A class of group-like objects

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Abstract
We introduce a class of group-like objects and prove that Cayley Theorem on groups has a counterpart in the class of group-like objects.

In the study of generalizing the Lie correspondence between linear Lie groups and linear Lie algebras, we have found a generalization of the notion of inverse in group theory. The generalization of the notion of inverse in group theory not only is indispensable in our method of generalizing the Lie correspondence between linear Lie groups and linear Lie algebras, but also plays a key role in our study of group-like, ring-like and field-like objects. In this paper, we use the generalization of the notion of inverse to introduce a class of group-like objects. The class of group-like objects share many properties with groups. The main result of this paper is that Cayley Theorem on groups has a counterpart in the class of group-like objects.

1 Basic Definitions
We first give the definition of the class of group-like objects.

Definition 1.1 Let $G$ be a nonempty set together with two binary operations $\cdot \rightsquigarrow$ and $\cdot \leftsquigarrow$ on $G$. $G$ is called a digroup with the identity $e$ under the two binary operations if $e \in G$ and the following three properties are satisfied

1. The two operations $\cdot \rightsquigarrow$ and $\cdot \leftsquigarrow$ are diassociative, that is,

$$x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow y) \cdot \leftsquigarrow z = x \cdot \leftsquigarrow (y \cdot \leftsquigarrow z)$$ (1)

$$\leftsquigarrow (x \cdot \leftsquigarrow y) \cdot z = x \cdot \leftsquigarrow (y \cdot \leftsquigarrow z)$$ (2)

$$\leftsquigarrow (x \cdot \leftsquigarrow y) \cdot z = (x \cdot \leftsquigarrow y) \cdot \leftsquigarrow z = x \cdot \leftsquigarrow (y \cdot \leftsquigarrow z)$$ (3)

for all $x, y, z \in G$. 

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2. For all \( x \in G \), we have

\[
\begin{align*}
x & \cdot e = e \cdot x = x \\
x & \cdot e = e \cdot x
\end{align*}
\] (4)

(5)

3. For each element \( x \) in \( G \), there is an element \( x^{-1} \) in \( G \), call the Liu inverse of \( x \), such that

\[
x^{-1} \cdot x = e = x \cdot x^{-1}
\] (6)

The binary operations \( \cdot \) and \( \cdot \) are called the left product and the right product, respectively. A digroup \( G \) is also denoted by \( (G, e) \) or \( (G, \cdot, \cdot, e) \). A digroup \( G \) is called a finite digroup (or an infinite digroup) if \( G \) is a finite (or infinite) set. The cardinal number of a set will be called the order of the set.

\[1\), \(2\) and \(3\) consist of the diassociative law, which was introduced by J.-L. Loday to study Leibniz algebras [5]. A set with two binary operations \( \cdot \) and \( \cdot \) is called a dimonoid if the two binary operations \( \cdot \) and \( \cdot \) satisfy the diassociative law [5]. An element \( e \) of a dimonoid satisfying \(4\) is called a bar-unit [6].

By Definition [1], a digroup \( G \) is defined with respect to an fixed element \( e \) called the identity of \( G \). An Liu inverse of an element of \( G \) is defined with respect to the identity \( e \). The following proposition establishes the uniqueness of an Liu inverse.

**Proposition 1.1** If \( x \) is an element of a digroup \( (G, e) \), then the Liu inverse of \( x \) is unique. In other words, if \( y_i \cdot x = e = x \cdot y_i \) for \( y_i \in G \) and \( i = 1, 2 \), then \( y_1 = y_2 \).

**Proof** This is a direct consequence of Definition [1].

It is clear that a digroup \( (G, \cdot, \cdot, e) \) becomes a group if \( \cdot = \cdot \). The inverse of an element \( \alpha \) of a group will be called the group inverse of \( \alpha \), and the ordinary notation \( \alpha^{-1} \) will be used to denote the group inverse of \( \alpha \).

