On Levi-flat hypersurfaces tangent to holomorphic webs

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Abstract. We investigate germs of real analytic Levi-flat hypersurfaces tangent to germs of codimension one holomorphic webs. We introduce the notion of first integrals for local webs. In particular, we prove that a \( k \)-web with finitely many invariant analytic subvarieties through the origin tangent to a Levi-flat hypersurface has a holomorphic first integral.

Résumé. Nous étudions les germes d'hypersurfaces réelles analytiques Levi-plate tangente à les germes d'webs holomorphes codimension un. Nous introduisons la notion des intégrales premières des webs locales. En particulier, nous montrons que une \( k \)-web avec un nombre fini de feuilles invariant analytic par l'origine, tangente à une hypersurface Levi-plate possède une intégrale première holomorphe.

1 Introduction

In very general terms, a germ of codimension one \( k \)-web is a collection of \( k \) germs of codimension one holomorphic foliations in "general position". The study of webs was initiated by Blaschke and his school in the late 1920s. For a recent account of the theory, we refer the reader to [12].

For instance, take \( \omega \in \text{Sym}^k \Omega^1(\mathbb{C}^2, 0) \) defined by
\[
\omega = (dy)^k + a_{k-1}(dy)^{k-1}dx + \ldots + a_0(dx)^k,
\]
where \( a_j \in \mathcal{O}_2 \) for all \( 0 \leq j \leq k - 1 \). Then \( \mathcal{W} : \omega = 0 \), define a non-trivial \( k \)-web on \((\mathbb{C}^2, 0)\). In this paper we study webs and its relation with Levi-flat hypersurfaces.

Let \( M \) be a germ at \( 0 \in \mathbb{C}^n \) of a real codimension one analytic irreducible analytic set. Since \( M \) is real analytic of codimension one, it can be decomposed into \( M_{\text{reg}} \) and \( \text{Sing}(M) \), where \( M_{\text{reg}} \) is a germ of smooth real analytic hypersurface in \( \mathbb{C}^n \) and \( \text{Sing}(M) \), the singular locus, is contained in a proper analytic subvariety of lower dimension. We shall say that \( M \) is Levi-flat if the complex distribution \( L \) on \( M_{\text{reg}} \)
\[
L_p := T_p M \cap iT_p M \subset T_p M, \quad \text{for any } p \in M_{\text{reg}}
\]
is integrable, in Frobenius sense. It follows that \( M_{\text{reg}} \) is smoothly foliated by immersed complex manifolds of complex dimension \( n - 1 \). The foliation defined by \( L \) is called the Levi foliation and will be denoted by \( \mathcal{L}_M \).

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If $M$ is a real analytic smooth Levi-flat hypersurface, by a classic result of E. Cartan there exists a local holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $M$ can be represented by $M = \{\text{Im}(z_n) = 0\}$. The situation if different if the hypersurface have singularities. Singular Levi-flat real analytic hypersurfaces have been studied by Burns and Gong [1], Brunella [2], Lebl [3], the author [6], [7] and many others.

Recently D.Cerveau and A. Lins Neto [5] have studied codimension one holomorphic foliations tangent to singular Levi-flat hypersurfaces. A codimension one holomorphic foliation $F$ is tangent to $M$, if any leaf of $L_M$ is also a leaf of $F$. In [5] it is proved that a germ of codimension one holomorphic foliation tangent to a real analytic Levi-flat hypersurface has a non-constant meromorphic first integral. In the same spirit, the authors propose a problem for webs, which is as follows:

**Problem.** Let $M$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of real analytic hypersurface Levi-flat. Assume that there exists a singular codimension one $k$-web, $k \geq 2$, such that any leaf of the Levi foliation $L_M$ on $M_{reg}$ is also a leaf of the web. Does the web has a non-constant meromorphic first integral?

By a meromorphic first integral we mean something like $f_0(x) + z.f_1(x) + \ldots + z^k.f_k(x) = 0$, where $f_0, f_1, \ldots, f_k \in \mathcal{O}_n$. In this situation, the web is obtained by the elimination of $z$ in the system given by

$$
\begin{align*}
&f_0 + z.f_1 + z^2.f_2 + \ldots + z^k.f_k = 0 \\
&d_0 + z.d_1 + z^2.d_2 + \ldots + z^k.d_k = 0.
\end{align*}
$$

In this work, we organize some results on singular Levi-flat hypersurfaces and holomorphic foliations which provide a best approach to study of webs and Levi-flats. Concerning the problem, we obtain an interesting result in a case very special (Theorem [1]), the problem remains open in general.

### 1.1 Local singular webs

It is customary to define a germ of singular holomorphic foliation as an equivalence class $[\omega]$ of germs of holomorphic 1-forms in $\Omega^1(\mathbb{C}^n, 0)$ modulo multiplication by elements of $\mathcal{O}(\mathbb{C}^n, 0)$ such that any representative $\omega$ is integrable ($\omega \wedge d\omega = 0$) and with singular set $\text{Sing}(\omega) = \{p \in (\mathbb{C}^n, 0) : \omega(p) = 0\}$ of codimension at least two.

