ON THE ZILBER-PINK CONJECTURE FOR COMPLEX ABELIAN VARIETIES

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Abstract. In this article, we prove that the Zilber-Pink conjecture for abelian varieties over an arbitrary field of characteristic 0 is implied by the same statement for abelian varieties over the algebraic numbers.

More precisely, the conjecture holds for subvarieties of dimension at most \( m \) in the abelian variety \( A \) if it holds for subvarieties of dimension at most \( m \) in the largest abelian subvariety of \( A \) that is isomorphic to an abelian variety defined over \( \bar{\mathbb{Q}} \).

1. Introduction

For us, varieties and curves are irreducible and subvarieties are always irreducible and closed in the ambient variety. Fields are always of characteristic 0.

Let \( A \) be an abelian variety defined over an algebraically closed field \( K \). A special subvariety of \( A \) is an irreducible component of an algebraic subgroup of \( A \) or, equivalently, a translate of an abelian subvariety by a torsion point. Arbitrary translates of abelian subvarieties are called cosets or weakly special subvarieties. Special subvarieties are also called torsion cosets.

The Manin-Mumford conjecture, proved by Raynaud [Ray83], states that a subvariety of an abelian variety contains at most finitely many maximal special subvarieties. In particular, a non-special curve contains at most finitely many torsion points.

On the other hand, given a curve in an abelian variety, a dimension count suggests that it should not intersect a special subvariety of codimension at least 2. If one considers the union of all special subvarieties of codimension at least 2 and intersects it with a curve that is not contained in a proper special subvariety, one expects the intersection to be finite.

The pioneering work [BMZ99] of Bombieri, Masser and Zannier was one of the first to study this kind of problems and to pass from considering torsion points in subvarieties of algebraic groups to points lying in algebraic subgroups of appropriate codimension.

Indeed, Bombieri, Masser and Zannier proved that, given a curve defined over the algebraic numbers and contained in \( G_m^n \) but not in any of its proper (not necessarily torsion) cosets, it contains at most finitely many points that lie in an algebraic subgroup of \( G_m^n \) of codimension at least 2. The condition of not being contained in a proper coset was replaced by the necessary one of not lying in a proper torsion coset by Maurin [Mau08] and independently by Bombieri, Habegger, Masser and Zannier in [BHMZ10].

In the same paper [BMZ99], Bombieri, Masser and Zannier suggest that a possible analogue of their result for curves in \( G_m^n \) could hold for (families of) abelian varieties and...
that one could consider higher dimensional subvarieties and intersect them with algebraic subgroups of higher codimension.

A couple of years later, Zilber [Zil02] independently stated a conjecture for semiabelian varieties of which the result of Bombieri, Masser and Zannier is a consequence. This is formulated in a slightly different language and we are going to state it later. Similar conjectures for $G_m^n$ were formulated by Bombieri, Masser and Zannier in [BMZ07].

We now consider an apparently weaker formulation of the same principle due to Pink. We introduce the following notation: For a non-negative integer $k$, we denote by $A[k]$ the union of all special subvarieties of $A$ of codimension at least $k$.

Pink conjectured in [Pin05] that, if $V \cap A^{[\dim V+1]}$ is Zariski dense in $V$ for a subvariety $V$ of $A$, then $V$ is contained in a proper special subvariety of $A$. The conjecture in its full generality is still open. If $V$ is a curve and $K = \overline{\mathbb{Q}}$, it has been proven by Habegger and Pila in [HP16]. Previously, partial results have been obtained by Viada [Via03, Via08], Rémond and Viada [RV03], Ratatzzi [Rat08], Carrizosa [Car08, Car09] in combination with Rémond [Rém03, Rém07, Rém09], and Galateau [Gal10]. If $V$ is a hypersurface, Pink’s conjecture follows from the Manin-Mumford conjecture. If the dimension and codimension of $V$ are at least 2, then all known results place additional restrictions on $V$ or $A$, see for instance [CVV14, CV14] and [HV19].

In this article, we use a recent result of Gao in [Gao18a], which generalizes work by Rémond in [Rém09], to reduce Zilber’s conjecture to the case where everything is defined over $\overline{\mathbb{Q}}$. We even show that it can be reduced to Pink’s formulation of the conjecture over $\overline{\mathbb{Q}}$. Furthermore, we prove the full conjecture in Corollary 1.7 if no abelian variety of dimension greater than 4 that is defined over $\overline{\mathbb{Q}}$ embeds into $A$. For example, the conjecture holds in a power of an elliptic curve with transcendental $j$-invariant. Combining our result with Theorem 1.1 of [HP16] yields the following theorem.

**Theorem 1.1.** Let $A$ be an abelian variety defined over an algebraically closed field $K$ (of characteristic 0) and let $V \subset A$ be a curve. Then $V \cap A^{[2]}$ is finite unless $V$ is contained in a proper algebraic subgroup of $A$.

As mentioned before, Pink’s conjecture is implied by the following Conjecture 1.2 on unlikely or atypical intersections that was formulated by Zilber in [Zil02] for semiabelian varieties. An overview of the topic of unlikely intersections is given in the book [Zan12].

In order to state Conjecture 1.2, we introduce the notion of an atypical subvariety: Let $A$ be an abelian variety defined over an algebraically closed field $K$ and let $V$ be a subvariety of $A$. A subvariety $W$ of $V$ is called atypical (for $V$ in $A$) if $W$ is an irreducible component of the intersection of $V$ with a special subvariety of codimension at least $\dim V - \dim W + 1$. It is called maximal if it is not contained in any larger atypical subvariety.

**Conjecture 1.2.** Let $K$ be an algebraically closed field. Let $A$ be an abelian variety defined over $K$ and let $V$ be a subvariety of $A$. Then $V$ contains at most finitely many maximal atypical subvarieties.

If $V$ is a curve, then Conjecture 1.2 and Pink’s conjecture are obviously equivalent.
It turns out that another equivalent formulation of Conjecture 1.2 is more suited to our proof strategy. In order to state it, we have to introduce the notions of defect and optimality of a subvariety.

**Definition 1.3.** If $V$ is a subvariety of $A$, then there is a smallest special subvariety $\langle V \rangle$ containing $V$. We define the defect $\delta(V)$ of $V$ to be $\dim(\langle V \rangle) - \dim V$. A subvariety $W$ of $V$ is called optimal for $V$ in $A$ if $\delta(U) > \delta(W)$ for every subvariety $U$ with $W \subseteq U \subset V$.

Pink introduced the notion of defect in [Pin05], while the concept of optimality was introduced in [HP16] by Habegger and Pila. The latter is motivated by Poizat’s notion of $cd$-maximality in [Poi01]. $cd$-maximality is the toric analogue of the notion of geodesic optimality that we will introduce later. Using the concept of optimality, Habegger and Pila formulated the following conjecture, which is equivalent to Conjecture 1.2 by Lemma 2.7 in [HP16].

**Conjecture 1.4.** Let $K$ be an algebraically closed field and let $d$ be a non-negative integer. Let $A$ be an abelian variety defined over $K$ and let $V$ be a subvariety of $A$. Then $V$ contains at most finitely many optimal subvarieties of defect at most $d$.

In the statement of our results, we use the trace of an abelian variety with respect to a field extension of algebraically closed fields. This can be thought of as the largest abelian subvariety defined over the smaller field. See Definition 2.3 for a formal definition.

**Theorem 1.5.** Let $K$ be an algebraically closed field, let $m$ be a non-negative integer and $A$ an abelian variety defined over $K$ with $K/\overline{\mathbb{Q}}$-trace $(T, \text{Tr})$. Then, if Conjecture 1.4 holds for some non-negative integer $d$ and subvarieties of dimension at most $m$ in $T$ (over the field $\overline{\mathbb{Q}}$), it holds for the same $d$ and subvarieties of dimension at most $m$ in $A$ (over $K$).

Note that Habegger and Pila have shown in Corollary 9.10 of [HP16] that Conjecture 1.4 can be further reduced to the existence of sufficiently strong lower bounds for the size of the Galois orbits of optimal singletons over a field of definition that is finitely generated over $\mathbb{Q}$.

An analogue of Theorem 1.5 for powers of the multiplicative group was proved in [BMZ08] by Bombieri, Masser and Zannier. Note that in this case the ambient algebraic group is always defined over $\overline{\mathbb{Q}}$. In our situation, this corresponds to the special case where $A$ is isomorphic to the base change of an abelian variety over $\overline{\mathbb{Q}}$.

