On the quantum cohomology of the plane, old and new

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Abstract

We describe a method for counting maps of curves of given genus (and variable moduli) to \( \mathbb{P}^2 \), essentially by splitting the \( \mathbb{P}^2 \) in two; then specialising to the case of genus 0 we show that the method of quantum cohomology may be viewed as the 'mirror' of the former method where one splits the \( \mathbb{P}^1 \) rather than the \( \mathbb{P}^2 \), and we indicate a proof of the associativity of quantum multiplication based on this idea.

Recent work on Mirror Symmetry and Quantum Cohomology has contributed to a revival of interest in problems of a classical nature in Enumerative Geometry (cf. [F] and references therein). These problems involve (holomorphic) maps

\[ f : C \rightarrow X \]

where \( X \) is a fixed variety and \( C \) is a compact Riemann surface whose moduli are sometimes fixed (‘Gromov-Witten’) but here will not be, unless otherwise stated. While the case \( \text{dim } X = 1 \) is not entirely without interest (cf. [D]), the problem begins in earnest with \( \text{dim } X = 2 \) and naturally the simplest such \( X \) is \( \mathbb{P}^2 \). Here the problem specifically is to count the images \( f(C) \) of maps (1) where \( C \) has genus \( g \), \( f(C) \) has degree \( d \) and passes through \( 3d + g - 1 \) fixed points in \( \mathbb{P}^2 \). This problem has already, in essence, been solved in the author’s earlier paper [R] by means of a recursive method (we note however that the formula in [R], (3c.1), (3c.3) is trivially misstated and the factor \( c(\tilde{K}_1, \tilde{K}_2) \) should not be present).

Our purpose here is twofold. In Sect.1 we give a partial exposition of the method of [R] and illustrate it on a couple of new examples, namely the curves of degree \( d \) and genus \( g = \frac{(d-1)(d-2)}{2} - 2 \) (i.e. with 2 nodes); and the rational quartics. We recover classical formulae due, respectively, to Roberts [Ro] and Zeuthen [Z]. Hopefully, this will help make the method of [R] more accessible. In Sect.2 we show that the method of Kontsevich et al., at least as exposed in [F], may be viewed as none other than the 'dual' of that of [R] for the case of rational curves, 'dual' meaning 'interchanging source and target'; in particular, we sketch a proof from...
this viewpoint of the associativity of quantum multiplication.

This paper owes its existence to the unfailing encouragement of Bill Fulton, who believed all along in [R]; it is indeed a pleasure to thank him here.

1.Old.

We find it technically convenient here to work with possibly reducible curves; the modifications or 'correction terms' needed to treat the irreducible case are a routine matter.

Consider the locus $V_{d,\delta}$ of (not necessarily irreducible) curves of degree $d$ in $\mathbb{P}^2$ having $\delta$ ordinary nodes. This is well known to be a smooth locally closed subvariety of pure codimension $\delta$ in $\mathbb{P}^{(d+2)-1}$ and we are interested in its degree as such, which may be interpreted as the number of curves of $V_{d,\delta}$ passing through $\binom{d+2}{2} - \delta - 1$ general points in $\mathbb{P}^2$, a number which we denote by $N_{d,\delta}$. The idea is to get at $N_{d,\delta}$ by a recursive procedure, based on specializing $\mathbb{P}^2$ to a surface (called a 'fan')

$$S_0 = S_1 \cup S_2$$

where $S_1 = Bl_0(\mathbb{P}^2)$ (the 'bottom' component), $S_2 = \mathbb{P}^2$ (the 'top' component) and $E = S_1 \cap S_2$ (the 'axis') embedded in $S_i$ with self-intersection $2i - 3$, $i = 1, 2$. Corresponding to this is a specialization

$$V_{d,\delta} \to \sum m(\pi) V_{(d,e),(\delta_1,\delta_2),\pi},$$

where $V_{(d,e),(\delta_1,\delta_2),\pi}$ is a family of Cartier divisors on $S_0$ whose general member $C_0$ may be described as follows:

- $C_0 = C_1 \cup C_2$,
- $C_1 \in |dH - eE|_{S_1}, C_2 \in |eE|_{S_2}$ nodal curves with $\delta_1$ (resp. $\delta_2$) nodes, smooth near $E$;
- the divisor $D = C_1.E = C_2.E$ has shape $\pi$, i.e. $\pi$ is a partition having $\ell_i$ blocks of size $i$ (to be written as $\pi = [\ell_i]$) and $D = \sum_{i=1}^r \sum_{j=1}^{\ell_i} iQ_{ij}, Q_{ij} \in E$ distinct.

