Optimal Boundary Control of Reaction Diffusion Equations via Al’brekht’s Method

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Abstract

The two contributions of this paper are as follows. The first is the solution of an infinite dimensional, boundary controlled Linear Quadratic Regulator by the simple and constructive method of completing the square. The second contribution is the extension of Al’brekht’s method to the optimal stabilization of a boundary controlled, nonlinear Reaction Diffusion system.

1 Introduction

In 1961 Al’brekht [1] showed how one could compute degree by degree the Taylor polynomials of the optimal cost and optimal feedback of a smooth, nonlinear, infinite horizon, finite dimensional optimal control problem provided the linear part of the dynamics and the quadratic part of the running cost satisfied the standard Linear Quadratic Regulator (LQR) conditions.

Recently [13] we showed how Al’brekht’s method could be generalized to infinite dimensional problems with distributed control. In this note we show how Al’brekht’s method can be generalized to infinite dimensional problems with boundary control. In the next section we present and solve an LQR for the boundary control of a heated rod. We do this in a novel way by completing the square in infinite dimensions. In Section Three we show how Al’brekht’s method can be used to stabilize a nonlinear reaction diffusion equation using boundary control.

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We are not the first to use Al’brekh’s method on infinite dimensional systems, see the works of Kunisch and coauthors [2], [3], [15]. Krstic and coauthors have had great success stabilizing infinite dimensional systems through boundary control where the the nonlinearities are expressed by Volterra integral operators of increasing degrees using backstepping techniques, [14], [18]. In our extension of Al’brekh, we assume that the nonlinearities are given by Fredholm integral expressions of increasing degrees.

2 Boundary Control of a Heated Rod

We consider a modification of Example 3.5.5 of Curtain and Zwart [7]. We have a rod of length one insulated at one end and heated/cooled at the other. The goal is to control the temperature to a constant set point which we conveniently take to be zero.

Let $0 \leq x \leq 1$ be distance along the rod, $z(x,t)$ be the temperature of the rod at $x, t$ and $z^0(x)$ be the initial temperature distribution of the rod at $t = 0$. The goal is to stabilize the temperature to $z = 0$ as $t \to \infty$ using boundary control at $x = 1$.

The rod is modeled by these equations

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) \quad (2.1)$$

$$z(x, 0) = z^0(x) \quad (2.2)$$

$$\frac{\partial z}{\partial x}(0, t) = 0 \quad (2.3)$$

$$\frac{\partial z}{\partial x}(1, t) = \beta(u(t) - z(1, t)) \quad (2.4)$$

for some positive constant $\beta$ where the control is $u(t)$, the temperature applied to the end of the rod.

First we study the open loop eigenvalues and eigenvectors of the one dimensional Laplacian subject Neumann boundary conditions at $x = 0$ and Robin boundary conditions at $x = 1$,

$$\frac{\partial^2 \phi}{\partial x^2}(x) = \lambda \phi(x) \quad \frac{\partial \phi}{\partial x}(0) = 0$$
\[
\frac{\partial \phi}{\partial x}(1) = -\beta \phi(1)
\]

The eigenvectors are of form

\[
\phi(x) = c \cos \nu x
\]

where \(\nu\) is a root of the equation

\[
\nu = \beta \cot \nu
\]

There is one root, \(\nu_n\), of this equation in each open interval \((n\pi, (n+1/2)\pi)\) for \(n = 0, 1, \ldots\). The \(n^{th}\) root \(\nu_n \to n\pi\) as \(n \to \infty\). If \(\beta = 1\), the zeroth root is \(\nu_0 = 0.8603\). As \(\beta \to 0\) the \(n^{th}\) root \(\nu_n \to n\pi\). If \(\beta = 0\), \(\nu_n = n\pi\) and we have an uncontrolled rod with no heat flux at either end. As \(\beta \to \infty\) the \(n^{th}\) root \(\nu_n \to (n + 1/2)\pi\). The corresponding eigenvalues are \(\lambda_n = -\nu_n^2\). If \(\beta = 1\) the least stable eigenvalue is \(\lambda_0 = -0.7402\). As \(\beta \to 0\) the eigenvalues move to right and as \(\beta \to \infty\) the eigenvalues move to the left.

Because the Laplacian is self adjoint with respect to these boundary conditions, the eigenfunctions are orthogonal. We normalize them

\[
\phi_n(x) = c_n \cos \nu_n x
\]

where

\[
c_n = \text{sign}(\cos \nu_n) \sqrt{\frac{4}{2 + \sin 2\nu_n}}
\]

(2.6)

to get an orthonormal family satisfying \(\phi_n(1) > 0\).

The open loop system is asymptotically stable because all its eigenvalues are in the open left half plane. We seek a feedback control law of the form

\[
u(t) = \int_0^1 K^{[1]}(x)z(x, t) \, dx
\]

to speed up the stabilization. To find \(K^{[1]}(x)\) we solve an LQR problem of minimizing

\[
\left(\int_0^\infty \int_0^1 \int_0^1 Q(x_1, x_2)z(x_1, t)z(x_2, t) \, dx_1 dx_2 dt + \int_0^\infty Ru^2(t) \, dt\right)
\]

(2.7)
where $R > 0$ and $Q(x_1, x_2)$ is a symmetric function, $Q(x_1, x_2) = Q(x_2, x_1)$, satisfying
\[ 0 \leq \int_0^1 \int_0^1 Q(x_1, x_2) \xi(x_1) \xi(x_2) \, dx_1 dx_2 \]
for any function $\xi(x)$.

