Effective Meson Field Theory from QCD

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We give a simple and straightforward procedure of how to construct an effective meson Lagrangian from QCD Lagrangian. We integrate the methods of Gasser, Leutwyler, Alkofer and Reinhardt and use the derivative expansion scheme to derive the low energy effective Lagrangian for meson fields to $O(p^4)$. In this paper, why the meson particle can be treated as the goldstone mode is very clear. In our calculation the result in $O(p^2)$ is the same as in the chiral perturbation theory, but the result in $O(p^4)$ is different from that in literature. We will discuss the discrepancies and give some remarks on our result.

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I. INTRODUCTION

In the late fifties it was found that the perturbative method of quantum field theory always fails when it was applied to the strong interactions of hadrons (especially for the light meson systems). Although the remarkable success in QED, it was recognized that perhaps we should have a suitable formulation which is independent of perturbative approach to solve this. In fact the scaling behavior of the structure functions in the deep inelastic region tells us that the perturbative method can only be implemented in the high energy region. Now we know this is because QCD is a non-Abelian Yang-Mills theory with gauge group $SU_c(3)$. Due to the property of asymptotic freedom in this theory, we can safely use perturbative method to handle high energy reaction processes. But in the low energy region, the coupling constant is too large to use perturbative expansion. So if we can find an effective Lagrangian with new degrees of freedom (DOFs) and enough small coupling constant in the low energy region, then we still can use perturbative method. The question is how it can be done.

Free quarks are not observed in nature. It is commonly assumed that quarks are confined (color confinement), i.e., the DOFs can only be hadrons. If we want to find an effective Lagrangian which describes adequate low energy behavior of particle physics, the quark DOF must be replaced by the hadron DOF. In 1984, Gasser and Leutwyler [1, 2] proposed a method about how to systematically extract the low-energy structure of strong interactions from experiments. They considered only light pseudoscalar particles and used the spontaneously chiral symmetry breaking property in the low energy region. It is an effective chiral Lagrangian with the Nambu-Goldstone-bosons (NGB) DOF only. The form of each term in the chiral Lagrangian is determined by chiral symmetry requirements.

The theoretic derivative method also have been investigated. In the literature [3, 4] we can see how the procedure is implemented. A critical step is to introduce some NGB depended terms (by symmetry consideration) into the fermionic determinant and make a coordinate-space expansion of it. In general, the proper-time regularization scheme has been used. By this technique the meson DOF can be extracted and the effective Lagrangian with only meson DOF are obtained. Although the physical results obtained by the effective Lagrangian which derived from these methods indeed coincide with the experiment well. We still have two questions need to overcome. How the effective Lagrangian can be derived from the first principle and why the meson particle can be treated as the goldstone mode. In this article we will try to solve this question and derive an effective meson Lagrangian from QCD Lagrangian. We shall use a different method, derivative expansion method [5], which is a general expansion scheme used in effective field theory, to derive the low-energy effective Lagrangian. Using this method, we find that the loop effect in the underlying theory can be represented naturally in all types of terms in effective Lagrangian and the obtained low energy coefficients (LECs) have been renormalized. However, some of our results in $O(p^4)$ are different from that in others’ [1, 2, 3, 4]. Latter we will discuss the discrepancies and give some remarks on our results.

In this paper we first give a brief introduction to the effective field theory and illustrate how to use the derivative expansion method to obtain an effective Lagrangian from its underlying theory. These will be done in Sec. II. Afterward we briefly review some results derived by Alkofer and Reinhardt [6], who obtained an effective QFD Lagrangian by integrating out the gluon DOF in the QCD Lagrangian. We then mesonize this Lagrangian and
rewrite it as a constituent quark model (CQM) Lagrangian. This procedure will be illustrated in Sect. III. In this section we can see how the quark-gluon dynamics are represented by the constituent quarks propagating in background meson fields. Furthermore, we can find that as the running energy going down the potential of background meson fields will have spontaneously symmetry breaking which is essential condition of linear sigma model. The obtained Lagrangian is similar to that in the nonlinear sigma model except that the meson fields are background fields. Because we want to have an effective Lagrangian with only the meson DOFs. The quark fields should be integrated out. In Sec. IV we will illustrate how to use the derivative expansion method introduced in Sec. II to obtain what we want and discuss the difference between our result and others’. There we also give some remarks on our result. Finally, our conclusion is given in Sec. V.

II. DERIVATIVE EXPANSION METHOD

The aim of an effective field theory is to find an adequate and effective Lagrangian to describe the physical systems of interest. For example, owing to the property of asymptotic freedom the QCD Lagrangian can only be used in high energy region. So if we want to cope with the low-energy physical problems we must find an effective low-energy Lagrangian to describe it. Since effective field theories should be based on the fundamental field theory. They must be the quantum field theoretical implementation of the quantum ladder of the fundamental theory. As the energy changes, the new DOFs become relevant and must be included in the theory. Then we should ask: how these new DOFs are related to the old DOFs and how these new DOFs can be extracted from the underlying theory and made into an effective theory of its own. In this section we shall focus on how to extract the new DOFs from the underlying theory. Below we shall use a beautiful method, derivative expansion method, which mentioned in [5] to do that.

