Water-Waves Problem with Surface Tension in a Corner Domain II: The Local Well-Posedness

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Abstract
Based on the a priori estimates in our previous work, we continue to investigate the water-waves problem in a bounded two-dimensional corner domain in this paper. We prove the local well-posedness of the solution to the water-waves system when the contact angles are less than $\pi/16$. © 2020 Wiley Periodicals LLC

1 Introduction
We consider the irrotational incompressible water-waves problem in a two-dimensional bounded corner domain $\Omega_t$ with an upper free surface $\Gamma_t$ and a fixed bottom $\Gamma_b$. This domain contains two contact points, $p_l, p_r$, with contact angles $\omega_l, \omega_r$, which are the intersections of $\Gamma_t$ and $\Gamma_b$.

The water-waves problem on $\Omega_t$ can be expressed as the system of velocity $v$ and pressure $P$:

\[
\begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla P + g, \\
\text{div } v = 0, \quad \text{curl } v = 0 & \text{on } \Omega_t \\
P|_{\Gamma_t} = \sigma \kappa, \\
\partial_t v + v \cdot \nabla \text{ is tangent to } \bigcup_s \Gamma_s, \\
v \cdot N_{\Gamma_b}|_{\Gamma_b} = 0, \\
\beta_{\epsilon_i} v_i = \sigma (\cos \omega_s - \cos \omega_i) & \text{at } p_i \ (i = l, r),
\end{cases}
\]

(WW)

where $\kappa$ is the mean curvature of the free surface, $\sigma$ is the surface tension coefficient, $\omega_i \ (i = l, r)$ are the contact angles between $\Gamma_t$ and $\Gamma_b$, and $g$ is the gravity coefficient with $g = -g e_z$ the gravity vector. Moreover, we denote by $v_i$ the upward tangential component of the velocity at the corner points along $\Gamma_b$:

\[
v_l = -v \cdot \tau_b \quad \text{at } p_l \quad \text{and} \quad v_r = v \cdot \tau_b \quad \text{at } p_r.
\]
The last condition in (WW) describes the motion at the contact points, which was studied in [46] and was used in our previous work [41]. Here the stationary contact angle $\omega_s$ is a physical constant depending on the materials of the bottom and the fluid, and $\beta_e$ denotes the effective friction coefficient. In fact, this condition tells us that slip velocity is dominated by the unbalanced Young stress, which is an effective variation of Young’s law (1805) for stationary contact angles [61]. These kinds of conditions are very common and widely discussed; see [11, 12, 22, 50]. One can see in our previous work [41] that there is some dissipation corresponding to this condition at the contact points naturally. Mathematically, this condition turns out to be some kind of Lopatinsky condition, and it is necessary for solving the linear system for the iteration as well as for proving the energy estimates. Moreover, a similar condition was used in [25] for the Stokes flow.

1.1 Some Known Results

Let us recall some previous works on the well-posedness for the water-waves problems. When we say “classical” water-waves problems, we refer to the water-waves problems with a smooth free surface, and the fluid boundaries satisfy $\Gamma_{\tau} \cap \Gamma_b = \emptyset$. There is a rich literature on the classical water-waves problems.

First, we recall the results on the local well-posedness for the classical water-waves problems. To begin with, we have a quick review on some works about the irrotational case. Some early works such as Nalimov [43], Yoshihara [59, 60], and Craig [19] established the local well-posedness with small data in the two-dimensional case. Wu [54, 55] made a breakthrough that removed the smallness condition and proved that the Taylor sign condition

$$-\nabla_{N_{\tau}} P |_{\Gamma_{\tau}} \geq c_0 > 0$$

always holds as long as $\Gamma_{\tau}$ is not self-intersecting. Later on, some different methods were applied to prove the local well-posedness. Iguchi [29] and Ambrose [6] studied the local well-posedness in the two-dimensional case. In [34], Lannes proved the finite-depth case under Eulerian coordinates. Later, Ming and Zhang [42] generalized Lannes’s paper to the case with surface tension. Alazard, Burq, and Zuily in [1–3] used the tools of paradifferential operators to prove the local well-posedness in a low-regularity case. Moreover, Alvarez-Samaniego and
Lannes [5] considered the large-time existence for the problem under the shallow-water regime.

Concerning the rotational case, there are also many works on the local well-posedness using various methods. Christodoulou and Lindblad [16] were the first to prove a priori estimates based on the geometry of the moving domain, and later Lindblad [39] proved the existence of solutions using Nash-Moser iteration. Coutand and Shkoller [17] proved the local well-posedness under Lagrangian coordinates. Zhang and Zhang [62] used the Clifford analysis introduced by Wu [55] to solve the problem. Shatah and Zeng [48,49] treated the problem in a geometric way, where they used the equation of the mean curvature. Meanwhile, a similar geometric approach was also used by Beyer and Günther [9,10] to study the irrotational problem for some star-shaped domains. Recently Wang, Zhang, Zhao, and Zheng [53] proved the local well-posedness in the low-regularity case. For more results on local well-posedness, the reader can check the book by Lannes [35], and Iguchi, Tanaka, and Tani [30]. Ogawa and Tani [44,45], Schweizer [47], and Ambrose and Masmoudi [7,8].

For the global well-posedness, the first result was given by Wu [56], which proved the almost-global existence for the gravity problem in two dimensions. Later, Wu [57] and Germain, Masmoudi, and Shatah [23] showed the global existence of gravity waves in three dimensions by different methods. Moreover, Alazard and Delort [4] and Ionescu and Pusateri [32] studied the global regularity for gravitational water-waves systems in two dimensions independently. Recently Hunter, Ifrim, and Tataru [26–28] used the conformal-mapping method to give another proof of the global existence for the gravitational problem in two dimensions. For more results on the global well-posedness, the reader can check [21,52] and their references.

Compared to the classical water-waves case, when we say “nonsmooth” water-waves problems, we mean that there are contact points on the fluid boundaries, i.e., \( \Gamma_t \cap \Gamma_b \neq \emptyset \), or the free surface is not smooth. In fact, theoretical research on this field only started several years ago and there remains a lot of open problems. Alazard, Burq, and Zuily [3] proved the local well-posedness for the special case when the contact angle is equal to \( \pi/2 \). In this case, they used symmetrizing and periodizing methods to turn this problem into a classical case. Later, Kinsey and Wu proved the local well-posedness for two-dimensional water waves with angled crests when the wall is vertical; see [33,58]. Recently de Poyferré [20] gave a priori estimates for the water-waves problem in a bounded corner domain without surface tension under the assumption of small contact angles. Meanwhile, under the assumption of small contact angles, the authors proved a priori estimates for the water-waves problem in a corner domain with surface tension; see [41]. For both results [20,41], one important observation is that small contact angles can prevent the appearance of singularities from the corners.

On the other hand, Lannes and Métivier [38] solved the local well-posedness for the Green-Naghdi equations in a beach-type domain, which is a shallow-water
model of the water-waves problem. Lannes [36] addressed the floating-body problem and proposed a new formulation of the water-waves problem that can be easily generalized in order to take into account the presence of a floating body. Very recently Lannes and Iguchi [37] proved some sharp results for the initial boundary value problem with a free boundary arising in wave-structure interaction, and it contains the floating problem in the shallow-water regime. In addition, Guo and Tice considered a priori estimates for the contact line problem in the case of the Stokes equations; see [25]. Later Tice and Zheng proved the local well-posedness of the contact line problem in 2D Stokes flow; see [51].

At the end of this section, we also mention some results concerning geometric singularities on the free surfaces for the water-waves problems. In [14], the authors showed the existence of a wave that is given initially as the graph of a function and then can overturn at a later time. Later on, the authors in [13] proved the existence of some “splash” singularities. Moreover, this result was extended in [18] to the three-dimensional case and some other models.

1.2 Main Results and Ideas

In this paper, we prove the local well-posedness of system (WW) on a bounded two-dimensional corner domain, which is based on our previous work [41]. The following theorem states our main result:

**Theorem 1.1.** Assume that the initial data \((\Gamma_0, v_0) \in H^{8.5} \times H^{7.5}(\Omega_0)\) and the initial contact angles \(\omega_{i0} \in (0, \pi/16)\) for \(i = l, r\). When the compatibility conditions at \(t = 0\), namely

\[
\beta_c \sigma^k_i v_i(0) = \sigma \sigma^k_i (\cos \omega_s - \cos \omega_f(0))
\]

at \(p_i (i = l, r)\) for \(k = 0, 1, 2, 3, 4\), are satisfied, there exists a small constant \(T > 0\) depending on the initial data such that system (WW) has a unique solution \((\Gamma_t, v) \in C([0, T]; H^{8.5}) \times C([0, T]; H^{7.5}(\Omega_t))\). Moreover, the solution \((\Gamma_t, v)\) is locally well-posed.

**Remark 1.2.** According to our previous paper [41] and compactness arguments, we can get \((\Gamma_t, v) \in C([0, T]; H^s) \times C([0, T]; H^{s-1}(\Omega_t))\) with \(4 \leq s \leq 8.5\).

Similarly as in [48, 49], the pressure \(P\) in (WW) is regained by the velocity \(v\) and the mean curvature \(\kappa\). In fact, we decompose the pressure into two parts,

\[ P = \sigma \kappa + P_{v,v}, \]

where the first part is the harmonic extension of \(\kappa\) and the second part \(P_{v,v}\) is decided by \(v\) and \(g\):

\[
\begin{align*}
\Delta P_{v,v} &= -tr(\nabla v)^2 \quad \text{on } \Omega_t, \\

P_{v,v}|_{\Gamma_t} &= 0, \quad \nabla g P_{v,v} |_{\Gamma_t} = \nabla v N_b + \nabla v N_b \cdot v + N_b \cdot g.
\end{align*}
\]

Therefore, as long as we have \((\Gamma_t, v)\), the whole water-waves system (WW) is recovered immediately.
Remark 1.3. In this paper, we need smaller contact angles compared to [41]. The reason is that when we construct the Cauchy sequence, we need a higher regularity. To avoid the singularity from the corners, the range of the angles is determined by $P_{v,v}$ by using remark 5.20 [40] for the related mixed-boundary elliptic problem. The local well-posedness of system $\text{(WW)}$ with general contact angles still remains an open problem.

Now, we explain the main ideas of this paper. First, we need to choose a good formulation to construct approximate solutions. Inspired by [49], we introduce a universal coordinate map $\hat{S}_t \in \mathbb{R}$ that can reduce our system into a system defined on a fixed domain. We use $d_{\Gamma_t}$ for the “distance” between $\mathbb{E}$ and $\mathbb{E}_t$, where $\mathbb{E}$ is some reference upper surface.

Second, based on the mean curvature of $\mathbb{E}_t$, we introduce a new quantity on $\mathbb{E}$:

$$N_a \frac{\partial}{\partial \mathbb{E}_t \frac{\partial}{\partial \mathbb{E}_t}} = N \mathbb{E}_t \circ \Phi_{S_t} \circ \Phi_{S_t}^{-1} \circ \Phi_{S_t} \cdot d_{\Gamma_t}$$

which is different from the modified mean curvature $a$ in [49]. Here $N$ stands for the Dirichlet-Neumann operator. We derive the evolution equation and the boundary conditions for $N_a$ from $(\text{WW})$:

$$\begin{cases}
D_{\Gamma_t}^2 N_a + \sigma A(d_{\Gamma_t}) N_a = R_0 & \text{on } \Gamma_*, \\
D_{\Gamma_t} A(d_{\Gamma_t}) N_a + \frac{\sigma^2}{\sigma^2} \Phi_{S_t}^2 (\Delta_{\mathbb{E}_t} (A(d_{\Gamma_t}) N_a \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = R_{c,i} & \text{at } p_{i,*} (i = l, r),
\end{cases}$$

where $R_0, R_{c,i}$ are remainder terms defined by $d_{\Gamma_t}, \partial_t d_{\Gamma_t},$ and $v$, and the third-order operator $A(d_{\Gamma_t})$ is defined by

$$A(d_{\Gamma_t}) f = -(\mathbb{N} \Delta_{\mathbb{E}_t} (f \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t}$$

for some $f$ on $\Gamma_*$. As mentioned before, the boundary conditions at $p_i$ play a key role in the energy estimates and in the iteration, which have been used in a different version for the equation of $J = \nabla \mathbb{E}_t$ in [41]. The part involving the boundary conditions in our papers is completely new compared to the classical water waves or the other works on nonsmooth water waves.

Moreover, the velocity $v$ is recovered by $\partial_t d_{\Gamma_t}$ in the iteration, so the energy estimates and the iteration depend on the free surface $d_{\Gamma_t}$ and its boundary conditions. The boundary conditions for $d_{\Gamma_t}$ (i.e., information of the contact points) take the form of an ODE:

$$d_{\Gamma_t}''(t) = \mathbb{B}_t, \quad i = l, r,$$

where $d_i = d_{\Gamma_t} \big|_{p_i}$, and $\mathbb{B}_t$ depends on $d_{\Gamma_t}, \partial_t d_{\Gamma_t}, v$ as well. These conditions come naturally from the definition of the velocity on $\Gamma_t$ and the Euler equation, which are very important when one wants to retrieve $d_{\Gamma_t}$ from $N_a$.

As a result, the above system of $(N_a, d_i, d_r)$ provides a closed system to construct approximate solutions. Due to the presence of the contact points, we do not expect the system to be smooth even if the initial data is smooth enough. In [41],
we need small contact angles to avoid singularities from the corresponding elliptic systems. In this paper, we assume that the contact angles are smaller than $\pi/16$, which ensures that a related mixed-boundary elliptic system (see Lemma 2.1) has a solution in $H^3(\Omega_t)$. Compared to the regularity considered in [1–3], $H^3(\Omega_t)$ is a much higher regularity, and a lower regularity is still desirable.

Even with small contact angles, the choice of proper energy functionals is still made very carefully. To prove the energy estimates of the linear system for the iteration, we need to use the material derivative $D_t$ more often than $r_t$. We choose the following energy and dissipation functionals

$$E_h(t, \bar{f}, \partial_t \bar{f}) = \| \nabla_{\Gamma_t} \mathcal{N}_{\Gamma_t} D_t \bar{f} \|_{L^2(\Gamma_t)}^2 + \sigma \| \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathcal{N}_{\Gamma_t} \bar{f}) \|_{L^2(\Omega_t)}^2$$

$$+ \| D_t \bar{f} \|_{L^2(\Gamma_t)}^2 + \| \bar{f} \|_{L^2(\Gamma_t)}^2,$$

$$F_{h,i}(t, \bar{f}, \partial_t \bar{f}) = (\sin \omega_i)^2 \| \nabla_{\Gamma_t} \mathcal{N}_{\Gamma_t} D_t \bar{f} \|_{L^2(\Omega_t)}^2$$

at $p_i (i = l, r)$.

Notice that the dissipation only takes place at the contact points, and one can check our previous work [41] for more details.

In the end, we emphasize the differences between our paper with [49]. We use the geometric approach introduced by [49]. Compared to [48, 49], some new difficulties appear due to the presence of the corners. First, during the construction of approximate solutions, we choose to use the equation for a new quantity $\mathcal{N}_a$, while the modified mean curvature $\hat{\kappa} d \Gamma_t$ is used in [49]. The reason of using $\mathcal{N}_a$ is that it’s more convenient to derive the boundary conditions for $\mathcal{N}_a$ at the contact points, and we do not have the information for $\kappa_a$ at the same time. Besides, if we choose $\mathcal{N}(\kappa_a)$ instead of $\mathcal{N}_a$, we need to maintain $\int_{\Gamma_t} \mathcal{N}(\kappa_a) \circ \Phi_{S_t}^{-1} ds = 0$ in the iteration, which makes the iteration much more complicated. But by choosing $\mathcal{N}_a$ we need no restriction. Second, when we recover the free surface from $\mathcal{N}_a$, the boundary conditions $d_i = D_{\Gamma_t} \|_{p_i} (i = l, r)$ are needed essentially to solve the related elliptic equation; see Proposition 4.2. The system of $(\mathcal{N}_a, d_l, d_r)$ makes sure that our iteration sequence converges and goes back to the solution to system (WW). Third, the definition of the energy functionals as well as the dissipations are totally different in our paper, and the details involving the contact points in the energy estimates are completely new.

1.3 Organization of the Paper

In Section 2, we give some useful lemmas. In Section 3, the free surfaces and the domains are defined. In Section 4, we recover the velocity from the free surface. Meanwhile, we also give the equivalent formulation of the problem. Section 5 deals with the existence of the solution to the linear problem and proves higher-order energy estimates. In Section 6, we use an iteration scheme to finish the proof for the local well-posedness.
1.4 Notations

- $\Omega_*$ is the reference domain with the boundary $S_* = \Gamma_* \cup \Gamma_{b*}$. Here $\Gamma_*$ is the upper boundary and $\Gamma_{b*}$ is the fixed bottom. $\tau_*$ is the unit tangential vector of $\Gamma_*$. 
- $X$ and $p$ are both used to denote a point in $\Omega_t$ or sometimes $\Omega_*$. 
- The entire fixed bottom is denoted as $\Gamma_{\text{fix}}$. 
- $\Pi$: the second fundamental form where $\Pi(w) = \nabla w N_t \in T_X \Gamma_t$ for $w \in T_X \Gamma_t$. 
- $\Pi(v, w)$ denotes $\Pi(v) \cdot w$. 
- $\kappa = t r \Pi = \nabla_{\tau_t} N_t \cdot \tau_t$ is the mean curvature. 
- $\mathcal{H}(f)$ or $f_{\mathcal{H}}$ is the harmonic extension for some function $f$ on $\Gamma_t$, which is defined by the elliptic system
  \[
  \begin{cases}
  \Delta \mathcal{H}(f) = 0 \text{ on } \Omega_t, \\
  \mathcal{H}(f)|_{\Gamma_t} = f, \\
  \nabla_{N_b} \mathcal{H}(f)|_{\Gamma_b} = 0.
  \end{cases}
  \]
- $\nabla$ denotes the tangential component of a vector. 
- $\Delta^{-1}(h, g)$ is defined as the solution $u$ to the system
  \[
  \begin{cases}
  \Delta u = h \text{ on } \Omega_t, \\
  u|_{\Gamma_t} = 0, \\
  \nabla_{N_b} u|_{\Gamma_b} = g.
  \end{cases}
  \]
- $\overline{D}$ is the covariant derivative on $T \Gamma_t$ under Lagrangian coordinates, and $D$ is the corresponding derivative under Eulerian coordinates. 
- $M^*$ denotes the transport of a matrix $M$. 
- $w^\perp$ on $\Gamma_t$: the normal component $(w \cdot N_t) N_t$. 
- $w^\top$ on $\Gamma_t$: the tangential component $(w \cdot \tau_t) \tau_t$. Sometimes we also use $w^\top$ on $\Gamma_b$ with a similar definition. 
- $d_i = d_i(t)$ stands for the value of $d_{\Gamma_t}$ at the contact points $p_i$ ($i = l, r$). 
- Sometimes we need to identify the signs between the left and the right contact point in boundary conditions; We always take $+$ for $i = l$, and $-$ for $i = r$; see (4.26). 
- $F = F(u_1, u_2, \ldots, u_m)$ denotes that the higher-order terms in function $F$ are $u_1, u_2, \ldots, u_m$. 
- $C = C(\|u_1\|, \|u_2\|, \ldots, \|u_m\|)$ denotes a constant $C$ in the form of a polynomial of some norms for $u_1, u_2, \ldots, u_m$. 
- $D_{t*} = \partial_t + v^* \cdot \nabla$ is defined by $v^* = D \Phi_{S_t}^{-1}(v \circ S_t - \partial_t d_{\Gamma_t} \mu)$ on $\Gamma_*$. 
- $D_t = \partial_t + v_t \cdot \nabla$ is defined on $\Gamma_t$ and $\Omega_t$ with some parameter $\tau$, and $v_t$ is induced by $\Phi_{S_t}$ with $v_t = (\partial_t d_{\Gamma_t} \mu) \circ \Phi_{S_t}^{-1}$ on $\Gamma_t$. 
- $[\tilde{s}]$ denotes the smallest integer satisfying $[\tilde{s}] \geq s$. 

2 Preliminaries

First, we present some useful lemmas on elliptic systems adjusted from [40,41].
LEMMA 2.1. Assume that \(2 \leq s \leq 9\). Let \(f \in H^{s-1/2}(\Gamma_*)\) and \(g \in H^{s-1/2}(\Gamma_{b*})\) satisfy
\[
f|_{\Gamma_i} = g|_{\Gamma_i}, \quad i = l, r.
\]
If the contact angles of \(\Omega_*\) are less than \(\frac{\pi}{2(\frac{1}{s} - 1)}\), then the system
\[
\begin{aligned}
\Delta u &= 0 & & \text{on } \Omega_*, \\
\big| u \big|_{\Gamma_i} &= f, & & u \big|_{\Gamma_{b*}} = g,
\end{aligned}
\]
admits a unique solution \(u \in H^s(\Omega_*)\) satisfying the estimate
\[
\| u \|_{H^s(\Omega_*)} \leq C (\| f \|_{H^{s-1/2}(\Gamma_*)} + \| g \|_{H^{s-1/2}(\Gamma_{b*})}),
\]
with the constant \(C\) depending on \(\Omega_*\) and \(s\).

PROOF. Checking theorem 4.7, proposition 5.19, and remark 5.20 in [40] for the Dirichlet problem, one can obtain the desired results.

LEMMA 2.2 (Lemma 5.8 in [41]). We have the following embeddings:
\[
\| u \|_{L^4(\Omega_*)} \leq C (\| \Gamma_t \|_{H^{s/2}})\| u \|_{H^{1/2}(\Omega_*)}
\]
for any \(u \in H^{1/2}(\Omega_*)\), and
\[
\| u \|_{L^\infty(\Omega_*)} \leq C (\| \Gamma_t \|_{H^{s/2}})\| u \|_{H^s(\Omega_*)}
\]
for any \(u \in H^s(\Omega_*)\) with \(s_1 = 1 + \epsilon\) (\(\epsilon > 0\) is a small constant). Moreover, for any \(f \in H^{s_2}(\Gamma_t)\) with \(s_2 = 1/2 + \epsilon\), the embedding holds:
\[
\| f \|_{L^\infty(\Gamma_t)} \leq C (\| \Gamma_t \|_{H^{s_2}})\| f \|_{H^{s_2}(\Gamma_t)}
\]
A similar result holds for the case of \(\Gamma_b\).

LEMMA 2.3 (Lemma 5.9 in [41]). Let \(H(f)\) be the harmonic extension of a function \(f \in H^{1/2}(\Gamma_*)\). Then one has \(H(f) \in H^1(\Omega_*)\) satisfying the following estimate:
\[
\| H(f) \|_{H^1(\Omega_*)} \leq C (\| \Gamma_t \|_{H^{s/2}})\| f \|_{H^{1/2}(\Gamma_*)}.
\]

In addition, we sometimes need to consider the following mixed boundary problem:
\[
(2.1) \quad \begin{cases} 
\Delta u = h & \text{on } \Omega_t, \\
\big| u \big|_{\Gamma_t} = f. & \text{on } \Gamma_b,
\end{cases}
\]

LEMMA 2.4 (Elliptic estimates). Let the contact angles \(\omega_i \in (0, \frac{\pi}{2(\frac{1}{s} - 1)})\) and \(s \geq 2\). Then the following estimate holds for system \((2.1)\):
\[
\| u \|_{H^s(\Omega_*)} \leq C (\| \Gamma_t \|_{H^{s-1/2}})\(\| h \|_{H^{s-2}(\Omega_*)} + \| f \|_{H^{s-1/2}(\Gamma_*)} + \| g \|_{H^{s-1/2}(\Gamma_{b*})})\).
\]

PROOF. For \(s \leq 4\), the case was proved in theorem 5.1 in [41]. For the higher-order estimates, we apply proposition 5.19 in [40]. When contact angles are less than \(\frac{\pi}{2(\frac{1}{s} - 1)}\), there is no singular part in our elliptic estimate thanks to remark 5.20 in [40].
Moreover, the trace theorem on $\Gamma_t$ and $\Gamma_b$ is quoted directly from theorem 5.3 in [41].

**Lemma 2.5 (Traces on $\Gamma_t$ or $\Gamma_b$).** Let the integer $l \in [0, s - 1/2)$ with $s - l > 1/2$. We define the map

$$u \rightarrow \{u, \nabla N_j u, \ldots, \nabla^l N_j u\}_{|\Gamma_j},$$

for $u \in \mathcal{D}(\tilde{\Omega}_t)$ where $N_j$ is the unit outward normal vector on $\Gamma_j$ with $\Gamma_j$ taking $\Gamma_b$ or $\Gamma_t$. Then the map has a unique continuous extension as an operator from $H^s(\Omega_t)$ onto $\prod_{k=0}^{l} H^{s-k-1/2}(\Gamma_j)$.

Moreover, one has the estimate for $0 \leq k \leq l$:

$$\|\nabla^k N_j u\|_{H^{s-k-1/2}(\Gamma_b)} + \|\nabla^k N_j u\|_{H^{s-k-1/2}(\Gamma_t)} \leq C(\|\Gamma_i\|_{H^{s-1/2}}) \|u\|_{H^s(\Omega_t)}.$$

The Dirichlet-Neumann (D-N) operator $\mathcal{N}$ is defined by

$$\mathcal{N}(f) = \nabla N_i \mathcal{H}(f) \quad \text{on } \Gamma_t$$

for a function $f$ defined on $\Gamma_t$, which is an important operator in water waves. We would like to recall some useful properties of the Dirichlet-Neumann operator here.

