Integral-Type Operators from $F(p, q, s)$ Space to $\alpha$-Bloch–Orlicz and $\beta$-Zygmund–Orlicz Spaces

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Abstract Equivalent characterizations for the boundedness and compactness of several integral-type operators from $F(p, q, s)$ space to $\alpha$-Bloch–Orlicz and $\beta$-Zygmund–Orlicz spaces were developed in this paper.

Keywords Integral-type operator · $\alpha$-Bloch–Orlicz space · $\beta$-Zygmund–Orlicz space · Boundedness · Compactness

Mathematics Subject Classification Primary 46E30; Secondary 47G10

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ (or $S(\mathbb{D})$) be the collection of all analytic functions (or all analytic self-maps) on $\mathbb{D}$. Given $\phi \in S(\mathbb{D})$, the composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$C_\phi f = f \circ \phi,$$

where $f \in H(\mathbb{D})$.

Composition operators acting on various spaces of holomorphic functions have been the popular object in recent years. In particular, the problems of relating operator-theoretic properties of $C_\phi$ to function-theoretic properties of $\phi$ are interesting and
have been widely discussed. We refer the interested readers to [2, 16, 18, 28, 29] and recent articles [8–12, 26, 27] and their references therein, which are good literature on operator theory and holomorphic function spaces.

It’s of very interest to investigate the generalized composition operator $C^h_\phi$ with $\phi \in S(D)$ and $h \in H(D)$, defined as

$$C^h_\phi f(z) = \int_0^z f'(\phi(t))h(t)dt, \quad f \in H(D), \quad z \in D.$$ 

In fact, when $h(z) = \phi'(z)$, the integral-type operator $C^h_\phi$ is reduced to the difference of a composition operator and a point evaluation operator, more precisely $C^\phi_\phi = C_\phi - \delta_{\phi(0)}$. This operator was introduced in [7, 19, 21] and attracted much attention, mainly due to its link between classical function theory and operator theory. More generally, in this paper we will describe some equivalent characterizations for the boundedness and compactness of some relevant integral-type operators defined below.

(a) Let $h \in H(D)$, the integral-type operator $L^h$ is defined as

$$L^h f(z) = \int_0^z f'(t)h(t)dt, \quad f \in H(D), \quad z \in D.$$ 

(b) Let $h \in H(D)$, the integral-type operator $L_h$ is defined as

$$L_h f(z) = \int_0^z f'(t)h'(t)dt, \quad f \in H(D), \quad z \in D.$$ 

(c) Let $\phi \in S(D)$ and $h \in H(D)$, the integral-type operator $C^h_\phi$ is defined as

$$C^h_\phi f(z) = \int_0^z f'(\phi(t))h(t)dt, \quad f \in H(D), \quad z \in D.$$ 

(d) Let $\phi \in S(D)$ and $h \in H(D)$, the integral-type operator $V^h_\phi$ (Riemann-Stieltjes operator) is defined as

$$V^h_\phi f(z) = \int_0^z f'(\phi(t))h'(t)dt, \quad f \in H(D), \quad z \in D.$$ 

It turns out that these integral-type operators have close connections. On the one hand, when $\phi = id$ the identity map, then

$$C^h_{id} = L^h \quad \text{and} \quad V^h_{id} = L_h.$$ 

That is, the operators $L^h$ and $L_h$ are special cases of $C^h_\phi$ and $V^h_\phi$, respectively. On the other hand, if we let $h = k' \in H(D)$ in $C^h_\phi$, then $C^k_\phi = V^k_\phi$. Due to the above observations, we mainly provide some interesting results for $C^h_\phi$ acting from the general function space $F(p, q, s)$ to $\alpha$-Bloch–Orlicz and $\beta$-Zygmund–Orlicz spaces, then the
analogous results for other integral-type operators follow immediately. We refer the readers to \[7, 19, 20, 22–24\] for the properties of integral-type operators. For readers’ convenience, we introduced some holomorphic function spaces as below.

For \(0 < p, s < \infty\), \(-2 < q < \infty\), a function \(f \in H(\mathbb{D})\) is said to be in the general function space \(F(p, q, s) = F(p, q, s)(\mathbb{D})\) if

\[
\|f\|_{F(p, q, s)}^p = |f(0)|^p + \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s \, dA(z) < \infty,
\]

where \(\phi_w(z) = (w - z)/(1 - \overline{w}z)\), \(w \in \mathbb{D}\). The family of spaces \(F(p, q, s)\) on the unit disk was first introduced by Zhao \[25\]. It is called general function space, which contains, as special cases, many classical holomorphic function spaces, such as \(F(p, 0, 0)\) is the classical Bergman space \(A^p\), \(F(2, 1, 0)\) is just the Hardy space \(H^2\) and \(F(2, 0, 1) = \text{BMOA}\). Notice that \(F(p, q, s)\) is the space of constant functions if \(q + s \leq -1\). For the definition of these spaces described above, we recommend the readers to \[9–11, 13, 22, 24, 25\] and their references therein.

Let \(\mu\) be a positive continuous function on \(\mathbb{D}\) (a weight). We recall that the \(\mu\)-Bloch space \(B_\mu = B_\mu(\mathbb{D})\) consists of all \(f \in H(\mathbb{D})\) such that

\[
\|f\|_{B_\mu} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty.
\]

It’s a well-known fact that the \(\mu\)-Bloch space \(B_\mu\) is a Banach space endowed with the norm \(\|f\|_{B_\mu}\). In particular, if \(\mu(z) = (1 - |z|^2)^{\alpha}\), it yields that

\[
B^\alpha = \left\{ f \in H(\mathbb{D}), \|f\|_{B^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |f'(z)| < \infty \right\},
\]

which degenerates to the classical Bloch space \(\mathcal{B}\) for \(\alpha = 1\). In the analogous way, we recall that the \(\mu\)-Zygmund space \(Z_\mu = Z_\mu(\mathbb{D})\) includes those functions \(f \in H(\mathbb{D})\) satisfying

\[
\|f\|_{Z_\mu} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f''(z)| < \infty,
\]

which is a complete norm on \(Z_\mu\).

Recently, Ramos Fernández used Young’s functions to define the Bloch–Orlicz space \([14]\), which is a generalization of the classical Bloch space (see \[15\] for definitions and some information on the Orlicz spaces). More precisely, let \(\varphi : [0, +\infty) \to [0, +\infty)\) be an \(N\)-function, that is, \(\varphi\) is a strictly increasing convex function such that \(\varphi(0) = 0\), which implies that \(\lim_{t \to +\infty} \varphi(t) = +\infty\). The Bloch–Orlicz space linked with the function \(\varphi\), denoted by \(B^\varphi\), is the collection of all \(f \in H(\mathbb{D})\) fulfilling

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f'(z)|) < \infty,
\]
for some \( \lambda > 0 \) depending on \( f \). We can further suppose that \( \varphi^{-1} \) is continuously differentiable. If \( \varphi^{-1} \) is not differentiable everywhere, we define the function

\[
\psi(t) = \int_0^t \frac{\varphi(x)}{x} \, dx, \quad t \geq 0,
\]

then \( \psi \) is differentiable, whence \( \varphi^{-1} \) is differentiable everywhere on \([0, \infty)\). Since \( \varphi \) is a strictly increasing, convex function satisfying \( \varphi(0) = 0 \), therefore the function \( \varphi(t)/t, \ t > 0 \), is increasing and

\[
\varphi(t) \geq \psi(t) \geq \int_{t/2}^t \frac{\varphi(x)}{x} \, dx \geq \varphi\left(\frac{t}{2}\right) \quad \text{for all} \quad t \geq 0.
\]

As a consequence, \( B^\varphi = B^\psi \). Employing the convexity of \( \varphi \), we can prove that the Minkowski’s functional

\[
\|f\|_\varphi = \inf \left\{ k > 0 : S_\varphi\left(\frac{f'}{k}\right) \leq 1 \right\},
\]

defines a seminorm for \( B^\varphi \), where

\[
S_\varphi(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(|f(z)|).
\]

It’s easy to verify that \( B^\varphi \) is a Banach space under the norm \( \|f\|_{B^\varphi} = |f(0)| + \|f\|_\varphi \).

For any \( f \in B^\varphi \setminus \{0\} \), it holds that

\[
S_\varphi\left(\frac{f'}{\|f\|_{B^\varphi}}\right) \leq 1,
\]

which leads to the following lemma.

