WEIGHTED MULTILINEAR HARDY OPERATORS AND COMMUTATORS

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Abstract. In this paper, we introduce a type of weighted multilinear Hardy operators and obtain their sharp bounds on the product of Lebesgue spaces and central Morrey spaces. In addition, we obtain sufficient and necessary conditions of the weight functions so that the commutators of the weighted multilinear Hardy operators (with symbols in central BMO space) are bounded on the product of central Morrey spaces. These results are further used to prove sharp estimates of some inequalities due to Riemann-Liouville and Weyl.

1. Introduction

Let $\omega : [0,1] \to [0, \infty)$ be a measurable function. The \textit{weighted Hardy operator} $H_\omega$ is defined on all complex-valued measurable functions $f$ on $\mathbb{R}^n$ as follows:

$$H_\omega f(x) := \int_0^1 f(tx)\omega(t) \, dt, \quad x \in \mathbb{R}^n.$$  

Under certain conditions on $\omega$, Carton-Lebrun and Fosset \cite{4} proved that $H_\omega$ maps $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$. They also pointed out that the operator $H_\omega$ commutes with the Hilbert transform when $n = 1$, and with certain Calderón-Zygmund singular integral operators including the Riesz transform when $n \geq 2$. A further extension of the results obtained in \cite{4} was due to Xiao in \cite{24}.

\textbf{Theorem A} \cite{24}. Let $1 < p < \infty$ and $\omega : [0,1] \to [0, \infty)$ be a measurable function. Then, $H_\omega$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$A := \int_0^1 t^{-n/p} \omega(t) \, dt < \infty. \quad (1.1)$$

Moreover,

$$\|H_\omega f\|_{L^p(\mathbb{R}^n)} = A. \quad (1.2)$$

Notice that the condition (1.1) implies that $\omega$ is integrable on $[0,1]$. The constant $A$ seems to be of interest as it equals to $\frac{p}{p-1}$ if $\omega \equiv 1$ and $n = 1$. In this case, $H_\omega$ is reduced to the classical \textit{Hardy operator} $H$ defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) \, dt, \quad x \neq 0,$$

which is the most fundamental averaging operator in analysis. Also, a celebrated integral inequality, due to Hardy \cite{13}, can be deduced from Theorem A immediately

$$\|Hf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R})}, \quad (1.3)$$

where $1 < p < \infty$ and the constant $\frac{p}{p-1}$ is the best possible.

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\end{itemize}
Another interesting application of Theorem A is the sharp estimate of the Riemann-Liouville integral operator on the Lebesgue spaces. To be precise, let $n = 1$ and we take

$$\omega(t) := \frac{1}{\Gamma(\alpha)(1-t)^{1-\alpha}}, \quad t \in [0, 1],$$

where $0 < \alpha < 1$. Then

$$H_\omega f(x) = x^{-\alpha} I_\alpha f(x), \quad x > 0,$$

where $I_\alpha$ is the Riemann-Liouville integral operator defined by

$$I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0.$$

Note that the operator $I_\alpha$ is exactly the one-sided version of the well-known Riesz potential

$$I_\alpha f(x) := C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(t)}{|x-t|^{n+\alpha}} dt, \quad x \in \mathbb{R}.$$  

Clearly, Theorem A implies the celebrated result of Hardy, Littlewood and Polya in [8, Theorem 329], namely, for all $0 < \alpha < 1$ and $1 < p < \infty$,

$$\|I_\alpha\|_{L^p(\mathbb{R}) \to L^p(x^{-\alpha}dx)} = \frac{\Gamma(1 - 1/p)}{\Gamma(1 + \alpha - 1/p)}, \quad (1.4)$$

Now we recall the commutators of weighted Hardy operators introduced in [7]. For any locally integrable function $b$ on $\mathbb{R}^n$ and integrable function $\omega : [0, 1] \to [0, \infty)$, the commutator of the weighted Hardy operator $H_\omega^b$ is defined by

$$H_\omega^b f := b H_\omega f - H_\omega (bf).$$

It is easy to see that the commutator $H_\omega^b$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ when $b \in L^\infty(\mathbb{R}^n)$ and $\omega$ satisfies the condition (1.1). An interesting choice of $b$ is that it belongs to the class of BMO($\mathbb{R}^n$). Recall that BMO($\mathbb{R}^n$) is defined to be the space of all $b \in L_{loc}(\mathbb{R}^n)$ such that

$$\|b\|_{BMO} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty,$$

where $b_Q := \frac{1}{|Q|} \int_Q b$ and the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ with sides parallel to the axes. It is well known that $L^\infty(\mathbb{R}^n) \not\subset$ BMO($\mathbb{R}^n$) since BMO($\mathbb{R}^n$) contains unbounded functions such as $\log |x|$.

When symbols $b \in$ BMO($\mathbb{R}^n$), the condition (1.1) on weight functions $\omega$ does not ensure the boundedness of $H_\omega^b$ on $L^p(\mathbb{R}^n)$. Via controlling $H_\omega^b$ by the Hardy-Littlewood maximal operators instead of sharp maximal functions, Fu, Liu and Lu [7] established sufficient and necessary conditions on weight functions $\omega$ which ensure that $H_\omega^b$ is bounded on $L^p(\mathbb{R}^n)$ when $1 < p < \infty$. Precisely, they obtain the following conclusion.

**Theorem B.** Let

$$\mathbb{C} := \int_0^1 t^{-n/p} \omega(t) \log \frac{2}{t} dt$$

and $1 < p < \infty$. Then following statements are equivalent:

(i) $\omega$ is integrable and $H_\omega^b$ is bounded on $L^p(\mathbb{R}^n)$ for all $b \in$ BMO($\mathbb{R}^n$);

(ii) $\mathbb{C} < \infty$.

We remark that the condition (1.1), i.e., $\mathcal{A} < \infty$, is weaker than $\mathbb{C} < \infty$ in Theorem B. In fact, if we let

$$\mathbb{B} := \int_0^1 t^{-n/p} \omega(t) \log \frac{1}{t} dt,$$
then $C = A \log 2 + B$. Hence $C < \infty$ implies $A < \infty$. However, $A < \infty$ can not imply $C < \infty$. To see this, for $0 < \alpha < 1$, let

$$e^{s(n/p-1)}\omega(s) = \begin{cases} s^{-1+\alpha}, & 0 < s \leq 1, \\ s^{-1-\alpha}, & 1 < s < \infty, \\ 0, & s = 0, \infty \end{cases}$$

(1.5)

and $\omega(t) := \tilde{\omega}(\log \frac{1}{t})$, $0 \leq t \leq 1$. Then it is not difficult to verify $A < \infty$ and $C = \infty$.

Later on in [3], the conclusions in Theorems A and B were further generalized to the central Morrey spaces $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ and the central BMO space $\dot{C}MO^q(\mathbb{R}^n)$. Here the space $\dot{C}MO^q(\mathbb{R}^n)$ was first introduced by Lu and Yang in [19], and the space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is a generalization of $\dot{C}MO^q(\mathbb{R}^n)$ introduced by Alvarez, Guzman-Partida and Lakey in [1]; see also [2].

**Definition 1.1.** Let $\lambda \in \mathbb{R}$ and $1 < p < \infty$. The central Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all locally $p$-integrable functions $f$ satisfying that

$$\|f\|_{\dot{B}^{p,\lambda}} = \sup_{R > 0} \left( \frac{1}{|B(0, R)|^{1+\lambda/p}} \int_{B(0, R)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Obviously, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is a Banach space. One can easily check that $\dot{B}^{p,\lambda}(\mathbb{R}^n) = \emptyset$ if $\lambda < -1/p$, $\dot{B}^{p,0}(\mathbb{R}^n) = B^p(\mathbb{R}^n)$, $\dot{B}^{p,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, and $\dot{B}^{p,\lambda}(\mathbb{R}^n) \supset \dot{B}^{p,\lambda}(\mathbb{R}^n)$ if $\lambda > -1/p$, where the space $\dot{B}^{p}(\mathbb{R}^n)$ was introduced by Beurling in [3]. Similar to the classical Morrey space, we only consider the case $-1/p < \lambda \leq 0$ in this paper. In the past few years, there is an increasing interest on the study of Morrey-type spaces and their various generalizations and the related theory of operators; see, for example, [1, 12, 9, 20, 15].

