A unified treatment for ODEs under Osgood and Sobolev type conditions

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Abstract

In this paper we present a unified treatment for the ordinary differential equations under
the Osgood and Sobolev type conditions, following Crippa and de Lellis’s direct method.
More precisely, we prove the existence, uniqueness and regularity of the DiPerna–Lions flow
generated by a vector field which is “almost everywhere Osgood continuous”.

Keywords: DiPerna–Lions theory, Sobolev regularity, Osgood condition, regular Lagrangian
flow, transport equation

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1 Introduction

In the seminal paper [7], DiPerna and Lions established the existence and uniqueness of the quasi-
invariant flow of measurable maps generated by Sobolev vector fields with bounded divergence.
Their method is quite indirect in the sense that they first established the well-posedness of
the corresponding transport equation, from which they deduced the results on ODE. Their
methodology is now called the DiPerna–Lions theory and can be regarded as a generalization
of the classical method of characteristics. It has subsequently been extended to the case of BV
vector fields with bounded divergence by Ambrosio [1], via the well-posedness of the continuity
equation (cf. [2, 3] for a detailed account of these results). This theory was generalized in [4, 9] to
the infinite dimensional Wiener space. Recently, Crippa and de Lellis [6] obtained some a-priori
estimates on the flow (called regular Lagrangian flow there) directly in Lagrangian formulation,
namely without exploiting the connection of the ODE with the transport or continuity equation.
Applying this method, they can give a direct construction of the flow (see the extension to the
case of stochastic differential equations in [21, 10]).

To introduce the setting of the present work, we recall the key ingredient in Crippa–de
Lellis’s direct method, namely, a Sobolev vector field $b \in W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ ($p \geq 1$) is “almost everywhere
Lipschitz continuous” (it holds even for BV vector fields, see [6, Lemma A.3]). More precisely,
there are a negligible subset $N \subset \mathbb{R}^d$ and a constant $C_d$ depending only on the dimension $d$,
such that for all $x, y \notin N$ and $|x - y| \leq R$, one has

$$|b(x) - b(y)| \leq C_d|x - y|\left(M_R|\nabla b|(x) + M_R|\nabla b|(y)\right),$$

(1.1)
On the other hand, if we take continuous (see the end of Section 2 for an example of this kind of functions). It is clear that if \( \rho \) is essentially bounded, then (1.4) becomes the general Osgood condition

\[
|b(x) - b(y)| \leq C|x - y|^r r, \quad |x - y| \leq c_0, \tag{1.3}
\]

where \( r : (0, c_0] \to [1, \infty) \) is a continuous function defined on a neighborhood of 0 and satisfies

\[
\int_0^{c_0} s \frac{ds}{s^r(s)} = \infty. \tag{1.6}
\]

Under this condition and assuming that the ODE

\[
\frac{dX_t}{dt} = b(X_t), \quad X_0 = x
\]

has no explosion, they proved that the solution \( X_t \) is a flow of homeomorphisms on \( \mathbb{R}^d \) (see [11, Theorem 2.7]). If in addition \( r(s) = \log \frac{1}{s} \) and the generalized divergence of \( b \) is bounded, then it is shown in [8, Theorem 1.8] that the Lebesgue measure \( \mathcal{L}^d \) is also quasi-invariant under the flow \( X_t \). In a recent paper [16], the second named author extended this result to the Stratonovich SDE with smooth diffusion coefficients, using Kunita’s formula for the Radon–Nikodym derivative of the stochastic flow (see [13, Lemma 4.3.1]).

Inspired by these two types of conditions (1.1) and (1.3), we consider in this work the following assumption on the time dependent measurable vector field \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \):

(H) there exist a nonnegative function \( g \in L^1([0, T], L^1_{\text{loc}}(\mathbb{R}^d)) \) and negligible subsets \( N_t \), such that for all \( t \in [0, T] \) and \( x, y \notin N_t \), one has

\[
|b_t(x) - b_t(y)| \leq (g_t(x) + g_t(y)) \rho(|x - y|), \tag{1.4}
\]

where \( \rho \in C(\mathbb{R}_+, \mathbb{R}_+) \) is strictly increasing, \( \rho(0) = 0 \) and \( \int_0^\infty \frac{ds}{\rho(s)} = \infty \).

The typical examples of the function \( \rho \) are \( \rho(s) = s, s \log \frac{1}{s} \), \( s \log \frac{1}{s} \), \( s \log \log \frac{1}{s} \), \ldots. Notice that the latter two functions are only well defined on some small interval \([0, c_0]\), but we can extend their domain of definition by piecing them together with radials. In this paper we fix such a function \( \rho \). Similarly we may call a function satisfying (1.4) “almost everywhere Osgood continuous” (see the end of Section 2 for an example of this kind of functions). It is clear that if we take \( g_t = C_d M_{Bt} |\nabla b| \) and \( \rho(s) = s \) for all \( s \geq 0 \), then the inequality (1.4) is reduced to (1.1). On the other hand, if \( g \) is essentially bounded, then (1.4) becomes the general Osgood condition
(1.3), except on the negligible set $N_t$ (we can redefine $b_t$ on this null set to get a continuous vector field). Therefore, this paper can be seen as a unified treatment of the two different types of conditions. We would like to mention that Professor L. Ambrosio once told Professor S. Fang and the second author (by private communication) that there might be no unified framework for the DiPerna–Lions theory and the Osgood type conditions. The assumptions like (H) were considered in [19], but the function $\rho$ was always taken as $\rho(s) = s$ for all $s \geq 0$.

The paper is organized as follows. In Section 2, we recall the definition of the regular Lagrangian flow and some important facts. An example of a vector field $b$ satisfying the condition (H) is also given there. Then in section 3, we construct a unique regular Lagrangian flow under the condition (H) and the boundedness of the divergence of $b$, following the direct method of Crippa and de Lellis. Finally in Section 4, we show a regularity property of the flow, which is weaker than the approximate differentiability discussed in [6] and prove a compactness result on the flow. To avoid technical complexities, we assume that the vector fields are bounded throughout this paper.

2 Preparations and an example

We first give the definition of the flow associated to a vector field $b$ (also called regular Lagrangian flow in [1, 6]).

