HIGH FREQUENCY PERTURBATION OF CNOIDAL WAVES IN KdV

M. B. ERDOĞAN†, N. TZIRAKIS†, AND V. ZHARNITSKY†

Abstract. The KdV equation with periodic boundary conditions is considered. The interaction of a periodic solitary wave (cnoidal wave) with high frequency radiation of finite energy ($L^2$ norm) is studied. It is proved that the interaction of a low frequency component (cnoidal wave) and high frequency radiation is weak for finite time in the following sense: the radiation approximately satisfies the Airy equation.

Key words. Korteweg–de Vries, KdV cnoidal waves, high frequency

AMS subject classifications. 35Q53, 37K45

DOI. 10.1137/120868220

1. Introduction. The KdV equation

$$q_t + q_{xxx} + q^2 q = 0$$

is one of the most basic dispersive partial differential equations (PDEs) with solitary wave solutions. There are two types of solitary waves in KdV posed on the real line: exponentially decaying and spatially periodic waves. This paper deals exclusively with the periodic case, and even more restrictively, we consider KdV with periodic boundary conditions. The periodic traveling waves in KdV (already known to Korteweg and de Vries) are called cnoidal waves, as they may be expressed in terms of the elliptic Jacobi function (see, e.g., [16])

$$(1) \quad \phi_c(z) = \beta_2 + (\beta_3 - \beta_2) \csc^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{12}} z; k \right),$$

where

$$z = x - ct, \beta_1 < \beta_2 < \beta_3, \beta_1 + \beta_2 + \beta_3 = 3c, \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}.$$
We, on the other hand, are interested in the behavior of solutions in the case of high frequency perturbation of finite energy. We consider the evolution of solitary wave and the high frequency perturbation and prove that the perturbation evolves almost linearly. Within the context of our previous results in [11, 12], where we proved that the evolution of high frequency solutions of KdV is near-linear, our result in this paper can be considered as a superposition principle for a nonlinear dispersive PDE. It has been long suggested in the physics literature that in the regime below collapse, high frequency solutions evolve almost linearly and interact little with the low frequencies. This mechanism has also been used, perhaps implicitly, to prove low regularity results; see, e.g., [5, 7, 8]. In a way, our proof of nearly independent evolution of the cnoidal wave and high frequency radiation provides some support of this heuristics; see, e.g., [17, p. 118]. This nearly independent evolution is due to a subtle averaging effect in the nonlinear dispersive dynamics, making these results possible. Recently KdV was studied with respect to this averaging effect. In [11, 12], nearly linear dynamics was established for high frequency initial data and in [1] a new elegant proof of well-posedness in \( H^s \), \( s \geq 0 \), was found using explicitly high frequency averaging effects. The approach had been originated in [2].

In contrast to the Lyapunov problem, where infinite time stability is usually established, our result is valid only on finite times, which is related to adiabatic invariance phenomena: high frequency wave oscillations are averaged out to produce effective slow evolution. Since KdV as a model is valid only on a finite time scale, it is meaningful to consider the dynamics of the solutions for finite times. Given that even the classical adiabatic invariance theorem (conservation of action of the pendulum with slowly changing frequency) requires careful analysis, our task becomes even harder because of the infinite dimensionality.

Now we state our main theorem.

**Theorem 1.1.** Let \( \phi(x - ct) \) be a 2\( \pi \)-periodic cnoidal wave solution of KdV. Fix \( s \in (0, 1/2) \). Consider the real valued solution of KdV on \( T \times \mathbb{R} \) with the initial data \( q(x,0) = \phi(x) + g(x) \) satisfying, for some \( 0 < s < 1/2 \),

\[
\|g\|_2 \leq C_0, \quad \|g\|_{H^{-s}} = \varepsilon \ll 1.
\]

Then, for each \( t > 0 \), we have

\[
\|q(x,t) - \phi(x - ct) - (e^{Lt}g)(x)\|_2 \leq Ce^{C_1 t} \varepsilon,
\]

where \( L = -\partial_x^3 - \langle \phi \rangle \partial_x \), and \( C \) depends only on \( s \) and \( \phi \).

As usual, \( H^{-s} \) is the completion of \( L^2 \) under the norm \( \|u\|_{H^{-s}} = \|\hat{u}(k)/(1 + |k|^2)^{s/2}\|_{\ell^2} \) and we use the notation \( \langle \phi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi(r) dr \).

**Remark 1.2.** The theorem is proved by first deriving the equation for the perturbation \( q(x,t) - \phi(x - ct) \) on the Fourier side. The obtained equation is of the form

\[
\partial_t v_k = -\frac{ik}{2} \sum_{k_1 + k_2 = k} e^{-i\delta k_1 k_2 t} (v_{k_1} + 2S_k)v_{k_2}.
\]

Next we prove that for time of order 1, the perturbation \( v \) evolves little \( \|v(t) - v(0)\|_{L^2} \leq C \varepsilon \). One should compare our statement to the averaging theorem for ODEs: the solutions of the equations

\[
\dot{v} = F(v, t/\varepsilon) \quad \text{and} \quad \dot{w} = \langle F \rangle(w)
\]

are \( \varepsilon \) close for the time of order 1 with \( F \) periodic in the second argument.
The fast oscillation $t/\epsilon$ corresponds to the large frequencies in the exponent $e^{-i3k_1k_2t}$. Thus, one can hope to get the accuracy of order $\epsilon$ if the bulk of the energy is in the frequencies of order $1/\epsilon$ and higher.

**Remark 1.3.** The averaging argument applied to KdV can be extended to other equations, including nonintegrable ones. For example, our method applies to the equation

$$u_t + u_{xxxx} + uu_x = 0,$$

which is believed to be nonintegrable. In fact, higher dispersion makes the averaging effect even stronger than in KdV.

**Remark 1.4.** The statement of the theorem above can be extended to an arbitrary $H^4$ solution $\rho$ of KdV in the following sense. Given $T$, there is a constant $C = C(s, T, \rho)$ such that given $\epsilon > 0$ and $g$ as in the theorem we have

$$\|q(x, t) - \rho(x, t) - (e^{tL}g)(x)\|_2 \leq C\epsilon, \quad t \in [0, T],$$

where $L = -\partial_x^3 - (\rho)\partial_x$.

This variation follows from the proof of Theorem 1.1 by utilizing Remarks 2.8 and 4.1.

As is well-known, KdV is a completely integrable system with infinitely many conserved quantities. However, our methods in this paper do not rely on the integrability structure of KdV, and thus they can be applied to other dispersive models. On the other hand, we use the fact that the smooth solutions of KdV satisfy momentum conservation,

$$\int_{-\pi}^{\pi} u(x, t)dx = \int_{-\pi}^{\pi} u(x, 0)dx,$$

and the conservation of energy,

$$\int_{-\pi}^{\pi} u^2(x, t)dx = \int_{-\pi}^{\pi} u^2(x, 0)dx.$$

The KdV equation is locally well-posed in $L^2(\mathbb{T})$ [6]. Due to energy conservation it is globally well-posed and the solution is in $C(\mathbb{R}; L^2(\mathbb{T}))$. Kenig, Ponce, and Vega [14] improved Bourgain’s result and showed that the solution of the KdV is locally well-posed in $H^s(\mathbb{T})$ for any $s > -\frac{1}{2}$. Later, Colliander et al. [7] showed that the KdV is globally well-posed in $H^s(\mathbb{T})$ for any $s \geq -\frac{1}{2}$, thus adding a local well-posedness result for the endpoint $s = -\frac{1}{2}$. Recently Kappeler and Topalov [13] extended the latter result and prove that the KdV is globally well-posed in $H^s(\mathbb{T})$ for any $s \geq -1$. Since our statements concern $L^2$ functions, from the results listed above, we only use the global well-posedness in $L^2$ [6].

