Asymmetric quantum telecloning of multiqubit states

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We propose a scheme of $1\to 2$ optimal universal asymmetric quantum telecloning of pure multiqubit states. In particular, we first investigate the asymmetric telecloning of arbitrary 2-qubit states and then extend it to the case of multiqubit system. Many figures of merit for the telecloning process are checked, including the entanglement of the quantum channel and fidelities of the clones. Our scheme can be used for the $1\to 4$ universal telecloning of mixed multiqubit states.

I. INTRODUCTION

One of the most essential differences between classical and quantum-information theory (QIT) is the no cloning theorem [1, 2]. It forbids the perfect cloning of arbitrarily given quantum state, in both pure and mixed cases. It is then natural to ask how well one can copy quantum states, i.e., with the highest fidelity. This problem was firstly addressed by Buzek and Hillery [3], whose scheme proved to be optimal by [4]. The Buzek-Hillery theory actually exhibits a universal symmetric $1\to 2$ quantum cloning machine (QCM), which exports two identical clones closest to the input pure qubit state with a constant fidelity. The related work in past years has established the $N \to M$ universal symmetric QCM for both qubits [5] and qudits [6], (transforming $N$ identical input states into $M > N$ identical output copies), as well as the continuous-variable systems [8]. Correspondingly, the $N \to M$ asymmetric QCM generates $M$ output states with different fidelities from $N$ input copies [9, 10, 11, 12]. Some experimental progress on quantum cloning has also been made [13].

The essentiality of quantum cloning is to broadcast information to certain distributed objects, so it is regarded as a widely useful quantum-information transmission, e.g., the eavesdropping on implementation of quantum key distribution [14]. It is well-known that quantum teleportation [15, 16] is the most effective technique for remotely broadcasting information. Murao et al. [17] has advanced the $1 \to M$ quantum teleportation which combines the tricks of both quantum teleportation and cloning. In this scheme, the sender Alice holds an unknown input state and she previously shares an entangled state with $M$ receivers, which resembles the scenario of quantum teleportation. The object is to duplicate the input at the location of every receiver as well as possible, since the no-cloning theorem precludes the faithful copy of unknown quantum state. Similarly, there exist symmetric and asymmetric quantum teleporting with identical and different fidelities of the clones respectively. The technique of symmetric teleportation has been extended to the case of $N \to M$ for qubit states [18] and $1 \to M$ for qudits [19], while the $1 \to 2$ universal optimal asymmetric teleporting was realized by [20, 21]. Of all these traditional schemes, the input states are restricted in the local scenario, namely the sender can arbitrarily perform the unitary operation on its system. This is no longer correct when the input states are entangled, and some primary investigation for entanglement cloning has been made recently [20]. However, they merely found out the condition on which a universal QCM can be optimal for the input of maximally entangled states, and this problem proves exceedingly difficult. As the entanglement plays the essential role in QIT, it is significant to explore the cloning and telecloning of entanglement. Unlike the broadcasting of entangled states [22], entanglement telecloning is optimal in the sense that it achieves the best fidelity as those of universal symmetric QCMs for qudits and so on. Recently, [23] proposed the scheme of telecloning for the entangled inputs $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$, which is a small set of the two-qubit states (we refer to $|j\rangle$, $j = 0, 1, ...$ as the computational basis in this paper, see below). It is then interesting to extend this scheme to the case of general two-qubit inputs. However, we doubt that whether such a scheme can be universal for any input, namely with a constant fidelity. If so, can it reach the optimal fidelity of the universal QCMs such as Werner’s bound [24]? Furthermore, as the extensive use of multipartite entanglement, it is important to explore the telecloning of multiqubit states.

In the present work, we propose a scheme of $1\to 2$ optimal universal asymmetric quantum telecloning of pure multiqubit states, by virtue of the Heisenberg QCM in [10, 11]. Based on this motivation, we firstly investigate the asymmetric telecloning of arbitrary two-qubit states and compare the achievable fidelity with the existing universal QCMs. The required entanglement in this scheme is shown to be optimal for the 4-dimensional input states. We also explicitly prove that the telecloner never creates more entanglement than that contained in the input qubits. Furthermore, we extend the above scheme to the case of multiqubit inputs. Thus we have realized for the first time the universal telecloning of arbitrarily nonlocal multiqubit states. As a $d$-level system can be composed of many qubits, one can hence optimally teleclone any state by our scheme. An important application of this technique is to perform the $1\to 4$ telecloning of mixed multiqubit states, which is a greatly puzzling problem in QIT [2]. Strikingly, we find that such a scheme can be realized with a higher fidelity than that of the optimal telecloning of pure states. We thus make resultful progress to get insight into the field of telecloning of mixed states.

The paper is organized as follows. In Sec. II we present the explicit protocol for the case of 2-qubit input states,
and investigate the properties of the telecloning process. In Sec. III we extend it to the case of multiqubit inputs and apply it to the 1→4 telecloning of arbitrary mixed states. We present our conclusion in Sec. IV.

