EIKONAL QUANTUM GRAVITY AND PLANCKIAN SCATTERING

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ABSTRACT

Various approaches to high energy forward scattering in quantum gravity are compared using the eikonal approximation. The massless limit of the eikonal is shown to be equivalent to other approximations for the same process, specifically the semiclassical calculation due to G. ’t Hooft and the topological field theory due to H. and E. Verlinde. This comparison clarifies these previous results, as it is seen that the amplitude arises purely from a linearized gravitational interaction. The interpretation of poles in the scattering amplitude is also clarified.
I. INTRODUCTION

Quantization of General Relativity leads to a perturbatively non-renormalizeable theory\(^1\); hence the general view that this approach to quantum gravity is without predictive power. It is therefore interesting that predictive calculations do exist for the two particle forward scattering amplitude at energies of order the Planck scale. G. ’t Hooft\(^2\) evaluated the amplitude semiclassically by solving the Klein-Gordon equation for one particle in the gravitational shock wave due to the other particle. Since there is no particle production in a shock wave spacetime, this calculation may also be thought of as solving a free scalar field in the curved background (Aichelburg–Sexl metric\(^3\)) arising from a classical ultrarelativistic source\(^4\). Subsequently H. and E. Verlinde\(^5\) showed that in this kinematic regime, quantum gravity separates into weakly and strongly coupled sectors. The weakly coupled sector can be treated classically, while the strongly coupled sector is a topological theory which may be solved nonperturbatively to reproduce ’t Hooft’s scattering amplitude.

The eikonal approximation is a technique in quantum field theory for evaluating the leading behavior of a forward scattering amplitude in the limit of large center of mass energy by summing an infinite class of Feynman graphs\(^6,7,8\). It has been used in string models of quantum gravity by Muzinich and Soldate\(^9\) and by Amati et. al.\(^10\) to obtain results similar to those of ’t Hooft for Planckian scattering (see also Ref. 11). In this paper we apply the eikonal approximation to linearized quantum gravity. We first use the approximation to derive an amplitude which is identical to that of Refs. 2, 5. This equivalence has previously been noted in the context of string theory\(^9,10\) and for electromagnetic scattering\(^12\). We then show how, in the massless limit, the summation of Feynman graphs in the eikonal approximation is
equivalent to the methods employed by both ’t Hooft and H. and E. Verlinde to evaluate the
Planckian scattering amplitude. The calculation of the eikonal approximation away from the
massless limit allows us to clarify the issue of poles in the scattering amplitude first discussed
by ’t Hooft\textsuperscript{2,25}.

The paper is organized as follows. Part II is a calculation of the eikonal scattering am-
plitude for linearized gravity. Part III shows the equivalence of the eikonal approximation to
’t Hooft’s semiclassical calculation, and part IV discusses the origin of the poles in the scat-
tering amplitude. Finally, part V shows how graviton exchange in the eikonal approximation
is governed by a topological theory in the high energy limit.

II. EIKONAL AMPLITUDE FOR LINEARIZED GRAVITY

We present the eikonal calculation of the two particle scattering amplitude in linearized
gravity. Similar calculations for the scattering of massless particles in a string theory model
of quantum gravity may be found in Refs. 9 and 10.

The action for Einstein gravity coupled to a real scalar field is

\[
S = \int d^4x \sqrt{-\text{det}g} \left\{ -\frac{1}{16\pi G} \left( R + \frac{1}{2} g_{\mu\nu} C^{\mu\nu} C_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 \right\}.
\]  

The metric is to be expanded about a Minkowski background, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \). The gauge
fixing term is \( C_\mu = \partial_\nu h^\nu_{\mu} - \frac{1}{2} \partial_\mu h^\nu_{\nu} \); the ghosts associated with this gauge choice are ignored,
since we expect ghost contributions to be sub-dominant in forward scattering. Expanding in
the leading terms give the action for linearized gravity.

\[
S = \int d^4x \frac{1}{16\pi G} \frac{1}{8} h_{\alpha\beta} \left[ \eta^{\alpha\gamma} \eta^{\beta\delta} + \eta^{\alpha\delta} \eta^{\beta\gamma} - \eta^{\alpha\beta} \eta^{\gamma\delta} \right] \Box^2 h_{\gamma\delta} + \frac{1}{2} \phi(\Box^2 - m^2)\phi
+ \frac{1}{2} h_{\mu\nu} \left[ \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^\mu^\nu \left( \partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2 \right) \right].
\] (2.2)

Here and subsequently all indices are to be raised and lowered with \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \).

This action leads to the propagators and vertices in Fig. 1.

Perturbative quantum gravity has been investigated by many authors\cite{13}. It is well known that the theory suffers from non-renormalizeable infinities, which for the process we are considering are present even at one loop\cite{1,14}. Within this approach, the two particle scattering amplitude may apparently only be calculated in the Born approximation, a calculation which may be found in De Witt’s seminal paper on quantum gravity\cite{13}. However, as we shall now explain, it is possible to go beyond this first order approximation if one wishes to probe only the leading behaviour of high energy forward scattering.

We are interested in the elastic forward scattering amplitude of two scalar particles with initial momenta \( p_1, p_2 \) and final momenta \( p_3, p_4 \), in the regime where \( s/t \) is large (here \( s \) and \( t \) are the usual Mandelstam variables \( s = -(p_1 + p_2)^2 \) and \( t = -(p_1 - p_3)^2 \)). We proceed to evaluate the scattering amplitude using the eikonal approximation, which may be described as follows\cite{7}. Consider the sum of ladder and crossed ladder graphs (Fig. 2), and make the following approximations. In evaluating the vertex factors, ignore the recoil of the matter field,

\[
p_{\mu}p'_{\nu} + p_{\nu}p'_{\mu} - \eta_{\mu\nu} \left( p \cdot p' + m^2 \right) \approx 2p_{\mu}p_{\nu}.
\]

In the matter propagators, ignore \( k^2 \) relative to \( p \cdot k \),

\[
\frac{1}{(p + k)^2 + m^2 - i\epsilon} \approx \frac{1}{2p \cdot k - i\epsilon}.
\]
This procedure is expected to give the leading behaviour of the ladder diagrams for large centre of mass energy\textsuperscript{15}. To preserve Bose symmetry we should include diagrams where $p_3$ and $p_4$ are interchanged, but such diagrams are clearly subdominant. The question of whether the ladder diagrams dominate over other exchange diagrams in high energy scattering in quantum gravity is a more subtle question\textsuperscript{16,17}.