An analogous definition can be made for codimension one $k$-webs. A germ at $(\mathbb{C}^n, 0)$, $n \geq 2$ of codimension one $k$-web $W$ is an equivalence class $[\omega]$ of germs of $k$-symmetric 1-forms, that is sections of $\text{Sym}^k \Omega^1(\mathbb{C}^n, 0)$, modulo multiplication by $\mathcal{O}(\mathbb{C}^n, 0)$ such that a suitable representative $\omega$ defined in a connected neighborhood $U$ of the origin satisfies the following conditions:

1. The zero set of $\omega$ has codimension at least two.
2. The 1-form $\omega$, seen as a homogeneous polynomial of degree $k$ in the ring $O_n[dx_1, \ldots, dx_n]$, is square-free.

3. (Brill’s condition) For a generic $p \in U$, $\omega(p)$ is a product of $k$ linear forms.

4. (Frobenius’s condition) For a generic $p \in U$, the germ of $\omega$ at $p$ is the product of $k$ germs of integrable 1-forms.

Both conditions (3) and (4) are automatic for germs at $(C^2,0)$ of webs and non-trivial for germs at $(C^n,0)$ when $n \geq 3$.

We can think $k$-webs as first order differential equations of degree $k$.

The idea is to consider the germ of web as a meromorphic section of the projectivization of the cotangent bundle of $(C^n,0)$. This is a classical point view in the theory of differential equations, which has been recently explored in Web-geometry. For instance see [3], [4], [14].

1.2 The contact distribution

Let us denote $P := PT^*(C^n,0)$ the projectivization of the cotangent bundle of $(C^n,0)$ and $\pi : PT^*(C^n,0) \to (C^n,0)$ the natural projection. Over a point $p$ the fiber $\pi^{-1}(p)$ parametrizes the one-dimensional subspaces of $T^*_p(C^n,0)$. On $P$ there is a canonical codimension one distribution, the so called contact distribution $D$. Its description in terms of a system of coordinates $x = (x_1, \ldots, x_n)$ of $(C^n,0)$ goes as follows: let $dx_1, \ldots, dx_n$ be the basis of $T^*(C^n,0)$ associated to the coordinate system $(x_1, \ldots, x_n)$. Given a point $(x,y) \in T^*(C^n,0)$, we can write $y = \sum_{j=1}^{n} y_j dx_j$, $(y_1, \ldots, y_n) \in C^n$. In this way, if $(y_1, \ldots, y_n) \neq 0$ then we set $[y] = [y_1, \ldots, y_n] \in \mathbb{P}^{n-1}$ and $(x,[y]) \in (C^n,0) \times \mathbb{P}^{n-1} \cong P$. In the affine coordinate system $y_n \neq 0$ of $P$, the distribution $D$ is defined by $\alpha = 0$, where

$$\alpha = dx_n - \sum_{j=1}^{n-1} p_j dx_j, \quad p_j = -\frac{y_j}{y_n} \quad (1 \leq j \leq n - 1). \quad (1.2)$$

The 1-form $\alpha$ is called the contact form.

1.3 Webs as closures of meromorphic multi-sections

Let us consider $X \subset P$ a subvariety, not necessarily irreducible, but of pure dimension $n$. Let $\pi_X : X \to (C^n,0)$ be the restriction to $X$ of the projection $\pi$. Suppose also that $X$ satisfies the following conditions:

1. The image under $\pi$ of every irreducible component of $X$ has dimension $n$. 

2. The generic fiber of $\pi$ intersects $X$ in $k$ distinct smooth points and at these the differential $d\pi_X : T_pX \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is surjective. Note that $k = \deg(\pi_X)$.

3. The restriction of the contact form $\alpha$ to the smooth part of every irreducible component of $X$ is integrable. We denote $\mathcal{F}_X$ the foliation defined by $\alpha|_X = 0$.

We can define a germ $\mathcal{W}$ at $0 \in \mathbb{C}^n$ of $k$-web as a triple $(X, \pi_X, \mathcal{F}_X)$. This definition is equivalent to one given in Section 1. In the sequel, $X$ will always be the variety associated to $\mathcal{W}$, the singular set of $X$ will be denoted by $\text{Sing}(X)$ and its the smooth part will be denoted by $X_{\text{reg}}$.

**Definition 1.1.** Let $R$ be the set of points $p \in X$ where
- either $X$ is singular,
- or the differential $d\pi_X : T_pX_{\text{reg}} \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is not an isomorphism.

The analytic set $R$ is called the criminant set of $\mathcal{W}$ and $\Delta_{\mathcal{W}} = \pi(R)$ the discriminant of $\mathcal{W}$. Note that $\dim(R) \leq n - 1$.

**Remark 1.2.** Let $\omega \in \text{Sym}^k\Omega_1(\mathbb{C}^n, 0)$ and assume that it defines a $k$-web $\mathcal{W}$ with variety $X$. Then $X$ is irreducible if, and only if, $\omega$ is irreducible in the ring $\mathcal{O}_n[dx_1, \ldots, dx_n]$. In this case we say that the web is irreducible.

Let $M$ be a germ at $0 \in \mathbb{C}^n$ of a real analytic Levi-flat hypersurface.

**Definition 1.3.** We say that $M$ is tangent to $\mathcal{W}$ if any leaf of the Levi foliation $\mathcal{L}_M$ on $M_{\text{reg}}$ is also a leaf of $\mathcal{W}$.