**Lemma 2.6.** Let $s \in (1, 6.5]$ and $\omega_i \in (0, \frac{\pi}{2(\Gamma_i-1)})$ for $i = l, r$.

1. The D-N operator $\mathcal{N}$ is an order-1 operator on $\Gamma_t$:

$$\|\mathcal{N}(f)\|_{H^{s-1}(\Gamma_i)} \leq C(\|\Gamma_i\|_{H^s}) \|f\|_{H^s(\Gamma_i)}.$$

2. The following estimate holds:

$$\|f\|_{H^s(\Gamma_i)} \leq C(\|\Gamma_i\|_{H^s}) (\|\mathcal{N}(f)\|_{H^{s-1}(\Gamma_i)} + \|f\|_{L^2(\Gamma_i)}).$$

Moreover, when $\mathcal{N}(f) \in (H^{1/2}(\Gamma_i))^*$ (the dual space of $H^{1/2}(\Gamma_i)$), one has

$$\|f\|_{H^{1/2}(\Gamma_i)} \leq C(\|\Gamma_i\|_{H^2}) (\|\mathcal{N}(f)\|_{H^{1/2}(\Gamma_i)} + \|f\|_{L^2(\Gamma_i)}).$$

3. When $\int_{\Gamma_t} f \, ds = 0$ and $\int_{\Gamma_t} g \, ds = 0$, the inverse D-N operator $\mathcal{N}^{-1}$ makes sense, i.e.,

$$\mathcal{N}(f) = g \quad \text{implies} \quad f = \mathcal{N}^{-1}(g),$$

and the above two inequalities in (2) hold without the term $\|f\|_{L^2(\Gamma_i)}$ on the right side.

**Proof.** Although similar results as in this lemma have been proved in [20, 48], we still provide some details here. The first result comes directly from Lemma 2.5. For the second and third results, we need to consider the following system:

$$\begin{cases}
\Delta u = 0 \quad \text{on } \Omega_t, \\
\nabla N_i u|_{\Gamma_i} = g, \quad \nabla N_b u|_{\Gamma_b} = 0,
\end{cases}$$

where $N_i$ is the unit outward normal vector on $\Gamma_i$ with $\Gamma_i$ taking $\Gamma_b$ or $\Gamma_t$. Then the map has a unique continuous extension as an operator from $H^s(\Omega_t)$ onto $\prod_{k=0}^{l} H^{s-k-1/2}(\Gamma_j)$. Moreover, one has the estimate for $0 \leq k \leq l$:

$$\|\nabla^k N_j u\|_{H^{s-k-1/2}(\Gamma_b)} + \|\nabla^k N_j u\|_{H^{s-k-1/2}(\Gamma_t)} \leq C(\|\Gamma_i\|_{H^{s-1/2}}) \|u\|_{H^s(\Omega_t)}.$$
where \( f = u|_{\Gamma_t} \) and the compatibility condition holds:
\[
\int_{\Gamma_t} g \, ds = 0.
\]
Since the elliptic estimates have been proved in theorem 5.10 of [41], the proof lies in the existence of the variational solution \( u \in H^1(\Omega_t) \).

**Step 1.** The existence of the solution \( u \) and the \((H^{1/2}(\Gamma_t))^*\) case. Defining the variation space
\[
\mathcal{V} = \left\{ v \in H^1(\Omega_t) \bigg| \int_{\Omega_t} v \, dX < \infty \right\},
\]
one writes the variation equation for \( u \) as
\[
\int_{\Omega_t} \nabla u : \nabla v \, dX = \int_{\Gamma_t} g v \, ds,
\]
with \( \forall v \in \mathcal{V} \).

Due to the compatibility condition, one has
\[
\int_{\Gamma_t} g v \, ds = \int_{\Gamma_t} g(v - \overline{v})ds \leq \|g\|_{H^{1/2}(\Gamma_t)}^* \|v - \overline{v}\|_{H^{1/2}(\Gamma_t)},
\]
where \( \overline{v} = (\int_{\Omega_t} dX)^{-1} \int_{\Omega_t} v \, dX \). Applying Lemma 2.5 and Poincaré’s inequality, one obtains
\[
\int_{\Gamma_t} g v \, ds \leq C(\|\Gamma_t\|_{H^{1/2}}) \|g\|_{H^{1/2}(\Gamma_t)}^* \|v\|_{H^1(\Omega_t)},
\]
as long as \( g \in (H^{1/2}(\Gamma_t))^* \).

Consequently, applying Lax-Milgram’s theorem, one concludes that there exists a variation solution \( u \in \mathcal{V} \) to the Neumann problem satisfying the estimate
\[
\|\nabla u\|_{L^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{1/2}}) \|g\|_{H^{1/2}(\Gamma_t)}^* \|v\|_{H^1(\Omega_t)}.
\]

On the other hand, since \( u = \mathcal{H}(f) \), one knows directly that
\[
\|\mathcal{H}(f)\|_{L^2(\Omega_t)} \leq C \left( \|\nabla \mathcal{H}(f)\|_{L^2(\Omega_t)} + \|f\|_{L^2(\Gamma_t)} \right),
\]
where the constant \( C \) depends on the size of the domain \( \Omega_t \). Combining these two inequalities above with Lemma 2.5, one derives the second estimate in (2).

In order to prove (3), when \( \int_{\Gamma_t} f \, ds = 0 \), one has
\[
\|\mathcal{H}(f)\|_{L^2(\Omega_t)} = \|\mathcal{H}(f) - \mathcal{H}(\overline{f})\|_{L^2(\Omega_t)} \leq C \|\nabla \mathcal{H}(f)\|_{L^2(\Omega_t)}.
\]
where \( \overline{f} = (\int_{\Gamma_t} ds)^{-1} \int_{\Gamma_t} f \, ds \). As a result, one derives by Lemma 2.5 that
\[
\|f\|_{H^{1/2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{1/2}}) \|\mathcal{N}(f)\|_{H^{1/2}(\Gamma_t)}^*.
\]
which implies that the D-N operator \( \mathcal{N} \) is invertible in this case and the lower-order estimate in (3) follows.
Step 2. The higher-order estimates for (2) and (3). The estimates follow from Step 1 and theorem 5.10 in [41] as long as one checks carefully from Remark 5.20 [40] or [24] that the contact angle \(\omega \in (0, \frac{\pi}{2(s-1)})\) for an integer \(s\) under the required regularity of this lemma.

\[\square\]

3 Definitions of Surfaces and Domains

In this section, we first define a coordinate system based on a reference domain \(\Omega_\ast\), and then we construct surfaces according to the coordinate system.

To begin with, we fix a reference domain \(\Omega_\ast\), where \(\Omega_\ast\) can be taken as the initial domain \(\Omega_0\). The boundary of the reference domain \(\Omega_\ast\) is denoted by \(S_\ast\), which contains two parts indeed: the upper surface \(\Gamma_\ast\) and the bottom \(\Gamma_{b*}\). The corresponding contact points are noted as \(p_{i*} (i = l, r)\) with contact angles \(\omega_{i*} \in (0, \pi/16)\).

The unit outward normal vectors and tangent vector are denoted by \(N_{\ast*}\), \(N_{b*}\) and \(\tau_{\ast*}\), \(\tau_{b*}\) accordingly.

3.1 Definition for Surfaces

Since the domain will be fixed if the boundary is fixed, we consider about how to define the boundary, or the upper free surface. We use some oblique coordinate system on \(\Gamma_\ast\) to define surfaces \(S\) near \(S_\ast\). In fact, these surfaces are set to be in a neighborhood of \(S_\ast\).

First, we introduce a unit upward vector field \(\mu \in H^s(\Gamma_\ast, S^1)\) for some large \(s\) such that

\[\mu \cdot N_{\ast*} \geq c_0 \quad \text{on} \quad \Gamma_\ast \quad \text{and} \quad \mu|_{p_{l*}} = -\tau_{b*}|_{p_{l*}}, \quad \mu|_{p_{r*}} = \tau_{b*}|_{p_{r*}}\]

with some fixed constant \(c_0 \in (0, 1)\). Note that this condition holds at \(p_{l*}, \ p_{r*}\) since the contact angle stays in \((0, \pi/2)\) in this paper.

From the implicit function theorem, there exists a small constant \(d_0 > 0\) such that the map

\[\Phi : \Gamma_\ast \times [-d_0, d_0] \to \mathbb{R}^2 \quad \text{where} \quad \Phi(p, d) \triangleq p + d \mu(p)\]

is an \(H^s\) diffeomorphism from its domain to a neighborhood of \(\Gamma_\ast\). As a result, this coordinate system identifies each upper surface \(\Gamma\) close to \(\Gamma_\ast\) with a unique function \(d_\Gamma : \Gamma_\ast \to \mathbb{R}\). Plugging \(d_\Gamma\) into \(\Phi\), one writes

\[\Phi_S : \Gamma_\ast \to \Gamma \subset \mathbb{R}^2 \quad \text{where} \quad \Phi_S(p) = p + d_\Gamma(p)\mu(p)\]

Sometimes, the function \(d_\Gamma(p)\) is also used as the expression of the upper surface \(\Gamma\). Moreover, we extend \(\Phi_S\) to be defined on the whole boundary \(S_\ast\). In fact, let

\[\Phi_S : \Gamma_{b*} \to \Gamma_b \subset \Gamma_{\text{fix}}\]
satisfy
\[ \Phi_S \in H^2(\Gamma_{b*}, \Gamma_b), \quad \Phi_S(p_i^*) = p_i^* + d_\Gamma(p_i^*) \mu(p_i^*) \ (i = l, r). \]

Notice that \( \Gamma_{b*} \) and \( \Gamma_b = \Phi_S(\Gamma_{b*}) \) are both parts from the entire fixed bottom \( \Gamma_{fix} \).

Consequently, we denote the surface of the domain by
\[ S = \Gamma \cup \Gamma_b, \]
which is defined by \( \Phi_S \). One can see that in our case, as long as we know the free upper surface \( \Gamma \), the whole boundary \( S \), and the domain \( \Omega \) are fixed. So all we need is to concentrate on the upper free surface \( \Gamma \).

We are going to consider free surfaces varying near the reference surface \( \Gamma_* \) in the following set.

**Definition 3.1.** Let \( \delta > 0 \) and \( s \in (1.5, 8.5] \). One defines the *set of upper free surfaces*
\[ \Lambda(S_*, s, \delta, \pi/16) \triangleq \{ \Gamma \mid d_\Gamma \| H^s(\Gamma_*) < \delta, \| \Phi_S - Id_{S_*} \| H^{s+0.5}(\Gamma_{b*}) \leq \delta, \omega_i \in (0, \pi/16), i = l, r \} \]
as a neighborhood of \( \Gamma_* \).

When the constant \( \delta \) is taken small enough, for any \( \Gamma \in \Lambda(S_*, s, \delta, \pi/16) \), \( \Phi_S \) is a diffeomorphism both in \( H^s(\Gamma_*, \Gamma) \) and \( H^{s}(\Gamma_{b*}, \Gamma_b) \), and the contact angles \( \omega_i \) lie in \((0, \pi/16)\). Moreover, from the definition of \( \Phi_S \), one can see that the norm \( \| \Phi_S - Id_{S_*} \| H^{s+0.5}(\Gamma_{b*}) \) on the bottom can be controlled by \( d_\Gamma \) as well.

### 3.2 Harmonic Coordinates

Let
\[ \mathcal{T}_S : \Omega_* \to \Omega \quad \text{with} \quad \mathcal{T}_S = \mathcal{H}_* (\Phi_S - Id_{S_*}) + Id. \]

Here \( \mathcal{H}_* (\Phi_S - Id_{S_*}) \) is the harmonic extension of \( \Phi_S - Id_{S_*} \) satisfying the system with Dirichlet boundary conditions
\[ \begin{aligned}
\Delta \mathcal{H}_* (\Phi_S - Id_{S_*}) &= 0 \quad \text{on} \ \Omega_*, \\
\mathcal{H}_* (\Phi_S - Id_{S_*}) |_{\Gamma_*} &= d_\Gamma \mu, \quad \mathcal{H}_* (\Phi_S - Id_{S_*}) |_{\Gamma_{b*}} = \Phi_S |_{\Gamma_{b*}} - Id_{\Gamma_{b*}}.
\end{aligned} \tag{3.1} \]

One can immediately see from this definition that, the boundary of \( \Omega \) is the surface \( S = \Gamma \cup \Gamma_b \).

Recalling the definition for \( \Phi_S \), one knows that the compatibility condition for the Dirichlet boundary conditions is satisfied, i.e.
\[ d_\Gamma \mu(p_i^*) = (\Phi_S |_{\Gamma_{b*}} - Id_{\Gamma_{b*}})(p_i^*) \ (i = l, r). \]

Applying Lemma 2.1 on \( \mathcal{H}_* (\Phi_S - Id_{S_*}) \), one finds immediately that \( \mathcal{H}_* (\Phi_S - Id_{S_*}) \in H^{s+0.5}(\Omega_*, \mathbb{R}^2) \) for \( 1.5 \leq s \leq 8.5 \) with corresponding estimate
\[ \| \mathcal{H}_* (\Phi_S - Id_{S_*}) \|_{H^{s+0.5}(\Omega_*)} \leq C \left( \| d_\Gamma \mu \|_{H^s(\Gamma_*)} + \| \Phi_S - Id \|_{H^s(\Gamma_{b*})} \right) \]
\[ \leq C \left( \| d_\Gamma \|_{H^s(\Gamma_*)} + \| \Phi_S - Id \|_{H^s(\Gamma_{b*})} \right). \]
where the constant $C = C(\Omega_*, \mu)$ is uniform in $\Lambda(S_*, S, \delta, \pi/16)$. Moreover, one has
\[
\| \nabla T_S - I \|_{H^{s-0.5}(\Omega_*)} = \| \nabla H_* (\Phi_S - I d S_*) \|_{H^{s-1/2}(\Omega_*)} \\
\leq C \left( \| d \Gamma \|_{H^s(\Gamma_*)} + \| \Phi_S - I d \|_{H^s(\Gamma_*)} \right),
\]
which implies that $T_S$ is a diffeomorphism from $\Omega_*$ to $\Omega = T_S(\Omega_*)$.

As a result, the map $T_S$ can be used as coordinates on $\Omega_*$.

4 Reformulation of the Problem

We are going to introduce the new quantity $N$ and derive an equivalent system of $(WW)$. From now on, we consider a family of upper free surfaces $\Gamma_t$ with time variable $t$ in $\Lambda(S_*, S, \delta, \pi/16)$. The unit outward normal vector is denoted by $N_t$ and the unit tangent vector is $t$, while the domain is $\Omega_t$.

To begin with, denoting by $\Omega_0$ and $\Gamma_0$ the initial domain and upper surface respectively, the velocity field $v$ induces a flow map $U(t, \cdot) : S_0 \rightarrow S_t$ by
\[
U(0, \cdot) = I d S_0, \quad \partial_t U(t, \cdot) = v(t, U(t, \cdot)),
\]
and the material derivative is
\[
D_t \triangleq \partial_t + \nabla v.
\]

4.1 Some Commutators

We recall some commutators involving $D_t$ from [41, 48] on the surface $\Gamma_t$ or in the domain $\Omega_t$.

To start with, we recall directly from [41, 48] that
\[
D_t t = (\nabla t v \cdot N_t) N_t, \quad D_t N_t = -((\nabla v)^* N_t)^\top \quad \text{on } \Gamma_t.
\]

Moreover, we have
\[
(4.1) \quad D_t \kappa = -\Delta_{\Gamma_t} v \cdot N_t - 2\Pi(t) \cdot \nabla t v \quad \text{on } \Gamma_t.
\]

The commutators are listed here:

(1) $[D_t, \mathcal{H}]$. Recalling directly from [41], we obtain for a function $f$ on $\Gamma_t$ that
\[
[D_t, \mathcal{H}]f = \Delta^{-1} \left( 2 \nabla v \cdot \nabla^2 f_{\mathcal{H}} + \Delta v \cdot \nabla f_{\mathcal{H}} - (\nabla N_b v - \nabla v N_b) \cdot \nabla f_{\mathcal{H}} \right) \quad \text{on } \Omega_t,
\]
where $\Delta^{-1}(h, g)$ and $\mathcal{H}$ are already defined in the Notations subsection.

(2) $[D_t, N]$. Again, we quote directly that
\[
[D_t, N]f = \nabla N_t \Delta^{-1} \left( 2 \nabla v \cdot \nabla^2 f_{\mathcal{H}} + \Delta v \cdot \nabla f_{\mathcal{H}} - (\nabla N_b v - \nabla v N_b) \cdot \nabla f_{\mathcal{H}} \right) - \Delta v \cdot \nabla f_{\mathcal{H}} - (\nabla f_{\mathcal{H}})^\top v \cdot N_t \quad \text{on } \Gamma_t.
\]
\( [D_t, \Delta \Gamma_t] \). For a function \( f \) on \( \Gamma_t \), one has
\[
[D_t, \Delta \Gamma_t] f = 2D^2 f (\tau_t, (\nabla_d v)^T) - (\nabla f)^T : \Delta \Gamma_t v + \kappa \nabla (\nabla f)^\top v \cdot N_t \quad \text{on } \Gamma_t.
\]

\( [D_t, \Delta^{-1}] \). We have
\[
D_t \Delta^{-1} (h, g) = \Delta^{-1} (D_t h, D_t g) + \Delta^{-1} (h_1, g_1)
\]
with
\[
h_1 = 2\nabla v \cdot \nabla^2 \Delta^{-1} (h, g) + \Delta v \cdot \nabla \Delta^{-1} (h, g),
g_1 = (\nabla_N h v - \nabla_y N_h) \cdot \nabla \Delta^{-1} (h, g).
\]

### 4.2 Recovery of the Velocity

Based on the construction of the free surface, one can recover the velocity fields \( v \). In fact, we start with the evolution of the boundary \( \Gamma_t \) expressed by \( d\Gamma_t \).

To begin with, the normal component of \( \partial_t \Phi_{S_t} \) represents the normal component of velocity \( v \) on \( \Gamma_t \), which means
\[
\partial_t \Phi_{S_t} \cdot (N_t \circ \Phi_{S_t}) = (v \cdot N_t) \circ \Phi_{S_t},
\]
where
\[
\partial_t \Phi_{S_t} = \partial_t d\Gamma_t \mu \quad \text{on } \Gamma_*.
\]
Therefore we have
\[
\partial_t d\Gamma_t = \frac{(v \cdot N_t) \circ \Phi_{S_t}}{\mu \cdot (N_t \circ \Phi_{S_t})}, \quad \text{i.e.,} \quad v \cdot N_t = (\partial_t d\Gamma_t \mu) \circ \Phi_{S_t}^{-1} \cdot N_t.
\]

Since the velocity \( v \) in our paper is assumed to be irrotational, we define
\[
v = \nabla \phi
\]
with \( \phi \) satisfying
\[
\begin{cases}
\Delta \phi = \xi \gamma & \text{on } \Omega_t, \\
\nabla N_t \phi |_{\Gamma_t} = (\partial_t d\Gamma_t \mu) \circ \Phi_{S_t}^{-1} \cdot N_t, \quad \nabla N_{\nu} \phi |_{\Gamma_{\nu}} = 0,
\end{cases}
\]
where
\[
\gamma = \left( \int_{\Omega_t} dX \right)^{-1} \quad \text{and} \quad \xi = \int_{\Gamma_t} v : N_t \, ds.
\]

The definition of \( \phi \) is motivated by [49]. In Sections 5 and 6, we cannot use the Euler equation any more since we need to solve the system of our new quantity \( \mathcal{R}_d \) (which is of course derived from (WW)). As a result, when we need to deal with velocity \( v \) in the following sections, we always use this new definition (4.7) related to the free surface \( d\Gamma_t \), and we plan to recover the Euler equation for \( v \) after the iteration. Here, we must point out that by the definition of \( d\Gamma_t \), we cannot get \( \text{div } v = 0 \) right now. But we prove this in the last section as well.
On the other hand, the flow map $U(t)$ together with the diffeomorphism $\Phi_{S_t}$ induces a conjugate flow map $U_* : S_* \to S_*$ by

$$U_*(t, \cdot) \triangleq \Phi_{S_t}^{-1} \circ U(t, \cdot) \circ \Phi_{S_0},$$

and the corresponding velocity $v^*$ is given by

$$\partial_t U_*(t, \cdot) = v^*(t, U_*(t, \cdot)).$$

We express $v^*$ on the upper surface $\Gamma_*$ in terms of $v$ and $\Phi_{S_t}$. To begin with, we know from the definition of $U_*(t)$ that

$$\Phi_{S_t} \circ U_*(t) = U(t) \circ \Phi_{S_0}.$$

Applying $\partial_t$ on both sides and constraining the computation on the surfaces $\Gamma_t$ and $\Gamma_*$ leads to

$$\partial_t \Phi_{S_t} \circ U_*(t) + (D\Phi_{S_t} \circ U_*(t)) \partial_t U_*(t) = \partial_t U(t) \circ \Phi_{S_0}.$$

Consequently, one obtains

$$\tag{4.9} v^* = D\Phi_{S_t}^{-1}(v \circ \Phi_{S_t} - \partial_t d_{\Gamma_t} \mu) \quad \text{on } \Gamma_*.$$

Recalling from (4.6), one has

$$v^* = D\Phi_{S_t}^{-1}(v^T \circ \Phi_{S_t} - \partial_t d_{\Gamma_t} \mu^T).$$

Denoting the material derivative related to $U_*(t, \cdot)$ and $v^*$ by

$$D_{t*} \triangleq \partial_t + \nabla v^*,$$

a direct computation shows that

$$\tag{4.10} (D_{t*} f) \circ \Phi_{S_t} = D_{t*}(f \circ \Phi_{S_t})$$

for any function $f$ on $\Gamma_t$.

Next, we want to consider the variation of $v^*$ with respect to a parameter $\tau$. Therefore, $\Phi_{S_t}$ and $\Gamma_t$ depends on $\tau$. Rewriting (4.9) as

$$(D\Phi_{S_t}) v^* = v \circ \Phi_{S_t} - \partial_t d_{\Gamma_t} \mu$$

and taking $\partial_\tau$ on both sides leads to

$$\partial_\tau (D\Phi_{S_t}) v^* + (D\Phi_{S_t}) \partial_\tau v^* = \partial_\tau (v \circ \Phi_{S_t}) - \partial_\tau \partial_t d_{\Gamma_t} \mu,$$

where

$$\partial_\tau (v \circ \Phi_{S_t}) = (D_\tau v) \circ \Phi_{S_t}, \quad \partial_\tau (D\Phi_{S_t}) = D(\partial_t d_{\Gamma_t} \mu).$$

Consequently, we arrive at the expression

$$\partial_\tau v^* = (D\Phi_{S_t})^{-1} ((D_\tau v) \circ \Phi_{S_t} - \partial_\tau \partial_t d_{\Gamma_t} \mu - D(\partial_t d_{\Gamma_t} \mu))$$

$$\cdot D\Phi_{S_t}^{-1}(v \circ \Phi_{S_t} - \partial_t d_{\Gamma_t} \mu).$$

For the moment, we still need to rewrite $D_\tau v$ in (4.11). In fact, one has immediately by recalling from (4.7) that

$$D_\tau v = D_\tau \nabla \phi = \nabla D_\tau \phi + [D_\tau, \nabla] \phi.$$
with $[D_\tau, \nabla] = -\nabla v_\tau \cdot \nabla \phi$.

Applying $D_\tau$ on the system (4.8) for $\phi$, one can write

$$D_\tau \phi \triangleq u_1 + u_2,$$

where $u_1$ satisfies

$$\begin{align*}
    \Delta u_1 &= 0 \quad \text{on } \Omega_t, \\
    \partial N_\ell u_1 |_{\Gamma_\ell} &= (\partial_\tau \partial_\tau d_\Gamma \mu) \circ \Phi_{S_t}^{-1} \cdot N_t, \\
    \partial N_\ell u_1 |_{\Gamma_\ell} &= 0,
\end{align*}$$

and $u_2$ the remainder part satisfies

$$\begin{align*}
    \Delta u_2 &= 2\nabla v_\tau \cdot \nabla^2 \phi + \Delta v_\tau \cdot \nabla \phi + D_\tau (v_\tau^2) \quad \text{on } \Omega_t, \\
    \partial N_\ell u_2 |_{\Gamma_\ell} &= (\partial_\tau d_\Gamma \mu) \circ \Phi_{S_t}^{-1} \cdot D_\tau N_t - (D_\tau N_t - \nabla N_t v_\tau) \cdot \nabla \phi, \\
    \partial N_\ell u_2 |_{\Gamma_\ell} &= (\nabla N_\ell v_\tau - \nabla v_\tau N_\ell) \cdot \nabla \phi |_{\Gamma_\ell}
\end{align*}$$

with $D_\tau N_t = -((\nabla v_\tau)^* N_t)^T$.

