Curvature perturbations from dimensional decoupling

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Abstract

The scalar modes of the geometry induced by dimensional decoupling are investigated. In the context of the low-energy string effective action, solutions can be found where the spatial part of the background geometry is the direct product of two maximally symmetric Euclidean manifolds whose related scale factors evolve at a dual rate so that the expanding dimensions first accelerate and then decelerate, while the internal dimensions always contract. After introducing the perturbative treatment of the inhomogeneities, a class of five-dimensional geometries is discussed in detail. Quasi-normal modes of the system are derived and the numerical solution for the evolution of the metric inhomogeneities shows that the fluctuations of the internal dimensions provide a term that can be interpreted, in analogy with the well-known four-dimensional situation, as a non-adiabatic pressure density variation. Implications of this result are discussed with particular attention to string cosmological scenarios.

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1. Introduction and motivation(s)

It is not unreasonable to think that internal dimensions of Planckian size can play a role in the early stages of the life of the Universe [1]. The question then is to know how the internal dimensions became so small with respect to the external (expanding) ones. In the last 20 years various sorts of cosmological compactifications have been proposed [2–5] (see also [1] for a review). These mechanisms invoke, in some way or another, a long phase during which the scale factor of the internal manifold shrinks until it stabilizes at the wanted Planckian value. There are at least three independent problems associated with a mechanism of cosmological compactification:

• the nature of the sources driving the contraction of the internal dimensions;
• the nature of the mechanism stabilizing the radius of the internal manifold;
• the properties of the inhomogeneities induced by the process of compactification.
Even higher dimensional Einstein equations allow for vacuum solutions of the Kasner type. These solutions may well describe the contraction of the internal dimensions. The caveat with such a simple realization of cosmological compactification is the possible occurrence of physical singularities both in the external and internal manifolds, as repeatedly noted in connection with string cosmological models [5].

The properties of metric perturbations induced by a phase of compactification have been partially discussed by various authors. In particular, the Bardeen formalism [7] has been generalized to the case of anisotropic manifolds [8, 9] (see also [10]). The evolution equations for the Bardeen potentials have been solved in a number of explicit examples [11, 12]. Canonical normal modes of the fluctuations have been obtained in specific classes of anisotropic manifolds [9, 13].

The aim of the present paper is to argue that the contraction of the internal manifold is not a sufficient condition for the perturbative treatment of the curvature inhomogeneities induced by the process of compactification. In the purely four-dimensional case, curvature fluctuations are conserved over wavelengths larger than the Hubble radius in a number of relevant cases such as the one where a single scalar field determines the dynamics of the background geometry. If internal dimensions are present, their fluctuations will necessarily induce a term analogous to the non-adiabatic pressure density variation. Therefore, curvature fluctuations will no longer be conserved. This occurrence would not be, by itself, problematic. However, a problem may arise if the growth of curvature fluctuations is too sharp when the relevant wavelengths are larger than the Hubble radius.

To discuss this problem we will use, as a toy model, the low-energy string effective action typical of pre-big bang models supplemented by a (non-local) dilaton potential. Within this set-up, it is possible to obtain quite reasonable solutions describing the compactification of the internal dimensions and the simultaneous expansion of the external dimensions [14–17]. An interesting feature of these solutions is that the external dimensions expand in an accelerated fashion and then decelerate (but always expand). The transition from acceleration to deceleration is regular in a technical sense. Finally, these solutions can usefully be expressed in analytical terms and this aspect helps an unambiguous analytical treatment of the evolution of the fluctuations of the geometry.

The present paper is organized as follows. In section 2 the background evolution will be specifically described in the case of multidimensional cosmological models characterized by \(d\) expanding and \(n\) contracting (internal) dimensions. In section 3, the salient features of the evolution of the fluctuations of the geometry will be derived. Owing to the large number of scalar degrees of freedom present in higher dimensional (anisotropic) manifolds, the analysis will then be performed in a five-dimensional geometry. In section 4 the evolution equation of the generalized curvature fluctuations will be presented. Moreover, the evolution equations of the system will be recast in a useful first-order form. In section 5 the issue of quantum-mechanical normalization of the fluctuations will be discussed and quasi-normal modes of the system introduced. Section 6 concerns some numerical discussion while the concluding remarks are presented in section 7. The appendix contains a series of useful technical details on the derivation of the perturbed five-dimensional equations.

### 2. Dimensional decoupling

Multidimensional cosmological models whose line element can be expressed as

\[
\text{d}x^2 = G_{AB} \text{d}x^A \text{d}x^B = \text{d}t^2 - a^2(t)\delta_{ij} \text{d}x^i \text{d}x^j - b^2(t)\delta_{ab} \text{d}y^a \text{d}y^b.
\]  
(2.1)
will be considered in the present section. In equation (2.1) \( G_{AB} \) is the full metric; the indices \( A, B \) run over the full \( D = d + n + 1 \)-dimensional spacetime; the \( d \)-expanding dimensions are typically three and the \( n \)-contracting dimensions are generic, but the case \( n = 1 \) will be of particular interest; for notational convenience, in explicit formulae, the indices \( i, j, k, l, \ldots \) run over the \( d \)-expanding dimensions and the indices \( a, b, c, d, \ldots \) run over the \( n \) internal dimensions.

The background equations stemming from the low-energy string effective action can then be written, in natural string units, as \([5]\)

\[
\begin{align*}
\ddot{\varphi}^2 - dH^2 - nF^2 - V &= 0, \\
H &= H \overline{\varphi}, \\
F &= F \overline{\varphi}, \\
2\ddot{\varphi} - \dot{\varphi}^2 - dH^2 - nF^2 + V - \frac{\partial V}{\partial \varphi} &= 0,
\end{align*}
\]

where \( H = \dot{a}/a \) and \( F = \dot{b}/b \); \( \overline{\varphi} = \varphi - \log \sqrt{-G} \) is the so-called shifted dilaton; the overdot denotes a derivation with respect to the cosmic time coordinate \( t \). The potential \( V \) is only a function of \( \varphi \), i.e. \( V = V(\varphi) \). Equations (2.2)–(2.5) can be derived from the following (generally covariant) action\(^1\)

\[
S = -\frac{1}{2\lambda_{D}^{-1}} \int d^{D-1}x \sqrt{|G|} e^{-\varphi} \left[ R + G^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi + V(\varphi) \right],
\]

where \( V = V(e^{-\varphi}) \),

\[
e^{-\varphi(x)} = \frac{1}{\lambda_{D}^{-1}} \int d^Dw \sqrt{|G(w)|} e^{-\varphi(w)} \sqrt{G^{\alpha\beta} \partial_\alpha \varphi(w) \partial_\beta \varphi(w)} \delta(\varphi(x) - \varphi(w)).
\]

The variation of action (2.6) with respect to \( G_{\mu\nu} \) and \( \varphi \) leads, after a linear combination, to equations (2.2)–(2.5), which are valid in the case of a homogeneous dilaton field. The general \((D\text{-dimensional})\) derivation of equations (2.2)–(2.5) from equation (2.6) has been performed in [18] (see also [19]) and will not be repeated here. The combination of equations (2.2) and (2.5) and of equations (2.3) and (2.4) leads, respectively, to the following pair of equations:

\[
\ddot{\varphi}^2 - \ddot{\varphi} + \frac{1}{2} \frac{\partial V}{\partial \varphi} - V = 0,
\]

\[
(H - F) = (H - F) \overline{\varphi}.
\]

Using equations (2.8) and (2.9) it is rather easy to find solutions of the whole system for various sets of potentials. Consider, in particular, the case of exponential potential \( V(\varphi) = -V_0 e^{\varphi} \). In this case the solution of the system can be expressed as

\[
a(t) = (t + \sqrt{t^2 + t_0^2})^\frac{1}{D-2},
\]

\[
b(t) = (t + \sqrt{t^2 + t_0^2})^{-\frac{1}{D-2}},
\]

\[
\varphi = \varphi_0 - \frac{1}{2} \ln (t^2 + t_0^2).
\]

To satisfy system (2.2)–(2.5), the free parameters of the solution are subjected to the consistency relation \( e^{\varphi_0} t_0 V_0 = 1 \). Concerning the above solution a few comments are in order:

\(^1\) The choice of natural string units implies that \( 2\lambda_{D}^{-1} = 1 \).
the solutions are defined for \( t \) between \(-\infty\) and \(+\infty\);

- the solutions are regular (in a technical sense) since all the curvature invariants are regular for every \( t \);

- the class of solutions expressed by equations (2.10)--(2.12) describes a situation of dimensional decoupling. In fact while \( a(t) \) is always expanding (i.e. \( \dot{a} > 0 \)), \( b(t) \) is always contracting (i.e. \( \dot{b} < 0 \)) at a dual rate (i.e. \( b(t) \sim 1/a(t) \)).

