TORIC DEGENERATION OF SCHUBERT VARIETIES
AND GEL’FAND–CETLIN POLYTOPES

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Abstract. This note constructs the flat toric degeneration of the manifold $F_\ell_n$ of flags in $\mathbb{C}^n$ from [GL96] as an explicit GIT quotient of the Gröbner degeneration in [KM03]. This implies that Schubert varieties degenerate to reduced unions of toric varieties, associated to faces indexed by rc-graphs (reduced pipe dreams) in the Gel’fand–Cetlin polytope. Our explicit description of the toric degeneration of $F_\ell_n$ provides a simple explanation of how Gel’fand–Cetlin decompositions for irreducible polynomial representations of $GL_n$ arise via geometric quantization.

Introduction

A number of recent developments at the intersection of algebraic geometry and combinatorics have exploited degenerations of certain varieties related to linear algebraic groups. Sometimes the varieties involved have been classical flag and Schubert varieties [Kog00, Vak03a, Vak03b], and other times they have been closely related affine varieties [KM03, KMS03]. In all cases, it has been vital not only to construct an appropriate family degenerating the primal variety, but also to identify combinatorially all components occurring in the degenerate limit. Indeed, this is what geometrically produces (or reproduces) various combinatorial formulae (for classical objects such as Littlewood–Richardson coefficients [Ful97], or for universally defined polynomials discovered more recently [LS82a, LS82b, Ful99, BF99, Buc02]): components of the special fiber correspond to combinatorial summands in the desired formula.

Independently motivated degenerations of similar flavors have appeared in areas related to representation theory, particularly standard monomial theory [GL96, Chi00] and [Cal02]. The primal varieties there have been generalized flag and Schubert varieties in arbitrary type, with the limiting fibers being toric varieties, or reduced unions thereof.

Our goal in this note is to create a single geometric framework relating some of the more natural degenerations above. To this end, we express the flat toric degeneration of the manifold $F_\ell_n$ of flags in $\mathbb{C}^n$ from [GL96], which is a special case of the degenerations in [Cal02], as a quotient of the Gröbner degeneration of $n \times n$ matrices $M_n$ in [KM03]. The quotient is constructed by deforming the action of the lower triangular matrices $B \subset GL_n$ on the space $M_n$ of matrices. More precisely, we construct an explicit action of $B$ on $M_n \times \mathbb{C}$ and define the GIT quotient $B \backslash (M_n \times \mathbb{C})$, which is the total family $F$ of the desired degeneration and still fibers over the line $\mathbb{C}$.

This degeneration can also be thought of simply as a Plücker embedding of the Gröbner degeneration from [KM03]. The closure of the image of this embedding is

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the family $\mathcal{F}$ of projective varieties over the affine line $\mathbb{C}$, whose fiber $\mathcal{F}(1)$ over $1 \in \mathbb{C}$ is the flag variety $\mathcal{F}_\ell_n$, and whose fiber $\mathcal{F}(0)$ over $0 \in \mathbb{C}$ is the toric variety associated to the Gelfand–Cetlin polytope from representation theory \cite{GC50, GS83}.

Two consequences result from our explicit description of the family $\mathcal{F}$. First, since $\mathcal{F}$ is derived from Gröbner degeneration, it induces a subfamily degenerating every Schubert variety in $\mathcal{F}_\ell_n$. Therefore—and this is the main point—applying the combinatorial characterization of degenerated matrix Schubert varieties in \cite{KM03}, we characterize in Theorem 8 which faces of the Gelfand–Cetlin toric variety occur in degenerate Schubert varieties of $\mathcal{F}_\ell_n$. Namely, these faces correspond by \cite{Kog00} to combinatorial diagrams called rc-graphs (or reduced pipe dreams) \cite{FK96b, BB93}.

Our second consequence is a simple explanation (Section 5) for how the classical Gelfand–Cetlin decomposition of irreducible polynomial representations of $GL_n$ into one-dimensional weight spaces arises geometrically from toric degeneration $\mathcal{F}$ of the flag manifold. The idea is to think of Gelfand–Cetlin decomposition as geometric quantization on the total space of the family $\mathcal{F}$, directly extending the manner in which the Borel–Weil theorem is geometric quantization at the fiber $\mathcal{F}_\ell_n = \mathcal{F}(1)$. The point is that sections of line bundles over $\mathcal{F}_\ell_n$, which constitute irreducible representations of $GL_n$ by Borel–Weil, canonically acquire at the special fiber of $\mathcal{F}$ an action of the torus densely embedded in the toric variety $\mathcal{F}(0)$. This produces a basis for sections over $\mathcal{F}_\ell_n$ indexed by integer points of the Gelfand–Cetlin polytope.

The methods in this note can be extended to partial flag manifolds in type $\text{A}$, but we believe the most exciting prospects for future research lie in extensions to other types. In particular, \cite{GL96} and \cite{Cal02} describe a number of degenerations of generalized flag varieties to toric varieties. Under these degenerations, Schubert varieties become unions of toric subvarieties. In “nice” cases \cite{Lit98} (this is a technical term), the degenerate toric variety has an easily described moment polytope, in terms of generalizations of Gelfand–Cetlin patterns. Identification of the components in degenerations of Schubert varieties could therefore provide arbitrary-type combinatorial generalizations of pipe dreams, such as those suggested by \cite{FK96a} in type $\text{B}$.

The approach of degenerating generalized flag manifolds $P \backslash G$ instead of degenerating groups $G$ sidesteps the use of an equivariant partial compactification of $G$ to a vector space, which is suggested by \cite{KM03}, but is something we do not know how to do for arbitrary linear algebraic groups. Furthermore, one should be able to obtain positive combinatorial formulae for Schubert classes in arbitrary type (though perhaps not a definition of Schubert polynomial) by summing over components in torically degenerated Schubert varieties. The shapes taken by such combinatorial formulae would echo the manner in which \cite{Kog00} geometrically decomposes Schubert classes, agreeing with the combinatorially positive formula in \cite{BJS93, FS94} for the Schubert polynomials of Lascoux and Schützenberger \cite{LS82a}.

