Adaptive non-Zero Mean Gaussian Detection and Application to Hyperspectral Imaging

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Abstract—Classical target detection schemes are usually obtained deriving the likelihood ratio under Gaussian hypothesis and replacing the unknown background parameters by their estimates. In most applications, interference signals are assumed to be Gaussian with zero mean or with a known mean vector that can be removed and with unknown covariance matrix. When mean vector is unknown, it has to be jointly estimated with the covariance matrix, as it is the case for instance in hyperspectral imaging. In this paper, the adaptive versions of the classical Matched Filter and the Normalized Matched Filter, as well as two versions of the Kelly detector are first derived and then are analyzed for the case when the mean vector of the background is unknown. More precisely, theoretical closed-form expressions for false-alarm regulation are derived and the Constant False Alarm Rate property is pursued to allow the detector to be independent of nuisance parameters. Finally, the theoretical contribution is validated through simulations and on real hyperspectral scenes.

Index Terms—Hyperspectral Imaging, adaptive target detection, non-zero mean Gaussian distribution, false alarm regulation.

I. INTRODUCTION

HYPERSPECTRAL imaging (HSI) extends from the fact that for any given material, the amount of radiation emitted varies with wavelength. HSI sensors measure the radiance of the materials within each pixel area at a very large number of contiguous spectral bands and provide image data containing both spatial and spectral information (see for more details [1] and reference therein). Hyperspectral processing involves various applications such as unmixing, classification, detection, dimensionality reduction, ... Among them, hyperspectral detection is an active research topic that has led to many publications e.g. [2], [3], [4], [5]. More precisely, hyperspectral target detection methods are commonly used to detect targets embedded in background and that generally cannot be solved by spatial resolution [6]. Furthermore, Detection Theory [7] arises in many different military and civilian applications and has been widely investigated in several signal processing domains such as radar, sonar, communications, see [8] for the different references. There are two different methodologies for target detection purposes in the HSI literature [9]: Anomaly Detection [3], [4] and Target Detection [2].

In many practical situations, there is not enough information about the target to detect, thus Anomaly Detection methods are widely used. The most widespread detector, the RX detector [10] is based on the Mahalanobis distance [11]. This detector and most of its variants search for pixels in the image with spectral characteristics that differ from the background. On the other hand, when the spectral signature of the desired target is known, it can be used as steering vector in Target Detection techniques [9].

Interestingly, target detection methods have been extensively developed and analyzed in the signal processing and radar processing [8], [12], [13], [14]. In all these works as well as in several signal processing applications, signals are assumed to be Gaussian with zero mean or with a known mean vector (MV) that can be removed. In such context, Statistical Detection Theory [7] has led to several well-known algorithms, for instance the Matched Filter (MF) and its adaptive versions, the Kelly detector [12] or the Adaptive Normalized Matched Filter [15]. Other interesting approaches based on subspace projection methods have been derived and analyzed in [13]. However, when the mean vector of the noise background is unknown, these techniques are no longer adapted and improved methods have to be derived by taking into account the mean vector estimation. For this purpose, some preliminary results have been given in [16]. One of the contributions of this work is to extend and generalize these original results.

More precisely, this work deals with the classical Adaptive Matched Filter (AMF), the Kelly detection test and the Adaptive Normalized Matched Filter (ANMF). These detectors have been derived under Gaussian assumptions and benefit from great popularity in HSI target detection literature, see e.g. [17], [18]. To evaluate the detector performance, the classical process, according to the Neyman-Pearson criterion is first to regulate the false-alarm, by setting a detection threshold for a given probability of false-alarm (PFA). Since the PFA is the cumulative distribution function (CDF) of the detection test, this process is equivalent to the derivation of the detection test distribution. Then, the probability of detection is evaluated for different Signal-to-Noise Ratios (SNR). Therefore, keeping the false-alarm rate constant (CFAR) is essential to set a proper detection threshold [19], [20]. The aim is to build a CFAR detector which provides detection thresholds that are relatively immune to noise and background variation, and allow target detection with a
constant false-alarm rate. The theoretical analysis of CFAR methods for adaptive detectors is a challenging problem since in adaptive schemes, the statistical distribution of the detectors is not always available in a closed-form expression.

The theoretical contributions of this paper are twofold. First, we derive the expression of each adaptive detector under the Gaussian assumption where both the mean vector and the covariance matrix (CM) are assumed to be unknown. Then, the exact derivation of the distribution of each proposed detection scheme under null hypothesis, i.e. when no target is supposed to be present, is provided. Thus, through Gaussian assumption, closed-form expressions for the false-alarm regulation are obtained, which allow to theoretically set the detection threshold for a given PFA.

One the other hand, one difficulty for the background detection statistic is to assume a tractable model or at least to account for robustness to deviation from the assumed theoretical model in the detection scheme. Since Gaussian assumption is not always fulfilled for real hyperspectral data, alternative robust estimation techniques are proposed in [21]. However, it is essential to notice that the derivations for many results in robust detection contexts strongly rely on the results obtained in the Gaussian context. For instance, this is the case of [22] in which the derivation of a robust detector distribution is based on its Gaussian counterpart.