**Definition 2.1 (Regular Lagrangian flow).** Let $b \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. A map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is called the regular Lagrangian flow associated to the vector field $b$ if

(i) for a.e. $x \in \mathbb{R}^d$, the function $t \to X_t(x)$ is absolutely continuous and satisfies

$$X_t(x) = x + \int_0^t b_s(X_s(x)) \, ds,$$

for all $t \in [0,T]$;

(ii) there exists a constant $L > 0$ independent of $t \in [0,T]$ such that $(X_t)_# \mathcal{L}^d \leq LL^d$.

Recall that $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $(X_t)_# \mathcal{L}^d$ is the push-forward of $\mathcal{L}^d$ by the flow $X_t$. $L$ will be called the compressibility constant of the flow $X$.

Next we introduce some notations and results that will be used in the subsequent sections. Denote by $\Gamma_T = C([0,T], \mathbb{R}^d)$, i.e. the space of continuous paths in $\mathbb{R}^d$. For $\gamma \in \Gamma_T$, we write $\|\gamma\|_{\infty,T}$ for its supremum norm. Let $\delta > 0$, we define an auxiliary function by (cf. [11, (2.7)])

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{\rho(s) + \delta}, \quad \xi \geq 0.$$  \hspace{1cm} (2.1)

Note that if $\rho(s) = s$ for all $s \geq 0$, then

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{s + \delta} = \log \left( \frac{\xi}{\delta} + 1 \right)$$

which is the functional used in (1.2). Here are some properties of $\psi_\delta$.

**Lemma 2.2.**

(1) $\lim_{\delta \downarrow 0} \psi_\delta(\xi) = +\infty$ for all $\xi > 0$;

(2) for any $\delta > 0$, the function $\psi_\delta$ is concave.
Proof. Property (1) follows from the fact that \( \int_{0}^{\infty} \frac{ds}{\rho(s)} = \infty \). To prove (2), we notice that 
\[
\psi'_{\delta}(s) = \frac{1}{\rho(s)+\delta}.
\]
Since \( s \mapsto \rho(s) \) is increasing, we see that the derivative \( \psi'_{\delta}(s) \) is monotone decreasing, and hence \( \psi_{\delta} \) is concave. \( \square \)

The concavity of \( \psi_{\delta} \) will play an important role in the arguments of Section 4. Finally we give an inequality concerning the local maximal function (see [6, Lemma A.2]).

**Lemma 2.3.** Let \( R, \lambda \) and \( \alpha \) be positive constants. Then there is \( C_d \) depending only on the dimension \( d \), such that

\[
L^d \{ x \in B(R) : M_{\lambda} f(x) > \alpha \} \leq \frac{C_d}{\alpha} \int_{B(R+\lambda)} |f(y)| \, dy.
\]

Before finishing this section, we give an example of a vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \) which satisfies the condition (H), but at the same time satisfies neither (1.1) nor (1.3). The basic idea is to consider the sum of a Sobolev vector field and an Osgood continuous vector field.

**Example 2.4.** For \( t \in \mathbb{R} \), let

\[
V(t) = \sum_{k=1}^{\infty} \frac{|\sin kt|}{k^2}.
\]

Then by [11, (2.12)], we have

\[
|V(t) - V(s)| \leq C_1 |t - s| \log \frac{1}{|t - s|}, \quad \text{for } |t - s| \leq e^{-1},
\]

where \( C_1 > 0 \) is a constant. Define the function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) as follows:

\[
\rho(s) = \begin{cases} 
  s \log \frac{1}{s}, & 0 \leq s \leq e^{-2}; \\
  s + e^{-2}, & s > e^{-2}.
\end{cases}
\]

Then \( \rho \in C^1(\mathbb{R}_+) \) is strictly increasing and \( \rho(s) \geq s \log \frac{1}{s} \) for all \( s \in [0, e^{-1}] \), hence

\[
|V(t) - V(s)| \leq C_1 \rho(|t - s|), \quad \text{for } |t - s| \leq e^{-1}.
\]

It is clear that \( 0 \leq V(t) \leq \frac{\pi^2}{6} \), thus for \( |t - s| > e^{-1} \),

\[
|V(t) - V(s)| \leq \frac{\pi^2}{3} \leq \frac{\pi^2}{3} \rho(e^{-1}) \leq \frac{\pi^2}{3} \rho(|t - s|).
\]

Let \( C_2 := C_1 \vee \frac{\pi^2}{3} \); then

\[
|V(t) - V(s)| \leq C_2 \rho(|t - s|), \quad \text{for all } t, s \in \mathbb{R}.
\]  \( (2.2) \)

As mentioned in [11, Remark 2.10], \( V(t) \sim t \log \frac{1}{t} \) as \( t \downarrow 0 \), hence it is not locally Lipschitz continuous.

Now we take a Sobolev vector field \( b_1 \in W^{1,p}(\mathbb{R}^d, \mathbb{R}^d) \) \( (p > 1) \) which is not continuous. For \( x \in \mathbb{R}^d \), define \( b_2(x) = (V(x_1), \ldots, V(x_d)) \). Then by [12, THEOREM 1], there exist \( g \in L^p(\mathbb{R}^d) \) and a negligible set \( N \subset \mathbb{R}^d \) such that

\[
|b_1(x) - b_1(y)| \leq |x - y| (g(x) + g(y)), \quad \text{for all } x, y \notin N.
\]

Next by (2.2), it holds for all \( x, y \in \mathbb{R}^d \) that

\[
|b_2(x) - b_2(y)| \leq \sum_{i=1}^{d} |V(x_i) - V(y_i)| \leq \sum_{i=1}^{d} C_2 \rho(|x_i - y_i|) \leq C_2 d \rho(|x - y|).
\]
Finally we set $b = b_1 + b_2$. Then by the above two inequalities, for all $x, y \notin N$,

$$|b(x) - b(y)| \leq |b_1(x) - b_1(y)| + |b_2(x) - b_2(y)| \leq |x - y|(|g(x) + g(y)| + C_2d \rho(|x - y|) \leq C_2d \rho(|x - y|)((1 + g(x)) + (1 + g(y)))$$

where in the last inequality we have used the facts that $C_2d > 1$ and $\rho(s) \geq s$ for all $s \geq 0$. Note that the function $1 + g \in L^p_{loc}(\mathbb{R}^d)$, thus it is locally integrable.

