Non locality, closing the detection loophole and communication complexity

Serge Massar
Service de Physique Théorique, Université Libre de Bruxelles, CP 225, Bvd. du Triomphe, B1050 Bruxelles, Belgium.
(17-08-01)

It is shown that the detection loophole which arises when trying to rule out local realistic theories as alternatives for quantum mechanics can be closed if the detection efficiency \( \eta \) is larger than \( \eta > d^{1/2}2^{-0.0035d^2} \) where \( d \) is the dimension of the entangled system. Furthermore it is argued that this exponential decrease of the detector efficiency required to close the detection loophole is almost optimal. This argument is based on a close connection that exists between closing the detection loophole and the amount of classical communication required to simulate quantum correlation when the detectors are perfect.

Experimental tests of the entanglement of quantum systems are important for several reasons. They provide an experimental check of the validity of quantum mechanics, and in particular the surprising “non locality” exhibited by quantum mechanics. Furthermore they can be viewed as primitives from which one can build more complicated protocols of interest for quantum information processing and they provide a benchmark with which to compare the performance of different quantum systems, such as ion traps, photons, etc.

To test the entanglement of a quantum system one carries out measurements on each particle, and compares the correlations between the results of these measurements with the predictions of quantum mechanics. A crucial check of the quantumness of these correlations is whether they exhibit “non locality”, that is whether they cannot be reproduced by a classical local variable theory (also called local realistic theory) [1]. Formally this is done by inserting the joint probabilities of outcomes into an inequality, called a “Bell inequality”, which must be satisfied in the case of local variable theories but can be violated by quantum mechanics.

During the past decades successively more sophisticated tests of Bell inequalities have been carried out (for a review see [2]). Most experiments so far have involved entangled photons. By letting the photons propagate a large distance from their emission point it has been possible to spatially separate the two measurements and thereby close the so called “locality loophole”. However in optical experiments, because of losses and small detector efficiency, all tests of Bell inequalities so far leave open the so called “detection loophole”. This means that all experimental results that use pairs of photons can be explained by a classical local variable theory if the local variable theory can instruct the detectors either to click, i.e. register the presence of a particle, or not. The strongest theoretical result so far is that the detection loophole can be closed in the efficiency is \( \eta > 2/3 \) [3], but this is too stringent for optical experiments. Recently an experiment that closes the detection loophole has been carried out using trapped ions [4]. But in this experiment the ions where separated by a very small distance and the locality loophole was not closed.

In almost all experiments on entangled systems each system belongs to a Hilbert space of dimension 2. (One recent experiment tested the entanglement of systems of dimension 3 [5].) However when pairs of photons are produced (for instance by parametric down conversion), the photons are entangled in position-momentum and time-energy in addition to a possible entanglement in polarization. Thus entangled systems of large dimensionality can easily be produced in the laboratory. Can one exploit the large dimensionality of these entangled photons to carry out stronger tests of quantum non locality? This has been the subject of several recent theoretical works [6–10] in which it has been shown that using entangled systems of large dimensionality can be advantageous, but no spectacular improvements have been found.

In the present work it will be shown that using entangled systems of large dimensionality allows in principle a dramatic decrease in the detector efficiency required to close the detection loophole. More precisely, the minimum detector efficiency required to close the detection loophole decreases exponentially with the dimension \( d \). This is particularly relevant to possible experiments involving momentum or energy entangled photons since in this case it may be possible to devise an experiment in which photon losses and detector efficiency decrease only slowly with the dimension.

This result is obtained by explicitly describing a set of measurements carried out by Alice and Bob on an entangled system of large dimension and writing a Bell inequality adapted to this measurement scenario. It will be shown that this Bell inequality is violated even for exponentially small detector efficiencies. However this Bell inequality is extremely sensitive to noise and therefore does not constitute a realistic experimental proposal. A noteworthy feature of this measurement scenario is that the number of measurements between which Alice and Bob must choose is exponentially large.

