Frictional Impacts in Multibody Systems

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Abstract

A unifying slipping and sticking frictional impact model for multibody systems in contact with a frictional surface is presented. It is shown that the model can lead to energetic consistency in both slip state and stick state upon imposing specific constraints on the coefficient of friction (CoF) and the coefficient of restitution (CoR). A discriminator in the form of a quadratic function of the pre-impact velocity is introduced based on isotropic Coulomb constraint such that its sign determines whether the impact occurs in the sticking mode or in the slipping mode just prior to the contact. Solving the zero-crossings of such a function in terms of the CoF and the CoR variables leads to another discriminator called Critical CoF, which is the lowest static CoF required to prevent the subsequent impulse vector violating the isotropic friction cone constraint. Investigating conditions for the energetically consistent impact model reveals that the maximum values of either CoR or CoF should be limited depending on stick state or slip state. Furthermore, it is shown that these upper-bound limits in conjunction with the introduced Critical CoF variable can be used to specify the admissible set of CoR and CoF parameters, which can be represented by two distinct regions in the plan of CoF versus CoR.

1 Introduction

Nonlinear dynamics arises from slipping and sticking frictional impact phenomena occurs in many robotics applications involving multiple contacts or formations of closed-loop topology [1-8]. Examples include industrial manipulators performing complex contact tasks [9-11], force control of constrained robots [12-16], walking robots [17-20] and space robotics capturing free-floating objects [21]. Such systems can be generally treated as multibody systems (MBSs) with a time-variant structure, and hence they often exhibit a nonsmooth behaviour due to friction, unilateral constraint, and impact [6,22,26]. Consequently, there are two main challenges in dynamics formulation of these systems: (i) Discontinuity due to activation and deactivation of unilateral constraints and dynamics itself, (ii) friction in the contact, which causes fundamental change in the dynamic behavior of the systems when the friction characteristic changes from stick state to the slip state and vice versa.

Frictional impact is often modeled as instantaneous events because the system’s velocities are of bounded variation and hence they cannot be assumed continuous during impact events. Therefore, the system’s post-impact states cannot be calculated from an acceleration model.

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On the contrary, they have to be derived from an impact model by incorporating the impulse-momentum balance of the entire system together with a restitution law and a friction law. Therefore, modeling frictional impacts demand proper combination of a restitution law and friction law. The Newtonian restitution law is extensively used to model the normal direction supplemented by the Coulomb friction law in the tangential direction frictional impacts [27]. According to isotropic Coulomb friction, frictional impacts occur in the sticking mode if the magnitudes of the tangential and normal impulses do not violate the friction cone constraint, otherwise the frictional impacts occur in the slipping mode. Impact dynamics fundamentally changes when the MBS switches from sticking mode to slipping mode or vice versa. That is why the properties of the MBS during impact, such as its energy change during impact, depend on whether the system is in slip state or stick state. Simulation and analysis of frictional impact in MBSs calls for a unifying formulation which is energetically consistent, i.e. the model is always dissipative or energy preserving, for both slip state and stick state. Ideally, an unified formulation has to be general enough to give a closed-form and energetically-consistent solution for the velocity jump and the subsequent impulse in slipping and sticking frictional impacts.

