Abstract

We consider the Poisson equation \((I - P)u = g\), where \(P\) is the transition matrix of a Quasi-Birth-and-Death (QBD) process with infinitely many levels, \(g\) is a given infinite dimensional vector and \(u\) is the unknown. Our main result is to provide the general solution of this equation. To this purpose we use the block tridiagonal and block Toeplitz structure of the matrix \(P\) to obtain a set of matrix difference equations, which are solved by constructing suitable resolvent triples.

Keywords: Poisson equation, QBD process, Matrix difference equation, Jordan pairs, Resolvent triples, Group inverse

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1 Introduction

Given a row-stochastic matrix \(P\) and a vector \(g\), the Poisson equation is written as

\[(I - P)u = g,\] (1)
where $\mathbf{u}$ is the unknown vector. A matrix $P$ is row-stochastic if it has non-negative entries and $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the vector with components equal to 1. The Poisson equation is important in Markov chain theory, where $P$ represents the transition probability matrix of an irreducible homogeneous Markov chain. Some examples of applications are: heavy-traffic limit theory e.g. Asmussen [1], central limit theorem e.g. Glynn [7], variance constant analysis e.g. Asmussen and Bladt [2], mean and variance of mixing times of QBD processes e.g. Li and Cao [13], asymptotic variance of single-birth process e.g. Jiang et al. [11].

If the matrix $P = (p_{i,j})_{i,j=1,n}$ is finite and $\pi$ is the stationary distribution of the Markov chain, i.e., $\pi$ is the vector such that $\pi^T P = \pi^T$ and $\pi^T \mathbf{1} = 1$, then the Poisson equation has solutions if and only if $\pi^T g = 0$. In fact, this condition is equivalent to the property that $g$ belongs to the span of the columns of $I - P$. Moreover, the Poisson equation has a unique solution, up to an additive constant, given by

$$\mathbf{u} = (I - P)^\# g + \alpha \mathbf{1}$$

for an arbitrary scalar $\alpha$, where $H^\#$ denotes the group inverse of the matrix $H$ (see Meyer [15] and Campbell and Meyer [3]). In the case of infinite state Markov chains, $P = (p_{i,j})_{i,j\in\mathbb{N}}$ has infinite size, and uniqueness does not hold in general. This case has been studied by Makowski and Schwartz in [14], where the authors give some characteristics of the solutions.

A class of infinite dimensional Markov chains, having great importance in applications, is given by Quasi-Birth-and-Death processes. The transition matrix of these processes has the following block tridiagonal, almost block-Toeplitz structure

$$P = \begin{bmatrix} B & A_1 & & & \\
A_{-1} & A_0 & A_1 & & \\
& A_{-1} & A_0 & \ddots & \\
& & \ddots & \ddots & 
\end{bmatrix}$$

where $B, A_{-1}, A_0$ and $A_1$ are square matrices of order $m < \infty$. The QBD processes are described and analyzed for instance in Neuts [16] and Latouche and Ramaswami [12]. In Dendievel et al. [6], the authors derive particular solutions of the Poisson equation for QBDs in terms of the deviation matrix, and their solution is based on probabilistic arguments.

In this paper, we focus instead on finding all the solutions of the Poisson equation for a QBD, by exploiting the structure of the matrix (3). Indeed,
rewriting (1) in terms of the blocks of the transition matrix (3) yields the set of equations
\[(B - I)u_0 + A_1u_1 = -g_0, \tag{4}\]
\[A_{-1}u_r + (A_0 - I)u_{r+1} + A_1u_{r+2} = -g_{r+1}, \tag{5}\]
for \(r \geq 0\), where the infinite vectors \(u\) and \(g\) have been partitioned into blocks \(u_i, g_i, i \geq 0\), of length \(m\). Equation (5), for \(r \geq 0\), represents a matrix difference equation, while (4) provides the initial condition. Seen in this way, the Poisson equation for QBDs may be analyzed by relying on the theory of matrix difference equations developed by Gohberg et al. in [9].

We provide the general expression of the solution of the matrix difference equation (5) by means of resolvent triples of the matrix polynomial
\[\eta(\lambda) = A_{-1} + (A_0 - I)\lambda + A_1\lambda^2. \tag{6}\]
Due to the stochasticity of \(P\), the polynomial \(\det(\eta(\lambda))\) has a root \(\lambda\) equal to 1. If \(\lambda = 1\) is a simple zero, like for positive recurrent or transient QBDs, then we may construct a resolvent triple from the minimal nonnegative solutions \(\tilde{G}\) and \(\hat{G}\) of the quadratic matrix equations
\[A_{-1} + (A_0 - I)X + A_1X^2 = 0\]
\[A_{-1}X^2 + (A_0 - I)X + A_1 = 0\]
respectively. If the QBD is null recurrent, then \(\lambda = 1\) is not a simple zero. In this case we cannot construct a resolvent triple directly from \(G\) and \(\hat{G}\), and we follow a different approach which consists in transforming the matrix difference equation (5) into a modified difference equation
\[\tilde{A}_{-1}\tilde{u}_r + (\tilde{A}_0 - I)\tilde{u}_{r+1} + \tilde{A}_1\tilde{u}_{r+2} = -\tilde{g}_{r+1}, \tag{6'}\]
such that the polynomial \(\det(\tilde{A}_{-1} + (\tilde{A}_0 - I)\lambda + \tilde{A}_1\lambda^2)\) has a simple root at \(\lambda = 1\). This new difference equation can be solved by means of resolvent triples and we give an explicit expression relating the solutions of the modified difference equation to the solutions of the original equation. This transformation relies on the shift technique introduced by He et al. [10] and developed by Bini et al. in [4].

Once we have a general expression of the solution of the difference equation (5), we impose the initial condition (4). In the positive and null recurrent cases, the initial condition leads to a Poisson equation of finite size, which can be solved by means of the group inverse according to equation (2). In the transient case, the initial condition leads to a nonsingular linear system. In all cases, the expression of the general solution of the Poisson equation depends on an arbitrary vector \(y\). We show that the particular solution obtained in Dendievel et al. [6] by means of probabilistic arguments, corresponds to a specific choice of the vector \(y\).
The paper is organized as follows. In Section 2, we recall the fundamental elements on QBD processes and introduce some key matrices. In Section 3, we analyze the matrix difference equation adapted to our problem and introduce the notion of resolvent triple. The main results of this paper are given in Sections 4 and 5. In Section 4, Theorem 3 provides the general solution of the Poisson equation in the case of a positive recurrent or a transient QBD process. In Section 5, we deal with the null recurrent case and show in details two different approaches based on the shift technique. We compare in Section 6 the particular probabilistic solution given in Dendievel et al. [6] with the solution given in Section 4.

2 Quasi-Birth-and-Death process

Some properties on QBD processes will be used in the next sections. We consider a discrete-time QBD process with transition matrix \( P \) given in (3), on the state space \( S = \{(n, i) : n \in \mathbb{N}, i \in \mathcal{E}\}, \mathcal{E} = \{1, \ldots, m\}\), where \( n \) is called the level and \( i \) is the phase. We define the matrix \( G \) as the minimal nonnegative solution of the equation

\[ A_{-1} + (A_0 - I)X + A_1X^2 = 0, \tag{7} \]

and the matrix \( \hat{G} \) as the minimal nonnegative solution of the equation

\[ A_1 + (A_0 - I)X + A_{-1}X^2 = 0. \tag{8} \]

The component \( G_{ij} \) of the matrix \( G \) is the conditional probability that the process goes to the level \( n \) in a finite time and that \( j \) is the first phase visited in this level, given that the process starts from state \((n+1, i)\), for \( i, j \in \mathcal{E}, n \in \mathbb{N}\). The matrix \( \hat{G} \) corresponds to the matrix \( G \) of the level-reversed process.