**Lemma 1.1** The Bloch–Orlicz space is isometrically equal to \( \mu_1 \)-Bloch space, where

\[
\mu_1(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \quad z \in \mathbb{D}.
\]

Whence for any \( f \in B^\varphi \),

\[
\|f\|_{B^\varphi} = |f(0)| + \sup_{z \in \mathbb{D}} \mu_1(z) |f'(z)|.
\]

By an apparent generalization, the \( \alpha \)-Bloch–Orlicz space \( B^\varphi_\alpha = B^\varphi_\alpha(\mathbb{D}) \) for \( \alpha > 0 \) can be defined as the class of those \( f \in H(\mathbb{D}) \) satisfying

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \varphi(\lambda |f'(z)|) < \infty
\]
for some \( \lambda > 0 \) depending on \( f \). Furthermore, the \( \alpha \)-Bloch–Orlicz space \( B_\alpha^\varphi \) is also a Banach space under the norm \( \| f \|_{B_\alpha^\varphi} = |f(0)| + \| f \|_{\varphi, \alpha} \), where

\[
\| f \|_{\varphi, \alpha} = \inf \left\{ k > 0 : S_{\varphi, \alpha} \left( \frac{f'}{k} \right) \leq 1 \right\}
\]

and

\[
S_{\varphi, \alpha}(f) = \sup_{z \in D} (1 - |z|^2)^{\alpha} \varphi(|f(z)|).
\] (1.1)

Similarly, a standard fact is

\[
S_{\varphi, \alpha} \left( \frac{f'}{\| f \|_{B_\alpha^\varphi}} \right) \leq 1,
\] (1.2)

which yields a lemma analogous to Lemma 1.1.

**Lemma 1.2** The \( \alpha \)-Bloch–Orlicz space is isometrically equal to \( \mu_\alpha \)-Bloch space, where

\[
\mu_\alpha(z) = \frac{1}{\varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) }, \quad z \in D.
\]

Whence for any \( f \in B_\alpha^\varphi \),

\[
\| f \|_{B_\alpha^\varphi} = |f(0)| + \sup_{z \in D} \mu_\alpha(z) |f'(z)|.
\] (1.3)

Building on the Luxemburg seminorm and the display (1.2), we affirm that

\[
S_{\varphi, \alpha}(f') \leq 1 \iff \| f \|_{B_\alpha^\varphi} \leq 1, \quad \text{for } f \in B_\alpha^\varphi.
\] (1.4)

At present, we turn our attention to introduce the \( \beta \)-Zygmund–Orlicz space \( Z_\beta^\varphi = Z_\beta^\varphi(D) \) for \( \beta > 0 \), which is the class of all \( f \in H(D) \) satisfying

\[
\sup_{z \in D} (1 - |z|^2)^{\beta} \varphi(\lambda |f''(z)|) < \infty,
\]

for some \( \lambda > 0 \) depending on \( f \). Same as the \( \alpha \)-Bloch–Orlicz space, since \( \varphi \) is convex, the Minkowski functional

\[
\| f \|_{Z_\beta^\varphi} = \inf \left\{ k > 0 : S_{\varphi, \beta} \left( \frac{f''}{k} \right) \leq 1 \right\}
\]
defines a seminorm for $Z_{\beta}^{\phi}$ and $S_{\phi, \beta}$ has emerged in (1.1). In particular, $Z_{\beta}^{\phi}$ is a Banach space under the norm
\[ \| f \|_{Z_{\beta}^{\phi}} = |f(0)| + |f'(0)| + \| f \|_{Z_{\beta}^{\phi}}. \]

**Lemma 1.3** For any $f \in Z_{\beta}^{\phi} \setminus \{0\}$, the following relation holds
\[ S_{\phi, \beta} \left( \frac{f''}{\| f \|_{Z_{\beta}^{\phi}}} \right) \leq 1. \]

Furthermore,
\[ S_{\phi, \beta} (f'') \leq 1 \Leftrightarrow \| f \|_{Z_{\beta}^{\phi}} \leq 1, \text{ for } f \in Z_{\beta}^{\phi}. \tag{1.5} \]

Consequently, Lemma 1.3 tells that the $\beta$-Zygmund–Orlicz space is isometrically equal to $\mu_{\beta}$-Zygmund space with
\[ \mu_{\beta}(z) = \frac{1}{\phi^{-1} \left( \frac{1}{(1-|z|^2)^{\beta}} \right)}, \quad z \in \mathbb{D}. \]

That is, for any $f \in Z_{\beta}^{\phi}$,
\[ \| f \|_{Z_{\beta}^{\phi}} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu_{\beta}(z) |f''(z)|. \tag{1.6} \]

As far as we know, there are also other interesting Orlicz-type spaces, such as Bergman–Orlicz and Hardy–Orlicz spaces, see, e.g., [1, 4, 6, 17] and their references therein. It turns out that the boundedness and compactness of classical operators acting on Orlicz-spaces of analytic functions have attracted a lot of attention. In 2014, Yang etc. established the descriptions for generalized composition operators on Zygmund–Orlicz type spaces and Bloch–Orlicz type spaces in [23]. In 2015, Jiang etc. presented some characterizations for the properties of generalized product-type operators from weighted Bergman–Orlicz spaces to Bloch–Orlicz spaces in [5], and then for the product-type operator from weighted Bergman–Orlicz space to weighted type spaces. After that they proved the similar results for the generalized weighted composition operator from Zygmund spaces to Bloch–Orlicz spaces in [3]. However, there is no consideration on linear operators acting on $\alpha$-Bloch–Orlicz or $\beta$-Zygmund–Orlicz spaces. In the present paper, we firstly concentrate on the boundedness and compactness of the generalized composition operator $C_{\phi}^{\beta}$ from general space $F(p, q, s)$ to the $\alpha$-Bloch–Orlicz spaces or $\beta$-Zygmund–Orlicz spaces. After that we derive some corollaries for other integral-type operators, which can be deduced from generalized composition operator. The organization of this paper is as follows, we recall some lemmas in Sect. 2 for our further use. Then we provide the necessary and sufficient
conditions for the boundedness and compactness of the generalized composition operator \(C_{\varphi}^h\) acting from \(F(p, q, s)\) to \(B^\alpha_{\varphi}\) or \(Z^\beta_{\varphi}\) in Sects. 3 and 4, respectively. Finally, we deduce some corollaries for remanent integral-type operators with brief proof details.

Finally, note that we will often use the notation \(A \preceq B\) for two nonnegative quantities \(A\) and \(B\) if \(A \leq CB\) for an unimportant constant \(C > 0\). Moreover, the notation \(A \succeq B\) will have the similar meaning. The notation \(A \asymp B\) means that \(A \preceq B\) and \(A \succeq B\) hold simultaneously. In the rest of the paper, we always suppose that \(0 < p, s < \infty, -2 < q < \infty, q + s > -1\) and \(\alpha, \beta > 0\) for the sake of simplicity.

2 Some Lemmas

In this section, we presented some lemmas for our further use.

**Lemma 2.1** [10, Lemma 2.2] If \(f \in F(p, q, s)\), then \(f \in B^{(2+q)/p}\) and
\[
\| f \|_{B^{(2+q)/p}} \preceq \| f \|_{F(p,q,s)}.
\] (2.1)

On account of Lemma 2.1 and [20, Lemma 3], we develop a remark.

**Remark 2.2** For \(f \in F(p, q, s)\), there exists a constant \(C_1 > 0\) such that
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{2+q/p + 1} |f''(z)| \leq C_1 \| f \|_{F(p,q,s)}.
\] (2.2)

Furthermore, it turns out that
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{2+q/p + m-1} |f^{(m)}(z)| \leq C_{m-1} \| f \|_{F(p,q,s)},
\] (2.3)
where \(C_{m-1}\) is a constant only dependent on \(m \in \mathbb{Z}^+\).

**Lemma 2.3** Let \(\alpha > 0\), then there exists a constant \(C > 0\) such that
\[
| f(z) | \leq C G_{\alpha}(z) \| f \|_{B^\alpha},
\]
for all \(f \in B^\alpha\) and \(z \in \mathbb{D}\), where the function \(G_{\alpha}\) is defined by
\[
G_{\alpha}(z) = \begin{cases}
1, & 0 < \alpha < 1; \\
\log \frac{2}{1-|z|^2}, & \alpha = 1; \\
\left(\frac{1}{1-|z|^2}\right)^{\alpha-1}, & \alpha > 1.
\end{cases}
\]

In view of Lemma 2.1 and Lemma 2.3, the following remark holds.

