ON THE BEHAVIOR OF $F$-SIGNATURES, SPLITTING PRIMES, AND TEST MODULES UNDER FINITE COVERS

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Abstract. We give a comprehensive treatment on how $F$-signatures, splitting primes, splitting ratios, and test modules behave under finite covers. To this end, we expand on the notion of transposability along a section of the relative canonical module as first introduced by K. Schwede and K. Tucker.

1. Introduction

This work is concerned with natural invariants attached to Cartier modules over rings of positive characteristic. More precisely, it investigates their behavior under finite covers of the base ring. The prototypical Cartier module over an $\mathbb{F}_p$-algebra $R$ is a pair $(M, \varphi)$ where $M$ is a finitely generated $R$-module and $\varphi : M \rightarrow M$ is a $p^{-1}$-linear map, i.e. $r \varphi(m) = \varphi(r^p m)$ for all $r \in R$ and $m \in M$. For instance, $M$ can be taken to be $R$ and $\varphi$ to be a Frobenius splitting, i.e. a splitting of the Frobenius endomorphism $F : R \rightarrow R$ sending $r \mapsto r^p$. Another prominent example is the Cartier operator $\kappa : \omega_R \rightarrow \omega_R$ associated to the canonical module of $R$ of suitable rings $R$. More general Cartier modules are obtained by replacing $\varphi$ by a so-called Cartier algebra of $p^{-1}$-linear maps. See Section 2.2 and Section 2.3 for further details on Cartier modules and their invariants.

The concept of $F$-regularity is central in the study of Cartier modules. To many, the most important open problem in positive characteristic commutative algebra is the $F$-regularity of splinters. Roughly speaking, it is conjectured that $F$- regularity of an $\mathbb{F}_p$-algebra $R$ is equivalent to the property that all finite covers of $R$ are split, in this case, $R$ is called a splinter. In particular, this poses the problem of describing the behavior of the $F$-regularity of Cartier modules under finite covers of the ground-ring. There have been several occurrences of this in the literature [ST14] being the most prominent work. The same theme lies at the core of e.g. [CRST18, CR22, JS22].

The main results in the aforementioned papers are formulas (referred to as transformations rules) for the basic objects measuring the $F$-regularity of (certain) Cartier modules under finite maps. There is, however, no unified treatment of the phenomena expressed in those formulas. Further, it is often not clear what the key ingredients and underlying reasons for those formulas to work are. For instance, is normality an essential hypothesis? This is a truly important question as any birational geometer in positive characteristic can tell. In fact, by removing normality from the hypothesis, those formulas can be valuable tools for proving normality when this one is unknown (e.g. cyclic covers).

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In this work, we provide a formalism to treat those questions in a rather simple way. For instance, the desired transformation rules are exhibited as formal consequences of Grothendieck duality for finite covers. For such formalism to work the best, we need to work in a rather general framework. However, we do it seeking naturality in the proofs rather than generality in the statements. Concretely, we generalize the transformation rules for the $F$-signature in [CRST18, CR22] and the main results in [ST14] regarding the behavior of test ideals under finite covers. These generalizations are achieved by using the formalism of Cartier modules and functors $f^*$, $f^!$ introduced by M. Blickle and the second named author [BS19] as well as the formalism of transposability along sections of the relative canonical module introduced by Schwede–Tucker [ST14 ST15]. We develop the theory of transposability in Section 3 allowing us to drop normality from our hypothesis and express our results in a cleaner way.

Before stating our results, let us provide some background. Let $R$ be a $F$-finite noetherian $\mathbb{F}_p$-algebra and $\mathcal{C}$ be a Cartier $R$-algebra. To a Cartier $\mathcal{C}$-module $M$, we may associate different objects that are useful in understanding its simplicity. For example, the test module $\tau(M, \mathcal{C}) \subset M$ is the smallest among certain Cartier submodules; see [BS19]. If $R$ is a local domain and $M = R$, its splitting prime $\beta(R, \mathcal{C}) \subset R$ is either nonproper or the unique largest proper Cartier submodule of $R$ and turns out to be a prime ideal. The ideals $\tau(R, \mathcal{C})$, $\beta(R, \mathcal{C})$ bound the non-simplicity of $R$ as a Cartier module as follows. If $0 \subsetneq \mathfrak{r} \subsetneq R$ is a Cartier submodule of $R$ then:

$$\tau(R, \mathcal{C}) \subset \mathfrak{r} \subset \beta(R, \mathcal{C}).$$

In particular, $R$ is a simple Cartier $\mathcal{C}$-module if and only if $\tau(R, \mathcal{C}) = (1)$, or if and only if $\beta(R, \mathcal{C}) = (0)$. In fact, if $\beta(R, \mathcal{C})$ is proper, the Cartier submodules of $R$ form a finite sublattice; in the sense of [Jac85, 8], of radical ideals of $R$ which is bounded (i.e. it has a greatest and a least element); see [EH08, Sha07, Sch09, KM09, ST10, HW15]. Following [Sch10], we call the prime ideals in this lattice centers of $F$-purity for $(R, \mathcal{C})$ and $\beta(R, \mathcal{C})$ is the maximal center of $F$-purity of $(R, \mathcal{C})$. These notions have been traditionally studied in tight closure and $F$-singularity theory. For example, $(R, \mathcal{C})$ is said to be $F$-pure if $\beta(R, \mathcal{C}) \subset R$ is proper and $F$-regular if $R$ is simple as a $\mathcal{C}$-module. If $R$ is Cohen–Macaulay and $M = \omega_R$, analogous considerations lead to the notions of $F$-injectivity and $F$-rationality. By definition, a general Cartier module $(M, \mathcal{C})$ is $F$-regular if $\tau(M, \mathcal{C}) = M$ which is a subtler notion of simplicity.

Suppose that $(R, \mathcal{C})$ is $F$-pure, so $R/\beta(R, \mathcal{C})$ is a simple Cartier module under the induced action of $\mathcal{C}$. To $(R, \mathcal{C})$ we associate a number $r(R, \mathcal{C}) \in (0, 1]$ measuring such $F$-regularity. This number is called the splitting ratio of $(R, \mathcal{C})$ and the larger it is the “milder” the singularities of the $F$-regular pair $(R/\beta(R, \mathcal{C}), \mathcal{C})$ are. When $\beta(R, \mathcal{C}) = 0$, the splitting ratio becomes the $F$-signature of $(R, \mathcal{C})$ and is denoted by $s(R, \mathcal{C})$.

Let $f: \text{Spec} S \rightarrow \text{Spec} R$ be a finite cover, i.e. a dominant finite morphism (between $F$-finite and noetherian schemes). How does the $F$-regularity of $(M, \mathcal{C})$ pull-back along $f$? Following [BS19], we see that a precise, natural way to formulate this question is: How does the $F$-regularity of $(M, \mathcal{C})$ and $(f^!M, f^*\mathcal{C})$ compare to each other? See Section 2 for details on how this Cartier-theoretic pullback is defined. Our main result in this direction is the following:

**Theorem A** (Theorem 5.1). Let $f: \text{Spec} S \rightarrow \text{Spec} R$ be a finite cover between $F$-finite noetherian schemes. Let $\mathcal{C}$ be a Cartier $R$-algebra and $M$ be a Cartier $\mathcal{C}$-module. Then,

$$\tau(M, \mathcal{C}) = \langle f_*(f^!M, f^*\mathcal{C}) \rangle = \tau(M, \mathcal{C}),$$

where $\tau(M, \mathcal{C})$ is the $F$-signature of $(M, \mathcal{C})$.
where $\text{Tr}_M : f_*f^!M \to M$ is the Grothendieck trace map (evaluation at 1). In particular, $\text{Tr}_M$ is surjective if $(M, \mathcal{C})$ is $F$-regular. Conversely, if $\text{Tr}_M$ is surjective and $(f^!M, f^*\mathcal{C})$ is $F$-regular, then $(M, \mathcal{C})$ is $F$-regular.

This has the salient application of telling us why, for an $F$-regular Cartier module $(M, \mathcal{C})$, the trace maps $\text{Tr}_M : f_*f^!M \to M$ are surjective for all $f$, which for $M = R$ specializes to saying that $R$ is a splinter. See Section 5.0.1 for more on this.

In practice, very often the Cartier module $M$ of interest is divisorial, i.e. $M = R(D)$ for some (generalized) divisor $D$ on Spec $R$ (e.g. $D = K_R$ some canonical divisor). In that case, our interest is on $S(f_*D)$ rather than $f^!(R(D)) = S(f_*D - K_{S/R})$. However, it is then not clear what the right question should be as $S(f^*D)$ has no natural Cartier $f^*\mathcal{C}$-module structure. Our solution to this is inspired by [ST14]: we consider a natural transformation $f^\dagger \to f^!$ where $f^\dagger$ is the functor from $R$-modules to $S$-modules obtained by pullback followed by $S_2$-ification (we require some technical $G_i + S_j$ conditions on $R$ and $S$ for this to work). For example, $f^!(R(D)) = S(f_*D)$. The choice of such natural transformation is tantamount to the choice of global section of $\omega_{S/R}$, which is the choice of a map $T \in \text{Hom}_R(S, R)$. Then, we define a Cartier $\mathcal{C}$-module $M$ to be $T$-transposable so that $f^\dagger M \to f^!M$ is a morphism of Cartier $f^*\mathcal{C}$-modules. In that case, under suitable, mild conditions, $f^\dagger M \to f^!M$ will induce an isomorphism after applying the test module functor $\tau$; see Corollary 3.11. See Section 3 for the details about transposability and $f^\dagger$. In particular, we obtain the following result:

**Theorem B** (Corollary 5.5, cf. [ST14]). Let $f : \text{Spec } S \to \text{Spec } R$ be as in Theorem A and further assume that $R$ and $S$ satisfy the $G_1 + S_2$ condition (e.g. normal). Suppose that either $R$ is local or of finite type over a field (so that we can ensure $F^\omega \cong \omega$). Choose an element $T \in \omega_{S/R}$ such that the $S$-linear map $S \to \omega_{S/R}$ defined by $1 \mapsto T$ is generically an isomorphism. Then, for all almost Cartier divisors $D$ on Spec $R$ we have:

\[
(1.0.2) \quad T(f_*\tau(S(f_*D), f^*\mathcal{C})) = \tau(R(D), \mathcal{C})
\]

if $R(D)$ is a $T$-transposable Cartier $\mathcal{C}$-module.

Theorem B should be thought of as a generalization of the main results in [ST14] 6 by Schwede–Tucker; see Corollary 5.5 for the details. However, we have substantially weakened the normality hypothesis; the conditions we assume are the bare minimum for which such transformation rules are possible.

To know when $R(D)$ is a $T$-transposable, we extend the Schwede–Tucker criterion for $T$-transposability; see Theorem 3.13. We explain how the formula $\text{(1.0.2)}$ generalizes those in [ST14] when $D = 0$ and $\mathcal{C}$ is the Cartier algebra of triples; see Section 5. Likewise, we obtain analogous formulas for test ideals along closed subschemes (as treated in [Smo19b] §3.1, [SMo19a] §4); see Theorem 5.12. Similarly, Corollary 5.6 contains the special cases for canonical modules. Analogous results for non-$F$-pure modules are established in Section 5.1.

In the local case, $F$-regularity can be measured by invariants other than test modules. Our main result in this case is the following theorem.

**Theorem C** (Theorem 4.3 and Theorem 4.8, cf. CRST18 Theorems 3.1, 4.4, CR22 Theorem 4.11). Let $\theta : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a finite local extension between $F$-finite rings defining a cover $f : \text{Spec } S \to \text{Spec } R$. Suppose that $R$ is an integral domain with field of fractions $K$, set $L := S \otimes_R K$, and write $[L : K] := \dim_K L$. Suppose that there is a generic isomorphism $\sigma_R : S \to \omega_{S/R}$ of $S$-modules such that $T := \sigma_R(1)$ is surjective and $T(\mathfrak{n}) \subset \mathfrak{m}$.
Let $\mathcal{C}$ be a Cartier $R$-algebra acting on $R$ such that $R$ is a $T$-transposable Cartier $\mathcal{C}$-module (e.g., $\sigma$ is an isomorphism). Then:

\begin{equation}
\beta(S, f^*\mathcal{C}) \cap R = \beta(R, \mathcal{C})
\end{equation}

and

\begin{equation}
[k(n) : k(m)] \cdot r(S, f^*\mathcal{C}) = [k(\beta(S, f^*\mathcal{C})) : k(\beta(R, \mathcal{C}))] \cdot r(R, \mathcal{C}),
\end{equation}

where $k(-)$ denotes residue fields. In particular, $(R, \mathcal{C})$ is $F$-pure (resp. $F$-regular) if and only if so is $(S, f^*\mathcal{C})$. In the $F$-regular case:

\begin{equation}
[k(n) : k(m)] \cdot s(S, f^*\mathcal{C}) = [L : K] \cdot s(R, \mathcal{C}).
\end{equation}

As we explain in Remark 4.5, the transformation rule (1.0.5) generalizes those in [CRST18, Theorems 3.1, 4.4], [CR22, Theorem 4.11], which focus on rather specific setups.

These results illustrate the crucial role transposability plays in the behavior of Cartier modules under finite covers. Bearing this in mind, we revisit Schwede–Tucker’s transposability criterion in Section 6. There, we study how norm functions can be used to translate the criterion from effectiveness of divisor upstairs to effectiveness of divisors downstairs, which tend to be simpler as illustrated by Noether normalizations of Cohen–Macaulay singularities.

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2. Preliminaries

We recall the notions of Cartier modules, $F$-signature, splitting primes, splitting ratios, and test modules as we employ them here. However, we recall first some generalities about Hartshorne’s theory of generalized divisors [Har07, §2]. The main takeaway of Section 2.1 is Proposition 2.12 as well as our conventions (e.g., Convention 2.1, Remark 2.7, Terminology 2.8), and it could be skipped by experts.

Convention 2.1. All schemes and rings are defined over $\mathbb{F}_p$, and assumed to be noetherian and $F$-finite. We denote the $e$-th iterate of the Frobenius endomorphism by $F^e: X \to X$ if $X$ is a scheme or by $F^e: R \to F^e R$ if $R$ is a ring. We use the shorthand notation $q := p^e$. We refer to a finite dominant morphism as a finite cover or simply as a cover. We shall denote the category of finitely generated $R$-modules by $R$-fmod.
2.1. Generalized divisors, duality, and canonical modules. The following is the version of Grothendieck duality that we use throughout; see [Har77, III, Ex. 6.10].

Proposition 2.2 (Grothendieck duality for finite covers). Let \( f: \text{Spec} \, S \to \text{Spec} \, R \) be a finite cover and \( f^1 \) be the functor \( \text{R-fmod} \to \text{S-fmod} \) given by \( f^1 = \text{Hom}_R(S, -) \). Then, the morphism of \( R \)-modules

\[
\xi = \xi(M, N): f_* \text{Hom}_S(N, f^! M) \to \text{Hom}_R(f_* N, M), \quad \psi \mapsto \text{Tr}_M \circ f_* \psi
\]

is a natural isomorphism on both \( M \) and \( N \). Here, \( \text{Tr}_M: f_* f^! M \to M \) is the trace natural transformation given by the evaluation-at-1 map. The inverse \( \zeta = \zeta(M, N) \) is given by \( \zeta(\vartheta)(n) = \vartheta(- \cdot n) \) for all \( n \in N \). In other words, \( (\zeta(\vartheta)(n))(s) = \vartheta(s \cdot n) \) for all \( n \in N, s \in S \). By gluing on affine charts, the same duality applies to a general cover \( f: Y \to X \).

Remark 2.3. The maps \( \xi, \zeta \) in Proposition 2.2 preserve the \( S \)-linear structures. Thus, we may think of \( \xi \) as a natural \( S \)-linear isomorphism \( \xi': \text{Hom}_S(N, f^! M) \to \text{Hom}_R(f_* N, M) \), where the \( S \)-linear structure of the target is given by scalar pre-multiplication. We shall refer to these natural \( S \)-linear isomorphisms as Grothendieck duality too. Thus, \( \text{Hom}_S(N, f^! M) \to \text{Hom}_R(N, M), \psi \mapsto \text{Tr}_M \circ \psi \), is an isomorphism of abelian groups that preserves both \( R \)-linear and \( S \)-linear structures.

Terminology 2.4. Let \( X \) be a scheme. Following Hartshorne, we say that a coherent sheaf \( \mathcal{F} \) on \( X \) satisfies Serre’s condition \( S_k \) if depth \( \mathcal{F}_x \geq \min\{k, \dim \mathcal{O}_{X,x}\} \) for all \( x \in X \). We say that \( X \) satisfies \( S_k \) if so does \( \mathcal{O}_X \). We say that \( X \) satisfies the \( G_l \) condition if \( \mathcal{O}_{X,x} \) is a Gorenstein local ring for all points \( x \in X \) of codimension \( \leq l \). Naturally, the \( G_l + S_k \) condition means that \( X \) satisfies both \( G_l \) and \( S_k \). The same terminology is applied to the ring/module theoretic setting.

Definition 2.5 (Canonical modules). Let \( X \) be a noetherian equidimensional scheme. \( X \) admits a canonical module/sheaf if there are morphisms \( i \circ f: X \to S \to G \) where \( f \) is a finite cover, \( i \) is a closed embedding, and \( G \) is a Gorenstein scheme. If \( \omega_G \) is a fixed dualizing invertible sheaf on \( G \), the canonical module of \( X \) is defined by \( \omega_X := f^! \omega_S \) where \( \omega_S := \mathcal{O}_S^{\dim G}(\omega_S, \omega_G) = \mathcal{H}^{-\dim \delta}(i^! \omega_G)[\dim G] \), and \( \delta = \dim G - \dim S \). If \( g: Y \to X \) is a finite cover between schemes admitting canonical modules, we always make the choice of canonical modules compatible with \( g \), that is, we always set \( \omega_Y \cong g^! \omega_X \); see e.g. Remark 2.7.

Caveat 2.6. In [Har07], the scheme \( G \) in Definition 2.5 is assumed to be regular. However, the results in [Har07, §1, §2] hold in our more general setup by [HK71].

Remark 2.7. We say that an \( F \)-finite scheme \( X \) satisfies property (!) if it admits a canonical module satisfying \( F^i \omega_X \cong \omega_X \).

This includes the local case and the (essentially) of finite type case. Indeed, \( F^i \omega_X \) is a dualizing sheaf and thus by [Har66, Chapter V, Theorem 3.1] one has \( \omega_X \cong F^0 \omega_X \otimes \mathcal{L} \) for some invertible sheaf \( \mathcal{L} \), which implies the local case. In the essentially of finite type case, say \( f: X \to \text{Spec} \, \mathcal{O} \), we may take \( \omega_X := f^! \mathcal{O} \) since \( \mathcal{O} \cong F^0 \mathcal{O} \). It is however not always possible to satisfy (!) in the general \( F \)-finite case; see [ST14, §2.5].

Terminology 2.8 ([Har07]). Let \( X \) be a scheme admitting a canonical module \( \omega_X \). For a quasi-coherent sheaf \( \mathcal{F} \), one defines \( \mathcal{F}^\omega := \text{Hom}_X(\mathcal{F}, \omega_X) \) and refers to it as its \( \omega \)-dual. Consider the natural \( \mathcal{O}_X \)-linear maps \( \alpha_{\mathcal{F}}: \mathcal{F} \to \mathcal{F}^\omega \). One refers to \( \alpha \) as the \( \omega \)-reflexification.
or as the $S_2$-ification natural transformation when $\mathcal{F}$ satisfies $S_1$; see [Har07, Remark 1.8]. One says that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is $\omega$-reflexive if $\alpha_\mathcal{F}$ is an isomorphism and we denote the corresponding full subcategory by $\mathcal{O}_X$-mod$^{\omega}$. We apply the same terminology to the ring/module theoretic setting.

**Lemma 2.9.** Let $X$ be a scheme satisfying $S_1$ and admitting a canonical module. If $f : Y \to X$ is a cover, then $f^* \alpha \omega$ restricts to a functor $f^*: \mathcal{O}_X$-mod$^{\omega} \to \mathcal{O}_Y$-mod$^{\omega}$.

**Proof.** By [Har07] Lemma 1.3, $\omega_X$ satisfies $S_2$. Further, $\omega$-reflexivity and $S_2$ are equivalent conditions; see [Har07] Proposition 1.5. Moreover, the $\omega$-dual of any coherent sheaf is $\omega$-reflexive [Har07, Corollary 1.6]. In particular, $\text{Hom}_X(\mathcal{F}, \mathcal{G})$ is $\omega$-reflexive if so is $\mathcal{G}$ by $\otimes$-Hom adjointness. Additionally, we have natural isomorphisms $f_*(\mathcal{G}^{\omega}) \cong (f_* \mathcal{G})^{\omega}$—provided by Grothendieck duality—for all coherent $\mathcal{O}_Y$-modules $\mathcal{G}$. \hfill $\square$

Let $X$ be a scheme satisfying the $S_2$ condition and admitting a canonical module $\omega_X$. In particular, $\mathcal{O}_X$ is $\omega$-reflexive ([Har07, Proposition 1.5]). Let $\mathcal{K}(X)$ denote its sheaf of total fractions, which has good properties as $X$ satisfies $S_1$ ([Har94, Proposition 2.1]). For instance, $\mathcal{K}(X) = \bigoplus \mathcal{O}_{X, \eta}$ where the sum runs over the generic points $\eta$ of $X$ and $j : \{\eta\} \to X$ is the inclusion. In particular, a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is supported in codimension $\geq 1$ if and only if $\mathcal{K}(X) \otimes \mathcal{F} = 0$, and if and only if $\mathcal{F}$ is locally annihilated by regular elements of $\mathcal{O}_X$.

**Definition 2.10.** In the situation above we have the following notions:

(a) A fractional ideal is a coherent subsheaf $\mathcal{I} \subset \mathcal{K}(X)$. A fractional ideal $\mathcal{I}$ is said to be non-degenerate if $\mathcal{I}$ is supported in codimension $0$, i.e. $\mathcal{I}_\eta \neq 0$ for each generic point $\eta \in X$. In particular, $\mathcal{I}_\eta \cong \mathcal{O}_{X, \eta}$ for all generic points $\eta$ and so $\mathcal{K}(X) \otimes \mathcal{I} \cong \mathcal{K}(X)$.