Following a suggestion of Habegger, we prove Conjecture 1.4 in Theorem 6.1 if $K = \overline{\mathbb{Q}}$ and $d = 1$. This together with the preceding Theorem 1.5 and Theorem 1.1 in [HP16] implies the following corollary.

**Corollary 1.6.** Let $K$ be an algebraically closed field, let $m$ be a non-negative integer and $A$ an abelian variety defined over $K$. Then, Conjecture 1.4 holds for subvarieties of dimension at most $m$ in $A$ if either $m \leq 1$ or $d \leq 1$.

Note that Conjecture 1.4 trivially holds for subvarieties of $A$ of dimension or codimension 0. In codimension 1, every proper optimal subvariety is special, so Conjecture 1.4 follows from the theorem of Raynaud in [Ray83] (Manin-Mumford conjecture). By Corollary 1.6
Conjecture 1.4 also holds for subvarieties of codimension 2 since every proper optimal subvariety of a subvariety of codimension 2 has defect at most 1; for the toric analogue see [BMZ07] (there proven over $\bar{\mathbb{Q}}$, then extended to $\mathbb{C}$ in [BMZ08]). Conjecture 1.4 has previously been proven for $K = \bar{\mathbb{Q}}$ and subvarieties of codimension 2 in powers of elliptic curves with CM (in [CVV14]) and without CM (in [HV19]) as well as in arbitrary products of elliptic curves with CM (in [CV14]).

In particular, Conjecture 1.4 holds for abelian varieties of dimension at most 4 and by applying Theorem 1.5 we obtain the following corollary.

**Corollary 1.7.** Let $K$ be an algebraically closed field and $A$ an abelian variety defined over $K$ with $K/\bar{\mathbb{Q}}$-trace $(T, \text{Tr})$. If $\dim T \leq 4$, then Conjecture 1.4 holds for $A$.

As mentioned above, we show that Conjecture 1.4 over an arbitrary algebraically closed field $K$ can be reduced to Pink’s formulation of the conjecture over $\bar{\mathbb{Q}}$. For the precise statement of our result, we now give a more articulated version of Pink’s conjecture. Recall that we denote by $A^{[k]}$ the union of all special subvarieties of an abelian variety $A$ of codimension at least $k$.

**Conjecture 1.8.** Let $K$ be an algebraically closed field and let $d$ be a non-negative integer. Let $A$ be an abelian variety defined over $K$ and let $V$ be a subvariety of $A$. If $V \cap A^{[\max\{\dim V + 1, \dim A - d\}]}$ is Zariski dense in $V$, then $V$ is contained in a proper special subvariety of $A$.

**Theorem 1.9.** Let $K$ be an algebraically closed field, let $m$ be a non-negative integer and $A$ an abelian variety defined over $K$. Then, if Conjecture 1.8 holds for some non-negative integer $d$ and subvarieties of dimension at most $m$ in every abelian subvariety $B$ of $A$, Conjecture 1.4 holds for the same $d$ and subvarieties of dimension at most $m$ in $A$.

We prove Theorem 1.9 in Section 5. The proof is a direct application of Theorem 9.8(i) in [HP16]. As pointed out by Ulmo and Zannier, the analogous reduction can be done in the toric case by imposing additional multiplicative relations on the positive-dimensional atypical intersections. Combining Theorem 1.9 and Theorem 1.5 yields the following corollary.

**Corollary 1.10.** Let $K$ be an algebraically closed field, let $m$ be a non-negative integer and $A$ an abelian variety defined over $K$ with $K/\bar{\mathbb{Q}}$-trace $(T, \text{Tr})$. Then, if Conjecture 1.8 holds for some non-negative integer $d$ and subvarieties of dimension at most $m$ in every abelian subvariety $T'$ of $T$ (over the field $\bar{\mathbb{Q}}$), Conjecture 1.4 holds for the same $d$ and subvarieties of dimension at most $m$ in $A$ (over $K$).

In the proof of Theorem 1.5 we use a double induction firstly on the dimension of $A$ and secondly on the transcendence degree of its field of definition. If the transcendence degree of the field of definition is minimal in the sense that $A$ is obtained as a geometric fiber of a certain universal family $\mathbb{A}_{g,l} \to A_{g,l}$ of abelian varieties, then we apply Gao’s result to reduce to abelian varieties of smaller dimension in Proposition 3.2.

We then use Rémond’s results to increase the transcendence degree of the field of definition in Proposition 4.1. This part of the proof at some points resembles the proof of the main result in [BMZ08], albeit formulated rather differently.
The proofs of both propositions begin with the use of the fact that optimal subvarieties are geodesic-optimal, i.e., optimal with respect to the geodesic defect that is the analogue of the defect if one replaces special by weakly special subvarieties. For abelian varieties, this has been proved by Habegger and Pila. It turns out that their proof can be adapted to show that the same holds if one considers a slightly different definition of the geodesic defect, where one replaces weakly special subvarieties by translates of abelian subvarieties by a torsion point plus a \( \mathbb{Q} \)-point of the trace. We call it the \( \mathbb{Q} \)-geodesic defect and \( \mathbb{Q} \)-geodesic-optimal subvarieties are then the analogue of geodesic-optimal subvarieties for this defect.

To any \( (\mathbb{Q} \text{-}) \)geodesic-optimal subvariety, there is an associated abelian subvariety. Thanks to the results of Gao and Rémond, this abelian subvariety lies in a finite set. If its dimension is positive, we can quotient out by it and use the inductive hypothesis. Otherwise, we either use the full strength of Gao’s result to reduce to the trace or we use the inductive hypothesis on the transcendence degree of the field of definition.

2. Preliminaries

In this section, we collect some results that are going to be useful in the proof of Theorem

2.1. Definitions and a useful lemma.

**Definition 2.1.** Let \( V \) be a variety over a field \( K \) and let \( K \subset L \) be a field extension. Then \( V_L = V \times_K L \) is called the base change of \( V \) to \( L \). We use analogous notation for the base change of morphisms between varieties.

**Lemma 2.2.** Let \( K \subset L \) be an extension of algebraically closed fields such that \( L \) has transcendence degree 1 over \( K \). Let \( V \) be a variety over \( K \) and let \( W \) be a subvariety of \( V_L \). Then there exists a subvariety \( W' \) of \( V \) with \( \dim W' \leq \dim W + 1 \) such that \( W \subset W'_L \).

**Proof.** We can replace \( L \) by a finitely generated subextension of \( K \) of transcendence degree 1 over which \( W \) is defined. We can then find a curve \( C \) over \( K \) such that \( K(C) = L \). After maybe replacing \( C \) by a Zariski open subset, we can find a subvariety \( W \subset V \times_K C \) of dimension \( \dim W + 1 \) such that the generic fiber of \( W \) over \( C \) is \( W \). The closure of the projection of \( W \) onto \( V \) is our \( W' \). \( \square \)

**Definition 2.3.** ([Con06], Theorem 6.2) Let \( K \subset L \) be an extension of algebraically closed fields. Let \( A \) be an abelian variety defined over \( L \). The \( L/K \)-trace of \( A \) is a pair \( (T, \text{Tr}) \) of an abelian variety \( T \) that is defined over \( K \) and a homomorphism of algebraic groups \( \text{Tr} : T_L \to A \) that is characterized uniquely by the fact that for every abelian variety \( T' \) that is defined over \( K \) and every homomorphism of algebraic groups \( \sigma : T'_L \to A \), there is a homomorphism of algebraic groups \( \tau : T' \to T \) such that \( \sigma = \text{Tr} \circ \tau_L \). The map \( \text{Tr} \) is a closed embedding.

**Definition 2.4.** Let \( K \) be an algebraically closed field and let \( A \) be an abelian variety over \( K \). Let \( m \) and \( d \) be non-negative integers. We say that \( \text{ZP}(A, m, d) \) holds if Conjecture \( \text{L4} \) holds for \( d \) and for all subvarieties \( V \) of \( A \) with \( \dim V \leq m \).
2.2. Smoothness and optimality under homomorphisms.

**Lemma 2.5.** Let $K$ be an algebraically closed field and let $f : V \to W$ be a dominant morphism of algebraic varieties, defined over $K$. Then there exists $V_0 \subset V$ open and Zariski dense such that $f(V_0)$ is open and Zariski dense in $W$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth.

