mathcal{E}_0 = L^2(\mathbb{R}^d)$, and for $N \geq 1$,

$$\mathcal{E}_N = H^1(\mathbb{R}^d) \cap (H^N(\Omega) \oplus H^N(\mathbb{R}^d \setminus \overline{\Omega})).$$

In this setting $\mathcal{D}$ equals the subspace of $\mathcal{E}_2$ satisfying $\partial_\nu u + \partial_\nu u + Vu|_{\partial \Omega} = 0$.

**Lemma 7.2.** Suppose that $\partial \Omega$ is of regularity $C^\infty$, and $N \geq 0$. Then for all $\epsilon > 0$ there exists $R < \infty$, so that if $|\text{Re} \lambda| > R$, $|\text{Im} \lambda| \leq (\frac{1}{2}d_\Omega - \epsilon) \log(|\text{Re} \lambda|)$, and $\chi \in C^\infty(\mathbb{R}^d)$ equals 1 on a neighborhood of $\overline{\Omega}$, then

$$\|\chi(R_V(\lambda) - R_V(-\lambda)) \chi g\|_{\mathcal{E}_N} \leq C_N |\lambda|^{N-1} e^{2d_\chi(\text{Im} \lambda)} \|g\|_{L^2}.$$  

**Proof.** We proceed by induction on $N$. By Lemma 7.1, the result holds for $N = 0, 1, 2$. We assume then that the result is true for integers less than or equal to $N$.

Letting $u = (R_V(\lambda) - R_V(-\lambda)) \chi f$, we write

$$\Delta(\chi u) = 2(\nabla \chi) \cdot \nabla u + (\Delta \chi) u - \lambda^2 \chi u + (V \otimes \delta_{\partial \Omega}) u.$$  

By the induction hypothesis,

$$\|(\Delta \chi) u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} + \|\chi u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} \leq C_N |\lambda|^{N-2} e^{2d_\chi(\text{Im} \lambda)} \|g\|_{L^2},$$

$$\|(\nabla \chi) \cdot \nabla u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} + \|Vu\|_{H^{N-1}(\partial \Omega)} \leq C_N |\lambda|^{N-1} e^{2d_\chi(\text{Im} \lambda)} \|g\|_{L^2}.$$  

Lemma 7.1 then gives the desired result for $\mathcal{E}_{N+1}$. □

**Proof of Theorem 5.** We use the notation from the proof of Theorem 4 above. We first note that

$$\frac{1}{2\pi} \int_{\Sigma_A} e^{-it\lambda} R_V(-\lambda) d\lambda = -\sum_{\mu_j > A + \log 2} (2\mu_j)^{-1} e^{-it\mu_j} \Pi_{\mu_j},$$

where the completion of the contour to the lower half plane is justified by Lemma 7.1 and the rapid decrease of $e^{-it\lambda}$ for $t > 0$. We thus can write

$$\chi E_{\Sigma_A}(t) \chi g = \frac{1}{2\pi} \int_{\Sigma_A} e^{-it\lambda} \chi(R_V(\lambda) - R_V(-\lambda)) \chi g d\lambda - \sum_{\mu_j > A + \log 2} (2\mu_j)^{-1} e^{-it\mu_j} \chi \Pi_{\mu_j} \chi g.$$  

Assume $c(t - 2d_\chi) \geq N + 1$, the $\mathcal{E}_N$ norm of the integral term is dominated by

$$C_A e^{-A(t-2d_\chi)} \int_{-\infty}^\infty e^{-(N+1) \log(2+|R|)} (A + |R|)^{N-1} dR \leq C_{A,\chi,N} e^{-At} \|g\|_{L^2}.$$  

