On fractional harmonic functions

Huyuan Chen and Ying Wang

Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China

Abstract

Our concern in this paper is to study the qualitative properties for harmonic functions related to the fractional Laplacian. Firstly, we classify the polynomials in the whole space and in the half space for the fractional Laplacian defined in a principle value sense at infinity. Secondly, we study the fractional harmonic functions in half space with singularities on the boundary and the related distributional identities.

Keywords: Fractional Laplacian; s-Harmonic function, Poisson kernel.

AMS Subject Classifications: 35R11; 35B40.

1 Introduction

Let $N \in \mathbb{N}$, $s \in (0,1)$ satisfy $N > 2s$, and $(-\Delta)^s = \lim_{\epsilon \to 0^+} (-\Delta)^s_\epsilon$ be the fractional Laplacian, where

$$(-\Delta)^s_\epsilon u(x) = c_{N,s} \int_{\mathcal{O}_\epsilon} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz,$$

(1.1)

where $\mathcal{O}_\epsilon$ is a sequence of domains in $\mathbb{R}^N$ with certain symmetric properties such that

$$\bigcup_{\epsilon \in (0,1]} \mathcal{O}_\epsilon = \mathbb{R}^N$$

and

$$c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(N + 2s)}{\Gamma(1 - s)} > 0$$

is a normalized constant such that the Fourier transform holds

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi) \quad \text{for} \quad u \in L^1(\mathbb{R}^N)$$

and $\Gamma$ is the Gamma function. Here the domains with certain symmetric properties could be formed by the balls, the cubes and polyhedrons. In this article, we mainly take

$$\mathcal{O}_\epsilon = B_\frac{1}{\epsilon} \setminus B_\epsilon,$$

where $B_r(A)$ is the ball centered at $A$ and with the radius $r$, $B_\epsilon = B_\epsilon(0)$.

In the last decades, there has been a renewed and increasing interest in the study of Dirichlet problems involving linear and nonlinear integro-differential operators, especially for the simplest model — the fractional Laplacian. This growing interest is justified both by seminal advances in the understanding of nonlocal phenomena from a PDE or a probabilistic point of view, see e.g. [4, 6, 22, 24] and the references therein, and by important applications. The most important $s$-harmonic functions are the fractional Green kernels, which started as early as in 1961 in [9] working on a ball, and the fractional Poisson kernels in bounded domains in [14, 23]. For the fractional Green kernels we refer the readers to [8, 15, 20].

1 chenhuyuan@yeah.net
2 yingwang00@126.com
in bounded domains, to [17] in the half space. Furthermore, we refer to the s-harmonic functions in cones in [20] and blowing up on the boundary of a regular bounded domain in [25].

From the integral-differential form of the fractional Laplacian, a natural restriction for the function \( u \) in (1.1) is

\[
\|u\|_{L^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} \, dx < \infty.
\]

Under this restriction, the s-harmonic functions in the whole domain or half space can’t grow over \( |x|^{2s} \) at the infinity, see the references [14, 16]. However, the principle value sense at infinity in (1.1) allows the fractional Laplacian to work on some special functions not being in \( L^1_s(\mathbb{R}^N) \) such as the functions in \( \text{span}\{x_1, \cdots, x_N\} \) and in \( \text{span}\{x_i, x_j : i, j = 1, \cdots, N, i \neq j\} \), which is well-defined for the fractional laplacian in \( \mathbb{R}^N \) by the oddness. Our aim in this article is to study the s-harmonic functions in the whole space and in upper the half space.

In the whole space, let’s denote the \( m \)-order polynomials’ set with \( k \in \mathbb{N} \),

\[
P_k(\mathbb{R}^N) = \left\{ \sum_{|\alpha|=k} b_\alpha x^\alpha : \alpha \in \mathbb{N}^N, b_\alpha \in \mathbb{R} \right\},
\]

where \( \mathbb{N} \) is the set of all nonnegative integers and \( x^\alpha = \prod_{i=1}^N x_i^{\alpha_i} \). We first classify the s-harmonic polynomials in \( \mathbb{R}^N \) i.e. the classical solution of

\[
(-\Delta)^s u = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}
\]

For \( s \in (0, 1] \), we denote \( \mathcal{H}^s(\mathbb{R}^N) \) the set of all s-harmonic functions in \( \mathbb{R}^N \) and for \( m \in \mathbb{N} \)

\[
\mathcal{H}^s_m(\mathbb{R}^N) = \{ u \in P_m(\mathbb{R}^N) : u \text{ is } s \text{-harmonic in } \mathbb{R}^N \}. \tag{1.3}
\]

Particularly, \( \mathcal{H}^1_m(\mathbb{R}^N) \) is the set of \( m \) order harmonic polynomials in \( \mathbb{R}^N \), i.e.

\[-\Delta u = 0 \quad \text{in} \quad \mathbb{R}^N. \]

For \( P \in P_m \), we denote \( D_\rho = \sum_{|\alpha|=m} a_\alpha D^\alpha \) and the inner norm \( [P, Q]_m = D_\rho Q \), then

\[
\mathcal{H}^1_m = Q_m^\perp,
\]

where \( Q_m = \{ |x|^2 P : P \in P_{m-2} \} \) for \( m \geq 2 \), \( Q_m = \{ 0 \} \) for \( m = 0, 1 \). The dimension of \( \mathcal{H}^1_m \) is \( C_m^{m+N-1} - C_m^{m+N-3} \) for \( m \geq 2 \). We refer to the book [2] for more properties of \( \mathcal{H}_m(\mathbb{R}^N) \).

**Theorem 1.1** Let \( N \geq 2 \), \( s \in (0, 1] \) and \( \mathcal{H}^s_m(\mathbb{R}^N) \) be defined in (1.2). Then

(i) for any \( m \in \mathbb{N} \),

\[
\mathcal{H}^s_m(\mathbb{R}^N) = \mathcal{H}_m^1(\mathbb{R}^N),
\]

(ii) \( \mathcal{H}^1(\mathbb{R}^N) \subset \mathcal{H}^s(\mathbb{R}^N) \).

Our proof of Theorem 1.1 is based on the mean value property of harmonic functions, which allows us to get a same harmonic polynomials’ set for nonlocal operators with general kernels in Section 2 under suitable assumptions. From the mean value property, we obtain that all harmonic functions in \( \mathbb{R}^N \) are s-harmonic. However it is still open if \( \mathcal{H}_1(\mathbb{R}^N) = \mathcal{H}^s(\mathbb{R}^N) \).

The second part in this paper is devoted to the s-harmonic functions with isolated singularity on the boundary

\[
(-\Delta)^s u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N_+ \setminus \{0\}. \tag{1.4}
\]

Note that the prototype problem

\[-\Delta u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = 0 \quad \text{on} \quad \partial \mathbb{R}^N_+ \setminus \{0\}
\]
We remark that $\rho \int$ in order to state the distributional identity, we need the following test functions’ space:

**Theorem 1.2** Let $\Omega$ of $\mathbb{R}^N$ where, in the sequel, we use the following constants

in the distributional sense that

$$-\Delta u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = \delta_0 \quad \text{on} \quad \partial \mathbb{R}^N_+$$

in the upper half space with the formula:

$$\int_{\mathbb{R}^N} \mathcal{P}_1(x)(-\Delta)\varphi(x)dx = \frac{\partial}{\partial x_1}\varphi(0), \quad \forall \varphi \in C^{1,1}(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}^N_+}).$$

Note that the Dirac mass in (1.5) holds in the trace sense that

$$\lim_{t \to 0^+} \int_{\mathbb{R}^N_+} \mathcal{P}_1(t, x')(\zeta(x'))dx' = \zeta(0) \quad \text{for} \quad \zeta \in C_c(\mathbb{R}^{N-1}).$$

Based on the Green kernel and the Poisson kernel, the semilinear elliptic problem involving the Radon measure data $\mu, \nu$

$$-\Delta u + g(u) = \nu \quad \text{in} \quad \Omega, \quad u = \mu \quad \text{on} \quad \partial \Omega$$

has been studied extensively in [19,27] and a survey in [28]. Involving the nonlocal operator, the semilinear problems with measures could see [1[1, 21, 23].

Back to our problem (1.4), we want to study a fundamental solution $\mathcal{P}_s$ of the fractional Laplacian in the upper half space with the formula:

$$\mathcal{P}_s(x) = \left\{ \begin{array}{ll} \mathcal{K}_s |x|^{-N} x_1^s & \text{for} \quad x \in \mathbb{R}^N_+, \\ 0 & \text{for} \quad x \in \mathbb{R}^N_+ \setminus \{0\}, \end{array} \right.$$  

(1.7)

where, in the sequel, we use the following constants

$$\mathcal{K}_s = k_{N,s} 2^{2s-1} s^{-1} \quad \text{and} \quad \kappa_{N,s} = \pi^{-(N/2-1)} \Gamma(N/2) \sin(\pi s).$$

In order to state the distributional identity, we need the following test functions’ space: *Given a domain $\Omega$ of $\mathbb{R}^N$, let $\mathcal{X}_s(\Omega) \subset C(\mathbb{R}^N)$ be the space of functions $\xi$ satisfying:

(i) $\xi$ has the compact support in $\tilde{\Omega}$;
(ii) $\rho(x)^{-s} \xi$ is continuous in $\tilde{\Omega}$ and $\|(-\Delta)^s \xi\|_{L^\infty(\Omega)} < \infty$, where $\rho(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$;
(iii) there exist $\varphi \in L^1(\Omega, \rho dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^s \xi| \leq \varphi$ a.e. in $\mathbb{R}^N_+$, for all $\epsilon \in (0, \epsilon_0]$.

We remark that $\rho(x) = (x_1)_+$ if $\Omega = \mathbb{R}^N_+$ and $C^\infty_c(\Omega)$ is dense in $\mathcal{X}_s(\Omega)$, where $t_+ = \max\{t, 0\}$.

**Theorem 1.2** Let $\mathcal{P}_s$ be defined in (1.4) and $\epsilon_1 = 1, 0, \cdots, 0 \in \mathbb{R}^N$. Then $\mathcal{P}_s$ is a solution of (1.4) and verifies that

$$\int_{\mathbb{R}^N} \mathcal{P}_s(x)(-\Delta)^s \varphi(x)dx = \frac{\partial^s}{\partial x_1^s}\varphi(0) \quad \forall \varphi \in \mathcal{X}_s(\mathbb{R}^N_+),$$

(1.8)

where

$$\frac{\partial^s}{\partial x_1^s}\varphi(0) = \lim_{t \to 0^+} \frac{\varphi(t \epsilon_1)}{t^s}.$$  

It is remarkable that, unlike (1.5), in the trace sense, $\mathcal{P}_s$ doesn’t have the Dirac mass in the boundary trace, i.e. it isn’t a weak solution of

$$(-\Delta)^s u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = \delta_0 \quad \text{on} \quad \mathbb{R}^N_+.$$  

(1.9)
which, from [10], has no weak solutions. Thanks to the nonlocal property of the fractional Laplacian, the source outside of the domain could be taken in the calculation of the first equation of (1.3) directly, which means outside source in (1.3) plays a role different from (1.5).

