Lieb-Schultz-Mattis Theorem and its generalizations from the Perspective of the Symmetry Protected Topological phase

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We ask whether a local Hamiltonian with a featureless (fully gapped and nondegenerate) ground state could exist in certain quantum spin systems. We address this question by mapping the vicinity of certain quantum critical point (or gapless phase) of the $d$–dimensional spin system under study to the boundary of a $(d + 1)$–dimensional bulk state, and the lattice symmetry of the spin system acts as an on-site symmetry in the field theory that describes both the selected critical point of the spin system, and the corresponding boundary state of the $(d + 1)$–dimensional bulk. If the symmetry action of the field theory is nonanomalous, i.e. the corresponding bulk state is a trivial state instead of a bosonic symmetry protected topological (SPT) state, then a featureless ground state of the spin system is allowed; if the corresponding bulk state is indeed a nontrivial SPT state, then it likely excludes the existence of a featureless ground state of the spin system. From this perspective we identify the spin systems with $SU(N)$ and $SO(N)$ symmetries on one, two and three dimensional lattices that permit a featureless ground state. We also verify our conclusions by other methods, including an explicit construction of these featureless spin states.

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I. INTRODUCTION

The Lieb-Schultz-Mattis (LSM) theorem$^4$ and its higher dimensional generalization$^{23,24}$ state that if a quantum spin system defined on a lattice has odd number of spin-1/2s per unit cell, then any local spin Hamiltonian which preserves the spin and translation symmetry, cannot have a featureless (gapped and nondegenerate) ground state. This implies that any symmetry allowed Hamiltonian on the spin Hilbert space defined above can only have the following possible scenarios: 1. its ground state spontaneously breaks either the spin symmetry or the lattice symmetry, hence leads to degenerate ground states and possible gapless Goldstone modes; 2. it has gapped and degenerate ground states without breaking any symmetry, i.e. its ground state develops a topological order (the second possibility can only happen in two and higher dimensional systems); 3. its ground state has algebraic (power-law) correlation function of physical quantities, and the spectrum is again gapless (this scenario happens most often in 1d spin systems, while still possible in higher dimensions).

On the other hand, there are lattice spin systems for which one can very easily construct a local Hamiltonian with a featureless ground state that preserves all the symmetry. One class of such states are called the AKLT state$^5$, which can be constructed for an integer spin chain in 1d, the spin-2 antiferromagnet on the square lattice, and the spin-3/2 antiferromagnet on the honeycomb lattice, etc. Of course, these systems violate the crucial “odd number of spin-1/2s per unit cell” assumption of the LSM theorem.

However, there are also some spin systems in the “middle ground” where the answers are not so clear. These systems do not meet the key assumption of the LSM theorems, while a simple analogue of the AKLT state mentioned above does not obviously exist. For example, the honeycomb lattice has two sites per unit cell, thus a spin-1/2 system on the honeycomb lattice has even number of spin-1/2s per unit cell, and hence there is no LSM theorem to exclude a featureless ground state. But it has been a long standing problem whether a featureless spin-1/2 state exists or not on the honeycomb lattice. Another example is the spin-1 antiferromagnet on the square lattice. Depending on the Hamiltonian, possible states of this system include the Neel state which spontaneously breaks the spin symmetry, and a nematic type of valence bond solid state which breaks the lattice rotation symmetry, etc. But the existence of a featureless state is not obvious. However, recent progresses indicate that featureless states do exist in these two “middle ground” examples mentioned above$^{25,26}$, with a more sophisticated construction compared with the AKLT state.

Another seemingly very different subject is the symmetry protected topological (SPT) state$^{27}$, which is a generalization of topological insulators. By definition, the ground state of the $(d + 1)$–dimensional bulk of a SPT phase must be gapped and nondegenerate, while its $d$–dimensional boundary state must be either gapless or degenerate, as long as certain symmetries are preserved. In the last few years, the classification of bosonic SPT states with on-site internal symmetries has been well understood$^{27,28}$. The $d$–dimensional boundary of a $(d + 1)$–dimensional SPT state, just like those $d$–dimensional spin systems where the LSM theorem applies, cannot be trivially gapped. The key difference between these two systems is that, the former is (usually) protected by an on-site symmetry, while the latter is protected by the spin and lattice symmetries together. However, the fact that neither system permits a featureless
state suggests that we can potentially formulate both systems in a similar way. The connection to 3d bulk SPT states has been exploited in order to understand the fractional excitations of 2d topological orders with both spin and lattice symmetry.

Since we are comparing two $d$-dimensional systems with very different ultraviolet regularizations, their analogue can only be made precise when both systems are tuned close to a point where a low energy field theory description becomes available. Thus for our purpose, when we analyze a $d$-dimensional spin system, we will first tune it to a critical point described by a field theory, then interpret the lattice symmetry as an on-site symmetry, and interpret the $d$-dimensional field theory as the boundary state of a $(d+1)$-dimensional bulk. If the corresponding $(d+1)$-dimensional bulk is a trivial state instead of a nontrivial SPT state, then a featureless spin ground state must exist not too far from that critical point in the phase diagram; if the corresponding bulk is indeed a nontrivial SPT state, then it highly suggests that a featureless spin ground state does not exist.

However, the latter statement may not be necessarily true: if around that selected critical point of the spin system the field theory is formally equivalent to a SPT boundary state, it only rules out the featureless spin state at the vicinity of that critical point. But in principle a featureless state could be far away from the critical point in the phase diagram, and hence beyond the reach of the field theory.

In section II through V, we will discuss SU(N) and SO(N) systems on a 1d chain, 2d square lattice, 2d honeycomb lattice, and 3d cubic lattice respectively, by mapping them to the boundary of 2d, 3d and 4d bulk states. We will identify those spin systems that permit a featureless ground state. For all of these spin systems, we can explicitly construct a featureless tensor product state that is an analogue of the AKLT state. Some examples of these featureless states will be discussed in section VI. In section VI we will also verify our conclusions by making connection with a previous study on LSM theorem based on lattice homotopy class.

II. 1d SPIN CHAIN

A. SU(2) spin-1/2 chain

In this section we first discuss one dimensional spin chains with SU(2) symmetry. The low energy physics of the Heisenberg antiferromagnetic spin-1/2 chain with a SU(2) spin symmetry can be captured by the following nonlinear sigma model in $(1 + 1)d$ with a Wess-Zumino-Witten (WZW) term at level-1:

$$S = \int dx d\tau \frac{1}{g} (\partial_\mu \vec{n})^2 + \frac{2\pi i}{\Omega_3} \int_0^1 du \epsilon_{abcd} n^a \partial_\tau n^b \partial_u n^c \partial_u n^d$$

where $\vec{n}$ is a four component vector with unit length, and $\Omega_3$ is the volume of $S^3$ with unit radius. The physical meaning of $\vec{n}$ is that, $(n_1, n_2, n_3)$ are the three component Néel order parameter, while $n_4 \sim \phi$ is the valence bond solid (VBS) order parameter. If there is a SO(4) rotation symmetry of the four component vector $\vec{n}$, the coupling constant $g$ will flow to a fixed point, which corresponds to the SU(2)$_2$ conformal field theory.

The SO(4) symmetry becomes an emergent symmetry of the spin-1/2 Heisenberg chain in the infrared: the Néel and VBS order parameter both have the same scaling dimension $[n] = 1/2$. The key symmetry of the system, is the spin SU(2) symmetry, and the translation symmetry. $(n_1, n_2, n_3)$ transforms as a vector under spin SU(2), and $n_4 \sim \phi$ is a SU(2) singlet; and under translation by one lattice constant, $T_\tau: n \rightarrow -n$. The physical meaning of Eq. (1) is the intertwinement between the Néel and VBS order parameter: the domain wall of the Néel and VBS order parameter both have the same scaling dimension.

