Exactly Marginal Operators and Running Coupling Constants in 2D Gravity

Christof Schmidhuber*

California Institute of Technology, Pasadena, CA 91125

Abstract

The Liouville action for two-dimensional quantum gravity coupled to interacting matter contains terms that have not been considered previously. They are crucial for understanding the renormalization group flow and can be observed in recent matrix model results for the phase diagram of the Sine–Gordon model coupled to gravity. These terms ensure, order by order in the coupling constant, that the dressed interaction is exactly marginal. They are discussed up to second order.

Work supported in part by the U.S. Department of Energy under Contract No. DE-AC 0381-ER40050.

* christof@theory3.caltech.edu
1. Introduction

In the Liouville approach,[1,2] two dimensional quantum gravity coupled to \( c \leq 1 \) matter \( \cdot \cdot \cdot \) is formulated in terms of fields propagating on a fictitious background metric \( \hat{g}_{\alpha\beta} \). The action is the appropriate conformally invariant free action plus interaction terms which are usually assumed to be of the form

\[
\mathcal{L}_{\text{int}} = \text{cosmological constant} + \tau^i \int \Phi_i(x) e^{\alpha_i \phi}, \tag{1.1}
\]

where \( \Phi_i \) are primary fields of the matter theory, the \( \tau^i \) are small coupling constants, \( \phi \) is the Liouville mode and the \( \alpha_i \) are adjusted to make the dimensions of the operators equal to two.

However, (1.1) can not be the complete interaction, for at least two reasons:

1. The operators in (1.1) are not exactly marginal.\(^*\) They should be, because as a consequence of general covariance the Liouville theory must be background independent.[1,2] Therefore the beta functions of the theory must be zero to all orders in the couplings. Adjusting \( \alpha_i \) in (1.1) makes them zero to first order, but whenever there are nontrivial OPE’s, the beta functions have quadratic pieces.\(^\star\)

2. The renormalization group flow would be quite trivial with (1.1). As mentioned, there should be no flow with respect to the fictitious background scale \( \sqrt{\hat{g}} \). But, as explained in section 3, a constant shift of \( \phi \) should be interpreted as a rescaling of the physical cutoff,[3,4,5] and should in particular result in a mixing (flow) between different operators. This does not happen in (1.1).

As shown in section 2, the first problem can be solved by adding a term

\[
\propto -c^k_{ij} \tau^i \tau^j \int \Phi_k(x) \phi e^{\alpha_k \phi} \tag{1.2}
\]

to the interaction (1.1) where \( c^k_{ij} \) are the operator product coefficients. This, in fact, also resolves the second problem: the modified interaction displays the expected

---

\( ^* \) An operator is marginal if its dimension is two, and exactly marginal if its beta function is zero to all orders.

\( ^\star \) See section 2 for the issue of renormalization schemes and field redefinitions.
operator mixing under shifts of $\phi$ by a constant. Requiring that there be no flow w.r.t. the background scale $\sqrt{g}$ determines the flow w.r.t. the physical scale $\sqrt{g_\epsilon} e^{\alpha \phi}$. For the case of the Sine–Gordon model coupled to gravity it will be seen that this flow qualitatively agrees with recent matrix model results by Moore.$^6$

(1.2) should be viewed as a second order correction to the gravitational dressing of the $\Phi_i(x)$. It is conjectured that further modifications of (1.1)+(1.2) can be made order by order in the $\tau^i$, leading to an infinite dimensional space of exactly marginal perturbations.

The paper is organized as follows:

In section 2, the second order corrections (1.2) are discussed. First, it is shown in subsection 2.1 that the interaction (1.1) plus (1.2) is marginal up to second order. That the correction (1.2) is essentially unique is argued in appendix A by thinking of the marginality conditions as equations of motion of string theory. The $c = 1$ model coupled to gravity is discussed as an example, the interaction terms being the Sine–Gordon model near the Kosterlitz-Thouless momentum $p = \sqrt{2}$ and near $p = \frac{1}{2}\sqrt{2}$ in subsection 2.2, and the “discrete operators” in subsection 2.3. The effects of including the cosmological constant are studied in appendix B. The conclusions of appendices A and B are summarized in subsection 2.4.

In section 3, running coupling constants are discussed. They are defined in subsection 3.1 so that they absorb a constant shift of $\phi$. In subsections 3.2 and 3.3 this is applied to the Sine-Gordon model and the resulting phase boundaries are compared with those found with the nonperturbative matrix model techniques.$^6$ It is seen that the presence of the terms (1.2) is crucial even for qualitative agreement of the matrix model– and the Liouville approaches. A more detailed comparison of both is left for later. The one–loop beta functions for the discrete $c = 1$ operators are also obtained.

In section 4, possible extensions of this work are pointed out, as well as implications for black hole hair and correlation functions. In particular, it is argued that the relation between correlation functions in the matrix model and in the Liouville approach is more complicated than often assumed.
2. Exactly Marginal Operators

2.1. The Terms of Order $\tau^2$

In DDK’s approach, a conformal field theory with central charge $c$ and Lagrangian $L_m(x)$ coupled to 2D gravity is described by the action\cite{1,2}

$$S_0 = \frac{1}{8\pi} \int \sqrt{\hat{g}} \{ L_m(x) + \partial \phi^2 + Q \hat{R} \phi + \text{cosmological constant} + \text{ghosts} \} \quad (2.1)$$

with $Q = \sqrt{(25 - c)/3}$ and conformal factor $\phi$. The cosmological constant will be neglected first, but included later. (see subsection 2.4.)

When $L_m(x)$ is perturbed by operators $t^i \Phi_i(x)$, these operators get “dressed” upon coupling to gravity. As mentioned in the introduction, the dressed interaction must be an exactly marginal operator, not only an operator of dimension two. Exact marginality is needed because in DDK’s approach the background metric $\hat{g}$ is an arbitrary gauge choice that nothing should depend on. In particular, coupling constants should not run with respect to $\hat{g}$: all beta functions must be zero to all orders.

