A QUASICLASSICAL APPROXIMATION IN THE THEORY OF THE LANDAU–POMERANCHUK EFFECT

A.V. Tarasov, H.T. Torosyan, and O.O. Voskresenskaya

Joint Institute for Nuclear Researches, 141980 Dubna, Moscow region, Russia

Abstract

Using the improved value of the screening angular parameter in the quasiclassical approximation of the Molière multiple scattering theory we show that the best agreement between the Migdal theory of the LPM-effect and experiment is achieved if the multiple scattering of electrons by target atoms is described using the quasiclassical approximation instead of the traditionally used Born one.

1. Introduction. — In 1953 Landau and Pomeranchuk [1] predicted within classical electrodynamics that multiple scattering can considerably suppress bremsstrahlung of high energy charged particles in a medium. The effect due to the fact that multiple particle-atom scattering in a medium leads to destructive interference of photons emitted at different space points, provided that the formation (coherence) length \( l_{cr} \) of the bremsstrahlung is small in comparison with the mean free path of the particle in the medium \( (l_{cr} \ll L) \), where \( L \) is the thickness of a target. This result was obtained on the basis of the classical theory of electron radiation. However, the formula for a spectral density of radiation in substance, obtained by Landau and Pomeranchuk, has only evaluative character.

The quantitative theory of the multiple scattering effect on an electron radiation in an amorphous medium was offered by Migdal in [2, 3]. This theory was based on the application of the kinetic equation method to the given task. Owing to Migdal’s important contribution to the theory of the given effect now it is called the Landau–Pomeranchuk–Migdal (LPM) effect [4].

The further development of a quantitative LPM effect theory was achieved in [5] with use of the quasiclassical operator method in QCD [6]. One of the basic equations of this approach is a two dimensional Schrödinger equation in the impact parameter space with an imaginary potential obtained with use of kinetic equations describing a motion of electron in a medium in the presence of external field. The same equation (without external field) was rederived in [7]. The last derivation is based on the approach [8] results of which coincide basically with [6]. This equation can be solved using a transverse Green function based on a path integral. The criticism of the quasiclassical operator approach [9] was denied in [10].

In [11] it was shown that analogous effects are possible also at coherent radiation of relativistic electrons and positrons in a crystalline medium, and the theory of these effects must also be based on the quasiclassical methods. The LPM effect is relevant in many areas of physics but particularly in high energy cosmic ray air showers [12]. It has analogues in nuclear physics involving quarks and gluons moving through matter [13]. A LPM-type suppression also appears in stellar interiors [14].

The first tests of LPM suppression came shortly after Migdal’s paper appeared; these previous experiments have studied the LPM effect, mostly with cosmic rays [15]. They qualitatively confirmed the LPM effect, but with very limited statistics. The first quantitative measurement of the LPM effect for high energy electrons was performed at SLAC in a series of experiments [16]. These experiments were the challenge for the theory since in all the previous papers calculations are performed to logarithmic accuracy; but they are faced with an unexpected problem, the so-called ‘problem of normalization’. The experimental data obtained for 25 GeV electron beam interacting with a homogeneous gold target disagree with the theoretical predictions within the normalization factor 0.93 – 0.94 [16], and the reasons of this disagreement are not clear.

However, considering the fact that the calculations [7, 16] were performed within the Born approximation, the above-mentioned discrepancy can be explained at least qualitatively. The aim of this work is to shown that this discrepancy can be explained also quantitatively if the corrections to the results of the Born approximation are appropriately considered. In this work it is shown that the use of a revised Molière theory of multiple scattering [17] within the quasiclassical approach to the description of the particle-atom scattering allows overcoming the discrepancies [7, 16] between experiment and the Migdal LPM effect theory. Some results of this work are presented in [18].

