A spinor theory of gravity and the cosmological framework

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Abstract. Recently we have presented a new formulation of the theory of gravity based on an implementation of the Einstein equivalence principle distinct from general relativity. The kinetic part of the theory—that describes how matter is affected by the modified geometry due to the gravitational field—is the same as in general relativity. However, we do not consider the metric as an independent field. Instead, it is an effective one, constructed in terms of two fundamental spinor fields \( \Psi \) and \( \Upsilon \), and thus the metric does not have a dynamics of its own, but inherits its evolution through its relation with the fundamental spinors. In the first paper it was shown that the metric that describes the gravitational field generated by a compact static and spherically symmetric configuration is very similar to the Schwarzschild metric. In the present paper we describe the cosmological framework in the realm of the spinor theory of gravity.

Keywords: gravity, cosmology of theories beyond the SM

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1. Introduction

There is no doubt that the activity in the field of experimental gravitation has increased largely in the last decades. New space measurements and astronomical discoveries, including those of cosmological origin are mainly responsible for this. At the basis of any theory of gravity compatible with such observations, one has the Einstein equivalence principle (EEP) which can be described [1] as three conditions:

(a) the weak equivalence principle is valid (that is, all bodies fall precisely the same way in a gravitational field);
(b) the outcome of any local non-gravitational experiment is independent of the velocity of the freely falling reference frame in which it is performed;
(c) the outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed.

From the validity of this EEP one infers that ‘the gravitation must be a curved space–time phenomenon’. This was implemented by Einstein by assuming that the curvature of the space–time is related to the stress-energy–momentum tensor of matter in space–time and by postulating a specific form for such an equation. Taken together, the EEP and Einstein’s equation constitute the basis of a successful program of a theory of gravity.
Is this the unique way to deal with the universality of gravitational processes? Is the only way to implement the EEP? Recently [2] we proposed a new look into this old question by arguing that it is possible to treat the metric of space–time—that in general relativity (GR) describes the gravitational interaction—as an effective geometry, that is, the metric acting on matter is not an independent field and as such does not possess its own dynamics. Instead, it inherits one from the dynamics of two fundamental spinor fields Ψ and Υ which are responsible for the gravitational interaction and from which an effective geometry appears.

The non-linear character of gravity should be present already at the most basic level of these fundamental structures. It seems natural to describe this non-linearity in terms of the invariants constructed with the spinor fields. The simplest way to build a concrete model is to use the standard form of a contraction of the currents of these fields, e.g. $J_{\mu} J^{\mu}$, to construct the Lagrangian of the theory. We assume that these two fields (which are half-integer representations of the Poincaré group) interact universally with all other forms of matter and energy. As a consequence, this process can be viewed as nothing but a change of the metric of the space–time. In other words, the influence of these spinor fields on matter/energy is completely equivalent to a modification of the background geometry into an effective Riemannian geometry $g_{\mu\nu}$. In this aspect this theory agrees with the idea of general relativity theory which states that the equivalence principle implies a change on the geometry of space–time as a consequence of the gravitational interaction. However, the similarities between the spinor theory of gravity and general relativity stop here.

To summarize, let us stress the main steps of this program.

(a) there exist two fundamental spinor fields—which we will name Ψ and Υ;
(b) the interaction of Ψ and Υ is described by Fermi Lagrangian;
(c) the fields Ψ and Υ interact universally with all forms of matter and energy;
(d) as a consequence of this coupling with matter, the universal interaction produces an effective metric;
(e) the dynamics of the effective metric is already contained in the dynamics of Ψ and Υ: the metric does not have a dynamics of its own, but inherits its evolution through its relation with the fundamental spinors.

In [2] we presented a particular example of the effective metric in the case of a compact spherically static object, like a star and have shown that it is astonishingly similar to the Schwarzschild solution of GR.

Before entering the analysis of these questions let us briefly comment our motivation. As we shall see, the present proposal and the theory of general relativity have a common underlying idea: the characterization of gravitational forces as nothing but the effect on matter and energy of a modification of the geometry of space–time. This major property of general relativity remains unchanged. The main difference concerns the dynamics that this geometry obeys. In GR the dynamics of the gravitational field depends on the curvature invariants; in the spinor theory of gravity such a specific dynamics simply does not exist: the geometry evolves in space–time according to the dynamics of the spinors Ψ and Υ. The metric is not a field of its own, it does not have an independent reality but is just a consequence of the universal coupling of matter with the fundamental spinors. The motivation of walking down only half of Einstein’s path to general relativity is to
avoid certain known problems that still plague this theory, including its difficult passage to the quantum world and the questions put into evidence by astrophysics involving many discoveries such as the acceleration of the universe, the problems requiring dark matter and dark energy. It seems worthwhile to quote [3] here: ‘Dark energy appears to be the dominant component of the physical Universe, yet there is no persuasive theoretical explanation for its existence or magnitude. The acceleration of the Universe is, along with dark matter, the observed phenomenon that most directly demonstrates that our theories of fundamental particles and gravity are either incorrect or incomplete. Recent observations in cosmology are responsible for an unexpected attitude: to take seriously the possibility of altering Einstein’s theory of gravity’. The spinorial theory of gravity presents the possibility of a way out of these difficulties. The reason, which will be explained later on, can be understood from the fact that in the STG there is no direct relationship between the acceleration of the scale factor of the universe and the matter/energy distribution, contrary to the case of GR, in which the Friedmann equation that controls the dynamics of the universe relates the matter–energy content to the geometry through the evolution of the scale factor $a(t)$:

$$\frac{\dot{a}}{a} = -\frac{1}{6} (\rho + 3p).$$

It follows from this equation that if the universe is accelerating, then something very unusual must occur, like, for instance, a very negative pressure term dominating the evolution. As we shall see, nothing similar happens in STG, since the way in which matter influences the dynamics of the geometry does not take such form.

