Asymptotic and Assouad–Nagata dimension of finitely generated groups and their subgroups

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Abstract
We prove that for all $k, m, n \in \mathbb{N} \cup \{\infty\}$ with $4 \leq k \leq m \leq n$, there exists a finitely generated group $G$ with a finitely generated subgroup $H$ such that $\text{asdim}(G) = k$, $\text{asdim}_{\text{AN}}(G) = m$, and $\text{asdim}_{\text{AN}}(H) = n$. This simultaneously answers two open questions in asymptotic dimension theory.

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INTRODUCTION

This is the second in a series of two papers on asymptotic dimension and Assouad–Nagata dimension of finitely generated groups: the first is [1]. Asymptotic dimension ($\text{asdim}$) and asymptotic Assouad–Nagata dimension ($\text{asdim}_{\text{AN}}$) are two distinct but related ways of defining the large-scale dimension of a metric space. Each is invariant under quasi-isometry, and thus can be considered as an invariant of finitely generated groups. For countable groups with proper left-invariant metrics, asymptotic Assouad–Nagata dimension is equivalent to Assouad–Nagata dimension ($\text{dim}_{\text{AN}}$). So from now on, we use this shorter term when talking about groups, although we continue to denote it by $\text{asdim}_{\text{AN}}$.

Given a way of defining ‘dimension’ for an algebraic structure, it is natural to ask whether it is monotonic with respect to substructures: that is, whether $A \leq B$ implies that the dimension of $A$ is no greater than the dimension of $B$. Is our dimension like that of a vector space, where this natural monotonicity holds, or is it like the rank of a free group, where it fails spectacularly? Since $\text{asdim}$ is actually a coarse invariant, it follows that $\text{asdim}$ is well defined for all countable groups, and if $G$ is a countable group and $H \leq G$, then $\text{asdim}(H) \leq \text{asdim}(G)$. In this paper, we show that Assouad–Nagata dimension behaves quite differently. Namely, we prove the following theorem.
**Theorem 1.** For any $k, m, n \in \mathbb{N} \cup \{\infty\}$ with $4 \leq k \leq m \leq n$, there exist finitely generated, recursively presented groups $G$ and $H$ with $H \leq G$, such that

$$\operatorname{asdim}(G) = k,$$

$$\operatorname{asdim}_{\text{AN}}(G) = m,$$

$$\operatorname{asdim}_{\text{AN}}(H) = n.$$

In [2], Higes constructs an infinitely generated, locally finite abelian group, and a proper left-invariant metric with respect to which the group has asymptotic dimension 0 but infinite Assouad–Nagata dimension. In [3], Brodskiy, Dydak, and Lang construct finitely generated groups with a similar gap, showing that (for example) $\mathbb{Z}_2 \ast \mathbb{Z}^2$ has asymptotic dimension 2 but infinite Assouad–Nagata dimension. Previously it was not known whether a finitely generated group $G$ could satisfy $\operatorname{asdim}(G) < \operatorname{asdim}_{\text{AN}}(G) < \infty$ (Question (2) of [2]), nor was it known whether a finitely generated group could contain a finitely generated subgroup of greater Assouad–Nagata dimension (Questions 8.6 and 8.7 of [4]). With Theorem 1, we show that both these things are possible.

If $H \leq G$ but $\operatorname{asdim}_{\text{AN}}(H) > \operatorname{asdim}_{\text{AN}}(G)$, it must be that $H$ is distorted in $G$, and that this distortion collapses $H$ to a space of lesser Assouad–Nagata dimension in $G$. Note that in such a case, the distortion of $H$ in $G$ must be superpolynomial. However, superpolynomial distortion does not always affect the Assouad–Nagata dimension of the distorted subgroup. For example, in $BS(1, 2) = \langle a, b \mid b^{-1}aba^{-2} \rangle$, the subgroup $\langle a \rangle$ is exponentially distorted, but still has Assouad–Nagata dimension 1. The groups $G$ and $H$ from Theorem 1 are constructed specifically to satisfy those properties: as such, they are somewhat artificial, with long and rather unwieldy presentations. The author hopes that more natural examples can be found.

The paper is organized as follows. In Section 1, we fix countable group $K$, constructed as a direct sum of cyclic groups of increasing order. We then show that for each $m, n \in \mathbb{N} \cup \{\infty\}$ with $m < n$, there are two different proper left-invariant metrics on $K$ such that $\operatorname{asdim}_{\text{AN}}(K) = m$ with respect to one, and $\operatorname{asdim}_{\text{AN}}(K) = n$ with respect to the other.

In Section 2, we use techniques from small cancellation theory to establish a highly technical lemma. This lemma allows us to quasi-isometrically embed $K$, with respect to each proper left-invariant metric, into a finitely generated group.

In Section 3, we embed $K$ into finitely generated groups $A$ and $B$. This is done in such a way that, calling $K_A$ the copy of $K$ in $A$ and $K_B$ the copy of $K$ in $B$, we have that $\operatorname{asdim}_{\text{AN}}(K_A) = m$ and $\operatorname{asdim}_{\text{AN}}(K_B) = n$. We then identify the two with an isomorphism $\phi : K_A \to K_B$, and let $G = A \ast_{\phi} B$. Our technical small cancellation lemma comes back to help us a second time by showing that $\phi$ ‘crushes’ the image of $K_B$ in $G$ to the size of $K_A$. With a few calculations using well-known extension theorems for Assouad–Nagata dimension, we are able to prove the following.

**Proposition 1.** For any $m, n \in \mathbb{N} \cup \{\infty\}$ with $m < n$, there exists a group $G = A \ast_{\phi} B$ where $G$, $A$, and $B$ are finitely generated and recursively presented, such that

$$1 \leq \operatorname{asdim}(G) \leq 2,$$

$$m + 1 \leq \operatorname{asdim}_{\text{AN}}(G) \leq m + 2,$$

$$n + 1 \leq \operatorname{asdim}_{\text{AN}}(B) \leq n + 2.$$
Using the free product formulas for asymptotic and Assouad–Nagata dimension and the Morita theorem for Assouad–Nagata dimension, it is then easy to derive Theorem 1 from Proposition 1.

There are many technical restrictions placed on the presentations of $A$ and $B$ from Proposition 1. In Section 4, we give explicit presentations where these conditions are satisfied.

In our construction it is unclear how to establish the exact asymptotic and Assouad–Nagata dimensions of $G$ and $B$, so we are left with the upper and lower bounds given in Proposition 1. Part of the reason for this is that, although any $C′(1/6)$ group has asymptotic dimension at most 2, if the group is infinitely presented, it is unclear when the asymptotic dimension is 1 and when it is 2. And, while we can easily find a subgroup isomorphic to the abelian group $\mathbb{Z} \times K$ in each of $G$ and $B$ to establish the lower bounds on their Assouad–Nagata dimension, there is no apparent candidate subgroup (or other subset) in either $G$ or $B$ whose Assouad–Nagata dimension is known to be $m + 2$ or $n + 2$, respectively. Because the upper and lower bounds in Proposition 1 do not quite meet, we cannot give an explicit presentation of a group $G$, which satisfies the conclusion of Theorem 1; however, we can explicitly give presentations of two groups, one of which must be a group satisfying the conclusion of Theorem 1. This is also why we require $k \geq 4$ in Theorem 1, though we conjecture the theorem is true even when $k = 3$ or $k = 2$.

1 | ADAPTING A CONSTRUCTION OF HIGES

We refer the reader to [1, section 1] for basic conventions and notation regarding metric spaces, as well as definitions of the terms asymptotic dimension ($\text{asdim}$), asymptotic Assouad–Nagata dimension ($\text{asdim}_{\text{AN}}$), and control function. We assume that the reader is familiar with the notions of quasi-isometry and bi-Lipschitz equivalence. Occasionally we will mention terms such as ‘coarse’ map/embedding/equivalence. Since we will not need to work with these directly, we do not give a definition here, but one may be found in any text on coarse geometry, for example, [5, p. 9]. What matters to us is that, if $X$ and $Y$ are metric spaces which are coarsely equivalent, then $\text{asdim}(X) = \text{asdim}(Y)$, and that a quasi-isometry or bi-Lipschitz map is a special case of a coarse equivalence.

In this paper, we adopt the convention that the Cartesian product of two metric spaces is always endowed with the $\ell^1$ product metric. That is, if $X$ and $Y$ are metric spaces, then $X \times Y$ is equipped with the metric defined by

$$d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$

for all $x, x' \in X$ and $y, y' \in Y$. With this convention in mind, if $\sim$ stands for either ‘is coarsely equivalent to’, ‘is quasi-isometric to’, or ‘is bi-Lipschitz equivalent to’, then we have that $X \sim Y$ implies $X \times Y \sim X' \times Y'$. In addition, asdim and $\text{asdim}_{\text{AN}}$ are subadditive with respect to taking direct products, in a sense that is made precise by the following two theorems. We will use them often throughout this paper.

**Lemma 1.1** [4, 6]. Let $X, Y$ be metric spaces. Then,

$$\text{asdim}(X \times Y) \leq \text{asdim}(X) + \text{asdim}(Y),$$

$$\text{asdim}_{\text{AN}}(X \times Y) \leq \text{asdim}_{\text{AN}}(X) + \text{asdim}_{\text{AN}}(Y).$$
1.1 Normed groups

We denote the identity element of an arbitrary group by 1, and of an abelian group by 0. Let $G$ be a group. A norm on $G$ is a function $\| \cdot \| : G \to \mathbb{R}_0^+$ such that, for all $g, h \in G$,

- $\| g \| = 0$ if and only if $g = 1$.
- $\| g \| = \| g^{-1} \|$.
- $\| gh \| \leq \| g \| + \| h \|.$

Some authors call this a length function or weight function on $G$.

A norm is proper if $\{ g \in G \mid \| g \| \leq N \}$ is finite for all $N \geq 0$. There is a natural one-to-one correspondence between norms and left-invariant metrics, given by $d(g, h) = \| g^{-1} h \|$ and $\| g \| = d(1, g)$, and a left-invariant metric on a group is proper if and only if the corresponding norm is proper. Every countable group admits a proper norm, and any two proper norms on the same countable group are coarsely equivalent [7, Proposition 1.1]. Thus, $\operatorname{asdim}$ is an invariant of countable groups: in particular, if $G$ is a countable group and $H \leq G$, then $\operatorname{asdim}(H) \leq \operatorname{asdim}(G)$. It is easy to show that a countable group has asymptotic dimension zero if and only if it is locally finite, a fact which we will use many times.

Formally, a normed group should be an ordered pair $(G, \| \cdot \|_G)$. But from now on, whenever we say that $G$ is a normed group, it is understood that $G$ is equipped with a norm, which is always called $\| \cdot \|_G$. With this convention in mind, we eliminate the norm from the notation wherever possible.

If $G$ is a normed group and $s$ is a positive real number, then the function $s \| \cdot \|_G : G \to \mathbb{R}_0^+, g \mapsto s \| g \|_G$ is also a norm on $G$. We denote the normed group $(G, s \| \cdot \|_G)$ by $sG$. Direct products are assumed to have the $\ell^1$ product norm, so that $s_0 G_0 \times s_1 G_1$ denotes the group $G_0 \times G_1$ with norm $\| \cdot \|_{(s_0, s_1)}$ defined by

$$\|(g_0, g_1)\|_{(s_0, s_1)} = s_0 \| g_0 \|_{G_0} + s_1 \| g_1 \|_{G_1}$$

for all $g_0 \in G_0$ and $g_1 \in G_1$. We extend this notation for scaling to general direct sums in the following natural way.

**Definition 1.2.** Let $I$ be a set, let $(G_i)_{i \in I}$ be an $I$-tuple of normed groups, and let $s = (s_i)_{i \in I}$ an $I$-tuple of scaling constants. Let $G = \bigoplus_{i \in I} G_i$. Then, $\bigoplus_{i \in I} s_i G_i$ is defined to be the normed group $(G, \| \cdot \|_s)$, where $\| \cdot \|_s$ is given by

$$\| g \|_s = \sum_{i \in I} s_i \| g_i \|_{G_i}$$

for all $g \in G$. We call $\| \cdot \|_s$ the norm induced by $s$.

By convention, we declare $\bigoplus_{i \in I} G_i$ to be the trivial group.

**Lemma 1.3.** Let $I$ be a set, $s = (s_i)_{i \in I}$ an $I$-tuple of scaling constants bounded away from zero. Then, $\bigoplus_{i \in I} s_i G_i$ is bi-Lipschitz equivalent to $\bigoplus_{i \in I} s'_i G_i$, where $s'_i$ is a positive integer for all $i \in I$. 

Proof. Suppose $\varepsilon > 0$ is such that $s_i \geq \varepsilon$ for all $i \in I$. Let $s' = (s'_i)_{i \in I} = ([s_i]_{i \in I}$, and let $g = (g_i)_{i \in I} \in \bigoplus_{i \in I} G_i$. Then, clearly $\|g\|_s \leq \|g\|_{s'}$, and
\[
\|g\|_{s'} = \sum_{i \in I} \lceil s_i \rceil \|g_i\|_{G_i} \leq \left( 1 + \frac{1}{\varepsilon} \right) \sum_{i \in I} s_i \|g_i\|_{G_i} = \left( 1 + \frac{1}{\varepsilon} \right) \|g\|_s.
\]
\[\square\]

1.2 A fixed group with varying norms

The next set of lemmas deal specifically with direct sums of cyclic groups. Here, we assume that a cyclic group comes equipped with the natural norm, that is, $\|x\|_{\mathbb{Z}_\ell} = \min(x, \ell - x)$ for all $x \in \mathbb{Z}_\ell$, and $\|x\|_{\mathbb{Z}} = |x|$ for all $x \in \mathbb{Z}$. Unless otherwise noted, tuples are sequences indexed by $\mathbb{N}$, for example, $(s_i)$ stands for $(s_i)_{i \in \mathbb{N}}$.

Definition 1.4. Let $(x_i) \in \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_{\ell_i}$. The geodesic form of $(x_i)$ is the unique sequence of integers $(y_i)$ such that for all $i \in \mathbb{N}$,
\[\bullet \ y_i \equiv x_i \mod \ell_i \quad \text{and} \quad \bullet \ y_i \in \{-\lfloor \ell_i/2 \rfloor, \ldots, -1, 0, 1, \ldots, \lceil \ell_i/2 \rceil\}.
\]
Note that if $s = (s_i)$ is a sequence of scaling constants, $x = (x_i) \in \bigoplus_{i \in \mathbb{N}} s_i \mathbb{Z}_{\ell_i}$, and $(y_i)$ is the geodesic form of $x$, then we have
\[
\|x\|_s = \sum_{i \in \mathbb{N}} s_i |y_i|.
\]

Definition 1.5. An expanded $n$-dimensional cube is a space isometric to
\[
s\{0, 1, \ldots, k\}^n := \{0, s, 2s, \ldots, ks\}^n
\]
for some $s \geq 1$ and $k \in \mathbb{N}$, considered as a subspace of $\mathbb{R}^n$ equipped with the $\ell^1$ metric.

