New examples of systems of the Kowalevski type

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Abstract

A new examples of integrable dynamical systems are constructed. An integration procedure leading to genus two theta-functions is presented. It is based on a recent notion of discriminantly separable polynomials. They have appeared in a recent reconsideration of the celebrated Kowalevski top, and their role here is analogue to the situation with the classical Kowalevski integration procedure.
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1 Introduction

A notion of discriminantly separable polynomials has been introduced recently by one of the authors in [4]. It has been related there to a new view on the classical integration procedure of Kowalevski of her celebrated top (see the original work [7], [8], a classical presentation in [6], and for modern approach, see [2], [1], [5]). Following the way the Kowalevski integration procedure has been coded in [4], we construct here a new class of integrable dynamical systems of Kowalevski type. Their complete integration procedure parallels the classical one, and leads to the formulae in the genus two theta-functions.

Let us recall the definition of discriminantly separable polynomials from [4], here for polynomial \( F(x_1, x_2, s) \) of the second degree in each of three variables. Polynomial \( F(x_1, x_2, s) \) is discriminantly separable if there exist polynomials \( P_1, P_2, J \) of one variable each of degree not greater than four, such that
\[
D_{x_1}F(x_2, s) = P_2(x_2)J(s)
\]
\[
D_{x_2}F(x_1, s) = P_1(x_1)J(s)
\]
\[
D_sF(x_1, x_2) = P_1(x_1)P_2(x_2),
\]
where \( D_sF \) denotes the discriminant of \( F \) understood as a polynomial in \( s \). The discriminant is a polynomial in the other two variables. When polynomials \( P_1, P_2 \) and \( J \) coincide we say that \( F \) is strongly discriminantly separable. Here we distinguish lemma stated in [4] we will refer on later in the text.

Lemma 1 For an arbitrary discriminately separable polynomial \( F(x_3, x_1, x_2) \) of the second degree in each of the variables \( x_3, x_1, x_2 \), its differential is separable on the surface \( F(x_3, x_1, x_2) = 0 \):
\[
\frac{dF}{\sqrt{f_3(x_3)f_1(x_1)f_2(x_2)}} = \frac{dx_3}{\sqrt{f_3(x_3)}} + \frac{dx_1}{\sqrt{f_1(x_1)}} + \frac{dx_2}{\sqrt{f_2(x_2)}}.
\]

Recall now the Kowalevski fundamental equation:
\[
Q(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2) = 0 \tag{1}
\]
where
\[
R(x_1, x_2) = -x_1^2 x_2^2 + 6l_1 x_1 x_2 + 2lc(x_1 + x_2) + c^2 - k^2
\]
\[
R_1(x_1, x_2) = -6l_1 x_1 x_2 - (c^2 - k^2)(x_1 + x_2)^2 - 4lc x_1 x_2(x_1 + x_2) + 6l_1(c^2 - k^2) - 4c^2 l^2.
\]
\( Q(s, x_1, x_2) \) introduced with (1) as a polynomial in three variables degree two in each of them satisfies
\[
D_s(Q)(x_1, x_2) = 4P(x_1)P(x_2)
\]
\[
D_{x_1}(Q)(s, x_2) = -8J(s)P(x_2), D_{x_2}(Q)(s, x_1) = -8J(s)P(x_1)
\]
with
\[ P(x_i) = -x_i^4 + 6l_1 x_i^2 + 4l_2 x_i + c^2 - k^2, \quad i = 1, 2 \]
\[ J(s) = s^3 + 3l_1 s^2 + s(c^2 - k^2) + 3l_2 (c^2 - k^2) - 2l^2 c^2. \]

Finally, notice that equations of Kowalevski’s top in variables \( x_i = p \pm iq, \quad c_i = x_i^2 + c(\gamma_1 \pm \gamma_2), \quad i = 1, 2 \) which she introduces, may be rewritten in the form

\[
\begin{align*}
2 \dot{x}_1 &= -i(rx_1 + c\gamma_3) \\
2 \dot{x}_2 &= i(rx_2 + c\gamma_3) \\
\dot{e}_1 &= -ire_1 \\
\dot{e}_2 &= ire_2
\end{align*}
\tag{2}
\]

with two additional differential equations for \( \dot{r} \) and \( \dot{\gamma}_3 \). If we denote

\[
\begin{align*}
f_1 &= rx_1 + c\gamma_3, \\
f_2 &= rx_2 + c\gamma_3,
\end{align*}
\]

one can easily check that functions \( f_1, f_2 \) have following property

\[
\begin{align*}
f_1^2 &= P(x_1) + e_1(x_1 - x_2)^2 \\
f_2^2 &= P(x_2) + e_2(x_1 - x_2)^2.
\end{align*}
\tag{3}
\]

Generalization of (2) and (3) represent a base for systems of Kowalevski type we are going to introduce in next section.