In connection with the last point, it can easily be shown that \( a(t) \) expands in an accelerated way for \( t < 0 \) (i.e. \( \ddot{a} > 0 \)), while the expansion becomes decelerated for \( t > 0 \) (i.e. \( \ddot{a} < 0 \)). In fact, taking the limit of equations (2.10) and (2.11) for \( t \to \pm \infty \), the leading behaviour of the scale factors will be

\[
\begin{align*}
a(t) &\sim (t)^{-\frac{1}{\sqrt{D+n}}}, \quad b(t) \sim (t)^{\frac{1}{\sqrt{D+n}}}, \quad t < 0; \\
a(t) &\sim t^\frac{1}{\sqrt{D+n}}, \quad b(t) \sim t^{-\frac{1}{\sqrt{D+n}}}, \quad t > 0.
\end{align*}
\tag{2.13}
\]

On the basis of equations (2.13), the case of a five-dimensional geometry is particularly interesting since, for \( t \to +\infty \), the scale factor behaves as \( a(t) \sim \sqrt{t} \), i.e. as if the Universe were dominated by radiation.

### 3. Evolution equations of the fluctuations and gauge choices

In higher dimensional manifolds characterized by \( D \) spacetime dimensions, the perturbed metric \( \delta G_{AB} \) has, overall, \( D(D+1)/2 \) degrees of freedom. Since \( D \) gauge conditions can be imposed to fix the origin of the coordinate system, the total number of degrees of freedom becomes, after gauge-fixing, \( D(D-1)/2 \). If \( D = 4 \), it is rather obvious to classify fluctuations depending on their transformation properties under three-dimensional rotations. This is the choice usually employed in the context of the Bardeen formalism. If \( D > 4 \) one can, in principle, classify the metric fluctuations according to their transformation properties under some larger group of symmetries. If the spatial part of the background geometry is decomposed as in equation (2.1), it is practical to classify fluctuations with respect to the rotations in the expanding dimensions. This was at least a possibility studied in [9, 13] (see also [8]). In this approach, as soon as the number of internal dimensions increases, so does the number of scalar modes. For practical reasons, the analysis will then be conducted in the case of a five-dimensional geometry. Extensions of the formalism to \( D > 5 \) have also been proposed [9, 13].

To derive the evolution of the fluctuations of the geometry, it is practical to work directly in the conformal-time parametrization, i.e. \( a(\eta) \, d\eta = dt \). Within this parametrization, the fluctuations of a five-dimensional geometry can be written as

\[
\begin{align*}
\delta G_{00} &= 2a^2\phi, & \delta G_{0i} &= -a^2\partial_i P - a^2Q_i, & \delta G_{0y} &= -abC, \\
\delta G_{ij} &= 2a^2\psi\delta_{ij} - 2a^2\partial_i\partial_j E + a^2h_{ij} + a^2(\partial_i W_j + \partial_j W_i), \\
\delta G_{iy} &= -ab\partial_i D - abH_i, & \delta G_{yy} &= 2b^2\xi,
\end{align*}
\tag{3.1}
\]

where the tensor (i.e. \( h_{ij} \)) and vector (i.e. \( W_i, Q_i \) and \( H_i \)) modes satisfy

\[
\begin{align*}
\partial_j h^j_i &= 0, & h^i_i &= 0, & \partial_i W^j &= 0, & \partial_i Q^j &= 0, & \partial_i H^i &= 0. 
\end{align*}
\tag{3.2}
\]

So, \( h_{ij} \) carries two degrees of freedom; \( W_i, Q_i \) and \( H_i \) carry overall six degrees of freedom; and we also have seven scalars. This implies, as expected, that parametrization (3.1) respects the
total number of degrees of freedom of the perturbed five-dimensional metric. For infinitesimal coordinate transformations

$$x^A \rightarrow \hat{x}^A = x^A + \epsilon^A,$$

(3.3)

with gauge functions

$$\epsilon^A = (\epsilon^0, \epsilon^i, \epsilon^y), \quad \epsilon_A = (a^2\epsilon_0, -a^2\epsilon_i, -b^2\epsilon_y),$$

(3.4)

the tensor modes of the geometry are invariant, while the vectors and scalars transform, respectively, as

$$\hat{Q}_i = Q_i - \zeta'_i, \quad \hat{W}_i = W_i + \zeta_i, \quad \hat{H}_i = H_i - \frac{a}{b} \partial_y \zeta_i,$$

(3.5)

and

$$\hat{\phi} = \phi - \epsilon'_0 - \mathcal{H}\epsilon_0, \quad \hat{P} = P - \epsilon' + \epsilon_0, \quad \hat{C} = C - \frac{b}{a} \epsilon'_i + \frac{a}{b} \partial_y \epsilon_0,$$

(3.6)

where the notation

$$\epsilon_i = \partial_i \epsilon + \zeta_i$$

(3.7)

has been used, so that $\partial_i \zeta^i = 0$. On top of the fluctuations of the geometry, the fluctuations of the sources should also be taken into account: the gauge transformation for the dilaton fluctuation then is

$$\hat{\chi} = \chi - \psi' \epsilon_0,$$

(3.8)

where $\delta \phi = \chi$. For instance, the longitudinal gauge [7] can be generalized to the higher dimensional case. For the applications under study, it is practical to adopt the generalized uniform dilaton gauge stipulating that

$$\hat{W}_i = 0, \quad \hat{D} = 0, \quad \hat{P} = 0, \quad \hat{\chi} = 0,$$

(3.9)

which has already been exploited in various four-dimensional calculations [18, 19]. Note that equation (3.9) corresponds to five independent conditions, since $\hat{W}_i$ is divergenceless. Different gauge choices can be adopted (see, for instance, [9]). However, it has been shown [19] that these gauge choices are less suitable, already in four dimensions, since they lead to divergences in the evolution equations of the fluctuations. The expectation that the (generalized) uniform dilaton gauge leads to non-singular evolution equations will be confirmed by the present analysis.

In the uniform dilaton gauge the perturbed equations can be written in explicit terms (the details of the derivation are reported in the appendix). The perturbed form of the generally covariant equations can be written as

$$\delta \mathcal{G}^B_A + \delta G^{BM} \left[ \partial_M \partial_A \varphi - \Gamma^N_{MA} \partial_N \varphi \right] - \frac{1}{2} \delta \mathcal{G}^{MN} \partial_M \partial_N \varphi = -2 \delta \mathcal{G}^{MN} \left( \partial_M \partial_N \varphi - \Gamma^C_{MN} \partial_C \varphi \right) + 2 \delta \mathcal{G}^{MN} \partial_M \partial_N \varphi,$$

(3.10)

where $\delta \varphi = \varphi$. For instance, the longitudinal gauge [7] can be generalized to the higher dimensional case. For the applications under study, it is practical to adopt the generalized uniform dilaton gauge stipulating that

$$\hat{W}_i = 0, \quad \hat{D} = 0, \quad \hat{P} = 0, \quad \hat{\chi} = 0,$$

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which has already been exploited in various four-dimensional calculations [18, 19]. Note that equation (3.9) corresponds to five independent conditions, since $\hat{W}_i$ is divergenceless. Different gauge choices can be adopted (see, for instance, [9]). However, it has been shown [19] that these gauge choices are less suitable, already in four dimensions, since they lead to divergences in the evolution equations of the fluctuations. The expectation that the (generalized) uniform dilaton gauge leads to non-singular evolution equations will be confirmed by the present analysis.
where $\delta G_A^B$ denotes the fluctuation of the Einstein tensor and where
\[
\gamma_A^B = \delta_A^B - \frac{\delta^A_B \psi \partial_M \psi}{G^{MN} \partial_M \psi \partial_N \psi}.
\]  
(3.11)

Equation (3.10) holds in the case of the scalar modes of the geometry, since terms such as $\delta \gamma^0_0 \propto Q'$, relevant only in the analysis of the vector modes [20], have been omitted. Further details on the derivation of the evolution equations of the first-order fluctuations are reported in the appendix (see, in particular, equations (A.10) and (A.11)).

