LOWER $Q$-HOMEOMORPHISMS WITH RESPECT TO $p$-MODULUS AND ORLICZ-SOBOLEV CLASSES

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March 4, 2013

Abstract

We show that under a condition of the Calderon type on $\varphi$ the homeomorphisms $f$ with finite distortion in $W_{\text{loc}}^1,\varphi$ and, in particular, $f \in W_{\text{loc}}^{1,s}$ for $s > n - 1$ are the so-called lower $Q$-homeomorphisms with respect to $p$-modulus where $Q(x)$ is equal to its outer $p$-dilatation $K_{p,f}(x)$.

2000 Mathematics Subject Classification: Primary 30C65; Secondary 30C75
Key words: Sobolev classes, Orlicz-Sobolev classes, mappings of finite distortion, lower $Q$-homeomorphisms.

1 Introduction

In what follows, $D$ is a domain in a finite-dimensional Euclidean space. Following Orlicz, see [26], given a convex increasing function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$, denote by $L_\varphi$ the space of all functions $f : D \to \mathbb{R}$ such that

$$\int_D \varphi \left( \frac{|f(x)|}{\lambda} \right) \, dm(x) < \infty$$

(1.1)

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in $D$. $L_\varphi$ is called the Orlicz space. If $\varphi(t) = t^p$, then we write also $L_p$. In other words, $L_\varphi$ is the cone over the class of all functions $g : D \to \mathbb{R}$ such that

$$\int_D \varphi (|g(x)|) \, dm(x) < \infty$$

(1.2)

which is also called the Orlicz class, see [3].

The Orlicz-Sobolev class $W_{\text{loc}}^{1,\varphi}(D)$ is the class of locally integrable functions $f$ given in $D$ with the first distributional derivatives whose gradient $\nabla f$ belongs locally in $D$ to the Orlicz class. Note that by definition $W_{\text{loc}}^{1,\varphi} \subseteq W_{\text{loc}}^{1,1}$. As usual, we write $f \in W_{\text{loc}}^{1,p}$ if $\varphi(t) = t^p$, $p \geq 1$. It is known that a continuous function $f$ belongs to $W_{\text{loc}}^{1,p}$ if and only if $f \in ACL^p$, i.e., if $f$ is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis, and if the first partial derivatives
of $f$ are locally integrable with the power $p$, see, e.g., 1.1.3 in [24]. The concept of the distributional derivative was introduced by Sobolev [32] in $\mathbb{R}^n$, $n \geq 2$, and it is developed under wider settings at present, see, e.g., [28].

Later on, we also write $f \in W^{1,}\varphi\,_{\text{loc}}$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of $n$ real variables $x_1, \ldots, x_n$ if $f_i \in W^{1,1}_{\text{loc}}$ and

$$\int_D \varphi(|\nabla f(x)|) \, dm(x) < \infty \quad (1.3)$$

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left( \frac{\partial f_i}{\partial x_j} \right)^2}$. Note that in this paper we use the notation $W^{1,}\varphi\,_{\text{loc}}$ for more general functions $\varphi$ than in the classical Orlicz classes giving up the condition on convexity of $\varphi$. Note also that the Orlicz–Sobolev classes are intensively studied in various aspects at present.

Recall that a homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{R}^n$, $n \geq 2$, is called of finite distortion if $f \in W^{1,1}\,_{\text{loc}}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x) \quad (1.4)$$

with a.e. finite function $K$ where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix $f'$ of $f$ at $x \in D$, $\|f'(x)\| = \sup_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$, and $J_f(x) = \det f'(x)$ is its Jacobian. We set $K_{p,f}(x) = \|f'(x)\|^p / J_f(x)$ if $J_f(x) \neq 0$, $K_{p,f}(x) = 1$ if $f'(x) = 0$ and $K_{p,f}(x) = \infty$ at the rest points.

First this notion was introduced on the plane for $f \in W^{1,2}_{\text{loc}}$ in the work [16]. Later on, this condition was changed by $f \in W^{1,1}_{\text{loc}}$ but with the additional condition $J_f \in L^1_{\text{loc}}$ in the monograph [15]. The theory of the mappings with finite distortion had many successors, see, e.g., a number of references in the monograph [23]. They had as predecessors the mappings with bounded distortion, see [27] and [34], in other words, the quasiregular mappings, see, e.g., [4], [5], [13], [21], [29] and [35].

