Signature for the Shape of the Universe

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Abstract

If the universe has a nontrivial shape (topology) the sky may show multiple correlated images of cosmic objects. These correlations can be couched in terms of distance correlations. We propose a statistical quantity which can be used to reveal the topological signature of any Robertson-Walker (RW) spacetime with nontrivial topology. We also show through computer-aided simulations how one can extract the topological signatures of flat, elliptic, and hyperbolic RW universes with nontrivial topology.

1 Introduction

Whether we live in a finite or infinite space and what is the size and the shape of the universe are open problems, which modern cosmology seeks to solve (see, for example, [1] – [27] and references therein). The primary consequence of multiply-connectedness of the universe is the possibility of observing multiple images of cosmic objects, whose existence can be perceived by the simple reasoning presented below.

In the general relativity approach to cosmological modeling the Robertson-Walker (RW) spacetime manifolds \( \mathcal{M}_4 \) are decomposable into \( \mathcal{M}_4 = \mathcal{R} \times M \), and endowed with

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the metric
\[ ds^2 = dt^2 - R^2(t) \left\{ d\chi^2 + f^2(\chi) [d\theta^2 + \sin^2 \theta d\phi^2] \right\}, \]
(1.1)
where \( t \) is a cosmic time, the function \( f(\chi) \) is given by \( f(\chi) = \chi, \sin \chi, \) or \( \sinh \chi, \) depending on the sign of the constant spatial curvature \( (k = 0, \pm 1) \), and \( R(t) \) is the scale factor.

It is often assumed that the \( t = \text{const} \) spatial sections \( M \) of RW spacetime manifold \( \mathcal{M}_4 \) are one of the following simply connected spaces: Euclidean \( E^3 \) \((k = 0)\), elliptic \( S^3 \) \((k = 1)\), or the hyperbolic \( H^3 \) \((k = -1)\), depending on the sign of the curvature \( k \). However, the simply-connectedness for our 3-space has not been settled by cosmological observations. Thus, the space \( M \) where we live may be any one of the possible multiply-connected quotient 3-spaces \( M = \widetilde{M}/\Gamma \), where \( \widetilde{M} \) stands for \( E^3, S^3 \) or \( H^3 \), and \( \Gamma \) is a discrete group of isometries acting freely on the covering simply-connected 3-space \( \widetilde{M} \). The action of \( \Gamma \) tessellates \( \widetilde{M} \) into identical cells or domains which are copies of what is known as fundamental polyhedron (FP). In forming the quotient manifolds \( M \) the essential point is that they are obtained from \( \widetilde{M} \) by identifying points which are equivalent under the action of the group \( \Gamma \). Hence, each point on the quotient manifold \( M \) represents all the equivalent points on the covering manifold \( \widetilde{M} \). In this cosmological modeling context a given cosmic object is described by a point \( p \in M \), which represents, when \( M \) is multiply-connected, a set of equivalent points (images of \( p \)) on the covering manifold \( \widetilde{M} \). So, to figure out that multiple images of an object can indeed be observed if the universe is multiply-connected, consider that the observed universe is a ball \( B_{R_H} \subset \widetilde{M} \) whose radius \( R_H \) is the particle horizon, and denote by \( L \) the largest length of the fundamental polyhedron FP of \( M \) (FP \( \subset \widetilde{M} \)). Thus, when \( R_H > L/2 \), for example, the set of (multiple) images of a given object that lie in \( B_{R_H} \) can in principle be observed. Obviously the observable images of an object constitute a finite subset of the set of all equivalent images of the object.

