A note on the M2-M5 brane system and fuzzy spheres

David S. Berman

Queen Mary College, University of London,
Department of Physics,
Mile End Road,
London, E1 4NS, England.

and

Neil B. Copland

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
University of Cambridge,
Wilberforce Road,
Cambridge CB3 0WA,
England.

Abstract

This note covers various aspects of recent attempts to describe membranes ending on fivebranes using fuzzy geometry. In particular, we examine the Basu-Harvey equation and its relation to the Nahm equation as well as the consequences of using a non-associative algebra for the fuzzy three-sphere. This produces the tantalising result that the fuzzy funnel solution corresponding to $Q$ coincident membranes ending on a five-brane has $Q^{3/2}$ degrees of freedom.

1email: D.S.Berman@qmul.ac.uk
2email: N.B.Copland@damtp.cam.ac.uk
1 Introduction

This paper will be concerned with describing how multiple membranes end on five-branes. From the five-brane world volume point of view, membranes ending on a five-brane are described by the self-dual string solution [1] of the five-brane world volume theory. The same system may also be described from the membrane world volume point of view as a fuzzy three-funnel [2]. This has since been generalised to include membranes ending on five-branes wrapped on calibrated cycles [3] and membranes stretching between several five-branes [4].

To gain some intuition let us first consider the simpler case of D1 branes ending on D3 branes. This may be described in two equivalent ways. One is as a monopole in the D3 brane world volume theory and the other is as a fuzzy funnel solution to the Nahm equation,

$$\frac{dX^i}{d\sigma} - \frac{i}{2} \epsilon_{ijk} [X^j, X^k] = 0 , \quad (1)$$

which is the one half BPS equation in the D1 brane world volume theory. The fuzzy funnel solution [5] to (1) is essentially a fuzzy two-sphere whose radius depends on the transverse distance from the brane with the two-sphere opening up as it approaches the D3-brane. The Nahm transformation is the shift between the D3-brane and D1-brane points of view with the monopole describing the D1-brane from the D3-brane perspective and the fuzzy funnel describing the D3-brane from the D1-brane perspective [6, 7].

The Basu-Harvey equation [2]

$$\frac{dX^i}{d\sigma} + \frac{M_1^3}{8\pi\sqrt{2N}} \frac{1}{4!} \epsilon_{ijkl} [G_5, X^i, X^k, X^l] = 0. \quad (2)$$

is the M-theory analogue of the Nahm equation. It describes membranes ending on a five-brane from the membrane world volume point of view. Its solution is also a fuzzy funnel but for the membrane it is a fuzzy three-sphere that opens up into the five-brane. This has been shown to be consistent with what you would expect from comparing with the self-dual string solution of the five-brane.

Importantly though several issues remain unresolved for the Basu-Harvey equation. There is no analogue of the Nahm transformation which would then map solutions of the Basu-Harvey equation to the solutions of the five-brane world volume theory and worryingly there is no supersymmetric non-Abelian membrane theory that produces the Basu-Harvey equation as a one a half BPS equation.
Given its Nahm like role and the usual M-theory, string theory relationship one would expect there to be a reduction of the Basu-Harvey equation to the Nahm equation. This paper will describe the relationship between the Basu-Harvey equation and the Nahm equation. Effectively this will be a description of how to get fuzzy two-spheres from fuzzy three-spheres via a projection.

It has been noted that the appropriate algebra for a fuzzy three-sphere is non-associative [8–10]. We will examine the consequences of using such a non-associative algebra for the M2 brane geometry and find a tantalising result that the number of degrees of freedom scale as $N^{3/2}$ where $N$ is the number of coincident membranes. One can interpret this the following way. The fuzzy three-sphere is naturally endowed with an ultraviolet cutoff. Summing all the spherical harmonics on the three-sphere up to the cutoff provides one with a finite number of degrees of freedom; the ratio of the size of the sphere to the cutoff being $N$ dependent. It works out from the fuzzy sphere algebra that indeed the number of degrees then scale as $N^{3/2}$.

This is consistent with the view presented in [11] which advocates the presence of a non-associative algebra on five-branes in the presence of background C flux. The self-dual string produces flux at the core of the solution so it would be natural from this perspective to also expect a non-associative algebra for the membrane five-brane system. Alternative deformations of the five-brane geometry due to the presence of flux have also been described in [12].

Finally, it has been suggested [13] that a non-associative algebra may allow for a supersymmetric version of the coincident membrane theory. That non-associativity is different from that explored here but still it is interesting to consider novel algebras for the M2 brane system.

It should be noted that related work on fuzzy spheres has been studied recently from various perspectives in [14, 15]. The paper is structured as follows. We first describe the basics of the the fuzzy three-sphere and then examine its algebra and the relation to the membrane five-brane system. Finally we relate the Basu-Harvey equation to the Nahm equation through the projection from fuzzy three-sphere to fuzzy two-sphere.
2 The Fuzzy Three-Sphere and its Young Diagram Basis

Fuzzy odd spheres are more complicated than the fuzzy even spheres [8–10, 16]. The construction of fuzzy odd spheres is derived by starting from the fuzzy even sphere of one dimension higher and then applying a projection. For the fuzzy three-sphere the starting point is thus the fuzzy four-sphere. Given $V$, the four-dimensional spinor representation, on which the $Spin(5)$ $\Gamma$-matrices act, the fuzzy four-sphere co-ordinates are represented by operators which act on $n$'th symmetrised tensor product of $V$. The co-ordinates $\hat{G}^\mu$ are given by

$$\hat{G}^\mu = (\Gamma^\mu \otimes 1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes \Gamma^\mu \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \Gamma^\mu)_{sym}, \quad (3)$$