In accordance with Definition 1.5, whenever $s$ is a scaling constant and $s \geq 1$, we call $s$ an expansion constant. Sequences of expanded cubes are useful for establishing lower bounds on the asymptotic Assouad–Nagata dimension of a metric space.

Lemma 1.6 [8, Corollary 2.7]. Let $X$ be a metric space, $n \in \mathbb{N}$. If $X$ contains a sequence of expanded $n$-dimensional cubes $s_j\{0, \ldots, k_j\}^n$ where $\lim_{j \to \infty} k_j = \infty$, then asdim$_{AN}(X) \geq n$.

Suppose $P$ is a set with $|P| \geq n$, $(\ell_i)_{i \in P}$ is a $P$-tuple of natural numbers, and $s_P$ is an expansion constant. Let $k_P$ be a natural number with $k_P \leq \min\{\ell_i/2 \mid i \in P\}$. Then by (1), $s_P \bigoplus_{i \in P} \mathbb{Z}_{\ell_i}$ contains an expanded $n$-dimensional cube $s_P\{0, \ldots, k_P\}^n$. With this observation and Lemma 1.6, one can construct a group which can achieve any positive Assouad–Nagata dimension. The idea is to take a direct sum of cyclic groups, block every $n$ of them together, and scale the blocks appropriately. In [8], Higes uses this idea to construct, for any $n \in \mathbb{Z}^+ \cup \{\infty\}$, a normed group $G_n$ with
asymptotic dimension zero but Assouad–Nagata dimension \( n \). However, in Higes’ examples, if \( m \neq n \), then \( G_m \) and \( G_n \) are not isomorphic. For our purposes, it is important that the group be fixed, with only the norm varying. The rest of this section is devoted to working out the details of this construction. To smooth the process, we introduce the following ad hoc notation.

**Definition 1.7.** For each \( m \in \mathbb{Z}^+ \cup \{\infty\} \), let \( P_m = \{P_{(m,j)} \mid j \in \mathbb{N}\} \) be the partition of \( \mathbb{N} \) given by

\[
P_{(m,j)} = \begin{cases} 
jm, jm + 1, \ldots, (j + 1)m - 1 & \text{if } m \in \mathbb{Z}^+ \\
 j^2, j^2 + 1, \ldots, (j + 1)^2 - 1 & \text{if } m = \infty.
\end{cases}
\]

**Definition 1.8.** Let \( s = (s_i) \) be a sequence, \( m \in \mathbb{Z}^+ \cup \{\infty\} \). Let the \( m \)-inflation of \( s \), denoted \( m \times s \), be the sequence defined by

\[
(m \times s)_i = s_j \iff i \in P_{(m,j)}.
\]

For example, if \( s = (1, 2, 3, \ldots) \), then \( 2 \times s = (1, 1, 2, 2, 3, 3, \ldots) \) and \( \infty \times s = (1, 2, 2, 2, 3, 3, 3, 3, \ldots) \).

By definition,

\[
s_i = \begin{cases} 
(m \times s)_i & \text{if } m \in \mathbb{Z}^+ \\
(m \times s)_i^2 & \text{if } m = \infty.
\end{cases}
\]

(2)

**Lemma 1.9.** Let \( d \in \mathbb{N} \), and let \( (c_0, \ldots, c_{d-1}) \) be a finite sequence of scaling constants. Let \( m \in \mathbb{Z}^+ \cup \{\infty\} \) be fixed, let \( (s_i) \) be an increasing sequence of expansion constants, and let \( (\ell_i) \) be an increasing sequence of positive integers. Let

\[
Z_d = \bigoplus_{i=0}^{d-1} c_i \mathbb{Z}, \quad K_m = \bigoplus_{i \in \mathbb{N}} (m \times s)_i \mathbb{Z}_{\ell_i}.
\]

Then, \( \text{asdim}_{\text{AN}}(Z_d \times K_m) \geq d + m \).

**Proof.** By Lemma 1.3, we may assume without loss of generality that all \( s_i \) are positive integers. Since finite direct products preserve bi-Lipschitz equivalence, we may also assume that all \( c_i \) are equal to 1, so that \( Z_d = \mathbb{Z}^d \).

Now note

\[
Z_d \times K_m = \mathbb{Z}^d \times \bigoplus_{j \in \mathbb{N}} \left( s_j \bigoplus_{i \in \mathbb{P}_{(m,j)}} \mathbb{Z}_{\ell_i} \right),
\]

where \( \mathbb{Z}^d \times s_j \bigoplus_{i \in \mathbb{P}_{(m,j)}} \mathbb{Z}_{\ell_i} \) is an isometrically embedded subgroup for each \( j \in \mathbb{N} \). Let

\[
k_j = \min\{\lfloor \ell_i/2 \rfloor \mid i \in \mathbb{P}_{(m,j)}\} = \begin{cases} 
\lfloor \ell_j m/2 \rfloor & \text{if } m \in \mathbb{Z}^+ \\
\lfloor \ell_j \ell_j/2 \rfloor & \text{if } m = \infty.
\end{cases}
\]

Then \( \lim_{j \to \infty} k_j = \infty \).
If \( m \in \mathbb{Z}^+ \), then \( |P_{(m,j)}| = m \) for all \( j \in \mathbb{N} \). Then, since \( s_j \) is an integer, \( \mathbb{Z}^d \times s_j \bigoplus_{i \in P_{(m,j)}} \mathbb{Z} \epsilon_i \) contains the expanded \((d + m)\)-dimensional cube \( s_j\{0,\ldots,k_j\}^{d+m} \) for all \( j \in \mathbb{N} \). Since \( \lim_{j \to \infty} k_j = \infty \), by Lemma 1.6 we have \( \text{asdim}_{AN}(\mathbb{Z}^d \times K_m) \geq d + m \).

If \( m = \infty \), let \( n \in \mathbb{Z}^+ \). Then, \( |P_{(m,j)}| = (j + 1)^2 - j^2 = 2j + 1 \geq n \) for all \( j \geq n \). Therefore, \( s_j \bigoplus_{i \in P_{(m,j)}} \mathbb{Z} \ell_i \) contains the expanded \( n \)-dimensional cube \( s_j\{0,\ldots,k_j\}^n \) for all \( j \geq n \). Since \( \lim_{j \to \infty} k_j = \infty \), by Lemma 1.6, we have \( \text{asdim}_{AN}(K_\infty) \geq n \). Since \( n \in \mathbb{Z}^+ \) was chosen arbitrarily, \( \text{asdim}_{AN}(K_\infty) = \infty \), thus \( \text{asdim}_{AN}(\mathbb{Z}^d \times K_\infty) = \infty \).

Now, in the notation of Lemma 1.9, we wish to impose certain conditions on the sequence \((s_i)\) of expansion constants to guarantee \( \text{asdim}_{AN}(\mathbb{Z}^d \times K_m) = d + m \) exactly. We will use a lemma of Higes; in order to do so, we need to introduce a little notation, and consider a different norm on countable direct sums of normed groups.

**Definition 1.10.** Let \((G_i)\) be a sequence of normed groups and \( s = (s_i) \) a sequence of scaling constants. Let \( G = \bigoplus_{i \in \mathbb{N}} G_i \). For convenience, let us define the *height* function \( h : G \to \mathbb{N} \) by

\[
h(g) = \begin{cases} 
0 & \text{if } g = 1 \\
\max(\text{supp}(g)) & \text{otherwise.}
\end{cases}
\]

Now define the *quasi-ultranorm* on \( G \) induced by \( s \), denoted \( \| \cdot \|_{s}^{\text{qu}} \), by

\[
\|g\|_{s}^{\text{qu}} = s_h \|g_h\|_{G_h}
\]

for all \( g = (g_i) \in G \), where \( h = h(g) \).

In [8], Higes calls the metric associated to this norm the *quasi-ultrametric* generated by the sequence of metrics \( (d_{G_i}) \), where \( d_{G_i} \) is the metric associated to the scaled norm \( s_i \| \cdot \|_{G_i} \) for each \( i \in \mathbb{N} \). For this reason, we call the norm in (3) the quasi-ultranorm on \( G \) induced by \( s \), and put ‘qu’ in the superscript. The next lemma says that if all \( G_i \) are finite then, under mild assumptions about the growth of the sequence \( s \), the norms \( \| \cdot \|_{s} \) and \( \| \cdot \|_{s}^{\text{qu}} \) are, for our purposes, interchangeable.

**Lemma 1.11.** Let \((G_i)\) be a sequence of normed groups and \( s = (s_i) \) a sequence of scaling constants. Let \( G = \bigoplus_{i \in \mathbb{N}} G_i \). Suppose \( G_i, \| \cdot \|_{G_i}, s_i \) satisfy the following conditions for all \( i \in \mathbb{N} \):

- \( \|g_i\|_{G_i} \geq 1 \) for all \( g_i \in G_i \setminus \{1\} \).
- \( \text{diam}(G_{i+1}) \geq \text{diam}(G_i) \).
- \( s_{i+1} \geq 2 s_i \text{diam}(G_i) \).

Then, the norm \( \| \cdot \|_{s} \) and quasi-ultranorm \( \| \cdot \|_{s}^{\text{qu}} \) induced by \( s \) are bi-Lipschitz equivalent.

**Proof.** Clearly \( \|g\|_{s}^{\text{qu}} \leq \|g\|_{s} \) for all \( g \in G \).

We now prove by induction on \( h(g) \) that \( \|g\|_{s} \leq 2\|g\|_{s}^{\text{qu}} \). This is clear when \( h(g) = 0 \). Now suppose \( h(g) = k \geq 1 \). Write \( g \) as \( g'g'' \), where \( g'_j = g_j \) exactly when \( j = k \) and is equal to 1 otherwise,
and \( h(g'' \prime) = i < k \). Then, we have
\[
\|g\|_s \leq \|g'\|_s + \|g'' \prime\|_s = \|g'\|_{s_{qu}} + \|g'' \prime\|_s \leq \|g'\|_{s_{qu}} + 2\|g'' \prime\|_s \\
\leq \|g'\|_{s_{qu}} + 2s_k \text{diam}(G_i) \leq \|g'\|_{s_{qu}} + 2s_{k-1} \text{diam}(G_{k-1}) \\
\leq \|g'\|_{s_{qu}} + s_k \leq 2\|g'' \prime\|_s = 2\|g'' \prime\|_{s_{qu}}.
\]

\[\square\]

**Lemma 1.12** [8, Proof of Corollary 4.11]. Let \((\ell_i)\) be an increasing sequence of positive integers with \( \ell_0 \geq 2 \). Let \( m \) be a fixed positive integer. Let \( s = (s_i) \) be a sequence of expansion constants such that
\[
s_{i+1} \geq 1 + s_i \text{diam}(\mathbb{Z}^{m}_{\ell_i}) = 1 + (m\lfloor \ell_i / 2 \rfloor)s_i.
\]

Let \( K_{m}^{\text{qu}} = (\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{\ell_i}^m, \|\cdot\|_{s_{qu}}) \). Then, for any \( k \in \mathbb{N} \) we have \( \text{asdim}_{AN}(\mathbb{Z}^k \times K_{m}^{\text{qu}}) = k + m \).

We use this lemma in the case \( k = 0, m = 1 \) to obtain the slightly generalized lemma that we need.

**Lemma 1.13.** Let \( d \in \mathbb{N} \), and let \((c_0, \ldots, c_{d-1})\) be a finite sequence of scaling constants. Let \((\ell_i)\) be a sequence of positive integers, and let \( m \in \mathbb{Z}^+ \cup \{\infty\} \) be fixed. Let \((s_j)\) be an increasing sequence of expansion constants such that, if \( m \in \mathbb{Z}^+ \), we have
\[
s_{j+1} \geq \ell_{(j+1)m} s_j.
\]

for all \( j \in \mathbb{N} \). Now let
\[
Z_d = \bigoplus_{i=0}^{d-1} c_i \mathbb{Z} \quad \text{and} \quad K_{m} = \bigoplus_{i\in\mathbb{N}} (m \times s)_i \mathbb{Z}^{\ell_i}.
\]

Then, \( \text{asdim}_{AN}(Z_d \times K_{m}) = d + m \).

**Proof.** The lower bound is established in Lemma 1.9. For the upper bound, suppose \( m \in \mathbb{Z}^+ \). Then,
\[
K_{m} = \bigoplus_{r=0}^{m-1} \left( \bigoplus_{j\in\mathbb{N}} s_j \mathbb{Z}^{\ell_{jm+r}} \right).
\]

Since \((\ell_i)\) is increasing, for all \( j \in \mathbb{N} \) and \( r \in \{0,...,m-1\} \), we have
\[
s_{j+1} \geq \ell_{(j+1)m} s_j \geq \ell_{jm+r}s_j \geq (2\ell_{jm+r}/2)s_j = (2\text{diam}(\mathbb{Z}^{\ell_{jm+r}}))s_j \geq 1 + s_j \text{diam}(\mathbb{Z}^{\ell_{jm+r}}).
\]

Therefore, for any fixed \( r \in \{0,...,m-1\} \), the sequences \((\ell_{jm+r})\), \((\mathbb{Z}^{\ell_{jm+r}})\), and \((s_j)\) together satisfy the assumptions of Lemmas 1.11 and 1.12. Hence for all \( r \in \{0,...,m-1\} \),
\[
\text{asdim}_{AN} \left( \bigoplus_{j\in\mathbb{N}} s_j \mathbb{Z}^{jm+r} \right) = \text{asdim}_{AN} \left( \bigoplus_{j\in\mathbb{N}} \mathbb{Z}^{jm+r}, \|\cdot\|_{s} \right) = \text{asdim}_{AN} \left( \bigoplus_{j\in\mathbb{N}} \mathbb{Z}^{jm+r}, \|\cdot\|_{s_{qu}} \right) = 1.
\]
Thus, by Lemma 1.1,

\[ \text{asdim}_{\text{AN}}(K_m) \leq \sum_{r=0}^{m-1} \text{asdim}_{\text{AN}} \left( \bigoplus_{j \in \mathbb{N}} s_j \mathbb{Z}_{j^m + r} \right) \leq m, \]

and \( \text{asdim}_{\text{AN}}(Z_d) = \text{asdim}_{\text{AN}}(\mathbb{Z}^d) = d \). Therefore, \( \text{asdim}_{\text{AN}}(Z_d \times K_m) \leq d + m \).

The importance of Lemma 1.13 lies in the fact that if \((\ell_i)\) is fixed and \(m, n \in \mathbb{Z}^+ \cup \{\infty\}\) are distinct, then \(K_m\) and \(K_n\) are merely the same group with different norms. Later, we will construct two finitely generated groups \(A\) and \(B\) with subgroups that are isomorphic and bi-Lipschitz equivalent to \(K_m\) and \(K_n\), respectively. Since \(K_m\) and \(K_n\) are isomorphic, we construct a finitely generated group \(G\) which is the amalgamated product of \(A\) and \(B\) along an isomorphism between \(K_m\) and \(K_n\). The isomorphism ‘collapses’ \(K_n\), so that the Assouad–Nagata dimension of \(G\) is not much more than \(m\), while the Assouad–Nagata dimension of \(B\) is at least \(n\). To construct \(A, B,\) and \(G\) such that all of the aforementioned geometric properties hold, we use some small cancellation theory. This is the topic of the next section.

