A NOTE ON EIGENVALUE BOUNDS FOR SCHröDINGER OPERATORS

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Abstract. We obtain a new bound on the location of eigenvalues for a non-self-adjoint Schrödinger operator with complex-valued potentials by obtaining a weighted $L^2$ estimate for the resolvent of the Laplacian.

1. Introduction

In this paper we are concerned with bounds on the location of eigenvalues of a non-self-adjoint Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^d$ with complex-valued potentials $V$ in terms of the size of them.

When dimension $d = 1$ it turns out that every eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ satisfies

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{\mathbb{R}} |V(x)| dx$$

(1.1)

which shows that $\lambda$ lies in a disk whose radius is controlled by the $L^1$-norm of $V$ (see [1, 5]). There have been also attempts [10, 16, 19] to obtain this type of bounds in higher dimensions but away from the positive half-axis $[0, \infty)$. In [16], the following natural extension of (1.1),

$$|\lambda|^\gamma \leq C_{d, \gamma} \int_{\mathbb{R}^d} |V(x)|^{\gamma + d/2} dx,$$

(1.2)

was left unsolved for $d \geq 2$ and $0 < \gamma \leq d/2$. In fact, if $V$ is real-valued, it follows from Sobolev inequalities that (1.2) holds for $\gamma \geq 1/2$ if $d = 1$ and $\gamma > 0$ if $d \geq 2$ (see [13, 17]). However, the problem of obtaining (1.2) becomes much more difficult if $V$ is allowed to be complex-valued.

It is a remarkable observation of Frank [8] that the following resolvent estimate due to Kenig-Ruiz-Sogge [14] can be used to obtain the bound (1.2) based on the Birman-Schwinger principle:

$$\|(-\Delta - z)^{-1} f\|_{L^{p'}} \leq C|z|^{-\frac{d}{2} + \frac{d}{p} - 1} \|f\|_{L^p},$$

(1.3)

where $2d/(d + 2) < p \leq 2(d + 1)/(d + 3)$ and $d \geq 2$. Indeed, the range of $p$ in (1.3) implies (1.2) whenever $0 < \gamma \leq 1/2$. It should be noted here that (1.3) cannot hold...
for $2(d+1)/(d+3) < p < 2d/(d+1)$ corresponding to $1/2 < \gamma < d/2$ in (1.2). This means that (1.2) cannot be obtained for $\gamma > 1/2$ using $L^p - L^{p'}$ resolvent estimates. Recently, a different replacement of the bound (1.2) for the remaining case $\gamma > 1/2$ was given by Frank [9] again:

$$
\delta(\lambda)\gamma^{-1/2}|\lambda|^{1/2} \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} \, dx
$$

which is weaker than (1.2) since $\delta(\lambda) := \text{dist}(\lambda, [0, \infty)) \leq |\lambda|$. See also [11] for another replacement.

By the way, the class $L^p$ is too small to contain the inverse square potential $V(x) = 1/|x|^2$ which has attracted considerable interest from mathematical physics. This is because the Schrödinger operator $-\Delta + 1/|x|^2$ is physically related to the Hamiltonian of a spin-zero quantum particle in a Coulomb field ([3]). In this regard, we shall consider from now on a wider class of potentials where one can consider stronger singularities of the type $1/|x|^2$.

For this we first need to introduce the Kerman-Saywer class $\mathcal{KS}_\alpha$ which is defined for $0 < \alpha < d$ if

$$
\|V\|_{\mathcal{KS}_\alpha} := \sup_Q \left( \int_Q |V(x)| \, dx \right)^{-1/\alpha} \int_Q \int_Q \frac{|V(x)V(y)|}{|x-y|^{d-\alpha}} \, dx \, dy < \infty,
$$

where the sup is taken over all dyadic cubes $Q$ in $\mathbb{R}^d$. This class particularly when $\alpha = 2$ is closely related to the global Kato and Rollnik classes which are fundamental in spectral and scattering theory and satisfy

$$
\|V\|_\mathcal{T} := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-d}} \, dy < \infty
$$