### 3 Existence and uniqueness of the regular Lagrangian flow

In order to prove the existence and uniqueness of the flow generated by a vector field $b$ satisfying the assumption (H), we first establish an a-priori estimate.

**Theorem 3.1.** Let $b$ and $\tilde{b}$ be time dependent bounded vector fields satisfying (H) with $g$ and $\tilde{g}$ respectively. Let $X$ and $\tilde{X}$ be the regular Lagrangian flows associated to $b$ and $\tilde{b}$, with the compressibility constants $L$ and $\tilde{L}$ respectively. Then for any $R > 0$ and $t \leq T$,

$$\int_{B(R)} \psi_\delta(\|X(x) - \tilde{X}(x)\|_{\infty,T}) \, dx \leq (L + \tilde{L})\|g\|_{L^1([0,T] \times B(R))} + \frac{\tilde{L}}{\delta}\|b - \tilde{b}\|_{L^1([0,T] \times B(\tilde{R}))},$$

where $\psi_\delta$ is defined in (2.1), $\|\cdot\|_{\infty,T}$ is the supremum norm in $\Gamma_T$ and $\tilde{R} = R + T(\|b\|_{L^\infty} \vee \|\tilde{b}\|_{L^\infty})$.

**Proof.** By Definition 2.1(1), for a.e. $x \in \mathbb{R}^d$, the function $t \mapsto |X_t(x) - \tilde{X}_t(x)|$ is Lipschitz continuous, hence

$$\frac{d}{dt}[\psi_\delta(|X_t(x) - \tilde{X}_t(x)|)] = \psi_\delta'(|X_t(x) - \tilde{X}_t(x)|) \frac{d}{dt}|X_t(x) - \tilde{X}_t(x)| \leq \frac{|b_t(X_t(x)) - \tilde{b}_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|) + \delta}.$$

Integrating from 0 to $t$ and noticing that $\psi_\delta(0) = 0$, we get

$$\psi_\delta(|X_t(x) - \tilde{X}_t(x)|) \leq \int_0^t \frac{|b_s(X_s(x)) - \tilde{b}_s(\tilde{X}_s(x))|}{\rho(|X_s(x) - \tilde{X}_s(x)|) + \delta} \, ds.$$ 

As a result,

$$\sup_{0 \leq t \leq T} \psi_\delta(|X_t(x) - \tilde{X}_t(x)|) \leq \int_0^T \frac{|b_t(X_t(x)) - \tilde{b}_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|) + \delta} \, dt.$$ 

Since the function $\psi_\delta$ is continuous, we arrive at

$$\psi_\delta(\|X(x) - \tilde{X}(x)\|_{\infty,T}) \leq \int_0^T \frac{|b_t(X_t(x)) - \tilde{b}_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|) + \delta} \, dt.$$ 

Therefore

$$\int_{B(R)} \psi_\delta(\|X(x) - \tilde{X}(x)\|_{\infty,T}) \, dx \leq \int_0^T \int_{B(R)} \frac{|b_t(X_t(x)) - \tilde{b}_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|) + \delta} \, dx \, dt. \quad (3.1)$$
Denote by $I$ the integral on the right hand side of (3.1). Using the triangular inequality, we obtain

$$I \leq \int_0^T \int_{B(R)} \left| \frac{b_t(X_t(x)) - b_t(\tilde{X}_t(x))}{\rho(|X_t(x) - \tilde{X}_t(x)|)} \right| \, dx \, dt + \int_0^T \int_{B(R)} \frac{|b_t(\tilde{X}_t(x)) - \tilde{b}_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|)} \, dx \, dt. \quad (3.2)$$

Since the flows $X_t$ and $\tilde{X}_t$ leave the Lebesgue measure absolutely continuous, we can apply the condition (H) for $b_t$ and obtain that for a.e. $x \in B(R)$,

$$|b_t(X_t(x)) - b_t(\tilde{X}_t(x))| \leq (g_t(X_t(x)) + g_t(\tilde{X}_t(x)))\rho(|X_t(x) - \tilde{X}_t(x)|).$$

Next it is clear that from Definition 2.1(i), one has $\|X_t(x)\|_{\infty,T} \leq R + T\|b\|_{L^\infty}$ and $\|\tilde{X}_t(x)\|_{\infty,T} \leq R + T\|\tilde{b}\|_{L^\infty}$ for a.e. $x \in B(R)$. Therefore, by the definition of the compressibility constants $L$ and $\tilde{L}$, the first term on the right hand side of (3.2) can be estimated as follows:

$$\int_0^T \int_{B(R)} \frac{|b_t(X_t(x)) - b_t(\tilde{X}_t(x))|}{\rho(|X_t(x) - \tilde{X}_t(x)|)} \, dx \, dt \leq \int_0^T \int_{B(R)} (g_t(X_t(x)) + g_t(\tilde{X}_t(x))) \, dx \, dt$$

$$\leq L \int_0^T \int_{B(R)} g_t(y) \, dy \, dt + \tilde{L} \int_0^T \int_{B(R)} g_t(y) \, dy \, dt$$

$$= (L + \tilde{L}) \|g\|_{L^1([0,T] \times B(R))}. \quad (3.3)$$

Moreover, the second integral in (3.2) is dominated by

$$\frac{1}{\delta} \int_0^T \int_{B(R)} \left| b_t(\tilde{X}_t(x)) - \tilde{b}_t(\tilde{X}_t(x)) \right| \, dx \, dt \leq \frac{\tilde{L}}{\delta} \int_0^T \int_{B(R)} |b_t(y) - \tilde{b}_t(y)| \, dy \, dt$$

$$= \frac{\tilde{L}}{\delta} \|b - \tilde{b}\|_{L^1([0,T] \times B(R))}.$$

Combining this with (3.2) and (3.3), we obtain

$$I \leq (L + \tilde{L}) \|g\|_{L^1([0,T] \times B(R))} + \frac{\tilde{L}}{\delta} \|b - \tilde{b}\|_{L^1([0,T] \times B(R))}.$$

Substituting $I$ into (3.1), we complete the proof. \qed

Now we can prove the main result of this section.

