BIVARIATE HILBERT FUNCTIONS FOR THE TORSION FUNCTOR

EMANOIL THEODORESCU

Abstract. Let \((R, P)\) be a commutative, local Noetherian ring, \(I, J\) ideals, \(M\) and \(N\) finitely generated \(R\)-modules. Suppose \(J + \text{ann}_RM + \text{ann}_RN\) is \(P\)-primary. The main result of this paper is Theorem 6, which gives necessary and sufficient conditions for the length of \(\text{Tor}_i(M/I^nM, N/J^mN)\), to agree with a polynomial, for \(m, n \gg 0\). As a corollary, it is shown that the length of \(\text{Tor}_i(M/I^nM, N/I^nN)\) always agrees with a polynomial in \(n\), for \(n \gg 0\), provided \(I + \text{ann}_RM + \text{ann}_RN\) is \(P\)-primary.

Introduction

Throughout this paper, unless otherwise stated, \((R, P)\) is a commutative, Noetherian local ring with unit and \(I, J\) are (proper) ideals. Also, let \(M, N\) be finite \(R\)-modules, \(m, n\) be nonnegative integers, and let \(\lambda\) denote length. We would like to study the two-variable Hilbert function \(H(n, m) := \lambda(\text{Tor}_i(M/I^nM, N/J^mN))\). On the one hand, we have in mind extending results on \(H(n, m)\) of the authors of [BF], [KS] and [WCB], while on the other hand we seek two variable analogues of recent results concerning the Hilbert function \(H(n) := \lambda(\text{Tor}_i(M/I^nM, N))\). Previous work on \(H(n)\) appears in [TM], [VK]) and [ET]. In fact, in [ET] it is shown that \(H(n)\) agrees with a polynomial in \(n\) for \(n\) large, if we simply assume that the lengths \(\lambda(\text{Tor}_i(M/I^nM, N))\) are finite. Here we seek to give conditions under which \(H(n, m)\) has polynomial growth for \(n\) and \(m\) sufficiently large. In some special cases, we give a degree bound on the resulting polynomials in \(n\) and \(m\). Determining the exact degree of these polynomials seems to be a more difficult task. In the one variable case, [VK] and [ET] give upper bound estimates for the degree in general and while [ET], [DK] and [TM] determine the degree in some special cases.

In his Doctoral Thesis, Bruce Fields [BF] investigates two-variable functions of the form \(\lambda(\text{Tor}_i(R/I^n, R/J^m))\), where \(i \geq 0\), under the assumption that \(I + J\) is \(P\)-primary. For \(i \geq 2\), he proves that these lengths are eventually given by polynomials in two variables. Actually,
since $\text{Tor}_i(R/I^n, R/J^m) = \text{Tor}_{i-1}(I^n, R/J^m) = \text{Tor}_{i-2}(I^n, J^m)$ (by applying twice the shifting formula), his proof essentially shows that $\oplus_{m,n=0}^\infty \text{Tor}_j(I^n M, J^m N)$, $j \geq 0$, is a finite, bigraded module, over a suitable polynomial ring over $R$, where $M$, $N$ are two finite $R$-modules. It is then well-known that, if the lengths of homogeneous pieces of a finite bigraded module (over a suitable polynomial ring) are finite, then they are eventually given by a polynomial function (also see Notations and Conventions).

For $i = 0$ and $i = 1$, Fields only proves that polynomial growth holds under some rather restrictive conditions: he assumes that $R$ is regular local, and that $\oplus_{m,n=0}^\infty (I^n \cap J^m)$ is a finite bigraded module over some polynomial ring in two sets of variables. This is, in general, a very strong condition on two ideals $I$, $J$. The function $\lambda(R/(I^n + J^m))$ has also been studied by Kishor Shah [KS] and William C. Brown [WCB], who give sufficient conditions for it to be given by a polynomial, for $m, n \gg 0$.

The present paper gives a characterization of those cases for which the length of $\text{Tor}_i(M/I^n M, N/J^m N)$ has polynomial growth, provided the following condition is satisfied: $J + \text{ann}_R M + \text{ann}_R N$ is $P$-primary (see Theorem 6). It turns out that polynomial growth doesn’t always hold, even in the case $i \geq 2$, as Fields’ work might have suggested (see the Remark following Corollary 8). On the other hand, Proposition 3 shows that, provided $\text{Tor}_i(I^n M, N/J^m N)$ has finite length, for all large $m, n$, its length is always given by a polynomial, without any restrictive assumption.

As a corollary to the proof of Theorem 6, under the assumption that $I + \text{ann}_R M + \text{ann}_R N$ is $P$-primary, we prove that $\lambda(\text{Tor}_i(M/I^n M, N/I^n N))$ has always polynomial growth. Corollary 8 shows that, under the hypothesis that both $I + \text{ann}_R M + \text{ann}_R N$ and $J + \text{ann}_R M + \text{ann}_R N$ be $P$-primary, the length of $\text{Tor}_i(M/I^n M, N/J^m N)$ has polynomial growth if and only if both $\text{Tor}_i(M, N)$ and $\text{Tor}_{i-1}(M, N)$ have finite length. Finally, when $M \otimes N$ has finite length, Theorem 9 gives the formula $\lambda(\text{Tor}_i(M/I^n M, N/J^m N)) =$

$$\lambda(\text{Tor}_i(M, N)) + \lambda(\text{Tor}_{i-1}(I^n M, N)) + \lambda(\text{Tor}_{i-1}(M, J^m N)) + \lambda(\text{Tor}_{i-2}(I^n M, J^m N)),$$

which works for all $i \geq 0$, by assuming that all $\text{Tor}_i$ with $i < 0$ are zero.