Firstly, we assume that the relations between the new and old DOFs have been known and the corresponding effective Lagrangian with the new and old DOFs have been obtained. Then the next step is to derive the effective Lagrangian with only new DOFs. Consider a simple model with the Lagrangian

$$L(\phi, \Phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 + \frac{\alpha}{2} \phi^2 \Phi^2,$$

(1)

whose action is

$$S[\phi, \Phi] = \int d^4 x L(\phi, \Phi) \equiv S[\phi] + S_{\Phi}[\phi, \Phi],$$

(2)

where

$$S[\phi] = \int d^4 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right),$$

(3)

$$S[\phi, \Phi] = \int d^4 x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 + \frac{\alpha}{2} \phi^2 \Phi^2 \right),$$

(4)

and its generating functional is

$$Z = \int \mathcal{D}\Phi \mathcal{D}\Phi e^{iS[\phi, \Phi]} = \int \mathcal{D}\phi e^{i\Gamma_{\text{eff}}[\phi]},$$

(5)

where the $\phi$ and $\Phi$ indicate new and old DOFs respectively. Here the normalization constant was abbreviated by simplification and if no special indications we will keep this simplification in the following equations. Because the $\phi$ field is the DOF we want to keep, we integrated out the $\Phi$ field. Then we have

$$e^{i\Gamma_{\text{eff}}[\phi]} = \int \mathcal{D}\Phi e^{iS[\phi, \Phi]} = e^{iS[\phi]} \int \mathcal{D}\Phi e^{iS_{\Phi}[\phi, \Phi]} = e^{iS[\phi]} (\det \hat{O})^{-1/2},$$

(6)

$$\hat{O}_{xy} = (-\Box_x - M^2 + \alpha \phi_x^2) \delta_{xy},$$

(7)

$$\Gamma_{\text{eff}}[\phi] = S[\phi] + \frac{i}{2} \text{Tr} \ln \hat{O},$$

(8)
and now we can write
\[ \text{Tr} \ln \hat{O} = \text{Tr} \ln(-\Box - M^2) + \text{Tr} \ln \left[ 1 - \alpha(\Box + M^2)^{-1} \phi^2 \right]. \] (9)

We can ignore the first term in eq. (9) because it does not depend on \( \phi \) field and can be absorbed into the normalization constant. After expanding the logarithm term, we obtain
\[ \Gamma_{\text{eff}} = S[\phi] + \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Tr} \left[ \alpha(\Box + M^2)^{-1} \phi^2 \right]^n \equiv S[\phi] + \sum_{n=1}^{\infty} \Gamma^{(n)}[\phi]. \] (10)

The operator \((\Box + M^2)^{-1}\) is the propagator for the free \( \Phi \) field and can be defined as
\[ \Delta_{xy} \equiv \langle x | (\Box + M^2)^{-1} | y \rangle = \int \frac{d^4 q}{(2\pi)^4} e^{-iq(x-y)} \frac{1}{q^2 - M^2 + i\epsilon}. \] (11)

Hence all the terms involving \( \Phi \) were integrated out and we can find that each term in the effective action \( \Gamma_{\text{eff}} \) is composed of \( \phi \) field and the propagator of \( \Phi \) field. In other words, the loop contribution of \( \Phi \) field has been absorbed into the new terms in effective Lagrangian. For example, the \( \Gamma^{(1)}[\phi] \) contribution is
\[ \Gamma^{(1)}[\phi] = -\frac{i}{2} \alpha \int d^4 x \Delta_{xx} \phi_x^2, \] (12)
which corresponds to a one-loop effect with internal \( \Phi \) field line.

In the following sections we shall use this method to expand the fermionic determinant. At first we introduce how to obtain a constituent quark form Lagrangian from QCD.

### III. CONSTITUENT QUARK FORM

In QCD with three light flavors, u, d and s, the generating functional for Green functions is defined via the path-integral formula
\[ Z = \int \mathcal{D}q \mathcal{D}\overline{q} \mathcal{D}A \exp \left\{ i \int d^4 x L_0^{\text{QCD}} \right\}, \] (13)
where the \( L_0^{\text{QCD}} \) is QCD Lagrangian in the absence of quark masses,
\[ L_0^{\text{QCD}} = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + i \overline{q}_L \gamma^\mu D_\mu q_L + i \overline{q}_R \gamma^\mu D_\mu q_R, \] (14)
\( G_{\mu\nu}^a \) is the antisymmetric tensor of the gluon field. This Lagrangian is not adequate for perturbative calculations in the low energy region. Our aim is to find an effective Lagrangian \( \mathcal{L}_{\text{eff}} \) to describe the low energy physics and make the perturbative method work safely. We have known that in the low energy region the effective DOFs are hadron fields. So the first step is to extract the hadron DOFs from the QCD Lagrangian. For this purpose, we rewrite the generating functional such as
\[ Z = \int \mathcal{D}q \mathcal{D}\overline{q} \exp \left[ i \int d^4 x \overline{q}(i\gamma^\mu \partial_\mu)q + \Gamma[j] \right], \] (15)
where
\[ \Gamma[j] = -i \ln \int \mathcal{D}A \exp \left[ i \int d^4 x \left( -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + g A_{\mu}^a \overline{q}_L \gamma_\mu D_\mu q_L \right) \right]. \] (16)
The effective action of the gluon field \( \Gamma[j] \) can be expanded in power of the quark current \( j_\mu^a \) as
\[ \Gamma[j] = \Gamma[j = 0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int \left\{ \Gamma^{(n)}(x_1, x_2, \ldots, x_n; x_{\mu_1}, \ldots, x_{\mu_n}) \prod_{i=1}^{n} \left[ j^a_{\mu_i}(x_i) dx_i \right] \right\}, \] (17)
where the expansion coefficients are

\[
\Gamma^{(n)}(x_1, x_2, \ldots, x_n)^{\mu_1 \cdots \mu_n} = \left[ \frac{\delta^{(n)}(x)}{\prod_{j=1}^n \delta_{f_{\mu_j}}(x_j)} \right]_{j=0} = (-i) \cdot (ig)^n \langle 0 | \Gamma \left[ A_{\mu_1}(x_1) A_{\mu_2}(x_2) \cdots A_{\mu_n}(x_n) \right] | 0 \rangle,
\]

and the brackets \( \langle \cdots \rangle \) are the irreducible n-point correlation functions of the gluon field. Now let’s consider the second order term of the effective action of the gluon field, \( \Gamma_2[j] \) (the 0th and 1st order term have no contribution). Using the Fierz decomposition \([8]\) of spin, color, flavor matrices and by taking the Feynman gauge condition