Our next attempt is to understand the role of outside source in (1.8). To this end, we approximate the fundamental function $\mathcal{P}_s$ by the Green kernel and the Poisson kernel in the half space. Subject to zero Dirichlet boundary condition in $\mathbb{R}^N_+$, the Green kernel $G_{s,\infty}$ has the formula (see [17]):

$$G_{s,\infty}(x,y) = \begin{cases} \frac{k_s}{2} |x-y|^{2s-N} \int_0^{\frac{4x\cdot y}{|x-y|^2}} t^{s-1}(t+1)^{-\frac{N}{2}} \, dt & \text{if } x, y \in \mathbb{R}^N_+, \\ 0 & \text{if } x \text{ or } y \notin \mathbb{R}^N_+. \end{cases} \quad (1.10)$$

It is known that the Poisson kernel in half space is the following:

$$P_{s,\infty}(x,y) = K_s \left( \frac{x_1}{y_1} \right) |x-y|^{-N}, \quad \forall x \in \mathbb{R}^N_+, \, y \in \mathbb{R}^N_-, \quad (1.11)$$

where $\mathbb{R}^N_+ = (-\infty, 0) \times \mathbb{R}^{N-1}$.

**Theorem 1.3** Let $G_{s,\infty}$, $P_{s,\infty}$ be the Green kernel and the Poisson kernel in $\mathbb{R}^N_+$ defined in (1.10) and (1.11) respectively, then

$$e^{-s}G_{s,\infty}(\cdot, \varepsilon e_1) \to \mathcal{P}_s \quad \text{and} \quad e^sP_{s,\infty}(\cdot, -\varepsilon e_1) \to \mathcal{P}_s \quad \text{as } \varepsilon \to 0^+$$

in $L^1(\mathbb{R}^N_+)$ and uniformly in any compact set of $\mathbb{R}^N_+$.

Finally, we study the solution of

$$(-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^N_+, \quad u = 0 \quad \text{in } \mathbb{R}^N_+ \quad (1.12)$$

subject to the blowing up boundary condition on $\partial \mathbb{R}^N_+ := \{0\} \times \mathbb{R}^{N-1}$.

For a given Radon measure $\mu$ in $\mathbb{R}^N_+$, denote

$$\mathcal{P}_{s,\infty}[\mu](x) := \begin{cases} \int_{\mathbb{R}^N_+} P_{s,\infty}(x,y) d\mu(y) & \text{for } x \in \mathbb{R}^N_+, \\ \mu & \text{in } \mathbb{R}^N_+. \end{cases}$$

Particularly, let $\mu_\varepsilon = \delta_\varepsilon(y_1) d\omega_{\mathbb{R}^{N-1}}(y')$ for $\varepsilon > 0$ and $d\omega_{\mathbb{R}^{N-1}}$ is the Hausdorff measure in $\mathbb{R}^{N-1}$ and $\delta_\varepsilon(y_1)$ is the Dirac mass in variable $y_1$. For simplicity, we write it $d\omega_{\mathbb{R}^{N-1}}(y') = dy'$, then we have that for $x \in \mathbb{R}^N_+$

$$\mathcal{P}_{s,\infty}[\mu_\varepsilon](x) = \int_{\mathbb{R}^{N-1}} \mathcal{P}_{s,\infty}(x,(-\varepsilon,y')) dy' = K_s \int_{\mathbb{R}^{N-1}} \left( \frac{x_1}{\varepsilon} \right) |x-(-\varepsilon,y')|^{-N} dy'$$

**Theorem 1.4** Let

$$Q_s(x) = \begin{cases} x_1^{s-1} & \text{in } \mathbb{R}^N_+, \\ 0 & \text{in } \mathbb{R}^N_-. \end{cases}$$

Then $Q_s$ is a solution of (1.12) and it verifies that

$$\int_{\mathbb{R}^N_+} Q_s(x)(-\Delta)^s \varphi(x) dx = C_s \int_{\mathbb{R}^{N-1}} \frac{\partial^s}{\partial x_1^s} \varphi(0,x') dx' \quad \forall \varphi \in \mathcal{X}_s(\mathbb{R}^N_+), \quad (1.13)$$

where

$$C_s = 2\sqrt{\pi} \frac{s}{\sin(\pi s)} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)}.$$

Moreover, we have that

$$C_s \varepsilon^s \int_{\mathbb{R}^N_+} \mathcal{P}_{s,\infty}[\mu_\varepsilon] dx \to Q_s \quad \varepsilon \to 0^+$$

in $L^1(\mathbb{R}^N_+)$ and uniformly in any compact set of $\mathbb{R}^N_+$. 


Note that the $Q_s$ is known yet in the classical sense, and we provide a simple proof, which is the Poisson kernel in the half space. Also, we can also get the well-known $s$-harmonic function $(x_1)_+^s$ and we also refer to [14] Theorem 5.1 for the uniqueness.

Finally, we try to classify the $s$-harmonic functions in the half space in [11,12] without restriction of the continuity on the boundary. Denote

$$\mathcal{H}^s(\mathbb{R}^+_N) := \{ u \in C(\mathbb{R}^+_N) \text{ verifies } (1.12) \}.$$

**Theorem 1.5** Let $N \geq 3$, $s \in (0,1)$ and

$$A_{s,m}(\mathbb{R}^+_N) = \{(x_1)_+^s v(x') : v \in \mathcal{H}^s(\mathbb{R}^{N-1})\} \cup \{(x_1)_+^{s-1} v(x') : v \in \mathcal{H}^s(\mathbb{R}^{N-1})\}$$

for $m \in \mathbb{N}$.

Then

$$A_{s,m}(\mathbb{R}^+_N) \subset \mathcal{H}^s(\mathbb{R}^+_N) \quad \text{for } m \geq 0.$$ 

We remark that $\{(x_1)_+^s v(x') : v \in \mathcal{H}^s(\mathbb{R}^{N-1})\} \subset C(\mathbb{R}^N)$ and since $(x_1)_+^s$ is an $s$-harmonic function continuous up to the boundary.

The remainder of this paper is organized as follows. Section 2 is devoted to the $s$-harmonic functions in $\mathbb{R}^N$. In Section 3, we prove Theorem 1.2 by the Green kernel $G_{s,\infty}$ via passing to the limit with $y = \epsilon e_1$ as $\epsilon \to 0$. Section 4 is devoted to the derivation of the Poisson kernel $P_{s,\infty}$ and then show the approximation to the fundamental solution $P_s$ by this Poisson kernel. Finally, we obtain the whole boundary explosive solution $x_1^s$ by the Poisson kernel $P_{s,\infty}$ and give the proof of Theorem 1.4 and Theorem 1.5 in Section 5.

## 2 Harmonic polynomials

Here we involve general kernel for nonlocal problem

$$\mathcal{L}_K u = 0 \quad \text{in } \mathbb{R}^N,$$

where

$$\mathcal{L}_K u(x) = \lim_{\epsilon \to 0^+} \int_{B_1 \setminus B_\epsilon} (u(x) - u(x + z))K(|z|)dz$$

and the kernel $K : (0,+\infty) \to [0,+\infty)$ is continuous. We set

$$\mathcal{H}^K_m(\mathbb{R}^N) = \{ u \in \mathbb{P}_m(\mathbb{R}^N) : u \text{ verifies (2.1)} \}.$$

We have the following classifications:

**Proposition 2.1** Assume that $N \geq 2$, $m \in \mathbb{N}$, $K$ verifies that

$$\int_0^{+\infty} K(t)t^{N-1+i}dt \in [0,+\infty] \quad \text{for } i = 2,3,\ldots,m.$$ 

Let

$$\kappa_{i,j}(\epsilon) = \frac{\int_0^\epsilon K(t)t^{N-1+i}dt}{\int_\epsilon^{+\infty} K(t)t^{N-1+i}dt}, \quad i,j = 2,3,\ldots,m.$$ 

If either

$$\lim_{\epsilon \to 0^+} \kappa_{i,j}(\epsilon) = 0 \quad \text{for all } i,j = 2,3,\ldots,m, \quad i > j$$

or

$$\lim_{\epsilon \to 0^+} \kappa_{i,j}(\epsilon) = 0 \quad \text{for all } i,j = 2,3,\ldots,m, \quad i < j,$$

then

$$\mathcal{H}^K_m(\mathbb{R}^N) = \mathcal{H}^1_m(\mathbb{R}^N).$$

Particularly, for $m = 0,1$

$$\mathcal{H}^K_m(\mathbb{R}^N) = \mathcal{H}^1_m(\mathbb{R}^N) = \mathbb{P}_m(\mathbb{R}^N).$$
Proof. Part I: we show $\mathcal{H}_m^1(\mathbb{R}^N) \subset \mathcal{H}_m^K(\mathbb{R}^N)$. Particularly, for $m = 0, 1$

$$\mathcal{H}_m^K(\mathbb{R}^N) = \mathcal{H}_m^1(\mathbb{R}^N) = \mathbb{P}_m(\mathbb{R}^N).$$

Let $h_m \in \mathcal{H}_m^1(\mathbb{R}^N)$, then for any $z \in \mathbb{R}^N$, $h_m(\cdot + z)$ is also harmonic in $\mathbb{R}^N$. Note that

$$h_0(x) = c_0 \quad \text{and} \quad h_1(x + z) = \sum_{i=1}^{N} c_i(x_i + z_i)$$

for some $c_i \in \mathbb{R}$ with $i = 0, \ldots, N$, then

$$h_1(x + z) - h_1(x) = \sum_{i=1}^{N} c_i z_i.$$ 

Then for any $\epsilon \in (0, 1)$,

$$\int_{B_{1/\epsilon} \setminus B_{\epsilon}} (h_1(x) - h_1(x + z)) K(|z|) dz = \sum_{i=1}^{N} c_i \int_{B_{1/\epsilon} \setminus B_{\epsilon}} z_i K(|z|) dz = 0$$

by the oddness of $h_1$. Therefore, we obtain that for $m = 0, 1$

$$\mathcal{H}_m^K(\mathbb{R}^N) = \mathbb{P}_m(\mathbb{R}^N) = \mathcal{H}_m^1(\mathbb{R}^N).$$

Now we deal with the case: $m \geq 2$. For $h_m \in \mathcal{H}_m^1$, we have that

$$(-\Delta)z \left(h_m(x + z) - h_m(x)\right) = 0 \quad \text{for any} \quad z \in \mathbb{R}^N$$

and the mean value property implies that for any $r > 0$

$$(N|\mathbb{S}^N|)^{-1} r^{1-N} \int_{\partial B_r} (h_m(x + z) - h_m(x)) d\omega_r(z) = h_m(x + 0) - h_m(x) = 0,$$

that is,

$$\int_{\partial B_r} (h_m(x + z) - h_m(x)) d\omega_r(z) = 0.$$ 

Therefore, for any $\epsilon \in (0, 1)$,

$$\int_{B_{1/\epsilon} \setminus B_{\epsilon}} (h_m(x) - h_m(x + z)) K(|z|) dz = \int_{\epsilon}^{1/\epsilon} K(r) \int_{\partial B_r} (h_m(x) - h_m(x + z)) d\omega_r(z) = 0.$$ 

Now passing to the limit as $\epsilon \to 0^+$, we obtain that $h_m \in \mathcal{H}_m^K(\mathbb{R}^N)$.