This paper is organized as follows. Section II introduces background on classical detection techniques as well as the obtention of the adaptive detectors for both unknown MV and CM. Then, Section III provides the main theoretical contributions of the paper by deriving the exact \"PFA-threshold\" relationship for the AMF, the \"plug-in\" Kelly detector and the ANMF under Gaussian assumption while a generalized version of the Kelly detector is derived. Finally, in Section IV the theoretical analyses are validated through Monte-Carlo simulations and real HS data are processed to, first, extract homogeneous, let’s say Gaussian, data and then, highlight the agreement with the proposed theoretical results. Conclusions and perspectives are drawn in Section V.

In the following, vectors (resp. matrices) are denoted by bold-faced lowercase letters (resp. uppercase letters). \( T \) and \( H \) respectively represent the transpose and the Hermitian operators. \( |A| \) represents the determinant of the matrix \( A \) and \( \text{Tr}(A) \) its trace. \( j \) is used to denote the unit imaginary number. \( \sim \) means \"distributed as\". \( \Gamma(\cdot) \) denotes the gamma function. Eventually, \( \Re\{\cdot\} \) represents the real part of the complex vector \( x \).

II. BACKGROUND AND ADAPTIVE DETECTORS DERIVATION

After providing the general background in non-zero mean Gaussian detection, this section is devoted to the derivation of the expression of the adaptive detectors.

The problem of the detecting a known signal \( s \) corrupted by an additive noise \( b \) in a \( m \)-dimensional complex vector \( x \) can be stated as a the following binary hypothesis test:

\[
\begin{cases}
H_0 : x = b \\
H_1 : x = s + b,
\end{cases}
\]

and the signal \( s \) can be written in the form \( \alpha p \), where \( \alpha \) is an unknown complex scalar amplitude, and \( p \) is the steering vector describing the signal which is sought. Since the background statistics, i.e. the MV and the CM, are assumed to be unknown, they have to be estimated from \( x_1, \ldots, x_N \sim \mathcal{CN}(\mu, \Sigma) \) a sequence of \( N \) IID signal-free secondary data. Then, the adaptive detector is obtained by replacing the unknown parameters by their estimates. In practice, an estimate may be obtained from the range cells surrounding the cell under test, which play the role of the \( N \) IID signal-free secondary data. The sample size \( N \) has to be chosen large enough to ensure the invertibility of the covariance matrix and small enough to justify both spectral homogeneity (stationarity) and spatial homogeneity. The use of a sliding mask provides a more realistic scenario than when estimating the parameters using all the pixels in the image. Let us know recall the detectors under interest in this work.

A. Adaptive Matched Filter

The MF detector is the optimal linear filter for maximizing the SNR in the presence of additive Gaussian noise with known parameters [7]. Hence, the signal model can be written as:

\[
\begin{cases}
H_0 : \ x = b \sim \mathcal{CN}(\mu, \Sigma) \\
H_1 : \ x = \alpha p + b \sim \mathcal{CN}(\alpha p + \mu, \Sigma).
\end{cases}
\tag{1}
\]

The Likelihood Ratio (LR) is given by:

\[
L(\alpha) = \frac{f(x|H_1)}{f(x|H_0)} \geq \lambda
\]

or according to the signal model:

\[
L(\alpha) = \frac{\exp[-(x - (\alpha p + \mu))^H \Sigma^{-1} (x - (\alpha p + \mu))] \Sigma^{-1} (x - \mu)]}{\exp[-(x - \mu)^H \Sigma^{-1} (x - \mu)]} \geq \frac{H_1}{H_0} \lambda.
\tag{2}
\]

Since the complex amplitude is unknown, it has to be estimated from the observation vector \( x \) and the background parameters according to:

\[
\alpha = \frac{\Re\{p^H \Sigma^{-1} (x - \mu)\}}{p^H \Sigma^{-1} p}.
\tag{3}
\]

Replacing this value in (2) and after some manipulations, the resulting MF detection scheme is:

\[
\Lambda_{MFS} = \frac{|p^H \Sigma^{-1} (x - \mu)|^2 \Sigma^{-1} (x - \mu)}{(p^H \Sigma^{-1} p) \Sigma^{-1} (x - \mu)} \geq \lambda.
\tag{4}
\]

Note that it differs from the classical MF by the term \( \mu \), the background mean, but without any consequence since \( x - \mu \sim \mathcal{CN}(0, \Sigma) \). Moreover, the "PFA-threshold" relationship is given by [7]:

\[
PFA_{MFS} = \exp(-\lambda).
\]

The AMF, denoted \( \Lambda_{AMFS}^{(N)} \Sigma \) to underline the dependency with \( N \), is usually built replacing the covariance matrix \( \Sigma \) by its
estimate $\hat{\Sigma}$ obtained from the $N$ secondary data. The mean vector is generally supposed to be known. Thus, the adaptive version becomes:

$$
\Lambda_{AMF}^{(N)} = \frac{|p^H \Sigma^{-1} (x - \mu)|^2}{\left(\frac{p^H \Sigma^{-1} p}{2}\right)} \frac{H_0}{H_0} \geq \lambda.
$$

Then, the theoretical "PFA-threshold" relationship is given by [14] for $\Sigma = \Sigma_{SCM}$:

$$
PFA_{AMF} = 2F_1 \left( N - m + 1, N - m + 2; 1 + \frac{\lambda}{N} \right),
$$

where $2F_1(\cdot)$ is the hypergeometric function [23] defined as,

$$
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{(b-1)(1-t)^{-b-1}} \left(1 - tz\right)^{a} dt.
$$

This detector holds the CFAR properties in the sense that its false alarm expression only depends on the dimension of the vector $m$ and the number of secondary data used for the estimation $N$. Note that it is also independent of the noise covariance matrix $\Sigma$, therefore the detector is said to be CFAR-matrix. However, its performance strongly relies on the good fit of the Gaussian model and the false alarm rate is highly increased when normal assumption is not verified.