The multiple images of the cosmic objects are periodically distributed, and the periodicity, which arises from the correlations in their positions dictated by the group \( \Gamma \), are fundamentally related to the topology of the 3-space \( M \). The correlations among the images can be couched in terms of spatial distance correlations. Indeed, one may look for spatial distance correlations between cosmic images in multiply-connected universes by using pair separations histograms (PSH), which are functions \( \Phi(s_i) \) that count the number of pair of images separated by a distance \( s_i \) that lies in a given interval \( J_i \) (say). The embryonic expectation is certainly that the distance correlations manifest as topological spikes in PSH's, and that the spike spectrum of topological origin would be a definite signature of the topology. While simulations performed for specific flat manifolds
appeared to confirm the primary expectation [20], histograms subsequently generated for the Weeks and one of the Best hyperbolic manifolds revealed that the PSH’s corresponding to those specific 3-manifolds exhibit no spikes [21, 22]. Concomitantly, a theoretical statistical analysis of the distance correlations in the PSH’s was performed, and a formal proof was presented that the spikes of topological origin in PSH’s are due to just one type of isometry: the translations [23]. This result explains the absence of spikes in the PSH’s of hyperbolic manifolds, and also gives rise to the fact that Euclidean distinct manifolds which admit the same translations on their covering group present the same spike spectrum of topological origin (hereafter called topological spike, for short).

Although the set of topological spikes in PSH’s is not definite topological signature and is not sufficient for distinguishing even between some compact flat manifolds [27], the most striking evidence of multiply-connectedness in PSH’s is indeed the presence of topological spikes, which arise only when the covering group $\Gamma$ contains translational isometries $g_t$. The other isometries $g$, however, manifest as rather tiny deformations of the expected pair separation histogram $\Phi^{\text{sc}}_{\text{exp}}(s_i)$ corresponding to the underlying simply connected universe [23].

In computer-aided simulations the histograms contain statistical fluctuations (noises), which can give rise to sharp peaks of statistical origin, or can hide (or mask) the tiny deformations due to non-translational isometries. The most immediate approach to cope with fluctuation problems in PSH’s is by using the mean pair separation histogram (MPSH) scheme to obtain $<\Phi(s_i)>$ rather than a single PSH $\Phi(s_i)$. However, from computer simulations it becomes clear that for a reasonable number of images and when there is no topological spike (no translation) the graphs of the expected pair separation histograms (EPSH) $\Phi_{\text{exp}}(s_i) \simeq <\Phi(s_i)>$ of multiply-connected universes are practically indistinguishable from the graph of the EPSH $\Phi^{\text{sc}}_{\text{exp}}(s_i)$ corresponding to the underlying simply connected universe, making clear that even the noiseless quantity $\Phi_{\text{exp}}(s_i)$ (PSH without the statistical noise) is not a suitable quantity for revealing the topology of multiply-connected universes.

In this work we propose a way of extracting the topological signature of any multiply-connected universe of constant curvature by using a suitable new statistical quantity $\varphi^S(s_i)$. We also show through computer-aided simulations the strength of our proposal by extracting the topological signatures of a flat ($k = 0$), an elliptic ($k = 1$), and a hyperbolic ($k = -1$) multiply-connected RW universe.
2 Topological Signature and Simulations

Let us start by recalling that in dealing with discrete astrophysical sources in the context of multiply-connected RW spacetimes, the observables universe is the region or part of the universal covering manifold \( \widetilde{M} \) causally connected to an image of a given observer since the moment of matter-radiation decoupling. Clearly in the observable universe one has the set of observable images of the cosmic objects, denoted by \( O \). A catalog is a particular subset \( C \subset O \), of observed images, since by several observational limitations one can hardly record all the images present in the observable universe. The observed universe is the part of the observable universe which contains all the sources registered in the catalog. Our observational limitations can be formulated through selection rules which dictate how the subset \( C \) arises from \( O \). Catalogs whose images obey the same well-behaved distribution and that follow the same selection rules are said to be comparable catalogs. It should be noted that in the process of construction of catalogs it is assumed a RW geometry (needed to convert redshift into distance) and that a particular type of sources (clusters of galaxies, quasars, etc) is chosen from the outset.