Let $P^{[2]}(x_1, x_2)$ be any symmetric function of $(x_1, x_2)$. Assume that the control trajectory $u(t)$ results in $z(x, t) \to 0$ as $t \to \infty$. We know that there are such control trajectories because the open loop rod, $u(t) = 0$, is asymptotically stable. By the Fundamental Theorem of Calculus
\[ 0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 
+ \int_0^\infty \int_0^1 \int_0^1 \frac{d}{dt} \left( P^{[2]}(x_1, x_2) z(x_1, t) z(x_2, t) \right) \, dx_1 dx_2 dt \]
so
\[ 0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 \]
\[ + \int_0^\infty \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) \frac{\partial^2 z}{\partial x_1^2}(x_1, t) z(x_2, t) \, dx_1 dx_2 dt \]
\[ + P^{[2]}(x_1, x_2) z(x_1, t) \frac{\partial^2 z}{\partial x_2^2}(x_2, t) \, dx_1 dx_2 dt \]
(2.8)

If we assume that $P^{[2]}(x_1, x_2)$ satisfies Neumann boundary conditions at $x = 0$ and Robin boundary conditions at $x = 1$
\[ \frac{\partial P^{[2]}}{\partial x_1}(0, x_2) = 0, \quad \frac{\partial P^{[2]}}{\partial x_2}(x_1, 0) = 0 \]
\[ \frac{\partial P^{[2]}}{\partial x_1}(1, x_2) = -\beta P^{[2]}(1, x_2), \quad \frac{\partial P^{[2]}}{\partial x_2}(x_1, 1) = -\beta P^{[2]}(x_1, 1) \]
(2.9)
then when we integrate (2.8) by parts twice we get the equation
\[ 0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 
+ \int_0^\infty \int_0^1 \int_0^1 \nabla^2 P^{[2]}(x_1, x_2) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt \]
\[ + \beta \int_0^\infty \int_0^1 P^{[2]}(1, x_2) u(t) z(x_2, t) \, dx_2 dt \]
\[ + \beta \int_0^\infty \int_0^1 P^{[2]}(x_1, 1) u(t) z(x_1, t) \, dx_1 dt \]
(2.11)
where $\nabla^2$ is the two dimensional Laplacian.

We add (2.11) to the criterion (2.7) to be minimized to get the equivalent criterion

$$0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 + \int_0^\infty \int_0^1 \int_0^1 \left( \nabla^2 P^{[2]}(x_1, x_2) + Q(x_1, x_2) \right) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt + \beta \int_0^\infty \int_0^1 P^{[2]}(1, x_2) u(t) z(x_2, t) \, dx_2 dt + \beta \int_0^\infty \int_0^1 P^{[2]}(x_1, 1) z(x_1, t) u(t) \, dx_1 dt + \int_0^\infty Ru^2(t) \, dt$$

We would like to chose $K^{[1]}(x)$ so that the time integral in (2.12) is that of a perfect square, in other words we want (2.12) to become

$$0 = \int_0^1 \int_0^1 z^0(x_1) P^{[2]}(x_1, x_2) z^0(x_2) \, dx_1 dx_2 + \int_0^\infty \int_0^1 \int_0^1 R \left( u(t) - K^{[1]}(x_1) z(x_1, t) \right) \left( u(t) - K^{[1]}(x_2) z(x_2, t) \right) \, dx_1 dx_2 dt$$

Clearly the terms quadratic in $u(t)$ match so we equate the terms involving $u(t)$ and $z(x_1, t)$,

$$\beta \int_0^\infty \int_0^1 P^{[2]}(x_1, 1) z(x_1, t) u(t) \, dx_1 dt = - \int_0^\infty \int_0^1 K^{[1]}(x_1) R z(x_1, t) u(t) \, dx_1 dt$$

They will match if $K^{[1]}(x_1) R = -\beta P^{[2]}(x_1, 1)$ so we assume that

$$K^{[1]}(x_1) = -\beta P^{[2]}(x_1, 1) R^{-1}$$

By the symmetry of $P^{[2]}(x_1, x_2)$, $K^{[1]}(x_2) = -\beta P^{[2]}(1, x_2) R^{-1}$.

Finally we equate the terms involving $z(x_1, t)$ and $z(x_2, t)$,

$$\int_0^\infty \int_0^1 \int_0^1 \left( \nabla^2 P^{[2]}(x_1, x_2) + Q(x_1, x_2) \right) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt = R \int_0^\infty \int_0^1 K^{[1]}(x_1) K^{[1]}(x_2) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt$$

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This yields what we call the Riccati PDE
\[
\nabla^2 P[2](x_1, x_2) + Q(x_1, x_2) = RK[1](x_1)K[1](x_2) \\
= \beta^2 R^{-1}P[2](x_1, 1)P[2](1, x_2)
\]
(2.15)

This is to be interpreted in the weak sense, if \( \xi(x) \) is \( C^2 \) on \( 0 \leq x \leq 1 \) then
\[
\int_0^1 \int_0^1 \left( \nabla^2 P[2](x_1, x_2) + Q(x_1, x_2) \right) \xi(x_1)\xi(x_2) \, dx_1 \, dx_2 \\
= \beta^2 \int_0^1 \int_0^1 R^{-1}P[2](x_1, 1)P[2](1, x_2)\xi(x_1)\xi(x_2) \, dx_1 \, dx_2
\]

