Inverses of triangular matrices and bipartite graphs

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Abstract

To a given nonsingular triangular matrix $A$ with entries from a ring, we associate a weighted bipartite graph $G(A)$ and give a combinatorial description of the inverse of $A$ by employing paths in $G(A)$. Under a certain condition, nonsingular triangular matrices $A$ such that $A$ and $A^{-1}$ have the same zero-nonzero pattern are characterized. A combinatorial construction is given to construct outer inverses of the adjacency matrix of a weighted tree.

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1 Introduction

Let $A$ be a lower triangular matrix with entries from a ring, which is not necessarily commutative. In the first section of this paper we obtain a combinatorial formula for $A^{-1}$, when it exists. The formula is in terms of certain
paths in the bipartite graph associated with $A$. We note some consequences of this formula which include expressions for the inverse of a block triangular matrix and a formula for the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching.

In Section 3 we consider lower triangular, invertible, nonnegative matrices $A$ and characterize those such that $A$ and $A^{-1}$ have the same zero-nonzero pattern. This relates to a question posed by Godsil [5] for bipartite graphs. In the final section we provide a combinatorial construction of outer inverses of the adjacency matrix of a weighted tree.

## 2 Inverses of triangular matrices

Let $G$ be a bipartite graph and let $M$ be a matching in $G$. We assume that each edge $e$ of $G$ has a nonzero weight $w(e)$ from a ring (not necessarily commutative). A path in $G$ is said to be alternating if the edges are alternately in $M$ and $M^c$, with the first and the last edges being in $M$. A path with only one edge, the edge being in $M$, is alternating. Let $P$ be the alternating path consisting of the edges $e_1, e_2, \ldots, e_k$ in that order. The weight $w(P)$ of $P$ is defined to be $w(e_1)^{-1}w(e_2)^{-1}w(e_3)^{-1}\cdots w(e_{k-1})w(e_k)^{-1}$, assuming that the inverses exist. Thus, if the weights commute, then $w(P)$ is just the product of the weights of the edges in $P \cap M^c$ divided by the product of the weights of the edges in $P \cap M$. The length $\ell(P)$ of $P$ is the number of edges on that. For an alternating path $P$, we define

$$\epsilon(P) = (-1)^{(\ell(P)-1)/2}.$$ 

Let $A$ be an $n \times n$ matrix with entries from a ring. We associate a bipartite graph $G(A)$ with $A$ as usual: the vertex set is $\{R_1, \ldots, R_n\} \cup \{C_1, \ldots, C_n\}$ and there is an edge $e$ between $R_i$ to $C_j$ if and only if $a_{ij} \neq 0$, in which case we assign $e$ the weight $w(e) = a_{ij}$. We write vectors as row vectors. The transpose of $\mathbf{x}$ is denoted $\mathbf{x}^\top$.

**Theorem 1.** Let $A$ be a lower triangular $n \times n$ matrix with invertible diagonal elements and $M$ be the unique perfect matching in $G(A)$ consisting of the edges from $R_i$ to $C_i$, $i = 1, \ldots, n$. Then the entries of $B = A^{-1}$, for $1 \leq j \leq i \leq n$, are given by

$$b_{ij} = \sum_{P \in P_{ij}} \epsilon(P)w(P), \quad (1)$$
where \( \mathcal{P}_{ij} \) is the set of alternating paths from \( C_i \) to \( R_j \) in \( G(A) \).

**Proof.** We prove the result by induction on \( n \), the cases \( n = 1, 2 \) being easy. Assume the result for matrices of order less than \( n \). Partition \( A \) and \( B \) as

\[
A = \begin{pmatrix}
A_{11} & 0^\top \\
x & a_{nn}
\end{pmatrix}, \quad B = \begin{pmatrix}
B_{11} & 0^\top \\
y & b_{nn}
\end{pmatrix}.
\]

Note that \( b_{nn} = a_{nn}^{-1} \) and \( B_{11} = A_{11}^{-1} \).

