Small-Time Asymptotics for Gaussian Self-Similar Stochastic Volatility Models

Archil Gulisashvili¹ · Frederi Viens² · Xin Zhang³

Abstract We consider the class of Gaussian self-similar stochastic volatility models, and characterize the small-time (near-maturity) asymptotic behavior of the corresponding asset price density, the call and put pricing functions, and the implied volatility. Away from the money, we express the asymptotics explicitly using the volatility process’ self-similarity parameter $H$, and its Karhunen–Loève characteristics. Several model-free estimators for $H$ result. At the money, a separate study is required: the asymptotics for small time depend instead on the integrated variance’s moments of orders $\frac{1}{2}$ and $\frac{3}{2}$, and the estimator for $H$ sees an affine adjustment, while remaining model-free.

Keywords Stochastic volatility models · Gaussian self-similar volatility · Implied volatility · Small-time asymptotics · Karhunen–Loève expansions

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1 Introduction

In a stochastic volatility model, the underlying security’s volatility is a stochastic process. There are numerous books devoted to such models, for example, [8,33,34,45,54,55]. In [48], we introduced and studied stochastic volatility models, in which the volatility is described by the absolute value of a continuous Gaussian process. The asset price process $S$ in a Gaussian stochastic volatility model satisfies the following stochastic differential equation:

$$dS_t = rS_t dt + |X_t|S_t dW_t.$$  \hspace{1cm} (1)

In (1), $X$ is a continuous adapted Gaussian process on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, $W$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}$, $S_0 = s_0 > 0$ a.s., and $r \geq 0$ is the risk-free interest rate. We will assume throughout the paper that the processes $X$ and $W$ are independent. The case, where the processes $X$ and $W$ are correlated, is more complicated, and such models are not considered in the present paper. An important special example of a Gaussian stochastic volatility model is the Stein–Stein model (see [64]). In the Stein–Stein model, the process $X$ is the Ornstein–Uhlenbeck process.

The present paper deals with the small-time (near-maturity) asymptotics of the asset price density, the call and put pricing functions, and the implied volatility in Gaussian self-similar stochastic volatility models. The process $X$ in such a model is a self-similar Gaussian process with the self-similarity parameter $H$. Gaussian self-similar models are special cases of general fractional stochastic volatility models. The interested reader can find in [6,13,32,35,36,38–42,44,63,68] more detailed information about fractional models.

Small-time asymptotic behavior of densities, option pricing functions, and implied volatilities has been a popular topic of study. There are various model-independent results (see, e.g., [10,37,51,53,62]), explaining how the asymptotics of the implied volatility depend on those of option pricing functions. There are also papers discussing small-time asymptotics of the functions mentioned above in the case of stochastic volatility or local-stochastic volatility models (see [4,7,24,28,29,43,51,61]), and for special models (see [3,25,26,57,58,65] (models with jumps), [23,27,30,31] (the Heston model), [20,21] (the Stein–Stein model), [46,49,50,61] (the SABR model)).

The asymptotic formulas obtained in this paper allow us to investigate the question of long-memory SV calibration, since long-range dependence and self-similarity are proxies for each other in many known models, via their common Hurst parameter $H$. Based on a Gaussian long-memory model for log-volatility pioneered by Comte and Renault in [12], the work in [11] used an ad-hoc calibration method based on option prices to determine $H$ so as to best explain market prices. In the present paper, we show that calibration of $H$ near maturity can be given a stronger mathematical foundation under self-similarity assumptions for the volatility process. The parameter $H$ can also be a proxy for local regularity measurements, in the sense of their paths’ Hölder continuity parameter. Some recent papers and presentations appear to show that volatility is rough, in the sense that the log-volatility process is fractional and it is not Hölder continuous for $1/2 - \varepsilon < H < 1/2$, where $\varepsilon$ is a positive number (see [40–42]).
the other hand, [11] and many studies before it (see references therein) indicate that \( H > 1/2 \) in terms of memory length. This is a demonstration that the use of \( H \) to measure self-similarity and long memory and path regularity/roughness, such as in the case of fractional Brownian motion (fBm), might be a misspecification in volatility modeling. The authors of [42] indicate that classical long-memory tests detect this property in their Gaussian rough volatility model, which is a geometric fBm or a geometric OU process with shorter memory (\( H < 1/2 \)). The studies in [11] show on the other hand that no consistent memory estimation results in practice from any classical method when used on the non-self-similar stationary long-memory model of [12]. Our current work could help in elucidating the differences between these points of view; we do not comment on them further herein. An interesting discussion of the long memory vs short memory problem can be found in Sect. 1.2 of [42]. In any case, the numerics which we include in this paper and will discuss at the end of this introduction show that our model class allows for a very sharp calibration tool for both long and shorter-memory cases.

Arguably the best known Gaussian self-similar process is the fractional Brownian motion (fBm) \( B_H \), the centered Gaussian process whose law is defined by

\[
B_H (0) = 0
\]

and

\[
\mathbb{E} \left[ \left( B_H^t - B_H^s \right)^2 \right] = |t - s|^{2H}.
\]

It is the only (continuous) self-similar centered Gaussian process with stationary increments. Many texts can be consulted on \( B_H \), including, e.g., [59,60,63]. Among the many other centered Gaussian self-similar models, which are all necessarily non-stationary, the easiest to construct is the Riemann–Liouville fBm, defined as

\[
B_H^{RL}(t) = \int_0^t (t - s)^{H - 1/2} dW(s)
\]

where \( W \) is a standard Wiener process (see for instance [56]). This process, which is \( H \)-self-similar, has properties close to those of fBm, and can be more amenable to calculations. The so-called Bifractional Brownian motion depends on two similarity parameters \( H \) and \( K \), has a more complex representation, as the sum of an fBm with parameter \( HK \), and a process with \( C^\infty \) paths which is not adapted to a Brownian filtration: see [52], see also [5] and the references therein. This process, which is \( HK \)-self-similar, can model the effect of smoothly acquired exogenous information, and is an extension of the so-called sub-fractional Brownian motion (see [9]). Self-similar Gaussian processes can also be obtained as the solutions of stochastic partial differential equations: a class which includes solutions to fractional colored stochastic heat equations is studied in [66], which has the interesting property that its discrete quadratic variation has fluctuations which become non-Gaussian at a threshold of self-similarity which is lower than for fBm, and can be adjusted to be as low as desired. This can be helpful to model volatilities whose local behavior has heavier-tailed fluctuations than what standard fBm can allow, regardless of the volatility’s self-similarity. It also allows the modeler to choose regularity and self-similarity properties independently of each other, which offers more flexibility than the models considered in [11,12,42]. More examples of Gaussian self-similar process can be found in [9,19]. Interestingly, many of the Gaussian self-similar models share the same path regularity properties as fBm, because it can be shown that there are positive finite constants \( c, C \) for which
where the symbol $H$ stands for the self-similarity parameter of the model under consideration.

Finally, it bears noting that self-similarity implies that $X_0 = 0$ and that $\text{Var}[X_t]$ is proportional to $t^{2H}$. This is a strong assumption on $X$. An uncertainty level on volatility which increases with time is a reasonable conservative forecasting assumption. That the volatility starts at 0 is more restrictive, since, in our implied volatility context, it corresponds to saying that the underlying risky asset’s movements tend towards certainty near the derivative’s maturity. Such a behavior is characteristic of specific risky asset classes, such as fixed-income securities, e.g. treasury bonds, and the dividend streams in preferred stocks; it is atypical of common stocks. To soften the assumption that $X_0 = 0$, one can add a constant mean to each centered self-similar $X$. We have investigated this possibility; it appears that this will require additional non-trivial tools not contained herein. Given the length of the current article, we have opted to leave this improvement for another work. One may, however, include a non-zero mean for each $X_t$ which is proportional to $t^H$; this is the framework used herein throughout.

We will next provide a brief overview of the paper. In Sect. 2, we discuss the Karhunen–Loève decomposition of a Gaussian process and introduce Gaussian self-similar stochastic volatility models. In Sect. 3, the small-time asymptotic behavior of the mixing density in such models is characterized (see Theorem 2). The mixing density is an important object in the theory of uncorrelated stochastic volatility models since the asset price density in such a model can be represented as the image of the mixing density under a certain log-normal integral operator (see formula (2)).

Theorem 3 formulated in Sect. 4 provides a sharp small-time asymptotic formula with an error estimate for the asset price density in a Gaussian self-similar model under the assumption that $r = 0$ and the volatility is a centered Gaussian process. The small-time asymptotic behavior of the call and put pricing functions in out-of-the-money regime is also studied in Sect. 4 (see Theorems 4 and 6), while the small-time asymptotic behavior of the implied volatility in the away-from-the-money regime is characterized in Sect. 5 (see Theorems 7 and 8). The coefficients in the asymptotic formulas for the above-mentioned functions are all expressed in terms of the Karhunen–Loève characteristics of the volatility process. Using the asymptotic formulas for the call and put pricing functions and the implied volatility, we estimate the self-similarity parameter $H$ as empirical statistics:

$$H = \lim_{T \to 0} \frac{\log \log C(T,K)}{\log \frac{1}{T}} - \frac{1}{2}$$

$$= \lim_{T \to 0} \frac{\log \log \frac{1}{T}}{\log \frac{1}{T}} - \frac{1}{2}$$

$$= 2 \lim_{T \to 0} \frac{\log \frac{1}{T}}{\log \frac{1}{T}} + \frac{1}{2}.$$
where the first line holds for $K > s_0$, the second for $K < s_0$, and the third holds for all $K \neq s_0$ (see Corollaries 1–4). These expressions for $H$ do not depend on any of the model parameters and statistics, and are in this sense model free within the class of self-similar models. However, in practice, since the regime $T \to 0$ is limited by the ability to trade options in a liquid way sufficiently close to maturity, the full asymptotic formulas for the call and put pricing functions and the implied volatility will typically be needed to help control the estimation error.

The at-the-money asymptotic behavior of option pricing functions and the implied volatility is studied in Sects. 6 and 7, respectively. The resulting asymptotics seem to rely on model statistics which cannot be related to the Karhunen–Loève elements in any simple fashion, since they require computing the moments $\mu_{1/2}$ and $\mu_{3/2}$ of order $1/2$ and $3/2$ of the non-explicit integrated variance’s law. Again, simple $H$-estimators can result, which do not rely on the moments $\mu_{1/2}$ and $\mu_{3/2}$. For instance, we show in Theorem 11 that

$$H = \lim_{T \to 0} \frac{\log \frac{1}{I(T,s_0)}}{\log \frac{1}{T}}.$$  

To illustrate the usage of our various asymptotic formulas numerically, we provide simulated stock prices, with corresponding call prices and implied volatilities, from the self-similar volatility model, using a classical Monte-Carlo method. Utilizing market-realistic parameter choices, we show how close prices and IVs are to our asymptotic approximations, noting that the fit is good in the call price case, and is excellent in the implied volatility case, for time-to-maturity as large as 2 weeks. It is then not surprising when we show that our implied volatility-based model-free calibration formulas for $H$ are accurate to 2 decimals up to 7 days in most cases, and 14 days in some cases. Being able to use the longest-possible time to maturity is important in practice because of liquidity considerations. This is all explained in Sect. 8. The last section of the present paper is the Appendix, containing the proofs of our main results (Theorems 3 and 4).

2 Gaussian Self-Similar Stochastic Volatility Models

We will denote by $D_t$ the distribution density of the asset price $S_t$ in the model described by (1). The following formula is known for this density:

$$D_t(x) = \frac{\sqrt{s_0 e^r}}{\sqrt{2\pi t}} x^{-3/2} \int_0^\infty y^{-1} \exp \left\{-\frac{\log^2 \frac{x}{s_0 e^r}}{2ty^2} + \frac{ty^2}{8} \right\} \tilde{p}_t(y)dy. \quad (2)$$

In (2), $\tilde{p}_t$ is the distribution density of the random variable $\tilde{Y}_t = \left\{ \frac{1}{t} \int_0^t X_s^2 ds \right\}^{1/2}$. The function $\tilde{p}_t$ is called the mixing density (see [45]). Formula (2) also holds for more general uncorrelated stochastic volatility models. The proof of formula (2) can be found in [45,47].

