Quantum mechanics without spacetime
- A case for noncommutative geometry -

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Abstract
Quantum mechanics in its presently known formulation requires an external classical time for its description. A classical spacetime manifold and a classical spacetime metric are produced by classical matter fields. In the absence of such classical matter fields, quantum mechanics should be formulated without reference to a classical time. If such a new formulation exists, it follows as a consequence that standard linear quantum mechanics is a limiting case of an underlying non-linear quantum theory. A possible approach to the new formulation is through the use of noncommuting spacetime coordinates in noncommutative differential geometry. Here, the non-linear theory is described by a non-linear Schrodinger equation which belongs to the Doebner-Goldin class of equations, discovered some years ago. This mass-dependent non-linearity is significant when particle masses are comparable to Planck mass, and negligible otherwise. Such a non-linearity is in principle detectable through experimental tests of quantum mechanics for mesoscopic systems, and is a valuable empirical probe of theories of quantum gravity. We also briefly remark on the possible connection our approach could have with loop quantum gravity and string theory.

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“There is no doubt that quantum mechanics has seized hold of a beautiful element of truth and that it will be a touchstone for a future theoretical basis in that it must be deducible as a limiting case from that basis, just as electrostatics is deducible from the Maxwell equations of the electromagnetic field or as thermodynamics is deducible from statistical mechanics. I do not believe that quantum mechanics will be the starting point in the search for this basis, just as one cannot arrive at the foundations of mechanics from thermodynamics or statistical mechanics.”

- Einstein (1936)

1 Why quantum mechanics without classical spacetime?

There are several, well-known, ‘classical’ elements in the presently known formulation of quantum mechanics. The presence of these classical elements in the theory makes this formulation incomplete, because such elements are part of a classical world. The classical world is a limiting case of a quantum world, and a fundamental theory should not have to depend on its own limit, for its formulation. This suggests that a formulation more complete than the present one should exist, which does not refer to these classical elements.

These classical elements include: (i) a prior knowledge of the canonical position and momenta (so that canonical commutation relations can be set up) and Hamiltonian, of the corresponding classical mechanical system, (ii) the necessity to have a concept of time [which is part of a classical spacetime geometry] and of a spacetime metric, so as to be able to describe quantum evolution, and (iii) the necessity of having a classical measuring apparatus, so as to be able to interpret a quantum measurement.

These classical elements are part of the classical Universe in which we live, which consists of classical matter fields and the classical spacetime manifold, whose pseudo-Riemannian geometry is determined by the classical matter fields via the general theory of relativity. Because such a classical Universe is available to us today, we use its features (classical spacetime and classical matter) for describing a quantum system. In principle, however, such a classical Universe may not be available, and we should still be able to describe the dynamics of the quantum system.

For instance, soon after the Big Bang, there would have been a phase when neither the matter fields nor gravity were classical, and in this phase the laws of quantum mechanics could not be expressed in the form that we know them today. One could also construct a thought experiment where a similar situation would arise. Let us imagine a box of electrons, and imagine this to be the whole Universe. It is plausible to expect that the dynamics of the electrons should be described by quantum mechanics; however the spacetime metric inside the box undergoes quantum fluctuations, because
the metric is being produced by the electrons themselves. Thus one could not write down the quantum dynamics of the electrons in the usual manner, because an external classical spacetime geometry is no longer available.

One could ask if the presence of an underlying classical spacetime manifold could be assumed, even if the overlying spacetime metric is subject to quantum fluctuations. While this may be acceptable at an intermediate stage in the development towards the final formulation of quantum mechanics, it cannot be an acceptable assumption in the final formulation of the theory. This is because, in the ‘active’ picture of spacetime coordinate transformations, according to which coordinate transformations move points in spacetime, quantum fluctuations of the spacetime metric clearly destroy the point structure of the underlying spacetime manifold.

It hence follows, in view of the arguments presented above, that it is reasonable to demand that a formulation of quantum mechanics which does not refer to an external classical time ought to exist. In order to describe the quantum dynamics, this new formulation will have to also incorporate an appropriate notion of a ‘quantum spacetime’, and hence will be intimately related to a quantum theory of gravity.

The new formulation should have the following two properties. Firstly, in the limit in which the quantum system under consideration becomes macroscopic, the ‘quantum spacetime’ should become the standard classical spacetime described by the laws of special and general relativity; and the quantum dynamics should reduce to standard classical dynamics on this spacetime. Secondly, consider the situation in which a dominant part of the quantum system becomes macroscopic and classical, and a sub-dominant part remains microscopic and quantum. By virtue of the first property, the dominant part should look like our classical Universe - classical matter existing in a classical spacetime. Seen from the viewpoint of this dominant part, the quantum dynamics of the sub-dominant part should be the same as the standard quantum mechanics on an external classical spacetime.

It is important to note that the need for such a new formulation of quantum mechanics arises not only at the Planck energy scale. The need arises in any situation when a background classical spacetime is not available - like in the case of the thought experiment about the box of electrons discussed above. This situation is not necessarily tied to the Planck scale, and the energy scale of the electrons in question can be much below the Planck scale. In this sense, quantum gravity, which is closely tied to the new formulation, is not exclusively a Planck scale phenomenon.

Another motivation for developing a new formulation of quantum mechanics comes from the search for the correct theory of quantum gravity. Attempts to develop a quantum theory of gravity by ‘quantizing’ a classical theory of gravity do face an issue of principle, because one is in this case using the rules of quantum theory to quantize the very spacetime geometry whose existence is a pre-requisite for formulating these rules. While this may still lead to the correct quantum gravity one is nonetheless left with
a sense of discomfort with regard to such an attempt, because it does not seem to be the most logical way to approach the problem. In particular, as is well-known, one is confronted with the severe unresolved ‘problem of time’ in quantum gravity. It is much more desirable that the new formulation of quantum mechanics, and the related quantum gravity, be constructed from first principles [without resorting to ‘quantization’ of a classical system] and that standard quantum theory as well as classical general relativity should emerge as approximations to the new formulation, in a suitable limit. Eventually, only experiments can decide as to which of the approaches to quantum gravity is the correct one.

One could object that quantum mechanics, which considers systems with finitely many degrees of freedom, is only an approximation to quantum field theory. Hence, one should address the issue of a new formulation also in quantum field theory, and not just in quantum mechanics. While this would be a valid objection, one would naturally like to address the simpler problem first, and only then attempt a generalization to quantum field theory.

The paper is organized as follows. In Section 2 we give an argument as to why quantum gravity should be a non-linear theory - this argument is independent of any specific mathematical structure one might use to describe a quantum theory of gravity, and is hence generic. In the third section we propose that noncommutative differential geometry is the appropriate language for the new formulation of quantum mechanics. Noncommutative geometry provides a natural generalization of general covariance to include noncommuting coordinate systems. In Section 4 we propose a description of the quantum Minkowski spacetime using noncommuting coordinates and explain how standard quantum dynamics can be recovered as an approximation to the dynamics on the noncommutative Minkowski spacetime.

In Section 5 we develop a description for a nonlinear quantum mechanics, to which the linear theory is an approximation. A key feature is the introduction of an antisymmetric tensor field, which vanishes in the classical limit, and hence explains why Riemannian geometry is a natural approximation to noncommutative geometry, on macroscopic scales. The non-linear Schrodinger equation we arrive at belongs to the Doebner-Goldin class of non-linear Schrodinger equations - this aspect is discussed in Section 6. In Section 7, we briefly emphasize the importance of experimental tests of quantum mechanics for mesoscopic systems.

We now show that if a formulation of quantum mechanics which does not refer to an external classical time exists, then such a formulation is an approximation to an underlying non-linear quantum theory. It follows as a consequence that standard quantum mechanics on a background classical spacetime is also an approximation to a more general, non-linear quantum mechanics.
2 Why a quantum theory of gravity should be nonlinear?

Consider the aforesaid thought experiment, wherein we consider a box of point particles, having masses $m_i$. The only known fundamental mass with which these masses can be compared is the Planck mass $m_{Pl} \equiv (\hbar c/G)^{1/2} \sim 10^{-5}$ grams. Since we know from observations that microscopic masses obey quantum mechanics and macroscopic masses obey classical mechanics, we will assume a particle behaves quantum mechanically if its mass is much smaller than Planck mass, and classically if its mass is much greater than Planck mass. In order to argue that quantum gravity should be a non-linear theory we construct a series of thought experiments where the values of the masses are made to vary, from one experiment to the next.

We shall first consider the case where each of the masses, as well as the total mass of the system, is much, much smaller than Planck mass; say typically in the atomic mass range, or smaller. If we observe this box from an external classical spacetime, the dynamics of the particles will obey the rules of quantum mechanics. We also assume that the total mean energy associated with the system is much smaller than the Planck energy scale $E_p \sim 10^{19} \text{GeV}$. Since both Planck-mass and Planck-energy scale inversely with the gravitational constant, the approximation we are considering is equivalent to considering the limit $m_{Pl} \to \infty$, or letting $G \to 0$. It is thus reasonable, in this approximation, to neglect the gravitational field produced by the particles inside the box. The reason for doing so is the same, for example, as to why we ignore the gravitational field of the hydrogen atom while studying its spectrum. What we thus have in the box is a collection of particles obeying quantum dynamics in an external spacetime, and the gravitation of these particles can be neglected.

Let us imagine now that the external classical spacetime is not there, and that the ‘box’ of particles is the whole universe. The arguments of the previous section imply that there is no longer any classical spacetime manifold available, and one should describe the quantum dynamics without reference to it. We assume that there is a concept of ‘quantum spacetime’ associated with the system, and since the associated gravitational field can be neglected we call this quantum spacetime the ‘quantum Minkowski spacetime’. The classical analog of this situation is a set of particles of small mass, whose gravitation can be neglected, existing in Minkowski spacetime.