By the induction assumption, (1) holds for \( 1 \leq j \leq i \leq n - 1 \). Thus we need to verify (1) for the pairs \((n, 1), \ldots, (n, n - 1)\).

From \( BA = I \) we see that \( yA_{11} + b_{nn}x = 0 \) and hence \( y = -a_{nn}^{-1}xA_{11}^{-1} \).

Therefore

\[
y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_ib_{ij}, \quad j = 1, \ldots, n - 1. \tag{2}
\]

Consider any alternating path from \( C_n \) to \( R_j \) in \( G(A) \). Any such path must be composed of the edge from \( C_n \) to \( R_n \), followed by an edge from \( R_n \) to \( C_i \) for some \( i \in \{1, \ldots, n - 1\} \), and then an alternating path from \( C_i \) to \( R_j \).

If \( P \) is an alternating path from \( C_i \) to \( R_j \), then denote by \( P' \) the alternating path from \( C_n \) to \( R_j \) obtained by concatenating the edge from \( C_n \) to \( R_n \), then the edge from \( R_n \) to \( C_i \), followed by \( P \). Note that

\[
\epsilon(P')w(P') = -\epsilon(P)a_{nn}^{-1}x_iw(P). \tag{3}
\]

By the induction assumption, \( b_{ij} = \sum \epsilon(P)w(P) \), where the summation is over all alternating paths from \( C_i \) to \( R_j \). Hence it follows from (2) and (3) that for \( j = 1, \ldots, n - 1 \),

\[
b_{nj} = y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_ib_{ij} = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i \left( \sum_{P \in \mathcal{P}_{nj}} \epsilon(P)w(P) \right) = \sum_{P \in \mathcal{P}_{nj}} \epsilon(P)w(P),
\]

completing the proof. \( \square \)

We note some consequences of Theorem 1. Since the weights are noncommutative, we may take the weights to be square matrices of a fixed order.
This leads to combinatorial formulas for inverses of block triangular matrices. For example, the usual formula
\[
\begin{pmatrix}
A & O \\
C & B
\end{pmatrix}^{-1} =
\begin{pmatrix}
A^{-1} & O \\
-B^{-1}CA^{-1} & B^{-1}
\end{pmatrix}
\]
is a consequence of Theorem 1. Another example is the identity
\[
\begin{pmatrix}
A & O & O & O \\
W & B & O & O \\
X & O & C & O \\
O & Y & Z & D
\end{pmatrix}^{-1}
\]
\[
= \begin{pmatrix}
A^{-1} & O & O & O \\
-B^{-1}WA^{-1} & B^{-1} & O & O \\
-C^{-1}XA^{-1} & O & C^{-1} & O \\
D^{-1}YB^{-1}WA^{-1} + D^{-1}ZC^{-1}XA^{-1} & -D^{-1}YB^{-1} & -D^{-1}ZC^{-1} & D^{-1}
\end{pmatrix}.
\]

We note yet another consequence of Theorem 1. Let GF(2) denote the Galois field of order 2. The following result easily follows from Theorem 1.

**Corollary 2.** Let \(A\) be an \(n \times n\) lower triangular matrix over GF(2) such that \(a_{ii} = 1, i = 1, \ldots, n;\) and let \(B = A^{-1}\). Let \(G(A)\) be the graph associated with \(A\). Then \(b_{ij} = 1\) if and only if there are an odd number of alternating paths from \(C_i\) to \(R_j\) in \(G(A)\).