and

$$
\|V\|_\mathcal{R} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|V(x)V(y)|}{|x-y|^2} \, dx \, dy < \infty,
$$

respectively. Indeed, $\mathcal{T} \subset \mathcal{KS}_2$ and $\mathcal{R} \subset \mathcal{KS}_2$. Also, $\mathcal{KS}_\alpha$ is wider than the Morrey-Campanato class $\mathcal{LC}_{\alpha,p}$ which is defined for $\alpha > 0$ and $1 < p \leq d/\alpha$ by

$$
\|f\|_{\mathcal{LC}_{\alpha,p}} := \sup_{x,r} r^{-\alpha} \left( r^{-d} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p} < \infty.
$$

In general, $1/|x|^\alpha \subset L^{d/\alpha,\infty} \subset L_{\alpha,p} \subset \mathcal{KS}_\alpha$ for $1 < p < d/\alpha$ (see [2], Subsection 2.2).

Our result below gives bounds like (1.2) for the eigenvalues of the Schrödinger operator in terms of the size $\|V\|_{\mathcal{LS}_\alpha}^{1/\beta}$ with $\beta = (2\alpha - d + 1)/2$.

**Theorem 1.1.** Let $d \geq 2$. If $d-1 \leq \alpha < d$ for $d \geq 3$ and if $3/2 \leq \alpha < 2$ for $d = 2$, then any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of the Schrödinger operator $-\Delta + V(x)$ satisfies

$$
|\lambda|^{\beta+1/\beta} \leq C\|V\|_{\mathcal{KS}_\alpha}^{1/\beta}
$$

where $\beta = (2\alpha - d + 1)/2$. 


Remark 1.2. Particularly when $\beta = 1$, $\alpha = 2$ if $d = 3$ and $\alpha = 3/2$ if $d = 2$. In this case, (1.4) becomes
\[
|\lambda|^{1/4} \leq C\|V\|_{KS_{3/2}}
\]
when $d = 2$, and
\[
1 \leq C\|V\|_{KS_{3/2}} \quad (1.5)
\]
when $d = 3$. In particular, the bound (1.5) has the following consequence: if $\|V\|_{KS_3}$ is sufficiently small, then the Schrödinger operator has no eigenvalue in $\mathbb{C} \setminus [0, \infty)$.

Remark 1.3. Our theorem considers singularities of the type $a/|x|^2$ with a small $a > 0$ (this smallness condition is a natural restriction - see [18], p. 172). Indeed, when $2\beta = \alpha$, the condition $\beta = (2\alpha - d + 1)/2$ becomes $\alpha = d - 1$, which implies
\[
1 \leq Ca\|x^{-\alpha}\|_{KS_{\alpha}}^{1/\beta} \leq Ca
\]
when applying (1.4) with $d \geq 3$ to $a/|x|^2$. This means that the Schrödinger operator $-\Delta + a/|x|^2$ with a small $a > 0$ has no eigenvalue in $\mathbb{C} \setminus [0, \infty)$.

Remark 1.4. It should be noted that the above theorem improves the following previous result due to Frank [5] that
\[
|\lambda|^{2\gamma/\alpha} \leq C\|V\|_{L^{2d/(2\gamma+d),p}} \quad (1.6)
\]
for $0 < \gamma < 1/2$ and $\frac{(d-1)(2\gamma+d)}{2(d-2\gamma)} < p \leq \gamma + \frac{d}{2}$ (see Theorem 3 there). Indeed, if $V \in L^{k,p}$, then $V^\beta \in L^{k\beta,p/\beta} \subset KS_{\delta\beta}$ when $p > \beta$, and
\[
\|V^\beta\|_{KS_{\delta\beta}}^{1/\beta} \leq \|V\|_{L^{k,\beta}}^{1/\beta} = \|V\|_{L^{k,p}}.
\]
Using this fact and setting $\alpha = \delta\beta$ with $\delta = 2d/(2\gamma+d)$ and $\beta = (2\alpha - d + 1)/2$, one can easily check that (1.4) implies (1.6), as follows:
\[
|\lambda|^{2\gamma/\alpha} = |\lambda|^{\frac{\alpha-d+1}{\alpha}} \leq C\|V^\beta\|_{KS_{\alpha}}^{1/\beta} \leq C\|V\|_{L^{2d/(2\gamma+d),p}}
\]
under the same conditions on $\gamma$ and $p$ as in (1.6).