Part II: we show that

$$\mathcal{H}_m^K(\mathbb{R}^N) \subset \mathcal{H}_m^1(\mathbb{R}^N).$$

Let

$$h_m(x) = \sum_{|\alpha| = m} b_\alpha x^\alpha \in \mathcal{H}_m^K(\mathbb{R}^N),$$

where $m \geq 2$ and $b_\alpha \neq 0$. Then direct computation shows that

$$h_m(x + z) = \sum_{|\alpha_1| + |\alpha_2| = m} u_{\alpha_1}(x)v_{\alpha_2}(z),$$

where $\alpha_1, \alpha_2 \in \mathbb{N}^m$,

$$u_{\alpha_1} \in \mathbb{P}_{|\alpha_1|}(\mathbb{R}^N) \quad \text{and} \quad v_{\alpha_2} \in \mathbb{P}_{|\alpha_2|}(\mathbb{R}^N).$$
Moreover,\[ h_m(x + z) - h_m(x) = \sum_{i=1}^{N} u_i(x)z_i + \sum_{|\alpha_1| + |\alpha_2| = m, |\alpha_2| \geq 2} u_{\alpha_1}(x)u_{\alpha_2}(z), \tag{2.5} \]
where \( u_i \in \mathbb{P}_{m-1}(\mathbb{R}^N) \).

Therefore, from the definition of fractional laplacian we obtain that for \( m \geq 2 \)
\[
0 = \lim_{\epsilon \to 0^+} \int_{B_1^+ \setminus B_\epsilon} (h_m(x) - h_m(x + z)) K(|z|)dz
\]
\[
= \lim_{\epsilon \to 0^+} \left( \sum_{|\alpha_1| + |\alpha_2| = m, |\alpha_2| \geq 2} u_{\alpha_1}(x) \int_{B_1^+ \setminus B_\epsilon} v_{\alpha_2}(z)K(|z|)dz \right)
\]
\[
= \lim_{\epsilon \to 0^+} \left( \sum_{|\alpha_1| + |\alpha_2| = m, |\alpha_2| \geq 2} u_{\alpha_1}(x) \int_{\mathbb{R}^N} v_{\alpha_2}(z) d\omega(z) \int_{\epsilon}^{1} K(r)r^{N-1} dr \right)
\]
\[
= \lim_{\epsilon \to 0^+} \sum_{j=2}^{m} \left( \int_{\epsilon}^{1} K(r)r^{N-1+j} dr \left( \sum_{|\alpha_1| = m-j, |\alpha_2| = j} u_{\alpha_1}(x) \int_{\mathbb{R}^N} v_{\alpha_2}(z) d\omega(z) \right) \right). \tag{2.6}
\]

Let \( \sigma_j(\epsilon) = \int_{\epsilon}^{1} K(r)r^{N-1+j} dr, \quad j \in \mathbb{N} \)
and
\[
Z_j = \sum_{|\alpha_1| = m-j, |\alpha_2| = j} u_{\alpha_1}(x) \int_{\mathbb{R}^N} v_{\alpha_2}(z) d\omega(z),
\]
then (2.6) could be written as
\[
\lim_{\epsilon \to 0^+} \sum_{j=2}^{m} Z_j \sigma_j(\epsilon) = 0. \tag{2.7}
\]

Under the assumption (2.4), we prove that \( Z_j = 0 \) for \( j = 2, \cdots, m \), i.e.
\[
\sum_{|\alpha_1| = m-j, |\alpha_2| = j} u_{\alpha_1}(x) \int_{\mathbb{R}^N} v_{\alpha_2}(z) d\omega(z) = 0. \tag{2.8}
\]
In fact, if \( Z_m \neq 0 \), then (2.7) implies that
\[
\lim_{\epsilon \to 0^+} \sigma_m(\epsilon) \left( Z_m + \sum_{j=2}^{m-1} Z_j \kappa_{m,j}(\epsilon) \right) = 0
\]
which implies \( Z_m = 0 \), thanks to
\[
\lim_{\epsilon \to 0^+} \kappa_{m,j}(\epsilon) = 0.
\]
Inductively, we can obtain \( Z_j = 0 \) for \( j = 2, \cdots, m-1 \).

Under the assumption (2.4), we show that \( Z_j = 0 \) for \( j = 2, \cdots, m \). Indeed, if \( Z_2 \neq 0 \), then (2.7) implies that
\[
\lim_{\epsilon \to 0^+} \sigma_2(\epsilon) \left( Z_2 + \sum_{j=4}^{m} Z_j \kappa_{2,j}(\epsilon) \right) = 0,
\]
which implies \( Z_1 = 0 \), thanks to
\[
\lim_{\epsilon \to 0^+} \kappa_{2,j}(\epsilon) = 0.
\]

7
Inductively, we can obtain $Z_j = 0$ for $j = 3, \ldots, m$.

Therefore, we obtain that for any $\epsilon > 0$

$$\int_{\partial B_r} (h_m(x + z) - h_m(x))d\omega_r(z)$$

$$= \sum_{|\alpha_1| + |\alpha_2| = m, |\alpha_2| \geq 2} u_{\alpha_1}(x) \int_{\partial B_r} v_{\alpha_2}(z)d\omega_r(z)$$

$$= \sum_{j=2}^m r^{j+N-1} \left( \sum_{|\alpha_1| = m-j, |\alpha_2| = j} u_{\alpha_1}(x) \int_{\mathbb{R}^N} v_{\alpha_2}(z)d\omega(z) \right)$$

$$= 0$$

$$= h_m(x + 0) - h_m(x),$$

and from the converse of the mean value property we deduce that

$$-\Delta z h_m(x) = -\Delta z h_m(x + 0) = -\Delta z (h_m(x + 0) - h_m(x)) = 0.$$

By the arbitrary of $x$, we obtain $h_m \in H^1_m(\mathbb{R}^N)$. We complete the proof. \qed

**Remark 2.1** (i) Let

$$\frac{1}{c}(1 + t)^{-N-\zeta} \leq K_{1,\zeta}(t) \leq ct^{-N-\zeta}$$

for some $c \geq 1$ and $\zeta \leq 2$, then $K_{1,\zeta}$ satisfies (2.3).

(ii) Let

$$\frac{1}{c}t^{-N-\zeta}\chi_{(0,1)}(t) \leq K_{2,\zeta}(t) \leq ct^{-N-\zeta}e^{-t}$$

for $c \geq 1$, where $\chi_{(0,1)}(t) = 1$ if $t \in (0, 1)$ and $\chi_{(0,1)}(t) = 0$ if $t \geq 1$.

Then for $\zeta \geq m$, $K_{2,\zeta}$ satisfies (2.4).

**Proof of Theorem 1.1** Part (i). When $K(|z|) = c_{N,s}|z|^{-N-2s}$ with $s \in (0, 1)$, then (2.3) holds true and $L_K = (-\Delta)^s$, $H^1_m(\mathbb{R}^N) = H^s_m(\mathbb{R}^N)$, then Theorem 1.1 follows by Proposition 2.1.

Part (ii). For any $h \in H^1(\mathbb{R}^N)$, we have that

$$(-\Delta)_z (h(x + z) - h(x)) = 0 \text{ for any } z \in \mathbb{R}^N$$

and the mean value property implies that for any $r > 0$

$$(N|\mathbb{S}^N|)^{-1}r^{1-N} \int_{\partial B_r} (h(x + z) - h(x))d\omega_r(z) = h(x + 0) - h(x) = 0,$$

then for any $\epsilon \in (0, 1)$,

$$\int_{\partial B_{\epsilon} \setminus B_r} (h(x) - h(x + z))K(|z|)dz = \int_{\epsilon}^1 K(r) \int_{\partial B_r} (h(x) - h(x + z))d\omega_r(z) = 0$$

with $K(t) = t^{-N-2s}$. Now passing to the limit as $\epsilon \to 0^+$, we obtain that $h \in H^s(\mathbb{R}^N)$. \qed
3 Approximation by the Green kernel

Note that the Green function associated with $(-\Delta)^s$ in the unit ball $B_1$ was computed by Blumenthal, Getoor and Ray in [7] with the formula

$$G_{s,1}(x, y) = \kappa_{N,s}|x - y|^{2s-N} \int_1^{(\psi(x,y)+1)^{1/2}} \left( \frac{z^2 - 1}{z^{N-1}} \right)^{s-1} dz$$

$$= \frac{k_{N,s}}{2}|x - y|^{2s-N} \int_0^{\psi(x,y)} \frac{z^{s-1}}{(z + 1)^{N/2}} dz$$

for $x, y \in B_1$ and $G_{s,1}(x, y) = 0$ if $x \notin B_1$ or $y \notin B_1$, where

$$\psi(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}$$

and $k_{N,s}$ is the normalization constant

$$k_{N,s} = \pi^{-(N/2+1)}\Gamma(N/2)\sin(\pi s).$$

If $N = 1 = 2s$, then direct computations give

$$\int_0^{\psi(x,y)} \frac{z^{-1/2}}{(z + 1)^{1/2}} dz = 2\ln \frac{1 - xy + (1 - x^2)^{1/2}(1 - y^2)^{1/2}}{|x - y|}$$

and in this case we have that

$$G_{s,1}(x, y) = \frac{1}{\pi} \ln \frac{1 - xy + (1 - x^2)^{1/2}(1 - y^2)^{1/2}}{|x - y|}.$$  

By the dilation, the Green function for the ball $B_R$ with $R > 0$ is given by

$$G_{s,R}(x, y) = R^{2s-N}G_{s,1}\left(\frac{x}{R}, \frac{y}{R}\right)$$

$$= \frac{k_{N,s}}{2}|x - y|^{2s-N} \int_0^{\psi_R(x,y)} \frac{z^{s-1}}{(z + 1)^{N/2}} dz$$

with $\psi_R(x, y) = \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2|x - y|^2}$. From [17] (3.1), the Green function $G_{s,\infty}^+$ in the half space $\mathbb{R}_+^N$ has the formula

$$G_{s,\infty}(x, y) = \frac{k_{N,s}}{2}|x - y|^{2s-N} \int_0^{\psi_\infty(x,y)} \frac{r^{s-1}}{(r + 1)^{N/2}} dr \quad \text{for } x, y \in \mathbb{R}_+^N, \tag{3.1}$$

where

$$\psi_\infty(x, y) = \frac{4x_1y_1}{|x - y|^2}.$$  

Next we obtain $\mathcal{P}_s$ by passing to the limit of Green kernel $G_{s,\infty}^+(x, y)$.