If \(A\) is a lower triangular matrix, then
\[
\begin{pmatrix}
O & A \\
A^\top & O
\end{pmatrix}
\]
is the (weighted) adjacency matrix of a bipartite graph with a unique perfect matching. Conversely the adjacency matrix of a bipartite graph with a unique perfect matching can be put in the form (4) after a relabeling of the vertices. In view of this observation, the unweighted case of Theorem 1 can be seen to be equivalent to Lemma 2.1 of Barik, Neumann and Pati [2]. Our proof technique is different. In the same spirit, Theorem 1 leads to a formula for the inverse of the adjacency matrix of a weighted tree (see Section 4) when the tree has a perfect matching, generalizing a well-known result from [4, 7] (see also [1, Section 3.6]).
Remark 3. Let $T$ be tree with nonsingular weighted adjacency matrix $A$. Then $A^{-1}$ is the weighted adjacency matrix of a bipartite graph. The graphs that can occur as inverses of nonsingular trees were characterized in [6]. Namely, a graph $G$ is the inverse of some tree if and only if $G \in \mathcal{F}_k$ where $\mathcal{F}_k$ is the family of graphs defined recursively as follows. Set $\mathcal{F}_1 = \{P_2\}$ and for $k \geq 2$ any $G \in \mathcal{F}_k$ is obtained from some $H \in \mathcal{F}_{k-1}$ by taking any vertex $u$ of $H$ and adding two new vertices $u'$ and $v$ where $u'$ is joined to all the neighbors of $u$ and $v$ (a pendant vertex) is joined to $u'$. The characterization remains valid in the more general setting when the weights of the edges come from a ring (provided the required inverses of the weights exist).

3 Matrices with isomorphic inverses

In this section we consider real matrices. It is an interesting problem to determine the triangular matrices $A$ for which $G(A)$ is isomorphic to $G(A^{-1})$. This problem is in close connection with the one posed by Godsil [5] as described below.

Let $G$ be a bipartite graph on $2n$ vertices which has a unique perfect matching $\mathcal{M}$. Then there is a lower triangular matrix $A$ such that $G = G(A)$. With the additional hypothesis that the graph $G/\mathcal{M}$, obtained from $G$ by contracting the edges in $\mathcal{M}$, is bipartite, Godsil [5] showed that $A^{-1}$ is diagonally similar to a matrix $A^+$ whose entries are nonnegative and which dominates $A$, that is $A^+(i,j) \geq A(i,j)$ for all $1 \leq i, j \leq n$. In turn, $A^+$ can be regarded as the adjacency matrix of a bipartite multigraph $G^+$ in which $G$ appears as a subgraph. In this framework, Godsil asked for a characterization of the graphs $G$ such that $G^+$ is isomorphic to $G$. This was answered in [8], by showing that $G$ and $G^+$ are isomorphic if and only if $G$ is a corona of a bipartite graph. The corona of a graph is obtained by creating a new vertex $v'$ for each vertex $v$ such that $v'$ is adjacent to $v$. The following theorem is a generalization of this result.

Theorem 4. Let $A$ be a lower triangular matrix with nonnegative entries, $\mathcal{M}$ being the unique matching of $G = G(A)$ and such that $G/\mathcal{M}$ is bipartite. Then $A$ and $A^{-1}$ have the same zero-nonzero pattern if and only if $G$ is a corona of a bipartite graph.
Proof. If $G$ is a corona, by some rearranging, we may write $A$ as

$$A = \begin{pmatrix} I & O \\ A_0 & I \end{pmatrix},$$

for some $A_0$. Hence

$$A^{-1} = \begin{pmatrix} I & O \\ -A_0 & I \end{pmatrix},$$

proving the ‘if’ part of the theorem.

Next, assume that $A$ and $A^{-1}$ have the same zero-nonzero pattern. To show that $G$ is a corona, it suffices to prove that the alternating paths of $G$ are of length at most 3. By contradiction, suppose that $G$ has an alternating path of length larger than 3 and so it has an alternating path of length 5 between $R_j$ and $C_i$, say. Since $G/M$ is bipartite, all the alternating paths between $R_j$ and $C_i$ must have the same length mod 4 (note that two alternating paths with different lengths mod 4 between two vertices give rise to an odd cycle in $G/M$). So, by Theorem [1], the $(i, j)$ entry of $A^{-1}$ is nonzero. Since $A$ and $A^{-1}$ have the same zero-nonzero pattern, the $(i, j)$ entry of $A$ is nonzero and hence $R_j$ and $C_i$ are adjacent. This implies the existence of a triangle in $G/M$, a contradiction. □