The main result of this paper shows that, at least when $J + \text{ann}_R M + \text{ann}_R N$ is $P$-primary, the nature of $\lambda(\text{Tor}_i(M/I^n M, N/J^m N))$ is controlled by modules of the form
Therefore, a study of modules of this kind would deepen our understanding of \( \lambda(\text{Tor}_i(M/I^n M, N/J^m N)) \).

**Notation and Conventions**

We will be using (free) resolutions of modules over several different rings. There will be resolutions of modules over \( R \), graded resolutions of graded modules over the polynomial ring in \( r \) variables, \( S_1 := R[X_1, ..., X_r] \), as well as bigraded resolutions of bigraded modules over the polynomial ring in two sets of variables, \( S_2 := R[X_1, ..., X_r, Y_1, ..., Y_s] \). Unless otherwise stated, the Tor’s are over \( R \).

To further simplify notation, we denote \( M = \bigoplus_{n=0}^{\infty} M_n \), which is an (infinitely generated) graded module over the Rees ring \( R_I := \bigoplus_{n=0}^{\infty} I^n \). If \( I \) is generated by \( x_1, ..., x_r \), then \( M \) is naturally an infinitely generated \( S_1 \)-graded module, via the canonical ring homomorphism \( S_1 \to R_I \), given by \( X_i \mapsto x_i \), for all \( i \). The action of \( S_1 \) on \( M \) is given by \( X_i v_k = x_i v_k \), where \( v_k \) denotes a homogeneous vector of degree \( k \). Also, if we denote \( IM := \bigoplus_{n=0}^{\infty} I^n M \), then this is a finitely generated graded module over \( R_I \), and hence over \( S_1 \), as before. It follows that \( M/IM = \bigoplus_{n=0}^{\infty} (M/I^n M) \) is a graded module over both \( R_I \) and \( S_1 \).

Similarly, if we assume \( J = (y_1, ..., y_s) \), \( \bigoplus_{m,n=0}^{\infty} J^m M \) is a bigraded module over the bigraded Rees ring \( R_{I,J} := \bigoplus_{m,n=0}^{\infty} I^n J^m M \), and hence over the polynomial ring \( S_2 \), via a similar map \( S_2 \to R_{I,J} \).

Note that any graded free resolution over \( S_1 \) or \( S_2 \) of some graded module, is also a free resolution of that module over \( R \).

We will be making use of the fact that, in a (bi)graded resolution of some \( S_1 \) (or \( S_2 \))-graded module, say \( \mathcal{I}M \), by considering just its homogeneous part of degree \( k \), we obtain a free resolution, over \( R \), of the module \( I^k M \), the \( k \)-th homogeneous component of \( \mathcal{I}M \).

We will be making repeated use of the fact that, if \( \mathcal{P} := \bigoplus_{m,n=0}^{\infty} P_{m,n} \) is a finite bigraded \( S_2 \)-module, whose homogeneous pieces have finite length, then \( \lambda(P_{m,n}) \) is eventually given by a polynomial. In particular, \( \lambda(\text{Tor}_i(I^n M, J^m N)) \) is eventually given by a polynomial. Indeed, we can take \( \mathcal{C} \) a \( S_1 \)-graded free resolution (consisting of finite free \( S_1 \)-modules) of \( \bigoplus_{n=0}^{\infty} I^n M \) and, similarly, \( \mathcal{D} \) a \( S_1' \)-graded free resolution of \( \bigoplus_{m=0}^{\infty} J^m N \), also consisting of finite free \( S_1' \)-modules. (Here, \( S_1' = R[Y_1, ..., Y_s] \).) Then the modules in \( \mathcal{C} \otimes_R \mathcal{D} \) have
a natural structure of $S_1 \otimes_R S'_1 \cong S_2$-modules. Actually, $C \otimes_R D$ is a complex of finite, free, $S_2$-modules, whose i-th homology is $\text{Tor}_i^R(\oplus_{n=0}^{\infty} I^n M, \oplus_{m=0}^{\infty} J^m N)$. Of course, this is a finitely generated bigraded $S_2$-module. Since the homogeneous components of this are just $\text{Tor}_i^R(I^n M, J^m N)$, it follows that, if their lengths are finite, then these lengths are eventually given by a polynomial in $m, n$.

**The main result**

In an attempt to study the length of $\text{Tor}_i(M/I^n M, N/J^m N)$ in as great generality as possible, we first investigate $\text{Tor}_i(I^n M, N/J^m N)$. It turns out that in this case polynomial growth follows from the simplest assumption that these Tor’s have finite length. The following few results are essentially given without proof, as their proofs parallel those of corresponding one-variable statements (see [ET]).