So far, this condition has been exploited only to first order\cite{1,2}. Here it will be exploited up to second order. Generally,\cite{7} the beta functions for a perturbed conformally invariant theory $S_0$,

$$S = S_0 + \tau^i \int d^2 r \ V_i, \quad ^*$$

are

$$\beta^i = (\Delta_i^j - 2 \delta_i^j) \ \tau^j + \pi \epsilon_{ijk} \tau^j \tau^k + o(\tau^3), \quad (2.2)$$

if the $V_i$ are primary fields of dimension $\Delta_i$ close to two. $\Delta_i^j$ is the dimension matrix computed with $S_0$. If the operators $V_k$ on the RHS of the operator algebra\footnote{David, Distler and Kawai}

$$V_i(r)V_j(0) \sim \sum_k |r|^{-\Delta_i - \Delta_j + \Delta_k} \epsilon_{ij}^k V_k(0)$$

also have dimension close to two, the coefficients $\epsilon_{ij}^k$ are universal constants, inde-
pendent of the renormalization scheme used to compute them. Operators of other dimensions also appear on the RHS. For them, the $c_{jk}^i$ are scheme–dependent, that is, not invariant under coupling constant redefinitions. Let us ignore them here.*

We now show that $\beta^i = 0 + o(\tau^3)$ for the perturbation (1.1) plus (1.2):°

$$\delta S = \tau^i \int V_i(x, \phi) \equiv \tau^i \int \tilde{V}_i(x, \phi) - \pi c_{ij}^k \tau^i \tau^j \int X_k(x, \phi), \quad (2.3)$$

$$V_i = \tilde{V}_i - \pi c_{ij}^k \tau^j X_k, \quad \tilde{V}_i \equiv \Phi_i(x) e^{\alpha_i \phi}, \quad X_k \equiv -\frac{1}{Q + 2\alpha_k} \Phi_k(x) \phi e^{\alpha_k \phi}. \quad (2.4)$$

$\alpha_i$ is adjusted to make the dimension of $\tilde{V}_i$ exactly two. Without the $o(\tau)$–corrections in $V_i$, we would thus have $\Delta^i_j = 2\delta^i_j$ and $\beta = 0 + o(\tau^2)$ from (2.2). With them,

$$\Delta^i_j = 2\delta^i_j - \pi c_{ij}^k \tau^k + o(\tau^2), \quad (2.5)$$

hence $\beta = 0 + o(\tau^3)$ in (2.2). (2.5) can be derived by writing

$$X_k = -\frac{1}{Q + 2\alpha_k} \Phi_k(x) \frac{\partial}{\partial \alpha_k} e^{\alpha_k \phi},$$

defining the generator $L_0 + \bar{L}_0$ of global scale transformations and differentiating with respect to $\alpha_k$ the dimension formula

$$(L_0 + \bar{L}_0)e^{\alpha_k \phi} = -\alpha_k(\alpha_k + Q) e^{\alpha_k \phi},$$

$$\Rightarrow \quad (L_0 + \bar{L}_0)X_k = 2X_k + \tilde{V}_k,$$

$$\Rightarrow \quad (L_0 + \bar{L}_0)V_k = 2V_k - \pi c_{ij}^k \tau^j V_i + o(\tau^2).$$

As a simple check of all this, one can consider rescaling $\psi \rightarrow (1 + \lambda)\psi$ in

$$S_{toy\ model} = \frac{1}{8\pi} \int \partial \psi^2 + \gamma \cos \sqrt{2} \psi \quad \text{with} \quad \lambda \ll \gamma (a^2).$$

This should keep the interaction marginal at $o(\gamma \lambda)$ and is equivalent to adding the terms $2\lambda \partial \psi^2 - \sqrt{2} \lambda \gamma \psi \sin \sqrt{2} \psi$ to the Lagrangian. Using the above method, one can check that the second term indeed arises as the correction to the first term.○

* Presumably the scheme can be chosen so that they vanish.
° The question of the uniqueness of (1.2) is deferred to subsection 4.4.
○ Here, $8\pi \tilde{V}_1 = \cos \sqrt{2} \psi$, $8\pi \tilde{V}_2 = 2\partial \psi^2$, $c_{12}^1 = c_{21}^1 = -2/\pi$ and $8\pi X_1 = -1/(2\sqrt{2}) \psi \sin \sqrt{2} \psi$. 

4
2.2. The Sine–Gordon model

As an example, consider an uncompactified scalar field $x$ coupled to gravity. Then $c = 1$ and $Q = 2\sqrt{2}$. First, we perturb this model by the Sine–Gordon interaction,

$$S = \frac{1}{8\pi} \int \sqrt{g} \{ \partial x^2 + \partial \phi^2 + 2\sqrt{2}R\phi + \text{ghosts} \} + m \int \cos px \ e^{(p-\sqrt{2})\phi},$$

and determine the $o(m^2)$ corrections (2.3). To find the coefficients $c^k_{ij}$, consider the operator product expansion (OPE), using the propagator $-\log r^2$:

$$\cos px \ e^{(p-\sqrt{2})\phi(r)} \cos px \ e^{(p-\sqrt{2})\phi(0)} 
\sim |r|^{-2-4(p-\sqrt{2})^2} \ e^{2(p-\sqrt{2})\phi} \left\{ \frac{1}{2} - |r|^2 \frac{P^2}{8} \partial x^2 + \ldots \right\}$$

$$+ |r|^{-2+(4\sqrt{2}p-2)} \cos 2px \ e^{2(p-\sqrt{2})\phi} \left\{ \frac{1}{2} - |r|^2 \frac{P^2}{8} \partial x^2 + \ldots \right\}$$

(2.7)