The components of equation (3.10), computed explicitly in equations (A.11), give the set of perturbed equations for the evolution of the fluctuations. More specifically, from the perturbed (00) and (0i) components of equation (3.10), the Hamiltonian and momentum constraints are, respectively,
\[
\phi [\mathcal{F}^2 - \mathcal{V}^2 + 3 \mathcal{H}^2] + (\mathcal{V}^2 + \mathcal{F}) \mathcal{E}^i + 3 (\mathcal{V}^2 + \mathcal{H}) \psi' + \frac{a}{b} (\mathcal{V}^2 + \mathcal{F}) \partial_i \mathcal{C}
\]
\[
+ 3 \frac{a^2}{b^2} \nabla^2 \psi + \nabla^2 \left[ \mathcal{E}^i - 2 \mathcal{F} - (\mathcal{V}^2 + \mathcal{H}) E' - \frac{a^2}{b^2} \nabla^2 \mathcal{E} \right] = 0.
\]  
(3.12)

and
\[
- \partial_i \left[ 2 \psi' + \xi' + (\mathcal{F} - \mathcal{H}) \mathcal{E} - (\mathcal{V}^2 + \mathcal{H}) \phi + \frac{a}{2b} \partial_i \mathcal{C} \right] = 0.
\]  
(3.13)

In equations (3.12) and (3.13) and in the following equations, $\nabla^2_y$ denotes the Laplacian with respect to the internal coordinate $y$.

From the $(i = j)$ component of equation (3.10) we have
\[
\phi \left[ 2 \mathcal{V}^{\prime} - \mathcal{V}^2 + 2 \mathcal{H} - 5 \mathcal{H}^2 + \mathcal{F}^2 - 4 \mathcal{H} \mathcal{V} - \frac{a^2 \partial V}{2} \right] - 2 \mathcal{V}'' - \xi'' + (\mathcal{V}^2 + \mathcal{H}) \phi' + 2 (\mathcal{V}^2 + \mathcal{H}) \psi' + (\mathcal{V}^2 + 2 \mathcal{H} - \mathcal{F}) \xi'
\]
\[
+ \nabla^2 \left[ \mathcal{E}'' - (\mathcal{V}^2 + \mathcal{H}) E' + \psi + \xi - \phi - \frac{a^2}{b^2} \nabla^2 \mathcal{E} \right]
\]
\[
+ \frac{a^2}{b^2} \nabla^2_y (2 \psi - \phi) - \frac{a}{b} \partial_y [\mathcal{C}' - (\mathcal{V}^2 + \mathcal{H}) \mathcal{C}] = 0.
\]  
(3.14)

Finally the $(i \neq j)$, $(yy)$, $(0y)$ and $(iy)$ components of equation (3.10) are, respectively,
\[
\partial_i \partial_j \left[ \mathcal{E}'' - (\mathcal{V}^2 + \mathcal{H}) E' + \psi + \xi - \phi - \frac{a^2}{b^2} \nabla^2 \mathcal{E} \right] = 0,
\]  
(3.15)

\[
-3 \mathcal{V}'' + (\mathcal{V}^2 + \mathcal{F}) (\phi' + 3 \psi') + \phi \left[ 2 \mathcal{V}'' - \mathcal{V}^2 + 2 \mathcal{F} - 2 (\mathcal{F} + \mathcal{H}) \mathcal{V} - 2 \mathcal{H} \mathcal{F} \right]
\]
\[
- 3 \mathcal{H}^2 - \mathcal{F}^2 - \frac{a^2 \partial V}{2} \left[ 2 \mathcal{V}'' - \mathcal{V}^2 + 2 \mathcal{F} - 2 (\mathcal{H} + \mathcal{F}) \mathcal{V} - 2 \mathcal{H} \mathcal{F} \right]
\]
\[
+ \frac{3}{b^2} \delta^2 C - \frac{a^2}{b^2} \partial^3 [3 \psi' - (\mathcal{V}^2 + \mathcal{F}) \phi + 3 (\mathcal{H} - \mathcal{F}) \psi - \nabla^2 \left[ E' + (\mathcal{H} - \mathcal{F}) E \right] = 0,
\]  
(3.16)

\[
\frac{a}{2b} \partial^3 C - \frac{a^2}{b^2} \partial^5 [2 \psi - \phi] = 0.
\]  
(3.17)

Concerning the above system of equations, the following remarks are in order:

- if the gradients with respect to $y$ are neglected, the evolution of $C$ decouples;
- not all the equations are independent;
- the equations for the zero modes with respect to $y$ have a slightly more tractable form.
Let us now prove, in detail, each of the above statements. Since equations (3.17) and (3.18) imply, respectively, $\nabla^2 C = 0$ and $C' = (\phi' + 2(H - F))C$, indeed the evolution of $C$ decouples.

Concerning the second remark, from equation (3.13) the variable $\phi$ can be expressed as

$$\phi = \frac{2\psi' + \xi' + (F - H)\xi}{(\phi' + H)} + a \frac{\partial_y C}{2b (\phi' + H)}.$$  \hspace{1cm} (3.19)

Furthermore, from equation (3.15), the variable $E''$ can be expressed as

$$E'' = (\phi' + H)E' + \frac{a^2}{b^2} \nabla^2 E + \frac{2\psi' + \xi' + (F - H)\xi}{(\phi' + H)} + a \frac{\partial_y C}{2b (\phi' + H)} - \psi - \xi,$$  \hspace{1cm} (3.20)

where the dependence upon $\phi$ has been eliminated through equation (3.19).

Equations (3.19) and (3.20) can be used to eliminate $\phi$ and $E''$ in equation (3.14). The remaining dependence on $C$ can be simplified by means of equation (3.18). Then, repeated use of the background equations implies that equation (3.14) is identically satisfied.

It is appropriate to recall here a useful form of the five-dimensional background equations in the conformal-time parametrization, namely

$$\bar{\phi}'' = \bar{\phi}'(\phi' + H) + f(V),$$  \hspace{1cm} (3.21)

$$H' - F' = (H - F)(\phi' + H),$$  \hspace{1cm} (3.22)

$$\bar{\phi}^2 - 3H^2 - F^2 = Va^2,$$  \hspace{1cm} (3.23)

where

$$f(V) = \frac{1}{2} \left( \frac{\partial V}{\partial \phi} - 2V \right) a^2$$  \hspace{1cm} (3.24)

depends on the specific form of the potential. Equations (3.21)-(3.23) can easily be obtained from equations (2.2)–(2.5) and by taking a combination of equations (2.2) and (2.5). Note, finally, that in the case of the potential leading to the class of solutions described in equations (2.10)–(2.12), $f(V) = Va^2$.