The new quantum description of the dynamics of the particles in the box should become equivalent to standard quantum mechanics as and when a classical spacetime manifold becomes externally available. A classical spacetime would become externally available if outside the box there are classical matter fields which dominate the Universe.

Consider next the box universe, in the case in which, to a first order approximation, the values of the masses $m_i$ (as well as the total mass and the
mean energy of the system) in the box are no longer negligible, compared to Planck mass. We need to now take into account the ‘quantum gravitational field’ produced by these masses, and denoted by a set of variables, say $\eta$. There is still no background classical spacetime manifold, but only a background quantum Minkowski spacetime. Let us associate with the system a physical state $\Psi(\eta, m_i)$. It is plausible that to this order of approximation the physical state is determined by a linear equation

$$\hat{O}\Psi(\eta, m_i) = 0$$

where the operator $\hat{O}$, defined on the background quantum Minkowski spacetime, depends on the gravitational field variables $\eta$ only via the linearized departure of $\eta$ from its ‘quantum Minkowski limit’ and furthermore, does not have any dependence on the physical state $\Psi$. The classical analog of this situation is a linearized description of gravity, obtained from general relativity, when the spacetime metric is a small departure from Minkowski spacetime, and the gravitation of the matter sources cannot be entirely neglected.

We now consider the case of central interest to us, where we see departure from linear quantum theory, and argue that quantum gravity should be non-linear. Let the masses $m_i$ in the box (as well as the total mass and the mean energy) be comparable to Planck mass. The behavior of the particles will still be quantum mechanical and there is still no background classical spacetime manifold available. Furthermore, the ‘quantum gravitational field’ of the system can no longer be neglected, nor is it a first order departure from the ‘quantum Minkowski spacetime’. As a result, the quantum gravitational field described by the physical state $\Psi(\eta, m_i)$ will have to be taken into account, and will act as a source for itself. This will happen recursively, so that as a consequence the operator $\hat{O}$ in (1) will itself depend non-linearly on the state $\Psi$. In this sense quantum gravity should be a non-linear theory, because the quantum gravitational field will act as a source for itself.

We now carefully examine various aspects of this argument. There is a well-known, useful, parallel of this argument in classical gravitation. Starting from a linear theory of gravitation, and allowing gravity to be a source for itself, and doing this iteratively, one concludes that classical gravity is a non-linear theory, and one arrives at the general theory of relativity. The Einstein tensor of course depends non-linearly on the spacetime metric. The above argument for a non-linear quantum gravity is indeed analogous to the classical argument.

In this argument, the role played by the requirement for a new formulation of quantum mechanics (which does not refer to a classical spacetime manifold) is indirect, but crucial. We are no longer constrained by an a priori knowledge or requirement of the standard linear quantum theory; instead we allow for the theory to be built from first principles, without recourse to the aforementioned classical elements. If we stick to the linearity constraint, then the quantization of general relativity naturally leads to the Wheeler-DeWitt
equation - here the non-linearity of the gravitational field is contained, as in the classical theory, in the three-metric appearing in the Hamiltonian constraint. However, in our non-linearity argument above, wherein there is no classical spacetime manifold, nor a classical spacetime metric, the only way to capture the information that ‘the quantum gravitational field acts as a source for itself’ is via the physical state $\Psi(\eta, m_i)$ - leading to a non-linear quantum gravity.

One should contrast this situation with that for a quantized non-abelian gauge theory. In the case of gravity, the operator $\hat{O}$ captures information about the ‘evolution’ of the quantized gravitational field, and hence should depend on $\Psi(\eta, m_i)$, because the gravitational field also plays the role of describing spacetime structure. In the case of a non-abelian gauge theory, the analog of the operator $\hat{O}$ again describes evolution of the quantized gauge field, but the wave-functional $\Psi_A(A_i)$ describing the gauge field will not contribute to $\hat{O}$, because the gauge-field does not describe spacetime structure. $\Psi_A(A_i)$ can, on the other hand, be thought of as contributing non-linearly to a description of the quantized geometry of the internal space on which the gauge field lives.

In approaches to quantum gravity wherein one quantizes a classical theory of gravitation, using the standard rules of quantum theory, linearity is inherent, by construction. Such a treatment could by itself yield a self-consistent theory of quantum gravity. However, by requiring that there be a formulation of quantum mechanics which does not refer to a spacetime manifold, one is led to conclude that quantum gravity should be a non-linear theory. This happens because we have introduced the notion of a quantum Minkowski spacetime (unavoidable when there is no classical spacetime manifold available); iterative corrections to this quantum Minkowski spacetime because of gravity bring about the non-linear dependence on the physical quantum gravitational state. Eventually, only experiments can decide as to whether the linear theory is the right one, or the non-linear theory is.

The final case of the ‘box’ Universe that should be mentioned is when all the masses $m_i$ have values far exceeding Planck mass, in which limit the ‘quantum spacetime’ should reduce to standard classical spacetime described by general relativity.

We now present an argument to show that if the quantum theory of gravity is non-linear, the equation of motion which describes the quantum dynamics of the particles in the box is also non-linear, and provides a non-linear generalization of the Schrodinger equation. We do this, like above, by considering thought experiments with different values for masses of the particles.

An analogy with classical general relativity will be helpful. On a classical background Minkowski spacetime, the equations of motion of particles are those provided by special relativity. On a curved classical background, the motion of a test particle is geodesic. If the particle is not a test particle but contributes to the gravity of the spacetime, the motion of the particle,
and the gravitational field of the spacetime, have to be determined self-consistently, and this implies corrections to the geodesic equation of motion. A similar argument for the case of quantum mechanics without a classical spacetime manifold implies that there should be non-linear corrections to the Schrödinger equation, as we demonstrate below.

Consider the dynamical equation which describes the motion of a particle $m_1$ in the box universe, in the absence of a classical spacetime manifold. We assume that there are also present other particles, and in general $m_1$ and all the other particles together determine the ‘quantum gravitational field’ of this ‘quantum spacetime’.

The simplest situation is the one in which the masses of all the particles are much smaller than Planck mass, so that the spacetime is a quantum version of Minkowski spacetime, and there is no gravity. In this case, the equation of motion of $m_1$ will be the ‘classical background independent’ analog of the flat spacetime Klein-Gordon equation.

Keeping the mass $m_1$ small, increase the mass of the other particles so that $m_1$ becomes a test particle, while the ‘quantum gravitational field’ of the other particles obeys the linear equation (1). In this case, the equation of motion of $m_1$, which is a test particle, will be the ‘classical background independent’ analog of the curved spacetime Klein-Gordon equation, where the ‘quantum gravitational field’ depends linearly on the source.

Still keeping the mass $m_1$ a test particle, increase the mass of the other particles further, to the Planck mass range, so that the ‘quantum gravitational field’ of the other particles now obeys a non-linear equation, where the quantum gravitational field depends non-linearly on the quantum state of these particles. In this case, the equation of motion of $m_1$ will be the ‘classical background independent’ analog of the curved spacetime Klein-Gordon equation, where the ‘quantum gravitational field’ now depends non-linearly on the quantum state of the source, the source being all the particles other than $m_1$.

Now increase the mass $m_1$ also, to the Planck mass range, so that it is no longer a test particle. The equation of motion of $m_1$ thus now depends non-linearly on the quantum gravitational state of the system, where system now includes the mass $m_1$ as well. A special case would be one where we remove all particles, except $m_1$, from the system. The equation of motion for $m_1$ would then depend non-linearly on its own quantum state. This is what gives rise to non-linear quantum mechanics, which when seen from a classical spacetime manifold (as and when the latter becomes available) would be a non-linear Schrödinger equation. The possibility of non-linear modifications to the Schrödinger equation arises here because one has not a priori imposed the constraint that quantum mechanics is a linear theory on a background classical manifold. Instead, the quantum mechanics is being inferred from first principles.

It can be inferred that the introduction of non-linearity in quantum gravity and quantum mechanics is related to the introduction of the (Planck)
mass (equivalently energy) scale. When the masses in question are much smaller than Planck scale, the situation is equivalent to sending Planck mass to infinity, and also, the theory is then linear. There is thus a direct connection between non-linearity in quantum mechanics, and the presence of a fundamental energy scale in the theory.

Although the following is not a consequence of the above discussion, nor needed for further discussion, we would like to suggest a structure for the quantum Minkowski spacetime. This spacetime does not have any information about the masses present, does not have gravity, nor is it classical. Hence a plausible structure for it is that it is a grid made of cells of Planck length size. A coarse graining on scales larger than Planck length gives makes it appear as the classical Minkowski spacetime.

If this picture is correct, it implies a nice parallel between classical and quantum gravity. Classical gravity, as described by general relativity, imposes the gravitational field, produced by classical matter, on Minkowski spacetime. Quantum gravity, in our picture, imposes a ‘quantum gravitational field’, produced by quantum matter, on the quantum Minkowski spacetime possessing a Planck size grid. Coarse graining of this grid, along with the classical limit for matter, should reduce quantum gravity to classical general relativity. Thus, for us, quantum gravity is not the ‘quantization’ of classical general relativity, but a theory completely parallel to the classical theory, and built on the quantum Minkowski spacetime, instead of the classical Minkowski spacetime. Both the theories of gravity, classical as well as quantum, are non-linear.

