Efficient Information Aggregation Strategies for Distributed Control and Signal Processing

by

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Abstract

This thesis will be concerned with distributed control and coordination of networks consisting of multiple, potentially mobile, agents. This is motivated mainly by the emergence of large scale networks characterized by the lack of centralized access to information and time-varying connectivity. Control and optimization algorithms deployed in such networks should be completely distributed, relying only on local observations and information, and robust against unexpected changes in topology such as link failures.

We will describe protocols to solve certain control and signal processing problems in this setting. We will demonstrate that a key challenge for such systems is the problem of computing averages in a decentralized way. Namely, we will show that a number of distributed control and signal processing problems can be solved straightforwardly if solutions to the averaging problem are available.

The rest of the thesis will be concerned with algorithms for the averaging problem and its generalizations. We will (i) derive the fastest known averaging algorithms in a variety of settings and subject to a variety of communication and storage constraints (ii) prove a lower bound identifying a fundamental barrier for averaging algorithms (iii) propose a new model for distributed function computation which reflects the constraints facing many large-scale networks, and nearly characterize the general class of functions which can be computed in this model.

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Chapter 1

Introduction

This thesis is about certain control and signal processing problems over networks with unreliable communication links. Some motivating scenarios are:

a. Distributed estimation: a collection of sensors are trying to estimate an unknown parameter from observations at each sensor.

b. Distributed state estimation: a collection of sensors are trying to estimate the (constantly evolving) state of a linear dynamical system from observations of its output at each sensor.

c. Coverage control: A group of robots wish to position themselves so as to optimally monitor an environment of interest.

d. Formation control: Several UAVs or vehicles are attempting to maintain a formation against random disturbances to their positions.

e. Distributed task assignment: allocate a collection of tasks among agents with individual preferences in a distributed way.

f. Clock synchronization. A collection of clocks are constantly drifting apart, and would like to maintain a common time as much as this is possible. Various pairs of clocks can measure noisy versions of the time offsets between them.

We will use “nodes” as a common word for sensors, vehicles, UAVs, and so on. A key assumption we will be making is that the communication links by means of which the nodes exchange messages are unreliable. A variety of simple and standard techniques can be used for any of the above problems if the communication links never fail. By contrast, we will be interested in the case when the links may “die” and “come online” unpredictably. We are interested in algorithms for the above problems which work even in the face of this uncertainty.

It turns out that a key problem for systems of this type is the problem of computing averages, described next. Nodes 1, . . . , n each begin with a real number \( x_i \). We are given a discrete sequence of times \( t = 1, 2, 3, \ldots \), and at each time step a communication graph \( G(t) = (\{1, \ldots, n\}, E(t)) \) is exogenously provided by “nature”
determining which nodes can communicate: node $i$ can send messages to node $j$ at time $t$ if and only if $(i, j) \in E(t)$. For simplicity, no restrictions are placed on the messages nodes can send to each other, and in particular, the nodes may broadcast their initial values to each other. The nodes need to compute $(1/n) \sum_{i=1}^{n} x_i$, subject to as few assumptions as possible about the communication sequence $G(t)$. We will call this the *averaging problem*, and we will call any algorithm for it an *averaging algorithm*.

The averaging problem is key in the sense that a variety of results are available describing how to use averaging algorithms to solve many other distributed problems with unreliable communication links. In particular, for each of the above problems (distributed estimation, distributed state estimation, coverage control, formation control, clock synchronization), averaging algorithms are a cornerstone of the best currently known solutions.

The remainder of this chapter will begin by giving a historical survey of averaging algorithms, followed by a list of the applications of averaging, including the problems on the previous page.

### 1.1 A history of averaging algorithms

The first paper to introduce distributed averaging algorithms was by DeGroot [36]. DeGroot considered a simple model of “attaining agreement:” $n$ individuals are on a team or committee and would like to come to a consensus about the probability distribution of a certain parameter $\theta$. Each individual $i$ begins with a probability distribution $F_i$ which they believe is the correct distribution of $\theta$. For simplicity, we assume that $\theta$ takes on a value in some finite set $\Omega$, so that each $F_i$ can be described by $|\Omega|$ numbers.

The individuals now update their probability distributions as a result of interacting with each other. Letting $F_i(t)$ be the distribution believed by $i$ at time $t$, the agents update as $F_i(t + 1) = \sum_j a_{ij} F_j(t)$, with the initialization $F_i(0) = F_i$. Here, $a_{ij}$ are weights chosen by the individuals. Intuitively, people may give high weights to a subset of the people they interact with, for example if a certain person is believed to be an expert in the subject. On the other hand, some weights may be zero, which corresponds to the possibility that some individual ignore each other’s opinions. It is assumed, however, that all the weights $a_{ij}$ are nonnegative, and $a_{11}, \ldots, a_{nn}$ add up to 1 for every $i$. Note that the coefficients $a_{ij}$ are independent of time, corresponding to a “static” communication pattern: individuals do not change how much they trust the opinions of others.

DeGroot gave a condition for these dynamics to converge, as well as a formula for the limiting opinion; subject to some natural symmetry conditions, the limiting opinion distribution will equal the average of the initial opinion distributions $F_i$. DeGroot’s work was later extended by Chatterjee and Seneta [30] to the case where the weights $a_{ij}$ vary with time. The paper [30] gave some conditions on the time-varying sequence $a_{ij}(t)$ required for convergence to agreement on a single distribution among the individuals.
The same problem of finding conditions on \( a_{ij}(t) \) necessary for agreement was addressed in the works \([91, 92, 14]\), which were motivated by problems in parallel computation. Here, the problem was phrased slightly differently: \( n \) processors each begin with a number \( x_i \) stored in memory, and the processors need to agree on a single number within the convex hull \([\min x_i, \max x_i]\). This is accomplished by iterating as \( x_i(t+1) = \sum_j a_{ij} x_j(t) \). This problem was a subroutine of several parallel optimization algorithms \([91]\). It is easy to see that it is equivalent to the formulation in terms of probability distributions addressed by DeGroot \([36]\) and Chatterjee and Seneta \([30]\).

The works \([91, 92, 14]\) gave some conditions necessary for the estimates \( x_i(t) \) to converge to a common value. These were in the same spirit as \([30]\), but were more combinatorial than \([30]\) being expressed directly in terms of the coefficients \( a_{ij}(t) \). These conditions boiled down to a series of requirements suggesting that the agents have repeated and nonvanishing influence over each other. For example, the coefficients \( a_{ij}(t) \) should not be allowed to decay to zero, and the graph sequence \( G(t) \) containing the edges \((i, j)\) for which \( a_{ji}(t) > 0 \) needs to be “repeatedly connected.”

Several years later, a similar problem was studied by Cybenko \([35]\) motivated by load balancing problems. In this context, \( n \) processors each begin with a certain number of jobs \( x_i \). The variable \( x_i \) can only be an integer, but assuming a large number of jobs in the system, this assumption may be dispensed with. The processors would like to equalize the load. To that end, they pass around jobs: processors with many jobs try to offload their jobs on their neighbors, and processors with few job ask for more requests from their neighbors. The number of jobs of processor \( i \) behave approximately as \( x_i(t+1) = x_i(t) + \sum_j a_{ij}(t) \). This subject to some conditions on the coefficients \( a_{ij} \), may be viewed as a special case of the iterations considered in \([30, 91, 92, 14]\).

Cybenko showed that when the neighborhood structure is a hypercube (i.e., we associate with each processor a string of \( \log n \) bits, and \( a_{ij}(t) \neq 0 \) whenever processors \( i \) and \( j \) differ by at most 1 bit), the above processes may converge quite fast: for some natural simple processes, an appropriately defined convergence time is on the order of \( \log n \) iterations.

Several years later, a variation on the above algorithms was studied by Vicsek et al. \([94]\). Vicsek et al. simulated the following scenario: \( n \) particles were placed randomly on a torus with random initial direction and constant velocity. Periodically, each particle would try to align its angle with the angles of all the particles within a certain radius. Vicsek et al. reported that the end result was that the particles aligned on a single direction.

The paper \([52]\) provided a theoretical justification of the results in \([94]\) by proving the convergence of a linearized version of the update model of \([94]\). The results of \([52]\) are very similar to the results in \([91, 92]\), modulo a number of minor modifications. The main difference appears to be that \([91, 92]\) makes certain assumptions on the sequence \( G(t) \) that are not made in \([52]\); these assumptions, however, are never actually used in the proofs of \([91, 92]\). We refer the reader to \([13]\) for a discussion.

The paper \([52]\) has created an explosion of interest in averaging algorithms, and the subsequent literature expanded in a number of directions. It is impossible to give a complete account of the literature since \([52]\) in a reasonable amount of space, so we
give only a brief overview of several research directions.

**Convergence in some natural geometric settings.** One interesting direction of research has been to analyze the convergence of some plausible geometric processes, for which there is no guarantee that any of the sufficient conditions for consensus (e.g. from [92] or [52]) hold. Notable in this direction was [34], which proved convergence of one such process featuring all-to-all communication with decaying strength. In a different direction, [27, 26] give tight bounds on the convergence of averaging dynamics when the communication graph corresponds to nearest neighbors of points in $R^k$.

**General conditions for averaging.** A vast generalization of the consensus conditions in [92] and [52] was given in [73]. Using set-valued generalizations of the Lyapunov functions in [92, 52], it was shown that a large class of possible nonlinear maps lead to consensus. Further investigation of these results was given in [2] and [70].

**Quantized consensus.** The above models assume that nodes can transmit real numbers to each other. It is natural to consider quantized versions of the above schemes. One may then ask about the tradeoffs between storage and performance. A number of papers explored various aspects of this tradeoffs. In [74], a simple randomized scheme for achieving approximate averaging was proposed. Further research along the same lines can be found in [103] and [10]. A dynamic scheme which allows us to approximately compute the average as the nodes communicate more and more bits with each other can be found in [21].

We will also consider this issue in this thesis, namely in Chapters 7 and 8 which are based on the papers [76] and [50], respectively. In Chapter 7, we will give a recipe for quantizing any linear averaging scheme to compute the average approximately. In Chapter 8, we will consider the problem of computing the averaging approximately with a deterministic algorithm, when each node can store only a constant number of bits for each link it maintains.

**Averaging with coordinates on geometric random graphs.** Geographic random graphs are common models for sensor networks. It is therefore of interest to try to specialize results for averaging to the case of geometric random graphs. Under the assumption that every node knows its own exact coordinates, an averaging algorithm with a lower than expected averaging cost was developed in [37]. Further research in [10, 93] reduced the energy cost even further. Substantial progress towards removing the assumption that each node knows its coordinates was recently made in [80].

**Design of fast averaging algorithms on fixed graphs.** It is interesting to consider the fastest averaging algorithm for a given, fixed graph. It is hoped that an answer would give some insight into averaging algorithms: what their optimal speed is, how it relates to graph structure, and so on. In [97], the authors showed how to compute optimal symmetric linear averaging algorithms with semidefinite programming. Some further results for specific graphs were given in [18, 19]. Optimization over a larger class of averaging algorithms was consider in [85] and also in [58].
Analysis of social networks. We have already mentioned the work of DeGroot [36] which was aimed at modeling the interactions of individual via consensus-like updates. A number of recent works has taken this line of analysis further by analyzing how the combinatorial structure of social networks affects the outcome. In particular, [47] studied how good social networks are at aggregating distributed information in terms of various graph-cut related quantities. The recent work [1] quantified the extent to which “forceful” agents which are not influenced by others interfere with information aggregation.

1.2 Applications of averaging

We give an incomplete list of the use of averaging algorithms in various applications.

a. Consider the following distributed estimation problem: a sensor network would like to estimate some unknown vector of parameters. At a discrete set of times, some sensors make noise-corrupted measurements of a linear function of the unknown vector. The sensors would like to combine the measurements that are coming in into a maximum likelihood estimate.

We discuss a simpler version of this problem in the following chapter, and describe some averaging-based algorithms. In brief, other known techniques (flooding, fusion along a spanning tree) suffer from either high storage requirements or lack of robustness to link failures. The use of averaging-based algorithms allows us to avoid these downfalls, as we will explain in Chapter 2. For some literature on this subject, we refer the reader to [98, 99] and [24].

b. Distributed state estimation: a collection of sensors are trying to estimate the (constantly evolving) state of a linear dynamical system. Each sensor is able to periodically make a noise corrupted measurement of the system output. The sensors would like to cooperate on synthesizing a Kalman filter estimate.

There are a variety of challenges involving in such a problem, not least of which is the delay involved in a receiving at one node the measurements from other nodes which are many hops away. Several ideas have been presented in the literature for solving this sort of problem based on averaging algorithms. We refer the reader to [81, 5, 82, 23, 39].

c. Coverage control is the problem of optimally positioning a set of robots to monitor an area. A typical case involves a polygon-shaped area along with robots which can measure distances to the boundary as well as to each other. Based on these distances, it is desirable to construct controllers which cover the entire area, yet assign as little area as possible to each robot. A common addition to this setup involves associating a number $f(x)$ to each point $x$ in the polygon, representing the importance of monitoring this point. The robots then optimize a corresponding objective function which weights regions according to the importance of the points in them.
It turned out that averaging algorithms have proven very useful in designing distributed controllers for such systems. We refer the reader to [43] for the connection between distributed controllers for these systems and averaging over a certain class of graphs defined by Voronoi diagrams. A similar approach was adopted in [89]. Note, also, the related paper [68] and the extension to nonuniform coverage in [69].

d. Formation control is the problem of maintaining a set formation, defined by a collection of relative distances, against random or adversarial disturbances. Every once in a while pairs of agents manage to measure the relative offset between them. The challenge is for the agents to use these measurements and take actions so that in the end everyone gets into formation.

We discuss this problem in greater detail in Chapter 2, where we explain several formation control ideas originating in [83]. Averaging theorems can show the possibility of working formation control in this setting, subject only to very intermittent communication.

e. The task assignment problem consists in distributing a set of tasks among a collection of agents. This arises, for example, in the case of a group of aircraft or robots who would like to make decisions autonomously without communication with a common base. A typical example is dividing a list of locations to be monitored among a group of aircraft.

In such cases, various auction-based methods are often used to allocate tasks. Averaging and consensus algorithms provide the means by which these auctions are implemented in a distributed way; we refer the reader to the papers [29, 96] for details.

f. Consider a collection of clocks which are constantly drifting apart. This is a common scenario, because clocks drift randomly depending on various factors within their environment (e.g., temperature), and also because clocks have a (nonzero) drift relative to the “true” time. Maintaining a common time as much as possible is important for a number of estimation problems (for example, direction of arrival problems).

An important problem is to design distributed protocols to keep the clocks synchronized. These try to keep clock drift to a minimum, at least between time periods when an outside source can inform each node of the correct time. This problem has a natural similarity to averaging, except that one does not care very much getting the average right, but rather agreement on any time will do. Moreover, the constantly evolving times present a further challenge.

A natural approach, explored in some of the recent literature, is to adopt averaging techniques to work in this setting. We refer the reader to [87, 22, 9, 41, 101].
1.3 Main contributions

This thesis is devoted to the analysis of the convergence time of averaging schemes and to the degradation in performance as a result of quantized communication. What follows is a brief summary of our contributions by chapter.

Chapters 2 and 3 are introductory. We begin with Chapter 2 which seeks to motivate the practical use of averaging algorithms. We compare averaging algorithms to other ways of aggregating information, such as flooding and leader-election based methods, and discuss the various advantages and disadvantages. Our main point is that schemes based on distributed averaging possess two unique strengths: robustness to link failures and economical storage requirements at each node.

Next, in Chapter 3, we discuss the most elementary known results on the convergence of averaging methods. The rest of this thesis will be spent on improving and refining the basic results in this chapter.

In Chapter 4 we give an exposition of the first polynomial-time convergence bound on the convergence time of averaging algorithms. Previously known bounds, such as those described in Chapter 3, took exponentially many steps in the number of nodes $n$ to converge, in the worst case. In the subsequent Chapter 5 we give an averaging algorithm whose convergence time scales as $O(n^2)$ steps on nearly arbitrary time-varying graph sequences. This is the currently best averaging algorithm in terms of convergence time bounds.

We next wonder if it is possible to design averaging algorithms which improve on this quadratic scaling. In Chapter 6 we prove that it is in fact impossible to beat the $n^2$ time steps bound within a large class of (possibly nonlinear) update schemes. The schemes we consider do not exhaust all possible averaging algorithms, but they do encompass the majority of averaging schemes proposed thus far in the literature.

We then move on to study the effect of quantized communication and storage. Chapter 7 gives a recipe for quantizing any linear averaging scheme. The quantization performs averaging while storing and transmitting only $c \log n$ bits. It is shown that this quantization preserves the convergence time bounds of the scheme, and moreover allows one to compute the average to any desired accuracy: by picking $c$ large (but not dependent on $n$), one can make the final result be as close to the average as desired.

In Chapter 8 we investigate whether it is possible to push down the $\log n$ storage down even further; in particular, we show how to the average may be approximately with a deterministic algorithm in which each node stores only a constant number of bits per every connection it maintains. An algorithm for fixed graphs is given; the dynamic graph case remains an open question.

Finally, Chapter 9 tackles the more general question: which functions can be computed with a decentralized algorithm which uses a constant number of bits per link? The chapter assumes a consensus-like termination requirement in which the nodes only have to get the right answer eventually, but are not required to know when they have done so. The main result is a nearly tight characterization of the functions which can be computed deterministically in this setting.
Chapter 2

Why averaging?

Our goal in this chapter is to motivate the study of distributed averaging algorithms. We will describe two settings in which averaging turns out to be singularly useful. The first is an estimation problem in a sensor network setting; we will describe an averaging-based solution which avoids the pitfalls which plague alternative schemes. The second is a formation maintenance problem; we will show how basic theorems on averaging allow us to establish the satisfactory performance of some formation control schemes.

2.1 A motivating example: distributed estimation in sensor networks

Figure 2-1: The set of online links at some given time.
Consider a large collection of sensors, $1, \ldots, n$, that want to estimate an unknown parameter $\theta \in \mathbb{R}^k$. Some of these sensors are able to measure a noise corrupted version of $\theta$; in particular, all nodes $i$ in some subset $S \subset \{1, \ldots, n\}$ measure

$$x_i = \theta + w_i.$$ 

We will assume, for simplicity, that the noises $w_i$ are jointly Gaussian and independent at different sensors. Moreover, only node $i$ knows the statistics of its noise $w_i \sim N(0, \sigma_i)$.

It is easy to see that the maximum likelihood estimate is given by

$$\hat{\theta} = \frac{\sum_{i \in S} x_i / \sigma_i^2}{\sum_{i \in S} 1 / \sigma_i^2}.$$ 

Note that if $S = \{1, \ldots, n\}$ (i.e. every node makes a measurement), and all the variances $\sigma_i^2$ are equal, the maximum likelihood estimate is just the average $\hat{\theta} = (1/n) \sum_{i=1}^n x_i$.

The sensors would like to compute $\hat{\theta}$ in a distributed way. We do not assume the existence of a “fusion center” to which the sensors can transmit measurements; rather, the sensors have to compute the answer by exchanging messages with their neighbors and performing computations.

The sensors face an additional problem: there are communication links available through which they can exchange messages, but these links are unreliable. In particular, links fail and come online in unpredictable ways. For example, there is no guarantee that any link will come online if the sensors wait long enough. It is possible for a link to be online for some time, and then fail forever. Figure 2.1 shows an example of what may happen: at any given time, only some pairs of nodes may exchange messages, and the network is effectively split into disconnected clusters.

More concretely, we will assume a discrete sequence of times $t = 1, 2, 3, \ldots$, during which the sensors may exchange messages. At time $t$, sensor $i$ may send a message to its neighbors in the undirected graph $G(t) = (\{1, \ldots, n\}, E(t))$. We will also assume that the graph $G(t)$ includes all self loops $(i, i)$. The problem is to devise good algorithms for computing $\hat{\theta}$, and to identify minimal connectivity assumptions on the sequence $G(t)$ under which such a computation is possible.

### 2.1.1 Flooding

We now describe a possible answer. It is very plausible to make an additional assumption that sensors possess unique identifiers; this is the case in almost any wireless system. The sensors can use these identifiers to “flood” the network so that eventually, every sensor knows every single measurement that has been made.

At time 1, sensor $i$ sends its own triplet $(\text{id}_i, x_i, \sigma_i^2)$ to each of its neighbors. Each sensor stores all the messages it has received. Moreover, a sensor maintains a “to broadcast” queue, and each time it hears a message with an id it has not heard before, it adds it to the tail of the queue. At times $t = 2, 3, \ldots$, each sensor broadcasts the
top message from its queue.

If $G(t)$ is constant with time, and connected, then eventually each sensor learns all the measurements that have been made. Once that happens, each sensor has all the information it needs to compute the maximum likelihood estimate. Moreover, the sensors do not even need to try to detect whether they have learned everything; each sensor can simply maintain an estimate of $\hat{\theta}$, and revise that estimate each time it learns of a new measurement.

If $G(t)$ is not fixed, flooding can still be expected to work. Indeed, each time a link appears, there is opportunity for a piece of information to be learned. One can show that subject to only very minimal requirements on connectivity, every sensor eventually does learn every measurement.

Let us state a theorem to this effect. We will use the notation $\cup_{t \in X} G(t)$ to mean the graph obtained by forming the union of the edge sets of $G(t)$, $t \in X$, i.e.

$$\bigcup_{t \in X} G(t) = \left( \{1, \ldots, n\}, \bigcup_{t \in X} E(t) \right).$$

A relatively light connectivity assumption is the following.

**Assumption 2.1.** (Connectivity) The graph $\cup_{s \geq t} G(s)$ is connected for every $t$.

In words, this assumption says that the graph sequence $G(t)$ has enough edges for connectivity, and that moreover this remains true after some finite set of graphs is removed from the sequences.

**Theorem 2.1.** If Assumption 2.1 holds, then under the flooding protocol every node eventually learns each triplet $(\text{id}_j, x_i, \sigma_i^2)$.

*Proof. (Sketch).* Suppose that some triplet $(\text{id}_j, x_i, \sigma_i^2)$ is not learned by node $j$. Let $A$ be the nonempty set of nodes that do learn this triplet; one can easily argue that the number of edges in the graphs $E(t)$ between $A$ and $A^c$ is finite. But this contradicts Assumption 2.1. \qed

It is possible to relax the assumption of this theorem: $\cup_{s \geq t} G(s)$ actually only needs to be connected for a sufficiently long but finite time interval.

The problem with flooding, however, lies with its storage requirements: each sensor needs to store $n$ pieces of information, i.e., it needs to store a list of id’s whose measurements it has already seen. This means that the total amount of storage throughout the network is at least on the order of $n^2$.

We count the storage requirements of the numbers $x_i, \sigma_i^2$ as constant. When dealing with estimation problems, it is convenient to assume that $x_i, \sigma_i^2$ are real.
numbers, and we will maintain this technical assumption for now. Nevertheless, in practice, these numbers will be truncated to some fixed number of bits (independent of $n$). Thus it makes sense to think of transmitting each of the $x_i, \sigma_i^2$ as incurring a fixed cost independent of $n$.

Thus tracing out the dependence on $n$, we have that at least $n^2$ bits must be stored, in addition to a fixed number of bits at each node to maintain truncated versions of $x_i, \sigma_i^2$ and the estimate $\hat{\theta}$. One hopes for the existence of a scheme whose storage requirements scale more gracefully with the number of nodes $n$.

### 2.1.2 Leader election based protocols

The protocol we outline next has considerably nicer storage requirements. On the other hand, it will require some stringent assumptions on connectivity. We describe it next for the case of a fixed communication graph, i.e., when the graph sequence $G(t)$ does not depend on $t$.

First, the sensors elect one of them as a leader. There are various protocols for doing this. If sensors have unique identifiers they may pick (in a distributed way) the sensor with the largest or smallest id as the leader. Even in the absence of identifiers, there are randomized protocols for leader election (see [4]) which take on the order of the diameter of the network time steps, provided that messages of size $O(\log n)$ bits can be sent at each time step.

Next, the sensors can build a spanning tree with the leader as the root. For example, each sensor may pick as its parent the node one hop closer to the root. Finally, the sensors may forward all of their information (i.e., their $(x_i, \sigma_i^2)$) to the root, which can compute the answer and forward it back.

Our description of the algorithm is deliberately vague, as the details do not particularly matter (e.g., which leader election algorithm is used). We would like to mention, however, that it is even possible to avoid having the leader learn all the information $(x_i, \sigma_i^2)$. For example, once a spanning tree is in place, each node may wait to hear from all of its children, and then forward to the leader a sufficient statistic for the measurements in its subtree.

We state the existence of such protocols as a theorem:

**Theorem 2.2.** If $G(t)$ does not depend on time, i.e., $G(t) = G$ for all $t$, it is possible to compute $\hat{\theta}$ with high probability in $O(d(G))$ time steps. The nodes need to store and forward messages of size $O(\log n)$ bits in each time step, as well a constant number of real numbers which are smooth functions of the measurements $x_i, \sigma_i^2$.

Observe that that the theorem allows for the existence of protocols which (only) work with high probability; this is due to the necessarily randomized nature of leader election protocols in the absence of identifiers.

The above theorem is nearly optimal for any fixed graph. In other words, if the communication links are reliable, there is no reason to choose anything but the above algorithm.

---

1The diameter of the graph $G$, denoted by $d(G)$, is the largest distance between any two nodes.
On the other hand, if the graph sequence $G(t)$ is changing unpredictably, this sort of approach immediately runs into problems. Maintaining a spanning tree appears to be impossible if the graph sequence changes dramatically from step to step, and other approaches are needed.

### 2.2 Using distributed averaging

We now describe a scheme which is both robust to link failures (it only needs Assumption 2.1 to work) and has nice storage requirements (only requires nodes to maintain a constant number of real numbers which are smooth functions of the data $x_i, \sigma_i^2$). However, it needs the additional assumption that the graphs $G(t)$ are undirected. This condition is often satisfied in practice, for example if the sensors are connected by links whenever they are within a certain distance of each other.

First, let us introduce some notation. Let $N_i(t)$ be the set of neighbors of node $i$ in the graph $G(t)$. Recall that we assume self-arcs are always present, i.e., $(i, i) \in E(t)$ for all $i, t$, so that $i \in N_i(t)$ for all $i, t$. Let $d_i(t)$ be the degree of node $i$ in $G(t)$.

Let us first describe the scheme for the case where $S = \{1, \ldots, n\}$, i.e., every node makes a measurement, and all $\sigma_i^2$ are the same. In this case, the maximum likelihood estimate is just the average of the numbers $x_i$: $\hat{\theta} = (1/n) \sum_{i=1}^n x_i$.

The scheme is as follows. Each node sets $x_i(0) = x_i$ and updates as

$$x_i(t + 1) = \sum_{j \in N_i(t)} a_{ij}(t)x_j(t), \quad (2.1)$$

where

$$a_{ij}(t) = \min \left( \frac{1}{d_i(t)}, \frac{1}{d_j(t)} \right), \quad \text{for } j \in N_i(t), j \neq i$$

$$= 0, \quad \text{otherwise.}$$

$$a_{ii}(t) = 1 - \sum_{j \in N_i(t)} a_{ij}. $$

Then:

**Proposition 2.1.** If Assumption 2.1 holds and all the graphs $G(t)$ are undirected, then

$$\lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{i=1}^n x_i = \hat{\theta}. $$

The above proposition is true because it is a special case of the following theorem:

**Theorem 2.3.** Consider the iteration

$$x(t + 1) = A(t)x(t)$$

where:
1. A(t) are doubly stochastic matrices.

2. If (i, j) ∈ G(t), then a_{ij} > 0 and a_{ji} > 0, and if (i, j) ∉ G(t), then a_{ij} = a_{ji} = 0.

3. There is some η > 0 such that if a_{ij} > 0 then a_{ij} > η.

4. The graph sequence G(t) is undirected.

5. Assumption 2.1 on the graph sequence G(t) holds.

Then:

\[ \lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{j=1}^{n} x_j(0), \]

for all i.

We will prove this theorem in Chapter 3. In this form, this theorem is a trivial modification of the results in [67, 25, 73, 49, 17] which themselves are based on the earlier results in [92, 52].

Accepting the truth of this theorem for now, we can conclude that we have described a simple way to compute \( \hat{\theta} \). Every node just needs to store and update a single real number \( x_i(t) \). Nevertheless, subject to only the weak connectivity Assumption 2.1, every \( x_i \) approaches the correct \( \theta \). This scheme thus manages to avoid the downsides that plague flooding and leader election (high storage, lack of robustness to link failures).

Let us describe next how to use this idea in the general case where \( S \) is a proper subset of \( \{1, \ldots, n\} \) and the \( \sigma_i^2 \) are not all equal. Each node sets \( x_i(0) = x_i / \sigma_i^2 \), \( y_i(0) = 1 / \sigma_i^2 \), and \( z_i(0) = x_i(0) / y_i(0) \). If the node did not make a measurement, it sets \( x_i(0) = y_i(0) = 0 \), and leaves \( z_i(0) \) undefined. Each node updates as

\[
\begin{align*}
x_i(t + 1) &= \sum_{j \in N_i(t)} a_{ij}(t)x_j(t) \\
y_i(t + 1) &= \sum_{j \in N_i(t)} a_{ij}(t)y_j(t) \\
z_i(t + 1) &= \frac{x_i(t)}{y_i(t)}
\end{align*}
\]

Observe that and we have:

**Proposition 2.2.** If Assumption 2.1 holds, then

\[ \lim_{t \to \infty} z_i(t) = \hat{\theta}. \]

---

2A matrix is called doubly stochastic if it is nonnegative and all of its rows and columns add up to 1.
Proof. By Theorem 2.1

\[
\lim_{t \to \infty} x_i(t) = (1/n) \sum_{i \in S} x_i/\sigma_i^2
\]

\[
\lim_{t \to \infty} y_i(t) = (1/n) \sum_{i \in S} 1/\sigma_i^2
\]

and so

\[
\lim_{t \to \infty} z_i(t) = \frac{\lim_{t \to \infty} x_i(t)}{\lim_{t \to \infty} y_i(t)} = \frac{\sum_{i \in S} x_i/\sigma_i^2}{\sum_{i \in S} 1/\sigma_i^2} = \hat{\theta}.
\]

The punchline is that even this somewhat more general problem can be solved by an algorithm that relies on Theorem 2.1. This solution has nice storage requirements (nodes store only a constant number of real numbers which are smooth functions of \(y_i, \sigma_i^2\)) and is robust to link failures.

2.3 A second motivating example: formation control

We now give another application of Theorem 2.1, this time to a certain formation control problem. Our exposition is based on [83].

Suppose that the nodes have real positions \(x_i(t)\) in \(\mathbb{R}^k\); the initial positions \(x_i(0)\) are arbitrary, and the nodes want to move into a formation characterized by positions \(p_1, \ldots, p_n\) in \(\mathbb{R}^k\) (formations are defined up to translation). This formation is uniquely characterized by the offset vectors \(r_{ij} = p_i - p_j\). We assume that at every \(t\), various pairs of nodes \(i, j\) succeed in measuring the offsets \(x_i(t) - x_j(t)\). Let \(E(t)\) be the set of (undirected) edges \((i, j)\) corresponding to the measurements. The problem is how to use these intermittent measurements to get into the desired formation.

Let us assume for simplicity that our \(x_i(t)\) lie in \(\mathbb{R}\) (we will dispense with this assumption shortly). A very natural idea is to perform gradient descent on the function

\[
\sum_{(i,j) \in E(t)} (x_i(t) - x_j(t) - r_{ij})^2.
\]

This leads to the following control law:

\[
x_i(t + 1) = x_i(t) - 2\Delta \sum_{j \in N_i(t)} (x_i(t) - x_j(t)) + 2\Delta \sum_{j \in N_i(t)} r_{ij}, \tag{2.2}
\]

where \(\Delta\) is the stepsize. Essentially, every node repeatedly “looks around” and moves to a new position depending on the positions of its neighbors and its desired offset vectors \(r_{ij}\).
Defining \( b_i(t) = 2\Delta \sum_{j \in N_i(t)} r_{ij} \), the above equations may be rewritten as

\[
x(t + 1) = A(t)x(t) + b(t).
\]

Now let \( z \) be any translate of the given positions \((p_1, \ldots, p_n)\). Then, \( z \) satisfies

\[
z = A(t)z + b(t),
\]

because the gradient at \( x_i = z_i \) equals 0. Subtracting the two equations we get

\[
x(t + 1) - z = A(t)(x(t) - z).
\]

Observe that if \( \Delta \leq 1/(2n) \), then the above matrix is nonnegative, symmetric, and has rows that add up to 1. Applying now Theorem 2.1 we get the following statement:

**Proposition 2.3.** Suppose that the nodes implement the iteration of Eq. (2.2). If:

1. \( z' \) is the translate of \( p \) whose average is the same as the average of \( x_i(0) \).
2. The communication graph sequence \( G(t) \) satisfies Assumption 1 (connectivity).
3. \( \Delta \leq 1/(2n) \)

Then,

\[
\lim_{t \to \infty} x_i(t) = z'_i,
\]

for all \( i \).

The proof is a straightforward application of Theorem 2.1. This theorem tells us that subject to only minimal conditions on connectivity, the scheme of Eq. (2.2) will converge to the formation in question.