**1.1. Solitary waves description.** Without loss of generality we restrict ourselves to the case of periodic waves with prescribed period $2\pi$. The proof can be readily extended to arbitrary period.

To make the presentation self-consistent, we directly show that there are such waves, which can also be found by using properties of Jacobi functions (1). Let $q = f(x - ct)$, and substitute this in KdV,

$$cf' = ff' + f'''.$$
Integrating once, we obtain
\[ a + cf = \frac{f^2}{2} + f'', \]
which can be written in the potential form
\[ f'' + W_f(f) = 0, \]
where
\[ W(f) = \frac{f^3}{6} - \frac{c^2 f^2}{2} - af. \]
Under the assumption \( c^2 + 2a > 0 \), this cubic polynomial has one local maximum and one local minimum:
\[ f_- = -\sqrt{c^2 + 2a} + c, \quad f_+ = \sqrt{c^2 + 2a} + c. \]
Taking the second derivative of \( W \) at \( f_+ \),
\[ W_{ff}(f_+) = f_+ - c = \sqrt{c^2 + 2a}, \]
we obtain the period of small oscillations:
\[ T_0(a, c) = \frac{2\pi}{\sqrt{W_{ff}(f_+)}} = \frac{2\pi}{\sqrt{c^2 + 2a}}. \]
Therefore, moving through the family of periodic solutions nested between the minimum and the separatrix, we will see the period assuming all intermediate values between \( T_0(a, c) \) and \( \infty \). Thus, if \( T_0(a, c) < 1 \leftrightarrow W'(f_+) < 1 \) (which can be achieved by taking \( a \) or \( c \) sufficiently large), by continuity, somewhere between the critical point and the separatrix, there will be a \( 2\pi \)-periodic solution. In particular, this can be done by setting \( c = 0 \) and taking \( a \) sufficiently large.

We conclude by noting the well-known fact that the cnoidal wave \( \phi \) is real analytic with exponentially decaying Fourier coefficients.

1.2. Notation. To avoid the use of multiple constants, we write \( A \lesssim B \) to denote that there is an absolute constant \( C \) such that \( A \leq CB \). We also write \( A \sim B \) to denote both \( A \lesssim B \) and \( B \lesssim A \).

We define the Fourier sequence of a \( 2\pi \)-periodic \( L^2 \) function \( u \) as
\[ u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx, \quad k \in \mathbb{Z}. \]
With this normalization we have
\[ u(x) = \sum_k e^{ikx} u_k, \quad \text{and} \quad (uv)_k = u_k * v_k = \sum_{m+n=k} u_n v_m. \]

2. Proof of the main theorem. First we discuss that it suffices to prove the theorem for time-independent cnoidal waves. Consider KdV with periodic boundary conditions (on the circle) \( q(x + 2\pi) = q(x) \). Note that if \( q(x, t) \) is a solution, then \( q(x + ct, t) + c \) is the solution with initial data \( q(x, 0) + c \). In particular, for a cnoidal
wave \( \phi(x - ct) \), the function \( \phi(x) + c \) is also a cnoidal wave. Applying the statement of the theorem with initial data \( \phi(x) + c + g(x) \), we obtain at time \( t \) that the solution is of the form
\[
q(x + ct, t) + c = \phi(x) + c + (e^{L_c t} g)(x) + O_{L^2}(\varepsilon e^{Ct}),
\]
where \( L_c = -\partial_x^2 - \left( \frac{1}{2} \int_0^{2\pi} (r \phi) dr + c \right) \partial_x \). Noting that \( (e^{L_c t} g)(x) = (e^{Lt} g)(x + ct) \), we obtain
\[
q(x, t) = \phi(x - ct) + (e^{Lt} g)(x) + O_{L^2}(\varepsilon e^{Ct}).
\]
Thus, it suffices to prove the theorem for the stationary cnoidal wave.

Let \( \phi(x) \) be a time-independent, \( 2\pi \)-periodic cnoidal wave. Consider a solution of KdV of the form \( q(x, t) = \phi(x) + u(x, t) \). Substituting in KdV, we obtain
\[
(3) \quad u_t + u_{xxx} + \langle \phi \rangle u_x + (\Phi u)_x + uu_x = 0, \quad u(x, 0) = g(x),
\]
where \( \langle \phi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi(x) dx \) and \( \Phi = \phi - \langle \phi \rangle \).

**Remark 2.1.** Our assumption on the \( H^{-s} \) norm of the initial data \( g \) and the momentum conservation imply that \( \langle u(x, t) \rangle = \tilde{g}(0) = \mathcal{O}(\varepsilon) \). In the proof of our theorem we will restrict ourselves to the case when \( \langle u(x, t) \rangle = 0 \). This makes the proof more presentable. Removing this assumption introduces more terms in the differentiation by parts formulas which are smaller than the ones we have. In particular, in Theorem 2.3 below, the formulas for \( B(v) \) and \( \mathcal{R}(v) \) would have additional terms which satisfy the a priori estimates given in Proposition 2.4.

**Remark 2.2.** Note that for a mean-zero \( L^2 \) function \( u \), \( \|u\|_{H^{-s}} \sim \|u_k/|k|^s\|_{L^2} \), we will use this formula without further comments. For a sequence \( u_k \), with \( u_0 = 0 \), we will use \( \|u\|_{H^{-s}} \) notation to denote \( \|u_k/|k|^s\|_{L^2} \).

Using the notation
\[
u(x, t) = \sum_k u_k(t)e^{ikx} \quad \text{and} \quad \Phi(x, t) = \sum_k \Phi_k(t)e^{ikx},
\]
we write (3) on the Fourier side,
\[
\partial_t u_k = -\frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2} - ik \sum_{k_1+k_2=k} \Phi_{k_1} u_{k_2} + i(k^3 - ak)u_k, \quad u_k(0) = \tilde{g}(k),
\]
where \( a = \langle \phi \rangle \). Because of the mean-zero assumption on \( \Phi \) and \( u \), and conservation of momentum, there are no zero harmonics in this equation. Without the mean-zero assumption this equation would have an additional term of the form \( iu_0 k \Phi_k \) which is of order \( \varepsilon \) and has fast decay in \( k \).

Using the transformations
\[
u_k(t) = \nu_k(t)e^{i(k^3 - ak)t},
\]
\[
\Phi_k(t) = S_k(t)e^{i(k^3 - ak)t}
\]
and the identity
\[
(k_1 + k_2)^3 - k_1^3 - k_2^3 - (k_1^3 - k_2^3) - a(k_1 + k_2) = ak_1 + ak_2 = 3(k_1 + k_2)k_1 k_2,
\]
the equation can be written in the form

\[ \partial_t v_k = -\frac{i k}{2} \sum_{k_1 + k_2 = k} e^{-i k k_1 k_2 t} (v_{k_1} + 2S_{k_1})v_{k_2}. \]

The following theorem will be proved in section 3 by distinguishing the resonant and nonresonant sets and using differentiation by parts.