The field theory Eq. (1) also describes the boundary of a 2d bosonic SPT state with SO(3)$\times$Z$_2$ symmetry, where Z$_2$ acts as $n \rightarrow -n$. This SPT state can be understood as the decorated domain wall construction: we decorate every Z$_2$ symmetry breaking domain wall in the 2d bulk with a Haldane phase with SO(3) symmetry (Fig. 1), then proliferate the Z$_2$ domain walls to restore the Z$_2$ symmetry. The so-constructed phase in the bulk is the desired SO(3)$\times$Z$_2$ SPT phase. And at the 1d boundary of the system, there is a spin-1/2 degree of freedom localized at every Z$_2$ domain wall, which is also the boundary state of the Haldane phase decorated at each Z$_2$ domain wall in the bulk. This is consistent with the physics of the spin-1/2 chain.

This simple example demonstrates that the lattice translation symmetry, once interpreted as an on-site symmetry in a field theory, is equivalent to an “anomalous” symmetry of the boundary of a higher dimensional SPT state. And by definition the boundary of a SPT state cannot be trivially gapped without degeneracy, which is consistent with the LSM theorem of the spin-1/2 chain.

Here we stress that, the 1d SPT phase decorated at a Z$_2$ domain wall must have a Z$_2$ classification as long as the symmetry G of the 1d SPT phase commutes with the Z$_2$, i.e. two of the 1d SPT phases must fuse into a trivial state. One way to see this is that, after gauging the Z$_2$ symmetry, the vison ($\pi$-flux introduced by the Z$_2$ gauge symmetry) preserves the symmetry G as long as G commutes with Z$_2$, and the vison is the boundary of the 1d decorated SPT state. Since two visons fuse into a local excitation, the 1d SPT state must have a Z$_2$ classification. But at a Z$_2^T$ (time-reversal) domain wall one can decorate a lower-dimensional SPT phase with (for example) Z classification, because the anti-domain wall of Z$_2^T$ is the time-reversal conjugate of a Z$_2^T$ domain wall, which is automatically decorated with the “inverse” state of the SPT state decorated at the Z$_2^T$ domain wall. This observation is consistent with many known facts about SPT phases. For instance, in three dimensional space, there is a Z$_3^T$ SPT which can be viewed as Z$_3^T$ domain walls decorated with the $E_8$ invertible topological order but
there is no such decorated domain wall construction for 3d SPT phases with a $Z_2$ symmetry.

B. spin chain with reduced symmetry

Now one can exploit the connection between 1d spin chains and the boundary of 2d SPT states even further, and consider a spin chain with a reduced spin symmetry. For example we can start with a spin-1/2 chain, and break the SO(3) symmetry down to its subgroup $G \times Z_2$, where $Z_2$ is the spin $\pi$-rotation $S^z \rightarrow -S^z$, $S^y \rightarrow -S^y$, and $G$ is a subgroup of the inplane U(1) spin symmetry. Whether the spin chain can be featureless or not, is equivalent to the problem of whether the corresponding bulk state with $(G \times Z_2) \times Z_2$ symmetry is a nontrivial SPT state or not; and based on the “decorated domain wall” picture mentioned above, this again is equivalent to the problem of whether the 1d $Z_2$ domain wall is a nontrivial 1d SPT state with $G \times Z_2$ symmetry or not, and if it is indeed a nontrivial SPT, whether it has a $Z_2$ classification.

Now we can look up the classification in Ref. 8[10]. For example, when $G = Z_{2n+1}$ with integer $n$, since there is no nontrivial 1d SPT state with $Z_{2n+1} \times Z_2$ symmetry, the bulk SO(3)$\times Z_2$ SPT state discussed previously must be trivialized by breaking the SO(3) spin symmetry down to $Z_{2n+1} \times Z_2$, thus its boundary can in principle be gapped and nondegenerate. This observation already gives us a meaningful conclusion:

A spin chain with translation and $(Z_{2n+1} \times Z_2)$ spin symmetry can have a featureless ground state.

By contrast, for $G = U(1)$ or $Z_{2n}$, a nontrivial 1d SPT state with $G \times Z_2$ does exist, and it does have a $Z_2$ classification. Hence the Haldane phase with SO(3) spin symmetry remains a nontrivial SPT state under the symmetry reduction to $G \times Z_2$. Thus the 2d bulk SPT state with $(G \times Z_2) \times Z_2$ remains nontrivial, and hence the 1d boundary cannot be trivially gapped. This observation leads to the following conclusion:

A 1d spin-1/2 chain cannot have a featureless ground state, even if we break the SU(2) spin symmetry down to $(Z_{2n} \times Z_2)$ symmetry.

C. SU(2N) spin chain

Now let’s consider spin chains with higher spin symmetries. A natural generalization of the spin-1/2 chain with translation symmetry, is a SU(2N) spin chain with self-conjugate representation on each site (Young tableau with $N$ boxes in one column, Fig. 2). The analogue of the “Néel” order parameter of this SU(2N) spin chain, is a $2N \times 2N$ Hermitian matrix order parameter $\mathcal{P}$, and it can be represented in the form

$$\mathcal{P} = V \Omega V^\dagger, \quad \Omega \equiv \begin{pmatrix} 1_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & -1_{N \times N} \end{pmatrix} \quad (2)$$

where $V$ is a SU(2N) matrix. All the configurations of $\mathcal{P}$ belong to the Grassmanian manifold $\mathcal{M} = U(2N)/[U(N) \times U(N)]^\odot$. To see that $\mathcal{P}$ is a natural generalization of the ordinary SU(2) Néel order parameter, we can take $N = 1$, then this Grassmanian is precisely $S^2$, which is the manifold of the ordinary SU(2) Néel order parameter. We can also define matrix order parameter $\mathcal{P} = \vec{n} \cdot \vec{\sigma}$ for the SU(2) spin chain, where $\vec{n}$ is the SU(2) Néel order parameter.

The effective field theory for the SU(2N) spin chain described above, can be written as:

$$S = \int dx d\tau \left[ -\frac{1}{g} \text{tr}[\partial_\mu \mathcal{P} \partial_\mu \mathcal{P}] + \frac{\Theta}{16\pi} \varepsilon_{\mu\nu\rho} \text{tr}[\mathcal{P} \partial_\mu \mathcal{P} \partial_\nu \mathcal{P}] \right]. \quad (3)$$

This is the analogue of the Nonlinear sigma model for the SU(2) spin chain [20][29], with a $\Theta$-term which comes...
from the fact that for all \( N \), the Grassmanian \( \mathcal{M} \) satisfies \( \pi_2[\mathcal{M}] = \mathbb{Z} \). Under translation by one lattice constant, \( \mathcal{P} \) transforms as \( T_x : \mathcal{P} \to -\mathcal{P} \) and \( -\mathcal{P} \) both belong to the same Grassmanian target manifold, and the coefficient \( \Theta \) transforms as \( T_x : \Theta \to -\Theta \), which guarantees that \( \Theta \) is quantized to be multiple of \( \pi \). The same field theory as Eq. 3 with a topological \( \Theta \) term has been used to describe the phase diagram of the integer quantum Hall systems\(^{10-12}\), while there the theory is written in the \( 2d \) real space instead of space-time. A proposed renormalization group flow for Eq. 3 is that, \( \Theta = 2\pi k \) are stable fixed points, while \( \Theta = \pi(2k + 1) \) are instable fixed points which correspond to transitions between stable fixed points \( \Theta = 2k\pi, 2k\pi + \pi \).