Throughout the paper we assume that \( P \) is irreducible, that \( A_{-1} + A_0 + A_1 \) is irreducible and that the following property holds.

**Assumption 1.** The doubly infinite QBD on \( \mathbb{Z} \times \mathcal{E} \) has only one final class \( \mathbb{Z} \times \mathcal{E}_* \), where \( \mathcal{E}_* \subseteq \mathcal{E} \). Every other state is on a path to the final class. Moreover, the set \( \mathcal{E}_* \) is not empty.

Assumption 1 is Condition 5.2 in [3, Page 111] where it is implicitly assumed that \( \mathcal{E}_* \) is not empty.
The roots of the polynomial $\phi(\lambda) = \det \eta(\lambda)$, where $\eta(\lambda)$ is defined in $[\text{0}]$, have a useful property that we give now for future reference (Bini et al. [3, Theorem 4.9], Govorun et al. [9, Theorem 3.2]).

**Lemma 1.** Denote by $\xi_1, \ldots, \xi_{2m}$ the roots of $\phi(\lambda)$, organized so that $|\xi_1| \leq |\xi_2| \leq \cdots \leq |\xi_{2m}|$, with $\xi_{d+1} = \cdots = \xi_m = \infty$ if the degree of $\phi(\lambda)$ is $d < 2m$. The roots $\xi_1, \ldots, \xi_m$ are the eigenvalues of the matrix $G$ and $\xi_{m+1}, \ldots, \xi_{2m}$ are the reciprocals of the eigenvalues of $\hat{G}$ with the convention that $1/\infty = 0$ and $1/0 = \infty$. The roots $\xi_m, \xi_{m+1}$ are real and one has

$$|\xi_{m-1}| < \xi_m \leq 1 \leq \xi_{m+1} < |\xi_{m+2}|.$$

Moreover

- if the QBD is positive recurrent then $\xi_m = 1 < \xi_{m+1}$, $G$ is stochastic and $\hat{G}$ is sub-stochastic,

- if the QBD is transient then $\xi_m < 1 = \xi_{m+1}$, $G$ is sub-stochastic and $\hat{G}$ is stochastic,

- if the QBD is null recurrent then $\xi_m = 1 = \xi_{m+1}$, $G$ and $\hat{G}$ are stochastic.

Furthermore, thanks to the repetitive structure of $P$, the stationary distribution $\pi$ of the process, partitioned as $\pi = (\pi_i)_{i \geq 0}$, where $\pi_i$ is an $m$-dimensional vector representing the stationary probability of the level $i$, has a matrix-geometric structure, that is, $\pi_i^T = \pi_0^TR^i$, where $R$ is the minimal nonnegative solution of the equation

$$A_1 + X(A_0 - I) + X^2A_{-1} = 0 \quad \text{(9)}$$

(Latouche and Ramaswami [12, Theorems 6.2.1 and 6.2.10]).

We assume in this section and the next two that the QBD process is positive recurrent or transient (not null recurrent for short). Under this assumption the series

$$W = \sum_{j=0}^{\infty} G^j(U - I)^{-1}R^j \quad \text{(10)}$$

is convergent, where $U = A_0 + RA_{-1}$, and it is shown in [4] that the matrices $R$ and $\hat{G}$ are related by

$$WR = \hat{GW}. \quad \text{(11)}$$
Note for later reference that from [12, Theorem 6.2.9] we have the relations
\[ R = A_1(I - U)^{-1}, \quad (12) \]
\[ U = A_0 + A_1 G. \quad (13) \]

In addition, the vector \( \pi_0 \) is a solution of
\[ \pi_0^T(I - B - A_1 G) = 0, \quad (14) \]
normalized by \( \pi_0^T(I - R)^{-1}1 = 1. \)

We show in Lemma 2 that \( W \) is invertible, so that \( R \) and \( \hat{G} \) are actually similar matrices. The subsequent results give additional characterizations of the matrix \( W \).

**Lemma 2.** If the QBD is not null recurrent, then the matrix \( W \) defined in (10) and the matrix \( G\hat{G} - I \) are nonsingular, moreover
\[ W^{-1} = (I - U) \left( G\hat{G} - I \right). \quad (15) \]

**Proof.** By (11), we obtain
\[ \left( G\hat{G} - I \right) W = \sum_{j=0}^{\infty} G^{j+1}(U - I)^{-1}R^{j+1} - W = -(U - I)^{-1}. \]
It follows that \( W \) and \( G\hat{G} - I \) are nonsingular so that (15) holds. \( \square \)

**Lemma 3.** If the QBD is not null recurrent, then the matrix \( W \) of (10) is such that
\[ WA_1(G\hat{G} - I) = \hat{G}. \quad (16) \]

**Proof.** Since the QBD is not null recurrent, \( W \) is well defined and invertible, and it follows from (11) that equation (16) is equivalent to
\[ WA_1(G\hat{G} - I)W = WR, \quad \text{or to} \quad A_1GWR - A_1W = R. \]
Replacing \( W \) by its definition leads to
\[ A_1 \sum_{j=0}^{\infty} G^{j+1}(U - I)^{-1}R^{j+1} - A_1(U - I)^{-1} - A_1 \sum_{j=1}^{\infty} G^{j}(U - I)^{-1}R^j = R, \]
which simplifies to \( A_1(I - U)^{-1} = R \), a true relation given by (12). \( \square \)
We introduce some notation. Let $M$ be a nonsingular matrix such that
\[ \hat{G}M = MJ, \]  
(17)
with
\[ J = \begin{bmatrix} \ V_1 & 0 \\ 0 & V_0 \end{bmatrix}, \]  
(18)
where $V_1$ is a nonsingular square matrix of order $p$, $0 \leq p \leq m$, and $V_0$ is square matrix of order $m - p$ with $\rho(V_0) = 0$. For instance, we may choose $M$ such that $J$ is the Jordan normal form of $\hat{G}$, with $V_1$ containing all the blocks for the non-zero eigenvalues and $V_0$ containing all the blocks for the zero eigenvalues. The matrix $M$ may be written with corresponding dimensions as
\[ M = [L \mid K], \]  
(19)
here, $L$ is a matrix with dimensions $m \times p$ and $K$ is a matrix with dimensions $m \times (m - p)$. As a consequence, we have that (17) may be equivalently written as the system of equations
\[ \hat{G}L = LV_1, \quad \hat{G}K = KV_0. \]  
(20)

As a corollary of (8), we have
\[ A_1L + (A_0 - I)LV_1 + A_{-1}LV_1^2 = 0, \]
\[ A_1K + (A_0 - I)KV_0 + A_{-1}KV_0^2 = 0. \]  
(21)

The next proposition will be useful in Section 3.

**Lemma 4.** Assume that the QBD is not null recurrent. Take
\[ Y = [A_1LV_1^{-1} \mid -A_{-1}KV_0 - (A_0 - I)K], \]  
(22)
with $M, L, K, V_1$ and $V_0$ as defined above. The matrix $Y - A_1GM$ is nonsingular and the matrix $W$ defined in (10) is equal to
\[ W = M (A_1GM - Y)^{-1}. \]  
(23)

**Proof.** We prove that
\[ W(A_1GM - Y) = M, \]  
(24)
from which the nonsingularity of $A_1GM - Y$ and (23) follow, since $W$ and $M$ are nonsingular. By Lemma 2, (24) will follow from

$$Y = (I - U) \left( I - G\hat{G} \right) M + A_1GM.$$  \hfill (25)

By [12, Theorem 6.2.9] and (17),

$$\begin{align*}
(I - U)(I - G\hat{G})M + A_1GM \\
= (I - (A_0 + A_1G)) (M - GMJ) + A_1GM, \\
= M - A_0M + ((A_0 - I)G + A_1G^2) MJ, \\
= M - A_0M - A_{-1}MJ, \quad \text{by (7)}
\end{align*}$$

since $M$ and $J$ may be replaced, respectively with (19) and (18). From (21), it follows that

$$(I - A_0)L - A_{-1}LV_1 = A_1LV_1^{-1}$$

so that (25) is satisfied.

