Construction of Orthonormal Bases from the Vectors of the Problem in Minkowski Space

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Abstract
We propose to use the modified Gram – Schmidt orthonormalization process in Minkowski space for construction of orthonormal bases from the vectors of the problem.

1 Introduction
Nowadays, methods of direct calculation of the amplitudes of processes are intensively developed (see e.g. [1] and references there). In this connection the task of construction of orthonormal bases from the vectors of the problem (in particular, polarization bases for vector bosons) becomes actual.

In the present work, we propose to use the modified Gram – Schmidt orthonormalization process in Minkowski space for these purposes.

We use Feynman metrics:
\[ \mu = 0, 1, 2, 3, \quad a^\mu = (a_0, \vec{a}), \quad a_\mu = (a_0, -\vec{a}), \quad ab = a_\mu b^\mu = a_0 b_0 - \vec{a} \vec{b}; \]
the sign of the Levi-Civita tensor is determined as \( \varepsilon_{0123} = +1 \).

Moreover, the generalized Gram determinants are used (see also [2]). For example:
\[
G \left( \begin{array}{c}
\mu \\
\alpha \\
\beta
\end{array} \begin{array}{c}
\nu \\
\alpha \\
\beta
\end{array} \right) = \begin{vmatrix} g_{\mu\alpha} & g_{\mu\beta} \\ g_{\nu\alpha} & g_{\nu\beta} \end{vmatrix} = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}, \quad g_{\mu\alpha} = \begin{cases} 1, & \mu = \alpha = 0 \\ -1, & \mu = \alpha = 1, 2, 3 \\ 0, & \mu \neq \alpha \end{cases}
\]

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\[ G \left( \begin{array}{cc} a & b \\ c & \beta \end{array} \right) = G \left( \begin{array}{cc} \alpha & \nu \\ \beta & \beta \end{array} \right) a^\mu b^\nu c^\alpha = \left| \begin{array}{cc} (ac) \\ (bc) \end{array} \right| a_\beta b_\beta = (ac)b_\beta - (bc)a_\beta \], and so on.

2 Gram–Schmidt orthonormalization process in the \( n \)-dimensional space

We consider briefly the Gram–Schmidt orthonormalization process, with the help of which one can construct an orthonormal basis from the vectors of the problem (see e.g. [3]).

Let a system of linearly independent vectors \( \{x_1, x_2, \ldots, x_n\} \) is in the \( n \)-dimensional space. Then the following system of vectors is orthogonal:

\[ y_1 = x_1, \]
\[ y_2 = x_2 - \frac{(x_2 y_1)}{(y_1 y_1)} y_1, \]
\[ y_3 = x_3 - \frac{(x_3 y_1)}{(y_1 y_1)} y_1 - \frac{(x_3 y_2)}{(y_2 y_2)} y_2, \]
\[ y_n = x_n - \frac{(x_n y_1)}{(y_1 y_1)} y_1 - \frac{(x_n y_2)}{(y_2 y_2)} y_2 - \cdots - \frac{(x_n y_{n-1})}{(y_{n-1} y_{n-1})} y_{n-1}. \]

Note that

\[ (y_k)_\rho = G \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_{k-1} \end{array} \right)_{\rho} \]
\[ G \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_{k-1} \end{array} \right), \quad k = 1, 2, \ldots, n. \]

The orthonormal system is formed by the vectors:

\[ (z_k)_\rho = \frac{(y_k)_\rho}{\sqrt{(y_k y_k)}} = G \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_{k-1} \end{array} \right)_{\rho} \left( G \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_{k-1} \end{array} \right) G \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_{k-1} \end{array} \right) \right)^{1/2}. \] (1)

3 Modified orthonormalization process in Minkowski space

In the Minkowski space, the Gram–Schmidt orthonormalization process has certain features.

\[ ^1 \text{Here the Greek indices denote the components of vectors, } \rho = 1, 2, \ldots, n. \]
The first, one of the vectors of the basis should be time-like. Therefore, the 4-momentum of a massive particle can be chosen as the first vector (further we also consider the situation where only the massless particles participate in the reaction).

The second, because of some symmetric Gram determinants entering into the denominator (1) are negative (see Appendix), their signs should be taken into account.

