LIPSCHITZ REGULARITY OF THE INVARIANT MEASURE OF RANDOM DYNAMICAL SYSTEMS

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To our great friend Helton Ferreira Albuquerque Medeiros (in memorian)

Abstract. In this article we derive a regularity result for the disintegration of the invariant measure associated to a class of Random Dynamical Systems - RDS. The results of this work are obtained by constructing a suitable anisotropic normed space defined by the Wasserstein-Kantorovich-like metric and understanding the dynamics of the associated transfer operator in a neighborhood of its fixed point. Precisely, we employ functional analytic techniques to demonstrate a spectral gap for its action on suitable spaces of signed measures. We apply this analysis to prove an exponential decay of correlation statement for Lipschitz observables and statistical properties of the RDS.

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1. Introduction

Regarding relevant properties of invariant measures, assessing their degree of regularity is useful. This type of problem can be approached from different perspectives. For instance, if the invariant measure of the system is absolutely continuous with respect to a reference measure, its regularity can be evaluated based on the regularity of its Radon-Nikodym derivative (see [35]). On the other hand, in the absence of a reference measure, understanding the regularity of the disintegration with respect to a measurable partition can be valuable for various applications (see [19], [20], and [21]).

In this work, we demonstrate that the disintegration of the invariant measure of a Random Dynamical System (RDS) is Lipschitz continuous, provided the fiber map is contracting. Some recent research has focused on the regularity of disintegrations and works in the field include [19], [20], [21], [24], and [7]. In [19], the authors established bounded variation regularity for the disintegration of the invariant measure of Lorenz-like maps with respect to a specific topology, which they used to demonstrate the statistical stability of the invariant measure under certain deterministic perturbations. Similarly, in [20], the authors proved Hölder regularity for the disintegration in a class of piecewise partially hyperbolic maps semi-conjugated to non-uniformly expanding systems. This finding enabled them to obtain exponential decay of correlations for Hölder observables and other limit theorems. In [21], Hölder regularity was leveraged to prove statistical stability results for the equilibrium states of partially hyperbolic maps.

We examine the action of the transfer operator associated with a class of random dynamical systems. We establish appropriate anisotropic spaces by extending a spectral gap from a subshift of finite type to the transfer operator of the skew-product map. Following the general approach outlined in [19] and [20], we generalize the techniques and apply them to the system under study. In essence, we achieve a fixed point of the transfer operator of the skew-product within a specific anisotropic space by lifting to the RDS a known spectral gap from the base system.

This type of study is often carried out using the Ionescu-Tulcea and Marinescu Theorem by constructing a pair of suitable function spaces—a stronger space and an auxiliary weaker space—such that the action of the Perron-Frobenius operator on the stronger space exhibits a spectral gap (see [6], [35], [11], [17], and [40] for introductory texts). Commonly, these approaches achieve regularity through compact inclusion arguments. However, our approach differs; we obtain regularity by examining the behavior of the transfer operator in the neighbourhood of the constructed fixed point. Specifically, we achieve uniqueness and regularity as a result of convergence to the equilibrium and the presence of a spectral gap.

We use these facts to obtain exponential decay of correlations over Lipschitz observables. This approach gives stronger convergence results than [19] and open the possibility to obtain stronger statistical statements. In the examples, as a simplest case of the main system of this work, some results on hyperbolic iterated functions systems (IFS) are also provided, where we conclude some properties for its invariant measure and obtain limit theorems.

We apply these results to achieve exponential decay of correlations for Lipschitz observables. Moreover, this approach yields stronger convergence results than those in [19] (although for different class of maps) and opens up opportunities for making more robust statistical statements. In the examples, as a particular case, we provide
results on hyperbolic iterated function systems (IFS), leading to conclusions about the properties of its invariant measure and the derivation of limit theorems.

### Organization of the article
The paper is structured as follows:

- **Section 2**: we introduce the kind of systems we consider in the paper. Essentially, it is a system of the type \( F(x, y) = (\sigma(x), G(x, y)) \). Here and until section 7, where more regularity is required, we do not ask any kind of regularity on \( G \) in the horizontal direction (for the functions \( x \mapsto G(x, y) \), \( y \) fixed);

- **Section 3**: we introduce the functional spaces used in the paper and discussed in the last paragraph;

- **Section 4**: we show the basic properties of the transfer operator of \( F \) when applied to these spaces. In particular we see that there is a useful “Perron-Frobenius”-like formula (see Proposition 4.2);

- **Section 5**: we discuss the basic properties of the iteration of the transfer operator on the spaces we consider. In particular, we prove a Lasota-Yorke inequality and a convergence to equilibrium statement (see Propositions 5.7 and 5.8);

- **Section 6**: we use the convergence to equilibrium and the Lasota-Yorke inequalities to prove the spectral gap for the transfer operator associated to the system restricted to a suitable strong space (see Theorem B);

- **Section 7**: we consider a similar system with some more regularity on the family of functions \( \{G(\cdot, y)\}_{y \in K}, \theta \mapsto G(x, y) \): there exists a partition (cylinders) \( P_i \) such that the restriction of the function \( x \mapsto G(x, y) \) to \( P_i \) is \( k_{y,i} \)-Lipschitz, where the family \( \{k_{y,i}\}_{y \in K} \) is bounded, and it holds for all \( i \). For this sort of system, we prove a stronger regularity result for the iteration of probability measures (see Theorem 7.7 and Remark 7.8) and show that the \( F \)-invariant invariant measure has a Lipschitz disintegration along the stable fibers (see Theorem C);

- **Section 8**: we use the Lipschitz regularity of the invariant measure established in section 7, to prove that the abstract set of functions on which the system has decay of correlations (see Proposition 8.1) contains all Lipschitz functions (see Proposition 8.3) and we finalize the section with the main decay of correlations statement, Theorem D.

## 2. Settings

### 2.0.1. Sub-shifts of finite type
Let \( A \) be an aperiodic matrix and \( \Sigma^+_A \) the subshift of finite type associated to \( A \), i.e.

\[
\Sigma^+_A = \{ \mathbf{z} = \{x_i\}_{i \in \mathbb{N}} ; A_{x_i, x_{i+1}} = 1 \ \forall i \},
\]

where \( x_i \in \{1, \cdots, N\} \) for all \( i \). On \( \Sigma^+_A \) we consider the metric

\[
d_\delta(\mathbf{z}, \mathbf{y}) = \sum_{i=0}^\infty \theta^i(1 - \delta(x_i, y_i)),
\]

where \( \theta \in (0, 1) \) and \( \delta \) is defined by \( \delta(x, y) = 1 \) if \( x = y \) and \( \delta(x, y) = 0 \) if \( x \neq y \).
function $\varphi : \Sigma_A^+ \to \mathbb{R}$ we denote its Lipschitz constant by $|\varphi|_{\sigma}$, i.e.

$$|\varphi|_{\sigma} := \sup_{x \neq y \in \Sigma_A^+} \left\{ \frac{|\varphi(x) - \varphi(y)|}{d_{\sigma}(x, y)} \right\}.$$

Let $\mathcal{F}_\sigma(\Sigma_A^+)$ be the real vector space of the Lipschitz functions on $\Sigma_A^+$, i.e.

$$\mathcal{F}_\sigma(\Sigma_A^+) := \{ \varphi : \Sigma_A^+ \to \mathbb{R} : |\varphi|_{\sigma} < \infty \}$$

endowed with the norm $|| \cdot ||_{\sigma}$, defined by $||\varphi||_{\sigma} = ||\varphi||_{\infty} + |\varphi|_{\sigma}$. In this case, we know that the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$, defined by $\sigma(x)_i = x_{i+1}$ for all $i \geq 0$, is such that its Perron-Frobenius operator $P_\sigma : \mathcal{F}_\sigma(\Sigma_A^+) \to \mathcal{F}_\sigma(\Sigma_A^+)$ has spectral gap. It means that $P_\sigma : \mathcal{F}_\sigma(\Sigma_A^+) \to \mathcal{F}_\sigma(\Sigma_A^+)$ can be written (see [39]) as

$$P_\sigma = \Pi_\sigma + N_\sigma$$

where the spectral radius of $N_\sigma$ is smaller than 1, that is, $\rho(N_\sigma) < 1$, and $\Pi_\sigma$ is a projection ($\Pi_\sigma^2 = \Pi_\sigma$). So, there are $D > 0$ and $r \in (0, 1)$ such that for every $\varphi \in \ker(\Pi_\sigma)$ it holds

$$||P_\sigma \varphi||_{\sigma} \leq Dr^n ||\varphi||_{\sigma}, \quad \forall n \geq 0.$$

### 2.0.2. Contracting Fiber Maps

Fix a compact metric space $(K, d)$ with its Borel’s sigma algebra and let $F$ be the map $F : \Sigma_A^+ \times K \to \Sigma_A^+ \times K$ given by

$$F(x, z) = (\sigma(x), G(x, z)),$$

where $G : \Sigma_A^+ \times K \to K$ will be uniformly contracting on $m$-a.e fiber $\gamma_x = \{x\} \times K$. By simplicity, henceforth $\gamma$ stands for a generic leaf, instead of $\gamma_x$. Moreover, to avoid multiplicative constants, we suppose that diam$(K) = 1$.

Throughout the paper, we shall denote by $\pi_1$ and $\pi_2$ the projections on $\Sigma_A^+$ and $K$, respectively. We denote by $f \mu$ the measure $f \mu(E) = \int_E f d\mu$.

### 2.0.3. Properties of $G$.

**G1:** Consider the $F$-invariant lamination

$$\mathcal{F}^s := \{ \gamma_x \}_{x \in \Sigma_A^+}.$$

We suppose that $\mathcal{F}^s$ is contracted: there exists $0 < \alpha < 1$ such that for $m$-almost all $x \in \Sigma_A^+$ it holds

$$d(G(x, z_1), G(x, z_2)) \leq \alpha d(z_1, z_2), \quad \forall z_1, z_2 \in K.$$

**Example 1.** Let $I_1, \ldots, I_d$ be closed and disjoint intervals in $\mathbb{R}$ and let $I$ be the convex hull of them. Consider $g : I_1 \cup \ldots \cup I_d \to I$ such that $|g'| \geq \lambda > 1$ and $g(I_i) = I$ diffeomorphically for all $i = 1, \ldots, d$. Set $f_i := (g|_{I_i})^{-1} : I_i \to I_i$, $A = (a_{i,j})_{i,j=1}^d$ where $a_{i,j} = 1$ for all $i, j \in \{1, 2, \ldots, d\}$ and $G : \Sigma_A^+ \times I \to I$, by $G(x, z) := f_{x_0}(z)$, if $x = (x_0, x_1, \ldots)$. Define $F : \Sigma_A^+ \to \Sigma_A^+$ by $F(x, z) := (\sigma(x), G(x, z))$. We note that by the assumptions, the Mean Value Theorem implies

$$d(G(x, z_1), G(x, z_2)) = d(f_{x_0}(z_1), f_{x_0}(z_2)) \leq \lambda^{-1} d(z_1, z_2)$$

this shows that $\textbf{G1}$ is satisfied, where $K = I$ and $\alpha = \lambda^{-1}$.

---

1The unique operator $P_\sigma : L_1(m) \to L_1(m)$ such that for all $\varphi \in L_1(m)$ and for all $\psi \in L_\infty(m)$ it holds $\int \varphi \cdot \psi \circ \sigma dm = \int P_\sigma(\varphi) \cdot \psi dm$. 


2.1. Statements of the Main Results. Here we expose the main results of this work. The first one guaranties existence and uniqueness for the $F$-invariant measure in the space $S^\infty$ (see equation (12)).

**Theorem A.** If $F$ satisfies (G1), then the unique invariant probability for the system $F: \Sigma^+_A \times K \rightarrow \Sigma^+_A \times K$ in $S^\infty$ is $\mu_0$. Moreover, $||\mu_0||_\infty = 1$ and $||\mu_0||_{S^\infty} = 2$.

Next result shows that the transfer operator acting on the space $S^\infty$ is quasi-compact. This sort of result has many consequences for the dynamic and it implies several limit theorems. For instance, we obtain an exponential rate of convergence for the limit

$$\lim C_n(f,g) = 0,$$

where

$$C_n(f,g) := \left| \int (g \circ F^n) f d\mu_0 - \int g d\mu_0 \int f d\mu_0 \right|,$$

$g: \Sigma^+_A \times K \rightarrow \mathbb{R}$ is a Lipschitz function and $f \in \Theta_{\mu_0}$. The set $\Theta_{\mu_0}$ is defined as

(4) \[ \Theta_{\mu_0} := \{ f: \Sigma^+_A \times K \rightarrow \mathbb{R}; f\mu_0 \in S^\infty \}, \]

where the measure $f\mu_0$ is defined by $f\mu_0(E) := \int_E f d\mu_0$, for all measurable set $E$.

**Theorem B (Spectral gap).** If $F$ satisfies (G1), then the operator $F^*: S^\infty \rightarrow S^\infty$ can be written as

$$F^* = P + N,$$

where

a) $P$ is a projection i.e. $P^2 = P$ and $\dim \text{Im}(P) = 1$;

b) there are $0 < \xi < 1$ and $K > 0$ such that $\forall \mu \in S^\infty$

$$||N^n(\mu)||_{S^\infty} \leq ||\mu||_{S^\infty} \xi^n K;$$

c) $PN = NP = 0$.

The following theorem is an estimate for the Lipschitz constant (see equation (27) in Definition 7.2) of the disintegration of the unique $F$-invariant measure $\mu_0$ in $S^\infty$. This kind of result has many applications and similar estimations (for other systems) were given in [19], [7] and [20]. In [19], for instance, they use the regularity of the disintegration to prove stability of the $F$-invariant measure under a kind of ad-hoc perturbation. Here, we use this result to show that the abstract set $\Theta_{\mu_0}$, defined by equation (4) contains the Lipschitz functions.

**Theorem C.** Suppose that $F$ satisfies (G1) and (G2). Let $\mu_0$ be the unique $F$-invariant probability measure in $S^\infty$. Then $\mu_0 \in L^+_\theta$ and

$$|\mu_0|_\theta \leq \frac{C_1}{1 - \theta},$$

where $C_1$ was defined in Theorem 7.7 by $C_1 = \max\{H\theta + \theta N|g|_\theta, 2\}$.

The following is a consequence of all previous theorems. It shows that the system $F$ has exponential decay of correlations ($\lim C_n(f,g) = 0$ exponentially fast) for Lipschitz observables $f,g$. 
Theorem D. Suppose that $F$ satisfies (G1) and (G2). For all Lipschitz functions $g, f \in \mathcal{F}(\Sigma_A^+ \times K)$, it holds
\[
\left| \int (g \circ F^n) f \, d\mu_0 - \int g \, d\mu_{0} \int f \, d\mu_0 \right| \leq \|f\|_{S^\infty} K L_\phi(g) \xi^n \quad \forall n \geq 1,
\]
where $\xi$ and $K$ are from Theorem 2 and $|g|_\theta := |g|_\infty + L_\phi(g)$.