**Definition 1.2.** Let $1 < q < \infty$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the central bounded mean oscillation space $\dot{C}MO^q(\mathbb{R}^n)$ if

$$\|f\|_{\dot{C}MO^q} = \sup_{R > 0} \left( \frac{1}{|B(0, R)|} \int_{B(0, R)} |f(x) - f_{B(0, R)}|^q \, dx \right)^{1/q} < \infty. \quad (1.6)$$

The space $\dot{C}MO^q(\mathbb{R}^n)$ is a Banach space in the sense that two functions that differ by a constant are regarded as a function in this space. Moreover, (1.6) is equivalent to the following condition

$$\sup_{R > 0} \inf_{c \in \mathbb{C}} \left( \frac{1}{|B(0, R)|} \int_{B(0, R)} |f(x) - c|^q \, dx \right)^{1/q} < \infty.$$

For more detailed properties of these two spaces, we refer to [3].

For $1 < p < \infty$ and $-1/p < \lambda \leq 0$, it was proved in [9] Theorem 2.1] that the norm

$$\|H_w\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n) \rightarrow \dot{B}^{p,\lambda}(\mathbb{R}^n)} = \int_0^1 t^{n+\lambda} w(t) \, dt.$$

Moreover, if $1 < p_1 < p < \infty$, $1/p_1 = 1/p + 1/q$ and $-1/p < \lambda < 0$, then it was proved in [9] Theorem 3.1] that $H_w^{p_1}$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{n+\lambda} w(t) \log \frac{2}{t} \, dt < \infty,$$

where the symbol $b \in C\dot{M}O^q(\mathbb{R}^n)$.

In this paper, we consider the multilinear version of the above results. Recall that the weighted multilinear Hardy operator is defined as follows.
Definition 1.3. Let $m \in \mathbb{N}$, and

$$ω : [0, 1] \times [0, 1] \times \cdots \times [0, 1] \to [0, \infty)$$

be an integrable function. The \textit{weighted multilinear Hardy operator} $\mathcal{H}_ω^m$ is defined as

$$\mathcal{H}_ω^m(\vec{f})(x) := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} f_i(t_i x) \right) \omega(\vec{t}) \, d\vec{t}, \quad x \in \mathbb{R}^n,$$

where $\vec{f} := (f_1, \ldots, f_m)$, $\omega(\vec{t}) := \omega(t_1, t_2, \ldots, t_m)$, $d\vec{t} := dt_1 \cdots dt_m$, and $f_i$ ($i = 1, \ldots, m$) are complex-valued measurable functions in $\mathbb{R}^n$. When $m = 2$, $\mathcal{H}_ω^m$ is referred to as bilinear.

The study of multilinear averaging operators is traced to the multilinear singular integral operator theory (see, for example, [5]), and motivated not only the generalization of the theory of linear ones but also their natural appearance in analysis. For a more complete account on multilinear operators, we refer to [6], [11], [17] and the references therein.

The main aim of the paper is to establish the sharp bounds of weighted multilinear Hardy operators on the product of Lebesgue spaces and central Morrey spaces. In addition, we find sufficient and necessary conditions of the weight functions so that commutators of such weighted multilinear Hardy operators (with symbols in $\lambda$-central BMO space) are bounded on the product of central Morrey spaces.

The paper is organized as follows: Section 2 is devoted to the sharp estimates of $\mathcal{H}_ω^m$ on the products of Lebesgue spaces and also central Morrey spaces. In Section 3, we present the sharp estimates of the commutator generated by $\mathcal{H}_ω^m$ with symbols in $C\tilde{M}O^2(\mathbb{R}^n)$. Section 4 focuses on weighted Cesàro operators of multilinear type related to weighted multilinear Hardy operators.

## 2. Sharp boundedness of $\mathcal{H}_ω^m$ on the product of central Morrey spaces

We begin with the following sharp boundedness of $\mathcal{H}_ω^m$ on the product of Lebesgue spaces, which when $m = 1$ goes back to Theorem A.

**Theorem 2.1.** Let $1 < p, p_i < \infty$, $i = 1, \ldots, m$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then, $\mathcal{H}_ω^m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if

$$A_m := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-n/p_i} \right) \omega(\vec{t}) \, d\vec{t} < \infty.$$  (2.1)

Moreover,

$$\|\mathcal{H}_ω^m\|_{L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = A_m.$$

**Proof.** In order to simplify the proof, we only consider the case that $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$.

Suppose that (2.1) holds. Using Minkowski’s inequality yields

$$\|\mathcal{H}_ω^2(f_1, f_2)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left( \int_{0 < t_1, t_2 < 1} f_1(t_1 x) f_2(t_2 x) \omega(t_1, t_2) \, dt_1 dt_2 \right)^p \, dx \right)^{1/p}$$

$$\leq \int_{0 < t_1, t_2 < 1} \left( \int_{\mathbb{R}^n} |f_1(t_1 x) f_2(t_2 x)|^p \, dx \right)^{1/p} \omega(t_1, t_2) \, dt_1 dt_2.$$
By Hölder’s inequality with $1/p = 1/p_1 + 1/p_2$, we see that
\[
\|\mathcal{H}_\omega^{(2)}(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq \int \prod_{i=1}^2 \left( \int_{\mathbb{R}^n} |f_i(t_i, x)|^{p_i} \, dx \right)^{1/p_i} \omega(t_1, t_2) \, dt_1 \, dt_2
\]
\[
\leq \left( \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right) \int \prod_{i=1}^2 \left( t_i^{-n/p_i} \right) \omega(t_1, t_2) \, dt_1 \, dt_2.
\]
Thus, $\mathcal{H}_\omega^{(2)}$ maps the product of Lebesgue spaces $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and
\[
\|\mathcal{H}_\omega^{(2)}\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq A_2.
\] (2.2)

To see the necessity, for sufficiently small $\varepsilon \in (0, 1)$, we set
\[
f_1^\varepsilon(x) := \begin{cases} 0, & |x| \leq \frac{\sqrt{2}}{2}, \\ |x| - \frac{n-\varepsilon}{p_1}, & |x| > \frac{\sqrt{2}}{2}, \end{cases}
\] (2.3) and
\[
f_2^\varepsilon(x) := \begin{cases} 0, & |x| \leq \frac{\sqrt{2}}{2}, \\ |x| \frac{n-\varepsilon}{p_2}, & |x| > \frac{\sqrt{2}}{2}. \end{cases}
\] (2.4)

An elementary calculation gives that
\[
\|f_1^\varepsilon\|_{L^{p_1}(\mathbb{R}^n)} = \|f_2^\varepsilon\|_{L^{p_2}(\mathbb{R}^n)} = \frac{\omega_n}{p_2\varepsilon} \left( \frac{\sqrt{2}}{2} \right)^{-p_2\varepsilon},
\]
where $\omega_n = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the unit sphere. Consequently, we have
\[
\|\mathcal{H}_\omega^{(2)}(f_1^\varepsilon, f_2^\varepsilon)\|_{L^p(\mathbb{R}^n)}
\]
\[
= \left\{ \int_{\mathbb{R}^n} |x|^{-n-2\varepsilon} \left[ \int_{E_\varepsilon(t_1, t_2)} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2} - \varepsilon} \omega(t_1, t_2) \, dt_1 \, dt_2 \right] \, dx \right\}^{1/p},
\]
where
\[
E_\varepsilon(t_1, t_2) := \left\{ (t_1, t_2) \big| 0 < t_1, t_2 < 1; t_1 > \frac{\sqrt{2}}{2|x|}; t_2 > \frac{\sqrt{2}}{2|x|} \right\}.
\]

