ON WELLPOSEDNESS OF GENERALIZED NEURAL FIELD EQUATIONS WITH DELAY

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Abstract. We obtain conditions for existence of unique global or maximally extended solutions to generalized neural field equations. We also study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate and prehistory functions.

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1 Introduction

Firing rate models are used in the investigation of the properties of strongly interconnected cortical networks. In neural field models the cortical tissue has in addition been modeled as continuous lines or sheets of neurons. In such models the spatiotemporally varying neural activity is described by a single or several scalar fields, one for each neuron type incorporated in the model. These models are formulated in terms of differential, integro-differential equations and integral equations. The most well-known and simplest model in that respect is the Amari model (see e.g. [2])

\[ u_t(t,x) = -u_t(t,x) + \int_{\mathbb{R}} \omega(x-y)f(u(t,y))dy + I(t,x) + h, \quad t \geq 0, x \in \mathbb{R}. \]  

Here the function \( u(t,x) \) denotes the activity of a neural element at time \( t \) and position \( x \). The connectivity function (spatial convolution kernel) \( \omega(x) \) determines the coupling between the elements and the non-negative function \( f(u) \) gives the firing rate of a neuron with activity \( u \). Neurons at a position \( x \) and time \( t \) are said to be active if \( f(u(t,x)) > 0 \). The function \( I(t,x) \) and the parameter \( h \) represent a variable and a constant external inputs, respectively.

The literature on the Amari model (1.1) and its extensions is vast. The key issues in most of the published papers on these models are existence and stability of coherent structures like localized stationary solutions (so-called \textit{bumps}) and traveling fronts/pulses, pattern formation as the outcome of a Turing type of instability and issues like wellposedness of the actual models. See e.g. the reviews [12], [9] and [8] (and the references therein) for more details.

This is a draft of the paper containing the main results with the proofs. Full-text version is available at http://math-res-pub.org/jadea/6/1/wellposedness-generalized-neural-field-equations-delay

\[ Au_t(t,x) = -u_t(t,x) + \int_{\mathbb{R}} \mathbb{H}(x-y)f(u(t,y))dy, \quad t \geq 0, x \in \mathbb{R}, \]  

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathbb{H}(x) = \begin{pmatrix} \omega_{ee}(x) & -\omega_{e}(x) \\ \omega_{e}(x) & -\omega_{ii}(x) \end{pmatrix}, \]  

\[ u(t,x) = \begin{pmatrix} u_e(t,x) \\ u_t(t,x) \end{pmatrix}, \quad f(u(t,x)) = \begin{pmatrix} f_e(u_e(t,x)) \\ f_t(u_t(t,x)) \end{pmatrix}. \]  

\[ u_t(t,x) = -Lu_t(t,x) + \int_{\Omega} \omega(t,x,y)f(u(t-\tau(x,y),y))dy + I(t,x), \quad t \in [a, \infty), \ x \in \Omega \subset \mathbb{R}^m \]  

\[ u^0_t(t,x) = -u^0_t(t,x) + \int_{\mathbb{R}} \omega^0(x-y)f(u^0(t,y))dy, \quad t \geq 0, x \in \mathbb{R}, \]
Taking $\Omega = R^m \times \mathcal{Y}$ in (1.8) has been investigated by Venkov.

The Volterra formulation

$$u_t(t, x, y) = -u_0(t, x, y) + \int_{\mathcal{Y}} \int_{\Omega} \omega(x - y, x_f - y_f) f(u(t, y, y_f)) dy dy_f,$$

$$t > 0, \quad x \in R^m, \quad x_f \in \mathcal{Y} \subset R^m.$$  \hfill (1.5)

We do not consider external inputs $I(t, x)$ and $h$ (unlike [2], [13]) in our models, as they do not involve any nonlinearities and, hence, only make statements and proofs more cumbersome. We stress, however, that all the results below remain valid in the presence of the external inputs as well.

Note that we get (1.2) from (1.8) by taking

$$W(t, x, y) = \eta(t, s) \omega(x - y)$$

with

$$\eta(t, s) = \text{diag} \{ \exp(- (t-s)), \alpha \exp(- \alpha(t-s)) \} \text{ and } \tau(t, x, y) \equiv 0.$$  \hfill (1.7)

If we neglect $I(t, x)$ in (1.3), we can obtain (1.3) from (1.8) with

$$W(t, x, y) = \eta(t, s) \omega(t, x, y),$$

$$\eta(t, s) = \text{diag} \{ l_1 \exp(- l_1(t-s)), \ldots, l_n \exp(- l_n(t-s)) \}, \quad \tau(t, x, y) = \tau(x, y).$$

Taking $\Omega = R^m \times \mathcal{Y}$ ($\mathcal{Y}$ is some $m$-dimensional torus $[25]$),

$$x = (x_e, x_f), \quad y = (y_e, y_f),$$

$$W(t, x, y) = \exp(- (t - s)) \omega(x_e - y_e, x_f - y_f)$$

in (1.8) with

$$\tau(t, x, y) \equiv 0,$$  \hfill (1.6)
we get the model (1.5). Finally, with
\[ W(t, s, x, y) = \eta(t - s)\omega(x - y) \]
and
\[ \tau(t, x, y) = |x - y|/v \]
in (1.7), we obtain (1.6), which covers, in turn, the model (1.1) without the external inputs.

Our results generalize the results obtained by Potthast et al. [18] and Faye et al. [13] concerning existence of a unique solution to the Amari model (1.1) in the Banach space of continuous bounded functions and to the model (1.3) in the space of square integrable functions on a bounded domain, respectively. Here we also study dependence of solutions on the parameters.

The paper is organized in the following way. Section 2 is devoted to the study of local solvability, extendability and continuous dependence of solutions to operator Volterra equations on parameters. Building on these general results we investigate the models (1.7) and (1.8) in Section 3. Section 4 contains conclusions and an outlook.

We stress that one of the challenging parts of our study is application of the general theory of Volterra operators to the integral equations (1.7) and (1.8), which are defined on unbounded spatial and temporal domains. This general setting requires some conditions which are difficult to verify (see main theorems in Section 3). In two special cases, which are highly relevant for the neural field theory, we can however relax these conditions. The analogues of the main theorems for these special cases are formulated as remarks in Section 3 and their proofs are given in Appendix.

2 Existence, uniqueness and continuous dependence of solutions on parameters: the case of Volterra operator equations

Let us introduce the following notation:

- \( R^n \) is the space of vectors consisting of \( n \) real components with the norm \(| \cdot |\);
- \( \Omega \) is some closed subset of \( R^m \);
- \( \mathcal{B} \) is some Banach space with the norm \( \| \cdot \|_{\mathcal{B}} \);
- \( Y([a, b], \mathcal{B}) \) is a Banach space of functions \( y : [a, b] \to \mathcal{B} \) with the norm \( \| \cdot \|_Y \);
- \( \mathcal{B}(\Omega, R^n) \) is some Banach space of functions \( v : \Omega \to R^n \) with the norm \( \| \cdot \|_{\mathcal{B}(\Omega, R^n)} \);
- \( \Lambda \) is some metric space;
- \( \mu \) is the Lebesgue measure;
- \( L^p(\Omega, \mu, R^n) \) is the space of all measurable and integrable with \( p \)-th degree functions \( \chi : \Omega \to R^n \) with the norm \( \| \chi \|_{L^p(\Omega, \mu, R^n)} = \left( \int_{\Omega} |\chi(s)|^p \, ds \right)^{1/p}, 1 \leq p < \infty \);
- \( BC(\Omega, R^n) \) is the space of all continuous bounded functions \( \vartheta : \Omega \to R^n \) with the norm \( \| \vartheta \|_{BC(\Omega, R^n)} = \sup_{x \in \Omega} |\vartheta(x)| \);
- \( C_0(\Omega, R^n) \) is the space of all continuous functions \( \tilde{\vartheta} : \Omega \to R^n \) satisfying the additional condition \( \lim_{|x| \to \infty} |\tilde{\vartheta}(x)| = 0 \) in the case if \( \Omega \) is unbounded, with the norm \( \| \tilde{\vartheta} \|_{C_0(\Omega, R^n)} = \max_{x \in \Omega} |\tilde{\vartheta}(x)| \).
Definition 2.1. An operator \(\Psi : Y \to Y\) is said to be a Volterra operator (in the sense of A.N. Tikhonov \([20]\)) if for any \(\xi \in (0,b-a)\) and any \(y_1,y_2 \in Y\) the fact that \(y_1(t) = y_2(t)\) on \([a,a+\xi]\) implies that \((\Psi y_1)(t) = (\Psi y_2)(t)\) on \([a,a+\xi]\).

In what follows we assume that in the space \(Y\) the following condition is fulfilled:

\begin{enumerate}
\item \textbf{\(V\)-condition \([28]\):} For arbitrary \(y \in Y\), \(\{y_i\} \subset Y\) such that \(\|y_i - y\|_Y \to 0\) and for any \(\xi \in (0,b-a)\) if \(y_i(t) = 0\) on \([a,a+\xi]\), then \(y(t,x) = 0\) on \([a,a+\xi]\).
\end{enumerate}

For any \(\xi \in (0,b-a)\) let \(Y_\xi = Y((a,a+\xi],\mathcal{B})\) denote the linear space of restrictions \(y_\xi\) of functions \(y \in Y\) to \([a,a+\xi]\) which implies that for each \(y_\xi \in Y_\xi\) there exists at least one extension \(y \in Y\) of the function \(y_\xi\). Then we can define the norm of \(Y_\xi\) by \(\|y_\xi\|_Y = \inf\|y\|_Y\), where the infimum is taken over all extensions \(y \in Y\) of the function \(y_\xi\). Hence, the space \(Y_\xi\) becomes a Banach space.

For an arbitrary \(\xi \in (0,b-a)\) let an operator \(P_\xi : Y \to Y\) takes each \(y_\xi \in Y_\xi\) to some extension \(y \in Y\) of \(y_\xi\). Moreover, we define the operators \(E_\xi : Y \to Y_\xi\) by \((E_\xi y)(t) = y(t), t \in [a,a+\xi]\) and \(\Psi_\xi : Y_\xi \to Y_\xi\) by \(\Psi_\xi y_\xi = E_\xi \Psi E_\xi y_\xi\), respectively. Note that for any Volterra operator \(\Psi : Y \to Y\) the operator \(\Psi_\xi : Y_\xi \to Y_\xi\) is also a Volterra operator and it is independent of the way \(y = P_\xi y_\xi\) extends \(y_\xi\).

Definition 2.2. A Volterra operator \(\Psi : Y \to Y\) is called locally contracting if there exists \(q < 1\) such that for any \(r > 0\) one can find \(\delta > 0\) such that the following two conditions are satisfied for all \(y_1,y_2 \in Y\), such that \(\|y_1\|_Y \leq r, \|y_2\|_Y \leq r\):

\begin{enumerate}
\item \(q_1) \quad \|E_\delta \Psi y_1 - E_\delta \Psi y_2\|_{y_\delta} \leq q\|E_\delta y_1 - E_\delta y_2\|_{y_\delta},\)
\item \(q_2)\) for any \(y \in (0,\delta-a\],\) the condition \(E_\gamma y_1 = E_\gamma y_2\) implies that \(\|E_{\gamma+\delta} \Psi y_1 - E_{\gamma+\delta} \Psi y_2\|_{y_{\gamma+\delta}} \leq q\|E_{\gamma+\delta} y_1 - E_{\gamma+\delta} y_2\|_{y_{\gamma+\delta}}.\)
\end{enumerate}

The class of locally contracting operators is rather wide. It includes not only contracting operators, but also, e.g. \(\tau\)-Volterra operators.

