Closed string tachyon driving $f(R)$ cosmology

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Abstract. To study quantum effects on the bulk tachyon dynamics, we replace $R$ with $f(R)$ in the low-energy effective action that couples gravity, the dilaton, and the bulk closed string tachyon of bosonic closed string theory and study properties of their classical solutions. The $\alpha'$ corrections of the graviton-dilaton-tachyon system are implemented in the $f(R)$. We obtain the tachyon-induced rolling solutions and show that the string metric does not need to remain fixed in some cases. In the case with $H(t = -\infty) = 0$, only the $R$ and $R^2$ terms in $f(R)$ play a role in obtaining the rolling solutions with nontrivial metric. The singular behavior of more classical solutions are investigated and found to be modified by quantum effects. In particular, there could exist some classical solutions, in which the tachyon field rolls down from a maximum of the tachyon potential while the dilaton expectation value is always bounded from above during the rolling process.

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1 Introduction

In the standard cosmological model, the Friedmann equations are derived from Hilbert action, coupling the metric with ad hoc matter sources. In string theory, a low energy effective action arises from the requirement of quantum conformal invariance. The tree level of this action couples the metric, the dilaton and the axion. This action, implying the dynamics of gravitational field and other background fields, has been used in many areas. One of the most important applications is to develop a string cosmology [1, 2].

There exist two ways to improve this string effective action [3, 4]. The first one is $\alpha'$-controlled expansion, which includes higher derivatives of the metric and background fields. Another one is string coupling $g_s$-controlled expansion (higher-genus expansion) and which reflects the higher loop string interactions. The $\alpha'$-controlled expansion, which becomes significant in Planck and high curvature region, had been discussed in [5, 6]. When curvature and energy scale grow up, the usual simplest perturbative expansion of the string effective action becomes no-go theorem. In this condition, the higher order expansions should be taken into account. A good review of string cosmology is given by [7] and references therein.

The instabilities associated with open string tachyon is well understood presently. The stable vacuum is the vacuum of closed string without open string excitations and D-branes. The open string tachyon rolls down from the perturbative unstable vacuum to an excited state of closed string carrying the energy of the D-brane. On the other hand, the instabilities associated with closed string tachyon is a much more difficult problem since the action is non-polynomial. Some progresses have been achieved in the calculation of the effective potential [8–11] by truncating the polynomial to quintic order. Those calculations indicate that there do exist a local minimum and several saddle points in the closed string tachyon and
dilaton effective potential. The conjecture that the action has to vanish at the closed string field theory (CSFT) vacuum, made in [8, 12], is also implied. At the closed string vacuum, parallel to the conclusions in open string theory, one is led to believe that spacetime itself ceases to exist. To make these conjectures more reliable, higher order calculations or more desirable analytical methods are necessary.

For these reasons, studying the string effective action that couples the metric, the dilaton and the tachyon is of importance. One can expect that this action will reveal some features of string theory and related problems. Some progresses have been made in recent developments. The author of [13] discussed the solutions of the effective action with tachyon and B-field and analyzed the solutions in $AdS_3$ background. In [14], Brandenberger et al. found a nonsingular and static tachyon condensation by discussing an effective theory with a non-vanishing dilaton potential. The quantum effects of graviton-dilaton-tachyon system is investigated in [15] where it shows that the singular behavior of classical solutions should be modified by quantum effects. The author of [16] uses some constraints to fix the form of action without computing the beta functions. Some other progresses of graviton-dilaton-tachyon system refer to [17–27].

The aim of this paper is an extension of [8, 12] to study the $\alpha'$ corrections of the graviton-dilaton-tachyon system. The $\alpha'$ correction should also satisfy the quantum conformal invariance and then brings higher derivative terms of gravitational field as well as other background fields. To preserve the covariance and gauge invariance, the same expansion order of $\alpha'$ could lead to different effective actions [7, 28]. To simplify the story, we only implement corrections to the graviton by replacing $R$ with $f(R)$. That is, after the low-energy action is obtained by integrating out other massive fields in the action, we discard the higher-genus expansions and the higher-derivative expansions for the dilaton and the tachyon. In our toy model, we only consider one kind of $\alpha'$ corrections to the graviton, which only depend on the Ricci scalar $R$. Since different higher order terms have a widely different effect in cosmology, one should be aware that our results might be not truly perturbative in the sense that they might be completely changed by considering, say, $R_{\mu\nu}R^{\mu\nu}$ terms.\footnote{We thank the anonymous referee for this comment.}

In [12], the rolling process triggered by the tachyon was investigated for tachyonic potentials by considering only the leading terms of the graviton-dilaton-tachyon system. It was found that during the rolling process, the string metric did not evolve while the string coupling always became infinity at an finite time in the string frame, which is consistent with the conjecture of the closed string tachyon condensation. In the framework of $f(R)$ gravity, our analysis shows some interesting results:

1. In section 3, we find for tachyon-induced rolling solutions that time evolution of the string metric could be nontrivial in some cases. Specially in the case with $H (t = -\infty) = 0$, only the $R$ and $R^2$ terms in $f(R)$ play a role in obtaining the rolling solutions with nontrivial metric while the higher order ones do not.

2. In section 4, assuming the Hubble parameter of the string metric is constant, we obtain a set of close string tachyon rolling solutions, in which the tachyon field rolls down from the top of some tachyon potential while the string coupling is always finite during the rolling process.

This paper is organized as follows. In section 2, we derive the equations of motion of graviton, dilaton, and tachyon field in $f(R)$ theories. In section 3, we define tachyon-induced
rolling solutions and solve the equations of motion for them. More classical solutions are discussed in section 4. The section 5 is our conclusion.

2 Coupled system of fields

The low energy effective action for the metric, the dilaton, and the tachyon is given by

\[ S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} e^{-2\Phi} \left[ f(R) + 4(\nabla\Phi)^2 - (\nabla T)^2 - 2V(T) \right], \]  

(2.1)

where \( g_{\mu\nu} \) is string metric, \( \Phi \) is dilaton, \( T \) is tachyon with potential \( V(T) \), and \( f(R) \) is the generalized gravity. The number of spatial dimensions is \( d \). Since here we focus on the tachyon driving solutions, the dilaton potential is set to zero for simplicity. Varying the action (2.1) with respect to \( g_{\mu\nu} \), \( \Phi \), and \( T \), we find that the equations of motion for graviton, dilaton and tachyon are

\[
F(R) R_{\mu\nu} - \nabla_\mu T \nabla_\nu T + 4g_{\mu\nu} [f(R) - 1] (\nabla \Phi)^2 - 4g_{\mu\nu} \nabla^a \Phi \nabla_a F(R) \\
- 2g_{\mu\nu} [f(R) - 1] \nabla^2 \Phi + g_{\mu\nu} \nabla^2 F(R) + 2F(R) \nabla_\mu \nabla_\nu \Phi \\
- \nabla_\mu F(R) - 4 [f(R) - 1] \nabla_\mu \Phi \nabla_\nu \Phi + 2 \nabla_\mu \Phi \nabla_\nu F(R) + 2 \nabla_\nu \Phi \nabla_\mu F(R) = 0, 
\]