We remark that we can replace the assumption that the \( x_i \) are real numbers with the assumption that the \( x_i \) belong to \( \mathbb{R}^k \). In this case, we can apply the control law of Eq. (2.2), which decouples along each component of \( x_i(t) \), and apply Proposition 2.3 to each component of \( x_i(t) \).

Finally, we note that continuous-time versions of these updates may be presented; see [83].

### 2.4 Concluding remarks

Our goal in this chapter has been to explain why averaging algorithms are useful. We have described averaging-based algorithms for estimation and formation problems which are robust to link failures and have very light storage requirements at each node.

Understanding the tradeoff between various types of distributed algorithms is still very much an open question, and the discussion in this chapter has only scratched the surface of it. One might additionally wonder how averaging algorithms perform...
on a variety of other dimensions: convergence time, energy expenditure, robustness to node failures, performance degradation with noise, and so on.

Most of this thesis will be dedicated to exploring the question of convergence time. In the next chapter we will give a basic introduction to averaging algorithms, and in particular we will furnish a proof of Theorem 2.1. In the subsequent chapters, will turn to the question of designing averaging algorithms with good convergence time guarantees.
Chapter 3

The basic convergence theorems

Here we begin our analysis of averaging algorithm by proving some basic convergence results. Our main goal is to prove Theorem 2.1 from the previous chapter, as well as some natural variants of it. Almost all of this thesis will be spent refining and improving the simple results obtained by elementary means in this chapter. The results we will present may be found in [91, 92, 14, 52, 67, 17]. We also recommend the paper [73] for a considerable generalization of the results found here. The material presented here appeared earlier in the paper [17] and the M.S. thesis [77].

3.1 Setup and assumptions

We consider a set \( N = \{1, \ldots, n\} \) of nodes, each starting with a real number stored in memory. The nodes attempt to compute the average of these numbers by broadcasting these numbers and repeatedly combining them by forming convex combination. We will first only be concerned with the convergence of this process.

Each node \( i \) starts with a scalar value \( x_i(0) \). The vector \( x(t) = (x_1(t), \ldots, x_n(t)) \) with the values held by the nodes at time \( t \), is updated according to the equation \( x(t + 1) = A(t)x(t) \), or

\[
x_i(t + 1) = \sum_{j=1}^{n} a_{ij}(t)x_j(t),
\]

where \( A(t) \) is a nonnegative matrix with entries \( a_{ij}(t) \), and where the updates are carried out at some discrete set of times which we will take, for simplicity, to be the nonnegative integers. We will refer to this scheme as the agreement algorithm.

We will assume that the row-sums of \( A(t) \) are equal to 1, so that \( A(t) \) is a stochastic matrix. In particular, \( x_i(t + 1) \) is a weighted average of the values \( x_j(t) \) held by the nodes at time \( t \). We are interested in conditions that guarantee the convergence of each \( x_i(t) \) to a constant, independent of \( i \).

Throughout, we assume the following.

**Assumption 3.1 (non-vanishing weights).** The matrix \( A(t) \) is nonnegative, stochastic, and has positive diagonal. Moreover, there exists some \( \eta > 0 \) such that if \( a_{ij}(t) > 0 \) then \( a_{ij}(t) > \eta \).

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Intuitively, whenever \( a_{ij}(t) > 0 \), node \( j \) communicates its current value \( x_j(t) \) to node \( i \). Each node \( i \) updates its own value by forming a weighted average of its own value and the values it has just received from other nodes.

The communication pattern at each time step can be described in terms of a directed graph \( G(t) = (N, E(t)) \), where \((j, i) \in E(t)\) if and only if \( a_{ij}(t) > 0 \). Note that \((i, i) \in E(t)\) for all \( i, t \) since \( A(t) \) has positive diagonal. A minimal assumption is that starting at an arbitrary time \( t \), and for any \( i, j \), there is a sequence of communications through which node \( i \) will influence (directly or indirectly) the value held by node \( j \). This is Assumption 2.1 from the previous chapter.

We note various special cases of possible interest.

**Fixed coefficients:** There is a fixed matrix \( A \), with entries \( a_{ij} \) such that, for each \( t \), and for each \( i \neq j \), we have \( a_{ij}(t) \in \{0\} \cup \{a_{ij}\} \) (depending on whether there is a communication from \( j \) to \( i \) at that time). This is the case presented in [14].

**Symmetric model:** If \((i, j) \in E(t)\) then \((j, i) \in E(t)\). That is, whenever \( i \) communicates to \( j \), there is a simultaneous communication from \( j \) to \( i \).

**Equal neighbor model:** Here,

\[
a_{ij}(t) = \begin{cases} 
1/d_i(t), & \text{if } j \in N_i(t), \\
0, & \text{if } j \notin N_i(t), 
\end{cases}
\]

This model is a linear version of a model considered by Vicsek et al. [94]. Note that here the constant \( \eta \) of Assumption 3.1 is equal to \( 1/n \).

**Metropolis model:** Here,

\[
a_{ij}(t) = \begin{cases} 
1/\max(d_i(t), d_j(t)), & \text{if } j \in N_i(t), i \neq j \\
0, & \text{if } j \notin N_i(t), 
\end{cases}
\]

and

\[
a_{ii}(t) = 1 - \sum_{j=1}^{n} a_{ij}(t).
\]

The Metropolis model is similar to the equal-neighbor model, but has the advantage of symmetry: \( a_{ij}(t) = a_{ji}(t) \).

**Pairwise averaging model ([20]):** This is the special case of both the symmetric model and of the equal neighbor model in which, at each time, there is a set of disjoint pairs of nodes who communicate with each other. If \( i \) communicates with \( j \), then \( x_i(t+1) = x_j(t+1) = (x_i(t) + x_j(t))/2 \). Note that the sum \( x_1(t) + \cdots + x_n(t) \) is conserved; therefore, if consensus is reached, it has to be on the average of the initial values of the nodes.

The assumption below is a strengthening of Assumption 2.1 on connectivity. We will see that it is sometimes necessary for convergence.
Assumption 3.2 (B-connectivity). There exists an integer $B > 0$ such that the directed graph 

$$(N, E(kB) \cup E((k + 1)B) \cup \cdots \cup E((k + 1)B - 1))$$

is strongly connected for all integer $k \geq 0$.

3.2 Convergence results in the absence of delays.

We say that the agreement algorithm guarantees asymptotic consensus if the following holds: for every $x(0)$, and for every sequence $\{A(t)\}$ allowed by whatever assumptions have been placed, there exists some $c$ such that $\lim_{t \to \infty} x_i(t) = c$, for all $i$.

Theorem 3.1. Under Assumptions 3.1 (non-vanishing weights) and 3.2 (B-connectivity), the agreement algorithm guarantees asymptotic consensus.

Theorem 3.1 may be found in [52]; a slightly different version is in [92, 91]. We next give an informal account of its proof.

Sketch of proof. The proof has several steps.

Step 1: Let us define the notion of a path in the time-varying graph $G(t)$. A path $p$ from $a$ to $b$ of length $l$ starting at time $t$ is a sequence of edges $(k_0, k_1), (k_1, k_2), \ldots, (k_{l-1}, k_l)$ such that $k_0 = a$, $k_l = b$, and $(k_0, k_1) \in E(t), (k_1, k_2) \in E(t + 1), \ldots$, and so on. We will use $c(p)$ to denote the product

$$c(p) = \prod_{i=0}^{l-1} a_{k_i, k_{i+1}}.$$

Define

$$\Phi(t_1, t_2) = A(t_2 - 1)A(t_2 - 2) \cdots A(t_1),$$

and let the $(i, j)$’th entry of this matrix be denoted by $\phi_{i,j}(t_1, t_2)$. The following fact can be established by induction:

$$\phi_{i,j}(t_1, t_2) = \sum_{\text{paths } p \text{ from } i \text{ to } j \text{ of length } t_2 - t_1 \text{ starting at time } t_1} c(p).$$

A consequence is that if $\phi_{i,j}(t_1, t_2) > 0$, then Assumption 3.1 implies $\phi_{i,j}(t_1, t_2) \geq \eta^{t_2 - t_1}$.

Step 2: Assumptions 3.1 and 3.2 have the following implication: for any two nodes $i, j$, there is a path of length $nB$ that begins at $i$ and ends at $j$.

This implication may be proven by induction on the following statement: for any node $i$ there are at least $m$ distinct nodes $j$ such that there is a path of length $mB$ from $i$ to $j$. The proof crucially relies on Assumption 3.1 which implies that all the self loops $(i, i)$ belong to every edge set $E(t)$. 
Step 3: Putting Steps 1 and 2 together, we get that $\Phi(kB,(k+n)B)$ is a matrix whose every entry is bounded below by $\eta^{nB}$. The final step is to argue that

$$Q_nQ_{n-1}\cdots Q_1x$$

converges to a multiple of the all-ones vector $1$ for any sequence of matrices $Q_i$ having this property and any initial vector $x$. This is true because for any such matrix $Q_i$,

$$\max_k(Q_i x)_k - \min_k(Q_i x)_k \leq (1 - \eta^{nB}) (\max_k x_k - \min_k x_k).$$

\[\square\]

In the absence of $B$-connectivity, the algorithm does not guarantee asymptotic consensus, as shown by Example 1 below (Exercise 3.1, in p. 517 of [14]). In particular, convergence to consensus fails even in the special case of the equal neighbor model. The main idea is that the agreement algorithm can closely emulate a nonconvergent algorithm that keeps executing the three instructions $x_1 := x_3$, $x_3 := x_2$, $x_2 := x_1$, one after the other.

Example 1. Let $n = 3$, and suppose that $x(0) = (0, 0, 1)$. Let $\epsilon_1$ be a small positive constant. Consider the following sequence of events. Node 3 communicates to node 1; node 1 forms the average of its own value and the received value. This is repeated $t_1$ times, where $t_1$ is large enough so that $x_1(t_1) \geq 1 - \epsilon_1$. Thus, $x(t_1) \approx (1, 0, 1)$. We now let node 2 communicates to node 3, $t_2$ times, where $t_2$ is large enough so that $x_3(t_1 + t_2) \leq \epsilon_1$. In particular, $x(t_1 + t_2) \approx (1, 0, 0)$. We now repeat the above two processes, infinitely many times. During the $k$th repetition, $\epsilon_1$ is replaced by $\epsilon_k$ (and $t_1, t_2$ get adjusted accordingly). Furthermore, by permuting the nodes at each repetition, we can ensure that Assumption 2.1 is satisfied. After $k$ repetitions, it can be checked that $x(t)$ will be within $1 - \epsilon_1 - \cdots - \epsilon_k$ of a unit vector. Thus, if we choose the $\epsilon_k$ so that $\sum_{k=1}^{\infty} \epsilon_k < 1/2$, asymptotic consensus will not be obtained.

On the other hand, in the presence of symmetry, the $B$-connectivity Assumption 3.2 is unnecessary. This result is proved in [67] and [25] for the special case of the symmetric equal neighbor model and in [73, 49], for the more general symmetric model. A more general result will be established in Theorem 3.4 below.

Theorem 3.2. Under Assumptions 2.1 and 3.1, and for the symmetric model, the agreement algorithm guarantees asymptotic consensus.

3.3 Convergence in the presence of delays.

The model considered so far assumes that messages from one node to another are immediately delivered. However, in a distributed environment, and in the presence of communication delays, it is conceivable that a node will end up averaging its own value with an outdated value of another node. A situation of this type falls within the framework of distributed asynchronous computation developed in [14].
Communication delays are incorporated into the model as follows: when node $i$, at time $t$, uses the value $x_j$ from another node, that value is not necessarily the most recent one, $x_j(t)$, but rather an outdated one, $x_j(\tau^i_j(t))$, where $0 \leq \tau^i_j(t) \leq t$, and where $t - \tau^i_j(t)$ represents communication and possibly other types of delay. In particular, $x_i(t)$ is updated according to the following formula:

$$x_i(t + 1) = \sum_{j=1}^{n} a_{ij}(t)x_j(\tau^i_j(t)). \quad (3.2)$$

We make the following assumption on the $\tau^i_j(t)$.

Assumption 3.3. (Bounded delays) (a) If $a_{ij}(t) = 0$, then $\tau^i_j(t) = t$.
(b) $\tau^i_i(t) = t$, for all $i$, $t$.
(c) There exists some $B > 0$ such that $t - B + 1 \leq \tau^i_j(t) \leq t$, for all $i$, $j$, $t$.

Assumption 3.3(a) is just a convention: when $a_{ij}(t) = 0$, the value of $\tau^i_j(t)$ has no effect on the update. Assumption 3.3(b) is quite natural, since a node generally has access to its own most recent value. Assumption 3.3(c) requires delays to be bounded by some constant $B$.

The next result, from [91, 92], is a generalization of Theorem 3.1. The idea of the proof is similar to the one outlined for Theorem 3.1, except that we now define $m(t) = \min_{s=t,t-1,...,t-B+1} x_i(s)$ and $M(t) = \max_{s=t,t-1,...,t-B+1} x_i(s)$. For convenience, we will adopt the definition that $x_i(t) = x_i(0)$ for all negative $t$. Once more, one shows that the difference $M(t) - m(t)$ decreases by a constant factor after a bounded amount of time.

Theorem 3.3. Under Assumptions 3.1, 3.2, 3.3 (non-vanishing weights, bounded intercommunication intervals, and bounded delays), the agreement algorithm with delays [cf. Eq. (3.2)] guarantees asymptotic consensus.

Theorem 3.3 assumes bounded intercommunication intervals and bounded delays. The example that follows (Example 1.2, in p. 485 of [14]) shows that Assumption 3.3(d) (bounded delays) cannot be relaxed. This is the case even for a symmetric model, or the further special case where $E(t)$ has exactly four arcs $(i, i)$, $(j, j)$, $(i, j)$, and $(j, i)$ at any given time $t$, and these satisfy $a_{ij}(t) = a_{ji}(t) = 1/2$, as in the pairwise averaging model.

Example 2. We have two nodes who initially hold the values $x_1(0) = 0$ and $x_2(0) = 1$, respectively. Let $t_k$ be an increasing sequence of times, with $t_0 = 0$ and $t_{k+1} - t_k \to \infty$. If $t_k \leq t < t_{k+1}$, the nodes update according to

$$x_1(t + 1) = (x_1(t) + x_2(t_k))/2,$$
$$x_2(t + 1) = (x_1(t_k) + x_2(t))/2.$$  

We will then have $x_1(t_1) = 1 - \epsilon_1$ and $x_2(t_1) = \epsilon_1$, where $\epsilon_1 > 0$ can be made arbitrarily small, by choosing $t_1$ large enough. More generally, between time $t_k$ and
the absolute difference $|x_1(t) - x_2(t)|$ contracts by a factor of $1 - 2\epsilon_k$, where the corresponding contraction factors $1 - 2\epsilon_k$ approach 1. If the $\epsilon_k$ are chosen so that $\sum_k \epsilon_k < \infty$, then $\prod_{k=1}^{\infty} (1 - 2\epsilon_k) > 0$, and the disagreement $|x_1(t) - x_2(t)|$ does not converge to zero.

According to the preceding example, the assumption of bounded delays cannot be relaxed. On the other hand, the assumption of bounded intercommunication intervals can be relaxed, in the presence of symmetry, leading to the following generalization of Theorem 3.2

Theorem 3.4. Under Assumptions 2.1 (connectivity), 3.1 (non-vanishing weights), and 3.3 (bounded delays), and for the symmetric model, the agreement algorithm with delays [cf. Eq. (3.2)] guarantees asymptotic consensus.

Proof. Let

$$M_i(t) = \max\{x_i(t), x_i(t-1), \ldots, x_i(t-B+1)\},$$
$$M(t) = \max_i M_i(t),$$
$$m_i(t) = \min\{x_i(t), x_i(t-1), \ldots, x_i(t-B+1)\},$$
$$m(t) = \min_i m_i(t).$$

Recall that we are using the convention that $x_i(t) = x_i(0)$ for all negative $t$. An easy inductive argument, as in p. 512 of [14], shows that the sequences $m(t)$ and $M(t)$ are nondecreasing and nonincreasing, respectively. The convergence proof rests on the following lemma.

Lemma 3.1. If $m(\tau - B) = 0$ and $M(\tau) = 1$, then there exists a time $\tau' \geq \tau$ such that $M(\tau') - m(\tau' - B) \leq 1 - \eta^B$.

Given Lemma 1, the convergence proof is completed as follows. Using the linearity of the algorithm, there exists a time $\tau_1$ such that $M(\tau_1) - m(\tau_1 - B) \leq (1 - \eta^B)(M(B) - m(0))$. By applying Lemma 1, with $\tau$ replaced by $\tau_1$, and using induction, we see that for every $k$ there exists a time $\tau_k$ such that $M(\tau_k) - m(\tau_k - B) \leq (1 - \eta^B)^k(M(B) - m(0))$, which converges to zero. This, together with the monotonicity properties of $m(t)$ and $M(t)$, implies that $m(t)$ and $M(t)$ converge to a common limit, which is equivalent to asymptotic consensus.

Proof of Lemma 3.1. For $k = 1, \ldots, n$, we say that “Property $P_k$ holds at time $t$” if there exist at least $k$ indices $i$ for which $m_i(t) \geq \eta^k B$.

We assume, without loss of generality, that $m(\tau - B) = 0$ and $M(\tau) = 1$. Then, $m(t) \geq 0$ for all $t \geq \tau - B$ by the monotonicity of $m(t)$. Furthermore, there exists some $i$ and some $t' \in \{\tau - B + 1, \tau - B + 2, \ldots, \tau\}$ such that $x_i(t') = 1$. Using the inequality $x_i(t+1) \geq \eta x_i(t)$, we obtain $m_i(t' + B) \geq \eta^B$. This shows that there exists a time at which property $P_1$ holds.

We continue inductively. Suppose that $k < n$ and that Property $P_k$ holds at some time $t$. Let $S$ be a set of cardinality $k$ containing indices $i$ for which $m_i(t) \geq \eta^k B$, and...
let $S^c$ be the complement of $S$. Let $t'$ be the first time, greater than or equal to $t$, at which $a_{ij}(t') 
eq 0$, for some $j \in S$ and $i \in S^c$ (i.e., an node $j$ in $S$ gets to influence the value of an node $i$ in $S^c$). Such a time exists by the connectivity assumption (Assumption 2.1).

Note that between times $t$ and $t'$, the nodes $\ell$ in the set $S$ only form convex combinations between the values of the nodes in the set $S$ (this is a consequence of the symmetry assumption). Since all of these values are bounded below by $\eta^kB$, it follows that this lower bound remains in effect, and that $m_\ell(t') \geq \eta^kB$, for all $\ell \in S$.

For times $s \geq t'$, and for every $\ell \in S$, we have $x_\ell(s+1) \geq \eta x_\ell(s)$, which implies that $x_\ell(s) \geq \eta^{kB}\eta^B$, for $s \in \{t'+1, \ldots, t'+B\}$. Therefore, $m_\ell(t'+B) \geq \eta^{(k+1)B}$, for all $\ell \in S$.

Consider now an node $i \in S^c$ for which $a_{ij}(t') 
eq 0$. We have

$$x_i(t'+1) \geq a_{ij}(t')x_j(\tau_j(t')) \geq \eta m_i(t') \geq \eta^{kB+1}.$$ 

Using also the fact $x_i(s+1) \geq \eta x_i(s)$, we obtain that $m_i(t'+B) \geq \eta^{(k+1)B}$. Therefore, at time $t'+B$, we have $k+1$ nodes with $m_\ell(t'+B) \geq \eta^{(k+1)B}$ (namely, the nodes in $S$, together with node $i$). It follows that Property $P_{k+1}$ is satisfied at time $t'+B$.

This inductive argument shows that there is a time $\tau'$ at which Property $P_k$ is satisfied. At that time $m_i(\tau') \geq \eta^nB$ for all $i$, which implies that $m(\tau') \geq \eta^nB$. On the other hand, $M(\tau'+B) \leq M(0) = 1$, which proves that $M(\tau'+B) - m(\tau') \leq 1 - \eta^nB$.

Now that we have proved Theorem 3.4, let us give a variation of it which will ensure not only convergence but convergence to the average.

**Assumption 3.4.** (Double stochasticity) The matrix $A(t)$ is column-stochastic for all $t$, i.e.,

$$\sum_{i=1}^n a_{ij}(t) = 1,$$

for all $j$ and $t$.

Note that Assumption 3.1 ensures that the matrix $A(t)$ is only row-stochastic. The above assumption together with Assumption 3.1 ensures that $A(t)$ is actually doubly stochastic.

**Theorem 3.5.** Under Assumptions 2.1 (connectivity), 3.1 (non-vanishing weights), and 3.4 (double stochasticity), and for the symmetric model, the agreement algorithm (without delays) satisfies

$$\lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{i=1}^n x_i(0).$$

**Proof.** By Theorem 3.4, every $x_i(t)$ converges to the same value. Assumption 3.4 ensures that $\sum_{i=1}^n x_i(t)$ is preserved from iteration to iteration:

$$\sum_{i=1}^n x_i(t+1) = 1^T x(t+1) = 1^T A(t)x(t) = 1^T x(t) = \sum_{i=1}^n x_i(t),$$

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where we use the double stochasticity of \( A(t) \) to conclude that \( 1^T A(t) = 1^T \). This immediately implies that the final limit is the average of the initial values.

Finally, let us observe that Theorem 2.3 from the previous chapter is a special case of the theorem we have just proved.

**Proof of Theorem 2.4.** Observe that the assumptions of Theorem 3.3 are present among the assumptions of Theorem 2.1 except for Assumption 3.4 which needs to be verified. To argue that the matrices \( A(t) \) in Theorem 2.1 are doubly stochastic, we just observe that they are symmetric and stochastic.

### 3.4 Relaxing symmetry

The symmetry condition \([i, j] \in E(t) \text{ iff } (j, i) \in E(t)\) used in Theorem 3.4 is somewhat unnatural in the presence of communication delays, as it requires perfect synchronization of the update times. A looser and more natural assumption is the following.

**Assumption 3.5 (Bounded round-trip times).** There exists some \( B > 0 \) such that whenever \( (i, j) \in E(t) \), then there exists some \( \tau \) that satisfies \(|t - \tau| < B\) and \((j, i) \in E(\tau)\).

Assumption 3.5 allows for protocols such as the following. Node \( i \) sends its value to node \( j \). Node \( j \) responds by sending its own value to node \( i \). Both nodes update their values (taking into account the received messages), within a bounded time from receiving the other node’s value. In a realistic setting, with unreliable communications, even this loose symmetry condition may be impossible to enforce with absolute certainty. One can imagine more complicated protocols based on an exchange of acknowledgments, but fundamental obstacles remain (see the discussion of the “two-army problem” in pp. 32-34 of [12]). A more realistic model would introduce a positive probability that some of the updates are never carried out. (A simple possibility is to assume that each \( a_{ij}(t) \), with \( i \neq j \), is changed to a zero, independently, and with a fixed probability.) The convergence result that follows remains valid in such a probabilistic setting (with probability 1). Since no essential new insights are provided, we only sketch a proof for the deterministic case.

**Theorem 3.6.** Under Assumptions 2.1 (connectivity), 3.1 (non-vanishing weights), 3.3 (delays) and 3.5 (bounded round-trip times) the agreement algorithm with delays [cf. Eq. (3.2)] guarantees asymptotic consensus.

**Proof outline.** A minor change is needed in the proof of Lemma 1. In particular, we define \( P_k \) as the event that there exist at least \( k \) indices \( l \) for which \( m_l(t) \geq \eta^{2kB} \). It follows that \( P_1 \) holds at time \( t = 2B \).

By induction, let \( P_k \) hold at time \( t \), and let \( S \) be the set of cardinality \( k \) containing indices \( l \) for which \( m_l(t) \geq \eta^{2kB} \). Furthermore, let \( \tau \) be the first time after time \( t \) that \( a_{ij}(\tau) \neq 0 \) where exactly one of \( i, j \) is in \( S \). Along the same lines as in the
proof of Lemma 1, \( m_l(\tau) \geq \eta^{2kB} \) for \( l \in S \); since \( x_l(t+1) \geq \eta x_l(t) \), it follows that \( m_l(\tau + 2B) \geq \eta^{2(k+1)B} \) for each \( l \in S \). By our assumptions, exactly one of \( i, j \) is in \( S^c \). If \( i \in S^c \), then \( x_i(\tau + 1) \geq a_{ij}(\tau) x_j(\tau) \geq \eta^{2kB+1} \) and consequently \( x_i(\tau + 2B) \geq \eta^{2B-1} \eta^{2kB+1} = \eta^{2(k+1)B} \). If \( j \in S^c \), then there must exist a time \( \tau_j \in \{ \tau + 1, \tau + 2, \ldots, \tau + B - 1 \} \) with \( a_{ji}(\tau_j) > 0 \). It follows that:

\[
m_j(\tau + 2B) \geq \eta^{\tau + 2B - (\tau_j + 1)} x_j(\tau_j + 1) \\
\geq \eta^{\tau + 2B - \tau_j - 1} \eta x_i(\tau_j) \\
\geq \eta^{\tau + 2B - \tau_j - 1} \eta \eta^{2B} \\
= \eta^{2(k+1)B}
\]

Therefore, \( P_{k+1} \) holds at time \( \tau + 2B \) and the induction is complete.

\[ \square \]

### 3.5 Concluding remarks

In this chapter, we have presented some basic convergence results on the averaging iterations \( x(t+1) = A(t)x(t) \). In particular, we proved Theorem 2.3 from the previous chapter, as well as several variations of it involving asymmetry and delays.

There is, however, one troubling feature of the results so far: the convergence time bounds which follow from our proofs are quite large. We have shown that after \( nB \) steps, an appropriately defined measure of convergence shrinks by a factor of \( 1 - \eta^{nB} \). Considering that \( \eta \) can be as small as \( 1/n \) (for example, in the Metropolis model), this means that one must wait \( n^n B \log(1/\epsilon) \) steps for the same measure of convergence to shrink by \( \epsilon \). It goes without saying that this is an enormous number even for relatively small \( n, B \).

One might hope that the bounds we have derived are lax. Unfortunately, one can actually construct examples of graph sequences on which convergence takes time exponential in \( n \). An example may be found in [85]; an example with an undirected graph may be found in an unpublished manuscript by Cao, Spielman, and Morse.

In the next several chapters, we will be concerned with the possibility of designing averaging algorithms with better guarantees. A first goal is to replace the exponential scaling with \( n \) by a polynomial one.
Chapter 4

Averaging in polynomial time

4.1 Convergence time

In the previous chapter, we proved Theorem 3.5 which states that subject to a few natural conditions the iteration

$$x(t+1) = A(t)x(t)$$

results in

$$\lim_{t \to \infty} x(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0).$$

A special case of this result is Theorem 2.3 stated earlier. The assumptions include: Assumption 2.1 ensuring that the graphs $G(t)$ contain enough links, Assumption 3.1 on the weights, and the “double stochasticity” Assumption 3.4. Finally, we also had to assume we were in the “symmetric model.” Here our goal will be to reproduce the result but with better bounds on convergence time, which scale polynomially, rather than exponentially, in $n$.

For this, we will need to slightly strengthen our connectivity assumptions. It is obvious that with Assumption 2.1 no effective bounds on convergence can hold since the sequence $G(t)$ may contain arbitrarily many empty graphs. Thus we will replace Assumption 2.1 with the slightly stronger Assumption 3.2. On the positive side, with the stronger Assumption 3.2 we can dispense with the assumption that we are in the “symmetric model.”

As a convergence measure, we use the “sample variance” of a vector $x \in \mathbb{R}^n$, defined as

$$V(x) = \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

where $\bar{x}$ is the average of the entries of $x$:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$
We are interested in providing an upper bound on the number of iterations it takes for the “sample variance” \( V(x(t)) \) to decrease to a small fraction of its initial value \( V(x(0)) \). The main result of this chapter is the following theorem.

**Theorem 4.1.** Let Assumptions 3.1 (non-vanishing weights), 3.2 (\( B \)-conneectivity), and 3.4 (double stochasticity) hold. Then there exists an absolute constant \( c \) such that we have

\[
V(t) \leq \epsilon V(0) \quad \text{for all } t \geq c\left(\frac{n^{2}/\eta}{B \log(1/\epsilon)}\right).
\]

This is the “polynomial time averaging” result alluded to in the title of this chapter. Our exposition follows the paper [76] where this material first appeared. Note that this bound is exponentially better than the convergence time bound of \( O\left(\frac{1}{\eta}nBnB \log 1/\epsilon\right) \) which follow straightforwardly from the arguments of the previous chapter.

We now proceed to the task of proving this theorem. We first establish some technical preliminaries that will be key in the subsequent analysis. In particular, in the next subsection, we explore several implications of the double stochasticity assumption on the weight matrix \( A(t) \).

### 4.1.1 Preliminaries on doubly stochastic matrices

We begin by analyzing how the sample variance \( V(x) \) changes when the vector \( x \) is multiplied by a doubly stochastic matrix \( A \). The next lemma shows that \( V(Ax) \leq V(x) \). Thus, under Assumptions 3.1 and 3.4 the sample variance \( V(x(t)) \) is nonincreasing in \( t \), and \( V(x(t)) \) can be used as a Lyapunov function.

**Lemma 4.1.** Let \( A \) be a doubly stochastic matrix. Then, \( ^2 \) for all \( x \in \mathbb{R}^n \),

\[
V(Ax) = V(x) - \sum_{i<j} w_{ij}(x_i - x_j)^2,
\]

where \( w_{ij} \) is the \((i, j)\)-th entry of the matrix \( A^T A \).

**Proof.** Let \( \mathbf{1} \) denote the vector in \( \mathbb{R}^n \) with all entries equal to 1. The double stochasticity of \( A \) implies

\[
A\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^T A = \mathbf{1}^T.
\]

Note that multiplication by a doubly stochastic matrix \( A \) preserves the average of the entries of a vector, i.e., for any \( x \in \mathbb{R}^n \), there holds

\[
\overline{Ax} = \frac{1}{n} \mathbf{1}^T Ax = \frac{1}{n} \mathbf{1}^T x = \bar{x}.
\]

---

\(^1\) We say \( c \) is an absolute constant when it does not depend on any of the parameters in the problem, in this case \( n, B, \eta, \epsilon \).

\(^2\) In the sequel, the notation \( \sum_{i<j} \) will be used to denote the double sum \( \sum_{j=1}^{n} \sum_{i=1}^{j-1} \).
We now write the quadratic form $V(x) - V(Ax)$ explicitly, as follows:

$$
V(x) - V(Ax) = (x - \bar{x}1)^T(x - \bar{x}1) - (Ax - \bar{Ax}1)^T(Ax - \bar{Ax}1)
= (x - \bar{x}1)^T(x - \bar{x}1) - (Ax - \bar{Ax}1)^T(Ax - \bar{Ax}1)
= (x - \bar{x}1)^T(I - A^T A)(x - \bar{x}1).
$$

(4.1)

Let $w_{ij}$ be the $(i, j)$-th entry of $A^T A$. Note that $A^T A$ is symmetric and stochastic, so that $w_{ij} = w_{ji}$ and $w_{ii} = 1 - \sum_{j \neq i} w_{ij}$. Then, it can be verified that

$$
A^T A = I - \sum_{i<j} w_{ij}(e_i - e_j)(e_i - e_j)^T,
$$

(4.2)

where $e_i$ is a unit vector with the $i$-th entry equal to 1, and all other entries equal to 0 (see also [100] where a similar decomposition was used).

By combining Eqs. (4.1) and (4.2), we obtain

$$
V(x) - V(Ax) = (x - \bar{x}1)^T\left(\sum_{i<j} w_{ij}(e_i - e_j)(e_i - e_j)^T\right)(x - \bar{x}1)
= \sum_{i<j} w_{ij}(x_i - x_j)^2.
$$

We will use this result to provide a lower bound on the amount of decrease of the sample variance $V(x(t))$ in between iterations.

Moreover, the amount of variance decrease is given by

$$
V(x(t)) - V(x(t + 1)) = \sum_{i<j} w_{ij}(t)(x_i(t) - x_j(t))^2.
$$

We will use this result to provide a lower bound on the amount of decrease of the sample variance $V(x(t))$ in between iterations.

Since every positive entry of $A(t)$ is at least $\eta$, it follows that every positive entry of $A(t)^T A(t)$ is at least $\eta^2$. Therefore, it is immediate that

$$
V(x(t + 1)) \leq V(x(t)) \quad \text{for all } t.
$$

Moreover, the amount of variance decrease is given by

$$
V(x(t)) - V(x(t + 1)) = \sum_{i<j} w_{ij}(t)(x_i(t) - x_j(t))^2.
$$

In our next lemma, we establish a stronger lower bound. In particular, we find it useful to focus not on an individual $w_{ij}$, but rather on all $w_{ij}$ associated with edges $(i, j)$ that cross a particular cut in the graph $(N, E(A^T A))$. For such groups of $w_{ij}$, we prove a lower bound which is linear in $\eta$, as seen in the following.