As a result, one arrives at

$$D_\tau v = \nabla u_1 + \nabla u_2 - \nabla v_\tau \cdot \nabla \phi.$$

Substituting this expression together with (4.9) back into (4.11), one finally derives

$$\begin{align*}
    \partial_\tau v^* &= (D\Phi_{S_t})^{-1} \left( (\nabla u_1 + \nabla u_2 - \nabla v_\tau \cdot \nabla \phi) \circ \Phi_{S_t} \right. \\
    &\quad \left. - \partial_\tau \partial_\tau d_\Gamma \mu - D(\partial_\tau d_\Gamma \mu) v^* \right).
\end{align*}$$

**Lemma 4.1.** Let the surface $\Gamma_t \in \Lambda(S_*, s, \delta, \pi/6)$. Then we have

$$\begin{align*}
    \|v^*\|_{H^2(\Gamma_t)} &\leq C \left( \|d_\Gamma_{\nabla} \|_{H^{s+1}(\Gamma_t)} \cdot \|\partial_\tau d_\Gamma \|_{H^s(\Gamma_t)} \cdot \|v\|_{H^s(\Gamma_t)} \right), \\
    \|\partial_\tau v^*\|_{H^2(\Gamma_t)} &\leq C \left( \|d_\Gamma_{\nabla} \|_{H^{s+1}(\Gamma_t)} \cdot \|\partial_\tau d_\Gamma \|_{H^s(\Gamma_t)} \cdot \|v\|_{H^s(\Gamma_t)} \right) \\
    &\quad \times \left( \|\partial_\tau d_\Gamma \|_{H^{s+1}(\Gamma_t)} + \|\partial_\tau \partial_\tau d_\Gamma \|_{H^s(\Gamma_t)} + \|D_\tau v\|_{H^s(\Gamma_t)} \right).
\end{align*}$$

**Proof.** Applying Lemma 2.5 and Lemma 2.4 directly, the proof can be finished.

**4.3 The New Quantity $\mathfrak{N}_a$**

We introduce the new quantity $\mathfrak{N}_a$ related to the mean curvature $\kappa$ for some large constant $a > 0$:

$$\mathfrak{N}_a = N(\kappa) \circ \Phi_S + a^3 d_{\nabla} \Gamma_t,$$

which is used to rewrite the water-waves problem (WW). The constant $a$ is taken large enough, which is useful in the following proposition and also in the iteration section.

Before we derive the equation for $\mathfrak{N}_a$, we need to make sure that we can recover the upper surface $\Gamma_t$ from it. In order to do this, we also need the boundary condition at the two endpoints $p_{l*}, p_{r*}$:

$$d_{\Gamma_t}(p_{l*}) = d_l, \quad i = l, r,$$

where $d_i = d_i(t)$ is some function to be given later.
We define the operator $K$ by
\[ K : d \Gamma \rightarrow (\mathcal{N}_a, d_l, d_r). \]
When $a$ is chosen to be big enough, we prove that the operator $K$ is invertible.

**Proposition 4.2.** Let $s \in [1, 5.5]$ and the set $B_{\delta_1}$ be defined by
\[ B_{\delta_1} \triangleq \{(\mathcal{N}_a, d_l, d_r) \mid \|\mathcal{N}_a - \mathcal{N}_{a*}\|_{H^s(\Gamma_*)}, |d_l|, |d_r| < \delta_1\}, \]
where $\mathcal{N}_{a*}$ is the value of $\mathcal{N}_a$ taken at $\Gamma_*$. Then, there exists $A > 0$ such that when $a \geq A > 0$, the operator $K$ is a diffeomorphism from $\Lambda(S_*, s + 3, \delta, \pi/16) \subset H^{s+3}(\Gamma_*)$ to $K(\Lambda(S_*, s + 3, \delta, \pi/16)) \subset B_{\delta_1}$ and there holds
\[ \|d\Gamma_t\|_{H^{s+3}(\Gamma_*)} \leq C(\|\mathcal{N}_a - \mathcal{N}_{a*}\|_{H^s(\Gamma_*)} + |d_l| + |d_r|), \]
where the constant $C$ depends on $s, \delta, A, a,$ and $\Gamma_*$. Moreover, one also has the following estimate involving the constant $A^{-1}$:
\[ \|d\Gamma_t\|_{H^{s+1}(\Gamma_*)} \leq a^{s_1-s_2-3} C(\|\mathcal{N}_a - \mathcal{N}_{a*}\|_{H^s(\Gamma_*)} + |d_l| + |d_r|) \]
with $8.5 \geq s_1 \geq s_2 \geq 1$ and $s_1 \leq s_2 + 3$.

**Proof.** Since the problem is quasilinear, we start by linearizing the operator $K$ around $\Gamma_*$ (where $d\Gamma_*=0$) as in (2.4) in [49] to obtain the linearized operator $L(\Gamma_*):$
\[ L(\Gamma_*)d \triangleq \left( \frac{\delta \mathcal{N}_a}{\delta d \Gamma}(\Gamma_*)d, d(p_{l*}), d(p_{r*}) \right), \]
where the linearization for $\mathcal{N}_a$ is derived by a variation with parameter $\tau$:
\[ \frac{\delta \mathcal{N}_a}{\delta d \Gamma}(\Gamma_*)d = a^3 d + D_{\tau}N(\kappa)|_{\tau=0} \]
(4.16)
\[ = a^3 d + N_*(D_{\tau}\kappa)|_{\tau=0} - \nabla N_*(\mu d)_{ex} \cdot \nabla \mathcal{H}(\kappa_*) \]
\[ - \nabla(\nabla \mathcal{H}(\kappa_*))^{-1}(\mu d)_{ex} \cdot N_* + \nabla N_* \Delta^{-1}(f_d, g_d) , \]
with
\[ f_d = 2\nabla(\mu d)_{ex} \cdot \nabla^2 \mathcal{H}(\kappa_*) + \Delta(\mu d)_{ex} \cdot \nabla \mathcal{H}(\kappa_*), \]
\[ g_d = (\nabla N_*(\mu d)_{ex} - \nabla(\mu d)_{ex} N_b) \cdot \nabla \mathcal{H}(\kappa_*), \]
where (4.3) is applied and $N_*, \kappa_*$, and $N_*$ are defined on $\Gamma_*$. In addition, $v|_{\tau=0} = (\mu d)_{ex}$ denotes the extension of $\mu d$ from $\Gamma_t$ to $\Omega_I$ (for example, use the trace theorem in [40]). The variation is defined with the velocity field $v(\tau) : S_t \rightarrow \mathbb{R}^2$ where $S_t$ is a family of surfaces.

Moreover, one has
\[ D_{\tau} \kappa|_{\tau=0} = \frac{\delta(\kappa \circ \Phi_S)}{\delta d \Gamma}(\Gamma_*)d . \]
where we quote the computation from \cite{49} directly:
\[
\frac{\delta (\kappa \circ \Phi_S)}{\delta d \Gamma} (\Gamma_* d) = -(\mu \cdot N_*) \Delta_{\Gamma_*} d - 2(\nabla_{N_*} \mu \cdot \tau_*) \nabla_{\tau_*} d - (N_* \cdot \Delta_{\Gamma_*} \mu) d - 2(\Pi_* \cdot \tau_*) \tau_* \cdot \nabla_{\tau_*} (d \mu).
\]

Summing up these expressions above, we can conclude that
\[
\frac{\delta \mathcal{N}_d}{\delta d \Gamma} (\Gamma_* d) = a^3 d - \mathcal{N}_* ((\mu \cdot N_*) \Delta_{\Gamma_*} d) + r_d
\]
with the remainder term
\[
r_d = r_d (\hat{\partial}^2 d)
\]
\[
= -\mathcal{N}_* (2(\nabla_{N_*} \mu \cdot \tau_*) \nabla_{\tau_*} d + (N_* \cdot \Delta_{\Gamma_*} \mu) d + 2(\Pi_* \cdot \tau_*) \tau_* \cdot \nabla_{\tau_*} (d \mu))
- \nabla_{N_*} (\mu d) \cdot \nabla \mathcal{H}(\kappa_*) - \nabla (\nabla \mathcal{H}(\kappa_*)) \cdot (\mu d) \nabla \mathcal{N}_* + \nabla_{N_*} \Delta^{-1} (f_d, g_d, \eta_d).
\]

As a result, if one can prove that this linear operator \( \mathcal{L}(\Gamma_*) \) is invertible and continuous, one immediately has that \( \mathcal{K} \) is a diffeomorphism near \( \mathcal{S} \) by the inverse function theorem (see, for example, \cite[theorem 1.2.3]{15}) and the desired estimate will follow.

So all we need to prove now is the invertibility of \( \mathcal{L}(\Gamma_*) \), which can be done by applying standard elliptic analysis on the following system:
\[
\mathcal{L}(\Gamma_*) d = (h, d_1, d_r)
\]
for some \( h \in H^2(\Gamma_*) \) and \( d_1, d_r \in \mathbb{R} \).

First, we consider the variation solution in the space
\[
\mathcal{V} = \{ d \in H^{2, 5}(\Gamma_*) \mid d(p_{i_*}) = 0, i = l, r \}.
\]

Here a standard analysis is used to obtain zero Dirichlet boundary condition at \( p_i \) for \( d = d - f \), where
\[
f(p_{i_*}) = d_i (i = l, r) \quad \text{and} \quad \| f \|_{H^4(\Gamma_*)} \leq C (|d_l| + |d_r|).
\]

In order to apply Lax-Milgram’s theorem, we define
\[
A(d_1, d_2) = \int_{\Gamma_*} \left(a^3 d_1 - \mathcal{N}_* ((\mu \cdot N_*) \Delta_{\Gamma_*} d_1) + r_d_1 \right) \left(1 - a^{-1} (\mu \cdot N_*) \Delta_{\Gamma_*} d_2 \right) ds,
\]
and
\[
F(d_2) = \int_{\Gamma_*} h (1 - a^{-1} (\mu \cdot N_*) \Delta_{\Gamma_*} d_2) ds.
\]

One easily shows that the conditions in the Lax-Milgram theorem are all satisfied when \( a \) is large enough, so the linear system admits a unique solution \( d \in \mathcal{V} \) and the estimate in \( \mathcal{V} \) follows. Here we remark that the boundary conditions \( d_i \) are used for the operator \( \Delta_{\Gamma_*} \), while no boundary conditions are needed due to the symmetry property of \( \mathcal{N} \).
Second, we prove the higher-order estimates. For example, setting
\[ u_1 = \mathcal{N}_a((\mu \cdot N_a)\Delta_{\tau_a} d) - \mathcal{N}_a(2((\nabla_{N_a\mu} \cdot \tau_a) + (\Pi_a \cdot \tau_a) \tau_a \cdot \mu)\nabla_{\tau_a} d) \]
and taking \( \Delta_{\tau_a} \) on the equation of \( d \), we derive the linear system for \( u_1 \) with Dirichlet boundary conditions. Therefore, the higher-order estimates are proved.

Moreover, noticing that there is \( a^3 \in \mathcal{A}_a \) and the linearized equation as well, so one can obtain elliptic estimates with weight \( a \) for the linearized equation using interpolations. Consequently, one can prove the desired weighted estimate and the proof is finished.

\[ \square \]

Remark 4.3. Following the proof above, we know immediately that the elliptic operator
\[ \mathcal{A}_a \triangleq d^3 + \mathcal{A}(d_{\Gamma}) \]
is invertible with Dirichlet boundary conditions at \( p_i \), and corresponding elliptic estimates follow naturally.

As a result, as long as we have \( d_{\Gamma} \), we retrieve \( \Phi_{\Sigma} \) to define the surface \( \Sigma \) and the corresponding domain \( \Omega \).

4.4 Evolution of \( \mathcal{A}_a \) and the Boundary Conditions at \( p_i \)

Now, we are in a position to derive the evolution equation of \( \mathcal{A}_a \) on \( \Gamma_n \) from the water-waves problem (WW). The boundary conditions at the corner points are adjusted to the version for the new quantity \( \mathcal{A}_a \). In the end, the evolution equations for \( d_i = d_{\Gamma_i}(p_i) \) (\( i = l, r \)) are derived.

First, since \( \mathcal{A}_a \) is related to \( \kappa \) defined on \( \Gamma_n \), we begin with the equation of \( \kappa \). In fact, it has been proved and used in [41, 49] that the mean curvature \( \kappa \) satisfies the equation
\[ D_\tau^2 \kappa = -N_t \cdot \Delta_{\Gamma_t} D_\tau v + 2\sigma \Pi(\tau_t) \cdot \nabla_{\tau_t} \nabla \kappa_H + r_1, \]
where \( r_1 \) is the lower-order term
\[
\begin{align*}
    r_1 &= 2\left( D((\nabla v)^* N_t)^T + \Pi((\nabla_{\tau_t} v)^T) \cdot \nabla_{\tau_t} v + \Delta_{\Gamma_t} v \cdot ((\nabla v)^* N_t)^T \right) \\
    &+ 2\Pi(\tau_t) \cdot (\nabla v)^2 \tau_t - 2(\nabla_{\tau_t} v \cdot N_t)(\Pi(\tau_t) \cdot \nabla_{\tau_t} v + \Pi(\tau_t) \cdot (\nabla v)^* N_t) \\
    &- 2N_t \cdot D^2 v(\tau_t, (\nabla v)^T) - 2(\nabla_{N_t} v \cdot N_t)(\Pi(\tau_t) \cdot \nabla_{\tau_t} v) \\
    &+ \nabla v((\Delta_{\Gamma_t} v)^T) - \kappa |((\nabla v)^* N_t)^T|^2 \\
    &+ 2\Pi(\tau_t) \cdot \nabla_{\tau_t} \nabla P_{\tau_t} \\
\end{align*}
\]
One can tell that the highest-order terms in \( r_1 \) are like \( \partial^2 v, \partial N_t \).

Substituting the Euler equation
\[ D_t v = -\sigma \nabla \kappa_H - \nabla P_{\tau_t} + g \]
into the equation above, a direct computation leads to
\[ D_\tau^2 \kappa = \sigma \Delta_{\Gamma_t} N(\kappa) + \tilde{R}_1, \]
where

\[
\tilde{R}_1 = \sigma [N_t, \Delta \gamma_t] \cdot \nabla \kappa + N_t \cdot \Delta \gamma_t \nabla P_{v,v} + 2\sigma \Pi (\tau) \cdot \nabla \tau \nabla \kappa + r_1
\]

and one can tell that the highest-order terms in \( \tilde{R}_1 \) are like \( \partial \kappa, \partial^2 v \).

Second, applying \( N \) to both sides of equation (4.18) leads to

\[
D_t^2 N(k) = \sigma \Delta \gamma_t N(k) + \tilde{R}_1 \circ \Phi_{S_t}^{-1} \quad \text{on } \Gamma_t,
\]

where \( \tilde{R}_1 \) is defined by

\[
\tilde{R}_1 = (D_t [D_t, N] \kappa + [D_t, N] D_t \kappa + N \tilde{R}_1) \circ \Phi_{S_t}
\]

with the commutator expressed by (4.3) and \( D_t v \) replaced by \( -\sigma \nabla \kappa - \nabla P_{v,v} + g \) from Euler equation.

Moreover, substituting \( N(k) = N_a \circ \Phi_{S_t}^{-1} - a^3 d_{\gamma_t} \circ \Phi_{S_t}^{-1} \) into both sides of the equation above, we obtain immediately that

\[
D_t^2 (N_a \circ \Phi_{S_t}^{-1}) = \sigma \Delta \gamma_t (N_a \circ \Phi_{S_t}^{-1}) + R_0 \circ \Phi_{S_t}^{-1}
\]

with the remainder terms

\[
R_0 = R_a + \tilde{R}_1,
\]

and

\[
R_a = (a^3 D_t^2 (d_{\gamma_t} \circ \Phi_{S_t}^{-1}) - a^3 \sigma \Delta \gamma_t (d_{\gamma_t} \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t}.
\]

To remove the second-order time derivative of \( d_{\gamma_t} \circ \Phi_{S_t}^{-1} \) in \( R_a \), we rewrite

\[
D_t^2 (d_{\gamma_t} \circ \Phi_{S_t}^{-1}) = (D_t^2 d_{\gamma_t} \circ \Phi_{S_t}^{-1}),
\]

where

\[
D_t^2 d_{\gamma_t} = \partial_t^2 d_{\gamma_t} + \nabla \partial_t v \cdot d_{\gamma_t} + 2 \nabla v \cdot \partial_t d_{\gamma_t} + \nabla v \cdot \nabla v \cdot d_{\gamma_t}.
\]

and one has from (4.9) that

\[
\partial_t v = (D \Phi_{S_t})^{-1} [(D_t v) \circ \Phi_{S_t} - \partial_t^2 d_{\gamma_t} \mu
+ ((\partial_t d_{\gamma_t} \mu \circ \Phi_{S_t}^{-1} - v) \cdot \nabla v) \circ \Phi_{S_t}]
\]

Moreover, we need to express \( \partial_t^2 d_{\gamma_t} \) in terms of \( \nabla v, \partial^2 d_{\gamma_t}, \partial_t d_{\gamma_t} \). In fact, recalling from (4.6) and (4.10), one obtains immediately

\[
D_t (v \cdot N_t) = D_t ((\partial_t d_{\gamma_t} \mu) \circ \Phi_{S_t}^{-1} \cdot N_t),
\]

or equivalently

\[
D_t v \cdot N_t = (\mu \circ \Phi_{S_t}^{-1} \cdot N_t)(\partial_t^2 d_{\gamma_t} + \nabla v \cdot \partial_t d_{\gamma_t}) \circ \Phi_{S_t}^{-1}
\]

\[
+ (\nabla v \cdot \mu \circ \Phi_{S_t}^{-1} \cdot N_t) \partial_t d_{\gamma_t} \circ \Phi_{S_t}^{-1}
\]

\[
+ ((\partial_t d_{\gamma_t} \mu) \circ \Phi_{S_t}^{-1} - v) \cdot D_t N_t.
\]
Substituting the Euler equation from \( \text{WW} \) into (4.22) and expressing \( N_a \) with \( N_a \), one arrives at

\[
\begin{align*}
\partial_t^2 d_{\Gamma_1} &= -\frac{1}{\mu \cdot (N_t \circ \Phi_{S_t})} \left( \mu \cdot (N_t \circ \Phi_{S_t}) \nabla_{v^*} \partial_t d_{\Gamma_1} + \nabla_{v^*} \mu \cdot (N_t \circ \Phi_{S_t}) \partial_t d_{\Gamma_1} \right) \\
&\quad + (\partial_t d_{\Gamma_1} \mu - v \circ \Phi_{S_t}) \cdot (D_t N_t \circ \Phi_{S_t}) \\
&\quad + (\sigma N_a - \sigma a^3 d_{\Gamma_1} + \nabla P_{v,v \cdot N_t + g \cdot N_t} \circ \Phi_{S_t})
\end{align*}
\]

(4.23)

As a result, \( R_a \) is expressed by

\[
R_a = a^3 \left( \partial_t^2 d_{\Gamma_1} + \nabla_{\partial_t v^*} d_{\Gamma_1} + 2 \nabla_{v^*} \partial_t d_{\Gamma_1} + \nabla_{v^*} \nabla_{\partial_t v^*} d_{\Gamma_1} \right)
\]

- \( a^3 \sigma \Delta_{\Gamma_1} N (d_{\Gamma_1} \circ \Phi_{S_t}^{-1}) \circ \Phi_{S_t} \).

where \( \partial_t^2 d_{\Gamma_1}, \partial_t v^* \) are given by (4.23), (4.21). One knows immediately that \( R_a = R_a (a^3 \partial_t^2 d_{\Gamma_1}, a^3 \partial_t v^* \partial_t d_{\Gamma_1}, a^3 \partial_t v^* \).

Now summing up all these computations above, we get the following lemma.

**Lemma 4.4.** We have that \( N_a \) satisfies the equation

\[
D_{t_*}^2 N_a + \sigma \mathcal{A}(d_{\Gamma_1}) N_a = R_0 \quad \text{on } \Gamma_*.
\]

where

\[
R_0 = R_0 \left( \partial_t^2 d_{\Gamma_1}, \partial_t \nabla_{\partial_t v^*} d_{\Gamma_1}, \nabla_{\partial_t v^*} \partial_t d_{\Gamma_1}, a^3 \nabla_{\partial_t v^*} d_{\Gamma_1}, a^3 \partial_t v^* \right)
\]

is defined in (4.19) and the operator \( \mathcal{A}(d_{\Gamma_1}) \) is defined by

\[
\mathcal{A}(d_{\Gamma_1}) f \triangleq -\left( N \Delta_{\Gamma_1} (f \circ \Phi_{S_t}^{-1}) \right) \circ \Phi_{S_t} \quad \text{for some } f \text{ on } \Gamma_*.
\]

The boundary conditions at the contact points in \( \text{WW} \) were rewritten in lemma 7.1 of [31], which gives the boundary condition at the left contact point \( p_l \) for \( J = \nabla \phi_{H} \). By a similar argument, we find the corresponding boundary conditions on \( p_{l_*} \) for \( N_a \).

**Lemma 4.5.** We have the conditions below for \( N_a \) at the contact points \( p_{l_*} \) (i = l, r) on \( \Gamma_* \):

\[
\begin{align*}
D_{t_*} \mathcal{A}(d_{\Gamma_1}) N_a + \frac{\sigma^2}{\beta c} (\sin \omega_l)^2 \circ \Phi_{S_t} \left( \nabla_{\partial_t} \mathcal{A}(d_{\Gamma_1}) N_a \circ \Phi_{S_t}^{-1} \right) \circ \Phi_{S_t} &\quad = R_{c,i} \text{ at } p_{l_*}, \\
D_{t_*} \mathcal{A}(d_{\Gamma_1}) N_a - \frac{\sigma^2}{\beta c} (\sin \omega_l)^2 \circ \Phi_{S_t} \left( \nabla_{\partial_t} \mathcal{A}(d_{\Gamma_1}) N_a \circ \Phi_{S_t}^{-1} \right) \circ \Phi_{S_t} &\quad = R_{c,i} \text{ at } p_{r_*}.
\end{align*}
\]

(4.26)
where the remainder term \( R_{c,i} = R_{c,i}(\partial^5 d_{\Gamma}, \partial^4 \partial_t d_{\Gamma}, \partial^3 v, \partial^2 \Pi_a, \partial_t \Pi_a) \) is expressed in detail as below:

\[
R_{c,i} = R_{ca} - \frac{1}{\sigma} D_t \cdot R_0 - \frac{\sigma}{\beta_c}(\sin \omega_t)^2(\nabla_{\tau_t}(R_0 \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} + (D_t^2 r_c) \circ \Phi_{S_t}
\]

\[
- \frac{\sigma}{\beta_c}[(D_t \cdot (\sin \omega_t)^2 \circ \Phi_{S_t})(\nabla_{\tau_t}(D_t \cdot \Pi_a \circ \Phi_{S_t}^{-1}))] \circ \Phi_{S_t}
\]

\[
- \frac{\sigma}{\beta_c}[(\sin \omega_t)^2((D_t \tau_t - \nabla_{\tau_t} v) \cdot \nabla(D_t \cdot \Pi_a \circ \Phi_{S_t}^{-1}))] \circ \Phi_{S_t}
\]

\[
- \frac{\sigma^2}{\beta_c} D_t[(\sin \omega_t)^2(D_t \tau_t - \nabla_{\tau_t} v) \cdot \nabla(\Pi_a \circ \Phi_{S_t}^{-1})] \circ \Phi_{S_t}
\]

\[
+ \frac{\sigma^4}{\beta_c} D_t[(\sin \omega_t)^2(D_t \tau_t - \nabla_{\tau_t} v) \cdot \nabla(\Pi_a \circ \Phi_{S_t}^{-1})] \circ \Phi_{S_t},
\]

with

\[
R_{ca} = a^2 D_t^3 d_{\Gamma_t} + a^3 D_t^2 \left( \frac{\sigma^2}{\beta_c}(\sin \omega_t)^2(\nabla_{\tau_t}(d_{\Gamma_t} \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} \right)
\]

\[
= R_{ca}(a^3 \partial^4 d_{\Gamma_t}, a^3 \partial^3 \partial_t d_{\Gamma_t}, a^3 \partial^2 v)
\]

and

\[
r_c = (\sigma \sin \omega_t(\nabla_{\tau_t} \nabla P_{v,v} - \nabla_{\tau_t} v \cdot \Pi_t + \nabla_{\tau_t} v \cdot \nabla v) \cdot N_t) \mathbf{t}_b \cdot N_t
\]

\[
+ (\sigma \sin \omega_t(\nabla_{\tau_t} v \cdot ((\nabla v)^* N_t)^\top)) \mathbf{t}_b \cdot N_t
\]

\[
- \sigma \mathbf{t}_b \cdot ((\nabla v)^* N_t)^\top(\nabla_{\tau_t} v \cdot N_t)(\mathbf{t}_b \cdot N_t) - ((\nabla v)^* N_t)^\top \cdot \nabla \kappa_H
\]

\[
+ \frac{\sigma^2}{\beta_c}(\sin \omega_t)^2 \nabla_{\tau_t} N_t \cdot \nabla \kappa_H + \beta_c(\mathbf{t}_b \cdot N_t) \nabla_{\tau_t} v \cdot \nabla P_{v,v}
\]

\[
- \beta_c(\mathbf{t}_b \cdot N_t) \nabla_{\tau_t} \Delta^{-1}(2 \nabla v \cdot \nabla^2 P_{v,v} + \Delta v \cdot \nabla P_{v,v}, (\nabla \Pi_n - \nabla v \cdot \mathbf{N}_b) \cdot \nabla P_{v,v}).
\]

Here \( D_t^2 d_{\Gamma_t}, \partial_t v^* \) are expressed by (4.20), (4.21), and (4.22), and \( D_t v \) is replaced by \(-\sigma \nabla \kappa_H - \nabla P_{v,v} + g\).