Let us fix the time horizon $T > 0$, and denote by $m$ and $K$ the mean function and the covariance function of the process $X$ given by $m(t) = \mathbb{E}[X_t]$, $t \in [0,T]$, and $K(t,s) = \mathbb{E}[(X_t - m(t))(X_s - m(s))]$, $t \in [0,T]^2$, respectively. It will be assumed
that \( K(s, s) > 0 \) if \( 0 \leq s \leq T \). The covariance operator associated with the process \( X \) is defined by

\[
K(f)(t) = \int_0^T f(s) K(t, s) ds, \quad f \in L^2([0, T]), \quad 0 \leq t \leq T.
\]

We will next discuss the Karhunen–Loève decomposition of a Gaussian process. All the statements formulated below can be found in \([2,70]\). The operator \( K \) is a nonnegative compact self-adjoint operator on \( L^2([0, T]) \). The non-zero eigenvalues of the operator \( K \) are of finite multiplicity, and we assume that they and the corresponding eigenfunctions are rearranged so that \( \lambda_1 = \lambda_2 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \lambda_{n_1+2} = \cdots = \lambda_{n_1+n_2} > \cdots \). In particular, \( \lambda_1 \) is the top eigenvalue, and \( n_1 \) is its multiplicity. It is known that \( \sum_{n=1}^{\infty} \lambda_n < \infty \). The system of eigenfunctions \( \{e_n\}_{n \geq 1} \), associated with the system \( \{\lambda_n\}_{n \geq 1} \), is orthonormal, and each function \( e_n \) is continuous on \([0, T]\). These eigenvalues and eigenfunctions are called the Karhunen–Loève characteristics of the process \( X \). A celebrated Karhunen–Loève theorem states that the centered Gaussian process \( \tilde{X}_t = X_t - m(t), \quad t \geq 0 \), can be represented as follows: \( \tilde{X}_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n \), where the symbols \( Z_n = Z_{n,T}, \quad n \geq 1 \), stand for a system of iid \( \mathcal{N}(0, 1) \) random variables.

**Remark 1** There exist explicit formulas for the Karhunen–Loève characteristics of various Gaussian processes. For Brownian motion, Brownian bridge, and OU processes, such formulas can be found in \([17]\). For OU bridges, one can consult \([16,18]\), and for the Gaussian process introduced in \([19]\), the Karhunen–Loève decomposition can be found in the same paper. Unfortunately, even for classical fractional Gaussian processes, e.g., fBm or fOU, the Karhunen–Loève characteristics are not known. In \([15]\) (see also \([14]\)), Corlay developed a powerful numerical method to approximate Karhunen–Loève eigenvalues and eigenfunctions. Corlay uses the Nyström method associated with the trapezoidal integration rule combined with the Richardson-Romberg extrapolation in his work.

The number \( \lambda_0 = 0 \) always belongs to the spectrum of the covariance operator, and it may happen so that \( \lambda_0 \) is an eigenvalue of \( K \). The spectral subspace associated with \( \lambda_0 \) may be infinite-dimensional, and we choose a basis \( \tilde{E} \) in this subspace. Then \( (E, \tilde{E}) \) is a complete orthonormal system in \( L^2([0, T]) \). Note that the eigenvalues and eigenfunctions of \( K \) depend on \( T \).

Our paper \([48]\) is mostly devoted to the extreme strike asymptotics of option pricing functions and the implied volatility in Gaussian stochastic volatility models. The present paper deals with Gaussian models, in which the volatility process is self-similar, and also with small-time asymptotic behavior of option pricing functions and the implied volatility in such models. For the sake of shortness, we introduce the following notation:

\[
\delta_n = \delta_n(T) = \int_0^T m(t)e_n(t) dt, \quad n \geq 1,
\]  

(3)
\[ s = s(T) = \int_0^T m(t)^2 dt, \quad \delta = \delta(T) = \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2. \] (4)

The coefficients in the asymptotic formulas obtained in the present paper and [48] are expressed in terms of the parameters defined in (3) and (4).

**Definition 1** Let \( 0 < H < 1 \). A stochastic process \( X^{(H)} \) is called \( H \)-self-similar if for every \( a > 0 \), \( X_{at}^{(H)} \overset{\text{law}}{=} a^H X_t^{(H)} \). Here \( \overset{\text{law}}{=} \) means the equality of all finite-dimensional distributions.

It is easy to see that if the process \( X^{(H)} \) is \( H \)-self-similar, then \( X_0^{(H)} = 0 \). It will always be assumed in the sequel that the self-similar process \( X^{(H)} \) is stochastically continuous. For a Gaussian process \( X \), the \( H \)-self-similarity condition is expressed in terms of the covariance function \( K \) as follows: \( K(at, as) = a^{2H} K(t, s), (t, s) \in [0, T]^2 \). We refer the interested reader to [22,67] for more information on self-similar stochastic processes.

Let us consider the following special Gaussian stochastic volatility model:

\[ dS_t = rS_t dt + |X_t^{(H)}| S_t dW_t, \quad S_0 = s_0, \] (5)

where \( s_0 > 0 \) is the initial condition for the asset price process \( S \), \( W \) is a standard Brownian motion, and \( X^{(H)} \) is a continuous \( H \)-self-similar adapted Gaussian process. The process \( S \) characterizes the dynamics of the asset price in the stochastic volatility model, where the volatility is described by the absolute value of a self-similar Gaussian process. We will call the model in (5) a Gaussian self-similar stochastic volatility model. It will be assumed throughout the paper that the model in (5) is uncorrelated, which means that the processes \( X^{(H)} \) and \( W \) are independent. We will often suppress the parameter \( H \) in various symbols used in the paper.

### 3 Small-Time Asymptotic Behavior of the Mixing Density

In the paper [48], we studied the asymptotic behavior of the distribution density \( p_T \) of the integrated variance \( \Gamma_T = \int_0^T X_t^2 dt \) in a general Gaussian stochastic volatility model. It is assumed in the present section that \( X \) is a continuous adapted Gaussian process.

The next theorem was established in [48]. We include its formulation for the sake of completeness.

**Theorem 1** (i) If \( \delta > 0 \), then

\[ p_T(x) = C x^{n_1^{-3}} \exp \left\{ \sqrt{\frac{\delta}{\lambda_1}} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left\{ 1 + O \left( x^{-\frac{1}{2}} \right) \right\} \] (6)
as \( x \to \infty \), where

\[
C = \frac{1}{2\sqrt{2\pi}} \lambda_1^{-n_1/2} \frac{\lambda_1^{-1/n_1}}{\delta} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \lambda_1^2}{2\lambda_1} - \delta \right\} \times \prod_{j > n_1} \left( \frac{\lambda_1}{\lambda_1 - \lambda_j} \right)^{1/2} \exp \left\{ \frac{\delta_j^2}{2(\lambda_1 - \lambda_j)} \right\}.
\]

(7)

(ii) If \( \delta = 0 \), then

\[
p_T(x) = C x^{n_1-2} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left( 1 + O \left( x^{-1} \right) \right)
\]

as \( x \to \infty \), where

\[
C = \frac{1}{2^{n_1/2} \Gamma \left( \frac{n_1}{2} \right)} \lambda_1^{-n_1/2} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \lambda_1^2}{2\lambda_1} \prod_{j > n_1} \left( \frac{\lambda_1}{\lambda_1 - \lambda_j} \right)^{1/2} \exp \left\{ \frac{\delta_j^2}{2(\lambda_1 - \lambda_j)} \right\} \right. \]

In particular, if the process \( X \) is centered, then (8) holds with

\[
C = \frac{1}{2^{n_1/2} \Gamma \left( \frac{n_1}{2} \right)} \lambda_1^{-n_1/2} \prod_{j > n_1} \left( \frac{\lambda_1}{\lambda_1 - \lambda_j} \right)^{1/2}.
\]

Let us next assume that the Gaussian model is self-similar (see (5)). Recall that we denoted by \( \tilde{p}_t \) the mixing density, that is, the density of the random variable

\[
\tilde{Y}_t = \left[ \frac{1}{t} \int_0^t \left( X_s^{(H)} \right)^2 ds \right]^{1/2}.
\]

It is not hard to see, using the self-similarity of the process \( X^{(H)} \), that for all \( x > 0 \),

\[
\tilde{p}_t(x) = t^{-H} \tilde{p}_1 \left( t^{-H} x \right).
\]

(9)

Therefore,

\[
\tilde{p}_t(x) = 2t^{-2H} x^p \left( t^{-2H} x^2 \right).
\]

(10)

The next assertion characterizes the small-time asymptotic behavior of the mixing density in a Gaussian self-similar stochastic volatility model.

**Theorem 2**  
(i) Suppose \( \delta > 0 \). Then, for every \( x > 0 \), the following asymptotic formula holds as \( T \to 0 \):
\[ \tilde{p}_T(x) = 2CT^{-\frac{H(n_1(1)+1)}{2}}x^{-\frac{n_1(1)-1}{2}} \exp \left\{ \frac{\sqrt{2} \delta(1)}{\lambda_1(1) T H} x \right\} \exp \left\{-\frac{x^2}{2T^2 H \lambda_1(1)} \right\} \times \left(1 + O_x \left(T^H\right)\right), \]

where

\[ C = \frac{1}{2\sqrt{2\pi}} \lambda_1(1)^{-\frac{n_1(1)+1}{4}} \varpi(1)^{-\frac{n_1(1)-1}{4}} \exp \left\{ \frac{s(1) - \sum_{n=1}^{\infty} \delta_n(1)^2}{2\lambda_1(1)} - \frac{\delta(1)}{2} \right\} \times \prod_{j>n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_j(1)} \right)^{1/2} \exp \left\{ \frac{\delta_j(1)^2}{2(\lambda_1(1) - \lambda_j(1))} \right\}. \]  

(ii) Suppose \( \delta = 0 \). Then, for every \( x > 0 \), the following asymptotic formula holds as \( T \to 0 \):

\[ \tilde{p}_T(x) = 2CT^{-Hn_1(1)}x^{n_1(1)-1} \exp \left\{-\frac{x^2}{2T^2 H \lambda_1(1)} \right\} \left(1 + O_x \left(T^H\right)\right), \]  

where

\[ C = \frac{1}{2} \frac{1}{\Gamma \left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{2}} \exp \left\{ \frac{s(1) - \sum_{n=1}^{\infty} \delta_n(1)^2}{2\lambda_1(1)} \right\} \times \prod_{j>n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_j(1)} \right)^{1/2} \exp \left\{ \frac{\delta_j(1)^2}{2(\lambda_1(1) - \lambda_j(1))} \right\}. \]

In particular, if the process \( X \) is centered, then (12) holds with \( C \) given by

\[ C = \frac{1}{2} \frac{1}{\Gamma \left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{2}} \prod_{j>n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_j(1)} \right)^{1/2}. \]

**Proof** Theorem 2 follows from (10) and Theorem 1. \( \square \)

### 4 Small-Time Asymptotic Behavior of Densities and Option Prices. Out-of-the-Money Regime

In this section, we restrict ourselves to the case where the process \( X^{(H)} \) in (5) is an adapted continuous \( H \)-self-similar centered Gaussian process.

Our first goal in the present section is to find asymptotic estimates of the density \( DT(x) \) as \( T \to 0 \), which are uniform with respect to the values of \( x > 0 \) separated from \( s_0 \) (away-from-the-money regime). Here we distinguish among two special cases. In the first case, we fix \( \varepsilon > 0 \), and consider asymptotic expansions as \( t \to 0 \), which are uniform with respect to \( x > s_0 + \varepsilon \). The notation \( O_\varepsilon(\phi(t, x)) \) as \( t \to 0 \), where \( \phi \)
is a positive function of two variables, means that the big-\(O\) estimate holds as \(t \to 0\) uniformly with respect to \(x > s_0 + \varepsilon\). In the second case, we fix \(\varepsilon\) with \(0 < \varepsilon < s_0\), and assume that \(0 < x < s_0 - \varepsilon\). The same notation \(O_\varepsilon(\phi(t, x))\) will be used in the second case.

Let us assume that \(r = 0\). Then it follows from (2) and (9) that

\[
D_T(x) = \frac{\sqrt{s_0}}{\sqrt{2\pi}} T^{-\frac{H}{2} - \frac{1}{2}} x^{-3} \int_0^{\infty} u^{-1} \exp \left\{ - \frac{\log^2 x}{2T^2 H + 1} + \frac{T^{2H+1} u^2}{8} \right\} \tilde{p}_1(u)du. 
\]

The next assertion is one of the main results of the present paper. It characterizes the small-time asymptotic behavior of the asset price density in a Gaussian model with a centered self-similar volatility process under the restriction \(r = 0\).