**Lemma 3.1** Let $N \geq 2s$, $G_{s,\infty}$ be defined in (3.1), then there holds that

$$\epsilon^{-s}G_{s,\infty}(\cdot, \epsilon e_1) \to \mathcal{P}_s \quad \text{as } \epsilon \to 0^+$$  

in $L_1^1(\mathbb{R}_+^N)$ and uniformly in any compact set of $\mathbb{R}_+^N$.

Moreover, for $\varphi \in X_s(\mathbb{R}_+^N)$

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}_+^N} \epsilon^{-s}G_{s,\infty}(x, \epsilon e_1)(-\Delta)^s \varphi(x) dx = \int_{\mathbb{R}_+^N} \mathcal{P}_s(x)(-\Delta)^s \varphi(x) dx. \tag{3.3}$$
Proof. We first claim that for any bounded open set \( O \subseteq \mathbb{R}^N \setminus \{0\} \), there exists \( \epsilon_0 \in (0, \frac{1}{2}) \) such that
\[
2\epsilon_0 e_1 \not\in O.
\]

Now we want to prove that
\[
\epsilon^{-s} \mathcal{G}_{s, \infty}(\cdot, \epsilon e_1) \rightarrow \mathcal{P}_s \text{ as } \epsilon \to 0^+ \text{ uniformly in } O.
\]

(3.4)

Note that
\[
|x - \epsilon e_1|^2 = |x|^2 \left(1 - 2|x|^{-2} x_1 \epsilon + |x|^{-2} \epsilon^2\right)
\]
and for \( x \in O \subseteq \mathbb{R}^N \setminus \{0\} \), \( |x|^{-2}, |x|^{-2} x_1 \) are positive and bounded, taking \( \epsilon > 0 \) small enough,
\[
\frac{4x_1}{|x|^2} \epsilon < \psi_{\infty}(x, \epsilon e_1) = \frac{4x_1 \epsilon}{|x|^2} \frac{1}{1 - 2|x|^{-2} x_1 + |x|^{-2} \epsilon^2}
\]
\[
< \frac{4x_1}{|x|^2} \frac{1}{1 - 2|x|^{-2} x_1}
\]
\[
< \frac{4x_1}{|x|^2} \left(1 + 2\epsilon |x|^{-2} x_1\right) \epsilon,
\]
then
\[
|x|^{2s-N} < |x - \epsilon e_1|^{2s-N} < |x|^{2s-N} \left(1 + 4(N - 2s)|x|^{-2} x_1\right),
\]
\[
\int_0^{\psi_{\infty}(x, \epsilon e_1)} \frac{r^{s-1}}{(r+1)^{\frac{s}{2}}} \, dr < \int_0^{\psi_{\infty}(x, \epsilon e_1)} r^{s-1} \, dr = s^{-1} \psi_{\infty}(x, \epsilon e_1)
\]
\[
< 4^s s^{-1} \frac{x_1^4}{|x|^{2s}} \left(1 + 2\epsilon |x|^{-2} x_1\right) \epsilon^s
\]
\[
< 4^s s^{-1} \frac{x_1^4}{|x|^{2s}} \left(1 + 2^s \epsilon^s |x|^{-s}\right) \epsilon^s
\]
and
\[
\int_0^{\psi_{\infty}(x, \epsilon e_1)} \frac{r^{s-1}}{(r+1)^{\frac{s}{2}}} \, dr > \int_0^{\psi_{\infty}(x, \epsilon e_1)} \left(r^{s-1} - N r^s\right) \, dr
\]
\[
= s^{-1} \psi_{\infty}(x, \epsilon e_1)^s - \frac{N}{1 + s} \psi_{\infty}(x, \epsilon e_1)^{1+s}
\]
\[
> 4^s s^{-1} \frac{x_1^4}{|x|^{2s}} \epsilon^s - \frac{N}{1 + s} \frac{4x_1}{|x|^2} \left(1 + 2\epsilon |x|^{-2} x_1\right)^{1+s} \epsilon^{1+s}
\]
\[
> 4^s s^{-1} \frac{x_1^4}{|x|^{2s}} \epsilon^s \left(1 - \frac{N}{1 + s} 2^{1+s} \epsilon^s\right).
\]

Thus we conclude that
\[
\epsilon^{-s} \mathcal{G}_{s, \infty}(x, \epsilon e_1) > \frac{k N s}{2} 4^s s^{-1} \left(1 - 2^{1+s} \epsilon\right)|x|^{-N} x_1^s
\]
and for some \( c > 0 \)
\[
\epsilon^{-s} \mathcal{G}_{s, \infty}(x, \epsilon e_1) < \frac{k N s}{2} 4^s s^{-1} \left(1 + c \epsilon^s |x|^{-s}\right)|x|^{-N} x_1^s,
\]
which imply that for \( \epsilon > 0 \) small
\[
\left|\epsilon^{-s} \mathcal{G}_{s, \infty}(x, \epsilon e_1) - \mathcal{P}(x)\right| \leq c \epsilon^s |x|^{-N} x_1^s \leq c \epsilon^s |x|^{s-N},
\]
where $c > 0$ is independent of $R$. Thus, (3.2) holds uniformly in any compact set of $\mathbb{R}_+^N$.

Now for any $R > 1 > \sigma > 0$ large enough

$$
\int_{B_R^+ \setminus B_\sigma} \left| \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) - \mathcal{P}_s(x) \right| \, dx < c e^s \int_{B_R^+ \setminus B_\sigma} \left| x \right|^{s-N} \, dx \\
\leq c e^s (R^s - \sigma^s)
$$

and

$$
\int_{B_R^+} \left| \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) - \mathcal{P}_s(x) \right| \, dx \leq \int_{B_R^+} \left( \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) + \mathcal{P}_s(x) \right) \, dx \\
\leq c (\varepsilon R^s + \varepsilon^{-s} \sigma^s + \sigma^s).
$$

Taking $\sigma = 4 \epsilon$ and $R = \varepsilon^{-\frac{1}{2}}$ with $\epsilon \in (0, \frac{1}{8})$, we can see that

$$
\int_{B_R^+} \left| \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) - \mathcal{P}_s(x) \right| \, dx \leq c \left( e^s R^s + \varepsilon^{-s} \sigma^s + \sigma \right) \\
\leq c \left( \varepsilon^{-\frac{1}{2}} + e^s \right) \\
\to 0 \text{ as } \epsilon \to 0^+
$$

and

$$
\int_{\mathbb{R}_+^N \setminus B_R^+} \left| \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) - \mathcal{P}_s(x) \right| \frac{1}{(1 + |x|)^{N+2s}} \, dx \\
\leq \int_{\mathbb{R}_+^N \setminus B_R^+} \left( \varepsilon e^{s} \mathcal{G}_{s, \infty}(x, e \epsilon e_1) + \mathcal{P}_s(x) \right) \frac{1}{(1 + |x|)^{N+2s}} \, dx \\
\leq c \int_{\mathbb{R}_+^N \setminus B_R^+} \left( \varepsilon^{-s} |x - e \epsilon e_1|^{2s-N} + |x|^{s-N} \right) \frac{1}{(1 + |x|)^{N+2s}} \, dx \\
\leq c \left( \varepsilon^{-s} R^{2s-N} + R^{s-N} \right) R^{-2s} \\
\leq c \left( \varepsilon^{-\frac{1}{2}} N^{-s} + \varepsilon^{-\frac{1}{2}} s \right) \\
\to 0 \text{ as } \epsilon \to 0^+.
$$

For $\varphi \in \mathcal{X}_s(\mathbb{R}_+^N)$, we have that $(-\Delta)^s \varphi$ is bounded in $\mathbb{R}_+^N$ and

$$
\left| (-\Delta)^s \varphi(x) \right| \leq \frac{c}{(1 + |x|)^{N+2s}}
$$

for some $c > 0$, then we deduce (3.3).

Therefore, (3.2) holds in $L^1_1(\mathbb{R}_+^N)$. We complete the proof. \hfill \square

In the approximation of the fundamental solution $\mathcal{P}_s$, we need the following regularity results:

**Lemma 3.2** Assume that $w \in C^{2s+\epsilon}(B_1)$ with $\epsilon > 0$ satisfies

$$
(-\Delta)^s w = f \quad \text{in} \quad B_1,
$$

where $f \in C^1(B_1)$. Then for $\beta \in (0, s)$, there exist $c_1, c_2 > 0$ such that

$$
\|w\|_{C^{\beta}(B_{1/2})} \leq c_1 \left( \|w\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|w\|_{L^1_1(\mathbb{R}^N)} \right) \tag{3.5}
$$

and

$$
\|w\|_{C^{2s+\beta}(B_{1/4})} \leq c_2 \left( \|w\|_{L^\infty(B_1)} + \|f\|_{C^{\beta}(B_{1/2})} + \|w\|_{L^1_1(\mathbb{R}^N)} \right). \tag{3.6}
$$
The proof is postponed in the appendix.