**Proof.** The boundary conditions will be proved for the left corner point \( p_l \), and the case for the right one is similar. To begin with, we know from lemma 7.1 in [11] that the conditions at the corner points can be written under the form of \( J = \nabla \kappa_H \):

\[
D_t J = \frac{\sigma^2}{\beta_c} \sin \omega_t(\nabla_{\tau_t} J \cdot N_t) \mathbf{t}_b + r_J \quad \text{at} \quad p_l,
\]

where

\[
r_J = (\sigma \sin \omega_t(\nabla_{\tau_t} \nabla P_{v,v} - [D_t, \nabla_{\tau_t}] v) \cdot N_t - \sigma \sin \omega_t(\nabla_{\tau_t} v \cdot D_t N_t)
\]

\[
+ \sigma \mathbf{t}_b \cdot D_t N_t (\nabla_{\tau_t} v \cdot N_t) - \beta_c D_t \nabla_{\tau_t} v \cdot \mathbf{t}_b) \mathbf{t}_b.
\]
Applying the inner product with \( N_t \) on both sides of the equation above and noticing that 
\[
\tau_b \cdot N_t = -\sin \omega t,
\]
one obtains 
\[
D_t \mathcal{N}(\kappa) = -\frac{\sigma^2}{\beta_c} (\sin \omega t)^2 \nabla_{\tau_t} \mathcal{N}(\kappa) + r_c,
\]
where 
\[
r_c = D_t N_t \cdot \nabla \kappa_H + \frac{\sigma^2}{\beta_c} (\sin \omega t)^2 \nabla_{\tau_t} N_t \cdot \nabla \kappa_H + r_f \cdot N_t.
\]
In addition, substituting \( \mathcal{N}(\kappa) = \mathcal{V}_\alpha \circ \Phi_{S_t}^{-1} - a^2 \mathcal{D}_{\Gamma_t} \circ \Phi_{S_t}^{-1} \) into the equation above and noticing that 
\[
(\nabla_{\tau_t} f) \circ \Phi_{S_t} = \nabla (\nabla \Phi_{S_t})^* \tau_t \circ \Phi_{S_t} (f \circ \Phi_{S_t}),
\]
one can change the equation for \( \mathcal{N}(\kappa) \) into an equation for \( \mathcal{V}_\alpha \): 
\[
D_t \mathcal{V}_\alpha = -\frac{\sigma^2}{\beta_c} (\sin \omega t)^2 (\nabla_{\tau_t} (\mathcal{V}_\alpha \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} + R_{c0} \quad \text{at } p_{i*},
\]
where 
\[
R_{c0} = r_c \circ \Phi_{S_t} + \left( a^2 D_{t*} d_{\Gamma_t} + a^2 \frac{\sigma^2}{\beta_c} (\sin \omega t)^2 (\nabla_{\tau_t} (d_{\Gamma_t} \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} \right)
\]
with \( r_c \) expressed in the statement of this lemma. Moreover, one uses the Euler equation from (WW) to replace \( D_t v \) by \( -\sigma \nabla \kappa_H - \nabla P_{v,v} + g \) and applies (4.5) and the expressions for \( D_t N_t, D_t \tau_t \) in the beginning of Section 4.

Second, applying \( D_{t*} \) twice on the equation above, we get 
\[
D_{t*}^2 \mathcal{V}_\alpha = -D_{t*}^2 \left( \frac{\sigma^2}{\beta_c} (\sin \omega t)^2 (\nabla_{\tau_t} (\mathcal{V}_\alpha \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} \right) + D_{t*}^2 R_{c0}.
\]
Checking term by term and applying (4.25), we finally derive the desired condition. Again, one needs to notice that the remainder terms involving \( D_t v \) should be replaced by \( -\sigma \nabla \kappa_H - \nabla P_{v,v} + g \) from Euler equation, and \( D_{t*}^2 d_{\Gamma_t}, \partial^2_{t} d_{\Gamma_t}, \partial_{t} v^* \) are expressed by (4.20), (4.21) and (4.23).

\[\square\]

**Remark 4.6.** In the following text, the choices of signs at the left and the right corner points \( p_{i*} \) are the same as in (4.26).

**Remark 4.7.** From the compatibility conditions (1.1), it is straightforward to get 
\[
D_{t*} \mathcal{A}(d_{\Gamma_t}) \mathcal{V}_\alpha |_{l=0} = \left( \frac{\sigma^2}{\beta_c} (\sin \omega t)^2 \circ \Phi_{S_t} (\nabla_{\tau_t} (\mathcal{A}(d_{\Gamma_t}) \mathcal{V}_\alpha \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} \right) |_{l=0}
= R_{c,l} |_{l=0} \quad \text{at } p_{i*} (i = l, r).
In the end, except for the equation of \( \mathcal{N}_a \) and the boundary conditions, we also need the evolution equation for \( d_{\Gamma_i} \) at the corner points in the iteration.

Notice that at the corner points \( p_1, p_r \), the velocity \( v \) is tangential along the bottom \( \Gamma_b \) and the unit vector \( \mu \) is defined to be tangential along \( \Gamma_{b*} \) as well. Combining with (4.6), one has

\[
v = (\partial_t d_{\Gamma_i} \mu) \circ \Phi_{S_i}^{-1} \quad \text{at } p_i \left( i = l, r \right).
\]

Substituting this equality into (4.23), one derives the evolution equation for \( d_i(t) = d_{\Gamma_i}(p_{i*}) \left( i = l, r \right) \):

\[
d_i''(t) = \mathfrak{B}_i, \quad i = l, r,
\]

where

\[
\mathfrak{B}_i \triangleq - \frac{1}{\mu \cdot (N_t \circ \Phi_{S_i})} \left( \mu \cdot (N_t \circ \Phi_{S_i}) \nabla_{v^*} \partial_t d_{\Gamma_i} + \nabla_{v^*} \mu \cdot (N_t \circ \Phi_{S_i}) \partial_t d_{\Gamma_i} + \sigma \mathcal{N}_a - \sigma \mathcal{A}_D \right) d_{\Gamma_i} + \nabla P_{v,v} \cdot N_t + g \cdot N_t \big|_{p_{i*}}.
\]

Consequently, we specify the boundary conditions for \( d_{\Gamma_i} \) in (4.15), if the right-side term \( \mathfrak{B}_i \) and \( d_i(0) \), \( d_i'(0) \) are known in the iteration section:

\[
d_i = d_i(0) + d_i'(0)t + \int_0^t \int_0^s \mathfrak{B}_i(\tau) \, d\tau \, ds \quad \text{at } p_{i*} \left( i = l, r \right).
\]

Remark 4.8. According to (4.22) and the equation above it, the boundary conditions (4.28) are in fact corresponding to \( D_t \nabla \cdot N_t = -\sigma \mathcal{N}(\kappa) - \nabla_{N_t} P_{v,v} + g \cdot N_t \) at \( p_i \) where \( v \) is given by (4.7). One will find that these boundary conditions play an important role in the last section, where we go back to the water-waves system from the iteration.

### 4.5 Estimates for the Remainder Terms \( R_0 \) and \( R_{e,i} \)

We consider the estimates for the remainder terms here. In the iteration, the estimates of the remainder terms \( R_0, R_{e,i} \) depend on the norms of \( d_{\Gamma_i}, \partial_t d_{\Gamma_i} \). Meanwhile, notice that one can only use the velocity \( v = \nabla \phi \) defined in (4.7) and relate \( N_t, \tau_t, \kappa \) with \( d_{\Gamma_i} \) as well. Therefore, \( D_t \nabla v, D_t N_t, D_t \tau_t, D_t \kappa \) should be estimated in terms of \( d_{\Gamma_i}, \partial_t d_{\Gamma_i} \) in order to go back to \( \mathcal{N}_a, \partial_t \mathcal{N}_a, d_i, \partial_t d_i \) by using the operator \( \mathcal{K} \).

To begin with, we consider the estimates for \( v, N_t, \tau_t \).

**Lemma 4.9.** Let \( v = \nabla \phi \) be defined in (4.7) and \( N_t, \tau_t \) be the unit normal and tangential vectors defined on \( \Gamma_t \in \Lambda(S^*, 8.5, \delta, \pi/16) \) from Definition 3.1.

1. The following estimates hold for \( N_t, \tau_t, \kappa \) when \( s \geq 3 \):

\[
\| N_t \|_{H^s(\Gamma_t)}, \| \tau_t \|_{H^s(\Gamma_t)}, \| \kappa \|_{H^{s-1}(\Gamma_t)} \leq C \left( \| d_{\Gamma_i} \|_{H^{s+1}(\Gamma_{a*})}, \| d_{\Gamma_i} \|_{H^{s+1}(\Gamma_{b*})} \right).
\]

\[
\| D_t N_t \|_{H^{s-1.5}(\Gamma_t)}, \| D_t \tau_t \|_{H^{s-1.5}(\Gamma_t)}, \| D_t \kappa \|_{H^{s-2.5}(\Gamma_t)} \leq C \left( \| d_{\Gamma_i} \|_{H^{s+0.5}(\Gamma_{a*})}, \| d_{\Gamma_i} \|_{H^{s+0.5}(\Gamma_{b*})}, \| v \|_{H^{s-1}(\Omega_i)} \right).
\]


Moreover, if \( N_\hat{t} \) denotes this parametric curve reads
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
Consequently, one obtains from (4.10) that
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
One can see that the higher-order terms in \( D \) handled in a similar way.

\[k\frac{D}{v} \frac{D}{v} \frac{D}{v}\]
For a parameter \( t \), the following estimates hold when \( s \geq 3 \):
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
Moreover, if \( s \geq 4 \), one has
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
**Proof.**

**Step 1.** Estimates for \( N, \tau, \) and \( \kappa \). First, we recall from Section 2 that the free surface \( \Gamma_t \) is defined by \( \Phi_{\Gamma_0}(p) = p - \hat{d}_t(p) \hat{\gamma}(p) \) for \( p \in \Gamma_\ast \). Meanwhile, \( N_t, \tau_t \) are defined on \( \Gamma_t \). Parameterizing \( \Gamma_t \) with the arc length parameter \( s \) and denoting \( \Phi_{\Gamma_t} = \Phi_{\Gamma_t}(s) \), one knows immediately that the unit tangential vector of this parametric curve reads
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
Consequently, one obtains from (4.10) that
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
with \( \frac{k}{v} \frac{D}{v} \frac{D}{v} \frac{D}{v} \frac{D}{v} \frac{D}{v} \) for any function \( f \) on \( \Gamma_\ast \), where \( \tau_\ast \) is the unit tangential vector of \( \Gamma_\ast \), one rewrites
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
One can see that the higher-order terms in \( D_t \tau_t \) are \( \partial_s \partial_t d_G, \partial_s^2 d_G \), and \( v \), or we simply write \( D_t \tau_t = h(\partial_s \partial_t d_G, \partial_s^2 d_G, v \) ), which implies immediately the desired estimate for \( \tau_t \), \( D_t \tau_t \) with Lemma 2.5. Meanwhile, \( N_t, \tau_t, N_t, \kappa, D_t \kappa \) can be handled in a similar way.

**Step 2.** Estimates for \( v \) and \( D_t^2 \tau_t \). In fact, recalling that \( v = -\nabla \phi \) with \( \phi \) satisfying system (4.8), one obtains immediately by Lemma 2.4 that
\[
\frac{k}{v} \frac{D}{v} \frac{D}{v}
\]
which implies the desired estimate.
On the other hand, we turn to the estimate for \( D_t v \). One has similarly as in (4.12) that
\[
D_t v = \nabla D_t \phi + [D_t, \nabla] \phi.
\]
Checking term by term and applying Lemma 2.4, one obtains the desired estimate. Meanwhile, a similar computation as in Step 1 leads to
\[
D_t^2 \tau_t = h \left( \partial_t d_{\tau_t}, \partial_t d_{\Gamma_t}, \partial_t^2 d_{\Gamma_t}, v, D_t v \right)
\]
for some polynomial function \( h \). Therefore, combining with the estimate for \( D_t v \), one obtains the desired estimate for \( D_t^2 \tau_t \), and \( D_t^2 N_t \) can be handled similarly as well.

In the end, analogous analysis and applying Lemma 2.5 and Lemma 2.4, one can prove the desired estimates for \( D_t v \) and \( D_t D_t v \).

Now we turn to the remainder term \( R_0 \) from equation (4.25), where the highest-order terms in \( R_0 \) are like \( \partial^4 d_{\Gamma_t}, \partial^3 \partial_t d_{\Gamma_t}, \partial^2 v \), and \( \partial \mathcal{M}_t \).

**Lemma 4.10.** Let \( \Gamma_t \in \Delta(S_*), 8.5, 8, \pi/16 \). Then the remainder term \( R_0 \) satisfies
\[
\| R_0 \|_{H^4(\Gamma_0)} \leq a^{3/2} C(\| \mathcal{M}_a \|_{H^{5.5}(\Gamma_0)}, \| \partial_t \mathcal{M}_a \|_{H^4(\Gamma_0)}, |d_t|, |d_t'|)
\]
and
\[
\| R_0 \|_{H^{2.5}(\Gamma_0)} \leq C(\| \mathcal{M}_a \|_{H^{5.5}(\Gamma_0)}, \| \partial_t \mathcal{M}_a \|_{H^4(\Gamma_0)}, |d_t|, |d_t'|).
\]
where \(|d_t| \) means \(|d_t| + |d_t'|\) for the sake of convenience.
Moreover, one also has for a parameter \( \tau \) the following estimate:
\[
\| \partial_t R_0 \|_{H^{2.5}(\Gamma_0)} \leq a^{3/2} C(\| \mathcal{M}_a \|_{H^{5.5}(\Gamma_0)}, \| \partial_t \mathcal{M}_a \|_{H^4(\Gamma_0)}, |d_t|, |d_t'|) \times \left( \| \partial_t \mathcal{M}_a \|_{H^4(\Gamma_0)} + \| \partial_t \partial_t \mathcal{M}_a \|_{H^{2.5}(\Gamma_0)} + \| \partial_t d_t \| + |\partial_t \partial_t d_t| \right).
\]
**Proof.**

**Step 1.** Estimate for \( R_0 \). Recalling the definition of \( R_0 \), we have that
\[
\| R_0 \circ \Phi_{s_t}^{-1} \|_{H^4(\Gamma_t)} \leq \| R a \|_{H^4(\Gamma_0)} + \| [D_t, \mathcal{N}] D_t \kappa \|_{H^4(\Gamma_t)} + \| D_t [D_t, \mathcal{N}] \kappa \|_{H^4(\Gamma_t)} + \| \mathcal{N} \mathcal{R}_1 \|_{H^4(\Gamma_t)}.
\]
For the first term in the inequality above, one has directly from the expression of \( R a \) the following estimate:
\[
\| R a \|_{H^4(\Gamma_0)} \leq C \left( a^3 \| d_{\Gamma_t} \|_{H^7(\Gamma_0)}, a^3 \| \partial_t d_{\Gamma_t} \|_{H^5(\Gamma_0)}, a^3 \| v \|_{H^{5.5}(\Omega_t)} \right),
\]
where the polynomial \( C \) is linear with \( a^3 \).
Recalling \([4.3]\) and \([4.1]\), we have
\[
\|D_t, \mathcal{N}\|_{H^4(\Gamma_t)}^2 \\
\leq \|D_t, \mathcal{N}\|_{H^4(\Gamma_t)}^2 + \|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)}^2 + \|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)}^2 \\
\leq C(\|\nabla_t, \mathcal{N}\|_{H^4(\Gamma_t)}) \\
\leq C(\|\nabla_t, \mathcal{N}\|_{H^4(\Gamma_t)} + \|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)}).
\]

where \([2.5]\), \([2.4]\), and \([4.9]\) are applied. Consequently, remembering that we used \(D_t v = -\sigma \nabla_k H - \nabla P_e v + g\), one has
\[
\|D_t[D_t, \mathcal{N}]\|_{H^4(\Gamma_t)}^2 \\
\leq C(\|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)} + \|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)} + \|\nabla_t(D_t, \mathcal{N})\|_{H^4(\Gamma_t)}).
\]

where one uses the following estimate according to \([4.5]\):
\[
\|D_t w\|_{H^5(\Omega_t)}^2 \\
\leq C(\|D_t w\|_{H^5(\Omega_t)}^2 + \|D_t w\|_{H^5(\Omega_t)}^2 + \|D_t w\|_{H^5(\Omega_t)}^2).
\]

and also the estimate
\[
\|D_t^k H\|_{H^{k}(\Omega_t)}^2 \\
\leq C(\|D_t^k H\|_{H^{k}(\Omega_t)}^2 + \|D_t^k H\|_{H^{k}(\Omega_t)}^2 + \|D_t^k H\|_{H^{k}(\Omega_t)}^2).
\]

while \([4.2]\) and \([2.4]\) are applied. Therefore, applying \([4.9]\) again, we arrive at
\[
\|D_t[D_t, \mathcal{N}]\|_{H^4(\Gamma_t)} + \|\partial_t(D_t[\mathcal{N}])\|_{H^4(\Gamma_t)} \\
\leq C(\|\partial_t(D_t[\mathcal{N}])\|_{H^4(\Gamma_t)} + \|\partial_t(D_t[\mathcal{N}])\|_{H^4(\Gamma_t)}).
\]
To finish the estimate for \( \| R_0 \|_{H^4(\Gamma)} \), we still need to deal with \( \| \mathcal{N} \tilde{R}_1 \|_{H^4(\Gamma)} \). In fact, from the expression of \( \tilde{R}_1 \), we know that the higher-order terms are like \( \partial_t d_{\Gamma, \alpha}, \partial^2 d_{\Gamma, \alpha}, \nabla \kappa, \partial^2 v, \Delta_{\Gamma} N_t, \) and \( \Delta_{\Gamma} \nabla P_{v, v} \). For example, from the definition of \( P_{v, v} \), we have
\[
\| \nabla P_{v, v} \|_{H^7(\Gamma)} \leq C (\| d_{\Gamma} \|_{H^8(\Gamma)}, \| \partial_t d_{\Gamma} \|_{H^7(\Gamma)})(1 + \| v \|_{H^7(\Gamma)})^2 \leq C (\| d_{\Gamma} \|_{H^8(\Gamma, \rho)}, \| \partial_t d_{\Gamma} \|_{H^7(\Gamma, \rho)}).
\]

Then, checking term by term, it is easy to show that
\[
\| \mathcal{N} \tilde{R}_1 \|_{H^4(\Gamma)} \leq C (\| d_{\Gamma} \|_{H^8(\Gamma)}, \| \partial_t d_{\Gamma} \|_{H^7(\Gamma)}).
\]

Summing these estimates up, we have the estimate for \( R_0 \):
\[
\| R_0 \|_{H^4(\Gamma)} \leq C (\| d_{\Gamma} \|_{H^8, \rho}, \| \partial_t d_{\Gamma} \|_{H^7(\Gamma)}, a^3 \| d_{\Gamma} \|_{H^7(\Gamma)},
\]
\[
a^3 \| \partial_t d_{\Gamma} \|_{H^5(\Gamma), \rho}, a^3 \| v \|_{H^5(\Omega)}).
\]

and we notice here that the right side is linear with respect to \( a^3 \). The desired estimate for \( \| R_0 \|_{H^4(\Gamma)} \) follows from Proposition 4.2, Proposition 4.11, and Lemma 4.9.

Moreover, one has by an analogous analysis that
\[
\| R_0 \|_{H^{2.5}(\Gamma)} \leq C (\| d_{\Gamma} \|_{H^6, \rho}, \| \partial_t d_{\Gamma} \|_{H^5(\Gamma, \rho), \rho}, a^3 \| d_{\Gamma} \|_{H^5(\Gamma, \rho)},
\]
\[
a^3 \| \partial_t d_{\Gamma} \|_{H^3(\Gamma, \rho), \rho}, a^3 \| v \|_{H^4(\Gamma)}).
\]

which leads to the desired estimate by using again Proposition 4.2, Proposition 4.12, and Lemma 4.9.

Step 2. Estimate for \( \partial_t R_0 \). The analysis is similar to that before, so we omit the details. In fact, since
\[
R_0 = R_0 (\partial^4 d_{\Gamma}, \partial^3 \partial_t d_{\Gamma}, \partial^2 v, \partial\mathcal{N}_d, a^3 \partial^3 d_{\Gamma}, a^3 \partial_t \partial_t d_{\Gamma}, a^3 v)
\]

and \( \partial_t \) acting on \( R_0 \) results in an extra \( \partial_t \) (or \( D_t \) on \( \Gamma \)), a similar analysis as before leads to the desired estimates.

On the other hand, we prove the estimate of the reminder term \( R_{c, i} \).

**Lemma 4.11.** Let \( \Gamma \in \Lambda(S, 8.5, \delta, \pi/16) \) and \( \tau \) be a parameter. Then the reminder term \( R_{c, i} \) (i = 1, r) from (4.26) satisfies
\[
\| D_t R_{c, i} \|_{H^1(\Gamma)} \leq a C (\| \mathcal{N}_d \|_{H^{5.5}(\Gamma)}, \| \partial_t \mathcal{N}_d \|_{H^4(\Gamma)}, \| \partial_t^2 \mathcal{N}_d \|_{H^{2.5}(\Gamma)},
\]
\[
|d_i|, |d_i'|, |d_i''|),
\]

and
\[
\| \partial_t R_{c, i} \|_{H^1(\Gamma)} \leq a C (\| \mathcal{N}_d \|_{H^{5.5}(\Gamma)}, \| \partial_t \mathcal{N}_d \|_{H^4(\Gamma)}, \| |d_i|, |\partial_t d_i|,
\]
\[
\times (\| \partial_t \mathcal{N}_d \|_{H^4(\Gamma)} + |\partial_t |d_i| + |\partial_t d_i'| + |\partial_t |d_i''|).
\]
PROOF. The proof is analogous to the proof of the previous lemma, so we omit most of the details. First, since one knows from Lemma 4.5 that

\[ R_{c,i} = R_{c,i}(\partial^2 \gamma, \partial^2 \gamma, \partial^3 v, \partial^2 \gamma, \partial \gamma, \partial \gamma, \partial^2 \gamma, a^3 \partial \gamma) \]

one has

\[
\| R_{c,i} \|_{H^1(\Gamma_\alpha)} \leq C \left( \| \partial^2 \gamma \|_{H^6(\Gamma_\alpha)} + \| \partial \gamma \|_{H^5(\Gamma_\alpha)} + \| v \|_{H^4(\Gamma_\alpha)} \right),
\]

(4.29)

\[
\| \partial v \|_{H^4(\Gamma_\alpha)} + \| \partial \gamma \|_{H^3(\Gamma_\alpha)} + \| \partial \gamma \|_{H^2(\Gamma_\alpha)} + \| a^3 \partial \gamma \|_{H^1(\Gamma_\alpha)} + \| a^3 \partial^2 \gamma \|_{H^1(\Gamma_\alpha)} + \| a^3 \partial^2 \gamma \|_{H^1(\Gamma_\alpha)}
\]

where the right side is linear with respect to \( a^3 \).

On the other hand, the estimate for \( D_{t*} R_{c,i} \) results in extra \( \partial \gamma \) and \( \nabla \gamma \) terms in each term of \( R_{c,i} \), which implies

\[
\| D_{t*} R_{c,i} \|_{H^1(\Gamma_\alpha)} \\
\leq C \left( \| \partial^2 \gamma \|_{H^6(\Gamma_\alpha)} + \| \partial \gamma \|_{H^5(\Gamma_\alpha)} + \| \partial^2 \gamma \|_{H^5(\Gamma_\alpha)} + \| \partial \gamma \|_{H^4(\Gamma_\alpha)} + \| \partial \gamma \|_{H^3(\Gamma_\alpha)} + \| \partial \gamma \|_{H^2(\Gamma_\alpha)} + \| a^3 \partial \gamma \|_{H^1(\Gamma_\alpha)} + \| a^3 \partial^2 \gamma \|_{H^1(\Gamma_\alpha)} + \| a^3 \partial^2 \gamma \|_{H^1(\Gamma_\alpha)} \right)
\]

and the right side is linear with \( a^3 \) as well. Consequently, applying Proposition 4.2, Proposition 4.12, and Lemma 4.9 we finish the proof for the first estimate.

On the other hand, the estimate for \( \partial \gamma R_{c,i} \) follows in a similar way, while notice that \( \partial \gamma \) only adds extra \( \partial \gamma \) to each term in (4.29) (except \( D_r \) on \( v \)).

\[ \square \]

4.6 Estimates for Time Derivatives of \( \Gamma_\alpha \)

Since the norms involving time derivatives of \( \partial \gamma, \partial^2 \gamma \) will be used in the energy estimates, we need to consider the estimates for \( \partial \gamma, \partial^2 \gamma, \partial^3 \gamma \) in terms of time derivatives of \( \partial \gamma \) and \( \partial^2 \gamma \).

**Proposition 4.12.** Let \( \Gamma_\alpha \in \Lambda(S_\alpha, 8.5, \delta, \pi/16) \) for \( t \in [0, T] \) and \( s_1, s_2 \) satisfy

\[ 1 \leq s_2 \leq s_1 \leq 8.5 \quad \text{and} \quad s_1 \leq s_2 + 3. \]

Then the following estimates hold:

\[
\| \partial \gamma \|_{H^s(\Gamma_\alpha)} \leq a^{s_2-s_1+3} C (\| \partial \gamma \|_{H^s(\Gamma_\alpha)} + \| \partial^2 \gamma \|_{H^s(\Gamma_\alpha)} + \| \partial^3 \gamma \|_{H^s(\Gamma_\alpha)} + \| \partial \gamma \|_{H^s(\Gamma_\alpha)} + \| \partial^2 \gamma \|_{H^s(\Gamma_\alpha)})
\]

where \( \| \partial \gamma \| \) means \( |\partial \gamma| \).
PROOF. The proof follows from the proof of Proposition 4.2. In fact, denoting
\[ v_1 = (\partial_t d_{\Gamma_*} \mu) \circ \Phi_{S_t}^{-1} \quad \text{and} \quad D_{t1} = \partial_t + \nabla v_1, \]
one has from the definition of \( \partial_t \mathcal{N}_a \) that
\[
\partial_t \mathcal{N}_a = a^3 \partial_t d_{\Gamma_t} + D_{t1} \mathcal{N}(\kappa) \circ \Phi_{S_t} \\
= a^3 \partial_t d_{\Gamma_t} + \mathcal{N}(D_{t1} \kappa) \circ \Phi_{S_t} + ([D_{t1}, \mathcal{N}] \kappa \circ \Phi_{S_t}^{-1}) \circ \Phi_{S_t}.
\]

On the other hand, expressing \( D_{t1} \kappa \) similarly as in Proposition 4.2 and substituting it into the expression of \( \partial_t \mathcal{N}_a \), one obtains a linear system for \( \partial_t d_{\Gamma_t} \) using (4.15):
\[
\begin{cases}
a^3 \partial_t d_{\Gamma_t} - \mathcal{N}((\mu \circ \Phi_{S_t}^{-1}) \cdot N_t) \Delta_{\Gamma_t}(\partial_t d_{\Gamma_t} \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} + r_{\partial_t d} \\
\partial_t d_{\Gamma_t}(p_{i*}) = \partial_t d_i, \quad i = l, r,
\end{cases}
\]
where \( r_{\partial_t d} \) contains remainder terms of \( \partial_t d_{\Gamma_t} \) and \( d_{\Gamma_t} \), which can be written explicitly.