Part of our purpose in writing this note was to make toric degenerations of flag and Schubert varieties as in \cite{GL96, Chi00, Cal02} accessible to an audience unaccustomed to specialized language surrounding arbitrary finite type root systems and standard monomial theory. In particular, we thought it important to include an equivalent characterization of the family $\mathcal{F}$ in the language of sagbi bases (Theorem 5), whose
elementary definition and properties we review in Section 2. This version of the toric
degeneration of $F_{\ell n}$ is implicit in [GL96], but so much so as to be difficult to locate.
In addition, although the identification of the limiting fiber $F(0)$ as the Gel’fand–
Cetlin toric variety can be derived using results of [Cal02] and [Lit98], we include the
appropriate combinatorial arguments for the reader’s convenience (Section 3).

**Organization.** Our deformation of the Borel action on $M_n$ is constructed in Sec-
tion 1. Then Section 2 uses the results of [GL96] to show that the quotient $F = B \backslash (M_n \times \mathbb{C})$ is a flat family. Section 3 proves that the zero fiber of $F$ is the Gel’fand–
Cetlin toric variety. The connection to [KM03] is exhibited in Section 4 by explaining
how Schubert varieties behave inside of $F$. The final section deals with geomet-
ric quantization of the degeneration, by analyzing sections of line bundles over the
family $F$, thereby constructing Gel’fand–Cetlin decompositions geometrically.

1. **Degenerating the Borel action**

Thinking about $GL_n$ as a subset of $n \times n$ matrices $M_n$ allows us to think about the
flag manifold $F_{\ell n} = B \backslash GL_n$ as a GIT quotient of $M_n$ by $B$ whose precise definition
is given in Section 2. In this section we construct a degeneration of the action of $B
on M_n$, and in the next section we explain what happens to the GIT quotient under
this degeneration.

The group $(GL_n)^n$ has a left action on $M_n$ columnwise: if $Z \in M_n$ has columns
$Z_1, \ldots, Z_n$, then $\gamma = (\gamma_1, \ldots, \gamma_n) \in (GL_n)^n$ acts via

$$\gamma Z = \text{matrix with columns } \gamma_1 Z_1, \ldots, \gamma_n Z_n.$$  

The torus of $(GL_n)^n$ under the left action coincides with the standard torus inside
$GL(M_n)$, scaling separately each entry of any given matrix.

Let $B_\Delta \subset (GL_n)^n$ be the image of the lower triangular Borel subgroup $B \subset GL_n$
under the $n$-fold diagonal embedding in $(GL_n)^n$, so $B_\Delta = \{(b, \ldots, b) \mid b \in B\}$.

For every one-parameter subgroup $T \cong \mathbb{C}^*$ inside the torus of $(GL_n)^n$ consisting of
sequences of diagonal matrices, denote by $\tilde{\tau} \in T$ the element corresponding to the
complex number $\tau \in \mathbb{C}^*$. Given a matrix $\omega = (\omega_{ij})$ of integers, let the one-parameter
subgroup $T(\omega)$ consist of sequences of diagonal matrices, with the $j$th component of $\tilde{\tau}$
being the diagonal matrix $\tilde{\tau}_j = \text{diag}(\tau_{i_1}, \ldots, \tau_{i_j})$ for the $j$th column of $\omega$.

For the rest of the paper, fix the matrix $\omega$ whose entries equal

$$\omega_{ij} = \begin{cases} 3^{n-i-j} & \text{if } i + j \leq n, \\ 0 & \text{if } i + j > n. \end{cases}$$

For instance, when $n = 5$, we get the following $5 \times 5$ matrix:

$$\omega = \begin{bmatrix} 27 & 9 & 3 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the family $B^* \subset B^n \times \mathbb{C}^*$ of subgroups of $B^n$ with fiber

$$B(\tau) = \tilde{\tau}^{-1} B_\Delta \tilde{\tau}$$
over $\tau \in \mathbb{C}^*$, where $\tilde{\tau}$ lies in the one-parameter subgroup $T(\omega)$ corresponding to $\omega$. When $n = 5$, for instance, this multiplies the entries in $B^n$ by powers of $\tau$ as follows.

\[
\begin{pmatrix}
1 & \tau & \tau & \tau & \tau \\
\tau & 1 & \tau & \tau & \tau \\
\tau & \tau & 1 & \tau & \tau \\
\tau & \tau & \tau & 1 & \tau \\
\tau & \tau & \tau & \tau & 1
\end{pmatrix}
\]

The family $B^\tau$ extends to a family over all of $\mathbb{C}$:

**Definition 1.** The family $B \subset B^n \times \mathbb{C}$ has fiber $B(\tau)$ over $\tau \in \mathbb{C}^*$, and fiber $B(0)$ consisting of sequences $(b_1, \ldots, b_n) \in B^n$, where $b_j$ is obtained from the matrix $b_n$ by setting to 0 all entries in columns 1, $\ldots$, $n - j$ strictly below the main diagonal.

When $n = 5$, elements in the special fiber $B(0)$ look heuristically like:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

**Lemma 2.** There is a canonical algebraic group isomorphism $B \times \mathbb{C} \to B$ over $\mathbb{C}$.

**Proof.** Use that $B \cong B_{\Delta}$ by sending $b \mapsto b_{\Delta} = (b, \ldots, b)$. For $\tau \neq 0$ the isomorphism is now by $(\mathbb{Z})$, sending $b_{\Delta}$ to $\tilde{\tau}^{-1} b_{\Delta} \tilde{\tau}$. For $\tau = 0$, the map sets to 0 all entries in columns 1, $\ldots$, $n - j$ strictly below the main diagonal in the $j$th entry of $(b, \ldots, b)$.

Elementary computation shows that if $b$ is a lower triangular matrix, then the matrix $\tilde{\tau}^{-1} b \tilde{\tau}$ has no negative powers of $\tau$, and setting $\tau$ to 0 has the effect of setting to 0 all entries in columns 1, $\ldots$, $n - j$ strictly below the main diagonal of $b$. Hence the $\tau = 0$ case above really is obtained from the $\tau \neq 0$ case by taking limits as $\tau \to 0$. \qed

The family $B$ of groups acts fiberwise on $M_n \times \mathbb{C}$, but Lemma 2 allows us to view this fiberwise action as a single action of $B$ on the total space $M_n \times \mathbb{C}$. The actions on all fibers $M_n \times \tau$ are isomorphic for $\tau \in \mathbb{C}^*$, in the sense that the map $Z \times 1 \mapsto \tilde{\tau}^{-1} Z \times \tau$ identifies $M_n \times 1$ with $M_n \times \tau$ equivariantly with respect to the actions of $B$ on the fibers over 1 and $\tau$. However, when $\tau$ equals zero, $B$ acts on the $j$th column as the product of an $n - j$ dimensional torus (in the upper-left corner) and a smaller Borel group with $j$ columns (in the lower-right corner).