**Lemma 4.2.** Let $A$ be a row-stochastic matrix with positive diagonal entries, and assume that the smallest positive entry in $A$ is at least $\eta$. Also, let $(S^-, S^+)$ be a
partition of the set \( N = \{1, \ldots, n\} \) into two disjoint sets. If 
\[
\sum_{i \in S^-, \ j \in S^+} w_{ij} > 0,
\]
then 
\[
\sum_{i \in S^-, \ j \in S^+} w_{ij} \geq \frac{\eta}{2}.
\]

Proof. Let \( \sum_{i \in S^-, \ j \in S^+} w_{ij} > 0 \). From the definition of the weights \( w_{ij} \), we have 
\( w_{ij} = \sum_k a_{ki} a_{kj} \), which shows that there exist \( i \in S^- \), \( j \in S^+ \), and some \( k \) such that 
\( a_{ki} > 0 \) and \( a_{kj} > 0 \). For either case where \( k \) belongs to \( S^- \) or \( S^+ \), we see that there 
exists an edge in the set \( \mathcal{E}(A) \) that crosses the cut \( (S^-, S^+) \). Let \( (i^*, j^*) \) be such an 
edge. Without loss of generality, we assume that \( i^* \in S^- \) and \( j^* \in S^+ \).

We define
\[
C_{j^*}^+ = \sum_{i \in S^+} a_{j^*i},
\]
\[
C_{j^*}^- = \sum_{i \in S^-} a_{j^*i}.
\]

See Figure 4-1(a) for an illustration. Since \( A \) is a row-stochastic matrix, we have
\[
C_{j^*}^- + C_{j^*}^+ = 1,
\]
implying that at least one of the following is true:

Case (a): \( C_{j^*}^- \geq \frac{1}{2} \),

Case (b): \( C_{j^*}^+ \geq \frac{1}{2} \).

We consider these two cases separately. In both cases, we focus on a subset of the 
edges and we use the fact that the elements \( w_{ij} \) correspond to paths of length 2, with 
one step in \( \mathcal{E}(A) \) and another in \( \mathcal{E}(A^T) \).

Case (a): \( C_{j^*}^- \geq 1/2 \).

We focus on those \( w_{ij} \) with \( i \in S^- \) and \( j = j^* \). Indeed, since all \( w_{ij} \) are nonnegative, we have
\[
\sum_{i \in S^-, \ j \in S^+} w_{ij} \geq \sum_{i \in S^-} w_{ij}. \tag{4.3}
\]

For each element in the sum on the right-hand side, we have
\[
w_{ij^*} = \sum_{k=1}^{n} a_{ki} a_{kj^*} \geq a_{j^*i} a_{j^*j^*} \geq a_{j^*i} \eta,
\]

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Figure 4-1: (a) Intuitively, $C_{j^+}$ measures how much weight $j^*$ assigns to nodes in $S^+$ (including itself), and $C_{j^-}$ measures how much weight $j^*$ assigns to nodes in $S^-$. Note that the edge $(j^*, j^*)$ is also present, but not shown. (b) For the case where $C_{j^-} \geq 1/2$, we only focus on two-hop paths between $j^*$ and elements $i \in S^-$ obtained by taking $(i, j^*)$ as the first step and the self-edge $(j^*, j^*)$ as the second step. (c) For the case where $C_{j^+} \geq 1/2$, we only focus on two-hop paths between $i^*$ and elements $j \in S^+$ obtained by taking $(i^*, j^*)$ as the first step in $\mathcal{E}(A)$ and $(j^*, j)$ as the second step in $\mathcal{E}(A^T)$.

where the inequalities follow from the facts that $A$ has nonnegative entries, its diagonal entries are positive, and its positive entries are at least $\eta$. Consequently,

$$\sum_{i \in S^-} w_{ij^*} \geq \eta \sum_{i \in S^-} a_{j^*i} = \eta C_{j^-}.$$  

(4.4)

Combining Eqs. (4.3) and (4.4), and recalling the assumption $C_{j^-} \geq 1/2$, the result follows. An illustration of this argument can be found in Figure 4-1(b).

Case (b): $C_{j^+} \geq 1/2$.

We focus on those $w_{ij}$ with $i = i^*$ and $j \in S^+$. We have

$$\sum_{i \in S^-, j \in S^+} w_{ij} \geq \sum_{j \in S^+} w_{i^*j},$$  

(4.5)

since all $w_{ij}$ are nonnegative. For each element in the sum on the right-hand side, we have

$$w_{i^*j} = \sum_{k=1}^n a_{kii^*} a_{kj} \geq a_{j^*i^*} a_{j^*j} \geq \eta a_{j^*j},$$

where the inequalities follow because all entries of $A$ are nonnegative, and because the choice $(i^*, j^*) \in \mathcal{E}(A)$ implies that $a_{j^*i^*} \geq \eta$. Consequently,

$$\sum_{j \in S^+} w_{i^*j} \geq \eta \sum_{j \in S^+} a_{j^*j} = \eta C_{j^+}.$$  

(4.6)

Combining Eqs. (4.5) and (4.6), and recalling the assumption $C_{j^+} \geq 1/2$, the result follows. An illustration of this argument can be found in Figure 4-1(c).
4.1.2 A bound on convergence time

With the preliminaries on doubly stochastic matrices in place, we can now proceed to derive bounds on the decrease of \( V(x(t)) \) in between iterations. We will first somewhat relax our connectivity assumptions. In particular, we consider the following relaxation of Assumption 3.2.

**Assumption 4.1 (Relaxed connectivity).** Given an integer \( t \geq 0 \), suppose that the components of \( x(tB) \) have been reordered so that they are in nonincreasing order. We assume that for every \( d \in \{1, \ldots, n-1\} \), we either have \( x_d(tB) = x_{d+1}(tB) \), or there exist some time \( t \in \{tB, \ldots, (t+1)B-1\} \) and some \( i \in \{1, \ldots, d\}, j \in \{d+1, \ldots, n\} \) such that \((i, j)\) or \((j, i)\) belongs to \( \mathcal{E}(A(t)) \).

**Lemma 4.3.** Assumption 3.2 implies Assumption 4.1, with the same value of \( B \).

**Proof.** If Assumption 4.1 does not hold, then there must exist an index \( d \) [for which \( x_d(tB) \neq x_{d+1}(tB) \) holds] such that there are no edges between nodes \( 1, 2, \ldots, d \) and nodes \( d+1, \ldots, n \) during times \( t = tB, \ldots, (t+1)B-1 \). But this implies that the graph

\[
\big( N, \mathcal{E}(A(tB)) \big) \cup \mathcal{E}(A(tB+1)) \cup \cdots \cup \mathcal{E}(A((t+1)B-1))
\]

is disconnected, which violates Assumption 2. \( \square \)

For our convergence time results, we will use the weaker Assumption 4.1 rather than the stronger Assumption 3.2. Later on, in Chapter 5 we will exploit the sufficiency of Assumption 4.1 to design a decentralized algorithm for selecting the weights \( a_{ij}(t) \), which satisfies Assumption 4.1 but not Assumption 3.2.

We now proceed to bound the decrease of our Lyapunov function \( V(x(t)) \) during the interval \([tB, (t+1)B-1]\). In what follows, we denote by \( V(t) \) the sample variance \( V(x(t)) \) at time \( t \).

**Lemma 4.4.** Let Assumptions 3.1 (non-vanishing weights), 3.4 (doubly stochasticity) and 4.1 (relaxed connectivity) hold. Let \( \{x(t)\} \) be generated by the update rule (3.1). Suppose that the components \( x_i(tB) \) of the vector \( x(tB) \) have been ordered from largest to smallest, with ties broken arbitrarily. Then,

\[
V(tB) - V((t+1)B) \geq \eta \frac{n-1}{2} \sum_{i=1}^{n-1} (x_i(tB) - x_{i+1}(tB))^2.
\]

**Proof.** By Lemma 4.1 we have for all \( t \),

\[
V(t) - V(t + 1) = \sum_{i < j} w_{ij}(t)(x_i(t) - x_j(t))^2,
\]

where \( w_{ij}(t) \) is the \((i, j)\)-th entry of \( A(t)^T A(t) \). Summing up the variance differences
We next introduce some notation.

(a) For all \(d \in \{1, \ldots, n-1\}\), let \(t_d\) be the first time larger than or equal to \(tB\) (if it exists) at which there is a communication between two nodes belonging to the two sets \(\{1, \ldots, d\}\) and \(\{d+1, \ldots, n\}\), to be referred to as a communication across the cut \(d\).

(b) For all \(t \in \{tB, \ldots, (t+1)B-1\}\), let \(D(t) = \{d \mid t_d = t\}\), i.e., \(D(t)\) consists of “cuts” \(d \in \{1, \ldots, n-1\}\) such that time \(t\) is the first communication time larger than or equal to \(tB\) between nodes in the sets \(\{1, \ldots, d\}\) and \(\{d+1, \ldots, n\}\). Because of Assumption 4.1, the union of the sets \(D(t)\) includes all indices \(1, \ldots, n-1\), except possibly for indices for which \(x_d(tB) = x_{d+1}(tB)\).

(c) For all \(d \in \{1, \ldots, n-1\}\), let \(C_d = \{(i,j), (j,i) \mid i \leq d, \ d+1 \leq j\}\).

(d) For all \(t \in \{tB, \ldots, (t+1)B-1\}\), let \(F_{ij}(t) = \{d \in D(t) \mid (i,j) \text{ or } (j,i) \in C_d\}\), i.e., \(F_{ij}(t)\) consists of all cuts \(d\) such that the edge \((i,j)\) or \((j,i)\) at time \(t\) is the first communication across the cut at a time larger than or equal to \(tB\).

(e) To simplify notation, let \(y_i = x_i(tB)\). By assumption, we have \(y_1 \geq \cdots \geq y_n\).

We make two observations, as follows:

(1) Suppose that \(d \in D(t)\). Then, for some \((i,j) \in C_d\), we have either \(a_{ij}(t) > 0\) or \(a_{ji}(t) > 0\). Because \(A(t)\) is nonnegative with positive diagonal entries, we have

\[
    w_{ij}(t) = \sum_{k=1}^{n} a_{ki}a_{kj} \geq a_{ii}(t)a_{ij}(t) + a_{ji}(t)a_{jj}(t) > 0,
\]

and by Lemma 4.2 we obtain

\[
    \sum_{(i,j) \in C_d} w_{ij}(t) \geq \frac{\eta}{2}. \tag{4.9}
\]

(2) Fix some \((i,j)\), with \(i < j\), and time \(t' \in \{tB, \ldots, (t+1)B-1\}\), and suppose that \(F_{ij}(t')\) is nonempty. Let \(F_{ij}(t') = \{d_1, \ldots, d_k\}\), where the \(d_j\) are arranged in increasing order. Since \(d_1 \in F_{ij}(t')\), we have \(d_1 \in D(t)\) and therefore \(t_{d_1} = t'\).

By the definition of \(t_{d_i}\), this implies that there has been no communication between a node in \(\{1, \ldots, d_1\}\) and a node in \(\{d_1 + 1, \ldots, n\}\) during the time interval \([tB, t' - 1]\). It follows that \(x_i(t') \geq y_{d_1}\). By a symmetrical argument, we also have

\[
    x_j(t') \leq y_{d_k+1}. \tag{4.10}
\]
These relations imply that
\[ x_i(t') - x_j(t') \geq y_{d_1} - y_{d_{k+1}} \geq \sum_{d \in F_{ij}(t')} (y_d - y_{d+1}), \]

Since the components of \( y \) are sorted in nonincreasing order, we have \( y_d - y_{d+1} \geq 0 \), for every \( d \in F_{ij}(t') \). For any nonnegative numbers \( z_i \), we have
\[ (z_1 + \cdots + z_k)^2 \geq z_1^2 + \cdots + z_k^2, \]
which implies that
\[ (x_i(t') - x_j(t'))^2 \geq \sum_{d \in F_{ij}(t')} (y_d - y_{d+1})^2. \] (4.11)

We now use these two observations to provide a lower bound on the expression on the right-hand side of Eq. (4.7) at time \( t' \). We use Eq. (4.11) and then Eq. (4.9), to obtain
\[
\sum_{i<j} w_{ij}(t')(x_i(t') - x_j(t'))^2 \geq \sum_{i<j} w_{ij}(t') \sum_{d \in F_{ij}(t')} (y_d - y_{d+1})^2 \\
= \sum_{d \in D(t')} \sum_{(i,j) \in C_d} w_{ij}(t')(y_d - y_{d+1})^2 \\
\geq \frac{\eta}{2} \sum_{d \in D(t')} (y_d - y_{d+1})^2.
\]

We now sum both sides of the above inequality for different values of \( t \), and use Eq. (4.8), to obtain
\[
V(tB) - V((t+1)B) = \sum_{k=tB}^{(t+1)B-1} \sum_{i<j} w_{ij}(k)(x_i(k) - x_j(k))^2 \\
\geq \frac{\eta}{2} \sum_{k=tB}^{(t+1)B-1} \sum_{d \in D(k)} (y_d - y_{d+1})^2 \\
= \frac{\eta}{2} \sum_{d=1}^{n-1} (y_d - y_{d+1})^2,
\]
where the last inequality follows from the fact that the union of the sets \( D(k) \) is only missing those \( d \) for which \( y_d = y_{d+1} \).

We next establish a bound on the variance decrease that plays a key role in our convergence analysis.
Lemma 4.5. Let Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), and 4.1 (connectivity relaxation) hold, and suppose that \( V(tB) > 0 \). Then,
\[
\frac{V(tB) - V((t+1)B)}{V(tB)} \geq \frac{\eta}{2n^2} \quad \text{for all } t.
\]

Proof. Without loss of generality, we assume that the components of \( x(tB) \) have been sorted in nonincreasing order. By Lemma 4.4, we have
\[
V(tB) - V((t+1)B) \geq \frac{\eta}{2} \sum_{i=1}^{n-1} (x_i(tB) - x_{i+1}(tB))^2.
\]
This implies that
\[
\frac{V(tB) - V((t+1)B)}{V(tB)} \geq \frac{\eta}{2} \sum_{i=1}^{n-1} (x_i(tB) - x_{i+1}(tB))^2
\]
\[
\sum_{i=1}^{n} (x_i(B) - \bar{x}(B))^2.
\]
Observe that the right-hand side does not change when we add a constant to every \( x_i(tB) \). We can therefore assume, without loss of generality, that \( \bar{x}(tB) = 0 \), so that
\[
V(tB) - V((t+1)B) \geq \frac{\eta}{2} \min_{x_1 \geq x_2 \geq \ldots \geq x_n} \sum_{i=1}^{n-1} (x_i - x_{i+1})^2.
\]
Note that the right-hand side is unchanged if we multiply each \( x_i \) by the same constant. Therefore, we can assume, without loss of generality, that \( \sum_{i=1}^{n} x_i^2 = 1 \), so that
\[
\frac{V(tB) - V((t+1)B)}{V(tB)} \geq \frac{\eta}{2} \min_{x_1 \geq x_2 \geq \ldots \geq x_n} \sum_{i=1}^{n-1} (x_i - x_{i+1})^2.
\]
(4.12)
The requirement \( \sum_{i} x_i^2 = 1 \) implies that the average value of \( x_i^2 \) is \( 1/n \), which implies that there exists some \( j \) such that \( |x_j| \geq 1/\sqrt{n} \). Without loss of generality, let us suppose that this \( x_j \) is positive.\(^3\)

The rest of the proof relies on a technique from [66] to provide a lower bound on the right-hand side of Eq. (4.12). Let
\[
z_i = x_i - x_{i+1} \quad \text{for } i < n, \quad \text{and } z_n = 0.
\]
Note that \( z_i \geq 0 \) for all \( i \) and
\[
\sum_{i=1}^{n} z_i = x_1 - x_n.
\]
Since \( x_j \geq 1/\sqrt{n} \) for some \( j \), we have that \( x_1 \geq 1/\sqrt{n} \); since \( \sum_{i=1}^{n} x_i = 0 \), it follows

\(^3\)Otherwise, we can replace \( x \) with \( -x \) and subsequently reorder to maintain the property that the components of \( x \) are in descending order. It can be seen that these operations do not affect the objective value.
that at least one \(x_i\) is negative, and therefore \(x_n < 0\). This gives us

\[
\sum_{i=1}^{n} z_i \geq \frac{1}{\sqrt{n}}
\]

Combining with Eq. (4.12), we obtain

\[
\frac{V(tB) - V((t + 1)B)}{V(tB)} \geq \frac{\eta}{2} \min_{z_i \geq 0} \sum_{i=1}^{n} z_i \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i^2.
\]

The minimization problem on the right-hand side is a symmetric convex optimization problem, and therefore has a symmetric optimal solution, namely \(z_i = 1/n^{1.5}\) for all \(i\). This results in an optimal value of \(1/n^2\). Therefore,

\[
\frac{V(tB) - V((t + 1)B)}{V(kB)} \geq \frac{\eta}{2n^2},
\]

which is the desired result.

We are now ready for our main result, which establishes that the convergence time of the sequence of vectors \(x(k)\) generated by Eq. (3.1) is of order \(O(n^2B/\eta)\).

**Theorem 4.2.** Let Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), and 4.1 (connectivity relaxation) hold. Then there exists an absolute constant \(c\) such that we have

\[
V(t) \leq \epsilon V(0) \quad \text{for all} \quad t \geq c(n^2/\eta)B\log(1/\epsilon).
\]

**Proof.** The result follows immediately from Lemma 4.5.

Recall that, according to Lemma 4.3, Assumption 3.2 implies Assumption 4.1. In view of this, the convergence time bound of Theorem 4.2 holds for any \(n\) and any sequence of weights satisfying Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), and 3.2 (\(B\)-connectivity). This proves Theorem 4.1 from the beginning of this chapter.

### 4.2 Concluding remarks

In this chapter, we have presented a polynomial convergence-time bound on the performance of a class of averaging algorithms. Several open research directions naturally present themselves.

First, is it possible to design faster algorithms which nevertheless compute averages correctly on arbitrary (time-varying, undirected) graph sequences? We make some headway on this question in Chapter 5 where we design an algorithm whose
convergence time scales as $O(n^2)$ with $n$, and in Chapter 6 we prove a lower bound of at least $\Omega(n^2)$ for a somewhat restricted class of algorithms. However, the general question of how fast averaging algorithms can scale with $n$ is still open.

Secondly, where is the dividing line before polynomial and exponential convergence time? In particular, how far may we relax the double stochasticity Assumption 3.4 while still having polynomial convergence time? For example, does polynomial time convergence still hold if we replace Assumption 3.4 with the requirement that the matrices $A(t)$ be (row) stochastic and each column sum is in $[1 - \epsilon, 1 + \epsilon]$ for some small $\epsilon > 0$?


Chapter 5

Averaging in quadratic time

In the previous section, we have shown that a large class of averaging algorithms have $O(B(n^2/\eta) \log 1/\epsilon)$ convergence time.

In this section, we consider decentralized ways of synthesizing the weights $a_{ij}(t)$ while satisfying Assumptions 3.1, 3.4, and 4.1. We assume that the sequence of (undirected) graphs $G(t) = (N, E(t))$ is given exogenously, but the nodes can pick the coefficients $a_{ij}(t)$. Our focus is on improving convergence time bounds by constructing “good” schemes.

Naturally, several ways to pick the coefficients present themselves. For example, each node may assign

$$a_{ij}(t) = \epsilon,$$

if $(j, i) \in E(t)$ and $i \neq j$,

$$a_{ii}(t) = 1 - \epsilon \cdot \text{deg}(i),$$

where $\text{deg}(i)$ is the degree of $i$ in $G(t)$. If $\epsilon$ is small enough and the graph $G(t)$ is undirected [i.e., $(i, j) \in E(t)$ if and only if $(j, i) \in E(t)$], this results in a nonnegative, doubly stochastic matrix (see [84]). The Metropolis algorithm from Chapter 2 is a special case of this method. However, if a node has $\Theta(n)$ neighbors, $\eta$ will be of order $\Theta(1/n)$, resulting in $\Theta(n^3)$ convergence time. Moreover, this argument applies to all protocols in which nodes assign equal weights to all their neighbors; see [97] and [16] for more examples.

In this section, we examine whether it is possible to synthesize the weights $a_{ij}(t)$ in a decentralized manner, so that the above convergence time is reduced. Our main result shaves a factor of $n$ off the convergence time of the previous paragraph.

**Theorem 5.1.** Suppose $G(t) = (N, E(t))$ is a sequence of undirected graphs such that $(N, E(tB) \cup E(tB + 1) \cup \cdots \cup E((t + 1)B - 1))$ is connected, for all integers $t$. Then, there exists a decentralized way to pick the coefficients $a_{ij}(t)$ such that

$$V(t) \leq \epsilon V(0) \quad \text{for all } t \geq cn^2 B \log(1/\epsilon).$$

This theorem has appeared in the paper [76], which we will follow here.

Our approach we will be to pick $a_{ij}(t)$ so that $a_{ij}(t) \geq \eta$ whenever $a_{ij}(t) \neq 0$, where $\eta$ is a positive constant independent of $n$ and $B$. We show that this is indeed
possible, under the additional assumption that the graphs \( G(t) \) are undirected (in Chapter 3, we referred to this case as the “symmetric model.”). Our algorithm is data-dependent, in that \( a_{ij}(t) \) depends not only on the graph \( G(t) \), but also on the data vector \( x(t) \). Furthermore, it is a decentralized 3-hop algorithm, in that \( a_{ij}(t) \) depends only on the data at nodes within a distance of at most 3 from \( i \). Our algorithm is such that the resulting sequences of vectors \( x(t) \) and graphs \( G(t) = (N, E(t)) \), with \( E(t) = \{(j, i) \mid a_{ij}(t) > 0\} \), satisfy Assumptions 3.1, 3.4 and 4.1. Thus, a convergence time result can be obtained from Theorem 4.2.

5.1 The algorithm

The algorithm we present here is a variation of an old load balancing algorithm (see [35] and Chapter 7.3 of [14]).

At each step of the algorithm, each node offers some of its value to its neighbors, and accepts or rejects such offers from its neighbors. Once an offer from \( i \) to \( j \), of size \( \delta > 0 \), has been accepted, the updates \( x_i \leftarrow x_i - \delta \) and \( x_j \leftarrow x_j + \delta \) are executed.

We next describe the formal steps the nodes execute at each time \( t \). For clarity, we refer to the node executing the steps below as node \( C \). Moreover, the instructions below sometimes refer to the neighbors of node \( C \); this always means current neighbors at time \( t \), when the step is being executed, as determined by the current graph \( G(t) \). We assume that at each time \( t \), all nodes execute these steps in the order described below, while the graph remains unchanged.

Balancing Algorithm:

1. Node \( C \) broadcasts its current value \( x_C \) to all its neighbors.
2. Going through the values it just received from its neighbors, Node \( C \) finds the smallest value that is less than its own. Let \( D \) be a neighbor with this value. Node \( C \) makes an offer of \((x_C - x_D)/3\) to node \( D \).
   
   If no neighbor of \( C \) has a value smaller than \( x_C \), node \( C \) does nothing at this stage.
3. Node \( C \) goes through the incoming offers. It sends an acceptance to the sender of a largest offer, and a rejection to all the other senders. It updates the value of \( x_C \) by adding the value of the accepted offer.
   
   If node \( C \) did not receive any offers, it does nothing at this stage.
4. If an acceptance arrives for the offer made by node \( C \), node \( C \) updates \( x_C \) by subtracting the value of the offer.

Note that the new value of each node is a linear combination of the values of its neighbors. Furthermore, the weights \( a_{ij}(t) \) are completely determined by the data and the graph at most 3 hops from node \( i \) in \( G(t) \). This is true because in the course

\footnote{This algorithm was also considered in [35], but in the absence of a result such as Theorem 4.2 a weaker convergence time bound was derived.}
of execution of the above steps, each node makes at most three transmissions to its neighbors, so the new value of node \( C \) cannot depend on information more than 3 hops away from \( C \).

## 5.2 Performance analysis

The following theorem (stated at the beginning of the chapter as Theorem 5.1) allows us to remove a factor of \( n \) from the worst-case convergence time bounds of Theorem 4.2.

**Theorem 5.2.** Consider the balancing algorithm, and suppose that \( G(t) = (N, E(t)) \) is a sequence of undirected graphs such that \( (N, E(tB) \cup E(tB+1) \cup \cdots \cup E((t+1)B-1)) \) is connected, for all integers \( t \). There exists an absolute constant \( c \) such that we have

\[
V(t) \leq \epsilon V(0) \quad \text{for all } t \geq cn^2B \log(1/\epsilon).
\]

**Proof.** Note that with this algorithm, the new value at some node \( i \) is a convex combination of the previous values of itself and its neighbors. Furthermore, the algorithm keeps the sum of the nodes’ values constant, because every accepted offer involves an increase at the receiving node equal to the decrease at the offering node. These two properties imply that the algorithm can be written in the form

\[
x(t + 1) = A(t)x(t),
\]

where \( A(t) \) is a doubly stochastic matrix, determined by \( G(t) \) and \( x(t) \). It can be seen that the diagonal entries of \( A(t) \) are positive and, furthermore, all nonzero entries of \( A(t) \) are larger than or equal to 1/3; thus, \( \eta = 1/3 \).

We claim that the algorithm [in particular, the sequence \( E(A(t)) \)] satisfies Assumption 4.1. Indeed, suppose that at time \( tB \), the nodes are reordered so that the values \( x_i(tB) \) are nonincreasing in \( i \). Fix some \( d \in \{1, \ldots, n-1\} \), and suppose that \( x_d(tB) \neq x_{d+1}(tB) \). Let \( S^+ = \{1, \ldots, d\} \) and \( S^- = \{d+1, \ldots, n\} \).

Because of our assumptions on the graphs \( G(t) \), there will be a first time \( t' \) in the interval \( \{tB, \ldots, (t+1)B-1\} \), at which there is an edge in \( E(t) \) between some \( i^* \in S^+ \) and \( j^* \in S^- \). Note that between times \( tB \) and \( t' \), the two sets of nodes, \( S^+ \) and \( S^- \), do not interact, which implies that \( x_i(t') \geq x_d(tB) \), for \( i \in S^+ \), and \( x_j(t') < x_d(tB) \), for \( j \in S^- \).

At time \( t \), node \( i^* \) sends an offer to a neighbor with the smallest value; let us denote that neighbor by \( k^* \). Since \( (i^*, j^*) \in E(t') \), we have \( x_{k^*}(t') \leq x_{j^*}(t') \leq x_d(tB) \), which implies that \( k^* \in S^- \). Node \( k^* \) will accept the largest offer it receives, which must come from a node with a value no smaller than \( x_{j^*}(t') \), and therefore no smaller than \( x_d(tB) \); hence the latter node belongs to \( S^+ \). It follows that \( E(A(t')) \) contains an edge between \( k^* \) and some node in \( S^+ \), showing that Assumption 4.1 is satisfied.

The claimed result follows from Theorem 4.2, because we have shown that all of the assumptions in that theorem are satisfied with \( \eta = 1/3 \). \( \square \)
5.3 Concluding remarks

In this chapter, we have presented a specific averaging algorithm whose scaling with the number of nodes $n$ is $O(n^2)$. An interesting direction to explore is to what extent, if any, this can be reduced. The next chapter provides a partial negative answer to this question, showing that one cannot improve on quadratic scaling with a limited class of algorithms.
Chapter 6

On the optimality of quadratic time

The goal of this chapter is to analyze the fundamental limitations of the kind of distributed averaging algorithms we have been studying. The previous chapter described a class of averaging algorithms whose convergence time scales with the number of agents \( n \) as \( O(n^2) \). Our aim in this chapter is to show that this is the best scaling for a common class of such algorithms; namely, that any distributed averaging algorithm that uses a single scalar state variable at each agent and satisfies a natural “smoothness” condition will have this property. Our exposition will follow the preprint [79] where these results have previously appeared.

We next proceed to define the class of distributed averaging algorithms we are considering and informally state our result.

6.1 Background and basic definitions.

Definition of local averaging algorithms: Agents 1, \ldots, n begin with real numbers \( x_1(0), \ldots, x_n(0) \) stored in memory. At each round \( t = 0, 1, 2, \ldots \), agent \( i \) broadcasts \( x_i(t) \) to each of its neighbors in some undirected graph \( G(t) = (\{1, \ldots, n\}, E(t)) \), and then sets \( x_i(t+1) \) to be some function of \( x_i(t) \) and of the values \( x_{i'}(t), x_{i''}(t), \ldots \) it has just received from its own neighbors:

\[
x_i(t+1) = f_{i,G(t)}(x_i(t), x_{i'}(t), x_{i''}(t), \ldots).
\]

(6.1)

We require each \( f_{i,G(t)} \) to be a differentiable function. Each agent uses the incoming messages \( x_{i'}(t), x_{i''}(t), \ldots \) as the arguments of \( f_{i,G(t)} \) in some arbitrary order; we assume that this order does not change, i.e. if \( G(t_1) = G(t_2) \), then the message coming from the same neighbor of agent \( i \) is mapped to the same argument of \( f_{i,G(t)} \) for \( t = t_1 \) and \( t = t_2 \). It is desired that

\[
\lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0),
\]

(6.2)
for every $i$, for every sequence of graphs $G(t)$ having the property that

$$\text{the graph } \{(1, \ldots, n), \cup_{s \geq t} E(s)\} \text{ is connected for every } t, \quad (6.3)$$

and for every possible way for the agents to map incoming messages to arguments of $f_{i,G(t)}$.

In words, as the number of rounds $t$ approaches infinity, iteration (6.1) must converge to the average of the numbers $x_1(0), \ldots, x_n(0)$. Note that the agents have no control over the communication graph sequence $G(t)$, which is exogenously provided by “nature.” However, as we stated previously, every element of the sequence $G(t)$ must be undirected: this corresponds to symmetric models of communication between agents. Moreover, the sequence $G(t)$ must satisfy the mild connectivity condition of Eq. (6.3), which says that the network cannot become disconnected after a finite period.

Local averaging algorithms are useful tools for information fusion due to their efficient utilization of resources (each agent stores only a single number in memory) as well as their robustness properties (the sequence of graphs $G(t)$ is time-varying, and it only needs to satisfy the relatively weak connectivity condition in Eq. (6.3) for the convergence in Eq. (6.2) to hold). As explained in Chapter 2, no other class of schemes for averaging (e.g., flooding, fusion along a spanning tree) is known to produce similar results under the same assumptions.

**Remark:** As can be seen from the subscripts, the update function $f_{i,G(t)}$ is allowed to depend on the agent and on the graph. Some dependence on the graph is unavoidable since in different graphs an agent may have a different number of neighbors, in which case nodes will receive a different number of messages, so that even the number of arguments of $f_{i,G(t)}$ will depend on $G(t)$. It is often practically desired that $f_{i,G(t)}$ depend only weakly on the graph, as the entire graph may be unknown to agent $i$. For example, we might require that $f_{i,G(t)}$ be completely determined by the degree of $i$ in $G(t)$. However, since our focus is on what distributed algorithms cannot do, it does not hurt to assume the agents have unrealistically rich information; thus we will not assume any restrictions on how $f_{i,G(t)}$ depends on $G(t)$.

**Remark:** We require the functions $f_{i,G(t)}$ to be smooth, for the following reason. First, we need to exclude unnatural algorithms that encode vector information in the infinitely many bits of a single real number. Second, although we make the convenient technical assumption that agents can transmit and store real numbers, we must be aware that in practice agents will transmit and store a quantized version of $x_i(t)$. Thus, we are mostly interested in algorithms that are not disrupted much by quantization. For this reason, we must prohibit the agents from using discontinuous update functions $f_{i,G(t)}$. For technical reasons, we actually go a little further, and prohibit the agents from using non-smooth update functions $f_{i,G(t)}$. 

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6.1.1 Examples.

In order to provide some context, let us mention just a few of the distributed averaging schemes that have been proposed in the literature:

1. The max-degree method \[84\] involves picking \( \epsilon(t) \) with the property \( \epsilon(t) \leq 1/(d(t) + 1) \), where \( d(t) \) is the largest degree of any agent in \( G(t) \), and updating by

\[
x_i(t + 1) = x_i(t) + \epsilon(t) \sum_{i \in N_i(t)} (x_j(t) - x_i(t)) .
\]

Here we use \( N_i(t) \) to denote the set of neighbors of agent \( i \) in \( G(t) \). In practice, a satisfactory \( \epsilon(t) \) may not be known to all of the agents, because this requires some global information. However, in some cases a satisfactory choice for \( \epsilon(t) \) may be available, for example when an a priori upper bound on \( d(G(t)) \) is known.

2. The Metropolis method (see Chapter 2) from \[97\] involves setting \( \epsilon_{ij}(t) \) to satisfy \( \epsilon_{ij}(t) \leq \min(1/(d_i(t)), 1/(d_j(t))) \), where \( d_i(t), d_j(t) \) are the degrees of agents \( i \) and \( j \) in \( G(t) \), and updating by

\[
x_i(t + 1) = x_i(t) + \sum_{j \in N_i(t)} \epsilon_{ij}(t) (x_j(t) - x_i(t)) .
\]

3. The load-balancing algorithm of Chapter 5 involves updating by

\[
x_i(t + 1) = x_i(t) + \sum_{i \in N_i(t)} a_{ij}(t) (x_j(t) - x_i(t)) ,
\]

where \( a_{ij}(t) \) is determined by the following rule: each agent selects exactly two neighbors, the neighbor with the largest value above its own and with the smallest value below its own. If \( i, j \) have both selected each other, then \( a_{ij}(t) = 1/3 \); else \( a_{ij}(t) = 0 \). The intuition comes from load-balancing: agents think of \( x_i(t) \) as load to be equalized among their neighbors; they try to offload on their lightest neighbor and take from their heaviest neighbor.