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** From (3.5), for any compact set $O$ in $\mathbb{R}^N$, there exists $c > 0$ such that

$$\|\varepsilon^{-s}G_{s,\infty}(\cdot, \epsilon e_1)\|_{C^{2s+\beta}(O)} \leq c_2\|\varepsilon^{-s}G_{s,\infty}(\cdot, \epsilon e_1)\|_{L^1(\mathbb{R}^N, d\mu_s)}.$$ 

Therefore, up to subsequence, there holds

$$\varepsilon^{-s}G_{s,\infty}(\cdot, \epsilon e_1) \to P_s$$

in $C^{2s+\beta}(O)$ as $\epsilon \to 0^+$,

which, together with (3.2) in $L^1(\mathbb{R}^N, d\mu_s)$, implies that for any $x \in \mathbb{R}^N$

$$(-\Delta)^s P_s(x) = \lim_{\epsilon \to 0^+} \varepsilon^{-s}(-\Delta)^s G_{s,\infty}(x, \epsilon e_1) = 0.$$ 

Therefore, we obtain that

$$(-\Delta)^s P_s(x) = 0 \text{ in } \mathbb{R}^N, \quad P_s(x) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$ 

From (3.3), we have that

$$\int_{\mathbb{R}^N} P_s(x)(-\Delta)^s \varphi(x) dx = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \varepsilon^{-s} G_{s,\infty}(x, \epsilon e_1)(-\Delta)^s \varphi(x) dx$$

$$= \lim_{\epsilon \to 0^+} \varepsilon^{-s} \varphi(\epsilon e_1) = \lim_{\epsilon \to 0^+} \frac{\varphi(\epsilon e_1) - \varphi(0)}{\epsilon^s}$$

$$= \frac{\partial^s}{\partial x_1^s} \varphi(0).$$

We complete the proof.

**4 Approximation by the Poisson kernel**

In this section, we approximate the fundamental solution $P_s$ by the Poisson kernel and we consider the solution of

$$\left\{ \begin{array}{ll}
(-\Delta)^s u = 0 & \text{in } \mathbb{R}^N, \\
u = \varepsilon^s \delta_{-\epsilon e_1} & \text{in } \mathbb{R}^N_+ 
\end{array} \right.$$ 

(4.1)

for $\epsilon \in (0, 1)$, where we recall $\mathbb{R}^N_+ = (-\infty, 0] \times \mathbb{R}^{N-1}$.

**Theorem 4.1** Problem (4.1) has a unique nonnegative solution

$$u_\epsilon(x) = \left\{ \begin{array}{ll}
K_s \varepsilon^{-x} x_1^N |x + \epsilon e_1|^{-N} & \text{in } \mathbb{R}^N, \\
\varepsilon^s \delta_{-\epsilon e_1} & \text{in } \mathbb{R}^N_+ 
\end{array} \right.$$ 

Moreover, there holds

$$\int_{\mathbb{R}^N} u_\epsilon(x)(-\Delta)^s \xi(x) dx = \varepsilon^s \int_{\mathbb{R}^N} \xi(x) \Gamma_\epsilon(x) dx, \quad \forall \xi \in C_c(\mathbb{R}^N),$$ 

(4.2)

where

$$\Gamma_\epsilon(x) = \frac{c_{N,s}}{|x + \epsilon e_1|^{N+2s}}, \quad \forall x \in \mathbb{R}^N_+.$$ 

(4.3)

Before the proof of Theorem 4.1, we consider the solution of

$$\left\{ \begin{array}{ll}
(-\Delta)^s u = 0 & \text{in } B_r(\epsilon e_1), \\
u = t^s \delta_{-t e_1} & \text{in } \mathbb{R}^N \setminus B_r(\epsilon e_1), 
\end{array} \right.$$ 

(4.4)

where $t \in (0, 1)$.
Proposition 4.1 Problem \([4.1]\) has a unique nonnegative solution

\[ u_{t,r}(x) = K_{s} \left( \frac{2r x_1 - |x|^2}{2tr + t^2} \right)^s |x + t e_1|^{-N}, \quad \forall x \in B_r(re_1) \]  

(4.5)

and

\[ \int_{B_r(re_1)} u_{t,r}(x)(-\Delta)^s \xi(x)dx = t^s \int_{B_r(re_1)} \xi(x) \Gamma_t(x)dx, \quad \forall \xi \in C^\infty_0(B_r(re_1)). \]  

(4.6)

Proof. Step 1: approximation of the Dirac mass. Let \( g_0 : \mathbb{R}^N \rightarrow [0,1] \) be a radially symmetric decreasing C\(^2\) function with the support in \( B_{\frac{1}{2}}(0) \) such that \( \int_{\mathbb{R}^N} g_0(x)dx = 1 \). For any \( n \in \mathbb{N} \) and \( t \in (0,1) \), we denote \( g_n(x) = n^N g_0(n(x + te_1)) \), \( \forall x \in \mathbb{R}^N \).

Then we certainly have that

\[ g_n \rightharpoonup \delta_{-te_1} \text{ as } n \to +\infty \]

in the distribution sense and for any \( t > 0 \), there exists \( m_t > 0 \) such that for any \( n \geq m_t \),

\[ \text{supp}(g_n) \subset B_{\frac{1}{2}}(-te_1). \]

For \( t > 0 \), problem

\[
\left\{ \begin{array}{ll}
  (-\Delta)^s u = 0 & \text{in } B_r(re_1), \\
  u = t^s g_n & \text{in } \mathbb{R}^N \setminus B_r(re_1)
\end{array} \right.
\]

admits a unique solution \( w_n \).

Denote that

\[ \tilde{g}_n(x) := c_{N,s} \int_{\mathbb{R}^N} \frac{g_n(y)}{|x - y|^{N + 2s}}dy, \quad \forall x \in B_r(re_1). \]

For \( n \geq m_t \), we have that \( \text{supp}(g_n) \subset B_{\frac{1}{2}}(-te_1) \) and then \( \tilde{g}_n \in C^1(B_1(e_1)) \) and

\[ \tilde{w}_n = w_n - t^s g_n \quad \text{in } \mathbb{R}^N. \]

By the definition of the fractional Laplacian, it implies that

\[
(-\Delta)^s \tilde{w}_n(x) = (-\Delta)^s w_n(x) - t^s (-\Delta)^s g_n(x) = c_{N,s} t^s \int_{\mathbb{R}^N} \frac{g_n(z)}{|z - x|^{N + 2s}}dz = t^s \tilde{g}_n(x).
\]

Then \( \tilde{w}_n \) is the unique solution of

\[
\left\{ \begin{array}{ll}
  (-\Delta)^s u = t^s \tilde{g}_n & \text{in } B_r(re_1), \\
  u = 0 & \text{in } \mathbb{R}^N \setminus B_r(re_1)
\end{array} \right.
\]

and

\[ \int_{B_r(re_1)} \tilde{w}_n(x)(-\Delta)^s \xi(x)dx = t^s \int_{B_r(re_1)} \xi(x) \tilde{g}_n(x)dx, \quad \forall \xi \in C^\infty_0(B_r(re_1)). \]  

(4.7)

Step 2: we prove that \( \tilde{g}_n \) converges to \( \Gamma_i \) uniformly in \( B_r(re_1) \) and in \( C^0(B_r(re_1)) \) for \( \theta \in (0,1) \).

It is obvious that \( \tilde{g}_n \) converges to \( \Gamma_x \) every point in \( B_1(e_1) \). For \( x, y \in B_r(re_1) \) and any \( n \in \mathbb{N} \), we have that

\[
|\tilde{g}_n(x) - \tilde{g}_n(y)| = c_{N,s} \int_{B_{\frac{1}{2}}(-te_1)} \left| \frac{1}{|x - z|^{N + 2s}} - \frac{1}{|y - z|^{N + 2s}} \right| g_n(z)dz.
\]
\[
\leq c_{N,s} \int_{B_{3/2}(\cdot - t \epsilon_1)} \frac{|x - z|^{N + 2s} - |y - z|^{N + 2s}}{|x - z|^{N + 2s} - |y - z|^{N + 2s}} g_n(z) \, dz \\
\leq c_{N,s}(N + 2s)|x - y| \int_{B_{3/2}(\cdot - t \epsilon_1)} \frac{|x - z|^{N + 2s - 1} + |y - z|^{N + 2s - 1}}{|x - z|^{N + 2s} - |y - z|^{N + 2s}} g_n(z) \, dz \\
\leq c_3|x - y| \int_{B_{3/2}(\cdot - t \epsilon_1)} g_n(z) \, dz \\
= c_3|x - y|,
\]

where \(c_3 > 0\) independent of \(n\). So \(\{\tilde{g}_n\}_n\) is uniformly bounded in \(C^{0,1}(B_r(\epsilon_1))\). Combining the converging

\[ \tilde{g}_n \to \Gamma_r \text{ every point in } B_r(\epsilon_1). \]

We conclude that \(\tilde{g}_n\) converges to \(\Gamma_r\) uniformly in \(B_r(\epsilon_1)\) and in \(C^0(B_r(\epsilon_1))\) for \(\theta \in (0, 1)\).

Step 3: passing to the limit. We denote \(O_i\) the open sets with \(i = 1, 2, 3\) such that

\[ O_1 \subset \bar{O}_1 \subset O_2 \subset \bar{O}_2 \subset O_3 \subset \bar{O}_3 \subset \mathbb{R}^N_+. \]

By Lemma 3.2, for \(\beta \in (0, s)\), there exist \(c, c' > 0\) independent of \(n\) such that

\[ \|w_n\|_{C^\beta(O_2)} \leq c [\|w_n\|_{L^1(B_1(\epsilon N))} + \|\tilde{g}_n\|_{L^\infty(O_3)} + \|w_n\|_{L^\infty(O_3)}] \leq c' \]

and

\[ \|w_n\|_{C^{2s+\beta}(O_1)} \leq c [\|w_n\|_{L^1(B_1(\epsilon N))} + \|\tilde{g}_n\|_{C^\beta(O_2)} + \|w_n\|_{C^\beta(O_2)}] \leq c'. \]

Therefore, by the Arzela-Ascoli Theorem, there exist \(u_{t,r} \in C^{2s+\epsilon}_{loc} \) in \(B_r(\epsilon_1)\) for some \(\epsilon \in (0, \beta)\) and a subsequence \(\{w_{n_k}\}\) such that

\[ w_{n_k} \to u_{t,r} \text{ in } C^{2s+\epsilon} \text{ locally in } \mathbb{R}^N_+, \text{ as } n_k \to \infty. \]

Passing the limit of (4.7) as \(n_k \to \infty\), we obtain (4.6).