Consider a catalog \( C \) with \( n \) cosmic images and denote by \( \eta(s) \) the number of pairs of images whose spatial separation is \( s \). Consider also that our observed universe is a ball of radius \( a \) and divide the interval \((0, 2a]\) in a number \( m \) of equal subintervals \( J_i \) of length \( \delta s = 2a/m \). Each of such subintervals can always be taken to be in the form

\[ J_i = \left(s_i - \frac{\delta s}{2}, s_i + \frac{\delta s}{2}\right), \]

with \( i = 1, 2, \ldots, m \), and centered at

\[ s_i = (i - \frac{1}{2}) \delta s. \]

The PSH is a normalized function which counts the number \( \eta(s) \) of pair of images separated by a spatial distance \( s \) that lies in a given subinterval \( J_i \). Thus the function PSH is given by

\[ \Phi(s_i) = \frac{2}{n(n-1)} \frac{1}{\delta s} \sum_{s \in J_i} \eta(s), \]

where \( n \) is the number of cosmic images in \( C \), and the PSH is clearly subjected to the normalizing condition

\[ \sum_{i=1}^{m} \Phi(s_i) \delta s = 1. \]

1 In the coordinate system relative to which the line element (1.1) is written this ball is defined by \( \chi \leq a \) for any RW metric.
If one considers an ensemble of comparable catalogs with the same number $n$ of images, and corresponding to the same 3-manifold $M$ of constant curvature, one can compute, e.g. probabilities, expected and mean values of quantities which depend on the images in the catalogs of the ensemble. In particular, one can compute the expected (and normalized) pair separation histogram which clearly is given by

$$\Phi_{\text{exp}}(s_i) = \frac{1}{N} \frac{1}{\delta s} \eta_{\text{exp}}(s_i) = \frac{1}{\delta s} F(s_i) ,$$

where $\eta_{\text{exp}}(s_i)$ is the number of images with separation in the interval $J_i$, the normalizing number $N = n(n - 1)/2$ is the total number of pairs of cosmic images in a typical catalog of the ensemble $C$, and $F(s_i)$ is the probability that two images listed in $C$ be separated by a distance that lies in $J_i = (s_i - \frac{\delta s}{2}, s_i + \frac{\delta s}{2}]$.

In multiply-connected universes there are two types of pairs, namely $\Gamma$-pairs or correlated pairs, and $U$-pairs or uncorrelated pairs. A $g$-pair is a pair of the form $(p, gp)$ for any (fixed) isometry $g$. A $u$-pair is a pair $(p, q)$ which is not of the form $(p, gp)$ for any $g \in \Gamma$, that is to say the elements $p$ and $q$ of the $U$-pairs are not related (correlated) by any isometry $g \in \Gamma$.

Now, if one denotes by $F_g(s_i)$ and $F_u(s_i)$, respectively, the probability that the elements of a $g$-pair and of a $u$-pair be separated by a distance that lies in $J_i$, the probability $F(s_i)$ that a pair in a typical catalog of the ensemble be separated by a spatial distance in $J_i$ is given by

$$F(s_i) = \frac{N_u}{N} F_u(s_i) + \frac{1}{2} \sum_{g \in \tilde{\Gamma}} \frac{N_g}{N} F_g(s_i) .$$

where $\tilde{\Gamma}$ denotes the covering group $\Gamma$ without the identity map, and where $N_u$ and $N_g$ denote, respectively, the (total) expected number of uncorrelated pairs and the (total) expected number of $g$-pairs in a typical catalog $C$ of the ensemble. It should be noticed that since the pairs of cosmic images are either correlated ($\Gamma$-pairs) or uncorrelated ($U$-pairs) we must have

$$N_u + \frac{1}{2} \sum_{g \in \Gamma} N_g = N .$$

Inserting eq. (2.5) into eq. (2.4) one obtains

$$\Phi_{\text{exp}}(s_i) = \frac{N_u}{N} \Phi_{\text{exp}}^u(s_i) + \frac{1}{2} \sum_{g \in \Gamma} \frac{N_g}{N} \Phi_{\text{exp}}^g(s_i) ,$$