4 Generalized inverses and matchings

Let $A$ be an $m \times n$ matrix with entries from a ring such that $T = G(A)$ is a tree and let $\mathcal{M}$ be a matching in $T$. When $\mathcal{M}$ is perfect, $A$ is nonsingular and a formula for $A^{-1}$ may be given in terms of alternating paths, as noted at the end of Section 2. When $\mathcal{M}$ is not perfect, we still may define an $n \times m$ matrix $B = (b_{ij})$ using the alternating paths of $\mathcal{M}$ in the same fashion as when $\mathcal{M}$ is a perfect matching. More precisely, if $\{R_1, \ldots, R_m\}$ and $\{C_1, \ldots, C_n\}$ are color classes of $T$, then for $1 \leq j \leq i \leq n$,

$$b_{ij} = \sum \epsilon(P)w(P),$$

where the summation is over all alternating paths $P$ from $C_i$ to $R_j$ in $G(A)$. We call such a matrix the path matrix of $T$ with respect to $\mathcal{M}$. We show that the path matrix turns out be an outer inverse of the adjacency matrix.
Theorem 5. Let $A$ be an $m \times n$ matrix such that $T = G(A)$ is a tree and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matchings in $T$ with $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Let $B_1$ and $B_2$ be $n \times m$ path matrices of $T$ with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then

$$B_1 A B_2 = B_2 A B_1 = B_2.$$ 

Proof. Let $F_1$ and $F_2$ be the induced forests by $T$ on the vertices saturated by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let $A_1$ and $A_2$ be the submatrices of $A$ such that $F_1 = G(A_1)$ and $F_2 = G(A_2)$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are perfect matchings for $F_1$ and $F_2$, respectively. Let $|\mathcal{M}_1| = p$ and $|\mathcal{M}_2| = q$. It turns out that, with an appropriate ordering of the vertices,

$$B_1 = \begin{pmatrix} A_1^{-1} & O_{p \times (m-p)} \\ O_{(n-p) \times p} & O_{(n-p) \times (m-p)} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} A_2^{-1} & O_{q \times (m-q)} \\ O_{(n-q) \times q} & O_{(n-q) \times (m-q)} \end{pmatrix}.$$ 

Note that $A_2^{-1}$ is also a submatrix of $A_1^{-1}$, so $B_1$ is in fact of the form

$$B_1 = \begin{pmatrix} A_2^{-1} & O & O \\ * & * & O \\ O & O & O \end{pmatrix}.$$ 

Then

$$AB_1 = \begin{pmatrix} I_{p \times p} & O_{p \times (m-p)} \\ * & O_{(n-p) \times (m-p)} \end{pmatrix}.$$ 

It follows that

$$B_2 A B_1 = \begin{pmatrix} A_2^{-1} & O & O \\ O & O & O \end{pmatrix} = B_2.$$ 

The equality $B_1 A B_2 = B_2$ is proved similarly. $\square$

With the same proof as the theorem above, we can prove even a more general statement as follows.

Theorem 6. Let $A$ be an $m \times n$ matrix such that $T = G(A)$ is a tree and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matchings in $T$. If $B_1$ and $B_2$ be $n \times m$ path matrices of $T$ with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then

$$B_1 A B_2 = B_2 A B_1 = C,$$

where $C$ is the path matrix of $T$ with respect to $\mathcal{M}_1 \cap \mathcal{M}_2$. 

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Recall that the matrix $B$ is called a 2-inverse (or an outer inverse) of the matrix $A$ if $BAB = B$ (see, for example, [3]). The next result is an immediate consequence of Theorem 5.

**Corollary 7.** Let $A$ be a matrix such that $T = G(A)$ is a tree and let $\mathcal{M}$ be a matching in $T$. If $B$ is the path matrix of $T$ with respect to $\mathcal{M}$, then $B$ is an outer inverse of $A$.

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