The Poisson kernel of \(B_r\) (see (9)) has the formula

\[
P_{s,r}(x, y) = \begin{cases} 
K_s \left( \frac{x^2 - |y|^2}{|y|^2 - r^2} \right)^s |x - y|^{-N} & \text{if } |y| > r, |x| < r, \\
0 & \text{if not}. 
\end{cases}
\]

(4.8)

The constant \(K_s\) is chosen such that

\[
\int_{\mathbb{R}^N} P_{s,r}(0, y) \, dy = \int_{\mathbb{R}^N \setminus B_r} P_{s,r}(0, y) \, dy = 1.
\]

Then

\[
u_{t,r}(x) = P_{s,r}(x - re_1, -(t + r)e_1) = K_s \left( \frac{2tx_1 - |x|^2}{2tr + t^2} \right)^s |x + te_1|^{-N}.
\]

We complete the proof. \(\square\)

**Proof of Theorem 4.1** Since

\[ B_r(\epsilon_1) \subset B_R(\epsilon_1) \text{ for } R \geq r \text{ and } \mathbb{R}^N_+ = \bigcup_{r > 1} B_r(\epsilon_1). \]

Taking \(t = \epsilon\) in (4.9), we have that

\[
u_{\epsilon,r}(x) = K_s \left( \frac{2x_1 - |x|^2}{2\epsilon + t^2} \right)^s |x + \epsilon e_1|^{-N}, \quad \forall x \in B_r(\epsilon_1),
\]

(4.9)
Finally, this identity holds for any \( \epsilon \) and \( x \). Let
\[
u_{\epsilon}(x) = \begin{cases} K_s \epsilon^{-s} x_1^+ |x + \epsilon e_1|^{-N} & \text{in } \mathbb{R}_1^N, \\ \epsilon^s \delta_{-\epsilon e_1} & \text{in } \mathbb{R}_1^N \end{cases} \tag{4.10}
\]
and then we note that the sequence \( \{\nu_{\epsilon,r}\}_{r > 0} \) has an upper bound \( \nu_{\epsilon} \) in \( \mathbb{R}_1^N \).

By the direct computation, as \( r \to +\infty \), \( u_{\epsilon,r} \) converges to \( u_{\epsilon} \in C^{2s+\theta} \) in any compact set of \( \overline{\mathbb{R}_1^N} \) and in \( L_1^{1}(\mathbb{R}_1^N) \). Therefore, \( u_{\epsilon} \) is a solution of (4.4).

Then for any \( \varphi \in C_c^{\infty}(\mathbb{R}_1^N) \), there exists \( r > 0 \) such that the support of \( \varphi \) is a subset of \( B_r(\epsilon_1) \) and passing to the limit of (4.6), we obtain that
\[
\int_{\mathbb{R}_1^N} \nu_{\epsilon}(x)(-\Delta)^s \xi(x) dx = \epsilon^s \int_{\mathbb{R}_1^N} \xi(x) \Gamma_\epsilon(x) dx, \quad \forall \xi \in C_c^{\infty}(\mathbb{R}_1^N).
\]

Finally, this identity holds for \( \xi \in X_s(\mathbb{R}_1^N) \) since \( C_c^{\infty}(\mathbb{R}_1^N) \) is dense in \( X_s(\mathbb{R}_1^N) \). \( \square \)

From the above proof, we have the following corollary.

**Corollary 4.1** Let \( \mu \) be a Radon measure with the support in \( \mathbb{R}_1^N \) and
\[
\mathcal{P}_{s,\infty}(x,y) = K_s \left( \frac{x_1}{y-1} \right)^s |x-y|^{-N} \quad \text{for } x \in \mathbb{R}_1^N, \ y \in \mathbb{R}_1^N.
\]

Then
\[
\mathcal{P}_{s,\infty}[\mu](x) := \left\{ \begin{array}{ll} \mu & \text{in } \mathbb{R}_1^N \\
\end{array} \right.
\]
is the unique solution of
\[
\left\{ \begin{array}{ll} (-\Delta)^s u = 0 & \text{in } \mathbb{R}_1^N, \\
u = \mu & \text{in } \mathbb{R}_1^N \end{array} \right. \tag{4.11}
\]
and the following distributional identity holds
\[
\int_{\mathbb{R}_1^N} \mathcal{P}_{s,\infty}[\mu](x)(-\Delta)^s \xi(x) dx = \int_{\mathbb{R}_1^N} \xi(x) \Gamma_\mu(x) dx, \quad \forall \xi \in X_s(\mathbb{R}_1^N), \tag{4.12}
\]
where
\[
\Gamma_\mu(x) = c_{N,s} \int_{\mathbb{R}_1^N} \frac{d\mu(y)}{|x-y|^{N+2s}}.
\]

**Proof of Theorem 1.3.** From Lemma 3.1
\[
\epsilon^{-s} G_{s,\infty}(\cdot, \epsilon e_1) \to \mathcal{P}_{s} \quad \text{as } \epsilon \to 0^+
\]
in \( L_1^{1}(\mathbb{R}_1^N) \) and uniformly in any compact set of \( \mathbb{R}_1^N \).

We next show
\[
\epsilon^s \mathcal{P}_{s,\infty}(\cdot, -\epsilon e_1) \to \mathcal{P}_{s} \quad \text{as } \epsilon \to 0^+
\]
in \( L_1^{1}(\mathbb{R}_1^N) \) and uniformly in any compact set of \( \mathbb{R}_1^N \).

Recall that \( u_{\epsilon} \), defined in (4.10), is the solution of (4.4) and
\[
u_{\epsilon}(x) = \epsilon^s \mathcal{P}_{s,\infty}(x, -\epsilon e_1) = K_s x_1^+ |x + \epsilon e_1|^{-N},
\]
then \( \epsilon^s \mathcal{P}_{s,\infty}(\cdot, -\epsilon e_1) \in L_1^{1}(\mathbb{R}_1^N) \) and
\[
\epsilon^s \mathcal{P}_{s,\infty}(x, -\epsilon e_1) - \mathcal{P}_{s}(x) = \frac{x_1^N + \epsilon x}{|x|^N |x + \epsilon e_1|^N} x_1^s.
\]
Thus, for any open set $O, \overline{O} \subset \mathbb{R}^N_+$,

$$e^s \mathcal{P}_{s, \infty}(\cdot, -\epsilon e_1) \to \mathcal{P}_s \quad \text{as} \quad \epsilon \to 0^+ \quad \text{uniformly in} \quad O.$$  \hfill (4.14)

Moreover, let $\sigma = \epsilon^\theta$ with $\theta \in (0, \frac{N}{2N-2})$ and we see that

$$\int_{\mathbb{R}^N_+} x_1^s |x + \epsilon e_1|^{-N} - x_1^s |x|^{-N} \left(1 + |x|\right)^{-N-2s} \, dx \leq 4^N \epsilon^\theta \int_{\mathbb{R}^N_+} \frac{x_1^s}{|x|} + \frac{\epsilon^\theta}{|x|} x_1^s (1 + |x|)^{-N-2s} \, dx$$

$$\leq 4^N \epsilon^\theta \left( \int_{B^+_0(N\epsilon^\theta)} \frac{x_1^s}{|x|} + \frac{\epsilon^\theta}{|x|} dx + \int_{\mathbb{R}^N_+ \setminus B^+_0(0)} \frac{1 + |x|}{|x|} \left(1 + |x|\right)^{-N-2s} \, dx \right)$$

$$\leq 4^N \epsilon^\theta \left( c^s - \frac{\epsilon^\theta}{|x|} \frac{\epsilon^\theta}{|x|} \right) \int_{\mathbb{R}^N} (1 + |x|)^{-N-2s} \, dx$$

$$\to 0 \quad \text{as} \quad \epsilon \to 0^+.$$  

Moreover, we recall that

$$\int_{\mathbb{R}^N_+} u_s(x)(-\Delta)^s \xi(x) \, dx = e^s \int_{\mathbb{R}^N_+} \xi(x) \Delta_s^e(x) \, dx, \quad \forall \xi \in \mathcal{K}_s(\mathbb{R}^N_+).$$  \hfill (4.15)

The left hand side of (4.15) verifies the convergence

$$\int_{\mathbb{R}^N_+} u_s(x)(-\Delta)^s \xi(x) \, dx \to \int_{\mathbb{R}^N_+} \mathcal{P}_s(-\Delta)^s \xi(x) \, dx \quad \text{as} \quad \epsilon \to 0^+$$

and the right hand has

$$e^s \int_{\mathbb{R}^N_+} \xi(x) \Delta_s^e(x) \, dx \to \frac{\partial_s}{\partial x_1^s} \xi(0) \quad \text{as} \quad \epsilon \to 0^+,$$

then we deduce the identity (4.8) from the Poisson kernel.  \hfill \qedsymbol

## 5 The solution blowing up on whole boundary

### 5.1 Distributional indentity

**Proof of Theorem 1.4**  

Let

$$\mu_t = \delta_{-t} (y_1) (t^s \omega_{\mathbb{R}^{N-1}}(y')) \quad \text{with} \quad t > 0,$$

where $\omega_{\mathbb{R}^{N-1}}(y')$ is the Hausdorff measure of $\mathbb{R}^{N-1}$ and $d\omega_{\mathbb{R}^{N-1}}(y') = dy'$ and $\delta_t(h) = h(t)$ for $h \in C_0(\mathbb{R})$.  

From the Poisson kernel expression, we obtain that

$$Q_{s,t}(x_1, x') = \int_{\mathbb{R}^N} \mathcal{P}_{s, \infty}(x, y) d\mu_t$$

$$= K_s x_1^s \int_{\mathbb{R}^{N-1}} \left( (x_1 + t)^2 + |x' - y'|^2 \right)^{-\frac{N}{2}} \, dy'$$