In the first subsection we present the mathematical background used in the paper and in particular the very important Pauli–Kofinki identity. These relations allow us to obtain a set of products of currents which will be very useful to simplify our calculations. In section 2 we recall the definition of the effective metric and some of its properties and compare with the field theory formulation of general relativity. In section 3 we present the dynamics, separated in two parts: (i) the kinetic part, which tells us how particles move in a given gravitational field; and (ii) the influence of matter on the formation of the gravitational field. We shall see that in what concerns the first part, the spinor theory of gravity is completely identical to general relativity. They differ in the second part, once in STG there is no independent dynamics for the geometry. In the field theory formulation of general relativity as it was described in the fifties by Gupta, Feynman [6] and others, and more recently in [7], the gravitational field can be described alternatively either as the metric of space–time—as in Einstein’s original version—or as a field $\varphi_{\mu\nu}$ in an arbitrary unobservable background geometry, which is chosen to be Minkowski$^1$. We shall see that by universally coupling the spinor fields to all forms of matter and energy, a metric structure appears, in a similar way to the field theoretical description of GR. The main distinction between these two approaches concerns the status of this metric. In general relativity it has a dynamics provided by a Lagrangian constructed in terms of the curvature invariants. In our proposal, this is not the case. The metric is an effective way to describe gravity and it appears because of the universal form of the coupling of matter/energy of any form and the fundamental spinors. Section 4 deals with the induced

$^1$ In [7] another choice is investigated, concerning the de Sitter geometry.
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In section 5 we start the new cosmological program. Section 6 contains the conclusions and comments.

1.1. Definitions

In this paper we deal with two spinor fields $\Psi$ and $\Upsilon$. We use capital symbols to represent the vector and axial currents constructed with $\Psi$ as above and lower case to represent the corresponding terms of the spinor $\Upsilon$, namely,

\[ J^\mu \equiv \overline{\Psi} \gamma^\mu \Psi \]
\[ I^\mu \equiv \overline{\Psi} \gamma^\mu \gamma^5 \Psi. \]
\[ j^\mu \equiv \overline{\Upsilon} \gamma^\mu \Upsilon \]
\[ i^\mu \equiv \overline{\Upsilon} \gamma^\mu \gamma^5 \Upsilon. \]

We use the standard convention and definitions (cf [4]). For completeness we recall:

\[ \overline{\Psi} \equiv \Psi^+ \gamma^0. \]

The $\gamma^5$ is Hermitian and the others $\gamma^\mu$ obey the Hermiticity relation

\[ \gamma^+ \mu \equiv \gamma^0 \gamma^\mu \gamma^0. \]

The properties needed to analyse non-linear spinors are contained in the Pauli–Kofink (PK) relation. These are identities that establish a set of relations concerning elements of the four-dimensional Clifford algebra. The main property states that, for any element $Q$ of this algebra, the PK relation ensures the validity of the identity:

\[ (\overline{\Psi} Q \gamma^\lambda \Psi) \gamma^\lambda \Psi = (\overline{\Psi} Q \Psi) \Psi - (\overline{\Psi} Q \gamma_5 \Psi) \gamma_5 \Psi \]

for $Q$ equal to $I$, $\gamma^\mu$, $\gamma_5$ and $\gamma^\mu \gamma_5$, respectively, where $I$ is the identity of the Clifford algebra. As a consequence of this relation we obtain two extremely important facts:

- the norm of the currents $J_\mu$ and $I_\mu$ have the same value and opposite sign;
- vectors $J_\mu$ and $I_\mu$ are orthogonal.

This formula implies some identities which will be used later on to simplify our calculations:

\[ J_\mu \gamma^\mu \Psi \equiv (A + i B \gamma^5) \Psi \]
\[ I_\mu \gamma^\mu \gamma^5 \Psi \equiv -(A + i B \gamma^5) \Psi \]
\[ I_\mu \gamma^\mu \Psi \equiv (A + i B \gamma^5) \gamma^5 \Psi \]
\[ J_\mu \gamma^\mu \gamma^5 \Psi \equiv -(A + i B \gamma^5) \gamma^5 \Psi, \]

where $A \equiv \overline{\Psi} \Psi$ and $B \equiv i \overline{\Psi} \gamma^5 \Psi$. 

