CHAPTER 1

Field Theoretical Approaches to the Superconducting Phase Transition

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Several field theoretical approaches to the superconducting phase transition are discussed. Emphasis is given to theories of scaling and renormalization group in the context of the Ginzburg-Landau theory and its variants. Also discussed is the duality approach, which allows to access the strong-coupling limit of the Ginzburg-Landau theory.

1. Introduction

The Ginzburg-Landau (GL) phenomenological theory of superconductivity was proposed long before the famous Bardeen-Cooper-Schrieffer (BCS) microscopic theory of superconductivity. A few years after the BCS theory, Gorkov derived the GL theory from the BCS theory. This has been done by deriving an effective theory for the Cooper pairs valid in the neighborhood of the critical temperature $T_c$. The modern derivation proceeds most elegantly via functional integrals, introducing a collective quantum field $\Delta(x,t)$ for the Cooper pairs via a Hubbard-Stratonovich transformation and integrating out the fermions in the partition function.

Amazingly, the GL theory has kept great actuality up to now. It is highly relevant for the description of high-$T_c$ superconductors, even though the original BCS theory is inadequate to treat these materials. The success of the GL theory in the study of modern problems of superconductivity lies on its universal effective character, the details of the microscopic model being unimportant. In the neighborhood of the critical point the GL theory possesses a wide range of application, many of them outside the field of the superconductivity. Perhaps, the most famous example in condensed
mater physics is application to the smectic-A to nematic phase transition in liquid crystals.\textsuperscript{1} In elementary particle physics, Gorkov’s derivation of the GL theory has been imitated starting out from a four-dimensional relativistic version of the BCS model, the so-called Nambu-Jona-Lasinio model. The GL field in the resulting GL theory describes quark-antiquark bound states, the mesons $\pi$, $\sigma$, $\rho$, and $A_1$.\textsuperscript{6} This model is still of wide use in nuclear physics. Another relativistic GL model is needed to generate the masses of the vector bosons $W$ and $Z$ in the unified theory of electromagnetic and weak interactions in a renormalizable way.\textsuperscript{7} This is done by a nonabelian analog of the Meissner effect, which in this context is called Higgs mechanism.

In 1973 Coleman and Weinberg\textsuperscript{8} showed that the four-dimensional abelian GL model (or scalar electrodynamics, in the language of elementary particle physics) exhibits a spontaneous mass generation in an initially massless theory. In the language of statistical mechanics, this implies a first-order transition if the mass of the scalar field passes through zero. One year later, Halperin, Lubensky, and Ma (HLM) observed a similar phenomenon in the three-dimensional GL theory of superconductivity.\textsuperscript{9} These papers inaugurated a new era in the study of the GL model. It was the first time that renormalization group (RG) methods were used to study the superconducting phase transition.

At the mean-field level, the GL model exhibits a second-order phase transition. The HLM analysis, however, concludes that fluctuations change the order of the transition to first. The phase transition would be of second-order only if the number of complex components of the order parameter is absurdly high. If the number of complex components is $N/2$, the one-loop RG analysis of HLM gave the lower bound $N > N_c = 365$ for a second-order transition. No infrared stable charged fixed point was found for $N \leq N_c$. However, years later Dasgupta and Halperin\textsuperscript{10} raised doubts on this result by using duality arguments and a Monte Carlo simulation of a lattice model in the London limit, to demonstrate that the transition for $N = 2$ in the type II regime is of second order. The RG result that the phase transition is always of first order seems therefore to be an artifact of the $\epsilon$-expansion. Shortly after this, Kleinert\textsuperscript{11} performed a quantitative duality on the lattice and found a disorder field theory\textsuperscript{12} which clarified the discrepancy between the HLM RG result and the numerical simulations of Dasgupta and Halperin. He showed that there exists a tricritical point in the phase diagram of the superconductor on the separation line between first- and second-order regimes. This result and the numerical prediction\textsuperscript{11}
were fully confirmed only recently by large scale Monte Carlo simulations of Mo et al.\cite{13}

In the last ten years the RG approach to the GL model was revisited by several groups.\cite{14,15,16,17,18} The aim was to improve the RG analysis in such a way as to obtain the charged fixed point for $N = 2$ predicted by the duality scenario. Despite considerable progress, our understanding of this problem is far from satisfactory.

In this paper we shall review the basic field theoretic ideas relevant for the understanding of the superconducting phase transition. The reader is supposed to have some familiarity with RG theory and duality transformations. For a recent review on the RG approach to the GL model which focus more on resummation of $\epsilon$-expansion results, including calculations of amplitude ratios, see Ref.\cite{19}.

2. Review of the HLM theory

The GL Hamiltonian is given by

$$H = \frac{1}{2}(\nabla \times A)^2 + |(\nabla - ieA)\psi|^2 + m^2|\psi|^2 + \frac{u}{2}|\psi|^4, \quad (1)$$

where $\psi$ is the order field and $A$ is the fluctuating vector potential. The partition function is written in the form of a functional integral as

$$Z = \int D\!A \!D\!\psi \!D\!\psi^{\dagger} \det(-\nabla^2) \delta(\nabla \cdot A) \exp \left(-\int d^3 r H\right), \quad (2)$$

where a delta functional in the measure of integration enforces the Coulomb gauge $\nabla \cdot A = 0$. The factor $\det(-\nabla^2)$ is the associated Faddeev-Popov determinant.\cite{7} The above Hamiltonian coincides with the euclidian version of the Abelian Higgs model in particle physics. The Coulomb gauge is the euclidian counterpart of the relativistic Lorentz gauge. The Faddeev-Popov determinant should be included in order cancel a contribution $1/\det(-\nabla^2)$ that arises upon integration over $A$ taking into account the constraint $\nabla \cdot A = 0$.\cite{7,30}

2.1. HLM mean-field theory

Let us make the following change of variables in the partition function \cite{2}:

$$\psi = \frac{1}{\sqrt{2}} \rho e^{i\theta}, \quad A = a + \frac{1}{e} \nabla \theta, \quad (3)$$
which brings the partition function to the form

\[
Z = \int \mathcal{D}a \mathcal{D}\rho \mathcal{D}\theta \rho \det(-\nabla^2)\delta(\nabla \cdot a + e^{-1}\nabla^2\theta) \exp \left( -\int d^3r \mathcal{H} \right),
\]

with

\[
\mathcal{H} = \frac{1}{2}(\nabla \times a)^2 + \frac{e^2}{2}\rho^2 a^2 + \frac{1}{2}(\nabla \rho)^2 + \frac{m^2}{2}\rho^2 + \frac{u}{8}\rho^4.
\]

In these variables, the Hamiltonian does not depend on \(\theta\). This change of variables is allowed only in a region where the system has few vortex lines, since otherwise the cyclic nature of the \(\theta\)-field becomes relevant, and \(\nabla e^{i\theta}\) is no longer equal to \(i(\nabla \theta - 2\pi n) e^{i\theta}\), where \(n\) is a vortex gauge field. This problem of the Hamiltonian \(\mathcal{H}\) which is said to be in the unitary gauge will need special attention.

Now we can use the delta function to integrate out \(\theta\). The result of this integration cancels out the Faddeev-Popov determinant. Next we assume that \(\rho\) is uniform, say \(\rho = \bar{\rho} = \text{const.}\) Since the Hamiltonian is quadratic in \(a\), the \(a\) integration can be done straightforwardly to obtain the free energy density:

\[
\mathcal{F} = \frac{1}{2V} \text{Tr} \ln\left[ (-\nabla^2 + e^2\bar{\rho}^2)\delta_{\mu\nu} + \partial_\mu \partial_\nu \right] - \frac{1}{2V} \delta^3(0) \text{Tr} \ln(e^2\bar{\rho}^2) + \frac{m^2}{2}\rho^2 + \frac{u}{8}\rho^4,
\]

where \(V\) is the (infinite) volume and the term \(\delta^3(0)\text{Tr} \ln(e^2\bar{\rho}^2)/2V\) comes from the exponentiation of the functional Jacobian in Eq. (4). In order to evaluate the \(\text{Tr} \ln\) in Eq. (6) we have to use the Fourier transform of the operator

\[
M(\mathbf{r}, \mathbf{r}') = [(-\nabla^2 + e^2\bar{\rho}^2)\delta_{\mu\nu} + \partial_\mu \partial_\nu] \delta^3(\mathbf{r} - \mathbf{r}').
\]