We remark that the above load-balancing algorithm is not a "local averaging algorithm" according to our definition because \( x_i(t + 1) \) does not depend only on \( x_i(t) \) and its neighbors; for example, agents \( i \) and \( j \) may not match up because \( j \) has a neighbor \( k \) with \( x_k(t) > x_j(t) \). By contrast, the max-degree and Metropolis algorithm are indeed "local averaging algorithms."

6.1.2 Our results

Our goal is to study the worst-case convergence time over all graph sequences. This convergence time may be arbitrarily bad since one can insert arbitrarily many empty
graphs into the sequence $G(t)$ without violating Eq. (6.3). To avoid this trivial situation, we require that there exist some integer $B$ such that the graphs
\[
\left\{ 1, \ldots, n \right\}, \bigcup_{i=kB}^{(k+1)B} E(k) \tag{6.4}
\]
are connected for every integer $k$.

Let $x(t)$ be the vector in $\mathbb{R}^n$ whose $i$th component is $x_i(t)$. We define the convergence time $T(n, \epsilon)$ of a local averaging algorithm as the time until “sample variance”
\[
V(x(t)) = \sum_{i=1}^{n} \left( x_i(t) - \frac{1}{n} \sum_{j=1}^{n} x_j(0) \right)^2
\]
permatically shrinks by a factor of $\epsilon$, i.e., $V(x(t)) \leq \epsilon V(x(0))$ for all $t \geq T(n, \epsilon)$, for all possible $n$-node graph sequences satisfying Eq. (6.4), and all initial vectors $x(0)$ for which not all $x_i(0)$ are equal; $T(n, \epsilon)$ is defined to be the smallest number with this property. We are interested in how $T(n, \epsilon)$ scales with $n$ and $\epsilon$.

Currently, the best available upper bound for the convergence time is obtained with the load-balancing algorithm of Chapter 5, where it was proven
\[
T(n, \epsilon) \leq C n^2 B \log \frac{1}{\epsilon},
\]
for some absolute constant $C$. We are primarily interested in whether it is possible to improve the scaling with $n$ to below $n^2$. Are there nonlinear update functions $f_{i,G(t)}$ which speed up the convergence time?

Our main result is that the answer to this question is “no” within the class of local averaging algorithms. For such algorithms we prove a general lower bound of the form
\[
T(n, \epsilon) \geq c n^2 B \log \frac{1}{\epsilon},
\]
for some absolute constant $c$. Moreover, this lower bound holds even if we assume that the graph sequence $G(t)$ is the same for all $t$; in fact, we prove it for the case where $G(t)$ is a fixed “line graph.”

### 6.2 Formal statement and proof of main result

We next state our main theorem. The theorem begins by specializing our definition of local averaging algorithm to the case of a fixed line graph, and states a lower bound on the convergence time in this setting.

We will use the notation $\mathbf{1}$ to denote the vector in $\mathbb{R}^n$ whose entries are all ones, and $\mathbf{0}$ to denote the vector whose entries are all 0. The average of the initial values $x_1(0), \ldots, x_n(0)$ will be denoted by $\bar{x}$.

---

\[\text{1By “absolute constant” we mean that } C \text{ does not depend on the problem parameters } n, B, \epsilon.\]
Theorem 6.1. Let $f_1, f_n$ be two differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$, and let $f_2, f_3, \ldots, f_{n-1}$ be differentiable functions from $\mathbb{R}^3$ to $\mathbb{R}$. Consider the dynamical system
\begin{align*}
x_1(t+1) &= f_1(x_1(t), x_2(t)), \\
x_i(t+1) &= f_i(x_i(t), x_{i-1}(t), x_{i+1}(t)), \quad i = 2, \ldots, n-1, \\
x_n(t+1) &= f_n(x_{n-1}(t), x_n(t)).
\end{align*}
(6.5)

Suppose that there exists a function $\tau(n, \epsilon)$ such that
$$\frac{\|x(t) - \bar{x}\|_2}{\|x(0) - \bar{x}\|_2} < \epsilon,$$
for all $n$ and $\epsilon$, all $t \geq \tau(n, \epsilon)$, and all initial conditions $x_1(0), \ldots, x_n(0)$ for which not all $x_i(0)$ are equal. Then,
$$\tau(n, \epsilon) \geq \frac{n^2}{30} \log \frac{1}{\epsilon},$$
(6.6)
for all $\epsilon > 0$ and $n \geq 3$.

Remark: The dynamical system described in the theorem statement is simply what a local averaging algorithm looks like on a line graph. The functions $f_1, f_n$ are the update functions at the left and right endpoints of the line (which have only a single neighbor), while the update functions $f_2, f_3, \ldots, f_{n-1}$ are the ones used by the middle agents (which have two neighbors). As a corollary, the convergence time of any local averaging algorithm must satisfy the lower bound $T(n, \epsilon) \geq (1/30)n^2 \log(1/\epsilon)$.

Remark: Fix some $n \geq 3$. A corollary of our theorem is that there are no “local averaging algorithms” which compute the average in finite time. More precisely, there is no local averaging algorithm which, starting from initial conditions $x(0)$ in some ball around the origin, always results in $x(t) = \bar{x}$ for all times $t$ larger than some $T$.

We will sketch a proof of this after proving Theorem 1. By contrast, the existence of such algorithms in slightly different models of agent interactions was demonstrated in [31] and [88].

6.2.1 Proof.

We first briefly sketch the proof strategy. We will begin by pointing out that $\mathbf{0}$ must be an equilibrium of Eq. (6.5); then, we will argue that an upper bound on the convergence time of Eq. (6.5) would imply a similar convergence time bound on the linearization of Eq. (6.5) around the equilibrium of $\mathbf{0}$. This will allow us to apply a previous $\Omega(n^2)$ convergence time lower bound for linear schemes, proved by the authors in [85].

Let $f$ (without a subscript) be the mapping from $\mathbb{R}^n$ to itself that maps $x(t)$ to $x(t+1)$ according to Eq. (6.5).
Lemma 6.1. $f(a1) = a1$, for any $a \in \mathbb{R}$.

Proof. Suppose that $x(0) = a1$. Then, the initial average is $a$, so that

$$a1 = \lim_{t} x(t) = \lim_{t} x(t + 1) = \lim_{t} f(x(t)).$$

We use the continuity of $f$ to get

$$a1 = f(\lim_{t} x(t)) = f(a1).$$

For $i, j = 1, \ldots, n$, we define $a_{ij} = \frac{\partial f_i(0)}{\partial x_j}$, and the matrix

$$A = f'(0) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & a_{n,n-1} & a_{nn} \end{pmatrix}.$$ 

Lemma 6.2. For any integer $k \geq 1$,

$$\lim_{x \to 0} \frac{\|f^k(x) - A^k x\|_2}{\|x\|_2} = 0,$$

where $f^k$ refers to the $k$-fold composition of $f$ with itself.

Proof. The fact that $f(0) = 0$ implies by the chain rule that the derivative of $f^k$ at $x = 0$ is $A^k$. The above equation is a restatement of this fact. \hfill \Box

Lemma 6.3. Suppose that $x^T 1 = 0$. Then,

$$\lim_{m \to \infty} A^m x = 0.$$

Proof. Let $B$ be a ball around the origin such that for all $x \in B$, with $x \neq 0$, we have

$$\frac{\|f^k(x) - A^k x\|_2}{\|x\|_2} \leq \frac{1}{4}, \quad \text{for } k = \tau(n, 1/2).$$

Such a ball can be found due to Lemma 6.2. Since we can scale $x$ without affecting the assumptions or conclusions of the lemma we are trying to prove, we can assume
that $x \in B$. It follows that that for $k = \tau(n, 1/2)$, we have

$$\frac{\|A^k x\|_2}{\|x\|_2} = \frac{\|A^k x - f^k(x) + f^k(x)\|_2}{\|x\|_2} \leq \frac{1}{4} + \frac{\|f^k(x)\|_2}{\|x\|_2} \leq \frac{1}{4} + \frac{1}{2} \leq \frac{3}{4}.$$ 

Since this inequality implies that $A^k x \in B$, we can apply the same argument recursively to get

$$\lim_{m \to \infty} (A^k)^m x = 0,$$

which implies the conclusion of the lemma.

\begin{lemma}
$A1 = 1$.
\end{lemma}

\begin{proof}
We have

$$A1 = \lim_{h \to 0} \frac{f(0 + h1) - f(0)}{h} = \lim_{h \to 0} \frac{h1}{h} = 1,$$

where we used Lemma 6.4. \qed

\begin{lemma}
For every vector $x \in \mathbb{R}^n$,

$$\lim_{k \to \infty} A^k x = \bar{x}1,$$

where $\bar{x} = (\sum_{i=1}^n x_i)/n$.
\end{lemma}

\begin{proof}
Every vector $x$ can be written as

$$x = \bar{x}1 + y,$$

where $y^T1 = 0$. Thus,

$$\lim_{k \to \infty} A^k x = \lim_{k \to \infty} A^k (\bar{x}1 + y) = \bar{x}1 + \lim_{k \to \infty} A^k y = \bar{x}1,$$

where we used Lemmas 6.3 and 6.4. \qed

\begin{lemma}
The matrix $A$ has the following properties:

1. $a_{ij} = 0$ whenever $|i - j| > 1$.

2. The graph $G = (\{1, \ldots, n\}, E)$, with $E = \{(i, j) \mid a_{ij} \neq 0\}$, is strongly connected.
\end{lemma}
3. $A1 = 1$ and $1^T A = 1$.

4. An eigenvalue of $A$ of largest modulus has modulus 1.

Proof. 1. True because of the definitions of $f$ and $A$.

2. Suppose not. Then, there is a nonempty set $S \subset \{1, \ldots, n\}$ with the property that $a_{ij} = 0$ whenever $i \in S$ and $j \in S^c$. Consider the vector $x$ with $x_i = 0$ for $i \in S$, and $x_j = 1$ for $j \in S^c$. Clearly, $(1/n) \sum_i x_i > 0$, but $(A^k x)_i = 0$ for $i \in S$. This contradicts Lemma 6.5.

3. The first equality was already proven in Lemma 6.4. For the second, let $b = 1^T A$. Consider the vector

$$z = \lim_{k \to \infty} A^k e_i,$$

where $e_i$ is the $i$th unit vector. By Lemma 6.5

$$z = \frac{1^T e_i}{n} 1 = \frac{1}{n} 1.$$

On the other hand,

$$\lim_{k \to \infty} A^k e_i = \lim_{k \to \infty} A^{k+1} e_i = \lim_{k \to \infty} A^k (A e_i).$$

Applying Lemma 6.5 again, we get

$$z = \frac{1^T \cdot (A e_i)}{n} 1 = \frac{b_i}{n} 1,$$

where $b_i$ is the $i$th component of $b$. We conclude that $b_i = 1$; since no assumption was made on $i$, this implies that $b = 1$, which is what we needed to show.

4. We already know that $A1 = 1$, so that an eigenvalue with modulus 1 exists. Now suppose there is an eigenvalue with larger modulus, that is, there is some vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$ and $|\lambda| > 1$. Then $\lim_k \|A^k x\|_2 = \infty$. By writing $x = x_{\text{real}} + ix_{\text{imaginary}}$, we immediately have that $A^k x = A^k x_{\text{real}} + i A^k x_{\text{imaginary}}$. But by Lemma 6.5 both $A^k x_{\text{real}}$ and $A^k x_{\text{imaginary}}$ approach some finite multiple of 1 as $k \to \infty$, so $\|A^k x\|_2$ is bounded above. This is a contradiction.

Theorem 6.2 (Eigenvalue lemma). If $A$ satisfies all of the conclusions of Lemma 6.6, then $A$ has an eigenvector $v$, with real eigenvalue $\lambda \in (1 - \frac{6}{n^2}, 1)$, such that $v^T 1 = 0$.

This proof of this fact needs its own section.

6.3 Proof of the eigenvalue lemma

Lemma 6.7. Consider an $n \times n$ matrix $A$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$, be its eigenvalues, sorted in order of decreasing magnitude. Suppose that the following conditions hold.
(a) We have $\lambda_1 = 1$ and $A1 = 1$.

(b) There exists a positive vector $\pi$ such that $\pi^T A = \pi^T$.

(c) For every $i$ and $j$, we have $\pi_i a_{ij} = \pi_j a_{ji}$.

Let

$$S = \{ x \mid \sum_{i=1}^n \pi_i x_i = 0, \sum_{i=1}^n \pi_i x_i^2 = 1 \}$$

Then, all eigenvalues of $A$ are real, and

$$\lambda_2 = 1 - \frac{1}{2} \min_{x \in S} \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} (x_i - x_j)^2.$$  \hfill (6.8)

In particular, for any vector $y$ that satisfies $\sum_{i=1}^n \pi_i y_i = 0$, we have

$$\lambda_2 \geq 1 - \frac{\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} (y_i - y_j)^2}{2 \sum_{i=1}^n \pi_i y_i^2}.$$  \hfill (6.9)

**Proof.** Let $D$ be a diagonal matrix whose $i$th diagonal entry is $\pi_i$. Condition (c) yields $DA = A^T D$. We define the inner product $\langle \cdot, \cdot \rangle_\pi$ by $\langle x, y \rangle_\pi = x^T Dy$. We then have

$$\langle x, Ay \rangle_\pi = x^T D A y = (Ax, y)_\pi.$$  

Therefore, $A$ is self-adjoint with respect to this inner product, which proves that $A$ has real eigenvalues.

Since the largest eigenvalue is 1, with an eigenvector of 1, we use the variational characterization of the eigenvalues of a self-adjoint matrix (Chapter 7, Theorem 4.3 of [90]) to obtain

$$\lambda_2 = \max_{x \in S} \langle x, Ax \rangle_\pi$$

$$= \max_{x \in S} \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} x_i x_j$$

$$= \frac{1}{2} \max_{x \in S} \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} (x_i^2 + x_j^2 - (x_i - x_j)^2).$$

For $x \in S$, we have

$$\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} (x_i^2 + x_j^2) = 2 \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} x_i^2 = 2 \sum_{i=1}^n \pi_i x_i^2 = 2 \langle x, x \rangle_\pi = 2,$$
which yields
\[ \lambda_2 = 1 - \frac{1}{2} \min_{x \in S} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} (x_i - x_j)^2. \]

Finally, Eq. (6.9) follows from (6.8) by considering the vector \( x_i = y_i / \sqrt{\sum_{j=1}^{n} \pi_j y_j^2} \).

With the previous lemma in place, we can now prove Theorem 6.2.

**Proof of Theorem 6.2.** If the entries of \( A \) were all nonnegative, we would be dealing with a birth-death Markov chain. Such a chain is reversible, i.e., satisfies the detailed balance equations \( \pi_i a_{ij} = \pi_j a_{ji} \) (condition (c) in Theorem 6.7). In fact the derivation of the detailed balance equations does not make use of nonnegativity; thus, detailed balance holds in our case as well. Since \( \pi_i = 1 \) by assumption, we have that our matrix \( A \) is symmetric.

For \( i = 1, \ldots, n \), let \( y_i = i - (n + 1)/2 \); observe that \( \sum_{i=1}^{n} y_i = 0 \). We will make use of the inequality (6.9). Since \( a_{ij} = 0 \) whenever \( |i - j| > 1 \), we have
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} (y_i - y_j)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n. \] (6.10)

Furthermore,
\[ \sum_{i=1}^{n} \pi_i y_i^2 = \sum_{i=1}^{n} \left( i - \frac{n + 1}{2} \right)^2 \geq \frac{n^3}{12}. \] (6.11)

The last inequality follows from the well known fact \( \text{var}(X) = (n^2 - 1)/12 \) for a discrete uniform random variable \( X \). Using the inequality (6.9) and Eqs. (6.10)-(6.11), we obtain the desired bound on \( \rho \).  

**Remark:** Note that if the matrix \( A \) is as in the previous theorem, it is possible for the iteration \( x(t + 1) = Ax(t) \) not to converge at all. Indeed, nothing in the argument precludes the possibility that the smallest eigenvalue is \(-1\), for example. In such a case, the lower bounds of the theorem — derived based on bounding the second largest eigenvalue — still hold as the convergence rate and time are infinite.

**Remark:** We could have saved ourselves a few lines by appealing to the results of [18] once we showed \( A \) is symmetric.

### 6.4 Proof of the main theorem

We are now in a position to finally prove the main result of this chapter.

**Proof of Theorem 6.1.** Let \( v \) be an eigenvector of \( A \) with the properties in part 5 of Lemma 6.6. Fix a positive integer \( k \). Let \( \epsilon > 0 \) and pick \( x \neq 0 \) to be a small
enough multiple of \( v \) so that
\[
\frac{\| f^k(x) - A^k(x) \|_2}{\| x \|_2} \leq \epsilon.
\]
This is possible by Lemma 6.2. Then, we have
\[
\frac{\| f^k(x) \|_2}{\| x \|_2} \geq \frac{\| A^k x \|_2}{\| x \|_2} - \epsilon \geq \left( 1 - \frac{6}{n^2} \right)^k - \epsilon.
\]
Using the orthogonality property \( x^T 1 = 0 \), we have \( \bar{x} = 0 \). Since we placed no restriction on \( \epsilon \), this implies that
\[
\inf_{x \neq 0} \frac{\| f^k(x) - \bar{x} 1 \|_2}{\| x - \bar{x} 1 \|_2} = \inf_{x \neq 0} \frac{\| f^k(x) \|_2}{\| x \|_2} \geq \left( 1 - \frac{6}{n^2} \right)^k
\]
Plugging \( k = \tau(n, \epsilon) \) into this equation, we see that
\[
\left( 1 - \frac{6}{n^2} \right)^{\tau(n, \epsilon)} \leq \epsilon.
\]
Since \( n \geq 3 \), we have \( 1 - 6/n^2 \in (0, 1) \), and
\[
\tau(n, \epsilon) \geq \frac{1}{\log(1 - 6/n^2)} \log \epsilon.
\]
Now using the bound \( \log(1 - \alpha) \geq 5(\alpha - 1) \) for \( \alpha \in [0, 2/3] \), we get
\[
\tau(n, \epsilon) \geq \frac{n^2}{30} \log \frac{1}{\epsilon}.
\]

Remark: We now sketch the proof of the claim we made earlier that a local averaging algorithm cannot average in finite time. Fix \( n \geq 3 \). Suppose that for any \( x(0) \) in some ball \( B \) around the origin, a local averaging algorithm results in \( x(t) = \bar{x} 1 \) for all \( t \geq T \).

The proof of Theorem 1 shows that given any \( k, \epsilon > 0 \), one can pick a vector \( v(\epsilon) \) so that if \( x(0) = v(\epsilon) \) then \( V(x(k))/V(x(0)) \geq (1 - 6/n^2)^k - \epsilon \). Moreover, the vectors \( v(\epsilon) \) can be chosen to be arbitrarily small. One simply picks \( k = T \) and \( \epsilon < (1 - 6/n^2)^k \) to get that \( x(T) \) is not a multiple of \( 1 \); and furthermore, picking \( v(\epsilon) \) small enough in norm to be in \( B \) results in a contradiction.

Remark: Theorem 6.1 gives a lower bound on how long we must wait for the 2-norm \( \| x(t) - \bar{x} 1 \|_2 \) to shrink by a factor of \( \epsilon \). What if we replace the 2-norm with other norms, for example with the \( \infty \)-norm? Since \( B_\infty(0, r/\sqrt{n}) \subset B_2(0, r) \subset B_\infty(0, r) \), it follows that if the \( \infty \)-norm shrinks by a factor of \( \epsilon \), then the 2-norm must shrink by at least \( \sqrt{n} \epsilon \). Since \( \epsilon \) only enters the lower bound of Theorem 6.1 logarithmically,
the answer only changes by a factor of \( \log n \) in passing to the \( \infty \)-norm. A similar argument shows that, modulo some logarithmic factors, it makes no difference which \( p \)-norm is used.

6.5 Concluding remarks

We have proved a lower bound on the convergence time of local averaging algorithms which scales quadratically in the number of agents. This lower bound holds even if all the communication graphs are equal to a fixed line graph. Our work points to a number of open questions.

1. Is it possible to loosen the definition of local averaging algorithms to encompass a wider class of algorithms? In particular, is it possible to weaken the requirement that each \( f_{i,G(t)} \) be smooth, perhaps only to the requirement that it be piecewise-smooth or continuous, and still obtain a \( \Omega(n^2) \) lower bound?

2. Does the worst-case convergence time change if we introduce some memory and allow \( x_i(t+1) \) to depend on the last \( k \) sets of messages received by agent \( i \)? Alternatively, there is the broader question of how much is there to be gained if every agent is allowed to keep track of extra variables. Some positive results in this direction were obtained in [53].

3. What if each node maintains a small number of update functions, and is allowed to choose which of them to apply? Our lower bound does not apply to such schemes, so it is an open question whether its possible to design practical algorithms along these lines with worst-case convergence time scaling better than \( n^2 \).

In general, it would be nice to understand the relationship between the structure of classes of averaging algorithms (e.g., how much memory they use, whether the updates are linear) and the best achievable performance.
Chapter 7

Quantized averaging

In this chapter, we consider a quantized version of the update rule in Eq. (3.1). This model is a good approximation for a network of nodes communicating through finite bandwidth channels, so that at each time instant, only a finite number of bits can be transmitted. We incorporate this constraint in our algorithm by assuming that each node, upon receiving the values of its neighbors, computes the convex combination \( \sum_{j=1}^{n} a_{ij}(k)x_j(k) \) and quantizes it. This update rule also captures a constraint that each node can only store quantized values.

Unfortunately, under Assumptions 3.1, 3.2, and 3.4 if the output of Eq. (3.1) is rounded to the nearest integer, the sequence \( x(k) \) is not guaranteed to converge to consensus; see [54] for an example. We therefore choose a quantization rule that rounds the values down, according to

\[
x_i(k+1) = \left\lfloor \sum_{j=1}^{n} a_{ij}(k)x_j(k) \right\rfloor,
\]

where \( \lfloor \cdot \rfloor \) represents rounding down to the nearest multiple of \( 1/Q \), and where \( Q \) is some positive integer.

We adopt the natural assumption that the initial values are already quantized.

**Assumption 7.1.** For all \( i \), \( x_i(0) \) is a multiple of \( 1/Q \).

We will next demonstrate that starting from multiples of \( 1/Q \), arbitrarily accurate quantized computation of the average is possible provided the number of bits used to quantize is at least on the order of \( \log n \). Moreover, the time scaling with \( n \) required to do this is still on the order of \( n^2 \), as in the previous chapters. Thus the above Eq. (7.1), despite its simplicity, turns out to have excellent performance in the quantized setting.

Our exposition in this chapter will follow the paper [76], where the results described here have previously appeared.
7.1 A quantization level dependent bound

For convenience we define

\[ U = \max_i x_i(0), \quad L = \min_i x_i(0). \]

We use \( K \) to denote the total number of relevant quantization levels, i.e.,

\[ K = (U - L)Q, \]

which is an integer by Assumption 7.1.

We first present a convergence time bound that depends on the quantization level \( Q \).

**Proposition 7.1.** Let Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), 3.2 (\( B \)-connectivity), and 7.1 (quantized initial values) hold. Let \( \{x(k)\} \) be generated by the update rule (7.1). If \( k \geq nBK \), then all components of \( x(k) \) are equal.

**Proof.** Consider the nodes whose initial value is \( U \). There are at most \( n \) of them. As long as not all entries of \( x(k) \) are equal, then every \( B \) iterations, at least one node must use a value strictly less than \( U \) in an update; such a node will have its value decreased to \( U - 1/Q \) or less. It follows that after \( nB \) iterations, the largest node value will be at most \( U - 1/Q \). Repeating this argument, we see that at most \( nBK \) iterations are possible before all the nodes have the same value. \( \Box \)

Although the above bound gives informative results for small \( K \), it becomes weaker as \( Q \) (and, therefore, \( K \)) increases. On the other hand, as \( Q \) approaches infinity, the quantized system approaches the unquantized system; the availability of convergence time bounds for the unquantized system suggests that similar bounds should be possible for the quantized one. Indeed, in the next section, we adopt a notion of convergence time parallel to our notion of convergence time for the unquantized algorithm; as a result, we obtain a bound on the convergence time which is independent of the total number of quantization levels.

7.2 A quantization level independent bound

We adopt a slightly different measure of convergence for the analysis of the quantized consensus algorithm. For any \( x \in \mathbb{R}^n \), we define \( m(x) = \min_i x_i \) and

\[ V(x) = \sum_{i=1}^{n} (x_i - m(x))^2. \]

We will also use the simpler notation \( m(k) \) and \( V(k) \) to denote \( m(x(k)) \) and \( V(x(k)) \), respectively, where it is more convenient to do so. The function \( V \) will be our Lyapunov function for the analysis of the quantized consensus algorithm. The reason for
not using our earlier Lyapunov function, $V$, is that for the quantized algorithm, $V$ is not guaranteed to be monotonically nonincreasing in time. On the other hand, we have that $V(x) \leq \sqrt{4n}V(x)$ for any $x \in \mathbb{R}^n$. As a consequence, any convergence time bounds expressed in terms of $\sqrt{V}$ translate to essentially the same bounds expressed in terms of $V$, up to a logarithmic factor.

Before proceeding, we record an elementary fact which will allow us to relate the variance decrease $V(x) - V(y)$ to the decrease, $\sqrt{V(x)} - \sqrt{V(y)}$, of our new Lyapunov function. The proof involves simple algebra, and is therefore omitted.

**Lemma 7.1.** Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be real numbers satisfying

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} w_i.$$  

Then, the expression

$$f(z) = \sum_{i=1}^{n} (u_i - z)^2 - \sum_{i=1}^{n} (w_i - z)^2$$

is a constant, independent of the scalar $z$.

Our next lemma places a bound on the decrease of the Lyapunov function $V(t)$ between times $kB$ and $(k+1)B - 1$.

**Lemma 7.2.** Let Assumptions 3.1, 3.4, 4.1, and 7.1 hold. Let $\{x(k)\}$ be generated by the update rule (7.1). Suppose that the components $x_i(kB)$ of the vector $x(kB)$ have been ordered from largest to smallest, with ties broken arbitrarily. Then, we have

$$V(kB) - V((k+1)B) \geq \frac{\eta}{2} \sum_{i=1}^{n-1} (x_i(kB) - x_{i+1}(kB))^2.$$  

**Proof.** For all $k$, we view Eq. (7.1) as the composition of two operators:

$$y(k) = A(k)x(k),$$

where $A(k)$ is a doubly stochastic matrix, and

$$x(k+1) = \lfloor y(k) \rfloor,$$

where the quantization $\lfloor \cdot \rfloor$ is carried out componentwise.

We apply Lemma 7.1 with the identification $u_i = x_i(k)$, $w_i = y_i(k)$. Since multiplication by a doubly stochastic matrix preserves the mean, the condition $\sum_i u_i = \sum_i w_i$
is satisfied. By considering two different choices for the scalar $z$, namely, $z_1 = \bar{x}(k) = \bar{y}(k)$ and $z_2 = m(k)$, we obtain

$$V(x(k)) - V(y(k)) = V(x(k)) - \sum_{i=1}^{n}(y_i(k) - m(k))^2. \quad (7.2)$$

Note that $x_i(k+1) - m(k) \leq y_i(k) - m(k)$. Therefore,

$$V(x(k)) - \sum_{i=1}^{n}(y_i(k) - m(k))^2 \leq V(x(k)) - \sum_{i=1}^{n}(x_i(k+1) - m(k))^2. \quad (7.3)$$

Furthermore, note that since $x_i(k+1) \geq m(k+1) \geq m(k)$ for all $i$, we have that $x_i(k+1) - m(k+1) \leq x_i(k+1) - m(k)$. Therefore,

$$V(x(k)) - \sum_{i=1}^{n}(x_i(k+1) - m(k))^2 \leq V(x(k)) - V(x(k+1)). \quad (7.4)$$

By combining Eqs. (7.2), (7.3), and (7.4), we obtain

$$V(x(t)) - V(y(t)) \leq V(x(t)) - V(x(t+1)) \quad \text{for all } t.$$ 

Summing the preceding relations over $t = kB, \ldots, (k+1)B - 1$, we further obtain

$$\sum_{t=kB}^{(k+1)B-1} \left( V(x(t)) - V(y(t)) \right) \leq V(x(kB)) - V(x((k+1)B)).$$

To complete the proof, we provide a lower bound on the expression

$$\sum_{t=kB}^{(k+1)B-1} \left( V(x(t)) - V(y(t)) \right).$$

Since $y(t) = A(t)x(t)$ for all $t$, it follows from Lemma 4.1 that for any $t$,

$$V(x(t)) - V(y(t)) = \sum_{i<j} w_{ij}(t)(x_i(t) - x_j(t))^2,$$

where $w_{ij}(t)$ is the $(i, j)$-th entry of $A(t)^T A(t)$. Using this relation and following the same line of analysis used in the proof of Lemma 4.4 where the relation $x_i(t) \geq y_{di}$ holds in view of the assumption that $x_i(kB)$ is a multiple of $1/Q$ for all $k \geq 0$, cf. Assumption 7.1, we obtain the desired result. \hfill $\square$

The next theorem contains our main result on the convergence time of the quantized algorithm.
Theorem 7.1. Let Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), 4.1 (connectivity relaxation), and 7.1 hold. Let \{x(k)\} be generated by the update rule (7.1). Then, there exists an absolute constant c such that we have

\[ V(k) \leq \epsilon V(0) \quad \text{for all } k \geq c(n^2/\eta)B \log(1/\epsilon). \]

Proof. Let us assume that \( V(kB) > 0 \). From Lemma 7.2, we have

\[ V(kB) - V((k + 1)B) \geq \frac{\eta}{2} \sum_{i=1}^{n-1} (x_i(kB) - x_{i+1}(kB))^2, \]

where the components \( x_i(kB) \) are ordered from largest to smallest. Since \( V(kB) = \sum_{i=1}^{n} (x_i(kB) - x_n(kB))^2 \), we have

\[ \frac{V(kB) - V((k + 1)B)}{V(kB)} \geq \frac{\eta}{2} \frac{\sum_{i=1}^{n-1} (x_i(kB) - x_{i+1}(kB))^2}{\sum_{i=1}^{n} (x_i(kB) - x_n(kB))^2}. \]

Let \( y_i = x_i(kB) - x_n(kB) \). Clearly, \( y_i \geq 0 \) for all \( i \), and \( y_n = 0 \). Moreover, the monotonicity of \( x_i(kB) \) implies the monotonicity of \( y_i \):

\[ y_1 \geq y_2 \geq \cdots \geq y_n = 0. \]

Thus,

\[ \frac{V(kB) - V((k + 1)B)}{V(kB)} \geq \frac{\eta}{2} \min_{y_1 \geq y_2 \geq \cdots \geq y_n \geq 0} \frac{\sum_{i=1}^{n-1} (y_i - y_{i+1})^2}{\sum_{i=1}^{n} y_i^2}. \]

Next, we simply repeat the steps of Lemma 4.5. We can assume without loss of generality that \( \sum_{i=1}^{n} y_i^2 = 1 \). Define \( z_i = y_i - y_{i+1} \) for \( i = 1, \ldots, n-1 \) and \( z_n = 0 \). We have that \( z_i \) are all nonnegative and \( \sum_i z_i = y_1 - y_n \geq 1/\sqrt{n} \). Therefore,

\[ \frac{\eta}{2} \min_{\sum z_i^2 \geq y_n} \sum_{i=1}^{n-1} (y_i - y_{i+1})^2 \geq \frac{\eta}{2} \min_{\sum z_i \geq 1/\sqrt{n}} \sum_{i=1}^{n} z_i^2. \]

The minimization problem on the right-hand side has an optimal value of at least \( 1/n^2 \), and the desired result follows.

7.3 Extensions and modifications

In this section, we comment briefly on some corollaries of Theorem 7.1.

First, we note that the results of Section 5 immediately carry over to the quantized case. Indeed, in Section 5, we showed how to pick the weights \( a_{ij}(k) \) in a decentralized manner, based only on local information, so that Assumptions 1 and 4.1 are satisfied, with \( \eta \geq 1/3 \). When using a quantized version of the load-balancing algorithm of Chapter 5, we once again manage to remove the factor of \( 1/\eta \) from our upper bound.
Proposition 7.2. For the quantized version of the load-balancing algorithm of Chapter 5 and under the same assumptions as in Theorem 5.2, if \( k \geq c n^2 B \log(1/\epsilon) \), then \( V(k) \leq c V(0) \), where \( c \) is an absolute constant.

Second, we note that Theorem 7.1 can be used to obtain a bound on the time until the values of all nodes are equal. Indeed, we observe that in the presence of quantization, once the condition \( V(k) < 1/Q^2 \) is satisfied, all components of \( x(k) \) must be equal.