It remains to show that if $\mu_j > A$, and if $\text{Im} \lambda < -A$, then

$$e^{-it\mu_j} \|\chi \Pi_{\mu_j} \chi g\|_{\mathcal{E}_N} + \|\text{Res}(e^{-it\lambda} R_V(\lambda), \lambda_j) \chi g\|_{\mathcal{E}_N} \leq C_{A,\chi,N} e^{-tA} \|g\|_{L^2},$$

since the difference of $\chi E_A(t) \chi$ and $\chi E_{\Sigma_A}(t) \chi$ is a sum of such terms.
A similar argument to the proof of Lemma 7.2 gives the bound
\[ \| \Pi_{\mu_j} f \|_{E_N} \leq C_N \| \mu_j \|^{N} \| f \|_{L^2}, \]
which handles the eigenvalues. To handle the resonances in the lower half plane, consider first the case that \(-\lambda_j\) is not a pole. We can then write
\[ \text{Res}(e^{-it\lambda} R_V(\lambda), \lambda_j) = \frac{1}{2\pi i} \oint_{\gamma_j} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) \, d\lambda, \]
and the estimate follows from Lemma 7.2 by choosing a small contour about \(\lambda_j\) which is contained in \(\text{Im}\lambda < -A\). In the case that \(-\lambda_j\) is a pole, hence an eigenvalue, then the term \(R_V(-\lambda)\) contributes an eigenvalue projection, which is handled as above. \(\square\)

We now complete the proof of Theorem 4 by considering the case \(N \geq 2\). Eigenfunctions clearly belong to \(D_N\), and by an induction argument we have \(\| \chi \Pi_{\mu_j} \chi g \|_{D_N} \leq C_N \| \mu_j \|^{2N} \| g \|_{L^2}\). The proof then follows from that of Theorem 5 using the following

**Lemma 7.3.** Suppose that \(\Gamma\) is a finite union of \(C^{1,1}\) hypersurfaces, and \(N \geq 1\). Then for all \(\epsilon > 0\) there exists \(R < \infty\) so that if \(|\text{Re}\lambda| > R, \text{Im}\lambda| < (\frac{1}{2} d_{\Gamma}^{-1} - \epsilon) \log(|\text{Re}\lambda|)\), and \(\chi \in C^\infty_c(\mathbb{R}^d)\) equals 1 on a neighborhood of \(\Gamma\), then
\[ \| \chi (R_V(\lambda) - R_V(-\lambda)) \chi g \|_{D_N} \leq C \langle \lambda \rangle^{2N-1} e^{2d_{\chi}(\text{Im}\lambda)-} \| g \|_{L^2}. \]

**Proof.** The result was proven above for \(N = 1\). We then proceed by induction, writing
\[ \Delta_{V,\Gamma} \chi (R_V(\lambda) - R_V(-\lambda)) \chi g = \left( [\Delta, \chi] - \lambda^2 \chi \right) (R_V(\lambda) - R_V(-\lambda)) \chi g = (2\nabla \chi \cdot \nabla + (\Delta \chi) - \lambda^2 \chi) (R_V(\lambda) - R_V(-\lambda)) \chi g. \]
By induction, and since \(\text{supp}(\Delta \chi) \subset \text{supp}(\chi)\),
\[ \| (\Delta \chi) - \lambda^2 \chi \|_{D_{\infty}} \leq C \langle \lambda \rangle^{2N-1} e^{2d_{\chi}(\text{Im}\lambda)-} \| g \|_{L^2}. \]
On the complement of \(\Gamma\), the function \(u = (R_V(\lambda) - R_V(-\lambda)) \chi g\) satisfies \(-\Delta u = \lambda^2 u\), and by Lemma 7.1 if \(\chi_1 \in C^\infty_c\) with \(\text{supp}(\chi_1) \subset \text{supp}(\chi)\),
\[ \langle \lambda \rangle \| \chi_1 u \|_{L^2} + \| \chi_1 u \|_{H^1} \leq C e^{2d_{\chi}(\text{Im}\lambda)-} \| g \|_{L^2}. \]
Since \(\nabla \chi\) vanishes on a neighborhood of \(\Gamma\), an induction argument and elliptic regularity yields
\[ \| \nabla \chi \cdot \nabla (R_V(\lambda) - R_V(-\lambda)) \chi g \|_{H^{2N-1}} \leq C \langle \lambda \rangle^{2N-1} e^{2d_{\chi}(\text{Im}\lambda)-} \| g \|_{L^2}, \quad N \geq 1. \]
Since \(H^{2N-1}_{\text{comp}}(\mathbb{R}^d \setminus \Gamma) \subset D_{\infty-1}\) with continuous inclusion, this term also satisfies the bound of (7.11), and the result follows. \(\square\)