Note that the above additional condition $J_f \in L^1_{\text{loc}}$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{R}^n$ with the first partial derivatives a.e. in $D$, there is a set $E$ of the Lebesgue measure zero such that $f$ satisfies $(N)$-property by Lusin on $D \setminus E$ and

$$\int_A J_f(x) \, dm(x) = |f(A)| \quad (1.5)$$

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [8]. On this base, it is easy by the Hölder inequality to verify, in particular, that if $f \in W^{1,1}_{\text{loc}}$ is a homeomorphism and $K_f \in L^q_{\text{loc}}$ for some $q > n - 1$, then also $f \in W^{1,p}_{\text{loc}}$ for some $p > n - 1$, that we use further to obtain corollaries.
In this paper $H^k(A)$, $k \geq 0$, $\dim_H A$ denote the $k$-dimensional Hausdorff measure and the Hausdorff dimension, correspondingly, of a set $A$ in $\mathbb{R}^n$, $n \geq 1$. It was shown in [11] that a set $A$ with $\dim_H A = p$ can be transformed into a set $B = f(A)$ with $\dim_H B = q$ for each pair of numbers $p$ and $q \in (0, n)$ under a quasiconformal mapping $f$ of $\mathbb{R}^n$ onto itself, cf. also [1] and [2].

2 Preliminaries

First of all, the following fine property of functions $f$ in the Sobolev classes $W^{1,p}_{loc}$ was proved in the monograph [12], Theorem 5.5, and can be extended to the Orlicz-Sobolev classes. The statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev’s class $W^{1,1}_{loc}$ in terms of ACL (absolute continuity on lines), see, e.g., Section 1.1.3 in [24].

**Theorem 2.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping in the class $W^{1,\phi}_{loc}(\Omega)$ where $\phi : [0, \infty) \to [0, \infty)$ is increasing with the condition

$$\int_1^\infty \left[ \frac{t}{\phi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \quad (2.1)$$

Then $f$ has a total differential a.e. in $\Omega$.

**Corollary 2.1.** If $f : \Omega \to \mathbb{R}^n$ is a homeomorphism in $W^{1,1}_{loc}$ with $K_f \in L^p_{loc}$ for $p > n - 1$, then $f$ is differentiable a.e.

**Theorem 2.2.** Let $U$ be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $\phi : [0, \infty) \to [0, \infty)$ is increasing with the condition (2.1). Then each continuous mapping $f : U \to \mathbb{R}^m$, $m \geq 1$, in the class $W^{1,\phi}_{loc}$ has the $(N)$-property (furthermore, it is locally absolutely continuous) with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane $\mathcal{P}$ which is parallel to a fixed coordinate hyperplane $\mathcal{P}_0$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq \mathcal{P}$ for a.e. such $\mathcal{P}$.

Note that, if the condition (2.1) holds for an increasing function $\phi$, then the function $\phi_* = \phi(ct)$ for $c > 0$ also satisfies (2.1). Moreover, the Hausdorff measures are quasi-invariant under quasi-isometries. By the Lindelöf property of $\mathbb{R}^n$, $U \setminus \{x_0\}$ can be covered by a countable collection of open segments of spherical rings in $U \setminus \{x_0\}$ centered at $x_0$ and each such segment can be mapped onto a rectangular oriented segment of $\mathbb{R}^n$ by some quasi-isometry, see, e.g., I.5.XI in [20] for the Lindelöf theorem. Thus, applying piecewise Theorem 2.2, we obtain the following.

**Corollary 2.2.** Under (2.1) each $f \in W^{1,\phi}_{loc}$ has the $(N)$-property (furthermore, it is locally absolutely continuous) on a.e. sphere $S$ centered at a prescribed point $x_0 \in \mathbb{R}^n$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq S$ for a.e. such sphere $S$. 
3 Moduli of families of surfaces

The recent development of the moduli method in the connection with modern classes of mappings can be found in the monograph [23] and further references therein.

Let \( \omega \) be an open set in \( \mathbb{R}^k \), \( k = 1, \ldots, n-1 \). A (continuous) mapping \( S : \omega \to \mathbb{R}^n \) is called a \( k \)-dimensional surface \( S \) in \( \mathbb{R}^n \). Sometimes we call the image \( S(\omega) \subseteq \mathbb{R}^n \) the surface \( S \), too. The number of preimages

\[
N(S, y) = \text{card} S^{-1}(y) = \text{card} \{ x \in \omega : S(x) = y \}, \quad y \in \mathbb{R}^n \tag{3.1}
\]
is said to be a \textbf{multiplicity function} of the surface \( S \). In other words, \( N(S, y) \) denotes the multiplicity of covering of the point \( y \) by the surface \( S \). It is known that the multiplicity function is lower semicontinuous, i.e.,

\[
N(S, y) \geq \lim \inf_{m \to \infty} N(S, y_m)
\]
for every sequence \( y_m \in \mathbb{R}^n, m = 1, 2, \ldots \), such that \( y_m \to y \in \mathbb{R}^n \) as \( m \to \infty \), see e.g. [?] , p. 160. Thus, the function \( N(S, y) \) is Borel measurable and hence measurable with respect to every Hausdorff measure \( H^k \); see e.g. [31], p. 52.