Lastly in this section, we would like to comment on the significance of Planck mass ($\sim 10^{-5}$ grams), versus Planck energy ($\sim 10^{19}$ Gev). Because of the mass-energy equivalence, they are both of course the same concept. In the literature however, an impression is sometimes conveyed that Planck energy is a fundamental scale, whereas Planck mass is not. This viewpoint is partly unavoidable because, as masses go, Planck mass is a rather large mass - it is comparable to the mass of a small grain of dust, pretty much in the classical range. Nonetheless, if new physics arises at the Planck energy scale, it should definitely arise also at the Planck mass scale. Considering that the value of Planck mass is embarrassingly large, what is likely is that if new physics does arise, it comes about at a fundamental scale $m_e$ which is a few orders below the Planck mass scale (and equivalently a few orders below the Planck energy scale). Why $m_e$ should be not the same as Planck mass, but smaller, is not clear, and we will continue to refer to the fundamental mass scale as Planck mass $m_{Pl}$, although what we really have in mind is the mass scale $m_e$.

Continuing on this issue, it is said that quantum gravity effects can be probed by accelerating an elementary particle (whose rest mass is much smaller than Planck mass) to Planck energies, so that it can probe length scales as small as Planck length. While this is of course true, it would certainly also be possible to probe quantum gravity effects with an elementary
particle (if such a particle existed) whose rest mass becomes comparable to Planck mass, so that its Compton wavelength becomes comparable to Planck length. Known objects with mass in the Planck mass range are much, much larger than Planck length, in physical size, because of non-gravitational effects (the nuclear and the electromagnetic force). This rules them out as probes of physics on the Planck length scale. (It is also interesting to note that Planck mass is perhaps the mass scale where classicality sets in, because for objects with a mass larger than Planck mass the Schwarzschild radius becomes larger than the Compton wavelength, so that the quantum nature of the object is hidden behind the Schwarzschild radius.)

However, one cannot a priori rule out the possibility that Planck mass objects can serve as probes of non-linear corrections that might arise in quantum mechanics, near the Planck mass scale. This is especially significant in light of the fact that quantum mechanics has indeed not been experimentally tested for isolated mesoscopic systems whose mass lies in the intermediate range (say an object having a mass of a billionth of a gram) - in between the mass of an atomic system and the mass of a macroscopic object. The non-linear corrections could show up, for instance, as corrections to the energy levels of a system, which could happen independent of the fact that a realistic Planck mass object may not probe physics at the Planck length scale.

The arguments and conclusions presented in these two sections are generic, and do not depend on the specific mathematical structure used to develop a quantum theory of gravity. The only key assumptions made are that there should be a new formulation of quantum mechanics which does not refer to a classical spacetime manifold, and that the quantum gravitational field acts as a source for itself, like the classical gravitational field does, in the classical theory. In the next section we outline the proposal that the new formulation should be developed using the language of noncommutative differential geometry.

3 A formulation based on noncommutative differential geometry

Noncommutative differential geometry is an abstract, but in retrospect a rather natural, extension of Riemannian geometry, that includes the latter as a special case. (A short but highly readable account, on which the present brief discussion is based, is by Martinetti [1]). The starting point of the development is the famous theorem of Gelfand, Naimark and Segal which shows that the topological properties of a compact space $X$ can be described completely in algebraic terms, i.e. in terms of the algebra of complex functions on $X$. Conversely, given an algebra of functions $\mathcal{A}$ (with a few basic additional properties) one can build a compact topological space $X$, for which $\mathcal{A}$ can be interpreted as the algebra of functions. One could also generalize the algebra
A to a noncommutative algebra, and build from it a noncommutative space Y, for which A can be interpreted as the algebra of functions.

One could next ask about the coding of the differential structure of a manifold in algebraic terms. Connes observed that, given a Riemannian spin manifold, its geometrical information can be recovered from the so-called spectral triple, which consists of an algebra \( \mathcal{A} \), a Hilbert space \( \mathcal{H} \) and a self-adjoint operator \( \mathcal{D} \). Thus one has a map from a Riemannian spin geometry to an algebra, which is commutative. Conversely, given a commutative algebra \( \mathcal{A} \) as part of a spectral triple, one can build from it a Riemannian spin manifold.

The central point now is that the features used in mapping from the algebra to geometry do not depend on the commutativity of the algebra. One can make the algebra of functions noncommutative - the geometry to which the algebra then gets mapped is called noncommutative differential geometry. Riemannian geometry becomes a special of Connes’ noncommutative geometry. Also, a concept of distance and metric can be developed for the noncommutative space, given the operator \( \mathcal{D} \).

Given the map between a differentiable manifold and the corresponding commutative algebra, diffeomorphisms on the manifold can be mapped to the automorphisms of the algebra. When the algebra is noncommutative, the automorphisms of the algebra represent corresponding diffeomorphisms on the noncommutative space. Thus if the noncommutative space possesses a symmetry representing invariance under diffeomorphisms, this symmetry can be expressed as an invariance under the algebra automorphisms, and could be called automorphism invariance. Of special interest are the inner automorphisms, which are implemented by unitary elements of the algebra - these form a group, which is trivial in the commutative case. The inner automorphisms can hence be interpreted as the noncommutative part of the diffeomorphisms on the noncommutative space.

Our proposal is that noncommutative differential geometry is an appropriate framework for a formulation of quantum mechanics which does not refer to a classical (i.e. Riemannian) spacetime manifold. We will motivate this approach, and describe its present status, in the following sections. Our present understanding, though well-motivated on physical grounds, is still partly heuristic, and does not yet make contact with the rigorous formulation of noncommutative geometry, for instance with regard to construction of the spectral triple, and relating the noncommutative metric introduced below to the one formally defined in noncommutative geometry. Nonetheless the ideas developed below hold out the promise that the formal connections of these ideas with noncommutative geometry will eventually get developed.

We begin by suggesting the notion of a noncommuting coordinate system, which is to be thought of as ‘covering’ a noncommutative manifold, and wherein the commutation relations between the coordinates are to be introduced on physical grounds. The new formulation of quantum dynamics is to be given in such a coordinate system, and is to be invariant under trans-
formation (an automorphism) from the given coordinate system to different noncommuting coordinate systems. This is a generalization of general covariance to the noncommutative case, and one is proposing that a physical theory should be invariant under transformations of noncommuting coordinates.

By drawing analogies with special and general relativity, we will argue that the new formulation of quantum mechanics is essentially equivalent to a generalization of relativity to a noncommutative spacetime. That is, when one makes a transition from a commutative spacetime to a noncommutative spacetime, one simultaneously also makes a transition from classical dynamics to quantum dynamics. The quantum Minkowski spacetime is arrived at by introducing noncommutativity on a Minkowski spacetime, and quantum gravity is arrived at by introducing noncommutativity on a curved classical spacetime.

This formulation on a noncommutative spacetime satisfies the two properties mentioned in the introduction. When the system being described becomes macroscopic, the dynamics coincides with classical dynamics. Also, if the quantum dynamics (expressed in terms of noncommuting coordinates) is viewed from an external, classical spacetime (described by ordinary coordinates) it looks the same as standard quantum dynamics.

It may appear surprising as to how spacetime noncommutativity, for which the natural scale is the Planck length scale, can give rise to the standard quantum dynamics, which becomes relevant already on length scales much larger than Planck length. The answer, as we will see, lies in noting that in addition to spacetime noncommutativity, one also introduces momentum space noncommutativity. The noncommutativity between coordinates (which occurs at a very small length scale) together with noncommutativity between momentum components (which occurs at a very high energy scale), gives rise to a dynamics which looks like the standard quantum dynamics, when seen from our classical spacetime.

One could also ask as to why noncommutative differential geometry is the right arena for a mathematical formulation of the physical ideas described in the previous two sections. One could of course not give a proof for this, but only a motivation, and eventually the predictions of the new formulation have to be tested, and confirmed or ruled out, by laboratory experiments. The motivation is that noncommutative geometry is a natural extension of Riemannian geometry, and incorporates the generalization of diffeomorphisms to include transformations of noncommuting coordinates, and automorphism invariance is a plausible physical symmetry. It is then reasonable to ask what new physics could possibly relate to automorphism invariance. On the other hand, while noncommutativity is intrinsic to quantum mechanics, this noncommutativity is introduced in an ad hoc manner, building on a pre-existing classical theory. One would like to explore whether this ad hoc element can be avoided by relating the feature of noncommutativity inherent in noncommutative geometry to the noncommutativity that is central in quantum mechanics. And if one wants to avoid dependence on a classical, commuting
spacetime manifold, and yet have a concept of ‘quantum spacetime’ from which ordinary spacetime emerges, a noncommutative manifold appears to be the most likely starting point.

We should also compare our approach with other investigations of applications of noncommutative geometry to physics. While these other studies undoubtedly explore some fascinating avenues, our approach differs from them in that we are proposing noncommutative differential geometry as the appropriate mathematical language for describing quantum dynamics. Amongst other investigations, a prominent line of research, inspired in part by string theory, is quantum field theory on a noncommutative spacetime. Here, spacetime noncommutativity (usually only space-space noncommutativity) is assumed at the outset, as resulting from an underlying quantum structure of spacetime. One then investigates how this noncommutativity affects quantization of fields, with regards to issues like renormalizability, and avoidance of divergences [2]. Another important line of development is concerned with investigating consequences of noncommutativity for the structure of spacetime [3]. Spacetime noncommutativity appears to also play an important role in studies of Doubly Special Relativity [4] and the possibility of violation of Lorentz symmetry [5]. Investigation of noncommutative gravity at the classical level has also received serious attention [6].