We illustrate this procedure order by order (Fig. 3). The Born amplitude is simply

\[
\begin{align*}
\imath M_{\text{Born}} &= \imath p_{1\alpha}p_{1\beta} \frac{-\imath 16\pi G}{(p_1-p_3)^2 - i\epsilon} \left( \eta^{\alpha\gamma} \eta^{\beta\delta} + \eta^{\alpha\delta} \eta^{\beta\gamma} - \eta^{\alpha\beta} \eta^{\gamma\delta} \right) \imath p_{2\gamma}p_{2\delta} \\
&= \imath 16\pi G \gamma(s) \frac{1}{(p_1-p_3)^2 - i\epsilon}
\end{align*}
\]

where

\[
\gamma(s) \equiv 2(p_1 \cdot p_2)^2 - m^4 = \frac{1}{2} \left[ (s - 2m^2)^2 - 2m^4 \right].
\]

In terms of the Mandelstam variables, it may be written as

\[
i M_{\text{Born}} = \frac{i 16\pi G \gamma(s)}{-t}
\]

which agrees with the expressions derived by De Witt\textsuperscript{13} and in a more recent analysis by Diebel and Schücker\textsuperscript{18} when $t/s \to 0$.

In the higher order ladder diagrams one has a choice of which graviton momentum to fix by momentum conservation. We proceed by averaging over this choice, as illustrated in Fig.
3. The one loop terms are then
\[ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} ip_{1\alpha}p_{1\beta} \frac{-i16\pi G}{k^2 - i\epsilon} (\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}) ip_{2\gamma}p_{2\delta} \]
\[ - \frac{-2p_1 \cdot k - i\epsilon}{2p_2 \cdot k - i\epsilon} \]
\[ i p_{1\mu}p_{1\nu} \frac{-i16\pi G}{(p_1 - p_3 - k)^2 - i\epsilon} (\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\sigma}) ip_{2\lambda}p_{2\sigma} \]
\[ + \text{crossed and symmetrized graphs} \]
\[ = (16\pi G)^2 \gamma^2(s) \int \frac{d^4k}{(2\pi)^4} k^2 - i\epsilon \]
\[ \frac{1}{(p_1 - p_3 - k)^2 - i\epsilon} \]
\[ \left( \frac{1}{-2p_1 \cdot k - i\epsilon} \frac{1}{2p_2 \cdot k - i\epsilon} + \frac{1}{-2p_1 \cdot k - i\epsilon} \frac{1}{-2p_4 \cdot k - i\epsilon} \right) \]
\[ + \frac{1}{2p_3 \cdot k - i\epsilon} \frac{1}{2p_2 \cdot k - i\epsilon} + \frac{1}{2p_3 \cdot k - i\epsilon} \frac{1}{-2p_4 \cdot k - i\epsilon} \].

Note that the eikonal approximation to the one loop integral is finite, as the expected infinity appears in a portion of the integral sub-dominant in $s/t$. Now in QED, for example, there is a well defined procedure for renormalizing the divergent contributions. The absence of such a procedure in quantum gravity means that we must simply ignore the subdominant terms and be satisfied with the observation that the leading order terms arise from finite integrals.

To write the amplitude to all orders, we first introduce
\[ \frac{-1}{k^2 - i\epsilon} = \int d^4x e^{-ik \cdot x} \Delta(x) \]
so that the infinite sum of ladder graphs is given by
\[ i\mathcal{M} = -16\pi G \gamma(s) \int d^4x e^{-i(p_1 - p_3) \cdot x} \Delta(x) \left\{ i - \frac{1}{2} \chi + \cdots \right\} \]
where
\[ \chi \equiv -16\pi G \gamma(s) \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \frac{1}{k^2 - i\epsilon} \]
\[ \left( \frac{1}{-2p_1 \cdot k - i\epsilon} \frac{1}{2p_2 \cdot k - i\epsilon} + \frac{1}{-2p_1 \cdot k - i\epsilon} \frac{1}{-2p_4 \cdot k - i\epsilon} \right) \]
\[ + \frac{1}{2p_3 \cdot k - i\epsilon} \frac{1}{2p_2 \cdot k - i\epsilon} + \frac{1}{2p_3 \cdot k - i\epsilon} \frac{1}{-2p_4 \cdot k - i\epsilon} \].
The combinatorics are such that the higher order terms form an exponential series. Summing the series gives the eikonal expression for the amplitude,

\[ iM = -16\pi G\gamma(s) \int d^4xe^{-i(p_1-p_3)\cdot x} \Delta(x) \frac{e^{ix\chi}-1}{\chi}. \]  

(2.4)

We now make the further approximation of taking \( p_1 \approx p_3, p_2 \approx p_4 \) in \( \chi \), so that

\[
\chi = -16\pi G\gamma(s) \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \frac{1}{k^2 - i\epsilon} \frac{1}{2p_1 \cdot k + i\epsilon} \frac{1}{2p_1 \cdot k - i\epsilon} \frac{1}{2p_2 \cdot k + i\epsilon} \frac{1}{2p_2 \cdot k - i\epsilon} (-2\pi i)^2 \delta(2p_1 \cdot k) \delta(2p_2 \cdot k) \\
= -16\pi G\gamma(s) \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \frac{1}{k^2 - i\epsilon} (-1)^k \frac{1}{2\pi} \delta(2p_1 \cdot k) \delta(2p_2 \cdot k) \\
= \frac{2\pi G\gamma(s)}{Ep} \int \frac{d^2k_{\perp}}{(2\pi)^2} e^{ik_{\perp} \cdot x_{\perp}} \frac{1}{k_{\perp}^2 + \mu^2 - i\epsilon}.
\]

This is in the center of mass frame, where \( p^\mu_1 = (E, 0, 0, p) \) and \( p^\mu_2 = (E, 0, 0, -p) \). The \( x_{\perp} \) are coordinates in the transverse \((x, y)\) plane, and \( \mu \) is a graviton mass inserted as an infrared regulator. The integral over \( k_{\perp} \) may be carried out, yielding

\[
\chi = \frac{G\gamma(s)}{Ep} K_0(\mu x_{\perp}) \\
\approx -\frac{G\gamma(s)}{Ep} \log(\mu x_{\perp}),
\]

for \( \mu x_{\perp} \ll 1 \) (with a numerical constant absorbed in \( \mu \)).