\section{Subclass of systems of the Kowalevski type}

We will start with a modal example.

Let us consider the next system of ordinary differential equations in variables \( x_1, x_2, e_1, e_2, r, \gamma_3 \) with constant parameter \( g_2 \) and condition \( p \neq 0 \):

\[
\begin{align*}
\dot{p} &= pqr \\
\dot{q} &= -\frac{1}{2} \left( (p^2 - q^2)r + \gamma_3 \right) \\
\dot{r} &= -\frac{1}{2} \frac{pq - q(p^2 + q^2) + q\gamma_1 - p\gamma_2}{p^2} \\
\dot{\gamma}_1 &= (p^2 + q^2)qr + 2p\gamma_2 - q\gamma_3 \\
\dot{\gamma}_2 &= -2pq^2r + p\gamma_3 - pr\gamma_1 \\
\dot{\gamma}_3 &= \frac{g_2 pq + 4q(p^2 + q^2)^2 + 4q\gamma_1(3p^2 - q^2) + 4p\gamma_2(p^2 - 3q^2)}{8p^2}.
\end{align*}
\tag{4}
\]

Before we start with an analysis of the first integrals of the system (4), let us consider the existence of an invariant measure.
Lemma 2  The system \( \text{(4)} \) for \( p \neq 0 \) possesses an invariant measure with density
\[
\rho = \frac{1}{4p^2}. \tag{5}
\]

Proof. Rewrite system of equations \( \text{(4)} \) in the form:
\[
\begin{align*}
\frac{dp}{X_1} &= \frac{dq}{X_2} = \frac{dr}{X_3} = \frac{d\gamma_1}{X_4} = \frac{d\gamma_2}{X_5} = \frac{d\gamma_3}{X_6} = dt
\end{align*}
\]
where
\[
\begin{align*}
X_1 &= pqr \\
X_2 &= -1 \left( \frac{1}{2} \left( (p^2 - q^2)r + \gamma_3 \right) \right) \\
X_3 &= -1 \frac{1}{2} \left( (p^2 - q^2)r + \gamma_3 \right) \\
X_4 &= \frac{(p^2 - q^2)}{p^2}q + 2p\gamma_2 - q\gamma_3 \\
X_5 &= -2p^2r + p\gamma_3 - pr\gamma_1 \\
X_6 &= \frac{g_2pq + 4q(p^2 - q^2)^2 + 4q\gamma_1(3p^2 - q^2) + 4p\gamma_2(p^2 - 3q^2)}{8p^2}.
\end{align*}
\]

Divergence of \( \mathbf{X} = (X_1, X_2, X_3, X_4, X_5, X_6) \) is nonzero:
\[
\frac{\partial X_1}{\partial p} + \frac{\partial X_2}{\partial q} + \frac{\partial X_3}{\partial r} + \frac{\partial X_4}{\partial \gamma_1} + \frac{\partial X_5}{\partial \gamma_2} + \frac{\partial X_6}{\partial \gamma_3} = 2qr.
\]

Simple check shows that density function \( \rho \) such that
\[
\frac{\partial \rho X_1}{\partial p} + \frac{\partial \rho X_2}{\partial q} + \frac{\partial \rho X_3}{\partial r} + \frac{\partial \rho X_4}{\partial \gamma_1} + \frac{\partial \rho X_5}{\partial \gamma_2} + \frac{\partial \rho X_6}{\partial \gamma_3} = 0
\]
is given by \( \text{(5)} \). \( \square \)

Now, we are going to focus on the structure of the first integrals of the system \( \text{(4)} \). In order to put this question in a wider context, let us first make a change of variables:
\[
\begin{align*}
x_1 &= p + uq \\
x_2 &= p - uq \\
e_1 &= x_1^2 + \gamma_1 + v\gamma_2 \\
e_2 &= x_2^2 + \gamma_1 - v\gamma_2.
\end{align*} \tag{6}
\]

The system \( \text{(4)} \) after change \( \text{(6)} \) becomes
\[ \begin{align*}
\dot{x}_1 &= -\frac{1}{2}(x_1^2 r + \gamma_3) \\
\dot{x}_2 &= \frac{1}{2}(x_2^2 r + \gamma_3) \\
\dot{e}_1 &= -v(x_1 + x_2)re_1 \\
\dot{e}_2 &= v(x_1 + x_2)re_2 \\
\dot{r} &= \frac{v}{2} \left( \frac{x_1^2 - x_2^2 + 2e_2x_1 - 2e_1x_2}{(x_1 + x_2)^2} \right) \\
\dot{\gamma}_3 &= -\frac{v}{8} \left( \frac{g_2(x_1^2 - x_2^2) + 8e_2x_1^3 - 8e_1x_2^3}{(x_1 + x_2)^2} \right).
\end{align*} \tag{7} \]

Now, let’s make assumptions for a subclass of systems of ordinary differential equations that will also include our modal example. After introducing such systems and establish relations that will hold for them in Theorem 1 we will return to (7) and show how one can apply Kowalevski’s procedure from [7] on modal example and in the same way on a whole class of systems we are going to introduce.