The operator \(M(\mathbf{r}, \mathbf{r}')\) is diagonal in momentum space. Indeed, we can write

\[
M(\mathbf{r}, \mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{r} - \mathbf{r}')} \hat{M}(\mathbf{p}),
\]

where \(\hat{M}(\mathbf{p}) = (\mathbf{p}^2 + e^2\bar{\rho}^2)P^T_{\mu\nu}(\mathbf{p}) + e^2\bar{\rho}^2 P^L_{\mu\nu}(\mathbf{p})\), and \(P^T_{\mu\nu}(\mathbf{p}) = \delta_{\mu\nu} - p_\mu p_\nu/\mathbf{p}^2\) and \(P^L_{\mu\nu}(\mathbf{p}) = p_\mu p_\nu/\mathbf{p}^2\) are the transverse and longitudinal projectors, respectively. Now, the \(\text{Tr} \ln M(\mathbf{r}, \mathbf{r}')\) is obtained.
first by taking the logarithm of the transversal and longitudinal parts of 
\( \hat{M}(p) \) and tracing over the vector indices. The second step is to trace out
the coordinates by integrating over \( r \) for \( r = r' \). This produces an overall
volume factor \( V \) which cancels out exactly the \( 1/V \) factor in Eq. (5). Note 
that the term proportional to \( \delta^3(0) \ln(e^2 \bar{\rho}^2) \) drops out in the calculations.
If we write \( \bar{\rho} = |\bar{\psi}| \), the end result is the celebrated HLM mean field free
energy:

\[
F_{HLM} = -\frac{e^3 |\bar{\psi}|^{3}}{6\pi} + \frac{m^2}{2} |\bar{\psi}|^2 + \frac{u}{8} |\bar{\psi}|^4. \tag{10}
\]

Due to the cubic term in Eq. (10), the transition is of first order. The basic
problem with this argument was pointed out in Ref. [11]. In the critical
regime of a type-II superconductor, the order field contains numerous lines
of zeros which make it impossible to use uniformity assumption \( \rho = \bar{\rho} = \text{const.} \)

2.2. Renormalization group in \( d = 4 - \epsilon \) dimensions

In the original HLM work, the RG calculations were performed using the
Wilson version of the renormalization group [21] where fast modes are inte-
grated out to obtain an effective theory in terms of the slow modes. The field
theoretical approach using the Callan-Symanzik equation gives an equiva-
 lent result in a perturbative setting. We shall discuss the RG calculation in
\( d = 4 - \epsilon \) dimensions in this context.

The dimensionless renormalized couplings are defined by

\[
f \equiv \mu^{-\epsilon} Z_A e^2 e^0, \quad g \equiv \mu^{-\epsilon} Z^2 Z_g u_0, \tag{11}
\]

where \( \mu \) gives the mass scale of the problem. Note that in Eq. (11) we have
denoted the bare couplings by a zero subindex. The bare fields are denoted
by \( \psi_0 \) and \( A_0 \). Accordingly, the bare mass is denoted by \( m_0 \). We shall use
this notation from now on.

The renormalization constants are defined such that the “renormalized
Hamiltonian” \( \mathcal{H}_r \) is given by the following rewriting of the bare Hamilto-
nian:

\[
\mathcal{H}_r(A, \psi; m^2, u, e) = \mathcal{H}(Z_A^{1/2} A, Z_\psi^{1/2} \psi; Z_m, Z_\psi^{-1} m^2, Z_g, Z_g^{-1} u, Z_A^{1/2} e). \tag{12}
\]

From Eq. (12) we see that the renormalized fields are given by \( \psi = Z_\psi^{-1/2} \psi_0 \)
and \( A = Z_A^{-1/2} A_0 \).
The calculations are more easily done if we set $m = 0$ and evaluate the Feynman graphs at nonzero external momenta to avoid infrared divergences. The external momenta in a four leg graph will be taken at the symmetry point:

$$p_i \cdot p_j = \frac{\mu^2}{4} (4 \delta_{ij} - 1).$$

(13)

Even if $m \neq 0$ we have to face severe infrared divergences in this problem. This is due to the masslessness of the vector potential field. The free $A$ propagator is given in the Coulomb gauge by

$$D_{\mu\nu}(p) = \frac{1}{p^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right).$$

(14)

Thus, the graph in Fig. 1 is proportional to the integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(p-k)^2} \bigg|_{\text{SP}} = \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{\Gamma(d-2)} \mu^{d-4},$$

(15)

evaluated at the symmetry point (SP) as prescribed in Eq. (13).

Fig. 1. Feynman graph contributing to the $f^2$ term in $\beta_g$. The dashed lines represent the vector potential propagator, while the external lines represent the order parameter.

The $\beta$-functions are defined by

$$\beta_f \equiv \mu \frac{\partial f}{\partial \mu}, \quad \beta_g \equiv \mu \frac{\partial g}{\partial \mu}.$$

(16)

These $\beta$-functions are defined for the dimension $d$ in the interval $(2, 4]$. For $d > 4$ the theory is no longer renormalizable and the $\beta$-functions are not defined. At $d = 2$ the infrared divergences are very strong and require a special treatment.
Let us assume that the order parameter field has \( \frac{N}{2} \) complex components. Then, for any dimension \( d \in (2, 4] \) the \( \beta \)-functions are given at one-loop order by

\[
\begin{align*}
\beta_f &= (4 - d)\{-f + NA(d)f^2\}, \\
\beta_g &= (4 - d)\left\{-g + B(d)\left[-2(d - 1)f g + \frac{N+8}{2}g^2 + 2(d-1)f^2\right]\right\},
\end{align*}
\] 

(17)

where

\[
A(d) = -\frac{\Gamma(1 - d/2)\Gamma^2(\frac{d}{2})}{(4\pi)^{d/2}\Gamma(d)},
\]

(19)

\[
B(d) = \frac{\Gamma(2 - d/2)\Gamma^2(\frac{d}{2}-1)}{(4\pi)^{d/2}\Gamma(d-2)}.
\]

(20)

If we set \( d = 4 - \epsilon \) and expand to first order in \( \epsilon \), we obtain

\[
\begin{align*}
\beta_f &= -\epsilon f + \frac{N}{48\pi^2}f^2, \\
\beta_g &= -\epsilon g - \frac{3fg}{4\pi^2} + \frac{N+8}{8\pi^2}g^2 + \frac{3}{8\pi^2}f^2.
\end{align*}
\]

(21)

(22)

From Eqs. (21) and (22) we see easily that charged fixed points exist only if \( N > N_c = 365.9 \). Thus, if \( N \) is large enough to allow for the existence of charged fixed points we obtain the flow diagram shown schematically in Fig. 2. In the figure the arrows correspond to \( \mu \to 0 \). There are four fixed points. The Gaussian fixed point is trivial and corresponds to \( f_\ast = g_\ast = 0 \). It governs the ordinary mean-field behavior. There is one non-trivial uncharged fixed point, labeled “Heisenberg” in the figure, which governs the \( N \)-component Heisenberg model universality class. For \( N = 2 \) the superfluid \( ^4 \)He belongs to this universality class (in this case we speak of a \( XY \) universality class). The Heisenberg fixed point is unstable for non-zero charge. There are two charged fixed points. The one labeled \( SC \) in the figure is infrared stable and governs the superconducting phase transition. Its infrared stability ensures that the phase transition is second-order. The second charged fixed point is labeled with a \( T \) and is called the tricritical fixed point. It is infrared stable along the line starting in the Gaussian fixed
point and unstable along the \( g \)-direction. The line of stability of the tricritical fixed point is called the tricritical line. The tricritical line separates the regions in the flow diagram corresponding to first- and second-order phase transition.

As we shall see later on in these lectures, the flow diagram shown in Fig. 2 should also be valid for \( N = 2 \). For the moment let us remark that if we use the \( \beta \)-functions in fixed dimension as given in Eqs. (17) and (18) the value of \( N_c \) is considerably smaller at \( d = 3 \). Indeed, by setting \( d = 3 \) in Eqs. (17) and (18), we obtain that charged fixed points exist for \( N > N_c = 103.4 \).