While it is perhaps fair to say that at present a clear picture of the relation between noncommutative geometry and quantum gravity is not available, we would like to mention two important ideas which in our view could likely become center-stage in the future. The first is the possibility that noncommutative geometry could in a natural way provide an explanation for the origin of time, something for which it is difficult to find an explanation in the standard approaches to quantum gravity. A key property of von Neumann algebras is given by the Tomita-Takesaki theorem, which defines a one-parameter group of automorphisms of the algebra, and hence allows for a time-flow - this is the ‘thermal time hypothesis’ [7]. An alternative explanation for the spontaneous generation of time has recently been put forward by Majid [8] who shows this to be a natural consequence of properties of the noncommutative differential calculus, when one starts with space-space noncommutativity.

The second key idea is that by taking the product of a manifold with a suitable finite dimensional geometry, the standard model of particle physics can be described in the language of noncommutative geometry [9].

The application of noncommutative geometry to physical situations is generally segregated from quantum theory. Often, spacetime (or space-space) noncommutativity is introduced first, and quantization of the system is undertaken as a separate step. We now try to make the case that it may be fruitful to allow spacetime noncommutativity and quantum mechanics to go hand in hand, the latter being a consequence of the former.
4  A possible description of quantum Minkowski spacetime

As mentioned above, a quantum Minkowski spacetime will arise if no external classical spacetime is available, and if the masses of all the particles in the box Universe (as well as the total mass and energy scale of the box) are much smaller than Planck mass. In order to motivate our construction of the quantum Minkowski spacetime using noncommuting coordinates, we first briefly recall the relativistic Schrödinger equation for a free particle in quantum mechanics, and for simplicity of presentation we assume only one space dimension. Subsequently we will generalize to the case of many particles, and also to 3+1 spacetime. We chose to consider the relativistic case, as opposed to the non-relativistic one, only because the available space-time symmetry makes the analysis more transparent. Later, we will consider also the non-relativistic limit.

The relativistic Schrödinger equation for a particle of mass \( m \) in 2-d spacetime

\[
-\hbar^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi = m^2 \psi
\]  

(2)

can be rewritten, after the substitution \( \psi = e^{iS/\hbar} \), as

\[
\left( \frac{\partial S}{\partial t} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 - i\hbar \left( \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2} \right) = m^2.
\]  

(3)

The object \( S(t, x) \) introduced above, which we will call the complex action, will play a central role for us. In the classical limit, it will become real and coincide with the classical action. We would like to suggest that quantum dynamics could alternately be described in terms of this complex action, via Equation (3), which should be thought of as the quantum Hamilton-Jacobi equation. The quantum corrections of course come from the \( \hbar \)-dependent terms, which correct the classical Hamilton-Jacobi equation. In terms of the complex action \( S(t, x) \) we define the generalized momenta

\[
p^t = -\frac{\partial S}{\partial t}, \quad p^x = \frac{\partial S}{\partial x}
\]  

(4)

in terms of which Eqn. (3) reads

\[
p^\mu p_\mu + i\hbar \frac{\partial p^\mu}{\partial x^\mu} = m^2.
\]  

(5)

where the index \( \mu \) takes the values 1 and 2. The momenta defined above as gradients of the complex action have an obvious parallel with classical dynamics. In terms of the complex action \( S(t, x) \) we have a uniform description of classical and quantum dynamics. Below, we will make the case that Eqn. (3) can be derived from the fundamental assumptions of an underlying noncommutative theory.
In standard quantum mechanics the origin of the $\hbar$-dependent term in (5) of course lies in the position-momentum commutation relation, and this term provides a correction to the classical energy-momentum relation $E^2 - p^2 = m^2$. This, along with the consideration that (5) should be written in a more symmetric manner so that spatial gradients do not appear, suggests that from the viewpoint of dynamics on a noncommutative spacetime, these correction terms maybe expressible in terms of noncommutativity of momenta.

Following this lead, we now suggest a model for the dynamics of the ‘quantum Minkowski spacetime’ by assuming that there exist two noncommuting coordinates $\hat{x}$ and $\hat{t}$, and that the quantum mechanical particle lives in this noncommutative spacetime. We ascribe to the particle a ‘generalized momentum’ $\hat{p}$, having two components $\hat{p}^t$ and $\hat{p}^x$, which do not commute with each other. The noncommutativity of these momentum components is assumed to be a consequence of the noncommutativity of the coordinates, as the momenta are defined to be the partial derivatives of a complex action $S(\hat{x}, \hat{t})$, with respect to the corresponding noncommuting coordinates. We will propose a dynamics for the particle, analogous to classical special relativity, in these noncommuting coordinates. We will then argue that seen from an external classical spacetime (as and when the latter exists) this noncommutative dynamics looks the same as standard quantum mechanics as described by Equations (2) and (3).

On the quantum Minkowski spacetime we introduce the following noncommutative flat metric

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

and assume that there exists a corresponding noncommutative line-element

$$ds^2 = \hat{\eta}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu = d\hat{t}^2 - d\hat{x}^2 + d\hat{t}d\hat{x} - d\hat{x}d\hat{t}.\hspace{1cm} (7)$$

In analogy with special relativity, the metric and the line-element are assumed to be invariant under a class of coordinate transformations which generalize Lorentz transformations. Thus this line-element is left invariant by the transformation

$$\hat{x}' = \gamma(\hat{x} - \beta \hat{t}), \quad \hat{t}' = \gamma(\hat{t} - \beta \hat{x})$$

where $\gamma = (1 - \beta^2)^{-1/2}$. In the commutative case, $\beta$ has the interpretation of velocity: $\beta = v/c$. On a noncommutative space, $\beta$ could be thought of as defining a ‘rotation’ in the noncommutative space, by an angle $\theta$ such that $\beta = \tanh \theta$.

The above metric, which is obtained by adding an antisymmetric component to the standard Minkowski metric, is assumed to generalize the Minkowski metric to the noncommutative case. It is non-Hermitean, and has zero determinant - below we consider also a Hermitean modification of
this metric. Our discussion here, though, is not obstructed by the fact that the metric is not Hermitean.

In order to motivate a noncommutative dynamics, we note that, from (7), one could heuristically define a velocity \( \hat{u}^i = d\hat{x}^i / ds \), which satisfies the relation
\[
1 = \hat{\eta}_{\mu\nu} \frac{d\hat{x}^\mu}{ds} \frac{d\hat{x}^\nu}{ds} = (\hat{u}^t)^2 - (\hat{u}^x)^2 + \hat{u}^t \hat{u}^x - \hat{u}^x \hat{u}^t. \tag{9}
\]
This suggests a definition of the generalized momentum as \( \hat{p}^i = m \hat{u}^i \), which hence satisfies
\[
\hat{p}^\mu \hat{p}_\mu = m^2. \tag{10}
\]
Here, \( \hat{p}_\mu = \hat{\eta}_{\mu\nu} \hat{p}^\nu \) is well-defined. Written explicitly, this equation becomes
\[
(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = m^2. \tag{11}
\]
Eqn. (10) appears an interesting and plausible proposal for the dynamics, because it generalizes the corresponding special relativistic equation to the noncommutative case. The noncommutative Hamilton-Jacobi equation is constructed from (11) by defining the momentum as gradient of the complex action function. We are proposing here that in the absence of a classical spacetime manifold, quantum dynamics should be described using this Hamilton-Jacobi equation - this is the noncommutative analog of the quantum Hamilton-Jacobi equation (3).

Let us consider now that an external classical Universe becomes available (in the next section we will discuss how a classical Universe could arise from the quantum dynamics). Seen from this classical Universe the quantum dynamics should be as described by Eqn. (3). Thus, we need to justify the following correspondence rule, when the noncommutative Hamilton-Jacobi equation is examined from our standard spacetime point of view:
\[
(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = (p^t)^2 - (p^x)^2 + i\hbar \frac{\partial p^\mu}{\partial x^\mu}. \tag{12}
\]
Here, \( p \) is the ‘generalized momentum’ of the particle as seen from the commuting coordinate system, and is related to the complex action by Eqn. (4). This equality should be understood as an equality between the two equivalent equations of motion for the complex action function \( S \) - one written in the noncommuting coordinate system, and the other written in the standard commuting coordinate system.

The idea here is that by using the Minkowski metric of ordinary spacetime one does not correctly measure the length of the ‘momentum’ vector, because the noncommuting off-diagonal part is missed out. The last, \( \hbar \) dependent term in (12) provides the correction - the origin of this term’s relation to the commutator \( \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t \) can be understood as follows.

Let us write this commutator by scaling all momenta with respect to Planck momenta: define \( \hat{P}^\mu = \hat{p}^\mu / P_{pl} \). Also, all lengths are scaled with respect to Planck length: \( \hat{X}^\mu = \hat{x}^\mu / L_{pl} \). Let the components of \( X^\mu \) be denoted
as \((\hat{T}, \hat{X})\). The commutator \(\hat{P}^T \hat{P}^X - \hat{P}^X \hat{P}^T\) represents the ‘non-closing’ of the basic quadrilateral when one compares (i) the operator obtained by moving first along \(\hat{X}\) and then along \(\hat{T}\), with (ii) the operator obtained by moving first along \(\hat{T}\) and then along \(\hat{X}\). When seen from a commuting coordinate system, this deficit (i.e. non-closing) can be interpreted as a result of moving to a neighboring point, and in the infinitesimal limit the deficit will be the sum of the momentum gradients in the various directions. The deficit is thus given by \(i\partial P^\mu / \partial X^\mu = i(L_{PL}/P_{PL})\partial p^\mu / \partial x^\mu\). This gives that \(\hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = i\hbar \partial p^\mu / \partial x^\mu\). Hence, since the relation (12) holds, there is equivalence between the noncommutative description (10) and standard quantum dynamics given by (5).