In the second part of this letter we consider whether it is possible to improve this Bell inequality. Can one decrease the number of measurements between which Alice and Bob must choose, or decrease the dimensionality of the entangled system, while keeping the same low sensitivity to detector inefficiency? We argue that this is not the case and that our Bell inequality is close to optimal. These latter results follow from a close connection be-
between the detection loophole and the minimum amount of classical communication required to perfectly simulate measurements on an entangled quantum system. Suppose measurements are carried out on an entangled quantum system (with perfect detectors \( \eta = 1 \)). The correlations exhibited by such measurements will in general violate a Bell inequality and therefore cannot be reproduced by local variable theories. However by supplementing the local variable theory by classical communication one can reproduce the quantum correlations. Recently there have been several works that attempted to understand how much classical communication is necessary to bridge the gap between quantum mechanics and local variable theories [11][13][14]. Intuitively one would expect that the more communication is required to recover the quantum correlations, the stronger the quantum correlations test non locality. This intuition will be made precise below in the context of the detection loophole. It will be shown that the minimum amount of classical communication \( C_{min} \) required to recover the quantum correlations is anti-correlated to the minimum detection efficiency \( \eta^* \) required to close the detection loophole.

We begin with some definitions.

A measurement scenario is defined by a bipartite quantum state \( \psi \) belonging to the tensor product of two Hilbert spaces \( \mathcal{H}_A \otimes \mathcal{H}_B \), and by two ensembles of measurements, \( M_A \) acting on \( \mathcal{H}_A \) and \( M_B \) acting on \( \mathcal{H}_B \). For instance \( \psi = \sum_{k=1}^{d_1} |k\rangle_A |k\rangle_B / \sqrt{d} \) can be the maximally entangled state of \( d \) dimensions. The elements \( x \in M_A \) are a basis of \( \mathcal{H}_A \): \( x = \{x_1, ..., x_{d_1}\} \) with \( \langle x_i | x_j \rangle = \delta_{ij} \).

Similarly the elements \( y \in M_B \) are a basis of \( \mathcal{H}_B \). Party A is given as input a random element \( x \in M_A \) and party B is given as input a random element \( y \in M_B \).

In a measurement scenario with perfect detectors (\( \eta = 1 \)), both parties must give as output one of \( d \) possible outcomes. Denote Alice’s output by \( a \) and Bob’s output by \( b \). The joint probabilities of the outcomes are \( P(a, b | x, y) = \langle \psi | x_i \rangle \langle y_j | \psi \rangle^2 \).

In a measurement scenario with detectors of finite efficiency \( \eta \), both parties must give as output one of \( d + 1 \) possible outcomes. Output 0 occurs with probability \( 1 - \eta \) and corresponds to the detector not detecting the particle whereas outcomes \( 1 \) to \( d \) occur with probability \( \eta \) and correspond to a specific result of the measurement when the particle is detected. The probability that one of the detectors gives outcome 0 is independent of the other detector. Thus the joint probabilities of outcomes are:

\[
\begin{align*}
P(a = 0, b = 0 | x, y) &= (1 - \eta)^2, \\
P(a = i, b = 0 | x, y) &= \eta(1 - \eta) \text{Tr} | x_i \rangle \langle x_i | \otimes \mathbb{1}_B | \psi \rangle \langle \psi |, \\
P(a = 0, b = j | x, y) &= \eta(1 - \eta) \text{Tr} \mathbb{1}_A \otimes | y_j \rangle \langle y_j | | \psi \rangle \langle \psi |, \\
P(a = i, b = j | x, y) &= \eta^2 | \langle \psi | x_i \rangle \langle y_j | \psi \rangle^2. \\
\end{align*}
\]