Contact dynamics was formulated by Moreau [28] through a non-smooth mechanics modelling approach, which allows the time evolutions of the positions and of the velocities to be non-smooth and discontinuous. A good survey of impact analysis in the framework of non-smooth mechanics can be found in [29]. Numerical methods for nonsmooth dynamical systems applicable to both mechanics and electronics are described in [30]. Stability theory is developed in [31] for non-smooth dynamical models, which arise from mechanical systems with unilateral constraints, such as unilateral contact, impact and friction. The frictional contact problem is cast in a mixed penalty-duality formulation, known as the prox formulation, by Alart et al. [32] as an alternative to the well-known linear and nonlinear complementarity problem formulations. However, it was later shown in [33] that these two formulations are indeed equivalent. A unified theory and application of dynamical systems that incorporate hard inequality constraint such as mechanical systems with impact or electrical circuits with diodes are comprehensively addressed in [34]. A new friction model based on a regularization of the Coulomb friction model is incorporated in the collision micro-dynamics hard contact models for simulation of the macroscopic behaviour of granular matter [7]. The earliest analytical investigation into robot dynamics during impact based on modelling of collisions between the robot and the environment was reported in [35]. It followed by design and evaluation of impact control schemes for robots during collisions [10, 36]. In the literature, the primary approach to modelling frictional impact in MBSs is based on combining the equation of momentum-balance together with a restitution and friction laws [23, 37]. In this approach, the impact is assumed to be an infinitesimal event and subsequently the set of the impulse-momentum equations are algebraically solved to produce velocity jump rather than updating the velocity from integration of the acceleration vector. In this formulation, the impact is characterized by the coefficient of restitution (CoR), which is defined as the ratio of the local velocities after and before collision. Other approaches assume smooth compliant modeling of impact where the impulsive force is typically presented by a linear or nonlinear spring-damper model [38, 39], e.g., the Haunt-Crossley nonlinear spring-dashpot model. However, it has been established that a linear or non-linear compliant model becomes equivalent to the momentum-balance model if the coefficient of restitution is specifically selected according to the damping and stiffness properties of the compliance model [17, 40, 41]. It is known that frictionless impacts for single or multiple contacts based
on Newton’s restitution law are always dissipative \[27,42,43\] if a global CoR is selected to be within \([0, 1]\). However, the Newton’s restitution law may potentially produce energetically inconsistent results when combined with friction in some contact situations \[27,43–45\]. Glocker et al. \[23\] introduced an impact model for two-dimensional contacts under Coulomb friction and Poisson’s hypothesis that is based on a linear complementarity formulation. Although this formulation of impact has been designed to be energy dissipative, it does not correspond to fundamental physical properties \[46,47\], the normal impulse may sometimes be too small to prevent penetration yielding an unrealistic solution \[23\], and finally, likewise Newton’s impact law, Poisson’s hypothesis may predict an increase in the kinetic energy \[44,46,48\]. Comparisons between different impact laws in terms of their respective abilities to correctly model dissipation and dispersion of energy are shown in \[49\]. The Kane’s example in which a double pendulum strikes a flat surface by a frictional impact has been thoroughly analyzed in \[27\] by decomposing the friction elements into its basic primitives. This technique allowed to reveal the mechanism leading to energy inconsistency in the Kane’s example. The Poisson’s and Coulomb law are combined in \[44\] to solve the impact problem at force level for two-dimensional single contact cases. A dissipation principle for resolving post-impact tangential velocities after simultaneous oblique impact events on a MBS is proposed in \[50\]. The proposed dissipation principle allows for changes in the dynamic coefficient of friction depending on the orientation of the impacting surfaces.

In this paper, a unified frictional impact model which is consistent in both slip state and stick state along with comprehensive analysis of energetic consistency leading to admissible values of CoR and CoF are presented. Derivation of the unified model is made possible by introducing a quadratic function of pre-impact velocity, the sign of which determines whether a frictional impact occurs in slip state or in stick state. It will be shown that zero-crossings of such a implicit function can be solved in terms of a new variable, called Critical CoF, which is an explicit function of CoR. It turns out that the Critical CoF is a convenient discriminator function, which determines the minimum required static CoF to prevent slipping during an impact. The condition for energetically consistent description of the impact model for both stick state and slip state is comprehensively investigated leading to identification of two exclusive regions in the plan of CoR and CoF variables, where a MBS becomes energetically consistent during slip and stick frictional impact. Finally, a case study along with simulation results underpins the impact model and the analytical results.