**Theorem 3** Fix \(\varepsilon > 0\) and let \(x > s_0 + \varepsilon\). Then as \(T \to 0\), the following asymptotic formula holds for the asset price density \(D_T\) in the model described by (5):

\[
D_T(x) = \frac{\sqrt{s_0}}{2^{n_1(1)-2} \Gamma \left( \frac{n_1(1)}{2} \right) \lambda_1(1) - \frac{n_1(1)}{4}} \prod_{k>n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_k(1)} \right)^{\frac{1}{2}} x^{-\frac{3}{2}} 
\times \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} T^{-\frac{(2H+1)n_1(1)}{4}} \left( \frac{x}{s_0} \right)^{-\frac{\sqrt{4 + \lambda_1(1) T^{2H+1}}}{2 \sqrt{\lambda_1(1) T^{2H+1}}}} 
\times \left( 1 + O \left( T^{2H+1} \right) \right) \left( 1 + O_\varepsilon \left( T^{\frac{2H+1}{4}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right). 
\]

**Remark 2** The case where the process \(X^{(H)}\) is noncentered is more complicated and will be addressed in future publications. We will assume throughout the paper that \(r = 0\). This assumption is not restrictive. In the case where \(r > 0\), we can obtain an asymptotic formula similar to that in (14), by replacing the process \(T \mapsto S_T\) by the process \(T \mapsto e^{-rt} S_T\). More precisely, if \(r > 0\), then the expression on the right-hand side of (14) must be multiplied by \(e^{rt}\), and the variable \(x\) in it must be replaced by the variable \(e^{rt} x\). The previous statement can be justified by taking into account the fact that the formula in (14) is uniform with respect to \(x > s_0 + \varepsilon\). A similar remark applies to the other asymptotic formulas established in the present paper.

**Remark 3** The constant in the first big-\(O\) error estimate in formula (14) does not depend on \(\varepsilon\) or \(x\), while the constant in the second one depends on a fixed \(\varepsilon\), but does not depend on \(x\) with \(x > s_0 + \varepsilon\). This allows us to integrate the estimate in (14) over the set \([s_0 + \varepsilon, \infty)\).

It was established in [48] that Gaussian stochastic volatility models are risk-neutral. More precisely, in such a model, the discounted asset price process \(t \mapsto e^{-rt} S_t\) is a \(\mathcal{F}_t\)-martingale. Define the call and the put pricing functions by

\[
C(T, K) = \mathbb{E} \left[ (S_T - K)^+ \right] \quad \text{and} \quad P(T, K) = \mathbb{E} \left[ (K - S_T)^+ \right],
\]
where \( T \) is the maturity and \( K \) is the strike price. It follows from the risk-neutrality condition that the put/call parity formula \( C(T, K) = P(T, K) + s_0 - K \) holds.

The call price \( C(T, K) \) equals the price \( C_{BS}(T, K; \sigma) \) in the Black-Scholes model with the volatility \( \sigma \) depending on \( T \) and \( K \). That value of \( \sigma \) is called the implied volatility and is denoted by \( I(T, K) \). In the present paper, we concentrate on the behavior of the functions \( C, P, \) and \( I \) for small \( T \) when \( K \) is fixed; consequently, we typically drop the dependence of the pricing functions and the implied volatility on \( K \).

Our next goal is to characterize the asymptotic behavior as \( T \to 0 \) of the function \( T \mapsto C(T) \) for \( K > s_0 \) (out-of-the-money call) and of the function \( T \mapsto P(T) \) for \( 0 < K < s_0 \) (out-of-the-money put) in a Gaussian self-similar stochastic volatility model.

**Theorem 4** Let \( K > s_0 \). Then the following asymptotic formula holds for the call pricing function in the model described by (5):

\[
C(T) = MT^{(2H+1)/(4-\eta_1(1))} \left( \frac{s_0}{K} \right)^{\frac{\lambda_1(1)}{2}} T^{-H - \frac{1}{2}} \left( 1 + O \left( T^{\frac{2H+1}{4}} \right) \right) \tag{15}
\]

as \( T \to 0 \), where

\[
M = \frac{s_0}{2^{\frac{\eta_1(1)}{4}} \Gamma \left( \frac{\eta_1(1)}{2} \right)} \lambda_1(1) \left( \log \frac{K}{s_0} \right)^{\frac{\eta_1(1)-2}{4}} \prod_{k>n_1(1)} \frac{\lambda_1(1)-\lambda_k(1)}{\lambda_1(1)} \frac{1}{2}.
\]

The following statement allows us to recover the self-similarity index \( H \) from the asymptotics of the call pricing function.

**Corollary 1** Under the conditions in Theorem 4, for every \( K > s_0 \),

\[
H = \lim_{T \to 0} \frac{\log \log \frac{1}{C(T,K)}}{\log \frac{1}{T}} - \frac{1}{2}. \tag{16}
\]

**Proof** It follows from (15) that

\[
\log \frac{1}{C(T)} = \log \frac{1}{M} + \frac{(2H+1)(4-\eta_1(1))}{4} \log \frac{1}{T} + \lambda_1(1)\frac{1}{2} T^{-H - \frac{1}{2}} \log \frac{K}{s_0} + O \left( T^{\frac{2H+1}{4}} \right) \tag{17}
\]

as \( T \to 0 \). Hence,

\[
\log \log \frac{1}{C(T)} = \log \left[ \lambda_1(1)\frac{1}{2} T^{-H - \frac{1}{2}} \log \frac{K}{s_0} \right] + \log \left( 1 + O \left( T^{H + \frac{1}{2}} + T^{H + \frac{1}{2}} \log \frac{1}{T} + T^{H + \frac{1}{2}} O \left( T^{\frac{2H+1}{4}} \right) \right) \right)
\]
\[
= \left( H + \frac{1}{2} \right) \log \frac{1}{T} + \log \left[ \lambda_1(1)^{-\frac{1}{2}} \log \frac{K}{s_0} \right] + O \left( T^{H + \frac{1}{2}} \log \frac{1}{T} \right)
\]

(18)
as \( T \to 0 \).

Now, it is clear that (16) follows from the previous formula.

Next, we turn our attention to the out-of-the-money put pricing function

\[ T \mapsto P(T) \text{ with } 0 < K < s_0. \]

The asymptotic behavior of the put pricing function with \( 0 < K < s_0 \) will be characterized using the symmetry properties of the model in (5). In ([45], Lemma 9.25), several equivalent conditions are given for the symmetry of a stochastic volatility model. One of them is as follows (see (9.79) in [45]):

\[
D_T(x) = \left( \frac{s_0}{x} \right)^3 D_T \left( \frac{s_0^2}{x} \right)
\]

(19)

for all \( x > 0 \) and \( T > 0 \). It is clear that for the model described by (5), the previous equality can be derived from formula (13). Next, using Theorem 3 and (19), we establish the following proposition.

**Theorem 5** Let \( 0 < \varepsilon < s_0 \) and \( 0 < x < s_0 - \varepsilon \). Then as \( T \to 0 \), the following asymptotic formula holds:

\[
D_T(x) = \frac{s_0^{\frac{1}{2} \gamma}}{\Gamma \left( \frac{n_1(1)}{2} \right) \sqrt{\lambda_1(1)}} \left( \frac{s_0}{x} \right) \frac{1}{\varepsilon} T^{-\frac{1}{2}} \lambda_1(1)^{-\frac{1}{2}} \prod_{k=2}^{\infty} \left( \frac{\lambda_1(1) - \rho_k(1)}{\lambda_1(1) - \frac{1}{2}} \right)^{\frac{n_k}{2}} x^{-\frac{3}{2}}
\]

\[
\times \left( \log \frac{s_0}{x} \right)^{\frac{n_1(1)-2}{2}} T^{-(2H+1)n_1(1)} \left( \frac{s_0}{x} \right)^{\frac{1}{2}} \lambda_1(1)^{H+\frac{1}{2}}
\]

\[
\times \left( 1 + O \left( T^{2H+1} \right) \right)^{\frac{1}{2}} \left( 1 + O_{\varepsilon} \left( T^{2H+1} \right) \right)^{\frac{1}{2}}
\]

Since the model that we are studying is symmetric, the following equality holds:

\[
P(T, K) = \frac{K}{s_0} C \left( T, \frac{s_0^2}{K} \right)
\]

(20)

(see condition 3 in Lemma 9.25 in [45]). The next assertion can be derived from Theorem 4 and (20).

**Theorem 6** Let \( 0 < K < s_0 \). Then the following asymptotic formula holds for the put pricing function in the model described by (5):

\[
P(T) = \tilde{M} T^{\frac{(2H+1)(4-n_1(1))}{4}} \lambda_1(1)^{-\frac{1}{2}} T^{-\frac{1}{2}} \left( 1 + O \left( T^{2H+1} \right) \right)^{\frac{1}{2}}
\]

\[
\times \left( \frac{K}{s_0} \right) \lambda_1(1)^{-\frac{1}{2}} T^{-\frac{1}{2}} \left( 1 + O \left( T^{2H+1} \right) \right)
\]

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as $T \to 0$, where the constant $\tilde{M}$ is given by

$$\tilde{M} = \frac{(s_0 K)^{\frac{1}{2}}}{2 \frac{n_1(1)}{\lambda_1(1)} \frac{\lambda_1(1)}{\lambda_1(1)} \frac{4 - n_1(1)}{4} \left(\frac{s_0}{K}\right)^{\frac{1}{2}} \prod_{k > n_1(1)} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \lambda_k(1)}\right)^{\frac{1}{2}}}.$$ 

The next statement can be established, using the same reasoning as in the proof of Corollary 1.

**Corollary 2** Under the conditions in Theorem 6, for every $0 < K < s_0$,

$$H = \lim_{T \to 0} \frac{\log \log \frac{1}{P(T, K)}}{\log \frac{1}{T}} - \frac{1}{2}.$$ 

### 5 Small-Time Asymptotic Behavior of the Implied Volatility

Theorems 4 and 6 characterize the small-time behavior of the call and put pricing functions in a stochastic volatility model defined in (5), under the assumption that the process $X^{(H)}$ is an adapted continuous $H$-self-similar centered Gaussian process. In the present section, we study the small-time behavior of the implied volatility in such a model. We will use some of the results obtained by Gao and Lee [37]. Gao and Lee establish certain asymptotic relations between the implied volatility and the call pricing function under very general conditions. They consider various asymptotic regimes, e.g., the extreme strike, the small/large time, or mixed regimes. Of our interest is formula (7.11) in Corollary 7.3 in [37]. It follows from this formula that if $K \neq s_0$, then

$$\sqrt{T} I(T, K) = \frac{\log \frac{K}{s_0}}{2 \log \frac{1}{C(T, K)}} \left(1 + O\left(\frac{\log \log \frac{1}{C(T, K)}}{\log \frac{1}{C(T, K)}}\right)\right)$$

as $T \to 0$. Therefore,

$$I(T, K) = \frac{\log \frac{K}{s_0}}{2T \log \frac{1}{C(T, K)}} + O\left(\frac{\log \log \frac{1}{C(T, K)}}{\sqrt{T} \log \frac{1}{C(T, K)}}\right)^{\frac{1}{2}}$$

as $T \to 0$.

The following assertion can be derived from (15) and (21).

**Theorem 7** Let $K > s_0$. Then the following asymptotic formula holds for the implied volatility in the model described by (5):

$$I(T) = \frac{\lambda_1(1)^{\frac{1}{2}} \sqrt{\log \frac{K}{s_0}} T^{\frac{2H - 1}{4}}}{\sqrt{2}} + O\left(T^{\frac{2H - 1}{4} - \frac{1}{2}} \log \frac{1}{T}\right)$$

(22)
as \( T \to 0 \).