(2.2)

\[
\frac{1}{2} [f(R) R - f(R)] + 2 \{d[f(R) - 1] + 1\} (\nabla \Phi)^2 - 2d \nabla^a \Phi \nabla_a F(R) \\
- \{d[f(R) - 1] + 1\} \nabla^2 \Phi + \frac{1}{2} d \nabla^2 F(R) + V(T) = 0, 
\]

(2.3)

\[
\nabla^2 T - 2 \nabla^a \Phi \nabla_a T - V'(T) = 0, 
\]

(2.4)

where \( F(R) \equiv df(R)/dR \). As in [12], we make the following ansatz

\[
ds^2 = -dt^2 + a(t)^2 \delta_{ij}dx^i dx^j, \\
\Phi = \Phi(t), \\
T = T(t). 
\]

(2.5)

For the string metric in eq. (2.5), the Ricci scalar is

\[ R = 2d \dot{H} + d(d + 1) H^2, \]

(2.6)

where \( H = \frac{\dot{a}}{a} \). In this case, the 00 and \( ij \) components of gravitational equations become

\[
F(R) \left( dH + dH^2 \right) + \dot{T}^2 - 2 \dot{\Phi} + \left\{ 2[f(R) - 1] \dot{\Phi} - \ddot{F}(R) \right\} dH = 0, 
\]

(2.7)

\[
F(R) \left( \dot{H} + dH^2 \right) - 4 [f(R) - 1] \dot{\Phi}^2 + 4 \dot{\Phi} \ddot{F}(R) + 2 [f(R) - 1] \dot{\Phi} \\
+ 2 [f(R) (d - 1) - d] H \dot{\Phi} - \ddot{F}(R) - (d - 1) H \ddot{F}(R) = 0, 
\]

(2.8)

which can be rearranged into two equivalent equations

\[
\frac{1}{2} F(R) (d - 1) \dot{H} + \frac{1}{2} \dot{T}^2 + 2 [f(R) - 1] \dot{\Phi}^2 - 2 \dot{\Phi} \ddot{F}(R) \\
- F(R) \ddot{\Phi} + F(R) H \dot{\Phi} + \frac{1}{2} \dddot{F}(R) - \frac{1}{2} H \dddot{F}(R) = 0, 
\]

(2.9)

\[
\frac{1}{2} F(R) (d - 1) dH^2 - 2d [f(R) - 1] \dot{\Phi}^2 + 2d \dot{\Phi} \ddot{F}(R) + \{d [f(R) - 1] + 1\} \dot{\Phi} \\
+ \left\{ d [f(R) - 1] - 2F(R) + 1 \right\} \dot{\Phi} \frac{1}{2} (d - 2) \dddot{F}(R) \right\} dH - \frac{1}{2} d \dddot{F}(R) - \frac{1}{2} \dot{T}^2 = 0. 
\]

(2.10)
The equations of motion for the dilaton and the tachyon are
\[
\frac{1}{2} [F(R) R - f(R)] - 2 [d (F(R) - 1) + 1] \dot{\Phi}^2 + 2 d \dot{\Phi} \dot{F}(R) \\
+ \{d [F(R) - 1] + 1\} \left( \ddot{\Phi} + d H \dot{\Phi} \right) - \frac{1}{2} d \left[ \ddot{F}(R) + d H \dot{F}(R) \right] + V(T) = 0, \tag{2.11}
\]
\[
\ddot{T} + (d H - 2 \dot{\Phi}) \dot{T} + V'(T) = 0. \tag{2.12}
\]

Note that there are four differential equations for three unknown dynamical variables: \(a(t)\), \(\Phi(t)\), and \(T(t)\). Therefore, one of the four equations should be redundant. In fact, it shows in the appendix that eqs. (2.7), (2.8), and (2.12) could guarantee that eq. (2.11) holds whenever \(\dot{T} \neq 0\). It is noteworthy that the redundance of cosmological equations is a common property of all the Einstein equations in FLRW metrics, and hence our result is expected.

3 Tachyon-driven rolling solutions

In this section, we consider a general class of potentials \(V(T)\) for a tachyon \(T\) that has a local maximum at \(T = 0\), which can be written as
\[
V(T) = V(0) - \frac{1}{2} m^2 T^2 + \mathcal{O}(T^3). \tag{3.1}
\]

In [12] where \(f(R) = R\), the rolling solutions driven by the tachyon were discussed. The ansatzes for \(T(t)\) and \(\Phi(t)\) were assumed to be
\[
T(t) = e^{mt} + \sum_{n \geq 2} t_n e^{mnt}, \\
\Phi(t) = \sum_{n \geq 2} \phi_n e^{mnt}, \tag{3.2}
\]
\[
H(t) = \sum_{n \geq 2} h_n e^{m_\gamma t},
\]
where \(T \to 0\) for \(t \to -\infty\), and \(\Phi(t)\) has exponentials subleading to \(e^{mt}\) since the tachyon drives the rolling in the very early time. For the case with \(f(R) = R\), eq. (2.10) becomes
\[
\frac{1}{2} (d - 1) d H^2 = \frac{1}{2} \ddot{T}^2 - \ddot{\Phi} + d H \dot{\Phi}, \tag{3.3}
\]
which gives that \(H \to 0\) when \(t \to -\infty\), which is consistent with the ansatz for \(H(t)\) proposed in eq. (3.2). In fact, it showed in [12] that \(H(t)\) vanished identically for the tachyon-driven rolling solutions.

For the case with a general form of \(f(R)\), we build an analogous tachyon-driven rolling solution:
\[
T(t) = e^{\gamma t} + \sum_{n \geq 2} t_n e^{\gamma n t}, \\
\Phi(t) = \sum_{n \geq 2} \phi_n e^{\gamma n t}, \tag{3.4}
\]
where $\gamma$ is a positive real number, and the first term in $T(t)$ is the solution to the linearized tachyon equation of motion. The arbitrary constant multiplying the first term in $T(t)$ can be absorbed, as we did, by a redefinition of time. Unlike in the case with $f(R) = R$, eq. (2.10) fails to require that $H \to 0$ when $t \to -\infty$. There might exist possible solutions with $H(t)$ which does not go to zero when $t \to -\infty$. Thus, one might consider a more general ansatz for $H(t)$:

$$H(t) = H_0 + \sum_{n \geq 1} h_n e^{n\gamma t}. \quad (3.5)$$

If the rolling process is solely triggered by the tachyon field, one needs that $H_0 = h_1 = 0$. As we show below, for such ansatz nontrivial $H(t)$ could exist in some cases. However to explore more possibilities, we now do not require $H_0 = h_1 = 0$. Plugging $T(t)$ and $\Phi(t)$ from eqs. (3.4) into the tachyon equation (2.12), we find

$$\gamma^2 + dH_0 \gamma - m^2 = 0, \quad (3.6)$$

which gives

$$\gamma = m \left( \sqrt{1 + \frac{d^2 H_0^2}{4m^2} - \frac{4H_0}{2m}} \right).$$

The corresponding Ricci scalar is

$$R = R_0 + \sum_{n \geq 1} r_n e^{n\gamma t}, \quad (3.7)$$

where $R_0 = d(d+1)H_0^2$ and $r_n$ depends on $h_m \leq n$. We assume $f(R)$ is analytic along the real axis except certain poles.