2.3. Critical exponents

The critical exponents can be evaluated using standard methods.\(^{22}\) The \( \eta \) exponent is obtained as the value of the RG function

\[
\gamma_\psi = \mu \frac{\partial \ln Z_\psi}{\partial \mu}, \tag{23}
\]

\(^{a}\)In the context of the \( \epsilon \)-expansion, see for example the two-loops resummed RG analysis by Folk and Holovatch.\(^{15,19}\)
at the infrared stable fixed point. The $\eta$ exponent governs the large distance behavior of the order field correlation function at the critical point:

$$\langle \psi(\mathbf{r})\psi^\dagger(\mathbf{r}') \rangle \sim \frac{1}{|\mathbf{r} - \mathbf{r}'|^{d - 2 + \eta}}.$$  \hfill (24)

The critical exponent $\nu$, governing the scaling of the correlation length $\xi = m^{-1} \sim t^{-\nu}$, with $t$ being the reduced temperature, is obtained as the infrared stable fixed point value of the RG function $\nu_\psi$ defined by the equation

$$\frac{1}{\nu_\psi} - 2 = \gamma_m,$$  \hfill (25)

where

$$\gamma_m = \mu \frac{\partial}{\partial \mu} \ln \left( \frac{Z_m}{Z_\psi} \right).$$  \hfill (26)

In an approach where the correlation functions are computed at the critical point the mass renormalization $Z_m$ must be computed through $|\psi|^2$ insertions in the 2-point function. \footnote{For any dimension $d \in (2, 4]$, we obtain the one-loop result:

$$\gamma_\psi = (1 - d)(4 - d)B(d)f,$$  \hfill (27)

$$\gamma_m = (N + 2)(d - 4)B(d)g/2 - \gamma_\psi.$$  \hfill (28)

Once the critical exponents $\eta$ and $\nu$ are evaluated, all the other critical exponents can be obtained using the standard scaling relations. We shall prove later that the standard scaling relations apply also in the case of the GL model.

2.4. $1/N$ expansion

The $1/N$ expansion is one of the most popular non-perturbative methods in the field theoretical and statistical physics literature. The critical exponents of the GL model at $O(1/N)$ were calculated at $d = 3$ by HLM. Since the original paper does not contain any detail of the calculation, we shall outline it here. The large $N$ limit in the GL model is taken at $Nu$ and $Ne^2$ fixed.
Let us calculate the critical exponent \( \eta \). The relevant graphs are shown in Fig. 3. The idea is to pick up the \( p^2 \ln |p| \) contribution of \( \Gamma^{(2)}(p) = G^{-1}(p) \), where \( G \) is the order field propagator. To keep the theory massless, we have to subtract the \( p = 0 \) contribution of the self-energy. The subtracted contribution coming from the graph (a) in Fig. 3 is

\[
\Sigma_a(p) = \int \frac{d^3k}{(2\pi)^3} V(k) \left[ \frac{1}{(p-k)^2} - \frac{1}{k^2} \right],
\]

where

\[
V(k) = \frac{2u}{1+uN\Pi(k)},
\]

and \( \Pi(k) \) is the polarization bubble:

\[
\Pi(p) = \int \frac{d^3k}{(2\pi)^3} G_0(p-k)G_0(k),
\]

where \( G_0 \) is the free scalar propagator. In the massless case we have simply

\[
\Pi(p) = \frac{1}{8|p|}.
\]

When \( e = 0 \) Eq. (30) gives the effective interaction of the \( O(N) \) model. The \( -\eta a p^2 \ln |p| \) contribution from Eq. (29) gives

\[
\eta_a = \frac{8}{3\pi^2 N},
\]

which is just the \( \eta \)-exponent of the \( O(N) \) model at \( O(1/N) \).

In order to compute the contribution from the graph (b) of Fig. 3, we need to calculate the dressed vector potential propagator. The dressed propagator is obtained by summing the chain of bubbles shown in Fig. 3. The result is

\[
D_{\mu\nu}(p) = \frac{1}{p^2 + \Sigma_A(p)} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right),
\]

where in the massless case

\[
\Sigma_A(p) = \frac{Ne^2}{32} |p|.
\]
The self-energy contribution from graph (b) is then

$$\Sigma_b(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(2p_\mu - k_\mu)(2p_\nu - k_\nu)}{(p - k)^2} D_{\mu\nu}(k).$$

(36)

Thus, we obtain the contribution $-\eta_b p^2 \ln |p|$, where

$$\eta_b = -\frac{128}{3\pi^2 N}. \quad (37)$$

The $\eta$-exponent of the GL model at $O(1/N)$ is obtained by adding the contributions from Eqs. (33) and (37):

$$\eta = \eta_a + \eta_b = -\frac{40}{\pi^2 N}. \quad (38)$$

The critical exponent $\nu$ at $O(1/N)$ can be computed through similar calculations, except that one needs to consider massive propagators for the order field. Instead picking up the contribution $p^2 \ln |p|$, we take the $m^2 \ln m$ one from the self-energy at zero momentum. This gives in fact the critical exponent $\gamma$. The critical exponent $\nu$ is obtained straightforwardly.
by using the scaling relation $\gamma = \nu(2 - \eta)$. The end result is:

$$\nu = 1 - \frac{96}{\pi^2 N}. \quad (39)$$

Note that in the $1/N$ expansion no tricritical fixed point is generated at $O(1/N)$. The reason for that comes from the fact that the graph in Fig. 1 is $O(1/N^2)$ and does not contribute to the above calculations. This graph is the main obstacle against attaining the charged fixed point.

3. Existence of the charged fixed point

3.1. Scaling near the charged fixed point

By assuming the existence of the charged fixed point, it is possible to derive exact scaling relations for the GL theory. For example, it is easy to prove that if a charged fixed point exists then $\nu' = \nu$, where $\nu'$ is the exponent of the penetration depth $\lambda$. To this end, it is necessary to consider the GL model in the ordered phase, that is, at $T < T_c$. In this situation the vector potential becomes massive through the Higgs mechanism, with a mass $m_A = \lambda^{-1}$. The Higgs mechanism implies that there are no massless modes in the superconductor in the ordered phase. This shows a fundamental difference between a superconductor and a superfluid: The ordered phase of a superfluid contains a massless mode or Goldstone boson. The Ginzburg parameter $\kappa$ is defined by the ratio between the Higgs mass and the vector potential mass: $\kappa = m/m_A$. It can be shown that $m^2 = u\rho_s/2$ and $m_A^2 = e^2\rho_s$, with $\rho_s$ being the superfluid density. These formulas are easily obtained in a tree level analysis of the Higgs mechanism. Their application in the renormalized case follows from imposing suitable renormalization conditions and the Ward identities.

Thus, $\kappa^2 = u/2e^2 = g/2f$. Since both $f$ and $g$ tend to a nonzero fixed point value in the infrared ($m \to 0$), it follows that the Ginzburg parameter $\kappa^2 = \lambda^2/\xi^2 = g/2f \to \text{const}$ as $m \to 0$. As a consequence, the scaling relation $\lambda \sim \xi$ holds and therefore the equality between the corresponding critical exponents.

In the following it will be useful to use $m$ as the running RG scale. Thus, $f = m^{d-4}Z_Ae_0^2$ and $g = m^{d-4}Z_\psi^2Z_g^{-1}u_0$. Therefore,

$$\beta_f \equiv m \frac{\partial f}{\partial m} = (\gamma_A + d - 4)f, \quad (40)$$
where
\[ \gamma_A \equiv m \frac{\partial \ln Z_A}{\partial m} \]  
(41)

Thus, under the assumption of existence of the charged fixed point we obtain that
\[ \eta_A \equiv \gamma_A(f_*, g_*) = 4 - d. \]  
(42)

The above equation gives the exact anomalous dimension of the vector potential. This means that the large distance behavior at the critical point:
\[ \langle A_\mu(r)A_\mu(r') \rangle \sim \frac{1}{|r - r'|^{d-2+\eta_A}} \sim \frac{1}{|r - r'|^2}, \]  
(43)

which holds for all \( d \in (2, 4] \).