**1.4 First integrals for webs**

**Definition 1.4.** We say that $\mathcal{W}$ a $k$-web has a meromorphic first integral if, and only if, there exists

$$P(z) = f_0 + z.f_1 + \ldots + z^k.f_k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_k \in \mathcal{O}_n$, such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$.

**Definition 1.5.** We say that $\mathcal{W}$ a $k$-web has a holomorphic first integral if, and only if, there exists

$$P(z) = f_0 + z.f_1 + \ldots + z^{k-1}.f_{k-1} + z^k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_{k-1} \in \mathcal{O}_n$, such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$. 


We will prove a result concerning the situation of definitions 1.3 and 1.5.

**Theorem 1.** Let $W$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$ of $k$-web defined by

$$
\omega = \sum_{i_1 + \ldots + i_n = k} a_{i_1, \ldots, i_n}(z)dz_{i_1}^1 \ldots dz_{i_n}^n,
$$

where $a_{i_1, \ldots, i_n} \in \mathcal{O}_n$ and $a_{0,0,\ldots,0,k}(0) \neq 0$. Suppose that $W$ is tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real-analytic Levi-flat hypersurface $M$. Furthermore, assume that $W$ is irreducible and has finitely many invariant analytic subvarieties through the origin. Let $X$ be the variety associated to $W$. Then $W$ has a non-constant holomorphic first integral, if one of the following conditions is fulfilled:

1. If $n = 2$.

2. If $n \geq 3$ and $\text{cod}_{X_{\text{reg}}}(\text{Sing}(X)) \geq 2$.

Moreover, if $P(z) = f_0 + z.f_1 + \ldots + z^{k-1}.f_{k-1} + z^k \in \mathcal{O}_n[z]$ is a holomorphic first integral for $W$, then $M = (F = 0)$, where $F$ is obtained by the elimination of $z$ in the system given by

$$
\begin{cases}
    f_0 + z.f_1 + z^2.f_2 + \ldots + z^{k-1}.f_{k-1} + z^k = 0 \\
    f_0 + z.\tilde{f}_1 + z^2.\tilde{f}_2 + \ldots + z^{k-1}.\tilde{f}_{k-1} + z^k = 0.
\end{cases}
$$

**Remark 1.6.** Under the hypotheses of Theorem 1 if $n = 2$ and $k = 1$, $W$ is a non-dicritical holomorphic foliation at $(\mathbb{C}^2,0)$ tangent to a germ of an irreducible real analytic Levi-flat hypersurface $M$, then Theorem 1 given by Cerveau and Lins Neto [5] assures that $W$ has a non-constant holomorphic first integral. In this sense, our theorem is a generalization of result of Cerveau and Lins Neto.

**Remark 1.7.** Let $W$ a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of a smooth $k$-web tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real codimension one submanifold $M$. In other words, $W = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$ is a generic superposition of $k$ germs at $0 \in \mathbb{C}^n$ of smooth foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. In this case the irreducibility and tangency conditions to $M$ implies the existence of a unique $i \in \{1, \ldots, k\}$ such that $\mathcal{F}_i$ is tangent to $M$. Therefore we can find a coordinates system $z_1, \ldots, z_n$ of $\mathbb{C}^n$ such that $\mathcal{F}_i$ is defined by $dz_n = 0$ and $M = (\text{Im}(z_n) = 0)$.

## 2 The foliation associated to a web

In this section, we prove a key lemma which will be used in the proof of main theorem.

Since the restriction of $\mathcal{D}$ to $X_{\text{reg}}$ is integrable, it defines a foliation $\mathcal{F}_X$, which in general is a singular foliation. Given $p \in (\mathbb{C}^n,0) \setminus \Delta_W$, $\pi_X^{-1}(p) =$
\{q_1, \ldots, q_k\}, where q_i \neq q_j, \text{ if } i \neq j, (\deg(\pi_X) = k), denote by \mathcal{F}_X^i, the germ of \mathcal{F}_X \text{ at } q_i, i = 1, \ldots, k.

The projections \pi_*(\mathcal{F}_X) := \mathcal{F}_p define \( k \) germs of codimension one foliations at \( p \).

**Definition 2.1.** A leaf of the web \( \mathcal{W} \) is, by definition, the projection on \((\mathbb{C}^n, 0)\) of a leaf of \( \mathcal{F}_X \).

**Remark 2.2.** Given \( p \in (\mathbb{C}^n, 0) \setminus \Delta_W \), and \( q_i \in \pi_X^{-1}(p) \), the projection \( \pi_X(L_i) \) of the leaf \( L_i \) of \( \mathcal{F}_X \) through \( q_i \), gives rise to a leaf of \( \mathcal{W} \) through \( p \). In particular, \( \mathcal{W} \) has at most \( k \) leaves through \( p \).

We will use the following proposition (cf. [8] Th. 5, pg. 32). Let \( \mathcal{O}(X) \) denote the ring of holomorphic functions on \( X \).

**Proposition 2.3.** Let \( V \) be an analytic variety. If \( \pi: V \to W \) is a finite branched holomorphic covering of pure order \( k \) over an open subset \( W \subseteq \mathbb{C}^n \), then to each holomorphic function \( f \in \mathcal{O}(V) \) there is a canonically associated monic polynomial \( P_f(z) \in \mathcal{O}_n[z] \subseteq \mathcal{O}(V)[z] \) of degree \( k \) such that \( P_f(f) = 0 \) in \( \mathcal{O}(V) \).