**Theorem 3.2** (Existence and uniqueness of the regular Lagrangian flow). Assume that $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded vector field satisfying (H) with $g \in L^1([0,T], L^1_{loc}(\mathbb{R}^d))$. Moreover, the distributional divergence $\text{div}(b)$ of $b$ exists and $|\text{div}(b)|^{-} \in L^1([0,T], L^\infty(\mathbb{R}^d))$. Then there exists a unique regular Lagrangian flow generated by $b$.

**Proof.** Step 1: Uniqueness. Suppose there are two regular Lagrangian flows $X_t$ and $\tilde{X}_t$ associated to $b$ with compressibility constants $L$ and $\tilde{L}$ respectively. Applying Theorem 3.1, we have

$$\int_{B(R)} \psi_\delta \left( \|X(x) - \tilde{X}(x)\|_{\infty,T} \right) \, dx \leq (L + \tilde{L}) \|g\|_{L^1([0,T] \times B(\tilde{R}))}, \quad (3.4)$$

where $\tilde{R} = R + T\|b\|_{L^\infty}$. If $\mathcal{L}^d \{ x \in B(R) : X(x) \neq \tilde{X}(x) \} > 0$, then there is $\varepsilon_0 > 0$ such that $K_{\varepsilon_0} := \{ x \in B(R) : \|X(x) - \tilde{X}(x)\|_{\infty,T} \geq \varepsilon_0 \}$ has positive measure. Thus by (3.4),

$$(L + \tilde{L}) \|g\|_{L^1([0,T] \times B(\tilde{R}))} \geq \int_{K_{\varepsilon_0}} \psi_\delta \left( \|X(x) - \tilde{X}(x)\|_{\infty,T} \right) \, dx \geq \psi_\delta(\varepsilon_0) \mathcal{L}^d(K_{\varepsilon_0}).$$
Letting $\delta \downarrow 0$, we deduce from Lemma 2.2(1) that

$$\infty > (L + \hat{L})\|g\|_{L^1([0,T] \times B(R))} \geq \infty,$$

which is a contradiction. Hence $N = \{x \in B(R) : X(x) \neq \hat{X}(x)\}$ is $\mathcal{L}^d$-negligible. We conclude that the two flows $X_t$ and $\hat{X}_t$ coincide with each other on the interval $[0,T]$.

**Step 2: Existence.** Let $\{\chi_n : n \geq 1\}$ be a sequence of standard convolution kernels. For $t \in [0,T]$, define $b^n_t = b_t * \chi_n$, i.e. the convolution of $b_t$ and $\chi_n$. Then for every $n \geq 1$, $b^n$ is a time dependent smooth vector field, and

$$\|b^n_t\|_{L^\infty} \leq \|b_t\|_{L^\infty}, \quad \|\text{div}(b^n_t)\|_{L^\infty} \leq \|\text{div}(b_t)\|_{L^\infty}, \quad t \in [0,T].$$

Let $X^n_t$ be the smooth flow generated by $b^n_t$, then it is easy to know that $(X^n_t)_{#} \mathcal{L}^d \leq L_n \mathcal{L}^d$, where

$$L_n = \exp\left(\int_0^T \|\text{div}(b^n_t)\|_{L^\infty} dt\right) \leq \exp\left(\int_0^T \|\text{div}(b_t)\|_{L^\infty} dt\right) =: L.$$

Now we show that $b^n_t$ satisfies (H) with $g^n_t = g_t * \chi_n$ for all $n \geq 1$. To this end, we fix any two points $x, y \in \mathbb{R}^d$. We have by the definition of convolution,

$$|b^n_t(x) - b^n_t(y)| \leq \int_{\mathbb{R}^d} |b_t(x - z) - b_t(y - z)| \chi_n(z) \, dz.$$

Now we shall make use of the condition (H). Notice that $(x - N_t) \cup (y - N_t)$ is a negligible subset. When $z \notin (x - N_t) \cup (y - N_t)$, one has $x - z \notin N_t$ and $y - z \notin N_t$, hence by (H),

$$|b_t(x - z) - b_t(y - z)| \leq (g_t(x - z) + g_t(y - z)) \rho(|x - y|).$$

As a result,

$$|b^n_t(x) - b^n_t(y)| \leq \int_{\mathbb{R}^d} (g_t(x - z) + g_t(y - z)) \rho(|x - y|) \chi_n(z) \, dz = (g^n_t(x) + g^n_t(y)) \rho(|x - y|). \tag{3.5}$$

Thus $b^n_t$ satisfies (H) with the function $g^n_t$. Notice that (3.5) holds for all $x, y \in \mathbb{R}^d$.

From the above arguments, we can apply Theorem 3.1 to the sequence of smooth flows $\{X^n_t : n \geq 1\}$ and get

$$\int_{B(R)} \psi_d\left(\|X^n(x) - X^m(x)\|_{\infty,T}\right) \, dx$$

$$\leq (L_n + L_m)\|g_n\|_{L^1([0,T] \times B(R))} + \frac{L_m}{\delta}\|b^n - b^m\|_{L^1([0,T] \times B(R))}$$

$$\leq 2L\|g\|_{L^1([0,T] \times B(R))} + \frac{L}{\delta}\|b^n - b^m\|_{L^1([0,T] \times B(R))}. \tag{3.6}$$

Set

$$\delta = \delta_{n,m} := \|b^n - b^m\|_{L^1([0,T] \times B(R))}$$

which tends to 0 as $n, m \to \infty$, since $b^n$ converges to $b$ in $L^1([0,T], L^1_{\text{loc}}(\mathbb{R}^d))$. Then we obtain

$$\int_{B(R)} \psi_{\delta_{n,m}}\left(\|X^n(x) - X^m(x)\|_{\infty,T}\right) \, dx \leq 2L\|g\|_{L^1([0,T] \times B(R))} + L =: C < \infty. \tag{3.7}$$
We will show that \( \{X^n : n \geq 1\} \) is a Cauchy sequence in \( L^1(B(R), \Gamma_T) \). For any \( \eta > 0 \), let
\[
K_{n,m} = \{ x \in B(R) : \|X^n(x) - X^m(x)\|_{\infty,T} \leq \eta \} = \{ x \in B(R) : \psi_{\delta_{n,m}}(\|X^n(x) - X^m(x)\|_{\infty,T}) \leq \psi_{\delta_{n,m}}(\eta) \}.
\]