B. Adaptive Kelly detector

The Kelly detector was derived in [12]. It is based on the Generalized Likelihood Ratio Test (GLRT) assuming Gaussian distribution and the same signal model than the AMF in [1]. In this case, only the covariance matrix $\Sigma$ is unknown, the mean vector is assumed to be known. Thus, the joint probability density function (p.d.f.) of the $N$ secondary data and the observation vector $x$ under the two hypotheses $\mathcal{H}_i$ can be written as:

$$
f_i(x) = \left(\frac{1}{\pi^N |\Sigma|} \exp[-Tr(\Sigma^{-1}T_0)]\right)^{-N+1},
$$

where $T_0$ is the composite sample covariance matrix constructed from both the secondary data and observation vector:

$$
T_0 = \frac{1}{N+1} \left( (x - \mu)(x - \mu)^H + \hat{\Sigma} \right)
$$

$$
T_1 = \frac{1}{N+1} \left( (x - (\alpha p + \mu))(x - (\alpha p + \mu))^H + \hat{\Sigma} \right)
$$

and $\hat{\Sigma} = N \Sigma_{SCM}$, where $\Sigma_{SCM}$ represents the well-known Sample Covariance Matrix (SCM) recalled in Appendix A. Then, by maximizing the p.d.f. under both hypotheses and by maximizing the LR with respect to (w.r.t) the complex, and after some manipulations, the resulting adaptive Kelly detector scheme takes the following form:

$$
\Lambda_{Kelly}^{(N)} = \frac{|p^H \Sigma_{SCM}^{-1} (x - \mu)|^2}{\left(\frac{p^H \Sigma_{SCM}^{-1} p}{2}\right) \left( \frac{N + (x - \mu)^H \Sigma_{SCM}^{-1} (x - \mu)}{2} \right)} \frac{H_0}{H_0} \geq \lambda,
$$

where $\lambda = 1 - \eta^{-\frac{1}{m}}$. As shown in [12], the PFA for the Kelly test is given by:

$$
PFA_{Kelly} = (1 - \lambda)^{N-m+1}.
$$

The Kelly detector is a CFAR test, in which the PFA is independent of the true covariance matrix. However, it has no known optimality property in the sense of maximizing the probability of detection for a given probability of false alarm. The AMF and the Kelly detector are based on the same assumptions about the nature of the observations. It is therefore interesting to compare their detection performance for a given PFA. Note that for large values of $N$ the performances are substantially the same.

C. Adaptive Normalized Matched Filter

The Normalized Matched Filter (NMF) is obtained when considering that the covariance matrix is different under the two hypotheses. That is to say that the clutter has the same covariance structure but different variance.

$$
\begin{align*}
\mathcal{H}_0 : & \quad x = b \sim \mathcal{CN}(\mu, \sigma_0^2 \Sigma) \\
\mathcal{H}_1 : & \quad x = \alpha p + b \sim \mathcal{CN}(\alpha p + \mu, \sigma_1^2 \Sigma).
\end{align*}
$$

Thus, the ML estimates of $\sigma_j^2$ are easily derived from $\hat{\sigma}_j^2 = \arg \max x \{f(x|\sigma, \mathcal{H}_j)\}$, $(j = 0, 1)$ and assuming normal distribution, one has:

$$
\begin{align*}
\sigma_0^2 &= \frac{1}{2m} (x - \mu)^H \Sigma^{-1} (x - \mu) \\
\sigma_1^2 &= \frac{1}{2m} (x - (\alpha p + \mu))^H \Sigma^{-1} (x - (\alpha p + \mu)).
\end{align*}
$$

After replacing complex amplitude $\alpha$ by its estimate $\hat{\alpha}$ when building the LR and after some manipulations, one obtains [24]:

$$
\Lambda_{NMF} = \frac{|p^H \Sigma_{SCM}^{-1} (x - \mu)|^2}{\left(\frac{p^H \Sigma_{SCM}^{-1} p}{2}\right) \left( (x - \mu)^H \Sigma_{SCM}^{-1} (x - \mu) \right)} \frac{H_0}{H_0} \geq \lambda,
$$

where $\lambda = 1 - \eta^{-\frac{1}{m}}$ and for which one has [24]:

$$
PFA_{NMF} = (1 - \lambda)^{(m-1)}.
$$

The ANMF is generally obtained when the unknown noise covariance matrix is replaced by an estimate [13]:

$$
\Lambda_{ANMF}^{(N)} = \frac{|p^H \hat{\Sigma}_{SCM}^{-1} (x - \mu)|^2}{\left(\frac{p^H \hat{\Sigma}_{SCM}^{-1} p}{2}\right) \left( (x - \mu)^H \hat{\Sigma}_{SCM}^{-1} (x - \mu) \right)} \frac{H_0}{H_0} \geq \lambda.
$$

And the PFA follows [13] for $\hat{\Sigma} = \hat{\Sigma}_{SCM}'$:

$$
PFA_{ANMF} = (1 - \lambda)^{\frac{1}{m}} 2F_1(a, a - 1; b - 1; \lambda),
$$

where $a = N - m + 2$ and $b = N + 2$.