In the rest of this section, we will use eqs. (2.7), (2.8) and (2.12) to solve for $R_0$, $h_n$, $\phi_n$ and $t_n$. And eq. (2.11) could be used to determine the value of $V(0)$. Note that in [12] where $f(R) = R$, it showed that $V(0) = 0$, $R_0 = 0$ and $h_n = 0$.

### 3.1 Analytic at $R_0$

Now $f(R)$ is assumed to be analytic at $R_0$. Thus, we can expand $f(R)$ at $R = R_0$:

$$f(R) = f(R_0) + \sum_{l \geq 1} \alpha_l (R - R_0)^l. \quad (3.8)$$

Eq. (3.7) gives $F(R) = F(R_0) + O(e^{\gamma t})$. Thus, the leading terms of the 00 and $ij$ components of gravitational equations (2.7) and (2.8) both become $F(R_0) \, dH_0^2 = 0$. Given $R_0 = d(d+1)H_0^2$, one finds that either $F(R_0) = 0$ or $R_0 = 0$. The tachyon equation (2.12) is trivial at the leading order. The leading order of the dilation equation (2.11) gives

$$V(0) = -\frac{f(R_0)}{2}, \quad (3.9)$$

where we use $F(R_0) R_0 = 0$. Note that eq. (3.9) must be satisfied to admit the tachyon rolling solution. For example, if $f(R) = R - \Lambda$ where $\Lambda$ is the cosmological constant and $R_0 = 0$, one has $V(0) = \frac{\Lambda}{2}$. In what follows, we will solve for $h_n$, $\phi_n$ and $t_n$ in the two cases with $R_0 = 0$ and $F(R_0) = 0$. 

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3.1.1 \( R_0 = 0 \)

If \( f(R) \) is analytic at \( R = 0 \), \( F(R) = 1 + \sum_{l\geq 2} l\alpha_l R^{l-1} \). Plugging eqs. (3.4) and (3.5) with \( H_0 = 0 \) into the equations of motion, one could determine the coefficients \( t_n, \phi_n \) and \( h_n \) order by order. Note that \( \gamma = m \) for \( H_0 = 0 \).

At \( O(e^{nt}) \), eqs. (2.7) and (2.8) give

\[
dm_1 e^m = 0, \\
(1 - 4dm^2 \alpha_2) mh_1 e^m = 0, \tag{3.10}
\]

which leads to \( h_1 = 0 \). Tachyon equation is also trivial at this order. At \( O(e^{2nt}) \), eqs. (2.7) and (2.8) give

\[
(m^2 + 2dmh_2 - 8m^2 \phi_2) e^{2mt} = 0, \\
2m (1 - 16dm^2 \alpha_2) h_2 e^{2mt} = 0, \tag{3.11}
\]

which yield

\[
\phi_2 = \frac{1}{8} + \frac{dh_2}{4m}, \\
h_2 = 0 \text{ or } \alpha_2 = \frac{1}{16dm^2}. \tag{3.12}
\]

Note that we have \( h_2 = 0 \) if \( \alpha_2 \neq \frac{1}{16dm^2} \). Now we prove by induction that \( H(t) \) vanishes identically if \( \alpha_2 \neq \frac{1}{4dN^2 m^2} \) for \( N = 2, 3 \cdots \). First assume that \( h_2 = h_3 = \cdots = h_N = 0 \). Since \( \dot{\Phi}(t) \sim e^{2mt} \), eq. (2.8) gives

\[
(N + 1) \left[ 1 - 4d\alpha_2 (N + 1)^2 m^2 \right] mh_{N+1} e^{(N+1)mt} + O(e^{(N+2)mt}) = 0, \tag{3.13}
\]

which yields \( 1 - 4d\alpha_2 (N + 1)^2 m^2 \) \( h_{N+1} = 0 \). Therefore, if \( \alpha_2 \neq \frac{1}{4dN^2 m^2} \) for \( N = 2, 3 \cdots \), \( h_{N+1} = 0 \).

On the other hand, if \( \alpha_2 = \frac{1}{4dN^2 m^2} \) for some positive integer \( N > 1 \), \( h_N \) could be nonzero. Therefore, there might be solutions with nonzero \( H(t) \) in such cases. Consider an example with

\[
f(R) = R + \frac{R^2}{16dm^2} + \sum_{l\geq 3} \alpha_l R^l \text{ and } V(T) = -\frac{1}{2} m^2 T^2,
\]

we find solutions up to \( O(e^{4mt}) \)

\[
T(t) = e^{mt} + \frac{e^{3mt}}{16} + O(e^{4mt}), \\
\Phi(t) = \left( \frac{1}{8} + \frac{dh}{4m} \right) e^{2mt} + O(e^{4mt}), \tag{3.14}
\]

\[
H(t) = he^{2mt} + O(e^{4mt}),
\]

where \( h \) is a free parameter. As far as the tachyon rolling ansatz is concerned, the parameter \( h \) could be arbitrary.
3.1.2 \( F (R_0) = 0 \)

