Tessellations and Descartes disk configurations

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Abstract

An intriguing correspondence between certain finite planar tessellations and the Descartes circle arrangements is presented. This correspondence may be viewed as a visualization of the spinor structure underlying Descartes circles.

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1 Dodecagonal tessellations

René Descartes found a beautiful relation for four mutually tangent disks (such arrangement is called Descartes configuration):

$$2(A^2 + B^2 + C^2 + D^2) = (A + B + C + D)^2$$

(1)

where \(A, B, C,\) and \(D\) are the curvatures of the disks, i.e., reciprocals of the radii. He shared it with Elizabeth II, princes of Bohemia, in a form of a problem: given three mutually tangent circles, find the fourth tangent to the three. (For more on Descartes configuration and Apollonian disk packing, see [2, 3, 8, 9, 10, 11].) Since (1) is quadratic, there are two solutions; here is an example of such pair:

\((2, 3, 6, 23)\) and \((2, 3, 6, -1)\)

Finding other integral solutions requires certain parametrization [1, 6].

It turns out that there is an intriguing duality between certain tessellations and Descartes arrangements of four circles:

Areas in a tessellation \(\longleftrightarrow\) Curvatures in Descartes configuration

This duality is the main topic of the present notes. But there is a pleasant byproduct: one can easily construct an integral tessellation and obtain therefore effortlessly an integral Descartes disk configuration.

Here is the construction. Consider three vectors \(\mathbf{a}, \mathbf{b}, \mathbf{c}\) in a two dimensional vector space \(\mathbb{R}^2\), such that

\(\mathbf{a} + \mathbf{b} + \mathbf{c} = 0\)

Construct a tessellated dodecagon from this triple of vectors as described in Figure [1]. It consists of 15 parallelograms, three of which are squares and the remaining twelve consists of pairs of congruent pieces. Here are the steps.
Start with three vectors with vanishing sum (first panel). Next, construct squares on these vectors (yellow squares in panel 2). Add parallelograms in spaces between the squares (red, panel 3). Continue with adding the next layer of parallelograms as in Panel 4 (green). Finally complete the construction with light-red parallelograms as in panel 5.

Now the unexpected happens: interpret the areas of the three red tiles at the center as curvatures of three disks, here 2, 3, and 6 (Figure 2). The curvature of the fourth disk, the solution to Descartes problem, can be read off from the above tessellation as the sum of the areas of three red and two green tiles, here $2 + 3 + 2 \times 6 = 23$. One may easily check that the Descartes formula holds:

$$2(2^2 + 3^2 + 6^2 + 23^2) = (2 + 3 + 6 + 23)^2$$

The curvature of the fourth disk may be read off from the picture in a few ways, for instance as the area of any of the three “butterflies” shown in Figure 3.

Remark 1: The example above deals with integral vectors, which makes the task of finding the areas easy. In particular, one can use Pickâ€™s Theorem [12].
Figure 3: Three butterflies (the area of either equals to the curvature of the fourth disk)

Remark 2: We witness here a non-intuitive duality. Since area is measured in $cm^2$ while curvatures in $cm^1$, the correspondence interchanges these units: $cm^2 \leftrightarrow cm^1$.

Observations: The following geometric facts are visible in the tessellation:

1. All green pieces have the same area (here equal to 6).
2. Each shape among green pieces appears twice.
3. The three exterior light-red pieces are congruent to the three red central pieces.
4. Yellow squares have area equal to the sum of the adjacent red tiles.
5. The sum of a square and a red tile touching it at a vertex is the same for each square (here 11)

There is more information on the Descartes configuration hidden in the tessellation. A **mid-circle** $(ABC)$ is defined as the unique circle that passes through the tangency points of a given 3 mutually tangent circles $A$, $B$, and $C$.

1. The mid-circle of the three main disks (here 2, 3, 6) equals to the area the green tile (here 6).
2. The other mid-circles shown in Figure 4 have curvatures equal the area of a yellow tile plus the green, correspondingly. (Here $5 + 6 = 11$, $8 + 6 = 14$, and $9 + 6 = 15$)

Figure 4: Mid-circles: $5+6=11$, $8+6=14$, $9+6=15$.