**Proposition 1.** Let $R$ be a Noetherian ring (not necessarily local), and $J \subset R$ an ideal. Let $S_1$ be the polynomial ring over $R$ in $r$ variables, and let

$$C : \quad F_2 \xrightarrow{\psi} F_1 \xrightarrow{\phi} F_0$$

be a graded complex of graded $S_1$-modules, graded by total degree. Assume that $F_1, F_0$ are finitely generated $S_1$-modules. Then, there is $l \geq 0$, such that, for all $m \geq l$

$$H_1(C \otimes_R R_{J^m}) = \frac{U + J^{m-l} V}{Z + J^{m-l} W},$$

where $Z \subseteq U$ and $W \subseteq V$ are finite, graded $S_1$-modules.

Proof. It essentially goes as in Proposition 3 in [ET]. □

**Proposition 2.** Let $R, S_1, J$ be as in Proposition 1. Let $T$ be a graded $S_1$-module, and $U, V, W, Z$ be finite graded $S_1$-submodules of $T$. Assume that $Z \subseteq U$, and that $W \subseteq V$, and denote

$$L_m := \frac{U + J^m V}{Z + J^m W}.$$
Then, if \((L_m)_n\), the \(n\)-th degree homogeneous component of \(L_m\), has finite length for all large values of \(m\) and \(n\), \(\lambda((L_m)_n)\) is eventually given by a polynomial in \(m\) and \(n\).

**Proof.** It follows the same path as Lemma 2, (b) in [ET]. □

**Proposition 3.** Let \(R\) be a Noetherian ring, \(I, J \subseteq R\) ideals, \(M, N\) be finite \(R\)-modules, and \(i \geq 0\). If \(\text{Tor}_i(I^n M, N/J^m N)\) has finite length for all \(m, n \gg 0\), then this length is eventually given by a polynomial in \(m, n\).

**Proof.** Take an \(S_1\)-graded resolution by finite free \(S_1\)-modules of the finite graded \(S_1\)-module \(\bigoplus_{n=0}^{\infty} I^n M\). Tensor it with \(N/J^m N\), in two steps, first with \(N\), (call the resulting \(S_1\)-complex \(C\)), then with \(R/J^m\). The part giving \(\text{Tor}_i(I^n M, N/J^m N)\), looks just like the situation described in Proposition 1. Therefore, by Proposition 1, we see that

\[
\text{Tor}_i^R(\bigoplus_{n=0}^{\infty} I^n M, N/J^m N) = U + J^{m-l} V / Z + J^{m-l} W,
\]

for some \(l, m \geq l\), where \(U, V, Z\) and \(W\) are all finite graded \(S_1\)-modules. It follows that

\[
\text{Tor}_i^R(I^n M, N/J^m N) = \frac{U_n + J^{m-l} V_n}{Z_n + J^{m-l} W_n},
\]

by looking at homogeneous pieces of degree \(n\) in the previous Tor formula. Thus, the conclusion follows from Proposition 2. □

**Lemma 4.** Let \((R, P)\) be Noetherian, local, \(I, J \subset R\) ideals, \(i \geq 0\). Then, for two finite \(R\)-modules \(M, N\), we have:

(a) The image of the induced map

\[ \text{Tor}_i(I^n M, N) \xrightarrow{H(f_i)} \text{Tor}_i(M, N) \]

is of the form \(I^{n-k} A\), for some \(k \geq 0\) and \(n \geq k\), where \(A\) is the image of the map

\[ \text{Tor}_i(I^k M, N) \xrightarrow{H(f_i)} \text{Tor}_i(M, N) . \]

(b) The image of the induced map

\[ \text{Tor}_i(M, N) \xrightarrow{H(g_i)} \text{Tor}_i(M, N/J^m N) \]
has the form\[
\frac{\text{Tor}_i(M, N) + J^m B}{J^m B},
\]
for some module $B$, such that $\text{Tor}_i(M, N) \subseteq B$.

**Proof.** (a) Let
\[
\ldots \rightarrow R^{\beta_{i+1}} \rightarrow R^{\beta_{i}} \rightarrow R^{\beta_{i-1}} \rightarrow \ldots
\]
be a free resolution of $N$. Then we have the following commutative diagram
\[
\begin{array}{cccccccc}
\ldots & \rightarrow & I^n M^{\beta_{i+1}} & \psi & I^n M^{\beta_{i}} & \phi & I^n M^{\beta_{i-1}} & \rightarrow & \ldots \\
\downarrow & & \downarrow f_i & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & M^{\beta_{i+1}} & \psi & M^{\beta_{i}} & \phi & M^{\beta_{i-1}} & \rightarrow & \ldots 
\end{array}
\]
Let $K = \ker \phi$ and $L = \text{im} \psi$, so $\text{Tor}_i(M, N) = K/L$. We also have that $\ker \phi_n = K \cap I^n M^{\beta_i}$ and $\text{im} \psi_n = I^n L$, and thus $\text{Tor}_i(I^n M, N) = (K \cap I^n M^{\beta_i})/I^n L$. It follows that
\[
\text{im} (H(f_i)) = \frac{K \cap I^n M^{\beta_i} + L}{L} = \frac{I^{n-k}(K \cap I^k M^{\beta_i}) + L}{L},
\]
for some $k$ and all $n \geq k$. Note that this is of the form $I^{n-k} A$, where $A$ is the image of the map $\text{Tor}_i(I^k M, N) \xrightarrow{H(f_i)} \text{Tor}_i(M, N)$, as stated.

