Flow management in optimal resource exchange tasks

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Abstract. The paper presents a method for optimal initial allocation of resources of a dynamical system and reallocation of resources at the control interval. The main types of practical tasks on the exchange of material, energy and information resources are considered. A general scheme is given for their formalization using a special extension of the dynamical system of the original problem. The results of verification simulation experiments are presented.

Keywords: resource management, flows, switching, simulation modelling

1. Introduction
The paper describes a new approach to formalizing the tasks of optimal initial allocation or optimal real-time reallocation of resources of a particular technical system. It is assumed that the initial control problem (without optimization of the allocation or reallocation of resources) is formalized as a standard problem of optimal control of a smooth dynamical system. The proposed approach is to add a controlled switch of resource flows into the initial system of equations. A controlled switch in this case means the current values of the system state vector modules. The switch makes it possible to randomly allocate the current resources of the system, saving their amount. The addition of the switch to the initial system allows one to formalize three types of exchange tasks: finding the optimal initial allocation of resources, optimal reallocation of current resources at the control interval, and a mixed task. In this case, the exchange task sets the majorant of the switch control vector components. (For example, in optimal initial allocation tasks, the majoring function allows the system nodes to exchange resources within a given intensity over a small initial control subinterval, and then disables the exchange.) The proposed approach is verified fully enough through developed simulation programs and has been used in planning experiments on the research subjects [1,2].

2. Flow switch
Consider the system

$$\dot{Z}_k(t) = -U_k(t) \cdot Z_k(t) + \frac{1}{N} \sum_{i=1}^{N} U_i(t) \cdot Z_i(t),$$

$$1 \leq k \leq N,$$

(1)

with positive initial states $Z_k(0)$ and controls $U_k(t) > 0$.
If states $Z_k(t)$ are characteristics of some objects, the system (1) describes a switch providing reallocation of a given initial resource between the objects. While

$$\sum_{i=1}^{N} Z_i(t) = \text{const},$$

(2)

because the sum of the right parts in (1) is identically zero.

It is logical to show that for arbitrary positive $Z_i(0)$ and a constant control vector, there are no fluctuations in the system (1), and the values of $Z_i(t)$ asymptotically approach some constants [3]. This means that the corresponding real system reaches constant $Z_i(t) = Z_k$ in finite time. The results of modelling the simplest case of coinciding controls in the simulation program [4] are shown in Figure 1.

**Figure 1.** Change in the state vector

Let us find the values of $Z_k$ set for constant controls $U_k$. Since derivatives $\dot{Z}_k(t)$ in (1) become zero, one can write:

$$-U_k \cdot Z_k + \frac{1}{N} \sum_{i=1}^{N} U_i \cdot Z_i = 0, 1 \leq k \leq N,$$

(3)

or

$$\begin{cases} N \cdot U_1 \cdot Z_1 = U_1 \cdot Z_1 + \ldots + U_N \cdot Z_N, \\
\ldots \\
N \cdot U_N \cdot Z_N = U_1 \cdot Z_1 + \ldots + U_N \cdot Z_N, 
\end{cases}$$

(4)

In the obtained system (3) and (4), the constant $Z_k$ is determined as the solution of the linear system of equations along with the initial conditions $Z_k(0)$.
thus, it follows that set values of $Z_k$ depend only on the control vector $U = \{U_1, \ldots, U_N\}$ and the initial sum of states $\sum_{i=1}^{N} Z_i(0)$.

The results of the experiment for two different sets $Z(0) = \{Z_1(0), \ldots, Z_N(0)\}$ are given in Figures 2 and 3.

**Figure 2.** The system switches to static mode.

From the system of equations (4), we obtain:

$$\begin{cases}
U_i \cdot Z_i = \ldots = U_i \cdot Z_i = \ldots = U_N \cdot Z_N, \\
Z_1 + \ldots + Z_i + \ldots + Z_N = R,
\end{cases}$$

where $R$ is a given retained resource.

From (5) it is easy to get:

$$Z_i = \left(\frac{U_i \cdot \sum_{j=1}^{N} U_j}{U_j} \right)^{-1} \cdot R, 1 \leq i \leq N,$$

which describes state vector $Z = \{Z_1, \ldots, Z_N\}$ set at a constant control vector in the control interval $(0, T)$ $U = \{U_1, \ldots, U_N\}$ and a given resource $R = \sum_{i=1}^{N} Z_i(0) = \sum_{i=1}^{N} Z_i$. 