Suppose \( F (R) \) is analytic at \( R_0 \) where \( F (R_0) = 0 \). Then, one can expand \( F (R) \) at \( R_0 \):

\[
F (R) = \sum_{l \geq 1} \beta_l (R - R_0)^l, \tag{3.15}
\]

where we assume \( \beta_1 \neq 0 \) for simplicity. Plugging eqs. (3.4) and (3.5) into eqs. (2.7), (2.8), and (2.12) gives the recurrence relations for \( h_n, \phi_n, \) and \( t_n \) with necessary initial values. In fact, after putting the ansatzes into eqs. (2.7), (2.8), and (2.12), one has at \( \mathcal{O} (e^{n \gamma t}) \) that

\[
\begin{align*}
2n \gamma \left[ (n \gamma + dH_0) \phi_n - d^2 H_0^2 \beta_1 h_n \right] &= G_n (h_{i<n}, \phi_{i<n}, t_{i<n}), \\
2n \gamma (n \gamma + dH_0) \phi_n + 2n \gamma d \left[ (n^2 \gamma^2 - H_0^2) + (d-1) H_0 n \gamma \right] \beta_1 h_n &= F_n (h_{i<n}, \phi_{i<n}, t_{i<n}), \\
\gamma \left[ (n^2 - 1) \gamma + dH_0 (n-1) \right] t_n &= H_n (h_{i<n}, \phi_{i<n}, t_{i<n}),
\end{align*}
\]

where \( G_n, F_n \) and \( H_n \) are functions of only \( h_{i<n}, \phi_{i<n}, \) and \( t_{i<n} \). Solving the above equations for \( h_n, \phi_n, \) and \( t_n \), we find that the recurrence relations for \( h_n, \phi_n, \) and \( t_n \) for \( n \geq 2 \) are

\[
\begin{align*}
\phi_n &= \frac{G_n (h_{i<n}, \phi_{i<n}, t_{i<n}) + dH_0^2 F_n (h_{i<n}, \phi_{i<n}, t_{i<n}) - G_n (h_{i<n}, \phi_{i<n}, t_{i<n})}{2n \gamma (n \gamma + dH_0)}, \\
h_n &= \frac{F_n (h_{i<n}, \phi_{i<n}, t_{i<n}) - G_n (h_{i<n}, \phi_{i<n}, t_{i<n})}{2d n^2 \gamma^2 \beta_1 (n \gamma + (d-1) H_0)}, \\
t_n &= \frac{1}{\gamma (n^2 - 1) \gamma + dH_0 (n-1)} H_n (h_{i<n}, \phi_{i<n}, t_{i<n}).
\end{align*} \tag{3.17}
\]

From eqs. (3.4), one obtains that \( t_1 = 1 \) and \( \phi_1 = 0 \). Since \( \dot{H}, F (R) \sim \mathcal{O} (e^{\gamma t}) \) and \( \dot{T}^2, \Phi \sim \mathcal{O} (e^{2 \gamma t}) \), all terms of eq. (2.9) except \( \frac{1}{2} \ddot{F} (R) \) are at \( \mathcal{O} (e^{2 \gamma t}) \). Given \( \ddot{F} (R) = 2d \beta_1 h_1 \gamma^3 e^{\gamma t} + \mathcal{O} (e^{2 \gamma t}) \), one has \( h_1 = 0 \) since \( \beta_1 \neq 0 \) by assumption. After the initial conditions \( h_1 = 0, \phi_1 = 0, \) and \( t_1 = 1 \) are obtained, one could use the recurrence relations (3.17) to find values of \( h_n, \phi_n, \) and \( t_n \). For example, we have for \( n = 2 \) that

\[
\begin{align*}
\phi_2 &= \frac{1}{8}, \\
h_2 &= \frac{\gamma}{4d \beta_1 [(d+1) H_0^2 - 2dH_0 \gamma - 4 \gamma^2]}, \\
t_2 &= 0,
\end{align*} \tag{3.18}
\]

where \( h_2 \) is generally not zero.

3.2 Pole at \( R_0 \)

We now study the scenario in which \( f (R) \) has a pole of order \( |L| \) at \( R_0 \). Therefore, the Laurent series expansion of \( f (R) \) at \( R_0 \) is

\[
f (R) = \sum_{l \geq L} \alpha_l (R - R_0)^l, \tag{3.19}
\]

where \( \alpha_L \neq 0 \) and \( L \leq -1 \) is some negative integer. We now consider two cases: \( R_0 = 0 \) and \( R_0 \neq 0 \).
3.2.1 \( R_0 = 0 \)

Since \( H_0 = 0 \), the ansatz for \( H(t) \) (3.5) becomes

\[
H(t) = \sum_{n \geq N} h_n e^{nmt}, \tag{3.20}
\]

where \( h_N e^{Nmt} \) is assumed to be the nonzero leading term. If \( H(t) \neq 0 \), one has some integer \( N \geq 1 \) for which \( h_N \neq 0 \). Plugging eq. (3.20) into \( R = 2d \dot{H} + d(d + 1) H^2 \) gives for nonzero integer \( l \) that

\[
R^l = r_N^l e^{Nlmt} + O \left( e^{(NI+1)mt} \right), \tag{3.21}
\]

where \( r_N = 2dN mh_N \). Thus, we have

\[
F(R) = \sum_{l \geq L} l \alpha_l R^{l-1} = Lr_N^{L-1} \alpha_L e^{N(L-1)mt} + O \left( e^{(N(L-1)+1)mt} \right). \tag{3.22}
\]

Given eq. (3.22), the \( ij \) components of gravitational equation (2.8) becomes

\[
-Lr_N^{L-1} \alpha_L N^2 (L - 1)^2 m^2 e^{N(L-1)mt} + O \left( e^{(N(L-1)+1)mt} \right) = 0,
\]

which gives \( h_N e^{N\gamma t} = 0 \). Since \( \alpha_L \neq 0 \), one obtains \( h_N = 0 \) and hence a contradiction, which means \( H(t) = 0 \). However, \( f(R) \) blows up at \( R = 0 \), and hence the tachyon rolling ansatzes (3.4) and (3.5) do not solve the equations of motion (2.7), (2.8) and (2.12) in this case.

3.2.2 \( R_0 \neq 0 \)

The ansatz for \( H(t) \) is

\[
H(t) = H_0 + \sum_{n \geq N} h_n e^{n\gamma t}, \tag{3.23}
\]

where \( h_N e^{N\gamma t} \) is the first nonzero term in the series. Using eq. (3.19), one has

\[
f(R) = \alpha_L r_N^L e^{NL\gamma t} + O \left( e^{(NL+1)\gamma t} \right), \tag{3.24}
\]

where \( r_N = 2d \left[ N \gamma + (d + 1) H_0 \right] h_N \). Therefore, eq. (2.7) gives

\[
[H_0 - N (L - 1) \gamma] dH_0 Lr_N^{L-1} \alpha_L e^{N(L-1)\gamma t} + O \left( e^{(N(L-1)+1)\gamma t} \right) = 0. \tag{3.25}
\]

If \( N \gamma + (d + 1) H_0 \neq 0 \) and \( H_0 - N (L - 1) \gamma \neq 0 \) for any positive integer \( N \), eq. (3.25) gives \( \alpha_L = 0 \), which means \( H(t) = H_0 \). However, \( f(R) \) blows up at \( R = R_0 \), and hence the tachyon rolling ansatz does not solve the equations of motion. If \( N \gamma + (d + 1) H_0 = 0 \) or \( H_0 - N (L - 1) \gamma = 0 \) some \( N \), the coefficient of \( e^{N(L-1)\gamma t} \) in eq. (3.25) is zero without making \( \alpha_L = 0 \). In principle, one could put the ansatz into the equations of motion and solve for \( h_n, \phi_n, \) and \( t_n \). However, one could not obtain the recurrence relations as in the \( F(R_0) = 0 \) case. One might need other means to find values of \( h_n, \phi_n, \) and \( t_n \).
4 Cosmological solutions