Suppose, that a given system in variables \( x_1, x_2, e_1, e_2, r, \gamma_3 \), after some transformations reduces to

\[ \begin{align*}
2\dot{x}_1 &= -if_1 \\
2\dot{x}_2 &= if_2 \\
\dot{e}_1 &= -me_1 \\
\dot{e}_2 &= me_2 \tag{8} \\
\end{align*} \]

where

\[ \begin{align*}
f_1^2 &= P(x_1) + e_1A(x_1, x_2) \\
f_2^2 &= P(x_2) + e_2A(x_1, x_2). \tag{9} \end{align*} \]

Here \( f_i, i = 1, 2 \) and \( m \) represent functions of system’s variables. Notice here that all systems of this type will have the first integral

\[ e_1e_2 = k^2. \]

Suppose additionally, that the first integrals of the initial system reduce to a relation

\[ P(x_2)e_1 + P(x_1)e_2 = C(x_1, x_2) - e_1e_2A(x_1, x_2) \tag{10} \]

with \( A(x_1, x_2), C(x_1, x_2) \) polynomials in two variables and \( P(x_i) \) in one.

Systems satisfying above assumptions \[8], \[9] and \[10] we will call systems of Kowalevski type.

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Theorem 1

For a system which reduces to (8), (9), (10) with functions $f_i$ introduced in (12), and at least one of conditions $n_1 \neq n_2$ or $m_2 \neq n_2$ is valid, relations in the form (11) are satisfied for expressions:

$$
p_1 = \frac{Ax_1^{2n_1}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2}, \quad p_2 = \frac{Ax_2^{2n_2}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2},
$$

$$
q_1 = \frac{Ax_1^{n_1+m_1}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2}, \quad q_2 = \frac{Ax_2^{n_2+m_2}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2},
$$

$$
r_1 = \frac{Ax_1^{2m_1}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2}, \quad r_2 = \frac{Ax_2^{2m_2}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2},
$$

$$
E_i = \frac{x_2^{m_2} P(x_1) + x_1^{m_1} P(x_2) \pm B(x_1, x_2)x_1^{m_1} x_2^{m_2}}{(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1})^2}, \quad i = 1, 2
$$

$$
F_i = \frac{E_i(x_1^{m_1} x_2^{n_2} - x_2^{m_2} x_1^{n_1}) + x_1^{2m_1} P(x_2) - x_2^{2n_2} P(x_1)}{2x_1^{m_1} x_2^{n_2} (x_1^{m_1} x_2^{n_2} - x_1^{m_2} x_2^{n_1})}, \quad i = 1, 2
$$

$$
G_i = \frac{E_i x_1^{m_1} x_2^{n_2} (x_1^{m_1} x_2^{n_2} - x_1^{m_2} x_2^{n_1}) + x_1^{m_1+n_1} P(x_2) - x_2^{m_2+n_2} P(x_1)}{x_1^{m_1} x_2^{n_2} (x_1^{m_1} x_2^{n_2} - x_1^{m_2} x_2^{n_1})}, \quad i = 1, 2.
$$

Here by $B(x_1, x_2)$ we denoted

$$
B^2(x_1, x_2) = 4A(x_1, x_2)C(x_1, x_2) + 4P(x_1)P(x_2).
$$

Proof. Replacing (11) into condition (10) with $f_i = x_i^{m_i} \cdot r + x_i^{n_i} \cdot \gamma_3$, we get

$$
r^2 x_1^{2m_1} + 2r \gamma_3 x_1^{m_1+n_1} + \gamma_3^2 x_1^{2n_1} = P(x_1) + e_i A(x_1, x_2).
$$