In a local variable theory for the measurement scenario \( \{\psi, M_A, M_B\} \) with detector efficiency \( \eta \), Alice and Bob are both given the same element \( \lambda \in \Lambda \) drawn with probability \( p(\lambda) \) (often called the “local hidden variable”). Alice knows \( x \) but does not know \( y \). From her knowledge of \( \lambda \) and \( x \), Alice selects an outcome \( a = f(x, \lambda) \). Similarly Bob knows \( y \) but does not know \( x \) and chooses an outcome \( b = g(y, \lambda) \). We can suppose that the functions \( f \) and \( g \) are deterministic since all local randomness can be put in \( \lambda \). The joint probabilities \( P(a, b | x, y) = \int_d d\lambda \ p(\lambda) \delta(f(x, \lambda) - a)\delta(g(y, \lambda) - b) \) are identical with the predictions of quantum mechanics eq. (4).

A local variable theory will only exist if the detector efficiency is sufficiently small. The maximum detector efficiency for which a local variable theory exists will be denoted \( \eta^*(\psi, M_A, M_B) \).

We are now in a position to state our main result:

**Theorem 1:** There exists a measurement scenario for which the state is the maximally entangled state of dimension \( d = 2^n \) with \( n \geq 2 \) an integer, and for which the number of measurements carried out by Alice and Bob are exponentially large \( |M_A| = |M_B| = 2^d \), and such that the detection loophole is closed if \( \eta \geq d^{1/2 - 0.0035d} \).

**Proof:** We consider the same measurement scenario as that described in Theorem 4 of [11] (which is inspired by the Deutsch-Jozsa problem, see [12]). The state is \( \psi = \sum_{k=1}^{d_1} |k\rangle \langle k| / \sqrt{d} \). The sets of measurements \( M_A \) and \( M_B \) are identical. The measurements \( x \in M_A \) are parameterized by a string of \( d \) bits: \( x = x_1 x_2 ... x_d \) where \( x_i \in \{0,1\} \) and similarly for \( y \in M_B \). Hence \( |M_A| = |M_B| = 2^d \). The measurements are described in detail in [11]. They have the important properties that

1. if \( x = y \), then Alice and Bob’s outcome are identical (\( a = b \)),
2. if the Hamming distance \( \Delta(x, y) \) between \( x \) and \( y \) is \( \Delta(x, y) = d/2 \), then Alice and Bob’s outcomes are always different (\( a \neq b \)).

Let us define \( \alpha(x, y) = \delta(x = y) - \delta(\Delta(x, y) = d/2) \) which is equal to +1 if \( x = y \), equal to −1 if \( \Delta(x, y) = d/2 \), and equals zero otherwise. Consider the following Bell expression

\[
I = \sum_{x=1}^{2^d} \sum_{y=1}^{2^d} P(a = b \text{ AND } a \neq 0) \alpha(x = y). \quad (2)
\]

It is immediate to compute the value of \( I \) predicted by quantum mechanics for the above measurement scenario since from properties 1 and 2 above, only the term proportional to \( \delta(x = y) \) contributes:

\[
I(QM) = \eta^2 2^d. \quad (3)
\]

It is more difficult to compute the maximum value of \( I \) in the case of local variable theories. Let \( Z \) be the largest subset of \( \{0,1\}^d \) such that if \( z, z' \in Z \), then \( \Delta(z, z') \neq d/2 \) (i.e. no two elements of \( Z \) are Hamming distance \( d/2 \) one from the other). We shall show below that

\[
I(\text{local variable}) \leq d |Z| \quad (4)
\]
independently of \(\eta\). Frankl and Rödl have given bounds on \(|Z|\). Theorem 1.10 of [13] states that \(|Z| < (2 - \epsilon)^d\) for some constant \(\epsilon > 0\). And from corollary 1.2 of [13] one can deduce a more precise bound: \(|Z| < 2.993d\). Combining this with eq. (3) implies that one can close the detection loophole if \(\eta \geq d^{1/2} - 0.0035d \geq d^{1/2} |Z|^{1/2} / 2^{d/2}\).