When \( \Theta = \pi \), Eq. 3 describes the SU(2\(N \)) spin chain with self-conjugate representation on each site; when \( \Theta = 2\pi \), Eq. 3 describes the Haldane phase of a SU(2\(N \)) spin chain, or more precisely it is the Haldane phase of a PSU(2\(N \)) spin chain, as \( \mathcal{P} \) is invariant under the center of SU(2\(N \)). The PSU(2\(N \)) Haldane phase should have \( Z_{2N} \) classification\(^{13} \), as its boundary could be \( 2N \) different projective representation of PSU(2\(N \)), which are also the \( 2N \) different representation of the \( Z_{2,N} \) center of SU(2\(N \)). But the particular state described by Eq. 2 and Eq. 3 is the “\( N \)th” PSU(2\(N \)) Haldane phase, whose 0\(d \) boundary is a self-conjugate projective representation of PSU(2\(N \)). This state has a \( Z_2 \) nature, namely two copies of this state will be a trivial state, i.e. its boundary is no longer a projective representation of PSU(2\(N \)).

As we discussed before, the spin-1/2 SU(2) chain can also be described by Eq. 1, where a VBS order parameter is introduced. For the SU(2\(N \)) spin chain with self-conjugate representation, the analogue of Eq. 1 is

\[
S = \int dx d\tau \frac{1}{g} \text{tr}[\partial_\mu U^\dagger \partial_\mu U] + \int_0^1 du \frac{i2\pi}{24\pi^2} \text{tr}[U^\dagger dU]^3, (4)
\]

where \( U = I_{2N \times 2N} \cos(\theta) + i \sin(\theta) P \) is a SU(2\(N \)) unitary matrix. Once again, when \( N = 1 \), \( U \) is a SU(2) matrix, whose manifold is \( S^3 \), the same as the target manifold of Eq. 1. For arbitrary \( N \), under translation, \( T_x : \theta \to \pi - \theta, T_x : U \to -U \). Thus \( \cos(\theta) \sim \phi \) is the VBS order parameter.

The same field theory Eq. 3 describes the boundary of a \( 2d \) SPT state with PSU(2\(N \)) \( \times \) Z\(_2\) symmetry, where Z\(_2\) plays the role of \( T_x \). And the physical picture of this \( 2d \) SPT is that, we decorate every Z\(_2\) domain wall with a Haldane phase with PSU(2\(N \)) symmetry. Thus as one would naively expect, the SU(2\(N \)) spin chain with self-conjugate representation cannot have a featureless ground state, because it can be mapped to the boundary of a nontrivial \( 2d \) SPT state.

D. SO(\(N \)) spin chain

A SO(\(N \)) spin chain with a translation symmetry may still obey a generalization of the LSM theorem. But first let us review the current understanding of the Haldane phase of \( 1d \) SO(\(N \)) spin chain. When \( N \) is an odd integer, the double covering group of SO(\(N \)), i.e. Spin(\(N \)), has a representation which is a spinor of SO(\(N \)). Thus when \( N \) is odd, there is a Haldane phase with SO(\(N \)) symmetry with \( Z_2 \) classification, as two spinors of SO(\(N \)) will merge into a linear representation of SO(\(N \))\(^{14} \). Thus in \( 2d \) space, there is a SPT state with SO(\(N \)) \( \times \) Z\(_2\) symmetry, which is constructed by decorating the \( 1d \) SO(\(N \)) Haldane phase in each Z\(_2\) domain wall. Then the 1\(d \) boundary of this \( 2d \) SPT state with SO(\(N \)) \( \times \) Z\(_2\) symmetry, has the feature that, at every Z\(_2\) domain wall there must be a SO(\(N \)) spinor, and this 1\(d \) boundary cannot be trivially gapped without breaking the Z\(_2\) symmetry.

Now let’s consider a Spin(\(N \)) spin chain with a spinor on every site. Two Spin(\(N \)) spinors with odd \( N \) can always form a singlet, thus this spin chain naturally hosts two fold degenerate VBS states, which transform into each other through translation by one lattice constant. The domain wall of these two VBS states is a Spin(\(N \)) SPT state. But this Spin(\(N \)) SPT state is trivially gapped without breaking the Spin(\(N \)) symmetry, which is constructed by decorating the \( 1d \) Spin(\(N \)) Haldane phase in each Z\(_2\) domain wall. Thus for both odd and even \( N \), a Spin(\(N \)) spin chain with spinor representation on every site, does not permit a featureless gapped state.

For even \( N \), let’s take \( N = 2n \), then the Haldane phase has a richer structure. SO(\(2n \)) has a Z\(_2\) center which commutes with all the other elements, thus we can actually consider the Haldane phase with symmetry PSO(\(2n \)) = SO(\(2n \))/Z\(_2\). Then according to Ref. 33, the center of Spin(\(2n \)) can be either Z\(_4\) or Z\(_2\) \( \times \) Z\(_2\), for odd and even integer \( n \) respectively. But in either case, a Haldane phase with PSO(\(2n \)) symmetry could have either spinor or vector representation at the boundary, both cases are nontrivial Haldane phase. And we can construct a \( 2d \) SPT with PSO(\(2n \)) \( \times \) Z\(_2\) symmetry, by decorating the Z\(_2\) domain wall with a PSO(\(2n \)) Haldane phase. But this PSO(\(2n \)) Haldane phase must have a Z\(_2\) nature, in the sense that two copies of the Haldane phase must be a trivial state, because two Z\(_2\) domain walls will fuse into a trivial defect. Thus for both odd and even \( n \), we can always decorate the Z\(_2\) domain wall with the PSO(\(2n \)) Haldane phase with a SO(\(2n \)) vector at the boundary, which leads to the following LSM theorem:

A \( 1d \) SO(\(2n \)) spin chain with vector representation on every site does not permit a featureless gapped state.

This conclusion is consistent with the result of Ref. 35.

III. SPIN SYSTEMS ON THE SQUARE LATTICE

A. SU(2) spin systems

The generalized LSM theorem in higher dimensions does apply to the 2\(d \) spin-1/2 system on the square lattice\(^{24} \), i.e. there cannot be a featureless spin state on the
square lattice for a spin-1/2 system with SU(2) spin symmetry. This conclusion is consistent with many observations, including a generalization of Eq. 1 to (2 + 1)D.

\[
S = \int d^2x d\tau \frac{1}{g} (\partial_\mu \vec{n})^2 + \frac{2\pi i}{\Omega_4} \int_0^1 du \epsilon_{abcd} n^a \partial_x n^b \partial_x n^c \partial_y n^d \partial_y n^e, \tag{5}
\]

where \(\vec{n}\) is a five component unit vector, which forms the target manifold \(S^4\) with volume \(\Omega_4\). \((n_1, n_2, n_3)\) is still the three component Néel order parameter on the square lattice, while \(n_4\) and \(n_5\) are the columnar VBS states along the \(x\) and \(y\) directions respectively. The site-centered 90 degree rotation of the square lattice acts on \((n_4, n_5)\) as a \(Z_4\) rotation, and close to the deconfined quantum critical point \((n_1, n_2, n_3)\) would carry nonzero boson number of \(b\). Thus if we destroy the ordinary Néel order by condensing the Skyrmions of the Néel order parameter, the system automatically develops a columnar VBS order; and if we destroy the VBS order by condensing the \((Z_4)\) vortex of the columnar VBS order parameter, the system automatically breaks the spin symmetry and develops the Néel order.