In section 3, we considered a special class of solutions to the equations of motion, whose ansatzes are given in eqs. (3.4) and (3.5). For the case with \( f(R) = R \), the string metric remains fixed for such solutions. However for some more general form of \( f(R) \), there might exist tachyon-driven rolling solutions with nonzero \( H(t) \). It appears that the behavior of the classical solutions is richer in the \( f(R) \) theory. In this section, we study the classical solutions beyond the ansatzes (3.4) and (3.5). We are still interested in the form of solutions in eqs. (2.5), in which the metric is the spatially flat FRW metric. The equations of motion then reduce to eqs. (2.7), (2.8), (2.11), and (2.12). Note that if \( V(T) = V(-T) \), the solutions are invariant under the “time-reversal” transformations \( t \to -t \), for which

\[
H \to -H, \; R \to R, \; \Phi \to \Phi, \; T \to -T.
\]

However, the equations of motion might become nonlinear higher order differential equations, which are difficult to solve. To investigate the properties of their solutions, we consider two simple scenarios, the one with constant \( H \), and the other with constant \( T \).

4.1 Constant \( H \)

If we assume \( H(t) = H_0 \) which is a constant, there are three independent equations of motion for \( \Phi(t) \) and \( T(t) \). Therefore, one of these equations determines the form of the tachyon potential \( V(T) \), which depends on \( H_0 \). In other words, only some particular potentials \( V(T) \) admit the classical solutions with \( H(t) = H_0 \). In this case, the \( ij \) components of gravitational equation (2.8) becomes

\[
dF(R_0)H_0^2 - 4[F(R_0) - 1] \dot{\Phi}^2 + 2[F(R_0) - 1] \dddot{\Phi} + 2dH_0[F(R_0) - 1] \dddot{\Phi} - 2H_0 F(R_0) \dot{\Phi} = 0,
\]

where \( R(t) = R_0 = d(d+1)H_0^2 \). To solve this equation, we can introduce the variable \( \tau = dH_0t \). In what follows, the dot denotes \( t \)-derivative, and the prime denotes \( \tau \)-derivative.

If \( F(R_0) = 1 \) and \( H_0 = 0 \), eq. (4.2) becomes trivial and hence there are only two independent equations of motion for \( \Phi(t) \) and \( T(t) \). In this case, \( \Phi(t) \) and \( T(t) \) can be solved for any \( V(T) \), and the properties of these solutions have been discussed in [12]. It has been found that the string coupling always became divergent at some time.

If \( F(R_0) = 1 \) and \( H_0 \neq 0 \), the solution to eq. (4.2) is

\[
\Phi(\tau) = \frac{\tau - \tau_0}{2},
\]

where the integration constant \( \tau_0 \) is a constant time translation and can be set to zero for simplicity from now on. The dilaton \( \Phi \) goes to infinity at \( t = +\infty \). This solution evolves to a singular configuration with a strongly coupled background at infinite string time. The dilaton equation (2.11) gives \( V(T) = \frac{1}{2}[f(R_0) - R_0] \). However, eq. (2.7) becomes

\[
\dot{T}^2 = -dH_0^2 < 0,
\]

which contradicts the tachyon field \( T \) being real. Thus, there are no classical solutions with constant \( H \) in this case.

If \( F(R_0) \neq 1 \), solving eq. (4.2) for \( \Phi \) gives

\[
\Phi(\tau) = \frac{1}{2} \ln \left| \frac{e^\tau}{1 \pm \gamma e^{\lambda \tau}} \right| + C_\Phi,
\]

where \( \gamma \) and \( \lambda \) are determined by the initial conditions.
where $C_\Phi$ is an integration constant, $A = 1 + \frac{F(R_0)}{d[F(R_0) - 1]}$, and $\gamma = |1 - F(R_0)|$. Plugging $\Phi(t)$ into the dilaton equation (2.11), one could solve it for $V(T(\tau))$:

$$V(\tau) = -\frac{F(R_0) R_0 - f(R_0)}{2d^2 R^2} + \text{sgn}(F(R_0) - 1) \frac{\rho A e^{A \tau}}{1 + \gamma e^{A \tau}},$$

(4.6)

where $\rho = \frac{F(R_0)d[F(R_0) - 1]}{2d}$, and $\text{sgn}(x)$ is the sign function with $\text{sgn}(x) = \frac{x}{|x|}$. Plugging $\Phi(t)$ into the gravitational equation (2.7) gives:

$$T^2_\pm = \frac{\pm 1 + \beta e^{A \tau}}{1 + \gamma e^{A \tau}})^2,$$

(4.7)

where $B = d + F(R_0) - 3dF(R_0) + 2dF^2(R_0)$, $\alpha = \frac{F(R_0)(d+1)-d}{2d^2 - F(R_0)}$, and $\beta = \frac{B}{2d^2 - F(R_0)}$. When $F(R_0) = \frac{d}{d+1}$, one has that $\alpha = A = 0$. In this case, $T$ stays constant, and

$$\Phi(\tau) = \frac{\tau}{2} + C_\Phi,$$

(4.8)

which is the linear dilaton solution. When $F(R_0) \neq \frac{d}{d+1}$, we list the sign of $T^2_\pm$ for the possible values of $F(R_0)$ in table 1 for $d < 6$ and table 2 for $d \geq 6$, where we define $F_\pm = \frac{-1 + 3d \pm \sqrt{1 - 6d + d^2}}{4d}$ such that $B = 0$ when $F(R_0) = F_\pm$. Note that $0 < F_- < F_+ < \frac{d}{d+1}$. When $T^2_\pm \geq 0$, we integrate eq. (4.7) and obtain

$$T_\pm(\tau) = \frac{2\sqrt{\alpha}}{A} \text{Re} \left[ \sqrt{\frac{\gamma - \beta}{\gamma}} \text{arctanh} \left( \sqrt{\frac{\gamma}{\gamma - \beta}} \sqrt{\pm 1 + \beta e^{A \tau}} \right) - \text{arctanh} \left( \sqrt{\pm 1 + \beta e^{A \tau}} \right) \right] + C_T,$$

(4.9)

where $C_T$ is an integration constant. In principle, one could use eqs. (4.6) and (4.9) to find $V(T)$.