### 2 Dynamics model

Consider a frictional contact surface with static and dynamic coefficients of friction \(\mu_s\) and \(\mu_d\). The vector of contact force

\[
\lambda = \begin{bmatrix} \lambda_n \\ \lambda_t \end{bmatrix}
\]

consists of normal force \(\lambda_n \in \mathbb{R}\) and tangential force \(\lambda_t \in \mathbb{R}^2\). The contact friction has two modes: sticking and slipping. The sticking refers to a situation where the magnitude of the tangential force in the contact is not sufficient to overcome the static friction and hence causing the relative motion in the contact, i.e.,

\[
\|\lambda_t\| \leq \mu_s \lambda_n.
\]

(1)
Otherwise, slipping friction force occurs where the friction force vector has the magnitude of normal force times $\mu_d$ and it opposes the vector of tangential velocity $v_t$. Inequality (1) is called friction cone constraint and can be equivalently transcribed by the following quadratic inequality on the entire vector of contact force: $\lambda^T U \lambda \geq 0$, where

$$U = \text{diag}(\mu^2, -1, -1).$$

From the preceding discussion, we can generalize the description of the contact force vector in a frictional surface by

$$\lambda = \begin{cases} 
\begin{bmatrix} \lambda_n \\ \lambda_t \end{bmatrix} & \text{if } \lambda^T U \lambda \geq 0 \\
-\mu_d \frac{v_t}{\|v_t\|} \lambda_n & \text{otherwise}
\end{cases}$$

Now, consider a multibody system with generalized coordinate $q \in \mathbb{R}^n$ subject to the surface constraint with frictional contact. Dynamics equation of such system in the sticking friction mode can be described by

$$M(q) \ddot{q} + h = A^T \lambda$$

subject to:

$$A \dot{q} = 0$$

$$\lambda^T U \lambda \geq 0$$

$$\lambda_n \geq 0.$$
during an impact event, the velocity instantaneously jumps and thus requiring infinitely large acceleration and constraint force. Therefore, the acceleration models (4) or (5) are not adequate to determine post-impact velocities rather an impact model is required to deal with the impulsive constraint force and discontinuities in the velocities. Since the impact is assumed to be an infinitesimal event, it is also reasonable to assume either stick friction or slip friction occurs during the impact. That is either (4) or (5) governs the impact dynamics. Although it is not possible to calculate the velocity change at the time of impact through integration of the equations of motion, it is possible to calculate the velocity change using the Newton’s impact law. Suppose impacts occur at time interval $[t^-, t^+]$, where the impact duration $\delta t = t^+ - t^-$ is infinitesimal. Since the generalized coordinate $q$ is assumed to be constant over the impact, the mass matrix $M(q)$ and the Jacobian $A(q)$ remain unchanged during the impact. Therefore, one can carry out integration of the differential equation (4) over $[t^-, t^+]$ to obtain impact equation of the system as

$$M(\dot{q}^+ - \dot{q}^-) = A^T i. \quad (7)$$

Here, $i$ is the impact or the Dirac integral of the contact forces, i.e.,

$$i = \lim_{\delta t \to 0} \int_{t^-}^{t^-+\delta t} \lambda \, dt, \quad (8)$$

and $\dot{q}^- = \dot{q}(t^-)$ and $\dot{q}^+ = \dot{q}(t^+)$ are pre-impact and post-impact velocities. Notice that in derivation of (7), we assumed $h$ to be a continuous function consisting of nonimpulsive terms, and thus it vanishes by the integration. The velocity jump can be captured by the restitution model