We now prove eq. (4). Recall that in the case of local variable model, Alice’s output is a function \(a(\lambda, x)\) of the local variable and of her measurement, and similarly for Bob. Using \(P(a = b \text{ AND } a \neq 0) = \sum_{k=1}^{d} P(a = k \text{ AND } b = k)\), the value of \(I\) for a local variable model can be written as

\[
I(lv) = \sum_{\lambda} p(\lambda) \sum_{x} \sum_{y} \sum_{k=1}^{d} P[a(\lambda, x) = k \text{ AND } b(\lambda, y) = k] \alpha(x, y)
\]

where \(X_{k\lambda}\) is the set of \(x\) such that \(a(\lambda, x) = k\) and \(Y_{k\lambda}\) is the set of \(y\) such that \(b(\lambda, y) = k\). Let us denote by \(Z_{k\lambda}\) the largest set such that 1) \(Z_{k\lambda} \subset X_{k\lambda}\); 2) \(Z_{k\lambda} \subset Y_{k\lambda}\); 3) if \(z, z' \in Z_{k\lambda}\) then \(\Delta(z, z') \neq d/2\). This implies that \(|Z_{k\lambda}| \leq |Z|\). Consider the sum \(\beta(x) = \sum_{y \in Y_{k\lambda}} \alpha(x, y)\). \(\beta(x)\) is an integer less or equal to 1. Let us show that if \(x \notin Z_{k\lambda}\), then \(\beta(x) \leq 0\). Suppose this is not true (i.e. \(x \notin Z_{k\lambda}\) and \(\beta(x) = 1\)), then necessarily \(x \in Y_{k\lambda}\) and there is no \(y \in Y_{k\lambda}\) such that \(\Delta(x, y) = d/2\). But then we could increase \(Z_{k\lambda}\) by adding \(x\) to \(Z_{k\lambda}\). But \(Z_{k\lambda}\) is maximal, hence there is a contradiction. We therefore obtain that \(\sum_{x \in X_{k\lambda}} \beta(x) \leq \sum_{x \in Z_{k\lambda}} \beta(x) \leq |Z_{k\lambda}| \leq |Z|\). Inserting this in eq. (4) yields eq. (4). □

Note that the Bell expression eq. (2) is extremely sensitive to noise. This is because in the presence of noise the term in \(\alpha\) proportional to \(\delta(\Delta(x, y) = d/2)\) receives a very large contribution, and therefore leads to a much reduced value of \(I\).

We now turn to the relation between the detection loophole and communication complexity. We begin with a definition:

**In a local variable theory supplemented by \(C\) bits of classical communication for the measurement scenario \(\{\psi, M_A, M_B\}\) with perfect detectors \((\eta = 1)\) the parties, in addition to sharing the random variable \(\lambda\), are allowed to communicate \(C\) bits before choosing their output. Note that one should distinguish whether \(C\) is the absolute bound on the amount of communication, or whether \(C\) is the average amount of communication between the parties, where the average is taken over many repetitions of the protocol, see [4].**

For a given measurement scenario \(\{\psi, M_A, M_B\}\) with perfect detectors one can try to minimize the amount of communication required to reproduce the quantum probabilities. The minimum amount of communication required to simulate the measurement scenario in the average communication model will be denoted \(C_{min}(\psi, M_A, M_B)\).

We shall now show that the minimum detector efficiency \(\eta^*\) required to close the detection loophole and the minimum amount of communication \(C_{min}\) required to simulate a measurement scenario with perfect detectors are closely related. We begin by showing that if a measurement scenario is difficult to simulate classically, then the minimum detector efficiency required to close the detection loophole is small. In fact this result was the inspiration for Theorem 1: the measurement scenario considered in Theorem 1 is difficult to simulate classically [11], hence \(\eta^*\) must be small. Further investigations led to the strong result of Theorem 1.

**Theorem 2:** For all measurement scenarios \(\{\psi, M_A, M_B\}\), the relation \(\eta^*(\psi, M_A, M_B) \leq \sqrt{2/C_{min}(\psi, M_A, M_B)}\) holds.