Collecting coefficients with $e_i$ we obtain system

$$
p_2 x_1^{2m_1} - 2q_2 x_1^{m_1+n_1} + r_2 x_1^{2n_1} = A(x_1, x_2)
$$

$$
p_1 x_1^{2m_1} - 2q_1 x_1^{m_1+n_1} + r_1 x_1^{2n_1} = 0
$$

$$
E x_1^{m_1} + 2F x_1^{m_1+n_1} + G x_1^{2n_1} = P(x_1)
$$

$$
p_1 x_2^{2m_2} - 2q_1 x_2^{m_2+n_2} + r_1 x_2^{2n_2} = A(x_1, x_2)
$$

$$
p_2 x_2^{2m_2} - 2q_2 x_2^{m_2+n_2} + r_2 x_2^{2n_2} = 0
$$

$$
E x_2^{m_2} + 2F x_2^{m_2+n_2} + G x_2^{2n_2} = P(x_2)
$$

(13)
with solutions:

\[ p_1 = \frac{A(x_1, x_2)x_1^{2n_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2} \]
\[ p_2 = \frac{A(x_1, x_2)x_2^{2n_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2} \]
\[ r_1 = \frac{A(x_1, x_2)x_1^{m_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2} \]
\[ r_2 = \frac{A(x_1, x_2)x_2^{m_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2} \]

\[ F = \frac{1}{2}(x_1^{2n_1} - x_2^{2n_2})E + x_1^{2n_1}P(x_2) - x_2^{2n_2}P(x_1) \]
\[ G = -x_2^{2n_2}P(x_1) + x_1^{m_1+n_1}P(x_2) \]

The second assumption is that the relation

\[(E + p_2e_1 + p_1e_2)(G + r_2e_1 + r_1e_2) - (F - q_2e_1 - q_1e_2)^2 = 0 \] (14)

is in the form (10). According to (10), the coefficients of \( e_1^2 \) should vanish, so we get:

\[ q_1 = \frac{A(x_1, x_2)x_1^{n_1 + m_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2} \]
\[ q_2 = \frac{A(x_1, x_2)x_1^{n_2 + m_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2}. \]

Replacing these results into (14) it becomes

\[ \frac{A}{(x_1^{m_1}x_2^{n_2} - x_1^{m_2}x_2^{n_1})^2} (Ae_1 e_2 + P(x_2)e_1 + P(x_1)e_2 + \varphi(E)) = 0 \]

with \( \varphi(E) \), a quadratic function of \( E \)

\[ \varphi(E) = -\frac{A}{4} \left( \frac{x_1^{m_1-n_1} - x_2^{m_2-n_2}}{2} \right)^2 + E \frac{P(x_1)}{x_1^{2n_1}} + \frac{P(x_2)}{x_2^{2n_2}} - \frac{(P(x_1)x_2^{2n_2} - P(x_2)x_1^{2n_1})^2}{4x_1^{2n_1}x_2^{2n_2}(x_1^{m_1}x_2^{n_2} - x_1^{m_2}x_2^{n_1})^2}. \]

Finally, solving the quadratic equation

\[ \varphi(E) = -C \cdot \frac{A}{(x_1^{m_1}x_2^{n_2} - x_1^{m_2}x_2^{n_1})^2}, \]

we get the solutions

\[ E_i = \frac{x_2^{2n_2}P(x_1) + x_1^{2n_1}P(x_2) \pm B(x_1, x_2)x_1^{n_1}x_2^{n_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2}, \quad i = 1, 2. \] (15)
Remark 1 The discriminant of
\[ F(x_1, x_2, s) = A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2) \]  
(16)
as a polynomial in \( s \) is factorizable
\[ D_s F(x_1, x_2) = B^2(x_1, x_2) - 4A(x_1, x_2)C(x_1, x_2) = 4P(x_1)P(x_2). \]

If we choose \( A, B, C \) to be coefficients of discriminantly separable polynomial of degree two in each variable then integration of the systems which satisfy assumption of Theorem 1, will be performed following Kowalevski’s procedure in terms of theta-function of genus two.

3 Examples

Our modal example \([6]\) after change of variables \([6]\) fits into introduced subclass of the systems of the Kowalevski type as a system that reduces to \([8], [9], [10]\) with
\[ f_1 = rx_1^2 + \gamma_3, \quad f_2 = rx_2^2 + \gamma_3 \]  
(17)
and
\[ A(x_1, x_2) = (x_1 - x_2)^2, \]
\[ B(x_1, x_2) = -2x_1x_2(x_1 + x_2) + \frac{g_2}{2}(x_1 + x_2) + g_3, \]
\[ C(x_1, x_2) = x_1^2x_2^2 + \frac{g_2}{2}x_1x_2 + g_3(x_1 + x_2) + \frac{g_2^2}{16}, \]
\[ P(x) = 2x^3 - \frac{g_2}{2}x - \frac{g_3}{2}. \]  
(18)
Notice here that
\[ F(x_1, x_2, s) = A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2) \]
is strongly discriminately separable polynomial with the discriminants
\[ D_s F(x_1, x_2) = B^2(x_1, x_2) - 4A(x_1, x_2)C(x_1, x_2) = 4P(x_1)P(x_2) \]
\[ D_x F(x_2, s) = 4P(x_2)P(s) \]
\[ D_{x_2} F(x_1, s) = 4P(x_1)P(s). \]

