FITTING THE QUARK AND LEPTON MASSES IN STRING THEORIES

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Abstract

The capability of string theories to reproduce at low energy the observed pattern of quark and lepton masses and mixing angles is examined, focusing the attention on orbifold constructions, where the magnitude of Yukawa couplings depends on the values of the deformation parameters which describe the size and shape of the compactified space. A systematic exploration shows that for $Z_3$, $Z_4$, $Z_6$–I and possibly $Z_7$ orbifolds a correct fit of the physical fermion masses is feasible. In this way the experimental masses, which are low–energy quantities, select a particular size and shape of the compactified space, which turns out to be very reasonable (in particular the modulus $T$ defining the former is $T = O(1)$). The rest of the $Z_N$ orbifolds are rather hopeless and should be discarded on the assumption of a minimal $SU(3) \times SU(2) \times U(1)_Y$ scenario. On the other hand, due to stringy selection rules, there is no possibility of fitting the Kobayashi–Maskawa parameters at the renormalizable level, although it is remarked that this job might well be done by non–renormalizable couplings.
1 Introduction

One of the most intriguing facts of particle physics is the peculiar experimental pattern of quark and lepton masses and mixing angles. In the framework of the Standard Model these are just initial parameters put by hand without any possible hint about their origin. Grand Unification theories (GUTs) impose certain relations between them. For instance, in the minimal SU(5) model, $m_e = m_d$, $m_\mu = m_s$, $m_\tau = m_b$ at $M_{GUT}$. Only the third equality is compatible with experiments. This is in fact a major shortcoming of GUTs (also shared by supersymmetric GUTs) which may only be bypassed by complicating the Higgs sector in an artificial way. On the other hand, if Superstring Theories are the fundamental theory from which the Standard Model is derived as an energy limit, they should be able to give an answer to this fundamental question. This is the main motivation of the present work. In this sense a crucial ingredient to relate theory and observation is the knowledge of the theoretical Yukawa couplings predicted by Superstrings. Actually, there are several ways to construct four–dimensional strings, but perhaps the most complete study of the Yukawa couplings has been carried out for orbifold compactifications [1–6], which on the other hand have proved to possess very interesting properties from the phenomenological point of view [7]. So, we will focus in this letter on this kind of scenarios. Furthermore, orbifold Yukawa couplings for twisted matter (see below) present a very rich range, which is extremely attractive as the geometrical origin of the observed variety of fermion masses [2,8,9]. We will assume throughout that the effective four–dimensional field theory has $N = 1$ supersymmetry, $SU(3) \times SU(2) \times U(1)_Y$ observable gauge group and three generations of particles with the correct gauge representations. (We do not consider a GUT theory to avoid the above mentioned problems.) All these properties have been obtained in explicit orbifold constructions [10]. Moreover we will assume a unique generation of Higgses $\{H_1, H_2\}$ (necessary to get a correct Weinberg angle [11]) and that all the observable matter is of the twisted type (as was argued in ref.[9], observable untwisted matter is not phenomenologically viable).

We have made a study, as systematic as possible, for the complete set of Abelian $Z_N$ orbifolds, i.e. $Z_3, Z_4, Z_6$–I, $Z_6$–II, $Z_7, Z_8$–I, $Z_8$–II, $Z_{12}$–I, $Z_{12}$–II. We will not enter here into the details about the construction of these schemes (these can be found in refs.[1]). Let us recall, however, that a $Z_N$ orbifold is constructed by dividing $R^6$ by a six–dimensional lattice $\Lambda$ modded by some $Z_N$ symmetry, called the point group $P$. The space group $S$ is defined as $S = \Lambda \times P$, i.e. $S = \{(\gamma, u); \ \gamma \in P, \ u \in \Lambda\}$. A twisted string satisfies $x(\sigma = 2\pi) = gx(\sigma = 0)$ as the boundary condition, where $g$ is an element (more precisely a conjugation class) of the space group whose point group component is non–trivial. Owing
to the boundary condition, a twisted string is attached to a fixed point (sometimes to a fixed torus) \( f \) of \( g \). Roughly speaking, the form of \( g \) is \( g = (\theta^k, (1 - \theta^k)(f + v)) \), where \( \theta \) is the generator of \( Z_N (\theta^N = 1) \) and \( v \in \Lambda \). It is said that the string belongs to the \( \theta^k \) sector. For a Yukawa coupling to be allowed the product of the three relevant space group elements, say \( g_1 g_2 g_3 \), must contain the identity. This implies two important equalities:

\[
k_1 + k_2 + k_3 = 0 \pmod{N} \tag{1}
\]

\[
(1 - \theta^{k_1})(f_1 + v_1) + \theta^{k_1}(1 - \theta^{k_2})(f_2 + v_2) - (1 - \theta^{k_1+k_2})(f_3 + v_3) = 0, \quad v_i \in \Lambda \tag{2}
\]

The first one is the so-called point group selection rule, which implies that the coupling must be of the \( \theta^{k_1} \theta^{k_2} \theta^{-(k_1+k_2)} \) type. The second one is the so-called space group selection rule, which can have different characteristics depending on the orbifold under consideration. Some additional complications appear when a fixed point \( f \) under \( \theta^k \) is not fixed under \( \theta \) \([12,6]\). The space group selection rules for all the \( Z_N \) orbifolds have been classified in refs.\([5,6]\). Likewise, the expressions for the different Yukawa couplings have been calculated in refs.\([2–6]\). Their characteristics are summarized in Table 1. They contain suppression factors that depend on the relative positions of the fixed points to which the fields involved in the coupling are attached (i.e. \( f_1, f_2, f_3 \)), and on the size and shape of the orbifold\([\Box]\). As mentioned above, this has been suggested as the possible origin of the observed hierarchy of fermion masses. We have explored that possibility in this letter, finding that for certain schemes it can be successfully realized. In section 2 we discuss the possibility of getting the correct Kobayashi–Maskawa parameters at the renormalizable level. This turns out to be out of reach even for non–prime orbifolds, where the mass matrices are allowed to be non–diagonal. It is remarked, however, that this job might well be done by non–renormalizable couplings, at the same time as they account for the masses of the first generation (which should come from off–diagonal entries in the mass matrices). However, renormalizable couplings should still be responsible for the masses of the second and third generations. Whether this is possible or not is studied in section 3. As a first step, a renormalization group analysis is performed, which presents (slight) differences from the ordinary GUT one. Then it is shown that for a reasonable size and shape of the compactified space, the \( Z_3, Z_4, Z_6–I \) and possibly \( Z_7 \) orbifolds can fit the physical quark and lepton masses adequately. The rest of the \( Z_N \) orbifolds, however, should be discarded under the previous minimal assumptions. We present our conclusions in section 4.