II. OPTIMAL UNIVERSAL 1→2 TELECLONING OF 2-QUBIT STATES

As shown in Ref. 13, either of quantum teleportation and telecloning requires an unknown input state, which is to be reconstructed in several remotely distributed places. In the present situation, the input state shared by two parties $A_1$, $A_2$ has the following form,

$$|\psi\rangle_{A_1A_2} = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle,$$ (1)

where $\alpha_i \in \mathbb{C}, \forall i$, and $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$ by the normalization condition. The aim of the telecloning is to respectively transmit two copies of this state to two groups of receivers $B_1$, $B_2$ and $C_1$, $C_2$ with the highest fidelity, where every party can only operate on their states locally with the help of classical communication. To find out the appropriate quantum channel between senders and receivers, we recall the optimal universal asymmetric Heisenberg QCM 10, 19.

$$U(j)_B |00\rangle_{C,Anc} = \frac{1}{\sqrt{1 + (d-1)(p^2 + q^2)}} \times \left( |j\rangle B |j\rangle + p \sum_{r=1}^{d-1} |j\rangle |j+r\rangle \right) + q \sum_{r=1}^{d-1} |j+r\rangle |j+r\rangle \right),$$

0 ≤ j ≤ d - 1, (2)

which indeed represents series of QCMs by altering j. Here, $j + r = j + r$ modulo d and d is the dimension of the input state. The real constants $p, q$ satisfy $p + q = 1$ and their concrete meaning is to generate a universal QCM and to keep the optimality of it, so they can be properly defined previously. By superposition of the QCMs in expression (2), one can set an arbitrary state of system $B$ as input to obtain two clones at systems $B, C$ respectively, and the third qubit is the ancilla.

In a d-dimension Hilbert space, the computational basis can be expressed as a composition of the qubits, e.g., when $d = 4$ we denote that $|0\rangle \rightarrow |00\rangle, |1\rangle \rightarrow |01\rangle, |2\rangle \rightarrow |10\rangle$ and $|3\rangle \rightarrow |11\rangle$. For simplicity, let $|\eta_j\rangle = U(j)_B |00\rangle_{C,Anc}, j = 0, 1, ...$ In our scheme, either of $B, C$ and the ancilla should be a composite system of two separated qubits, whose dimension is at most $d = 4$.

Concretely, we write out $|\eta_j\rangle$'s from expression (2),

$$|\eta_0\rangle = [1 + 3(p^2 + q^2)]^{-1/2} (|00\rangle |00\rangle |00\rangle + p |00\rangle |01\rangle |01\rangle + p |00\rangle |10\rangle |10\rangle + p |00\rangle |11\rangle |11\rangle + q |01\rangle |00\rangle |01\rangle + q |01\rangle |00\rangle |10\rangle + q |01\rangle |00\rangle |11\rangle + q |01\rangle |01\rangle |00\rangle + q |01\rangle |01\rangle |10\rangle + q |01\rangle |01\rangle |11\rangle + q |01\rangle |10\rangle |00\rangle + q |01\rangle |10\rangle |10\rangle + q |01\rangle |10\rangle |11\rangle + q |01\rangle |11\rangle |00\rangle + q |01\rangle |11\rangle |10\rangle + q |01\rangle |11\rangle |11\rangle) \tag{3}$$

Then we propose that the quantum channel shared by all parties is

$$|\Omega\rangle_{A_1'A_2'B_1B_2C_1C_2\eta_1\eta_2} = \frac{1}{2} \left( |00\rangle_{A_1'A_2'} |\eta_0\rangle |00\rangle_{B_1B_2C_1C_2\eta_1\eta_2} + |01\rangle_{A_1'} |\eta_1\rangle |01\rangle_{B_1B_2C_1C_2\eta_1\eta_2} + |10\rangle_{A_2'} |\eta_2\rangle |10\rangle_{B_1B_2C_1C_2\eta_1\eta_2} + |11\rangle_{A_1'A_2'} |\eta_3\rangle |11\rangle_{B_1B_2C_1C_2\eta_1\eta_2} \right) \tag{4}$$

where $A_1'$ and $A_2'$ are two particles belonging to the senders $A_1$ and $A_2$ respectively. Notice the two ancillas $\eta_1, \eta_2$ are held by some separated observers. The ancilla particles are necessary for the Heisenberg QCM, otherwise it cannot reach the optimal fidelity 10. Although the ancillas do not play the role of clones, we will see that they actually join the realization of optimal telecloning of entanglement. For example, there are some useful relations with respect to the states $|\eta_j\rangle$'s, which involves all the participants in the system

