A class of inexact modified Newton-type iteration methods for solving the generalized absolute value equations

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Abstract: In Wang et al. (J. Optim. Theory Appl., 181: 216–230, 2019), a class of effective modified Newton-type (MN) iteration methods are proposed for solving the generalized absolute value equations (GAVE) and it has been found that the MN iteration method involves the classical Picard iteration method as a special case. In the present paper, it will be claimed that a Douglas-Rachford splitting method for AVE is also a special case of the MN method. In addition, a class of inexact MN (IMN) iteration methods are developed to solve GAVE. Linear convergence of the IMN method is established and some specific sufficient conditions are presented for symmetric positive definite coefficient matrix. Numerical results are given to demonstrate the efficiency of the IMN iteration method.

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1 Introduction

The system of generalized absolute value equations (GAVE) is of finding a vector \( x \in \mathbb{R}^n \) such that

\[
F(x) := Ax - B|x| - b = 0,
\]

where \( A, B \in \mathbb{R}^{n \times n} \) are known matrices, \( b \in \mathbb{R}^n \) is known vector and \( x \in \mathbb{R}^n \) is unknown. Here, \(|x|\) denotes the vector in \( \mathbb{R}^n \) with absolute values of components of \( x \). To our knowledge,
GAVE (1.1) dates back to [31] and is further investigated in [18, 21]. Particularly, if \( B = I \) or \( B \) is nonsingular, GAVE (1.1) can be reduced to the absolute value equations (AVE)

\[
Ax - |x| - b = 0.
\] (1.2)

GAVE (1.1) and AVE (1.2) have been attracted by more and more researchers and the main reason is that many well-known optimization problems such as mixed integer programming problems, linear complementarity problems (LCP), the generalized LCP and the horizontal LCP can be reformulated as GAVE (1.1) or AVE (1.2), see [1, 18, 21, 29] and references therein.

There are two active topics in the study of GAVE (1.1) and AVE (1.2). On one hand, some scholars are interested in theoretical analysis, especially in presenting sufficient conditions or sufficient and necessary conditions to guarantee the existence of the solution for GAVE (1.1) and AVE (1.2), see [18, 21, 24, 29, 35–37] and references therein. In particular, due to the combinatorial character introduced by the absolute value operator, determining the existence of a solution to AVE (1.2) is NP-hard [18]. Moreover, if AVE (1.2) is solvable, checking whether AVE (1.2) has a unique solution or multiple solutions is NP-complete [29]. Throughout this paper, we will assume that the solution set of GAVE (1.1) is nonempty. On the other hand, many scholars devote to designing some efficient iteration methods for solving GAVE (1.1) and AVE (1.2). For example, the exact and inexact Newton-like methods [4, 6, 17, 20, 25], the SOR-like iteration method [5, 10, 15], the generalization of the Gauss-Seidel iteration method [8], the conjugate gradient method [30], the Picard iteration method [32], the CSCS-based iteration method [9] and others; see [1, 14, 19, 22, 23, 26, 27, 34, 38] and references therein.

Our work here is mainly inspired by recent studies on GAVE (1.1) [33] and by the inexact semi-smooth Newton method for AVE (1.2) [6].

Recently, by separating GAVE (1.1) with the sum of a differentiable function and a non-differential function, a class of modified Newton-type (MN) iteration methods are established to solve GAVE (1.1) [33]. Convergence properties of the MN iteration method are analyzed in detail and numerical results are given to illustrate the effectiveness of the MN iteration method. However, each step of the MN iteration method requires the exact solution of a system of linear equations with \( A + \Omega \) (where \( \Omega \) is a given positive semi-definite matrix) being the coefficient matrix, which may be either expensive or impossible, especially for large scale problems. This motivates us to adopt the idea in [6, 7] and develop an inexact MN iteration method by inexacty solving the involved system of linear equations. Global linear convergence of the IMN method is minutely analyzed. In addition, the new scheme involves the well-known Picard iteration method [32], the exact MN iteration method [33] and the Douglas-Rachford splitting method (when \( A \) is positive definite and \( B = I \)) as the special cases.

The rest of this paper is organized as follows. Section 2 introduces IMN iteration method for solving GAVE (1.1) and its convergence properties are explored in detail in Section 3. In Section 4, two numerical examples are given to demonstrate our claims. Finally, some concluding remarks are given in Section 5.

**Notations.** \( \mathbb{R}^{n \times n} \) denotes the set of all \( n \times n \) real matrices and \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \). \( I \) is the identity matrix with suitable dimension. The transposition of a matrix or vector is denoted by \( \cdot^\top \). For a vector \( x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n \), \( x_i \) refers to its \( i \)-th entry, \( |x| \) is in \( \mathbb{R}^n \) with its \( i \)-th entry \( |x_i| \), and \( | \cdot | \) denotes the absolute value for real scalar. For \( x \in \mathbb{R}^n \), \( \|x\| \) denotes its 2-norm and \( \text{diag}(x) \) represents a diagonal matrix with \( x_i \) as its diagonal entries for every \( i = 1, 2, \ldots, n \). For \( M \in \mathbb{R}^{n \times n} \), \( \|M\| \) denotes the spectral norm of \( M \) and is defined by \( \|M\| = \max \{\|Mx\| : x \in \mathbb{R}^n, \|x\| = 1\} \).
2 Inexact modified Newton-type iteration method

In this section, an inexact version of the MN iteration method for solving GAVE (1.1) is developed. Before this, the MN iteration method proposed in [33] is reviewed.