1Note that Goldman has taken into account also the edge effects [4].
The paper is organized as follows: Section 2 provides the essential background for our results. In this section, we get the basic formulae of the LPM effect theory for finite-size targets in the small-angle approximation adhering to the classical works by Migdal and Goldman. The expressions for the intensity \(dI(\omega)/d\omega\) of the bremsstrahlung produced the electron moving in matter and the kinetic equation for the electron distribution function are given within this theory in the small-angle approximation. In Section 3 on the basis of the Molière theory and the Fokker–Planck approximation, the analytical solution of the equation for \(dI(\omega)/d\omega\) at the emitted photon frequency \(\omega \gg \omega_{cr}\) is obtained. This solution and the exact relation between the values of the screening angle \(\theta_0\) of the Molière theory in the Born and quasiclassical approximations are used to calculate \((dI(\omega)/d\omega)_{\text{get}}\) and \((dI(\omega)/d\omega)_{\text{Born}}\) and estimate their ratio \(R(\omega, L)\) for \(\omega \gg \omega_{cr}\). It is shown that the \(R(\omega, L)\) value coincides with the normalization constant \(R\) from [7, 10]; however, the latter ignores the dependence of the ratio on \(\omega\) and \(L\). In Section 4 brief conclusions are given.

2. Formalism of the LPM effect theory for finite-size targets. — Simple but cumbersome calculations based on the results of [2, 4] yield the following formula for the electron bremsstrahlung intensity averaged over various trajectories of electron motion in the medium (hereafter the units \(\hbar = c = 1\), \(e^2 = 1/137\) are used):

\[
\begin{align*}
\frac{dI}{d\omega} &= 2\sum_{e} \left\{ n_0 L \int f^*(\vec{n}_2) \nu(\vec{n}_2 - \vec{n}_1) f(\vec{n}_1) d\vec{n}_1 d\vec{n}_2 \right. \\
& \quad - \left( n_0 v \right)^2 \int_0^T dt_1 \int_0^T dt_2 \text{Re} \left[ \int f^*(\vec{n}_2) \nu(\vec{n}_2 - \vec{n}_2') w(t_2, t_1, \vec{n}_2', \vec{n}_1', \vec{k}) \\
& \quad \times \nu(\vec{n}_1') \nu(\vec{n}_1' - \vec{n}_1) d\vec{n}_1' d\vec{n}_2' d\vec{n}_2 d\vec{n}_1 \right] \right\}, \tag{1}
\end{align*}
\]

where

\[
f(\vec{n}_{1,2}) = \frac{e}{2\pi} \cdot \frac{\vec{v}_{1,2}}{1 - \vec{n} \cdot \vec{v}_{1,2}},
\]

\[
\vec{v}_{1,2} = v \cdot \vec{n}_{1,2}, \quad \vec{n} = \frac{\vec{k}}{\omega}, \quad d\vec{n}_{1,2} \equiv d\omega_{1,2}, \quad T = \frac{L}{v},
\]

\[
\nu(\vec{n}_2 - \vec{n}_1) = \delta(\vec{n}_2 - \vec{n}_1) \int \sigma_0(\vec{n}_2' - \vec{n}_1) d\vec{n}_2' - \sigma_0(\vec{n}_2 - \vec{n}_1),
\]

\[
w(t_2, t_1, \vec{n}_2, \vec{n}_1, \vec{k}) = \int w(t_2, t_1, \vec{r}_2 - \vec{r}_1, \vec{n}_2, \vec{n}_1) \\
\times \exp[i\omega(t_2 - t_1) - i\vec{k}(\vec{r}_2 - \vec{r}_1)] d\vec{r}_2.
\]

Here \(\vec{k}\) and \(\vec{v}\) are the wave vector and the polarization vector of the emitted photon, \(\vec{n}_{1,2}\) are the unit vectors in the electron motion direction, \(v\) is the electron velocity assumed to be invariant during the interaction with the target (the quantum-mechanical recoil effect is negligibly small), \(e\) is the electron charge, \(\sigma_0(\vec{n}_2 - \vec{n}_1) = d\sigma/d\omega_{1,2}\) is the differential cross-section of the electron scattering by target atoms, \(n_0\) is the number of atoms in an unit volume of the medium, \(L\) is the thickness of the target, \(w(t_2, t_1, r_2 - r_1, \vec{n}_2, \vec{n}_1)\) is the electron distribution function in the coordinates \(\vec{r}_2\), and the direction of motion \(\vec{n}_2\) at the time \(t_2\) provided that at time \(t_1\) the electron had the coordinate \(\vec{r}_1\) and moved in the direction characterized by the unit vector \(\vec{n}_1\).