With previous assumptions, from Theorem \([11]\) with \( m_1 = m_2 = 2, n_1 = 0 = n_2 \) follows that next relations are satisfied:
\[ r^2 = \frac{2}{x_1 + x_2} + \frac{e_1}{(x_1 + x_2)^2} + \frac{e_2}{(x_1 + x_2)^2} \]
\[ r^2 \gamma_3 = \frac{4x_1 x_2 - g_2}{4(x_1 + x_2)} - \frac{x_2^3 e_1}{(x_1 + x_2)^2} - \frac{x_1^3 e_2}{(x_1 + x_2)^2} + \frac{g_3}{2} \]  
\[ \gamma_3^2 = -\frac{x_1 x_2 g_2}{2(x_1 + x_2)} + \frac{x_2^3 e_1}{(x_1 + x_2)^2} + \frac{x_1^3 e_2}{(x_1 + x_2)^2} + g_3 \]  
\[ e_1 \cdot e_2 = k^2. \]

Notice here that singular hyperplane
\[ \alpha : p = 0 \]

or after a change of variables \[ \tilde{\alpha} : x_1 + x_2 = 0 \]
divides \( \mathbb{R}^6 \) into two half-spaces, which are simply connected. Denote those half-spaces with \( H^+ \) for \( p > 0 \) and \( H^- \) for \( p < 0 \).

From the Jacobi theorem, and using Lemma 2, we come to the following

**Proposition 1** The system of equations (7) is completely integrable on two invariant simply-connected sets \( H^+ \) and \( H^- \), since it has the first integrals and invariant relations (19), and an invariant measure with density \( \mu = \frac{1}{(x_1 + x_2)^2} \).

In extension we will show how this subclass of Kowalevski type systems fits into Kowalevski’s transforming procedure, (see [7]).

Multiplying the first and the third relation from (19) and deducting square of the second one obtains relation in the form (10):

\[ \frac{(x_1 - x_2)^2 e_1 e_2 + P(x_2)e_1 + P(x_1)e_2 - C(x_1, x_2)}{(x_1 + x_2)^2} = 0, \]  
\[ (\sqrt{e_1} \sqrt{P(x_2)} \pm \sqrt{e_2} \sqrt{P(x_1)})^2 = -k^2(x_1 - x_2)^2 + C(x_1, x_2) \pm 2\sqrt{P(x_1)P(x_2)}k \]

The last relations lead to

\[ \left( \frac{\sqrt{e_1} \sqrt{P(x_2)}}{x_1 - x_2} + \frac{\sqrt{e_2} \sqrt{P(x_1)}}{x_1 - x_2} \right)^2 = (s_1 - k)(s_2 + k) \]  
\[ \left( \frac{\sqrt{e_1} \sqrt{P(x_2)}}{x_1 - x_2} - \frac{\sqrt{e_2} \sqrt{P(x_1)}}{x_1 - x_2} \right)^2 = (s_1 + k)(s_2 - k) \]
where $s_1, s_2$ are the solutions of the quadratic equation

$$F(x_1, x_2, s) = A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2) = 0,$$

with $A, B, C$ introduced in (18).

Notice also that relations (21) and (22) lead to a morphism between two two-valued Buchstaber-Novikov groups, as it has been explained in [4] (for basic notions and examples of the theory of $n$-valued Buchstaber-Novikov groups see [3]).

From (21) and (22) we get

$$2\sqrt{e_1}\frac{\sqrt{P(x_2)}}{x_1 - x_2} = \sqrt{(s_1 - k)(s_2 + k) + \sqrt{(s_1 + k)(s_2 - k)}}$$

$$2\sqrt{e_2}\frac{\sqrt{P(x_1)}}{x_1 - x_2} = \sqrt{(s_1 - k)(s_2 + k) - \sqrt{(s_1 + k)(s_2 - k)}},$$

Solutions $s_i$ of the quadratic equation $F(x_1, x_2, s)$ are

$$s_1 = \frac{-B(x_1, x_2) - \sqrt{B^2(x_1, x_2) - 4(x_1 - x_2)^2C(x_1, x_2)}}{2(x_1 - x_2)^2}$$

$$s_2 = \frac{-B(x_1, x_2) + \sqrt{B^2(x_1, x_2) - 4(x_1 - x_2)^2C(x_1, x_2)}}{2(x_1 - x_2)^2}.$$ 