Hence,
\[
\|\mathcal{H}_\omega^{(2)}(f_1^\varepsilon, f_2^\varepsilon)(x)\|_{L^p(\mathbb{R}^n)}
\]
\[
\geq \int_{|x|>1/\varepsilon} |x|^{-n-2\varepsilon} \left( \int_{E_\varepsilon(t_1, t_2)} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2} - \varepsilon} \omega(t_1, t_2) \, dt_1 \, dt_2 \right) \, dx
\]
\[
= \frac{\varepsilon^{p_2\varepsilon} \omega_n}{p_2\varepsilon} \left( \int_{E_\varepsilon(t_1, t_2)} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2} - \varepsilon} \omega(t_1, t_2) \, dt_1 \, dt_2 \right)^{1/p}
\]
\[
= \left( \frac{\sqrt{2}}{2} \right)^{p_2\varepsilon} 2 \prod_{i=1}^2 \|f_i^\varepsilon\|_{L^{p_i}(\mathbb{R}^n)} \left( \int_{E_\varepsilon(t_1, t_2)} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2} - \varepsilon} \omega(t_1, t_2) \, dt_1 \, dt_2 \right)^{p}.
\]
Therefore,
\[
\|H^2_\omega\|_{L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \geq \left(\frac{\sqrt{2}}{2^\alpha} \right)^{p_0^\alpha/p} \int_{E_{\frac{1}{2}}(t_1, t_2)} t_1^{-\frac{m_1}{p_1}} t_2^{-\frac{m_2}{p_2}} \omega(t_1, t_2) \, dt_1 \, dt_2.
\]

Since \((\sqrt{2}/2)^{p_0^\alpha/p} \to 1\) as \(\varepsilon \to 0^+\), by letting \(\varepsilon \to 0^+\), we know that
\[
\|H^2_\omega\|_{L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \geq A_2.
\] (2.5)

Combining (2.2) and (2.6) then finishes the proof. \(\square\)

Observe that when \(n = 1\) and \(\alpha \in (0, m)\), if we take
\[
\omega(t) := \frac{1}{\Gamma(\alpha)(1-t_1, \ldots, 1-t_m)^{m-\alpha}},
\]
then
\[
H^m_\omega(\tilde{\mathcal{I}}(\hat{f}))(x) = x^{-\alpha} I^m_\alpha \tilde{\mathcal{I}}(\hat{f})(x), \quad x > 0,
\]
where
\[
I^m_\alpha \hat{f}(x) := \frac{1}{\Gamma(\alpha)} \int_{0 < t_1, t_2, \ldots, t_m < 1} \frac{\prod_{i=1}^m f_i(t_i)}{|(x-t_1, \ldots, x-t_m)|^{m-\alpha}} \, dt_i.
\]
The operator \(I^m_\alpha\) turns out to be the one-sided analogous to the one-dimensional multilinear Riesz operator \(\mathcal{I}^m_\alpha\) studied by Kenig and Stein in [7], where
\[
\mathcal{I}^m_\alpha \hat{f}(x) := \int_{t_1, t_2, \ldots, t_m \in \mathbb{R}} \frac{\prod_{i=1}^m f_i(t_i)}{|(x-t_1, \ldots, x-t_m)|^{m-\alpha}} \, dt_i, \quad x \in \mathbb{R}.
\]

As an application of Theorem 2.1 we obtain the following sharp estimate of the boundedness of \(I^m_\alpha\).

**Corollary 2.1.** Let \(0 < \alpha < m\). With the same assumptions as in Theorem 2.1, the operator \(I^m_\alpha\) maps \(L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_m}(\mathbb{R})\) to \(L^p(x^{-\alpha_\mathcal{I}} \, dx)\) and the operator norm equals to
\[
\frac{1}{\Gamma(\alpha)} \int_{0 < t_1, t_2, \ldots, t_m < 1} \left(\prod_{i=1}^m t_i^{1/p_i} \right)^{1/(m-\alpha)} \, dt_i.
\]

Next we extend the result in Theorem 2.1 to the product of central Morrey spaces.

**Theorem 2.2.** Let \(1 < p < p_i < \infty, 1/p = 1/p_1 + \cdots + 1/p_m, \lambda = \lambda_1 + \cdots + \lambda_m\) and \(-1/p_i \leq \lambda_i < 0, i = 1, 2, \ldots, m\).

(i) If
\[
\bar{A}_m := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left(\prod_{i=1}^m t_i^{-\lambda_i} \right) \omega(t) \, dt < \infty,
\] (2.6)
then \(H^m_\omega\) is bounded from \(\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m, \lambda_m}(\mathbb{R}^n)\) to \(\dot{B}^{p, \lambda}(\mathbb{R}^n)\) with its operator norm not more than \(\bar{A}_m\).

(ii) Assume that \(\lambda_1 p_1 = \cdots = \lambda_m p_m\). In this case the condition (2.6) is also necessary for the boundedness of \(H^m_\omega\) : \(\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m, \lambda_m}(\mathbb{R}^n) \to \dot{B}^{p, \lambda}(\mathbb{R}^n)\). Moreover,
\[
\|H^m_\omega\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m, \lambda_m}(\mathbb{R}^n) \to \dot{B}^{p, \lambda}(\mathbb{R}^n)} = \bar{A}_m.
\]
Proof. By similarity, we only give the proof in the case \( m = 2 \).

When \(-1/p_i = \lambda_i, \ i = 1, 2\), then Theorem 2.2 is just Theorem 2.1

Next we consider the case that \(-1/p_i < \lambda_i < 0, \ i = 1, 2\).

First, we assume \( \tilde{A}_2 < \infty \). Since \( 1/p = 1/p_1 + 1/p_2 \), by Minkowski’s inequality and Hölder’s inequality, we see that, for all balls \( B = B(0, R) \),

\[
\left( \frac{1}{|B|^{1+\lambda_p}} \int_B |\mathcal{H}_\omega^2(\tilde{f})(x)|^p dx \right)^{1/p} 
\leq \int_{0<t_1,t_2<1} \left( \frac{1}{|B|^{1+\lambda_p}} \int_B \left| \prod_{i=1}^2 f_i(t_i x) \right|^p dx \right)^{1/p} \omega(t) dt \cdot 
\leq \int_{0<t_1,t_2<1} \prod_{i=1}^2 \left( \frac{1}{|B|^{1+\lambda_p}} \int_B |f_i(t_i x)|^p dx \right)^{1/p_i} \omega(t) dt \cdot 
= \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \prod_{i=1}^2 \left( \frac{1}{|t_i |^{1+\lambda_p}} \int_{t_i B} |f_i(x)|^p dx \right)^{1/p_i} \omega(t) dt \cdot 
\leq \|f_1\|_{\tilde{B}^{p_1,\lambda_1}} \|f_2\|_{\tilde{B}^{p_2,\lambda_2}} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t) dt. 
\tag{2.7}
\]

This means that \( \|\mathcal{H}_\omega^2\|_{\tilde{B}^{p_1,\lambda_1}(\mathbb{R}^n) \times \tilde{B}^{p_2,\lambda_2}(\mathbb{R}^n) \rightarrow \tilde{B}^{p,\lambda}(\mathbb{R}^n)} \leq \tilde{A}_2 \).