When referring collectively to correlated pairs we use the terminology $\Gamma$-pairs, leaving the name $g$-pair for particular correlated pair, i.e. a pair corresponding to a specific isometry $g \in \Gamma$. Similarly, we shall use the terminology $U$-pairs when referring collectively to uncorrelated pairs.
where from (2.4) we have been led to define the EPSH’s corresponding to uncorrelated pairs and associated to an isometry $g$, respectively, as

$$\Phi^u_{\text{exp}}(s_i) = \frac{1}{\delta s} F_u(s_i) = \frac{1}{N_u} \frac{1}{\delta s} \eta^u_{\text{exp}}(s_i), \quad (2.8)$$

$$\Phi^g_{\text{exp}}(s_i) = \frac{1}{\delta s} F_g(s_i) = \frac{1}{N_g} \frac{1}{\delta s} \eta^g_{\text{exp}}(s_i), \quad (2.9)$$

with $N_u = \sum_{s_i} \eta^u_{\text{exp}}(s_i)$ and $N_g = \sum_{s_i} \eta^g_{\text{exp}}(s_i)$.

Similarly for simply connected universe with $N$ pairs of cosmic images, since all pairs are uncorrelated, equation (2.4) reduces to

$$\Phi^{sc}_{\text{exp}}(s_i) = \frac{1}{N} \frac{1}{\delta s} \eta^{sc}_{\text{exp}}(s_i) = \frac{1}{\delta s} F_{sc}(s_i), \quad (2.10)$$

where $F_{sc}(s_i)$ is the probability that two objects in the universe be separated by a distance that lies in $J_i$.

An alternative expression for the EPSH of multiply-connected universe $\Phi_{\text{exp}}(s_i)$ in terms of $\Phi^{sc}_{\text{exp}}$ can be obtained. Indeed, using (2.7) and (2.6) one easily obtains

$$\Phi_{\text{exp}}(s_i) = \Phi^{sc}_{\text{exp}}(s_i) + \frac{N_u}{N} \left[ \Phi^u_{\text{exp}}(s_i) - \Phi^{sc}_{\text{exp}}(s_i) \right] + \frac{1}{2} \sum_{g \in \Gamma} \frac{N_g}{N} \left[ \Phi^g_{\text{exp}}(s_i) - \Phi^{sc}_{\text{exp}}(s_i) \right]. \quad (2.11)$$

Using the expression of $N$ in terms of the number of cosmic images $n$ from eq. (2.11) one finally obtains the following expression for what we will define as the topological signature of multiply-connected universes, namely

$$\varphi^S(s_i) \equiv (n - 1) \left[ \Phi_{\text{exp}}(s_i) - \Phi^{sc}_{\text{exp}}(s_i) \right] = \varphi^U(s_i) + \varphi^\Gamma(s_i), \quad (2.12)$$

where

$$\varphi^U(s_i) = \nu_u \left[ \Phi^u_{\text{exp}}(s_i) - \Phi^{sc}_{\text{exp}}(s_i) \right] \quad (2.13)$$

and

$$\varphi^\Gamma(s_i) = \sum_{g \in \Gamma} \nu_g \left[ \Phi^g_{\text{exp}}(s_i) - \Phi^{sc}_{\text{exp}}(s_i) \right], \quad (2.14)$$

and where $\nu_u = 2N_u/n$ and $\nu_g = N_g/n$.

To extract the topological signature $\varphi^S(s_i)$ of multiply-connected universes, an important point to bear in mind is that the EPSH is essentially a typical PSH from which the statistical noise has been withdrawn. Hence we have

$$\Phi(s_i) = \Phi_{\text{exp}}(s_i) + \rho(s_i), \quad (2.15)$$
where $\Phi(s_i)$ is a typical PSH constructed from $C$ and $\rho(s_i)$ represents the statistical fluctuation that arises in the PSH $\Phi(s_i)$.