$$\sigma_{x'B_1} \otimes \sigma_{zC_1} \otimes \sigma_{zA_1} |\eta_i\rangle = |\eta_i\rangle, i = 0, 1, 2, 3 \tag{5}$$
$$\sigma_{x'B_1} \otimes \sigma_{zC_1} \otimes \sigma_{zA_1} |\eta_i\rangle = - |\eta_i\rangle, i = 2, 3 \tag{6}$$
$$\sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{zA_2} |\eta_i\rangle = |\eta_i\rangle, i = 0, 2 \tag{7}$$
$$\sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{zA_2} |\eta_i\rangle = - |\eta_i\rangle, i = 1, 3 \tag{8}$$
$$\sigma_{xB_1} \otimes \sigma_{zC_1} \otimes \sigma_{zA_1} |\eta_2\rangle = |\eta_2\rangle, \tag{9}$$
$$\sigma_{xB_1} \otimes \sigma_{zC_1} \otimes \sigma_{zA_1} |\eta_3\rangle = |\eta_3\rangle, \tag{10}$$
$$\sigma_{xB_2} \otimes \sigma_{zC_2} \otimes \sigma_{zA_2} |\eta_1\rangle = |\eta_1\rangle, \tag{11}$$
$$\sigma_{xB_2} \otimes \sigma_{zC_2} \otimes \sigma_{zA_2} |\eta_2\rangle = |\eta_3\rangle. \tag{12}$$

These equations can be easily checked by using of the expressions of $|\eta_j\rangle$'s. Specially, the first four equations represent the change of the sign while the last four represent the change between the states $|\eta_j\rangle$'s. We thus call them parity-transformation and state-transformation respectively.

In what follows we show how to carry out the universal optimal 1→2 telecloning of the two-qubit state.
Similarly, one can check that the resulting state derived and the target state is

\( |\Phi^+\rangle_{A_1A_2} = |00\rangle + |11\rangle \)

\( |\Phi^-\rangle_{A_1A_2} = |01\rangle + |10\rangle \)

\( |\eta_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \)

\( |\eta_2\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \)

and the target state is

\[ |\omega\rangle_{B_1B_2C_1C_2a_1a_2} = \sum_{j=0}^{3} \alpha_j |\eta_j\rangle, \]

which contains the optimal two clones of system \( B_1B_2 \) and \( C_1C_2 \) respectively, as well as one ancilla of system \( a_1a_2 \) due to the universal Heisenberg QCM [10]. Since either of the senders \( A_1 \) and \( A_2 \) holds two particles being in the state \( |\Psi\rangle_{tot} \), they can individually perform a joint measurement on its 2-qubit system in the Bell basis

\[ |\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle), \]

\[ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle). \]

Evidently, the resulting state is \( \langle \Phi^\pm \rangle_{A_1A_1'} \langle \Phi^\pm \rangle_{A_2A_2'} |\Psi\rangle_{tot} \), etc, and there are in all 16 cases here. To simplify the situation, we call the superscript “+” or “-” of the Bell basis the parity of it. It is easy to show that any Bell projection can be turned into one of the cases \( \langle \Phi^\pm \rangle_{A_1A_1'} \langle \Phi^\pm \rangle_{A_2A_2'} |\Psi\rangle_{tot} \) with the same parity as the former one, by using of the state-transformations (9)-(12). For example, if the measurement is taken in \( \langle \Phi^- \rangle_{A_1A_1'} \langle \Psi^- \rangle_{A_2A_2'} \), the resulting state is

\[ |\Psi\rangle = a_0 |\eta_1\rangle - a_1 |\eta_0\rangle - a_2 |\eta_2\rangle + a_3 |\eta_3\rangle. \]

By using of the state-transformation \( |\eta_0\rangle \leftrightarrow |\eta_1\rangle \) and \( |\eta_2\rangle \leftrightarrow |\eta_3\rangle \) (it requires the classical communication between the participants), one can obtain

\[ |\Psi\rangle_{res} = a_0 |\eta_0\rangle - a_1 |\eta_1\rangle - a_2 |\eta_2\rangle + a_3 |\eta_3\rangle, \]

which is the resulting state by measuring \( |\Psi\rangle_{tot} \) in \( \langle \Phi^- \rangle_{A_1A_1'} \langle \Phi^- \rangle_{A_2A_2'} \), and its parity is unchanged. Similarly, one can check that the resulting state derived from other Bell measurement can be turned with the same identical parity by the state-transformation operators. So it suffices to merely consider the cases of measurements in \{ \langle \Phi^\pm \rangle_{A_1A_1'}, \langle \Phi^\pm \rangle_{A_2A_2'} \}.

In particular, there are four subcases such that \{ \langle \Phi^+ \rangle_{A_1A_1'}, \langle \Phi^+ \rangle_{A_2A_2'} \}, \{ \langle \Phi^- \rangle_{A_1A_1'}, \langle \Phi^- \rangle_{A_2A_2'} \}, \{ \langle \Phi^+ \rangle_{A_1A_1'}, \langle \Phi^- \rangle_{A_2A_2'} \} and \{ \langle \Phi^- \rangle_{A_1A_1'}, \langle \Phi^+ \rangle_{A_2A_2'} \} here. Besides, the senders need broadcast the results of the measurement to the receivers and ancillals so that they can perform the unitary operations to modify the shared states locally. The result by the measurement \{ \langle \Phi^+ \rangle_{A_1A_1'}, \langle \Phi^+ \rangle_{A_2A_2'} \} is precisely \( |\omega\rangle_{B_1B_2C_1C_2a_1a_2} \).