Using the Viète formulae and the discriminant separability condition we get

$$s_1 + s_2 = -\frac{B(x_1, x_2)}{(x_1 - x_2)^2}$$

$$s_2 - s_1 = \frac{\sqrt{4P(x_1)P(x_2)}}{(x_1 - x_2)^2}.$$ 

Since

$$\dot{x}_1 = -\frac{1}{2}(x_1^2r + \gamma_3),$$

then

$$-4\dot{x}_1^2 = x_1^2r^2 + 2x_1^2r\gamma_3 + \gamma_3^2.$$ 

Substituting $r^2, r\gamma_3, \gamma_3^2$ from (19) into (26) we obtain

$$-4\dot{x}_1^2 = 2x_1^3 - \frac{9a}{2}x_1 - \frac{9a}{2} + (x_1 - x_2)^2e_1$$

$$= P(x_1) + (x_1 - x_2)^2e_1.$$ 

The same is

$$-4\dot{x}_2^2 = P(x_2) + (x_1 - x_2)^2e_2.$$
Using (24) and (25) finally we get

\[ -4 \dot{x}_1^2 = P(x_1) + (x_1 - x_2)^2 e_1 \]

\[ = \frac{(x_1 - x_2)^4}{4P(x_2)} [(s_1 - s_2)^2 + (\sqrt{(s_1 - k)(s_2 + k)} + (s_1 + k)(s_2 - k))^2] \]

\[ = \frac{P(x_1)}{(s_1 - s_2)^2} [(s_1 - k)(s_1 + k) + (s_2 - k)(s_2 + k)]^2 \]

and

\[ -4 \dot{x}_2^2 = \frac{P(x_2)}{(s_1 - s_2)^2} [(s_1 - k)(s_1 + k) - (s_2 - k)(s_2 + k)]^2. \]

From the last two equations, it follows

\[ \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} = \frac{s_1}{s_1 - s_2} \sqrt{(s_1 - k)(s_1 + k)} dt \]

\[ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} = \frac{s_2}{s_1 - s_2} \sqrt{(s_2 - k)(s_2 + k)} dt. \]

Now we will make use of a Lemma stated in Introduction.

The strong discriminant separability of polynomial \( F(x_1, x_2, s) \) implies

\[ \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} = \frac{ds_1}{\sqrt{P(s_1)}} \] \hspace{1cm} (28)

\[ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} = -\frac{ds_2}{\sqrt{P(s_2)}}. \]

This way, we can finally conclude

**Proposition 2** The system of differential equations defined by (7) is integrated through the solutions of the system

\[ \frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} = 0 \]

\[ \frac{1}{s_1 ds_1} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} = i dt, \] \hspace{1cm} (29)

where

\[ \Phi(s) = P(s)(s - k)(s + k). \]

It is linearized on the Jacobian of the curve \( \Gamma : y^2 = \Phi(s) \).

As a result of Theorem there is another system, this time with two constant parameters \( g_2 \) and \( g_3 \) which also reduces to (3), (4), (10) and satisfies relations
in the form (19):

\[ r^2 = \frac{2(x_1 + x_2)(x_1^2 + x_2^2 - \frac{2}{r^2}) - 2g_3}{(x_1^2 - x_2^2)^2} + \frac{e_1}{(x_1 + x_2)^2} + \frac{e_2}{(x_1 + x_2)^2} \]

\[ r\gamma^3 = \frac{(x_1 + x_2)^3g_2 + 4(x_1^2 + x_2^2)g_3 - 4x_1x_2(x_1 + x_2)^3}{4(x_1^2 - x_2^2)^2} - \frac{x_2^2e_1}{(x_1 + x_2)^2} - \frac{x_1^2e_2}{(x_1 + x_2)^2} \]

\[ \gamma^3 = \frac{-x_1x_2(x_1 + x_2)(x_1^2 + x_2^2)g_2 - (x_1^2 + x_2^2)^3g_3 + 8x_1^2x_2^3(x_1 + x_2)}{2(x_1^2 - x_2^2)^2} - \frac{x_2^2e_1}{(x_1 + x_2)^2} - \frac{x_1^2e_2}{(x_1 + x_2)^2}. \]

