ON ROBUST THEOREMS DUE TO BOLZANO, WEIERSTRASS, JORDAN, AND CANTOR

DAG NORMANN AND SAM SANDERS

Abstract. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics where the aim is to identify the minimal axioms needed to prove a given theorem from ordinary, i.e. non-set theoretic, mathematics. This program has unveiled surprising regularities: the minimal axioms are very often equivalent to the theorem over the base theory, a weak system of ‘computable mathematics’, while most theorems are either provable in this base theory, or equivalent to one of only four logical systems. The latter plus the base theory are called the ‘Big Five’ and the associated equivalences are robust following Montalbán, i.e. stable under small variations of the theorems at hand. Working in Kohlenbach’s higher-order RM, we obtain two new and long series of equivalences based on theorems due to Bolzano, Weierstrass, Jordan, and Cantor; these equivalences are extremely robust and have no counterpart among the Big Five systems. Thus, higher-order RM is much richer than its second-order cousin, boasting at least two extra ‘Big’ systems.

1. Introduction

1.1. Motivation and caveat. Like Hilbert ([34]), we believe the infinite to be a central object of study in mathematics. That the infinite comes in ‘different sizes’ is a relatively new insight, due to Cantor around 1874 ([9]), in the guise of the uncountability of the real numbers, also known simply as Cantor’s theorem.

With the notion ‘countable versus uncountable’ in place, it is an empirical observation, witnessed by many textbooks, that to show that a set is countable one often constructs an injection (or bijection) to \( \mathbb{N} \). When given a countable set, one (additionally) assumes that this set can be enumerated, i.e. represented by some sequence. In this light, implicit in much of mathematical practise is the following most basic principle about countable sets:

\[
\text{a set that can be mapped to } \mathbb{N} \text{ via an injection (or bijection) can be enumerated.}
\]

This principle was studied in [79, 92, 94] as part of the study of the uncountability of \( \mathbb{R} \). In this paper, we continue the study of this principle in Reverse Mathematics (RM hereafter) and connect it to well-known ‘household name’ theorems due to
Bolzano-Weierstrass, Cantor, Jordan, and Heine-Borel, as discussed in detail in 
Section 1.2. We assume basic familiarity with RM, also sketched in Section 1.3.1. 
In particular, working in Kohlenbach’s higher-order RM, we obtain two new long 
series of extremely robust equivalences involving the aforementioned theorems. In 
this concrete way, third-order arithmetic is much richer than its second-order cousin 
in that the former boasts (at least) two extra ‘Big’ systems\footnote{A logical system is called ‘Big’ if it boasts many equivalences involving robust principles.} compared to the latter.

For all the aforementioned reasons, our results provide new answers to one of 
the driving questions behind RM, formulated as follows by Montalbán.

The way I view it, gaining a greater understanding of [the big five] 
phenomenon is currently one of the driving questions behind reverse 
mathematics. To study [this] phenomenon, one distinction that 
I think is worth making is the one between robust systems and 
non-robust systems. A system is robust if it is equivalent to small 
perturbations of itself. This is not a precise notion yet, but we can 
still recognize some robust systems. All the big five systems are 
very robust. For example, most theorems about ordinals, stated in 
different possible ways, are all equivalent to each other and to ATR₀. 
Apart from those systems, weak weak König’s Lemma WWKL₀ is 
also robust, and we know no more than one or two other systems 
that may be robust. (\cite{63, p. 432}, emphasis in original)

Finally, the uncountability of \( \mathbb{R} \) deals with arbitrary mappings with domain \( \mathbb{R} \) and 
is therefore best studied in a language that has such objects as first-class citizens. 
Obviousness, much more than beauty, is however in the eye of the beholder. Lest 
we be misunderstood, we formulate a blanket caveat: all notions (computation, 
continuity, function, open set, et cetera) used in this paper are to be interpreted 
via their higher-order definitions, also listed below, unless explicitly stated otherwise.

1.2. From Bolzano-Weierstrass to Heine-Borel and Jordan. In this section, 
we provide an overview of our results; in a nutshell, we obtain a large number of 
robust equivalences involving the Bolzano-Weierstrass theorem for countable sets 
and many theorems concerned with countable sets and related notions. We also 
obtain equivalences for theorems that do not involve countable sets in any obvious 
or direct way at all, namely the Jordan decomposition theorem and similar results 
on functions of bounded variation and related notions.

First of all, the Bolzano-Weierstrass theorem comes in different formulations. 
Weierstrass formulates this theorem around 1860 in \cite{115, p. 77} as follows, while 
Bolzano \cite{87, p. 174} states the existence of suprema rather than just limit points.

If a function has a definite property infinitely often within a finite 
domain, then there is a point such that in any neighbourhood of 
this point there are infinitely many points with the property.

We start by studying the Bolzano-Weierstrass theorem for countable sets as in 
Principle \ref{pr:bwc}. Precise definitions of all notions involved can be found in Section 1.3.2 
while motivation for our choice of definitions is provided in Section 3.3.3.

Principle 1.1 (BWC). For a countable set \( A \subset 2^{\mathbb{N}} \), the supremum \( \sup A \) exists.
Unless explicitly stated otherwise, the supremum is taken relative to the lexicographic ordering. A number of variations $\text{BWC}_l$ of Principle [1,1] are possible, which we shall express via the indicated super- and sub-scripts as follows.

- For $i = 0$, *countable sets* are defined via *injections* to $\mathbb{N}$ (Definition [1,4]).
- For $i = 1$, we restrict to *strongly countable sets*, which are defined via *bijections* to $\mathbb{N}$ (Definition [1,4]).
- For $j$ including seq, we additionally have that a *sequence* $(f_n)_{n \in \mathbb{N}}$ in $A$ is given with $\lim_{n \to \infty} f_n = \sup A$.
- For $j$ including fun, we additionally have that $\sup_{f \in A} F(f)$ exists for arbitrary *functionals* $F : 2^\mathbb{N} \to 2^\mathbb{N}$.
- For $j$ including pwo, the supremum is relative to the pointwise ordering.

Since Cantor space with the lexicographic ordering and $[0,1]$ with its usual ordering are intimately connected, we take the former ordering to be fundamental. We have shown in [79] that $\text{BWC}_0^\text{fun}$ is ‘explosive’ in that it yields the much stronger $\Pi^1_2$-CA$_0$ when combined with the Suslin functional, i.e. higher-order $\Pi^1_2$-CA$_0$. Previously, metrisation theorems from topology were needed to reach $\Pi^1_2$-CA$_0$ ([68],[70]), while Rathjen states in [82] that $\Pi^1_1$-CA$_0$ *dwarfs* $\Pi^1_1$-CA$_0$ and Martin-Löf talks of a *chasm* and *abyss* between these two systems in [60]. Analogous results hold at the level of computability theory, in the sense of Kleene’s S1-S9 ([47]), while we even obtain $\Pi^0_3$, and hence full second-order arithmetic, if we assume $V=L$, by [79] Theorem 4.6. Thus, the following natural questions arise.

(Q0) Is the ‘extra information’ as in ‘fun’ or ‘seq’ necessary for explosions?
(Q1) Is it possible to ‘split’ e.g. $\text{BWC}_0$ in ‘less explosive’ components?
(Q2) Since $\text{BWC}_0$ is formulated using injections, is there an equivalent formulation only based on *bijections*?
(Q3) Is the explosive nature of $\text{BWC}_0$ caused by the use of injections or bijections?
(Q4) Are there equivalences involving $\text{BWC}_0$ from ordinary mathematics, especially involving theorems not related to countability in any obvious way?

Secondly, to answer (Q0), we connect $\text{BWC}_0$ to the other variations $\text{BWC}_l$, as part of Kohlenbach’s *higher-order Reverse Mathematics*, briefly introduced in Section [1,3.1]. We assume basic familiarity with Reverse Mathematics (RM hereafter), to which [100] provides an introduction. We establish the series of equivalences in (EQ) in Section 2.2, where IND$_1$ are fragments of the induction axiom.

$$\begin{align*}
\text{BWC}_0^\text{fun} & \leftrightarrow \text{BWC}_0^\text{seq} \leftrightarrow \text{BWC}_0^\text{pwo} \leftrightarrow [\text{BWC}_0 + \text{IND}_0] \leftrightarrow \text{BWC}_0^\text{fun,pwo}. \\
\text{cocode}_1 & \leftrightarrow \Delta^1_0\text{-CA}_0 \leftrightarrow \text{BW}_1^\text{seq} \leftrightarrow [\text{BW}_1 + \text{IND}_1] \leftrightarrow \text{BW}_1^\text{pwo}.
\end{align*}$$

(EQ)

Here, $\Delta^1_0\text{-CA}_0^-$ is a peculiar axiom inspired by $\Delta^0_1$-comprehension while $\text{cocode}_1$ expresses that *strongly countable* sets, i.e. boasting bijections to $\mathbb{N}$, can be enumerated. We point out that $\text{BWC}_0 \leftrightarrow \text{BWC}_0^\text{seq}$ is interesting as follows: to obtain the extra sequence in the latter, the only method seems to use countable choice, while the equivalence is provable without the latter. Thus, the extra sequence from $\text{BWC}_0^\text{seq}$, while seemingly a choice function, can be defined explicitly in terms of the other

---

2The pointwise ordering $\leq_1$ is defined as $f \leq_1 g \equiv (\forall n \in \mathbb{N})(f(n) \leq g(n))$ for any $f,g \in 2^{\mathbb{N}}$. The sequence $\sup A$ is the supremum of $A \subset 2^{\mathbb{N}}$ for this ordering if $(\forall f \in A)(f \leq_1 \sup A)$ and $(\forall k \in \mathbb{N})(\sup A)(k) = 1 \Rightarrow (\exists f \in A)(f(k) = 1)$.

3Apply countable choice to $(\forall n \in \mathbb{N})(\exists f \in A)(d(f, \sup A) < \frac{1}{2^n})$ which holds by definition.
data, i.e. without the Axiom of Choice. By Remark 2.8 the second line of \((\text{EQ})\) is connected to hyperarithmetical analysis.

Thirdly, in answer to (Q3), the principles from \((\text{EQ})\) are formulated using injections and bijections to \(\mathbb{N}\), while items \(([a]-[c])\) below are basic theorems about the real line \(\mathbb{R}\) based on enumerable sets, i.e. listed by (possibly infinite) sequences, which is essentially the notion of countable set used in second-order RM:

(a) \(\text{accu}\): a non-enumerable closed set in \(\mathbb{R}\) has a limit point,
(b) \(\text{accu}'\): a non-enumerable set in \(\mathbb{R}\) contains a limit point,
(c) \(\text{ccc}\): a collection of disjoint open intervals in \(\mathbb{R}\) is enumerable.
(d) \(\text{cloq}\): a countable linear ordering is order-isomorphic to a subset of \(\mathbb{Q}\).

Closed sets are defined as in Definition 1.2, which generalises the second-order notion ([103, II.5.6]). The principles \(\text{ccc}_i\) and \(\text{accu}_i\) for \(i = 0, 1\) are defined as for \(\text{BWC}_i\) above. We establish the following series of implications in Section 2.4.

\[ \text{accu} \leftrightarrow \text{accu}' \leftrightarrow \text{ccc} \leftrightarrow \text{BWC}_0 \leftrightarrow \left[ \text{CBN} + \text{BW}_1 \right] \leftrightarrow \left[ \text{CWO}^\omega + \text{IND}_0 \right]. \]  

\((\text{EQ2})\)

\[ \text{ccc}_1 \leftrightarrow \text{CBN} \leftrightarrow \text{accu}_1, \quad \text{and} \quad \text{cocode}_0 \leftrightarrow \left[ \text{cloq} + \text{IND}_0 \right] \leftrightarrow \left[ \text{cloq}' + \text{IND}_0 \right]. \]

Here, \(\text{CBN}\) is the Cantor-Bernstein theorem for \(\mathbb{N}\) as in Principle 2.14, which is independent of \(\text{BWC}_1\) by Theorem 2.16, thus answering (Q2). The principle \(\text{CWO}^\omega\) expresses that countable well-orderings are comparable, while \(\text{cloq}'\) is Cantor’s theorem characterising the order type \(\eta\) of \(\mathbb{Q}\). The notion of limit point goes back to Cantor ([14, p. 98]) in 1872; he also proved the first instance of the countable chain condition \(\text{ccc}\) in [14, p. 161] and introduced order types, including \(\eta\), in [12, 13].

Fourth, following (Q4), we also study \(\text{BWC}_0\) and \(\text{BWC}_1\) in the grand(er) scheme of things, namely how they connect to set theory and ordinary mathematics. In Section 3.2 we obtain equivalences between \(\text{BWC}_0\) and \(\text{BWC}_1\), and fragments of the well-known countable union theorem from set theory (see e.g. [32, §3.1]). As to ordinary mathematics, in Section 3.1 we establish equivalences between \(\text{BWC}_0\) and versions of the Lindelöf lemma and Heine-Borel theorem as studied in [74, 79]. In Section 3.3 we establish equivalences between \(\text{BWC}_0\), the Jordan decomposition theorem, and related results from [80, 81]. The latter theorem and its ilk have no obvious or direct connection to countability at all.

Finally, we discuss how these results provide detailed answers to (Q0)-(Q4) in the below sections. In light of all the aforementioned equivalences, we believe the following quote by Friedman to be apt:

When a theorem is proved from the right axioms, the axioms can be proved from the theorem. ([26])

Next, Section 1.3 details the definitions used in this paper while a neat motivation for our choice of definitions is provided in Section 3.3.3 with the gift of hindsight.

1.3. Preliminaries and definitions. We briefly introduce Reverse Mathematics and higher-order computability theory in Section 1.3.1. We introduce some essential definitions in Section 1.3.2. A full introduction may be found in e.g. [79, §2]. In Section 3.3.3 we motivate our choice of definitions, Definition 1.2 in particular.

1.3.1. Reverse Mathematics and higher-order computability theory. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([26, 27]) and developed extensively by Simpson ([103]).
The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [106] for a basic introduction to RM and to [102,103] for an overview of RM. We expect basic familiarity with RM, in particular Kohlenbach's higher-order RM ([49]) essential to this paper, including the base theory \( \text{RCA}_0 \). An extensive introduction can be found in e.g. [74, 77, 79]. We have chosen to include a brief introduction as a technical appendix, namely Section A. All undefined notions may be found in the latter.

Next, some of our main results will be proved using techniques from computability theory. Thus, we first make our notion of ‘computability’ precise as follows.

(I) We adopt \( \text{ZFC} \), i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.

(II) We adopt Kleene’s notion of higher-order computation as given by his nine clauses S1-S9 (see [58, Ch. 5] or [47]) as our official notion of ‘computable’.

We refer to [58] for a thorough overview of higher-order computability theory.

1.3.2. Some definitions in higher-order arithmetic. We introduce the standard definitions for countable set and related notions.

First of all, the main topic of [79] is the logical and computational properties of the uncountability of \( \mathbb{R} \), established in 1874 by Cantor in his first set theory paper [9], in the guise of the following natural principles:

- NIN: there is no injection from \([0, 1]\) to \(\mathbb{N}\),
- NBI: there is no bijection from \([0, 1]\) to \(\mathbb{N}\).

As it happens, NIN and NBI are among the weakest principles that require a lot of comprehension for a proof. An overview may be found in [79, Figure 1].

Secondly, we shall make use of the following notion of (open) set, which was studied in detail in [76,79,94]. We motivate this choice in detail in Section 3.3.3.

**Definition 1.2.** [Sets in \( \text{RCA}_0 \)] We let \( Y : \mathbb{R} \to \mathbb{R} \) represent subsets of \( \mathbb{R} \) as follows: we write ‘\( x \in Y \)’ for ‘\( Y(x) > R \)' and call a set \( Y \subseteq \mathbb{R} \) ‘open’ if for every \( x \in Y \), there is an open ball \( B(x, r) \subset Y \) with \( r > 0 \). A set \( Y \) is called ‘closed’ if the complement, denoted \( Y^c = \{ x \in \mathbb{R} : x \notin Y \} \), is open.

Note that for open \( Y \) as in the previous definition, the formula ‘\( x \in Y \)’ has the same complexity (modulo higher types) as in second-order RM (see [103, II.5.6]), while given (\( \exists^2 \)) from Section A.1.4 the former becomes a ‘proper’ characteristic function, only taking values ‘0’ and ‘1’. Hereafter, an ‘(open) set’ refers to Definition 1.2 while ‘RM-open set’ refers to the second-order definition from RM.

The attentive reader has of course noted that e.g. the unit interval is only a set in the sense of Definition 1.2 in case we assume \( \text{ACA}_0 \equiv \text{RCA}_0 + (\exists^2) \). For this reason, we shall adopt the latter as our base theory in our paper. We discuss how the reader may obtain equivalences over \( \text{RCA}_0 \) in Remark 2.1.

Thirdly, the notion of ‘countable set’ can be formalised in various ways, namely via Definitions 1.3 and 1.4.

**Definition 1.3.** [Enumerable sets of reals] A set \( A \subseteq \mathbb{R} \) is enumerable if there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) such that \( (\forall x \in \mathbb{R})(x \in A \rightarrow (\exists n \in \mathbb{N})(x = R x_n)) \).
This definition reflects the RM-notion of ‘countable set’ from [103, V.4.2]. We note that given $\mu^2$ from Section A.1.4 we may replace the final implication in Definition 1.4 by an equivalence.

The usual definition of ‘countable set’ is as follows in $\text{RCA}_0$.

**Definition 1.4.** [Countable subset of $\mathbb{R}$] A set $A \subseteq \mathbb{R}$ is countable if there exists a function $f : \mathbb{R} \to \mathbb{N}$ such that $(\forall y)(\exists x)(f(x) = y)$. If $f$ is bijective, we call $A$ strongly countable.

The first part of Definition 1.4 is from Kunen’s set theory textbook ([54, p. 63]) and the second part is taken from Hrbacek-Jech’s set theory textbook [38] (where the term ‘countable’ is used instead of ‘strongly countable’). For the rest of this paper, ‘strongly countable’ and ‘countable’ shall exclusively refer to Definition 1.4.

Finally, we shall use the following definition of finite and infinite set.

**Definition 1.5.** [Finite and infinite sets] A set $A \subseteq \mathbb{R}$ is called finite if $|A| < \infty$, and infinite if $|A| = \infty$. A set $A \subseteq \mathbb{R}$ is finite if it is not infinite.

The exact definition of (in)finite set plays a minor role in most of this paper, but a major role in the study of the Jordan decomposition theorem and related topics in Section 3.3. This observation is explained at length in Remark 3.31. In a nutshell, the notion of finite set as in Definition 1.5 is suitable for the RM-study of functions of bounded variation, whereas the ‘usual’ definitions of finite set, involving injections or bijections to $\mathbb{N}$, are not.

### 1.3.3. Some axioms of higher-order arithmetic.

We introduce a number of axioms of higher-order arithmetic, including the ‘higher-order counterparts’ of WKL and ACA$^\omega$. We motivate the latter term in detail based on Remark 1.12.

First of all, with Definitions 1.2 and 1.4 in place, the following principle has interesting properties, as studied in [79, 92, 94].

**Principle 1.6 (cocode).** For any non-empty countable set $A \subseteq [0, 1]$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that $(\forall x)(x \in A \leftrightarrow (\exists n)(x_n = x))$.

Indeed, as explored in [79, 94], we have cocode$\leftrightarrow$BWC$^\text{fun}$ over ACA$^\omega$, while another interesting equivalence is based on the ‘projection’ axiom studied in [89].

$$(\forall Y)(\exists X \subseteq \mathbb{N})(\forall n)(n \in X \leftrightarrow (\exists f)(Y(f, n) = 0)) \quad \text{(BOOT)}$$

We mention that BOOT is equivalent to e.g. the monotone convergence theorem for nets indexed by Baire space (see [89, §3]), while it is essentially Feferman’s (Proj1) from [24] without set parameters. The axiom BOOT$^-$ results from restricting BOOT to functionals $Y$ with the following ‘at most one’ condition:

$$(\forall n)(\exists f)(Y(f, n) = 0), \quad (1.1)$$

where similar constructs appear in the RM of ATR by [103, V.5.2]. The weaker BOOT$^-$ appears prominently in the RM-study of open sets given as (third-order) characteristic functions ([76]). In turn, BOOT$^\circ$ is BOOT$^-$ with $\mathbb{N}$ replaced by $\omega$ everywhere; BOOT$^\circ$ was introduced in [79, §3.1] in the study of BWC$^\text{fun}$, and we
have \( \text{BOOT}_C \leftrightarrow \text{BWC}^\text{up}_0 \) over \( \text{RCA}_0^\omega \) by [94] Theorem 3.12. In light of [103] V.5.2, \( \text{ACA}_0^\omega + \text{BOOT}_C \) proves \( \text{ATR}_0 \).

Secondly, related to \( \text{BOOT}_C \) and \( \text{cocode}_0 \) is the following principle.

**Principle 1.7** (range of \( \text{A}_0 \)). For \( Y : 2^\omega \to \omega \) an injection on \( A \subset 2^\omega \), we have
\[
(\exists X \subset \omega)(\forall n \in \omega)[(\exists f \in A)(Y(f) = n) \leftrightarrow n \in X],
\]
i.e. the range of \( Y \) restricted to \( A \) exists.

With the gift of hindsight\(^4\) from [79, 92, 94], we see that \( \text{cocode}_0 \) is equivalent to:
\[
\text{a linear order } (A, \preceq_A) \text{ for countable } A \subset \mathbb{R} \text{ can be enumerated.} \quad (1.2)
\]

In second-order RM, countable linear orders are represented by sequences (see [103] V.1.1), i.e. the previous principle seems essential if one wants to interpret theorems about countable linear orders in higher-order arithmetic or set theory. Another useful fragment of \( \text{BOOT} \) is \( \Delta^\omega - \text{CA} \), which is central to ‘lifting’ second-order reversals to higher-order arithmetic (see [91, 93]).

**Principle 1.8** (\( \Delta^\omega - \text{CA} \)). For \( i = 0, 1 \), \( Y_i^2 \), and \( A_i(n) \equiv (\exists f \in \omega^\omega)(Y_i(f, n) = 0) \):
\[
(\forall n \in \omega)(A_0(n) \leftrightarrow \neg A_1(n)) \to (\exists X \subset \omega)(\forall n \in \omega)(n \in X \leftrightarrow A_0(n)).
\]

This principle borrows its name from the fact that the \( \text{ECF} \)-translation (see Remark 1.12) converts \( \Delta^\omega - \text{CA} \) into \( \Delta^0_1 \)-comprehension. As will become clear below, \( \Delta^\omega - \text{CA} \) with the ‘at most one’ condition \( 1.1 \) plays an important role in the RM of the Bolzano-Weierstrass theorem.

Thirdly, the Heine-Borel theorem states the existence of a finite sub-covering for an open covering of certain spaces. Now, a functional \( \Psi : \mathbb{R} \to \mathbb{R}^+ \) gives rise to the canonical covering \( \bigcup_{x \in I} I_x^\Psi \) for \( I \equiv [0, 1] \), where \( I_x^\Psi \) is the open interval \((x - \Psi(x), x + \Psi(x))\). Hence, the uncountable covering \( \bigcup_{x \in I} I_x^\Psi \) has a finite sub-covering by the Heine-Borel theorem, which yields the following principle.

**Principle 1.9** (HBU). \((\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_0, \ldots, y_k \in I)(\forall x \in I)(\exists i \leq k)(x \in I_{y_i}^\Psi)\).

Note that HBU is essentially Cousin’s lemma (see [18] p. 223), i.e. the Heine-Borel theorem for canonical coverings. By [74, 77], \( Z_2^1 \) proves HBU, but \( Z_2^1 + \text{QF-AC}^{0,1} \) cannot. Basic properties of the gauge integral (67, 107) are equivalent to HBU. By [74] Theorem 3.3, HBU is equivalent to the same compactness property for \( 2^\omega \).

**Principle 1.10** (HBU\(_C\)). \((\forall G^2)(\exists f_1, \ldots, f_k \in 2^\omega)(\forall f \in 2^\omega)(\exists i \leq k)(f \in \overline{\bigcup}_{i=1}^k G(f_i)))\).

As studied in [90] §3.1, canonical coverings in HBU are not suitable for the study of basic topological notions like paracompactness and dimension. This suggests the need for a more general notion of covering; the solution adopted in [90] it to allow \( \psi : I \to \mathbb{R} \), i.e. \( I_x^\psi \) is empty in case \( \psi(x) \leq 0 \). In this way, we say that \( \bigcup_{x \in I} I_x^\psi \) covers \([0, 1]\) if \((\forall x \in [0, 1])(\exists y \in [0, 1])(x \in I_y^\psi)\). Thus, we obtain the Heine-Borel theorem as in HBT, going back to Lebesgue in 1898 (see [56] p. 133).

**Principle 1.11** (HBT). For \( \psi : [0, 1] \to \mathbb{R}_+ \), if \( \bigcup_{x \in I} I_x^\psi \) covers \([0, 1]\), then there are \( y_1, \ldots, y_k \in [0, 1] \) such that \( \bigcup_{i \leq k} I_{y_i}^\psi \) covers \([0, 1]\).

\(^4\)A countable \( A \subset \mathbb{R} \) yields a linear order via \( x \preceq y \equiv Y(x) \leq Y(y) \), where \( Y \) is injective on \( A \).
As shown in [91, §3], we have HBU $\leftrightarrow$ HBT over various natural base theories, some of which we shall discuss and use in Section 3.1.4.

Finally, as discussed in detail in [49, §2], the base theories $\text{RCA}_0^\omega$ and $\text{RCA}_0$ prove the same $L_2$-sentences ‘up to language’ as the latter is set-based (the $L_2$-language) and the former function-based (the $L_\omega$-language). Here, $L_2$ is the language of second-order arithmetic, while $L_\omega$ is the language of all finite types. This conservation result is obtained via the so-called ECF-interpretation, discussed next.

Remark 1.12 (The ECF-interpretation). The (rather) technical definition of ECF may be found in [111, p. 138, §2.6]. Intuitively, the ECF-interpretation $[A]_{\text{ECF}}$ of a formula $A \in L_\omega$ is just $A$ with all variables of type two and higher replaced by type one variables ranging over so-called ‘associates’ or ‘RM-codes’ (see [49, §4]); the latter are (countable) representations of continuous functionals. The ECF-interpretation connects $\text{RCA}_0^\omega$ and $\text{RCA}_0$ (see [49, Prop. 3.1]) in that if $\text{RCA}_0^\omega$ proves $A$, then $\text{RCA}_0$ proves $[A]_{\text{ECF}}$, again ‘up to language’, as $\text{RCA}_0$ is formulated using sets, and $[A]_{\text{ECF}}$ is formulated using types, i.e. using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the canonical embedding of higher-order into second-order arithmetic. Moreover, $\text{RCA}_0^\omega + \text{BOOT}$ is called the ‘higher-order counterpart’ of $\text{ACA}_0$ as the former is a conservative extension of the latter, and ECF maps $\text{BOOT}$ to $\text{ACA}_0$. Similarly, $\text{RCA}_0^\omega + \text{HBT}$ is the ‘higher-order counterpart’ of $\text{WKL}_0$.