Let $p$ is an arbitrary 4-momentum, such that $p^2 = m^2 \neq 0$, $a$, $b$ and $c$ are arbitrary vectors. Using (1), we obtain that four vectors $l_0$, $l_1$, $l_2$, $l_3$ form an orthonormal basis:

$$l_0 = \frac{p}{m},$$

$$l_1(\rho) = \frac{G(p \ a \ b)}{m} \left[ -G(p \ a \ b) \left( \frac{m^2 a_\rho - (p a)p_\rho}{m(p a)^2 - m^2 a_\rho} \right) \right]^{1/2},$$

$$l_2(\rho) = \frac{G(p \ a \ b)}{m} \left[ G(p \ a \ b) \left( \frac{[a^2(pb) - (pa)(ab)]a_\rho + [m^2(ab) - (pa)(pb)]a_\rho + [(pa)^2 - m^2 a^2]b_\rho}{\sqrt{(pa)^2 - m^2 a^2}} \right) \right]^{1/2},$$

$$l_3(\rho) = \frac{G(p \ a \ b \ c)}{m} \left[ -G(p \ a \ b \ c) G(p \ a \ b \ c) \right]^{1/2}.$$ 

Finally we take into account that in the Minkowski space

$$G(p \ a \ b \ c) = -\varepsilon_{\mu\nu\sigma\rho}p^\mu a^\nu b^\sigma c^\tau \varepsilon_{\alpha\beta\lambda\kappa}p^\alpha a^\beta b^\lambda c^\kappa,$$

$$G(p \ a \ b \ c) = \varepsilon_{\mu\nu\sigma\rho}p^\mu a^\nu b^\sigma c^\tau \varepsilon_{\rho\alpha\beta\lambda}p^\alpha a^\beta b^\lambda.$$

As result we have

$$l_3(\rho) = \frac{\varepsilon_{\rho\alpha\beta\lambda}p^\alpha a^\beta b^\lambda}{\sqrt{2(pa)(pb)(ab) + m^2 a^2 b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}}.$$ 

Note that in the final formulae the vector $c$ is absent.
Thus, an orthonormal basis in the Minkowski space, using three vectors of the problem (one of which is a 4-momentum of a massive particle) and a total antisymmetric Levi-Civita tensor, can be always constructed.

Construction of a basis for the reaction with massless particles can be performed in a following way. Let \( p \) is easy to show that following way. Let \( p \) is easy to show that Civita tensor, can be always constructed. (one of which is a 4-momentum of a massive particle) and a total antisymmetric Levi-Civita tensor, can be always constructed.

In particular when \( p_1 \) is an arbitrary massless vector, \( Q \) is an arbitrary 4-momentum of a photon, vectors

\[
l_0 = \frac{p_1 + p_2}{\sqrt{2(p_1 p_2)}},
\]

\[
l_1 = \frac{-p_1 + p_2}{\sqrt{2(p_1 p_2)}},
\]

\[
(l_2)_\rho = \frac{-(p_2 b)(p_1)_\rho - (p_1 b)(p_2)_\rho + (p_1 p_2) b_\rho}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b)} - b^2(p_1 p_2)^2},
\]

\[
(l_3)_\rho = \frac{\varepsilon_{\rho\alpha\beta\lambda} p_1^\alpha p_2^\beta b^\lambda}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b)} - b^2(p_1 p_2)^2}.
\]

In particular when \( p_1 \) is the 4-momentum of a photon, vectors

\[
e_\rho^\pm(p_1, p_2, b) = -(l_2)_\rho \pm i(l_3)_\rho = \frac{\text{Tr} \left[ (1 \pm \gamma_5) \gamma_\rho \hat{p}_1 \hat{b} \hat{p}_2 \right]}{4\sqrt{2(p_1 p_2)} \sqrt{2(p_1 b)(p_2 b)} - b^2(p_1 p_2)}\]

(10)

can be used as the polarization basis for this photon. Using the algebra of \( \gamma \)-matrices, it is easy to show that

\[
\hat{e}^\pm(p_1, p_2, b) = \frac{(1 \mp \gamma_5) \hat{p}_1 \hat{b} \hat{p}_2 + (1 \pm \gamma_5) \hat{p}_2 \hat{b} \hat{p}_1}{2\sqrt{2(p_1 p_2)} \sqrt{2(p_1 b)(p_2 b)} - b^2(p_1 p_2)}.
\]

(11)

Note that replacement of the vector of the problem \( b \) in \( (\hat{e})_\rho \) by another vector \( c \) leads to the appearance of the phase factor. Really, using the identity (see e.g. [1])

\[
(1 \pm \gamma_5) \hat{q} Q (1 \pm \gamma_5) \hat{q} = \text{Tr} \left[ (1 \pm \gamma_5) \hat{q} Q \right] (1 \pm \gamma_5) \hat{q},
\]

where \( q \) is an arbitrary massless vector, \( Q \) is an arbitrary \( 4 \times 4 \)-matrix, we have

\[
e_\rho^\pm(p_1, p_2, b) = e_\rho^\pm(p_1, p_2, c) e^{\pm i \varphi(p_1, p_2, b, c)} ,
\]

(12)

\[
e^{\pm i \varphi(p_1, p_2, b, c)} = \frac{\text{Tr} \left[ (1 \mp \gamma_5) \hat{p}_1 \hat{b} \hat{p}_2 \hat{c} \right]}{4\sqrt{2(p_1 b)(p_2 b)} - b^2(p_1 p_2) \sqrt{2(p_1 c)(p_2 c)} - c^2(p_1 p_2)}.
\]

(13)

Some details of calculations within the considered bases may be found in [1].
Appendix
Consider the signs of symmetric Gram determinants in denominators of the formulae (4) – (3).