Noticing that \((\mu \circ \Phi_{S_t}^{-1}) \cdot N_t \geq c_1 > 0 \) for \( t \in [0, T] \) and for some constant \( c_1 > 0 \), so when \( a \) is large enough, standard elliptic analysis as before leads to the desired estimates for \( \partial_t d_{\Gamma_t} \). In the end, one can also prove similar estimates for \( \partial_t^2 d_{\Gamma_t} \), and the proof is finished. \( \square \)

In the following sections, one also needs to consider the estimates for \( \partial_t d_{\Gamma_t} \), \( \partial_t \partial_t d_{\Gamma_t} \), and \( \partial_t^2 \partial_t d_{\Gamma_t} \) with a parameter \( \tau \).

**Corollary 4.13.** Let \( \Gamma_t \in \Lambda(S_*, 8.5, \delta, T) \) for \( t \in [0, T] \) and \( s_1, s_2 \) satisfy
\[ 1 \leq s_2 \leq s_1 \leq 8.5 \quad \text{and} \quad s_1 \leq s_2 + 3. \]

Then the following inequalities hold:
\[
\| \partial_t d_{\Gamma_t} \|_{H^{s_1}(\Gamma_*)} \leq a^{s_1-s_2-3} C \left( \| \mathcal{N}_a - \mathcal{N}_{a*} \|_{H^{s_2}(\Gamma_*)}, |d_i| \right) \\
\times \left( \| \partial_t \mathcal{N}_a \|_{H^{s_2}(\Gamma_*)} + |\partial_t d_i| \right),
\]
\[
\| \partial_t \partial_t d_{\Gamma_t} \|_{H^{s_1}(\Gamma_*)} \leq a^{s_1-s_2-3} \\
\times C \left( \| \mathcal{N}_a - \mathcal{N}_{a*} \|_{H^{s_2}(\Gamma_*)}, \| \partial_t \mathcal{N}_a \|_{H^{s_2}(\Gamma_*)}, |d_i|, |\partial_t d_i| \right) \\
\times \left( \| \partial_t \mathcal{N}_a \|_{H^{s_2}(\Gamma_*)} + \| \partial_t \partial_t \mathcal{N}_a \|_{H^{s_2}(\Gamma_*)} + |\partial_t d_i| + |\partial_t \partial_t d_i| \right).
\]
Moreover, one has
\[
\| \partial_t^2 \partial_t d \|_{H^1(\Gamma_\sigma)} \\
\leq a^{\sigma_1 - \sigma_2} C \left( \| \partial H \partial_0 \|_{H^2(\Gamma_\sigma), \| \partial \partial H \partial \|_{H^2(\Gamma_\sigma)}}, \| \partial_t^2 \partial_0 \|_{H^2(\Gamma_\sigma)} \right) \\
\times (\| \partial_t \partial H \partial \|_{H^2(\Gamma_\sigma)} + \| \partial_t \partial H \partial \|_{H^2(\Gamma_\sigma)} + \| \partial_t^2 \partial H \partial \|_{H^2(\Gamma_\sigma)}) \\
+ |\partial_t d | + |\partial_t \partial d | + | \partial_t^2 \partial d |). 
\]

PROOF. The proof is analogous to the proof of Proposition 4.12 and is left to the reader.

5 The Linear Problem on the Free Surface

Assuming that the free surface \( \Gamma_t \in \Lambda(S_*, S, \delta, 1/40) \) is known already, we consider the following linear system of (\( f(t, p), d_i(t), d_T(t) \)) with \( p \in \Gamma_* \):

\[
\begin{cases}
D_{t*}^2 f + \sigma A(d_{\Gamma}) f = g_1 & \text{on } \Gamma_* \\
D_{t*} d \partial_{\Gamma} d = \frac{c_0}{2} (\sin \omega t)^2 \\
\circ \Phi_{S_t} (\nabla_{T} (A(d_{\Gamma}) f \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = g_{2*,} & \text{at } p_t.* \\
d_i''(t) = B_i, & i = l, r.
\end{cases}
\]

(5.1)

with initial data

\[
f|_{t=0} = f_0, \quad D_{t*} f|_{t=0} = f_1, \quad d_i(0) = d_{i,0}, \quad d_i'(0) = d_{i,1}.
\]

Here the right sides \( g_1, g_{2, i}, B_i \) and the initial data \( f_0, f_1, d_{i,0}, d_{i,1} \) are given \((i = l, r)\).

To solve the linear system above, we introduce

\[
F \triangleq (a^3 + A(d_{\Gamma})) f = A_d f.
\]

This makes sense thanks to Remark 4.3, so we could retrieve \( f \) from \( F \) as long as we have Dirichlet boundary conditions for \( f \).

By direct calculations, we get

\[
\begin{cases}
D_{t*}^2 F + \sigma A(d_{\Gamma}) F = A_d g_1 + [A_d, D_{t*}^2] f & \text{on } \Gamma_* \\
D_{t*} F = \frac{c_0}{2} (\sin \omega t)^2 \circ \Phi_{S_t} (\nabla_{T} (F \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = g_{2, i} & \text{at } p_t.* \\
\end{cases}
\]

(5.2)

Consequently, system (5.1) is equivalent to the following system:

\[
\begin{cases}
D_{t*}^2 F + \sigma A(d_{\Gamma}) F = A_d g_1 + [A_d, D_{t*}^2] f & \text{on } \Gamma_* \\
D_{t*} F = \frac{c_0}{2} (\sin \omega t)^2 \circ \Phi_{S_t} (\nabla_{T} (F \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = g_{2, i} & \text{at } p_t.* \\
(a^3 + A(d_{\Gamma})) f = F, \\
d_i''(t) = B_i, & i = l, r.
\end{cases}
\]

(5.3)
Remark 5.1. The reason of using higher-order boundary conditions in (5.1) is that we need high-order energy estimates of \( f \) later; meanwhile, second- or higher-order time derivatives on the right side should be avoided. In fact, we derived the high-order boundary conditions containing \( D_t v \) and \( \partial_t^2 d_{\Gamma} \) in the right-side part \( g_2 \) in the previous section, where the Euler equation was applied to replace them. In contrast, we do not have the Euler equation in the iteration scheme, which means we have to avoid second- or higher-order of time derivatives on the right side from now on. As a result, we give the high-order boundary conditions in the first place.

We first prove the existence of the solution to our linear problem.

**Proposition 5.2.** Let \( (f_0, f_1) \in H^{5.5}(\Gamma_*) \times H^4(\Gamma_*) \), \( g_1 \in C([0, T]; H^4(\Gamma_*)) \), \( g_{2,i} \in C^1([0, T]; H^1(\Gamma_*)) \), and \( d_{i,0}, d_{i,1} \in \mathbb{R} \) be given and \( \omega_{i,*} \in (0, \pi/6) \). If the compatibility conditions at \( t = 0 \) hold, namely

\[
D_t A(d_{\Gamma}) f \big|_{t=0} = \pm \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 \circ \Phi_{S_i} (\nabla_{\Gamma} (T_n (T^*_{S_i}))) \circ \Phi_{S_i} |_{t=0}
\]

\[
= g_{2,i} \big|_{t=0} \text{ at } p_i^*,
\]

then there exists some \( T > 0 \) such that the system (5.1) has a unique solution on \( [0, T] \).

**Proof.** To solve system (5.1), we need to solve system (5.3) equivalently.

First of all, thanks to Remark 4.3, we notice that the commutator \([A_a, D_t^2 f] \) is a lower-order term of \( T \), \( D_t F \), and we can write it as \( R_1 F + R_0 D_t F \), where \( R_1 \) means at most a first-order derivative of \( F \) and \( R_0 \) means at most a zeroth-order derivative of \( D_t F \). The operator \( A_a \) is invertible since Dirichlet boundary conditions for \( f \) are given in (5.3) by \( D_t^2 f \pm \sigma A(d_{\Gamma}) f = g_1 \).

Consequently, to finish the proof, we only need to prove the existence of the solution of the following system:

\[
\begin{align*}
D_t^2 F + \sigma A_a F &= R_1 F + R_0 D_t F + A_a g_1 & \text{on } \Gamma_* \\
D_t F &\pm \sigma (\sin \omega_i)^2 \circ \Phi_{S_i} (\nabla_{\Gamma} (T_n (T^*_{S_i}))) \circ \Phi_{S_i} = g_{2,i} & \text{at } p_i^*,
\end{align*}
\]

with initial data (which can be derived from (5.1) and are omitted here)

\[
F |_{t=0} = F_0, \quad D_t F |_{t=0} = F_1.
\]

The equation of \( F \) is a linear hyperbolic equation of third order with mixed boundary conditions. To solve this system, we follow the proof of [31], while we keep in mind that the difference between our paper and [31] lies in the third-order elliptic operator \( A_a = A(d_{\Gamma}) + a^2 \) studied in Remark 4.3. As a result, we mainly focus on the estimates related to \( A_a \), and we only outline the proof for the remainder parts.

Introducing

\[
U(t) = (F(t, p), D_t F(t, p))^* \]
we rewrite system (5.4) into an equivalent system again:

\[
\frac{d}{dt} U(t) = \tilde{A}(t) U(t) + G(t) \quad \text{on } \Gamma_*,
\]

(5.5)

\[
B(t) U(t) = D_{t*} g_{2,i} \quad \text{at } p_i*, \quad U(0) = U_0,
\]

where

\[
G(t) = \begin{pmatrix} 0 & -\nabla v^* \\ \sigma A_a g_1 & -\sigma A_a + R_1 - \nabla v^* + R_0 \end{pmatrix},
\]

and

\[
B(t) = (-\sigma A_a + R_1 c_0 (\nabla v (\cdot \circ \Phi_S^{-1})) \circ \Phi_S)
\]

with the notation

\[
c_0 = c_0(t) = \frac{\sigma^2}{\beta c} (\sin \omega_1)^2 \circ \Phi_S > 0.
\]

We recall that \( R_1 = R_1(\partial) \) is some first-order operator, and \( R_0 = R_0(\partial) \) is some zeroth-order operator on \( \Gamma_* \) (the details here can be computed in a direct way and are omitted since they are lower-order terms as mentioned in the beginning of the proof). Moreover, the boundary conditions \( B(t) U(t) = D_{t*} g_{2,i} \) are derived from the boundary condition of \( F \) by taking \( D_t \) and applying the equation of \( F \), since we use the \( H^{2,5} (\Gamma_*) \) norm for \( F \) as a starting point for proving the existence.

We plan to prove first the existence of the solution to system (5.5) when the time \( t = t_0 \) is fixed in \( \tilde{A}(t) \) and the boundary satisfies \( B(t) U = 0 \), where we can use the standard semigroup method as in [31, lemma 2.4]. As the second step, we can build energy inequality for this system and prove the existence of the solution to our system (5.4) of \( F \) when the time is fixed in the operators as well. Finally, we can use the method of Cauchy’s polygonal line to recover the existence of the solution to system (5.4) as in section 4 of [31].

To finish the first step and use the semigroup method for proving the existence, the key point lies in the estimates for \( \tilde{A}(t) \) under a proper norm. In fact, for \( U = (u, v)^* \in H^{2,5} (\Gamma_*) \times H^1 (\Gamma_*) \), we define the following norm:

\[
\| U \|_H = (U, U)_H,
\]

where

\[
(U_1, U_2)_H \triangleq \sigma ((\nabla \Delta_{\Gamma_*} u_1) \circ \Phi_{S_i}, (\nabla \Delta_{\Gamma_*} u_2) \circ \Phi_{S_i}) + (u_1, u_2)
\]

\[
+ ((\nabla v_1 \Phi_S^{-1}) \circ \Phi_{S_i}, (\nabla v_1 \Phi_S^{-1}) \circ \Phi_{S_i}) + (v_1, v_2)
\]

with \( U_j = (u_j, v_j)^* \), \( \tilde{U}_j = U_j \circ \Phi_{S_i}^{-1} \), and \( (\cdot, \cdot) \) the \( L^2 \) inner product on \( \Gamma_* \).

Similarly as in [31, lemma 2.2], we want to show that the following estimate holds for some constant \( C \) depending on \( \Gamma_*, \Gamma_t \):

\[
(5.6) \quad (\tilde{A}(t) U, U)_H + (U, \tilde{A}(t) U)_H \leq C \left( (U, U)_H + \sum_i |B(t) U|^2_{p_i*} \right).
\]
As long as we have this estimate, we can prove as in [31] lemma 2.3 that
\[
\| (\lambda I - \tilde{A}(t))^{-1} \|_H \leq \frac{1}{\lambda - \lambda_0}
\]
for any \(\lambda > \lambda_0\) with some constant \(\lambda_0 > 0\). As a result, when \(B(t) U = 0\) at \(p_i\), we can show the existence of the solution to system \(5.5\) when the time \(t = t_0\) is fixed in \(\tilde{A}(t)\) by Hille-Yosida’s theorem.

Now it remains to focus on the proof for \(5.6\). In fact, one has by a direct calculation that
\[
(\tilde{A}(t) U, U)_H + (U, \tilde{A}(t) U)_H
\]
\[
= 2\sigma \int_{\Gamma_t} \mathcal{N} \Delta \nabla_x \Delta \nabla_x \bar{u} \, ds - 2\sigma \int_{\Gamma_t} \mathcal{N} \Delta \nabla_x \Delta \nabla_x \bar{u} \, ds + 2 \int_{\Gamma_*} u \bar{v} \, ds
\]
\[
- 2 \int_{\Gamma_*} \nabla \Delta \nabla_x \bar{u} \, ds
\]
\[
+ 2 \int_{\Gamma_t} \nabla x \nabla_x \nabla_x (-\sigma A_{\alpha} u + \mathcal{R}_1 u - 2\nabla \bar{v} + \mathcal{R}_0 v) \circ \Phi_{S_t}^{-1} \, ds
\]
\[
+ 2 \int_{\Gamma_*} \bar{v} (-\sigma A_{\alpha} u + \mathcal{R}_1 u - 2\nabla \bar{v} + \mathcal{R}_0 v) \, ds
\]
\[
= I_1 + I_2 + \cdots + I_6
\]
where we need to pay attention to the higher-order terms \(I_1, I_2, I_5\), since the boundary condition will be used there. The other terms can be handled directly and hence are omitted here.

For \(I_5\), the main part is the \(A_{\alpha}\) part, so an integration by parts leads to
\[
I_5 = 2\sigma \int_{\Gamma_t} \nabla x \nabla_x \nabla_x \nabla \Delta \nabla_x \bar{u} \, ds + \text{l.o.t.}
\]
\[
= 2\sigma \nabla x \nabla_x \nabla_x \nabla \Delta \nabla_x \bar{u} \Big|_{p_l}^{p_r} + \text{l.o.t.}
\]
Due to the boundary condition \(B(t) U = D_{1*} g_{2,i}\), we know immediately that
\[
\nabla x \nabla_x \nabla_x \nabla \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r} = \frac{1}{c_0} (\sigma A_{\alpha} u - \mathcal{R}_1 u + B(t) U) \circ \Phi_{S_t}^{-1} \text{ at } p_i \text{ (} i = l, r \text{)}
\]
which implies
\[
2\sigma \nabla x \nabla_x \nabla \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r} = -\frac{\sigma^2}{c_0} \sum_i \frac{\mathcal{N} \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r}}{c_0} - \frac{1}{c_0} \sum_i \mathcal{R}_1 u \nabla \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r}
\]
\[
+ \frac{1}{c_0} \sum_i B(t) U \circ \Phi_{S_t}^{-1} \nabla \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r}
\]
Substituting this into \(I_5\), we arrive at
\[
I_5 \leq -I_1 - \frac{1}{2} \frac{\sigma^2}{c_0} \sum_i \frac{(\mathcal{N} \Delta \nabla_x \bar{u} \big|_{p_l}^{p_r})^2}{c_0} + C \left( \| u \|_{H^{2.5}(\Gamma_\ast)}^2 + \sum_i \| B(t) U \|_{p_i}^2 \right)
\]
for some constant $C$ depending on $\Gamma_\ast$ and $\Gamma_\ast$, so it means

$$I_1 + I_5 \leq -\frac{1}{2} \sigma^2 \sum_i (\mathcal{N}^8_{i}) (u)^2 \|_{\mathcal{S}_i} + C \left(\|u\|_{H^{2.5}(\Gamma)_{\ast}}^2 + \sum_i |B(t)U|^2 \|_{\mathcal{S}_i}^2 \right).$$

Moreover, $I_2$ can also be handled by using integration by parts, the trace theorem, and combining with the inequality above.

As a result, we can conclude the estimate (5.6), where we also use elliptic properties for $A_{\alpha}$ from Remark 4.3.

In this way, we can continue to follow the proof in [31], and our proof can be finished.

Next, we give the energy estimates for system (5.1).

Denoting by

$$\bar{f} = f \circ \Phi_{\ast}^{-1},$$

one has $\bar{f}$ defined on $\Gamma_t$ and

$$D_{t \ast} f = (D_{t} \bar{f}) \circ \Phi_{\ast}, \quad D_{t \ast}^2 f = (D_{t}^2 \bar{f}) \circ \Phi_{\ast},$$

$$A(d_{t \ast}) f = - (\mathcal{N}^8_{i}) (\bar{f}) \circ \Phi_{\ast}.$$

Consequently, (5.1) is equivalent to the following linear problem of $\bar{f}$, $d_i$, $d_r$:

$$\begin{cases}
D_{t \ast}^2 \bar{f} - \sigma \mathcal{N}^8_{i} \bar{f} = \bar{g}_1 & \text{on } \Gamma_t, \\
D_{t \ast} \mathcal{N}^8_{i} \bar{f} = \frac{a^2}{b^2} (\sin \omega t)^2 \nabla r_i \mathcal{N}^8_{i} \bar{f} = \bar{g}_2 & \text{at } p_i, \\
d_i''(t) = B_i, & i = 1, r.
\end{cases}$$

(5.7)

with initial data

$$\bar{f}|_{t=0} = f_0 \circ \Phi_{\ast}^{-1}, \quad D_{t \ast} \bar{f}|_{t=0} = f_1 \circ \Phi_{\ast}, \quad d_i(0) = d_{i,0}, \quad d_i'(0) = d_{i,1}.$$

Here we use the notations

$$\bar{g}_1 = g_1 \circ \Phi_{\ast}^{-1}, \quad \bar{g}_2 = g_2 \circ \Phi_{\ast}^{-1}.$$

The energy $E_h(t, \bar{f}, \partial_t \bar{f})$ and the dissipation $F_h(t, \bar{f}, \partial_t \bar{f})$ for system (5.1) are defined by

$$E_h(t, \bar{f}, \partial_t \bar{f}) = \| \nabla r_i \mathcal{N}^8_{i} \partial_t \bar{f} \|_{L^2(\Omega_t)}^2 + \sigma \| \mathcal{H}(\mathcal{N}^8_{i} \partial_t \bar{f}) \|_{L^2(\Omega_t)}^2$$

$$+ \| D_{t \ast} \bar{f} \|_{L^2(\Gamma_t)}^2 + \| \bar{f} \|_{L^2(\Gamma_t)}^2,$$

$$F_h(t, \bar{f}, \partial_t \bar{f}) = (\sin \omega t)^2 |\nabla r_i \mathcal{N}^8_{i} D_{t \ast} \bar{f}|^2 \quad \text{at } p_i \ (i = l, r).$$

Meanwhile, we also define the lower-order energy and dissipation that are used in the iteration part:

$$E_l(t, \bar{f}, \partial_t \bar{f}) = \| \mathcal{H}(D_{t \ast} \mathcal{N}^8_{i} \bar{f}) \|_{L^2(\Omega_t)}^2 + \sigma \| \nabla r_i \mathcal{N}^8_{i} \bar{f} \|_{L^2(\Gamma_t)}^2$$

$$+ \| D_{t \ast} \bar{f} \|_{L^2(\Gamma_t)}^2 + \| \bar{f} \|_{L^2(\Gamma_t)}^2,$$

(5.8)

$$F_l(t, \bar{f}, \partial_t \bar{f}) = (\sin \omega t)^2 |\nabla r_i \mathcal{N}^8_{i} \bar{f}|^2 \quad \text{at } p_i \ (i = l, r).$$
PROPOSITION 5.3 (Higher-order energy estimates). Let
\[ \mathcal{N}_d \in C^0([0, T]; H^{5.5}(\Gamma_\ast)) \cap C^1([0, T]; H^4(\Gamma_\ast)) \]
and \( d_i, d'_i \in \mathbb{R} \) (\( i = l, r \)) be given, and
\[ g_1 \in C([0, T]; H^4(\Gamma_\ast)), \quad g_{2,i} \in C^1([0, T]; H^1(\Gamma_\ast)). \]

Assume moreover that the contact angles \( \omega_i \) satisfy \( \min_i \sin \omega_i \geq c_0 \) for some \( c_0 > 0 \).

Then we have the following energy estimates for the linear problem (5.1):
\[
\| f \|_{H^{5.5}(\Gamma_\ast)} + \| \partial_t f \|_{H^4(\Gamma_\ast)} \\
\leq e^{Q_1t} Q_2(0) \left( \| f_0 \|_{H^{5.5}(\Gamma_\ast)} + \| f_1 \|_{H^4(\Gamma_\ast)} \right) \\
+ e^{Q_1t} \int_0^t Q_1(\| g_1 \|_{H^4(\Gamma_\ast)} + \| D_t g_{2,i} \|_{H^1(\Gamma_\ast)}) dt',
\]
and
\[
| d_i(t) | + | d'_i(t) | \leq | d_i(0) | + (1 + t) | d'_i(0) | + \int_0^t \int_0^{t'} | B_i(\tau) | d \tau dt' + \int_0^t | B_i(t') | dt',
\]
where \( Q_1 \) is a polynomial of the norms \( \| d_\Gamma \|_{H^{5.5}(\Gamma_\ast)}, \| \partial_t d_\Gamma \|_{H^{5.5}(\Gamma_\ast)}, \| \partial_\tau^2 d_\Gamma \|_{H^4(\Gamma_\ast)}, \| v \|_{H^6(\Omega)}, \| D_t v \|_{H^{4.5}(\Omega)}, \)
and \( Q_2(t) \) is a polynomial of \( \| d_\Gamma(t) \|_{H^{5.5}(\Gamma_\ast)}, \| \partial_t d_\Gamma(t) \|_{H^4(\Gamma_\ast)}, \) and \( \| v(t) \|_{H^{4.5}(\Omega)}. \)

In order to prove these energy estimates, we first prove the estimate in forms of \( E_h(t, \bar{f}, \partial_t \bar{f}) \) and \( F_{h,i}(t, \bar{f}, \partial_t \bar{f}). \)

LEMMA 5.4. Under the assumptions of Proposition 5.3 we have for system (5.7) that
\[
\partial_t E_h(t, \bar{f}, \partial_t \bar{f}) + \sum_i F_{h,i}(t, \bar{f}, \partial_t \bar{f}) \\
\leq Q_1 \left( E_h(t, \bar{f}, \partial_t \bar{f}) + \| \bar{f} \|_{H^{5.5}(\Gamma_\ast)}^2 + \| D_t \bar{f} \|_{H^4(\Gamma_\ast)}^2 \right) + \| \bar{g}_1 \|_{H^4(\Gamma_\ast)}^2 + \| D_t \bar{g}_{2,i} \|_{H^1(\Gamma_\ast)}^2.
\]

PROOF FOR LEMMA 5.4. Apply \( \mathcal{N} \Delta \Gamma_\ast \) on both sides of the equation of \( \bar{f} \) in (5.7) to obtain
\[
\mathcal{N} \Delta \Gamma_\ast D_t^2 \bar{f} - \sigma(\mathcal{N} \Delta \Gamma_\ast) \bar{f} = \mathcal{N} \Delta \Gamma_\ast \bar{g}_1 \quad \text{on } \Gamma_\ast.
\]
Multiplying $-\Delta_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f}$ on both sides of this equation and integrating on $\Gamma_t$, one has

$$
- \int_{\Gamma_t} N \Delta_{\Gamma_t} D_t^2 \vec{f} \cdot \Delta_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds \\
+ \sigma \int_{\Gamma_t} (N \Delta_{\Gamma_t})^2 \vec{f} \cdot \Delta_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds \\
= - \int_{\Gamma_t} N \Delta_{\Gamma_t} \vec{g} \cdot \Delta_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds.
$$

(5.9)

Next, we deal with the above integrals one by one.