As a neat application of the ECF-interpretation, Remark 3.28 establishes that the Jordan decomposition theorem (see Section 3.3.1) does not imply $(\exists^2)$, although the former theorem applies to discontinuous functions.

2. Equivalences for the Bolzano-Weierstrass theorem

2.1. Introduction. We establish the results sketched in Section 1.2 and [EQ].

In Section 2.2.1 we establish the equivalence between $\text{cocode}_1$ and the Bolzano-Weierstrass theorem for strongly countable sets in Cantor space in various guises, including $\text{BWC}_1$. In Section 2.2.2 we do the same for $\text{cocode}_0$ and $\text{BWC}_0$ and variations. In Section 2.3 we study $\text{CBN}$, the Cantor-Berstein theorem for $\mathbb{N}$, and show that it is strictly weaker than $\text{BWC}_0$ in that $Z_2^3 + \text{CBN}$ cannot even prove $\text{NBI}$. In Section 2.4 we study items (11)-(14) from Section 1.2 which are basic theorems about limit points in $\mathbb{R}$ and related concepts, all going back to Cantor somehow.

We establish equivalences between versions of some of these items on one hand, and $\text{CBN}$ and $\text{cocode}_0$ on the other hand; unlike the latter, items (11)-(14) do not mention ‘injections’ or ‘bijections’.

As to technical machinery, we mention the ‘excluded middle trick’ pioneered in [77]. While we adopt $\text{ACA}_0^\omega$ as our base theory, the following trick can be used to replace the latter theory by $\text{RCA}_0^\omega$ if the reader so desires.

Remark 2.1 (Excluded middle trick). The law of excluded middle as in $(\exists^2) \lor \neg (\exists^2)$ is quite useful as follows: suppose we are proving $T \rightarrow \text{cocode}_0$ over $\text{RCA}_0^\omega$. Now, in case $\neg (\exists^2)$, all functions on $\mathbb{R}$ are continuous by [49, §3] and $\text{cocode}_0$ trivially holds. Hence, what remains is to establish $T \rightarrow \text{cocode}_0$ in case we have $(\exists^2)$. However, the latter axiom e.g. implies $\text{ACA}_0$ and can uniformly convert reals to their binary representations. In this way, finding a proof in $\text{RCA}_0^\omega + (\exists^2)$ is ‘much easier’ than
finding a proof in $\text{RCA}_0^\omega$. In a nutshell, we may wlog assume $(\exists^2)$ when proving theorems that are trivial (or readily proved) when all functions (on $\mathbb{R}$ or $\mathbb{N}^\mathbb{N}$) are continuous, like $\text{cocode}_0$.

We stress that the previous trick should be used sparingly: the unit interval is not a set in the sense of Definition 1.2 in the absence of $(\exists^2)$.

In addition to the previous remark, we shall need a coding trick based on the well-known lexicographic ordering $\prec_{\text{lex}}$, as described in Notation 2.2. For brevity, we sometimes abbreviate $(n) * w^0 * f^1$ as $nwf$ if all types are clear from context.

**Notation 2.2** (Sequences with information). For a finite binary sequence $s^{o^s}$, define $w_1$ by replacing 0 in $s$ with the word 1001 and 1 in $s$ with 101. Conversely, if $w^{o^w}$ is a finite conjunction of words 1001 and 101, we let $s_w$ be the finite binary sequence $s$ such that $w_{s_w} = w$. This coding and decoding transfers directly to infinite binary sequences and infinite conjunctions of the words 1001 and 101. A sequence with information is any coded presentation $g = w_s0f$ of a pair $(s, f)$ where $s^{o^s}$ is a finite binary sequence and $f \in 2^\mathbb{N}$.

This notation is convenient when trying to define the set $X$ of binary sequences $s^{o^s}$ such that $(\exists f \in 2^\mathbb{N})\{Y(s, f) = 0\}$ for some fixed $Y^2$. Indeed, one point is that the coding as in Notation 2.2 preserves the lexicographic ordering of the sequences. Another point is that if $s_1$ is a strict subsequence of $s_2$, and $w_{s_1}0f_1$ and $w_{s_2}0f_2$ are two sequences with information, then $w_{s_1}0f_1 \prec_{\text{lex}} w_{s_2}0f_2$. In this way, the above versions of the Bolzano-Weierstrass are applied to sets of sequences with information in such a way that the information parts do not show up in the supremum.

### 2.2. Bolzano-Weierstrass theorem and (strongly) countable sets.

In this section, we study the RM of the Bolzano-Weierstrass theorem in the guise of $\text{BWC}_i^j$ from Section 1.2. In particular, we provide a positive answer to question (Q0) from the latter by establishing the equivalences in (EQ).

#### 2.2.1. Strongly countable sets.

We connect the Bolzano-Weierstrass theorem for strongly countable sets to $\text{cocode}_1$, which is $\text{cocode}_0$ restricted to strongly countable sets. We discuss the connection to hyperarithmetical analysis in Remark 2.8.

First of all, we need a little bit of the induction axiom, formulated as in $\text{IND}_1$ in Principle 2.3. The equivalence between induction and bounded comprehension is well-known in second-order RM ([103, X.4.4]).

**Principle 2.3** ($\text{IND}_1$). Let $Y^2$ satisfy $(\forall n \in \mathbb{N})(\exists f \in 2^\mathbb{N})\{Y(n, f) = 0\}$. Then $(\forall n \in \mathbb{N})(\exists w^{w^1})\{\forall w = n \land (\forall i < n)(Y(i, w(i)) = 0)\}$.

Note that $\text{IND}_1$ is a special case of the axiom of finite choice, and is valid in all models considered in [72–79]. Moreover, $\text{IND}_1$ is trivial in case $(\exists^2)$ since the condition on $Y$ is then false.

**Lemma 2.4.** The system $\text{ACA}_0^\omega$ proves $\text{cocode}_1 \rightarrow \text{IND}_1$.

**Proof.** To show that $\text{cocode}_1 \rightarrow \text{IND}_1$, assume $(\forall n \in \mathbb{N})(\exists f \in 2^\mathbb{N})A_0(n, f)$ where $A_0$ is quantifier-free. Let $(n) * f \in A$ if $A_0(n, f)$ and define $F(g) := g(0)$, i.e. $F((n) * f) = n$. Modulo coding, we may view $A$ as a subset of $2^{\mathbb{N}}$. By assumption, $F$ is a bijection from $A$ to $\mathbb{N}$, and by $\text{cocode}_1$, $A$ is enumerable as $\{g_1\}_{i \in \mathbb{N}}$. From this enumeration, we can (Turing) compute $n \mapsto f_n$ where $f_n$ is the unique $f$ with $A_0(n, f)$ for any $n \in \mathbb{N}$, and in particular an object as claimed to exist by $\text{IND}_1$. \[\square\]
Secondly, the following theorem completes most of the results in \([\text{EQ}]\) for \(\text{BWC}_1\).

**Theorem 2.5 (\(\text{ACA}_0^\omega\)).** \([\text{BWC}_1 + \text{IND}_1] \iff \text{BWC}_1^{\text{pwo}} \iff \text{cocode}_1\).

*Proof.* We have already established that \(\text{cocode}_1 \rightarrow \text{IND}_1\) in Lemma 2.4. Moreover, it is straightforward to prove both \(\text{BWC}_1^{\text{pwo}}\) and \(\text{BWC}_1\) from \(\text{cocode}_1\). We first prove that \(\text{BWC}_1^{\text{pwo}} \rightarrow \text{cocode}_1\) in \(\text{RCA}_0^\omega\). To this end, let \(F : 2^\omega \rightarrow \omega\) be a bijection on \(\omega \subseteq 2^\omega\). Define the set \(B \subseteq 2^\omega\) as follows: \(g \in B\) if the following items are satisfied:

- for all \(n, m, a, b \in \omega\), \(g(\langle n, a \rangle) = g(\langle m, b \rangle) = 1 \rightarrow n = m\),
- for a unique \(n_0 \in \omega\), \(g(\langle n_0, 0 \rangle) = 1\),
- for this \(n_0\), the function \(\lambda a.g(\langle n_0, a + 1 \rangle)\) is in \(A\) and maps to \(n_0\) under \(F\).

Clearly, \(B\) is strongly countable and \(\text{BWC}_1^{\text{pwo}}\) yields a pointwise least upper bound for \(B\). This is essentially the characteristic function of the disjoint union of the sets (with characteristic functions) in \(A\), and we can recover an enumeration of \(A\).

Next, we prove that \(\text{BWC}_1 \rightarrow \text{cocode}_1\), using \(\text{IND}_1\). Let \(F\) be bijective on \(A \subseteq 2^\omega\). We will construct a strongly countable set \(B\) such that \(g(\langle i, j \rangle) = F^{-1}(i)(j)\) is coded as the lexicographic supremum of \(B\). Let \(w0f \in B\) if \(f = g_0 \oplus g_1 \oplus \cdots \oplus g_{k-1}\) where \(k\) is the length of \(s_w\), where \(F(g_i) = i\) for \(i < k\), and where \(s_w(\langle i, j \rangle) = g_i(j)\) whenever \(\langle i, j \rangle < k\). We let \(G(w0f)\) be the length of \(s_w\). Then \(G\) is a bijection on \(B\). We need \(\text{IND}_1\) to establish the unique existence of \(g_0 \oplus g_1 \oplus \cdots \oplus g_{k-1}\) for each \(k\) for this otherwise trivial fact. The supremum of \(B\) in the lexicographic ordering now codes the enumeration of \(A\) via the inverse of \(F\) and the \(0 \leftrightarrow 1001\) and \(1 \leftrightarrow 101\) translation from Notation 2.2. 

Thirdly, by the following, \(\text{ACA}_0^\omega + \text{BWC}_1^{\text{pwo}}\) and \(\text{ACA}_0^\omega + \text{BWC}_1 + \text{IND}_1\) are connected to hyperarithmetical analysis. We discuss this connection in Remark 2.8.

**Corollary 2.6.** The system \(\text{ACA}_0^\omega + \text{BWC}_1^{\text{pwo}}\) proves weak-\(\Sigma_1^1\)-\(\text{AC}_0\); the former yields a conservative extension when added to \(\Sigma_1^1\)-\(\text{AC}_0\).

*Proof.* By [94 Theorem 3.17], \(\text{QF-AC}^{0,1} \rightarrow \text{cocode}_1 \rightarrow \text{QF-AC}^{0,1}\), where the final principle is the first principle with a uniqueness condition. Now, \(\text{ACA}_0^\omega + \text{QF-AC}^{0,1}\) is a conservative extension of \(\Sigma_1^1\)-\(\text{AC}_0\) by [10 Cor. 2.7], while \(\text{ACA}_0^\omega + \text{QF-AC}^{0,1}\) clearly proves weak-\(\Sigma_1^1\)-\(\text{AC}_0\). 

We note that the monotone convergence theorem for nets with strongly countable index set, called \(\text{MCT}^\text{rst}\) in [94], is equivalent to \(\text{cocode}_1\) over \(\text{RCA}_0^\omega\) by [94 Theorem 3.12]. Hence, this theorem has the same status as e.g. \(\text{BWC}_1 + \text{IND}_1\).

Finally, the previous results suggest a connection between \(\text{cocode}_1\) and hyperarithmetical analysis. A well-known system here is \(\text{\Delta}^1_1\)-comprehension (see [103 Table 4, p. 54]) and we now connect the latter to \(\text{cocode}_1\). To this end, let \(\Delta\text{-CA}_C\) be \(\Delta\text{-CA}\) restricted to formulas \(A_i(n) \equiv (\exists f \in 2^n)(Y_i(f, n) = 0)\) also satisfying \((\forall n \in \omega)(\exists at least one f \in 2^n)(Y_i(f, n) = 0)\) for \(i = 0, 1\). In this way, \(\Delta\text{-CA}_C\) is similar in role and form to \(\text{BOOT}_C\). We have the following surprising result.

**Theorem 2.7.** The system \(\text{ACA}_0^\omega\) proves that the following are equivalent:

(a) \(\text{cocode}_1\): any strongly countable set can be enumerated,
(b) For strongly countable \(A \subseteq [0, 1]\), any subset of \(A\) can be enumerated,
(c) \(\Delta\text{-CA}_C\): the axiom \(\Delta\text{-CA}\) with an ‘at most one’ condition for \(2^\omega\).
Proof. For the implication \((\exists f \in 2^N) (\forall n \in \mathbb{N}) (Y_0(f, n) = 0)] \rightarrow [\exists f \in 2^N : (\exists m \in \mathbb{N}) (Y_1(g, m) = 0)]\), let \(A \subset [0, 1]\) be strongly countable and use \(\text{cocode}_1\) to obtain a sequence listing all elements of \(A\). For \(B \subset A\), use \(\mu^2\) to remove all elements in \(A \setminus B\) from this sequence. For \([\exists f \in 2^N] \rightarrow [\exists h \in 2^N]\), fix \(Y^2_i\) for \(i = 0, 1\) as in \(\Delta - \text{CA}_C\) and define the following subsets of Cantor space:

\[ A := \{ f \in 2^N : (\exists n \in \mathbb{N}) (Y_0(f, n) = 0) \} \quad \text{and} \quad B := \{ g \in 2^N : (\exists m \in \mathbb{N}) (Y_1(g, m) = 0) \}. \]

Define \(Z, W : 2^N \rightarrow \mathbb{N}\) as \(Z(f) := (\mu n) (Y_0(f, n) = 0)\) and \(W(g) := (\mu m) (Y_1(g, m) = 0)\). By the assumption on \(Y_0\) (resp. \(Y_1\)), \(Z\) (resp. \(W\)) is injective on \(A\) (resp. \(B\)). Now let \(A \cup B\) be the disjoint union of \(A\) and \(B\) and define the following:

\[ V(h) := \begin{cases} Z(h(1) \ast h(2) \ast \ldots) & h(0) = 0 \land h(1) \ast h(2) \ast \ldots \in A \\ W(h(1) \ast h(2) \ast \ldots) & h(0) = 1 \land h(1) \ast h(2) \ast \ldots \in B \\ 0 & \text{otherwise} \end{cases} \tag{2.1} \]

Now, \(V : 2^N \rightarrow \mathbb{N}\) defined as in (2.1) is bijective on \(A \cup B\), which is readily verified via a tedious-but-straightforward case distinction. Hence, \(A \cup B\) is strongly countable and applying item \([\exists f_n \in \mathbb{N}]/(\exists n \in \mathbb{N})\) of \(A\). By the definition of \(A\), we have \((\exists f \in 2^N) (Y_0(f, n) = 0) \leftrightarrow (\exists m \in \mathbb{N}) (Y_0(f, m) = 0)\), for any \(n \in \mathbb{N}\). Now define \(X \subset \mathbb{N}\) as follows: \(n \in X \leftrightarrow (\exists m \in \mathbb{N}) (Y_0(f, m) = 0)\). This set is exactly as needed for \(\Delta - \text{CA}_C\), and we are done.

For the implication \(\Delta - \text{CA}_C \rightarrow \text{cocode}_1\), let \(Y : 2^N \rightarrow \mathbb{N}\) be bijective on \(A \subset 2^N\). Now consider, for any \(n, m \in \mathbb{N}\) and \(i = 0, 1\), the following:

\[(\exists g \in A)(g(m) = i \land Y(g) = n) \leftrightarrow (\forall f \in A)(f(m) \neq i \land Y(f) \neq n),\]

which follows by definition and satisfies the required ‘at most one’ conditions. Then \(\Delta - \text{CA}_C\) provides \(X \subset \mathbb{N}^3\) such that

\[(n, m, i) \in X \leftrightarrow (\exists g \in A)(g(m) = i \land Y(g) = n)\]

for any \(n, m \in \mathbb{N}\) and \(i = 0, 1\). The enumeration of \(A\) is given by \(f_n(m) = i\) for the unique \(i\) such that \((n, m, i) \in X\), and we are done.

The ‘at most one’ conditions in \(\Delta - \text{CA}_C\) may seem strange, but similar constructs exist in second-order RM: as discussed in [103, p. 181], a version of Suslin’s classical result that the Borel sets are exactly the \(\Delta_1^1\)-sets can be proved in \(\text{ATR}_0\). However, Borel sets in second-order RM are in fact given by \(\Delta_1^1\)-formulas that satisfy an ‘at most one’ condition, in light of [103, V.3.3-4].

We finish this section with a remark on hyperarithmetical analysis.

**Remark 2.8.** The notion of hyperarithmetical set ([103, VIII.3]) gives rise to the (second-order) definition of system/statement of hyperarithmetical analysis (see e.g. [62] for the exact definition), which includes systems like \(\Sigma_1^1 - \text{CA}_0\) (see [103, VII.6.1]). Montalbán claims in [62] that \(\text{INDEC}\), a special case of [103, IV.3.3], is the first ‘mathematical’ statement of hyperarithmetical analysis. The latter theorem by Julien can be found in [25, 6.3.4.(3)] and [84, Lemma 10.3].

The monographs [25, 45, 84] are all ‘rather logical’ in nature and \(\text{INDEC}\) is the restriction of a higher-order statement to countable linear orders in the sense of RM ([103, V.1.1]), i.e. such orders are given by sequences. In our opinion, the statements \(\text{MCT}_1^\text{ext}\) and \(\text{BWC}_1\) introduced above are (much) more natural than \(\text{INDEC}\) as they are obtained from theorems of mainstream mathematics by a (similar to the case

\[\text{The disjoint union } A \cup B \text{ can be defined as } \{(\langle n \rangle \ast f) \in 2^N : (n = 0 \land f \in A) \lor (n = 1 \land f \in B)\} \].
of INDEC) restriction, namely to strongly countable sets. Now consider, ACAω ω + X
where X is either MCT1,[1], cocode1, ∆-CA0, or BWC1 + IND1. By the above,
ACAω ω + X is a rather natural system in the range of hyperarithmetical analysis,
namely sitting between RCAω ω + weak-Σ1-CA0 and ACAω ω + QF-AC0.1 ≡L2 Σ1-CA0.

2.2.2. Countable sets. We study the Bolzano-Weierstrass for countable sets in its
various guises and connect it to cocode0.

Firstly, as perhaps expected in light of the use of IND1 above, we also need a
fragment of the induction axiom, as follows.

**Definition 2.9. [IND0]** Let Y2 satisfy (∀n ∈ N)(∃ at most one f ∈ 2N)((Y(f, n) = 0).
For k ∈ N, there is w′ with |w| = k such that for m ≤ k, we have:

\[(w(m) ∈ 2N ∧ Y(w(m), m) = 0) ↔ (∃f ∈ 2N)(Y(f, m) = 0).\]

Note that IND0 → IND1 by definition. The following theorem is a first approxi-
mation of the results in EQ.

**Theorem 2.10.** The system ACAω 0 proves BWC0pse ↔ cocode0.

**Proof.** The reverse implication is immediate as cocode0 converts A into a sequence.
Of course, (EQ) implies ACA0 and hence the second-order Bolzano-Weierstrass theorem
by [103, II.2]. For the forward implication, the construction in the proof of
Theorem 2.5 is readily adapted. □

Secondly, what remains to establish EQ is the following.

**Theorem 2.11.** The system ACAω 0 proves

\[\text{cocode0} ↔ \text{[BWC0 + IND0]} ↔ \text{range0} ↔ \text{BWC0pse}.\] (2.2)

**Proof.** The implication BWC0pse → [BWC0 + IND0] follows in the same way as for
BWC1pse → [BWC1 + IND1] in the proof of Theorem 2.5, i.e. via cocode0. To prove
[BWC0 + IND0] → range0, let F : 2N → N be injective on A ⊂ 2N. Define the set B
of sequences with information w0g such that w0g ∈ B if g is of the form g0 ⊕ ⋯ ⊕ gk−1,
where k is the length of sω, and such that F(gi) = i whenever sω(i) = 1. Then B
is clearly countable since A is countable. Using IND0 we see that for each k there
is a w̄k such that sω has length k and approximates the characteristic function of
the range of F. Using IND0 again, there is g = g0 ⊕ ⋯ ⊕ gk−1 such that w̄kg ∈ B.
This object is the lexicographicly largest object w′0g′ ∈ B such that the length of sω ≤ k.
It follows that sup B will approximate a coded representation of the
characteristic function of the range of F, and range0 follows.

To prove range0 → BWC0pse, let F : 2N → N be injective on A ⊂ 2N. Let nf ∈ B
if f ∈ A and f(n) = 1 and let G(nf) = ⟨n, F(f)⟩. Then G is injective on B, so
let X be the range of B under G. The pointwise least upper bound f of A is then
definable from X and ∃2 by f(n) = 1 ↔ (∃k)((⟨n, k⟩) ∈ X). □

Thirdly, we also obtain some nice equivalences for IND0, which can be proved as
well for IND1 and strongly countable sets. Note that the third item uses the ‘set
theoretic’ definition of finite sets of reals, also discussed in Section 3.3.3.

**Theorem 2.12 (ACAω ω + QF-AC0.1).** The following are equivalent.

- IND0.
- A countable and finite set can be enumerated (by a finite sequence).

\[\begin{align*}
\text{cocode0} & \iff \text{[BWC0 + IND0]} \iff \text{range0} \iff \text{BWC0pse}. \\
\end{align*}\]
• A set \( A \subset [0,1] \) with \( Y : [0,1] \to \mathbb{N} \) injective and bounded on \( A \), can be enumerated (by a finite sequence).

We only need QF-AC\(^{0,1} \) to obtain the second item.

**Proof.** The second item readily implies the third one. We now prove the second item from the first one. To this end, assume \( A,Y \) are as in the second item and suppose \( (\forall n \in \mathbb{N})(\exists x \in A)(Y(x) > n) \). Apply QF-AC\(^{0,1} \) and let \((x_n)_{n \in \mathbb{N}} \) be the resulting sequence. Define \( g : \mathbb{N} \to \mathbb{N} \) as follows:

\[
g(0) := 0 \quad \text{and} \quad g(n+1) := Y(x_{g(n)})
\]

for which we use the primitive recursion scheme in RCA\(^{0} \). Now note that \((x_{g(n)})_{n \in \mathbb{N}} \) is a sequence of distinct reals in \( A \), contradicting the assumption that it is finite (as in Definition 1.5). The previous contradiction implies that there is \( N \in \mathbb{N} \) such that \((\forall x \in A)(Y(x) \leq N) \). Since \( Y \) is injective on \( A \), we also have \((\forall n \leq N)(\exists \text{at most one } x \in A)(Y(x) = n) \). Now apply IND\(_0 \) to obtain the desired enumeration of \( A \). To prove IND\(_0 \) from the third item, let \( Y \) be as in the former and fix \( k \in \mathbb{N} \). Define the set \( A := \{ f \in 2^\mathbb{N} : (\exists n \leq k)(Y(f,n) = 0) \} \) and define \( Z(f) := (\mu n \leq k)(Y(f,n) = 0) \), if such there is, and 0 otherwise. Clearly, \( Z \) is injective and bounded (by \( k \)) on \( A \). Applying the third item, we can enumerate \( A \), yielding \( w^{1^*} \) as required by IND\(_0 \). \( \square \)

The previous theorem suggests that IND\(_0 \) (and even cocode\(_0 \)) cannot prove that a finite set is enumerable, due to the absence of an injection. However, finite sets (that come without any obvious injection) do occur 'in the wild', namely in the study of functions of bounded variation, as discussed in detail in Section 3.3.2.

Finally, we discuss equivalences for cocode\(_0 \) from other parts of mathematics.

**Remark 2.13 (Lifting results).** Firstly, consider the following algebra statement:

*any countable sub-field of \( \mathbb{R} \) is isomorphic to a sub-field of an algebraically closed countable field.*

The second-order version of the latter is equivalent to ACA\(_0 \) by [103, III.3.2]. Now, the centred statement with 'countable' removed everywhere is (equivalent to) ALCL from [93, §3.6.1]; it is shown in [93, Theorem 3.31] that

\[
\text{ACA}\(_{0}^{\omega} \) + \Delta-\text{CA} \text{ proves ALCL} \rightarrow \text{BOOT}. \tag{2.3}
\]

*without any essential modification* to the proof of [113, III.3.2], i.e. the proof of the latter is ‘lifted’ to the proof in (2.3) by ‘bumping up’ all the relevant types by one. Now let ALCL\(_0 \) be the above centred statement in italics with 'countable' interpreted as in Definition 1.4. One readily modifies the proof from (2.3) to yield:

\[
\text{ACA}\(_{0}^{\omega} \) + \Delta-\text{CA}^*_C \text{ proves ALCL}_0 \rightarrow \text{BOOT}^*_C, \tag{2.4}
\]

therewith yielding \([\text{ALCL}_0 + \text{cocode}_1] \leftrightarrow \text{cocode}_0 \) over RCA\(_{0}^{\omega} \) by Theorem 2.7. One can obtain similar results for the other proofs in [91, 93], and most likely for any second-order reversal involving countable algebra (and beyond).

**2.3. The Cantor-Bernstein theorem.** We connect BWC\(_{0}^{\text{pwo}} \) to the Cantor-Bernstein theorem for \( \mathbb{N} \) as studied in [94] and defined as in Principle 2.14. As it happens, this theorem is studied in second-order RM as [15, Problem 1] and was studied by Cantor already in 1878 in [10]. Our results provide an answer to (Q1).
**Principle 2.14 (CBN).** A countable set $A \subseteq \mathbb{R}$ is strongly countable if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of pairwise distinct reals such that $(\forall n \in \mathbb{N})(x_n \in A)$.

First of all, the equivalence $[\text{CBN} + \text{cocode}_1] \leftrightarrow \text{cocode}_0$ is proved in [79, Theorem 3.12]. We have the following corollary to Theorems 2.5 and 2.10 which provides an answer to (Q1), as $\text{BW}_0^{\text{pwo}}$ can be split further.

**Corollary 2.15.** The system $\text{ACA}_0^{\omega}$ proves $\text{BW}_0^{\text{pwo}} \leftrightarrow [\text{CBN} + \text{BW}_0^{\text{pwo}}]$.

**Proof.** Immediate form $[\text{CBN} + \text{cocode}_1] \leftrightarrow \text{cocode}_0$ and Theorems 2.5 and 2.10.

Secondly, we show that $\text{CBN}$ does not imply $\text{cocode}_1$, based on the proof of [79, Theorem 3.26]. This establishes that the statements inside the same square brackets in (2.5) are independent, even relative to $\mathbb{Z}_2^+$:

$$\text{cocode}_0 \leftrightarrow [\text{BW}_1^{\text{pwo}} + \text{CBN}] \leftrightarrow [\text{cocode}_1 + \text{CBN}] \leftrightarrow [\Delta^0_2 \text{-CA}_C + \text{CBN}].$$

Note that (2.5) follows from Corollary 2.15 while trivially $\text{cocode}_1 \to \text{NBI}$.

**Theorem 2.16.** The system $\mathbb{Z}_2^+ + \text{CBN} + \text{IND}_0$ cannot prove $\text{NBI}$.

**Proof.** The proof of [79, Theorem 3.28] discusses a model $Q^*$ of $\mathbb{Z}_2^+ + \neg \text{NBI}$, which implies that $\mathbb{Z}_2^*$ cannot prove $\text{NBI}$ (or $\text{cocode}_1$). This model is defined in [79, Definition 2.28] and its properties are based on [79, Lemma 2.16 and Theorem 2.17]. Here, we will explain the properties of $Q^*$ essential for the proof of our theorem, namely that this model satisfies $\text{CBN} + \text{IND}_0$. For the proofs of (most of) these properties, we refer to [79].