(throughout $\mu, \nu, \ldots$ run from 1 to 5, $i, j, \ldots$ from 1 to 4 and $a, b, \ldots$ from 1 to 3). Some intuition can be gained from the $n = 1$ case where these are just the $Spin(5)$ Gamma matrices. $\rho_m(\Gamma^\mu)$ will denote the action of $\Gamma^\mu$ on the $m$'th factor of the tensor product and $e_i$, $i = 1, \ldots, 4$, is the basis of $V$,

$$\rho_m(\Gamma^\mu)(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_m} \otimes \ldots \otimes e_{i_n}) = (e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes (\Gamma^\mu_{j_m,i_m} e_{j_m}) \otimes \ldots \otimes e_{i_n}) \quad (4)$$

and $P_n$ denotes symmetrisation so

$$\hat{G}^\mu = P_n \sum_m \rho_m(\Gamma^\mu) P_n. \quad (5)$$

In their non-Abelian algebra the $\hat{G}^\mu$ obey a version of the equation of a sphere, $\sum_\mu \hat{G}^\mu \hat{G}^\mu = R^2 I$, where $R$ is a function of $n$. They also obey a higher Poisson bracket equation [17, 18] and commute with the generators of $SO(5)$, which are given by $P_n \sum_m \rho_m(\Gamma^\mu \Gamma^\nu) P_n$.

To get the fuzzy three-sphere we use the projection $P_\pm = \frac{1}{2}(1 \pm \Gamma^5)$ to decompose $V$ into $V_+$ and $V_-$, the positive and negative chirality two-dimensional spinor representations of $SO(4)$. The co-ordinates act in a reducible representation $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$. To get $\mathcal{R}^+$ we take the symmetrised tensor product of $\frac{n+1}{2}$ factors of $V_+$ and $\frac{n-1}{2}$ factors of $V_-$, similarly $\mathcal{R}^-$ is the symmetrised tensor product of $\frac{n-1}{2}$ factors of $V_+$ and $\frac{n+1}{2}$ of $V_-$. $\mathcal{R}^+$ is the irreducible representation of $SO(4)$ with $(2j_L, 2j_R) = (\frac{n+1}{2}, \frac{n-1}{2})$ and $\mathcal{R}^-$ is the irreducible representation with $(2j_L, 2j_R) = (\frac{n-1}{2}, \frac{n+1}{2})$. The projector $\mathcal{P}_{\mathcal{R}^\pm} = \left( P_{\mathcal{R}^+}^{\otimes (n+1)/2} \otimes P_{\mathcal{R}^-}^{\otimes (n-1)/2} \right)_{sym}$ projects the fuzzy four-sphere onto $\mathcal{R}^\pm$. $\mathcal{P}_{\mathcal{R}} = \mathcal{P}_{\mathcal{R}^+} + \mathcal{P}_{\mathcal{R}^-}$ and the co-ordinates of the fuzzy three-sphere are given by

$$G^i = \mathcal{P}_{\mathcal{R}} \hat{G}^\mu \mathcal{P}_{\mathcal{R}}. \quad (6)$$
We can treat the $G^i$ as $N \times N$ matrices acting linearly on the $N = \frac{(n+1)(n+3)}{2}$ dimensional representation $\mathcal{R}$. It can be shown that $\sum_i G^i G^i = \frac{(n+1)(n+3)}{2} \mathbf{1}$ and also \cite{2} that

$$G^i + \frac{1}{2(n+2)} \epsilon_{ijkl} G^j G^k G^l = 0,$$

(7)

where $G_5 = \mathcal{P}_R \hat{G}^5 \mathcal{P}_R = \mathcal{P}_R^+ - \mathcal{P}_R^-.$

The space of $N \times N$ matrices acting on $N$-dimensional $\mathcal{R}$, $\text{Mat}_N(\mathbb{C})$, can be decomposed into representations of $SO(4)$, this is a basis of operators corresponding to Young diagrams \cite{9}. Young diagrams for $SO(4)$ have a maximum of two rows (as $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = \Gamma_5$ so all products of more than two $\Gamma$’s can be rewritten as products or two or less) and can be represented by the row lengths $(r_1, r_2)$, where $r_2$ can be positive or negative. Each column is either one or two boxes long and we represent these by factors of $\rho_m(\Gamma)$ or $\rho_m(\Gamma \Gamma)$ respectively, each acting on different factors of the tensor product. We suppress the indices on the $\Gamma$’s. (They will be contracted with a tensor of appropriate Young diagram symmetry). If $r_2$ is positive, the $\rho_m(\Gamma \Gamma)$ all act on $V_+$ factors, if it is negative they all act on $V_-$ factors. If $r_1 - r_2$ is divisible by two them we are in $\text{End}(\mathcal{R}^\pm)$ and half of the $\rho_m(\Gamma)$’s come with $P_+$ projector ($\rho_m(\Gamma P_+)$) and half come with a $P_-$ to make sure that we stay within $\mathcal{R}^\pm$. If $r_1 - r_2$ is not divisible by two them we are in $\text{Hom}(\mathcal{R}^\pm, \mathcal{R}^\mp)$ and $(r_1 - r_2 \pm 1)/2$ of the $\rho_m(\Gamma)$’s come with a $P_+$ and $(r_1 - r_2 \mp 1)/2$ with a $P_-$. For example a diagram in $\text{Hom}(\mathcal{R}^+, \mathcal{R}^-)$ with row lengths $(4, 1)$ the operators would be of the form

$$\sum_{\vec{r}, \vec{s}} \rho_{r_1}(\Gamma P_+) \rho_{s_1}(\Gamma P_-) \rho_{s_2}(\Gamma P_+) \rho_{s_3}(\Gamma P_+)$$