1. \( G \left( \begin{array}{cc} p & a \\ a & a \end{array} \right) = p^2 a^2 - (pa)^2 \), \( p^2 = p_0^2 - \vec{p}^2 = m^2 \), \( (p_0 > |\vec{p}|) \).
   When \( a^2 \leq 0 \), it is evident that
   \[
   G \left( \begin{array}{cc} p & a \\ a & a \end{array} \right) \leq 0. \tag{14}
   \]

   Now we consider the situation when \( a^2 = \alpha_0^2 - \vec{a}^2 = m'^2 \), \( \alpha_0 > |\vec{a}| \):
   \[
   p^2 a^2 - (pa)^2 = (p_0^2 - \vec{p}^2)(\alpha_0^2 - \vec{a}^2) - (p_0 \alpha_0 - |\vec{p}| |\vec{a}| \cos \phi)^2
   = \vec{p}^2 \vec{a}^2 (1 - \cos^2 \phi) - p_0^2 \vec{a}^2 - \alpha_0^2 \vec{p}^2 + 2p_0 \alpha_0 |\vec{p}| |\vec{a}| \cos \phi
   \leq \vec{p}^2 \vec{a}^2 (1 - \cos^2 \phi) - (p_0 |\vec{a}| - \alpha_0 |\vec{p}|)^2 - 2p_0 \alpha_0 |\vec{p}| |\vec{a}| (1 - \cos \phi)
   = -\vec{p}^2 \vec{a}^2 (1 - \cos^2 \phi)^2 - (p_0 |\vec{a}| - \alpha_0 |\vec{p}|)^2 \leq 0.
   \]
   Thus, for any vector \( a \), (14) is fulfill always.

2. By the direct calculation it can be shown that
   \[
   G \left( \begin{array}{ccc} p & a & b \\ p & a & a \\ a & a & b \end{array} \right)
   = \left| \begin{array}{ccc} p_0 & p_x & p_y \\ a_0 & a_x & a_y \\ b_0 & b_x & b_y \end{array} \right|^2 + \left| \begin{array}{ccc} p_0 & p_x & p_z \\ a_0 & a_x & a_z \\ b_0 & b_x & b_z \end{array} \right|^2 + \left| \begin{array}{ccc} p_0 & p_y & p_z \\ a_0 & a_y & a_z \\ b_0 & b_y & b_z \end{array} \right|^2 - \left| \begin{array}{ccc} p_x & p_y & p_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{array} \right|^2.
   \]
   Under a Lorentz transformation of \( p^\mu = (p_0, p_x, p_y, p_z) \) into \((m, 0, 0, 0)\), the last term reduced to zero. Due to Gram determinant is Lorentz invariant, we have
   \[
   G \left( \begin{array}{ccc} p & a & b \\ p & a & a \\ a & a & b \end{array} \right) \geq 0. \tag{15}
   \]

3. \[
   G \left( \begin{array}{ccc} p & a & b & c \\ p & a & a & b \\ a & a & b & c \end{array} \right) = -\left| \begin{array}{cccc} p_0 & p_x & p_y & p_z \\ a_0 & a_x & a_y & a_z \\ b_0 & b_x & b_y & b_z \\ c_0 & c_x & c_y & c_z \end{array} \right|^2 \leq 0. \tag{16}
   \]
   Anyway, as it is well known, the equals signs in (14) – (16) can be only when the vectors entering into the Gram determinants are linearly dependent.
References

[1] A.L. Bondarev, Teor. Mat. Fiz., v.96, p.96 (1993) (in Russian), translated in: Theor. Math. Phys., v.96, p.837 (1993), hep-ph/9701333; A.L. Bondarev, in Proceedings of the Joint International Workshop: VIII Workshop on High Energy Physics and Quantum Field Theory & III Workshop on Physics at VLEPP. Zvenigorod, Russia, September 15–21, 1993, Moscow University Press, 1994, p.181, hep-ph/9701331; A.L. Bondarev, hep-ph/9710398

[2] A.L. Bondarev, Teor. Mat. Fiz., v.101, p.315 (1994) (in Russian), translated in: Theor. Math. Phys., v.101, p.1376 (1994), hep-ph/9701329; A.L. Bondarev, in Proceedings of the XVIII International Workshop on High Energy Physics and Field Theory: Relativity, Gravity, Quantum Mechanics and Contemporary Fundamental Physics. Protvino, Russia, June 26–30, 1995 (IHEP, Protvino, 1996) p.242, hep-ph/9701332

[3] Roger A. Horn, Charles R. Johnson, Matrix Analysis, Cambridge University Press, 1986

[4] A.L. Bondarev, hep-ph/9710399 v.3 (in preparation)