In the eikonal regime, the momentum transfer \( q = p_1 - p_3 \) has components predominantly in the transverse directions. This follows from the on–shell relations \( q^2 + 2q \cdot p_3 = q^2 - 2q \cdot p_1 = 0 \), and the eikonal kinematics \( t = -q^2 \ll s = 4E^2 \). Note that

\[
\int dt \, dz \Delta(x) = \int \frac{d^2q_{\perp}}{(2\pi)^2} e^{iq_{\perp} \cdot x_{\perp}} \frac{-1}{q_{\perp}^2 - i\epsilon} = \frac{-Ep}{2\pi G\gamma(s)} \chi.
\]
so that setting $q^0 = q^3 = 0$ in (2.4) gives

$$iM = 8Ep \int d^2x_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} (e^{i\chi} - 1).$$

The integral over $\mathbf{x}_\perp$ may be carried out, leading to

$$iM = \frac{8\pi Ep}{\mu^2} \Gamma \left( 1 - i\frac{G\gamma(s)}{2Ep} \right) \left( \frac{4\mu^2}{q_\perp^2} \right)^{1 - i\frac{G\gamma(s)}{2Ep}}$$

where the delta function in $iM$ has been dropped. The final result for the eikonal amplitude may be rewritten in terms of Lorentz invariant Mandelstam variables:

$$iM = \frac{2\pi}{\mu^2} \sqrt{s(s - 4m^2)} \frac{\Gamma(1 - i\alpha(s))}{\Gamma(i\alpha(s))} \left( \frac{4\mu^2}{-t} \right)^{1 - i\alpha(s)}$$

(2.5)

where

$$\alpha(s) = G \frac{(s - 2m^2)^2 - 2m^4}{\sqrt{s(s - 4m^2)}}.$$

(2.6)

This expression is very similar to the electrodynamical eikonal expression\(^{10}\), but with additional powers of $s$ in $\alpha(s)$ arising from the gravitational couplings. Provided that $s > 0$ one may set $m = 0$ in (2.6) to obtain ’t Hooft’s scattering amplitude\(^2\). Note that ’t Hooft scales $\mu$ to unity, and evidently includes a kinematic factor $1/8\pi^2 s$ for the external lines.

The eikonal amplitude may also be written in the form

$$iM = \frac{i16\pi G\gamma(s)}{-t} \frac{\Gamma(1 - i\alpha(s))}{\Gamma(1 + i\alpha(s))} \left( \frac{4\mu^2}{-t} \right)^{-i\alpha(s)}$$

from which it is evident that $iM$ is just the Born amplitude (2.3) multiplied by

$$\frac{\Gamma(1 - i\alpha(s))}{\Gamma(1 + i\alpha(s))} \left( \frac{4\mu^2}{-t} \right)^{-i\alpha(s)}.$$
The effect of the additional factor, which is just a phase for $\alpha(s)$ real, is to introduce additional poles in the scattering amplitude. These poles will be discussed further in section IV.

III. QUANTUM MECHANICS FROM THE EIKONAL

In this section we discuss the relationship between quantum mechanics and the eikonal approximation in quantum field theory. In the course of our discussion, we shall show that the technique employed by 't Hooft for deriving the forward scattering amplitude is completely equivalent to performing an eikonal approximation. Our approach in this section applies the work in Refs. 6, 20, and 21 on the QED eikonal approximation to gravity.

Let us begin by writing the four point Green's function in path integral language:

$$G(x_1, x_1'; x_2, x_2') =$$

$$\int \mathcal{D}h_{\mu\nu} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \phi_1(x_1)\phi_1(x_1')\phi_2(x_2)\phi_2(x_2') \exp \left\{ i \int d^4x \sqrt{-\det g(h)} \left[ -\frac{1}{16\pi G} \left( R(h) + \frac{1}{2} g_{\mu\nu} C^\mu C^\nu \right) - \frac{1}{2} \nabla_\mu \phi_i \nabla^\mu \phi_i - \frac{1}{2} m_i^2 \phi_i \phi_i \right] \right\}. $$

Here $i$ labels the two scalar fields which have been introduced to avoid the unnecessary complication of identical particles. This may also be written

$$\int \mathcal{D}h_{\mu\nu} G_1(x_1, x_1'|h_{\mu\nu}) G_2(x_2, x_2'|h_{\mu\nu}) \exp \left\{ i \int d^4x \sqrt{-\det g(h)} \left[ -\frac{1}{16\pi G} \left( R(h) + \frac{1}{2} g_{\mu\nu} C^\mu C^\nu \right) \right] \right\}. \tag{3.1}$$

Here $G_1(x_1, x_1'|h_{\mu\nu}), G_2(x_2, x_2'|h_{\mu\nu})$ are two point Green’s functions for scalar fields of mass $m_1, m_2$ respectively in the presence of a background gravitational field $h_{\mu\nu}$.
As we are not interested in the full amplitude, we truncate the functional integral (3.1) to generate only the sum of generalized ladder graphs of Fig. 2. Following Abarbanel and Itzykson\textsuperscript{6}, first consider the truncation

$$\int D h_{\mu\nu} G^c_1(x_1, x'_1|h_{\mu\nu}) G^c_2(x_2, x'_2|h_{\mu\nu}) \exp \left\{ i \int d^4 x \frac{1}{2} h_{\alpha\beta} (D^{-1})^{\alpha\beta\gamma\delta} h_{\gamma\delta} \right\} (3.2)$$

where $G^c_i(x_i, x'_i|h_{\mu\nu})$ is the connected part of the two point Green’s function for a free scalar field in the linearized gravitational background $h_{\mu\nu}$. That is, $G^c_i$ is simply a Green’s function for the single particle Klein–Gordon equation in a linearized background metric. In (3.2) we have eliminated scalar loops and scalar–multi-graviton vertices by taking only the connected linearized two point functions. We have also eliminated graviton self–couplings by linearizing the gravitational action, so all that remains are the generalized ladder graphs of Fig. 2, in which all gravitons are exchanged between the two scalar lines, along with graphs of the type shown in Fig. 4, in which one or more gravitons couple back to a single scalar line.