\section{Resolvent triple}

We report from Gohberg et al. [8] some definitions and results concerning the resolution of matrix difference equations. We apply these results to the solution of the Poisson equation.

Given the $m \times m$ matrix polynomial $B(\lambda) = \sum_{i=0}^{l} B_i\lambda^i$ of degree $l$, a pair of matrices $(X, T)$, with $X$ of size $m \times ml$ and $T$ of size $ml \times ml$, is called a decomposable pair for $B(\lambda)$ if:

1. $X = [X_1 \mid X_2]$, and $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$, where $X_1$ is an $m \times q$ matrix, $T_1$ is a $q \times q$ matrix, for some $0 \leq q \leq ml$;

2. the matrix

$$\begin{bmatrix}
X_1 & X_2T_2^{l-1} \\
X_1T_1 & X_2T_2^{l-2} \\
\vdots & \vdots \\
X_1T_1^{l-1} & X_2
\end{bmatrix}$$

is nonsingular;
3. \( \sum_{i=0}^{l} B_i X_1 T_1^i = 0 \) and \( \sum_{i=0}^{l} B_i X_2 T_1^{l-i} = 0 \).

Furthermore, the triple \((X, T, Z)\) is a resolvent triple of \(B(\lambda)\) if \((X, T)\) is a decomposable pair of \(B(\lambda)\) and \(Z\) is a matrix such that \(B^{-1}(\lambda) = XT^{-1}(\lambda)Z\), where \(T(\lambda) = \text{diag}(\lambda I - T_1, \lambda T_2 - I)\).

We state next a finite difference equation theorem for general matrix equations ([8, Theorem 8.3]).

**Theorem 1.** Let \((X, T, Z)\) be a resolvent triple of the \(m \times m\) matrix polynomial \(B(\lambda) = \sum_{i=0}^{l} B_i \lambda^i\), where \(X = [X_1 | X_2]\), and \(T = \text{diag}(T_1, T_2)\), and let \(Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\) be the corresponding partition of \(Z\). The general solution of the homogeneous difference equation

\[
B_0 u_r + B_1 u_{r+1} + \cdots + B_l u_{r+l} = 0,
\]

is \(h_r = X_1 T_1^r z\), for \(r \geq 0\), where \(z \in \mathbb{C}^q\) is arbitrary.

Let \(\{f_r\}_{r \in \mathbb{N}}\) be a sequence of vectors in \(\mathbb{C}^m\). A particular solution of the non-homogeneous difference equation

\[
B_0 u_r + B_1 u_{r+1} + \cdots + B_l u_{r+l} = f_r, \quad (26)
\]

is given by

\[
\sigma_r = -\sum_{i=0}^{\nu-1} X_2 T_2^i Z_2 f_{i+r} + \sum_{j=0}^{r-1} X_1 T_1^{r-j} Z_1 f_j, \quad r \geq 0,
\]

for some positive integer \(\nu\) such that \(T_2^\nu = 0\). The general solution of the non-homogeneous equation \(26\) is

\[
u_r = h_r + \sigma_r, \quad r \geq 0.
\]

In our QBD problem, \(26\) reduces to \(5\) and the matrix polynomial \(\eta(\lambda)\) defined in \(6\) plays the role of the matrix polynomial \(B(\lambda)\) defined in Theorem 11.

We use known properties of the blocks of the transition matrix to construct a resolvent triple of \(\eta(\lambda)\) and we obtain in the next section a general solution of the Poisson equation. Equation \(4\) furnishes a supplementary condition on the vector called \(z\) in Theorem 11. First, we give a decomposable pair of \(\eta(\lambda)\) in the following lemma. In its proof, we find it helpful to indicate explicitly the dimensions of the identity matrix and in such cases we indicate it as an index.
Lemma 5. Assume that the QBD (3) is not null recurrent. Let $G$ be the minimal nonnegative solution of (7), let $\hat{G}$ be the minimal nonnegative solution of (8) and let $L, K, V_1$ and $V_0$ be defined as in (17)–(19).

Define $X = [X_1 \mid X_2]$ with $X_1 = [I_m \mid L]$, $X_2 = K$, and define $T = \text{diag}(T_1, T_2)$, with $T_1 = \text{diag}(G, V_1^{-1})$, $T_2 = V_0$.

The pair $(X, T)$ is a decomposable pair of $\eta(\lambda)$.

Proof. We check the conditions (i), (ii) and (iii) of the definition of decomposable pairs. Conditions (i) and (iii) are obvious by construction. For condition (ii), we have to verify that the matrix

\[
\begin{bmatrix}
X_1 & X_2 T_2 \\
X_1 T_1 & X_2
\end{bmatrix} = \begin{bmatrix}
I_m & LKV_0 \\
G & KV_1^{-1} K
\end{bmatrix} \begin{bmatrix}
I_m & 0 & 0 \\
0 & V_1^{-1} & 0 \\
0 & 0 & I_{m-p}
\end{bmatrix},
\]

is nonsingular. By (17), we write

\[
\begin{bmatrix}
X_1 & X_2 T_2 \\
X_1 T_1 & X_2
\end{bmatrix} = \begin{bmatrix}
I_m & \hat{G} \\
G & \hat{G}
\end{bmatrix} \begin{bmatrix}
I_m & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix}
I_m & 0 & 0 \\
0 & V_1^{-1} & 0 \\
0 & 0 & I_{m-p}
\end{bmatrix},
\]

and this is a product of nonsingular matrices. In fact, the first factor is nonsingular since its determinant coincides with the determinant of $I - G\hat{G}$, which is nonsingular in view of Lemma 2. The other two factors are nonsingular by construction.

Given a decomposable pair $(X, T)$ of a matrix polynomial, Theorem 7.7 in Gohberg et al. [8] gives an explicit expression for a matrix $Z$ such that $(X, T, Z)$ is a resolvent triple for the same matrix polynomial. In the next theorem, we adapt this directly to our special case where the matrix polynomial is quadratic. The construction of such a triple $(X, T, Z)$ will help us to build the solution of the Poisson equation, relying on Theorem 1.

We partition the inverse of the matrix $M$ defined in (17) as

\[
M^{-1} = \begin{bmatrix}
E \\
F
\end{bmatrix},
\]

(27)

where $E$ is $p \times m$ and $F$ is $(m - p) \times m$. 

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Theorem 2. Assume that the QBD (3) is not null recurrent. Let \( Z \) be the matrix defined by
\[
Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} W \\ -EW \end{bmatrix}, \quad Z_2 = -V_0FW,
\]
where \( V_0 \) is given in (18), and \( E \) and \( F \) are defined in (27). The triple \( (X,T,Z) \), where \( X \) and \( T \) are given in Lemma 5, is a resolvent triple of \( \eta(\lambda) \).