For the necessity when \( \lambda_1 p_1 = \lambda_2 p_2 \), let \( f_1(x) := |x|^{n\lambda_1} \) and \( f_2(x) := |x|^{n\lambda_2} \) for all \( x \in \mathbb{R}^n \setminus \{0\} \), and \( f_1(0) = f_2(0) := 0 \). Then for any \( B := B(0, R) \),

\[
\left( \frac{1}{|B|^{1+\lambda_p}} \int_B |f_i(x)|^p dx \right)^{1/p_i} = \left( \frac{1}{|B|^{1+\lambda_p}} \int_B |x|^{n\lambda_i} dx \right)^{1/p_i} = \left( \frac{\omega_n}{n} \right)^{-\lambda_i} \left( \frac{1}{1+\lambda_i p_i} \right)^{1/p_i}.
\]

Hence \( \|f_i\|_{\tilde{B}^{p_i,\lambda_i}} = (\omega_n/n)^{-\lambda_i} \left( \frac{1}{n^{\lambda_1 p_i}} \right)^{1/p_i} \), \( i = 1, 2 \). Since \( \lambda = \lambda_1 + \lambda_2 \) and \(-1/p_i < \lambda_i < 0, 1 < p < p_i < \infty, \ i = 1, 2\), we have

\[
\left( \frac{1}{|B|^{1+\lambda_p}} \int_B |\mathcal{H}_\omega^2(\tilde{f})(x)|^p dx \right)^{1/p} 
= \left( \frac{1}{|B|^{1+\lambda_p}} \int_B |x|^{n\lambda_p} dx \right)^{1/p} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t) dt \cdot 
= \left( \frac{\omega_n}{n} \right)^{-\lambda} \left( \frac{1}{1+\lambda_p} \right)^{1/p} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t) dt \cdot 
= \|f_1\|_{\tilde{B}^{p_1,\lambda_1}} \|f_2\|_{\tilde{B}^{p_2,\lambda_2}} \frac{(1+\lambda_1 p_1)^{1/p_1}(1+\lambda_2 p_2)^{1/p_2}}{(1+\lambda_p)^{1/p}} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t) dt. 
\tag{2.8}
\]

since \( \lambda_1 p_1 = \lambda_2 p_2 \). Then, \( \tilde{A}_2 \leq \|\mathcal{H}_\omega^2\|_{\tilde{B}^{p_1,\lambda_1} \times \tilde{B}^{p_2,\lambda_2} \rightarrow \tilde{B}^{p,\lambda}} < \infty \). Combining (2.7) and (2.8) then concludes the proof. This finishes the proof of the Theorem 2.2 \( \square \).

We remark that Theorem 2.2 when \( m = 1 \) goes back to [9] Theorem 2.1.
A corresponding conclusion of $I^m_\alpha$ is also true.

**Corollary 2.2.** Let $0 < \alpha < m$. With the same assumptions as in Theorem 2.2, the operator $I^m_\alpha$ maps $\dot{B}^{p_1, \lambda_1}(\mathbb{R}) \times \cdots \times \dot{B}^{p_m, \lambda_m}(\mathbb{R})$ to $\dot{B}^{p, \lambda}(x^{-\alpha}dx)$ with the operator norm not more than

$$\frac{1}{\Gamma(\alpha)} \int_{0\leq t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^m t_i^{\lambda_i} \right) \frac{1}{|(1-t_1, \ldots, 1-t_m)|^{m-\alpha}} dt.$$ 

In particular, when $\lambda_1 p_1 = \cdots = \lambda_m p_m$, then the operator norm of $I^m_\alpha$ equals to the above quantity.

**Remark 2.3.** Notice that in the necessary part of Theorem 2.2, we need an additional condition $\lambda_1 p_1 = \cdots = \lambda_m p_m$. In the case of Lebesgue spaces, this condition holds true automatically. For the case of Morrey spaces, such condition has known to be the necessary and sufficient condition for the interpolation properties of Morrey spaces; see, for example, [10].

### 3. Commutators of weighted multilinear Hardy operators

In this section, we consider the sharp estimates of the multilinear commutator generated by $\mathcal{H}^m_\omega$ with symbols in $CMO^g(\mathbb{R}^n)$. Before presenting the main results of this section, we first introduce the following well-known Riemann-Lebesgue-type Lemma, which plays a key role in the below proof. For completeness, we give a detailed proof.

**Lemma 3.1.** Let $m \in \mathbb{N}$ and $\omega : [a, b]^m \to [0, \infty)$ be an integrable function. Then

$$\lim_{r \to \infty} \int_{[a,b]^m} \omega(t_1, \cdots, t_m) \prod_{i \in E} \sin(\pi r t_i) \, dt_1 \cdots dt_m = 0,$$

where $E$ is an arbitrary nonempty subset of $\{1, \cdots, m\}$.

**Proof.** For simplicity, we only give the proof for the case that $m = 2$ and $E = \{1\}$, namely, to show

$$\lim_{r \to \infty} \int_{[a,b]^2} \omega(t_1, t_2) \sin(\pi r t_1) \, dt_1 \, dt_2 = 0.$$ 

Since $\omega$ is integrable, for any $\varepsilon > 0$, there exists a partition $\{I_i \times J_j : i = 1, \cdots, k \text{ and } j = 1, \cdots, l\}$ such that $I_i = [a_i, b_i], J_j = [a_j, b_j], [a, b] = \bigcup_{i=1}^k I_i = \bigcup_{j=1}^l J_j, I_i \cap I_j = \emptyset = J_i \cap J_j$ if $i \neq j$, and

$$0 \leq \int_a^b \int_a^b \omega(t_1, t_2) \, dt_1 \, dt_2 - \sum_{i=1}^k \sum_{j=1}^l m_{ij} |I_i| |J_j| < \varepsilon/2,$$

where $m_{ij}$ is the minimum value of $\omega$ on $I_i \times J_j$. Let

$$g(t_1, t_2) := \sum_{i=1}^k \sum_{j=1}^l m_{ij} \chi_{I_i}(t_1) \chi_{J_j}(t_2), \quad t_1, t_2 \in [a, b].$$

Then

$$\int_a^b \int_a^b g(t_1, t_2) \, dt_1 \, dt_2 = \sum_{i=1}^k \sum_{j=1}^l m_{ij} |I_i| |J_j|$$

and

$$0 \leq \int_a^b \int_a^b [\omega(t_1, t_2) - g(t_1, t_2)] \, dt_1 \, dt_2 < \varepsilon/2.$$
It follows from $\omega - g \geq 0$ that
\[
\left| \int_{[a,b]^2} \omega(t_1, t_2) \sin(\pi rt_1) \, dt_1 \, dt_2 \right|
\leq \left| \int_{[a,b]^2} [\omega(t_1, t_2) - g(t_1, t_2)] \sin(\pi rt_1) \, dt_1 \, dt_2 \right| + \left| \int_{[a,b]^2} g(t_1, t_2) \sin(\pi rt_1) \, dt_1 \, dt_2 \right|
\leq \left| \int_{[a,b]^2} [\omega(t_1, t_2) - g(t_1, t_2)] \, dt_1 \, dt_2 \right| + \left| \int_{[a,b]^2} g(t_1, t_2) \sin(\pi rt_1) \, dt_1 \, dt_2 \right|
\leq \varepsilon/2 + \frac{1}{\pi r} \sum_{i=1}^{k} \sum_{j=1}^{m} m_{ij} |J_j| |(\cos(\pi r a_i) - \cos(\pi r b_i))|.
\]
Choosing $r$ large enough such that
\[
\left| \frac{1}{\pi r} \sum_{i=1}^{k} \sum_{j=1}^{m} m_{ij} |J_j| |(\cos(\pi r a_i) - \cos(\pi r b_i))| \right| < \varepsilon/2,
\]
we then know that
\[
\left| \int_{[a,b]^2} \omega(t_1, t_2) \sin(\pi rt_1) \, dt_1 \, dt_2 \right| < \varepsilon.
\]
This finishes the proof. \hfill \Box

Now we recall the definition of the multilinear version of the commutator of the weighted Hardy operators. Let $m \geq 2$, $\omega : [0,1] \times [0,1]^m \to [0, \infty)$ be an integrable function, and $b_i \ (1 \leq i \leq m)$ be locally integrable functions on $\mathbb{R}^n$. We define
\[
\mathcal{H}_\omega^\delta(f)(x) := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} f_i(t_1 x) \right) \left( \prod_{i=1}^{m} (b_i(x) - b_i(t_1 x)) \right) \omega(\vec{t}) \, d\vec{t}, \quad x \in \mathbb{R}^n.
\]
In what follows, we set
\[
\mathcal{B}_m := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{m \lambda_i} \right) \omega(\vec{t}) \prod_{i=1}^{m} \log \frac{1}{t_i} \, d\vec{t}
\]
and
\[
\mathcal{C}_m := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{m \lambda_i} \right) \omega(\vec{t}) \prod_{i=1}^{m} \log 2 \, d\vec{t}.
\]
Then we have the following multilinear generalization of Theorem B.

