Counting conjugacy classes of cyclic subgroups for fusion systems

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Abstract. Thévenaz [6] made an interesting observation that the number of conjugacy classes of cyclic subgroups in a finite group $G$ is equal to the rank of the matrix of the numbers of double cosets in $G$. We give another proof of this fact and present a fusion system version of it. In particular we use finite groups realizing the fusion system $\mathcal{F}$ as in our previous work [3].

1 Statements of the results

In [6], Thévenaz observed the ‘curiosity’ that a finite cyclic group $G$ can be characterized by the nonsingularity of the matrix of the numbers of double cosets in $G$. In fact he proved a stronger proposition that for an arbitrary finite group $G$ the number of the conjugacy classes of cyclic subgroups of $G$ is equal to the rank of that matrix. This can be stated slightly more generally by introducing a subgroup $H$ of $G$ and considering the $G$-conjugacy classes of subgroups of $H$ as follows.

Theorem 1.1. Let $G$ be a finite group and let $H \leq G$. The rank of the matrix $(|P \backslash G/Q|)_{P,Q \leq G,H}$, whose rows and columns are indexed by the $G$-conjugacy classes of subgroups of $H$ and whose entries are the numbers of the corresponding double cosets in $G$, is equal to the number of the $G$-conjugacy classes of cyclic subgroups of $H$.

In the above theorem, the matrix $(|P \backslash G/Q|)_{P,Q \leq G,H}$ is determined by the $(H,H)$-biset $H \times H$, i.e., the set $G$ with left and right $H$-action (induced by multiplication in the group $G$) which are compatible. In particular, when $H = S$ is a Sylow $p$-subgroup of $G$, the $(S,S)$-biset $S \times S$, viewed as a left $S \times S$-set, decomposes as

$$S \times S \cong \bigsqcup_{x \in [S \backslash G/S]} S \times S/\Delta(c_x, S \cap S^x).$$

See Section 3 for the notation. Consequently the $(S,S)$-biset $S \times S$ determines the $p$-fusion of $G$, i.e., the conjugacy relation between $p$-subgroups of $G$. The es-
sential feature of the \( p \)-fusion in finite groups is generalized to categories called \textit{saturated fusion systems}, which connect the \( p \)-local aspects of group theory, representation theory and homotopy theory. We refer the reader to the book [1] for an introduction to the subject.

In [3], we observed that every saturated fusion system \( \mathcal{F} \) on a finite \( p \)-group \( S \) can be realized by a finite group \( G \) containing \( S \) as a (not necessarily Sylow) \( p \)-subgroup. Thus the above theorem yields a fusion system version as follows.

\textbf{Theorem 1.2.} Let \( \mathcal{F} \) be a saturated fusion system on a finite \( p \)-group \( S \). Let \( G \) be a finite group which contains \( S \) as a subgroup and realizes \( \mathcal{F} \). Then the rank of the matrix \( (|P \backslash G/\mathcal{Q}|)_{\mathcal{P},\mathcal{Q} \leq \mathcal{F} \mathcal{S}} \) is equal to the number of the \( \mathcal{F} \)-conjugacy classes of cyclic subgroups of \( S \).

By a result of Broto, Levi and Oliver [2, Proposition 5.5], every saturated fusion system \( \mathcal{F} \) on a finite \( p \)-group \( S \) has a (non-unique) characteristic biset \( \Omega \). See Section 3 for a precise definition; in particular, \( \Omega \) is a finite \((S, S)\)-biset. If \( \mathcal{F} \) is the fusion system of a finite group \( G \) on its Sylow \( p \)-subgroup \( S \), then \( G \) is a characteristic biset for \( \mathcal{F} \) with the obvious \( S \)-action on the left and right. So we may well expect that the matrix of the above theorem, with \( G \) replaced by \( \Omega \), has the same rank. Indeed this is the case.