\footnote{The size and shape of the orbifold are given by the vacuum expectation values (VEVs) of certain fields (moduli) and, consequently, they are dynamical parameters. It has been shown \([13,14]\) that supersymmetry breaking effects could determine their actual values.}
2 Mixing angles and geometrical selection rules

In order to reproduce the mixing angles and the CP violating phase of the experimental Kobayashi–Maskawa (KM) matrix the quark mass matrices must have off–diagonal entries. Consequently, the first question is whether it is possible or not to get non–diagonal mass matrices. The answer to this question is intimately related to the space–group selection rule (see eq.(2)). For prime orbifolds (\(Z_3, Z_7\)), this is of the so–called diagonal type. This means that given two fields associated with two fixed points \(f_1, f_2\), they can only couple to a unique third fixed point \(f_3\). Of course the coupling, to be allowed, must satisfy other requirements, in particular gauge invariance. On the other hand, the couplings must satisfy the point group selection rule, eq.(1), which is also diagonal and, in addition, in a \(\theta^k\) sector the matter associated with a given fixed point is not degenerate, i.e. all fields have different gauge quantum numbers. Consequently, for \(Z_3\) and \(Z_7\) orbifolds the mass matrices are diagonal. For instance, \(Q_u H_2\) (where \(Q_u\) denotes the \((u, d)_L\) doublet) can only couple to a unique field with the gauge quantum numbers of \(u^c\), although this does not mean that such a coupling must be present. If this were the whole story we should conclude that mixing angles cannot be obtained within the \(Z_3\) and \(Z_7\) frameworks. Fortunately, things are quite different when the gauge group is spontaneously broken after compactification and, in fact, this is what happens in all the phenomenologically interesting models so far constructed [10\(^2\)]. Then there appear new effective trilinear couplings coming from higher order operators in which some of the fields get non–vanishing VEVs[3]. These couplings have a strong exponential damping [17], but they are no longer subjected to the trilinear selection rule (examples of this can be found in ref.[10]). This leads to a natural ansatz for quark and lepton mass matrices:

\[
M = \begin{pmatrix}
\epsilon & a & b \\
\tilde{a} & A & c \\
\tilde{b} & \tilde{c} & B
\end{pmatrix}
\]  

(3)

where \(\epsilon, a, \tilde{a}, b, \tilde{b}, c, \tilde{c} << A << B\) in magnitude, lower–case letters denoting entries generated by higher order operators. Here we have assumed that the \((1,1)\) entry is zero at the renormalizable level. As is known, this is extremely convenient to obtain the Cabibbo angle in a natural manner (more precisely, \(\sin \theta_c \sim \sqrt{m_d/m_s}\)). Notice that in this way the masses of the first generation should also be caused by higher order operators. It is

\[^2\]Several sources for this breakdown have been explored, namely Fayet–Iliopoulos breaking [10], flat directions, and gaugino condensation induced breaking [14,15].

\[^3\]Another (model–dependent) mechanism for mixings, after the breaking, is explained in refs.[9,16]
not difficult to construct explicit models with this property (see e.g. ref.[18]). On the other hand, the entries $A, B$ should essentially be generated by renormalizable couplings since non–renormalizable ones are too small to fit the second and third generation masses properly. Of course, one has to require $A, B$ to be the correct ones in order to reproduce those masses. Whether this is possible or not will be studied in the next section. The ansatz (3) was obtained in ref.[9] in the context of the $Z_3$ orbifold. It was shown there that it gives correct KM parameters and first generation masses for reasonable values of the off–diagonal entries (in particular it is highly desirable that $\epsilon = 0$). Of course, the precise values of these have to be calculated in each particular case, but at least this shows that there is no incompatibility ab initio between prime orbifolds and the observed KM parameters. In some sense eq.(3) (with $\epsilon = 0$) is a ”stringy” alternative to the Fritzsch ansatz [19]

\[
M = \begin{pmatrix}
0 & A & 0 \\
A & 0 & B \\
0 & B & C
\end{pmatrix}
\] (4) (with $|A| << |B| << |C|$), which is the most extensively discussed form for $u$ and $d$–type quark mass matrices.

Things go in a different way for even orbifolds. The reason is twofold. First, it is clear from Table 1 that, for an even orbifold, Yukawa couplings are not necessarily of a unique $\theta^{k_1}\theta^{k_2}\theta^{k_3}$ type. Second, the space group selection rule for a given $\theta^{k_1}\theta^{k_2}\theta^{k_3}$ coupling is not, in general, of the diagonal type [6], i.e. for two given fixed points $(f_1, f_2), f_3$ is not uniquely selected. These two features in principle open the possibility of having non–diagonal mass matrices at the renormalizable level, and this is indeed what happens. However, we will argue now that the structure of these matrices is still strongly constrained by the selection rules, so that, as for prime orbifolds, no realistic prediction for the KM parameters can emerge at the renormalizable level.

