Creation of Stable Multipartite Entangled States in Spin Chains with Defects

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We show how defects in a spin chain described by the XXZ model may be used to generate entangled states, such as Bell and W states, and how to maintain them with high fidelity. In the presence of several excitations, we also discuss how the anisotropy of the system may be combined with defects to effectively assist in the creation of the desired states.

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I. INTRODUCTION

Applying the superposition principle to product states of composite systems leads to one of the most striking properties of quantum mechanics: entanglement. What used to be a subject within the foundations of quantum mechanics is now a resource for the development of new technologies, playing a central role in quantum computation and quantum communication.

The generation of highly entangled states is a prerequisite for quantum teleportation, various quantum cryptographic protocols, and is also needed in quantum computation. In the bipartite setting, maximal entanglement appears in the Bell [or also called Einstein-Podolsky-Rosen (EPR)] state (1/√2)(|10⟩ + |01⟩). Dür et al. showed that there are two different kinds of genuine tripartite pure state entanglement: the maximally entangled Greenberger-Horne-Zeilinger (GHZ) state and the so called W state. The later is the state of three qubits that retains a maximal amount of bipartite entanglement when any one of the three qubits is traced out. It is written as (1/√3)(|100⟩ + |010⟩ + |001⟩). Here, we show how Bell and W states may be created in spin chains with defects.

Spin chains provide ideal systems for the study of entanglement. They are naturally used to model quantum computers: the two states of a spin-1/2 particle correspond to the two levels of a qubit and the exchange interaction corresponds to the qubit-qubit interaction. We focus here on a spin chain described by the XXZ model, where two different kinds of interaction are identified: the Ising part, proportional to the anisotropy coupling, and the XY part, which can be used to create entanglement between two or more qubits.

In principle, the qubit level spacings may be controllable, allowing the creation of entanglement between precisely selected qubits. As it turns out, the anisotropy of the system can also be used to our advantage, for states with different numbers of neighboring excited qubits are not coupled and a large system may then be treated as consisting of several uncoupled small chains.

The paper is organized as follows. The model is described in the next section. Section 3 discusses how defects can be used to create Bell and W states when the system contains just one excited qubit. Once the desired state is generated, we outline a way to preserve it with high fidelity for a long time, which requires quickly detuning the defects involved in the process. Section 4 shows how the anisotropy may be used as an extra tool toward the creation of entanglement.

II. THE MODEL

A spin chain with nearest neighbor coupling is considered. The Hamiltonian describing the system has two parts: one corresponds to the Zeeman energy of each spin, $H_d$, and the other is related to the spin-spin interaction and is given by the XXZ model, $H_{XXZ}$. In units where $\hbar = 1$, we have

$$H = H_d + H_{XXZ},$$

where

$$H_d = \sum_{n=1}^{L} \frac{\varepsilon_n}{2} \sigma_n^z,$$

$$H_{XXZ} = \sum_{n=1}^{L} \left[ \frac{J}{4} \sigma_n^+ \sigma_{n+1}^- + H_{hop} \right],$$

$$H_{hop} = \frac{J}{8} (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+).$$

Above $\sigma_n^+,\sigma_n^-$ are Pauli matrices. There are $L$ sites and we deal with a periodic (or closed) chain, that is, sites $n + L$ and $n$ are the same. Each site $n$ is subjected to a magnetic field in the $z$ direction, giving the energy splitting $\varepsilon_n$. In terms of qubits, $\varepsilon_n$ is the level spacing of qubit $n$. A spin pointing up corresponds to an excited qubit, or an excitation. The parameter $J$ is the hopping integral and $\Delta$ is a dimensionless parameter related to the anisotropy coupling. The non-diagonal term $H_{hop}$ is responsible for propagating the excitations and delocalizing the states. The diagonal term $\sigma_n^z \sigma_{n+1}^z$ gives the Ising interaction. It is only relevant when at least two excitations are present, being therefore associated with many-body problems. We set $J$ and $\Delta > 0.$

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We note that some recent works have discussed entanglement in anisotropic systems with impurities \[11,12\], but their anisotropy differs from the one in Hamiltonian \[11\]. The hopping part of the Hamiltonian considered here can be equivalently written as \( H_{\text{hopp}} \propto J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y \). Here \( J_x = J_y = J \), but in the models cited above, the degree of anisotropy comes from the difference between \( J_x \) and \( J_y \). The anisotropy in our case originates from the extra Ising interaction. In terms of energy spectra, the term \( d \) may play a role in the level spacing \( \varepsilon \) and assume that all qubits have the same level spacing except for some defects that have all the same level spacing for \( d \). In terms of energy spectra, we form two well separated bands. It is then possible to treat the spin chain as two smaller and decoupled chains.