**Theorem 2.3.** The system (4) can be written in the form

\[ \partial_t[v + K(v) + B(v)]_k = L_0(v)_k + R(v)_k, \]

where we define \( K(v)_0 = B(v)_0 = L_0(v)_0 + R(v)_0 = 0 \), and for \( k \neq 0 \) we define

\[ K(v)_k = -\sum_{k_1 + k_2 = k} \frac{e^{-i k k_1 k_2 t} S_{k_1} v_{k_2}}{3k_1 k_2}, \]

\[ B(v)_k = -\sum_{k_1 + k_2 = k} \frac{e^{-i k k_1 k_2 t} v_{k_1} v_{k_2}}{6k_1 k_2} - \frac{1}{18} \sum_{k_1 + k_2 + k_3 = k}^* \frac{e^{-i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1 (k_1 + k_2)(k_2 + k_3)(k_3 + k_1)} (v_{k_1} + S_{k_1})v_{k_2}v_{k_3}, \]

\[ L_0(v)_k = \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} S_{k_1} S_{k_2} v_{k_3}, \]

\[ R(v)_k = -\frac{i}{6} \frac{v_k |v_k|^2}{k} - \frac{i}{6} S_{-k} v_k v_k + \frac{i}{3} \frac{v_k}{k} \sum_{j \neq k} \frac{S_j v_{-j}}{j} - \frac{2i}{3} \frac{S_k}{k} \sum_j S_{-j} v_j \]

\[ + \sum_{k_1 + k_2 = k} \frac{e^{-i k k_1 k_2 t} (\partial_t S_{k_1})v_{k_2}}{3k_1 k_2} \]

\[ + \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k \neq 0} \frac{e^{-i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1 (k_1 + k_2)(k_2 + k_3)(k_3 + k_1)} v_{k_1} S_{k_2} v_{k_3} \]

\[ + \frac{1}{18} \sum_{k_1 + k_2 + k_3 = k}^* \frac{e^{-i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1 (k_1 + k_2)(k_2 + k_3)(k_3 + k_1)} v_{k_1} v_{k_2} v_{k_3} \]

\[ - \frac{i}{18} \sum_{k_1 + k_2 + k_3 + k_4 = k}^* \frac{e^{i t(k_1 + k_2 + k_3 + k_4)}}{k_1 (k_1 + k_2)(k_2 + k_3 + k_4)(k_2 + k_3 + k_4)} v_{k_1} v_{k_2} v_{k_3} v_{k_4} \]

\[ + \frac{i}{18} \sum_{k_1 + k_2 + k_3 + k_4 = k}^* \frac{e^{i t(k_1 + k_2 + k_3 + k_4)}}{k_1 (k_1 + k_2)(k_2 + k_3 + k_4)(k_2 + k_3 + k_4)} v_{k_1} v_{k_2} v_{k_3} v_{k_4} \]

Here \( \sum^* \) means the sum does not contain the terms which makes the denominator zero.

**Proposition 2.4.** Assume that \( \|v\|_2 \lesssim 1 \) and \( 0 < s < 1/2 \); then

\[ \|K(v)\|_{H^{-s}} \lesssim \|K(v)\|_2 \lesssim \|v\|_{H^{-s}}, \]

\[ \|B(v)\|_{H^{-s}} \lesssim \|B(v)\|_2 \lesssim \|v\|_{H^{-s}}, \]

\[ \|L_0(v)\|_{H^{-s}} \lesssim \|v\|_{H^{-s}}, \quad \|L_0(v)\|_2 \lesssim \|v\|_2, \]

\[ \|R(v)\|_{H^{-s}} \lesssim \|R(v)\|_2 \lesssim \|v\|_{H^{-s}}. \]
We will prove this proposition in section 4. Now, we continue with the proof of the main theorem. First we will prove the near-linear behavior using a modified linear operator (Theorem 2.5), then we will prove that the modified linear evolution is close to the Airy evolution (Theorem 2.6). These two theorems imply Theorem 1.1.

**Theorem 2.5.** Let $0 < s < 1/2$. Let $u$ be a mean-zero solution of (3) with $u(\cdot,0) = g$, where

$$\|g\|_{L^2} \lesssim 1, \text{ and } \|g\|_{H^{-s}} \lesssim \varepsilon \ll 1.$$  

Then, for each $t > 0$, we have

$$\|u(\cdot,t) - e^{tL_1}g\|_2 \leq C e^{Ct} \varepsilon.$$  

Here $L_1 = L + P$, where $L = -\partial_x^3 - \langle \phi(\cdot) \rangle \partial_x$ and $P$ is defined in the Fourier side as

$$(Pu)_k = \left[ e^{Lt} L_0(e^{-Lt}u) \right]_k = \frac{i}{3} \sum_{k_1+k_2+k_3=k} \Phi_{k_1} \Phi_{k_2} u_{k_3}, \quad k \neq 0,$$

and $(Pu)_0 = 0$.

**Theorem 2.6.** Let $g, L_1, L$ be as in the previous theorem. Then, for each $t > 0$, we have

$$\|e^{tL_1}g - e^{tL}g\|_2 \leq C e^{Ct} \varepsilon.$$

**Proof of Theorem 2.5.** First we will prove that the norm assumptions on the initial data remain intact up to times of order $\log(1/\varepsilon)$. By $L^2$ conservation in KdV, we have

$$\|u(\cdot,t)\|_{L^2} \leq \|q(\cdot,t)\|_{L^2} + \|\phi(\cdot)\|_{L^2} = \|q(\cdot,0)\|_{L^2} + \|\phi(\cdot)\|_{L^2} \lesssim 1.$$  

Now we prove that for some $C$,

$$\|u(\cdot,t)\|_{H^{-s}} \leq C e^{Ct} \varepsilon.$$  

To prove this, integrate (5) from 0 to $T$ to obtain

$$(I + K)v(T) = (I + K)v(0) + B(v)(0) - B(v)(T) + \int_0^T (L_0(v) + R(v))dt.$$  

Since $u_k = v_k e^{i\psi(k)t}$ and $\Phi_k = S_k e^{i\psi(k)t}$ with $\psi(k) = k^3 - ak$, we have

$$(I + \tilde{K})u(T) = e^{i\psi(k)T} (I + K)u(0) + e^{i\psi(k)T} B(u)(0) - e^{i\psi(k)T} B(e^{-i\psi(k)T}u)(T) + e^{i\psi(k)T} \int_0^T (L_0(e^{-i\psi(k)t}u) + R(e^{-i\psi(k)t}u)dt, $$

where $\tilde{K}$ is the time-independent operator:

$$\tilde{K}(u)_k = - \sum_{k_1+k_2=k} \frac{e^{-3k_1k_2} - e^{3k_1k_2}}{3k_1k_2} \Phi_{k_1} u_{k_2} = - \sum_{k_1+k_2=k} \frac{\Phi_{k_1} u_{k_2}}{3k_1k_2}.$$  