The boundary conditions (2.9, 2.10) are also to be interpreted in the weak sense
\[
0 = \int_0^1 \frac{\partial P[2]}{\partial x_1}(0, x_2)\xi(x_2) \, dx_2 \\
0 = \int_0^1 \frac{\partial P[2]}{\partial x_2}(x_1, 0)\xi(x_1) \, dx_1 \\
0 = \int_0^1 \left( \frac{\partial P[2]}{\partial x_1}(1, x_2) + \beta P[2](1, x_2) \right) \xi(x_2) \, dx_2 \\
0 = \int_0^1 \left( \frac{\partial P[2]}{\partial x_2}(x_1, 1) + \beta P[2](x_1, 1) \right) \xi(x_1) \, dx_1
\]

If we can solve the Riccati PDE subject to these boundary conditions then clearly the optimal cost starting from \( z^0(x) \) is
\[
\int_0^1 \int_0^1 P[2](x_1, x_2)z^0(x_1)z^0(x_2) \, dx_1 \, dx_2
\]
and the optimal linear feedback is
\[
u(t) = \int_0^1 K[1](x)z(x, t) \, dx = -\beta \int_0^1 P[2](x, 1)R^{-1}z(x, t) \, dx \quad (2.16)
\]

We assume that the solution to the Riccati PDE has an expansion of the form
\[
P[2](x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Pi[m,n]^2 \phi_m(x_1)\phi_n(x_2)
\]
where $\phi_n(x)$ is given by (2.5). We also assume that $Q(x_1, x_2)$ has a similar expansion,

$$Q(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{m,n} \phi_m(x) \phi_n(x)$$

We plug these into (2.15) and get an algebraic Riccati equation for the infinite dimensional matrix $\Pi_{m,n}^{[2]}$,

$$(\lambda_m + \lambda_n) \Pi_{m,n}^{[2]} + Q_{m,n} = \beta^2 R^{-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Pi_{m,j}^{[2]} \Pi_{i,n}^{[2]} \phi_i(1) \phi_j(1) \quad (2.17)$$

In the particular case where $R = 1$ and $Q(x_1, x_2) = \delta(x_1 - x_2)$ we guess that $\Pi_{m,n}^{[2]} = \delta(m - n) \Pi_{n,n}^{[2]}$ and get a sequence of quadratic equations

$$0 = \beta^2 \phi_n^2(1) \left( \Pi_{n,n}^{[2]} \right)^2 - 2\lambda_n \Pi_{n,n}^{[2]} - 1$$

whose roots are

$$\Pi_{n,n}^{[2]} = \frac{\lambda_n \pm \sqrt{\lambda_n^2 + \beta^2 \phi_n^2(1)}}{\beta^2 \phi_n^2(1)}$$

Clearly we wish to take the positive sign so

$$\Pi_{m,n}^{[2]} = \delta(m - n) \frac{\lambda_n + \sqrt{\lambda_n^2 + \beta^2 \phi_n^2(1)}}{\beta^2 \phi_n^2(1)} \quad (2.18)$$

This infinite dimensional matrix is positive definite, $\Pi_{n,n}^{[2]} > 0$, but not coercive as $\Pi_{n,n}^{[2]} \to 0$ as $n \to \infty$.

The optimal feedback gain is

$$K^{[1]}(x) = -\beta \sum_{n=0}^{\infty} \Pi_{n,n}^{[2]} \phi_n(1) \phi_n(x) \quad (2.19)$$

and the closed loop poles are

$$\mu_n = \lambda_n - \beta \Pi_{n,n}^{[2]} \phi_n(1) \quad (2.20)$$

Because both $\Pi_{n,n}^{[2]}$ and $\phi_n(1)$ are positive the feedback shifts the poles to the left. When $\beta = 1$ the least stable pole moves from $\lambda_0 = -0.7402$ to $\mu_0 = -1.2274$. The closed loop eigenvectors are the same as the open loop eigenvectors, $\phi_n(x)$.
3 Boundary Control of a Nonlinear Reaction Diffusion Equation

To the above system we add a destabilizing nonlinear term to obtain the boundary controlled reaction diffusion system

\[
\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + \alpha z^2(x, t) \tag{3.21}
\]

\[
z(x, 0) = z^0(x) \tag{3.22}
\]

\[
\frac{\partial z}{\partial x}(0, t) = 0 \tag{3.23}
\]

\[
\frac{\partial z}{\partial x}(1, t) = \beta(u(t) - z(1, t)) \tag{3.24}
\]

for some positive constants \(\alpha\) and \(\beta\). To find a feedback to stabilize this system we consider the nonlinear quadratic optimal control of minimizing (2.7) subject to (3.21), (3.22), (3.23), (3.24).

Let \(P^{[2]}(x_1, x_2)\) be the solution of the Riccati PDE (2.15) and \(K^{[1]}(x)\) be the gain of the optimal linear feedback (2.19). Let \(P^{[3]}(x_1, x_2, x_3)\) be any symmetric function of three variables. By symmetric we mean that the value of the function is invariant under any permutation of the three variables. Assume that the optimal feedback takes the form

\[
u(t) = \int_0^1 K^{[1]}(x_1) z(x_1, t) \, dx_1 + \int_0^1 \int_0^1 K^{[2]}(x_1, x_2) z(x_1, t) z(x_2, t) \, dx_1 \, dx_2 + O(z(x, t))^3\tag{3.25}
\]

where the omitted terms are of degrees three and higher in \(z(x, t)\).