$$= C_s x_1^s (x_1 + t)^{-1}$$

$$\to C_s x_1^{s-1} \quad \text{as} \quad t \to 0^+$$

16
and the above convergence holds in $L^1_1(\mathbb{R}^N)$ and uniformly in any compact set of $\mathbb{R}^N_+$, where
\[
C_1 = K_s \int_{\mathbb{R}^{N-1}} (1 + |z'|^2)^{-\frac{N}{2}} dz'.
\]
Recall that
\[
\Gamma_f(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N+2s}} d\mu(y)
= c_{N,s} \int_{\mathbb{R}^{N-1}} \frac{t^s}{((x_1 + t)^2 + |x' - y'|^2)^{\frac{N-s}{2}}} dy' = C_2 t^s (x_1 + t)^{-2s},
\]
where
\[
C_2 = c_{N,s} \int_{\mathbb{R}^{N-1}} (1 + |z'|^2)^{-\frac{N-s}{2}} dz'.
\]
Then (4.12) with $\mu = \mu_t$, implies that for any $\xi \in \mathcal{D}_0(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} Q_{s,t}(x_1, x') (-\Delta)^s \xi(x) dx = C_2 t^s \int_{\mathbb{R}^N} \xi(x)(x_1 + t)^{-2s} dx.
\]
Note that
\[
C_2 t^s \int_{\mathbb{R}^N} \xi(x)(x_1 + t)^{-2s} dx \rightarrow C_2 \int_{\mathbb{R}^{N-1}} \frac{\partial^s}{\partial x_1^s} \xi(0, x') dx'.
\]
Therefore, we obtain that
\[
\int_{\mathbb{R}^N} x_1^{-1} (-\Delta)^s \xi(x) dx = \frac{C_2}{C_1} \int_{\mathbb{R}^{N-1}} \frac{\partial^s}{\partial x_1^s} \xi(0, x') dx' \quad \text{for} \quad \xi \in \mathcal{D}_0(\mathbb{R}^N),
\]
where
\[
\frac{C_2}{C_1} = \frac{c_{N,s} \int_{\mathbb{R}^{N-1}} (1 + |z'|^2)^{-\frac{N-s}{2}} dz'}{K_s \int_{\mathbb{R}^{N-1}} (1 + |z'|^2)^{-\frac{N}{2}} dz'}
= \frac{c_{N,s} \omega_{N-2} B \left( \frac{N+2s}{2}, \frac{N-1}{2} \right)}{K_s \omega_{N-2} B \left( \frac{N-s}{2}, \frac{N-1}{2} \right)} = 2\sqrt{\pi} \frac{s}{\sin(\pi s)} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)},
\]
here $B$ is the beta function and we use the fact that for $M \in \mathbb{N}$ and $\tau < -\frac{M}{2}$,
\[
\int_{\mathbb{R}^M} (1 + |z|^2)^{\tau} dz = \omega_M \int_0^{+\infty} (1 + r^2)^{\tau} r^{M-1} dr
= \omega_M \int_0^1 t^{-\frac{M}{2}} \gamma(M-1, \frac{M}{2}) t^{M-1} dt = \omega_M B(-\tau - \frac{M}{2}, \frac{M}{2})
= \omega_M \frac{\Gamma(-\tau - \frac{M}{2}) \Gamma(\frac{M}{2})}{\Gamma(-\tau)},
\]
where $\omega_M$ is the volume of the unit sphere in $\mathbb{R}^M$. \hfill \Box

Let
\[
\mathcal{R}_s(x) = (x_1)^s
\]
and it is known that $\mathcal{R}_s$ is a solution of
\[
(-\Delta)^s u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N_+.
\]
Next we use our approximation to show this fact.
Corollary 5.1 The function $R_s$ is a solution of (5.1) and it verifies that
\[
\int_{\mathbb{R}^N_+} R_s(x)(-\Delta)^s \varphi(x)dx = 0 \quad \text{for any } \varphi \in \mathcal{X}_s(\mathbb{R}^N_+).
\] (5.2)

Proof. We take
\[
\nu_t = \delta_{-t}(y_1) \left(t^{1+s} \omega_\mathbb{R}^N_-(y')\right) \quad \text{with } t > 0.
\]

From the Poisson kernel expression, we obtain that
\[
\int_{\mathbb{R}^N_+} P_s(x,y)du_t = K_s x^*_t \int_{\mathbb{R}^N_+} (|x_1 + t|^2 + |x' - y'|^2)^{-\frac{N}{2}} dx'
\]
\[
= C_1 x^*_t(x_1 + t)^{-1}
\]
\[
\to C_1 x^*_1 \quad \text{as } t \to 0^+
\]
and the above convergence holds in $L^1(\mathbb{R}^N_+)$ and uniformly in any compact set of $\mathbb{R}^N_+$, where
\[
C_1 = K_s \int_{\mathbb{R}^N_+} (1 + |z'|^2)^{-\frac{N}{2}} dz'.
\]

Recall that for $x \in \mathbb{R}^N_+$
\[
\Gamma_{\nu_t}(x) = c_{N,s} \int_{\mathbb{R}^N_+} \frac{t^{1+s}}{((x_1 + t)^2 + |x' - y'|^2)^{\frac{N}{2}}} dy' = C_4 t^{1+s}(x_1 + t)^{-2s}.
\]
From (4.12) with $\mu = \nu_t$, we obtain that for any $\xi \in \mathcal{X}_s(\mathbb{R}^N_+)$,
\[
\int_{\mathbb{R}^N_+} Q_{s,t}(x_1,x')(\Delta)^s \xi(x)dx = C_4 t^{1+s} \int_{\mathbb{R}^N_+} \xi(x)(x_1 + t)^{-2s}dx.
\]

Note that
\[
C_4 t^{1+s} \int_{\mathbb{R}^N_+} \xi(x)(x_1 + t)^{-2s}dx \to 0 \quad \text{as } t \to 0^+.
\]

Therefore, we obtain that for any $\xi \in \mathcal{X}_s(\mathbb{R}^N_+)$
\[
\int_{\mathbb{R}^N_+} x^*_1(\Delta)^s \xi(x)dx = 0.
\]
We complete the proof. \hfill \Box

5.2 More $s$-harmonic functions in $\mathbb{R}^N_+$

It is shown in [14, Theorem 5.1] that the nonnegative solution of (5.1) only has the form
\[
u = cR_s \quad \text{with } c \geq 0.
\]

In this subsection, we will show more $s$-harmonic functions in $\mathbb{R}^N_+$. Without the restriction of positivity.

To this end, we consider the functions with the separable variables with the form $u(x_1,x') = x^*_1 h(x')$ and we can take an equivalent definition of the fractional Laplacian in the principle value sense
\[
(\Delta)^s u(x_1,x') = c_{N,s} \lim_{\epsilon \to 0^+} \int_{(-\epsilon,\epsilon)_x \cup (\epsilon,\infty)} \int_{B^*_x \setminus B^*_2} \frac{u(x_1,x') - u(x_1 + z_1, x' + z'_1)}{|z|^{N+2s}} dz'dz_1,
\] (5.3)

where $B^*_r$ is the ball centered at the origin with radius $r$ in $\mathbb{R}^{N-1}$. 

18
Proof of Theorem 1.5. Let $h_0 \in \mathcal{H}^s(\mathbb{R}^{N-1})$ with $N \geq 3,$

$$u_0(x_1, x') = (x_1)_+^s h_0(x')$$

and we need prove that

$$(-\Delta)^s u_0 = 0 \quad \text{in} \quad \mathbb{R}^N_+.$$ 
We use the definition (5.3) to obtain that

$$\frac{1}{c_{N,s}} (-\Delta)^s u_0(x_1, x')$$

$$= \lim_{\epsilon \to 0^+} \int_{(-\frac{1}{2}, -\epsilon) \cup (\epsilon, \frac{1}{2})} \left( (x_1)_+^s - (x_1 + z_1)_+^s \right) \int_{B_1^+ \setminus B_\epsilon^+} \frac{h_0(x')}{(1 + |z'|^2)^{\frac{N+s}{2}}} dz' dz_1$$

$$+ \lim_{\epsilon \to 0^+} \int_{(-\frac{1}{2}, -\epsilon) \cup (\epsilon, \frac{1}{2})} (x_1 - z_1)_+^s \int_{B_1^+ \setminus B_\epsilon^+} \frac{h_0(x') - h_0(x' + z')}{{|z'|^2}^{\frac{N+s}{2}}} dz' dz_1$$

$$= h_0(x') \lim_{\epsilon \to 0^+} \int_{(-\frac{1}{2}, -\epsilon) \cup (\epsilon, \frac{1}{2})} \left( (x_1)_+^s - (x_1 + z_1)_+^s \right) \int_{B_1^+ \setminus B_\epsilon^+} \frac{1}{{(1 + |z'|^2)^{\frac{N+s}{2}}}} dz'$$

$$+ \lim_{\epsilon \to 0^+} \int_{(-\frac{1}{2}, -\epsilon) \cup (\epsilon, \frac{1}{2})} (x_1 - z_1)_+^s \int_{B_1^+ \setminus B_\epsilon^+} \left( h_0(x') - h_0(x' + z') \right) K_{z_1}(|z'|) dz' dz_1.$$ 

Note that

$$\int_{(-\frac{1}{2}, -\epsilon) \cup (\epsilon, \frac{1}{2})} \frac{(x_1)_+^s - (x_1 + z_1)_+^s}{{|z_1|^{1+2s}}} dz_1 \to 0 \quad \text{as} \quad \epsilon \to 0^+$$

and

$$\int_{B_1^+ \setminus B_\epsilon^+} \frac{1}{{(1 + |z'|^2)^{\frac{N+s}{2}}}} dz' \to \int_{\mathbb{R}^{N-1}} \frac{1}{{(1 + |z'|^2)^{\frac{N+s}{2}}}} dz' \quad \text{as} \quad \epsilon \to 0^+.$$ 

For any $z_1, K_{z_1}$ verifies (2.3) in $\mathbb{R}^{N-1},$ see Remark 2.1 part (i), and from the proof of Proposition 2.1 we have that

$$\int_{B_1^+ \setminus B_\epsilon^+} \left( h_0(x') - h_0(x' + z') \right) K_{z_1}(|z'|) dz' = 0 \quad \text{for} \quad \epsilon \in (0, 1).$$

Then passing to the limit, we obtain that

$$(-\Delta)^s u_0 = 0 \quad \text{in} \quad \mathbb{R}^N_+.$$ 

Set

$$v_0(x_1, x') = (x_1)_+^{s-1} h_0(x'),$$

via a similar proof, we can obtain that

$$(-\Delta)^s v_0 = 0 \quad \text{in} \quad \mathbb{R}^N_+.$$ 

We obtain that $\{ (x_1)_+^s v(x') : \ v \in \mathcal{H}^s(\mathbb{R}^{N-1}) \} \cup \{ (x_1)_+^{s-1} v(x') : \ v \in \mathcal{H}^s(\mathbb{R}^{N-1}) \} \subset \mathcal{H}^s(\mathbb{R}^N_+).$ We complete the proof. \hfill \Box

A Appendix: Proof of Theorem 3.1

The regularities of fractional Poisson problems have been studied in [3, 21, 25].
Lemma A.1 Let $f \in L^\infty(B_1)$ and $u : \mathbb{R}^N \to \mathbb{R}$ be a bounded function verifying

$$(-\Delta)^s u = f \text{ in } B_1.$$ 

Then for $\beta \in (0, s)$

$$\|u\|_{C^{\beta}(B_{\frac{1}{2}})} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(\mathbb{R}^N)}).$$

Proof. It follows by [5, Theorem 12.1] replacing the maximal and a minimal operator by the fractional Laplacian. \(\square\)

For $k \in \mathbb{N}$ and $\beta \in (0, 1)$, denote

$$\|u\|_{C^{2s, \beta}(\Omega)}^s = \sum_{|\gamma| = k} \sup_{x, y \in \Omega} d_{x,y}^{k+\beta} \frac{D^\gamma u(x) - D^\gamma u(y)}{|x - y|^\beta}$$

and for $b > 0$

$$\|u\|_{C^{\beta}(\Omega)}^{(b)} = \sup_{x \in \Omega} d_x^b |u(x)| + \sup_{x, y \in \Omega} d_{x,y}^{\beta} \frac{D^\gamma u(x) - D^\gamma u(y)}{|x - y|^\beta},$$

where $\Omega$ is a bounded $C^{k+1}$ domain and $d_{x,y}^{\beta} = \min\{\rho(x), \rho(y)\}$.