(b) A generalized divisor on $X$ is a non-degenerate fractional ideal $\mathcal{I} \subset \mathcal{K}(X)$ satisfying the $S_2$ condition. We will follow the usual convention of denoting a generalized divisor by $D$ and refer to $\mathcal{I}_D$ as the fractional ideal associated to $D$. Further, we denote

$$\mathcal{O}_X(D) := \mathcal{I}_D^\vee = \text{Hom}_X(\mathcal{I}_D, \mathcal{O}_X) \cong \mathcal{I}_D^{-1} = \mathcal{I}_{-D},$$

and $\omega_X(D) := \text{Hom}_X(\mathcal{I}_D, \omega_X)$, which Hartshorne denotes by $\mathcal{L}(D)$ and $\mathcal{M}(D)$; respectively. A pair of generalized divisors $D, E$ are added as follows:

$$\mathcal{I}_{D+E} := (\mathcal{I}_D \cdot \mathcal{I}_E)^{\omega}. $$

Following Hartshorne, we define another generalized divisor by

$$(D)(-E) := \text{Hom}_X(\mathcal{I}_E, \mathcal{I}_D)^{\omega}. $$

(c) A generalized divisor $D$ is said to be Cartier (resp. principal) if $\mathcal{I}_D$ is an invertible sheaf (resp. $\mathcal{I}_D \cong \mathcal{O}_X$). If $D$ is principal and $\mathcal{I}_D = \mathcal{O}_X \cdot g$ for some invertible global section $g \in K(X)$, we write $D = \text{div}_X g = \text{div} g$.

(d) A generalized divisor $D$ is said to be almost Cartier if there is a big open $U \subset X$ (i.e. $U$ contains every codimension $1$ point of $X$) such that $(\mathcal{I}_D)|_U$ is Cartier. We denote by $\text{ADiv}X$ the group of almost Cartier divisors on $X$ (under the addition law from (b)).

(e) We say that two divisors $D, E$ are linearly equivalent if $\mathcal{I}_D \cdot g = \mathcal{I}_E$, where $g \in \mathcal{K}(X)$ is an invertible global section. We denote the group of almost Cartier divisors modulo linear equivalence by $\text{APic}X$.

(f) Let $f : Y \to X$ be a cover, so that $Y$ satisfies the $S_2$ condition and admits a canonical module $\omega_Y \cong f^* \omega_X$. We define the pullback homomorphism of almost Cartier divisors

$$f^*: \text{ADiv}X \to \text{ADiv}Y$$
as follows: Let $D$ be an almost Cartier divisor on $X$. Let $i: U \to X$ be a big open such that $i^*\mathcal{I}_D$ is Cartier and $j: V \to Y$ be the pullback of $i$ along $f$, which is a big open too. Then, we define $f^*D$ by $\mathcal{I}_{f^*D} := j_!f_U^*(i^*\mathcal{I}_D)$, where $f_U: V \to U$ is the pullback of $f$ to $U$, which is an almost Cartier divisor. We note that $f^*$ is compatible with linear equivalence. Furthermore, we obtain a homomorphism of abelian groups $f^*: \text{APic}_X \to \text{APic}_Y$; see [Har07, Definition after Corollary 2.3].

(g) For a given abelian group $A$ (such as $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$), we define an almost Cartier $A$-divisor to be an element of $A \otimes_{\mathbb{Z}} \text{ADiv} X := \text{ADiv}_A X$. We define $\text{APic}_A X$ similarly, so that we have a surjective homomorphism $\text{ADiv}_A X \to \text{APic}_A X$ (obtained by tensoring $\text{ADiv} X \to \text{APic}_X$ with $A$).

We say that two almost Cartier $A$-divisors $D_1$, $D_2$ are $A$-linearly equivalent and write $D_1 \sim_A D_2$ if the difference $D_1 - D_2$ is in $\ker(\text{ADiv}_A X \to \text{APic}_A X)$. If $A = \mathbb{Q}$ (resp. $A = \mathbb{Z}_{(p)}$), $D_1 \sim_{\mathbb{Q}} D_2$ (resp. $D_1 \sim_{\mathbb{Z}_{(p)}} D_2$) if and only if there is $n \in \mathbb{Z}$ nonzero (resp. prime-to-$p$) such that $nD_1 \sim nD_2$ in $\text{ADiv}_X$.

The pullback of almost Cartier $A$-divisors $f^*: \text{ADiv}_A X \to \text{ADiv}_A Y$ is defined by twisting the pullback of almost Cartier divisors by $A$.

**Lemma 2.11.** Let $X$ be a scheme satisfying the $G_0 + S_2$ condition and admitting a canonical module $\omega_X$.

(a) There is an embedding $\omega_X \to \mathcal{K}(X)$ realizing $\omega_X$ as a non-degenerate fractional ideal, say $\mathcal{I}_{-\mathcal{K}_X}$, which defines an anticanonical (generalized) divisor $-\mathcal{K}_X$. More generally, if $D$ is a generalized divisor then $\omega_X(D)$ is also a generalized divisor. Further, if either $X$ is $G_1$ or $D$ is almost Cartier, we have the following (canonical) relation

\[(2.11.1) \quad \mathcal{O}_X(D) \cong \omega_X(D + (-\mathcal{K}_X)).\]

(b) Assume now that $X$ also satisfies $G_1$, then one defines a canonical divisor $\mathcal{K}_X$ on $X$ by $\mathcal{K}_X := -(-\mathcal{K}_X)$, which is an almost Cartier divisor and $\mathcal{O}_X(\mathcal{K}_X) \cong \omega_X$. Furthermore, $\omega_X(D)$ is a reflexive generalized divisor for any generalized divisor $D$.

**Proof.** For (a), since $X$ is $G_0$, the sheaf $\omega_X$ is generically free of rank 1 ($i.e.$ free of rank 1 at every generic point of $X$). Using [Har07, 2.4], we find an embedding $\omega_X \to \mathcal{K}(X)$ realizing $\omega_X$ as a non-degenerate fractional ideal; see [Har07, Definition–Remark 2.7]. Note that $\omega_X(D)$ is $\omega$-reflexive and generically free of rank 1. Hence, by [Har94, Proposition 2.8], $\omega_X(D)$ is isomorphic to $\mathcal{I}_E$ for some generalized divisor $E$, namely $E = (-\mathcal{K}_X)(-D))$. Since $X$ satisfies $S_2$, $\mathcal{O}_X$ is $\omega$-reflexive; see [Har07, Proposition 1.5]. In other words, $\mathcal{O}_X$ defines a generalized divisor. Further, $\mathcal{O}_X(D)$ is $\omega$-reflexive for all generalized divisors. Hence, we have a canonical isomorphism $\mathcal{O}_X(D) \to \mathcal{H}om_X(\omega_X, \mathcal{H}om(\mathcal{I}_{-\mathcal{K}_X}, \omega_X))$. However, $\mathcal{H}om_X(\mathcal{I}_{-\mathcal{K}_X}, \omega_X) = \mathcal{I}_{D + (-\mathcal{K}_X)}$ if either $D$ or $-\mathcal{K}_X$ is almost Cartier by [Har07, Proposition 2.8.(c)]. Putting these together yields $\mathcal{O}_X(D) \cong \mathcal{H}om_X(\omega_X, \mathcal{I}_{D + (-\mathcal{K}_X)}) = \omega(D + (-\mathcal{K}_X))$.

For (b), we have $\mathcal{O}_X(\mathcal{K}_X) \cong \omega_X$ by (2.11.1) Using [Har07, Proposition 2.2 (d)], we have $\omega_X(D) \cong \mathcal{O}_X(D + \mathcal{K}_X)$, which is reflexive.

Next proposition establishes the connection between generalized divisors and maps.

**Proposition 2.12.** Let $f: Y \to X$ be a finite cover between schemes satisfying $G_0 + S_2$. Let $E$ be a reflexive divisor on $Y$ and $D$ be an almost Cartier divisor on $X$. Then, there is a canonical isomorphism of $\mathcal{O}_X$-modules:

\[(2.12.1) \quad \mathcal{H}om_X(f_*\omega_Y(E), \omega_X(D)) \cong f_*\mathcal{O}_Y(f^*D - E),\]
which can also be thought of as an isomorphism of $\Omega_X$-modules

\[(2.12.2) \quad \hom_X (f_*\omega_Y(E), \omega_X(D)) \cong \Omega_Y(f^*D - E), \]

as in Remark 2.3. In particular, to any non-degenerate map $\varphi: f_*\omega_Y(E) \to \omega_X(D)$ there corresponds an effective generalized divisor $D_\varphi \sim f^*D - E$ on $Y$.

Further, let $q: Y' \to Y$ be a finite cover of $X$-schemes satisfying $G_0 + S_2$, and set $f' = f \circ g: Y' \to X$. Let $E'$ be a reflexive generalized divisor on $Y'$, and let $E$ and $D$ be almost Cartier divisors on $Y$ and $X$ respectively. Then the following equality holds for all non-degenerate $\psi \in \hom_Y (g_*\omega_Y(E'), \omega_Y(E))$ and $\varphi \in \hom_X (f_*\omega_Y(E), \omega_X(D))$:

\[(2.12.3) \quad D_{\varphi \circ f, \psi} = D_\psi + g^*D_\varphi. \]

**Proof.** Work in the affine setting. Set $X = \mathrm{Spec} R$, $Y = \mathrm{Spec} S$, $Y' = \mathrm{Spec} S'$. By the projection formula, the canonical morphism $f_*\omega_S \otimes_R \mathcal{F}_D \to f_* (\omega_S(E) \otimes_S f^*\mathcal{F}_D)$ is an isomorphism on a big open $U \subset X$ on which $D$ is Cartier. Hence, its $\omega$-dual is an isomorphism and so:

\[
\hom_R (f_*\omega_S(E), \omega_R(D)) \cong \hom_R (f_*\omega_S(E) \otimes_R \mathcal{F}_D, \omega_R) \cong \hom_R (f_* (\omega_S(E) \otimes_S f^*(\mathcal{F}_D)), \omega_R)
\]

Next, observe that

\[
(\omega_S(E) \otimes_S f^*(\mathcal{F}_D))^\omega \cong \mathcal{F}_D^{(-\omega_S)} \otimes_R f^*\mathcal{F}_D = \mathcal{F}_D^{(-\omega_S)} \otimes_R f^*\mathcal{F}_D \cong \omega_S(E - f^*D),
\]

where the middle equality follows from [Har07, Proposition 2.8 (b)] and that $f^*D$ is almost Cartier. Summing up, we have canonical isomorphisms

\[(2.12.4) \quad \hom_R (f_*\omega_S(E), \omega_R(D)) \cong \hom_R (f_*\omega_S(E - f^*D), \omega_R). \]

By denoting $E^* := E - f^*D$, we may assume $D = 0$. Note that $E^*$ is reflexive by [Har07, Proposition 2.2 (c), (e)]. Notice that the isomorphism (2.12.4) is trivial at codimension 1 points or whenever $\mathcal{F}_D \cong \Omega_X$. Thus, we may use Grothendieck duality to conclude:

\[(2.12.5) \quad \hom_R (f_*\omega_S(E^*), \omega_R) \cong f_* \hom_S (\omega_S(E^*), \omega_S) \cong f_* \mathcal{F}_{E^*} = f_* \mathcal{F}_{-E^*} \cong f_*(-E^*). \]

Note that a non-degenerate element of $S(-E^*)$ is an embedding $\mathcal{F}_{-E^*} \to S$ and so an effective divisor linearly equivalent to $-E^*$. Thus, we obtain a mapping $\varphi \mapsto D_\varphi \in f^*(D - E)$.

We show now (2.12.3), i.e. the naturality of $\varphi \mapsto D_\varphi$. As explained above, we may assume $D = 0$—this will simplify the notation. Let us consider the following commutative triangle.

\[
\begin{array}{ccc}
  f'_*\omega_S(E') & \xrightarrow{f_*\psi} & f_*\omega_S(E) \\
  \downarrow{\varphi'} & & \downarrow{\varphi} \\
  \omega_R & & \omega_S
\end{array}
\]

By naturality (on the source) of the first isomorphism in (2.12.5) this diagram becomes:

\[
\begin{array}{ccc}
  g_*\omega_S'(E') & \xrightarrow{\psi} & \omega_S(E) \\
  \downarrow{\varphi''} & & \downarrow{x \in \mathcal{F}_E} \\
  \omega_S & & \omega_S
\end{array}
\]

\[1\text{The non-degeneracy of } \varphi \text{ means that the corresponding morphism } \Omega_Y \to \hom_X (f_*\omega_Y(E), \omega_X(D)) \text{ is injective. In other words, } \varphi, \text{ as a global section of the } \Omega_Y \text{-module } \hom_X (f_*\omega_Y(E), \omega_X(D)), \text{ is supported generically (i.e. } \varphi \text{ is locally a regular element). Recall we assume } Y \text{ satisfies } S_1. \]
By naturality of (2.12.5) (on the target) and (2.12.4), we obtain the commutative diagram

\[
\begin{array}{ccc}
\omega_{S'}(E' - g^*E) & \xrightarrow{x} & \omega_{S'}(E') \\
y \in \mathcal{I} & \downarrow & y \in \mathcal{I} \\
\omega_{S'} & \downarrow & \omega_{S'}
\end{array}
\]

Writing this down additively yields the claimed divisorial equality. \qed

**Example 2.13.** Work in the setup of Proposition 2.12. Setting \(D, E = 0\), (2.12.2) means that \(\text{Tr}_{\omega_f} : f_!\omega_Y \to \omega_X\) is a free generator of \(\text{Hom}_X(f_!\omega_Y, \omega_X)\) as an \(\mathcal{O}_Y\)-module. Assuming \(X\) satisfies \(G_1\) and setting \(D = -K_R\) and \(E = -K_S\) in (2.11.1), then \(\omega_{Y/X} \cong \mathcal{O}_Y(f^*(-K_X - (-K_Y)))\) where \(\omega_{Y/X} := f^!\mathcal{O}_X\). Set \(K_{Y/X} := f^*(-K_X - (-K_Y))\) to be the relative canonical divisor of \(f\). Thus, an injective global section \(\sigma : \mathcal{O}_Y \to \omega_{Y/X}\) (i.e. a non-degenerate map \(T : f_*\mathcal{O}_Y \to \mathcal{O}_X\)) defines an effective divisor \(D_T \sim K_{Y/X}\). We often write \(\text{Ram}_T = D_T\).

**Example 2.14** (Schwede’s correspondence). Let \(X\) be a scheme satisfying the conditions \(G_0 + S_2\) and \(\boxed{4}\); see Remark 2.7. Applying Proposition 2.12 to \(F^e : X \to X\) yields:

\[
\text{Hom}_X(F^e_!\omega_X(D), \omega_X(D)) \cong F^e_!\mathcal{O}_X((q - 1)D)
\]

for all almost Cartier divisors \(D\) on \(X\). Equivalently,

\[(2.14.1) \quad \text{Hom}_X(F^e_!\omega_X(D), \omega_X(D)) \cong \mathcal{O}_X((q - 1)D)\]

where the \(\mathcal{O}_X\)-linear structure on the left-hand side module is given by pre-multiplication. These isomorphisms can be described via Cartier operators. Indeed, let \(\kappa_X^e : F^e_!\omega_X \to \omega_X\) be the map defined by (2.14.1) when \(D = 0\). We refer to it as the \(e\)-th Cartier operator of \(X\). We will see below that \(\kappa_X^e\) is the \(e\)-th power of \(\kappa_X^1 = \kappa_X\). Using the projection formula, we may twist \(\kappa_X^e\) by an almost Cartier divisor \(D\) to obtain a map \(\kappa_{X,D}^e : F^e_!\omega_X(qD) \to \omega_X(D)\). For instance, \(\kappa_{X,-K_X}^e\) is the so-called Frobenius trace. Locally, a section \(x\) of \(\mathcal{O}_X((q - 1)D)\) defines a map \(\mathcal{O}_X(D) \to \mathcal{O}_X(qD)\) and further a map \(\omega_X(D) \to \omega_X(qD)\) which when composed with \(\kappa_{X,D}^e\) gives \(\kappa_{X,D}^e \cdot x : F^e_!\omega_X(D) \to F^e_!\omega_X(qD) \to \omega_X(D)\). Thus, to any non-degenerate map \(\varphi : F^e_!\omega_X(D) \to \omega_X(D)\) we associate an effective almost Cartier divisor \(D_\varphi \sim (q - 1)D\). One defines the normalized divisor of \(\varphi\) by \(\Delta_\varphi := \frac{1}{q - 1}D_\varphi\), which is an effective almost Cartier \(\mathbb{Z}_{(p)}\)-divisor such that \(\Delta_{\varphi} \sim_{\mathbb{Z}_{(p)}} D\). Let us look at the case \(\mathcal{O}_X(D) = \omega_R(D + (-K_X))\). Then, to any non-degenerate map \(\varphi : F^e_!\mathcal{O}_X(D) \to \mathcal{O}_X(D)\) there corresponds an effective almost Cartier \(\mathbb{Z}_{(p)}\)-divisor \(\Delta_{\varphi} \sim_{\mathbb{Z}_{(p)}} D + (-K_X)\). If \(X\) also satisfies \(G_1\), then \(\Delta_{\varphi} + K_X \sim_{\mathbb{Z}_{(p)}} D\). We then recover Schwede’s correspondence [Sch09]. Concretely, if \(\Delta\) is an effective almost Cartier \(\mathbb{Z}_{(p)}\)-divisor such that \(\Delta + K_X \sim_{\mathbb{Z}_{(p)}} D\), then there is a non-degenerate map \(\varphi : F^e_!\mathcal{O}_X(D) \to \mathcal{O}_X(D)\) for some \(e > 0\) such that \(\Delta_{\varphi} = \Delta\) and \(\varphi\) is unique up to pre-multiplication by global units and by powers of \(\varphi\) (where \(\varphi^n : F^e_!\mathcal{O}_X(D) \to \mathcal{O}_X(D)\) is defined inductively by \(\varphi^{n+1} = \varphi \circ F^e_!\varphi^n\)). In particular, \(\Delta_{\varphi^n} = \Delta_{\varphi}\) for all \(n\) by (2.12.3). We denote the minimal such \(e > 0\) as \(e_\Delta\). Thus, Schwede’s correspondence holds in this generality (using [Har94 Proposition 2.9], [Har07 Remark 2.9], and \(\mathbb{Z}_{(p)} = \text{colim}_e \mathbb{Z}_{q-1}\)) by the arguments in [Sch09]; see [BS13 §4].

### 2.2. Cartier modules and the functors \(f^*, f^!\)

Let us define Cartier algebras and modules as well as the functors \(f^*, f^!\) introduced in [BS13], which is the formalism we need to express our transformation rules naturally. For simplicity, we will work in the affine setting yet everything generalizes to the general scheme-theoretic setting by gluing on affine charts.
**Definition 2.15** (Cartier algebras). Let $R$ be a ring. A Cartier $R$-algebra $C$ is an $\mathbb{N}$-graded ring $C = \bigoplus_r C_r$ with $C_0 = R$, and equipped with a graded $R$-bimodule structure such that $r \cdot \kappa_e = \kappa_e \cdot r^0$ for all $r \in R$, $\kappa_e \in C_e$. Cartier $R$-algebras form a category in the obvious way.

**Example 2.16.** Let $M$ be an $R$-module. Its full Cartier algebra is $C_M := \bigoplus_e \text{Hom}_R(F^e_\ast M, M)$ (setting $C_{0,M} := R$). Write $C_{e,M} = \text{Hom}_R(F^e_\ast M, M)$. The (graded) left $R$-module structure is given by post-multiplication: $(r \cdot \varphi_e)(-):= r \cdot \varphi_e(-)$ for all $r \in R, \varphi_e \in C_{e,M}$. The (graded) right $R$-module structure is given by pre-multiplication: $(\varphi_e \cdot r)(-):= \varphi_e(\cdot r \cdot -)$ for all $r \in R, \varphi_e \in C_{e,M}$. The (graded) ring structure is given by: if $\varphi_e \in C_{e,M}$ and $\varphi \in C_{d,M}$, $\varphi_e \cdot \varphi_d \in C_{e+d,M}$ is the composition $\varphi_e \circ F^e_\ast \varphi_d: F^e_\ast d \rightarrow F^e_\ast M \rightarrow M$.

Given map $\varphi: F^e_\ast M \rightarrow M$, we write $C^e_M \subset C_M$ for the Cartier subalgebra of $C_M$ generated by $\varphi$. That is, $C^e_{M,e} = \varphi^n \cdot R$ whereas $C_{d,M}^e = 0$ if $e \nmid d$.

Let $X = \text{Spec } R$ be as in Example 2.14 and $M = R(D)$ for some almost Cartier divisor $D$ on $X$. Then, $C_{RD} := \bigoplus_e R\left(\langle q-1 \rangle (D-K_X)\right)$ and if we treat elements of $R\left(\langle q-1 \rangle (D-K_X)\right)$ as invertible elements $x_e \in \mathcal{K}(R)$ such that div $x_e + \langle q-1 \rangle (D-K_X) \geq 0$, then $x_e x_d := x_e^q x_d$ by (2.12.3). Further, such an element $x_e$ acts on $x \in R(D)$ as $x_e \cdot x := \kappa^e_{D-K_X}(F^e_\ast x e x) \in R(D)$.

**Example 2.17.** Let $X = \text{Spec } R$ be as in Example 2.14 and $D$ be an almost Cartier divisor. Fixing an effective almost Cartier $\mathbb{Q}$-divisor $\Delta$, we define $C^e_{\Delta,R(D)}$ to be the Cartier subalgebra of $C_{\omega_{R(D)}}$ consisting of degree-$e$ homogeneous elements $\varphi$ such that $\Delta - \Delta \geq \lceil(q-1)\Delta\rceil$. Thus (assuming $R$ further satisfies $G_1$),

$$C^e_{\Delta,R(D)} = \text{Hom}_R\left(F^e_\ast R\left(D + \lceil(q-1)\Delta\rceil\right), R(D)\right) \subset \text{Hom}_R(F^e_\ast R(D), R(D)),$$

as in [BST12] 4.3.1. One also defines $C^e_{\omega_{R(D)}}$ as customary. When $\Delta = \Delta_{\varphi}$ for some $\varphi: F^e_\ast R(D) \rightarrow R(D)$ (which is unique up to pre-multiplication by units), we have $C^e_{\Delta,R(D),\omega_{\Delta}} = C^e_{\Delta,R(D),\Delta}$ for all $e \in \mathbb{N}$. As in the last paragraph of Example 2.16,

$$C^e_{\Delta,R(D)} \cong \bigoplus_{e \in \mathbb{N}} R\left(\langle q-1 \rangle (D-K_X) - \lceil(q-1)\Delta\rceil\right).$$

**Definition 2.18** (Cartier modules). Work in the setup of Definition 2.15. A Cartier $C$-module is a left $C$-module that is finite as an $R$-module, i.e., it is a module $M$ in $R$-fmod together with a homomorphism of Cartier $R$-algebras $\Xi: C \rightarrow C_M$. The category of Cartier modules is defined in the obvious way.