**Proof** It follows from (17) and (18) that 

\[
\log \frac{1}{C(T)} \approx T^{-H-\frac{1}{2}} \quad \text{and} \quad \log \log \frac{1}{C(T)} \approx \log \frac{1}{T}
\]
as \( T \to 0 \). Moreover, the mean value theorem implies that

\[
\left( \log \frac{1}{C(T)} \right)^{-\frac{1}{2}} = \left( \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \log \frac{K}{s_0} \right)^{-\frac{1}{2}} + O \left( T^{\frac{6H+3}{4} \log \frac{1}{T}} \right)
\]
as \( T \to 0 \). Now it is not hard to see that (22) follows from (21) and the previous formulas. \( \square \)

**Remark 4** Assume \( K > s_0 \). It follows from Theorem 7 that if the Hurst index satisfies \( 0 < H < \frac{1}{2} \), then the implied volatility \( T \mapsto I(K, T) \) is singular at \( T = 0 \), and it behaves near zero like the function \( T \mapsto T^{\frac{2H-1}{4}} \). For standard Brownian motion, \( H = \frac{1}{2} \), and we have

\[
\lim_{T \to 0} I(K, T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\log \frac{K}{s_0}}}{\sqrt{2}}.
\]
Finally, for \( \frac{1}{2} < H < 1 \), the implied volatility \( T \mapsto I(K, T) \) tends to zero like the function \( T \mapsto T^{\frac{2H-1}{4}} \).

The next statement is a corollary to Theorem 7. It provides a representation of the self-similarity index in terms of the implied volatility.

**Corollary 3** Let \( K > s_0 \). Then the following equality holds:

\[
H = 2 \lim_{T \to 0} \frac{\log \frac{1}{I(T, K)}}{\log \frac{1}{T}} + \frac{1}{2}.
\] (23)

In the case where \( 0 < K < s_0 \), Theorem 7, Corollary 3, and the symmetry condition

\[
I(T, K) = I \left( T, \frac{s_0^2}{K} \right)
\]
(see [45], Lemma 9.25) imply the following assertions.

**Theorem 8** Let \( 0 < K < s_0 \). Then the following asymptotic formula holds for the implied volatility in the model described by (5):

\[
I(T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\log \frac{s_0}{K}}}{\sqrt{2}} T^{\frac{2H-1}{4}} + O \left( T^{\frac{6H+3}{4} \log \frac{1}{T}} \right)
\]
as \( T \to 0 \).
Corollary 4 Let $0 < K < s_0$. Then equality (23) holds for the self-similarity index $H$.

6 At-the-Money Options

In this section, we study the asymptotic behavior of the call pricing function in at-the-money regime, that is, the regime where $K = s_0$. It is assumed in the present section that $X^{(H)}$ is an adapted $H$-self-similar Gaussian process. However, we do not assume that the process $X^{(H)}$ is centered.

Using (13) and the formula $C(T, K) = \int_K^{\infty} (x - K) d\tau_{T}(x) dx$, we obtain the following equalities for at-the-money call:

$$C(T, s_0) = \sqrt{\frac{s_0}{2\pi}} T^{-H-\frac{1}{2}} \int_{0}^{\infty} u^{-1} \exp \left\{-\frac{T^{2H+1} u^2}{8} \right\} \tilde{p}_1(u) du \times \int_{s_0}^{\infty} (x - s_0) x^{-\frac{3}{2}} \exp \left\{-\frac{\log^2 \frac{x}{s_0}}{2T^2H+1} u^2 \right\} dx$$

$$= \sqrt{\frac{s_0}{2\pi}} T^{-H-\frac{1}{2}} \int_{0}^{\infty} u^{-1} \exp \left\{-\frac{T^{2H+1} u^2}{8} \right\} \tilde{p}_1(u) du$$

$$\times \left[ \int_{s_0}^{\infty} x^{-\frac{1}{2}} \exp \left\{-\frac{\log^2 \frac{x}{s_0}}{2T^2H+1} u^2 \right\} dx \right.$$ $$- s_0 \int_{s_0}^{\infty} x^{-\frac{3}{2}} \exp \left\{-\frac{\log^2 \frac{x}{s_0}}{2T^2H+1} u^2 \right\} dx \left. \right].$$

It follows from the previous formula that

$$C(T, s_0) = \frac{s_0}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} \int_{0}^{\infty} u^{-1} \exp \left\{-\frac{T^{2H+1} u^2}{8} \right\} \tilde{p}_1(u)$$

$$\times [\Phi_1(T, u) - \Phi_2(T, u)] du,$$

where

$$\Phi_1(T, u) = \int_{1}^{\infty} y^{-\frac{1}{2}} \exp \left\{-\frac{\log^2 y}{2T^2H+1} u^2 \right\} dy$$

(25)

and

$$\Phi_2(T, u) = \int_{1}^{\infty} y^{-\frac{3}{2}} \exp \left\{-\frac{\log^2 y}{2T^2H+1} u^2 \right\} dy.$$ (26)

Our next goal is to estimate the functions $\Phi_1$ and $\Phi_2$ defined in (25) and (26). We have
\[ \Phi_1(T, u) = \int_0^\infty \exp \left\{ - \left[ \frac{w^2}{2T^2H^2u^2} - \frac{w}{2} \right] \right\} \, dw \]
\[ = \exp \left\{ \frac{T^2H^2u^2}{8} \right\} \int_0^\infty \exp \left\{ - \frac{1}{2T^2H^2u^2} \left( w - \frac{T^2H^2u^2}{2} \right)^2 \right\} \, dw \]
\[ = \exp \left\{ \frac{T^2H^2u^2}{8} \right\} \int_{-\frac{1}{2T^2H^2u^2}}^{\infty} \exp \left\{ - \frac{1}{2T^2H^2u^2} \right\} \, dz \]
\[ = T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^2H^2u^2}{8} \right\} \int_{-\frac{1}{2T^{H+\frac{1}{2}}u}}^{\infty} \exp \left\{ - \frac{y^2}{2} \right\} \, dy. \]

Similarly,
\[ \Phi_2(T, u) = T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^2H^2u^2}{8} \right\} \int_{\frac{1}{2T^{H+\frac{1}{2}}u}}^{\infty} \exp \left\{ - \frac{y^2}{2} \right\} \, dy. \]

Therefore
\[ \Phi_1(T, u) - \Phi_2(T, u) = 2T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^2H^2u^2}{8} \right\} \int_0^{\frac{1}{2T^{H+\frac{1}{2}}u}} \exp \left\{ - \frac{y^2}{2} \right\} \, dy. \tag{27} \]

The next lemma will allow us to estimate the integral in (27).

**Lemma 1** Let \(0 < a < 1\). Then the following inequalities are valid:
\[ a - \frac{a^3}{6} \leq \int_0^a \exp \left\{ - \frac{y^2}{2} \right\} \, dy \leq a - \frac{a^3}{6} + \frac{a^5}{40}. \tag{28} \]

On the other hand, if \(a \geq 1\), then
\[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{1}{a} \exp \left\{ - \frac{a^2}{2} \right\} \leq \int_0^a \exp \left\{ - \frac{y^2}{2} \right\} \, dy \leq \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{a}{a^2 + 1} \exp \left\{ - \frac{a^2}{2} \right\}. \tag{29} \]

**Proof** The inequalities in (28) can be established using the Taylor expansion with two and three terms. To prove the estimates in (29), we use the following known inequalities:
\[ \frac{x}{x^2 + 1} \exp \left\{ - \frac{x^2}{2} \right\} \leq \int_x^{\infty} \exp \left\{ - \frac{y^2}{2} \right\} \, dy \leq \frac{1}{x} \exp \left\{ - \frac{x^2}{2} \right\}. \tag{30} \]
for all \( x > 0 \). The previous inequalities follow from stronger estimates formulated in [1], 7.1.13. Now, (29) can be derived from (30) and the equality

\[
\int_0^a \exp \left\{ -\frac{y^2}{2} \right\} dy = \frac{\sqrt{\pi}}{\sqrt{2}} - \int_a^\infty \exp \left\{ -\frac{y^2}{2} \right\} dy.
\]

This completes the proof of Lemma 1. \( \square \)

The next assertion provides estimates for the at-the-money call.

**Theorem 9** The following inequalities are true for every \( T > 0 \):

\[
U_1(T) \leq C(T, s_0) \leq U_2(T),
\]

where

\[
U_1(T) = \frac{s_0}{\sqrt{2\pi}} T^{H+\frac{1}{2}} \int_0^\infty \tilde{p}_1(u)udu - \frac{s_0}{24\sqrt{2\pi}} T^{3H+\frac{3}{2}} \int_0^\infty \tilde{p}_1(u)u^3 du
\]
\[+ \frac{2s_0}{\sqrt{2\pi} T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1\left( \frac{v}{T^{H+\frac{1}{2}}} \right) \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{v}{2} + \frac{v^3}{48} - \frac{2}{v} \exp \left\{ -\frac{v^2}{8} \right\} \right] dv
\]

and

\[
U_2(T) = \frac{s_0}{\sqrt{2\pi}} T^{H+\frac{1}{2}} \int_0^\infty \tilde{p}_1(u)udu - \frac{s_0}{24\sqrt{2\pi}} T^{3H+\frac{3}{2}} \int_0^\infty \tilde{p}_1(u)u^3 du
\]
\[+ \frac{s_0}{640\sqrt{2\pi}} T^{5H+\frac{5}{2}} \int_0^\infty \tilde{p}_1(u)u^5 du + \frac{2s_0}{\sqrt{2\pi} T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1\left( \frac{v}{T^{H+\frac{1}{2}}} \right) \times \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{v}{2} + \frac{v^3}{48} - \frac{v^5}{1280} - \frac{2v}{v^2 + 4} \exp \left\{ -\frac{v^2}{8} \right\} \right] dv
\]

**Proof** Using (24), (27) and Lemma 1, we can obtain the following upper and lower estimates for the call pricing function:

\[
C(T, s_0) \leq \frac{s_0}{\sqrt{2\pi}} \int_0^{\frac{2}{T^{H+\frac{1}{2}}}} \tilde{p}_1(u) \left[ T^{H+\frac{1}{2}} u - \frac{1}{24} T^{3H+\frac{3}{2}} u^3 + \frac{1}{640} T^{5H+\frac{5}{2}} u^5 \right] du
\]
\[+ \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^\infty \tilde{p}_1(u) \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2T^{H+\frac{1}{2}} u}{T^{2H+1} u^2 + 4} \exp \left\{ -\frac{T^{2H+1} u^2}{8} \right\} \right] du
\]
\[= \frac{s_0}{\sqrt{2\pi}} \int_0^\infty \tilde{p}_1(u) \left[ T^{H+\frac{1}{2}} u - \frac{1}{24} T^{3H+\frac{3}{2}} u^3 + \frac{1}{640} T^{5H+\frac{5}{2}} u^5 \right] du
\]
\begin{align}
- \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[ \frac{1}{2} T^{H+\frac{1}{2}} u - \frac{1}{48} T^{3H+\frac{3}{2}} u^3 + \frac{1}{1280} T^{5H+\frac{5}{2}} u^5 \right] du \\
+ \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2 T^{H+\frac{1}{2}} u}{T^{2H+1} u^2 + 4} \exp \left\{ -T^{2H+1} u^2 \right\} \right] du
\end{align}

and

\begin{align}
C(T, s_0) \geq \frac{s_0}{\sqrt{2\pi}} \int_{0}^{\frac{2}{T^{H+\frac{1}{2}}}} \tilde{p}_1(u) \left[ T^{H+\frac{1}{2}} u - \frac{1}{24} T^{3H+\frac{3}{2}} u^3 \right] du \\
+ \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2}{T^{H+\frac{1}{2}}} \exp \left\{ -T^{2H+1} u^2 \right\} \right] du \\
- \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[ \frac{1}{2} T^{H+\frac{1}{2}} u - \frac{1}{48} T^{3H+\frac{3}{2}} u^3 \right] du \\
+ \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[ \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2}{T^{H+\frac{1}{2}}} \exp \left\{ -T^{2H+1} u^2 \right\} \right] du. \tag{32}
\end{align}

Now, it is not hard to see, making the substitution \( v = T^{H+\frac{1}{2}} u \), that Theorem 9 follows from (31) and (32).

The next statement characterizes the small-time asymptotic behavior of the at-the-money call pricing function in a Gaussian self-similar stochastic volatility model.

\begin{corollary}
The following formula holds as \( T \to 0 \):
\begin{align}
C(T, s_0) = c_1 T^{H+\frac{1}{2}} - c_2 T^{3H+\frac{3}{2}} + O \left( T^{5H+\frac{5}{2}} \right), \tag{33}
\end{align}

where
\begin{align}
c_1 = \frac{s_0}{\sqrt{2\pi}} \int_{0}^{\infty} p_1(u) u \frac{1}{2} du \tag{34}
\end{align}

and
\begin{align}
c_2 = \frac{s_0}{24\sqrt{2\pi}} \int_{0}^{\infty} p_1(u) u^2 du. \tag{35}
\end{align}
\end{corollary}
Proof For a centered volatility process $X$, we will use formula (50). In the case of a noncentered volatility process $X$, we will need the following formula:

$$
\tilde{p}_1(x) = 2C x^{n_1(1)-1} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} x \right\} \exp \left\{ -\frac{x^2}{2\lambda_1(1)} \right\} \left( 1 + O(x^{-1}) \right) \quad (36)
$$
as $x \to \infty$, where the constant $C$ is given by (11). Formula (36) derives easily from (6) and (10).