**Theorem 3.2.** Let $1 < p < p_i < \infty, 1 < q_i < \infty, -1/p_1 = \lambda_i < 0, i = 1, \ldots, m$, such that $1/p = 1/p_1 + \cdots + 1/p_m + 1/q_1 + \cdots + 1/q_m, \lambda = \lambda_1 + \cdots + \lambda_m$. Assume further that $\omega$ is a non-negative integrable function on $[0, 1] \times \cdots \times [0, 1]$.

(i) If $\mathcal{C}_m < \infty$, then $\mathcal{H}_\omega^\delta$ is bounded from $B_{\mathcal{P}_1, \lambda_1}(\mathbb{R}^n) \times \cdots \times B_{\mathcal{P}_m, \lambda_m}(\mathbb{R}^n)$ to $\dot{B}_{\mathcal{P}, \lambda}(\mathbb{R}^n)$ for all $\vec{b} = (b_1, b_2, \ldots, b_m) \in \text{CMO}_{q_1}^{\chi}(\mathbb{R}^n) \times \cdots \times \text{CMO}_{q_m}^{\chi}(\mathbb{R}^n)$.

(ii) Assume that $\lambda_1 p_1 = \cdots = \lambda_m p_m$. In this case the condition $\mathcal{C}_m < \infty$ in (i) is also necessary.

**Remark 3.3.** It is easy to verify that condition (ii) in Theorem 3.2 is weaker than the condition ($2.6$) in Theorem 2.2.
Proof. By similarity, we only consider the case that \( m = 2 \).

We first show (i). That is, we assume \( \mathbb{C}_2 < \infty \) and show that
\[
\|\mathcal{H}_\delta^\varepsilon\|_{\mathcal{B}^p_{\lambda_1}(\mathbb{R}^n) \times \mathcal{B}^p_{\lambda_2}(\mathbb{R}^n) \to \mathcal{B}^p_{\lambda}(\mathbb{R}^n)} < \infty
\]
whenever \( \vec{b} = (b_1, b_2) \in \text{CMO}^{q_1}(\mathbb{R}^n) \times \text{CMO}^{q_2}(\mathbb{R}^n) \). By Minkowski’s inequality we have
\[
\left( \frac{1}{|B|} \int_B \| \mathcal{H}_\delta^\varepsilon(f)(x) \|^p \right)^{1/p} 
\leq \left( \frac{1}{|B|} \int_B \left( \int_0^1 \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_i(t_i x)|\omega(t_1, t_2)dt_1 dt_2 \right)^p dx \right)^{1/p} 
\leq \int_0^1 \int_0^1 \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_i(t_i x)| \right)^p dx \right)^{1/p} \omega(t_1, t_2)dt_1 dt_2
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]
where
\[
I_1 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
\[
I_2 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(t_i x) - b_i,t_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
\[
I_3 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i,B - b_i,t_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
\[
I_4 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \sum_{D(i,j)} |b_i(x) - b_i,B||b_j,B - b_j,t_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
\[
I_5 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \sum_{D(i,j)} |b_i(x) - b_i,B||b_j(t_j x) - b_j,t_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
\[
I_6 := \int_{0<t_1,t_2<1} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^2 |f_i(t_i x)| \sum_{D(i,j)} |b_i,B - b_i,t_i,B||b_j(t_j x) - b_j,t_i,B| \right)^p dx \right)^{1/p} \omega(t) dt,
\]
and
\[
D(i,j) := \{(i,j)|(1,2); (2,1)\}, \quad b_{i,B} := \frac{1}{|B|} \int_B b_i, \quad i = 1, 2.
\]

Choose \( p < s_1 < \infty, p < s_2 < \infty \) such that \( 1/s_1 = 1/p_1 + 1/q_1, 1/s_2 = 1/p_2 + 1/q_2 \). Then by Hölder’s inequality, we know that
\[
I_1 \leq \int_{0<t_1,t_2<1} \prod_{i=1}^2 \left( \frac{1}{|B|} \int_B |f_i(t_i x)|^{p_i} dx \right)^{1/p_i} \prod_{i=1}^2 \left( \frac{1}{|B|} \int_B |b_i(x) - b_i,B|^{q_i} dx \right)^{1/q_i} \omega(t) dt.
\]
It follows from \(1 + \frac{1}{p_i} + \frac{1}{q_i} \leq \frac{1}{q} \leq \frac{1}{p} \leq |C|_q \leq |C|_0 < tB \leq |B| \leq |B| \times |B|_3 = |B| \leq |i\lambda|, t_1 \leq |B| \leq \|B\| \leq |\sum_{i=1}^{\infty} \prod_{i=1}^{\infty} b_{i, B} x^i | \leq |2^{\lambda_1} \times \prod_{i=1}^{\infty} |t_i\lambda_i, |\omega(\vec{t})| \|d\vec{t}.

Similarly, we obtain

\[ I_2 \leq \int_{0 < t_1, t_2 < 1} 2 \prod_{i=1}^{\infty} \left( \frac{1}{|B|} \int_B |f_i(t_i x)|^{p_i} \, dx \right)^{1/p_i} 2 \prod_{i=1}^{\infty} \left( \frac{1}{|B|} \int_B |b_i(t_i x) - b_{i, t_i B}|^{q_i} \, dx \right)^{1/q_i} \omega(\vec{t}) \|d\vec{t}.

\]
\[
\times \left( \sum_{j=0}^{k} |b_{2,2-j} - b_{2,2-j-1}| + |b_{2,2-k-1} - b_{2,t_2}| \right) \omega(\tilde{t}) \, dt
\]

\[
\leq C|B|^{\lambda} \||b_1||_{CMO^{\alpha}} \||b_2||_{CMO^{\alpha}} \|f_1\|_{B^{\alpha_1}, \lambda_1} \|f_2\|_{B^{\alpha_2}, \lambda_2}
\times \int_{0 < t_1, t_2 < 1} \frac{2}{t_1} \int_{t_1}^{t_2} \int_{t_1}^{t_2} t_1^{\lambda_1} t_2^{\lambda_2} \omega(\tilde{t}) \log \frac{2}{t_1} \log \frac{2}{t_1} \, dt_1 \, dt_2 \, \omega(\tilde{t}) \, dt_2,
\]

where we use the fact that

\[
|b_{1,t} - b_{1,t_1}| \leq \sum_{j=0}^{k} |b_{1,2-j} - b_{1,2-j-1}| + |b_{1,2-k-1} - b_{1,t_1}|
\leq C(k + 1)||b||_{CMO^{\alpha}} \leq C \log \frac{2}{t_1} ||b||_{CMO^{\alpha}}
\]

and

\[
|b_{2,t} - b_{2,t_2}| \leq C \log \frac{2}{t_2} ||b||_{CMO^{\alpha}}
\]

We now estimate \(I_4\). Similarly, we choose \(1 < s < \infty\) such that \(1/p = 1/p_1 + 1/p_2 + 1/s\) and \(1/s = 1/q_1 + 1/q_2\). Using Minkowski’s inequality and Hölder’s inequality yields