As mentioned, we must look for nearly quadratic singularities, so that the $c^j_{jk}$ are universal constants. The second line in (2.7) has quadratic singularities at the “discrete momenta” $p \in \{\ldots, 0, \frac{1}{2}\sqrt{2}, \sqrt{2}, \ldots\}^*$ and the third line at $p \in \{\ldots, 0, \frac{1}{4}\sqrt{2} \}^\star$. Let us study the neighborhoods of $p = \frac{1}{2}\sqrt{2}$ and $p = \sqrt{2}$. There the induced operators are:

at $p = \frac{1}{2}\sqrt{2} + \delta$: $\hat{V}_1 = e^{(2\delta-\sqrt{2})\phi}$ with $c^1_{mm} = \frac{1}{2}$

at $p = \sqrt{2} + \epsilon$: $\hat{V}_2 = \partial x^2 \ e^{2\epsilon\phi}$ with $c^2_{mm} = -\frac{P^2}{8}$

From (2.3) and (2.4), the leading order corrections to (2.6) are obtained:

near $p = \frac{1}{2}\sqrt{2}$: $\delta S = \frac{m^2 \pi}{8\delta} \int \phi \ e^{-\sqrt{2}\phi}$

near $p = \sqrt{2}$: $\delta S = -\frac{m^2 \pi}{8\sqrt{2}} \int \phi \ \partial x^2$

(2.8)

This will be further discussed in section 3. Note that from the string theory point of view, (2.8) describes the backreaction of the tachyon onto itself and the graviton.

* corresponding to the discrete tachyons $\Phi_{j, \pm j}$ of the next subsection

* However, for $p < \frac{1}{2}\sqrt{2}$ the operators on the RHS do not exist. As a consequence, $\cos 2px$—terms will not be induced and nothing happens at those momenta, as argued in appendix B.
2.3. The Discrete $c = 1$–Operators

As a second example, consider perturbing the $c = 1$ model with the nonrenormalizable so–called (chiral) discrete primaries $\Phi_{jm}(x),[8]$

$$\Phi_{jm} = f_{jm}[\partial x, \partial^2 x, ...] e^{im\sqrt{2}x} \equiv (\oint e^{-i\sqrt{2}x})^m e^{ij\sqrt{2}x} \tag{2.9}$$

with dimension $j^2$ and $SU(2)$ indices $j, m$, the $SU(2)$ algebra being generated by

$$H^\pm \sim \oint e^{\pm i\sqrt{2}x} = \oint \Phi_{1, \pm 1}, \quad H^3 \sim \oint i\sqrt{2}\partial x = \oint \Phi_{1, 0}.$$

If an interaction $t^{jm}\Phi_{jm}[x]\bar{\Phi}_{jm}[\bar{x}]$ is added to the matter Lagrangian, the dressed interaction is given to first order in the coupling constants by

$$\mathcal{L}_{int} = \tau^{jm} \hat{V}_{jm}, \quad \hat{V}_{jm} \equiv \Phi_{jm}(x)\bar{\Phi}_{jm}(\bar{x}) e^{\alpha_j(\phi + \bar{\phi})}$$

with $\alpha_j = (j - 1)\sqrt{2}$. The $\Phi_{jm}$ can be rescaled such that the operator algebra of the $\hat{V}_{jm}$ has the $w_\infty$ structure$[9,10]$

$$c^{jm}_{kn k'n'} = (kn' - k'n)^2 \delta_{j, k+k'-1} \delta_{m, n+n'}.$$

From (2.3) and (2.4) one obtains the second order interaction term

$$\delta \mathcal{L} = \sum_{j, m} \Phi_{jm} \Phi_{jm} \phi e^{\alpha_j \phi} \times \frac{\pi}{2\sqrt{2} j} \sum_{j', j'', m' + m'' = m} (j' m'' - j'' m')^2 \hat{\tau}^{j' m''} \hat{\tau}^{j'' m'} \tau^{j m}. \tag{2.10}$$

$\mathcal{L}_{int} + \delta \mathcal{L}$ is marginal up to order $\tau^2$. Again, with different renormalization schemes operators whose dimension is not two may also appear in $\delta \mathcal{L}$.

2.4. Uniqueness and the Cosmological Constant

Next, we must ask whether the modifications (1.2) of the operators (1.1) are the unique modifications that achieve marginality up to order $\tau^2$. The situation is greatly clarified by thinking of the marginality conditions as equations of motion of string theory, as in ref. [3]. One concludes the following (more details are in appendix A):
The marginality conditions are second order differential equations in $\phi$ and $x$. Their solutions are unique after two boundary conditions are imposed, namely: (i): the modifications must vanish at $\phi = 0$, and (ii): the second, more negative Liouville dressing (as e.g., in (A.5)) does not appear. Boundary condition (ii) arises because operators with the more negative Liouville dressing do not exist.

Boundary condition (i) comes about because the Liouville mode $\phi$ lives on a half line:[3] The sum over geometries can be covariantly regularized as a sum over random lattices. Then no two points can come closer to each other than the lattice spacing $a$:

\[\hat{g}_{\mu\nu} e^{\alpha\phi} d\xi^\mu d\xi^\nu \geq a^2 \Rightarrow \phi \leq \phi_0 \text{ with } e^{\alpha\phi_0} \propto a^2 \quad (2.11)\]

(recall that $\alpha < 0$.) After shifting $\phi$ so that $\phi_0 = 0$, boundary condition (i) states that the action $S(\phi = 0)$ at the cutoff scale is the bare action (see e.g., (A.4)).[3]

The correction terms found above obey the boundary conditions (i) and (ii) and are therefore unique. Of course, there is always an ambiguity due to field redefinitions, that is, to choosing different renormalization schemes when computing the beta function. There is no problem as long as we stick to one scheme.*

Another important question is how the cosmological constant modifies our results. The problem with the cosmological constant operator is that it cannot be made small in the IR ($\phi \to -\infty$). It can only be shifted in the $\phi$-direction. Thus it cannot be treated perturbatively, rather it should be included from the start in $S_0$ of (2.1). In its presence the OPE’s used above are modified. Applying the discussions in refs. [3,11,12], one tentatively concludes the following (more details are in appendix B):

1. The effects of the cosmological constant on gravitational dressings can be neglected in the ultraviolet ($\phi \sim 0$), but not in the infrared ($\phi \to -\infty$).