The action of $B = B(0)$ on $M_n \times 0$ commutes with an $\binom{n}{2}$ dimensional torus action, which scales all entries lying strictly above the main antidiagonal in each $n \times n$ matrix.

We shall see that this torus acts on the degenerated $\mathcal{F} \ell_n$ to make it a toric variety.

2. Degeneration of Plücker coordinates

Degenerating the action of $B$ on $M_n$ via the action of $B \times \mathbb{C}$ in Lemma 2 on $M_n \times \mathbb{C}$ induces a degeneration of the GIT quotient of $M_n$ by $B$. Using the results of Gonciulea and Lakshmibai [G196], we show that the GIT quotient $B \backslash (M_n \times \mathbb{C})$, when defined appropriately, flatly degenerates the flag manifold $\mathcal{F} \ell_n$ to a toric variety.

Let $U$ be the lower triangular matrices with 1’s on the diagonal (the unipotent radical of $B$). As can be done in the general setting of $B$ actions, we define the GIT
equation or term that is not clear.

Lemma 3. If every variable dividing the antidiagonal term of the minor \( \Delta_{I,J}(Z) \) in the generic matrix \( Z \) lies on or above the main antidiagonal of \( Z \), then the unique \( z \)-monomial in \( \Delta_{I,J}(Z) \) with the lowest weight is its antidiagonal term.

Proof. It suffices to prove the lemma when \( I \) and \( J \) have cardinality 2, because every \( z \)-monomial in each minor can be made into the antidiagonal term by successively replacing 2 \( \times \) 2 diagonals with 2 \( \times \) 2 antidiagonals. In the 2 \( \times \) 2 case, let \( I = \{ i, i+k \} \) and \( J = \{ j, j+\ell \} \) with \( i, j, k, \ell \geq 1 \). The weights on the two terms in \( \Delta_{I,J}(Z) \) satisfy

\[
3^{n-i-j} + 3^{n-i-k-j-\ell} > 3^{n-i-k-j} + 3^{n-i-j-\ell},
\]

which proves the lemma.

Denote by \( \tilde{t} \) the \( n \)-tuple of \( n \times n \) diagonal matrices whose \( j \)-th diagonal entry in the \( i \)-th matrix is \( t^{a_{ij}} \), and define \( \tilde{Z} \) for \( \tilde{\gamma} = \gamma \) as in (1) for the matrix \( Z \) of variables. In addition, for \( J = \{ j_1, \ldots, j_k \} \), let

\[
\omega_{J} = \sum_{i=1}^{k} \omega_{i,n+1-j_i}.
\]
be the sum of weights along the antidiagonal of the square submatrix in rows 1, \ldots, k and columns J of \omega. Then, as an immediate consequence of Lemma 3, we conclude that the polynomials

\[(4) \quad q_J = t^{-\omega_J} \Delta_J(\tilde{t} \tilde{Z})\]

are U-invariants in \(\mathbb{C}[z, t]\) under the action of \(B\) resulting from Lemma 2. The power \(t^{-\omega_J}\) precisely makes the antidiagonal term of \(q_J\) have coefficient \(\pm 1\).

**Definition 4.** Define the family \(\mathcal{F}\) inside the product \(\prod_{k=1}^{n} \mathbb{P}(\Lambda^k \mathbb{C}^n) \times \mathbb{C}\) over the line \(\mathbb{C} = \text{Spec}(\mathbb{C}[t])\) as the multiple Proj of the subalgebra \(\mathbb{C}[q_J | J \subseteq \{1, \ldots, n\}]\) of \(\mathbb{C}[z, t]\).

We wish to state the main result in this section in terms of sagbi bases. Recall that a *term order* on \(\mathbb{C}[z]\) is a multiplicative total order on monomials with \(1 \in \mathbb{C}[z]\) being smaller than any other monomial; see [Eis95, Chapter 15]. A set \(\{f_1, \ldots, f_r\} \subseteq \mathbb{C}[z]\) is a *sagbi basis* if the initial term \(\text{in}(f)\) of every polynomial \(f\) in the subalgebra \(\mathbb{C}[f_1, \ldots, f_r]\) lies inside the *initial subalgebra* generated by the initial terms \(\text{in}(f_1), \ldots, \text{in}(f_r)\). The initial subalgebra is generated by monomials, so its multiple Proj is a toric variety. Choosing a weight order inducing the given term order [Eis95, Chapter 15] allows us to express the original algebra and its initial algebra as the fibers over 0 and 1 of a flat family of subalgebras of \(\mathbb{C}[z]\). In fact, \(\{f_1, \ldots, f_r\}\) form a sagbi basis if and only if this degeneration to the initial subalgebra is flat.

The terms orders on the coordinate ring \(\mathbb{C}[z]\) of \(M_n\) that interest us are *antidiagonal* and *diagonal*. By definition, the leading term of any minor in the matrix of variables under an (anti)diagonal term order is its (anti)diagonal term, namely the product of all entries on the (anti)diagonal of the corresponding square submatrix. Initial terms of polynomials other than minors will not be important in what follows.

**Theorem 5.** The polynomials \(q_J\) from (4) generate the \(\mathbb{C}[t]\)-algebra of U-invariant functions inside \(\mathbb{C}[z, t]\), so \(\mathcal{F}\) is the GIT quotient family \(B \backslash (M_n \times \mathbb{C})\) flatly degenerating the flag manifold \(\mathcal{F}\ell_n = \mathcal{F}(1)\) to a toric variety \(\mathcal{F}(0)\). In fact, the Plücker coordinates \(p_J\) constitute a sagbi basis for any diagonal or antidiagonal term order.

**Proof.** The second sentence implies the first; to show this, we may as well assume by symmetry (reflecting left-to-right) that the term order is antidiagonal.