The distribution function \(w\) satisfies the kinetic equation

\[
\frac{\partial w(t_2, t_1, \vec{r}_2 - \vec{r}_1, \vec{n}_2, \vec{n}_1)}{\partial t_2} = -\vec{v}_2 \cdot \nabla \vec{r}_2 \cdot w(t_2, t_1, \vec{r}_2 - \vec{r}_1, \vec{n}_2, \vec{n}_1) \\
- n_0 \int \nu(\vec{n}_2 - \vec{n}_1') \tilde{w}(t_2, t_1, \vec{r}_2 - \vec{r}_1, \vec{n}_2', \vec{n}_1) d\vec{n}_2'. \tag{2}
\]
with the boundary condition

\[ \tilde{w}(t_2, t_1, \tilde{r}_2 - \tilde{r}_1, \tilde{n}_2, \tilde{n}_1)|_{t_2 = t_1} = \delta(\tilde{r}_2 - \tilde{r}_1)\delta(\tilde{n}_2 - \tilde{n}_1). \] (3)

The term in (1) linear in \( n_0 \) is ‘usual’ (incoherent) contribution to intensity of the electron bremsstrahlung in the medium derived by summation of the radiation intensities of the electron interaction with separate atoms of the target.

The term quadratic in \( n_0 \) includes the contribution from the interference of the bremsstrahlung amplitudes on various atoms. The destructive character of this interference leads to suppression of the soft radiation intensity, i.e., to the Landau–Pomeranchuk effect.

For \( \omega \) larger than \( \omega_{cr} = 2I/m^2L \), where \( m \) is the electron mass (for estimation of \( \omega_{cr} \) see [1, 2, 16]), the interference term becomes negligibly small, and radiation is of pure incoherent character.

For ultrarelativistic particles \((1-v \ll 1)\) it is convenient to pass in (1) to the small-angle approximation [2, 11] according to the scheme

\[ \tilde{n}_{1,2} = \left(1 - \frac{\vartheta_{1,2}^2}{2}\right)\tilde{n} + \vartheta_{1,2}, \quad d\tilde{n}_{1,2} = d\tilde{\vartheta}_{1,2}; \]

\[ f(\tilde{n}_{1,2}) = f(\vartheta_{1,2}) = \frac{c}{\pi} \cdot \frac{\tilde{\vartheta}_{1,2}}{\tilde{\vartheta}_{1,2}^2 + \lambda^2}, \quad \lambda = \frac{m}{E} = \gamma^{-1}; \] (4)

\[ \sigma_0(\tilde{n}_{2} - \tilde{n}_1) = \sigma_0(\tilde{\vartheta}_2 - \tilde{\vartheta}_1), \quad \delta_0(\tilde{n}_{2} - \tilde{n}_1) = \delta_0(\tilde{\vartheta}_2 - \tilde{\vartheta}_1), \]

\[ \nu_0(\tilde{n}_2 - \tilde{n}_1) = \nu_0(\tilde{\vartheta}_2 - \tilde{\vartheta}_1); \quad w(t_2, t_1, \tilde{n}_2, \tilde{n}_1, \tilde{k}) = w(t_2, t_1, \tilde{\vartheta}_2, \tilde{\vartheta}_1, \omega) \]

and further to the Fourier transforms of \( f, \nu, w \)