2. In the Sine–Gordon model coupled to gravity, no unwanted terms with $\cos 2px$ are induced because the OPE’s are “softer” than in free field theory (see (A.4-5)).

This will be confirmed in section 3 by the agreement with matrix model results.

---

* Actually, the scheme used subsection 2.1. is not the same as the one used for the string equations of motion in appendix A, but this does not affect the above conclusions.
3. Running Coupling Constants

3.1. Renormalization Group Transformations

In subsections 3.1 and 3.2, the cosmological term will be assumed to be $\mu e^{-\sqrt{2}\phi}$ to simplify the discussion, which can be generalized to more complicated forms like $T_\mu(\phi)$ in (B.2).

Consider rescaling the cutoff $a \rightarrow ae^\rho$ in the path integral of 2D gravity,

$$\int_{\phi \geq \phi_0} D\phi \, Dx \, Db \, Dc \, e^{-S(\phi,x,b,c)}$$

From (2.11) one sees that this induces a shift of the bound $\phi_0 \rightarrow \phi_0 + \lambda$, in addition to an ordinary RG transformation. Here,

$$\lambda = \phi_0(\phi e^\rho) - \phi_0(a) = \frac{2}{\alpha} \rho = -\sqrt{2} \rho. \quad (3.1)$$

In fact, since ordinary RG transformations are irrelevant (all beta functions are zero), only the shift of the bound remains. The constant shift of the bound is equivalent to a constant shift of the Liouville mode, $\phi \rightarrow \phi + \lambda$. Let us absorb this shift in “running coupling constants” $\bar{\tau}(\lambda), \bar{\tau}_0 \equiv \bar{\tau}(0)$, defined by:

$$S[\bar{\tau}(\lambda), x, \phi + \lambda] = S[\bar{\tau}_0, x, \phi]. \quad (3.2)$$

After expressing $\lambda$ in terms of $\rho$, one obtains the renormalization group flow $\bar{\tau}(\rho)$ (for similar conclusions, see [4,13].)

As mentioned above, the action (3.2) corresponds to a classical solution of string theory with two–dimensional target space $(x, \phi)$. The equations of motion of classical string theory thus play the role of the Gell–Mann–Low equations in the presence of gravity.$^{[4]}$ They contain second (and higher) order derivatives of $\phi$, i.e. $\rho$. It has been suggested that those are due to the contribution of pinched spheres in the functional integral over metrics.$^{[14]}$

More generally, $\rho = \frac{1}{2} \log \frac{T_\mu(\lambda)}{T_\mu(0)}$ with cosmological constant $T_\mu$ as in (B.2)

This is more complicated with $T_\mu$. But to find phase boundaries, $\bar{\tau}(\lambda)$ will be good enough.
3.2. Sine–Gordon Model near $p = \sqrt{2}$

We now apply the preceding to the examples worked out in section 2, starting with the Sine–Gordon model. In flat space, at $p = \sqrt{2}$ the Kosterlitz-Thouless phase transition occurs. With gravity, at $p = \sqrt{2} + \epsilon$ the action is to order $(m, \epsilon)^2$ (see (2.8); we ignore $o(\mu)$–corrections of the Sine–Gordon interaction):

$$S = \frac{1}{8\pi} \int \sqrt{g} \left\{ \partial x^2 + \partial \phi^2 + 2\sqrt{2} \hat{R} \phi + \text{ghosts} + \mu e^{-\sqrt{2} \phi} \right\}$$

$$+ m \int \cos(\sqrt{2} + \epsilon) x e^{\epsilon \phi} - \frac{\pi}{8\sqrt{2}} m^2 \int \phi \partial x^2$$

(3.3)

To $o(m, \epsilon)^2$, a shift $\phi \rightarrow \phi + \lambda$ can be absorbed in the $\lambda$–dependent couplings

$$m(\lambda) = m_0 e^{-\epsilon \lambda}, \quad \epsilon(\lambda) = \epsilon_0 - \frac{\pi^2}{2} \lambda m^2, \quad \mu(\lambda) = \mu_0 e^{\sqrt{2} \lambda}.$$

In deriving $\epsilon(\lambda)$, the $\lambda m^2 \partial x^2$ term has been absorbed in a redefinition of $x$ and then in a shift of $\epsilon$. Defining ‘prime’ as $\frac{d}{d\lambda}$, we get

$$\epsilon' = -\frac{\pi^2}{2} m^2 + .., \quad m' = -\epsilon m + .., \quad \mu' = \sqrt{2} \mu + ..$$

Defining ‘dot’ as $\frac{d}{d\rho} = -\sqrt{2} \frac{d}{d\lambda}$ yields the beta functions

$$\dot{\epsilon} = \frac{\pi^2}{\sqrt{2}} m^2, \quad \dot{m} = \sqrt{2} \epsilon m, \quad \dot{\mu} = -2\mu.$$

(3.4)

$\dot{\mu}$ serves as a check: $\mu$ decays in the UV according to its dimension two. The coupling constant flow is qualitatively the same as in flat space and is given by the Kosterlitz-Thouless diagram (Figure 1). We see that the $m^2 \phi \partial x^2$ correction (1.2) of (1.1) plays a crucial role: ignoring it would be like forgetting about field renormalization in the ordinary Sine-Gordon model.
From (3.4), the phase boundary for \( p > \sqrt{2} \) is linear, \( m \propto \epsilon \). To this order, this agrees with the matrix model result \([6]\)

\[
m \propto \epsilon e^{\frac{1}{2}\sqrt{2} \log \epsilon}.
\]

(3.5)

With the normalization of \( m \) and \( \epsilon \) as in (3.3), we obtain the slope \( \sqrt{2}/\pi \) for the phase boundary. After comparing the normalizations, this should also be checked with the matrix model. It will also be interesting to see if the logarithm in (3.5) follows from the modifications of higher order in \( m \), needed to keep the interaction near \( p = \sqrt{2} \) marginal beyond \( o(m^2) \).