Proposition 7.3. Consider the quantized algorithm (7.1), and assume that Assumptions 3.1 (non-vanishing weights), 3.4 (double stochasticity), 4.1 (connectivity relaxation), and 7.1 (quantized initial values) hold. If \( k \geq c (n^2/\eta) B \left[ \log Q + \log V(x(0)) \right] \), then all components of \( x(k) \) are equal, where \( c \) is an absolute constant.

7.4 Quantization error

Despite favorable convergence properties of our quantized averaging algorithm (7.1), the update rule does not preserve the average of the values at each iteration. Therefore, the common limit of the sequences \( x_i(k) \), denoted by \( x_f \), need not be equal to the exact average of the initial values. We next provide an upper bound on the error between \( x_f \) and the initial average, as a function of the number of quantization levels.

Proposition 7.4. There is an absolute constant \( c \) such that for the common limit \( x_f \) of the values \( x_i(k) \) generated by the quantized algorithm (7.1), we have

\[
\left| x_f - \frac{1}{n} \sum_{i=1}^{n} x_i(0) \right| \leq \frac{c}{Q} \frac{n^2}{\eta} B \log(Qn(U-L)).
\]

Proof. By Proposition 7.3 after \( O\left( (n^2/\eta) B \log(QV(x(0))) \right) \) iterations, all nodes will have the same value. Since \( V(x(0)) \leq n(U-L)^2 \) and the average decreases by at most \( 1/Q \) at each iteration, the result follows.

Let us assume that the parameters \( B, \eta, \) and \( U-L \) are fixed. Proposition 7.4 implies that as \( n \) increases, the number of bits used for each communication, which is proportional to \( \log Q \), needs to grow only as \( O(\log n) \) to make the error negligible. Furthermore, this is true even if the parameters \( B, 1/\eta, \) and \( U-L \) grow polynomially in \( n \).

For a converse, it can be seen that \( \Omega(\log n) \) bits are needed. Indeed, consider \( n \) nodes, with \( n/2 \) nodes initialized at 0, and \( n/2 \) nodes initialized at 1. Suppose that \( Q < n/2 \); we connect the nodes by forming a complete subgraph over all the nodes with value 0 and exactly one node with value 1; see Figure 7.1 for an example with \( n = 6 \). Then,
Figure 7-1: Initial configuration. Each node takes the average value of its neighbors.

Each node forms the average of its neighbors. This brings one of the nodes with an initial value of 1 down to 0, without raising the value of any other nodes. We can repeat this process, to bring all of the nodes with an initial value of 1 down to 0. Since the true average is 1/2, the final result is 1/2 away from the true average. Note now that $Q$ can grow linearly with $n$, and still satisfy the inequality $Q < n/2$. Thus, the number of bits can grow as $\Omega(\log n)$, and yet, independent of $n$, the error remains 1/2.

### 7.5 Concluding remarks

The high-level summary of this chapter is that $c \log n$ bits suffice for quantized averaging. The answers obtained will not be exact, but the error can be made arbitrarily small by picking $c$ large, and the favorable convergence times from the earlier chapters retained.

An interesting direction is whether its possible to reduce the number of bits even further, to a constant number per each link maintained by a node (at least one bit per link is necessary merely to store incoming messages). A positive answer in the case of fixed graphs is provided by the following chapter. On dynamic graphs, the number of bits required by deterministic averaging algorithms is still not understood.
Chapter 8

Averaging with a constant number of bits per link

The previous chapter analyzed the performance of a quantized update rule. The “punchline” was that if the number of bits involved in the quantization is on the order of \( \log n \), then arbitrarily accurate averaging is possible.

This chapter analyzes what happens if we try to push down the number of bits stored at each node to a constant. Our exposition in this chapter will follow the preprint [50], where the results described here have previously appeared.

We will assume that at any time step, nodes can exchange binary messages with their neighbors in an interconnection graph which is undirected and unchanging with time. Naturally, a node needs to store at least one bit for each of the links just to store the message arriving on that link. We will allow the nodes to maintain a constant number of bits for each of their links. Thus in a constant-degree network, this translates to a constant number of bits at each node.

Supposing that the nodes begins with numbers \( x_i \in \{0, 1, \ldots, K\} \), we will say that a function of the initial values is *computable with constant storage* if it computable subject to the restrictions in the previous paragraph. The exact average is not computable with constant storage because just storing the average takes \( \Omega(\log n) \) bits. Thus we will turn to the related, but weaker, question of deciding whether the 0’s or 1’s are in the majority.

In fact, we will study a more general problem called “interval averaging,” which asks to return the set among \( \{0\}, (0, 1), \{1\}, (1, 2), \ldots, (K-1, K), \{K\} \) within which the average of the initial numbers lies. If interval averaging is possible, then majority computation with binary initial conditions is possible as well. Indeed, the argument for this is simple: to do majority computation, every node beginning with \( x_i = 1 \) instead sets \( x_i = 2 \) and runs interval averaging with \( K = 2 \). Depending on whether the average is in \( \{0, \}, (0, 1), \{1\}, (1, 2), \{2\} \) each node knows which initial condition (if any) had the majority.

The interval averaging problem has trivial solutions with randomized algorithms (see Chapter 2). These algorithms “centralize” the problem by electing a leader and streaming information towards the leader. Decentralized randomized algorithms are also possible [54, 40]. In this chapter, we address the question of whether it is possible
to solve this problem with deterministic algorithms.

The main result of this chapter is the following theorem.

**Theorem 8.1.** Interval averaging is computable with constant storage. Moreover, there exists a (constant storage) algorithm for it under which every node has the correct answer after $O(n^2K^2 \log K)$ rounds of communication.

The remainder of this chapter will be devoted to the proof of this fact. Let us briefly summarize the main idea behind the computation of interval averages. Imagine the integer input value $x_i$ as represented by a number of $x_i$ pebbles at node $i$. The algorithm attempts to exchange pebbles between nodes with unequal numbers so that the overall distribution becomes more even. Eventually, either all nodes will have the same number of pebbles, or some will have a certain number and others just one more. The nodes will try to detect which of these two possibilities have occurred, and will estimate the interval average accordingly.

The remainder of this chapter is structured as follows. First, we discuss the problem of tracking, in a distributed way, the maximum of time-varying values at each node. This is a seemingly unrelated problem, but it will be useful in the proof of Theorem 8.1. After describing a solution to this problem, we give a formal algorithm which implements the pebble matching idea above. This algorithm will use the maximum tracking algorithm we developed as a subroutine to match up nodes with few pebbles to nodes with a large number of pebbles.

### 8.1 Related literature

A number of papers explored various tradeoffs associated with quantization of consensus schemes. We will not attempt a survey of the entire literature, but only mention papers that are most closely relevant to the algorithms presented here. The paper [54] proposed randomized *gossip-type* quantized averaging algorithms under the assumption that each agent value is an integer. They showed that these algorithms preserve the average of the values at each iteration and converge to approximate consensus. They also provided bounds on the convergence time of these algorithms for specific static topologies (fully connected and linear networks). We refer the reader also to the later papers [103] and [40]. A dynamic scheme which allows us to approximately compute the average as the nodes communicate more and more bits with each other can be found in [21]. In the recent work [32], Carli *et al.* proposed a distributed algorithm that uses quantized values and preserves the average at each iteration. They showed favorable convergence properties using simulations on some static topologies, and provided performance bounds for the limit points of the generated iterates.

### 8.2 Computing and tracking maximal values

We now describe an algorithm that tracks the maximum (over all nodes) of time-varying inputs at each node. It will be used as a subroutine later. The basic idea is
simple: every node keeps track of the largest value it has heard so far, and forwards this “intermediate result” to its neighbors. However, when an input value changes, the existing intermediate results need to be invalidated, and this is done by sending “restart” messages. A complication arises because invalidated intermediate results might keep circulating in the network, always one step ahead of the restart messages. We deal with this difficulty by “slowing down” the intermediate results, so that they travel at half the speed of the restart messages. In this manner, restart messages are guaranteed to eventually catch up with and remove invalidated intermediate results.

We start by giving the specifications of the algorithm. Suppose that each node $i$ has a time-varying input $u_i(t)$ stored in memory at time $t$, belonging to a finite set of numbers $\mathcal{U}$. We assume that, for each $i$, the sequence $u_i(t)$ must eventually stop changing, i.e., that there exists some $T'$ such that
\[ u_i(t) = u_i(T'), \quad \text{for all } i \text{ and } t \geq T'. \]
(However, node $i$ need not ever be aware that $u_i(t)$ has reached its final value.) Our goal is to develop a distributed algorithm whose output eventually settles on the value $\max_i u_i(T')$. More precisely, each node $i$ is to maintain a number $M_i(t)$ which must satisfy the following condition: for every network and any allowed sequences $u_i(t)$, there exists some $T''$ with
\[ M_i(t) = \max_{j=1,\ldots,n} u_j(t), \quad \text{for all } i \text{ and } t \geq T''. \]

Moreover, each node $i$ must also maintain a pointer $P_i(t)$ to a neighbor or to itself. We will use the notation $P_i^2(t) = P_{P_i(t)}(t)$, $P_i^3(t) = P_{P_i^2(t)}(t)$, etc. We require the following additional property, for all $t$ larger than $T''$: for each node $i$ there exists a node $j$ and a power $K$ such that for all $k \geq K$ we have $P_i^k(t) = j$ and $M_i(t) = u_j(t)$. In words, by successively following the pointers $P_i(t)$, one can arrive at a node with a maximal value.

We next describe the algorithm. We will use the term slot $t$ to refer, loosely speaking, to the interval between times $t$ and $t+1$. More precisely, during slot $t$ each node processes the messages that have arrived at time $t$ and computes the state at time $t+1$ as well as the messages it will send at time $t+1$.

The variables $M_i(t)$ and $P_i(t)$ are a complete description of the state of node $i$ at time $t$. Our algorithm has only two types of messages that a node can send to its neighbors. Both are broadcasts, in the sense that the node sends them to every neighbor:

1. “Restart!”
2. “My estimate of the maximum is $y_i$,” where $y_i$ is some number in $\mathcal{U}$ chosen by the node.
Figure 8-1: Flowchart of the procedure used by node \( i \) during slot \( t \) in the maximum tracking algorithm. The subscript \( i \) is omitted, but \( u(t), M(t), \) and \( P(t) \) should be understood as \( u_i(t), M_i(t), \) and \( P_i(t) \). In those cases where an updated value of \( M \) or \( P \) is not indicated, it is assumed that \( M(t+1) = M(t) \) and \( P(t+1) = P(t) \). The symbol \( \emptyset \) is used to indicate no action. Note that the various actions indicated are taken during slot \( t \), but the messages determined by these actions are sent (and instantaneously received) at time \( t+1 \). Finally, observe that every node sends an identical message to all its neighbors at every time \( t > 0 \). We note that the apparent non-determinism in instruction O2 can be removed by picking a node with, say, the smallest port label.
Initially, each node sets $M_i(0) = u_i(0)$ and $P_i(0) = i$. At time $t = 0, 1, \ldots$, nodes exchange messages, which then determine their state at time $t+1$, i.e., the pair $M_i(t+1), P_i(t+1)$, as well as the messages to be sent at time $t+1$. The procedure node $i$ uses to do this is described in Figure 8.1. One can verify that a memory size of $C \log |U| + C \log d(i)$ at each node $i$ suffices, where $C$ is an absolute constant. (This is because $M_i$ and $P_i$ can take one of $|U|$ and $d(i)$ possible values, respectively.)

Note that while we assumed that agents can only transmit binary messages to each other, the above algorithm requires transmissions of the values in $\{0, \ldots, K\}$. This means that each step of the above algorithm takes $\log K$ real-time steps to implement.

The result that follows asserts the correctness of the algorithm. The idea of the proof is quite simple. Nodes maintain estimates $M_i(t)$ which track the largest among all the $u_i(t)$ in the graph; these estimates are “slowly” forwarded by the nodes to their neighbors, with many artificial delays along the way. Should some value $u_j(t)$ change, restart messages traveling without artificial delays are forwarded to any node which thought $j$ had a maximal value, causing those nodes to start over. The possibility of cycling between restarts and forwards is avoided because restarts travel faster. Eventually, the variables $u_i(t)$ stop changing, and the algorithm settles on the correct answer.

**Theorem 8.2** (Correctness of the maximum tracking algorithm). Suppose that the $u_i(t)$ stop changing after some finite time. Then, for every network, there is a time after which the variables $P_i(t)$ and $M_i(t)$ stop changing and satisfy $M_i(t) = \max_i u_i(t)$; furthermore, after that time, and for every $i$, the node $j = P_i^*(t)$ satisfies $M_i(t) = u_j(t)$.

We now turn to the proof of Theorem 8.2.

In the following, we will occasionally use the following convenient shorthand: we will say that a statement $S(t)$ holds eventually if there exists some $T$ so that $S(t)$ is true for all $t \geq T$.

The analysis is made easier by introducing the time-varying directed graph $G(t) = ([1, \ldots, n], E(t))$ where $(i, j) \in E(t)$ if $i \neq j$, $P_i(t) = j$, and there is no “restart” message sent by $j$ (and therefore received by $i$) during time $t$. We will abuse notation by writing $(i, j) \in G(t)$ to mean that the edge $(i, j)$ belongs to the set of edges $E(t)$ at time $n$.

**Lemma 8.1.** Suppose that $(i, j) \notin G(t-1)$ and $(i, j) \in G(t)$. Then, $i$ executes O2 during slot $t - 1$.

**Proof.** Suppose that $(i, j) \notin G(t-1)$ and $(i, j) \in G(t)$. If $P_i(t-1) = j$, the definition of $G(t-1)$ implies that $j$ sent a restart message to $i$ at time $t-1$. Moreover, $i$ cannot execute O3 during the time slot $t - 1$ as this would require $P_i(t-1) = i$. Therefore, during this time slot, node $i$ either executes O2 (and we are done) or it executes one of O1 and O4a. For both of the latter two cases, we will have $P_i(t) = i$, so that $(i, j)$ will not be in $G(t)$, a contradiction. Thus, it must be that $P_i(t-1) \neq j$. We now observe that the only place in Figure 8.1 that can change $P_i(t-1) \neq j$ to $P_i(t) = j$ is O2. \hfill \[1\]

\[1\]In fact, it can be shown that this case never occurs.

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Lemma 8.2. In each of the following three cases, node $i$ has no incoming edges in either graph $G(t)$ or $G(t + 1)$:

(a) Node $i$ executes O1, O2, or O4a during time slot $t - 1$;
(b) $M_i(t) \neq M_i(t - 1)$;
(c) For some $j$, $(i, j) \in G(t)$ but $(i, j) \notin G(t - 1)$.

Proof. (a) If $i$ executes O1, O2, or O4a during slot $t - 1$, then it sends a restart message to each of its neighbors at time $t$. Then, for any neighbor $j$ of $i$, the definition of $G(t)$ implies that $(j, i)$ is not in $G(t)$. Moreover, by Lemma 8.1, in order for $(j, i)$ to be in $G(t + 1)$, node $j$ must execute O2 during slot $t$. But the execution of O2 during slot $t$ cannot result in the addition of the edge $(j, i)$ at time $t + 1$, because the message broadcast by $i$ at time $t$ to its neighbors was a restart. So, $(j, i) \notin G(t + 1)$.

(b) If $M_i(t) \neq M_i(t - 1)$, then $i$ executes O1, O2, or O4a during slot $t - 1$, so the claim follows from part (a).

(c) By Lemma 8.1, it must be the case that node $i$ executes O2 during slot $t - 1$, and part (a) implies the result. \qed

Lemma 8.3. The graph $G(t)$ is acyclic, for all $t$.

Proof. The initial graph $G(0)$ does not contain a cycle. Let $t$ be the first time a cycle is present, and let $(i, j)$ be an edge in a cycle that is added at time $t$, i.e., $(i, j)$ belongs to $G(t)$ but not $G(t - 1)$. Lemma 8.2(c) implies that $i$ has no incoming edges in $G(t)$, so $(i, j)$ cannot be an edge of the cycle—a contradiction. \qed

Note that every node has out-degree at most one, because $P_i(t)$ is a single-valued variable. Thus, the acyclic graph $G(t)$ must be a forest, specifically, a collection of disjoint trees, with all arcs of a tree directed so that they point towards a root of the tree (i.e., a node with zero out-degree). The next lemma establishes that $M_i$ is constant on any path of $G(t)$.

Lemma 8.4. If $(i, j) \in G(t)$, then $M_i(t) = M_j(t)$.

Proof. Let $t'$ be a time when $(i, j)$ is added to the graph, or more precisely, a time such that $(i, j) \in G(t')$ but $(i, j) \notin G(t' - 1)$. First, we argue that the statement we want to prove holds at time $t'$. Indeed, Lemma 8.1 implies that during slot $t' - 1$, node $i$ executed O2, so that $M_i(t') = M_j(t' - 1)$. Moreover, $M_j(t' - 1) = M_j(t')$, because otherwise case (b) in Lemma 8.2 would imply that $j$ has no incoming edges at time $t'$, contradicting our assumption that $(i, j) \in G(t')$.

Next, we argue that the property $M_i(t) = M_j(t)$ continues to hold, starting from time $t'$ and for as long as $(i, j) \in G(t)$. Indeed, as long as $(i, j) \in G(t)$, then $M_j(t)$ remains unchanged, by case (b) of Lemma 8.2. To argue that $M_i(t)$ also remains unchanged, we simply observe that in Figure 8-1 every box which leads to a change in $M_i$ also sets $P_i$ either to $i$ or to the sender of a message with value strictly larger than $M_i$; this latter message cannot come from $j$ because as we just argued, increases in $M_j$ lead to removal of the edge $(i, j)$ from $G(t)$. So, changes in $M_i$ are also accompanied by removal of the edge $(i, j)$ from $G(t)$. \qed

For the purposes of the next lemma, we use the convention $u_i(-1) = u_i(0)$. 78
Lemma 8.5. If \( P_i(t) = i \), then \( M_i(t) = u_i(t - 1) \); if \( P_i(t) \neq i \), then \( M_i(t) > u_i(t - 1) \).

Proof. We prove this result by induction. Because of the convention \( u_i(-1) = u_i(0) \), and the initialization \( P_i(0) = i \), \( M_i(0) = u_i(0) \), the result trivially holds at time \( t = 0 \). Suppose now that the result holds at time \( t \). During time slot \( t \), we have three possibilities for node \( i \):

(i) Node \( i \) executes O1 or O4a. In this case, \( M_i(t + 1) = u_i(t) \), \( P_i(t + 1) = i \), so the result holds at time \( t + 1 \).

(ii) Node \( i \) executes O2. In this case \( P_i(t) \neq i \) and \( M_i(t + 1) > M_i(t) \geq u_i(t - 1) = u_i(t) \). The first inequality follows from the condition for entering step O2. The second follows from the induction hypothesis. The last equality follows because if \( u_i(t) \neq u_i(t - 1) \), node \( i \) would have executed O1 rather than O2. So, once again, the result holds at time \( t + 1 \).

(iii) Node \( i \) executes O3 or O4b. The result holds at time \( t + 1 \) because neither \( u_i \) nor \( M_i \) changes.

In the sequel, we will use \( T' \) to refer to a time after which all the \( u_i \) are constant.

The following lemma shows that, after \( T' \), the largest estimate does not increase.

Lemma 8.6. Suppose that at some time \( t' > T' \) we have \( \hat{M} > \max_i M_i(t') \). Then

\[
\hat{M} > \max_i M_i(t) \tag{8.1}
\]

for all \( t \geq t' \).

Proof. We prove Eq. (8.1) by induction. By assumption it holds at time \( t = t' \). Suppose now Eq. (8.1) holds at time \( t \); we will show that it holds at time \( t + 1 \).

Consider a node \( i \). If it executes O2 during the slot \( t \), it sets \( M_i(t + 1) \) to the value contained in a message sent at time \( t \) by some node \( j \). It follows from the rules of our algorithm that the value in this message is \( M_j(t) \) and therefore, \( M_i(t + 1) = M_j(t) < \hat{M} \).

Any operation other than O2 that modifies \( M_i \) sets \( M_i(t + 1) = u_i(t) \), and since \( u_i(t) \) does not change after time \( T' \), we have \( M_i(t + 1) = u_i(t - 1) \). By Lemma 8.5 \( M_i(t) \geq u_i(t - 1) \), so that \( M_i(t + 1) \leq M_i(t) \). We conclude that \( M_i(t + 1) < \hat{M} \) holds for this case as well.

We now introduce some terminology used to specify whether the estimate \( M_i(t) \) held by a node has been invalidated or not. Formally, we say that node \( i \) has a valid estimate at time \( t \) if by following the path in \( G(t) \) that starts at \( i \), we eventually arrive at a node \( r \) with \( P_r(t) = r \) and \( M_r(t) = u_r(t - 1) \). In any other case, we say that a node has an invalid estimate at time \( t \).

Remark: Because of the acyclicity property, a path in \( G(t) \), starting from a node \( i \), eventually leads to a node \( r \) with out-degree 0; it follows from Lemma 8.4.
that \( M_i(t) = M_i(t) \). Moreover, Lemma 8.4 implies that if \( P_i(t) = r \), then \( M_i(t) = M_r(t) = u_r(t - 1) \), so that the estimate is valid. Therefore, if \( i \) has an invalid estimate, the corresponding node \( r \) must have \( P_r(t) \neq r \); on the other hand, since \( r \) has out-degree 0 in \( G(t) \), the definition of \( G(t) \) implies that there is a “restart” message from \( P_r(t) \) to \( r \) sent at time \( t \).

The following lemma gives some conditions which allow us to conclude that a given node has reached a final state.

**Lemma 8.7.** Fix some \( t' > T' \) and let \( M^* \) be the largest estimate at time \( t' \), i.e., \( M^* = \max_i M_i(t') \). If \( M_i(t') = M^* \), and this estimate is valid, then for all \( t \geq t' \):

(a) \( M_i(t) = M^* \), \( P_i(t) = P_i(t') \), and node \( i \) has a valid estimate at time \( t \).
(b) Node \( i \) executes either O3 or O4b at time \( t \).

**Proof.** We will prove this result by induction on \( t \). Fix some node \( i \). By assumption, part (a) holds at time \( t = t' \). To show part (b) at time \( t = t' \), we first argue that \( i \) does not execute O2 during the time slot \( t \). Indeed, this would require \( i \) to have received a message with an estimate strictly larger than \( M^* \), sent by some node \( j \) who executed O3 or O4b during the slot \( t - 1 \). In either case, \( M^* < M_j(t - 1) = M_j(t) \), contradicting the definition of \( M^* \). Because of the definition of \( T' \), \( u_i(t) = u_i(t - 1) \) for \( t > T' \), so that \( i \) does not execute O1. This concludes the proof of the base case.

Next, we suppose that our result holds at time \( t \), and we will argue that it holds at time \( t + 1 \). If \( P_i(t) = i \), then \( i \) executes O3 during slot \( t \), so that \( M_i(t + 1) = M_i(t) \) and \( P_i(t + 1) = P_i(t) \), completes the induction step for this case.

It remains to consider the case where \( P_i(t) = j \neq i \). It follows from the definition of a valid estimate that \((i, j) \in E(t) \). Using the definition of \( E(t) \), we conclude that there is no restart message sent from \( j \) to \( i \) at time \( t \). By the induction hypothesis, during the slot \( t - 1 \), \( j \) has thus executed O3 or O4b, so that \( M_j(t - 1) = M_j(t) \); in fact, Lemma 8.4 gives that \( M_j(t) = M_i(t) = M^* \). Thus during slot \( t \), \( i \) reads a message from \( j = P_i(t) \) with the estimate \( M^* \), and executes O4b, consequently leaving its \( M_i \) or \( P_i \) unchanged.

We finally argue that node \( i \)'s estimate remains valid. This is the case because since we can apply the arguments of the previous paragraph to every node \( j \) on the path from \( i \) to a node with out-degree 0; we obtain that all of these nodes both (i) keep \( P_j(t + 1) = P_j(t) \) and (ii) execute O3 or O4b, and consequently do not send out any restart messages. \( \square \)

Recall (see the comments following Lemma 8.3) that \( G(t) \) consists of a collection of disjoint in-trees (trees in which all edges are oriented towards a root node). Furthermore, by Lemma 8.4, the value of \( M_i(t) \) is constant on each of these trees. Finally, all nodes on a particular tree have either a valid or invalid estimate (the estimate being valid if and only if \( P_r(t) = r \) and \( M_r(t) = u_r(t - 1) \) at the root node \( r \) of the tree.) For any \( z \in U \), we let \( G_z(t) \) be the subgraph of \( G(t) \) consisting of those trees at which all nodes have \( M_i(t) = z \) and for which the estimate \( z \) on that tree is invalid. We refer to \( G_z(t) \) as the *invalidity graph* of \( z \) at time \( t \). In the sequel we will say that \( i \) is in \( G_z(t) \), and abuse notation by writing \( i \in G_z(t) \), to mean that \( i \) belongs to the set of nodes of \( G_z(t) \). The lemmas that follow aim at showing that the
invali
dity graph of the largest estimate eventually becomes empty. Loosely speaking, the first lemma a
derts that after the $u_i(t)$ have stopped changing, it takes essentially two time steps for a maximal estimate $M^*$ to propagate to a neighboring node.

**Lemma 8.8.** Fix some time $t > T'$. Let $M^*$ be the largest estimate at that time, i.e., $M^* = \max_i M_i(t)$. Suppose that $i$ is in $G_{M^*}(t + 2)$ but not in $G_{M^*}(t)$. Then $P_i(t + 2) \in G_{M^*}(t)$.

**Proof.** The fact that $i \not\in G_{M^*}(t)$ implies that either (i) $M_i(t) \neq M^*$ or (ii) $M_i(t) = M^*$ and $i$ has a valid estimate at time $t$. In the latter case, it follows from Lemma 8.7 that $i$ also has a valid estimate at time $t + 2$, contradicting the assumption $i \in G_{M^*}(t + 2)$. Therefore, we can and will assume that $M_i(t) < M^*$. Since $t > T'$, no node ever executes O1. The difference between $M_i(t)$ and $M_i(t + 2) = M^*$ can only result from the execution of O2 or O4a by $i$ during time slot $t$ or $t + 1$.

Node $i$ cannot have executed O4a during slot $t + 1$, because this would result in $P_i(t + 2) = i$, and $i$ would have a valid estimate at time $t + 2$, contradicting the assumption $i \in G_{M^*}(t + 2)$. Similarly, if $i$ executes O4a during slot $t$ it sets $P_i(t + 1) = i$. Unless it executes O2 during slot $t + 1$, we have again $P_i(t + 2) = i$ contradicting the assumption $i \in G_{M^*}(t + 2)$. Therefore, $i$ must have executed O2 during either slot $t + 1$ or slot $t$, and in the latter case it must not have executed O4a during slot $t + 1$.

Let us suppose that $i$ executes O2 during slot $t + 1$, and sets thus $P_i(t + 2) = j$ for some $j$ that sent at time $t + 1$ a message with the estimate $M^* = M_i(t + 2)$. The rules of the algorithm imply that $M^* = M_j(t + 1)$. We can also conclude that $M_j(t + 1) = M_j(t)$, since if this were not true, node $j$ would have sent out a restart at time $t + 1$. Thus $M_j(t) = M^*$. It remains to prove that the estimate $M^*$ of $j$ at time $t$ is not valid. Suppose, to obtain a contradiction, that it is valid. Then it follows from Lemma 8.7 that $j$ also has a valid estimate at time $t + 2$, and from the definition of validity that the estimate of $i$ is also valid at $t + 2$, in contradiction with the assumption $i \in G_{M^*}(t + 2)$. Thus we have established that $P_i(t + 2) = j \in G_{M^*}(t)$ if $i$ executes O2 during slot $t + 1$. The same argument applies if $i$ executes O2 during slot $t$, without executing O4a or O2 during the slot $t + 1$, using the fact that in this case $P_i(t + 2) = P_i(t + 1)$.

Loosely speaking, the next lemma asserts that the removal of an invalid maximal estimate $M^*$, through the propagation of restarts, takes place at unit speed.

**Lemma 8.9.** Fix some time $t > T'$, and let $M^*$ be the largest estimate at that time. Suppose that $i$ is a root (i.e., has zero out-degree) in the forest $G_{M^*}(t + 2)$. Then, either (i) $i$ is the root of an one-element tree in $G_{M^*}(t + 2)$ consisting only of $i$, or (ii) $i$ is at least “two levels down in $G_{M^*}(t)$”, i.e., there exist nodes $i', i''$ with $(i,i'),(i',i'') \in G_{M^*}(t)$.

**Proof.** Consider such a node $i$ and assume that (i) does not hold. Then,

$$
M_i(t) = M_i(t + 1) = M_i(t + 2) = M^* \\
P_i(t) = P_i(t + 1) = P_i(t + 2)
$$

(8.2)
This is because otherwise, cases (a) and (b) of Lemma 8.2 imply that \(i\) has zero indegree in \(G_{M^*}(t+2) \subseteq G(t+2)\), in addition to having a zero out-degree, contradicting our assumption that (i) does not hold. Moreover, the estimate of \(i\) is not valid at \(t\), because it would then also be valid at \(t+2\) by Lemma 8.7 in contradiction with \(i \in G_{M^*}(t+2)\). Therefore, \(i\) belongs to the forest \(G_{M^*}(t)\). Let \(r\) be the root of the connected component to which \(i\) belongs. We will prove that \(i \neq r\) and \(P_i(t) \neq r\), and thus that (ii) holds.

Since \(r \in G_{M^*}(t)\), we have \(M_r(t) = M^*\) and thus \(r\) does not execute O2 during slot \(t\). Moreover, \(r\) is a root and has an invalid estimate, so \(P_t(t) \neq r\) and there is a “restart” message from \(P_r(t)\) to \(r\) at time \(t\). Therefore, \(r\) executes O4a during slot \(t\), setting \(P_r(t+1) = r\) and sending “restart” messages to all its neighbors at time \(t+1\). This implies that \(i \neq r\), as we have seen that \(P_i(t) = P_i(t+1)\). Let us now assume, to obtain a contradiction, that \(P_i(t) = r\) and thus by Eq. 8.2, \(P_i(t+2) = P_i(t+1) = r\). In that case, we have just seen that there is at time \(t+1\) a “restart” message from \(r = P_i(t+1) \neq i\) to \(i\), so \(i\) executes O4a during slot \(t+1\) and sets \(P_i(t+2) = i\). This however contradicts the fact \(P_i(t+2) = P_i(t+1)\). Therefore, \(r \neq i\) and \(r \neq P_i(t)\), i.e., \(i\) is “at least two levels down” in \(G_{M^*}(t)\).

Let the depth of a tree be the largest distance between a leaf of the tree and the root; the depth of a forest is the largest depth of any tree in the forest. We will use \(g(\cdot)\) to denote depth. The following lemma uses the previous two lemmas to assert that a forest carrying an invalid maximal estimate has its depth decrease by at least one over a time interval of length two.

**Lemma 8.10.** Fix some time \(t > T'\), and let \(M^*\) be the largest estimate value at that time. If \(g(G_{M^*}(t+2)) > 0\), then \(g(G_{M^*}(t+2)) \leq g(G_{M^*}(t)) - 1\).

**Proof.** Suppose that \(g(G_{M^*}(t+2)) > 0\). Let us fix a leaf \(i\) and a root \(j\) in the forest \(G_{M^*}(t+2)\) such that the length of the path from \(i\) to \(j\) is equal to the depth of \(G_{M^*}(t+2)\). Let \(i'\) be the single neighbor of node \(i\) in \(G_{M^*}(t+2)\). We first claim that every edge \((k, k')\) on the path from \(i'\) to \(j\) in \(G_{M^*}(t+2)\) was also present in \(G_{M^*}(t)\). Indeed, by Lemma 8.2 the appearance of a new edge \((k, k')\) at time \(t+1\) or \(t+2\) implies that node \(k\) has in-degree 0 in \(G(t+2)\), which contradicts \(k\) being an intermediate node on the path from \(i\) to \(j\) in \(G_{M^*}(t+2)\). The same argument establishes that \(M_k(t) = M_k(t+1) = M^*\). Finally, the estimate of \(k\) at time \(t\) is invalid, for if it were valid, it would still be valid at time \(t+2\) by Lemma 8.7, so \(i\) would also have a valid estimate at time \(t+2\), which is false by assumption. Thus we have just established that both the node \(k\) and its edge \((k, k')\) at time \(t+2\) belong to \(G_{M^*}(t)\).

Thus the graph \(G_{M^*}(t)\) includes a path from \(i'\) to \(j\) of length \(g(G_{M^*}(t+2)) - 1\). Moreover, by Lemma 8.9, we know that at time \(t\) some edges \((j, j')\) and \((j', j'')\) were present in \(G_{M^*}(t)\), so the path length from \(i'\) to \(j''\) is at least \(g(G_{M^*}(t+2)) + 1\). This proves that \(g(G_{M^*}(t)) \geq g(G_{M^*}(t+2)) + 1\) and the lemma.

The following lemma analyzes the remaining case of invalidity graphs with zero depth. It shows that the invalidity graph will be empty two steps after its depth reaches zero.
Lemma 8.11. Fix some time $t > T'$, and let $M^*$ be the largest estimate value at that time. If $G_{M^*}(t + 2)$ is not empty, then $g(G_{M^*}(t + 1)) > 0$ or $g(G_{M^*}(t)) > 0$.

