Finite time stability analysis for T-S fuzzy positive systems and its application in pest control

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Abstract. This paper is considered with the finite time stability (FTS) for positive nonlinear systems. First, the fuzzy copositive Lyapunov function (FCLF) is established to reduce the conservatism of FTS conditions. Then, by using FCLF, a new FTS criteria of the studied systems is established. Since, the time-derivatives (TD) of membership functions are considered in the stability analysis, our results are less conservative. Besides, the obtained conditions are expressed via the linear programming technique, that can be verified effectively. Finally, the pest-predator system is used to reflect the effectiveness of the results.

1. Introduction

A system can be called a positive system as long as it satisfies the condition that the initial condition is nonnegative and the state and output remain nonnegative. In real word life, the nonnegative characteristics of states are common in many kinds of systems. For example, Blood pressure in the circulatory system, the height of liquid level in tank connected system and the population densities in pest-predator system. Hence, positive systems have received extensive attentions. Recently, some important results on positive systems have been published, such as controller synthesis [1-3], stability analysis [4-5] and $L_1$-gain analysis [6-7].

Finite time stability (FTS) is an important research direction which is different from asymptotic stability. A system wants to have FTS, its state must not exceed the given upper bound within the specified time. FTS can well reflect the transient performance, so FTS has important research significance in missile system, robot operating system and other systems with short working time. For positive systems, some meaningful results on FTS have been published [8-9].

Note that almost all dynamical systems in real life have nonlinear characteristics and they cannot be described by linear differential equations. Hence, the results of linear systems are difficult to be directly applied to nonlinear systems. Fortunately, the emergence of T-S fuzzy theory solves this problem well [10]. Now, T-S fuzzy method has been widely used to analyse the stability of various nonlinear systems [11]. Actually, the positive systems we encounter in life have certain nonlinear characteristics. Besides, if we do not consider the nonlinearity of the Lyapunov function when analysing the stability of nonlinear positive systems, the results will be more conservative. However, as far as we know, the results of analysing FTS of positive system with FCLF have not appeared. Therefore, it is necessary to use the T-S fuzzy method and FCLF to analyse the FTS of nonlinear positive systems. This prompted us to carry out this study.
Inspired by the above discussion, this paper studies the FTS of T-S fuzzy positive system. Our main works in this paper are as follows: 1) A new FTS criteria for studied system is established, since the FCLF is selected, our criterion is more general than [8-10]. 2) The upper bounds of TD of membership functions is considered, hence our results are less conservative. 3) The obtained conditions are expressed via the linear programming technique, that can be verified effectively. 4) The results are extended to pest-predator system to judge whether the population density is within the expected range.

The other parts of this paper are as follows: We give some a lemma and a definition in section 2. The FTS criterion is proved in Section 3. The results obtained are applied to pest-predator system in Section 4. Section 5 is the conclusion.

Notations: $\mathbb{R}^{l \times p}$: a set containing all $l \times p$ real matrices. $A \in M$: all off-diagonal elements of $A$ are nonnegative.

2. Problem formulation
Consider a nonlinear system as follow:

$$\chi(t) = \sum_{i=1}^{r} h_i(\nu(t)) (A_i \chi(t) + B_i \mu(t)), \quad (1)$$

where $\chi(t) \in \mathbb{R}^{n}$ is the state; $\mu(t) \in \mathbb{R}^{r}$ is the input; $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times r}$ are system matrices. $\nu^T(t) = [\nu_1(t), \nu_2(t), \cdots, \nu_r(t)]$; $h_i(\nu(t))$ are the membership functions with satisfying

$$h_i(\nu(t)) \geq 0, \sum_{i=1}^{r} h_i(\nu(t)) = 1. \quad (2)$$

Lemma 1. [9] System (1) is a positive, if $A_i \in M$ and $B_i \succcurlyeq 0$.

Definition 1. [9] For given positive scalars $\sigma_1$, $\sigma_2$, $T$ with $\sigma_2 > \sigma_1 > 0$, and vector $\zeta > 0$, if

$$\chi_0^T \zeta < \sigma_1 \Rightarrow \chi^T(t) \zeta < \sigma_2, \forall t \in [0, T].$$

then positive T-S fuzzy system (1) is said to be FTS on $(\sigma_i, \sigma_2, T, \zeta)$.