It follows from Theorem 9 that

$$
C(T, s_0) - U_1(T) \leq U_2(T) - U_1(T) \leq \frac{s_0}{640 \sqrt{2\pi}} T^{5H+\frac{5}{2}} \int_0^\infty \tilde{p}_1(u) u^5 \, du
\]

$$
+ \frac{2s_0}{\sqrt{2\pi}T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1 \left( \frac{v}{T^{H+\frac{1}{2}}} \right) \left[ \frac{2}{v} \exp \left\{ -\frac{v^2}{8} \right\} \right]
\]

$$
+ \frac{2v}{v^2 + 4} \exp \left\{ -\frac{v^2}{8} \right\} + \frac{v^5}{1280} \right] dv. \quad (37)
$$

Let us next suppose the process $X$ is centered. Then, using (50), we see that for $v > 2$ and for sufficiently small values of $T$,

$$
\frac{1}{T^{H+\frac{1}{2}}} \tilde{p}_1 \left( \frac{v}{T^{H+\frac{1}{2}}} \right) \leq \alpha \left( \frac{v}{T^{H+\frac{1}{2}}} \right)^{n_1(1)-1} \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{2\lambda_1(1)T^{2H+1}} \right\}
\]

$$
\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{4\lambda_1(1)T^{2H+1}} \right\}
\]

$$
\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{1}{2\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\}
\]

$$
\leq \alpha \exp \left\{ -\frac{1}{4\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\}. \quad (38)
$$

Here $\alpha > 0$ is a constant that may change from line to line.

Now, assume the process $X$ is noncentered. Then for $v > 2$ and for sufficiently small $T$,

$$
\frac{1}{T^{H+\frac{1}{2}}} \tilde{p}_1 \left( \frac{v}{T^{H+\frac{1}{2}}} \right) \leq \alpha \left( \frac{v}{T^{H+\frac{1}{2}}} \right)^{n_1(1)-1} \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} \frac{v}{T^{H+\frac{1}{2}}} \right\}
\]

$$
\times \exp \left\{ -\frac{v^2}{2\lambda_1(1)T^{2H+1}} \right\}
\]

$$
\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{4\lambda_1(1)T^{2H+1}} \right\}
\]

$$
\leq \alpha \exp \left\{ -\frac{1}{4\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\}. \quad (39)
$$
Finally, taking into account (37), (38), and (39), we obtain

\[ C(T, s_0) - U_1(T) = O \left( T^{5H + \frac{5}{2}} \right) \] (40)

as \( T \to 0 \). Now, it is not hard to see that the definition of \( U_1 \), (38), and (40) imply that

\[ C(T, s_0) = b_1 T^{H + \frac{1}{2}} - b_2 T^{3H + \frac{3}{2}} + O \left( T^{5H + \frac{5}{2}} \right), \]

where

\[ b_1 = \frac{s_0}{\sqrt{2\pi}} \int_0^\infty \tilde{p}_1(u) u du \]

and

\[ b_2 = \frac{s_0}{24\sqrt{2\pi}} \int_0^\infty \tilde{p}_1(u) u^3 du. \]

Finally, using the equality \( \tilde{p}_1(u) = 2u \tilde{p}_1(u^2) \), we obtain \( b_i = c_i \) for \( i = 1, 2 \).

This completes the proof of Corollary 5. \( \square \)

### 7 Implied Volatility in At-the-Money Regime

The restrictions on the process \( X^{(H)} \) in the present section are the same as in Sect. 6.

Recall that the Black-Scholes call pricing function in the case where \( r = 0 \) and \( K = s_0 \) is given by

\[ C_{BS}(T, s_0, \sigma) = \frac{s_0}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} dy \]

Hence,

\[ C_{BS}(T, s_0, \sigma) = s_0 \text{erf} \left( \frac{\sigma \sqrt{T}}{2\sqrt{2}} \right), \] (41)

where erf is the error function defined by \( \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2} dx \). The error function is a strictly increasing continuous function from \([0, \infty)\) onto \([0, 1]\). Its inverse function is denoted by \( \text{erf}^{-1} \). It is known that the inverse error function has the following Maclaurin expansion:

\[ \text{erf}^{-1}(z) = \frac{\sqrt{\pi}}{2} \left( z + \frac{\pi}{12} z^3 + \frac{7\pi^2}{480} z^5 + \cdots \right), \quad 0 \leq z \leq 1 \] (42)

(see, e.g., [69]). It follows from the definition of the implied volatility that
CBS \( (T, s_0, I(T, s_0)) = C(T, s_0) \).

Therefore, (41) implies

\[
I(T, s_0) = \frac{2 \sqrt{2}}{\sqrt{T}} \text{erf}^{-1} \left( \frac{C(T, s_0)}{s_0} \right).
\]

Next, using (42), we obtain

\[
I(T, s_0) = \frac{\sqrt{2\pi}}{\sqrt{T}} \left[ \frac{C(T, s_0)}{s_0} + \frac{\pi}{12} \frac{C(T, s_0)^3}{s_0^3} + O \left( C(T, s_0)^5 \right) \right]
\]

as \( T \to 0 \).

Now, we are ready to characterize the small-time asymptotic behavior of the implied volatility in at-the-money regime.

**Theorem 10** The following asymptotic formula holds as \( T \to 0 \):

\[
I(T, s_0) = T^H \int_0^\infty p_1(u)u^{\frac{1}{2}} du + T^{3H+1} \frac{1}{24} \left[ \left( \int_0^\infty p_1(u)u^{\frac{1}{2}} du \right)^3 - \int_0^\infty p_1(u)u^{\frac{3}{2}} du \right] + O \left( T^{5H+2} \right).
\]

**Proof** Our first goal is to obtain an asymptotic formula for the implied volatility with error term of the order \( O \left( T^{5H+2} \right) \), by using formula (33) in (43). Following this plan, we obtain

\[
I(T, s_0) = \frac{\sqrt{2\pi}}{s_0 \sqrt{T}} \left( c_1 T^{H+\frac{3}{2}} - c_2 T^{3H+\frac{3}{2}} + O \left( T^{5H+\frac{5}{2}} \right) \right) + \frac{\pi \sqrt{2\pi}}{12s_0^{\frac{3}{2}} \sqrt{T}} \left( c_1 T^{H+\frac{3}{2}} - c_2 T^{3H+\frac{3}{2}} + O \left( T^{5H+\frac{5}{2}} \right) \right)^3 + O \left( T^{5H+2} \right)
\]

as \( T \to 0 \). Now, it is not difficult to see that formula (44) follows from (34), (35), and (45).

This completes the proof of Theorem 10. \( \Box \)

**Remark 5** It is clear that the following formulas are valid for the integrals in (44):

\[
\mu_{1/2} := \int_0^\infty p_1(u)u^{\frac{1}{2}} du = \mathbb{E} \left[ \left( \int_0^1 X_s^2 ds \right)^{\frac{1}{2}} \right]
\]
and

\[ \mu_{3/2} := \int_0^\infty p_1(u)u^3du = \mathbb{E}\left[ \left( \int_0^1 X_s^2ds \right)^{3/2} \right]. \]

Theorem 10 allows us to recover the self-similarity index \( H \), knowing the small-time behavior of the implied volatility in at-the-money regime.

**Theorem 11** The following formula holds:

\[ H = \lim_{T \to 0} \frac{1}{\log T} \log \frac{I(T,s_0)}{T}. \]

## 8 Numerical Illustration

To illustrate the numerical potential of our asymptotic formulas in practice, we finish this article with a brief section comparing exact (Monte-Carlo-simulated) option prices and IVs with the asymptotics we have derived. Formulas such as (22) can be used to calibrate various parameters which might be linked explicitly or empirically to \( \lambda_1(1) \), assuming \( H \) is known. We refer to the numerics in our prior work in [48] for details on what can be done, leaving to the interested reader any details of how to translate the ideas therein which are for extreme strike asymptotics to the small time case. Gulisashvili et al. [48] also contains a description of how to simulate the fBm-driven models of interest to us, for Monte-Carlo purposes, as alluded to in Remark 1; we do not repeat this information here.

Our results in the at-the-money case are presumably harder to exploit along these lines because they depend on moment statistics \( \mu_{1/2} \) and \( \mu_{3/2} \) (Remark 5), which are not explicitly related to model parameters. An exception to this observation is in the case of models with a volatility scale parameter \( \sigma \), by which we mean that one replaces model (1) with

\[ dS_t = rS_t dt + \sigma |X_t| S_t dW_t. \]

(46)

Here the parameter \( \sigma \) is rather innocuous since, by self-similarity of \( |X| \), this \( \sigma \) can be absorbed as a linear time change, but it represents a convenient parameter for tuning a model to realistic time-scales and volatility levels. We will use this device in this section. In particular, at the money, it is easy to see from Theorem 10 that one has

\[ I(T,s_0) = \sigma \mu_{1/2} T^H + \frac{\sigma^3}{24} T^{3H+1} \left[ (\mu_{1/2})^3 - \mu_{3/2} \right] + O(T^{5H+2}) \]

where \( \mu_{1/2} \) and \( \mu_{3/2} \) are given in Remark 5. Thus at-the-money implied volatility asymptotics can be used to calibrate \( \sigma \) in model (46). We do not comment on this further herein.

Instead, we provide a numerical analysis of our results’ use in \( H \)’s calibration. Indeed, the reference [48] contains an effort to calibrate \( H \) itself, when other parameters have been estimated by other means, but left some stones unturned. We found...
therein that $H$ calibration can be relatively successful in some cases in practice, though this is not necessarily backed up by any asymptotic theory. In this section we show instead how model-free results such as Corollary 3 and Theorem 11 provide excellent calibration of $H$ in many cases. We choose to present this in the at-the-money case for two reasons. First, it illustrates the model-free framework, since the results we obtain are not sensitive to the values of $\mu_{1/2}$ and $\mu_{3/2}$. Second, in practice, liquidity is low for options away from the money near maturity, which all but dictates the use of at-the-money implied volatility.

The setup we use is that of model (46) with $X = fBm$, $r = 0$, and $\sigma = 2$ or $\sigma = 3$. The choice of $\sigma$ is tailored to provide a realistic volatility level after 1 or 2 weeks, with time measured in years. Specifically, a practitioner may simply select the desired magnitude of $\sigma$ by matching it to the mean magnitude of volatility in (46) via the formula

$$E[\sigma | X_t |] = \sigma t^H \sqrt{2/\pi}.$$ 

For example, with $H = 0.6$ and $\sigma = 3$ we get $E[\sigma | X_t |] \approx 0.22$ after one week ($t = 7/365 \approx 0.019$), and $E[\sigma | X_t |] \approx 0.34$ after two weeks ($t = 14/365 \approx 0.038$), which could represent a realistic scenario for a rather volatile short-term bond market. The value $\sigma = 2$ provides for a less volatile market, and as we will see, results in very sharp calibration even with two-week maturity. The exercise could be repeated for the lower volatility scenario $\sigma = 1$, and would result in even sharper calibration; we do not present the results for $\sigma = 1$ here. Significantly smaller values of $\sigma$, which we do not explore here, allow for an extremely sharp fit between theoretical call and implied volatility values and our asymptotics, but would typically be unrealistic in a near-maturity self-similar context.