In practice one can approach the topological signature $\varphi^S(s_i)$ by reducing the statistical fluctuations through any suitable method to lower the noise $\rho(s_i)$. In computer simulations the simplest way to accomplish this is to use several comparable catalogs to generate a mean pair separation histogram (MPSH). In other words, the use of the MPSH to extract the topological signature $\varphi^S(s_i)$ consists in the use of $K$ (say) computer-generated comparable catalogs, with approximately the same number $n$ of images and corresponding to the same manifold $M$, to obtain the mean pair separation histogram $< \Phi(s_i) >$ (over the $K$ catalogs), and analogously to have $< \Phi^{sc}(s_i) >$; and use them as approximations for $\Phi_{exp}(s_i)$ and $\Phi_{exp}^{sc}(s_i)$, to construct the topological signature $\varphi^S(s_i) \simeq (n - 1) \left[ < \Phi(s_i) > - < \Phi^{sc}(s_i) > \right]$.

An improvement of the above procedure to extract the topological signature $\varphi^S(s_i)$ comes out for the cases one can derive the expression for the PSH’s $\Phi_{exp}^{sc}(s_i)$ corresponding to the simply connected covering universes. The explicit formulae for $\Phi_{exp}^{sc}(s_i)$ corresponding to an uniform distribution of objects in the covering universes endowed with the Euclidean, elliptic and hyperbolic geometries can indeed be obtained [24, 25]. Thus, for multiply-connected universes with homogeneous distribution of objects, which we will be concerned with in the computer simulations, the topological signature clearly has the form $\varphi^S(s_i) \simeq (n - 1) \left[ < \Phi(s_i) > - \Phi_{exp}^{sc}(s_i) > \right]$, where $\Phi_{exp}^{sc}(s_i)$ is known from the outset.

The first series of computer-aided simulations concerns a compact orientable Euclidean manifold of class $G_6$ in Wolf’s classification [28]. We shall denote this Euclidean cubic manifold by $T_4$ in agreement with the notation used in [30], wherein the cubic fundamental polyhedron FP and the pairwise faces identification are shown. Relative to a coordinate system whose origin coincides with the center of the FP, the actions of the generators $\alpha$, $\beta$ and $\delta$ on a generic point $p = (x, y, z)$ were shown to be described by [28, 29]

$$\alpha p = (x + L, -y, -z), \quad \beta p = (-x, z + L, y), \quad \delta p = (-x, z, y + L),$$

where $L$ is the edge of the cubic FP. In the simulations corresponding to $T_4$ the center of the FP was taken to be the origin of the coordinate system, and to coincide with the center of the observed universe $B_a$, whose diameter is $2a = L \sqrt{2} \simeq 1.41 L$. It should be noted that with this ratio for $a/L$ and for $s \in (0, 2a)$ one has only the contribution of
non-translational isometries for the topological signature.

In the computer simulation we have used a program whose input are the number \( K \) of catalogs, the radius \( a \) of the observed universe \( B_0 \), the number \( m \) of subinterval (bins), and the number \( n_s \) of objects inside the FP (seeds). The program generates \( K \) different catalogs, starting (each) from the same number \( n_s \) of homogeneously distributed seeds inside the FP, and then using the generators \( \alpha, \beta \) and \( \delta \) [whose actions are given by (2.16) \( \cdots \) (2.18)] and their inverses \( \alpha^{-1}, \beta^{-1} \) and \( \delta^{-1} \). For each bin \( J_i \) of width \( \delta s = 2a/m \) it counts the normalized number of pairs \( \sum \hat{\eta}_k(s) \) for all catalogs \( k \) from 1 to \( K \). Finally, it calculates the normalized average numbers of pairs for all \( s_i \in (0, 2a) \), finding therefore the mean pair separations histogram \(< \Phi(s_i) > \) over \( K \) catalogs.

We have performed simulations corresponding to the manifold \( T_4 \) with \( L = 1 \) and in an observed universe of radius \( a = \sqrt{2}/2 \simeq 0.71 \), \( \delta s = 0.01 \), and with different number \( n_s \) of seed objects uniformly distributed in the FP. Figure 1a is the graph of the topological signature \( \varphi^S(s_i) \) \( (\simeq (\not n > -1) [\not < \Phi(s_i) > > \not \Phi_{\text{exp}}(s_i)]) \) and was obtained using the MPSH procedure for \( K = 16000 \) catalogs, and \( n_s = 15 \), which corresponds to an average number of images per catalog \( < n > \simeq 23 \). Figure 1b shows a graph of the topological signature for the same universe and manifold, and was obtained through the MPSH scheme for identical number of catalogs, but now the number of seeds was taken to be \( n_s = 100 \), which corresponds to \( < n > \simeq 153 \) images. These figures make clear that the topological signature \( \varphi(s_i) \) for \( n_s = 15 \) is essentially the same obtained for \( n_s = 100 \), and that the plain topological signature arises in simulations where there are just a few images for each object.