Differentiating first and third of previous relations, with \( \dot{x}_i \) and \( \dot{\gamma}_3 \) given by \( \mathbf{8}, \mathbf{17} \), we get expressions for \( \dot{r} \) and \( \dot{\gamma}_3 \). Replacing these values into differentiated second relation we get so far unknown function \( m \) and finally we get system of equations:

\[ \dot{x}_1 = \frac{i}{2}(x_1^2r + \gamma_3) \]
\[ \dot{x}_2 = \frac{i}{2}(x_2^2r + \gamma_3) \]
\[ \dot{e}_1 = -me_1 \]
\[ \dot{e}_2 = me_2 \]

\[ \dot{r} = -\frac{i\epsilon_1(r(x_2 - x_1) - im)}{2(x_1 + x_2)^2r} - \frac{i\epsilon_2(r(x_2 - x_1) + im)}{2(x_1 + x_2)^2r} + \frac{i\gamma_3(x_1 + x_2)}{4(x_2 - x_1)^4(x_1 + x_1)r} + \frac{2i((x_1^2 + x_2^2)^2)}{r(x_1 + x_2)^2(x_2 - x_1)^3} \]

\[ \dot{\gamma}_3 = \frac{i((x_1 + x_2)(x_1 + x_2) + m\epsilon_2 + 2\gamma_3)x_2^3e_1}{\gamma_3(x_1 + x_2)^2} - \frac{i((x_1 + x_2)(x_1 + x_2) + m\epsilon_1 + 2\gamma_3)x_1^3e_2}{\gamma_3(x_1 + x_2)^2} \]

\[ - \frac{i(2x_1x_2 + \gamma_3)(x_1 - x_2)(x_1 + x_2)x_1x_2g_3}{\gamma_3(x_1 + x_2)^2(x_1 - x_2)^3} \]
\[ - \frac{i((x_1^2 + x_2^2 + x_3^3)(x_1 + x_2)^2g_3 + 2x_1^2x_2^2(x_1 + x_2)^2r)g_2}{8\gamma_3(x_1 + x_2)(x_1 - x_2)^3} \]
\[ + \frac{i(3(x_1^2 + x_2^2)(x_1 + x_2)^2 - 2x_1x_2\gamma_3 + 4x_1^2x_2^2)}{\gamma_3(x_1 + x_2)(x_1 - x_2)^3}. \]
where
\[
m = (x_1 + x_2)r_1 + \frac{1}{4((x_1 + x_2)^2(x_1 - x_2)^3((x_2^2r + \gamma_3)^2 + (x_1^2r + \gamma_3)^2)}
\]
\[
\cdot \left[ ((x_1 + x_2)^2(2x_1^2 x_2^3(x_1 + x_2)^2r^3 + \gamma_3(6x_1^2 x_2(1 + x_2) + 4x_1 x_2(x_1^2 + x_2^2) + 5x_2^3 - 3x_2^2 + 2x_1^2)r^2 + 2\gamma_3^2(5x_1^2 + x_2^2) + 2x_1 x_2)r + 8\gamma_3^3 g_2 + (8x_1^2 x_2^2(x_1^2 + x_2^2) - 3x_1 x_2^2 + 3x_1^2 x_2 + 6x_1 x_2^2 + 2x_1^2 x_2 - x_1 x_2^2 - x_1 x_2^2 + 2x_1^2 x_2 + 3x_1^2 x_2 + x_1^2 + 2x_1^2 x_2 + 3x_1^2 x_2^2 + x_1^2 x_2^2 - x_1 x_2^2 - x_1 x_2^2) - 4(x_1 + x_2)^2(8r^3 x_1 x_2^3 + 2x_1 x_2^3 \gamma_3(-2x_1 x_2 + 5x_1 x_2^2 - 2x_2 + 2x_2^3 + 5x_1^2 + 4x_2^3)r^2 + \gamma_3^2(2x_1^3 + 4x_2^3 x_2 + x_1^2 + x_2^2 - 2x_1^2 x_2 + 4x_1 x_2^2 + 14x_2^2 r^2 + 2\gamma_3(3x_1 + x_2)^2))\right]
\]

We will now give one more example of system belonging to previously introduced subclass for which relations obtained in Theorem 1 represent actually set of the first integrals. We will replace Kowalevski’s fundamental equation \([1]\) by strongly discriminantly separable polynomial \([10]\) with coefficients

\[
A(x_1, x_2) = (x_1 - x_2)^2, \\
B(x_1, x_2) = 2x_1 x_2(x_1 + x_2) + 2ax_1 x_2 + b(x_1 + x_2) + 2c, \\
C(x_1, x_2) = x_1^2 x_2^2 - bx_1 x_2 - 2c(x_1 + x_2) + \frac{b^2}{4} - ac.
\]

Then, discriminants \(D\) are:

\[
D_x F(x_1, x_2) = P(x_1)P(x_2), \\
D_x F(x_2, s) = P(x_2)P(s), \\
D_x F(x_1, s) = P(x_1)P(s),
\]

with polynomial \(P\) of third degree

\[
P(x) = 2x^3 + ax^2 + bx + c.
\]