**Definition 1.2** Let \( (G, \cdot, \cdot, e) \) be a digroup.

(i) Two elements \( x \) and \( y \) of \( G \) are said to be commutative if

\[
x \cdot y = y \cdot x.
\]

(ii) \( G \) is said to be a commutative digroup if any two elements of \( G \) are commutative.
If a digroup $G$ is commutative, then the additive notations $\rightarrow +$ and $\leftarrow -$ are also used to denote the two binary operations on $G$. Also in this additive notations, we write $0$ for the identity, and we write $\sim x$ for the Liu inverse of $x$ in $G$.

**Example 1** Let $M := \{0, a\}$ be a set of two distinct elements. We define two binary operations $\rightarrow +$ and $\leftarrow -$ on $M$ as follows:

\[
\begin{array}{c|ccc}
\rightarrow + & 0 & a & a \\
\hline
0 & 0 & 0 & a \\
a & a & a & a \\
a & a & a & a \\
\end{array}
\quad
\begin{array}{c|ccc}
\leftarrow - & 0 & a & a \\
\hline
0 & 0 & a & a \\
a & a & a & a \\
a & a & a & a \\
\end{array}
\]

It is easy to check that $M$ is a commutative digroup with the identity $0$, and $0$ is the Liu inverse for both $0$ and $a$. Since a digroup consisting of a single element must be the identity group, $M$ is the smallest digroup which is not a group.

**Example 2** Let $N := \{e, \alpha, \beta, \gamma, \delta, \epsilon\}$ be a set of six distinct elements. We define two binary operations $\rightarrow \cdot$ and $\leftarrow \cdot$ on $N$ as follows:

\[
\begin{array}{c|cccccc}
\rightarrow \cdot & e & \alpha & \beta & \gamma & \delta & \epsilon \\
\hline
e & e & \alpha & \beta & \gamma & \delta & \epsilon \\
\alpha & \alpha & e & e & \alpha & \alpha & \alpha \\
\beta & \beta & \delta & \delta & \beta & \beta & \beta \\
\gamma & \gamma & \epsilon & \epsilon & \epsilon & \gamma & \gamma \\
\delta & \delta & \beta & \beta & \delta & \delta & \delta \\
\epsilon & \epsilon & \gamma & \gamma & \epsilon & \epsilon & \epsilon \\
\end{array}
\quad
\begin{array}{c|cccccc}
\leftarrow \cdot & e & \alpha & \beta & \gamma & \delta & \epsilon \\
\hline
e & e & \alpha & \beta & \gamma & \delta & \epsilon \\
\alpha & \alpha & e & e & \delta & \gamma & \beta \\
\beta & \beta & \alpha & e & \epsilon & \gamma & \beta \\
\gamma & \gamma & \alpha & e & \epsilon & \delta & \gamma & \beta \\
\delta & \delta & \epsilon & \alpha & \beta & \gamma & \delta & \epsilon \\
\epsilon & \epsilon & \alpha & \beta & \gamma & \delta & \epsilon \\
\end{array}
\]

One can check that $N$ is a digroup with the identity $e$. Since

$$\beta \leftarrow \beta = \delta \neq \epsilon = \beta \rightarrow \beta,$$

$N$ is not commutative. By the way, we indicate that $N$ is the unique non-commutative digroup with order $6$, and the number $6$, which is the smallest perfect number, is the smallest order among the orders of non-commutative digroups. Hence, $N$ is the smallest non-commutative digroup which is not a group.

**Definition 1.3** Let $H$ be a nonempty subset of a digroup $(G, \rightarrow, \leftarrow, e)$. If $e \in H$ and $(H, \rightarrow, \leftarrow, e)$ is itself a digroup with the identity $e$, we say that $H$ is a subdigroup of $G$. $H \leq G$ or $G \geq H$ is used to signify that $H$ is a subdigroup of $G$.