Again by the Fundamental Theorem of Calculus if the control trajectory \(u(t)\) takes \(z(x, t) \to 0\) as \(t \to \infty\) then

\[
0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 \, dx_2 \\
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 \, dx_2 \\
+ \int_0^\infty \int_0^1 \int_0^1 \frac{d}{dt} \left( P^{[2]}(x_1, x_2) z(x_1, t) z(x_2, t) \right) \, dx_1 \, dx_2 \, dt \\
+ \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \frac{d}{dt} \left( P^{[3]}(x_1, x_2) z(x_1, t) z(x_2, t) z(x_3, t) \right) \, dx_1 \, dx_2 \, dx_3 \, dt \\
+ O(z(x, t))^4
\]
Because $P[x_1, x_2]$ is the solution of the Riccati PDE, the terms quadratic in $z$ in the time integral drop out. But we pick up cubic terms from the boundary when we integrate (2.3) by parts twice,

$$
\begin{align*}
\beta & \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P[3](1, x_2, x_3) K[1](x_1) + P[3](1, x_3) K[1](x_2) \\
& + P[3](x_1, 1) K[1](x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3 dt \\
& + \beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P[2](1, x_2) K[2](x_1, x_3) + P[2](1, x_3) K[2](x_2, x_3) \right) \\
& \times z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3 dt
\end{align*}
$$

So we obtain

$$
0 = \int_0^1 \int_0^1 P[2](x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2
$$

(3.26)

$$
+ \int_0^1 \int_0^1 \int_0^1 P[3](x_1, x_2) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 dx_2
$$

$$
+ \alpha \int_0^\infty \int_0^1 \int_0^1 P[2](x_1, x_2) \left( z^2(x_1, t) z(x_2, t) + z(x_1, t) z^2(x_2, t) \right) \, dx_1 dx_2 dt
$$

$$
+ \int_0^\infty \int_0^1 \int_0^1 \int_0^1 P[3](x_1, x_2, x_3) \frac{\partial^2 z}{\partial x_1^2}(x_1, t) z(x_2, t) z(x_3, t)
$$

$$
+ P[3](x_1, x_2, x_3) z(x_1, t) \frac{\partial^2 z}{\partial x_2^2}(x_2, t) z(x_3, t)
$$

$$
+ P[3](x_1, x_2, x_3) z(x_1, t) z(x_2, t) \frac{\partial^2 z}{\partial x_3^2}(x_3, t) \, dx_1 dx_2 dx_3 dt
$$

$$
+ \beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P[3](1, x_2, x_3) K[1](x_1) + P[3](1, x_3) K[1](x_2) \\
& + P[3](x_1, 1) K[1](x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3 dt
$$

$$
+ \beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P[2](1, x_2) K[2](x_1, x_3) + P[2](1, x_3) K[2](x_2, x_3) \right)
$$

$$
\times z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3 dt + O(z(x, t))^4
$$

If we assume that $P[3](x_1, x_2, x_3)$ weakly satisfies Neumann boundary conditions at $x = 0$

$$
\frac{\partial P[3]}{\partial x_1}(0, x_2, x_3) = 0, \quad \frac{\partial P[3]}{\partial x_2}(x_1, 0, x_3) = 0, \quad \frac{\partial P[3]}{\partial x_3}(x_1, x_2, 0) = 0 \quad (3.27)
$$
then when we integrate (3.26) by parts twice we get the equation

\[
0 = \int_0^1 \int_0^1 P^{[2]}(x, y) z^0(x) z^0(y) \, dx \, dy
\]

and Robin boundary conditions at \( x = 1 \)

\[
\frac{\partial P^{[3]}_3}{\partial x_1}(1, x_2, x_3) = -\beta P^{[3]}_3(1, x_2, x_3)
\]

\[
\frac{\partial P^{[3]}_1}{\partial x_1}(x_1, 1, x_3) = -\beta P^{[3]}_1(x_1, 1, x_3)
\]

\[
\frac{\partial P^{[3]}_2}{\partial x_1}(x_1, x_2, 1) = -\beta P^{[3]}_2(x_1, x_2, 1)
\]

then when we integrate (3.26) by parts twice we get the equation

\[
0 = \int_0^1 \int_0^1 P^{[2]}(x, y) z^0(x) z^0(y) \, dx \, dy
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x, y, z) z^0(x) z^0(y) z^0(z) \, dx \, dy \, dz
\]

\[
+ \int_0^\infty \int_0^1 \int_0^1 \nabla^2 P^{[3]}(x, y, z) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \, dx_1 \, dy_1 \, dz_1
\]

\[
+ \alpha \int_0^\infty \int_0^1 \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^2(x_1, t) z^2(x_2, t) + z(x_1, t) z^2(x_2, t) \, dx_1 \, dy_1 \, dz_1 \, dt
\]

\[
+ \beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P^{[3]}_3(x_1, x_2, x_3) K^{[1]}(x_1) + P^{[3]}_3(x_1, 1, x_3) K^{[1]}(x_2) + P^{[3]}_3(x_1, x_2, 1) K^{[1]}(x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 \, dy_1 \, dz_1 \, dt
\]

\[
+ \beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P^{[2]}_3(x_1, x_2) K^{[2]}_3(x_1, x_3) + P^{[2]}_3(x_1, 1, x_3) K^{[2]}_2(x_2, x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 \, dy_1 \, dz_1 \, dt
\]

where \( \nabla^2 \) is now the three dimensional Laplacian.