Using equations (3.21)–(3.23) and neglecting the gradients with respect to the internal coordinate, the system of perturbed equations can be reduced to a more tractable form. In particular, equations (3.12) and (3.13) lead, respectively, to

$$\nabla^2 (\xi + 2\psi) - (\phi' + H) \nabla^2 E' - Va^2 \phi + (\phi' + F)\xi' + 3(\phi' + H)\psi' = 0,$$  \hspace{1cm} (3.25)

and

$$2\psi' + \xi' + (F - H)\xi - (\phi' + H)\phi = 0,$$  \hspace{1cm} (3.26)

while equation (3.14) implies

$$-2\psi'' - \xi'' + (\phi' + H)\phi' + 2(\phi' + H)\psi' + (\phi' + 2H - F)\xi' + f(V)\phi = 0.$$  \hspace{1cm} (3.27)

Finally, equations (3.15) and (3.16) become, respectively:

$$E'' - (\phi' + H)E' + \psi + \xi - \phi = 0,$$  \hspace{1cm} (3.28)

$$-3\psi'' + (\phi' + F)(\phi' + 3\psi') - f(V)\phi + \nabla^2 (\psi - \xi) - (F - H) \nabla^2 E' = 0.$$  \hspace{1cm} (3.29)
4. Generalized curvature fluctuations

For the purposes of the subsequent investigation, it is practical to define the variable

\[ \lambda = \psi + \frac{\xi}{2}. \]  \hspace{1cm} (4.1)

In terms of \( \lambda \), equation (3.19) becomes

\[ \phi = \frac{2\lambda'}{(\varphi' + H)} + \frac{F - H}{\varphi' + H} \xi. \]  \hspace{1cm} (4.2)

Inserting equation (4.2) into equation (3.25) and using equations (3.21)–(3.23) the generalized evolution equation for the curvature perturbations can be obtained:

\[ \lambda' = \frac{\varphi'}{(\varphi + 3H) + 2F^2} \left[ \frac{(F - H)(3H^2 + F^2 - \varphi^2)}{(\varphi + 3H)^2 + 2F^2} \xi \right. \]
\[ \left. - \frac{2(\varphi' + H)}{(\varphi + 3H)^2 + 2F^2} \nabla^2 \lambda + \frac{2(\varphi' + H + F)}{(\varphi + 3H)^2 + 2F^2} \nabla^2 E' \right]. \]  \hspace{1cm} (4.3)

Clearly, in the limit \( F \to 0 \) and \( \xi \to 0 \) (i.e. \( \lambda \to \psi \)), equation (4.3) reduces to

\[ \psi' = -\frac{2(\varphi' + H + F)}{(\varphi + 3H)^2 + 2F^2} \nabla^2 \psi + \frac{(\varphi' + H)^2}{(\varphi + 3H)^2 + 2F^2} \nabla^2 E', \]  \hspace{1cm} (4.4)

which is the typical form for the evolution of curvature perturbations in the uniform dilaton gauge [19]. Equation (4.4) suggests that if \( \varphi' = (\varphi' + 3H) \neq 0 \) (as in the class of models discussed in [19]), then \( \psi' \approx 0 \) for long wavelengths larger than the Hubble radius. If internal dimensions are dynamical, equation (4.3) generalizes the evolution of the curvature perturbations.

Using definition (4.1) and inserting equation (4.2) into equation (3.28), equation (4.3) allows us to eliminate \( \lambda' \), and the resulting equation will be

\[ E'' = \left[ \frac{2(\varphi' + H)}{(\varphi + 3H)^2 + 2F^2} \nabla^2 E' \right] - \left[ \frac{\lambda}{(\varphi + 3H)^2 + 2F^2} + \frac{4}{\varphi + 3H} \right] \nabla^2 \lambda \]
\[ + \frac{(\varphi' + 3H - 2F)}{(\varphi + 3H)^2 + 2F^2} \xi' - \frac{2(\varphi' + 3H + 2F^2)}{\varphi + 3H + 2F^2} \left[ \frac{\varphi'' + 15H^2 + 12H^2 + 2F^2 - 6H' \varphi - 6H'F}{2(\varphi + 3H)^2 + 2F^2} \right] \xi. \]  \hspace{1cm} (4.5)

Since the background evolution has a simple analytical form in \( t \) (but not in \( \eta \)), equations (4.3) and (4.5) can be written, for future simplicity, in the cosmic time coordinate and in terms of the auxiliary variables

\[ E = a^2 \dot{E}, \quad \mathcal{P} = \dot{\xi}, \]  \hspace{1cm} (4.6)

whose convenience will become clear in the context of the numerical analysis. The result of this simple algebra will be, in Fourier space:

\[ \dot{\lambda}_k = \frac{(\varphi + H)(\varphi + 3H - 2F)}{2(\varphi + 3H)^2 + 2F^2} \dot{P}_k - \frac{(F - H)(3H^2 + F^2 - \varphi^2)}{(\varphi + 3H)^2 + 2F^2} \xi_k \]
\[ - \frac{(\varphi + H)^2}{(\varphi + 3H)^2 + 2F^2} \alpha^2 \mathcal{E}_k + \frac{2(\varphi + H)}{(\varphi + 3H)^2 + 2F^2} \alpha^2 \lambda_k. \]  \hspace{1cm} (4.7)
where the physical frequency of the fluctuation, i.e. $\omega = k/a$ has been introduced. With similar manipulations, the evolution equation for $\dot{E}$ will be, instead

$$
\dot{E}_k = \left[ \ddot{\xi} + 2H - \frac{2\omega^2(\ddot{\xi} + H)}{(\dddot{\xi} + 3H)^2 + 2F^2} \right] \dot{E}_k + \left[ \frac{\ddot{\xi} + 3H - 2F}{(\dddot{\xi} + 3H)^2 + 2F^2} \right] P_k
$$

$$
- \frac{\ddot{\xi}^2 + H(15H + 12\ddot{\xi} + 2F^2 - 6F(\dddot{\xi} + H))}{2(\dddot{\xi} + 3H)^2 + 2F^2} \dot{\xi}_k - \left[ 1 - \frac{4\omega^2}{(\dddot{\xi} + 3H)^2 + 2F^2} \right] \lambda_k.
$$

(4.8)

Equations (4.7) and (4.8) define a first-order linear differential system (with time-dependent coefficients) in the variables, $E_k, \lambda_k, P_k$ and $\xi_k$. Having already specified the equation for $\dot{\xi}_k$, i.e.

$$
\dot{\xi}_k = P_k,
$$

(4.9)

we just need a further relation determining $\dot{P}_k$. After some algebra (which will be swiftly mentioned in the following section, in connection with the problem of the canonical normal modes) the wanted evolution equation turns out to be

$$
\dot{P}_k = \left[ \frac{\ddot{\xi} + 2V}{(\dddot{\xi} + 3H + F)^2((\dddot{\xi} + 3H)^2 + 2F^2)} \right] P_k
$$

$$
- 4V\omega^2 \frac{(\dddot{\xi} + 3H)(\ddot{\xi} + 3H - 2F)}{(\dddot{\xi} + 3H + F)^2[(\dddot{\xi} + 3H)^2 + 2F^2]} \lambda_k
$$

$$
+ \left[ 6V \frac{(F - H)(\dddot{\xi} + 3H - 2F)(3H^2 + F^2 + H \dddot{\xi})}{(\dddot{\xi} + 3H + F)^2[(\dddot{\xi} + 3H)^2 + 2F^2]} + \omega^2 \right] \xi_k
$$

$$
- 4V \frac{\ddot{\xi} + 3H - 2F}{(\dddot{\xi} + 3H + F)^2[(\dddot{\xi} + 3H)^2 + 2F^2]} \omega^2 \xi_k,
$$

(4.10)

where the explicit form of the potential mentioned before equation (2.10) has been used.

Concerning equations (4.7), (4.8) and (4.10), it is appropriate to recall that the time-dependent coefficients are always regular, for any finite value of the cosmic-time coordinate as it can be argued by noting that, in the denominators, the only combinations that appear are

$$(\dddot{\xi} + 3H + F), \quad (\dddot{\xi} + 3H)^2 + 2F^2.$$  

(4.11)

It is clear that the second expression in equation (4.11) never goes to zero since it is the sum of two positive-definite quantities. Moreover, the first quantity, corresponding to $\dddot{\xi}$, is also positive-definite for the class of backgrounds obtained by setting $d = 3$ and $n = 1$ in equations (2.10) and (2.11).

Equations (4.7), (4.8) and (4.10) can then numerically be integrated once the initial conditions for the fluctuations are set in the limit $t \to -\infty$. Consequently, it is mandatory to determine which are the correct canonical normal modes that have to be used to enforce, for instance, quantum-mechanical initial conditions for the fluctuations.