Next, we define the functors $f^*$ and $f^!$ associated to a cover $f: \text{Spec } S \rightarrow \text{Spec } R$. Let $C$ be a Cartier $R$-algebra. One defines a Cartier $S$-algebra $f^*C$ as follows. As a right $S$-module, $f^*C$ is equal to $C \otimes_R S = \bigoplus_e C_e \otimes_R S$, which is a graded ring in the obvious way. The graded left $S$-module structure is defined by the rule: $s \cdot (\kappa \otimes s') := \kappa \otimes s's'$ for all $s, s' \in S, \kappa \in C_e$. This defines a functor $f^*$ from Cartier $R$-algebras to Cartier $S$-algebras.

For the “upper-shriek” functor, consider $f^! = \text{Hom}_R(S, -): R$-fmod $\rightarrow S$-fmod. It is extended to a functor from Cartier $S$-modules to Cartier $f^*C$-modules as follows. Let $M$ be a Cartier $C$-module and $\mu \in f^!M = \text{Hom}_R(S, M)$, $\kappa \otimes s' \in f^*C_e$. Define $(\kappa \otimes s') \cdot \mu \in f^!M$ as: $((\kappa \otimes s') \cdot \mu)(s) := \kappa \cdot \mu(s's^0)$. Alternatively, given a homomorphism $\Xi: C \rightarrow C_M$, define a natural homomorphism $f^!\Xi: f^*C \rightarrow C_{f^!M}$ by setting in degree $e$:

$$f^!\Xi(\kappa \otimes s') := \left(\Xi(\kappa)\right)^0 \cdot s'$$

for all $\kappa \in C_e, s' \in S$. 


where, for a given \( \varphi \in \text{Hom}_R(F^*_e M, M) \), \( \varphi' \) is defined by Grothendieck duality as the only element of \( \text{Hom}_R(F^*_e f^! M, f^! M) \) making the following diagram commutative

\[
\begin{array}{c}
F^*_e f^! M \\
\downarrow F^*_e \text{Tr}_M \\
F^*_e M
\end{array} \xrightarrow{f^* \varphi'} \begin{array}{c}
f^* f^! M \\
\downarrow \text{Tr}_M \\
M
\end{array}
\]

That is, \( \varphi'(F^*_e \mu)(s) = \varphi(F^*_e \mu(s^q)) \) for all \( \mu \in f^! M, s \in S \).

**Remark 2.19.** The following observations are in order:

\( (a) \) If \( N \subset f^! M \) is a Cartier \( f^* \mathcal{C} \)-submodule, \( \text{Tr}_M(f_* N) \subset M \) is a Cartier \( \mathcal{C} \)-submodule.

\( (b) \) If \( \Xi: \mathcal{C} \to \mathcal{C}_M \) is an inclusion, the image of \( f^! \Xi: f^* \mathcal{C} \to \mathcal{C}_{f^! M} \) is the right \( S \)-span of \( \{ \varphi' \mid \varphi \in \mathcal{C} \} \) in \( \mathcal{C}_{f^! M} \). In fact, we have an isomorphism \( f^* \mathcal{C}_{f^! M} \cong \mathcal{C}^{\varphi'}_{f^! M} \) for all non-degenerate \( \varphi: F^*_e M \to M \) (i.e., \( \varphi \cdot r \neq 0 \) for all regular elements \( r \in R \)).

\( (c) \) If \( \mathcal{C} \subset \mathcal{C}_R \) and \( S \to \omega_{S/R} := f^! R; 1 \mapsto T, \) is an isomorphism, Grothendieck duality yields natural isomorphisms on \( N: f_* \text{Hom}_S(N, S) \to \text{Hom}_R(f_* N, R), \psi \mapsto T \circ \psi \).

Plugging in \( N = F^*_e S \) gives that, for every \( \varphi \in \mathcal{C}_R \), there is a unique \( \varphi^\top \in \mathcal{C}_S \) such that

\[
\begin{array}{c}
F^*_e S \\
\downarrow F^*_e T \\
F^*_e R
\end{array} \xrightarrow{\varphi^\top} \begin{array}{c}
S \\
\downarrow T \\
R
\end{array}
\]

is commutative. Hence, the image of \( f^* \mathcal{C}_R \to \mathcal{C}_S \) is the right \( S \)-span of \( \{ \varphi^\top \mid \varphi \in \mathcal{C} \} \) in \( \mathcal{C}_S \) which we shall often denote by \( f^* \mathcal{C}_R \) by abuse of notation.

### 2.3. \( F \)-signatures, splitting primes, and test modules

For the reader’s convenience, we recall the notions \( F \)-signature, splitting ratio, splitting prime, and test module.

#### 2.3.1. Test modules, \( F \)-purity, and \( F \)-regularity

**Definition 2.20.** Let \( (M, \mathcal{C}) \) be a Cartier module over a ring \( R \) and \( \mathcal{C}_+ := \bigoplus_{e \geq 1} \mathcal{C}_e \).

\( (a) \) A morphism \( \Phi: N \to M \) of Cartier modules is a *nil-isomorphism* if \( \ker \Phi \) and \( \operatorname{coker} \Phi \) are annihilated by some power of \( \mathcal{C}_+ \).

\( (b) \) The *test module* \( \tau(M, \mathcal{C}) \) is defined as the smallest Cartier submodule \( N \subset M \) for which \( H^0_{\eta}(N_{\eta}) \to H^0_{\eta}(M_{\eta}) \) is a nil-isomorphism for all \( \eta \in \text{Ass} M \). \( M \) is *\( F \)-regular* if \( \tau(M, \mathcal{C}) = M \). See \([BS19, \text{§}1]\) for more details.

\( (c) \) If \( R \) is a Cohen–Macaulay ring satisfying \([\dagger]\) with Cartier operator \( \kappa_R: F^*_e \omega_R \to \omega_R \), we say that \( R \) is *\( F \)-rational* (resp. *\( F \)-injective*) if \( \omega_R, \kappa_R \) is \( F \)-regular (resp. \( F \)-pure); where \( (\omega_R, \kappa_R) \) is the Cartier module \( \omega_R \) with respect to its full Cartier algebra (as \( \kappa_R \cdot R = \text{Hom}_R(F^*_e \omega_R, \omega_R) \)). Equivalently, there are no non-zero proper submodules \( M \subset \omega_R \) stable under \( \kappa_R \); cf. \([ST12, \text{§}8.1, 8.2]\), cf. \([Smi97, Fed83, FW89, V´el95]\).

We also note that \( F \)-rationality can be defined without a Cohen–Macaulay assumption but we do not use this here.
2.3.2. Splitting primes and ratios. Let \((R, m, \mathcal{E})\) be a local ring and \(\mathcal{C}\) be a Cartier \(R\)-algebra acting on \(R\). Following [BST12], we define the \(e\)-th splitting number of \((R, \mathcal{C})\) to be
\[
a_{e}(R, \mathcal{C}) := \lambda_{R}(\mathcal{C}_{e}/\mathcal{C}_{e}^{\text{ns}}),
\]
where lengths are computed as left \(R\)-modules, and \(m \cdot \mathcal{C}_{e} \subset \mathcal{C}_{e}^{\text{ns}} \subset \mathcal{C}_{e}\) is the submodule of \(\mathcal{C}_{e}\) given by \(\mathcal{C}_{e}^{\text{ns}} := \{ \kappa \in \mathcal{C}_{e} | \kappa \cdot R \subset m \}\). Since \(\mathcal{C}_{e}^{\text{ns}}\) contains \(\ker(\mathcal{C}_{e} \to \mathcal{C}_{e,R})\) and surjectivity/nonsurjectivity is preserved under this map, splitting numbers can be computed by replacing \(\mathcal{C}\) for its image along \(\mathcal{C} \to \mathcal{C}_{R}\). The \(F\)-signature of \((R, \mathcal{C})\) is defined as
\[
s(R, \mathcal{C}) := \lim_{e \to \infty} a_{e,n}/q^{\delta},
\]
where \(n := \gcd\{e \in \mathbb{N} | a_{e}(R, \mathcal{C}) \neq 0\}\) and \(\delta = \dim R + \log_{p}[\mathcal{E}_{1/p} : \mathcal{E}]\). This limit exists ([Tuc12]) and its positivity characterizes the \(F\)-regularity of \((R, \mathcal{C})\); see [BST12] §3.

Following [AE05], we define the splitting prime to be the ideal
\[
\beta(R, \mathcal{C}) := \{ r \in R | \kappa \cdot r \in m \text{ for all } e \in \mathbb{N}, \kappa \in \mathcal{C}_{e} \}.
\]
Observe that \((R, \mathcal{C})\) is \(F\)-pure if and only if \(\beta(R, \mathcal{C})\) is proper. In that case, \(\beta(R, \mathcal{C})\) is a prime ideal and is zero if and only if \((R, \mathcal{C})\) is \(F\)-regular; see [AE05] Theorem 1.1, [BST12] Proposition 2.12, Lemma 2.13. Further, \(\beta(R, \mathcal{C}) \subset \mathcal{C}\) is the largest proper Cartier \(\mathcal{C}\)-submodule [Sch10] Remark 4.4, [BST12] Proposition 2.12. Thus, a map \(\varphi \in \text{Image}(\mathcal{C}_{e} \to \mathcal{C}_{e,R})\) induces a unique map \(\mathcal{C} \in \mathcal{C}_{e,R/\beta(\mathcal{C})}\) making the following diagram commutative:

\[
\begin{array}{ccc}
F_{s}^{e}R & \xrightarrow{\varphi} & R \\
\downarrow & & \downarrow \\
F_{s}^{e}(R/\beta(\mathcal{C})) & \xrightarrow{\varphi} & R/\beta(\mathcal{C})
\end{array}
\]

That is, \(\mathcal{C}\) induces a Cartier \(R/\beta(\mathcal{C})\)-algebra, say \(\mathcal{C}\); see [BST12] Definition 2.10. It follows that \((R/\beta(R, \mathcal{C}), \mathcal{C})\) is \(F\)-regular; see [AE05] Theorem 4.7, [Sch10] Corollary 7.8, [BST12] Lemma 2.13, and one defines the splitting ratio of \((R, \mathcal{C})\) to be
\[
r(R, \mathcal{C}) := s(R/\beta(\mathcal{C}), \mathcal{C}).
\]

See [BST12] Theorem 4.2, cf. [Tuc12] Theorem 4.9, [AE05].

The following is well-known to experts but the key to recover previous results from ours.

**Lemma 2.21.** Let \(\mathcal{C} \subset \mathcal{D}\) be Cartier algebras over a ring \(R\) and \(M\) be a Cartier \(\mathcal{D}\)-module. Suppose that there is an \(M\)-regular element \(c \in R\) such that \(\mathcal{D}_{e} \cdot c \subset \mathcal{C}_{e} \subset \mathcal{D}_{e}\) for all \(e > 0\). Then, \(\tau(M, \mathcal{C}) \supset \tau(M, \mathcal{D})\) is an equality. If \(R\) is local and \(M = R\), then \(s(R, \mathcal{C}) = s(R, \mathcal{D})\).

**Proof.** By assumption, \(c\) is not contained in any \(\eta \in \text{Ass} M\) and so \(\mathcal{C}_{\eta} = \mathcal{D}_{\eta}\). We have an inclusion \(\tau(M, \mathcal{C}) \subset \tau(M, \mathcal{D})\) and an inclusion \(\mathcal{D}_{\eta} \cdot \tau(M, \mathcal{C}) \subset \tau(M, \mathcal{C})\) where the former is a \(\mathcal{D}\)-module. Hence,
\[
\begin{align*}
H_{\eta}^{0}(\mathcal{D}_{\eta} \cdot \tau(M, \mathcal{C}), \tau(M, \mathcal{C}))_{\eta} &= H_{\eta}^{0}(\tau(M, \mathcal{C}), \tau(M, \mathcal{C}))_{\eta} = H_{\eta}^{0}(\tau(M_{\eta}, \mathcal{C}_{\eta})) = H_{\eta}^{0}(\tau(M_{\eta}, \mathcal{D}_{\eta})) = H_{\eta}^{0}(\tau(M, \mathcal{D}), \mathcal{C}_{\eta})
\end{align*}
\]
using [BS19] Proposition 1.19 (b), Remark 2.4. By definition, \(\tau(M, \mathcal{D}) \subset M\) is the smallest \(\mathcal{D}\)-submodule such that \(H_{\eta}^{0}(\tau(M, \mathcal{D}), \mathcal{C}_{\eta}) \subset H_{\eta}^{0}(M_{\eta})\) is a nil-isomorphism for all \(\eta\). Then, \(\tau(M, \mathcal{D}) \subset \mathcal{D}_{\eta} \cdot \tau(M, \mathcal{C})\) which yields the other inclusion. The claim about \(F\)-signatures is shown as in [BST12] Lemma 4.17, [CRST18] §2.2. \(\square\)
3. Transposability along a section of the relative canonical module

Let us consider the following observation.

**Lemma 3.1.** Let \( f : Y \to X \) be a finite cover and \( \text{Hom}(f^*, f^!_1) \) be the set of natural transformations \( \sigma : f^* \to f^!_1 \). Then, the mapping \( \sigma \mapsto (\sigma_\circ X : \mathcal{O}_Y \to \omega_{Y/X}) \) defines a bijection \( \text{Hom}(f^*, f^!_1) \to \Gamma(Y, \omega_{Y/X}) \). Further, splittings of \( \mathcal{O}_X \to f_\circ \mathcal{O}_Y \) in \( \omega_{Y/X} \) correspond to natural transformations \( \sigma : f^* \to f^!_1 \) such that the composition of natural transformations

\[
\text{id} \xrightarrow{\eta} f_\circ \circ f^* \xrightarrow{f_\circ \circ f^!_1} f_\circ \circ f^!_1 \xrightarrow{\text{id}} \text{id}
\]

is the identity, where \( \eta \) is the unit functor of the adjointness \( (f^*, f_\circ) \).

**Proof.** Work in the affine case and set \( X = \text{Spec} R \) and \( Y = \text{Spec} S \). The mapping \( \omega_{S/R} \to \text{Hom}(f^*, f^!_1) \) sending \( S \to \omega_{S/R} \) to

\[
M \mapsto (\omega_{S/R} \otimes_R M \xrightarrow{\text{can}} f^!_1 M) \circ ((S \to \omega_{S/R}) \otimes_R M)
\]

is the required inverse. More succinctly, if \( T \in \omega_{S/R} \), then one defines a natural transformation \( \sigma : f^* \to f^!_1 \) by declaring \( \sigma_M : f^* M \to f^!_1 M \) to be the \( S \)-linear map adjoint to the \( R \)-linear map \( M \to f_\circ f^!_1 M \) given by \( m \mapsto (s \mapsto T(s)m) \) for all \( s \in S, m \in M \), and all \( R \)-modules \( M \). In other words, \( \sigma_M \) sends \( s \otimes m \) to the \( R \)-linear map \( T(s \cdot -)m : S \to M \) given by \( s' \mapsto T(ss')m \).

We readily see that \( \omega_{S/R} \to \text{Hom}(f^*, f^!_1) \to \omega_{S/R} \) is the identity. To demonstrate that \( \text{Hom}(f^*, f^!_1) \to \omega_{S/R} \to \text{Hom}(f^*, f^!_1) \) is the identity, we use that a natural transformation \( \sigma : f^* \to f^!_1 \) is compatible with the maps in \( \text{Hom}_R(R, M) \) to write down commutative squares

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma_R} & \omega_{S/R} \\
\downarrow{s \mapsto s \otimes m} & & \downarrow{\sigma \mapsto (s \mapsto \sigma(s)m)} \\
f^* M & \xrightarrow{\sigma_M} & f^!_1 M
\end{array}
\]

for all \( m \in M \). Equivalently, \( \sigma_M = (\omega_{S/R} \otimes_R M \to f^!_1 M) \circ (\sigma_R \otimes_R M) \). The last statement follows directly from the definition of the bijections. \( \square \)

Following [ST14], given a finite cover \( f : Y \to X \), we choose a natural transformation \( \sigma : f^* \to f^!_1 \), and set \( T := \text{Tr}_X \circ \sigma_X \) to be the corresponding global section of \( \omega_{Y/X} \). Before considering the general case, we illustrate with a key example what our goal is. Set \( X = \text{Spec} R \) and \( Y = \text{Spec} S \). Suppose that \( \sigma_R : S \to \omega_{S/R} \) is injective (i.e. \( T \) is non-degenerate). We say that \( \varphi \in \mathcal{C}_{e,R} \) is \( T \)-transposable if \( \varphi^! \circ F^e \sigma_R : F^e S \to \omega_{S/R} \) belongs to the image of \( \text{Hom}_S(F^e S, S) \) under the embedding \( \text{Hom}_S(F^e S, S) : \text{Hom}_S(F^e S, \omega_{S/R}) \). That is, \( \varphi \) is \( T \)-transposable if there is a necessarily unique map \( \varphi^\top \in \mathcal{C}_{e,S} \) such that

\[
\begin{array}{ccc}
F^e S & \xrightarrow{\varphi^\top} & S \\
\downarrow{F^e \sigma_R} & & \downarrow{\sigma_R} \\
F^e \omega_{S/R} & \xrightarrow{\varphi^!} & \omega_{S/R}
\end{array}
\]

is commutative. We refer to \( \varphi^\top \) as the \( T \)-transpose of \( \varphi \) (suppressing \( T \) from the notation hoping the context will make it clear). Note that \( \varphi^\top \) is characterized by the equality

\[
(3.1.1) \quad \varphi(F^e T(ss')) = T(\varphi^\top(F^e s)s')
\]
for all \(s, s' \in S\). In particular, \(\varphi \circ F_s^* T = T \circ \varphi^\top\); obtained by setting \(s' = 1\) in (3.1.1)\.

However, \(T(\varphi^\top(F_s s')s') = T(\varphi^\top(s'F_s s)) = T(\varphi^\top(F_s s s'^\top))\) and so (3.1.1) holds if so does \(\varphi \circ F_s^* T = T \circ \varphi^\top\). In other words, \(\varphi^\top\) is characterized by the equality

\[
(3.1.2) \quad \varphi \circ F_s^* T = T \circ \varphi^\top.
\]

We thank Anne Fayolle for pointing this out to us.

**Proposition 3.2.** With notation as above, suppose that \(\sigma_R\) is an isomorphism in codimension 0 and that \(S\) satisfies \(S_1\). Then, the injective morphism of Cartier modules \(\sigma_R: (S, \varphi^\top) \to (\omega_{S/R}, \varphi^\top)\) induces an isomorphism on test modules.

**Proof.** See Corollary 3.11 below. \(\square\)

If \(S\) satisfies \(S_1\) then so does \(R\). The converse holds if \(\sigma_R\) is injective. Moreover, if \(\sigma_R\) is an isomorphism in codimension 0 and \(S\) satisfies \(S_1\), then \(\sigma_R\) is injective. Thus, in view of Proposition 3.2, the natural setup for transposability is that both \(R\) and \(S\) satisfy \(S_1\) and that \(\sigma_R: S \to \omega_{S/R}\) is an injective generic isomorphism. If \(R\) and \(S\) also satisfy \(G_0\), then \(\sigma_R\) being injective implies that it is an isomorphism in codimension 0.

We aim to generalize Proposition 3.2 and the notions behind it to all Cartier modules. The first hurdle we face is controlling \(\text{ker } \sigma_R\). However, if \(\omega\) is a \(\omega\)-reflexive module, which is well-behaved assuming the underlying spaces satisfy \(S_1\); see [Har07, Har94]. We introduce the following setup.

**Setup 3.3.** Let \(f: Y = \text{Spec } S \to X = \text{Spec } R\) be a finite cover of \(S_1\) schemes admitting canonical modules. Choose \(T \in \omega_{S/R}\) and consider the corresponding natural transformation \(\sigma: f^* \to f^!\). We assume that \(\sigma_R\) is a generic isomorphism. Let \(K := \prod_{\text{ht } p = 0} R_p\) be the total ring of fractions of \(R\); see [Har94, Proposition 2.1]. Since \(\phi: R \to S\) is integral, \(L := K \otimes_R S\) is the total ring of fractions of \(S\). In particular, proving something generically (or in codimension 0) means pulling back along \(\text{Spec } K \to X\) and replacing \(f\) by \(f_K: \text{Spec } L \to \text{Spec } K\) when \(f\) is generically flat (i.e. \(f_K\) is flat), we write \([L : K]\) for the free rank of \(L/K\).

**Definition-Proposition 3.4.** Work in Setup 3.3. Then, \(\sigma: f^* \to f^!\) factors through \(\omega\)-reflexifications. That is, \(\sigma_M: f^*M \to f^!M\) factors naturally as

\[
\sigma_M: f^*M \xrightarrow{\alpha_M} (f^*M)^{\omega_{\omega}} \xrightarrow{\zeta_M} f^!M
\]

for all \(M\) in \(R\text{-mod}^{\omega}\). Thus, \(M \mapsto (f^*M)^{\omega_{\omega}}\) defines a functor \(f^!:\text{R-mod}^{\omega} \to \text{S-mod}^{\omega}\) together with a natural transformation \(\zeta: f^! \to f^!\) factoring \(\sigma: f^* \to f^!\) through \(\omega\)-reflexifications.

The advantage of working with \(\zeta\) instead of \(\sigma\) is the following.

**Proposition 3.5.** Work in Setup 3.3. Then, \(\sigma_M^{\omega_{\omega}}\) is an (injective) generic isomorphism for all generically flat finite \(R\text{-modules } M\) satisfying \(S_1\). In particular, \(\zeta_M: f^!M \to f^!M\) is an (injective) generic isomorphism for all generically flat modules \(M\) in \(R\text{-mod}^{\omega}\).