\[
\langle A_\mu^a(x) A_\nu^b(y) \rangle = -i\delta^{ab}g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m_g^2 + i\epsilon} = -i\delta^{ab}g_{\mu\nu} \Delta(x - y),
\]

where the effective gluon mass \( m_g \) originated in higher order correctional effect \([3]\). Then the \( \Gamma_2[j] \) is \([9, 10]\)

\[
\Gamma_2[j] = \int \int d^4x d^4y \int d^4k \frac{e^{-ik(x-y)}}{k^2 - m_g^2 + i\epsilon} j^{\alpha\beta}(y)
\]

\[
= \frac{g^2}{3} \int d^4x d^4y \eta^\alpha q(x) \Delta'(x - y) \bar{\eta} q(y),
\]

where the matrix \( \Lambda^\alpha \) is tensor product of color, flavor and Dirac matrices

\[
\Lambda_\alpha = \frac{\lambda^A}{2} \gamma_\alpha, \quad A = 0, \ldots, 8
\]

and

\[
\Gamma_\alpha \in \left\{ 1, i\gamma_5, \frac{i}{\sqrt{2}} \gamma_\mu, \frac{i}{\sqrt{2}} \gamma_\mu \gamma_5 \right\}.
\]

Applying the eqs. \([14]\) and \([20]\) in eq. \([15]\) one obtains

\[
Z = \int D\eta D\bar{\eta} \exp \left\{ i \int d^4x \bar{\eta} (i\gamma^\mu \partial_\mu) \eta - \frac{1}{2} \int d^4x d^4y \eta^\alpha q(x) \Delta'(x - y) \bar{\eta} q(y) \Lambda^\alpha q(y) + \cdots \right\},
\]

with

\[
\Delta'(x - y) = \frac{2}{3} g^2 \Delta(x - y) = -i \cdot \frac{g^2}{6\pi} \cdot \frac{m_g^2}{l} K_{-1}(m_g l),
\]

where \( l = \sqrt{(x^0 - y^0)^2 + (\vec{x} - \vec{y})^2} \) and \( K_{-1}(m_g l) \) is modified Bessel function. This is a NJL-like Lagrangian. It can be mesonized by introducing an auxiliary bilocal field for which \( \eta(x, y) = \eta_\alpha(x, y) \Lambda^\alpha \) and \( \bar{\eta}(x, y) = \bar{\eta}(y, x) \) via the identity

\[
\exp \left[ -\frac{1}{2} \int d^4x d^4y \bar{\eta} \eta^\alpha q(x) \Delta'(x - y) \bar{\eta} q(y) \right] = \frac{(\det \Delta')^{-1/2}}{(2\pi)^2} \int D\eta \exp \left\{ -\int d^4x d^4y \left[ \frac{1}{2} \eta_\alpha^* \Delta^{-1} \eta_\alpha - \bar{\eta}(y) \Lambda^\alpha q(y) \eta_\alpha \right] \right\}
\]

The generating functional then becomes

\[
Z = \int D\eta D\bar{\eta} D\bar{\eta} \exp \left\{ i \int d^4x \bar{\eta} (i\gamma^\alpha \partial_\alpha) q - \int d^4x d^4y \left[ \frac{1}{2} \eta_\alpha^* \Delta^{-1} \eta_\alpha - \bar{\eta}(y) \Lambda^\alpha q(x) \eta_\alpha \right] + \cdots \right\}.
\]

In the eq. \([20]\) we can find that the strong interaction effect in the original Lagrangian eq. \([13]\) has been replaced by the background meson field \( \eta \). Furthermore, the vacuum expectation value of the auxiliary field is

\[
\langle \eta_\alpha(x, y) \rangle = i \Delta'(x - y) \langle \bar{\eta}(y) \Lambda^\alpha q(y) \rangle.
\]

[\ast] Here the original result has meson and diquark parts. Since we only want to discuss the meson system, we ignore the diquark part.
This indeed demonstrate that the meson ($\eta$ field) is composed of quark and anti-quark fields. Because the bilocal property of the $\eta$ field should complicate the problem, we will try to localize it. We know that the DOFs in the low energy region are meson particles. So we can make an assumption that for small momentum the positions of the meson’s constituent particles, $\vec{q}(x)$ and $q(y)$, are undifferentiated and by inserting the Dirac delta function $\delta(x^0 - y^0) \cdot \delta^3(\vec{l} - |\vec{x} - \vec{y}|)$ into the generating functional eq.(26) to constraint it, where $l_0$ is the space between the quarks and $\vec{l}$ approaches to zero. Then we have

$$\int \int d^4xd^4y \delta(x^0 - y^0) \cdot \delta^3(\vec{l} - |\vec{x} - \vec{y}|) \cdot \left[ \frac{1}{2} \eta_\alpha A^{\alpha-1} \eta^\alpha - i \overline{\eta}(y) A^\alpha q(x) \eta_\alpha \right]$$

$$\approx -i \int d^4x \left[ \frac{1}{g^2} \eta_\alpha \eta^\alpha + \overline{q} A^\alpha q \eta_\alpha \right], \quad (28)$$

where we have absorbed all other coefficients into the renormalized coupling constant $g_r$. After localizing the bilocal term we can further decompose the generic meson field $\eta$ according to its properties under Lorentz transformation

$$\eta = S + i \gamma_5 P - i \overline{\eta} - A \gamma_5, \quad (29)$$

where $S$ is a scalar, $P$ is a pseudoscalar, $V$ is a vector and $A$ is a axialvector field. All these fields are flavor matrices, $S = S^\alpha \frac{\lambda_\alpha}{\sqrt{2}} F$, etc. Then we can discuss the behavior of all kinds of meson multiplets. For simply we only consider scalar and pseudoscalar particles. We choose $V_\mu = A_\mu = 0$, $S = g_5 \sigma_1/F \sqrt{N}$, $P = g_5 \overline{\lambda} \overline{\phi} / \sqrt{2}$, with $N$ is its flavor number and $g_5$ is Yukawa coupling constant, then the effective Lagrangian in eq.(29) is

$$\mathcal{L} = i \overline{\eta} \gamma q + g_5 \left( \sigma \overline{\eta} q / \sqrt{N} + i \overline{\overline{\eta}} \cdot \overline{\eta} \lambda q / \sqrt{2} \right) - V(\sigma^2 + \overline{\phi}^2). \quad (30)$$