\[ f(\eta) = \frac{1}{2\pi} \int \tilde{f}(\tilde{\vartheta}) \exp[i\tilde{\vartheta}\eta] d\tilde{\vartheta} = \frac{i\epsilon\lambda\eta}{\pi\eta} K_1(\lambda\eta); \]

\[ \nu(\eta) = \int \tilde{\nu}(\tilde{\vartheta}) e^{i\tilde{\vartheta}\eta} d\tilde{\vartheta} = 2\pi \int \sigma_0(\tilde{\vartheta}) [1 - J_0(\eta\theta)]\tilde{\vartheta} d\tilde{\vartheta}; \] (5)

\[ w(t_2, t_1, \tilde{n}_2, \tilde{n}_1, \omega) = \frac{1}{(2\pi)^2} \int \tilde{w}(t_2, t_1, \tilde{\vartheta}_2, \tilde{\vartheta}_1, \omega) \times \exp[i\tilde{\vartheta}_2\tilde{n}_2 - i\tilde{n}_1 \tilde{\vartheta}_1] d\tilde{\vartheta}_1 d\tilde{\vartheta}_2, \]

where \( \tilde{\vartheta}_{1(2)} \) are the two-dimensional electron scattering angles in the plane orthogonal to the electron direction at the instants of time \( t_{1(2)} \), \( m \) and \( E \) are the electron mass and its energy, \( \tilde{\vartheta} \) denotes the electron multiple scattering angle over the interval \( t_2 - t_1 \), \( \lambda \) is the characteristic frequency of the emitted photon, \( J_0 \) and \( K_1 \) are the Bessel and Macdonald functions, respectively.

Consequently, expression (1) is reduced to

\[ \frac{dI}{d\omega} = \frac{2\lambda^2c^2}{\pi^2} \left\{ n_0 L \int K_1^2(\lambda\eta)\nu(\eta)d\eta \right. \]

\[ - n_0^2 \int_0^L dt_1 \int_0^L dt_2 \int \frac{\tilde{n}_1 \tilde{n}_2}{\eta_1 \eta_2} K_1(\lambda\eta_1) K_1(\lambda\eta_2) \nu(\eta_1)\nu(\eta_2) \]

\[ \times \operatorname{Re}[w(t_2, t_1, \tilde{n}_2, \tilde{n}_1, \omega)] d\eta_1 d\eta_2 \}, \] (6)

where \( w \) satisfies the kinetic equation

\[ \frac{\partial w(t_2, t_1, \tilde{n}_2, \tilde{n}_1, \omega)}{\partial t_2} = \frac{i\omega}{2} (\lambda^2 - \Delta_{\tilde{\vartheta}_2}) w(t_2, t_1, \tilde{n}_2, \tilde{n}_1, \omega) \]
The analytical solution of equation (7) with arbitrary values of $\omega$ approximation is possible:

$$w(t_2, t_1, \bar{\eta}_2, \bar{\eta}_1, \omega) = \delta(\bar{\eta}_2 - \bar{\eta}_1).$$

The form of equation (7) is similar to the equation for function of the two-dimensional Schrödinger equation with the mass $\omega^{-1}$ and the complex potential

$$U(\eta) = -\frac{\omega \lambda^2}{2} - in_0 \nu(\eta)$$

and therefore admits of a formal solution in the form of a continual integral (see, e.g., [19]).

3. Applying the quasiclassical approximation of the Molière theory to the theory of the LPM effect. — The analytical solution of equation (7) with arbitrary values of $\omega$ is only possible within the Fokker–Planck approximation is possible\footnote{An explicit expression for $w$ obtained in this approach can be found in [4].}

$$\nu(\eta) = a \cdot \eta^2,$$

but at $\omega = 0$ it is also possible for arbitrary $\nu(\eta)$.