Setting \(t = 1\) in (4) obviously yields the Plücker coordinates \(p_J\). Therefore the polynomials \(q_J\) generate the U-invariants in \(\mathbb{C}[z, t^{\pm 1}]\) over the coordinate ring \(\mathbb{C}[t^{\pm 1}]\) of \(\mathbb{C}^*\), because the Plücker coordinates \(p_J\) generate the U-invariants over \(t = 1\), and the family of U-invariants is trivial by scaling outside \(t = 0\). Intersecting the U-invariants in \(\mathbb{C}[z, t^{\pm 1}]\) with \(\mathbb{C}[z, t]\) yields U-invariants in \(\mathbb{C}[z, t]\), because a polynomial function on \(M_n \times \mathbb{C}\) is U-invariant if and only if its restriction to \(M_n \times \mathbb{C}^*\) is. Therefore, we must show that the polynomials \(q_J\) generate as a \(\mathbb{C}[t]\) algebra the intersection with \(\mathbb{C}[z, t]\) of the subalgebra they generate inside \(\mathbb{C}[z, t^{\pm 1}]\). This follows from the sagbi property.

For the proof of the second sentence, we assume the term order is diagonal, to agree with [GL96]. Let \(H\) be the set of all subsets of \(\{1, \ldots, n\}\). Following [GL96], define a partial order on \(H\) as follows. For \(I = \{i_1 < \cdots < i_k\}\) and \(J = \{j_1 < \cdots < j_\ell\}\) set

\[I \geq J \iff k \leq \ell \text{ and } i_s \geq j_s \text{ for } 1 \leq s \leq k.\]
This makes $H$ a distributive lattice. It is shown in [GL96] that if $k \leq \ell$, then the meet and join of $I$ and $J$ are characterized by

$$I \land J = (\min(i_1, j_1), \ldots, \min(i_k, j_k), j_{\ell+1}, \ldots, j_\ell)$$
and
$$I \lor J = (\max(i_1, j_1), \ldots, \max(i_k, j_k)).$$

In the degeneration from [GL96, Theorems 5.2 and 10.6], the fiber over 1 equals the subalgebra of $C[z]$ generated by Plücker coordinates. The initial algebra in [GL96] maps surjectively onto the 'diagonal' semigroup algebra generated by diagonals, because the degenerated Plücker relations $x_I x_J = x_{I \land J} x_{I \lor J}$ in [GL96] hold on diagonals of Plücker coordinates. But the initial algebra in [GL96] and the diagonal semigroup algebra are integral domains of equal Krull dimension $\binom{n+1}{2}$ so the surjection must be an isomorphism. It follows that the diagonal semigroup algebra has the same Hilbert series as the Plücker algebra, by flatness of the degeneration in [GL96]. Since the diagonal semigroup algebra is certainly contained inside the initial algebra, whose Hilbert series equals that of the Plücker algebra by flatness again, the diagonal semigroup algebra must equal the initial algebra.

\[ \square \]

**Remark 6.** For each nonzero $\tau$, the fiber $\mathcal{F}(\tau)$ equals $B(\tau) \setminus GL_n(\tau)$, where $GL_n(\tau)$ is the set of $n \times n$ matrices $Z$ with nonzero $\tau$-determinant $\det(\tilde{\tau}Z)$. When $\tau = 0$, we can define $GL_n(0)$ as the set of matrices with nonzero entries along the antidiagonal. The quotient $B(0) \setminus GL_n(0)$ is still well defined, but no longer equal to $\mathcal{F}(0)$: geometrically, under the zeroth Plücker map, whose coordinates are obtained by setting $t = 0$ in (4), the variety $GL_n(0)$ has image equal to an open cell inside the toric variety $\mathcal{F}(0)$. \[ \square \]

### 3. Gel'fand–Cetlin polytopes

Next we show that the zero fiber $\mathcal{F}(0)$ is the toric variety associated to a Gel'fand–Cetlin polytope, defined as follows. Let $\lambda = (\lambda_1 > \cdots > \lambda_n)$ be a nonincreasing sequence of nonnegative integers. An array $\Lambda = (\lambda_{i,j})_{i+j \leq n+1}$ of real numbers is a Gel'fand–Cetlin pattern for $\lambda$ if $\lambda_{i,1} = \lambda_i$ for all $i = 1, \ldots, n$, and $\lambda_{i,j} \geq \lambda_{i,j+1} \geq \lambda_{i+1,j}$ for $i, j = 1, \ldots, n$. Equivalently, entries in Gel'fand–Cetlin patterns $\Lambda$ decrease in the directions indicated by the arrows in diagram below, whose left column is $\lambda$:

\[
\begin{align*}
\lambda_{1,1} &\rightarrow \lambda_{1,2} \rightarrow \lambda_{1,3} \rightarrow \cdots \\
\downarrow &\swarrow \downarrow \swarrow \\
\lambda_{2,1} &\rightarrow \lambda_{2,2} \rightarrow \cdots \\
\downarrow &\swarrow \\
\lambda_{3,1} &\rightarrow \cdots \\
\downarrow \\
&\vdots
\end{align*}
\]

(5)

The Gel'fand–Cetlin polytope $P_{\lambda}$ is the convex hull of all integer Gel'fand–Cetlin patterns for $\lambda$. This polytope defines the Gel'fand–Cetlin toric variety together with its projective embedding. For background on toric varieties, see [Ful93].

Set $a_k = \lambda_k - \lambda_{k+1}$ for $k = 1, \ldots, n$, where by convention $\lambda_{n+1} = 0$, and assume $a_k \geq 1$ for all $k$. Recall that we expressed the flag manifold $\mathcal{F} \ell_n$ as a subvariety of the product $\mathbb{P}_1 \times \cdots \times \mathbb{P}_n$, where $\mathbb{P}_k = \mathbb{P}(\wedge^k \mathbb{C}^n)$. Since all $a_k$ are strictly positive, the
integer sequence \( a = (a_1, \ldots, a_n) \) corresponds to a choice of very ample line bundle \( \mathcal{O}_{\mathcal{F} \ell_n}(a) \) on \( \mathcal{F} \ell_n \), namely the result of tensoring together the pullbacks of the bundles \( \mathcal{O}_{\mathcal{F}_i}(a_1), \ldots, \mathcal{O}_{\mathcal{F}_n}(a_n) \) to \( \mathcal{F} \ell_n \). In fact, we get a choice of very ample line bundle \( \mathcal{O}_{\mathcal{F}}(a) \) on the entire family \( \mathcal{F} \) from Definition 4, to get an embedding of the family

\[
\mathcal{F} \to \mathbb{P}_\lambda \times \mathbb{C} \quad \text{where} \quad \mathbb{P}_\lambda = \mathbb{P}\left( \otimes_{k=1}^n \text{Sym}^{a_k}(\Lambda^k \mathbb{C}^n) \right).
\]

The image of the zero fiber \( \mathcal{F}(0) \) is a toric variety projectively embedded inside \( \mathbb{P}_\lambda \) by a line bundle \( \mathcal{O}_{\mathcal{F}(0)}(a) \).