We have now the following lemma.

**Lemma 2.4.** Suppose that \( (X, \pi_X, \mathcal{F}_X) \) defines a \( k \)-web \( \mathcal{W} \) on \((\mathbb{C}^n, 0)\), \( n \geq 2 \), where \( X \) is an irreducible subvariety of \( \mathbb{P}^k \). If \( \mathcal{F}_X \) has a non-constant holomorphic first integral then \( \mathcal{W} \) also has a holomorphic first integral.

**Proof.** Let \( g \in \mathcal{O}(X) \) be the first integral for \( \mathcal{F}_X \). By Proposition 2.3 there exists a monic polynomial \( P_g(z) \in \mathcal{O}_n[z] \subseteq \mathcal{O}(V)[z] \) of degree \( k \) such that \( P_g(g) = 0 \) in \( \mathcal{O}(V) \). Write

\[ P_g(z) = g_0 + zg_1 + \ldots + z^{k-1}g_{k-1} + z^k, \]

where \( g_0, \ldots, g_{k-1} \in \mathcal{O}_n \).

**Assertion.** \( P_g \) define a holomorphic first integral for \( \mathcal{W} \).

Let \( U \subseteq (\mathbb{C}^n, 0) \setminus \Delta_W \) be an open subset and let \( \varphi: X \to (\mathbb{C}^n, 0) \times \mathbb{C} \) be defined by \( \varphi = (\pi_X, g) \). Take a leaf \( L \) of \( \mathcal{W}|_U \). Then there is \( z \in \mathbb{C} \) such that the following diagram

\[
\begin{array}{ccc}
\pi_X^{-1}(U) \cap \varphi^{-1}(L \times \{z\}) & \xrightarrow{\varphi} & L \times \{z\} \\
\downarrow \pi_X & & \downarrow \text{pr}_1 \\
L & & \end{array}
\]

is commutative, where \( \text{pr}_1 \) is the projection on the first coordinate. It follows that \( L \) is a leaf of \( \mathcal{W} \) if and only if \( g \) is constant along of each connected component of \( \pi_X^{-1}(L) \) contained in \( \varphi^{-1}(L \times \{z\}) \).
Consider now the hypersurface \( G = \varphi(X) \subset (\mathbb{C}^n, 0) \times \mathbb{C} \) which is the closure of set

\[
\{(x, s) \in U \times \mathbb{C} : g_0(x) + s.g_1(x) + \ldots + s^{k-1}.g_{k-1}(x) + s^k = 0\}.
\]

Let \( \psi : (\mathbb{C}^n, 0) \times \mathbb{C} \to (\mathbb{C}^n, 0) \) be the usual projection and denote by \( Z \subset (\mathbb{C}^n, 0) \) the analytic subset such that the restriction to \( G \) of \( \psi \) not is a finite branched covering. Notice that for all \( x_0 \in (\mathbb{C}^n, 0) \setminus Z \), the equation

\[
g_0(x) + s.g_1(x) + \ldots + s^{k-1}.g_{k-1}(x) + s^k = 0
\]

defines \( k \) analytic hypersurfaces pairwise transverse in \( x_0 \) and therefore correspond to leaves of \( \mathcal{W} \).

\[ \square \]

3 Examples

This section is devoted to give some examples of Levi-flat hypersurfaces tangent to holomorphic foliations or webs.

**Example 3.1.** Take a non constant holomorphic function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) and set \( M = (\mathcal{I}m(f) = 0) \). Then \( M \) is Levi-flat and \( M_{\text{sing}} \) is the set of critical points of \( f \) lying on \( M \). Leaves of the Levi foliation on \( M_{\text{reg}} \) are given by \( \{f = c\} \), \( c \in \mathbb{R} \). Of course, \( M \) is tangent to a singular holomorphic foliation generated by the kernel of \( df \).

**Example 3.2.** Let \( f_0, f_1, \ldots, f_k \in \mathcal{O}_n, n \geq 2 \), be irreducible germs of holomorphic functions, where \( k \geq 2 \). Consider the family of hypersurfaces

\[
G := \{G_s := f_0 + sf_1 + \ldots + s^k f_k / s \in \mathbb{R}\}.
\]

By eliminating the real variable \( s \) in the system \( G_s = G_{s+} = 0 \), we obtain a real analytic germ \( F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0) \) such that any complex hypersurface \( (G_s = 0) \) is contained in the real hypersurface \( (F = 0) \). For instance, in the case \( k = 2 \), we obtain

\[
F = \det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & 0 \\ 0 & \bar{f}_0 & \bar{f}_1 & \bar{f}_2 \end{pmatrix} =
\]

\[
f_0^2 \bar{f}_2^2 + \bar{f}_0^2 f_2^2 + f_0 f_2 \bar{f}_1^2 + \bar{f}_0 \bar{f}_2 f_1^2 - |f_1|^2 (f_0 \bar{f}_2 + \bar{f}_0 f_2) - 2 |f_0|^2 |f_2|^2. \quad (3.1)
\]