Proof. Let us take a node $i \in G_{M^*}(t + 2)$ and let $j = P_i(t + 2)$. It follows from the definition of a valid estimate that $j \neq i$. This implies that $i$ did not execute O4a (or O1) during slot $t + 1$. We treat two cases separately:

(i) Node $i$ did not execute O2 during slot $t + 1$. In this case, $P_i(t + 1) = P_i(t + 2) = j$ and $M_j(t + 1) = M_j(t + 2) = M^*$. Besides, there is no “restart” message from $j = P_i(t + 1)$ to $i$ at time $t + 1$, for otherwise $i$ would have executed O4a during slot $t + 1$, which we know it did not. Therefore, $(i, j) \in E(t + 1)$ by definition of $G(t)$, and $M_j(t + 1) = M_j(t + 1) = M^*$ by Lemma 8.4. Moreover, neither $i$ nor $j$ have a valid estimate, for otherwise Lemma 8.7 would imply that they both hold the same valid estimate at $t + 2$, in contradiction with $i \in G_{M^*}(t + 2)$. So the edge $(i, j)$ is present in $G_{M^*}(t + 1)$, which has thus a positive minimal depth.

(ii) Node $i$ did execute O2 during slot $t + 1$: In that case, there was a message with the value $M_j(t + 2) = M^*$ from $j$ to $i$ at time $t + 1$, which implies that $M_j(t + 1) = M_j(t + 2) = M^*$. This implies that $j$ did not execute operation O2 during slot $t$. Moreover, node $j$ did not have a valid estimate at time $t + 1$. Otherwise, part (a) of Lemma 8.7 implies that $j$ has a valid estimate at time $t + 2$, and part (b) of the same lemma implies there was not a “restart” message from $j$ at $t + 2$, so that $(i, j) \in E(t + 2)$. This would in turn imply that $i$ has a valid estimate at time $t + 2$, contradicting $i \in G_{M^*}(t + 2)$. To summarize, $j$ has an invalid estimate $M^*$ at time $t + 1$ and did not execute O2 during slot $t$. We now simply observe that the argument of case (i) applies to $j$ at time $t + 1$.

The next lemma asserts that the largest invalid estimates are eventually purged, and thus that eventually, all remaining largest estimates are valid.

Lemma 8.12. Fix some time $t > T'$, and let $M^*$ be the largest estimate value at that time. Eventually $G_{M^*}(t)$ is empty.

Proof. Lemma 8.10 implies there is a time $t' > T'$ after which $g(G_{M^*}(t'')) = 0$ for all $t'' > t'$. Lemma 8.11 then implies that $G_{M^*}(t)$ is empty for all $t > t' + 2$.

We are now ready for the proof of the main theorem.

Proof of Theorem 8.2. Let $\overline{M} = \max_i u_i(T')$. It follows from the definition of a valid estimate that any node holding an estimate $M_i(t) > \overline{M}$ at time $t \geq T'$ has an invalid estimate. Applying Lemma 8.12 repeatedly shows the existence of some time $T \geq T'$ such that when $t \geq T$, no node has an estimate larger than $\overline{M}$, and every node having an estimate $\overline{M}$ has a valid estimate.

We will assume that the time $t$ in every statement we make below satisfies $t \geq T$. Define $Z(t)$ as the set of nodes having the estimate $\overline{M}$ at time $t$. Every node in $Z(t)$ holds a valid estimate, and $Z(t)$ is never empty because Lemma 8.5 implies that $M_i(t) \geq \overline{M}$ for every $i$ with $u_i = \overline{M}$. Moreover, it follows from Lemma 8.7 and the definition of validity that any node belonging to some $Z(t)$ will forever afterwards maintain $M_i(t) = \overline{M}$ and will satisfy the conclusion of Theorem 8.2.
We conclude the proof by arguing that eventually every node is in $Z(t)$. In particular, we will argue that a node $i$ adjacent to a node $j \in Z(t)$ necessarily belongs to $Z(t + 2)$. Indeed, it follows from Lemma 8.7 that node $j$ sends to $i$ a message with the estimate $M$ at time $t + 1$. If $i \in Z(t + 1)$, then $i \in Z(t + 2)$; else $M_i(t + 1) < M$, $i$ executes O2 during slot $t + 1$, and sets $M_i(t + 2) = M$, so indeed $i \in Z(t + 2)$.

8.3 Interval-averaging

In this section, we present an interval-averaging algorithm and prove its correctness. We start by repeating the informal discussion of its main idea from the beginning of the chapter. Imagine the integer input value $x_i$ as represented by a number of $x_i$ pebbles at node $i$. The algorithm attempts to exchange pebbles between nodes with unequal numbers so that the overall distribution becomes more even. Eventually, either all nodes will have the same number of pebbles, or some will have a certain number and others just one more. We let $u_i(t)$ be the current number of pebbles at node $i$; in particular, $u_i(0) = x_i$. An important property of the algorithm will be that the total number of pebbles is conserved.

To match nodes with unequal number of pebbles, we use the maximum tracking algorithm of Section 8.2. Recall that the algorithm provides nodes with pointers which attempt to track the location of the maximal values. When a node with $u_i$ pebbles comes to believe in this way that a node with at least $u_i + 2$ pebbles exists, it sends a request in the direction of the latter node to obtain one or more pebbles. This request follows a path to a node with a maximal number of pebbles until the request either gets denied, or gets accepted by a node with at least $u_i + 2$ pebbles.

8.3.1 The algorithm

The algorithm uses two types of messages. Each type of message can be either originated at a node or forwarded by a node.

(a) (Request, $r$): This is a request for a transfer of pebbles. Here, $r$ is an integer that represents the number of pebbles $u_i(t)$ at the node $i$ that first originated the request, at the time $t$ that the request was originated. (Note, however, that this request is actually sent at time $t + 1$.)

(b) (Accept, $w$): This corresponds to acceptance of a request, and a transfer of $w$ pebbles towards the node that originated the request. An acceptance with a value $w = 0$ represents a request denial.

As part of the algorithm, the nodes run the maximum tracking algorithm of Section 8.2 as well as a minimum tracking counterpart. In particular, each node $i$ has access to the variables $M_i(t)$ and $P_i(t)$ of the maximum tracking algorithm (recall that these are, respectively, the estimated maximum and a pointer to a neighbor or to itself). Furthermore, each node maintains three additional variables.

(a) “Mode($t$)$\in \{\text{Free, Blocked}\}$. Initially, the mode of every node is free. A node is blocked if it has originated or forwarded a request, and is still waiting to hear whether the request is accepted (or denied).
(b) “Rin$_i(t)$” and “Rout$_i(t)$” are pointers to a neighbor of $i$, or to $i$ itself. The meaning of these pointers when in blocked mode are as follows. If Rout$_i(t) = j$, then node $i$ has sent (either originated or forwarded) a request to node $j$, and is still in blocked mode, waiting to hear whether the request is accepted or denied. If Rin$_i(t) = k$, and $k \neq i$, then node $i$ has received a request from node $k$ but has not yet responded to node $k$. If Rin$_i(t) = i$, then node $i$ has originated a request and is still in blocked mode, waiting to hear whether the request is accepted or denied.

A precise description of the algorithm is given in Figure 8.2. The proof of correctness is given in the next subsection, thus also establishing Theorem 8.1. Furthermore, we will show that the time until the algorithm settles on the correct output is of order $O(n^2K^2 \log K)$.

### 8.3.2 Proof of correctness

We begin by arguing that the rules of the algorithm preclude one potential obstacle; we will show that nodes will not get stuck sending requests to themselves.

**Lemma 8.13.** A node never sends (originates or forwards) a request to itself. More precisely, Rout$_i(t) \neq i$, for all $i$ and $t$.

**Proof.** By inspecting the first two cases for the free mode, we observe that if node $i$ originates a request during time slot $t$ (and sends a request message at time $t+1$), then $P_i(t) \neq i$. Indeed, to send a message, it must be true $M_i(t) > u_i(t) = u_i(t-1)$. However, any action of the maximum tracking algorithm that sets $P_i(t) = i$ also sets $M_i(t) = u_i(t-1)$, and moreover, as long as $P_i$ doesn’t change neither does $M_i$. So the recipient $P_i(t)$ of the request originated by $i$ is different than $i$, and accordingly, Rout$_i(t+1)$ is set to a value different than $i$. We argue that the same is true for the case where Rout$_i$ is set by the the “Forward request” box of the free mode. Indeed, that box is enabled only when $u_i(t) = u_i(t-1)$ and $u_i(t) - 1 \leq r < M_i(t) - 1$, so that $u_i(t-1) < M_i(t)$. As in the previous case, this implies that $P_i(t) \neq i$ and that Rout$_i(t+1)$ is again set to a value other than $i$. We conclude that Rout$_i(t) \neq i$ for all $i$ and $t$. \qed

We will now analyze the evolution of the requests. A request is originated at some time $\tau$ by some originator node $\ell$ who sets Rin$_\ell(\tau + 1) = \ell$ and sends the request to some node $i = \text{Rout}_\ell(\tau + 1) = P_i(\tau)$. The recipient $i$ of the request either accepts/denies it, in which case Rin$_i$ remains unchanged, or forwards it while also setting Rin$_i(\tau + 2)$ to $\ell$. The process then continues similarly. The end result is that at any given time $t$, a request initiated by node $\ell$ has resulted in a “request path of node $\ell$ at time $t$,” which is a maximal sequence of nodes $\ell, i_1, \ldots, i_k$ with Rin$_\ell(t) = \ell$, Rin$_{i_1}(t) = \ell$, and Rin$_{i_m}(t) = i_{m-1}$ for $m \leq k$. 

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Lemma 8.14. At any given time, different request paths cannot intersect (they involve disjoint sets of nodes). Furthermore, at any given time, a request path cannot visit the same node more than once.

Proof. For any time $t$, we form a graph that consists of all edges that lie on some request path. Once a node $i$ is added to some request path, and as long as that request path includes $i$, node $i$ remains in blocked mode and the value of $R_{in_i}$ cannot change. This means that adding a new edge that points into $i$ is impossible. This readily implies that cycles cannot be formed and also that two request paths cannot involve a common node.

We use $p_{\ell}(t)$ to denote the request path of node $\ell$ at time $t$, and $s_{\ell}(t)$ to denote the last node on this path. We will say that a request originated by node $\ell$ terminates when node $\ell$ receives an (Accept, $w$) message, with any value $w$.

Lemma 8.15. Every request eventually terminates. Specifically, if node $\ell$ originates a request at time $t'$ (and sends a request message at time $t'+1$), then there exists a later time $t'' \leq t' + n$ at which node $s_{\ell}(t'')$ receives an “accept request” message (perhaps with $w = 0$), which is forwarded until it reaches $\ell$, no later than time $t'' + n$.

Proof. By the rules of our algorithm, node $\ell$ sends a request message to node $P_{\ell}(t')$ at time $t'+1$. If node $P_{\ell}(t')$ replies at time $t'+2$ with a “deny request” response to $\ell$’s request, then the claim is true; otherwise, observe that $p_{\ell}(t'+2)$ is nonempty and until $s_{\ell}(t)$ receives an “accept request” message, the length of $p_{\ell}(t)$ increases at each time step. Since this length cannot be larger than $n-1$, by Lemma 8.14 it follows that $s_{\ell}(t)$ receives an “accept request” message at most $n$ steps after $\ell$ initiated the request. One can then easily show that this acceptance message is forwarded backwards along the path (and the request path keeps shrinking) until the acceptance message reaches $\ell$, at most $n$ steps later.

The arguments so far had mostly to do with deadlock avoidance. The next lemma concerns the progress made by the algorithm. Recall that a central idea of the algorithm is to conserve the total number of “pebbles,” but this must include both pebbles possessed by nodes and pebbles in transit. We capture the idea of “pebbles in transit” by defining a new variable. If $i$ is the originator of some request path that is present at time $t$, and if the final node $s_i(t)$ of that path receives an (Accept, $w$) message at time $t$, we let $w_i(t)$ be the value $w$ in that message. (This convention includes the special case where $w = 0$, corresponding to a denial of the request). In all other cases, we set $w_i(t) = 0$. Intuitively, $w_i(t)$ is the value that has already been given away by a node who answered a request originated by node $i$, and that will eventually be added to $u_i$, once the answer reaches $i$.

We now define

$$\hat{u}_i(t) = u_i(t) + w_i(t).$$

By the rules of our algorithm, if $w_i(t) = w > 0$, an amount $w$ will eventually be added to $u_i$, once the acceptance message is forwarded back to $i$. The value $\hat{u}_i$ can thus be seen as a future value of $u_i$, that includes its present value and the value that has been sent to $i$ but has not yet reached it.
Figure 8-2: Flowchart of the procedure used by node $i$ during slot $t$ in the interval-averaging algorithm. The subscript $i$ is omitted from variables such as $\text{Mode}(t)$, $M(t)$, etc. Variables for which an update is not explicitly indicated are assumed to remain unchanged. “Denying a request” is a shorthand for $i$ sending a message of the form $(\text{Accept}, 0)$ at time $t + 1$ to a node from which $i$ received a request at time $t$. Note also that “forward the acceptance” in the blocked mode includes the case where the answer had $w = 0$ (i.e., it was a request denial), in which case the denial is forwarded.
The rules of our algorithm imply that the sum of the \( \hat{u}_i \) remains constant. Let \( \bar{x} \) be the average of the initial values \( x_i \). Then,

\[
\frac{1}{n} \sum_{i=1}^{n} \hat{u}_i(t) = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}.
\]

We define the variance function \( V \) as

\[
V(t) = \sum_{i=1}^{n} (\hat{u}_i(t) - \bar{x})^2.
\]

**Lemma 8.16.** The number of times that a node can send an acceptance message (Accept, \( w \)) with \( w \neq 0 \), is finite.

**Proof.** Let us first describe the idea behind the proof. Suppose that nodes could instantaneously transfer value to each other. It is easily checked that if a node \( i \) transfers an amount \( w^* \) to a node \( j \) with \( u_i - u_j \geq 2 \) and \( 1 \leq w^* \leq \frac{1}{2} (u_i - u_j) \), the variance \( \sum_i (u_i - \bar{x})^2 \) decreases by at least 2. Thus, there can only be a finite number of such transfers. In our model, the situation is more complicated because transfers are not immediate and involve a process of requests and acceptances. A key element of the argument is to realize that the algorithm can be interpreted as if it only involved instantaneous exchanges involving disjoint pairs of nodes.

Let us consider the difference \( V(t+1) - V(t) \) at some typical time \( t \). Changes in \( V \) are solely due to changes in the \( \hat{u}_i \). Note that if a node \( i \) executes the “fulfill the acceptance” instruction at time \( t \), node \( i \) was the originator of the request and the request path has length zero, so that it is also the final node on the path, and \( s_i(t) = i \). According to our definition, \( w_i(t) \) is the value \( w \) in the message received by node \( s_i(t) = i \). At the next time step, we have \( w_i(t+1) = 0 \) but \( u_i(t+1) = u_i(t) + w \). Thus, \( \hat{u}_i \) does not change, and the function \( V \) is unaffected.

By inspecting the algorithm, we see that a nonzero difference \( V(t+1) - V(t) \) is possible only if some node \( i \) executes the “accept request” instruction at slot \( t \), with some particular value \( w^* \neq 0 \), in which case \( u_i(t+1) = u_i(t) - w^* \). For this to happen, node \( i \) received a message (Request, \( r \)) at time \( t \) from a node \( k \) for which \( Rout_k(t) = i \), and with \( u_i(t) - r \geq 2 \). That node \( k \) was the last node, \( s_i(t) \), on the request path of some originator node \( \ell \). Node \( k \) receives an (Accept, \( w^* \)) message at time \( t+1 \) and, therefore, according to our definition, this sets \( w_i(t+1) = w^* \).

It follows from the rules of our algorithm that \( \ell \) had originated a request with value \( r = u_\ell(t') \) at some previous time \( t' \). Subsequently, node \( \ell \) entered the blocked mode, preventing any modification of \( u_\ell \), so that \( r = u_\ell(t) = u_\ell(t+1) \). Moreover, observe that \( w_\ell(t) \) was 0 because by time \( t \), no node had answered \( \ell \)'s request. Furthermore, \( w_i(t+1) = w_i(t) = 0 \) because having a positive \( w_i \) requires \( i \) to be in blocked mode, preventing the execution of “accept request”. It follows that

\[
\hat{u}_i(t+1) = u_i(t+1) = u_i(t) - w^* = \hat{u}_i(t) - w^*.
\]
and
\[ \hat{u}_\ell(t + 1) = r + w^* = \hat{u}_\ell(t) + w^*. \]

Using the update equation \( w^* = \lfloor (u_i(t) - r)/2 \rfloor \), and the fact \( u_i(t) - r \geq 2 \), we obtain
\[ 1 \leq w^* \leq \frac{1}{2}(u_i(t) - r) = \frac{1}{2}(\hat{u}_i(t) - \hat{u}_\ell(t)). \]

Combining with the previous equalities, we have
\[ \hat{u}_\ell(t) + 1 \leq \hat{u}_\ell(t + 1) \leq \hat{u}_i(t + 1) \leq \hat{u}_i(t) - 1. \]

Assume for a moment that node \( i \) was the only one that executed the “accept request” instruction at time \( t \). Then, all of the variables \( \hat{u}_j \), for \( j \neq i, \ell \), remain unchanged. Simple algebraic manipulations then show that \( V \) decreases by at least 2. If there was another pair of nodes, say \( j \) and \( k \), that were involved in a transfer of value at time \( t \), it is not hard to see that the transfer of value was related to a different request, involving a separate request path. In particular, the pairs \( \ell, i \) and \( j, k \) do not overlap. This implies that the cumulative effect of multiple transfers on the difference \( V(t + 1) - V(t) \) is the sum of the effects of individual transfers. Thus, at every time for which at least one “accept request” step is executed, \( V \) decreases by at least 2. We also see that no operation can ever result in an increase of \( V \). It follows that the instruction “accept request” can be executed only a finite number of times. \( \square \)

**Proposition 8.1.** There is a time \( t' \) such that \( u_i(t) = u_i(t') \), for all \( i \) and all \( t \geq t' \). Moreover,
\[ \sum_i u_i(t') = \sum_i x_i, \quad \max_i u_i(t') - \min_i u_i(t') \leq 1. \]

**Proof.** It follows from Lemma 8.16 that there is a time \( t' \) after which no more requests are accepted with \( w \neq 0 \). By Lemma 8.15, this implies that after at most \( n \) additional time steps, the system will never again contain any “accept request” messages with \( w \neq 0 \), so no node will change its value \( u_i(t) \) thereafter.

We have already argued that the sum (and therefore the average) of the variables \( \hat{u}_i(t) \) does not change. Once there are no more “accept request” messages in the system with \( w \neq 0 \), we must have \( w_i(t) = 0 \), for all \( i \). Thus, at this stage the average of the \( u_i(t) \) is the same as the average of the \( x_i \).

It remains to show that once the \( u_i(t) \) stop changing, the maximum and minimum \( u_i(t) \) differ by at most 1. Recall (cf. Theorem that 8.2) that at some time after the \( u_i(t) \) stop changing, all estimates \( M_i(t) \) of the maximum will be equal to \( M(t) \), the true maximum of the \( u_i(t) \); moreover, starting at any node and following the pointers \( P_i(t) \) leads to a node \( j \) whose value \( u_j(t) \) is the true maximum, \( M(t) \). Now let \( A \) be the set of nodes whose value at this stage is at most \( \max_i u_i(t) - 2 \). To derive a contradiction, let us suppose that \( A \) is nonempty.

Because only nodes in \( A \) will originate requests, and because every request eventu-
ally terminates (cf. Lemma 8.15), if we wait some finite amount of time, we will have the additional property that all requests in the system originated from \( A \). Moreover, nodes in \( A \) originate requests every time they are in the free mode, which is infinitely often.

Consider now a request originating at a node in the set \( A \). The value \( r \) of such a request satisfies \( M(t) - r \geq 2 \), which implies that every node that receives it either accepts it (contradicting the fact that no more requests are accepted after time \( t' \)), or forwards it, or denies it. But a node \( i \) will deny a request only if it is in blocked mode, that is, if it has already forwarded some other request to node \( P_i(t) \). This shows that requests will keep propagating along links of the form \( (i, P_i(t)) \), and therefore will eventually reach a node at which \( u_i(t) = M(t) \geq r + 2 \), at which point they will be accepted—a contradiction.

We are now ready to conclude.

Proof of Theorem 8.1. Let \( u^*_i \) be the value that \( u_i(t) \) eventually settles on. Proposition 8.1 readily implies that if the average \( \bar{x} \) of the \( x_i \) is an integer, then \( u_i(t) = u^*_i = \bar{x} \) will eventually hold for every \( i \). If \( \bar{x} \) is not an integer, then some nodes will eventually have \( u_i(t) = u^*_i = [\bar{x}] \) and some other nodes \( u_i(t) = u^*_i = [\bar{x}] \). Besides, using the maximum and minimum computation algorithm, nodes will eventually have a correct estimate of \( \max u^*_i \) and \( \min u^*_i \), since all \( u_i(t) \) settle on the fixed values \( u^*_i \). This allows the nodes to determine whether the average is exactly \( u^*_i \) (integer average), or whether it lies in \( (u^*_i, u^*_i + 1) \) or \( (u^*_i - 1, u^*_i) \) (fractional average). Thus, with some simple post-processing at each node (which can be done using finite automata), the nodes can produce the correct output for the interval-averaging problem. The proof of Theorem 8.1 is complete.

Next, we give a convergence time bound for the algorithms we have just described.

The general idea is quite simple. We have just argued that the nonnegative function \( V(t) \) decreases by at least 2 each time a request is accepted. It also satisfies \( V(0) = O(nK^2) \). Thus there are at most \( O(nK^2) \) acceptances. To complete the argument, one needs to argue that if the algorithm has not terminated, there will be an acceptance within \( O(n) \) iterations (which corresponds to \( O(n \log K) \) real-time rounds of communication due to the \( O(\log K) \) slowdown from transmitting elements in \( \{0, \ldots, K\} \)). This should be fairly clear from the proof of Theorem 8.1. A formal argument is given in the next section. It is also shown there that the running time of our algorithm, for many graphs, satisfies a \( \Omega(n^2) \) lower bound, in the worst case over all initial conditions.

### 8.4 The Time to Termination

The aim of this section is to prove an \( O(n^2K^2 \log K) \) upper bound on the time to termination of the interval-averaging algorithm from Section 8.3.
We will use the notation $M'(t)$ to denote the largest estimate held by any node at time $t$ or in the $n$ time steps preceding it:

$$M'(t) = \max_{i = 1, \ldots, n} \max_{k = t, t - 1, \ldots, t - n} M_i(k).$$

For $M'(t)$ to be well defined, we will adopt the convention that for all negative times $k$, $M_i(k) = u_i(0)$.

**Lemma 8.17.** In the course of the execution of the interval-averaging algorithm, $M'(t)$ never increases.

**Proof.** Fix a time $t$. We will argue that

$$M_i(t + 1) \leq M'(t) \quad (8.3)$$

for each $i$. This clearly implies $M'(t + 1) \leq M'(t)$.

If $M_i(t + 1) \leq M'_i(t)$, then Eq. (8.3) is obvious. We can thus suppose that $M_i(t + 1) > M'_i(t)$. There are only three boxes in Figure 8-1 which result in a change between $M_i(t)$ and $M_i(t + 1)$. These are $O2$, $O1$, and $O4a$. We can rule out the possibility that node $i$ executes $O2$, since that merely sets $M_i(t + 1)$ to some $M_j(t)$, and thus cannot result in $M_i(t + 1) > M'(t)$.

Consider, then, the possibility that node $i$ executes $O1$ or $O4a$, and as a consequence $M_i(t + 1) = u_i(t)$. If $u_i(t) \leq u_i(t - 1)$, then we are finished because

$$M_i(t + 1) = u_i(t) \leq u_i(t - 1) \leq M_i(t),$$

which contradicts the assumption $M_i(t + 1) > M'_i(t)$. Note that the last step of the above chain of inequalities used Lemma 8.5.

Thus we can assume that $u_i(t) > u_i(t - 1)$. In this case, $i$ must have fulfilled acceptance from some node $j$ during slot $t - 1$. Let $\hat{t}$ the time when node $j$ received the corresponding request message from node $i$. The rules of our algorithm imply that $u_i(\hat{t}) = u_i(t - 1)$, and that the quantity $w$ sent by $j$ to $i$ in response to the request is no greater than $\frac{1}{2}(u_j(\hat{t}) - u_i(\hat{t}))$. This implies that $u_i(t) = u_i(t - 1) + w < u_j(\hat{t})$.

Crucially, we have that $\hat{t} \in [t - 1 - n, t - 1]$, since at most $n + 1$ time steps pass between the time node $j$ receives the request message it will accept and the time when node $i$ fulfills $j$’s acceptance. So

$$M_i(t + 1) = u_i(t) < u_j(\hat{t}) \leq M_j(\hat{t} + 1) \leq M'(t).$$

We have thus showed that $M_i(t + 1) \leq M'(t)$ in every possible case, which implies that $M'(t + 1) \leq M'(t)$.

**Lemma 8.18.** Consider the maximum tracking algorithm. If each $u_i(t)$ is constant for $t \in [t_0, t_0 + 4n]$, then at least one of the following two statements is true: (a) $M'(t_0 + 3n) < M'(t_0)$. (b) $M_i(t_0 + 4n) = \max_j u_j(t_0)$ for every $i$. 

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Proof. Suppose first that no node holds an estimate equal to \( M'(t_0) \) at some time between \( t_0 + 2n \) and \( t_0 + 3n \). Then it follows from the definition of \( M'(t) \) and its monotonicity (8.17) that condition (a) holds. Suppose now that some node holds an estimate equal to \( M'(t_0) \) at some time between \( t_0 + 2n \) and \( t_0 + 3n \). The definition of \( M'(t) \) and the monotonicity of \( \max_i M_i(t) \) when all \( u_i \) are constant (Lemma 8.10) imply that \( M'(t_0) = \max_i M_i(t) \) for all \( t \in [t_0, t_0 + 3n] \). It follows from repeated application of Lemmas 8.10 and 8.11 (similarly to what is done in the proof of Proposition 8.12) that every estimate \( M'(t_0) \) at time \( t_0 + 2n \) is valid, which by definition implies the existence of at least one node \( i \) with \( u_i(t_0) = M'(t_0) \). Besides, since \( M_i(t) \geq u_i(t) \) holds for all \( i \) and \( t \) by Lemma 8.5 and since we know that \( M'(t_0) = \max_i M_i(t) \) for all \( t \in [t_0, t_0 + 3n] \), we have \( M'(t_0) = \max_i u_i(t_0) \). As described in the proof of Theorem 8.2 this implies that after at most \( 2n \) more time steps, \( M_i(t) = M'(t_0) \) holds for every \( i \), and so (b) holds.

\[ \]
Theorem 8.3. The interval-averaging algorithm described in Section 8.3 terminates after at most $O(n^2K^2\log K)$ time steps.

Proof. Consider the function $V(t) = \sum_{i=1}^{n} (\hat{u}_i(t) - \bar{u})^2$, where $\bar{u}$ is the average of the $x_i$, which is also the average of the $\hat{u}_i$, and where the $\hat{u}_i(t)$ are as defined before Lemma 8.16. Since $\hat{u}_i(t) \in \{0, 1, \ldots, K\}$ for all $i$, one can verify that $V(0) \leq \frac{1}{4}nK^2$. Moreover, as explained in the proof of Lemma 8.16, $V(t)$ is non-increasing, and decreases by at least 2 with every request acceptance. Therefore, a total of at most $\frac{1}{8}nK^2$ requests can be accepted. Furthermore, we showed that $M'(t)$ is non-increasing, and since $M'(t)$ always belongs to $\{0, 1, \ldots, K\}$, it can strictly decrease at most $K$ times. It follows then from Lemma 8.19 that condition (c) must hold after at most $\frac{1}{8}nK^2 \cdot 8n + K \cdot 3n + 5$ time steps.

Recall that in parallel with the maximum-tracking and averaging algorithm, we also run a minimum tracking algorithm. In the previous paragraph, we demonstrated that condition (c) of Lemma 8.19 holds, i.e. $u_i$ remain fixed forever, after $n^2K^2 + K \cdot 3n$ time steps. A similar argument to Lemma 8.18 implies that the minimum algorithm will reach a fixed point after an additional $(3K + 4)n$ steps.

Putting it all together, the algorithm reaches a fixed point after $n^2K^2 + (6K + 4) \cdot n$ steps. Accounting in addition for the log $K$ slowdown from transmitting values in $\{0, \ldots, K\}$, we obtain a convergence time of $O(n^2K^2 \log K)$.

We note that there are cases where the running time of interval averaging is quadratic in $n$. For example, consider the network in Figure 8.4 consisting of two arbitrary connected graphs $G_1, G_2$ with $n/3$ nodes each, connected by a line graph of $n/3$ nodes. Suppose that $K = 2$, and that $x_i = 0$ if $i \in G_1$, $x_i = 2$ if $i \in G_2$, and $x_i = 1$ otherwise. The algorithm will have the nodes of $G_1$ with $u_i = 0$ send requests to nodes $j$ in $G_2$ with $u_j = 2$, and each successful request will result in the pair of nodes changing their values, $u_i$ and $u_j$, to 1. The system will reach its final state after $n/3$ such successful requests. Observe now that each successful request must cross the line graph, which takes $n/3$ time steps in each direction. Moreover, since nodes cannot simultaneously treat multiple requests, once a request begins crossing the line graph, all other requests are denied until the response to the first request reaches $G_1$, which takes at least $2n/3$ time steps. Therefore, in this example, it takes at least $2n^2/9$ time steps until the algorithm terminates.

8.5 Simulations

We report here on simulations involving our algorithm on several natural graphs. Figures 8.4 and 8.5 describe the results for a complete graph and a line. Initial conditions were random integers between 1 and 30, and each data point represents the average of two hundred runs. As expected, convergence is faster on the complete graph. Moreover, convergence time in both simulations appears to be approximately linear.
8.6 Concluding remarks

In this chapter, we have given a deterministic algorithm for averaging which stores a constant number of bits per each link. Unfortunately, this algorithm works only for static graphs; whether such an algorithm exists for time-varying graph sequences is still an open question.

As discussed in chapter 2, low storage is one of the main reasons to pick an averaging algorithm over competitors such as flooding. It is therefore interesting to consider just how low the storage requirements of averaging algorithms are. In the next chapter, we will ask the analogous question for the problem of computing arbitrary functions.
Figure 8-4: The number of iterations as a function of the number of nodes for a complete graph.

Figure 8-5: The number of iterations as a function of the number of nodes for a line graph.
Chapter 9

A framework for distributed function computation

We now wish to take a broader perspective and consider general distributed function computation problems modeled on consensus. Indeed, the goal of many multi-agent systems, distributed computation algorithms, and decentralized data fusion methods is to have a set of nodes compute a common value based on initial values or observations at each node. Towards this purpose, the nodes, which we will sometimes refer to as agents, perform some internal computations and repeatedly communicate with each other. The objective of this chapter is to understand the fundamental limitations and capabilities of such systems and algorithms when the available information and computational resources at each node are limited.

Our exposition in this chapter will follow the preprint [50], where the results described here have previously appeared.

9.1 Motivation

(a) Quantized consensus: Suppose that each node begins with an integer value $x_i(0) \in \{0, \ldots, K\}$. We would like the nodes to end up, at some later time, with values $y_i$ that are almost equal, i.e., $|y_i - y_j| \leq 1$, for all $i, j$, while preserving the sum of the values, i.e., $\sum_{i=1}^{n} x_i(0) = \sum_{i=1}^{n} y_i$. This is the so-called quantized averaging problem, which has received considerable attention recently; see, e.g., [55 40 8 59 11], and we have talked about it at some length in the previous Chapters 7 and 8. It may be viewed as the problem of computing the function $(1/n) \sum_{i=1}^{n} x_i$, rounded to an integer value.

(b) Distributed hypothesis testing and majority voting: Consider $n$ sensors interested in deciding between two hypotheses, $H_0$ and $H_1$. Each sensor collects measurements and makes a preliminary decision $x_i \in \{0, 1\}$ in favor of one of the hypotheses. The sensors would like to make a final decision by majority vote, in which case they need to compute the indicator function of the event $\sum_{i=1}^{n} x_i \geq n/2$, in a distributed way. Alternatively, in a weighted majority vote, they may be interested in computing the indicator function of an event such as $\sum_{i=1}^{n} x_i \geq 3n/4$. A variation of
this problem involves the possibility that some sensors abstain from the vote, perhaps due to their inability to gather sufficiently reliable information.