Applying result of Theorem \([1]\) for \(m_i = 1, n_i = 0, i = 1, 2\) like in Kowalevski’s case, but on previously introduced polynomials \(A, C\) and \(P\) we get that system reduces to \([8], [9], [10]\) also satisfies relations \([11]\) with

\[
p_1 = 1, p_2 = 1 \\
q_1 = x_1, q_2 = x_2 \\
r_1 = x_1^2, r_2 = x_2^2
\]

and

\[
E_1 = 2x_1 + 2x_2 + a \\
F_1 = -x_1 x_2 + \frac{b}{2} \\
G_1 = c,
\]

(31)
or
\[
E_2 = \frac{2(x_1 + x_2)(x_1^2 + x_2^2) + a(x_1 + x_2)^2 + 2b(x_1 + x_2) + 4c}{(x_1 - x_2)^2}
\]
\[
F_2 = -\frac{2x_1x_2(3x_1^2 + 2x_1x_2 + 3x_2^2) + 4ax_1x_2(x_1 + x_2)}{2(x_1 - x_2)^2}
+ \frac{b(x_1^2 + 6x_1x_2 + x_2^2) + 4c(x_1 + x_2)}{2(x_1 - x_2)^2}
\]
\[
G_2 = \frac{4x_1^2x_2(x_1 + x_2) + 2bx_1x_2(x_1 + x_2) + c(x_1 + x_2)^2}{(x_1 - x_2)^2}.
\]

Lemma 3 For choice of values \(E_1, F_1, G_1\) relations (32) are
\[
r^2 = 2(x_1 + x_2) + e_1 + e_2 + a
\]
\[
r\gamma_3 = -x_1x_2 + \frac{b}{2} - x_2e_1 - x_1e_2
\]
\[
\gamma_3^2 = x_1^2e_1 + x_2^2e_2 + c
\]
\[
e_1e_2 = d^2
\]
and represent the first integrals of following system:
\[
\dot{x}_1 = -\frac{i}{2}(rx_1 + \gamma_3)
\]
\[
\dot{x}_2 = \frac{i}{2}(rx_2 + \gamma_3)
\]
\[
\dot{e}_1 = -ire_1
\]
\[
\dot{e}_2 = ire_2
\]
\[
\dot{r} = \frac{i}{2}(x_2 - x_1 + e_2 - e_1)
\]
\[
\dot{\gamma}_3 = \frac{i}{2}(e_1x_2 - e_2x_1).
\]

Proof. We start with assumptions that system of equations is of the form (8) with \(f_i = rx_i + \gamma_3, i = 1, 2\). Like in previous example, by differentiating the first and third of relations (33) we get
\[
\dot{r} = \frac{i}{2}(x_2 - x_1 + e_2 - e_1)
\]
\[
\dot{\gamma}_3 = \frac{i}{2}(e_1x_2 - e_2x_1).
\]
Then replacing previously obtained values for \(\dot{r}\) and \(\dot{\gamma}_3\) into differentiated second relation from (33), we get that it will be identically satisfied for function
\[
m = ir
\]
what brings us to system (34). \(\square\)
Lemma 4 The system (34) preserves the standard measure.

Proof. As usually, the system (34) can be rewritten in a more compact form:

\[
\begin{align*}
\frac{dx_1}{X_1} &= \frac{dx_2}{X_2} = \frac{dr}{X_3} = \frac{de_1}{X_4} = \frac{de_2}{X_5} = \frac{d\gamma_3}{X_6} = dt
\end{align*}
\]

with

\[
\begin{align*}
X_1 &= -\frac{i}{2}(rx_1 + \gamma_3) \\
X_2 &= \frac{i}{2}(rx_2 + \gamma_3) \\
X_3 &= \frac{i}{2}(x_2 - x_1 + e_2 - e_1) \\
X_4 &= -ivre_1 \\
X_5 &= ire_2 \\
X_6 &= \frac{i}{2}(e_1x_2 - e_2x_1).
\end{align*}
\]

Then the divergence of \( X = (X_1, X_2, X_3, X_4, X_5, X_6) \) is zero:

\[
\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial r} + \frac{\partial X_4}{\partial e_1} + \frac{\partial X_5}{\partial e_2} + \frac{\partial X_6}{\partial \gamma_3} = 0
\]

so, the standard measure is preserved. \( \square \)

The standard measure is invariant under the flow associated with (34), which makes the system of ordinary differential equations (34) with four first integrals (33) completely integrable by Jacobi’s theorem.

We have shown that our modal example and the class of systems we have considered in this paper share many interesting properties, typical for completely integrable Hamiltonian systems. Through the connection with discriminantly separable polynomials, they are particularly close to the celebrated Kowalevski top. Thus, as an important question, it remains to be seen if they admit a Poisson structure in which they are Hamiltonian. The question of physical or mechanical interpretation of such systems is not less interesting.

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