**Step 1.** The first term on the left side. First, one rewrites the first term in (5.9) as

$$
- \int_{\Gamma_t} N \Delta_{\Gamma_t} D_t^2 \vec{f} \cdot \Delta_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds \\
= \int_{\Gamma_t} \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t^2 \vec{f} \cdot \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds \\
- \sum_i N \Delta_{\Gamma_t} D_t^2 \vec{f} \cdot \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} |_{p_i} \\
= \frac{1}{2} \partial_t \| \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \|^2_{L^2(\Gamma_t)} + A_1 + A_2,
$$

where

$$A_1 = \int_{\Gamma_t} [\nabla_{\Gamma_t} N \Delta_{\Gamma_t}, D_t] D_t \vec{f} \cdot \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} \, ds \\
- \frac{1}{2} \int_{\Gamma_t} (D \cdot v) [\nabla_{\Gamma_t} N \Delta_{\Gamma_t}, D_t] D_t \vec{f} \, ds,
$$

$$A_2 = -N \Delta_{\Gamma_t} D_t^2 \vec{f} \cdot \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} |_{p_i}.
$$

**Term A1.** In fact, a direct computation leads to

$$[\nabla_{\Gamma_t} N \Delta_{\Gamma_t}, D_t] D_t \vec{f} = [\nabla_{\Gamma_t}, D_t] N \Delta_{\Gamma_t} D_t \vec{f} + \nabla_{\Gamma_t} [N, D_t] \Delta_{\Gamma_t} D_t \vec{f} + \nabla_{\Gamma_t} N [\Delta_{\Gamma_t}, D_t] D_t \vec{f}.
$$

Applying (4.3), (4.4), Lemma 2.4 and Lemma 4.9(1), we have

$$|A_1| \leq C \left( \| d_{\Gamma_t} \|_{H^4(\Gamma_r)} \cdot \| \partial_t d_{\Gamma_t} \|_{H^2(\Gamma_r)} \cdot \| v \|_{H^{4.5}(\Omega_t)} \right)
\times \left( \| D_t \vec{f} \|^2_{H^4(\Gamma_t)} + E_h(t, \vec{f}, \partial_t \vec{f}) \right).
$$

(5.10)

**Term A2.** Recalling the conditions at the corner points from system (5.7) and applying $D_t$ on both sides again, one derives

$$N \Delta_{\Gamma_t} D_t^2 \vec{f} \pm \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 \nabla_{\Gamma_t} N \Delta_{\Gamma_t} D_t \vec{f} = R_{c2,i} \text{ at } p_i \ (i = l, r).
$$
with
\[ R_{c2,i} = D_t \bar{g}_{2,i} - \left[ D_t^2, \mathbf{N}^\Delta \Gamma_i \right] \bar{f} - \frac{\sigma^2}{\beta_c} D_t (\sin \omega_i)^2 \nabla_{\nu} \mathbf{N}^\Delta \Gamma_i \bar{f} \]
\[ - \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 \left[ D_t, \nabla_{\nu} \mathbf{N}^\Delta \Gamma_i \right] \bar{f}. \]
Substituting this equality into \( A_2 \), one obtains
\[ (5.11) \quad A_2 = \frac{\sigma^2}{\beta_c} \sum_i F_{h,i} (t, \bar{f}, \partial_t \bar{f}) + A_{2R} \]
with
\[ A_{2R} = -R_{c2,i} \cdot \nabla_{\nu} \mathbf{N}^\Delta \Gamma_i D_t \bar{f} \big|_{\partial \Omega}. \]
Moreover, analogous to the analysis used before shows that the remainder term \( A_{2R} \) satisfies the estimate
\[ |A_{2R}| \leq \delta F_h(t, \bar{f}, \partial_t \bar{f}) + (\sin \omega_i)^{-2} C_\delta \| R_{c2} \|^2_{L^\infty(\Gamma)} \]
\[ + (\sin \omega_i)^{-2} C_\delta C \left( \| \partial_t^2 \|_{H^5(\Gamma)} , \| \partial_t^3 \|_{H^3(\Gamma)} , \| v \|_{H^{6.5}(\Omega)} , \| D_t v \|_{H^{4.5}(\Omega)} \right) \]
\[ \times \left( \| \bar{f} \|^2_{H^5(\Gamma)} + \| D_t \bar{f} \|^2_{H^4(\Gamma)} + \| D_t \bar{g}_{2,i} \|^2_{H^1(\Gamma)} \right), \]
with \( C_\delta \) a constant depending on \( \delta^{-1}, \beta_c, \sigma \).

**Step 2.** The second term on the left side. We first rewrite this integral by Green’s formula as follows:
\[ \sigma \int_{\Gamma_t} \left( \mathbf{N}^\Delta \Gamma_i \right)^2 \bar{f} \cdot \Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i D_t \bar{f} \, ds \]
\[ = \sigma \int_{\Omega_t} \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \cdot \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, D_t \bar{f}) \, dX \]
\[ \triangleq \frac{\sigma}{2} \partial_t \| \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \|^2_{L^2(\Omega_t)} + A_3 + A_4, \]
where the remainder terms
\[ A_3 = \sigma \int_{\Gamma_t} \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \cdot [\nabla \mathcal{H} \Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, D_t] \bar{f} \, dX \]
\[ = \sigma \int_{\Gamma_t} \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \cdot [\nabla \mathcal{H}, D_t] \Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i \bar{f} \, dX \]
\[ + \sigma \int_{\Gamma_t} \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \cdot \nabla \mathcal{H}[\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, D_t] \bar{f} \, dX \]
\[ \triangleq A_{31} + A_{32} \]
and
\[ A_4 = -\frac{\sigma}{2} \int_{\Omega_t} (\text{div } v) \left| \nabla \mathcal{H}(\Delta_{\Gamma_t} \mathbf{N}^\Delta \Gamma_i, \bar{f}) \right|^2 \, dX. \]
**TERM $A_{31}$**. To handle $A_{31}$, denoting 

$$g = \Delta_{\Gamma_I} \mathcal{N} \Delta_{\Gamma_I} \vec{f},$$

a direct computation leads to

$$A_{31} = \int_{\Omega_t} \nabla g_{\mathcal{H}} \cdot [\nabla, D_t]g_{\mathcal{H}} dX + \int_{\Omega_t} \nabla g_{\mathcal{H}} \cdot \nabla[D_{\mathcal{H}}, D_t]g dX.$$  

Noticing that

$$[\nabla, D_t]g_{\mathcal{H}} = -\nabla v \cdot \nabla g_{\mathcal{H}}.$$ 

and due to the definition of $g_{\mathcal{H}}$ and

$$[\mathcal{H}, D_t]g = \Delta^{-1}(2\nabla v \cdot \nabla^2 g_{\mathcal{H}} + \Delta v \cdot \nabla g_{\mathcal{H}}, (\nabla v, D_t) \cdot \nabla g_{\mathcal{H}}),$$

we know

$$\int_{\Omega_t} \nabla g_{\mathcal{H}} \cdot \nabla[D_{\mathcal{H}}, D_t]g dX = -\int_{\Omega_t} \Delta g_{\mathcal{H}} \cdot [\mathcal{H}, D_t]g dX + \int_{\mathcal{H}_t} \nabla_{N_t} g_{\mathcal{H}} \cdot [\mathcal{H}, D_t]g dS$$

$$+ \int_{\mathcal{H}_b} \nabla_{N_b} g_{\mathcal{H}} \cdot [\mathcal{H}, D_t]g dS = 0.$$ 

Consequently, one derives

$$|A_{31}| \leq C \left( \|d_{\Gamma_I}\|_{H^{2,5}(\Gamma_1)}, \|v\|_{H^{2,5}(\Omega_t)} \right) E_h(t, \vec{f}, \partial_t \vec{f}).$$

**TERMS $A_{32}$ AND $A_4$**. Applying Lemma 2.3 and Lemma 2.4 and recalling the commutators (4.3) and (4.4), one obtains

$$|A_{32}| + |A_4|$$

$$ \leq C \left( \|d_{\Gamma_I}\|_{H^{2,5}(\Gamma_1)}, \|v\|_{H^{2,5}(\Omega_t)} \right) \left( \|\mathcal{H}^0_{\mathcal{H}}(\Gamma_1)\|_{H^{0,5}(\Gamma_1)} E_h(t, \vec{f}, \partial_t \vec{f}) \right)^{1/2}$$

$$\leq C \left( \|d_{\Gamma_I}\|_{H^{5,5}(\Gamma_1)}, \|v\|_{H^{5,5}(\Omega_t)} \right) \left( \|\mathcal{H}^0_{\mathcal{H}}(\Gamma_1)\|_{H^{0,5}(\Gamma_1)} + E_h(t, \vec{f}, \partial_t \vec{f}) \right).$$

As a result, combining (5.13) with (5.10), (5.11), and (5.12), we finally conclude that the left side of equation (5.9) becomes

$$\text{Left side} = \frac{1}{2} \partial_t E_h(t, \vec{f}, \partial_t \vec{f}) + \frac{\sigma^2}{\rho_c} F_h(t, \vec{f}, \partial_t \vec{f}) + R_L,$$

where the remainder term

$$R_L = A_1 + A_2 + A_{31} + A_{32} + A_4.$$
satisfies the estimate
\[ |R_L| \leq \delta F_h(t, \vec{f}, \partial_t \vec{f}) + (\min\{\sin \omega_i\})^{-2}C_{\delta} C (\|d_{\Gamma_r} \|_{H^{5.5}(\Gamma_\ast)}, \|\partial_t d_{\Gamma_r} \|_{H^3(\Gamma_\ast)}) \]
\[ \|v\|_{H^6(\Omega_t)}, \|D_t v\|_{H^{4.5}(\Omega_t)} \cdot (E_h(t, \vec{f}, \partial_t \vec{f}) + \|\vec{f}\|_{H^{5.5}(\Gamma_r)}^2 + \|D_t \vec{f}\|_{H^4(\Gamma_r)}^2) \]
\[ + \|D_t \vec{f}\|_{H^4(\Gamma_r)}^2 + \|D_t \vec{g}_{2,1}\|_{H^4(\Gamma_r)}^2 \]  

**Step 3.** The right side. Integrating by parts on the right side leads to

Right side \[ = \int_{\Gamma_r} \nabla_t \nabla \Delta_{\Gamma_r} \vec{g}_1 \cdot \nabla_t \nabla \Delta_{\Gamma_r} D_t \vec{f} \, ds - \nabla \Delta_{\Gamma_r} \vec{g}_1 \cdot \nabla_t \nabla \Delta_{\Gamma_r} D_t \vec{f} \, \left|_{t_t}^{p_t} \right. \]

Therefore, we have

Right side \[ \leq \delta F_h(t, \vec{f}, \partial_t \vec{f}) + (\min\{\sin \omega_i\})^{-2}C_{\delta} \|\nabla \Delta_{\Gamma_r} \vec{g}_1\|_{H^4(\Gamma_r)}^2 \]
\[ + \|\nabla_t \nabla \Delta_{\Gamma_r} \vec{g}_1\|_{L^2(\Gamma_r)} \|E_h(t, \vec{f}, \partial_t \vec{f})\|_{L^2(\Gamma_r)}^{1/2} \]
\[ \leq C_{\delta} (\min\{\sin \omega_i\})^{-2}C (\|d_{\Gamma_r} \|_{H^4(\Gamma_\ast)} \|E_h(t, \vec{f}, \partial_t \vec{f})\|_{L^2(\Gamma_r)} + \|\vec{g}_1\|_{L^2(\Gamma_r)}^2) \]
\[ + \delta F_h(t, \vec{f}, \partial_t \vec{f}) \]

**Step 4.** \(L^2\) energy. Multiply \(D_t \vec{f}\) on the both sides of (5.7) and integrate on \(\Gamma_r\) to get

\[ \partial_t \|D_t \vec{f}\|_{L^2(\Gamma_r)}^2 \leq C (\|\nabla \Delta_{\Gamma_r} \vec{f}\|_{L^2(\Gamma_r)} + \|\vec{g}_1\|_{L^2(\Gamma_r)}) \|D_t \vec{f}\|_{L^2(\Gamma_r)} \]
\[ + C (\|d_{\Gamma_r} \|_{H^{2.5}(\Gamma_\ast)}, \|v\|_{H^{2.5}(\Omega_t)}) \|D_t \vec{f}\|_{L^2(\Gamma_r)} \]
\[ \leq C (\|d_{\Gamma_r} \|_{H^3(\Gamma_\ast)} \|v\|_{H^{2.5}(\Omega_t)}) \]
\[ \times (\|\vec{f}\|_{H^3(\Gamma_r)}^2 + \|\vec{g}_1\|_{L^2(\Gamma_r)}^2 + E_h(t, \vec{f}, \partial_t \vec{f})) \]  

Moreover, using

\[ \partial_t \|\vec{f}\|_{L^2(\Gamma_r)}^2 = 2 \int_{\Gamma_r} D_t \vec{f} \cdot \vec{f} \, ds + \int_{\Gamma_r} (\mathcal{D} \cdot v) \|\vec{f}\|^2 \, ds, \]

we immediately have that

\[ \partial_t \|\vec{f}\|_{L^2(\Gamma_r)}^2 \leq C (\|d_{\Gamma_r} \|_{H^{2.5}(\Gamma_\ast)}, \|v\|_{H^{2.5}(\Omega_t)}) E_h(t, \vec{f}, \partial_t \vec{f}). \]

Combining all the estimates from Step 1 to Step 4, we get the desired estimate.

Moreover, we replace the energy \(E_h(t, \vec{f}, \partial_t \vec{f})\) by

\[ \|\vec{f}\|_{H^{5.5}(\Gamma_r)} \quad \text{and} \quad \|D_t \vec{f}\|_{H^4(\Gamma_r)} \]

which is more convenient to use.
Lemma 5.5.

(1) There exists a polynomial \( Q_I \) of \( \| d \Gamma_t \|_{H^{4} (\Gamma_s)}, \| v \|_{H^{3} (\Omega_s)} \) such that
\[
E_I (t, \bar{f}, \partial_t \bar{f}) \leq Q_I (\| d \Gamma_t \|_{H^{4} (\Gamma_s)}, \| v \|_{H^{3} (\Omega_s)}) \left( \| D_t \bar{f} \|_{H^{2.5} (\Gamma_s)}^2 + \| \bar{f} \|_{H^{4} (\Gamma_s)}^2 \right),
\]
and
\[
\| D_t \bar{f} \|_{H^{2.5} (\Gamma_s)}^2 + \| \bar{f} \|_{H^{4} (\Gamma_s)}^2 \leq Q_I (\| d \Gamma_t \|_{H^{4} (\Gamma_s)}, \| v \|_{H^{3} (\Omega_s)}) E_I (t, \bar{f}, \partial_t \bar{f}).
\]

(2) There exists a polynomial \( Q_h \) of \( \| d \Gamma_t \|_{H^{5} (\Gamma_s)}, \| v \|_{H^{4.5} (\Omega_s)} \) such that
\[
E_h (t, \bar{f}, \partial_t \bar{f}) \leq Q_h (\| d \Gamma_t \|_{H^{5} (\Gamma_s)}, \| v \|_{H^{4.5} (\Omega_s)}) \left( \| D_t \bar{f} \|_{H^{4} (\Gamma_s)}^2 + \| \bar{f} \|_{H^{5.5} (\Gamma_s)}^2 \right),
\]
and
\[
\| D_t \bar{f} \|_{H^{4} (\Gamma_s)}^2 + \| \bar{f} \|_{H^{5.5} (\Gamma_s)}^2 \leq Q_h (\| d \Gamma_t \|_{H^{5} (\Gamma_s)}, \| v \|_{H^{4.5} (\Omega_s)}) E_h (t, \bar{f}, \partial_t \bar{f}).
\]

Proof. In fact, the first inequality in (1) can be proved directly by applying Lemma 2.3 (4.4), Lemma 2.5 and Lemma 2.4.

We focus on the second inequality in (1). Due to the definition of the norm \( H^{2.5} (\Gamma_s) \), one has
\[
\| D_t \bar{f} \|_{H^{2.5} (\Gamma_s)} \leq C (\| d \Gamma_t \|_{H^{2.5} (\Gamma_s)} \left( \| \Delta \Gamma_t D_t \bar{f} \|_{H^{0.5} (\Gamma_s)} + \| D_t \bar{f} \|_{L^2 (\Gamma_s)} \right).
\]
In addition, one has by Lemma 2.5 that
\[
\| \Delta \Gamma_t D_t \bar{f} \|_{H^{0.5} (\Gamma_s)} \leq \| \Delta \Gamma_t D_t \bar{f} \|_{H^{0.5} (\Gamma_s)} + \| [\Delta \Gamma_t, D_t] \bar{f} \|_{H^{0.5} (\Gamma_s)} \leq C (\| d \Gamma_t \|_{H^{2.5} (\Gamma_s)} \left( \| \Delta \Gamma_t D_t \bar{f} \|_{H^{1} (\Omega_s)} + \| v \|_{H^{3} (\Omega_s)} \| \bar{f} \|_{H^{2.5} (\Gamma_s)} \right).
\]
Noticing that
\[
\| \Delta \Gamma_t D_t \bar{f} \|_{L^2 (\Gamma_s)} \leq C (\| \nabla \Delta \Gamma_t D_t \bar{f} \|_{L^2 (\Omega_s)} + \| D_t \Delta \Gamma_t \bar{f} \|_{L^2 (\Gamma_s)}),
\]
and combining the above inequality with the two estimates above, one derives the following estimate:
\[
\| D_t \bar{f} \|_{H^{2.5} (\Gamma_s)} \leq C (\| d \Gamma_t \|_{H^{2.5} (\Gamma_s)} + \| v \|_{H^{3} (\Omega_s)}) \left( E (t, \bar{f}, \partial_t \bar{f})^{1/2} + \epsilon \| \bar{f} \|_{H^{4} (\Gamma_s)} \right),
\]
where one uses an interpolation for \( \| \bar{f} \|_{H^{2.5} (\Gamma_s)} \) and \( \epsilon \) is a small constant.

Second, one writes that
\[
\| \bar{f} \|_{H^{4} (\Gamma_s)} \leq C (\| d \Gamma_t \|_{H^{4} (\Gamma_s)} \left( \| \bar{f} \|_{L^2 (\Gamma_s)} + \| \Delta \Gamma_t \bar{f} \|_{H^{2} (\Gamma_s)} \right).
\]
where the term \( \| \Delta \Gamma_t \bar{f} \|_{H^{2} (\Gamma_s)} \) needs to be handled. Applying Lemma 2.6 (2), we obtain immediately
\[
\| \Delta \Gamma_t \bar{f} \|_{H^{2} (\Gamma_s)} \leq C (\| d \Gamma_t \|_{H^{2.5} (\Gamma_s)} \left( \| \nabla \Delta \Gamma_t \bar{f} \|_{H^1 (\Gamma_s)} + \| \Delta \Gamma_t \bar{f} \|_{L^2 (\Gamma_s)} \right),
\]
which leads to the following inequality:

\[
\| \bar{f} \|_{H^4(\Gamma_t)} \leq C(\| d \Gamma_t \|_{H^4(\Gamma_s)})(\| \bar{f} \|_{L^2(\Gamma_t)} + \| \Delta \bar{f} \|_{H^4(\Gamma_t)})
\]

(5.15)

\[
\leq C(\| d \Gamma_t \|_{H^4(\Gamma_s)})E_t(t, \bar{f}, \partial_t \bar{f})^{1/2}.
\]

In the end, one only needs to combine (5.14) with (5.15) and choose \( \epsilon \) small enough to finish the proof for (1). Moreover, the proof for (2) follows in the same way. \( \square \)

Now, we are in a position to prove Proposition 5.3.

**PROOF FOR PROPOSITION 5.3.** In fact, multiplying \( Q_2 \) defined in Proposition 5.3 on both sides of the inequality in Lemma 5.4 and using Lemma 5.5 lead to

\[
Q_2 \partial_t \bar{e}_h(t, \bar{f}, \partial_t \bar{f}) + Q_2 \sum_i F_{h,i}(t, \bar{f}, \partial_t \bar{f}) \\
\leq Q_1 Q_2 (E_h(t, \bar{f}, \partial_t \bar{f}) + \| \bar{g}_1 \|_{H^4(\Gamma_t)}^2 + \| D_t \bar{g}_2, i \|_{H^4(\Gamma_t)})
\]

On the other hand, acting \( \partial_t \) on \( Q_2 \), one has

\[
(\partial_t Q_2) E_h(t, \bar{f}, \partial_t \bar{f}) \leq Q_1 E_h(t, \bar{f}, \partial_t \bar{f}) \leq Q_1 Q_2 E_h(t, \bar{f}, \partial_t \bar{f}),
\]

where \( Q_1 \) is defined in Proposition 5.3. Summing up these two inequalities and applying Lemma 5.5 again, one derives

\[
Q_2 E_h(t, \bar{f}, \partial_t \bar{f}) + \int_0^t e^{Q_1 t'} Q_2 \sum_i F_{h,i}(t', \bar{f}, \partial_t \bar{f}) dt' \\
\leq e^{Q_1 t} Q_2 \big|_{t=0}^{t} E_h(t, \bar{f}, \partial_t \bar{f}) \big|_{t=0} + \int_0^t e^{-Q_1 t'} Q_1 (\| \bar{g}_1 \|_{H^4(\Gamma_t)}^2 + \| D_t \bar{g}_2, i \|_{H^4(\Gamma_t)}^2) dt'.
\]

On the other hand, one knows that

\[
\| f \|_{H^{5.5}(\Gamma_s)} + \| \partial_t f \|_{H^4(\Gamma_s)} \\
\leq C(\| d \Gamma_t \|_{H^{5.5}(\Gamma_s)}, \| \partial_t d \Gamma_t \|_{H^4(\Gamma_s)}, \| v \|_{H^4(\Omega_t)}) \\
\times (\| f \|_{H^{5.5}(\Gamma_s)} + \| D_t \bar{f} \|_{H^4(\Gamma_t)}) \\
\leq Q_2 (\| \bar{f} \|_{H^{5.5}(\Gamma_s)} + \| D_t \bar{f} \|_{H^4(\Gamma_t)}).
\]

Combining this with Lemma 5.5, we derive immediately the desired energy estimate for \( f \). Moreover, the estimate for \( d(t) \) follows directly by integrating with respect to time twice on its equation. \( \square \)

In the end, the estimate for the second-order time derivatives of \( f \) is considered here, which is based on the linear system (5.1).
COROLLARY 5.6. Under the assumptions of Proposition 5.3, one has the following estimate for $f$ in system (5.1) when $s \geq 3$:

$$\left\| \partial_t^2 f \right\|_{H^s(\Gamma_s)} \leq \|g_1\|_{H^s(\Gamma_s)} + C \left( \|\mathcal{M}_a\|_{H^{s+1}(\Gamma_s)}, \|\partial_t \mathcal{M}_a\|_{H^{s+2}(\Gamma_s)}, \|\partial_t^2 \mathcal{M}_a\|_{H^{s+2}(\Gamma_s)}, \right)$$

where $|d_t|, |d_t'|, a^{-1}|d_t''| : \left( \|f\|_{H^{s+3}(\Gamma_s)} + \|\partial_t f\|_{H^{s+1}(\Gamma_s)} \right)$.

where $|d_t| = |d_t'| = |d_t''|$. When $1 \leq s \leq 3$, the norms for $\mathcal{M}_a, \partial_t \mathcal{M}_a, \partial_t^2 \mathcal{M}_a$ on the right side are always taken higher than $H^1(\Gamma_s)$.

PROOF. To begin with, one knows from the equation of $f$ in system (5.1) that

$$\partial_t^2 f = -\nabla \partial_t v^* f - 2 \nabla v^* \partial_t f - \nabla v^2 f + \sigma A(d_{\Gamma_t}) f + g_1,$$

where $v^*$ is defined by (4.9) and $\partial_t v^*$ is expressed by (4.21).

Immediately, one has

$$\left\| \partial_t^2 f \right\|_{H^s(\Gamma_s)} \leq \|g_1\|_{H^s(\Gamma_s)} + C \left( \|d_{\Gamma_t}\|_{H^{s+2}(\Gamma_s)}, \|\partial_t^2 v^*\|_{H^s(\Gamma_s)}, \|v^*\|_{H^{s+1}(\Gamma_s)} \right),$$

where $|d_t| = |d_t'| = |d_t''| : \left( \|f\|_{H^{s+3}(\Gamma_s)} + \|\partial_t f\|_{H^{s+1}(\Gamma_s)} \right)$.

Moreover, (4.21) leads to

$$\left\| \partial_t v^* \right\|_{H^s(\Gamma_s)} \leq C \left( \|d_{\Gamma_t}\|_{H^{s+2}(\Gamma_s)}, \|\partial_t d_{\Gamma_t}\|_{H^{s+1}(\Gamma_s)}, \|\partial_t^2 d_{\Gamma_t}\|_{H^s(\Gamma_s)}, \right),$$

$$\|v\|_{H^{s+1.5}(\Omega_t)}, \|D_t v\|_{H^{s+0.5}(\Omega_t)} \right)$$

and

$$\left\| v^* \right\|_{H^{s+1}(\Gamma_s)} \leq C \left( \|d_{\Gamma_t}\|_{H^{s+2}(\Gamma_s)}, \|\partial_t d_{\Gamma_t}\|_{H^{s+1}(\Gamma_s)}, \|v\|_{H^{s+1.5}(\Omega_t)} \right),$$

so one obtains

$$\left\| \partial_t^2 f \right\|_{H^s(\Gamma_s)} \leq \|g_1\|_{H^s(\Gamma_s)} + C \left( \|d_{\Gamma_t}\|_{H^{s+2}(\Gamma_s)}, \|\partial_t d_{\Gamma_t}\|_{H^{s+1}(\Gamma_s)}, \|\partial_t^2 d_{\Gamma_t}\|_{H^s(\Gamma_s)}, \right),$$

$$\|v\|_{H^{s+1.5}(\Omega_t)} \right) \cdot \left( \|f\|_{H^{s+3}(\Gamma_s)} + \|\partial_t f\|_{H^{s+1}(\Gamma_s)} \right).$$

As a result, the desired estimate follows from Proposition 4.2 and Proposition 4.12.