Moreover $m(\pi) = \prod_{i=1}^r i^{\ell_i}$ and the sum is extended over all data $((d,e),(\delta_1,\delta_2),\pi)$ satisfying

$$\delta_1 + \delta_2 + \sum_{i=1}^r (i - 1)\ell_i = \delta$$
(i.e. each i-tacnode $iQ_{ij}$ ‘counts as $i - 1$ nodes’).

Now to apply the specialization (2) to the degree question, we specialize our point set on $\mathbb{P}^2$ to a collection of points on $S_0$, which a priori we may distribute at will among $S_1$ and $S_2$, with each distribution giving rise to some formula which, however, may or may not be usable. For the purpose of the present discussion we will make the important simplifying assumption

$$\delta < d,$$

and put $d + 1$ points on $S_1$ and the remaining $\left(\frac{d+1}{2}\right) - 1 - \delta$ on $S_2$. It is then easy to see that the only limit components $V$ that will contribute to the resulting formula will be ones with $e = d - 1$.

For those, we can write

$$C_1 = C_{1,0} + \sum_{i=1}^{\delta_1} R_i$$

with $C_{1,0}$ a smooth (rational) curve of ‘type’ $(d - \delta_1, d - \delta_1 - 1)$ (i.e. $C_{1,0} \in \left|(d - \delta_1)H - (d - \delta_1 - 1)E\right|$) and $R_i$ distinct rulings.

Now let us say that a partition $\pi' = [\ell_i'] \leq \pi = [\ell_i]$ if $\ell_i' \leq \ell_i \forall i$, in which case we may define the complementary partition $\pi - \pi' = [\ell_i - \ell_i']$; also put $|\pi| = \sum i\ell_i, s(\pi) = \sum \ell_i, n(\pi) = \frac{s(\pi)!}{\ell_1! \cdots \ell_r!}$. Counting the degree of a limit component $V_1 = \{C_1 \cup C_2\}$ in terms of those of $\{C_1\}$ and $\{C_2\}$ is basically a matter of decomposing the ‘diagonal’ condition $C_1.E = C_2.E$ correspondingly to the standard Kunneth decomposition of the diagonal class on the product of $\Pi\mathbb{P}^{\ell_i}$ with itself; this leads to a sum of conditions corresponding to partitions $\pi' \leq \pi$, each amounting to fixing the location on $E$ of a portion $D'$ of $C_1.E$ corresponding to $\pi'$ and the complementary portion $D''$ of $C_2.E$ corresponding to $\pi - \pi'$. The resulting formula is as follows.

$$N_{d,\delta} = \sum_{|\pi|=d-1} m(\pi) \sum_{\pi'=[\ell_i']} m(\pi - \pi')n(\pi - \pi')N_{d-1,\delta-s(\pi-\pi') \pm s(\pi) - d + 1, \pi - \pi', \pi'}$$

$$\times \sum_{j=0}^{\ell_i'} \left(\frac{\ell_i'}{j}\right)\left(\frac{d + 1}{s(\pi - \pi') - j}\right).$$

Here $N_{e,\delta,\pi',\pi''}$ denotes the degree of the locus of nodal curves of degree $e$ with $\delta_2$ nodes meeting a fixed line $E$ in a fixed divisor of shape $\pi''$ plus a divisor of shape $\pi'$. 


We have used the fact that \( \delta_1 = s(\pi - \pi') \), which comes from the observation that the number of ‘axis’ conditions on the bottom curve \( C_1 \), i.e. \( |\pi| - s(\pi - \pi') = d - 1 - s(\pi - \pi') \), plus the number of ‘interior’ points imposed, i.e. \( d + 1 \), must equal the dimension of the family (4), i.e. \( 2d - \delta_1 \). Also, the factor \( m(\pi - \pi')n(\pi - \pi') \) is simply the degree of the ‘discriminant’ variety of divisors of shape \( \pi - \pi' \) on \( E = \mathbb{P}^1 \), while the binomial factors correspond to letting \( j \) of the rulings go through some of the multiplicity -1 part of \( D' \) with the remaining \( \delta_1 - j \) going through some of the \( d + 1 \) interior points.