For the second and third cases, by using of the parity-transformations \sigma_{2B_2} \otimes \sigma_{2C_2} \otimes \sigma_{z_2a_2} \) and \sigma_{z_1B_1} \otimes \sigma_{z_1C_1} \otimes \sigma_{z_1a_1} \) respectively, it is known from the relations (5)-(8) that the receivers recover the correct state again. In case of the final situation, it requires the collective rotations \sigma_{2B_2} \otimes \sigma_{C_1} \otimes \sigma_{z_2a_2} \otimes \sigma_{B_2} \otimes \sigma_{C_2} \otimes \sigma_{z_2a_2} \) by all parties. Hence, one can always recover the target state and thereby explicitly realize the optimal universal asymmetric 1 \rightarrow 2 telecloning of arbitrary two-qubit state by LOCC.

We investigate the scheme in terms of some figures of merit. First, the required entanglement between senders and receivers is \( E(\langle \omega\rangle_{A_1A_1'\'A_1''B_1B_1''B_2B_2''C_1C_1''C_2C_2''a_1a_1''a_2a_2'') = 2 \) ebits. Besides, the classical cost informing the receivers and ancillals is 4 ebits in all. Although the protocol in our paper is sufficient to treat the optimal 1 \rightarrow 2 asymmetric telecloning of any 2-qubit input, the quantum cost here is not always necessary for it has turned out that by using of only 1 ebit one can complete the optimal telecloning of a special family of two-qubit states as described in the introduction. We readily prove that for the case of genuine 4-dimensional space, namely \( a_0 \neq a_1 \neq a_2 \neq a_3 \), the cost of 2 ebits is also necessary for the telecloning scheme. Suppose that the input state is maximally entangled with another qudit:

\[ |\psi\rangle_{A_1A_2A_3} = \frac{1}{2} \left( |000\rangle + |011\rangle + |102\rangle + |113\rangle \right). \]

Following the formal procedure described above, we can obtain the resulting state

\[ |\omega\rangle_{A_3B_3B_2C_1C_2a_1a_2} = \sum_{j=0}^{3} \frac{1}{2} |j\rangle |\eta_j\rangle. \]
Second, the fidelity of our teleportation scheme is optimal. Due to the optimal universal asymmetric Heisenberg QCM \[10, 19\], for a \(d\)-level input state \(|\psi\rangle\) the clones have the form
\[
\rho_B = [1+(d-1)(p^2+q^2)]^{-1}\{[1-q^2+(d-1)p^2]|\psi\rangle\langle\psi|+q^2 I\},
\]
and
\[
\rho_C = [1+(d-1)(p^2+q^2)]^{-1}\{[1-p^2+(d-1)q^2]|\psi\rangle\langle\psi|+p^2 I\}.
\]
Then we can easily obtain the corresponding fidelities
\[
F_B(\rho_\psi, \rho_B) = \frac{1+(d-1)p^2}{1+(d-1)(p^2+q^2)},
\]
\[
F_C(\rho_\psi, \rho_C) = \frac{1+(d-1)q^2}{1+(d-1)(p^2+q^2)}.
\]
Explicitly, they reach Werner’s fidelity bound \([10]\) when \(p = q = 1/2\). As our protocol derives from the case of \(d = 4\) of the Heisenberg QCM, it is universal and not dependent on what the input state is. Recently, N. Cerf et al. \([20]\) has proposed an optimal universal 1 \(\rightarrow\) 2 QCM for maximally entangled inputs. Their fidelity is a little higher than Werner’s bound, since the set of maximal entanglement is a small part of the whole \(d\)-dimensional states. One can thus expect to get a more efficient scheme of teleporting by following \([20]\), as well as other special QCMs such as the phase covariant cloning \([24]\) and real cloner \([23]\), for both of them contribute a higher fidelity than Werner’s bound. However, as all these potential schemes of teleporting are remarkably restricted in the input states, our protocol gives a more universal plan. On the other hand, it is difficult to create a better entanglement QCM scheme for the unique characters it holds. As described by N. Cerf et al. \([20]\), when the input state is separable, the clones through the entanglement QCM should still be separable. Moreover, such a protocol has to maximize the entanglement of the clones, since it is regarded that some amounts of entanglement of the initial state will lose during the cloning process. Unfortunately, so far there is little progress for these problems. The main difficulty originates in the mathematical skills because of many variables in the deduction of optimal QCM, and it is also hard to explore the asymmetric case \([27]\). Moreover, so far all QCMs of entanglement require the bipartite inputs, which becomes fairly sophisticated if generalized to the multipartite case. So it is difficult to create the teleporting schemes by employing the universal QCMs of entangled states. Comparatively speaking, we will show that our scheme can be readily extended to the situation of multiqubit inputs and even the mixed setting is also included.