2 van KAMPEN DIAGRAMS AND THE \(C'(\lambda)\) CONDITION

The goal of this section is to prove Lemma 2.20, which states that words of a certain form are quasi-geodesic in certain central extensions of \(C'(\lambda)\) groups, where \(0 < \lambda < 1/12\). This is a generalization of [9, Lemma 5.10], originally used to construct finitely generated groups with circle-tree asymptotic cones. The proof of Lemma 2.20 is a technical argument that involves performing surgery on van Kampen diagrams.

We assume that the reader is familiar with the \(C'(\lambda)\) condition and the notion of a van Kampen diagram. However, there are myriad definitions of van Kampen diagram in the literature, and for our purposes, it is necessary to define the \(C'(\lambda)\) condition in a way which, though clearly equivalent to the usual definition, is slightly nonstandard. Therefore, in Sections 2.1 and 2.2, we fix terminology and notation, and provide all necessary definitions for the following sections.

In Section 2.3, we define signed and unsigned \(r\)-face counts, where \(r\) is a relation of a presentation. We also introduce various operations on van Kampen diagrams, and examine how each of these operations affects the signed and unsigned \(r\)-face counts. Our approach is to treat van Kampen diagrams as graphs embedded in the plane, so that the 2-cells are simply the bounded faces enclosed by the graph. In this way, we manipulate van Kampen diagrams directly in the plane and keep topological considerations to a minimum. In Section 2.4, we collect some facts about van Kampen diagrams over \(C'(1/6)\) presentations that are used in the proof of Lemma 2.20. Finally, in Section 2.5, we prove Lemma 2.20.

2.1 The \(C'(\lambda)\) condition

Let \(S\) be a set. Let \(S^{-1}\) be the set of formal inverses of \(S\), let \(1\) be a new symbol not in \(S\), and declare \(1^{-1} = 1\). Let

\[ S_1 = S \cup \{1\}, \]

\[ S_o = S \cup S^{-1} \cup \{1\}. \]
The length of a word \( w \) in the free monoid \( S^*_o \) is denoted \( |w| \). There is a unique word of length 0 called the empty word and denoted \( \varepsilon \). We define \( w^0 \) to be \( \varepsilon \) for any \( w \in S^*_o \). A word \( w \in S^*_o \) is reduced if \( w \) does not contain a subword of the form \( ss^{-1} \), or \( s^{-1}s \) for any \( s \in S \), and cyclically reduced if every cyclic shift of \( w \) (including \( w \) itself) is reduced.

Let \( R \) be a language over the alphabet \( S_o \), that is, \( R \subseteq S^*_o \). Then, \( R_* \) denotes the closure of \( R \) under taking cyclic shifts and formal inverses of its elements. We say that \( R \) is reduced if every element of \( R \) is reduced, and cyclically reduced if \( R_* \) is reduced. We say that \( R \) is cyclically minimal if it does not contain two distinct words, one of which is a cyclic shift of the other word or its inverse. That is, \( R \) is cyclically minimal if \( R \cap \{r\}_* = \{r\} \) for each \( r \in R \).

A presentation is a pair \( \langle S \mid R \rangle \), where \( S \) is a set and \( R \subseteq S^*_o \). The notation \( G = \langle S \mid R \rangle \) means that \( \langle S \mid R \rangle \) is a presentation and \( G \equiv F(S)/\langle\langle R\rangle\rangle \), where \( F(S) \) is the free group with basis \( S \), and \( \langle\langle R\rangle\rangle \) is the normal closure of \( R \) as a subset of \( F(S) \).

If \( G \) is a group generated by a set \( S \), there is a natural monoid homomorphism from \( S^*_o \) to \( G \) that evaluates a word in \( S^*_o \) as a product of generators and their inverses, and sends \( 1 \) to the identity element. If both \( G \) and \( S \) are understood, then for a word \( w \in S^*_o \) we denote the image of \( w \) under this homomorphism by \( \hat{w} \). For a group element \( g \in G \), we write \( w =_G g \) to abbreviate that \( \hat{w} = g \).

The word norm on \( G \) with respect to \( S \) is defined by

\[
\| g \|_G = \min\{|w| \mid w \in S^*_o, w =_G g\}.
\]

We omit the generating set from the notation since any other choice of finite generating set yields a norm which is bi-Lipschitz equivalent. What matters is that the generating set is fixed throughout in which the word norm plays a role. A word \( w \in S^*_o \) is called geodesic in \( G \) if \( |w| = \| \hat{w} \|_G \). If \( u, w \in S^*_o \), \( g \in G \), \( w \) is geodesic, and \( w =_G u =_G g \), then \( w \) is called a geodesic representative of \( u \) or of \( g \) in \( G \).

Given two words \( u, v \in S^*_o \), we say that \( p \) is a piece (of \( u \) and of \( v \)) if there exists \( u' \in \{u\}_*, v' \in \{v\}_* \) such that \( p \) is a common prefix of \( u' \) and \( v' \).

**Definition 2.1.** Let \( S \) be a set, \( R \subseteq S^*_o \) a language, and \( \lambda \) a real number with \( 0 < \lambda < 1 \). Then, \( R \) satisfies \( C'(\lambda) \) if, whenever \( u, v \in R \) and \( u' \in \{u\}_*, v' \in \{v\}_* \) witness that \( p \) is a piece of \( u \) and \( v \), then either \( u' = v' \) or \( |p| < \lambda \min(|u|, |v|) \).

In this case, we say that \( R \) is a \( C'(\lambda) \) language. If \( G \) is a group and \( G = \langle S \mid R \rangle \) for some \( C'(\lambda) \) language \( R \), then \( \langle S \mid R \rangle \) is called a \( C'(\lambda) \) presentation and \( G \) is called a \( C'(\lambda) \) group.

In most treatments of the \( C'(\lambda) \) condition, it is assumed that \( R = R_* \), and a piece is defined to be a common prefix of two distinct words in \( R \). In our case, however, it is important to assume that \( R \) is cyclically minimal (in particular \( R \neq R_* \)), in order to ensure that the signed \( r \)-face count (Definition 2.5 below) is well defined. For this reason, we give the definition above, which, though not the usual definition of the \( C'(\lambda) \) condition, is clearly equivalent.

### 2.2 van Kampen diagrams

Let \( \Gamma \) be a connected graph. By a path in \( \Gamma \) we mean a combinatorial path, which may have repeated edges or vertices: in graph-theoretic terms, our ‘path’ is really a walk. Since points in the interiors of edges generally do not matter to us, we write \( x \in \Gamma \) to mean \( x \in V(\Gamma) \). Likewise, if \( \alpha \) is a path in \( \Gamma \), then \( x \in \alpha \) means that \( x \) is a vertex visited by \( \alpha \).
Let $\Gamma$ be any directed graph, and suppose $\text{Lab} : E(\Gamma) \rightarrow S_1$ (see (4) above) is a function which assigns labels from $S_1$ to the edges of $\Gamma$. Then, we extend $\text{Lab}$ to a map from the set of all paths in $\Gamma$ to $S_\ast$ in the following natural way.

- If $e = (x, y)$ is a directed edge labeled $s$, then $\text{Lab}(x, e, y) = s$ and $\text{Lab}(y, e, x) = s^{-1}$.
- If $\alpha = (x_0, e_1, x_1, \ldots, x_{n-1}, e_n, x_n)$ is a path, then

$$\text{Lab}(\alpha) = \text{Lab}(x_0, e_1, x_1) \text{Lab}(x_1, e_2, x_2) \cdots \text{Lab}(x_{n-1}, e_n, x_n).$$

For a path $\alpha$ we define $\ell(\alpha)$, the length of $\alpha$, to be the number of edges traversed by $\alpha$, counting multiplicity. Equivalently, $\ell(\alpha) = |\text{Lab}(\alpha)|$.

A plane graph is a graph which is topologically embedded in $\mathbb{R}^2$. A face of a plane graph $M$ is the closure of a connected component of $\mathbb{R}^2 \setminus M$. Let $F$ be a face of a finite directed plane graph with edges labeled by elements of $S_1$. Choosing a base point $x \in \partial F$ and an orientation counterclockwise (+) or clockwise (−), there is a unique circuit which traverses $\partial F$ exactly once, called the boundary path and denoted $(\partial F, x, \pm)$. If all properties of $(\partial F, x, \pm)$ that we care about are preserved after changing its base point and orientation, then we leave these choices out of the notation and write $\partial F$. The boundary label of $F$ is $\text{Lab}(\partial F, x, \pm)$, sometimes denoted by just $\text{Lab}(\partial F)$. We write $\partial M$ instead of $\partial F$ if $F$ is the unbounded face; from now on, ‘face’ will mean ‘bounded face’ unless otherwise stated.

**Definition 2.2.** A van Kampen diagram over a presentation $\langle S \mid R \rangle$ is a finite, connected, directed plane graph $M$ with edges labeled by elements of $S_1$, such that if $F$ is a face of $M$, then either $\text{Lab}(\partial F) \in R_\ast$ or $\text{Lab}(\partial F) = F(S)^1$. An edge is essential if it is labeled by an element of $S$, and inessential if it is labeled by 1. A face $F$ is called essential if $\text{Lab}(\partial F) \in R_\ast$ and inessential if $\text{Lab}(\partial F) = F(S)^1$. If $R$ is cyclically reduced, then these cases are mutually exclusive. A face with boundary label $r \in R$ is called an $r$-face. We call a van Kampen diagram bare if it contains no inessential faces, and padded otherwise.

A subdiagram of a van Kampen diagram $M$ is a simply connected union of faces of $M$. If $M$ is a van Kampen diagram and $D$ is a subdiagram of $M$, then we call $D$ simple if $\partial D$ is a simple closed curve in the plane. Likewise, a face $F$ of $M$ is called simple if $\partial F$ is a simple closed curve.

Let $\alpha$ and $\beta$ be two paths in a van Kampen diagram. Then, we say that $\alpha \cap \beta$ is trivial if it contains at most one vertex, and nontrivial otherwise. We say that $\alpha$ and $\beta$ intersect simply if $\alpha \cap \beta$ a single subpath of both $\alpha$ and $\beta$ or the reverse path of $\beta$. Note that this is not the same as saying that $\alpha \cap \beta$ is connected. We apply this terminology to faces as well. For example, if we say that $F$ and $\alpha$ intersect simply, it means that there is a choice of base point $x \in \partial F$ such that $(\partial F, x, +)$ and $\alpha$ intersect simply. If we say that two faces $F$ and $F'$ intersect simply, it means that $(\partial F, x, +)$ and $(\partial F', x, -)$ intersect simply for some $x \in \partial F \cap \partial F'$. Let $M$ be a van Kampen diagram, and suppose $F$ and $F'$ are distinct faces of $M$. Then, we say that $F$ and $F'$ cancel if there exists an edge $e = (x, y)$ in $\partial F \cap \partial F'$ such that $\text{Lab}(\partial F, x, +) = \text{Lab}(\partial F', x, -)$. A van Kampen diagram is called reduced if no two of its faces cancel. We have the following geometric interpretation of the $C'(\lambda)$ condition, which follows immediately from the definition.
Lemma 2.3. Let $\langle S \mid R \rangle$ be a presentation where $R$ satisfies $C'(\lambda)$, and let $M$ be a van Kampen diagram over $(S \mid R)$. Suppose $F, F'$ are essential faces of $M$ and $\alpha$ is a common subpath of $\partial F$ and $\partial F'$. Then, either $F$ and $F'$ cancel, or $\ell(\alpha) < \lambda \min(\ell(\partial F), \ell(\partial F'))$.

Whenever $G$ is a group generated by $S$, the Cayley graph of $G$ with respect to $S$ is denoted $\Gamma(G, S)$.

Lemma 2.4 (van Kampen Lemma [10, chapter V, section 1]). Let $G = \langle S \mid R \rangle$ and $w \in S^*$. Then, $w = G_1$ if and only if there exists a van Kampen diagram $M$ over $(S \mid R)$ and $x \in \partial M$ such that $\text{Lab}(\partial M, x, +) = w$. Furthermore, given $g \in G$, there exists a combinatorial map $f : M \to \Gamma(G, S)$ preserving labels and orientations of edges, such that $f(x) = g$. In particular, $f$ does not increase distances, that is, is $1$-Lipschitz.

2.3 | Operations on van Kampen diagrams

Given a van Kampen diagram $M$ over a presentation $(S \mid R)$, there are various ways to deform $M$ within the plane to get another van Kampen diagram $M'$. To check that the resulting graph $M'$ is really a van Kampen diagram, it suffices to show that the operation preserves connectedness and produces a planar embedding of $M'$. If one also requires that $M'$ is a van Kampen diagram over the same presentation, one needs to check that any new faces enclosed by the operation have a boundary label which is either in $R_*$ or equal to the identity in $F(S)$. In this section, we list a few operations on van Kampen diagrams that are needed for the proof of Lemma 2.20. In our case, it will be necessary to keep track of how each operation affects the boundary label $\text{Lab}(\partial M)$, as well as two quantities that we call the signed and unsigned $r$-face counts.

Definition 2.5. Let $M$ be a van Kampen diagram over a presentation $(S \mid R)$. Let $r \in R$. Then, the (unsigned) $r$-face count $\kappa(M, r)$ is the total number of $r$-faces of $M$.

Note that a face $F$ of $M$ contributes to the $r$-face count of at most one $r \in R$ if and only if $R$ is cyclically minimal. In particular, we have the equality $\sum_{r \in R} \kappa(M, r) = |\{F \mid F$ is a face of $M\}|$ if and only if $R$ is cyclically minimal.

Definition 2.6. Let $M$ be a van Kampen diagram over a presentation $(S \mid R)$ where $R$ is cyclically reduced, and let $r \in R$. Then, the signed $r$-face count $\sigma(M, r)$ is defined as follows.

- If $F$ is a face of $M$, then

$$\sigma(F, r) = \begin{cases} 1 & \text{if } \text{Lab}(\partial F, x, +) = r \text{ for some } x \in \partial F \\ -1 & \text{if } \text{Lab}(\partial F, x, -) = r \text{ for some } x \in \partial F \\ 0 & \text{otherwise.} \end{cases}$$

- $\sigma(M, r) = \sum\{\sigma(F, r) \mid F \text{ is a face of } M\}$.

It is an easy exercise to show that for any word $r \in S^*_\circ$, $r$ is equal to a cyclic shift of $r^{-1}$ if and only if $r = F(S)_1$. Thus, the assumption that $R$ is cyclically reduced ensures that each of the cases in the
definition of \( \sigma(F, r) \) are mutually exclusive, so \( \sigma(M, r) \) is well defined. If in addition we assume that \( R \) is cyclically minimal, then each face \( F \) of \( M \) contributes to \( \sigma(M, r) \) for at most one \( r \in R \).

Note that if \( F \) and \( F' \) are two faces of \( M \) that cancel with each other, then \( \sigma(F, r) = -\sigma(F', r) \) for all \( r \in R \).