$$A\dot{q}^+ = -EA\dot{q}^- \quad (9)$$

where the restitution matrix. An impact law of Newton’s type imposes a kinematic condition on the impact via scalar restitution $e$ on the pre- and post-impact relative velocities [27]. The restitution matrix can be written in a general form $E = \text{diag}[e_n, e_t, e_t]$, where $e_n$ and $e_t$ are CoRs associated with normal and tangential impacts. Tangential compliance is typically negligible at contacts between smooth and hard materials such as steel and glass, while it may become more prominent with materials such as rubber, clay, and wood [46]. Therefore, the restitution matrix typically takes the form $E = \text{diag}[e, 0, 0]$ with $e$ being the coefficient of restitution [27]. It is worth mentioning that the CoR can be derived from a linear or nonlinear spring-dashpot model or Hertz’s model to make CoR physically sound [17,39–41,49]. For instance, it has been shown that the at low impact velocity and for materials having linear elastic behaviour the CoR can be effectively approximated by the equation $e = 1 - \alpha v_n$, where parameter $\alpha = 2c/3k$ with $c$ and $k$ being the damping and stiffness constants of the spring-dashpot model and $v_n$ being the relative normal velocity [17,39,41]. The equations of momentum balance (7) combined with the restitution equation (9) are sufficient to solve for the vectors of post-impact velocity and impulse. To this end, by pre-multiplying both sides of (7) by $AM^{-1}$ we get:

$$A\dot{q}^+ - A\dot{q}^- = AM^{-1}A^T i \quad (10)$$

Substituting $A\dot{q}^+$ from (9) into (10) and then solving the resultant equation for $i$, we arrive at

$$i = -G(E + I)A\dot{q}^-, \quad (11a)$$
where \( G = (AM^{-1}A^T)^{-1} \). Finally, upon substitution of the expression of impulse from (11a) into (10), we can write the expression of the post-impact velocity by
\[
\dot{q}^+ = \dot{q}^- - M^{-1}A^TG(I + E)Aq^-.
\] (11b)

An isotropic Coulomb constraint restricts the magnitude of the friction impulse by the following inequality
\[
i^TU_i \geq 0.
\] (12)

Upon substitution of \( i \) from (11a) into (12), the latter inequality can be equivalently written in terms of the following *discriminator function* of pre-impact velocity
\[
\sigma := \dot{q}^{-T}Q\dot{q}^{-} \geq 0,
\] (13)
where the matrix is defined as
\[
Q := A^TG(I + E)U(I + E)GA.
\] (14)

In other words, if the pre-impact velocity satisfies \( \sigma \geq 0 \), then the impact is with sticking friction mode and hence solutions (11) is valid. Otherwise, the frictional impact involves slipping and then the solution should take the following steps. Suppose \( i = [i_n \ i_t]^T \), where \( i_n \in \mathbb{R} \) and \( i_t \in \mathbb{R}^2 \) are normal and tangential impulses. Then, integration of equation (5a) gives the governing slipping impact model as
\[
M(\dot{q}^+ - \dot{q}^-) = (a_n^T - \mu_d\bar{\vec{b}}^T)n_i,
\] (15)
where
\[
\int_{t^-}^{t^+ + \delta t} b(q, \dot{q})\lambda_n \ dt = \bar{b}i_n
\] (16)

and \( \bar{b} = b(q, \bar{\dot{q}}) \) with \( \bar{\dot{q}} \) being the average velocity during the slipping impact. Notice that the computation in (16) does not assume the slip velocity being constant during the impact, rather \( \bar{b} \) is computed based on average of pre- and post-impact velocities. In other words, we assume the average value of \( b \) calculated at the average velocity \( \bar{\dot{q}} \) in order to be able to evaluate the integral in (16). Alternatively, one may use one of the proposed trajectories during the compression and expansion phases of an impact \[26\] to find the integration of \( b \) over the short time of impact. The restitution model in the case of slipping friction mode is simply described by the following equation
\[
a_n\dot{q}^+ = -e_n a_n \dot{q}^-
\] (17)