First of all, the construction of the model is based on Kleene-computability relative to the functionals $S^2_k$, where $S^2_k$ is the characteristic function of some complete $\Pi^1_k$-subset of $\mathbb{N}^\mathbb{N}$. Using the Löwenheim-Skolem theorem, we let $A \subseteq \mathbb{N}^\mathbb{N}$ be a countable set such that all $\Pi^1_k$-formulas are absolute for $A$ for all $k$. We let $(g_k)_{k \in \mathbb{N}}$ be an enumeration of $A$ and we let $A_k$ be the set of functions computable in $S^2_k$ and $\{g_0, \ldots, g_{k-1}\}$. The key properties are that $A_k \subseteq A_{k+1} \subseteq A$ and that $A_{k+1}$ contains an enumeration of $A_k$ for each $k$.

Secondly, we let $Q[1] = A$ be the elements of the model of pure type 1. The definition of $Q[2]$ is as follows: If $F : A \to \mathbb{N}$ we let $F \in Q[2]$ if there is a $k_0$ such that for all $k \geq k_0$, the restriction of $F$ to $A_k$ is partially computable in $S^2_k$ and $\{g_0, \ldots, g_{k-1}\}$. No uniformity is required. On top of this, we close $Q[1]$ and $Q[2]$ under Kleene computability hereditarily for each pure type. As proved in [79], this will not add new elements of type 1 or type 2 to the structure. Finally, we use a canonical extension to interpretations of all finite types. The resulting type structure, named $Q^*$ in [79], is a model of $\mathbb{Z}_2^*$ and satisfies our weak induction axioms $\text{IND}_0$ and $\text{IND}_1$. Indeed, the models are constructed as computational closures, implying that for any sequence $f_0, \ldots, f_n$ of elements in the model, the coded sequence $(f_0, \ldots, f_n)$ is also in the model, and the two induction axioms $\text{IND}_0$ and $\text{IND}_1$ readily follow.

Thirdly, having witnessed the construction of the model $Q^*$, we now show that it satisfies that all infinite subsets of $\mathbb{N}^\mathbb{N}$ are strongly countable. In particular, we have that $\text{CBN}$ holds in $Q^*$. To this end, fix some arbitrary $B \in Q[2]$ that is (the

---

The forward implication is trivial, assuming $3^\mathbb{N}$. For the reverse implication: if $A$ is countable, consider a set $B$ isomorphic to $\mathbb{N} \aleph A$. Apply $\text{CBN}$ to this set to show that it is strongly countable, and then $\text{cocode}_1$ to show that it is enumerable. Thus, $A$ is enumerable.
characteristic function of) an infinite subset of $A = \mathbb{Q}[1]$. We have established in the proof of \cite{[7]} Theorem 3.26 that $\mathbb{Q}[2]$ contains a bijection $\phi : \mathbb{Q}[1] \rightarrow \mathbb{N}$ with the extra property that $\phi_k$, the restriction of $\phi$ to $A_k$, is partially computable in $S_k^2$ and $g_0, \ldots, g_{k-1}$. We do not need the explicit construction of $\phi$; it suffices to split the argument for finding a bijection from $B$ to $\mathbb{N}$ in two cases, as follows.

- If $B \subseteq A_k$ for some $k$, then $B$ is enumerable in $A_{k'}$ for some $k' > k$ (property of the model $\mathbb{Q}^*$), and the inverse can be found directly.
- In the ‘otherwise’ case, we construct an increasing sequence of functionals $\psi_k : (B \cap A_k) \rightarrow \mathbb{N}$ as being equal to the restriction of $\phi$ to $B \cap A_k$ except at finitely many points; we use the finite set of exceptions to make $\psi := \lim_{k \to \infty} \psi_k$ surjective. Now, for infinitely many $k$ we have that $B \cap (A_{k+1} \setminus A_k) \neq \emptyset$. At each stage where this is the case, and where the range of $A_k \cap B$ under $\psi_k$ is a proper subset of the range of $A_k$ under $\phi$, we define $\psi_{k+1}$ as follows.
  - Choose one element $f$ in $B \cap (A_{k+1} \setminus A_k)$. Let $n$ be the least element in the range of $A_k$ under $\phi$ that is not in the range of $B \cap A_k$ under $\psi_k$, and define $\psi_{k+1}(f) := n$.
  - We let $\psi_{k+1}$ be equal to $\phi$ on the rest of $B \cap (A_{k+1} \setminus A_k)$, noticing that the injectivity of $\phi$ ensures that the value $n$ used above will not be in the range of $A_{k+1} \setminus A_k$ under $\phi$, so injectivity is preserved.

Since $B$ and $\phi$ are elements in $\mathbb{Q}[2]$ and $\psi$ differs from the restriction of $\phi$ to $B$ at only finitely many points in each $A_k$, it follows that $\psi \in \mathbb{Q}[2]$.

The previous case distinction finishes the proof.

We now list some other interesting properties of the model $\mathbb{Q}^*$ constructed above. If $A$ and $A_k$ are as in the construction, and $B \subset A$ is such that $B \cap A_k$ is finite for each $k$, then automatically $B \in \mathbb{Q}[2]$. Since each set $A_{k+1} \setminus A_k$ is dense in $\mathbb{N}^k$, this opens up numerous possibilities for counter-intuitive properties consistent with $\mathbb{Z}_2^\omega$. A few examples are as follows.

- There is a strongly countable set such that all enumerable subsets are finite.
- There is an infinite subset of $[0, 1]$ with no cluster-point.
- There is an infinite subset of $[0, 1]$ with one cluster-point 0, but with no sequence from the set converging to 0.

By the above, $\text{CB}\mathbb{N}$ is weaker than $\text{BWC}_0$ and we also conjecture that the former is ‘less explosive’ than the latter as follows.

**Conjecture 2.17.** The system $\Pi_1^1\text{-CA}_0^\omega + \text{CB}\mathbb{N}$ cannot prove $\Pi_1^1\text{-CA}_0$.

Proving the previous conjecture may be difficult, as Theorem 2.17 suggests that $\text{CB}\mathbb{N}$ is ‘very close’ to $\text{BWC}_0$ in explosive power.

Now, the Cantor-Bernstein theorem is a standard exercise in axiomatic set theory (see e.g. \cite{[38]} p. 69]). Experience bears out that when the students are asked to construct a bijection $H : A \rightarrow B$ from given injections $F : A \rightarrow B$ and $G : B \rightarrow A$, the successful solutions will all have the property that for each $a \in A$, either $H(a) = F(a)$ or $a = G(H(a))$. Let $H$ with this property be called a canonical witness to the Cantor-Bernstein theorem. Let $\text{CB}\mathbb{N}^+$ be $\text{CB}\mathbb{N}$ augmented with the existence of such a canonical witness. Such witnesses are assumed in \cite{[13]} Problem 1] as part of the study of the Cantor-Bernstein theorem in second-order RM.

**Theorem 2.18.** The system $\Pi_1^1\text{-CA}_0^\omega + \text{CB}\mathbb{N}^+$ proves $\Pi_1^1\text{-CA}_0$. 
Principle 2.20 is a sentence of second-order arithmetic that is interesting as the latter does not mention bijections or injections. In particular, open intervals in $\mathbb{R}$. The following principle

**Principle 2.20.** A non-enumerable and closed set in $\mathbb{R}$ has a limit point.

Proof. We prove $\text{CBN}^+ \rightarrow \text{BOOT}_{\mathcal{C}}$ and note that [79] Theorem 3.23] yield $\Pi^1_2\text{-CA}_0$ via $\Pi^1_2\text{-CA}_0$. Let $Y^2$ be such that $(\forall n \in \mathbb{N})(\exists$ at most one $f \in 2^\mathbb{N})(Y(f, n) = 0)$. Let $f_0$ be the constant zero function and define $A \subset \mathbb{N} \times 2^\mathbb{N}$ as follows:

$$(m, f) \in A \leftrightarrow (\exists n \in \mathbb{N})(m = 2n + 1 \land Y(n, f) = 0) \lor (m = 2n \land f = f_0).$$

Modulo coding, we can view $A$ as a subset of $2^\mathbb{N}$. Define $F : A \rightarrow \mathbb{N}$ and $G : \mathbb{N} \rightarrow A$ as follows: $F((k, f)) := k$ and let $G(n) := (2n, f_0)$. Both functions are injective, so let $H : A \rightarrow \mathbb{N}$ be a canonical witness as in $\text{CBN}^+$. Now consider the following:

$$(\exists f \in 2^\mathbb{N})(Y(f, n) = 0) \leftrightarrow [H(2(2n + 1), f_0) = 2(2n + 1)]. \tag{2.6}$$

To prove (2.6), assume the left-hand side of (2.6) for fixed $n \in \mathbb{N}$. Then there is $f \in 2^\mathbb{N}$ such that $2n + 1, f) \in A$. Since this $(2n + 1, f)$ is not in the range of $G$, we must have that $H(2n + 1, f) = F(2n + 1, f) = 2n + 1$ and $G(2n + 1) = (2(2n + 1), f_0)$. Since the case that $H(2(2n + 1), f_0) = 2n + 1$ (the inverse of $G$) violates that $H$ is injective, we must have that $H(2(2n + 1), f_0) = 2(2n + 1)$. Now assume the left-hand side of (2.6) is false for fixed $n \in \mathbb{N}$. Then there is no $f \in 2^\mathbb{N}$ such that $F(f) = 2n + 1$. Since there is an $m \in \mathbb{N}$ and a $g \in 2^\mathbb{N}$ such that $(m, g) \in A$ and $H((m, g) = 2n + 1$, we must have used the $G^{-1}$-part of $H$ and have that $m = 2(2n + 1)$ and $g = f_0$. This contradicts that $H(2(2n + 1), f_0) = 2(2n + 1)$. \hfill \Box

**Corollary 2.19.** The system $\text{ACA}^+_0$ proves $\text{BOOT}_{\mathcal{C}} \leftrightarrow \text{CBN}^+$. Proof. Immediate from the proof of the theorem, the above results, and the fact that $\text{BOOT}_{\mathcal{C}} \leftrightarrow \text{BW}^+_0$ over $\text{RCA}^+_0$ by [91] Theorem 3.12]. \hfill \Box

2.4. Theorems going back to Cantor. In this section, we establish [EQ2] from Section 1.2. In particular, we extend Theorem 2.11 via a number of equivalences involving basic theorems about the real line or limit points, all going back to Cantor one way or the other. While interesting in their own right, our results also provide (positive) answers to questions (Q2)-(Q3) from Section 1.2. On a conceptual note, the order type $\eta$ of $\mathbb{Q}$ appears throughout the second-order RM, but Cantor’s characterisation of $\eta$ as in clo’ below is quite explosive by Corollary 2.19.6

First of all, the perfect set theorem or the Cantor-Bendixson theorem (see [103] V and VI) for the RM-study) imply that a nonempty uncountable and closed set has a perfect subset, and therefore the original set has at least one limit point. We shall study the latter for closed sets as in Definition 1.2. We note that the modern notion of limit/accumulation point was first articulated by Cantor in [14] p. 98].

**Principle 2.20.** A non-enumerable and closed set in $\mathbb{R}$ has a limit point.

Theorem 2.30] shows that $\text{BW}^+_0$ is equivalent to a version of Principle 2.20 which is interesting as the latter does not mention bijections or injections. In particular, Principle 2.20 is a sentence of second-order arithmetic with one single modification, namely the use of Definition 1.2 rather than RM-closed sets.

Secondly, Cantor shows in [14] p. 161, Hilfsatz II] that a collection of disjoint open intervals in $\mathbb{R}$ is countable; this is the first instance of the well-known countable chain condition. The following principle $\text{ccc}$ expresses the former property without mentioning the words ‘injection’ or ‘bijection’.

6Let $\text{acc}_{\text{RM}}^+$ be Principle 2.20 formulated with RM-closed sets. Since $\text{ATR}_0$ implies the perfect set theorem [103] 1.11.5), we have the first implication in $\text{ATR}_0 \rightarrow \text{acc}_{\text{RM}}^+ \rightarrow \text{ACA}_0$, while the second one follows via the proof of [103] III.2.2]. We believe that the final implication reverses.
Principle 2.21 (ccc). Let \( A \subset \mathbb{R}^2 \) be such that for any non-identical intervals \((a,b)\) and \((c,d)\) in \( A \), the intersection is empty. Then \( A \) can be enumerated.

Let \( \text{ccc}^0 \) be \( \text{ccc} \) with the conclusion ‘\( A \) is countable’. As will become clear in the proof of Theorem 2.30 \( \text{ccc}^0 \) is provable in \( \text{RCA}_0^\omega \), akin to how Cantor’s theorem is provable in \( \text{RCA}_0 \) by [103, II.4.7].

Thirdly, the countable chain condition is found in the original version of Suslin’s hypothesis, first formulated by Suslin in [103]. In this context, Cantor contributed the following theorem (for any countable set), as discussed in [83, p. 122-123].

Principle 2.22 (cloq). A countable linear ordering \((X, \preceq_X)\) for \( X \subset \mathbb{R} \) is order-isomorphic to a subset of \( \mathbb{Q} \).

Moreover, Cantor introduces the notion of order type in [12] and characterises the order type \( \eta \) of \( \mathbb{Q} \) in [13] based on the following (for any countable set).

Principle 2.23 (cloq’). A countable and dense linear ordering without endpoints \((X, \preceq_X)\) for \( X \subset \mathbb{R} \) is order-isomorphic to \( \mathbb{Q} \).

We use the usual definition of linear ordering where ‘\( \preceq_X \)’ is given by a characteristic function \( F_X : \mathbb{R}^2 \rightarrow \mathbb{N} \), i.e. \( x \preceq_X y \equiv F_X(x,y) = 0 \), while \( (X, \preceq_X) \) is called countable if \( X \subset \mathbb{R} \) is. Similarly, an order-isomorphism from \( (X, \preceq_X) \) to \( (Y, \preceq_Y) \) is a surjective \( Y : X \rightarrow Y \) that respects the order relation (see [103, Def. V.2.7]), i.e.

\[
(\forall x, x' \in X)(x \preceq_X x' \iff Y(x) \preceq_Y Y(x')), \tag{2.7}
\]

while a well-founded linear order (well-order) has no strictly descending sequences. The reader should verify that using a stronger definition of order-isomorphism does not change the below equivalences.

As to well-orders, Simpson calls the comparability of countable well-orders ‘indispensable’ for a decent theory of ordinals, pioneered by Cantor in [11]. We agree that it would be very indecent indeed to have incomparable countable well-orders, suggesting the following principle, which is just the second-order CWO from [103, V.6] formulated for linear orders over \( \mathbb{R} \) that are countable.

Principle 2.24 (CWO*). For countable well-orders \((X, \preceq_X)\) and \((Y, \preceq_Y)\) where \( X, Y \subset \mathbb{R} \), the former order is order-isomorphic to the latter order or an initial segment of the latter order, or vice versa.

Thirdly, we present a preliminary result that got everything started.

Theorem 2.25 (ACA^\omega_0). Principle [2,30] implies the uncountability of \( \mathbb{R} \) as in NIN.

**Proof.** Let \( Y : [0,1] \rightarrow \mathbb{N} \) be an injection and use \( \exists^2 \) to define \( A \subset \mathbb{R} \) as follows:

\[
x \in A \iff (\exists n \in \mathbb{N})(n \leq x < n + 1 \land Y(x - n) = n). \tag{2.8}
\]

Intuitively, \( A \) is the set \( \{z + Y(z) : z \in [0,1]\} \), although the latter need not exist (as a set) in ACA^\omega_0. Each \( [m, m+1) \cap A \) has at most one element as \( x, y \in ([m, m+1) \cap A) \) implies \( Y(x-m) = Y(y-m) \) by (2.8) and hence \( x =_R y \) by the injectivity of \( Y \).

In this light, \( A \) does not have a limit point, while this set is trivially closed.

\[\text{Footnotes:} \quad ^8\text{Namely that the relation } \preceq_X \text{ is transitive, anti-symmetric, and connex, just like in [103, V.1.1].} \]

\[\text{Footnotes:} \quad ^9\text{Note that (2.7) implies that } (\forall x, x' \in X)(Y(x) =_X Y(x') \rightarrow x =_X x'), \text{ i.e. } Y \text{ is injective relative to the equalities } =_X \text{ and } =_Y, \text{ i.e. ‘surjective’ may be replaced by ‘bijective’}.
\]
Towards a contradiction, we now show that \( A \) is non-enumerable. Suppose
\((x_n)_{n \in \mathbb{N}}\) lists all elements of \( A \), i.e. \((\forall x \in A)(\exists n \in \mathbb{N})(x = x_n)\). Since we have
\( x \in A \leftrightarrow (Y(x - \lceil x \rceil) = \lfloor x \rfloor) \) for non-negative \( x \in \mathbb{R} \), the sequence \((x_n - \lceil x_n \rceil)_{n \in \mathbb{N}}\)
lists all elements of \([0, 1]\). Indeed, for \( y_0 \in [0, 1] \), \((y_0 + Y(y_0)) \in A\) by definition and
suppose \( x_{n_0} = y_0 + Y(y_0) \). Hence, \([x_{n_0}] = Y(y_0)\), and hence \( y_0 = x_{n_0} - \lceil x_{n_0} \rceil \). A
sequence listing the reals in \([0, 1]\) yields a contradiction by \([103, \text{II.4.7}]\). \( \square \)

As is often the case (see e.g. \([79, 92, 93]\)), the previous proof can be generalised to
yield \( \text{cocode}_0 \). As noted above, there is however a fundamental difference between
\( \text{NN} \) and \( \text{cocode}_0 \): the latter combined with \( \Pi^1_1-\text{CA}_0^\omega \) proves \( \Pi^1_1-\text{CA}_0 \), while the former
does not (see to go beyond \( \Pi^1_1-\text{CA}_0 \)).

**Corollary 2.26** (\( \text{ACA}_0^\omega \)). \( \Pi^1_1-\text{CA}_0^\omega \text{ implies } \text{cocode}_0 \).

**Proof.** Note that \( \text{cocode}_0 \) is trivial in case \( \neg (\exists^2) \), as all functions on \( \mathbb{R} \) are then
continuous by \([49, \text{§3}]\). Hence, for the rest of the proof, we may assume \( (\exists^2) \).

Let \( B \subset [0, 1] \) be a countable set, i.e. there exists \( Y : [0, 1] \to \mathbb{N} \) such that \( Y \) is
injective on \( B \). Similar to \( (2.8) \), we define the following set using \( \exists^2 \):
\[
x \in A \leftrightarrow (\exists n \in \mathbb{N})(n \leq x < n + 1 \land Y(x - n) = n \land (x - n) \in B).
\]
(2.9)

Since \( Y \) is an injection on \( B \), \((m, m + 1] \cap A \) has at most one element. Thus, \( A \) is a
close subset with no limit point. By the contraposition of Principle \( 2.20 \)
there is a sequence \((x_n)_{n \in \mathbb{N}}\) such that
\( x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n)\). Clearly, \((x_n - \lfloor x_n \rfloor)_{n \in \mathbb{N}}\)
similarly enumerates \( B \), and we are done.

The previous corollary is interesting as follows: let \( \text{PST} \) and \( \text{CBT} \) be the perfect
set theorem and the Cantor-Bendixson theorem formulated as in \([76]\), i.e. for closed
sets as in Definition \( 1.2 \) that are not enumerable. Note that \( \Pi^1_1-\text{CA}_0 \) proves these
theorems formulated for \( \text{RM}\)-closed sets (and in \( L_2 \)) by \([103, \text{V and VI}]\).

**Corollary 2.27.** The system \( \Pi^1_1-\text{CA}_0^\omega \) cannot prove \( \text{PST} \) or \( \text{CBT} \).

**Proof.** By Theorem \( 2.20 \) both \( \text{PST} \) and \( \text{CBT} \) imply Principle \( 2.20 \). If \( \Pi^1_1-\text{CA}_0^\omega \) could
prove e.g. \( \text{PST} \), we would obtain \( \Pi^1_2-\text{CA}_0 \) by \([76, \text{Theorem 4.22}]\). However, \( \Pi^1_1-\text{CA}_0^\omega \)
is \( \Pi^1_1 \)-conservative over \( \Pi^1_1-\text{CA}_0 \) by \([88, \text{Theorem 2.2}]\). \( \square \)

By the previous proof, \( \Pi^1_1-\text{CA}_0^\omega + \text{PST} \) proves \( \Pi^1_2-\text{CA}_0 \) (and the same for \( \text{CBT} \)),
i.e. Definition \( 1.2 \) makes these theorems quite explosive.

Unfortunately, we could not find a way to obtain the reversal of Corollary \( 2.26 \).
On the other hand, assuming Principle \( 2.20 \) in case \( A \subset \mathbb{R} \) is closed and has no
limit points, one readily defines \( G : \mathbb{R} \to \mathbb{N} \) (using \( \exists^2 \)) such that
\( (\forall x \in A)(\exists n \leq G(x))(B(x, \frac{1}{2^n}) \cap A = \{x\}) \),
(2.10)
where \( (2.10) \) expresses that \( G \) is a witnessing functional for ‘\( A \) has no limit points’.
In other words, Principle \( 2.20 \) ‘enriches itself’ with a witnessing functional \( G \), while
the set \( A \) from \( (2.9) \) has an almost trivial such witnessing functional (again using \( \exists^2 \)).
All this suggests the latter witnessing construct merits further study.

Fourth, to obtain the equivalences in Theorem \( 2.30 \), we seem to need the following
slight constructive enrichment of Principle \( 2.20 \) as provided by \( (2.10) \).

**Principle 2.28** (\text{accu}). For any closed \( A \subset \mathbb{R} \) and \( G : \mathbb{R} \to \mathbb{N} \) such that \( (2.10) \),
there is \((x_n)_{n \in \mathbb{N}} \) in \( A \) with \((\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n)) \).
We also study the following, apparently stronger, variation in Theorem 2.30

**Principle 2.29 (accu').** For any $A \subseteq \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{N}$ such that (2.10), there is $(x_n)_{n \in \mathbb{N}}$ in $A$ with $(\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n))$.

We note that Theorem 2.30 provides a positive answer to (Q3) from Section 1.2 as accu and ccc do not involve the notions ‘injection’ or ‘bijection’. Moreover, accu' $\leftrightarrow$ accu is a nice robustness result, showing that quantifying over all sub-sets of $\mathbb{R}$ (rather than just the closed ones) need not be problematic.

**Theorem 2.30.** The following are equivalent over ACA$_0^\omega$:

(a) cocode$_0$,

(b) BW$\rhd$ (Bolzano-Weierstrass),

(c) accu',

(d) accu,

(e) ccc.

**Proof.** The equivalence (a) $\leftrightarrow$ (b) can be found in Theorem 2.11. We note that $\frac{1}{1+x}$ defines an injection from $\mathbb{R}$ to $(0,1)$. Hence, using $\exists^2$, one readily extends cocode$_0$ to subsets $A \subseteq \mathbb{R}$.

The implication accu $\to$ cocode$_0$ is (essentially) proved in Corollary 2.20 as the functional $G$ as in (2.10) is readily defined in this case. To prove ccc $\to$ accu, let $A, G$ be as in the latter, i.e. satisfying (2.10). By the latter, we have that for $x, y \in A$, the intersection $B(x, \frac{1}{2^{G(x)+1}}) \cap B(y, \frac{1}{2^{G(y)+1}})$ is empty in case $x \neq y$. We shall now define the set consisting of $B(x, \frac{1}{2^{G(x)+1}})$ for $x \in A$. To this end, define $B \subseteq \mathbb{R}^2$ as:

$$(a, b) \in B \leftrightarrow \left[ \frac{a+b}{2} \in A \land a = R \frac{a+b}{2} - \frac{1}{2^{G(\frac{a+b}{2})+1}} \land b = R \frac{a+b}{2} + \frac{1}{2^{G(\frac{a+b}{2})+1}} \right]. \quad (2.11)$$

Applying ccc to the set $B$ to yield sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ with

$$(\forall a, b \in \mathbb{R})((a, b) \in B \leftrightarrow (\exists n \in \mathbb{N})(a_n = a \land b_n = b))$$

Then the sequence $(\frac{a_n+b_n}{2})_{n \in \mathbb{N}}$ enumerates $A$, and this implication is done.

For cocode$_0$ $\to$ ccc, we first prove cccc$_0$ in ACA$_0^\omega$. To the latter end, let $A \subseteq \mathbb{R}^2$ be as in cccc$_0$ and fix some enumeration $(q_n)_{n \in \mathbb{N}}$ of $\mathbb{Q}$. Define $Y((a, b))$ as the least $n \in \mathbb{N}$ such that $q_n \in (a, b)$ if such there is, and 0 otherwise. Clearly, $Y$ is injective on $A$ and the latter is countable, i.e. cccc$_0$ follows inside ACA$_0^\omega$. Clearly, the combination cocode$_0 + ccc_0$ implies ccc. Thus, cocode$_0$ $\to$ ccc over ACA$_0^\omega$ follows.

Finally, the reverse implication in accu $\leftrightarrow$ accu' is trivial. Now fix $A \subseteq \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{N}$ satisfying (2.10). Then (2.11) yields a collection of open disjoint intervals in $\mathbb{R}$. Since accu $\to$ ccc, this collection can be enumerated, yielding accu'.

We shall obtain an equivalence between Principle 2.20 and cocode$_0$ over an elegant base theory in Section 3.3.2.

Fifth, the theorem has some interesting corollaries as follows. Let accu$_0$ be accu with the consequent weakened to stating that $A$ is countable. In contrast to BW$\rhd$, the former principles for countable sets are weak, as follows.

**Corollary 2.31.** The system ACA$_0^\omega$ proves accu$_0$ and ccc$_0$.

**Proof.** Note that cccc$_0$ was proved in ACA$_0^\omega$ in the proof of the theorem. To prove accu$_0$, note that (2.10) yields an injection to $\mathbb{Q}$. □
Let accu$_1$ be the restriction of accu to infinite sets $A \subset \mathbb{R}$ and with conclusion weakened to: there is a bijection from $A$ to $\mathbb{N}$. Similarly, let ccc$_1$ be ccc with the weaker conclusion ‘$A$ is strongly countable’ for infinite $A \subset \mathbb{R}$.

**Corollary 2.32.** The system ACA$_0^\omega$ proves cocode$_0 \leftrightarrow [\text{accu}_1 + \text{cocode}_1]$ and ccc$_1 \leftrightarrow \text{CBN} \leftrightarrow \text{accu}_1$.