By Chebyshev’s inequality and (3.7),
\[
\mathcal{L}^d(B(R) \setminus K_{n,m}) \leq \frac{1}{\psi_{\delta_{n,m}}(\eta)} \int_{B(R)} \psi_{\delta_{n,m}}(\|X^n(x) - X^m(x)\|_{\infty,T}) \, dx \leq \frac{C}{\psi_{\delta_{n,m}}(\eta)}.
\]

Therefore
\[
\int_{B(R)} \|X^n(x) - X^m(x)\|_{\infty,T} \, dx = \left( \int_{K_{n,m}} + \int_{B(R) \setminus K_{n,m}} \right) \|X^n(x) - X^m(x)\|_{\infty,T} \, dx \\
\leq \eta \mathcal{L}^d(K_{n,m}) + 2(R + T \|b\|_{L^\infty}) \mathcal{L}^d(B(R) \setminus K_{n,m}) \\
\leq \eta \mathcal{L}^d(B(R)) + \frac{2R}{\psi_{\delta_{n,m}}(\eta)} C.
\]

Note that as \( n, m \to \infty, \delta_{n,m} \to 0 \), thus by Lemma 2.2(1), \( \psi_{\delta_{n,m}}(\eta) \to \infty \) for any \( \eta > 0 \). Consequently,
\[
\lim_{n,m \to \infty} \int_{B(R)} \|X^n(x) - X^m(x)\|_{\infty,T} \, dx \leq \eta \mathcal{L}^d(B(R)).
\]

By the arbitrariness of \( \eta > 0 \), we conclude that \( \{X^n : n \geq 1\} \) is a Cauchy sequence in \( L^1(B(R), \Gamma_T) \) for any \( R > 0 \). Therefore, there exists a measurable map \( X : \mathbb{R}^d \to \Gamma_T \) which is the limit in \( L^1_{\text{loc}}(\mathbb{R}^d, \Gamma_T) \) of \( X^n \). We can find a subsequence \( \{n_k : k \geq 1\} \), such that for a.e. \( x \in \mathbb{R}^d, X^n_{n_k}(x) \) converges to \( X_t(x) \) uniformly in \( t \in [0, T] \). Hence we still have
\[
\|X_t(x)\|_{\infty,T} \leq R + T \|b\|_{L^\infty} = \bar{R}, \quad \text{for a.e. } x \in B(R). \tag{3.8}
\]

Now we prove that \( X_t \) is a regular Lagrangian flow generated by \( b \). Firstly, for any \( \phi \in C_c(\mathbb{R}^d, \mathbb{R}_+) \), we have by the Fatou lemma,
\[
\int_{\mathbb{R}^d} \phi(X_t(x)) \, dx \leq \liminf_{k \to \infty} \int_{\mathbb{R}^d} \phi(X^n_{n_k}(x)) \, dx \leq L \int_{\mathbb{R}^d} \phi(y) \, dy.
\]

This implies
\[
(X_t)_{\#} \mathcal{L}^d \leq L \mathcal{L}^d, \quad \text{for all } t \in [0, T]; \tag{3.9}
\]
thus Definition 2.1(ii) is satisfied. Secondly, we show that for \( \mathcal{L}^d \)-a.e. \( x \in \mathbb{R}^d, t \to X_t(x) \) is an integral curve of the vector field \( b_t \). To this end, we estimate the quantity
\[
J^n := \int_{B(R)} \sup_{0 \leq s \leq T} \left| \int_0^t b^n_s(X^n_s(x)) \, ds - \int_0^t b_s(X_s(x)) \, ds \right| \, dx.
\]

By the triangular inequality, \( J^n \) is dominated by the sum of
\[
J^n_1 := \int_{B(R)} \int_0^T \left| b^n_s(X^n_s(x)) - b_s(X^n_s(x)) \right| \, ds \, dx
\]
and
\[
J^n_2 := \int_{B(R)} \int_0^T \left| b_s(X^n_s(x)) - b_s(X_s(x)) \right| \, ds \, dx.
\]
For the first term, we have

\[ J^n_1 = \int_0^T \int_{B(R)} |b^n_s(x_t^n(x)) - b_s(X^n_t(x))| \, dx \, ds \leq L \int_0^T \int_{B(R)} |b^n_s(y) - b_s(y)| \, dy \, ds. \]

Hence

\[ \lim_{n \to \infty} J^n_1 = 0. \tag{3.10} \]

For any \( \varepsilon > 0 \), we take a vector field \( \hat{b} \in C^1([0, T] \times B(\bar{R}), \mathbb{R}^d) \) such that

\[ \int_0^T \int_{B(R)} |\hat{b}_s(x) - b_s(x)| \, dx \, ds < \varepsilon. \]

Again by the triangular inequality,

\[ J^n_2 \leq \int_0^T \int_{B(R)} |b_s(X^n_s(x)) - \hat{b}_s(X^n_s(x))| \, dx \, ds + \int_0^T \int_{B(R)} |\hat{b}_s(X^n_s(x)) - \hat{b}_s(X_s(x))| \, dx \, ds \]

\[ \quad + \int_0^T \int_{B(R)} |\hat{b}_s(X_s(x)) - b_s(X_s(x))| \, dx \, ds \]

\[ =: J^n_{2,1} + J^n_{2,2} + J^n_{2,3}. \]

Since \( (X^n_s) \# \mathcal{L}^d \leq L \mathcal{L}^d \) for all \( s \in [0, T] \), we have

\[ J^n_{2,1} \leq L \int_0^T \int_{B(R)} |b_s(y) - \hat{b}_s(y)| \, dy \, ds \leq L \varepsilon. \]

By (3.8) and (3.9), the same argument leads to

\[ J^n_{2,3} < L \varepsilon. \]