The mid-circles relate to symmetries of the original Descartes configuration and its completion to the Apollonian Disk packing. For instance the circles 2, 3, 6 are invariant under inversions through circle 6. The image of the smallest disk 23 is the external disk of curvature (-1).

There is, of course, the other solution of the Descartes problem. It may also be easily read off the tessellation, and the corresponding mid-circles as well, namely as a difference of the three red minus two green. Or, equivalently, the sum of yellow and red that share point, minus two green. (Here it is is $(-1)$, indicating an unbounded disk.)
The next two figures are just a few more examples of tiling, drawn with the help of Cinderella [7]. In general, any starting vectors are admissible, there is a slight problem that for some choices of the initial vectors, some tiles will have negative areas and therefore will overlap with other tiles.

Figure 5: The fourth circle $D' = -1$ and the mid-circles: $5 - 6 = -1$, $8 - 6 = 2$, $9 - 6 = 3$ (numbers in red font).

Figure 6: Another example of tiling and the corresponding Descartes configuration: $(2 + 3 + 3 + 15)^2 = 2(2^2 + 2^2 + 3^2 + 15^2)$.

Figure 7: Additional examples: $(3, 14, 6, 47)$ and $(11, 14, 23, 102)$.
2 Spinor space for tangent disks

In his section we review briefly the notion of spinor description of disks in configurations. For details, proofs, and motivation see [5].

2.1 Spinor space

First, general notions. In the following, by the spinor space we will mean a two dimensional real vector space, which may be also emulated with 1-dimensional complex space (Argand plane), \( \mathbb{T} = \mathbb{R}^2 \cong \mathbb{C} \).

Typical vectors:

\[
\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]

The space is equipped with two structures, the “dot product” and the “cross-product”, both with values in real numbers:

inner product: \( \mathbf{a}, \mathbf{b} \mapsto \mathbf{a} \cdot \mathbf{b} = xx' + yy' \)

symplectic product: \( \mathbf{a}, \mathbf{b} \mapsto \mathbf{a} \times \mathbf{b} = xy' - x'y \)

We also define “symplectic conjugation”

\[
\mathbf{a}^* = \begin{bmatrix} x \\ y \end{bmatrix}^* = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

The two structures are related

\[
\mathbf{a} \times \mathbf{b} = \mathbf{a}^* \cdot \mathbf{b}
\]

There are other identities that are readily implied, e.g.,

\[
(a^*)^* = -a, \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}^*, \quad \mathbf{a}^* \cdot \mathbf{b}^* = \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a}^* \times \mathbf{b}^* = \mathbf{a} \times \mathbf{b}, \quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}.
\]

The squares are:

\[
\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 = x^2 + y^2, \quad \mathbf{a} \times \mathbf{a} = 0, \quad \mathbf{a} \cdot \mathbf{a}^* = 0.
\]

**Interpretation via complex numbers.** When the spinor space \( \mathbb{T} \) is represented by complex numbers, the above formulas obtain the following forms. If

\[
\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow a = x + yi \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} x' \\ y' \end{bmatrix} \leftrightarrow b = x' + y'i
\]

then the structures are expressed as follows:

inner product: \( a, b \mapsto a \cdot b = \frac{1}{2}(\bar{a}b + a\bar{b}) \)

symplectic product: \( a, b \mapsto a \times b = \frac{1}{2i}(\bar{a}b - a\bar{b}) \)

conjugation: \( a \mapsto a^* = ai \)

Note that by “conjugation” we mean “symplectic conjugation” (denoted by star). Not to be confused with “complex conjugation”, always called by its full name. The two descriptions, vector and complex, will be used interchangeably.
2.2 Spinors and Descartes

In this section we review the basic facts concerning the “spinor structure” of Descartes configurations. The idea of tangency spinor was introduced in [4]. For details see [5].