Figure 3. The system switches to static mode.

1. A proportional change in the components of the constant control vector \( \{U_1, \ldots, U_N\} \rightarrow \{r \cdot U_1, \ldots, r \cdot U_N\} \) preserves the vector of the set states \( \{Z_1, \ldots, Z_N\} \).

Figure 4 shows the switch to the static mode for the system with the initial conditions corresponding to Figure 2 and a proportionally changed control vector.

2. A proportional change in the components of the variable control vector \( U(t) = \{U_1, \ldots, U_N\} \rightarrow rU(t) = \{r \cdot U_1, \ldots, r \cdot U_N\}, r > 0 \), with a simultaneous inversely proportional change in the control time scale \( t \rightarrow \frac{r}{t} \), preserves the state vector \( Z(t) = Z(U(t), t) \), i.e.

\[
Z_k(U(t), t) = Z_k \left( rU \left( \frac{t}{r} \right) \right), 1 \leq k \leq N. \tag{7}
\]

3. The auxiliary task of resource exchange management

Consider a smooth dynamical system

\[
\dot{P}_k(t) = f_k(P(t), \ldots, P_N(t), x(t), t), 1 \leq k \leq N \tag{8}
\]

with the control vector \( x(t) \).

Having set the initial conditions, the control interval \((0, T)\) and the set of controls \(X\), we write:

\[
J(x^*) = \inf_{x \in X} J(x), \tag{9}
\]

\[
J(x) = F(P_1(T), \ldots, P_N(T)), \tag{10}
\]

where the functional \(J(x)\) is a continuously differentiable function of the final states of the system.
Let us now consider the following extension of the original system (8) with the help of the system (1):

\[
P_k(t) = f_k(P_k(t), \ldots, P_N(t), x(t), t) - U_k \tilde{P}_k(t) + \frac{1}{N} \sum_{i=1}^{N} U_i(t) \tilde{P}_i(t), 1 \leq k \leq N
\]  

(11)

with the initial conditions and the control interval from the problem (8) – (10) and the set of controls \( \tilde{X} = X \otimes U \), where \( U_k(0) \) is given and the set continuous function \( j(t) \) majorizes the components of vector \( U \).

Let us write for the auxiliary system (11):

\[
J(\tilde{x}) = \inf_{\tilde{x}} J(\tilde{x}), \quad \tilde{x} \in \tilde{X}
\]  

(12)

\[
J(\tilde{x}) = F(\tilde{P}_1(T), \ldots, \tilde{P}_N(T)).
\]  

(13)

Now return to the original optimal control problem (8) – (10) and indicate three types of resource exchange problems, i.e. the components of the state vector, for it:

1. Finding the optimal values of \( P_k(0) \) under the constraint \( \sum_{k=1}^{N} P_k(0) = C \).
2. Optimally reallocating current resources at the control interval \( (O, T) \).
3. Combining tasks 1 and 2.

The properties of the switch (1) – (7) allow us to formalize these exchange tasks for the original problem (8) – (10), setting the necessary majoring functions \( \phi(t) \) in (11). In this case, the optimal values of \( P_k(0) \) and the optimal controls of tasks 2 and 3 with a given accuracy are obtained from the controls of the auxiliary problem (11) – (13).

**Figure 4.** The system switches to the static mode at a proportional change in the control vector components.
4. Conclusion
1. If the resource exchange is prohibited for any components of the state vector of the initial problem (8) – (10), the corresponding differential equations of the auxiliary problem (11) – (13) are not modified.

2. The problem of finding the optimal initial values of the state vector of the initial problem (8) – (10) is trivially extended to the case of an uncontrolled dynamical system.

3. System (1) describes the simplest manageable switch of fictitious or real resource flows connecting some physical objects. At the same time, the due consideration of the technical constraints on the speed of change in controls and object states complicates the algorithms for exchange tasks significantly. The developed program [5] allows us to investigate modifications of the switch (1) with built-in feedback that smoothes the system's response to the specified controls.

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