Before using Theorem 11, a first question might be whether it would not be sufficient to use an asymptotic theory for call prices to estimate parameters. The use of implied volatility over option prices has been advocated in many articles, including many of the ones cited herein, but the question is still legitimate since one rarely sees evidence in the literature that this is indeed preferable in practice. The following two images compare the fit between our asymptotic formulas (Corollary 5 and Theorem 10) and exact (simulated) call and implied volatility values for times from 1 day to 2 weeks.
We chose the extreme case $H = 0.51$ because, as it turns out, the asymptotics’ accuracy increase as $H$ increases. We see from the above that the implied volatility asymptotics are accurate at a roughly 5%-error level for more than 10 days, and remains fairly accurate up to 2 weeks, while the call asymptotics are only accurate at a 5%-error level for 2 days, and deteriorate significantly thereafter. Other values of $H$ show similar pictures. The choice to use implied volatility over call prices for calibration purposes in small time is clear. This can of course be verified rigorously on our formulas since our coefficients can be computed numerically as well; this is omitted from our study. The next four pictures show the extremely sharp fit of implied volatility asymptotics over two weeks as $H$ increases above 0.51, for $\sigma = 3$, as we mentioned. Ten more pictures, at the end of the article, show an even sharper fit as $H$ ranges from 0.25 to 0.85, for $\sigma = 2$. Only the lowest value of $H = 0.25$ shows any difficulty in the weekly approximation.

![Graphs showing the fit of implied volatility asymptotics](image1)

IV with $\sigma = 3$, $t \in [1 \text{ day, 2 weeks}]$, $H = 0.55$

IV with $\sigma = 3$, $t \in [1 \text{ day, 2 weeks}]$, $H = 0.60$

IV with $\sigma = 3$, $t \in [1 \text{ day, 2 weeks}]$, $H = 0.75$

IV with $\sigma = 3$, $t \in [1 \text{ day, 2 weeks}]$, $H = 0.85$

Since liquidity decreases as time to maturity decreases, it is desirable to use the largest possible time $t_0$ such that the relative error in implied volatility approximation does not exceed a given error level, say 1% which would be a high level of accuracy. The table below give an idea of what this means in practice, by computing $t_0$ for a 1% level in the above realistic cases: with

$$t_0 = \max \left\{ t \in [0, 14] : \frac{\text{simulated IV}(t) - \text{asymptotic IV}(t)}{\text{simulated IV}(t)} < 0.01 \right\}$$
we find for $\sigma = 3$:

\[
\begin{array}{ccccccccccc}
H & 0.25 & 0.35 & 0.4 & 0.45 & 0.49 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
t_0 \text{ in days} & 0 & 3.7 & 6.4 & 7.7 & 10.0 & 2.8 & 3.7 & 10.4 & 14 & 14
\end{array}
\]

and for $\sigma = 2$:

\[
\begin{array}{ccccccccccc}
H & 0.25 & 0.35 & 0.4 & 0.45 & 0.49 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
t_0 \text{ in days} & 0 & 5.9 & 10.4 & 14 & 14 & 3.7 & 11.3 & 14 & 14 & 14
\end{array}
\]

These values of $t_0$ could be considered as rather conservative, due to the choice of 1% accuracy; practitioners may decide to choose a slightly more liberal level. This is evident from the last tables below, in which we show the result of the calibration of $H$ from exact (simulated) option prices, via Theorem 11. For $\sigma = 3$ our calibration yields:

$T = 1$ day, $\sigma = 3$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85
\end{array}
\]

$T = 2$ days, $\sigma = 3$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.24 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85
\end{array}
\]

$T = 7$ days, $\sigma = 3$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.23 & 0.34 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85
\end{array}
\]

$T = 14$ days, $\sigma = 3$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.16 & 0.32 & 0.38 & 0.44 & 0.48 & 0.49 & 0.50 & 0.54 & 0.59 & 0.75 & 0.85
\end{array}
\]

For $\sigma = 2$, the calibration is even better; in particular one notes a marked improvement with the 14-day options, and the remaining calibrations are essentially perfect:

$T = 1$ day, $\sigma = 2$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85
\end{array}
\]

$T = 2$ days, $\sigma = 2$, $H$ calibrated from IV via Theorem 11:

\[
\begin{array}{ccccccccccc}
H \text{ used in simulation} & 0.25 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85 \\
H \text{ calibrated} & 0.24 & 0.35 & 0.40 & 0.45 & 0.49 & 0.50 & 0.51 & 0.55 & 0.60 & 0.75 & 0.85
\end{array}
\]
$T = 7 \text{ days, } \sigma = 2, H$ calibrated from IV via Theorem 11

| $H$ used in simulation | 0.25 | 0.35 | 0.40 | 0.45 | 0.49 | 0.50 | 0.51 | 0.55 | 0.60 | 0.75 | 0.85 |
|------------------------|------|------|------|------|------|------|------|------|------|------|------|
| $H$ calibrated          | 0.24 | 0.35 | 0.40 | 0.45 | 0.49 | 0.50 | 0.51 | 0.55 | 0.60 | 0.75 | 0.85 |

$T = 14 \text{ days, } \sigma = 2, H$ calibrated from IV via Theorem 11

| $H$ used in simulation | 0.25 | 0.35 | 0.40 | 0.45 | 0.49 | 0.50 | 0.51 | 0.55 | 0.60 | 0.75 | 0.85 |
|------------------------|------|------|------|------|------|------|------|------|------|------|------|
| $H$ calibrated          | 0.23 | 0.34 | 0.39 | 0.45 | 0.49 | 0.50 | 0.50 | 0.55 | 0.60 | 0.75 | 0.85 |

In all cases except for $H = 0.25$, even with a 14-day time to maturity, the error in $H$-calibration is no greater than one hundredth (less than 2% relative error). Other than the case $H = 0.25$, the only difficulty we experience appears to be in differentiating between a model with Brownian scaling ($H = 0.50$, no memory in the volatility) and a model with $H \neq 0.50$ : when $\sigma = 3$, this fails except for the very short times to maturity $t = 1, 2$ days (for $\sigma = 2$, it succeeds for all but the case $t = 14$ days). If liquidity at those levels is adequate for the high-volatility case of $\sigma = 3$, as it may be in heavily traded bond markets, then our calibration can be used with such short horizons. Otherwise a maturity of one week is preferable, particularly for self-similarity indices which are not too close to 0.50. A maturity of two weeks can be recommended in all cases for scenarios where one is satisfied with a possible error of one hundredth on $H$ calibration (except for very low $H$); this could be a realistic accuracy level for many users of stochastic volatility models who are currently not using any self-similarity or long-memory assumptions.

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Appendix: Proofs of the Main Theorems

Proof of Theorem 3 Fix $x > 0$, and denote

$$J_x(T) = \int_0^\infty u^{-1} \exp\left\{ - \left[ \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} \tilde{p}_1(u)du. \quad (47)$$

It is clear from (13) that the small-time asymptotic behavior of the density $D_T(x)$ is determined by that of the integral $J_x(T)$.

The next lemma will allow us to use Theorem 2 to estimate the integral in (47). □

Lemma 2 Fix $\alpha \in \mathbb{R}$, $b > 0$, and $\varepsilon > 0$. Let $x > s_0 + \varepsilon$, and suppose $f$ is an integrable function on $[0, b]$. Then

$$\int_0^b u^\alpha \exp\left\{ - \left[ \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)|du = O_\varepsilon \left( \exp\left\{ - \frac{\log^2 \frac{x}{s_0}}{2b^2T^{2H+1}} \right\} \right)$$

as $T \to 0$. □
Proof The lemma is trivial if $\alpha \geq 0$. For $\alpha < 0$, we have

\[
\int_0^b u^\alpha \exp \left\{ - \left[ \frac{\log^2 \frac{x}{s_0}}{2T^2H+1} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)|du
\leq \int_0^b u^\alpha \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2T^2H+1} u^2 \right\} |f(u)|du.
\]  

(48)

The following equality holds for every $A > 0$:

\[
\left( u^\alpha \exp \left\{ - \frac{A}{u^2} \right\} \right)' = \left[ 2Au^{\alpha - 3} + \alpha u^{\alpha - 1} \right] \exp \left\{ - \frac{A}{u^2} \right\}.
\]

It follows that for $2A > -\alpha b^2$, the function $u \mapsto u^\alpha \exp \left\{ - \frac{A}{u^2} \right\}$ is increasing on the interval $(0, b]$. Set $A = \frac{\log^2 \frac{x}{s_0}}{2T^2H+1}$. Next, using (48), we obtain

\[
\int_0^b u^\alpha \exp \left\{ - \left[ \frac{\log^2 \frac{x}{s_0}}{2T^2H+1} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)|du
\leq b^\alpha \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2b^2T^2H+1} \right\} \int_0^b |f(u)|du,
\]  

(49)

provided that $\log^2 \frac{x}{s_0} > b^2T^{2H+1}$. It is clear that the previous inequality holds for small enough values of $T$ provided that $x > s_0 + \varepsilon$.

Finally, Lemma 2 follows from (49).

Using (10) with $t = 1$ and formula (8) for centered processes, we obtain

\[
\tilde{p}_1(y) = \tilde{A}y^{n_1(1)-1} \exp \left\{ - \frac{y^2}{2\lambda_1(1)} \right\} \left( 1 + O \left( y^{-1} \right) \right)
\]  

(50)

as $y \to \infty$, where

\[
\tilde{A} = \frac{2^{1-n_1(1)}}{\Gamma \left( \frac{n_1(1)}{2} \right)} \lambda_1^{-\frac{n_1(1)}{2}} \prod_{k>n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_k(1)} \right)^{\frac{1}{2}}.
\]  

(51)

Let $y_0 > 0$ be a constant such that the big-$O$ estimate in (50) is valid. Our next goal is to replace the function $\tilde{p}_1$ in (47) by its approximation from (50), using Lemma 2. The resulting formula is

\[
J_\chi(T) = \tilde{A} \int_0^\infty u^{n_1(1)-2} \exp \left\{ - \left[ \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} \right] \right\} du,
\]

(51)
\[ + \left( \frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)} u^2 \right) \right] \left( 1 + O\left( u^{-1} \right) \right) du \\
+ O_x \left( \exp \left\{ - \frac{\log \frac{x_{\leq 0}}{s_0}}{2y_0^2 T^{2H+1}} \right\} \right) \]  \quad (52)

as \( T \to 0 \). Formula (52) can be obtained as follows. Applying Lemma 2 with \( \alpha = -1 \), \( b = y_0 \), and \( f = \tilde{p}_1 \), to the integral with same integrand as in (47), we can include this integral in the error term in (52). Next, we can replace the function \( \tilde{p}_1 \) in the integral over \([y_0, \infty)\) by the expression on the right-hand side of (50). Finally, we observe that

\[
\int_{y_0}^{y_0} u^1 \left( 1 + 2 + \log \left( \frac{2T^{2H+1} + \frac{1}{2\lambda_1(1)}}{2y_0^2 T^{2H+1}} \right) \right) \left( 1 + O\left( u^{-1} \right) \right) du \\
= O_x \left( \exp \left\{ - \frac{\log \frac{x_{\leq 0}}{s_0}}{2y_0^2 T^{2H+1}} \right\} \right) \]  \quad (53)

as \( T \to 0 \). Indeed, formula (53) can be established by applying Lemma 2 to the expression on the left-hand side of (53) twice, first with \( \alpha = n_1(1) - 2 \), \( b = y_0 \), and \( f(u) = \exp\left\{ -\frac{u^2}{2\lambda_1(1)} \right\} \), and then with \( \alpha = n_1(1) - 3 \), \( b = y_0 \), and \( f(u) = \exp\left\{ -\frac{u^2}{2\lambda_1(1)} \right\} \). This proves formula (52).