Let \( \alpha_I \) be the exponent vector on the antidiagonal monomial of the Plücker coordinate \( p_I \). Such exponent vectors are elements in \( \mathbb{Z}^{n^2} \) that look, for example, like

\[
\begin{array}{|c|c|}
\hline
1 & 1 \\
1 & 1 \\
\hline
\end{array}
\quad \text{for } p_{124} \quad \text{and} \quad
\begin{array}{|c|c|}
\hline
1 & 1 \\
1 & 1 \\
\hline
\end{array}
\quad \text{for } p_{13}.
\]

The vector space of global sections of \( \mathcal{O}_{\mathcal{F}(0)}(a) \) decomposes into a multiplicity-one direct sum of weight spaces. The set of weights occurring in this decomposition is

\[
\mathcal{T}_\lambda = \left\{ \text{sums } \sum \alpha_I \text{ in which } a_k \text{ of the indices } I \subseteq \{1, \ldots, n\} \text{ have size } k \right\}.
\]

**Proposition 7.** The projective embedding \( \mathcal{F}(0) \to \mathbb{P}_\lambda \) is the projective embedding of the Gel’fand–Cetlin toric variety associated to the polytope \( P_\lambda \).

**Proof.** By standard results about projective embeddings of toric varieties [Ful93], it suffices to identify \( P_\lambda \) with the convex hull \( \text{conv}(\mathcal{T}_\lambda) \) of all points in \( \mathcal{T}_\lambda \). This we shall prove for all \( \lambda \), not just those producing strictly positive \( a_1, \ldots, a_n \).

Denote the set of integer points of the polytope \( P_\lambda \) by \( \Pi_\lambda \). Then \( \Pi_\lambda \) sits inside an integer lattice of rank \( \frac{n(n-1)}{2} \), and \( P_\lambda = \text{conv}(\Pi_\lambda) \).

We claim that the linear map \( \phi : \mathbb{Z}^{n^2} \to \mathbb{Z}^{\frac{n(n-1)}{2}} \) given by

\[
\lambda_{ij} = a_{ij} + a_{i,j+1} + \cdots + a_{i,n+1-i}
\]

provides a bijection between \( \mathcal{T}_\lambda \) and \( \Pi_\lambda \), implying the required identification of \( \text{conv}(\mathcal{T}_\lambda) \) with \( P_\lambda \).

To check this, consider the map \( \psi : \mathbb{Z}^{\frac{n(n-1)}{2}} \to \mathbb{Z}^{n^2} \) given by

\[
a_{ij} = \lambda_{i,j} - \lambda_{i-1,j}.
\]

Notice that the composite maps \( \phi \circ \psi \) and \( \psi \circ \phi \) are identities on \( \Pi_\lambda \) and \( \mathcal{T}_\lambda \) respectively. It remains to check \( \phi(\mathcal{T}_\lambda) \subseteq \Pi_\lambda \) and \( \psi(\Pi_\lambda) \subseteq \mathcal{T}_\lambda \); these are simple exercises in linear algebra that go as follows.

To check \( \phi(\mathcal{T}_\lambda) \subseteq \Pi_\lambda \), consider an element \( \alpha = \sum \alpha_I \) of \( \mathcal{T}_\lambda \). The \( \lambda_{i,1} \) coordinate of \( \phi(\alpha) \) is the sum of all \( a \)'s in row \( i \), and this equals \( a_i + \cdots + a_n = \lambda_i \), which is the number of indices \( I \) of size at least \( i \). The \( \lambda_{i,j} \) coordinate of \( \phi(\alpha) \) is the sum of the \( a \)'s in the horizontal strip between \( a_{i,j} \) and the antidiagonal. This is not greater than \( \lambda_{i-1,j} \), which is the sum of the one-longer horizontal strip starting at \( a_{i-1,j} \). On the other hand, \( \lambda_{i,j} \) is at least \( \lambda_{i-1,j+1} \). Indeed, \( \lambda_{i-1,j+1} \) is the sum of the entries in the horizontal strip of the same length as for \( \lambda_{i,j} \), but shifted down one row and moved one column to the left. Since \( \alpha \) is the sum of \( \alpha_I \)'s, the sum of entries for \( \lambda_{i-1,j+1} \) is at most that for \( \lambda_{i,j} \). Thus \( \phi(\mathcal{T}_\lambda) \) is a subset of \( \Pi_\lambda \).
Conversely, given a Gel’fand–Cetlin pattern $\Lambda$ for $\lambda$ we need to show that $\psi(\lambda)$ can be written as a sum $\sum \alpha_I$. (It will follow immediately that there are $a_k$ indices $I$ of cardinality $k$, since the number of indices of cardinality at least $k$ must be $\lambda_k$.) Derive a pattern $\Lambda'$ from $\Lambda$ by decreasing the last nonzero entry $\lambda_{k,i}$ in each row by 1. Then $\Lambda'$ is still a Gel’fand–Cetlin pattern. At the same time it is clear that $\psi(\Lambda) = \psi(\Lambda') + \alpha_I$ for the set $I = \{i_1 > \ldots > i_{\ell}\}$, where row $\ell$ of $\Lambda$ is not zero but row $\ell+1$ of $\Lambda$ is zero. Induction on the sum of the entries in Gel’fand–Cetlin patterns finishes the proof.

4. Degenerating Schubert varieties

In this section we present our main theorem, which says that Schubert varieties degenerate inside the family $\mathcal{F}$ to unions of toric subvarieties given by rc-faces of the Gel’fand–Cetlin polytope. First, we must review the combinatorics involved.

