Hybrid model for determining the stiffness of a spatial rod with consideration geometric nonlinearity

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Abstract. The mathematical model for elastoplastic calculation of spatial rod is considered with taking into account shear deformation. It is supposed to use this model in hybrid method of finite elements for determination of elastoplastic deformation in rod systems with geometric nonlinearity. In this case the rod stiffness matrix will be determined by force method or by functions of element’s form using the matrix of tangent stiffness of cross section. This matrix counts the stress state in each elementary area of cross section given the shear. The hybrid approach allows to eliminate the finite elements along the rod length and replace them with integration sites. The calculation algorithm of elastoplastic deformation for a spatial rod is provided considering the geometric nonlinearity. The proposed nonlinear model can be used in finite element method in form of displacement and force methods to elaborate a special mechanism for building a system of resolving equations. The results are compared based on the example of spatial console rod calculation.

1. Introduction
The task of load-bearing structures calculation is focused on more accurate prediction of the construction behavior at all stages of its work and most frequently can not be resolved by linear structural mechanics methods. Deviation from Hooke’s law (physical nonlinearity), refusal to consider equilibrium conditions for the undeformed state of the design scheme (geometric nonlinearity), accounting for possible changes in the design scheme during deformation (constructive nonlinearity) constitute a set of nonlinear tasks to which the most part of modern software developments belongs. It is necessary to consider geometric nonlinearity while significant displacements comparable with structure dimensions have occurred. For instance in case of longitudinal and longitudinal-transverse bending of rods and changing the coordinates of construction points due to relatively significant displacements. To perform nonlinear construction’s calculations corresponding to reality it is required to use complex software systems for their analysis and create finite element models of large dimension. Most of nonlinear tasks can be calculated using the computer simulation and finite element method (FEM). However it is necessary to use new mathematical models and algorithms to provide high speed and accuracy of calculations.

The analytical solution of a statistical determined rod system with physical nonlinearity is considered [1]. The research of a bending of a geometrically nonlinear rod using the Cosserat – Timoshenko theory is provided in [2]. The article [3] contains an analytical solution for determining the force diagram of console elements dissipating energy due to plastic deformation. In clause [4] there are presented functions of the shape of the spatial rod’s element according to Tymoshenko’s
theory, which takes into account shear deformation. In most of works stress distribution over the cross-sectional area at deformation is ignored.

In this research a geometrically nonlinear model based on a generalized More’s formula with a tangential stiffness matrix is considered. This matrix characterizes the stress state of all points of the rod section.

The aim of this research was to develop more efficient algorithm for elastoplastic calculation of rod systems considering the geometric nonlinearity. Geometrically nonlinear work is related to the need to consider the change in the geometry of the system when it is deformed under load [5, 6]. Account of the nonlinear components in deformation expressions by displacement is necessary to obtain more accurate results of the calculating experiment, inasmuch as the usage of geometrically linear models for the calculation of stress-strain state gives a significant inaccuracy at high loads.

A numerical model of geometrically nonlinear deformation of a spatial cantilever rod for a hybrid finite element method is provided in this article. The model is created based on the tangent stiffness matrix [7], in which shear deformations are considered. The account of physical and geometric nonlinearity was carried out by usage of the generalized More’s formula [8] with tangent stiffness matrix. The difference from nonlinear approaches in classical finite element method consists on the fact that the rod can be considered as a whole without dividing by finite elements. In this case the accuracy will depend on the number of integration sites in using More’s formula. Further this algorithm will be used in the discrete-analytical method (hybrid finite element method) for calculating nonlinear tasks. The proposed algorithm in contrast the classical Euler-Bernoulli bending theory allows to consider the effect of shear deformations on total displacements. In some software systems it is used the Tymoshenko theory of bending beam, in which the shear also taken into account, but only within the finite element. The developed nonlinear model can be used in finite element method in form of displacement method and force method working on the development of a special mechanism of building a resolving equations system. There is considered the algorithm for determination of elastoplastic deformations in the rod system taking into account geometric nonlinearity (i.e. using deformed scheme).

2. Methods
At the first stage it is necessary to set the number of integration sites and determine the initial coordinates of the rod nodes. Further it is required to determine the value of internal forces and moments in the rod nodes from the applied load, using the coordinate transformation matrix with the guiding cosines:

\[
[D] = \begin{bmatrix}
\lambda
0
0
\end{bmatrix},
\]

\[
[\lambda] = \begin{bmatrix}
\cos(xx') & \cos(xy') & \cos(xz') \\
\cos(yx') & \cos(yy') & \cos(yz') \\
\cos(zx') & \cos(zy') & \cos(zz')
\end{bmatrix},
\]

where \(xx', yy', zz'\) – the angles from the global axes, accordingly \(x, y, z\) to the axes \(x', y', z'\) in a movable coordinate system.