Definition 2.3. An operator \(\Psi : Y \to Y\) is called \(\tau\)-Volterra if for any \(y_1,y_2 \in Y\) the condition \((\Psi y_1)(t) = (\Psi y_2)(t)\) holds true on \([a,a+\tau]\) and for any \(\xi \in [0,\delta-a\],\) if \(y_1(t) = y_2(t)\) on \([a,a+\xi]\), then \((\Psi y_1)(t) = (\Psi y_2)(t)\) on \([a,a+\xi+\tau]\).
Notice that τ-Volterra operators satisfy conditions \( q_1 \) and \( q_2 \) with \( q = 0 \) and \( δ = τ \), which are independent of a choice of \( r \).

Let us now consider the equation

\[
y(t) = (Ψy)(t), \ t \in [a, b],
\]

where \( Ψ : Y → Y \) is a Volterra operator.

**Definition 2.4.** We define a local solution to Eq. (2.1) on \([a, a+γ], \ γ \in (0, b-a)\) to be a function \( y_γ \in Y_γ \) that satisfies the equation \( Ψ_γy_γ = y_γ \) on \([a, a+γ]\). We define a maximally extended solution to Eq. (2.1) on \([a, a+ζ], \ ζ \in (0, b-a)\) to be a function \( y_ζ : [a, a+ζ) → B \), whose restriction \( y_γ \) to \([a, a+γ]\) is a local solution of Eq. (2.1) for any \( γ < ζ \) and \( \lim_{{γ → ζ-0}} \|y_γ\|_{Y_γ} = ∞ \). We define a global solution to Eq. (2.1) to be a function \( y \in Y \) that satisfies this equation on the entire interval \([a, b]\).

Let us now consider the equation

\[
y(t) = (F(y, λ))(t), \ t \in [a, b]
\]

with a parameter \( λ \in Λ \), where for each \( λ \in Λ \) a Volterra operator \( F(\cdot, λ) : Y → Y \) satisfies the property: \( F(\cdot, λ_0) = Ψ \) for some \( λ_0 \in Λ \). Our aim is to formulate conditions for existence and uniqueness of solutions to Eq. (2.2) on a certain fixed set \([a, a+ξ] \subset [a, b]\) (We, naturally, also apply Definition 4 to Eq. (2.2) at each fixed \( λ \in Λ \); and convergence of these solutions to solution to Eq. (2.1) in the norm of \( Y_ξ \) as \( λ \) approaches \( λ_0 \). This means, that the problem (2.2) is wellposed.

**Definition 2.5.** For any \( λ \in Λ_0 \subseteq Λ \), let the Volterra operator \( F(\cdot, λ) : Y → Y \) be given. This family of operators is called uniformly locally contracting if there exist \( q ≥ 0 \) and \( δ > 0 \), such that for each \( λ \in Λ_0 \subseteq Λ \) the operator \( F(\cdot, λ) : Y → Y \) is locally contracting with the constants \( q \) and \( δ \).

The following theorem represents our main tool to study of the wellposedness of the models (1.7) and (1.8). Minding future applications, we formulate this theorem here in a more general form than it is needed for the classical neural field theory.

**Theorem 2.1.** Assume that the following two conditions are satisfied:

1) There is a neighborhood \( U_0 \) of \( λ_0 \) where the operators \( F(\cdot, λ) : Y → Y, \ λ \in U_0 \) are uniformly locally contracting;

2) For arbitrary \( y \in Y \), the mapping \( F : Y × Λ → Y \) is continuous at \((y, λ_0)\).

Then for each \( λ \in U_0 \), Eq. (2.2) has a unique global or maximally extended solution, and each local solution is a restriction of this solution.

If Eq. (2.2) has a global solution \( y_0 \) at \( λ = λ_0 \), then for each \( λ \) (sufficiently close to \( λ_0 \)) it also has a global solution \( y = y(λ) \), and \( \|y(λ) − y_0\|_{Y} → 0 \) as \( λ → λ_0 \).

If Eq. (2.2) has a maximally extended solution \( y_{0ξ} \) defined on \([a, a+ξ) \) at \( λ = λ_0 \), then for any \( γ \in (0, ξ) \) one can find a neighborhood of \( λ_0 \) such that for any \( λ \) in this neighborhood Eq. (2.2) has a local solution \( y_γ = y_γ(λ) \) defined on \([a, a+γ] \) and \( \|y_γ(λ) − y_{0γ}\|_{Y_γ} → 0 \) as \( λ → λ_0 \).

**Proof.** Choose a fixed \( λ \in U_0 \). Let \( r > 0, ξ \in (0, b-a) \), \( y_ξ \in Y_ξ \), \( y_ξ \in Y \). Let \( B_Y(y_ξ, r) \) denote the set of functions \( y \in Y \) such that \( \|y砟\|_Y < r \) and \( Y([a, b], B, y_ξ) \) denote the set of functions \( y \in Y \) such that \( E_y = y_ξ \). Put \( B_Y([a, b], y_ξ, r) = B_Y(y_ξ, r) \cap Y([a, b], B, y_ξ) \).
We construct the solution in the following way. We set \( r_1 = (1-q)^{-1}||F(0,\lambda)||_y + 1 \) and find all \( \delta > 0 \) that satisfy the condition 1) with \( r = r_1 \). For \( \delta_1 = \frac{1}{2} \sup \{ \delta \} \), we have

\[
||E_{\delta_1} F(y,\lambda) - E_{\delta_1} F(u,\lambda)||_{y,\delta_1} \leq q ||E_{\delta_1} y - E_{\delta_1} u||_{y,\delta_1}
\]
at any \( y, u \in B_Y(0, r_1) \). Then \( F((B_Y(0, r)), \lambda) \subset B_Y(0, r) \) for \( B_Y(0, r) \) with \( r \geq r_1 \). By the Banach fixed point theorem \(^{[14]}\), p. 43\) the mapping \( E_{\delta_1} (\cdot, \lambda) \) has a fixed point \( \bar{y}_{\delta_1} \) in the ball \( B_{y_{\delta_1}} (0, r_1) \). This fixed point is a local solution to Eq. (2.2). Using the Banach theorem, one can also prove that for arbitrary \( \vartheta_1 \in (0, \delta_1) \) and any local solution \( \bar{y}_{\vartheta_1} \) to Eq. (2.2) defined on \([a, a+\vartheta_1]\) it holds that \( \bar{y}_{\vartheta_1} (t) = \bar{y}_{\vartheta_1} (t) \) for all \( t \in [a, a+\vartheta_1] \).

Choose \( r_2 = (1-q)^{-1}||F(P_{\delta_1}, y_{\delta_1}, \lambda)||_y + 1 \) and find all possible \( \delta > 0 \) that satisfy the condition 1) with \( r = r_2 \). For \( \delta_2 = \frac{1}{2} \sup \{ \delta \} \) at any \( y, u \in B_Y([a, a_0], y_{\delta_1}, \vartheta_2) \) we have

\[
||E_{\delta_1 + \delta_2} F(y,\lambda) P_{\delta_1} y_{\delta_1} - E_{\delta_1 + \delta_2} F(u,\lambda) y_{\delta_1 + \delta_2} || \leq q ||E_{\delta_1 + \delta_2} y - E_{\delta_1 + \delta_2} u||_{y_{\delta_1 + \delta_2}}
\]

According to the Banach theorem there exists a fixed point \( y_{\delta_1 + \delta_2} \) of the mapping \( F_{\delta_1 + \delta_2} (\cdot, \lambda) \) in \( B_Y([a, a + \delta_1 + \delta_2], y_{\delta_1}) \) and \( E_{\delta_1 + \delta_2} P_{\delta_1} y_{\delta_1}, r_2 ) \). This fixed point is a local solution to Eq. (2.2) defined on \([a, a + \delta_1 + \delta_2] \). It is an extension of the local solution \( y_{\delta_1} \). For any \( \vartheta_2 \in (0, \delta_2) \) and any local solution \( \bar{y}_{\delta_1 + \vartheta_2} \) at \( \bar{y}_{\vartheta_2} \) to Eq. (2.2) defined on \([a, a + \delta_1 + \vartheta_2] \), it holds that \( \bar{y}_{\vartheta_2} (t) = \bar{y}_{\vartheta_2} (t) \) for all \( t \in [a, a + \delta_1 + \vartheta_2] \).

Next, let us choose \( r_3 = (1-q)^{-1}||F(P_{\delta_1 + \delta_2}, y_{\delta_1 + \delta_2}, \lambda)||_y + 1 \), find all possible \( \delta > 0 \) that satisfy the condition 1) with \( r = r_3 \), repeat the procedure, etc.

If the norms of the obtained local solutions are uniformly bounded by some \( M \in R \), then for \( r = M + 1 \) due to the local contractivity of the operator \( F(\cdot, \lambda) : Y \rightarrow Y \) we find \( \delta \) such that \( \delta_1 \geq \frac{1}{2} \) at each of the steps described above. Therefore, in a finite number of steps we will obtain a unique global solution to Eq. (2.2). But if such \( M \) does not exist, then the number of steps becomes infinite. As a result, we obtain a unique maximally extended solution to Eq. (2.2).

We now prove the continuous dependence of solutions on a parameter \( \lambda \). Consider the case when, Eq. (2.2) has global solution \( y_0 = y(\lambda_0) \in Y \) at \( \lambda = \lambda_0 \). Let us find \( \delta > 0 \) satisfying the condition 1) at \( r = ||y_0||_y + 1 \), and any \( \lambda \in U_0 \). For \( k = \lfloor \frac{b l}{\delta} \rfloor + 1 \) denote \( \Delta_l = b l, l = 1, 2, \ldots, k \). Since the condition 2) holds true, for any \( \varepsilon > 0 \) one can find \( \sigma_1 > 0 \) and a neighborhood \( U_1 \) such that for each \( \lambda \in U_1 \) we have

\[
||F(u,\lambda) - F(y,\lambda)||_y < \frac{(1-q)\varepsilon}{6}
\]

for all \( u \in Y \) such that \( ||u - y||_y < \sigma_1 \). Assume that \( \sigma_1 < \frac{(1-q)\varepsilon}{6} \). Let us find \( \sigma_2 > 0 \) and \( U_2 \) such that for arbitrary \( \lambda \in U_2 \) it holds that

\[
||F_{\Delta_{k-1}} (u_{\Delta_{k-1}}, \lambda) - F_{\Delta_{k-1}} (y_{\Delta_{k-1}}, \lambda_0)||_{y_{\Delta_{k-1}}} < \frac{(1-q)\sigma_1}{6}
\]

for all \( u_{\Delta_{k-1}} \in Y_{\Delta_{k-1}}, ||u_{\Delta_{k-1}} - y_{\Delta_{k-1}}||_{y_{\Delta_{k-1}}} < \sigma_2 \). Assume that \( \sigma_2 < \frac{(1-q)\varepsilon}{6} \). There exist \( \sigma_3 > 0 \) and \( U_3 \) such that for any \( \lambda \in U_3 \) it holds true that

\[
||F_{\Delta_{k-2}} (u_{\Delta_{k-2}}, \lambda) - F_{\Delta_{k-2}} (y_{\Delta_{k-2}}, \lambda_0)||_{y_{\Delta_{k-2}}} < \frac{(1-q)\sigma_2}{6}
\]

for any \( u_{\Delta_{k-2}} \in Y_{\Delta_{k-2}}, ||u_{\Delta_{k-2}} - y_{\Delta_{k-2}}||_{y_{\Delta_{k-2}}} < \sigma_3 ; \sigma_3 < \frac{(1-q)\varepsilon}{6}, U_3 \subseteq U_2 \) etc. We perform \( k \) iterations and at the last step find \( \sigma_k \) and \( U_k, 0 < \sigma_k < \frac{(1-q)\varepsilon}{6}, U_k \subseteq U_{k-1} \).