**Proof.** For the second part, RCA$_0^\omega$ proves ccc$_0$ by the proof of the theorem. Applying CBN to the conclusion of the former for infinite sets, one obtains ccc$_1$. The proof of ccc$_1 \rightarrow \text{accu}_1$ follows from the proof of ccc $\rightarrow$ accu. Finally, the proof of Corollary 2.20 is readily adapted to accu$_1 \rightarrow \text{CBN}$. For the first part, we note that accu$_1$ only deals with infinite sets. To obtain the same results for finite sets $A \subset \mathbb{R}$, consider the infinite set $B := A \cup \mathbb{Q}$ and note that $\mu^2$ can enumerate all elements in $A \cap \mathbb{Q}$. Given an enumeration of $B$, one similarly obtains an enumeration of $A$. \(\square\)

The first equivalence is interesting as the left-hand side (only) deals with injections, while the right-hand side (only) deals with bijections. Similarly, we have BWC$_0 \leftrightarrow [\text{ccc}_1 + \text{BW}]$ by Corollary 2.15. Thus, we have provided an answer to question (Q2) from Section 1.2. Next, we consider CWO$^\omega$ as follows.

**Theorem 2.33.** The system ACA$_0^\omega$ proves cocode$_0 \leftrightarrow [\text{CWO}^\omega + \text{IND}_0]$.

**Proof.** For cocode$_0 \rightarrow$ CWO$^\omega$, use the proof that ATR$_0 \rightarrow$ CWO over RCA$_0$ from [103, V.6.8]. Note that ACA$_0^\omega + \text{BOOT}_C$ proves ATR$_0$ by [103, V.5.2]. Recall that cocode$_0 \rightarrow$ IND$_0$ is proved in Theorem 2.4

For $[\text{CWO}^\omega + \text{IND}_0] \rightarrow$ cocode$_0$, let $Y : \mathbb{R} \rightarrow \mathbb{N}$ be injective on $A \subset [0, 1]$. In case $(\exists m \in \mathbb{N})(\forall x \in A)(Y(x) \leq m)$, IND$_0$ provides an enumeration of $A$ as we have $(\forall n \in \mathbb{N})(\exists \text{at most one } x \in [0, 1])(x \in A \land Y(x) = n)$.

Hence, we may assume $(\forall m \in \mathbb{N})(\exists x \in A)(Y(x) \geq m)$. Now define the linear order $(A, \preceq_A)$ via the following formula:

$$x \preceq_A y \equiv [Y(y) = n_0 \lor [Y(x) \neq n_0 \land Y(x) \leq n \land Y(y)]],$$

where $n_0 \in \mathbb{N}$ is the least $n \in \mathbb{N}$ such that $(\exists x \in A)(Y(x) = n)$; this number is readily defined using IND$_0$. Let $y_0 \in A$ be such that $Y(y_0) = n_0$. Intuitively, $(A, \preceq_A)$ has order type $\omega + 1$, i.e. the order of $\mathbb{N}$ followed by one element. Hence, of the four different possibilities provided by the consequent of CWO$^\omega$, three lead to contradiction. Indeed, a finite initial segment of either $(\mathbb{N}, \leq \mathbb{N})$ or $(A, \preceq_A)$ has only got finitely many elements (since $Y$ is an injection), while $\mathbb{N}$ is infinite and $A$ satisfies $(\forall m \in \mathbb{N})(\exists x \in A)(Y(x) \geq m)$. Similarly, an order-isomorphism $W : A \rightarrow \mathbb{N}$ leads to contradiction as follows: since there is $y_0 \in A$ such that $Y(y_0) = n_0$, there cannot be a bijection from $A \setminus \{y_0\}$ to $\{0, 1, \ldots, W(y_0)\}$, as the latter set is finite, while the former is not. Similarly, an order-isomorphism $Z : \mathbb{N} \rightarrow A$ yields a contradiction as any $n \geq n_0$ is mapped below $Z(n_0) \in A$ (relative to $\preceq_A$), which is not possible as $Y$ is an injection. The only remaining possibility is that CWO$^\omega$ provides an order-isomorphism $Z : \mathbb{N} \rightarrow A \setminus \{y_0\}$, where $A \setminus \{y_0\} = \{y \in A : y \prec y_0\}$ is an initial segment of $A$. The morphism $Z$ is then a sequence satisfying $(\forall x \in A \setminus \{y_0\})(\exists n \in \mathbb{N})(Z(n) =_R x)$, i.e. we obtain an enumeration of $A$. \(\square\)

**Theorem 2.34.** The system ACA$_0^\omega$ proves cocode$_0 \leftrightarrow [\text{cloq} + \text{IND}_0]$.
Proof: To prove \( \text{cocode}_0 \to \text{cloq} \), use the well-known ‘back-and-forth’ proof based on the enumeration of \( A \) (see [33, p. 123]). By Theorem 2.11, we only need to prove \( \text{cloq} \to \text{range}_0 \) in \( \text{ACA}_0^\omega \). To this end, fix \( A \subseteq [0,1] \) and let \( Y: [0,1] \to \mathbb{N} \) be countable in \( A \). Wlog we may assume that \( 0, 1 \notin A \). Now define the set \( R \subseteq \mathbb{R} \) as follows: \( y \in R \) if and only if either \( (\exists n \in \mathbb{N})(y =_R n) \), or the following holds

\[
(\exists q \in \mathbb{Q})(|y - q| < 1) \wedge (\forall m \in \mathbb{N})(m < |y| < m + 1 \rightarrow Y(|y - q|) = m).
\]

Clearly, the set \( R \) is countable and \( (R, \leq_R) \) is a linear order. Apply \text{cloq} to obtain \( Q \subseteq \mathbb{Q} \) and \( Z: R \to Q \) such that \( Z \) is an order-isomorphism from \( (R, \leq_R) \) to \( (Q, \leq_Q) \). Now consider the following formula where \( n \in \mathbb{N} \):

\[
(\exists x \in A)(Y(x) = n) \leftrightarrow (\exists y \in (n, n + 1))(y \in R) \leftrightarrow (\exists q \in Q)(Z(n) <_Q q <_Q Z(n + 1)).
\]

(2.12)

The first equivalence holds by the definition of \( R \), while the second equivalence follows from the fact that \( Z \) is an order-isomorphism. Since (2.12) is decidable given \( (\exists^?) \), \( \text{range}_0 \) is now immediate. □

Inspired by the previous proof, a version of Hausdorff’s decomposition theorem for countable linear orders (see [16, Theorem 12] for the second-order RM version) should imply \( \text{cocode}_0 \). In turn, the previous proof inspires the following corollary.

Corollary 2.35. The system \( \text{ACA}_0^\omega \) proves \( \text{cocode}_0 \leftrightarrow [\text{cloq}' + \text{IND}_0] \).

Proof. To prove \( \text{cocode}_0 \to \text{cloq}' \), use the well-known ‘back-and-forth’ proof based on the enumeration of \( A \) (see [33, p. 123]). To prove \( \text{cloq}' \to \text{range}_0 \), fix \( A \subseteq [0,1] \) and let \( Y: [0,1] \to \mathbb{N} \) be countable in \( A \). Wlog we may assume that \( A \cap \mathbb{Q} = \emptyset \) as Feferman’s \( \mu^2 \) allows us to list the rationals in \( A \). Now define the set \( R' \subseteq \mathbb{R} \) as follows: \( y \in R' \) if and only if we have either \( (\exists q \in \mathbb{Q})(y =_R q) \), or the following holds

\[
(\exists q \in \mathbb{Q})(|y - q| < 1) \wedge (\forall m \in \mathbb{N})(m < |y| < m + 1 \rightarrow Y(|y - q|) = m).
\]

Clearly, the set \( R' \) is countable and \( (R', \leq_R) \) is a dense linear order without end points. Apply \text{cloq}' to obtain an order-isomorphism \( Z \) from \( (R', \leq_R) \) to \( (Q, \leq_Q) \). Now consider the following formula where \( n \in \mathbb{N} \):

\[
(\exists x \in A)(Y(x) = n) \leftrightarrow (\exists y \in (n, n + 1))(y \in R' \wedge y \text{ is irrational}) \leftrightarrow (\exists q \in Q \cap (Z(n), Z(n + 1))) \forall r \in Q \cap (n, n + 1) \exists r (Z(r) \neq q).
\]

(2.13)

The first equivalence holds by definition while the second equivalence follows from the fact that \( Z \) is an order-isomorphism. As for the theorem, \( \text{range}_0 \) follows. □

Restricting \text{cloq}' to strongly countable sets, one readily obtains an equivalence to \( \text{cocode}_1 + \text{IND}_1 \) by introducing an extra condition ‘\( x \rangle \) in (2.13) with \( p \in \mathbb{Q} \).

Finally, as to related research, Mal’tsev’s theorem on countable ordered groups (54) is studied in second-order RM (103), and seems to imply \( \text{cocode}_0 \).

3. The bigger picture

Section 2 yields many (robust) equivalences for the Bolzano-Weierstrass theorem as in \( \text{BW}_0 \) and \( \text{BWC}_1 \). With these in place, it is time to connect the latter to the bigger picture, namely ordinary mathematics and set theory, as follows.

• In Section 3.1 we connect the Bolzano-Weierstrass theorem as in \( \text{BWC}_0 \) to the Heine-Borel theorem and the Lindelöf lemma as studied in [74,75].
We connect \( \text{BWC}_0 \) to the *countable union theorem* from set theory (Section 3.2); a natural restriction of the latter is equivalent to the former.

In Section 3.3 we show that \( \text{BWC}_0 \) is equivalent to the *Jordan decomposition theorem* and similar results on functions of *bounded variation*. We also consider theorems on *regulated* functions.

In Section 3.4, we show that \( \text{BWC}_0 \) is equivalent to basic properties of *unordered sums*, which are a device for bestowing meaning upon sums over uncountable index sets.

Regarding the final item, the Jordan decomposition theorem and its ilk have no obvious or direct connection to countability at all, and have been studied in second-order RM (\([53, 71]\)).

### 3.1. Heine-Borel and Lindelöf.

#### 3.1.1. Introduction.

In this section, we connect the Bolzano-Weierstrass theorem as in \( \text{BWC}_0 \) to the Heine-Borel theorem and the Lindelöf lemma. An overview of our results is as follows.

In Section 3.1.2, we identify weak/countable versions of the Heine-Borel theorem and Lindelöf lemma that are equivalent to \( \text{BW}_0 \). In Section 3.1.3, we show that \( \text{LIN} \), a most general version of the Lindelöf lemma for \( \mathbb{N} \), we have \( \text{BOOT} + \text{QF-AC}_0^1 \rightarrow \text{LIN} \rightarrow \text{BWC}_0 \), working over \( \text{ACA}_0^\omega \). In Section 3.1.4, assuming a fragment of the induction axiom, we similarly establish:

\[
\text{BOOT} \rightarrow \text{HBT} \rightarrow \text{BWC}_0 \rightarrow \text{BWC}_1.
\]  

Recall that \( \text{BOOT} \) and \( \text{HBT} \) are the higher-order counterparts of \( \text{ACA}_0 \) and \( \text{WKL}_0 \) (see Remark 1.12). In this light, higher-order RM yields a much richer picture than its second-order counterpart, in that there are at least two extra ‘Big’ systems.

Next, the following series of implications is also established in Section 3.1.4 without the use of extra induction:

\[
\text{BOOT} \rightarrow \Sigma^0 \text{-SEP} \rightarrow \text{BWC}_0 \rightarrow \text{BWC}_1,
\]  

where \( \Sigma^0 \text{-SEP} \) is the higher-order counterpart of \( \Sigma^1_1 \)-separation. The latter is equivalent to \( \text{WKL}_0 \) by \( [103, \text{IV.4.4}] \) and ECF maps \( \Sigma^0 \text{-SEP} \) to \( \Sigma^0 \text{-SEP} \); we believe that \( \text{HBU} \) ‘speaks more to the imagination’ than \( \Sigma^0 \text{-SEP} \). Moreover, \( \text{HBU} \leftrightarrow \text{HBT} \leftrightarrow \Sigma^0 \text{-SEP} \) is established in Section 3.1.4 assuming extra axioms discussed next.

Finally, we should say a few words on the *neighbourhood function principle* \( \text{NFP} \) from \( [112, \text{p. 215}] \). Restricted to the \( \mathcal{L}_2 \)-language, \( \text{NFP} \) is equivalent to the usual comprehension principle of \( \mathcal{Z}_2 \). Now, the higher-order generalisation of comprehension, in the form of the functionals \( \mathcal{S}^2_\lambda \), does not provide a satisfactory classification of e.g. \( \text{HBU} \). Indeed, we know that \( \mathcal{Z}^2_2 \) proves \( \text{BOOT}, \text{HBU}, \text{BWC}_0, \text{BWC}_1 \) and \( \mathcal{Z}^0_2 \) does not, while of course \( \mathcal{Z}^\omega_2 \equiv_{\mathcal{L}_2} \mathcal{Z}^{\omega}_2 \equiv_{\mathcal{L}_2} \mathcal{Z}^{\omega}_2 \). As explored in \( [79, 89, 90] \), the higher-order generalisation of \( \text{NFP} \) provides a more satisfying classification of these principles: there are natural fragments of \( \text{NFP} \) equivalent to \( \text{BOOT}, \text{HBU}, \text{BWC}_0 \), and the Lindelöf lemma, *assuming a fragment of \( \text{NFP} \) called \( \mathcal{A}_0 \), discussed in Section 3.1.3* By Theorem 3.10 and Corollary 3.11 we have \( \text{HBU} \leftrightarrow \Sigma^0 \text{-SEP} \leftrightarrow \text{HBT} \) working over \( \text{ACA}_0^\omega \) + \( \mathcal{A}_0 \).

We should not have to point out that second-order RM assumes/needs \( \Delta^0_1 \)-comprehension in the base theory. Thus, it stands to reason that the development of RM based on \( \text{NFP} \) requires a fragment of the latter, like the \( \mathcal{A}_0 \) axiom, in the base
theory. This argument is explored at length in [89,80]. Moreover, by Corollary 3.12
there is even a fragment of NFP, similar to $A_0$, that is equivalent to BWC$_0$.

3.1.2. Countable coverings. We connect BWC$_0$ to versions of the Heine-Borel theorem
and Lindelöf lemma for coverings that are countable as in Definition 1.1.

First of all, the following version of the ‘countable’ Heine-Borel theorem implies
NIN by [79, Cor. 3.20], but no reversal is known.

Principle 3.1 (HBC$_0$). For countable $A \subset \mathbb{R}^2$ with $(\forall x \in [0,1]) (\exists (a, b) \in A)(x \in (a,b))$, there are $(a_0, b_0), \ldots (a_k, b_k) \in A$ with $(\forall x \in [0,1]) (\exists i \leq k) (x \in (a_i, b_i))$.

The Heine-Borel theorem for different representations of open coverings is studied
in RM (103), i.e. the motivation for HBC$_0$ is already present in second-order RM.
Moreover, Borel in [6, p. 42] uses ‘countable infinity of intervals’ and not ‘sequence
of intervals’ in his formulation of the Heine-Borel theorem. He also mentions in
[6, p. 42, Footnote (1)] a ‘theoretical method’ for ‘effectively determining’ the finite
sub-covering at hand. In this light, we may assume that the finite sub-covering in
HBC$_0$ is given by a finite sequence of reals without fear of adding ‘extra data’.

We shall study a ‘sequential’ version of HBC$_0$ involving sequences of (sub-)coverings.
Such sequential theorems are well-studied in RM, starting with [103, IV.2.12], and
also in [21, 22, 29, 30, 33, 37, 116].

Principle 3.2 (HBC$_0^{\text{seq}}$). Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets in $\mathbb{R}^2$ with countable
union. Then there is $(b_n)_{n \in \mathbb{N}}$ such that for $n \in \mathbb{N}$, $b_n$ is a finite sequence of elements
of $A_n$ and if the intervals in $A_n$ cover $[0,1]$, then so do the intervals in $b_n$.

On a related note, let LIN$_0$ be the Lindelöf lemma for countable sets $A \subset \mathbb{R}$, i.e.
for $\Psi : \mathbb{R} \to \mathbb{R}^+$, there is $(x_n)_{n \in \mathbb{N}}$ in $A$ such that $\bigcup_{n \in \mathbb{N}} B(x_n, \Psi(x_n))$ covers $A$. We
have the following theorem connecting the aforementioned principles.

Theorem 3.3. The system ACA$_0^\omega$ proves LIN$_0 \leftrightarrow$ cocode$_0 \leftrightarrow$ HBC$_0^{\text{seq}}$.

Proof. The implication LIN$_0 \leftrightarrow$ cocode$_0$ is trivial, while the reversal follows from
LIN$_0 \to$ accu which in turn follows from applying LIN$_0$ to the covering provided by
\begin{equation}
\begin{cases}
(a, b) \in A_n \leftrightarrow \frac{a+b}{2} \in A \land Y(\frac{a+b}{2}) = n \land b - a = 4 \max \{|1 - \frac{a+b}{2}|, \frac{a+b}{2}\}.
\end{cases}
\end{equation}

By definition, each $A_n$ has at most one element and the union is countable as
$\bigcup_{n \in \mathbb{N}} A_n$ is a variation of $A$. Let $(b_n)_{n \in \mathbb{N}}$ be as provided by HBC$_0^{\text{seq}}$ and note that

\begin{equation}
(3x \in A)(Y(x) = n) \leftrightarrow (3(a, b) \in b_n)(Y(\frac{a+b}{2}) = n),
\end{equation}

which immediately yields cocode$_0$, and we are done. \hfill \square

\textsuperscript{10}In fact, Borel’s explicitly mentions a version of cocode$_1$ in [89, p. 6] while the proof of the
Heine-Borel theorem in [6, p. 42] starts with an application of cocode$_1$ and then proceeds with
the usual ‘interval-halving’ proof, similar to Cousin’s proof in [13].
Note that the formulation of HBC" avoids the countable union theorem, which happens to be the topic of Section 3.2. Theorem 3.3 also has a certain robustness: the second equivalence still goes through if we let \((b_n)_{n\in\mathbb{N}}\) be a sequence of non-empty finite sets, while assuming cocode. Moreover, we believe that many sequential versions of theorems are equivalent to cocode, like e.g. ADS and RT\(^2\) from the RM zoo (see 55). An exception is cloq', as shown in Section 3.2.

Finally, by Theorem 3.3 the general Lindelöf lemma for any set \(A \subseteq \mathbb{R}\) is quite explosive, yielding \(\Pi^0_1\)-BOOT. Regarding (3.1), weakening 3.1.3. A general Lindelöf lemma. We show that a most general formulation of the Lindelöf lemma still follows from BOOT. We have established a similar result for the Heine-Borel theorem for uncountable coverings of closed sets in [76, Theorem 4.5]. We note that Lindelöf proves his eponymous lemma for any set in \(\mathbb{R}\) in [57].

**Principle 3.4 (LIN).** For any \(G^2\) and \(D \subseteq \mathbb{N}^\mathbb{N}\), there is \((f_n)_{n\in\mathbb{N}}\) in \(D\) such that \(\bigcup_{n\in\mathbb{N}}[f_nG(f_n)]\) covers \(D\).

The following theorem is the main result of this section.

**Theorem 3.5.** The system \(\mathbb{ACA}_0^\omega + \mathbb{QF-AC}^{0,1} + \mathbb{BOOT}\) proves LIN.

**Proof.** Fix a non-empty set \(D \subseteq \mathbb{N}^\mathbb{N}\) and \(G^2\) and let \((\sigma_n)_{n\in\mathbb{N}}\) be a list of all finite sequences. Use BOOT to define \(X \subseteq \mathbb{N}\) such that

\[ n \in X \iff (\exists f \in D)(f \in [\sigma_n] \land \sigma_n = 0, G(f)). \quad (3.3) \]

Define \(\tau_n\) as \(\sigma_n\), where \(\tau_0 \overset{\text{def}}{=} (\mu n)(n \in X)\), and define \(\tau_{n+1}\) as \(\sigma_{n+1}\) if \(n+1 \in X\), and \(\tau_n\) otherwise. Then \([\tau_n]\) also covers \(D\), but we still need to ‘identify’ the associated \(f \in D\) from (3.3). To this end, apply \(\mathbb{QF-AC}^{0,1}\) to

\[ (\forall n \in X)(\exists f \in D)(f \in [\sigma_n] \land \sigma_n = 0, G(f)). \]

The resulting sequence provides the countable sub-covering as required by the conclusion of Principle 3.4. \(\square\)

As shown in [74,77], the Lindelöf lemma for the full Baire space yields \(\Pi^1_1\)-CA\(_0\) when combined with (3.2). Moreover, by Theorem 3.3 LIN → cocode\(_0\) is immediate, implying that that \(\Pi^1_1\)-CA\(_0\) + LIN proves \(\Pi^1_1\)-CA\(_0\). We also have

\[ [\mathbb{BOOT} + \mathbb{QF-AC}^{0,1}] \rightarrow \mathbb{LIN} \rightarrow \mathbb{BWC}_0 \rightarrow \mathbb{BWC}_1. \]

Finally, since Baire space is not \(\sigma\)-compact, we believe the use of countable choice in the previous proof to be essential.

3.1.4. Uncountable coverings. In this section, we connect HBT and related principles to \(\mathbb{BWC}_0\) as sketched in Section 3.1.3.

First of all, in more detail, our main result is HBU → \(\mathbb{BWC}_0\) assuming an extra axiom \(A_0\) introduced in [89,90] and discussed below. This implication is established using the intermediate principle \(\Sigma\)-SEP as in Principle 3.6. The latter is the third-order counterpart of the \(\Sigma^0_1\)-separation principle, which is equivalent to WKL\(_0\) by 103, IV.4.4]. Since HBU is the higher-order counterpart of WKL\(_0\), one expects HBU ↔ \(\Sigma\)-SEP, which is indeed proved in Theorem 3.10 also assuming \(A_0\). Regarding 3.1, weakening \(A_0\) is possible as in (3.12). We note that ECF maps both
HBU and $\Sigma$-SEP to WK$L_0$, while $A_0$, $BWC_0$, $BWC_1$ are trivial under ECF. Moreover, a version of $A_0$ turns out to be equivalent to cocode$_0$ by Corollary 3.12.

Secondly, we have previously considered a separation principle in connection to HBU in [89], namely as follows.

**Principle 3.6 ($\Sigma$-SEP).** For $i = 0, 1$, $Y^i_0$, and $\varphi_i(n) \equiv (\exists f_i \in N^n)(Y_i(f_i, n) = 0)$, $$(\forall n \in N)((\neg \varphi_0(n) \lor \neg \varphi_1(n)) \rightarrow (\exists Z \subset N)(\forall n \in N)[\varphi_0(n) \rightarrow n \in Z \land \varphi_1(n) \rightarrow n \notin Z].$$

The following theorem implies that $\Pi^1_1$-CA$_0^\omega + \Sigma$-SEP proves $\Pi^1_2$-$CA_0$, which also follows immediately from [103, VII.6.14].

**Theorem 3.7.** The system $ACA^\omega_0$ proves $\Sigma$-SEP $\rightarrow$ cocode$_0$.

*Proof.* Let $Y : R \rightarrow N$ be injective on the non-empty set $A \subset [0, 1]$. Define the formula $\varphi_i(n, q)$ as follows where $n \in N$ and $q \in Q \cap (0, 1)$:

$$\varphi_0(n, q) \equiv (\exists x \in A)(Y(x) = n \land x >_R q) \quad (3.4)$$

$$\varphi_1(n, q) \equiv (\exists x \in A)(Y(x) = n \land x \leq_R q). \quad (3.5)$$

Since $Y$ is injective on $A$, we have $(\forall n \in N, q \in Q \cap (0, 1))(\neg \varphi_0(n, q) \lor \neg \varphi_1(n, q)).$

Let $Z \subset N \times Q$ be as in $\Sigma$-SEP and note that for $n \in N, q \in Q \cap (0, 1)$, we have

$$(n, q) \in Z \rightarrow (\forall x \in A)(Y(x) = n \rightarrow x >_R q), \quad (3.6)$$

$$(n, q) \notin Z \rightarrow (\forall x \in A)(Y(x) = n \rightarrow x \leq_R q). \quad (3.7)$$

Based on (3.6) and (3.7), define a sequence $(x_n)_{n \in N}$ of reals in $[0, 1]$ as follows:

$$(n, q)(0) = \frac{1}{2} \text{ if } (n, \frac{1}{2}) \in Z, \text{ and } 0 \text{ otherwise; } [x_n](k+1) = [x_n](k) + \frac{1}{2^k} \text{ if } (n, [x_n](k) + \frac{1}{2^k}) \in Z, \text{ and } [x_n](k) \text{ otherwise. Using Feferman's } \mu \text{, define } (y_n)_{n \in N} \text{ as a sub-sequence (possibly with repetitions) of } (x_n)_{n \in N} \text{ such that } (\forall n \in N)(y_n \in A).$$

Thus, $(y_n)_{n \in N}$ is an enumeration of $A$ such that for all $k \in N$:

$$(\exists x \in A)(Y(x) = k) \leftrightarrow (\exists m \in N)(Y(y_m) = k). \quad (3.8)$$

Indeed, the reverse implication in (3.8) is immediate by the definition of $(y_n)_{n \in N}$.

For the forward implication if $(\exists x \in A)(Y(x) = k)$ for fixed $k \in N$, then $Y(x_k) = k$ and $x_k \in A$, by the definition of $(x_n)_{n \in N}$. Hence, the right-hand side of (3.8) follows, and we observe that $(y_n)_{n \in N}$ enumerates $A$.

We can obtain an equivalent via the following ‘at most one’ condition:

$$(\forall i \in \{0, 1\})(\forall n \in N)(\exists \text{ at most one } f \in 2^N)(Y_i(f, n) = 0). \quad (3.9)$$

Let $\Sigma$-SEP$_C$ be $\Sigma$-SEP with all type 1 quantifiers restricted to $2^N$ and (3.9).

**Corollary 3.8.** The system $ACA^\omega_0$ proves $\Sigma$-SEP$_C^- \leftrightarrow \text{cocode}_0$.

*Proof.* The forward implication is immediate from the proof of the theorem as (3.4) and (3.5) satisfy the required ‘at most one’ conditions. For the reverse implication, let $Y^i_0$ be as in $\Sigma$-SEP$_C$ and define $A_i := \{f \in 2^N : (\exists n \in N)(Y_i(f, n) = 0)\}$. Clearly, this set is countable as $Z_i(f) := (\mu n)Y_i(f, n) = 0)$ yields an injection on $A_i$. Hence, cocode$_0$ provides an enumeration $(f_m)_{m \in N}$ of $A_0$, implying

$$\varphi_0(n) \leftrightarrow (\exists f \in 2^N)(Y_0(f, n) = 0) \leftrightarrow (\exists m \in N)(Y(f_m, n) = 0),$$

i.e. $\varphi_0(n)$ is decidable modulo $\exists^2$. The same holds for $\varphi_1(n)$ and we are done. \(\square\)
Next, as shown in [39], §5 and [10], HBU, BOOT, and the Lindelöf lemma are equivalent to elegant fragments of the neighborhood function principle NFP from [112]. In the same way as \( \Delta_0^1 \)-comprehension is included in RCA_0, the RM of NFP warrants a base theory that includes the following fragment of NFP, as discussed at length and in minute detail in [39], §5 and [10], §3.5.