It is instructive and experimentally relevant to compare the scaling near a charged fixed point with the one governed by the \( XY \) fixed point. To this end, let us first note that the flow equation for \( \kappa^2 \) is written exactly as
\[ m \frac{\partial \kappa^2}{\partial m} = \kappa^2 \left( \frac{\beta g}{g} + 4 - d - \gamma_A \right). \]  
(44)

Since \( \kappa^2 = m^2/m_A^2 \), it follows the following exact evolution equation for \( m_A^2 \):
\[ m \frac{\partial m_A^2}{\partial m} = m_A^2 \left( d - 2 + \gamma_A - \frac{\beta g}{g} \right). \]  
(45)

From Eq. (46) we obtain that near a fixed point \( m_A \) behaves as
\[ m_A \sim m^{(d-2+\eta_A)/2}. \]  
(46)

The above constitutes a rederivation of a result due to Herbut and Tesanovic. For the charged fixed point \( \eta_A = 4 - d \) and we obtain once more that \( \nu' = \nu \) (remember: \( m = \xi^{-1} \) and \( m_A = \lambda^{-1} \)). For the \( XY \) fixed point, on the other hand, \( \eta_A = 0 \) and we obtain from Eq. (46) the scaling relation for the superconducting \( XY \) universality class:
\[ \nu' = \frac{\nu(d-2)}{2}. \]  
(47)
A further interesting consequence of the scaling relation (46) is the following. The vector potential mass is given in terms of the superfluid density $\rho_s$ as $m_A^2 = e^2 \rho_s$. Eq. (46) implies the scaling $e^2 \sim m_A^{d/2}$. Near the charged fixed point, therefore, from Eq. (46) we obtain

$$
\rho_s \sim m^{d-2} \sim |t|^{\nu(d-2)},
$$

(48)

where $t$ is the reduced temperature. Eq. (48) is just the Josephson relation and we have thus shown that it is also valid near the charged fixed point. In his original paper, Josephson has obtained the above relation for the superfluid in the form $\rho_s \sim t^{2\beta-\nu\eta}$. Then he derived the result (48) by assuming that hyperscaling holds, that is, $d\nu = 2 - \alpha$, which together with the scaling relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \nu(2-\eta)$ imply $2\beta - \nu\eta = \nu(d-2)$. We have obtained the result (48) without using these supplementary scaling relations. Our result follows from the exact evolution equation for the vector potential mass, Eq. (45). The form $\rho_s \sim t^{2\beta - \nu\eta}$ can be proven without any reference to the charge and is therefore valid also for the superconductor. From this statement and Eq. (48) we prove

$$
2\beta - \nu\eta = \nu(d-2)
$$

(49)

for the superconductor.

Experiments in high quality crystals of $YBa_2Cu_3O_{7-\delta}$ (YBCO) performed at zero field verify very well Eq. (47) for the $d = 3$ case. Most experiments are unable to probe the charged critical region. However, in experiments involving critical dynamics the situation is not clear. In this case we have again that different scaling relations are obtained near the charged fixed point. In general the AC conductivity scales as $\sigma(\omega) \sim e^2 \rho_s/(-i\omega)$. Since $\rho_s \sim \xi^{2-d}, e^2 \sim \xi^{-\eta_A}$, and $\omega \sim \xi^{-z}$, where $z$ is the dynamical critical exponent, we derive the scaling relation

$$
\sigma(\omega) \sim \xi^{2-d+z-\eta_A} \sim |t|^{\nu(d-2-z-\eta_A)}.
$$

(50)

For the XY universality class where $\eta_A = 0$ we have $\sigma(\omega) \sim \xi^{2-d+z}$. Near the charged fixed point, however, we obtain $\sigma(\omega) \sim |t|^{\nu(2-z)}$.

### 3.2. Duality

Duality is a powerful tool in physics. It allows the mapping of a weak coupling problem on a strong-coupling one. In the context of statistical
physics, it maps the low temperature expansion into a high temperature expansion. In some cases, duality allows to obtain exact information on the physical system. The classical example is the two-dimensional Ising model, where the exact critical temperature was obtained\cite{31} before the exact solution appeared.\cite{32} The exact determination of the critical temperature was possible because the Ising model has the self-duality property, that is, the duality transformation has a fixed point. The self-duality property is also verified in other systems and for $d > 3$, like in the $Z_2$ lattice gauge theories. However, the discreteness of the gauge group makes these theories very similar from the point of view of duality to the two-dimensional Ising model. Self-duality is more difficult to be found in continuous gauge groups. The GL model, for example, has no such property, but as we shall see, it is nevertheless almost self-dual.

In this section we shall discuss the field theoretical approach to duality in the GL model. We shall show in detail how scaling works in a disorder field theory (DFT) for the superconducting phase transition. The DFT to be discussed here was proposed first by Kleinert nearly twenty years ago.\cite{11} This formulation allowed to demonstrate that a tricritical point exists in the phase diagram of the superconductor. The existence of this tricritical point allowed to build a consistent picture where the strong-coupling limit – which exhibits a second-order phase transition\cite{10} – and the weak coupling limit, with its weak first order scenario, coexist with the normal phase, meeting at the tricritical point. On the basis of the DFT, the estimated value of $\kappa$ at the tricritical point was $\kappa_t \approx 0.8/\sqrt{2}$\cite{11}. Early Monte Carlo simulation\cite{33} gives, on the other hand, the estimate $\kappa_t \approx 0.42/\sqrt{2}$. Remarkably, Kleinert’s estimate agree within 5 % with a recent, more precise, Monte Carlo simulation by Mo, Hove, and Sudbo.\cite{13}

The first scaling analysis of the DFT was made by Kiometzis, Kleinert, and Schakel.\cite{14} From the analysis in Ref.\cite{14} it was possible to establish the value of the critical exponent $\nu$ as having a $\text{XY}$ value, $\nu \simeq 0.67$. However, as we shall see, the scaling analysis of the DFT contains some ambiguities which are not yet completely solved.

### 3.2.1. Duality in the lattice Ginzburg-Landau model

A lattice version of the GL model has the Hamiltonian

$$
H = -\beta \sum_{i,\mu} \cos(\nabla_\mu \theta_i - eA_{i\mu}) + \frac{1}{2} \sum_i (\nabla \times A_i)^2,
$$  

(51)
where $\nabla_{\mu}$ is the lattice derivative, $\nabla_{\mu} f_i \equiv f_{i+\mu} - f_i$, and $\beta = 1/T$. The partition function is then given by

$$Z = \int_{-\pi}^{\pi} \left[ \prod_i \frac{d\theta_i}{2\pi} \right] \int_{-\infty}^{\infty} \left[ \prod_{i,\mu} dA_{i\mu} \right] \exp(-H).$$

(52)

The duality transformation can be done exactly when the Villain form of the Hamiltonian is used. The Villain approximation corresponds to the replacement

$$e^{\beta \cos x} \rightarrow \sum_{n=-\infty}^{\infty} e^{-\frac{2}{\beta}(x-2\pi n)^2},$$

(53)

which turns out to be very accurate near the critical region.

Assuming from now on the Villain approximation, we introduce an auxiliary integer field $m_{i\mu}$ such that

$$\sum_{\{n_{i\mu}\}} \exp \left[ -\frac{\beta}{2} \left( \nabla_{\mu} \theta_i - eA_{i\mu} - 2\pi n_{i\mu} \right)^2 \right]$$

$$\propto \sum_{\{m_{i\mu}\}} \exp \left[ -\frac{1}{2\beta} m_{i\mu}^2 + i(\nabla_{\mu} \theta_i - eA_{i\mu})m_{i\mu} \right].$$

(54)

The proportionality factor above is not important in the following. All such proportionality factors will be neglected in the foregoing manipulations. They correspond to smooth factors in the temperature. Eq. (54) was obtained using the identity

$$\sum_{m=-\infty}^{\infty} e^{-(t/2)m^2+ixm} = \sqrt{\frac{2\pi}{t}} \sum_{n=-\infty}^{\infty} e^{(-1/2t)(x-2\pi n)^2}.$$  

(55)

The summation notation in Eq. (54) with a $\{n_{i\mu}\}$ means a multiple summation, analogous to multiple integration.

By integrating out the angular variables $\theta_i$ we obtain the partition function:
Field Theoretical Approaches to the Superconducting Phase Transition

Z = \int_{-\infty}^{\infty} \left[ \prod_{i,\mu} dA_{i\mu} \right] \sum_{\{m_i\}} \delta_{\nabla \cdot m_i, 0} \exp \left\{ \sum_i \left[ -\frac{1}{2\beta} m_i^2 + ie A_i \cdot m_i - \frac{e^2}{2} h_i^2 + 2\pi i m_i \cdot h_i \right] \right\}.