**Remark 2.4** If \(f \in F(p, q, s)\), then \(f \in B^{(2+q)/p}\) and hence
\[
| f(z) | \leq G_{(2+q)/p}(z) \| f \|_{B^{(2+q)/p}} \leq G_{(2+q)/p}(z) \| f \|_{F(p,q,s)}.
\] (2.4)
Lemma 2.5 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function and $Y$ be the $\alpha$-Bloch–Orlicz space $B^\varphi_\alpha$ (or $\beta$-Zygmund–Orlicz space $Z^\varphi_\beta$). Then $C^h_\varphi : F(p, q, s) \to Y$ is compact if and only if $C^h_\varphi : F(p, q, s) \to Y$ is bounded and, for any uniformly bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $F(p, q, s)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, one has $\|C^h_\varphi f_k\|_Y \to 0$ as $k \to \infty$.

Proof Without loss of generality, we will show the characterization for the compactness of $C^h_\varphi : F(p, q, s) \to B^\varphi_\alpha$. The proof for the compactness of $C^h_\varphi : F(p, q, s) \to Z^\varphi_\beta$ follows from the similar lines.

Necessity Assume that the operator $C^h_\varphi : F(p, q, s) \to B^\varphi_\alpha$ is compact and suppose $\{f_k\}_{k \in \mathbb{N}}$ is a sequence in $F(p, q, s)$ with sup $\|f_k\|_{F(p,q,s)} < \infty$ and $f_k \to 0$ on compact subsets of $\mathbb{D}$ as $k \to \infty$. Then the boundedness of $C^h_\varphi : F(p, q, s) \to B^\varphi_\alpha$ is trivial. Due to the compactness of $C^h_\varphi : F(p, q, s) \to B^\varphi_\alpha$, we have that $\{C^h_\varphi f_k\}$ has a subsequence $\{C^h_\varphi f_{k_l}\}$ which converges in $B^\varphi_\alpha$, say to $u$. On the one hand, by Remark 2.4, it follows that for any compact set $K \subseteq \mathbb{D}$, there is a positive constant $C_K$ independent of $f$ such that $|C^h_\varphi f_{k_l}(z) - u(z)| \leq C_K \|C^h_\varphi f_{k_l} - u\|_{F(p,q,s)}$ for all $z \in K$. That is to say that $C^h_\varphi f_{k_l}(z) - u(z) \to 0$ uniformly on compact set $K \subseteq \mathbb{D}$. On the other hand, employing the hypothesis and the definition of $C^h_\varphi$, it turns out that the $C^h_\varphi f_{k_l}(z)$ converges to zero uniformly on $K$. Considering that $K$ is arbitrary, it yields that the limit function $u \equiv 0$. Indeed the above fact is true for arbitrary subsequence of $\{C^h_\varphi f_{k_l}\}$, hence we verify that $\|C^h_\varphi f_k\|_{B^\varphi_\alpha} \to 0$ as $k \to \infty$.

Sufficiency We suppose that $C^h_\varphi : F(p, q, s) \to B^\varphi_\alpha$ is bounded and let $\{f_k\}_{k \in \mathbb{N}} \subseteq K_r = D_{F(p,q,s)}(0, r)$, where $D_{F(p,q,s)}(0, r)$ is a disk in $F(p, q, s)$ and $f_k$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. In the sequel, we will show that $C^h_\varphi(K_r)$ is relatively compact. By Remark 2.4, it yields that $\{f_k\}$ is uniformly bounded in arbitrary compact subsets of $\mathbb{D}$. Therefore, Montel’s Lemma implies that $\{f_k\}$ is a normal family, and then there is a subsequence $\{f_{k_l}\}$ which converges uniformly to an $f \in H(\mathbb{D})$ on compact subsets of $\mathbb{D}$. By Cauchy estimate, it turns out that $f_{k_l}' \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $l \to \infty$.

Denote $D_j = D \left(0, 1 - \frac{1}{j} \right) \subseteq \mathbb{C}$, then

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z) = \lim_{j \to \infty} \int_{D_j} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z) \leq \lim_{j \to \infty} \lim_{l \to \infty} \int_{D_j} |f'_{k_l}(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z).$$

Hence

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z) \leq \lim_{j \to \infty} \lim_{l \to \infty} \sup_{w \in \mathbb{D}} \int_{D_j} |f'_{k_l}(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z).$$
The fact \( \{ f_{k_l} \} \subset D_{F(p,q,s)}(0, r) \) tells that
\[
\sup_{w \in D_j} \int_{D_j} |f'_{k_l}(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s \, dA(z) < r^p,
\]
so that
\[
\sup_{w \in D_j} \int |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s \, dA(z) < r^p.
\]
As a consequence, \( \| f \|_{F(p,q,s)} \leq r \) and then \( f \in F(p,q,s) \). Furthermore, \( \| f_{k_l} - f \|_{F(p,q,s)} \leq 2r < \infty \) and the sequence \( \{ f_{k_l} - f \} \) converges to 0 on compact subsets of \( \mathbb{D} \). By the hypothesis of this lemma, it yields that \( C^{h}_{\phi} f_{k_l} \rightarrow C^{h}_{\phi} f \) in \( \mathcal{B}^{q}_{\alpha} \) as \( l \rightarrow \infty \).
That is to say the set \( C^{h}_{\phi}(K_r) \) is relatively compact and the proof is finished. \( \square \)

### 3 \( C^{h}_{\phi} \) from \( F(p,q,s) \) to \( \alpha\)-Bloch–Orlicz Space

In this section, we provide the sufficient and necessary conditions ensuring the boundedness and compactness of the operator \( C^{h}_{\phi} : F(p,q,s) \rightarrow \mathcal{B}^{q}_{\alpha} \).

**Theorem 3.1** Let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be an \( N \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then the operator \( C^{h}_{\phi} : F(p,q,s) \rightarrow \mathcal{B}^{q}_{\alpha} \) is bounded if and only if
\[
M := \sup_{z \in \mathbb{D}} \left| \frac{h(z)}{\varphi^{-1} \left( \frac{1}{(1 - |z|^2)^\alpha} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} \right| < \infty. \tag{3.1}
\]

**Proof** Sufficient Suppose that (3.1) is true. Hence for any \( f \in F(p,q,s) \), by the fact \( \varphi \) is a strictly increasing convex function and (2.1), we conclude that
\[
S_{\varphi, \alpha} \left( \frac{(C^{h}_{\phi} f)'(z)}{M \| f \|_{F(p,q,s)}} \right) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \varphi \left( \frac{|f'(\phi(z)) h(z)|}{M \| f \|_{F(p,q,s)}} \right)
\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \varphi \left( \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^\alpha} \right) \right.
\times \left. (1 - |\phi(z)|^2)^{\frac{2+q}{p}} \right| f'(\phi(z)) \right| \| f \|_{F(p,q,s)} \n\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \varphi \left( \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^\alpha} \right) \right) \leq 1. \tag{3.2}
\]

Employing (1.4), it turns out that
\[
\left\| \frac{C^{h}_{\phi} f}{M \| f \|_{F(p,q,s)}} \right\| \leq 1.
\]
That is to say \( \| C^h_{\phi} f \|_{B^q_\alpha} \leq M \| f \|_{F(p,q,s)} \) for any \( f \in F(p,q,s) \), which implies the boundedness of \( C^h_{\phi} : F(p,q,s) \to B^q_\alpha \).