2. The effective metric

In [2] we showed how to treat gravity as the universal interaction of two fundamental
spinors $\Psi$ and $\Upsilon$ with matter. This led to the identification of gravity as a geometric
phenomenon, at least for the kinematic part, that is, when dealing with the question of
how gravity influences matter. This form of interaction leads to the introduction of a
Riemannian geometry in an analogous way as it is done in the field theoretical description
of general relativity, namely, in terms of the metric of the background $\eta_{\mu\nu}$ and a symmetric
second order tensor $\varphi_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}. \quad (3)$$

The field $\varphi_{\mu\nu}$ is chosen to be non-dimensional, that is, we set:

$$\varphi_{\mu\nu} = -\frac{g_F g_m}{4} \frac{1}{\sqrt{X}} (c_{\mu\nu} + c_{\nu\mu}), \quad (4)$$

where

$$c_{\mu\nu} = \Sigma_\mu \Pi_\nu$$

and the vectors $\Sigma_\mu$, $\Pi_\mu$ are constructed in terms of the currents of the spinors, through
the definitions

$$\Sigma_\mu \equiv J_\mu + j_\mu + I_\mu + i_\mu$$

and

$$\Pi_\mu \equiv J_\mu + j_\mu + \beta (I_\mu + i_\mu).$$

$X$ is defined as the trace$^2$

$$X \equiv c^{\mu\nu} \eta_{\mu\nu} = \Sigma^\mu \Pi_\mu.$$

Note that Minkowski geometry is the unique metric that we deal with in what concerns
$\Psi$ and $\Upsilon$. The effective metric $g_{\mu\nu}$ appears only when considering the evolution of
matter. The asymmetry exhibited by vectors $\Sigma_\mu$ and $\Pi_\mu$ is a consequence of the way we
constructed these terms: from the association of a self-interacting Heisenberg potential
with a modification of the internal connection, as demonstrated in the previous paper.
This is the only free parameter of the theory. We shall see in the next section that the case
in which $\beta = 1$ a number of interesting special properties appear. To quote some: it was
shown in [2] that the gravitational field of a static spherically symmetric configuration
coincides with the Schwarzschild solution of general relativity; the Heisenberg self-
interaction disappears; the coupling between the fundamental spinor fields coincides with
low energy weak interaction. The constant $g_F$ has the same dimensionality as Fermi
constant and $g_m$ has the dimensionality of $(\text{energy})^{-1}$. We note that in the previous
article the constant $g_m$ was named $\lambda$.

Let us emphasize that this form of the construction of the metric tensor is an exact
one: it is not an approximation. This implies that the inverse contravariant expression
$g^{\mu\nu}$ defined by $g^{\mu\nu} g_{\alpha\nu} = \delta^\mu_\alpha$ is an infinite series:

$$g^{\mu\nu} = \eta^{\mu\nu} - \varphi^{\mu\nu} + \varphi^{\mu\alpha} \varphi^\alpha_{\nu} + \cdots. \quad (5)$$

$^2$ In the previous work I used the restricted definition $X \equiv J_\mu J^\mu + j_\mu j^\mu$. I decided to change for the present
definition in order to include in the same expression the case in which the field $\Psi$ is an eigenstate of $\gamma^5$. 
In order to be clearly understood, let us pause for a while to very briefly describe the so-called field theoretical description of general relativity while emphasizing that such a description is not related to our proposal.

2.1. The universal coupling: gravity

The field theoretical approach of GR goes back to the fact that Einstein dynamics of the curvature of the Riemannian metric of space–time can be obtained as a sort of iterative process, starting from a linear theory of a symmetric second order tensor $\varphi_{\mu\nu}$ and by an infinite sequence of self-interacting processes leading to a geometrical description. The definition of the metric is the same as above. Other definitions were also used (see for instance [7] for an analysis of the convenience of these alternative non-equivalent definitions). Although these theories can be named ‘field theories’, they contain the same metric content of general relativity, disguised in a non-geometrical form. The framework of the spinor theory of gravity is totally different. Let us emphasize that we are not presenting a dynamics for the metric in the sense of such field theories. Instead, the geometry is understood as an effective one, in the sense that it is the way gravity appears for all forms of matter and energy. However, its evolution is provided by the fundamental spinor fields $\Psi$ and $\Upsilon$. These field theories of gravitation teach us how to couple the tensor field $\varphi_{\mu\nu}$ with matter terms in order to guarantee that the net effect of this interaction produces the desired modification of the metric structure. This idea will guide us when coupling the two fundamental spinors with all forms of matter and energy in order to obtain the same equivalent interpretation of the identification of the gravitational field with the metric of the space–time.

3. Dynamics

The coupling of matter to gravity is provided by the identification of the gravitational field with the geometry. This means that we have to modify the matter Lagrangian in the Minkowski background by changing $\eta_{\mu\nu}$ to $g_{\mu\nu} \equiv \eta_{\mu\nu} + \varphi_{\mu\nu}$. This part of the action—which answers the question of how gravity acts on matter—has the same structure as in general relativity. However, in STG the geometry is an effective one and does not have a dynamics of its own. In order to obtain the evolution of the metric we have to look into the dynamics of the spinors.