We can now interpret the phase diagram of \([6]\) (figure 2) near \( m, \epsilon = 0 \): For \( \epsilon < 0 \) (regions II and V of \([6]\)), \( m \) grows exponentially towards the IR. The model thus flows to (infinitely many copies of) the \( c = 0 \), pure gravity model.\([13,15]\)

For \( \epsilon > 0 \) but \( m \) greater than a critical value \( m_c(\epsilon) \) (region VI of \([6]\),) we flow again towards the \( c = 0 \) model in the IR. For \( m < m_c(\epsilon) \) (region III of \([6]\)) the flow seems to take us to the free \( c = 1 \) model. However, the domain of small \( \epsilon, m \) is now the IR domain. As noted in subsection 2.4, the cosmological constant can not be neglected there and further investigation is needed.

3.3. Sine–Gordon Model near \( p = \frac{1}{2} \sqrt{2} \)

At \( p = \frac{1}{2} \sqrt{2} + \delta \), the situation is less clear. From (2.8), instead of the \( \partial x^2 \)–term a \( "1" \)–term is induced. That is, the cosmological constant is modified by the induced operator \( \phi e^{-\sqrt{2} \phi} \). The latter becomes comparable with the background cosmological constant at \( \delta \sim \frac{m^2}{\mu} \). Let us tentatively\(*\) write the action to leading order as:

\[
S = \frac{1}{8\pi} \int \sqrt{g} \{ \partial x^2 + \partial \phi^2 + 2\sqrt{2} \bar{R} \phi + \text{ghosts} \} + m \int \cos\left(\frac{1}{2} \sqrt{2} + \delta \right) x e^{-\frac{1}{2} \sqrt{2} + \delta} \phi + \left( m^2 + \frac{m^2}{8\delta} \right) \int \phi e^{-\sqrt{2} \phi} \]

(3.6)

With our normalizations, the effective cosmological constant is now \( \mu + \frac{m^2}{8\pi} \). For

\(*\)

Here we use \( \phi e^{-\sqrt{2} \phi} \) instead of the simple form \( e^{-\sqrt{2} \phi} \) for the cosmological constant.
fixed $\mu$, it blows up as $|\delta| \to 0$. For $\delta < 0$ and $m \geq \frac{1}{2} \sqrt{|\mu \delta|}$, it is negative. Indeed, in the matrix model a singularity of the free energy has been found at

$$\delta < 0, \quad m \propto \sqrt{|\mu \delta|} e^{\frac{1}{2} \sqrt{2} \log |\delta|}. \quad (3.7)$$

Let us therefore identify the region where $\mu + \frac{m^2}{2} \pi^2$ is negative with region IV of [6]. We leave a further interpretation of the situation near $p = \frac{1}{2} \sqrt{2}$ for the future.

### 3.4. The Discrete Operators

We can also determine the one–loop beta function for the “discrete” interactions (2.9) of subsection 2.3. From (2.10),

$$\mathcal{L} + \delta \mathcal{L} = \sum_{j,m} \Phi \bar{\Phi} \langle jm \rangle e^{(j-1)\sqrt{2}} \phi \left\{ \tau_{jm} + \phi \frac{\pi}{2 \sqrt{2} j} \sum_{j', j'' = j, j'' = j+1 \atop m' + m' = m} (j'' m'' - j m'')^2 \tau_{j' m'} \tau_{j'' m''} \right\}. \quad (3.8)$$

Constant shifts $\phi \to \phi + \lambda$ are absorbed up to $o(\tau^2)$ in:

$$\tau_{jm}(\bar{\tau}_0, \lambda) = \left\{ \tau_{0 jm} - \lambda \times \frac{\pi}{2 \sqrt{2} j} \sum_{j', j'' = j, j'' = j+1 \atop m' + m' = m} (j'' m'' - j m'')^2 \tau_{j' m'} \tau_{j'' m''} \right\} e^{-(j-1)\sqrt{2}\lambda}. \quad (3.9)$$

From this we find the one loop beta function (using (3.1)):

$$\dot{\tau}_{jm} = 2(j - 1) \tau_{jm} + \frac{\pi}{2 j} \sum_{j', j'' = j, j'' = j+1 \atop m' + m' = m} (j'' m'' - j m'')^2 \tau_{j' m'} \tau_{j'' m''} + o(\tau^3). \quad (3.10)$$

Thus, turning on the operators $\Phi \bar{\Phi} j' m'$ with $j' > 1$ will in general induce an infinite set of higher spin operators $\Phi \bar{\Phi} jm$ at $o(\tau^2)$, whose couplings were originally turned off. This is what one expects from these nonrenormalizable operators, but it would not happen without the $o(\tau^2)$ modification $\delta \mathcal{L}$.
4. Outlook

4.1. Correlation Functions

The modifications (1.2) are important not only for understanding the renormalization group flow but also for computing correlation functions in Liouville theory. They imply the identification (the notation is as in (2.4)):

\[ < \exp \left\{ \int t^i \Phi_i \right\} >_G \sim < \exp \left\{ \int (\tau^i \hat{V}_i + \kappa_l c_{ij} \tau^i \tau^j \phi \hat{V}_l + \ldots) \right\} >_L. \]

(5.1)

where \(< \ldots >_G\) and \(< \ldots >_L\) denote correlation functions computed in the matrix model (Gravity) and in Liouville theory, respectively, and \(\kappa_l = \pi / (Q + 2 \alpha_l)\). \(\tau^i\) is related to the \(t^j\) in some nontrivial way.\(^{[16]}\)

Geometrically, the extra terms on the RHS can be interpreted as arising from pinched spheres in the sum over surfaces. (5.1) has consequences for the correspondence of matrix model and Liouville correlation functions. Expanding both sides and temporarily identifying \(t\) and \(\tau^*\) yields, e.g., for the two-point function:

\[ < \int \Phi_i \int \Phi_j >_G = \int d^2 z \int d^2 w < \hat{V}_i(z) \hat{V}_j(w) >_L + 2 \kappa_l c_{ij} \int d^2 w < \phi \hat{V}_l(w) >_L \]

(5.2)

In fact, the last term is necessary for background invariance: Inserting a covariant regulator like \(\Theta(\sqrt{g} e^\phi |z - w|^2 - a^2)\) into the two-point function induces new background dependence, coming from the integration region \(z \sim w\).\(^{[7]}\) By construction, the one–point functions added in (5.2) are precisely the ones needed to cancel this dependence.