Let \( y_{\Delta_1} \) denote a local solution to Eq. (2.2) at \( \lambda = \lambda_0 \), that is a fixed point of the operator \( F_{\Delta_1} (\cdot, \lambda_0) : Y_{\Delta_1} \rightarrow Y_{\Delta_1} \). If \( ||y_{\Delta_1} - y_{\Delta_0}||_{y_{\Delta_1}} < \sigma_k \), then

\[
||F_{\Delta_1} (u_{\Delta_1}, \lambda) - F_{\Delta_1} (y_{\Delta_1}, \lambda_0)||_{y_{\Delta_1}} < \frac{(1-q)\sigma_{k-1}}{6}
\]
for all $\lambda \in U_k$. Taking into account the condition 1), we get for any natural number $m$ that

$$\|F_{\lambda_1}(y_{\lambda_1}, \lambda) - y_{\lambda_1}\|_{U_{\lambda_1}} \leq \|F_{\lambda_1}(y_{\lambda_1}, \lambda) - F_{\lambda_2}(y_{\lambda_2}, \lambda)\|_{U_{\lambda_1}} + \ldots$$

$$\ldots + \|F_{\lambda_1}(y_{\lambda_1}, \lambda) - y_{\lambda_1}\|_{U_{\lambda_1}} \leq (q^{m-1} + \ldots + q + 1)(1 - q)\sigma_{k-1} \leq \frac{\sigma_{k-1}}{6}.$$  

Due to the convergence of the approximations $F_{\lambda}^m(y_{\lambda_1}, \lambda)$ to the fixed point $y_{\lambda_1} = y_{\lambda_1}(\lambda)$ of the operator $F_{\lambda_1}(\cdot): Y_{\lambda_1} \to Y_{\lambda_1}$ we obtain that $\|y_{\lambda_1} - y_{\lambda_1}\|_{U_{\lambda_1}} \leq \frac{\sigma_{k-1}}{6}$ for each $\lambda \in U_k$. Further, let $y_{\lambda_2}$ be a local solution to Eq. (2.2) at $\lambda = \lambda_0$ defined on $[a, a + \Delta_2] \times R^n$. Then, for all $\lambda \in U_k \subseteq U_{k-1}$ and any $u_{\lambda_2} \in B_{Y([a, a + \Delta_2], y_{\lambda_1})}(y_{\lambda_2}, \sigma_{k-1})$ we get

$$\|F_{\lambda}(u_{\lambda_2}, \lambda) - y_{\lambda_2}\|_{U_{\lambda_2}} = \|F_{\lambda}(u_{\lambda_2}, \lambda) - F_{\lambda}(y_{\lambda_2}, \lambda_0)\|_{U_{\lambda_2}} \leq \frac{(1 - q)\sigma_{k-2}}{6}.$$  

Then

$$\|F_{\lambda}(u_{\lambda_2}, \lambda) - u_{\lambda_2}\|_{U_{\lambda_2}} \leq \frac{(1 - q)\sigma_{k-2}}{6} < \frac{\sigma_{k-2}}{3}.$$  

For all $m = 1, 2, \ldots$ we have

$$\|F_{\lambda}^m(u_{\lambda_2}, \lambda) - u_{\lambda_2}\|_{U_{\lambda_2}} \leq \|F_{\lambda}^m(u_{\lambda_2}, \lambda) - F_{\lambda}^{m-1}(u_{\lambda_2}, \lambda)\|_{U_{\lambda_2}} + \ldots$$

$$\ldots + \|F_{\lambda}^m(u_{\lambda_2}, \lambda) - u_{\lambda_2}\|_{U_{\lambda_2}} \leq (q^{m-1} + \ldots + q + 1)(1 - q)\sigma_{k-2} \leq \frac{\sigma_{k-2}}{3}.$$  

Taking into account the convergence of the approximations $F_{\lambda}^m(u_{\lambda_2}, \lambda)$ to $y_{\lambda_2} = y_{\lambda_2}(\lambda)$ we obtain

$$\|y_{\lambda_2} - y_{\lambda_2}\|_{U_{\lambda_2}} \leq \|y_{\lambda_2} - F_{\lambda}^m(u_{\lambda_2}, \lambda)\|_{U_{\lambda_2}} +$$

$$+\|F_{\lambda}^m(u_{\lambda_2}, \lambda) - u_{\lambda_2}\|_{U_{\lambda_2}} + \|u_{\lambda_2} - y_{\lambda_2}\|_{U_{\lambda_2}} \leq \frac{\sigma_{k-2}}{3} + \sigma_{k-1} \leq \frac{\sigma_{k-2}}{2}.$$  

Using the convergence of sequential approximations $F_{\lambda}^m(u_{\lambda_1}, \lambda)$ to a fixed point $y_{\lambda_1} = y_{\lambda_1}(\lambda)$ of the operator $F_{\lambda_1}(\cdot): Y_{\lambda_1} \to Y_{\lambda_1}$ for any $u_{\lambda_2} \in B_{Y([a, a + \Delta_2], y_{\lambda_1})}(y_{\lambda_2}, \sigma_{k-2})$ and each $\lambda \in U_k \subseteq U_{k-1} \subseteq U_{k-2}$, we obtain the estimate $\|y_{\lambda_2} - y_{\lambda_2}\|_{U_{\lambda_2}} \leq \frac{\sigma_{k-2}}{3}$. We, then, repeat this procedure. At the $k$-th step we prove in an analogous way that the inequality $\|y(\lambda) - y_0\|_Y < \varepsilon$ holds true for all $\lambda \in U_k$. Therefore, $\|y(\lambda) - y_0\|_Y \to 0$ as $\lambda \to \lambda_0$.

Let now a solution $y_{\lambda_0}$ to Eq. (2.2) at $\lambda = \lambda_0$ be maximally extended. Fix arbitrary $\gamma \in (0, \eta)$ and let $y_{0\gamma}$ denote the restriction of the solution $y_{0\gamma}$ to $[a, a + \gamma] \times R^n$. For the equation $u_{\gamma} = F_{\gamma}(u_{\gamma}, \lambda_0)$ the function $y_{0\gamma} \in Y([a, a + \gamma] \times \Omega, R^n)$ is a global solution. As is shown above, for all $\lambda$ from some neighborhood of $\lambda_0$ the equations $u_{\gamma} = F_{\gamma}(u_{\gamma}, \lambda)$ have global solutions $y_{\gamma}(\lambda)$, and $\|y_{\gamma}(\lambda) - y_{0\gamma}\|_Y \to 0$ as $\lambda \to \lambda_0$.

The proof of Theorem 1 has several corollaries which are summarized in the following remarks:

**Remark 2.2.** If the constant $\delta$ in the condition 1) of Theorem 2.1 is independent of $r$, then Eq. (2.2) has a global solution. This is the case e.g. for $\tau$-Volterra operators.

**Remark 2.3.** In case of a priori boundedness of the solution, it is possible to extend the solution beyond the point $b$ in the same way as it was done in the proof of Theorem 2.1. This will give a unique solution defined on $[a, \infty)$. 


Notice that the existence of a maximally extended solution to Eq. (2.2) at $\lambda = \lambda_0$ does not guarantee the existence of maximally extended solutions to the equation (2.2) at $\lambda$ arbitrarily close to $\lambda_0$. The following example illustrates this fact.

**Example 2.1.** Let operators $\Phi(\cdot, \lambda) : L^1([0, \pi], \mu, R) \rightarrow L^1([0, \pi], \mu, R)$, $\lambda \in [0, \pi]$, be defined as

$$(\Phi(y, \lambda))(t) = \begin{cases} 0, & \text{if } t \in [0, \lambda); \\ \left( \frac{t - \lambda}{t} \right)^2 + 1, & \text{if } t \in [\lambda, \pi]. \end{cases}$$

These operators are Volterra operators and satisfy the condition 1) of Theorem 2.1: For $q = \frac{1}{2}$ and any $r > 0$ one can choose $\delta = \frac{1}{4r}$ and condition 1) becomes fulfilled for all $t \in [0, \pi]$ and any $\lambda \in [0, \pi])$. Condition 2) of the Theorem 2.1 is also fulfilled. The equation $y(t) = (\Phi(y, 0))(t)$, $t \in [0, \pi]$ has a unique maximally extended solution $y(t) = \frac{1}{\cos t}$ defined on $[0, \frac{\pi}{2}]$. Now, since for any $\lambda \in (0, \pi]$ the operator $\Phi(\cdot, \lambda)$ is a $\tau$-Volterra operator, the equation $y(t) = (\Phi(y, \lambda))(t)$, $t \in [0, \pi]$ is globally solvable for each $\lambda \in (0, \pi]$.

When analyzing Theorem 2.1, it is natural to ask the question whether the maximally extended solutions to Eq. (2.2) are defined on time intervals with arbitrarily small length. The following two remarks give answers to that question:

**Remark 2.4.** Let the assumptions of Theorem 2.1 be fulfilled and let there exist some neighborhood $\tilde{U} \subset U_0$ of $\lambda_0$ such that Eq. (2.2) has maximally extended solutions $y_{\xi}$, defined on $[a, a + \xi]$ for any $\lambda \in \tilde{U}$. Then $\inf_{\lambda \in \tilde{U}} y_{\xi} > 0$. Since for all $\lambda \in U_0$ operators $F(\cdot, \lambda)$ are uniformly locally contracting, we get $\inf_{\lambda \in \tilde{U}} y_{\xi} > 0$.

**Remark 2.5.** Let the assumptions of Theorem 2.1 be fulfilled and let for $\lambda = \lambda_0$ and some sequence $\lambda_i \subset U_0$ equation (2.2) have maximally extended solutions $y_{\gamma_i}$ and $y_{\zeta_i}$ defined on $[a, a + \gamma_i]$ and $[a, a + \zeta_i)$, respectively. Then $\beta = \min_{\gamma_i} \{\zeta_i \mid \inf_{\gamma_i} y_{\gamma_i} \geq 0\}$, and either $\beta = \zeta_i$, or $\beta = \zeta_i$, or $\beta = \zeta_i$, at some $i_0$.

The positivity of $\beta$ follows from Remark 3. Next, we choose arbitrary $\varepsilon > 0$ and a sequence $\gamma_j \ni (0, \beta)$, $\gamma_j \rightarrow \beta$, $j \rightarrow \infty$. For each $\gamma_j \ni (0, \beta)$ there exists a finite $\sup_{\gamma_j} \|y_{\gamma_j}\|_{\gamma_j}$ otherwise $\beta = \gamma_j$. Let us associate the number $\gamma_1$ with the corresponding local solution $y_{\gamma_1}$ to Eq. (2.2) at $\lambda = \lambda_1$, where $i_1$ is the least number such that $\max_j \{\|y_{\gamma_1}\|_{\gamma_1}, \sup_{\gamma_j} \|y_{\gamma_j}\|_{\gamma_j} - \|y_{\gamma_1}\|_{\gamma_1}\} < \varepsilon$; we associate the number $\gamma_2$ with the corresponding local solution $y_{\gamma_2}$ to Eq. (2.2) at $\lambda = \lambda_2$, where $i_2$ is the least number such that $\max_j \{\|y_{\gamma_2}\|_{\gamma_2}, \sup_{\gamma_j} \|y_{\gamma_j}\|_{\gamma_j} - \|y_{\gamma_2}\|_{\gamma_2}\} < \varepsilon$ etc. We obtain a subsequence $\{i_j\}$ of numbers of local solutions $y_{\gamma_j}$ to Eq. (2.2) such that $\|y_{\gamma_j}\|_{\gamma_j} \rightarrow \infty$ as $j \rightarrow \infty$. If the subsequence $\{i_j\}$ is bounded, then one can find a number $i_{j_0}$ such that $\lim_{j \rightarrow j_0} \|y_{\gamma_j}\|_{\gamma_j} = \infty$, i.e. $\zeta_{j_0} = \beta$. Otherwise, using the fact that $\|y_{\gamma_j} - y_{0}\|_{\gamma_j} \rightarrow 0$ as $j \rightarrow \infty$ for any $\gamma \ni (0, \beta)$ we obtain $\lim_{j \rightarrow j_0} \|y_{\gamma_j}\|_{\gamma_j} = \infty$, i.e. $\zeta = \beta$. 