In the latter case ($\omega = 0$)

$$w(t_2, t_1, \bar{\eta}_2, \bar{\eta}_1, 0) = \delta(\bar{\eta}_2 - \bar{\eta}_1) \exp\{-n_0 \nu(\eta_2)(t_2 - t_1)\},$$

and integration over $t_1$, $t_2$ in (6) is carried out trivially, leading to the simple result

$$\frac{dI(\omega)}{d\omega} \bigg|_{\omega=0} = \frac{4\lambda^2 \nu^2}{\pi} \int K_1^2(\lambda \eta)(1 - \exp[-n_0 \nu(\eta)L]) \eta d\eta.$$

Considering the aforesaid, in the other limiting case ($\omega \gg \omega_{cr}$) we get

$$\frac{dI(\omega)}{d\omega} \bigg|_{\omega\gg\omega_{cr}} = n_0 L \lambda^2 e^2 \int K_1^2(\lambda \eta) \nu(\eta) \eta d\eta.$$

Due to the properties of the Macdonald functions, the main contribution to the integrals (12), (13) comes from the area $0 \leq \eta \leq 1/\lambda$. As was shown in classical works of Molière [20] on the theory of multiple scattering of charged particles in a medium, the quantity $\nu(\eta)$ can be represented in this area as \footnote{Here the units $\hbar = c = 1$, $\beta = v/c = 1$ are used.}

$$\nu(\eta) = 4\pi \left(\frac{Z\alpha}{m}\right)^2 \eta^2 \left[ \ln\left(\frac{2}{\eta \theta_a}\right) + \frac{1}{2} - C \right],$$

where $C = 0.57721$ is Euler’s constant, and $\theta_a$, referred to ‘a screening angle’, depends both on the screening properties of the atom and on the $\sigma_0(\theta)$-approximation used for its calculation.

Using the Thomas–Fermi model of atom [21], Molière obtained $\theta_a$ for the cases where $\sigma_0$ is calculated within the Born and quasiclassical approximations:

$$(\theta_a)_{Born} = 1.2 \cdot \alpha \cdot Z^{1/3},$$

$$(\theta_a)_{qcl} = (\theta_a)_{Born} \sqrt{1 + 3.44 (Z\alpha)^2}.$$

The latter result is approximate (see critical remarks on its deviation in [22]).

With the technique developed in [23], it is possible to show [17] that for any model of the atom the following relation holds:

$$\ln\left[(\theta_a)_{qcl}\right] = \ln\left[(\theta_a)_{Born}\right] + \Re\{\psi(1 + iZ\alpha)\} + C$$

or, equivalently,

$$\ln\left[(\theta_a)_{qcl}\right] - \ln\left[(\theta_a)_{Born}\right] = f(Z\alpha),$$

\footnote{Here the units $\hbar = c = 1$, $\beta = v/c = 1$ are used.}
where $\psi$ is a logarithmic derivative of the $\Gamma$-function, and the Bethe–Maximon function reads

$$f(Z\alpha) = (Z\alpha)^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + (Z\alpha)^2)}.$$  

(18)

In the second order in the parameter $a = Z\alpha$, this result is as follows

$$\langle \theta_a \rangle_{qcl} = \langle \theta_a \rangle_{Born} \sqrt{1 + 2.13(Z\alpha)^2}.$$  

(19)

After the substitution of $\nu(\eta)$ (14) into (13), the integration is carried out analytically, leading to the following result:

$$\frac{dI(\omega)}{d\omega} \bigg|_{\omega \gg \omega_{cr}} = 16 \pi^3 \cdot Z^2 \alpha^3 \cdot \left( \ln \frac{\lambda}{\theta_a} + \frac{7}{12} \right) \cdot n_0 L.$$  

(20)

Substituting here the numerical values of parameter $\theta_a$ from (16) and (17) corresponding to $Z = 79$, and introducing the ratio

$$R(\omega) = \frac{\left[ \frac{dI(\omega)}{d\omega} \right]_{qcl}}{\left[ \frac{dI(\omega)}{d\omega} \right]_{Born}},$$  

(21)

we get

$$R(\omega) \bigg|_{\omega \gg \omega_{cr}} = 0.922,$$

which practically coincides with the normalization factor value $0.93 \div 0.94$ introduced in [7, 16] for obtaining agreement of the calculation for $\sigma_0$ in the Born approximation with experiment.