**Proof.** For \(M\) in \(R\text{-fmod}\), we have natural isomorphisms of \(R\)-modules:

\[
f_*f^!(M^{\omega_{\omega}}) = \text{Hom}_R(S, M^{\omega_{\omega}}) \cong \text{Hom}_R(S \otimes_R M^{\omega}, \omega_R) = \text{Hom}_R(f_*f^*(M^{\omega}), \omega_R) \\
\cong f_* \text{Hom}_S(f^*(M^{\omega}), \omega_S) \\
= f_* (f^!(M^{\omega}))^{\omega_{\omega}},
\]
We claim that it is commutative. This is the case for $c$ where $M = M^\omega$ (3.5.3)
and $f$ is the (natural transformation given by $\delta_M := (\omega_M) = (f(M^\omega))_\omega$ and $\delta_M$ is the $(f^*, f_*)$-adjoint of the $R$-linear map $M^\omega \to f_*(f^1M)^\omega$ given by $\mu \mapsto f_\mu^1$. Of course, the outer rectangle in (3.5.1) is commutative for all $M$. A straightforward computation verifies that the upper triangle in (3.5.1) is commutative as well (for all $M$). The lower triangle, however, is not necessarily commutative unless $\alpha_{f^1M}$ is surjective. Thus, if $M$ and so $f^1M$ satisfy $S_1$ then $\alpha_{f^1M}$ is an isomorphism in codimension 1 thereby the lower triangle commutes in codimension 1. Therefore, since all modules in the lower triangle are reflexive, this triangle is commutative if $M$ satisfies $S_1$; see [Har07, Remark 1.8]. Moreover, in that case, it would be a commutative diagram of natural isomorphisms.

In conclusion, $\delta_M$ naturally realizes $\beta_M$ as the $S_2$-ification of $f^1M$ in the (full) subcategory of modules $M$ that satisfy $S_1$. This will be crucial below.

On the other hand, specializing to $M = R$ gives us an injective map:

$$S \xrightarrow{\beta_R} \omega_M \xrightarrow{\beta_R(T)} \omega_M = \operatorname{Hom}_S(f^*\omega_R, \omega_S).$$

Note that $(\beta_R \circ \sigma_R)(1) = \beta_R(T) = \sigma_{\omega_R}$. Then, we may consider the following composition:

$$f^* \operatorname{Hom}_R(M, \omega_R) \xleftarrow{c} \operatorname{Hom}_S(f^*M, f^*\omega_R) \xrightarrow{\operatorname{Hom}_S(f^*M, \omega_R)} \operatorname{Hom}_S(f^*M, \omega_R)$$

where $c$ is the $(f^*, f_*)$-adjoint of the $R$-linear map $\operatorname{Hom}_R(M, \omega_R) \to f_* \operatorname{Hom}_S(f^*M, f^*\omega_R)$ given by $\mu \mapsto f_\mu^1$ for all $\mu \in \operatorname{Hom}_R(M, \omega_R)$. This gives us a natural transformation $\gamma_M : f^*(\omega^\omega) \to (f^*(M^\omega))^\omega$. Dualizing $\gamma_M$ gives $\gamma_M^\omega : f^1M \to (f^*(M^\omega))^\omega$. Further, consider the following diagram:

$$f^1M \xrightarrow{\gamma_M^\omega} (f^*(M^\omega))^\omega \xrightarrow{\beta_M} f^1M$$

$$f^*M \xrightarrow{\sigma_M} f^1M$$

We claim that it is commutative. This is the case for $M = R$ by construction. Noting that all natural transformations are compatible with $\operatorname{Hom}_R(R, M)$, we obtain the commutativity for general $M$. Next, we prove that this diagram (naturally) realizes $\gamma_M^\omega$ as $\sigma_M^\omega$ via $\delta_M$.

**Claim 3.6.** $\gamma_M^\omega = \delta_M^\omega \circ \sigma_M^\omega$ for all $M$. In fact, $\gamma_M = \sigma_M \circ \delta_M$ for all $M$.

\[\text{Since } f_*((f^*M)^\omega) \cong (f_*f^*M)^\omega \text{ by Grothendieck duality, } \delta_M \text{ is the } (f^*, f_*)\text{-adjoint of } \mathcal{R}_M : M^\omega \to (f_*f^1M)^\omega.\]

\[\text{Use [The22] Lemma 0AV0 and the fact that depth is invariant under pushforwards along finite morphisms.}\]
Proof of claim. We verify the second equality which implies the first one. Both sides are natural transformations $f^*(M^\omega) \to (f^*)^\omega M$. Thus, it suffices to check that both sides agree at an arbitrary element $1 \otimes \mu \in f^*(M^\omega)$ with $\mu \in M^\omega$. On one side, $\gamma_M : 1 \otimes \mu \mapsto \sigma_{\omega R} \circ f^* \mu$. On the other, $\sigma_M^\omega \circ \delta_M : 1 \otimes \mu \mapsto f^* \mu \circ \sigma_M$. Thus, we just need to show $\sigma_{\omega R} \circ f^* \mu = f^* \mu \circ \sigma_M$ for all $\mu$. However, this is clear as both send $1 \otimes m$ to $s \mapsto T(s)\mu(m)$.

In particular, $\sigma_{\omega R}^\omega = \delta_M^{\omega - 1} \circ \gamma_M^\omega$ if $M$ satisfies $S_1$, and $\zeta_M = \beta_M^{\omega - 1} \circ \gamma_M^\omega$ if $M$ satisfies $S_2$. Consequently, in either case, we just need to prove that $\gamma_M^\omega$ is an injective generic isomorphism for all $M$. Since the domain of $\gamma_M^\omega$ satisfies $S_1$, it suffices to prove that $\gamma_M^\omega$ is a generic isomorphism as then its injectivity is automatic. Then, it is enough to prove that $\gamma_M$ is generically an isomorphism. To this end, recall that $\gamma_M$ is the composition of (3.5.3). Note that the map $c$ in (3.5.3) is generically an isomorphism if $M$ is generically flat (or if $f$ is generically flat). Hence, it suffices to prove that $\sigma_{\omega R}$ is generically an isomorphism. To this end, recall that $\sigma_{\omega R}$ is the image of 1 under the composition (3.5.2) which is generically an isomorphism (by hypothesis). Generically, we then have an isomorphism $L \to \text{Hom}_L (f^*_K\omega_K, \omega_L)$ of $L$-modules. Taking $\omega$-duals, we obtain an isomorphism $\rho : f^*_K \omega_K \to \omega_L$ using that in dimension 0 all modules are $\omega$-reflexive [Har07, Lemma 1.1]. Therefore, there is (a unique) $l \in L$ such that $\rho = l \cdot \sigma_{\omega_K}$ and $l$ cannot be a zerodivisor as $\rho$ is an isomorphism, whence $\sigma_{\omega_K} = l^{-1} \cdot \rho$ is an isomorphism; as desired.

Remark 3.7. In Proposition 3.5, we may add the hypothesis of $f$ being generically flat and remove the one of generic flatness on $M$ (as pointed out in its proof). Such condition is determined by $T$ as $\sigma_R$ is a generic isomorphism by assumption. Indeed, if $(R, m, \mathfrak{m}) \subset (S, n, \ell)$ is a local extension of artinian rings such that $S \to \omega_S/R; 1 \mapsto T$, is an isomorphism, then $S/R$ is free if and only if the inclusion of ideals $mS \subset f^! \mathfrak{m} : T = \{ s \in S \mid T(sS) \subset \mathfrak{m} \}$ is an equality. To see this, note that $\dim_k S/\mathfrak{m}S$ computes the minimal number of generators of $S$ as an $R$-module whereas $\dim_k S/(f^! \mathfrak{m} : T)$ computes its free rank as $1 \mapsto T$ is an isomorphism; cf. [BST12, Lemma 3.6]. This is all irrelevant if $R$ satisfies $R_0$.

Definition 3.8 (Transposability). Working in Setup 3.3, let $M$ be a generically flat module in $R$-mod$^\omega$. Consider the following definitions:

- A map $\varphi \in C_{e, M}$ is said to be $T$-transposable if $\varphi^! \circ F^e_{\omega M} \in \text{Hom}_S (F^e_* f^! M, f^! M)$ belongs to $\text{Hom}_S (F^e_* f^! M, f^! M)$, i.e. if there is $\psi \in C_{e, f^! M}$ such that the diagram

$$
\begin{array}{ccc}
F^e_* f^! M & \xrightarrow{\psi} & f^! M \\
\downarrow \sigma_{\omega M} & & \downarrow \varsigma_M \\
F^e_* f^! M & \xrightarrow{\varphi^!} & f^! M
\end{array}
$$

is commutative. Since $\varsigma_M$ is injective, any such $\psi$ is unique and we denote it by $\varphi^T$ (suppressing $T$ from the notation hoping that it is clear from the context) and refer to it as the $T$-transpose of $\varphi$. We denote the set of $T$-transposable maps by $C^T_{e, M}$, which is a Cartier subalgebra of $C_{e, M}$.

- Let $C$ be a Cartier $R$-algebra acting on $M$ via $\Xi : C \to C_M$. We say that $M$ is a $T$-transposable Cartier module if $\Xi$ factors through $C^T_M \subset C_M$. In particular, $\varsigma_M : f^! M \to f^! M$ is a morphism of Cartier $f^* C$-modules where $f^* C$ acts on $f^! M$ via $\varphi \otimes s \mapsto \varphi^T \cdot s$. 

The above definitions do not apply to \( M = R \) unless \( R \) satisfies \( \textbf{S}_1 \), which is only assumed to be \( \textbf{S}_1 \). We define the same transposability notions for \( M = R \) by replacing \( M \) by \( R \) and \( \varsigma_M \) by \( \sigma_R \) in the previous two items—this generalizes the preliminary definitions we gave in the paragraphs that followed \textbf{Lemma 3.1}. Of course, if \( R \) and \( S \) satisfy \( \textbf{S}_2 \), these two notions of transposability coincide for \( M = R \).

Remark 3.9. (Degeneracy) Work in the setup of \textbf{Definition 3.8}. Let \( \varphi : F^*_r M \to M \) be a \( T \)-transposable map. If any map in the set \( \{ \varphi, \varphi^i, \varphi^\tau \} \) is nondegenerate then so are the other two. Indeed, \( \varphi^i \) and \( \varphi^\tau \) are generically the same map up to isomorphism (of both source and target) as \( \varsigma_M \) is generically an isomorphism by \textbf{Proposition 3.5}. To see why \( \varphi \) and \( \varphi^i \) share degeneracy, observe that if \( \varphi = 0 \) then \( \varphi^i = 0 \) in general and so generically. Next, suppose \( \varphi^i \) is generically zero. Since \( M \) is generically flat (so free), we may assume that \( M = R \). Thus, to show \( \varphi \) is generically zero, we show that \( \text{Tr}_{R} : f^* \omega_{S/R} \to R \) is generically surjective. This follows because \( R \to S \) is a finite cover which implies \( \omega_{S/R} \) is generically nonzero and so there is a map in \( \omega_{S/R} \) whose image contains a regular element. However, it is not clear to the authors whether degenerate maps are \( T \)-transposable, unless \( R \) and \( S \) are integral domains.

Remark 3.10. By definition, a \( T \)-transposable Cartier module \( M \neq R \) is \( \omega \)-reflexive. However, we may also work with \( \textbf{S}_1 \)-modules (e.g. \( M = R \)) when studying their test modules as \( \alpha_M : M \to M^{\omega} \) then realizes \( M \) as a Cartier submodule of \( M^{\omega} \). Indeed, if \( C \) acts on \( M \) via \( \Xi : C \to C_M \), then \( C \) acts on \( M^{\omega} \) by \( \Xi^{\omega} : k_e \mapsto \Xi(k_e)^{\omega} \) using the canonical isomorphism \( (F^*_r M)^{\omega} \cong F^*_r (M^{\omega}) \). Since \( \alpha_M \) is a generic isomorphism, \( \tau(\alpha_M) \) is an isomorphism. Thus, in studying test modules, we may replace \( M \) with its \( \textbf{S}_2 \)-ification and consider transposability on \( M^{\omega} \).

Corollary 3.11. Work in \textbf{Setup 3.3}. Let \( C \) be a Cartier \( R \)-algebra and \( M \) be an \( \omega \)-reflexive \( T \)-transposable Cartier \( C \)-module. Then \( \varsigma_M : f^1 M \to f^1 M \) induces an isomorphism on test modules, i.e. \( \tau(\varsigma_M) : \tau(f^1 M, f^* C) \to \tau(f^1 M, f^* C) \) is an isomorphism. Further, the same holds with \( R \) in place of \( M \) and \( \sigma_M \) in place of \( \varsigma_M \).

Proof. Suppose first that \( M \) is \( \omega \)-reflexive. Observe that \( \varsigma_M \) is an injective generic isomorphism between modules having no embedded primes. Since \( \varsigma_M \) is an injective \( f^* C \)-linear map, \( \varsigma_M(\tau(f^1 M, f^* C)) \) is an \( f^* C \)-submodule of \( f^1 M \) that generically agrees with \( f^1 M \). Thus, \( \tau(f^1 M, f^* C) \subset \varsigma_M(\tau(f^1 M, f^* C)) \) by minimality of \( \tau(f^1 M, f^* C) \). Conversely, since \( \varsigma_M(f^1 M) \) is a \( f^* C \)-submodule of \( f^1 M \) that generically agrees with \( f^1 M \), it must contain \( \tau(f^1 M, f^* C) \). Then, \( \tau_M(\tau(f^1 M, f^* C)) \subset \tau(f^1 M, f^* C) \) by minimality of \( \tau(f^1 M, f^* C) \).

If \( M = R \) is not necessarily \( \omega \)-reflexive, the same proof applies with \( \sigma_M \) instead of \( \varsigma_M \). Alternatively, one may apply \textbf{Remark 3.10}.

Remark 3.12. Working in \textbf{Setup 3.3} assume that \( R \) and \( S \) satisfy \( \textbf{G}_0 \). Let \( C \) be a Cartier \( R \)-algebra acting on \( \omega_R(D) \) with \( D \) an almost Cartier divisor. Notice that the isomorphism \( \beta_{\omega_R(D)} : f^* \omega_R(D) \to f^* (\omega_R(D)^{\omega}) \) (as in the proof of \textbf{Proposition 3.5}) yields a natural isomorphism \( f^1 \omega_R(D) \to \omega_S(f^* D) \). Further, assuming that \( R \) and \( S \) satisfy \( \textbf{G}_1 \), we have:

\[ f^1 \omega_R(D) = (f^* \mathcal{J}_{-K_R-D})^{\omega} = \mathcal{J}_{f^*(-K_R)-f^*D} = \mathcal{J}_{K_S-R-f^*D-K_S} = \omega_S(f^* D - K_{S/R}). \]

Moreover, the map \( \varsigma_{\omega_R(D)} : f^1 \omega_R(D) \to f^* \omega_R(D) \) is the embedding \( \omega_S(f^* D - K_{S/R}) \to \omega_S(f^* D) \) given by \( \text{Ram}_T \sim K_{S/R} \)—this is how the map \( \chi_{\omega_R(D)} = \varsigma_{\omega_R(D)} \) was constructed in the proof of \textbf{Proposition 3.5}. In particular, \( f^1 R(D) \cong S(f^* D) \) whereas \( f^* R(D) \cong S(f^* D + K_{S/R}) \) and \( \varsigma_{R(D)} \) corresponds to the embedding \( S(f^* D) \to S(f^* D + K_{S/R}) \) defined by \( \text{Ram}_T \sim K_{S/R} \).
\textbf{Theorem 3.13} (Schwede–Tucker’s transposability criterion). Working in \textbf{Setup 3.3}, assume that $R$ and $S$ satisfy $G_1 + S_2$ and $\Box$. Let $D$ be an almost Cartier divisor on $X$. Then, a nondegenerate map $\varphi : F^e_s \omega_R(D) \to \omega_R(D)$ is $T$-transposable if and only if $f^* \Delta_\varphi - \text{Ram}_T$ is effective, in which case $\Delta_{\varphi^T} = f^* \Delta_\varphi - \text{Ram}_T$.

\textit{Proof.} By Remark 3.9 if $\varphi$ is nondegenerate then so are $\varphi^!$ and $\varphi^\top$. By (2.12.3) $D_{\varphi^!} = f^* D_{\varphi}$ and so $\Delta_{\varphi^!} = f^* \Delta_\varphi$. By (2.12.3) again, the divisor associated to

$$\vartheta : F^e_s \omega_S(f^* D - K_{S/R}) \to F^e_s \omega_S(f^* D) \xrightarrow{\varphi^!} \omega_S(f^* D)$$

is $D_{\varphi^!} + \text{Ram}_T$. However, $D_{\varphi^!} + q \text{Ram}_T$ is the divisor associated to a composition

$$F^e_s \omega_S(f^* D - K_{S/R}) \xrightarrow{\varphi^\top} \omega_S(f^* D - K_{S/R}) \to \omega_S(f^* D)$$

Therefore, $\vartheta$ restricts to $\omega_S(f^* D - K_{S/R})$ if and only if $D_{\varphi^!} - (q-1) \text{Ram}_T$ is effective, i.e. if $\Delta_{\varphi^!} - \text{Ram}_T$ is effective. In that case, $\Delta_{\varphi^T} = \Delta_{\varphi^!} - \text{Ram}_T$; as required. \hfill $\Box$

\textbf{Proposition 3.14.} Work in the setup of \textbf{Theorem 3.13} and suppose that $f$ is generically flat. Let $\Delta$ be an effective almost Cartier $\mathbb{Q}$-divisor on $X$ and choose $a \subset R$, $t \in \mathbb{R}_{\geq 0}$. Suppose that $\Delta^* := f^* \Delta - \text{Ram}_T \geq 0$ and let $\Xi : f^* \mathcal{C}^\Delta_{(a,S)^t} \to \mathcal{C}^\Delta_{(a,S)^t}$ be the homomorphism defined by \textbf{Theorem 3.13}. Then, there exists a regular element $c \in S$ such that

$$\mathcal{C}^\Delta_{(a,S)^t} \cdot c \subset \Xi \left( f^* \mathcal{C}^\Delta_{(a,S)^t} \right) \subset \mathcal{C}^\Delta_{(a,S)^t}.$$ 

In particular,

$$\tau \left( S(f^* D), f^* \mathcal{C}^\Delta_{(a,S)^t} \right) = \tau \left( S(f^* D), \mathcal{C}^\Delta_{(a,S)^t} \right) := \tau \left( S(f^* D), \Delta^*, (a \cdot S)^t \right)$$

Likewise, if $f$ is local and $D = 0$, then

$$s \left( S, f^* \mathcal{C}^\Delta_{(a,S)^t} \right) = s \left( S, \Delta^*, (a \cdot S)^t \right).$$

\textit{Proof.} We may assume $a = 0$ as the general case readily follows from this case. The hypothesis $\Delta^* \geq 0$ implies that every map $\varphi : F^e_s R(D) \to R(D)$ with $\Delta_\varphi \geq \Delta$ satisfies $f^* \Delta_\varphi - \text{Ram}_T \geq 0$, which means that $R(D)$ is a $T$-transposable Cartier $\mathcal{C}^\Delta_{R(D)}$-module by \textbf{Theorem 3.13}. Moreover, the structural homomorphism $f^* \mathcal{C}^\Delta_{R(D)} \to \mathcal{C}_S(f^* D)$; given by $\varphi \otimes s \mapsto \varphi^! \cdot s$, factors through the inclusion $\mathcal{C}^\Delta_{S(f^* D)} \subset \mathcal{C}_S(f^* D)$. This defines the homomorphism $\Xi : f^* \mathcal{C}^\Delta_{R(D)} \to \mathcal{C}^\Delta_{S(f^* D)}$.

Writing $\mathcal{C}^\Delta_{R(D)} \cong \bigoplus_n R((1-n)(D-K_X))$ we use Cartier operators (see Example 2.17 and Example 2.16), the $S_2$-ification of $f^* \mathcal{C}^\Delta_{e,R(D)}$ corresponds to

$$S\left( (a,S)^t, f^* \mathcal{C}^\Delta_{e,R(D)} \right)$$

\footnote{It is enough to assume $R$ satisfies $\Box$ as then so does $S$, for $F^e_s \omega_S = F^e_s f^! \omega_R = f^! F^e_s \omega_R \cong f^! \omega_R = \omega_S$.}

Writing $\Xi \cong \bigoplus_n R((1-n)(D-K_X))$ we use Cartier operators (see Example 2.17 and Example 2.16), the $S_2$-ification of $f^* \mathcal{C}^\Delta_{e,R(D)}$ corresponds to

$$S\left( (a,S)^t, f^* \mathcal{C}^\Delta_{e,R(D)} \right)$$

\footnote{$\Xi$ is an isomorphism if $\Delta = \Delta_\varphi$ for some $\varphi \in \mathcal{C}_{e,R(D)}$ as then both $f^* \mathcal{C}^\Delta_{R(D)}$ and $\mathcal{C}^\Delta_{S(f^* D)}$ are $\mathcal{C}^\Delta_{S(f^* D)}$.}
Further, by the naturality of Cartier operators, $\Xi_e$ corresponds to
\[
f^*e_{\varepsilon,R(D)} \xrightarrow{\cong} S\left((q-1)(f^*D - K_Y) - \left(f^*[\langle q-1 \rangle \Delta] - (q-1)K_{Y/X}\right)\right)
\]
\[
\xrightarrow{\cong: \text{Ram}_T \sim K_{Y/X}} S\left((q-1)(f^*D - K_Y) - \left(f^*[\langle q-1 \rangle \Delta] - (q-1)\text{Ram}_T\right)\right)
\]
\[
\subset S\left((q-1)(f^*D - K_Y) - \left([\langle q-1 \rangle f^*\Delta] - (q-1)\text{Ram}_T\right)\right)
\]
\[
= S((q-1)(f^*D - K_Y) - [\langle q-1 \rangle \Delta^*]) \cong C^e_{\varepsilon,S(f^*D)}.
\]
where the inclusion is defined by the inequality $0 \leq f^*[\langle q-1 \rangle \Delta] - [\langle q-1 \rangle f^*\Delta]$.  

**Claim 3.15.** There exists a regular element $a \in R$ that belongs to the annihilator of $\text{coker}(S \otimes_R R(D') \to S(f^*D'))$ for all almost Cartier divisors $D'$ on $X$.  

**Proof of claim.** Since $f$ is generically flat, we may find a free $R$-submodule $G \subset S$ of rank $[L : K]$ and such that $a \cdot S \subset G \subset S$ for some regular element $a \in R$. In particular, there is an induced commutative diagram for all almost Cartier divisor $D'$ on $X$

\[
\begin{array}{ccc}
(a \cdot S) \otimes R(D') & \longrightarrow & G \otimes R(D') \\
\downarrow & & \downarrow \cong \\
(a \cdot S)(f^*D') & \longrightarrow & G' \longrightarrow S(f^*D')
\end{array}
\]

where the vertical arrows are $S_2$-ifications either as $R$-modules or $S$-modules when appropriate (which does not matter as both $R$ and $S$ satisfy $S_2$) and the lower horizontal arrows are inclusions. In particular, $G' \cong R(D')^{[L : K]}$ is an $R$-submodule of $S(f^*D')$ that contains $a \cdot S(f^*D')$. This proves the claim. \qed

Let $a \in R$ be as in **Claim 3.15** so $a \in \bigcap_e \text{Ann coker} \alpha_e$. Let $b \in S$ a regular element such that $0 \leq f^*[\langle q-1 \rangle \Delta] - [\langle q-1 \rangle f^*\Delta] \leq \text{div}_S b$ (see [ST14, §2.2] and [CR18, Lemma 3.10]). Then,

\[
C^e_{\varepsilon,S(f^*D)} \cdot b \subset S\left((q-1)(f^*D - K_Y) - \left(f^*[\langle q-1 \rangle \Delta] - (q-1)\text{Ram}_T\right)\right) \subset C^e_{\varepsilon,S(f^*D)}
\]

for all $e$. Thus, we may take $c := a \cdot b$. The remaining statements follow from **Lemma 2.21** \qed

4. Transformation rule for the $F$-signature and splitting ratio

In this section, we prove transformation rules for $F$-signature and splitting ratio under finite covers that generalize those in [CRST18, Theorems 3.1, 4.4], [CR22, Theorem 4.11].