Proof. The pair \( (X,T) \) in Lemma 5 is a decomposable pair of \( \eta(\lambda) \). By [8, Theorem 7.7], \( (X,T,Z) \) is a resolvent triple of \( \eta(\lambda) \) if \( Z \) takes the form
\[
Z = \begin{bmatrix} I_{m+p} & 0 \\ 0 & V_0 \end{bmatrix} \Gamma^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix},
\]
where
\[
\Gamma = \begin{bmatrix} I_m & L \\ A_1G & -A_1KV_0 - (A_0 - I)K \end{bmatrix} = \begin{bmatrix} I_m & M \\ A_1G & Y \end{bmatrix},
\]
with \( Y \) defined in (22). The matrix \( S = Y - A_1GM \) is the Schur complement of \( I_m \) in \( \Gamma \). By Lemma 4, the matrix \( S \) is invertible and \( S = -W^{-1}M \). We have
\[
\Gamma^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \begin{bmatrix} -MS^{-1} \\ S^{-1} \end{bmatrix} = \begin{bmatrix} W \\ -M^{-1}W \end{bmatrix} = \begin{bmatrix} W \\ -EW \\ -FW \end{bmatrix}
\]
by (27). Replacing this in (28) completes the proof. \( \square \)

4 The general solution: non null recurrent case

We need to recall the concept of group inverse of a matrix. When it exists, the group inverse \( H^\# \) of a square matrix \( H \) is the matrix solving the three equations \( HH^\# = H^\#H, HH^\#H = H, H^\#HH^\# = H^\# \); if \( H \) is nonsingular, then \( H^\# = H^{-1} \). For an irreducible finite Markov process with transition matrix \( P \), the group inverse of the matrix \( H = I - P \) always exists, it is uniquely characterized by the set of equations
\[
I - (I - P)(I - P)^\# = 1\pi^T, \quad \pi^T(I - P)^\# = 0
\]
(29)
where $\pi$ is the stationary distribution vector of the Markov process (see Theorem 8.5.5 in [5]). As indicated in the introduction, if $g$ belongs to the columns span of $I - P$ then the equation $(I - P)u = g$ has the solution (2).

Relying on the results of the previous section, we provide an explicit representation for the general solution of the Poisson equation in the case of a non null recurrent QBD. Under this assumption, the matrix $W$ in (10) exists and is nonsingular, moreover, by Theorem 2 there exists a resolvent triple $(X, T, Z)$ of $\eta(\lambda)$. In the case of a homogeneous equation, the next lemma provides the general solution, it is an immediate consequence of Theorems 1 and 2.

**Lemma 6.** Let $G$ be the minimal nonnegative solution of (7), let $L$ and $V_1$ be the matrices of size $m \times p$ and $p \times p$, respectively, defined through (17), (18), (19). The general solution of the homogeneous equation

$$A_{-1}u_{r} + (A_0 - I)u_{r+1} + A_1u_{r+2} = 0,$$

is given by

$$h_r = G^r x + LV_1^{-r} y, \quad r \geq 0,$$

where $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^p$ are arbitrary.

The following result provides a particular solution of the non-homogeneous equation together with the general solution. It immediately follows from Theorems 1 and 2.

**Lemma 7.** Let $G$ be the minimal nonnegative solution of (7), let $L$, $K$, $V_1$ and $V_0$ be the matrices of size $m \times p$, $m \times (m-p)$, $p \times p$ and $(m-p) \times (m-p)$, respectively, defined through (17), (18), (19). Let $E$ and $F$ be the matrices defined in (27), let $W$ be defined in (10). A particular solution of

$$A_{-1}u_{r} + (A_0 - I)u_{r+1} + A_1u_{r+2} = -g_{r+1}, \quad (30)$$

is given by

$$\sigma_r = -\sum_{k=1}^{r} (G^{r-k} - LV_1^{k-r} E) W g_k - \sum_{j=1}^{\nu-1} K V_0^j F W g_{j+r}, \quad r \geq 0, \quad (31)$$

where $\nu$ is the smallest integer such that $V_0^\nu = 0$. The general solution of (30) is

$$u_r = G^r x + LV_1^{-r} y + \sigma_r, \quad r \geq 0, \quad (32)$$

where $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^p$ are arbitrary.
Lemma 7 characterizes all the solutions of the difference equation (5). If we consider also the boundary condition (4), we arrive at the following result, which expresses the general solution of the Poisson equation.

In the positive recurrent case, we need to assume that the series \( \sum_{k=0}^{\infty} R^k g_k \) converges. As \( \rho(R) < 1 \) for positive recurrent QBDs, this allows some flexibility for asymptotic properties of the \( g_k \)s.

**Theorem 3.** The general solution of the Poisson equation (1) is given by

\[
U_r = G^r x + LV_1^{-r} y + \sigma_r, \quad r \geq 0,
\]

where \( \sigma_r \) is defined in (31) and \( x \) and \( y \) satisfy the following constraints.

If the QBD is transient, then \( y \in \mathbb{C}^p \) is arbitrary and

\[
x = (I - P_*)^{-1} \left( \left( (B - I)\hat{G} + A_1 \right) \left( \sigma_1 + LV_1^{-1}y \right) + g_0 \right)
\]

where \( P_* = B + A_1G \) and

\[
\sigma_1 = -\sum_{j=0}^{\nu-1} KV_0^j FW g_{j+1},
\]

with \( \nu \) being the smallest positive integer such that \( V_0^\nu = 0 \).

If the QBD is positive recurrent and if the series \( \sum_{k=0}^{\infty} R^k g_k \) converges, then

\[
y = y^* + y_\perp
\]

\[
x = (I - P_*)^{-1} \left( \left( (B - I)\hat{G} + A_1 \right) \left( \sigma_1 + LV_1^{-1}y \right) + g_0 \right) + \alpha 1
\]

where

\[
y^* = -\sum_{k=1}^{\infty} V_1^k EW g_k,
\]

the vector \( y_\perp \in \mathbb{C}^p \) is any vector in the hyperplane \( \pi_0^T W^{-1}Ly = \pi^T g \), and \( \alpha \) is an arbitrary constant.

**Proof.** Firstly, we show that \( \sigma_1 \) is given by (35): from (31), we have

\[
\sigma_1 = -(I - LE)W g_1 - \sum_{j=1}^{\nu-1} KV_0^j FW g_j = \sum_{j=0}^{\nu-1} KV_0^j FW g_j,
\]

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since \( I = MM^{-1} = LE + KF \). Furthermore,

\[
\sigma_0 = \sum_{j=1}^{\nu-1} KV_0^j FW g_j = KV_0 F \sigma_1.
\]

The boundary equation (41), together with (32) gives

\[
(I - B - A_1 G) x = (B - I)(L y + \sigma_0) + A_1 (LV_1^{-1} y + \sigma_1) + g_0
\]
or

\[
(I - P_*) x = (B - I) \hat{G} + A_1 (\sigma_1 + LV_1^{-1} y) + g_0.
\]

(39)

To see this, we observe that \( \hat{GL} = LV_1 \) and \( \hat{GK} = KV_0 \) so that \( Ly = \hat{GLV}_1^{-1} y \) on the one hand and that

\[
\sigma_0 = KV_0 F \sigma_1 = \hat{GKF} \sigma_1 = \hat{G} \sigma_1
\]
on the other hand since \( FK = I \).

If the QBD is transient, then \( G \) is sub-stochastic, the matrix \( I - P_* \) is nonsingular, and the constraint (34) on \( x \) immediately results from (39) while there is no constraint on \( y \).

If the QBD is recurrent, then \( G \) is stochastic and \( P_* \) is the transition matrix of an irreducible finite Markov process, so that (39) is a finite Poisson equation and (37) follows, provided that the right-hand side is in the span of the columns of \( I - P_* \), that is, provided that

\[
\pi_0^T \left( (B - I) \hat{G} + A_1 \right) (\sigma_1 + LV_1^{-1} y) + \pi_0^T g_0 = 0, \tag{40}
\]
as \( \pi_0^T(I - P_*) = 0 \) by (14). We have

\[
\pi_0^T \left( (B - I) \hat{G} + A_1 \right) = \pi_0^T A_1 (I - \hat{G} \hat{G}) \quad \text{as} \quad \pi_0^T(I - B - A_1 G) = 0,
\]

\[
= -\pi_0^T W^{-1} \hat{G} \quad \text{by (16)},
\]

and, as we have seen earlier that \( \hat{GLV}_1^{-1} = L \), the constraint (40) may be written as

\[
\pi_0^T W^{-1} L y = \pi_0^T g_0 - \pi_0^T W^{-1} \hat{G} \sigma_1.
\]