Let us first show that the point group selection rule implies that any viable form for the quark mass matrices should be built up with Yukawa couplings of a unique $\theta^{k_1}\theta^{k_2}\theta^{k_3}$ type. Consider for example the $d$–quark mass matrix and suppose that $H_1, d^c, s^c, b^c$ correspond to the $\theta^l, \theta^{m_1}, \theta^{m_2}, \theta^{m_3}$ sectors respectively. Notice now that if one row of the mass matrix contains more than one entry different from zero, say $M_{ij_1}, M_{ij_2} \neq 0$, then the point group selection rule (1) requires $l + m_{j_1} = l + m_{j_2} \rightarrow m_{j_1} = m_{j_2}$, otherwise the two $SU(2)$ singlet quarks involved here could not be coupled to the same quark doublet. Now, it is easy to apply this rule to check that a mass matrix of the Fritzsch type, eq.(4), or of the type of eq.(3) cannot be obtained unless all the Yukawa couplings involved are of the
same $\theta^k \theta^k \theta^k$ class. In fact, it is hardly conceivable a phenomenologically viable mass matrix which does not contain rows with more than one non–vanishing entry involving the three generations, so this rule is general.

Now, we will show that the space group selection rule induces an important property in the mass matrix which we call "box–closing" for short. This property means that if we have a $2 \times 2$ box in the mass matrix with three entries different from zero, then the fourth entry must also be different from zero, e.g.

$$\begin{bmatrix} \times \\ \times \end{bmatrix} \rightarrow \begin{bmatrix} \times \\ \times \end{bmatrix}$$

To see this, suppose that the three initial entries correspond to the couplings

$$Q_a H_1 q_1^c, \quad Q_a H_1 q_2^c, \quad Q_b H_1 q_2^c$$

Calling $\theta^p$, $\theta^l$ the sectors to which $Q_{a,b}$, $H_1$ belong, the space group selection rule (2) implies

$$\begin{align*}
(1 - \theta^p)(Q_a + v_1) + \theta^p(1 - \theta^l)(H_1 + v_2) - (1 - \theta^{l+p})(q_1^c + v_3) &= 0 \\
(1 - \theta^p)(Q_a + \tilde{v}_1) + \theta^p(1 - \theta^l)(H_1 + \tilde{v}_2) - (1 - \theta^{l+p})(q_2^c + v_4) &= 0 \\
(1 - \theta^p)(Q_b + v_5) + \theta^p(1 - \theta^l)(H_1 + \tilde{\tilde{v}}_2) - (1 - \theta^{l+p})(q_2^c + \tilde{v}_4) &= 0
\end{align*}$$

where $v_i, \tilde{v}_i, \tilde{\tilde{v}}_i \in \Lambda$ and we have denoted, for simplicity, a field and its corresponding fixed point by the same symbol. Now, (7)+(9)−(8) reads

$$\begin{align*}
(1 - \theta^p)(Q_b + v_1 + v_5 - \tilde{v}_1) + \theta^p(1 - \theta^l)(H_1 + v_2 + \tilde{v}_2 - \tilde{\tilde{v}}_2) - (1 - \theta^{l+p})(q_1^c + v_3 + \tilde{v}_4 - v_4) &= 0
\end{align*}$$

which implies that the coupling $Q_b H_1 q_1^c$ is also allowed. This excludes the possibility of obtaining the Fritzsch matrix (4) at the renormalizable level (starting with (4) and applying the box–closing property four times we fill all the entries). Also the matrix of eq.(3) with $\epsilon = 0$ is not allowed. Again, it is hard to imagine any viable mass matrix satisfying the box–closing property.

This seems to exclude any possibility of having a reasonable mass matrix at the renormalizable level. One could still try to get something similar to the Fritzsch matrix, for example, but with very suppressed couplings instead of zeros. However, this is hopeless since the selection rule not only imposes the ”closing” of any $2 \times 2$ box with three non–vanishing entries, see (5), but it usually relates the value of the fourth entry to those of

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4 The above analysis is more involved when some of the fixed points are not invariant under $\theta$. However, after an exhaustive study, it turns out that the box–closing property holds in all cases.
the three initial ones. When this is not so, the corresponding suppression factors for the would–be zero entries should be very strong, thus requiring high values for the moduli (see next section) and making virtually impossible to fit the fermion masses correctly. In view of these results we have to give up fitting the KM parameters at the renormalizable level. As was mentioned above, this job can be realized by the (model–dependent) non–renormalizable operators for the mass matrix of eq.(2), at the same time as they account for the first generation masses. However, the renormalizable couplings should be able to fit the fermion masses of the second and third generations, which is still extremely restrictive. This is what we study in the next section.

3 Fermion masses

3.1 Renormalization group analysis

It is customary to give the experimental values of fermion masses [22] (except $m_t$) at 1 GeV, see first row of Table 2. On the other hand, the Yukawa couplings in orbifolds are calculated at the string scale $M_{Str} = 0.527 \times g \times 10^{18}$ GeV [23], where $g \simeq 1/\sqrt{2}$ is the corresponding value of the gauge coupling constant. Thus, in order to compare theory and experiment a renormalization group (RG) running between these two scales is necessary. This RG analysis differs from the ordinary GUT one since, in GUTs, the running of Yukawa couplings is performed between $M_{GUT}$ and 1 GeV, where $M_{GUT}$ is the scale at which gauge interactions are unified. On the other hand, in string theories, there are ”stringy” (no GUT) threshold corrections on the value of the gauge coupling constants, shifting the actual scale at which they are unified. More precisely, the value of these threshold corrections depends on the VEVs of some moduli that, in general, are not the ones involved in the Yukawa couplings (see next subsection) [6]. It has been shown in ref.[24] that, for appropriate VEVs of these moduli the gauge couplings still unify at an effective unification scale $M_X \simeq 10^{16}$ GeV, as is phenomenologically required [25]. However, the running of the Yukawa couplings has still to be made from $M_{Str}$. This fact, for example, modifies (slightly) the traditional relation $m_b/m_\tau$ at low energy when one sets $m_b = m_\tau$ at tree level, as will be seen shortly.