4.1. Generalizing a Theorem of Tucker. Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local ring. Given $M$ in $R$-fmod, we define $M^p := \bigoplus_e M^p_e := \bigoplus_e \text{Hom}_R(F^p_e M, R)$, which is endowed with a graded left $C_R$-module structure: $\varphi \cdot \vartheta := \varphi \circ F^p_e \vartheta$ for all $\varphi \in C_{d,R}$, $\vartheta \in M^p_e$. Denote the category of graded left modules over a Cartier algebra $C$ by $C$-glmod. Note that $M \mapsto M^p$ is a contravariant functor $R$-fmod $\to$ $C$-glmod.[6] Given a Cartier algebra $C \subset C_R$ and $\mathcal{M} \subset M^p$ in $C$-glmod, we define the splitting numbers

\[
a_e(\mathcal{M}) := \lambda_R(\mathcal{M}_e/\mathcal{M}^\text{ns}_e),
\]

[6]However, $M^p$ is not necessarily finitely generated as an $R$-module, therefore not a Cartier module.
where lengths are computed as left $R$-modules and $\mathcal{M}^\text{ns} := \mathcal{M} \cap \text{Hom}_R(F_*^e M, m)$. Note that $\mathcal{M}^\text{ns} \supset m \cdot \mathcal{M}$ and so $\alpha_e(\mathcal{M})$ is finite. Set $n_\mathcal{M} := \gcd\{e \mid \alpha_e(\mathcal{M}) \neq 0\}$ and $\delta = \dim R + \log_{\mathcal{K}^{1/p}}[\mathcal{K} : \mathcal{K}]$ as in $[\text{BST}12]$. Following $[\text{Yao}05]$, we define the splitting ratio of $\mathcal{M}$

\[
\#(\mathcal{M}) := \lim_{e \to \infty} \frac{\alpha_{e-n_\mathcal{M}}(\mathcal{M})}{q^\delta}.
\]

**Theorem 4.1** (cf. $[\text{Tuc}12$, Theorem 4.11]). If $(R, m, \mathcal{K}, K)$ is a local domain and $M$ is a finitely generated $R$-module, then $\#(M^\sigma) = \text{rank } M \cdot s(R, \mathcal{C}_R)$.

Let $\mathcal{C} \subset \mathcal{C}_R$ be a Cartier algebra and $M$ in $R$-fmod. Define $M^\sigma \mathcal{C} \subset M^\mathcal{C}$ in degree $e$ as:

\[
M^\sigma_e := \{ \varphi \circ F_*^e \rho \in M^\mathcal{C} \mid \varphi \in \mathcal{C}_e \text{ and } \rho \in M^\mathcal{C} = \text{Hom}_R(M, R) \}.
\]

Observe that $M \mapsto M^\sigma$ defines a contravariant functor $R$-fmod $\to \mathcal{C}$-glmod. Consider the functor $R$-fmod $\to \mathcal{C}$-glmod, $M \mapsto \mathcal{C} \otimes_R M^\mathcal{C}$. Then, $\eta_M : \mathcal{C} \otimes_R M^\mathcal{C} \to M^\sigma$, $\varphi \otimes \rho \mapsto \varphi \circ F_*^e \rho$ is a surjective natural transformation (although not injective in general).

**Theorem 4.2.** Let $(R, m, \mathcal{K}, K)$ be a local domain and $\mathcal{C} \subset \mathcal{C}_R$ a Cartier $R$-algebra. Then, $\#(M^\sigma) = \text{rank } M \cdot s(R, \mathcal{C})$ for all finite $R$-modules $M$.

**Proof.** Let $g := \text{rank } M$ and consider a short exact sequence of $R$-modules

\[
0 \to L \xrightarrow{i} M \to M/L \to 0,
\]

where $L$ is free of rank $g$ and $T := M/L$ is torsion; i.e. $\text{Ann}_R T \neq 0$. Applying the exact functor $F_*^e$ followed by the left exact functor $\text{Hom}_R(-, R)$ yields an exact sequence

\[
0 \to M^\sigma_e \xrightarrow{\iota} L^\sigma_e,
\]

as $\text{Hom}_R(F_*^e T, R) = 0$ (for $R$ is a domain and $F_*^e T$ is torsion). The map $\iota$ is none other than restriction. Thus, a map $\vartheta : F_*^e M \to T$ is determined by its values at $F_*^e L \subset F_*^e M$. Let us think of $\iota$ as an inclusion $M^\sigma_e \subset L^\sigma_e$ by realizing $M^\sigma_e$ inside $L^\sigma_e$ as the maps $\vartheta : F_*^e L \to R$ admitting a (necessarily unique) extension to a map $F_*^e M \to R$. Note that $\iota$ respects nonsurjectivity, i.e. $(M^\sigma_e)^\text{ns} \subset (L^\sigma_e)^\text{ns}$. For a nonzero $c \in \text{Ann}_R T$, we have $L^\sigma_e \cdot c \subset M^\sigma_e \subset L^\sigma_e$ and we readily see that it restricts to

\[
(4.2.1) \quad L^\sigma_e \cdot c \subset M^\sigma_e \subset L^\sigma_e.
\]

Consequently, one may conclude as usual (using the well-known argument to experts):

\[
a_e(M^\sigma) = \lambda_R(F_*^e L) = \{ F_*^e l \mid \vartheta(F_*^e l) \in m \text{ for all } \vartheta \in M^\sigma_e \}
\]

\[
= \sum_{i=1}^g \lambda_R(F_*^e R / \{ F_*^e r \mid (\vartheta \circ \sigma_i)(F_*^e r) \in m \text{ for all } \vartheta \in M^\sigma_e \})
\]

where $\sigma_i : F_*^e R \to F_*^e L$ are the direct sum structural maps. Consider the ideals $J_{i,e} := \{ r \mid (\vartheta \circ \sigma_i)(F_*^e r) \in m \text{ for all } \vartheta \in M^\sigma_e \} \subset R$ and $I_e := \{ r \mid \varphi(F_*^e r) \in m \text{ for all } \varphi \in \mathcal{C}_e \} \subset R$. Then

(4.2.1) implies $I_e \subset J_{i,e} \subset I_e : c$ for all $i = 1, \ldots, g$. One then argues as in e.g. $[\text{BST}12$, Lemma 4.17], $[\text{CRST}18$, Lemma 2.7], $[\text{Tuc}12$, Theorem 4.11].

---

It is worth noting that $J_{i,e}$ is an ideal because $\vartheta(\sigma(F_*^e r s)) = \vartheta(F_*^e r \cdot \sigma_i(F_*^e s)) = ((\vartheta \cdot r) \circ \sigma_i)(F_*^e s)$.
4.2. Transformation rule for $F$-signature. The desired formula is the following.

**Theorem 4.3.** Let $\theta : (R, m, k) \to (S, n, \ell)$ be a finite local extension defining a cover $f : \text{Spec} S \to \text{Spec} R$ and $C$ be a Cartier $R$-algebra. Suppose that $R$ is an integral domain with field of fractions $K$, set $L := S \otimes_R K$, and write $[L : K] := \dim_k L$. Suppose that there is a generic isomorphism $\sigma_R : S \to \omega_{S/R}$ of $S$-modules such that $T := \sigma_R(1)$ is surjective and $T(n) \subseteq m$. If $R$ is a $T$-transposable Cartier $C$-module, then

$$[\ell : k] \cdot s(S, f^*C) = [L : K] \cdot s(R, C).$$

In particular, $(R, C)$ is $F$-regular if and only if so is $(S, f^*C)$.

**Proof.** Note that $\theta$ is an extension of $S_1$ rings. Let $d = \dim R = \dim S$ and $\delta = d + \log_p[\ell^{1/p} : k] = d + \log_p[\ell^{1/p} : \ell]$. Note that:

$$[\ell : k] \cdot a_e(f^*C) = [\ell : k] \cdot \lambda_S(f^*C_\ell / (f^*C_\ell)^{\text{ns}}) = \lambda_R \left( f_*(f^*C_\ell / (f^*C_\ell)^{\text{ns}}) \right).$$

**Claim 4.4.** If $\sigma_R$ is an isomorphism, the isomorphism given by Proposition 2.3 (cf. Remark 2.19) $\xi = \xi(R, F^*_eS) : f_*\text{Hom}_S(F^*_eS, S) \to \text{Hom}_R(F^*_eF^*_eS, R)$, $\psi \mapsto T \circ f_*\psi$, induces an isomorphism

$$f_*\left( f^*C_\ell / (f^*C_\ell)^{\text{ns}} \right) \to \left( f_*S \right)^{\text{c}}_e / \left( (f_*S)^{\text{c}}_e \right)^{\text{ns}}.$$

**Proof of claim.** We must prove the equality $\xi(f_*f^*C_\ell) = (f_*S)^{\text{c}}_e$ and that $\psi$ is surjective if and only if so is $\xi(\psi)$. Let us recall that an element of $f^*C_\ell$ is a finite sum $\sum_i \varphi_i^\top \cdot s_i$ where all $\varphi_i$ are in $C_\ell$; see Remark 2.19. Thus, for such an element, we have:

$$\xi\left( \sum_i \varphi_i^\top \cdot s_i \right) = \sum_i T \circ (\varphi_i^\top \cdot s_i) = \sum_i \varphi_i \circ F^*_e(s_i \cdot T) \in (f_*S)^{\text{c}}_e.$$

In other words, $\xi(f_*f^*C_\ell) \subseteq (f_*S)^{\text{c}}_e$. Conversely, on the right-hand side of the above equality, we hit all the elements of $(f_*S)^{\text{c}}_e$ as any $\rho \in \text{Hom}_R(S, R)$ is of the form $s \cdot T$ for some $s \in S$, for $\sigma_R$ is an isomorphism. The equivalence between the surjectivity of $\psi$ and $\xi(\psi) = T \circ \psi$ follows from the remaining two hypothesis on $T$. Indeed, since $T$ is surjective, then $\xi(\psi)$ is surjective if so is $\psi$. Conversely, suppose $\psi$ is not surjective, meaning that it maps $F^*_eS$ into $n$. Then, since $T(n) \subseteq m$, we have that $\xi(\psi)$ maps $F^*_eS$ into $m$. This proves the claim.

If $\sigma_R$ were an isomorphism, we could use Claim 4.4 directly to write

$$[\ell : k] \cdot a_e(f^*C) = \lambda_R \left( f_*\left( f^*C_\ell / (f^*C_\ell)^{\text{ns}} \right) \right) = \lambda_R \left( \left( f_*S \right)^{\text{c}}_e / \left( (f_*S)^{\text{c}}_e \right)^{\text{ns}} \right) = a_e \left( (f_*S)^{\text{c}}_e \right).$$

Dividing by $q^\delta$, letting $e \to \infty$, and using Theorem 4.2 yields desired result. We are, however, assuming $\sigma_R$ to be an isomorphism only generically and so Claim 4.4 and its proof still hold generically. Let $c \in S$ be a regular element in $\text{Ann}_S \text{coker} \sigma_R$. Thus, $\xi$ still sends $f_*f^*C_\ell$ injectively into $(f_*S)^{\text{c}}_e$ preserving surjectivity/nonsurjectivity in the process and

$$(f_*S)^{\text{c}}_e \cdot c \subseteq \xi(f_*f^*C_\ell) \subseteq (f_*S)^{\text{c}}_e.$$

As in the proof of Theorem 4.2 this suffices to conclude that

$$\lambda_R \left( \xi(f_*f^*C_\ell) / \xi(f_*f^*C_\ell)^{\text{ns}} \right) / q^\delta \text{ and } a_e \left( (f_*S)^{\text{c}}_e \right) / q^\delta$$

have the same limit as $e$ goes to $\infty$, which is all we need to conclude as before.
Remark 4.5. Using Proposition 3.14, we see that Theorem 4.3 recovers the transformation rules in [CRST18, CR22].

Remark 4.6. If $\sigma_R$ is an isomorphism in Theorem 4.3, $[\ell : \mathcal{H}]$ is the free rank of $S$ as an $R$-module: $\text{frk}_R f_* S = \lambda_R(\text{Hom}_R(S, R)/\text{Hom}_R(S, \mathfrak{m})) = \lambda_R(S \cdot T/\mathfrak{n} \cdot T) = \lambda_R(\ell) = [\ell : \mathcal{H}]$.

Example 4.7. The following examples show why the hypothesis in Theorem 4.3 about $T$ are necessary; see [CR18, Example 3.15] for further details.

(a) For the surjectivity of $T$, consider [ST14, Example 7.12] of a a quasi-étale degree-2 extension $R \subset S = \mathbb{F}_2[u, v]$ of 2-dimensional $\mathbb{F}_2$-algebras such that $\text{Tr}_{S/R}$ is not surjective.

In fact, $R$ is a log terminal singularity that is $F$-pure but not strongly $F$-regular. $R$ is the ring of invariants of $S$ under certain non-linear action of $\mathbb{Z}/2\mathbb{Z}$ due to M. Artin [Art76].

(b) For $T(n) \subset \mathfrak{m}$, consider any Noether normalization $R \subset S$ of a singular Gorenstein local ring $S$ such that $\text{char } \mathcal{H} + [\mathcal{H}(S) : \mathcal{H}(R)]$. For instance, take $R = \mathcal{H}[x^2, y^2] \subset \mathcal{H}[x^2, xy, y^2] = S$ with $\text{char } \mathcal{H} \neq 2$. Then, $\omega_{S/R} \cong S$, say with free generator $T$. Then $T$ is surjective as $R \subset S$ splits. However, Theorem 4.3 fails as otherwise it would imply $s(S) \geq s(R) = 1$ (but $S$ is singular). A free basis for $\mathcal{H}[x^2, y^2] \subset \mathcal{H}[x^2, xy, y^2]$ is 1, $xy$ and $T$ may be taken to be $(xy)^\gamma$. Hence, $T(n) \not\subset \mathfrak{m}$ as $T(xy) = 1$.

4.3. Splitting primes and splitting ratios under finite covers. Since splitting ratios are $F$-signatures of Cartier algebras [BST12], it is natural to expect them to satisfy transformation rules. In this regard, we have the following.

Theorem 4.8. With hypothesis as in Theorem 4.3, the contraction of $\beta(S, f^* \mathcal{C})$ along $\theta$, i.e. $\beta(S, f^* \mathcal{C}) \cap R$, is $\beta(R, \mathcal{C})$. Furthermore:

$$[\kappa(n) : \kappa(m)] \cdot r(S, f^* \mathcal{C}) = [\kappa\left(\beta(S, f^* \mathcal{C}) \cap R\right) : \kappa(\beta(R, \mathcal{C}))] \cdot r(R, \mathcal{C}),$$

where $\kappa(-)$ denotes the residue field at the respective prime ideal. In particular, $(R, \mathcal{C})$ is $F$-pure (resp. strongly $F$-regular) if and only if $(S, f^* \mathcal{C})$ is so.

Proof. We show first $\beta(S, f^* \mathcal{C}) \cap R = \beta(R, \mathcal{C})$. We must prove that for $r \in R$, the existence of $\varphi \in \mathcal{C}_e$ such that $\varphi(F^e_r) = 1$ is equivalent to the existence of $\psi \in f^* \mathcal{C}_e$ such that $\psi(F^e_r) = 1$.

Let $v \in S$ be a unit such that $T(v) = 1$, which exists because $T$ is surjective and $T(n) \subset \mathfrak{m}$. Let $\varphi \in \mathcal{C}_e$ such that $\varphi(F^e_r) = 1$, we claim that $\varphi^T \cdot r$ maps $F^e_r$ to a unit in $S$. Indeed, if $\varphi^T(F^e_r) \in n$, then $T((\varphi^T(F^e_r \cdot v)) \in T(n) \subset \mathfrak{m}$. However, $T(\varphi^T(F^e_r \cdot v)) = \varphi(F^e_r \cdot v) = \varphi(F^e_r) = 1$.

Conversely, say there is $\psi \in f^* \mathcal{C}_e$ such that $\psi(F^e_r) = 1$, so $\psi(F^e_{sv^\alpha}) = v$. Since $\psi$ is a sum of elements of the form $\varphi^T \cdot s$, we may assume $\psi = \varphi^T \cdot s$ with $\varphi \in \mathcal{C}_e$, $s \in S$. Thus, $\varphi^T(F^e_{sv^\alpha}) = v$.

Hitting this equality with $T$ and using $T \circ \varphi^T = \varphi \circ F^e_r T$ gives:

$$\varphi(F^e_r T(sv^\alpha \cdot r)) = \varphi(F^e_r T(sv^\alpha) \cdot r) = 1.$$

In other words, $\varphi \cdot T(sv^\alpha)$ sends $F^e_r$ to 1. This shows $\beta(S, f^* \mathcal{C}) \cap R = \beta(R, \mathcal{C})$.

In this manner, to prove the transformation rule, we may assume $(R, \mathcal{C})$ and therefore $(S, f^* \mathcal{C})$ are $F$-pure as otherwise the transformation rule is trivially true ($0 = 0$). Observe that $R \subset S$ restricts to a local inclusion of domains

$$R/\beta(R, \mathcal{C}) \subset S/\beta(S, f^* \mathcal{C}),$$
with corresponding morphism of schemes denoted by $\mathcal{F}$. Further, let us set $p := \beta(R, \mathcal{C})$, $q := \beta(S, f^*\mathcal{C})$, $R := R/p$, and $S := S/q$. In order to apply the transformation rule in Theorem 4.3 to (4.8.1), we show that $\mathcal{F}$ inherits the properties of $f$. To this end, we note that

$$T(q) \subset p,$$

as $\varphi(F_{q}T(s)) = T(\varphi^T(F^*_s s)) \in T(n) \subset m$ for all $\varphi \in \mathcal{C}e$ if $s \in q = \beta(S, f^*\mathcal{C})$. In other words, $T$ restricts to a unique map $\overline{T} \in \text{Hom}_R(\overline{S}, \overline{R})$ such that the square

$$\begin{array}{ccc}
S & \xrightarrow{T} & R \\
\downarrow & & \downarrow \\
\overline{S} & \xrightarrow{T} & \overline{R}
\end{array}$$

is commutative. The same holds for any $S$-multiple of $T$, so that we have an $\overline{S}$-linear map

$$\overline{\sigma} : \overline{S} \to \text{Hom}_R(\overline{S}, \overline{R}) = \omega_{\overline{S}/\overline{R}}, \quad \overline{s} \mapsto \overline{T} \cdot \overline{s} = \overline{T \cdot s}.$$

**Claim 4.9.** $\sigma$ is injective and generically an isomorphism. $\mathcal{T}$ is surjective and $\mathcal{T}(\overline{p}) \subset \overline{m}$

**Proof of claim.** The last two statements are clear. For the injectivity of $\sigma$, consider the following. If $\mathcal{T} \cdot s = 0$, then $T(ss') \in p$ for all $s' \in S$. Equivalently,

$$\varphi(F^*_s T(ss')) = T(\varphi^T(s's)) \in m,$$

for all $\varphi \in \mathcal{C}e$. We must conclude that $s \in q$. Assume to the contrary: there is $\varphi \in \mathcal{C}e$ and $s' \in S$ such that $\varphi^T(s's) \in S \setminus m = S^\times$. Rescaling $s'$, we may assume that $\varphi^T(s's) = v$ which contradicts (4.9.1) as $T(v) = 1$. Therefore, $\varphi^T(s's) \in m$ for all $\varphi \in \mathcal{C}e$ and $s' \in S$. Hence $s \in q$, as required. To show that $\overline{\sigma}$ is generically surjective, it suffices to prove that $\mathcal{T} \neq 0$, i.e. $T(S) \not\subset p$. This follows from the surjectivity of $T$ and the $F$-purity of $(R, \mathcal{C})$. \qed

**Claim 4.10.** $(\overline{R}, \overline{\mathcal{C}})$ is $\mathcal{T}$-transposable. In fact, $\overline{\varphi^T} = \overline{\varphi^T}$ and in particular $\overline{\mathcal{F}^* \mathcal{C}} = \overline{f^* \mathcal{C}}$.

**Proof of claim.** Recall that $\varphi^T$ is the unique map fitting in (3.1.2) and similarly for $\varphi^T$. Reducing (3.1.2) modulo $q$ yields that $\varphi^T$ satisfies the condition characterizing $\varphi^T$. \hfill \Box

Combining Theorem 4.3 with the above gives the desired transformation rule. \hfill \Box

5. Test modules under finite covers

We come now to our generalizations of the results in [ST14] (cf. [MS21, Theorem D]) of the form $T(\tau(B, \Delta_B)) = \tau(A, \Delta_A)$ for a cover $\text{Spec } B \to \text{Spec } A$ between normal varieties. We prove a more general version for Cartier modules which says that $\mathcal{T}M(f_\ast \tau(f^!M, f^*\mathcal{C})) = \tau(M, \mathcal{C})$ for any cover $f$ (this was proved for flat morphisms in [Stä17, Lemma 4.17] under an additional technical assumption) and then utilize transposability to obtain more special results yet recovering the classical ones. In subsequent sections, we shall remark the form these results take in the case of canonical modules as well as providing analogous results for non-$F$-pure ideals and test ideals along closed subschemes.

Let us recall some concepts from [BS19]. Fix a Cartier algebra $\mathcal{C}$ over a ring $R$ and consider Cartier modules for this fixed algebra. For a Cartier module $M$, one denotes by $M := \mathcal{C}^e_M$ the stable image of $M$ under $\mathcal{C}_+$. This exists by [Bli13, Proposition 2.13] and one can show that $M$ does not admit any nilpotent quotients. In particular, if $N \subset M$ is a nil-isomorphism it is an equality. Moreover, the inclusion $M \subset M$ is always a nil-isomorphism. Hence, to
is always a radical ideal. We use freely and using that

\[ \text{Tr} \]

Let

\[ \text{Therefore}, \quad \text{Tr} \quad (1) \text{ holds in (5.1.2) above.} \]

Claim 5.2. Proof of claim.