**Lemma 2.7.** For $0 < s \leq 1$,

$$\|(I + \tilde{K})u\|_{H^{-s}} \gtrsim \|u\|_{H^{-s}},$$
Proof. This follows from the Fredholm alternative. First note that for \( u \in L^2 \) with mean-zero
\[
\| \tilde{K} u \|_2 = \left\| \sum_{k_1 + k_2 = k} \frac{\Phi_{k_1} u_{k_2}}{3k_1 k_2} \right\|_2 \leq \left\| \frac{\Phi_k}{3k} \right\|_\ell \left\| \frac{u_k}{k} \right\|_2 \lesssim \| u \|_{H^{-s}}.
\]
By the density of \( L^2 \) in \( H^{-s} \), this inequality holds for each \( u \in H^{-s} \). Therefore, by Rellich’s theorem, \( \tilde{K} \) is a compact operator on \( H^{-s} \). It suffices to show that the kernel of \( I + \tilde{K} \) is trivial. Note that if \( (I + \tilde{K}) u = 0 \) for some \( u \in H^{-s} \), then by the discussion above, \( \tilde{K} u \in L^2 \), and hence \( u \in L^2 \). Using the definition of \( \tilde{K} \), we have
\[
(I + \tilde{K}) u = 0 \iff u(x) - \frac{1}{3} \Phi_{-1}(x) u_{-1}(x) = 0,
\]
where \( f_{-1}(x) \) denotes the mean-zero antiderivative of a mean-zero function,
\[
f_{-1}(x) = \partial_x^{-1} f(x) - \langle \partial_x^{-1} f \rangle = \int_0^x f(r) dr + \frac{1}{2\pi} \int_0^{2\pi} r f(r) dr.
\]
Let \( U(x) = u_{-1}(x) \); then we obtain equivalent first order linear ODE
\[
U' - \frac{1}{3} \Phi_{-1}(x) U(x) = 0,
\]
which has the general solution \( U(x) = U(0) \exp(\int_0^x \Phi_{-1}(x) dx) \), which can be mean-zero only if \( U(x) \equiv 0 \). This implies \( u(x) \equiv 0 \).

Remark 2.8. Let
\[
\tilde{K}_i(u) := - \sum_{k_1 + k_2 = k} \frac{\rho_k(t) u_{k_2}}{3 k_1 k_2},
\]
where \( \rho_k(t) = \rho(\cdot, t)(k) \), and \( \rho \) is an \( L^2 \) solution of KdV. Then, the statement of the lemma is valid for \( \tilde{K}_i \)
\( \text{(i) for } t \in [0, T] \text{ with a constant } C_T \text{ depending on } T, \)
\( \text{(ii) on } \mathbb{R} \text{ with a constant independent of } t \text{ if } \| \rho(\cdot, t) \|_2 = \| \rho(\cdot, 0) \|_2 < 1/2. \)
To prove these statements first note that
\[
\| \tilde{K}_i \|_{H^{-s} \to H^{-s}} \leq 2 \| \rho(\cdot, t) \|_2.
\]
Then, using the resolvent identity,
\[
(I + \tilde{K}_i)^{-1} - (I + \tilde{K})^{-1} = (I + \tilde{K}_i)(I + \tilde{K}_i)^{-1},
\]
the linearity of \( \tilde{K}_i \) in \( \rho \), and (13), we see that the operator \( (I + \tilde{K}_i)^{-1} \) is continuous in time in the operator norm \( H^{-s} \to H^{-s} \). Thus, (i) follows from the proof of the lemma and compactness. To see (ii), note that by (13) the operator norm of \( \tilde{K} \) in \( H^{-s} \) is \( < 1 \), and invert the operator using Neumann series.

Using Lemma 2.7 and Proposition 2.4 in (11), we obtain
\[
\| u(t) \|_{H^{-s}} \lesssim \| u(0) \|_{H^{-s}} + \| u(0) \|_{H^{-s}}^2 + \| u(T) \|_{H^{-s}}^2 + \int_0^T \| u(t) \|_{H^{-s}} dt
\]
\[
\leq C \varepsilon + C \| u(T) \|_{H^{-s}}^2 + C \int_0^T \| u(t) \|_{H^{-s}} dt.
\]
Therefore, on \([0, T_0]\), where \(T_0 = \inf\{T : \|u(t)\|_{H^{-s}} \geq \frac{1}{2C}\}\), we have

\[
\|u(T)\|_{H^{-s}} \leq 2C\varepsilon + 2C \int_0^T \|u(t)\|_{H^{-s}} dt.
\]

This implies by Gronwall that \(\|u(T)\|_{H^{-s}} \leq 2C\varepsilon e^{2CT}\) as claimed.

Now, note that \(\tilde{K}(u) = e^{-i\psi(k)T}K(e^{i\psi(k)T}u)\). Therefore, using Proposition 2.4 (for \(K, B, \text{ and } R\)) and (10) in (11), we obtain

\[
(14) \quad u(T) = e^{i\psi(k)T}u(0) + e^{i\psi(k)T} \int_0^T L_0(e^{-i\psi(k)T}u)dt + O_{L^2}(\varepsilon e^{CT}).
\]

Using the definition of \(L_0\), we have

\[
L_0(e^{-i\psi(k)t}u) = \sum_{k_1+k_2+k_3 = k} \frac{1}{k_1} \Phi_{k_1}\Phi_{k_2}\Phi_{k_3} u_{k_3}
\]

\[
= \sum_{k_1+k_2+k_3 = k} \Phi_{k_1}\Phi_{k_2}\Phi_{k_3} u_{k_3}
\]

In the last line we used the fact that for \(k = k_1 + k_2 + k_3\),

\[-\psi(k) = -3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) - \psi(k_1) - \psi(k_2) - \psi(k_3),\]

and the definition of \(L\) and \(P\).

Using this in (14), we have

\[
u(T) = e^{LT}u(0) + e^{LT} \int_0^T e^{-Lt}P(u(t))dt + O_{L^2}(\varepsilon e^{CT})
\]

\[
= e^{LT}u(0) + \int_0^T e^{L(T-t)}P(e^{L(t)}u(0))dt \\
+ \int_0^T e^{L(T-t)}P[u(t) - e^{L(t)}u(0)]dt + O_{L^2}(\varepsilon e^{CT})
\]

\[
= e^{L_1T}u(0) + \int_0^T e^{L(T-t)}P[u(t) - e^{L(t)}u(0)]dt + O_{L^2}(\varepsilon e^{CT}).
\]

Let \(h(t) := \|u(t) - e^{L(t)}u(0)\|_2\). Using the equality above and the bound for \(L_0\) in Proposition 2.4 for the operator \(P(u(t)) = e^{LT}L_0(e^{-LT}u(t))\), we obtain

\[
h(T) \lesssim \varepsilon e^{CT} + \int_0^T h(t)dt.
\]

The theorem follows from this by Gronwall. \(\Box\)

**Proof of Theorem 2.6.** First we prove that our assumptions on the initial data remain intact for times of order \(\log(1/\varepsilon)\):

**Lemma 2.9.** For \(s \in [-1, 1]\), the operator \(L_1\) defined above satisfies

\[
\|e^{L_1}\|_{H^s \to H^s} \lesssim \varepsilon e^{Ct}.
\]
Proof. First note that we can rewrite $P(u)$ (for mean-zero $u$ and $\Phi$) in the following form, which is valid for each $k \in \mathbb{Z}$:

$$P(u)_k = -\frac{1}{3} \sum_{k_1 + k_2 + k_3 = k} \frac{\Phi_{k_1} \Phi_{k_2} u_{k_3}}{i k_1} + \frac{1}{3} \sum_{k_1 + k_2 + k_3 = 0} \frac{\Phi_{k_1} \Phi_{k_2} u_{k_3}}{i k_1}.$$