Besides rotation it is also indispensably to consider the coordinate system relocation to each node (figure 1). It is proposed to define the moments and internal forces in the cantilever rod from the concentrated forces through the projections of the integration areas \(ds_k\) on the main axes after deformation:

\[
M_x(k) = \sum_k P_y \cdot dz_k + \sum_k P_z \cdot dy_k,
\]

\[
M_y(k) = \sum_k P_z \cdot dx_k + \sum_k P_x \cdot dz_k,
\]

\[
M_z(k) = \sum_k P_x \cdot dx_k + \sum_k P_y \cdot dy_k,
\]
\[
\begin{bmatrix}
N_x \\
Q_y \\
Q_z \\
P_x \\
P_y \\
P_z
\end{bmatrix} = \begin{bmatrix}
\lambda
\end{bmatrix}
\begin{bmatrix}
P_x \\
P_y \\
P_z
\end{bmatrix},
\]

\[
[\lambda] = \begin{bmatrix}
1 & 0 & 0 & \cos(\beta) & 0 & -\sin(\beta) \\
0 & \cos(\alpha) & \sin(\alpha) & 0 & 1 & 0 \\
0 & -\sin(\alpha) & \cos(\alpha) & \sin(\beta) & 0 & \cos(\beta) \\
\end{bmatrix},
\]

where \( [\lambda] \) – the matrix of coordinate transformation, expressed through the relative rotation angles \( \alpha, \beta, \gamma \) (figure 2).

The relative angles of rotation \( \{ \phi \} = [\alpha, \beta, \gamma]^T \) between the Ox axis and areas of integration in the movable coordinate system with the time step is determined by the generalized More’s formula[8].

\[
\{ \phi(s) \} = \int_0^s \{ \Psi \} ds \cdot \Delta t,
\]

\[
\{ \Psi \} = [T]^{-1} \{ \psi \},
\]

\[
\{ \psi \} = \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\frac{d\psi}{dx} \dot{x}_3 - \dot{\gamma}_2
\end{bmatrix}^T,
\]

\[
\{ u \} = \begin{bmatrix}
\frac{\partial M_1}{\partial t} \\
\frac{\partial M_2}{\partial t} \\
\frac{\partial M_3}{\partial t} \\
\frac{\partial Q_1}{\partial t} \\
\frac{\partial Q_2}{\partial t} \\
\frac{\partial Q_3}{\partial t}
\end{bmatrix}^T,
\]

and accordingly the displacement projection

\[
\{ \Delta x(s) \} = \int_0^s \{ \psi \} ds \cdot \Delta t,
\]

where \( M_1, M_2, M_3, Q_1, Q_2, Q_3 \) – vectors components \( M \) and \( Q \) in coordinate system \( \xi, \eta, \zeta \) (figure 3); \( [u_{ele23}] \), \( [u_{emle23}] \) – matrix 3x6 dimension with the components of the vectors of moments and internal stresses \( M_e, Q_e \) in coordinate system \( \xi, \eta, \zeta \) from the application of unit forces and unit moments; \( \frac{\partial \gamma}{\partial t} \) – the vector of changes in time of the generalized curvatures; \( \gamma_2, \gamma_3 \) – the shear velocity cross-section; \( \frac{d\psi}{dx} \) – velocity in axial direction; \( T \) – the tangential stiffness of the cross section; \( \Delta t \) – time step.

Analogously operation can be performed with concentrated moments, using the coordinate transformation matrix:

\[
\{ M' \} = [\lambda] \{ M \},
\]

\[
\{ M \} = \begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix},
\]

After calculation of the internal forces values and moments in the rod nodes it is determined the stresses at each point of the cross section with the coordinates in the plane \( \xi, \eta \) via formula

\[
\{ \sigma_k \} = [A] \{ S_k \} [\Psi_k]^T,
\]