Consider a finite subset $R$ of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ and think of it as a network of pipes, which intersect at each $(i,j) \in R$ and do not intersect otherwise. Such subsets are called diagrams (but are also known as pipe dreams). For example, see Figure 1.

![Figure 1. Diagrams that are given by \{(2,1),(2,2),(2,3),(3,1),(3,2)\}, \{(1,2),(2,1),(2,2),(3,1)\} and \{(1,1),(1,4),(2,1),(2,2)\}.](image)

Associate to each diagram $R$ the permutation $w_R \in S_n$ such that the pipe entering row $i$ exits column $w_R(i)$. For example, the permutations associated to the diagrams from Figure 1 are 15423, 14235 and 21534. The diagram $R$ is an rc-graph (or reduced pipe dream) if no two strands intersect twice. The first and the third diagrams from Figure 1 are rc-graphs, while the second one is not. These diagrams were originally introduced in [FK96b]. They index the monomials in Schubert polynomials the same way that semistandard Young tableaux index monomials in Schur polynomials; see [BJS93], [FS94] and [FK96b] for details.

For an rc-graph $R$, let

- $L_R$ be the coordinate subspace of $M_n$ consisting of all matrices whose coordinates $z_{ij}$ are zero for every crossing $(i,j) \in R$;
- $F_R$ be the rc-face of the Gel’fand–Cetlin polytope given by setting $\lambda_{i,j} = \lambda_{i+1,j}$ for each $(i,j) \in R$; and
- $T_R$ be the rc-toric subvariety of the Gel’fand–Cetlin toric variety with face $F_R$.

Next let us review some geometric ingredients for our main theorem and its proof. For a permutation $w \in S_n$, the Schubert determinantal ideal $I_w \subseteq \mathbb{C}[z]$, defined by Fulton in [Ful92], is generated by all minors of size $1 + wp$ in the top left $q \times p$
submatrix $Z_{p \times q}$ of $Z = (z_{ij})$ for all $q, p$, where $w_{qp}$ is the number of $i \leq q$ such that $w(i) \leq p$. The matrix Schubert variety for $w$ \cite{KM03} is by definition the zero set $X_w$ of $I_w$. We denote by $F_w \subseteq \mathcal{F}_n$ the Schubert variety obtained by projecting $X_w \cap GL_n$ to $\mathcal{F}_n$. This Schubert variety is the closure in $\mathcal{F}_n$ of the $B^+$ orbit through the coset $B\bar{w}$, where the permutation matrix $\bar{w}$ has its nonzero entries at $(i, w(i))$, and $B^+$ is the Borel group of upper triangular matrices acting on the right of $\mathcal{F}_n = B \setminus GL_n$.

In general, a flat family degenerating a variety $Y$ does not induce degenerations on its subvarieties. However, Gröbner and sagbi degenerations of $Y$ are canonically isomorphic to trivial families over $\mathbb{C}^*$. Any subvariety of $Y$ defines an (isomorphically trivial) subfamily over $\mathbb{C}^*$, and hence a flat subfamily over all of $\mathbb{C}$ by taking the closure of this subfamily.

**Theorem 8.** The quotient family $\mathcal{F} = B \setminus (M_n \times \mathbb{C})$ induces flat degenerations of Schubert subvarieties $X_w$ of the complete flag manifold $\mathcal{F}_n = \mathcal{F}(1)$ to reduced unions $\bigcup_{w(R) = w} T_R$ of toric subvarieties of the Gel'fand–Cetlin toric variety $\mathcal{F}(0)$.

**Proof.** It was shown in \cite{KM03} that the minors generating $I_w$ constitute a Gröbner basis for any antidiagonal term order, and hence define a Gröbner degeneration of $X_w$. Moreover, it was shown that any such degeneration—in particular (by Lemma 3) the one given by $\omega$—degenerates the matrix Schubert variety $X_w$ to the reduced union $\bigcup_{w(R) = w} L_R$ of rc-subspaces for $w$ inside $M_n$. The image of the total space of this Gröbner degeneration under the family of Plücker embeddings given by coordinates \cite{Z} equals our family $\mathcal{F}$ by Theorem $\S$. On the other hand, the closure of the image of an rc-subspace $L_R$ under the degenerated Plücker map obtained by setting $t = 0$ in \cite{Z} equals the corresponding rc-toric subvariety $T_R$ by definition.

The argument in the proof can be summarized as: the GIT quotient by $B$ of the Gröbner degeneration in \cite{KM03} equals the sagbi degeneration in Theorem $\S$.

**Remark 9.** The Schubert variety $X_w \subseteq \mathcal{F}_n$ equals the intersection of the embedded subvariety $\mathcal{F}_n \subseteq \prod \mathbb{P}(\Lambda^k \mathbb{C}^n)$ with a set of hyperplanes, one hyperplane $p_I = 0$ for each subset $I = \{i_1, \ldots, i_k\}$ satisfying $k > w_{k,i_k}$, where $w_{k,i_k}$ is the number of $i \leq k$ with $w(i) \leq i_k$. Intersecting the family $\mathcal{F}$ with the same set of hyperplanes produces the degeneration of $X_w$ in Theorem $\S$.

**Remark 10.** Using the involution on $\mathcal{F}_n$ that switches the Plücker coordinate $p_I$ with $p_{\bar{I}}$, where $\bar{I}$ is the complement of $I$, it can be shown that opposite Schubert varieties degenerate to unions of toric subvarieties associated to opposite rc-walls of the Gel'fand–Cetlin polytope.

---

5. Gel'fand–Cetlin decomposition

This final section gives a geometric construction of the Gel'fand–Cetlin basis of an irreducible $GL_n$ representation, by extending the Borel–Weil construction to the whole family $\mathcal{F}$. Our proof logically depends only on the Borel–Weil theorem. To introduce notation, we begin by reviewing the construction of Gel'fand and Cetlin \cite{GC50}.