And the potential $V(\sigma^2 + \overline{\phi}^2)$ is $[\|]

$$V(\sigma^2 + \overline{\phi}^2) = - \frac{g_5^2}{2 g^2} (\sigma^2 + \overline{\phi}^2) + \frac{\lambda}{4!} (\sigma^2 + \overline{\phi}^2)^2. \quad (31)$$

Here we only take the quadratic and quartic terms into account because that in the low energy region the expansion coefficient is so small that there is no need to consider higher order terms. Besides, we also discard the cubic term$[^1]$, for we still don’t know how to deal with it. Now let’s look at the form of this potential. Because the strong coupling constant $g_r$ is in denominator and it is a running energy depended parameter. For the high energy region the expanding coefficient will be too large to implement the perturbative expansion. But as the running energy going down there will be spontaneous chiral symmetry breaking emerged from this potential and then the pseudoscalar particles can be treated as its goldstone modes. In order to see this the ground state should be decided firstly. For the stable vacuum we have the constraint condition

$$\sigma^2 + \overline{\phi}^2 = \frac{6 g_5^2}{g^2 \lambda} = N f^2 \quad (32)$$

with $f$ is a running-energy depended parameter which will be defined latter. The constraint can be resolved by choosing

$$\sigma(x) = \sqrt{N} f \cos \left( \frac{\sqrt{2} |\phi(x)|}{\sqrt{N} f} \right), \quad \overline{\phi}(x) = \sqrt{N} f \phi \sin \left( \frac{\sqrt{2} |\phi(x)|}{\sqrt{N} f} \right) \quad (33)$$

[1] We have known that the $\eta$ field can be treated as the composite field of quark and anti-quark. And the $\eta^2$ term can be derived from the $j^2$ contribution, so we expect that the $\eta^4$ term also can be derived from the $j^4$ contribution and the coefficient $\lambda$ should be proportional to $\frac{g_5^4}{g^2}$.  

[2] In fact, this term is an explicit chiral symmetry breaking term.
with \( \tilde{\phi}(x) = |\tilde{\phi}(x)| \hat{\phi} \) and \( \hat{\phi} \cdot \hat{\phi} = 1 \). Substituting eq. (33) into the eq. (30) we find that the Lagrangian can be rewrote as

\[
\mathcal{L} = i \bar{\psi} \gamma_q q + g_\phi f \bar{q} q \left[ \cos \left( \frac{\sqrt{2} |\tilde{\phi}(x)|}{\sqrt{N_f}} \right) + i \gamma_5 \frac{\tilde{\phi} \sqrt{N}}{\sqrt{2}} \sin \left( \frac{\sqrt{2} |\tilde{\phi}(x)|}{\sqrt{N_f}} \right) \right] q
= \bar{\psi}(i \hat{\partial} - \nabla - A \gamma_5 + m_\psi) \psi \equiv \mathcal{L}_{\text{CQM}}
\]

with

\[
\nabla = - \frac{i}{2} (\xi^\dagger \xi + \xi \xi^\dagger), \quad A = - \frac{i}{2} (\xi^\dagger \xi - \xi \xi^\dagger), \quad \psi = \Lambda q, \quad \bar{\psi} = \bar{\eta} \Lambda, \quad m_\psi = g_\phi f,
\]

\[
\Lambda = \exp \left\{ i \sqrt{2} \gamma_5 \Phi / (2 f) \right\} = P_R \xi + P_L \xi^\dagger, \quad \xi = \exp \left\{ i \sqrt{2} \Phi / (2 f) \right\},
\]

and the \( \Phi \) is the conventional parametrization matrix of meson field:

\[
\Phi(x) \equiv \frac{\tilde{\phi} \cdot \phi}{\sqrt{2}} = \left[ \begin{array}{ccc}
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+\\
\pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0\\nK^- & \frac{1}{\sqrt{2}} \pi^0 & -\frac{2}{\sqrt{6}} \eta
\end{array} \right].
\]

The Lagrangian in eq. (34), which reveals constituent quarks propagating in the background meson fields, can be regarded as describing a constituent quark model. It is noted that the sign of the constituent quark’s mass term is positive. It means that in our model the constituent quark should have negative mass. Although this can be adjusted by change the sign of the parameter \( g_\phi \). Later we will find that if we want to obtain an effective meson Lagrangian with positive mass, the constituent quark’s mass must be negative. This is different from the result of Espriu, de Rafael and Taron [2].

So far we obtain an effective Lagrangian with the new DOFs, constituent quark fields \( \psi \) and meson fields \( \tilde{\phi} \). Although we start with a massless QCD Lagrangian, the constituent quark fields are massive. We can treat this as a remnant effect of the strong interactions after integrating out the gauge fields.