\[
I_4 = \int_{0 < t_1, t_2 < 1} \left( \frac{1}{|B|} \int_{B} \left( \left( \prod_{i=1}^{2} |f_i(t_i,x)| \left( \sum_{D(i,j)} |b_i(x) - b_{i,B}||b_j,B - b_{j,t_1,B}| \right) \right) dx \right)^{1/p} \omega(\tilde{t}) \, dt
\leq C \int_{0 < t_1, t_2 < 1} \left( \frac{1}{|B|} \int_{B} \left( \left( \prod_{i=1}^{2} |f_i(t_i,x)| \right) \left( |b_{2,t} - b_{2,t_2}| \right) \right) dx \right)^{1/p} \omega(\tilde{t}) \, dt
\leq C \int_{0 < t_1, t_2 < 1} \left( \frac{1}{|B|} \int_{B} |f_i(t_i,x)|^{p_i} \, dx \right)^{1/p_i} \left( \frac{1}{|B|} \int_{B} |b_i(x) - b_{i,B}|^{s} \, dx \right)^{1/s} \omega(\tilde{t}) \, dt
\leq C |B|^{\lambda} \prod_{i=1}^{2} \left( \frac{1}{|B|^{1+\lambda_{p_i}} t_{i,\lambda_i}} \int_{t_i,B} |f_i(t_i,x)|^{p_i} \, dx \right)^{1/p_i} \omega(\tilde{t}) \, dt
\leq C |B|^{\lambda} \prod_{i=1}^{2} \left( \frac{1}{|B|^{1+\lambda_{p_i}} t_{i,\lambda_i}} \int_{t_i,B} |f_i(t_i,x)|^{p_i} \, dx \right)^{1/p_i} \omega(\tilde{t}) \, dt
\leq C |B|^{\lambda} \prod_{i=1}^{2} \left( \frac{1}{|B|^{1+\lambda_{p_i}} t_{i,\lambda_i}} \int_{t_i,B} |f_i(t_i,x)|^{p_i} \, dx \right)^{1/p_i} \omega(\tilde{t}) \, dt
\leq C |B|^{\lambda} \prod_{i=1}^{2} \left( \frac{1}{|B|^{1+\lambda_{p_i}} t_{i,\lambda_i}} \int_{t_i,B} |f_i(t_i,x)|^{p_i} \, dx \right)^{1/p_i} \omega(\tilde{t}) \, dt
\leq C \log \frac{2}{t_1} \log \frac{2}{t_1} \, \omega(\tilde{t}) \, dt
\]
\begin{align*}
&\leq C|B|^\lambda \|f_1\|_{B^{p_1,\lambda_1}} \|f_2\|_{B^{p_2,\lambda_2}} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \left\{ \frac{1}{|B|} \left| \int_B |b_1(x) - b_{1,B}|^p \, dx \right|^s \right\}^{1/s} \\
&\quad \times |b_{2,B} - b_{2,t_2,B}| + \left( \frac{1}{|B|} \int_B |b_2(x) - b_{2,B}|^p \, dx \right)^{1/s} |b_{1,B} - b_{1,t_1,B}| \right\} \omega(\vec{t}) \, d\vec{t}.
\end{align*}

From the estimates of \( I_1 \) and \( I_3 \), we deduce that

\begin{align*}
I_4 &\leq C|B|^\lambda \|f_1\|_{B^{p_1,\lambda_1}} \|f_2\|_{B^{p_2,\lambda_2}} \|b_1\|_{CMO^{\lambda_1}} \|b_2\|_{CMO^{\lambda_2}} \\
&\quad \times \int_0^1 \int_0^1 t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t_1,t_2) \left( 1 + \sum_{i=1}^s \log \frac{1}{t_i} \right) \, dt_1 \, dt_2.
\end{align*}

It can be deduced from the estimates of \( I_1, I_2, I_3 \) and \( I_4 \) that

\begin{align*}
I_5 &\leq C|B|^\lambda \|f_1\|_{B^{p_1,\lambda_1}} \|f_2\|_{B^{p_2,\lambda_2}} \|b_1\|_{CMO^{\lambda_1}} \|b_2\|_{CMO^{\lambda_2}} \int_{0<t_1,t_2<1} t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(\vec{t}) \, d\vec{t}.
\end{align*}

and

\begin{align*}
I_6 &\leq C|B|^\lambda \|f_1\|_{B^{p_1,\lambda_1}} \|f_2\|_{B^{p_2,\lambda_2}} \|b_1\|_{CMO^{\lambda_1}} \|b_2\|_{CMO^{\lambda_2}} \\
&\quad \times \int_0^1 \int_0^1 t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t_1,t_2) \left( 1 + \sum_{i=1}^s \log \frac{1}{t_i} \right) \, dt_1 \, dt_2.
\end{align*}

Combining the estimates of \( I_1, I_2, I_3, I_4, I_5 \) and \( I_6 \) gives

\begin{align*}
\left( \frac{1}{|B|^{1+\lambda_2}} \int_B |\mathcal{H}_B \tilde{f}(x)|^p \, dx \right)^{1/p} &\leq C|B|^\lambda \|f_1\|_{B^{p_1,\lambda_1}} \|f_2\|_{B^{p_2,\lambda_2}} \|b_1\|_{CMO^{\lambda_1}} \|b_2\|_{CMO^{\lambda_2}} \\
&\quad \times \int_0^1 \int_0^1 t_1^{n\lambda_1} t_2^{n\lambda_2} \omega(t_1,t_2) \prod_{i=1}^s \log \frac{2}{t_i} \, dt_1 \, dt_2.
\end{align*}

This proves (i).

Now we prove the necessity in (ii). Assume that

\[ \|\mathcal{H}_B \tilde{f}\|_{B^{p_1,\lambda_1}(\mathbb{R}^n) \times B^{p_2,\lambda_2}(\mathbb{R}^n) \to B^{p,\lambda}(\mathbb{R}^n)} < \infty \]

whenever \( \tilde{b} = (b_1, b_2) \in CMO^{\lambda_1}(\mathbb{R}^n) \times CMO^{\lambda_2}(\mathbb{R}^n) \). To show \( C_2 < \infty \), it suffices to prove that \( A_2 < \infty, B_2 < \infty \),

\[ D := \int_{0<t_1,t_2<1} \left( \prod_{i=1}^s t_i^{n\lambda_i} \right) \omega(t_1,t_2) \log \frac{1}{t_1} \, dt_1 \, dt_2 < \infty, \]

and

\[ E := \int_{0<t_1,t_2<1} \left( \prod_{i=1}^s t_i^{n\lambda_i} \right) \omega(t_1,t_2) \log \frac{1}{t_2} \, dt_1 \, dt_2 < \infty. \]

To prove \( B_2 < \infty \), we set \( b_1(x) := \log |x| \in BMO(\mathbb{R}^n) \subset CMO^{\lambda_1}(\mathbb{R}^n) \), and \( b_2(x) := \log |x| \in BMO(\mathbb{R}^n) \subset CMO^{\lambda_2}(\mathbb{R}^n) \). Define \( f_1 := |x|^{n\lambda_1} \) and \( f_2 := |x|^{n\lambda_2} \) if \( x \in \mathbb{R}^n \setminus \{0\} \), and \( f_1(0) = f_2(0) := 0 \). Then

\[ \|f_1\|_{B^{p_1,\lambda_1}} = \left( \frac{\omega_n}{n} \right)^{-\lambda_1} \left( \frac{1}{1 + \lambda_1 p_1} \right)^{1/p_1}, \quad \|f_2\|_{B^{p_2,\lambda_2}} = \left( \frac{\omega_n}{n} \right)^{-\lambda_2} \left( \frac{1}{1 + \lambda_2 p_2} \right)^{1/p_2}. \]
and
\[ \mathcal{H}_0^{\hat{F}}(\hat{f})(x) = |x|^{\lambda_1} |x|^{\lambda_2} \int_0^1 \int_0^1 t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \log \frac{1}{t_1} \log \frac{1}{t_2} dt_1 dt_2. \]

Since \( 1 < p < p_i < \infty, -1/p_i < \lambda_i < 0 \) and \( \lambda = \lambda_1 + \lambda_2 \) \((i = 1, 2)\), we see that, for all \( B = B(0, R) \),
\[
\left( \frac{1}{|B|^{1+\lambda p}} \int_B |\mathcal{H}_0^{\hat{F}}(\hat{f})(x)|^p dx \right)^{1/p} = \left( \frac{1}{|B|^{1+\lambda p}} \int_B |x|^{|\lambda| p} dx \right)^{1/p} \int_0^1 \int_0^1 t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \log \frac{1}{t_1} \log \frac{1}{t_2} dt_1 dt_2 = \left( \frac{\omega_p}{n} \right)^{-\lambda} \left( \frac{1}{1+\lambda p} \right)^{1/p} \int_0^1 \int_0^1 t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \log \frac{1}{t_1} \log \frac{1}{t_2} dt_1 dt_2 = \| f_1 \|_{\dot{B}^{\lambda_1,1} \times \dot{B}^{\lambda_2,2}} \| f_2 \|_{\dot{B}^{\lambda_1,1} \times \dot{B}^{\lambda_2,2}} \int_0^1 \int_0^1 t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \log \frac{1}{t_1} \log \frac{1}{t_2} dt_1 dt_2.
\]

Thus \( B_2 \leq \| \mathcal{H}_0^{\hat{F}} \|_{\dot{B}^{\lambda_1,1} \times \dot{B}^{\lambda_2,2}(\mathbb{R}^n)} < \infty \).