Finally, we prove that our scheme does not create more entanglement than that contained in the input state. The case of maximally entangled input by the optimal QCM has been checked in \([20]\), i.e., when \(\mu \equiv |\alpha_0\alpha_3 - \alpha_1\alpha_2| = 1/2\). Here, we show that this is a universal result for any \(\mu\) of entangled input. Due to the normalization condition of \(|\psi\rangle_{A_1A_2}\), we have \(\mu \in [0, 1/2]\). Let
\[
H(x) \equiv -(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2})\log_2(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2})
- (\frac{1}{2} - \frac{1}{2}\sqrt{1-x^2})\log_2(\frac{1}{2} - \frac{1}{2}\sqrt{1-x^2}),
\]
which is monotonically increasing with \(x \in [0, 1]\). One can simply obtain the entanglement of the input state is \(E(|\psi\rangle_{A_1A_2}) = H(2\mu)\). We employ the entanglement of formation \(E = H(C)\) \([26]\), where \(C = C_B(p)\) or \(C_C(p)\) is the concurrence \([24]\), to calculate the entanglement of the clones. Replace \(|\psi\rangle\) in \(\rho_B\) with \(\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle\), and calculate the eigenvalues \(\lambda_i\)'s of 
\[
\rho_B(\sigma_y \otimes \sigma_y)\rho_B^\dagger(\sigma_y \otimes \sigma_y).
\]
Notice \(F_B(\rho_\psi, \rho_B) = F_B(p) = 1 + 3\mu^2\), some simple algebra leads to
\[
C_B(p) = \max\{0, \sqrt{\lambda_0} - \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3}\}
= \max\{0, \frac{8}{3}F_B - \frac{2}{3}\mu - \frac{2}{3}(1-F_B)\},
\]
where \(\lambda_i\)'s are decreasingly ordered. Similarly, let 
\[
C_C(p) = \max\{0, \frac{8}{3}F_C - \frac{2}{3}\mu - \frac{2}{3}(1-F_C)\}.
\]
Let \(\Delta(\mu) = H(2\mu) - H(C_B(p)) - H(C_C(p))\), then our assertion is \(\Delta(\mu) \geq 0, \mu \in [0, 1/2]\). We have analytically proven it in appendix, so our scheme will never create more entanglement than that contained in the original state. In addition, \(F_B\) and \(F_C\) are not less than \(1/4\) from their expressions. When one of them reaches this lowest value, the other must be explicitly unit. For the symmetric case namely \(p = q = 1/2\), we have \(F_B = F_C = 7/10\), which reaches Werner’s bound. Hence, \(C_B(1/2) = C_C(1/2) = \max\{0, \frac{2}{3}\mu - \frac{1}{3}\}\) and the maximal amount of entanglement created in either of the clones is \(H(C(1/2)) = 0.250225\) ebits. This is less than that in \([20]\), which is a special set of the two-qubit states. Generally, the relation between entanglements created in the clones constitute a tesseract board due to the monotonicity of \(H(C_B(p))\) and \(H(C_C(p))\), i.e., if one of them decreases then the other must increases, and vice versa.

III. OPTIMAL UNIVERSAL 1 \(\rightarrow\) 2 TELECLONING OF N-QUBIT STATES AND 1 \(\rightarrow\) 4 TELECLONING OF MIXED STATES

In this section we extend the 1 \(\rightarrow\) 2 teleporting to the case of n-qubit pure states, and many properties of the above scheme works here. Subsequently, we apply this universal scheme to the 1 \(\rightarrow\) 4 teleporting of arbitrary mixed state, which is an interesting and difficult subject in QIT.
For convenience, we define the \( n \)-bit binary form of integer \( N \). Let \( N = 2^{n-1} \cdot c_{n-1} + \cdots + 2^i \cdot c_i + 2^0 \cdot c_0 \), where \( 2^n > N \) and \( c_i = 0 \) or 1, \( \forall i \). Then the unique binary form is \( N = c_{n-1} \cdots c_1 c_0 \) (we also write \( N = \overline{c_{n-1} \cdots c_1 c_0} \)). The situation here is that \( n \) separated senders \( A_1, A_2, \ldots, A_n \) share an arbitrary multiqubit state

\[
|\psi\rangle_{A_1, A_2, \ldots, A_n} = \sum_{k=0}^{2^n-1} \alpha_k |k\rangle_{A_1, A_2, \ldots, A_n},
\]

where the coefficients \( \alpha_k \)’s satisfy \( \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1 \), and the senders know nothing about the state. They plan to optimally teleclone this state at two remote locations, where two groups of uncorrelated receivers \( B_1, B_2, \ldots, B_n \) and \( C_1, C_2, \ldots, C_n \) make up the system in the clone respectively. Again, either of the participants in the whole system can only operate locally and they can communicate with each other. Consider the state \( |\eta\rangle_{BC,anc} \) in the last section. We write \( j \) in its \( n \)-bit binary form and each of the bit represents a party \( B_j \) or \( C_j \), namely

\[
|\eta\rangle_{BC,anc} \sim |\eta\rangle_{B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n, a_2, a_3, \ldots, a_{n-1}}.
\]

Having explained the form of \( |\eta\rangle_{BC,anc} \), we can propose the feasible quantum channel for the telecloning as follows

\[
|\varOmega\rangle_{A',BC,anc} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \left( \sum_{m=0}^{2^n-1} \alpha_k |k\rangle_{A_1, A_2, \ldots, A_n} \right) |\eta\rangle_{BC,anc}.
\]