**Operation 2.7** (Removing an inessential edge). Suppose \( e = (x, y) \) is an inessential edge of a van Kampen diagram \( M \) over a presentation \( \langle S \mid R \rangle \), where \( R \) is cyclically reduced and cyclically minimal, and \( \text{Lab}(\partial M) \) is cyclically reduced. Then, \( e \) is on the boundary of exactly two inessential bounded faces. There are two possibilities.

(a) If \( x \neq y \), contract \( e \) to remove it. This will produce a connected, planar embedding of the new graph. This changes two inessential faces with labels \( 1u \) and \( 1v \) to two inessential faces with labels \( u \) and \( v \). Since \( R \) is cyclically reduced, this does not affect the \( r \)-face count for any \( r \in R \).

(b) If \( x = y \), delete \( e \) to remove it. Since \( e \) is a loop, this will leave the graph connected. This replaces two inessential faces on either side of \( e \) with labels \( u1 \) and \( 1v \) with a single inessential face labeled \( uv \). Again since \( R \) is cyclically reduced, this operation does not affect \( \sigma(M, r) \) for any \( r \in R \).

Note that (b) cannot introduce new self-intersections in the boundary path of any face of \( M \). Also, since \( \text{Lab}(\partial M) \) is cyclically reduced, neither (a) nor (b) affects \( \text{Lab}(\partial M) \).

**Operation 2.8** (Removing a simple subdiagram with trivial boundary label). Let \( M \) be a van Kampen diagram over \( \langle S \mid R \rangle \), where \( R \) and \( \text{Lab}(\partial M) \) are both cyclically reduced. Suppose \( M \) contains a simple subdiagram \( D \) such that \( \partial D \) contains no inessential edges and \( \text{Lab}(\partial D) =_{F(S)} 1 \). Then, \( \partial D = \alpha_+ \alpha_- \), where \( \text{Lab}(\alpha_-) = \text{Lab}(\alpha_+)^{-1} \). We may then remove \( D \) by replacing \( D \) with a simple inessential face \( F \) and deforming \( \alpha_+ \) onto \( \alpha_- \) through the interior of \( F \). This does not affect the boundary label of \( M \) (Figure 1).

Note that if \( F \) and \( F' \) are simple faces that intersect simply, and \( F \) cancels with \( F' \), then \( F \cup F' \) is a simple subdiagram of \( M \) with trivial boundary label, which may be removed by applying Operation 2.8. Perhaps surprisingly, Operation 2.8 does not always preserve the signed \( r \)-face count, as the following example shows.

**Example 2.9.** Figure 2 depicts a van Kampen diagram \( M \) over the presentation \( \langle a, b \mid a^2, aba^{-1}b \rangle \) with boundary label \( bb^{-1} \), such that \( \sigma(M, aba^{-1}b) = 2 \).
FIGURE 2 A van Kampen diagram in Operation 2.8 does not preserve the signed \( r \)-face count.

However, Operation 2.8 does preserve the signed \( r \)-face count of van Kampen diagrams over \( C'(1/6) \) presentations. This is because \( C'(1/6) \) presentations are aspherical. The definition of a spherical van Kampen diagram is the same as that of a van Kampen diagram with \( \mathbb{R}^2 \) replaced by \( S^2 \): in particular, every face is bounded. A presentation \( \langle S \mid R \rangle \) is aspherical if every bare spherical van Kampen diagram over \( \langle S \mid R \rangle \) contains a pair of faces that cancel. The following is a special case of a lemma of Olshanskii.

**Lemma 2.10** [11, Lemma 31.1 part 2)]. *Let \( \langle S \mid R \rangle \) be an aspherical presentation, and suppose \( M \) is a van Kampen diagram over \( \langle S \mid R \rangle \) with boundary label \( w \), where \( w = F(S) \). Then, \( \sigma(M, r) = 0 \) for all \( r \in R \).*

**Operation 2.11** (Padding a vertex). In this construction, we take a vertex and ‘blow up’ the neighborhood around it by adding inessential faces. If the vertex is a point of self-intersection of a face \( F \) of a van Kampen diagram, we can use this operation to take the boundary path \( \partial F \) and ‘push it off of itself’ near \( x \). The resulting van Kampen diagram may still have faces with self-intersecting boundary paths, but these will all be inessential faces with a small number of possible boundary labels, and are thus easier to reason about. While the following few paragraphs of formalism are necessary to define the operation precisely, to understand the operation it is better to look at Figure 3.

Suppose \( x \) is a vertex of \( M \) which appears twice in the boundary path of some face (bounded or unbounded) of \( M \). Choose \( \varepsilon > 0 \) small enough so that \( B(x, \varepsilon) \subset \mathbb{R}^2 \) contains only the half-edges
incident to $x$, and no other edges or vertices. Now $B(x, \epsilon) \setminus M$ consists of finitely many connected components: let these be denoted $C_0, C_1, \ldots, C_k$, where $C_0$ is chosen arbitrarily and $C_0, C_1, \ldots, C_k$ are numbered counterclockwise around $x$. From the perspective of one standing at $x$ and facing $C_i$, each component $C_i$ is bounded by exactly two half-edges on the right and left, call them $h_{(i,R)}$ and $h_{(i,L)}$. (In case these half-edges are equal, that is, when $k = 0$ and $x$ is a degree-one vertex, we still distinguish between $h_{(0,L)}$ and $h_{(0,R)}$ by thinking of them as the same half-edge approach from the left and right, respectively). Note that $h_{(i,R)} = h_{(i+1,L)}$ for each $i$ taken modulo $k + 1$. Let us also denote the full edge that $h_{(i,R)}$ is a half-edge of by $e_{(i,R)}$. For this operation, the distinction between half-edges and full edges is important because one full edge may represent two distinct half-edges, that is, $e_i$ may be equal to $e_j$ for some $i \neq j$, and this occurs exactly when $e_i = e_j$ is a loop at $x$.

For each $i \in \{0, \ldots, k\}$, insert a vertex $x_i$ into $C_i$, and connect it to $x$ with an inessential edge. Now insert two half-edges $h'_{(i,R)}$ and $h'_{(i,L)}$, each having one endpoint at $x_i$ and running parallel to $h_{(i,R)}$ and $h_{(i,L)}$ on the left and right, respectively, in a sufficiently small tubular neighborhood within $C_i$. Regarding the next step of extending $h'_{(i,L)}$ and $h'_{(i,R)}$ to full edges and connecting them to the rest of the graph, there are two cases. We state how to extend $h'_{(i,R)}$ to a full edge $e_{(i,R)}$, after which extending $h'_{(i,L)}$ to $e_{(i,L)}$ will follow easily.

(a) If $e_i$ has distinct endpoints $x$ and $y$, then connect $h'_{(i,R)}$ to $y$ to form $e_{(i,R)}$. We then set $	ext{Lab}(x, e_{(i,R)}, y) = \text{Lab}(x, e_i, y)$.

(b) If $e_i$ is a loop, then $e_i = e_j$ for some $j \neq i$, and $e_i$ includes half-edges $h_{(i,R)}$ and $h_{(j,R)} = h_{(j-1,L)}$. Extend $h'_{(i,R)}$ outside of $B(x, \epsilon)$ following $e_i$ on the left in a sufficiently small neighborhood. Now when $e_i$ returns to $B(x, \epsilon)$, from the perspective of one standing at $x$ and facing $C_{j-1}$, the newly extended $h'_{(i,R)}$ appears to be coming in slightly to the right of $e_j$. Therefore, we connect it to $h'_{(j-1,L)}$, which is also tracking $e_j$ on the right and has endpoint $x_{j-1}$, to form $e_{(i,R)}$, and declare this to be $e_{(j-1,L)}$ as well. We have that $e_{(i,R)} = e_{(j-1,L)}$ has distinct endpoints $x_i$ and $x_{j-1}$, and extends both $h'_{(i,R)}$ and $h'_{(j-1,L)}$. We then set $	ext{Lab}(x_i, e_{(i,R)}, x_j) = \text{Lab}(x, e_i, x)$.

We use the same process up to symmetry to extend $h'_{(i,L)}$ to $e_{(i,L)}$ for each $i \in \{0, \ldots, k\}$, but we do not duplicate edges already formed in Case (b).

This operation preserves the boundary label of $M$, and each face originally incident to $x$ has the same boundary label as before. However, $x$ is no longer a point of self-intersection of any face originally in $M$. All new faces are inessential, with boundary label $1ss^{-1}$ in Case (a), or $1111$ or $1ss^{-1}$ for some $s \in S$ in Case (b).

**Operation 2.12** (Quotienting simple faces). Suppose $G = \langle S \mid R_G \rangle$ and $H = \langle S \mid R_H \rangle$ is a quotient of $G$, so every word in $R_G$ represents the identity element of $H$. Suppose $M_G$ is a van Kampen diagram over $\langle S \mid R_G \rangle$. Let $F$ be a simple face of $M_G$, and let $M_F$ be a chosen van Kampen diagram over $\langle S \mid R_H \rangle$ with boundary label $\text{Lab}(\partial F)$. Then, we may quotient $F$ to a copy of $M_F$ without affecting the boundary label of $M_G$; see Figure 4. Applying this operation once produces a van Kampen diagram over $\langle S \mid R_G \cup R_H \rangle$ with boundary label $\text{Lab}(\partial F)$. Then, we may quotient $F$ to a copy of $M_F$ without affecting the boundary label of $M_G$; see Figure 4. Applying this operation once produces a van Kampen diagram over $\langle S \mid R_G \cup R_H \rangle$. If $F$ is the last face of $M_G$ with label in $R_G \setminus R_H$, then this results in a van Kampen diagram over $\langle S \mid R_H \rangle$. Thus, if this operation can be applied to every essential face of $M_G$ in sequence, then we obtain a ‘quotient van Kampen diagram’ $M_H$ over $\langle S \mid R_H \rangle$ with the same boundary label as $M_G$. 


Operation 2.13 (Excising a subpath of $\partial M$). Let $M$ be a van Kampen diagram over a presentation $\langle S \mid R \rangle$, where $R$ is cyclically minimal and cyclically reduced. Let $z \in \partial M$, and suppose we can write $(\partial M, z, +)$ as $\alpha * \beta$, where $\alpha$ and $\beta$ are paths of positive length. Suppose $\alpha = \alpha_0 * \rho * \alpha_1$, where $\text{Lab}(\rho)$ is a cyclic shift of $r \pm 1$ for some $r \in R$. Let $x$ be the initial and $y$ the terminal vertex of $\rho$, and suppose $x \neq y$. Then, we may contract $x$ to $y$ through the unbounded face, identifying the two vertices to obtain a new van Kampen diagram $M'$. Now $M'$ has exactly one new face $F'$, where $(\partial F', x, -) = \rho$, so $M'$ is a van Kampen diagram over the same presentation $\langle S \mid R \rangle$. Also, $(\partial M', z, +) = \alpha' * \beta$, where $\alpha' = \alpha_0 * \alpha_1$: see Figure 5. Note that $\rho$ may intersect itself, in which case $\partial F'$ will have self-intersections in $M'$, but this is fine. The only topological feature of $M$ which is essential to this operation is that $x$ and $y$ are distinct.

Now $\ell(\alpha') = \ell(\alpha) - |r|$, and $\beta$ is unaffected by the operation. Also, for all $r' \in R$,

$$\kappa(M', r') = \begin{cases} \kappa(M, r^\pm) + 1 & \text{if } r' = r \\ \kappa(M, r') & \text{otherwise.} \end{cases}$$

$$\sigma(M', r') = \begin{cases} \sigma(M, r') - 1 & \text{if } r' = r \text{ and } \text{Lab}(\rho) \text{ is a cyclic shift of } r \\ \sigma(M, r') + 1 & \text{if } r' = r \text{ and } \text{Lab}(\rho) \text{ is a cyclic shift of } r^{-1} \\ \sigma(M, r') & \text{otherwise.} \end{cases}$$

The first equality uses the assumption that $R$ is cyclically minimal, while the second uses the assumption that $R$ is cyclically reduced.
2.4 Reductions that preserve signed $r$-face counts

Later we will need to use Lemma 2.25, a result that applies only to bare, reduced van Kampen diagrams over $C'(1/6)$ presentations. At the same time, we would like to apply this result to van Kampen diagrams with signed $r$-face counts that are carefully controlled. Thus, we need to establish a method of taking a van Kampen diagram over a $C'(1/6)$ presentation, and making it bare and reduced without affecting the signed $r$-face counts. In this subsection, we develop such a process, which is encapsulated in Lemma 2.18. We then prove Lemma 2.19, which allows us to construct certain ‘quotient’ van Kampen diagrams with controlled $r$-face counts.

**Lemma 2.14.** Let $M$ be a van Kampen diagram over $\langle S \mid R \rangle$ such that $R$ is cyclically reduced, and $\text{Lab}(\partial M)$ is cyclically reduced. Then, there exists a van Kampen diagram $M'$ such that all of the following conditions hold.

(a) $\text{Lab}(\partial M') = \text{Lab}(\partial M)$.
(b) $\sigma(M', r) = \sigma(M, r)$ for all $r \in R$.
(c) Every inessential face of $M'$ has boundary label $1$, $11$, $111$, $1111$, $1s1s^{-1}$, $1ss^{-1}$, or $ss^{-1}$ for some $s \in S$.
(d) All inessential edges of $M'$ are loops.

**Proof.** Let $I$ be the set of all inessential faces of $M$ whose boundary labels are not equal to $1$, $11$, $111$, $1s1s^{-1}$, $1ss^{-1}$ or $ss^{-1}$ for some $s \in S$. Let $F \in I$. Pad vertices of $\partial F$ (Operation 2.11) until $F$ is simple. Since each inessential face added in the process has boundary label $1111$, $1s1s^{-1}$, or $1ss^{-1}$ for some $s \in S$, this does not increase $|I|$.

Suppose $\partial F$ contains an inessential edge $e$. If $\ell(\partial F) = 1$, then $\text{Lab}(\partial F) = 1$ and $F \notin I$, so assume $\ell(\partial F) \geq 2$. Since $\partial M$ and $R$ are both cyclically reduced, $e$ lies on the boundary of exactly two inessential, bounded faces, one of which is $F$: call the other one $F'$. Then, for some $u, u' \in S_o^*$, we have $\text{Lab}(F) = 1u$ and $\text{Lab}(F') = 1u'$. We know that $e$ must have distinct endpoints since $\partial F$ is a simple closed curve and $\ell(\partial F) \geq 2$. Therefore, we may remove $e$ using Operation 2.7(a). This changes the boundary label of $F$ from $1u$ to $u$, and the boundary label of $F'$ from $1u'$ to $u'$. Note that this can only move $F$ or $F'$ out of $I$ if either $F$ or $F'$ were previously in $I$, but not the other way around. Since this operation does not increase $|I|$, and we may repeat until either $F \notin I$ or $\partial F$ contains no inessential edges.