The Kronecker delta constraint \( \nabla \cdot m_i = 0 \) generated by the \( \theta_i \) integrations implies that the links variables \( m_{i\mu} \) form closed loops. These closed loops are interpreted as magnetic vortices. When \( e = 0 \) the vector potential decouples and we have, up to proportionality factor, the partition function for the XY model in terms of link variables:

\[ Z_{XY} = \sum_{\{m_i\}} \delta_{\nabla \cdot m_i, 0} \exp \left( -\frac{1}{2\beta} \sum_i m_i^2 \right). \]

We can solve the constraint on \( m_i \) by introducing a new integer link variable through \( m_i = \nabla \times l_i \). After integrating out \( A_i \) and using the Poisson formula

\[ \sum_{n=-\infty}^{\infty} F(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dx F(x) e^{2\pi i m x} \]

to go from the integer variables \( l_i \) to continuum variables \( h_i \), we obtain

\[ Z = \int_{-\infty}^{\infty} \left[ \prod_{i,\mu} dh_{i\mu} \right] \sum_{\{m_i\}} \delta_{\nabla \cdot m_i, 0} \exp \left\{ \sum_i \left[ -\frac{1}{2\beta} (\nabla \times h_i)^2 - \frac{e^2}{2} h_i^2 + 2\pi i m_i \cdot h_i \right] \right\}. \]

Eq. (59) corresponds to the dually transformed lattice GL model.

By performing the \( h_i \) integration in Eq. (59) we obtain

\[ Z = \sum_{\{m_i\}} \delta_{\nabla \cdot m_i, 0} \exp \left[ -2\pi^2 \beta \sum_{i,j,\mu} m_{i\mu} G(r_i - r_j) m_{j\mu} \right], \]

where the Green function \( G \) has the following behavior at large distances:

\[ G(r_i - r_j) \sim \frac{e^{-\sqrt{\beta e |r_i - r_j|}}}{4\pi |r_i - r_j|}. \]
Thus, in the superconductor the magnetic vortex loops interact with a screened long range interaction.

Let us consider now the “frozen” superconductor limit \( T \to 0 \) of the dual representation. The “frozen” superconductor corresponds to the zero temperature limit, \( T \to 0 \). In this case, after integrating out \( h_i \), we obtain

\[
Z_{\text{frozen}} = \sum_{\{m_i\}} \delta \nabla \cdot m_i \exp \left( -\frac{2\pi^2}{e^2} \sum_i m_i^2 \right). \tag{62}
\]

Eq. (62) has the same form as Eq. (57). Thus, the “frozen” superconductor is the same as a \( XY \) model provided we identify

\[
e^2 = \frac{4\pi^2}{T_c}. \tag{63}
\]

The above result allows us to localize a critical point on the \( e^2 \)-axis in the phase diagram in the \( e^2 - T \)-plane. Using Eq. (63) and the fact that \( T_c \approx 3 \) for the \( XY \) model in the Villain approximation, we obtain \( e_c^2 = 4\pi^2 / T_c \approx 13.159 \) on the \( e^2 \)-axis. Thus, we can locate two limiting critical points in the phase diagram, since we have in the \( T \)-axis the Villain-\( XY \) critical point at \( T_c \approx 3 \).

The vortex-vortex interaction in Eq. (60) is singular at short distances. Therefore, it is natural to introduce a vortex core term with energy \( \epsilon_0 \) in the dual lattice Hamiltonian:

\[
H_{\text{dual}} = \sum_i \left[ \frac{1}{2\beta} (\nabla \times h_i)^2 + \frac{e^2}{2} h_i^2 - 2\pi i m_i \cdot h_i + \frac{\epsilon_0}{2} m_i^2 \right]. \tag{64}
\]

Using Eq. (55) and the integral representation of the Kronecker delta

\[
\delta \nabla \cdot m_{i,0} = \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} e^{i\theta_i (\nabla \cdot m_i)}, \tag{65}
\]

we obtain

\[
H'_{\text{dual}} = \sum_i \left[ \frac{1}{2\beta} (\nabla \times h_i)^2 + \frac{e^2}{2} h_i^2 \right] + \sum_{i,\mu} \frac{1}{2\epsilon_c} (\nabla_\mu \theta_i - 2\pi i n_{i\mu} - 2\pi i h_{i\mu})^2. \tag{66}
\]
The Hamiltonian in Eq. (66) has the same form as the lattice GL Hamiltonian in the Villain approximation, except that in Eq. (66) the vector field is massive. Thus, we see that there is almost a self-duality between them. When \( e = 0 \) the Hamiltonian (66) is the dual of the (Villain) lattice \( XY \) Hamiltonian. Note that in this duality transformation a locally gauge invariant model is mapped on a globally invariant one.

Let us set \( e = 0 \) in Eq. (64) and look for the phase diagram in the \( \beta - \epsilon_0 \) plane. In such a phase diagram the point \((\beta_c, 0)\) corresponds to the \( XY \) critical point. Integrating out \( h_i \) we obtain the partition function:

\[
Z'|_{e=0} = \sum_{\{m_i\}} \delta \nabla \cdot m_i, 0 \exp \left[ -2\pi^2 \beta \sum_{i,j,\mu} m_{i\mu} \bar{G}(r_i - r_j)m_{j\mu} - \frac{\epsilon_0}{2} m_i^2 \right],
\]

(67)

where \( \bar{G} \) behaves at large distances as

\[
\bar{G}(r_i - r_j) \sim \frac{1}{4\pi |r_i - r_j|}.
\]

(68)

From Eqs. (67) and (68) we see that the point \((0, 1/2\beta_c)\) in the \( \beta - \epsilon_0 \) plane corresponds to an “inverted” \( XY \) (IXY) transition. By performing the \( A_i \) integration in Eq. (56) we obtain

\[
Z = \sum_{\{m_i\}} \delta \nabla \cdot m_i, 0 \exp \left[ -\frac{e^2}{2} \sum_{i,j,\mu} m_{i\mu} \bar{G}(r_i - r_j)m_{j\mu} - \frac{1}{2\beta} m_i^2 \right],
\]

(69)

and we see that the IXY critical point corresponds to \( e_c^2 = 4\pi^2 \beta_c \).

Clearly the IXY transition is a second-order phase transition. Note that this transition arises in a lattice GL model where the amplitude fluctuations are frozen (London limit). Therefore, we should not expect to find a first-order phase transition in this case. This London limit is appropriate when magnetic fluctuations are relevant in the type II regime. In Fig. 4 we the approximate phase diagram.

3.2.2. The disorder field theory

The IXY universality class must have the same thermodynamic exponents as the \( XY \) model. For instance, we expect \( \nu \approx 0.67 \) and, from the scaling analysis of Section 3.1, \( \nu' = \nu \). This last scaling relation has been confirmed
in Monte Carlo simulations of the lattice model \(^{51,12}\). Although the thermodynamic exponents are the same as in the XY universality class, critical exponents as \(\eta\) and \(\eta_A\) are not the same. We have seen in Section 3.1 that \(\eta_A = 4 - d\) near the charged fixed point. The \(d = 3\) value, \(\eta_A = 1\), corresponds to the IXY universality class discussed in the preceding subsection. The XY universality class, on the other hand, has \(\eta_A = 0\). Also, we have \(-1 < \eta < 0\) in the IXY universality class, while \(\eta > 0\) in the XY one.

Useful information from different universality classes and crossovers in superconductors can be obtained from the DFT. The DFT is constructed out the dual lattice Hamiltonians discussed in this section. The \textit{bare} DFT associated to the lattice Hamiltonian \(^{10}\) is given by \(^{13,14}\)

\[
\mathcal{H}_{\text{DFT}} = \frac{1}{2} (\nabla \times \mathbf{h}_0)^2 + \frac{m_{A,0}^2}{2} \mathbf{h}_0^2 + |(\nabla - i \tilde{e}_0 \mathbf{h}_0) \phi_0|^2 + \tilde{m}_0^2 |\phi_0|^2 + \frac{\tilde{u}_0}{2} |\phi_0|^4, \tag{70}
\]

where the bare dual charge \(\tilde{e}_0 = 2\pi m_{A,0}/e_0\). \(m_{A,0}\) is the bare mass of the vector potential in the original theory. We see that the disordered phase of the DFT corresponds to the ordered phase of the GL model. This is a general feature of all duality transformations: the low temperature phase is mapped in the high temperature phase. This justifies the denomination “disorder field theory” for the continuum limit of the lattice dual model. The field \(\phi_0\) is the bare \textit{disorder parameter field}. In the superconducting phase \(\langle \phi_0 \rangle = 0\), while the \textit{order parameter} in the original GL model \(\langle \psi_0 \rangle \neq 0\). Conversely, the normal phase corresponds to \(\langle \phi_0 \rangle \neq 0\) and \(\langle \psi_0 \rangle = 0\).