(c) Direction coordination on a ring: Consider \( n \) vehicles placed on a ring, each with some arbitrarily chosen direction of motion (clockwise or counterclockwise). We would like the \( n \) vehicles to agree on a single direction of motion. A variation of this problem was considered in [71], where, however, additional requirements on the vehicles were imposed which we do not consider here. The solution provided in [71] was semi-centralized in the sense that vehicles had unique numerical identifiers, and the final direction of most vehicles was set to the direction of the vehicle with the largest identifier. We wonder whether the direction coordination problem can be solved in a completely decentralized way. Furthermore, we would like the final direction of motion to correspond to the initial direction of the majority of the vehicles: if, say, 90% of the vehicles are moving counterclockwise, we would like the other 10% to turn around. If we define \( x_i \) to be 1 when the \( i \)th vehicle is initially oriented clockwise, and 0 if it is oriented counterclockwise, then, coordinating on a direction involves the distributed computation of the indicator function of the event \( \sum_{i=1}^{n} x_i \geq n/2 \).

(d) Solitude verification: This is the problem of checking whether exactly one node has a given state. This problem is of interest if we want to avoid simultaneous transmissions over a common channel [48], or if we want to maintain a single leader (as in motion coordination — see for example [52]) Given states \( x_i \in \{0, 1, \ldots, K\} \), solitude verification is equivalent to the problem of computing the binary function which is equal to 1 if and only if \( |\{i : x_i = 0\}| = 1 \).

There are numerous methods that have been proposed for solving problems such as the above. Oftentimes, different algorithms involve different computational capabilities on the part of the nodes, which makes it hard to talk about a “best” algorithm. At the same time, simple algorithms (such as setting up a spanning tree and aggregating information by progressive summations over the tree, as in Chapter 2) are often considered undesirable because they require too much coordination or global information. It should be clear that a sound discussion of such issues requires the specification of a precise model of computation, followed by a systematic analysis of fundamental limitations under a given model. This is precisely the objective of this chapter: to propose a particular model, and to characterize the class of functions computable under this model.

9.1.1 The features of our model

Our model provides an abstraction for common requirements for distributed algorithms in the wireless sensor network literature. We model the nodes as interacting deterministic finite automata that exchange messages on a fixed undirected network, with no time delays or unreliable transmissions. Some important qualitative features of our model are the following.

Identical nodes: Any two nodes with the same number of neighbors must run the same algorithm.
**Anonymity:** A node can distinguish its neighbors using its own, private, local identifiers. However, nodes do not have global identifiers.

**Determinism:** Randomization is not allowed. This restriction is imposed in order to preclude essentially centralized solutions that rely on randomly generated distinct identifiers and thus bypass the anonymity requirement. Clearly, developing algorithms is much harder, and sometimes impossible, when randomization is disallowed.

**Limited memory:** We focus on the case where the nodes can be described by finite automata, and pay special attention to the required memory size. Ideally, the number of memory bits required at each node should be bounded above by a slowly growing function of the degree of a node.

**Absence of global information:** Nodes have no global information, and do not even have an upper bound on the total number of nodes. Accordingly, the algorithm that each node is running is independent of the network size and topology.

**Convergence requirements:** Nodes hold an estimated output that must converge to a desired value which is a function of all nodes’ initial observations or values. In particular, for the case of discrete outputs, all nodes must eventually settle on the desired value. On the other hand, the nodes do not need to become aware of such termination, which is anyway impossible in the absence of any global information [7].

In this chapter, we only consider the special case of fixed graph topologies, where the underlying (and unknown) interconnection graph does not change with time. Developing a meaningful model for the time-varying case and extending our algorithms to that case is an interesting topic, but outside the scope of this thesis.

### 9.1.2 Literature review

There is a very large literature on distributed function computation in related models of computation [14, 61]. This literature can be broadly divided into two strands, although the separation is not sharp: works that address general computability issues for various models, and works that focus on the computation of specific functions, such as the majority function or the average. We start by discussing the first strand.

A common model in the distributed computing literature involves the requirement that all processes terminate once the desired output is produced and that nodes become aware that termination has occurred. A consequence of the termination requirement is that nodes typically need to know the network size $n$ (or an upper bound on $n$) to compute non-trivial functions. We refer the reader to [3, 7, 10, 60, 75] for some fundamental results in this setting, and to [12] for a comprehensive summary of known results. Closest to our work is the reference [33] which provides an impossibility result very similar to our Theorem 9.1 for a closely related model computation.

The biologically-inspired “population algorithm” model of distributed computation has some features in common with our model, namely, anonymous, bounded-resource nodes, and no requirement of termination awareness; see [6] for an overview of available results. However, this model involves a different type of node interactions.
from the ones we consider; in particular, nodes interact pairwise at times that may be chosen adversarially.

Regarding the computation of specific functions, [62] shows the impossibility of majority voting if the nodes are limited to a binary state. Some experimental memoryless algorithms (which are not guaranteed to always converge to the correct answer) have been proposed in the physics literature [46]. Several papers have quantified the performance of simple heuristics for computing specific functions, typically in randomized settings. We refer the reader to [51], which studied simple heuristics for computing the majority function, and to [86], which provides a heuristic that has guarantees only for the case of complete graphs.

The large literature on quantized averaging often tends to involve themes similar to those addressed in this chapter [10, 59, 8, 28, 55]. However, the underlying models of computation are typically more powerful than ours, as they allow for randomization and unbounded memory. Closer to the current chapter, [76] develops an algorithm with $O(n^2)$ convergence time for a variant of the quantized averaging problem, but requires unbounded memory. Reference [11] provides an algorithm for the particular quantized averaging problem that we consider in Section 9.4 (called in [11] the “interval consensus problem”), which uses randomization but only bounded memory (a total of two bits at each node). Its convergence time is addressed in [38], but a precise convergence time bound, as a function of $n$, is not available. Nevertheless, it appears to be significantly higher than $O(n^2)$. Similarly, the algorithm in [103] runs in $O(n^5)$ time for the case of fixed graphs. (However, we note that [103] also addresses the case of time-varying graphs.) Roughly speaking, the algorithms in [11, 103] work by having positive and negative “load tokens” circulate randomly in the network until they meet and annihilate each other. Our algorithm involves a similar idea. However, at the cost of some algorithmic complexity, our algorithm is deterministic. This allows for fast progress, in contrast to the slow progress of algorithms that need to wait until the coalescence time of two independent random walks. Finally, a deterministic algorithm for computing the majority function (and some more general functions) was proposed in [63]. However, the algorithm appears to rely on the computation of shortest path lengths, and thus requires unbounded memory at each node.

Semi-centralized versions of the problem, in which the nodes ultimately transmit to a fusion center, have often been considered in the literature, e.g., for distributed statistical inference [74] or detection [56]. The papers [13, 57], and [65] consider the complexity of computing a function and communicating its value to a sink node. We refer the reader to the references therein for an overview of existing results in such semi-centralized settings. However, the underlying model is fundamentally different from ours, because the presence of a fusion center violates our anonymity assumption.

Broadly speaking, our results differ from previous works in several key respects: (i) Our model, which involves totally decentralized computation, deterministic algorithms, and constraints on memory and computation resources at the nodes, but does not require the nodes to know when the computation is over, is different from that considered in almost all of the relevant literature. (ii) Our focus is on identifying computable and non-computable functions under our model, and we achieve a nearly tight separation, as evidenced by Theorem 9.1 and Corollary 9.4.
9.1.3 Summary and Contributions

We provide a general model of decentralized anonymous computation on fixed graphs, with the features described in Section 9.1.1, and characterize the type of functions of the initial values that can be computed.

We prove that if a function is computable under our model, then its value can only depend on the frequencies of the different possible initial values. For example, if the initial values $x_i$ are binary, a computable function can only depend on $p_0 := |\{i : x_i = 0\}|/n$ and $p_1 := |\{i : x_i = 1\}|/n$. In particular, determining the number of nodes, or whether at least two nodes have an initial value of 1, is impossible.

Conversely, we prove that if a function only depends on the frequencies of the different possible initial values (and is measurable), then the function can be approximated with any given precision, except possibly on a set of frequency vectors of arbitrarily small volume. Moreover, if the dependence on these frequencies can be expressed through a combination of linear inequalities with rational coefficients, then the function is computable exactly. In particular, the functions involved in the quantized consensus, distributed hypothesis testing, and direction coordination examples are computable, whereas the function involved in solitude verification is not. Similarly, statistical measures such as the standard deviation of the distribution of the initial values can be approximated with arbitrary precision. Finally, we show that with infinite memory, the frequencies of the different initial values (i.e., $p_0, p_1$ in the binary case) are computable exactly, thus obtaining a precise characterization of the computable functions in this case.

The key to our positive results is the algorithm for calculating the (suitably quantized) average of the initial values described in the previous Chapter.

9.1.4 Outline

In Section 9.2 we describe formally our model of computation. In Section 9.3 we establish necessary conditions for a function to be computable. In Section 9.4 we provide sufficient conditions for a function to be computable or approximable. Our positive results rely on an algorithm that keeps track of nodes with maximal values, and an algorithm that calculates a suitably rounded average of the nodes’ initial values; these were described in the previous Chapter. We we end with some concluding remarks, in Section 9.5.

9.2 Formal description of the model

Under our model, a distributed computing system consists of three elements:

(a) A network: A network is a triple $(n, G, \mathcal{L})$, where $n$ is the number of nodes, and $G = (V, E)$ is a connected undirected graph $G = (V, E)$ with $n$ nodes (and no self-arcs). We define $d(i)$ as the of node $i$. Finally, $\mathcal{L}$ is a port labeling which assigns a port number (a distinct integer in the set $\{0, 1, \ldots, d(i)\}$) to each outgoing edge of any node $i$.  

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We are interested in algorithms that work for arbitrary port labelings. However, in the case of wireless networks, port labelings can be assumed to have some additional structure. For example, if two neighboring nodes \(i\) and \(j\) have coordinated so as to communicate over a distinct frequency, they should be able to coordinate their port numbers as well, so that the port number assigned by any node \(i\) to an edge \((i, j)\) is the same as the port number assigned by \(j\) to edge \((j, i)\). We will say that a network is edge-labeled if port numbers have this property. In Section 9.3 we will note that our negative results also apply to edge-labeled networks.

(b) Input and output sets: The input set is a finite set \(X = \{0, 1, \ldots, K\}\) to which the initial value of each node belongs. The output set is a finite set \(Y\) to which the output of each node belongs.

(c) An algorithm: An algorithm is defined as a family of finite automata \((A_d)_{d=1,2,\ldots}\), where the automaton \(A_d\) describes the behavior of a node with degree \(d\). The state of the automaton \(A_d\) is a tuple \([x, z, y; (m_1, \ldots, m_d)]\); we will call \(x \in X\) the initial value, \(z \in Z_d\) the internal memory state, \(y \in Y\) the output or estimated answer, and \(m_1, \ldots, m_d \in M\) the outgoing messages. The sets \(Z_d\) and \(M\) are assumed finite. We allow the cardinality of \(Z_d\) to increase with \(d\). Clearly, this would be necessary for any algorithm that needs to store the messages received in the previous time step. Each automaton \(A_d\) is identified with a transition law from \(X \times Z_d \times Y \times M^d\) into itself, which maps each \([x, z, y; (m_1, \ldots, m_d)]\) to some \([x', z', y'; (m'_1, \ldots, m'_d)]\). In words, at each iteration, the automaton takes \(x, z, y\), and incoming messages into account, to create a new memory state, output, and (outgoing) messages, but does not change the initial value.

Given the above elements of a distributed computing system, an algorithm proceeds as follows. For convenience, we assume that the above defined sets \(Y\), \(Z_d\), and \(M\) contain a special element, denoted by \(\emptyset\). Each node \(i\) begins with an initial value \(x_i \in X\) and implements the automaton \(A_{d(i)}\), initialized with \(x = x_i\) and \(z = y = m_1 = \cdots = m_d = \emptyset\). We use \(S_i(t) = [x_i, y_i(t), z_i(t), m_{i,1}(t), \ldots, m_{i,d(i)}(t)]\) to denote the state of node \(i\)'s automaton at time \(t\). Consider a particular node \(i\). Let \(j_1, \ldots, j_{d(i)}\) be an enumeration of its neighbors, according to the port numbers. (Thus, \(j_k\) is the node at the other end of the \(k\)th outgoing edge at node \(i\).) Let \(p_k\) be the port number assigned to link \((j_k, i)\) according to the port labeling at node \(j_k\). At each time step, node \(i\) carries out the following update:

\[
[x_i, z_i(t + 1), y_i(t + 1); m_{i,1}(t + 1), \ldots, m_{i,d(i)}(t + 1)]
= A_{d(i)}[x_i, z_i(t), y_i(t); m_{j_1,p_1}(t), \ldots, m_{j_{d(i)},p_{d(i)}}(t)]
\]

In words, the messages \(m_{j_k,p_k}(t)\), \(k = 1, \ldots, d(i)\), "sent" by the neighbors of \(i\) into the ports leading to \(i\) are used to transition to a new state and create new messages \(m_{i,k}(t + 1)\), \(k = 1, \ldots, d(i)\), that \(i\) "sends" to its neighbors at time \(t + 1\). We say that the algorithm terminates if there exists some \(y^* \in Y\) (called the final output of the algorithm) and a time \(t'\) such that \(y_i(t) = y^*\) for every \(i\) and \(t \geq t'\).

Consider now a family of functions \((f_n)_{n=1,2,\ldots}\) where \(f_n : X^n \to Y\). We say that
such a family is *computable* if there exists a family of automata \((A_d)_{d=1,2,...}\) such that for any \(n\), for any network \((n,G,L)\), and any set of initial conditions \(x_1,\ldots,x_n\), the resulting algorithm terminates and the final output is \(f_n(x_1,\ldots,x_n)\).

As an exception to the above definitions, we note that although we primarily focus on the finite case, we will briefly consider in Section 9.4.1 function families \((f_n)_{n=1,2,...}\) *computable with infinite memory*, by which we mean that the internal memory sets \(Z_d\) and the output set \(Y\) are countably infinite, the rest of the model remaining the same.

The rest of the chapter focuses on the following general question: what families of functions are computable, and how can we design a corresponding algorithm \((A_d)_{d=1,2,...}\)? To illustrate the nature of our model and the type of algorithms that it supports, we provide a simple example.

**Detection problem:** In this problem, all nodes start with a binary initial value \(x_i \in \{0,1\} = X\). We wish to detect whether at least one node has an initial value equal to 1. We are thus dealing with the function family \((f_n)_{n=1,2,...}\), where \(f_n(x_1,\ldots,x_n) = \max\{x_1,\ldots,x_n\}\). This function family is computable by a family of automata with binary messages, binary internal state, and with the following transition rule:

\[
\begin{align*}
\text{if } x_i = 1 \text{ or } z_i(t) = 1 \text{ or } \max_{j:(i,j) \in E} m_{ji}(t) = 1 \text{ then} \\
& \quad \text{set } z_i(t+1) = y_i(t+1) = 1 \\
& \quad \text{send } m_{ij}(t+1) = 1 \text{ to every neighbor } j \text{ of } i \\
\text{else} \\
& \quad \text{set } z_i(t+1) = y_i(t+1) = 0 \\
& \quad \text{send } m_{ij}(t+1) = 0 \text{ to every neighbor } j \text{ of } i \\
\end{align*}
\]

In the above algorithm, we initialize by setting \(m_{ij}(0), y_i(0), \text{ and } z_i(0)\) to zero instead of the special symbol \(\emptyset\). One can easily verify that if \(x_i = 0\) for every \(i\), then \(y_i(t) = 0\) for all \(i\) and \(t\). If on the other hand \(x_k = 1\) for some \(k\), then at each time step \(t\), those nodes \(i\) at distance less than \(t\) from \(k\) will have \(y_i(t) = 1\). Thus, for connected graphs, the algorithm will terminate within \(n\) steps, with the correct output. It is important to note, however, that because \(n\) is unknown, a node \(i\) can never know whether its current output \(y_i(t)\) is the final one. In particular, if \(y_i(t) = 0\), node \(i\) cannot exclude the possibility that \(x_k = 1\) for some node whose distance from \(i\) is larger than \(t\).

### 9.3 Necessary condition for computability

In this section we establish our main negative result, namely, that if a function family is computable, then the final output can only depend on the frequencies of the different possible initial values. Furthermore, this remains true even if we allow for infinite memory, or restrict to networks in which neighboring nodes share a common label for the edges that join them. This result is quite similar to Theorem 3 of [33], and so is the proof. Nevertheless, we provide a proof in order to keep the chapter self-contained.

We first need some definitions. Recall that \(X = \{0,1,\ldots,K\}\). We let \(D\) be the
unit simplex, that is, \( D = \{ (p_0, \ldots, p_K) \in [0,1]^{K+1} : \sum_{k=0}^{K} p_k = 1 \} \). We say that a function \( h : D \to Y \) corresponds to a function family \((f_n)_{n=1,2,\ldots}\) if for every \( n \) and every \( x \in X^n \), we have
\[
f(x_1, \ldots, x_n) = h(p_0(x_1, \ldots, x_n), p_1(x_1, \ldots, x_n), \ldots, p_K(x_1, \ldots, x_n)),
\]
where
\[
p_k(x_1, \ldots, x_n) = |\{i \mid x_i = k\}|/n,
\]
so that \( p_k(x_1, \ldots, x_n) \) is the frequency of occurrence of the initial value \( k \). In this case, we say that the family \((f_n)\) is frequency-based.

**Theorem 9.1.** Suppose that the family \((f_n)\) is computable with infinite memory. Then, this family is frequency-based. The result remains true even if we only require computability over edge-labeled networks.

The following are some applications of Theorem 9.1.

(a) The parity function \( \sum_{i=1}^{n} x_i \pmod{k} \) is not computable, for any \( k > 1 \).
(b) In a binary setting \( (X = \{0,1\}) \), checking whether the number of nodes with \( x_i = 1 \) is larger than or equal to the number of nodes with \( x_i = 0 \) plus 10 is not computable.
(c) Solitude verification, i.e., checking whether \( |i : \{x_i = 0\}| = 1 \), is not computable.
(d) An aggregate difference function such as \( \sum_{i<j} |x_i - x_j| \) is not computable, even if it is to be calculated modulo \( k \).

### 9.3.1 Proof of Theorem 9.1

The proof of Theorem 9.1 involves a particular degree-two network (a ring), in which all port numbers take values in the set \( \{0,1,2\} \), and in which any two edges \((i,j)\) and \((j,i)\) have the same port number. The proof proceeds through a sequence of intermediate results, starting with the following lemma, which can be easily proved by induction on time; its proof is omitted.

**Lemma 9.1.** Suppose that \( G = (\{1, \ldots, n\}, E) \) and \( G' = (\{1, \ldots, n\}, E') \) are isomorphic; that is, there exists a permutation \( \pi \) such that \((i,j) \in E \) if and only if \((\pi(i), \pi(j)) \in E' \). Furthermore, suppose that the port label at node \( i \) for the edge leading to \( j \) in \( G \) is the same as the port label at node \( \pi(i) \) for the edge leading to \( \pi(j) \) in \( G' \). Then, the state \( S_i(t) \) resulting from the initial values \( x_1, \ldots, x_n \) on the graph \( G \) is the same as the state \( S_{\pi(i)}(t) \) resulting from the initial values \( x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)} \) on the graph \( G' \).

**Lemma 9.2.** Suppose that the family \((f_n)_{n=1,2,\ldots}\) is computable with infinite memory on edge-labeled networks. Then, each \( f_i \) is invariant under permutations of its arguments.

**Proof.** Let \( \pi_{ij} \) be the permutation that swaps \( i \) with \( j \) (leaving the other nodes intact); with a slight abuse of notation, we also denote by \( \pi_{ij} \) the mapping from \( X^n \) to \( X^n \).
that swaps the $i$th and $j$th elements of a vector. (Note that $\pi^{-1}_{ij} = \pi_{ij}$. ) We show that for all $x \in X^n$, $f_n(x) = f_n(\pi_{ij}(x))$.

We run our distributed algorithm on the $n$-node complete graph with an edge labeling. Note that at least one edge labeling for the complete graph exists: for example, nodes $i$ and $j$ can use port number $(i + j) \mod n$ for the edge connecting them.

Consider two different sets of initial values, namely the vectors (i) $x$, and (ii) $\pi_{ij}(x)$. Let the port labeling in case (i) be arbitrary; in case (ii), let the port labeling be such that the conditions in Lemma 9.1 are satisfied (which is easily accomplished). Since the final value is $f(x)$ in case (i) and $f(\pi_{ij}(x))$ in case (ii), we obtain $f(x) = f(\pi_{ij}(x))$. Since the permutations $\pi_{ij}$ generate the group of permutations, permutation invariance follows.

Let $x \in X^n$. We will denote by $x^2$ the concatenation of $x$ with itself, and, generally, by $x^k$ the concatenation of $k$ copies of $x$. We now prove that self-concatenation does not affect the value of a computable family of functions.

**Lemma 9.3.** Suppose that the family $(f_n)_{n=1,2,...}$ is computable with infinite memory on edge-labeled networks. Then, for every $n \geq 2$, every sequence $x \in X^n$, and every positive integer $m$,

$$f_n(x) = f_{mn}(x^m).$$

**Proof.** Consider a ring of $n$ nodes, where the $i$th node clockwise begins with the $i$th element of $x$; and consider a ring of $mn$ nodes, where the nodes $i, i + n, i + 2n, \ldots$ (clockwise) begin with the $i$th element of $x$. Suppose that the labels in the first ring are $0, 1, 2, 1, 2, \ldots$. That is, the label of the edge $(1, 2)$ is 0 and the labels of the subsequent edges alternate between 1 and 2. In the second ring, we simply repeat $m$ times the labels in the first ring. See Figure 9-1 for an example with $n = 5$, $m = 2$.

![Figure 9-1](image.png)

Figure 9-1: Example of two situations that are algorithmically indistinguishable. The numbers next to each edge are the edge labels.

Initially, the state $S_i(t) = [x_i, y_i(t), z_i(t), m_{i,1}(t), m_{i,2}(t)]$, with $t = 0$, of node $i$ in the first ring is exactly the same as the state of the nodes $j = i, i + n, i + 2n, \ldots$ in
Thus, because of periodicity of the edge labels, nodes $i$ in the second ring receive identical messages through identically labeled ports at time $t$. Since $i$ and $j$ were in the same state at time $t$, they must be in the same state at time $t + 1$. This proves that they are always in the same state. It follows that $y_i(t) = y_j(t)$ for all $t$, whenever $j \equiv i \pmod n$, and therefore $f_n(x) = f_{mn}(x^m)$. □

Proof of Theorem 9.1. Let $x$ and $y$ be two sequences of $n$ and $m$ elements, respectively, such that $p_k(x_1, \ldots, x_n)$ and $p_k(y_1, \ldots, y_m)$ are equal to a common value $\hat{p}_k$, for $k \in X$; thus, the number of occurrences of $k$ in $x$ and $y$ are $n\hat{p}_k$ and $m\hat{p}_k$, respectively. Observe that for any $k \in X$, the vectors $x^m$ and $y^n$ have the same number $mn$ of elements, and both contain $mn\hat{p}_k$ occurrences of $k$. The sequences $y^n$ and $x^m$ can thus be obtained from each other by a permutation, which by Lemma 9.2 implies that $f_{nm}(x^m) = f_{nm}(y^n)$. From Lemma 9.3, we have that $f_{nm}(x^m) = f_n(x)$ and $f_{nm}(y^n) = f_m(y)$. Therefore, $f_n(x) = f_m(y)$. This proves that the value of $f_n(x)$ is determined by the values of $p_k(x_1, \ldots, x_n)$, $k = 0, 1, \ldots, K$. □

9.4 Reduction of generic functions to the computation of averages

In this section, we turn to positive results, aiming at a converse of Theorem 9.1. The centerpiece of our development is Theorem 9.2, proved in the previous chapter, which states that a certain average-like function is computable. Theorem 9.2 then implies the computability of a large class of functions, yielding an approximate converse to Theorem 9.1.

The average-like functions that we consider correspond to the “interval consensus” problem studied in [11]. They are defined as follows. Let $X = \{0, \ldots, K\}$. Let $Y$ be the following set of single-point sets and intervals:

$$Y = \{\{0\}, (0, 1), \{1\}, (1, 2), \ldots, \{K - 1\}, (K - 1, K), \{K\}\}$$

(or equivalently, an indexing of this finite collection of intervals). For any $n$, let $f_n$ be the function that maps $(x_1, x_2, \ldots, x_n)$ to the element of $Y$ which contains the average $\sum_i x_i/n$. We refer to the function family $(f_n)_{n=1,2,\ldots}$ as the interval-averaging family.

The motivation for this function family comes from the fact that the exact average $\sum_i x_i/n$ takes values in a countably infinite set, and cannot be computed when the
set $Y$ is finite. In the quantized averaging problem considered in the literature, one settles for an approximation of the average. However, such approximations do not necessarily define a single-valued function from $X^n$ into $Y$. In contrast, the above defined function $f_n$ is both single-valued and delivers an approximation with an error of size at most one. Note also that once the interval-average is computed, we can readily determine the value of the average rounded down to an integer.

**Theorem 9.2.** The interval-averaging function family is computable.

The proof of Theorem 9.2 (and the corresponding algorithm) was given in Chapter 8. In this section, we show that the computation of a broad class of functions can be reduced to interval-averaging.

Since only frequency-based function families can be computable (Theorem 9.1), we can restrict attention to the corresponding functions $h$. We will say that a function $h$ on the unit simplex $D$ is computable if it corresponds to a frequency-based computable family $(f_n)$. The level sets of $h$ are defined as the sets $L(y) = \{ p \in D \mid h(p) = y \}$, for $y \in Y$.

**Theorem 9.3** (Sufficient condition for computability). Let $h$ be a function from the unit simplex $D$ to $Y$. Suppose that every level set $L(y)$ can be written as a finite union, 

$$L(y) = \bigcup_k C_{i,k},$$

where each $C_{i,k}$ can in turn be written as a finite intersection of linear inequalities of the form

$$\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_K p_K \leq \alpha,$$

or

$$\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_K p_K < \alpha,$$

with rational coefficients $\alpha, \alpha_0, \alpha_1, \ldots, \alpha_K$. Then, $h$ is computable.

**Proof.** Consider one such linear inequality, which we assume, for concreteness, to be of the first type. Let $P$ be the set of indices $i$ for which $\alpha_k \geq 0$. Since all coefficients are rational, we can clear their denominators and rewrite the inequality as

$$\sum_{k \in P} \beta_k p_k - \sum_{k \in P^c} \beta_k p_k \leq \beta,$$

for nonnegative integers $\beta_k$ and $\beta$. Let $\chi_k$ be the indicator function associated with initial value $k$, i.e., $\chi_k(i) = 1$ if $x_i = k$, and $\chi_k(i) = 0$ otherwise, so that $p_k = \frac{1}{n} \sum_i \chi_k(i)$. Then, (9.1) becomes

$$\frac{1}{n} \sum_{i=1}^n \left( \sum_{k \in P} \beta_k \chi_k(i) + \sum_{k \in P^c} \beta_k (1 - \chi_k(i)) \right) \leq \beta + \sum_{k \in P^c} \beta_k,$$

or

$$\frac{1}{n} \sum_{i=1}^n q_i \leq q^*.$$
where \( q_i = \sum_{k \in P} \beta_k \chi_k(i) + \sum_{k \in P^c} \beta_k(1 - \chi_k(i)) \) and \( q^* = \beta + \sum_{k \in P^c} \beta_k \).

To determine whether the last inequality is satisfied, each node can compute \( q_i \) and \( q^* \), and then apply a distributed algorithm that computes the integer part of \( \frac{1}{n} \sum_{i=1}^{n} q_i \); this is possible by virtue of Theorem 9.2 with \( K \) set to \( \sum_k \beta_k \) (the largest possible value of \( q_i \)). To check any finite collection of inequalities, the nodes can perform the computations for each inequality in parallel.

To compute \( h \), the nodes simply need to check which set \( L(y) \) the frequencies \( p_0, p_1, \ldots, p_K \) lie in, and this can be done by checking the inequalities defining each \( L(y) \). All of these computations can be accomplished with finite automata: indeed, we do nothing more than run finitely many copies of the automata provided by Theorem 9.2 one for each inequality. The total memory used by the automata depends on the number of sets \( C_{i,k} \) and the magnitude of the coefficients \( \beta_k \), but not on \( n \), as required.

Theorem 9.3 shows the computability of functions \( h \) whose level-sets can be defined by linear inequalities with rational coefficients. On the other hand, it is clear that not every function \( h \) can be computable. (This can be shown by a counting argument: there are uncountably many possible functions \( h \) on the rational elements of \( D \), but for the special case of bounded degree graphs, only countably many possible algorithms.) Still, the next result shows that the set of computable functions is rich enough, in the sense that computable functions can approximate any measurable function, everywhere except possibly on a low-volume set.

We will call a set of the form \( \prod_{k=0}^{K} (a_k, b_k) \), with every \( a_k, b_k \) rational, a rational open box, where \( \prod \) stands for Cartesian product. A function that can be written as a finite sum \( \sum_i a_i \text{1}_{B_i} \), where the \( B_i \) are rational open boxes and the \( \text{1}_{B_i} \) are the associated indicator functions, will be referred to as a box function. Note that box functions are computable by Theorem 9.3.

**Corollary 9.4.** If every level set of a function \( h : D \to Y \) on the unit simplex \( D \) is Lebesgue measurable, then, for every \( \epsilon > 0 \), there exists a computable box function \( h_\epsilon : D \to Y \) such that the set \( \{ p \in D \mid h(p) \neq h_\epsilon(p) \} \) has measure at most \( \epsilon \).

**Proof.** (Outline) The proof relies on the following elementary result from measure theory. Given a Lebesgue measurable set \( E \subseteq D \) and some \( \epsilon > 0 \), there exists a set \( E' \) which is a finite union of disjoint open boxes, and which satisfies

\[
\mu(E \Delta E') < \epsilon,
\]

where \( \mu \) is the Lebesgue measure. By a routine argument, these boxes can be taken to be rational. By applying this fact to the level sets of the function \( h \) (assumed measurable), the function \( h \) can be approximated by a box function \( h_\epsilon \). Since box functions are computable, the result follows.

The following corollary states that continuous functions are approximable.

**Corollary 9.5.** If a function \( h : D \to [L, U] \subseteq \mathbb{R} \) is continuous, then for every \( \epsilon > 0 \) there exists a computable function \( h_\epsilon : D \to [L, U] \) such that \( \| h - h_\epsilon \|_\infty < \epsilon \)
Proof. Since $D$ is compact, $h$ is uniformly continuous. One can therefore partition $D$ into a finite number of subsets, $A_1, A_2, \ldots, A_q$, that can be described by linear inequalities with rational coefficients, so that $\max_{p \in A_j} h(p) - \min_{p \in A_j} h(p) < \epsilon$ holds for all $A_j$. The function $h_\epsilon$ is then built by assigning to each $A_j$ an appropriate value in $\{L, L+\epsilon, L+2\epsilon, \ldots, U\}$.

To illustrate these results, let us consider again some examples.

(a) Majority voting between two options is equivalent to checking whether $p_1 \leq 1/2$, with alphabet $\{0, 1\}$, and is therefore computable.

(b) Majority voting when some nodes can “abstain” amounts to checking whether $p_1 - p_0 \geq 0$, with input set $X = \{0, 1, \text{abstain}\}$. This function family is computable.

(c) We can ask for the second most popular value out of four, for example. In this case, the sets $A_i$ can be decomposed into constituent sets defined by inequalities such as $p_2 \leq p_3 \leq p_4 \leq p_1$, each of which obviously has rational coefficients.

(d) For any subsets $I, I'$ of $\{0, 1, \ldots, K\}$, the indicator function of the set where $\sum_{i \in I} p_i > \sum_{i \in I'} p_i$ is computable. This is equivalent to checking whether more nodes have a value in $I$ than do in $I'$.

(e) The indicator functions of the sets defined by $p_1^2 \leq 1/2$ and $p_1 \leq \pi/4$ are measurable, so they are approximable. We are unable to say whether they are computable.

(f) The indicator function of the set defined by $p_1 p_2 \leq 1/8$ is approximable, but we are unable to say whether it is computable.

9.4.1 Computability with infinite memory

Finally, we show that with infinite memory, it is possible to recover the exact frequencies $p_k$. (Note that this is impossible with finite memory, because $n$ is unbounded, and the number of bits needed to represent $p_k$ is also unbounded.) The main difficulty is that $p_k$ is a rational number whose denominator can be arbitrarily large, depending on the unknown value of $n$. The idea is to run separate algorithms for each possible value of the denominator (which is possible with infinite memory), and reconcile their results.

Theorem 9.6. The vector $(p_0, p_1, \ldots, p_K)$ is computable with infinite memory.

Proof. We show that $p_1$ is computable exactly, which is sufficient to prove the theorem. Consider the following algorithm, to be referred to as $Q_m$, parametrized by a positive integer $m$. The input set $X_m$ is $\{0, 1, \ldots, m\}$ and the output set $Y_m$ is the same as in the interval-averaging problem: $Y_m = \{\emptyset, \{0, 1\}, \{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \ldots, \{m-1\}, \{m-1, m\}, \{m\}\}$. If $x_i = 1$, then node sets its initial value $x_{i,m}$ to $m$; else, the node sets its initial value $x_{i,m}$ to 0. The algorithm computes the function family $(f_n)$ which maps $X_m^n$ to the element of $Y_m$ containing $(1/n) \sum_{i=1}^{n} x_{i,m}$, which is possible, by Theorem 9.2.