Since $F$ is simple, $\text{Lab}(\partial F') = F(S)$, and $\partial F$ contains no inessential edges, we may remove $F$ using Operation 2.8. This reduces $|I|$ by 1. Since $\text{Lab}(\partial M)$ is cyclically reduced, none of the previous operations affect $\text{Lab}(\partial M)$. Since only inessential faces were removed, and $R$ is cyclically reduced, $\sigma(M, r)$ is also preserved for all $r \in R$. Repeating this process, we obtain a diagram $M'$ for which (a) and (b) hold and $|I| = 0$, that is, such that (a)–(c) hold. At this point we may repeatedly apply Operation 2.7(a) to remove all inessential edges of $M'$ with distinct endpoints, so that (d) holds in $M'$. Reasoning as in the previous paragraph, one can see that this does not interfere with conditions (a)–(c). Thus, (a)–(d) hold in $M'$, finishing the construction. □

**Lemma 2.15.** Let $G$ be a group given by presentation $\langle S \mid R \rangle$, where $R$ is cyclically reduced, and $s \not\in G$ 1 for any $s \in S$. Let $M$ be a van Kampen diagram over $\langle S \mid R \rangle$ such that $\text{Lab}(\partial M)$ is cyclically reduced. Then, there exists a van Kampen diagram $M'$ over $\langle S \mid R \rangle$ such that all of the following conditions hold.
(a) Lab(∂M′) = Lab(∂M).
(b) σ(M′, r) = σ(M, r) for all r ∈ R.
(c) Every inessential face of M′ is contained in a simple subdiagram whose boundary label is equal to ss⁻¹ for some s ∈ S.

Proof. We may assume that we have a van Kampen diagram M′ that satisfies (a)–(d) of Lemma 2.14. We prove here that, in the presence of the assumption that s ≠ G 1 for all s ∈ S, it follows that M′ also satisfies conclusion (c) of the current lemma.

By (d) of Lemma 2.14, every inessential edge of M is a loop. Let us call an inessential edge outermost if it is not enclosed by any other inessential edge. All inessential edges of M are enclosed by some outermost inessential edge.

Let e be an outermost inessential edge. Then, since Lab(∂M) and R are reduced, e must be on the boundary of two inessential, bounded faces. Since e is a loop, one of these faces is external to e, and one is enclosed by e. Let F be the face on the exterior of e. Since e is outermost by assumption, it must be the case that Lab(∂F) = 1s1s⁻¹ or 1ss⁻¹ for some s ∈ S; otherwise, e would be enclosed by another inessential edge. Since all inessential edges are loops and all s-labeled edges have distinct endpoints, this implies that F is enclosed in a simple subdiagram with boundary ss⁻¹: see Figure 6a,b. Since any outermost inessential edge is enclosed in a simple subdiagram with boundary label ss⁻¹ for some s ∈ S, it follows that any inessential edge, and thus any inessential face containing an inessential edge, is also contained in such a subdiagram.

On the other hand, suppose F is an inessential face of M that does not contain an inessential edge. Then, Lab(∂F) = ss⁻¹ by (c) of Lemma 2.14. But all s-labeled edges have distinct endpoints since s ≠ G 1, therefore F itself is a simple subdiagram, as in Figure 6c.

Corollary 2.16. Let G be a group given by an aspherical presentation (S | R), where R is cyclically reduced and s ≠ G 1 for all s ∈ S. Let M be a van Kampen diagram over (S | R) such that Lab(∂M) is cyclically reduced. Then, there exists a van Kampen diagram M′ such that all of the following conditions hold.

(a) Lab(∂M′) = Lab(∂M).
(b) σ(M′, r) = σ(M, r) for all r ∈ R.
(c) M′ is bare.
Proof. We may assume that $M'$ satisfies (a)–(c) of Lemma 2.15. Now all inessential faces of $M'$ are contained in simple subdiagrams of $M'$ with boundary label $ss^{-1}$ for some $s \in S$. Thus, we may make $M'$ bare by repeatedly applying Operation 2.8. Operation 2.8 always preserves the boundary label of a van Kampen diagram, so (a) holds. Since $\langle S \mid R \rangle$ is aspherical, it follows from Lemma 2.10 that each application of Operation 2.8 preserves $\sigma(M', r)$ for all $r \in R$. Thus, (a)–(c) hold for $M'$, and we are done.

Often one would like to take a van Kampen diagram $M$ which is not reduced, and reduce it using Operation 2.8. However, the canceling faces may not be simple, or may not intersect each other simply, so their union might not be a simple subdiagram of $M$. A common solution is to pad the van Kampen diagram with inessential faces. However, if $M$ is a van Kampen diagram over a $C'(1/6)$ presentation, then $M$ is topologically well behaved enough to perform this operation without the use of inessential faces. We make this claim precise in the following lemma and corollary, which collect some well-known but useful facts about van Kampen diagrams over $C'(1/6)$ presentations.

**Lemma 2.17.** Let $G$ be a group given by presentation $\langle S \mid R \rangle$, where $R$ is cyclically reduced and satisfies $C'(1/6)$, and let $M$ be a bare van Kampen diagram over $\langle S \mid R \rangle$. Then,

(a) every face of $M$ is simple;
(b) (Greendlinger Lemma) if $M$ is reduced, then there exists a face $F$ of $M$ such that $\partial F$ intersects $\partial M$ in a common subpath of length at least $1/2c(\partial F)$;
(c) if $M$ is reduced, then every two faces of $M$ that intersect nontrivially, intersect simply;
(d) if $M$ is not reduced, then there exist a pair of faces that cancel and intersect simply;
(e) if $r \in R$ and $u$ is a proper prefix of some $r' \in \{r\}_*$, then $u \neq_1 1$. In particular, for all generators $s \in S$, if $s \not\in R$ then $s \neq_1 1$.

Proof. For a proof of (a) and the Greendlinger Lemma, we refer the reader to [10, chapter V], Lemma 4.1 and Theorem 4.4, respectively. Parts (c)–(e) are left as an exercise to the reader (hint: let $M$ be a minimal counterexample, and use the Greendlinger Lemma).

With this in mind, we return to the task of this subsection, namely to prove Lemma 2.19.

**Lemma 2.18.** Let $M$ be a van Kampen diagram over a $C'(1/6)$ presentation $\langle S \mid R \rangle$, where $R$ is cyclically reduced and $|r| \geq 2$ for all $r \in R$. Then, there exists a van Kampen diagram $M'$ over $\langle S \mid R \rangle$ such that

(a) $\text{Lab}(\partial M') = \text{Lab}(\partial M)$,
(b) $\sigma(M', r) = \sigma(M, r)$ for all $r \in R$,
(c) $M'$ is bare and reduced.

Proof. We may assume that $M'$ satisfies (a)–(c) of Corollary 2.16. Thus, we only have to show that it is possible to transform $M'$ so that it is reduced, while preserving the boundary label and signed $r$-face count for each $r \in R$, and without adding any inessential faces.

Suppose $M'$ is not reduced. Since $M'$ is bare, by Lemma 2.17(a) and (d) there exist two simple faces $F$ and $F'$ that cancel and intersect simply. Thus, $F \cup F'$ is a simple subdiagram of $M$ with trivial boundary label. Now remove $F \cup F'$ with Operation 2.8. Since $\langle S \mid R \rangle$ is $C'(1/6)$, and
therefore aspherical, this operation preserves $\sigma(M', r)$ for all $r \in R$. Repeating, we end up with a reduced van Kampen diagram.

Lemma 2.19. Let $G, H$ be groups given by presentations

$$G = \langle S \mid R_G \rangle,$$

$$H = \langle S \mid R_H \rangle,$$

where $\langle S \mid R_H \rangle$ is a cyclically reduced $C'(1/6)$ presentation, and $|r_H| \geq 2$ for all $r_H \in R_H$. Suppose $r_G =_H 1$ for all $r_G \in R_G$, so $H$ is a quotient of $G$. Let $M_G$ be a van Kampen diagram over $\langle S \mid R_G \rangle$, and for each essential face $F$ of $M_G$, let $M_F$ be a van Kampen diagram over $\langle S \mid R_G \rangle$ with boundary label Lab(∂$F$). Then, there exists a ‘quotient van Kampen diagram’ $M_H$ over $\langle S \mid R_H \rangle$ such that

(a) $\text{Lab}(\partial M_G) = \text{Lab}(\partial M_H)$,

(b) for all $r_H \in R_H$, $\sigma(M_H, r_H) = \sum \{\sigma(M_F, r_H) \mid F$ is an essential face of $M_G\}$,

(c) $M_H$ is bare and reduced.

Note that $R_G$ is not even assumed to be cyclically reduced, so in fact $M_G$ may have essential faces that are also inessential. This is important because later, Lemmas 3.2 and 3.7 will both be proved by applying Lemma 2.20 (which depends on this lemma) to a certain presentation, which in Lemma 3.2 is obviously not cyclically reduced. Removing this natural restriction on $R_G$ thus allows us to hit two birds with one stone, avoiding the need for a separate proof of Lemma 3.2.

Proof. Let $F$ be an essential face of $M_G$. Repeatedly pad vertices of $\partial F$ until $F$ is simple, then quotient $F$ to $M_F$. Repeat this process as many times as necessary to quotient all essential faces that were originally in $M_G$. Since padding vertices and quotienting simple faces preserve the boundary label, we obtain a van Kampen diagram $M_H$ over $\langle S \mid R_H \rangle$, possibly with many inessential faces, such that $\text{Lab}(\partial M_H) = \text{Lab}(\partial M_G)$. Thus, (a) holds.

Now $R_H$ is cyclically reduced, so $\sigma(M_H, r_H)$ is well defined for all $r_H \in R_H$. Since each essential face $F_H$ of $M_H$ belongs to $M_F$ for exactly one essential face $F$ of $M_G$, it follows that (b) holds as well.

Now since $R_H$ satisfies the hypotheses of Lemma 2.18, we may ensure that (c) holds, without interfering with conditions (a) or (b).

2.5 | A technical lemma

This section is devoted to proving the following lemma. In essence it is similar to [9, Lemma 5.10], but for our purposes we need the more general version stated here. In order to avoid constantly reiterating the assumptions, the notation used in this lemma will be ‘globally fixed’ for this section. Thus, until the next section, $G$ will always refer to the group with presentation given in Lemma 2.20, etc. Any new notation introduced in the body of this section will also remain fixed until the beginning of the next section.
Lemma 2.20. Let $\lambda$ be a real number such that $0 < \lambda < 1/12$. Let $\{ c_i \mid i \in \mathbb{N} \}$ be a set of positive integers, where each $c_i \geq 2$. Let $S$ be a finite set. Let

$$U = \{ u_i \mid i \in \mathbb{N} \} \subset S^*$$

$$V = \{ v_i \mid i \in \mathbb{N} \} \subset S^*$$

be languages, and let $\bar{u} \in S^*$ be a word, such that the following conditions are satisfied for all $i, i' \in \mathbb{N}$.

(a) $U \cup V$ is cyclically minimal and cyclically reduced, and satisfies $C'(\lambda)$.
(b) $2 \leq |u_i| \leq |v_i|$.
(c) If $p$ is a piece of $\bar{u}$ and $u_i$, then $|p| < \lambda |u_i|$, and the same statement holds if $u_i$ is replaced with $v_i$.
(d) If $u_i = u_{i'}$, $v_i = v_{i'}$, or $u_i = v_{i'}$, then $i = i'$.

Now let

$$G = \langle S \mid R_G \rangle := \langle S \mid [s, u_i], u_i^{c_i}, u_i v_i^{-1} : s \in S, i \in \mathbb{N} \rangle,$$

$$H = \langle S \mid R_H \rangle := \langle S \mid U \cup V \rangle = \langle S \mid u_i, v_i : i \in \mathbb{N} \rangle.$$

Let $(k_i)$ be a sequence of integers where $|k_i| \leq c_i/2$ for all $i \in \mathbb{N}$, and $k_i = 0$ for all but finitely many $i \in \mathbb{N}$. Let $u \in S^*$ be a word of the form

$$u = \bar{u} \prod_{i=0}^{\infty} u_i^{k_i}.$$ 

Then, $|u| \leq \frac{3}{1-12\lambda} \| u \|_G$.

Note that condition (c) is weaker than the assertion that $U \cup V \cup \{ \bar{u} \}$ satisfies $C'(\lambda)$, since it only goes one way. For example, it may happen that $|p| = |\bar{u}|$, and indeed this does occur in our final construction in Section 4. If condition (d) seems strange, it is because we really have two extremal cases in mind: one where $U = V$ and $u_i = v_i$ for all $i \in \mathbb{N}$, and one where $U$ and $V$ are disjoint.

The former will be applied later to prove Lemma 3.2, and the latter to prove Lemma 3.7. As stated, Lemma 2.20 generalizes both extremes and includes many cases in between.

We now begin our proof. If $U \cup V$ is $C'(\lambda)$, then so is $(U \cup V \cup \{ u_i^{-1} \}) \setminus \{ u_i \}$. Therefore, assume without loss of generality that all $k_i$ are nonnegative.

Let $w$ be a geodesic representative of $u$ in $G$. Then, $uw^{-1} \rightarrow_1 1$, so by the van Kampen Lemma, there exists a van Kampen diagram $M_G$ with $\text{Lab}(\partial M_G) = uw^{-1}$.

Lemma 2.21. There exists a van Kampen diagram $M_H$ over $\langle S \mid U \cup V \rangle$ such that

(a) $M_H$ is bare and reduced,
(b) $\partial M_H = \alpha \ast \beta$, where $\text{Lab}(\alpha) = u$ and $\text{Lab}(\beta) = w^{-1}$,
(c) $\sigma(M_H, u_i) + \sigma(M_H, v_i) \equiv 0 \mod c_i$ for all $i \in \mathbb{N}$,
(d) $\sigma(M_H, u_i) \equiv 0 \mod c_i$ for all $i \in \mathbb{N}$ such that $u_i = v_i$. 


FIGURE 7  Chosen quotient van Kampen diagrams for faces of $M_G$.

Proof. Each face $F$ of $M_G$ has boundary label equal to either $[s, u_i], u_i^{\ell_i}$, or $u_i v_i^{-1}$. Each of these words represents the trivial element of $H$. For each face $F$ of $M_G$, choose a van Kampen diagram $M_F$ over $\langle S \mid R_H \rangle$, of one of forms depicted in Figure 7.

Applying Lemma 2.19, there exists a bare, reduced van Kampen diagram $M_H$ over $\langle S \mid R_H \rangle$ with $\text{Lab}(M_H) = \text{Lab}(M_G) = uw^{-1}$ (i.e., satisfying (a) and (b)), such that

$$\sigma(M_H, r) = \sum \{ \sigma(M_F, r) \mid F \text{ is an essential face of } M_G \}$$

for all $r \in R_H := U \cup V$.