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3. Preliminares

Rokhlin’s Disintegration Theorem. Now we give a brief introduction about disintegration of measures.

Consider a probability space $(\Sigma, \mathcal{B}, \mu)$ and a partition $\Gamma$ of $\Sigma$ by measurable sets $\gamma \in \mathcal{B}$. Denote by $\pi : \Sigma \to \Gamma$ the projection that associates to each point $x \in \Sigma$ the element $\gamma_x$ of $\Gamma$ which contains $x$, i.e. $\pi(x) = \gamma_x$. Let $\mathcal{B}$ be the $\sigma$-algebra of $\Gamma$ provided by $\pi$. Precisely, a subset $Q \subset \Gamma$ is measurable if, and only if, $\pi^{-1}(Q) \in \mathcal{B}$.

We define the quotient measure $\hat{\mu}$ on $\Gamma$ by $\hat{\mu}(Q) = \mu(\pi^{-1}(Q))$.

The proof of the following theorem can be found in [38], Theorem 5.1.11.

Theorem 3.1. (Rokhlin’s Disintegration Theorem) Suppose that $\Sigma$ is a complete and separable metric space, $\Gamma$ is a measurable partition$^2$ of $\Sigma$ and $\mu$ is a probability on $\Sigma$. Then, $\mu$ admits a disintegration relatively to $\Gamma$, i.e. a family $\{\mu_\gamma\}_{\gamma \in \Gamma}$ of probabilities on $\Sigma$ and a quotient measure $\hat{\mu} = \pi^*\mu$ such that:

\(\begin{align*}
(a) & \quad \mu_\gamma(\gamma) = 1 \text{ for } \hat{\mu}\text{-a.e. } \gamma \in \Gamma; \\
(b) & \quad \text{for all measurable set } E \subset \Sigma, \text{ the function } \Gamma \to \mathbb{R}, \text{ defined by } \gamma \mapsto \mu_\gamma(E) \\
& \quad \text{is measurable;}
\end{align*}\)

\(\begin{align*}
(c) & \quad \text{for all measurable set } E \subset \Sigma, \text{ it holds } \mu(E) = \int \mu_\gamma(E) \, d\hat{\mu}(\gamma).
\end{align*}\)

The proof of the following lemma can be found in [38], proposition 5.1.7.

Lemma 3.2. Suppose the $\sigma$-algebra $\mathcal{B}$, of $\Sigma$, has a countable generator. If $\{\{\mu_\gamma\}_{\gamma \in \Gamma}, \hat{\mu}\}$ and $\{\{\mu_\gamma\}_{\gamma \in \Gamma}, \hat{\mu}\}$ are disintegrations of the measure $\mu$ relatively to $\Gamma$, then $\mu_\gamma = \mu'_\gamma$, $\hat{\mu}$-almost every $\gamma \in \Gamma$.

3.1. The $L^\infty$ and $S^\infty$ spaces. Fix $\Sigma = \Sigma_A^+ \times K$ and let $\mathcal{SB}(\Sigma)$ be the space of Borel signed measures on $\Sigma$. Given $\mu \in \mathcal{SB}(\Sigma)$ denote by $\mu^+$ and $\mu^-$ the positive and the negative parts of its Jordan decomposition, $\mu = \mu^+ - \mu^-$ (see remark $^3$). Let $\mathcal{M}$ be the set
\[
\mathcal{M} = \{\mu \in \mathcal{SB}(\Sigma) : \pi_1^+\mu^+ \ll m \text{ and } \pi_1^-\mu^- \ll m\},
\]
where $m$ is the Markov measure.

$^2$We say that a partition $\Gamma$ is measurable if there exists a full measure set $M_0 \subset \Sigma$ s.t. restricted to $M_0$, $\Gamma = \bigvee_{n=1}^{\infty} \Gamma_n$, for some increasing sequence $\Gamma_1 \prec \Gamma_2 \prec \cdots \prec \Gamma_n \prec \cdots$ of countable partitions of $\Sigma$. Furthermore, $\Gamma_i \prec \Gamma_{i+1}$ means that each element of $\mathcal{P}_{i+1}$ is a subset of some element of $\Gamma_i$. 

$^3$Remark 3.1.
Given a probability measure \( \mu \in \mathcal{AB} \), theorem \( \S 1 \) describes a disintegration \( \{ \mu_\gamma \}_\gamma \) on \( F^s \) (see \( \S 2 \)) \(^3\) by a family \( \{ \mu_\gamma \}_\gamma \) of probability measures on the stable leaves and, since \( \mu \in \mathcal{AB} \), \( \hat{\mu} \) can be identified with a non negative marginal density \( \phi_\gamma : \Sigma^+_A \to \mathbb{R} \), defined almost everywhere and satisfying \( |\phi_\gamma|_1 = 1 \).

For a positive measure \( \mu \in \mathcal{AB} \) we define its disintegration by disintegrating the normalization of \( \mu \). In this case, it holds \( |\phi_\gamma|_1 = \mu(\Sigma) \).

Let \( \pi_{2,\gamma} : \gamma \to K \) be the restriction \( \pi_{2,\gamma} \), where \( \pi_2 : \Sigma \to K \) is the projection on \( K \) and \( \gamma \in F^s \).

Definition 3.3. Given a positive measure \( \mu \in \mathcal{AB} \) and its disintegration along the stable leaves \( F^s \), \( \{ \mu_\gamma \}_\gamma \) we define the restriction of \( \mu \) on \( \gamma \) as the positive measure \( \mu_\gamma \) on \( K \) (not on the leaf \( \gamma \)) defined for all measurable set \( E \subset K \) by

\[
\mu_\gamma(E) := \pi_{2,\gamma}(\phi_\gamma \mu_\gamma)(E).
\]

Lemma 3.4. If \( \mu \in \mathcal{AB} \), then for each measurable set \( E \subset K \), the function \( \hat{c} : \Sigma^+_A \to \mathbb{R} \), given by \( \hat{c}(\gamma) = \mu_\gamma(E) \), is measurable.

Proof. For a given \( \mu \in \mathcal{AB} \), by Theorem \( \S 1 \) we have that the function \( \gamma \mapsto \mu_\gamma(E) \) is measurable. Moreover, \( \phi_\gamma \) is measurable. Since,

\[
\hat{c}(\gamma) = \mu_\gamma(E) = \phi_\gamma \mu_\gamma(\pi_{2,\gamma}(E)) = \phi_\gamma(\gamma) \mu_\gamma(\pi_{2,\gamma}(E) \cap \gamma) = \phi_\gamma(\gamma) \mu_\gamma(\pi_{2,\gamma}(E))
\]

we have \( \hat{c} \) is a measurable function.

\( \square \)

For a given signed measure \( \mu \in \mathcal{AB} \) and its Jordan decomposition \( \mu = \mu^+ - \mu^- \), define the restriction of \( \mu \) on \( \gamma \) by

\[
\mu_\gamma := \mu^+_\gamma - \mu^-_\gamma.
\]

Remark 3.5. As we prove in Corollary 11.7, the restriction \( \mu_\gamma \) does not depend on the decomposition. Precisely, if \( \mu = \mu_1 - \mu_2 \), where \( \mu_1 \) and \( \mu_2 \) are any positive measures, then \( \mu_\gamma := \mu_1^+_\gamma - \mu_2^+_\gamma = \mu^+_\gamma - \mu^-_\gamma \), m.a.e. \( \gamma \in \Sigma^+_A \).

Let \( (X,d) \) be a compact metric space, \( g : X \to \mathbb{R} \) be a Lipschitz function and let \( L(g) \) be its best Lipschitz constant, i.e.

\[
L(g) = \sup_{x \neq y} \left\{ \frac{\left| g(x) - g(y) \right|}{d(x,y)} \right\}.
\]

Definition 3.6. Given two signed measures \( \mu \) and \( \nu \) on \( X \), we define a Wasserstein-Kantorovich Like distance between \( \mu \) and \( \nu \) by

\[
W_1^1(\mu, \nu) = \sup_{L(g) \leq 1, ||g||_{\infty} \leq 1} \left| \int gd\mu - \int gd\nu \right|.
\]

\(^3\) By lemma \( \S 2 \), the disintegration of a measure \( \mu \) is the \( \hat{\mu} \)-unique \( (\hat{\mu}, \phi_\gamma m) \) measurable family \( (\mu_\gamma, \phi_\gamma m) \) such that, for every measurable set \( E \subset \Sigma \) it holds

\[
\mu(E) = \int_{\Sigma^+_A} \mu_\gamma(E \cap \gamma) d\phi_\gamma m(\gamma).
\]

We also remark that, in our context, \( \Gamma \) and \( \pi \) of theorem \( \S 3.1 \) are respectively equal to \( F^s \) and \( \pi_1 \), defined by \( \pi_1(\gamma, y) = \gamma \), where \( \gamma \in \Sigma^+_A \) and \( y \in K \).
From now on, we denote
\[(9) \quad ||\mu||_W := W^0_1(0, \mu).\]
As a matter of fact, \(|| \cdot ||_W\) defines a norm on the vector space of signed measures defined on a compact metric space. We remark that this norm is equivalent to the dual of the Lipschitz norm.

Other applications of this metric to obtain limit theorems can be seen in [24], [20] and [32]. For instance, in [32] the author apply this metric to a more general case of shrinking fibers systems.

**Remark 3.7.** From now on, we denote \(||\mu|_{\gamma}||_W := W^0_1(\mu^+|_{\gamma}, \mu^-|_{\gamma}).\)

**Definition 3.8.** Let \(L^\infty \subseteq AB(\Sigma)\) be defined as
\[(10) \quad L^\infty := \{\mu \in AB: \text{ess sup}_\gamma \{||\mu|_{\gamma}||_W : \gamma \in \Sigma^+_A\} < \infty\},\]
where the essential supremum is taken over \(\Sigma^+_A\) with respect to \(m\). Define the function \(|| \cdot ||_\infty : L^\infty \rightarrow \mathbb{R}\) by
\[(11) \quad ||\mu||_\infty = \text{ess sup}\{W^0_1(\mu^+|_{\gamma}, \mu^-|_{\gamma}); \gamma \in \Sigma^+_A\}.\]
Finally, consider the following set of signed measures on \(\Sigma\)
\[(12) \quad S^\infty = \{\mu \in L^\infty; \phi_1 \in \mathcal{F}_\theta(\Sigma^+_A)\},\]
and the function, \(|| \cdot ||_{S^\infty} : S^\infty \rightarrow \mathbb{R}\), defined by
\[(13) \quad ||\mu||_{S^\infty} = ||\phi_1||_\theta + ||\mu||_\infty,\]
where \(\phi_1 = \frac{d(\pi_1^*\mu)}{dm} \).

It is straightforward to show that \((L^\infty, || \cdot ||_\infty)\) and \((S^\infty, || \cdot ||_{S^\infty})\) are normed vector spaces, see for instance [57]. Consider \(SB(K)\) with the Borel’s sigma algebra generating with the Wasserstein-Kantorovich Like metric. We have that the map \(\tilde{c} : \Sigma^+_A \rightarrow SB(K), \gamma \mapsto |\mu|_{\gamma}\) is a measurable function. In fact, just note that by 3.1 the map \(\gamma \mapsto \mu|_{\gamma}\) from \(\Sigma^+_A\) to \(SB(\Sigma)\) is measurable, \(\phi\) is a measurable function and \(\pi_{2, \gamma} : SB(\Sigma) \rightarrow SB(K)\) is a contraction.

4. Transfer operator associated to \(F\)

Consider the transfer operator \(F^*\) associated with \(F\), that is, \(F^*\) is given by
\[|F^* \mu|(E) = F^{-1}(E),\]
for each signed measure \(\mu \in SB(\Sigma)\) and for each measurable set \(E \subset \Sigma\).

If \(\mu \in AB\) we write \((\{\mu_\gamma\}, \phi_1)\) to denote is disintegration, where \(\phi_1 = \frac{d(\pi_1^*\mu)}{dm}\).

**Lemma 4.1.** If \(\mu \in AB\) is a probability measure and \(\phi_1 := \frac{d(\pi_1^*\mu)}{dm}\), then \(F^* \mu \in AB\) and
\[(14) \quad \frac{d\pi_1^*(F^* \mu)}{dm} = P_\sigma(\phi_1).\]
Moreover,
\[(15) \quad (F^* \mu)|_{\gamma} = \nu_{\gamma} := \frac{1}{P_\sigma(\phi_1)(\gamma)} \sum_{i=1}^{N} \phi_1 \sigma_i^{-1}(\gamma) \cdot \chi_{\sigma_i^*(\mu_\gamma)} \cdot F^* \mu|_{\sigma_i^{-1}(\gamma)}\]
when $P_\sigma(\phi_1)(\gamma) \neq 0$. Otherwise, if $P_\sigma(\phi_1)(\gamma) = 0$, then $\nu_\gamma$ is the Lebesgue measure on $\gamma$ (the expression $\int_{m,\sigma} \frac{\phi_1}{\rho} \circ \sigma_i^{-1}(\gamma) \cdot \frac{X_{\sigma_i(P_i)}(\gamma)}{P_\sigma(\phi_1)(\gamma)} \cdot F^* \mu_{\sigma_i^{-1}(\gamma)}$ is understood to be zero outside $\sigma_i(P_i)$ for all $i = 1, \ldots, N$). Here and above, $\chi_A$ is the characteristic function of the set $A$.

Proof. By the uniqueness of the disintegration (see Lemma 3.2) to prove Lemma 4.1, it is enough to prove the following equation

$$F^* \mu(E) = \int_{\Sigma} \nu_\gamma(E \cap \gamma) P_T(\phi_x)(\gamma) d\gamma,$$

for a measurable set $E \subset \Sigma$. To do it, let us define the sets $B_1 = \{ \gamma \in N_1; T^{-1}(\gamma) = \emptyset \}$, $B_2 = \{ \gamma \in B_1^c; P_T(\phi_x)(\gamma) = 0 \}$ and $B_3 = (B_1 \cup B_2)^c$. The following properties can be easily proven:

1. $B_i \cap B_j = \emptyset$, $\sigma_i^{-1}(B_i) \cap \sigma_j^{-1}(B_j) = \emptyset$, for all $1 \leq i, j \leq 3$ such that $i \neq j$ and $\bigcup_{i=1}^3 B_i = \bigcup_{i=1}^3 \sigma_i^{-1}(B_i) = N_1$;  
2. $m(\sigma_i^{-1}(B_1)) = m(\sigma_i^{-1}(B_2)) = 0$.