**On Wellposedness of Generalized Neural Field Equations with Delay**
3 Existence, uniqueness and continuous dependence of solutions on parameters: the case of neural field equations

In this section we apply the results obtained in the previous section to a class of nonlinear integral equations, typical representatives of which can be found in the neural field theory. For the sake of convenience, we consider the following generalization of the model (1.8):

\[
\begin{align*}
  u(t, x) &= \varphi(a, x) + \int_\Omega \int_a^t W(t, s, x, y) f(u(t - \tau(s, x, y), y)) dy ds, \\
  &\quad t \in [a, \infty), x \in \Omega; \\
  u(\xi, x) &= \varphi(\xi, x), \quad \xi \leq a, x \in \Omega.
\end{align*}
\]

(3.1)

under the following assumptions on the functions involved:

(A1) For any \( b > a, (t, x) \in [a, b] \times \Omega \), the function \( W(t, \cdot, x, \cdot) : [a, b] \times \Omega \to \mathbb{R}^n \) is measurable.

(A2) For any \( b > a \), at almost all \((s, y) \in [a, b] \times \Omega \), the function \( W(\cdot, s, \cdot, y) : [a, b] \times \Omega \to \mathbb{R}^n \) is uniformly continuous.

(A3) For any \( b > a, t \in [a, b] \), \( \int_{\Omega} \sup_{s \in \Omega} |W(t, s, x, y)| dy \leq G(s) \), where \( G \in L^1([a, b], \mu, \mathbb{R}^n) \).

(A4) The function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is measurable and for any \( r > 0 \) one can find \( f_r > 0 \), such that for all \( u \in \mathbb{R}^n, |u| \leq r \), it holds true that \( |f(u)| \leq f_r \).

(A5) The delay function \( \tau : \mathbb{R} \times \Omega \times \Omega \to [0, \infty) \) is continuous on \( \mathbb{R} \times \Omega \times \Omega \).

(A6) The prehistory function \( \varphi \) belongs to \( C((-\infty, a), BC(\Omega, \mathbb{R}^n)) \).

The model (3.1) with \( \varphi(\xi, x) \equiv 0 \) can be obtained from (1.7) by taking \( W(t, s, x, y) = \eta(t, s) \omega(x, y) \), where, e.g.

\[
\eta(t, s) = \begin{cases} 
  \kappa \exp(-\kappa(t - s)), & \text{if } t \geq a; \\
  0, & \text{if } t < a;
\end{cases}
\]

or

\[
\eta(t, s) = \begin{cases} 
  \kappa(t - s) \exp(-\kappa(t - s)), & \text{if } t \geq a; \\
  0, & \text{if } t < a
\end{cases}
\]

and \( \omega \) can be represented by the "Mexican hat"

\[
\omega(x, y) = M \exp(-m|x - y|) - K \exp(-k|x - y|)
\]

or the "wizard hat"

\[
\omega(x, y) = M(1 - |x - y|) \exp(-m|x - y|),
\]

and

\[
f(u) = \begin{cases} 
  u^\kappa/(\theta^\kappa + u^\kappa), & \text{if } u \geq 0; \\
  0, & \text{if } u < 0,
\end{cases}
\]

for some \( \kappa > 0, \theta > 0, M > K > 0, \text{and } m > k > 0 \). These functions satisfy the conditions (A1) – (A4). The condition (A4) is also fulfilled e.g. for the sigmoidal functions

\[
f(u) = \frac{1}{2}(1 + \tanh(\kappa(u - \theta)))
\]
or

\[
f(u) = \frac{1}{1 + \exp(-\kappa(u - \theta))}
\]

with some positive \(\kappa\) and \(\theta\). We do not assume in (A4) that function \(f\) is bounded (as in the classical neural field theory), because it allows us to obtain more general results which may have other applications. If we take the delay functions \(\tau(t,x,y) = |x - y|/v\) for some positive velocity \(v\) or \(\tau(t,x,y) = d(x,y)\) with continuous function \(d : \mathbb{R} \times \mathbb{R} \to [0,\infty)\) from [24] and [13], respectively, we find out that the condition (A5) is also satisfied.

We introduce the definition of local, maximally extended and global solutions just as in the previous section (Definition 2.4).

**Definition 3.1.** We define a *local solution* to Eq. (3.1) on \([a,a+\gamma] \times \mathbb{R}^n\), \(\gamma \in (0,\infty)\) to be a function \(u_{\gamma} \in C([a,a+\gamma], BC(\Omega, \mathbb{R}^n))\) that satisfies the equation (3.1) on \([a,a+\gamma] \times \mathbb{R}^n\). We define a *maximally extended solution* to Eq. (3.1) on \([a,a+\zeta] \times \Omega\), \(\zeta \in (0,\infty)\) to be a function \(u_{\zeta} : [a,a+\zeta] \times \Omega \to \mathbb{R}^n\), whose restriction \(u_{\gamma}\) to \([a,a+\gamma] \times \Omega\) for any \(\gamma < \zeta\) is a local solution of Eq. (3.1) and \(\lim_{\gamma \to \zeta^-} \|u_{\gamma}\|_{C([a,a+\gamma], BC(\Omega, \mathbb{R}^n))} = \infty\). We define a *global solution* to Eq. (3.1) to be a function \(u : [a,\infty) \times \Omega \to \mathbb{R}^n\), whose restriction \(u_{\gamma}\) to \([a,a+\gamma] \times \Omega\) is its local solution for any \(\gamma \in (0,\infty)\).

**Theorem 3.1.** Let the assumptions (A1) – (A6) hold true. If for any \(r > 0\) there exists \(\overline{f}_r \in \mathbb{R}\) such that for all \(u_1, u_2 \in \mathbb{R}^n\), \(|u_1| \leq r\), \(|u_2| \leq r\), we have \(|f(u_1) - f(u_2)| \leq \overline{f}_r|u_1 - u_2|\), then Eq. (3.1) has a unique global or maximally extended solution and each local solution is a restriction of this solution.

**Proof.** We will use Theorem 2.1, namely, the condition 1), which is responsible for solvability of the Eq. (2.2)) and Remark 2.2 of the previous section to prove the solvability of (3.1).

First, we choose an arbitrary \(b \in (a,\infty)\), define the following operator

\[
(Fu)(t,x) = \varphi(a,x) + \int_a^t \int_\Omega W(t,s,x,y) f((S_{\tau}u)(s,x,y)) dy ds,
\]

and show that

\[
F : C([a,b], BC(\Omega, \mathbb{R}^n)) \to C([a,b], BC(\Omega, \mathbb{R}^n)).
\]

For any \(t \in [a,b]\) and \(u \in C([a,b], BC(\Omega, \mathbb{R}^n))\) we have

\[
|(Fu)(t,x_1) - (Fu)(t,x_2)| \leq \\
\leq |\varphi(a,x_1) + \int_a^t \int_\Omega W(t,s,x_1,y) f((S_{\tau}u)(s,x_1,y)) dy ds - \\
-\varphi(a,x_2) + \int_a^t \int_\Omega W(t,s,x_2,y) f((S_{\tau}u)(s,x_2,y)) dy ds| \leq \\
\leq |\varphi(a,x_1) - \varphi(a,x_2)|
\]
By the virtue of the assumption (A6), the first term goes to 0 as \(|x_1 - x_2| \to 0\). The assumptions (A2) – (A4) and (A6) guarantee convergence to 0 of the second term on the right hand side of this inequality as \(|x_1 - x_2| \to 0\). The superposition \(f((S\tau u)(s, \cdot, y))\) is continuous as the assumptions (A4) – (A6) hold true. This fact and the assumption (A3) imply convergence of the last term to 0 as \(|x_1 - x_2| \to 0\). This proves continuity of \((Fu)(t, \cdot)\).

For each \(t \in [a, b]\) and any \(u \in C([a, b], \text{BC}(\Omega, \mathbb{R}^n))\) the function \((Fu)(t, \cdot)\) is bounded by the virtue of the assumptions (A3), (A4) and (A6).

Finally, we choose an arbitrary \(u \in C([a, b], \text{BC}(\Omega, \mathbb{R}^n))\) and, assuming that \(t_2 > t_1\), check that \((Fu)(\cdot, x)\) is continuous:

\[
\sup_{x \in \Omega} |(Fu)(t_1, x) - (Fu)(t_2, x)| \leq
\]

\[
\leq \sup_{x \in \Omega} \int_a^{t_1} \int_{\Omega} W(t_1, s, x, y) f((S\tau u)(s, x, y)) dyds -
\]

\[
- \int_a^{t_2} \int_{\Omega} W(t_2, s, x, y) f((S\tau u)(s, x, y)) dyds | 
\]

\[
\leq \sup_{x \in \Omega} \int_a^{t_1} \int_{\Omega} (W(t_1, s, x, y) - W(t_2, s, x, y)) f((S\tau u)(s, x, y)) dyds | +
\]

\[
+ \sup_{x \in \Omega} \int_{t_1}^{t_2} \int_{\Omega} W(t_2, s, x, y) f((S\tau u)(s, x, y)) dyds | \leq
\]

\[
\leq \int_a^{t_1} \sup_{x \in \Omega} |W(t_1, s, x, y) - W(t_2, s, x, y)| \sup_{x \in \Omega} |f((S\tau u)(s, x, y))| dyds +
\]

\[
+ \int_{t_1}^{t_2} \sup_{x \in \Omega} |W(t_2, s, x, y)| \sup_{x \in \Omega} |f((S\tau u)(s, x, y))| dyds.
\]

We note that by the virtue of the assumptions (A2) – (A4) and (A6), the first term converges to 0 as \(t_1 - t_2 \to 0\). The second summand goes to 0 as the assumptions (A3), (A4) and (A6) hold true and \(t_1 - t_2 \to 0\).

Thus we proved that \(F : C([a, b], \text{BC}(\Omega, \mathbb{R}^n)) \to C([a, b], \text{BC}(\Omega, \mathbb{R}^n))\).

Next, we examine the fulfilment of Theorem 2.1 condition for the defined above operator \(F : C([a, b], \text{BC}(\Omega, \mathbb{R}^n)) \to C([a, b], \text{BC}(\Omega, \mathbb{R}^n))\). Choose an arbitrary \(q_0 < 1, r > 0\). Let \(\gamma \in (0, b - a)\)
and \( u_1(t, \cdot) = u_2(t, \cdot), \ t \in [a, a+\gamma], \) where \( \|u_1\|_{C([a,b],BC(\Omega,R^n))} \leq r \) and \( \|u_2\|_{C([a,b],BC(\Omega,R^n))} \leq r. \) By assumption, we get the estimates

\[
\sup_{t \in [a, a+\gamma+\delta], x \in \Omega} \int_0^t \left| \int_\Omega W(t, s, x, y) f\left( (S^p_t u_1)(s, x, y) \right) dy \right| ds - \int_0^t \int_\Omega W(t, s, x, y) f\left( (S^p_t u_2)(s, x, y) \right) dy ds \leq 0.
\]

Remark 3.1. If in the Theorem 3.1 condition \( \bar{f}_r = \bar{f} \) is independent of \( r \) (as e.g. in classical neural field models, where \( 0 \leq f(u) \leq 1 \)), then according to Remark 2.1 we will get a global solution to the Eq. (3.1). In this case, if we take \( \tau(t, x, y) \equiv 0, \) Theorem 3.1 becomes analogous to the results concerning solvability of the Amari model obtained by Potthast et al. [13].