3.1. The behaviour of matter in a given gravitational field

Assume that in the special theory of relativity the dynamics of a certain matter distribution is given by a Lagrangian $L_m$. The field theoretical description of general relativity, describes its interaction with gravity using the equivalence principle, also known as the minimal coupling principle. This means substituting all terms in the action $S_0$ in which the Minkowski metric$^3$ $\gamma_{\mu\nu}$ appears by the effective metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. As an example consider a scalar field $\Phi$. In the Minkowski background, its dynamics is

$^3$ We note that when Minkowski metric is written in an arbitrary system of coordinates it is represented by $\gamma_{\mu\nu}$. 

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provided by
\[ S_0 = \int \sqrt{-\gamma} B^\mu{}^\nu \gamma_{\mu\nu} = \int \sqrt{-\gamma} \partial^\mu \Phi \partial^\nu \Phi \gamma_{\mu\nu}, \]
where \( \gamma \equiv \text{det} \gamma_{\mu\nu} \). In this case \( B^\mu{}^\nu \) can be written in terms of the energy–momentum tensor defined as
\[ E^\mu{}^\nu = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma_{\mu\nu}} \left( \sqrt{-\gamma} L \right). \]
Indeed, a direct calculation yields
\[ E^\mu{}^\nu = \partial_\alpha \Phi \partial_\beta \Phi \gamma^{\alpha\mu} \gamma^{\beta\nu} - \frac{1}{2} \partial_\lambda \Phi \partial_\sigma \Phi \gamma^{\lambda\sigma} \gamma^{\mu\nu} \]
immediately implying
\[ B^\mu{}^\nu = E^\mu{}^\nu - \frac{1}{2} E \gamma_{\mu\nu}, \]
where \( E \equiv E^\mu{}^\nu \gamma_{\mu\nu} \). The corresponding action, including the gravitational interaction, is obtained by replacing \( \gamma_{\mu\nu} \) and its inverse \( \gamma_{\mu\nu} \) with the corresponding \( g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu} \) which yields
\[ S = \int \sqrt{-\omega} \omega \partial^\mu \Phi \partial^\nu \Phi \ g_{\mu\nu}, \]
where we have used the same definition as in the field theory of gravity, namely, \( \omega \equiv \sqrt{-g}/\sqrt{-\gamma} \), and \( g = \text{det} g_{\mu\nu} \). In this case
\[ B^\mu{}^\nu = \omega \left[ E^\mu{}^\nu - \frac{1}{2} E g^\mu{}^\nu \right]. \]
Thus, for any kind of matter interacting with the gravitational field, the action is provided by the golden rule of GR, namely
\[ S = \int \sqrt{-\omega} \omega L_M = \int \sqrt{-g} L_M, \tag{6} \]
where the corresponding energy–momentum tensor is given by
\[ T^\mu{}^\nu = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left( \sqrt{-g} L \right). \]
It follows [5] that this quantity is divergence free, in the effective metric \( g_{\mu\nu} \), that is, \( T^\mu{}^\nu = 0 \).

This way of coupling matter with the fundamental spinors guarantees that, kinematically, the behaviour of any kind of matter (and energy) in the spinor theory of gravity is the same as in general relativity: free particles follow geodesics in a prescribed geometry, as the manifestation of gravitational interaction.

Let us now turn to the influence of matter on the gravitational field. The dynamics of the gravitational field is completely distinct in these two theories. In general relativity, the metric obeys a dynamics generated by the Hilbert–Einstein Lagrangian
\[ S_{\text{HE}} = \frac{1}{k_e} \int \sqrt{-g} R \ d^4x. \]
Nothing similar occurs in the spinorial theory of gravity. The metric does not have a specific dynamics, but instead obeys the evolution dictated by its relationship with the dynamics of the fundamental spinors.
4. Generating the gravitational field

The dynamics presented in [2] contains the following terms:
\[ L = L(\Psi) + L(\Upsilon) + L_{\text{int}}(\Psi, \Upsilon) + L_{\text{mat}}. \]  
(7)
We concentrate our analysis on the equation for the spinor \( \Psi \). The corresponding equation for the other field \( \Upsilon \) is obtained similarly by substituting \( \Psi \) by \( \Upsilon \). We have:
\[ i\gamma^\mu \partial_\mu \Psi + g_F \gamma^\mu \left( C^\mu + D^\mu \gamma^5 \right) \Psi = 0. \]  
(8)
We write in the equivalent compact form:
\[ i\gamma^\mu \partial_\mu \Psi + g_F \mathcal{H} \Psi = 0. \]  
(9)
We separate the interaction in three parts:
\[ \mathcal{H} = \mathcal{H}_s + \mathcal{H}_o + \mathcal{H}_m, \]  
(10)
which represents the self-interaction \( \mathcal{H}_s \), the interaction with the other spinor \( \mathcal{H}_o \) and the influence of matter \( \mathcal{H}_m \). Thus, the quantities \( C^\mu \) and \( D^\mu \) are separated in three parts, according to their origin in the process of interaction. Let us summarize what was pointed out in the previous paper [2].