(8)

and in $\text{End}(\mathcal{R}^-)$ with row lengths $(5, -3)$ they are of the form

$$\sum_{\vec{r}, \vec{s}} \rho_{r_1}(\Gamma P_-) \rho_{r_2}(\Gamma P_-) \rho_{r_3}(\Gamma P_-) \rho_{s_1}(\Gamma P_+) \rho_{s_2}(\Gamma P_-).$$

(9)

The sum over $\vec{r}, \vec{s}$ is such that $r_1 \neq r_2 \neq \ldots s_1 \neq s_2 \neq \ldots$ and all indices run form 1 to $n$. The vector index on the $\Gamma$’s is contracted with a tensor of appropriate Young diagram symmetry. The product of two $\Gamma$’s ensures that we have anti-symmetry down columns and the symmetrisation of $\mathcal{R}$ ensures we have the correct symmetry along rows. Group theoretical formulae for the number of independent traceless tensors of this form for given row lengths are given in \cite{9}. Summing over allowed row lengths one finds the number of independent operators is exactly $N^2$. (Diagrams with both $\rho_r(\Gamma P_+$ and $\rho_r(\Gamma P_-)$ factors vanish, again see \cite{9}.)

4
3 Projection to Fuzzy Spherical Harmonics

In the fuzzy three-sphere construction used in [2] the co-ordinate matrices were taken to be in \( \text{Mat}_N(\mathbb{C}) \). As mentioned there, this was not the only choice. We want an algebra which reproduces the classical algebra of functions on the \( S^3 \) in the large \( N \) limit, \( \text{Mat}_N(\mathbb{C}) \) does not. To see this let \( x^i = G^i / n \), so that \( x^i x^i \sim O(1) \). Then \( x^{[i} x^{j]} \sim O(1) \) as well, and the anti-symmetric part of the product of co-ordinates persists in the large \( N \) limit. This behaviour is caused by the second term of equation (17) below. For fuzzy even spheres the symmetry over all \( n \) tensor factors causes such a term to vanish. Here we have a division between \( V_+ \) and \( V_- \) factors.

The projection which does give the correct limit is detailed in [9]. It is a projection onto operators corresponding to Young diagrams with only one row (i.e. completely symmetric), the algebra of these operators is called \( \mathcal{A}_n(S^3)^3 \). We can extract the terms corresponding to these operators from the general sums given in [9]. One finds that the number of operators surviving in \( \text{End}(\mathcal{R}^\pm) \) is \( n(n+1)(n+2)/6 \) and the number in \( \text{Hom}(\mathcal{R}^\pm, \mathcal{R}^\pm) \) is \( (n+1)(n+2)(n+3)/6 \). The other condition of the projection is that matrices should act in the same manner on \( \mathcal{R}^+ \) and \( \mathcal{R}^- \) so we sum each state of \( \text{End}(\mathcal{R}^+) \) with the corresponding one form \( \text{End}(\mathcal{R}^-) \) and so on, giving the total number of degrees of freedom as

\[
D = (n+1)(n+2)(2n+3)/6. \tag{10}
\]

However, the algebra \( \mathcal{A}_n(S^3) \) does not close under multiplication, and we have to project back onto \( \mathcal{A}_n(S^3) \) after multiplying. We denote the projected product by \( A \bullet B \) or \( (AB)_+ \) for \( A, B \in \mathcal{A}_n(S^3) \). This new product is non-associative. (In the large \( n \) limit one recovers associativity [9, 19]). The mechanism for non-associativity can be easily seen for the simple \( n = 1 \) example where

\[
(\Gamma^1 \bullet \Gamma^1) \bullet \Gamma^2 = \mathbb{1} \bullet \Gamma^2 = \Gamma^2 \neq 0 = \Gamma^1 \bullet 0 = \Gamma^1 \bullet (\Gamma^1 \bullet \Gamma^2). \tag{11}
\]

Generalising to higher \( n \) we must be careful to decompose into the Young diagram basis and keep only the proscribed set. In particular we should remember to keep traces of tensor operators that correspond to smaller symmetric diagrams. Firstly taking the projected

\[\text{End}(\mathcal{R}^\pm) \supset \mathcal{A}_n(S^3) \supset \mathcal{A}_n(S^3)\].

\[\text{Hom}(\mathcal{R}^\pm, \mathcal{R}^\pm) \supset \mathcal{A}_n(S^3) \supset \mathcal{A}_n(S^3)\].

\[\text{End}(\mathcal{R}^\pm) \supset \mathcal{A}_n(S^3) \supset \mathcal{A}_n(S^3)\].
product of two of the co-ordinates we get

\[ G^i \bullet G^j = \mathcal{P}_{R^+} \left[ \sum_r \rho_r(\delta^{ij} P_+) + \sum_{r \neq s} \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j P_+ - P_+ \right] \mathcal{P}_{R^+} + (+ \leftrightarrow -). \] (12)