Remark 3.2. If in Theorem 3.1 the condition \( \bar{f}_r = \bar{f} \) is independent of \( r, \) Theorem 3.1 can be compared to the theorem on solvability of Eq. (1.3) in \( C([a,b],L^2(\Omega,R^n)) \) for any \( b > a \) proved in Faye et al. [13]. Here we obtained the same result for the more general model (3.1) in \( C([a,b],BC(\Omega,R^n)). \) We note that in case when the delay \( \tau(t, x, y) = \tau(x, y) \) is independent of \( t, \) it is possible to prove Theorem 3.1 for the space \( C([a,b],L^2(\Omega,R^n)) \) using our technique as well thus getting the main theoretical result of [13].
Note that the remarks 3 and 4 on maximally extended solutions are valid for the problem (3.1) as well.

It is also worth mentioning that our approach to delayed functional-differential equations is based on the idea to include the prehistory function in the inner superposition operator. It allows us to consider the operator equation (2.1) with the operator (3.2) defined on \([a,b]\) instead of \((-\infty,b]\).

The same approach to functional-differential equations with delay was implemented e.g. in [5], [6].

Next we complete the study of wellposedness of the problem (3.1) by investigating continuous dependence of solutions to the associated problem

\[ u(t,x) = \varphi_{\lambda}(a,x) + \int_a^t \int_{\Omega} W_{\lambda}(t,s,x,y)f_{\lambda}(u(t-\tau_{\lambda}(s,x,y),y))dyds, \quad t \in [a,\infty), x \in \Omega; \]

\[ u(\xi,x) = \varphi_{\lambda}(\xi,x), \xi \leq a, x \in \Omega \]

on a parameter \(\lambda \in \Lambda\).

The assumptions (A_{\lambda}1) – (A_{\lambda}6) imposed on the functions in the model (3.3) for each \(\lambda \in \Lambda\) repeat the assumptions (A1) – (A6), respectively.

We will naturally apply Definition 3.1 to the model (3.3) at each \(\lambda \in \Lambda\).

The following theorem gives conditions that guarantee wellposedness of the problem (3.3).

**Theorem 3.2.** Let the assumptions (A_{\lambda}1) – (A_{\lambda}6) hold true. Assume that the following conditions are satisfied:

1) There is a neighborhood \(U_0\) of \(\lambda_0\) such that for any \(r > 0\) there exists \(\overline{f}_r \in R\) (independent of \(\lambda \in U_0\)) such that for which \(|f_{\lambda}(u_1) - f_{\lambda}(u_2)| \leq \overline{f}_r|u_1 - u_2|\) for all \(u_1, u_2 \in \mathbb{R}^n, |u_1| \leq r, |u_2| \leq r\).

For any \(\lambda_i \in \Lambda, \lambda_i \to \lambda_0\) it holds true that:

2) For any \(b > a,\)

\[
\sup_{r \in [a,b], x \in \Omega} \left| \int_a^t \int_{\Omega} W_{\lambda_i}(t,s,x,y)dyds - \int_a^t \int_{\Omega} W_b(t,s,x,y)dyds \right| \to 0;
\]

3) For any \(b,a,\) if \(|u_i(\cdot) - u(\cdot)| \to 0\) in measure on \([a,b] \times \Omega\) as \(i \to \infty,\) then \(|f_{\lambda_i}(u_i(\cdot)) - f_{\lambda_0}(u(\cdot))| \to 0\) in measure on \([a,b] \times \Omega\) as \(i \to \infty;\)

4) For any \(b > a,\) \(\sup_{x \in \Omega} |\tau_{\lambda_i}(\cdot,x) - \tau_{\lambda_0}(\cdot,x)| \to 0\) in measure on \([a,b] \times \Omega;\)

5) \(\|\varphi_{\lambda_i} - \varphi_{\lambda_0}\|_{C((-\infty,a],BC(\Omega,\mathbb{R}^n))} \to 0.\)

Then there is a neighborhood \(U\) of \(\lambda_0\) such that for each element \(\lambda \in U, Eq. (3.3)\) has a unique global or maximally extended solution, and each local solution is a restriction of this solution. Moreover, if at \(\lambda = \lambda_0\) Eq. (3.3) has a local solution \(u_{0\gamma}\) defined on \([a,a+\gamma] \times \Omega,\) then for any
\{\lambda_i\} \subset \Lambda, \lambda_i \to \lambda_0 \text{ one can find number } I \text{ such that for all } i > 1 \text{ Eq. (3.3) has a local solution } u_{\gamma} = u_{\gamma}(\lambda_i) \text{ defined on } [a, a+\gamma) \times \Omega \text{ and } \|u_{\gamma}(\lambda_i) - u_{\gamma}\|_{C([a,a+\gamma],BC(\Omega,R^n))} \to 0.

Proof. Choose an arbitrary \( b \in (a, \infty) \). In order to use Theorem 2.1, we need to bring the Eq. (3.3) to the form \( u(t, \cdot) = (F(u, \lambda))(t) \), \( t \in [a, b] \). Using the same technique as in the proof of Theorem 3.1 and the corresponding assumptions (A_1) - (A_6), we get here

\[ (F(u, \lambda))(t, x) = \varphi_{\lambda}(a, x) + \int_{a}^{t} \int_{\Omega} W_{\lambda}(t, s, x, y) f_{\lambda}(S_{\lambda}^{\psi} u)(s, x, y) \, dy \, ds, \]

\( t \in [a, b], x \in \Omega, \)

\[ (S_{\lambda}^{\psi} u)(t, x, y) = \left\{ \begin{array}{ll} \varphi_{\lambda}(t - \tau_{\lambda}(t, x, y), y), & \text{if } t - \tau_{\lambda}(t, x, y) < a; \\ u(t - \tau_{\lambda}(t, x, y), y), & \text{if } t - \tau_{\lambda}(t, x, y) \geq a \end{array} \right. \]

for all \( \lambda \in \Lambda. \)

The condition 1) of this theorem allows us to verify the assumption 1) of Theorem 2.1 for each \( \lambda \in U_0 \) by the same procedure as we used in the proof of Theorem 2. So, we only need to verify the condition 2) of Theorem 1.

Choose an arbitrary \( u \in C([a, b], BC(\Omega, R^n)) \). Let \( \|u_i - u\|_{Y} \to 0 \), i.e., \( \|u_i - u\|_{C([a, b], BC(\Omega, R^n))} \to 0 \), \( i \to \infty \), and \( \lambda \to \lambda_0 \).

We have the following estimates:

\[ |(S_{\lambda}^{\psi} u_i)(t, x, y) - (S_{\lambda_0}^{\psi} u)(t, x, y)| \leq |(S_{\lambda}^{\psi} u_i)(t, x, y) - (S_{\lambda}^{\psi} u)(t, x, y)| + |(S_{\lambda}^{\psi} u)(t, x, y) - (S_{\lambda_0}^{\psi} u)(t, x, y)|.\]

If \( \lambda \to \lambda_0 \), then the first term on the right-hand side of this inequality goes to 0 uniformly as \( \|u_i - u\|_{C([a, b], BC(\Omega, R^n))} \to 0 \). By the virtue of the condition 4), the second term on the right-hand side goes to 0 in measure on \((a, b) \times \Omega\), uniformly in \( x \in \Omega \), as \( \lambda \to \lambda_0 \). The third term on the right-hand side of the inequality goes to 0 uniformly when \( \lambda \to \lambda_0 \) as the condition 5) holds true. Thus, we have

\[ |(S_{\lambda}^{\psi} u_i)(\cdot, x, \cdot) - (S_{\lambda_0}^{\psi} u)(\cdot, x, \cdot)| \to 0 \]

in measure, uniformly in \( x \in \Omega \), as \( \|u_i - u\|_{C([a, b], BC(\Omega, R^n))} \to 0 \) and \( \lambda \to \lambda_0 \).

Using this convergence, we can make the following estimates

\[ \sup_{\tau \in [a, b]} \left| \int_{a}^{t} \int_{\Omega} W_{\lambda}(t, s, x, y) f_{\lambda}(S_{\lambda}^{\psi} u_i)(s, x, y) \, dy \, ds \right| \leq \]

\[ \int_{a}^{t} \int_{\Omega} W_{\lambda}(t, s, x, y) f_{\lambda}(S_{\lambda_0}^{\psi} u)(s, x, y) \, dy \, ds \leq \]

\[ \sup_{\tau \in [a, b]} \left| \int_{a}^{t} \int_{\Omega} W_{\lambda}(t, s, x, y) f_{\lambda}(S_{\lambda}^{\psi} u_i)(s, x, y) \, dy \, ds \right| \]
Taking into account the condition 3), we conclude that the first term on the right-hand side of the inequality goes to 0 as $\lambda \to \lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as $\lambda \to \lambda_0$.

Thus, the condition 2) of Theorem 2.1 is satisfied and Theorem 3.2 is proved. $\square$

We emphasize here that our aim was to formulate the assumptions on the functions involved in the model (3.3) (see conditions 2) – 5) of Theorem 3.2) as general as it possible. Of course, we can strengthen these assumptions in order to make them more conventional e.g. in the following way.

**Remark 3.3.** If the estimate in the assumption ($A_{\lambda}$3) holds true uniformly with respect to $\lambda \in \Lambda$, then it is possible to get the conclusion of Theorem 3.2 by claiming that for any $b > a$ the functions

$$W_{(\cdot)} : \Lambda \times [a,b] \times [a,b] \times \Omega \times \Omega \to R^n,$$

$$f_{(\cdot)} : \Lambda \times R^n \to R^n,$$

$$\tau_{(\cdot)} : \Lambda \times [a,b] \times \Omega \times \Omega \to [0,\infty),$$

$$\varphi_{(\cdot)} : \Lambda \times (\infty,b] \times \Omega \to R^n$$

are continuous instead of claiming the conditions 2) – 5) of Theorem 3.2.

We now consider two important special cases of the model (3.3).

As the neural field theory studies processes in cortical tissue, it is realistic to assume that $\Omega$ is bounded (see e.g. [13]). The following remark represents the result, analogous to Theorem 3.2 for this case.

**Remark 3.4.** If $\Omega$ is bounded, we can substitute ($A_{\lambda}$6) by

($A_{\lambda}^*$6) For any $a^* < a$ and each $\varphi_{\lambda} \in C([a^*, a], C_0(\Omega, R^n)), \lambda \in \Lambda$.

In order to get the conclusion of Theorem 3.2, we need the following conditions instead of the conditions 3), 4), and 5), respectively:

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

3*) For any $u \in R^n$ we have $|f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0$;

4*) For all $x \in \Omega$, $|\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0$ in measure on $[a,b] \times \Omega$;

5*) For any $a^* < a$ and all $(t, x) \in [a^*, a] \times \Omega$, $|\varphi_{\lambda_i}(t, x) - \varphi_{\lambda_0}(t, x)| \to 0$. 

Proof of the statement in Remark 3.4 is given in Appendix A.

In neural field modeling special attention is paid to spatially localized solutions, so-called "bumps". If \( \Omega \) is unbounded, but the solution to (3.3) is spatially localized, we can relax Theorem 3.2 conditions in the following way.

**Remark 3.5.** If we replace (A.6) by

\( (A'_6) \) For each \( \lambda \in \Lambda \), the prehistory function \( \varphi_{\lambda} \in C((-\infty,a], C(\Omega, R^n)) \);
and impose the additional condition, corresponding to localization in the spatial variable,

\( (A'_7) \) For each \( \lambda \in \Lambda \) and any \( b > a \) \( \lim_{|x| \to \infty} |W_{\lambda}(t,s,x,y)| = 0 \) for all \( (t,s,y) \in [a,b] \times [a,b] \times \Omega \),
then, in order to get the conclusion of Theorem 3.2 holds true for spatially localized solutions, we need the following conditions, instead of 2), 3), 4), and 5) respectively:

For any \( \{ \lambda_i \} \subset \Lambda \), \( \lambda_i \to \lambda_0 \) it holds true that:

2') For any \( b > a \), \( r > 0 \), and each \( t \in [a,b] \), \( x \in \Omega \), \( |x| \leq r \) it holds true that

\[
\left| \int_a^t \int_{\Omega} \left( W_{\lambda_i}(t,s,x,y) dy - W_{\lambda_0}(t,s,x,y) \right) dy ds \right| \to 0;
\]

3') For any \( u \in R^n \) we have \( |f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0 \);

4') For all \( x \in \Omega \), \( |\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0 \) in measure on \([a,b] \times \Omega \);

5') For any \( (t,x) \in (-\infty,a] \times \Omega \), \( |\varphi_{\lambda_i}(t,x) - \varphi_{\lambda_0}(t,x)| \to 0 \).