5. Quasi-normal modes

The fluctuations of the geometry have to be normalized in the limit $\eta \to -\infty$ (or $t \to -\infty$) and, for this purpose, the canonical normal modes have to be determined. In the system under
consideration, when \( t \to -\infty \) the potential term becomes negligible with respect to \( \dot{\phi}^2 \), i.e. \( V \ll \dot{\phi}^2 \). In this situation, two normal modes of the system can be identified. They will be called quasi-normal modes since, for \( t \sim t_0 \) (when the potential dominates) the two modes mix non-trivially.

As far as the evolution of the system is concerned, it is necessary to discuss directly the system derived in equations (4.7), (4.8) and (4.10). The reason why this choice is not arbitrary stems from the particular form of the equations characterizing the normal modes of the system. These equations contain time-dependent coefficients whose denominators include powers of either \((\dot{\phi} + H)\) or \((\dot{\phi} + F)\). Now, these terms lead, generically, to poles in the time-dependent coefficients of the system. For instance, in the specific case of the class of solutions (2.10)–(2.12), the zeros of \((\dot{\phi} + H)\) and \((\dot{\phi} + F)\) are, respectively, in \((d+n-1) \frac{1}{2}\) and \(- (d+n-1) \frac{1}{2}\). For the specific case of the numerical analysis to be presented in the following section, this would imply poles (of various degree) for either \(1/\sqrt{3}\) or \(-1/\sqrt{3}\).

To prove the statements of the previous two paragraphs, let us derive the evolution equations for the normal modes of the system, first by looking at the evolution equations and then by looking directly at the action perturbed to second order in the amplitude of the fluctuations. Summing up equations (3.25) and (3.27) and eliminating \( \phi \) by means of equation (3.28), the following equation can be derived:

\[
\nabla^2 \lambda - \lambda'' + 3(\gamma' + H)\lambda' + g(V)\phi = - \frac{\gamma' + H}{2} [E''' - (\gamma' + H)E'' - (\gamma'' + H')E' - \nabla^2 E'],
\]

where equation (4.1) has been used and where

\[
g(V) = \frac{1}{4} \left( \frac{\delta V}{\delta \phi} - 4V \right)
\]

is a new function that depends on the specific form of the potential. Note that \(g(V)\) vanishes for the class of solutions described in section 2. Taking the difference between equations (3.29) and (3.27), the resulting equation is, in some way, similar to equation (5.1):

\[
(\xi'' - \psi'') - \nabla^2 (\xi - \psi) + \psi'(\gamma' + 4F - 3H) - \xi'(\gamma' + 3H - 2F) = -(F + H)[E''' - (\gamma' + H)E'' - (\gamma'' + H')E' - \nabla^2 E'].
\]

Hence, combining equations (5.1) and (5.3) to get rid of the terms containing the derivatives of \( E \), the following simple relation appears

\[
\nabla^2 \xi - \xi'' + (\gamma' + H)\xi' + \frac{4}{3} \frac{(F - H)}{(\gamma' + H)} g(V)\phi = \frac{2}{3} \left( \frac{\gamma' + 3H - 2F}{\gamma' + H} \right) [\nabla^2 \lambda - \lambda'' + (\gamma' + H)\lambda'].
\]

This equation can be used either to eliminate the Laplacian of \( \xi \) in favour of a Laplacian of \( \lambda \) or vice versa.

To get a quasi-decoupled equation for \( \lambda \), the first step is to take the conformal time derivative of equation (4.3). Terms such as \( \nabla^2 E'' \) will naturally appear, and their presence can be eliminated by means of equation (3.20) or (4.5). As a result of this manipulation, terms proportional to \( \nabla^2 \xi \) and \( \nabla^2 E' \) will arise. They can both be eliminated using equations (5.4)
and (4.3). The final result for the evolution equation of $\lambda$ then reads

$$
\lambda'' - \left[ 1 + 3 \frac{a^2}{\partial V \frac{1}{\partial \varphi}} \frac{1}{(\varphi')^2} \left( \frac{\partial V}{\partial \varphi} + \mathcal{H} \right) + \frac{6 f(V) (3 \mathcal{H}^2 + F^2 \varphi')}{(\varphi' + \mathcal{H}) (\varphi' + 3 \mathcal{H} + F)^2} \right] \nabla^2 - \lambda' \left[ \left( \frac{\partial V}{\partial \varphi} + \mathcal{H} \right) + \frac{6 f(V) (3 \mathcal{H}^2 + F^2 \varphi')}{(\varphi' + \mathcal{H}) (\varphi' + 3 \mathcal{H} + F)^2} \right]
$$

$$
= \frac{\left( \mathcal{F} - \mathcal{H} \right) (\varphi' + 3 \mathcal{H} - 2 \mathcal{F})}{(\varphi' + 3 \mathcal{H} + F)^2} g(V) \psi + 3 \frac{\mathcal{F} - \mathcal{H}}{2 (\varphi' + 3 \mathcal{H} + F)^2} a^2 \frac{\partial V}{\partial \varphi} \xi'
$$

$$
+ 3 \frac{\left( \mathcal{F} - \mathcal{H} \right) (3 \mathcal{H}^2 + F^2 + \mathcal{H} \varphi')}{(\varphi' + \mathcal{H}) (\varphi' + 3 \mathcal{H} + F)^2} f(V) \xi.
$$

As previously remarked, in the limit $\eta \to -\infty$ the potential terms are negligible with respect to the other terms of comparable dimensions such as $\varphi^2$, $\mathcal{H}^2$, $F^2$ and their mutual combinations. Consequently, the evolution equation for $\lambda$ becomes, in this limit

$$
\lambda'' + 2 z_s' \lambda' - \nabla^2 \lambda = 0,
$$

where

$$
z_s = -2 \sqrt{\frac{2}{3}} \sqrt{\frac{e^{-\mathcal{H}/2} \varphi' + 3 \mathcal{H} + F}{\varphi' + \mathcal{H}}}.
$$

Note that, as anticipated, if we ought to use equation (5.5) for explicit numerical integration over the whole range of $\eta$, poles arise for $(\varphi' + \mathcal{H})$. This confirms that second-order equations such as (5.6) are mandatory for the determination of the initial conditions, but not appropriate for the description of the evolution of the fluctuations.

The same type of algebra leading to equation (5.6) can be applied to find the second (asymptotic) normal mode of the system. Indeed, the variable

$$
\theta = \left( \frac{\varphi' + \mathcal{F}}{\varphi' + 3 \mathcal{H} + F} \right) \xi = \left( \frac{\varphi' + 3 \mathcal{H} - 2 \mathcal{F}}{\varphi' + 3 \mathcal{H} + F} \right) \psi,
$$

also obeys the following equation

$$
\theta'' - (\varphi' + \mathcal{H}) \theta' - \nabla^2 \theta = \frac{2 V a^2}{(\varphi' + \mathcal{H}) (\varphi' + 3 \mathcal{H} + F)} \left[ 2 (\mathcal{F} - \mathcal{H}) \lambda' - (2 \mathcal{H} + F) \theta' \right]
$$

$$
- \frac{V a^2 (2 \mathcal{H} + F)}{(\varphi' + 3 \mathcal{H} + F) (\varphi' + \mathcal{H})^2} \left[ \varphi'^2 - \mathcal{H}^2 - 2 V a^2 \right] \theta,
$$

whose specific form becomes, in the limit $\eta \to -\infty$,

$$
\theta'' + 2 z_s' \theta' - \nabla^2 \theta = 0.
$$

Hence the canonical normal modes are

$$
v = z_s \lambda, \quad w = z_s \theta,
$$

and their Lagrangian density can be written as

$$
L^{(2)} = \frac{1}{2} \left[ v^2 + \frac{z_s''}{z_s} v^2 - (\partial_i v)^2 + w^2 + \frac{z_s''}{z_s} w^2 - (\partial_i w)^2 \right].
$$
It is relevant to stress that equation (5.12) holds, strictly speaking, only in the limit of vanishing potential (i.e. \( \eta \to -\infty \)) and agrees with the Lagrangian for the normal modes of a dimensionally reduced geometry discussed in [13]. As a side remark, we should conclude the present section by noting that, inserting equation (5.5) into equation (5.4), the equation of a dimensionally reduced geometry discussed in [13].

\[
\xi'' = \left\{ \frac{(\varphi' + \chi) + 2 Va^2}{(\varphi' + 3 \varphi + 2F)} \right\} \xi' \\
+ 4Va^2 \frac{(\varphi' + 3 \varphi + 2F)}{(\varphi' + 3 \varphi + 2F)^2} \nabla^2 \lambda \\
+ 6Va^2 \frac{(F - \varphi)(\varphi' + 3 \varphi + 2F)(3 \varphi^2 + 2F + \varphi' \varphi)}{(\varphi' + 3 \varphi + 2F)^2} \xi - \nabla^2 \xi \\
+ 4Va^2 \frac{(\varphi' + 3 \varphi - 2F)(3 \varphi^2 + 2F + \varphi' \varphi)}{(\varphi' + 3 \varphi + 2F)^2} \nabla^2 E' 
\]  
(5.13)

can be obtained. This is the equation already anticipated in section 4 (see equation (4.10)).