(41)
Now, having assumed that the series converges, we have
\[
\sum_{k=0}^{\infty} R^k g_k = g_0 + \sum_{k=1}^{\infty} W^{-1}\tilde{G}^k W g_k \quad \text{by (41)),}
\]
\[
= g_0 + W^{-1}\tilde{G} \sum_{k=0}^{\infty} KV_0^k F W g_{k+1} + W^{-1}\tilde{G} \sum_{k=0}^{\infty} LV_1^k E W g_{k+1}
\]
since \(\tilde{G}^k = KV_0^k F + LV_1^k E\),
\[
= g_0 - W^{-1}\tilde{G}\sigma_1 - W^{-1}\tilde{G}LV_1^{-1}y^*.
\]
Thus, (41) may be written as
\[
\pi_0^T W^{-1} L y = \pi_0^T \sum_{k=0}^{\infty} R^k g_k + \pi_0^T W^{-1}\tilde{G}LV_1^{-1}y^* = \pi^T g + \pi_0^T W^{-1} L y^*.
\]
This proves (36). \(\square\)

The expression of \(u_r\) given in (33) can be equivalently rewritten in a numerically more convenient form as follows
\[
u_r = G^r x - \sum_{k=0}^{r-1} G^k W g_{r-k} + LV_1^{-r} \left( y + \sum_{k=1}^{r} V_1^r E W g_k \right) - \sum_{j=1}^{\nu-1} KV_0^j FW g_{j+r}.
\]
Some simplification occurs when the matrix \(A_1\) is nonsingular: then the matrix \(R\) is nonsingular and the expression for the general solution simplifies as follows.

**Corollary 1.** Assume that \(\det A_1 \neq 0\). Let \(G\) and \(R\) be the minimal nonnegative solutions of (7) and (9). The general solution of the Poisson equation (1) is given by
\[
u_r = G^r x + WR^{-r} \tilde{y} - \sum_{k=1}^{r} \left( G^{r-k} W - WR^{k-r} \right) g_k, \quad r \geq 0,
\]
where the vectors \(x\) and \(\tilde{y}\) satisfy the following constraints.
If the QBD is transient, then \(\tilde{y} \in \mathbb{C}^n\) is arbitrary and
\[
x = (I - P_\tau)^{-1} \left( (B - I)W\tilde{y} + A_1WR^{-1}\tilde{y} + g_0 \right),
\]
with \( P_* = B + A_1G \).

If the QBD is positive recurrent and the series \( \sum_{k=0}^{\infty} R^k g_k \) converges, then \( \tilde{y} = \tilde{y}^* + \tilde{y}_\perp \) and

\[
x = (I - P_*)^\#((B - I)W\tilde{y} + A_1WR^{-1}\tilde{y} + g_0) + \alpha 1,
\]

where \( \tilde{y}^* = -\sum_{k=1}^{\infty} R^k g_k \), \( \tilde{y}_\perp \) is any vector in the hyperplane \( \pi_0^T\tilde{y}_\perp = \pi^T g \), and \( \alpha \) is arbitrary.

**Proof.** Since \( R \) is nonsingular, \( \mathcal{G} \) is nonsingular as well, \( V_1 \) is an \( m \times m \) matrix, \( V_0 \) does not exist, and we may take \( M = I, V_1 = \mathcal{G}, \) and \( L = E = I \).

In view of (11), we have \( V_1^k = \mathcal{G}^k = WR^kW^{-1} \) for any integer \( k \). With this choice of matrices, we have

\[
V_1^{k-r}EWg_k = WR^{k-r}W^{-1}g_k = WR^{k-r}g_k
\]

and (31) becomes

\[
\sigma_r = -\sum_{k=1}^{r} (G^{r-k}W - WR^{k-r})g_k.
\]

Set \( \tilde{y} = W^{-1}y \) so that \( V_1^{r-r}y = WR^{-r}\tilde{y} \). Replace the latter expression in (33) and get

\[
u_r = G^r x + WR^{-r}\tilde{y} + \sigma_r.
\]

The remainder of the proof results from (34, 36–38).

The expression of \( u_r \) given in the above corollary can be equivalently rewritten in a numerically more convenient form as follows

\[
u_r = G^r x - \sum_{k=0}^{r-1} G^kWg_{r-k} + WR^{-r}\left(y + \sum_{k=1}^{r} R^kg_k\right).
\]

To conclude this section, we briefly examine the asymptotics of \( u_r \) in (33) as \( r \to \infty \) and we discuss the effect of the powers of \( G \) and of \( V_1^{-1} \) for different choices of \( y \) — note that \( x \) is actually a function of the arbitrary vector \( y \).

The powers of \( G \) are bounded, since \( G1 \leq 1 \) and so, \( \lim_{r\to\infty} G^r x \) is bounded for any given \( y \). Concerning the powers of \( V_1^{-1} \), recall that the eigenvalues of \( V_1 \) coincide with the nonzero eigenvalues of \( R \). Thus, in the
positive recurrent case where the spectral radius $\rho(R)$ of the matrix $R$ is such that $\rho(R) < 1$, all eigenvalues of $V_1^{-1}$ are strictly greater than one in absolute value and the powers of $V_1^{-1}$ diverge. In the transient case where $\rho(R) = 1$, the powers of $V_1^{-1}$ diverge as well if $p > 1$. The term in the general solution (44) which involves $V_1$ is

$$s_r = LV_1^{-r} \left( y + \sum_{k=1}^{r} V_1^k EW g_k \right).$$

Assume that the series $\sum_{k=0}^{\infty} R^k g_k$ is convergent. Under this assumption, the series $\sum_{k=1}^{\infty} V_1^k EW g_k$ is convergent as well. Thus, choosing $y = y^*$ from (38) implies that

$$s_r = -LV_1^{-r} \sum_{k=r+1}^{\infty} V_1^k EW g_k = -L \sum_{k=r+1}^{\infty} V_1^{k-r} EW g_k.$$

Whence $s_r$ is bounded.

We discuss further the significance of the vector $y^*$ in Section 6.

5 The general solution: null recurrent case

If the QBD is null recurrent, then $\xi_m = \xi_{m+1} = 1$ and we cannot directly apply the arguments of the previous section. Indeed, both $G$ and $R$ have spectral radius equal to one, the matrix $W$ in (10) Theorem 2 is not defined, and the standard triple which allowed us to build a solution cannot be constructed.

However, after a suitable manipulation, we can transform the original difference equation into a new one where we can express the solution through a standard triple. This manipulation is based on the shift technique of [10, 4], which enables us to construct a new matrix polynomial having the same eigenvalues as the original polynomial except for $\xi_m$ and $\xi_{m+1} = 1$ which are replaced by zero or by infinity. The new quadratic matrix polynomial is associated with a new matrix difference equation which can be solved by means of the resolvent triples as in Section 4. We will prove that from the solution of the transformed matrix difference equation we can recover the solution of the original equation.

This transformation can be performed in different ways, say, by applying a left shift, or a right shift, or combining together the two transformations.
In this section we recall the shift technique from [4], while in the next section we describe the transformation which relates the solutions of the matrix difference equations obtained this way.

In addition to $G$, $R$, and $\hat{G}$, define $\hat{H} = A_0 - I + A_{-1}\hat{G}$, together with $\hat{R} = A_{-1}(I - \hat{H})^{-1}$; the matrix $\hat{R}$ coincides with the minimal nonnegative solution of the matrix equation

$$A_{-1} + X(A_0 - I) + X^2 A_1 = 0.$$

For a null recurrent QBD we have $\rho(G) = \rho(\hat{G}) = \rho(R) = \rho(\hat{R}) = 1$, and 1 is a simple eigenvalue in each case. If $X$ is any of the four matrices, denote by $w_X$ and $v_X$ a right and a left nonnegative eigenvectors, respectively, of $X$ corresponding to the eigenvalue 1. The main results of [4] concerning the right and the left shift are as follows.