Proof of the statement in Remark 3.5 is given in Appendix B.

As Theorems 2 and 3 are valid for each \( a \in R \) in the model (3.1), it is natural to address the question, what happens in the case when \( a = -\infty \) (i.e., when (3.1) becomes (1.7)).

**Remark 3.6.** Solution to (1.7) is not necessarily unique.

The following example illustrates this fact.

**Example 3.1.** Consider the equation

\[
u(t,x) = \int_{-\infty}^t \int_{R} \exp(-s)\omega(x)u(s,y)dy ds, \ t \in R, \ x \in R
\]

with some Gaussian function \( \omega \). Define the function \( u \in C((-\infty,b], BC(R,R)) \) as follows:

\[
u(t,x) = v(t)\omega(x),
\]
where

\[ v(t) = V \exp(-\exp(-t)), V \in \mathbb{R}, \]

is a solution to

\[ \dot{v}(t) = \exp(-t)v(t), \]

satisfying the property \( v(t) \to 0 \) as \( t \to -\infty \). Thus, for any \( V \in \mathbb{R} \) we get a solution to (1.7) which belongs to \( C((\infty, b], BC(\Omega, R^n)) \).

Nevertheless, it is possible to find conditions, which guarantee wellposedness of the model (1.7). The last part of the present paper is devoted to this problem. We have the following assumptions on the functions involved:

\( (\mathcal{A}1) \) For any \( a, b \in R, a < b, t \in [a,b], x \in \Omega, \) the function \( W(t, \cdot, x, \cdot) : [a,b] \times \Omega \to R^n \) is measurable.

\( (\mathcal{A}2) \) For any \( a, b \in R, a < b, \) at almost all \( (s, y) \in [a,b] \times \Omega, \) the function \( \dot{W}(\cdot, s, \cdot, y) : (-\infty, b] \times \Omega \to R^n \) is uniformly continuous.

\( (\mathcal{A}3) \) For any \( b \in R, t \in (-\infty, b], \int \sup \Omega W(t, s, x, y) dy = G(s), \) where \( G \in L^1((-\infty, b], \mu, R^n) \).

Assumptions \( (\mathcal{A}4) \) and \( (\mathcal{A}5) \) are the same as the corresponding assumptions \( (A4) \) and \( (A5) \).

Now, we need to give the definitions of local, maximally extended and global solutions to Eq. (1.7).

**Definition 3.2.** We define a local solution to Eq. (1.7) on \( (-\infty, \gamma) \times \Omega \), \( \gamma \in R \), to be a function \( u_\gamma \in C((-\infty, \gamma], BC(\Omega, R^n)) \) that satisfies the equation (1.7) on \( (-\infty, \gamma) \times \Omega \). We define a maximally extended solution to Eq. (1.7) on \( (-\infty, \zeta) \times \Omega \), \( \zeta \in R \) to be a function \( u_\zeta : (-\infty, \zeta) \times \Omega \to R^n \), whose restriction \( u_\gamma \) to \( (-\infty, \gamma) \times \Omega \) is a local solution to Eq. (1.7) for any \( \gamma < \zeta \) and \( \lim_{\gamma \to -\infty} \| u_\gamma \|_{C((-\infty, \gamma], BC(\Omega, R^n))} = \infty \). We define a global solution to Eq. (1.7) to be a function \( u : R \times \Omega \to R^n \), whose restriction \( u_\gamma \) to \( (-\infty, \gamma) \times \Omega \) is its local solution for any \( \gamma \in R \).

**Theorem 3.3.** Let the assumptions \( (\mathcal{A}4) - (\mathcal{A}5) \) hold true. If for any \( r > 0 \) there exists \( \bar{f}_r \in R \) such that for all \( u_1, u_2 \in R^n, |u_1| \leq r, |u_2| \leq r, \) we have \( \| f(u_1) - f(u_2) \| \leq \bar{f}_r, |u_1 - u_2| \), then Eq. (1.7) has a unique global or maximally extended solution and each local solution is a restriction of this global or maximally extended solution (all types of solutions are meant in the sense of Definition 3.2).

**Proof.** First, we prove existence of a unique local solution to (1.7). Choose arbitrary \( b \in R \). Using the same estimation technique as in the proof of Theorem 3.1 and the corresponding assumptions \( (\mathcal{A}1) - (\mathcal{A}5) \), we rewrite Eq. (1.7) as the operator equation \( u(t, \cdot) = (Fu)(t) \), and consider it on \( (-\infty, b] \), where

\[ F : C((-\infty, b], BC(\Omega, R^n)) \to C((-\infty, b], BC(\Omega, R^n)), \]

\[ (Fu)(t) = \int_{-\infty}^{t} \int \Omega W(t, s, x, y)f(u(s - \tau(s, x, y), y)) dy ds, \]

\( t \in [a, b], x \in \Omega. \)

Choose arbitrary \( q_0 < 1, r > 0, \| u_1 \|_{C((-\infty, b], BC(\Omega, R^n))} \leq r \) and \( \| u_2 \|_{C((-\infty, b], BC(\Omega, R^n))} \leq r \). In order to prove existence of a unique local solution to (1.7) using the Banach fixed point theorem, we need to
find $\delta \in R$ such that

$$\max_{t \in [−\infty, \delta]} ||(Fu_1)(t) − (Fu_2)(t)||_{BC(\Omega, R^d)} \leq q_0 \max_{t \in [−\infty, \delta]} ||(u_1)(t) − (u_2)(t)||_{BC(\Omega, R^d)}.$$  

For any $\delta < b$, we get the estimates

$$\sup_{t \in (−\infty, \delta), x \in \Omega} \left| \int_{−\infty}^{t} \int_{\Omega} W(t, s, x, y) f\left(u_1(s − \tau(s, x, y), y)\right) dyds − \right.$$

$$\left. − \int_{−\infty}^{t} \int_{\Omega} W(t, s, x, y) f\left(u_2(s − \tau(s, x, y), y)\right) dyds \right| \leq$$

$$\leq \sup_{t \in (−\infty, \delta), x \in \Omega} \left| \int_{−\infty}^{t} \int_{\Omega} W(t, s, x, y) \left(f\left(u_1(s − \tau(s, x, y), y)\right) − \right.ight.$$  

$$\left. − f\left(u_2(s − \tau(s, x, y), y)\right)\right) dyds \right| \leq$$

$$\leq \sup_{t \in (−\infty, \delta), x \in \Omega} \int_{−\infty}^{t} \int_{\Omega} |W(t, s, x, y)| \tilde{f}_r dyds \| u_1 − u_2 \|_{BC(−\infty, \delta) \times \Omega, R^d)} \leq$$

$$\leq q \| u_1 − u_2 \|_{BC(−\infty, \delta) \times \Omega, R^d)}.$$  

Here

$$q = \tilde{f}_r \sup_{t \in (−\infty, \delta), x \in \Omega} \int_{−\infty}^{t} \int_{\Omega} |W(t, s, x, y)| dyds.$$  

Using the assumption (\(A_3\)), we can find $\delta > 0$ such that $q \leq q_0$. Thus, the equation (1.7) has a unique local solution, defined on $(-\infty, \delta] \times \Omega$. Now, regarding this solution as a prehistory function for the model (3.1) and taking $a = \delta$, we use Theorem 3.1 and obtain the conclusion of the theorem.  

In order to approach the problem of wellposedness of (1.7), we consider its parameterized version:

$$u(t, x) = \int_{−\infty}^{t} \int_{\Omega} W_\lambda(t, s, x, y) f_\lambda(u(s − \tau_\lambda(s, x, y), y)) dyds,$$

$$t \in R, x \in \Omega,$$

(3.4)

with a parameter $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$, the assumptions (\(A_1\)) – (\(A_5\)), imposed on the functions involved in the model (3.4) repeat the assumptions (\(A_1\)) – (\(A_5\)), respectively.

At each $\lambda \in \Lambda$ we define the types of solutions to (3.4) according to Definition 3.2.

**Theorem 3.4.** Let the assumptions (\(A_1\)) – (\(A_5\)) hold true. Assume that the following conditions are satisfied:
1) There is a neighborhood $U_0$ of $\lambda_0$ such that for any for any $r > 0$ there exists $f_\lambda \in R$ (independent of $\lambda \in U_0$), for which $|f_\lambda(u_1) - f_\lambda(u_2)| \leq f_\lambda|u_1 - u_2|$ for all $u_1, u_2 \in R^n$, $|u_1| \leq r$, $|u_2| \leq r$.

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

2) For any $b \in R$, $\sup_{(0, b)} \int_{\Omega} W_\lambda(t, s, x, y) dy - \int_{\Omega} W_{\lambda_0}(t, s, x, y) dy \to 0$;

3) For any $b \in R$, if $|u_i(\cdot, \cdot) - u(\cdot, \cdot)| \to 0$ in measure on $(-\infty, b] \times \Omega$ as $i \to \infty$, then $|f_\lambda(u_i(\cdot, \cdot)) - f_{\lambda_0}(u(\cdot, \cdot))| \to 0$ in measure on $(-\infty, b] \times \Omega$ as $i \to \infty$;

4) For any $b \in R$, $\sup_{x \in \Omega} |\tau_\lambda(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0$ in measure on $(-\infty, b] \times \Omega$.

Then there is a neighborhood $U$ of $\lambda_0$, such that for each $\lambda \in U$, Eq. (3.4) has a unique global or maximally extended solution, and each local solution is a restriction of this solution. Moreover, if at $\lambda = \lambda_0$ Eq. (3.4) has a local solution $u_0(\cdot, \cdot)$ defined on $(-\infty, \gamma) \times \Omega$, then for any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ one can find number $I$ such that for all $i > I$ Eq. (3.4) has a local solution $u_i(\cdot, \cdot)$ defined on $(-\infty, \gamma) \times \Omega$ and $\|u_i(\lambda_i) - u_0(\cdot, \cdot)\|_{C((-\infty, \gamma), BC(\Omega, R^n))} \to 0$ as $\lambda \to \lambda_0$.

**Proof.** Choose an arbitrary $b \in R$. Consider the following operator equation

$$u(t, \cdot) = F(u, \lambda)(t), \ t \in (-\infty, b],$$

where at each $\lambda \in \Lambda$, by the virtue of the assumptions $(\mathcal{A}_1)$ – $(\mathcal{A}_5)$,

$$F(\cdot, \lambda) : C((-\infty, b], BC(\Omega, R^n)) \to C((-\infty, b], BC(\Omega, R^n)),$$

$$(F(u, \lambda))(t, x) = \int_{-\infty}^{t} \int_{\Omega} W_\lambda(t, s, x, y) f_\lambda(u(t - \tau_\lambda(t, x, y), y)) dy ds, \ t \in (-\infty, b], x \in \Omega.$$

Note that by Theorem 3.3 we have a unique solution to Eq. (3.4) defined on $(-\infty, \delta] \times \Omega$ for each $\lambda \in U_0$. We need to prove continuous dependence of these solutions on $\lambda$. First, we prove that the operator $F$ is continuous in $(u, \lambda_0)$ for any fixed $u \in C((-\infty, b], BC(\Omega, R^n))$.