Analogously, additional terms like the ones in (5.2) are also present in higher point functions. They can be determined by background invariance. It should be possible to see them in matrix model computations, e.g., of higher–point functions of

* The nontrivial relation between \(\tau\) and \(t\) noted in [16] corresponds to the appearance of operators \(\hat{V}_l\) (instead of \(\phi \hat{V}_l\)) on the RHS of (5.1). They are also present, but let us here focus on the new type of operators \(\phi \hat{V}_l\).
tachyons at the “discrete” momenta. It then needs to be better understood why we can recover some of the matrix model correlators with the method of Goulian and Li from the Liouville correlators,[17,18] without the extra terms in (5.2).

4.2. Black Hole Hair

The conjecture that all the discrete operators, in particular the ‘static’ ones $\Phi_{j,0}$ with zero $x$–momentum can be turned into exactly marginal ones implies that each $\Phi_{j,0}$ adds a new dimension to the space of black hole solutions of classical 2D string theory, corresponding to higher spin (not only metric) hair. It will be very interesting to better understand in how far this is significant for the issue of information loss in black holes.[19]

4.3. Four Dimensions

It would also be interesting to extend this work to four dimensions. Four–dimensional Euclidean quantum geometry is, at the least, an interesting statistical mechanical model. At the ultraviolet fixed point of infinite Weyl coupling, where the theory is asymptotically free[20] it can be solved with the methods of two dimensional quantum gravity in conformal gauge.[21] Perturbing away from this limit is similar to adding perturbations to the free $c = 1$ theory. One might be able to find a phase diagram for Euclidean quantum gravity by generalizing the method suggested in this paper.
4.4. Summary

In the Liouville theory approach to 2D quantum gravity coupled to an interacting scalar field, new terms appear in the Lagrangian at higher orders in the coupling constants. They are required by background invariance and cannot be eliminated by a field redefinition when the interaction is given by one of the discrete tachyons or higher-spin operators.

The new terms are crucial for obtaining the correct phase diagram, as found with the nonperturbative matrix model techniques in the case of the Sine–Gordon model. We have partly interpreted this diagram, but the transition below $\frac{1}{2}$ of the Kosterlitz–Thouless momentum must be clarified more, the cosmological constant must be treated more rigorously, and the cubic terms in the beta function (2.2), which are also universal, should be derived. The new terms have various other implications and should in particular be important for the correct computation of higher-point correlation functions.

Acknowledgements

I would like to thank J. H. Schwarz for questions and advice, A. M. Polyakov for a discussion in Les Houches and K. Li for critical comments. I also thank D. Kutasov and many others for comments on the first draft.

Note added:

Some other aspects of the Sine–Gordon model coupled to gravity have been studied recently in ref. [22].
Appendix A: String Equations of Motion and Boundary Conditions

The question is whether (1.2) are the unique modifications that make the interaction (1.1) marginal up to order $\tau^2$. It is useful to think of 2D quantum gravity as classical string theory.\cite{3} Let us first discuss the example of the Sine–Gordon model. The discussion will be restricted to genus zero.

It is well known that, for genus zero, exactly marginal perturbations of the world sheet action correspond to classical solutions of string theory. Some of them can be found by expanding the dilaton $\Phi$, the graviton $G_{\mu\nu}$ and the tachyon $T$ in the sigma model

\begin{equation}
S = \frac{1}{8\pi} \int \sqrt{g} \{ G_{\mu\nu}(x, \phi) \partial x^\mu \partial x^\nu + \hat{R}(x, \phi) + T(x, \phi) \} \tag{A.1} \end{equation}

and by then solving the equations of motion derived from the low-energy effective action of two-dimensional string theory,\cite{23,24}

\begin{align*}
T(x, \phi) &= m \cos px e^{(p - \sqrt{2})\phi} + m^2 T^{(1)}(x, \phi) + .. \\
\Phi(x, \phi) &= 2\sqrt{2}\phi + m^2 \Phi^{(1)}(x, \phi) + .. \\
G_{\mu\nu}(x, \phi) &= \delta_{\mu\nu} + m^2 h_{\mu\nu}(x, \phi) + ..
\end{align*}

and by then solving the equations of motion derived from the low-energy effective action of two-dimensional string theory,\cite{23,24}

\begin{equation}
\int dx \, d\phi \sqrt{G} e^{\Phi} \{ R + \nabla \Phi^2 + 8 + \nabla T^2 - 2T^2 + \frac{4}{3} T^3 + o(m^4) \}. \end{equation}

The corrections to $G, \Phi$ in (A.2) are of order $m^2$ because $T$ appears in the Hilbert-Einstein equations only in the tachyon stress tensor, which is quadratic in $T$. The $T^3$–term is ambiguous,\cite{24,25} but this will not be important here. It is useful to choose a gauge in which the dilaton is linear, i.e., $\Phi^{(1)} = 0$. To $o(m^2)$, the equations

\begin{itemize}
\item[\triangleright] The cosmological constant will be included in the tachyon in appendix B.
\item[\triangleright] Its presence justifies the expansion in $m$.
\item[\triangleright] This is always possible at least at order $m^2$ and $m^3$.
\end{itemize}
of motion are second order differential equations:

\[
\nabla_\mu \nabla_\nu \Phi - \frac{1}{2} G_{\mu\nu} (\nabla^2 \Phi + 2 \Box \Phi - 8) = \Theta_{\mu\nu}
\]

\[
\Box T + \nabla^2 \Phi \nabla T + 2T - 2T^2 = 0
\]

(A.3)

with tachyon stress tensor \( \Theta_{\mu\nu} \).