Since $U \cup V$ is cyclically minimal, each face of $M_F$ contributes to $\sigma(M_F, r)$ for at most one $r \in U \cup V$. Combining this with condition (d) of Lemma 2.20, the sets $\{u_i, v_i\}_i$, $i \in \mathbb{N}$, are pairwise disjoint. Therefore, for all $F \in M_G$ we have that $M_F$ affects the signed $u_i$- or $v_i$-face count of $M_H$ for at most one $i \in \mathbb{N}$, and we may consider each $i$ independently. Now fixing $i \in \mathbb{N}$, we have

$$\sigma(M_H, u_i) + \sigma(M_H, v_i) = \sum \{ \sigma(M_F, u_i) + \sigma(M_F, v_i) \mid F \text{ is an essential face of } M_G \}.$$

Now notice that for all $M_F$ depicted in Figure 7, $\sigma(M_F, u_i) + \sigma(M_F, v_i) \equiv 0 \mod \ell_i$. Also, if $u_i = v_i$, then $\sigma(M_F, u_i) \equiv 0 \mod \ell_i$. Since (c) and (d) hold for each summand, they hold for the sum as well. □

Let $M_H$ be the van Kampen diagram from Lemma 2.21. Let $k = \sum_{i \in \mathbb{N}} k_i$. Then, we may write

$$\alpha = \tilde{\alpha} * \alpha_0 * \cdots * \alpha_k * \beta,$$

where $\text{Lab}(\tilde{\alpha}) = \tilde{u}$, and for all $j \in \{0, \ldots, k\}$, we have $\text{Lab}(\alpha_j) = u_i$ for some $i \in \mathbb{N}$.

Lemma 2.22. There exists a van Kampen diagram $M'_H$ over $\langle S \mid U \cup V \rangle$ and natural numbers $\{h_i \mid i \in \mathbb{N}\}$ satisfying all of the following conditions for all $i \in \mathbb{N}$.

(a) $M'_H$ is bare and reduced.
(b) $\kappa(M'_H, u_i) = \kappa(M_H, u_i) - h_i$.
(c) $\partial M'_H = \alpha' * \beta'$, where $\text{Lab}(\alpha') = \tilde{u} \prod_{i=0}^{\infty} u_i^{k_i-h_i}$ and $\text{Lab}(\beta') = w$.
(d) No face $F$ of $M'_H$ intersects $\alpha'$ in a common subpath of length at least $2\lambda \ell_i(\partial F)$.
(e) $0 \leq h_i \leq k_i \leq \ell_i/2$. 
Proof. If \( M_H \) already satisfies (d), then all conditions are satisfied by setting \( M'_H = M_H \) and \( h_i = 0 \) for all \( i \in \mathbb{N} \). Therefore, suppose \( M_H \) does not satisfy (d), that is, there exists a face \( F \) of \( M_H \) such that \( \partial F \) intersects \( \alpha \) in a common subpath of length at least \( 2\lambda \ell(\partial F) \). Then, there must be a common subpath of \( \partial F \) and \( \bar{\alpha} \) or \( \alpha_j \) for some \( j \in \{0, \ldots, k\} \), of length at least \( \lambda \ell(\partial F) \). The former possibility is excluded by condition (c) of Lemma 2.20. Thus, \( \partial F \) intersects \( \alpha_j \) in a common subpath of length at least \( \lambda \ell(\partial F) \) for some \( j \in \{0, \ldots, k\} \). Call this common subpath \( \gamma \).

Now apply Operation 2.13 to excise \( \alpha_j \) from \( \alpha \). Let \( F' \) be the new \( u_i \)-face created by this operation. Then, \( \gamma \) is a common subpath of \( \partial F \) and \( \partial F' \) of length at least \( \lambda \ell(\partial F) \), so \( F \) and \( F' \) cancel. Since \( M_H \) was reduced, \( F \) and \( F' \) are the only pair of faces that cancel at this stage. Therefore, by Lemma 2.17(a) and (d), \( F \) and \( F' \) are simple and intersect simply. Thus, \( F \cup F' \) is a simple subdiagram of \( M \) with trivial boundary label, which we may remove with Operation 2.8.

Let \( \hat{M}_H \) be the van Kampen diagram obtained in this way. Then, clearly \( \hat{M}_H \) satisfies (a). We added one \( u_i \)-face and removed two, so \( \kappa(\hat{M}_H, u_i) = \kappa(M_H, u_i) - 1 \). Since \( U \cup V \) is cyclically minimal and \( u_i \neq u_i' \) whenever \( i \neq i' \), no \( u_i \)-face counts were affected for any \( i' \neq i \). Therefore, (b) is satisfied with \( h_i = 1 \). Now after excising \( \alpha_j \), the boundary path becomes \( \bar{\alpha}^* \beta = \bar{\alpha} * \alpha_0 * \cdots * \alpha_{j-1} * \alpha_{j+1} * \cdots * \alpha_k * \beta \). Removing \( F \cup F' \) does not change the boundary label, so (c) is satisfied with \( h_i = 1 \). Because of this, we may iterate the process. By construction, \( \hat{M}_H \) has one fewer face than \( M_H \) which fails to satisfy (d). Therefore, repeat as many times as there are faces in \( \hat{M}_H \) failing to satisfy (d) to get \( M'_H \). Each such face must be a \( u_i \)-face for some \( i \in \mathbb{N} \), so for each \( i \in \mathbb{N} \), let \( h_i \) be the number of \( u_i \)-faces in \( M_H \) failing to satisfy (d). Since the boundary label becomes shorter at each step, by (c) it follows that \( h_i \leq k_i \) for all \( i \in \mathbb{N} \). Therefore, \( M'_H \) satisfies (e), and we are done.

For the next step in the proof, the following ad hoc lemma is useful.

**Lemma 2.23.** Let \( M \) be a bare, reduced van Kampen diagram over a cyclically reduced \( C'(1/6) \) presentation. Let \( \alpha \) be a subpath of \( \partial M \) such that no face \( F \) of \( M \) intersects \( \alpha \) in a common subpath of length at least \( \frac{1}{4} \ell(\partial F) \). Then, every face of \( M \) that intersects \( \alpha \) nontrivially, intersects \( \alpha \) simply.

**Proof.** Suppose \( F \) is a face of \( M \) such that \( \partial F \) shares more than one vertex with \( \alpha \), but \( \partial F \) does not intersect \( \alpha \) simply. Then, there exist subpaths of \( \alpha \) and \( \partial F \) that together enclose a simple subdiagram \( D \) of \( M \). Since \( M \), and therefore \( D \), is reduced, by the Greendlinger Lemma there exists a face \( F' \) of \( D \) such that \( \partial F' \) shares a common subpath of length at least \( \frac{1}{4} \ell(\partial F') \) with \( \partial D \). Thus, \( \partial F' \) intersects either \( \partial F \) or \( \alpha \) in a common subpath of length at least \( \frac{1}{2} \ell(\partial F') \). The latter possibility is ruled out by assumption, so \( F \) cancels with \( F' \) by the \( C'(1/6) \) condition. But this contradicts our assumption that \( M \) is reduced.

**Definition 2.24.** Let \( M \) be a van Kampen diagram over a presentation \( \langle S \mid R \rangle \). Then, the **perimeter sum** of \( M \), denoted \( \text{PS}(M) \), is defined by

\[
\text{PS}(M) = \sum \{ \ell(\partial F) \mid F \text{ is a face of } M \}.
\]

Note that if \( M \) is bare and \( R \) is cyclically minimal, then

\[
\text{PS}(M) = \sum_{r \in R} |r| \kappa(M, r).
\]
To obtain bounds on \( \ell(\alpha') \) in terms of \( \ell(\beta) \), and on \( \ell(\alpha) \) in terms of \( \ell(\alpha') \), we use the following fact about van Kampen diagrams over \( C'(1/6) \) presentations. It is the final puzzle piece in the proof.

**Lemma 2.25** [9, Lemma 3.8]. *Let \( M \) be a bare and reduced van Kampen diagram over a cyclically reduced \( C'(\lambda) \) presentation, where \( \lambda \leq 1/6 \). Then, \( (1 - 6\lambda) \text{PS}(M) \leq \ell(\partial M) \).*

With this in mind, we resume our proof.

**Lemma 2.26.** \( \ell(\alpha') < 2\ell(\beta) \).

**Proof.** Note that an edge of \( \alpha' \) is shared by the boundary path of some face of \( M_H' \) if and only if it is not also an edge of \( \beta' \). We have by Lemma 2.22(d) that no face \( F \) intersects \( \alpha' \) in a common subpath of length at least \( 2\lambda \ell(\partial F) < \frac{1}{5} \ell(\partial F) < \frac{1}{4} \ell(\partial F) \). Therefore, by Lemma 2.23, every face whose boundary path shares an edge with \( \alpha' \) intersects \( \alpha' \) in a single common subpath. Thus,

\[
\text{PS}(M_H) > \frac{1}{2\lambda} (\ell(\alpha') - \ell(\beta')) = \frac{1}{2\lambda} (\ell(\alpha') - \ell(\beta')).
\]

On the other hand, \( \ell(\partial M_H') = \ell(\alpha') + \ell(\beta') \). Thus, by Lemma 2.25,

\[
\frac{1}{2\lambda} (\ell(\alpha') - \ell(\beta')) < \text{PS}(M_H') \leq \frac{1}{1 - 6\lambda} \ell(\partial M_H) = \frac{1}{1 - 6\lambda} (\ell(\alpha') + \ell(\beta')),
\]

\[
(1 - 6\lambda)(\ell(\alpha') - \ell(\beta')) < 2\lambda(\ell(\alpha') + \ell(\beta')),
\]

\[
(1 - 8\lambda)\ell(\alpha') < (1 - 4\lambda)\ell(\beta'),
\]

\[
\ell(\alpha') < \frac{1 - 4\lambda}{1 - 8\lambda} \ell(\beta') < 2\ell(\beta') = 2\ell(\beta),
\]

since \( 0 < \lambda < 1/12 \). \( \square \)

**Lemma 2.27.** \( \text{PS}(M_H) \geq 2(\ell(\alpha) - \ell(\alpha')) \).

**Proof.** Let \( I = \{i \in \mathbb{N} | u_i = v_i\} \). By Lemma 2.21, if \( i \in I \), then \( \sigma(M_H, u_i) \equiv 0 \mod \ell_i \). If \( i \notin I \), then \( \sigma(M_H, u_i) + \sigma(M_H, v_i) \equiv 0 \mod \ell_i \). Note that \( \kappa(M_H, u_i) + \kappa(M_H, v_i) \geq \kappa(M_H, u_i) \geq h_i \) by Lemma 2.22(b). Since \( h_i \leq \ell_i/2 \), it follows that there are at least \( 2h_i \) faces in \( M_H \) with boundary label either \( u_i^{\pm 1} \) or \( v_i^{\pm 1} \). If \( u_i = v_i \), this says that \( \kappa(M_H, u_i) \geq 2h_i \). If \( u_i \neq v_i \), this means \( \kappa(M_H, u_i) + \kappa(M_H, v_i) \geq 2h_i \). Therefore,

\[
\text{PS}(M_H) = \sum_{r \in R_H} |r| \kappa(M_H, r) = \sum_{i \in I} |u_i| |\kappa(M_H, u_i)| + \sum_{i \notin I} (|u_i| + |v_i|) \kappa(M_H, u_i) + \kappa(M_H, v_i) \geq \sum_{i \in I} |u_i| + \sum_{i \notin I} (|u_i| + |v_i|) \kappa(M_H, u_i) + \kappa(M_H, v_i) \geq \sum_{i \in I} 2h_i |u_i| + \sum_{i \notin I} 2h_i |v_i| = \sum_{i \in \mathbb{N}} 2h_i |u_i| = 2(\ell(\alpha) - \ell(\alpha')),
\]
where the last equality follows from Lemma 2.22(c).

Now we are ready to prove Lemma 2.20.

Proof of Lemma 2.20. Continuing to use the terminology and notation built up in this section, since \( w \) is a geodesic representative of \( u \), \( |u| = \ell(\alpha) \), and \( |w| = \ell(\beta) \), it suffices to prove that \( \ell(\alpha) < \frac{3}{1 - 12\lambda} \ell(\beta) \). By Lemmas 2.25, 2.26, and 2.27,

\[
2(\ell(\alpha) - \ell(\alpha')) \leq \text{PS}(M_H) \leq \frac{1}{1 - 6\lambda} \ell(\partial M_H) = \frac{1}{1 - 6\lambda}(\ell(\alpha) + \ell(\beta)),
\]

\[
(1 - 12\lambda)\ell(\alpha) \leq (2 - 12\lambda)\ell(\alpha') + \ell(\beta) < \ell(\alpha') + \ell(\beta) < 3\ell(\beta),
\]

\[
\ell(\alpha) < \frac{3}{1 - 12\lambda} \ell(\beta).
\]

Therefore \( |u| \leq \frac{3}{1 - 12\lambda} \|u\|_G \), as desired.

For this to be a meaningful bound, we must have \( 0 < \lambda < 1/12 \), explaining our initial choice of \( \lambda \).

3 | PROOF OF THE MAIN RESULT

In this section, we prove the following proposition.

**Proposition 3.1.** Let \( m, n \in \mathbb{Z}^+ \cup \{\infty\} \) with \( m < n \). Then, there exist finitely generated, recursively presented groups \( G \) and \( B \) such that \( B \leq G \) and

\[
1 \leq \text{asdim}(G) \leq 2,
\]

\[
m + 1 \leq \text{asdim}_{AN}(G) \leq m + 2,
\]

\[
n + 1 \leq \text{asdim}_{AN}(B) \leq n + 2.
\]

Since the proof requires many auxiliary lemmas, we again ‘globally fix’ all notation in this section.

Let \( m \) be a fixed positive integer, and let \( n \in \mathbb{Z}^+ \cup \{\infty\} \) with \( m < n \). Let \( (\ell) \) be an increasing sequence of positive integers with \( \ell_0 \geq 2 \). Let \( S_A, S_B \) be disjoint finite sets, and let \( 0 < \lambda < 1/12 \). Suppose we have two languages

\[
U_A = \{u_i \mid i \in \mathbb{N}\} \subset (S_A)_o^*, \quad V_B = \{v_i \mid i \in \mathbb{N}\} \subset (S_B)_o^*
\]

satisfying all of the following conditions for all \( i, i', j \in \mathbb{N} \).

(a) \( U_A, V_B \) are cyclically minimal and cyclically reduced, and satisfy \( C'(\lambda) \).
(b) There exists a nonempty word \( y \in (S_B)_o^* \) such that, for all \( h \in \mathbb{Z} \), if \( p \) is a piece of \( y^h \) and \( v_i \), then \( |p| < \lambda |v_i| \).
(c) \( 2 \leq |u_i| \leq |v_i| \).
(d) If \( u_i = u_{i'} \) or \( v_i = v_{i'} \), then \( i = i' \).
(e) The sequence of word lengths \(|u_i|\) is constant on blocks of the partition \(P_m\) and \(|v_i|\) is constant on blocks of \(P_n\) (see Definition 1.7).

(f) \(|u_{(j+1)m}| \geq \ell_{(j+1)m}|u_{jm}|\). If \(n \in \mathbb{Z}^+\), then \(|v_{(j+1)n}| \geq \ell_{(j+1)n}|v_{jn}|\), and if \(n = \infty\), then \(|v_{(j+1)\infty}| \geq \ell_{(j+1)\infty}|v_{j\infty}|\).

(g) \(U_A, V_B\) are recursive.

