The geometric complex of a Morse–Bott–Smale pair and an extension of a theorem by Bismut–Zhang

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Ray–Singer torsion

$(M, g)$ . . . closed Riemannian manifold

$(E, µ)$ . . . flat Euclidean vector bundle over $M$

$\leadsto$ deRham complex $(Ω^*(M; E), d)$

$\leadsto$ Laplacian $Δ^q : Ω^q(M; E) → Ω^q(M; E)$

$\leadsto$ zeta regularized $\text{det}' Δ^q$

$\leadsto$ Ray–Singer torsion

$$\log T^E_{\text{an}}^{g,µ} := \frac{1}{2} \sum_q (-)^{q+1} q \log \text{det}' Δ^q$$

$\leadsto$ deRham cohomology $H^q(M; E) = \ker Δ^q$

$\leadsto$ scalar product $\| \cdot \|_{H^q(M; E)}$
Recall a theorem of Cheeger, Müller, and Bismut–Zhang.

**Theorem.** $X$ Morse–Smale vector field. Then:

$$\log T_{\text{an}}^{E,g,\mu} = \log T_{\text{met}} + \log T_{\text{comb}} + \int_M \theta \wedge (-X)^* \Psi$$

$T_{\text{comb}}$ ... torsion of Morse complex $C^*(X; E)$ with scalar product induced from $\mu$.

$T_{\text{met}}$ ... volume of the integration isomorphism $\text{Int} : H^*(M; E) \rightarrow HC^*(X; E)$.

$\theta := -\frac{1}{2} \text{tr}_\mu \nabla \mu \in \Omega^1(M; \mathbb{R})$ ... closed 1–form

$\Theta := [\theta] \in H^1(M; \mathbb{R})$ holonomy of $\text{det} E \otimes \mathcal{O}_E$

$\Psi \in \Omega^{n-1}(TM\backslash M; \mathcal{O}_M)$ ... Mathai–Quillen form
Analysis of the ODE
\[
X = -\nabla\tilde{g}(f) \ldots \text{Morse–Bott–Smale}
\]
\( \leadsto \) critical manifold \( \Sigma := \text{Zero}(X) \)
\( \leadsto \) decomposition \( TM|_{\Sigma} = T\Sigma \oplus N^+ \oplus N^- \)
\( \leadsto \) Morse index \( \text{ind} = \text{rank}(N^-) : \Sigma \to \mathbb{N}_0 \)
\( \leadsto \) unstable manifold
\[
\Sigma \xleftarrow{p} W \xrightarrow{i} M
\]
p diffeomorphic to \( N^- \), \( i : W \to M \) smooth
\( \leadsto \) vertical bundle \( V_W \to W \)
\[
\text{rank}(V_W) = p^* \text{ind}
\]
\( \leadsto \) space of unparametrized trajectories
\[
\Sigma \xleftarrow{\pi_-} T \xrightarrow{\pi_+} \Sigma
\]
\( \pi_- \) smooth fiber bundle, \( \pi_+ \) smooth
\( \leadsto \) vertical bundle \( V_T \to T \)
\[
\text{rank}(V_T) = \pi_+^* \text{ind} - \pi_-^* \text{ind} - 1
\]
canonic compactification to manifold with corners

\[ \Sigma \xleftarrow{\hat{\pi}_-} \hat{T} \xrightarrow{\hat{\pi}_+} \Sigma \]

\(\hat{\pi}_-\) smooth fiber bundle, \(\hat{\pi}_+\) smooth,

\[ \partial_1 \hat{T} = \mathcal{T} \times_\Sigma \mathcal{T} \]

\[ \sim \text{canonic compactification to manifold with corners} \]

\[ \Sigma \xleftarrow{\hat{p}} \hat{W} \xrightarrow{\hat{i}} M \]

\(\hat{p}\) smooth fiber bundle, \(\hat{i}\) smooth,

\[ \partial_1 \hat{W} = \mathcal{T} \times_\Sigma \mathcal{W} \]
Geometric complex

$X$ . . . Morse–Bott–Smale vector field
$E$ . . . flat vector bundle
$E_S := (E \otimes \mathcal{O}_{N-})|_S$

$\leadsto \mathbb{Z}$–graded Morse–Bott complex

$$C^q(X; E) := \bigoplus_{S \subseteq \Sigma} \Omega^{q-\text{ind}(S)}(S; E_S)$$

$\delta := d \pm (\hat{\pi}_-)_* \circ (\hat{\pi}_+)^* \quad \delta^2 = 0$

$\leadsto$ cohomology $HC^*(X; E)$

$\leadsto$ homomorphism

$$\Omega^q(M; E) \xrightarrow{\text{Int}} C^q(X; E)$$

$\text{Int} := \hat{p}_* \circ \hat{i}^*$

$\text{Int} \circ d = \delta \circ \text{Int}$

$\leadsto$ isomorphism

$$H^*(M; E) \xrightarrow{\text{Int}} HC^*(X; E)$$
\[ C^q_{\geq p}(X; E) := \bigoplus_{\text{ind}(S) \geq p} \Omega^{q-\text{ind}(S)}(S; E_S) \]

\[ E_1 C^q_p(X; E) = \bigoplus_{\text{ind}(S) = p} H^{q-p}(S; E_S) \]