To specify a solution, we need two boundary conditions. Following ref. [3], we will adopt boundary conditions given (i) in the ultraviolet by the bare action and (ii) in the infrared by the requirement of regularity. Let us for now assume the simple form \( e^{\alpha \phi} \) for the cosmological constant. ‘Infrared’ means \( \phi \to -\infty \) since \( \alpha = -\sqrt{2} < 0 \).

(i) UV: As pointed out in (2.11), the Liouville coordinate is bounded:

\[
\hat{g}_{\mu\nu} e^{\alpha \phi} \, d\xi^\mu d\xi^\nu \geq a^2 \Rightarrow \phi \leq \phi_0 \sim \frac{1}{\alpha} \log a^2.
\]

This bound on \( \phi \) does not modify the Einstein equations. It just requires specifying the action at the cutoff, \( S(\phi = \phi_0) \). As in [3], we identify it with the unperturbed action \( S_0 \) plus the bare matter interaction (\( \Delta \) is the bare cosmological constant:)

\[
S(\phi = \phi_0) = S_0 + \frac{1}{8\pi} \int \Delta + m_B \cos px \leftrightarrow \begin{cases} T(\phi_0) = \Delta + m_B \cos px \\ G_{\mu\nu}(\phi_0) = \delta_{\mu\nu} \end{cases} \quad (A.4)
\]

(ii) IR: It has been pointed out \([11, 12]\) that operators that diverge faster than \( e^{-Q/2} \phi \) as \( \phi \to -\infty \) do not exist in the Liouville theory (2.1). This provides the second boundary condition. Given one solution of (A.3) for \( T^{(1)} \) and \( h_{\mu\nu} \), the other solutions are obtained by adding linear combinations of \( o(m^2) \) of the on-shell tachyons and the two discrete gravitons

\[
\cos px \, e^{(p-\sqrt{2})\phi}, \quad \cos px \, e^{(-p-\sqrt{2})\phi}, \quad \partial x^2 \quad \text{and} \quad \partial x^2 \, e^{-2\sqrt{2}\phi}.
\]

(A.5)

Boundary condition (ii) means essentially that the operators with the more negative Liouville dressing must be dropped. For a more precise statement, see appendix B.

\( \Diamond \) The implicit assumption here is that the term \( \log \sqrt{\hat{g}} \) in the definition of \( \phi_0 \) is absorbed in the gravitational dressing of the operators. Otherwise \( \phi_0 \) varies with \( \hat{g} \) and we can no longer expect that the perturbations are (1,1), let alone exactly marginal.
The discussion has been restricted to the tachyon and the graviton. Including the discrete operators of subsection 2.3 as interactions corresponds to turning on higher spin backgrounds in the sigma model, and the same arguments seem to apply. That two boundary conditions still suffice to specify a solution is suggested by the fact that there are only two possible Liouville dressings for each of the discrete operators of the \( c = 1 \) model.

Setting the bound \( \phi_0 = 0 \), we see that the operators found in section 2 already satisfy the boundary conditions (i) and (ii), and are thus the unique marginal perturbations.

Appendix B: The Cosmological Constant

Gravitational dressings in the presence of a cosmological constant \( \mu \) can in principle be found as as follows (See [3,12] for some details):

One includes the cosmological constant in the tachyon of string theory, replacing, e.g. for the Sine–Gordon model, the ansatz (A.2) by

\[
T(x, \phi) = T_\mu(\phi) + m \cos p x \ f_\mu(p, \phi) + m^2 T_\mu^{(1)}(x, \phi) + ..
\]

\[
G_{\alpha\beta}(x, \phi) = \delta_{\alpha\beta}^{\mu}(\phi) + m^2 h_{\alpha\beta}^{\mu}(x, \phi) + ..
\]

where \( T_\mu \) is the cosmological constant and \( f_\mu, T_\mu^{(1)} \) and \( h_\mu \) are the modified dressings, exact in \( \mu \) order by order in \( m \). Let us assume that \( m < \mu \) but both are small.

First, one must find \( T_\mu \) and \( \delta_\mu \) exactly. \( T_\mu \) has the form of a kink centered at a free parameter \( \tilde{\phi} \), related to \( \mu \) by \( \mu = e^{\sqrt{2} \tilde{\phi}} \) and to the bare cosmological constant \( \Delta \) by \( \Delta \propto \tilde{\phi} e^{\sqrt{2} \tilde{\phi}} \). [3]

\[
T_\mu(\phi) = T_0(\phi - \tilde{\phi}), \quad T_0(\phi) = \begin{cases} 1 & \text{for } \phi \to -\infty \\ \infty \phi e^{-\sqrt{2} \phi} & \text{for } \phi \to \infty \end{cases} \quad (B.2)
\]

(B.2) satisfies the boundary conditions (i), \( T_\mu(0) = \Delta \) and (ii), \( T_\mu \) is regular as

* Although we cannot expand in \( \mu \), for \( \mu > 0 \) we can expand in \( m \).
\( \phi \to -\infty \) \((T \to 1)\). \( \delta^\mu \) differs from \( \delta \) by the backreaction of the tachyon \( T_\mu \) on the metric. Like \( T_\mu \), this difference will decay exponentially in the UV.