(b) Now assume that (1) gives a free resolution of $M$, and tensor it with $N/J^m N$. We get
\[
\begin{array}{cccccccc}
\ldots & \rightarrow & N^{\beta_{i+1}} & \psi & N^{\beta_{i}} & \phi & N^{\beta_{i-1}} & \rightarrow & \ldots \\
\downarrow & & \downarrow g_i & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & N^{\beta_{i+1}}/J^m N^{\beta_{i+1}} & \psi & N^{\beta_{i}}/J^m N^{\beta_{i}} & \phi & N^{\beta_{i-1}}/J^m N^{\beta_{i-1}} & \rightarrow & \ldots 
\end{array}
\]
Again, if we denote $K = \ker \phi$ and $L = \text{im} \psi$, then $\text{Tor}_i(M, N) = K/L$ and, moreover, we obtain that
\[
\ker \phi_m = \frac{K + J^{m-1}(\phi^{-1}(J^i N^{\beta_{i-1}}))}{J^m N^{\beta_i}},
\]
for some \( l \) and \( m \geq l \).

We also get

\[
\text{im } \psi_m = L + J^m N^{\beta_i} / J^m N^{\beta_i},
\]

so

\[
\text{Tor}_i(M, N/J^m N) = \frac{K + J^{m-l}(\phi^{-1}(J^l N^{\beta_i}))}{L + J^m N^{\beta_i}}.
\]

It follows that

\[
\text{im } H(g_i) = \frac{K + J^m N^{\beta_i}}{L + J^m N^{\beta_i}} \cong \frac{\text{Tor}_i(M, N) + J^m B}{J^m B},
\]

where \( B = N^{\beta_i}/L \). Of course, \( \text{Tor}_i(M, N) \subseteq B \). \( \square \)

The next Proposition is an extended version of the following well-known result: Let \((R, P)\) be Noetherian, local, and \( I \subseteq R \) an ideal. If \( L, M \) are finitely generated modules, \( L \) of finite length, then, for any \( i \geq 0 \), the natural map \( \text{Tor}_i(I^n M, L) \rightarrow \text{Tor}_i(M, L) \) is zero, for \( n \gg 0 \) (see [GL]).

**Proposition 5.** Let \((R, P)\) be a Noetherian, local ring. Let \( I \subseteq R \) be an ideal, \( M, N \) two finite \( R \)-modules and \( i \geq 0 \), fixed. Then the following are equivalent:

(a) \( I \subseteq \text{rad}(\text{ann}_R \text{Tor}_i(M, N)) \).
(b) \( I \subseteq \text{rad}(\text{ann}_R \text{Tor}_i(I^k M, N)) \), for some \( k \geq 0 \).
(c) \( I \subseteq \text{rad}(\text{ann}_R \text{Tor}_i(I^n M, N)) \), for all \( n \geq 0 \).
(d) \( I \subseteq \text{rad}(\text{ann}_R \text{im } (\text{Tor}_i(I^n M, N) \rightarrow \text{Tor}_i(M, N))) \), for all \( n \geq 0 \).
(e) \( \text{im } (\text{Tor}_i(I^n M, N) \rightarrow \text{Tor}_i(M, N)) = 0 \), for all \( n \gg 0 \).

**Proof.** Clearly, (c) implies (a) and (b). Conversely, consider the long exact sequence

\[
\cdots \rightarrow \text{Tor}_{i+1}(M/I^n M, N) \xrightarrow{\partial} \text{Tor}_i(I^n M, N) \xrightarrow{\alpha} \text{Tor}_i(M, N) \xrightarrow{\beta} \text{Tor}_i(M/I^n M, N) \rightarrow \cdots
\]

(a) implies (b), (c) follows by considering \( \alpha \) and \( \partial \), since \( I \subseteq \text{rad}(\text{ann}_R \text{Tor}_j(M/I^n M, N)) \) for all \( n \geq 0 \). (b) implies (a) follows from (c) implies (a).

(a) implies (d) and (d) implies (a) are immediate, considering \( \alpha \).

(e) implies (a): if \( \alpha = 0 \), then \( \beta \) is an injection, so the conclusion follows.

(a) implies (e) follows from Lemma 4(a). \( \square \)
Here is the main result of this paper:

**Theorem 6.** Let \((R, P)\) be Noetherian, local, \(I, J \subseteq R\) two ideals, \(M, N\) finitely generated \(R\)-modules, \(i \geq 0\). Assume that \(\text{ann}_R M + \text{ann}_R N + J\) is \(P\)-primary. Then,

\[
\lambda(\text{Tor}_i(M/I^nM, N/J^mN))
\]

is eventually given by a polynomial in \(m\) and \(n\) if and only if \(I \subseteq \text{rad}(\text{ann}_R \text{Tor}_j(M, N))\), for \(j \in \{i - 1, i\}\).