8. Appendix: The Transmission Property for \(C^{1,1}\) Domains

We provide here a proof of the transmission estimate that we need to establish \(H^2\) regularity of solutions away from \(\partial \Omega\). In the case of smooth boundaries, the following is well known; see [5], and in particular Theorems 9 and 10 of [11].
Lemma 8.1. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $\partial \Omega$ is locally the graph of a $C^\infty$ function. Let $G_0(x,y)$ be Green’s kernel for $\Delta^{-1}$. Then for $N \geq -1$ the single layer potential map

$$S\ell f(x) = \int_{\partial \Omega} G_0(x,y) f(y) \, d\sigma(y)$$

induces a continuous map from $H^{N+\frac{1}{2}}(\partial \Omega)$ to $H^{N+2}(\Omega) \oplus H^N_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega})$.

Additionally, for $N \geq 0$ the map

$$G_0 g(x) = \int G_0(x,y) g(y) \, dy$$

induces a continuous map from $H^N(\Omega) \oplus H^N_{\text{comp}}(\mathbb{R}^d \setminus \overline{\Omega})$ to $H^{N+2}(\Omega) \oplus H^N_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega})$.

We need the same result for $N = 0$ and $\partial \Omega$ of $C^{1,1}$ regularity, in which case just the single layer potential result is nontrivial.

Lemma 8.2. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $\partial \Omega$ is locally the graph of a $C^{1,1}$ function. Let $G_0(x,y)$ be Green’s kernel for $\Delta^{-1}$. Then the single layer potential map

$$S\ell f(x) = \int_{\partial \Omega} G_0(x,y) f(y) \, d\sigma(y)$$

induces a continuous map from $H^{\frac{1}{2}}(\partial \Omega)$ to $H^2(\Omega) \oplus H^2_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega})$.

Proof. Since the kernel is smooth away from the diagonal we may work locally, and assume that $\partial \Omega$ is given as a graph $x_n = F(x')$, with $F \in C^{1,1}(\mathbb{R}^{d-1})$. Since surface measure $d\sigma(y) = m(y') \, dy'$ where $m$ is Lipschitz, we can absorb the $m$ into $f$. Assuming then that $f \in C^1_c(\mathbb{R}^{d-1})$, consider the maps

$$T' f(x) = (\nabla_{x'} S\ell f)(x', F(x') + x_d) = c_d \int \frac{(x' - y') f(y') \, dy'}{|x' - y'|^2 + |x_d + F(x') - F(y')|^2}$$

$$T_d f(x) = (\partial_{x_d} S\ell f)(x', F(x') + x_d) = c_d \int \frac{(x_d + F(x') - F(y')) f(y') \, dy'}{|x' - y'|^2 + |x_d + F(x') - F(y')|^2}$$

We seek $H^{\frac{1}{2}} \to H^1_{\text{loc}}(x_d \neq 0)$ bounds for both terms. We have $\partial_{x_d} T' = \nabla_{x'} T_d - (\nabla_{x'} F) \partial_{x_d} T_d$, and since $\Delta S\ell f = 0$, for $x_d \neq 0$ we can write

$$(1 + |\nabla_{x'} F|^2) \partial_{x_d} T_d f = \nabla_{x'} T' f - (\nabla_{x'} F) \nabla_{x'} T_d f.$$ 

Thus it suffices to prove $H^{\frac{1}{2}} \to L^2$ bounds for $\chi \nabla_{x'} T'$ and $\chi \nabla_{x'} T_d$.