The main idea of the MN iteration method [33] is to separate the originally non-differentiable function $F(x)$ into the sum of a differentiable function $G(x)$ and a non-differentiable (but Lipschitz continuous) function $H(x)$. The similar idea has been used in the literatures [3,11]. Specially, let

$$F(x) = G(x) + H(x) \quad \text{with} \quad G(x) := Ax + \Omega x \quad \text{and} \quad H(x) := -\Omega x - B|x| - b,$$

where $\Omega \in \mathbb{R}^{n \times n}$ is a positive semi-definite matrix. By separating $G(x)$ and $H(x)$, the MN iteration method for solving the GAVE (1.1) is presented (provided that the Jacobian matrix $G'(x) = A + \Omega$ is nonsingular), which is totally described in the following Algorithm 2.1.

**Algorithm 2.1** ([33]). *(The MN iteration method)* Given an initial guess $x^0 \in \mathbb{R}^n$ and a positive semi-definite matrix $\Omega \in \mathbb{R}^{n \times n}$ such that $A + \Omega$ is invertible. For $k = 0, 1, 2, \cdots$ until the generated sequence $\{x^k\}$ is convergent, compute

$$x^{k+1} = (A + \Omega)^{-1}(\Omega x^k + B|x^k| + b). \quad (2.1)$$

**Remark 2.1.** It has been mentioned in [33] that the MN iteration scheme (2.1) will reduce to

$$x^{k+1} = A^{-1}(B|x^k| + b) \quad (2.2)$$

when $\Omega = 0$, which is known as the Picard iteration scheme [32]. In addition, if $A \in \mathbb{R}^{n \times n}$ is a positive definite matrix, $B = I$ and let $\Omega = \left(\frac{2}{\gamma} - 1\right)A$ with $\gamma \in (0,2)$, then the MN iteration scheme (2.1) becomes

$$x^{k+1} = (1 - \frac{1}{2}\gamma)x^k + \frac{1}{2}\gamma A^{-1}(|x^k| + b), \quad (2.3)$$

a special case of the Douglas-Rachford splitting method for solving AVE (1.2). The Douglas-Rachford splitting method has been discussed in detail in our other paper.

The iteration sequence $\{x^k\}$ generated by Algorithm 2.1 has the following general convergence property.

**Theorem 2.1** ([33]). Let the solution set of GAVE (1.1) be nonempty. Assume that $A, B \in \mathbb{R}^{n \times n}$ and $\Omega \in \mathbb{R}^{n \times n}$ is a positive semi-definite matrix such that $A + \Omega$ is nonsingular. If

$$\|(A + \Omega)^{-1}\| < \frac{1}{\|\Omega\| + \|B\|}, \quad (2.4)$$

then the sequence $\{x^k\}$ generated by Algorithm 2.1 converges linearly from any starting point to a solution $x^*$ of GAVE (1.1).

From Algorithm 2.1, each step of the MN iteration method requires the exact solution of the linear system with coefficient matrix $A + \Omega$. However, in many cases, solving a system of linear equations exactly is either expensive or impossible. In order to alleviate the burden of each step for the MN iteration method, a new algorithm adopting approximate solution of the above linear system is much desirable. This motivates us to develop an inexact MN (IMN) iteration method for solving GAVE (1.1), which is described in the following Algorithm 2.2.
Algorithm 2.2. (The IMN iteration method)

Given an initial guess $x^0 \in \mathbb{R}^n$, a residual relative error tolerance $\theta \in [0, 1)$ and a positive semi-definite matrix $\Omega \in \mathbb{R}^{n \times n}$ such that $A + \Omega$ is invertible. For $k = 0, 1, 2, \cdots$ until the iteration sequence $\{x^k\}$ is convergent, find some $x^{k+1}$ which satisfies

$$||(A + \Omega)x^{k+1} - (\Omega x^k + B|x^k| + b)|| \leq \theta\|F(x^k)\|.$$  \hfill (2.5)

Remark 2.2. It is obvious that the IMN iteration method will retrieve the MN iteration method provided that $\theta = 0$. In addition, from Remark 2.1, the IMN iteration method also contains the inexact versions of the Picard iteration method (2.2) and the special Douglas-Rachford splitting method (2.3) as its special cases.

3 Convergence analysis

In this section, the general convergence analysis of the IMN iteration method for solving GAVE (1.1) will be explored. The analysis here is inspired by that of [6].

Before establishing the convergence of the sequence $\{x^k\}$ generated by Algorithm 2.2, a family of mappings are defined according to (2.5) and their properties are studied.

Definition 3.1. For $\theta \in [0, 1)$, $\mathcal{N}_\theta$ is the family of mappings $\mathcal{N}_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$||(A + \Omega)\mathcal{N}_\theta(x) - (\Omega x + B|x| + b)|| \leq \theta \|F(x)\|, \ \forall x \in \mathbb{R}^n.$$  

If $A + \Omega$ is invertible, then the family $\mathcal{N}_0$ only has a single element for all $x \in \mathbb{R}^n$, that is, the exact MN iteration map $N_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$N_0(x) = (A + \Omega)^{-1}(\Omega x + B|x| + b).$$

According to Definition 3.1, $\mathcal{N}_0 \subseteq \mathcal{N}_\theta \subseteq \mathcal{N}_{\theta'}$ if $0 \leq \theta \leq \theta' < 1$. Hence $\mathcal{N}_\theta$ is nonempty for all $\theta \in [0, 1)$. Furthermore, for $x \in \mathbb{R}^n$, any $\theta \in [0, 1)$ and $\mathcal{N}_\theta \in \mathcal{N}_\theta$ we have $N_\theta(x) = x$ if and only if $F(x) = 0$.