We have cancelled the quadratic terms in the criterion by our choice \( P^{[2]}_3(x_1, x_2) \) and \( K^{[1]}(x) \) but the quadratic term in the feedback generates a cubic term in the criterion

\[
R \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( K^{[1]}_3(x_1) K^{[2]}_3(x_2, x_3) + K^{[1]}_3(x_2) K^{[2]}_3(x_1, x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 \, dy_1 \, dz_1 \, dt
\]

Because of (2.19) this equals

\[
-\beta \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( P^{[2]}_3(x_1, x_2) K^{[2]}_3(x_1, x_3) + P^{[2]}_3(x_1, 1, x_3) K^{[2]}_2(x_2, x_3) \right) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 \, dy_1 \, dz_1 \, dt
\]
and so this cancels out the last term in (3.29). So the new criterion to be minimized is
\[
0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 \\
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 dx_2 dx_3 \\
+ \int_0^\infty \int_0^1 \int_0^1 \nabla^2 P^{[3]}(x_1, x_2) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3 dt \\
+ \beta (P^{[3]}(1, x_2, x_3) K^{[1]}(x_1)) + P^{[3]}(x_1, 1, x_3) K^{[1]}(x_2) \\
+ P^{[3]}(x_1, x_2, 1) K^{[1]}(x_3)) z(x_1) z(x_2) z(x_3) \, dx_1 dx_2 dx_3 \\
+ \alpha \int_0^1 \int_0^1 \int_0^1 P^{[2]}(x_1, x_2, x_3) (z^2(x_1, t) z(x_2, t) + z(x_1, t) z^2(x_2, t)) \, dx_1 dx_2 dt
\]

We assume that \(P^{[3]}(x_1, x_2, x_3)\) is a weak solution to linear elliptic PDE
\[
0 = \nabla^2 P^{[3]}(x_1, x_2, x_3) + \beta P^{[3]}(1, x_2, x_3) K^{[1]}(x_1) \\
+ \beta P^{[3]}(x_1, 1, x_3) K^{[1]}(x_2) + \beta P^{[3]}(x_1, x_2, 1) K^{[1]}(x_3) \\
+ \alpha P^{[2]}(x_1, x_2) \delta(x_1 - x_3) + \alpha P^{[2]}(x_1, x_2) \delta(x_2 - x_3)
\]

subject to the weak boundary conditions (3.27, 3.28) then the optimal cost is
\[
\pi(z^0(\cdot)) = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 \\
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, x_3) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 dx_2 dx_3 \\
+ O(z^0(x))
\]

We symmetrize \(P^{[3]}(x_1, x_2, x_3)\) by averaging over all permutations of its three arguments.

Consider a cubic polynomial
\[
\int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, x_3) \zeta(x_1) \zeta(x_2) \zeta(x_3) \, dx_1 dx_2 dx_3
\]
whose argument is \(C^2\) functions \(\zeta(x)\) which satisfy at \(x = 0\) the Neumann boundary condition \(\frac{\partial \zeta}{\partial x}(0) = 0\) and satisfy at \(x = 1\) the Robin boundary condition \(\frac{\partial \zeta}{\partial x}(1) = -\beta \zeta(1)\). Consider the linear operator that maps such a polynomial into
the cubic polynomial
\[
\int_0^1 \int_0^1 \int_0^1 \nabla^2 P[3](x_1, x_2, x_3) \\
+ \beta \left( P[3](1, x_2, x_3) K[1](x_1) + P[3](x_1, 1, x_3) K[1](x_2) \\
+ P[3](x_1, x_2, 1) K[1](x_3) \right) \zeta(x_1) \zeta(x_2) \zeta(x_3) \, dx_1 dx_2 dx_3
\]
where \( \nabla^2 \) is the three dimensional Laplacian. It is not hard to show that the eigenvectors of this operator are of the form \( \phi_{n_1}(x_1) \phi_{n_2}(x_2) \phi_{n_3}(x_3) \) where \( \phi_n(x) \) are eigenfunctions of the open and closed loop linear dynamics and the corresponding eigenvalues are \( \mu_{n_1} + \mu_{n_2} + \mu_{n_3} \) where again \( \mu_{n-i} \) is an eigenvalue of the closed loop linear dynamics. Since all the eigenvalues are in the open left half plane, \( \mu_{n_1} + \mu_{n_2} + \mu_{n_3} \neq 0 \) and this operator is invertible. Therefore (3.31) always has a weak solution.

We use a standard argument to find the quadratic part of the optimal feedback (3.25). By the Principal of Optimality the optimal cost starting at \( z^0 \) at \( t = 0 \) is for small \( \tau \geq 0 \)
\[
\pi(z^0(\cdot)) = \min_{u(\cdot)} \left\{ \pi(z(\tau)) + \int_0^\tau Ru^2(t) \, dt \\
+ \int_0^\tau \int_0^1 \int_0^1 Q(x_1, x_2) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt \right\} \tag{3.33}
\]
Now for small \( \tau \geq 0 \)
\[
\int_0^\tau Ru^2(t) \, dt = \tau Ru^2(0) + O(\tau)^2
\]
\[
\int_0^\tau \int_0^1 \int_0^1 Q(x_1, x_2) z(x_1, t) z(x_2, t) \, dx_1 dx_2 dt
\]
\[
= \tau \int_0^1 \int_0^1 Q(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 + O(\tau)^2
\]
and by arguments similar to the above
\[
\pi(z(\tau)) - \pi(z(0)) = \tau \left. \frac{\partial \pi(z(\cdot), t)}{\partial t} \right|_{t=0} + O(\tau)^2
\]
\[
= \tau \left( \int_0^1 \int_0^1 \nabla P[2](x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2 \\
+ \alpha \int_0^1 \int_0^1 P[2](x_1, x_2) \left( (z^0(x_1))^2 z^0(x_2) + z^0(x_1)(z^0(x_2))^2 \right) \, dx_1 dx_2 \right)
\]
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\[ + \beta \int_0^1 P^{[2]}(1, x_2)u(0)z^0(x_2) \, dx_1 + \beta \int_0^1 P^{[2]}(x_1, 1)z^0(x_1)u(0) \, dx_2 \\
+ \int_0^1 \int_0^1 \nabla P^{[3]}(x_1, x_2, x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dx_1 dx_2 dx_3 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(1, x_2, x_3)u(0)z^0(x_2)z^0(x_3) \, dx_2 dx_3 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(x_1, 1, x_3)u(0)z^0(x_1)u(0)z^0(x_3) \, dx_1 dx_3 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, 1)z^0(x_1)z^0(x_2)u(0) \, dx_1 dx_2 \\
+ O(z^0(x))^4 + O(\tau)^2 \]