Let us write, for the sake of definiteness, the Yukawa Lagrangian for the second and

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5 This mechanism has also been considered in ref.[20] in the context of a flipped string model [21].
6 Explicit expressions for the threshold corrections can be found in ref.[23].
third generations, at a scale $\mu$

\[
\mathcal{L}_{\text{Yuk}} = h_c(\mu)Q_cH_2c^c + h_s(\mu)Q_cH_1s^c + h_\mu(\mu)L_\muH_1\mu^c \\
+ h_t(\mu)Q_tH_2t^c + h_b(\mu)Q_tH_1b^c + h_\tau(\mu)L_\tauH_1\tau^c
\]

(11) where the capital letters denote $SU(2)$ doublets and the $h$’s are the Yukawa couplings. The physical masses at $1 \text{ GeV}$ are then given by

\[
m_\alpha = h_\alpha(1 \text{ GeV})\nu_1, \ m_\beta = h_\beta(1 \text{ GeV})\nu_2
\]

(12) where $\alpha = s, b, \mu, \tau$ ; $\beta = c, t$ and $\nu_{1,2} = \langle H_{1,2} \rangle$ are subjected to the bound

\[
\nu_1^2 + \nu_2^2 = 2 \left( \frac{M_W}{g_2} \right)^2 = (175 \text{ GeV})^2
\]

(13) Moreover, electroweak symmetry breaking in the context of the minimal supersymmetric standard model suggests $\nu_2 > \nu_1$ [26]. In order to relate $h_{\alpha,\beta}(1 \text{ GeV})$ to $h_{\alpha,\beta}(M_{Str})$ we have to make use of the RG equations for the Yukawa couplings between these two scales (see e.g. [27,28]). This has to be done in several steps since the matter content is not the same at any intermediate scale. In particular, we assume as usual a unique supersymmetric mass $M_S$ for all the supersymmetric partners of the standard matter. (Allowing for a differentiation of the various supersymmetric masses does not modify the results substantially.) Besides this, there are, of course, the ordinary quark thresholds. Following a standard RG analysis and working in the usual limit $h_b, h_\tau << h_t$, we find, for the first two generations of quarks and the three generations of leptons, the following expressions

\[
\begin{align*}
\hspace{1cm}
 h_{u,c}(1 \text{ GeV}) &= F_u h_{u,c}(M_{Str}) = \left( \frac{\alpha_3(1 \text{ GeV})}{\alpha_3(m_c)} \right)^{\frac{4}{7}} \left( \frac{\alpha_3(m_s)}{\alpha_3(m_b)} \right)^{\frac{3}{12}} \left( \frac{\alpha_3(m_t)}{\alpha_3(M_Z)} \right)^{\frac{1}{16}} \\
&\times \left( \frac{\alpha_3(M_Z)}{\alpha_3(M_S)} \right)^{\frac{17}{168}} \left( \frac{\alpha_3(M_S)}{\alpha_3(M_{Str})} \right)^{\frac{12}{168}} h_{u,c}(M_{Str}) \\
&= \left( \frac{\alpha_1(M_Z)}{\alpha_1(M_S)} \right)^{\frac{12}{168}} \left( \frac{\alpha_1(M_S)}{\alpha_1(M_{Str})} \right)^{\frac{6}{168}} h_{u,c}(M_{Str})
\end{align*}
\]

(14)

\[
\begin{align*}
\hspace{1cm}
 h_{d,s}(1 \text{ GeV}) &= F_u \left( \frac{\alpha_1(M_Z)}{\alpha_1(M_S)} \right)^{\frac{12}{168}} \left( \frac{\alpha_1(M_S)}{\alpha_1(M_{Str})} \right)^{\frac{6}{168}} h_{d,s}(M_{Str})
\end{align*}
\]

(15)

\[
\begin{align*}
\hspace{1cm}
 h_{e,\mu,\tau}(1 \text{ GeV}) &= \left( \frac{\alpha_2(M_Z)}{\alpha_2(M_S)} \right)^{\frac{2}{7}} \left( \frac{\alpha_2(M_S)}{\alpha_2(M_{Str})} \right)^{\frac{2}{12}} \\
&\times \left( \frac{\alpha_1(M_Z)}{\alpha_1(M_S)} \right)^{\frac{4}{168}} \left( \frac{\alpha_1(M_S)}{\alpha_1(M_{Str})} \right)^{\frac{2}{12}} h_{e,\mu,\tau}(M_{Str})
\end{align*}
\]

(16)
where \( \alpha_1, \alpha_2, \alpha_3 \) are the gauge couplings of \( U(1)_Y, SU(2) \) and \( SU(3) \) respectively. For \( h_t, h_b \) the RG equations are more complicated since the effect of the top Yukawa interactions are not negligible here (see e.g. [27]). After some algebra one arrives at

\[
h_t(M_Z) = \left( \frac{E_1(M_Z)}{1 + \frac{9}{16\pi^2} h_t^2(M_S) F_1(M_Z)} \right)^{\frac{1}{2}} \times \left( \frac{E_1(M_S)}{1 + \frac{6}{16\pi^2} h_t^2(M_S) F_1(M_S)} \right) \frac{1}{2} h_t(M_{Str})
\]

(17)

\[
h_b(1 \text{ GeV}) = \left( \frac{\alpha_3(1 \text{ GeV})}{\alpha_3(m_c)} \right)^{\frac{4}{9}} \left( \frac{\alpha_3(m_c)}{\alpha_3(m_b)} \right)^{\frac{12}{25}} \left( \frac{\alpha_3(m_b)}{\alpha_3(M_Z)} \right)^{\frac{12}{25}} \times \left( \frac{E_2(M_S)}{1 + \frac{6}{16\pi^2} h_t^2(M_S) F_1(M_S)} \right)^{\frac{1}{6}} \frac{1}{2} h_b(M_{Str})
\]