We have not yet said anything about the all-important commutation relations on the non-commutative spacetime. The following is a plausible, natural structure:

\[
[\hat{t}, \hat{x}] = iL_{PL}^2, \quad [\hat{p}^t, \hat{p}^x] = iP_{PL}^2.
\] (13)

These relations look plausible because in the quantum Minkowski spacetime gravity is ignorable; so the commutation relations should not carry any information about the matter content of the spacetime, and hence they should not depend on the mass \(m\). It is generally not considered likely that spacetime noncommutativity on the Planck length scale could account for quantum effects on the much larger atomic scales. However, here we note that we have also introduced noncommutativity on the Planck momentum scale, which of course is much larger than the momenta encountered on atomic scales. We could picture that on the noncommutative 4-d ‘phase space’ corresponding to the two noncommuting coordinates \(\hat{x}, \hat{t}\) there is a natural grid (which is a consequence of noncommutativity) which when projected on the \(\hat{x}, \hat{p}^x\) plane, has an area \(L_{PL} \times P_{PL} = \hbar\). When we choose to examine this noncommutative quantum dynamics from our ordinary spacetime, we take the limit \(L_{PL} \to 0, P_{PL} \to \infty\), while keeping their product (the area of the fundamental phase space cell) constant at \(\hbar\). From the viewpoint of our classical spacetime, quantum dynamics is then recovered by imposing the commutator \([q, p] = i\hbar\) which preserves the granular structure of the phase space in quantum mechanics.

This completes our construction for the quantum dynamics of a particle on a 2-d noncommutative spacetime. It remains to be seen whether this dynamics can be obtained from an action principle. In the next section we will see how the off-diagonal part of the noncommutative metric (6) introduced above gets suppressed when the particle has a mass much larger than Planck mass, so that this construction can no longer be distinguished from classical dynamics.

It is straightforward to generalize this construction to 4-d spacetime. The noncommutative metric is constructed in analogy with (6) - the diagonal part is \((1, -1, -1, -1)\); the off-diagonal entries are 1 and \(-1\) - those above the diagonal are 1 and those below the diagonal are \(-1\) - so that the off-diagonal
part is antisymmetric. The energy-momentum relation is the same as in Eqn. (10) above, with the index $\mu$ running from one to four. In Eqn. (12) the net off-diagonal contribution coming from the commutators is to be equated to the gradient of the complex action on the right hand side, the reasoning for doing so being the same as in the 2-d case. Furthermore, commutation relations analogous to those in (13) are assumed to hold for each coordinate pair and each momentum pair. However an important change which comes about is that the 4-d line-element analogous to (7) is not invariant the Lorentz transformation (8), suggesting that there is a new class of transformations which leave this line-element invariant, and which reduce to Lorentz transformations in the limit of a commutative spacetime. (In fact, the presence of a second fundamental scale, namely Planck length, suggests the possibility of a connection with Doubly Special relativity).

The multi-particle case can be described in analogy with the way it is dealt with in special relativity. We will continue to assume that the generalized momentum of a particle is defined as the gradient of the complex action with respect to the coordinate of the corresponding particle, and that the energy-momentum relation (10) holds separately for each of the particles. The total momentum is the sum of the momenta of the individual particles, and the quantum Hamilton-Jacobi equation is constructed by squaring the total momentum and using (10) for each of the particle. In this regard, the treatment is no different from that of the multi-particle case in special relativity.

Lastly in this section we comment on the possibility of using a Hermitean metric

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

instead of the metric (14). By following the construction given above, it is easy to see that the same dynamics can alternately be described with this metric. What is not clear though is the implication of the fact that this metric can be diagonalized, via a complex transformation, to Minkowski-like coordinates. In the following, we will work with a generalization of the metric (6) - which will have a nice interpretation that the off-diagonal components become negligible in the macroscopic limit, so that the metric reduces to the standard metric on a curved space.

In summary, we have tried to make the case that dynamics in noncommutative special relativity is the same as quantum dynamics, if one assumes the above form for the noncommutative metric, along with the proposed commutation relations. Our discussion in this section has been largely heuristic, and further work needs to be done to make rigorous contact with noncommutative differential geometry. One could also criticize the construction by noting that there is no suggestion here for experimentally verifying whether the noncommutative description is indeed the right one. We will argue now
that an experimental prediction can be made, by considering the quantum dynamics of a mass whose value approaches the Planck mass.

5 A non-linear Schrödinger equation

In Section 2 we argued that when the particle masses become comparable to Planck mass, the quantum equation of motion for the particles should become non-linear. We now present an approximate construction for such a non-linear equation, based on the dynamics for the ‘quantum Minkowski spacetime’ proposed above. Since the quantum Minkowski spacetime and its metric were obtained as a noncommutative generalization of special relativity one can expect that there will also be a generalization of the quantum Minkowski spacetime to a ‘quantum curved spacetime’ which will be equivalent to a noncommutative generalization of general relativity. This will be the origin of non-linearity; and since standard quantum dynamics corresponds to the dynamics on the quantum Minkowski spacetime, one can expect that the dynamics on the quantum curved spacetime will be a non-linear generalization of standard quantum dynamics.

We once again start by considering the case of a single particle in 2-d spacetime. The starting point for our discussion will be Eqn. (10) above - we will assume that a natural generalization of this equation describes quantum dynamics when the particle mass $m$ is comparable to the Planck mass $m_{Pl}$, and the effects of the particle’s own gravity cannot be ignored. In this case, we no longer expect the metric $\eta$ to have the ‘flat’ form given in (6) above. The symmetric components of the metric are of course expected to start depending on $m$ (that is gravity) and in general the antisymmetric components can also be expected to depend on $m$ - so long as the antisymmetric components are non-zero, we can say that quantum effects are present. As $m$ goes to infinity, the antisymmetric component should go to zero - since in that limit we should recover classical mechanics. In fact the antisymmetric part (we will call it $\theta_{\mu\nu}$) should already start becoming ignorable when $m$ exceeds $m_{Pl}$. It is interesting to note that the symmetric part should grow with $m$, while the antisymmetric part should fall with increasing $m$. There probably is some deep reason why this is so.

If $\hat{h}_{\mu\nu}$ is the noncommutative metric which generalizes $\hat{\eta}_{\mu\nu}$ we can write it as

$$\hat{h}_{\mu\nu} = \begin{pmatrix} \hat{g}_{tt} & \hat{\theta} \\ -\hat{\theta} & \hat{g}_{xx} \end{pmatrix}$$

(15)

There is a corresponding generalization of the line-element,

$$ds^2 = \hat{h}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = \hat{g}_{tt} d\hat{t}^2 - \hat{g}_{xx} d\hat{x}^2 + \hat{\theta} [d\hat{t} d\hat{x} - d\hat{x} d\hat{t}]$$

(16)

while the velocity and momentum are defined as in the previous section. The
energy-momentum relation will now be
\[ \hat{h}_{\mu\nu} \hat{p}^\mu \hat{p}^\nu = m^2 \] (17)
and instead of Eqn. (12), we will have the dynamical equation
\[ \hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta} \left( \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t \right) = m^2 \] (18)
and the correspondence rule
\[ \hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hbar \theta \frac{\partial \hat{p}^\mu}{\partial x^\mu} = m^2 \] (19)
which replaces Eqn. (3) as the dynamical equation.

Here, \( g \) is the symmetric part of the metric, and \( \theta \) is the value of a component of \( \theta_{\mu\nu} \). The quantum Hamilton-Jacobi equation is constructed from here as before, by defining the momentum as gradient of the complex action. It also appears reasonable to assume that the fundamental commutation relations (13), as well as the correspondence rule, remain unchanged. The new element, on the Planck mass scale, is that the noncommutative metric now becomes dynamical, and depends on the masses present.

We outline here the overall picture for the dynamics of the quantum curved spacetime, described by the noncommutative metric (15). This is the spacetime produced by the particles in the ‘box Universe’ when there is no external classical spacetime, and the total mass (and energy) of the particles in the box is of the order of the Planck mass scale. This is the mesoscopic domain, as contrasted to the microscopic (quantum) domain considered in the previous section, and as contrasted to the classical (macroscopic) domain produced by a system with a mass much larger than the Planck scale. In analogy with classical general relativity, the metric of the ‘quantum gravitational spacetime’ (15) will be determined by the quantum distribution of particles whose physical state is described by the complex action \( S(\hat{t}, \hat{x}) \). The evolution of this complex action function will in turn depend on the noncommutative metric, as for instance in the case of one-particle dynamics described by (18) - this makes the resultant quantum theory of gravity nonlinear. The metric and the dynamical equations are assumed to be covariant under general coordinate transformations (automorphisms) of the noncommuting coordinates.

A central feature of this dynamics is that the off-diagonal part of the metric, \( \theta_{\mu\nu} \), is assumed to go to zero for large masses. When this happens, the quantum dynamics described by Eqns. (15)-(18) is indistinguishable from the classical dynamics described using commuting coordinates on a classical spacetime manifold. Thus we could say that at the fundamental level
dynamics is actually always described using noncommuting coordinates, but the use of commuting coordinates in the macroscopic world is an excellent approximation. It is an excellent approximation in the same spirit in which Galilean transformations are an excellent approximation to Lorentz transformations at speeds much smaller than the speed of light, or flat spacetime is an excellent approximation to curved spacetime when the curvature is negligible.