The constant term makes the right-hand side vanish for $k = 0$, which makes $P(u)$ mean-zero in the space side. Using the formula (12) for the function $\Phi$, we write $L_1 = L + P$ in the space side as

$$L_1 u = -\partial_3^3 u - a \partial_x u + Gu - \frac{1}{2\pi} \langle u, G \rangle,$$

where

$$G(x) = -\frac{1}{3} \Phi(x) \left( \int_0^x \Phi(r) dr + \frac{1}{2\pi} \int_0^{2\pi} r \Phi(r) dr \right).$$

Also note that

$$L_1^* u = \partial_3^3 u + a \partial_x u + Gu - \frac{1}{2\pi} G \langle u, 1 \rangle.$$

Note that by duality and interpolation it suffices to prove the assertion of the lemma for $s = 1$ for $L_1$ and $L_1^*$. We will give the proof for $L_1$; for $L_1^*$ the proof is essentially the same since they have very similar forms. Consider the equation

$$u_t = L_1 u = -\partial_3^3 u - a \partial_x u + Gu - \frac{1}{2\pi} \langle u, G \rangle.$$

First, we calculate

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_2 \leq \int_0^{2\pi} u_t u dx$$

$$= -\int_0^{2\pi} u_{xxx} u dx - a \int_0^{2\pi} u_x u dx + \int_0^{2\pi} Gu^2 dx - \frac{1}{2\pi} \langle u, G \rangle \int_0^{2\pi} u dx$$

$$\leq \int_0^{2\pi} Gu^2 dx - \frac{1}{2\pi} \langle u, G \rangle \int_0^{2\pi} u dx.$$

This implies that

$$\left| \frac{d}{dt} \|u\|^2_2 \right| \leq 2\|G\|_\infty \|u\|^2_2 + \|G\|_2 \|u\|^2_2 \lesssim \|u\|^2_2.$$

Similarly,

$$\frac{1}{2} \frac{d}{dt} \|u_x\|^2_2 = \int_0^{2\pi} u_{xx} u_x dx$$

$$= -\int_0^{2\pi} u_{xxx} u_x dx - a \int_0^{2\pi} u_{xx} u_x dx + \int_0^{2\pi} (Gu)_x u_x dx$$

$$= \int_0^{2\pi} Gu^2_x dx - \frac{1}{2} \int_0^{2\pi} u^2 G_{xx} dx$$

$$= O\left( \|u\|^2_2 \|G\|_\infty + \|u\|^2_2 \|G_{xx}\|_\infty \right) = O\left( \|u\|^2_{H_1} \right).$$
Combining the two inequalities, we obtain

\[ \left| \frac{d}{dt} \| u \|_{H^1}^2 \right| \lesssim \| u \|_{H^1}^2, \]

which finishes the proof.

We return to the proof of Theorem 2.6. Consider the equation

\[ u_t = L_1 u, \quad u(0, x) = g(x). \]

Repeating the discussion in the beginning of section 2, and introducing the variables \( v_k \) and \( S_k \) as above, we have

\[ \partial_t v_k = \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k} e^{-3it(k_1 + k_2)(k_2 + k_3)(k_1 + k_3)} \frac{S_{k_1}}{k_1} S_{k_2} S_{k_3} v_{k_3}. \tag{15} \]

We will prove the following proposition, using differentiation by parts, in section 3.

**Proposition 2.10.** The system (15) is equivalent to

\[ \partial_t (v_k + D(v)_k) = E(v)_k, \tag{16} \]

where

\[ D(v)_k = -\frac{i}{3} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-3it(k_1 + k_2)(k_2 + k_3)(k_1 + k_3)}}{k_1(k_1 + k_2)(k_2 + k_3)(k_1 + k_3)} S_{k_1} S_{k_2} S_{k_3} v_{k_3}, \]

\[ E(v)_k = \frac{2i}{3} S_k \sum_{j \neq k} S_j v_j \frac{S_j}{j} + \frac{2i}{3} S_k \sum_j S_j v_j - \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-3it(k_1 + k_2)(k_2 + k_3)(k_1 + k_3)}}{k_1(k_1 + k_2)(k_2 + k_3)(k_1 + k_3)} \partial_t (S_{k_1} S_{k_2} v_{k_3}) \]

\[ + \frac{2}{9} \sum_{k_1 + k_2 + k_3 + k_4 + k_5 = k} \frac{e^{it\tilde{\psi}(k_1, k_2, k_3, k_4, k_5)}}{k_1 k_3 (k_1 + k_2)(k_1 + k_3)(k_1 + k_3)} S_{k_1} S_{k_2} S_{k_3} S_{k_4} S_{k_5} v_{k_5}. \]

Here \( \tilde{\psi} \) is a real valued phase function which is irrelevant for the proof of the theorem.

**Proposition 2.11.** The following a priori estimates hold:

\[ \| D(v) \|_2 \lesssim \| v \|_{H^{-1}}, \quad \| E(v) \|_2 \lesssim \| v \|_{H^{-1}}. \]

**Proof.** Using \( |k_1| + |k_2| \geq |k_3| \) for nonzero integer values of \( k_1 \) and \( k_1 + k_3 \), and eliminating the product \( |(k_1 + k_2)(k_2 + k_3)| \) from the denominator, we have

\[ |D(v)_k| \lesssim \sum_{k_1 + k_2 + k_3 = k} |S_{k_1} S_{k_2} | \frac{|v_{k_3}|}{|k_3|}. \]

This implies by Young’s inequality

\[ \| D(v) \|_2 \lesssim \| S \|_{\ell^1} \| S \|_{\ell^1} \| v_k / k \|_2 \lesssim \| v \|_{H^{-1}}. \]

The proof for the contribution of the second line in the definition of \( E(v) \) is exactly the same using the fast decay of \( |\partial_t S_k| \lesssim |k^2| |S_k| \). The \( L^2 \) norm of the first line is

\[ \lesssim \|S\|_2 \sum_j \frac{|S_j| |v_j|}{|j|} + \|S_k / k\|_2 \sum_j |j| S_j |v_j| |j| \]

\[ \lesssim \|S\|_2^2 + \|S_k / k\|_2 \|k S_k\|_2 \|v_k / k\|_2 \lesssim \|v\|_{H^{-1}}. \]
The second inequality follows from Cauchy–Schwarz. To estimate the third line in the definition of $E(v)$, note that
\[ |k - k_2|k_1k_3| = \frac{|k_1 + k_3 + k_4||k_1||k_3|}{|k_4|} \approx \frac{|k_5|}{|k_4|} \]
for nonzero integer values of the factors. Using this in the sum and eliminating the rest of the factors, we estimate the third line as
\[ \lesssim \sum_{k_1 + k_2 + k_4 + k_5 = k} |S_{k_1}S_{k_2}S_{k_3}k_4k_5| |v_{k_5}|. \]
As above, this implies that the $L^2$ norm is
\[ \lesssim \|S\|_2^3 \|kS_k\|_{L^1} \|v_k/k\|_2 \lesssim \|v\|_{H^{-1}}. \]