It should be noted that the Hamiltonian \(^{70}\) is a generalization of the
London model. The field $h_0$ is in fact the magnetic induction, while $|\phi_0|^2$ gives the vortex loop density.

The renormalization of the Hamiltonian (70) is similar to the one of the GL model, up to the following subtlety. From the Ward identities we obtain that the mass term for the vector field is not renormalized, that is, $m_{A,0}^2 h_0^2 / 2 = m_A^2 h^2 / 2$, where the absence of the zeroes subindices indicate renormalized quantities. Since the renormalized induction field is given by $h = Z_h^{-1/2} h_0$, we obtain

$$m_A^2 = Z_h m_{A,0}^2. \quad (71)$$

The dual charge renormalizes in a similar way:

$$\tilde{\epsilon}^2 = Z_h \epsilon_0^2. \quad (72)$$

Since $\tilde{\epsilon}_0^2 = 4\pi^2 m_{A,0}^2 / \epsilon_0^2$, it follows from Eqs. (71) and (72) that the Cooper pair charge $e_0$ is not renormalized in the DFT, $\tilde{e} = e_0$.

It is important to remark that the vector potential mass renormalization in the DFT involves only one renormalization constant, while the same is not true in the GL model.18

Due to the presence of a massive vector field, the DFT has an ambiguous scaling.23 This ambiguity was a matter of debate recently.37,38,39 Let’s see how it works. In Ref. 14 the scaling chosen was in principle very natural, with the bare masses behaving in the same way with mean field exponents: $\tilde{m}_0^2 \sim |t|$ and $m_{A,0}^2 \sim |t|$. Renormalization is employed as usual. We have,

$$\tilde{m}_0^2 = Z_{\tilde{m}} Z_0^{-1} \tilde{m}_0^2, \quad (73)$$

and therefore,

$$\tilde{m} \frac{\partial \tilde{m}_0^2}{\partial m} = (2 + \gamma_{\tilde{m}}) \tilde{m}_0^2, \quad (74)$$

where $\gamma_{\tilde{m}}$ is defined in a way similar to $\gamma_m$ in Eq. 28. Since $\tilde{m}_0^2 / m_{A,0}^2 = \text{const}$, we obtain

$$\tilde{m} \frac{\partial m_{A,0}^2}{\partial m} = (2 + \gamma_{\tilde{m}}) m_{A,0}^2. \quad (75)$$
Let us define the dimensionless renormalized dual coupling by \( \tilde{f} = \tilde{e}^2 / \tilde{m} \). From Eqs. (72) and (75), we obtain the \( \beta \)-function:

\[
\beta_{\tilde{f}} \equiv \tilde{m} \frac{\partial \tilde{f}}{\partial \tilde{m}} = (\gamma_h + \gamma_{\tilde{m}} + 1) \tilde{f},
\]

(76)

where

\[
\gamma_h \equiv \tilde{m} \frac{\partial \ln Z_h}{\partial \tilde{m}}.
\]

(77)

Now, we can easily obtain the following bound

\[
\gamma_{\tilde{m}} + 1 \leq \frac{1}{\nu} - 1 \leq 1.
\]

(78)

It is also straightforward to show that \( \gamma_h \geq 0 \). Therefore, the infrared stable fixed point to (76) is at \( \tilde{f}^* = 0 \). This implies that the critical exponent \( \nu \) has a \( XY \) value and that \( \eta_h \equiv \gamma_{\tilde{h}}^* = 0 \).

From Eqs. (71) and (75) we obtain

\[
\tilde{m} \frac{\partial \tilde{m}^2}{\partial \tilde{m}} = (\gamma_h + \gamma_{\tilde{m}} + 2) \tilde{m}^2.
\]

(79)

Near the fixed point the above equation becomes

\[
\tilde{m} \frac{\partial \tilde{m}^2}{\partial \tilde{m}} \approx \frac{1}{\nu} \tilde{m}^2_{A},
\]

(80)

which implies the scaling

\[
m^2_{A} \sim \tilde{m}^{1/\nu}.
\]

(81)

Since \( \tilde{m} \sim |t|^\nu \) the above scaling implies that the penetration depth exponent is given exactly by \( \nu' = 1/2 \).

Another possible scaling is the one considered by Herbut where it is assumed that \( m^2_{A,0} = \text{const} \). Within this scaling we obtain instead Eq. (76) the following \( \beta \)-function:

\[
\beta_{\tilde{f}} = (\gamma_h - 1) \tilde{f},
\]

(82)
which is similar to the \( \beta \)-function of the coupling \( f \) in the GL model at \( d = 3 \). From Eq. (71) we obtain

\[
\tilde{m} \frac{\partial m_A^2}{\partial \tilde{m}} = \gamma h m_A^2.
\] (83)

The \( \beta \)-function for the coupling \( \tilde{g} = \tilde{u}/\tilde{m} \) contains functions of the ratio \( \tilde{m}/m_A \) multiplying every power of \( \tilde{f} \).\footnote{Due to the evolution equation (83), we see that \( m_A^2 \sim \tilde{m}^{\gamma} \sim \tilde{m} \). Therefore, \( \tilde{m}/m_A \to 0 \) as the critical point is approached. Thus, fixed point \( \tilde{g}_* \) is the same as in the XY model and once more the critical exponent \( \nu \) has a XY value.\footnote{However, from the exact scaling behavior \( m_A^2 \sim \tilde{m} \) we see that the penetration depth exponent is given by \( \nu' = \nu/2 \), which corresponds to the same value as in the 3D XY superconducting universality class. Therefore, the scaling considered in Ref.\footnote{It was shown in Ref.\footnote{It seems that if we want that the dual model implies the result \( \nu' = \nu \), we have to make this assumption.}\footnote{An alternative scenario for the scaling behavior in the DFT is the following. We have shown that the XY model dualizes in a GL model. The GL model, on the other hand, dualizes on the DFT whose Hamiltonian is given in Eq. (71). Now, the dual of the dual must be of course the original model. This means that the GL model should dualize in a XY model and therefore the DFT Hamiltonian should be equivalent to it. On the basis of this argument we should expect a scaling consistent with the XY universality class, instead of IXY. If we accept this argument, we are led to the conclusion that the correct scaling behavior should assume \( m_{A,0}^2 = \text{const} \) as in Ref.\footnote{to obtain the XY scaling of the penetration depth, \( \nu' = \nu/2 \).}}.\footnote{}}\footnote{In the superconducting phase transition only the exponents \( \nu, \nu' \), and \( \alpha \) are measured. Here \( \alpha \) is the specific heat exponent, which is related to \( \nu \) by the hyperscaling relation \( d \nu = 2 - \alpha \). At present the critical exponent \( \eta \) is not measured and we can even doubt of its physical significance. We can argue that the superconducting order parameter cannot be considered to be a physical measurable quantity because its conjugate field has no physical meaning. In a ferromagnet the field conjugate to the magnetization}}}

\section{4. The physical meaning of the critical exponent \( \eta \)}

In the superconducting phase transition only the exponents \( \nu, \nu' \), and \( \alpha \) are measured. Here \( \alpha \) is the specific heat exponent, which is related to \( \nu \) by the hyperscaling relation \( d \nu = 2 - \alpha \). At present the critical exponent \( \eta \) is not measured and we can even doubt of its physical significance. We can argue that the superconducting order parameter cannot be considered to be a physical measurable quantity because its conjugate field has no physical meaning. In a ferromagnet the field conjugate to the magnetization
is just the external magnetic field, which can be controlled by experiments. Another problem is that the order parameter $\langle \psi \rangle$ is not gauge invariant. Thus, a calculation of $\langle \psi(r) \psi^\dagger(r') \rangle$ will depend on the gauge choice and, as a consequence, $\eta$ will also be gauge dependent.