Choose an arbitrary $u \in C((-\infty, b], BC(\Omega, R^n))$. Let $\|u_i - u\|_{C((-\infty, b], BC(\Omega, R^n))} \to 0$, $i \to \infty$, and $\lambda \to \lambda_0$.

We have the following estimates:

$$|u_i(t - \tau_\lambda(t, x, y), y) - u(t - \tau_{\lambda_0}(t, x, y), y)| \leq$$

$$\leq |u_i(t - \tau_\lambda(t, x, y), y) - u(t - \tau_\lambda(t, x, y), y)| +$$

$$+ |u(t - \tau_\lambda(t, x, y), y) - u(t - \tau_{\lambda_0}(t, x, y), y)|.$$

If $\lambda \to \lambda_0$, then the first term on the right-hand side of this inequality goes to 0 uniformly as $\|u_i - u\|_{C((-\infty, b], BC(\Omega, R^n))} \to 0$. By virtue of the condition 4), the second term on the right-hand side goes to 0 in measure on $((-\infty, b] \times \Omega)$, uniformly in $x \in \Omega$, as $\lambda \to \lambda_0$. So,

$$|u_i(\cdot - \tau_\lambda(\cdot, x, \cdot), \cdot) - u(t - \tau_{\lambda_0}(\cdot, x, \cdot), \cdot)| \to 0$$

in measure, uniformly in $x \in \Omega$, as $\|u_i - u\|_{C((-\infty, b], BC(\Omega, R^n))} \to 0$ and $\lambda \to \lambda_0$. 
Using this convergence, we obtain

$$\sup_{t \in (-\infty, b], x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{d}(t, s, x, y) f_{d}(u(t - \tau_{d}(s, x, y), y)) dy ds \right| \leq$$

$$\sup_{t \in (-\infty, b], x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{d0}(t, s, x, y) f_{d0}(u(t - \tau_{d0}(s, x, y), y)) dy ds \right| +$$

$$\sup_{t \in (-\infty, b], x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{d}(t, s, x, y) f_{d}(u(t - \tau_{d}(s, x, y), y)) dy ds \right| .$$

Taking into account the condition 3), we conclude that the first term on the right-hand side of the inequality goes to 0 as $\lambda \to \lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as $\lambda \to \lambda_0$. Thus, the operator $F$ is continuous in $(u, \lambda_0)$ for any chosen $u \in C((-\infty, b], BC(\Omega, R^n))$. Using this fact, for any $\varepsilon > 0$ we can find such $\varepsilon_1 > 0$ and neighborhood $U_1$ of $\lambda_0$, that

$$\|F(u_\delta, \lambda) - F(u_{0\delta}, \lambda)\|_{C((-\infty, b], BC(\Omega, R^n))} \leq \varepsilon$$

for all $\lambda \in U_1$ and any $u_\delta \in C((-\infty, \delta], BC(\Omega, R^n))$, satisfying the estimate

$$\|u_\delta - u_{0\delta}\|_{C((-\infty, \delta], BC(\Omega, R^n))} \leq \varepsilon_1 .$$

As the mapping $F(\cdot, \lambda)$ is contracting with the constant $q_0 < 1$ (see Theorem 3.3) for any $\lambda \in U_0$, for any $m = 1, 2, \ldots$ we have

$$\|F^m(u_{0\delta}, \lambda) - u_{0\delta}\|_{C((-\infty, \delta], BC(\Omega, R^n))} \leq \|F^m(u_{0\delta}, \lambda) - F^{m-1}(u_{0\delta}, \lambda)\|_{C((-\infty, \delta], BC(\Omega, R^n))} + \ldots + \|F(u_{0\delta}, \lambda) - u_{0\delta}\|_{C((-\infty, \delta], BC(\Omega, R^n))} \leq$$

$$(q_0^{m-1} + \ldots + q_0 + 1)(1 - q_0)\varepsilon \leq \varepsilon .$$

Due to the convergence of the approximations $F^m(u_{0\delta}, \lambda)$ to the fixed point $u_\delta = u_\delta(\lambda)$ of the operator $F(\cdot, \lambda) : C((-\infty, \delta], BC(\Omega, R^n)) \to C((-\infty, \delta], BC(\Omega, R^n))$ we get $\|u_\delta(\lambda) - u_{0\delta}\|_{C((-\infty, \delta], BC(\Omega, R^n))} \leq \varepsilon$ for each $\lambda \in U_0 \cap U_1$ and $\varepsilon \to 0$ as $\lambda \to \lambda_0$.

Now, addressing the model (3.3) and Theorem 3.2, and taking $\varphi, \lambda = u_\delta(\lambda)$ and $a = \delta$, we prove this theorem. □
We note here that the remark, analogous to Remark 3.3, is valid for Theorem 3.4 as well.

Remark 3.7. If \( \Omega \) is bounded, we can get the conclusion of Theorem 3.4 replacing 3) and 4) by the following conditions:

For any \( \{\lambda_i\} \subset \Lambda, \lambda_i \to \lambda_0 \) it holds true that:

3') For any \( u \in \mathbb{R}^n \) we have \( |f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0 \);  

4') For all \( x \in \Omega, |\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0 \) in measure on \((\infty, b) \times \Omega \).

Proof of the statement in Remark 3.7 is given in Appendix C.

In case of spatially localized solutions to the (1.7) and (3.4), we have the following remark to Theorem 3.4.

Remark 3.8. If in (3.4) we add the condition, corresponding to localization in the spatial variable,

\[ (\mathcal{A}^{' \delta}) \text{ For each } \lambda \in \Lambda \text{ and any } b \in \mathbb{R}, \lim_{|x| \to \infty} |W_{\lambda}(t, s, x, y)| = 0 \text{ for all } (t, s, y) \in (\infty, b) \times (\infty, b) \times \Omega, \]

then, in order to get the conclusion of Theorem 3.4 for spatially localized solutions, we need the following conditions instead of 2), 3), and 4), respectively:

For any \( \{\lambda_i\} \subset \Lambda, \lambda_i \to \lambda_0 \) it holds true that:

2') For any \( b \in \mathbb{R}, r > 0 \) and each \( t \in (\infty, b), x \in \Omega, |x| \leq r \) it holds true that

\[ \left| \int_{\infty}^{t} \int_{\Omega} (W_{\lambda_i}(t, s, x, y) - W_{\lambda_0}(t, s, x, y)) dy ds \right| \to 0; \]

3') For any \( u \in \mathbb{R}^n \) we have \( |f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0 \);

4') For all \( x \in \Omega, |\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0 \) in measure on \((\infty, b) \times \Omega \).

Proof of the statement in Remark 3.8 is given in Appendix D.

4 Conclusions and Outlook

For the nonlinear Volterra integral equations (1.7) and (3.1), which generalize the commonly used in the neural field theory models (1.1) - (1.6), we have defined the notions of local, global and maximally extended solutions. We have obtained conditions which guarantee existence of a unique global or maximally extended solution and its continuous dependence on the equation parameters. These results can also serve as a starting point for the development of numerical schemes for a broad class of neural field models. A key word in this context is justification of such schemes. We will emphasize that our results shed light on the problem of structural stability in nonlocal field models in, e.g. systems biology.
Appendix A. Proof of The Statement in Remark 3.4

We refer here to the proof of Theorem 3.2 and note that conditions in Remark 3.4 imply that

$$|(S_{\tau_l}^\varphi u_i)(\cdot, x, \cdot) - (S_{\tau_l}^\varphi u)(\cdot, x, \cdot)| \to 0$$

uniformly on $$((a, b) \times \Omega) \setminus \Theta_l) \times R^n (\mu(\Theta_l) \to 0$$, for each $$x \in \Omega$$, as $$\|u_i - u\|_{C((a, b), C_0(\Omega, R^n))} \to 0$$ and $$\lambda \to A_0$$.

Choose arbitrary $$\varepsilon > 0$$, for the $$b$$ chosen in the proof of Theorem 3.2 we find

$$a^* = \min_{r \in [a, b]; (x, y) \in \Omega^2} (t - \tau_l(t, x, y)).$$

Define the piecewise constant functions $$\overline{u} : [a, b] \times R^n \to R^n$$ and $$\overline{\varphi}_{\lambda_0} : [a^*, a] \times R^n \to R^n$$ as $$\overline{u}(t, x) \in R^n$$ for $$t \in [a, b]$$, $$\xi \in [a^*, a]$$, $$x \in \Omega$$ such that

$$\begin{cases}
|\overline{u}(t, x) - u(t, x)| \leq \varepsilon/2, \text{ if } |\overline{u}(t, x)| > |u(t, x)|; \\
|\overline{u}(t, x) - u(t, x)| < \varepsilon/2, \text{ if } |\overline{u}(t, x)| < |u(t, x)|;
\end{cases}$$

$$\begin{cases}
|\overline{\varphi}_{\lambda_0}(\xi, x) - \varphi_{\lambda_0}(\xi, x)| \leq \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi, x)| > |\varphi_{\lambda_0}(\xi, x)|; \\
|\overline{\varphi}_{\lambda_0}(\xi, x) - \varphi_{\lambda_0}(\xi, x)| < \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi, x)| < |\varphi_{\lambda_0}(\xi, x)|.
\end{cases}$$

We get the estimate

$$\left| f_\lambda \left((S_{\tau_l}^\varphi u_i)(t, x, y) - f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(t, x, y)\right) \right| \leq$$

$$\leq \left| f_\lambda \left((S_{\tau_l}^\varphi u_i)(t, x, y) \right) - f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(t, x, y)\right) \right| +$$

$$+ \left| f_\lambda \left((S_{\tau_l}^\varphi u)(t, x, y) \right) - f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(t, x, y)\right) \right| +$$

$$+ \left| f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(t, x, y) \right) - f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(t, x, y)\right) \right|.$$

Using the functions $$\overline{u}$$ and $$\overline{\varphi}_{\lambda_0}$$, it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to $$2\varepsilon$$ and $$\varepsilon$$, respectively, on $$((a, b) \times \Omega) \setminus \Theta_l) \times \Omega$$, where $$\mu(\Theta_l) \to 0$$ as $$\lambda \to A_0$$. In addition, the condition 4”) provide convergence to 0 of the second term on the right-hand side of the inequality as $$\lambda \to A_0$$.

Using the convergence obtained above, we get

$$\max_{(a, b), x \in \Omega} \left| \int_a^t \int_\Omega W_d(t, s, x, y) f_\lambda \left((S_{\tau_l}^\varphi u_i)(s, x, y)\right) dy ds -$$

$$- \int_a^t \int_\Omega W_{\lambda_0}(t, s, x, y) f_{\lambda_0} \left((S_{\tau_l}^\varphi u)(s, x, y)\right) dy ds \right| \leq$$
\[
\max_{t \in [a, b], x \in \Omega} \int_a^t \int_\Omega W_A(t, s, x, y) f_{\lambda}(S^{\phi_A}_{t, s} u)(s, x, y) dy ds - \\
\int_a^t \int_\Omega W_A(t, s, x, y) f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} u)(s, x, y) dy ds + \\
+ \max_{t \in [a, b], x \in \Omega} \int_a^t \int_\Omega W_A(t, s, x, y) f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} u)(s, x, y) dy ds
\]

Taking into account the condition 3'), we have the first term on the right-hand side of this inequality going to 0 as \( \lambda \to \lambda_0 \). The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as \( \lambda \to \lambda_0 \). Thus, the statement in Remark 3.4 is valid.

**Appendix B. Proof of The Statement in Remark 3.5**

Conditions in Remark 3.5 imply the following changes in the proof of Theorem 3:

\[ \|(S^{\phi_A}_{t, s} u)(\cdot, x, \cdot) - (S^{\phi_{\lambda_0}}_{t, s} u)(\cdot, x, \cdot)\| \to 0 \]

uniformly on \(((a, b] \times \Omega) \setminus \Theta_A) \times R^n (\mu(\Theta_A) \to 0)\), for each \( x \in \Omega \), as \( \|u - u\|_{C((a, b] \times \Omega \setminus \Theta_A) \times R^n} \to 0 \) and \( \lambda \to \lambda_0 \).