**Proposition 1.2** Let $H$ be a subset of a digroup $(G, \rightarrow, \leftarrow, e)$. The following are equivalent.

(i) $H$ is a subdigroup of $G$.

(ii) $e \in H$ and $(H \rightarrow H^{-1}) \cup (H^{-1} \leftarrow H) \subseteq H$. 


(iii) $H$ is nonempty and $(H * H) \cup H^{-1} \subseteq H$, where $\ast = \cdot$ or $\cdot$.

**Proof** (i) $\Rightarrow$ (ii): This is clear by Definition 1.3.

(ii) $\Rightarrow$ (iii): For $h \in H$, we have $h^{-1} = e \cdot h^{-1} \in H \cdot H^{-1} \subseteq H$. Hence, we get $H^{-1} \subseteq H$.

For $h_1, h_2 \in H$, we have

$$h_1 \cdot h_2 = h_1 \cdot (e \cdot h_2) = h_1 \cdot (e \cdot h_2) = h_1 \cdot (h_1^{-1})^{-1} \in H,$$

which implies that $H \cdot H \subseteq H$. Similarly, we have $H \cdot H \subseteq H$.

This proves that (iii) is true.

(iii) $\Rightarrow$ (i): Since $H \neq \emptyset$, there is an element $h \in H$. Hence, $h^{-1} \in H$. It follows that $e = h^{-1} \cdot h \in H \cdot H \subseteq H$. By $H * H \subseteq H$, $H$ is closed under both the left product $\cdot$ and the right product $\cdot$. This proves that $(H, \cdot, \cdot, e)$ is a digroup.  

## 2 The counterpart of Cayley Theorem

Let $\Omega$ be a set. The set of all maps from $\Omega$ to $\Omega$ is denoted by $T(\Omega)$. Thus

$$T(\Omega) := \{ f : \Omega \to \Omega \text{ is a map} \}.$$

It is well known that $T(\Omega)$ is a semigroup with the identity $1$ under the product $fg := f \cdot g$, where $1$ is the identity map, and the product $f \cdot g$ is defined by

$$(f \cdot g)(x) := f(g(x)) \text{ for } x \in \Omega.$$

Let $(G, \cdot, \cdot, e)$ be a digroup. For $a \in G$, we define two maps $La$ and $\tilde{La}$ as follows:

$$La(x) := a \cdot x, \quad \tilde{La}(x) := a \cdot x \text{ for all } x \in G.$$ 

$La$ and $\tilde{La}$ are called the left translations determined by $a$. Let

$$L_G := \{ La | a \in G \}, \quad \tilde{L}_G := \{ \tilde{La} | a \in G \}.$$

Then both $\tilde{L}_G$ and $L_G$ are subsets of $T(G)$.

**Proposition 2.1** Let $(G, \cdot, \cdot, e)$ be a digroup. If $a, b \in G$, then

$$\tilde{L}_{a+b} = \tilde{L}_a \tilde{L}_b, \quad \tilde{L}_{a^{-1}} = \tilde{L}_a \tilde{L}_b = \tilde{L}_a \tilde{L}_b, \quad \tilde{L}_{a \cdot b} = \tilde{L}_a \tilde{L}_b, \quad (7)$$

$$\tilde{L}_a = 1, \quad \tilde{L}_a \tilde{L}_a = \tilde{L}_a, \quad \tilde{L}_a \tilde{L}_a = \tilde{L}_a \tilde{L}_a, \quad (8)$$

where $* = \cdot$ or $\cdot$.  

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Proof Each of the equations above follows from the definition of a digroup. For example, let us prove that $L_e L_a = L_a L_e$. For $x \in G$, we have

\[ (L_e L_a)(x) = L_e (L_a(x)) = e \cdot (a \cdot x) = (a \cdot x) \cdot e \]

which implies that $L_e L_a = L_a L_e$. $\|$

The following three propositions summarize the main properties of $L_G$ and $\L_G$.