**Definition 3.9.** \([A_0]\) For \( Y^2 \) and \( A(\sigma^0) \equiv (\exists f \in 2^\mathbb{N})(Y(f, \sigma) = 0) \), we have

\[(\forall f \in 2^\mathbb{N})(\exists n \in \mathbb{N})A(\mathcal{F}n) \rightarrow (\exists \Phi^2)(\forall f \in 2^\mathbb{N})A(\Phi^2(f)).\]

Recall the equivalence from [103], X.4.4 between \( \Sigma_0^2 \)-induction and bounded \( \Sigma_0^2 \)-comprehension. As noted above, \( \text{IND}_0 \) occupies the same category as the latter axiom, while an equivalence between HBU and \( \Sigma \)-SEP needs bounded separation, as follows. The axiom ‘bounded-\( \Sigma \)-SEP’ is \( \Sigma \)-SEP weakened such that for any \( k \in \mathbb{N} \):

\[(\forall n \leq k)(\neg \varphi_0(n) \lor \neg \varphi_1(n)) \rightarrow (\exists Z \subset \mathbb{N})(\forall n \leq k)[\varphi_0(n) \rightarrow n \in Z \land \varphi_1(n) \rightarrow n \notin Z].\]

Clearly, bounded-\( \Sigma \)-SEP only provides a finite/bounded fragment of the separating set from \( \Sigma \)-SEP, and the former follows from the induction axiom. We now have the following theorem which establishes (3.1).

**Theorem 3.10.** The system \( \text{ACA}_0^\omega + A_0 \) proves [HBU + bounded-\( \Sigma \)-SEP] \( \leftrightarrow \) \( \Sigma \)-SEP; the reverse implication holds over \( \text{ACA}_0^\omega \).

**Proof.** Assume HBU and suppose \( \neg \Sigma \)-SEP. Fix \( Y_0, Y_1 \) as in the latter and let \( A(\mathcal{Z}n) \) be the following, i.e. the formula in square brackets in \( \Sigma \)-SEP:

\[(\varphi_0(n) \rightarrow n \in Z) \land (\varphi_1(n) \rightarrow n \notin Z), \tag{3.10}\]

where the notation \( \mathcal{Z}n \) in \( A(\mathcal{Z}n) \) is justified by noting that the set \( Z \) is only invoked in (3.11) in the form ‘\( n \in Z \)’. By assumption, we have \( (\forall Z \subset \mathbb{N})(\exists n \in \mathbb{N})\neg A(\mathcal{Z}n) \), which has the right form to apply \( A_0 \). Hence, there is \( G : 2^\mathbb{N} \rightarrow \mathbb{N} \) such that

\[(\forall Z \subset \mathbb{N})\neg A(\mathcal{Z}G(Z)). \] Apply HBU to obtain \( f_1, \ldots, f_k \in 2^\mathbb{N} \), a finite sub-covering of the canonical covering \( \cup_{f \in 2^\mathbb{N}}(\mathcal{Z}f) \). Define \( n_0 := \max_{i \leq k} G(f_i) \) and note that

\[(\forall Z \subset \mathbb{N})(\exists n \leq n_0)\neg A(\mathcal{Z}n). \]

However, bounded-\( \Sigma \)-SEP provides a set \( Z_0 \subset \mathbb{N} \) such that for \( m \leq n_0 + 1 \), we have \( A(\mathcal{Z}n_0m) \), a contradiction, and we are done.

For the reverse implication, \( \Sigma \)-SEP implies bounded-\( \Sigma \)-SEP. Now assume \( \Sigma \)-SEP and suppose HBU fails for \( \Psi_0 : [0, 1] \rightarrow \mathbb{R}^+ \). Consider the following for \( q \in \mathbb{Q} \cap (0, 1) \):

\[
\varphi_0(q) \equiv (\exists w^{1^*})[(\forall i < |w|)(w(i) \in [0, 1]) \land [0, q] \subset \cup_{i < |w|} I_{w(i)}^{\Psi_0}],
\]

\[
\varphi_1(q) \equiv (\exists w^{1^*})[(\forall j < |v|)(v(j) \in [0, 1]) \land [q, 1] \subset \cup_{j < |v|} I_{v(j)}^{\Psi_0}],
\]

where \( (\forall q \in \mathbb{Q} \cap (0, 1)) (\neg \varphi_0(q) \lor \neg \varphi_1(q)) \) by assumption. Let \( Z_0 \subset \mathbb{N} \) be as provided by \( \Sigma \)-SEP and define a real \( x_0 \in [0, 1] \) as follows. Define \( [x_0](0) \) as \( \frac{1}{2} \) if \( \frac{1}{2} \in Z_0 \), and 0 otherwise; define \( [x_0](k + 1) \) as \( [x_0](k) + \frac{1}{2^{k+1}} \) if \( [x_0](k) + \frac{1}{2^{k+1}} \in Z \), and \( [x_0](k) \) otherwise. By definition, the real \( x_0 \) satisfies the following:

\[(\forall w^{1^*})[(\forall i < |w|)(w(i) \in [0, 1]) \rightarrow ([x_0](k), [x_0](k) + \frac{1}{2^{k+1}}) \not\subset \cup_{i < |w|} I_{w(i)}^{\Psi_0}], \tag{3.11}\]

which immediately yields a contradiction as \( ([x_0](k), [x_0](k) + \frac{1}{2^{k+1}}) \subset I_{x_0}^{\Psi_0} \) for \( k \) large enough, and we are done.

**Corollary 3.11.** The system \( \text{ACA}_0^\omega \) proves [HBT + bounded-\( \Sigma \)-SEP] \( \leftrightarrow \) \( \Sigma \)-SEP.
Proof. The reverse implication readily follows from the second part of the proof of the theorem. For the forward implication, consider \((\forall Z \subseteq \mathbb{N})((\exists n \in \mathbb{N})\neg A(Zn))\) as in the proof of the theorem. As noted above, we may use \(2^2\) to code \(\mathbb{N} \to \mathbb{N}\) sequences as binary sequences. Let \(Y\) be the characteristic function of the formula obtained by omitting the leading existential quantifiers (over \(2^N\)) of \(\sim A(\sigma)\). Define the function \(\psi : [0, 1] \to \mathbb{R}\) as follows: \(\psi(x) := 0\) if there is no initial segment \(\sigma^f\) of the binary expansion \(\sigma = f(x)\) such that \(Y(f, \sigma) = 0\); otherwise \(\psi(x) := \frac{1}{k}\) where \(k\) is the length of the shortest such initial segment. Then \(\psi\) yields a covering of \([0, 1]\) to which HBT applies. In the same was as in the proof of the theorem, one obtains a contradiction using bounded-\(\Sigma\)-SEP.

It is straightforward to show that HBT implies the fragment of \(A_0\) needed to prove \(HBU \to HBT\). Another interesting exercise is to consider \(A_0\) with the extra condition \((\forall \sigma^f \leq \alpha)(1)(\exists \text{ at most one } f \in 2^N)(Y(f, \sigma) = 0)\). Using the above results, one readily shows that over \(\text{ACA}_0\):\n\begin{align*}
\text{BOOT} \to [\text{HBU} + A_0] \to \text{cocode}_0 \to A_0^-, \quad \text{(3.12)}
\end{align*}
\begin{align*}
[\text{HBU} + \text{bounded-\(\Sigma\)-SEP} + A_0^-] \leftrightarrow \Sigma\text{-SEP}. \quad \text{(3.13)}
\end{align*}
What is more important is the following corollary to Theorem 3.10 related to \(A_0\).\n
Let \(\Sigma\text{-NFP}^-\) be \(A_0\) with the conclusion strengthened as in NFP, i.e. \((\exists \gamma \in K_0)(\forall f \in 2^N)A(\gamma(f))\). Note that \(\gamma \in K_0\) is the notation used in NFP from [112] for \(\gamma\) being a total RM-code/associate. Let bounded-\(\Sigma\)-SEP\(_C\) be bounded-\(\Sigma\)-SEP with the same restrictions as \(\Sigma\text{-SEP}\(_C\).\)

Corollary 3.12. The system \(\text{ACA}_0\) proves cocode\(_0\) \(\leftrightarrow [\Sigma\text{-NFP}^- + \text{bounded-\(\Sigma\)-SEP}^-]\).\n
Proof. The forward implication is straightforward: \(\text{BOOT}^-\) makes \(A(\sigma)\) from \(\Sigma\text{-NFP}^-\) decidable, i.e. there is \(X\), up to coding a subset of \(\mathbb{N}\), such that\n\[(\forall \sigma^f \leq 1)[\sigma \in X \leftrightarrow (\exists f \in 2^N)(Y(f, \sigma) = 0)].\]
Using QF-AC\(_{1,0}\) (and induction), we obtain \(G^2\) such that \((\forall f \in 2^N)A(\gamma G(f))\), where \(G(f)\) is the least such number. Clearly, \(G^2\) has an RM-code, and NFP\(_C\) follows.

For the reverse implication, we prove \([\Sigma\text{-NFP}^- + \text{bounded-\(\Sigma\)-SEP}^-] \to \Sigma\text{-SEP}\(_C\)\) and Corollary 3.8 finishes the proof. To obtain \(\Sigma\text{-SEP}\(_C\), consider \(A(\sigma)\) as in (3.10). Note that \((\forall Z \subseteq \mathbb{N})(\exists n \in \mathbb{N})\neg A(Zn)\) has the right form to apply \(\Sigma\text{-NFP}^-\). The resulting function \(\gamma \in K_0\) has an upper bound given WKL by [113] IV.2.2. Now use bounded-\(\Sigma\)-SEP\(_C\) to obtain a contradiction in the same way as in the proof of Theorem 3.10. Note that Corollary 3.8 yields cocode\(_0\).\n
Finally, \(A_1\) is \(A_0\) but for formulas \(A(\sigma^f) \equiv (\forall f \in 2^N)(Y(f, \sigma) = 0)\) and proves the equivalence between accu and Principle 2.20. The axiom \(A_1\) implies that any continuous function on \(\mathbb{N}^N\) has an associate/RM-code, as explored in [89] §5.

3.1.5. More on separation. In this section, we show that \(\Pi\)-SEP, a separation principle much weaker than \(\Sigma\)-SEP, implies cocode\(_1\). We also obtain an equivalence based on a weakening of \(\Pi\)-SEP.

First of all, note that the following principle is readily proved by applying QF-AC\(_{0,1}\) to the antecedent (see also [113] V.5.7). Theorem 3.14 is reminiscent of the fact that \(\Pi^1_1\)-separation implies \(\Delta^1_1\)-comprehension.
Principle 3.13 (II-SEP). For \( i = 0, 1, Y_i^2 \), and \( \varphi_i(n) \equiv (\forall f_i \in \mathbb{N}^n)(Y_i(f_i, n) = 0) \),
\[
(\forall n \in \mathbb{N})(\neg \varphi_0(n) \vee \neg \varphi_1(n)) \to (\exists Z \subset \mathbb{N})(\forall n \in \mathbb{N})[\varphi_0(n) \to n \in Z \land \varphi_1(n) \to n \notin Z].
\]

Theorem 3.14. The system \( \text{ACA}_0^\omega \) proves II-SEP \( \rightarrow \text{cocode}_1 \).

Proof. Let \( Y : \mathbb{R} \to \mathbb{N} \) be bijective on the non-empty set \( A \subset [0, 1] \). Define the formula \( \varphi_i(n, q) \) as follows where \( n \in \mathbb{N} \) and \( q \in \mathbb{Q} \cap (0, 1) \):
\[
\varphi_0(n, q) \equiv (\forall x \in A)(Y(x) = n \to x >_{\mathbb{R}} q) \quad (3.14)
\]
\[
\varphi_1(n, q) \equiv (\forall x \in A)(Y(x) = n \to x \leq_{\mathbb{R}} q). \quad (3.15)
\]
Since \( Y \) is bijective on \( A \), we have \( (\forall n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1))(\neg \varphi_0(n, q) \vee \neg \varphi_1(n, q)) \).

Let \( Z \subset \mathbb{N} \times \mathbb{Q} \) be as in II-SEP and note that for \( n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1) \), we have
\[
(n, q) \in Z \to (\exists x \in A)(Y(x) = n \land x >_{\mathbb{R}} q), \quad (3.16)
\]
\[
(n, q) \notin Z \to (\exists x \in A)(Y(x) = n \land x \leq_{\mathbb{R}} q). \quad (3.17)
\]
Now proceed as in the proof of Theorem 3.7 to define an enumeration of \( A \). \( \square \)

Finally, let II-SEP! be II-SEP restricted to \( Y_i^2 \) such that
\[
(\forall n \in \mathbb{N})(\exists f \in 2^\mathbb{N})[Y_0(f, n) \neq 0 \lor Y_1(f, n) \neq 0]. \quad (3.18)
\]
and all type 1 quantifiers restricted to \( 2^\mathbb{N} \). We have the following corollary.

Corollary 3.15. The system \( \text{ACA}_0^\omega \) proves II-SEP! \( \leftrightarrow \) cocode_1.

Proof. The forward direction is immediate from the proof of the theorem as \((3.18)\) is satisfied by the formulas \((3.14)\) and \((3.15)\). For the reverse implication, the set \( \{ f \in 2^\mathbb{N} : Y_0(f, n) \neq 0 \lor Y_1(f, n) \neq 0 \} \) is strongly countable. The enumeration provided by cocode_1 readily provides the set \( Z \) from II-SEP! and we are done. \( \square \)

3.2. Countable unions and the Axiom of Choice. In this section, we study the connection between the Bolzano-Weierstrass theorem, the countable union theorem for \( \mathbb{R} \), and the existence of sets not in the class \( \mathbb{F}_\sigma \). By Corollary 3.20 there are natural versions of the countable union theorem equivalent to \( \text{BWC}_i \) for \( i = 0, 1 \).

First of all, the Axiom of Choice (AC for short) is perhaps the most (in)famous axiom of the usual foundations of mathematics, i.e. ZFC set theory. It is known that very weak fragments of AC are independent of \( \text{ZF} \), like the countable union theorem which expresses that a countable union of countable (or even 2-element) sets is again countable. We refer to [32] for an overview of this kind of results on AC, while we note that Cantor already considered the countable union theorem in 1878, namely in [10] p. 243. The countable union theorem involving enumerations and (codes of) analytic sets may be found in second-order RM as [103] V.4.10, i.e. the following principle is a quite natural object of study in higher-order RM. We discuss the naturalness and generality of CUC in Remark 3.24.

Principle 3.16 (CUC). Let \((A_n)_{n \in \mathbb{N}} \) be a sequence of sets in \( \mathbb{R} \) such that for all \( n \in \mathbb{N} \), there is an enumeration of \( A_n \). Then there is an enumeration of \( \bigcup_{n \in \mathbb{N}} A_n \).

Note that we need \((\exists^2)\) to guarantee that the union in CUC exists. As noted above, the countable union theorem for 2-element sets is still unprovable in \( \text{ZF} \). In this light, define \( \text{CUC}(2) \) as CUC where each \( A_n \) has exactly two elements, i.e.
\[
(\forall x, y, z \in A_n)(x =_{\mathbb{R}} y \lor x =_{\mathbb{R}} z) \land (\exists w, v \in A_n)(w \neq_{\mathbb{R}} v). \quad (3.19)
\]
The following principle is (possibly) weaker than the countable union theorem according to [32 Diagram 3.4, p. 23]: \( \mathbb{R} \) is not a countable union of countable sets. We distill the following principle from the latter.

**Principle 3.17 (RUC).** Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of sets in \(\mathbb{R}\) such that for all \(n \in \mathbb{N}\), there exists an enumeration of \(A_n\). Then there is \(y \in \mathbb{R}\) not in \(\cup_{n \in \mathbb{N}} A_n\).

Note that RUC fails in the model \(\mathbb{Q}^*\) constructed in the proof of Theorem 2.16; i.e., \(\neg\text{RUC}\) is consistent with \(\mathbb{Z}_2^\ast\). By [103 II.4.7], Cantor’s theorem (that the reals cannot be enumerated) is provable in \(\text{RCA}_0\), and hence \(\text{CUC} \rightarrow \text{RUC}\) over \(\text{ACA}_0\).

The connection between RUC and the following principle is however more interesting:

**Principle 3.18 (NF}_\sigma).** There exists a subset of \(\mathbb{R}\) that is not \(F}_\sigma\).

To be precise, we let \(F}_\sigma\) be the class of sets obtained by closing the class of closed sets under unions of countable subclasses, always assuming that the unions exist. The following theorem connects CUC and RUC to \(\text{BW}_0\) and \(\text{BW}_1\).

**Theorem 3.19.** The system \(\text{ACA}_0^\omega\) proves \(\text{CUC} \rightarrow \text{cocode}_0 \rightarrow \text{CUC}(2) \rightarrow \text{cocode}_1\) and \(\text{NF}_\sigma \rightarrow \text{RUC} \rightarrow \text{NIN}\).

**Proof.** For the first part, fix non-empty \(A \subseteq [0, 1]\) and \(Y : [0, 1] \rightarrow \mathbb{N}\) such that the latter is injective on the former. Let \(x_0 \in A\) be some element in \(A\) and define the sequence of sets \((A_n)_{n \in \mathbb{N}}\) as follows:

\[
x \in A_n = \{x \in A \land Y(x) = n \land x =_R x_0\}.\tag{3.20}
\]

Clearly, for each \(n \in \mathbb{N}\), there exists an enumeration of \(A_n\), namely either the sequence \(x_0, x_0, \ldots\) or the sequence \(x_0, y, x_0, y, \ldots\) where \(y \in [0, 1]\) satisfies \(Y(y) = n\), if such there is. By CUC, there is an enumeration of \(A = \cup_{n \in \mathbb{N}} A_n\), yielding \(\text{cocode}_0\). Now assume the latter and fix a sequence \((A_n)_{n \in \mathbb{N}}\) satisfying (3.19). By the latter, we have the following

\[
(\forall n \in \mathbb{N})(\exists! x \in [0, 1])(\exists! y \in [0, 1])(x, y \in A_n \land x <_R y),\tag{3.21}
\]

as \(A_n\) has exactly two elements. Recall that \(\text{cocode}_1 \leftrightarrow \text{QF-AC}_0^1\) by [32 Theorem 3.17]. Modulo some coding \(\text{QF-AC}_0^1\) applies to (3.21), and let \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) be the resulting sequences. Use \(\mathbb{Z}^2\) to remove any reals from \((y_n)_{n \in \mathbb{N}}\) already in \((x_n)_{n \in \mathbb{N}}\). Then \(Y : \mathbb{R} \rightarrow \mathbb{N}\) is injective on \(\cup_{n \in \mathbb{N}} A_n\):

\[
Y(x) := \begin{cases} 0 & x \notin \cup_{n \in \mathbb{N}} A_n \\ 2P(x) & (\exists n \in \mathbb{N})(x =_R x_n) \\ 2P(x) + 1 & (\exists n \in \mathbb{N})(x =_R y_n) \end{cases}\tag{3.22}
\]

where \(P(x) := (\mu n)(x \in A_n)\). Then \(\text{cocode}_0\) yields \(\text{CUC}(2)\), as required. For the implication \(\text{CUC}(2) \rightarrow \text{cocode}_1\), fix \(A \subseteq [0, 1]\) such that \(Y : [0, 1] \rightarrow \mathbb{N}\) is bijective on \(A\). Define the set \(A_n := \{x \in A : Y(x) = n \lor Y(x) = n + 1\}\) and note that (3.19) is satisfied. Applying \(\text{CUC}(2)\) yields an enumeration of \(A = \cup_{n \in \mathbb{N}} A_n\), as required.

For the second part, suppose \(\mathbb{R} = \cup A_n\), where for each \(n \in \mathbb{N}\) there exists an enumeration of \(A_n\). Then all subsets of \(\mathbb{R}\) are \(F}_\sigma\) as follows: for \(E \subseteq \mathbb{R}\), one defines an enumeration of \(E \cap A_n\) by checking each element in the enumeration of \(A_n\) for elementhood in \(E\). Hence, \(E = \cup_{n \in \mathbb{N}} (A_n \cap E)\) is a countable union of enumerable sets, and therefore \(F}_\sigma\). For \(\text{RUC} \rightarrow \text{NIN}\), suppose \(Y : \mathbb{R} \rightarrow \mathbb{N}\) is an injection. Define a sequence \((A_n)_{n \in \mathbb{N}}\) as follows \(x \in A_n = [Y(x) = 0 \lor x =_R 0]\). Clearly, for each
n ∈ N, there exists an enumeration of Aₙ. By RUC, there is y ∈ R not in ∪ₙ∈ℕAₙ. However, R = ∪ₙ∈ℕAₙ by definition, yielding RUC → NIN.

Assuming ACA₀ + ¬RUC, the previous proof implies that all subsets of R are Fₗₐₜ, and considering complements implies that all subsets are also Gₜₐₜ. In stronger systems, the class Fₗₐₜ ∩ Gₜₐₜ corresponds to Δ²₀-formulas with function parameters.

Let CUC₀(2) be CUC(2) without the second conjunct of (3.19) and let CUC₁(2) be CUC(2) where we additionally assume the sets Aₙ to be pairwise disjoint.

Corollary 3.20 (ACA₀). We have cocode₀ ↔ CUC₀(2) and cocode₁ ↔ CUC₁(2).

Proof. The proof of CUC(2) → cocode₁ from the theorem yields CUC₁(2) → cocode₁ as Aₙ := \{x ∈ A : Y(x) = 2n \lor Y(x) = 2n + 1\} are indeed pairwise disjoint. The proof of cocode₀ → CUC(2) yields cocode₁ → CUC₁(2) as the extra ‘pairwise disjoint’ condition in CUC₁(2) guarantees that Y defined in (3.22) is bijective on ∪ₙ∈ℕAₙ. The proof of CUC → cocode₀ from the theorem yields a proof of CUC₀(2) → cocode₀ as the sets from (3.20) have at most two elements. The proof of cocode₀ → CUC(2) from the theorem can be adapted as follows: consider the following formula, where the boldface text is different from (3.21):

\[(∀n ∈ N)(∃\text{ at most one } (x, y) ∈ R^2)(x, y ∈ Aₙ \land x \prec R y),\]

to which BOOTₐᵣ applies modulo coding. For the resulting set X ⊆ N we have

\[(∀n ∈ X)(∃x ∈ [0, 1])(∃y ∈ [0, 1])(x, y ∈ Aₙ \land x \prec R y),\]

One now readily modifies (3.22) to the case at hand, which yields an enumeration of all Aₙ that have exactly two elements. To enumerate the Aₙ that are singletons, consider the following:

\[(∀n ∈ N \setminus X)(∃\text{ at most one } x ∈ R)(x ∈ Aₙ),\]

to which BOOTₐᵣ applies modulo coding. For the resulting set Z ⊆ N we have

\[(∀n ∈ Z)(∃x ∈ [0, 1])(x ∈ Aₙ),\]

which readily yields the required enumeration.

By the previous, one can view CUC as the sequential version of cocode₀. However, the sequential version of e.g. \text{BWC}_0 is readily proved in \text{Z}_2^0 \text{ (and hence ZF). By contrast, the sequential version of cloq' is equivalent to CUC by Corollary 3.21.}

Principle 3.21 (cloq' seq). Let \(Xₙ, ≤ₙ\)ₙ∈ℕ be a sequence of dense linear orderings without endpoints, with each \(Xₙ ⊆ R\) countable. Then there is a sequence \((Zₙ)ₙ∈ℕ\) with \(Zₙ : R → \mathbb{Q}\) an order-isomorphism from \(Xₙ, ≤ₙ\) to \(\mathbb{Q}\) for each \(n ∈ N\).

Corollary 3.22. The system ACA₀ proves [cloq' seq + IND₀] ↔ CUC.

Proof. For the reverse implication, CUC yields cocode₀ by Theorem 3.19. Hence, if \((Xₙ, ≤ₙ)ₙ∈ℕ\) is as in the antecedent of cloq' seq, cocode₀ implies that for each \(Xₙ\), there is an enumeration. By CUC, there is a ‘master’ enumeration of ∪ₙ∈ℕXₙ. Use the well-known ‘back-and-forth’ proof (see [33, p. 123]) for each \((Xₙ, ≤ₙ)\), uniformly in N and based on the master enumeration, to yield a sequence as in the consequence of cloq' seq.

For the forward implication, we have access to cocode₀ by Corollary 2.35. Let \((Aₙ)ₙ∈ℕ\) be a sequence as in CUC and define \(A := ∪ₙ∈ℕAₙ\). Note that (3²) shows that each \(Aₙ\) is countable via an obvious injection. Without loss of generality, we
may assume that \( \mathbb{Q} \cap A = \emptyset \), since Feferman’s \( \mu^2 \) can list all the rationals in a given set of reals. Now define \( X_n := \mathbb{Q} \cup A_n \) and \( \preceq_n \) the usual ordering of the reals. Let \( (Z_n)_{n \in \mathbb{N}} \) be as provided by \( \text{cloq}_{\text{seq}} \), let \( (p_n)_{n \in \mathbb{N}} \) be the usual list of primes, and let \( G : \mathbb{Q} \rightarrow (\mathbb{N} \setminus \{0\}) \) be an injection. Define \( H(x) := (\mu y)(x \in A_n) \) and define \( Y : \mathbb{R} \rightarrow \mathbb{N} \) as \( Y(x) := p_{H(x)}(G(H(x))) \). By definition, \( Y \) is an injection on \( A \); the latter is therefore countable, and enumerable by Corollary 2.35. \( \square \)

For the negative result, let \( (\mathbb{Q} \cup A_n) \) be as provided by \( \text{cloq}_{\text{seq}} \), let \( (p_n)_{n \in \mathbb{N}} \) be the usual list of primes, and let \( G : \mathbb{Q} \rightarrow (\mathbb{N} \setminus \{0\}) \) be an injection. Define \( H(x) := (\mu y)(x \in A_n) \) and define \( Y : \mathbb{R} \rightarrow \mathbb{N} \) as \( Y(x) := p_{H(x)}(G(H(x))) \). By definition, \( Y \) is an injection on \( A \); the latter is therefore countable, and enumerable by Corollary 2.35.

We note in passing that the weak choice principle \( \text{WCC} \) from [5] is intermediate between \( \text{cocode}_0 \) and \( \text{cocode}_1 \) by the previous. We also have the following corollary.

**Corollary 3.23.** \( Z_2^0 + \text{QF-AC}^{0,1} \) proves \( \text{CUC} \); \( Z_2^\omega + \text{QF-AC}^{0,1} \) cannot prove \( \text{RUC} \).

**Proof.** For the negative result, \( \text{fin} \) is not provable in \( Z_2^\omega + \text{QF-AC}^{0,1} \) by [79] Theorem 3.2], while \( \text{RUC} \rightarrow \text{fin} \) over \( \text{ACA}_0^\omega \) by Theorem 3.19. For the positive result, the antecedent of \( \text{CUC} \) expresses the following:

\[
(\forall n \in \mathbb{N}) (\exists (x_m)_{m \in \mathbb{N}})(\forall y \in \mathbb{R}) (y \in A_n \iff (\exists k \in \mathbb{N})(x_k =_\mathbb{R} y)).
\]

Using \( \exists \exists \) and \( \text{QF-AC}^{0,1} \), there is a ‘master’ sequence, yielding \( \text{CUC} \). \( \square \)

We finish this section with a remark on the naturalness and generality of \( \text{CUC} \).