**Proof.** By Corollary II.8.16 of [Har77], we can find $V_1 \subset V$ open, Zariski dense and nonsingular. By Theorem 10.19 of [GW10], $f(V_1)$ contains $V_2$ open and Zariski dense in $f(V) = W$. By generic smoothness (Corollary III.10.7 in [Har77]), we can find $V_3 \subset V_2$ open and Zariski dense in $V_2$ and hence in $W$ such that $V_0 = f|_{V_1}(V_3)$ is open and Zariski dense in $V$ and $f|_{V_0} : V_0 \to V_3 = f(V_0)$ is smooth. □

**Lemma 2.6.** Let $K$ be an algebraically closed field. Let $f : A \to A'$ be a homomorphism of algebraic groups between abelian varieties defined over $K$. Then the following hold:

1. Let $V$ be a subvariety of $A$. Suppose that $V_0 \subset V$ is open and Zariski dense in $V$ such that $f(V_0)$ is open and Zariski dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth. Let $W$ be a subvariety of $V$ that is optimal for $V$ in $A$ and intersects $V_0$. If $\langle W \rangle$ contains a translate of a component of ker $f$, then $f(W)$ is optimal for $f(V)$ in $A'$ of defect at most $\delta(W)$.

2. If $f$ has finite kernel and $ZP(A', m, d)$ holds, then $ZP(A, m, d)$ holds.

**Proof.** For (1), let $W$ be optimal for $V$ in $A$ such that $W \cap V_0 \neq \emptyset$, where $f|_{V_0} : V_0 \to f(V_0)$ is smooth of relative dimension $n$. Suppose that there exists $U$ such that $f(W) \subset U \subset f(V)$ and $\delta(U) \leq \delta(f(W))$. Since $f(\langle W \rangle)$ contains $\langle f(W) \rangle$ and $\langle W \rangle$ contains a translate of a component of ker $f$, we have

$$\delta(U) \leq \delta(f(W)) \leq \dim f(\langle W \rangle) - \dim f(W) = \dim(W) - \dim \ker f - \dim f(W).$$

Let now $U'$ be an irreducible component of $f^{-1}(U) \cap V = f|_{V_0}^{-1}(U)$ that contains $W$. We have $U' \cap V_0 \neq \emptyset$ since it contains $W \cap V_0$. Furthermore, $U' \cap V_0$ is an irreducible component of $f|_{V_0}^{-1}(U) \cap f(V_0))$. Since $f|_{V_0} : V_0 \to f(V_0)$ is smooth of relative dimension $n$, it follows that $\dim U' = \dim(U' \cap V_0) = \dim(U \cap f(V_0)) + n = \dim U + n$. An analogous reasoning with $W$ and $f(W)$ in place of $U'$ and $U$ shows that $\dim W \leq \dim f(W) + n$. Therefore,

$$\delta(U) \leq \delta(f(W)) \leq \dim(W) - \dim \ker f - \dim W + n.$$

After noting that $n \leq \dim \ker f$, we deduce that the defect of $f(W)$ is at most the defect of $W$.

Since $\dim U \geq \dim f(W)$, we have $W \subset U'$ and so $\delta(U') > \delta(W)$ by the optimality of $W$. Moreover, we certainly have $\dim(U') \leq \dim(U) + \dim \ker f$, so it follows that $\delta(U') \leq \delta(U) + \dim \ker f - n$. Therefore,

$$\delta(U) \leq \delta(W) - \dim \ker f + n < \delta(U') - \dim \ker f + n \leq \delta(U),$$

a contradiction.

We can now prove (2) by induction on $m$. For the base step of the induction, note that $ZP(A, 0, d)$ holds trivially for all $d$. Let now $V$ be a subvariety of $A$ of dimension $m$ and
let $W$ be an optimal subvariety of defect at most $d$ for $V$ in $A$. By Lemma 2.5, we can find $V_0 \subset V$ open and Zariski dense such that $f(V_0)$ is open and Zariski dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth.

If $W \subset V \setminus V_0$, then $W$ is contained in one of the finitely many components of $V \setminus V_0$. The dimension of that component is at most $m - 1$ and $W$ is of course optimal of defect at most $d$ for that component in $A$, so we are done by induction. Otherwise, we have $W \cap V_0 \neq \emptyset$. Since $\ker f$ is finite, a translate of one of its components is trivially contained in $(W)$, so we can apply (1) to find that $f(W)$ is optimal of defect at most $d$ for $f(V)$ in $A'$. As $f$ has finite kernel, we have $\dim f(V) = \dim V$, so it follows from $ZP(A', m, d)$ that $f(W)$ belongs to a finite set of varieties. The same follows for $W$ since $W$ is an irreducible component of $f^{-1}(f(W))$ because of the equality $\dim W = \dim f(W) = \dim f^{-1}(f(W))$. □

2.3. Abelian schemes. We denote by $A_{g,l}$ the moduli space of principally polarized abelian varieties of dimension $g$ with symplectic level $l$-structure over $\overline{Q}$. For $l \geq 3$, this is a fine moduli space and we denote by $\mathfrak{A}_{g,l}$ the corresponding universal family over $A_{g,l}$.

Lemma 2.7. Let $K$ be an algebraically closed field. Let $A$ be an abelian variety of dimension $g$ defined over $K$. Then there exists a subvariety $S \subset A_{g,l}$ with the following property: Let $A = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$ and let $A'$ be the generic fiber of $A \to S$. There exists a field embedding $\overline{Q}(S) \hookrightarrow K$ such that $A$ is isogenous to $A'_{K}$.

Proof. Over $K$, the abelian variety $A$ is isogenous to a principally polarized abelian variety by Corollary 1 on p. 234 of [Mum70] and so we can assume without loss of generality that $A$ is itself principally polarized. We can find a field $K_0 \subset K$ and an abelian variety $B$ defined over $K_0$ such that $\overline{Q} \subset K_0$, $K_0$ is finitely generated over $\overline{Q}$ and $A = B_{K}$. We fix a natural number $l \geq 3$. Without loss of generality, we can assume that all torsion points of $B$ of order $l$ are $K_0$-rational. We can find a normal variety $V$, defined over $\overline{Q}$, with $\overline{Q}(V) = K_0$. By spreading out (see Theorem 3.2.1 and Table 1 on pp. 306–307 of [Poo17]), we find an abelian scheme $B \to V$ with generic fiber $B$. The principal polarization of $B$ gives a principal polarization of $B \to V$ by the argument on p. 6 of [FC90]. The $l$-torsion points of $B$ extend to $l$-torsion sections of $B \to V$. By choosing among these sections a symplectic basis with respect to the Weil pairing, $B \to V$ becomes a principally polarized abelian scheme with symplectic level $l$-structure as defined in the Appendix to Chapter 7 of [MFK94].

Since $A_{g,l}$ is a fine moduli space by the Appendix to Chapter 7 of [MFK94], we get a map $\iota : V \to A_{g,l}$, defined over $\overline{Q}$, such that $B$ is isomorphic to $\mathfrak{A}_{g,l} \times_{A_{g,l}} V$. Let $S$ be the Zariski closure of $\iota(V)$, let $A = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$ and let $A'$ be the generic fiber of $A \to S$. It follows from the universal property of the fiber product that $B$ is isomorphic to $A \times_{S} V$. The dominant morphism $\iota : V \to S$ yields a field embedding $\overline{Q}(S) \hookrightarrow K_0$. Passing to the generic fiber over $V$ shows that $B$ is isomorphic to $A'_{K_0}$. □

Definition 2.8. Let $S$ be a variety and $A \to S$ an abelian scheme, both defined over an algebraically closed field $K$. An irreducible subgroup scheme $B$ of $A$ is called an abelian subscheme if $B \to S$ is flat, proper and dominant.
Equivalently, an abelian subscheme is an irreducible closed subgroup scheme that is flat over $S$. An abelian subscheme $B$ is itself an abelian scheme over $S$: For each natural number $N$, the multiplication-by-$N$ morphism from $B$ to $B$ is dominant and proper, hence surjective. It follows that the geometric fibers of $B$ must be connected as desired.

**Lemma 2.9.** Let $S$ be a normal variety and $A \to S$ an abelian scheme, both defined over an algebraically closed field $K$. Let $\xi$ be the generic point of $S$. Suppose that every $l$-torsion point of $(A_\xi)_{K(S)}$ is $K(S)$-rational for some natural number $l \geq 3$, where $K(S)$ denotes a fixed algebraic closure of $K(S)$. Then every abelian subvariety $B$ of $(A_\xi)_{K(S)}$ is the geometric generic fiber of an abelian subscheme $B \subset A$.