Using the change of variables $\gamma = \sigma_i(\beta)$ and the definition of $\nu_\gamma$ (see (15)), we have

$$\int_{\Sigma} \nu_\gamma(E \cap \gamma) P_\sigma(\phi_1)(\gamma) dm(\gamma) = \int_{B_3} \sum_{i=1}^N \phi_i \int_{m,\sigma_i} \frac{\phi_1}{\rho} \circ \sigma_i^{-1}(\gamma) F^* \mu_{\sigma_i^{-1}(\gamma)}(E) X_{\sigma_i(P_i)}(\gamma) dm(\gamma)$$

$$= \sum_{i=1}^N \int_{\sigma_i^{-1}(B_3) \cap B_3} \phi_1(\beta) \mu_{\beta}(F^{-1}(E)) dm(\beta)$$

$$= \int_{\sigma_i^{-1}(\sigma_1^{-1}(B_3) \cap B_3)} \phi_1(\beta) \mu_0(F^{-1}(E)) dm(\beta)$$

$$= \int_{\sigma_i^{-1}(B_3) \sigma_1^{-1}(B_3)} \mu_0(F^{-1}(E)) d(\phi_1 m)(\beta)$$

$$= \mu(F^{-1}(E))$$

$$= \mu(E).$$

And the proof is done. \hfill \square

As said in Remark 3.1, Proposition 4.2 yields that the restriction $\mu \mid_\gamma$ does not depend on the decomposition. Thus, for each $\mu \in \mathcal{L}_\infty$, since $F^* \mu$ can be decomposed as $F^* \mu = F^*(\mu^+) - F^*(\mu^-)$, we can apply the above Lemma to $F^*(\mu^+)$ and $F^*(\mu^-)$ to get the following

**Proposition 4.2.** Let $\gamma \in \mathcal{F}^*$ be a stable leaf. Let us define the map $F_\gamma : K \to K$ by

$$F_\gamma = \pi_2 \circ F \mid_\gamma \circ \pi_2^{-1}.$$
Then, for each \( \mu \in \mathcal{L}^\infty \) and for almost all \( \gamma \in \Sigma_A^+ \) (interpreted as the quotient space of leaves) it holds
\[
(F^* \mu)|_\gamma = \sum_{i=1}^N \frac{F^*_{\sigma_i^{-1}(\gamma)} \mu}_{\partial \sigma_i^{-1}(\gamma)} \chi_{\sigma_i(p_i)}(\gamma) \text{ m-a.e. } \gamma \in \Sigma_A^+.
\]

5. Basic properties of the norms and convergence to equilibrium

In this section, we show important properties of the norms and their behavior with respect to the transfer operator. In particular, we show that the \( \mathcal{L}^\infty \) norm is weakly contracted by the transfer operator. We prove a Lasota-Yorke like inequality and an exponential convergence to equilibrium statement. All these properties will be used in next section to prove a spectral gap statement for the transfer operator.

Proposition 5.1 (The weak norm is weakly contracted by \( F^* \)). Suppose that \( F \) satisfies (G1). If \( \mu \in \mathcal{L}^\infty \) then
\[
||F^* \mu||_\infty \leq ||\mu||_\infty.
\]

In the proof of the proposition we will use the following lemma about the behavior of the \( || \cdot ||_W \) norm (see equation (9)) after a contraction. It says that a contraction cannot increase the \( || \cdot ||_W \) norm.

Lemma 5.2. For every \( \mu \in AB \) and and \( m \)-almost all stable leaf \( \gamma \in F^s \), it holds
\[
||F_\gamma^* (\mu|_\gamma)||_W \leq ||\mu|_\gamma||_W,
\]
where \( F_\gamma : K \to K \) is defined in Proposition 4.2. Moreover, if \( \mu \) is a probability measure on \( K \) it holds
\[
||\mu||_W = 1, \quad \forall \ n \geq 1.
\]

In particular, since \( F^* \mu \) is also a probability we have that \( ||F^* \mu||_W = 1 \).

Proof. (of Lemma 5.2) Indeed, since \( F_\gamma \) is an \( \alpha \)-contraction, if \( |g|_\infty \leq 1 \) and \( \text{Lip}(g) \leq 1 \) the same holds for \( g \circ F_\gamma \). Since
\[
\left| \int g \, dF_\gamma^* (\mu|_\gamma) \right| = \left| \int g \circ F_\gamma \, d\mu|_\gamma \right|,
\]
taking the supremum over \( |g|_\infty \leq 1 \) and \( \text{Lip}(g) \leq 1 \) we finish the proof of the inequality.

In order to prove equation (20), consider a probability measure \( \mu \) on \( K \) and a Lipschitz function \( g : K \to \mathbb{R} \), such that \( |g|_\infty \leq 1 \) and \( L(g) \leq 1 \). Therefore, \( |\int g \, d\mu| \leq ||g||_\infty \leq 1 \), which yields \( ||\mu||_W \leq 1 \). Reciprocally, consider the constant function \( g \equiv 1 \). Then \( 1 = |\int g \, d\mu| \leq ||\mu||_W \). These two facts proves equation (20). \( \square \)

Now we are ready to prove Proposition 5.1.

Proof. (of Proposition 5.1) In the following, we consider for all \( i \), the change of variable \( y = \sigma_i(z) \), where \( \sigma_i \) is the restriction of \( \sigma \) on the cylinder \([0|i] \). Moreover, for a given signed measure \( \mu \),
we denote the function $\gamma \mapsto \|\mu_\gamma\|_W$ by $c(\gamma)$, i.e., $c(\gamma) = \|\mu_\gamma\|_W$. Thus, Lemma 5.2 and equation (17) yield

$$\|F^* \mu\|_{\infty} = \text{ess sup}\{\|(F^* \mu)_\gamma\|_W : \gamma \in \Sigma_A^+\}$$

$$\leq \text{ess sup}\{\sum_{i=1}^{N} \left\| \frac{F_{\sigma_i^{-1}(\gamma)}^* \mu_{\sigma_i^{-1}(\gamma)}}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} \right\|_W : \gamma \in \Sigma_A^+\}$$

$$= \text{ess sup}\{\sum_{i=1}^{N} \frac{\|\mu_{\sigma_i^{-1}(\gamma)}\|_W}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} : \gamma \in \Sigma_A^+\}$$

$$= \text{ess sup}\{\sum_{i=1}^{N} \frac{c(\sigma_i^{-1}(\gamma))}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} : \gamma \in \Sigma_A^+\}$$

$$= \|P_\sigma(c)\|_{\infty}$$

$$\leq |c|_{\infty}$$

$$= \|\mu\|_{\infty}.$$

□

The following proposition shows a regularizing action of the transfer operator with respect to the strong norm. Such inequalities are usually called Lasota-Yorke or Doeblin-Fortet inequalities.

5.1. **Convergence to equilibrium.** In general, we say that the transfer operator $F^*$ has convergence to equilibrium with at least speed $\Phi$ and with respect to the norms $\| \cdot \|_{S^\infty}$ and $\| \cdot \|_{\infty}$, if for each $\mu \in \mathcal{V}$, where

(21) $\mathcal{V} = \{\mu \in S^\infty; \frac{d\pi^1_\mu}{dm} \in \text{ker}(\Pi_\sigma)\},$

it holds

(22) $\|F^n \mu\|_{\infty} \leq \Phi(n)\|\mu\|_{S^\infty},$

and $\Phi(n) \rightarrow 0$ as $n \rightarrow \infty$.

**Remark 5.3.** We observe that, $\mathcal{V}$ contains all zero average signed measure ($\mu(\Sigma) = 0$). Indeed, if $\mu(\Sigma) = 0$ (we remember that $\phi_1 = \frac{d(\pi^1_\mu)}{dm}$), then

$$\Pi_\sigma(\phi_1) = \int_{\Sigma_A^+} \phi_1 dm = \int_{\Sigma_A^+} \frac{d(\pi^1_\mu)}{dm} dm = \int_{\Sigma_A^+} d(\pi^1_\mu) = \mu(\pi^{-1}(\Sigma_A^+)) = \mu(\Sigma) = 0.$$

Next, we prove that $F^*$ has exponential convergence to equilibrium. This is weaker with respect to spectral gap. However, the spectral gap follows from the above Lasota-Yorke inequality and the convergence to equilibrium. To do it, we need some preliminary lemma and the following is somewhat similar to Lemma 5.2 considering the behaviour of the $\| \cdot \|_W$ norm after a contraction. It gives a finer estimate for zero average measures. The following Lemma is useful to estimate the behaviour of the $W$ norms under contractions.

**Lemma 5.4.** If $F$ satisfies (G1), then for all signed measures $\mu$ on $K$ and for $m$-almost all $\gamma \in \Sigma_A^+ (= F^*)$, it holds

$$\|F^*_\gamma \mu\|_W \leq \alpha\|\mu\|_W + |\mu(K)|$$
(α is the rate of contraction of G). In particular, if μ(K) = 0, then
\[ \|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W. \]

**Proof.** If \( \text{Lip}(g) \leq 1 \) and \( \|g\|_\infty \leq 1 \), then \( g \circ F_\gamma \) is \( \alpha \)-Lipschitz. Moreover, since \( \|g\|_\infty \leq 1 \), then \( \|g \circ F_\gamma - u\|_\infty \leq \alpha \), for some \( u \) s.t. \( |u| \leq 1 \). Indeed, let \( z \in K \) be such that \( |g \circ F_\gamma(z)| \leq 1 \), set \( u = g \circ F_\gamma(z) \) and let \( d \) be the metric of \( K \). Thus, we have
\[ |g \circ F_\gamma(y) - u| \leq \alpha d(y, z) \leq \alpha \]
and consequently \( \|g \circ F_\gamma - u\|_\infty \leq \alpha \).

This implies,
\[ \left| \int_K gdF_\gamma^* \mu \right| = \left| \int_K g \circ F_\gamma d\mu \right| \leq \left| \int_K g \circ F_\gamma - u d\mu \right| + \left| \int_K ud\mu \right| = \alpha \left| \int_K \frac{g \circ F_\gamma - u}{\alpha} d\mu \right| + \alpha|\mu(K)|. \]

And taking the supremum over \( |g|_\infty \leq 1 \) and \( \text{Lip}(g) \leq 1 \), we have \( \|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W + \mu(K) \). In particular, if \( \mu(K) = 0 \), we get the second part. \( \square \)

5.2. \( L^\infty \) norms. In this section we consider an \( L^\infty \) like anisotropic norm. We show how a Lasota Yorke inequality can be proved for this norm too.

**Lemma 5.5.** If \( F \) satisfies (G1), for all signed measure \( \mu \in S^\infty \) with marginal density \( \phi_1 \) it holds
\[ \|F_\gamma \mu\|_\infty \leq \alpha \|\mu\|_\infty + \|\phi_1\|_\infty. \]

**Proof.** Let \( \sigma_i \) be the branches of \( \sigma \), for all \( i = 1 \cdots N \). Applying Lemma 5.4 on the third line below and using the facts that \( \mu|_{\sigma_i^{-1}(\gamma)}(K) = \phi_1(\sigma_i^{-1}(\gamma)) \), \( P_\sigma(1) = 1 \), \( |P_\sigma(\phi_1)|_\infty \leq |\phi|_\infty \) we have
\[
\|F_\gamma \mu\|_W = \left\| \sum_{i=1}^N \frac{F_\gamma^* \mu|_{\sigma_i^{-1}(\gamma)}}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} \right\|_W \leq \sum_{i=1}^N \frac{\alpha \|\mu|_{\sigma_i^{-1}(\gamma)}\|_W + \alpha \|\phi_1(\sigma_i^{-1}(\gamma))\|_W}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} \leq \sum_{i=1}^N \frac{\alpha \|\mu|_{\sigma_i^{-1}(\gamma)}\|_W + \phi_1(\sigma_i^{-1}(\gamma))}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} \leq \frac{\alpha \|\mu\|_\infty \sum_{i=1}^N \frac{1}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} + \sum_{i=1}^N \phi_1(\sigma_i^{-1}(\gamma))}{J_{m,\sigma_i}(\sigma_i^{-1}(\gamma))} \leq \alpha \|\mu\|_\infty P_\sigma(1)(\gamma) + |P_\sigma(\phi_1)(\gamma)| \leq \alpha \|\mu\|_\infty + |P_\sigma(\phi_1)|_\infty \leq \alpha \|\mu\|_\infty + |\phi_1|_\infty. \]

\( \square \)
Iterating the main inequality of the previous lemma one obtains

**Corollary 5.6.** If $F$ satisfies (G1), then for all signed measure $\mu \in S^\infty$ it holds

$$\|F^n\mu\|_\infty \leq \alpha^n\|\mu\|_\infty + \frac{1}{1-\alpha}\|\phi_1\|_\infty,$$

where $\phi_1$ is the marginal density of $\mu$.

**Proposition 5.7** (Lasota-Yorke inequality for $S^\infty$). Suppose $F$ satisfies the (...). Then, there are $0 < \alpha_1 < 1$ and $B_4 \in \mathbb{R}$ such that for all $\mu \in S^\infty$, it holds

$$\|F^n\mu\|_{S^\infty} \leq 2\alpha_1^n\|\mu\|_{S^\infty} + B_4\|\mu\|_\infty.$$

**Proof.** Below we shall use the spectral gap for $P_\sigma$

$$\|F^n\mu\|_{S^\infty} = \|P_\sigma^n\phi_1\|_\theta + \|F^n\mu\|_{S^\infty} \leq \theta^n\|\phi_1\|_\theta + C_2\|\phi_1\|_\infty + \alpha^n\|\mu\|_\infty + \frac{1}{1-\alpha}\|\phi_1\|_\infty \leq 2\alpha_1^n\|\mu\|_{S^\infty} + (C_2 + \frac{1}{1-\alpha})\|\phi_1\|_\infty$$

where $|\phi_1|_\infty \leq \|\mu\|_\infty$, $|\phi_1|_\theta \leq \|\mu\|_{S^\infty}$, $\|\mu\|_\infty \leq \|\mu\|_{S^\infty}$ and we set $\alpha_1 = \max\{\theta, \alpha\}$. \qed

**Proposition 5.8** (Exponential convergence to equilibrium). Suppose that $F$ satisfies (G1). There exist $D_2 \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that for every signed measure $\mu \in V$ (see equation (21)), it holds

$$\|F^n\mu\|_{S^\infty} \leq D_2\beta_1^n\|\mu\|_{S^\infty},$$

for all $n \geq 1$.

**Proof.** Given $\mu \in V$ and denoting $\phi_1 := \frac{\pi_1^1\mu}{dm}$, it holds that $\phi \in \ker(\Pi_\sigma)$. Thus, $\|P_\sigma^n(\phi_1)\|_\theta \leq D\sigma^n\|\phi_1\|_\theta$ for all $n \geq 1$, therefore $\|P_\sigma^n(\phi_1)\|_\theta \leq D\sigma^n\|\mu\|_{S^\infty}$ for all $n \geq 1$.