III. MAIN RESULTS

In this section, let us now assume that the mean parameter is unknown as it is the case for instance in HSI and let us derive the new corresponding detection schemes. Then, using standard calculus on Wishart distributions, recapped in Appendix B, the distributions of each detection test is provided.
A. Adaptive Matched Filter Detector

When both covariance matrix and mean vector are unknown, they are replaced by their estimates from the secondary data in (4) leading to the AMF detector of the following form:

\[ \Lambda_{AMF}^{(N)} \Sigma, \mu = \frac{|p^H \Sigma^{-1} (x - \hat{\mu})|^2}{(p^H \Sigma^{-1} p)} \geq \lambda, \]

where the notation \( \Lambda_{AMF}^{(N)} \Sigma, \mu \) is used to stretch now the dependency on the estimated mean vector \( \hat{\mu} \). The distribution of this detection test is given in the next Proposition, through its PFA.

**Proposition III.1** Under Gaussian assumptions, the theoretical relationship between the PFA and the threshold is given by

\[ PFA_{AMF} \Sigma, \mu = 2F_1 \left( N - m, N - m + 1; N; \frac{\lambda'}{N - 1} \right), \]

where \( \lambda' = \frac{(N-1)}{(N+1)} \lambda, \Sigma = \Sigma_{SCM} \) and \( \mu = \mu_{SMV} \), recapped in Appendix [A].

Before turning into the proof, let us comment on this result.

- Interestingly, this detector also holds the CFAR property in the sense that its false-alarm expression depends only on the dimension \( m \) and on the number of secondary data \( N \), but not on the noise parameters \( \mu \) and \( \Sigma \).

- Note that the only effect of estimating the mean is the loss of one degree of freedom and the modification of the threshold compared to eq. (5). Obviously, the impact of these modification decreases as the number of secondary data \( N \) used to estimate the unknown parameters increases.

- Moreover, the result has been obtained when using the MLEs of the unknown parameters but the proof can be easily extended to other covariance matrix estimators such as \( \hat{\Sigma} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \hat{\mu})(x_i - \hat{\mu})^H \)

which the unbiased covariance matrix estimate or

\[ \hat{\Sigma} = \frac{1}{N+1} \sum_{i=1}^{N} (x_i - \hat{\mu})(x_i - \hat{\mu})^H. \]

**Proof:** For simplicity matters, the following notations are used: \( \Sigma = \Sigma_{SCM} \) and \( \mu = \mu_{SMV} \).

Since the derivation of the PFA is done under hypothesis \( H_0 \), let us set \( \forall i = 1, ..., N, x_i \sim \mathcal{CN}(\mu, \Sigma) \) and \( x \sim \mathcal{CN}(\mu, \Sigma) \), where all these vectors are independent. Now, let us denote

\[ \hat{\Sigma}_{N-1} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \hat{\mu})(x_i - \hat{\mu})^H \sim \mathcal{CW}(N-1, \Sigma), \]

Since \( \hat{\mu} \sim \mathcal{CN}(\mu, \frac{1}{N} \Sigma) \), one has \( x - \hat{\mu} \sim \mathcal{CN}(0, \frac{N+1}{N} \Sigma) \).

This can be equivalently rewritten as

\[ \sqrt{N/(N+1)}(x - \hat{\mu}) \sim \mathcal{CN}(0, \Sigma). \]

Now, let us set \( y = \sqrt{\frac{N}{N+1}}(x - \hat{\mu}) \) with \( y \sim \mathcal{CN}(0, \Sigma) \).

When computing the SCM, one has

\[ \hat{\Sigma}_{SCM} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)(x_i - \mu)^H = \frac{1}{N} \hat{W}_{N-1}. \]

As we jointly estimate the mean and the covariance matrix, a degree of freedom is lost, compared with the only covariance matrix estimation problem.

Let us now consider the classical AMF test (i.e. \( \mu \) known) built from \( N-1 \) secondary data, rewritten in terms of \( \hat{W}_{N-1} \):

\[ \Lambda_{AMF}^{(N-1)} = (N-1) \frac{|p^H \hat{W}_{N-1}^{-1} y|^2}{(p^H \hat{W}_{N-1}^{-1} p)}, \]

where \( y \sim \mathcal{CN}(0, \Sigma) \) and whose “PFA-threshold” relationship is given by eq. (5) where \( N \) is replaced by \( N - 1 \).