When $\tau \to -\infty$, $\Phi_\pm(\tau)$ always goes to $-\infty$. Defining $\tau = \tau_a$ such that $\gamma e^{A \tau_a} = 1$, one finds that $\Phi_-(\tau_a) = +\infty$. As $\tau \to +\infty$, one has for $\Phi_\pm(\tau)$ that

$$\Phi_\pm(\tau) \to \begin{cases} +\infty, & \text{for } A < 1 \\ C_\Phi, & \text{for } A = 1 \\ -\infty, & \text{for } A > 1 \end{cases}$$

(4.10)
For $\Phi_+ (\tau)$ with $A < 1$ and $\Phi_- (\tau)$, the string coupling always diverges at some time. However for $\Phi_+ (\tau)$ with $A \geq 1$, the string coupling always stay finite. In figure 1(a), we plot $\Phi_+ (\tau)$ for $F (R_0) = 2$. Note that $A \geq 1$ implies that $F (R_0) \leq 0$ or $F (R_0) > 1$. Tables 1 and 2 show that $T^2_0 < 0$ when $F (R_0) \leq 0$. Thus when $A \geq 1$, the solution $T_+ (\tau)$ only exists for $F (R_0) > 1$. When $F (R_0) > 1$, eqs. (4.6) and (4.9) shows that as $\tau \rightarrow -\infty$

$$T_+ (\tau) \sim \sqrt{\alpha \tau}, \quad \frac{dV (T)}{dT} \sim \frac{d^2 H_0^2}{\sqrt{\alpha}} e^{A \tau}, \quad \text{and} \quad \frac{d^2 V (T)}{dT^2} \sim \frac{d^2 H_0^2}{\alpha} e^{A \tau},$$

and as $\tau \rightarrow +\infty$

$$T_+ (\tau) \sim C_T, \quad \frac{dV (T)}{dT} \sim \frac{d^2 H_0^2}{\sqrt{\alpha \beta \gamma}} e^{-A \tau/2} \sim 0, \quad \text{and} \quad \frac{d^2 V (T)}{dT^2} \sim -\frac{d^2 H_0^2}{2\alpha \beta} < 0,$$

where we use $\alpha > 0, \beta > 0$, and $\rho > 0$ for $F (R_0) > 1$. Therefore when $\tau = +\infty$, the tachyon field $T$ stays at a maximum of $V (T)$. Since $T_+ (\tau)$ goes to $-\infty$ as $\tau \rightarrow -\infty$, one finds for $T \rightarrow -\infty$ that the tachyon potential becomes

$$V (T) \sim -\frac{F (R_0) R_0 - f (R_0)}{2} + \rho A d^2 H_0^2 e^{A T/\sqrt{\pi}}. \quad (4.11)$$

In figure 1(b), we plot $V (T)$ for $F (R_0) = 2$. Recalling that $\tau = dH_0 t$, we find for $H_0 < 0$ and $F (R_0) > 1$, that the solution $T_+ (t)$ in eq. (4.9) describes the scenario in which the tachyon field is at a maximum of $V (T)$ at $t = -\infty$ and begins to roll down $V (T)$ afterwards. At $t = \infty$, $T_+ (t)$ goes to $-\infty$, and $V (T)$ is given in eq. (4.11). Moreover, this tachyon rolling down scenario is free of singularities since the Ricci scalar is constant, and the string coupling is always finite.

### 4.2 Constant $T$

If $T = T_0$ such that $V' (T_0) = 0$, the tachyon equation (2.12) becomes trivial. As a result, one could solve the gravitational equations (2.7) and (2.8) for $H (t)$ and $\Phi (t)$. The dilaton equation (2.11) impose constraints on the integration constants. This scenario could provide some insights into the possible final state of bulk tachyon condensation. The case with $f (R) = R$ has been discussed in [15], where the classical solutions always were found to evolve from or to singular configurations. We here investigate the singular behavior of the solutions in the case with $f (R) = R + \frac{a}{4} R^2$, in which the perturbation method is used to find solutions. The perturbation method gives how these solutions are altered for non-zero
but small $\alpha$. In doing so, we assume that the altered solutions can be Taylor expanded in $\alpha$. In addition, it turns out that the forms of the solutions depend on the sign of $V (T_0)$. We calculate the perturbative solutions to $\mathcal{O} (\alpha)$ for $V (T_0) = 0$ and discuss the singular behavior of the solutions for $V (T_0) > 0$ and $V (T_0) < 0$.

Substituting the Taylor expansions of $H (t)$ and $\Phi (t)$ in powers of $\alpha$

$$
H (t) = H_0 (t) + \alpha H_1 (t) + \mathcal{O} (\alpha^2),
$$

$$
\Phi (t) = \Phi_0 (t) + \alpha \Phi_1 (t) + \mathcal{O} (\alpha^2),
$$

(4.12)

into the gravitational equations (2.7) and (2.8), one finds

$$
dH_0 + dH_0^2 - 2\Phi_0 = 0,
$$

$$\dot{H}_0 + dH_0^2 - 2H_0 \Phi_0 = 0,
$$

$$d\dot{H}_1 + 2dH_0 H_1 - 2\dot{\Phi}_1 = F (t),
$$

$$\dot{H}_1 + 2dH_0 H_1 - 2H_0 \dot{\Phi}_0 - 2H_0 \Phi_1 = G (t),
$$

(4.13)

where we define

$$F (t) \equiv -d \left[ R_0 \left( \dot{H}_0 + H_0^2 \right) + \left( 2R_0 \Phi_0 - \dot{R}_0 \right) H_0 \right],
$$

$$G (t) \equiv -R_0 \left( H_0 + dH_0^2 \right) + 4R_0 \Phi_0^2 - 4\Phi_0 \dot{R}_0 - 2R_0 \Phi_0 - 2 (d - 1) R_0 \Phi_0,\Phi_0 + \dot{R}_0 + (d - 1) H_0 \dot{R}_0.
$$

(4.14)

If $V (T_0) = 0$, solving eqs. (4.13) and using eq. (2.11) to constrain the integration constants, one finds that the solutions to $\mathcal{O} (\alpha)$ are

$$H_{\pm} (t) = \pm \frac{1}{\sqrt{d} (t - t_0)} + \frac{\alpha h_{\pm}}{4 (t - t_0)^3} + \alpha c (t - t_0)^{-2},
$$

$$\Phi_{\pm} (t) = \pm \frac{\sqrt{d} - 1}{2} \ln |t - t_0| + \Phi_0 + \frac{\alpha \varphi_{\pm}}{(t - t_0)^2} - \frac{\alpha c (d - \sqrt{d})}{2 (t - t_0)},
$$

(4.15)

where $c$ and $\Phi_0$ and are integration constants, and we have

$$h_{\pm} = \mp \frac{\left( \mp \sqrt{d} + 1 \right)^2 \left( 5 \pm 6\sqrt{d} + 5d \right)}{\sqrt{d}},
$$

$$\varphi_{\pm} = \frac{\left( \pm \sqrt{d} - 1 \right)^2 \left( \pm 5d^2 + 4d \pm \sqrt{d} - 2 \right)}{16}.
$$