Now of course in general the formula (5) is not by itself sufficient as one needs a recursive formula starting and ending with the \( N_{e,\delta_2,\pi',\pi} \) or something similar. Such a formula is indeed given in [R], and it is not our purpose to reproduce it here. In the examples worked out below the necessary further recursion is relatively straightforward, and will be indicated.

**Example 1.** \( N_{d,2} \)

There are seven relevant limit components and we proceed to list them and their contributions.

A. \( V(d,d-1),(0,2),(d-1) \); multiplicity \( m = 1 \); contribution \( N_{d-1,2} \)

B. \( V(d,d-1),(1,1),(d-1) \); \( m = 1 \). As \( \delta_1 = 1 \) we must take \( \pi' = [d - 2], \pi - \pi' = [1] \) so \( j = 0 \) or \( 1 \) and the contribution is \((d + 1 + d - 2).N_{d-1,1,[1],[d-2]} = 3(2d-1)(d-2)^2 \).

C. \( V(d,d-1),(2,0),(d-1) \); \( m = 1 \); \( \pi' = [d - 3], j = 0, 1, 2 \), contribution = \( (\binom{d-3}{2} + (d - 3)(d + 1) + \binom{d+1}{2})N_{d-1,0,[2],[d-3]} = 2d^2 - 5d + 3 \).

D. \( V(d,d-1),(0,1),(d-3,1) \); \( m = 2, \delta_1 = 0 \Rightarrow \pi' = \pi, \) so contribution is \( 2N_{d-1,1,0,[d-3,1]} \).

By an easier but simpler recursion (involving 1 node and 1 tangency), the latter evaluates to \( 12(d - 1)(d - 2)(d - 3) \).

E. \( V(d,d-1),(1,0),(d-3,1) \); \( m = 2, \pi' = [d - 3] \) or \( [d - 4, 1] \), contribution = \( 8(d - 1)(d - 3) \).

F. \( V(d,d-1),(0,0),(d-4,0,1) \); \( m = 3, \pi' = \pi \), contribution \( 9d - 27 \).

G. \( V(d,d-1),(0,0),(d-5,2) \); \( m = 4, \pi' = \pi \), contribution \( 4.4 \cdot \binom{d-3}{2} = 9d^2 - 56d + 96 \).

Summing up, we get

\[
N_{d,2} = N_{d-1,1,0} = 18d^3 - 81d^2 + 84d + 12
\]
Moreover it is easy to see that \( N_{3,2} = \binom{7}{2} = 21 \) so by integrating we get
\[
N_{d,2} = \frac{9}{2}d^4 - 18d^3 + 6d^2 + \frac{81}{2}d - 33.
\]
This is a classical formula due to S. Roberts [Ro], which has been given modern treatment by I. Vainsencher [V]. Note that the curves are automatically irreducible if \( d \geq 4 \).

**Example 2.** \( N_{4,3} \)

Here we have seven limit components.

A. \( V_{(4,3),(0,3),[3]} m = 1 \), contribution 15.

B. \( V_{(4,3),(1,2),[3]} m = 1 \) contribution 21.7 = 147.

C. \( V_{(4,3),(2,1),[3]} m = 1 \), contribution 15 \( N_{3,1,[2],[1]} = 180 \).

D. \( V_{(4,3),(3,0),[3]} m = 1 \), contribution \( \binom{5}{2} = 10 \).

E. \( V_{(4,3),(1,1),[1,1]} m = 2, \pi' = [1] \) or \([0, 1] \). Contribution 2.2 \( N_{3,1,[0,1],[1]} + 2.5 \cdot N_{3,1,[1],[0,1]} \).

By a similar but simpler recursion the latter \( N \)'s evaluate respectively to 10, 16, so the total contribution is 200.

F. \( V_{(4,3),(0,2),[1,1]} m = 2, \pi = \pi' = [1, 1] \), contribution 2.15.2 = 60.

G. \( V_{(4,3),(0,1),[0,0,1]} m = 3, \pi = \pi' = [0, 0, 1] \), contribution 3 \( N_{3,1,[0],[0,0,1]} \). By a similar but simpler recursion, the latter \( N \) is 21, so the contribution is 63.

Summing up, we get
\[
N_{4,3} = 675 = 5^2 \cdot 3^3.
\]

As the \{cubic + line\} locus clearly has degree \( \binom{11}{2} = 55 \), we obtain 620 as the number of irreducible rational quartics through 11 points. (cf. [Z]).