**Remark 3.24** (\( \text{CUC} \), old and new). First of all, an \( L_2 \)-version of \( \text{CUC} \) for sets represented by analytic codes is proved in [103] V.4.10, inside \( \text{ATR}_0 \). Note that enumerable sets are automatically Borel, and therefore analytic. Similarly, (codes for) Borel sets are closed under countable unions in second-order RM by [103] V.3.3], also working in \( \text{ATR}_0 \). Modulo coding, there is thus antecedent for the study of \( \text{CUC} \) in second-order RM.

Secondly, in contrast to the second-order principles from the previous paragraph, \( \text{CUC} \) does (seem to) quantify over all enumerable subsets of \( \mathbb{R} \). This apparent generality of \( \text{CUC} \) should not be overstated: an enumerated set is of course measurable (provably having measure zero in \( \text{ACA}_0^\omega \)), and the class of (codes for) measurable sets is closed under countable unions in second-order RM, as mentioned in [103] X.1.17]. Similarly, enumerated sets are clearly Borel sets (of low level) in \( \text{ACA}_0^\omega \). Hence, \( \text{CUC} \) is of a level of generality comparable to what one studies in RM, but formulated with third-order characteristic functions rather than second-order codes.

Thirdly, in Section 3.3.2 we connect \( \text{cocode}_0 \) to theorems pertaining to bounded variation (and related concepts), like the Jordan decomposition theorem as in Theorem 3.27. On one hand, this theorem readily implies \( \text{cocode}_0 \), while the reversal should go through, seeing as though functions of bounded variation only have countably many points of discontinuity. Indeed, an enumeration of the latter set even guarantees that Jordan’s original proof (13) of the Jordan decomposition theorem goes through. Try as we might, the aforementioned reversal only goes through assuming the following (seemingly trivial) fragment of the countable union theorem, which however does not even imply \( \text{fin} \) over \( Z_2^\omega + \text{QF-AC}^{0,1} \).

**Principle 3.25** (\( \text{CUC}_{\text{fin}} \)). Let \( (X_n)_{n \in \mathbb{N}} \) be subsets of \( \mathbb{R} \) such that \( \bigcup_{n \in \mathbb{N}} X_n \) is not countable. Then \( X_m \) is not finite for some \( m \in \mathbb{N} \).

Recall our notion of ‘finite set’ from Definition 1.5 to be discussed in detail in Section 3.3.2. In the below, we even obtain equivalences involving \( \text{CUC}_{\text{fin}} \), i.e. the countable union theorem is a natural/useful object of study in this context.

\^1Note that \( \text{ fin } \) implies \( \text{CUC}_{\text{fin}} \), while \( Z_2^\omega + \text{QF-AC}^{0,1} \) does not prove \( \text{fin} \) by [79] §3].
3.3. **Bounded variation and related concepts.** In this section, we establish an equivalence between BWC\(_0\) and the well-known Jordan decomposition theorem as in Theorem 3.27. We also obtain other equivalences involving theorems about bounded variation and regulated functions. We introduce definitions for the previous italicised notions in Section 3.3.1, while our main results are in Section 3.3.2. The latter results provide some non-trivial motivation for our choice of definition of closed and finite set, as discussed in Section 3.3.3.

### 3.3.1. Definitions: bounded variation and related notions.

We formulate the definitions of bounded variation and regulated functions, as well as some background.

Firstly, the notion of bounded variation (often abbreviated BV below) was first explicitly introduced by Jordan around 1881 ([43]) yielding a generalisation of Dirichlet’s convergence theorems for Fourier series. Indeed, Dirichlet’s convergence results are restricted to functions that are continuous except at a finite number of points, while BV-functions can have infinitely many points of discontinuity, as already studied by Jordan, namely in [43, p. 230]. Nowadays, the total variation of a function \( f : [a, b] \to \mathbb{R} \) is defined as follows:

\[
V^b_a(f) := \sup_{a \leq x_0 < \cdots < x_n \leq b} \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|.
\]  

(3.25)

If this quantity exists and is finite, one says that \( f \) has bounded variation on \([a, b]\).

Now, the notion of bounded variation is defined in [71] without mentioning the supremum in (3.25); this approach can also be found in [3, 4, 53]. Hence, we shall distinguish between the two notions in Definition 3.26. As it happens, Jordan seems to use item (a) of Definition 3.26 in [43, p. 228-229]. This definition suggests a two-fold variation for any result on functions of bounded variation, namely depending on whether the supremum (3.25) is given, or only an upper bound on the latter.

**Definition 3.26.** [Variations on variation]

(a) The function \( f : [a, b] \to \mathbb{R} \) has bounded variation on \([a, b]\) if there is \( k_0 \in \mathbb{N} \) such that \( k_0 \geq \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})| \) for any partition \( x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b \).

(b) The function \( f : [a, b] \to \mathbb{R} \) has a variation on \([a, b]\) if the supremum in (3.25) exists and is finite.

Secondly, the fundamental theorem about BV-functions is formulated as follows.

**Theorem 3.27** (Jordan decomposition theorem, [43, p. 229]). A BV-function \( f : [0, 1] \to \mathbb{R} \) is the difference of two non-decreasing functions \( g, h : [0, 1] \to \mathbb{R} \).

Theorem 3.27 has been studied via second-order representations in [31, 53, 71, 117]. The same holds for constructive analysis by [31, 63, 85], involving different (but related) constructive enrichments. Now, ACA\(_0\) suffices to derive Theorem 3.27 for various kinds of second-order representations of BV-functions in [53, 71]. By contrast, our results imply that \( \mathbb{Z}_2^+ + \text{QF-AC}^{0,1} \) cannot prove the third-order version of Theorem 3.27 as the latter is equivalent to BWC\(_0\) over a suitable base theory (see Theorem 3.34). Nonetheless, the third-order Jordan decomposition theorem does not imply much comprehension, by the following remark.

\[\text{Lakatos in [55, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [43], but rather discovered it in Dirichlet's 1829 paper [19].}\]
Remark 3.28 (Comprehension and Jordan decompositions). The third-order version of the Jordan decomposition theorem (Theorem 3.27) implies neither (Σ²) nor any theorem of Z₂ not provable in ACA₀, working over RCA₀. Indeed, the ECF-translation (Remark 1.12) of the former is implied by Jordan, the second-order version of Theorem 3.27 from [71] and provable in ACA₀. By contrast, ECF translates (Σ²) to ‘0 = 1’ while second-order sentences are translated to themselves.

Thirdly, Jordan proves in [44] §105 that BV-functions are exactly those for which the notion of ‘length of the graph of the function’ makes sense. In particular, f ∈ BV if and only if the ‘length of the graph of f’, defined as follows:

\[ L(f, [0, 1]) := \sup_{t_0 < t_1 < \ldots < t_m = 1} \sum_{i=0}^{m-1} \sqrt{(t_i - t_{i+1})^2 + (f(t_i) - f(t_{i+1}))^2} \quad (3.26) \]

exists and is finite by [1] Thm. 3.28(c). In case the supremum in (3.26) exists (and is finite), f is also called rectifiable. Rectifiable curves predate BV-functions: in [99] §1-2, it is claimed that (3.26) is essentially equivalent to Duhamel’s 1866 approach from [23, Ch. VI]. Around 1833, Dirksen, the PhD supervisor of Jacobi and Heine, already provides a definition of arc length that is (very) similar to (3.26) (see [20, §2, p. 128]), but with some conceptual problems as discussed in [17] §3.

Fourth, a function is regulated (called ‘regular’ in [1]) if for every x₀ in the domain, the ‘left’ and ‘right’ limit f(x₀−) = \lim_{x \to x₀−} f(x) and f(x₀+) = \lim_{x \to x₀+} f(x) exist. Scheeffer studies discontinuous regulated functions in [99] (without using the term ‘regulated’), while Bourbaki develops Riemann integration based on regulated functions in [7]. Now, BV-functions are regulated (see Theorem 3.33), while Weierstrass’ ‘monster’ function is a natural example of a regulated function not in BV. An interesting observation about regular functions and continuity is as follows.

Remark 3.29 (Continuity and the Axiom of Choice). As discussed in [49] §3, the local equivalence for functions on Baire space between sequential and ‘epsilon-delta’ continuity can be proved in RCA₀ + QF-AC⁰¹, but not in ZF. By the final item in Theorem 3.33 this equivalence for regulated functions is provable in ACA₀.

Finally, the Jordan decomposition theorem as in Theorem 3.27 shows that a BV-function can be ‘decomposed’ as the difference of monotone functions. This is however not the only result of its kind: Sierpiński e.g. establishes in [101] that for regulated f: [0, 1] → ℝ, there are g, h such that f = g ∘ h with g continuous and h strictly increasing on their respective domains.

3.3.2. Bounded variation and Reverse Mathematics. In this section, we develop the RM of the Jordan decomposition theorem and related results on bounded variation and regulated functions. As will become clear, the principle CUCfin from Remark 3.24 is central to this enterprise.

First of all, we recall our particular notion of ‘finite set’ to be used in CUCfin and provide some motivation in Remark 3.31 right below. On a historical note, the study of various definitions of finite set (in set theory) was the topic of Mostowski’s dissertation, as suggested by Tarski ([66, p. 18-19]).

Definition 3.30 (Finite). Any X ⊂ ℝ is finite if there is N ∈ ℕ such that for any finite sequence (x₀, . . . , x_N) of distinct reals, there is i ≤ N such that x_i ∉ X.

The number N ∈ ℕ from the previous definition is called an upper bound on the size of the finite set X ⊂ ℝ, and we use |X| ≤ N as purely symbolic notation for
this. Note that Definition \ref{def:finite_set_3.30} is not circular as ‘finite sequences of reals’ are just objects of type 1, modulo coding using $\mathbb{P}^2$. We now motivate Definition \ref{def:finite_set_3.30}.

\textbf{Remark 3.31} (Finite sets by any other name). First of all, working in set theory, the various definitions \ref{def:finite_set_3.30} of ‘finite set’ are not equivalent over ZF, while countable choice suffices to establish the equivalence \ref{eq:QF-AC}. Hence, it should not be a surprise that studying finite sets in weak systems requires one to choose a specific definition.

Secondly, consider the following set where $f$ is a function of bounded variation:

$$A_n := \{ x \in (0, 1) : \left| f(x^+) - f(x^-) \right| > \frac{1}{2^n} \lor \left| f(x^-) - f(x^+) \right| > \frac{1}{2^n} \}$$

(3.27)

This set is finite as each element of $A_n$ contributes at least $\frac{1}{2^n}$ to the total variation. Finite as $A_n$ may be, we are unable to exhibit an injection from $A_n$ to $\{0, 1, \ldots, k\}$ for some $k \in \mathbb{N}$, say computable in some $S_m^k$ (see Remark \ref{rem:finite_set_3.30} for details). By contrast, $A_n$ is trivially finite in the sense of Definition \ref{def:finite_set_3.30} in ACA$_0^\omega$.

In conclusion, if one wants to work in a weak logical system, then (certain) finite sets that ‘appear in the wild’ are best studied via Definition \ref{def:finite_set_3.30} and not the definition from Footnote \ref{footnote:finite_set_3.30} involving bijections or injections. Moreover, Theorem \ref{thm:finite_set_2.12} suggests that IND$_0$ (and cocode$_0$) does not suffice to study finite sets as in Definition \ref{def:finite_set_3.30} as noted in Remark \ref{rem:finite_set_3.30} we indeed seem to need CU$_{\mathrm{fin}}$.

Secondly, we need Theorem \ref{thm:finite_set_3.30} to establish basic properties of BV and regulated functions. We shall make (seemingly essential) use of the following fragment of the induction axiom, which also follows from QF-AC$^{0,1}$.

\textbf{Definition 3.32.} [IND$_2$] Let $Y^2, k^0$ satisfy $(\forall n \leq k)(\exists f \in 2^\mathbb{N})(Y(f, n) = 0)$. There is $w^1$ such that $(\forall n \leq k)(\exists i < |w|)(Y(w(i), n) = 0)$.

Note that we use the ‘standard’ definition of left and right limits, i.e. as in \ref{def:finite_set_3.29}.

\textbf{Theorem 3.33} (ACA$_0^\omega$).

- \textbf{Assuming IND$_2$}, any BV-function $f : [0, 1] \rightarrow \mathbb{R}$ is regulated.
- Any monotone function $f : [0, 1] \rightarrow \mathbb{R}$ has bounded variation.
- For any monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ that enumerates all $x \in [0, 1]$ such that $f$ is discontinuous at $x$.
- For regulated $f : [0, 1] \rightarrow \mathbb{R}$ and $x \in [0, 1]$, $f$ is sequentially continuous at $x$ if and only if $f$ is epsilon-delta continuous at $x$.
- For finite $X \subset [0, 1]$, the function $\mathbb{I}_X$ has bounded variation.

\textbf{Proof}. For the first item, assume $f(c^-)$ does not exist for $c \in (0, 1)$. We obtain a contradiction using QF-AC$^{0,1}$ and then using IND$_2$. Hence, there is $\varepsilon > 0$ with

$$\left( \forall k \in \mathbb{N} \right)(\exists x, y \in (c-\frac{1}{2^{m+1}}, c))(x < y \land |f(x) - f(y)| > \varepsilon).$$

(3.28)

Apply QF-AC$^{0,1}$ to (3.28); modify the resulting sequence $(x_n, y_n)_{n \in \mathbb{N}}$ to guarantee

$$x_m < y_m < c - \frac{1}{2^{m+1}} < x_{m+1} < y_{m+1} < c - \frac{1}{2^{m+1}}$$

for large enough $m \in \mathbb{N}$. By definition, $|f(x_k) - f(y_k)| > \varepsilon$ for large enough $k \in \mathbb{N}$, i.e. collecting enough such points in a partition, the associated variation is arbitrary large. We now observe how the previous proof is readily modified: apply IND$_2$ to

---

\footnote{In ZF, a set $A$ is ‘finite’ if there is a bijection to $\{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$; a set $A$ is ‘Dedekind finite’ if any injective mapping from $A$ to $A$ is also surjective.}
where (3.25) as follows: use $g(3.25)$.

For the second part, assume $f : [0, 1] \to \mathbb{R}$ is monotone. Then the usual telescoping sum trick implies that the total variation of $f$ as in (3.25) exists and equals $|f(0) - f(1)|$. The third part is follows from [30, Lemma 7], which applies to $[0, 1]$ but trivially generalises to $\mathbb{R}$.

For the fourth item, let $f : [0, 1] \to \mathbb{R}$ be regulated and fix $x_0 \in [0, 1]$. We only need to prove the forward implication, i.e. assume $f$ is sequentially continuous at $x_0$. To show that $f(x_0-) = f(x_0)$, consider $y_n := x_0 - \frac{1}{2^n}$ and note that $(y_n)_{n \in \mathbb{N}}$ converges to $x_0$, implying that $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x_0)$. Now consider the definition of 'the left limit $f(x_0-)$ exists' as follows:

$$\exists y \in \mathbb{R}(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall z \in (x_0 - \frac{1}{2^k}, x_0))(|f(z) - y| < \frac{1}{2^k}) \tag{3.29}$$

Since $(y_n)_{n \in \mathbb{N}}$ converges to $x_0$ and $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x_0)$, we have $y = f(x_0)$ in (3.29). In the same way, one shows that $f(x_0+) = f(x_0)$. Then (3.29) and the associated 'right limit' version imply that $f$ is epsilon-delta continuous at $x_0$.

For the fifth item, fix finite $X \subset [0, 1]$ with $N \in \mathbb{N}$ as in Definition 3.30. Now suppose $f(x) := 1_x(x)$ does not have bounded variation, i.e. for any $n \in \mathbb{N}$, there is a partition $x_0 = 0, x_1, \ldots, x_k, x_{k+1} = 1$ of $[0, 1]$ such that $n + 5 \leq \sum_{i=0}^{k} |f(x_{i+1}) - f(x_i)|$. By the definition of $f$, the latter inequality implies that there are $i_0, \ldots, i_n \leq k$ such that $x_{i_j} \in X$ for $j \leq n$. Taking $n = N + 1$, we obtain a contradiction. \(\square\)

Thirdly, we can now connect the Jordan decomposition theorem and \text{cocode} \text{e}_0. Note that 'bounded variation' refers to item (iii) in Definition 3.26.

**Theorem 3.34 (ACA\text{\textsuperscript{0}} + IN\text{D}_2 + CUC\text{\textsubscript{em}}).** The following are equivalent.

(i) The principle \text{cocode} \text{e}_0.

(ii) The Jordan decomposition theorem (Theorem 3.27).

(iii) For a BV-function $f : [0, 1] \to \mathbb{R}$, there is a sequence enumerating all points where $f$ is discontinuous.

The previous upward implications are provable over ACA\text{\textsuperscript{0}}. Assuming QF-AC\text{\textsuperscript{0,1}}, the above are equivalent to the following.

(iv) For regulated $f : [0, 1] \to \mathbb{R}$, there is a sequence enumerating all points where $f$ is discontinuous.

(v) (Sierpiński) For regulated $f : [0, 1] \to \mathbb{R}$, there are $g, h$ such that $f = g \circ h$ with $g$ continuous and $h$ strictly increasing on their interval domains.

The previous upward implications are provable over ACA\text{\textsuperscript{0}} + IN\text{D}_2.

**Proof.** The equivalence (ii) $\leftrightarrow$ (iii) follows from Theorem 3.33 and the usual proof of the Jordan decomposition theorem. Indeed, we can 'imitate' the supremum in (3.26) as follows: use $\mu^2$ to define, for any $x \in [0, 1]$, the following:

$$V(x) := \sup_{0 \leq y_0 < \cdots < y_n \leq x} \sum_{i=0}^{n} |f(y_i) - f(y_{i+1})|, \tag{3.30}$$

where $(y_i)_{i \in \mathbb{N}}$ is the sequence consisting of $\mathbb{Q} \cap [0, 1]$ together with the sequence provided by item (iii). Trivially, $g(x) := \lambda x.V(x)$ is increasing on $[0, 1]$ and the same holds for $h(x) := V(x) - f(x)$. Indeed, for $0 \leq y < z \leq 1$, we have

$$h(z) - h(y) = V(z) - f(z) - V(y) + f(y) = (V(z) - V(y)) - (f(z) - f(y)) \geq 0,$$

where the final inequality follows from the definition of $V$. We now have $f(x) - g(x) = h(x)$ for all $x \in [0, 1]$, yielding the Jordan decomposition theorem.
For the implication (iii) → (i), fix $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{N}$ injective on $A$. Define $f(x)$ as $\frac{1}{2^n}$ in case $x \in A$, and 0 otherwise. Clearly, $f \in BV$ as any sum $\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ is at most $\sum_{i=0}^{n} \frac{1}{2^n}$, which is bounded by 1 for any $n \in \mathbb{N}$. The points of discontinuity for $f$ are exactly the points of $A$, and $\text{cocode}_0$ follows.

For the implication (i) → (iii), fix a $BV$-function $f : [0,1] \to \mathbb{R}$ and $n \in \mathbb{N}$. We may assume that the upper bound as in item (iii) in Def. 3.26 is 1. The first item of Theorem 3.33 guarantees that $f$ is regular. Now define the following set

$$A_n := \{ x \in (0,1) : |f(x+) - f(x)| > \frac{1}{2^n} \lor |f(x-) - f(x)| > \frac{1}{2^n} \}$$

(3.31)

which is finite (in the sense of Definition 3.30). Indeed, assuming $A_n$ were not finite, there are arbitrary long finite sequences of elements of $A_n$. However, each element of $A_n$ contributes at least $\frac{1}{2^n}$ to the variation of $f$, a contradiction. Hence, $A_n$ is finite (and has at most $2^n$ elements). Using the contraposition of $\text{CUC}_\text{fin}$, the union $A := \bigcup_{n \in \mathbb{N}} A_n$ is countable. This union can now be enumerated thanks to $\text{cocode}_0$, yielding a sequence listing all points of discontinuity of $f$.

The implications (iv)→(iii)→(i) are immediate by Theorem 3.33. For (i)→(iv), fix regulated $f : [0,1] \to \mathbb{R}$ and consider the proof of [1] Theorem 0.36, p. 28, going back to [101]. This proof establishes the existence of $g,h$ such that $f = g \circ h$ with $g$ continuous and $h$ strictly increasing. Moreover, one finds an explicit construction (modulo $\exists^2$) of the function $h$ required, assuming a sequence listing all points of discontinuity of $f$ on $[0,1]$. The function $g$ is then defined as $\lambda y.f(h^{-1}(y))$ where $h^{-1}$ is the inverse of $h$, definable using $\exists^2$.

Finally, we shall make use of $\text{QF-AC}_0^{0,1}$ to prove (i)→(iv): fix regulated $f : [0,1] \to \mathbb{R}$ and $n \in \mathbb{N}$ and note that $A_n$ as in (3.31) is again finite. Indeed, assuming $A_n$ were not finite, $\text{QF-AC}_0^{0,1}$ provides a sequence $(x_j)_{j \in \mathbb{N}}$ of elements of $A_n$. By the Bolzano-Weierstrass theorem, this sequence has a convergent sub-sequence, say with limit $c \in [0,1]$. However, $f(c+)$ and $f(c-)$ do not exist by the definition of $A_n$ (via the usual epsilon-delta argument), a contradiction. In conclusion, the union $A := \bigcup_{n \in \mathbb{N}} A_n$ can now be enumerated, thanks to item (i) and CUC$_\text{fin}$.

The use of $\text{QF-AC}_0^{0,1}$ in the theorem can be avoided in various ways, one of which is the principle NCC from [79]. We will explore this in a follow-up paper.

Fourth, we establish a (more) elegant result as in Theorem 3.35. In the latter, the uniform finite union theorem expresses the existence of $h : \mathbb{N} \to \mathbb{N}$ such that $|X_n| \leq h(n)$ for a sequence of finite sets $(X_n)_{n \in \mathbb{N}}$ in $[0,1]$. The finite union theorem expresses (only) that for such a sequence, each $\bigcup_{n \leq k} X_n$ is finite for $k \in \mathbb{N}$. Regarding item (i), Principle 2.20 was studied in Corollary 2.26 and we can now obtain an equivalence involving the former and $\text{cocode}_0$.

**Theorem 3.35** (ACA$^0_0$ + QF-AC$^{0,1}$). The following are equivalent.

(a) The combination CUC$_\text{fin}$ + $\text{cocode}_0$.

(b) For regulated $f : \mathbb{R} \to \mathbb{R}$, there is a sequence enumerating the points of discontinuity.

(c) For regulated $f : [0,1] \to \mathbb{R}$, there is a sequence enumerating the points of discontinuity.

(d) The uniform finite union theorem plus the Jordan decomposition theorem.

(e) The uniform finite union theorem plus: for $f : [0,1] \to \mathbb{R}$ in BV, there is a sequence enumerating the points of discontinuity.
The finite union theorem plus the Jordan decomposition theorem on the half-line: for $f : \mathbb{R} \to \mathbb{R}$ with bounded variation on $[0, y]$ for any $y \in \mathbb{R}^+$, there are monotone $g, h$ such that $f(x) = g(x) - h(x)$ for any $x \geq 0$.

The finite union theorem plus: for $f : \mathbb{R} \to \mathbb{R}$ with bounded variation on $[0, y]$ for any $y \in \mathbb{R}^+$, there is a sequence enumerating the points of discontinuity of $f$ on $[0, +\infty)$.

A non-enumerable and closed set in $\mathbb{R}$ has a limit point (Principle 2.20).

Proof. First of all, we derive the following basic properties concerning finite sets, working in our base theory $\text{ACA}_0^+ + \text{QF-AC}^{0,1}$.

1. Any item (a)-(h) implies that a finite set of reals can be enumerated.
2. Item (g) implies the uniform finite union theorem and CUC$_{\mathbb{N}}$.

For item (x1), a finite set has characteristic function that is in $BV$ and regulated by Theorem 3.33 assuming $\text{IND}_2$ which follows from $\text{QF-AC}^{0,1}$. Hence, over our base theory, items (a)-(g) imply that a finite set can be enumerated (as a finite sequence, using $\mu^2$), where we note the third item of Theorem 3.33. Since finite sets do not have limit points, item (x1) also holds for item (h).

For item (x2), let $(X_n)_{n \in \mathbb{N}}$ be a sequence of finite sets. We may assume $0, 1 \notin \cup_{n \in \mathbb{N}} X_n$. Now consider $f_k : [0, 1] \to \mathbb{N}$ for $k \geq 2$ defined as follows: define $Y_i := \{y \in (\frac{i}{k}, \frac{i+1}{k}) : k(y - \frac{i}{k}) \in X_i\}$ and $f_k(x) := \sum_{i=0}^{k-1} 1_{Y_i}(x)$. By definition, $Y_i$ is the set $X_i$ for $i \leq k$, but shrunk by a factor $\frac{1}{k}$ and moved to $(\frac{i}{k}, \frac{i+1}{k})$. Hence, $Y_i$ is finite for $i \leq k$ and since $f_k(x)$ equals $\sum_{i=0}^{k-1} 1_{Y_i}(x)$ for $x \in (\frac{i}{k}, \frac{i+1}{k})$, the function $f_k$ is regulated by Theorem 3.33. Thus, item (x) implies that the points of discontinuity of $f_k$ can be enumerated, which means $\cup_{i \leq k} Y_i$ can be enumerated. Using $\mu^2$ and the latter enumeration, one finds an upper bound $N_i \in \mathbb{N}$ for each $Y_i$. Taking the sum, $\cup_{i \leq k} Y_i$ (and hence $\cup_{i \leq k} X_i$) is finite. One obtains item (x3) in the same way: let $Z_i$ be the set $X_i$ moved to $(i + 1, i + 2)$ without shrinking for $i \in \mathbb{N}$. Then the function $\sum_{i \in \mathbb{N}} 1_{Z_i}$ is regular on $\mathbb{R}$ and item (x) provides an enumeration of $\cup_{i \in \mathbb{N}} Z_i$, which readily yields CUC$_{\mathbb{N}}$. Using this enumeration and $\mu^2$, one obtains the function $h : \mathbb{N} \to \mathbb{N}$ as in the uniform finite union theorem.