**IV. EFFECTIVE MESON LAGRANGIAN**

We shall now attempt to derive an effective Lagrangian with meson DOFs only. In the latest section we have obtained an effective CQM Lagrangian. We can find that the form is the same as in the nonlinear sigma model except that the meson fields are background fields. In fact we can already safely use the perturbative expansion method in this effective CQM Lagrangian to handle the low energy problems. But this is not our aim here. Our purpose is to transform the QCD theory into an effective theory with that the DOFs are completely in the meson form. So let’s get down to business. Firstly, let us add some external current source terms to the massless QCD Lagrangian eq. (14). Then the corresponding constituent quark form Lagrangian in eq. (34) can be rewritten such this

\[
\mathcal{L}_{\text{CQM}}^{(\text{new})} = \mathcal{L}_{\text{CQM}} + \bar{\eta} \tilde{\gamma}_\mu \left( v_\mu + \gamma_5 a_\mu \right) q - \bar{\eta} (s - i \gamma_5 p) q
= \bar{\psi}(i \hat{D}' + m_\psi) \psi,
\]

with

\[
i \hat{D}' = i \hat{\partial} + \xi^\dagger
= i \hat{\partial} - \nabla - A \gamma_5 + \gamma_\mu (P_L \xi l_\mu + P_R \xi^\dagger r_\mu \xi)
- \frac{1}{2 B_0} (P_R \xi^\dagger \chi + P_L \xi^\dagger \chi^\dagger)
\]

\[
l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu, \quad \chi = 2 B_0 (s + i p),
\]

where \( v_\mu, a_\mu, s \) and \( p \) are the corresponding current source fields which are hermitian \( 3 \times 3 \) matrices in flavor space and \( B_0 \) is an arbitrary constant. Because when we analyze low energy physics, the effective realization of quark current in
meson form must be understood. By this elegant technique the matrix elements of currents can be calculated in a very straightforward way. We note that it is started from this constituent quark form Lagrangian, eq.(35), the effective chiral Lagrangian in the chiral perturbation theory can be obtained by the proper-time regularization method [3]. But we don’t use this method here. Now by means of the derivative expansion method introduced in Sec. II, we can write the generating functional as

\[ Z = e^{i \int d^4 x [L_{\text{QCD}} + \text{source term}]} \]

\[ = e^{i \int d^4 x [\psi U]} \exp \{ i \int d^4 \bar{\psi} D \psi / \bar{U} \} J(U) \]

\[ = \int DU e^{i \Gamma_{\text{eff}}[U]} G \text{WZW} \]

Here \( U = \xi \), which is the same as the exponential parametrization of the NGB fields using in the chiral perturbation theory (ChPT) and the effective action \( \Gamma_{\text{eff}}[U] \) is

\[ \Gamma_{\text{eff}}[U] = \int d^4 x L_{\text{eff}}(U) + \Gamma_{\text{WZW}} \]

\[ = i \sum_{n=1}^{\infty} (-1)^n n \text{Tr}[(i \not{\partial} + m_\psi)^{-1} \not{\mathcal{G}}^n] + \Gamma_{\text{WZW}} \]

\[ = \sum_{n=1}^{\infty} \Gamma^{(n)} \].

The Jacobian \( J(U) \) comes from the chiral transformation \( q \rightarrow \Lambda q \) and \( \Gamma_{\text{WZW}} \) is the corresponding action, which is the Wess-Zumino-Witten(WZW) term [11, 12]. We neglect this term for simplicity in this article. By this expansion formula, the effective Lagrangian can be expanded in the powers of \( \mathcal{G} \). But the expanded form is not adequate for discussion. So we rearrange it according to the powers of momentum such that

\[ L_{\text{eff}}(U) = L_2 + L_4 + L_6 + \cdots \]  

(43)

The details of the calculation are given in the appendix. The result expanding to \( O(p^2) \) is

\[ L_2 = -i m_\psi^2 N_c I_3 (D_\mu U^\dagger D^\mu U) - i \frac{m_\psi}{B_0} N_c I_0 \langle \chi^\dagger U^\dagger \chi \rangle \]

(44)

where \( \langle O \rangle \) denotes the trace of the matrix \( O \), \( N_c \) is color number and

\[ D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu, \quad D_\mu U^\dagger = \partial_\mu U^\dagger + i U^\dagger r_\mu - i l_\mu U^\dagger. \]

(45)

\[ I_0 = \int \frac{d^4 q}{q^2 - m_\psi^2 + i \epsilon} = \frac{2m_\psi^2}{(4\pi)^2} \left( N_c + 1 - \ln \frac{m_\psi^2}{\mu^2} \right), \]

(46)

\[ I_3 = \int \frac{d^4 q}{(q^2 - m_\psi^2 + i \epsilon)(q + p)^2 - m_\psi^2 + i \epsilon} = \frac{I_0}{m_\psi^2}, \]

(47)

\[ N_c = \frac{2}{\epsilon} + \ln 4\pi - \gamma + O(\epsilon), \]

(48)

where \( \gamma \) is Euler’s constant. The eq.(46) is a divergent integral and still need to be renormalized. If we substituting \( m_\psi = g_\phi f \) into the eq.(46) and renormalized the strong coupling constant again (re-absorbed the divergent part into the strong coupling constant in the parameter \( f \), eq.(52)), and let \( B_0 = m_\psi, g_\phi = \frac{2\pi}{\sqrt{N_c}} \) then we obtain

\[-i I_0 N_c = \frac{(g_\phi f)^2 N_c}{(4\pi)^2} = \frac{f^2}{4}, \]

(49)
and
\[ \mathcal{L}_2 = \frac{f_2^2}{4} (D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U). \]  
(50)

We note that the form is totally the same as in ChPT [2]. Using the Lagrangian \( \mathcal{L}_2 \) the currents can be calculated by taking the appropriate derivatives with respect to the external sources:

\[ J_L^{\mu i} = \frac{\delta S_2}{\delta l_\mu^i} = -\frac{if_2^2}{2} \langle \lambda^i U \partial^\mu U \rangle = -f \partial^\mu \phi^i + O(\phi^2), \]  
(51a)

\[ J_R^{\mu i} = \frac{\delta S_2}{\delta r_\mu^i} = \frac{if_2^2}{2} \langle \lambda^i U \partial^\mu U \rangle = f \partial^\mu \phi^i + O(\phi^2). \]  
(51b)