2. New.

The new approach works for maps from a fixed curve \( C \), say to \( \mathbb{P}^2 \). For simplicity we will assume \( C = \mathbb{P}^1 \). Considering rational curves of degree \( d \) in \( \mathbb{P}^2 \) amounts to considering curves of bidegree \((1, d)\) in \( \mathbb{P}^1 \times \mathbb{P}^2 \), and the old method to count them is by specialising the \( \mathbb{P}^2 \) factor to a fan; the new approach on the other hand is to specialise the \( \mathbb{P}^1 \) factor to a '1-dimensional fan', i.e. to
\[
C_1 = C + C_0. C_1 = \mathbb{P}^1, C_0 \cap C_1 = \{ c \}.
\]
Because $\mathbb{P}^1$ is simpler than $\mathbb{P}^2$ this approach works better in this case; on the other hand it is apparently unknown how to make it work when the source curve is allowed to vary with moduli.

To be precise, fix a pair of points $y_1, y_2$ and a pair of lines $L_3, L_4$ in $\mathbb{P}^2$ and 4 points $x_1, ..., x_4 \in \mathbb{P}^1 = C$ and consider curves of bidegree $(1, d)$ in $C \times \mathbb{P}^2$ containing $(x_1, y_1), (x_2, y_2)$ and meeting $x_3 \times L_3, x_4 \times L_4$, as well as a further collection of $3d - 4$ 'horizontal' lines $C \times z_j$. We then specialise this to $C_0 \times \mathbb{P}^2$ in two ways: (A) $x_1, ..., x_4$ specialise to $x_1, 1, x_2, 1 \in C_1, x_3, 2, x_4, 2 \in C_2$; (B) $x_1, x_3, x_2, x_4$ specialise to $x_1, 1, x_3, 1 \in C_1, x_2, 1, x_4, 2 \in C_2$. In the (A) limit it is possible to have a component of bidegree $(1, 0)$ in $C_2 \times (L_3 \cap L_4)$, while in the (B) limit all curves have bidegrees $(1, d_1) \cup (1, d_2), d_1 + d_2 = d, d_i > 0$. Thus letting $n_d$ denote the number of rational curves in $\mathbb{P}^2$ through $3d - 1$ points, writing $(A) = (B)$ we get an equation of the form

$$n_d + f(n_1, ..., n_{d-1}) = g(n_1, ..., n_{d-1})$$

for suitable quadratic expressions $f, g$, which may be solved for $n_d$.

**Example:** $d = 4$.

$$f = \binom{8}{2}.12.1.1.1.3 + \binom{8}{3}.1.1.2.2.4 + 1.1.12.3.3.3 = 2228$$

with the summands corresponding to $d_1 = 3, 2, 1$ and, e.g. in the first product the factors corresponding to: choosing 2 of the 8 points $z_j$ for the image of $C_2$ to go through; the number of possible images of $C_1, C_2, x_3, x_4, x$;

$$g = 8.12.1.3.1.3 + \binom{8}{4}.1.1.2.2.4 + 8.1.12.1.3.3 = 2848$$

$$n_4 = 620.$$
\( \bar{M}_{0,n+1}(X, \beta_1) \times \bar{M}_{0,n+1}(X, \beta_2) \) given by the pullback of the diagonal \( \Delta \subset X \times X \) via a suitable projection. Now fixing any decomposition of \( n \) as \( n_1 + n_2 \), a divisor \( D \) in the space \( \bar{M}_{0,n+4}(X, \beta) \), defined by fixing the position of the first four points, say, may be specialised to the union (with multiplicity 1) of the various \( \bar{M}_{0,0;n_1+2,n_2+2}(X, \beta_1, \beta_2) \) with \( \beta_1 + \beta_2 = \beta \) (because fixing the position of three points on each component of \( C_0 \) is vacuous); using the formula

\[
[\Delta] = \sum g^{e,f} T_e \otimes T_f,
\]

we conclude:

\[
I'_\beta(\gamma^n T_i T_j T_k T_l) = \sum I_{\beta_1}(\gamma^{n_1} T_i T_j T_e) I_{\beta_2}(\gamma^{n_2} T_k T_l T_f) g^{e,f}
\]

where the LHS is the appropriate Gromov-Witten integral over \( D \) and the sum is over all \( \beta_1 + \beta_2 = \beta \) and all \( e, f \). As the LHS is symmetric in \( i, j, k, l \), this is certainly stronger than the relation (2.9) on p.16 of [F], which is equivalent to associativity.

References

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