4.1. Self-interaction

We have:
\[ \mathcal{H}_s \Psi = (1 - \beta)(A + iB \gamma^5) \Psi \]  
(11)
which implies
\[ C_s^\mu \equiv J^\mu + \frac{1 + \beta}{2} I^\mu \]  
\[ D_s^\mu \equiv \frac{1 + \beta}{2} J^\mu + \beta I^\mu. \]  
(12)
This term, which contains only quantities constructed with the spinor \( \Psi \) itself, is given by the quartic Heisenberg Lagrangian [8], the simplest non-linear covariant term which can be constructed with a spinor field. The Lagrangian is
\[ L_s = \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{i}{2} \partial_\mu \bar{\Psi} \gamma^\mu \Psi - V(\Psi). \]  
(13)
Potential \( V \) is constructed with the two scalars that can be formed with \( \Psi \), which are \( A \) and \( B \). Thus the Heisenberg potential is
\[ V = \frac{1 - \beta}{2} g_F \left( A^2 + B^2 \right). \]  
(14)
Note that Pauli–Kofink identity implies that
\[ A^2 + B^2 = J_\mu J^\mu. \]
For the case \( \beta = 1 \), the self-interacting Heisenberg term vanishes.
4.2. Interaction with the other fundamental spinor $\Upsilon$

We have:

$$H_0 \Psi = \gamma_\mu \left( j^\mu + \frac{(1 + \beta)}{2} i^\mu \right) \Psi + \gamma_\mu \gamma^5 \left( \frac{(1 + \beta)}{2} j^\mu + \beta i^\mu \right) \Psi. \quad (15)$$

The interacting Lagrangian is provided by

$$L_o = g_F \{ J_\mu j^\mu + \beta I_\mu i^\mu \} + \frac{g_F}{2} (1 + \beta) (J^\mu i_\mu + I^\mu j_\mu). \quad (16)$$

In the case $\beta = 1$ the interaction assumes the reduced form

$$L_F = g_F \overline{\Psi} \gamma^\mu (1 + \gamma^5) \gamma_\mu (1 + \gamma^5) \Upsilon.$$

This term is similar to the Lagrangian of weak interaction processes in the ancient Fermi treatment. It appears here as the natural covariant interaction between the two fundamental spinors. The Fermi constant $g_F$ appears for dimensionality reasons. The presence of such constant in the realm of gravitational world may seem very unusual. However, an interesting remark attributed to W Pauli makes this identification less strange. It is generally argued that, as far as gravity is concerned, the quantity $10^{-33}$ cm is an important one. This number appears very naturally by simple dimensional analysis and it is generally associated to the appearance of quantum gravitational processes. Its expression contains three ingredients: the relativistic quantity $c$ (the light velocity), the Heisenberg constant $\hbar$ and a typical gravity representative provided by Newton’s constant $g_N$, yielding the Planck–Newton constant:

$$L_{PN} = \sqrt{\frac{\hbar g_N}{c^3}}.$$

A similar quantity cannot be constructed with the other known long range field (electrodynamics), but it can be defined for the weak interaction. In this case we have only to exchange $g_N$ by the Fermi constant, yielding the definition of what we call the Planck–Fermi length:

$$L_{PF} = \sqrt{\frac{g_F}{\hbar c}}.$$

Now Pauli remarks that this quantity is equal to $10^{-16}$ cm, the square root of the Planck–Newton value. It is clear that such a coincidence depends on the units used. The original argument, which in a sense was re-taken by Dicke in 1957 deals with the so-called ‘natural system of units’ for the high energy physics community, that is for $\hbar = c = 1$ and by taking a specific unit of mass (the electron mass in the Dicke’s choice).

In the case of the interaction of the fundamental spinors, the vectors $C^\mu, D^\mu$ are given by

$$C^\mu_o = j^\mu + \frac{1 + \beta}{2} i^\mu,$$

$$D^\mu_o = \frac{1 + \beta}{2} j^\mu + \beta i^\mu. \quad (17)$$

4 This remark of Pauli was commented to me in a private conversation with professor J M Jauch in the seventies.

5 Actually the argument was given in the system of units in which $\hbar = c = 1$. 

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4.3. The effect of matter in the generation of gravity

This term is provided by (6) inspired by the equivalence principle that states that the matter interacts only through the effective metric $g_{\mu\nu}$. Variation of the spinor $\Psi$ in equation (6) yields

$$\delta S = -\frac{1}{2} \int \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu},$$

$$= -\frac{1}{2} \int \sqrt{-g} T^{\mu\nu} \delta \varphi_{\mu\nu},$$

$$= \frac{g_F m}{2} \int \sqrt{-g} T^{\mu\nu} \delta \left( \frac{\Sigma_{\mu\nu}}{\sqrt{X}} \right),$$

(18)

where we used (3) and (4). Then we can write

$$\delta S = \frac{g_F m}{2} (I_1 + I_2 + I_3),$$

where

$$I_1 = \int \sqrt{-g} T^{\mu\nu} \Sigma_{\mu} \Pi_{\nu} \frac{1}{\sqrt{X}} \delta 1,$$

$$I_2 = \int \sqrt{-g} T^{\mu\nu} \frac{1}{\sqrt{X}} \Sigma_{\mu} \delta \Pi_{\nu},$$

$$I_3 = \int \sqrt{-g} T^{\mu\nu} \frac{1}{\sqrt{X}} \Pi_{\nu} \delta \Sigma_{\mu}.$$

(19)