Eq. 5 can be derived explicitly by starting with the \(\pi\)-flux spin liquid state on the square lattice\(^{11}\) and it was proposed as an effective field theory\(^{13}\) that describes the deconfined quantum critical point between Néel and VBS order on the square lattice\(^{36}\), and this is the critical point whose vicinity we will study and map to the boundary of a 3d system, as we discussed in the introduction. The key physics of the intertwinement between the Néel and VBS order parameter is encoded in the WZW term. Eq. 5 is capable of encapsulating a large SO(5)\(\times\)\(Z_2^5\) symmetry, and it also describes the boundary state of a 3d bosonic SPT state whose symmetry can be as large as SO(5)\(\times\)\(Z_2^5\). Eq. 5 can also describe the boundary of 3d SPT states with a symmetry that is a subgroup of SO(5)\(\times\)\(Z_2^5\)\(^{10,11}\). According to the definition of SPT states, if the 3d bulk is a nontrivial SPT state, then the boundary cannot be a featureless state; while if the 3d bulk is a trivial direct product state after breaking the SO(5)\(\times\)\(Z_2^5\) to its subgroup, then the boundary in principle can be trivially gapped without degeneracy.

It is clear that if the symmetry SO(5)\(\times\)\(Z_2^5\) is reduced to SO(3)\(\times\)U(1), where \((n_1, n_2, n_3)\) rotates as a vector of SO(3) and singlet under U(1), while \((n_4, n_5)\) transforms as a vector of U(1) and singlet of SO(3), the bulk is still a nontrivial SPT state. And this state can be understood as the “decorated vortex line” construction introduced in Ref. 10, one first breaks the U(1) symmetry by condensing the two component vector \((n_4, n_5)\), and decorate a Haldane phase with the SO(3) spin symmetry on each vortex loop of \((n_4, n_5)\) with odd vorticity, then proliferate the vortex loops to restore the U(1) symmetry. The SPT state so-constructed has a \(Z_2\) classification, which is consistent with the \(Z_2\) classification of the Haldane phase decorated in each vortex loop, and also consistent with the \(Z_2\) nature of the fourth Steifel-Whitney class of the SO(5) gauge bundle\(^{10}\). This implies that two copies of the 3d SPT states with SO(3)\(\times\)U(1) symmetry weakly coupled together will become a trivial 3d bulk state.

The site-centered rotation symmetry of the square lattice acts on \((n_4, n_5)\) as the \(Z_4\) subgroup of U(1). The 3d nontrivial SPT state with SO(3)\(\times\)U(1) symmetry survives under the further symmetry breaking of U(1) to \(Z_4\), as a \(Z_4\) vortex loop is still a well-defined object in the bulk and can be decorated with a 1d Haldane phase. The same conclusion still holds if we consider a spin-1/2 system on the rectangular lattice. Now this system corresponds to the boundary of a 3d bulk SPT with SO(3)\(\times\)\(Z_2^5\)\(\times\)\(Z_2^4\) symmetry, or equivalently decorating the \(Z_2^4\) domain wall with the 2d SPT with SO(3)\(\times\)\(Z_2^5\) symmetry, or equivalently decorating the \(Z_2^4\) domain wall with the 2d SPT with SO(3)\(\times\)\(Z_2^4\)\(\times\)\(Z_2^5\) state. This observation is consistent with the generalized LSM theorem which states that a spin-1/2 system on the rectangular lattice cannot have a featureless state.

Just like the previous section, if we break the spin symmetry down to \(G\times\)\(Z_2\), when \(G = Z_{2n+1}\) the spin system on the square lattice allows a featureless state, because the Haldane phase that we decorated in the vortex loop becomes a trivial state with only \(Z_{2n+1}\times\)\(Z_2\) spin symmetry.

Now suppose we consider a spin-1 system on the square lattice, then a similar deconfined quantum critical point corresponds to Eq. 5 with a level-2 WZW term: the coefficient of the WZW term doubles. This equation with a level-2 WZW term can be derived using the \(\pi\)-flux spin liquid state of a spin-1 system on the square lattice: there are twice as many Dirac fermions in the Brilloin zone compared with the case derived in Ref. 10 thus the level of the WZW term also doubles (the difference from the spin-1/2 \(\pi\)-flux state is that, the spin-1 \(\pi\)-flux state has a Sp(4) gauge fluctuation\(^{44}\), while the spin-1/2 \(\pi\)-flux state has a SU(2) gauge fluctuation). The physical meaning of this term is that, the vortex of \((n_1, n_2, n_3)\) now carries a spin-1 instead of spin-1/2, which is equivalent to the physics of the boundary of two weakly coupled 3d SPT states with SO(3)\(\times\)U(1) symmetry, and as we discussed above, this state is generically a trivial state in the bulk. Thus its boundary could be a featureless gapped state. This observation implies that a spin-1 system on the square lattice permits a featureless state, which is consistent with the conclusion of Ref. 10.
B. SU($N$) and SO($N$) spin systems

Now let us consider a SU($N$) spin system on the square lattice, with fundamental representation (FR) on sublattice A, and anti-fundamental representation (AFR) on sublattice B. Since the spins on two nearest neighbor sites can still form a SU($N$) spin singlet, the columnar VBS order parameter and its $Z_4$ structure still naturally hold: the site-centered lattice rotation acts as a $Z_4$ rotation of the columnar VBS order parameter in this system. The $Z_4$ vortex (antivortex) of the VBS order parameter always has a vacant sublattice A (B) in the core, hence it always carries SU($N$) FR (AFR). This is consistent with the fact that a vortex-antivortex pair can always annihilate, hence the quantum spin they carry must together form a spin singlet. An analogous effect on the honeycomb lattice is depicted in Fig. 3.4

With large enough $N$, a Heisenberg model with the representation described above should have the four fold degenerate VBS state. Now we ask whether a featureless ground state of this spin system is in principle allowed or not. Once again, we first view the $Z_4$ lattice rotation as an onsite internal symmetry, and enlarge it to U(1). Then the 2d spin system on the square lattice can be potentially viewed as the boundary of a 3d bosonic SPT state with PSU($N$) × U(1) symmetry.

The bosonic SPT states with PSU($N$) × U(1) symmetry do exist in 3d, and they can be interpreted as the decorated vortex loop construction, i.e. we decorate every U(1) unit vortex loop with a 1d PSU($N$) Haldane phase, whose boundary is a projective representation of the PSU($N$), or a faithful representation of SU($N$). As we have discussed, 1d PSU($N$) Haldane phase has a $Z_N$ classification, which corresponds to $N$ different projective representations of the PSU($N$) group, or $N$ different representation of the $Z_N$ center of SU($N$).

In general, the $N-1$ different nontrivial Haldane phases of PSU($N$) can be described by Eq. 3 with $\Theta = 2\pi$, and $\mathcal{P}$ replaced by $\mathcal{P} = V\Omega V^\dagger$, $\Omega \equiv \begin{pmatrix} 1_{m \times m} & 0_{m \times N - m} \\ 0_{N - m \times m} & -1_{N - m \times N - m} \end{pmatrix}$ (6)

with $m = 1, \cdots N - 1$, and V is a SU($N$) matrix. All the configurations of $\mathcal{P}$ belong to the Grassmanian manifold $U(N)/[U(m) \times U(N - m)]$. In our case, when the vortex line terminates at the boundary, the vortex at the boundary will carry a FR of SU($N$), hence for our case we need to choose $m = 1$, and $\mathcal{P}$ becomes the CP$^{N-1}$ manifold.