(4.16)

Integrating $H_{\pm} (t)$, we find

$$a_{\pm} (t) = a_0 |t - t_0|^\frac{\mp}{\sqrt{d}} \exp \left[ - \frac{\alpha h_{\pm}}{8 (t - t_0)^3} - \frac{\alpha c}{t - t_0} \right],
$$

(4.17)

where $a_0$ is a constant. The $\mathcal{O} (\alpha)$ corrections would not change the asymptotic behavior of the solutions at $t = \pm \infty$. Around $t = t_0$, the $\mathcal{O} (\alpha)$ corrections could dramatically change the asymptotic behavior of the these solutions. We plot $a_{\pm} (t)$ and $\Phi_{\pm} (t)$ in figure 2, where we have $d = 3$, $\alpha = 1$, $t_0 = 0$, $\Phi_0 = 0$, $a_0 = 1$ and $c = 0$. For example, the solutions $a_+ (t)$
Figure 2. Plots of $a_{\pm}(t)$ and $\Phi_{\pm}(t)$, where we have $d = 3$, $\alpha = 1$, $t_0 = 0$, $\Phi_0 = 0$, $a_0 = 1$ and $c = 0$.

and $\Phi_{\pm}(t)$ with $\alpha = 0$ evolve from (to) big bang (big crunch) at $t = t_0$, while they have a very weakly string coupled background. However for $\alpha > 0$, $a_{\pm}(t)$ goes to infinity at $t = t_0$, while the string coupling becomes divergent. Note that these perturbative solution are valid when $\alpha (t - t_0)^{-2} \ll 1$. Therefore, the higher order corrections are necessary to study the singular behavior of classical solutions. However, our analysis gives a sense of how quantum effects modify the singular behavior of classical solutions.

For $V(T_0) > 0$, the leading terms of the solutions are

$$H_0^\pm (t) = \pm \frac{C}{\sqrt{d} \sinh C (t - t_0)} ,$$

$$\Phi_0^\pm (t) = \frac{1}{2} \left( \pm \sqrt{d} - 1 \right) \ln \left| \sinh \frac{C}{2} (t - t_0) \right| - \frac{1}{2} \left( \pm \sqrt{d} + 1 \right) \ln \left| \cosh \frac{C}{2} (t - t_0) \right| + \Phi_0 ,$$

(4.18)

where $C = \sqrt{2V(T_0)}$, and $t_0$ and $\Phi_0$ are integration constants. For $V(T_0) < 0$, the leading terms are

$$H_0^\pm (t) = \pm \frac{C}{\sqrt{d} \sin C (t - t_0)} ,$$

$$\Phi_0^\pm (t) = \frac{1}{2} \left( \pm \sqrt{d} - 1 \right) \ln \left| \sin \frac{C}{2} (t - t_0) \right| - \frac{1}{2} \left( \pm \sqrt{d} + 1 \right) \ln \left| \cos \frac{C}{2} (t - t_0) \right| + \Phi_0 ,$$

(4.19)
where \( C = \sqrt{-2V(T_0)} \), and \( t_0 \) and \( \Phi_0 \) are integration constants. These solutions have a singularity at \( t = t_0 \). Around \( t = t_0 \), their singular behaviors are the same as in the case with \( V(T_0) = 0 \). Therefore, the last two equations in eqs. (4.13) give that the singular behaviors of \( H_1(t) \) and \( \Phi_1(t) \) at \( t = t_0 \) are also the same as in the case with \( V(T_0) = 0 \).

5 Conclusion

In [12], the tachyon-induced rolling solutions have been considered using the low-energy effective field equations, which were derived from the effective action (2.1) with \( f(R) = R \). To gain some insight into quantum effects on the tachyon dynamics, in this paper we investigated the behavior of the classical solutions of the low-energy effective action (2.1) of the graviton-dilaton-tachyon system, in which quantum corrections are included only in \( f(R) \) for simplicity. After the equations of motion were obtained in section 2, we solved them for the tachyon-induced rolling ansatzes (3.4) and (3.5) in section 3. Finally, more classical solutions were discussed in section 4.

In [12] where \( f(R) = R \), it showed that \( H(t) \) vanished identically for the tachyon-induced rolling solutions. For more general forms of \( f(R) \), we found in subsection 3.1 that there were some cases in which \( H(t) \) could be nonzero. In the case with \( H(t = -\infty) = 0 \), we found that only the \( R \) and \( R^2 \) terms in \( f(R) \) were responsible for obtaining the rolling solutions with nontrivial metric. This is quite surprising since in many cosmological models in \( f(R) \) gravity, all the higher order corrections play a role in modifying the dynamics. Moreover, we solved the equations of motion assuming \( H(t) = H_0 \) which was a constant. When \( F(R_0) = 1 \) and \( H_0 = 0 \), the properties of classical solutions have been discussed in [12], and it has been found that the dilaton always rolled toward stronger coupling. However for \( F(R_0) > 1 \), we found that there existed some solutions in which the string coupling could always stay finite. In the case of \( f(R) = R + \frac{a}{2} R^2 \), we also solved the equations of motion assuming \( T(t) = T_0 \), whose scenario is related to the possible final state of bulk tachyon condensation. It turned out for some solutions that higher order terms in \( f(R) \) could dramatically change their singular behavior. Since we only used an effective model, the \( f(R) \) gravity theory, to investigate quantum effects on tachyon dynamics, one might not take our analysis too seriously. However, our analysis suggests that quantum corrections should be important to understand tachyon dynamics.

A Dependence of the EOM

On defining

\[
G_1 \equiv F(R) \left( d\dot{H} + dH^2 \right) + \dot{F}^2 - 2\dot{H} + \left[ 2 \left( F(R) - 1 \right) \dot{\Phi} - \dot{F}(R) \right] dH, \tag{A.1}
\]
\[
G_2 \equiv F(R) \left( \dot{H} + dH^2 \right) - 4 \left[ F(R) - 1 \right] \dot{\Phi}^2 + 4\dot{\Phi} \dot{F}(R) + 2 \left( F(R) - 1 \right) \dot{\Phi}
+ 2 \left[ F(R) (d-1) - d \right] H \dot{\Phi} - \dot{F}(R) - (d-1) H \dot{F}(R), \tag{A.2}
\]
\[
G_3 \equiv \frac{1}{2} \left[ F(R) R - f(R) \right] - 2 \left[ d \left( F(R) - 1 \right) + 1 \right] \dot{\Phi}^2 + 2 d \dot{\Phi} \dot{F}(R)
+ \left[ d \left( F(R) - 1 \right) + 1 \right] \left( \dot{\Phi} + dH \dot{\Phi} \right) - \frac{1}{2} d \left[ \dot{F}(R) + dH \dot{F}(R) \right] + V(T), \tag{A.3}
\]
\[
G_4 \equiv \dot{T} + \left( dH - 2\dot{\Phi} \right) \dot{T} + V'(T), \tag{A.4}
\]
the equations of motion (2.7), (2.8), (2.11), and (2.12) become \( G_i = 0 \). Supposing that \( G_1 = G_2 = G_3 = 0 \), we now show that this leads to \( G_4 = 0 \) whenever \( \dot{T} \neq 0 \).