Moreover, by the choice of \( \hat{b} \), there is \( C_1 > 0 \) such that \( \sup_{0 \leq s \leq T} \| \nabla \hat{b}_s \|_{L^\infty(B(\bar{R}))} \leq C_1 \). Therefore

\[ J^n_{2,2} \leq C_1 \int_0^T \int_{B(R)} |X^n_s(x) - X_s(x)| \, dx \, ds \leq C_1 T \int_{B(R)} \|X^n_s(x) - X_s(x)\|_{\infty,T} \, dx \to 0 \]

as \( n \) goes to \( \infty \). Summing up the above arguments, we get

\[ \limsup_{n \to \infty} J^n_2 \leq 2L \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain \( \lim_{n \to \infty} J^n_2 = 0 \). Combining this with (3.10), we finally obtain \( \lim_{n \to \infty} J^n = 0 \). Therefore, letting \( n \to \infty \) in the equality

\[ X^n_t(x) = x + \int_0^t b^n_s(X^n_s(x)) \, ds \quad \text{for all } t \leq T, \]

we see that both sides converge in \( L^1_{loc}(\mathbb{R}^d, \Gamma_T) \) to \( X_t(x) \) and \( x + \int_0^t b_s(X_s(x)) \, ds \) respectively. Hence for \( \mathcal{L}^d \)-a.e. \( x \in \mathbb{R}^d \), it holds

\[ X_t(x) = x + \int_0^t b_s(X_s(x)) \, ds \quad \text{for all } t \in [0, T]; \]

that is, \( t \to X_t(x) \) is an integral curve of the vector field \( b_t \). To sum up, \( X_t \) is a regular Lagrangian flow generated by \( b \).

\[ \square \]
Remark 3.3. The condition $[\text{div}(b)]^- \in L^1([0,T],L^\infty(\mathbb{R}^d))$ can be relaxed as $[\text{div}(b)]^- \in L^1([0,T],L^\infty_{\text{loc}}(\mathbb{R}^d))$, since we have good estimate on the growth of the flow $X_t$.

Remark 3.4. Under the condition (H), it seems to the authors that one is unable to prove the well posedness of the transport equation

$$\frac{\partial}{\partial t} u_t + b_t \cdot \nabla u_t + c_t u_t = 0, \quad u_{t=0} = u_0.$$ 

by following DiPerna-Lions’s original approach, that is, by showing that the commutator

$$r_n(b_t, u_t) = (b_t \cdot \nabla u_t) \ast \chi_n - b_t \cdot \nabla (u_t \ast \chi_n)$$

converges to 0 strongly in $L^1_{\text{loc}}$. Here $\chi_n$ is the standard convolution kernel. This can be seen from the proof of [7, Lemma II.1] (or [2, Proposition 4.1]), which essentially relies on the “almost everywhere Lipschitz continuity” of Sobolev vector fields.

4 Regularity of the flow

In this section, we first prove a regularity result on the regular Lagrangian flow, a property much weaker than the approximate differentiability discussed in [6]. We need the following notation: for a bounded measurable subset $U$ with positive measure, define the average of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ on $U$ by

$$\int_U f \, dx = \frac{1}{\mathcal{L}^d(U)} \int_U f \, dx.$$ 

Then the local maximal function

$$M_R f(x) = \sup_{0 < r \leq R} \int_{B(x,r)} |f(y)| \, dy.$$ 

Now we can prove

**Theorem 4.1.** Let $b$ be a bounded vector field satisfying (H), and $[\text{div}(b)]^- \in L^1([0,T],L^\infty(\mathbb{R}^d))$. Let $X$ be the unique regular Lagrangian flow associated to $b$. Then for any $R > 0$ and sufficiently small $\varepsilon$, there are a measurable subset $E \subset B(R)$ and some constant $C$ depending on $R, d$ and $g$, such that $\mathcal{L}^d(B(R) \setminus E) \leq \varepsilon$ and for all $t \in [0,T]$ and $x, y \in E$, one has

$$|X_t(x) - X_t(y)| \leq \psi_{|x-y|}^{-1}(C/\varepsilon).$$

Here $\psi_{|x-y|}^{-1}$ is the inverse function of $\psi_{|x-y|}$. Note that by Lemma 2.2(1), we have $\lim_{r \to 0} \psi_{|x-y|}^{-1}(\xi) = 0$ for all $\xi > 0$. Therefore this theorem implies that $X_t$ is uniformly continuous in $E$, since when $y \to x$ in the subset $E$, the quantity $\psi_{|x-y|}^{-1}(C/\varepsilon)$ decreases to 0. Unfortunately, the function $\psi_{|x-y|}^{-1}$ does not have an explicit expression, unless $\rho(s) = s$ for all $s \geq 0$ (see Remark 4.2).

**Proof of Theorem 4.1.** We follow the ideas of [6, Remark 2.4] (see also [14, Proposition 5.2]). For $0 \leq t \leq T$, $0 < r \leq 2R$ and $x \in B(R)$, define

$$Q(t, x, r) = \int_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|) \, dy.$$ 

Then

$$Q(0, x, r) = \int_{B(x,r)} \psi_r(|x - y|) \, dy \leq \int_{B(x,r)} \psi_r(r) \, dy \leq 1.$$
By Definition 2.1(i), we see that \( t \mapsto Q(t, x, r) \) is Lipschitz and
\[
\frac{d}{dt}Q(t, x, r) = \int_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|) \frac{d}{dt}|X_t(x) - X_t(y)| \, dy 
\leq \int_{B(x,r)} \frac{|b_t(X_t(x)) - b_t(X_t(y))|}{\rho(|X_t(x) - X_t(y)|)} + r \, dy.
\]

Using the assumption (H) on \( b \), we have
\[
\frac{d}{dt}Q(t, x, r) \leq \int_{B(x,r)} (g_t(X_t(x)) + g_t(X_t(y))) \, dy = g_t(X_t(x)) + \int_{B(x,r)} g_t(X_t(y)) \, dy.
\]

Integrating both sides with respect to time from 0 to \( t \), we arrive at
\[
Q(t, x, r) \leq Q(0, x, r) + \int_0^t g_s(X_s(x)) \, ds + \int_0^t \int_{B(x,r)} g_s(X_s(y)) \, dy \, ds 
\leq 1 + \int_0^T g_s(X_s(x)) \, ds + \int_0^T \int_{B(x,r)} g_s(X_s(y)) \, dy \, ds. \quad (4.1)
\]

Denote by \( \Phi(x) = \int_0^T g_s(X_s(x)) \, ds \) for a.e. \( x \in \mathbb{R}^d \). Then for all \( t \leq T \),
\[
Q(t, x, r) \leq 1 + \Phi(x) + \int_{B(x,r)} \Phi(y) \, dy.
\]