Since the proof for \( \mathcal{D} < \infty \) is similar to that for \( \mathcal{E} < \infty \), we only show \( \mathcal{E} < \infty \). To this end, for any \( r \in \mathbb{N} \) and \( R \in (0, +\infty) \), we choose \( b_1(x) := \chi_{[B(0, R/2)]^c}(x) \sin(\pi r |x|) \), and \( b_2(x) := \log |x| \), where \( [B(0, R/2)]^c := \mathbb{R}^n \setminus B(0, R/2) \). Obviously, we have \( \bar{b} = (b_1, b_2) \in \text{CMO}^{\lambda_1}(\mathbb{R}^n) \times \text{CMO}^{\lambda_2}(\mathbb{R}^n) \), and hence,
\[
\| \mathcal{H}_0^{\hat{F}} \|_{\dot{B}^{\lambda_1,1} \times \dot{B}^{\lambda_2,2}(\mathbb{R}^n)} < \infty.
\]

Let
\[
f_1(x) := \begin{cases} 0, & |x| \leq \frac{R}{2}, \\ |x|^{\lambda_1}, & |x| > \frac{R}{2}, \end{cases}
\]
and
\[
f_2(x) := \begin{cases} 0, & |x| \leq \frac{R}{2}, \\ |x|^{\lambda_2}, & |x| > \frac{R}{2}. \end{cases}
\]

Then, we have
\[
\mathcal{H}_0^{\hat{F}}(\hat{f})(x) = \int_{0 < t_1, t_2 < 1} \left( \prod_{i=1}^2 f_i(t_i x) \right) \left( \prod_{i=1}^2 (b_i(x) - b_i(t_i x)) \right) \omega(t) dt = |x|^{\lambda_1} \int_{0 < t_1, t_2 < 1} \int_{\mathbb{R}^n} t_1^{\lambda_1} t_2^{\lambda_2} (b_1(x) - b_1(t_1 x)) \omega(t_1, t_2) \frac{1}{t_1} dt_1 dt_2 = |x|^{\lambda_1} b_1(x) \int_{0 < t_1, t_2 < 1} \int_{\mathbb{R}^n} t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \frac{1}{t_1} dt_1 dt_2 - \eta_d,
\]
whenever \( R/2 < |x| < R \),
\[
\eta_d = |x|^{\lambda_1} \int_{0 < t_1, t_2 < 1} \int_{\mathbb{R}^n} t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) b_1(t_1 x) \frac{1}{t_1} dt_1 dt_2 = |x|^{\lambda_1} \int_{0 < t_1, t_2 < 1} \int_{\mathbb{R}^n} t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \sin(\pi r t_1 |x|) \frac{1}{t_1} dt_1 dt_2.
\]

Since \( \omega \) is integrable on \([0, 1] \times [0, 1] \) and \( B_2 < \infty \), we know that
\[
t_1^{\lambda_1} t_2^{\lambda_2} \omega(t_1, t_2) \frac{1}{t_1} dc.
\]
is integrable on \((\frac{1}{2}, 1)\) \times (\frac{1}{2}, 1). Then, it follows from Lemma 5.1 that for any \(\delta > 0\), there exists a positive constant \(C_{R, \delta}\) that depends on \(R\) and \(\delta\) such that

\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \sin(\pi r t_1) \log \frac{1}{t_2} dt_1 dt_2 < \delta/2
\]

for all \(r > C_{R, \delta}\). Now we choose \(r > \max(1/R, 1)C_{R, \delta}\). Then for any \(R/2 < |x| < R\), \(r|x| > C_{R, \delta}\), and hence

\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \sin(\pi r t_1 |x|) \log \frac{1}{t_2} dt_1 dt_2 < \delta/2
\]

which further implies that \(\eta_B < \frac{1}{2} |x|^{n_1}\). Therefore, for any \(R/2 < |x| < R\),

\[
|\mathcal{H}_\omega^B (f(x))| \geq |x|^{n_1} \left( \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2 - \frac{\delta}{2} \right).
\]

Let \(\varepsilon > 0\) be small enough and choose \(\delta > 0\) such that

\[
\delta < \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2.
\]

Then, for all balls \(B = B(0, R)\),

\[
\left( \frac{1}{|B|^{1+\lambda p}} \int_B |\mathcal{H}_\omega^B (f(x))|^p dx \right)^{1/p} \geq \left( \frac{1}{|B|^{1+\lambda p}} \int_{R/2 < |x| < R} |x|^{n_1} \left( \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2 - \frac{\delta}{2} \right)^p dx \right)^{1/p} \geq C \left( \frac{1}{|B|^{1+\lambda p}} \int_{R/2 < |x| < R} |x|^{n_1} \left( \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2 \right)^p dx \right)^{1/p}
\]

\[
greater C \left( \frac{1}{|B|^{1+\lambda p}} \int_{R/2 < |x| < R} |x|^{n_1} \left( \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2 \right)^p dx \right)^{1/p} = C \prod_{i=1}^{2} ||f_i||_{\tilde{B}^{n_1, \lambda_1}(\mathbb{R}^n)} \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2,
\]

which further implies that

\[
||\mathcal{H}_\omega^B||_{\tilde{B}^{n_1, \lambda_1} \times \tilde{B}^{n_2, \lambda_2}(\mathbb{R}^n)} \geq C \prod_{i=1}^{2} ||f_i||_{\tilde{B}^{n_1, \lambda_1}(\mathbb{R}^n)} \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} t_1^{n_1} t_2^{n_2} \omega(t_1, t_2) \log \frac{1}{t_2} dt_1 dt_2.
\]

Letting \(\varepsilon \to 0^+\) concludes \(E < \infty\).

To show that \(k_2 < \infty\), we let

\[
b_1(x) = b_2(x) := \chi_{[B(0, R/2)]^c}(x) \sin(\pi r |x|),
\]
where $R \in (0, +\infty)$ and $r \in \mathbb{N}$, and let $f_1, f_2$ be as in [3.1], [3.2], respectively. Repeating the proof for $E < \infty$, we also obtain that $A_2 < \infty$. Combining all above estimates then yields $C_2 < \infty$. This finishes the proof of the Theorem 3.2. 

We remark that Theorem 3.2 when $m = 1$ is just [9, Theorem 3.1]. In particular, when $n = 1$ and $\omega(\vec{t}) := 1/\Gamma(\alpha)(1 - t_1, \ldots, 1 - t_m)]^{m-\alpha}$, we know that

$$H_{\omega}^{\vec{b}}(f)(x) = x^{-\alpha}I_{m, \alpha, \vec{b}}^{\vec{f}}(x), \quad x > 0,$$

where

$$I_{m, \alpha, \vec{b}}^{\vec{f}}(x) := \frac{1}{\Gamma(\alpha)} \int_{0 < t_1, t_2, \ldots, t_m < x} \frac{\prod_{i=1}^{m} f_i(t_i) \prod_{i=1}^{m} (b_i(x) - b_i(t_i x))}{|x - t_1, \ldots, x - t_m]|^{m-\alpha} dt_i.$$

Then, as an immediate consequence of Theorem 3.2, we have the following corollary.

**Corollary 3.1.** Let $0 < \alpha < m$. Under the assumptions of Theorem 3.2, the operator $I_{m, \alpha, \vec{b}}^{\vec{f}}$ maps the product of central Morrey spaces $\dot{B}^{p_1, \lambda_1}(\mathbb{R}) \times \cdots \times \dot{B}^{p_m, \lambda_m}(\mathbb{R})$ to $\dot{B}^{p, \lambda}(x^{-p_{\alpha}}dx)$.

### 4. Weighted Cesàro operator of multilinear type and its commutator

In this section, we focus on the corresponding results for the adjoint operators of weighted multilinear Hardy operators.