Using the identity

$$\int D V e^{\frac{1}{2} V \Delta^{-1} V} F[V] = \exp \left\{ \frac{i}{2} \frac{\delta}{\delta V} \Delta \frac{\delta}{\delta V} \right\} F[V] \bigg|_{V=0}$$

we can further rewrite (3.2) as

$$\exp \left\{ i \int d^4 x d^4 y \frac{\delta}{\delta h_1^{\alpha\beta}(x)} D^{\alpha\beta\gamma\delta}(x - y) \frac{\delta}{\delta h_2^{\gamma\delta}(y)} \right\}$$

$$G^c_1(x_1, x'_1|h_{1\mu\nu}) G^c_2(x_2, x'_2|h_{2\mu\nu}) \bigg|_{h_1=h_2=0} (3.3)$$

Here we have introduced distinct $h_{1\mu\nu}$ and $h_{2\mu\nu}$ to prohibit gravitons from coupling to only one scalar line, so that (3.3) generates only the sum of generalized ladder graphs. To evaluate this sum, it remains to determine the two point functions $G^c_i(x_i, x'_i|h_{\mu\nu})$. 
We proceed by studying the quantum mechanical Green’s function $G(x, x'|h_{\mu\nu})$ for the linearized, curved space Klein–Gordon equation, along with the free Feynman propagator $\Delta(x, x')$.

\[
(-\Box^2 + m^2 + V(x')) G(x, x'|h_{\mu\nu}) = -\delta^4(x - x')
\]

\[
(-\Box^2 + m^2) \Delta(x, x') = -\delta^4(x - x').
\]

Here $V(x') = h_{\mu\nu}(x') \partial_{x'\mu} \partial_{x'\nu}$ is the potential in the linearized Klein–Gordon equation in De Donder gauge, $\partial_{\mu} h_{\nu}^{\mu} - \frac{1}{2} \partial_{\nu} h = 0$. Also introduce an auxiliary kernel $\phi(x, x')$ defined by

\[
G^c(x, x') \equiv \Delta^{-1}(G - \Delta)\Delta^{-1} = V(x')\phi(x, x').
\]

$G^c$ is the connected amputated Green’s function. It follows that $\phi$ obeys the homogeneous Klein–Gordon equation once the incoming line is put on–shell,

\[
(-\Delta^{-1} + V) \phi = 0.
\]

(Off–shell one has in general $(-\Delta^{-1} + V) \phi = -\Delta^{-1}$). Transforming to

\[
\phi_p(x') = \int d^4 x e^{ip \cdot x} \phi(x, x')
\]

we make the eikonal approximation by setting $\phi_p(x') = e^{ip \cdot x'} \tilde{\phi}_p(x')$ and assuming that $\tilde{\phi}_p(x')$ is slowly varying compared to $e^{ip \cdot x'}$. Keeping leading terms in the linearized Klein–Gordon equation, $\tilde{\phi}_p$ is seen to obey

\[
(2ip^\mu \partial_{\mu} + h_{\mu\nu}p^\mu p^\nu) \tilde{\phi}_p = 0.
\]

This equation may be integrated to give the eikonal wavefunction

\[
\tilde{\phi}_p(x') = \exp \left\{ \frac{i}{2m} \int_{-\infty}^{0} d\tau p^\mu p^\nu h_{\mu\nu} \left( x' + \frac{p}{m} \tau \right) \right\}.
\]
which satisfies the boundary condition $\tilde{\phi}_p \to 1$ in the incoming direction. This is just a
WKB approximation with the particle taken to travel along a straight line. The eikonal
approximation to the amputated momentum space Green’s function is then

$$G_{\text{eik}}^c(p, p') = \int d^4x' e^{-i p'^\mu x'^\mu} V(x') \phi_p(x')$$

$$\approx - \int d^4x' e^{-i (p' - p) \cdot x'} h_{\mu\nu} p'^\mu p'^\nu \tilde{\phi}_p(x')$$

$$= \int d^4x' e^{-i (p' - p) \cdot x'} 2ip'^\mu \frac{\partial}{\partial x'^\mu} \exp \left\{ \frac{i}{2m} \int_{-\infty}^0 d\tau p'^\mu p'^\nu h_{\mu\nu} \left( x' + \frac{p}{m} \tau \right) \right\}.$$  \hspace{1cm} (3.5)

Changing variables of integration:

$$x' = \frac{p}{m} \sigma + z,$$

$$\tau = \tau' - \sigma,$$

(3.5) becomes

$$G_{\text{eik}}^c = \frac{p^0}{m} \int d\sigma d^3z e^{-i (p' - p) \cdot z} 2im \frac{\partial}{\partial \sigma} \exp \left\{ \frac{i}{2m} \int_{-\infty}^\sigma d\tau' p'^\mu p'^\nu h_{\mu\nu} \left( z + \frac{p}{m} \tau' \right) \right\}.$$ \hspace{1cm} (3.6)

In the eikonal regime, $q = p - p'$ has components predominantly in the transverse directions,
so we have dropped the $\sigma$ contribution to $e^{iq \cdot z}$ in (3.6). Performing the $\sigma$ integration and
dropping the disconnected piece gives

$$G_{\text{eik}}^c = 2ip^0 \int d^3z e^{iq \cdot z} \exp \left\{ \frac{i}{2m} \int_{-\infty}^\infty d\tau p'^\mu p'^\nu h_{\mu\nu} \left( z + \frac{p}{m} \tau \right) \right\}$$

which may be written in the simple form

$$G_{\text{eik}}^c = 2ip^0 \int d^3z e^{iq \cdot z} \exp \left\{ \frac{i}{2} \int d^4x T^\mu\nu(x) h_{\mu\nu}(x) \right\}$$

where

$$T^\mu\nu(x) = \int_{-\infty}^\infty d\tau \frac{1}{m} p'^\mu p'^\nu \delta^4 \left( x - z - \frac{p}{m} \tau \right)$$

is the classical energy-momentum tensor for a point particle with momentum $p$. 