Now the physical meaning of the parameter \( f \) is obvious. Up to the order \( O(p^2) \), \( f = f_{\pi^+} \approx 130.7 \text{ MeV} \), defined as \( \left| \left\langle 0 \mid A^\mu_\mu (0) \mid \pi^+ (q) \right\rangle \right| = if_{\pi^+} q_\mu \),

\[ \langle 0 | A^\mu_\mu (0) | \pi^+ (q) \rangle = if_{\pi^+} q_\mu, \]  
(52)

with

\[ A^{\mu i} = \frac{1}{2} (J_R^{\mu i} - J_L^{\mu i}), \quad A^\mu_\mu = \frac{1}{\sqrt{2}} (A^1_\mu \mp i A^2_\mu) \]  
(53)

And the constituent quark mass \( m_\psi \) is

\[ m_\psi = g_\phi f = \frac{2\pi f}{\sqrt{N_c}} \approx 474.13 \text{ MeV}. \]  
(54)

Similarly, the constant \( B_0 \) can be related to the quark condensate:

\[ \langle 0 | \bar{q}^i q^j | 0 \rangle = -f^2 B_0 \delta^{ij} \approx -(200.8 \text{ MeV})^3 \delta^{ij} \]  
(55)

If we take \( s = M \) and \( p = 0 \), where \( M \) denotes the quark-mass matrices such that

\[ M = \text{diag}(m_u, m_d, m_s), \]  
(56)

then the relations between the physical meson masses and the quark masses are:

\[ M_{\pi^\pm}^2 = 2\hat{m} B_0, \quad M_{\pi^0}^2 = (2\hat{m} - \varepsilon) B_0 + O(\varepsilon^2), \]  
(57)

\[ M_{K^\pm}^2 = (m_u + m_s) B_0, \quad M_{K^0}^2 = (m_d + m_s) B_0, \]  
\[ M_{\eta}^2 = \frac{2}{3} (\hat{m} + 2m_s + \frac{3}{2} \varepsilon) B_0 + O(\varepsilon^2), \]  

where

\[ \hat{m} = \frac{1}{2} (m_u + m_d), \quad \varepsilon = \frac{1}{4} \frac{(m_u - m_d)^2}{(m_s - \hat{m})}. \]  
(58)

Use \( B_0 = m_\psi \approx 474.13 \text{ MeV} \) and the input \( M_{\pi^\pm} = 139.6 \text{ MeV}, M_{K^\pm} = 493.677 \text{ MeV}, M_{K^0} = 497.672 \text{ MeV} \), we obtain that

\[ m_u \approx 16.37 \text{ MeV}, \quad m_d \approx 24.73 \text{ MeV}, \quad m_s \approx 497.66 \text{ MeV}, \]  
\[ M_{\pi^0} \approx 138.97 \text{ MeV}, \quad M_{\eta} \approx 566.8 \text{ MeV}. \]  
(59)

[5] We must emphasized that the definition, \( f = f_{\pi^+} \), was established in the running energy is low enough. If the running energy is so large so that we can not ignore the next leading term contribution, then the parameter \( f \) must be redefined.
Compared with the experimental values $M_{\pi^0} = 134.98$ MeV and $M_{\eta} = 547.45$ MeV. Our results are acceptable. It is noted that the light quark masses given by particle data group (PDG) are $m_u = 2$ to 8 MeV, $m_d = 5$ to 15 MeV, $m_s = 100$ to 300 MeV at scale $\mu \approx 1$ Gev. It is meaning that the effective running scale in the order $O(p^2)$ in our theory should be smaller than 1 Gev. In other words, the suitable energy region in the order $O(p^2)$ in our theory also should be smaller than 1 Gev.

Although our result in the order of $O(p^2)$ does coincide with that in the literature, that in the order of $O(p^4)$ does not. Our result is

$$\mathcal{L}_4 = -\frac{f^2}{4m_{\psi}^2} \left\{ -\frac{1}{12} (D_\mu U U^\dagger D_\mu U)^2 - \frac{1}{36} (D_\mu U U^\dagger D_\rho U)(D_\mu U U^\dagger D_\rho U) + \frac{1}{12} [(D_\mu U U^\dagger D_\mu U)^2] 
+ \frac{i}{6} (F_{\mu\nu} F_{\mu\nu}^\dagger + F_{\mu\nu}^\dagger F_{\mu\nu}) + \frac{i}{18} (F_{\mu\nu}^\dagger D_\mu U D_\nu U + F_{\mu\nu} D_\mu U D_\nu U^\dagger) - \frac{1}{16} (U \chi^\dagger - \chi U)^2 
+ \frac{5}{8} (\chi^\dagger \chi) + \frac{3}{16} (\chi^\dagger U \chi U + U^\dagger \chi U^\dagger \chi) + \frac{1}{2} (D_\mu U U^\dagger D_\mu U(U^\dagger \chi + \chi^\dagger U)) 
+ \frac{1}{4} (|\Pi^\mu, \Delta_U| E) - \frac{i}{4} (\partial_\mu \Pi^\mu F) - \frac{4}{9} ([\Pi^\mu, \Pi^\nu] [\Delta^\mu, \Delta^\nu] - \frac{2}{9} ([\Pi^\mu, \Pi^\nu])^2 
+ \frac{i}{9} (\xi^\dagger F_{\mu\nu} \xi [2\Delta^\mu + \Pi^\mu, \Pi^\nu] - \xi F_{\mu\nu}^\dagger [2\Delta^\mu - \Pi^\mu, \Pi^\nu] 
- \frac{2}{9} (\Pi^\nu \Delta^\mu + \Delta^\nu \Pi^\mu) + \frac{10}{9} ([\Pi^\mu, \Pi^\nu] [\Pi^\mu, \Pi^\nu])) \right\}, \quad (60)$$

with

$$\Pi^\mu = \frac{i}{2} [\xi^\dagger, \partial_\mu \xi] + \frac{1}{2} \xi^\dagger U \xi + \frac{1}{2} \xi^\dagger \partial_\mu \xi; \quad (61a)$$

$$\Delta^\mu = \frac{i}{2} \{\xi^\dagger, \partial_\mu \xi\} - \frac{1}{2} \xi^\dagger \partial_\mu \xi + \frac{1}{2} \xi^\dagger \partial_\mu \xi = \frac{i}{2} \xi^\dagger D_\mu U \xi^\dagger = -\frac{i}{2} \xi^\dagger D_\mu U \xi, \quad (61b)$$