We then use this to multiply a third co-ordinate giving

\[ (G^i \bullet G^j)G^k = \mathcal{P}_{R^+} \left[ \sum_{r \neq s \neq t} \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j P_+)\rho_t(\Gamma^k P_-) + \sum_{r \neq s} \rho_r(\delta^{ij} P_+)\rho_s(\Gamma^k P_-) \right. \\
\left. + \sum_r \rho_r(\delta^{ij} \Gamma^k P_-) + \frac{1}{2} \sum_{r \neq s} \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j \Gamma^k P_-) + \right. \\
\left. \frac{1}{2} \sum_{r \neq s} \rho_r(\Gamma^j P_-)\rho_s(\Gamma^i \Gamma^k P_-) \right] \mathcal{P}_{R^-} + (+ \leftrightarrow -). \] (13)

Now we must be careful to keep the traces of the last term when we project. The traceless combination is

\[ \frac{1}{2} \sum_{r \neq s} \left( \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j \Gamma^k P_-) + \rho_r(\Gamma^j P_-)\rho_s(\Gamma^i \Gamma^k P_-) \right) - \frac{n-1}{6} \left( \delta^{ij} G^k + \delta^{ik} G^j + \delta^{jk} G^i \right), \] (14)

as can be checked by contracting with a delta function. The trace parts correspond to \((r_1, r_2) = (1, 0)\) diagrams so we keep them. Projecting the second product therefore gives

\[ (G^i \bullet G^j) \bullet G^k = \mathcal{P}_{R^+} \left[ \sum_{r \neq s \neq t} \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j P_+)\rho_t(\Gamma^k P_-) + \right. \\
\left. \frac{2n+1}{3} \delta^{ij} G^k + \frac{n-1}{6} \left( \delta^{ik} G^j + \delta^{jk} G^i \right) \right] \mathcal{P}_{R^-} + (+ \leftrightarrow -). \] (15)

A similar calculation can be done for \(G^i \bullet (G^j \bullet G^k)\) and the non associativity can be expressed by

\[ (G^i \bullet G^j) \bullet G^k - G^i \bullet (G^j \bullet G^k) = \frac{n+1}{2} (\delta^{ij} G^k - \delta^{ik} G^j), \] (16)

where we have a different overall factor to [19] due to making only keeping the symmetric part after the first projection.

One can ask if the Basu-Harvey equation still holds for this projection, and we find that it does not. As one might expect the antisymmetric bracket vanishes when we project onto symmetric representations. The co-ordinates \(G^i\) are contained in \(\mathcal{A}_n(S^3)\). The commutator
of two of these co-ordinate matrices is given by

\[
[G^i, G^j]_{PR^\pm} = 2 \sum_r \rho_r (\Gamma^{ij} P_\pm)_{PR^\pm} + \sum_{r \neq s} 2 \rho_r (\Gamma^{ij} P_\pm) \rho_s (\Gamma^{ji} P_\pm)_{PR^\pm}
\]

(17)

\[
= 2 \sum_r \rho_r (\Gamma^{ij} P_\pm)_{PR^\pm} - \frac{n + 1}{2} \sum_r \rho_r (\Gamma^{ij} P_\pm)_{PR^\pm}
\]

\[
+ \frac{n - 1}{2} \sum_r \rho_r (\Gamma^{ij} P_\pm)_{PR^\pm}
\]

(18)

where in the second line we have written everything in terms of the Young diagram basis of [9]. We see that written in this basis every term contains a product of two \(\Gamma\)'s acting on the same tensor product factor. This corresponds to Young diagrams with \(r_2 \neq 0\) so after applying the projection there are no surviving terms.

Since the two bracket vanishes, the anti-symmetric four bracket also does. It seems therefore the algebra \(A_n(S^3)\) is not compatible with the Basu-Harvey equation. However the algebra \(A_n(S^3)\) has a tantalising property. If we take the solutions to the Basu-Harvey equation

\[
X^i(\sigma) = \frac{i\sqrt{2\pi}}{M^3_{11}} \frac{1}{\sqrt{\sigma}} G^i
\]

(19)

and look at the physical radius,

\[
R = \sqrt{\frac{Tr \sum (X^i)^2}{Tr \frac{1}{2}}}
\]

(20)

we get

\[
\sigma = \frac{2\pi N}{M^2_{11} R^2}.
\]

(21)

(Notice that in (20) we no longer have a matrix trace with our modified algebra, however because \(\sum_i X^i X^i\) is proportional to the identity in the algebra, and trace of the identity is precisely what we divide by to obtain the physical radius, the form of the trace is unimportant here). If we still identify \(N\) with \(Q\), the number of membranes, then we reproduce the profile expected from the self-dual string.

However, now the number of degrees of freedom in the co-ordinates is no longer \(N^2\), but

\[
D = (n + 1)(n + 2)(2n + 3)/6 \sim n^3
\]

so that for large \(N\) (thus \(n\)) we have that

\[
D \sim Q^\frac{3}{2}
\]

(22)

exactly as expected for \(Q\) coincident membranes in the large \(Q\) limit. This is interpreted as the result of the fuzzy three-sphere being endowed with an ultraviolet cutoff and the scaling
of the cutoff to the size of sphere depends in the right way on $Q$ to give the correct number of degrees of freedom. This means that one can interpret the $Q^{3/2}$ degrees of freedom corresponding to the non-Abelian membrane theory as coming from modes on the fuzzy sphere.

This is encouraging but we must reconcile the non-associative projection and the four-bracket. One possibility is that the projection does not act inside the bracket, which is thought of as an operator $[G_5, \ldots, :] : (A_n(S^3))^3 \rightarrow A_n(S^3)$. In this case obviously $X^i$ will still provide a solution. (See the following section for how something similar happens for dimensional reduction.)