In a development similar to (10)-(11b), one can solve equations (15) and (17) for
\[
\dot{i}_n = -(1 + e)g a_n \dot{q}^-,
\] (18a)
\[
\dot{\dot{q}}^+ = \dot{q}^- - (1 + e)gM^{-1}(a_n^T a_n - \mu_d\bar{\bar{b}}^T a_n)q^-,
\] (18b)

where \( g = 1/(a_n^T M^{-1}a_n - \mu_d a_n M^{-1}\bar{\bar{b}}^T) \). On the other hand, the tangential impact in the case of slip state is related to normal impact by
\[
i_t = -\mu_d \frac{A_t \bar{\dot{q}}}{\|A_t \bar{\dot{q}}\|} i_n.
\] (18c)
Thus, from (18a) and (18c), we can describe the overall vector of impulse by

$$i = -(1 + e)g \left[ \frac{a_n}{\| A_t \hat{\dot{q}}_n \|} A_t \hat{\dot{q}}_n \right] \hat{q}^- \quad (18d)$$

It is important to point out that solution (18a) is kinematically consistent if

$$i_n \geq 0 \quad (19)$$

Since the pre-impact normal velocity is always negative, i.e., $v_n^- = a_n \hat{\dot{q}}^- < 0$, one can conclude from expression (18a) that $i_n \geq 0$ is held if and only if $g > 0$. In other words, the kinematically consistency is tantamount to satisfying the following inequality

$$a_n M^{-1} a_n^T - \mu_d a_n M^{-1} A_t^T \hat{u} > 0 \quad (20)$$

where $\hat{u} = \frac{A_t \hat{\dot{q}}}{\| A_t \hat{\dot{q}} \|}$ is the unit directional vector along with the slipping velocity. It will be shown later in Section 2.2 that the kinematically consistency condition is tantamount to energetic consistency of impacts.

In summary, computation of sticking/slipping frictional impact may proceed as follows:

i. For a given pre-impact velocity, compute the discriminator $\sigma = \hat{\dot{q}}^- Q \hat{q}^-$ to check the subsequent inequality condition;

ii. if $\sigma \geq 0$, then use the set of equations (11) to determine the post-impact velocity and impulsive force;

iii. if $\sigma < 0$, then use set of equations (18) to determine the post-impact velocity and impulsive force.

### 2.1 Critical coefficient of friction

It is evident from (2), (13), and (14) that the discriminator function $\sigma$ is a quadratic function of static CoF, CoR, and pre-impact velocity, i.e., $\sigma = \sigma(\mu_s, e, \hat{\dot{q}}^-)$. A natural question arises that what values of the CoF will make $\sigma = 0$. Let us define the Critical CoF, denoted by $\mu_{cr}$, as the root of the discriminator function, i.e.,

$$\sigma(\mu_{cr}, e, \hat{\dot{q}}^-) = 0. \quad (21)$$

Then, the variable $\mu_{cr}$ must be a function of CoR and pre-impact velocity, i.e., $\mu_{cr} = \mu_{cr}(\hat{\dot{q}}^-, e)$. Defining auxiliary vector $\xi = [\xi_n, \xi_i]^T = GA \hat{\dot{q}}^-$, we can write the quadratic expression of $\sigma$ in (13) in terms of the elements of vector $\xi$ as follows

$$\sigma = \mu_s^2 \xi_n^2 (e + 1)^2 - \| \xi_i \|^2, \quad (22)$$

Then, the valid solution to the quadratic equation $\sigma = 0$ takes the form

$$\mu_{cr} = \frac{a}{e + 1}, \quad \text{where} \quad a = \| \xi_i \|/|\xi_n|. \quad (23)$$
It is clear from the above second-order polynomial that \( \mu_s \geq \mu_{cr} \) implies \( \sigma \geq 0 \) and conversely \( \mu_s < \mu_{cr} \) implies \( \sigma < 0 \). In other words, \( \mu_{cr} \) is also a discriminator which determines the lowest static CoF required to prevent slipping during impacts. That is to say

\[
\begin{align*}
\mu_s \geq \mu_{cr} & \quad \text{sticking impact} \\
\mu_s < \mu_{cr} & \quad \text{slipping impact}
\end{align*}
\]  

(24)

It worths noting that there is always an algebraic solution for the critical CoF. However, from a physics point of view, the possibility of CoF being greater than 1 is not very practical. In that case, one can conclude from (23) that if

\[
a > 1 \quad \land \quad e < a - 1
\]

then there is no physically feasible solution for the critical CoF, meaning that slipping impact can not be avoided for all CoF inside \([0, 1)\).