**Proof.** Consider the long exact sequence

\[
\cdots \rightarrow \text{Tor}_i(I^n M, N/J^m N) \xrightarrow[\alpha_{i,m,n}]{\sim} \text{Tor}_i(M, N/J^m N) \rightarrow \text{Tor}_i(M/I^n M, N/J^m N) \rightarrow \text{Tor}_{i-1}(I^n M, N/J^m N) \xrightarrow[\alpha_{i-1,m,n}]{\sim} \text{Tor}_{i-1}(M, N/J^m N) \rightarrow \cdots
\]

We already know that the lengths of the modules above, save the one in the middle, are (eventually) given by polynomials in one or two variables (see Proposition 3). Thus, we have

\[
\lambda(\text{Tor}_i(M/I^n M, N/J^m N)) = \lambda(\text{Tor}_i(M, N/J^m N)) - \lambda(\text{im} \alpha_{i,m,n}) + \lambda(\ker \alpha_{i,m,n})
\]

(2)

\[
= [\lambda(\text{Tor}_i(M, N/J^m N)) - \lambda(\text{im} \alpha_{i,m,n})] + [\lambda(\text{Tor}_{i-1}(I^n M, N/J^m N)) - \lambda(\text{im} \alpha_{i-1,m,n})].
\]

Therefore, we need to examine \(\lambda(\text{im} \alpha_{j,m,n})\), for \(j \in \{i - 1, i\}\). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Tor}_i(I^n M, N) & \xrightarrow[\psi_{i,m,n}]{\sim} & \text{Tor}_i(I^n M, N/J^m N) & \xrightarrow[\phi_{i,m,n}]{\sim} & \text{Tor}_{i-1}(I^n M, J^m N) \\
\downarrow[\sigma_{i,m,n}] & & \downarrow[\alpha_{i,m,n}] & & \downarrow[\tau_{i,m,n}]
\end{array}
\]

\[
\begin{array}{ccc}
\text{Tor}_i(M, N) & \xrightarrow[\theta_{i,m,n}]{\sim} & \text{Tor}_i(M, N/J^m N) & \xrightarrow[\phi_{i,m,n}]{\sim} & \text{Tor}_{i-1}(M, J^m N) \\
\downarrow[\gamma_{i,m,n}] & & \downarrow[\psi_{i,m,n}] & & \\
\text{Tor}_i(M, N/J^m N)/\alpha_{i,m,n}(L_{m,n}) & & & & 0
\end{array}
\]

where, \(L_{m,n} = \text{im} \phi_{i,m,n} = \ker \phi_{i,m,n}\).
Note that the commutative diagram (3) is a homogeneous piece of the diagram (3′) below. That’s because $\text{Tor}_R^i$ is additive, and the natural maps in (3) commute with the action of $I$ and $J$ on the modules occurring in this diagram. It follows that the diagram (3′) is a commutative diagram of bigraded $S_2$-modules and maps.

\[
\begin{array}{ccccccc}
\text{Tor}_R^i(IM,N) & \xrightarrow{\psi} & \text{Tor}_R^i(IM,N/JN) & \xrightarrow{\phi} & \text{Tor}_R^{i-1}(IM,JN) \\
\downarrow{\sigma} & & \downarrow{\alpha_i} & & \downarrow{\tau} \\
\text{Tor}_R^i(M,N) & \xrightarrow{\theta} & \text{Tor}_R^i(M,N/JN) & \rightarrow & \text{Tor}_R^{i-1}(M,JN) \\
\downarrow{\pi_i} & & \downarrow{\tau} \\
\text{Tor}_R^i(M,N/JN)/\alpha_i(L) & \rightarrow & 0 \\
\end{array}
\] (3′)

where $L = \bigoplus_{m,n=0}^{\infty} L_{m,n}$.

Observe now that $\pi_i \circ \alpha_i$ factors through the image of $\phi$, which is a finitely generated, bigraded $S_2$-module (since $\text{Tor}_R^i(IM,N/JN)$ is so), hence $\text{im} (\pi_i \circ \alpha_i)$ is a finite, bigraded $S_2$-module. Then $\lambda (\text{im} (\pi_i \circ \alpha_i)^{m,n})$ is eventually given by a polynomial, by classical theory.

Note that $\lambda (\text{im} \alpha_i^{m,n}) = \lambda (\text{im} (\pi_i \circ \alpha_i)^{m,n}) + \lambda (\alpha_i^{m,n}(L_{m,n}))$, and a similar equality holds for $i-1$ in place of $i$. From (2) and what we have just seen, it follows that $\lambda (\text{Tor}_i(M/I^nM,N/J^mN))$ is eventually given by a polynomial, if and only if the same is true of $\lambda (\alpha_i^{m,n}(L_{m,n})) + \lambda (\alpha_i^{m,n}(L_{m,n}))$.

We now examine $\lambda (\alpha_i^{m,n}(L_{m,n}))$. From (3), we find that

\[
\alpha_i^{m,n}(L_{m,n}) = \alpha_i^{m,n}(\psi^{m,n}(\text{Tor}_i(I^nM,N))) = (\theta \circ \sigma)^{m,n}(\text{Tor}_i(I^nM,N)). \quad (4)
\]

From Lemma 4, (a) and (b), we get that

\[
(\theta \circ \sigma)^{m,n}(\text{Tor}_i(I^nM,N)) = \frac{I^{n-k}A + J^mB}{J^mB} = \frac{I^{n-k}A}{I^{n-k}A \cap J^mB}, \quad (5)
\]

for some $k \geq 0$ and $n \geq k$, where $A = \text{im} (\text{Tor}_i(I^kM,N) \rightarrow \text{Tor}_i(M,N))$.