Here, the particle \( A'_j \) belongs to the sender \( A_j \). So the total system is in the state

\[
|\Psi\rangle_{tot} = |\psi\rangle_{A_1, A_2, \ldots, A_n} \otimes |\varOmega\rangle_{A',BC,anc}
\]

\[
= \sum_{k=0}^{2^n-1} \alpha_k |k\rangle_{A_1, A_2, \ldots, A_n} \otimes \left( \sum_{m=0}^{2^n-1} \alpha_m |m\rangle_{A'_1, A'_2, \ldots, A'_n} \right) \langle \eta |_{BC,anc}
\]

\[
= \frac{\alpha_0}{2^n} \left( \sum_{k=0}^{2^n-1} |\psi\rangle_{A_1, A_2, \ldots, A_n} \langle \eta |_{BC,anc} \right)
\]

\[
+ \frac{\alpha_1}{2^n} \left( \sum_{k=0}^{2^n-1} |\psi\rangle_{A_1, A_2, \ldots, A_n} \langle \eta |_{BC,anc} \right)
\]

\[
\ldots
\]

\[
+ \frac{\alpha_{2^n-1}}{2^n} \left( \sum_{k=0}^{2^n-1} |\psi\rangle_{A_1, A_2, \ldots, A_n} \langle \eta |_{BC,anc} \right)
\]

\[
\otimes |\eta\rangle_{BC,anc}.
\]

Next, the senders take measurement in the Bell basis on their two-qubit systems respectively, and it is similar to that in the telecloning of two-qubit states. The target state in the present protocol is

\[
|\varOmega\rangle_{BC,anc} = \sum_{j=0}^{2^n-1} \alpha_j |\eta_j\rangle,
\]

which contains two optimal asymmetric clones of system \( B \) and \( C \). However, we face a more complicated situation here, and there are two main steps required for reaching the target state, which also resembles two-qubit’s case. First, we prove that any Bell measurement can be turned into one of the cases \( \{|\Phi^\pm\rangle_{A_1, A'_1}, |\Phi^\pm\rangle_{A_2, A'_2}, \ldots, |\Phi^\pm\rangle_{A_{n-1}, A'_{n-1}} \} \) with the same parity as the former one by certain collective unitary operations. That is, one can always obtain the state

\[
|\eta\rangle_{res} = \sum_{j=0}^{2^n-1} \alpha_j (1 - n) |\eta_j\rangle,
\]

where the sign \((-1)^n\) originates from the parity of Bell projection. In order to get this result, we notice that every term in \( |\Psi\rangle_{tot} \) can be written as (the coefficient is omitted)

\[
T_{lk} = |a_1, a'_1, a_2, a'_2, a_3, a'_3, \ldots, a_n, a'_n, A_{n-1}, A'_{n-1}, |\eta\rangle,
\]

\[
\text{where } k = a'_2 \cdots a'_n, \text{ and } l = a_1 \cdots a_{n-1}, \text{ denotes the sequence number of } a_{l_1} \text{ out of the bracket including this term. We call the factor } |a_{l_1}, a'_{l_1}, a_{l_2}, a'_{l_2}, \ldots, a'_{l_n}, A_{n-1}, A'_{n-1}, |\eta\rangle \text{ the secondary term of } T_{lk}. \text{ Moreover, if } a_{l_1} = a'_n, \text{ then this second term is even, otherwise it is odd. Evidently, the even secondary term is the sum (or subtraction) of the Bell basis } |\Phi^\pm\rangle, \text{ while the odd one is the sum (or subtraction) of the Bell basis } |\Phi^\pm\rangle. \text{ This implies that a single Bell projection only operates on an even or odd secondary term.}
\]

Observe the terms in the \( l \)-th bracket in \( |\Psi\rangle_{tot}, T_{00}, T_{11}, \ldots \). One can find that no two terms contain completely the same secondary terms with respect to the position of every secondary term, since the sequence number \( l \) is unchanged. Hence, there must be a uniquely residual term in every bracket after the Bell-measurement by the senders. Denote \( e_i \) the parity “+” or “−”, and suppose the measurement is taken in the sequence \( \{|\Psi^{e_1}_{a_1, A'_1}, \ldots, |\Psi^{e_n}_{a_n, A'_{n-1}}\} \), and other two-qubit systems are projected onto the basis \( \{|\Phi_{a_{l_1}, A_{n-1}}^{e_{l_1}}, \ldots, |\Phi_{a_{l_n}, A'_{n-1}}^{e_{l_n}}\} \). Such a projection eliminates all but one term in every bracket, which has \( s \) odd secondary terms. Concretely for the \( l \)-th bracket, the residual term is

\[
T_{lk} = |a_1, a'_1, a_2, a'_2, a_3, a'_3, \ldots, a_n, a'_n, A_{n-1}, A'_{n-1}, |\eta\rangle,
\]

\[
\text{where the tilde means the bit-shift, } \tilde{0} = 1, \tilde{1} = 0. \text{ Thus } k = a'_1 \cdots a'_{n-1}, \text{ and we must transform the term } |\eta\rangle \text{ into } |\eta_j\rangle \text{ for the } l \text{-th bracket simultaneously. We can realize it by virtue of a collective operation}
\]
Following the preceding scheme, we can obtain the asymmetric telecloning of state $|\Psi\rangle$

\[
\rho_{BB'} = [1+(d-1)(p^2+q^2)]^{-1}\{[1-q^2+(d-1)p^2]|\Psi\rangle\langle\Psi|+p^2I\},
\]

and

\[
\rho_{CC'} = [1+(d-1)(p^2+q^2)]^{-1}\{[1-p^2+(d-1)q^2]|\Psi\rangle\langle\Psi|+q^2I\}.
\]