From (2.5) and Definition 3.1, the outcome of IMN iteration method is

$$x^{k+1} = \mathcal{N}_\theta(x^k), \quad k = 0, 1, 2, \cdots$$  \hfill (3.1)

with some $\mathcal{N}_\theta \in \mathcal{N}_\theta$ and $\theta \in [0, 1)$. Thus, the following lemma lays the foundation of our convergence analysis.

Lemma 3.1. Assume that $A + \Omega$ is invertible. Let $\theta \in [0, 1)$ and $\mathcal{N}_\theta \in \mathcal{N}_\theta$ defined as in Definition 3.1. If $F(x^*) = 0$, then for every $x \in \mathbb{R}^n$ we have

$$\|N_\theta(x) - x^*\| \leq \|(A + \Omega)^{-1}\| \theta(\|A + \Omega\| + \|B\| + \|\Omega\|) + \|B\| + \|\Omega\|) \|x - x^*\|. \hfill (3.2)$$

Proof. Let $x \in \mathbb{R}^n$. It follows from $F(x^*) = 0$ and some simple algebraic manipulations that

$$N_\theta(x) - x^* = (A + \Omega)^{-1}(\{A + \Omega\}N_\theta(x) - \Omega x - B|x| - b$$

$$+ |F(x^*) - F(x) + (A + \Omega)(x - x^*)|).$$

Then

$$\|N_\theta(x) - x^*\| \leq \|(A + \Omega)^{-1}\|\|\{A + \Omega\}N_\theta(x) - \Omega x - B|x| - b\|$$

$$+ \|F(x^*) - F(x) + (A + \Omega)(x - x^*)\|.$$  \hfill (3.3)
The combination of Definition 3.1 and the inequality (3.3) implies
\[ \|N_0(x) - x^*\| \leq \|(A + \Omega)^{-1}\|\|\theta\|F(x)\| + \|F(x^*) - F(x) - (A + \Omega)(x^* - x)\|. \] (3.4)

On the other hand, note that \( F(x^*) = 0 \) means that
\[ F(x) = (A + \Omega)(x - x^*) - [F(x^*) - F(x) - (A + \Omega)(x^* - x)]. \]

Hence,
\[ \|F(x)\| \leq \|A + \Omega\|\|x - x^*\| + \|F(x^*) - F(x) - (A + \Omega)(x^* - x)\|. \] (3.5)

Furthermore, by some calculations, it holds that
\[ F(x^*) - F(x) - (A + \Omega)(x^* - x) = Ax^* - B|x^*| - b - Ax + B|x| + b - (A + \Omega)x^* + (A + \Omega)x \]
\[ = \Omega(x - x^*). \]

Then it can be concluded that
\[ \|F(x^*) - F(x) - (A + \Omega)(x^* - x)\| \leq (\|B\| + \|\Omega\|)\|x - x^*\|, \] (3.6)
in which \( \|x - x^*\| \leq \|x - x^*\| \) is used.

Combining (3.5) and (3.6), it holds that
\[ \|F(x)\| \leq (\|A + \Omega\| + \|B\| + \|\Omega\|)\|x - x^*\|. \] (3.7)

Substituting the inequalities (3.6) and (3.7) into (3.4), it has
\[ \|N_0(x) - x^*\| \leq \|(A + \Omega)^{-1}\|\|\theta\|\|A + \Omega\| + \|B\| + \|\Omega\|\|x - x^*\| + (\|B\| + \|\Omega\|)\|x - x^*\| \]
\[ = \|(A + \Omega)^{-1}\|\|\theta\|\|A + \Omega\| + \|B\| + \|\Omega\|\|B\| + \|\Omega\|\|x - x^*\|, \]
the assertion then follows immediately.

Now, we turn attention to prove the main results of this section.

**Theorem 3.1.** Assume that the solution set of GAVE (1.1) is nonempty. Let \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, \theta \in [0, 1) \) and \( \Omega \in \mathbb{R}^{n \times n} \) be a positive semi-definite matrix such that \( A + \Omega \) is invertible. Then, the sequence \( \{x^k\} \) generated by the IMN iteration method with any starting point \( x^0 \in \mathbb{R}^n \) is well defined and for \( k = 1, 2, \cdots \), it has
\[ \|x^{k+1} - x^*\| \leq \|(A + \Omega)^{-1}\|\|\theta\|\|A + \Omega\| + \|B\| + \|\Omega\|\|x^k - x^*\|. \] (3.8)

Moreover, if
\[ \|(A + \Omega)^{-1}\| < \frac{1}{\theta\|A + \Omega\| + \|B\| + \|\Omega\| + \|\Omega\|}, \] (3.9)
then \( \{x^k\} \) converges linearly to \( x^* \in \mathbb{R}^n \), a solution of GAVE (1.1).