**Proof.** Fix an abelian subvariety $B$ of $(A_\xi)_{K(S)}$. It is a consequence of Poincaré’s reducibility theorem that there exists an endomorphism $\psi$ of $(A_\xi)_{K(S)}$ such that $B$ is the irreducible component of $\ker(\psi)$ containing the neutral element. By Theorem 2.4 of [Sil92], every endomorphism of $(A_\xi)_{K(S)}$ is the base change of an endomorphism of $A_\xi$. It follows that every abelian subvariety of $(A_\xi)_{K(S)}$ is the base change of an abelian subvariety of $A_\xi$. We identify $\psi$ and $B$ with the corresponding endomorphism and abelian subvariety of $A_\xi$ respectively.

By Theorem 3.2.1(iii) of [Poo17] (cf. Théorème 8.8.2(i) in [Gro66]), the endomorphism $\psi$ spreads out to a morphism from $A_U = A \times_S U$ to itself for $U \subset S$ open and Zariski dense. This morphism has to be an endomorphism by Corollary 6.4 on p. 117 of [MFK94]. By Proposition 2.7 of Chapter I in [FC90], this endomorphism extends to an endomorphism $\Psi : A \to A$.

We get a closed subgroup scheme $\ker(\psi)$ of $(A_\xi)_{K(S)}$. Let $\ker(\psi)^0$ be the functor defined in Section 3 of Exposé VIB in [ABD+65]. We want to apply Corollaire 4.4 of Exposé VIB in [ABD+65] by verifying condition (ii).

Fix $s \in S$. The fiber $\ker(\psi)_s$ is an algebraic group over a field and hence smooth. Furthermore, we have $\dim A_s = \dim \ker(\psi)_s + \dim \Psi(A_s)$. Since $\Psi$ is proper, $\Psi(A)$ is a closed irreducible subscheme of $A$ and it follows from the Fiber Dimension Theorem (Lemma 14.109 in [GW10]) that $\dim \Psi(A_\xi) \geq \dim \Psi(A_\xi)$. Similarly, we have $\dim \ker(\psi)_s \geq \dim B$. But then it follows from $\dim \Psi(A_\xi) + \dim B = \dim A_\xi = \dim A_s$ that $\dim \ker(\psi)_s = \dim B$. This means that the function $s \mapsto \dim \ker(\psi)_s$ is constant on $S$.

Therefore, we can apply Corollaire 4.4 of Exposé VIB in [ABD+65] to find that $\ker(\psi)^0$ is represented by an open subgroup scheme of $\ker(\psi)$, which we also denote by $\ker(\psi)^0$. By the same Corollaire, $\ker(\psi)^0$ is smooth over $S$. By definition, the generic fiber of $\ker(\psi)^0$ is equal to $B$.

The number of geometrically irreducible components of the fibers of $\ker(\psi)$ is uniformly bounded. Therefore, for some large $N$, we have that $\ker(\psi)^0$ is equal to the image of $\ker(\psi)$ under the multiplication-by-$N$ morphism. As this morphism is proper, it follows that $\ker(\psi)^0$ is closed in $A$ and therefore the morphism $\ker(\psi)^0 \to S$ is proper. Since $\ker(\psi)^0$ is smooth over $S$, it is flat over $S$. As its generic fiber is irreducible and it is flat over $S$, $\ker(\psi)^0$ is irreducible as well. Since $\ker(\psi)^0$ is a subgroup scheme that dominates $S$, this shows that $\ker(\psi)^0$ is an abelian subscheme of $A$ with generic fiber equal to $B$ as desired. \qed
Lemma 2.9 does not hold if the base variety $S$ is allowed to be arbitrary. We provide a counterexample: Let $S \subseteq \mathbb{A}^1_\mathbb{K} \setminus \{0,1\} \times \mathbb{A}^1_\mathbb{K}$ be defined by the equation $(\lambda + 1)^2\lambda = \mu^2$ in the affine coordinates $(\lambda, \mu)$ on $\mathbb{A}^2_\mathbb{K}$. Let $\xi$ be the generic point of $S$. We consider two elliptic schemes $\mathcal{E}$ and $\mathcal{E}'$ over $S$ that are defined in $\mathbb{P}^2_{\mathbb{K}} \times_{\mathbb{K}} S$ by equations $y^2z = x(x-z)(x-\lambda z)$ and $\lambda y^2z' = x'(x'-z')(x'-\lambda z')$ in the projective coordinates $[x : y : z]$ and $[x' : y' : z']$ respectively. We set $A = \mathcal{E} \times_S \mathcal{E}'$ and let $B$ be the abelian subvariety of $A \subset \mathcal{E} \times_{\mathbb{K}} S$ defined by the equations $xz = x'z'$ and $yz = \frac{z}{x+y}z'$. If $p = (-1,0) \in S(K) \subset A^2_{\mathbb{K}}(K) = K^2$ is the singular point of $S$, then $B$ extends to an abelian subscheme of $A \times_S (S \setminus \{p\})$, but the fiber over $p$ of the Zariski closure of $B$ in $A$ has two irreducible components.

2.4. Defect and optimality. We now recall the notions of weakly special subvariety and of geodesic defect.

**Definition 2.10.** Let $A$ be an abelian variety defined over an algebraically closed field $K$. A weakly special subvariety of $A$ is a translate of an abelian subvariety (by any point). If $V$ is a subvariety of $A$, then $\langle V \rangle_{\text{geo}}$ is defined to be the smallest weakly special subvariety containing $V$. The difference $\dim(\langle V \rangle_{\text{geo}}) - \dim V$ is called the geodesic defect $\delta_{\text{geo}}(V)$ of $V$. A subvariety $W$ of $V$ is called geodesic-optimal for $V$ in $A$ if $\delta_{\text{geo}}(U) > \delta_{\text{geo}}(W)$ for every subvariety $U$ with $W \subset U \subset V$.

**Lemma 2.11.** Let $K \subset L$ be an extension of algebraically closed fields. Let $A$ be an abelian variety defined over $K$ and let $V$ be a subvariety of $A$. If $W$ is an optimal subvariety for $V_L$ in $A_L$, then there exists an optimal subvariety $W'$ for $V$ in $A$ such that $W = (W')_L$ and $\delta(W) = \delta(W')$.

**Proof.** Since $A$ is defined over $K$, which is algebraically closed, we have that any special subvariety of $A_L$ is the base change of a special subvariety of $A$. Therefore, if $V$ is a subvariety of $A$, any optimal subvariety of $V_L$ in $A_L$ is an irreducible component of an intersection $V_L \cap H_L$ for some special subvariety $H$ of $A$ and is then the base change of a subvariety $W \subset V$ that must also be optimal in $V$ of the same defect. \qed

We also introduce a new kind of defect.

**Definition 2.12.** Let $K \subset L$ be an extension of algebraically closed fields. Let $A$ be an abelian variety defined over $L$ with $L/K$-trace $(T, \text{Tr})$. For a subvariety $V$ of $A$ we define $\langle V \rangle_{K,\text{geo}}$ to be the smallest translate of an abelian subvariety of $A$ by a point in $\text{Tr}(T(K)) + A_{\text{tors}}$ that contains $V$. We call $K$-geodesic defect the difference $\delta_{K,\text{geo}}(V) = \dim(\langle V \rangle_{K,\text{geo}}) - \dim V$. If $W \subset V \subset A$ we say that $W$ is $K$-geodesic-optimal for $V$ in $A$ if $\delta_{K,\text{geo}}(U) > \delta_{K,\text{geo}}(W)$ for every subvariety $U$ with $W \subset U \subset V$.

Note that $A$ is isogenous to $T_L \times B$ for an abelian variety $B$ such that there exists no nontrivial homomorphism between $T_L$ and $B$. Therefore the intersection of two translates of abelian subvarieties of $A$ by points in $\text{Tr}(T(K)) + A_{\text{tors}}$ is again a finite union of translates of abelian subvarieties of $A$ by points in $\text{Tr}(T(K)) + A_{\text{tors}}$, so $\langle V \rangle_{K,\text{geo}}$ is well defined.

In [HPT16], Habegger and Pila defined the defect condition for subvarieties of complex abelian varieties. If $W$ and $V$ are subvarieties of $A$ such that $W \subset V$, then the condition
says that $\delta(V) - \delta_{\geo}(V) \leq \delta(W) - \delta_{\geo}(W)$. They then showed that the fact that this holds in this setting implies that optimal subvarieties are also geodesic-optimal. Here we do the same for $\delta_{K,\geo}$ in place of $\delta_{\geo}$ and show that optimal subvarieties are also $K$-geodesic-optimal.