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix},
\]

\[
[S] = \begin{bmatrix}
0 & -\xi & \eta & 1 & 0 & 0 \\
\xi & 0 & 0 & 0 & 0 & 1 \\
-\eta & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\{ \sigma \} = \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}^T,
\]

\[
\sigma_1 = \sigma_{\xi\xi}; \sigma_2 = \tau_{\xi\eta}; \sigma_3 = \tau_{\xi\zeta},
\]

where \( [A] \) – matrix with pseudoelastic coefficients describing the bilinear diagram \( \sigma-\epsilon \) [7]. For the stress state in the rod \( \sigma_{\eta\eta} = \sigma_{\xi\zeta} = \tau_{\xi\eta} = 0 \).
Pseudo-elastic coefficients $a_{ij}$ are calculated using formulas:

$$a_{11} = 2 \cdot (1 + \nu) \cdot G \cdot \left[ 1 - 2 \cdot (1 + \nu) \cdot \kappa \frac{\sigma_{i}^2}{\sigma_{i}^2} \right]; \quad a_{jj} = G \cdot \left[ 1 - 9 \cdot \kappa \frac{\sigma_{j}^2 \cdot \sigma_{j}^2}{\sigma_{j}^2} \right]; \quad j = 2, 3, \quad (19)$$

$$a_{ij} = a_{ji} = -2 \cdot (1 + \nu) \cdot G \cdot 3 \kappa \frac{\sigma_{i} \cdot \sigma_{j}}{\sigma_{i}^2}; \quad a_{23} = a_{32} = -G \cdot 9 \kappa \frac{\sigma_{2} \cdot \sigma_{3}}{\sigma_{2}^2}, \quad (20)$$

$$\kappa = \frac{G}{3G + \phi} \frac{1}{1 - (1 - 2\nu) \cdot \frac{G}{3G + \phi} \frac{\sigma_{1}^2}{\sigma_{s}^2}}, \quad (21)$$

where $G$ – shear modulus; \quad $\phi = \frac{E \cdot E_{pl}}{E - E_{pl}}$; \quad $E_{pl}$ – tangent modulus; \quad $\langle \Phi \rangle = 0$ in terms of the elastic deformation.

**Figure 1.** Rod node.  \quad **Figure 2.** Relative rotation angles.  \quad **Figure 3.** Coordinates of section.

Knowing the stresses at each point of the rod section, the matrix $[A]$ can be defined using the conditions of transition from elastic to elastoplastic loading:

- elastic stress $\sigma_{sum}^2 \leq \sigma_{i}^2$
- elastoplastic stress $\sigma_{sum}^2 > \sigma_{i}^2, \ d\sigma_{sum}/dt > 0$

$$\sigma_{sum}^2 = \sigma_{i}^2 + 3\sigma_{2}^2 + 3\sigma_{3}^2. \quad (22)$$

Via $\sigma_{i}$ the value of the total stress at the beginning of the $n$-th semicircle of elastic loading is indicated (if $t = 1$, that $\sigma_{i}$ will be equal the yield strength $\sigma_s$). Matrix components $S$ depend on coordinates of considering rod section point.

Further the value of section stiffness in each node is defined taking into account the stress distribution over the cross-section area:

$$[r] = \int_{L} [L][A][S]dF, \quad (23)$$

where $L$ – transition matrix from stresses to internal forces:

$$[L] = \begin{bmatrix} \eta & 0 & 0 \\ -\xi & 0 & 0 \\ 0 & \xi & -\eta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

Then a new vector of internal forces and moments for the rod is formed. Using the force method along with the developed mathematical model it is possible to obtain a stiffness matrix of the rod, which considers the nonlinear geometric and plastic for the calculation via the finite element method. Computation algorithm of plane problems is presented [9, 10].

Obtained dependencies can be used for an automated calculation of rod systems of any complexity by using modern software systems.
3. Determination of stresses and strains in the cantilever rod

A calculation program for the spatial rod in Mathcad was created to verify the developed mathematical model. A cantilever bar of square section with concentrated forces at the end is considered (figure 4). Elastoplastic calculation given geometric nonlinearity was carried out explicitly in 20 time steps. Different search procedures were not considered.

The calculation results were compared with the finite element method in ANSYS. The automatic time step feature has been disabled. The following initial data were accepted in the task:
- elastic modulus $E = 2.1 \times 10^{11}$ Pa; the tangent elastic modulus $E_{\text{pl}} = 2.1 \times 10^{10}$ Pa;
- Poisson’s ratio $\nu = 0.3$; yield strength $\sigma_s = 240$ MPa;
- square section width $a = 0.1$ m; $l = 2$ m; force $F = 5 \times 10^4$ N.