Recall that a \( k \)-dimensional Hausdorff area in \( \mathbb{R}^n \) (or simply \textbf{area}) associated with a surface \( S : \omega \to \mathbb{R}^n \) is given by

\[
\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S, y) \, dH^k y \tag{3.2}
\]
for every Borel set \( B \subseteq \mathbb{R}^n \) and, more generally, for an arbitrary set that is measurable with respect to \( H^k \) in \( \mathbb{R}^n \), cf. 3.2.1 in [8]. The surface \( S \) is called \textbf{rectifiable} if \( \mathcal{A}_S(\mathbb{R}^n) < \infty \), see 9.2 in [23].

If \( \varrho : \mathbb{R}^n \to [0, \infty] \) is a Borel function, then its \textbf{integral over} \( S \) is defined by the equality

\[
\int_S \varrho \, d\mathcal{A} := \int_{\mathbb{R}^n} \varrho(y) \, N(S, y) \, dH^k y . \tag{3.3}
\]

Given a family \( \Gamma \) of \( k \)-dimensional surfaces \( S \), a Borel function \( \varrho : \mathbb{R}^n \to [0, \infty] \) is called \textbf{admissible} for \( \Gamma \), abbr. \( \varrho \in \text{adm} \, \Gamma \), if

\[
\int_S \varrho^k \, d\mathcal{A} \geq 1 \tag{3.4}
\]
for every \( S \in \Gamma \). Given \( p \in (0, \infty) \), the \textbf{\( p \)-modulus} of \( \Gamma \) is the quantity

\[
M_p(\Gamma) = \inf_{\varrho \in \text{adm} \, \Gamma} \int_{\mathbb{R}^n} \varrho^p(x) \, dm(x) . \tag{3.5}
\]

We also set

\[
M(\Gamma) = M_n(\Gamma) \tag{3.6}
\]
and call the quantity $M(\Gamma)$ the **modulus of the family** $\Gamma$. The modulus is itself an outer measure in the space of all $k$-dimensional surfaces.

We say that $\Gamma_2$ is **minorized** by $\Gamma_1$ and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to $\Gamma_1$. It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$, see [?], p. 176-178. We also say that a property $P$ holds for $p$-a.e. (almost every) $k$-dimensional surface $S$ in a family $\Gamma$ if a subfamily of all surfaces of $\Gamma$, for which $P$ fails, has the $p$-modulus zero. If $0 < q < p$, then $P$ also holds for $q$-a.e. $S$, see Theorem 3 in [?]. In the case $p = n$, we write simply a.e.

**Remark 3.1.** The definition of the modulus immediately implies that, for every $p \in (0, \infty)$ and $k = 1, \ldots, n - 1$

(1) $p$-a.e. $k$-dimensional surface in $\mathbb{R}^n$ is rectifiable,

(2) given a Borel set $B$ in $\mathbb{R}^n$ of (Lebesgue) measure zero,

$$A_S(B) = 0$$

for $p$-a.e. $k$-dimensional surface $S$ in $\mathbb{R}^n$.

The following lemma was first proved in [17], see also Lemma 9.1 in [23].

**Lemma 3.1.** Let $k = 1, \ldots, n - 1, p \in [k, \infty)$, and let $C$ be an open cube in $\mathbb{R}^n$, $n \geq 2$, whose edges are parallel to coordinate axis. If a property $P$ holds for $p$-a.e. $k$-dimensional surface $S$ in $C$, then $P$ also holds for a.e. $k$-dimensional plane in $C$ that is parallel to a $k$-dimensional coordinate plane $H$.

The latter a.e. is related to the Lebesgue measure in the corresponding $(n - k)$-dimensional coordinate plane $H^\perp$ that is perpendicular to $H$.

The following statement, see Theorem 2.11 in [18] or Theorem 9.1 in [23], is an analog of the Fubini theorem, cf. e.g. [31], p. 77. It extends Theorem 33.1 in [?], cf. also Theorem 3 in [?], Lemma 2.13 in [?] and Lemma 8.1 in [23].