**Lemma 2.13.** In the setting of Definition 2.12, let $W$ and $V$ be subvarieties of $A$ such that $W \subset V$. Then, the following hold:

1. We have $\delta(V) - \delta_{K,\geo}(V) \leq \delta(W) - \delta_{K,\geo}(W)$.
2. If $W \subset V$ is optimal for $V$ in $A$, then it is $K$-geodesic-optimal for $V$ in $A$.

*Proof.* The deduction of (2) from (1) is done analogously to the proof of Proposition 4.5 in [HP16].

It remains to prove (1): the defect condition in this case amounts to proving that

$$\dim\langle V \rangle - \dim\langle V \rangle_{K,\geo} \leq \dim\langle W \rangle - \dim\langle W \rangle_{K,\geo}.$$ 

For this, we can just copy the proof of Proposition 4.3(ii) of [HP16]. □

The geodesic defect of a subvariety of the connected mixed Shimura variety $(A_{g,l})_{C}$, which we define below, is linked to the $C$-geodesic defect of the components of its geometric generic fiber.

**Lemma 2.14.** Suppose that $U$ is a subvariety of $(A_{g,l})_{C}$ and $S$ is its image under the projection to $(A_{g,l})_{C}$. Let $U$ be an irreducible component of the fiber of $U$ over the geometric generic point of $S$. We define the geodesic defect $\delta_{\geo}(U)$ of $U$ to be $\dim \langle S \rangle_{\geo} - \dim S + \dim (U)_{C,\geo} - \dim U$, where $\langle S \rangle_{\geo}$ is the smallest weakly special subvariety of $(A_{g,l})_{C}$ that contains $S$. This definition of geodesic defect is independent of the choice of $U$ and agrees with $\delta_{ws}$ as in Definition 8.1(i) of [Gao18a].

*Proof.* The independence from the choice of $U$ follows from the fact that the irreducible components of the fiber of $U$ over the geometric generic point of $S$ form one orbit under the action of the Galois group Gal $(\overline{C}(S)/C(S))$ and $\langle \cdot \rangle_{C,\geo}$ commutes with the action of Gal $(\overline{C}(S)/C(S))$.

In Definition 8.1(i) of [Gao18a], the defect $\delta_{ws}(U)$ is defined as $\dim U^{biZar} - \dim U$, where $U^{biZar}$ is the smallest bi-algebraic subvariety of $(A_{g,l})_{C}$ containing $U$. By Proposition 5.3 of [Gao18b], this is equal to $\dim \langle S \rangle_{\geo} - \dim S + \dim W - \dim U$, where $W$ is the smallest generically special subvariety of sg type (as defined in Definition 1.5 of [Gao18b]) of $(A_{g,l})_{C} \times (A_{g,l})_{C}$ containing $U$.

It is enough to show that $\dim (U)_{C,\geo} - \dim U = \dim W - \dim U$. By looking at the generic fiber of $W$, we see that $\dim (U)_{C,\geo} - \dim U \leq \dim W - \dim U$. For the inequality in the other direction to hold, we need to know that after a finite surjective base change we can extend abelian subvarieties and torsion points of the geometric generic fiber and $C$-points of the trace to abelian subschemes, torsion sections and constant sections respectively. This is ensured by Lemma 2.9 – note that after a finite surjective base change we can assume the base to be normal. □
Definition 2.15. If \( \mathcal{W} \subset \mathcal{V} \) are subvarieties of \( (\mathfrak{A}_{g,l})_{\mathbb{C}} \), we say that \( \mathcal{W} \) is geodesic-optimal for \( \mathcal{V} \) in \( (\mathfrak{A}_{g,l})_{\mathbb{C}} \) if \( \delta_{\text{geo}}(\mathcal{U}) > \delta_{\text{geo}}(\mathcal{W}) \) for every subvariety \( \mathcal{U} \) with \( \mathcal{W} \subseteq \mathcal{U} \subset \mathcal{V} \).

3. A statement in the universal family

The following result is a fundamental tool for our proof. It relies on a result of Gao, which is formulated in the language of mixed Shimura varieties. We show that, in the special setting of an abelian variety that is the geometric generic fiber of a “subfamily” of the universal family, Gao’s result gives a strengthening of what is sometimes called a “Structure Theorem” proved by Rémond for abelian varieties in [Rém09]. The analogous statement for powers of the multiplicative group was proved by Poizat in Corollaire 3.7 of [Poi01] and independently by Bombieri, Masser and Zannier in [BMZ07].

Theorem 3.1. Let \( S \) be a subvariety of \( A_{g,l} \) and \( A = \mathfrak{A}_{g,l} \times_{A_{g,l}} S \). Let \( A \) be the geometric generic fiber of \( A \) and \( V \) a subvariety of \( A \) and let \( (T, \text{Tr}) \) be the \( K/\mathbb{Q} \)-trace of \( A \), where \( K \) is a fixed algebraic closure of \( \mathbb{Q}(S) \). There is a finite set of pairs \( (q_0, H) \), where \( q_0 \in A(K) \) is a torsion point and \( H \) is an abelian subvariety of \( A \), and a finite union \( Z \) of proper subvarieties of \( V \) such that for every optimal \( W \subset V \) one of the following holds:

1. \( W \) is contained in \( Z \), or
2. there exists some point \( t \in \text{Tr}(T(\mathbb{Q})) \) such that \( W \) is an irreducible component of \( (t + q_0 + H) \cap V \) and \( t + q_0 + H \subset \langle W \rangle \).

Proof. We want to apply a result of Gao (Theorem 8.2 of [Gao18a]). For this, we need to make a base change. Let \( K' \) be a fixed algebraic closure of the function field of \( S_{\mathbb{C}} \). Both \( K \) and \( \mathbb{C} \) embed into \( K' \) and the intersection of their images is \( \bar{\mathbb{Q}} \) since the transcendence degree of \( K'/\mathbb{C} \) is equal to the transcendence degree of \( K/\bar{\mathbb{Q}} \).

Let \( W \) be optimal for \( V \) in \( A \). We want to show that \( W_{K'} \) is optimal for \( V_{K'} \) in \( A_{K'} \). If \( U \) is an optimal subvariety for \( V_{K'} \) containing \( W_{K'} \) and satisfying \( \delta(U) \leq \delta(W_{K'}) \), then, by Lemma 2.11, \( U \) is the base change of a subvariety of \( V \) of the same defect that is optimal for \( V \) in \( A \). Because of the optimality of \( W \) for \( V \) in \( A \), that subvariety has to be equal to \( W \), so \( U = W_{K'} \).

By Lemma 2.13, \( W_{K'} \) is also \( \mathbb{C} \)-geodesic-optimal for \( V_{K'} \) in \( A_{K'} \).

Suppose first that \( W_{K'} \subset \sigma(V_{K'}) \) for some \( \sigma \in \text{Gal}(K'/\mathbb{C}(S_{\mathbb{C}})) \) with \( \sigma(V_{K'}) \neq V_{K'} \). Then \( W_{K'} \) is contained in \( V_{K'} \cap \sigma(V_{K'}) \subseteq V_{K'} \). Let \( Z_{\sigma} \subset A \) be maximal among all finite unions of subvarieties \( Z' \subset A \) with \( Z_{K'} \subset V_{K'} \cap \sigma(V_{K'}) \) (and equal to the Zariski closure of the union of all such \( Z' \)). Then \( Z_{\sigma} \) is a finite union of proper subvarieties of \( V \) and contains \( W \). We set \( Z = \bigcup_{\sigma(V_{K'}) \neq V_{K'}} Z_{\sigma} \) — the union is finite since \( \sigma(V_{K'}) \) varies in a finite set. We deduce that \( W \subset Z \), so (1) is satisfied.

From now on, we assume that \( W_{K'} \subset \sigma(V_{K'}) \) for some \( \sigma \in \text{Gal}(K'/\mathbb{C}(S_{\mathbb{C}})) \) only holds if \( \sigma(V_{K'}) = V_{K'} \) and we want to prove that (2) holds.