We construct an example of languages \(U_A, V_B\) satisfying (a)–(f) in the next section, and show that they can be recursive in the process. Assuming we already have \(U_A, V_B\) satisfying (a)–(g), let \(H_A, H_B\) be given by the presentations

\[H_A = \langle S_A \mid U_A \rangle \quad H_B = \langle S_B \mid V_B \rangle\]

and let \(A, B\) be central extensions of \(H_A, H_B\), respectively, defined by

\[A = \langle S_A \mid R_A \rangle := \langle S_A \mid [a, u_i], u_i^{\ell_i} : a \in S_A, i \in \mathbb{N} \rangle,\]
\[B = \langle S_B \mid R_B \rangle := \langle S_B \mid [b, v_i], v_i^{\ell_i} : b \in S_B, i \in \mathbb{N} \rangle.\]

Since all elements in \(R_A, R_B\) represent the trivial element in \(H_A, H_B\), respectively, there are natural epimorphisms \(\pi_A : A \rightarrow H_A\) and \(\pi_B : B \rightarrow H_B\). Recall that for a word \(w\) in \((S_A)^*\) or \((S_B)^*\), we denote by \(\hat{w}\) the element of \(A\) or \(B\), respectively, that \(w\) represents. Let

\[K_A = \text{Ker}(\pi_A) = \langle \hat{u}_i : i \in \mathbb{N} \rangle \leq Z(A),\]
\[K_B = \text{Ker}(\pi_B) = \langle \hat{v}_i : i \in \mathbb{N} \rangle \leq Z(B),\]

where we consider \(K_A\) as a normed group, equipped with the restriction to \(K_A\) of the word norm on \(A\) with respect to the generating set \(S_A\), which we will denote \(\| \cdot \|_A\): similarly for \(K_B\).

By condition (c), there exist sequences \(s = (s_j), t = (t_j)\) such that \(|u_i| = (m \times s)_i\) and \(|v_i| = (n \times t)_i\) for all \(i \in \mathbb{N}\). Define normed groups \(K_m, K_n\) similar to the normed group defined in Lemma 1.13, as follows:

\[K_m = \bigoplus_{i \in \mathbb{N}} |u_i| \mathbb{Z}_{\ell_i} = \bigoplus_{i \in \mathbb{N}} (m \times s)_i \mathbb{Z}_{\ell_i}, \quad K_n = \bigoplus_{i \in \mathbb{N}} |v_i| \mathbb{Z}_{\ell_i} = \bigoplus_{i \in \mathbb{N}} (n \times t)_i \mathbb{Z}_{\ell_i}.\]

Suppose \(x\) is a word over \(S_A\) satisfying (b) with respect to \(U_A\), except possibly the condition that \(x\) not be the empty word. Now condition (d) guarantees that \(s\) and \(t\) are increasing sequences of positive integers, such that for all \(j \in \mathbb{N}\), \(s_{j+1} \geq s_j \ell_{(j+1)m}\) and \(t_{j+1} \geq t_j \ell_{(j+1)n}\) if \(n \in \mathbb{Z}^+\). Condition (e) guarantees that \(s_0 \geq 2\) and \(t_0 \geq 2\). Therefore, all hypotheses of Lemmas 1.9 and 1.13 are satisfied, and we have

\[\text{asdim}_{AN}(|x| \mathbb{Z} \times K_m) = \begin{cases} m & \text{if } x = \varepsilon \\ m + 1 & \text{otherwise} \end{cases}, \quad \text{asdim}_{AN}(|y| \mathbb{Z} \times K_n) = n + 1.\]

Now \(K_A\) is abelian, \(K_A\) satisfies \(\hat{u}_i^{\ell_i} = 1\) for all \(i \in \mathbb{N}\), and, since \(K_A\) is central in \(A\), we have \(\langle \hat{x}, K_A \rangle \cong \langle \hat{x} \rangle \times K_A\). All the corresponding statements hold for \(y\) and \(K_B\). Therefore, there exist
natural epimorphisms $\phi_A$ and $\phi_B$ defined by

$$
\phi_A : \langle x \rangle \times \mathbb{Z} \times K_m \to \langle \bar{x}, K_A \rangle \quad \phi_B : \langle y \rangle \times \mathbb{Z} \times K_n \to \langle \bar{y}, K_B \rangle
$$

$$(h, z) \mapsto x^h \prod_{i \in \mathbb{N}} \bar{u}_i^{z_i} \quad (h, z) \mapsto y^h \prod_{i \in \mathbb{N}} \bar{v}_i^{z_i}$$

for all $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$ or $K_n$. In the case that $x = \epsilon$, we have that $|x| = 0$ and $0\mathbb{Z} = \{0\}$, so we can consider $\phi_A$ as an epimorphism of $K_m$ onto $K_A$.

**Lemma 3.2.** Each of the epimorphisms $\phi_A, \phi_B$ is bi-Lipschitz, hence each is a quasi-isometry and an isomorphism.

**Proof.** We prove the statement for $\phi_A$. Let $\| \cdot \|$ be the norm on $K_m$. Let $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$. Let $(k_i)$ be the geodesic form of $z$ (see Definition 1.4). Then,

$$
\|\phi_A(h, z)\|_A = \left\| x^h \prod_{i \in \mathbb{N}} \bar{u}_i^{k_i} \right\|_A \leq \left\| x^h \prod_{i \in \mathbb{N}} u_i^{k_i} \right\|_A = h|x| + \sum_{i \in \mathbb{N}} |k_i| |u_i| = \|(h, z)\|.
$$

Now $k_i \leq \ell_i/2$ for all $i \in \mathbb{N}$, and $x^h$ satisfies condition (c) of Lemma 2.20. Furthermore,

$$
A = \langle S_A \mid [a, u_i], u_i^{\ell_i} : a \in S_A, i \in \mathbb{N} \rangle = \langle S_A \mid [a, u_i], u_i^{\ell_i}, u_i(u_i)^{-1} : a \in S_A, i \in \mathbb{N} \rangle
$$

and $U_A \cup U_A = U_A = \{u_i \mid i \in \mathbb{N}\}$ is a cyclically reduced, cyclically minimal $C'(\lambda)$ language, where $2 \leq |u_i| \leq |u_i|$ and $u_i = u_{i'}$ implies $i = i'$ for all $i, i' \in \mathbb{N}$. Thus, we may apply Lemma 2.20 with $G = A, U = U_A, V = U_A$, and $\bar{u} = x^h$. This yields

$$
\|(h, z)\| = h|x| + \sum_{i \in \mathbb{N}} |u_i||k_i| = \left\| x^h \prod_{i \in \mathbb{N}} u_i^{k_i} \right\| \leq \left( 1 - \frac{3}{1 - 12\lambda} \right) \left\| \phi_A(h, z) \right\|_A,
$$

hence $(1 - \frac{3}{3})(\|(h, z)\| \leq \|\phi_A(k, z)\|_A \leq \|(h, z)\|)$ and $\phi_A$ is bi-Lipschitz. \qed

By replacing $x$ or $y$ with $\epsilon$, we obtain the following.

**Corollary 3.3.** Both $\phi_A|_{K_m} : K_m \to K_A$ and $\phi_B|_{K_n} : K_n \to K_B$ are bi-Lipschitz maps. Therefore, $\text{asdim}_{\text{AN}}(K_A) = \text{asdim}_{\text{AN}}(K_m) = m$ and $\text{asdim}_{\text{AN}}(K_B) = \text{asdim}_{\text{AN}}(K_n) = n$.

In order to get our bounds on $\text{asdim}_{\text{AN}}(G)$ and $\text{asdim}_{\text{AN}}(B)$, we use the extension theorems for asymptotic and Assouad–Nagata dimension.

**Lemma 3.4 (Extension Theorems [4, 12]).** Let

$$
1 \to K \to G \to H \to 1
$$

be a short exact sequence, where $G$ and $H$ are finitely generated groups equipped with the word norm with respect to some finite generating set, and the norm on $K$ is the restriction to $K$ of the norm on $G$. 


Then,

\[ \text{asdim}(G) \leq \text{asdim}(K) + \text{asdim}(H), \]

\[ \text{asdim}_{AN}(G) \leq \text{asdim}_{AN}(K) + \text{asdim}_{AN}(H). \]

**Lemma 3.5** [1]. Let H be a finitely generated \( C'(1/6) \) group. Then, \( \text{asdim}_{AN}(H) \leq 2 \).

**Corollary 3.6.** We have

\[ 1 \leq \text{asdim}(A) \leq 2, \quad m \leq \text{asdim}_{AN}(A) \leq m + 2 \]
\[ n + 1 \leq \text{asdim}_{AN}(B) \leq n + 2. \]

Also, if \( x \neq \varepsilon \), then \( \text{asdim}_{AN}(A) \geq m + 1 \).

**Proof.** We establish the bounds for \( A \): the argument for \( B \) is similar. Since \( A \) is finitely generated and infinite, \( \text{asdim}_{AN}(A) \geq 1 \). By Corollary 3.3, \( \text{asdim}_{AN}(A) \geq \text{asdim}_{AN}(K_A) = m \). If \( x \neq \varepsilon \),

\[ \text{asdim}_{AN}(A) \geq \text{asdim}_{AN}(\langle \bar{x}, K_A \rangle) = \text{asdim}_{AN}(|x|\mathbb{Z} \times K_m) = m + 1 \]

since \(|x| > 0\). This gives the lower bounds on the asymptotic and Assouad-Nagata dimension of \( A \). For the upper bounds, note that \( A \) is constructed so that there is a short exact sequence

\[ 1 \to K_A \to A \to H_A \to 1, \]

where \( H_A \) is a finitely generated \( C'(1/6) \) group and hence \( \text{asdim}(H_A) \leq \text{asdim}_{AN}(H_A) \leq 2 \). Since \( K_A \) is locally finite, \( \text{asdim}(K_A) = 0 \) [7]. Now by Lemma 3.4,

\[ \text{asdim}(A) \leq \text{asdim}(K_A) + \text{asdim}(H_A) \leq 2, \]
\[ \text{asdim}_{AN}(A) \leq \text{asdim}_{AN}(K_A) + \text{asdim}_{AN}(H_A) \leq m + 2. \]

By Corollary 3.3, the maps \( \phi_A|_{K_m} : K_m \to K_A \) and \( \phi_B|_{K_n} : K_n \to K_B \) are isomorphisms. Therefore, the map defined by \( \bar{u}_i \mapsto \bar{v}_i \) for all \( i \in \mathbb{N} \) extends to an isomorphism from \( K_A \) to \( K_B \). Let \( \phi : K_A \to K_B \) be this isomorphism. Let

\[ G = A *_{\phi} B := \langle A \sqcup B \mid a\phi(a)^{-1} : a \in A \rangle. \]

Let \( S = S_A \sqcup S_B \). Then, \( G \) admits the presentation

\[ G = \langle S \mid R_G \rangle := \langle S \mid [s, u_i], u_i^{f_i}, u_i u_i^{-1} : s \in S, i \in \mathbb{N} \rangle, \]

which is recursive if \( U_A \) and \( V_B \) are recursive. Let

\[ H = \langle S \mid R_H \rangle := \langle S_A \sqcup S_B \mid U_A \sqcup V_B \rangle = \langle S_A \sqcup S_B \mid u_i, v_i : i \in \mathbb{N} \rangle. \]
Since $U_A$ and $V_B$ are $C'(\lambda)$ languages over disjoint alphabets, $H$ is a $C'(\lambda)$ group. Furthermore, all words in $R_G$ represent the trivial element of $H$, so there is a natural epimorphism $\pi : G \to H$. Let $K = \text{Ker}(\pi)$. Then,

$$K = \langle \tilde{a}_i : i \in \mathbb{N} \rangle \leq Z(G).$$

We consider $K$ as a normed group, where the norm on $K$ is the restriction to $K$ of the word norm on $G$ with respect to $S$. Thus, we have a short exact sequence

$$1 \to K \to G \to H \to 1.$$

Let $b \in S_B$. Considering the relations of $K$ and the fact that $K$ is central in $G$, there exists a natural epimorphism $\phi_K : \mathbb{Z} \times K_m \to \langle \tilde{b}, K \rangle$ given by

$$\phi_K(h, z) = \tilde{b}^h \prod_{i \in \mathbb{N}} \tilde{u}_i^{z_i}$$

for all $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$. Now we have the following.

**Lemma 3.7.** The epimorphism $\phi_K$ is bi-Lipschitz, in particular $\phi_K$ is a quasi-isometry and an isomorphism.

**Proof.** The proof is similar to that of Lemma 3.2. The only difference is that now we apply Lemma 2.20 with $U = U_A, V = V_B$, and $\tilde{u}_i = b^h$. Since $b$ is a word over an alphabet disjoint from $S_A$, clearly condition (c) of Lemma 2.20 is satisfied with $\tilde{u}_i = b^h$ for any $h \in \mathbb{N}$. Since $2 \leq |u_i| \leq |v_i|$ and $u_i \neq v_{i'}$ for all $i, i' \in \mathbb{N}$, all hypotheses of Lemma 2.20 are satisfied. □

We are now ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let $B, G$ be defined as in this section. The bounds on $\text{asdim}_{\text{AN}}(B)$ are established in Corollary 3.6. Since $G$ is finitely generated and infinite, $\text{asdim}(G) \geq 1$. For the lower bound on the Assouad–Nagata dimension of $G$, note that

$$\text{asdim}_{\text{AN}}(G) \geq \text{asdim}_{\text{AN}}(\langle \tilde{b}, K \rangle) = \text{asdim}_{\text{AN}}(\mathbb{Z} \times K_m) = m + 1.$$

By Lemma 3.5, we have $\text{asdim}(H) \leq \text{asdim}(H) \leq 2$. On the other hand, since $B$ is locally finite, $\text{asdim}(B) = 0$ [7]. Applying the extension theorems to the short exact sequence $1 \to K \to G \to H \to 1$ yields that $\text{asdim}(G) \leq 2$ and $\text{asdim}_{\text{AN}}(G) \leq m + 2$. □

We give a presentation of a group $G$ satisfying the conditions of Proposition 3.1 in the next section. For now, we derive the main result of this paper as a corollary. To do this, we will need to recall two theorems of asymptotic dimension theory. The first a theorem of Dranishnikov and Smith, known as the Morita theorem for asymptotic Assouad–Nagata dimension. We state a special case of it here.

**Theorem 3.8** (Morita theorem for $\text{asdim}_{\text{AN}}$ [13]). Let $G$ be a finitely generated group. Then, $\text{asdim}_{\text{AN}}(G \times \mathbb{Z}) = \text{asdim}_{\text{AN}}(G) + 1$. 
The second is the free product formulas for asymptotic and Assouad–Nagata dimension. The theorem for asdim is due to Dranishnikov, and its counterpart for asdim\textsubscript{AN} is due to Brodskiy and Higes.

**Theorem 3.9** [14, 15]. Let $A$ and $B$ be finitely generated groups. Then,

$$\text{asdim}(A * B) = \max\{\text{asdim}(A), \text{asdim}(B), 1\},$$

$$\text{asdim}_{\text{AN}}(A * B) = \max\{\text{asdim}_{\text{AN}}(A), \text{asdim}_{\text{AN}}(B), 1\}.$$

We are now ready to prove the main theorem.