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Now return to the truncated four–point function (3.3), and consider the kinematical regime discussed by ’t Hooft in Ref. 2. The forward scattering process is viewed in a frame in which particle 1 is highly energetic compared to particle 2, so one may regard the deflection of particle 1 to be negligible. It is therefore natural in this frame to treat the two point Green’s function for particle 1 in the eikonal approximation, so we replace $G^c_1(x_1, x'_1)$ with $G^c_{1\text{ eik}}(p_1, p'_1)$ in (3.3). Note that we work in position space for particle 2 and momentum space for particle 1. Carrying out the functional differentiation in the resulting expression yields

$$G(p_1, p'_1 = p_1 + q; x_2, x'_2) = 2i p_1^0 \int d^3z e^{iq \cdot z} G^c_2(x_2, x'_2|h_{\mu\nu})$$

for the four point Green’s function, where

$$h_{\mu\nu}(x) = \frac{1}{2} \int d^4y D_{\mu\nu\alpha\beta}(x - y) T^\alpha_\beta(y)$$

is the linearized spacetime due to the classical source $T^\alpha_\beta$. By treating one of the two point Green’s functions of (3.3) in the eikonal approximation we have therefore rewritten the four point function in terms of a quantum mechanical Green’s function in a gravitational background.

Recall that in section II we made the eikonal approximation to both matter lines, which is equivalent to solving the eikonalized Klein–Gordon equation (3.4) for $m_2$ in the linearized metric due to $m_1$. However, the eikonal approximation is exact for quantum mechanics in the $\frac{1}{r}$ potential of a linearized metric, so the amplitude obtained in section II will agree with the amplitude we will obtain in section IV by solving the full (non-eikonalized) Klein–Gordon equation.
In the massless case, the 4-momentum of particle 1 is given by \( p_1^\mu = (E, 0, 0, E) \) and \( h_{\mu\nu} \) is the Aichelburg-Sexl metric\(^3\). Hence 't Hooft’s approach\(^2\) of solving the Klein–Gordon equation to obtain \( G_2^c(x_2, x'_2|h_{\mu\nu}) \) and the scattering amplitude will reproduce the eikonal amplitude obtained by summing ladder graphs (in the massless limit). Alternatively, one may work in the rest frame of particle 1, \( p_1^\mu = (m_1, 0, 0, 0) \), where one sees that \( h_{\mu\nu}(x) \) is the linearized Schwarzschild metric created by a mass \( m_1 \) located at rest at \( z \), and that \( G_2^c(x_2, x'_2|h_{\mu\nu}) \) is the Green’s function for the Klein–Gordon equation in the metric produced by \( m_1 \). This is the calculation we will perform in section IV.

### IV. POLES IN THE SCATTERING AMPLITUDE

In this section we proceed to solve the Klein–Gordon equation in a linearized metric. As expected from section III, the resulting scattering amplitude will exactly reproduce the eikonal sum of ladder graphs amplitude (2.5). The motivation for this calculation is to understand the poles in the eikonal amplitude. We shall show in the context of the Klein–Gordon equation that the physical poles correspond to the bound states expected in the \( \frac{1}{r} \) potential of a linearized Schwarzschild metric.

We choose to work in the rest frame of a mass \( M \), which gives rise in linearized gravity to the metric
\[
    ds^2 = \left( -1 + \frac{2GM}{r} \right) dt^2 + \left( 1 + \frac{2GM}{r} \right) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right).
\]
This is in De Donder gauge, i.e. \( \partial_\nu h^\nu_{\mu} - \frac{1}{2} \partial_\mu h^\nu_{\nu} = 0 \). The linearized Klein–Gordon equation describing a particle of mass \( m \) is then
\[
    \left( \left( 1 + \frac{2GM}{r} \right) \frac{\partial^2}{\partial t^2} - \left( 1 - \frac{2GM}{r} \right) \nabla^2 + m^2 \right) \phi = 0.
\]
Note that a calculation of scattering states will be completely equivalent to ’t Hooft’s, provided that one is careful in taking the limit \( M \to 0 \) at the end of the calculation. This follows from the fact that by boosting the metric (4.1) up to the speed of light and taking the limit \( M \to 0 \), one obtains the Aichelburg-Sexl metric that ’t Hooft considers*. The advantage of working in the rest frame of \( M \) rather than on the light cone is that we can have \( M \neq 0 \), which will be necessary to get the correct poles in the scattering amplitude below.

Separate variables \( \phi(t, x) = e^{-iE t} \Phi(x) \), and define

\[
\begin{align*}
  k &= \sqrt{E^2 - m^2} \\
  \tilde{\alpha} &= GM \frac{2E^2 - m^2}{\sqrt{E^2 - m^2}} \\
  &= G \frac{(s - m^2 - M^2)^2 - 2m^2M^2}{\sqrt{s - (m + M)^2} \sqrt{s - (m - M)^2}}.
\end{align*}
\]

The branch cut for the square root is taken to lie just below the real axis, so that \( k \) is positive imaginary for bound states. The resulting cut \( s \) plane defines the physical sheet of the amplitude.

The Klein–Gordon equation is then

\[
\left( \nabla^2 + k^2 + \frac{2\tilde{\alpha}k}{r} \right) \Phi = 0
\]

which has solutions with the asymptotic form

\[
\Phi \to e^{i[kz - \tilde{\alpha} \log k(r - z)]} + f(k, \theta) e^{i[kr + \tilde{\alpha} \log 2kr]} \left( \frac{r}{\sin^2(\theta/2)} \right)^{i\tilde{\alpha}}.
\]

* Although the Aichelburg–Sexl metric is a solution to the fully non-linear Einstein equations in the presence of a light-like source, it is also co-incidentally a solution for the same source in the linearized theory, and can therefore be obtained by boosting the linearized Schwarzschild solution\(^3^{,23}\).
We may write the amplitude in terms of $t = -4k^2 \sin^2(\theta/2)$ instead of the lab frame scattering angle

$$f(k, \theta) = -\frac{i}{2k} \frac{\Gamma(1 - i\tilde{\alpha})}{\Gamma(i\tilde{\alpha})} \left(\frac{4k^2}{-t}\right)^{1-i\tilde{\alpha}}. \quad (4.5)$$

Upon setting $M = m$, so that $\tilde{\alpha} = \alpha$, this amplitude equals the previous eikonal result in $(2.5)$ times a factor $\frac{1}{8\pi m} \left(\frac{k^2}{\mu^2}\right)^{-i\alpha}$. The $\frac{1}{8\pi m}$ is a kinematic normalization, while the $\left(\frac{k^2}{\mu^2}\right)^{-i\alpha}$ is the infrared regulator which has been absorbed into the asymptotic form of the scattering wavefunction (4.4).