\textbf{Theorem 1.3.} Let \( \mathcal{F} \) be a saturated fusion system on a finite \( p \)-group \( S \). Let \( \Omega \) be a characteristic biset for \( \mathcal{F} \). Then the rank of the matrix \( (|P \backslash \Omega/\mathcal{Q}|)_{\mathcal{P},\mathcal{Q} \leq \mathcal{F} \mathcal{S}} \) is equal to the number of \((P, Q)\)-orbits of \( \Omega \) indexed by the \( \mathcal{F} \)-conjugacy classes of subgroups of \( S \) is equal to the number of the \( \mathcal{F} \)-conjugacy classes of cyclic subgroups of \( S \).

Finally, one can replace the characteristic biset \( \Omega \) in the above theorem by the characteristic idempotent \( \omega_{\mathcal{F}} \) (which is a virtual \((S, S)\)-biset; see Section 3) with \( |P \backslash \omega_{\mathcal{F}}/\mathcal{Q}| \) as the linearized number of \((P, Q)\)-orbits.

\textbf{Theorem 1.4.} Let \( \mathcal{F} \) be a saturated fusion system on a finite \( p \)-group \( S \). Let \( \omega_{\mathcal{F}} \) be the characteristic idempotent for \( \mathcal{F} \). The rank of the matrix \( (|P \backslash \omega_{\mathcal{F}}/\mathcal{Q}|)_{\mathcal{P},\mathcal{Q} \leq \mathcal{F} \mathcal{S}} \) is equal to the number of the \( \mathcal{F} \)-conjugacy classes of cyclic subgroups of \( S \).

The bisets appearing in Theorems 1.3 and 1.4 play an important role in the theory of fusion systems. See [4] for more details.

We will give a proof of Theorem 1.1 (and hence obtain Theorem 1.2 as a corollary), which is slightly different from that of [6]. This new proof uses (at least explicitly) only the Burnside ring \( B(G) \) of \( G \), not the rational representation ring \( R_{\mathbb{Q}}(G) \) as in [6]. Therefore it is better suited for adapting to the fusion system case (Theorems 1.3 and 1.4), which we do subsequently.
\section{The group case}

We prove Theorem 1.1. As remarked in Section 1, Theorem 1.2 then immediately follows as a corollary by \cite{3}.

Let $G$ be a finite group. Let $B(G)$ be the Burnside ring of $G$, i.e., the Grothendieck ring of the isomorphism classes $[X]$ of finite $G$-sets $X$. As an additive group, $B(G)$ is a free abelian group with the canonical basis $\{[G/P] \mid P \leq G\}$. Let $\mathbb{Q}B(G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ and regard $B(G)$ as a subring of $\mathbb{Q}B(G)$. In particular the canonical basis for $B(G)$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$.

It is a well-known fact that for each $P \leq G$ the fixed-point map

$$\chi_P : B(G) \to \mathbb{Z}, \quad [X] \mapsto |X^P|,$$

is a ring homomorphism which depends only on the $G$-conjugacy class of $P$, and the product of these homomorphisms (tensored with $\mathbb{Q}$),

$$\chi = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{P \leq G} \chi_P : \mathbb{Q}B(G) \to \prod_{P \leq G} \mathbb{Q},$$

is a $\mathbb{Q}$-algebra isomorphism. For each subgroup $P \leq G$, let $e^G_P$ denote the element of $\mathbb{Q}B(G)$ such that

$$\chi_\mathbb{Q}(e^G_P) = \begin{cases} 1, & P = G Q, \\ 0, & \text{otherwise}. \end{cases}$$

Then again the element $e^G_P$ depends only on the $G$-conjugacy class of $P$ and $\{e^G_P \mid P \leq G\}$ is a set of pairwise orthogonal primitive idempotents of $\mathbb{Q}B(G)$ whose sum is equal to 1; in particular it is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$. Furthermore, for $H \leq G$, let $B(G)_H$ be the subgroup of $B(G)$ generated by the elements $[G/P]$ with $P \leq G$ such that $B(G)_H = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)_H$ is a subalgebra of $\mathbb{Q}B(G)$ with $\mathbb{Q}$-basis $\{[G/P] \mid P \leq G H\}$. Note that the elements $e^G_P$ with $P \leq G H$ belong to $\mathbb{Q}B(G)_H$ and hence $\{e^G_H \mid P \leq G H\}$ is another basis for $\mathbb{Q}B(G)_H$.