6. Numerical analysis

To solve the evolution numerically, it is essential to have a set of evolution equations whose time-dependent coefficients are all non-singular. The relevant set of equations follows from equations (4.7), (4.8) and (4.10), recalling the remark related to equation (4.11).

To solve this system numerically, quantum-mechanical initial conditions have to be imposed for \( t \to -\infty \). According to the results of the previous section, the Fourier modes of \( \theta \) and \( \lambda \) will be normalized as

\[
\theta_k = \frac{1}{\sqrt{2\omega(\sqrt{a}_z)}} e^{-i\omega(\sqrt{a}_z)\delta}, \quad \lambda_k = \frac{1}{\sqrt{2\omega(\sqrt{a}_z)}} e^{-i\omega(\sqrt{a}_z)\delta}. \]  
(6.1)

From equation (6.1), the asymptotic expressions of \( \theta_k \) and \( \lambda_k \) can be deduced by taking the derivative of the two expressions with respect to \( t \). Having determined the values of \( \theta_k, \lambda_k \) and of their derivatives it is possible to set the initial conditions for the other variables. The function \( \lambda_k \) directly appears in the system. Recalling the explicit expressions of \( \lambda_k \) and \( \theta_k \) in terms of \( \xi_k \) and \( \psi_k \), i.e. equations (4.1) and (5.8), the initial conditions for \( \xi_k \) and \( \varphi_k \) can easily be obtained:

\[
\xi_k = 2 \left[ \frac{(\varphi + 3H + F)}{\varphi + H} \right] \theta_k + \left( \frac{\varphi + 3H - 2F}{\varphi + H} \right) \lambda_k, \\
\varphi_k = 2 \left[ \frac{(\varphi + 3H + F)}{\varphi + H} \right] \theta_k + \left( \frac{\varphi + 3H - 2F}{\varphi + H} \right) \lambda_k + \frac{2}{3} \frac{V(2(F - H)\lambda_k - (2H + F)\theta_k)}{(\varphi + H)^2}. \]  
(6.2)

The last quantity to be determined is the initial condition for \( \psi_k \), whose expression, in terms of the initial conditions of the other variables, is

\[
\psi_k = \frac{(\varphi + 3H - 2F)}{2\omega^2(\varphi + H)} \varphi_k - \frac{(F - H)(3H^2 + F^2 - \varphi^2)}{\omega^2(\varphi + H)^2} \xi_k - \frac{(\varphi + 3H)^2 + 2F^2}{\omega^2(\varphi + H)^2} \lambda_k + \frac{2}{(\varphi + H)} \lambda_k. \]  
(6.3)
The initial time of integration, $t_i$, is fixed in such a way that the given physical frequency is larger than the Hubble rate, i.e. the corresponding wavelength is smaller than the Hubble radius. In the case of the class of solutions introduced in equations (2.10)–(2.12) this requirement, implies, for a given Fourier mode $k$, $t_i < -(2\kappa)^{-2/3} t_0$ where $\kappa = k t_0$. Thus, as $\kappa$ becomes smaller, the initial integration time becomes larger in absolute value.

Consider now the five-dimensional background geometry obtained from equations (2.10)–(2.12) by setting $d = 3$ and $n = 1$. In this case equations (4.7), (4.8) and (4.10) can be integrated by setting quantum-mechanical initial conditions as discussed in the previous paragraphs of the present section. The result of the numerical integration is illustrated in figure 1, where the logarithm (in base ten) of the spectrum of $\lambda$, i.e.

$$\delta_\lambda = k^{3/2} |\lambda_k|^2,$$

is reported. In analogy with equation (6.4) we will also define, in similar terms, the spectrum of $\xi$, i.e. $\delta_\xi$ and so on.

It is clear from figure 1 as well as from the equivalent plots obtained for $\delta_\xi$ and $\delta_\xi$ that, in spite of the initial conditions, for $t > 0$ the following approximate relation holds

$$\delta_\lambda(t) \sim \delta_\xi(t) \sim \delta_\xi(t) \sim A(k) e^{\alpha(t/t_0)},$$

where $\alpha$ is a constant and it is numerically determined to be 4.9; the symbol of similarity in equation (6.5) means that the relation holds up to constant factors of order 1. This occurrence signals that there is a kind of late-time attractor in equations (4.7), (4.8) and (4.10). According to equations (6.5), the plots for the other power spectra will be quantitatively and qualitatively similar to that of $\lambda$. The sharp (exponential) amplification of the fluctuations is rather dangerous for the validity of the perturbative expansion, as can be argued from figure 1.

To gain analytical understanding of the behaviour reported in figure 1, let us write the system provided by equations (4.7), (4.8) and (4.10) in the limit of large (positive) times, i.e.
\( t \gg t_0 \). Denoting \( \tau = t/t_0 \) and \( \kappa = k t_0 \), the system becomes

\[
\dot{\lambda}_k \approx \left( \frac{2}{3\tau^2} - \frac{1}{2} \right) \mathcal{P}_k + \left( \frac{4}{3\tau^3} - \frac{26}{9\tau^5} \right) \dot{\xi}_k + \left( \frac{17}{8\tau^2} - \frac{2}{3\kappa^2} \right) \lambda_k + \left( \frac{35}{72\tau^3} - \frac{\kappa^2}{6\tau} \right) \xi_k,
\]

\[
\dot{\xi}_k \approx \left( \frac{2}{3\tau^2} - \frac{17}{18\tau^2} \right) \mathcal{E}_k + \left( \frac{2}{3} - \frac{10}{3\tau^2} \right) \dot{\lambda}_k + \left( \frac{8}{3\tau^2} - \frac{2\kappa^2}{9\tau} - 1 \right) \dot{\xi}_k.
\]

(6.6)

In equation (6.6) we kept all the leading terms in the limit \( \tau \gg 1 \) as well as the first subleading correction. System (6.6) can be solved in the limit \( \kappa \to 0 \), i.e. when all the modes have wavelengths larger than the Hubble radius at \( t_0 \). In this limit the system becomes

\[
\dot{\xi}_k \simeq \mathcal{P}_k,
\]

\[
\dot{\xi}_k \simeq \mathcal{P}_k,
\]

\[
\dot{\xi}_k \simeq \mathcal{P}_k,
\]

\[
\dot{\lambda}_k \simeq -\frac{1}{2} \mathcal{P}_k.
\]

(6.7)

System (6.7) can easily be solved and the result is that an exponential mode

\[
\xi_k(t) \simeq \xi_k(0) e^{2\sqrt{\mathcal{E}_k}/\kappa},
\]

\[
\lambda_k(t) \simeq -\sqrt{\xi_k(0)} e^{2\sqrt{\mathcal{E}_k}/\kappa},
\]

\[
\mathcal{E}_k(t) \simeq \frac{\sqrt{3}}{4\sqrt{2}} \xi_k(0) e^{2\sqrt{\mathcal{E}_k}/\kappa},
\]

(6.8)

may arise. Equation (6.8) is the rationale for the result of equation (6.5) and, indeed, \( 2\sqrt{6} \simeq 4.8989 \). In figure 1, the dot-dashed line is the slope of the logarithm (in base ten) of the function \( e^{2\sqrt{\mathcal{E}_k}/\kappa} \). This example shows that the fluctuations of the internal dimensions provide an effective non-adiabatic pressure variation whose role is to provide a source term for curvature perturbations. As is evident from this example, the continued contraction of the internal scale factor is not sufficient for a perturbative description of the fluctuations. In other words, it is the fact that \( F \) does not vanish for \( t > t_0 \) that triggers the exponential growth of the curvature fluctuations.