To complete the proof of Theorem 2.6, integrate (16) from 0 to $T$:
\[ v_k(T) = v_k(0) - D(v)_k(T) + D(v)_k(0) + \int_0^T E(v)_k(t)dt. \]
Using the transformation $u_k = v_k e^{i\psi(k)t}$, we have
\[ u_k(T) = e^{i\psi(k)T} u_k(0) - e^{i\psi(k)T} D(e^{-i\psi(k)T} u)_k(T) + e^{i\psi(k)T} D(u)_k(0) \]
\[ + \int_0^T e^{i\psi(k)T} E(e^{-i\psi(k)T} u)_k(t)dt. \]
Noting that $(e^{LT}u)_k = e^{i\psi(k)T} u_k(0)$, and using the a priori estimates in Proposition 2.11, we have
\[ \|u(T) - e^{LT}u(0)\|_2 \lesssim \|u(T)\|_{H^{-1}} + \|u(0)\|_{H^{-1}} + \int_0^T \|u(t)\|_{H^{-1}} dt \]
\[ \lesssim \varepsilon e^{CT}. \]
In the last line we used Lemma 2.9 and the hypothesis on $\|u(0)\|_{H^{-1}} \geq \|u(0)\|_{H^{-1}}$. \]

3. Differentiation by parts. In this section we prove Theorem 2.3 and Proposition 2.10.

Proof of Theorem 2.3. We need to obtain (5) from (4) by using differentiation by parts. It will be useful to name the terms appearing in the formula (5). We will denote the terms in the first and second lines of the definition of $B(v)$ by $B_1(v)$ and $B_2(v)$, respectively. We also denote the term in the $j$th line of the definition of $R(v)$ in the statement of the theorem by $R_j(v)$, $j = 1, 2, \ldots, 6$, and we further denote the four summands in $R_1(v)$ by $R_{1,m}(v)$, $m = 1, 2, 3, 4$.

Since $e^{-3ik_1k_2t} = \partial_t \left( \frac{1}{3k_1k_2} e^{-3ik_1k_2t} \right)$, using differentiation by parts we can rewrite (4) as
\[ \partial_t v_k = \partial_t \left( \frac{1}{2} \sum_{k_1 + k_2 = k} \frac{e^{-3ik_1k_2t}(v_{k_1} + 2S_{k_1})v_{k_2}}{3kk_1k_2} \right) \]
\[ - \frac{1}{2} \sum_{k_1 + k_2 = k} \frac{e^{-3ik_1k_2t}}{3kk_1k_2} \partial_t [(v_{k_1} + 2S_{k_1})v_{k_2}]. \]
Recalling the definition of $K(v)$ and $B_1(v)$ from (5), we can rewrite this equation in the form

$$\partial_t[v_k + K(v)_k + B_1(v)_k] = - \sum_{k_1 + k_2 = k} \frac{e^{-3i k_1 k_2 t}}{6k_1 k_2} \partial_t[(v_{k_1} + 2S_{k_1})v_{k_2}].$$

Note that since $v_0 = 0$ and $S_0 = 0$, in the sums above $k_1$ and $k_2$ are not zero. We now handle the term when the derivative hits $v_{k_1}v_{k_2}$. By symmetry and (4), we have

$$\sum_{k_1 + k_2 = k} \frac{e^{-3i k_1 k_2 t}}{k_1 k_2} \partial_t(v_{k_1}v_{k_2})$$

$$= -i \sum_{k_1 + k_2 = k} \frac{e^{-3i k_1 k_2 t}}{k_1} v_{k_1} \left( \sum_{\mu + \lambda = k_2} e^{-3i t k_2 \mu \lambda} (v_{\mu} + 2S_{\mu})v_{\lambda} \right)$$

$$= -i \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} v_{k_1} (v_{\mu} + 2S_{\mu})v_{\lambda} e^{-3i t[k_1(\mu + \lambda) + \mu \lambda(\mu + \lambda)]}.$$

We note that $\mu + \lambda$ cannot be zero since $\mu + \lambda = k_2$. Using the identity $kk_1 + \mu \lambda = (k_1 + \mu + \lambda)k_1 + \mu \lambda = (k_1 + \mu)(k_1 + \lambda)$

and by renaming the variables $k_2 = \mu, k_3 = \lambda$, we have that

$$\sum_{k_1 + k_2 = k} \frac{e^{-3i k_1 k_2 t}}{k_1 k_2} \partial_t(v_{k_1}v_{k_2})$$

$$= -i \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} \frac{e^{-3i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} v_{k_1} (v_{k_2} + 2S_{k_2})v_{k_3}.$$ 

Calculating the term when the derivative hits $S_{k_1}v_{k_2}$ similarly, we have

$$\partial_t[v_k + K(v)_k + B_1(v)_k] = \sum_{j=1}^{5} Y_j(v)_k,$$

where

$$Y_1(v)_k = \frac{i}{6} \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} \frac{e^{-3i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} v_{k_1}v_{k_2}v_{k_3},$$

$$Y_2(v)_k = \frac{i}{6} \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} \frac{e^{-3i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} S_{k_1}v_{k_2}v_{k_3},$$

$$Y_3(v)_k = \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} \frac{e^{-3i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} v_{k_1}S_{k_2}v_{k_3},$$

$$Y_4(v)_k = \frac{i}{3} \sum_{k_1 + k_2 + k_3 = k \atop k_2 + k_3 \neq 0} \frac{e^{-3i t(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} S_{k_1}S_{k_2}v_{k_3},$$

$$Y_5(v)_k = - \sum_{k_1 + k_2 = k} \frac{e^{-3i k_1 k_2 t}}{3k_1 k_2} (\partial_t S_{k_1})v_{k_2}.$$
Note that
\begin{equation}
Y_3(v) = R_3(v), \quad Y_4(v) = L_0(v) + R_{1,4}, \quad \text{and} \quad Y_5(v) = R_2(v).
\end{equation}

Due to the fast decay of $S_k$ and $\partial_x S_k$, the terms $R_2(v)$, $R_3(v)$, and $R_{1,4}$ are small (as stated in Proposition 2.4). The term $L_0(v)$ is not small but linear in $v$, and it is handled separately; see the proof of Theorem 2.5. On the other hand, one cannot directly estimate the terms $Y_1$ and $Y_2$ without performing another differentiation by parts. To do that we need to check the resonant terms in $Y_1$ and $Y_2$:
\begin{equation}
(k_1 + k_2)(k_3 + k_1) = 0.
\end{equation}

The set for which (19) holds is the disjoint union of the following three sets (taking into account that $k_2 + k_3 \neq 0$):
\begin{align*}
A_1 &= \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 = 0\} \Leftrightarrow \{k_1 = -k, \ k_2 = k, \ k_3 = k\}, \\
A_2 &= \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 \neq 0\} \Leftrightarrow \{k_1 = j, \ k_2 = -j, \ k_3 = k, \ |j| \neq |k|\}, \\
A_3 &= \{k_3 + k_1 = 0\} \cap \{k_1 + k_2 \neq 0\} \Leftrightarrow \{k_1 = j, \ k_2 = k, \ k_3 = -j, \ |j| \neq |k|\}.
\end{align*}

We write
\begin{equation*}
Y_2(v)_k = Y_{2r}(v)_k + Y_{2nr}(v)_k,
\end{equation*}
where the subscript $r$ and $nr$ stand for the resonant and nonresonant terms, respectively. We have
\begin{equation*}
Y_{2r}(v)_k = \frac{i}{6} \sum_{\lambda=1}^3 \frac{S_{k_1} v_{k_2} v_{k_3}}{k_1} = \frac{i}{6} \frac{S_{-k} v_k}{-k} + \frac{i}{3} \frac{v_k}{|j| \neq |k|} \sum_{j \in \mathbb{Z}} \frac{S_j v_{-j}}{j}
\end{equation*}
\begin{equation*}
= R_{1,2}(v)_k + R_{1,3}(v)_k
\end{equation*}
and
\begin{equation*}
Y_{2nr}(v)_k = \frac{i}{6} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-3it(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1} S_{k_1} v_{k_2} v_{k_3}.
\end{equation*}