which comes from the elimination of \( s \) in the system

\[
f_0 + s.f_1 + s^2.f_2 = \bar{f}_0 + s.\bar{f}_1 + s^2.\bar{f}_2 = 0.
\]
We would like to observe that the examples of this type are tangent to singular webs. The web is obtained by the elimination of $s$ in the system given by
\[
\begin{align*}
\{ & f_0 + s.f_1 + s^2.f_2 + \ldots + s^k.f_k = 0 \\
& df_0 + s.df_1 + s^2.df_2 + \ldots + s^k.df_k = 0
\end{align*}
\]
In the case we get a 2-web given by the implicit differential equation $\Omega = 0$, where
\[
\Omega = \det \begin{pmatrix}
f_0 & f_1 & f_2 & 0 \\
0 & f_0 & f_1 & f_2 \\
df_0 & df_1 & df_2 & 0 \\
0 & df_0 & df_1 & df_2
\end{pmatrix}
\]
This example shows that, although $\mathcal{L}_M$ is a foliation on $M_{\text{reg}} \subset M = (F = 0)$, in general it is not tangent to a germ of holomorphic foliation at $(\mathbb{C}^n,0)$.

**Example 3.3.** [Clairaut’s equations] Clairaut’s equations are tangent to Levi-flat hypersurfaces. Consider the first-order implicit differential equation
\[
y = xp + f(p),
\]
where $(x,y) \in \mathbb{C}^2$, $p = \frac{dy}{dx}$ and $f \in \mathbb{C}[p]$ is a polynomial of degree $k$, the equation defines a $k$-web $\mathcal{W}$ on $(\mathbb{C}^2,0)$. The variety $S$ associated to $\mathcal{W}$ is given by $(y - xp - f(p) = 0)$ and the foliation $\mathcal{F}_S$ is defined by $\alpha|_S = 0$, where $\alpha = dy - pdx$. In the chart $(x,p)$ of $S$, we get $\alpha|_S = (x + f'(p))dp$. The criminant set of $\mathcal{W}$ is given by $R = (y - xp - f(p) = x + f'(p))$.

Observe that $\mathcal{F}_S$ is tangent to $S$ along $R$ and has a non-constant first integral $q(x,p) = p$. Denote by $\pi_S : S \to (\mathbb{C}^2,0)$ the restriction to $S$ of the usual projection $\pi : \mathbb{P} \to (\mathbb{C}^2,0)$, then the leaves of $\mathcal{F}_S$ project by $\pi_S$ in leaves of $\mathcal{W}$. Those leaves are as follows
\[
-y + s.x + f(s) = 0,
\]
where $s$ is a constant. By the elimination of the variable $s \in \mathbb{R}$ in the system
\[
\begin{align*}
\{ & -y + s.x + f(s) = 0 \\
& -\tilde{y} + s.\tilde{x} + \tilde{f}(s) = 0
\end{align*}
\]
we obtain a Levi-flat hypersurface tangent to $\mathcal{W}$. In particular, Clairaut’s equation has a holomorphic first integral.
4 Lifting of Levi-flat hypersurfaces to the cotangent bundle

In this section we give some remarks about the lifting of a Levi-flat hypersurface to the cotangent bundle of $(\mathbb{C}^n, 0)$.

Let $P$ be as before, the projectivized cotangent bundle of $(\mathbb{C}^n, 0)$ and $M$ an irreducible real analytic Levi-flat at $(\mathbb{C}^n, 0)$, $n \geq 2$. Note that $P$ is a $\mathbb{P}^{n-1}$-bundle over $(\mathbb{C}^n, 0)$, whose fiber $PT_z^*\mathbb{C}^n$ over $z \in \mathbb{C}^n$ will be thought of as the set of complex hyperplanes in $T^*_z\mathbb{C}^n$. Let $\pi : P \to (\mathbb{C}^n, 0)$ be the usual projection.

The regular part $M_{\text{reg}}$ of $M$ can be lifted to $P$: just take, for every $z \in M_{\text{reg}}$, the complex hyperplane

$$ T_z^\mathbb{C}M_{\text{reg}} = T_zM_{\text{reg}} \cap i(T_zM_{\text{reg}}) \subset T_z\mathbb{C}^n. \quad (4.1) $$

We call

$$ M'_{\text{reg}} \subset P \quad (4.2) $$

this lifting of $M_{\text{reg}}$. We remark that it is no more a hypersurface: its (real) dimension $2n - 1$ is half of the real dimension of $\mathbb{P}T^*\mathbb{C}^n$. However, it is still “Levi-flat”, in a sense which will be precised below.

Take now a point $y$ in the closure $\overline{M_{\text{reg}}}$ projecting on $\mathbb{C}^n$ to a point $x \in \overline{M}$. Now, we shall consider the following results, which are adapted from [2].

**Lemma 4.1.** There exist, in a germ of neighborhood $U_y \subset \mathbb{P}T^*(\mathbb{C}^n, 0)$ of $y$, a germ of real analytic subset $N_y$ of dimension $2n - 1$ containing $M'_{\text{reg}} \cap U_y$.

**Proposition 4.2.** Under the above conditions, in a germ of neighborhood $V_y \subset U_y$ of $y$, there exists a germ of complex analytic subset $Y_y$ of (complex) dimension $n$ containing $N_y \cap V_y$.