To study the asymptotics of the function \( T \mapsto J_x(T) \) defined by (52), we consider the following two integrals:

\[
\tilde{J}_x(T) = \tilde{A} \int_{y_0}^{\infty} u^1 \left( 1 + 2 + \log \left( \frac{2T^{2H+1} + \frac{1}{2\lambda_1(1)}}{2y_0^2 T^{2H+1}} \right) \right) \left( 1 + O\left( u^{-1} \right) \right) du \\
\]  and

\[
\hat{J}_x(T) = \hat{A} \int_{y_0}^{\infty} u^1 \left( 1 + 2 + \log \left( \frac{2T^{2H+1} + \frac{1}{2\lambda_1(1)}}{2y_0^2 T^{2H+1}} \right) \right) \left( 1 + O\left( u^{-1} \right) \right) du. 
\]  

Set

\[
\beta_T = \frac{\log \frac{x_{\leq 0}}{s_0}}{2T^{2H+1}}, \quad \gamma_T = \frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)}. 
\]  

Note that \( \beta_T \) depends on \( x \), while \( \gamma_T \) does not. Then we have

\[
\tilde{J}_x(T) = \tilde{A} \int_{y_0}^{\infty} u^1 \left( 1 + 2 + \log \left( \frac{2T^{2H+1} + \frac{1}{2\lambda_1(1)}}{2y_0^2 T^{2H+1}} \right) \right) \left( 1 + O\left( u^{-1} \right) \right) du \\
\]  and

\[
\hat{J}_x(T) = \hat{A} \int_{y_0}^{\infty} u^1 \left( 1 + 2 + \log \left( \frac{2T^{2H+1} + \frac{1}{2\lambda_1(1)}}{2y_0^2 T^{2H+1}} \right) \right) \left( 1 + O\left( u^{-1} \right) \right) du. 
\]
Next, making a substitution

\[ u = \left( \frac{\beta_T}{\gamma_T} \right) ^ \frac{1}{4} \nu, \]

we transform the previous integrals as follows:

\[ \tilde{J}_x(T) = \tilde{A} \left( \frac{\beta_T}{\gamma_T} \right) \frac{n_1(T)}{4} \int_0^\infty \nu^{n_1(T)-2} \exp \left\{ -\sqrt{\beta_T \gamma_T} \left[ \frac{1}{\nu^2} + \nu^2 \right] \right\} d\nu \]

and

\[ \hat{J}_x(T) = \tilde{A} \left( \frac{\beta_T}{\gamma_T} \right) \frac{n_1(T)-2}{4} \int_0^\infty \nu^{n_1(T)-3} \exp \left\{ -\sqrt{\beta_T \gamma_T} \left[ \frac{1}{\nu^2} + \nu^2 \right] \right\} d\nu. \]

Let us denote

\[ z(T) = \frac{1}{4} \sqrt{\frac{\lambda_1(T) T^{2H+1} + 4}{\lambda_1(T) T^{2H+1}}}. \] (54)

Then we have \( \sqrt{\beta_T \gamma_T} = z(T) \log \frac{x}{s_0} \). Therefore,

\[ \tilde{J}_x(T) = \tilde{A} \left( \frac{\beta_T}{\gamma_T} \right) \frac{n_1(T)}{4} \int_0^\infty \nu^{n_1(T)-2} \exp \left\{ -z(T) \log \frac{x}{s_0} \left[ \frac{1}{\nu^2} + \nu^2 \right] \right\} d\nu \] (55)

and

\[ \hat{J}_x(T) = \tilde{A} \left( \frac{\beta_T}{\gamma_T} \right) \frac{n_1(T)-2}{4} \int_0^\infty \nu^{n_1(T)-3} \exp \left\{ -z(T) \log \frac{x}{s_0} \left[ \frac{1}{\nu^2} + \nu^2 \right] \right\} d\nu. \] (56)

It follows from (54) that \( z(T) \to \infty \) as \( T \to 0 \). Our next goal is to apply Laplace’s method to study the asymptotic behavior of the functions \( T \mapsto \tilde{J}_x(T) \) and \( T \mapsto \hat{J}_x(T) \) as \( T \to 0 \). Note that the unique critical point of the function \( \psi(v) = v^{-2} + v^2 \) is at \( v = 1 \). Moreover, we have \( \psi''(1) = 8 > 0 \).

We will first reduce the integrals in (55) and (56) to the integrals over the interval \([0, 2]\) and give an error estimate. The next assertion will be helpful.

**Lemma 3** Suppose \( a \in \mathbb{R} \) and \( 0 < \varepsilon < s_0 \). Then

\[ \int_2^\infty \nu^a \exp \left\{ -\sqrt{\beta_T \gamma_T} \left[ \frac{1}{\nu^2} + \nu^2 \right] \right\} d\nu \sim O_\varepsilon \left( \exp \left\{ -4\sqrt{\beta_T \gamma_T} \right\} \right) \]

as \( t \to 0 \).
Proof  Fix a small number \( r > 0 \). Then for \( 0 < T < T_0 \), we have

\[
\int_2^\infty v^a \exp \left\{ -\sqrt{\beta T} \gamma T \left[ \frac{1}{v^2} + v^2 \right] \right\} dv \leq \int_2^\infty v^a \exp \left\{ -\sqrt{\beta T} \gamma T v^2 \right\} dv
\]

\[
\leq c_r \int_2^\infty \exp \left\{ - \left( \sqrt{\beta T} \gamma T - r \right) v^2 \right\} dv
\]

\[
= c_r \left( \sqrt{\beta T} \gamma T - r \right)^{-1/2} \int_2^{\infty} e^{-u^2} du \leq c_r \exp \left\{ -4 \left( \sqrt{\beta T} \gamma T - r \right) \right\}.
\]

The proof of Lemma 3 is thus completed. \( \square \)

Now, we are ready to apply Laplace’s method to the integrals in (55) and (56). The dependence of the parameter \( x \) in (55) and (56) is very simple. This allows us to obtain uniform error estimates. By taking into account Lemma 3, we see that for every \( \varepsilon > 0 \) and all \( x > s_0 + \varepsilon \),

\[
\tilde{J}_x(T) = \frac{\tilde{A} \sqrt{\pi}}{2} \left( \frac{\beta T}{\gamma T} \right)^{n_1(1)-1} \left( z(T) \log \frac{x}{s_0} \right)^{-1/2} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\}
\]

\[
\times \left( 1 + O_\varepsilon \left( \frac{1}{z(T) \log \frac{x}{s_0}} \right) \right) + O_\varepsilon \left( \exp \left\{ -4z(T) \log \frac{x}{s_0} \right\} \right)
\]

(57)

and

\[
\hat{J}_x(T) = \frac{\hat{A} \sqrt{\pi}}{2} \left( \frac{\beta T}{\gamma T} \right)^{n_2(1)-2} \left( z(T) \log \frac{x}{s_0} \right)^{-1/2} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\}
\]

\[
\times \left( 1 + O_\varepsilon \left( \frac{1}{z(T) \log \frac{x}{s_0}} \right) \right) + O_\varepsilon \left( \exp \left\{ -4z(T) \log \frac{x}{s_0} \right\} \right)
\]

(58)

as \( T \to 0 \). Recall that the \( O_\varepsilon \) estimates in (57) and (58) are uniform with respect to \( x > s_0 + \varepsilon \). Since

\[
J_x(T) = \tilde{J}_x(T) + O_\varepsilon \left( \hat{J}_x(T) \right) + O_\varepsilon \left( \exp \left\{ -\frac{\log^2 \frac{x}{s_0}}{2\gamma_0 T^{2H+1}} \right\} \right),
\]

as \( T \to 0 \), formulas (57) and (58) imply that

\[
J_x(T) = \frac{\tilde{A} \sqrt{\pi}}{2} \left( \frac{\beta T}{\gamma T} \right)^{n_1(1)-1} \left( z(T) \log \frac{x}{s_0} \right)^{-1/2} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\}
\]

\[
\times \left( 1 + O_\varepsilon \left( \left[ \frac{\beta T}{\gamma T} \right]^{-1/2} \right) \right) \left( 1 + O_\varepsilon \left( \frac{1}{z(T) \log \frac{x}{s_0}} \right) \right)
\]
\[ + O_\varepsilon \left( \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2y_0^2T^{2H+1}} \right\} \right) + O_\varepsilon \left( \exp \left\{ -4z(T) \log \frac{x}{s_0} \right\} \right) \]

as \( T \to 0 \). Since for \( T < 1 \),

\[
\frac{1}{4} \sqrt{\frac{\lambda_1(1) + 4}{\lambda_1(1)}} T^{-H - \frac{1}{2}} > z(T) > \frac{1}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H - \frac{1}{2}},
\]

we have

\[
O_\varepsilon \left( \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2y_0^2T^{2H+1}} \right\} \right) + O_\varepsilon \left( \exp \left\{ -4z(T) \log \frac{x}{s_0} \right\} \right) = O_\varepsilon \left( \exp \left\{ -2\lambda_1(1)^{-\frac{1}{2}} T^{-H - \frac{1}{2}} \log \frac{x}{s_0} \right\} \right)
\]

as \( T \to 0 \), and therefore,

\[
J_x(T) = \frac{\tilde{\Delta} \sqrt{\pi}}{2} \left( \frac{\beta_T}{\gamma_T} \right)^{\frac{H-1}{4}} \left( z(T) \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\}
\times \left( 1 + O_\varepsilon \left( \frac{\beta_T}{\gamma_T} \right)^{-\frac{1}{2}} (z(T) \log \frac{x}{s_0}) \right) \left( 1 + O_\varepsilon \left( \frac{1}{z(T) \log \frac{x}{s_0}} \right) \right)
+ O_\varepsilon \left( \exp \left\{ -2\lambda_1(1)^{-\frac{1}{2}} T^{-H - \frac{1}{2}} \log \frac{x}{s_0} \right\} \right)
\]

as \( T \to 0 \). Moreover, for all \( T < 1 \) and \( x > s_0 + \varepsilon \),

\[
\left( \frac{\beta_T}{\gamma_T} \right)^{-\frac{1}{2}} \leq c_1 \frac{T^{\frac{2H+1}{4}}}{\log \frac{x}{s_0}} \leq c_2 \frac{T^{\frac{2H+1}{2}}}{\log \frac{x}{s_0}} \geq c_3 \frac{1}{z(T) \log \frac{x}{s_0}},
\]

and hence

\[
\left( 1 + O_\varepsilon \left( \frac{\beta_T}{\gamma_T} \right)^{-\frac{1}{2}} \right) \left( 1 + O_\varepsilon \left( \frac{1}{z(T) \log \frac{x}{s_0}} \right) \right) = 1 + O_\varepsilon \left( \frac{T^{\frac{2H+1}{4}}}{\log \frac{x}{s_0}} \right)^{-\frac{1}{2}}
\]
as $T \to 0$. Finally,

$$
J_x(T) = \frac{\tilde{A} \sqrt{\pi}}{2} \left( \frac{\beta_T}{\gamma T} \right)^{\frac{n_1(1)-1}{4}} \left( z(T) \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} \\
\times \left( 1 + O_\varepsilon \left( T^{\frac{2n_1(1)-1}{4}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right) \\
+ O_\varepsilon \left( \exp \left\{ -2\lambda_1(1)^{-\frac{1}{2}} T^{-\frac{4}{2}} \log \frac{x}{s_0} \right\} \right)
$$

as $T \to 0$.

Recall that we assumed $r = 0$. It follows from (13) and (47) that

$$
D_T(x) = \frac{\sqrt{s_0} \tilde{A}}{2\sqrt{2}} T^{-\frac{H}{2}} x^{-\frac{1}{2}} \left( \frac{\beta_T}{\gamma T} \right)^{\frac{n_1(1)-1}{4}} \left( z(T) \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} \left( 1 + O_\varepsilon \left( T^{\frac{2n_1(1)-1}{4}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right) \\
+ O_\varepsilon \left( \exp \left\{ -2\lambda_1(1)^{-\frac{1}{2}} T^{-\frac{4}{2}} \log \frac{x}{s_0} \right\} \right)
$$

as $T \to 0$.

Our next goal is to remove the last $O_\varepsilon$-term from formula (60). Analyzing the expressions in (60), we see that in order to prove the statement formulated above, it suffices to show that there exists a constant $c > 0$ independent of $T < T_0$ and $x > s_0 + \varepsilon$ and such that

$$
\left( \frac{x}{s_0} \right)^{\frac{1}{2}} \leq cT^{-\frac{H}{2}} x^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{2}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}}
\right).
$$

The previous inequality is equivalent to the following:

$$
\left( \frac{x}{s_0} \right)^{-\frac{2}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}}
\right).
$$

Using (54), we see that the inequality in (62) follows from the inequality

$$
\left( \frac{x}{s_0} \right)^{-\frac{2}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \\
\times \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \frac{x}{s_0} \right)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}}
\right).
$$
To prove the inequality in (63), we observe that for every small enough number \( \tau > 0 \) there exists a constant \( c_{\tau,\varepsilon} \) such that

\[
c_{\tau,\varepsilon} \left( \frac{x}{s_0} \right)^{-\tau} \leq \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-1}{4}-1}
\]

for all \( x > s_0 + \varepsilon \). Moreover, there exists \( T_{\tau,\varepsilon} > 0 \) such that

\[
\left( \frac{x_0}{s_0} \right)^{-\tau T^{-H-\frac{1}{2}}} \leq \left( \frac{s_0 + \varepsilon}{s_0} \right)^{-\tau T^{-H-\frac{1}{2}}} \leq T^{-\frac{(2H+1)(n_1(1)-1)}{8}}
\]

for all \( T < T_{\tau,\varepsilon} \). Now, it is clear that (63) follows from the estimate

\[
\left( 2\lambda_1(1)^{-\frac{1}{2}} - \tau \right) T^{-H-\frac{1}{2}} \geq \frac{3}{2} + \frac{1}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \sqrt{\lambda_1(1)T^{2H+1} + 4 + \tau},
\]

for all \( T < T_{\tau} \). It is not hard to see that there exist numbers \( \tau \) and \( T_{\tau} \), for which the inequality in (64) holds. This establishes (61), and it follows that

\[
D_T(x) = \frac{\sqrt{s_0 \tilde{A}}}{2\sqrt{2}} T^{-H-\frac{1}{2}} x^{-\frac{1}{2}} \left( \frac{\beta_T}{\gamma T} \right)^{\frac{n_1(1)-1}{4}} \left( z(T) \log \frac{x}{s_0} \right)^{-\frac{1}{2}}
\]

\[
\times \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} \left( 1 + O_\varepsilon \left( T^{\frac{2H+1}{4}} \left( \log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right)
\]

as \( T \to 0 \), where \( \tilde{A} \) is given by (51). Formula (65) will help us to characterize the asymptotic behavior of the function \( T \mapsto D_T(x) \).