Since the exponent in $Y_{2nr}(v)$ is not zero we can differentiate by parts one more time and obtain that
\begin{equation*}
Y_{2nr}(v)_k = \partial_t M_3(v)_k + M_4(v)_k,
\end{equation*}
where
\begin{equation}
M_3(v)_k = -\frac{1}{18} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-3it(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)} S_{k_1} v_{k_2} v_{k_3}
\end{equation}
and
\begin{equation*}
M_4(v)_k = \frac{1}{18} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{-3it(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}}{k_1(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)}
\times (v_{k_2} v_{k_3} \partial_t S_{k_1} + S_{k_1} v_{k_3} \partial_t v_{k_2} + S_{k_1} v_{k_2} \partial_t v_{k_3}).
\end{equation*}
A calculation as before, by expressing time derivatives using (4), reveals that
\[ M_4(v)_k = \frac{1}{18} \sum_{k_1+k_2+k_3=k}^* e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \]
\[ - \frac{i}{18} \sum_{k_1+k_2+k_3+k_4=k}^* e^{it\tilde{\psi}(k_1,k_2,k_3,k_4)}(k_3+k_4)S_{k_1}v_{k_2}(v_{k_3} + 2S_{k_3}v_{k_4}) \]
\[ = R_4(v)_k + R_5(v)_k. \]

The phase function \( \tilde{\psi} \) is irrelevant for our calculations since it is going to be estimated out by taking absolute values inside the sums. For completeness we note that it can be expressed as
\[(k_1 + k_2 + k_3 + k_4)^3 - k_1^3 - k_2^3 - k_3^3 - k_4^3.\]

Hence
\[(21) \quad Y_2(v) = R_{1,2}(v)_k + R_{1,3}(v)_k + \partial_t M_3(v) + R_4(v)_k + R_5(v)_k.\]

Similarly,
\[(22) \quad Y_1(v) = R_{1,1} + \partial_t N_3(v) + R_6(v),\]
where
\[(23) \quad N_3(v)_k := -\frac{1}{18} \sum_{k_1+k_2+k_3=k}^* e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} .\]

Using (20) and (23), note that \( B_2(v) = M_3(v) + N_3(v) \). Therefore, substituting (18), (21), and (22) in (17), we obtain (5).

Proof of Proposition 2.10. We write the right-hand side of (15) by distinguishing the resonant and nonresonant terms. The resonant set corresponding to the terms with \( k_2 + k_3 \neq 0 \) is the same as above, and thus we get the following three terms:
\[ \frac{i}{3} \left( \sum_j S_{j-k} v_{k} v_{j-k} - \sum_{j \neq k} S_j S_{j-k} + \sum_{j \neq k} S_j v_{j-k} \right). \]

Note that, by symmetry, the second term is zero. Combining the other terms we obtain the first summand in the definition of \( E(v) \). The terms with \( k_2 + k_3 = 0 \) give the second summand.

For the nonresonant terms we differentiate by parts as above, obtaining
\[ \frac{i}{3} \partial_t \left( \sum_{k_1+k_2+k_3=k}^* e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)} \frac{v_{k_2}v_{k_3}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} S_{k_1} \right) \]
\[ - \frac{i}{3} \sum_{k_1+k_2+k_3=k}^* \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \partial_t (S_{k_1}v_{k_2}v_{k_3}). \]

The first line gives \( D(v) \), and the second line gives the remaining terms in the definition of \( E(v) \) after using the formula for \( \partial_t v_{k_3} \) and renaming the variables. \( \Box \)
4. Proof of Proposition 2.4. We start with the term \(K\):
\[
\|K(v)\|_2 \lesssim \|S_k/k\|_{\ell^1} \|v_k/k\|_2 \lesssim \|v\|_{H^{-s}}.
\]
Similarly, the \(L^2\) norm of the first summand in the definition of \(B(v)\) is
\[
\lesssim \|v_k/k\|_{\ell^1} \|v_k/k\|_2 \lesssim \|v_k/k\|_2 \lesssim \|v\|_{H^{-s}}
\]
since \(0 < s < 1/2\). Note that the second summand is
\[
\lesssim \sum_{k_1+k_2+k_3=k} \frac{|v_{k_1}| + |S_{k_1}|}{|k_1||k_2+k_3||k_3+k_1|} |v_{k_2}| |v_{k_3}|
\]
\[
\lesssim \sum_{k_1+k_2+k_3=k} \frac{|v_{k_1}| |v_{k_2}| |v_{k_3}|}{|k_1||k_2|} + \frac{|k_1| |S_{k_1}| |v_{k_2}| |v_{k_3}|}{|k_2||k_3|}.
\]
In the second line we use the following inequalities, which are valid for the nonzero integral values of the factors:
\[
|k_1 + k_2||k_2 + k_3||k_3 + k_1| \gtrsim |k_2|, \quad |k_1||k_1 + n| \gtrsim |n|.
\]
Therefore, by Young’s inequality, the \(L^2\) norm of the second summand is
\[
\lesssim \|v_k/k\|_{\ell^1}^2 \|v_k\|_2 + \|kS_k/k\|_{\ell^1} \|v_k/k\|_{\ell^1} \|v_k/k\|_2 \lesssim \|v\|_{H^{-s}}.
\]
The \(L^2\) bound for \(L_0(v)\) follows from Young’s inequality:
\[
\|L_0(v)\|_2 \lesssim \|S_k/k\|_{\ell^1} \|S\|_{\ell^1} \|v\|_2 \lesssim \|v\|_2.
\]
For the \(H^{-s}\) bound, using the inequality \(|k_3|^s \lesssim |k_1|^s |k_2|^s |k_1 + k_2 + k_3|^s = |k_1 k_2 k|^s\) (for nonzero integral values of the factors), we obtain
\[
\frac{|L_0(v)_{k_1}|}{|k|^s} \lesssim \sum_{k_1+k_2+k_3=k} \frac{|S_{k_1}| |k_1| |k_2|^s |S_{k_2}|}{|k_1| |k_1 + k_2|^s |k_3|^s} |v_{k_3}|.
\]
Therefore,
\[
\|L_0(v)\|_{H^{-s}} \lesssim \|S_k/k\|_{\ell^1} |k|^{1-s} \|S\|_{\ell^1} \|v_k/k\|_{\ell^1} \|v_k/k\|_2 \lesssim \|v\|_{H^{-s}}.
\]
We now estimate \(R(v)\). Denote the terms in the \(j\)th line of the definition of \(R(v)\) by \(R_j(v), j = 1, 2, \ldots, 6\). The estimate for \(R_1(v)\) follows as in the proof of Proposition 2.11. The estimate for \(R_2(v)\) is the same as the estimate for \(K(v)\) with \(S\) replaced with \(\partial_t S\). For \(R_3(v)\), note that by Young’s inequality
\[
\|R_3(v)\|_2 \lesssim \|v_k/k\|_{\ell^1} \|S\|_1 \|v\|_2 \lesssim \|v\|_{H^{-s}}.
\]
For \(R_4(v)\), using \(|k_1||k_1 + k_2| \gtrsim |k_2|\), we have
\[
\|R_4(v)\|_2 \lesssim \|\partial_t S\|_{\ell^1} \|v_k/k\|_{\ell^1} \|v\|_2 \lesssim \|v\|_{H^{-s}}.
\]
For \(R_5(v)\), using \(|k_1||k_1 + k_2| \gtrsim |k_2|, \quad |k_3 + k_4| \lesssim |k_1||k_1 + k_3 + k_4|, \text{and} \quad |k_1||k_2 + k_3 + k_4| \gtrsim |k_1 + k_2 + k_3 + k_4| = |k|\), we have
\[
|R_5(v)_{k_1}| \lesssim \frac{1}{|k|} \sum_{k_1+k_2+k_3+k_4=k} k_1^2 |S_{k_1}| \frac{|v_{k_2}|}{|k_2|} |v_{k_3} + 2S_{k_3}| |v_{k_4}|.
\]
Therefore, by Young’s inequality