**Theorem 4** (Right shift). Take $Q = w_G v_G^T$, with $v_G^T w_G = 1$, and define

$$\tilde{A}_{-1} = A_{-1}(I - Q), \quad \tilde{A}_0 = A_0 + A_1 Q, \quad \tilde{A}_1 = A_1.$$

Normalize $w_{\hat{R}}$ so that $v_G^T \hat{H}^{-1} w_{\hat{R}} = -1$.

The matrix equations

$$\tilde{A}_{-1} + (\tilde{A}_0 - I) X + \tilde{A}_1 X^2 = 0 \quad \text{and} \quad \tilde{A}_1 + (\tilde{A}_0 - I) X + \tilde{A}_{-1} X^2 = 0$$

have the solutions $\tilde{G} = G - Q$ and $\hat{G} = \hat{G} + (w_G + \hat{H}^{-1} w_{\hat{R}}) v_G^T$, respectively. Moreover, $\rho(\hat{G}) < 1$ and $\rho(\tilde{G}) = 1$, $\det(I - \tilde{G}\hat{G}) \neq 0$, and the matrix

$$\tilde{W} = \sum_{i=0}^{\infty} \tilde{G}^i (U - I)^{-1} R_i$$

is nonsingular.

We recall that in the above theorem, the scalar product $v_G^T \hat{H}^{-1} w_{\hat{R}}$ is always non zero [4].

**Theorem 5** (Left shift). Take $S = w_{\hat{R}} v_{\hat{R}}^T$, with $v_{\hat{R}}^T w_{\hat{R}} = 1$ and define

$$\tilde{A}_{-1} = A_{-1}, \quad \tilde{A}_0 = A_0 + SA_{-1}, \quad \tilde{A}_1 = (I - S)A_1.$$

Normalize $v_\hat{G}$ so that $v_\hat{G}^T \hat{H}^{-1} w_{\hat{R}} = -1$. 

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The matrix equations
\[ \tilde{A}_{-1} + (\tilde{A}_0 - I)X + \tilde{A}_1X^2 = 0, \quad \text{and} \quad \tilde{A}_1 + (\tilde{A}_0 - I)X + \tilde{A}_{-1}X^2 = 0 \]
have the solutions \( G \) and \( \tilde{G} = \tilde{G} + \hat{H}^{-1}w_\hat{R}v_\hat{G}^T \), respectively. Moreover, \( \rho(G) = 1 \) and \( \rho(\tilde{G}) < 1 \).

For the next developments, it is useful to reformulate the difference equation (5) in the following functional form
\[ \eta(\lambda)u(\lambda^{-1}) = k_{-1}\lambda^2 + k_0\lambda - \sum_{j\geq 1} g_j\lambda^{-j+1}, \quad (45) \]
where \( u(\lambda) = \sum_{i=0}^{\infty} u_i\lambda^i \), with \( k_{-1} = A_1u_0 \) and \( k_0 = (A_0 - I)u_0 + A_1u_1 \).

5.1 Solution based on the right shift

Define \( \tilde{\eta}(\lambda) = \eta(\lambda)(I - \frac{1}{1-\lambda}Q) \). It follows from (4) that
\[ \tilde{\eta}(\lambda) = \tilde{A}_{-1} + \lambda(\tilde{A}_0 - I) + \lambda^2\tilde{A}_1, \]
where the matrices \( \tilde{A}_i, i = -1,0,1 \), are defined in Theorem 4. We may associate with the matrix polynomial \( \tilde{\eta}(\lambda) \) the matrix difference equation
\[ \tilde{A}_{-1}\tilde{u}_r + (\tilde{A}_0 - I)\tilde{u}_{r+1} + \tilde{A}_1\tilde{u}_{r+2} = -g_{r+1} \quad (46) \]

Our goal is to express the solutions \( \tilde{u}_r \) of the above difference equation by means of standard triples, using the solutions \( \tilde{G} \) and \( \hat{G} \) given in Theorem 4 and to relate these solutions to the general solution of the original matrix difference equation (5).

By Theorem 4, the matrices \( I - \tilde{G}\hat{G} \) and \( \hat{W} \) are both nonsingular. Knowing this, we follow the steps in Lemma 5 and Theorem 2 and obtain a resolvent triple for \( \tilde{\eta}(\lambda) \). We apply Theorem 1 and obtain the general solution of (46).

The solutions of (46) and those of (5) are related in a simple manner. Observe that the product \( \eta(\lambda)u(\lambda^{-1}) \) is such that
\[ \eta(\lambda)u(\lambda^{-1}) = \eta(\lambda)(I - \frac{1}{1-\lambda}Q)(I - \frac{1}{1-\lambda}Q)^{-1}u(\lambda^{-1}) = \tilde{\eta}(\lambda)\tilde{u}(\lambda^{-1}), \]
where \( \tilde{u}(\lambda) \) is defined by
\[ u(\lambda) = (I - \frac{\lambda}{\lambda - 1}Q)\tilde{u}(\lambda). \quad (47) \]
Since $\tilde{\eta}(\lambda)\tilde{u}(\lambda^{-1}) = \eta(\lambda)u(\lambda^{-1})$, from (45) we deduce that

$$\tilde{\eta}(\lambda)\tilde{u}(\lambda^{-1}) = k_{-1}\lambda^2 + k_0\lambda - \sum_{j \geq 1} g_j \lambda^{-j+1}.$$  

Multiplying both sides of (47) by $\lambda - 1$ and comparing the terms with the same degree in $\lambda$ yields

$$u_0 = \tilde{u}_0, \quad u_k = \tilde{u}_k + Q \sum_{i=0}^{k-1} \tilde{u}_i, \quad k \geq 1. \quad (48)$$

and so $k_{-1} = \tilde{A}_1\tilde{u}_0$ and $k_0 = (\tilde{A}_0 - I)\tilde{u}_0 + \tilde{A}_1\tilde{u}_1$, that is, the vector sequence $\tilde{u}_r$ solves (46). This proves that we may recover the general solution of the original equation from (48).

In view of Lemma 7, the general solution of the matrix difference equation (46) may be expressed in the following form

$$\tilde{u}_r = G^r x + \tilde{L}\tilde{V}_1^{-r} y + \tilde{\sigma}_r, \quad r \geq 0$$

$$\tilde{\sigma}_r = -\sum_{k=1}^{r} \left( G^{r-k} - \tilde{L}\tilde{V}_1^{k-r} E \right) \tilde{W} g_k - \sum_{j=1}^{r-1} \tilde{K}\tilde{V}_0^j \tilde{F}\tilde{V}_j g_{j+r}, \quad r \geq 0,$$

for any vectors $x$ and $y$ where

$$\tilde{G}\tilde{M} = \tilde{M}\tilde{J}, \quad \tilde{J} = \begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & \tilde{V}_0 \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} \tilde{L} & \tilde{K} \end{bmatrix}$$

with $\rho(\tilde{V}_0) = 0$, det $\tilde{V}_1 \neq 0$, and $\tilde{L}$ and $\tilde{K}$ are matrices of size $m \times p$ and $m \times (m - p)$, respectively; moreover

$$\tilde{M}^{-1} = \begin{bmatrix} \tilde{E} \\ \tilde{F} \end{bmatrix}$$

where $\tilde{E}$ and $\tilde{F}$ are matrices of size $p \times m$ and $(m - p) \times m$, respectively.

Now it remains to analyze the solution which satisfies the initial conditions (4). To this end we assume that the series $\sum_{k=0}^{\infty} R^k g_k$ is convergent.