**Proposition 2.2** If $(G, \cdot, \cdot, e)$ is a digroup, then $\L_G$ is a subgroup of $\mathcal{T}(G)$ such that the group inverse of $\L_a$ in the group $\L_G$ is $\L_a^{-1}$, where $a \in G$ and $a^{-1}$ is the Liu inverse of $a$ in the digroup $G$.

**Proof** By (7), $L_a L_b = L_{a b} \in L_G$ for $a, b \in G$. Hence, $L_G$ is closed under the product in $\mathcal{T}(G)$. By (9), $L_a^{-1}$ is the group inverse of $L_a$. Thus $L_G$ is a subgroup of $\mathcal{T}(G)$. $\|$

**Proposition 2.3** If $(G, \cdot, \cdot, e)$ is a digroup, then $L_G$ is a subsemigroup of $\mathcal{T}(G)$ such that

(i) $L_e$ is a right unit of $L_G$, i.e.,

\[ L_a L_e = L_a \quad \text{for } a \in G, \]

(ii) $\L_a^{-1}$ is a left inverse of $\L_a$ with respect to $\L_e$, i.e.,

\[ \L_a^{-1} L_a = L_e. \]

**Proof** For $a, b \in G$, we have $L_a L_b = L_{a b} \in L_G$ by (7). Hence, $L_G$ is a subsemigroup of $\mathcal{T}(G)$. (i) and (ii) follow from (8) and (9), respectively. $\|$

**Proposition 2.4** If $(G, \cdot, \cdot, e)$ is a digroup, then the map $\phi : L_G \rightarrow L_G$ defined by

\[ \phi(L_a) := L_a \quad \text{for } a \in G \]

is a semigroup homomorphism satisfying

\[ \phi(L_e) = 1, \quad L_e L_a = \phi(L_a) L_e, \]

\[ \phi(L_a) L_a^{-1} = L_e, \]

\[ L_a \phi(L_b) = L_a L_b, \quad \phi(\phi(L_a) L_b) = \phi(L_a) \phi(L_b), \]

where $a, b \in G$. 

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**Proof** $\phi$ is well-defined. In fact, if $\tilde{L}_a = \tilde{L}_b$ for $a, b \in G$, then $a = a \cdot e = \tilde{L}_a (e) = \tilde{L}_b (e) = b \cdot e = b$. Hence, $\phi(\tilde{L}_a) = \phi(\tilde{L}_b)$.

By (7), $\phi(\tilde{L}_a \tilde{L}_b) = \phi(\tilde{L}_a \cdot \tilde{L}_b) = \tilde{L}_a \tilde{L}_b = \phi(\tilde{L}_a) \phi(\tilde{L}_b)$. Thus, $\phi$ is a semigroup homomorphism.

$\phi(\tilde{L}_e) = \tilde{L}_e = 1$ by (5). Also, we have $\tilde{L}_e \tilde{L}_a = \tilde{L}_e \cdot \tilde{L}_a = \tilde{L}_a \tilde{L}_e = \phi(\tilde{L}_a)$.

This proves (10).

Since $\phi(\tilde{L}_a) \tilde{L}_{a^{-1}} = \tilde{L}_a \tilde{L}_{a^{-1}} = \tilde{L}_e$, (11) holds.

Finally, by (7), we have

$$\tilde{L}_a \phi(\tilde{L}_b) = \tilde{L}_a \tilde{L}_b = \tilde{L}_a \cdot \tilde{L}_b = \tilde{L}_a \tilde{L}_b$$

and

$$\phi(\phi(\tilde{L}_a) \tilde{L}_b) = \phi(\tilde{L}_a \tilde{L}_b) = \phi(\tilde{L}_a \cdot \tilde{L}_b) = \tilde{L}_a \tilde{L}_b = \phi(\tilde{L}_a) \phi(\tilde{L}_b),$$

which proves that (12) holds.