For each $P \leq G$ consider the $\mathbb{Q}$-linear map

$$\rho_P : \mathbb{Q}B(G) \to \mathbb{Q}, \quad [X] \mapsto |P \setminus X|,$$

which counts the $P$-orbits. By Burnside’s orbit counting lemma, we have

$$\rho_P(x) = \frac{1}{|P|} \sum_{u \in P} \chi_{\langle u \rangle}(x), \quad x \in \mathbb{Q}B(G).$$

Thus

$$\rho_P(e^G_Q) \neq 0 \iff Q \text{ is cyclic and } Q \leq G P.$$  \hfill (2.1)
Now the given matrix in Theorem 1.1 is equal to \((\rho_P(G/Q))_{P,Q \leq G,H}\). By change of basis, this matrix has the same rank as \((\rho_P(e_Q^G))_{P,Q \leq G,H}\). List the subgroups of \(H\) (up to \(G\)-conjugacy) in two families, the first consisting of cyclic subgroups and the second of noncyclic subgroups, and with nondecreasing order in each family. Then by (2.1) the above matrix has the form \(\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\) where \(A\) is a lower triangular matrix with nonzero diagonal entries. Thus Theorem 1.1 follows.

3 The fusion system case

We first prove Theorem 1.3. In fact, we prove a slightly generalized version of it.

**Proposition 3.1.** Let \(\mathcal{F}\) be a saturated fusion system on a finite \(p\)-group \(S\). Let \(\Omega\) be a finite \((S,S)\)-biset which is \(\mathcal{F}\)-stable and \(\mathcal{F}\)-generated and which contains the obvious \((S,S)\)-biset \(S\). Then the rank of the matrix \((|P\backslash \Omega/Q|)_{P,Q \leq \mathcal{F} S}\) is equal to the number of the \(\mathcal{F}\)-conjugacy classes of cyclic subgroups of \(S\).

We first explain the terminology. Let \(\mathcal{F}\) be a saturated fusion system on a finite \(p\)-group \(S\). An \((S,S)\)-set \(X\) is \(\mathcal{F}\)-stable if, for all \(P \leq S\) and all morphisms \(\varphi: P \to S\) in \(\mathcal{F}\), the restrictions of the \(S\)-action on \(X\) to \(P\) via the inclusion \(P \hookrightarrow S\) and via \(\varphi: P \to S\) give isomorphic \(P\)-sets. We say that an \((S,S)\)-biset is \(\mathcal{F}\)-stable if it is \(\mathcal{F}\times \mathcal{F}\)-stable viewed as a left \(S \times S\)-set by inverting the right action of \(S\). An \((S,S)\)-biset is \(\mathcal{F}\)-generated if, viewed as a left \(S \times S\)-set, all its isotropy subgroups are of the form \(\Delta(\varphi, P) = \{(\varphi(u), u) \mid u \in P\}\) with \(P \leq S\), \(\varphi: P \to S\) in \(\mathcal{F}\). A finite \((S,S)\)-biset \(\Omega\) is called a characteristic biset for \(\mathcal{F}\) if it is \(\mathcal{F}\)-stable and \(\mathcal{F}\)-generated and such that \(|\Omega|/|S|\) is not divisible by \(p\). It is easy to see that every characteristic biset \(\Omega\) contains the \((S,S)\)-biset \(S\).