4 Reducing the Fuzzy Three-Sphere to the Fuzzy Two-Sphere

Reducing the fuzzy three-sphere to the fuzzy two-sphere comes down to finding a projection on the fuzzy three-sphere so that three of the fuzzy three-sphere matrices obey the fuzzy two-sphere algebra.

We begin at $n=1$ and look for a projector $\bar{P}$ such that it commutes with $\Gamma^a$, $a = 1, 2, 3$ and $[\Gamma^a, \Gamma^b] \bar{P} = 2i\epsilon_{abc} \Gamma^c \bar{P}$. Looking ahead to the Basu-Harvey equation, we will also require that $\bar{P} \Gamma^4 \bar{P} = 0$ to satisfy the fuzzy three-sphere equation. We use a basis given by Appendix A and assume that the projector is made of $2 \times 2$ blocks proportional to the identity, i.e. constructed from $\mathbb{1}, \Gamma^4, \Gamma^5, \Gamma^{45}$. The solution is given by

$$\bar{P} = \frac{1}{2}(1 + i\Gamma^4\Gamma^5).$$

One can then clearly see $\bar{P}^2 = \bar{P}$, $\bar{P}$ commutes with $\Gamma^a$ and $\bar{P} \Gamma^4 \bar{P} = \bar{P} \Gamma^5 \bar{P} = 0$.

Defining $\tilde{\Gamma}^a = \bar{P} \Gamma^a \bar{P}$ we now have

$$[\tilde{\Gamma}^a, \tilde{\Gamma}^b] = 2i\epsilon_{abc} \tilde{\Gamma}^c,$$

in other words, when restricted to the subspace which $\bar{P}$ projects onto, the $\Gamma^a$ form an $SU(2)$ algebra. This is easy to see when we choose \{\bar{e}_1 = \frac{1}{2}(e_1 + ie_3), \bar{e}_2 = \frac{1}{2}(e_2 + ie_4)\} as a basis for $\bar{P}V$. Then

$$\tilde{\Gamma}^a(\bar{e}_i) \equiv \sigma^a_{ij}\bar{e}_j$$

where $\sigma^a$ are the Pauli matrices.

We can generalise to any $n$ by introducing $\bar{P} = \bar{P}^{\otimes n}$ which projects onto $\mathcal{R} = (\bar{P}V)^{\otimes n}$.
We denote the original fuzzy four-sphere matrices, which act on $V^\otimes n$, by $\hat{G}^\mu$. Then we set

$$\bar{G}^\mu = \mathcal{P}\hat{G}^\mu\mathcal{P} = \mathcal{P}\sum_r \rho_r(\Gamma^\mu)\mathcal{P}$$

(26)

which is just saying that we have restricted $\hat{G}^\mu$ to $\bar{R}$. Now because of (25) we will recover the construction of the fuzzy two-sphere given in Appendix B of [18]. Thus one can check that

$$[\bar{G}^a, \bar{G}^b] = 2i\epsilon_{abc}\bar{G}^c,$$

(27)

$$\bar{G}^a\bar{G}^a = n(n + 2)\mathbb{1}.$$  

(28)

Notice that if we define the co-ordinate matrices of the fuzzy three-sphere with a projector onto $\mathcal{R}$ either side of them $(G^i = P_{\mathcal{R}}\hat{G}^iP_{\mathcal{R}})$ then $\bar{G}^a \neq \mathcal{P}G^a\mathcal{P}$. However $\bar{G}^a \propto \mathcal{P}G^a\mathcal{P}$ because $\mathcal{P}P_{\pm}\mathcal{P} = \mathcal{P}/2$. Thus the constant of proportionality has a power of 2. It also has combinatoric factor dependent as there are $\binom{n-1}{(n-1)/2}$ ways of choosing which tensor product factors are acted upon by the $\frac{n-1}{2}$ $P_+$'s in $P_{\mathcal{R}}$ which do not act on the same factor as the $\Gamma$. In $\mathcal{P}$ we just have $n - 1$ $\mathcal{P}$ factors so only one choice. Hence the constant of proportionality is given by

$$\bar{G}^a = \left(\frac{(n - 1)}{(n - 1)/2}\right)^{-1}2^{n-1}\mathcal{P}G^a\mathcal{P}.$$

(29)

The projectors are there to indicate that we project on to $\mathcal{R}$ or $\bar{\mathcal{R}}$, so this is not a problem; we can think of acting with the same original $\hat{G}^\mu$ of the fuzzy $S^4$ in both cases before we project back to the representation we are dealing with using $P_{\mathcal{R}}$ or $\mathcal{P}$.

To make sure we can project from the fuzzy two-sphere to the fuzzy three-sphere what we should check is that for any state, $\Psi$, in $\bar{\mathcal{R}}$ we can find state in $\mathcal{R}$ such that $\mathcal{P}$ projects it onto $\Psi$. Indeed we can find many such states. Similarly for any operator on these states in the fuzzy two-sphere we can find operators in the full fuzzy three-sphere algebra, $Mat_N(\mathbb{C})$, that project on to it. In fact if we restrict ourselves to the non-associative algebra $\mathcal{A}_n(S^3)$ there is a unique operator that projects onto each operator in the fuzzy two-sphere, up to addition of operators in the kernel of $\mathcal{P}$. A general operator of the form

$$\bar{\mathcal{P}}\sum_{\overrightarrow{r} \neq \overrightarrow{s} \neq \overrightarrow{t}} \rho_{r_1}(\Gamma^1)\rho_{r_2}(\Gamma^1)\ldots\rho_{r_i}(\Gamma^1)\rho_{s_1}(\Gamma^2)\ldots\rho_{s_j}(\Gamma^2)\rho_{t_1}(\Gamma^3)\ldots\rho_{t_k}(\Gamma^3)\mathcal{P}$$