Let $(G, \tilde{\cdot}, \tilde{\cdot}, e)$ be a digroup. We define two binary operations $\tilde{\cdot}$ and $\tilde{\cdot}$ on the set $\tilde{L}_G \times \tilde{L}_G$ by

$$\tilde{(L_a, L_b)} \cdot \tilde{(L_c, L_d)} = \tilde{(L_a \cdot_c L_b \cdot_d)}$$

(13)

and

$$\tilde{(L_a, L_b)} \cdot \tilde{(L_c, L_d)} = \tilde{(L_a \cdot_c L_b \cdot_d)}$$

(14)

where $a, b, c, d \in G$.

It is clear that under the binary operations above, $\tilde{L}_G \times \tilde{L}_G$ becomes a digroup with the identity $(\tilde{L}_e, \tilde{L}_e) = (1, \tilde{L}_e)$, and the Liu inverse of $(\tilde{L}_a, \tilde{L}_b)$ is $(\tilde{L}_{a^{-1}}, \tilde{L}_{b^{-1}})$.

The digroup $\tilde{L}_G \times \tilde{L}_G$ obtained from a digroup $G$ has a natural action on the set $G \times G$:

$$\tilde{(L_a, L_b)} (x, y) := (\tilde{L}_a (x), \tilde{L}_b (y)) \quad \text{for} \quad (x, y) \in G \times G.$$ 

Hence, $\tilde{L}_G \times \tilde{L}_G$ can be regarded as a subset of

$$\mathcal{T}(G \times G) := \{ f : G \times G \to G \times G \mid f \text{ is a map} \}.$$ 

Note that the identity $(1, \tilde{L}_e)$ of the digroup $\tilde{L}_G \times \tilde{L}_G$ does not act like the identity map on the set $G \times G$.

**Definition 2.1** A **homomorphism** $\eta$ from a digroup $(G, e_G)$ to a digroup $(H, e_H)$ is a map from $G$ to $H$ such that

$$\eta(e_G) = e_H,$$

$$\eta(x \star y) = \eta(x) \star \eta(y),$$

where $x, y \in G$ and $\star = \tilde{\cdot}, \tilde{\cdot}$. We say that two digroups are **isomorphic** if there is a bijective homomorphism from one to the other.
The following proposition gives a counterpart of Cayley Theorem.

**Proposition 2.5** Every digroup \((G, e)\) is isomorphic a subdigroup of \(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G\).

**Proof** Let 
\[\mathcal{D}(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G) := \{ (\overset{\leftarrow}{L}_a, \overset{\rightarrow}{L}_a) \mid a \in G \}.\]

By [13] and [14], \(\mathcal{D}(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G)\) is a subdigroup of \(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G\).

Define \(\eta : G \to \mathcal{D}(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G)\) by 
\[\eta(a) := (\overset{\leftarrow}{L}_a, \overset{\rightarrow}{L}_a) \quad a \in G.\]

First, it is clear that \(\eta(e) = (\overset{\leftarrow}{L}_e, \overset{\rightarrow}{L}_e) = (1, 1)\) is the identity of the digroup \(\mathcal{D}(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G)\), and \(\eta\) is surjective.

Next, we have
\[\eta(a \ast b) = (\overset{\leftarrow}{L}_{a \ast b}, \overset{\rightarrow}{L}_{a \ast b}) = (\overset{\leftarrow}{L}_a, \overset{\rightarrow}{L}_a) \ast (\overset{\leftarrow}{L}_b, \overset{\rightarrow}{L}_b) = \eta(a) \ast \eta(b).\]

Hence, \(\eta\) preserves the binary operations.

Finally, \(\eta(a) = \eta(b) \Rightarrow \overset{\leftarrow}{L}_a = \overset{\leftarrow}{L}_b \Rightarrow a = b.\) Thus, \(\eta\) is injective.