Let

\[ \text{B} \]

free with 1 being part of some basis

trace map \( \text{Hom} \)

much like that one. Roughly speaking, the stable image of \( \eta \)

\( \eta \) is any prime in \( R \) \cite{BS19}, Proposition 1.19 (b)]. We also use that the (underived) local cohomology functor \( H^0_\eta \) is a functor of \( \mathcal{C} \)-modules (cf. paragraph after \cite{BS19}, Example 1.4)).

**Theorem 5.1.** Let \( f \colon \text{Spec} S \to \text{Spec} R \) be a finite cover. Let \( \mathcal{C} \) be a Cartier \( R \)-algebra and \( M \) a Cartier \( \mathcal{C} \)-module. Then,

\[ \text{Tr}_M \left( f_\ast \tau(f^!M, f^*\mathcal{C}) \right) = \tau(M, \mathcal{C}). \]

In particular, \( \text{Tr}_M \) is surjective if \( (M, \mathcal{C}) \) is \( F \)-regular. Conversely, if \( \text{Tr}_M \) is surjective and \( (f^!M, f^*\mathcal{C}) \) is \( F \)-regular, then \( (M, \mathcal{C}) \) is \( F \)-regular.

**Proof.** By \cite{BS19}, Proposition 6.13], there is a natural inclusion \( \tau \circ f^! \hookrightarrow f^! \circ \tau \) and so

\[ \tau\left(f^!M, f^*\mathcal{C}\right) \subset \text{Hom}_R(S, \tau(M, \mathcal{C})) \subset \text{Hom}_R(S, M). \]

Therefore, \( \text{Tr}_M \left( f_\ast \tau(f^!M, f^*\mathcal{C}) \right) \subset \tau(M, \mathcal{C}). \) Set \( \mathcal{F} := \text{Tr}_M \left( f_\ast \tau(f^!M, f^*\mathcal{C}) \right) \). For the converse inclusion, recall that for any \( f^*\mathcal{C} \)-submodule \( N \subset f^!M \) we have that \( \text{Tr}_M(f_\ast N) \) is a \( \mathcal{C} \)-submodule of \( M \); see \cite[Remark 2.19]{BS19}. Then, by definition of \( \tau(M, \mathcal{C}) \), it suffices to show that \( H^0_\eta(\mathcal{F}) \subset H^0_\eta(M) \) is a nil-isomorphism for all \( \eta \in \text{Ass} \ M \). By applying \( \ast \) and using that for any Cartier module \( N \) the inclusion \( N \subset N \) is a nil-isomorphism, it suffices to show that

\[ H^0_\eta(\mathcal{F}) \subset H^0_\eta(M) \]  

is a nil-isomorphism (equivalently, an equality). Since localization is flat, \( (\text{Tr}_M(N))_\eta = \text{Tr}_{M_\eta}(N_\eta) \) for all submodules \( N \subset f^!M \), where \( \text{Tr}_{M_\eta} \) is the trace map of the base change of \( f \) along \( \text{Spec} R_\eta \to \text{Spec} R \). Note that \( \text{Tr} \) is compatible with Cartier structures and so preserves nil-isomorphisms. We will show the following equalities and inclusions thereby showing that \( (5.1.1) \) is an equality.

\[ H^0_\eta(M) \overset{(1)}{=} \text{Tr}_{M_\eta}\left( H^0_{\eta S}(f^!M) \right) \overset{(2)}{=} \text{Tr}_{M_\eta}\left( H^0_{\eta S}(\tau(f^!M, f^*\mathcal{C})) \right) \overset{(3)}{=} H^0_\eta(\mathcal{F}) \]

Most readers will be more familiar with the case where \( R \) and \( S \) are domains, \( \eta \) is the generic point of \( R \), and \( M \) has rank 1. Up to technicalities, the general case proceeds very much like that one. Roughly speaking, the stable image of \( H^0_\eta(f^!M) \) may be identified with \( \text{Hom}_{R/\eta}(S/\eta, M) \) and the stable image of \( H^0_\eta(M) \) may be identified with the \( \eta \)-torsion of \( M_\eta \). Denoting the \( \eta \)-torsion of \( M \) by \( N \), we therefore have to consider the Cartier module \( N \) on \( \text{Spec} R/\eta \) where \( \eta \) is now a generic point. Since \( S \) is not assumed to be a domain, we need to work with the points in the fiber of \( \eta \) separately. We now show \( (5.1.2) \).

**Claim 5.2.** (1) holds in \( (5.1.2) \) above.

**Proof of claim.** The restriction of the trace map to \( H^0_{\eta S}(f^!M) \) may be identified with the trace map \( \text{Hom}_{R_\eta/\eta}(S/\eta S, M) \to M_\eta \). In particular, \( R_\eta/\eta \) is a field and thus \( (S/\eta S)_\eta \) is free with 1 being part of some basis \( B \). The containment from right to left is clear. Let \( m \) belong to the left hand side. Fix \( e \gg 0 \) such that \( C^e_+ H^0_\eta(M) \eta = H^0_\eta(M) \eta \) and such that it also
computes $H_{\eta S}^0(f^iM)$. Let $\varphi \in \mathcal{C}_+^*$ be homogeneous of degree $a$ and $n \in H_{\eta}^0(M)_\eta$ be such that $\varphi(n) = m$. Our goal is to construct an element $\alpha \in (f^iM)_\eta$ such that $\varphi(\alpha) := \varphi \circ \alpha \circ F^a$ is $(\eta S)^k$-torsion and $\varphi(\alpha(1)) = m$. To this end, we define $\alpha : (S/\eta S)_\eta \to M_\eta$ by setting $\alpha(1) = n$ and $\alpha(b) = 0$ for all other $b \in B$. Write $\varphi = \sum_i \varphi_i$, where $\varphi_i$ is homogeneous of degree $i$. Then, for any $b \in B$, we have

$$\varphi(\alpha)(b) = \sum_i \varphi_i \alpha(b^i) = \sum_i \varphi_i \alpha(r_{1,i}) = \sum_i \varphi_i(r_{1,i}n),$$

where we write $b^i = \sum_{b\in B} r_{b,i}b$. If $b = 1$ in $(5.2.1)$ then $r_{1,i} = \delta_{i1}$ and the expression yields $m$. For general $b$, we still have $(\eta S)^k$-torsion since $n$ is $\eta^k$-torsion.

**Claim 5.3.** (2) holds in $(5.1.2)$ above.

**Proof of claim.** We show that $H_{\eta S}^0(f^iM)$ and $H_{\eta S}^0(\tau(f^iM, f^*\mathcal{C}))$ agree. To this end, recall that $\text{Ass } f^iM = f^{-1}(\text{Ass } M)$ by [BS19, Lemma 6.12]. Note that the support of $H_{\eta S}^0(f^iM)$ is contained in $f^{-1}(\eta)$, which consists of a finite number of points. Hence,

$$(5.3.1) \quad H_{\eta S}^0(f^iM) = \bigoplus_{\nu \in f^{-1}(\eta)} H_{\nu}^0(f^iM)_{\nu},$$

where we note that a direct summand on the right hand side is zero whenever $\nu \not\in \text{Supp } H_{\eta S}^0(f^iM)$. A similar statement holds for $\tau(f^iM, f^*\mathcal{C})$ instead of $f^iM$. By definition of $\tau$, we have for each $\nu \in f^{-1}(\eta)$ a nil-isomorphism $H_{\nu}^0(\tau(f^iM, f^*\mathcal{C}))_{\nu} \subset H_{\nu}^0(f^iM_{\nu})$, where again we may pass to the stable image on both sides and take direct sums to obtain:

$$(5.3.2) \quad \bigoplus_{\nu \in f^{-1}(\eta)} H_{\nu}^0(\tau(f^iM, f^*\mathcal{C}))_{\nu} = \bigoplus_{\nu \in f^{-1}(\eta)} H_{\nu}^0(f^iM)_{\nu}.$$ 

Putting $(5.3.1)$ and $(5.3.2)$ together, we obtain

$$(5.3.3) \quad H_{\eta S}^0(\tau(f^iM, f^*\mathcal{C})) = H_{\eta S}^0(f^iM).$$

**Claim 5.4.** (3) holds in $(5.1.2)$ above.

**Proof of claim.** We may drop the $\bigoplus$ on the right hand side. Indeed, $(\text{Tr}_{M_{\eta}})_\eta$ is a morphism of Cartier modules and quite generally, if $g$ is a Cartier morphism and $g(N) \subset M$, then $M = \mathcal{C}_+^*M \supset \mathcal{C}_+^*g(N) = g(\mathcal{C}_+^*N) = g(N)$. By construction, the left hand side is $\eta$-torsion. Hence, it suffices to show that it is contained in $\mathcal{T}_{\eta}$. Since $(\text{Tr}_{M})_{\eta}$ commutes with localization:

$$\mathcal{T}_{\eta} = (\text{Tr}_{M_{\eta}}) \left( (f_{\ast}\tau(f^iM, f^*\mathcal{C}))_{\eta} \right).$$

By definition of $(f^iM, f^*\mathcal{C})$, for any $\nu \in \text{Spec } S$, the inclusion $H_{\nu}^0(\tau(f^iM, f^*\mathcal{C}))_{\nu} \subset H_{\nu}^0(f^iM)_{\nu}$ is a nil-isomorphism. This entails

$$H_{\nu}^0(f^iM)_{\nu} = H_{\nu}^0(\tau(f^iM, f^*\mathcal{C}))_{\nu} \subset H_{\nu}^0(\tau(f^iM, f^*\mathcal{C}))_{\nu}.$$
Taking the (direct) sum over all \( \nu \in f^{-1}(\eta) \) in the inclusion above, we get
\[
\bigoplus_{\nu \in f^{-1}(\eta)} H^0_{\nu}(\tau(f^1M, f^*\mathcal{C}))_{\nu} = \bigoplus_{\nu \in f^{-1}(\eta)} H^0_{\eta S}(\tau(f^1M, f^*\mathcal{C}))_{\eta} \subset \bigoplus_{\nu \in f^{-1}(\eta)} H^0_{\nu}(\tau(f^1M, f^*\mathcal{C}))_{\nu} = H^0_{\eta S}(\tau(f^1M, f^*\mathcal{C}))_{\eta} \subset \tau(f^1M, f^*\mathcal{C})_{\eta}.
\]

Applying Tr shows the desired inclusion. \( \square \)

This demonstrates the result.

**Corollary 5.5** (cf. [ST14]). Work in Setup 3.3. Let \( \mathcal{C} \) be a Cartier \( R \)-algebra. Then,
\[
(5.5.1) \quad T_M(f_*\tau(f^1M, f^*\mathcal{C})) = \tau(M, \mathcal{C})
\]
for all \( T \)-transposable Cartier \( \mathcal{C} \)-modules \( M \), where \( T_M = \text{Tr}_M \circ \varsigma_M \). In particular, if \( R \) is a \( T \)-transposable Cartier \( \mathcal{C} \)-module, we have
\[
(5.5.2) \quad T(f_*\tau(S, f^*\mathcal{C})) = \tau(R, \mathcal{C}).
\]

Further, working in the setup of Theorem 3.13 if \( R(D) \) is \( T \)-transposable then
\[
(5.5.3) \quad T_K(f_*\tau(S(f^*D), f^*\mathcal{C})) = \tau(R(D), \mathcal{C})
\]
where \( T_K : L \rightarrow K \) coincides with \( K \otimes_R T \). Moreover, if \( f \) is generically flat and \( \Delta^* := f^*\Delta - \text{Ram}_T \geq 0 \) for some effective almost Cartier \( \mathcal{Q} \)-divisor on \( X \), then
\[
(5.5.4) \quad T_K(f_*\tau(S(f^*D), \Delta^*, (aS)^t)) = \tau(R(D), \Delta, a^t).
\]

**Proof.** By Corollary 3.11 \( \varsigma_M(f_*\tau(f^1M, f^*\mathcal{C})) = \tau(f^1M, f^*\mathcal{C}) \). Applying \( \text{Tr}_M \) and using [Theorem 5.1] shows (5.5.1). For (5.5.3) Remark 3.12 gives that \( f^1R(D) \cong S(f^*D) \) and, under this isomorphism, \( \text{Tr}_{R(D)} \) corresponds to the composition
\[
S(f^*D) \xrightarrow{\text{Ram}_T \sim K_{S/R}} S(f^*D + K_{S/R}) \rightarrow R(D)
\]
where, by naturality, the latter map is the restriction of \( T_K : L \rightarrow K \) under the inclusions \( R(D) \subset K \) and \( S(f^*D) \subset L \). Lastly, (5.5.4) follows from (5.5.3) and [Proposition 3.14]. \( \square \)

We illustrate next the simpler form of Theorem 5.1 takes in case \( M = \omega_R \). Recall that if \( \kappa_R : F^\dagger \omega_R \rightarrow \omega_R \) is a Cartier operator of \( R \) then \( \kappa_S : = \kappa^t_R : F^\dagger \omega_S \rightarrow \omega_S \) is a Cartier operator of \( S \) as \( \Delta_{\kappa_S} = \Delta f^\dagger \kappa_R = f^*0 = 0 \).

**Corollary 5.6** (cf. [BST15]). Work in the setup of Theorem 3.13 assuming \( R \) and \( S \) are Cohen–Macaulay. Then, \( \text{Tr}_{\omega_R}(f_*\omega(S, \kappa_S)) = \tau(\omega_R, \kappa_R) \). In particular, \( \text{Tr}_{\omega_R} : f_*\omega_S \rightarrow \omega_R \) is surjective if \( R \) is \( F \)-rational. Conversely, if \( \text{Tr}_{\omega_R} : f_*\omega_S \rightarrow \omega_R \) is surjective and \( S \) is \( F \)-rational, then \( R \) is \( F \)-rational.

A. Singh constructed a \( \mathcal{Q} \)-Gorenstein \( F \)-rational singularity whose canonical cover is not \( F \)-rational [Sin83]. Therefore, we cannot expect in Corollary 5.6 (resp. Theorem 5.1) that the \( F \)-injectivity of \( R \) (resp. the \( F \)-regularity of \( M \)) implies the \( F \)-injectivity of \( S \) (resp. the \( F \)-regularity of \( f^1M \)).
5.0.1. **Relationship with splinters.** By plugging in \( \mathcal{C} = \mathcal{C}_R \) and \( M = R \) in Theorem 5.1, we see why strongly \( F \)-regular rings are splinters; see [Ma88, Hoc73]. Moreover, we may take a splitting of the finite cover \( R \subset S \) to be an element of \( \tau(\omega_{S/R}, f^*\mathcal{C}_R) \) when \( R \) is strongly \( F \)-regular. In other words, while a splinter guarantees a splitting for any finite extension \( R \subset S \), a strongly \( F \)-regular ring guarantees that splitting to be an element of \( \tau(\omega_{S/R}, f^*\mathcal{C}_R) \).

**Corollary 5.7.** Let \( R \) be a domain. Then, \( R \) is strongly \( F \)-regular if and only if every (equivalently some) finite extension \( R \subset S \) splits by an element in \( \tau(\omega_{S/R}, f^*\mathcal{C}_R) \).

The \( F \)-regularity of splinters is to many the main open problem in positive characteristic singularity theory. The answer is known to be affirmative for \( \mathbb{Q} \)-Gorenstein rings by [HH94, Sin99] yet far open beyond this case. Let us fix a domain \( R \) and say that a finite \( R \)-module \( M \) is a **splinter** if \( \text{Tr}_M: f_* f^! M \to M \) is surjective for all covers \( f: \text{Spec } S \to \text{Spec } R \). Thus, [Theorem 5.1] says that \( F \)-regular Cartier modules are splinters. We may wonder about the converse: Is a splinter module \( F \)-regular with respect to its full Cartier algebra? Of course, the case of interest is \( M = R \). We even got the following inclusion of submodules of \( M \)

\[
(5.7.1) \quad \tau(M, \mathcal{C}_M) \subset \bigcap_f \text{Tr}_M \left( f_* f^! M \right),
\]

where the intersection traverses over all covers \( f: \text{Spec}(S) \to \text{Spec}(R) \). For what (families of) modules is \([5.7.1]\) an equality? For example, this is known to hold for the class of canonical modules over Cohen–Macaulay rings by [BST15], where \( F \)-rationality of Cohen–Macaulay rings is characterized by the canonical module being a splitter module. Does equality hold in \([5.7.1]\) for \( M = \omega_R(\varepsilon D) \) if \( 0 < \varepsilon \ll 1 \)?

5.1. **Non-\( F \)-pure modules under finite covers.** Given a Cartier module \( (M, \mathcal{C}) \), let us write \( \sigma(M, \mathcal{C}) := M \) and refer to it as the **non-\( F \)-pure module** (this terminology is borrowed from the case of non-\( F \)-pure ideals [FST11]). It is natural to ask whether the formula \( \text{Tr}_M \left( f_* \sigma(f^! M, f^* \mathcal{C}) \right) = \sigma(M, \mathcal{C}) \) holds. We readily see that the inclusion “\( \subset \)” holds, whereas the converse requires the surjectivity of \( \text{Tr}_M \) to hold.

**Proposition 5.8.** Work in the setup of [Theorem 5.1]. Then, the following inclusion holds

\[
\text{Tr}_M \left( f_* \sigma(f^! M, f^* \mathcal{C}) \right) \subset \sigma(M, \mathcal{C})
\]

and equality holds if \( \text{Tr}_M: f_* f^! M \to M \) is surjective.

**Proof.** We already observed in [Theorem 5.1] that \( \text{Tr}_M \) is a \( \mathcal{C} \)-morphism, which implies

\[
\text{Tr}_M \left( f_* f^* \mathcal{C}^e_+ f^! M \right) = \mathcal{C}^e_+ \text{Tr}_M \left( f_* f^! M \right) \subset \mathcal{C}^e_+ M.
\]

The result follows by taking \( \varepsilon \) sufficiently large and noting that if \( \text{Tr}_M \) is surjective the displayed inclusion is an equality. \( \square \)

**Corollary 5.9.** Work in the setup of [Corollary 5.6]. If \( \text{Tr}: f_* \omega_S \to \omega_R \) is surjective, then \( \text{Tr}(f_* \sigma(\omega_S, \kappa_S)) = \sigma(\omega_R, \kappa_R) \). In particular, \( R \) is \( F \)-injective if so is \( S \).

**Proof.** Use [Proposition 5.8] \( \square \)

The following example shows that surjectivity of the trace is necessary for the equality in [Proposition 5.8] to hold. Compare this to the discussion in [ST14 §8].
Example 5.10. Consider the example (a) in Example 4.7. Note that \( R \) and \( S \) are \( F \)-pure Gorenstein local rings; use Fedder’s criterion for \( R \) [Fed83]. Then, \( \sigma(R, \kappa_R) = R \) and \( \sigma(S, \kappa_S) = S \), yet \( \text{Tr}_R(S) = (x, y, z) \). Nevertheless, \( \tau(R, \kappa_R) = (x, y, z) \) and \( \tau(S, \kappa_S) = S \), which verifies Theorem 5.1.

5.2. Test ideals along closed subschemes under finite covers. In this section, we explain how test ideals along closed subschemes—as treated in [Smo19b, §3.1], [Smo19a, §4]—behave under finite covers. These were introduced as positive characteristic analogs of adjoint ideals in characteristic zero by Takagi [Tak08, Tak10].

Fix a Cartier subalgebra \( C \subset C_R \) and a radical \( C \)-compatible ideal \( a \) with irredundant primary decomposition \( a = p_1 \cap \cdots \cap p_r \). All \( p_i \) are prime since \( a \) is radical and further \( C \)-compatible by [Sch10, Corollary 4.8]. Set \( P = \bigcup_{i=1}^r p_i \). The adjoint ideal \( \tau_a(R, C) \) is defined as the smallest Cartier \( C \)-submodule of \( R \) not contained in \( P \). The Cartier algebra \( C \) is called non-degenerate with respect to \( a \) if there is \( e > 0 \) such that \( C_e(R) \) is not contained in \( P \). By prime avoidance, this is equivalent to \( C_e(R) \not\subset p_i \) for all \( i \). In other words, \( (R, C) \) is non-degenerate with respect to \( a \) if and only if \( (R, C) \) is non-degenerate with respect to \( p_i \) for all \( i \), i.e., \( p_i \) is a center of \( F \)-purity of \( (R, C) \) for all \( i \). Smolkin proved that, for a prime \( \tau_a(R, C) \) exists if \( C \) is non-degenerate, assuming \( R \) is a domain (as well as noetherian and \( F \)-finite; as we do)—see [Smo19a, §4.1] and [Smo19b, §3.1]. Smolkin explained to us that a similar theory can be developed assuming \( a \) radical and \( R \) noetherian and \( F \)-finite ([Smo19c]). Nevertheless, we will be able to reduce to the case \( a \) is prime using Lemma 5.11 below but we will assume that \( R \) is a domain to cite [Smo19b, §3.1].

One says that a pair \( (R, C) \) is purely \( F \)-regular along a radical ideal \( a \subset R \) if \( \tau_a(R, C) = R \) (implicitly assuming that \( \tau_a(R, C) \) exists). For these ideals, we have the following.

Lemma 5.11. Let \( R \) be a domain and \( a \subset R \) be a radical ideal that is a Cartier \( C \)-submodule. Let \( p_1, \ldots, p_k \) be the minimal primes of \( a \). Assume that \( (R, C) \) is non-degenerate with respect to \( a \). Then, \( \tau_a(R, C) = \sum_{i=1}^k \tau_{p_i}(R, C) \).

Proof. First, note that \( \tau_{p_i}(R, C) \) exists by [Smo19b, Section 3.1.1]. Write \( P = \bigcup_{i=1}^r p_i \). By definition, \( \tau_{p_i}(R, C) \) is the smallest \( C \)-compatible ideal not contained in \( P \). If \( I \) is any \( C \)-compatible ideal not contained in \( P \), then it is not contained in any \( p_i \) whence \( \tau_{p_i} \subset I \) for all \( i \). To finish the proof, assume that \( \sum_i \tau_{p_i}(R, C) \subset P \). By prime avoidance, the sum is then contained in, say, \( p_1 \). Then, \( \tau_{p_1}(R, C) \subset \sum_i \tau_{p_i}(R, C) \subset p_1 \), which is a contradiction. Thus, \( \sum_{i=1}^k \tau_{p_i}(R, C) \) is the smallest \( C \)-compatible ideal not contained in \( P \) and so coincides with \( \tau_a(R, C) \).