$$E = \xi^\dagger \chi \xi - \xi^\dagger \chi^\dagger \xi, \quad F = \xi^\dagger \xi + \xi^\dagger \chi \xi^\dagger, \quad (61c)$$

while the result in the literature [2] is

$$\mathcal{L}_4 = L_1 (D_\mu U^\dagger D_\mu U)^2 + L_2 (D_\mu U^\dagger D_\rho U)(D_\mu U^\dagger D_\rho U) 
+ L_3 (D_\mu U^\dagger D_\mu U D_\rho U^\dagger D_\rho U) + L_4 (D_\rho U^\dagger D_\rho U)(U^\dagger \chi + \chi^\dagger U) 
+ L_5 (D_\mu U^\dagger D_\mu U(U^\dagger \chi + \chi^\dagger U)) + L_6 (U^\dagger \chi + \chi^\dagger U)^2 
+ L_7 (U^\dagger \chi - \chi^\dagger U)^2 + L_8 (\chi^\dagger U \chi U + U^\dagger \chi U^\dagger \chi) 
- iL_9 (F_{\mu\nu}^\dagger D_\mu U D_\nu U + F_{\mu\nu} D_\mu U D_\nu U^\dagger) + L_{10} (U^\dagger F_{\mu\nu} U F_{\mu\nu}) 
+ H_i (F_{\mu\nu} F_{\mu\nu} + F_{\mu\nu}^\dagger F_{\mu\nu}^\dagger) + H_2 (\chi^\dagger \chi). \quad (62)$$

Our result is not so compact and can not be written in the covariant form as in the literature. This is because that our starting point, the CQM Lagrangian in eq. (34), is too simplistic. In above, in order to obtain the CQM Lagrangian we made so many simplification. In eq. (34) we absorbed all the nonperturbative effect to the effective gluon mass. This procedure is very rough. In eq. (60) we ignored the cubic and other higher order terms. The former is an explicit chiral symmetry breaking term which should have contribution to the meson mass. And the latter have something to do with high energy effect. In ChPT, if the running energy is such high that have to take the $O(p^4)$ contribution into account then the high order contribution in eq. (60) must also be considered.

After counting the $O(p^4)$ contribution in we can obtain that

$$f_{\pi^\pm} = f - 2g_{\phi\bar{\psi}} \tilde{m} \quad (63)$$

and

$$M_{\phi}^2 = b_\phi / a_\phi. \quad (64)$$
The other fields such as weak or lepton particles still need to be renormalized.

The original Lagrangian terms but from the higher order terms of the effective Lagrangian. In fact all the divergent quantities obtained from eq.(67) we can see that the higher order contribution of effective action doesn’t come from the series expansion but from the higher order terms of the effective Lagrangian. It is noted that in eq.(63) and eq.(65) we didn’t count the loop contribution in. This is because that in our method all other fields should have been absorbed into the LECs (see Sec. II), there is no need to consider their contribution.

It is about twenty years since the ChPT came into being. Although it does obtain very good results in low energy physics, still it can not be treated as a complete theory. There are still some problems should be considered. In

\[ L(\phi, \partial \phi) = \frac{\epsilon^2}{4!} \int L_0(d^4x)^2 + \ldots \]

From eq. (67) we can see that the higher order contribution of effective action doesn’t come from the series expansion terms but from the higher order terms of the effective Lagrangian. In fact all the divergent quantities obtained from the original Lagrangian \( L_0 \) have been renormalized and absorbed into the LECs in the effective Lagrangian. This characteristic is very conspicuous in our method.

\[ \exp i \int L_0(d^4x) = 1 + i \int L_0(d^4x) + \frac{1}{2!} \left( i \int L_0(d^4x)^2 + \ldots \right) \]

\[ = 1 + i \int L_{\text{eff}}(d^4x) \rightarrow \exp i \int L_{\text{eff}}(d^4x). \]
above context, we have shown that how to solve these problems. We have described a simple procedure to derive the low-energy effective Lagrangian of the strong interactions between the octet of pseudoscalar states from QCD. We see that the complete procedure is straightforward with an obvious physical concept of transition process. Firstly, by expanding the action of gluon fields in power of the quark current, the NJL-type Lagrangian can be obtained easily, which is that most of theoretical derivative techniques are started in. Then, by mesonizing this Lagrangian, we see how the constituent quarks were generated and how the strong interaction effect was represented in the background meson field. This step also shows how the dynamical masses of constituent quarks were generated. Finally, we integrated out the constituent quark DOF and use derivative expansion method to expand the fermionic determinant. In this process we find that if the meson particles have positive mass (for \(B_0 > 0\)) then the constituent quarks must has negative mass. We know that a particle will be a virtual particle if its mass is negative. And a virtual particle is invisible. Then our result can just explain why the quarks are not observed in natural.

Finally, we mention that the Gasser and Leutwyler’s ChPT is still a very successful method when dealing with the low energy physics. Although we think that there is no need to take into account the loop effect. By their method, due to that the LECs are obtained by fit in with the experimental values. They can obtain a very good result even though they count the loop contribution in. They consider the loop effect of each order in effective Lagrangian and absorbed it into the LECs. We absorbed the loop effect to the parameter \(f\) which contained in the LECs when construct the effective Lagrangian. In some respects these two concepts are the same.

Our model is still roughly. In eq. (31), in order to obtain a potential with spontaneous symmetry breaking property we didn’t consider the cubic term, but this term should have more contribution than the quartic term. So our constituent quark form Lagrangian, eq. (34), and the effective meson Lagrangian which derived from this potential are oversimplification. In our calculation the result which contain \(O(p^2)\) and \(O(p^4)\) is even worse than the result which only contain \(O(p^2)\) contribution. It is a fatal defect in perturbation theory. Furthermore, The vector and pseudovector states of meson fields should also be included. The question of how to translate the diquark part of interaction to baryon field remains to be investigated.