The rest of this paper is organized as follows: In Section 2 we prove Theorem 1.1 based on the Birman-Schwinger principle. In order to make use of this principle, we obtain weighted $L^2$ resolvent estimates with weights in the class $KS_{\alpha}$. The key point here is that the integral kernel of $(-\Delta - z)^{-\gamma}$ can be controlled by that of the fractional integral operator which will be shown in the final section, Section 3. This observation leads us to apply the known weighted estimates for this operator to obtain the desired resolvent estimates.

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2. Proof of Theorem 1.1

Following [8], we shall use the following Birman-Schwinger principle: if $\lambda \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of the Schrödinger operator $-\Delta + V$, then $-1$ is an eigenvalue of the Birman-Schwinger operator

$$A := V^{1/2} (-\Delta - \lambda)^{-1} |V|^{1/2},$$

where $V^{1/2} = \frac{V}{|V|^{1/2}}|V|^{1/2}$.

This principle implies that the norm of this operator in $L^2$ is at least 1. Indeed,

$$\|A\| = \sup_{\|\psi\| \leq 1} \|A\psi\|_2 \geq \sup_{\|\psi\| \leq 1} \|\lambda\psi\|_2 \geq \sup_{\|\psi\| \leq 1} |\lambda| \geq |1| = 1,$$

where $\psi_\lambda$ is an eigenfunction corresponding to an eigenvalue $\lambda$. Hence the proof of (1.4) is now reduced to showing that

$$\|V^{1/2} (-\Delta - \lambda)^{-1} |V|^{1/2}\| \leq C|\lambda|^{-\frac{2\alpha-d+1}{d2}} \|V\|_{KS_n}^{\frac{1}{d}} \|f\|_{L^2}$$

which follows immediately from the following resolvent estimate:

**Proposition 2.1.** Let $d \geq 2$. If $d-1 \leq \alpha < d$ for $d \geq 3$ and if $3/2 \leq \alpha < 2$ for $d = 2$, then

$$\|\|V|^{\frac{1}{2}} (-\Delta - \lambda)^{-1} |V|^{\frac{1}{2}} f\|_{L^2} \leq C|\lambda|^{-\frac{2\alpha-d+1}{d2}} \|V\|_{KS_n}^{\frac{1}{d}} \|f\|_{L^2}$$

(2.1)

where $\beta = (2\alpha - d + 1)/2$.

**Proof of Proposition 2.1.** We only consider the case where $d \geq 3$ because the case $d = 2$ would follow obviously from the same argument.

We will use Stein’s complex interpolation (cf. [21]). Following [14], we first consider the analytic family of operators,

$$T_\zeta = |V|^{\frac{\zeta}{2}} (-\Delta - \lambda)^{-\zeta} |V|^{\frac{\zeta}{2}},$$

with $0 \leq \Re \zeta \leq (d+1)/2$. Then we shall obtain (2.1) with the assumption that $|\lambda| = 1$, as follows:

$$\|\|V|^{\frac{1}{2}} (-\Delta - \lambda)^{-1} |V|^{\frac{1}{2}} f\|_{L^2} \leq C\|V\|_{KS_n}^{\frac{1}{d}} \|f\|_{L^2}$$

(2.2)

under the same conditions on $\alpha$ and $\beta$ as in (2.1). Once we obtain (2.2), the desired estimate (2.1) follows from the scaling. Indeed, note first that

$$R(\lambda) f(x) := (-\Delta - \lambda)^{-1} f(x) = |\lambda|^{-1} (-\Delta - \lambda/|\lambda|)^{-1} [f(|\lambda|^{-1/2})](|\lambda|^{1/2} x).$$