Secondly, we establish $(\omega) \to (\beta) \to (\gamma) \to (\delta)$. Now, $(\delta) \to (\gamma)$ follows from the proof of $(\beta) \to (\delta)$ in Theorem 3.34 by replacing $[0, 1]$ by $\mathbb{R}$. In turn, $(\omega) \to (\beta)$ is trivial while $(\gamma) \to (\delta)$ is proved as follows: let $(X_n)_{n \in \mathbb{N}}$ be a sequence of finite sets in $[0, 1]$ and define the following function:

$$g(x) := \begin{cases} \frac{1}{2^n} & x \in X_n \text{ and } n \text{ is the least such number} \\ 0 & \text{otherwise} \end{cases}.$$ (3.32)

To show that $g : [0, 1] \to \mathbb{R}$ is regulated, fix $x \in [0, 1]$ and $n \in \mathbb{N}$. Then $\cup_{i \leq k} X_i$ is finite by the finite union theorem, which is available due to item (x2) from the first paragraph of this proof. Then $\exists m(x)(\forall y \in B(x, \frac{1}{2^m}) \setminus \{x\}) (y \notin \cup_{i \leq k} X_i)$ readily follows by contradiction. By definition, $g(x) \leq \frac{1}{2^{m(x)}}$ on this punctured disc, i.e. $g$ becomes arbitrarily small near $x$, implying $g(x+) = 0 = g(x-)$. Item (g) now provides a list $(x_n)_{n \in \mathbb{N}}$ with all points where $g$ is discontinuous; this sequence also

14Suppose $(\forall y \in B(x, \frac{1}{2^m}) \setminus \{x\}) (y \notin \cup_{i \leq k} X_i)$ and apply $\text{QF-AC}^{0,1}$ to obtain a sequence in $\cup_{i \leq k} X_i$ converging to $x$. Using $\mu^2$, one modifies this sequence to guarantee it consists of pairwise disjoint reals. This however contradicts the finiteness of $\cup_{i \leq k} X_i$. 


enumerates $\bigcup_{n\in\mathbb{N}}X_n$. Indeed, $g(x_m-) = 0 = g(x_m+)$ implies that $g(x_m) > 0$ as $g$ must be discontinuous at $x_m$; by (3.32), $x_m$ is in $\bigcup_{n\in\mathbb{N}}X_n$. Similarly, if $y$ is in the latter union, we have $g(y) > 0$ by (3.32); hence $g$ is discontinuous at $y$, implying there is $m \in \mathbb{N}$ with $y = x_m$. Hence, $\bigcup_{n\in\mathbb{N}}X_n$ can be enumerated, which immediately implies $\text{cocode}_{\Theta}$ and $\text{CUC}_{\text{fin}}$. We have established $(\text{a}) \leftrightarrow (\text{b}) \leftrightarrow (\text{w})$.

Thirdly, we show that $(\text{b}) \rightarrow (\text{d}) \rightarrow (\text{e}) \rightarrow (\text{f})$. The implication $(\text{b}) \rightarrow (\text{d})$ follows from item (x3) and Theorem 3.34. The implication $(\text{d}) \rightarrow (\text{e})$ follows by the third item of Theorem 3.33. The set $1_{\bigcup_{n\in\mathbb{N}}X_n}$ is in $BV$, with variation bounded by 1. Applying item (3), one obtains an enumeration of $\bigcup_{n\in\mathbb{N}}X_n$, as required for item $(\text{b})$. By the previous paragraph, we obtain $(\text{a}) \leftrightarrow (\text{b}) \leftrightarrow (\text{c}) \leftrightarrow (\text{d}) \leftrightarrow (\text{e})$.

For each $i$, we show that $(\text{b}) \rightarrow (\text{f}) \rightarrow (\text{g}) \rightarrow (\text{h})$. The implication $(\text{b}) \rightarrow (\text{f})$ follows from item (x3) and the generalisation of (3.30) to arbitrary intervals $[0, y]$ for $y > 0$; the second part is essentially the same as the proof of $(\text{iii}) \rightarrow (\text{ii})$ in Theorem 3.34. The implication $(\text{f}) \rightarrow (\text{g})$ follows by the third item of Theorem 3.33. To prove that item (g) implies item (a), note that $1_{\bigcup_{n\in\mathbb{N}}Z_n}$ satisfies the conditions of item (g). Indeed, on the interval $[0, y]$ with $0 < y \leq m \in \mathbb{N}$, the function $1_{\bigcup_{n\in\mathbb{N}}Z_n}$ reduces to $1_{\bigcup_{n\leq m}Z_n}$, and the latter has bounded variation by the finite union theorem and the final item in Theorem 3.34. An enumeration of the points of discontinuity of $1_{\bigcup_{n\in\mathbb{N}}Z_n}$ readily yields an enumeration of $\bigcup_{n\in\mathbb{N}}X_n$, as required for item (a). By the previous paragraph, we obtain $(\text{a}) \leftrightarrow (\text{f}) \leftrightarrow \ldots \leftrightarrow (\text{g})$, i.e. all that remains is item (h).

Finally, we prove $(\text{h}) \rightarrow (\text{f}) \rightarrow (\text{g})$, finishing the theorem. Hence, assume (h) and fix $f : [0, 1] \rightarrow \mathbb{R}$ in $BV$ and consider $A_n$ a in (3.34), which is well-defined thanks to Theorem 3.33. The set $A_n$ is also finite as in the proof of Theorem 3.34 and we may assume $0, 1 \notin A_n$ for $n \in \mathbb{N}$. Now let $B_n$ be a copy of $A_n$ translated from $[0, 1]$ to $[n + 1, n + 2]$ for $n \in \mathbb{N}$. Then $B := \bigcup_{n\in\mathbb{N}}B_n$ has no limit points, which one proves (by contradiction) using $\text{QF-AC}^{0,1}$ and the Bolzano-Weierstrass theorem. Hence, item (h) yields an enumeration of $B$ and hence a sequence listing all points where $f$ is discontinuous. Similarly, item (h) implies the uniform finite union theorem. For the implication $(\text{e}) \rightarrow (\text{h})$, fix closed $A \subset \mathbb{R}$ with no limit points. Then $A_n := A \cap [-n, n]$ is finite for any $n \in \mathbb{N}$, which one proves (by contradiction) using $\text{QF-AC}^{0,1}$ and the Bolzano-Weierstrass theorem. By $\text{CUC}_{\text{fin}}$, $\bigcup_{n\in\mathbb{N}}A_n$ is countable, and can be enumerated using $\text{cocode}_{\Theta}$, and we are done.

By the proof of Theorem 3.35, item (f) implies the finite union theorem, while the same does not seem to hold for the Jordan decomposition theorem. We believe this is due to fact that ‘regulated’ is a local property while ‘bounded variation’ is a global property (of the domain). Moreover, there are many and very different intermediate spaces (see [1] or [90, Remark 4.13]) between the space of regulated and of $BV$-functions; each of these intermediate spaces yields an equivalent generalisation of e.g. item (g) in Theorem 3.35, also showcasing a certain robustness.
Next, by the following theorem, we may replace the finite union theorem in Theorem 3.35 by ‘more mathematical’ principles.

**Theorem 3.36 (ACA₀ + QF-AC⁰¹).** The higher items imply the lower items.

- The combination CUCₙ₀ + cocode₀.
- For f₁, ..., fₖ : [0, 1] → ℜ in BV, the sum ∑ₖᵢ₌₁ fᵢ is in BV.
- The finite union theorem.

**Proof.** For the first downward implication, the following set

\[ A_{n,i} := \{ x ∈ [0, 1] : |f_i(x⁺) - f_i(x)| > \frac{1}{2^n} ⋁ |f_i(x⁻) - f_i(x)| > \frac{1}{2^n} \} \]

is finite for all n ∈ N and i ≤ k if f₁, ..., fₖ ∈ BV. Clearly, the set \( B_n := \bigcup_{i ≤ k} A_{n,i} \) is finite. Hence, there is an enumeration of \( ∪_{n ∈ N} B_n \), yielding a sequence \( (x_n)_{n ∈ N} \) that lists all points of discontinuity of the functions \( f_i \) for \( i ≤ k \). Using (µ²), we can compute \( V_0^1(f_i) \) for \( i ≤ q \) as we can replace the usual supremum by one over \( N \) (and \( Q \)). The proof that \( V_0^1(f + g) ≤ V_0^1(f) + V_0^1(g) \) in [10] p. 57 essentially amounts to the triangle inequality over \( ℜ \), i.e. that \( \sum_{i=1}^{k} f_i \) is in BV now follows.

The second downward implication is straightforward as a characteristic function \( 1_X \) is in BV if \( X ⊂ [0, 1] \) is finite by Theorem 3.38.

We could replace the second item in Theorem 3.36 by the following statement:

for \( f \) in BV and \( 0 = x₀ < x₁ < \cdots < xₖ < xₖ₊₁ = 1 \), \( V_0^1(f) = \sum_{i=0}^{k} V_{x_i}^{x_{i+1}}(f) \), but this would entail a number of technical details. The same division property for the arc length of rectifiable functions would of course be rather natural.

Next, Jordan’s original motivation for introducing BV-functions in [43] was the convergence of Fourier series. Now, the latter always converges to \( \frac{f(x⁺) + f(x⁻)}{2} \) for \( f ∈ BV \). In this light, item (vi) from Theorem is equivalent to the following.

The uniform finite union theorem plus: for \( f ∈ BV \), there is a sequence enumerating all points where the Fourier series does not equal the function value.

To derive the centred statement, it is a somewhat tedious verification that ACA₀ + QF-AC⁰¹ can formalise the proof that the Fourier series of \( f ∈ BV \) always converges to \( \frac{f(x⁺) + f(x⁻)}{2} \). We refer to [92] for an elementary proof of this convergence result.

A more detailed discussion is in [96] §3, including various textbook proofs.

Finally, we have used ACA₀ (plus extensions) as our base theory in the above; the following theorem implies that (3²) can be expressed in terms of basic properties of regulated functions as well. We use ‘usco’ to abbreviate ‘upper semi-continuous’.

**Theorem 3.37 (RCA₀ + WKL).** The following are equivalent to (3²).

(i) There exists Riemann integrable \( f : [0, 1] → [0, 1] \), \( g : [0, 1] → ℜ \) such that \( g ∘ f \) is not Riemann integrable.

(ii) There exists a function that is not Riemann integrable.

(iii) There exists regulated \( f : [0, 1] → [0, 1] \), \( g : [0, 1] → ℜ \) such that \( g ∘ f \) is not regulated.

(iv) There exists a function that is not regulated.

(v) There exists \( f : [0, 1] → [0, 1] \), \( g : [0, 1] → ℜ \) in Baire 1 such that \( g ∘ f \) is not in Baire 1.

(vi) There exists a function \( f : [0, 1] → ℜ \) that is not Baire 1.

(vii) There exists usco \( f : [0, 1] → [0, 1] \), \( g : [0, 1] → ℜ \) such that \( g ∘ f \) is not usco.

(viii) There exists a function that is not everywhere usco.
There exists a function that is not everywhere quasi-continuous.
There exists a function that is not everywhere cliquish.

We only need WKL for the first and second items.

Proof. First of all, assume \( \exists^2 \) and define \( f : [0,1] \to [0,1] \) as follows:

\[
f(x) := \begin{cases} 
0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\
\frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } p, q \text{ are co-prime}.
\end{cases}
\]

(3.34)

Thomae introduces this function around 1875 in [110, p. 14, §20]; one readily verifies that Thomae’s function is Riemann integrable (with integral equal to zero) and regulated (with zero as left and right limits) on any interval. Now define \( g : [0,1] \to \mathbb{R} \) as 0 in case \( x = 0 \), and 1 otherwise; this function is trivially Riemann integrable and regulated. However, \( g \circ f \) is Dirichlet’s function \( 1_\mathbb{Q} \), i.e. the characteristic function of the rationals, which is trivially shown to be not Riemann integrable and not regulated. Thus, \( \exists^2 \) implies items (i)-(iv).

Secondly, assume item (iii) (similar for item (iv)) and note that \( g \circ f \) must be discontinuous, as continuous functions are trivially Baire 1; in this way, we obtain a functional that returns the numbers \( 0 \) in case \( x = 0 \) and \( 1 \) otherwise. We now show that

\[
\exists \exists^2 \text{ implies } \forall \exists^2
\]

To establish this result, fix a non-empty interval \( [c,d] \subset [0,1] \) such that \( f_N([c,d]) = [\frac{1}{4}, \frac{3}{4}] \).

To establish this result, fix a non-empty interval \( [a,b] \subset [0,1] \) and fix \( x < y \) such that \( x \in \mathbb{Q} \cap [a,b] \) and \( y \in [a,b] \setminus \mathbb{Q} \). Since \( (f_n)_{n \in \mathbb{N}} \) converges pointwise to \( 1_\mathbb{Q} \), there exists an interval \( [c,d] \subset [x,y] \subset [a,b] \) such that \( f_N([c,d]) = [\frac{1}{4}, \frac{3}{4}] \).

By [237, §3], \( \exists^2 \) is equivalent to the existence of a functional witnessing the intermediate value theorem. Hence, following the previous paragraph, \( \exists^2 \) readily yields a functional that returns the numbers \( N \in \mathbb{N} \) and \( c,d \in [0,1] \) as in the centred statement on input \( [a,b] \) and \( m \in \mathbb{N} \), where \( N \geq m \). Using the latter functional, one readily obtains sequences \( (c_n)_{n \in \mathbb{N}}, (d_n)_{n \in \mathbb{N}} \), and \( g \in \mathbb{N}^{\mathbb{N}} \) such that \( g(n) \geq m \), \( f_N(c_n)(c_n,d_n) = [\frac{1}{4}, \frac{3}{4}] \), and \( c_n - d_n < \frac{1}{m} \) for all \( n \in \mathbb{N} \). However, if \( c = \lim_{n \to \infty} c_n \), then \( 1_\mathbb{Q}(c) = \lim_{n \to \infty} f_{g(n)}(c) \in [\frac{1}{4}, \frac{3}{4}] \), a contradiction. Hence, we have proved item (iii) and (iv). Since \( 1_\mathbb{Q} \) is not usco (or quasic-continuous or
cliquish) by definition, the equivalence between \((\exists^2)\) and items \(\text{(vii)-(x)}\) follows in the same way.

The previous theorem yields the following strange result by contraposition: if all functions on \(\mathbb{R}\) are Baire 1, then all functions on \(\mathbb{R}\) are continuous. In this light, Brouwer’s theorem is not an isolated event, but rather the limit of a certain process. One cannot push the previous equivalences much beyond Baire 1, as follows.

**Theorem 3.38** \((\text{ACA}_0^{\omega} + \text{IND}_0)\). The principle \(\text{NIN}\) follows from the statement: there is a \([0, 1] \to \mathbb{R}\) function that is not Baire 2.

**Proof.** Fix \(f : [0, 1] \to \mathbb{R}\) and let \(Y : [0, 1] \to \mathbb{N}\) be injective. Now define \(f_n(x)\) as \(f(x)\) in case \(Y(x) \leq n\), and 0 otherwise. Clearly, \(f\) is the pointwise limit of the sequence \((f_n)_{n \in \mathbb{N}}\). Now fix some \(n_0 \in \mathbb{N}\) use \(\text{IND}_0\) to enumerate all \(x \in [0, 1]\) such that \(Y(x) \leq n_0\). With this finite sequence, one readily defines a sequence of continuous functions converging to \(f_{n_0}\), which shows that the latter is Baire 1. □

In conclusion, we note that the insights in this section (esp. regarding Definition 3.30) came about after a recent FOM-discussion initiated by Friedman [28].

3.3.3. **On the choice of definitions.** In this section, we discuss our choice of definitions and provide some motivation.

First of all, the following remark provides some motivation for the use of our definitions of finite and closed set as in Definitions 1.2 and 3.30.

**Remark 3.39.** As discussed above, the sets \(A_n\) from (3.27) are finite and hence closed. In particular, working in \(\text{ZF}\) (or even \(\text{Z}_2\) from Section A.1.4), the following objects can be constructed:

- for \(n \in \mathbb{N}\), an injection \(Y_n\) from \(A_n\) to some \(\{0, 1, \ldots, k\}\) with \(k \in \mathbb{N}\),
- for \(m \in \mathbb{N}\), an RM-code \(C_m\) (see [103, II.5.6]) for the closed sets \(A_m\).

However, it is shown in [97, 98] that neither \(Y_n\) nor \(C_n\) are computable (in the sense of Kleene S1-S9) in terms of any \(S_m^2\) and the other data. Hence, it seems \(Z_2\) cannot prove the general existence of \(Y_n\) and \(C_n\) as in the previous items. By contrast, the system \(\text{ACA}_0^{\omega}\) (and even fragments) suffice to show that \(A_n\) from (3.27) is finite in the sense of Definition 3.30 and closed in the sense of Definition 1.2.

In conclusion, the study of \(BV\)-functions readily yields finite (resp. closed) sets for which there is no reasonable injection to some fragment of \(\mathbb{N}\) (resp. RM-code). This observation justifies our choice of definitions of closed and finite set as in Definitions 1.2 and 3.30.

Secondly, Remark 3.39 has some ramifications for our choice of the definition of ‘countable set’, as follows. Indeed, one could reformulate \(\text{CUC}_m^\infty + \text{cocode}_0\) as:

- **a height countable set in the unit interval can be enumerated**, where the boldface notion is defined as follows.

**Definition 3.40.** [Height countable] A set \(A \subset \mathbb{R}\) is **height countable** if there is a height \(H : \mathbb{R} \to \mathbb{N}\) for \(A\), i.e. for all \(n \in \mathbb{N}\), \(A_n := \{x \in A : H(x) < n\}\) is finite.

The notion of ‘height’ is mentioned in e.g. [41, 51, 61, 86, 114] in connection to countability. Now, as to the naturalness of Definition 3.40 consider the set of discontinuities of a function \(f \in BV\) (or even regular), definable in \(\text{ACA}_0^{\omega}\):

\[
A := \{x \in [0, 1] : f(x^+) \neq f(x^-)\}. 
\]
The set $A$ is trivially height countable and central to many proofs in [1]. As discussed in [95], no $S^2_m$ suffices to compute an injection from $A$ to $\mathbb{N}$ in general.

In conclusion, the textbook study of $BV$-functions yields height countable sets occurring ‘in the wild’ but with no ‘reasonable’ injection (or bijection) to $\mathbb{N}$. Hence, it seems we have a choice between using CUC$_{fin}$ or adopting Definition 3.40 as our definition of countable set. We choose the former option as e.g. Theorem 3.35 is still quite elegant. By contrast, Definition 3.40 is used in [95, 96], as this seems to be the only way of obtaining elegant equivalences for the uncountability of $\mathbb{R}$. To be absolutely clear, as documented in [95, 96], the statement the unit interval is not height countable readily gives rise to many interesting equivalences while NIN does not (seem to), say working over ACA$^\omega_0$ + QF-AC$^{0,1}$ or fragments.

Thirdly, our notion of ‘finite set’ as in Definition 3.30 is different from the mainstream set theory definition (see Footnote 13), for reasons discussed in Remark 3.31. Nonetheless, the reader may desire an equivalence in Theorem 3.35 involving a (more) mainstream definition of finite set. To this end, let CUC$_{fin}$ be CUC$_{fin}$ formulated with the following finiteness notion.

**Definition 3.41.** [Set theory finite] A set $X \subset \mathbb{R}$ is set theory finite if there are $k \in \mathbb{N}$ and $Y : [0,1] \to \mathbb{N}$ such that on $X$, $Y$ is bounded by $k$ and injective.

One readily shows that the following are equivalent, say over ACA$^\omega_0$ + QF-AC$^{0,1}$.

- (Bolzano-Weierstrass) For $X \subset [0,1]$ which is not set theory finite, there is a limit point $y \in [0,1]$, i.e. $(\forall k \in \mathbb{N})(\exists x \in X)(|x - y| < \frac{1}{k})$.
- A finite set (in the sense of Definition 3.30) is set theory finite.

Letting $BW$ be the first item, we note that item (a) from Theorem 3.35 is equivalent to $BW + CUC'_{fin} + cocode_0$, and where the latter uses Definition 3.41 exclusively. Jordan mentions $BW$ in e.g. [44] p. 23, §27. We intend to explore the content of the previous remark in a future paper.

Fourth, a regulated function has bounded Waterman variation ([1, Prop. 2.24]). The latter notion amounts to replacing $|f(x_{i+1}) - f(x_i)|$ by $\lambda_i |f(x_{i+1}) - f(x_i)|$ in (2.38), for a Waterman sequence $(\lambda_k)_{k \in \mathbb{N}}$ as in [1, Def. 2.15]. Now, for $BV$-functions with variation bounded by 1, (3.31) can have at most $2^n$ elements. Functions of bounded Waterman variation similarly have explicit upper bounds -defined in terms of $(\lambda_k)_{k \in \mathbb{N}}$ and $\exists^*$- on the set (3.31). In this way, the regulated function $g$ from (3.32) has bounded Waterman variation and this readily yields an upper bound function for $(X_n)_{n \in \mathbb{N}}$ as in the uniform finite union theorem. Hence, we can avoid the use of the latter (and perhaps even QF-AC$^{0,1}$) if we have access to the information provided by the bounded Waterman variation of a regulated function.

3.4. **Unordered sums.** We develop the RM-study of unordered sums, which are a device for bestowing meaning upon sums involving uncountable index sets. We first introduce the relevant definitions and prove the equivalence between cocode$_0$ and basic properties of unordered sums in Theorem 3.43.

First of all, unordered sums are essentially ‘uncountable sums’ $\sum_{x \in I} f(x)$ for any index set $I$ and $f : I \to \mathbb{R}$. A central result is that if $\sum_{x \in I} f(x)$ somehow exists, it must be a ‘normal’ series of the form $\sum_{i \in \mathbb{N}} f(y_i)$, i.e. $f(x) = 0$ for all but countably many $x \in [0,1]$. Tao mentions this theorem in [109], p. xiii.

By way of motivation, there is considerable historical and conceptual interest in this topic: Kelley notes in [46] p. 64] that E.H. Moore’s study of unordered sums in
led to the concept of net with his student H.L. Smith ([65]). Unordered sums can be found in (self-proclaimed) basic or applied textbooks ([39][104]) and can be used to develop measure theory ([46] p. 79). Moreover, Tukey shows in [113] that topology can be developed using phalanxes, which are nets with the same index sets as unordered sums.

Now, unordered sums are just a special kind of net and \( a : [0, 1] \to \mathbb{R} \) is therefore written \((a_x)_{x \in [0, 1]}\) in this context to suggest the connection to nets. The associated notation \(\sum_{x \in [0, 1]} a_x\) is purely symbolic. We only need the following notions in the below. Let \(|\mathbb{R}|\) be the set of all finite sequences of reals without repetitions.

**Definition 3.42.** Let \( a : [0, 1] \to \mathbb{R} \) be any mapping, also denoted \((a_x)_{x \in [0, 1]}\).
- We say that \(\sum_{x \in [0, 1]} a_x\) is bounded if there is \( N_0 \in \mathbb{N} \) such that for any \( J \in \text{fin}(\mathbb{R}) \), \( N_0 \geq |\sum_{x \in J} a_x| \).
- We say that \((a_x)_{x \in [0, 1]}\) is convergent to \( a \in \mathbb{R} \) if for all \( k \in \mathbb{N} \), there is \( I \in \text{fin}(\mathbb{R}) \) such that for \( J \subseteq I \), we have \(|a - \sum_{x \in J} a_x| < \frac{1}{k}\).

Note that in the first item, \( \Phi \) is called a Cauchy modulus. For simplicity, we focus on positive unordered sums, i.e. \((a_x)_{x \in [0, 1]}\) such that \( a_x \geq 0 \) for \( x \in [0, 1] \).

Secondly, we establish equivalences basic properties of unordered sums. We note that QF-AC\(^0,1\) is no longer needed in the base theory, while \( \text{cocode}_0 \) is equivalent to the first item in Theorem 3.33 given CUC\(_{\text{fin}}\).

**Theorem 3.33 (ACA\(_0^\omega\)).** The following are equivalent.

(i) Let \((X_n)_{n \in \mathbb{N}}\) and \( g \in \mathbb{N}^\mathbb{N} \) be such that \( g(n) \) is an upper bound on the size of \( X_n \), for all \( n \in \mathbb{N} \). Then \( \bigcup_{n \in \mathbb{N}} X_n \), can be enumerated.

(ii) For a positive and bounded unordered sum \(\sum_{x \in [0, 1]} a_x\), there is a sequence \((y_n)_{n \in \mathbb{N}}\) of reals such that \( a_y = 0 \) for all \( y \) not in this sequence.

Assuming QF-AC\(^0,1\), the above are equivalent to:

(iii) A positive bounded unordered sum \(\sum_{x \in [0, 1]} a_x\) is convergent to some \( a \in \mathbb{R} \).

**Proof.** The equivalence between items (i) and (ii) is as follows: assume the latter and let \((X_n)_{n \in \mathbb{N}}\) and \( g : \mathbb{N} \to \mathbb{N} \) be as in item (i). Define \((a_x)_{x \in [0, 1]}\) as follows:

\[
a_x := \begin{cases} 
0 & x \notin \bigcup_{n \in \mathbb{N}} X_n \\
\frac{1}{2^{n+1}} & x \in X_n \text{ and } n \text{ is the least such natural number}
\end{cases}
\]

Clearly, this unordered sum has upper bound 1. If \((y_n)_{n \in \mathbb{N}}\) is as in item (ii), we obtain an enumeration of \( \bigcup_{n \in \mathbb{N}} X_n \). Now assume item (i) and let \((a_x)_{x \in [0, 1]}\) be an unordered sum that is Cauchy, and consider the following set:

\[
X_n := \{ x \in [0, 1] : a_x > 1/2^n \}. \tag{3.36}
\]

Let \( N_0 \in \mathbb{N} \) be an upper bound for \( \sum_{x \in K} a_x \) for any \( K \in \text{fin}(\mathbb{R}) \). Hence, the finite set \( X_n \) in (3.36) has size at most \( 2^n N_0 \). An enumeration of \( \bigcup_{n \in \mathbb{N}} X_n \) immediately yields the sequence as in item (ii).

The implication (ii) \(\Rightarrow\) (iii) is straightforward: the former guarantees that an unordered sum is a ‘normal’ series, which must converge by the monotone convergence theorem (provable in ACA\(_0\) by [103 III.2]). Now assume item (iii) and note that convergence of an unordered sum to some \( a \in \mathbb{R} \) implies

\[
(\forall k \in \mathbb{N})(\exists I \in \text{fin}(\mathbb{R}))(|a - \sum_{x \in I} a_x| < \frac{1}{2^k}). \tag{3.37}
\]
Apply QF-AC\(^0\) to (3.37) to obtain a sequence \((I_n)_{n\in\mathbb{N}}\) of finite sequences of reals. This sequence must contain all \(y \in \mathbb{R}\) such that \(a_y \neq 0\), otherwise (3.37) would be false. Use Feferman’s \(\mu^2\) to remove all other reals, and we are done. \(\square\)

The following result is perhaps more surprising. Note that the second item also follows from item (i) in Theorem 3.43.

**Corollary 3.44** (\(\text{ACA}_\omega^0 + \text{IND}_1\)). The higher items imply the lower ones.

- QF-AC\(^0\).
- For a positive and convergent unordered sum \(\sum_{x\in[0,1]} a_x\), there is a sequence \((y_n)_{n\in\mathbb{N}}\) of reals such that \(a_y = 0\) for all \(y\) not in this sequence.
- \(\text{cocode}_1\).

**Proof.** The first downward implication is proved as in the proof of the theorem. For the second downward application, let \(A \subset [0,1]\) and \(Y : [0,1] \to \mathbb{R}\) be such that the latter is bijective on the former. Define \(a_x := \frac{1}{Y(x)}\) if \(x \in A\), and 0 otherwise. One readily proves that \(\sum_{x\in[0,1]} a_x\) is convergent to 1, for which \(\text{IND}_1\) is needed. The sequence from the second item now yields the enumeration of the set \(A\) required by \(\text{cocode}_1\). \(\square\)

One can derive a version of QF-AC\(^0\) involving an ‘at most finitely many’ condition on the existential quantifier in the antecedent.