Let us assume that \( x > s_0 + \varepsilon \). Then we have

\[
\left( \frac{\beta_T}{\gamma T} \right)^{\frac{n_1(1)-1}{4}} = \lambda_1(1)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-1}{2}} T^{-\frac{(2H+1)(n_1(1)-1)}{4}} (1 + \bar{h})^{-\frac{n_1(1)-1}{4}}
\]

where \( \bar{h} = \frac{\lambda_1(1)T^{2H+1}}{4} \). Therefore,

\[
\left( \frac{\beta_T}{\gamma T} \right)^{\frac{n_1(1)-1}{4}} = \lambda_1(1)^{-\frac{1}{2}} \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-1}{2}} T^{-\frac{(2H+1)(n_1(1)-1)}{4}}
\]

\[
\times \left( 1 + O \left( T^{2H+1} \right) \right)
\]
as \( T \to 0 \). Moreover,

\[
z(T)^{-\frac{1}{2}} = 2 \left[ \frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}} \right]^{-\frac{1}{4}} = \sqrt{2\lambda_1(1)} \frac{1}{4} T^{\frac{2H+1}{4}} \left( 1 + O \left( T^{2H+1} \right) \right)
\]  

(67)

and

\[
\exp\left\{ -2z(T) \log \frac{x}{s_0} \right\} = \left( \frac{x}{s_0} \right)^{-\frac{\sqrt{4+\lambda_1(1)T^{2H+1}}}{2\sqrt{T^{1H+\frac{1}{2}}}}}
\]

(68)

as \( T \to 0 \). Next, combining (51), (65), (66), (67), and (68), and simplifying the resulting expressions, we obtain formula (14).

This completes the proof of Theorem 3.

**Proof of Theorem 4**  Let us consider the call pricing function \( T \mapsto C(T) \) with \( K > s_0 \). It is known that

\[
C(T) = \int_K^\infty (x - K)D_T(x)dx.
\]

(69)

Therefore, we can use the uniform estimate in formula (14) to characterize the small-time behavior of the call pricing function. Let us consider the following integrals:

\[
I_1(T) = \int_K^\infty (x - K)x^{-\frac{3}{2}} \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp\left\{ -2z(T) \log \frac{x}{s_0} \right\} dx
\]

\[
= -s_0^{-\frac{1}{2}} \int_K^\infty \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp\left\{ -\left( \frac{1}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx
\]

\[
- s_0^{-\frac{3}{2}} K \int_K^\infty \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp\left\{ -\left( \frac{3}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx
\]

(70)

and

\[
I_2(T) = \int_K^\infty (x - K)x^{-\frac{3}{2}} \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp\left\{ -2z(T) \log \frac{x}{s_0} \right\} dx
\]

\[
= -s_0^{-\frac{1}{2}} \int_K^\infty \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp\left\{ -\left( \frac{1}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx
\]

\[
- s_0^{-\frac{3}{2}} K \int_K^\infty \left( \log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp\left\{ -\left( \frac{3}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx,
\]

(71)

where we use the notation in (54) for the sake of shortness.
We will next make a substitution $u = (2z(T) - \frac{1}{2}) \log \frac{x}{s_0}$ in the integral on the second line in (70). The resulting expression is as follows:

$$s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \frac{1}{x} \int_{2z(T) - \frac{1}{2}}^{\infty} u^{-\frac{n_1(1)-2}{2}} e^{-u} du,$$

which is equal to

$$s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \Gamma \left(\frac{n_1(1)}{2}, \left(2z(T) - \frac{1}{2}\right) \log \frac{K}{s_0}\right),$$

where the symbol $\Gamma$ stands for the upper incomplete gamma function defined by

$$\Gamma(s, x) = \int_x^{\infty} v^{s-1} e^{-v} dv.$$

Making similar transformations in the other integrals in (70) and (71), we finally obtain

$$I_1(T) = s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \Gamma \left(\frac{n_1(1)}{2}, \left(2z(T) - \frac{1}{2}\right) \log \frac{K}{s_0}\right) - s_0^{-\frac{1}{2}} K \left(2z(T) + \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \Gamma \left(\frac{n_1(1)}{2}, \left(2z(T) + \frac{1}{2}\right) \log \frac{K}{s_0}\right),$$

and

$$I_2(T) = s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)-1}{2}} \Gamma \left(\frac{n_1(1) - 1}{2}, \left(2z(T) - \frac{1}{2}\right) \log \frac{K}{s_0}\right) - s_0^{-\frac{1}{2}} K \left(2z(T) + \frac{1}{2}\right)^{-\frac{n_1(1)-1}{2}} \Gamma \left(\frac{n_1(1) - 1}{2}, \left(2z(T) + \frac{1}{2}\right) \log \frac{K}{s_0}\right).$$

It is known that

$$\Gamma(s, x) = x^{s-1} e^{-x} \left(1 + (s - 1)x^{-1} + O \left(x^{-2}\right)\right) \quad (72)$$

as $x \to \infty$. Formula (72) can be easily derived from the recurrence relation

$$\Gamma(s, x) = (s - 1)\Gamma(s - 1, x) + x^{s-1} e^{-x}$$

for the upper incomplete gamma function. It follows that

$$I_1(T) = s_0^{2z(T)} K^{-2z(T) + \frac{1}{2}} \left(\log \frac{K}{s_0}\right)^{\frac{n_1(1)-2}{2}}$$
\[
\times \left[ \frac{1}{2z(T) - \frac{1}{2}} \left( 1 + \frac{n_1(1) - 2}{2(2z(T) - \frac{1}{2}) \log \frac{K}{s_0}} + O(T^{2H+1}) \right) \right] \\
- \frac{1}{2z(T) + \frac{1}{2}} \left( 1 + \frac{n_1(1) - 2}{2(2z(T) + \frac{1}{2}) \log \frac{K}{s_0}} + O(T^{2H+1}) \right) \\
= s_0^{2z(T)} K^{-2z(T) + \frac{1}{2}} \left( \log \frac{K}{s_0} \right)^{n_1(1) - \frac{2}{2}} \left( \frac{1}{4z(T)^2 - \frac{1}{4}} + O \left(T^{3H+\frac{3}{2}}\right) \right)
\]
as \(T \to 0\). Therefore,

\[
I_1(T) = s_0^{2z(T)} K^{-2z(T) + \frac{1}{2}} \left( \log \frac{K}{s_0} \right)^{n_1(1) - \frac{2}{2}} \left( 4z(T)^2 - \frac{1}{4} \right)^{-1} \\
\times \left( 1 + O \left( T^{H+\frac{1}{2}} \right) \right)
\]
as \(T \to 0\). Similarly,

\[
I_2(T) = s_0^{2z(T)} K^{-2z(T) + \frac{1}{2}} \left( \log \frac{K}{s_0} \right)^{n_1(1) - \frac{3}{2}} \left( 4z(T)^2 - \frac{1}{4} \right)^{-1} \\
\times \left( 1 + O \left( T^{H+\frac{1}{2}} \right) \right)
\]
as \(T \to 0\). It is not hard to see that

\[
\left( 4z(T)^2 - \frac{1}{4} \right)^{-1} = \lambda_1(1) T^{2H+1}.
\]

It follows from (73) and (74) that

\[
I_1(T) = \lambda_1(1) K^{\frac{1}{2}} \left( \log \frac{K}{s_0} \right)^{n_1(1) - \frac{2}{2}} \left( \frac{s_0}{K} \right)^{2z(T)} T^{2H+1} \left( 1 + O \left( T^{H+\frac{1}{2}} \right) \right)
\]
as \(T \to 0\). Similarly,

\[
I_2(T) = \lambda_1(1) K^{\frac{1}{2}} \left( \log \frac{K}{s_0} \right)^{n_1(1) - \frac{3}{2}} \left( \frac{s_0}{K} \right)^{2z(T)} T^{2H+1} \left( 1 + O \left( T^{H+\frac{1}{2}} \right) \right)
\]
as \(T \to 0\). Using (14), (69), (70) and (71), we see that

\[
C(T) = \frac{\sqrt{s_0}}{2^{\frac{n_1(1)}{2}} - \Gamma \left( \frac{n_1(1)}{2} \right)} \lambda_1(1)^{-\frac{n_1(1)}{4}} \prod_{k > n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_k(1) - \lambda_k(1)} \right)^{\frac{1}{2}} \\
\times T^{-\frac{(2H+1)n_1(1)}{4}} \left( 1 + O \left( T^{2H+1} \right) \right) \left[ I_1(T) + O \left( T^{\frac{2H+1}{4}} I_2(T) \right) \right]
\]
as \( T \to 0 \). Next, (75) and (76), imply

\[
C(T) = \left( \frac{s_0 K}{2} \right)^{\frac{1}{2} \lambda_1(1) - \frac{n_1(1) - 4}{4}} \prod_{k > n_1(1)} \left( \frac{\lambda_1(1)}{\lambda_1(1) - \lambda_k(1)} \right)^{\frac{1}{2}} \\
\times \left( \log \frac{K}{s_0} \right)^{\frac{n_1(1) - 2}{2}} \frac{1}{T^{\frac{(2H+1)(4 - n_1(1))}{4}}} \left( \frac{s_0}{K} \right)^{2z(T)} \left( 1 + O \left( T^{-\frac{2H+1}{4}} \right) \right)
\]

(77)
as \( T \to 0 \). We also have

\[
\sqrt{\frac{\lambda_1(1) T^{2H+1} + 4}{\lambda_1(1) T^{2H+1}}} - \sqrt{\frac{4}{\lambda_1(1) T^{2H+1}}} = O \left( T^{H+\frac{1}{2}} \right)
\]

(78)
as \( T \to 0 \). Therefore,

\[
\left( \frac{s_0}{K} \right)^{2z(T)} = \exp \left\{ -2z(T) \log \frac{K}{s_0} \right\}
\]

\[
= \exp \left\{ -\frac{1}{2} \sqrt{\frac{\lambda_1(1) T^{2H+1} + 4}{\lambda_1(1) T^{2H+1}}} \log \frac{K}{s_0} \right\}
\]

\[
= \exp \left\{ -\frac{1}{2} \sqrt{\frac{4}{\lambda_1(1) T^{2H+1}}} \log \frac{K}{s_0} \right\}
\]

\[
\times \exp \left\{ -\frac{1}{2} \left[ \sqrt{\frac{\lambda_1(1) T^{2H+1} + 4}{\lambda_1(1) T^{2H+1}}} - \sqrt{\frac{4}{\lambda_1(1) T^{2H+1}}} \right] \log \frac{K}{s_0} \right\}
\]
as \( T \to 0 \). Using (78), we obtain

\[
\left( \frac{s_0}{K} \right)^{2z(T)} = \left( \frac{s_0}{K} \right)^{\lambda_1(1) - \frac{1}{2} T^{-H - \frac{1}{2}}} \left( 1 + O \left( T^{-H - \frac{1}{2}} \right) \right)
\]

(79)
as \( T \to 0 \).

Now, it is clear that Theorem 4 follows from (77) and (79). \( \square \)
From top to bottom and left to right: IV with $\sigma = 2$, $t \in [1\ day, 2\ weeks]$, $H = 0.25$, 0.35, 0.40, 0.45, 0.49, 0.51, 0.55, 0.60, 0.75, 0.85.

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