A direct calculation yields:

$$C_{m}^{\mu} \equiv \frac{g_m}{4} \omega \left( -\frac{1}{2} \Phi \xi^{\mu} + E^{\mu} + H^{\mu} \right),$$

$$D_{m}^{\mu} \equiv \frac{g_m}{4} \omega \left( -\frac{1}{2} \Phi \eta^{\mu} + \beta E^{\mu} + H^{\mu} \right),$$

(20)

where

$$E^{\mu} = \frac{1}{\sqrt{X}} T^{\mu\nu} \Sigma_{\nu},$$

$$H^{\mu} = \frac{1}{\sqrt{X}} T^{\mu\nu} \Pi_{\nu},$$

$$\Phi = \frac{1}{X^{3/2}} T^{\mu\nu} \Sigma_{\mu} \Pi_{\nu},$$

$$\xi^{\mu} = \Pi^{\mu} + \Sigma^{\mu},$$

$$\eta^{\mu} = \Pi^{\mu} + \beta \Sigma^{\mu}.$$
For later use it is useful to separate this matter influence into three parts using the notation of equation (10):
\[ \mathcal{H}_m = \mathcal{T}_s + \mathcal{T}_o + \mathcal{T}_m, \]  
(21)

where
\[ \mathcal{T}_s = -\frac{g_m}{2} \omega \Phi (1 - \beta) (A + i B \gamma^5) \]
(22)
\[ \mathcal{T}_o = -\frac{g_m}{2} \omega \Phi j^\mu \left( \gamma_\mu + \frac{1 + \beta}{2} \gamma_\mu \gamma^5 \right) - \frac{g_m}{2} \omega \Phi i^\mu \left( \frac{1 + \beta}{2} \gamma_\mu + \beta \gamma_\mu \gamma^5 \right) \]
(23)
\[ \mathcal{T}_m = \frac{g_m}{4} \omega \gamma_\mu (E^\mu + H^\mu) + \frac{g_m}{4} \omega \gamma_\mu \gamma^5 (\beta E^\mu + H^\mu). \]
(24)

The origin of these terms is very similar to the other expression. Indeed, \( \mathcal{T}_s \) is proportional to \( \mathcal{H}_s \); the term \( \mathcal{T}_o \) is proportional to \( \mathcal{H}_o \). This suggests treating the third term in such a way that it can be reduced to a combination of both terms. We postpone this analysis to another place. Here we concentrate on a cosmological scenario in which matter is not important for the generation of the gravitational field.

5. Cosmology in the STG

We now present a solution of the spinor theory of gravity which represents an empty spatially homogeneous and isotropic expanding universe. This case shows a net distinction between the properties of the STG and GR. Indeed, in the case of general relativity it is not possible to conciliate an empty universe with a FRW type geometry. On the other hand, we shall show that if we neglect the influence of matter, the spinor theory of gravity allows for a cosmological solution which represents a geometry that is spatially homogeneous and isotropic.

5.1. The Milne expanding universe

In order to find a solution to the STG equations of motion, the first step is to make a convenient choice of the coordinate system that describes the unobserved auxiliary geometry where the spinors live\(^6\). In the case of cosmology, the structure of the effective geometry should have the property of being non-static, spatially homogeneous and isotropic. In order to simplify our calculations it is convenient to choose a coordinate system that represents Minkowski background geometry which already exhibits such

\(^6\) This choice of coordinate system will simplify our calculations. Note however that the induced geometry depends not only the physical information of the gravitational field but also on the parametrical realization of the unobserved background Minkowski geometry. There is a gauge invariance that leaves the physical content of the gravitational field invariant. We will come back to this in a future paper. The interested reader could consult the paper of Grishchuch et al quoted above, once this gauge invariance occurs in a very similar way in the field theoretical formulation of general relativity.
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property. This led us to deal with Milne description of the flat space–time manifold. We set

\[ ds^2 = dt^2 - t^2 \left( d\chi^2 + \sinh^2 \chi \left( d\theta^2 - \sin^2 \theta \, d\varphi^2 \right) \right). \tag{25} \]

In consequence, the \( \gamma^\mu \)'s are given in terms of the constant \( \tilde{\gamma}^\mu \) as follows:

\[
\begin{align*}
\gamma_0 &= \tilde{\gamma}_0 \\
\gamma_1 &= t \tilde{\gamma}_1 \\
\gamma_2 &= t \sinh \chi \tilde{\gamma}_2 \\
\gamma_3 &= t \sinh \chi \sin \theta \tilde{\gamma}_3.
\end{align*}
\]

For later use we display our convention of the constant \( \gamma^\mu \)'s:

\[
\begin{align*}
\tilde{\gamma}^0 &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \\
\tilde{\gamma}^k &= \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \\
\gamma^5 &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.
\end{align*}
\]

This form was obtained by using the property

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu \mathbb{I}. \tag{26} \]

Note that, from now on, we write 1 instead of \( \mathbb{I} \) to represent the identity of the Clifford algebra. Since we are led to use a coordinate system that is not Euclidean, we must use the generalized covariant derivative. This is defined in the standard way \cite{9} as

\[ \nabla_\mu \Psi = \partial_\mu \Psi - i\Gamma_\mu \Psi. \tag{27} \]