**Proof.** For any starting point \( x^0 \in \mathbb{R}^n \), by Definition 3.1 and (2.5), the well-definedness of \( \{x^k\} \) follows from invertibility of \( A + \Omega \). Since \( x^* \) is the solution of (1.1), \( F(x^*) = 0 \). Hence, using (3.1) and (3.2), it is immediate to conclude that \( \{x^k\} \) satisfies (3.8) for \( k = 1, 2, \cdots \). On the other hand, (3.9) implies that
\[ \|(A + \Omega)^{-1}\|\|\theta\|\|A + \Omega\| + \|B\| + \|\Omega\| + \|\Omega\| < 1, \]
which combining with (3.8) means that \( \{x^k\} \) converges linearly to \( x^* \).
Remark 3.1. If $\theta = 0$, (3.9) reduces to (2.4), a sufficient convergence condition for the MN iteration method proposed in [33]. However, our proof here is different from that of Theorem 3.1 in [33].

The proof of the following theorem is similar to that of Theorem 3.2 in [33] and thus we omit the detail. Particularly, it reduces to Theorem 3.1 in [33] when $\theta = 0$.

**Theorem 3.2.** Let the solution set of GAVE (1.1) be nonempty, $A \in \mathbb{R}^{n \times n}$ be invertible, $B \in \mathbb{R}^{n \times n}$ and $\Omega \in \mathbb{R}^{n \times n}$ be positive semi-definite such that $A + \Omega$ is invertible. If

$$
\|A^{-1}\| < \frac{1}{2\|\Omega\| + \|B\| + \theta(\|A + \Omega\| + \|B\| + \|\Omega\|)},
$$

then the IMN iteration method converges linearly from any starting point to a solution $x^*$ of GAVE (1.1).

Since GAVE (1.1) reduces to AVE (1.2) by simply letting $B = I$, the IMN iteration method can be directly used to solve AVE (1.2) and the following corollary can be obtained, which reduces to Corollary 3.1 in [33] when $\theta = 0$.

**Corollary 3.1.** Let the solution set of AVE (1.2) be nonempty, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $\theta \in [0, 1)$. Assume that $A + \Omega$ is invertible. Then if

$$
\|(A + \Omega)^{-1}\| < \frac{1}{\theta(\|A + \Omega\| + \|\Omega\| + 1) + \|\Omega\| + 1},
$$

or if $A$ is invertible and

$$
\|A^{-1}\| < \frac{1}{\theta(\|A + \Omega\| + \|\Omega\| + 1) + 2\|\Omega\| + 1} < 1,
$$

the IMN iteration method converges linearly from any starting point to a solution $x^*$ of AVE (1.2).

Finally, if $\Omega = 0$, then the IMN iteration method becomes an inexact version of the Picard iteration method and the following two corollaries are generalizations of Corollary 3.2 and Corollary 3.3 of [33], respectively.

**Corollary 3.2.** Let the solution set of GAVE (1.1) be nonempty, $A \in \mathbb{R}^{n \times n}$ be invertible and $B \in \mathbb{R}^{n \times n}$. If

$$
\|A^{-1}\| < \frac{1}{\|B\| + \theta(\|A\| + \|B\|)},
$$

then the inexact Picard iteration method converges linearly from any starting point to a solution $x^*$ of GAVE (1.1).

**Corollary 3.3.** Let $A \in \mathbb{R}^{n \times n}$ be invertible. If

$$
\|A^{-1}\| < \frac{1}{1 + \theta(\|A\| + 1)},
$$

then the inexact Picard iteration method converges linearly from any starting point to the unique solution $x^*$ of AVE (1.2).

In the following, let $\Omega = \omega I (\omega \geq 0)$ be a nonnegative scalar matrix. Then the corresponding results in subsection 3.2 of [33] can be briefly extended to the inexact circumstance.
Theorem 3.3. Assume that the solution set of GAVE (1.1) is nonempty. Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix, \( \Omega = \omega I (\omega \geq 0) \in \mathbb{R}^{n \times n} \). Let \( \mu_{\min} \) and \( \mu_{\max} \) be the smallest and the largest eigenvalue of the matrix \( A \), respectively, and \( \| B \| = \tau \). If
\[
\tau + \theta(2\omega + \mu_{\max} + \tau) < \mu_{\min},
\]
then the IMN iteration method converges linearly from any starting point to a solution \( x^{*} \) of GAVE (1.1).

In particular, if \( B = I \), the following corollary can be obtained.

Corollary 3.4. Assume that the solution set of AVE (1.2) is nonempty. Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix, \( \Omega = \omega I (\omega \geq 0) \in \mathbb{R}^{n \times n} \). Let \( \mu_{\min} \) and \( \mu_{\max} \) be the smallest and the largest eigenvalue of the matrix \( A \), respectively, and \( B = I \). If
\[
1 + \theta(2\omega + \mu_{\max} + 1) < \mu_{\min},
\]
then the IMN iteration method converges linearly from any starting point to the unique solution \( x^{*} \) of AVE (1.2).