2.2 Energetic consistency

Energy lost during an impact is an important quantity not only to gain insight into complex physical phenomenon during contact but to examine whether an impact model is physically consistent [51, 52]. The energy dissipation done by the contact force during impacts can be expressed as

\[
W_{loss} = \int_{t^-}^{t^- + \delta t} \mathbf{v} \cdot \mathbf{\lambda} \, dt = \bar{\mathbf{v}} \cdot \mathbf{i}
\]

(26)

where \( \bar{\mathbf{v}} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-) \) is the average velocity during the contact. In the following analysis, we compute the equation of power (26) and drive the conditions for its negativity for two district modes of frictional impact, i.e., the sticking mode where \( \mu \geq \mu_{cr} \) and slipping mode where \( \mu < \mu_{cr} \).

2.2.1 Sticking mode

Consider stick frictional impacts: From the restitution law \( \mathbf{v}^+ = -E \mathbf{v}^- \) we can write the expression of average contact velocity as

\[
\bar{\mathbf{v}} = \frac{1}{2}(I - E)\mathbf{A} \dot{\mathbf{q}}^- \quad \iff \quad \mu_s \geq \mu_{cr}
\]

(27)

Substituting expression of \( \bar{\mathbf{v}} \) and \( \mathbf{i} \) from (27) and (11a) into (26) yields

\[
W_{loss} = -\frac{1}{2} \dot{\mathbf{q}}^- (I - E)^T G (I + E) \mathbf{A} \dot{\mathbf{q}}^- \\
= -\frac{1}{2} \dot{\mathbf{q}}^- (G - EGE) \mathbf{A} \dot{\mathbf{q}}^- \quad \iff \quad \mu_s \geq \mu_{cr}
\]

(28)

in which (28) is obtained by using the fact that \( G - (GE)^T \) is a skew symmetric matrix and hence the corresponding quadratic term is eliminated in expression of RHS of (28). Therefore, one can infer from (28) that a sticking frictional impact is energetically consistent if the value of the CoR is upper-bounded as

\[
e \leq e^*,
\]

(29)
where $e^*$ is root of the following quadratic function
\[
v^{-T}(G - E(e^*)GE(e^*))v^- = 0. \tag{30}\]

Alternatively, one can conclude from (28) that the power loss during sticking frictional impact is always negative semidefinite if the following symmetric matrix is positive semidefinite
\[
G - EGE \succeq 0 \tag{31}\]

In other words, if $\lambda_{\min}(G - EGE) \geq 0$, where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix, the energy consistency of the impact is ensured. Therefore, the upper-bound limit of CoR can be specified by
\[
e^* := \arg \min \lambda_{\min}(G - EGE), \tag{32}\]

regardless of the pre-impact velocity. Notice that (32) gives a conservative upper-bound limit, however (32) is independent of the pre-impact velocity unlike (30). From (24) and (29), we can say the loci of all CoR and CoF pairs leading to energetically consistent sticking impacts can be described by the following set:
\[
\forall e, \mu_s \ni \mu_s \geq \mu_{cr}(e) \land e \leq e^* \tag{33}\]

Suppose $K^- = \frac{1}{2}q^{-T}Mq^-$ and $K^+ = \frac{1}{2}q^{+T}Mq^+$ are the pre-impact and post-impact values of the kinetic energy of the constrained system. Then, from the expression of the post-impact velocity (11b), one can verify that
\[
W_{\text{loss}} = K^+ - K^- \tag{34}\]