In the case of the original Fock–Ivanenko condition (i.e., vanishing of the covariant derivative of the \( \gamma^\mu \)) one obtains:

\[ \Gamma^0_\mu = \frac{1}{8} \left[ \gamma^\alpha \gamma^\mu, \gamma^\alpha - \gamma^\mu, \gamma^\alpha + \Gamma^e_\mu \left( \gamma^e \gamma^n - \gamma^n \gamma^e \right) \right]. \tag{28} \]

The index \( (0) \) in \( \Gamma^0_\mu \) is just a reminder that we are dealing with a Minkowski background in an arbitrary system of coordinates. We can globally annihilate such a connection by moving to a Euclidean constant coordinate system. Using these quantities, we obtain the unique non-identically null background FI connection:

\[
\begin{align*}
\Gamma^{(0)}_1 &= -\frac{1}{4} \tilde{\gamma}_0 \tilde{\gamma}_1 \\
\Gamma^{(0)}_2 &= -\frac{1}{4} \sinh \chi \tilde{\gamma}_0 \tilde{\gamma}_2 + \frac{1}{4} \cosh \chi \tilde{\gamma}_1 \tilde{\gamma}_2 \\
\Gamma^{(0)}_3 &= -\frac{1}{4} \sinh \chi \sin \theta \tilde{\gamma}_0 \tilde{\gamma}_3 + \frac{1}{4} \cosh \chi \sin \theta \tilde{\gamma}_1 \tilde{\gamma}_3 + \frac{1}{4} \cos \theta \tilde{\gamma}_2 \tilde{\gamma}_3.
\end{align*}
\]
5.2. The cosmological metric

To simplify our presentation we will consider the particular case in which $\beta = 1$. The equations of motion in this case are
\[
i \gamma^\mu \partial_\mu \Psi + \gamma^\mu \Gamma_\mu^{(0)} \Psi + g_F \gamma_\mu \left( C^\mu + D^\mu \gamma^5 \right) \Psi = 0,
\]
(29)
where
\[
C^\mu_0 = D^\mu_0 = j^\mu + i^\mu
\]
and an analogous equation for $\Upsilon$.

This is a non-linear system that must be solved in order to obtain the effective metric. We look for a solution of the form
\[
\Psi = f(t) e^{i R(t)} e^{i H(\chi)} e^{i L(\theta)} \Psi^0,
\]
(30)
where $\Psi^0$ and $\Upsilon^0$ are constant spinors. This choice implies immediately that the currents constructed with these spinors may depend only on time. Substituting the above expressions in the fundamental equation we obtain
\[
\mathcal{W} \Psi = 0,
\]
(31)
where we defined
\[
\mathcal{W} = i\bar{\gamma}_0 \left( \frac{1}{f} \frac{df}{dt} + \frac{1}{R} \frac{dR}{dt} \right) + \frac{1}{t} \frac{1}{\gamma_1} \frac{dH}{d\chi} + \frac{1}{t} \frac{1}{\sinh \chi} \frac{dL}{d\theta} + \frac{3}{4t} \frac{1}{\gamma_0} - \frac{1}{2t} \cot(\chi) \frac{1}{\gamma_1} - \frac{1}{4t} \frac{1}{\sinh \chi} \frac{1}{\gamma_2} - g_F \gamma_\mu (1 + \gamma^5) (j^\mu + i^\mu).
\]
(32)
Thus, $R, H$ and $L$ are given by
\[
R = \epsilon \ln \sqrt{t},\quad H = \frac{1}{4} \ln \sin \chi + i\alpha \gamma^0,\quad L = \frac{1}{4} \ln \sin \theta.
\]
(33)
which yield
\[
\Psi = f(t) e^{i \epsilon \ln \sqrt{t}} e^{i(\chi/2) \ln \sin \chi + i\alpha \gamma^0} e^{i(\theta/4) \ln \sin \theta} \Psi^0
\]
(34)
\[
\Upsilon = f'(t) e^{i \epsilon' \ln \sqrt{t}} e^{i(\chi/2) \ln \sin \chi + i\alpha' \gamma^0} e^{i(\theta/4) \ln \sin \theta} \Upsilon^0,
\]
(35)
where $\epsilon, \epsilon', \alpha$ and $\alpha'$ are constants. From this form it follows immediately that the dependence on $\chi$ and $\theta$ disappear in both (vector and axial) currents. The functions $f$ and $f'$ differ by a constant factor and both satisfy the equation
\[
\frac{1}{f^3} \frac{df}{dt} = \mu_0,
\]
(36)
where $\mu_0$ is a constant. Thus, the function $f$ is proportional to $t^{-1/2}$.

Since we are solely interested in the behaviour of the geometry, we note that only function $f$ is of interest. In order to find a solution such that the form of the geometry represents a spatially homogeneous and isotropic geometry all the components of the
currents but $\Sigma_0$ and $\Pi_0$ must be zero. This is achieved by imposing the restrictions $J_k + j_k = 0$ and $I_k + i_k = 0$. The constant spinor $\Psi^0$ (respectively $\Upsilon^0$) satisfies the equation:

$$i\mu_0 \gamma^0 \Psi^0 + \mathcal{H}_0 \Psi^0 = 0,$$

which is the compatibility condition that must be satisfied for the constant spinor $\Psi^0$.