However, let us not forget that eventually we need to break the U(1) symmetry down to $Z_4$. Then for the 3d SPT state to survive under this symmetry breaking, the $Z_N$ classification of the PSU($N$) Haldane phase must be compatible with the $Z_4$ vortex. If $N$ and 4 are coprime, then this bulk state definitely becomes trivial after breaking the U(1) to $Z_4$. For example, when $N = 3$, there is no consistent way we can decorate the $Z_4$ vortex with a PSU(3) Haldane phase. Because four $Z_4$ vortex loops merge together will no longer be a well-defined defect, while four PSU(3) Haldane phases merge together is still a nontrivial Haldane phase. Thus for odd integer $N$, the 3d SPT phase with PSU($N$) × U(1) symmetry becomes a trivial phase once U(1) is broken down to $Z_4$.

To further demonstrate that for odd integer $N$, the 3d SPT phase with PSU($N$) × U(1) symmetry is trivialized with U(1) broken down to $Z_4$, we need to show that its 2d boundary can be trivially gapped out when U(1) is broken down to $Z_4$. One of the 2d boundary states of the 3d PSU($N$) × U(1) SPT phase, is a $Z_N$ topological order, which can be constructed by starting with a superfluid order with spontaneous U(1) symmetry breaking at the 2d boundary, and then condense the $N-$fold vortex (a vortex with $2\pi$ N vorticity) of the superfluid order. The single vortex of the superfluid phase carries a FR of SU($N$), hence a $N-$fold vortex can carry a SU($N$) singlet, and its condensate is a $Z_N$ topological order which preserves all the symmetries. A 2d $Z_N$ topological order has bosonic $e$ and $m$ excitations, while $e$ and $m$ have mutual statistics with statistical angle $\theta_{e,m} = 2\pi/N$. In our construction, the $e$ excitation carries $1/N$ charge of the U(1) symmetry, and the $m$ excitation carries a FR of SU($N$).

Once U(1) is broken down to $Z_4$, in order to gap out the $Z_N$ topological order, we can condense the bound state of a $e$ particle and $3N$ $Z_4$ charges. This bound state carries $\frac{3N^2 + 1}{N} Z_4$ charges. Under the $Z_4$ transformation, it acquires a phase $\exp(\frac{2\pi(3N^2 + 1)i}{4N})$, which can always be cancelled/compensated by a gauge transformation with odd integer $N$ (the numerator of the phase angle is always a multiple of $8\pi$ with odd integer $N$). Thus the condensate of this bound state will drive the $Z_N$ topological order into a completely featureless gapped state without any anyons, and all the global symmetries are preserved. This is only possible when $N$ is odd.

As a contrast, for even integer $N$, we can always construct a nontrivial 3d SPT by decorating the $Z_4$ vortex loop with the 1d SPT state with SU($N$)/$Z_2$ symmetry, which has a $Z_2$ classification.

Now we can make the following conclusion:

A SU($N$) spin system on the square lattice with fundamental and anti-fundamental representation on the two sublattices, permit a featureless gapped ground state for odd integer $N$.

We can also consider SO($N$) spin systems on the square lattice. The analysis is very similar to the previous case. We can make the following conclusion:

A SO(2n) spin system with vector representation on every site does not permit a featureless gapped state on the square lattice.

A SO(2n + 1) spin system with spinor representation on every site does not permit a featureless gapped state on the square lattice.

On the other hand, A SO(2n + 1) spin system with vec-
A spin-1/2 system on the honeycomb lattice, when tuned close to certain point, can also be described by Eq. 5. Eq. 5 can be derived with the SU(2) spin liquid on the honeycomb lattice, like the one discussed in Ref. [44]. Now the lattice symmetry, both the translation $T_x$ and a site-centered 120 degree rotation, acts as a $Z_3$ subgroup of the U(1) transformation on $(n_4, n_5)$.

Once again, the question of whether a featureless spin-1/2 state exists on the honeycomb lattice is equivalent to whether the 3d SPT state with SO(3) $\times$ U(1) symmetry is stable against breaking the U(1) down to $Z_3$. It turns out that this time the 3d bulk becomes a trivial state. The vortex loop decoration picture fails with a $Z_3$ symmetry. Suppose we decorate a Haldane phase on each $Z_3$ vortex loop, then three of the $Z_3$ vortex loops would be decorated with three Haldane phases, and due to the $Z_3$ classification of the 1d Haldane phase, three Haldane phases is still a nontrivial 1d SPT state. However, a three fold $Z_3$ vortex loop is no longer a well-defined defect any more. Thus the decorated vortex loop picture is incompatible with the $Z_3$ symmetry. Thus the bulk becomes a trivial state once we break the U(1) down to $Z_3$. This implies that the 2d boundary, which corresponds to the spin-1/2 system on the honeycomb lattice, permits a featureless spin state. This is consistent with the previous result on the honeycomb lattice [44].

We can also add other symmetries of the honeycomb lattice, such as reflection $P_x : y \rightarrow -y$. Under this reflection, $P_x : (n_1, n_2, n_3) \rightarrow -(n_1, n_2, n_3)$, while $(n_4, n_5)$ is unchanged. In the Euclidean space-time, a reflection symmetry can be treated equivalently as the time-reversal symmetry. Thus with both translation $T_x$ and reflection $P_x$, we need to study whether the 3d SPT state with SO(3) $\times$ $Z_3^T$ $\times$ U(1) symmetry is stable against symmetry breaking down to SO(3) $\times$ $Z_3^T$ $\times$ $Z_3$. The analysis is the same as before: the 3d SPT state with SO(3) $\times$ $Z_3^T$ $\times$ U(1) symmetry is constructed with proliferated vortex loops decorated with a 1d Haldane phase with SO(3) $\times$ $Z_3^T$ symmetry. However, this construction is still incompatible with the $Z_3$ vortex loops, because the classification of the Haldane phase with SO(3) $\times$ $Z_3^T$ symmetry is $Z_2 \times Z_2$.

B. SU(N) and SO(N) spin systems

Now let us consider a SU(N) spin system on the honeycomb lattice, again with FR on sublattice A, and AFR on sublattice B. This system can still form the three fold degenerate VBS states, and the vortex (antivortex) of the VBS order parameter has a vacant site in sublattice A (B), which carries a FR (AFR) of SU(N) (Fig. 3). Now we want to ask whether the 3d SPT state with PSU(N) $\times$ U(1) symmetry is stable against breaking the U(1) down to $Z_3$. This depends on whether the PSU(N) SPT state decorated on the vortex line is compatible with the $Z_3$ nature of the vortex line, i.e. N at least cannot be coprime with 3. Thus when N is coprime with 3, the 3d SPT state PSU(N) $\times$ U(1) symmetry is trivialized by breaking U(1) down to $Z_3$.

Just like the case in the previous section, the boundary of a 3d SPT with PSU(N) $\times$ U(1) symmetry could be a 2d $Z_N$ topological order, whose e particle carries 1/N charge of U(1), and m particle carries a FR of SU(N). Once U(1) is broken down to $Z_3$, if N is coprime with 3, by condensing a bound state of e and certain number of $Z_3$ charges, this 2d boundary $Z_N$ topological order is
driven into a featureless gapped state.

We can now make the following conclusion:

**SU(N) spin systems on the honeycomb lattice with fundamental and anti-fundamental representation on the two sublattices, permit a featureless gapped ground state when N is not a multiple of 3.**

Also, similar conclusions can be made for SO(N) spin systems:

- A SO(2n) spin system with vector representation on every site permits a featureless state on the honeycomb lattice.
- A SO(2n + 1) spin system with spinor or vector representation on every site also permits a featureless state on the honeycomb lattice.