Let $l$ and $0 \leq d \leq 1$ be the coefficients of the division of $n$ by 2, i.e. $n = 2l + d$. Thus, $l = \frac{n-d}{2}$ (by Proposition 5.1) we have $\|F^s\mu\|_\infty \leq \|\mu\|_\infty$ for all $s \in \mathbb{N}$, and $\|\mu\|_{\infty} \leq \|\mu\|_{S^\infty}$ and by Corollary 5.6 it holds (below, set $\beta_1 = \max\{\sqrt{\theta}, \sqrt{\alpha}\}$ and $\alpha = \frac{1}{1-\alpha}$)
\[ \| F^{*n} \mu \|_\infty \leq \| F^{*l+d} \mu \|_\infty \]
\[ \leq \alpha l \| F^{*l+d} \mu \|_\infty + \frac{1}{1 - \alpha} \left| \frac{d(\pi_1^l(F^{*l+d} \mu))}{dm} \right|_\infty \]
\[ \leq \alpha l \| \mu \|_\infty + \frac{1}{1 - \alpha} |P_0^l(\phi_1)|_\infty \]
\[ \leq \alpha l \| \mu \|_\infty + \frac{1}{1 - \alpha} \| P_0^l(\phi_1) \|_\theta \]
\[ \leq \alpha l \| \mu \|_\infty + \frac{1}{1 - \alpha} D_r t \| \mu \|_{S^\infty} \]
\[ \leq (1 + \pi D) \beta_1^{-d} \beta_1^n \| \mu \|_{S^\infty} \]
\[ \leq D_2 \beta_1^n \| \mu \|_{S^\infty}, \]

where \( D_2 = \frac{1 + \pi D}{\beta_1} \), which does not depend on \( n \).

**Remark 5.9.** We remark that the rate of convergence to equilibrium, \( \beta_1 \), for the map \( F \) found above, is directly related to the rate of contraction, \( \alpha \), of the stable foliation, and to the rate of convergence to equilibrium, \( r \), of the induced basis map \( T \). More precisely, \( \beta_1 = \max\{\sqrt{\alpha}, \sqrt{r}\} \). Similarly, we have an explicit estimate for the constant \( D_2 \), provided we have an estimate for \( D \) in the basis map \( T \).

Let \( \mu_0 \) be the \( F \)-invariant probability measure constructed by lifting \( m \) as an application of Theorem 10.3. By construction, it holds \( d(\pi_1^l \mu_0)/dm = 1 \in F_\theta^+ \).

With this fact in hands, let us prove theorem \( A \).

**Proof.** (of Theorem \( A \))

Let \( \mu_0 \) be the \( F \)-invariant measure such that \( d(\pi_1^l \mu_0)/dm = 1 \in F_\theta^+ \). Besides that, \( \| \mu_0 \|_{S^\infty} = 1 \) (it is a probability), thus \( \| \mu_0 \|_{S^\infty} = 1 \). Therefore, \( \mu_0 \in S^\infty \).

For the uniqueness, if \( \mu_0, \mu_1 \in S^\infty \) are \( F \)-invariant probabilities, i.e. \( \mu_0(\Sigma) = \mu_1(\Sigma) = 1 \), then by Remark 5.3 \( \mu_0 - \mu_1 \in V \). By Proposition 5.8 \( F^{*n}(\mu_0 - \mu_1) \to 0 \) in \( L^\infty \). Therefore, \( \mu_0 - \mu_1 = 0 \).

**Example 2.** (Iterated Function Systems) Let \( \phi_1, \phi_2, \ldots, \phi_N \) be a finite family of contractions \( \phi_i : K \to K, i = 1, \ldots, N \). In [18], Hutchinson, J.E. proved the existence of an invariant measure: a measure \( \mu \) on \( K \) which satisfies the relation

\[ \mu = \sum_{i=1}^N p_i \phi_i \mu, \]

where \( (p_1, p_2, \ldots, p_N) \) is a probability vector, \( \sum_{i=1}^N p_i = 1 \). Moreover, if \( m \) is the Bernoulli measure defined by \( p_1, \ldots, p_N \), then the product \( m \times \mu \) is invariant by the skew product \( F : \Sigma_A \times K \to \Sigma_A \times K \), defined by \( F(\sigma(x), z) = (\sigma(x), G(x, z)) \), where \( G(x, z) := \phi_{x_0}(z) \). Since the measure \( m \times \mu \) belongs to \( S^\infty \), the Proposition \( A \) yields that

\[ (23) \]
\[ \mu_0 = m \times \mu. \]

\( ^4 \)It can be difficult to find a sharp estimate for \( D \). An approach allowing to find some useful upper estimates is shown in [23].
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We believe that, following the ideas of [19], it is possible to prove statistical stability (computing the modulus of continuity), for the Hutchinson’s measure \( \mu \), under deterministic perturbations of the IFS.

6. Spectral gap

In this section, we prove a spectral gap statement for the transfer operator applied to our strong spaces. For this, we will directly use the properties proved in the previous section, and this will give a kind of constructive proof. We remark that, like in [19], we cannot apply the traditional Hennion, or Ionescu-Tulcea and Marinescu’s approach to our function spaces because there is no compact immersion of the strong space into the weak one. This comes from the fact that we are considering the same “dual of Lipschitz” distance in the contracting direction for both spaces.

Proof, (of theorem [13]) First, let us show there exist \( 0 < \xi < 1 \) and \( K_1 > 0 \) such that, for all \( n \geq 1 \), it holds

\[
\| F^n \|_{\mathcal{V} \to \mathcal{V}} \leq \xi^n K_1.
\]

Indeed, consider \( \mu \in \mathcal{V} \) (see equation (21)) s.t. \( \| \mu \|_{S^\infty} \leq 1 \) and for a given \( n \in \mathbb{N} \) let \( m \) and \( 0 \leq d \leq 1 \) be the coefficients of the division of \( n \) by 2, i.e. \( n = 2m + d \). Thus \( m = \frac{n - d}{2} \). By the Lasota-Yorke inequality (Proposition 5.7) we have the uniform bound \( \| F^n \|_{S^1} \leq 2 + B_4 \) for all \( n \geq 1 \). Moreover, by Propositions 5.8 and 5.1 there is some \( D_2 \) such that it holds (below, let \( \lambda_0 \) be defined by \( \lambda_0 = \max(\beta_1, \theta) \))

\[
\| F^n \mu \|_{S^\infty} \leq 2\theta^m \| F^{m+d} \mu \|_{S^\infty} + B_4 \| F^{m+d} \mu \|_{\mathcal{V}^\infty} \\
\leq \theta^m 2(2 + B_4) + B_4 \| F^{m+d} \mu \|_{\mathcal{V}^\infty} \\
\leq \theta^m 2(2 + B_4) + B_4 D_2 \beta_1^m \\
\leq \lambda_0^m [2(2 + B_4) + B_4 D_2] \\
\leq \lambda_0^m 2(2 + B_4) + B_4 D_2 \\
\leq \left( \frac{\sqrt{\lambda_0}}{\lambda_0} \right)^n [2(2 + B_4) + B_4 D_2] \\
= \xi^n K_1,
\]

where \( \xi = \sqrt{\lambda_0} \) and \( K_1 = \left( \frac{\lambda_0}{\lambda_0} \right)^n [2(2 + B_4) + B_4 D_2] \). Thus, we arrive at

\[
\| (F^* |_{\mathcal{V}})^n \|_{S^\infty \to S^\infty} \leq \xi^n K_1.
\]

Now, recall that \( F^* : S^\infty \to S^\infty \) has an unique fixed point \( \mu_0 \in S^\infty \), which is a probability (see Proposition [3]). Consider the operator \( P : S^\infty \to [\mu_0] \) ([\mu_0] is the space spanned by \( \mu_0 \)), defined by \( P(\mu) = \mu(\Sigma) \mu_0 \). By definition, \( P \) is a projection and \( \dim \text{Im} (P) = 1 \). Define the operator

\[
S : S^\infty \to \mathcal{V},
\]

by

\[
S(\mu) = \mu - P(\mu), \quad \forall \; \mu \in S^\infty.
\]

Thus, we set \( N = F^* \circ S \) and observe that, by definition, \( PN = NP = 0 \) and \( F^* = P + N \). Moreover, \( N^n(\mu) = F^* n(S(\mu)) \) for all \( n \geq 1 \). Since \( S \) is bounded and \( S(\mu) \in \mathcal{V} \), we get by (25), \( \| N^n(\mu) \|_{S^\infty} \leq \xi^n K \| \mu \|_{S^\infty} \), for all \( n \geq 1 \), where \( K = K_1 \| S \|_{S^\infty \to S^\infty} \). \( \square \)
Remark 6.1. We remark, the constant $\xi$ for the map $F$, found in Theorem [B] is directly related to the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of $F$ found before (see Remark [5.9]). More precisely,

$$\xi = \max\{\sqrt{\lambda}, \sqrt{\beta_1}\}.$$ We remark that, from the above proof we also have an explicit estimate for $K$ in the exponential convergence, while many classical approaches are not suitable for this.

7. Consequences

7.1. Lipschitz regularity of the disintegration of the invariant measure. We have seen that a positive measure on $\Sigma_A^+ \times K$, can be disintegrated along the stable leaves $F_s$ in a way that we can see it as a family of positive measures on $K$, $\{\mu_\gamma\}_{\gamma \in F_s}$. Since there is a one-to-one correspondence between $F_s$ and $\Sigma_A^+$, this defines a path in the metric space of positive measures, $\Sigma_A^+ \hookrightarrow SB(K)$, where $SB(\Sigma_A^+)$ is endowed with the Wasserstein-Kantorovich like metric (see definition [3.6]). It will be convenient to use a functional notation and denote such a path by $\Gamma_\mu : \Sigma_A^+ \rightarrow SB(K)$ defined almost everywhere by $\Gamma_\mu(\gamma) = \mu_\gamma$, where $\{|\mu_\gamma\}_{\gamma \in \Sigma_A^+}$ is some disintegration for $\mu_1$-a.e. $\gamma \in \Sigma_A^+$, the path $\Gamma_\mu$ is not unique. For this reason we define more precisely $\Gamma_\mu$ as the class of almost everywhere equivalent paths corresponding to $\mu$.

Definition 7.1. Consider a positive Borel measure $\mu$ and a disintegration $\omega = (\{\mu_\gamma\}_{\gamma \in \Sigma_A^+}, \phi_1)$, where $\{\mu_\gamma\}_{\gamma \in \Sigma_A^+}$ is a family of probabilities on $\Sigma$ defined $\mu_1$-a.e. $\gamma \in \Sigma_A^+$ (where $\mu_1 = \phi_1 m$) and $\phi_1 : \Sigma_A^+ \rightarrow \mathbb{R}$ is a non-negative marginal density. Denote by $\Gamma_\mu$ the class of almost everywhere equivalent paths associated to $\mu$.

Definition 7.2. Given a disintegration $\omega$ of $\mu$ and its functional representation $\Gamma_\mu^\omega$ we define the Lipschitz constant of $\mu$ associated to $\omega$ by

\begin{equation}
|M|_{\theta}^\omega := \sup_{\gamma_1, \gamma_2 \in I_{\Gamma_\mu}} \{\frac{||\mu_{\gamma_1} - \mu_{\gamma_2}||_{\theta}}{d_{\theta}(\gamma_1, \gamma_2)}\}. \tag{26}
\end{equation}

By the end, we define the Lipschitz constant of the positive measure $\mu$ by

\begin{equation}
|M|_{\theta} := \inf_{\Gamma_\mu^\omega \in \Gamma_\mu} \{|M|_{\theta}^\omega\}. \tag{27}
\end{equation}

Remark 7.3. When no confusion can be done, to simplify the notation, we denote $\Gamma_\mu^\omega(\gamma)$ just by $\mu_\gamma$.

From now on, we suppose that $G$ satisfies the additional hypothesis...
\textbf{G2}: \(G(.,z): \Sigma^+_A \to K\) satisfies

\[d(G(x, z), G(y, z)) \leq k_z d_\theta(x, y)\]

and

\[(28) \quad H := \text{ess sup}_{z \in K} k_z < \infty.\]

\textbf{Example 3 (Example 1 revisited).} Consider the system introduced in the Example 1 If \(x, y\) belong to a same cylinder \([0, j]\) then

\[d(G(x, z), G(y, z)) = d(f_{x_0}(z), f_{y_0}(z)) = 0\]

while otherwise

\[d(G(x, z), G(y, z)) = d(f_{x_0}(z), f_{y_0}(z)) \leq \text{diam}(I) = \frac{\theta \text{diam}(I)}{\theta}.\]

but \(d(x, y) = \sum_{i=0}^{\infty} \theta^i(1 - \delta(x_i, y_i)) \geq 1 > \theta\), because \(x_0 \neq y_0\). Therefore,

\[d(G(x, z), G(y, z)) \leq H d(x, y)\]

where \(H = \frac{\text{diam}(I)}{\theta} < \infty\). Since the constant \(H\) found here does not depend on \(z\), the previous computation shows that \textbf{G2} is also satisfied.

\textbf{Definition 7.4.} From the definition \textbf{G2} we define the set of Lipschitz positive measures \(L^+_\theta\) as

\[(29) \quad L^+_\theta = \{\mu \in AB : \mu \geq 0, |\mu|_\theta < \infty\}.\]

For the next lemma, for a given path, \(\Gamma_\mu\) which represents the measure \(\mu\), we define for each \(\gamma \in I_{\Gamma_\mu} \subset \Sigma^+_A\), the map

\[(30) \quad \mu_f(\gamma) := F_\gamma^* \mu|_{\gamma},\]

where \(F_\gamma : K \to K\) is defined as \(F_\gamma(y) = \pi_2 \circ F \circ (\pi_2|_{\gamma})^{-1}(y)\) and \(\pi_2 : \Sigma^+_A \times K \to \Sigma^+_A \times K\) is the projection \(\pi_2(x, y) = y\).