We plug these approximations into (3.33) and to find the minimum we set the derivative with respect to \( u(0) \) to zero to obtain

\[-2Ru(0) = \beta \int_0^1 P^{[2]}(1, x_2)z^0(x_2) \, dx_1 + \beta \int_0^1 P^{[2]}(x_1, 1)z^0(x_1) \, dx_2 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(1, x_2, x_3)z^0(x_2)z^0(x_3) \, dx_2 dx_3 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(x_1, 1, x_3)z^0(x_1)z^0(x_3) \, dx_2 dx_3 \\
+ \beta \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, 1)z^0(x_1)z^0(x_2) \, dx_2 dx_3 \\
+ O(z^0(x))^4 + O(\tau)^2 \]

So we conclude again that the linear part of the optimal feedback is given by (2.16). Using the symmetry of \( P^{[3]}(x_1, x_2, x_3) \), the kernel of the quadratic part of the optimal feedback is

\[ K^{[2]}(x_1, x_2) = -\frac{3}{2}R^{-1}P^{[3]}(x_1, x_2, 1) \quad (3.34) \]

Because \( P^{[3]}(x_1, x_2, x_3) \) is symmetric in its three arguments, \( K^{[2]}(x_1, x_2) \) is symmetric in its two arguments.

We assume that the solution \( P^{[3]}(x_1, x_2, x_3) \) of (3.31) has an expansion in the eigenfunctions of the closed loop linear dynamics. As we have seen before these are also eigenfunctions \( \phi_n(x) \) of the open loop linear dynamics (2.5).

\[ P^{[3]}(x_1, x_2, x_3) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \Pi^{[3]}_{n_1,n_2,n_3} \phi_{n_1}(x_1)\phi_{n_2}(x_2)\phi_{n_3}(x_3) \]
The symmetry of $P^3(x_1, x_2, x_3)$ implies that $\Pi_{n_1, n_2, n_3}^3$ is symmetric in $n_1, n_2, n_3$.

We again make the simplifying assumptions $R = 1$ and $Q(x_1, x_2) = \delta(x_1 - x_2)$ so (2.18) holds. We expand the other terms in (3.31) in these eigenfunctions, to compute the coefficients we evaluate these integrals

$$
\int_0^1 \int_0^1 \int_0^1 P^2(x_1, x_2) \delta(x_1 - x_3) \phi_{n_1}(x_1) \phi_{n_2}(x_2) \phi_{n_3}(x_3) \, dx_1 \, dx_2 \, dx_3 \\
= \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^\infty \Pi_{n,n}^2 \phi_{n_1}(x_1) \phi_{n_2}(x_2) \delta(x_1 - x_3) \phi_{n_1}(x_1) \phi_{n_2}(x_2) \phi_{n_3}(x_3) \, dx_1 \, dx_2 \, dx_3 \\
= \int_0^1 \Pi_{n_2,n_2} \phi_{n_1}(x_1) \phi_{n_2}(x_1) \phi_{n_3}(x_1) \, dx_1 \\
= \frac{c_{n_1} c_{n_2} c_{n_3}}{4} \Pi_{n_2,n_2}^2 \left( \frac{\sin(\nu_{n_1} + \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} + \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} - \nu_{n_3}} \\
+ \frac{\sin(\nu_{n_1} - \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} - \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} - \nu_{n_3}} \right)
$$

where $c_n$ are given by (2.6). By symmetry

$$
\int_0^1 \int_0^1 \int_0^1 P^2(x_1, x_2) \delta(x_1 - x_3) \phi_{n_1}(x_1) \phi_{n_2}(x_2) \phi_{n_3}(x_3) \, dx_1 \, dx_2 \, dx_3 \\
= \frac{c_{n_1} c_{n_2} c_{n_3}}{4} \Pi_{n_1,n_1}^2 \left( \frac{\sin(\nu_{n_1} + \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} + \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} - \nu_{n_3}} \\
+ \frac{\sin(\nu_{n_1} - \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} - \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} - \nu_{n_3}} \right)
$$

Then (3.31) becomes

$$
(\mu_{n_1} + \mu_{n_2} + \mu_{n_3}) \Pi_{n_1,n_1,n_3}^3 \\
= -\frac{\alpha c_{n_1} c_{n_2} c_{n_3}}{4} \left( \frac{\sin(\nu_{n_1} + \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} + \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} + \nu_{n_2} - \nu_{n_3}} \\
+ \frac{\sin(\nu_{n_1} - \nu_{n_2} + \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} + \nu_{n_3}} + \frac{\sin(\nu_{n_1} - \nu_{n_2} - \nu_{n_3})}{\nu_{n_1} - \nu_{n_2} - \nu_{n_3}} \right) \left( \Pi_{n_2,n_1}^2 + \Pi_{n_2,n_2}^2 \right)
$$

The solutions $\Pi_{n_1,n_2,n_3}^3$ of these equations are not necessarily symmetric in the indices $n_1, n_2, n_3$. They need to be symmetrized by averaging over all permutations of $n_1, n_2, n_3$. 