6 Local Well-Posedness

In this section, we use a standard iteration method and a fixed point theorem to prove that the system of (4.25), (4.27) admits a unique solution $(\mathcal{M}_a, d_t, d_r)$ satisfying

$$\mathcal{M}_a \in C^0([0, T]; H^{5.5}(\Gamma_s)) \cap C^1([0, T]; H^4(\Gamma_s))$$

and

$$|d_t| \leq L_0, \quad |d_t'| \leq L_1 (i = l, r).$$
where the constants \( L_0, L_1 \) are specified later. Consequently, we can construct the free surface \( \Phi_{S_t} \) by \( d_{\Gamma_t} \in C([0, T]; H^{8.5}(\Gamma_*)) \cap C^1([0, T]; H^{7}(\Gamma_*)) \) from Proposition 4.2 and Proposition 4.12, and we define the velocity \( v \in H^{7.5}(\Omega_t) \) by (4.7). In the end, we prove that the system of \((\mathcal{H}_t, d_l, d_r)\) is equal to the system (WW) to finish the proof for the local well-posedness of the water-waves problem.

6.1 Settings

Before we start the iteration scheme, we need to clarify some settings. First of all, we use the set of surfaces \( \Lambda_* = \Lambda(S_*, 8.5, \pi/16) \) from Definition 3.1. In addition, the following set \( \Sigma \) is defined for the bounds related to \( \mathcal{H}_t \) and \( d_l, d_r \).

**Definition 6.1.** The set \( \Sigma \) is defined as the collection of \((\mathcal{H}_t, d_l, d_r)\) that satisfies the conditions for the initial data:

\[
\begin{align*}
\| \mathcal{H}_t(0, \cdot) - \mathcal{H}_t^* \|_{H^{5.5}(\Gamma_*)} & \leq \delta_1, \\
\partial_t \mathcal{H}_t(0, \cdot) & \in H^{4}(\Gamma_*), \\
|d_l^i(0)| & \leq L \quad (i = l, r),
\end{align*}
\]

as well as the higher-order bounds for \( \mathcal{H}_t \) and \( d_l \):

\[
\begin{align*}
\| \partial_t \mathcal{H}_t \|_{C([0,T]; H^{5.5}(\Gamma_*))} & \leq L_0, \\
\| \partial_t \mathcal{H}_t \|_{C([0,T]; H^{4}(\Gamma_*))} & \leq L_1, \\
\| \partial_t^2 \mathcal{H}_t \|_{C([0,T]; H^{2.5}(\Gamma_*))} & \leq L_2,
\end{align*}
\]

where the constants \( T, \delta_1, L, L_0, L_1, \) and \( L_2 \) are given, and \( \delta_1 \) is small.

Meanwhile, we introduce the set of initial data

\[
\mathcal{I}(\epsilon, A_1) \triangleq \left\{ \left((\mathcal{H}_t)_I, (\partial_t \mathcal{H}_t)_I, (d_l)_I, (d_r)_I \right) \mid \begin{array}{l}
|\mathcal{H}_t|_I = 0, \quad |\partial_t \mathcal{H}_t|_I = 0, \\
|d_l^i|_I = \epsilon, \\
|d_r^i|_I = \epsilon, \\
\| \partial_t \mathcal{H}_t \|_{H^{4}(\Gamma_*)} + |\partial_t \mathcal{H}_t|_I \leq \epsilon, \\
\| \partial_t^2 \mathcal{H}_t \|_{H^{2.5}(\Gamma_*)} + |\partial_t \mathcal{H}_t|^2_I \leq A_1, \quad (\omega_l)_I \in (0, \pi/16), \quad i = l, r,
\end{array} \right\}
\]

where \( 0 < \epsilon \ll \delta_1, A_1 > 0 \) is some large constant, and \((\omega_l)_I\) are the corresponding contact angles. When \( \epsilon \) is taken sufficiently small, \((\omega_l)_I \in (0, \pi/16)\) will be satisfied naturally.

6.2 The Iteration Scheme

Fixing the initial data \((\mathcal{H}_t)_I, (\partial_t \mathcal{H}_t)_I, (d_l)_I, (d_r)_I \) \( \in \mathcal{I}(\epsilon, A_1) \), we plan to use the linear system (5.1) to generate an iteration sequence of \((\mathcal{H}_t, d_l, d_r)\). Then we show that this sequence converges to \((\mathcal{H}_t, d_l, d_r)\) by a fixed point theorem.

To get started, for any given \((\mathcal{H}_t, d_l, d_r) \in \Sigma\), using Proposition 4.2 we define the free surface \( \Gamma_t \) (i.e., \( \Phi_{S_t} \)) by the function \( d_{\Gamma_t} \in C([0, T]; H^{8.5}(\Gamma_*)) \) with the conditions at the corner point \( p_i^* \)

\[
d_{\Gamma_t}(p_i^*) = d_i, \quad i = l, r,
\]
and the velocity $v$ is defined by (4.7).

Let $(\mathcal{N}_a, \tilde{d}_l, \tilde{d}_r)$ be the solution to the following linear system:

$$\begin{cases}
D^2_{t*} \mathcal{N}_a + \sigma A(d_{t*}) \mathcal{N}_a = R_0 & \text{on } \Gamma_*, \\
D_{t*} A(d_{t*}) \mathcal{N}_a + \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 (\nabla_{t*} (A(d_{t*}) \mathcal{N}_a) \circ \Phi_{S_*}^{-1})) \circ \Phi_{S_*} = R_{c,i} & \text{at } p_{i*}, \\
\tilde{d}_i''(t) = B_i & (i = l, r),
\end{cases}
$$

(6.1)

with initial data from $\mathcal{I}(\epsilon, A_1)$:

$$\mathcal{N}_a |_{t=0} = (\mathcal{N}_a)_I, \quad \partial_t \mathcal{N}_a |_{t=0} = (\partial_t \mathcal{N}_a)_I, \quad \tilde{d}_i(0) = (d_i)_I, \quad \tilde{d}'_i(0) = (d'_i)_I.$$

Here the right sides $R_0, R_{c,i}$, and $B_i$ are defined in (4.25), (4.26), and (4.27), which depend on the known functions $(\mathcal{N}_a, d_i, d_r) \in \Sigma$.

From Remark 4.7 it is straightforward to get that the initial data satisfy the following compatibility conditions:

$$D_{t*} A(d_{t*}) \mathcal{N}_a |_{t=0} = \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 (\nabla_{t*} (A(d_{t*}) \mathcal{N}_a) \circ \Phi_{S_*}^{-1})) \circ \Phi_{S_*} |_{t=0}$$

$$= R_{c,i} |_{t=0} \quad \text{at } p_{i*}.$$

Applying Proposition 5.2 and Proposition 5.3 we prove immediately that (6.1) admits a unique solution $(\mathcal{N}_a, \tilde{d}_l, \tilde{d}_r)$ satisfying the initial data above.

Moreover, the following higher-order energy estimates hold for $\forall t \in [0, T]$:

$$\| \mathcal{N}_a \|_{H^{5.5}(\Gamma_a)}^2 + \| \partial_t \mathcal{N}_a \|_{H^4(\Gamma_a)}^2$$

$$\leq e^{Q_1 t} \int_0^t Q_2(\sigma) \| (\mathcal{N}_a)_I \|_{H^{5.5}(\Gamma_a)}^2 + \| (\partial_t \mathcal{N}_a)_I \|_{H^4(\Gamma_a)}^2$$

$$+ e^{Q_1 t} \int_0^t Q_1(\| R_0 \|_{H^4(\Gamma_a)}^2 + | D_{t*} R_{c,i} |_{H^1(\Gamma_a)}^2) dt',$$

and

$$|\tilde{d}_i| + |\tilde{d}_r|' \leq |(d_i)_I| + (1 + t) |(d'_i)_I|$$

$$+ \int_0^t \int_0^t |B_i(\tau)| d\tau d\tau' + \int_0^t |B_i(t')| dt' \quad (i = l, r),$$

where $Q_1$ is a polynomial of the norms

$$\| d_{\Gamma_a} \|_{H^{5.5}(\Gamma_a)}, \quad \| \partial_t d_{\Gamma_a} \|_{H^{5.5}(\Gamma_a)}, \quad \| d_{t*}^2 d_{\Gamma_a} \|_{H^4(\Gamma_a)},$$

$$\| v \|_{H^6(\Omega)}, \quad \text{and} \quad \| D_{t*} v \|_{H^{4.5}(\Omega)},$$

and $Q_2(\sigma)$ is a polynomial of

$$\| d_{\Gamma_a}(0) \|_{H^{5.5}(\Gamma_a)}, \quad \| \partial_t d_{\Gamma_a}(0) \|_{H^4(\Gamma_a)}, \quad \text{and} \quad \| v(0) \|_{H^{4.5}(\Omega)}.$$
Applying Lemma 4.9, Proposition 4.2, and Proposition 4.12, one has immediately that

\[
Q_1 = Q_1 \left( \| \partial_t \delta_m \|_{H^5(\Gamma_\omega)}, \| \partial_t d_{\Gamma} \|_{H^5(\Gamma_\omega)}, \| \partial_t^2 d_{\Gamma} \|_{H^4(\Gamma_\omega)} \right)
\]

\[
\leq C \left( \| \partial_t \delta_m \|_{H^7(\Gamma_\omega)}, \| \partial_t d_{\Gamma} \|_{H^5(\Gamma_\omega)}, \| \partial_t^2 d_{\Gamma} \|_{H^4(\Gamma_\omega)} \right)
\]

\[
\leq C \left( \| \delta_m - \delta_m \|_{H^4(\Gamma_\omega)}, \| \delta_t \delta_m \|_{H^{2.5}(\Gamma_\omega)}, \| \delta_t^2 \delta_m \|_{H^{1}(\Gamma_\omega)}, |d_i|, |d_i'|, |d_i''| \right)
\]

\[
\leq C(L_0, L_1, L_2).
\]

On the other hand, using system (6.1), Lemma 4.10, and Corollary 5.6 with \(L_1\), we arrive at

\[
\| R_0 \|_{H^4(\Gamma_\omega)} \leq C \left( \delta_1, L \right).
\]

Therefore, combining with these estimates above and going back to (6.2), one obtains that

\[
\| R_0 \|_{H^4(\Gamma_\omega)} + \| D_t \ast R_c, i \|_{H^1(\Gamma_\omega)} \leq a^{3/2} C \left( \| \delta_m \|_{H^5(\Gamma_\omega)}, \| \delta_t \delta_m \|_{H^{2.5}(\Gamma_\omega)}, \| \delta_t^2 \delta_m \|_{H^{1}(\Gamma_\omega)}, |d_i|, |d_i'|, |d_i''| \right)
\]

\[
\leq a^{3/2} C \left( L_0, L_1, L_2 \right).
\]

Moreover, we have from Lemma 4.9, Proposition 4.2, and Proposition 4.12 that

\[
Q_2(0) = C \left( \delta_1, L \right).
\]

Choosing \(T\) small enough compared to \(a, L_0, L_1, L_2, \) and \(L_0\) and \(L_1\) large compared to \(L\) and \(A_1\) in the inequalities above, we have

\[
\| \delta_m \|_{H^5(\Gamma_\omega)}, \| \delta_t \delta_m \|_{H^{2.5}(\Gamma_\omega)}, \| \delta_t^2 \delta_m \|_{H^{1}(\Gamma_\omega)}, \| \delta_t^2 \delta_m \|_{H^{1}(\Gamma_\omega)} \leq L_0.
\]

On the other hand, using system (6.1), Lemma 4.10, and Corollary 5.6 with \(s = 2.5\), we arrive at

\[
\| \delta_t^2 \delta_m \|_{H^{2.5}(\Gamma_\omega)} \leq C \left( \| \delta_m \|_{H^{1.5}(\Gamma_\omega)}, \| \delta_t \delta_m \|_{H^1(\Gamma_\omega)}, \| \delta_t^2 \delta_m \|_{H^1(\Gamma_\omega)}, |d_i|, |d_i'|, |d_i''| \right)
\]

\[
\times \left( \| \delta_m \|_{H^{5.5}(\Gamma_\omega)} + \| \delta_t \delta_m \|_{H^{3.5}(\Gamma_\omega)} \right)
\]

\[
\leq C \left( L_0, L_1, a^{-1} L_2 \right).
\]
In addition, one has directly the estimates
\[ |\tilde{d}_i''| \leq C(L_0, L_1). \]

Therefore, when \( a \) is taken sufficiently large, we have
\[ \| \partial^2_2 \wt{\m}_a \|_{H^{2.5}(\Gamma_a)} \leq L_2, \]
with large \( L_2 \) compared to \( L_0 \) and \( L_1 \).

Consequently, checking the initial data for \( \m_a, d_l, d_r \), one concludes that when \( L \geq A_1 \) is sufficiently large, one has \( (\wt{\m}_a, \wt{d}_l, \wt{d}_r) \in \Sigma \).

Based on the analysis above, we define the iteration map \( F \) on \( \Sigma \) by solving the linear system (6.1):
\[ \mathcal{F}(\m_a, d_l, d_r) \triangleq (\wt{\m}_a, \wt{d}_l, \wt{d}_r), \]
with initial data \( ((\m_a)_I, (\partial_t \m_a)_I, (d_l)_I, (d_r)_I) \in \mathcal{I}(\epsilon, A_1) \). When \( (\m_a, d_l, d_r) \in \Sigma \), we have proved that
\[ (\wt{\m}_a, \wt{d}_l, \wt{d}_r) = \mathcal{F}(\m_a, d_l, d_r) \in \Sigma. \]

**Remark 6.2.** Thanks to the assumptions on the reference domain \( \Omega_\ast \) such that \( \omega_{\ast \ast} \in (0, \pi/16) \), we know that when \( T \) and the constant \( \delta_1 \) in \( \Sigma \) are chosen small enough, the contact angles generated by both \( (\m_a, d_l, d_r) \) and \( (\wt{\m}_a, \wt{d}_l, \wt{d}_r) \) lie in the same interval \( (0, \pi/16] \) as well.

### 6.3 The Contraction Mapping

First, we introduce a lower-order norm \( \| \cdot \|_\Sigma \) for \( \Sigma \):
\[
\| (\m_a, d_l, d_r) \|_\Sigma \triangleq \| \m_a \|_{C([0,T]; H^4(\Gamma_a))} + \| \partial_t \m_a \|_{C([0,T]; H^{2.5}(\Gamma_a))} + \sum_i \left( |d_i''|_{C([0,T])} + |d_i''|_{C([0,T])} + a^{-3/2} |d_i''|_{C([0,T])} \right).
\]

To prove that \( \mathcal{F} \) is a contraction mapping, we use the Banach fixed point theorem in a variational way. In fact, for a parameter \( \tau \), let’s consider a one-parameter family
\[ (\m_a(\tau), d_l(\tau), d_r(\tau)) \in \Sigma \]
with initial data
\[ ((\m_a)_I(\tau), (\partial_t \m_a)_I(\tau), (d_l)_I(\tau), (d_r)_I(\tau)). \]

Correspondingly, we write
\[ \mathcal{F}(\m_a(\tau), d_l(\tau), d_r(\tau)) = (\wt{\m}_a(\tau), \wt{d}_l(\tau), \wt{d}_r(\tau)). \]
Taking the variation with respect to $\tau$ on both sides of system (6.1) leads to a system for $(\partial_\tau \tilde{u}_a, \partial_\tau \tilde{d}_i, \partial_\tau \tilde{d}_r)$ immediately:

$$
\begin{cases}
D_{\tau*}^2 \partial_\tau \tilde{u}_a + \sigma A(d_{\Gamma_r}) \partial_\tau \tilde{u}_a = G_1 & \text{on } \Gamma_* , \\
D_{\tau*} A(d_{\Gamma_r}) \partial_\tau \tilde{u}_a \mp \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 \\
\partial_\tau^2 \partial_\tau d_i = \partial_\tau B_i & (i = l, r)
\end{cases}
$$

(6.3)

with

$$
\begin{align*}
G_1 \circ \Phi_{S_t}^{-1} &= (\partial_\tau R_0) \circ \Phi_{S_t}^{-1} - [D_{\tau}, D_{\tau}^2] \tilde{u}_a \circ \Phi_{S_t}^{-1} + \sigma [D_{\tau}, N \Delta_{\Gamma_r}] \tilde{u}_a \circ \Phi_{S_t}^{-1}, \\
G_{2,i} \circ \Phi_{S_t}^{-1} &= (\partial_\tau R_{c,i}) \circ \Phi_{S_t}^{-1} + [D_{\tau}, N \Delta_{\Gamma_r}, D_{\tau}] \tilde{u}_a \circ \Phi_{S_t}^{-1}
\end{align*}
$$

and moreover

$$
\partial_\tau B = \partial_\tau B \left( a^3 \partial_\tau \partial d_{\Gamma_r}, \nabla v \partial_\tau d_{\Gamma_r}, \partial_\tau v^*, \partial_\tau u, \partial_\tau \nabla u, \partial_\tau \nabla \Psi_v \right).
$$

In fact, the conditions at the corner point in (6.3) are rewritten from the conditions in system (6.1):

$$
D_{\tau*} A(d_{\Gamma_r}) \tilde{u}_a \mp \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 (\nabla_{\tau} (A(d_{\Gamma_r}) \tilde{u}_a \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = R_{c,i} & \text{at } p_i \ (i = l, r).
$$

which is equivalent to

$$
D_{\tau} N \Delta_{\Gamma_r} (\tilde{u}_a \circ \Phi_{S_t}^{-1}) \mp \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 (\nabla_{\tau} (A(d_{\Gamma_r}) \tilde{u}_a \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = R_{c,i} \circ \Phi_{S_t}^{-1} & \text{at } p_i \ (i = l, r).
$$

Taking $D_{\tau}$ on both sides of the equation above, one can derive the desired condition in (6.3).

To simplify the notations, we denote

$$
F(\tau) = (\partial_\tau \tilde{u}_a) \circ \Phi_{S_t}^{-1}(\tau)
$$

and rewrite (6.3) as

$$
\begin{cases}
D_{\tau}^2 F - \sigma N \Delta_{\Gamma_r} F = G_1 \circ \Phi_{S_t}^{-1} & \text{on } \Gamma_r , \\
D_{\tau}^2 N \Delta_{\Gamma_r} F \mp \frac{\sigma^2}{\beta_c} (\sin \omega_i)^2 \nabla_{\tau} (A(d_{\Gamma_r}) \tilde{u}_a \circ \Phi_{S_t}^{-1}) \circ \Phi_{S_t} = G_{2,i} \circ \Phi_{S_t}^{-1} & \text{at } p_i , \\
\partial_\tau^2 \partial_\tau d_i = \partial_\tau B_i & (i = l, r).
\end{cases}
$$

(6.4)

In order to prove the contraction mapping, we need a lower-order energy estimate for $(F, \partial_\tau \tilde{d}_i, \partial_\tau \tilde{d}_r)$ under the norm $\| \cdot \|_\Sigma$, which is proved similarly to the energy estimates obtained in Section 5.
**Proposition 6.3 (Lower-order energy estimates).** Assume \((\mathcal{N}_\alpha, d \xi, d \eta) \in \Sigma\) and that the contact angles \(\omega_i\) satisfy
\[
\min_i \sin \omega_i \geq c_0 \quad \text{for some } c_0 > 0.
\]

Then we have the following energy estimates for the linear problem (6.4):
\[
\| F \|_{H^4(\Gamma_t)}^2 + \| D_t F \|_{H^{2.5}(\Gamma_t)}^2 \\
\leq e^{C(L_0, L_1) t} C(\delta_1, L) \left( \| F(0) \|_{H^4(\Gamma_0)}^2 + \| D_t F(0) \|_{H^{2.5}(\Gamma_0)}^2 \right) \\
+ a^{3/2} e^{C(L_0, L_1) t} C(L_0, L_1) \\
\times \int_0^t \left( \| \partial_\tau \mathcal{N}_\alpha \|_{H^4(\Gamma_{s \tau})}^2 + \| \partial_\tau \partial_\tau \mathcal{N}_\alpha \|_{H^{2.5}(\Gamma_{s \tau})}^2 + a^{-3} \| \partial_\tau^2 \partial_\tau \mathcal{N}_\alpha \|_{H^1(\Gamma_{s \tau})}^2 \right) \\
+ |\partial_\tau d_i|^2 + |\partial_\tau \partial_\tau d_i|^2 + a^{-3} |\partial_\tau^2 \partial_\tau d_i|^2 \, dt'
\]
and
\[
|\partial_\tau d_i(t)| + |\partial_\tau \partial_\tau d_i(t)| \leq |\partial_\tau d(0)| + (1 + t) |\partial_\tau \partial_\tau d_i(0)| \\
+ C(L_0, L_1) \int_0^t \left( \| \partial_\tau \mathcal{N}_\alpha \|_{H^1(\Gamma_{s \tau})} + \| \partial_\tau \partial_\tau \mathcal{N}_\alpha \|_{H^1(\Gamma_{s \tau})} \right) \\
+ |\partial_\tau d_i| + |\partial_\tau \partial_\tau d_i| \right) \, dt',
\]
where the constant \(C\) depends on \(L_0, L_1\) and where \(|d_i|\) means \(|d_\tau| + |d_i|\).

**Proof.** The proof is very similar to the proof for the higher-order energy estimates, while the lower-order energy and dissipation (5.8) are used here.

To begin with, we apply \(\mathcal{N} \Delta \Gamma_t\) to both sides of the equation of \(F\) from (6.4) to obtain
\[
\mathcal{N} \Delta \Gamma_t \, D_t^2 F - \sigma (\mathcal{N} \Delta \Gamma_t)^2 F = \mathcal{N} \Delta \Gamma_t \left( G_1 \circ \Phi_{S_t}^{-1} \right) \quad \text{on } \Gamma_t.
\]
Multiplying \(D_t \Delta \Gamma_t F\) on both sides of the equation above and integrating on \(\Gamma_t\), one has
\[
\int_{\Gamma_t} \mathcal{N} \Delta \Gamma_t \, D_t^2 F \cdot D_t \Delta \Gamma_t F \, ds - \sigma \int_{\Gamma_t} \mathcal{N} \Delta \Gamma_t \, \mathcal{N} \Delta \Gamma_t \cdot D_t \Delta \Gamma_t F \, ds \\
= \int_{\Gamma_t} \mathcal{N} \Delta \Gamma_t \left( G_1 \circ \Phi_{S_t}^{-1} \right) \cdot D_t \Delta \Gamma_t F \, ds.
\]
(6.5)

First, for the first term on the left side, one derives
\[
\int_{\Gamma_t} \mathcal{N} \Delta \Gamma_t \, D_t^2 F \cdot D_t \Delta \Gamma_t F \, ds = \frac{1}{2} \partial_\tau \| \nabla (D_t \Delta \Gamma_t F) \|^2_{L^2(\Omega_t)} + A_{11},
\]
where

\[
A_{l1} = \int_{\Omega_t} [\nabla H, D_t] D_t \Delta_{\Gamma_t} F \cdot \nabla H (D_t \Delta_{\Gamma_t} F) dX \\
- \frac{1}{2} \int_{\Omega_t} (\text{div } v) |\nabla H (D_t \Delta_{\Gamma_t} F)|^2 dX \\
+ \int_{\Gamma_t} [\Delta_{\Gamma_t}, D_t^2] F \cdot \mathcal{N} D_t \Delta_{\Gamma_t} F ds.
\]

Applying (4.2), (4.4), Proposition 4.2, Corollary 4.13, Lemma 4.9, and the bounds in \( \Sigma \) implies

\[
|A_{l1}| \leq C \left( \|d_{\Gamma_t} + \Omega^{3.5}(\Omega_t), \|v\|_{H^3}^{3.5}(\Omega_t), \|D_t v\|_{H^{2.5}(\Omega_t)} \right) \times (E_t(l, F, \partial_t F) + \|F\|_{H^{2.5}(\Gamma_t)}^2 + \|D_t F\|_{H^{2.5}(\Gamma_t)}^2) \\
\leq C(L_0, L_1) (E_t(l, F, \partial_t F) + \|F\|_{H^{2.5}(\Gamma_t)}^2 + \|D_t F\|_{H^{2.5}(\Gamma_t)}^2).
\]

Second, for the second term on the left side, one shows that

\[
-\sigma \int_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F \cdot D_t \Delta_{\Gamma_t} F ds = \frac{\sigma}{2} \partial_t \|\nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F\|_{L^2(\Gamma_t)}^2 + A_{l2} + A_{l3},
\]

with the remainder term

\[
A_{l2} = \sigma \int_{\Gamma_t} \nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F \cdot [\nabla_{\Gamma_t} \mathcal{N}, D_t] \Delta_{\Gamma_t} F ds \\
- \frac{\sigma}{2} \int_{\Gamma_t} (D \cdot v) |\nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F|^2 ds \\
\leq C(\|d_{\Gamma_t}\|_{H^4(\Gamma_t)}, \|v\|_{H^{2.5}(\Omega_t)}) (E_t(l, F, \partial_t F) + \|F\|_{H^{3.5}(\Gamma_t)}^2) \\
\leq C(L_0, L_1) (E_t(l, F, \partial_t F) + \|F\|_{H^{3.5}(\Gamma_t)}^2).
\]

and the term at the corner points

\[
A_{l3} = -\sigma \nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F \cdot \mathcal{N} D_t \Delta_{\Gamma_t} F \bigg|_{p_i}^{p_i}.
\]