Remark 3.2. Theoretically, it follows from (3.9) that
\[
0 \leq \theta < \frac{1 - \|(A + \Omega)^{-1}\|\|B\| + \|\Omega\|}{\|(A + \Omega)^{-1}\|\|A + \Omega\| + \|B\| + \|\Omega\|} < 1.
\]
However, \( \frac{1 - \|(A + \Omega)^{-1}\|\|B\| + \|\Omega\|}{\|(A + \Omega)^{-1}\|\|A + \Omega\| + \|B\| + \|\Omega\|} \) is generally expensive to compute or hard to estimate. In practice, based on this theoretical guidance, \( \theta = \min \left\{ 0.5, \frac{1}{\max\{1,k-l_{\max}\}} \right\} \) with \( l_{\max} = 10 \) is used in the next section. Here, \( k \) counts the number of outer iteration step.

4 Numerical results

In this section, two numerical examples will be presented to illustrate the efficiency of Algorithm 2.2 for solving GAVE (1.1) and AVE (1.2). Six algorithms will be tested.

1. GN: Mangasarian’s generalized Newton method for AVE (1.2) [20]. Given an initial guess \( x^{0} \in \mathbb{R}^{n} \), for \( k = 0, 1, 2 \cdots \) until the iteration sequence \( \{x^{k}\} \) is convergent, compute
\[
\left[ A - D(x^{k}) \right] x^{k+1} = b, \tag{4.1}
\]
where \( D(x) \doteq \text{diag}(\text{sign}(x)) \) with \( \text{sign}(x) \) denoting a vector with components equal to \(-1, 0 \text{ or } 1\), respectively, depending on whether the corresponding component of the vector \( x \) is negative, zero or positive. For GAVE (1.1), the iteration scheme (4.1) becomes [33]
\[
\left[ A - BD(x^{k}) \right] x^{k+1} = b.
\]
It should be mentioned that a generalized Newton method is proposed for solving GAVE with second order cones in [13].
2. MGN: The modified generalized Newton method for AVE (1.2) [16]. Given an initial guess \( x^0 \in \mathbb{R}^n \), for \( k = 0, 1, 2, \cdots \) until the iteration sequence \( \{x^k\} \) is convergent, compute

\[
\begin{bmatrix} A + I - D(x^k) \end{bmatrix} x^{k+1} = x^k + b. \tag{4.2}
\]

For GAVE (1.1), the iteration scheme (4.2) changes to [33]

\[
\begin{bmatrix} A + I - BD(x^k) \end{bmatrix} x^{k+1} = x^k + b.
\]

3. Picard: The Picard iteration method [32], namely, Algorithm 2.1 with \( \Omega = 0 \).

4. MN: Algorithm 2.1.

5. IGN: The inexact semi-smooth Newton method for AVE (1.2) [6]. Given an initial guess \( x^0 \in \mathbb{R}^n \), for \( k = 0, 1, 2, \cdots \), if \( F(x^k) \neq 0 \), then find some \( x^{k+1} \) such that

\[
\begin{bmatrix} A - D(x^k) \end{bmatrix} x^{k+1} = b + r_k \quad \text{with} \quad \|r_k\| \leq \theta \left\| A x^k - |x^k| - b \right\|. \tag{4.3}
\]

Here, \( \theta \geq 0 \) is the residual relative error tolerance. For GAVE (1.1), (4.3) is slightly modified as

\[
\begin{bmatrix} A - BD(x^k) \end{bmatrix} x^{k+1} = b + r_k \quad \text{with} \quad \|r_k\| \leq \theta \left\| F(x^k) \right\|.
\]

6. IMN: Algorithm 2.2.

At each iteration step of GN, MGN, Picard and MN methods, the main task is to exactly solve a system of linear equations with the coefficient matrix \( A - BD(x^k) \), \( A + I - BD(x^k) \), \( A \) and \( A + \omega I \), respectively. In addition, the coefficient matrices \( A - BD(x^k) \) and \( A + I - BD(x^k) \) may vary with iteration while \( A \) and \( A + \omega I \) (provided that \( \omega \) is given) are fixed. This implies that the factorization of the coefficient matrix can be used in a ‘once-for-all’ manner for Picard and MN methods but not for GN and MGN methods. In the following, these systems of linear equations are solved by the sparse Cholesky factorization in combination with approximate minimum degree (AMD) reordering (if all coefficient matrices are symmetric positive definite) or the sparse LU factorization in combination with column AMD reordering. Specifically, for \( Ax = b \), the MATLAB expression

\[
p = \text{amd}(A);
L = \text{chol}(A(p,p), \text{`lower'});
x = L' \backslash (L \setminus b(p));
x(p) = x;
\]

or

\[
p = \text{colamd}(A);
[L, U] = \text{lu}(A(p,p));
x = U \setminus (L \setminus b(p));
x(p) = x;
\]
is used. At each iteration step of IGN and IMN methods, the approximate solution of a system of linear equations is needed, where \( A - BD(x^k) \) and \( A + \omega I \) is the coefficient matrix, respectively. The CG [12] (if both coefficient matrices are symmetric positive definite) or LSQR [28] is used to approximately solve these systems of linear equations.