We now claim that $\lambda (I^{n-k}A/I^{n-k}A \cap J^mB)$ is identically zero, for $m, n \gg 0$ if and only if it is polynomial for $m, n \gg 0$, if and only if $I \subseteq \text{rad}(\text{ann}_R \text{Tor}_i(M,N))$. To prove
this claim, assume \( I \subseteq rad(ann_R \Tor_i(M, N)) \). Then \( I^{n-k}A = 0 \), for large \( n \), and so 
\[ \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) = 0, \]
hence polynomial, for \( n \gg 0 \) and all \( m \). It remains to check 
that, if \( I \not\subseteq rad(ann_R \Tor_i(M, N)) \), then \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \) is nonzero and not given 
by a polynomial, for all \( m, n \gg 0 \). Indeed, by Proposition 5, (1) \( \Leftrightarrow \) (3), we know that 
\( I \not\subseteq rad(ann_R \Tor_i(M, N)) \), for all \( n \), so \( I^{n-k}A \neq 0 \) for all \( n \geq k \).

Now, since \( ann_RM + ann_RN + J \) is \( P \)-primary, there is a \( l \geq 0 \), such that \( I^l \subseteq ann_RM + 
ann_RN + J \). It follows that, for \( n \geq lm + k \), we have 
\[ I^{n-k} \subseteq J^m + ann_RM + ann_RN, \]
so 
\[ I^{n-k}A \subseteq J^mA \subseteq J^mB, \]
since we know that \( A \subseteq B \).

Thus, for \( n \geq lm + k \), \( l \) and \( k \) fixed, \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \) vanishes. On the other 
hand, note that, for every \( n \geq k \), \( I^{n-k}A/I^{n-k}A \cap J^mB \neq 0 \), for all \( m \gg 0 \). This is 
so since, for every \( n \geq k \), \( n \) fixed, \( I^{n-k}A \cap J^mB \nsubseteq I^{n-k}A \) for all large \( m \), by Krull’s 
Intersection Theorem. Hence \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \neq 0 \), for every \( n \geq k \) and \( m \gg 0 \). 
This proves the claim, since we proved that, above the line \( d : n = lm + k \) in the \((m,n)\)-
plane, \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \) always vanishes, for large \( m \) and \( n \), while below this line, 
the length in question is nonzero, in case \( I \not\subseteq rad(ann_R \Tor_i(M, N)) \).

Finally, note that both terms of the form \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \) occurring in the 
formula (2) of \( \lambda(\Tor_i(M/I^nM, N/J^mN)) \) (also see (4) and (5)), actually occur with the 
same sign. By the claim, it follows that the sum of these two terms vanishes for all large 
\( m \) and \( n \), if \( I \subseteq rad(ann_R \Tor_i(M, N)) \cap rad(ann_R \Tor_{i-1}(M, N)) \). On the other hand, if 
\( I \not\subseteq rad(ann_R \Tor_i(M, N)) \cap rad(ann_R \Tor_{i-1}(M, N)) \), then the sum in question vanishes 
above both lines \( d : n = lm + k \), \( d' : n = lm + k' \), (one line for each term), but it is nonzero 
below both these lines, \( d \) and \( d' \). This means that \( \lambda(\Tor_i(M/I^nM, N/J^mN)) \) can only then 
be (eventually) polynomial, when both terms of the form \( \lambda(I^{n-k}A/I^{n-k}A \cap J^mB) \) vanish. 
And this happens if and only if \( I \subseteq rad(ann_R \Tor_j(M, N)) \), for \( j \in \{i - 1, i\} \), as stated. \( \square \)

The proof of Theorem 6 yields the following interesting corollary:
Corollary 7. Let \((R, P)\) be Noetherian, local, \(I\) an ideal \(M, N\) two finite \(R\)-modules and \(i \geq 0\). Assume that \(I + \text{ann}_RM + \text{ann}_RN\) is \(P\)-primary. Then

\[
\lambda(\text{Tor}_i(M/I^nM, N/I^nN))
\]

is given by a polynomial, for \(n \gg 0\).

Proof. Note that, by the proof of Theorem 6, we only have to look at each of the two (similar) terms in \(\lambda(\text{Tor}_i(M/I^nM, N/J^mN))\), that turned out not to be polynomial, in general. If in each of them we set \(J = I\) and \(m = n\), we get two terms, each of which looks like

\[
\lambda(I^n - kA/I^n - kA\cap I^nB).
\]

It is immediate, by the Artin-Rees Lemma, that \(\oplus_{n=0}^{\infty}I^n - kA/I^n - kA\cap I^nB\) is a finite graded module over the Rees ring \(R_I = \oplus_{n=0}^{\infty}I^n\), hence the conclusion. \(\square\)

Corollary 8. Assume that both \(I + \text{ann}_RM + \text{ann}_RN\) and \(J + \text{ann}_RM + \text{ann}_RN\) are \(P\)-primary, in the statement of Theorem 6. Then \(\lambda(\text{Tor}_i(M/I^nM, N/J^mN))\) is eventually given by a polynomial if and only if \(\text{Tor}_j(M,N)\) has finite length for both \(j = i\), \(j = i - 1\).