By the dual of the trace estimate we have

$$\|\chi S\ell f\|_{H^1} \leq C \|f\|_{H^{-\frac{1}{2}}(\partial \Omega)},$$

and hence we can bound

$$\|\chi T'(\nabla_{y'} f)\|_{L^2} + \|\chi T_d(\nabla_{y'} f)\|_{L^2} \leq C \|f\|_{H^{\frac{1}{2}}(\partial \Omega)}.$$ 

The desired bound will thus follow from showing that

$$\|\chi [\nabla_{x'}, T'] f\|_{L^2} + \|\chi [\nabla_{x'}, T_d] f\|_{L^2} \leq C \|f\|_{L^2(\partial \Omega)}.$$
Both maps can be written in the form $\int K(x',x_d,y') f(y') \, dy'$, where

$$\sup_{x'} \int_{|y'| \leq L} |K(x',x_d,y')| \, dy' + \sup_{y'} \int_{|x'| \leq L} |K(x',x_d,y')| \, dx' \leq C_L \langle \ln |x_d| \rangle,$$

from which the result follows by the Schur test. □

REFERENCES

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover Publications, New York, 1972.
[2] A. A. Aligia and A. M. Lobos, Mirages and many-body effects in quantum corals. J. Phys.: Condens. Matter 17 2005.
[3] M. Blair, $L^q$ bounds on restrictions of spectral clusters to submanifolds for low regularity metrics. To appear, Analysis and PDE.
[4] M. Barr, M. Zalatel, and E. Heller, Quantum corral resonance widths: lossy scattering as acoustics. Nano Letters (2010), no. 10, p. 3253–3260.
[5] L. Boutet de Monvel, Comportement d’un opérateur pseudo-différentiel sur une variété à bord. I. La propriété de transmission. (French) J. Analyse Math. 17 (1966), 241–253.
[6] N. Burq, P. Gérard, and N. Tzvetkov, Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Math. J. 138 (2007), no. 3, 445–486.
[7] P. G. Burke, *Potential scattering in atomic physics*. Plenum Press, New York, 1977.
[8] A. Calderón, Lebesgue spaces of differentiable functions. Proc. Symp. in Pure Math. 4 (1961), 33–49.
[9] F. Cardoso, G. Popov, and G. Vodev, Distribution of resonances and local energy decay in the transmission problem. II. Math. Res. Lett. 6 (1999), no. 3–4, 377–396.
[10] M. Crommie, C. Lutz, D. Eigler, and E. Heller, Quantum corals. Physica D: Nonlinear Phenomena 83 (1995), no. 1, 98–108.
[11] C. Epstein, Pseudodifferential methods for boundary value problems. *Pseudo-differential operators: partial differential equations and time-frequency analysis*, 171–200, Fields Inst. Commun., 52, Amer. Math. Soc., Providence, RI, 2007.
[12] J. Galkowski, Distribution of resonances in lossy scattering. In preparation.
[13] A. Greenleaf and A. Seeger, Fourier integral operators with fold singularities. J. Reine Angew. Math. 455 (1994), 35–56.
[14] P. D. Lax and R. S. Phillips, *Scattering theory. Second edition*. Pure and Applied Mathematics, 26. Academic Press, Inc., Boston, MA, 1989.
[15] G. Popov and G. Vodev, Distribution of the resonances and local energy decay in the transmission problem. Asymptot. Anal. 19 (1999), no. 3–4, 253–265.
[16] M. Reed and B. Simon, *Methods of modern mathematical physics I : functional analysis*. Academic Press, New York, 1980.
[17] E. Stein, *Singular integrals and differentiability properties of functions*. Princeton University Press, 1971.
[18] S. Tang and M. Zworski, Resonance expansions of scattered waves. Comm. Pure Appl. Math. 53 (2000), no. 10, 1305–1334.
[19] D. Tataru, On the regularity of boundary traces for the wave equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26 (1998), no. 1, 185–206.
[20] M. Taylor, *Partial differential equations II. Qualitative studies of linear equations. Second edition..*. Applied Mathematical Sciences, 116. Springer, New York, 2011.
[21] B. Vainberg, *Asymptotic methods in equations of mathematical physics*, Gordon & Breach, 1989.
[22] M. Zworski, Resonances in physics and geometry. Notices Amer. Math. Soc. 46 (1999), no. 3, 319–328.
[23] M. Zworski, Lectures on scattering resonances. [http://math.berkeley.edu/~zworski/res.pdf](http://math.berkeley.edu/~zworski/res.pdf)
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