As the developed mathematical model takes into account the stress distribution over the cross-section area, so in ANSYS the rod cross-section was divided into 20 sites by height and width. The number of finite elements on the rod is 100. For the new model calculation the rod was given 100 sites of integration. Calculation results are shown on figure 5.

![Figure 4. Cantilever rod.](image)

![Figure 5. Stress distribution in a spatial rod.](image)

With the enabled option transverse-shear strain (Timoshenko theory including shear deformation) ANSYS showed the same results. Shear Timoshenko theory for discrete cross-section is ignored.

According to the preliminary calculation results the error is in the area of several percent. It is related to incorrect transition from elastic to plastic area at the time step. The results of the calculation of plane physically nonlinear tasks are successfully correlated with the numerical experiment [9]. The special program block in MathCAD, which considers only geometric nonlinearity, is presented in the Appendix.

4. Conclusions

In this article a mathematical model for determining the deformation of a rod element was considered with geometric nonlinearity, stress distribution over the cross-sectional area and with deformation of the shear.

The developed mathematical model allows to increase the accuracy in the determination of stresses in rod systems and significantly reduce the time of elastoplastic calculation with the exception of finite elements along the rod length. By the use of generalized More’s formula with the tangent stiffness matrix, finite elements are replaced by integration sites. Due to the influence of shear deformation on the total displacement it is able to increase the accuracy of determination the stresses at calculation of the geometrically nonlinear tasks. The proposed model will be used in a hybrid finite element method.
in the form of a displacement method. This approach can significantly extend the variety of design nonlinear tasks solved.

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Appendices
Determination of deformation of a cantilever rod with geometric nonlinearity

| Source date: | ORIGIN := 0 |
|-------------|-------------|
| Loads:     | PX := 1000  |
|            | PY := 1000  |
|            | PZ := 1000  |
|            | x - long axis  |
|            | y - vertical axis  |
|            | z - horizontal axis  |
| Mechanical property: | E := 2.1·10^11 - modulus of elasticity, Pa  |
|            | ν := 0.3 - Poisson’s ratio  |
| G := E 2(1 + ν) - shear modulus, Pa  |
| Section characteristics: | a := 0.1 - side of the square, m  |
|            | F := a^2 - area of a square, m^2  |
|            | Iy := a^4/12 - moment of inertia  |
|            | Iz := a^4/12 - moment of inertia  |
|            | Ip := 2a^4/12 - moment of inertia  |
| Rigidities of section: | T1 := G·Ip  |
|            | T2 := E·Iy  |
|            | T3 := E·Iz  |
|            | T4 := E·Fη  |
|            | T5 := G·F/η  |
|            | T6 := G·F/η  |
|            | η := 0.7 - shear coefficient  |
|            | T6 := (T1 T2 T3 T4 T5 T6)^T - stiffness vector  |
|            | L := 1 - length of the rod  |
|            | t := 10 - number of time steps  |
|            | n := 11 - number of nodes  |
|            | k := 1...n - node number  |
|            | ΔL := L/n - length of the segment  |
Computation program:

Initial coordinates of the nodes:
\[ x_0 := k \Delta L - \Delta L, \quad y_0 := 0, \quad z_0 := 0, \quad \alpha_0 := 0, \quad \beta_0 := 0, \quad \gamma_0 := 0. \]

\( (t,n) := \)
\[ \begin{align*}
\Delta(t,n) := & \quad \chi \leftarrow x_0,y \leftarrow y_0,z \leftarrow z_0, \alpha \leftarrow \alpha_0, \beta \leftarrow \beta_0, \gamma \leftarrow \gamma_0 \\
& \text{for } i \in 1..t; \text{ for } k \in 1..n; \text{ for } p \in 1..n \]
\[ \begin{align*}
\Delta x_k := & \quad (\frac{U_{i}^k + U_{i-1}^k}{2T} - \Delta L) \cdot \Delta x_k, \\
\Delta y_k := & \quad (\frac{U_{i}^k + U_{i-1}^k}{2T} - \Delta L) \cdot \Delta y_k, \\
\Delta z_k := & \quad (\frac{U_{i}^k + U_{i-1}^k}{2T} - \Delta L) \cdot \Delta z_k \\
\Delta x & := \Delta x, y := \Delta y, \quad z := z + \Delta z, \quad \alpha := \alpha + \Delta \alpha, \quad \beta := \beta + \Delta \beta, \quad \gamma := \gamma + \Delta \gamma. \quad \text{Return } x
\end{align*} \]