\textbf{Lemma 7.5.} If \(F\) satisfies (G1) and (G2), then for all representation \(\Gamma_\mu\) of a positive measure \(\mu \in L^+_\theta\), it holds

\[(31) \quad |\mu_f|_\theta \leq |\mu|_\theta + H||\mu||_\infty.\]

\textit{Proof.}

\[
\begin{align*}
||F_\gamma^* \mu|_{\gamma_1} - F_\gamma^* \mu|_{\gamma_2}||_W & \leq ||F_\gamma^* \mu|_{\gamma_1} - F_\gamma^* \mu|_{\gamma_2}||_W + ||F_\gamma^* \mu|_{\gamma_1} - F_\gamma^* \mu|_{\gamma_2}||_W \\
& \leq ||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W + ||F_\gamma^* \mu|_{\gamma_2} - F_\gamma^* \mu|_{\gamma_2}||_W \\
& \leq ||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W + \int d_2(G(\gamma_1, y), G(\gamma_2, y))d(\mu|_{\gamma_2})(y).
\end{align*}
\]

Thus,

\[
\begin{align*}
\frac{||F_\gamma^* \mu|_{\gamma_1} - F_\gamma^* \mu|_{\gamma_2}||_W}{d_\theta(\gamma_1, \gamma_2)} & \leq \frac{||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W}{d_\theta(\gamma_1, \gamma_2)} + \frac{\text{ess sup}_y d_2(G(\gamma_1, y), G(\gamma_2, y))}{d_\theta(\gamma_1, \gamma_2)} \int 1d(\mu|_{\gamma_2})(y) \\
& \leq |\mu|_\theta + H||\mu|_{\gamma_2}||_W.
\end{align*}
\]

Therefore,
(32) \[ |\mu F|_\theta \leq |\mu|_\theta^\omega + H||\mu||_\infty. \]

□

For the next proposition and henceforth, for a given path \( \Gamma^\omega_\mu \in \Gamma_\mu \) (associated with the disintegration \( \omega = (\{\mu_i\}, \phi_1) \), of \( \mu \)), unless written otherwise, we consider the particular path \( \Gamma^*_ F_\mu \in \Gamma^*_ F_\mu \) defined by the Proposition 4.2 by the expression

\[
(33) \quad \Gamma^*_ F_\mu(\gamma) = \sum_{i=1}^{N} F_{\gamma_i} \Gamma^\omega_\mu(\gamma_i) g_i(\gamma_i) \quad m-a.e. \quad \gamma \in \Sigma^+_A,
\]

where \( g_i(\gamma) = \frac{1}{J_{m,\sigma_i}(\gamma)} \) for all \( i = 1, \ldots, N \). We recall that, \( \Gamma^\omega_\mu(\gamma) = \mu|\gamma := \pi_{2*}(\phi_1(\gamma)\mu_\gamma) \) and in particular \( \Gamma^*_ F_\mu(\gamma) = (F_\mu)|\gamma = \pi_{2*}(P_\sigma \phi_1(\gamma)\mu_\gamma) \), where \( \phi_1 = \frac{d\pi_{1*}\mu}{dm} \) and \( P_\sigma \) is the Perron-Frobenius operator of \( f \).

**Proposition 7.6.** If \( F \) satisfies (G1) and (G2), then for all representation \( \Gamma_\mu \) of a positive measure \( \mu \in L^+_\theta \), it holds

\[
(34) \quad ||\Gamma^*_ F_\mu||_\theta^\omega \leq \theta|\mu|_\theta^\omega + C_1||\mu||_\infty,
\]

where \( C_1 = \max\{H\theta + \theta N|g_\theta, 2\} \).

**Proof.** Below we use the notation \( g_i(\gamma) = \frac{1}{J_{m,\sigma_i}(\gamma)} \) and for a given sequence \( \gamma = (x_i)_{i \in \mathbb{Z}^+} \in \Sigma^+_A \) we denote by \( j\gamma = (y_i)_{i \in \mathbb{Z}^+} \) the sequence defined by \( y_0 = x_0 \) and \( y_i = x_{i-1} \) for all \( i \geq 1 \). In this case, it is easy to see that \( d_\theta(j\gamma_1, j\gamma_2) = \theta d_\theta(\gamma_1, \gamma_2) \) for all \( \gamma_1, \gamma_2 \in \Sigma^+_A \).

We divide the proof of the proposition in two parts. First we consider two sequences \( \gamma_1 = (x_i)_{i \in \mathbb{Z}^+} \) and \( \gamma_2 = (y_i)_{i \in \mathbb{Z}^+} \) such that \( x_0 = y_0 \). By Proposition 4.2 (and equation (33)), we have

\[
||\Gamma^*_ F_\mu(\gamma_1) - \Gamma^*_ F_\mu(\gamma_2)||_W \leq \sum_{j:A_{j,x_0}=1} ||F_{j,\gamma_1}^* \mu|_{j,\gamma_1}||_W|g_j(j\gamma_1) - g_j(j\gamma_2)| + \sum_{j:A_{j,x_0}=1} |g_j(j\gamma_2)||F_{j,\gamma_1}^* \mu|_{j,\gamma_1} - F_{j,\gamma_2}^* \mu|_{j,\gamma_2}||_W.
\]

Thus,
\[
\frac{||\Gamma_{F*}^{\omega}(\gamma_1) - \Gamma_{F*}^{\omega}(\gamma_2)||_W}{d_\theta(\gamma_1, \gamma_2)} \leq \theta \sum_j ||F_{j\gamma_1}^* \mu|_{\gamma_1}||_W \frac{|g_j(j_1) - g_j(j_2)|}{\theta d_\theta(\gamma_1, \gamma_2)} \\
+ \theta \sum_j |g_j(j_2)| \frac{||F_{j\gamma_1}^* \mu|_{\gamma_1} - F_{j\gamma_2}^* \mu|_{\gamma_2}||_W}{\theta d_\theta(\gamma_1, \gamma_2)} \\
\leq \theta \sum_j ||F_{j\gamma_1}^* \mu|_{\gamma_1}||_W \frac{|g_j(j_1) - g_j(j_2)|}{d_\theta(j_1, j_2)} \\
+ \theta \sum_j |g_j(j_2)| \frac{||F_{j\gamma_1}^* \mu|_{\gamma_1} - F_{j\gamma_2}^* \mu|_{\gamma_2}||_W}{d_\theta(j_1, j_2)} \\
\leq \theta N|g_j| ||\mu||_\infty + \theta \sum_j |g_j(j_2)||\mu|_{\theta}^{\omega}.
\]

Applying Lemma 7.5 we get
\[
\frac{||\Gamma_{F*}^{\omega}(\gamma_1) - \Gamma_{F*}^{\omega}(\gamma_2)||_W}{d_\theta(\gamma_1, \gamma_2)} \leq \theta N|g| ||\mu||_\infty + \theta \sum_j |g_j(j_2)||\mu_{\theta}^{\omega}
\]
\[
\leq \theta N|g| ||\mu||_\infty + \theta \sum_j |g_j(j_2)|(||\mu||_{\theta}^{\omega} + \mu||\mu||_\infty)
\]
\[
= \theta ||\mu||_{\theta}^{\omega} + (H\theta + N\theta|g| ||\mu||_\infty).
\]

Taking the supremum over \( \gamma_1 \) and \( \gamma_2 \) we get
\[
(35) \quad ||F^* \mu||_{\theta}^{\omega} \leq \theta ||\mu||_{\theta}^{\omega} + (H\theta + N\theta|g| ||\mu||_\infty),
\]
where \( H \) was defined by equation (28).

In the remained case, where \( x_0 \neq y_0 \) it holds \( d_\theta(\gamma_1, \gamma_2) \geq 1 \). Then, by Lemma 7.2 we have
\[
\frac{||\Gamma_{F*}^{\omega}(\gamma_1) - \Gamma_{F*}^{\omega}(\gamma_2)||_W}{d_\theta(\gamma_1, \gamma_2)} \leq ||(F^* \mu)|_{\gamma_1} - (F^* \mu)|_{\gamma_2}||_W \\
\leq 2||\mu||_\infty.
\]

We finish the proof by setting \( C_1 = \max\{H\theta + N\theta|g|, 2\} \).

\[\square\]

**Theorem 7.7.** If \( F \) satisfies (G1) and (G2), then for all representation \( \Gamma_{\mu} \) of a positive measure \( \mu \in \mathcal{L}_\theta^+ \) and all \( n \geq 1 \), it holds
\[
(36) \quad ||\Gamma_{F^*}^{\omega n} \mu||_{\theta}^{\omega} \leq \theta^n ||\mu||_{\theta}^{\omega} + \frac{C_1}{1 - \theta} ||\mu||_\infty,
\]
where \( C_1 \) was defined in Proposition 7.6 by \( C_1 = \max\{H\theta + N\theta|g|, 2\} \).

**Remark 7.8.** Taking the infimum (with respect to \( \omega \)) on both sides of inequality 36 we get for each \( \mu \in \mathcal{L}_\theta^+ \)
Remark 7.10. We remark that, Theorem C is an estimation of the regularity (38) \( \Gamma \) this disintegration. By Proposition 4.2, we have (37) \( \mu \) where 

\[
\nu = \sum_{i=1}^{\deg(\sigma^n)} \frac{\text{deg}(\sigma^n)}{\gamma} \nu^n \cdot 1 \quad \text{a.e.} \quad \gamma \in \Sigma^+,
\]

for all \( n \geq 1 \).

Remark 7.9. For a given probability measure \( \nu \) on \( K \). Denote by \( m_1 \), the product \( m_1 = m \times \nu \), where \( m \) is the Markov measure fixed in the subsection. Besides that, consider its trivial disintegration \( \omega_0 = (\{m_{1,\gamma}\}, \phi_1) \), given by \( m_{1,\gamma} = \pi_{2,\gamma} \cdot \nu \), for all \( \gamma \) and \( \phi_1 \equiv 1 \). According to this definition, it holds that

\[
m_{1,\gamma} = \nu, \quad \forall \gamma.
\]

In other words, the path \( \Gamma_{m_1}^{\omega_0} \) is constant: \( \Gamma_{m_1}^{\omega_0}(\gamma) = \nu \) for all \( \gamma \). Moreover, for each \( n \in \mathbb{N} \), let \( \omega_n \) be the particular disintegration for the measure \( F_{\ast}^n m_1 \), defined from \( \omega_0 \) as an application of Lemma 11, and consider the path \( \Gamma_{F_{\ast}^n m_1}^{\omega_n} \) associated with this disintegration. By Proposition 12, we have

(38) \[
\Gamma_{F_{\ast}^n m_1}^{\omega_n}(\gamma) = \sum_{i=1}^{\deg(\sigma^n)} \frac{\text{deg}(\sigma^n)}{\gamma} \nu^n \cdot 1 \quad \text{a.e.} \quad \gamma \in \Sigma^+,
\]

where \( P_i, i = 1, \ldots, \deg(\sigma^n) \), ranges over the partition \( \mathcal{P}^{(n)} \) defined in the following way: for all \( n \geq 1 \), let \( \mathcal{P}^{(n)} \) be the partition of \( I \) s.t. \( \mathcal{P}^{(n)}(x) = \mathcal{P}^{(n+1)}(y) \) if and only if \( \mathcal{P}^{(n)}(\sigma_j(x)) = \mathcal{P}^{(n+1)}(\sigma_j(y)) \) for all \( j = 0, \ldots, n-1 \), where \( \mathcal{P}^{(1)} = \mathcal{P} \). This path will be used in the proof of the next proposition.

Proof. (of theorem C)

According to Proposition 13, let \( \mu_0 \in S^\infty \) be the unique \( F \)-invariant probability measure in \( S^\infty \). Consider the path \( \Gamma_{F_{\ast}^n m_1}^{\omega_n} \), defined in Remark 7.9, which represents the measure \( F_{\ast}^n m_1 \). By Theorem 14, these iterates converge to \( \mu_0 \) in \( \mathcal{L}^\infty \). It means that the sequence \( \{\Gamma_{F_{\ast}^n m_1}^{\omega_n}\}_{n=1}^{\infty} \) converges \( m \)-a.e. to \( \Gamma_{\mu_0}^{\omega} = \Gamma_{\mu_0}^{\omega} \) (in \( SB(K) \)) with respect to the metric defined in definition 3.6, where \( \Gamma_{\mu_0}^{\omega} \) is a path given by the Rokhlin Disintegration Theorem and \( \{\Gamma_{F_{\ast}^n m_1}^{\omega_n}\}_{n=1}^{\infty} \) is given by Remark 12. It implies that \( \{\Gamma_{F_{\ast}^n m_1}^{\omega_n}\}_{n=1}^{\infty} \) converges pointwise to \( \Gamma_{\mu_0}^{\omega} \) on a full measure set \( T \subset \Sigma^+ \). Let us denote \( \Gamma_{\mu_0}^{\omega} = \Gamma_{F_{\ast}^n m_1}^{\omega_n} \) and \( \Gamma_{\mu_0}^{\omega} = \Gamma_{\mu_0}^{\omega_n} \). Since \( \{\Gamma_{\mu_0}^{\omega_n}\}_{n=1}^{\infty} \) converges pointwise to \( \Gamma_{\mu_0}^{\omega} \), it holds

\[
\lim_{n \to \infty} \frac{||\Gamma_{\mu_0}^{\omega_n}(\gamma_1) - \Gamma_{\mu_0}^{\omega_n}(\gamma_2)||_W}{d_{\theta}(\gamma_1, \gamma_2)} = \frac{||\Gamma_{\mu_0}^{\omega_n}(\gamma_1) - \Gamma_{\mu_0}^{\omega_n}(\gamma_2)||_W}{d_{\theta}(\gamma_1, \gamma_2)}.
\]

On the other hand, by Theorem 17, we have

\[
|\Gamma_{\mu_0}^{\omega_n}|_\theta \leq \frac{C_1}{1 - \theta} \quad \text{for all } n \geq 1.
\]

Then \( \mu_0 \) holds

\[
|\Gamma_{\mu_0}^{\omega_n}|_\theta \leq \frac{C_1}{1 - \theta} \quad \text{and hence } |\mu_0|_\theta \leq \frac{C_1}{1 - \theta}.
\]

\[\square\]

Remark 7.10. We remark that, Theorem C is an estimation of the regularity of the disintegration of \( \mu_0 \). Similar results, for other sort of skew products are presented in [19, 7, 20] and [21].
8. Exponential decay of correlations

In this section, we will show how Theorem \[1\] implies an exponential rate of convergence for the limit
\[
\lim C_n(f, g) = 0,
\]
where
\[
C_n(f, g) := \left| \int (g \circ F^n) fd\mu_0 - \int gd\mu_0 \int fd\mu_0 \right|,
\]
g : \(\Sigma_A^+ \times K \to \mathbb{R}\) is a Lipschitz function and \(f \in \Theta_{\mu_0}\). The set \(\Theta_{\mu_0}\) is defined as
\[
\Theta_{\mu_0} := \{f : \Sigma_A^+ \times K \to \mathbb{R} : f \mu_0 \in S^\infty\},
\]
where the measure \(f \mu_0\) is defined by \(f \mu_0(E) := \int_E fd\mu_0\) for all measurable set \(E\).

Denote by \(\tilde{F}(\Sigma_A^+ \times K)\) the set of real Lipschitz functions, \(f : \Sigma_A^+ \times K \to \mathbb{R}\), with respect to the metric \(d_\theta + d\). And for such a function we denote by \(L_\theta(f)\) its Lipschitz constant.

**Proposition 8.1.** Consider \(F\) satisfying (G1) and (G2). For all Lipschitz function \(g : \Sigma_A^+ \times K \to \mathbb{R}\) and all \(f \in \Theta_{\mu_0}\), it holds
\[
\left| \int (g \circ F^n) fd\mu_0 - \int gd\mu_0 \int fd\mu_0 \right| \leq \|f\mu_0\|_{S^\infty} K L_\theta(g) \xi^n \quad \forall n \geq 1,
\]
where \(\xi\) and \(K\) are from Theorem \[1\] and \(|g|_\theta := |g|_\infty + L_\theta(g)\).