Theorem 5.12. Let \( f \colon \text{Spec} \, S \to \text{Spec} \, R \) be a cover between integral schemes. Let \( a \subset R \) be a radical ideal with minimal primes \( p_1, \ldots, p_k \), and set \( b := \sqrt{a \cdot S} \). Let \( \sigma_R \colon S \to \omega_{S/R} \) be a generic isomorphism such that \( T = \sigma_R(1) \) satisfies

\[
(5.12.1) \quad \sqrt{p_i \cdot S} = f^* p_i \cdot S, \quad \text{for all } i = 1, \ldots, k.
\]

Let \( C \) be a Cartier \( R \)-algebra acting on \( R \) such that \( R \) is \( T \)-transposable and \( a \subset R \) is a Cartier submodule. The following statements hold.

(a) The ideal \( b \subset S \) is a Cartier \( f^* C \)-submodule.

(b) The pair \( (R, C) \) is non-degenerate with respect to \( a \) if and only if \( (S, f^* C) \) is non-degenerate with respect to \( b \). In that case, the following equality holds

\[
(5.12.2) \quad T(f_* \tau_b(S, f^* C)) = \tau_a(R, C).
\]
(c) If \(a = p\) is a prime ideal, then \((R, \mathcal{C})\) is non-degenerate with respect to \(a\) if and only if 
\((S, f^*\mathcal{C})\) is non-degenerate with respect to all prime ideals \(q \subset S\) lying over \(p\). Further, 
for any such prime \(q\) the following equality holds

\[ T(f_\tau q(S, f^*\mathcal{C})) = \tau p(R, \mathcal{C}). \]

(d) If \((R, \mathcal{C})\) is purely \(F\)-regular along \(a\), then \(T\) is surjective. The converse holds provided 
that \((S, f^*\mathcal{C})\) is purely \(F\)-regular along \(b\).

**Proof.** For any ideal \(I \subset R\), the ideal \(f^I S :_S T\) equals \(\{s \in S \mid T(sS) \subset I\}\) and is the largest 
ideal \(J \subset S\) such that \(T(J) \subset I\). Further,

\[ (f^I S :_S T) \cap (f^J S :_S T) = (f^I \cap f^J) S :_S T = f^J (I \cap J) :_S T \]

for any two ideals \(I, J \subset R\). In particular, (5.12.1) implies

\[ b = f^I a :_S T. \]

For notation ease, we assume \(\mathcal{C} \subset \mathcal{C}_R\) and \(f^*\mathcal{C} \subset \mathcal{C}_S\); see [Remark 2.19].

**Proof of (a):** Next, we explain why \(b \subset S\) is a \(f^*\mathcal{C}\)-submodule. It suffices to show 
\(\varphi^\top(F^*_e b) \subset b\) for all \(\varphi \in \mathcal{C}_e\). Note that

\[ T(\varphi^\top(F^*_e b)) = \varphi(F^*_e T(b)) \subset \varphi(F^*_e a) \subset a, \]

where the inclusion \(T(b) \subset a\) is a consequence of (5.12.4), and the last one follows from
\(a \subset R\) being a \(\mathcal{C}\)-submodule. Thus, \(\varphi^\top(F^*_e b) \subset T^{-1}(a)\). Since \(\varphi^\top(F^*_e b)\) is an ideal of \(S\), this
implies \(\varphi^\top(F^*_e b) \subset f^\top a :_S T\), and so \(\varphi^\top(F^*_e b) \subset b\) by (5.12.4).

**Proof of (b) and (c):** We prove next the statements regarding non-degeneracy. We may
assume that \(a = p\) is prime; see [Lemma 5.11]. Let \(q_1, \ldots, q_n\) be the primes of \(S\) lying over \(p\), 
which are the minimal primes of \(b = \sqrt{pS} = \bigcap_i q_i\), and are all compatible prime ideals
of \((S, f^*\mathcal{C})\) by part (a). We show that \((R, \mathcal{C})\) is degenerate with respect to \(p\) if and only if
\((S, f^*\mathcal{C})\) is degenerate with respect to \(\sqrt{pS}\) if and only if \((S, f^*\mathcal{C})\) is degenerate with respect to \(q_i\) for all \(i \in \{1, \ldots, n\}\).

Suppose that \((R, \mathcal{C})\) is degenerate with respect to \(p\), that is: \(\varphi(F^*_e R) \subset p\) for all \(\varphi \in \mathcal{C}_e\). In particular, 
\(T(\varphi^\top(F^*_e S)) = \varphi(F^*_e T(S)) \subset p\) for all \(\varphi \in \mathcal{C}_e\). In other words, 
\(\varphi^\top(F^*_e S) \subset T^{-1}(p)\) for all \(\varphi \in \mathcal{C}_e\). Since \(\varphi^\top(F^*_e S) \subset S\) is an ideal, this implies that 
\(\varphi^\top(F^*_e S) \subset f^\top p : T = \sqrt{pS} = \bigcap_i q_i\) for all \(\varphi \in \mathcal{C}_e\). Therefore, since \(f^*\mathcal{C}\) is generated as a right \(S\)-module by maps \(\varphi^\top\), we conclude that 
\((S, f^*\mathcal{C})\) is degenerate with respect to all \(q_i\).

Suppose that \((S, f^*\mathcal{C})\) is degenerate with respect to \(\sqrt{pS} = f^\top p : T\). That is, \((S, f^*\mathcal{C})\) is
degenerate with respect to some prime lying over \(p\), say \(q_{i_0}\). In particular, \(\varphi^\top(F^*_e S) \subset q_{i_0}\) for all \(\varphi \in \mathcal{C}_e\). Therefore,

\[ \varphi^\top\left(F^*_e \bigcap_{i \neq i_0} q_i \right) \subset q_{i_0} \cap \bigcap_{i \neq i_0} q_i = \sqrt{pS} = f^\top p : T \]

for all \(\varphi \in \mathcal{C}_e\). For notation ease, let us set \(S_0 := \bigcap_{i \neq i_0} q_i\). In this way, for all \(\varphi \in \mathcal{C}_e\), we have 
\(T(\varphi^\top(F^*_e S_0)) \subset p\) and so \(\varphi(F^*_e T(S_0)) \subset p\) as \(\varphi \circ F^*_e T = T \circ \varphi^\top\). However, \(T(S_0) \not\subset p\) as
\(f^\top p : T = \sqrt{pS} \subset S_0\). Thus, we can take an element \(r \in T(S_0) \setminus p\) so that \(r^\top R \subset T(S_0)\).
Then, for all \(\varphi \in \mathcal{C}_e\), we have:

\[ r^\top \varphi(F^*_e R) = \varphi(F^*_e r^\top R) \subset \varphi(F^*_e T(S_0)) \subset p, \]

and so \(\varphi(F^*_e R) \subset p\) as \(r \notin p\). Therefore, \((R, \mathcal{C})\) is degenerate with respect to \(p\).
With the above in place, we explain why \((5.12.3)\) and so \((5.12.2)\) hold. In fact, we show that
\[
T\left(f_\ast \tau_q(S, f^\ast \mathcal{C})\right) = \tau_p(R, \mathcal{C})
\]
holds assuming both adjoint test ideals exist, where \(p\) is prime and \(q \subset S\) is any prime ideal lying over \(p\). We start with the containment \(\supset\). It suffices to show that \(T\left(f_\ast \tau_q(S, f^\ast \mathcal{C})\right)\) is \(\varphi\)-compatible for all \(\varphi \in \mathcal{C}_e\) and that it is not contained in \(p\). The compatibility follows once again using that \(\tau_q(S, f^\ast \mathcal{C})\) is \(\varphi^\top\)-compatible for all \(\varphi \in \mathcal{C}_e\) and employing \(T \circ \varphi^\top = \varphi \circ T\). On the other hand, \(T\left(f_\ast \tau_q(S, f^\ast \mathcal{C})\right)\) cannot be contained in \(p\) because else \(\tau_q(S, f^\ast \mathcal{C}) \subset \sqrt{pS} \subset q\) by \((5.12.1)\), which contradicts its non-degeneracy with respect to \(q\).

Conversely, for the inclusion \(\subset\), we use the description of adjoint ideals in terms of test elements; see \cite{Smo19b} 3.1.11, 3.1.12, 3.1.16. Let \(c \in \tau_q(S, f^\ast \mathcal{C}) \setminus q\) so that \(c\) is an \(f^\ast \mathcal{C}\)-test element along \(q\) using the terminology of \cite{Smo19b} Definition 3.1.15. Hence, we can write
\[
\tau_q(S, f^\ast \mathcal{C}) = \sum_{\varphi \in \mathcal{C}_e} \varphi^\top(F^e_cS).
\]

Hitting this equality by \(T\) and using \(T \circ \varphi^\top = \varphi \circ F^e_cT\), we see that it suffices to show \(T(cS) \subset \tau_p(R, \mathcal{C})\). Since \(c \notin q\), we have \(T(c \cdot S) \notin p\) by \((5.12.1)\). In particular, we find \(s \in S\) such that \(T(c \cdot S) \notin p\). Take \(s' \in S, r \in R \setminus p\) arbitrary. First of all, \(r \notin q\) as \(q\) contracts to \(p\) along \(R \to S\). Then, there are \(s_1, \ldots, s_n \in S\) and \(\varphi_1, \ldots, \varphi_n \in \mathcal{C}_e\); for some \(e > 0\), such that \(\sum_i \varphi_i^\top(F^e_c(s_i \cdot r)) = s'c\), for \(c\) is an \(f^\ast \mathcal{C}\)-test element along \(q\). In particular, \(\sum_i \varphi_i(F^e_cT(s_i)r) = T(s'c)\). In other words, for all \(s' \in S, r \in R \setminus p\), there exists \(\varphi \in \mathcal{C}_e\) for some \(e > 0\) such that \(\varphi(F^e_c r) = T(s'c)\). Therefore, specializing to \(s' = s\), we have that \(T(sc)\) is a \(\mathcal{C}\)-test element along \(p\). Hence,
\begin{equation}
(5.12.5) \quad \tau_p(R, \mathcal{C}) = \sum_{\varphi \in \mathcal{C}_e} \varphi(F^e_cT(sc)R).
\end{equation}

Now, let \(s' \in S\) be arbitrary, and let \(r = T(sc)\). By our previous observation, we may find \(\varphi \in \mathcal{C}_e\) such that \(\varphi(F^e_c r) = T(s'c)\), and so \(T(s'c) \in \tau_p(R, \mathcal{C})\) as a consequence of \((5.12.5)\).

Thus, we have established \(T(cS) \subset \tau_p(R, \mathcal{C})\), as desired.

**Proof of (d):** This follows directly from \((5.12.2)\). \(\square\)

As an application of Theorem 5.12 and the restriction theorem for adjoint ideals, we have:

**Corollary 5.13.** Work in the setup of Theorem 5.12. If \(a = p\) and \(b = q\) are prime, then
\[
\overline{T}\left(\overline{f}_\ast \tau(S/q, \overline{f}^\ast \mathcal{C})\right) = \tau(R/p, \overline{\mathcal{C}})
\]
where \(\overline{T}: S/q \to R/p\) is the restriction of \(T: S \to R\) given by the inclusion \(T(q) \subset p\), and \(\overline{f}: \Spec(S/q) \to \Spec(R/p)\) is the spectrum of the induced homomorphism \(R/p \to S/q\).

**Proof.** Reduce \((5.12.3)\) modulo \(p\) and use \cite{Smo19b} Proposition 3.1.14] as well as \(\overline{f}^\ast \mathcal{C} = \overline{f}^\ast \mathcal{C}\), which follows as in Claim 4.10. \(\square\)

**Remark 5.14.** Let us discuss the meaning of \((5.12.1)\) for small heights of \(p\) when \(T = \Tr_{S/R}\) and \(R \subset S\) is separable extension of normal domains. We always have \(\sqrt{pS} \subset f^\ast p S \Tr_{S/R}\) since \(\Tr_{S/R}\left(\sqrt{pS}\right) \subset p\); see \cite{Spe20} Lemma 9. If \(ht p = 0\), then \(p = 0\) and \(\sqrt{pS} = 0\) as we assume \(R\) and \(S\) to be integral. Then \((5.12.1)\) means that \(\sigma_R: S \to \omega_{S/R}\) is injective. This, however, is a consequence of \(\sigma_R\) being a generic isomorphism and \(S\) satisfying \(S_1\). Hence, in height 0, \((5.12.2)\) recovers \((5.5.2)\) in the integral case. Let us suppose now that \(ht p = 1\). We
explain what $\sqrt{pS} \supset f^!p : S \text{Tr}_{S/R}$ means in terms of divisors. Note that $\text{ht } p = 1$ amounts
to saying that $p = R(-P)$ with $P$ is an effective reduced divisor on $\text{Spec } R$. Similarly,
$\sqrt{pS} = S(-Q)$ for some effective reduced divisor $Q$ on $\text{Spec } S$. Thus, $s \in f^!p : S \text{Tr}_{S/R}$ if and
only if

$$\text{Tr}_{S/R} : s \in f^!p = \text{Hom}_R (S, R(-P)) = \text{Hom}_R (S \otimes_R R(P), R) = \text{Hom}_R (S(f^*P), R),$$

which means that $S(f^*P) \subset S(\text{Ram} + \text{div } s)$, equivalently $s \in S(\text{Ram} - f^*P) = S(-P^*)$
where $P^* := f^*P - \text{Ram}$. Hence, $q \supset f^!p : S \text{Tr}_{S/R}$ is equivalent to $P^* \geq Q$. The same applies
for general effective reduced divisors. Concretely, let $a := R(-D)$ and $b := S(-E)$ for reduced
effective divisors $D$ and $E$ on $\text{Spec } R$ and $\text{Spec } S$; respectively, such that $E = (f^{-1}D)_{\text{red}}$. Then,
$b = f^!a : S \text{Tr}_{S/R}$ is equivalent to $f^*D - \text{Ram} = E$. Using the terminology [ST14, Definition 3.9],
such divisorial equality is equivalent to having that for all height-1 prime ideals $q' \subset S$ the DVRs extension $R_{q' \cap R} \subset S_{q'}$ is: étale if $q'$ does not support $E$ and tamely
ramified otherwise [Har77, IV, Proposition 2.2]; see [CR18, Remark 2.9], [ST14, Remark 4.6].

Remark 5.15. In the forthcoming preprint [CRF22], Anne Fayolle and the first named author
expand upon the ideas behind Theorem 5.12 and describe how centers of $F$-purity behave
under transpositions and finite covers. Moreover, they clarify further the prominent role
played by the condition [5.12.1] in their theory. In particular, they provide better conceptual
interpretations for this condition as well as numerical rank-conditions it is equivalent to. For
instance, if $\sigma$ is an isomorphism over $p$, then [5.12.1] is equivalent to the residual degree
$$\sum_{i=1}^{k} [\kappa(q_i) : \kappa(p)]$$
being equal to the free rank of $S_{q}$ as an $R_{q}$-module, which will let us see that Theorem 5.12
holds for $T = \text{Tr}$ if $f$ is étale over $p$.

6. Schwede–Tucker’s transposability criterion revisited

Let $f : Y \to X$ be a finite cover of normal integral schemes and $T$ be a nonzero global
section of $\omega_{Y/X}$. Schwede–Tucker’s transposability criterion establishes that $\mathcal{E}_{e,X}^+ = \{ \varphi \in \mathcal{E}_{e,X} \mid f^*\Delta_{\varphi} \geq \text{Ram}_T \}$. We aim to describe $\mathcal{E}_{e,X}^+$ in terms of divisors on $X$ rather than
divisors on $Y$. To this end, we need the following facts regarding norm functions.

6.1. Norm functions. For a detailed exposition on norms, see [The22, Tag 0BCX]. A multiplicative function $N_f : f_!\mathcal{O}_Y \to \mathcal{O}_X$ is a norm of degree $d$ if $N_f \circ f^! : \mathcal{O}_X \to f_!\mathcal{O}_Y \to \mathcal{O}_X$
is given by raising local sections to the $d$-th power and whenever $v \in \mathcal{O}_Y (f^{-1}(U))$ vanishes at
$y \in f^{-1}(U)$ so does $N_f(v)$ at $f(y) \in U$. Let $K$ and $L$ be the fields of functions of $X$ and $Y$
respectively. Then, there exists a norm $N_f : f_!\mathcal{O}_Y \to \mathcal{O}_X$ of degree $n := [L : K]$. Indeed, we
take $N_f$ to be the integral restriction of the field extension norm $N_{L/K} : L \to K$ [The22, Tag 0BD3]. We often use the notation $N_f = N_{Y/X}$ in that case.

Applying $H^1(X, -)$ to $N_f : f_!\mathcal{O}_Y \to \mathcal{O}_X$ gives a homomorphism $N_f : \text{Pic } Y \to \text{Pic } X$ such
that the composition $N_f \circ f^* : \text{Pic } X \to \text{Pic } Y \to \text{Pic } X$ is multiplication-by-$n$ (in additive
notation); see [The22, Tag 0BCY]. This shows that if $\mathcal{L}$ is nontorsion, then so is $f^* \mathcal{L}$. Moreover,
if $\mathcal{L} \cong \mathcal{O}_Y(D) \subset L$ for some Cartier divisor $D$, then $N_f(\mathcal{L})$ is realized as the rank
1 subsheaf of $K$ given by $N_{L/K}(\mathcal{O}_Y(D))$. We define $N_f(D)$ to be the Cartier divisor on $X$
determined by the equality $N_{L/K}(\mathcal{O}_Y(D)) = \mathcal{O}_X (N_f(D))$. This extends to a homomorphism
$N_f : \text{Cl } Y \to \text{Cl } X$ as follows: $N_{L/K}(\mathcal{O}_Y(D)) = \mathcal{O}_X (N_f(D))$ for all Weil divisor $D$ on $Y$. It is
worth noticing that $N_f : \text{Div } Y \to \text{Div } X$ is given by $N_f : \text{Pic } f^{-1}(X_{\text{reg}}) \to \text{Pic } X_{\text{reg}}$. Finally,
we define $N_f$ on $\mathbb{Q}$-divisors by $N_f \otimes \mathbb{Q}$. 
Lemma 6.1. Let $f: Y \to X$ be a finite cover of normal integral schemes, let $\Delta$ be a $\mathbb{Q}$-divisor on $Y$. Then, $N_f(\Delta) \ge 0$ if $\Delta \ge 0$.

Proof. We may assume that $\Delta$ is integral. Effectiveness of a divisor $D$ on an integral normal scheme $S$ is equivalent to the inclusion $\mathcal{O}_S \subset \mathcal{O}_S(D)$ in $K(S)$. Thus, if $\mathcal{O}_Y \subset \mathcal{O}_Y(\Delta)$ then $\mathcal{O}_X \subset \mathcal{O}_X(N_f(\Delta))$ as $N_f(\mathcal{O}_Y) = \mathcal{O}_X$ (since $\mathcal{O}_Y = f^*\mathcal{O}_X$, the norm of $\mathcal{O}_Y$ is $\mathcal{O}_Y^{\mathbb{Q}} = \mathcal{O}_X$).

Definition 6.2. With notation as in Definition 6.1 a divisor $D$ on $Y$ is $f$-torsion if $mD = f^*D'$ for some $m \neq 0$ and some divisor $D'$ on $X$.

Proposition 6.3. With notation as in Lemma 6.1 a divisor $D$ on $Y$ is $f$-torsion if and only if $f^*N_f(D) = n \cdot D$.

Proof. Clearly, if $n \cdot D = f^*N_f(D)$ then $D$ is $f$-torsion. Conversely, if $mD = f^*D'$ for some $m \neq 0$, then applying $N_f$ gives $mN_f(D) = nD'$. Pulling back yields $mf^*N_f(D) = nf^*D' = mnD$, and dividing by $m$ gives $f^*N_f(D) = n \cdot D$.

Lemma 6.4. With notation as in Lemma 6.1, let $D$ be an effective divisor on $Y$. Then, $f^*N_f(D) \ge k \cdot D$ for some integer $1 \le k \le n$. Further, if $f$ is generically Galois and $\sigma(\mathcal{O}_Y(D)) \subset \mathcal{O}_Y(D)$ for all $\sigma \in \text{Gal}(L/K)$, we may take $k = n$.

Proof. Since the effectiveness and triviality of a divisor can be checked at finitely many codimension 1 points, we may assume that $X = \text{Spec} R$ is the spectrum of a DVR and $Y = \text{Spec} S$ is the spectrum of a semi-local Dedekind domain and so a PID. Indeed, once we have $1 \le k_P \le n$ that works for a prime component $P$ of $D$, we may take $k = \min_P \{k_P\}$. In particular, we may write $D = \text{div}_S s$ for some $s \in S^\times$. Further, $N_f(D) = \text{div}_R N_{L/K}(s)$. By setting $r := N_{L/K}(s) \in R^\times$, we are required to prove that $f^* \text{div}_R r = \text{div}_S r \ge k \cdot \text{div}_S s = \text{div}_S s^k$ for some integer $1 \le k \le n$. In other words, we must prove that $r/s^k \in L$ belongs to $S$ for some integer $k \in [1, n]$ (provided that $s \in S$). Note that if $t^m + r_{m-1}t^{m-1} + \cdots + r_1t + r_0 \in R[t]$ is the minimal polynomial of $s$ then $r = (-1)^n r_0^{n/m}$, where $n = [L : K]$. Indeed,

$$r = N_{L/K}(s) = N_{K(s)/K}(N_{L/K}(s)) = N_{K(s)/K}(N_{L/K}(s)) = N_{K(s)/K}(s^{n/m}) = N_{K(s)/K}(s^{n/m}) = N_{K(s)/K}(s)^{n/m},$$

and $N_{K(s)/K}(s) = (-1)^m r_0$, $n/m = [L : K(s)]$. Next, use the equation

$$s^m + r_{m-1}s^{m-1} + \cdots + r_1s + r_0 = 0$$

to find $s' \in R[s]$ such that $s^l s' = r_0$ with $l$ as large as possible. Also, notice that $1 \le l \le m \le n$. Hence, we conclude that if $k = l(n/m)$ then $s^k$ divides $r$; as needed.

6.2. Transposability along finite covers and branching divisors. As in the previous section $f: Y \to X$ is a finite cover of normal integral schemes and $T$ a nonzero global section of $\omega_{Y/X}$. We think of $\text{Ram}_T \sim K_{Y/X}$ as an effective divisor measuring the failure of $T$ in generating $\omega_{Y/X}$ in codimension 1 and call it the ramification divisor of $T$. In fact, $\mathcal{O}_Y(\text{Ram}_T) \to \omega_{Y/X}$, $v \mapsto T(v \cdot -)$, is an isomorphism of $\mathcal{O}_Y$-modules. By using norms, we may define a divisor on $X$ measuring such failure.