The subvarieties \( W_{K'} \) and \( V_{K'} \) are irreducible components of the base change of the geometric fiber of \( A_{\mathbb{C}} \). We define \( \mathcal{W} \) and \( \mathcal{V} \) to be the Zariski closures of these two subvarieties in \( A_{\mathbb{C}} \). Note that they are subvarieties of dimension \( \dim S + \dim W \) and \( \dim S + \dim V \) respectively and that they dominate \( S_{\mathbb{C}} \).
In Lemma 2.14, we defined the geodesic defect of subvarieties of \((\mathfrak{A}_{g,l})_\mathbb{C}\) and we have seen that it coincides with \(\delta_{\text{geo}}\) of \(\text{Gao18a}\). Let \(\mathcal{U} \subset \mathcal{V}\) be a geodesic-optimal subvariety for \(\mathcal{V}\) that contains \(\mathcal{W}\) and satisfies \(\delta_{\text{geo}}(\mathcal{U}) \leq \delta_{\text{geo}}(\mathcal{W})\). Using that \(\mathcal{W} \subset \mathcal{U}\), we find that there exists an irreducible component \(U\) of the geometric generic fiber of \(\mathcal{U}\) that contains \(W_{K'}\) and satisfies
\[
\delta_{\mathcal{C},\text{geo}}(U) = \delta_{\text{geo}}(\mathcal{U}) - \dim(S)_{\text{geo}} + \dim S \leq \delta_{\text{geo}}(\mathcal{W}) - \dim(S)_{\text{geo}} + \dim S = \delta_{\mathcal{C},\text{geo}}(W_{K'}). 
\]
Since \(\mathcal{U} \subset \mathcal{V}\), we can deduce that \(U \subset \sigma(V_{K'})\) for some \(\sigma \in \text{Gal}(K'/\mathbb{C}(S_\mathbb{C}))\). Since \(W_{K'} \subset U \subset \sigma(V_{K'})\), we must have \(\sigma(V_{K'}) = V_{K'}\) by our assumption from above.

Since \(W_{K'}\) is \(\mathcal{C}\)-geodesic-optimal for \(V_{K'}\) in \(A_{K'}\), it follows that \(W_{K'} = U\) and therefore \(\mathcal{W} = \mathcal{U}\). Hence, \(\mathcal{W}\) is geodesic-optimal for \(\mathcal{V}\) in the connected mixed Shimura variety \((\mathfrak{A}_{g,l})_\mathbb{C}\).

Let \(\mathcal{W}^{\text{biZar}}\) be the smallest bi-algebraic subvariety of \((\mathfrak{A}_{g,l})_\mathbb{C}\) that contains \(\mathcal{W}\). It is determined by a tuple \((Q, Y^+, N, \tilde{y})\), where \((Q, Y^+)\) is a connected mixed Shimura subdatum of \((\text{GSp}_{2g}, \mathbb{Q}^g, \mathbb{H}_g \times \mathbb{C}^g)\), \(N\) is a normal subgroup of \(Q^{\text{der}}\) and \(\tilde{y} \in Y^+\). Here \(\mathbb{H}_g\) denotes the Siegel upper half space. Thanks to Theorem 8.2 of \(\text{Gao18a}\), we know that the triple \((Q, Y^+, N)\) lies in a finite set that does not depend on \(\mathcal{W}\). By Proposition 5.3 of \(\text{Gao18b}\), \(\mathcal{W}^{\text{biZar}}\) is a generically special subvariety of \(\text{sg}\) type (as defined in Definition 1.5 of \(\text{Gao18b}\)) of \((\mathfrak{A}_{g,l})_\mathbb{C} \times (A_{g,l})_\mathbb{C} \xrightarrow{\pi} (\mathcal{W}^{\text{biZar}})\), where \(\pi : (\mathfrak{A}_{g,l})_\mathbb{C} \to (A_{g,l})_\mathbb{C}\) is the structural morphism, so up to finite surjective base change a translate of an abelian subvariety by a torsion section and a constant section.

Looking at the proof of Proposition 3.3 on p. 240 of \(\text{Gao17}\), we see that the abelian subscheme and the torsion section are uniquely determined by \((Q, Y^+, N)\). Note that \(\tilde{y}_{\mathcal{C}}\) in the proof of Proposition 3.3 of \(\text{Gao17}\) can be assumed fixed as \(\pi(\mathcal{W}) = S_\mathcal{C}\) is independent of \(\mathcal{W}\). After intersecting \(\mathcal{W}^{\text{biZar}}\) with \(A_\mathcal{C}\) and passing to the geometric generic fiber, we deduce that there exists a finite set of tuples \((q_0, H)\), where \(q_0 \in A(K)\) is a torsion point and \(H\) is an abelian subvariety of \(A\), such that there exists some point \(t \in \text{Tr}'(T'(\mathbb{C}))\) with \(W_{K'} \subset t + (q_0 + H)_{K'}\). Here, \((T', \text{Tr}')\) is the \(K'/\mathbb{C}\)-trace of \(A_{K'}\). Lemma 2.14 shows that we even have \((W_{K'})_{\mathcal{C},\text{geo}} = t + (q_0 + H)_{K'}\).

First of all, \(\text{Tr}' : T'_{K'} \to A_{K'}\) is the base change of a homomorphism \(T'_{K_\mathbb{C}} \to A_{K_\mathbb{C}}\) by Theorem 2.4 of \(\text{Sh92}\), because all torsion points of domain and codomain are \(K\)-C-rational. Hence, \((T', \text{Tr}')\) is equal to the base change of the \(K\mathbb{C}/\mathbb{C}\)-trace of \(A_{K_\mathbb{C}}\). Furthermore, this latter trace is equal to \((T_\mathbb{C}, \text{Tr}_{K_\mathbb{C}})\) by Theorem 6.8 of \(\text{Con06}\). It follows that \(T' = T_\mathbb{C}\) and \(\text{Tr}' = \text{Tr}_{K'}\).

Since it is \(\mathcal{C}\)-geodesic-optimal for \(V_{K'}\) in \(A_{K'}\), \(W_{K'}\) itself must be equal to an irreducible component of \((t + (q_0 + H)_{K'}) \cap W_{K'}\).

We now want to show that we can take \(t\) to be the image of the base change of a \(\mathbb{Q}\)-rational point of \(T\). Indeed, the image of any point in \(X = \text{Tr}^{-1}_{K'}(W_{K'} + (-q_0 + H)_{K'})\) that is the base change of a \(\mathbb{C}\)-rational point of \(T_\mathbb{C}\) can be chosen as \(t\). The subvariety \(X\) is equal to the base change of \(\text{Tr}^{-1}(W + (-q_0 + H)) \subset T_K\). On the other hand, one can see that \(X\) is equal to \(\text{Tr}^{-1}_{K'}(t + H_{K'})\). Since \(\text{Tr}\) is a homomorphism and every algebraic subgroup of \(T_{K'}\) is the base change of an algebraic subgroup of \(T\), this means that \(X\) is the base change of a union of translates of an abelian subvariety of \(T_\mathbb{C}\) by a point in \(T_\mathbb{C}(\mathbb{C})\).
Since $C \cap K = \bar{\mathbb{Q}}$, it follows from Corollaire 4.8.11 of [Gro65] that $X$ is equal to the base change of a union of algebraic subvarieties of $T$ and $t$ can be chosen as the image of the base change of a point of $T(\bar{\mathbb{Q}})$. If we denote this point also by $t$, we have that $W$ is an irreducible component of $(t + q_0 + H) \cap V$. Since $(t + q_0 + H)_{K'} = \langle W_{K'} \rangle_{\text{geo}} \subset \langle W \rangle_{K'}$, it also follows that $t + q_0 + H \subset (W)$.  

We now apply Theorem 3.1 to our problem.

**Proposition 3.2.** Let $S \subset A_{g,t}$ be a subvariety of positive dimension. Let $A = \mathbb{A}_{g,t} \times_{A_{g,t}} S$ and let $A$ be the geometric generic fiber of $A \to S$. If $\text{ZP}(B, m, d)$ holds for all quotients $B$ of $A$ by a positive-dimensional abelian subvariety, then $\text{ZP}(A, m, d)$ holds.

**Proof.** We induct on $m$. Clearly $\text{ZP}(A, 0, d)$ holds for all $d$.

Let $V$ be a subvariety of $A$ of dimension $m$ and let $W \subset V$ be an optimal subvariety of defect at most $d$. Let $(T, \text{Tr})$ denote the $K/\bar{\mathbb{Q}}$-trace of $A$, where $K$ is the algebraic closure of $\bar{\mathbb{Q}}(S)$. We have $\text{Tr}(T_K) \neq A$ since $\dim S > 0$.