**Proof.** It will be shown that any local variable model with detector efficiency \(\eta\) can be mapped into a communication protocol with an average of \(2/\eta^2\) bits of communication. Therefore \(C_{min} \leq 2/\eta^2\) for all detector efficiencies for which a local variable model exists, and this yields the upper bound on \(\eta^*\).

Recall that a local variable model is defined by the two functions \(f\) and \(g\) introduced above and the probability distribution \(p\) on the space \(A\). Now suppose that initially the parties share an infinite number of i.i.d. hidden variables \(\lambda_1, \lambda_2, \lambda_3, \ldots\) each drawn from the space \(A\) with probability \(p\). Consider the following protocol in which the two parties repeatedly simulate the local variable model and communicate whether the model predicts that the detectors work or not:

1. Set the index \(k = 1\).
2. Alice computes \(f(x, \lambda_k)\) and Bob computes \(g(y, \lambda_k)\).
3. Alice tells Bob whether \(f(x, \lambda_k) = 0\) or \(f(x, \lambda_k) \neq 0\) and Bob tells Alice whether \(g(y, \lambda_k) = 0\) or \(g(y, \lambda_k) \neq 0\).
4. If \(f(x, \lambda_k) = 0\) or \(g(y, \lambda_k) = 0\), Alice and Bob increase the index \(k\) by 1 and go back to step 2.
5. If \(f(x, \lambda_k) \neq 0\) and \(g(y, \lambda_k) \neq 0\), then Alice outputs \(f(x, \lambda_k)\) and Bob outputs \(g(y, \lambda_k)\).

This protocol reproduces exactly the correlations exhibited by quantum mechanics. The mean number of iterations of the protocol is \(1/\eta^2\). The number of bits communicated during each iteration is 2 (one bit from Alice to Bob and one from Bob to Alice). Hence the average amount of communication is \(2/\eta^2\). □

We now investigate whether a model with finite communication and perfect detectors can be mapped into a local variable model with inefficient detectors. We will give an argument, but not a proof, that suggests that such a mapping should exist.

Consider a measurement scenario. Suppose there is a classical protocol that simulates the quantum correlations with \(C\) bits of communication. In this protocol, Alice initially knows the local variable \(\lambda\) and her measurement \(x\), and Bob initially knows the local variable \(\lambda\) and his measurement \(y\). Denote the conversation by...
$C(x, y, \lambda) = c_1c_2 \ldots$ where $c_i \in \{0, 1\}$ is the $i$'th bit in the conversation. Alice and Bob's outputs are therefore given by functions $a = f(x, \lambda, C)$ and $b = g(y, \lambda, C)$.  

Now suppose that in addition to the local variable $\lambda$, Alice and Bob share a second local variable $\mu$ which consists of an infinite string of independent random bits $\mu_i \in \{0, 1\}$. The basic idea is that Alice and Bob will check whether the local variable $\mu$ is a possible conversation $\mu = C(x, y, \lambda)$. If it is then they give the corresponding output. If it is not then they give the outcome 0 corresponding to the detectors not working. The probability that $\mu = C$ is $2^{-C}$ which suggests that if $\eta \leq 2^{-C}$ a local variable model should exist.

Making the above argument precise is difficult because one wants to recover exactly the probability distribution eq. (1). For instance if some conversation are shorter than others, then they will be accepted with higher probability, yielding a skewed distribution. Nevertheless the above argument is very suggestive. For instance in [14] it was shown that if the entangled state has dimension $d$, then any measurement scenario can be simulated in the average communication model using less than $(6 + 3\log_2(d))d + 2$ bits on average. Combining this with the above argument suggests that if $\eta < O(2^{-6d^2d^{-3}})$ a local variable model should exist. This in turn suggests that theorem 1 is close to optimal.