### 2.2.2 Slipping mode

Now consider slip frictional impacts: In this case, the energy loss absorbed in the contact can be still expressed by (26), but the post-impact velocity and impulse are governed by equations (18b) and (18d). The average normal and tangential velocities during slipping impact are $\bar{v}_n = \frac{1}{2}(v^+_n + v^-_n) = \frac{1}{2}(1 - e)a_nq^-$ and $\bar{v}_t = \frac{1}{2}A_t(\dot{q}^- + \dot{q}^+)$, respectively. Thus, the expression of average contact velocity $\bar{v}^T = [\bar{v}_n \, \bar{v}_t^T]$ can be written as
\[
\bar{v} = \frac{1}{2} \left[ (1 - e)a_n \dot{q}^- \right] \quad \Leftarrow \quad \mu_s < \mu_{cr} \tag{35}\]

Upon substitution of (35) and (18d) in (36), the expression of power loss during slipping impact can be written as
\[
W_{\text{loss}} = -\frac{1}{2}g \left( (1 - e^2)q^-T(a_n^T a_n)\dot{q}^- - (1 + e)\mu_d \| A_t \dot{q}^- \| a_n \dot{q}^- \right)
\[
= -\frac{1}{2} \left( (1 - e^2)gq^-T(a_n^T a_n)\dot{q}^- + (1 + e)g\mu_d \| A_t \dot{q}^- \| v^-_n \right) \leq 0 \quad \Leftarrow \quad \mu_s < \mu_{cr} \tag{36}\]

Notice that the first term in the RHS of (36) is always positive semidefinite because $0 \leq e \leq 1$. Moreover, the pre-impact normal velocity is always negative, i.e.,
\[
a_n \dot{q}^- = v^-_n < 0 \tag{37}\]

Suppose $K^- = \frac{1}{2}q^{-T}Mq^-$ and $K^+ = \frac{1}{2}q^{+T}Mq^+$ are the pre-impact and post-impact values of the kinetic energy of the constrained system. Then, from the expression of the post-impact velocity (11b), one can verify that
\[
W_{\text{loss}} = K^+ - K^- \tag{34}\]
and hence the second term in the RHS of (36) is always positive definite. Therefore, the
condition for negative power losses boils down to the kinematic consistency \( g < 0 \), which has
been already stipulated in (20). Denoting

\[
\mu^* := \frac{a_n M^{-1} a^T_n}{\| a_n M^{-1} A^T_t \|},
\]

we can say \( g > 0 \) for all pre-impact velocities if

\[
\mu_s < \mu_{cr} \land \mu_d < \mu^*.
\]

Therefore, assuming \( \mu_s = \mu_d = \mu \), then the loci of energetically consistent CoR and CoF pairs
for slipping impact can be described by the following set

\[
\forall e, \mu \ni \mu \leq \min(\mu_{cr}(e), \mu^*)
\]

Finally, by virtue of (33) and (40), the conditions for energetically consistent of frictional impact
during both slipping and sticking modes can be described by

\[
\{ \mu, e : \mu \geq \mu_{cr}(e) \land e < e^* \cup \mu < \min(\mu_{cr}(e), \mu^*) \}
\]

where \( \mu_{cr}, \mu^* \), and \( e^* \) are obtained from (23), (38), and (30) or (32), respectively. It is important
to point out that these variables are not constant parameters as they change with changing the
states and configuration of the MBSs.

3 Conclusions

A unifying model slip and stick frictional impact for MBSs has been presented. It has been
proven that the model is energetically consistent in both slipping and sticking modes provided
that the values of CoR and CoF are within admissible regions in the variable plan. Unification of
the impact model has been complemented by introducing a quadratic function of the pre-impact
velocity whose sign determines whether the impact occurs in slip state or stick state. This allows
switching to the adequate impact model prior to the impact event. Parametrization of the
quadratic function in terms of CoR and CoF variable led to the Critical CoF, which determines
the minimum required CoF to prevent slipping during an impact. Energy consistency of the
impact model in both slipping and sticking friction modes has been analytically investigated and
the results revealed that there were upper-bound limits of CoR and CoF and in conjunction
with the introduced Critical CoF variables define the admissible set of CoR and CoF for a
consistent impact.

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