This proves that \(\eta\) is a isomorphism from \(G\) to \(\mathcal{D}(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G)\). \[\|

If \(a\) is an element of a digroup \((G, \overset{\leftarrow}{\cdot}, \overset{\rightarrow}{\cdot}, e)\), we may also define two maps \(\overset{\leftarrow}{R}_a\) and \(\overset{\rightarrow}{R}_a\) \(\in \mathcal{T}(G)\) as follows:
\[\overset{\leftarrow}{R}_a (x) := x \overset{\leftarrow}{\cdot} a, \quad \overset{\rightarrow}{R}_a (x) := x \overset{\rightarrow}{\cdot} a \quad \text{for all } x \in G.\]

\(\overset{\leftarrow}{R}_a\) and \(\overset{\rightarrow}{R}_a\) are called the right translations determined by \(a\). Let \(\overset{\leftarrow}{R}_G\) and \(\overset{\rightarrow}{R}_G\) be two subsets of \(\mathcal{T}(G)\) defined by
\[\overset{\leftarrow}{R}_G := \{ \overset{\leftarrow}{R}_a \mid a \in G \}, \quad \overset{\rightarrow}{R}_G := \{ \overset{\rightarrow}{R}_a \mid a \in G \}.\]

A similar argument will show that \(\overset{\leftarrow}{R}_G \times \overset{\rightarrow}{R}_G\) can be made into a digroup such that \(G\) is isomorphic to a subdigroup of \(\overset{\leftarrow}{R}_G \times \overset{\rightarrow}{R}_G\).

We finish this section with a construction of the digroup which generalizes the construction of the digroup \(\overset{\leftarrow}{L}_G \times \overset{\rightarrow}{L}_G\).

Let \(\Omega\) be a set. A triple \((\mathcal{S}, \mathcal{G}, \phi)\) is called a standard triple on \(\Omega\) if it satisfies the following three conditions:

1. \(\mathcal{G}\) is a subgroup of \(\mathcal{T}(\Omega)\) with the identity \(1\).
2. \(\mathcal{S}\) is a subsemigroup of \(\mathcal{T}(\Omega)\) such that
• $S$ has a right unit $e_S$, i.e.,
  \[ fe_S = f \quad \text{for } f \in S. \]  
  \[(15)\]

• Every element $f$ of $S$ has a left inverse $f^{-1}$ with respect to the right unit $e_S$, i.e.,
  \[ f^{-1} f = e_S. \]  
  \[(16)\]

3. $\phi : S \to G$ is a map satisfying
  \[ \phi(fg) = \phi(f)\phi(g), \quad \phi(S)S \subseteq S, \quad \phi(e_S)f = f, \quad e_Sf = \phi(f)e_S \]
  \[(17)\]

\[ \phi(f)f^{-1} = e_S, \]
\[(18)\]

\[ f\phi(g) = fg, \quad \phi(\phi(f)g) = \phi(f)\phi(g), \]
\[(19)\]

where $f, g \in S$.

By Proposition 2.2, Proposition 2.3 and Proposition 2.4, every digroup $G$ produces a standard triple $(\vec{L}_G, \vec{R}_G, \phi)$ on $G$.

Now let $(S, G, \phi)$ be a standard triple on $\Omega$. Consider the set
  \[ G \times S := \{ (\alpha, f) \mid \alpha \in G, \ f \in S \}. \]

Note that $G \times S$ can be regarded as a subset of $T(\Omega \times \Omega)$ in the following way:
  \[ (\alpha, f)(x, y) := (\alpha(x), f(y)) \quad \text{for } (x, y) \in \Omega \times \Omega. \]

We define two binary operations $\vec{\cdot}$ and $\vec{\cdot}$ on $G \times S$ by
  \[ (\alpha, f) \vec{\cdot} (\beta, g) : = (\alpha\beta, fg), \]
  \[(20)\]

  \[ (\alpha, f) \vec{\cdot} (\beta, g) : = (\alpha\beta, \phi(f)g), \]
  \[(21)\]

where $\alpha, \beta \in G$ and $f, g \in S$.