\[
\|R_5(v)\|_2 \lesssim \left\| \frac{1}{k} \right\|_2 \sum_{k_1+k_2+k_3+k_4=k} k_1^2 |S_{k_1}| \frac{|v_{k_2}|}{k_2} \frac{|v_{k_3} + 2S_{k_3}|}{k_3} |v_{k_4}| \left\|_\ell_\infty \lesssim k^2 S_k \|v_k/k\|_e \|v + 2S\|_2 \lesssim \|v\|_{H^{-\frac{1}{2}}}
\]

Finally, we consider \( R_6(v) \). Using \(|k_1 + 2k_3 + 2k_4| \leq |k_1| + 2|k_1 + k_3 + k_4| \), and then Cauchy–Schwarz, we have

\[
|R_6(v)|_k^2 \lesssim \left( \sum_{k_1+k_2+k_3+k_4=k} \frac{|v_{k_1}v_{k_2}(v_{k_3} + 2S_{k_3})v_{k_4}|}{|k_1|^2} \left( \frac{1}{|k_1|} + \frac{1}{|k_1 + k_3 + k_4|} \right) \right)^2
\]

\[
\lesssim \sum_{k_1+k_2+k_3+k_4=k} \frac{|v_{k_1}v_{k_2}(v_{k_3} + 2S_{k_3})^2}{|k_1|^2} \sum_{k_1+k_2+k_3+k_4=k} \frac{|k_1|^2 v_{k_4}^2}{|k_1 + k_2|^2 |k_2 + k_3 + k_4|^2} \left( \frac{1}{|k_1|^2} + \frac{1}{|k_1 + k_3 + k_4|} \right).
\]

Note that the first factor above is \( \lesssim \|v\|_{H^{-\frac{1}{2}}} \|v + 2S\|_2^2 \|v\|_2^2 \). Therefore it suffices to prove that the sum of the second factor in \( k \) is \( O(1) \). We write this sum as

\[
\sum_{k_1,k_2,k_3,k_4} \frac{v_{k_4}^2}{|k_1|^2 |k_1 + k_2|^2 |k_2 + k_3 + k_4|^2}
\]

\[
+ \sum_{k_1,k_2,k_3,k_4} \frac{|k_1|^{2s} v_{k_4}^2}{|k_1 + k_2|^2 |k_1 + k_3 + k_4|^2 |k_2 + k_3 + k_4|^2}.
\]

To estimate the first sum, take the sum first in \( k_3 \), then in \( k_2, k_1 \), and \( k_4 \) in the given order. The estimate for the second sum follows also by summing in the order given above and using the inequality

\[
\sum_{k_1,k_2,k_3,k_4} \frac{1}{|k_1 + k_2|^2 |k_1 + k_3 + k_4|^2 |k_2 + k_3 + k_4|^2} \lesssim \sum_{k_2} \frac{1}{|k_1 + k_2|^2 |k_1 - k_2|^2} \lesssim \frac{1}{|k_1|^2}.
\]

Here we used the inequality (for \( a > b > 1 \))

\[
\sum_m \frac{1}{|n_1 - m|^a |n_2 - m|^b} \lesssim \frac{1}{|n_1 - n_2|^b}.
\]

**Remark 4.1.** We note that in the proofs of Propositions 2.4 and 2.11, the strongest conditions we need for \( S_k = e^{-i\psi(k)t} \Phi_k \) are

\[
\partial_t S_k \in \ell^1 \text{ and } k^2 S_k \in \ell^1.
\]

It is easy to see, using the equation, that \( \partial_t \rho \in H^1 \) for an \( H^4 \) solution \( \rho \) of KdV. Therefore, the sequence \( e^{-i\psi(k)t} \rho_k \) satisfies the conditions above, and thus the assertions of Propositions 2.4 and 2.11 remain valid when \( \Phi \) is replaced with \( \rho(\cdot, t) \in H^1 \).
Acknowledgment. The authors would like to thank Michael Weinstein for a helpful discussion.

REFERENCES

[1] A. V. Babin, A. A. Ilyin, and E. S. Titi, On the regularization mechanism for the periodic Korteweg-de Vries equation, Commun. Pure Appl. Math., 64 (2011), pp. 591–648.
[2] A.V. Babin, A. Mahalov, and B. Nicolaenko, Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids, Asymptot. Anal., 15 (1997), pp. 103–150.
[3] T. B. Benjamin, Lectures on nonlinear wave motion, in Nonlinear Wave Motion, AMS, Providence, RI, 1974, pp. 3–47.
[4] N. Bottman and B. Deconinck, KdV cnoidal waves are spectrally stable, Discrete Contin. Dyn. Syst., 25 (2004), pp. 1163–1180.
[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part I: Schrödinger equations, Geom. Funct. Anal., 3 (1993), pp. 107–156.
[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part II: The KdV equation, Geom. Funct. Anal., 3 (1993), pp. 209–262.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp Global Well-Posedness for KdV and Modified KdV on R and T, J. Amer. Math. Soc., 16 (2003), pp. 705–749.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation, Invent. Math., 181 (2010), pp. 39–113.
[9] B. Deconinck and T. Kapitula, The orbital stability of the cnoidal waves of the Korteweg-de Vries equation, Phys. Lett. A, 374, (2010), pp. 4018–4022.
[10] B. Deconinck and M. Nivala, Periodic finite-genus solutions of the KdV equation are orbitally stable, Phys. D, 239 (2010), pp. 1147–1158.
[11] M. B. Erdoğan, N. Tzirakis, and V. Zharnitsky, Near-linear dynamics in KdV with periodic boundary conditions, Nonlinearity, 23 (2010), pp. 1675–1694.
[12] M. B. Erdoğan, N. Tzirakis, and V. Zharnitsky, Near-Linear Dynamics of Nonlinear Dispersive Waves, preprint, 2010.
[13] T. Kappeler and P. Topalov, Global wellposedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$, Duke Math. J., 135 (2006), pp. 327–360.
[14] C. E. Kenig, G. Ponce, and L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc., 9 (1996), pp. 573–603.
[15] H. P. McKean, Stability for the Korteweg-deVries equation, Commun. Pure Appl. Math., 30 (1977), pp. 347–353.
[16] J. A. Pava, J. L. Bona, and M. Scialom, Stability of cnoidal waves, Adv. Differential Equations, 11 (2006), pp. 1321–1374.
[17] T. Tao, Nonlinear dispersive equations: Local and global analysis, CBMS Reg. Conf. Ser. Math. 106, AMS, Providence, RI, 2006.