Observe that, since $u_0 = \tilde{u}_0$, and $u_1 = \tilde{u}_1 + Q\tilde{u}_0$, the initial condition (4) can be rewritten as

$$(\tilde{B} - I)\tilde{u}_0 + A_1\tilde{u}_1 = -g_0, \quad \tilde{B} = B + A_1 Q.$$
Rewriting the above equation in terms of the vectors $x$ and $y$ and exploiting the identities $\tilde{G} = G - Q$, $\tilde{L}y = \tilde{G}\tilde{L}^{-1}y$ and $\tilde{\sigma}_0 = \tilde{G}\tilde{\sigma}_1$ yields

$$(I - B - A_1G)x = g_0 + \left((\tilde{B} - I)\tilde{G} + A_1\right)(\tilde{\sigma}_1 + \tilde{L}\tilde{L}^{-1}y).$$

The matrix $P_\ast = B + A_1G$ is stochastic and $\pi_0^T(I - P_\ast) = 0$ so that the above system has a solution if and only if the following condition is satisfied

$$\pi_0^Tg_0 + \pi_0^T \left((\tilde{B} - I)\tilde{G} + A_1\right)(\tilde{\sigma}_1 + \tilde{L}\tilde{L}^{-1}y) = 0.$$  

Observe that $P_\ast = I - \tilde{B} - A_1\tilde{G}$ and therefore $\pi_0^T((\tilde{B} - I)\tilde{G} + A_1) = 0$ from which we get $\pi_0^T((\tilde{B} - I)\tilde{G} + A_1) = \pi_0^T A_1(I - \tilde{G}\tilde{G})$. Since $\det(I - \tilde{G}\tilde{G}) \neq 0$, we may proceed as in the proof of Theorem 3 and arrive at the following equivalent formulation of condition (49)

$$\pi_0^T \tilde{W}^{-1}\tilde{L}(y - \tilde{y}^*) = \pi^T g,$$

where $\tilde{y}^* = -\sum_{k=0}^\infty \tilde{V}_k \tilde{E}_k g_k$. Observe that the definition of $\tilde{y}^*$ is consistent since we assumed that $\sum_{k=0}^\infty R^k g_k$ is finite. The latter property implies also that $\pi^T g$ is finite.

### 5.2 Solution based on the left shift

Instead of shifting $\xi_m$ to 0 (or equivalently, replacing the maximal eigenvalue of $G$ by 0), we may shift $\xi_{m+1}$ to $\infty$, and so shift the maximal eigenvalue of $\tilde{G}$ to $1/\xi_{m+1} = 0$. This requires us to use the left shift and to modify the right-hand side of (5).

Rewrite (45) in the following form:

$$(I - \frac{\lambda}{\lambda - 1} S)\eta(\lambda)u(\lambda^{-1}) = (I - \frac{\lambda}{\lambda - 1} S)(k_{-1}\lambda^2 + k_0\lambda - \sum_{j \geq 1} g_j \lambda^{-j+1}),$$

where $S = w_\tilde{R}v_\tilde{R}^T$, $v_\tilde{R}^Tw_\tilde{R} = 1$, and get

$$\tilde{\eta}(\lambda)u(\lambda^{-1}) = \tilde{g}(\lambda)$$

with $\tilde{\eta}(\lambda) = (I - \frac{\lambda}{\lambda - 1} S)\eta(\lambda)$ and $\tilde{g}(\lambda)$ is the right-hand side of (50).
Recall that $R$ is the minimal nonnegative solution of (9) and that $v_R$ is its left eigenvector corresponding to the eigenvalue 1. Thus,

$$S(A_1 + (A_0 - I) + A_{-1}) = S(A_1 + R(A_0 - I) + R^2A_{-1}) = 0$$

and so $S(A_0 - I) = -S(A_2 + A_{-1})$. One readily verifies that $\tilde{\eta}(\lambda) = \tilde{A}_{-1} + \lambda(\tilde{A}_0 - I) + \lambda^2A_1$, where $A_i, i = -1, 0, 1$ are defined in Theorem 5.

If the function $\sum_{j \geq 1} g_j \lambda^{-j+1}$ is analytic for $|\lambda^{-1}| < c$ for some $c > 0$, since $\lambda/(\lambda - 1) = 1/(1 - \lambda^{-1})$ is analytic for $|\lambda^{-1}| < 1$, then the function $\tilde{g}(\lambda)$ is analytic for $|\lambda^{-1}| < \min(c, 1)$, and we may write $\tilde{g}(\lambda) = \sum_{i=-2}^{+\infty} \tilde{g}_i \lambda^{-i}$, where the coefficients $\tilde{g}_r$ can be explicitly expressed as functions of $S$, $k_{-1}$, $k_0$, and $g_j$ for $j \geq 1$.

Thus, we obtain a modified difference equation in the form

$$\tilde{A}_{-1} u_{r-1} + (\tilde{A}_0 - I) u_r + \tilde{A}_1 u_{r+1} = \tilde{g}_r.$$ 

In view of Theorem 5, the matrix equations associated with the matrix polynomial $\tilde{\eta}(\lambda)$ have solutions $\tilde{G}$ and $\tilde{\tilde{G}}$ such that $I - \tilde{G}\tilde{\tilde{G}}$ is nonsingular, we may apply the technique of standard triples and obtain the explicit solution of the difference equation.

Observe that in this case, unlike the shift to the right, the solution of the transformed matrix difference equation coincides with the solution of the original equation. Thus we do not need to reconstruct one solution from the other one. On the other hand, with the shift to the left we have to compute a different right-hand side.

The two techniques of shifting to the right and to the left can be combined together to obtain another possible representation of the solution. We leave the details to the reader.

6 Comparison with published results

Solutions of the Poisson equation are constructed in [6] under the assumptions that the QBD is positive recurrent, that $\pi_0^T \sum_{k=0}^{\infty} R^k \|g_k\| < \infty$, and that $\pi_0^T \sum_{k=0}^{\infty} R^k g_k = 0$. The approach there is based on a probabilistic argument, and a particular solution, up to an additive constant, is written, for $r \geq 0$, as

$$\omega_r = G^r \gamma + y_r + c1, \quad (51)$$
where \( c \) is any arbitrary constant,
\[
\gamma = (I - P_\star)^{#} \sum_{k=0}^{\infty} R^k g_k \quad \text{and} \quad y_r = - \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} G^j (U - I)^{-1} R^k g_{r+k-j}.
\]

(52)

This solution has a different aspect from (33); in particular, the right-hand sides of (52) are expressed as series while we have finite sums only in (33), which is more convenient for computational purposes. We show below that, for positive recurrent QBDs, the solution obtained by probabilistic reasoning is identical to the solution from Theorem 3, for the specific choice of \( y = y^\star \) defined in (38). This is proved in Lemma 8. What is more, we show in Lemma 9 that the vectors \( \omega_r \) actually form a solution of the Poisson equation for transient or null recurrent QBDs also.

Our condition throughout is that \( \sum_{k=0}^{\infty} \rho(R)^k \|g_k\| \) should be a convergent series for some vector norm \( \|\cdot\| \), and then automatically for any vector norm. For transient and for null recurrent QBDs, \( \rho(R) = 1 \) and this imposes a strong constraint on \( g \). One immediate advantage stemming from the constraint is that the series in (52) are all convergent, and so the \( y_r \)'s are all well defined. To see this, we choose a vector norm such that \( \|R\| = \rho(R) \) and we write
\[
\|y_r\| \leq \|U - I\|^{-1} \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} \|G\|^j \rho(R)^k \|g_{r+k-j}\|
\]
\[
= \|U - I\|^{-1} \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} \|G\|^j \rho(R)^k \|g_{r+k-j}\|
\]
\[
= \|U - I\|^{-1} \sum_{j=0}^{r-1} \|G\|^j \rho(R)^{j-r} \sum_{k=r-j}^{\infty} \rho(R)^k \|g_k\|
\]
which converges by assumption.