So, if we have (2.2) with $|\lambda| = 1$, then

$$\|R(\lambda) f\|^2_{L^2(|V|)} = |\lambda|^{-2} \|R(\lambda/|\lambda|) f(|\lambda|^{-1/2})\|^2_{L^2(|V|(|\lambda|^{-1/2})|^1)}$$

$$\leq C|\lambda|^{-2} \|f(|\lambda|^{-1/2})\|_{KS_n}^{\frac{1}{d}} \|f(|\lambda|^{-1/2})\|^2_{L^2(|V|(|\lambda|^{-1/2})|^{-1})}$$

$$\leq C|\lambda|^{-2} \|f(|\lambda|^{-1/2})\|_{KS_n}^{\frac{1}{d}} \|f\|^2_{L^2(|V|^{-1})}.$$
On the other hand,
\[
\|V_\lambda\|^\beta_{K,S_\alpha} = \sup_Q \left( \int_Q |V_\lambda|^\beta dx \right)^{-1} \int_Q \int_Q \frac{|V_\lambda|^\beta(x)|V_\lambda|^\beta(y)}{|x-y|^{d-\alpha}} dx dy \\
= \sup_Q \left( \int_Q |V|^\beta(x)|\lambda|^\frac{\beta}{2} dx \right)^{-1} \int_Q \int_Q \frac{|V|^\beta(x)|V|^\beta(y)}{|\lambda|^{(d-\alpha)/2}|x-y|^{d-\alpha}|\lambda|^d} dx dy \\
= |\lambda|^\frac{\beta}{2} \|V\|_{K,S_\alpha}^\beta 
\]
where $V_\lambda(\cdot) = V(|\lambda|^{-1/2} \cdot)$. Consequently, we get
\[
\|R(\lambda)f\|_{L^2(|V|)} \leq C|\lambda|^{\frac{\beta}{2} - 1} \|V\|_{K,S_\alpha}^\beta \|f\|_{L^2(|V|^{-1})}
\]
which is equivalent to (2.1).

Now we show (2.2). By Stein’s interpolation we only need to show the following two estimates
\[
\|T_\zeta f\|_{L^2} \leq Ce^{|\text{Im}\zeta|} \|f\|_{L^2} \quad \text{for} \quad \text{Re}\zeta = 0 \quad (2.3)
\]
and
\[
\|T_\zeta f\|_{L^2} \leq Ce^{\text{Re}\zeta} \|V\|_{K,S_\alpha} \|f\|_{L^2} \quad (2.4)
\]
for $\frac{d-1}{2} \leq \text{Re}\zeta < \frac{d+1}{2}$ and $\alpha = \frac{d-1}{2} + \text{Re}\zeta$. Indeed, by interpolating these estimates one can see
\[
\|T_1 f\|_{L^2} \leq C\|V\|_{K,S_\alpha} \|f\|_{L^2}
\]
for $\frac{d-1}{2} \leq \text{Re}\zeta < \frac{d+1}{2}$ and $\alpha = \frac{d-1}{2} + \text{Re}\zeta$. By setting $\beta = \text{Re}\zeta$, it is now easy to see that this is equivalent to the estimate (2.2) as desired.

Finally we show (2.3) and (2.4). The first estimate follows easily from Plancherel’s theorem. In fact, since $|V|^{\zeta/2} = 1$ for $\text{Re}\zeta = 0$, we get
\[
\|T_\zeta f\|_{L^2} = \left\| \frac{1}{(|\xi|^2 - \lambda)^\zeta} \hat{f}(\xi) \right\|_{L^2} \\
\leq \sup_{\xi \in \mathbb{R}^n} \left\| \frac{1}{(|\xi|^2 - \lambda)^\zeta} \right\|_{L^2} \|f\|_{L^2} \\
\leq \sup_{\xi \in \mathbb{R}^n} e^{\text{Re}\zeta |\text{arg}(|\xi|^2 - \lambda)|} \|f\|_{L^2} \\
\leq e^{e^{\text{Im}\zeta}} \|f\|_{L^2}
\]
by Plancherel’s theorem. It remains to show (2.4). The key point here is that the integral kernel $K_\zeta$ of $(-\Delta - \lambda)^{-\zeta}$ with $|\lambda| = 1$ can be controlled by that of the fractional integral operator $I_\alpha$ which is defined for $0 < \alpha < d$ by
\[
I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy,
\]
as follows:
\[
|K_\zeta(x)| \leq Ce^{\text{Im}\zeta} |x|^{\zeta\left(\frac{d-1}{2} + \text{Re}\zeta\right) - d} \quad (2.5)
\]
for $\frac{d-1}{2} \leq \text{Re}\zeta \leq \frac{d+1}{2}$, which will be shown later in the next section.
Now we use the following lemma, which gives the characterization of the weighted $L^2$ estimates for the fractional integrals, due to Kerman and Sawyer [15] (see Theorem 2.3 there and also Lemma 2.1 in [2]):