It should be pointed that although the expression of $|\Psi\rangle$ in (38) and (39) is given by (37), the particles held by the parties become $B, B'$ and $C, C'$ respectively. As the state $|\Psi\rangle$ is symmetric with the exchange of system $B$ and $B'$, so by tracing out the freedom of $B'$ one can get the clone of state $\rho_{\psi}$

\[
\rho_B = [1+(d-1)(p^2+q^2)]^{-1}\{[1-q^2+(d-1)p^2]\sum_{k=0}^{\sqrt{d}^{-1}}\alpha_k|\overline{k}\rangle_B|\overline{k}\rangle+\sqrt{d}q^2I\},
\]

and $\rho_C$ is similarly realized by $p \leftrightarrow q$. Generally, this scheme indeed realizes the cloning of $\rho_B$ to $\rho_{BB'}$, $\rho_C$, $\rho_{BB'}$ and $\rho_{CC'}$. Thus it is a $1 \rightarrow 4$ asymmetric telecloning of arbitrary mixed states, with the fidelities $F(\rho_{\psi}, \rho_B) = F(\rho_{\psi}, \rho_{BB'})$ and $F(\rho_{\psi}, \rho_{CC'}) = F(\rho_{\psi}, \rho_{CC'})$. Moreover, both of them are higher than that of the pure telecloning since the trace of a subsystem is a trace-preserving quantum operation $\mathcal{P}$, e.g.,

\[
F(\rho_{\psi}, \rho_B) = F(\text{Tr}[|\Psi\rangle\langle\Psi|], \text{Tr}[\rho_{BB'}]) \geq F(|\Psi\rangle\langle\Psi|, \rho_{BB'}).
\]

In the same way, we obtain $F(\rho_{\psi}, \rho_C) \geq F(|\Psi\rangle\langle\Psi|, \rho_{CC'})$. We explicitly calculate this fidelity as a most figure of the scheme. The fidelity of two mixed states is $\mathcal{F}$,

\[
F(\rho_1, \rho_2) = \left[\text{Tr}\sqrt{\rho_1\rho_2}\sqrt{\rho_1}\right]^2.
\]

By some simple algebra, it follows that

\[
F(\rho_{\psi}, \rho_B) = [1+(d-1)(p^2+q^2)]^{-1}\left[\sum_{k=0}^{\sqrt{d}^{-1}}\sqrt{[1-q^2+(d-1)p^2]\alpha_k^2+\sqrt{d}q^2\alpha_k}^2\right]^2.
\]

Here, the unique restriction is $\sum_{k=0}^{\sqrt{d}^{-1}}\alpha_k = 1, \forall \alpha_k \geq 0$. By employing the Lagrange multipliers, it is easy to prove that $F(\rho_{\psi}, \rho_B) \in [1-q^2+(d-1)p^2+\sqrt{d}q^2, 1]$, and similarly $F(\rho_{\psi}, \rho_C) \in [1-p^2+(d-1)q^2+\sqrt{d}p^2, 1]$. Comparing them with those for the pure states, we find that the $1 \rightarrow 4$ asymmetric telecloning of mixed states can be realized with a high fidelity by virtue of our scheme.

IV. CONCLUSIONS

In this paper, we addressed the problem of asymmetric quantum teleporting of arbitrary multipartite states.
in a universal case. Our 1→2 optimal scheme employed the Heisenberg QCM, which explicitly reaches Werner’s bound. We provided an important application of this scheme on the 1→4 universal telecloning of mixed multi-qubit states, with a fidelity higher than that of the pure states. It is interesting that there may exist some relations between the cloning of mixed states and multipartite states. The present scheme cannot create more entanglement than that of the original state. It is a problem to extend our scheme to the case of 1→M, so that the entanglement can be remotely cloned more generally.