Definition: Let $A$ and $B$ be an ordered pair of mutually tangent disks of radii $r_A$ and $r_B$ and centered at $C_A$ and $C_B$, respectively, in a plane identified with complex numbers, $\mathbb{C} \equiv \mathbb{R}^2$. Interpret the vector joining the centers as a complex number $z = (C_A C_B)$. The tangency spinor of the two disks is a square root defined

$$u = \pm \sqrt{\frac{z}{r_A r_B}} \in \mathbb{C} \equiv \mathbb{T}$$

and

$$u = \begin{bmatrix} \text{Re } u \\ \text{Im } u \end{bmatrix}$$

We shall view it as a vector of the spinor space $\mathbb{T}$ discussed in the previous subsection. The spinor is defined up to a sign since $(-u)^2 = u^2$. It depends on the order of disks: if $u$ is a spinor for $(AB)$, then the spinor for $(BA)$ is $u^* = iu$ (again, up to sign).

The geometric interpretation and motivation follows. Every disk in the Cartesian plane may be given a symbol, a fraction-like label that encodes the size and position of the disk: the curvature is indicated in the denominator while the positions of the centers may be read off by interpreting the symbol as a pair of fractions [3].

$$\text{symbol: } \frac{x, \bar{y}}{\beta} \implies \text{radius: } r = \frac{1}{\beta} \text{ center: } (x, y) = \left( \frac{x}{\beta}, \frac{y}{\beta} \right)$$

The numerator, called the reduced coordinates of the disk’s center, is denoted by dotted letters $(\hat{x}, \hat{y}) = (x/r, y/r)$. Unbounded disks extending outside a circle are given negative radius and curvature. Two tangent disks in a plane define a triangle with sides as follows:

$$\frac{x_1, \bar{y}_1}{\beta_1} \bowtie \frac{x_2, \bar{y}_2}{\beta_2} \implies \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \beta_1 x_2 - \beta_2 x_1 \\ \beta_1 y_2 - \beta_2 y_1 \\ \beta_1 + \beta_2 \end{bmatrix}$$

(2)

where $a^2 + b^2 = c^2$ (see Figure 8). The actual size of the triangle in the plane is scaled down by the factor of $\beta_1 \beta_2$ (gray triangles in Figure 8). The symbols in some disk packing are integral, then so are the triples $(a, b, c)$. Recall that Pythagorean triangles admit Euclidean parameters that determine them via the following prescription:

$$u = \begin{bmatrix} m \\ n \end{bmatrix} \rightarrow (a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$$

(see, e.g., [13, 15]). As explained in [5], Euclidean parameters can be viewed as a spinor, a vector of $\mathbb{T} \equiv u \in \mathbb{R}^2$. Equivalently, viewing the spinor as a complex number $u \in \mathbb{C} \equiv \mathbb{R}^2$ the above relations is defined by squaring:

$$u = m + ni \rightarrow u^2 = a + bi = (m^2 - n^2) + 2mn i$$

with $c = |u^2| = m^2 + n^2$. We extend this map to arbitrary oriented triangles, not necessarily integer.
The emergence of the **tangency spinor** for a pair of tangent disks is summarized in Figure 8. In graphical representation we shall mark a spinor by an arrow that indicates the order of circles, and will label it by its matrix value.

\[ u = \pm (3 + 2i) = \pm \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]

radius \( r = 1/11 \)
actual \( x = 8/11 \)
actual \( y = 6/11 \)

\[ u = \pm \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]

radius \( r = 1/2 \)
actual \( x = 1/2 \)
actual \( y = 0/2 \)

**Actual triangle**

\[ 5,12,13 \]

\[ u = [3,2]^T \]

\[ (a,b,c) \]

**Figure 8:** From two tangent circles to a spinor (not to scale)

Below, we state the main properties; for proofs see [5]. The capital letters will denote both circles and their curvatures.

**Proposition 1.** If \( u \) is the tangency spinor for two tangent disks of curvatures \( A \) and \( B \), respectively, (Figure 9, left) then

\[ |u|^2 = A + B \quad (3) \]

**Theorem 2 [curvatures from spinors]** In the system of three mutually tangent circles, the symplectic product of two spinors directed outward from (respectively inward into) one of the circles equals (up to sign) its curvature, e.g., following notation of Figure 9, center:

\[ \pm C = a \times b = \det[a|b] \quad (4) \]

\[ |u|^2 = A + B \]

\[ a \times b = \pm C \]

\[ a \cdot b = T \]

**Figure 9:** Spinors and disks.
Theorem 3 [mid-circles from spinors]. In the system of three mutually tangent circles, the dot product of two spinors directed outward (respectively, inward) from one of the circles equals to the curvature of the mid-circle $(ABC)$. Following the notation of Figure 9 right:

$$T = a \cdot b \quad \text{(respectively, } T = -a \cdot b)$$  \hspace{1cm} (5)

In particular, $a \cdot b = -a^* \cdot b^*$.