### A. Two defects: Bell states

Let us first examine the case where there are simply two defects with \( d \gg J \), placed on sites \( n_1 \) and \( n_2 \). The chain is broken into two, one with \( L-2 \) qubits, whose state energies lie within the band \( E_1 + J \Delta \), and the other with just two qubits, whose eigenstates are Bell states and have energies \( \sim E_1 + d \). An excitation created on one defect will hop between the two defects with a frequency determined by their distance. To find this value, we need to compute the energies \( E_{\pm} \) of the two eigenstates \( \psi_{\pm} = \frac{1}{\sqrt{2}} \left[ \phi(n_1) \pm \phi(n_2) \right] \) of the defect chain.

If the two defects are nearest neighbors, \( n_1 \) and \( n_1 + 1 \), they are coupled to first order of perturbation theory. We diagonalize a 2x2 submatrix whose diagonal elements are \( E_1 + d \) and off-diagonal elements give the effective hopping integral \( J_{\text{eff}} = J/2 \), so \( E_{\pm} = E_1 + d \pm J/2 \). If the defects are next-nearest neighbors, \( n_1 \) and \( n_1 + 2 \), they are coupled to second order of perturbation theory. Now the diagonal elements are \( E_1 + d + J/2(d) \) and the off-diagonal elements are \( J_{\text{eff}} = J^2/(4d) \), leading to states with much closer energies \( E_{\pm} = E_1 + d + J^2/(2d) \pm J^2/(4d) \). The larger the separation between the defects, the closer will be the energies of the two eigenstates.

An excited state initially prepared on site \( n_1 \) will have probability

\[
P_{\Phi(n_1)}(t) = \frac{1 + \cos[(E_+ - E_-)t]}{2},
\]

to be later found on the same site. The oscillation period of the excitation between the defects is inversely proportional to the energy difference of the two eigenstates. It therefore depends on the number \( \mu \) of sites between the defects as \( T_\mu = T_0 (2d/J)^\mu \), where \( T_0 = 2\pi/J \).

Bell states are created when \( P_{\phi(n_1)} = P_{\phi(n_2)} = 1/2 \), that is, at every instant \( t_B = \pi k / [2(E_+ - E_-)] \), where \( k \) is an odd number. The more separated are the defects the longer is the wait for a maximally entangled state to be created, but in principle we can entangle even remote qubits.

#### 1. Detuning procedure

Once the desired state has been generated, the two defects should be detuned for the state to be maintained unchanged. A scheme for a possible detuning procedure is shown in Fig. 14. The level spacing of one qubit is kept constant, while the other increases according to a certain function of time \( \delta(t) \), which depends on the experimental setting and on the fidelity we want to attain. At the end of the process, the energy difference between the two
defects should be much larger than their effective hopping integral.

\[ \delta(t) \]

\[ \varepsilon + d \]

\[ n_1 \]

\[ \varepsilon + d \]

\[ n_2 \]

FIG. 1: Scheme for the detuning procedure for two defects on sites \( n_1 \) and \( n_2 \).

The detuning should start at \( t = t_B \) and, apart from constants, it is well represented by the Hamiltonian

\[ H_{\text{detun}} = \delta(t) \frac{1 + \sigma_n^+}{2} + \frac{J_{\text{eff}}}{4} (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+) \]. (3)

In Fig. 2 we show the results of a linear detuning, \( \delta(t) = Dt \), and a quadratic one, \( \delta(t) = Dt^2 \). Two defects with \( d = 10J \) coupled to second order of perturbation theory are considered. The larger the ratio \( D/J_{\text{eff}} \), the closer the generated state will be to the Bell state at later times. Contrary to what happens when creating states, now more separated defects make it easier to preserve the state with high fidelity.

\[ \begin{array}{c c c}
D = 10J_{\text{eff}} & D = J_{\text{eff}} & D = 0.1J_{\text{eff}} \\
\end{array} \]

FIG. 2: Time evolution of the probability to find the excitation on site \( n_1 \) during the detuning of two defects placed on \( n_1 \) and \( n_1 + 2 \). The left panel has \( \delta(t) = Dt \) and the right panel has \( \delta(t) = Dt^2 \). The detuning is initiated at the moment a Bell state is created. The effective hopping integral is \( J_{\text{eff}} = J^2/(4d), \) \( d = 10J, \) and the values of \( D \) are shown in the figure.