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From (3.34) the gain of the quadratic part of the optimal feedback is

\[ K^{[2]}(x_1, x_2) = -\frac{3}{2} R^{-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \Pi_{n_1,n_2,n_3}^{[3]}(x_1)\phi_{n_1}(x_1)\phi_{n_2}(x_2)\phi_{n_3}(1) \]

Again this needs to be symmetrized by averaging over the interchange of \( x_1, x_2 \), \( K^{[2]}(x_1, x_2) \) is redefined to be

\[ \frac{1}{2} \left( K^{[2]}(x_1, x_2) + K^{[2]}(x_2, x_1) \right) \]

Next we sketch how to find the degree four part of the optimal cost and the degree three of the optimal feedback. We assume the Taylor polynomial of degree four of the optimal cost is of the form

\[
\int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 dx_2 dx_3
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 P^{[4]}(x_1, x_2, x_3, x_4) z^0(x_1) z^0(x_2) z^0(x_3) z^0(x_4) \, dx_1 dx_2 dx_3 dx_4
\]

and we assume the Taylor polynomial of degree three of the optimal feedback is of the form

\[
\int_0^1 K^{[1]}(x_1) z(x_1, t) \, dx_1
\]

\[
+ \int_0^1 \int_0^1 K^{[2]}(x_1, x_2) z(x_1, t) z(x_2, t) \, dx_1 dx_2
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 K^{[3]}(x_1, x_2, x_3) z(x_1, t) z(x_2, t) z(x_3, t) \, dx_1 dx_2 dx_3
\]

Assume a control trajectory \( u(\cdot) \) takes \( z(x, t) \rightarrow 0 \) as \( t \rightarrow \infty \) then

\[
0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2) z^0(x_1) z^0(x_2) \, dx_1 dx_2
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1, x_2) z^0(x_1) z^0(x_2) z^0(x_3) \, dx_1 dx_2 dx_3
\]

\[
+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 P^{[4]}(x_1, x_2, x_3, x_4) z^0(x_1) z^0(x_2) z^0(x_3) z^0(x_4) \, dx_1 dx_2 dx_3 dx_4
\]
\[
\int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} P^{[2]}(x_1, x_2) \frac{\partial}{\partial t} (z(x_1, t)z(x_2, t)) \ dx_1 dx_2 dt
\]
\[
+ \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} P^{[3]}(x_1, x_2) \frac{\partial}{\partial t} (z(x_1, t)z(x_2, t)z(x_3, t)) \ dx_1 dx_2 dx_3 dt
\]
\[
+ \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} P^{[4]}(x_1, x_2, x_3, x_4) \frac{\partial}{\partial t} (z(x_1, t)z(x_2, t)z(x_3, t)z(x_4, t)) \ dx_1 dx_2 dx_3 dx_4 dt
\]

We assume that \( P^{[2]}(x_1, x_2), P^{[3]}(x_1, x_2, x_3), K^{[1]}(x_1), K^{[2]}(x_1, x_2) \) are as above and \( P^{[4]}(x_1, x_2, x_3, x_4) \) satisfies Neumann boundary conditions at \( x = 0 \) and Robin boundary conditions at \( x = 1 \) similar to (3.27, 3.28). We integrate by parts twice and display the terms of (3.38) that are quartic in \( z(x, t) \)

\[
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} P^{[4]}(x_1, x_2, x_3, x_4) z^0(x_1)z^0(x_2)z^0(x_3)z^0(x_4) + \beta P^{[4]}(1, x_2, x_3, x_4) K^{[1]}(x_1)
\]
\[
+ \beta P^{[3]}(x_1, 1, x_3, x_4) K^{[1]}(x_2) + \beta P^{[4]}(x_1, x_2, 1, x_4) K^{[1]}(x_3)
\]
\[
+ \beta P^{[4]}(x_1, x_2, x_3, 1) K^{[1]}(x_4) \bigg) \bigg[ z(x_1, t)z(x_2, t)z(x_3, t)z(x_4, t) \ dx_1 dx_2 dx_3 dx_4 dt
\]
\[
+ \alpha \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} P^{[3]}(x_1, x_2, x_3) \bigg( z^2(x_1, t)z(x_2, t)z(x_3, t)
\]
\[
+ z(x_1, t)z^2(x_2, t)z(x_3, t) + z(x_1, t)z(x_2, t)z^2(x_3, t) \bigg) \ dx_1 dx_2 dx_3 dt
\]
\[
+ \beta \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \bigg( P^{[2]}(1, x_2) K^{[3]}(x_1, x_3, x_4) + P^{[2]}(1, 1) K^{[2]}(x_2, x_3, x_4) \bigg)
\]
\[
\times z(x_1, t)z(x_2, t)z(x_3, t)z(x_4, t) \ dx_1 dx_2 dx_3 dx_4 dt
\]

where \( \nabla^2 \) now denotes the four dimensional Laplacian.