The performance of IMN is compared with that of GN, MGN, Picard, MN and IGN in the sense of the number of iteration steps (denoted by “IT”) and the elapsed CPU time in seconds (denoted as “CPU”). For the sake of fairness, the reported CPU time is the mean value of ten tests for each method. All tests are started from the initial vector \( x^0 = [1, 0, 1, 0, \cdots, 1, 0, \cdots]^T \) and terminated if

\[
\text{RES}(x^k) := \frac{||A x^k - B [x^k] - b||_2}{||b||_2} \leq 10^{-7}
\]

or the prescribed maximal iteration number \( k_{\text{max}} = 1000 \) is exceeded (“-” is used in the following tables to illustrate this case). As it is known, for MN and IMN methods, the choice of \( \Omega \) will affect their performances. Hence, how to choose a better \( \Omega \) is an important question and needs further study. As in [33], let \( \Omega = \omega I \) and the experimentally optimal one \( \omega_{\text{exp}} \) (which leads to the smallest iteration step of MN iteration method) is chosen as the parameter \( \omega \). All runs are implemented in MATLAB R2017b with a machine precision \( 2.22 \times 10^{-16} \) on a personal computer with 2.60GHz central processing unit (Intel Core i7), 16GB memory and macOS (High Sierra) operating system.

**Example 4.1 ([33]).** Consider the AVE (1.2) with \( A \) being randomly generated by the following MATLAB procedure:

\[
A = R^\top \ast R + n \ast \text{eye}(n,n) \quad \text{with} \quad R = \text{rand}(n,n)
\]

and the right-hand side is chosen as \( b = (A - \text{eye}(n,n)) \ast \text{ones}(n,1) \). For reproducibility, the random seed is set to 456. Note that the generated dense matrix \( A \) is symmetric positive definite and all singular values of it exceed 1. Thus, AVE (1.2) has a unique solution [21].

Numerical results for Example 4.1 are reported in Table 1. It follows from Table 1 that GN, Picard and MN perform better in most cases in terms of IT. However, IMN is the most effective algorithm among all the tested algorithms in terms of CPU.

**Example 4.2 ([2]).** The second example arises from the following LCP: For a given matrix \( M \in \mathbb{R}^{n \times n} \) and a given vector \( q \in \mathbb{R}^n \), find \( z, w \in \mathbb{R}^n \) such that

\[
z \geq 0, \quad w = M z + q \geq 0 \quad \text{and} \quad z^\top w = 0,
\]

where \( M = \hat{M} + \mu I \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \) is given by \( q = -M z^* \). Here

\[
\hat{M} = \text{tridiag}(-I_m, S_m, -I_m) \in \mathbb{R}^{m^2 \times m^2}, \quad S_m = \text{tridiag}(-1,4,-1) \in \mathbb{R}^{m \times m},
\]

and \( z^* = [1.2, 1.2, \cdots, 1.2]^\top \in \mathbb{R}^n \) with \( n = m^2 \). From [18, 21], LCP (4.4) can be reduced to

\[
(M + I)x - (M - I)|x| = q \quad \text{with} \quad x = \frac{1}{2}((M - I)z + q),
\]

which belongs to GAVE (1.1). In particular, \( A = M + I, B = M - I \) and \( b = q \). In this paper, \( \mu = 4, -1, -4 \) are considered. Based on the construction, the solution set of GAVE (1.1) is nonempty (especially, GAVE (1.1) has a unique solution \( x^* = [-0.6, -0.6, \cdots, -0.6]^\top \) if \( \mu = 4 \)).
Numerical results for Example 4.2 are reported in Tables 2–4. When $\mu = 4$, it can be found from Table 2 that GN needs the smallest iteration step while IMN requires the minimal CPU time. When $\mu = -1$, Table 3 shows that GN needs the smallest iteration step while IGN requires the minimal CPU time. Especially, Picard method does not converge within 1000 iteration steps. Finally, for $\mu = -4$, Table 4 indicates that GN, Picard and IGN do not converge within 1000 iteration steps while MGN needs the smallest iteration step and IMN requires the minimal CPU time. In summary, in most cases, compared with other tested methods, IMN method is the most efficient and reliable one in terms of CPU time for this example.

5 Conclusions

A class of inexact modified Newton-type (IMN) iteration methods are developed for solving the generalized absolute value equations (GAVE) (1.1). IMN iteration method can be regarded as a generalization of the exact MN iteration method proposed in [33]. Linear convergence of IMN iteration method is studied in detail. In addition, some specific sufficient conditions are explored for one special coefficient matrix. Two numerical examples are given to illustrate the superior performance (in terms of CPU time) of IMN iteration method.

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References

[1] L. Abdallah, M. Haddou, T. Migot. Solving absolute value equation using complementarity and smoothing functions, J. Comput. Appl. Math., 327: 196–207, 2018.

[2] Z.-Z Bai. Modulus-based matrix splitting iteration methods for linear complementarity problems, Numer. Linear Algebra., 17(6): 917–933, 2010.

[3] Z.-Z Bai, X. Yang. On HSS-based iteration methods for weakly nonlinear systems, Appl. Numer. Math., 59: 2923–2936, 2009.

[4] L. Caccetta, B. Qu, G.-L. Zhou. A globally and quadratically convergent method for absolute value equations, Comput. Optim. Appl., 48(1): 45–58, 2011.

[5] C.-R. Chen, D.-M. Yu, D.-R. Han. Optimal parameter for the SOR-like iteration method for solving the system of absolute value equations, arXiv preprint, arXiv:2001.05781, 2020. https://arxiv.org/pdf/2001.05781.pdf.

[6] J.Y.B. Cruz, O.P. Ferreira, L.F. Prudente. On the global convergence of the inexact semi-smooth Newton method for absolute value equation, Comput. Optim. Appl., 65(1): 93–108, 2016.
[7] R.S. Dembo, E.T. Steihaug. Inexact Newton Methods, *SIAM J. Numer. Anal.*, 19(2): 400–408, 1982.