2. Measures of entanglement

Several measures of entanglement have been introduced. Here we briefly comment on how two of them, concurrence \[ \text{14} \] and global entanglement \[ \text{15, 16, 17}, \] can be affected by defects. They both vary from 0 (unentangled state) to 1 (entangled state), but concurrence is a measure of bipartite entanglement, while the other is able to quantify multipartite entanglement.

It is straightforward to conceive a system where all eigenvectors are Bell states and therefore have maximum individual (as well as the average over all states) concurrence. An example is a chain with an even number of qubits \( L \), where pairs of sites have the same level spacings, but differ from the others by more than the coupling strength between them. In this situation, all eigenvectors are Bell states each involving a particular pair of resonant sites. In terms of multipartite entanglement, designing a distribution of defects that would lead to large global entanglement is in general more demanding. The definition of global entanglement is

\[ 2 - (2/L) \sum_{n=1}^{L} \text{tr} \rho_n^2 \],

where \( \rho_n \) is the reduced density matrix for qubit \( n \) \[ \text{15, 16}. \] For the example described, it is maximum only for a two-qubit chain, decreasing with the chain size as \( 2 - 2/L - 2(L-2)/L \). Moreover, systems described by the XXZ model may not be ideal, because states that are known to have large global entanglement, such as the GHZ or products of Bell states, cannot be created. This is a consequence of the conservation of the total spin in the \( z \) direction, which prevents the coupling between basis states that have a different number of excitations.

B. Three defects: \( W \) states

To create a \( W \) state with one excitation, three defects are necessary. The procedure is very similar to the one described before. Let us assume three nearest neighbor defects, \( n_1, n_2 = n_1 + 1, \) and \( n_3 = n_1 + 2, \) with \( d \gg J \). The three eigenvectors of the defect chain are a good approximation to the actual corresponding vectors obtained with the total Hamiltonian. They can be simply computed by diagonalizing the submatrix

\[ \begin{pmatrix}
\mathcal{E}_1 + d & J/2 & 0 \\
J/2 & \mathcal{E}_1 + d & J/2 \\
0 & J/2 & \mathcal{E}_1 + d
\end{pmatrix}, \]

giving

\[ \psi_a = \frac{1}{\sqrt{3}} [\phi(n_1) + \sqrt{2} \phi(n_2) + \phi(n_3)] \]

\[ \psi_b = \frac{1}{\sqrt{3}} [-\phi(n_1) + \phi(n_2) + \phi(n_3)] \]

\[ \psi_c = \frac{1}{\sqrt{3}} [\phi(n_1) - \sqrt{2} \phi(n_2) + \phi(n_3)] \]

The \( W \) state is generated by preparing an initial state with an excitation in the middle site \( n_2 \), since this defect is coupled to the other two to first order of perturbation theory. The probability for the excitation to be found on this same site later in time is

\[ P_{\phi(n_2)}(t) = \frac{1 + \cos[(E_a - E_c)t]}{2}, \]  
\[ \text{(4)} \]
while the probabilities for sites $n_1$ and $n_3$ are the same

$$P_{\phi(n_1)}(t) = P_{\phi(n_3)}(t) = \frac{1 - \cos[(E_a - E_c)t]}{4}. \quad (5)$$

For the $W$ state we need $P_{\phi(n_1)}(t) = P_{\phi(n_2)}(t) = P_{\phi(n_3)}(t) = 1/3$, which happens at every instant

$$t_W = \frac{(-1)^k \arccos(-1/3) + 2\pi (k - |k/2|)}{E_a - E_c}, \quad (6)$$

where $k$ is an integer.