**Necessity** Assume the operator \( C^h_{\phi} : F(p,q,s) \to B^q_\alpha \) is bounded, consequently, there exists \( C > 0 \) such that

\[
\| C^h_{\phi} f \|_{B^q_\alpha} \leq C \| f \|_{F(p,q,s)}, \quad \text{for any } f \in F(p,q,s).
\]  

(3.3)

For \( a \in \mathbb{D} \), choose the function

\[
f_a(z) = \frac{1 - |\phi(a)|^2}{(1 - z\phi(a))^{2+q/p}}.
\]  

(3.4)

It can be proved that \( f_a \in F(p,q,s) \) with \( \sup_{a \in \mathbb{D}} \| f_a \|_{F(p,q,s)} \leq 1 \) by [16, Proposition 1.4.10]. By a direct calculation, we get that

\[
f'_a(z) = \frac{2 + q \phi(a)(1 - |\phi(a)|^2)}{p} \left(1 - z\phi(a)\right)^{2+q/p+1}.
\]

Additionally,

\[
f'_a(\phi(a)) = \frac{2 + q \phi(a)}{p} \left(1 - |\phi(a)|^2\right)^{2+q/p}.
\]

In view of (3.3), it holds that \( \| C^h_{\phi} f_a \|_{B^q_\alpha} \leq C \| f_a \|_{F(p,q,s)} \). In other words,

\[
\left\| \frac{C^h_{\phi} f_a}{C \| f_a \|_{F(p,q,s)}} \right\|_{B^q_\alpha} \leq 1,
\]

which is equivalent to

\[
S_{\psi,\alpha}\left(\left(\frac{C^h_{\phi} f_a}{C \| f_a \|_{F(p,q,s)}}\right)\right)' = S_{\psi,\alpha}\left(\frac{(C^h_{\phi} f_a)'}{C \| f_a \|_{F(p,q,s)}}\right) \leq 1.
\]

Therefore,

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)\alpha \phi \left(\frac{(C^h_{\phi} f_a)'}{C \| f_a \|_{F(p,q,s)}}\right) \leq 1.
\]

It’s trivial that

\[
(1 - |a|^2)\alpha \phi \left(\frac{(C^h_{\phi} f_a)'(a)}{C \| f_a \|_{F(p,q,s)}}\right) \leq 1.
\]
Consequently, we deduce that
\[
\varphi \left( \frac{|f_\alpha'(\phi(a)) h(a)|}{C \| f_\alpha \|_{F(p, q, s)}} \right) \leq \frac{1}{(1 - |a|^2)\alpha}.
\]

Taking \( \varphi^{-1} \) on both sides, it holds that
\[
\frac{|f_\alpha'(\phi(a)) h(a)|}{C \| f_\alpha \|_{F(p, q, s)}} \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)\alpha} \right).
\]

Hence
\[
\frac{2 + q}{p} \frac{|\phi(a)||h(a)|}{(1 - |\phi(a)|^2)^{\frac{2+q}{p}}} C \| f_\alpha \|_{F(p, q, s)} \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)\alpha} \right).
\]

Furthermore, it yields that
\[
\sup_{a \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1 - |a|^2)^{\alpha}} \right) (1 - |\phi(a)|^2)^{\frac{2+q}{p}} \leq \sup_{a \in \mathbb{D}} \| f_\alpha \|_{F(p, q, s)} < \infty.
\]

Thus we conclude that
\[
\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}} < \infty. \tag{3.5}
\]

On the other hand, let \( f_0(z) = z \in F(p, q, s) \) with \( \| f_0 \|_{F(p, q, s)} \leq 1 \). Applying similar arguments to \( f_0 \), it entails that
\[
L := \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) < \infty. \tag{3.6}
\]

Under the case \( |\phi(z)| \leq \frac{1}{2} \), we use (3.6) to obtain that
\[
\sup_{\{z \in \mathbb{D} : |\phi(z)| \leq 1/2\}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}} \leq \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) < \infty.
\]

Under the case \( |\phi(z)| > \frac{1}{2} \), from (3.5), one argue that
\[
\sup_{\{z \in \mathbb{D} : |\phi(z)| > 1/2\}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}} \leq \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}} < \infty.
\]

The desire result (3.1) can be deduced from the above two inequalities. This concludes the proof. \( \square \)
Theorem 3.2 Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( N \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then the operator \( C^h_\phi : F(p, q, s) \to \mathcal{B}_\alpha^p \) is compact if and only if \( C^h_\phi : F(p, q, s) \to \mathcal{B}_\alpha^p \) is bounded and

\[
\lim_{|\phi(z)| \to 1} \frac{|h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^s} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} = 0. \tag{3.7}
\]

Proof Sufficiency Suppose the operator \( C^h_\phi : F(p, q, s) \to \mathcal{B}_\alpha^p \) is bounded and (3.7) holds. Applying the boundedness of \( C^h_\phi : F(p, q, s) \to \mathcal{B}_\alpha^p \) and using the proof in Theorem 3.1, we can verify (3.6), that is, \( L < \infty \). In view of (3.7), for every \( \epsilon > 0 \), there exists \( 0 < r < 1 \) satisfying

\[
\frac{|h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^s} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} < \epsilon, \quad \text{for } |\phi(z)| > r. \tag{3.8}
\]

Let \( \{f_n\} \) be a sequence in \( F(p, q, s) \) with \( \sup_{n \in \mathbb{N}} \|f_n\|_{F(p, q, s)} \leq K \) and \( f_n \) converging to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). By Lemma 2.5, it suffices to show that \( \|C^h_\phi f_n\|_{\mathcal{B}_\alpha^p} \to 0 \) as \( n \to \infty \). Considering Lemma 1.2, we express the norm as follows,

\[
\|C^h_\phi f_n\|_{\mathcal{B}_\alpha^p} = |C^h_\phi f_n(0)| + \sup_{z \in \mathbb{D}} \frac{1}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^s} \right) ((C^h_\phi f_n)'(z)}
\]

\[
= \sup_{z \in \mathbb{D}} \frac{1}{\varphi^{-1} \left( \frac{1}{|\phi(z)|^2} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} |f_n'(\phi(z))h(z)|
\]

\[
= \sup_{\{z \in \mathbb{D} : |\phi(z)| \leq r\}} \frac{1}{\varphi^{-1} \left( \frac{1}{|\phi(z)|^2} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} |f_n'(\phi(z))h(z)|
\]

\[
+ \sup_{\{z \in \mathbb{D} : |\phi(z)| > r\}} \frac{1}{\varphi^{-1} \left( \frac{1}{|\phi(z)|^2} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} |f_n'(\phi(z))h(z)|
\]

\[
\leq L \sup_{\{w \in \mathbb{D} : |w| \leq r\}} |f_n'(w)| + K
\]

\[
\times \sup_{\{z \in \mathbb{D} : |\phi(z)| > r\}} \frac{|h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^s} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} < L \sup_{\{w \in \mathbb{D} : |w| \leq r\}} |f_n'(w)| + K \epsilon. \tag{3.9}
\]

In the chain of inequalities, we employ (3.8) in the last line. Since \( f_n \to 0 \) on compact subsets of \( \mathbb{D} \) and by the Cauchy estimate, it leads to \( f_n' \to 0 \) on compact subsets of \( \mathbb{D} \). Letting \( n \to \infty \) in (3.9), we arrive at \( \lim_{n \to \infty} \|C^h_\phi f_n\|_{\mathcal{B}_\alpha^p} \leq K \epsilon \). Since \( \epsilon \) is arbitrary, thus
\[ \| C_{\phi}^{h} f_n \|_{B^\psi_{\alpha}} \to 0 \text{ as } n \to \infty. \] Along with Lemma 2.5, the operator \( C_{\phi}^{h} : F(p, q, s) \to B^\psi_{\alpha} \) is compact.

**Necessity** Assume the operator \( C_{\phi}^{h} : F(p, q, s) \to B^\psi_{\alpha} \) is compact. The boundedness of the operator clearly follows. Let \( \{z_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( \lim_{k \to \infty} |\phi(z_k)| = 1 \). If such sequence does not exist, then the inequality (3.7) vacuously hold. Set

\[
C f_k = \frac{1 - |\phi(z_k)|^2}{(1 - z\phi(z_k))^{\frac{2+q}{p}}}.
\]

Then \( f_k \in F(p, q, s) \) for \( k \in \mathbb{N} \) and \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). By Lemma 2.5, we derive \( \lim_{k \to \infty} \| C_{\phi}^{h} f_k \|_{B^\psi_{\alpha}} = 0 \), which yields that

\[
\| C_{\phi}^{h} f_k \|_{B^\psi_{\alpha}} = |C_{\phi}^{h} f_k(0)| + \sup_{z \in \mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^p}\right)} |f_k'(\phi(z))h(z)|
\]

\[
\geq \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^p}\right)} |f_k'(\phi(z_k))h(z_k)|
\]

\[
\geq \frac{|\phi(z_k)h(z_k)|}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^p}\right)(1 - |\phi(z_k)|^2)^{\frac{2+q}{p}}}.
\]

Letting \( k \to \infty \) in the above inequality, we verify that

\[
\lim_{|\phi(z)| \to 1} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^p}\right)(1 - |\phi(z)|^2)^{\frac{2+q}{p}}} = \lim_{k \to \infty} \frac{|\phi(z_k)h(z_k)|}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^p}\right)(1 - |\phi(z_k)|^2)^{\frac{2+q}{p}}} = 0.
\]

That is to say (3.7) holds. The proof is finished. \( \Box \)

**4 C\( h \) from F\( (p, q, s) \) to \( \beta\)-Zygmund–Orlicz Space**

In this section, we describe the boundedness and compactness of the operator \( C_{\phi}^{h} : F(p, q, s) \to Z^\psi_{\beta} \).