We now discuss the poles in the eikonal amplitude (2.5) (or (4.5) for $M = m$). Since $\Gamma(z)$ is a nowhere vanishing meromorphic function, poles occur at the poles of the Gamma function in the numerator, that is at $i\alpha = N$, $N = 1, 2, 3, \cdots$. On the physical sheet these poles are located at

$$s^\text{pole}_N = 2m^2 \left\{ 1 \pm \sqrt{\frac{1}{2} + \frac{N^2}{8G^2 m^4} \left( -1 + \sqrt{1 + \frac{8G^2 m^4}{N^2}} \right)} \right\}. \quad (4.6)$$

These are the bound state poles expected in the $\frac{1}{r}$ potential of linearized gravity. This interpretation is clear since these poles correspond to normalizeable solutions of the Klein–Gordon equation (4.2). In (4.6) the $+$ sign yields the two particle bound state poles, and the $-$ sign yields the particle–antiparticle bound state poles. For the crossed channel the square of the center of mass energy is redefined as $s = -(p_1 - p_2)^2 = 2m^2 - 2mE_2$, and with this redefinition the particle–antiparticle bound state energies are equal to those for the two particle bound states as expected. The eikonal has been used previously to determine similar bound state energies in electrodynamics$^{20,22}$. 

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Note that the scattering amplitude has zeroes at $i\alpha = N$, $N = 0, -1, -2, \ldots$, where the Gamma function in the denominator has poles. On the physical sheet these zeroes are located at

$$s_N^{\text{zero}} = 2m^2 \left\{ 1 \pm \sqrt{\frac{1}{2} + \frac{N^2}{8G^2m^4}} \left(-1 - \sqrt{1 + \frac{8G^2m^4}{N^2}}\right) \right\}. \quad (4.7)$$

By analytically continuing the amplitude on to a second Riemann sheet, the square roots change sign, and we find that the locations of the poles and zeroes interchange. That is, there are poles on the second sheet located at $s_N^{\text{zero}}$ and zeroes located at $s_N^{\text{pole}}$.

These second sheet poles do not correspond to physical states. Their occurrence in linearized gravity may be understood as follows. The potential in the linearized Klein–Gordon equation (4.3) is

$$V(r) = -\frac{2\alpha k}{r} = -\frac{2GM}{r} \left(2E^2 - m^2\right). \quad (4.8)$$

In the region $2E^2 < m^2$ this potential is repulsive and no physical bound states or resonances exist. Still, it is scattering in this region that leads to the poles (4.7) on the second sheet. The reason is that continuing to the second sheet in the amplitude (4.5) changes $\alpha(s) \rightarrow -\alpha(s)$, which is equivalent staying on the first sheet but replacing $G \rightarrow -G$. This changes the sign of the potential (4.8) and leads to the poles (4.7). Similar second sheet poles arise in non–relativistic scattering from a repulsive Coulomb potential\(^{26}\).

Previous discussions of these amplitudes\(^{2,5,25}\) have necessarily worked in the massless limit from the beginning, and therefore obtained amplitudes with different analyticity properties. In particular some of the second sheet poles appeared on the physical sheet, and their significance was somewhat obscure. To correctly obtain the massless limit, one must first work with $m \neq 0$. 

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It should be noted that it is perhaps not surprising that the only physical poles in the scattering amplitude turn out to be no more than the gravitational analogues of the Coulomb bound states of electromagnetism. We have shown in section II that the eikonal scattering amplitude (2.5) arises from summing a perturbation expansion in powers of Newton’s constant $G$. Non-trivial features of quantum gravity related to black hole formation might be expected to be non-perturbative in $G$.

V. TOPOLOGICAL FORMULATION

To complete our investigation of the perturbative approach to high energy forward scattering in quantum gravity, we now show that the massless limit of the eikonal approximation to linearized gravity may be recast in the form of a topological theory equivalent to the Verlinde’s\(^5\). A similar reformulation has been performed in electrodynamics\(^12\).

V.1. THE BOUNDARY ACTION FROM THE EIKONAL APPROXIMATION

We start from the functional integral over the linearized Einstein action coupled to a scalar field,

$$\int \mathcal{D}h_{\mu\nu} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \phi_1(x_1)\phi_1(x'_1)\phi_2(x_2)\phi_2(x'_2)$$

$$\exp \left\{ i \int d^4x \frac{1}{2} h_{\alpha\beta} (D^{-1})^{\alpha\beta\gamma\delta} h_{\gamma\delta} + \frac{1}{2} \phi_i (\Box^2 - m^2) \phi_i + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}(\phi_i) \right\} \quad (5.1)$$

where $T^{\mu\nu}(\phi)$ is the energy–momentum tensor of the $\phi$ field, and $D^{-1}$ is $i$ times the inverse of the graviton propagator. Their explicit forms can be seen by comparing to (2.1). The above path integral yields the four point Green’s function.

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As we have seen in section III, we make the eikonal approximation for the matter degrees of freedom by letting the two particles travel undeflected along straight classical trajectories. For the massless case considered in Ref. 5, we take the two classical trajectories to be on the light cone, say

\[ x_1^\mu(t) = (t, x_1, y_1, t) \]
\[ x_2^\mu(t) = (t, x_2, y_2, -t) \]

so that (5.1) becomes

\[
\int \mathcal{D}h_{\mu\nu} \exp \left\{ i \int d^4x \frac{1}{2} h_{\alpha\beta} (D^{-1})^{\alpha\beta\gamma\delta} h_{\gamma\delta} + \frac{1}{2} h_{\mu\nu} T_1^{\mu\nu} + \frac{1}{2} h_{\mu\nu} T_2^{\mu\nu} \right\}. \tag{5.2}
\]

Here \( T_1^{\mu\nu} \) and \( T_2^{\mu\nu} \) are the classical energy–momentum tensors for the two particles. In the center of mass frame, if the particles carry momentum \( p_1^\mu = (E, 0, 0, E) \) and \( p_2^\mu = (E, 0, 0, -E) \), they are given by

\[
T_1^{\mu\nu}(t, x) = \frac{1}{E} p_1^\mu p_1^\nu \delta^3(x - x_1(t))
\]
\[
T_2^{\mu\nu}(t, x) = \frac{1}{E} p_2^\mu p_2^\nu \delta^3(x - x_2(t)). \tag{5.3}
\]

Note that we could evaluate the path integral (5.2) explicitly at this stage, which would yield the massless limit of the eikonal scattering amplitude (2.5), but with an additional dependence on the background gravitational field in the transverse directions. We proceed somewhat less directly in order to illustrate the equivalence of (5.2) to a topological theory.