(18)

where

\[
E_1(Q) = \left( 1 - 3 \frac{\alpha_3(M_{Str})}{4\pi} t \right)^{\frac{16}{19}} \left( 1 + \frac{\alpha_2(M_{Str})}{4\pi} t \right)^{\frac{3}{2}} \left( 1 + \frac{33 \alpha_1(M_{Str})}{4\pi} t \right)^{\frac{13}{27}}
\]

\[
E_1(Q') = \left( 1 - 7 \frac{\alpha_3(M_S)}{4\pi} t' \right)^{\frac{16}{19}} \left( 1 - 3 \frac{\alpha_2(M_S)}{4\pi} t' \right)^{\frac{3}{2}} \left( 1 + \frac{42 \alpha_1(M_S)}{10\pi} t' \right)^{\frac{17}{27}}
\]

\[
F_1(M_S) = \int_{Q = M_{Str}}^{Q = M_S} E_1(Q) dt' , \quad F_1'(Q) = \int_{Q' = M_S}^{Q' = M_Z} E_1'(Q') dt'
\]

\[
E_2(Q) = E_1(Q) \left( 1 + \frac{33 \alpha_1(M_{Str})}{4\pi} t \right)^{\frac{12}{19}}
\]

\[
E_2(Q') = E_1'(Q') \left( 1 + \frac{42 \alpha_1(M_S)}{10\pi} t' \right)^{\frac{24}{19}}
\]

with

\[
t = 2 \log \frac{M_{Str}}{Q}, \quad t' = 2 \log \frac{M_S}{Q'}
\]

(20)

The experimental values of \( \alpha_i(M_Z) \) are (see e.g. ref.[25]):

\[
\alpha_1(M_Z) = 0.016930(80), \quad \alpha_2(M_Z) = 0.03395(52), \quad \alpha_3(M_Z) = 0.125(5)
\]

(21)

from which \( \alpha_i \) can be obtained at any scale following a standard RG analysis. It has been shown in ref.[25] that a correct perturbative unification demands\footnote{It can be easily checked that with \( M_S = 10^3 \text{ GeV} \) the gauge coupling constants \( [21] \) are unified at \( M_X \simeq 2.3 \times 10^{16} \text{ GeV} \) with \( \alpha(M_X) \simeq 0.0393 \).} \( M_S \sim 10^3 \text{ GeV} \), which
is the value we insert in eqs.(14–20) (variations of $M_S$ within the errors are negligible for our purposes). It is interesting to calculate the values of $h_c, h_s, h_t, h_\mu, h_\tau$ at $M_{Str}$ which would give the measured values of the corresponding fermion masses. We have represented them in Fig.1 as functions of $\nu_2$. Of course, these values for the $h$’s should emerge from the theory. Whether this happens or not is studied in the next subsection.

Let us finally note that the relation $m_b(1 \, GeV)/m_\tau(1 \, GeV)$ is obtained from (18) and (16). If one imposes, following the usual GUT ansatz, $h_b(M_{Str}) = h_\tau(M_{Str})$, an expression similar to the GUT one is obtained, but only after substituting $M_{GUT}$ by $M_{Str}$ and taking into account that $\alpha_3(M_{Str}) \neq \alpha_2(M_{Str}) \neq \alpha_1(M_{Str})$. Of course, the numerical results are not substantially affected.

### 3.2 The fits

The theoretical Yukawa couplings $h(M_{Str})$ to be inserted in eqs.(14–18) have been calculated in refs.[2–6]. In order to get a feeling of their main characteristics, let us take the $Z_4$ orbifold based on an $[SO(4)]^3$ root lattice as a useful example. The action of $\theta$ on the lattice basis $(e_1, ..., e_6)$ is simply $\theta e_l = e_{l+1}$, $\theta e_{l+1} = -e_l$ with $l = 1, 3$ and $\theta e_5 = -e_5$, $\theta e_6 = -e_6$. Let us call $R_i \equiv |e_i|$ and $\alpha_{ij} = \cos \theta_{ij}$ with $e_i e_j = R_i R_j \cos \theta_{ij}$. In the orbifold without deformations $\alpha_{ij} = 0$ ($i \neq j$). However the orbifold can consistently be deformed by a modification of the values of the so–called deformation parameters [9]. For the $Z_4$ orbifold these are $R_1$, $R_3$, $R_5$, $R_6$, $\alpha_{13}$, $\alpha_{14}$, $\alpha_{56}$ with $(\alpha_{13} + \alpha_{14})^2 \leq 1$. The sizes of the Yukawa couplings depend on the values of some of them called effective deformation parameters [9]. For the $Z_4$ these are

$$R_1, R_3, \alpha_{13}, \alpha_{14} \quad (22)$$

On the other hand, for this orbifold all the twisted couplings are of the $\theta \theta \theta^2$ type and the selection rule reads

$$f_1 + f_2 - (1 + \theta) f_3 \in \Lambda, \quad (23)$$

where $f_3$ is the $\theta^2$ fixed point. The classification of the fixed points in terms of the lattice basis can be found in ref.[6]. The value of an allowed Yukawa coupling at $M_{Str}$ turns out to be

$$h_{\theta \theta \theta^2} = gN \sum_{v \in (f_2 - f_3 + \Lambda)_\perp} \exp\left[-\frac{1}{4\pi} \vec{v}^T M \vec{v}\right] = gN \begin{bmatrix} \vec{f}_{23}^T \\ 0 \end{bmatrix} [0, \Omega], \quad (24)$$

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8 The values of the deformation parameters correspond to the VEVs of certain singlet fields with perturbative flat potential, called moduli and usually denoted by $T_i$. 
where the subscript $\perp$ denotes projection on the $(e_1, \ldots, e_4)$ $D = 4$ space, $f_{23} = f_2 - f_3$, the arrow means the corresponding 4–plet of components and $\vartheta$ is the Jacobi theta function. Moreover

$$N = \sqrt{V_\perp \frac{1}{2\pi} \frac{\Gamma^2(\frac{1}{4})}{\Gamma^2(\frac{3}{4})}}$$

where $V_\perp = R_1^2 R_3^2 (1 - \alpha_{13}^2 - \alpha_{14}^2)$ is the volume of the unit cell of the $(e_1, \ldots, e_4)$ lattice. If $f_3$ is not fixed by $\theta$ the result for $h_{\theta \theta \theta}$ is exactly the same but multiplied by $\sqrt{2}$. Notice that $h_{\theta \theta \theta}$ depends on the relative positions in the lattice of the relevant fixed points to which the physical fields are attached. This information is condensed in $\vec{f}_{23}$. In addition, $h_{\theta \theta \theta}$ depends on the size and shape of the compactified space, which is reflected in the effective deformation parameters $(R_1, R_3, \alpha_{13}, \alpha_{14})$ appearing in $\Omega$ and in $V_\perp$. Note that both pieces of information appear in a completely distinguishable way from each other in eq.(24). It is also interesting to say that the number of allowed couplings is 160. The number of different Yukawa couplings is 10. These characteristics are summarized in Table 1 for all the orbifolds.