It is also interesting to combine the above argument with a result from [11] that states that it is always possible to simulate a measurement scenario with $C = \log_2 |M_A|$ bits of communication. Combining this with the above argument suggests that if $\eta > 1/|M_A|$ a local variable model should exist. This result (in a slightly weaker form, since the result in [11] depends only on $|M_A|$, independently of $|M_B|$) has been proven by S. Popescu [3] as follows:

**Theorem 3:** Consider a measurement scenario in which the number of possible measurements is $|M_A| = |M_B| = M$. Then a local hidden variable model exists if the detector efficiency is $\eta = 1/M$.

**Proof:** The local hidden variable consists of the quadruple $(x, i, j, y)$ where $x \in M_A$, $y \in M_B$, $i, j \in \{1, \ldots, d\}$ and $i, j$ have joint probabilities $P(i, j) = \langle |\psi||x_i||y_j\rangle|^2$. The protocol is as follows: Alice checks whether her measurement is equal to $x$, if so she outputs $i$, if not she outputs 0; Bob checks whether his measurement is equal to $y$, if so he outputs $j$, if not he outputs 0. This reproduces exactly the correlations eq. (1) with $\eta = 1/M$. □

In summary we have presented a measurement scenario that closes the detection loophole when the detector efficiency $\eta \geq 2^{-0.0035}\log_2 d$ is exponentially small. This should be contrasted to the best previous result that required $\eta > 0.5$. Our measurement scenario requires an entangled system of large dimension $d$, and it requires that Alice and Bob choose between exponentially many measurements. We have argued that it is not possible to substantially improve this measurement scenario, either by decreasing the number of measurements, or by decreasing the dimension, while keeping the same resistance to inefficient detectors.

The results reported here are inspired by recent work in communication complexity. Indeed the measurement scenario we consider in our main theorem is also known to require a large amount of communication in order to be simulated classically [11], and our general arguments concerning bounds on the minimum detector efficiency required to close the detection loophole follow from mappings between communication models and local variable models with inefficient detectors. This connection between two different approaches to entanglement, namely the point of view of computer scientists and the more pragmatic considerations of experimentalists will, we hope, continue to prove fruitful.

I would like to thank Nicolas Cerf, Richard Cleve, Thomas Durt, Nicolas Gisin, Jan-Ake Larsson, Noah Linden and Sandu Popescu for helpful discussions. Funding by the European Union under project EQUIP (IST-FET program) is gratefully acknowledged.

[1] J. S. Bell, Physics 1 (1964) 195
[2] A. Aspect, Nature 398 (1999) 189
[3] P. H. Eberhard, Phys. Rev. A 47 (1993) 47
[4] M.A. Rowe, D. Kielpinski, V. Meyer, C.A. Sackett, W.M. Itano, C. Monroe, and D.J. Wineland, Nature 409, 791-794 (2001).
[5] J. C. Howell, A. Lamas-Linares and D. Bouwmeester, quant-ph/0105132
[6] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, A. Zeilinger, Phys. Rev. Lett. 85, 4418 (2000)
[7] T. Durt, D. Kaszlikowski, M. Zukowski, quant-ph/0101082
[8] J.-L. Chen, D. Kaszlikowski, L. C. Kwek, M. Zukowski, C. H. Oh, quant-ph/0103096
[9] D. Kaszlikowski, L. C. Kwek, J.-L. Chen, M. Zukowski, C. H. Oh, quant-ph/0106010
[10] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, quant-ph/0106024
[11] G. Brassard, R. Cleve, A. Tapp, Phys. Rev. Lett. 83, 1874 (1999)
[12] H. Buhrman, R. Cleve and A. Wigderson, Proceedings of the 30th Annual ACM Symposium on Theory of Computing, May 1998, pp. 63 - 68
[13] M. Steiner, Phys. Lett. A 270, 239 (2000)
[14] S. Massar, D. Bacon, N. J. Cerf and R. Cleve, Phys. Rev. A 63, 052305 (2001)
[15] P. Frankl and V. Rödl, Trans. Am. Math. Soc., 300 (1987) 259
[16] S. Popescu, private communication.