## V. 3d SPIN SYSTEMS ON THE CUBIC LATTICE

A spin-1/2 system on the cubic lattice is subject to the generalized LSM theorem, thus it cannot have a featureless state. Besides the common Néel ordered state, another natural spin-1/2 state on the cubic lattice is the columnar VBS state. And the “hedgehog monopole” of the VBS order parameter, has a spin-1/2 state on the cubic lattice is subject to the generalized LSM theorem, thus it cannot have a featureless state. Besides the common Néel ordered state, another natural spin-1/2 state on the cubic lattice is the columnar VBS state. And the “hedgehog monopole” of the VBS order parameter, has a spin-1/2 state on the cubic lattice.

The picture above can again be generalized to the SU(N) spin system with FR and AFR on the two sublattices. Whether this spin system permits a featureless gapped state or not, is equivalent to whether the corresponding 4d bulk state is a trivial state or a SPT state. The CP$^{N-1}$ manifold, i.e. the SU(N) generalization of the Néel order parameter, has $\pi_2[\text{CP}^{N-1}] = \mathbb{Z}$, and hence also has a “hedgehog monopole” line in the 4d space. Thus we can again decorate the SO(3)$^m$ monopole line with the PSU(N) Haldane phase, and simultaneously decorate the PSU(N) monopole line with the SO(3)$^m$ Haldane phase. But now this 4d state is not always a nontrivial SPT state. Because the SO(3)$^m$ Haldane phase has a Z$_2$ classification, hence even-number copies of the 4d state must be a trivial state, while odd-number copies of the states is equivalent to the state itself. On the other hand, the PSU(N) Haldane phase has a Z$_N$ classification, namely N copies of the states must be trivial. Thus the 4d bulk state so constructed has a Z$_{2(N)}$ classification: the “mutual monopole line decoration” gives
us a nontrivial 4d SPT state only with even $N$.

The natural 3d boundary state of the 4d bulk based on the “mutual” monopole line decoration construction, is a U(1) photon phase whose $e$ excitations carry SU($N$) fundamental, and $m$ carries a spin-1/2 of SO(3). When $N$ is odd, we can drive the 3d boundary into a featureless state by condensing the dyon which is a bound state of $N$ e particles and two $m$ particles. We label this dyon as the $(N, 2)$ dyon. This $(N, 2)$ dyon is a boson, and its condensate will gap out the photons, while confining all the point particles, because there is no point particle that is mutual bosonic with this dyon, except for the dyon itself. Also, the $(N, 2)$ dyon could be a singlet of SU($N$), and singlet of SO(3), thus its condensate does not break any global symmetry. This means that for odd integer $N$, the 3d boundary of the 4d bulk state can be driven into a featureless gapped state, which again demonstrates that the 4d bulk state constructed above is trivial when $N$ is odd.

By contrast, if $N$ is even, then the $(N/2, 1)$ dyon (with nontrivial representation of SU($N$) and SO(3)) is still deconfined in the condensate of $(N, 2)$ dyon, and this condensate has topological order.

Now we can conclude that:

For odd $N$, the SU($N$) spin system on the cubic lattice with FR and AFR spins on two sublattices permits a featureless monopole line construction. We first define a U(2$N$) matrix field $U$ as

$$U = \cos(\theta) P \otimes I_{2 \times 2} + i \sin(\theta) I_{N \times N} \otimes \vec{n} \cdot \vec{r},$$

where $P$ is the CP$^{N-1}$ matrix field given by Eq. 6. The “mutual decoration” picture is captured by a topological term in the nonlinear sigma model of $U$ which reads

$$L_{5d}^{\text{topo}} = \int d^4x d\tau \frac{2\pi}{480\pi^3} \text{Tr} \left[ (U^\dagger dU)^5 \right].$$  \hspace{1cm} (9)

We will show that if we manually create a monopole line of $\vec{n}$, the topological term Eq. 9 precisely reduces to the topological term of the (1 + 1)d SU($N$) SPT. Let us parametrize the $(4 + 1)d$ space-time by Cartesian coordinates $(x, y, z, w, \tau)$ and consider a static monopole line of $\vec{n}$ whose core line lies on the $w$-axis. For any fixed $w$ and $\tau$, we will see a monopole configuration of $\vec{n}$ centered at origin in the $xyz$ space. For a monopole configuration in the $xyz$ space, we have

$$\theta(r = 0) = 0,$$

$$\theta(r \to \infty) = \pi/2$$

$$\int_{r = r_0 > 0} d^2\Omega \frac{1}{8\pi} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} n^i \partial_\alpha n^j \partial_\beta n^k = 1$$

where $r = \sqrt{x^2 + y^2 + z^2}$. We also assume $P$ is a function of $w$ and $\tau$. Now we plug in this configuration of $\vec{n}$ in to Eq. 9 and integrate over $x, y$ and $z$ directions. This topological term reduces to the following $(1 + 1)d$ topological term in the $(w, \tau)$ space:

$$L_{2d}^{\text{topo}} = \int dw d\tau \frac{2\pi}{16\pi} \epsilon_{\mu\nu} \text{Tr} \left( P \partial_\mu P \partial_\nu P \right),$$

which is precisely the topological $\Theta$-term for the PSU($N$) Haldane phase. This indicates that Eq. 9 implies there is a PSU($N$) SPT decorated on the monopole line of $\vec{n}$.

If we consider a monopole line of $P$ along $w$-axis, then in the $xyz$ directions we have

$$\theta(r = 0) = \pi/2,$$

$$\theta(r \to \infty) = 0$$

$$\int_{r = r_0 > 0} d^2\Omega \frac{1}{16\pi} \epsilon_{\mu\nu} \text{Tr} \left( P \partial_\mu P \partial_\nu P \right) = 1$$

Now integrating over $x, y$ and $z$ directions will give us the following topological term in the (1 + 1)d space-time of the monopole line world sheet:

$$L_{2d}^{\text{topo}} = \int dw d\tau \frac{2\pi i}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu\gamma} n^a \partial_\mu n^b \partial_\gamma n^c,$$

which exactly corresponds to the topological term of the (1 + 1)d SO(3) Haldane phase. Therefore, the topological term in Eq. 9 captures the “mutual decoration” construction of the $(4 + 1)d$ SPT phase with PSU($N$)$\times$SO(3) symmetry.

VI. FURTHER PROOF OF OUR CONCLUSIONS

A. Explicit construction of featureless spin states

Let us first restate our main conclusions about SU($N$) spin systems on the square, honeycomb, and cubic lattices:

1. A SU($N$) spin system on the square lattice with fundamental (FR) and anti-fundamental representation (AFR) on the two different sublattices respectively, permits a featureless gapped ground state when $N$ is an odd integer;

2. A SU($N$) spin system on the honeycomb lattice with FR and AFR on two different sublattices, permits a featureless gapped ground state when $N$ is coprime with 3.

3. A SU($N$) spin system on the cubic lattice with FR and AFR spins on two different sublattices, permits a featureless spin state when $N$ is odd.

For all the spin systems listed above, we can construct explicit featureless tensor product spin states similar to the AKLT states. All these states will be discussed in a future work. Here we discuss some of the examples of this construction.
On the honeycomb lattice, in the case of \( N = 3k + 1 \), we introduce \( 3k \) auxiliary spins on each site. We also introduce a tensor on each site:

\[
T_{i_1i_2...i_{3k}}^{\alpha} = \varepsilon_{\alpha i_1i_2...i_{3k}},
\]

where \( \varepsilon_{\alpha i_1i_2...i_{3k}} \) is the total anti-symmetric tensor with \( N = 3k + 1 \) indices. Here, the \( i_1, i_2, \cdots i_{3k} \) labels the \( 3k \) auxiliary FR (or AFR) spin degrees of freedom on each site in sublattice B (or A) before the projection. Each label \( i \) takes value in \( 1, 2, \cdots N \) representing the \( N \) states in each FR (or AFR). The label \( \alpha \), which also takes value \( 1, 2, \cdots N \), represents the physical states in AFR (or FR) spin degrees of freedom on each site in sublattice B (or A). Physically, on each site of sublattice A, the tensor in Eq. 14 projects the \( 3k \) auxiliary AFR spins into a totally anti-symmetric channel which, due to the nature of SU(\( 3k + 1 \)), becomes the physical FR spin. The analysis for sites in the sublattice B is similar. Now we can use the auxiliary spins to construct a featureless gapped state on the honeycomb lattice with \( k \) SU(N) singlet bonds along each link of the lattice, which is reminiscent of the AKLT state.