**Proof.** Let \(g : \Sigma \to \mathbb{R}\) be a Lipschitz function and \(f \in \Theta_{\mu_0}^1\). By Theorem \[1\] we have
\[
\left| \int (g \circ F^n) fd\mu_0 - \int gd\mu_0 \int fd\mu_0 \right| = \left| \int gd F^* (f \mu_0) - \int gd P(f \mu_0) \right|
\]
\[
= \|F^* (f \mu_0) - P(f \mu_0)\|_{W} \max\{L_\theta(g), |g|_\infty\}
\]
\[
= \|N^n (f \mu_0)\|_{W} \max\{L_\theta(g), |g|_\infty\}
\]
\[
\leq \|N^n (f \mu_0)\|_{S^\infty} \max\{L_\theta(g), |g|_\infty\}
\]
\[
\leq \|f \mu_0\|_{S^\infty} K L_\theta(g) \xi^n.
\]

\[\square\]

8.1. From a Space of Measures to a Space of Functions . In this section, we will show how the regularity of the \(F\)-invariant measure, given by Proposition \[3\] implies that \(\tilde{F}(\Sigma_A^+ \times K) \subset \Theta_{\mu_0}\). Where, \(\tilde{F}(\Sigma_A^+ \times K)\) was defined in the previous section.

Consider the vector space of functions, \(\Theta_{\mu_0}\), defined by
\[(39) \quad \Theta_{\mu_0} := \{f : \Sigma \to \mathbb{R} : f \mu_0 \in S^\infty\},\]
On \(\Theta_{\mu_0}\) we define the norm \(|\cdot|_\theta : \Theta_{\mu_0} \to \mathbb{R}\) by
\[(40) \quad |f|_\theta := |f|_{S^\infty}.\]

\(^5f\mu_0\) is defined by \(f \mu_0(E) = \int_E fd\mu_0\), for each measurable set \(E \subset \Sigma_A^+\).
Lemma 8.2. Let \( \{\{\mu_0,\gamma\}\} \gamma \in \Sigma^+_A \) be the disintegration of \( \mu_0 \) along the partition \( F^* := \{\{\gamma\} \times K : \gamma \in \Sigma^+_A \} \), and for a \( \mu_0 \) integrable function \( h : \Sigma^+_A \times K \to \mathbb{R} \), denote by \( \nu := h\mu_0 \). If \( \{\nu,\gamma\} \) is the disintegration of \( \nu \), where \( \overline{\nu} := \pi_{\gamma}\nu \), then \( \nu \ll m \) and \( \nu_\gamma \ll \mu_0_\gamma \). Moreover, denoting \( \overline{h} := \frac{d\overline{\nu}}{dm} \), it holds
\[
\overline{h}(\gamma) = \int_M h(\gamma,y)d(\mu_0|\gamma),
\]
and for \( \nu \)-a.e. \( \gamma \in \Sigma^+_A \)
\[
\frac{d\nu_\gamma}{d\mu_0,\gamma}(y) = \begin{cases} \frac{h|_\gamma(y)}{\int h|_\gamma(y)d\mu_0,\gamma(y)}, & \text{if } \gamma \in B^c & \text{for all } y \in K, \\ 0, & \text{if } \gamma \in B, \end{cases}
\]
where \( B := \overline{\nu}^{-1}(0) \).

Proposition 8.3. If \( F \) satisfies (G1) and (G2), then \( \overline{\nu}(\Sigma^+_A \times K) \subset \Theta_{\mu_0} \).

Proof. For a given \( f \in \overline{\nu}(\Sigma^+_A \times K) \), denote by \( k := \max\{L_\theta(f),|f|_\infty\} \) we have
\[
|\overline{f}(\gamma_2) - \overline{f}(\gamma_1)| = \left| \int_K f(\gamma_2,y)d(\mu_0|_{\gamma_2}) - \int_K f(\gamma_1,y)d(\mu_0|_{\gamma_1}) \right|
\leq \left| \int_K f(\gamma_2,y)d(\mu_0|_{\gamma_2}) - \int_K f(\gamma_1,y)d(\mu_0|_{\gamma_2}) \right|
+ \left| \int_K f(\gamma_1,y)d(\mu_0|_{\gamma_2}) - \int_K f(\gamma_1,y)d(\mu_0|_{\gamma_1}) \right|
\leq \int_K |f(\gamma_2,y) - f(\gamma_1,y)|d(\mu_0|_{\gamma_2})
+ k \left| \int_K \frac{f(\gamma_1,y)}{k}d(\mu_0|_{\gamma_1}) - \mu_0|_{\gamma_2} \right|
\leq L_\theta(f)d_\theta(\gamma_2,\gamma_1) \int_K 1d(\mu_0|_{\gamma_2}) + k \|\mu_0|_{\gamma_1} - \mu_0|_{\gamma_2}\|
\leq L_\theta(f)d_\theta(\gamma_2,\gamma_1)\|\mu_0\|_\infty + k \|\mu_0|_{\gamma_1} - \mu_0|_{\gamma_2}\|.
\]
Thus,
\[
|\overline{f}|_\theta \leq L_\theta(f)\|\mu_0\|_\infty + k\|\mu_0\|_\theta
\]
and so
\[
|\overline{f}|_\theta \leq L_\theta(f)\|\mu_0\|_\infty + k\|\mu_0\|_\theta.
\]
Therefore, \( \overline{f} \in \mathcal{F}_\theta(\Sigma^+_A) \).

It remains to show that \( \nu \in \mathcal{L}^\infty \). Indeed, since \( \nu|_{\gamma} = \pi^+_\gamma \nu_\gamma \) and by equations (41) and (42), we have
\[ \int gd(\nu|_\gamma) = \left| \int g \circ \pi_{2,\gamma} d(\nu|_\gamma) \right| \leq \|g\|_{\infty} \int K f(\gamma, y) d\mu \leq 1. \]

Hence, \( \nu \in S^\infty \) and \( f \in \Theta_{\mu_0} \).

**Proof.** (of theorem D)

It is a direct consequence of propositions 8.1 and 8.3. \( \square \)

### 8.2. Decay of Correlations for Random Dynamical Systems (IFS)

Consider an IFS (of Example 2) given by the finite family of contractions \( \mathcal{C} = \{ \varphi_1, \ldots, \varphi_N \} \), where \( \varphi_i : K \to K, i = 1, \ldots, N \) and \( \mu \) its Hutchinson’s invariant measure. Let \( (\tau_n)_n \) be a sequence such that \( \tau_n \in \mathcal{C} \), for all \( n \), i.e., \( \tau_n = \varphi_{\gamma_n} \) for some sequence \( \gamma \in \Sigma^+_A \), where \( \gamma = (\gamma_n)_{n \in \mathbb{N}} \). Define the sequence \( (\theta_n)_n \) by \( \theta_n = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_0 \).

\[ \theta_n = \varphi_{\gamma_n} \circ \varphi_{\gamma_{n-1}} \circ \cdots \circ \varphi_{\gamma_0}, \]

for some \( \gamma \in \Sigma^+_A \).

**Definition 8.4.** Let \( B_1 \) and \( B_2 \) be spaces of real valued functions \( K \to \mathbb{R} \). Define the \( \gamma \)-coefficient of correlation between \( g_1 \) and \( g_2 \), \( C_n(g_1, g_2)(\gamma) \), by

\[ C_n(g_1, g_2)(\gamma) := \int g_1(g_2 \circ \theta_n) d\mu - \int g_1 d\mu \int g_2 d\mu. \]

We say that the IFS, \( \mathcal{C} = \{ \varphi_1, \varphi_2, \ldots, \varphi_N \} \) has **Exponential Decay of Correlations** over \( B_1 \) and \( B_2 \) if there exists constants \( 0 < \xi < 1 \) and \( K > 0 \), such that

\[ \left| \int C_n(g_1, g_2)(\gamma) dm(\gamma) \right| \leq \|g_1\|_{B_1} \|g_2\|_{B_2} K \xi^n, \ \forall n \geq 1, \]

for all \( g_1 \in B_1 \) and \( g_2 \in B_2 \), where \( \| \cdot \|_{B_1} \) and \( \| \cdot \|_{B_2} \) are norms on \( B_1 \) and \( B_2 \), respectively.

**Theorem 8.5.** Let \( \mathcal{C} = \{ \varphi_1, \varphi_2, \ldots, \varphi_N \} \), a hyperbolic IFS and \( \mu \) its invariant measure. Then, \( \mathcal{C} \) has Exponential Decay of Correlations over the real Lipschitz functions, \( \hat{F}(K) \).

**Proof.** Let \( g_1, g_2 : K \to \mathbb{R} \) be Lipschitz observables and suppose that \( \overline{g}_1 \) and \( \overline{g}_2 \) are their trivial extensions to \( \Sigma^+_A \times K \), \( \overline{g}_i := g_i \circ \pi_2, i = 1, 2 \). They are still Lipschitz
functions, but defined on the set $\Sigma_A^+ \times K$.

\[ C_n(g_1, g_2)(\gamma) = \int_K g_1(y) g_2 \circ \varphi_n(\gamma, y) d\mu(y) - \int_K g_1(y) d\mu(y) \int_K g_2 \circ \varphi_n(\gamma, y) d\mu(y) \]

\[ = \int_K g_1 \circ \pi_2 \circ F^n(\gamma, y) d\mu(y) - \int_K g_1(y) d\mu(y) \int_K g_2 \circ \pi_2 \circ F^n(\gamma, y) d\mu(y) \]

\[ = \int_K g_1 \circ F^n(\gamma, y) d\mu(y) - \int_K g_1(y) d\mu(y) \int_K g_2 \circ F^n(\gamma, y) d\mu(y). \]

Integrating with respect to the Markov’s measure $m$ and applying Fubini’s Theorem, over $\Sigma_A^+$ we have

\[ \int C_n(g_1, g_2)(\gamma) d\mu = \int_{\Sigma_A^+} \int_K g_1 \circ \overline{F}_2 \circ F^n(\gamma, y) d\mu(y) d\mu \]

\[ - \int_{\Sigma_A^+} \int_K g_1(y) d\mu(y) \int_{\Sigma_A^+} \int_K \overline{F}_2 \circ F^n(\gamma, y) d\mu(y) d\mu \]

\[ = \int_{\Sigma_A^+} \int_K g_1 \circ \overline{F}_2 \circ F^n(\gamma, y) d\mu(y) d\mu \]

\[ - \int_{\Sigma_A^+} \int_K g_1(y) d\mu(y) \int_{\Sigma_A^+} \int_K \overline{F}_2 \circ F^n(\gamma, y) d\mu(y) d\mu \]

\[ = \int_{\Sigma_A^+ \times K} \overline{g}_1 \circ \overline{F}_2 \circ F^n(\gamma, y) d(m \times d\mu) \]

\[ - \int_{\Sigma_A^+ \times K} \overline{g}_1 (m \times \mu) \int_{\Sigma_A^+ \times K} \overline{F}_2 \circ F^n(\gamma, y) d(m \times \mu) \]

\[ = \int_{\Sigma_A^+ \times K} \overline{g}_1 \circ \overline{F}_2 \circ F^n(\gamma, y) d\mu_0 \]

\[ - \int_{\Sigma_A^+ \times K} \overline{g}_1 d\mu_0 \int_{\Sigma_A^+ \times K} \overline{F}_2 \circ F^n(\gamma, y) d\mu_0. \]

Where the last equality holds by the equation (23) of Example 2 (application of Proposition A).

Then,

\[ \int C_n(g_1, g_2)(\gamma) d\mu = \int_{\Sigma_A^+ \times K} \overline{g}_1 \circ \overline{F}_2 \circ F^n(\gamma, y) d\mu_0 - \int_{\Sigma_A^+ \times K} \overline{g}_1 d\mu_0 \int_{\Sigma_A^+ \times K} \overline{F}_2 \circ F^n(\gamma, y) d\mu_0 \]

and by Theorem D the proof is complete.

\[ \square \]

9. Appendix 1: On Disintegration of Measures

In this section, we prove some results on disintegration of absolutely continuous measures with respect to a measure $\mu_0 \in \mathcal{M}$. Precisely, we are going to prove Lemma 8.2.
Let us fix some notations. Denote by \((N_1, m_1)\) and \((N_2, m_2)\) the spaces defined in section 2.0.2. For a \(\mu_0\)-integrable function \(f : N_1 \times N_2 \rightarrow \mathbb{R}\) and a pair \((\gamma, y) \in N_1 \times N_2\) \((\gamma \in N_1\) and \(y \in N_2)\) we denote by \(f_\gamma : N_2 \rightarrow \mathbb{R}\), the function defined by \(f_\gamma(y) = f(\gamma, y)\) and \(f|_\gamma\) the restriction of \(f\) on the set \(\{\gamma\} \times N_2\). Then \(f_\gamma = f|_\gamma \circ \pi_{2, \gamma}^{-1}\) and \(f_\gamma \circ \pi_{2, \gamma} = f|_\gamma\), where \(\pi_{2, \gamma}\) is restriction of the projection \(\pi_2(\gamma, y) := y\) on the set \(\{\gamma\} \times N_2\). When no confusion is possible, we will denote the leaf \(\{\gamma\} \times N_2\), just by \(\gamma\).

From now on, for a given positive measure \(\mu \in \mathcal{AB}\), on \(N_1 \times N_2\), \(\mu\) stands for the measure \(\pi_1 \ast \mu\), where \(\pi_1\) is the projection on the first coordinate, \(\pi_1(x, y) = x\).