**Lemma 2.2.** Let $0 < \alpha < d$. Suppose that $w$ is a nonnegative measurable function on $\mathbb{R}^d$. Then there exists a constant $C_w$ depending on $w$ such that $w \in \mathcal{KS}_\alpha$ if and only if the following two equivalent estimates

$$\|I_{\alpha/2} f\|_{L^2(w)} \leq C_w \|f\|_{L^2}$$

and

$$\|I_{\alpha/2} f\|_{L^2(w^{-1})} \leq C_w \|f\|_{L^2(w)}$$

are valid for all measurable functions $f$ on $\mathbb{R}^d$. Furthermore, the constant $C_w$ may be taken to be a constant multiple of $\|w\|_{\mathcal{KS}_\alpha}^{1/2}$.

First, note that the two estimates in the lemma directly implies

$$\|w^{1/2} I_{\alpha/2} (w^{1/2} f)\|_{L^2} \leq C \|w\|_{\mathcal{KS}_\alpha} \|f\|_{L^2}.$$

Using (2.5) and this estimate with $w = |V|^{\text{Re} \zeta}$ and $\alpha = (d-1)/2 + \text{Re} \zeta$, we get the desired estimate (2.4).

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3. Appendix

In this final section we provide a proof of the estimate (2.5) for the integral kernel $K_{\zeta}$ of $(-\Delta - \lambda)^{-\zeta}$ with $|\lambda| = 1$. Its detailed proof for $\text{Re} \zeta = (d-1)/2$ can be also found in [20].

To show (2.5), we first recall the following formula for $K_{\zeta}$ (cf. [12] pp. 288-289, [14]):

$$K_{\zeta}(x) = \frac{e^{\zeta^2/2} x^{-\zeta+1}}{(2\pi)^{d/2} \Gamma(\zeta) \Gamma(\frac{d}{2} - \zeta)} \left(\frac{\lambda}{|x|^2}\right)^{\frac{d}{2} - \zeta} B_{d/2 - \zeta} \left(\sqrt{\lambda |x|^2}\right)$$

where $B_{\nu}(w)$ is the Bessel kernel of the third kind which satisfies for $\text{Re} w > 0$

$$|e^{-\nu^2} B_{\nu}(w)| \leq C |w|^{-|\text{Re} \nu|}, \quad |w| \leq 1,$$

and

$$|B_{\nu}(w)| \leq C_{\text{Re} \nu} e^{-\text{Re} w} |w|^{-1/2}, \quad |w| \geq 1.$$

See [14], p. 339 for details. To estimate $|K_{\zeta}|$ using these estimates with $\nu = d/2 - \zeta$ and $w = \sqrt{\lambda |x|^2}$, we then note that $\text{Re}(\sqrt{\lambda |x|^2}) = |x| \cos(\frac{1}{2} \arg \lambda) > 0$ for $x \neq 0$, since $-\pi < \arg \lambda \leq \pi$ by the principal branch, and $\lambda \not\in \mathbb{R}$. Also, the inequalities $|\sqrt{\lambda |x|^2}| \geq 1$ and $|\sqrt{\lambda |x|^2}| \leq 1$ correspond to $|x| \geq 1$ and $|x| \leq 1$, respectively, since we are assuming $|\lambda| = 1.$
It follows now from (3.2) that for \(|x| \geq 1\)