\( g \) \ldots \text{Riemannian metric on } M

\( \mu \) \ldots \text{fiber metric on } E

\( (\Sigma, g_S) \) \text{ Riemannian}

\( (E_S, \mu_S) \) \text{ flat Euclidean vector bundle}

\( \text{scalar product } \| \cdot \|_{E_1} \)

\( \text{inductively } \| \cdot \|_{E_k}, 1 \leq k \leq \infty \)
Combinatorial and metric torsion

\( \sim \) finite dimensional Laplacians \( \Delta_k^q \) on \( E_k C^q \)

\( \sim \) combinatorial torsion

\[ \log T_{\text{comb}}^{E,g,\mu,X} := \sum_k \frac{1}{2} \sum_q (-1)^{q+1} q \log \det' \Delta_k^q \]

\( \sim \) \( \text{GHC}^*(X; E) = E_\infty C^*(X; E) \)

\( \sim \) \( \det HC^* = \det \text{GHC}^* = \det E_\infty C^* \)

\( \sim \) geometric scalar product \( \| \cdot \|_{\det HC^*(X; E)} \)

\[ \det H^*(M; E) \xrightarrow{\det \text{Int}} \det HC^*(X; E) \]

\( \sim \) metric torsion

\[ \log T_{\text{met}}^{E,g,\mu,X} := \log \text{vol} \det \text{Int} \]

Note that \( H^*(M; E) = 0 \Rightarrow \log T_{\text{met}} = 0 \).
**Theorem A**

\[
\log T^M_{\text{an}} = \sum_{S \subseteq \Sigma} (-)^{\text{ind}(S)} \log T^S_{\text{an}} + \log T_{\text{met}} + \log T_{\text{comb}} \\
+ \int_M \theta \wedge (-X)^* \Psi
\]

**Remarks**

- localization theorem for analytic torsion

- specializes to Bismut–Zhang theorem if \( X \) Morse–Smale (\( \Sigma \) discrete, hence \( \log T^S_{\text{an}} = 0 \))

- implies (a slight generalization of) a result of Lück–Schick–Thielmann (analytic torsion of fiber bundles)
Euler structures (Turaev)

$M^n$ connected with base point $x_0$

consider equivalence classes $[X, c]$

$X$ vector field with non-degenerate zeros $\mathcal{X}$

$c \in C_1(M; \mathbb{R})$ ... Turaev spider (Euler chain)

$$\partial c = \sum_{x \in \mathcal{X}} \text{IND}(x)x - \chi(M)x_0$$

IND$(x)$ ... Hopf index

$[X_1, c_1] = [X_2, c_2]$ iff

$$c_2 - c_1 = c(X_1, X_2) \mod \partial(C_2(M; \mathbb{R}))$$

c$(X_1, X_2)$ ... one chain obtained as the zero set of a generic homotopy from $X_1$ to $X_2$

$\rightsquigarrow \text{Eul}_{x_0}(M)$ Euler structures over $\mathbb{R}$

affine version of $H_1(M; \mathbb{R})$
Euler structures (deRham approach)

$M^n$ connected with base point $x_0$

consider equivalence classes $[g, \alpha]$ where $g$ . . . Riemannian metric
$\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ with

$$d\alpha = E_g$$

$E_g \in \Omega^n(M; \mathcal{O}_M)$ . . . Euler form

$[g_1, \alpha_1] = [g_2, \alpha_2]$ iff

$$\alpha_2 - \alpha_1 = cs(g_1, g_2) \mod d(\Omega^{n-1}(M; \mathcal{O}_M))$$

$\leadsto \Eu^n_x(M)$ . . . (co)Euler structures over $\mathbb{R}$
affine version of $H^{n-1}(M; \mathcal{O}_M)$

$\leadsto$ isomorphism $\Eu^n_x(M) = \Eu^n_{x_0}(M)$
affine over Poincaré duality

$$H^{n-1}(M; \mathcal{O}_M) = H_1(M; \mathbb{R})$$
Analytic torsion

$E \ldots$ flat vector bundle over $M$

$\varepsilon = [g, \alpha] \in \text{Eul}_{x_0}(M) \ldots$ Euler structure

consider real line (one dim. vector space)

$\text{Det}_{x_0}(M; E) \coloneqq \det H^*(M; E) \otimes (\det E_{x_0})^{-\chi(M)}$

choose fiber metric $\mu$ on $E$ and define

$|| \cdot ||_{\text{an}} \coloneqq T_{\text{an}} \cdot e^{-\int_M \theta^\wedge \alpha} \cdot || \cdot ||_{\det H^*(M; E) \otimes || \cdot ||_{x_0}}$

anomaly formula for analytic torsion implies that this does only depend on $(E, \varepsilon)$.