Lemma A.2 [14, Theorem 12.2.1] (also see [24]) Let $f \in C^\beta(\Omega)$ and $u : \mathbb{R}^N \to \mathbb{R}$ be a function verifying

$$(-\Delta)^s u = f \text{ in } \Omega.$$ 

Then

$$\|u\|_{C^{2\alpha, \beta}(\Omega)} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{\beta}(\Omega)}).$$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1 We denote $v = u \eta$, where $\eta : \mathbb{R}^N \to [0, 1]$ is a $C^\infty$ function such that

$$\eta = 1 \text{ in } B_{\frac{1}{2}} \text{ and } \eta = 0 \text{ in } B_{\frac{1}{4}}^c.$$ 

Then $v \in C^{2s+\alpha}(\mathbb{R}^N)$ and for any $x \in B_{\frac{1}{2}}$, $\epsilon \in (0, \frac{1}{8}),$

$$(-\Delta)^{\alpha}_\epsilon v(x) = (-\Delta)^{\alpha}_\epsilon w(x) + c_{N, s} \int_{\mathbb{R}^N \setminus B_{\epsilon}} \frac{(1 - \eta(x + y))w(x + y)}{|y|^{N+2s}} dy.$$ 

Together with the fact of $\eta(x + y) = 1$ for $y \in B_{\epsilon}$, we derive that

$$\int_{\mathbb{R}^N \setminus B_{\epsilon}} \frac{(1 - \eta(x + y))w(x + y)}{|y|^{N+2s}} dy = \int_{\mathbb{R}^N} \frac{(1 - \eta(x + y))w(x + y)}{|y|^{N+2s}} dy =: h_1(x),$$

thus,

$$(-\Delta)^{\alpha} v = h + c_{N, s} h_1 \text{ in } B_{\frac{1}{2}}.$$ 

(i) For $x \in B_{\frac{1}{2}}$ and $z \in \mathbb{R}^N \setminus B_{\frac{1}{2}}$, there holds

$$|z - x| \geq |z| - |x| \geq |z| - \frac{1}{4} \geq \frac{1}{12}(1 + |z|),$$

which implies that

$$|h_1(x)| = \left| \int_{\mathbb{R}^N} \frac{(1 - \eta(z))w(z)}{|z - x|^{N+2s}} dz \right| \leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}} \frac{|w(z)|}{|z - x|^{N+2s}} dz \leq 12^{N+2s} \int_{\mathbb{R}^N} \frac{|w(z)|}{(1 + |z|)^{N+2s}} dz = 12^{N+2s} \|w\|_{L^1(\mathbb{R}^N)}.$$
From Lemma A.1, there exists $c_8 > 0$ such that
\[
\|v\|_{C^2(B_{1/4})} \leq c_8(\|v\|_{L^\infty(B_1)} + \|h + h_1\|_{L^\infty(B_{1/2})}) \\
\leq c_8(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|h_1\|_{L^\infty(B_{1/2})}) \\
\leq c_8(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|w\|_{L_j^2(\mathbb{R}^N)}),
\]
where $c_9 = 12^{N+2\alpha}c_8$. Combining with $w = v$ in $B_{1/2}$, we obtain (3.5).

(ii) For $x, y \in B_{1/2}$ and $|z| > 1/4$, there exists $c > 0$ such that
\[
\frac{1}{|z-x|^{N+2s}} \leq \frac{c}{|z|^{N+2s}} \quad \text{and} \quad \frac{1}{|z-y|^{N+2s}} \leq \frac{c}{|z|^{N+2s}}
\]
and
\[
|z-x|^{N+2s} - |z-y|^{N+2s} \leq c|z|^{N-2s-1}|x-y|,
\]
thus,
\[
|h_1(x) - h_1(y)| = \left| \int_{\mathbb{R}^N} \frac{\eta(x+z)w(x+z) - \eta(y+z)w(y+z)}{|z|^{N+2s}} dz \right|
\]
\[
= \left| \int_{\mathbb{R}^N \setminus B_{1/2}} \eta(z)w(z) \left( \frac{1}{|z-x|^{N+2s}} - \frac{1}{|z-y|^{N+2s}} \right) dz \right|
\]
\[
\leq \int_{\mathbb{R}^N \setminus B_{1/2}} |w(z)| \left[ \frac{|z-x|^{N+2s} - |z-y|^{N+2s}}{|z-x|^{N+2s} |z-x|^{N+2s}} \right] dz
\]
\[
\leq c \int_{\mathbb{R}^N \setminus B_{1/2}} \frac{|w(z)| |x-y|}{1+|z|^{N+2s+1}} dz
\]
\[
\leq c \|w\|_{L^1(\mathbb{R}^N)} |x-y|.
\]
Now we apply Lemma A.2 with $\Omega = B_{1/2}$ to obtain that
\[
\|v\|_{C^{2+\beta}(B_{1/5})} \leq c\|v\|_{C^{2+\beta}(B_{1/2})} \leq C(\|v\|_{L^\infty(\mathbb{R}^N)} + \|h + h_1\|_{C^\beta(B_{1/2})})
\]
\[
\leq c(\|w\|_{L^\infty(B_1)} + \|h\|_{C^\beta(B_{1/2})} + \|w\|_{L_j^2(\mathbb{R}^N)}).
\]
which implies that
\[
\|w\|_{C^{2+\beta}(B_{1/5})} \leq c(\|w\|_{L^\infty(B_1)} + \|h\|_{C^\beta(B_{1/2})} + \|w\|_{L_j^2(\mathbb{R}^N)})
\]
by the fact that $w = v$ in $B_{1/2}$. Then (3.6) is proved. \hfill \square

**Acknowledgement:** This work is supported by the Natural Science Foundation of China, No. 12071189, 12001252, by Jiangxi Province Science Funds, No. 20212ACB211005, 20202ACBL201001, by the Science and Technology Research Project of Jiangxi Provincial Department of Education, No. GJJ200307, GJJ200325.

**References**

[1] B. Abdellaoui, I. Peral, A. Primo, F. Soria, On the KPZ equation with fractional diffusion: global regularity and existence results, *J. Diff. Eq.* 312, 65–147 (2022).
[2] D. Armitage, S. Gardiner, Classical potential theory. *Springer Monographs in Mathematics*. Springer-Verlag London, Ltd., London, 2001.

[3] N. Abatangelo, Large $s$-harmonic functions and boundary blow-up solutions for the fractional Laplacian, *Discr. Contin. Dyn. Syst. 35*(12), 5555–5607 (2015).

[4] B. Barrios, A. Figalli, X. Ros-Oton, Free boundary regularity in the parabolic fractional obstacle problem, *Comm. Pure Appl. Math. 71*(10), 2129–2159 (2018).

[5] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math. 62*(5), 597–638 (2009).

[6] B. Barrios, A. Figalli, X. Ros-Oton, Global regularity for the free boundary in the obstacle problem for the fractional Laplacian. *Amer. J. Math. 140*(2), 415–447 (2018).

[7] M. Bhakta, P-T. Nguyen, On the existence and multiplicity of solutions to fractional Lane-Emden elliptic systems involving measures, *Adv. Nonlinear Anal. 9*(1), 1480–1503 (2020).

[8] K. Bogdan, T. Byczkowski, Potential theory for the $\alpha$-stable Schrödinger operator on bounded Lipschitz domains. *Studia Math. 133*(1), 53–92 (1999).

[9] R. M. Blumenthal, R. K. Getoor and D. B. Ray, On the distribution of first hits for the symmetric stable processes, *Trans. Amer. Math. Soc. 99*, 540–554 (1961).

[10] H. Chen, H. Hajaiej, Y. Wang, On a class of semilinear fractional elliptic equations involving outside Dirac data, *Nonlinear Anal. 125*, 639–668 (2015).

[11] H. Chen, T. Weth, The Poisson problem for the fractional Hardy operator: Distributional identities and singular solutions, *Trans. Amer. Math. Soc. 374*, 6881–6925 (2021).

[12] H. Chen, L. Véron, Initial trace of positive solutions to fractional diffusion equations with absorption, *J. Funct. Anal. 276*, 1145–1200 (2019).

[13] H. Chen, L. Véron, Weakly and strongly singular solutions of semilinear fractional elliptic equations. *Asymptot. Anal. 88*(3), 165–184 (2014).

[14] W. Chen, Y. Li, P. Ma, The fractional Laplacian, *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ* 2020.

[15] Z.-Q. Chen, R. Song: Estimates on Green functions and Poisson kernels for symmetric stable processes, *Math. Ann. 312*(3), 465–501 (1998).

[16] M. Fall, Entire $s$-harmonic functions are affine, *Proc. Amer. Math. Soc. 144*(6), 2587–2592 (2016).

[17] M. Fall, T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half-space, *Commun. Contemp. Math. 1550012* (2015).

[18] A. Gmira, L. Véron. Boundary singularities of solutions of nonlinear elliptic equations, *Duke Math. J. 64*, 271–324 (1991).

[19] T. Kulczycki, Reduced measures for semilinear elliptic equations involving Dirichlet operators, *Calc. Var. PDE 55*(4), Art. 78, 27 pp (2016).

[20] T. Kuusi, G. Mingione, Y. Sire, Nonlocal equations with measure data, *Comm. Math. Phys. 337*(3), 1317–1368 (2015).

[21] R. Musina and A.I. Nazarov, On fractional Laplacians. *Comm. Part. Diff. Eq. 39*, 1780–1790 (2014).

[22] P-T. Nguyen, L. Véron, Boundary singularities of solutions to semilinear fractional equations, *Adv. Nonlinear Stud. 18*(2), 237–267 (2018).

[23] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl. 101*(3), 275–302 (2014).
[25] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* 60, 67–112 (2007).

[26] S. Terracini, G. Tortone and S. Vita, On $s$-harmonic functions on cones, *Anal. PDE* 11(7), 1653–1691 (2018).

[27] L. Véron, C. Yarur, Boundary value problems with measures for elliptic equations with singular potentials, *J. Funct. Anal.* 262, 733–772 (2012).

[28] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, Vol. I, 593–712, *Handb. Differ. Equ., North-Holland, Amsterdam* (2004).