In momentum space, the only non–vanishing light cone components of the scalars’ energy–momentum tensors are \( (\pm \equiv \frac{1}{\sqrt{2}} (0 \pm 3)) \)

\[
T_1^{++}(k) = 4\pi E e^{-i k_\perp \cdot x_1} \delta (k^0 - k^3)
\]
\[
T_2^{--}(k) = 4\pi E e^{-i k_\perp \cdot x_2} \delta (k^0 + k^3). \tag{5.4}
\]

We see that only the ++ and -- components of \( h_{\mu\nu} \) are coupled to the sources in this limit. The values of the other components are determined by the boundary conditions imposed on
the path integral. For simplicity we choose boundary conditions that set all components of $h_{\mu\nu}$ except for $h_{++}$ and $h_{--}$ equal to zero, which in the language of Ref. 5 means taking $R_h$ to vanish. The relevant light cone components of the graviton propagator are then (from Fig. 1)

$$iD^{++++} = iD^{-----} = 0$$

$$iD^{++--} = iD^{----} = -i\frac{16\pi G}{k^2 - i\epsilon}.$$  

The fact that $D^{++++}$ and $D^{-----}$ vanish implies that a graviton cannot be reabsorbed by the same matter line it was emitted from – it must be exchanged to the other line. This provides some justification for neglecting gravitons which couple back to a single scalar line (as in Fig. 4) and considering only the generalized ladder graphs of Fig. 2 when we formulated the eikonal approximation in sections II and III.

Furthermore, because of the $\delta$-functions $\delta(k^0 - k^3)$ and $\delta(k^0 + k^3)$ in the sources (5.4), the exchanged gravitons have $k^0 = k^3 = 0$. Restricting our attention to such gravitons, we may write the effective inverse graviton propagator in position space as

$$(D^{-1})^{++++} = (D^{-1})^{-----} = 0$$

$$(D^{-1})^{++--} = (D^{-1})^{----} = \frac{1}{16\pi G} \frac{1}{2} \nabla_\perp^2$$

where $\nabla_\perp^2$ is the Laplace operator in the two transverse dimensions. Inserting this in (5.2) the functional integral becomes

$$\int \mathcal{D}h_{++} \mathcal{D}h_{--} \exp \left\{ i \int d^4x \frac{1}{16\pi G} \frac{1}{4} (h_{++} \nabla_\perp^2 h_{--} + h_{--} \nabla_\perp^2 h_{++})
\right.$$  

$$+ \frac{1}{2} h_{++} T_{1}^{++} + \frac{1}{2} h_{--} T_{2}^{--} \right\}. \tag{5.5}$$
This is a non–dynamical theory, since there are no longer any time derivatives in the Lagrangian. The $h_{++}$ and $h_{--}$ are constrained by the equations of motion
\begin{align*}
\frac{1}{16\pi G} \nabla^2 h_{++} + T_{2}^{--} &= 0 \\
\frac{1}{16\pi G} \nabla^2 h_{--} + T_{1}^{++} &= 0.
\end{align*}
These constraints can be immediately integrated.
\begin{align*}
h_{++}(x) &= -16\pi G \int \, d^2 x'_{\perp} \, \frac{1}{2\pi} \log |x_{\perp} - x'_{\perp}| \, T_{2}^{--}(t, x_{\perp}, z) \\
h_{--}(x) &= -16\pi G \int \, d^2 x'_{\perp} \, \frac{1}{2\pi} \log |x_{\perp} - x'_{\perp}| \, T_{1}^{++}(t, x_{\perp}, z)
\end{align*}
where we have used $\frac{1}{2\pi} \log |x_{\perp} - x'_{\perp}|$ as the Green’s function of the Laplace operator in two dimensions. Solving the constraints evaluates the entire functional integral, as (5.5) now reduces to
\begin{align*}
\exp \left\{ -4iG \int dt \, dz \, d^2 x_{\perp} \, d^2 x'_{\perp} \, T_{1}^{++}(t, x_{\perp}, z) \log |x_{\perp} - x'_{\perp}| \, T_{2}^{--}(t, x'_{\perp}, z) \right\}.
\end{align*}
Inserting the explicit form of the source current (5.3) yields
\begin{align*}
e^{-iGs \log |x_{1\perp} - x_{2\perp}|^2}.
\end{align*}
This is the final result for the two particle S–matrix obtained by H. and E. Verlinde (equation (6.2) in their paper). By construction, it is expected to be equivalent to the massless limit of the amplitudes (2.5), (4.5). This can be checked explicitly by the partial wave analysis of Ref. 5.

To see how (5.5) reduces to the topological action formulation of Ref. 5, we now gauge fix explicitly using the De Donder gauge condition
\begin{align*}
\partial_{\mu} h^{\mu}_{\nu} &= \frac{1}{2} \partial_{\nu} h.
\end{align*}
Since the transverse components of the metric have been eliminated from the problem, it follows from the gauge condition that

\[ \partial_+ h_- = \partial_- h_+ = 0. \]

This allows us to rewrite

\[ h_{++}(x^+, x_\perp) = \partial_+ X^+(x^+, x_\perp), \quad h_{--}(x^-, x_\perp) = \partial_- X^-(x^-, x_\perp). \]

Similarly, noting that \( T_{1}^{++} \) depends only on \( x^- \) and \( x_\perp \), while \( T_{2}^{--} \) depends only on \( x^+ \) and \( x_\perp \), one may introduce functions \( P^- \) and \( P^+ \) defined by \( T_{1}^{++} = \partial_- P^-(x^-, x_\perp) \) and \( T_{2}^{--} = -\partial_+ P^+(x^+, x_\perp) \). In terms of the variables \( X \) and \( P \), the action in (5.5) is seen to be a total derivative.