Now, for the joint estimation problem, the AMF can be rewritten as:

\[ \Lambda_{AMF}^{(N)} = N \frac{|p^H \hat{W}_{N-1}^{-1} (x - \hat{\mu})|^2}{(p^H \hat{W}_{N-1}^{-1} p)} = \frac{N}{N-1} \Lambda_{AMF}^{(N-1)} \]

where \( (x - \hat{\mu}) \) has been replaced by \( \sqrt{N+1/N} y \) with \( y \sim \mathcal{CN}(0, \Sigma) \), as previously.

Hence, one can determine the false-alarm relationship:

\[ PFA_{AMF} \Sigma, \mu = P \left( \Lambda_{AMF}^{(N)} \Sigma, \mu > \lambda | H_0 \right) \]

\[ = P \left( \frac{(N+1)}{(N-1)} \Lambda_{AMF}^{(N-1)} > \lambda | H_0 \right) \]

where \( \lambda' = \frac{(N-1)}{(N+1)} \lambda \), which leads to the conclusion.

B. Kelly Detector

The Kelly detector for both unknown mean vector and covariance matrix has now to be derived since it is not the previous Kelly in which an estimate of the mean is plugged. Following the same lines as in (12), we now assume that both the mean vector and the covariance matrix are unknown. The likelihood functions under \( H_0 \) and \( H_1 \) are given in [6]. Under \( H_0 \) and \( H_1 \), the maxima are achieved at

\[ \max_{\Sigma, \mu} f_i = \left( \frac{1}{(\pi e)^m |T_i|} \right)^{N+1}, \text{ for } i = 0, 1, \]

where

\[ (N+1)T_0 = (x - \mu_0)(x - \mu_0)^H + \sum_{i=1}^{N} (x_i - \mu_0)(x_i - \mu_0)^H, \]

\[ (N+1)T_1 = (x - \alpha p - \mu_1)(x - \alpha p - \mu_1)^H + \sum_{i=1}^{N} (x_i - \mu_1)(x_i - \mu_1)^H, \]
\[
\mu_0 = \frac{1}{N+1}(x + \sum_{i=1}^{N} x_i), \quad (13)
\]
\[
\mu_1 = \frac{1}{N+1}(x - \alpha p + \sum_{i=1}^{N} x_i). \quad (14)
\]

And neglecting the exponent \(N+1\), one obtains the following LR:

\[
L(\alpha) = \frac{|T_0|^H H_1 \mu_1}{|T_1|^H H_0 \mu_0}. \quad (16)
\]

Then, as this LR still depends on the unknown amplitude \(\alpha\) of the signal, thus, it has to be maximized w.r.t \(\alpha\), which is equivalent to minimize \(T_1\) w.r.t \(\alpha\). A way to do this is to introduce the following sample covariance matrix:

\[
S_0 = \sum_{i=1}^{N}(x_i - \mu_0)(x_i - \mu_0)^H. \quad (15)
\]

Then, \((N+1)|T_0\) can be written as

\[
(N+1)|T_0| = |S_0|(1 + (x - \mu_0)^H S_0^{-1}(x - \mu_0)).
\]

In the same way, and after some manipulations, \((N+1)|T_1|\) becomes

\[
(N+1)|T_1| = |S_0| \left( \sum_{i=1}^{N}(x_i - \mu_1)^H S_0^{-1}(x_i - \mu_1) 
+ (x - \alpha p - \mu_1)^H S_0^{-1}(x - \alpha p - \mu_1) \right) 
= |S_0|(A + B).
\]

Now, let us rewrite the two terms \(A\) and \(B\) to separate the terms involving \(\alpha\). By recalling that \(\mu_1 = \mu_0 - \frac{1}{N+1}\alpha p\), one obtains:

\[
A = 1 + \frac{N|\alpha|^2}{(N+1)^2} p^H S_0^{-1} p 
+ \frac{2}{N+1} \mathbb{R} \left\{ \bar{\alpha} p^H S_0^{-1} \sum_{i=1}^{N}(x_i - \mu_0) \right\},
\]

\[
B = (x - \mu_0)^H S_0^{-1}(x - \mu_0) + \frac{N^2|\alpha|^2}{(N+1)^2} p^H S_0^{-1} p 
- \frac{2N}{N+1} \mathbb{R} \left\{ \bar{\alpha} p^H S_0^{-1}(x - \mu_0) \right\}.
\]

Notice that \(x - \mu_0 = -\sum_{i=1}^{N}(x_i - \mu_0)\), then rearranging the expression of \((N+1)|T_1|\) leads to

\[
\frac{(N+1)|T_1|}{|S_0|} = \frac{(N+1)|T_0|}{|S_0|} + \frac{N|\alpha|^2}{(N+1)^2} p^H S_0^{-1} p 
- \frac{2N}{N+1} \mathbb{R} \left\{ \bar{\alpha} p^H S_0^{-1}(x - \mu_0) \right\}.
\]

Now, the term depending on \(\alpha\) can be rewritten as follows

\[
\frac{N}{(N+1)} p^H S_0^{-1} p \left[ \alpha - \frac{N+1}{N} \frac{p^H S_0^{-1}(x - \mu_0)^2}{p^H S_0^{-1} p} \right] 
\frac{-N+1}{N} \frac{p^H S_0^{-1}(x - \mu_0)^2}{p^H S_0^{-1} p}
\]

Minimizing \(|T_1|\) w.r.t \(\alpha\) is equivalent to cancel the square term in the previous equation. Thus, the GLRT can now be written according to the following definition.