For each measurable set \(A \subset N_1\), define \(g : N_1 \rightarrow \mathbb{R}\) by
\[
g(\gamma) = \phi_x(\gamma) \int \chi_{\pi_1^{-1}(A)}(\gamma) f|_\gamma(y) d\mu_0,\gamma(y)
\]
and note that
\[
g(\gamma) = \begin{cases} \phi_x(\gamma) \int f|_\gamma(y) d\mu_0,\gamma, & \text{if } \gamma \in A \\ 0, & \text{if } \gamma \notin A. \end{cases}
\]
Then, it holds
\[
g(\gamma) = \chi_A(\gamma) \phi_x(\gamma) \int f|_\gamma(y) d\mu_0,\gamma.
\]

Proof. (of Lemma 8.2)

For each measurable set \(A \subset N_1\), we have
\[
\int_A \frac{\pi_1^*(f \mu_0)}{dm_1} dm_1 = \int \chi_A \circ \pi_1 d(f \mu_0)
= \int \chi_{\pi_1^{-1}(A)} f d\mu_0
= \int \left[ \int \chi_{\pi_1^{-1}(A)}(\gamma) f|_\gamma(y) d\mu_0,\gamma(y) \right] d(\phi_x m_1)(\gamma)
= \int \left[ \phi_x(\gamma) \int \chi_{\pi_1^{-1}(A)}(\gamma) f|_\gamma(y) d\mu_0,\gamma(y) \right] d(m_1)(\gamma)
= \int g(\gamma) d(m_1)(\gamma)
= \int_A \left[ \int f_\gamma(y) d\mu_0|_\gamma(y) \right] d(m_1)(\gamma).
\]
Thus, it holds
\[
\frac{\pi_1^*(f \mu_0)}{dm_1}(\gamma) = \int f_\gamma(y) d\mu_0|_\gamma, \text{ for } m_1 \text{-a.e. } \gamma \in N_1.
\]
And by a straightforward computation
\[
\pi_1^*(f \mu_0)(\gamma) = \phi_x(\gamma) \int f|_\gamma(y) d\mu_0,\gamma, \text{ for } m_1 \text{-a.e. } \gamma \in N_1.
\]
Thus, equation (41) is established.
Remark 9.1. Setting

\[ f := \pi_1 \ast (f \mu_0) \]

we get, by equation (45), \( f(\gamma) = 0 \) iff \( \phi_x(\gamma) = 0 \) or \( \int f|_{\gamma}(y)d\mu_{0,\gamma}(y) = 0 \), for \( m_1 \)-a.e. \( \gamma \in N_1 \).

Now, let us see that, by the \( \hat{\nu} \)-uniqueness of the disintegration, equation (42) holds. To do it, define, for \( m_1 \)-a.e. \( \gamma \in N_1 \), de function \( h_\gamma : N_2 \rightarrow \mathbb{R} \), in a way that

\[ h_\gamma(y) = \begin{cases} \frac{f|_{\gamma}(y)}{\int f|_{\gamma}(y)d\mu_{0,\gamma}(y)}, & \text{if } \gamma \in B^c \\ 0, & \text{if } \gamma \in B. \end{cases} \]

Let us prove equation (42) by showing that, for all measurable set \( E \subset N_1 \times N_2 \), it holds

\[ f\mu_0(E) = \int_{N_1} \int_{E \cap \gamma} h_\gamma(y)d\mu_{0,\gamma}(y)d(\pi_1 \ast (f \mu_0))(\gamma). \]

In fact, by equations (45), (46), (47) and remark 9.1 we get

\[
\begin{align*}
f\mu_0(E) &= \int_E f d\mu_0 \\
&= \int_{N_1} \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma}(\phi_x m_1)(\gamma) \\
&= \int_{B^c} \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma}(\phi_x m_1)(\gamma) \\
&= \int_{B^c} \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma}(\phi_x m_1)(\gamma) \\
&= \int_{B^c} \left[ \frac{1}{\int f|_{\gamma}(y)d\mu_{0,\gamma}(y)} \right] \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma} d\mu_1(\gamma) \\
&= \int_{B^c} 7(\gamma) \left[ \frac{1}{\int f|_{\gamma}(y)d\mu_{0,\gamma}(y)} \right] \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma} d\mu_1(\gamma) \\
&= \int_{B^c} \left[ \frac{1}{\int f|_{\gamma}(y)d\mu_{0,\gamma}(y)} \right] \int_{E \cap \gamma} f|_{\gamma} d\mu_{0,\gamma} d\mu_1(\gamma) \\
&= \int_{B^c} \int_{E \cap \gamma} h_\gamma(y)d\mu_{0,\gamma}(y)d(\pi_1 \ast (f \mu_0))(\gamma) \\
&= \int_{N_1} \int_{E \cap \gamma} h_\gamma(y)d\mu_{0,\gamma}(y)d(\pi_1 \ast (f \mu_0))(\gamma).
\end{align*}
\]

And we are done. \( \square \)

10. Appendix 2: Lifting Invariant Measures

In this section, we will construct an invariant measure for a system \( F : M_1 \times M_2 \rightarrow M_1 \times M_2 \), defined by \( F(x, y) = (T(x), G(x, y)) \) where \( T : M_1 \rightarrow M_1 \) and \( G : M_1 \times M_2 \rightarrow M_2 \) are measurable transformations which satisfies some assumptions. Among them, we suppose that \( T : M_1 \rightarrow M_1 \) has an invariant measure \( \mu_1 \), on \( M_1 \).
We prove that \( F \) has an invariant measure, \( \mu_0 \), such that \( \pi^* \mu_0 = \mu_1 \) (defined by \( \pi^* \mu_0 (E) := \mu_0 (\pi^{-1} (E)) \)).

This result generalizes the ones, which can be found in the subsection 7.3.4 (page 225) of \( \text{[3]} \).

### 10.1. Lifting Measures.

Let \( (M_1, d_1) \) and \( (M_2, d_2) \) be two compact metric spaces and suppose that they are endowed with their Borelean sigma algebras \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), respectively. Moreover, let \( \mu_1 \) and \( \mu_2 \) be measures on \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), respectively, where \( \mu_1 \) is a \( T \)-invariant probability for the measurable transformation \( T : M_1 \to M_1 \).

Also denotes by \( \pi_1 : M_1 \times M_2 \to M_1 \) the projection on \( M_1 \), \( \pi_1 (x, y) = x \) for all \( x \in M_1 \) and all \( y \in M_2 \).

Consider a measurable map \( F : M_1 \times M_2 \to M_1 \times M_2 \), defined by \( F(x, y) = (T(x), G(x, y)) \), where \( G(x, y) : M_1 \times M_2 \to M_2 \) satisfies:

- There exists a measurable set \( A_1 \subset M_1 \), with \( \mu_1 (A^c) = 0 \), such that for each \( x \in A_1 \), it holds \( T^n (x) \in A_1 \) for all \( n \geq 0 \) and

\[
\lim_{n \to +\infty} \text{diam} \ F^n (\gamma_x) = 0
\]

uniformly on \( A_1 \), where \( \gamma_x := \{ x \} \times M_2 \) and the diameter of the set \( F^n (\gamma_x) \) is defined with respect to the metric \( d = d_1 + d_2 \).

For this sort of map, it holds \( F (\gamma_x) \subset \gamma_{T^n (x)} \), for all \( x \).

Let \( C^0 (M_1 \times M_2) \) be the space of the real continuous functions, \( \psi : M_1 \times M_2 \to \mathbb{R} \), endowed with the \( || \cdot ||_\infty \) norm.

For each \( \psi \in C^0 (M_1 \times M_2) \) define \( \psi_+ : M_1 \to \mathbb{R} \) and \( \psi_- : M_1 \to \mathbb{R} \) by

\[
\psi_+ (x) = \sup_{\gamma_x} \psi (x, y),
\]

and

\[
\psi_- (x) = \inf_{\gamma_x} \psi (x, y).
\]

**Lemma 10.1.** Both limits

\[
\lim \int (\psi \circ F^n) d\mu_1
\]

and

\[
\lim \int (\psi \circ F^n) d\mu_1
\]

do exist and they are equal.

**Proof.** Since \( (M_1 \times M_2, d) \) is a compact metric space and \( \psi \in C^0 (M_1 \times M_2) \), \( \psi \) is uniformly continuous.

Given \( \epsilon > 0 \), let \( \delta > 0 \) be such that \( |\psi (x_1, y_1) - \psi (x_2, y_2)| < \epsilon \) if \( d((x_1, y_1), (x_2, y_2)) < \delta \) and \( n_0 \in \mathbb{N} \) such that

\[
n > n_0 \Rightarrow \text{diam} \ F^n (\gamma_x) < \delta,
\]

for all \( x \in A_1 \).

For all \( k \in \mathbb{N} \), it holds

\[
(\psi \circ F^{n+k})_+ (x) - (\psi \circ F^n)_- (T^k (x)) = \inf_{\gamma_x} (\psi \circ F^n) - \inf_{\gamma_{T^k (x)}} (\psi \circ F^n) \geq 0.
\]
Moreover, for all $x \in A_1$

$$(\psi \circ F^{n+k})_-(x) - (\psi \circ F^n)_-(T^k(x)) = \inf_{F^k(\gamma_x)} \psi \circ F^n - \inf_{\gamma_{T^k(x)}} \psi \circ F^n$$

$$\leq \sup_{F^k(\gamma_x)} \psi \circ F^n - \inf_{\gamma_{T^k(x)}} \psi \circ F^n$$

$$\leq \sup_{\gamma_{T^k(x)}} \psi - \inf_{F^n(\gamma_{T^k(x)})} \psi$$

$$\leq \epsilon.$$

Thus,

$$(\psi \circ F^{n+k})_-(x) - (\psi \circ F^n)_-(T^k(x)) \leq \epsilon \forall x \in A_1.$$ Integrating over $M_1$, since $\mu_1(A_1) = 0$, we get

$$\int (\psi \circ F^{n+k})_-(x) - \int (\psi \circ F^n)_- \circ T^k \, d\mu_1 \leq \epsilon.$$ Since $\mu_1$ is $T$-invariant and by (51) we get

$$\left| \int (\psi \circ F^{n+k})_-(x) - \int (\psi \circ F^n)_- \, d\mu_1 \right| \leq \epsilon \forall n \geq n_0 \text{ and } \forall k \in \mathbb{N},$$ which shows that the sequence $(\int (\psi \circ F^n)_- \, d\mu_1)_n$ converges in $\mathbb{R}$. Let us denote

$$(52) \quad \mu(\psi) := \lim_{n \to +\infty} \int (\psi \circ F^n)_- \, d\mu_1.$$ It is remaining to show that, it also holds $\mu(\psi) = \lim_{n \to +\infty} \int (\psi \circ F^n)_+ \, d\mu_1$. Indeed, given $\epsilon > 0$, let $\delta > 0$ be such that $|\psi(x_1, y_1) - \psi(x_2, y_2)| < \frac{\epsilon}{2}$ if $d((x_1, y_1), (x_2, y_2)) < \delta$.

Consider $n_0 \in \mathbb{N}$ such that

$$(53) \quad n > n_0 \Rightarrow \text{diam } F^n(\gamma_x) < \delta,$$ for all $x \in A_1$ and

$$(54) \quad n > n_0 \Rightarrow \left| \mu(\psi) - \int (\psi \circ F^n)_- \, d\mu_1 \right| < \frac{\epsilon}{2}.$$ Then,

$$0 \leq (\psi \circ F^n)_+(x) - (\psi \circ F^n)_-(x)$$

$$= \sup_{F^n(\gamma_x)} \psi - \inf_{F^n(\gamma_x)} \psi$$

$$\leq \frac{\epsilon}{2}.$$ The above inequality yields, for all $n > n_0$
\[ (55) \quad \left| \int (\psi \circ F^n)_+ \, d\mu_1(x) - \int (\psi \circ F^n)_- \, d\mu_1 \right| \leq \frac{\epsilon}{2}. \]

Therefore, if \( n > n_0 \), by (54) and (55) it holds

\[
\left| \mu(\psi) - \int (\psi \circ F^n)_+ \, d\mu_1 \right| \leq \left| \mu(\psi) - \int (\psi \circ F^n)_- \, d\mu_1 \right| + \left| \int (\psi \circ F^n)_+ \, d\mu_1 - \int (\psi \circ F^n)_- \, d\mu_1 \right| < \epsilon.
\]

And the proof is complete. \( \square \)

**Lemma 10.2.** The function \( \mu : C^0(M_1 \times M_2) \to \mathbb{R} \) defined for each \( \psi \in C^0(M_1 \times M_2) \) by \( \mu(\psi) := \lim_{n \to +\infty} \int (\psi \circ F^n)_- \, d\mu_1 = \lim_{n \to +\infty} \int (\psi \circ F^n)_+ \, d\mu_1 \), defines a bounded linear functional such that \( \mu(1) = 1 \) and \( \mu(\psi) \geq 0 \) for all \( \psi \geq 0 \).

**Proof.** Is straightforward to show that \( \mu(1) = 1 \), \( \mu(\psi) \geq 0 \) for all \( \psi \geq 0 \) and \( \mu(\psi) \leq 1 \) for all \( \psi \) in the unit ball, \( ||\psi||_\infty \leq 1 \). Then \( \mu \) is bounded. Let us see, that \( \mu(\alpha \psi_1 + \psi_2) = \alpha \mu(\psi_1) + \mu(\psi_2) \), for all \( \psi_1, \psi_2 \in C^0(M_1 \times M_2) \) and all \( \alpha \in \mathbb{R} \).

Define \( A^\varepsilon_\psi := \{ \psi(x, y); (x, y) \in \gamma_x \} \) and note that \( \psi_+(x) = \inf A^\varepsilon_\psi \) and \( \psi_-(x) = \inf A^\varepsilon_\psi \). Moreover, \( A^\varepsilon_{\psi_1 + \psi_2} \subset A^\varepsilon_{\psi_1} + A^\varepsilon_{\psi_2} \). This implies

\[ \sup A^\varepsilon_{\psi_1 + \psi_2} \leq \sup A^\varepsilon_{\psi_1} + \sup A^\varepsilon_{\psi_2} \]

and

\[ \inf A^\varepsilon_{\psi_1 + \psi_2} \geq \inf A^\varepsilon_{\psi_1} + \inf A^\varepsilon_{\psi_2} \]

which gives \( (\psi_1 + \psi_2)_+ \leq (\psi_1)_+ + (\psi_2)_+ \) and \( (\psi_1 + \psi_2)_- \geq (\psi_1)_- + (\psi_2)_- \). Then,

\[ ((\psi_1 + \psi_2) \circ F^n)_+ \leq (\psi_1 \circ F^n)_+ + (\psi_2 \circ F^n)_+ \]

and

\[ ((\psi_1 + \psi_2) \circ F^n)_- \geq (\psi_1 \circ F^n)_- + (\psi_2 \circ F^n)_- \]

Integrating and taking the limit we have

\[ \lim_{n \to +\infty} \int ((\psi_1 + \psi_2) \circ F^n)_+ \, d\mu_1 \leq \lim_{n \to +\infty} \int (\psi_1 \circ F^n)_+ \, d\mu_1 + \lim_{n \to +\infty} \int (\psi_2 \circ F^n)_+ \, d\mu_1 \]

and

\[ \lim_{n \to +\infty} \int ((\psi_1 + \psi_2) \circ F^n)_- \, d\mu_1 \geq \lim_{n \to +\infty} \int (\psi_1 \circ F^n)_- \, d\mu_1 + \lim_{n \to +\infty} \int (\psi_2 \circ F^n)_- \, d\mu_1. \]

This implies \( \mu(\psi_1 + \psi_2) = \mu(\psi_1) + \mu(\psi_2) \).
For $\alpha \geq 0$, we have

$$
\mu(\alpha \psi) = \lim_{n \to +\infty} \int ((\alpha \psi) \circ F^n) + d\mu_1
$$

$$
= \alpha \lim_{n \to +\infty} \int (\psi \circ F^n) + d\mu_1
$$

$$
= \alpha \mu(\psi).
$$

For $\alpha < 0$, it holds

$$
\mu(\alpha \psi) = \lim_{n \to +\infty} \int ((\alpha \psi) \circ F^n) + d\mu_1
$$

$$
= \alpha \lim_{n \to +\infty} \int (\psi \circ F^n) - d\mu_1
$$

$$
= \alpha \mu(\psi).
$$

And the proof is complete. \hfill \Box

As a consequence of the above results and the Riez-Markov Lemma we get the following theorem.