We apply Theorem 3.1. If $W$ satisfies (1), then $W$ is contained in a component of $Z$ and optimal of defect at most $d$ for that component, so we are done by induction on $m$. If $W$ satisfies (2), then $W$ is an irreducible component of $(t + q_0 + H) \cap V$, where $t \in \text{Tr}(T(\bar{\mathbb{Q}}))$ and $(q_0, H)$ lies in a finite set of pairs of torsion points and abelian subvarieties of $A$ that does not depend on $W$. We can assume $H$ and $q_0$ fixed. We now quotient out by $H$. Let $f : A \to A/H$ be the corresponding morphism. We get a subvariety $f(V)$ of $A/H$ and a point $w$ such that $\{w\} = f(W)$. By Lemma 2.5, we can find $V_0 \subset V$ open and Zariski dense such that $f(V_0)$ is open and Zariski dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth of some relative dimension $n$.

If $W \subset V \setminus V_0$, then $W$ is contained in one of finitely many subvarieties of $V$ of dimension at most $m - 1$ (and of course optimal of defect at most $d$ for that subvariety in $A$), so we are done by induction. Hence we can assume that $W \cap V_0 \neq \emptyset$. Since $W \cap V_0$ is an irreducible component of $f|_{V_0}^{-1}(\{w\})$, it follows that $n = \dim(W \cap V_0) = \dim W$.

If $\dim H = 0$, then $W = \{t + q_0\}$. So the singleton $\{t\}$ is contained in $V' = \text{Tr}(T_K) \cap (-q_0 + V)$ and is optimal of defect at most $d$ for $V'$ in $\text{Tr}(T_K)$. Since $\text{Tr}(T_K) \neq A$, there is an isogeny between $\text{Tr}(T_K)$ and a quotient of $A$ by some positive-dimensional abelian subvariety. We can use our hypothesis and Lemma 2.6 (2) to deduce that $t$ and therefore $W$ belongs to a finite set. Hence, we can assume that $\dim H > 0$.

By Theorem 3.1, $\langle W \rangle$ contains a translate of a component of $\ker f = H$ since $t + q_0 + H \subset \langle W \rangle$. It follows from Lemma 2.6 (1) that $\{w\}$ is optimal of defect at most $d$ for $f(V)$ in $A/H$. Now we can use that $\text{ZP}(A/H, m, d)$ holds to deduce that $w$ and hence $W$ as an irreducible component of $f^{-1}(\{w\}) \cap V$ must lie in a finite set.  

4. **Proof of Theorem 1.5**

Fix non-negative integers $m$ and $d$. We argue by induction on the dimension of $A$. If the dimension of $A$ is at most 2, the statement follows from the Manin-Mumford conjecture. Let now $A$ be an abelian variety of dimension $> 2$ over an algebraically closed field $K$ and assume that Theorem 1.5 holds for the fixed $m$ and $d$ and all abelian varieties of smaller
dimension. In particular, it holds for all quotients of \( A \) by abelian subvarieties of positive dimension. Let \( V \) be a subvariety of \( A \) of dimension at most \( m \). We can find an algebraically closed subfield \( K_1 \) of \( K \) that has finite transcendence degree over \( \overline{\mathbb{Q}} \), an abelian variety \( A_1 \) defined over \( K_1 \) and a subvariety \( V_1 \) of \( A_1 \) such that \( A = (A_1)_K \) and \( V = (V_1)_K \). If \( W \) is any optimal subvariety of \( V \), it is equal to \((W_1)_K\) for an optimal subvariety \( W_1 \) of \( V_1 \) by Lemma \[2.11\]. Furthermore, the \( K/\overline{\mathbb{Q}} \)-trace of \( A \) is equal to the base change of the \( K_1/\overline{\mathbb{Q}} \)-trace of \( A_1 \) by Theorem 6.4(3) in [Con06]. Hence, we can assume without loss of generality that \( K = K_1 \), \( A = A_1 \) and \( V = V_1 \).

Applying Lemma \[2.7\] we find a subvariety \( S \) of \( A_{g,l} \) and an embedding of \( \overline{\mathbb{Q}}(S) \) into \( K \) such that \( A \) is isogenous to \( A'_{K} \), where \( A' \) is the generic fiber of \( A_{g,l} \times_{\mathfrak{A}_{g,l}} S \).

Moreover, the \( K/\overline{\mathbb{Q}} \)-traces of \( A \) and \( A'_{K} \) are isogenous. Therefore, by Lemma \[2.6(2)\], we only need to prove the statement of Theorem 1.5 for \( A'_{K} \).

The following proposition gives us the final reduction to the algebraic case or to what we proved in Proposition 3.2.

**Proposition 4.1.** Let \( K \subseteq L \) be an extension of algebraically closed fields such that \( L \) has finite transcendence degree over \( K \). Let \( A \) be an abelian variety defined over \( K \). If \( \text{ZP}(B, m, d) \) holds for all quotients \( B \) of \( A \) by an abelian subvariety, then \( \text{ZP}(A_L, m, d) \) holds.

**Proof.** Arguing by induction on the transcendence degree of \( L \) over \( K \), one can see that it is enough to prove our statement when \( L \) has transcendence degree 1 over \( K \).

We proceed by induction on \( m \). Clearly \( \text{ZP}(A_L, 0, d) \) holds for all \( d \), so, for some positive \( m \), we will deduce \( \text{ZP}(A_L, m, d) \) from \( \text{ZP}(A_L, m - 1, d) \) and \( \text{ZP}(B, m, d) \) for all quotients \( B \) of \( A \).

Let \( V \) be a subvariety of \( A_L \) of dimension \( m \). If \( V = V'_L \) for some \( V' \subset A \) then we are done by Lemma \[2.11\] and by our hypothesis. We will then assume that this is not the case.

Let \( V'_L \) be the smallest subvariety of \( A_L \) that is the base change of some \( V' \subset A \) and contains \( V \). This will have dimension \( m \) or \( m + 1 \) by Lemma \[2.2\] but the first case is not possible because it would imply that \( V = V'_L \).

Let \( W \subset V \) be an optimal subvariety of defect at most \( d \) for \( V \in A_L \). We can assume without loss of generality that \( W \neq V \).

We let \( W'_L \) be the smallest subvariety of \( A_L \) that is the base change of some \( W' \subset A \) and contains \( W \). By Lemma \[2.2\] we have either \( W = W'_L \) or \( \dim W'_L = \dim W + 1 \).

If \( W = W'_L \), then \( W \) is contained in \( Z'_L \subset V \) for \( Z' \subset A \) maximal among all finite unions of subvarieties \( Z'' \subset A \) with \( Z''_L \subset V \) (and equal to the Zariski closure of the union of all such \( Z'' \)). Since \( V \neq V'_L \), the dimension of \( Z'_L \) is at most \( m - 1 \). Of course, \( W \) is also optimal of defect at most \( d \) for the component of \( Z'_L \) that contains it and therefore lies in a finite set because \( \text{ZP}(A_L, m - 1, d) \) holds. We can therefore assume that \( W \subsetneq W'_L \) and so \( \dim W'_L = \dim W + 1 \).

Recall that, by Lemma \[2.11\] an optimal subvariety for \( V'_L \) in \( A_L \) is the base change of an optimal subvariety for \( V' \). Let \( U'_L \) be such an optimal subvariety for \( V'_L \) in \( A_L \) that contains
$W'_L$ and satisfies $\delta(U'_L) \leq \delta(W'_L)$. Note that $\langle W \rangle = \langle W'_L \rangle$ and $\langle V \rangle = \langle V'_L \rangle$ because, for instance, $V'_L \subset \langle V \rangle \cap V'_L$ by definition. It follows that $\delta(W'_L) = \delta(W) - 1$, so $\delta(U'_L) \leq d - 1$.

We claim that $U'_L \neq V'_L$. If not, we could deduce that $\delta(V'_L) = \delta(U'_L) \leq \delta(W'_L)$. It would then follow that $\delta(V) = \delta(V'_L) + 1 \leq \delta(W'_L) + 1 = \delta(W)$, which contradicts the optimality of $W \subseteq V$ for $V$.

We deduce that $U'_L \not\subseteq V'_L$ and hence $U'_L \cap V \not\subseteq V$, otherwise $U'_L \supset V$ would contradict the minimality of $V'_L$. Since $W \subset U'_L \cap V$ and $W$ is optimal of defect at most $d$ for a component of $U'_L \cap V$ in $A_L$, it suffices to show that $U'$ and therefore $U'_L$ belongs to a finite set and then we are done by the inductive hypothesis as $\dim(U'_L \cap V) < \dim V$.