5 Proof of Theorem [1]

The proof will be divided in two parts. First, we give the proof for $n = 2$. The proof in dimension $n \geq 3$ will be done by reduction to the case of dimension two.

First of all, we recall some results (cf. [3]) about foliations and Levi-flats. Let $M$ and $\mathcal{F}$ be germs at $(\mathbb{C}^2, 0)$ of a real analytic Levi-flat hypersurface and of a holomorphic foliation, respectively, where $\mathcal{F}$ is tangent to $M$. Assume that:

(i) $\mathcal{F}$ is defined by a germ at $0 \in \mathbb{C}^2$ of holomorphic vector field $X$ with an isolated singularity at $0$.

(ii) $M$ is irreducible.
Let us assume that 0 is a reduced singularity of \( X \), in the sense of Seidenberg [13]. Denote the eigenvalues of \( DX(0) \) by \( \lambda_1, \lambda_2 \).

**Proposition 5.1.** Suppose that \( X \) has a reduced singularity at \( 0 \in \mathbb{C}^2 \) and is tangent to a real analytic Levi-flat hypersurface \( M \). Then \( \lambda_1, \lambda_2 \neq 0 \), \( \lambda_2/\lambda_1 \in \mathbb{Q} \), and \( X \) has a holomorphic first integral.

In particular, in a suitable coordinates system \((x, y)\) around \( 0 \in \mathbb{C}^2 \), \( X = \phi Y \), where \( \phi(0) \neq 0 \) and

\[
Y = q.x\partial_x - p.y\partial_y, \quad g.c.d(p, q) = 1.
\] (5.1)

In this coordinate system, \( f(x, y) := x^{p}y^{q} \) is a first integral of \( X \).

We call this type of singularity of \( F \) a saddle with first integral, (cf. [10], pg. 162). Now we have the following lemma.

**Lemma 5.2.** For any \( z_0 \in M_{reg} \), the leaf \( L_{z_0} \) of \( L_M \) through \( z_0 \) is closed in \( M_{reg} \).

### 5.1 Planar webs

A \( k \)-web \( W \) on \((\mathbb{C}^2, 0)\) can be written in coordinates \((x, y) \in \mathbb{C}^2\) by

\[
\omega = a_0(x, y)(dy)^k + a_1(x, y)(dy)^{k-1}(dx) + \ldots + a_k(x, y)(dx)^k = 0,
\]

where the coefficients \( a_j \in \mathcal{O}_2 \), \( j = 1, \ldots, k \). We set

\[
U = \{(x, y, [adx + bdy]) \in \mathbb{PT}^*(\mathbb{C}^2, 0) : a \neq 0\}
\]

and

\[
V = \{(x, y, [adx + bdy]) \in \mathbb{PT}^*(\mathbb{C}^2, 0) : b \neq 0\}.
\]

Note that \( \mathbb{PT}^*(\mathbb{C}^2, 0) = U \cup V \). Suppose that \((S, \pi_S, F_S)\) define \( W \), in the coordinates \((x, y, p) \in U \), where \( p = \frac{dy}{dx} \), we have

1. \( S \cap U = \{(x, y, p) \in \mathbb{PT}^*(\mathbb{C}^2, 0) : F(x, y, p) = 0\} \), where

\[
F(x, y, p) = a_0(x, y)p^k + a_1(x, y)p^{k-1} + \ldots + a_k(x, y).
\]

Note that \( S \) is possibly singular at 0.

2. \( F_S \) is defined by \( \alpha|_S = 0 \), where \( \alpha = dy - pdx \).

3. The criminant set \( R \) is defined by the equations

\[
F(x, y, p) = F_p(x, y, p) = 0.
\]

In \( V \) the coordinate system is \((x, y, q) \in \mathbb{C}^3 \), where \( q = \frac{1}{p} \), the equations are similar.
5.2 Proof in dimension two

Let $W$ be a $k$-web tangent to $M$ Levi-flat and let us consider $S$, $π$ be as before. The idea is to use Lemma 2.4, assume that $W$ is defined by

$$\omega = a_0(x,y)(dy)^k + a_1(x,y)(dy)^{k-1}dx + \ldots + a_k(x,y)(dx)^k = 0,$$

where the coefficients $a_j \in \mathcal{O}_2$, $j = 1, \ldots, k$ and $a_0(0,0) = 1$. (5.2)

 Lemma 5.3. Under the hypotheses of Theorem 4 and above conditions, the surface $S$ is irreducible and $S \cap π^{-1}(0)$ contains just a number finite of points. See figure 1.

**Proof.** Since $W$ is irreducible so is $S$. On the other hand, $S \cap π^{-1}(0)$ is finite because $W$ has a finite number of invariant analytic leaves through the origin and is defined as in 5.2.

We can assume without lost of generality that $S \cap π^{-1}(0)$ contains just one point, in the case general, the idea of the proof is the same. Then in the coordinate system $(x, y, p) \in \mathbb{C}^3$, where $p = \frac{dy}{dx}$, we have $π^{-1}(0) \cap S = \{p_0 = (0, 0, 0)\}$, which implies that $S$ must be singular at $p_0 \in \mathbb{P}T^*(\mathbb{C}^2, 0)$. In particular, $(S, p_0)$ the germ of $S$ at $p_0$ is defined by $F^{-1}(0)$, where

$$F(x, y, p) = p^k + a_1(x,y)p^{k-1} + \ldots + a_k(x,y),$$

and $a_1, \ldots, a_k \in \mathcal{O}_2$. Let $\mathcal{F}_S$ be the foliation defined by $\alpha|_S = 0$. The assumptions implies that $\mathcal{F}_S$ is a non-dicritical foliation with an isolated singularity at $p_0$. 