**Theorem 3.10.** For all $k, m, n \in \mathbb{N} \cup \{\infty\}$ with $4 \leq k \leq m \leq n$, there exist finitely generated, recursively presented groups $G$ and $H$ with $H \leq G$, such that

$$\text{asdim}(G) = k,$$

$$\text{asdim}_{\text{AN}}(G) = m,$$

$$\text{asdim}_{\text{AN}}(H) = n.$$

**Proof.** Applying Proposition 3.1 with $m - 3$ and $n - 2$, there exist finitely generated, recursively presented groups $G_0$ and $B_0$ with $B_0 \leq G_0$, such that

$$1 \leq \text{asdim}(G_0) \leq 2,$$

$$m - 2 \leq \text{asdim}_{\text{AN}}(G_0) \leq m - 1,$$

$$n - 1 \leq \text{asdim}_{\text{AN}}(B_0) \leq n.$$

Let

$$G_1 = \begin{cases} G_0 \times \mathbb{Z}^2 & \text{if } \text{asdim}_{\text{AN}}(G_0) = m - 2, \\ G_0 \times \mathbb{Z} & \text{if } \text{asdim}_{\text{AN}}(G_0) = m - 1. \end{cases}$$

Then, by the Morita theorem for Assouad–Nagata dimension, we have $\text{asdim}(G_1) = m$. By the extension theorem for asymptotic dimension, we that $\text{asdim}(G_1) \leq \text{asdim}(G_0) + 2 \leq 4$. Now let $G = G_1 \ast \mathbb{Z}^k$. Then, since $4 \leq k \leq m$, by the free product formulas for asymptotic and Assouad–Nagata dimension, it follows that $\text{asdim}(G) = k$ and $\text{asdim}_{\text{AN}}(G) = m$. Note that $B_0$ and $B_0 \times \mathbb{Z}$ are both subgroups of $G$. Therefore, let

$$H = \begin{cases} B_0 \times \mathbb{Z} & \text{if } \text{asdim}_{\text{AN}}(B_0) = n - 1, \\ B_0 & \text{if } \text{asdim}_{\text{AN}}(B_0) = n. \end{cases}$$

Again by the Morita theorem, we have that $\text{asdim}_{\text{AN}}(H) = n$, and $H \leq G$. This completes the proof. \qed
4 | A CONCRETE EXAMPLE

In this section, we construct an example of a group of the sort described in Proposition 3.1. In doing so, we show that such a group can be given by an explicit presentation, that is, is recursively presented. The following lemma shows one way of constructing $C'(\lambda)$ languages, which was used by Bowditch in [16] to construct $2^{\aleph_0}$ small cancellation groups in distinct quasi-isometry classes.

**Lemma 4.1.** Let $U = \{u_i \mid i \in \mathbb{N}\} \subset \{a, x\}^*$ be a language where we define

$$u_i = (a^{m_i}x^{n_i})^{n_i}$$

for some positive integers $m_i, n_i$, for each $i \in \mathbb{N}$. Let $k \geq 2$ be an integer, and suppose all of the following conditions hold.

(a) $n_i \geq k$ for all $i \in \mathbb{N}$.
(b) $m_i \neq m_{i'}$ for all distinct $i, i' \in \mathbb{N}$.

Then all of the following conclusions hold for all $i \in \mathbb{N}$.

(i) $U$ is cyclically minimal and cyclically reduced, and satisfies $C'(\frac{1}{k-1})$.
(ii) For all $h \in \mathbb{Z}$, if $p$ is a piece of $x^h$ and $u_i$, then $|p| < \frac{1}{k-1}|u_i|$.
(iii) $2 \leq |u_i|$.
(iv) If $u_i = u_{i'}$ then $i = i'$.

**Proof.** If $u_i \in U$, then no cyclic shift of $u_i^{-1}$ is in $U$: if $\tilde{u}_i$ is a cyclic shift of $u_i$ that belongs to $U$, then $|\tilde{u}_i| = |u_i|$ and $\tilde{u}_i$ must begin with $a$ and end with $x$, in which case $\tilde{u}_i = u_i$. Therefore, $U$ is cyclically minimal. Since all $u_i$ are positive words (that is, do not contain letters $a^{-1}$ or $x^{-1}$), it is clear that $U$ is cyclically reduced. For the same reason, when talking about pieces of some $u_i$ and another positive word $w$, it suffices to consider only cyclic shifts of $u_i$ and $w$, and we may ignore cyclic shifts of $u_i^{-1}$ or $w^{-1}$. To show that $U$ satisfies $C'(\frac{1}{k-1})$, suppose $i, i' \in \mathbb{N}$ are distinct. Let $p$ be a maximal piece of $u_i$ and $u_{i'}$. Since $m_i \neq m_{i'}$, suppose without loss of generality that $m_i < m_{i'}$. Then, $p$ must have the form $a^{m_i}x^{m_i}$. But then $n_i|p| \leq |u_i|$ and $n_{i'}|p| \leq |u_{i'}|$. Since $n_i, n_{i'} \geq k$, we have $|p| \leq \frac{1}{k} \min(|u_i|, |u_{i'}|) < \frac{1}{k-1} \min(|u_i|, |u_{i'}|)$. Therefore, $U$ satisfies $C'(\frac{1}{k-1})$. Conclusion (ii) says only that any power of $x$ makes up less than $\frac{1}{k-1}$ of a cyclic shift of some $u_i$. But a maximal subword of a cyclic shift of $u_i$ of the form $x^h$ must be $x^{m_i}$, which has length at most $\frac{1}{2k}|u_i| \leq \frac{1}{2k}|u_i| < \frac{1}{k-1}|u_i|$, so this is clear. Parts (iii) and (iv) are obvious. \qed

**Lemma 4.2.** Let $m \in \mathbb{Z}^+ \cup \{\infty\}$, and let $P_m = \{P_{m,j} \mid j \in \mathbb{N}\}$ be the partition of $\mathbb{N}$ given in Definition 1.7. Let $k \geq 2$ be an integer. For each $i \in \mathbb{N}$, let $r_i = i - \min(P_{m,j})$ whenever $i \in P_{m,j}$. Let $(p_j)$ be an increasing sequence of positive integers. Let $U = \{u_i \mid i \in \mathbb{N}\} \subset \{a, x\}^*$ be given by

$$u_i = \left(a^{k^{(p_j-r_i)}}x^{k^{(p_j-r_i)}}\right)^{k^{(r_i+1)}}$$
whenever \( i \in P_{(m,j)} \). Let \((\ell'_i)\) be an increasing sequence of positive integers. Suppose the sequence \((p_j)\) satisfies

\[
\begin{align*}
    p_{j+1} &\geq p_j + \log_k(\ell'_{(j+1)m}) + |P_{(m,j+1)}| \text{ if } m \in \mathbb{Z}^+, \\
    p_{j+1} &\geq p_j + \log_k(\ell'_{(j+1)2}) + |P_{(m,j+1)}| \text{ if } m = \infty .
\end{align*}
\]

(5)

Then all of the following conclusions hold for all \( i \in \mathbb{N} \).

(i) \( U \) is cyclically minimal and cyclically reduced, and satisfies \( C'(\frac{1}{k-1}) \).

(ii) For all \( h \in \mathbb{N} \), if \( p \in \{a, x\}^*_o \) is a piece of \( x^h \) and \( \omega_j \), then \( |p| < \frac{1}{k-1} |u_i| \).

(iii) \( 2 \leq |u_i| \).

(iv) If \( u_i = u_{i'} \), then \( i = i' \).

(v) The sequence of word lengths \((|u_i|)\) is constant on blocks of \( P_m \).

(vi) If \( m \in \mathbb{Z}^+ \), then \( |u_{(j+1)m}| \geq \ell'_{(j+1)m}|u_{jm}| \), and if \( m = \infty \), then \( |u_{(j+1)2}| \geq \ell'_{(j+1)2}|u_{j2}| \).

Proof. Note that, if \( i \in P_{(m,j)} \), then \( |u_i| = 2k^{p_j - r_i}k^{r_i+1} = 2k^{p_j+1} \geq 2 \), which depends only on \( j \). This establishes (iv) and (v). Define the sequence \( s = (s_j) \) by

\[
s_j = 2k^{p_j+1}
\]

for all \( j \in \mathbb{N} \). Then, \( |u_i| = (m \times s) \).

For (vi), note that \( \log_k(s_{(j+1)m}) = \log_k(2) + p_{j+1} + 1 \). If \( m \in \mathbb{Z}^+ \), then we have \( p_{j+1} \geq p_j + \log_k(\ell'_{(j+1)m}) \), implying that \( s_{j+1} \geq \ell'_{(j+1)m}s_j \) for all \( j \in \mathbb{N} \). If \( m = \infty \), then \( \log_k(s_{j+1}) \geq p_{j+1} \geq p_j + \log_k(\ell'_{(j+1)2}) \), so \( s_{j+1} \geq \ell'_{(j+1)2}s_j \) for all \( j \in \mathbb{N} \). This establishes (vi).

For parts (i)–(iv), we use Lemma 4.1. Obviously part (a) of Lemma 4.1 is satisfied, so we only need to check part (b). For this it suffices to show that if \( i \in P_{(m,j)}, i' \in P_{(m,j')}, \) and \( i \neq i' \), then \( p_j - r_i \neq p_{j'} - r_{i'} \). If \( j' = j \), then this is immediate. If \( j' = j + 1 \), then we have

\[
p_{j'} - r_{i'} = p_{j+1} - r_{i'} \geq p_{j+1} - |P_{j+1}| + 1 > p_j \geq p_j - r_i.
\]

This shows that \( p_j - r_i \) increases with \( j \) no matter the choice of \( i \in P_{(m,j)} \), so we are done.

We are ready to construct our example. Let \( m, n \in \mathbb{Z}^+ \cup \{\infty\} \) with \( m < n \). Let \( S_A = \{a, x\}, S_B = \{b, y\} \) be disjoint two-element alphabets. Let \( k = 14 \) and let \( \ell'_i = 14^i \) for all \( i \in \mathbb{N} \). Let \((p_j), (q_j)\) be increasing sequences of positive integers. Let \( U_A = \{u_i \mid i \in \mathbb{N}\} \subset (S_A)^*_o \) be the language constructed with respect to \( m, k, (\ell'_i) \) and \((p_j)\) as in Lemma 4.2. Similarly define \( V_B = \{v_i \mid i \in \mathbb{N}\} \subset (S_B)^*_o \) with respect to \( n, k, (\ell'_i), \) and \((q_j)\).

**Lemma 4.3.** Suppose for all \( i, j \in \mathbb{N} \) we have

(a) \( p_{j+1} \geq p_j + (j+2)m \).

(b) \( q_{j+1} \geq q_j + (j+2)n \) if \( n \in \mathbb{Z}^+ \), and \( q_{j+1} \geq q_j + (j+2)^2 \) if \( n = \infty \).

(c) \( p_{\lfloor i/m \rfloor} \leq q_{\lfloor i/n \rfloor} \) if \( n \in \mathbb{Z}^+ \), and \( p_{\lfloor i/m \rfloor} \leq q_{\lfloor \sqrt{i} \rfloor} \) if \( n = \infty \).

Then, \( U_A, V_B \) satisfy conditions (a)–(f) listed in the proof of Proposition 3.1.
Proof. Note that
\[
\log_k (r_{(j+1)n}) + |P_{(n,j+1)}| = \log_{14} (14^{(j+1)n}) + n = (j + 2)n \quad \text{if } n \in \mathbb{Z}^+,
\]
\[
\log_k (r_{(j+1)^2}) + |P_{(n,j+1)}| = \log_{14} (14^{(j+1)^2}) + (2j + 1) \leq (j + 2)^2 \quad \text{if } n = \infty.
\]
Therefore, assumptions (a) and (b) guarantee that \((p_j)\) and \((q_j)\) satisfy (5) with respect to \((r_i)\) and \(m, n\), respectively, and so \(U_A, V_B\) satisfy all conditions listed in the proof of Proposition 3.1, except possibly that \(|u_i| \leq |v_i|\) for all \(i \in \mathbb{N}\). Now, if \(i \in P_{(n,j)} \in \mathcal{P}_n\), then \(j = \lfloor i/n \rfloor\) if \(n \in \mathbb{Z}^+\), and \(j = \lfloor \sqrt{i} \rfloor\) if \(n = \infty\). It follows that assumption (c) is necessary and sufficient to guarantee that \(|u_i| \leq |v_i|\) for all \(i \in \mathbb{N}\). \(\square\)

Example 4.4. Let
\[
p_j = m(j + 2)^2 \quad \quad q_j = \begin{cases} n^2(j + 3)^2 & \text{if } n \in \mathbb{Z}^+ \\ m(j + 3)^4 & \text{if } n = \infty \end{cases}
\]
Then, \((p_j), (q_j)\) satisfy the hypotheses of Lemma 4.3. The verification of this is no more than a tedious calculation, so we omit it. Note that, in the notation of Lemma 4.2,
\[
r_i = \begin{cases} i \mod n & \text{if } n \in \mathbb{Z}^+ \\ i^2 - \lfloor \sqrt{i} \rfloor^2 & \text{if } n = \infty \end{cases}
\]
Also, if \(i \in P_{(n,j)}\), then \(j = \lfloor i/n \rfloor\) if \(n \in \mathbb{Z}^+\), and \(j = \lfloor \sqrt{i} \rfloor\) if \(n = \infty\). So, expanding the forms of \(u_i\) and \(v_i\) according to Lemma 4.2 with respect to the sequences \((p_j)\) and \((q_j)\) given above yields
\[
u_i = \begin{cases} a^{14 \text{mod} \lfloor i/m \rfloor + 2 - (i \mod m)} x^{14 \text{mod} \lfloor i/m \rfloor + 2 - (i \mod m)} b^{14 \text{mod} \lfloor i/n \rfloor + 2 - (i \mod n)} y^{14 \text{mod} \lfloor i/n \rfloor + 2 - (i \mod n)} & \text{if } n \in \mathbb{Z}^+ \\ a^{14 \text{mod} \lfloor \sqrt{i} \rfloor + 2 - (i \mod \sqrt{i})} x^{14 \text{mod} \lfloor \sqrt{i} \rfloor + 2 - (i \mod \sqrt{i})} b^{14 \text{mod} \lfloor \sqrt{i} \rfloor + 2 - (i \mod \sqrt{i})} y^{14 \text{mod} \lfloor \sqrt{i} \rfloor + 2 - (i \mod \sqrt{i})} & \text{if } n = \infty \end{cases}
\]
Then, the languages \(\{u_i \mid i \in \mathbb{N}\}\) and \(\{v_i \mid i \in \mathbb{N}\}\) satisfy conditions (a)–(f) listed in the proof of Proposition 3.1, and are clearly recursive. Thus, the group \(G\) with presentation
\[
G = \langle a, b, x, y \mid [a, u_i], [x, u_i], [b, u_i], [y, u_i], u_i^{14}, u_i v_i^{-1} : i \in \mathbb{N} \rangle
\]
is a finitely generated, recursively presented group of Assouad–Nagata dimension at most \(m + 2\), containing a finitely generated subgroup of Assouad–Nagata dimension at least \(n + 1\).

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