In the other limiting case ($\omega = 0$) the performance of numerical integration in (12), we get the following result for three thicknesses of the experimental target [16]:

$$R(\omega) \bigg|_{\omega = 0} = \begin{cases} 
0.936, & L = 0.001X_0 \\
0.961, & L = 0.007X_0 \\
0.982, & L = 0.060X_0 
\end{cases}.$$  

(22)

where $X_0 \approx 0.33$ cm is the radiation length of the target material ($Z = 79$).

It is obvious from general consideration that when $0 < \omega < \omega_{cr}$,

$$R(\omega) \bigg|_{\omega \gg \omega_{cr}} \leq R(\omega) \leq R(\omega) \bigg|_{\omega = 0}.$$  

(23)

Finally, let us briefly discuss the accuracy of the Fokker–Planck approximation that allows an analytical expression to be derived for $w_0$ and the entire $dI(\omega)/d\omega$ range to be rather simply calculated (using numerical calculation of triple integrals).

To this end, we will fixe the parameter $a$ in expression (10) in such a way that the results of the exact calculation of $\left(\frac{dI(\omega)}{d\omega}\right) \bigg|_{\omega \gg \omega_{cr}}$ and its calculation in the Fokker–Planck approximation coincide. As a result, we get

$$a = 2\pi \frac{(Z\alpha)^2}{m} \left[ \ln \frac{\sigma}{\theta_a} + \frac{7}{12} \right].$$  

(24)

Now we calculate $\left(\frac{dI(\omega)}{d\omega}\right) \bigg|_{\omega = 0}$ using the relations (22) and (23) and compare the result with the result obtained using ‘realistic’ (Molière) expression (14) for $\nu(\eta)$. Then for the ratio

$$\tilde{R} = \frac{\left[ \frac{dI(\omega)}{d\omega} \right]_{FP} \bigg|_{\omega = 0}}{\left[ \frac{dI(\omega)}{d\omega} \right]_{M} \bigg|_{\omega = 0}},$$  

(25)

Since Migdal used a Gaussian approximation for the electron distribution, this underestimates the probability of large angle scatters. These occasional large angle scatters would produce some suppression for $\omega > \omega_{cr}$, where Migdal predicts no suppression and where the authors of [16] determine the normalization.
we get the following values:

\[
\bar{R} = \begin{cases} 
0.947, & L = 0.001X_0 \\
0.890, & L = 0.007X_0 \\
0.872, & L = 0.060X_0
\end{cases}
\]  

(26)

It is obvious that the difference (26) from unity is noticeable higher than the characteristic experimental error. Therefore, it is clear that Fokker–Planck approximation can be used only for the qualitative description of the \(dI(\omega)/d\omega\) behavior. For the accurate quantitative analysis it is necessary to use values of \(w\) obtained by the numerical solution of kinetic equation (7). The results of this analysis together with a detailed comparison with the experimental data, will be a subject of separate article.

4. Conclusion. — Accounting for the fact that the calculations for the description of the interaction of electrons with gold target atoms (\(Z_A \sim 0.6\)) in [7, 16] were performed using the Born approximation, we managed to show that the above-mentioned discrepancy between theory and experiment can be explained both qualitatively and quantitatively if the corrections to the results of the Born approximation are considered based on the quasiclassical approximation of the Molière multiple scattering theory with the improved value of the screening angular parameter. From (22) and (23) it follows that the results of the quasiclassical approach for \(dI/d\omega\) cannot be derived from Born approximation results by multiplying them by the normalization factor, which is independent of the frequency \(\omega\) and target thickness \(L\). However, considering nearly a 3% error of the experimental data [7, 16], it is clear why multiplication by the normalization factor helped the authors [7, 16] to get reasonable agreement of the calculations within the Born approximation with the experimental data.

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