3. Finite-Time Stability
In this section, we analyse the FTS of the studied system when $\mu(t) = 0$.

Assumption 1. Let $\left| h_i(\nu(t)) \right| \leq \rho_i$, where $\rho_i, (i = 1, 2 \cdots r)$ are nonnegative constants.

Theorem 1. For given constants $\lambda > 0$, $\sigma_2 > \sigma_1 > 0$, and a vector $\zeta > 0$ with $\chi_0^T \zeta < \sigma_1$, positive system (1) is said to be FTS on $(\sigma_i, \sigma_2, T, \zeta)$, if there exists vectors $P_i > 0, (i = 1, 2 \cdots r)$, and constants $d_1 > 0$, $d_2 > 0$ such that

$$A_i^T P_j + \sum_{i=1}^{r} \rho_i P_i - \lambda P_j \prec 0, \quad (3)$$

$$d_1 \zeta - P_i \prec 0, \quad (4)$$

$$P_i - d_2 \zeta \prec 0, \quad (5)$$
\[ d_1 \sigma_2 - d_2 \sigma_1 e^{\lambda T} > 0, \] 

(6)

hold.

**Proof.** Considering the following FCLF:

\[ V(\chi(t)) = \chi^T(t) \sum_{i=1}^{r} h_i(\nu(t)) P_i \] 

(7)

Then, by (3) and Assumption 1, one has

\[ V(\chi(t)) - \lambda V(\chi(t)) \leq \chi^T(t) \sum_{i=1}^{r} h_i(\nu(t)) h_j(\nu(t)) \left( A_i^T P_j + \sum_{i=1}^{r} \rho_i P - \lambda P_j \right) < 0 \] 

(8)

Multiplying (8) by \( e^{-\lambda t} \) and integrating (11) from 0 to \( t \), we have

\[ \int_{0}^{t} \frac{de^{-\lambda t}V(\chi(t))}{dt} dt < 0. \]  

(9)

By (9), we can obtain

\[ \chi^T(t) \sum_{i=1}^{r} h_i(\nu(t)) P_i < e^{\lambda t} \chi_0^T \sum_{i=1}^{r} h_i(\nu(0)) P_i \] 

(10)

According to (4) and (5), one has

\[ d_1 \chi^T(t) \varsigma < \chi^T(t) \sum_{i=1}^{r} h_i(\nu(t)) P_i < d_2 \chi^T(t) \varsigma. \] 

(11)

Considering inequalities (10) and (11), we can obtain

\[ d_1 \chi^T(t) \varsigma < \chi^T(t) \sum_{i=1}^{r} h_i(\nu(t)) P_i < e^{\lambda t} \chi_0^T \sum_{i=1}^{r} h_i(\nu(0)) P_i < d_2 e^{\lambda t} \chi_0^T \varsigma, \forall t \in [0, T]. \] 

(12)

Then, by (6) and (12), one has

\[ \chi^T(t) \varsigma < \frac{d_2}{d_1} e^{\lambda t} \chi_0^T \varsigma < \frac{d_2}{d_1} e^{\lambda t} \sigma_1 < \sigma_2, \forall t \in [0, T]. \]

Hence, by Definition 1, positive system (1) is FTS on \((\sigma_1, \sigma_2, T, \varsigma)\).

**4. Simulate results**

In this section, we apply the above results to pest control. Considering a nonlinear pest-predator system, which is described by Lotka-Volterra model:

\[
\begin{align*}
\dot{\chi}_1(t) &= \chi_1(t) \left( \sigma_1 - \sigma_3 \chi_2(t) \right), \\
\dot{\chi}_2(t) &= \chi_2(t) \left( \sigma_3 \chi_1(t) - \sigma_4 \right)
\end{align*}
\] 