In order to calculate the value of a specific Yukawa coupling, say $h_s$ (see eq.(11)) we need to know what the $\theta^k$ sectors are and fixed points to which $Q_c$, $H_1$ and $s_c$ are associated. Actually, it is an empirical fact that, because of the huge proliferation of scenarios within a given compactification scheme ($Z_4$ in this case), the observable fields can correspond to any choice of $\{\theta^k, f\}$ sectors, see e.g. ref.[18]. Consequently, we will take the freedom to assign the physical fields to $\theta^k$ sectors and fixed points at convenience. Of course, a particular assignment will only be realized in certain scenarios. It is interesting to notice, however, that not for all the assignments are the physical Yukawa couplings (see eq.(11)) allowed from the point group and space group selection rules. In order to illustrate this, suppose that $H_1$ and $H_2$ belong to the $\theta$ and $\theta^2$ sectors respectively. Then, writting the space group selection rule (23) for the quark couplings of $L_{Yuk}$ (see eq.(11)), one finds after some algebra

$$(H_2 - t^c) - (1 + \theta)(b^c - H_1) = (H_2 - c^c) - (1 + \theta)(s^c - H_1) + \Lambda$$

where we have denoted the fields and their corresponding fixed points by the same symbols. Eq.(26) sets severe restrictions on the possible assignments and, hence, on the possible
correspondences of the physical Yukawa couplings to the above mentioned 10 different Yukawa couplings. Similar expressions appear if we initially assign \( H_1, H_2 \) to other \( \theta^k \) sectors.

The final step is to let the effective deformation parameters vary in order to see whether for some choices of them the theoretical masses, calculated using eqs. (12, 14–18, 24), coincide with the experimental ones (see also Fig.1). In this fit \( \nu_1 \) has also to be considered as a free parameter (within the limits mentioned in the previous subsection) while \( \nu_2 \) is given by \( \nu_3 \). Of course, a different fit has to be made for each possible assignment.

If a satisfactory fit is found, this means that the corresponding orbifold scheme (in this case \( Z_4 \)) is compatible with the observed spectrum of fermion masses, which is highly non–trivial as will be seen shortly. On the other hand, if no such fit is found the orbifold scheme should be discarded. Obviously, orbifolds with a higher number of deformation parameters and different Yukawa couplings (see Table 1) are in a better position to fit the experimental masses, but this is not a guarantee. For the particular case of the \( Z_4 \) orbifold, we have found that, for most of the possible assignments, the fits are not satisfactory. However, provided

\[
\vec{f}_{23}(c) = (00\frac{1}{2}2) , \quad \vec{f}_{23}(s) = (0\frac{1}{2}0\frac{1}{2}) , \quad \vec{f}_{23}(t) = (\frac{1}{2}000) \\
\vec{f}_{23}(b) = (\frac{1}{2}00) , \quad \vec{f}_{23}(\mu) = (\frac{1}{2}0\frac{1}{2}) , \quad \vec{f}_{23}(\tau) = (\frac{1}{2}\frac{1}{2}0) \tag{27}
\]

where \( \vec{f}_{23}(\phi) \) is the corresponding \( f_{23} \) (see eq.(24)) for the \( h_\phi \) coupling, remarkably good fits can be found. Eq.(27) is satisfied (up to spurious lattice vectors) by the following assignment of physical fields to \( \{ \theta^k, f \} \) sectors

\begin{align*}
Q_c : (\frac{1}{2}00) , \quad Q_t : (0000) , \quad c^c : (\frac{1}{2}2\frac{1}{2}2) , \quad s^c : (\frac{1}{2}2\frac{1}{2}2) \\
b^c : (\frac{1}{2}00) , \quad t^c : (0\frac{1}{2}0) , \quad L_\mu : (00\frac{1}{2}\frac{1}{2}) , \quad L_\tau : (\frac{1}{2}\frac{1}{2}22) \\
\mu^c : (\frac{1}{2}0\frac{1}{2}) , \quad \tau^c : (0\frac{1}{2}\frac{1}{2}) , \quad H_1 : (0000) , \quad H_2 : (\frac{1}{2}20) \tag{28}
\end{align*}

where the \( \phi^c \) fields are understood to belong to the \( \theta^2 \) sector and the rest to the \( \theta \) one. The values for the deformation parameters (in string units) and \( \nu_1 \) for an illustrative fit are

\[
R_1 = 13.280 , \quad R_3 = 15.077 , \quad \alpha_{13} = -0.2395 , \quad \alpha_{14} = 0 , \quad \nu_1 = 71.8 \text{ GeV} \tag{29}
\]

\[9\text{There are other possible assignments consistent with (27).}\]
and the corresponding fermion masses are shown in the third row of Table 2. (Oscillations around these values with the subsequent variations of the fermions masses are of course possible.) This result is rather remarkable, specially when one notices that the number of free parameters is lower than the number of physical masses fitted. In some sense the experimental masses, which are low–energy quantities, are selecting a particular assignment of the physical fields to fixed points and the values of the deformation parameters that define the size and shape of the compactified space (e.g. the ”preferred” $\theta_{14}$ angle is the cartesian one). We find this quite encouraging. Notice also that the numbers of eq.(29) are quite sensible for a compactified space. The hierarchy of masses which emerges from them has to do with the exponential dependence of the Yukawa couplings (see eq.(24)).