Definition 6.5. The branching divisor of $T$ is $\text{Branch}_T = N_{Y/X}(\text{Ram}_T)$. If $f: Y \to X$ is separable and $T = T_{Y/X}$, we write $\mathfrak{A} = \mathfrak{A}_{Y/X} = \text{Ram}_T$, $\mathfrak{B} = \mathfrak{B}_{Y/X} = \text{Branch}_T$ and refer to them as the ramification divisor and branching divisor of $f$; respectively.

Lemma 6.6. With notation as in Definition 6.5, $\text{Branch}_T$ is supported at prime divisors whose generic points are images of the generic point of some prime divisor supporting $\text{Ram}_T$. 
Proof. We may assume that \(X = \text{Spec } R\) is the spectrum of a DVR and \(Y = \text{Spec } S\) is the spectrum of a semi-local Dedekind domain, so a PID. Since \(R\) and \(S\) are both Gorenstein, \(\omega_{S/R}\) is freely generated as an \(S\)-module by some \(R\)-linear map \(\Phi: S \to R\). Then, there exists a unique \(s \in S\) such that \(T = \Phi \cdot s\) and so \(\text{Ram}_T = \text{div}_S s\) and \(\text{Branch}_T = \text{div}_R N_{S/R}(s)\). The result then follows by observing that \(s\) is a unit if and only if so is \(N_{S/R}(s)\). The direction \(\Rightarrow\) is clear. The converse follows from \(N_{S/R}(s)\) being the determinant of the multiplication-by-\(s\) \(R\)-linear map and its surjectivity implying that \(s\) is a unit. \(\square\)

Proposition 6.7. With notation as in Definition 6.5, \(\text{Branch}_T\) is Cartier if \(f\) is flat.

Proof. Let \(\{U_i\}\) be an open covering of \(X\) trivializing \(f_*\mathcal{O}_Y\) on \(X\). Setting \(V_i := f^{-1}(U_i)\), then \(f_*\mathcal{O}_{V_i}\) is a free \(\mathcal{O}_{U_i}\)-module. Let \(\delta_i \in \mathcal{O}_{U_i}\) be the determinant of the \(\mathcal{O}_{U_i}\)-bilinear form \(T(- \cdot -)\) on \(f_*\mathcal{O}_{V_i}\). Then, \((U_i, \delta_i)\) defines a locally principal ideal sheaf \(\mathfrak{D}_T \subset \mathcal{O}_X\). We claim that \(\mathcal{O}_X(\text{Branch}_T) = \mathfrak{D}_T^{-1}\). This can be checked at codimension 1 points, and so we may work in the setup of the proof of Lemma 6.6. Let \(s_i\) be a free basis for \(S\) as an \(R\)-module, and let \(\delta := \det(T(s_i \cdot s_j))\) \(\in R\). Then, the result amounts to the equality \((\delta) = (N_{S/R}(s))\). This follows from the equality \(\delta = N_{S/R}(s) \cdot \det(\Phi(s_i \cdot s_j))\) and by noticing that \(\det(\Phi(s_i \cdot s_j))\) is a unit because \(\Phi\) is a free generator of \(\omega_{S/R}\). \(\square\)

Proposition 6.8. With notation as in Definition 6.5, \(\mathcal{R}_{Y/X}\) is \(f\)-torsion if \(f\) is generically Galois.

Proof. Work in the setup of the proof of Lemma 6.6. The statement then reduces to the equality \((s^n) = (N_{S/R}(s))\) of ideals in \(S\) if \(L/K\) is Galois. By Lemma 6.4 and its proof, \(N_{S/R}(s) \in (s^n)\) unconditionally. We suppose that \(L/K\) is Galois and show that \(s^n \in (N_{S/R}(s))\). By Proposition 6.3, it suffices to prove that \(s \in \sqrt{(N_{S/R}(s))}\), for which it is enough to show that \(s\) belongs to all the ramification primes (if any). This is granted by the Galois symmetry and \(\text{Tr} = \sum_{\sigma \in \text{Gal}(L/K)} \sigma\).

Remark 6.9. We cannot expect the converse of Proposition 6.8 to hold. Indeed, if \(f\) is quasi-étale then \(\mathcal{R}_{Y/X}\) is trivially \(f\)-torsion yet this has no bearing on \(f\) being Galois.

As a direct application of Lemma 6.1 and Lemma 6.4, Theorem 3.13 translates to:

Theorem 6.10. With notation as in Definition 6.5, there exists a rational number \(1 \leq c \leq n\) such that for all divisors \(D\) the following inclusions of Cartier algebras hold:

\[ \mathcal{C}_{\mathcal{O}_X(D)}^\Delta \subset \mathcal{C}_{\mathcal{O}_X(D)}^\top \subset \mathcal{C}_{\mathcal{O}_X(D)}^\Delta, \]

where \(\Delta := \frac{1}{n} \cdot \text{Branch}_T\). Furthermore, we may take \(c = 1\) if \(\text{Ram}_T\) is \(f\)-torsion.

Proof. Work in the affine case \(X = \text{Spec } R\). For \(\mathcal{C}_{R(D)}^\top \subset \mathcal{C}_{R(D)}^\Delta\), we must show that, for nonzero \(\varphi: F_*^R R(D) \to R(D)\), \(f^* \Delta_{\varphi} - \text{Ram}_T \geq 0\) implies \(\Delta_{\varphi} - \Delta \geq 0\). This follows from Lemma 6.1 by noticing that

\[ 0 \leq N_f(f^* \Delta_{\varphi} - \text{Ram}_T) = N_f(f^* \Delta_{\varphi}) - N_f(\text{Ram}_T) = n \cdot \Delta_{\varphi} - n \cdot \Delta, \]

and dividing by \(n\). Apply Lemma 6.4 to \(\text{Ram}_T\) to get \(1 \leq k \leq n\) and set \(c := n/k \in \mathbb{Q} \cap [1, n]\) (further, \(k\) can be taken to be \(n\) if \(\text{Ram}_T\) is \(f\)-torsion and so \(c = 1\)). To have \(\mathcal{C}_{R(D)}^\Delta \subset \mathcal{C}_{R(D)}^\top\), we must show that, for nonzero \(\varphi: F_*^R R(D) \to R(D)\), if \(\Delta_{\varphi} \geq c \cdot \Delta\) then \(f^* \Delta_{\varphi} \geq \text{Ram}_T\). This follows from:

\[ f^* \Delta_{\varphi} \geq f^*(c \cdot \Delta) = \frac{c}{n} \cdot f^* N_f(\text{Ram}_T) \geq \frac{c k}{n} \cdot \text{Ram}_T = \text{Ram}_T. \]
Let \((R, m, \mathfrak{k}, K)\) be a Cohen–Macaulay normal complete domain of dimension \(d\) and write \(X = \text{Spec} R\). By the Cohen–Gabber theorem (see [KS18]), \(R\) admits a generically étale Noether normalization \(f : X \to \mathbb{A}_K^d\) which is flat ([Eis95, Corollary 18.17]), say of rank \(n\). By [Ska16], we may assume that \(p \nmid n\) if \(\mathfrak{k} = \mathfrak{k}^{\text{alg}}\). Further, \(\omega_R = \omega_{R/A} \cong R(\mathfrak{R})\) and so \(\mathfrak{R}\) is a canonical divisor on \(X\). By [AK70, Theorem 6.8], the branch locus and ramification locus of \(f\) are divisors. Following [Mii80, I Exercise 3.9], denote the different ideal of \(f\) by \(\mathfrak{d}_{R/A} \subset R\) and its discriminant ideal by \(\mathfrak{D}_{R/A} \subset A\). These ideals are related by the equality \(N_{R/A}(\mathfrak{d}_{R/A}) = \mathfrak{D}_{R/A}\). In particular, \(\mathfrak{d}_{R/A} = R(-\mathfrak{R}), \mathfrak{D}_{R/A} = A(\mathfrak{B})\). Note that \(\mathfrak{B}\) is a Cartier divisor since \(f\) is flat and further principal since \(\mathbb{A}_K^d\) has trivial Picard group. In fact, \(\mathfrak{B} = \text{div} \delta\) where \(\delta\) is the discriminant of \(T\), e.g. \(\mathfrak{D}_{R/A} = (\delta)\). If \(R\) is Gorenstein, \(\omega_R\) is a free rank \(1\) \(R\)-module, and so there is a free generator \(T : R \to A\) for the \(R\)-module \(\text{Hom}_A(R, A)\). In particular, there is a unique \(\rho \in R\) such that \(\text{Tr}_{R/A} : R \to A\) equals \(T \cdot \rho\), and so \(\mathfrak{d}_{R/A} = (\rho), \mathfrak{R} = \text{div}_R \rho, \mathfrak{D}_{R/A} = (\delta),\) and \(\mathfrak{B} = \text{div}_A \delta\), where \(\delta := N_{R/A}(\rho)\).

**Corollary 6.11.** With notation as above, let \((\delta) = \mathfrak{D}_{R/A}\) and \(\Delta := \frac{1}{n} \cdot \text{div}_A \delta\). Then, there exists a rational number \(1 \leq c \leq n\) such that \(\mathbb{C}_A^c \Delta \subset \mathbb{C}_A^\top \subset \mathbb{C}_A^+\), where transposition is defined with respect to \(\text{Tr}_{R/A} : R \to A\). We may take \(c = 1\) if \(\mathfrak{R}\) is \(f\)-torsion. Assume that \(\mathfrak{k}\) algebraically closed and \(p \nmid n\). If the pair \((A, c \cdot \Delta)\) is \(F\)-regular (resp. \(F\)-pure) then so is \(R\) and \(s(R) \geq n \cdot s(A, c \cdot \Delta)\). The converse and equality hold if \(\mathfrak{R}\) is \(f\)-torsion.

**Proof.** The first part is a particular case of [Theorem 6.10] with \(T = \text{Tr}_{R/A}\). For the second part, observe that \(\text{Tr}_{R/A}\) is surjective, and \(\text{Tr}_{R/A}(m) \subset (x_1, \ldots, x_d)\) by [CRST18, Lemma 2.10]. Then, apply [Theorem 4.8 and Theorem 4.3]. For the final statement, note that if \(\mathfrak{R}\) is \(f\)-torsion then \(\mathbb{C}_A^c = \mathbb{C}_A^+\) and \(f^* \Delta = \mathfrak{R}\).

We end with some examples illustrating Corollary 6.11 beginning with two Gorenstein singularities known for their rather mysterious \(F\)-signature.

**Example 6.12.** Consider \(R = \mathbb{F}_2[x_0, \ldots, x_n]/(x_0^3 + \cdots + x_2^n)\), where \(\text{char} \mathbb{F}_2 \neq 2\). We can directly apply [Corollary 6.11] by taking \(A = \mathbb{F}_2[x_1^2, \ldots, x_d^2]\). The ramification divisor is given by \(\mathfrak{R} = \text{div}_R \rho\) where \(\rho = x_0 x_1 \cdots x_n\), and \(N_{R/A}(\rho) = \rho^2 = (x_1^2 + \cdots + x_n^2) \cdot x_1^2 \cdots x_n^2\). In particular, \(\mathfrak{R}\) is \(f\)-torsion. Thus, \(s(R) = 2^{n+1} \cdot s(A, \Delta)\), where \(\Delta = \frac{1}{2} \cdot \text{div}_A(x_1^2 + \cdots + x_n^2) \cdot x_1^2 \cdots x_n^2\). Let \(\Delta' = \frac{1}{2} \cdot \text{div}_A x_1^2 \cdots x_n^2\), then \(s(A, \Delta') = 1/2^n\) by [BST12, Example 4.19]. However, this gives no nontrivial upper bound for \(s(R)\). The \(F\)-signature of \(R\) (for large \(n\)) is quite involved and depends on \(p\). It can be computed by applying [WY04, Example 2.3] to the results in [GM10]. It turns out that \(\lim_{p \to \infty} s(R)\) is the coefficient of \(z^n\) in the Taylor series of \(sec z + tan z\).

**Example 6.13.** Set \(R = \mathbb{F}_2[x, y, z, u, v]/(x^3 + y^3 + xyz + uv)\). A Noether normalization is \(A = \mathbb{F}_2[y, z, u, v]\) and \(1, x^2\) is an \(A\)-basis of \(R\). A generator of \(\text{Hom}_A(R, A)\) is \((x^2)^c\nu\), for \((x^2)^c\nu \cdot x = x^c\nu\) and \((x^2)^c\nu \cdot (x^2 + yz) = 1^c\nu\). A short computation shows that \(\text{Tr}_{R/A} = 1^c\nu\). Hence, \(\mathfrak{R} = \text{div}_R \rho\) with \(\rho = x^2 + yz\). On the other hand, \(x \rho = y^3 + uw =: \epsilon\). Then, from \(x^3 + (yz)x + \epsilon = 0\), we obtain the minimal equation \(\rho^3 + (yz)\rho^2 + \epsilon^2 = 0\). Therefore, \(\delta := N_{R/A}(\rho) = \epsilon^2\) by a direct computation. Moreover, \(\rho^2(\rho + yz) = \rho^2 x^2 = \delta\). Hence, we may take \(c = 2/3\) in Corollary 6.11 to say that

\[
\mathbb{C}_A^{\frac{2}{3}} \text{div}_A \delta \subset \mathbb{C}_A^\top \subset \mathbb{C}_A^{\frac{2}{3}} \text{div}_A \delta.
\]
We claim that the left inclusion is an equality, i.e. $\mathcal{C}_A^\dagger = \mathcal{C}_A^\div A^\epsilon$. Indeed, let $\varphi \in \mathcal{C}_A^\dagger$ and write $\varphi = \Phi' \cdot a$, where $\Phi$ is the Frobenius trace of $A$. We show that $\Delta_{\varphi} \geq \div_A e$. We know that $\Delta_{\varphi} = \frac{1}{2^k - 1} \cdot \div_A a$ and $\frac{1}{2^k - 1} \cdot \div_R a = (\cdot f^{*}) \Delta_{\varphi} \geq \div_R \rho = \div_R e - \div_R x$. That is, $\div_R ax^{2e-1} \geq \div_R e^{2e-1}$, so $ax^{2e-1}/e^{2e-1} \in R$. To grasp this better, we need:

**Claim 6.14.** Let $a_e, b_e, c_e \in A$ be defined by $x^{2e-1} = a_e x^2 + b_e x + c_e$ in $R$. Then $a_e = 0$ for all $e$, $b_1 = 1$, $c_1 = 0$, and the recursive formulas hold: $b_{e+1} = (yz)^e b^e + c^e e$ and $c_{e+1} = b^e e$.

**Proof of claim.** The values for $a_e, b_e, c_e$ for $e = 1$ are clear. Next, we observe that

$$x^{2e+1-1} = (x^{2e-1})^2 \cdot x = (a_e x^2 + b_e x + c_e)^2 \cdot x = a^e x^2 + b^e x^3 + c^e x.$$ 

Using $x^3 = (yz)x + e$, we verify that $x^5 = (\epsilon x^2 + (yz)^2 x + (yz)e)$. Consequently,

$$x^{2e+1-1} = a^e (\epsilon x^2 + (yz)^2 x + (yz)e) + b^2 ((yz)x + e) + c^2 x$$

$$= \epsilon a^2 x^2 + ((yz)^2 a^e + (yz)b^e + c^e) x + (yz)e a^2 x + b^e e.$$

Hence, $a_{e+1} = a_e^2$ and so $a_e = 0$ for all $e$ as $a_1 = 0$. This also gives the desired formulas. □

With notation as in Claim 6.14, we have

$$\frac{ab_e}{e^{2e-1}} \cdot x + \frac{ac_e}{e^{2e-1}} = \frac{ax^{2e-1}}{e^{2e-1}} \in R.$$ 

Therefore, $ab_e/e^{2e-1}$ and $ac_e/e^{2e-1}$ belong to $A$. Since $ab_e/e^{2e-1} \in A$, we have $\div_A a + \div_A b_e \geq (2^e - 1) \div_A e$. Thus, to prove $\Delta_{\varphi} = \frac{1}{2^k - 1} \div_A a \geq \div_A e$, it suffices to show that $\div_A b_e$ has no support along the prime divisor $\div_A e$, i.e. $\val_{(e)} b_e \leq 0$. Note that Claim 6.14 implies $b_e \equiv (yz)^{2e-1} \mod e$. Hence, $\val_{(e)} b_e \leq (2e - 1) \val_{(e)} yz$ and we are left with showing $\val_{(e)} yz \leq 0$ which is clear as $y, z \not\in (e)$.

In summary, $\mathcal{C}_A^\dagger = \mathcal{C}_A^\div A^\epsilon$ which is principally generated by $\Phi' \cdot e^{2-1} = \Phi \cdot e$ and so not $F$-regular as its splitting prime contains $(e)$. Consequently, since $f^{*} \div_A e = \div_R \rho = \div_R x$, we have that $s(R, \div_R x) = 3 \cdot s(A, \div_A e) = 0$, which gives no information about the $F$-signature of $R$. However, we obtain $r(R, \div_R x) = r(A, \div_A e) > 0$ as $(A, \div_A e)$ is $F$-pure. Finally, we remark that $s(R)$ is conjectured to be $\frac{2}{3} - \frac{5\sqrt{7}}{98}$; see [Tul12] Proposition 4.22.

**Example 6.15.** Our final example exhibits how much simpler than $\mathfrak{R}$ the divisor $\mathfrak{B}$ could be. Let $(V_{n,d}, m)$ be the degree-$d$ Veronese subring of $K[x_1, \ldots, x_n]$ (with $K$ $F$-finite).

**Claim 6.16.** $A := K[x_1^{d}, \ldots, x_n^{d}] \subset V_{n,d}$ is a Noether normalization with basis $B = \{x_1^{r_1} \cdot \cdots x_n^{r_n} \mid d \mid \sum_i \nu_i \text{ and } 0 \leq \nu_i \leq d - 1\}$. Let $T_i$ be the dual of $x_1^{d-1} \cdot x_1 \cdot x_n^{d-1}$ with respect to $B$, where $n - 1 = \mu d + r$ and $0 \leq r < d - 1$. Then, $A_{x_i^{d}} \subset (V_{n,d})_{x_i^{d}}$ is a successive radical extension, $T_i \mid_{D(x_i^{d})}$ is a free generator of $\text{Hom}_{A_{x_i^{d}}} ((V_{n,d})_{x_i^{d}}, A_{x_i^{d}})$, and on $D(x_i^{d})$ we have

$$T := 1^Y = T_i \cdot \prod_{j \not= i} x_j^{((d-1)(n-1)+r)} \cdot \prod_{j \not= i} (x_j/x_i)^{d-1}.$$ 

In particular, $\text{Ram}_T \mid_{D(x_i^{d})} = \text{div} \prod_{j \not= i} x_j^{d-1}$. Further, $T(1) = 1$ and $T(m) \subset (x_1^{d}, \ldots, x_n^{d})$.

**Proof of claim.** By construction, $B$ is a generating set. Indeed, any monomial in $V_{n,d}$ of degree $< d$ in all $x_i$ is contained in this set. If some monomial has degree $\geq d$ in some $x_i$,
then we can extract a suitable monomial $x_1^{a_1} \cdots x_n^{a_n}$ and reduce to the previous case. To show that $B$ is linearly independent, we may invert $x_1^{i_1}$ (by symmetry). Then,

$$A[x_1^{-d}] \cong \mathcal{K}[x_1^{i_1}, (x_2/x_1)^{i_2}, \ldots, (x_n/x_1)^{i_n}] [x_1^{-d}] \subset \mathcal{K}[x_1, x_2, \ldots, x_n/x_1] [x_1^{-d}] \cong V_{n,d}[x_1^{-d}]$$

which is a successive radical extension with basis $C = \{(x_2/x_1)^{i_2} \cdots (x_n/x_1)^{i_n} \mid 0 \leq i_j \leq d-1\}$. We can transform $B$ into this basis by multiplying a monomial $x_1^{i_1} \cdots x_n^{i_n}$; with $\sum_i i_j = \lambda d$, by the unit $x_1^{-\lambda d}$. Hom$_A(V_{n,d}, A)$ restricted to $D(x_1^{d})$ is generated by $G := ((x_2/x_1)^{d-1} \cdots (x_n/x_1)^{d-1})^\vee$, where duals are taken with respect to $C$. In order to transfer this back to our original basis, note that $(x_1^{-a_1} \cdots x_n^{-a_n})$ is mapped to the monomial $x_1^{a_1} \cdots x_n^{a_n}$, where

$$(d-1)(n-1) + a = \lambda d$$

for some $0 \leq a \leq d-1$, and some $\lambda \geq 1$. Writing $n-1 = \mu d + r$ as above and taking (6.16.1) modulo $d$, we find that $a \equiv r \mod d$. Since $0 \leq a \leq d-1$, then $a = r$. Thus $T_1$ is a generator.

For the claim about $T$, note that the duals of $1$ with respect to $C$ and $B$ coincide. With respect to $C$: $1^\vee = G \cdot (x_2/x_1)^{d-1} \cdots (x_n/x_1)^{d-1}$. Clearly, $T(1) = 1$, and $T(m) = 0$. \qed

Note that if char $\mathcal{K} \not\equiv d$ then $T = 1/d^n \cdot \text{Tr}$. However, $T \neq 0$ even if $A \subset V_{n,d}$ is purely inseparable. By Claim 6.16, $A \subset V_{n,d}$ is locally a radical extension. Indeed, the pullback of $\text{Ram}_T$ to $\text{Spec} V_{n,d} \setminus \{m\}$ is given by the Cartier divisor $(D(x_i^d), \text{div} \prod_{j \neq i} x_j^{d-1})$. Denoting $\rho_i := \prod_{j \neq i} x_j^{d-1}$, we see that $\rho_i^d = (\prod_{j \neq i} x_j^d)^{d-1}$. In particular,

$$\text{Branch}_T = N_{R/A}(\text{Ram}_T) = \left(D(x_i^d), (d-1) \cdot \text{div} \prod_{j \neq i} x_j^d\right) = (d-1) \cdot \text{div} x_1^d \cdots x_n^d,$$

Thus, $\mathcal{G}_A = \mathcal{G}_A^\Delta$ where $\Delta = d^{n-1} / d \cdot \text{div} x_1^d \cdots x_n^d$, and moreover

$$s(V_{n,d}) = d^{n-1} \cdot s(A, \Delta) = d^{n-1} \cdot (1 - (d-1)/d)^n = 1/d$$

where the middle equality is [BST12] Example 4.19. Of course, the $F$-signature of the $V_{n,d}$ is already known by results of Singh [Sin05] and of Von Korff [VK12].

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