**Definition III.1 (The generalized Kelly detector)** Under Gaussian assumptions, the extension of the Kelly’s test when both the mean vector and the covariance matrix of the background are unknown takes the following form:

\[
\Lambda = \frac{\beta(N) \left( \frac{p^H S_0^{-1}(x - \mu_0)^2}{(p^H S_0^{-1} p)(1 + (x - \mu_0)^H S_0^{-1}(x - \mu_0))} \right)^{H_1}}{\lambda^{H_0}}, \quad (16)
\]

where \(\beta(N) = \frac{N+1}{N} \lambda = \frac{\eta - 1}{\eta} \) and

- \(S_0 = \sum_{i=1}^{N}(x_i - \mu_0)(x_i - \mu_0)^H\),
- \(\mu_0 = \frac{1}{N+1}(x + \sum_{i=1}^{N} x_i)\).

Let us now comment on this new detector. One can notice that both the covariance matrix \(S_0\) as well as the mean \(\mu_0\) estimates depend on the data \(x\) under test, which is not the case in other classical detectors where the unknown parameters are estimated from signal-free secondary data. Consequently, \(S_0\) and \(x - \mu_0\) are not independent. Moreover, the covariance matrix estimate \(S_0\) is not Wishart-distributed due to the non-standard mean estimate \(\mu_0\). Thus, the derivation of this ratio distribution is very difficult.

As for previous detector, it would be intuitive to think that the proposed test behaves as the classical Kelly’s test but for \(N - 1\) degrees of freedom. To prove that let us first rewrite \([16]\) as follows:

\[
\Lambda = \frac{\beta(N) \left( \frac{p^H S_0^{-1} y^2}{(p^H S_0^{-1} p)(1 + \frac{N}{N+1} y^H S_0^{-1} y)} \right)^{H_1}}{\lambda^{H_0}}
\]

where we use:

- \((x - \mu_0) = \frac{N}{N+1} (x - \mu_{SMV})\),
- \(\mu_{SMV} = 1/N \sum_{i=1}^{N} x_i\),
- \(y = \sqrt{\frac{N}{N+1}} (x - \mu_{SMV}) \sim \mathcal{CN}(0, \Sigma)\).

Now, let us set \(S_0^{(i)} = \sum_{i=1}^{N}(x_i - \mu_0^{(i)})(x_i - \mu_0^{(i)})^H\) where \(\mu_0^{(i)} = 1/N(\sum_{j \neq i} x_j + x)\). Then, the test becomes

\[
\frac{N+1}{N} \left( \frac{p^H(S_0^{(i)})^{-1}(x - \mu_{SMV})^2}{(p^H(S_0^{(i)})^{-1} p)(1 + (x - \mu_{SMV})^H(S_0^{(i)})^{-1}(x - \mu_{SMV}))} \right)^{H_1}
\]

One can notice that each \(x_i\) (including \(x\)) plays the same role, thus the distribution of this test is the same for every permutation of the \((N+1)\)-sample \((x, x_1, \ldots, x_N)\). However, the dependency between the covariance matrix estimate and the data under test \(x\) still remains.
To fill this gap, another way of taking advantage of the Kelly’s detector when the mean vector is unknown can be to use the classical scheme recalled in (7) and to plug the classical estimator of the mean, based only on the secondary data, i.e. $\mu_{SMV} = 1/N \sum_{i=1}^{N} x_i$. This leads to the plug-in Kelly’s detector:

$$
\Lambda_{Kelly, \Sigma, \mu}^{(N)} = \frac{\mathbf{p}^H \Sigma_{SCM}^{-1} (x - \mu_{SMV})^2}{\left(\mathbf{p}^H \Sigma_{SCM}^{-1} \mathbf{p} \right) \left( N + (x - \mu_{SMV})^H \Sigma_{SCM}^{-1} (x - \mu_{SMV}) \right)^{\frac{1}{2}}}
$$

In this case, the distribution can be derived. This is the purpose of the following proposition.

**Proposition III.2** The theoretical relationship between the PFA and the threshold is given by

$$
PFA_{Kelly, \Sigma, \mu} = \frac{\Gamma(N)}{\Gamma(N-m+1) \Gamma(m-1)} \times \int_{0}^{1} \left[ 1 + \frac{\lambda}{1-\lambda} \left( 1 - \frac{u}{N+1} \right) \right]^{m-N} u^{N-m} (1-u)^{m-2} du
$$