**Theorem 10.3.** There exists an unique measure $\mu_0$ on $M_1 \times M_2$ such that for every continuous function $\psi \in C^0(M_1 \times M_2)$ it holds

$$
(56) \quad \mu(\psi) = \int \psi d\mu_0.
$$

**Proposition 10.4.** The measure $\mu_0$ is $F$-invariant and $\pi_1 \ast \mu_0 = \mu_1$.

**Proof.** Denote by $F_\ast \mu_0$ the measure defined by $F_\ast \mu_0(E) = \mu_0(F^{-1}(E))$ for all measurable set $E \subset M_1 \times M_2$. For each $\psi \in C^0(M_1 \times M_2)$ we have

$$
\int \psi dF_\ast \mu_0 = \int \psi \circ F d\mu_0
$$

$$
= \mu(\psi)
$$

$$
= \lim_{n \to +\infty} \int (\psi \circ F^n \circ F) + d\mu_1
$$

$$
= \mu(\psi)
$$

$$
= \int \psi d\mu_0.
$$

It implies that $F_\ast \mu_0 = \mu_0$ and we are done.
To prove that \( \pi_1 \ast \mu_0 = \mu_1 \), consider a continuous function \( \phi : M_1 \to \mathbb{R} \). Then, \( \phi \circ \pi_1 \in C^0(M_1 \times M_2) \). Therefore,

\[
\int \phi d\pi_1 \ast \mu_0 = \int \phi \circ \pi_1 d\mu_0 \\
= \lim_{n \to +\infty} \int (\phi \circ \pi_1 \circ F^n)_+ d\mu_1 \\
= \lim_{n \to +\infty} \int (\phi \circ T^n)_+ d\mu_1 \\
= \lim_{n \to +\infty} \int \phi d\mu_1 \\
= \int \phi d\mu_1.
\]

Thus, \( \pi_1 \ast \mu_0 = \mu_1 \).

\[\square\]

11. Appendix 3: Linearity of the restriction

Let us consider the measurable spaces \((N_1, N_1')\) and \((N_2, N_2')\), where \(N_1\) and \(N_2\) are the Borel’s \(\sigma\)-algebra of \(N_1\) and \(N_2\) respectively. Let \(\mu \in \mathcal{B}\) be a positive measure on the measurable space \((\Sigma, \mathcal{B})\), where \(\Sigma = N_1 \times N_2\) and \(\mathcal{B} = \mathcal{N}_1 \times \mathcal{N}_2\) and consider its disintegration \(\{\mu_x\}_{x \in \mathcal{N}}\) along \(\mathcal{F}^s\), where \(\mu_x = \pi_1 \ast \mu\) and \(d(\pi_1 \ast \mu) = \phi_x dm_1\), for some \(\phi_x \in L^1(N_1, m_1)\). We will suppose that the \(\sigma\)-algebra \(\mathcal{B}\) has a countable generator.

**Proposition 11.1.** Suppose that \(\mathcal{B}\) has a countable generator, \(\Gamma\). If \(\{\mu_x\}_\gamma\) and \(\{\mu'_x\}_\gamma\) are disintegrations of a positive measure \(\mu\) relative to \(\mathcal{F}^s\), then \(\phi_x(\gamma) \mu_\gamma = \phi'_x(\gamma) \mu'_\gamma\) \(m_1\)-a.e. \(\gamma \in N_1\).

**Proof.** Let \(\mathcal{A}\) be the algebra generated by \(\Gamma\). \(\mathcal{A}\) is countable and \(\mathcal{A}\) generates \(\mathcal{B}\). For each \(A \in \mathcal{A}\) define the sets

\[
G_A = \{ \gamma \in N_1 | \phi_x(\gamma) \mu_\gamma(A) < \phi'_x(\gamma) \mu'_\gamma(A) \}
\]

and

\[
R_A = \{ \gamma \in N_1 | \phi_x(\gamma) \mu_\gamma(A) > \phi'_x(\gamma) \mu'_\gamma(A) \}.
\]

If \(\gamma \in G_A\) then \(\gamma \subset \pi_1^{-1}(G_A)\) and \(\mu_\gamma(A) = \mu_\gamma(A \cap \pi_1^{-1}(G_A))\). Otherwise, if \(\gamma \notin G_A\) then \(\gamma \cap \pi_1^{-1}(G_A) = \emptyset\) and \(\mu_\gamma(A \cap \pi_1^{-1}(G_A)) = 0\). The same holds for \(\mu'_\gamma\). Then, it holds

\[
\mu(A \cap \pi_1^{-1}(G_A)) = \begin{cases} \mu_\gamma(A \cap \pi_1^{-1}(Q)) \phi_x(\gamma) dm_1 = \int_{Q_A} \mu_\gamma(A) \phi_x(\gamma) dm_1 \\
\mu'_\gamma(A \cap \pi_1^{-1}(Q)) \phi'_x(\gamma) dm_1 = \int_{Q_A} \mu'_\gamma(A) \phi'_x(\gamma) dm_1. \end{cases}
\]

Since \(\phi_x(\gamma) \mu_\gamma(A) < \mu'_\gamma(A) \phi'_x(\gamma)\) for all \(\gamma \in G_A\), we get \(m_1(G_A) = 0\). The same holds for \(R_A\). Thus

\[
m_1 \left( \bigcup_{A \in \mathcal{A}} R_A \cup G_A \right) = 0.
\]
It means that, \( m_1 \)-a.e. \( \gamma \in N_1 \) the positive measures \( \phi_x(\gamma) \mu_x \) and \( \phi_x(\gamma) \mu' \) coincides for all measurable set \( A \) of an algebra which generates \( \mathcal{B} \). Therefore \( \phi_x(\gamma) \mu_x = \phi_x(\gamma) \mu' \) for \( m_1 \)-a.e. \( \gamma \in N_1 \).

**Proposition 11.2.** Let \( \mu_1, \mu_2 \in \mathcal{AB} \) be two positive measures and denote their marginal densities by \( d(\mu_1) = \phi_x dm_1 \) and \( d(\mu_2) = \psi_x dm_1 \), where \( \phi_x, \psi_x \in L^1(m_1) \) respectively. Then \( (\mu_1 + \mu_2)|\gamma = \mu_1|\gamma + \mu_2|\gamma \) \( m_1 \)-a.e. \( \gamma \in N_1 \).

**Proof.** Note that \( d(\mu_1 + \mu_2) = (\phi_x + \psi_x) dm_1 \). Moreover, consider the disintegration of \( \mu_1 + \mu_2 \) given by

\[
\{(\mu_1 + \mu_2)|\gamma, (\phi_x + \psi_x)m_1\},
\]

where

\[
(\mu_1 + \mu_2)|\gamma = \begin{cases} \frac{\phi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_{1,\gamma} + \frac{\psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_{2,\gamma}, & \text{if } \phi_x(\gamma) + \psi_x(\gamma) \neq 0 \\ 0, & \text{if } \phi_x(\gamma) + \psi_x(\gamma) = 0. \end{cases}
\]

Then, by Proposition 11.1 for \( m_1 \)-a.e. \( \gamma \in N_1 \), it holds

\[
(\phi_x + \psi_x)(\gamma)(\mu_1 + \mu_2)|\gamma = (\phi_x(\gamma) \mu_{1,\gamma} + \psi_x(\gamma) \mu_{2,\gamma}).
\]

Therefore, \( (\mu_1 + \mu_2)|\gamma = \mu_1|\gamma + \mu_2|\gamma \), \( m_1 \)-a.e. \( \gamma \in N_1 \).

**Definition 11.3.** We say that a positive measure \( \lambda_1 \) is disjoint from a positive measure \( \lambda_2 \) if \( (\lambda_1 - \lambda_2)^+ = \lambda_1 \) and \( (\lambda_1 - \lambda_2)^- = \lambda_2 \).

**Remark 11.4.** A straightforward computations yields that if \( \lambda_1 + \lambda_2 \) is disjoint from \( \lambda_3 \), then both \( \lambda_1 \) and \( \lambda_2 \) are disjoint from \( \lambda_3 \), where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are all positive measures.

**Lemma 11.5.** Suppose that \( \mu = \mu^+ - \mu^- \) and \( \nu = \nu^+ - \nu^- \) are the Jordan decompositions of the signed measures \( \mu \) and \( \nu \). Then, there exist positive measures \( \mu_1, \mu_2, \mu^+, \mu^-, \nu^+, \nu^- \) such that \( \mu^+ = \mu^+ + \mu_1 \mu^- = \mu^- + \mu_2 \) and \( \nu^+ = \nu^+ + \mu_2 \), \( \nu^- = \nu^- + \mu_1 \).

**Proof.** Suppose \( \mu = \nu_1 - \nu_2 \) with \( \nu_1 \) and \( \nu_2 \) positive measures. Let \( \mu^+ \) and \( \mu^- \) be the Jordan decomposition of \( \mu \). Let \( \mu' = \nu_1 - \mu^- \), then \( \nu_1 = \mu^- + \mu' \). Indeed \( \mu^+ = \mu^- = \nu_1 = \nu_2 \) which implies that \( \mu^+ - \nu_1 = \mu^- - \nu_2 \). Thus if \( \nu_1, \nu_2 \) is a decomposition of \( \mu \), then \( \nu_1 = \mu^+ + \mu' \) and \( \nu_2 = \mu^- + \mu' \) for some positive measure \( \mu' \). Now, consider \( \mu = \mu^+ - \mu^- \) and \( \nu = \nu^+ - \nu^- \). Since the pairs of positive measures \( \mu^+ \), \( \nu^- \) and \( (\mu^+ - \nu^-)^+ \), \( (\mu^+ - \nu^-)^- \) are both decompositions of \( \mu^+ - \nu^- \), by the above comments, we get that \( \mu^+ = (\mu^+ - \nu^-)^+ + \mu_1 \) and \( \nu^- = (\mu^+ - \nu^-)^- + \mu_1 \), for some positive measure \( \mu_1 \). Analogously, since the pairs of positive measures \( \mu^- \), \( \nu^+ \) and \( (\nu^+ - \mu^-)^+ \), \( (\nu^+ - \mu^-)^- \) are both decompositions of \( \nu^+ - \mu^- \), by the above comments, we get that \( \nu^+ = (\nu^+ - \mu^-)^+ + \mu_2 \) and \( \mu^- = (\nu^+ - \mu^-)^- + \mu_2 \), for some positive measure \( \mu_2 \). By definition 11.3, \( \mu^+ \) and \( \mu^- \) are disjoint, and so are \( (\mu^+ - \nu^-)^+ \) and \( (\nu^+ - \mu^-)^- \). Analogously, \( \nu^+ \) and \( \nu^- \) are disjoint, and so are \( (\mu^+ - \nu^-)^- \) and \( (\nu^+ - \mu^-)^+ \). Moreover, since \( (\mu^+ - \nu^-)^+ \) and \( (\mu^+ - \nu^-)^- \) are disjoint, so are \( (\nu^+ - \mu^-)^+ \) and \( (\nu^+ - \mu^-)^- \). This gives that, the pair \( (\mu^+ - \nu^-)^+ + (\nu^+ - \mu^-)^+ + (\nu^+ - \mu^-)^- + (\mu^+ - \nu^-)^- \) is a Jordan decomposition of \( \mu + \nu \) and we are done.
Proposition 11.6. Let $\mu, \nu \in AB$ be two signed measures. Then $(\mu + \nu)_{|\gamma} = \mu_{|\gamma} + \nu_{|\gamma}$ a.e. $\gamma \in N_1$.

Proof. Suppose that $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are the Jordan decompositions of $\mu$ and $\nu$ respectively. By definition, $\mu_{|\gamma} = \mu^+_{|\gamma} - \mu^-_{|\gamma}$, $\nu_{|\gamma} = \nu^+_{|\gamma} - \nu^-_{|\gamma}$.

By Lemma 11.5 suppose that $\mu^+ = \mu^{++} + \mu_1$, $\mu^- = \mu^{--} + \mu_2$ and $\nu^+ = \nu^{++} + \mu_2$, $\nu^- = \nu^{--} + \mu_1$. In a way that $(\mu + \nu)^+ = \mu^{++} + \nu^{++}$ and $(\mu + \nu)^- = \mu^{--} + \nu^{--}$.

By Proposition 11.2 it holds $\mu^+_{|\gamma} = \mu^{++}_{|\gamma} + \mu_1_{|\gamma}$, $\mu^-_{|\gamma} = \mu^{--}_{|\gamma} + \mu_2_{|\gamma}$, $\nu^+_{|\gamma} = \nu^{++}_{|\gamma} + \mu_2_{|\gamma}$ and $\nu^-_{|\gamma} = \nu^{--}_{|\gamma} + \mu_1_{|\gamma}$.

Moreover, $(\mu + \nu)^+_{|\gamma} = \mu^{++}_{|\gamma} + \nu^{++}_{|\gamma}$
$(\mu + \nu)^-_{|\gamma} = \mu^{--}_{|\gamma} + \nu^{--}_{|\gamma}$

Putting all together, we get:

\[
(\mu + \nu)_{|\gamma} = (\mu + \nu)^+_{|\gamma} - (\mu + \nu)^-_{|\gamma} = \\
= \mu^{++}_{|\gamma} + \nu^{++}_{|\gamma} - (\mu^{--}_{|\gamma} + \nu^{--}_{|\gamma}) = \\
= \mu^{++}_{|\gamma} + \mu_1_{|\gamma} + \nu^{++}_{|\gamma} + \mu_2_{|\gamma} - (\mu^{--}_{|\gamma} + \mu_2_{|\gamma} + \nu^{--}_{|\gamma} + \mu_1_{|\gamma}) = \\
= \mu^{++}_{|\gamma} - \mu^{--}_{|\gamma} + \nu^{++}_{|\gamma} - \nu^{--}_{|\gamma} = \\
= \mu_{|\gamma} + \nu_{|\gamma}.
\]

We immediately arrive at the following

Proposition 11.7. Let $\mu \in AB$ be a signed measure and $\mu = \mu^+ - \mu^-$ its Jordan decomposition. If $\mu_1$ and $\mu_2$ are positive measures such that $\mu = \mu_1 - \mu_2$, then $\mu_{|\gamma} = \mu_1_{|\gamma} - \mu_2_{|\gamma}$. It means that, the restriction does not depends on the decomposition of $\mu$.

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