**Theorem 4.1** Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( \mathcal{N} \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then the operator \( C_{\phi}^{h} : F(p, q, s) \to Z^\psi_{\beta} \) is bounded if and only if

\[
M_1 := \sup_{z \in \mathbb{D}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^p}\right)(1 - |\phi(z)|^2)^{\frac{2+q}{p}+1}} < \infty,
\]

(4.1)
Suppose (4.1) and (4.2) are true. For any $f \in F(p, q, s)$, by the fact $\varphi$ is a strictly increasing convex function and Lemma 2.1, we can prove that

$$S_{\varphi, \beta} \left( \frac{(C^h_{\varphi} f)^{''}(z)}{C \| f \|_{F(p,q,s)}} \right) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} \varphi \left( \frac{|f^{''}(\varphi(z))\phi'(z)h(z) + f'(\varphi(z))h'(z)|}{C \| f \|_{F(p,q,s)}} \right)$$

$$\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} \varphi \left( \frac{|f''(\varphi(z))\phi'(z)h(z)| + |f'(\varphi(z))h'(z)|}{C \| f \|_{F(p,q,s)}} \right)$$

$$= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} \varphi \left( \frac{(1 - |z|)^{\frac{2+q}{p}}|f''(\varphi(z))|\phi'(z)h(z)}{(1 - |\varphi(z)|^{\frac{2+q}{p}} + 1)C \| f \|_{F(p,q,s)}} \right)$$

$$\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} \varphi \left( \frac{1}{C} \left( \frac{1}{(1 - |z|^2)^{\beta}} \right)\left( M_2 C_1 + M_1 \right) \right)$$

$$\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} \varphi \left( \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\beta}} \right) \right) = 1,$$

where $C_1$ emerged in Remark 2.2 and the constant $C$ is chosen satisfying $M_2 C_1 + M_1 \leq C$. By the display (1.5), (4.3) implies $\|C^h_{\varphi} f \|_{Z^\varphi_{\beta}} \leq C \| f \|_{F(p,q,s)}$. Apparently, the operator $C^h_{\varphi} : F(p, q, s) \to Z^\varphi_{\beta}$ is bounded.

**Necessity** Suppose that the operator $C^h_{\varphi} : F(p, q, s) \to Z^\varphi_{\beta}$ is bounded. Hence there is a constant $C > 0$ such that

$$\|C^h_{\varphi} f \|_{Z^\varphi_{\beta}} \leq C \| f \|_{F(p,q,s)} \quad \text{for all } f \in F(p, q, s).$$

That is,

$$S_{\varphi, \beta} \left( \frac{(C^h_{\varphi} f)^{''}(z)}{C \| f \|_{F(p,q,s)}} \right) \leq 1.$$

(4.4)

Fix $a \in \mathbb{D}$, define

$$f_a(z) = \frac{(1 - |\phi(a)|^{\frac{2+q}{p}})^{\frac{1+\frac{2+q}{p}}{p}}}{(1 - z\phi(a))^{\frac{2+q}{p}}} - 2 \frac{1 - |\phi(a)|^{\frac{2+q}{p}}}{(1 - z\phi(a))^{\frac{2+q}{p}}}.$$
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which belongs to $F(p, q, s)$ with $\sup_{a \in D} \| f_a \|_{F(p, q, s)} \leq 1$. Besides

$$f'_a(\phi(a)) = 0 \quad \text{and} \quad f''_a(\phi(a)) = 2 \left( \frac{2 + q}{p} \right)^2 \frac{\phi(a)^2}{(1 - |\phi(a)|^2)^{\frac{2+q}{p} + 1}}. \quad (4.5)$$

Putting $f_a$ into (4.4), we show that

$$\begin{align*}
(1 - |a|^2)^\beta & \varphi \left( |f''_a(\phi(a))\phi'(a)h(a)| \right) \\
& = (1 - |a|^2)^\beta \varphi \left( |f''_a(\phi(a))\phi'(a)h(a) + f'_a(\phi(a))h'(a)| \right) \\
& \leq \sup_{z \in D} (1 - |z|^2)^\beta \varphi \left( |f''_a(\phi(z))\phi'(z)h(z) + f'_a(\phi(z))h'(z)| \right) \leq 1. \quad (4.6)
\end{align*}$$

Hence

$$\begin{align*}
\frac{|f''_a(\phi(a))\phi'(a)h(a)|}{C\| f_a \|_{F(p, q, s)}} & \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)^\beta} \right),
\end{align*}$$

which is equivalent to saying that

$$\begin{align*}
2 \left( \frac{2 + q}{p} \right)^2 \frac{|\phi(a)|^2|\phi'(a)h(a)|}{C\| f_a \|_{F(p, q, s)}(1 - |\phi(a)|^2)^{\frac{2+q}{p} + 1}} & \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)^\beta} \right).
\end{align*}$$

Then

$$\begin{align*}
\frac{|\phi(a)|^2|\phi'(a)h(a)|}{\varphi^{-1} \left( \frac{1}{(1 - |a|^2)^\beta} \right)} (1 - |\phi(a)|^2)^{\frac{2+q}{p} + 1} & \leq C\| f_a \|_{F(p, q, s)} < \infty.
\end{align*}$$

Generally speaking,

$$\begin{align*}
\sup_{z \in D} \varphi^{-1} \left( \frac{1}{(1 - |z|^2)^\beta} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p} + 1} & < \infty. \quad (4.7)
\end{align*}$$

Fix $a \in \mathbb{D}$, define another test function

$$\hat{f}_a = \frac{(1 - |\phi(a)|^2)^2}{(1 - z\phi(a))^{\frac{2+q}{p} + 1}} - \left( 1 + \frac{2p}{2 + q} \right) \frac{1 - |\phi(a)|^2}{(1 - z\phi(a))^{\frac{2+q}{p}}},$$

which belongs to $F(p, q, s)$ with $\sup_{a \in \mathbb{D}} \| f_a \|_{F(p, q, s)} \leq 1$. Besides
which is also an element in $F(p, q, s)$ with $\sup_{a \in D} \| \hat{f}_a \|_{F(p, q, s)} \leq 1$. Moreover,

$$\hat{f}_a'(\phi(a)) = -\frac{\overline{\phi}(a)}{(1 - |\phi(a)|^2)^{2+q/p}} \text{ and } \hat{f}_a''(\phi(a)) = 0. \quad (4.8)$$

By the analogous discussion above, we obtain that

$$\sup_{z \in D} \frac{|\phi(z)h'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{2+q/p}} < \infty. \quad (4.9)$$

After that pick two special test functions $f_0(z) = z$ and $\hat{f}_0(z) = z^2$. Replacing $f_a$ by $f_0$ and $\hat{f}_0$ in (4.6), respectively, we can state that

$$\hat{L}_1 := \sup_{z \in \mathbb{D}} \frac{|h'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right)} < \infty, \quad (4.10)$$

$$\sup_{z \in \mathbb{D}} \frac{|\phi'(z)h(z) + \phi(z)h'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right)} < \infty. \quad (4.11)$$

Then (4.10) together with (4.11) indicate

$$\hat{L}_2 := \sup_{z \in \mathbb{D}} \frac{|\phi'(z)h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right)} < \infty. \quad (4.12)$$

An application of (4.7) and (4.12) tell us (4.1), and (4.9) together with (4.10) entail (4.2), respectively. The proof is complete.