(13)

where \( \chi_1(t) \) and \( \chi_2(t) \) are the pest and predator population densities at time \( t \). \( \sigma_i, (i = 1, 2, 3, 4) \) are all known positive constants, and the specific meanings can be found in reference [9].
First, we need to build a T-S model for system (13). We denote \( \nu(t) \boxdot \chi(t) \) and define \( \theta_i = \min \{ \nu(t) \} \), \( \theta_i = \max \{ \nu(t) \} \) and \( \theta_i = \frac{\theta_1 + \theta_3}{2} \). Then, system (13) can be described by the following T-S fuzzy model:

\[
\text{IF } \nu(t) \text{ is } \theta_i, (i=1,2,3) \text{ THEN } \dot{\chi}(t) = A_i \chi(t).
\]

where \( A_i = \begin{bmatrix} \sigma_i - \sigma_2 \theta_1 & 0 \\ \sigma_3 \theta_1 & -\sigma_4 \end{bmatrix}, A_2 = \begin{bmatrix} \sigma_1 - \sigma_2 \theta_2 & 0 \\ \sigma_2 \theta_2 & -\sigma_4 \end{bmatrix}, A_3 = \begin{bmatrix} \sigma_1 - \sigma_2 \theta_3 & 0 \\ \sigma_3 \theta_3 & -\sigma_4 \end{bmatrix}. \]

We select the following membership functions: \( h_i(\nu(t)) = \max \left\{ 1 - \frac{\theta_i}{\theta_2} \chi(t), 0 \right\}, h_3(\nu(t)) = \max \left\{ \frac{\theta_1}{\theta_2} \chi(t) - 1, 0 \right\}, \)

\( h_2(\nu(t)) = 1 - h_1(\nu(t)) - h_3(\nu(t)). \)

Then, the global T-S fuzzy model is:

\[
\dot{\chi}(t) = \sum_{i=1}^{3} h_i(\nu(t)) A_i \chi(t). \quad (14)
\]

Let \( \sigma_1 = 0.21, \sigma_2 = 0.098, \sigma_3 = 0.3, \sigma_4 = 0.15, \theta_1 = 0, \theta_2 = 5, \theta_3 = 10 \) and \( \chi_0 = [1 \ 1.15]^T \). Then, the system matrices of the T-S model (14) are

\[
A_1 = \begin{bmatrix} 0.21 & 0 \\ 0 & -0.15 \end{bmatrix}, A_2 = \begin{bmatrix} -0.28 & 0 \\ 1.5 & -0.15 \end{bmatrix}, A_3 = \begin{bmatrix} -0.77 & 0 \\ 3 & -0.15 \end{bmatrix}.
\]

By Lemma 1, we can find that system (14) is positive.

Provided that the ideal population density of pests and predators is \( \chi(0) + 0.5 \chi_2(t) < 2.5, \forall t \in [0,2] \). Let \( \sigma_2 = 2.5, T = 2 \) and \( \zeta = [1 \ 0.5]^T \), then the desired population density of insect pest and predator can be represented by \( \chi(0) + 0.5 \chi_2(t) < 2.5, \forall t \in [0,2] \). Hence, we can use Theorem 1 to check whether the population density of insect pest and predator meets our expected value. Since
\[ \sigma_i > \chi_0^T \omega = 1.575, \] in order not to lose generality, let \( \sigma_1 = 1.6 \). Figure 1 shows the evolution of \( h_i(v(t)), (i = 1, 2, 3) \). By Figure 1, we obtain that \( \| h_1(v(t)) \| \leq 0.1, \| h_2(v(t)) \| \leq 0.1 \) and \( \| h_3(v(t)) \| = 0 \). Hence, let \( \rho_1 = \rho_2 = 0.1 \) and \( \rho_3 = 0 \).

Selecting \( \lambda = 0.9 \), and solving Theorem 1, we get \( P_1 = [723.4421 \, 354.4891]^T \), \( P_2 = [721.8694 \, 353.4571]^T \), \( P_3 = [719.8015 \, 352.3764]^T \), \( d_1 = 704.7439 \) and \( d_2 = 723.4452 \). Hence, positive system (14) is FTS on \( (1.6, 2.5, 2, [1 \, 0.5]^T) \) i.e. the number of pests and predators are within the range of our expected target area. Figure 2 also reflects the same fact.

5. Conclusions
In this study, by using FCLF, the problem of FTS analysis for T-S fuzzy positive systems have been investigated. The new FTS criteria for T-S fuzzy positive systems has been established. Because of the properties of the selected Lyapunov function, the criterion is more general. Moreover, the TD of membership functions have been considered in the FTS analysis. Finally, the obtained FTS criteria has been applied to the pest-predator system. And the effectiveness of the obtained FTS criteria has been proved by the simulation result.

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