On the technical side, let us comment that the fit has been performed with the help of a MINUIT program, choosing for the minimization function the total $\chi^2$. The major obstacle we have found was to control the convergence of the Jacobi $\vartheta$ function of eq.(24), particularly when the $\alpha_{13}$, $\alpha_{14}$ parameters are close to the boundary of their definition range. This requires to sum up to 10000 terms of the series: it is by no means significant to keep only a few terms. Finally, we have increased the usual experimental errors of $m_\mu$ and $m_\tau$ up to 1% to incorporate, to some extent, the errors attributable to the calculation. The ”experimental” error of $m_b$ was conservatively set at $\Delta m_b = 10\%$.

Let us now comment the results for the other $Z_N$ orbifolds. It turns out that, besides $Z_4$, the only ones that can work are the $Z_3$ and the $Z_6$–I. Our best fits for them are shown in the second and fourth rows of Table 2. For the $Z_6$–I one, all the couplings considered were of the $\theta^2\theta^2\theta^2$ type. Consequently, all the physical fields are understood to belong to the $\theta^2$ sector in this case. On the other hand, in the $Z_3$ orbifold there is a unique $\theta$ sector. The corresponding assignments, given in the respective lattice basis, are

$$ f_{23}(c) \begin{cases} Z_3 : (00\frac{1}{3}\frac{1}{3}\frac{1}{3}) \\ Z_6 : (0\frac{1}{3}0\frac{1}{3}) \end{cases} , \quad f_{23}(s) \begin{cases} Z_3 : (\frac{1}{3}\frac{2}{3}\frac{2}{3}\frac{1}{3}) \\ Z_6 : (0\frac{1}{3}00\frac{1}{3}) \end{cases} , \quad f_{23}(t) \begin{cases} Z_3 : (0000\frac{1}{3}) \\ Z_6 : (0000\frac{2}{3}) \end{cases} $$

$$ f_{23}(b) \begin{cases} Z_3 : (\frac{1}{3}\frac{2}{3}\frac{2}{3}00) \\ Z_6 : (0\frac{1}{3}0000) \end{cases} , \quad f_{23}(\mu) \begin{cases} Z_3 : (\frac{1}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}) \\ Z_6 : (000\frac{1}{3}\frac{2}{3}) \end{cases} , \quad f_{23}(\tau) \begin{cases} Z_3 : (\frac{1}{3}\frac{2}{3}\frac{2}{3}00) \\ Z_6 : (0000\frac{2}{3}) \end{cases} $$

where, again, $f_{23}(\phi)$ is the difference between two of the fixed points involved in the $h_\phi$ coupling. The corresponding values for the nine deformation parameters that the $Z_3$ orbifold possesses (see Table 1 and ref.[6]) and for $\nu_1$ are

$$ R_1 = 13.039 \ , \ R_3 = 30.460 \ , \ R_5 = 13.076 $$

$$ \alpha_{13} = -0.6055 \ , \ \alpha_{14} = -0.2395 \ , \ \nu_1 = 39.3 \text{ GeV} $$

$$ \alpha_{15} = \alpha_{16} = \alpha_{35} = \alpha_{36} = 0 \text{ (fixed)} \quad (31) $$
Notice that four of them have not been used in the fit. This has been done to improve the convergence of the MINUIT program. Clearly, a better fit could be obtained once these four parameters are also considered. Similarly, the values for the five deformation parameters of the $Z_6$–I orbifold and $\nu_1$ are

$$R_1 = 18.349, \quad R_3 = 17.588, \quad R_5 = 13.073$$

$$\alpha_{13} = -0.3438, \quad \alpha_{14} = 0.2978, \quad \nu_1 = 70 \text{ GeV}$$

(32)

For the rest of the orbifolds, after an exhaustive exploration, we have not found sensible fits. This should not be surprising since they have a smaller number of effective deformation parameters, see Table 1. This consideration makes the $Z_4$ case the most remarkable one. Just for completeness we have given in Table 2 our best fits for these orbifolds. It is worth noticing that the $Z_7$ orbifold (which has only four different couplings and three deformation parameters) works very acceptably for all fermions masses, except for the strange one, which on the other hand has a large experimental uncertainty.

Let us finally give the values of the moduli $T_i$ corresponding to eqs.(29,31,32). As usual, we define the normalization of $T_i$ in such a way that, under a duality transformation, they transform as $T_i \rightarrow 1/T_i$ [29]. This implies $\text{Re} \ T_i = \alpha R_i^2$ with $\alpha = \sqrt{3}/16\pi$ for $Z_3$ and $Z_6$, and $\alpha = \sqrt{2}/8\pi$ for $Z_4$. Hence, $T_1 = 1.86, \quad T_3 = 10.17, \quad T_5 = 1.87$ (for $Z_3$); $T_1 = 3.16, \quad T_3 = 4.07$ (for $Z_4$) and $T_1 = 3.69, \quad T_3 = 3.39, \quad T_5 = 1.87$ (for $Z_6$).

### 4 Summary and Conclusions

We have explored the capability of string theories to reproduce the observed pattern of quark and lepton masses and mixing angles. We have focused our attention on orbifold constructions since, apart from their phenomenological merits, there is at present a good knowledge of the theoretical Yukawa couplings in these scenarios. A first conclusion is that, due to stringy selection rules, there is no possibility of fitting the Kobayashi–Maskawa parameters at the renormalizable level. This is so even for non–prime orbifolds, where the mass matrices are allowed to have a non–diagonal structure. It is, however, argued that (model–dependent) non–renormalizable couplings might well do this job at the same time as they account for the masses of the first generation (which should come from off–diagonal entries in the mass matrices). On the other hand, non–renormalizable

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Notice that these values of $\alpha$ are consistent with the definition of our lattice. This is the usual one, coinciding with that of the first paper of ref.[2], while in the second paper of the same reference the lattice is redefined as $e_i \rightarrow 2\pi e_i$. Of course the values of $T_i$ are independent on these redefinitions.
couplings are too suppressed to adequately fit the fermion masses of the second and third generations (\(m_\mu, m_\tau, m_c\), etc.), which, in consequence, should be accounted for by renormalizable ones (this is still extremely restrictive). We then examined this issue.