Obviously, the so constructed tensor product state respects the translation symmetry of the lattice. Now we analyze the compatibility between the point group \( C_{3v} \) and the site tensor in Eq. 14. Here, notice that we include not only the \( C_3 \) rotation symmetry but also the mirror reflection symmetry of the honeycomb lattice into consideration. We notice that the point group only induces a permutation of the singlet bonds before the projection. Therefore, the action of the point group permutes the \( 3k \) spins on each site. Since we project the \( 3k \) spins into a totally anti-symmetric channel using the site tensor, the point group induced permutation keeps the site tensor invariant up to a global sign which is unimportant for the global tensor network wave function. Therefore, we can conclude that the choice of projection tensor in Eq. 14 preserves the space symmetries.

On the square lattice, in the case of \( N = 4k + 1 \), we introduce \( 4k \) auxiliary spins on each site and let the auxiliary spins form a state with \( k \) SU(N) singlet bonds along each link of the square lattice. We can choose the site tensors to be

\[
T_{i_1i_2...i_{4k}}^{\alpha} = \varepsilon_{\alpha i_1i_2...i_{4k}},
\]

where \( \varepsilon_{\alpha i_1i_2...i_{4k}} \) is the total anti-symmetric tensor with \( N = 4k + 1 \) indices. Based on analysis completely in parallel with the honeycomb lattice, we conclude that the physical spin carries AFR (FR) under SU(N) if the auxiliary spins transform as FR (AFR). Also, we can conclude that the tensors in Eq. 15 are invariant under the \( C_{4v} \) point group action up to an unimportant sign because the actions of the \( C_{4v} \) point group on the site tensor are only permutation of the tensor indices. Now we can use the \( 4k \) auxiliary spins on each site to construct a featureless spin state on the square lattice.

On the cubic lattice, for any odd integer \( N \) that is not 3, 5 or 11, we can write \( N \) as \( N = 8p + 6q + 1 \) with \( p \) and \( q \) non-negative integers. Again we introduce \( N-1 \) auxiliary spins, and an on-site tensor \( T_{i_1i_2...i_{N-1}}^{\alpha} = \varepsilon_{\alpha i_1i_2...i_{N-1}} \). Namely on sublattice B, we represent the AFR with \( N-1 \) auxiliary FRs, and on sublattice A we represent the FR with \( N-1 \) AFRs. Now these auxiliary spins can form a featureless states with valence bonds extended either along the link (for \( N = 6q + 1 \)) or the diagonal directions (for \( N = 6p + 1 \)), or both directions (when \( p \) and \( q \) are both nonzero) on the cubic lattice (Fig. 5). The point group \( O_h \) of the cubic lattice will induce a permutation among the \( N-1 \) auxiliary spins on each site which at most leads to an unimportant sign change of the site tensor. Therefore, this site tensor is compatible with the point group \( O_h \) symmetry. In fact, the \( O_h \) point group is isomorphic to \( S_4 \times Z_2 \). The \( Z_2 \) part is the spatial inversion which takes the point \((x, y, z)\) to \((-x, -y, -z)\). \( S_4 \) is the permutation group of 4 elements, which can be generated by a \( Z_3 \) cyclic permutation and a \( Z_4 \) cyclic permutation. In the language of the point group, the \( S_4 \) part is the part of \( O_h \) that preserves the spatial orientation. It can be generated by a \( C_3 \) rotation about the \((1, 1, 1)\)-axis and a \( C_4 \) rotation about the \(z\)-axis. This \( S_4 \) part alone (without the spatial inversion) is usually referred to the point group \( O \).

The construction of these featureless tensor product wave functions does provide strong evidence to our conclusions in previous sections. Nevertheless, we need to comment that, to eventually confirm the featureless-ness of these tensor product wave functions, numerical simulation of these states is demanded, in order to rule out possible spontaneous symmetry breaking, etc. For instance, it is known that the AKLT wave function on a three dimensional lattice could have long range spin order.

**B. Connection to “lattice homotopy class”**

In fact, we can also simplify all the discussions by just considering a \( Z_N \times Z_N \) subgroup of PSU(N) and analyzing how the FR and AFR of SU(N) transform under this...
we first consider two SU($Z_N \times Z_N$ subgroup. To specify this $Z_N \times Z_N$ subgroup, we first consider two SU($N$) matrices in the FR:

$$g_1 = e^{i\pi(N-1)} \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & \ddots & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & 0 \\
0 & \vdots & \ldots & 0 & 1 & 0 \\
1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}$$

$$g_2 = e^{i\pi(N-1)} \begin{pmatrix}
\begin{smallmatrix}
ie_{\frac{2\pi}{N}} & & & \\
& \begin{smallmatrix}i\pi
\end{smallmatrix} & & \\
& & \ddots & \\
& & & \begin{smallmatrix}i\pi
\end{smallmatrix}
\end{smallmatrix}
\end{pmatrix}
\begin{smallmatrix}
ie_{\frac{2\pi}{N}} \\
\vdots \\
1
\end{smallmatrix}$$

where $g_1$ only has non-zero entries on a subdiagonal and the bottom left corner, and $g_2$ is a diagonal matrix. It is straightforward to check that

$$g_1^N = g_2^N = 1_{N \times N}, \quad g_1 g_2 = e^{-i\pi/N} g_2 g_1.$$  \hspace{1cm} (17)

We denote the elements of PSU($N$) corresponding to $g_1$ and $g_2$ as $\tilde{g}_1$ and $\tilde{g}_2$. Obviously, $\tilde{g}_1, \tilde{g}_2$ are elements of order $N$. Since the phase factor $e^{-i\pi/N}$ in the commutation relation between $g_1$ and $g_2$ is one of the center elements in SU($N$), $\tilde{g}_1$ and $\tilde{g}_2$ should commute in PSU($N$). Therefore, $\tilde{g}_1$ and $\tilde{g}_2$ generate a $Z_N \times Z_N$ subgroup of PSU($N$). We will focus on this subgroup in the following. Notice that, a physical FR spin, which transforms according to $g_{1,2}$ under this $Z_N \times Z_N$ subgroup of PSU($N$), can be viewed as a projective representation of $Z_N \times Z_N$. In the classification of the projective representation $H^2(Z_N \times Z_N, U(1)) = Z_N$, the FR spins actually correspond to the generating element in $H^2(Z_N \times Z_N, U(1))$. The AFR spins then correspond to the conjugate of the FR spins in terms of projective representations of $Z_N \times Z_N$.

When we restrict to the global internal symmetry $Z_N \times Z_N$ (which is a subgroup of PSU($N$)), we can apply the lattice homotopy classification introduced in Ref.\[19]\ It was proven for 1$d$ and 2$d$, partially proven for 3$d$, and conjectured for general dimensions that the generalized Lieb-Schultz-Mattis (LSM) theorems will forbid the existence of any featureless states on lattices of “non-trivial lattice homotopy class”. In fact, the lattice homotopy classification proposed in Ref.\[19]\ also covers the cases with continuous internal symmetry group. However, the proof of the relations between non-trivial lattice homotopy classes and the existence of generalized LSM theorems is less comprehensive for the most general continuous symmetry group than for the general Abelian finite group. Therefore, we will focus on the lattice homotopy classification with Abelian finite group in this section.