For a dominant weight $\lambda$ of $GL_n$, which by definition is a decreasing sequence $(\lambda_1 > \cdots > \lambda_n)$ of positive integers, let $V^\lambda$ be the irreducible representation of $GL_n$.
with highest weight \( \lambda \). For \( n \geq i \geq 1 \) identify \( GL_i \) with the subgroup of \( GL_n \) sitting in the bottom right \( i \times i \) corner. As a \( GL_{n-1} \) representation, \( V^\lambda \) breaks up into a direct sum of irreducible components

\[
V^\lambda = \bigoplus_{\mu < \lambda} V^\mu,
\]

where the dominant weight \( \mu = (\mu_1 > \cdots > \mu_{n-1}) \) of \( GL_{n-1} \) satisfies

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots \geq \mu_{n-1} \geq \lambda_n,
\]

so \( \mu \) interpolates between \( \lambda \). Iterating defines partial Gel’fand–Cetlin decompositions

\[
V^\lambda = \bigoplus_{\Lambda_i} V^{\Lambda_i},
\]

of \( V^\lambda \) into irreducible components for the action of \( GL_i \), where \( \Lambda_i \) runs through chains \( (\lambda \succ \lambda_{i-1} \succ \cdots \succ \lambda^i) \) with \( \lambda_i \) being a weight of \( GL_j \). Gel’fand and Cetlin studied the decomposition of \( V^\lambda \) as a direct sum of one-dimensional subspaces \( V^\Lambda_i \), one for each chain \( \Lambda = (\lambda \succ \lambda_{i-1} \succ \cdots \succ \lambda^i) \). By definition, \( \Lambda \) lies in \( \Pi_\lambda \), the set of integer Gel’fand–Cetlin patterns for \( \lambda \). Hence we have the Gel’fand–Cetlin decomposition

\[
V^\lambda = \bigoplus_{\Lambda \in \Pi_\lambda} V^{\Lambda}.
\]

For a weight \( \lambda \), consider the very ample line bundle \( L = \mathcal{O}_F(a) \) from Section 3 over the family \( F \). Let \( L^\lambda_0 \) be the restriction of this line bundle to the fiber over \( \tau \in \mathbb{C} \). The Borel–Weil theorem states that the representation \( V^\lambda \) with highest weight \( \lambda \) is isomorphic to the space of algebraic sections of \( L^\lambda_0 \) as a representation of \( GL_n \):

\[
V^\lambda = \Gamma(L^\lambda_0).
\]

At the same time, we have already seen that the space \( \Gamma(L^\lambda_0) \) of sections over the toric variety \( F(0) \) carries an action of a torus \( T \) of dimension \( \binom{n}{2} \) under which \( \Gamma(L^\lambda_0) \) decomposes into one-dimensional weight spaces. Think of \( T \) as the product of one-dimensional tori \( T^i_{ij} \) for \( i + j \leq n \), each of which acts on \( M_n \) by scaling the \((i,j)\) entry, which lies strictly above the main antidiagonal. The weight spaces for the action of \( T \) on \( \Gamma(L^\lambda_0) \) are indexed by the set \( \Upsilon_\lambda \) from (7) in Section 3. In other words,

\[
\Gamma(L^\lambda_0) = \bigoplus_{A \in \Upsilon_\lambda} \mathbb{C}^A,
\]

where \( \mathbb{C}^A \) is a complex line on which \( T \) acts with weight \( A \).

Let \( T_{ij} \) be the one-dimensional torus scaling simultaneously the entries of an \( n \times n \) matrix in row \( i \), between column \( j + 1 \) and the antidiagonal column \( n + 1 - i \). Then \( T \) can be thought of as the product of all tori \( T_{ij} \) with \( i + j \leq n \). Under this direct product decomposition of \( T \), the discussion in the proof of Proposition 7 identifying \( \Upsilon_\lambda \) with \( \Pi_\lambda \) implies the weight space decomposition

\[
\Gamma(L^\lambda_0) = \bigoplus_{\Lambda \in \Pi_\lambda} \mathbb{C}^\Lambda,
\]
The weight space decompositions \([12]\) and \([13]\) are of course the same, and the two indexings correspond to two different choices of bases of the weight lattice of \(T\).

The family \(\mathcal{F}\) is projective over the affine complex line \(\text{Spec}(\mathbb{C}[t])\), so the algebraic sections \(\Gamma(\mathcal{L}^\lambda)\) form a finitely generated module over the coordinate ring \(\mathbb{C}[t]\) of the base. The localization \(\Gamma(\mathcal{L}^\lambda) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]\) is a finitely generated free module over the coordinate ring \(\mathbb{C}[t^{\pm 1}]\) of the complement of \(0 \in \mathbb{C}\), by triviality of the family \(\mathcal{F}\) outside the fiber over 0, and invariance of the vector space dimension of \(\Gamma(\mathcal{L}^\lambda)\) as a function of \(\tau\). On the other hand, \(\dim_{\mathbb{C}}(\mathcal{L}^\lambda) = \dim_{\mathbb{C}}(\mathcal{L}^\lambda)\) for \(\tau \neq 0\); indeed, both dimensions equal the number of lattice points in the Gel'fand–Cetlin polytope \(P_\lambda\), by \([10]\), \([11]\), and \([13]\). Hence we get the following.

**Proposition 11.** \(\Gamma(\mathcal{L}^\lambda)\) is free over \(\mathbb{C}[t]\), and possesses a \(\mathbb{C}[t]\)-basis of sections each of which is equivariant for the action of the \(n\)-dimensional diagonal torus in \(GL_n\). Restricting this basis to the 0 and 1 fibers results in a torus-equivariant isomorphism

\[
\Phi : \Gamma(\mathcal{L}^\lambda_1) = V^\lambda \longrightarrow \bigoplus_{\Lambda \in \Pi_\lambda} \mathbb{C}^\Lambda = \Gamma(\mathcal{L}^\lambda_0).
\]

To understand the map \(\Phi\) in terms of the inductive construction of Gel'fand and Cetlin, we present a construction of an \(n\)-parameter family extending \(\mathcal{F}\). Let \(\omega_i\) be the \(n \times n\) matrix whose \(i^{th}\) column has entries \(\omega_{i1}, \ldots, \omega_{in}\) and whose other columns are zero (the integers \(\omega_{ij}\) are defined in \([2]\)). Write \(T(\omega_i)\) for the one parameter subgroup of the torus of \(B^n\) associated to \(\omega_i\), and denote by \(\vec{\tau}_i\) the element of \(T(\omega_i)\) corresponding to the complex number \(\tau_i\). Define the family

\[
B(\tau_1, \ldots, \tau_n) = \vec{\tau}_1^{-1} \cdots \vec{\tau}_n^{-1} B_\Delta \vec{\tau}_1 \cdots \vec{\tau}_n
\]

of subgroups of \(B^n\). This family extends to zero values of \(\tau_i\) and defines an action of \(B\) on \(M_n \times \mathbb{C}^n\) as in Lemma \([2]\).