**Theorem 4.2** Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function, $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$. Then the operator $C^h_\phi : F(p, q, s) \to \mathcal{Z}_\beta^\varphi$ is compact if and only if $C^h_\phi : F(p, q, s) \to \mathcal{Z}_\beta^\varphi$ is bounded and

$$\lim_{|\phi(z)| \to 1} \frac{|\phi'(z)h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{2+q/p + 1}} = 0, \quad (4.13)$$

$$\lim_{|\phi(z)| \to 1} \frac{|h'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{2+q/p}} = 0. \quad (4.14)$$

**Proof** Sufficiency Assume the operator $C^h_\phi : F(p, q, s) \to \mathcal{Z}_\beta^\varphi$ is bounded and (4.13) and (4.14) hold. By the boundedness of $C^h_\phi : F(p, q, s) \to \mathcal{Z}_\beta^\varphi$ and making use of the proof in Theorem 4.1, we assert that $\hat{L}_1 < \infty$ and $\hat{L}_2 < \infty$ in (4.10) and (4.12),
respectively. Applying (4.13) and (4.14), we claim that, for every \( \epsilon > 0 \), there exists \( 0 < r < 1 \) such that

\[
\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2) \frac{2+q}{p} + 1 < \frac{\epsilon}{2},
\]

(4.15)

\[
|\phi'(z)h(z)| < \frac{\epsilon}{2},
\]

(4.16)

for \(|\phi(z)| > r\). Choose a sequence \( \{f_n\} \subset F(p, q, s) \) with \( \|f_n\|_{F(p, q, s)} \leq K \) and \( f_n \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). By Lemma 2.5, we will calculate \( \|C^h_{\phi} f_n\|_{\mathcal{Z}^q_{\beta}} \to 0 \) as \( n \to \infty \). By (1.6), we describe the norm into

\[
\|C^h_{\phi} f_n\|_{\mathcal{Z}^q_{\beta}} = |C^h_{\phi} f_n(0)| + |\{C^h_{\phi} f_n\}'(0)| + \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) |(C^h_{\phi} f_n)'(z)|
\]

\[
= |f_n'(\phi(0))h(0)| + \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) |f_n''(\phi(z))\phi'(z)h(z) + f_n'(\phi(z))h'(z)|
\]

\[
= |f_n'(\phi(0))h(0)| + \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) |f_n''(\phi(z))\phi'(z)h(z) + f_n'(\phi(z))h'(z)|
\]

\[
+ \sup_{\{z \in \mathbb{D}: |\phi(z)| > r\}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) |f_n''(\phi(z))\phi'(z)h(z) + f_n'(\phi(z))h'(z)|
\]

\[
\leq |f_n'(\phi(0))h(0)| + \hat{L}_2 \sup_{\{w \in \mathbb{D}: |w| \leq r\}} |f_n''(w)| + \hat{L}_1 \sup_{\{w \in \mathbb{D}: |w| \leq r\}} |f_n'(w)|
\]

\[
+ K \sup_{\{z \in \mathbb{D}: |\phi(z)| > r\}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2) \frac{2+q}{p} + 1 \cdot |\phi'(z)h(z)|
\]

\[
+ K \sup_{\{z \in \mathbb{D}: |\phi(z)| > r\}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2) \frac{2+q}{p} \cdot |h'(z)|
\]

\[
< |f_n'(\phi(0))h(0)| + \hat{L}_2 \sup_{\{w \in \mathbb{D}: |w| \leq r\}} |f_n''(w)| + \hat{L}_1 \sup_{\{w \in \mathbb{D}: |w| \leq r\}} |f_n'(w)| + 2K \epsilon.
\]

By Cauchy estimate, we claim that \( \lim_{n \to \infty} \|C^h_{\phi} f_n\|_{\mathcal{Z}^q_{\beta}} \leq 2K \epsilon \). Since \( \epsilon \) is arbitrary and then Lemma 2.5 implies the operator \( C^h_{\phi} : F(p, q, s) \to \mathcal{Z}^q_{\beta} \) is compact.

**Necessity** Assume the operator \( C^h_{\phi} : F(p, q, s) \to \mathcal{Z}^q_{\beta} \) is compact. The boundedness of \( C^h_{\phi} : F(p, q, s) \to \mathcal{Z}^q_{\beta} \) is trivial, we next prove (4.13) and (4.14). Let \( \{z_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( \lim_{k \to \infty} |\phi(z_k)| = 1 \). If such sequence does not exist, then the inequalities (4.13) and (4.14) vacuously hold. Setting the functions sequence

\[
f_k(z) = \frac{(1 - |\phi(z_k)|^2)^{1+\frac{2+q}{p}}}{(1 - z\phi(z_k))^2} - 2 \frac{1 - |\phi(z)|^2}{(1 - z\phi(z_k))^{2+q}}.
\]
we arrive at the sequence \( \{ f_k \} \) is uniformly bounded in \( F(p, q, s) \) and uniformly converges to zero on any compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Employing Lemma 2.5, it entails that \( \lim_{k \to \infty} \| C^h_{\phi} f_k \|_{Z^\beta} = 0 \). Combining (4.5), it turns out that

\[
\| C^h_{\phi} f_k \|_{Z^\beta} = |C^h_{\phi} f_k(0)| + |(C^h_{\phi} f_k)\prime(0)| + \sup_{z \in \mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} |(C^h_{\phi} f_k)''(z)|
\]

\[
\geq \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} |(C^h_{\phi} f_k)''(z_k)|
\]

\[
= \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} |f_k''(\phi(z_k))\phi'(z_k)h(z_k) + f_k'(\phi(z_k))h'(z_k)|
\]

\[
\geq \frac{\varphi'(z_k)h(z_k)}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} \frac{|\phi(z_k)|^2}{(1 - |\phi(z_k)|^2)^{\frac{2+q}{p}+1}}.
\]

Letting \( k \to \infty \) in the above inequality, it follows that

\[
\lim_{|\phi(z)| \to 1} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} \frac{|\phi(z_k)|^2}{(1 - |\phi(z_k)|^2)^{\frac{2+q}{p}+1}} = 0.
\]

That is to say (4.13) is true. On the other hand, we pick

\[
\hat{f}_k(z) = \frac{(1 - |\phi(z_k)|^2)^2}{(1 - z\phi(z_k))^\frac{2+q}{p}+1} - \left(1 + \frac{2p}{2+q}\right) \frac{1 - |\phi(z_k)|^2}{(1 - z\phi(z_k))^\frac{2+q}{p}},
\]

which is also uniformly bounded in \( F(p, q, s) \) and uniformly converges to zero on any compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Hence we arrive at \( \lim_{k \to \infty} \| C^h_{\phi} \hat{f}_k \|_{Z^\beta} = 0 \). Combining (4.10), it turns out

\[
\lim_{k \to \infty} \frac{|\phi(z_k)h'(z_k)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{p}}\right)} \frac{|\phi(z_k)|^2}{(1 - |\phi(z_k)|^2)^{\frac{2+q}{p}+1}} = 0,
\]

which implies (4.14). This ends the proof.

\[\square\]

5 Some Corollaries

In this section, we listed some corollaries for several classical integral-type operators acting from \( F(p, q, s) \) to \( \alpha\)-Bloch–Orlicz and \( \beta\)-Zygmund–Orlicz spaces.
Integral-Type Operators from $F(p, q, s)$ Space to α-Bloch–Orlicz...

(I) Let $\phi = id$, the identity map, then $C_{id}^h = L^h$, which provides the corollaries for the integral-type operator $L^h : F(p, q, s) \rightarrow \mathcal{B}_\alpha^\phi$ (or $\mathcal{Z}_\beta^\phi$). Indeed since $\phi(z) = z$, we have $\phi'(z) = 1$. Putting $\phi(z) = z$ into Theorems 3.1 and 3.2, we deduce Corollaries 5.1 and 5.2; and then we infer Corollaries 5.3 and 5.4 from Theorems 4.1 and 4.2 with $\phi(z) = z$ and $\phi'(z) = 1$ simultaneously.