Define

\[ B(\mathcal{F}) = \{x \in B(S) \mid \chi_P(x) = \chi_{P'}(x) \text{ for all } P, P' \leq S \text{ with } P = \mathcal{F} P'\}. \]

Clearly \(B(\mathcal{F})\) is a subring of \(B(S)\), which is called the Burnside ring of the fusion system \(\mathcal{F}\). For a finite \(S\)-set \(X\), we have \([X] \in B(\mathcal{F})\) if and only if \(X\) is \(\mathcal{F}\)-stable. The elements

\[ e_P^\mathcal{F} := \sum_{P' = \mathcal{F} P} e_{P'}, \]

where \(P \leq S\) and the sum is over the \(S\)-conjugacy classes of subgroups \(P'\) of \(S\) which are \(\mathcal{F}\)-conjugate to \(P\), belong to \(\mathbb{Q}B(\mathcal{F})\). The set \(\{e_P^\mathcal{F} \mid P \leq \mathcal{F} S\}\) is a set of pairwise orthogonal primitive idempotents of \(\mathbb{Q}B(\mathcal{F})\) whose sum is equal to 1; in particular it is a \(\mathbb{Q}\)-basis for \(\mathbb{Q}B(\mathcal{F})\). By (2.1), we have

\[ \rho_P(e_Q^\mathcal{F}) \neq 0 \iff Q \text{ is cyclic and } Q \leq \mathcal{F} P. \quad (3.1) \]
Let $\Omega$ be the $(S, S)$-biset given in the above proposition. By the $\mathcal{F}$-stability of $\Omega$, the left $S$-set $\Omega/P$ of the right $P$-orbits of $\Omega$ is also $\mathcal{F}$-stable for $P \leq S$. Moreover

$$\chi_Q(\Omega/P) \neq 0 \implies Q \leq \mathcal{F} P; \quad \chi_P(\Omega/P) \geq |NS(P)/P|.$$  

The former implication follows from the fact that $\Omega$ is $\mathcal{F}$-generated and the latter inequality comes from the fact that $\Omega$ contains $S$. Hence $\{\Omega/P \mid P \leq \mathcal{F} S\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B(\mathcal{F})$. Thus the matrix

$$(|P\backslash\Omega/Q|)_{P,Q \leq \mathcal{F} S} = (\rho_P(\Omega/Q))_{P,Q \leq \mathcal{F} S}$$

has the same rank as $(\rho_P(\epsilon_Q^\mathcal{F}))_{P,Q \leq \mathcal{F} S}$, which is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of $S$ by (3.1).

**Remark.** Note that the finite group $G$ in Theorem 1.2, viewed as an $(S, S)$-biset, satisfies the hypotheses for $\Omega$ in Proposition 3.1. Thus Theorem 1.2 can also be obtained from Proposition 3.1.

Now we address Theorem 1.4. In Proposition 3.1, the condition that the finite $(S, S)$-biset $\Omega$ contains the $(S, S)$-biset $S$ is equivalent to that $\chi_P(\Omega/P) \neq 0$ for all $P \leq S$, given the other conditions on $\Omega$. Proposition 3.1 then applies to all virtual $(S, S)$-bisets $\omega$ with coefficients in $\mathbb{Q}$ which are $\mathcal{F}$-stable, $\mathcal{F}$-generated and such that $\chi_P(\omega/P) \neq 0$ for all $P \leq S$, where $\omega/P$ denotes the linearized right $P$-orbits of $\omega$. The proof is identical to the one given above. In particular, Reeh [5, Proposition 4.5, Corollary 5.8] shows that if $\omega$ is the characteristic idempotent of $\mathcal{F}$, i.e., the unique virtual $(S, S)$-biset with coefficients in $\mathbb{Z}(p)$ which is $\mathcal{F}$-stable, $\mathcal{F}$-generated and which is an idempotent in the double Burnside ring $\mathbb{Z}(p)B(S, S)$ of $S$ (i.e., the Burnside ring of finite $(S, S)$-bisets), then the elements $\omega/P = \omega \circ_S [S/P] = \beta_P$ with $P \leq \mathcal{F} S$ form a basis of $\mathbb{Z}(p)B(\mathcal{F})$ such that $\chi_P(\omega/P) \neq 0$. This proves Theorem 1.4.

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