The nodes run the algorithms $Q_m$ for every positive integer value of $m$, in an interleaved manner. Namely, at each time step, a node runs one step of a particular
algorithm $Q_m$, according to the following order:

$$Q_1, Q_1, Q_2, Q_1, Q_2, Q_3, Q_1, Q_2, Q_3, Q_4, Q_1, Q_2, \ldots$$

At each time $t$, let $m_i(t)$ be the smallest $m$ (if it exists) such that the output $y_{i,m}(t)$ of $Q_m$ at node $i$ is a singleton (not an interval). We identify this singleton with the numerical value of its single element, and we set $y_i(t) = y_{i,m_i(t)}(t)/m_i(t)$. If $m_i(t)$ is undefined, then $y_i(t)$ is set to some default value, e.g., $\emptyset$.

Let us fix a value of $n$. For any $m \leq n$, the definition of $Q_m$ and Theorem 9.2 imply that there exists a time after which the outputs $y_{i,m}$ of $Q_m$ do not change, and are equal to a common value, denoted $y_m$, for every $i$. Moreover, at least one of the algorithms $Q_1, \ldots, Q_n$ has an integer output $y_m$. Indeed, observe that $Q_n$ computes $(1/n) \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} 1_{x_i=1}$, which is clearly an integer. In particular, $m_i(t)$ is eventually well-defined and bounded above by $n$. We conclude that there exists a time after which the output $y_i(t)$ of our overall algorithm is fixed, shared by all nodes, and different from the default value $\emptyset$.

We now argue that this value is indeed $p_1$. Let $m^*$ be the smallest $m$ for which the eventual output of $Q_m$ is a single integer $y_m$. Note that $y_{m^*}$ is the exact average of the $x_{i,m^*}$, i.e.,

$$y_{m^*} = \frac{1}{n} \sum_{i=1}^{n} m^* 1_{x_i=1} = m^* p_1.$$ 

For large $t$, we have $m_i(t) = m^*$ and therefore $y_i(t) = y_{i,m^*}(t)/m^* = p_1$, as desired.

Finally, it remains to argue that the algorithm described here can be implemented with a sequence of infinite memory automata. All the above algorithm does is run a copy of all the automata implementing $Q_1, Q_2, \ldots$ with time-dependent transitions. This can be accomplished with an automaton whose state space is the countable set $\mathcal{N} \times \bigcup_{m=1}^{\infty} \prod_{i=1}^{m} \mathcal{Q}_i$, where $\mathcal{Q}_i$ is the state space of $Q_i$, and the set $\mathcal{N}$ of integers is used to keep track of time.

\[\square\]

## 9.5 Conclusions

We have proposed a model of deterministic anonymous distributed computation, inspired by the wireless sensor network and multi-agent control literature. We have given an almost tight characterization of the functions that are computable in our model. We have shown that computable functions must depend only on the the frequencies with which the different initial conditions appear, and that if this dependence can be expressed in term of linear inequalities with rational coefficients, the function is indeed computable. Under weaker conditions, the function can be approximated with arbitrary precision. It remains open to exactly characterize the class of computable function families.

Our positive results are proved constructively, by providing a generic algorithm for computing the desired functions. Interestingly, the finite memory requirement is not used in our negative results, which remain thus valid in the infinite memory case. In particular, we have no examples of functions that can be computed with
infinite memory but are provably not computable with finite memory. We suspect though that simple examples exist; a good candidate could be the indicator function $1_{p_1 < 1/\pi}$, which checks whether the fraction of nodes with a particular initial condition is smaller than $1/\pi$. 
Chapter 10

Concluding remarks and a list of open problems

This thesis investigated several aspects of the convergence times of averaging algorithms and the effects of quantized communication and storage on performance. Our goal has been to try to understand some aspects of the tradeoffs between robustness, storage, and convergence speed in distributed systems.

The main results of this thesis are:

1. An $O((n^2/\eta)B \log(1/\epsilon))$ upper bound on the convergence time of products of doubly stochastic matrices (which lead to a class of averaging algorithms) in Chapter 4. This is the first polynomial upper bound on the convergence time of averaging algorithms.

2. An $O(n^2B \log 1/\epsilon)$ averaging algorithm in Chapter 5. This is currently the averaging algorithm with the best theoretical guarantees.

3. An $\Omega(n^2)$ lower bound in Chapter 6 on the convergence of any distributed averaging algorithm that uses a single scalar state variable at each agent and satisfies a natural “smoothness” condition.

4. The finding of Chapter 7 that storing and transmitting $c \log n$ bits can lead to arbitrarily accurate computation of the average by simply quantizing any linear, doubly stochastic averaging scheme.

5. The deterministic algorithm in Chapter 8 which, given initial values of 0 or 1 at each node, can compute which value has the majority with only a constant number of bits stored per link at each node.

6. The computability and non-computability results of Chapter 9 which tell us that with deterministic anonymous algorithms and a constant number of bits per link at each node we can compute only functions depending on proportions; and that all the “nice” functions of proportions are computable.

Many questions remain unanswered. We begin by listing several that are central to understanding the fundamentals of network information aggregation questions.
1. Is it possible to come up with local averaging algorithms which are robust to link failures, smoothly update a collection of real numbers of fixed size, and average faster than $O(n^2B \log 1/\epsilon)$ on $B$-connected graph sequences?

2. What if in addition to the above requirements we also ask that the convergence time be on the order of the diameter for a (time-invariant) geometric random graph?

3. What if in addition to the above requirements we also ask that the convergence time be on the order of the diameter for any fixed graph?

4. Suppose every node starts out with a 0 or 1. Is it possible to deterministically compute which node has the majority initially if the graph sequence $G(t)$ changes unpredictably? However, we do insist that each $G(t)$ be undirected, and we require that each node store only a constant number of bits per each link it maintains. Naturally, some additional connectivity assumptions on $G(t)$ will have to be imposed for any positive result.

5. More generally, characterize exactly which functions of binary initial values can be computed with a deterministic algorithm which maintains a constant number of bits per link at each node. The graph sequence $G(t)$ is undirected, but may be either fixed or change unpredictably.

We next list some other open questions motivated by the problems considered here.

1. For which classes of algorithms can an $\Omega(n^2)$ lower bound on convergence time be proven? In particular, is it possible to replace the assumption of Chapter 6 that the update function be differentiable by the weaker assumption that the update function be piecewise differentiable? What if we add some memory to the algorithm?

2. Here is a concrete instance of the previous question. Is it true that any doubly stochastic matrix “on the ring” mixes as $\Omega(n^2)$? That is, let $P$ be a doubly stochastic matrix such that $P_{ij} = 0$ if $|i - j| \mod n > 1$. Is it true that $\max_{\lambda(A) \neq 1} |\lambda(A)| \geq 1 - c/n^2$ for some constant $c$?

Major progress on this question was recently made in [44].

3. Where is the dividing line in Chapter 4 between polynomial and exponential convergence time? In particular, how far may we relax the double stochasticity Assumption 3.4 and still maintain polynomial convergence time? For example, does polynomial time convergence still hold if we replace Assumption 3.4 with the requirement that the matrices $A(t)$ be (row) stochastic and each column sum is in $[1 - \epsilon, 1 + \epsilon]$ for some small $\epsilon$?
4. Let $A_k$ be an irreducible, aperiodic stochastic matrix in $R^{k \times k}$. Let $\pi_k$ be its stationary distribution. Can we identify interesting classes of sequences $A_1, A_2, A_3, \ldots$, which satisfy $\lim_{k \to \infty} \pi_k(i) = 0$ for any $i$? In words, we are asking for each agent to have a negligible influence on the final result in the limit. This is sometimes a useful property in estimation problems; see [17].

5. Is there a decentralized way to pick a symmetric ($a_{ij} = a_{ji}$), stochastic linear update rule which minimizes

$$\sum_{\lambda(A) \neq 1} \frac{1}{1 - \lambda_i(A)^2}$$

This corresponds to handling the effect of white noise disturbances optimally; see [100] for a centralized algorithm based on convex optimization.

6. For which classes of correlated random variables can we design decentralized algorithms for maximum likelihood estimation as done in Chapter 2?

7. Suppose every node starts out with a 0 or 1. The communication graph is undirected, fixed, but unknown to the nodes. Each node can maintain a fixed number of bits per each link it maintains. It is possible to (deterministically) decide whether $\frac{1}{n} \sum_{i=0}^{n} x_i > \frac{1}{\sqrt{2}}$? What about $\frac{1}{n} \sum_{i=0}^{n} x_i > \frac{1}{\pi}$?

8. Is it possible to improve the running time of the interval-averaging algorithm of Chapter 8 to $O(n^2 K \log K)$ communication rounds?

9. Given a connected graph $G = (\{1, \ldots, n\}, E)$, suppose that every link $e$ is an erasure channel with erasure probability $p_e$. That is, every node can send a message on each link at each time step, but that the message is lost with probability $p_e$. Nodes do not know whether their messages are lost. What is the time complexity of average or sum computation in such a setting? How does it relate to various known graph connectivity measures?

10. How should one design averaging algorithms which send as few messages as possible? We ask that these algorithms work on arbitrary $B$-connected undirected graph sequences.
Appendix A

On quadratic Lyapunov functions for averaging

This appendix focuses on a technical issue appearing in Chapter 3: its content has previously appeared in the M.S. thesis [77] and the paper [78].

Specifically, in chapter 3 a number of theorems on the convergence of the process

$$x(t + 1) = A(t)x(t),$$

were proved, e.g. Theorems 3.1, 3.4, 3.3, 3.4. These theorems were proved by showing that the “span norm” $\max_i x_i(t) - \min_i x_i(t)$ is guaranteed to decrease after a certain number of iterations. Unfortunately, this proof method usually gives an overly conservative bound on the convergence time of the algorithm. Tighter bounds on the convergence time would have to rely on alternative Lyapunov functions, such as quadratic ones, of the form $x^T M x$, if they exist.

Moreover, in Chapter 4, we developed bounds for convergence of averaging algorithms based on quadratic Lyapunov functions. Thus we are led to the question: is it possible to find quadratic Lyapunov functions for the non-averaging convergence theorems of Chapter 3? A positive answer might lead to improved convergence times.

Although quadratic Lyapunov functions can always be found for linear systems, they may fail to exist when the system is allowed to switch between a fixed number of linear modes. On the other hand, there are classes of such switched linear systems that do admit quadratic Lyapunov functions. See [64] for a broad overview of the literature on this subject.

The simplest version of this question deals with the symmetric, equal-neighbor model and was investigated in [52]. The authors write:

“...no such common Lyapunov matrix $M$ exists. While we have not been able to construct a simple analytical example which demonstrates this, we have been able to determine, for example, that no common quadratic Lyapunov function exists for the class of all [graphs which have] 10 vertices and are connected. One can verify that this is so by using semidefinite programming...”
The aim of this appendix is to provide an analytical example that proves this fact.

A.1 The Example

Let us fix a positive integer \( n \). We start by defining a class \( \mathcal{Q} \) of functions with some minimal desired properties of quadratic Lyapunov functions. Let \( \mathbf{1} \) be the vector in \( \mathbb{R}^n \) with all components equal to 1. A square matrix is said to be stochastic if it is nonnegative and the sum of the entries in each row is equal to one. Let \( \mathcal{A} \subset \mathbb{R}^{n \times n} \) be the set of stochastic matrices \( A \) such that: (i) \( a_{ii} > 0 \), for all \( i \); (ii) all positive entries on any given row of \( A \) are equal; (iii) \( a_{ij} > 0 \) if and only if \( a_{ji} > 0 \); (iv) the graph associated with the set of edges \( \{(i, j) \mid a_{ij} > 0\} \) is connected. These are precisely the matrices that correspond to a single iteration of the equal-neighbor algorithm on symmetric, connected graphs.

**Definition A.1.** A function \( Q : \mathbb{R}^n \rightarrow \mathbb{R} \) belongs to the class \( \mathcal{Q} \) if it is of the form \( Q(x) = x^T M x \), where:

(a) The matrix \( M \in \mathbb{R}^{n \times n} \) is nonzero, symmetric, and nonnegative definite.

(b) For every \( A \in \mathcal{A} \), and \( x \in \mathbb{R}^n \), we have \( Q(Ax) \leq Q(x) \).

(c) We have \( Q(\mathbf{1}) = 0 \).

Note that condition (b) may be rewritten in matrix form as

\[
x^T A^T M A x \leq x^T M x, \quad \text{for all } A \in \mathcal{A}, \text{ and } x \in \mathbb{R}^n.
\]

The rationale behind condition (c) is as follows. Let \( S \) be the subspace spanned by the vector \( \mathbf{1} \). Since we are interested in convergence to the set \( S \), and every element of \( S \) is a fixed point of the algorithm, it is natural to require that \( Q(\mathbf{1}) = 0 \), or, equivalently,

\[
M \mathbf{1} = 0.
\]

Of course, for a Lyapunov function to be useful, additional properties would be desirable. For example we should require some additional condition that guarantees that \( Q(x(t)) \) eventually decreases. However, according to Theorem A.2 even the minimal requirements in Definition A.1 are sufficient to preclude the existence of a quadratic Lyapunov function.

**Theorem A.2.** Suppose that \( n \geq 8 \). Then, the class \( \mathcal{Q} \) (cf. Definition A.1) is empty.

The idea of the proof is as follows. Using the fact the dynamics of the system are essentially the same when we rename the components, we show that if \( x^T M x \) has the desired properties, so does \( x^T Z x \) for a matrix \( Z \) that has certain permutation-invariance properties. This leads us to the conclusion that there is essentially a single candidate Lyapunov function, for which a counterexample is easy to develop.

Recall that a permutation of \( n \) elements is a bijective mapping \( \sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \). Let \( \Pi \) be the set of all permutations of \( n \) elements. For any \( \sigma \in \Pi \), we
define a corresponding permutation matrix $P_\sigma$ by letting the $i$th component of $P_\sigma x$ be equal to $x_{\sigma(i)}$. Note that $P_\sigma^{-1} = P_\sigma^T$, for all $\sigma \in \Pi$. Let $\mathcal{P}$ be the set of all permutation matrices corresponding to permutations in $\Pi$.

**Lemma A.1.** Let $M \in \mathcal{Q}$. Define $Z$ as

$$Z = \sum_{P \in \mathcal{P}} P^T M P.$$  

Then, $Z \in \mathcal{Q}$.

**Proof:** For every matrix $A \in \mathcal{A}$, and any $P \in \mathcal{P}$, it is easily seen that $PAP^T \in \mathcal{A}$. This is because the transformation $A \mapsto PAP^T$ amounts to permuting the rows and columns of $A$, which is the same as permuting (renaming) the nodes of the graph.

We claim that if $M \in \mathcal{Q}$ and $P \in \mathcal{P}$, then $P^T MP \in \mathcal{Q}$. Indeed, if $M$ is nonzero, symmetric, and nonnegative definite, so is $P^T MP$. Furthermore, since $P1 = 1$, if $M1 = 0$, then $P^T MP1 = 0$. To establish condition (b) in Definition A.1, let us introduce the notation $Q_P(x) = x^T(P^T MP)x$. Fix a vector $x \in \mathbb{R}^n$, and $A \in \mathcal{A}$; define $B = PAP^T \in \mathcal{A}$. We have

$$Q_P(Ax) = x^T A^T P^T MP Ax,$$

$$= x^T P^T PA^T P^T MPAP^T P x,$$

$$= x^T P^T BP^T MBP x,$$

$$\leq x^T P^T MP x,$$

$$= Q_P(x),$$

where the inequality follows by applying Eq. (A.1), which is satisfied by $M$, to the vector $Px$ and the matrix $B$. We conclude that $Q_P \in \mathcal{Q}$.

Since the sum of matrices in $\mathcal{Q}$ remains in $\mathcal{Q}$, it follows that $Z = \sum_{P \in \mathcal{P}} P^T MP$ belongs to $\mathcal{Q}$. $\blacksquare$

We define the “sample variance” $V(x)$ of the values $x_1, \ldots, x_n$, by

$$V(x) = \sum_{i=1}^n (x_i - \bar{x})^2,$$

where $\bar{x} = (1/n) \sum_{i=1}^n x_i$. This is a nonnegative quadratic function of $x$, and therefore, $V(x) = x^T Cx$, for a suitable nonnegative definite, nonzero symmetric matrix $C \in \mathbb{R}^{n \times n}$.

**Lemma A.2.** There exists some $\alpha > 0$ such that

$$x^T Z x = \alpha V(x), \quad \text{for all } x \in \mathbb{R}^n.$$

**Proof:** We observe that the matrix $Z$ satisfies

$$R^T Z R = Z, \quad \text{for all } R \in \mathcal{P}. \quad (A.2)$$
To see this, fix $R$ and notice that the mapping $P \mapsto PR$ is a bijection of $\mathcal{P}$ onto itself, and therefore,

$$R^TZR = \sum_{P \in \mathcal{P}} (PR)^T M(PR) = \sum_{P \in \mathcal{P}} P^TMP = Z.$$ 

We will now show that condition (A.2) determines $Z$, up to a multiplicative factor. Let $z_{ij}$ be the $(i,j)$th entry of $Z$. Let $1^{(i)}$ be the $i$th unit vector, so that $1^{(i)^T}Z1^{(i)} = z_{ii}$. Let $P \in \mathcal{P}$ be a permutation matrix that satisfies $P1^{(i)} = 1^{(j)}$. Then, $z_{ii} = 1^{(i)^T}Z1^{(i)} = 1^{(i)^T}P^T Z P 1^{(i)} = 1^{(j)^T}Z1^{(j)} = z_{jj}$. Therefore, all diagonal entries of $Z$ have a common value, to be denoted by $z$.

Let us now fix three distinct indices $i, j, k$, and let $y = 1^{(i)} + 1^{(j)}, w = 1^{(i)} + 1^{(k)}$. Let $P \in \mathcal{P}$ be a permutation matrix such that $P1^{(i)} = 1^{(i)}$ and $P1^{(j)} = 1^{(k)}$, so that $Py = w$. We have

$$2z + 2z_{ij} = y^T Z y = y^T P^T Z P y = w^T Z w = 2z + 2z_{ik}.$$ 

By repeating this argument for different choices of $i, j, k$, it follows that all off-diagonal entries of $Z$ have a common value to be denoted by $r$. Using also the property that $Z1 = 0$, we obtain that $z + (n-1)r = 0$. This shows that the matrix $Z$ is uniquely determined, up to a multiplicative factor.

We now observe that permuting the components of a vector $x$ does not change the value of $V(x)$. Therefore, $V(x) = V(Px)$ for every $P \in \mathcal{P}$, which implies that $x^T P^T CP x = x^T C x$, for all $P \in \mathcal{P}$ and $x \in \mathbb{R}^n$. Thus, $C$ satisfies (A.2). Since all matrices that satisfy (A.2) are scalar multiples of each other, the desired result follows.

**Proof of Theorem A.2**: In view of Lemmas A.1 and A.2, if $\mathcal{Q}$ is nonempty, then $V \in \mathcal{Q}$. Thus, it suffices to show that $V \notin \mathcal{Q}$. Suppose that $n \geq 8$, and consider the vector $x$ with components $x_1 = 5$, $x_2 = x_3 = x_4 = 2$, $x_5 = 0$, $x_6 = x_7 = -3$, $x_8 = -5$, and $x_9 = \cdots = x_n = 0$. We then have $V(x) = 80$. Consider the outcome of one iteration of the symmetric, equal-neighbor algorithm, if the graph has the form shown in Figure A.1. After the iteration, we obtain the vector $y$ with components $y_1 = 11/5, y_2 = y_3 = y_4 = 7/2, y_5 = 0, y_6 = y_7 = -4, y_8 = -11/4, and y_9 = \cdots = y_n = 0$. We have

$$V_n(y) = \sum_{i=1}^{n} (y_i - \bar{y})^2 
\geq \sum_{i=1}^{8} (y_i - \bar{y})^2 
\geq \sum_{i=1}^{8} \left( y_i - \frac{1}{8} \sum_{i=1}^{8} y_i \right)^2, \quad (A.3)$$ 

where we used that $\sum_{i=1}^{k} (y_i - z)^2$ is minimized when $z = (1/k) \sum_{i=1}^{k} y_i$. A simple
calculation shows that the expression \( A.3 \) evaluates to \( \frac{10246}{127} \approx 80.68 \), which implies that \( V(y) > V(x) \). Thus, if \( n \geq 8 \), \( V \notin \mathbb{Q} \), and the set \( \mathbb{Q} \) is empty. ■

A.2 Conditions for the Existence of a Quadratic Lyapunov Function

Are there some additional conditions (e.g., restricting the matrices \( A \) to a set smaller than \( A \)), under which a quadratic Lyapunov function is guaranteed to exist? We start by showing that the answer is positive for the case of a fixed matrix (that is, if the graph \( G(t) \) is the same for all \( t \)).

Let \( A \) be a stochastic matrix, and suppose that there exists a positive vector \( \pi \) such that \( \pi^T A = \pi^T \). Without loss of generality, we can assume that \( \pi^T 1 = 1 \). It is known that in this case,

\[
x^T A^T D A x \leq x^T D x, \quad \forall \ x \in \mathbb{R}^n, \tag{A.4}
\]

where \( D \) is a diagonal matrix, whose \( i \)th diagonal entry is equal to \( \pi_i \) (cf. Lemma 6.4 in [15]). However, \( x^T D x \) cannot be used as a Lyapunov function because \( D 1 \neq 0 \) (cf. condition (c) in Definition A.1). To remedy this, we argue as in [17] and define the
matrix $H = I - 1 \pi^T$, and consider the choice $M = H^T D H$. Note that $M$ has rank $n - 1$.

We have $H 1 = (I - 1 \pi^T) 1 = 1 - 1(\pi^T 1) = 1 - 1 = 0$, as desired. Furthermore,

$$HA = A - 1 \pi^T A = A - A \pi^T A = AH.$$ 

Using this property, we obtain, for every $x \in \mathbb{R}^n$,

$$x^T A^T M A x = x^T A^T H^T D H A x = (x^T H^T) A^T D A (H x) \leq x^T H^T D H x = x^T M x,$$

where the inequality was obtained from (A.4), applied to $H x$. This shows that $H^T D H$ has the desired properties (a)-(c) of Definition A.1 provided that $A$ is replaced with $\{A\}$.

We have just shown that every stochastic matrix (with a positive left eigenvector associated to the eigenvalue 1) is guaranteed to admit a quadratic Lyapunov function, in the sense of Definition A.1. Moreover, our discussion implies that there are some classes of stochastic matrices $B$ for which the same Lyapunov function can be used for all matrices in the class.

(a) Let $B$ be a set of stochastic matrices. Suppose that there exists a positive vector $\pi$ such that $\pi^T 1 = 1$, and $\pi^T A = \pi^T$ for all $A \in B$. Then, there exists a nonzero, symmetric, nonnegative definite matrix $M$, of rank $n - 1$, such that $M 1 = 0$, and $x^T A^T M A x \leq x^T M x$, for all $x$ and $A \in B$.

(b) The condition in (a) above is automatically true if all the matrices in $B$ are doubly stochastic (recall that a matrix $A$ is doubly stochastic if both $A$ and $A^T$ are stochastic); in that case, we can take $\pi = 1$.

(c) The condition in (a) above holds if and only if there exists a positive vector $\pi$, such that $\pi^T A x = \pi^T x$, for all $A \in B$ and all $x$. In words, there must be a positive linear functional of the agents’ opinions which is conserved at each iteration. For the case of doubly stochastic matrices, this linear functional is any positive multiple of the sum $\sum_{i=1}^n x_i$ of the agents’ values (e.g., the average of these values).
Appendix B

Averaging with deadlock avoidance

In this appendix, we describe a solution of a certain averaging problem communicated to the author and J.N. Tsitsiklis by A.S. Morse. This is the problem of averaging with “deadlock avoidance,” namely averaging with the requirement that each node participate in at most one pairwise average at every time step.

More concretely, we will describe a deterministic distributed algorithm for nodes \(1, \ldots, n\), with starting values \(x_1(0), \ldots, x_n(0)\), to compute the average \((1/n) \sum_{i=1}^n x_i(0)\). Our algorithm selects pairs of nodes to average at each time step. We will allow the communication graphs \(G(t) = (\{1, \ldots, n\}, E(t))\) to change with time, but we will assume that each graph \(G(t)\) is undirected. Moreover, our algorithm will involve three rounds of message exchanges between nodes and their neighbors; we assume these messages can be sent before the graph \(G(t)\) changes. We do not assume nodes have unique identifiers; however, we do assume each node has a “port labeling” which allows it to tell neighbors apart, that is, if node \(i\) has \(d(i)\) neighbors in the graph \(G(t)\), it assigns them labels \(1, \ldots, d(i)\) in some arbitrary way.

B.1 Algorithm description

We will partition every time step into several synchronous “periods.” At the end of each period, nodes send each other messages, which all of them synchronously read at the beginning of the following period. During the first period, nodes exchange their values with their neighbors; after that, nodes will send messages from the binary alphabet \{+, −\}. Eventually, as a consequence of these messages, nodes will “pair up” and each node will average its value with that of a selected neighbor.

Intuitively, nodes will send “+”s to neighbors they would like to average with, and “−”s to decline averaging requests. Their goal will be to average with the neighbor whose value is the farthest from their own.

The following is a description of the actions undertaken by node \(i\) at time \(t\). At this point the nodes have values \(x_1(t), \ldots, x_n(t)\). Node \(i\) will keep track of a variable \(N(i)\), representing the node it would like to average with; and a variable \(g_i\), which will always be equal to \(|x_i(t) - x_{N(i)}(t)|\).

We find it convenient to label the nodes \(1, \ldots, n\) to make statements like “node
Of course, due to the absence of unique identifiers, node $i$ will set its local variable $N(i)$ to its port number for its neighbor $j$, and the above statement should be understood as such.

The algorithm (at node $i$, time $t$):

1. Initialize $N(i) = i$ and $g_i = 0$. Node $i$ broadcasts its value to all its neighbors.

2. Node $i$ reads the incoming messages. Next, node $i$ sends a “$+$” to a neighbor $k$ with the smallest value among neighbors with value smaller than $x_i(t)$. Ties can be broken arbitrarily. It sets $N(i) = k$ and $g_i = |x_i(t) - x_k(t)|$.

   If no such $k$ exists, node $i$ does nothing during this period.

3. node $i$ reads incoming messages. If node $i$ received at least one “$+$” at the previous step, it will compute the gap $|x_i(t) - x_j(t)|$ for every neighbor $j$ in the set $J$ of neighbors that have sent it a “$+$”:

   - If the node $m \in J$ with the largest gap $|x_m(t) - x_i(t)| > g_i$, $i$ will update $N(i) = m$, $g_i = |x_m(t) - x_i(t)|$; next, $i$ will send a “$+$” to $m$ and “$-$” to all the other nodes in $J$. Moreover, if $i$ sent a “$+$” to a node $k$ in step 2, it now sends $k$ a “$-$”.
   - On the other hand, if $|x_m(t) - x_i(t)| \leq g_i$, node $i$ will send a ”$-$” to everyone in $J$.

4. Node $i$ reads incoming messages. If $i$ receives a “$-$” from node $N(i)$, it sets $N(i) = i$, $g_i = 0$.

5. Finally, $i$ sets

   $$x_i(t + 1) \leftarrow \frac{x_i(t) + x_{N(i)}(t)}{2}.$$

Observe that if $N(i) = i$ at the execution of step 5, then the value of node $i$ is left unchanged.

B.2 Sketch of proof

It is not hard to see that this algorithm results in convergence to the average which is geometric with rate $1 - c/n^3$, for some constant $c$. To put it another way, if initial values $x_i(t)$ are in $[0, 1]$ then everyone is within $\epsilon$ of the average in $O(n^3 \log(n^3))$ time.

An informal outline of the proof of this statement follows.

Sketch of proof.

First, we informally describe the main idea. First, one needs to argue that the algorithm we just described results a collection of pairwise averages, i.e. if node $i$...
sets $x_i(t + 1) = (x_i(t) + x_j(t))/2$ in the final step, then node $j$ sets $x_j(t + 1) = (x_j(t) + x_i(t))/2$. Moreover, we will argue that at least one of these pairwise averages occurs across an edge with a “large” gap; more precisely, we will argue that among the edges $(i, j)$ maximizing $|x_i(t) - x_j(t)|$, at least one pair $i, j$ match up. This fact implies, after some simple analysis, that the “sample variance” defined as $\sum_{i=1}^{n}(x_i(t) - (1/n)\sum_j x_j(t))^2$ shrinks by at least $1 - 1/2n^3$ from time $t$ to $t + 1$. The desired convergence result then follows.

1. At the beginning of Step 5, if $N(i) = j$ then $N(j) = i$.
   In words, the last step always results in the execution of a number of pairwise averages for disjoint pairs. This can be proven with a case-by-case analysis.

2. Moreover, one of these averages has to happen on an edge with the largest “gap” $\max_{(i,j)\in E(t)} |x_i(t) - x_j(t)|$, where $E(t)$ is the set of edges in the communication graph at time $t$.
   Indeed, let $J$ be the set of nodes incident on one of these maximizing edges, and let $J'$ be the set of nodes in $J$ with the smallest value. Then, at least one node $j' \in J'$ will receive an offer at Step 2 that comes along a maximizing edge. Any offer $j'$ makes at Step 2 will be along an edge with strictly smaller gap. Consequently, at step 3, $j'$ will send a “+” to a sender of an offer along the maximizing edge — let us call this sender $i'$ — and a “−” to everyone else that sent $j'$ offers, as well as to any node which made $j'$ an offer in Step 2.
   Since $i'$ and $j'$ are connected by a maximizing edge, there is no way $i'$ receives a “+” in Step 2 associated with a gap larger than $|x_i'(t) - x_j'(t)|$, so that $i'$ does not send $j'$ a “−” at step 3. Finally, $i'$ and $j'$ average at Step 5.

3. Without loss of generality, we will assume that $\sum_i x_i(0) = 0$. Since every averaging operation preserves the sum, it follows that $\sum_i x_i(t) = 0$. Let $V(t) = \sum_i x_i^2(t)$. Its easy to see that $V(t)$ is nonincreasing, and moreover, an averaging operation of nodes $a$ and $b$ at time $t$ reduces $V$ by $(1/2)(x_a - x_b)^2$.

4. Let $U(t) = \max_i |x_i(t)|$. Clearly, $V(t) \leq nU^2$. However, $\sum_i x_i(t) = 0$ implies that at least one $x_i(t)$ is negative, so that

$$\max_{(i,j)\in E} |x_i(t) - x_j(t)| \geq \frac{U}{n},$$

which implies that

$$V(t + 1) \leq V(t) - \frac{U^2}{2n^2},$$

or

$$V(t + 1) \leq V(t) - \frac{1}{2n^3}V(t).$$

It follows that after $O(n^3)$ steps, $V(t)$ shrinks by a constant factor. Thus if the initial values $x_i(t)$ are in $[0, 1]$ then everyone is within $\epsilon$ of the average in $O(n^3 \log(n\epsilon))$ time.
Appendix C

List of assumptions

**Assumption 2.1** (connectivity) The graph
\[ \bigcup_{s \geq t} G(s) \]
is connected for every \( t \).

**Assumption 3.1** (non-vanishing weights) The matrix \( A(t) \) is nonnegative, stochastic, and has positive diagonal. Moreover, there exists some \( \eta > 0 \) such that if \( a_{ij} > 0 \) then \( a_{ij} > \eta \).

**Assumption 3.2** (\( B \)-connectivity) There exists an integer \( B > 0 \) such that the directed graph
\[ (N, E(kB) \cup E((k + 1)B) \cup \cdots \cup E((k + 1)B - 1)) \]
is strongly connected for all integer \( k \geq 0 \).

**Assumption 3.3** (bounded delays)
(a) If \( a_{ij}(t) = 0 \), then \( \tau_{ij}^i(t) = t \).
(b) \( \lim_{t \to \infty} \tau_{ij}^i(t) = \infty \), for all \( i, j \).
(c) \( \tau_{ij}^i(t) = t \), for all \( i, t \).
(d) There exists some \( B > 0 \) such that \( t - B + 1 \leq \tau_{ij}^i(t) \leq t \), for all \( i, j, t \).

**Assumption 3.4** (double stochasticity) matrix \( A(t) \) is column-stochastic for all \( t \), i.e.,
\[ \sum_{i=1}^{n} a_{ij}(t) = 1, \]
for all \( j \) and \( t \).
Assumption 3.5 (Bounded round-trip times) There exists some $B > 0$ such that whenever $(i, j) \in E(t)$, then there exists some $\tau$ that satisfies $|t - \tau| < B$ and $(j, i) \in E(\tau)$.

Assumption 4.1 (connectivity relaxation) Given an integer $t \geq 0$, suppose that the components of $x(tB)$ have been reordered so that they are in nonincreasing order. We assume that for every $d \in \{1, \ldots, n-1\}$, we either have $x_d(tB) = x_{d+1}(tB)$, or there exist some time $t \in \{tB, \ldots, (t+1)B-1\}$ and some $i \in \{1, \ldots, d\}$, $j \in \{d+1, \ldots, n\}$ such that $(i, j)$ or $(j, i)$ belongs to $E(A(t))$.

Assumption 7.1 For all $i$, $x_i(0)$ is a multiple of $1/Q$. 

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