**Corollary 5.1** Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then $L^h : F(p, q, s) \rightarrow \mathcal{B}_\alpha^\phi$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} < \infty.$$  

**Corollary 5.2** Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then $L^h : F(p, q, s) \rightarrow \mathcal{B}_\alpha^\phi$ is compact if and only if $L^h : F(p, q, s) \rightarrow \mathcal{B}_\alpha^\phi$ is bounded and

$$\lim_{|z| \to 1} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} = 0.$$  

**Corollary 5.3** Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then $L^h : F(p, q, s) \rightarrow \mathcal{Z}_\beta^\phi$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} < \infty,$$

and

$$\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h'(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} < \infty.$$  

**Corollary 5.4** Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then $L^h : F(p, q, s) \rightarrow \mathcal{Z}_\beta^\phi$ is compact if and only if $L^h : F(p, q, s) \rightarrow \mathcal{Z}_\beta^\phi$ is bounded and

$$\lim_{|z| \to 1} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} = 0,$$

and

$$\lim_{|z| \to 1} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( \frac{|h'(z)|}{1-|z|^2} \right)^{\frac{2+q}{p}} = 0.$$  

(II) Let $h(z) = k'(z) \in H(\mathbb{D})$, then $C_{\phi}^h = V_k^h$, which allows us to list analogous characterizations for the integral operator $V_k^h : F(p, q, s) \rightarrow \mathcal{B}_\alpha^\phi$ (or $\mathcal{Z}_\beta^\phi$). Putting $h(z) = k'(z)$ into Theorems 3.1 and 3.2, we deduce Corollaries 5.5 and 5.6; and then we can conclude Corollaries 5.7 and 5.8 from Theorems 4.1 and 4.2, respectively.
Corollary 5.5 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function, $\phi \in S(D)$ and $k \in H(D)$. Then $V^k_\phi : F(p, q, s) \to B^\varphi_\alpha$ is bounded if and only if

$$\sup_{z \in D} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} < \infty.$$  

Corollary 5.6 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function, $\phi \in S(D)$ and $k \in H(D)$. Then $V^k_\phi : F(p, q, s) \to B^\varphi_\alpha$ is compact if and only if $V^k_\phi : F(p, q, s) \to B^\varphi_\alpha$ is bounded and

$$\lim_{|\phi(z)| \to 1} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} = 0.$$  

Corollary 5.7 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function, $\phi \in S(D)$ and $k \in H(D)$. Then $V^k_\phi : F(p, q, s) \to \mathcal{Z}^\varphi_\beta$ is bounded if and only if

$$\sup_{z \in D} \frac{|\phi'(z)k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}+1}} < \infty,$$

$$\sup_{z \in D} \frac{|k''(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} < \infty.$$  

Corollary 5.8 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function, $\phi \in S(D)$ and $k \in H(D)$. Then $V^k_\phi : F(p, q, s) \to \mathcal{Z}^\varphi_\beta$ is compact if and only if $V^k_\phi : F(p, q, s) \to \mathcal{Z}^\varphi_\beta$ is bounded and

$$\lim_{|\phi(z)| \to 1} \frac{|\phi'(z)k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}+1}} = 0,$$

$$\lim_{|\phi(z)| \to 1} \frac{|k''(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p}}} = 0.$$  

(III) Let $\phi = id$, the identity map, then $V^k_{id} = L_k$, so we conclude the corollaries for the integral-type operator $L_k : F(p, q, s) \to B^\varphi_\alpha$ (or $\mathcal{Z}^\varphi_\beta$). Putting $\phi(z) = z$ and $\phi'(z) = 1$ into Corollaries 5.5—5.8, we deduce Corollaries 5.9—5.12, respectively.

Corollary 5.9 Let $\varphi : [0, \infty) \to [0, \infty)$ be an $N$-function and $k \in H(D)$. Then $L_k : F(p, q, s) \to B^\varphi_\alpha$ is bounded if and only if

$$\sup_{z \in D} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} \right) (1 - |z|^2)^{\frac{2+q}{p}}} < \infty.$$
Corollary 5.10 Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( \mathcal{N} \)-function and \( k \in H(\mathbb{D}) \). Then \( L_k : F(p, q, s) \to \mathcal{B}^p_\alpha \) is compact if and only if \( L_k : F(p, q, s) \to \mathcal{B}^q_\alpha \) is bounded and

\[
\lim_{|z| \to 1} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |z|^2 \right)^{\frac{2+q}{p} + 1}} = 0.
\]

Corollary 5.11 Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( \mathcal{N} \)-function and \( k \in H(\mathbb{D}) \). Then \( L_k : F(p, q, s) \to \mathcal{Z}^p_\beta \) is bounded if and only if

\[
\sup_{z \in \mathbb{D}} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |z|^2 \right)^{\frac{2+q}{p} + 1}} < \infty,
\]

\[
\sup_{z \in \mathbb{D}} \frac{|k''(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |z|^2 \right)^{\frac{2+q}{p} + 1}} < \infty.
\]

Corollary 5.12 Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( \mathcal{N} \)-function and \( k \in H(\mathbb{D}) \). Then \( L_k : F(p, q, s) \to \mathcal{Z}^q_\beta \) is compact if and only if \( L_k : F(p, q, s) \to \mathcal{Z}^q_\beta \) is bounded and

\[
\lim_{|z| \to 1} \frac{|k'(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |z|^2 \right)^{\frac{2+q}{p} + 1}} = 0,
\]

\[
\lim_{|z| \to 1} \frac{|k''(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |z|^2 \right)^{\frac{2+q}{p} + 1}} = 0.
\]

(IV) From another point of view, we extend again the generalized composition operator \( C^h_{\phi} \). Let \( h \in \mathbb{D} \), \( m \) be a nonnegative integer and \( \phi \in S(\mathbb{D}) \), then define

\[
(C^m_{\phi,h} f)(z) = \int_0^z f^{(m)}(\phi(t))h(t)dt, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

This integral-type operator was firstly introduced by Zhu in [30]. In particular, if \( m = 1 \), then \( C^1_{\phi,h} \) is the generalized composition operator \( C^h_{\phi} \). In the sequel, we derive corollaries for the boundedness and compactness of \( C^m_{\phi,h} : F(p, q, s) \to \mathcal{B}^p_\alpha \), in line with Theorems 3.1, 3.2 and 4.1, 4.2.

Theorem 5.13 Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( \mathcal{N} \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then \( C^m_{\phi,h} : F(p, q, s) \to \mathcal{B}^p_\alpha \) is bounded if and only if

\[
\sup_{z \in \mathbb{D}} \frac{|h(z)|}{\varphi^{-1} \left( \frac{1}{(1-|z|^2)^{\varphi}} \right) \left( 1 - |\phi(z)|^2 \right)^{\frac{2+q}{p} + m - 1}} < \infty.
\]
Proof Sufficiency We follow the similar lines in the proof of Theorem 3.1 and replace 
\((1 - |\phi(z)|^2)^{\frac{2+q}{p}}\) by \((1 - |\phi(z)|^2)^{\frac{2+q}{p} + m} - 1\) in the sufficient part.

Necessity Suppose the operator \(C_{\phi,h}^m : F(p, q, s) \to B^{\phi}_s\) is bounded. On the one hand, we still adopt the function emerged in (3.4). It holds that

\[ f_a^{(m)}(z) \approx \frac{(\phi(a))^m (1 - |\phi(a)|^2)}{(1 - z\phi(a))^{\frac{2+q}{p} + m}}, \]

for \(a \in \mathbb{D}\), hence we arrive at

\[ (1 - |a|^2)^{\alpha} \varphi \left( \frac{(C_{\phi,h}^m f_a)'(a)}{C\|f_a\|_{F(p,q,s)}} \right) \leq 1. \]

It yields that

\[ \varphi \left( \frac{(C_{\phi,h}^m f_a)'(a)}{C\|f_a\|_{F(p,q,s)}} \right) \leq \frac{1}{(1 - |a|^2)^{\alpha}}. \]

Taking \(\varphi^{-1}\) on both sides, it entails that

\[ \frac{|f_a^{(m)}(\phi(a))h(a)|}{C\|f_a\|_{F(p,q,s)}} \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)^{\alpha}} \right). \]

Furthermore,

\[ \frac{|\phi(a)|^m |h(a)|}{(1 - |\phi(a)|^2)^{\frac{2+q}{p} + m - 1} C\|f_a\|_{F(p,q,s)}} \leq \varphi^{-1} \left( \frac{1}{(1 - |a|^2)^{\alpha}} \right). \]

It follows that

\[ \sup_{a \in \mathbb{D}} \frac{|\phi(a)|^m |h(a)|}{\bar{\varphi}^{-1} \left( \frac{1}{(1 - |a|^2)^{\alpha}} \right) (1 - |\phi(a)|^2)^{\frac{2+q}{p} + m - 1}} \leq \sup_{a \in \mathbb{D}} \|f_a\|_{F(p,q,s)} < \infty. \]

That is,

\[ \sup_{z \in \mathbb{D}} \frac{|\phi(z)|^m |h(z)|}{\varphi^{-1} \left( \frac{1}{(1 - |z|^2)^{\alpha}} \right) (1 - |\phi(z)|^2)^{\frac{2+q}{p} + m - 1}} < \infty. \quad (5.2) \]

On the other hand, we choose the function \(z^m \in F(p, q, s)\), then (3.6) is true. The inequality (3.6) together with (5.2) reveal (5.1). This completes the proof.
\textbf{Theorem 5.14} Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( N \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then \( C^m_{\phi, h} : F(p, q, s) \to \mathcal{B}_{\alpha}^\phi \) is compact if and only if \( C^m_{\phi, h} : F(p, q, s) \to \mathcal{B}_{\alpha}^\phi \) is bounded and
\[
\lim_{|\varphi(z)| \to 1} \frac{|h(z)|}{\varphi^{-1} \left( \frac{1}{1-|z|^2} \right) \left( 1 - |\varphi(z)|^2 \right)^{\frac{2+q}{p} + m - 1}} = 0. \tag{5.3}
\]

\textbf{Proof} \textbf{Sufficiency} This part is similar to the proof in (3.9) just using the display (2.3) and Lemma 2.5.