Proof. \(\lambda(\text{Tor}_i(M/I^nM, N/J^mN))\) is eventually given by a polynomial if and only if \(I \subseteq \text{rad}(\text{ann}_R\text{Tor}_j(M,N))\), for \(j \in \{i - 1, i\}\), if and only if \(I + \text{ann}_RM + \text{ann}_RN \subseteq \text{rad}(\text{ann}_R\text{Tor}_j(M,N))\), for \(j \in \{i - 1, i\}\), if and only if \(\text{Tor}_j(M,N)\), has finite length for both \(j = i - 1\) and \(j = i\). \(\square\)

Remark. From this corollary alone we could construct numerous examples in which \(\lambda(\text{Tor}_i(M/I^nM, N/J^mN))\) is not eventually polynomial. It suffices to take \(I\) and \(J\) to be \(P\)-primary ideals and \(M, N\) two finite \(R\)-modules with at least one of the two modules \(\text{Tor}_i(M,N)\) and \(\text{Tor}_{i-1}(M,N)\) not having finite length. Let us give two such examples of \(\text{Tor}_i(M/I^nM, N/J^mN)\) that have non-polynomial length, the second of which works for any value of \(i\).

First, assume that \(R\) has positive depth and dimension at least two. Take \(x_1, x_2, \ldots, x_t, t \geq 1\) to be a regular sequence, such that the ideal generated by these elements is not \(P\)-primary. Take \(M = R/(x_1, \ldots, x_t)^s\) and \(N = R/(x_1, \ldots, x_t)^r\) for some \(s \geq r \geq 1\). Then
\[ \text{Tor}_1(M, N) = (x_1, \ldots, x_t)^s/(x_1, \ldots, x_t)^{s+r} \] has finite length if and only if \( R/(x_1, \ldots, x_t) \) has finite length. This is so because, by Rees’ theorem, \((x_1, \ldots, x_t)^j/(x_1, \ldots, x_t)^{j+1}\) is a free \( R/(x_1, \ldots, x_t) \)-module, for all \( j \geq 0 \). Therefore \( \text{Tor}_1(M, N) \) can’t have finite length by the choice of the regular sequence. Now take \( I \) and \( J \) any two \( P \)-primary ideals: by Corollary 8, the length of \( \text{Tor}_1(M/I^nM, N/J^mN) \) is not given by a polynomial, for \( i \in \{1, 2\} \).

Secondly, assume that \( R \) is neither regular, nor an isolated singularity. Then \( R_Q \) is not regular for some non-maximal prime \( Q \). Take \( M \) and \( N \) to be any two \( R \)-modules, such that their annihilator is \( Q \). Note that both \( M_Q \) and \( N_Q \) are direct sums of copies of the residue field of \( R_Q \). Then \( \text{Tor}_i(M, N) \) cannot have finite length for any \( i \). (For \( i \geq 1 \): this would imply that the localization at \( Q \) of \( \text{Tor}_i(M, N) \) vanishes, giving that \( R_Q \) is regular, contrary to the choice of \( R \).) Now, Corollary 8 says that for any choice of two primary ideals \( I \) and \( J \), the length of \( \text{Tor}_i(M/I^nM, N/J^mN) \) is not polynomial for all \( i \geq 0 \).

**Theorem 9.** Let \((R, P)\) be Noetherian local, \( I, J \subseteq R \) ideals, \( M, N \) finite \( R \)-modules and \( i \geq 0 \). Assume that \( M \otimes N \) has finite length. Then

\[
\lambda(\text{Tor}_i(M/I^nM, N/J^mN))
\]

is given by a polynomial, for \( m, n \gg 0 \).

Moreover, \( \lambda(\text{Tor}_i(M/I^nM, N/J^mN)) = \lambda(\text{Tor}_i(M, N)) + \lambda(\text{Tor}_{i-1}(I^nM, N)) + \lambda(\text{Tor}_{i-1}(M, J^mN)) + \lambda(\text{Tor}_{i-2}(I^nM, J^mN)) \)

**Proof.** The first statement follows immediately from Theorem 6, since, trivially, its hypotheses are met. For the last statement, let’s observe that, there is a \( k \geq 0 \), such that, for all \( m \geq 0 \), and \( n \geq k \), \( \sigma^{m,n} \) in (3) is the zero map, by Proposition 5. It follows that \( \alpha_i^{m,n}(\text{im } \psi^{m,n}) = \alpha_i^{m,n}(\text{ker } \phi^{m,n}) = 0 \), hence \( \alpha_i^{m,n} \) factors through \( \text{im } \phi^{m,n} \), and thus (as before) \( \lambda(\text{im } \alpha_i^{m,n}) \) is eventually given by a polynomial in \( m, n \). Finally, by Proposition 5 again, we see that for each fixed \( m \), \( \text{im } (\alpha_i^{m,n}) \) vanishes for \( n \gg 0 \). Therefore, \( \text{im } (\alpha_i^{m,n}) \) is identically zero, for all large \( m \) and \( n \).