![Figure 10: Spinor fundamental theorem](image)

Theorem 4 [spinor curl]. The signs of the three tangency spinors between be three mutually tangent circles (Figure 10 left) may be chosen so that

$$a + b + c = 0 \quad \text{["curl } u = 0"\text{]}$$  \hspace{1cm} (6)

The label in the above equation is to be understood figuratively meaning, and “$u$” in it stands for the triple of spinor $(a, b, c)$.

Theorem 5. Let $A$, $B$, $C$, and $D$ be four circles in a Descartes configuration.

[A. Vanishing divergence]: If $a$, $b$ and $c$ are tangency spinors for pairs $AD$, $BD$ and $CD$ (see Figure 10 center), then their signs may be chosen so that

$$a + b + c = 0 \quad \text{["div } u = 0"\text{]}$$  \hspace{1cm} (7)

The same property holds for the outward oriented spinors.

[B. Additivity]: If $a$ and $b$ are spinors of tangency for pairs $CA$ and $CB$ (see Figure 10 right), then there is a choice of signs so that the sum

$$c = a + b$$  \hspace{1cm} (8)

is the tangency spinor of $CD$. 
3 Proof of the correspondence

Here is the theorem that summarizes the observations of Section 1.

![Figure 11: Distribution of spinors in Descartes configuration (A,B,C,D) and (A,B,C,D')](image)

**Theorem 6:** Suppose $A$, $B$, $C$, and $D$ are disks in Descartes configuration. Let $a$, $b$, $c$ are tangency spinors joining $A$, $B$, and $C$, as shown in Figure 11. Then the “three-spinor tessellation” built on these spinors determines curvatures of disks and circles as follows:

1. The curvatures $A$, $B$ and $C$ are equal to the areas of the three red tiles. The fourth inscribed disk (solution to Descartes problem) has curvature equal to the butterfly (see Fig. 3):

   \[
   \text{Curv}(D) = \text{Area}(3 \text{ red } + \text{ two green}) \\
   = \text{Area}(\text{yellow } + \text{ aligned red } + \text{ two green}) .
   \]

   Equivalently, one may consider the ring of the three red (one of them light red) and two green tiles.

2. All green tiles have the same area, equal to the curvature of the mid-circle ($ABC$) that passes through the points of tangency of disks $A,B,C$. This, and the other mid-circles have curvatures:

   \[
   \begin{align*}
   \text{Curv}(ABC) &= \text{Area}(\text{any green}) \\
   \text{Curv}(ABD) &= \text{Area}(\text{yellow}(A + B) + \text{ green}) \\
   \text{Curv}(BCD) &= \text{Area}(\text{yellow}(B + C) + \text{ green}) \\
   \text{Curv}(CAD) &= \text{Area}(\text{yellow}(C + A) + \text{ green})
   \end{align*}
   \]

3. The second solution to the Descartes problem for disks $A$, $B$, and $C$, is the disk of curvature

   \[
   \begin{align*}
   \text{Curv}(D') &= \text{Area}(3 \text{ red } - \text{ two green}) \\
   &= \text{Area}(\text{yellow } + \text{ aligned red } - \text{ two green}).
   \end{align*}
   \]
Figure 12: Left: Areas of tiles. Right: Corresponding disk curvatures.

The mid-circles for this Descartes configuration are

\[
\text{Curv}(ABD') = \text{Area(yellow}(A + B) - \text{green}) \\
\text{Curv}(BCD') = \text{Area(yellow}(B + C) - \text{green}) \\
\text{Curv}(CAD') = \text{Area(yellow}(C + A) - \text{green})
\]

**Proof:** Let us start with three spinors between three tangent disks. Build the network of spinor as shown in Figure 11. The signs of the clockwise oriented spinors \(a\), \(b\), and \(c\) are chosen so that

\[a + b + c = 0\]

Figure 12 shows the tessellation based on these spinors.