In this Section we shall show that it is possible to define a gauge-independent $\eta$ exponent and discuss its physical significance. The physical meaning of $\eta$ arises due to a special feature which at first glance looks very much like a pathology: it has a negative sign. Indeed, we would expect from very general non-perturbative arguments that $\eta$ should be positive. In the case of a pure $|\psi|^4$ theory we can prove that $\eta \geq 0$ in the following way. In momentum space the correlation function $G(r - r') \equiv \langle \psi(r) \psi^\dagger(r') \rangle$ has the spectral representation:

$$\hat{G}(p) = \int_0^\infty d\mu \frac{\rho(\mu)}{p^2 + \mu^2}, \quad (84)$$

where the spectral weight $\rho(\mu)$ satisfies the sum rule

$$\int_0^\infty d\mu \rho(\mu) = 1. \quad (85)$$

The above representation is well known in quantum field theory and is called K"allen-Lehmann spectral representation.\[45 Now, because of the condition \[85 on the spectral weight, we have the inequality

$$\hat{G}(p) \leq \frac{1}{p^2}. \quad (86)$$

If one assumes the low momentum behavior at the critical point $\hat{G}(p) \sim 1/|p|^{2-\eta}$, we obtain from the inequality \[86 that $\eta \geq 0$.

In the case of the GL model, all calculations of $\eta$ give a negative value in the interval $-1 < \eta < 0$ for $d = 3$. $94, 15, 16, 18, 48, 46, 47, 18$ In general it is argued that since $\hat{G}(p) \sim 1/|p|^{2-\eta}$, we have in real space the large distance behavior at the critical point:

$$G(r - r') \sim \frac{1}{|r - r'|^{d-2+\eta}}. \quad (87)$$

The above will not diverge as $|r - r'| \to \infty$ provided $\eta > 2 - d$ and for this reason we could have in principle a negative $\eta$ exponent. Such an argument is certainly not correct in the case of pure $|\psi|^4$ theory where the
Källen-Lehmann representation holds which implies $\eta \geq 0$. In the case of the GL model the situation is much more subtle and the Källen-Lehmann representation does not apply, at least not in the above form [28,49].

In order to give a physical interpretation to the negative sign of $\eta$ in superconductors, let us consider a one-loop approximation at the critical point and fixed dimensionality $d = 3$. The calculation is uncontrolled but serves to illustrate the main point. Assuming $N = 2$, the vector potential propagator is then given by

$$D_{\mu\nu}(p) = \frac{1}{p^2 + e^2|p|/16} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right),$$

while the order parameter two-point correlation function is

$$G(p) = \frac{1}{p^2 - e^2|p|/4}.$$  

Now, we see from Eq. (88) that as $|p| \to 0$ the second term in the denominator dominates implying $\eta_A = 1$. However, the same argument does not apply to Eq. (89) because the second term in the denominator has a negative sign in front of it and therefore the $p^2$ term is still relevant. Thus, there is a momentum space instability in the problem similar to the one encountered in theories of magnetic systems exhibiting a Lifshitz point [50].

There the Hamiltonian already contains the momentum space instability from the very beginning due to the presence of higher order derivatives. Due to this, the susceptibility in those magnetic systems has a maximum at a nonzero value of $p$. This lead to the appearance of a modulated regime in the phase diagram, which is plotted in the $P - T$-plane, where $P$ is the ratio between two competing interactions. It was conjectured in Ref. [28] that a similar behavior occurs in the superconductor. The modulated regime would be associated to the type II behavior, which would be in this way analogous to the helical phase in magnetic systems exhibiting a Lifshitz point. In the case of magnetic systems, the Lifshitz point is the point in the phase diagram where the paramagnetic, ferromagnetic, and helical phases coexist. In the case of the superconductor it corresponds to the point where the type I, type II, and normal phases coexist. In straight analogy with magnetic systems, the phase diagram is plotted in the $\kappa^2 - T$-plane. Note that in the case of the GL model the Lifshitz point-like behavior would be generated by thermal fluctuations.

Further insight in this problem can be obtained by looking at the propagator in the $1/N$ expansion already discussed in this review. The self-energy
at the critical point and $O(1/N)$ is given by

$$\Sigma(p) = \frac{40}{\pi^2 N} p^2 \ln \left( \frac{|p|}{N e^2} \right). \quad (90)$$

Thus, besides the pole at $p = 0$ we have also a pole at

$$|p_0| = Ne^2 \exp \left( -\frac{\pi^2 N}{40} \right). \quad (91)$$

This instability is similar to the one leading to chiral symmetry breaking in three-dimensional QED (QED3). The difference is that in QED3 the instability occurs with respect to the mass, which is dynamically generated by spontaneous chiral symmetry breaking. Thus, if $M$ is the generated fermion mass, the pole of the propagator occurs at

$$M = Ne^2 \exp \left( -\frac{\pi^2 N}{8} \right). \quad (92)$$

We have not yet discussed how to cure the disease which results from the lack of gauge invariance of the order parameter correlation function $G(r - r')$. Our physical interpretation of the critical exponent $\eta$ given above has no value if $\eta$ is a gauge-dependent quantity. The best thing to do is to define a gauge-invariant correlation function for the order parameter. The choice of such a gauge-invariant correlation function is not unique but as we shall show, there is one whose value of $\eta$ coincides with the one which is obtained by computing it in the Coulomb gauge, which is the gauge we are using in this review.

A popular gauge-invariant correlation function which is often used in the literature is

$$G(r - r') = \langle \psi(r) \exp \left[ -ie \int_{r'} \, dr'' \cdot A(r'') \right] \psi^\dagger(r') \rangle. \quad (93)$$

A calculation of $\eta$ using the above correlation function was carried out recently by Kleinert and Schakel. They calculated this exponent using both the $\epsilon$-expansion and the $1/N$-expansion. In the former case the result is

$$\eta = \frac{36}{Ne^2}. \quad (94)$$
while in the latter case $\eta$ was computed for arbitrary dimensionality $d \in (2, 4)$ and up to order $1/N$,

$$\eta = -\frac{1}{N} \frac{(d^2 + 2d - 6)\Gamma(d-2)}{\Gamma(2-d/2)\Gamma^2(d/2-1)\Gamma(d/2)}.$$  \(\text{(95)}\)

By setting $d = 4 - \epsilon$ in Eq. (95) and expanding to order $\epsilon$ we obtain Eq. (94). In a GL Hamiltonian with a gauge-fixing term

$$H_{gf} = \frac{1}{2\alpha} (\nabla \cdot \mathbf{A})^2,$$  \(\text{(96)}\)

the above expressions of $\eta$ corresponds to values that would have been obtained by fixing the gauge $\alpha = -3$ in the $\epsilon$-expansion case and $\alpha = 1 - d$ in the case of the $1/N$-expansion. Thus, in each case the value of $\eta$ does not agree with the one that is obtained by fixing the Coulomb gauge, which corresponds to $\alpha = 0$. Note, however, that the above gauge-independent results both confirm that $\eta$ is indeed negative.