[8] V. Edalatpour, D. Hezari, D.K. Salkuyeh. A generalization of the Gauss-Seidel iteration method for solving absolute value equations, *Appl. Math. Comput.*, 293: 156–167, 2017.

[9] X.-M. Gu, T.-Z. Huang, H.-B. Li, S.-F. Wang, L. Li. Two CSCS-based iteration methods for solving absolute value equations, *J. Appl. Anal. Comput.*, 7(4): 1336–1356, 2017.

[10] P. Guo, S.-L Wu, C.-X Li. On the SOR-like iteration method for solving absolute value equations, *Appl. Math. Lett.*, 97: 107–113, 2019.

[11] D.-F. Han. The majorant method and convergence for solving nondifferentiable equations in Banach space, *Appl. Math. Comput.*, 118: 73–82, 2001.

[12] M.R. Hestenes, E. Stiefel. Methods of conjugate gradients for solving linear systems, *J. Res. N.B.S.*, 49: 409–436, 1952.

[13] S.-L Hu, Z.-H. Huang, Q. Zhang. A generalized Newton method for absolute value equations associated with second order cones, *J. Comput. Appl. Math.*, 235(5): 1490–1501, 2012.

[14] Y.-F. Ke. The new iteration algorithm for absolute value equation, *Appl. Math. Lett.*, 99: 10590, 2020.

[15] Y.-F. Ke, C.-F. Ma. SOR-like iteration method for solving absolute value equations, *Appl. Math. Comput.*, 311: 195–202, 2017.

[16] C.-X Li. A Modified Generalized Newton Method for Absolute Value Equations. *J. Optim. Theory Appl.*, 170: 1055–1059, 2016.

[17] Y.-Y. Lian, C.-X. Li, S.-L. Wu. Weaker convergent results of the generalized Newton method for the generalized absolute value equations, *J. Comput. Appl. Math.*, 338: 221–226, 2018.

[18] O.L. Mangasarian. Absolute value programming, *Comput. Optim. Appl.*, 36(1): 43–53, 2007.

[19] O.L. Mangasarian. Absolute value equation solution via concave minimization, *Optim. Lett.*, 1(1): 3–8, 2007.

[20] O.L. Mangasarian. A generalized Newton method for absolute value equations, *Optim. Lett.*, 3(1): 101–108, 2009.

[21] O.L. Mangasarian, R.R. Meyer. Absolute value equations, *Linear Algebra Appl.*, 419(2-3): 359–367, 2006.

[22] A. Mansoori, M. Erfanian. A dynamic model to solve the absolute value equations, *J. Comput. Appl. Math.*, 333: 28–35, 2018.

[23] A. Mansoori, M. Eshaghnezhad, S. Effati. An efficient neural network model for solving the absolute value equations, *IEEE T. Circuits-II*, 65(3): 391–395, 2017.

[24] F. Mezzadri. On the solution of general absolute value equations, *Appl. Math. Lett.*, 107: 106462, 2020.

[25] X.-H. Miao, J.-T. Yang, B. Saheya, J.-S. Chen. A smoothing Newton method for absolute value equation associated with second-order cone, *Appl. Numer. Math.*, 120: 82–96, 2017.

[26] M.A. Noor, J. Iqbal, K.I. Noor, E. Al-Said. On an iterative method for solving absolute value equations, *Optim. Lett.*, 6(5): 1027–1033, 2012.
[27] M.A. Noor, J. Iqbal, E. Al-Said. Residual Iterative Method for Solving Absolute Value Equations, Abstr. Appl. Anal., 2012: 1–9, 2012.

[28] C.C. Paige, M.A. Saunders. LSQR: An algorithm for sparse linear equations and sparse least squares, ACM Trans. Mathe. Softw. (TOMS), 8(1): 43–71, 1982.

[29] O. Prokopyev. On equivalent reformulations for absolute value equations, Comput. Optim. Appl., 44(3): 363–372, 2009.

[30] F. Rahpeymaii, K. Amini, T. Allahviranloo, M. R. Malkhalifeh. A new class of conjugate gradient methods for unconstrained smooth optimization and absolute value equations, Calcolo, 56(1): 1–28, 2019.

[31] J. Rohn. A theorem of the alternatives for the equation $Ax + B|x| = b$, Linear Multilinear Algebra, 52(6): 421–426, 2004.

[32] J. Rohn, V. Hooshyarbakhsh, R. Farhadsefat. An iterative method for solving absolute value equations and sufficient conditions for unique solvability. Optim. Lett. 8: 35–44, 2014.

[33] A. Wang, Y. Cao, J.-X. Chen. Modified Newton-Type Iteration Methods for Generalized Absolute Value Equations, J. Optim. Theory Appl., 181: 216–230, 2019.

[34] H.-J. Wang, D.-X. Cao, H. Liu, L. Qiu. Numerical validation for systems of absolute value equations, Calcolo, 54(3): 669-683, 2017.

[35] S.-L. Wu, C.-X. Li. The unique solvability of the absolute value equation, Appl. Math. Lett., 76: 195–200, 2018.

[36] S.-L. Wu, C.-X. Li. A note on unique solvability of the absolute value equation, Optim. Lett., 14: 1957–1960, 2020.