A possible detuning procedure for the defects for $t \geq t_W$ is the following: the level spacing of defect $n_3$ is kept fixed, while the level spacings of $n_1$ and $n_2$ increase in time according to the functions $\delta_1(t)$ and $\delta_2(t)$, respectively. The Hamiltonian describing the process is given by

$$H_{\text{detun}} = \delta_1(t) \frac{1 + \sigma_{n_1}^z}{2} + \delta_2(t) \frac{1 + \sigma_{n_2}^z}{2} + \frac{J}{2} \sum_{n=n_1, n_2} (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+). \quad (7)$$

For the three defects to become out of resonance, we can use for instance, $\delta_1(t) = D_1 t$, $\delta_2(t) = D_2 t$, and $D_3 \neq D_2$. The faster the detuning is performed the higher the fidelity for the desired state. Moreover, the final result becomes better if $D_2 > D_1$ instead of $D_1 > D_2$. An example is shown in Fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{Time evolution of the probability to find the excitation on site $n_1$ and on site $n_1 + 1$ during the detuning of three neighbor defects. The probability for the third defect placed on $n_1 + 2$ is very close to the probability for $n_1$. The detuning is initiated at the moment a $W$ state is created. For the left panel $D_1 = 10J$ and $D_2 = 100J$, for the right panel, $D_1 = 50J$ and $D_2 = 500J$.}
\end{figure}

\section{IV. Multiple Excitations}

When more than one excitation is present in the chain, the Ising part of the Hamiltonian starts having an effect. If the anisotropy parameter $\Delta$ is large, i.e. $J\Delta \gg J$, several well separated energy bands appear, each corresponding to states with a different number of excitations next to each other. The case of two excitations, for instance, is schematically shown in Fig. 4. The states where the excitations are far from each other have energy inside the band $2\varepsilon_1 \pm 2J$, while the states where they are next to each other have a much higher energy and form a much narrower band $2\varepsilon_1 + J\Delta + J/(2\Delta) \pm J/(2\Delta)$. Such bound pairs are coupled to second order of perturbation theory, taking much longer to propagate through the chain if compared to the free excitations. The two bands are not coupled and can be treated separately.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Energy spectrum of the considered spin chain with two excitations and one defect with $J\Delta \gg d \gg J$.}
\end{figure}

In addition, suppose that a single defect $n_1$ is added to the chain with two excitations, such that $J\Delta \gg d \gg J$. Two new bands appear [see Fig. 5]. One band has states with the two excitations far from each other, but one trapped on the defect, they have energy $2\varepsilon_1 + d \pm 2J$. The other band is made of just two (Bell) states having the largest energies. Here, the two excitations are next to each other and one of them is on the defect site. We have,

$$\psi_{\pm} = \frac{1}{\sqrt{2}} [\phi(n_1-1, n_1) \pm \phi(n_1, n_1+1)] \quad (8)$$

$$E_{\pm} = 2\varepsilon_1 + d + J\Delta + \frac{J}{4\Delta} \pm \frac{J^2}{4(J\Delta + d)} \pm \frac{J^2}{4(J\Delta + d)}.$$

Advantage may be taken of the large anisotropy to create entanglement. Preparing an initial bound pair on the sites $(n_1-1, n_1)$, for example, will lead to Bell states at every instant $t_{BP} = 2(J\Delta + d)\pi/2 + k\pi/J^2$, where $k$ is odd. As before, detuning sites $n_1 - 1$ or site $n_1 + 1$ should preserve the maximally entangled state. Moreover, since states $(n_1 - 1, n_1)$ and $(n_1, n_1 + 1)$ are coupled to second order, the interaction strength is small and the detuning rate required to keep the created Bell state with high fidelity does not need to be very large. Several other possibilities for Bell and $W$ states where use is made of the defects and anisotropy have been discussed elsewhere [3, 7].
V. CONCLUSION

We have investigated how bipartite and tripartite states can be generated in strongly anisotropic spin-1/2 chains described by the XXZ model by controlling the Zeeman energy of the spins [or equivalently, by manipulating the qubit level spacings]. Even though the interaction between the qubits is always on, it is effective only between states with very close energy. Selected resonant defects can then be used in the creation of entangled states. Once generated, the maintenance of the desired states depends on how fast the defects involved in the process can be moved away from resonance.

Our interest in the XXZ model emerges from its relevance to some proposals for quantum computers [6], but we now intend to extend the method developed here to other models, especially those that they may be used to create multipartite states with large global entanglement.

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