![Figure 1: $S \cap π^{-1}(0)$](image)
Recall that a germ of foliation $F$ at $p_0 \in S$ is dicritical if it has infinitely many analytic separatrices through $p_0$. Otherwise it is called non-dicritical.

Let $M'_{reg}$ be the lifting of $M_{reg}$ by $\pi_S$, and denote by $\sigma : (\hat{S}, D) \to (S, p_0)$ the resolution of singularities of $S$ at $p_0$. Let $\hat{F} = \sigma^*(F_S)$ be the pull-back of $F_S$ under $\sigma$. See figure 2.

**Lemma 5.4.** In the above situation. The foliation $\hat{F}$ has only singularities of saddle with first integral type in $D$.

**Proof.** Let $y \in M'_{reg}$, it follows from Lemma 4.1 the existence, in a neighborhood $U_y \subset \mathbb{P}^n(\mathbb{C}^2, 0)$ containing $y$, of a real analytic subset $N_y$ of dimension 3 containing $M'_{reg} \cap U_y$. Then by Proposition 4.2 there exists, in a neighborhood $V_y \subset U_y$ of $y$, a complex analytic subset $V_y$ of (complex) dimension 2 containing $N_y \cap V_y$. As germs at $y$, we get $Y_y = S_y$ then $N_y \cap V_y \subset S_y$, we have that $N_y \cap V_y$ is a real analytic hypersurface in $S_y$, and it is Levi-flat because each irreducible component contains a Levi-flat piece (cf. [1], Lemma 2.2).

Let us denote $M'_y = N_y \cap V_y$. The hypotheses implies that $F_S$ is tangent to $M'_y$. These local constructions are sufficiently canonical to be patched together, when $y$ varies on $M'_{reg}$; if $S_{y_1} \subset V_{y_1}$ and $S_{y_2} \subset V_{y_2}$ are as above, with $M'_{reg} \cap V_{y_1} \cap V_{y_2} \neq \emptyset$, then $S_{y_2} \cap (V_{y_1} \cap V_{y_2})$ and $S_{y_1} \cap (V_{y_1} \cap V_{y_2})$ have some common irreducible components containing $M'_{reg} \cap V_{y_1} \cap V_{y_2}$, so that $M'_y, M'_y'$ can be glued by identifying those components. In this way, we obtain a Levi-flat hypersurface $N$ on $S$ tangent to $F_S$.

By doing additional blowing-ups if necessary, we can suppose that $\hat{F}$ has reduced singularities. Since $F_S$ is non-dicritical, all irreducible components of $D$ are $\hat{F}$-invariants. Let $\hat{N}$ be the strict transform of $N$ under $\sigma$, then $\hat{N} \supset D$. In particular, $\hat{N}$ contains all singularities of $\hat{F}$ in $D$. It follows from Proposition 5.1 that all singularities of $\hat{F}$ are saddle with first integral. $\square$

### 5.3 End of the proof of Theorem 1 in dimension two

The idea is to prove that $F_S$ has a holomorphic first integral. Since $D$ is invariant by $\hat{F}$, i.e., it is the union of leaves and singularities of $\hat{F}$, we have $S := D \setminus \text{Sing}(\hat{F})$ is a leaf of $\hat{F}$. Now, fix $p \in S$ and a transverse section $\sum$ through $p$. By Lemma 5.3 the singularities of $\hat{F}$ in $D$ are saddle with first integral types. Therefore the transverse section $\sum$ is complete, (see [10], pg. 162). Let $G \subset \text{Diff}(\sum, p)$ be the Holonomy group of the leaf $S$ of $\hat{F}$. It follows from Lemma 5.2 that all leaves of $F_S$ through points of $N_{reg}$ are closed in $N_{reg}$. This implies that all transformations of $G$ have finite order and $G$ is linearizable. According to [11], $F_S$ has a non-constant holomorphic first integral. Finally from Lemma 2.3 $W$ has a first integral as follows:

$$P(z) = f_0(x, y) + z.f_1(x, y) + \ldots + z^{k-1}.f_{k-1}(x, y) + z^k,$$

where $f_0, f_1, \ldots, f_{k-1} \in \mathcal{O}_2$. 

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5.4 Proof in dimension $n \geq 3$

Let us give an idea of the proof. First of all, we will prove that there is a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ with the following properties:

(i) $i^{-1}(M)$ has real codimension one on $(\mathbb{C}^2, 0)$.

(ii) $i^*(\mathcal{W})$ is a $k$-web on $(\mathbb{C}^2, 0)$ tangent to $i^{-1}(M)$.

Set $E := i(\mathbb{C}^2, 0)$. The above conditions and Theorem [1] in dimension two imply that $\mathcal{W}|_E$ has a non-constant holomorphic first integral, say $g = f_0 + z.f_1 + \ldots + z^{k-1}.f_{k-1} + z^k$, where $f_0, \ldots, f_{k-1} \in \mathcal{O}_2$. After that we will use a lemma to prove that $g$ can be extended to a holomorphic germ $g_1$, which is a first integral of $\mathcal{W}$.

Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a holomorphic codimension one foliation, tangent to a real analytic hypersurface $M$. Let us suppose that $\mathcal{F}$ is defined by $\omega = 0$, where $\omega$ is a germ at $0 \in \mathbb{C}^n$ of an integrable holomorphic 1-form with $\text{cod}_{\mathbb{C}^n}(\text{Sing}(\omega)) \geq 2$. We say that a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ is transverse to $\omega$ if $\text{cod}_{\mathbb{C}^n}(\text{Sing}(i^*(\omega))) = 2$, which means in fact that, as a germ of set, we have $\text{Sing}(i^*(\omega)) = \{0\}$. Note that the definition is independent of the particular germ of holomorphic 1-form which represents $\mathcal{F}$. Therefore, we will say that the embedding $i$ is transverse to $\mathcal{F}$ if it is transverse to some holomorphic 1-form $\omega$ representing $\mathcal{F}$.

We will use the following lemma of [5].

**Lemma 5.5.** In the above situation, there exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{F}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.
We say that an embedding $i$ is transverse to $W$ if it is transverse to all $k$-foliations which defines $W$. Now, one deduces the following

**Lemma 5.6.** There exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $W$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.

**Proof.** First of all, note that outside of the discriminant set of $W$, we can suppose that $W = F_1 \boxtimes \ldots \boxtimes F_k$, where $F_1, \ldots, F_k$ are germs of codimension one smooth foliations. Since $W$ is tangent to $M$, there is a foliation $F_j$ such that it is tangent to a Levi foliation $L_M$ on $M_{reg}$. Lemma 5.5 implies that we can find a 2-plane $E_0$ transverse to $M$ and to $F_j$. Clearly the set of linear mappings transverse to $F_1, \ldots, F_k$ simultaneously is open and dense in the set of linear mappings from $\mathbb{C}^2$ to $\mathbb{C}^n$, by Transversality theory, there exists a linear embedding $i$ such that $E = i(\mathbb{C}^2, 0)$ is transverse to $M_{reg}$ and to $W$ simultaneously.

Let $E$ be a 2-plane as in Lemma 5.6. It easy to check that $W|_E$ satisfies the hypotheses of Theorem 12. By the two dimensional case $W|_E$ has a non-constant first integral:

$$g_0 + z.g_1 + \ldots + z^{k-1}.g_{k-1} + z^k,$$

where $g_0, \ldots, g_{k-1} \in \mathcal{O}_2$.

Let $X$ be the variety associated to $W$ and set $S$ be the surface associated to $W|_E$. Observe that $F_S$ has a non-constant holomorphic first integral $g$ defined on $S$.

**Lemma 5.7.** In the above situation, we have $F_X|_S = F_S$ and $F_X$ has a non-constant holomorphic first integral $g_1$ on $X$, such that $g_1|_S = g$.

**Proof.** It is easily seen that $S \subset X$ which implies that $F_X|_S = F_S$. Let us extend $g$ to $X$. Fix $p \in X_{reg} \setminus \text{Sing}(F_X)$. It is possible to find a small neighborhood $W_p \subset X$ of $p$ and a holomorphic coordinate chart $\varphi : W_p \to \Delta$, where $\Delta \subset \mathbb{C}^n$ is a polydisc, such that:

(i) $\varphi(S \cap W_p) = \{z_3 = \ldots = z_n = 0\} \cap \Delta$.

(ii) $\varphi_*(F_X)$ is given by $dz_n|_\Delta = 0$.

Let $\pi_n : \mathbb{C}^n \to \mathbb{C}^2$ be the projection defined by $\pi_n(z_1, \ldots, z_n) = (z_1, z_2)$ and set $\tilde{g}_p := g \circ \varphi^{-1} \circ \pi_n|_\Delta$. We obtain that $\tilde{g}$ is a holomorphic function defined in $\Delta$ and is a first integral of $\varphi_*(F_X)$. Let $g_p = \tilde{g}_p \circ \varphi$. Notice that, if $W_p \cap W_q \neq \emptyset$, $p$ and $q$ being regular points for $F_X$, then we have $g_p|_{W_p \cap W_q} = g_q|_{W_p \cap W_q}$. This follows easily form the identity principle for holomorphic functions. In particular, $g$ can be extended to

$$W = \bigcup_{p \in X_{reg} \setminus \text{Sing}(F_X)} W_p,$$
which is a neighborhood of $X_{\text{reg}} \setminus \text{Sing}(F_X)$. Call $g_W$ this extension.

Since $\text{codim}_{X_{\text{reg}}} \text{Sing}(F_X) \geq 2$, by a theorem of Levi (cf. [15]), $g_W$ can be extended to $X_{\text{reg}}$, as $\text{codim}_{X_{\text{reg}}} (\text{Sing}(X)) \geq 2$ this allows us to extend $g_W$ to $g_1$ as holomorphic first integral for $F_X$, in whole $X$. □

5.5 End of the proof of Theorem 1 in dimension $n \geq 3$

Since $F_X$ has a non-constant holomorphic first integral on $X$, Lemma 2.4 imply that $W$ has a non-constant holomorphic first integral.

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