Therefore
\[
\sup_{0 \leq t \leq T} \sup_{0 < r \leq 2R} Q(t, x, r) \leq 1 + \Phi(x) + M_2 \Phi(x). \quad (4.2)
\]

For \( \eta > 0 \) sufficiently small, we have
\[
I := \mathcal{L}^d \{ x \in B(R) : 1 + \Phi(x) + M_2 \Phi(x) > 1/\eta \} 
\leq \mathcal{L}^d \{ x \in B(R) : \Phi(x) > 1/(3\eta) \} + \mathcal{L}^d \{ x \in B(R) : M_2 \Phi(x) > 1/(3\eta) \}.
\]

By Chebyshev’s inequality and Lemma 2.3, we have
\[
I \leq 3\eta \int_{B(R)} \Phi(x) \, dx + 3\eta C_d \int_{B(3R)} \Phi(y) \, dy 
\leq 3\eta(1 + C_d) \int_{B(3R)} \Phi(y) \, dy.
\]

By the definition of \( \Phi \), one has
\[
I \leq 3\eta(1 + C_d) \int_0^T \int_{B(3R)} g_t(X_t(y)) \, dy \, dt
\leq 3\eta(1 + C_d) L \int_0^T \int_{B(3R+T||b||L^\infty)} g_t(x) \, dx \, dt.
\]

Let \( \bar{C} := 3(1 + C_d)L\|g\|_{L^1([0,T] \times B(3R+T||b||L^\infty))} \); then \( I \leq \eta \bar{C} \).

Now for any \( \varepsilon > 0 \), set \( \eta = \varepsilon / \bar{C} \). Then by (4.2) and the definition of \( I \),
\[
\mathcal{L}^d \{ x \in B(R) : \sup_{0 \leq t \leq T} \sup_{0 < r \leq 2R} Q(t, x, r) > \frac{\bar{C}}{\varepsilon} \} \leq I \leq \eta \bar{C} = \varepsilon.
\]
Let
\[ E = \left\{ x \in B(R) : \sup_{0 \leq t \leq T} \sup_{0 < r \leq 2R} Q(t, x, r) \leq \frac{\bar{C}}{\varepsilon} \right\}. \]

Then \( \mathcal{L}^d(B(R) \setminus E) \leq \varepsilon \) and for any \( x \in E, 0 \leq t \leq T \) and \( 0 < r \leq 2R \), one has
\[
\int_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|) \, dy \leq \frac{\bar{C}}{\varepsilon}. \tag{4.3}
\]

Now fix any \( x, y \in E \) and let \( r = |x - y| \) which is less than \( 2R \). Lemma 2.2(2) tells us that \( \psi_r \) is concave, hence \( \psi_r(a + b) \leq \psi_r(a) + \psi_r(b) \) for any \( a, b \geq 0 \). As a result,
\[
\psi_r(|X_t(x) - X_t(y)|) \leq \psi_r(|X_t(x) - X_t(z)|) + \psi_r(|X_t(z) - X_t(y)|).
\]

Therefore,
\[
\psi_r(|X_t(x) - X_t(y)|) = \int_{B(x,r) \cap B(y,r)} \psi_r(|X_t(x) - X_t(y)|) \, dz \leq \int_{B(x,r) \cap B(y,r)} \psi_r(|X_t(x) - X_t(z)|) \, dz + \int_{B(x,r) \cap B(y,r)} \psi_r(|X_t(z) - X_t(y)|) \, dz.
\]

Let \( \bar{C}_d = \mathcal{L}^d(B(x, r))/\mathcal{L}^d(B(x, r) \cap B(y, r)) \) which only depends on the dimension \( d \); then
\[
\psi_r(|X_t(x) - X_t(y)|) \leq \bar{C}_d \int_{B(x,r)} \psi_r(|X_t(x) - X_t(z)|) \, dz + \bar{C}_d \int_{B(y,r)} \psi_r(|X_t(z) - X_t(y)|) \, dz \leq 2\bar{C}_d \bar{C}/\varepsilon,
\]

where the last inequality follows from (4.3). Consequently, for all \( t \leq T \) and \( x, y \in E \),
\[
|X_t(x) - X_t(y)| \leq \psi_r^{-1}(2\bar{C}_d \bar{C}/\varepsilon) = \psi_{|x-y|}^{-1}(2\bar{C}_d \bar{C}/\varepsilon).
\]

\[ \square \]

**Remark 4.2.** If \( \rho(s) = s \) for all \( s \geq 0 \), then \( \psi_r(s) = \log \left( \frac{s}{r} + 1 \right) \) and \( \psi_r^{-1}(t) = r(e^t - 1) \). Thus the last inequality in the proof of Theorem 4.1 becomes
\[
|X_t(x) - X_t(y)| \leq |x - y| \left( e^{2\bar{C}_d \bar{C}/\varepsilon} - 1 \right) \leq |x - y| e^{2\bar{C}_d \bar{C}/\varepsilon},
\]

which is the estimate given in \( [6, \text{Proposition 2.3}] \) in the case \( p = 1 \).

We complete this section by discussing the compactness of the regular Lagrangian flow, following the ideas in \( [6, \text{Section 4}] \). For fixed \( R > 0 \) and \( 0 < r < R/2 \), set
\[
a(r, R, X) = \int_{B(R)} \sup_{0 \leq t \leq T} \int_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|) \, dy \, dx.
\]

**Proposition 4.3.** Let \( b \) be a bounded vector field satisfying the condition (H) with some function \( g \in L^1([0, T], L^1_{\text{loc}}(\mathbb{R}^d)) \). Let \( X \) be a regular Lagrangian flow associated to \( b \) with the compressibility constant \( L \). Then
\[
a(r, R, X) \leq \mathcal{L}^d(B(R)) + 2L\|g\|_{L^1([0, T] \times B(R))},
\]

where \( \bar{R} = 3R/2 + 2T\|b\|_{L^\infty} \).
**Proof.** Recall the definition of $Q(t, x, r)$ at the beginning of the proof of Theorem 4.1. We still have the estimate (4.1): for any $t \leq T$,

$$Q(t, x, r) \leq 1 + \int_{0}^{T} g_{s}(X_{s}(x)) \, ds + \int_{0}^{T} f_{B(x, r)} g_{s}(X_{s}(y)) \, dyds.$$ 