To stress this point, let us now consider the case when the solution is exactly that previously studied, but for \( t < t_1 \). For \( t > t_1 \) the Universe is described by a five-dimensional geometry where, however, \( b \sim \text{constant} \), and \( F \) is driven dynamically to zero. Under this assumption let us now solve the evolution equations of the fluctuations, i.e. equations (4.7), (4.8) and (4.10). Of course, as discussed above in the present and in the previous sections, quantum-mechanical normalization will be enforced for \( t \to -\infty \). In figure 2 the result of such an exercise is reported in the case \( t_1 = 10t_0 \). It is clear that the dangerous exponential amplification is absent and that the treatment of the inhomogeneities is perturbative, while the relevant wavelengths are larger than the Hubble radius. If \( b(t) \) freezes at an earlier epoch, the amplitude of the curvature fluctuations will be smaller. This observation is corroborated by figure 3, where the same quantity as illustrated in figure 2 is computed in the case \( t_1 \approx t_0 \). Since \( t_1 \) is smaller, \( F \to 0 \) earlier than in the case of figure 2. This implies that the (short) phase of exponential amplification present in the example of figure 2 is now absent.

The results discussed so far hold under the assumption that the low-energy string effective action can be used to model successfully the process of dimensional decoupling. Already in this framework, different possibilities can be imagined. For instance, it could happen that the internal manifold has a non-vanishing spatial curvature (see, for instance, [21]). Another possibility could be to allow more general sources, i.e. some gauge fields (or even relativistic fluids) that can make the dynamics richer.

In all these cases, the essential structure of the equations derived in the previous sections will remain the same. Therefore, we should also expect, qualitatively, that the evolution of the internal radii will always produce a source term for curvature perturbations. However, as
far as the quantitative evolution of the system is concerned, detailed analytical and numerical studies are needed. In this respect, the present analysis is just a first step.

In approaching a more general set-up two main problems may be envisaged. The first one has to do with the nature of dimensional decoupling. If a more general dynamical framework is allowed (i.e. more fields in the action) the solution of the background may be perfectly consistent but the evolution equations of the fluctuations may develop singularities. In this case it could happen that the effective dynamical variables used to follow the evolution of the fluctuations will not be exactly those employed in the present section. The second (technical) problem is associated with the proliferation of the degrees of freedom. If a genuine
ten-dimensional background is to be analysed, the number of scalar degrees of freedom will be so large that the symmetries of the internal manifolds will have to be used in order to simplify the problem. In these directions work is in progress.

7. Concluding remarks

In the present paper the possible occurrence of dimensional decoupling has been scrutinized in the light of the possible spacetime fluctuations of the internal dimensions. Dimensional decoupling arises in various classes of higher dimensional (anisotropic) manifolds characterized by the expansion of the observable dimensions and by the contraction of the internal ones. In the present paper, dimensional decoupling has been described by means of a class of solutions of the low-energy string effective action. The virtues of this description are that the background solutions are always regular. Therefore, a reasonably accurate description of the evolution of the inhomogeneities can be developed.

Various results have been obtained:

- the full system of equations for the perturbations has been derived in the string frame description;
- the evolution equations of the zero modes have been diagonalized in terms of a set of variables that are non-singular throughout the evolution of the background geometry;
- quasi-normal modes of the system have been found;
- a numerical analysis of the appropriate (first-order) evolution equations has been presented.

The obtained results show that the contraction of the internal dimensions is not a sufficient requirement for the perturbative description of the inhomogeneities. If the scale factor of the internal dimensions is not fixed to a constant (asymptotic) value, then the evolution of the internal fluctuations triggers the presence of a source term for the evolution of curvature fluctuations. This term may be interpreted as the analogue of a non-adiabatic pressure-density fluctuation whose role, in four spacetime dimensions, is known to imply a growth of curvature fluctuations. A possible problem arises, in this context, if the contraction of the internal manifold lasts for a long time. In this case the numerical results show that the evolution equations of the fluctuations may develop some kind of dynamical attractor, where the curvature fluctuations are exponentially amplified in spite of the initial conditions. The numerical analysis can be corroborated, in the specific case under consideration, by a precise analytical understanding. This potential problem of dimensional decoupling can be avoided, provided the internal scale factor sets, sufficiently early, to a constant value. In this case the evolution of the fluctuations may still be perturbative during the whole evolution.

The results reported in the present paper suggest that realistic models of multidimensional pre-big bang dynamics can be constructed. However, unlike the attitude taken so far, care must be taken in dealing with the fluctuations of the internal dimensions, as these can induce dangerous late-time attractors in the evolution equations of the fluctuations.

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Appendix. Technical details

In the case of the geometry given in equation (2.1), assuming that both the external and internal manifolds are spatially flat, the Christoffel connections can be written as

\[ \Gamma^0_{00} = \mathcal{H}, \quad \Gamma^0_{ij} = \mathcal{H} \delta_{ij}, \quad \Gamma^0_{0j} = \mathcal{H} \delta^j_i, \quad \Gamma^0_{ab} = \frac{b^2}{a^2} \mathcal{F} \delta_{ab}, \quad \Gamma^0_{\alpha \beta} = \mathcal{F} \delta^\alpha_\beta, \]  

(A.1)

while the components of the Ricci tensor are

\[ R^0_0 = -\frac{1}{a^2} [d \mathcal{H} + n \mathcal{F}^2 + n \mathcal{H} \mathcal{F}], \quad R^i_0 = -\frac{1}{a^2} [\mathcal{H} (d - 1) \mathcal{H}^2 + n \mathcal{H} \mathcal{F} \delta^i_j], \quad R^a_b = -\frac{1}{a^2} [\mathcal{F}^2 + (d - 1) \mathcal{H} \mathcal{F} + n \mathcal{F}^2] \delta^a_b. \]  

(A.2)

From equation (A.2), the Ricci scalar then is

\[ R = -\frac{1}{a^2} [2d \mathcal{H} + 2n \mathcal{F}^2 + n(1 + n) \mathcal{F}^2 + d(d - 1) \mathcal{H}^2 + 2n(d - 1) \mathcal{H} \mathcal{F}]. \]  

(A.3)

In the uniform dilaton gauge (see equation (3.9)) the non-vanishing components of the perturbed metric (3.1) become

\[ \delta G^0_0 = 2a^2 \phi, \quad \delta G^0_y = -ab \xi, \quad \delta G^0_i = 2a^2 \psi \delta_{ij} - 2b a \partial_i \partial_j E, \quad \delta G^{yy} = 2b^2 \xi \]  

(A.4)

and

\[ \delta G^{00} = -\frac{2a^2 \phi}{a^2}, \quad \delta G^{0y} = -\frac{C}{ab}, \quad \delta G^{ij} = -\frac{2a^2 \psi \delta_{ij}}{a^2}, \quad \delta G^{yy} = -\frac{2b^2 \xi}{a^2}. \]  

(A.5)

The fluctuations of the Christoffel connections then become

\[ \delta \Gamma^0_{00} = \phi', \quad \delta \Gamma^0_{0i} = \phi', \quad \delta \Gamma^0_{ij} = -[\psi' \delta_{ij} + 2 \mathcal{H} (\phi' + \psi')] \delta_{ij} + \partial_i \partial_j \mathcal{H}, \quad \delta \Gamma^0_{0y} = \partial_y \phi, \quad \delta \Gamma^0_{y0} = -\psi' \delta^y_i + \partial_i \partial_j E. \]  