(30)
is proportional to that obtained by projecting
\[ \mathcal{P}_R \sum_{\vec{r} \neq \vec{r} \neq \vec{t}} \rho_{r_1}(\Gamma^1 P_{\pm}) \rho_{r_2}(\Gamma^1 P_{\pm}) \ldots \rho_{r_i}(\Gamma^1 P_{\pm}) \rho_{s_1}(\Gamma^2 P_{\pm}) \ldots \rho_{s_j}(\Gamma^2 P_{\pm}) \rho_{t_1}(\Gamma^3 P_{\pm}) \ldots \rho_{t_k}(\Gamma^3 P_{\pm}) \mathcal{P}_R + (\leftrightarrow). \]  
(31)
where the signs of each \( P_{\pm} \) are chosen to alternate from right to left. In both the fuzzy three-sphere and the fuzzy two-sphere we are still effectively using the co-ordinate matrices of the fuzzy four-sphere, but we are restricting to a much reduced set of states.

Notice also if we plug the our projected fuzzy three-sphere matrices straight back into the fuzzy three-sphere equation (7) then it is not satisfied due to the vanishing of \( \bar{G}^4 \). This should be compared with taking the classical version of the three-sphere equation and reducing to two-sphere (or any other sphere reduction) where the Nambu bracket is replaced by a Poisson bracket: if we reduce to a sub-sphere of lower degree, say by fixing one of the co-ordinates, then the equation will not be satisfied. This occurs when the derivatives in the Poisson bracket act on the constant co-ordinate. However if we set the co-ordinate to its fixed value after evaluating the derivatives then the higher sphere equation will still be satisfied.

For the Nambu bracket there are no derivatives and the information is in the anti-commutation properties of the matrices. Hence we should make our projection after evaluating the Nambu-bracket in (7). In our case we see that our projected sphere satisfies the higher sphere equation trivially. The main change is that both terms vanish for \( i = 4 \), a necessity to obey the Basu-Harvey equation below.

We can also ask how the \( SU(2) \) of the fuzzy two-sphere fits inside the \( SU(2) \times SU(2) = SO(4) \) of the fuzzy three-sphere. The \( SO(4) \) has the six generators
\[ G^{ij} = \mathcal{P}_R \sum_r \rho_r(\Gamma^{ij}) \mathcal{P}_R. \]  
(32)

\( i \)From these we can construct two orthogonal \( SU(2) \)’s by
\[ \Sigma_{L/R}^{ij} = -i(G^{ij} \mp \epsilon_{ijkl}G^{kl}) = -i\mathcal{P}_R \sum_r \rho_r(\Gamma^{ij} P_{\pm}) \mathcal{P}_R. \]  
(33)
The \( \Sigma_L^{ij} \) and the \( \Sigma_R^{ij} \) form \( SU(2) \) algebras among themselves and commute with each other, we call these \( SU(2) \)’s \( SU(2)_L \) and \( SU(2)_R \) respectively. A general state in \( \mathcal{R}^\pm \) of the the fuzzy \( S^3 \) is given by
\[ \left( (e_1)^{\otimes p} \otimes (e_2)^{\otimes \frac{p+1}{2}} \otimes (e_3)^{\otimes q} \otimes (e_4)^{\otimes \frac{q+1}{2}} \right)_{sym}. \]  
(34)
where $e_i$ is the basis of $V$. If we label the generators of the two $SU(2)$’s by $\sigma^a_{L/R} = 1/2\epsilon_{abc}\Sigma^b_{L/R}$ then these basis states are eigenstates of $\sigma^3_{L/R}$, with eigenvalues $2m_{L/R}$. We can then apply $\bar{\mathcal{P}}$ to these states and examine the eigenvalues of $\bar{G}^3$ on these new $\bar{R}$ states, $2\bar{m}$ say. Then we find that $\bar{m} = m_L + m_R$. Since adding the size of the reps of $SU(2)_L$ and $SU(2)_R$ in $R^{\pm}$ gives $(n \mp 1)/2 + (n \mp 1)/2 = n$, which is the size of the rep of the fuzzy three-sphere $SU(2)$, we see that we are effectively taking the sum of $SU(2)_L$ and $SU(2)_R$.

5 Disappearance of Non-associativity after Dimensional Reduction

Given the non-associative nature of the fuzzy three-sphere algebra, it is natural to ask how it gives rise to the associative algebra of the fuzzy two-sphere. In the fuzzy two-sphere there exist only matrices corresponding to symmetric Young diagrams. This is because the $su(2)$ algebra $\sigma^i \sigma^j = \delta^{ij} \mathbb{1} + \epsilon^{ijk} \sigma^k$ implies that an anti-symmetrised product of $\sigma$’s can be written as a single $\sigma$.