A different point of view discussed in Ref. 53 focus instead in the flow of the gauge-fixing parameter $\alpha$. From the Ward identities it follows that the gauge-fixing parameter renormalizes as

$$\alpha = Z^{-1}_A \alpha_0,$$  \(\text{(97)}\)

which implies the $\beta$-function

$$\beta_\alpha = -\gamma_A \alpha.$$  \(\text{(98)}\)

Since at the charge fixed point we have $\gamma_A(f^*, g^*) = 4 - d$, the only way to get a fixed point to Eq. (98) when $d \in (2, 4)$ is to set $\alpha = 0$. Due to the negative sign in Eq. (98), the flow is unstable for arbitrary nonzero $\alpha$. Thus, it is clear that stable charged fixed points can be obtained only if the Coulomb gauge corresponds to the fixed point of the theory in $2 < d < 4$. For $d = 4$, which is the case of interest for particle physicists, any value of $\alpha$ can be chosen, since in this case $\gamma_A(f^*, g^*) = 0$. In Fig. 5 we show a schematic flow diagram in the $f - \alpha$-plane for the case where $d \in (2, 4)$. Based on this scenario, we are led to conclude that the above gauge-independent results for $\eta$ do not correspond to the infrared stable fixed point. This means that the correlation function (93) is not a good choice of gauge-invariant correlation function giving a gauge-independent value of the $\eta$ exponent. In fact, the correlation function (93) even fails to
lead to long-range order: It can be rigorously shown that in the dimensions of interest the correlation function always decay to zero. A better gauge-invariant correlation function is

\[ G(r - r') = \langle \psi(r) \exp \left[ -ie \int d^d r'' A(r'') \cdot b(r'') \right] \psi^\dagger(r') \rangle, \quad (99) \]

where

\[ b(r'') = \nabla V(r'' - r) - \nabla V(r'' - r'), \quad (100) \]

with

\[-\nabla^2 V(r) = \delta^d(r). \quad (101)\]

Since the above correlation function is gauge-invariant, any gauge can be fixed to calculate it. The end result will be always gauge-independent. It is easy to see that in the Coulomb gauge

\[ G(r - r')|_{\text{Coulomb}} = \langle \psi(r)\psi^\dagger(r') \rangle|_{\alpha=0}. \quad (102) \]

Therefore in such a scenario the \( \eta \) exponent corresponds precisely to the one we have calculated previously in the Coulomb gauge. Furthermore, it can be shown that the correlation function exhibits long-range order in contrast to the one given in Eq. (93).
5. Renormalization group calculation at fixed dimension
   and below $T_c$

In the $\epsilon$-expansion RG approach the same RG functions are obtained no
matter the calculation is carried out below or above $T_c$. As dictated by the
Ward identities the singular behavior is exactly the same above and below
$T_c$. However, if the calculation is done in fixed dimension $d = 3$ and below
$T_c$ the situation is different and the RG functions depend explicitly on the
Ginzburg parameter $\kappa$. We shall not give the details of this approach here.
Instead, we shall concentrate on the physical aspects of this new approach
which allows to obtain a charge fixed point at one-loop order. The interested
reader is referred to Ref. 18 for the technical details.

The approach we are going to discuss is not really perturbative. Actually,
only the powers of $f$ are being effectively counted and the powers of $g$
are counted only partially. Thus, by one-loop we mean first-order in $f$. The
point is that $\kappa$ arises in the calculations in two different ways: as the ratio
between the masses $\kappa = m/m_A$ and as the ratio between coupling constants,
$\kappa^2 = g/2f$. The coupling $g$, when it appears, is eliminated in favor of $\kappa$
and the RG flow is in this way parametrized in terms of $\kappa$ and $f$. This way of
working is of course more complicated than more usual RG approaches but
it has the advantage of being physically more appealing due to the explicit
presence of $\kappa$ in the RG functions. In the classical Abrikosov solution of
the GL model in an external magnetic field $\kappa$ appears explicitly and the
existence of two types of superconductivity is made evident. For instance,
the slope of the magnetization curve near $H_{c2}$ is given by

$$\frac{dM}{dH} = \frac{1}{4\pi\beta_A(2\kappa^2 - 1)},$$

(103)

where $\beta_A$ is the Abrikosov parameter. The above expression is singular at
$\kappa = 1/\sqrt{2}$, which corresponds to the point separating type I from type II
superconductivity. Such a singular behavior at $\kappa = 1/\sqrt{2}$ should be also
visible in the GL model with a thermally fluctuating vector potential. The
new approach introduced in Ref. 18 makes this feature explicit in a RG
context. As we shall see, this aspect of this new approach is crucial to
obtain the charged fixed point at $d = 3$ and $N = 2$.

The only RG function that is singular at $\kappa = 1/\sqrt{2}$ is $\gamma_A$:

$$\gamma_A = \frac{\sqrt{2}C(\kappa)f}{24\pi(2\kappa^2 - 1)^3},$$

(104)
where

\[ C(\kappa) = 4\kappa^6 + 10\kappa^4 - 24\sqrt{2}\kappa^3 + 27\kappa^2 + 4\sqrt{2}\kappa - 1/2. \]  \hspace{1cm} (105)

The $\beta$-function for $\kappa^2$ is given by

\[ \beta_{\kappa^2} = (2\gamma_\pi - \gamma_A - \zeta_\pi)\kappa^2, \]  \hspace{1cm} (106)

where

\[ \gamma_\pi = \frac{\kappa f}{12\pi} \frac{2\kappa^2 + \sqrt{2}\kappa - 8}{\sqrt{2}\kappa + 1}, \]  \hspace{1cm} (107)

\[ \zeta_\pi = -\frac{\sqrt{2}}{4\pi} f \left( \frac{3\kappa^2}{2} + \frac{1}{\sqrt{2}\kappa} \right). \]  \hspace{1cm} (108)

As before, the charged fixed point at $d = 3$ is determined by the condition $\gamma_A(f_*, \kappa_*) = 1$. This leads to the fixed points

\[ f_* \approx 0.3, \quad \kappa_* \approx 1.17/\sqrt{2}. \]  \hspace{1cm} (109)

Note that $\kappa_*$ is slightly above the value $1/\sqrt{2}$ and therefore the charged fixed point occurs in the type II regime.

The reason why a charged fixed point is obtained in the above analysis is similar to the reason why the $1/N$-expansion leads to a charged fixed point already at order $1/N$: The fixed point coupling $f_*$ is small enough such that a $f_*^2$-term is strongly suppressed in the other RG functions. It is a large $f^2$-term in $\beta_g$ that spoils at $N = 2$ the charged fixed point in the HLM theory. In order to explain why this new method is so successful, let us define an effective coupling $\bar{f}$ by

\[ \gamma_A(\bar{f}, \kappa) = 1. \]  \hspace{1cm} (110)

The above equation defines a critical line in the sense that the $\beta_f$ vanishes on this line. Note, however, that $\beta_{\kappa^2}$ does not vanish in general. From Eq. \hspace{1cm} (110) we obtain

\[ \bar{f}(\kappa) = \frac{24\pi(2\kappa^2 - 1)^3}{\sqrt{2}C(\kappa)}. \]  \hspace{1cm} (111)
From the above equation we see that $\tilde{f}(\kappa)$ becomes very small when $\kappa$ approaches $1/\sqrt{2}$ from the right. Precisely at $\kappa = 1/\sqrt{2}$ we have $\tilde{f} = 0$. This behavior suggests that the best approximation scheme should be one where the small parameter is given by $\Delta \kappa \equiv \kappa - 1/\sqrt{2}$. Now it is easy to see that the $\epsilon$-expansion based RG fails because it effectively expands around $\kappa = 0$ and therefore it corresponds to the deep type I regime where the transition is clearly first-order. Furthermore, $C(\kappa)$ vanishes at $\kappa = 0.096/\sqrt{2}$ and $\tilde{f}$ becomes very large as this value of $\kappa$ is approached from the left. Thus, a perturbation expansion around $\kappa = 0$ breaks down at $\kappa = 0.096/\sqrt{2}$. There is an “infinite barrier” separating the deep type I from the type II regime. In the interval $0.096/\sqrt{2} < \kappa < 1/\sqrt{2}$ the effective coupling $\tilde{f}$ is negative and thus unphysical. The coupling $\tilde{f}$ can be really small only for $\kappa > 1/\sqrt{2}$, i.e., in the type II regime.

Note that our one-loop approximation gives only one charged fixed point. The tricritical fixed point is absent in this approximation. This behavior also occurs in the $1/N$-expansion where only one charged fixed point is found. A higher order calculation is necessary to obtain the tricritical fixed point. At two loops the singular behavior in $\kappa$ is expected to change. Thus, instead finding a singularity at $\kappa = 1/\sqrt{2}$, which is the same as in the mean-field solution, we expect to find a singular behavior at $\kappa_t \approx 0.8/\sqrt{2}$, in agreement with Refs. 11 and 13.

6. Concluding remarks

In this paper we have reviewed several modern field-theoretic approaches in the superconducting phase transition. We have emphasized some special topics which are not extensively discussed in the literature. In particular, the scaling behavior of the continuum dual model was analysed in more detail than in the original publications. The duality scenario is physically and conceptually very important, but its scaling behavior is not yet fully understood. Another topic that deserves further attention is the recently conjectured Lifshitz point-like behavior in the GL model. Such a scenario provides an interesting possibility to understand physically the negative value of the critical exponent $\eta$.

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