We have cancelled the quadratic and cubic terms in the criterion by our choice \( P^{[2]}(x_1, x_2), P^{[3]}(x_1, x_2, x_3), K^{[1]}(x_1) \) and \( K^{[2]}(x_1, x_2) \) but the cubic term in the feedback generates a quartic term in the criterion

\[
R \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \bigg( K^{[1]}(x_1) K^{[3]}(x_2, x_3, x_4) + K^{[1]}(x_2) K^{[3]}(x_1, x_3, x_4) \bigg)
\]
\[
\times z(x_1, t)z(x_2, t)z(x_3, t)z(x_4, t) \ dx_1 dx_2 dx_3 dx_4 dt
\]

Because of (2.19) this equals

\[
-\beta \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \bigg( P^{[2]}(1, x_2) K^{[3]}(x_1, x_3, x_4) + P^{[2]}(1, 1) K^{[3]}(x_2, x_3, x_4) \bigg)
\]
\[
\times z(x_1, t)z(x_2, t)z(x_3, t)z(x_4, t) \ dx_1 dx_2 dx_3 dx_4 dt
\]
and so this cancels out the last term in (3.39). So the quartic terms in the new
criterion to be minimized are

\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 P^4(x_1, x_2, x_3, x_4) z^0(x_1) z^0(x_2) z^0(x_3) z^0(x_4) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \\
+ \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left( \nabla^2 P^4(x_1, x_2, x_3, x_4) + \beta P^4(1, x_2, x_3, x_4) K^{[1]}(x_1) \right) \\
+ \beta P^4(x_1, 1, x_3, x_4) K^{[1]}(x_2) + \beta P^4(x_1, x_2, 1, x_4) K^{[1]}(x_3) \\
+ \beta P^4(x_1, x_2, x_3, 1) K^{[1]}(x_4) \right) z(x_1, t) z(x_2, t) z(x_3, t) z(x_4, t) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dt \\
+ \alpha \int_0^\infty \int_0^1 \int_0^1 \int_0^1 P^3(x_1, x_2, x_3) \left( z^2(x_1, t) z(x_2, t) z(x_3, t) \\
+ z(x_1, t) z^2(x_2, t) z(x_3, t) + z(x_1, t) z(x_2, t) z^2(x_3, t) \right) \, dx_1 \, dx_2 \, dx_3 \, dt \\
\text{(3.40)}
\]

Suppose \( P^4(x_1, x_2, x_3, x_4) \) is a weak solution to the linear elliptic PDE

\[
0 = \nabla^2 P^4(x_1, x_2, x_3, x_4) + \beta P^4(1, x_2, x_3, x_4) K^{[1]}(x_1) \\
+ \beta P^4(x_1, 1, x_3, x_4) K^{[1]}(x_2) + \beta P^4(x_1, x_2, 1, x_4) K^{[1]}(x_3) \\
+ \beta P^4(x_1, x_2, x_3, 1) K^{[1]}(x_4) + \alpha P^3(x_1, x_2, x_3) \delta(x_1 - x_4) \\
+ \alpha P^3(x_1, x_2, x_3) \delta(x_2 - x_4) + \alpha P^3(x_1, x_2, x_3) \delta(x_3 - x_4) \\
\text{(3.41)}
\]

then the optimal cost through terms of degree four is given by (3.36). We symmetrize \( P^4(x_1, x_2, x_3, x_4) \) by averaging over all permutations of its four arguments.

We find the gain of the optimal feedback of degree three as before, using the symmetry of \( P^4(x_1, x_2, x_3, x_4) \) it is

\[
K^{[3]}(x_1, x_2, x_3) = -2R^{-1} P^4(x_1, x_2, x_3, 1) \\
\text{(3.42)}
\]

Because \( P^4(x_1, x_2, x_3, x_4) \) is symmetric in its four arguments, \( K^{[3]}(x_1, x_2, x_3) \) is symmetric in its three arguments.

We again make the simplifying assumptions \( R = 1 \) and \( Q(x_1, x_2) = \delta(x_1 - x_2) \) so (2.18) (3.35) and hold. We assume

\[
P^4(x_1, x_2, x_3, x_4) = \sum_{n_1, n_2, n_3, n_4} \prod_{n_1, n_2, n_3, n_4} \phi_{n_1}(x_1) \phi_{n_2}(x_2) \phi_{n_3}(x_3) \phi_{n_4}(x_4)
\]

then (3.41) yields the recursion

\[
0 = (\mu_{n_1} + \mu_{n_2} + \mu_{n_3} + \mu_{n_4}) \prod_{n_1, n_2, n_3, n_4} + \alpha \left( \prod_{n_1, n_2, n_3} \delta(n_1 - n_4) + \prod_{n_1, n_2, n_3} \delta(n_2 - n_4) + \prod_{n_1, n_2, n_3} \delta(n_3 - n_4) \right)
\]

\( 17 \)
This recursion is always solvable because \( \mu_{n_1} + \mu_{n_2} + \mu_{n_3} + \mu_{n_4} \neq 0 \). The solution needs to be symmetrized by replacing \( \Pi_{n_1,n_2,n_3,n_4} \) by its average over all permutations of its four indices. The the optimal feedback cubic gain is given in (3.42).

The higher degree terms in the optimal cost and optimal feedback are found in a similar fashion.

### 4 Conclusion

We solved the LQR problem for the boundary control of a infinite dimensional system by extending the finite dimensional technique of completing the square. We also optimally locally stabilized a reaction diffusion system by extending Al’brekht’s method to infinite dimensions.

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