One can check that the $\vec{\cdot}$ and $\vec{\cdot}$ satisfy the diassociative law, $(1, e_S)$ satisfies $(11)$ and $(12)$, and $(\alpha^{-1}, f^{-1})$ is the Liu inverse of $(\alpha, f)$ with respect to the identity $(1, e_S)$. This proves that $G \times S$ is a digroup with the identity $(1, e_S)$.

Therefore, we have the following proposition.

**Proposition 2.6** Every standard triple on a set $\Omega$ produces a digroup which is a subset of $T(\Omega \times \Omega)$.  

8
3 Three remarks about digroups

We finish this paper with three remarks about digroups.

1. It is challenging and interesting to develop the counterpart of group theory in the context of digroups. Even some very simple concepts in group theory may resist our effort. One of the examples is the order of an element of a group. Although different criteria can be used to judge whether a generalization of group theory is satisfactory, we believe that a satisfying generalization of group theory should contain a satisfying counterpart of the order of an element of a group. Finding a suitable counterpart of the order of an element of a group is time-consuming, but the solution we have got seems to be a satisfying solution (2).

2. Digroups arise automatically from our study of generalizing the Lie correspondence between linear Lie groups and linear Lie algebras. However, the non-associative algebra objects corresponding to digroups are not Leibniz algebras, and the group-like objects corresponding to Leibniz algebras are not digroups. The following table gives a simple sketch-map of the possible generalizations of the Lie correspondence.

| The possible generalizations of the Lie correspondence: |
|-------------------------------------------------------|
| \( G_0 = \{ \text{Groups} \} \leftrightarrow L_0 = \{ \text{Lie algebras} \} \) |
| The group-like objects | The generalizations of Lie algebras |
| \( G_1 = \{ ??? \} \leftrightarrow L_1 = \{ \text{Leibniz algebras} \} \) |
| \( G_2 = \{ \text{Digroups} \} \leftrightarrow L_2 = \{ ??? \} \) |
| ⋆ | ⋆ |
| ⋆ | ⋆ |
| ⋆ | ⋆ |

In the correspondence above, \( G_1 \) is a class of group-like objects which is different from digroups, and the notion of the Liu inverse is indispensable in the description of \( G_1 \). \( L_2 \) is a class of generalization of Lie algebras which is different from Leibniz algebras, and the notion of the Liu inverse is also indispensable in the description of \( L_2 \). The star part ⋆ is not empty and consists of other kinds of group-like objects. The corresponding bullet part ⋆ is also not empty and consists of other kinds of generalizations of Lie algebras. However, quantum groups do not belong to the star part and super Lie algebras do not belong the bullet part. The definitions of \( G_1 \) and \( L_2 \) will be given in [3].

3. Another application of the notion of the Liu inverse is in the search for the counterpart of commutative rings. Using the notion of the Liu inverse, we
have introduced a class of ring-like objects (\[4\]). A typical example of the class of ring-like objects is the set of endomorphisms of a commutative digroup. Some fundamental notions in commutative rings like integral domains, prime ideals and fields have natural counterparts in the class of ring-like objects. Hence, there seems to be a good foundation in the class of ring-like objects to reconsider the theory of commutative rings.

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References

[1] Marshall Hall, Jr, *The theory of groups*, The Macmillan Company, 1959

[2] Keqin Liu, *The classification of cyclic digroups*, in preparation

[3] Keqin Liu, *The Liu’s sketch of possible generalizations of the Lie correspondence*, in preparation

[4] Keqin Liu, *A class of ring-like objects*, in preparation

[5] J.-L. Loday, *Algèbres ayant deux operations associatives (digèbres)*, C. R. Acad. Sci. Paris 321 (1995), 141-146

[6] J.-L. Loday, A. Frabetti, F. Chapoton, F. Goichot, *Dialgebras and related operads*, Lecture Notes in Mathematics 1763, Springer, 2000