It is also likely that $\theta_{\mu\nu}$ is determined by the imaginary part of the complex matter action, since the imaginary part of the matter action also vanishes in the classical limit. Moreover the presence of the antisymmetric tensor field $\theta_{\mu\nu}$ for a mesoscopic system implies corrections to the classical gravitational metric $g$ predicted by general relativity.

Let us return now to the consideration of the dynamical equation (20) and its implications for quantum mechanics. We recall that the complex action $S(x,t)$ is related to the wave-function by the definition $\psi = e^{iS/\hbar}$, and now, because of gravitational corrections which become important at the Planck scale, the complex action satisfies the new dynamical equation (20) and not the equation (3). If we transform back from (20) by substituting for $S$ in terms of $\psi$ we will find that $\psi$ satisfies a non-linear equation, instead of the linear Klein-Gordon equation (2). This non-linearity is being caused by the presence of the non-trivial metric components $g$ and $\theta$, and is in principle observable.

We should now make simplifying assumptions, in order to construct a simple example of a non-linear Schrödinger equation. We are interested in the case $m \sim m_{Pl}$. The symmetric part of the metric - $g$ - should resemble the Schwarzschild metric, and assuming we are not looking at regions close to the Schwarzschild radius (which is certainly true for objects of such masses which we expect to encounter in the laboratory) we can approximately set $g$ to unity. The key quantity then is $\theta = \theta(m/m_{Pl})$ - we have no proof one way or the other whether $\theta$ should be retained. We will assume here that $\theta$ should be retained, and work out its consequences.

$\theta(m/m_{Pl})$ in principle should also depend on the quantum state via the complex action $S$, but we can know about the explicit dependence of $\theta$ on $m$ and $S$ only if we know the dynamical equations which relate $\theta$ to $m$, which at present we do not. It is like having to know the analog of the Einstein equations for $\theta$. But can we extract some useful conclusions just by retaining $\theta$ and knowing its asymptotic behavior? Retaining $\theta$, the above dynamical equation can be written in terms of the complex action $S$ as follows:

$$\left( \frac{\partial S}{\partial t} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 - i\hbar \theta(m/m_{Pl}) \left( \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2} \right) = m^2.$$  \hspace{1cm} (21)

This is the equation we would like to work with. We know that $\theta = 1$ is quantum mechanics, and $\theta = 0$ is classical mechanics. We expect $\theta$ to decrease from one to zero, as $m$ is increased. It is probably more natural
that $\theta$ continuously decreases from one to zero, as one goes from quantum mechanics to classical mechanics, instead of abruptly going from one to zero. In that case we should expect to find experimental signatures of $\theta$ when it departs from one. By substituting the definition $S = -i\hbar \ln \psi$ in (21) we get the following non-linear equation for the Klein-Gordon wave-function $\psi$:

$$-\hbar^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi + \frac{\hbar^2}{\psi} \left( 1 - \frac{1}{\theta} \right) \left( \dot{\psi}^2 + \psi'^2 \right) = m^2 \psi$$

(22)

We are interested in working out the consequences of the non-linearity induced by $\theta$, even though we do not know the explicit form of $\theta$.

Let us go back to the equation (21). In general $\theta$ will also depend on the state $S$ but for all states $\theta$ tends to zero for large masses, and if we are looking at large masses we may ignore the dependence on the state, and take $\theta = \theta(m)$. Let us define an effective Planck’s constant $h_{\text{eff}} = h\theta(m)$, i.e. the constant runs with the mass $m$. Next we define an effective wave-function $\psi_{\text{eff}} = e^{iS/\hbar_{\text{eff}}}$ . It is then easy to see from (21) that the effective wave function satisfies a linear Klein-Gordon equation

$$-\hbar_{\text{eff}}^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi_{\text{eff}} = m^2 \psi_{\text{eff}}$$

(23)

and is related to the usual wave function $\psi$ through

$$\psi_{\text{eff}} = \psi^{1/\theta(m)}$$

(24)

For small masses, the effective wave-function approaches the usual wave function, since $\theta$ goes to unity.

We would now like to obtain the non-relativistic limit for this equation. Evidently this limit is

$$i\hbar_{\text{eff}} \frac{\partial \psi_{\text{eff}}}{\partial t} = -\frac{\hbar_{\text{eff}}^2}{2m} \frac{\partial^2 \psi_{\text{eff}}}{\partial x^2}.$$ 

(25)

By rewriting $\psi_{\text{eff}}$ in terms of $\psi$ using the above relation we arrive at the following non-linear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m} (1 - \theta) \left( \frac{\partial^2 \psi}{\partial x^2} - [(\ln \psi)']^2 \psi \right).$$

(26)

It is reasonable to propose that if the particle is not free, a term proportional to the potential, $V(q)\psi$, can be added to the above non-linear equation. In the next section we will get a better insight into the relation between equations (25) and (26).

In terms of the complex action function $S$ defined above (3) as $\psi = e^{iS/\hbar}$ this non-linear Schrödinger equation is written as

$$\frac{\partial S}{\partial t} = -\frac{S'^2}{2m} + \frac{i\hbar}{2m} \theta(m) S''.$$ 

(27)
This equation is to be regarded as the non-relativistic limit of Eqn. (21).

It is clearly seen that the non-linear Schrödinger equation obtained in this particular example results from making the Planck constant mass-dependent, in the quantum mechanical Hamilton-Jacobi equation. Eqn. (26) is in principle falsifiable by laboratory tests of quantum mechanics, and its confirmation or otherwise will serve as a test of the various underlying assumptions of the noncommutative model. One might be tempted to say at this stage that the only slender contact that remains with noncommutativity of spacetime at this stage is the quantity $\theta(m)$ which corrects the Planck constant in Eqn. (21) and non-linearity of the Schrödinger equation could have been arrived at simply by invoking $\theta$, without making any reference to noncommutativity. Doing so would of course be ad hoc and noncommutativity of spacetime provides the underlying reason for the origin of $\theta$.

Conceptually, it is not difficult to generalize this non-linear Schrödinger equation to four dimensions, and to the many-particle case. In the following discussion we will continue to deal with the 2-d non-linear equation (26).

6 A comparison with the Doebner-Goldin equation

When we found the non-linear equation (26) we did not know that this equation already exists in the literature. Only subsequently we learned, from a review article by Svetlichny [10], that many years ago Doebner and Goldin arrived at a very similar equation from an apparently different (but possibly related) approach. Considering that the same equation has been arrived at independently by two different routes, and considering that the approach of Doebner and Goldin is on firmer ground (as compared to our partly heuristic analysis) we are led to believe that this non-linear equation deserves some serious attention, and should be tested in the laboratory. We will briefly review the Doebner-Goldin equation here, along with its possible implications, and its possible connection with our work. Actually there is an entire class of D-G equations, and we will begin by recalling the first non-linear equation derived by them.

Doebner and Goldin inferred their equation from a study [11] of representations of non-relativistic current algebras. This involves examining unitary representations of an infinite-dimensional Lie algebra of vector fields $Vect(R^3)$ and group of diffeomorphisms $Diff(R^3)$. These representations provide a way to classify physically distinct quantum systems. There is a one-parameter family, labeled by a real constant $D$, of mutually inequivalent one-particle representations of the Lie-algebra of probability and current densities. The usual one-particle Fock representation is the special case $D = 0$. The probability density $\rho$ and the current density $j$ satisfy, not the continuity
equation, but a Fokker-Planck equation

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} + D \nabla^2 \rho. \]  

(28)

A linear Schrodinger equation cannot be consistent with the above Fokker-Planck equation with \( D \neq 0 \), but Doebner and Goldin found that the following is one of the non-linear Schrodinger equations which leads to the above Fokker-Planck equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + i D \hbar \left( \nabla^2 \psi + \frac{|\nabla \psi|^2}{|\psi|^2} \psi \right). \]  

(29)

The Doebner-Goldin equation should be compared with the equation (26) found by us. Although we have considered a 2-d case, and although there are some differences, the similarity between the two equations is striking, considering that the two approaches to this non-linear equation are, at least on the face of it, quite different. It remains to be seen as to what is the connection between the representations of \( \text{Diff}(R^3) \), quantum mechanics, and the antisymmetric part \( \theta \) of the asymmetric metric introduced by us.

The comparison between the two non-linear equations suggests the following relation between the new constants \( D \) and \( \theta \)

\[ D \sim \frac{\hbar}{2m} (1 - \theta). \]  

(30)

There is a significant difference of an \( i \) factor in the correction terms in the two equations, and further, the relative sign of the two correction terms is different in the two equations, and in the last term we do not have absolute values in the numerator and denominator, unlike in the Doebner-Goldin equation. Despite these differences, the similarity between the two equations is noteworthy and we believe this aspect should be explored further. It is encouraging that there is a strong parallel between the limits \( D \to 0 \) and \( \theta \to 1 \) - both limits correspond to standard linear quantum mechanics. In their paper Doebner and Goldin note that the constant \( D \) could be different for different particle species. In the present analysis we clearly see that \( \theta(m) \) is labeled by the mass of the particle.