Recall that, as the adjoint operator of the weighted Hardy operator, the weighted Cesàro operator $G_{\omega}$ is defined by

$$G_{\omega} f(x) := \int_0^1 f(x/t) t^{-n} \omega(t) dt, \quad x \in \mathbb{R}^n.$$

In particular, when $\omega \equiv 1$ and $n = 1$, $G_{\omega}$ is the classical Cesàro operator defined as

$$G f(x) := \begin{cases} \int_x^\infty f(y) dy, & x > 0, \\ - \int_{-\infty}^x f(y) dy, & x < 0. \end{cases}$$

When $n = 1$ and $\omega(t) := \frac{1}{\Gamma(\alpha)(1 - t)^{1-\alpha}}$ with $0 < \alpha < 1$, the operator $G_{\omega} f(\cdot)$ is reduced to $(\cdot)^{1-\alpha}J_{\alpha} f(\cdot)$, where $J_{\alpha}$ is a variant of Weyl integral operator defined by

$$J_{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t - x)^{1-\alpha}} \frac{dt}{t}, \quad x > 0.$$

Moreover, it is well known that the weighted Hardy operator $H_{\omega}$ and the weighted Cesàro operator $G_{\omega}$ are adjoint mutually, namely,

$$\int_{\mathbb{R}^n} g(x) H_{\omega} f(x) dx = \int_{\mathbb{R}^n} f(x) G_{\omega} g(x) dx,$$

for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 < p < \infty, 1/p + 1/q = 1$. We refer to [24] for more details.
Let the integer \( m \geq 2 \), and \( \omega : [0, 1] \times [0, 1]^m \to [0, \infty) \) be an integrable function. Let \( f_i \) be measurable complex-valued functions on \( \mathbb{R}^n \), \( 1 \leq i \leq m \). Corresponding to the weighted multilinear Hardy operators, we define the following weighted multilinear Cesàro operator:

\[
G_\omega(\vec{f})(x) := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} f_i(x/t_i)(t_i)^{-n} \right) \omega(t) \, dt, \quad x \in \mathbb{R}^n.
\]

Notice that in general \( H \) and \( G_\omega \) do not obey the commutative rule (4.1).

We also point out that, when \( n = 1 \) and

\[
\omega(t) := \frac{1}{\Gamma(\alpha)(\frac{1}{t_1} - 1, \ldots, \frac{1}{t_m} - 1)^{m-\alpha}},
\]

the operator

\[
G_\omega(\vec{f})(x) = x^{m-\alpha}J_{\alpha}^m \vec{f}(x), \quad x > 0,
\]

where

\[
J_{\alpha}^m \vec{f}(x) := \frac{1}{\Gamma(\alpha)} \int_{x < t_1, t_2, \ldots, t_m < \infty} \left( \prod_{i=1}^{m} f_i(x_i) \right) \left| (t_1-x, \ldots, t_m-x) \right|^{m-\alpha} \frac{dt}{t}.
\]

Similar to the argument used in Section 2, we have the following conclusions.

**Theorem 4.1.** If \( f_i \in L^{p_i}(\mathbb{R}^n) \), \( 1 < p, p_i < \infty \), \( i = 1, \ldots, m \), and \( 1/p = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \), then \( G_\omega \) is bounded from \( L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) if and only if

\[
\mathcal{F} := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} \frac{dt}{t_i} \right) \omega(t) \, dt < \infty.
\]

Moreover,

\[
\| G_\omega \|_{L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = \mathcal{F}.
\]

We can also deduce from Theorem 4.1 that

**Corollary 4.1.** Let \( 0 < \alpha < m \). Under the assumptions of Theorem 4.1, we have \( J_{\alpha}^m \) maps the product of weighted Lebesgue spaces \( L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_m}(\mathbb{R}) \) to \( L^{p_n}(\mathbb{R}^n) \) with norm

\[
\frac{1}{\Gamma(\alpha)} \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-(1-1/p_i)} \right) \left| (\frac{1}{t_1} - 1, \ldots, \frac{1}{t_m} - 1) \right|^{m-\alpha} \frac{dt}{t}.
\]

Next, we define the commutator of weighted Cesàro operators of multilinear type as

\[
G_{\alpha, \vec{b}}^\vec{f}(x) := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} f_i(x/t_i)(t_i)^{-n} \right) \left( \prod_{i=1}^{m} \left( b_i(x) - b_i\left(\frac{x}{t_i}\right) \right) \right) \omega(t) \, dt, \quad x \in \mathbb{R}^n.
\]

In particular, we know that

\[
G_{\alpha, \vec{b}}^\vec{f}(x) = x^{m-\alpha}J_{\alpha, \vec{b}}^m \vec{f}(x), \quad x > 0,
\]

where

\[
J_{\alpha, \vec{b}}^m \vec{f}(x) := \frac{1}{\Gamma(\alpha)} \int_{x < t_1, t_2, \ldots, t_m < \infty} \left( \prod_{i=1}^{m} f_i(x_i) \right) \left( \prod_{i=1}^{m} (b_i(x) - b_i(x/t_i)) \right) \frac{dt}{t}.
\]

Let \( m \in \mathbb{N} \) and \( m \geq 2 \). Define

\[
F_m := \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-n} \right) \omega(t) \prod_{i=1}^{m} \log \frac{2}{t_i} \, dt.
\]
Similar to the arguments in Section 3, we have the following conclusion.

**Theorem 4.2.** If \( f_i \in L^{p_i} (\mathbb{R}^n) \), \( 1 < p < p_i < \infty, 1 < q_i < \infty, -1/p_i < \lambda_i < 0, \ i = 1, \ldots, m, \) and \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_m}, \lambda = \lambda_1 + \cdots + \lambda_m. \)

(i) If \( F_m < \infty, \) then \( G^p_{\vec{b}} \) is bounded from \( \dot{B}^{\lambda_1}_p (\mathbb{R}^n) \times \cdots \times \dot{B}^{\lambda_m}_p (\mathbb{R}^n) \) to \( \dot{B}^{\lambda}_p (\mathbb{R}^n) \), for all \( \vec{b} = (b_1, b_2, \ldots, b_m) \in \text{CMO}^{q_1} (\mathbb{R}^n) \times \cdots \times \text{CMO}^{q_m} (\mathbb{R}^n). \)

(ii) Assume that \( \lambda_1 p_1 = \cdots = \lambda_m p_m. \) In this case the condition \( F_m < \infty \) in (i) is also necessary.

As an immediate corollary, we have the following consequence.

**Corollary 4.2.** Let \( 0 < \alpha < m. \) Under the assumptions of Theorem 4.2 we have \( J_{a,b}^{m} \), maps the product of weighted Lebesgue spaces \( \dot{B}^{\lambda_1}_p (\mathbb{R}^n) \times \cdots \times \dot{B}^{\lambda_m}_p (\mathbb{R}^n) \) to \( \dot{B}^{\lambda}_p (x^{pm-p\alpha} dx) \)

Finally, we give some further comments on weighted product Hardy operators. Let \( \omega : [0,1] \to [0,\infty) \) be an integrable function. Let \( f(x_1, x_2) \) be measurable complex-valued functions on \( \mathbb{R}^n \times \mathbb{R}^m. \) The **weighted product Hardy operator** is defined as

\[
\mathbb{H}_\omega f(x_1, x_2) := \int_{0 < t_1, t_2 < 1} f(t_1 x_1, t_2 x_2) \omega(t_1, t_2) \, dt_1 dt_2, \quad (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m.
\]

If \( \omega \equiv 1 \) and \( n, m = 1, \) then \( \mathbb{H}_\omega f \) is reduced to the two dimensional Hardy operator \( \mathbb{H} \) defined by

\[
\mathbb{H} f(x_1, x_2) := \frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} f(t_1, t_1) \, dt_1 dt_2, \quad x_1, x_2 \neq 0,
\]

which is first introduced by Sawyer [21]. The sharp estimates for weighted product Hardy operators and their commutators on Lebesgue spaces will be interesting questions.

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