1. The red parallelograms are the curvatures of the Descartes by the virtue of (4):

\[A = b \times c^*, \quad B = c \times a^*, \quad C = a \times b^*\]

The areas of yellow squares that represent the squares of the the spinors:

\[|a|^2 = B + C, \quad |b|^2 = C + A, \quad |c|^2 = A + B,\]

This is consistent with

\[|a|^2 = a \times a^* = (-b - c) \cdot a^* = -b \times a^* - c \times a^* = a^* \times b + c^* \times a = B + C.\]

Using (4) again, one finds the curvature of the fourth inscribed circle:

\[D = (a + c^*) \times (b + a^*) = a \times b + a \times a^* + c^* \times b + c^* \times a^*\]

which indeed may be composed into a butterfly shape as in Figure 3.
2. All green tiles have the same area:

\[ a \times b = b \times c = c \times a = \text{curvature of the mid-circle}(A, B, C) \]

which follows from

\[ a \times b = a \times (-c - a) = -a \times c - a \times a = c \times a \]

and from \( a \times b = a^* \times b^* \). Using (5) for two spinors into \( D \) we obtain the curvature of the mid-circle \((ABC)\):

\[ b^* \cdot c = b \times c \]

which is one of the green tiles (see Figure 12). As to the other three other mid-circles, one of them is

\[ (ABD) = (c + b^*) \cdot c = c \cdot c + b^* \cdot c = |c|^2 + b \times c \]

which may be recognized as a yellow tile plus green.

3. As to the other solution for the Descartes problem, we use again (4) for two spinors into \( D' \):

\[
(a + b^*) \times (c + a^*) = a \times c + a^* \times a + b^* \times c + b^* \times a^*
\]

\[ = |a|^2 + b^* \times c - a \times b - c \times a \]

which is a yellow tile plus a red tile minus two green tiles. The symmetry mid-circles for the Descartes configuration \((ABCD')\) can be obtained similarly. For instance

\[ (BCD') = -(a + b^*) \cdot (c + a^*) = -a \cdot c - b^* \cdot c - b^* \cdot a^* \]

\[ = a \cdot (a + b) - b \times c - a \cdot b \]

\[ = |a|^2 - b \times c \]

which a difference between a yellow and a green tile. □

An obvious corollary to this is that if spinors are integral, so is the Descartes configuration.

4 Coda: Integral Descartes configurations from spinors

Stripping the content of the above construction from its geometric interpretation, we get effectively a method of obtaining integral Descartes configurations (and consequently integral Apollonian disk packings) from four arbitrary integers. The core is the curl property (6):

\[ a + b + c = 0 \] (9)

The implied curvatures of the disks in the Descartes configuration, \( A, B, C \), and the two alternative curvatures of the fourth, \( D_1 \) and \( D_2 \), are:

\[
\begin{align*}
A &= -b \cdot c \\
B &= -c \cdot a \\
C &= -a \cdot b \\
D_1 + D_2 &= |a|^2 + |b|^2 + |c|^2 \\
D_1 - D_2 &= 4a \times b = 4b \times c = 4c \times a
\end{align*}
\] (10)

Reducing the above to two spinors (4 integers) with (9) leads to the claim.
Proposition 7: Let \( a \) and \( b \) be arbitrary integral vectors in \( \mathbb{Z}^2 \). Then the following integers satisfy the Descartes Diophantine equation (1):

\[
\begin{align*}
A &= |b|^2 + a \cdot b \\
B &= |a|^2 + a \cdot b \\
C &= -a \cdot b \\
D &= |a|^2 + |b|^2 + a \cdot b \pm 2a \times b
\end{align*}
\]

(11)

A different parametrization of Descartes configurations from the spinors based on the divergence property (7) will be presented in a separate paper.

Note on software: Most of the figures were made with Tikz [16] except Figure 6–8, which were made with Cinderella [7].

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