**(17)**

**Proof:** The detection test rewritten with $\Sigma_{SCM}^{-1} = N W_{N-1}^{-1}$ becomes:

$$
\Lambda_{Kelly, \Sigma, \mu}^{(N)} = \frac{N^2 \mathbf{p}^H W_{N-1}^{-1} (x - \mu)^2}{N \left( \mathbf{p}^H W_{N-1}^{-1} \mathbf{p} \right) \left( N + N y^H W_{N-1}^{-1} (x - \mu) \right) N + N y^H W_{N-1}^{-1} (x - \mu)}
$$

and replacing $(x - \mu)$ by $\sqrt{\frac{N+1}{N}} y$, one obtains:

$$
\Lambda_{Kelly, \Sigma, \mu}^{(N)} = \frac{N+1}{N} N^2 \mathbf{p}^H W_{N-1}^{-1} y^2 = \frac{N \left( \mathbf{p}^H W_{N-1}^{-1} \mathbf{p} \right) \left( N + N y^H W_{N-1}^{-1} y \right)}{\left( \mathbf{p}^H W_{N-1}^{-1} \mathbf{p} \right) \left( N + N y^H W_{N-1}^{-1} y \right)}
$$

Solving the integral one obtains the "PFA-threshold" relationship:

$$
PFA_{Kelly, \Sigma, \mu} = \frac{\Gamma(N)}{\Gamma(N-m+1) \Gamma(m-1)} \times \int_{0}^{1} \left[ 1 + \frac{\lambda}{1-\lambda} \left( 1 - \frac{u}{N+1} \right) \right]^{m-N} u^{N-m} (1-u)^{m-2} du
$$

However, the final expression can not be further simplified and a closed-form expression as those obtained for the other detectors can not be determined.

### C. Adaptive Normalized Matched Filter

Similarly, the ANMF for both mean vector and covariance matrix estimation becomes:

$$
\Lambda_{ANMF, \Sigma, \mu} = \frac{\mathbf{p}^H \Sigma_{SCM}^{-1} (x - \mu)^2}{\left( \mathbf{p}^H \Sigma_{SCM}^{-1} \mathbf{p} \right) \left( \mathbf{p}^H \Sigma_{SCM}^{-1} (x - \mu) \right)^{\frac{1}{2}}}
$$

**Proposition III.3** The theoretical relationship between the PFA and the threshold is given by

$$
PFA_{ANMF, \Sigma, \mu} = (1-\lambda) a + b + 1 - \lambda
$$

where $a = (N-1) - m + 2$ and $b = (N-1) + 2$. **Proof:** The proof is similar to the proof of Proposition III.1. The main difference is due to the normalization term $(x - \mu)^H \Sigma_{SCM}^{-1} (x - \mu)$. Indeed, the correction factor $N/(N-1)$ appears both at the numerator and at the denominator, and
correspondingly, it disappears. The same argument is also true for the factor $N$ that arises from the covariance matrix estimates, i.e. since the detector is homogeneous in terms of covariance matrix estimates, this scalar also disappears. Thus, the distribution of the ANMF with an estimate of the mean is exactly the same as in eq. (11) where $N$ is replaced by $N - 1$.

IV. SIMULATIONS

In this section, we validate the theoretical analysis on simulated data. The experiments were conducted on $m = 5$ dimensional Gaussian vectors, for different values of $N$, the number of secondary data and the computations have been made through $10^6$ Monte-Carlo trials. The true covariance is chosen as a Toeplitz matrix whose entries are $\Sigma_{i,j} = \rho^{|i-j|}$ and where $\rho = 0.4$. The mean vector is arbitrary set to have all entries equal to $(3 + 4j)$.

A. False Alarm Regulation

The FA regulation is presented for previous detection schemes having a closed-form expression, i.e. for all except the generalized Kelly detector. Fig. 1 shows the false-alarm regulation for the MF, the AMF when only covariance matrix is unknown and the AMF for both covariance matrix and mean vector unknown. The perfect agreement of the green and yellow curves illustrates the results of Proposition III.1. Moreover, remark that when $N$ increases both AMF get closer to each other, and they approach the known parameters case MF.

Fig. 2 and Fig. 3 present the FA regulation for the Kelly detector and the ANMF respectively, under Gaussian assumption. For clarity purposes, the results are displayed in terms of the threshold $\eta$ from (7), $\eta = (1 - \lambda)^{-m}$, respectively and a logarithmic scale is used. This validates results of Proposition III.2 and III.3 for the SCM-SMV.

Remark that the derived relationships given by eqs. (12) and (19) are quite similar to those for which the mean is known. However, as illustrated in Fig. 1 and Fig. 3, there is an important difference for small values of $N$. It is worth pointing out that the theoretical "PFA-threshold" relationships presented above depend only on the size of the vectors $m$ and the number of secondary data used to estimate the parameters.
Thus, the detector outcome will not depend on the true value of the covariance matrix or the mean vector. These three detectors hold the CFAR property with respect to the background parameters. However, their distribution strongly relies on the underlying distribution of the background, i.e., if Gaussian assumption is not fulfilled the "PFA-threshold" relationship will divert from the theoretical results derived in this paper.

B. Performance Evaluation

![Figure 4: Probability of detection for different SNR values and $PFA = 10^{-3}$ in Gaussian case.](image)

The four detection schemes are compared in terms of probability of detection. Firstly, one sets the probability of false alarm to a specific value. Here we set $PFA = 10^{-3}$ with $m = 5$ and $N = 10$. Then, the threshold is adjusted to reach the desired PFA, according to the false alarm regulation curves described above. For the generalized Kelly detector, the threshold is empirically computed to ensure the same $PFA = 10^{-3}$. Fig. 4 presents the detection probability versus the SNR. When data follow a multivariate normal distribution, the detectors delivering the best performance results are the Kelly detectors ("Plug-in" and generalized). Actually, these detectors lead to very similar performance with a small improvement of the generalized (resp. "plug in") one at low (resp. high) SNR. As expected, the AMF and the ANMF require a higher SNR to achieve same performance.