We can perform a check that the product in the fuzzy two-sphere is associative,

$$
\bar{G}^a \bar{G}^b \bar{P} = \left( \sum_r \rho_r(\delta^{ab} \mathbb{1} + i\epsilon_{abc} \Gamma^c) + \sum_{r \neq s} \rho_r(\Gamma^a)\rho_s(\Gamma^b) \right) \bar{P}
$$

so that

$$(\bar{G}^a \bar{G}^b)\bar{G}^c \bar{P} = \left( \sum_r \rho_r(\delta^{ab} \Gamma^c + i\epsilon_{abc} \Gamma^d \Gamma^e) + \sum_{r \neq s} \rho_r(\Gamma^a)\rho_s(\Gamma^b) \right) \bar{P}
$$

$$
+ \sum_{r \neq s} \rho_r(\Gamma^a)\rho_s(\Gamma^b) \left( \sum_r \rho_r(\delta^{ab} \mathbb{1} + i\epsilon_{abc} \Gamma^e) + \sum_{r \neq t} \rho_r(\Gamma^a)\rho_t(\Gamma^c) \right) \bar{P}
$$

$$
+ \sum_{r \neq s} \rho_r(\Gamma^a)\rho_s(\Gamma^b) \left( \sum_r \rho_r(\delta^{ab} \mathbb{1} + i\epsilon_{abc} \Gamma^d) + \sum_{r \neq t} \rho_r(\Gamma^a)\rho_t(\Gamma^c) \right) \bar{P}
$$

$$
+ \sum_{r \neq s \neq t} \rho_r(\Gamma^a)\rho_s(\Gamma^b)\rho_t(\Gamma^c) \bar{P}
$$

\[11\]
\[
\begin{align*}
&= \left( in\epsilon_{abc} + n\delta^{ab}\bar{G}^c + (n-2)\delta^{ac}\bar{G}^b + n\delta^{bc}\bar{G}^a \\
&\quad + i\epsilon_{acd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^b) + i\epsilon_{bcd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^a) + i\epsilon_{abd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^c) \\
&\quad + \sum_{r\neq s\neq t} \rho_r(\Gamma^a)\rho_s(\Gamma^b)\rho_t(\Gamma^c) \right) \bar{P}
\end{align*}
\]

(36)

and similarly

\[
(\bar{G}^a\bar{G}^b)\bar{G}^c\bar{P} = \left( in\epsilon_{abc} + n\delta^{ab}\bar{G}^c + (n-2)\delta^{ac}\bar{G}^b + n\delta^{bc}\bar{G}^a \\
\quad + i\epsilon_{acd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^b) + i\epsilon_{bcd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^a) + i\epsilon_{abd}\sum_{r\neq s} \rho_r(\Gamma^d)\rho_s(\Gamma^c) \\
\quad + \sum_{r\neq s\neq t} \rho_r(\Gamma^a)\rho_s(\Gamma^b)\rho_t(\Gamma^c) \right) \bar{P}
\]

(37)

so we have an associative product as expected. All terms can be written in terms of symmetric operators as in the final line.

6 Reducing the Basu-Harvey Equation to the Nahm Equation

Consider the fuzzy three-sphere equation (7) but replace the \(G^i\) by unknowns \(\tilde{G}^i\) for which we must solve. Let us reduce to the fuzzy two-sphere so that the equation acts on \(\bar{P}\),

\[
\bar{P} \left( \tilde{G}^i + \frac{1}{2(n+2)}\epsilon_{ijkl}G^j\tilde{G}^k\tilde{G}^l \right) \bar{P} = 0.
\]

(38)

We let \(\tilde{G}^4 = G^4/c\) where \(c\) is the factor arising from projection we saw previously in equation (29), but fix the \(\tilde{G}^a\) to be matrices in the algebra of the fuzzy two-sphere (\(\tilde{G}^a = \bar{P}\tilde{G}^a\bar{P}\)). Then, because \(\bar{P}G^4\bar{P} = 0\) and \(\bar{P}G^5G^4\bar{P} = inc\bar{P}\), the \(\tilde{G}^a\) must obey

\[
\bar{P} \left( \tilde{G}^a + \frac{in}{2(n+2)}\epsilon_{abc}\tilde{G}^b\tilde{G}^c \right) \bar{P} = 0.
\]

(39)

In the large \(n\) limit this is the statement that the \(\tilde{G}^a\) must obey the fuzzy two-sphere \(SU(2)\) algebra

\[
[\tilde{G}^a, \tilde{G}^b]\bar{P} = 2i\epsilon_{abc}\tilde{G}^c\bar{P}
\]

(40)

which of course the \(\tilde{G}^a\) obey (27). Having to take the large \(n\) limit is expected because the fuzziness will make picking out a cross-section of the sphere difficult at small \(n\).
We can now follow a similar procedure for the Basu-Harvey equation. We must take into account the additional anti-symmetrisation of the 4-bracket and also use $R_{11} = M_{11}^{-3} \alpha'^{-1}$. We set $X^4 = \frac{32\pi R_{11} G^4}{3c}$ and restrict the equation and the $X^a$ to the fuzzy two-sphere. Then we get

$$\left(\frac{dX^a}{d\sigma} + \frac{in}{\alpha' \sqrt{2N}} \epsilon_{abc} X^b X^c\right) \mathcal{P} = 0. \quad (41)$$

(There is also the case when the free index is 4, here both terms vanish. This is required if we are to get the right $\sigma$ dependence.) Rearranging, again in the large $n$ limit, we get

$$\frac{dX^a}{d\sigma} + \frac{i}{2\alpha'} \epsilon_{abc} [X^b, X^c] = 0, \quad (42)$$

which is (11) but with the $X^a$ scaled by $\alpha'$ so that they have dimensions of length. It has solution

$$X^a = \frac{\alpha' G^a}{2\sigma}. \quad (43)$$

The appearance of $R_{11}$ is expected as the length scale in 11-dimensions is $1/M_{11}$ and in string theory it is $\sqrt{\alpha'}$. Changing from a three bracket for the Basu-Harvey equation to a two bracket for the Nahm produces a $M_{11}^{-3} \alpha'^{-1} = R_{11}$. Note, here we have considered either the case where the solutions before projection act in $Mat_N(\mathbb{C})$, or the case where they are in $A_n(S^3)$ but the projection does not act within the anti-symmetrised product.