**Definition 12.** The \(n\)-parameter family \(\tilde{\mathcal{F}} = B \langle (M_n \times \mathbb{C}^n) \rangle\) is the degeneration in stages. Denote by \(\tilde{F}_i\) its fiber over the point \((0, \ldots, 0, 1, \ldots, 1)\) with \(i\) entries equal to 1.

Observe that \(\tilde{F}_n = \mathcal{F}\ell_n\) is the flag manifold, and \(\tilde{F}_0\) is the Gel'fand–Cetlin toric variety by Theorem \([8]\).

Let \(T_i\) be the torus \(T_{n-1} \times \cdots \times T_i\) with \(T_j = T_{n-j} \times \cdots \times T_{j,n-j}\), and set \(G_i = GL_i\), thought of in the bottom right corner again. For the \(n = 5\) example of \(T_i\), each torus \(T_j\) scales the entries by its one-parameter subgroups \(T_{ij}\) in the indicated locations:

\[
\begin{array}{c|c|c|c}
\hline
T_4 & T_3 & T_2 & T_1 \\
\hline
T_{11} & T_{21} & T_{31} & T_{41} \\
T_{12} & T_{22} & T_{32} & \hline
T_{13} & T_{23} & T_{33} & T_{43} \\
\hline
T_{14} & T_{24} & T_{34} & \\
\hline
\end{array}
\]

Each fiber \(\tilde{F}_i\) carries the action of the group \(G_i \times T_i\). Moreover, for each \(i\), a subfamily of \(\tilde{F}\) degenerates \(\tilde{F}_i\) to \(\tilde{F}_{i-1}\) flatly and \(G_{i-1} \times T_i\) invariantly.

Associate to each dominant weight \(\lambda\) the very ample line bundle \(\tilde{L}^\lambda\) over \(\tilde{F}\), as we did for the family \(\mathcal{F}\). Then, as in \([8]\) for \(\mathcal{F}\), we get an embedding of \(\tilde{F}\) into \(\mathbb{P}_\lambda \times \mathbb{C}^n\). Let \(V_i^\lambda\) be the space of algebraic sections of the restriction of the line bundle \(\tilde{L}^\lambda\) to \(\tilde{F}_i\), treated as a representation of \(G_i \times T_i\). Since \(\tilde{F}_i\) degenerates to \(\tilde{F}_{i-1}\) flatly and
\[ G_{i-1} \times T_i \text{ invariantly, there are } G_{i-1} \times T_i \text{ invariant isomorphisms } \Phi_i : V^\lambda_i \rightarrow V^\lambda_{i-1} \text{ for } i = 1, \ldots, n. \text{ These are analogous to the isomorphism } \Phi \text{ from Proposition 11 and constructed geometrically in the same way. In fact } \Phi \text{ equals the composition } \Phi_1 \circ \cdots \circ \Phi_n, \text{ or equivalently } \Phi : V^\lambda = V^\lambda_n \xrightarrow{\Phi_n} V^\lambda_{n-1} \xrightarrow{\Phi_{n-1}} \cdots \xrightarrow{\Phi_1} V^\lambda_0. \]

In what follows, we write \( \Pi_\lambda(i) \) for the set of integer patterns \( \Lambda_i = (\lambda \succ \lambda^{n-1} \succ \cdots \succ \lambda^i) \), with \( \lambda^j \) being a weakly decreasing sequence of \( j \) nonnegative integers. For each pattern \( \Lambda_i \in \Pi_\lambda(i) \), let \( V^\Lambda_i \) be the irreducible representation of \( G_i \) with highest weight \( \lambda^i \), and declare the torus \( T_i \) to act on every vector in \( V^\Lambda_i \) with weight \( \Lambda^i \).

**Theorem 13.** The sections \( V^\lambda_i \) of the line bundle \( \tilde{L}^\lambda \) over the fiber \( \tilde{F} \) of the degeneration in stages decomposes into irreducible components for \( G_i \times T_i \) as
\[
V^\lambda_i = \bigoplus_{\Lambda_i \in \Pi_\lambda(i)} V^\Lambda_i,
\]
so
\[
V^\lambda = \bigoplus_{\Lambda_i \in \Pi_\lambda(i)} \Phi^{-1}_n \circ \cdots \circ \Phi^{-1}_{i+1}(V^\Lambda_i)
\]
is a partial Gel'fand–Cetlin decomposition. Thus the Gel'fand–Cetlin decomposition is
\[
V^\lambda = \bigoplus_{\Lambda \in \Pi_\lambda} \Phi^{-1}(C^\Lambda).
\]

**Proof.** It is enough to assume (14) is proved for \( i \), and then prove it for \( i - 1 \). Let \( \Lambda_{i-1} \mapsto \Lambda_i \) be the map \( \Pi_\lambda(i-1) \rightarrow \Pi_\lambda(i) \) forgetting \( \lambda^{i-1} \). Since \( \Phi_i \) is \( G_{i-1} \) equivariant, we get a decomposition of \( V^\lambda_{i-1} \) into irreducible components under \( G_{i-1} \):
\[
V^\lambda_{i-1} = \bigoplus_{\Lambda_{i-1} \in \Pi_\lambda(i-1)} \bigoplus_{\Lambda_{i-1} \rightarrow \Lambda_i} V^{\lambda^{i-1}}.
\]
It remains to show that \( T_{i-1} \) acts on each irreducible \( V^{\lambda^{i-1}} \) with weight \( \lambda^{i-1} \).

After the identification of \( V^\lambda \) with \( \Gamma(C^\lambda_1) \), every highest weight vector of the \( G_{i-1} \) action on \( V^\lambda \) can be thought of as a monomial \( \prod p_I \) in Plücker coordinates for subsets \( I \) whose columns in the range \( n - i + 2, \ldots, n \) are left justified. (These are the monomials invariant with respect to the right action of \( U^+_{i-1} \), the upper triangular matrices inside \( G_{i-1} \) with 1's on the diagonal.) Now simply note that the weight of \( T_{i-1} \) on such a monomial coincides with the weight of the diagonal torus in \( G_{i-1} \). \( \square \)

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