It follows from the optimality of $U'$ for $V'$ in $A$ and Proposition 4.5 of [HP16] that $U'$ is also geodesic-optimal for $V'$ in $A$. We can apply the results of Rémond in [Rém09] (the connection to geodesic optimality is explained in Section 6 of [HP16]) to deduce that there exists a finite set of abelian subvarieties of $A$ such that for each geodesic-optimal $U$ for $V'$ in $A$ there exists $H$ in this finite set such that for any $u \in U(K)$ we have $\langle U \rangle_{\text{geo}} = u + H$ (and $U$ is an irreducible component of $(u + H) \cap V'$ since it is geodesic-optimal in $V'$).

Since $H$ varies in a finite set, we can assume it fixed and divide out by it. Let $f : A \to A/H$ be the corresponding morphism. We get a subvariety $f(V')$ of $A/H$ and a point $u' \in (A/H)(K)$ such that $\{u'\} = f(U')$.

By Lemma 2.6, we can find $V'_0 \subset V'$ open and Zariski dense such that $f(V'_0)$ is open and Zariski dense in $f(V')$ and $f|_{V'_0} : V'_0 \to f(V'_0)$ is smooth of relative dimension $n = \dim V'_0 - \dim f(V'_0) = \dim V' - \dim f(V')$. If $U'$ is contained in $V' \setminus V'_0$, then $U'$ is contained in one of finitely many subvarieties of $V'$ of dimension at most $m$ and of course $U'$ is optimal of defect at most $d - 1$ for that subvariety. By our hypothesis, there are then only finitely many possibilities for $U'$.

Hence we can assume that $U' \cap V'_0 \neq \emptyset$. Since $U'$ is an irreducible component of $(u + H) \cap V'$ (for some $u \in U'(K)$), $U' \cap V'_0$ is then an irreducible component of $(u + H) \cap V'_0 = f|_{V'_0}^{-1}(\{u'\})$. It follows that $n = \dim(U' \cap V'_0) = \dim U'$.

Since we know that $W \not\subseteq W'_L$, we have $\dim W' > 0$ and hence $n = \dim U' > 0$.

It follows from Lemma 2.6(1) that $\{u'\}$ is optimal of defect at most $d - 1$ for $f(V')$ in $A/H$. Note that $\langle U' \rangle$ contains a translate of a component of $\ker f = H$ since $\langle U' \rangle_{\text{geo}} = u + H \subset \langle U' \rangle$ (for $u \in U'(K)$ arbitrary). Furthermore, $f(V')$ is a subvariety of $A/H$ of dimension $\dim V' - n \leq \dim V' - 1 = m$. It follows from our hypothesis that $u'$ lies in a finite set. As $U'$ is a component of $f^{-1}(\{u'\}) \cap V'$, it lies in a finite set as well. \hfill $\Box$

This last proposition tells us that $\ZP(A'_{K'}, m, d) \text{ follows if } \ZP(B, m, d)$ holds for all quotients $B$ of $A'_{K'}$, where $K'$ is the algebraic closure of $\overline{\mathbb{Q}}(S)$.

If $\dim S = 0$, we have nothing to do. We then assume that $S$ has positive dimension.

By the inductive hypothesis we know that for all quotients $B$ of $A'_{K'}$ by a positive-dimensional abelian subvariety with $K'/\overline{\mathbb{Q}}$-trace $(T_B, \text{Tr}_B)$, the implication

$$\ZP(T_B, m, d) \implies \ZP(B, m, d)$$

holds.
If we know that $ZP(T_{A'_K}, m, d)$ holds, then, for all $B$ quotients of $A'_K$, since there exists a homomorphism of algebraic groups with finite kernel from $T_B$ to $T_{A'_K}$, $ZP(T_B, m, d)$ holds as well because of Lemma 2.6(2).

The inductive hypothesis then tells us that $ZP(B, m, d)$ holds for all quotients $B$ of $A'_K$ by a positive-dimensional abelian subvariety and thus, by Proposition 3.2, $ZP(T_{A'_K}, m, d)$ holds as we wanted to prove. □

5. Proof of Theorem 1.9

We show that the hypotheses in Theorem 1.9 imply the following claim for every quotient $B$ of $A$: Every subvariety $V$ of $B$ of dimension at most $m$ contains at most finitely many optimal singletons (for $V$ in $B$) of defect at most $d$.

As every quotient of $A$ admits a homomorphism of algebraic groups with finite kernel to $A$, we can deduce as in the proof of Lemma 2.6(2) that it suffices to prove this claim for $A$ itself.

Let therefore $V$ be a subvariety of $A$ of dimension at most $m$. We show by induction on $j \in \{0, \ldots, \dim V\}$ that the optimal singletons for $V$ in $A$ of defect at most $d$ are contained in a finite union of subvarieties of $V$ of dimension at most $\dim V - j$. This is obvious for $j = 0$.

Suppose that this second claim has been proven for some $j < \dim V$. Let $W$ be one of the finitely many subvarieties of $V$ of dimension at most $\dim V - j$ that contain the optimal singletons for $V$ in $A$ of defect at most $d$. We can assume without loss of generality that $\dim W = \dim V - j$.

Any optimal singleton for $V$ in $A$ that is contained in $W$ is also optimal for $W$ in $A$. We want to show that the optimal singletons for $W$ in $A$ of defect at most $d$ are contained in a proper Zariski closed subset of $W$. This will establish the claim for $j + 1$.

Translating $W$ by a torsion point sends optimal singletons (for $W$ in $A$) to optimal singletons of the same defect, so we can assume without loss of generality that $B := \langle W \rangle$ is an abelian subvariety of $A$.

If $\{p\} \subset W$ is an optimal singleton for $W$ in $A$ of defect at most $d$, then $\langle \{p\} \rangle \subset B$. Since $\dim W = \dim V - j > 0$, we have that $\{p\} \subset W$ and therefore $\delta(\{p\}) = \dim(\{p\}) < \delta(W) = \dim B - \dim W$. It follows that the codimension of $\langle \{p\} \rangle$ in $B$ is greater than or equal to $k := \max\{\dim B - d, \dim W + 1\}$. So the optimal singletons for $W$ in $A$ of defect at most $d$ are contained in $W \cap B^k$.

As $B := \langle W \rangle$, no proper special subvariety of $B$ can contain $W$. It then follows from Conjecture 1.8 for $B$, $d$ and $W$ that $W \cap B^k$ is not Zariski dense in $W$. Together with the above, this implies that the optimal singletons for $W$ in $A$ of defect at most $d$ are contained in a proper Zariski closed subset of $W$ as desired. This establishes the second claim by induction.

Now taking $j = \dim V$ shows that the number of optimal singletons for $V$ in $A$ of defect at most $d$ is finite. This proves the first claim above. Theorem 1.9 now follows from the following theorem.
Theorem 5.1. Let $m$ and $d$ be non-negative integers and suppose that every subvariety of dimension at most $m$ of a quotient of $A$ contains at most finitely many optimal singletons of defect at most $d$. Then every subvariety of $A$ of dimension at most $m$ contains at most finitely many optimal subvarieties of defect at most $d$.

Proof. The hypotheses imply that every quotient of $A$ satisfies $LGO_d^m$ as defined in Definition 8.1 of [HP16] after fixing an arbitrary field of definition that is finitely generated over $\mathbb{Q}$. Theorem 5.1 then follows from Theorem 9.8(i) in [HP16]. □

6. The Zilber-Pink conjecture for subvarieties of defect $\leq 1$

Theorem 6.1. Let $A$ be an abelian variety over $\bar{\mathbb{Q}}$ and $V \subset A$ a subvariety. Then $V$ contains at most finitely many optimal subvarieties of defect at most 1.

The following proof was suggested to the authors in a private communication by Philipp Habegger.

Proof. Let $\{p\} \subset V$ be an optimal singleton for $V$ in $A$, contained in a torsion coset of dimension at most 1. By Proposition 4.5 of [HP16], $\{p\}$ is geodesic-optimal for $V$ in $A$ and therefore not contained in a coset of positive dimension that is contained in $V$. By the Theorem of [Hab09] with $s = \dim A - 1$, the height of $p$ with respect to any symmetric ample line bundle on $A$ is then bounded.

It then follows from Proposition 9.7 of [HP16] that (in the notation of [HP16]) $LGO_1(V)$ is satisfied after fixing a number field over which $V$ and $A$ are defined. As $V$ was arbitrary, this implies that every abelian variety $A$ over $\bar{\mathbb{Q}}$ satisfies $LGO_1^r$ for all integers $r \geq 0$; see Definition 8.1 of [HP16] for the definitions of $LGO_s(V)$ and $LGO_r^s$. The claim then follows from Theorem 9.8(i) in [HP16]. □

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