Differentiating \( G_1 = 0 \) with respect to time gives

\[
\ddot{T} = -\frac{\dot{F}(R) \left( dH + dH^2 \right)}{2} - \frac{F(R) \left( d\dot{H} + 2dH\dot{H} \right)}{2} + \ddot{\Phi}
\]

\[
- \left\{ \left[ F(R) - 1 \right] \dot{\Phi} - \frac{\dot{F}(R)}{2} \right\} d\dot{H} - \left\{ \left[ F(R) - 1 \right] \ddot{\Phi} + \dot{F}(R) \dot{\Phi} - \frac{\ddot{F}(R)}{2} \right\} dH
\]

where we use \( R = 2d\dot{H} + d(d + 1)H^2 \). By multiplying \( G_4 \) by \( \dot{T} \), eliminating \( \ddot{T} \) through the equation of motion \( G_1 = 0 \), we find

\[
\dot{T}G_4 - \left( d\dot{H} - 2\dot{\Phi} \right) G_1 - \frac{1}{2} \frac{dG_1}{dt} = -\frac{\dot{F}(R) \left( d\dot{H} + dH^2 \right)}{2} - \frac{F(R) \left( d\dot{H} + 2dH\dot{H} \right)}{2} + \ddot{\Phi}
\]

\[
- \left\{ \left[ F(R) - 1 \right] \dot{\Phi} - \frac{\dot{F}(R)}{2} \right\} d\dot{H} - \left\{ \left[ F(R) - 1 \right] \ddot{\Phi} + \dot{F}(R) \dot{\Phi} - \frac{\ddot{F}(R)}{2} \right\} dH \]

\[
- \left( d\dot{H} - 2\dot{\Phi} \right) \left\{ F(R) \left( d\dot{H} + dH^2 \right) - 2\ddot{\Phi} + \left[ 2 \left( F(R) - 1 \right) \dot{\Phi} - \dot{F}(R) \right] dH \right\} + V'(T) \dot{T} = 0.
\]

Differentiating \( G_3 = 0 \) with respect to time gives

\[
\frac{\dot{F}(R) R}{2} - 4 \left[ F(R) - 1 \right] \dot{\Phi} \ddot{\Phi} - 2d\dot{F}(R) \dot{\Phi}^2
\]

\[
+ 2d\dot{\Phi} \ddot{F}(R) + 2d\ddot{\Phi} \dot{F}(R) + d\dot{F}(R) \left( \dot{\Phi} + dH\dot{\Phi} \right)
\]

\[
+ \left\{ d \left[ F(R) - 1 \right] + 1 \right\} \left( \dot{\Phi} + dH\dot{\Phi} + dH\ddot{\Phi} \right) - \frac{1}{2} d \left[ \ddot{F}(R) + dH\ddot{F}(R) + dH\dot{F}(R) \right] = -V'(T) \dot{T}.
\]  

Differentiating \( G_2 = 0 \) with respect to time and multiplying by \( d/2 \) on both sides gives

\[
d \left[ F(R) - 1 \right] \ddot{\Phi} - \frac{d}{2} \frac{\dot{F}(R)}{2} - 4d \left[ F(R) - 1 \right] \dot{\Phi} \ddot{\Phi} - 2d\dot{F}(R) \dot{\Phi}^2 + 2d\ddot{\Phi} \dddot{F}(R)
\]

\[
+ 2d\dot{\Phi} \ddot{F}(R) + d\dot{F}(R) \ddot{\Phi} + dH\dot{\Phi} \dddot{F}(R) - \frac{d^2}{2} \dot{H}\ddot{F}(R) - \frac{d^2}{2} \dot{H}\dddot{F}(R)
\]

\[
= -\frac{d\dot{F}(R) \left( \dot{H} + dH^2 \right)}{2} - \frac{dF(R) \left( \ddot{H} + 2dH\dot{H} \right)}{2} + dH\dot{\Phi} \dddot{F}(R)
\]

\[
- d \left[ F(R) \left( d - 1 \right) - d \right] \dot{H}\ddot{\Phi} - d \left[ F(R) \left( d - 1 \right) - d \right] \dot{H}\dddot{\Phi} - \frac{dH\dot{F}(R)}{2} - \frac{dH\dddot{F}(R)}{2}.
\]

Subtracting eq. (A.8) from eq. (A.7) gives

\[
\frac{dG_3}{dt} - \frac{d}{2} \frac{dG_2}{dt} = \dddot{\Phi} - 4\ddot{\Phi} + [F(R) + 1] d\dot{H}\ddot{\Phi} - \frac{dF(R) \dddot{H}}{2}
\]

\[
+ \left[ F(R) + 1 \right] dH\ddot{\Phi} - \frac{dH\dddot{F}(R)}{2} + dH\dot{\Phi} \dddot{F}(R)
\]

\[
- d^2 F(R) H\dot{H} + \frac{d\ddot{F}(R) H^2}{2} + V'(T) \dot{T} = 0.
\]
Subtracting eq. (A.6) from eq. (A.9) gives
\[
\frac{dG_3}{dt} - \frac{dG_2}{dt} - \dot{T}G_4 + \left( \frac{dH}{dt} - 2\dot{\Phi} \right) G_1 + \frac{1}{2} \frac{dG_1}{dt} - 2F(R) - 1 \dot{\Phi} - \ddot{\Phi} + F(R) \left( \ddot{H} + dH^2 \right) + (d - 1) H \ddot{F}(R)
\]
\[-4 \left[ F(R) - 1 \right] \dot{\Phi}^2 + 4 \dot{\Phi} \ddot{F}(R) + 2 \left[ F(R) (d - 1) - d \right] H \dot{\Phi} \] .
\tag{A.10}
\]
Comparing eq. (A.10) with eq. (A.4), we obtain the relation for \(G_1, G_2, G_3\) and \(G_4\)
\[
\left( \frac{dH}{dt} - 2\dot{\Phi} \right) G_1 + \frac{1}{2} \frac{dG_1}{dt} - \frac{dG_2}{dt} = dH_2 + \frac{dG_3}{dt} - \dot{T}G_4 = 0,
\tag{A.11}
\]
which shows that \(G_4 = 0\) if \(G_1 = G_2 = G_3 = 0\) and \(\dot{T} \neq 0\).

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