**Theorem 3.1.** Let $k = 1, \ldots, n - 1, p \in [k, \infty)$, and let $E$ be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then $E$ is measurable by Lebesgue in $\mathbb{R}^n$ if and only if $E$ is measurable with respect to area on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$. Moreover, $|E| = 0$ if and only if

$$A_S(E) = 0$$

on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$.

**Remark 3.2.** Say by the Lusin theorem, see e.g. Section 2.3.5 in [8], for every measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$, there is a Borel function $\varrho^* : \mathbb{R}^n \to [0, \infty]$ such that $\varrho^* = \varrho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 3.1, $\varrho$ is measurable on $p$-a.e. $k$-dimensional surface $S$ in $\mathbb{R}^n$ for every $p \in (0, \infty)$ and $k = 1, \ldots, n - 1$. 
We say that a Lebesgue measurable function \( \rho : \mathbb{R}^n \to [0, \infty] \) is \( p \)-extensively admissible for a family \( \Gamma \) of \( k \)-dimensional surfaces \( S \) in \( \mathbb{R}^n \), abbr. \( \rho \in \text{ext}_p \text{adm} \Gamma \), if
\[
\int_S \rho^k \, dA \geq 1
\]for \( p \)-a.e. \( S \in \Gamma \). The \( p \)-extensive modulus \( \overline{M}_p(\Gamma) \) of \( \Gamma \) is the quantity
\[
\overline{M}_p(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^p(x) \, dm(x)
\]
where the infimum is taken over all \( \rho \in \text{ext}_p \text{adm} \Gamma \). In the case \( p = n \), we use the notations \( M(\Gamma) \) and \( \rho \in \text{ext adm} \Gamma \), respectively. For every \( p \in (0, \infty) \), \( k = 1, \ldots, n - 1 \), and every family \( \Gamma \) of \( k \)-dimensional surfaces in \( \mathbb{R}^n \),
\[
\overline{M}_p(\Gamma) = M_p(\Gamma) .
\]

4 Ring \( Q \)-homeomorphisms and their properties

Recall some necessary notions. Let \( E, F \subseteq \mathbb{R}^n \) be arbitrary domains. Denote by \( \Delta(E, F, G) \) the family of all curves \( \gamma : [a, b] \to \mathbb{R}^n \), which join \( E \) and \( F \) in \( G \), i.e. \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in G \) for \( a < t < b \). Set \( d_0 = \text{dist} (x_0, \partial G) \) and let \( Q : G \to [0, \infty] \) be a Lebesgue measurable function. Denote
\[
A(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \} ,
\]
and
\[
S_i = S(x_0, r_i) = \{ x \in \mathbb{R}^n : |x - x_0| = r_i \} , \quad i = 1, 2 .
\]

We say that a homeomorphism \( f : G \to \mathbb{R}^n \) is the ring \( Q \)-homeomorphism with respect to \( p \)-module at the point \( x_0 \in G \), \( 1 < p \leq n \) if the inequality
\[
\mathcal{M}_p \left( \Delta \left( f(S_1), f(S_2), f(G) \right) \right) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|) \, dx
\]
is fulfilled for any ring \( A = A(x_0, r_1, r_2) \), \( 0 < r_1 < r_2 < d_0 \) and for every measurable function \( \eta : (r_1, r_2) \to [0, \infty] \), satisfying
\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1 .
\]

The homeomorphism \( f : G \to \mathbb{R}^n \) is the ring \( Q \)-homeomorphism with respect to \( p \)-module in the domain \( G \), if inequality (4.2) holds for all points \( x_0 \in G \). The properties of the ring \( Q \)-homeomorphisms for \( p = n \) are studied in [?].
The ring $Q$-homeomorphisms are defined in fact locally and contain as a proper subclass of $Q$-homeomorphisms (see [?]). A necessary and sufficient condition for homeomorphisms to be ring $Q$-homeomorphisms with respect to $p$-module at a point given in [?], asserts:

**Proposition 4.1.** Let $G$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and let $Q : G \rightarrow [0, \infty]$ belong to $L^1_{loc}$. A homeomorphism $f : G \rightarrow \mathbb{R}^n$ is a ring $Q$-homeomorphism with respect to $p$-module at $x_0 \in G$ if and only if for any $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial G)$,

$$
\mathcal{M}_p(\Delta(f(S_1), f(S_2), f(G))) \leq \frac{\omega_{n-1}}{I^p_{p-1}},
$$

where $S_1$ and $S_2$ are the spheres defined in (4.1)

$$
I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{n-1/p}q_{x_0}(r)},
$$

and $q_{x_0}(r)$ is the mean value of $Q$ over $|x - x_0| = r$. Note that the infimum in the right-hand side of (4.2) over all admissible $\eta$ satisfying (4.3) is attained only for the function

$$
\eta_0(r) = \frac{1}{I r^{n-1/p}q_{x_0}(r)}.
$$

In this sections we establish the relationship between the ring and lower $Q$-homeomorphisms with respect to $p$-module.