Therefore

$$a(r, R, X) = \int_{B(R)} \sup_{0 \leq t \leq T} Q(t, x, r) \, dx$$

$$\leq L^{d}(B(R)) + \int_{B(R)} \int_{0}^{T} g_{s}(X_{s}(x)) \, dsdx + \int_{B(R)} \int_{0}^{T} \int_{B(x, r)} g_{s}(X_{s}(y)) \, dydsdx.$$ 

We denote by $I_{1}$ and $I_{2}$ the two integrals on the right hand side respectively. First

$$I_{1} = \int_{0}^{T} \int_{B(R)} g_{s}(X_{s}(x)) \, dxds \leq \int_{0}^{T} L \int_{B(R)} g_{s}(y) \, dyds = L\|g\|_{L^{1}([0, T] \times B(R))}.$$ 

For the second integral, by changing the order of integration, we have

$$I_{2} = \int_{B(R)} \int_{0}^{T} g_{s}(X_{s}(x + z)) \, dzdsdx = \int_{B(R)} \int_{0}^{T} \int_{B(x, r)} g_{s}(X_{s}(y)) \, dydsdz.$$ 

Therefore,

$$I_{2} \leq \int_{B(R)} \int_{0}^{T} L \int_{B(R)} g_{s}(y) \, dydsdz = L\|g\|_{L^{1}([0, T] \times B(R))}.$$ 

Combining the above two estimates, we arrive at the conclusion. \hfill \Box

Now we can prove

**Theorem 4.4** (Compactness of the flow). Let $\{b^{n} : n \geq 1\}$ be a sequence of vector fields equi-bounded in $L^{\infty}([0, T] \times \mathbb{R}^{d})$. For every $n \geq 1$, assume that $b^{n}$ satisfies (H) with the function $g^{n}$, and the family $\{g^{n} : n \geq 1\}$ is equi-bounded in $L^{1}([0, T], L^{1}_{loc}(\mathbb{R}^{d}))$. Let $X^{n}$ be a regular Lagrangian flow associated to $b^{n}$ with the compressibility constant $L_{n}$. Suppose $\sup_{n \geq 1} L_{n} \leq L < \infty$. Then the sequence $\{X^{n} : n \geq 1\}$ is strongly precompact in $L^{1}_{loc}([0, T] \times \mathbb{R}^{d})$.

**Proof.** Applying the estimate in Proposition 4.3 to the flow $X^{n}$, we get

$$a(r, R, X^{n}) \leq L^{d}(B(R)) + 2L_{n}\|g^{n}\|_{L^{1}([0, T] \times B(R_{n}))},$$

where $R_{n} = 3R/2 + 2T\|b^{n}\|_{L^{\infty}}$. Since $\{b^{n}\}$ is equi-bounded, we see that $\bar{R} := \sup_{n \geq 1} \bar{R}_{n} < \infty$. Moreover, by the boundedness of the sequence $\{g^{n}\}$ in $L^{1}([0, T], L^{1}_{loc}(\mathbb{R}^{d}))$, we obtain

$$\sup_{n \geq 1} a(r, R, X^{n}) \leq L^{d}(B(R)) + 2L \sup_{n \geq 1}\|g^{n}\|_{L^{1}([0, T] \times B(\bar{R}))} =: C_{d, R, T} < \infty. \quad (4.4)$$

For $0 < z \leq \bar{R}$, by Lemma 2.2(2), one has

$$\frac{\psi_{r}(z)}{z} \geq \frac{\psi_{r}(\bar{R})}{\bar{R}}, \quad \text{or equivalently,} \quad z \leq \frac{\bar{R}}{\psi_{r}(\bar{R})} \psi_{r}(z).$$
Since \( y \in B(x, r) \) and \( r \leq R/2 \), it holds \(|X^n_t(x) - X^n_t(y)| \leq \tilde{R}_n \leq \tilde{R} \). Hence by (4.4),
\[
\int_{B(R)} \sup_{0 \leq t \leq T} \int_{B(x, r)} |X^n_t(x) - X^n_t(y)| \, dy \, dx \\
\leq \frac{\tilde{R}}{\psi_r(\tilde{R})} \int_{B(R)} \sup_{0 \leq t \leq T} \int_{B(x, r)} \psi_r(|X^n_t(x) - X^n_t(y)|) \, dy \, dx \\
= \frac{\tilde{R}}{\psi_r(\tilde{R})} a(r, R, X^n) \leq \frac{\tilde{R}}{\psi_r(\tilde{R})} C_{d,R,T} =: g(r),
\]
where the function \( g(r) \) does not depend on \( n \) and satisfies \( g(r) \downarrow 0 \) as \( r \) decreases to 0, by Lemma 2.2(1). Similar to the estimate of \( I_2 \) in the proof of Proposition 4.3, we change the order of integration and obtain
\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \int_{B(r)} \int_{B(R)} |X^n_t(x) - X^n_t(x + z)| \, dx \, dz \leq g(r) \mathcal{L}^d(B(r)). \tag{4.5}
\]
The rest of the proof is similar to that of [6, Corollary 4.2], hence we omit it. \( \square \)

With the above compactness result in mind, we can give another proof of the existence of the regular Lagrangian flow.

**Corollary 4.5** (Existence of the flow). Let \( b \) be a bounded vector field satisfying (H) with the function \( g \). Assume that \([\text{div}(b)]^- \in L^1([0, T], L^\infty(\mathbb{R}^d))\). Then there exists a regular Lagrangian flow associated to \( b \).

**Proof.** We regularize the vector field \( b \) as in Step 2 of the proof of Theorem 3.2. It is clear that the conditions of Theorem 4.4 are satisfied by the smooth vector fields \( \{b^n\} \) and the corresponding flows \( \{X^n\} \). As a result, the sequence \( \{X^n\} \) is strongly precompact in \( L^1_{\text{loc}}([0, T] \times \mathbb{R}^d) \), and every limit point of \( \{X^n\} \) is a regular Lagrangian flow associated to \( b \). \( \square \)

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