**Acknowledgement 3.45.** We thank Anil Nerode for his valuable advice. We also thank the anonymous referee for the many detailed and helpful suggestions. Our research was supported by the Deutsche Forschungsgemeinschaft via the DFG grant SA3418/1-1. Initial results were obtained during the stimulating MFO workshop (ID 2046) on proof theory and constructive mathematics in Oberwolfach in early Nov. 2020. We express our gratitude towards the aforementioned institutions.

**Appendix A. Reverse Mathematics: introduction and definitions**

**A.1. Reverse Mathematics.** We discuss Reverse Mathematics (Section A.1.1) and introduce -in full detail- Kohlenbach’s base theory of higher-order Reverse Mathematics (Section A.1.2). Some essential axioms, functionals, and notations may be found in Sections A.1.3 and A.1.4.

**A.1.1. Introduction.** Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([26,27]) and developed extensively by Simpson ([103]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [106] for a basic introduction to RM and to [102,103] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach’s higher-order RM ([49]) essential to this paper, including the base theory \(\text{RCA}_\omega^0\) (Definition A.1).

First of all, in contrast to ‘classical’ RM based on second-order arithmetic \(\mathbb{Z}_2\), higher-order RM uses \(L_\omega\), the richer language of higher-order arithmetic. Indeed, while the former is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of all finite types \(T\), defined by the two clauses:
functions from numbers to numbers, and \( \sigma \) objects of type \( \sigma \) where 0 is the type of natural numbers, and \( \sigma \rightarrow \tau \) is the type of mappings from objects of type \( \sigma \) to objects of type \( \tau \). In this way, \( 1 \equiv 0 \rightarrow 0 \) is the type of functions from numbers to numbers, and \( n + 1 \equiv n \rightarrow 0 \). Viewing sets as given by characteristic functions, we note that \( \mathbb{Z} \in \rho \) constants of \( L \) in terms of ‘\( =_0 \)’ to have their usual meaning as operations on \( L \).

Secondly, the language \( L_\omega \) includes variables \( x^\rho, y^\rho, z^\rho, \ldots \) of any finite type \( \rho \in T \). Types may be omitted when they can be inferred from context. The constants of \( L_\omega \) include the type 0 objects 0, 1 and \( \prec_0, +_0, \times_0, =_0 \) which are intended to have their usual meaning as operations on \( \mathbb{N} \). Equality at higher types is defined in terms of \( '=' \) as follows: for any objects \( x^\tau, y^\tau \), we have

\[
[x =_\tau y] \equiv (\forall z^\tau_1 \ldots z^\tau_k)[xz_1 \ldots z_k =_0 yz_1 \ldots z_k],
\]

(A.1)

if the type \( \tau \) is composed as \( \tau \equiv (\tau_1 \rightarrow \ldots \rightarrow \tau_k \rightarrow 0) \). Furthermore, \( L_\omega \) also includes the recursor constant \( R_\sigma \) for any \( \sigma \in T \), which allows for iteration on type \( \sigma \)-objects as in the special case \( [A.2] \). Formulas and terms are defined as usual. One obtains the sub-language \( L_{n+2} \) by restricting the above type formation rule to produce only type \( n + 1 \) objects (and related types of similar complexity).

A.1.2. The base theory of higher-order Reverse Mathematics. We introduce Kohlenbach’s base theory \( \text{RCA}_0^\omega \), first introduced in [49, §2].

**Definition A.1.** The base theory \( \text{RCA}_0^\omega \) consists of the following axioms.

(a) Basic axioms expressing that 0, 1, \( \prec_0, +_0, \times_0 \) form an ordered semi-ring with equality \( =_0 \).

(b) Basic axioms defining the well-known \( \Pi \) and \( \Sigma \) combinators (aka \( K \) and \( S \) in [2]), which allow for the definition of \( \lambda \)-abstraction.

(c) The defining axiom of the recursor constant \( R_0 \): for \( m^0 \) and \( f^1 \):

\[
R_0(f, m, 0) := m \text{ and } R_0(f, m, n + 1) := f(n, R_0(f, m, n)).
\]

(A.2)

(d) The **axiom of extensionality**: for all \( \rho, \tau \in T \), we have:

\[
(\forall x^\rho, y^\rho, \varphi^{\rho \rightarrow \tau})[x =_\rho y \rightarrow \varphi(x) =_\tau \varphi(y)].
\]

(E\(\rho,\tau\))

(e) The induction axiom for quantifier-free formulas of \( L_\omega \).

(f) QF-AC\(1,0\): the quantifier-free Axiom of Choice as in Definition \( [A.2] \).

Note that variables (of any finite type) are allowed in quantifier-free formulas of the language \( L_\omega \): only quantifiers are banned. Recursion as in \( [A.2] \) is called *primitive recursion*: the class of functionals obtained from \( R_\rho \) for all \( \rho \in T \) is called *Gödel’s system* \( T \) of all (higher-order) primitive recursive functionals.

**Definition A.2.** The axiom QF-AC consists of the following for all \( \sigma, \tau \in T \):

\[
(\forall x^\sigma)(\exists y^\tau).A(x, y) \rightarrow (\exists Y^{\sigma \rightarrow \tau})(\forall x^\sigma).A(x, Y(x)),
\]

(QF-AC\(\sigma,\tau\))

for any quantifier-free formula \( A \) in the language of \( L_\omega \).

As discussed in [49, §2], \( \text{RCA}_0^\omega \) and \( \text{RCA}_0 \) prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. This conservation result is obtained via the so-called ECF-interpretation discussed in Remark \( [1,12] \).
A.1.3. Notations and the like. We introduce the usual notations for common mathematical notions, like real numbers, as also introduced in [49].

**Definition A.3** (Real numbers and related notions in RCAₙ).

(a) Natural numbers correspond to type zero objects, and we use ’n’ and ’n ∈ N’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ’q ∈ Q’ and ’<q’ have their usual meaning.

(b) Real numbers are coded by fast-converging Cauchy sequences q(·) : N → Q, i.e. such that (∀n, i)(|q_n - q_{n+i}| < q 1/n). We use Kohlenbach’s ’hat function’ from [49] p. 289] to guarantee that every q defines a real number.

(c) We write ’x ∈ R’ to express that x¹ := (q¹(·)) represents a real as in the previous item and write [x](k) := qk for the k-th approximation of x.

(d) Two reals x, y represented by q(·) and r(·) are equal, denoted x =y, if (∀n)(|q_n - r_n| ≤ 2⁻ⁿ⁺¹). Inequality ’<’ is defined similarly. We sometimes omit the subscript ’R’ if it is clear from context.

(e) Functions F : R → R are represented by Φ¹⁺¹ mapping equal reals to equal reals, i.e. extensionality as in (∀x, y ∈ R)(x =y → Φ(x) = Φ(y)).

(f) The relation ’x ≤ y’ is defined as in [49] but with ’<’ instead of ’=’.

(g) Binary sequences are denoted ’f’, ’g’, 0 ≤ 1', but also ’f, g ∈ C’ or ’f, g ∈ N²’. Elements of Baire space are given by f¹, g¹, but also denoted ’f, g ∈ N²’.

(h) Sets of type ρ objects Xρ=0, Yρ=0, ..., are given by their characteristic functions F_Xρ=0 ≤ρ→ 1, i.e. we write ’x ∈ X’ for F_X(x) = 1.

For completeness, we list the following notational convention for finite sequences.

**Notation A.4** (Finite sequences). The type for ’finite sequences of objects of type ρ’ is denoted ρ*, which we shall only use for ρ = 0, 1. Since the usual coding of pairs of numbers goes through in RCAₙ, we shall not always distinguish between 0 and 0’. Similarly, we assume a fixed coding for finite sequences of type 1 and shall make use of the type ’1*’. In general, we do not always distinguish between ’sρ’ and ’s(ρ’), where the former is ’the object s of type ρ’, and the latter is ’the sequence of type ρ* with only element sρ’. The empty sequence for the type ρ* is denoted by ’()', usually with the typing omitted.

Furthermore, we denote by ’|s| = n’ the length of the finite sequence sρ = ⟨s₀, s₁, ..., sₙ−₁⟩, where |()| = 0, i.e. the empty sequence has length zero. For sequences sρ¹, tρ², we denote by ’s∗t’ the concatenation of s and t, i.e. (s∗t)(i) = s(i) for i < |s| and (s∗t)(j) = t(|s|−j) for |s| ≤ j < |s|+|t|. For a sequence sρ, we define ΠN := ⟨s(0), s(1), ..., s(N−1)⟩ for N⁰ < |s|. For a sequence αₙ=ρ, we also write ΠN = ⟨α(0), α(1), ..., α(N−1)⟩ for any N⁰. By way of shorthand, (∀q∈Q)(A(q)) abbreviates (∀q∈Q)(A(Q(q))), which is (equivalent to) quantifier-free if A is.

A.1.4. Some comprehension functionals. As noted in Section 1.2 the logical hardness of a theorem is measured via what fragment of the comprehension axiom is needed for a proof. For this reason, we introduce some axioms and functionals related to higher-order comprehension in this section. We are mostly dealing with conventional comprehension here, i.e. only parameters over N and N^N are allowed in formula classes like Πₙ¹ and Σₙ¹.

First of all, the following functional is clearly discontinuous at f = 11;...; in fact, (Ω²) is equivalent to the existence of F : R → R such that F(x) = 1 if x ∈ R, 0,
and 0 otherwise ([19] §3). This fact shall be repeated often.

\[(∃\varphi^2 ≤ 2_1)(∀f^1) [ (∃n)(f(n) = 0) ↔ \varphi(f) = 0]. \tag{∃^2}\]

Related to (∃^2), the functional µ^2 in (µ^2) is also called Feferman’s μ ([2]).

\[(∃\mu^2)(∀f^1) [ (∃n)(f(n) = 0) → [f(\mu(f)) = 0 ∧ (∀i < \mu(f))(f(i) ≠ 0)] ∧ (∃n)(f(n) ≠ 0) → \mu(f) = 0], \tag{µ^2} \]

We have (∃^2) ↔ (µ^2) over RCA^ω_0 and ACA^ω_0 ≡ RCA^ω_0 + (∃^2) proves the same sentences as ACA_0 by [10] Theorem 2.5.

Secondly, the functional S^2 in (S^2) is called the Suslin functional ([19]).

\[(∃S^2 ≤ 2_1)(∀f^1) [ [ (∃n^1)(∀n^0)(f(\overline{n}) = 0) ↔ S(f) = 0], \tag{S^2} \]

The system Π^1_1-CACA^ω_0 ≡ RCA^ω_0 + (S^2) proves the same Π^1_1-sentences as Π^1_1-CACA_0 by [88, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ^1_1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S^2_k which decides the truth or falsity of Σ^1_k-formulas from L_2; we also define the system Π^1_1-CACA^ω_0 as RCA^ω_0 + (S^2_k), where (S^2_k) expresses that S^2_k exists. We note that the operators υ_3 from [8] p. 129] are essentially S^2_k strengthened to return a witness (if existant) to the Σ^1_k-formula at hand.

Thirdly, full second-order arithmetic Z_2 is readily derived from ι_kΠ^1_1 -CA^ω_0, or from:

\[(∃E^3 ≤ 3_1)(∀Y^2) [ (∃f^1)(Y(f) = 0) ↔ E(Y) = 0], \tag{∃^3} \]

and we therefore define Z^Ω_2 ≡ RCA^ω_0 + (∃^3) and Z^Ω_2 ≡ ι_kΠ^1_1 -CA^ω_0, which are conservative over Z_2 by [10] Cor. 2.6]. Despite this close connection, Z^Ω_2 and Z^Ω can behave quite differently, as discussed in e.g. [71] §2.2. The functional from (∃^3) is also called ‘∃^31’, and we use the same convention for other functionals.

References

[1] Jürgen Appell, Józef Banaś, and Nelson Merentes, Bounded variation and around, De Gruyter Series in Nonlinear Analysis and Applications, vol. 17, De Gruyter, Berlin, 2014.
[2] Jeremy Avigad and Solomon Feferman, Gödel’s functional (“Dialectica”) interpretation, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, 1998, pp. 337–405.
[3] Douglas Bridges, A constructive look at functions of bounded variation, Bull. London Math. Soc. 32 (2000), no. 3, 316–324.
[4] Douglas Bridges and Ayan Mahalanobis, Bounded variation implies regulated: a constructive proof, J. Symbolic Logic 66 (2001), no. 4, 1695–1700.
[5] Douglas Bridges, Fred Richman, and Peter Schuster, A weak countable choice principle, Proc. Amer. Math. Soc. 128 (2000), no. 9, 2749–2752.
[6] E. Borel, Leçons sur la théorie des fonctions, Gauthier-Villars, Paris, 1898.
[7] N. Bourbaki, Éléments de mathématique, Livre IV: Fonctions d’une variable réelle. (Théorie élémentaire), Actualités Sci. Ind., no. 1132, Hermann et Cie., Paris, 1951 (French).
[8] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, Iterated inductive definitions and subsystems of analysis, LNM 897, Springer, 1981.
[9] Georg Cantor, Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen, J. Reine Angew. Math. 77 (1874), 258–262.
[10] , Ein Beitrag zur Mannigfaltigkeitslehre., Journal für die reine und angewandte Mathematik 84 (1878), 242–258.
[11] , Ueber unendliche, lineare Punktmannichfaltigkeiten, Math. Ann. 21 (1883), no. 4, 545–591.
[12] , Zur Lehre vom Transfiniten: gesammelte Abhandlungen aus der Zeitschrift für Philosophie und Philosophische Kritik, vom Jahre 1887, Pfeffer, Halle, 1890.
ON ROBUST THEOREMS DUE TO BOLZANO, WEIERSTRASS, JORDAN, CANTOR

[13] __________, Beiträge zur Begründung der transfiniten Mengenlehre, Mathematische Annalen 46 (1895), 481-512.
[14] __________, Gesammelte Abhandlungen mathematischen und philosophischen Inhalts, Springer, 1980. Reprint of the 1932 original, vii+489.
[15] Douglas Cenzer and Jeffrey B. Remmel, Proof-theoretic strength of the stable marriage theorem and other problems, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 67–103.
[16] Peter Clote, The metamathematics of scattered linear orderings, Arch. Math. Logic 29 (1989), no. 1, 9–20.
[17] J. L. Coolidge, The lengths of curves, Amer. Math. Monthly 60 (1953), 89–93.
[18] Pierre Cousin, Sur les fonctions de n variables complexes, Acta Math. 19 (1895), 1–61.
[19] Lejeune P. G. Dirichlet, Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinusreihen, Repertorium der physik, von H.W. Dove und L. Moser, bd. 1, 1837.
[20] Enno Dirksen, Ueber die Anwendung der Analysis auf die Rectification der Curven, Akademie der Wissenschaften zu Berlin (1833), 123-168.
[21] François G. Dorais, Classical consequences of continuous choice principles from intuitionistic analysis, Notre Dame J. Form. Log. 55 (2014), no. 1, 25–39.
[22] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer, On uniform relationships between combinatorial problems, Trans. Amer. Math. Soc. 368 (2016), no. 2, 1321–1359.
[23] J. M. C. Dunham, Des méthodes dans les sciences de raisonnement. Application des méthodes générales à la science des nombres et à la science de l’étendue, Vol II, Gauthier-Villars, 1866.
[24] Solomon Feferman, How a Little Bit goes a Long Way: Predicative Foundations of Analysis, 2013. unpublished notes from 1977-1981 with updated introduction, https://math.stanford.edu/~feferman/papers/pfa(1).pdf.
[25] Roland Fraïssé, Theory of relations, Studies in Logic and the Foundations of Mathematics, vol. 145, North-Holland, 2000. With an appendix by Norbert Sauer.
[26] Harvey Friedman, Some systems of second order arithmetic and their use, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.
[27] __________, Systems of second order arithmetic with restricted induction, I & II (Abstracts), Journal of Symbolic Logic 41 (1976), 557–559.
[28] __________, Remarks on Reverse Mathematics /1, FOM mailing list (Sept. 21st, 2021). https://cs.nyu.edu/pipermail/fom/2021-September/022875.html.
[29] Makoto Fujita and Keita Yokoyama, A note on the sequential version of Π⁰₁ statements, Lecture Notes in Comput. Sci., vol. 7921, Springer, Heidelberg, 2013, pp. 171–180.
[30] Makoto Fujita, Kojiro Higuchi, and Takayuki Kihara, On the strength of marriage theorems and uniformity, MLQ Math. Log. Q. 60 (2014), no. 3.
[31] Noam Greenberg, Joseph S. Miller, and André Nies, Highness properties close to PA-completeness. To appear in Israel Journal of Mathematics (2019).
[32] Horst Herrlich, Axioms of choice, Lecture Notes in Mathematics, vol. 1876, Springer, 2006.
[33] Arend Heyting, Recent progress in intuitionistic analysis, Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968), North-Holland, Amsterdam, 1970, pp. 95–100.
[34] David Hilbert, Über das Unendliche, Math. Ann. 95 (1926), no. 1, 161–190 (German).
[35] Denis R. Hirschfeldt, Slicing the truth, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28, World Scientific Publishing, 2015.
[36] Jeffry L. Hirst and Carl Mummert, Reverse mathematics and uniformity in proofs without excluded middle, Notre Dame J. Form. Log. 52 (2011), no. 2, 149–162.
[37] __________, Representations of reals in reverse mathematics, Bull. Pol. Acad. Sci. Math. 55 (2007), no. 4, 303–316.
[38] Karel Hrbacek and Thomas Jech, Introduction to set theory, 3rd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 220, Marcel Dekker, Inc., New York, 1999.
[39] John K. Hunter and Bruno Nachtergaele, Applied analysis, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
[40] James Hunter, Higher-order reverse topology, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Wisconsin - Madison.
[41] M. N. Huxley, *Area, lattice points, and exponential sums*, London Mathematical Society Monographs. New Series, vol. 13, The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[42] Thomas J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland, 1973.

[43] Camille Jordan, *Sur la série de Fourier*, Comptes rendus de l’Académie des Sciences, Paris, Gauthier-Villars 92 (1881), 228–230.

[44] Camille Jordan, *Cours d’analyse de l’École polytechnique. Tome I*, Les Grands Classiques Gauthier-Villars, Éditions Jacques Gabay, 1991. Reprint of the third (1909) edition; first edition: 1883.

[45] Pierre Jullien, *Contribution à l’étude des types d’ordres dispersés*, PhD thesis, University of Marseilles, 1969.

[46] John L. Kelley, *General topology*, Springer-Verlag, 1975. Reprint of the 1955 edition; Graduate Texts in Mathematics, No. 27.

[47] Stephen C. Kleene, *Recursive functionals and quantifiers of finite types. I*, Trans. Amer. Math. Soc. 91 (1959), 1–52.

[48] Ulrich Kohlenbach, *Foundational and mathematical uses of higher types*, Reflections on the foundations of mathematics, Lect. Notes Log., vol. 15, ASL, 2002, pp. 92–116.

[49] ——, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295.

[50] ——, *Applied proof theory: proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.

[51] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis. Vol. 1. Metric and normed spaces*, Graylock Press, Rochester, N.Y., 1957. Translated from the first Russian edition by Leo F. Boron.

[52] T. W. Körner, *Fourier analysis*, Cambridge University Press, 1988.

[53] Alexander P. Kreuzer, *Bounded variation and the strength of Helly’s selection theorem*, Log. Methods Comput. Sci. 10 (2014), no. 4, 4:16, 15.

[54] Kenneth Kunen, *Set theory*, Studies in Logic, vol. 34, College Publications, London, 2011.

[55] Imre Lakatos, *Proofs and refutations*, Cambridge Philosophy Classics, Cambridge University Press, 2015. The logic of mathematical discovery; Edited by John Worrall and Elie Zahar; With a new preface by Paolo Mancosu; Originally published in 1976.

[56] Henri Lebesgue, *Comptes rendus et analyses: Review of Young and Young, The theory of sets of points*, Bulletin des sciences mathématiques 31 (1907), no. 2, 132–134.

[57] Ernst Lindelöf, *Sur Quelques Points De La Théorie Des Ensembles*, Comptes Rendus (1903), 697–700.

[58] John Longley and Dag Normann, *Higher-order Computability*, Theory and Applications of Computability, Springer, 2015.

[59] A. I. Mal’cev, *On ordered groups*, Izvestiya Akad. Nauk SSSR. Ser. Mat. 13 (1949), 473–482 (Russian).

[60] Per Martin-Löf, *The Hilbert-Brouwer controversy resolved?*, in: *One Hundred Years of Intuitionism (1907-2007)*, 1967, pp. 243–256.

[61] Victor H. Moll, *Numbers and functions*, Student Mathematical Library, vol. 65, American Mathematical Society, 2012.

[62] Antonio Montalbán, *Indecomposable linear orderings and hyperarithmetic analysis*, J. Math. Log. 6 (2006), no. 1, 89–120.

[63] Antonio Montalbán, *Open questions in reverse mathematics*, Bull. Sym. Logic 17 (2011), no. 3, 431–454.

[64] E. H. Moore, *Definition of Limit in General Integral Analysis*, Proceedings of the National Academy of Sciences of the United States of America 1 (1915), no. 12, 628–632.

[65] E. H. Moore and H. Smith, *A General Theory of Limits*, Amer. J. Math. 44 (1922), 102–121.

[66] Andrzej Mostowski, *Foundational studies. Selected works. Vol. I*, Studies in Logic and the Foundations of Mathematics, vol. 93, North-Holland; PWN—Polish Scientific Publishers, 1979.

[67] P. Muldowney, *A general theory of integration in function spaces, including Wiener and Feynman integration*, Vol. 153, Longman Scientific & Technical, Harlow; John Wiley, 1987.

[68] Carl Mummert and Stephen G. Simpson, *Reverse mathematics and II1 2 comprehension*, Bull. Symbolic Logic 11 (2005), no. 4, 526–533.
[69] Carl Mummert, On the reverse mathematics of general topology, ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The Pennsylvania State University.

[70] André Nies, Marcus A. Tripplett, and Keita Yokoyama, The reverse mathematics of theorems of Jordan and Lebesgue, The Journal of Symbolic Logic (2021), 1–18.

[71] Dag Normann and Sam Sanders, Nonstandard Analysis, Computability Theory, and their connections, Journal of Symbolic Logic 84 (2019), no. 4, 1422–1465.

[72] Michael Rathjen, The axiom of choice in Computability Theory and Reverse Mathematics, Journal of Logic and Computation 31 (2021), no. 1, 297–325.

[73] Michael Rathjen, The strength of compactness in Computability Theory and Nonstandard Analysis, Annals of Pure and Applied Logic, Article 102710 170 (2019), no. 11.

[74] Michael Rathjen, On the mathematical and foundational significance of the uncountable, Journal of Mathematical Logic, https://doi.org/10.1142/S0219061319500016 (2019).

[75] Michael Rathjen, Representations in measure theory, Submitted, arXiv: https://arxiv.org/abs/1902.02756 (2019).

[76] Michael Rathjen, Open sets in Reverse Mathematics and Computability Theory, Journal of Logic and Computation 30 (2020), no. 8, pp. 40.

[77] Michael Rathjen, Pincherle’s theorem in reverse mathematics and computability theory, Ann. Pure Appl. Logic 171 (2020), no. 5, 102788, 41.

[78] Michael Rathjen, The Axiom of Choice in Computability Theory and Reverse Mathematics, Journal of logic and computation 31 (2021), no. 1, 297–325.

[79] Michael Rathjen, On the uncountability of $\mathbb{R}$, Journal of Symbolic Logic, doi:10.1017/jsl.2022.27 (2022), 1-45.

[80] Michael Rathjen, Betwixt Turing and Kleene, LNCS 13137, proceedings of LFCS22 (2022), pp. 18.

[81] Michael Rathjen, On the computational properties of basic mathematical notions, Submitted, arxiv: https://arxiv.org/abs/2203.05250 (2022), pp. 43.

[82] Michael Rathjen, The art of ordinal analysis, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006.

[83] Judith Roitman, Introduction to modern set theory, Pure and Applied Mathematics (New York), John Wiley, 1990.

[84] Joseph G. Rosenstein, Linear orderings, Pure and Applied Mathematics, vol. 98, Academic Press, 1982.

[85] Fred Richman, Omniscience principles and functions of bounded variation, MLQ 48 (2002), 111–116.

[86] Halsey L. Royden, Real Analysis, Lecture Notes in Mathematics, Pearson Education, 1989.

[87] S.B. Russ, A translation of Bolzano’s paper on the intermediate value theorem., Hist. Math. 7 (1980), 156–185.

[88] Nobuyuki Sakamoto and Takeshi Yamazaki, Uniform versions of some axioms of second order arithmetic, MLQ Math. Log. Q. 50 (2004), no. 6, 587–593.

[89] Sam Sanders, Plato and the foundations of mathematics, Submitted, arxiv: https://arxiv.org/abs/1908.05676 (2019), pp. 40.

[90] Sam Sanders, Big in Reverse Mathematics: the uncountability of the reals, LNCS 13468, Proceedings of WoLLIC22, Springer (2022).

[91] Ludwig Scheeffer, Allgemeine Untersuchungen über Rectification der Curven, Acta Math. 5 (1884), no. 1, 49–82 (German).
[100] Paul Shafer, *The strength of compactness for countable complete linear orders*, Computability 9 (2020), no. 1, 25–36.
[101] Waclaw Sierpiński, *Sur une propriété des fonctions qui n’ont que des discontinuités de première espèce*, Bull. Acad. Sci. Roumaine 16 (1933), 1–4 (French).
[102] Stephen G. Simpson (ed.), *Reverse mathematics 2001*, Lecture Notes in Logic, vol. 21, ASL, La Jolla, CA, 2005.
[103] -, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, CUP, 2009.
[104] Reed Solomon, *Π₁¹-CA₀ and order types of countable ordered groups*, J. Symbolic Logic 66 (2001), no. 1, 192–206.
[105] J. Stillwell, *Reverse mathematics, proofs from the inside out*, Princeton Univ. Press, 2018.
[106] Charles Swartz, *Introduction to gauge integrals*, World Scientific, 2001.
[107] Michael Suslin, *Problème 3*, Fundamenta Mathematicae 1 (1920), 223.
[108] Terence Tao, *An introduction to measure theory*, Graduate Studies in Mathematics, vol. 126, American Mathematical Society, Providence, RI, 2011.
[109] Karl Weierstrass, *Einführung in die Theorie der bestimmten Integrale*, Halle a.S.: Louis Nebert, 1875.
[110] John W. Tukey, *Convergence and Uniformity in Topology*, Annals of Mathematics Studies, no. 2, Princeton University Press, Princeton, N. J., 1940.
[111] B.S. Vatssa, *Discrete Mathematics (4th edition)*, New Age International, 1993.
[112] Keita Yokoyama, *Standard and non-standard analysis in second order arithmetic*, Tohoku Mathematical Publications, vol. 34, Sendai, 2009. PhD Thesis, Tohoku University, 2007.
[113] Xizhong Zheng and Robert Rettinger, *Effective Jordan decomposition*, Theory Comput. Syst. 38 (2005), no. 2, 189–209.