**Lemma 8.** Assume that the QBD is positive recurrent, that \( \sum_{k=0}^{\infty} \rho(R)^k \|g_k\| < \infty \) and that \( \pi_0^T \sum_{k=0}^{\infty} R^k g_k = 0 \). Equation (51) may be written as
\[
\omega_r = G^r x + LV_1^{-r} y^\star + \sigma_r
\]
where \( y^\star \) is defined in (38), \( x \) is defined in (37) with \( y \) replaced by \( y^\star \), and \( \sigma_r \) is defined in (31).
Proof. We write

\[ y_r = -\sum_{k=0}^{\infty} \sum_{j=0}^{k+r-1} G^j (U - I)^{-1} R^k g_{k+r-j} + \sum_{k=0}^{\infty} \sum_{j=r}^{k+r-1} G^j (U - I)^{-1} R^k g_{k+r-j} \]

\[ = -\sum_{k=1}^{\infty} \sum_{j=0}^{k+r} G^j (U - I)^{-1} R^k g_{k+r-j} - \sum_{k=0}^{\infty} \sum_{j=k}^{k+r-1} G^j (U - I)^{-1} R^k g_{k+r-j} + G^r \zeta \]

where

\[ \zeta = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} G^j (U - I)^{-1} R^k g_{k-j} = \sum_{j=0}^{\infty} G^j (U - I)^{-1} \sum_{k=1}^{\infty} R^{k+j} g_k = W \sum_{k=1}^{\infty} R^k g_k. \]

We simplify the first term as

\[ \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} G^j (U - I)^{-1} R^k g_{k-r-j} = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} G^i (U - I)^{-1} R^{i+k} g_{k+r} \]

\[ = \sum_{k=r+1}^{\infty} WR^{k-r} g_k \]

\[ = \sum_{k=r+1}^{\infty} \tilde{G}^{k-r} W g_k, \]

by definition of \( W \) and (11). The second term becomes

\[ \sum_{k=0}^{\infty} \sum_{j=r}^{k+r-1} G^j (U - I)^{-1} R^k g_{k+r-j} = \sum_{k=1}^{r} \sum_{l=0}^{\infty} G^{r-k+l} (U - I)^{-1} R^l g_k \]

\[ = \sum_{k=1}^{r} G^{r-k} W g_k. \]

Thus,

\[ y_r = -\sum_{k=r+1}^{\infty} \tilde{G}^{k-r} W g_k - \sum_{k=1}^{r} G^{r-k} W g_k + G^r \zeta. \]
By (17–20, 27), since $V^j_0 = 0$ for, $j \geq \nu$, we may write
\begin{align*}
y_r &= -LV_1^{-r} \sum_{k=1}^{\infty} V^k_1 EW g_k + \sum_{k=1}^{r} LV_1^{k-r} EW g_k - \sum_{j=1}^{\nu-1} KV^j_0 FW g_{j+r} \\
& \quad - \sum_{k=1}^{r-1} G^{r-k} W g_k + G^r \zeta \\
& = LV_1^{-r} y^* + \sigma_r + G^r \zeta
\end{align*}
and so
\[ \omega_r = G^r (\gamma + \zeta) + LV_1^{-r} y^* + \sigma_r + c_1. \]
Finally, we verify that $G^r \mathbf{x} = G^r (\gamma + \zeta) + c_1 \mathbf{1}$, for some scalar $c_1$. The equation (43) may be written as
\[ -W^{-1} \tilde{G}(\sigma_1 + LV_1^{-1} y^*) = \sum_{k=1}^{\infty} R^k g_k = W^{-1} \zeta. \]
The vector $\mathbf{x}$ given in (37) may be rewritten as
\[ \mathbf{x} = (I - P_\sigma)^\# (- (B - I) \zeta + A_1 (\sigma_1 + LV_1^{-1} y^*) + g_0) + \alpha \mathbf{1}. \]
Furthermore, by repeating the argument (42–43), we find that
\[ \sigma_1 + LV_1^{-1} y^* = -W \sum_{k=0}^{\infty} R^k g_{k+1} \]
and so
\[ A_1 (\sigma_1 + LV_1^{-1} y^*) = (R - A_1 GW R) \sum_{k=0}^{\infty} R^k g_{k+1} \]
by Lemma 2
\[ = \sum_{k=1}^{\infty} R^k g_k - A_1 G \zeta, \]
and (53) becomes
\[ \mathbf{x} = (I - P_\sigma)^\# \left( (I - B - A_1 G) \zeta + \sum_{k=0}^{\infty} R^k g_k \right) + \alpha \mathbf{1} \\
= (I - (\pi_0^T 1)^{-1} 1 \pi_0^T) \zeta + \gamma + c_2 \mathbf{1} \]
by (29)
where $c_2$ is a scalar. Since $G$ is stochastic, this completes the proof. \qed
To prove that the vectors $\omega_r$ are always a solution of \((1)\), even if the QBD is null recurrent or transient, we may not refer to the vector $y^*$ since the definition \((38)\) depends on the matrix $W$, and the series in \((10)\) diverges in the null recurrent case. Instead, we prove by direct verification that \((51)\) is a solution of \((4,5)\).

Lemma 9. Assume that $\sum_{k=0}^{\infty} \rho(R)^k \|g_k\|$ converges, where $R$ is the minimal nonnegative solution of \((9)\).

If the QBD is recurrent, assume in addition that $\pi^T \sum_{k=0}^{\infty} R_k g_k = 0$, where $\pi_*$ is the stationary distribution of $P_*$. $\pi_*(I - P_*) = 0$, $\pi_*^T 1 = 1$.

Under these assumptions, one solution of the Poisson equation \((1)\) is given by \((51)\).

Proof. For positive recurrent QBDs, $\pi_0$ is proportional to $\pi_*$, and so the statement immediately results from Lemma 8.

For null recurrent QBDs, we have

\[
(B - I) \omega_0 + A_1 \omega_1 = (B - I) \gamma + A_1 G \gamma + A_1 y_1
\]

\[
= (P_* - I) \gamma - A_1 \sum_{k=0}^{\infty} (U - I)^{-1} R_k g_{k+1}
\]

\[
= (P_* - I) (I - P_*)^\# \sum_{k=0}^{\infty} R_k g_k + \sum_{k=1}^{\infty} R^k g_k
\]

\[
= (1 \pi_*^T - I) \sum_{k=0}^{\infty} R_k g_k + \sum_{k=1}^{\infty} R^k g_k \quad \text{by \((29)\)}
\]

\[
= -g_0.
\]

By \((52)\), $y_r = \sum_{j=0}^{r-1} G^j z_{r-j}$, with $z_n = (I - U)^{-1} \sum_{k=0}^{\infty} R_k g_{n+k}$, so that

\[
A_{-1} \omega_r + (A_0 - I) \omega_{r+1} + A_1 \omega_{r+2} = A_{-1} y_r + (A_0 - I) y_{r+1} + A_1 y_{r+2}
\]

\[
= (A_{-1} + (A_0 - I) G + A_1 G^2) \sum_{j=0}^{r-1} G^j z_j
\]

\[
+ (A_0 - I + A_1 G) z_{r+1} + A_1 z_{r+2}
\]

\[
= -\sum_{k=0}^{\infty} R_k g_{r+1+k} + A_1 (I - U)^{-1} \sum_{k=0}^{\infty} R_k g_{r+2+k}
\]

\[
= -g_{r+1}
\]
by equations (12, 13).

For transient QBDs, \( P_* \) is sub-stochastic, \( (I - P_*)^# = (I - P_*)^{-1} \) and the same calculation as above apply.

\[ \text{Remark 1.} \] It is interesting to note that the solution obtained by probabilistic reasoning, in the case of positive recurrent QBDs, corresponds to the solution (33, 36, 37) with \( y_\perp = 0 \); this is another emphasis placed on the role played by the vector \( y^* \). It is remarkable, in addition, that we should have found solutions of the Poisson equation, even when the process is null recurrent or even transient. This is another illustration of the nice properties stemming from the transition structure of QBDs.

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