We have also performed computer simulations for the specific elliptic 3-manifold \( S^3/\mathbb{Z}_5 \), whose volume \( 2 \pi^2 R^3/5 \) is one fifth of the volume of the three-sphere \( S^3 \). A FP (tetrahedron) together with the pairwise faces identifications is given by Weeks [31]. We have taken as the observed universe the whole covering space \( S^3 \), i.e. a solid sphere with radius \( a = \pi \) ( \( R = 1 \) in the corresponding RW metric, and the edge of tetrahedron \( L \simeq 1.82 \)). Thus all catalogs in our simulations for this manifold have the same number of images. Figure 2 shows the graph of the topological signature \( \varphi^S(s_i) \) for this multiply-connected universe, for \( m = 180, n = 100 \) images \( (n_s = 20) \), \( K = 3000 \) catalogs.

We have finally performed computer simulations for the specific compact hyperbolic 3-manifold known as Seifert-Weber dodecahedral space, which is obtained by identifying or glueing the opposite pentagonal faces of a dodecahedron after a rotation of \( 3 \pi/5 \). Figure 3 shows the graph of the topological signature \( \varphi^S(s_i) \) for this hyperbolic space where the center of the dodecahedron was taken to coincide with the center of the observed universe.
\(B_\alpha\), whose diameter is \(2a \simeq 2.88\). The length \(L\) of the edges of the pentagonal faces and the height \(H\) of the dodecahedron are \(L = H \simeq 1.99\), where the lengths are measured with the hyperbolic RW geometry with \(R = 1\). We have taken \(m = 100\) bins, \(n_s = 10\) seeds \((\langle n \rangle \simeq 18)\), \(K = 16000\) catalogs, and used the exact expression for \(\Phi_{\text{exp}}(s_i)\).

To close this work it is worthwhile mentioning that the ultimate step in most of such statistical approaches to extract the topological signature is the comparison of the graphs (signature) obtained from simulated catalogs against similar graphs generated from real catalogs. To do so one clearly has to have the simulated patterns of the topological signatures of the universes, which can be achieved by the method we have proposed in this article.

\section*{Captions for the figures}

\textbf{Figure 1} The topological signature \(\varphi^S(s_i)\) of an Euclidean multiply-connected universe of diameter \(2a \simeq 1.41\), with underlying topology of \(T_4\) with edge \(L = 1\). The horizontal axis gives the pair separation \(s\) while the vertical axis furnishes the normalized number of pairs. In (a) the number of seeds is \(n_s = 15\) and corresponds to an average number of images per catalog \(\langle n \rangle \simeq 23\). In (b) the number of seeds is \(n_s = 100\) and corresponds to an average number of images per catalog \(\langle n \rangle \simeq 153\). In both cases one arrives at essentially the same topological signature.

\textbf{Figure 2} The topological signature \(\varphi^S(s_i)\) of an elliptic multiply-connected universe with diameter \(2a = \pi\) and topology \(S^3/Z_5\). The edge of the tetrahedron (FP) is \(L \simeq 1.82\). The observed universe is whole unitary sphere \(S^3\). The horizontal axis gives the pair separation \(s\) while the vertical axis gives the normalized number of pairs.

\textbf{Figure 3} The topological signature \(\varphi^S(s_i)\) for a hyperbolic multiply-connected universe with diameter \(2a \simeq 2.88\) whose underlying topology is the Seifert-Weber dodecahedral space with edge and height \(L = H \simeq 1.99\). The horizontal axis gives the pair separation \(s\) while the vertical axis provides the normalized number of pairs.

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Figure 1
Figure 2