As a first step, a renormalization group running of the Yukawa couplings between the string scale (\(M_{Str}\)) and the low–energy scale (1 GeV) has to be performed. This running is slightly different from the ordinary GUT one since the gauge couplings are not unified at \(M_{Str}\) due to string threshold corrections. This modifies, for example, the traditional \(m_b/m_\tau\) relation at low energy, although not substantially. The magnitude of the orbifold Yukawa couplings depends on the values of the so–called deformation parameters, which describe the size and shape of the compactified space. A systematic exploration allows us to check whether there is a choice of these deformation parameters for which the physical fermion masses are properly fitted. Not all the \(Z_N\) orbifolds are here on a same footing. It turns out that this fit is possible only for the \(Z_3, Z_4\) and \(Z_6\)–I orbifolds. Besides these, the \(Z_7\) orbifold is able to fit all the fermion masses except the strange one, which on the other hand has a large experimental uncertainty.

The corresponding values of the deformation parameters are quite reasonable (they correspond to values for the moduli \(T_i = O(1)\)). The case of the \(Z_4\) orbifold is specially remarkable since the number of free parameters is lower than the number of physical masses fitted. In some sense the experimental masses, which are low–energy quantities, are selecting a particular size and shape of the compactified space. We find this quite encouraging. It should be stressed, however, that this only shows the compatibility of certain string schemes with the low–energy measurements, although this is certainly non–trivial. The rest of \(Z_N\) orbifolds, however, are rather hopeless and should be discarded on these grounds. Finally, let us remark that all these results have been obtained under the assumptions explained in the Introduction, in particular within a minimal \(SU(3) \times SU(2) \times U(1)_Y\) scenario.

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FIGURE CAPTION

FIG. 1: Values of $h_c, h_s, h_b, h_t, h_{\mu}, h_{\tau}$ at $M_{Str}$ versus $\nu_2 = \langle H_2 \rangle$ giving the measured values of the corresponding fermion masses. Errors are not included.
| Orb.  | Twist θ       | Lattice       | #DP | Coupling | #AC | #EDP | #DCR | #DCD |
|-------|---------------|---------------|-----|----------|-----|------|------|------|
| $Z_3$ | $(1, 1, −2)/3$ | $SU(3)^3$     | 9   | $θθθ$    | 729 | 9    | 4    | 14   |
| $Z_4$ | $(1, 1, −2)/4$ | $SU(4)^2$     | 7   | $θθθ^2$  | 160 | 4    | 6    | 10   |
|       |               | $SO(4)^3$     | 7   | $θθθ^2$  | 160 | 4    | 6    | 8    |
| $Z_6$ | $(1, 1, −2)/6$ | $G_2^2 × SU(3)$ | 5   | $θθθ^3$  | 90  | 4    | 10   | 30   |
|       |               |               |     | $θ^2θθ^2$ | 369 | 5    | 8    | 12   |
| $Z_8$ | $(1, 2, −3)/6$ | $SU(6) × SU(2)$ | 5   | $θθθ^3$  | 48  | 1    | 4    | 4    |
|       |               |               |     | $θθθ^4$  | 72  | 2    | 4    | 4    |
| $Z_7$ | $(1, 2, −3)/7$ | $SU(7)$       | 3   | $θθθ^4$  | 49  | 3    | 2    | 4    |
| $Z_8$ | $(1, 2, −3)/8$ | $SO(5) × SO(9)$ | 3   | $θ^2θθ^4$ | 80  | 2    | 8    | 8    |
|       |               |               |     | $θθθ^5$  | 40  | 3    | 8    | 9    |
| $Z_8$ | $(1, 3, −4)/8$ | $SO(4) × SO(8)$ | 5   | $θθθ^6$  | 40  | 3    | 8    | 9    |
|       |               |               |     | $θθθ^4$  | 72  | 2    | 4    | 4    |
| $Z_{12}$ | $(1, 4, −5)/12$ | $SU(3) × F_4$ | 3   | $θθθ^9$  | 6   | 2    | 2    | 2    |
|       |               |               |     | $θ^2θθ^7$ | "   | "    | "    | "    |
|       |               |               |     | $θθθ^4$  | 27  | 3    | 4    | 6    |
|       |               |               |     | $θθθ^6$  | 36  | 2    | 7    | 12   |
|       |               |               |     | $θθθ^4$  | 135 | 3    | 8    | 12   |
| $Z_{12}$ | $(1, 5, −6)/12$ | $SO(4) × F_4$ | 5   | $θθθ^{10}$ | 4   | 2    | 1    | 1    |
|       |               |               |     | $θ^2θθ^5$ | "   | "    | "    | "    |
|       |               |               |     | $θθθ^8$  | 24  | 2    | 6    | 6    |
|       |               |               |     | $θθθ^5$  | 24  | 2    | 6    | 6    |
|       |               |               |     | $θθθ^6$  | 40  | 2    | 6    | 8    |
|       |               |               |     | $θθθ^6$  | 16  | 2    | 3    | 4    |

Table 1: Characteristics of twisted Yukawa couplings for $Z_n$ orbifolds. The twist $θ$ is specified by the three $c_i$ parameters (one for each complex plane rotation) appearing in $θ = \exp(\sum c_iJ_i)$. 

$#DP \equiv$ No. of deformation parameters, $#AC \equiv$ No. of allowed couplings, $#EDP \equiv$ No. of effective deformation parameters, $#DCR \equiv$ No. of different Yukawa couplings for the non–deformed (rigid) orbifold, $#DCD \equiv$ No. of different Yukawa couplings when deformations are considered. Quotation marks denote equivalent couplings.
Table 2: Fits for each $Z_N$ orbifold of the first and second generation fermion masses and total $\chi^2$. The masses, given in GeV, are to be understood at the 1 GeV scale, except the top mass, which is at the $M_Z$ scale. The first row corresponds to the present central experimental values (errors are not shown). For the top mass, the recent estimations based on the size of the electroweak radiative corrections was considered. Only the $Z_3$, $Z_4$ and $Z_6$–I orbifolds are compatible with the experiment.