For a lattice with $n$ FR spins on each site of the sublattice $A$ and $n$ AFR spins on each site of the sublattice $B$, we will refer to it as the $(n, n)$-lattice. The fundamental-anti-fundamental lattics can then also be referred to as the $(1, 1)$-lattice. In addition to the global internal symmetry, the lattice homotopy classification depends on the choice of space group symmetry. Let’s specify the minimal space group symmetry for the $(1, 1)$-honeycomb, $(1, 1)$-square and $(1, 1)$-cubic lattices we want to consider. For the $(1, 1)$-honeycomb lattice, we want to at least include the $C_3$ spatial rotation symmetry into consideration. Therefore, the minimal choice of space group is the wallpaper group $p3$ (No. 13). For the $(1, 1)$-square lattice, we want to at least consider the $C_4$ spatial rotation symmetry. Therefore, the minimal choice of space group is the wallpaper group $p4$ (No. 10). For the $(1, 1)$-cubic lattice, we want to at least consider the symmetry of the point group $O$. Therefore, the minimal choice of the 3D space group is $F432$ (No. 209). The wallpaper group and 3D space group numbers can be found in Ref.\[49]\ With the global $Z_N \times Z_N$ internal symmetry and the minimal space groups symmetry given above, the $(1, 1)$-honeycomb lattice belongs to a non-trivial lattice homotopy class when $N$ is a multiple of 3. Similarly, $(1, 1)$-square and $(1, 1)$-cubic lattices are also non-trivial when $N$ is even. Therefore, according to Ref.\[19]\ there are generalized LSM theorems obstructing any featureless state compatible with the global and space group symmetries on these lattices. Of course, when we enlarge the $Z_N \times Z_N$ symmetry back to PSU($N$), such obstructions still exist.

Hence the analysis of lattice homotopy class also indicates that there is no featureless state with PSU($N$) global symmetry on the $(1, 1)$-honeycomb lattice with $N$ being a multiple of 3, or on $(1, 1)$-square or cubic lattices with even integer $N$. These conclusions are completely consistent with those obtained from the analysis in the previous sections.

One can perform a similar lattice homotopy analysis for SO($2N$) spin systems with spins carrying the vector representation per site and with the space group $p4$. When $N = 4k$, we construct the SO($4k$) matrices

$$g_1 = i\sigma^y \otimes I_{2k \times 2k}, \quad g_2 = \sigma^z \otimes I_{2k \times 2k},$$

and notice that

$$g_1^2 = -1, \quad g_2^2 = 1, \quad g_1 g_2 = -g_2 g_1.$$  \hspace{1cm} (19)

We denote the elements of PSO($4k$) that correspond to $g_1$ and $g_2$ as $\tilde{g}_1$ and $\tilde{g}_2$. Since $-I_{4k \times 4k}$ is a non-trivial center element of SO($4k$), the elements $\tilde{g}_1, \tilde{g}_2$ generate a $Z_{2k} \times Z_{2k}$ subgroup of PSO($4k$). The vector representation, which transforms according to $g_{1,2}$ under this $Z_{2k} \times Z_{2k}$ subgroup, can be viewed as a non-trivial projective representation of $Z_{2k} \times Z_{2k}$. If we restrict our attention to this $Z_{2k} \times Z_{2k}$ subgroup of PSO($4k$), we notice that a square lattice with a SO($4k$) spin in the vector representation per site and with the space group $p4$ belongs to a non-trivial lattice homotopy class.
When \( N = 4k + 2 \), we construct the SO(4k+2) matrices

\[
g_1 = \begin{pmatrix}
\sigma^z & i\sigma^y \\
-i\sigma^y & I_{2(k-1) \times 2(k-1)}
\end{pmatrix},
\]
\[
g_2 = \begin{pmatrix}
\sigma^z & i\sigma^y \\
i\sigma^y & \sigma^x
\end{pmatrix}
\]

which satisfy

\[
g_1^4 = g_2^4 = 1, \quad g_1g_2 = -g_2g_1. \tag{21}
\]

By similar reasoning in the SO(4k) case, we find that the vector presentation of SO(4k+2) can be viewed as a non-trivial projective representation of a \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) subgroup in PSO(4k + 2). In fact, the classification of projective representation of \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) is given by \( H^2(\mathbb{Z}_4 \times \mathbb{Z}_4, U(1)) = \mathbb{Z}_4 \), in which the vector representation belongs to the “second” non-trivial class. When we consider the space group \( p4 \) and the \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) subgroup of PSO(4k + 2) given above, we notice that the square lattice with a spin in the vector representation on each site also belongs to a non-trivial lattice homotopy class, just like that case of SO(4k).

Hence, we can conclude that \textit{A SO(2N) spin system with vector representation on every site does not permit a featureless gapped state on the square lattice.} This result completely agrees with the analysis in the previous sections.

Lastly, we consider SO(2N + 1) spin systems with spinor representations. SO(2N + 1) is the group of rotations in \( \mathbb{R}^{2N+1} \). Let \( x_1, \ldots, x_{2N+1} \) denote the \( 2N + 1 \) axes of \( \mathbb{R}^{2N+1} \). We’d like to focus on a \( Z_2 \times Z_2 \) subgroup of SO(2N + 1) generated by the \( \pi \)-rotation in the \( x_1-x_2 \) plane and the \( \pi \)-rotation in the \( x_1-x_3 \) plane. The spinor representation of SO(2N + 1) can be viewed as a non-trivial projective representation of this \( Z_2 \times Z_2 \) subgroup. When we consider the space group \( p4 \) and the \( Z_2 \times Z_2 \) subgroup of SO(2N + 1) given above, we notice that the square lattice with a spin in the spinor representation on each site belongs to a non-trivial lattice homotopy class. Therefore, \textit{a SO(2N + 1) spin system with spinor representation on every site does not permit a featureless gapped state on the square lattice.} Again, this statement is consistent with the analysis given in the previous sections.

\section{Summary}

In this work we made connection between two seemingly different subjects: the (generalized) Lieb-Shultz-Matthis theorem for a \( d \)-dimensional quantum spin systems, and the boundary of \( (d + 1) \)-dimensional symmetry protected topological states with on-site symmetries. This connection has led to fruitful results: we identified a series of quantum spin systems that permit a featureless spin state, as well as spin systems with a generalized LSM theorem \textit{i.e.} spin systems that do not permit a featureless spin state. The former cases correspond to trivial bulk states, while the latter correspond to nontrivial SPT states in one higher spatial dimensions. We have also tested and verified our conclusions by other methods. For example we explicitly constructed featureless tensor product spin states of those systems whose corresponding \( (d + 1) \)-dimensional bulk are trivial states (most of this construction will be presented in an upcoming paper). We expect the main logic and method used in this paper can be generalized to other related problems. For example, one can study SU(N) spin systems with more general representations.

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50 The authors thank Dung-Hai Lee for clarifying this important point for us.
51 The octahedral group O does not include the spatial mirror (reflection) symmetry. The mirror symmetry is equivalent to time-reversal symmetry in the analysis of SPT states, as we explained previously. Including the mirror symmetry does not change our conclusions, because the SO(3) Haldane phase with or without an extra time-reversal symmetry always has a $Z_2$ nature, i.e. two of these Haldane phases coupled together becomes a trivial state.