[37] S.-L. Wu, S.-Q. Shen. On the unique solution of the generalized absolute value equation, arXiv preprint, arXiv:2005.03287v1, 2020. https://arxiv.org/pdf/2005.03287.pdf.

[38] D.-M. Yu, C.-R. Chen, D.-R. Han. A modified fixed point iteration method for solving the system of absolute value equations, Optimization, 2020. https://doi.org/10.1080/02331934.2020.1804568.
Table 1: Numerical results for Example 4.1.

| Method | $\omega_{exp}$ | $n$          | $\times 10^{-15}$ | $\times 10^{-14}$ | $\times 10^{-13}$ | $\times 10^{-12}$ | $\times 10^{-11}$ | $\times 10^{-10}$ |
|--------|----------------|--------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| GN     | 0.8            | 2            | 1.2408            | 1.6640            | 3.3042            | 5.2210            |
|        | 0.3            | 2            | 1.5908            | 1.9868            | 4.4158            | 2.8268            |
| MGN    | 0.1            | 2            | 1.3040            | 3.4857            | 3.4337            | 5.4741            |
|        | 0.3            | 2            | 5.8109            | 1.7240            | 8.8312            | 5.6533            |
| Picard | 0.1124         | 2            | 0.1618            | 3.2161            | 5.0889            |
|        | 0.0889         | 2            | 2.9058            | 7.8486            | 4.4158            | 2.8268            |
| MN     | 0.1226         | 6            | 1.6497            | 3.3300            | 5.2255            |
|        | 5.4741         | 10           | 9.4112            | 8.6334            | 5.7405            | 3.6748            |
| IGN    | 0.01163        | 5            | 0.7140            | 1.2099            | 1.8437            |
|        | 5.4741         | 10           | 7.0875            | 4.3747            | 3.5640            | 2.8626            |
| IMN    | 0.0481         | 6            | 0.0134            | 0.9827            | 0.5428            |
|        | 0.0079         | 10           | 0.0249            | 0.5640            | 0.0889            |

Table 2: Numerical results for Example 4.2 with $\mu = 4$.

| Method | $\omega_{exp}$ | $n$          | $\times 10^{-15}$ | $\times 10^{-14}$ | $\times 10^{-13}$ | $\times 10^{-12}$ | $\times 10^{-11}$ | $\times 10^{-10}$ |
|--------|----------------|--------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| GN     | 2              | 9            | 1.2408            | 1.6640            | 3.3042            | 5.2210            |
|        | 2              | 9            | 1.5908            | 1.9868            | 4.4158            | 2.8268            |
| MGN    | 2              | 9            | 1.3040            | 3.4857            | 3.4337            | 5.4741            |
|        | 2              | 9            | 5.8109            | 1.7240            | 8.8312            | 5.6533            |
| Picard | 9              | 77           | 0.0249            | 0.0722            | 0.0918            |
|        | 9              | 77           | 0.0249            | 0.0722            | 0.0918            |
| MN     | 9              | 12           | 1.2408            | 1.6640            | 3.3042            | 5.2210            |
|        | 12             | 12           | 1.5908            | 1.9868            | 4.4158            | 2.8268            |
| IGN    | 9              | 12           | 1.3040            | 3.4857            | 3.4337            | 5.4741            |
|        | 12             | 12           | 5.8109            | 1.7240            | 8.8312            | 5.6533            |
| IMN    | 9              | 9.1124       | 0.0249            | 0.0722            | 0.0918            |
|        | 9              | 9.1124       | 0.0249            | 0.0722            | 0.0918            |
### Table 3: Numerical results for Example 4.2 with \( \mu = -1 \).

| Method | \( \omega_{\text{exp}} \) | \( n \) | IT | CPU | RES |
|--------|------------------------|-------|-----|-----|-----|
|        |                        | 3600  | 4900 | 6400 | 8100 | 10000 |
| GN     | 1.2                    | 1.2   | 1.2 | 1.2 | 1.2 | 1.2 |
| MGN    | 16                     | 16    | 16  | 16  | 16  | 16  |
| Picard |                       | –     | –   | –   | –   | –   |
| MN     | 19                     | 18    | 20  | 19  | 18  | 18  |
| IGN    | 45                     | 45    | 44  | 44  | 44  | 44  |
| IMN    | 46                     | 47    | 50  | 48  | 50  | 50  |

### Table 4: Numerical results for Example 4.2 with \( \mu = -4 \).

| Method | \( \omega_{\text{exp}} \) | \( n \) | IT | CPU | RES |
|--------|------------------------|-------|-----|-----|-----|
|        |                        | 3600  | 4900 | 6400 | 8100 | 10000 |
| GN     | 4.2                    | 4.2   | 4.2 | 4.2 | 4.2 | 4.2 |
| MGN    | 22                     | 22    | 22  | 23  | 23  | 23  |
| Picard |                       | –     | –   | –   | –   | –   |
| MN     | 42                     | 42    | 42  | 41  | 41  | 41  |
| IGN    | 38                     | 36    | 47  | 42  | 42  | 42  |
| IMN    | 0.0144                 | 0.0199| 0.0277| 0.0287| 0.0379| 7.6480 × 10^{-8} 9.4756 × 10^{-8} 9.7611 × 10^{-8} 9.6702 × 10^{-8} 7.9736 × 10^{-8} |