\[ |K_\zeta(x)| \leq \frac{e^{\zeta^2 - \zeta + 1}}{(2\pi)^{d/2} \Gamma(\zeta) \Gamma(d/2 - \zeta)} \left( \frac{\lambda}{|x|^2} \right)^{\frac{d}{2} - \zeta} \left| e^{-\text{Re}\lambda |x|^2} \sqrt{|\lambda |x|^2} \right|^{-\frac{d}{2}} \]

\[ \leq Ce^{-\text{Re}\lambda |x|^2} \left( \frac{\lambda}{|x|^2} \right)^{\frac{d}{2} - \zeta} \left| \sqrt{|\lambda |x|^2} \right|^{-\frac{d}{2}} \]

\[ \leq Ce^{-|x| \cos \left( \frac{1}{2} \arg \lambda \right)} \left( \frac{\lambda}{|x|^2} \right)^{\frac{d}{2} - \zeta} \left| \sqrt{|\lambda |x|^2} \right|^{-\frac{d}{2}} \]

\[ \leq Ce^{-|x| \cos \left( \frac{1}{2} \arg \lambda \right)} \zeta \arg \lambda |x| \zeta \text{Re} \zeta - \frac{d+1}{2} \]

as desired. (Here we used the fact that \(|\lambda| = 1\) and \(-\pi < \arg \lambda \leq \pi\).) On the other hand, when \(|x| \leq 1\), it follows from (3.1) that

\[ |K_\zeta(x)| \leq \frac{e^{\zeta^2 - \zeta + 1} e^{-((d/2 - \zeta)/2)^2}}{(2\pi)^{d/2} \Gamma(\zeta) \Gamma((d/2 - \zeta)/(d/2 - \zeta))} \left( \frac{\lambda}{|x|^2} \right)^{\frac{d}{2} - \zeta} \left| \sqrt{|\lambda |x|^2} \right|^{-\text{Re}(\frac{d}{2} - \zeta)} \]

\[ \leq C \left( \frac{\lambda}{|x|^2} \right)^{\frac{d}{2} - \zeta} \left| \sqrt{|\lambda |x|^2} \right|^{-\text{Re}(\frac{d}{2} - \zeta)} \]

\[ \leq Ce^{\frac{1}{2} \text{Im} \zeta \arg \lambda |x| - \frac{d}{2} + \text{Re} \zeta - |\text{Re}(\frac{d}{2} - \zeta)|} \]

\[ \leq Ce^{\frac{1}{2} \text{Im} \zeta |x| - \frac{d}{2} + \text{Re} \zeta - \left| \text{Re}(\frac{d}{2} - \zeta) \right|} . \quad (3.3) \]

When \(\text{Re} \zeta \geq d/2\), the estimate (3.3) implies

\[ |K_\zeta(x)| \leq Ce^{\frac{1}{2} \text{Im} \zeta |x| \text{Re} \zeta - \frac{d+1}{2}} \]

since \(|x| \text{Re} \zeta - \frac{d+1}{2} \geq 1\) if \(\text{Re} \zeta \leq (d+1)/2\) and \(|x| \leq 1\). When \(\text{Re} \zeta \leq d/2\), from (3.3) we have

\[ |K_\zeta(x)| \leq Ce^{\frac{1}{2} \text{Im} \zeta |x|^{-2(d/2 - \text{Re} \zeta)}} \]

\[ \leq Ce^{\frac{1}{2} \text{Im} \zeta |x| \text{Re} \zeta - \frac{d+1}{2}} \]

if \(\text{Re} \zeta \geq (d-1)/2\) and \(|x| \leq 1\). Consequently, we get for \(|x| \leq 1\)

\[ |K_\zeta(x)| \leq Ce^{\frac{1}{2} \text{Im} \zeta |x| \text{Re} \zeta - \frac{d+1}{2}} \]

if \((d-1)/2 \leq \text{Re} \zeta \leq (d+1)/2\), as desired.

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