To handle \( A_{l3} \), we use the condition for the corner points in (6.4):

\[
D_t \mathcal{N} \Delta_{\Gamma_t} F = \frac{\sigma^2}{\beta_c} \sin(\omega_i)^2 \nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F = G_{2,i} \circ \Phi_{S_i}^{-1} \quad \text{at } p_i \ (i = l, r),
\]

which can be rewritten as

\[
\mathcal{N} D_t \Delta_{\Gamma_t} F = \frac{\sigma^2}{\beta_c} \sin(\omega_i)^2 \nabla_{\Gamma_t} \mathcal{N} \Delta_{\Gamma_t} F + r_{4,i} \quad \text{at } p_i,
\]

where

\[
r_{4,i} = -[D_t, \mathcal{N}] \Delta_{\Gamma_t} F + G_{2,i} \circ \Phi_{S_i}^{-1}.
\]
Consequently, we have

$$A_{13} = \frac{\sigma^3}{\beta_c} F_l(t, F, \partial_t F) - \sigma \nabla_{\nabla} \nabla \Delta_{\Gamma_F} F \cdot r_{4,i}.$$  

where the remainder term satisfies

$$\left| \sigma \nabla_{\nabla} \nabla \Delta_{\Gamma_F} F \cdot r_{4,i} \right| \leq \delta F_l(t, F, \partial_t F) + (\sin \omega_i)^{-2} C_\delta \|r_{4,i}\|^2_{H^1(\Gamma_F)}.$$  

Moreover, checking the terms in $G_2$ carefully and applying Lemma 4.11, one has

$$\|r_{4,i}\|^2_{H^1(\Gamma_F)} \leq C \left( \|d_{\Gamma_F} \|_{H^5(\Gamma_F)}, \|\partial_t d_{\Gamma_F} \|_{H^4(\Gamma_F)}, \|v \|_{H^{5.5}(\Omega_F)}, \|\nabla_{\nabla} \|_{H^3(\Gamma_F)} , \right.$$  

$$\|\partial_t \nabla_{\nabla} \|_{H^2(\Gamma_F)} , \|\partial_t \nabla_{\nabla} \|_{H^4(\Gamma_F)} \right)$$  

$$\times \left( \|\partial_t d_{\Gamma_F} \|_{H^5(\Gamma_F)} + \|\partial_t \nabla_{\nabla} \|_{H^4(\Gamma_F)} + \|D_t v \|_{H^{5.5}(\Omega_F)} \right)$$  

$$+ a \|\partial_t \nabla_{\nabla} \|_{H^4(\Gamma_F)} + a \|\partial_t \partial_t \nabla_{\nabla} \|_{H^{2.5}(\Gamma_F)}$$  

$$+ a |\partial_t d_i| + a |\partial_t \partial_t d_i|);$$  

applying Proposition 4.2, Corollary 4.13, Lemma 4.9, and the bounds in $\Sigma$ leads to

$$\|r_{4,i}\|^2_{H^1(\Gamma_F)} \leq a C (L_0, L_1)$$  

$$\times \left( \|\partial_t \nabla_{\nabla} \|_{H^4(\Gamma_F)} + \|\partial_t \partial_t \nabla_{\nabla} \|_{H^{2.5}(\Gamma_F)} + |\partial_t d_i| + |\partial_t \partial_t d_i| \right).$$  

On the other hand, for the right side of (6.5), we have

$$\int_{\Gamma_F} \nabla \Delta_{\Gamma_F} (G_1 \circ \Phi_{\nabla}^{-1}) \cdot D_t \Delta_{\Gamma_F} F \ ds$$  

$$= \int_{\Omega_F} \nabla (\Delta_{\Gamma_F} (G_1 \circ \Phi_{\nabla}^{-1})) \cdot \nabla (D_t \Delta_{\Gamma_F} F) \ d X$$  

$$\leq C \left( \|d_{\Gamma_F} \|_{H^{2.5}(\Gamma_F)} \right) \|G_1 \circ \Phi_{\nabla}^{-1} \|_{H^{2.5}(\Gamma_F)} E_l (t, F, \partial_t F)^{1/2}.$$  

Checking the expression of $G_1$, one obtains from Lemma 4.9(1) and Lemma 4.10 that

$$\|G_1 \circ \Phi_{\nabla}^{-1} \|_{H^{2.5}(\Gamma_F)}$$  

$$\leq C \left( \|d_{\Gamma_F} \|_{H^{6.5}(\Gamma_F)}, \|\partial_t d_{\Gamma_F} \|_{H^{5.5}(\Gamma_F)}, \|v \|_{H^{5}(\Omega_F)}, \|\nabla_{\nabla} \|_{H^{3.5}(\Gamma_F)} , \right.$$  

$$\|\partial_t \nabla_{\nabla} \|_{H^{2.5}(\Gamma_F)} \right)$$  

$$\times \left( \|\partial_t d_{\Gamma_F} \|_{H^{6.5}(\Gamma_F)} + \|\partial_t \partial_t d_{\Gamma_F} \|_{H^{5.5}(\Gamma_F)} \right)$$  

$$+ a \|\partial_t \nabla_{\nabla} \|_{H^4(\Gamma_F)} + a \|\partial_t \partial_t \nabla_{\nabla} \|_{H^{2.5}(\Gamma_F)}$$  

$$+ a |\partial_t d_i| + a |\partial_t \partial_t d_i|),$$  

where $\|\partial_t ^2 d_{\Gamma_F} \|_{H^{2.5}(\Gamma_F)}$ comes from the term $[D_t, D_t^2] \Phi_{\nabla}^{-1}$ in $G_1$. Applying Proposition 4.2, Corollary 4.13, Lemma 4.9(2), and the bounds in $\Sigma$ again
leads to
\[ \| G_1 \circ \Phi_S^{-1} \|_{H^{2.5}(\Gamma)} \]
\[ \leq a^{3/2} C \left( \| \nabla a \|_{H^{3.5}(\Gamma)}, \| \partial_t \nabla a \|_{H^{2.5}(\Gamma)}, \| \nabla \partial_t \nabla a \|_{H^{2.5}(\Gamma)}, \| \partial_t \nabla a \|_{H^{3.5}(\Gamma)} \right) \times \left( \| \partial_t \nabla a \|_{H^4(\Gamma)}, \| \partial_t \partial_t \nabla a \|_{H^2(\Gamma)} \right) \]
\[ + a^{-3/2} \| \partial_t^2 \nabla a \|_{H^1(\Gamma)} + |\partial_t d_l| + |\partial_t \partial_t d_l| + a^{-3/2} |\partial_t^2 \partial_t d_l| \]
\[ \leq a^{3/2} C \left( L_0, L_1 \right) \times \left( \| \partial_t \nabla a \|_{H^4(\Gamma)} + \| \partial_t \partial_t \nabla a \|_{H^2(\Gamma)} + a^{-3/2} \| \partial_t^2 \partial_t \nabla a \|_{H^1(\Gamma)} \right)
\]
\[ + |\partial_t d_l| + |\partial_t \partial_t d_l| + a^{-3/2} |\partial_t^2 \partial_t d_l| \].

Consequently, since the energy estimates for \( \| D_t F \|_{L^2(\Gamma)} \) and \( \| F \|_{L^2(\Gamma)} \) are similar to those in the higher-order case, summing all the estimates above leads to
\[ \partial_t E_l(t, F, \partial_t F) + \sum_i F_{l,i}(t, F, \partial_t F) \]
\[ \leq C(L_0, L_1) \left( E_l(t, F, \partial_t F) + \| F \|_{H^{3.5}(\Gamma)}^2 + \| D_t F \|_{H^{2.5}(\Gamma)}^2 \right)
\]
\[ + a^{3/2} \left( \| \partial_t \nabla a \|_{H^{3.5}(\Gamma)} + \| \partial_t \partial_t \nabla a \|_{H^2(\Gamma)} \right)
\]
\[ + a^{-3/2} \| \partial_t^2 \partial_t \nabla a \|_{H^1(\Gamma)} + |\partial_t d_l| + |\partial_t \partial_t d_l| + a^{-3/2} |\partial_t^2 \partial_t d_l| \].

Applying Lemma 5.5 with the bounds in \( \Sigma \) for the coefficient, the desired energy estimate for \( F \) can be proved. Moreover, the energy estimate for \( \partial_t d_l \) (\( i = l, r \)) can be done similarly and more easily.

We now finish the iteration.

**Proposition 6.4.** For any \( 0 < \epsilon \ll \delta_1, A_1 > 0 \), and initial data
\[ \left( (\nabla a)_1, (\partial t \nabla a)_1, (d_l)_1, (d_r')_1 \right) \in I(\epsilon, A_1), \]
there exist \( L, L_0, \) and \( L_1 \) and small constants \( a^{-1} \) and \( T \) such that \( F \) defined on \( \Sigma \) has a fixed point.

**Proof.** Thanks to Proposition 6.3 when \( T \) is small enough we have the following estimate:
\[ \| \partial_t \nabla a \|_{H^4(\Gamma)}^2 + \| \partial_t \partial_t \nabla a \|_{H^2(\Gamma)}^2 + \sum_i \| \partial_t d_l \|^2 + \| \partial_t \partial_t d_l\|^2 \leq \]

\[ (6.6) \]
\[ \leq C(L_0, \delta_1, L) \left( \| \partial_t (\gamma_a) \|_{H^4(\Gamma_a)}^2 + \| \partial_t (\partial_t \gamma_a) \|_{H^{2.5}(\Gamma_a)}^2 \right. \\
+ \sum_i \left( |\partial_t (d_i)\|_{L^2}^2 + |\partial_t (\partial_t d_i)|_{L^2}^2 \right) \\
+ \left. Ta^3 C(L_0, L_1, \delta_1, L) \| (\partial_t \gamma_a, \partial_t d_i, \partial_t d_i) \|_{\Sigma}^2 \right], \]

where one can see that the estimates for the second-order time derivatives are still missing.

In fact, applying Corollary 5.6 with \( s = 1, f \) replaced by \( \partial_t \gamma_a \), and \( g_1 \) replaced by \( G_1 \), one obtains immediately that

\[ \| \partial_t^2 \partial_t \gamma_a \|_{H^1(\Gamma_a)} \leq \| G_1 \|_{H^1(\Gamma_a)} \]

 Moreover, one has similarly as before the estimate

\[ \| G_1 \|_{H^1(\Gamma_a)} \leq C(L_0, L_1) \]

\[ \times \left( \| \partial_t \gamma_a \|_{H^4(\Gamma_a)} + \| \partial_t \partial_t \gamma_a \|_{H^{2.5}(\Gamma_a)} + a^{-3/2} \| \partial_t^2 \partial_t \gamma_a \|_{H^1(\Gamma_a)} \\
+ |\partial_t d_i| + |\partial_t \partial_t d_i| + a^{-3/2} |\partial_t^2 \partial_t d_i| \right). \]

As a result, substituting this inequality and \( \Box (6.6) \) into (6.7), one arrives at

\[ \| \partial_t^2 \partial_t \gamma_a \|_{H^1(\Gamma_a)}^2 \leq C(L_0, L_1, \delta_1, L) \left( \| \partial_t \gamma_a \|_{H^4(\Gamma_a)}^2 + \| \partial_t \partial_t \gamma_a \|_{H^{2.5}(\Gamma_a)}^2 \\
+ \| \partial_t (\partial_t \gamma_a) \|_{H^4(\Gamma_a)}^2 \\
+ \sum_i \left( |\partial_t (d_i)|_{L^2}^2 + |\partial_t (\partial_t d_i)|_{L^2}^2 \right) \right). \]

On the other hand, using the equation for \( \partial_t \tilde{d}_i \) directly leads to the estimate

\[ |\partial_t^2 \partial_t \tilde{d}_i| \leq C(L_0, L_1) \left( \| \partial_t \gamma_a \|_{H^4(\Gamma_a)}^2 + \| \partial_t \partial_t \gamma_a \|_{H^2(\Gamma_a)}^2 + |\partial_t d_i|^2 + |\partial_t \partial_t d_i|^2 \right). \]
Summing up all the estimates above, we finally conclude that
\[
\| (\partial_t \mathcal{N}_a, \partial_t \bar{d}_l, \partial_t \bar{d}_r) \|_\Sigma^2 \\
\leq (T a^3 + a^{-3}) C(L_0, L_1, \delta_1, L) \| (\partial_t \mathcal{N}_a, \partial_t d_l, \partial_t d_r) \|_\Sigma^2 \\
+ C(L_0, L_1, \delta_1, L) (\| \partial_t (\mathcal{N}_a)_t \|_{H^4(\Gamma_\alpha)}^2 + \| \partial_t (\mathcal{N}_a)_t \|_{H^2(\Gamma_\alpha)}^2 \\
+ \sum_i (|\partial_t (d_i)_t|^2 + |\partial_t (d_i)_t|^2)).
\]

When we take \( a^{-1} \) and then \( T \) sufficiently small, we have
\[
\| (\partial_t \mathcal{N}_a, \partial_t \bar{d}_l, \partial_t \bar{d}_r) \|_\Sigma \\
\leq \frac{1}{2} \| (\partial_t \mathcal{N}_a, \partial_t d_l, \partial_t d_r) \|_\Sigma + C (\| \partial_t (\mathcal{N}_a)_t \|_{H^4(\Gamma_\alpha)} + \| \partial_t (\mathcal{N}_a)_t \|_{H^2(\Gamma_\alpha)} \\
+ \sum_i (|\partial_t (d_i)_t|^2 + |\partial_t (d_i)_t|^2)).
\]

Therefore, if we fix the initial data in \( \mathcal{I}(\varepsilon, A_1) \), we construct a Cauchy sequence \( \{(\mathcal{N}_a)^n, d^n_l, d^n_r\} \) in \( \Sigma \). Consequently, there exists a unique \((\mathcal{N}_a, d_l, d_r) \in \Sigma \) such that
\[
((\mathcal{N}_a)^n, d^n_l, d^n_r) \to (\mathcal{N}_a, d_l, d_r) \quad \text{in} \ \Sigma,
\]
which implies that \( \mathcal{F} \) defined on \( \Sigma \) admits a fixed point. \( \square \)

### 6.4 Back to the Euler Equation

In the previous section, we have proved the unique existence of the solution \((\mathcal{N}_a, d_l, d_r)\) of (4.25) and (4.27) for given initial data. As a result, we now have the moving domain \( \Omega_t \) and the velocity field \( v \) by \( d_l \) and (4.7).

For the moment, we are able to show that the water-waves system (WW) is satisfied by this velocity \( v \) and the pressure
\[
P = \sigma \kappa_H + P_{v,v}.
\]

To begin with, we first recall the definition of \( v = \nabla \phi \) by (4.7). So one has
\[
D_t v = D_t \nabla \phi = \nabla (D_t \phi) - \nabla v \cdot \nabla \phi = \nabla \left( D_t \phi - \frac{1}{2} |v|^2 \right).
\]

Now, we define the other pressure
\[
Q = -D_t \phi + \frac{1}{2} |v|^2,
\]
which implies
\[
D_t v + \nabla Q = 0.
\]

Moreover, we recall that
\[
\text{div } v = \gamma \xi
\]
We are going to prove

\[ P + g z = Q \quad \text{and} \quad \text{div} \ v = 0. \]

In order to do this, we first define

\[ V_0 \triangleq \nabla (Q - P - g z) = -D_t v - \nabla P + g. \]

so we have a slightly new equation for \( v \) compared to the Euler equation:

\[ D_t v = -\sigma \nabla \kappa_H - \nabla P_{v,v} + g - V_0. \tag{6.8} \]

We go through the computations for deriving (4.25) again to check the terms involving \( V_0 \), which means we need to trace the substitutions of \( D_t v \) and the Euler equation. To begin with, (4.18) is rewritten here:

\[ D_t^2 \kappa = -N_t \cdot \Delta \Gamma_t D_t v + 2\sigma \Pi(\tau_t) \cdot \nabla \tau_t \nabla \kappa_H + r_1. \]

where \( r_1 \) is the remainder term defined before in (4.18). Substituting the new equation (6.8) of \( v \) into this equation leads to

\[ D_t^2 \kappa = \sigma \Delta \Gamma_t N(\kappa) + \tilde{R}_1, \]

where

\[ \tilde{R}_1 = \tilde{R}_1 + N_t \cdot \Delta \Gamma_t V_0. \]

Moreover, noticing that the term \( R_\alpha \) in (4.19) contains \( D_t v \) by (4.23) and (4.21), one finds these extra terms in \( R_\alpha \) as below,

\[
\begin{align*}
-a^3 &\left( \frac{1}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} \right) V_0 \cdot N_t \\
+ &a^3 \frac{(\mu \circ \Phi^{-1}_{S_t}) \cdot \nabla (d_{\Gamma_t} \circ \Phi^{-1}_{S_t})}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t - a^3 V_0 \cdot \nabla (d_{\Gamma_t} \circ \Phi^{-1}_{S_t}).
\end{align*}
\]

Similarly, the term \( D_t [D_t, N] \kappa \) in \( \tilde{R}_1 \) from (4.19) also involves \( D_t v \). As a result, we finally show that

\[ D_t^2 \mathcal{M}_\alpha + \sigma \mathcal{A}(\kappa_o) \mathcal{M}_\alpha = \mathcal{R}_0 \quad \text{on} \ \Gamma_s. \]

where

\[
\begin{align*}
\mathcal{R}_0 \circ \Phi^{-1}_{S_t} &= R_0 \circ \Phi^{-1}_{S_t} + N(N_t \cdot \Delta \Gamma_t V_0) - a^3 \frac{1}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t \\
+ &a^3 \frac{(\mu \circ \Phi^{-1}_{S_t}) \cdot \nabla (d_{\Gamma_t} \circ \Phi^{-1}_{S_t})}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t - a^3 V_0 \cdot \nabla (d_{\Gamma_t} \circ \Phi^{-1}_{S_t}) \\
- &\nabla N_t \cdot \mathcal{M} + \nabla N_t V_0 \cdot \nabla g H + \nabla (\nabla g H) V_0 \cdot N_t.
\end{align*}
\]
with $R_0$ defined in (4.19) and
\[ w = \Delta^{-1} \left( 2\nabla V_0 \cdot \nabla^2 \kappa_H + \Delta V_0 \cdot \nabla \kappa_H, (\nabla N_b V_0 - \nabla V_0 N_b) \cdot \nabla \kappa_H \right). \]

Since we have proved that (4.25) holds, we know immediately that $V_0$ satisfies
\[ \mathcal{N}(N_t \cdot \Delta_{\Gamma_r} V_0) - \frac{1}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t \]
\[ + a^3 \frac{(\mu \circ \Phi^{-1}_{S_t}) \cdot \nabla (d_{\Gamma_r} \circ \Phi^{-1}_{S_t})}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t - a^3 V_0 \cdot \nabla (d_{\Gamma_r} \circ \Phi^{-1}_{S_t}) \]
\[ - \nabla_{N_t} w + \nabla_{N_t} V_0 \cdot \nabla g_H + \nabla (\nabla g_H)^T V_0 \cdot N_t = 0. \]

Moreover, denoting
\[ R_{V_0} = \mathcal{N}(2\nabla_{\Gamma_r} N_t \cdot \nabla_{\Gamma_r} V_0 + \Delta_{\Gamma_r} N_t \cdot V_0) \]
\[ - a^3 \frac{(\mu \circ \Phi^{-1}_{S_t}) \cdot \nabla (d_{\Gamma_r} \circ \Phi^{-1}_{S_t})}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t + a^3 V_0 \cdot \nabla (d_{\Gamma_r} \circ \Phi^{-1}_{S_t}) \]
\[ + \nabla_{N_t} w - \nabla_{N_t} V_0 \cdot \nabla g_H - \nabla (\nabla g_H)^T V_0 \cdot N_t, \]

the above equation for $V_0$ becomes
\[ (6.9) \quad a^3 \frac{1}{(\mu \circ \Phi^{-1}_{S_t}) \cdot N_t} V_0 \cdot N_t - \mathcal{N}(\Delta_{\Gamma_r} V_0 \cdot N_t) + R_{V_0} = 0, \]

which is analogous to the linearized equation of $\mathcal{N}_{\omega}$ with respect to $d_{\Gamma_r}$.

Second, recalling (4.22) (which leads to the equation (4.27) of $d_{l_1}$) and using (6.8), one obtains immediately with (4.27) and Remark 4.8 that
\[ V_0 \cdot N_t = 0 \quad \text{at } p_i \ (i = l, r). \]

On the other hand, a direct computation leads to the system for $P - Q + g \xi$:
\[ \begin{cases} \Delta(P - Q + g \xi) = D_t (\gamma \xi) = -\text{div} V_0 & \text{on } \Omega_t, \\ \nabla_{N_t}(P - Q + g \xi) |_{\Gamma_r} = -V_0 \cdot N_t, \quad \nabla_{N_b}(P - Q + g \xi) |_{\Gamma_b} = 0, \end{cases} \]

which admits the following elliptic estimate by [41] theorem 5.3 and Lemma 2.5:
\[ (6.10) \quad \|V_0\|_{H^3(\Omega_t)} \leq C \|P - Q + g \xi\|_{H^4(\Omega_t)} \]
\[ \leq C(L_0)(\|V_0 \cdot N_t\|_{H^{2,5}(\Gamma_r)} + |\beta_{\xi}\xi| + |\xi|). \]

Multiplying $(1 - a^{-1}\Delta_{\Gamma_r}) V_0 \cdot N_t$ on both sides of (6.9) and integrating by parts while using the boundary condition $V_0 \cdot N_t |_{\partial \Omega} = 0$, one has
\[ a^3 \|V_0 \cdot N_t\|_{L^2(\Gamma_r)}^2 + a^2 \|\nabla_{\Gamma_r} (V_0 \cdot N_t)\|_{L^2(\Gamma_r)}^2 + a^{-1}\|\Delta_{\Gamma_r} (V_0 \cdot N_t)\|_{H^{1/2}(\Gamma_r)}^2 \]
\[ \leq C(L_0)(\|V_0\|_{H^{2,5}(\Gamma_r)} + \|V_0\|_{H^2(\Omega_t)} + \|V_0\|_{H^{1,5}(\Gamma_b)}) \]
\[ \times (\|V_0 \cdot N_t\|_{L^2(\Gamma_r)} + a^{-1}\|\Delta_{\Gamma_r} (V_0 \cdot N_t)\|_{L^2(\Gamma_r)}). \]
Moreover, combining this estimate with (6.10), we conclude that
\[ a^3 \| V_0 \cdot N_t \|_{L^2(G)}^2 + a^2 \| \nabla_{\Gamma_t} (V_0 \cdot N_t) \|_{L^2(G)}^2 + a^{-1} \| \Delta_{\Gamma_t} (V_0 \cdot N_t) \|_{H^{1/2}(\Gamma_t)}^2 \]
\[ \leq C(L_0) (\| V_0 \cdot N_t \|_{H^{2.5}(\Gamma_t)} + |\partial_t \xi| + |\xi|) \]
\[ \times (\| V_0 \cdot N_t \|_{L^2(\Gamma_t)} + a^{-1} \| \Delta_{\Gamma_t} (V_0 \cdot N_t) \|_{L^2(\Gamma_t)}). \]
Consequently, we obtain
\[ \| V_0 \cdot N_t \|_{H^{2.5}(\Gamma_t)} \leq a^{-1} C(L_0) (|\partial_t \xi| + |\xi|). \]

For the moment, it remains to deal with $\xi, \partial_t \xi$. In fact, one has by a direct calculation that
\[
\partial_t \xi = \int_{\Omega_t} D_t (\gamma \xi) dX + \int_{\Omega_t} (\text{div } v) \gamma \xi \ dX = - \int_{\Omega_t} \text{div } V_0 \ dX + \gamma \xi^2 = - \int_{\Gamma_t} V_0 \cdot N_t \ ds + \gamma \xi^2.
\]
and
\[ \gamma |\xi|^2 \leq a^{-1} C(L_0, L_1)|\xi|. \]

Moreover, one knows
\[ \int_{\Gamma_t} V_0 \cdot N_t \ ds \leq C(L_0) \| V_0 \cdot N_t \|_{H^{2.5}(\Gamma_t)}, \]
due to the fact that the domain is bounded.

Combining these inequalities with the above estimate for $\| V_0 \cdot N_t \|_{H^{2.5}(\Gamma_t)}$ and taking $a^{-1}$ sufficiently small, we derive
\[ |\partial_t \xi| \leq a^{-1} C(L_0, L_1)|\xi|, \]
which implies
\[ \xi = 0 \quad \text{since} \quad \xi|_{t=0} = \int_{\Gamma_t} V_0 \cdot N_t \ ds \bigg|_{t=0} = 0. \]

As a result, we know immediately
\[ \text{div } v = 0 \quad \text{and} \quad V_0 = 0, \]
which implies that the Euler equation is satisfied by $v$:
\[ D_t v = -\nabla P + g \quad \text{on } \Omega_t. \]

In the end, we show that the condition for the corner points in (WW)
\[ \beta_c v_i = \sigma (\cos \omega_i - \cos \omega_l) \quad \text{at } p_i (i = l, r) \]
can be derived from
\[ D_{t,*} A(\kappa_d) \eta_d \pm \frac{\sigma^2}{\beta_c} (\sin \omega_l)^2 (\nabla_{\Gamma_t} (A(\kappa_d) \eta_d \circ \Phi_{S_t}^{-1})) \circ \Phi_{S_t} = R_{c,i} \quad \text{at } p_{i*}. \]
In fact, going back to the proofs for Lemma [4.3] and [4.1], lemma 7.1, we know that the equation above is obtained by taking $D_t$ three times on (6.11), while the Euler
equation and the equation of $\mathcal{H}_a$ are applied as well. As a result, integrating with respect to the time variable three times and remembering that $v \cdot N_b|_{\Gamma_b} = 0$, we retrieve conditions for the contact points.

Moreover, we have from Remark 6.2 that the contact angles $\omega_i$ stay in $(0, \pi/16)$ when $T$ is sufficiently small.

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