C. Hyperspectral Real Data

The same experiments have been conducted on a real hyperspectral image. The scene analyzed is the NASA Hyperion sensor dataset displayed in Fig. 5. The image is constituted of $798 \times 253$ pixels and 116 spectral bands after water absorption bands have been removed. The analysis has been done on a homogenous part of the image corresponding to the water region on the top left of the image. The part extracted consists on $60 \times 20$ pixels. In order to ensure the validity of the proposed methods, we show in Fig. 6 the outcome of a classical Gaussianity test "Q-Q plot" for the selected region over the band 42. However, these techniques allow to "validate" the Gaussianity of each band but cannot ensure the Gaussianity of the corresponding vector. Since hyperspectral data are real and positive, we propose to use a Hilbert filter in order to render them complex. A downsampling taking one over two consecutive bands is required to avoid redundant information that can reduce the covariance matrix rank. However, it is important to note that the real component after Hilbert transform is still the original signal. To avoid the well-known problem due to high dimensionality, we have chosen sequentially
six bands in the complex representation. In this approach, both covariance matrix and mean vector are estimated using a sliding window of size $5 \times 5$, having $N = 24$ secondary data.

The outcome of the detectors for this image are shown on the Fig. 7, Fig. 8, and Fig. 9 respectively. The results obtained on real HSI data on a Gaussian distributed region agree with the theoretical relationships presented above. Remark that the false-alarm rate that can be achieved depends on the number of points on which the detector is calculated (in a similar manner to the Monte-Carlo trials). As the homogenous area is bounded and the data set is small, the distribution of the detectors may divert for small values of the PFA directly related to the size of the region.

Depending on the underlying material, the distribution of the detector might divert from the expected behavior when Gaussian distribution is assumed. This is the case on these real data since the extracted area is not perfectly Gaussian. This suggests the use of non-Gaussian distributions to model the background for hyperspectral imaging.

V. CONCLUSION

Four adaptive detection schemes, the AMF, Kelly detectors with a "plug-in" and a generalized versions as well as the ANMF, have been analyzed in the case where both the covariance matrix and the mean vector are unknown and need to be estimated. In this context, theoretical closed-form expressions for false-alarm regulation have been derived under Gaussian assumptions for the SCM-SMV estimates for three detection schemes. The resulting "PFA-threshold" expressions highlight the CFARness of these detectors since they only depend on the size and the number of data, but not on the unknown parameters. The theoretical analysis has been validated through Monte Carlo simulations and the performances of the detectors have been compared in terms of probability of detection. Finally, the analysis on experimental hyperspectral data validates the
theoretical contribution through real application, in which a homogeneous subset of data has been extracted. But more generally, this work finds its purpose in signal processing methods for which both mean vector and covariance matrix are unknown.

APPENDIX A

COMPLEX NORMAL DISTRIBUTIONS

A \( m \)-dimensional vector \( \mathbf{x} = \mathbf{u} + j\mathbf{v} \) has a complex normal distribution with mean \( \mu \) and covariance matrix \( \Sigma = E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^H] \), denoted \( \mathcal{CN}(\mu, \Sigma) \), if \( x = (u^T, v^T)^T \in \mathbb{R}^{2m} \) has a normal distribution [26]. If \( \text{rank}(\Sigma) = m \), the probability density function exists and is of the form

\[
    f_{\mathbf{x}}(\mathbf{x}) = \pi^{-m/2} |\Sigma|^{-1} \exp\{-((\mathbf{x} - \mu)^H \Sigma^{-1} (\mathbf{x} - \mu))\}.
\]

The resulting Maximum Likelihood Estimates (MLE) are the well-known SCM and SMV defined as:

\[
    \mu_{SMV} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \quad \text{and} \quad \Sigma_{SCM} = \frac{1}{N} \sum_{i=1}^{N} (\mathbf{x}_i - \mu)(\mathbf{x}_i - \mu)^H
\]

where the \( x_i \) are independent and identically distributed (IID) \( \mathcal{CN}(\mu, \Sigma) \).

APPENDIX B

WISHART DISTRIBUTION

Let \( x_1, ..., x_N \) be an IID \( N \)-sample, where \( x_i \sim \mathcal{CN}(\mu, \Sigma) \). Let us define \( \mathbf{\mu} = \mu_{SMV} \) and \( \mathbf{W} = N \Sigma_{SCM} \) referred to as a Wishart matrix. Thus one has (see [27] for the real case):

- \( \mathbf{\mu} \) and \( \mathbf{W} \) are independently distributed;
- \( \mathbf{\mu} \sim \mathcal{CN}(\mu, \frac{1}{N} \Sigma) \);
- \( \mathbf{W} \sim \mathcal{CW}(N-1, \Sigma) \) is Wishart distributed with \( N-1 \) degrees of freedom

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