(A.6)
The first-order fluctuations of the Ricci tensor with mixed indices are

\[ \delta R_0^0 = \frac{1}{a^2} \nabla^2 \left[ \phi - E'' - H'E' \right] + \frac{1}{b^2} \nabla_j^2 \phi + \frac{1}{ab} \partial_y \left( C' + FC \right) + \frac{1}{a^2} [3 \psi'' + \xi'' + (3H \psi')^2 + (2F - H' \xi')^2] + \frac{1}{a^2} (6H' + 2F' - 2HF + 2F^2), \]

\[ \delta R_0^j = -\frac{1}{a^2} \partial_y \left[ 2 \psi' - \xi' + \xi (F - H) + (2H + F) \phi + \frac{b}{2a} C \right], \]

\[ \delta R_i^j = \frac{1}{a^2} \delta^j_i \left[ \psi'' + [2H' + 4H^2 + 2HF] \phi + H \phi' + (5H + F) \psi' + H \xi' - \nabla^2 \psi - \frac{a^2}{b^2} \nabla_j^2 \psi \right. \]

\[ - \nabla^2 E' + \frac{a}{b} \partial_y C + \partial_y \left[ -E'' - (2H + F)^E' + \phi - \xi - \psi - \frac{a^2}{b^2} \nabla_j^2 \psi \right], \]

\[ \delta R_i^j = \frac{1}{a^2} \left[ \psi'' + (2H + F) \xi' + 3F \psi' + \phi (2F^2 + 4HF + 2F') - \nabla^2 \xi - \nabla^2 E' \right] \]

\[ + \frac{1}{b^2} \nabla_j^2 \left[ \phi \nabla^2 E - 3 \psi \right] + \frac{1}{ab} \partial_y \left[ C' + (3H + F) C \right], \]

\[ \delta R_0^i = \frac{C}{ab} \left[ 3H \xi' - 2HF + 1 \nabla^2 C \right. \]

\[ - \frac{1}{b^2} \partial_i \left[ (3 \psi' + 3H \phi + 3H \xi - \nabla^2 [E' + (H - F) E] \right], \]

\[ \delta R_i^j = \frac{1}{2ab} \partial_y \left[ C' + (H + 2F) C \right] - \frac{1}{b^2} \partial_y \left[ 2 \psi - \phi \right]. \]  

(A.7)  

while the fluctuation of the Ricci scalar becomes

\[ \delta R = \frac{2}{a^2} \left\{ \nabla^2 \left[ \phi - \xi - 2 \psi - E'' - (3H + F) E' + \frac{a^2}{b^2} \nabla_j^2 E \right] + 3 \psi'' + \xi'' + (3H + F) \psi' \right. \]

\[ + (9H + 3F) \psi' + 2 \xi' (H + F) + (6H' + 2F' + 4HF + 6H^2 + 2F^2) \phi \]

\[ \left. + \frac{2}{b^2} \nabla_j^2 \left( \phi - 3 \psi \right) + \frac{2}{ab} \partial_y \left[ C' + (3H + F) \right] \right\}. \]  

(A.8)  

Defining now

\[ \mathcal{G}_A^A = \delta R_A^A - \frac{1}{2} R \delta A, \]  

the scalar fluctuation of the Einstein tensor leads to the following components:

\[ \delta \mathcal{G}_0^0 = \frac{1}{a^2} \nabla^2 \left[ \xi + 2 \psi + (2H + F) E' - \frac{a^2}{b^2} \nabla_j^2 E \right] - \frac{3}{ab} \nabla \partial_y \left[ C' + \frac{3}{b^2} \nabla_j^2 \psi \right. \]

\[ + \frac{1}{a^2} [3(2H + F) \psi' - 3H \xi' - 6H (H + F) \phi], \]

\[ \delta \mathcal{G}_i^j = \frac{1}{a^2} \delta^j_i \left[ -2 \psi'' - \xi'' - 2 [H^2 + F^2 + HF + F' + 2H'] \phi \right. \]

\[ - (F + 2H) \phi' - 2 (F + 2H) \psi' - (H + 2F) \xi' \]

\[ + \nabla^2 \left[ E'' + (2H + F) E' + \psi + \xi - \phi - \frac{a^2}{b^2} \nabla_j^2 E \right] \left\} \right. \]

\[ + \delta^j_i \nabla_j^2 (2 \psi - \phi) - \frac{1}{ab} \partial_y \left[ C' + (2H + F) \delta^j_i \right. \]

\[ - \frac{1}{a^2} \partial_y \left[ E'' + (2H + F) E' + \psi + \xi - \phi - \frac{a^2}{b^2} \nabla_j^2 E \right]. \]  

(A.9)
\[
\delta G^i_j = \frac{1}{a^2} \left( -3 \phi'' - 3 \mathcal{H} \phi' - 9 \mathcal{H} \psi' - 6 (\mathcal{H}' + \mathcal{H}^2) \phi + \nabla^2 [2 \psi - \phi + E' + 3 \mathcal{H} E'] \right), \tag{A.10}
\]

where, clearly, \( \delta G^0_0 = \delta R^0_0, \delta G^y_0 = \delta R^y_0, \delta G^y_i = \delta R^y_i. \)

Finally, the general form of the perturbed equations can usefully be expressed in a compact way as

\[
a^2 \delta G^0_0 + \phi \left[ 2 \mathcal{F} \psi' - \psi'^2 + 6 \mathcal{H} \phi' \right] + 3 \psi' \phi' + \phi' \nabla^2 E' + \frac{a}{b} \phi' \partial_i C = 0,
\]

\[
a^2 \delta G^i_j + \phi' \partial_i \partial^j E' + \delta^j_i \left\{ \phi \left[ 2 \phi'' - \phi'^2 + 2 \mathcal{H} \phi' + 2 \mathcal{F} \phi' - \frac{a^2}{2} \frac{\partial V}{\partial \phi} \right] \right. \\
+ 2 \phi' \phi' + 3 \psi' \psi' - \phi' \nabla^2 E' + \phi' \phi' + \frac{a}{b} \phi' \partial_i C \right\} = 0,
\]

\[
a^2 \delta G^y_y + \frac{a}{b} C \left[ -\phi'' + (\mathcal{H} + \mathcal{F}) \psi' + \frac{a^2}{2} \frac{\partial V}{\partial \phi} \right] + \psi' \phi' + \frac{a^2}{b^2} \partial^2 \phi = 0,
\]

\[
a^2 \delta G^y_0 + \frac{a}{2b} \phi' \partial_i C = 0, \quad a^2 \delta G^y_i + \phi' \partial^i \phi = 0. \tag{A.11}
\]

Equations (A.10) together with equations (A.11) can be used to derive the evolution equations of the fluctuations discussed in the bulk of the present paper.

References

[1] Appelquist T, Chodos A and Freund P 1986 Modern Kaluza–Klein Theories (Redwood City, CA: Addison-Wesley)
[2] Sahdev D 1984 Phys. Rev. D 30 2495
[3] Abbott R, Barr S and Ellis S 1984 Phys. Rev. D 30 720
[4] Kolb E W, Lindley D and Seckel D 1984 Phys. Rev. D 30 1205
[5] Gasperini M and Veneziano G 1993 Astropart. Phys. 1 317
[6] Sahdev D 1989 Phys. Rev. D 39 3155
[7] Bardeen J M 1980 Phys. Rev. D 22 1882
[8] Tomita K and Den M 1986 Phys. Rev. D 34 3570
[9] Giovannini M 1997 Phys. Rev. D 55 595
[10] Demianski M, Polnarev A and Naselski P 1993 Phys. Rev. D 47 5275
[11] Den M 1987 Prog. Theor. Phys. 77 653
[12] Abbott R, Bednarz B and Ellis S D 1986 Phys. Rev. D 33 2147
[13] Gasperini M and Giovannini M 1997 Class. Quantum Grav. 14 739
[14] Meissner K A and Veneziano G 1991 Phys. Lett. B 267 33
[15] Meissner K A and Veneziano G 1991 Mod. Phys. Lett. A 6 3397
[16] Gasperini M, Maharana J and Veneziano G 1996 Nucl. Phys. B 472 349
[17] Giovannini M 2004 Class. Quantum Grav. 21 4209
[18] Gasperini M, Giovannini M and Veneziano G 2004 Nucl. Phys. B 694 206
[19] Gasperini M, Giovannini M and Veneziano G 2003 Phys. Lett. B 569 113
[20] Giovannini M 2004 Phys. Rev. D 70 103509
[21] Randjbar-Daemi S, Salam A and Strathdee J 1984 Phys. Lett. B 135 388