For comparison with Doebner and Goldin we write down the corrections to the continuity equation which follow as a consequence of the non-linear terms in (26). These can be obtained by first noting, from (25), that the effective wave-function \( \psi_{eff} \) obeys the following continuity equation

\[ \frac{\partial}{\partial t} \left( \psi_{eff}^* \psi_{eff} \right) - \frac{i \hbar_{eff}}{2m} \left( \psi_{eff}^* \psi'_{eff} - \psi_{eff} \psi_{eff}^* \right)' = 0. \]  

(31)

By substituting \( \psi_{eff} = \psi^{1/\theta(m)} \) and \( \hbar_{eff} = \hbar \theta(m) \) in this equation we get the following corrections to the continuity equation for the probability and current density constructed from \( \psi(x) \)

\[ \frac{\partial}{\partial t} (\psi^* \psi) - \frac{i \hbar}{2m} \left( \psi^* \psi' - \psi \psi^* \right)' = \frac{\hbar (1 - \theta)}{m} |\psi|^2 \phi''. \]  

(32)
where $\phi$ is the phase of the wave-function $\psi$, i.e. $\phi = Re(S)/\hbar$. It is interesting that the phase enters in a significant manner in the correction to the continuity equation. This equation should be contrasted with Eqn. (28). The fact that the evolution is not norm-preserving when the mass becomes comparable to Planck mass suggests that the appropriate description should be in terms of the effective wave-function $\psi_{eff}$.

An equation similar to the Doebner-Goldin equation was also independently derived and studied by Schuch, Chung and Hartmann [12], [13] who suggested a modification of the continuity equation to include diffusion. They proposed a non-linear Schrödinger equation with a logarithmic non-linearity

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi - i\hbar \gamma (\ln \psi - \langle \ln \psi \rangle) \psi$$

(33)

The general Doebner-Goldin equations

The equation (29) is one of an entire class of the non-linear Doebner-Goldin equations which are consistent with the Fokker-Planck equation (28). This class is given by [14]

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi + iI[\psi] \psi + R[\psi] \psi$$

(34)

where $H_0$ is the usual linear Hamiltonian operator and $I[\psi], R[\psi]$ are real-valued non-linear functionals given by

$$R[\psi] = \hbar D' \sum_{j=1}^{5} c_j R_j[\psi], \quad I[\psi] = \frac{1}{2} \hbar D R_2[\psi]$$

(35)

where $R_2 = [\nabla^2 \rho]/\rho$ and the form of the other terms $R_i$ is given in Eqn. (1.5) of [14]. The coefficients $D$ and $D'$ are real numbers with the dimensions of diffusion coefficients. Eqn. (29) is a special case of the general class given by (34). It is clear that our non-linear Schrödinger equation (26), though very similar to the D-G equation, does not belong to the general Doebner-Goldin class (34) either, because the corrections to the continuity equation which our equation implies (Eqn. (32)) are not of the Fokker-Planck type. However, we will see below that there is a generalization of the D-G equation (34) proposed by Goldin [15], of which our equation (26) is indeed a special case.

An important concept introduced by Doebner and Goldin, which will be of relevance to us in understanding the relation between our Eqns. (26) and (25), is that of a non-linear gauge transformation [16]. It is important because such gauge transformations relate a set of distinct non-linear equations as belonging to the same equivalence class, in that all the equations within a class describe the same physics, with regard to time evolution. Yet, there are equivalence classes which are inequivalent to standard quantum mechanics. Another consequence of introducing the non-linear gauge
transformations is that a whole family of non-linear Schrödinger equations are actually physically equivalent to the standard linear quantum mechanics. Also, the coefficients arising in the general D-G equation (35) are in general not gauge-invariant, and gauge-invariant quantities must be constructed from them, before one can suggest that some of these quantities should be looked for in experiments, as a test of the proposed non-linearity.

The construction of the non-linear gauge transformation depends on the assumption that all quantum mechanical measurements are fundamentally positional measurements. Thus two quantum theories are considered equivalent if the corresponding wave functions give the same probability density in space, at all times. Doebner and Goldin define a two-parameter non-linear gauge transformation (given by Eqn. (2.2) of [16]) which leaves the probability density invariant. These gauge transformations form a two-parameter group, and they construct gauge-invariant parameters out of the coefficients which appear in their general non-linear equation (35).

A generalization of the Doebner-Goldin equation

Goldin has proposed [15] a generalization of the D-G equation, by introducing the terms $R_1$, $R_3$, $R_4$, and $R_5$ of (35) also in the imaginary part of (34). The resulting non-linear Schrödinger equation [Eqn. (10) of [15]] achieves complete symmetry between the real and imaginary parts of the non-linear correction terms, and as we will see shortly, our non-linear equation (26) is a special case of this Goldin equation. The Goldin equation is no longer restricted by the requirement that it has to be consistent with the Fokker-Planck equation (28). Thus one is now extending the class of nonlinear equations beyond what was obtained by studying the representations of $\text{Diff} R^3$. Nonetheless, a suitable generalization of the aforementioned concept of non-linear gauge transformations allows one to construct a conserved probability density which coincides with the standard definition in the linear theory.

Goldin starts by noting that the amplitude $R$ and the phase $\alpha$ of the wave function $\psi = R e^{i\alpha}$ are treated asymmetrically in standard quantum mechanics. $R$ is gauge invariant and physically observable, while $\alpha$ is not. (We use the notation $\alpha$ for the phase, instead of Goldin’s $S$, as we have already used $S$ to denote the complex action $S$ defined by $\psi = e^{iS/\hbar}$). This asymmetry is unnatural from the point of view of a non-linear gauge transformation because we are combining a gauge-dependent quantity $\alpha$ with a gauge-invariant quantity $R$ to construct the wave function $\psi$ and through the Schrodinger equation coupling both $R$ and $\alpha$ to gauge potentials. It would be more natural to couple gauge-dependent quantities to each other, and physical quantities to each other. This, we believe, is an important idea for us, as we have been dealing with the complex action $S$ and treating its real and imaginary parts on an equal footing.

The generalized four-parameter non-linear gauge transformation, after
defining $T = \ln R$ so that $\ln \psi = T + i\alpha$, is given by
\[
\begin{pmatrix}
\alpha' \\
T'
\end{pmatrix} =
\begin{pmatrix}
\Lambda & \gamma \\
\lambda & \kappa
\end{pmatrix}
\begin{pmatrix}
\alpha \\
T
\end{pmatrix}
\] (36)

The earlier group, which leaves the probability density invariant, is given by $\lambda \equiv 0, \kappa \equiv 1$. In terms of the functions $\alpha$ and $T$, Goldin’s non-linear equation can be elegantly rewritten as the Eqn. (25) in his paper [15]. It has a complete symmetry between the amplitude and the phase, or in our language, between the real and imaginary parts of the complex action $S$. Gauge-invariant quantities can be constructed from the coefficients in the non-linear equation, and in Goldin’s notation, the quantities $\tau_1$ and $\tau_2$ defined by $2\tau_1 = a_1 + b_2$ and $2\tau_2 = a_1b_2 - a_2b_1$ are gauge invariant. Furthermore, invariant combinations of $\alpha$ and $T$ exist, so that a gauge invariant probability density and a gauge invariant current can be constructed, and these reduce to the standard definitions in the linear case.

We can now examine our non-linear equation (26) as a special case of Goldin’s Eqn. (25). Our non-linear equation, when expressed in terms of the complex action $S$, is given by (27). By writing the latter equation in terms of its real and imaginary parts, and comparing with Goldin’s Eqn. (25), we find that our equation is indeed a special case of Goldin’s non-linear equation, with the non-vanishing coefficients given by
\[
a_3 = -a_5 = \hbar/2m, \quad a_2 = a_5\theta(m/m_{Pl}), \quad b_4 = -\hbar/m, \quad b_1 = -a_2
\] (37)

It is easily seen from here that for $\theta = 1$ our equation coincides with the linear Schrödinger equation, as it should.

We now perform Goldin’s non-linear gauge transformation, given in our (36) above, with the specific choice $\Lambda = \kappa = 1/\theta(m/m_{Pl}), \gamma = \lambda = 0$. In accordance with Goldin’s Eqns. (28) and (29) this transformation leads to a new evolution equation with the new non-vanishing coefficients given by
\[
a_3' = -a_5' = -a_3' = h_{eff}/2m, \quad b_4' = 2b_1' = -h_{eff}/m
\] (38)

where, as before, $h_{eff} = \hbar\theta(m/m_{Pl})$. Comparison with Goldin’s (27) shows that the new equation is the linear Schrödinger equation but with $\hbar$ replaced by $h_{eff}$. This transformation is of course the same as that in our Eqn. (24) and we now formally understand it as a non-linear gauge transformation in the sense of Goldin. A conserved probability density is hence given, as usual, by $|\psi_{eff}|^2$, or equivalently by $|\psi|^2/\theta(m)$.

Using the definitions of $\tau_1$ and $\tau_2$ and the values of $a'$s and $b'$s given above, the gauge-invariant combinations of coefficients are given, in our case, by $\tau_1 = 0$ and $\tau_2 = h_{eff}^2/8m^2$. Thus the quantity $h_{eff}^2/8m^2$ by itself is not gauge invariant. Furthermore, if our ideas are correct, an experiment designed to measure $\hbar/m$ for a mesoscopic system will show departure from the standard quantum mechanical value, in the vicinity of the Planck mass scale.
It should be noted that the reduction to a linear Schrödinger equation became possible only for our simplified model, in which $\theta$ depends only on the mass, without depending on the physical state through the complex action function $S$. Also, in the more general model described by Eqn. (20) there will be additional non-linear terms introduced by the symmetric metric components. In general, the non-linear Schrödinger equation in the mesoscopic domain will not be equivalent to a linear equation via a gauge transformation.

It is also interesting to ask whether the generalized equation introduced by Goldin can be arrived at by considerations of symmetry, analogous to considerations such as representations of $DiffR^3$ that led to the Doebner-Goldin equation. Considering that despite a striking similarity our non-linear equation differs from the D-G equation in a subtle and significant way we would like to conjecture that Goldin’s equation may be related to a study of noncommutative algebras on the manifold, unlike the commutative algebra that lead to the D-G equation. We hope to investigate this possibility (which may be paraphrased as a ‘Hopf generalization of Lie algebras’) in the near future.