5.3. The fundamental equation for the geometry

From these expressions and using the construction (4) the effective metric representing the gravitational interaction becomes

$$ds^2 = (1 - n^2 f^2) dt^2 - t^2 d\sigma^2.$$  (38)

In order to describe such geometry in the standard Gaussian global time we define the cosmical time $T$ by

$$dT = \sqrt{1 - n^2 f^2} dt.$$  (39)

In this way, the effective geometry takes the conventional FRW form

$$ds^2 = dT^2 - a^2 d\sigma^2,$$  (40)

in which the variable $t$ is now a function of the Gaussian time $T$ and is re-named by the substitution $a \equiv t(T)$. Using the solution of $f$ we can write the implicit dependence of the scale factor on the global time:

$$T = a \sqrt{1 - \frac{Q^2}{a} + \frac{Q^2}{2} \log \mathcal{G}},$$  (41)

where

$$\mathcal{G} \equiv \frac{\sqrt{a} - \sqrt{a - Q^2}}{\sqrt{a} + \sqrt{a - Q^2}}.$$

Before continuing let us comment on the properties of the expanding universe associated with this geometry as opposed to the similar result in general relativity. In GR the equations of motion relate the acceleration of the scale factor to the density and pressure through the Raychaudhuri equation

$$\frac{d^2 a}{dT^2} = -\frac{1}{6} (\rho + 3p).$$

This is precisely the basis of the recent difficulties of explaining the origin of the acceleration of the universe. Let us point out that since the spinor theory of gravity does not have a direct equation of motion relating the metric evolution to the matter sources this kind of difficulty does not appear. Thus one can expect that STG may in principle conciliate the observed acceleration with non-exotic matter. This will be treated in a future paper. We obtain from the above equation the expansion factor or Hubble parameter

$$\frac{da}{dT} = \frac{1}{\sqrt{1 - n^2 f^2}}.$$  (42)
and the acceleration
\[ \frac{\mathrm{d}^2 a}{\mathrm{d}T^2} = \left( \frac{n f^2}{1 - n^2 f^2} \right)^2 \mu_0. \] (43)

Thus the sign of the acceleration factor depends only on the constant \( \mu_0 \). For a real function \( f \) equation (36) implies that \( \mu_0 < 0 \) and in this case the scale factor is such that \( \frac{\mathrm{d}^2 a}{\mathrm{d}T^2} < 0 \). Finally, let us just point out that this solution, like the standard FRW geometry, contains a particle horizon. Indeed, setting the definition:
\[ \mathcal{J} \equiv \int \frac{\mathrm{d}T}{a} \] (44)

it follows
\[ \mathcal{J} = -\log \left( \frac{1 - \cos y}{1 + \cos y} \right) - 2 \cos y \] (45)

for \( y \equiv \arcsin \frac{Q}{\sqrt{a}} \). Thus the integral \( \mathcal{J} \) converges in the domain \( Q^2 < a < a_0 \), for finite \( a_0 \), showing that this geometry has a particle horizon.

6. Conclusion

In the present paper we have examined some properties of a new formalism to describe gravity. We have shown that there is an alternative way to implement the equivalence principle in which the geometry acting on matter is not an independent field, and as such does not possess its own dynamics. Instead, it inherits one from the dynamics of two fundamental spinor fields \( \Psi \) and \( \Upsilon \) which are responsible for the gravitational interaction and through which the effective geometry appears. We have presented a specific model by using Heisenberg equation of motion for the self-interacting spinors. This equation of motion can be understood in terms of a modification of the internal connection as seen by \( \Psi \) and \( \Upsilon \) and only by these two spinors, as it was described in [2]. This dynamics, which involves not only the self-terms but also a specific coupling among these two fields, provides an evolution for the effective metric (constructed in terms of these spinors) which is the way the fields \( \Psi \) and \( \Upsilon \) interact with all other forms of matter and energy. In a previous work we found a solution for the effective metric in the case of a static spherically symmetric configuration. The result is similar as in general relativity, showing the existence of horizon and the possibility of existence of black hole. In the present paper we have turned our analysis to cosmology. We found a solution of the fundamental spinor fields that corresponds to the interaction of \( \Psi \) to \( \Upsilon \) without any extra matter interaction. Such solution represents an empty spatially homogeneous and isotropic expanding universe\(^7\).

\(^7\) After this work was completed, some papers concerning related matters were pointed out to me. Among these we can quote the fusion theory of de Broglie; the non-linear spinor theory of Stumpf et al; the Rishon model of Harari et al and the interesting idea of composite graviton by C Heuson. Let us stress that, although the STG shares some similarities with the theories mentioned above, they differ in some crucial aspects: these theories aims to give dynamical equations for the gravitational field. Nothing similar is done in the present spinor theory of gravity. The gravitational field in STG is just the way fields \( \Psi \) and \( \Upsilon \) couple with matter and consequently there is no such thing ‘a dynamical equation for the gravitational field’. Instead, it comes as a by-product of the dynamical equation of the fundamental spinors. Gravity does not have a dynamics of its own. Gravity is just an effective theory.
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