**Theorem 4.1.** Every lower $Q$-homeomorphism with respect to $p$-module $f : G \rightarrow G^*$ at $x_0 \in G$, with $p > n - 1$ and $Q \in L^p_{loc}^{n-1/p}$, is a ring $\tilde{Q}$-homeomorphism with respect to $\alpha$-module at $x_0$ with $\tilde{Q} = Q^{n-1/p_{n+1}}$ and $\alpha = \frac{p}{p-n+1}$.

## 5 Lower $Q$-homeomorphisms and Orlicz-Sobolev classes

Let $D$ and $D'$ be two bounded domains in $\mathbb{R}^n$, $n \geq 2$ and $x_0 \in D$. Given a Lebesgue measurable function $Q : D \rightarrow [0, \infty]$, a homeomorphism $f : D \rightarrow D'$ is called the lower $Q$-homeomorphism with respect to $p$-modulus at $x_0$ if

$$
\mathcal{M}_p(f(\Sigma_{\varepsilon})) \geq \inf_{\rho \in \text{ext}_{p, \text{adm}} \Sigma} \int_{A_{\varepsilon}(x_0)} \frac{\rho^p(x)}{Q(x)} dm(x),
$$

(5.1)

where

$$
A_{\varepsilon}(x_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 < \varepsilon_0 < \sup_{x \in G} |x - x_0|,
$$

and $\Sigma_{\varepsilon}$ denotes the family of all spheres centered at $x_0$ of radii $r$, $\varepsilon < r < \varepsilon_0$, located in $D$. 
Theorem 5.1. Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi : [0, \infty) \to [0, \infty)$ be increasing with the condition (2.1). Then each homeomorphism $f : D \to D'$ of finite distortion in the class $W^{1,\varphi}_{\text{loc}}$ is a lower $Q$-homeomorphism at every point $x_0 \in D$ with $Q(x) = K_{p,f}(x)$.

Proof. Let $B$ be a (Borel) set of all points $x \in D$ where $f$ has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun’s theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [8], we see that $B$ is the union of a countable collection of Borel sets $B_l$, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, 3.2.2, 3.1.4 and 3.1.8 in [8]. With no loss of generality, we may assume that the $B_l$ are mutually disjoint. Denote also by $B^*$ the set of all points $x \in D$ where $f$ has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \cup B^*)$ has Lebesgue measure zero, see Theorem 2.1. Hence by Theorem 2.4 in [18] or by Theorem 9.1 in [23] the area $\mathcal{A}_{S^*(f(B_0))} = 0$ for a.e. hypersurface $S$ in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$. Thus, by Corollary 2.2 $\mathcal{A}_{S^*(f(B_r))} = 0$ for a.e. $S_r$ where $S^*_r = f(S_r)$.

Let $\Gamma$ be the family of all intersections of the spheres $S_r$, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0|$, with the domain $D$. Given $\rho_* \in \text{adm } f(\Gamma)$, $\rho_* \equiv 0$ outside of $f(D)$, set $\rho \equiv 0$ outside of $D$ and on $B_0$,

$$
\rho(x) : = \rho_*(f(x)) \|f'(x)\| \quad \text{for } x \in D \setminus B_0.
$$

Arguing piecewise on $B_l$, $l = 1, 2, \ldots$, we have by 1.7.6 and 3.2.2 in [8] that

$$
\int_{S_r} \sigma^{n-1} dA \geq \int_{S^*_r} \sigma_*^{n-1} dA \geq 1
$$

for a.e. $S_r$ and, thus, $\rho \in \text{ext}_p \text{adm } \Gamma$.

The change of variables on each $B_l$, $l = 1, 2, \ldots$, see, e.g., Theorem 3.2.5 in [8], and countable additivity of integrals give the estimate

$$
\int_D \frac{\rho^p(x)}{K_{p,f}(x)} dm(x) \leq \int_{f(D)} \rho_*^p(x) dm(x)
$$

and the proof is complete.

Corollary 5.1. Each homeomorphism $f$ of finite distortion in $\mathbb{R}^n$, $n \geq 3$, in the class $W^{1,s}_{\text{loc}}$ for $s > n - 1$ is a lower $Q$-homeomorphism at every point $x_0 \in D$ with $Q(x) = K_{p,f}(x)$. 

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