Next, one must find the dressings \( f_\mu \), \( T^{(1)}_\mu \) and \( h^\mu \) by solving the string equations of motion \((A.3)\) order by order in \( m \). E.g., the tachyon equation of motion, linearized around the background \( T_\mu \), determines \( f_\mu \):\[ \{ \partial_\phi^2 + 2\sqrt{2}\partial_\phi + 2 - p^2 - 4T_\mu \} f_\mu = 0. \quad (B.3) \]

Since \( T_\mu \) and \( \delta^\mu \) are very small in the UV \((\bar{\phi} \ll \phi < 0)\), the equations of motion for \( f_\mu \), \( T^{(1)}_\mu \) and \( h^\mu \) are the same as for \( \mu = 0 \) in this regime and the only role of the cosmological constant is to set the second boundary condition \((\text{ii})\) of appendix A. E.g., the solutions of \((B.3)\) in the UV are\[^3\]

\[ c_1e^{-(p-\sqrt{2})\phi} + c_2e^{(p-\sqrt{2})\phi} \propto e^{-\sqrt{2}\phi}\sinh(p\phi - \Theta). \]

In the IR region \( \phi \ll \bar{\phi} \), where \( T_\mu \sim \text{constant} \), the solution of \((B.3)\) that is regular at \( \phi \to -\infty \) grows exponentially. The other, divergent solution does not exist as an operator. To match the solutions for \( \phi < \bar{\phi} \) and \( \phi > \bar{\phi} \), one needs roughly \( \Theta \sim p\bar{\phi} \). In the UV, \( \phi - \bar{\phi} \) is large, of order \(|\log a^2|\). So unless \( p \) is close to zero, \( f_\mu \) is just \( e^{(p-\sqrt{2})\phi} \) there. Boundary condition \((\text{ii})\) then simply means dropping the term with the second Liouville dressing, as without cosmological constant. At \( o(m^2) \), the same arguments can be repeated for \( T^{(1)}_\mu \) and \( h^\mu \).

Next, let us discuss OPE’s in the presence of the cosmological constant. In free field theory, the OPE of two operators with Liouville momenta \( \alpha, \beta \) would produce an operator with Liouville momentum \( \alpha + \beta \). But in Liouville theory momentum is not conserved because of the exponential potential. Also, if \( \alpha + \beta < -Q/2 \), the operator \( e^{(\alpha+\beta)\phi} \) does not exist. Instead, new primary fields \( V_\sigma = e^{-Q/2}\phi \sin(\sigma\phi + \Theta) \) will be produced, with some weight \( f(\sigma) \) and less singular coefficients:\[^2\]

\[ e^{\alpha\phi(r)} e^{\beta\phi(0)} \sim \int_0^\infty d\sigma |r|^{-2\alpha\beta + (\alpha + \beta + Q/2)^2 + \sigma^2} f(\sigma) V_\sigma \quad (B.4) \]

instead of \( |r|^{-2\alpha\beta} e^{(\alpha+\beta)\phi} \).
For the Sine–Gordon model, this modification of the OPE’s seems to cure the problem of new $|r|^{-2}$ singularities that would naively appear in (2.7) below $p = \frac{1}{2}\sqrt{2}$. They would give rise to unwanted counterterms like $\cos 2px$ at $p = \frac{1}{4}\sqrt{2}$ and $\partial^2 x \cos 2px$ at $p = 0$. The modified OPE of $\cos px \ e^{\epsilon \phi}$ with itself produces

$$\int \frac{d\sigma}{2\pi} f(\sigma)|r|^{-2+4p^2+\sigma^2} \cos 2px \ V_\sigma(\phi) + ... \quad (B.5)$$

Except for the (negligible) case $p = \sigma = 0$, all singularities are milder than quadratic.

REFERENCES

1. F. David, Mod. Phys. Lett. A3, 1651 (1988)
2. J. Distler and H. Kawai, Nucl. Phys. B 321, 509 (1988)
3. J. Polchinski, Nucl. Phys. B 346, 253 (1990)
4. A. Cooper, L. Susskind and L. Thorlacius, Nucl. Phys. B363, 132 (1991)
5. A. M. Polyakov, Mod. Phys. Lett. A6, 3273 (1991)
6. G. Moore, Yale preprint YCTP-P1-92, hepth@xxx/9203061
7. e.g., J. H. Cardy, Les Houches session XLIX, 1988
8. G. Segal, Comm. Math. Phys. 80, 301 (1981)
9. E. Witten, Nucl. Phys. B 373, 187 (1991)
10. I. Klebanov and A. M. Polyakov, Mod. Phys. Lett. A6, 3273 (1991)
11. N. Seiberg, Prog. Theor. Phys. S 102, 319 (1990)
12. J. Polchinski, Nucl. Phys. 362, 125 (1991)
13. D. Kutasov, Princeton preprint PUPT-1334 (1992), hepth@xxx/9207064
14. A. M. Polyakov, Lectures in Les Houches, July 1992
15. D. Gross and I. Klebanov, Nucl. Phys. B 344, 475 (1990)
16. G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B 362, 665 (1991)
17. M. Goulian and M. Li, Phys. Rev. Lett. 66, 2051 (1991)
18. I. Klebanov, Lectures at ICTP Spring School, Trieste (1991)
19. J. H. Schwarz, Phys. Lett. B272, 239 (1991)
20. E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B 201, 469 (1982)
21. C. Schmidhuber, Nucl. Phys. B, to appear; Caltech preprint CALT-68-1745 (revised 5/92), hepth@xxx/9112005
22. E. Hsu and D. Kutasov, Princeton preprint PUPT-1357, hepth@xxx/9212023 (12/92)
23. E.g., M.B. Green, J.H. Schwarz and E. Witten, “Superstring Theory”, Cambridge University Press 1987
24. A.A. Tseytlin, Phys. Lett. B264, 311 (1991)
25. T. Banks, Nucl. Phys. B 361, 166 (1991)
Figures

Fig. 1
KT-transition with gravity at $o(m^2)$; arrows point towards IR

Fig. 2
Sine–Gordon model with gravity at $o(m^2)$