7 Discussion

There is still much to be learnt about the M2-M5 brane system. The tantalising appearance of $N^{3/2}$ for the non-associative algebra suggests there may be something in the algebra of the fuzzy three-sphere that is relevant for the M2-M5 system even though the projection discussed here appears not to be consistent with the Basu-Harvey equation. The relation to the Nahm equation has been clarified but until the many unresolved questions are answered concerning the supersymmetric non-Abelian membrane theory this discussion remains conjectural.

Acknowledgements

We wish to thank Anirban Basu, Neil Lambert, Costis Papageorgakis and Sanjaye Ramgoolam for discussions. DSB is supported by EPSRC grant GR/R75373/02 and would like
to thank DAMTP and Clare Hall college Cambridge for continued support. NBC is supported by a PPARC studentship. This work was in part supported by the EC Marie Curie Research Training Network, MRTN-CT-2004-512194.

A Conventions

We use the following basis for the $Spin(4)$ $\Gamma$ matrices:

\[
\Gamma^i = \begin{pmatrix}
0 & \sigma^i \\
\bar{\sigma}^i & 0
\end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix}
\mathbb{1}_{2\times2} & 0 \\
0 & -\mathbb{1}_{2\times2}
\end{pmatrix},
\]

(44)

where

\[
\sigma^i = (-i\bar{\sigma}_{\text{Pauli}}, \mathbb{1}_{2\times2}), \quad \bar{\sigma}^i = (i\bar{\sigma}_{\text{Pauli}}, \mathbb{1}_{2\times2})
\]

(45)

with $\bar{\sigma}_{\text{Pauli}}$ being the standard Pauli sigma matrices:

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(46)

Thus $\Gamma^5 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \frac{1}{4!}\epsilon_{ijkl}\Gamma^i\Gamma^j\Gamma^k\Gamma^l$.  

References

[1] P. S. Howe, N. D. Lambert and P. C. West, “The self-dual string soliton,” Nucl. Phys. B 515 (1998) 203 [arXiv:hep-th/9709014].

[2] A. Basu and J. A. Harvey, “The M2-M5 brane system and a generalized Nahm’s equation,” Nucl. Phys. B 713 (2005) 136 [arXiv:hep-th/0412310].

[3] D. S. Berman and N. B. Copland, “Five-brane calibrations and fuzzy funnels,” Nucl. Phys. B 723 (2005) 117 [arXiv:hep-th/0504044].

[4] D. Nogradi, “M2-branes stretching between M5-branes,” JHEP 0601 (2006) 010 [arXiv:hep-th/0511091].

[5] N. R. Constable, R. C. Myers and O. Tafjord, “The noncommutative bion core,” Phys. Rev. D 61 (2000) 106009 [arXiv:hep-th/9911136]. N. R. Constable, R. C. Myers and O. Tafjord, “Fuzzy funnels: Non-abelian brane intersections,” [arXiv:hep-th/0105035].

[6] D. E. Diaconescu, “D-branes, monopoles and Nahm equations,” Nucl. Phys. B 503, 220 (1997) [arXiv:hep-th/9608163].

[7] D. Tsimpis, “Nahm equations and boundary conditions,” Phys. Lett. B 433, 287 (1998) [arXiv:hep-th/9804081].

[8] Z. Guralnik and S. Ramgoolam, “On the polarization of unstable D0-branes into noncommutative odd spheres,” JHEP 0102, 032 (2001) [arXiv:hep-th/0101001].

[9] S. Ramgoolam, “On spherical harmonics for fuzzy spheres in diverse dimensions,” Nucl. Phys. B 610, 461 (2001) [arXiv:hep-th/0105006].

[10] S. Ramgoolam, “Higher dimensional geometries related to fuzzy odd-dimensional spheres,” JHEP 0210, 064 (2002) [arXiv:hep-th/0207111].

[11] C. M. Hofman and W. K. Ma, “Deformations of closed strings and topological open membranes,” JHEP 0106 (2001) 033 [arXiv:hep-th/0102201].

[12] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, “A noncommutative M-theory five-brane,” Nucl. Phys. B 590 (2000) 173 [arXiv:hep-th/0005026].
[13] J. Bagger and N. Lambert, private communication, paper to appear.

[14] S. Thomas and J. Ward, “Electrified fuzzy spheres and funnels in curved backgrounds,” arXiv:hep-th/0602071.

[15] R. Bhattacharyya and R. de Mello Koch, “Fluctuating fuzzy funnels,” JHEP 0510 (2005) 036 arXiv:hep-th/0508131.

[16] J. Castelino, S. M. Lee and W. I. Taylor, “Longitudinal 5-branes as 4-spheres in matrix theory,” Nucl. Phys. B 526 (1998) 334 arXiv:hep-th/9712105.

[17] M. M. Sheikh-Jabbari, “Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture,” JHEP 0409, 017 (2004) arXiv:hep-th/0406214.

[18] M. M. Sheikh-Jabbari and M. Torabian, “Classification of all 1/2 BPS solutions of the tiny graviton matrix theory,” arXiv:hep-th/0501001.

[19] C. Papageorgakis and S. Ramgoolam, “On time-dependent collapsing branes and fuzzy odd-dimensional spheres,” arXiv:hep-th/0603239.