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APPENDIX

Here we show that $\Delta(\mu) \geq 0, \mu \in [0,1/2]$. For the case of $\mu = 1/2$, $\Delta(1/2) = H(1) - H(\max \{0, 2F_B - 1\}) - H(\max \{0, 2F_C - 1\})$. When either of $2F_B - 1$ and $2F_C - 1$ is less than zero, the monotonicity of $H(x)$ makes $\Delta(\mu) \geq 0$. When both of them are positive, it is easy to recover the assertion by plotting the function $\Delta(1/2)$, whose independent variable is $p \in [1/3, 2/3]$. Moreover, when $\mu \in [0,1/6]$, there is at least one zero in $C_B(p)$ and $C_C(p)$, and the monotonicity of $H(x)$ makes $\Delta(\mu)$ ≥ 0 again. This implies that for the inputs with $\mu \leq 1/6$, it is impossible to create entanglement in both of the clones simultaneously by our scheme. So it suffices to investigate the case of $\mu \in (1/6, 1/2)$, where both $C_B(p)$ and $C_C(p)$ must be positive. Recall that $q = 1 - p$, we have

$$F_B(p) = \frac{1 + 3p^2}{4 - 6p + 6p^2} > \frac{\mu + 1}{4\mu + 1},$$

$$F_C(p) = \frac{4 - 6p + 3p^2}{4 - 6p + 6p^2} > \frac{\mu + 1}{4\mu + 1},$$

namely

$$p \in \left(1 + \frac{4\mu - \sqrt{4\mu + \mu^2}}{1 - 2\mu}, -3\mu + \frac{\sqrt{4\mu + \mu^2}}{1 - 2\mu}\right).$$

Mathematically, we only need calculate the derivative of $\Delta(\mu)$ with respect to $p$, but it is difficult to do it in this way because of the confused deduction. Notice that $F_B(p) = F_C(1-p)$, so $H(C_B(p)) = H(C_C(1-p))$, i.e., they are symmetric and the symmetry axis is $p = 1/2$. Thus we focus on the property of $H(C_B(p))$. As $C_B(p)$ is monotonically increasing with $p$, $H(C_B(p))$ is also monotonically increasing with $p$. Calculate the second derivative of $H(C_B(p))$ with respect to $p$,

$$\frac{d^2}{dp^2}H(C_B(p)) = \frac{d}{dp}\left(\frac{dH}{dC} \frac{dC}{dp}\right)
= \frac{d^2H}{dC^2} \left(\frac{dC}{dp}\right)^2 + \frac{dH}{dC} \frac{d^2C}{dp^2}
= \lambda \left[\left(\begin{array}{c}
\frac{8}{3} + \frac{2\sqrt{1 - C^2}}{3} \log_c \frac{1 - \sqrt{1 - C^2}}{1 + \sqrt{1 - C^2}} \\
4(2 - 3p + 3p^2)(5 - 9p - 9p^2 + 9p^3)
\end{array}\right) \left(\begin{array}{c}
\frac{1}{3(1 - 2p + 3p^2)^2}
\end{array}\right)
\right],$$

where $\lambda = \left(\frac{8}{3} + \frac{2\sqrt{1 - C^2}}{3}\right) \frac{dH}{dC} \frac{dC}{dp}$ is positive. As the first part and second part in the square bracket are monotonically decreasing with $C = C_B(p)$ and $p$ respectively, $\frac{d^2}{dp^2}H(C_B(p))$ is monotonically decreasing with $p$. By virtue of plotting it is easy to show that

$$\frac{d}{dp}H(C_B(p)) \bigg|_{p=0.56} > 0,$$

$$\frac{d}{dp}H(C_B(p)) \bigg|_{p=0.44} > 0.$$

Although the point $p = 2/3$ is usually not in the physical region $\left(\frac{1 + \mu - \sqrt{4\mu + \mu^2}}{1 - 2\mu}, \frac{3\mu + \sqrt{4\mu + \mu^2}}{1 - 2\mu}\right)$, the above argument mathematically applies to the region $\left(\frac{1 + \mu - \sqrt{4\mu + \mu^2}}{1 - 2\mu}, \frac{2}{3}\right)$. Thus the inflection point of $H(C_B(p))$ is $p_n > 0.56$. Consider the sum of the $H(C_B(p))$ and $H(C_C(p))$, where $H(C_B(p)) = H(C_C(1-p))$. When $p \in \left[p_n, \frac{-3\mu + \sqrt{4\mu + \mu^2}}{1 - 2\mu}\right]$,

$$\frac{d}{dp}H(C_B(p)) + H(C_C(p)) >$$

$$\frac{d}{dp}H(C_B(p)) \bigg|_{p=0.56} - \frac{d}{dp}H(C_B(p)) \bigg|_{p=0.44} > 0,$$

and when $p \in \left[1/2, p_n\right]$, one readily obtains $\frac{d}{dp}[H(C_B(p)) + H(C_C(p))] > 0$ as the reflection point $p_n > 0.56$. So $H(C_B(p)) + H(C_C(p))$ is monotonically increasing when $p \in \left[1/2, \frac{-3\mu + \sqrt{4\mu + \mu^2}}{1 - 2\mu}\right]$. Due to the symmetry of $H(C(p))$ and $H(C_C(p))$, the maximum of $\Delta(\mu)$ is in the bound $p = \frac{1 + \mu - \sqrt{4\mu + \mu^2}}{1 - 2\mu}$ or $\frac{-3\mu + \sqrt{4\mu + \mu^2}}{1 - 2\mu}$. This is just the case where $C_B(p)$ or $C_C(p)$ vanishes, and thus $\Delta(\mu) \geq 0$. So we conclude that our scheme of 2-qubit telecloning will never create more entanglement than that contained in the original state.
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