APPENDIX A

This appendix is a summary of the steps of the calculation in Sect. IV. From eq. (42) we know the effective action without the WZW term is

\[
\Gamma_{\text{eff}}[U] = i \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Tr}[(i\partial + m_\psi)^{-1} \mathcal{G}^\prime]^n, \tag{A1}
\]

and the propagator for \(\psi\) field is

\[
S_{xy} \equiv (i\partial + m_\psi)^{-1} = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{\partial - m_\psi}{q^2 - m_\psi^2 + i\epsilon}. \tag{A2}
\]

To simplify the calculation we rewrite the connection \(\mathcal{G}^\prime\) as

\[
\mathcal{G}^\prime = \gamma^\mu \Pi_\mu + \gamma^\mu \gamma_5 \Delta_\mu - \frac{1}{4B_0}\gamma_5 E - \frac{1}{4B_0}F; \tag{A3}
\]

then the first-order term of the effective action is

\[
\Gamma^{(1)} = -i \text{Tr} \int d^4x S_{xx} \mathcal{G}^\prime_x,
\]

\[
= -i N_c J_0 \frac{m_\psi}{B_0} \int d^4x \langle U^\dagger \chi + \chi^\dagger U \rangle. \tag{A4}
\]

This term belong to the contribution of \(O(p^2)\). Similarly, the rest can be deduced by analogy:

\[
\Gamma^{(2)} = \frac{i}{2} \text{Tr} \int d^4x d^4y S_{xy} \mathcal{G}^\prime_y S_{yx} \mathcal{G}^\prime_x, \tag{A5}
\]

\[
\Gamma^{(3)} = \frac{-i}{3} \text{Tr} \int d^4x d^4y d^4z S_{xy} \mathcal{G}^\prime_y S_{yz} \mathcal{G}^\prime_z S_{zx} \mathcal{G}^\prime_x. \tag{A6}
\]
Inserting the eq. \(A3\) into the expressions for the effective action and retaining terms up to \(O(p^4)\) we obtain

\[
\mathcal{L}_2 + \mathcal{L}_4 = \frac{f^2}{4} (D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U)
\]

\[
- \frac{f^2}{4m^2} \left\{ \frac{1}{12} (D_\mu U^\dagger D^\mu U)^2 - \frac{1}{36} (D_\mu U^\dagger D_\nu U) (D^\mu U^\dagger D^\nu U) + \frac{1}{12} ((D_\mu U^\dagger D^\mu U)^2)
\]

\[
+ \frac{1}{6} (F^\mu_{\nu\rho} F^{\nu\rho}_{\mu} + F^\mu_{\nu\rho} F^{\nu\rho}_{\mu}) + \frac{i}{18} (F^\mu_{\nu\rho} D^\mu U^\dagger D^\rho U + F^\mu_{\nu\rho} D^\rho U^\dagger D^\mu U) - \frac{1}{16} (U\chi^\dagger - \chi U^\dagger)^2
\]

\[
+ \frac{5}{8} \langle \chi^\dagger \chi \rangle + \frac{3}{16} \langle \chi^\dagger U^\dagger \chi U \rangle + \frac{1}{2} (D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U))
\]

\[
+ \frac{1}{4} \langle [\Pi^\mu, \Delta_\mu] E \rangle - \frac{i}{4} (H_\mu, \Pi^\mu) - \frac{4}{9} \langle [\Pi^\mu, \Pi^\nu] [\Delta^\mu, \Delta^\nu] \rangle - \frac{2}{9} \langle [\Pi^\mu, \Pi^\nu] \rangle^2
\]

\[
+ \frac{i}{9} (\xi^4 F^\mu_{\nu\rho} [2\Delta^\mu + \Pi^\mu, \Pi^\nu] - \xi F^\mu_{\nu\rho} \xi + \xi F^\mu_{\nu\rho} \xi + \xi [\Delta^\mu, \Delta^\nu])
\]

\[
- \langle \{2 \langle \Pi^\mu, \Delta_\mu \rangle + \Delta_\mu, \Pi^\mu \rangle + \frac{10}{9} \langle \Pi^\mu, \Pi^\nu \rangle \rangle \} [\Pi^\mu, \Pi^\nu] \rangle \right\}, \quad (A7)
\]

where we have used the following identities for simplification:

\[
[\Delta^\mu, \Delta^\nu] = \frac{1}{4} \xi [D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U] \xi^\dagger, \quad (A8)
\]

\[
\Pi^\mu_{\nu\rho} \equiv \partial_\mu \Pi^\nu - \partial_\nu \Pi_\rho - i [\Pi^\mu, \Pi^\nu] = \frac{1}{2} \xi [F^\mu_{\nu\rho} + \frac{1}{2} \xi F^\mu_{\nu\rho} \xi + i [\Delta^\mu, \Delta^\nu], \quad (A9)
\]

\[
(D_\mu D^\mu U)^\dagger - U (D_\mu D^\mu U) = \frac{1}{3} \langle U\chi^\dagger - \chi U \rangle - \langle U\chi^\dagger - \chi U \rangle^2, \quad (A10)
\]

\[
F^\mu_{\nu\rho} = \partial^\mu l_\nu - \partial^\nu l_\mu - i [l^\mu, l^\nu], \quad F^\mu_{\nu\rho} = \partial^\mu r_\nu - \partial^\nu r_\mu - i [r^\mu, r_\nu], \quad (A11)
\]

\[
(D_\mu D_\nu - D_\nu D_\mu) U = iU F^\mu_{\nu\rho} - iF^\mu_{\nu\rho} U, \quad (D_\mu D_\nu - D_\nu D_\mu) U^\dagger = iU^\dagger F^\mu_{\nu\rho} - iF^\mu_{\nu\rho} U^\dagger. \quad (A12)
\]

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