We also have the long exact sequence

\[
\cdots \to \text{Tor}_i(I^nM, N/J^mN) \xrightarrow{\alpha_i^{m,n}} \text{Tor}_i(M, N/J^mN) \to \text{Tor}_i(M/I^nM, N/J^mN) \to
\]
and we now know that \( \alpha_{i}^{m,n} = \alpha_{i-1}^{m,n} = 0 \) for \( m, n \gg 0 \). Then,

\[
\lambda(\text{Tor}_{i}(M/I^nM, N/J^mN)) = \lambda(\text{Tor}_{i}(M, N/J^mN)) + \lambda(\text{Tor}_{i-1}(I^nM, N/J^mN)).
\]

We apply this trick two more times. We have

\[
\cdots \rightarrow \text{Tor}_{i}(M, J^mN) \xrightarrow{0} \text{Tor}_{i}(M, N) \rightarrow \text{Tor}_{i}(M, N/J^mN) \rightarrow \text{Tor}_{i-1}(M, J^mN) \xrightarrow{0} \text{Tor}_{i-1}(M, N) \rightarrow \cdots,
\]

where the maps marked as 0 are so by Proposition 5. We get that

\[
\lambda(\text{Tor}_{i}(M, N/J^mN)) = \lambda(\text{Tor}_{i}(M, N)) + \lambda(\text{Tor}_{i-1}(M, J^mN)).
\]

Replacing \( M \) by \( I^nM \) in (7) and using the fact that \( \bigoplus_{m,n=0}^{\infty} \text{Tor}_{i}(I^nM, J^mN) \) is a finite bigraded \( S_2 \)-module, we see that the maps marked as 0 will remain so, for every \( n \) and large \( m \), again by Proposition 5. We then get that

\[
\lambda(\text{Tor}_{i-1}(I^nM, N/J^mN)) = \lambda(\text{Tor}_{i-1}(I^nM, N)) + \lambda(\text{Tor}_{i-2}(I^nM, J^mN)).
\]

Putting together (6), (8) and (9), we obtain

\[
\lambda(\text{Tor}_{i}(M/I^nM, N/J^mN)) = \lambda(\text{Tor}_{i}(M, N)) + \lambda(\text{Tor}_{i-1}(I^nM, N)) + \lambda(\text{Tor}_{i-1}(M, J^mN)) + \lambda(\text{Tor}_{i-2}(I^nM, J^mN)),
\]

as stated. \( \square \)

Note that this also yields a direct proof of the first statement of this theorem, since the four terms on the right-hand side of the equality above are eventually given by polynomials, by classical theory of finite (bi)graded modules.

Finally, we give an upper bound for the degree of the polynomial that arises in Corollary 8. Note that this estimate also applies to the case of Theorem 9.
Proposition 10. Assume the hypotheses in Corollary 8 and suppose that the length of $\text{Tor}_i(M/I^mM, J^mN)$ is given by a polynomial, for $m, n \gg 0$. Then

$$\text{deg} \lambda(\text{Tor}_i(M/I^mM, J^mN)) \leq \ell_M(I) + \ell_N(J) - 2.$$ 

Proof. This is a rather crude estimate, based on the one-variable case. We simply apply Corollary 4 in [ET], separately, for fixed, large enough values of $m$ and $n$, then add. For the exact degree in some special cases (in one variable, though), see [TM]. □

Acknowledgements

The work in this paper forms part of my dissertation written at the University of Kansas. I would particularly like to thank my advisor, Prof. D. Katz, for his constant patience and steady encouragements. I would also like to thank both D. Katz and A. Vraciu for their comments on this paper.

Special thanks also go to the referee for a number of valuable suggestions.

References

[WCB] Brown, William C., Hilbert Functions for Two Ideals, Journal of Algebra 225 (2000), 410-428.
[JBF] Fields, J. Bruce, Length Functions Determined by Killing Powers of Several Ideals in a Local Ring, Ph.D. Dissertation, University of Michigan (2000).
[GL] Levin, Gerson, Poincare Series of Modules over Local Rings, Proc. of the AMS 72, No. 1 (Oct. 1978), 6-10.
[DK] Kirby, David, Hilbert Functions and the Extension Functor, Math. Proc. Cambridge Philos. Soc. 105, No. 3 (1989), 441-446.
[VK] Kodiyalam, Vijay, Homological invariants of powers of an ideal., Proc. Amer. Math. Soc. 118, No. 3 (1993), 757-764.
[TM] Marley, Thomas, Hilbert Functions in Cohen-Macaulay Rings, Ph.D. Dissertation, Purdue University, (1989).
[KS] Shah, Kishor, On Equimultiple Ideals, Math. Z. 215, (1994), 13-24.
[ET] Theodorescu, Emanoil, Derived Functors and Hilbert Polynomials, Proc. Camb. Phil. Soc. 132 (2002), 75-88.

Department of Mathematics, University of Missouri, Columbia, MO, 65211
E-mail address: theodore@math.missouri.edu