Another interesting question pertains to the field theoretic generalization of the Doebner-Goldin equation. As we mentioned above, this equation arises from consideration of inequivalent one-particle representations, which are related to the representations of the configuration space $R^3$. Now $R^3$ is also one of the possible background spatial manifolds on which the 3-metric is defined, in the construction of the Wheeler-DeWitt equation, which of course is linear. The Wheeler-deWitt equation is then the $D=0$ analog of the Doebner-Goldin equation (29), and by considering other representations of $DiffR^3$ one would like to investigate what are the non-linear generalizations of the Wheeler-DeWitt equation, in the sense of Doebner and Goldin.

Lastly in this section we would like to briefly remark on some of the earlier results [17], [18] that non-linearity can lead to unphysical phenomena such as superluminal propagation. An interesting point is made by Doebner and Goldin [16] when they note that the physical equivalence of some of the non-linear equations and the linear Schrodinger equation (brought about by the non-linear gauge transformation) suggests that this argument against non-linearity cannot be as general as it might have been thought to be. More generally, the non-linearity proposed by us arises at the yet untested Planck mass/energy scale, and is a consequence of the underlying noncommutative structure of spacetime. Thus we could not a priori rule out the possibility that the causal structure of spacetime that we are familiar with may be an excellent approximation to an underlying quantum spacetime structure which may in some form permit acausality on the Planck scale.

Some of the other interesting and relevant discussions on non-linearity in quantum mechanics are those by Bialynicki-Birula and Mycielski [19], Parwani [20] and Adler [21].
7 The case for experimental tests of mesoscopic quantum mechanics

While there are very stringent bounds on non-linearity in the atomic mass range [22], [23], [24], it is not really a well-known fact that quantum mechanics has not been experimentally verified for objects of intermediate masses, where by intermediate we mean masses much larger than atomic or molecular masses, and much smaller than macroscopic masses. This range indeed spans many orders of magnitude, and an interesting range to explore would be say from $10^{-15}$ grams to $10^{-8}$ grams. Most people do not expect any departures from quantum mechanics in this domain, but should we pre-judge the results of experiments in a range of parameters that has not been explored, and for which there are now theoretical reasons for further investigation? There seems to be a general impression that enough is known about systems such as macromolecules, polymers, nanoparticles etc. so that any deviations from quantum mechanics would have been seen by now. However it should be emphasized that none of these systems fall in the requisite mass range, despite an impression to the contrary. Thus, if we could make an object of say a billionth of a gram, isolate it from its environment, and arrange things so that its internal degrees of freedom can be ignored (so that it behaves like a point particle), we do not know from any experiment whether or not its dynamics obeys the rules of standard quantum mechanics.

In Section 2 we have argued that, because of gravitational effects, quantum mechanics becomes non-linear at the Planck mass/energy scale. This argument is generic, and does not depend on the particular noncommutative model we have described subsequently. The genericity of our arguments suggests to us that there is a strong case for an experimental investigation; at the very least one will succeed in putting bounds on non-linearity in the mesoscopic domain. In the context of our specific model described in Section 5 we have predicted that Planck’s constant runs with mass (equivalently, energy) and $\hbar \theta(m/m_{Pl})/m$ is a gauge invariant quantity which can in principle be measured experimentally.

Another general impression is that since mesoscopic objects have physical sizes much, much larger than their associated de Broglie or Compton wavelengths, quantum effects, or departures therefrom, will be essentially impossible to detect. This is true to a large degree, but not entirely. Experiments, such as an interference experiment, which rely on having an experimental set-up with sizes comparable to the wavelength of the particle wave will indeed not be of help. But there are other possible measurements, such as the energy spectrum, or particle-particle scattering, where physical size need not play a direct role, and which could carry information about the non-linearity. Through future work we hope to be able to make predictions of more gauge-invariant quantities which carry information about the non-linearity and can be subjected to experiment. In the meanwhile, we feel there
is already a case for experimentally investigating if (and how) one can design a mesoscopic object which behaves like a point-particle (i.e. internal degrees of freedom can be ignored) and which can be isolated from its environment. It seems likely that such objects could be designed more easily by smashing macroscopic grains (as opposed to congregating atoms).

The antisymmetric tensor field $\theta_{\mu\nu}$ introduced by us, which is responsible for the quantum non-linearity, becomes weaker with increasing mass, unlike the usual gravitational field described by the metric $g_{\mu\nu}$. This antisymmetric field hence violates the equivalence principle, and search for such a field in the mesoscopic domain is another possible test for our ideas. The presence of a field of this nature also indicates there will be a departure from the inverse square law of gravitation of an object, as the value of the mass is brought closer to the mesoscopic range.

A remark on quantum measurement and wave-vector reduction. Penrose has suggested [25] that gravity may be responsible for the collapse of the wave function during a quantum measurement. While it is not clear to us that the non-linearity in our non-linear equation plays a role during a quantum measurement, it seems that such an equation, in which Planck mass scale non-linearities are induced as a result of self-gravity, may provide a suitable mathematical set-up for theoretically implementing Penrose’s idea.

8 Concluding Remarks

In this paper we have argued that there should exist a formulation of quantum mechanics which does not refer to a classical spacetime manifold. Physicists are likely to generally agree with this claim. Disagreement may arise with our subsequent inference that this implies quantum gravity to be a non-linear theory. It could be said that the non-linearity discussed here arises as a result of the gravitation self-interaction, and is not intrinsic to quantum mechanics. In a way that is right, but then mass-energy and gravitation are universal features of any physical system (classical or quantum), and we are saying that the dynamics on the Planck scale is a generalization of standard linear quantum dynamics, to which the linear dynamics is an excellent approximation at energy scales much smaller than the Planck scale. Perhaps the relation between the Planck scale dynamics and the linear quantum dynamics is similar to the relation between special relativity and general relativity. The geodesic equation of motion in the presence of a gravitational field is of course different from the corresponding equation in the absence of a gravitational field, though the former can certainly be called a generalization of the latter. Similarly, quantum dynamics in the presence of a quantum gravitational field is different from quantum dynamics in the absence of such a field. Furthermore, the linear dynamics and the non-linear dynamics will predict different results for the same experiment.

Despite appearances, our emphasis is not overly and exclusively on the
non-linearity, but in trying to apply noncommutative differential geometry to obtain a new formulation of quantum mechanics. Non-linearity is then a byproduct. Undoubtedly, our ideas on the use of noncommutative geometry in this context are still largely heuristic, and remain to be put on a proper mathematical foundation. Nonetheless, the physical content of these ideas appears attractive and persuasive. These ideas generalize special relativity and general relativity in much the same way that noncommutative differential geometry generalizes Riemannian geometry. That there is a connection with the Doebner-Goldin equation, which was derived completely independently and with a different motivation, is encouraging. An important task is to find out the field equations analogous to the Einstein equations, which the asymmetric tensor $h_{\mu\nu}$ should satisfy - equations that should reduce to Einstein equations in the classical limit.

Finally, a few remarks comparing our ideas with two of the much better developed approaches to quantum gravity - loop quantum gravity and string theory - which are both based on a linear quantum theory. LQG investigates whether standard quantum theory and general relativity can together be merged into a consistent theory of quantum general relativity - undoubtedly a well-defined question. It appears to us that from the point of view of the Doebner-Goldin equation, LQG is the D=0 limit of an entire class of inequivalent quantum gravity theories. It would be of major interest to ask how the linear theory would relate to the non-linear theory which would result from considering the feedback of the quantum gravitational field on itself, in the sense discussed in this article.

In the context of string theory, it is interesting that the Doebner-Goldin equation arises as an effective non-linear Schrodinger equation when one considers the quantum dynamics of a system of D0-branes [26]. An effective Fokker-Planck equation for the probability density is arrived at when one considers the quantum recoil due to the exchange of string states between the individual D-particles. To us the spirit here seems to be very much the same as that of our earlier question: how do quantum gravitational effects modify the quantum dynamics of the particles in our box Universe. That the Doebner-Goldin equation arises in both the analyses raises an interesting question - is one discussing the same physical situation in two different languages? Indeed, since one expects a new picture of a quantum spacetime to emerge from string theory, one could ask if noncommutative geometry is a natural language for an ab initio description of the quantum spacetime, from which classical spacetime emerges in an approximation. This would be a top-down approach (as advocated in this article) as opposed to the more conventional, and better understood, bottom-up approaches in which one starts with a linear quantum theory, and then allows for the possibility that feedback effects can introduce non-linearity. (An elementary example comparing top-down and bottom-up approaches could be had from general relativity. One can study non-linear gravitational effects by successive iterations of a linearized theory of gravitation, or one can obtain linearized gravity as an
approximation to the full theory. Obviously, the physical principles that go into the construction of general relativity are independent of our theoretical knowledge of linearized gravity). A possibility of a modification of quantum mechanics in string theory has been considered also in [27] as resulting from coupling to massive string states in the ‘spacetime foam’.

The work described in this article is partly based on a few earlier preliminary papers [28], [29], [30], [31] and was partly carried out in collaboration with Sashideep Gutti and Rakesh Tibrewalla [32].

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A note on the references: The literature on physical applications of noncommutative geometry is vast indeed, and there are also a considerable number of papers on non-linear quantum mechanics. The list of references below is certainly far from being complete, and is only suggestive of a few of the related ideas existing in the literature.

The statement of Einstein cited at the beginning of this article is from his paper in J. Franklin Inst. 221, 313 (1936). It is mentioned also in the book ‘Subtle is the Lord’ by Abraham Pais (p. 461).

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