Choose arbitrary \( \varepsilon > 0 \). Define the piecewise constant functions \( \overline{u}; [a, b] \times \Omega \to R^n \) and \( \overline{\varphi}_{\lambda_0}; (-\infty, a] \times \Omega \to R^n \) as \( \overline{u}(t, x) \in R^n \) for \( t \in [a, b] \), \( \xi \in (-\infty, a] \), \( x \in \Omega \) such that

\[
\begin{align*}
|\overline{u}(t, x) - u(t, x)| &\leq \varepsilon/2, \text{ if } |\overline{u}(t, x)| > |u(t, x)|; \\
|\overline{u}(t, x) - u(t, x)| &< \varepsilon/2, \text{ if } |\overline{u}(t, x)| < |u(t, x)|; \\
\end{align*}
\]

\[
\begin{align*}
|\overline{\varphi}_{\lambda_0}(\xi, x) - \varphi_{\lambda_0}(\xi, x)| &\leq \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi, x)| > |\varphi_{\lambda_0}(\xi, x)|; \\
|\overline{\varphi}_{\lambda_0}(\xi, x) - \varphi_{\lambda_0}(\xi, x)| &< \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi, x)| < |\varphi_{\lambda_0}(\xi, x)|.
\end{align*}
\]

We get the estimate

\[
\left| f_{\lambda}(S^{\phi_A}_{t, s} u)(t, x, y) - f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} u)(t, x, y) \right| \leq \\
\leq \left| f_{\lambda}(S^{\phi_A}_{t, s} u)(t, x, y) - f_{\lambda}(S^{\phi_{\lambda_0}}_{t, s} \overline{u})(t, x, y) \right| + \\
+ \left| f_{\lambda}(S^{\phi_{\lambda_0}}_{t, s} \overline{u})(t, x, y) - f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} \overline{u})(t, x, y) \right| + \\
+ \left| f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} \overline{u})(t, x, y) - f_{\lambda_0}(S^{\phi_{\lambda_0}}_{t, s} u)(t, x, y) \right|.
\]

Using the functions \( \overline{u} \) and \( \overline{\varphi}_{\lambda_0} \), it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to \( 2\varepsilon \) and \( \varepsilon \), respectively, on \(((a, b] \times \Omega) \setminus \Theta_A) \times \Omega \), where \( \mu(\Theta_A) \to 0 \) as \( \lambda \to \lambda_0 \). In addition to that, the condition 4') provide convergence to 0 of the second term on the right-hand side of the inequality as \( \lambda \to \lambda_0 \).
Using the convergence obtained above, \((A'_3), (A_1),_5\), and conditions \(2'\) and \(3'\), we get

\[
\max_{\tau \in [a,b], x \in \Omega} \left| \int_a^t \int_{\{x \in \Omega|x| \leq r'\}} W_d(t, s, x, y) f_d\left((S^{\varphi}_{\tau_s} u)(s, x, y)\right) dy ds - \right|
\]

\[
\int_a^t \int_{\{x \in \Omega|x| \leq r'\}} W_d(t, s, x, y) f_d\left((S^{\varphi}_{\tau_s} u)(s, x, y)\right) dy ds \leq
\]

\[
\max_{\tau \in [a,b], x \in \Omega} \left| \int_a^t \int_{\{x \in \Omega|x| \leq r'\}} W_d(t, s, x, y) f_d\left((S^{\varphi}_{\tau_s} u)(s, x, y)\right) dy ds \right| +
\]

\[
\int_a^t \int_{\{x \in \Omega|x| \leq r'\}} W_d(t, s, x, y) f_d\left((S^{\varphi}_{\tau_s} u)(s, x, y)\right) dy ds \left|+ \epsilon_r(t, x). \right.
\]

Here \(\epsilon_r(t, x) \to 0\) uniformly as \(r' \to \infty\). Taking into account the condition \(3'\), we have the first term on the right-hand side of this inequality going to 0 as \(\lambda \to \lambda_0\). The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition \(2'\) as \(\lambda \to \lambda_0\). Thus, the statement in Remark 3.5 is valid.

**Appendix C. Proof of The Statement in Remark 3.7**

The following changes in the proof of Theorem 3.4 stem from the conditions of Remark 3.7:

\[|u_i(t - \tau_d(t, x, y), y) - u(t - \tau_{\lambda_0}(t, x, y), y)| \to 0\]

uniformly on \(((\infty, b) \times \Theta_d) \times R^n (\mu(\Theta_d) \to 0)\) for each \(x \in \Omega\), as \(\|u_i - u\|_C((\infty, b) \times \Theta_d \times R^n) \to 0\) and \(\lambda \to \lambda_0\).

Choose an arbitrary \(\epsilon > 0\). Define the piecewise constant function \(\overline{u} : (-\infty, b] \times R^n \to R^n\) as \(\overline{u}(t, x) \in R^n\) for \(t \in (-\infty, b], x \in \Omega\) such that

\[
\overline{u}(t, x) - u(t, x) | \leq \epsilon/2, \text{ if } \overline{u}(t, x) > |u(t, x)|;
\]

\[
\overline{u}(t, x) - u(t, x) | \leq \epsilon/2, \text{ if } \overline{u}(t, x) < |u(t, x)|.
\]

Using the function introduced above, we get the estimate

\[
\left| f_d\left(u_i(t - \tau_d(t, x, y), y)\right) - f_d\left(u(t - \tau_{\lambda_0}(t, x, y), y)\right) \right| 
\]

\[
\leq \left| f_d\left(u_i(t - \tau_d(t, x, y), y)\right) - f_d(\overline{u}(t - \tau_{\lambda_0}(t, x, y), y)) \right| +
\]

\[
\left| f_d(\overline{u}(t - \tau_{\lambda_0}(t, x, y), y)) - f_d\left(u(t - \tau_{\lambda_0}(t, x, y), y)\right) \right|.
\]
Here, the first and the third terms on the right-hand side of this inequality are less or equal to $2\varepsilon$ and $\varepsilon$, respectively, on $(((-\infty, b) \times \Omega) \setminus \Theta_{i}) \times \mathbb{R}^{n}$, where $\mu(\Theta_{i}) \to 0$ as $\lambda \to \lambda_{0}$. In addition, the condition 4*) provide convergence to 0 of the second term on the right-hand side of the inequality as $\lambda \to \lambda_{0}$.

Using the convergence obtained above and (\mathcal{A}_{4})4), we get

$$
\max_{t \in (-\infty,b], \ x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{A}(t,s,x,y) f_{\lambda_{0}}(u_{i}(t-\tau_{A}(t,x,y),y)) \ dy \ ds \right|
$$

$$
+ \max_{t \in (-\infty,b], \ x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{A}(t,s,x,y) f_{\lambda_{0}}(u(t-\tau_{A}(t,x,y),y)) \ dy \ ds \right|
$$

$$
+ \max_{t \in (-\infty,b], \ x \in \Omega} \left| \int_{-\infty}^{t} \int_{\Omega} W_{b}(t,s,x,y) f_{\lambda_{0}}(u(t-\tau_{b}(t,x,y),y)) \ dy \ ds \right|
$$

Taking into account the condition 3*)2), we have the first term on the right-hand side of this inequality going to 0 as $\lambda \to \lambda_{0}$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the conditions 2) as $\lambda \to \lambda_{0}$. Thus, the statement in Remark 3.7 is valid.

**Appendix D. Proof of The Statement in Remark 3.8**

Referring to the proof of Theorem 3.4 we get the following changes caused by conditions of Remark 3.8:

$$
|u_{i}(t-\tau_{A}(t,x,y),y) - u(t-\tau_{b}(t,x,y),y)| \to 0
$$

uniformly on $(((-\infty, b) \times \Omega) \setminus \Theta_{i}) \times \mathbb{R}^{n}$ (\mu(\Theta_{i}) \to 0) for each $x \in \Omega$, as $\|u_{i} - u\|_{C((-\infty,b],[\mathcal{BC}(\Omega,R^{n})])} \to 0$ and $\lambda \to \lambda_{0}$.

Choose an arbitrary $\varepsilon > 0$. Define the piecewise constant function $\overline{u} : (-\infty, b] \times \mathbb{R}^{n} \to \mathbb{R}^{n}$ as $\overline{u}(t,x) \in \mathbb{R}^{n}$ for $t \in (-\infty,b]$, $x \in \Omega$ such that

$$
\begin{cases}
|\overline{u}(t,x) - u(t,x)| \leq \varepsilon / 2, \text{ if } |\overline{u}(t,x)| > |u(t,x)|; \\
|\overline{u}(t,x) - u(t,x)| < \varepsilon / 2, \text{ if } |\overline{u}(t,x)| < |u(t,x)|.
\end{cases}
$$
Using this function, we get the estimate
\[
\left| f_{3}(u_{i}(t - \tau_{3}(t, x, y)) - f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) \right| \leq \\
\leq \left| f_{3}(u_{i}(t - \tau_{3}(t, x, y))) - f_{3}(u(t - \tau_{3}_{0}(t, x, y))) \right| + \\
+ \left| f_{3}(u(t - \tau_{3}_{0}(t, x, y))) - f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) \right| + \\
+ \left| f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) - f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) \right|.
\]

Using the function \( \bar{\theta} \), it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to 2\( \varepsilon \) and \( \varepsilon \), respectively, on \( ((-\infty, b) \times \Omega) \setminus \Theta_{b} \times \mathbb{R}^{n} \), where \( \mu(\Theta_{b}) \to 0 \) as \( \lambda \to \lambda_{0} \). In addition, the condition 4’) provide convergence to 0 of the second term on the right-hand side of the inequality as \( \lambda \to \lambda_{0} \).

Using the convergence obtained above, (\( \mathcal{A}_{4} \), (\( \mathcal{A}_{4} \)), and conditions 2’) and 3’), we get
\[
\max_{t \in (-\infty, b), x \in \Omega} \int_{-\infty}^{t} \int_{\Omega} W_{3}(t, s, x, y) f_{4}(u_{i}(t - \tau_{4}(t, x, y))) dyds - \\
\int_{-\infty}^{t} \int_{\Omega} W_{3}_{0}(t, s, x, y) f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) dyds \leq \\
\max_{t \in (-\infty, b), x \in \Omega} \int_{-\infty}^{t} \int_{\{x \in \Omega, |x| \leq r'\}} W_{3}(t, s, x, y) f_{4}(u_{i}(t - \tau_{4}(t, x, y))) dyds - \\
\int_{-\infty}^{t} \int_{\{x \in \Omega, |x| \leq r'\}} W_{3}(t, s, x, y) f_{3}(u(t - \tau_{3}_{0}(t, x, y))) dyds + \\
+ \max_{t \in (-\infty, b), x \in \Omega} \int_{-\infty}^{t} \int_{\{x \in \Omega, |x| \leq r'\}} W_{3}_{0}(t, s, x, y) f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) dyds - \\
\int_{-\infty}^{t} \int_{\{x \in \Omega, |x| \leq r'\}} W_{3}_{0}(t, s, x, y) f_{3}_{0}(u(t - \tau_{3}_{0}(t, x, y))) dyds + \varepsilon_{r'}(t, x).
\]

Here \( \varepsilon_{r'}(t, x) \to 0 \) uniformly as \( r' \to \infty \). Taking into account the condition 3’), we have the first term on the right-hand side of this inequality going to 0 as \( \lambda \to \lambda_{0} \). The second term on the right-hand side of the inequality goes to 0 by the virtue of the conditions 2’) as \( \lambda \to \lambda_{0} \). Thus, the statement in Remark 3.8 is valid.

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