\[
\int d^4 x \frac{1}{16\pi G} \frac{1}{4} \left( \partial_+ X^+(x_\perp) \nabla^2 \partial_- X^-(x_\perp) + \partial_- X^-(x_\perp) \nabla^2 \partial_+ X^+(x_\perp) \right) \\
+ \frac{1}{2} \left( \partial_+ X^+(x_\perp) \partial_- P^-(x_\perp) - \partial_- X^-(x_\perp) \partial_+ P^+(x_\perp) \right) \\
= \int d^4 x \partial_+ \left[ \frac{1}{16\pi G} \frac{1}{4} X^+(x_\perp) \nabla^2 \partial_- X^-(x_\perp) + \frac{1}{2} X^+(x_\perp) \partial_- P^-(x_\perp) \right] \\
+ \partial_- \left[ \frac{1}{16\pi G} \frac{1}{4} X^-(x_\perp) \nabla^2 \partial_+ X^+(x_\perp) - \frac{1}{2} X^-(x_\perp) \partial_+ P^+(x_\perp) \right].
\]

Upon integrating over the entire transverse \( x_\perp \) space and a closed surface in the two dimensional \((t, z)\) Minkowski space, the action is given by the boundary contribution.

\[
\oint d\tau \int d^2 x_\perp \frac{1}{64\pi G} \left( X^+(x_\perp) \nabla^2 \dot{X}^-(x_\perp) - X^-(x_\perp) \nabla^2 \dot{X}^+(x_\perp) \right) \\
+ \frac{1}{2} \left( X^+(\dot{P}^-) + X^-(\dot{P}^+) \right).
\]

Here all quantities are evaluated along a closed contour \((t(\tau), z(\tau))\) bounding the surface in the \((t, z)\) plane. An overdot denotes \( \tau \)–differentiation. This is the topological boundary action of Ref. 5, equation (5.1) with a flat background metric on the transverse \((x, y)\) space.
In Ref. 5 this boundary action was derived from the full theory of quantum gravity. We note that the reduced action derived in Ref. 5 is actually equivalent to the linearized theory that was our starting point. This can be made evident by observing that the reduced action is quadratic, indicating that all non-linear terms in the Einstein action have decoupled. In the notation of Ref. 27, the reduced action takes the form

\[ S = -\frac{1}{32\pi G} \int \left( e^I_i e^J_j R^{ab}_{\alpha\beta} + 2e^a_\alpha e^I_i R^{bj}_{\beta j} \right) \varepsilon_{ab} \varepsilon_{IJ} \varepsilon_{ij\alpha\beta} \]

where \( I, J, i, j = 1, 2 \) and \( a, b, \alpha, \beta = 0, 3 \). This simplifies with the use of

\[ e^a_\alpha = \partial_\alpha X^a, \quad e^I_i = \delta^I_i \]

(we have again assumed that the spacetime transverse to the direction of motion of the particles is flat), to

\[ S = \frac{1}{16\pi G} \int e^a_\alpha \nabla_\perp^2 e^b_\beta \varepsilon_{ab} \varepsilon^{\alpha\beta}. \]

From this expression it is easy to see that if we expand the metric about flat space by writing

\[ e^a_\alpha = \delta^a_\alpha + h^a_\alpha \]

only quadratic terms in \( h^a_\alpha \) contribute. The action, rewritten in terms of \( h^a_\alpha \) takes precisely the form in equation (5.5) up to a gauge transformation. Thus the reduction performed in Ref. 5 implies that only linearized gravitons are exchanged in the kinematical regime under consideration. This explains how we have arrived at the same answer for the forward scattering amplitude as Ref. 5 by beginning with the linearized Einstein action (2.2).
VI. CONCLUSIONS

To summarize, we have demonstrated the equivalence of three approaches to computing high energy forward scattering in Quantum Gravity: using the eikonal approximation to sum ladder graphs, solving the linearized Klein–Gordon equation in the $\frac{1}{r}$ potential of a linearized metric, and reformulating the gravity action as an effective topological action for eikonal scattering of massless particles.

We have not addressed the question of the validity of these approximations in this paper. The topological formulation has been developed systematically from the full theory of quantum gravity\textsuperscript{5}, and is expected to be valid at energies of order the Planck scale provided the impact parameter is large enough. One would expect this to be reflected in the dominance of ladder graphs over other graviton exchanges in a fully non-linear perturbative analysis of Planckian scattering. However, the effect of scalar loops has not been considered in Ref. 5. It would be a welcome check to see the eikonal sum of ladder graphs giving the correct asymptotic behavior of the full perturbation series for the scattering amplitude\textsuperscript{17}. Corrections to the eikonal result have been discussed in string theory\textsuperscript{10}.

Can these results be used as hints toward an understanding of quantum gravity? Note that all these calculations have been performed purely in the linearized theory. Assuming that the approximations used above and in Refs. 2 and 5 are valid, it seems that high energy forward scattering is a regime in which all the complexities of full quantum gravity, as well as the divergencies of the linearized theory, are subdominant. This allows us, perhaps unexpectedly, to make predictive calculations. However, since these calculations do not confront
the non-linearities of gravity, they are unlikely to teach us anything about the full theory of quantum gravity away from this favoured kinematical regime.

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NOTE ADDED

After this work was completed, we received a preprint comparing various approaches to Planckian scattering: D. Amati, M. Ciafaloni, and G. Veneziano, Planckian Scattering Beyond the Semiclassical Approximation, CERN preprint CERN–TH.6395.92 (February 1992).
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\[ i\Delta = -\frac{i}{p^2 + m^2 - i\epsilon} \]
\[ iD^{\alpha\beta\sigma\tau} = -i \frac{16\pi G}{k^2 - i\epsilon} (\eta^{\alpha\sigma}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\sigma} - \eta^{\alpha\beta}\eta^{\sigma\tau}) \]
\[ \frac{i}{2} \left( p_\alpha p'_\beta + p_\beta p'_\alpha - \eta_{\alpha\beta} (p \cdot p' + m^2) \right) \]

Figure 1. Feynman rules in harmonic gauge.

Figure 2. Sum of ladder and crossed ladder graphs.

Figure 3. Symmetrized ladder and crossed ladder graphs up to one loop.

Figure 4. Graphs with gravitons which couple back to a single scalar line.