\textbf{Necessity} Assume the operator \( C^m_{\phi, h} : F(p, q, s) \to \mathcal{B}_{\alpha}^\phi \) is compact. Let \( \{z_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \). If such sequence does not exist, then the inequality (5.3) vacuously hold. We still employ the function sequence \( \{f_k\} \) defined in (3.10) and then it turns out that
\[
\|C^m_{\phi, h}f_k\|_{\mathcal{B}_{\alpha}^\phi} = |C^m_{\phi, h}f_k(0)| + \sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) |f_k^{(m)}(\varphi(z))h(z)|
\geq \varphi^{-1} \left( \frac{1}{1-|z_k|^2} \right) |f_k^{(m)}(\varphi(z_k))h(z_k)|
\geq \varphi^{-1} \left( \frac{1}{1-|z_k|^2} \right) (1 - |\varphi(z_k)|^2)^{\frac{2+q}{p}}.
\]

Letting \( k \to \infty \) in the above inequality and then the desired result follows from Lemma 2.5. This ends the proof. \( \square \)

\textbf{Theorem 5.15} Let \( \varphi : [0, \infty) \to [0, \infty) \) be an \( N \)-function, \( \phi \in S(\mathbb{D}) \) and \( h \in H(\mathbb{D}) \). Then \( C^m_{\phi, h} : F(p, q, s) \to \mathcal{Z}_{\beta}^\phi \) is bounded if and only if
\[
\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) (1 - |\varphi(z)|^2)^{\frac{2+q}{p} + m} \varphi'(z)h(z) < \infty,
\]
\[
\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{1-|z|^2} \right) (1 - |\varphi(z)|^2)^{\frac{2+q}{p} + m - 1} |h'(z)| < \infty.
\]

\textbf{Proof} \textbf{Sufficiency} This implication follows similarly from the sufficient part in Theorem 4.1 by use of the display (2.3).

\textbf{Necessity} We assume \( C^m_{\phi, h} : F(p, q, s) \to \mathcal{Z}_{\beta}^\phi \) is bounded. For \( a \in \mathbb{D} \), we define two test functions as
\[
f_a(z) = \frac{(1 - |\phi(a)|^2)^2}{(1 - z\phi(a))^{2 + \frac{q}{p} + 1}} - \left( 1 + \frac{pm}{2 + q} \right) \frac{1 - |\phi(a)|^2}{(1 - z\phi(a))^{\frac{2+q}{p}}},
\]
\[
\hat{f}_a(z) = \frac{(1 - |\phi(a)|^2)^2}{(1 - z\phi(a))^{2 + \frac{q}{p} + 1}} - \left( 1 + \frac{p(m+1)}{2 + q} \right) \frac{1 - |\phi(a)|^2}{(1 - z\phi(a))^{\frac{2+q}{p}}}.\]
Both of them belong to $F(p, q, s)$ with $\sup_{a \in \mathbb{D}} \left\{ \| f_a \|_{F(p, q, s)}, \| \hat{f}_a \|_{F(p, q, s)} \right\} \leq 1$. Besides,

$$f_a^{(m)}(\phi(a)) = 0 \quad \text{and} \quad f_a^{(m+1)}(\phi(a)) \asymp \frac{\phi(a)^{m+1}}{(1 - |\phi(a)|^2)^{\frac{2+q}{p}+m}}; \tag{5.4}$$

$$\hat{f}_a^{(m)}(\phi(a)) \asymp \frac{\phi(a)^{m}}{(1 - |\phi(a)|^2)^{\frac{2+q}{p}+m-1}} \quad \text{and} \quad \hat{f}_a^{(m+1)}(\phi(a)) = 0. \tag{5.5}$$

Employing similar methods in Theorem 4.1, we arrive at

$$\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} (1 - |\phi(z)|^2)^{\frac{2+q}{p}+m} \right) < \infty. \tag{5.6}$$

$$\sup_{z \in \mathbb{D}} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} (1 - |\phi(z)|^2)^{\frac{2+q}{p}+m-1} \right) < \infty. \tag{5.7}$$

After that, taking test functions $z^m$ and $z^{m+1}$, it follows that $\hat{L}_1 < \infty$ and

$$\sup_{z \in \mathbb{D}} \left| (m+1)! \phi'(z)h(z) + m! \phi(z)h'(z) \right| < \infty.$$ 

And then we can deduce $\hat{L}_2 < \infty$. Finally, (5.6), (5.7) together with the boundedness of $\hat{L}_1$ and $\hat{L}_2$ imply the desired results. This completes the proof. \hfill Q.E.D.

**Theorem 5.16** Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $N$-function, $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$. Then $C_{\phi, h}^m : F(p, q, s) \rightarrow Z_\beta^\varphi$ is compact if and only if $C_{\phi, h}^m : F(p, q, s) \rightarrow Z_\beta^\varphi$ is bounded and

$$\lim_{|\phi(z)| \rightarrow 1} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} (1 - |\phi(z)|^2)^{\frac{2+q}{p}+m} \right) \frac{|\phi'(z)h(z)|}{|h'(z)|} = 0, \tag{5.8}$$

$$\lim_{|\phi(z)| \rightarrow 1} \varphi^{-1} \left( \frac{1}{(1-|z|^2)^p} (1 - |\phi(z)|^2)^{\frac{2+q}{p}+m-1} \right) \frac{|\phi'(z)h(z)|}{|h'(z)|} = 0. \tag{5.9}$$

**Proof Sufficiency** This implication follows analogously from the sufficient part by the inequality (2.3) and Lemma 2.5.

**Necessity** The boundedness of $C_{\phi, h}^m : F(p, q, s) \rightarrow Z_\beta^\varphi$ is clear from the compactness. In order to show (5.8) and (5.9), we let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k \rightarrow \infty} |\phi(z_k)| = 1$. If such sequence does not exist, then the inequalities (5.8) and (5.9)
vacuously hold. Picking two function sequences

\[
\begin{align*}
    f_k(z) &= \frac{(1 - |\phi(z_k)|^2)^2}{(1 - z^2|\phi(z_k)|^2)^{\frac{2+q}{p}+1}} - \left(1 + \frac{pm}{2+q}\right) \frac{1 - |\phi(z_k)|^2}{(1 - z^2|\phi(z_k)|^2)^{\frac{2+q}{p}}}, \\
    \hat{f}_k(z) &= \frac{(1 - |\phi(z_k)|^2)^2}{(1 - z^2|\phi(z_k)|^2)^{\frac{2+q}{p}+1}} - \left(1 + \frac{p(m+1)}{2+q}\right) \frac{1 - |\phi(z_k)|^2}{(1 - z^2|\phi(z_k)|^2)^{\frac{2+q}{p}}},
\end{align*}
\]

which are both uniformly bounded in \( F(p, q, s) \) and uniformly converges to zero on any compact subsets of \( \mathbb{D} \) as \( k \to \infty \). On account of Lemma 2.5, we have that \( \lim_{k \to \infty} \|C_{\phi, h} f_k\|_{Z^\beta_p} = 0 \) and \( \lim_{k \to \infty} \|C_{\phi, h} \hat{f}_k\|_{Z^\beta_p} = 0 \). In other words, we conclude that

\[
\begin{align*}
    \lim_{k \to \infty} \frac{|\phi(z_k)|^{m+1}|\phi'(z_k)h(z_k)|}{\varphi^{-1} \left( \frac{1}{(1-|z_k|^2)^p} \right) \left(1 - |\phi(z_k)|^2\right)^{\frac{2+q}{p}+m}} &= 0; \\
    \lim_{k \to \infty} \frac{|\phi(z_k)|^m|h'(z_k)|}{\varphi^{-1} \left( \frac{1}{(1-|z_k|^2)^p} \right) \left(1 - |\phi(z_k)|^2\right)^{\frac{2+q}{p}+m-1}} &= 0.
\end{align*}
\]

The above displays imply (5.8) and (5.9) and the proof is finished. \( \square \)

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