$\leadsto$ analytic torsion $\pm \tau_{\text{an}}^{E, \varepsilon} \in \text{Det}_{x_0}(M)$

for $\sigma \in H_1(M; \mathbb{R})$ we have

$\tau_{\text{an}}^{E, \varepsilon + \sigma} = \tau_{\text{an}}^{E, \varepsilon} \cdot e^{-\langle \Theta, \sigma \rangle}$
Localization of Euler structures

consider equivalence classes \([X, c, \{e_S\}]\)

\(X \ldots\) Morse–Bott–Smale vector field

\(e_S \in \mathcal{E}ul_{x_S}(S) \ldots\) Euler structure on \(S \subseteq \Sigma\)

\(c \in C_1(M; \mathbb{R})\) with

\[
\partial c = \sum_{S \subseteq \Sigma} \text{IND}(S)x_S - \chi(M)x_0
\]

+ more complicated equivalence relation

\(\sim \mathcal{E}ul_{x_0}(M) \ldots\) affine version of \(H_1(M; \mathbb{R})\)
Geometric torsion

$X$ . . . Morse–Bott–Smale vector field
$\mathcal{e} = [-X, c, \{e_S\}]$ . . . Euler structure
$\text{Det}_{x_0}(X; E) := \det HC^*(X; E) \otimes (\det E_{x_0})^{-\chi(M)}$

spectral sequence provides canonic
$\det HC^* = \det GHC^* = \det E_\infty C^* = \cdots$
$= \det E_1 C^* = \bigotimes_{S \subseteq \Sigma} \left( \det H^*(S; E_S) \right)^{(-)^{\text{ind}(S)}}$

+ parallel transport along $c$ provides
$\text{Det}_{x_0}(X; E) = c \bigotimes_{S \subseteq \Sigma} \left( \text{Det}_{x_S}(S; E_S) \right)^{(-)^{\text{ind}(S)}}$

$\leadsto$ geometric torsion $\tau_{E, \mathcal{e}, X}^{\text{geom}} \in \text{Det}_{x_0}(X; E)$

for $\sigma \in H_1(M; \mathbb{R})$ we have
$\tau_{E, \mathcal{e} + \sigma, X}^{\text{geom}} = \tau_{E, \mathcal{e}, X}^{\text{geom}} \cdot e^{-\langle \Theta, \sigma \rangle}$
Reformulation of theorem A

\( M \) ... closed manifold
\( E \) ... flat vector bundle over \( M \)
\( X \) ... Morse–Bott–Smale vector field on \( M \)
\( e \in \text{Eul}_{x_0}(M) \) ... Euler structure

\[
\text{Int} : H^*(M; E) = HC^*(X; E) \quad \text{provides}
\]
\[
\text{Det}_{x_0}(M; E) = \text{Det}_{x_0}(X; E)
\]

**Theorem B.** \( \frac{E, e}{\text{tan}} = \frac{E, e, X}{\text{geom}} \)

Tracing back the definitions we easily find

theorem A \( \Leftrightarrow \) theorem B
Sketch of proof.

$X$ MBS with critical manifold $\Sigma$.

$Y$ MS on $\Sigma$. Then $X' := X + \varepsilon \tilde{Y}$ MS.

Apply Bismut–Zhang to $X'$ and obtain relation between $\tau_{\text{an}}^{E, \varepsilon}$ and $\tau_{\text{geom}}^{E, \varepsilon, X'}$.

$C^*(X'; E)$ inherits filtration from $X$. Obtain spectral sequence converging to $HC^*(X'; E')$ with $E_1$–term:

$$\bigoplus_{S \subseteq \Sigma} HC^{*-\text{ind}(S)}(Y_S; E_S)$$

Homological algebra (Maumary) thus provides relation between $\tau_{\text{geom}}^{E, \varepsilon, X'}$ and $\tau_{\text{geom}}^{E_S, \varepsilon_S, Y_S}$, $S \subseteq \Sigma$.

Applying Bismut–Zhang for every $Y_S$, $S \subseteq \Sigma$, we get relation between $\tau_{\text{geom}}^{E_S, \varepsilon_S, Y_S}$ and $\tau_{\text{an}}^{E_S, \varepsilon_S}$. 

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