REALISATION OF LUSZTIG CONES

PHILIPPE CALDERO, ROBERT MARSH, AND SOPHIE MORIER-GENOUD

Abstract. Let \( U_q(\mathfrak{g}) \) be the quantised enveloping algebra associated to a simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \). The negative part \( U^- \) of \( U_q(\mathfrak{g}) \) possesses a canonical basis \( B \) with favourable properties. Lusztig has associated a cone \([19]\) to a reduced expression \( i \) for the longest element \( w_0 \) in the Weyl group of \( \mathfrak{g} \), with good properties with respect to monomial elements of \( B \). The first author \([8]\) has associated a subalgebra \( A_i \) of \( U^- \), compatible with the dual basis \( B^* \), to each reduced expression \( i \). We show that, after a certain twisting, the string parametrisation of the adapted basis of this subalgebra coincides with the corresponding Lusztig cone. As an application, we give explicit expressions for the generators of the Lusztig cones.

1. Introduction

Let \( U = U_q(\mathfrak{g}) \) be the quantum group associated to a semisimple Lie algebra \( \mathfrak{g} \). The negative part \( U^- \) of \( U \) has a canonical basis \( B \) with favourable properties (see Kashiwara \([14]\) and Lusztig \([18, \S 14.4.6]\)). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible \( U^- \)-modules. The dual canonical basis \( B^* \) of the positive part \( U^+ \) has good multiplicative properties. Two elements of \( B^* \) are said to be multiplicative if their product also lies in \( B^* \) up to a power of \( q \).

The first author has shown that for each reduced expression \( i = (i_1, i_2, \ldots, i_N) \) for the longest element \( w_0 \) (see \([19]\) in the Weyl group of \( \mathfrak{g} \), there is a corresponding subalgebra \( A_i \) of \( U^+ \), known as a standard adapted subalgebra, with basis given by \( A_i \cap B^* \), consisting entirely of elements which are pairwise multiplicative. The subalgebras \( A_i \) are \( q \)-polynomial algebras, i.e. algebras given by generators and \( q \)-commuting relations, with GK-dimension \( N = l(w_0) \). Note that adapted algebras were introduced for the Berenstein-Zelevinsky conjecture and are connected with the larger theory of cluster algebras \([11]\).

By a Lusztig cone of \( U \), we mean the cone \( C_i \subseteq \mathbb{N}^N \) associated by Lusztig (see \([19, \S 16]\)) to each reduced expression \( i \) for \( w_0 \). In \([19]\) these cones arise naturally from the linear term of a nonhomogeneous quadratic form associated to \( i \) which is used by Lusztig to give a positivity condition for a monomial

\[
F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_N}^{(a_N)}
\]

to lie in the canonical basis. Here the \( F_i \) are the standard generators of \( U^- \). Monomials of this form with \((a_1, a_2, \ldots, a_N)\) lying in the Lusztig cone corresponding to \( i \) lie in the canonical basis in types \( A_1, A_2 \) and \( A_3 \) \([19]\), in type \( A_4 \) \([20]\) and in type \( B_2 \) \([29]\); see also \([25]\). Counter-examples of M. Reineke \([25]\) and N. H. Xi \([30]\) show that this fails in general in type \( A_5 \). Recently, R. Bedard \([2]\) has analysed the quadratic forms associated to these monomials and, as an application, was able to compute some interesting examples in types \( D_4, A_5 \) and affine \( A_1 \).

Note that Lusztig cones are used to describe regular functions on a reduced real double Bruhat cell of the corresponding algebraic group \([31]\), they have links with primitive elements in the dual canonical basis (this can be seen using \([4]\) ) and therefore with the representation theory of affine Hecke algebras \([17]\), and they are known to correspond to regions of linearity of the Lusztig reparametrisation functions (see \([10]\)).

Given a reduced expression \( i \) for \( w_0 \), elements of the dual canonical basis can be parametrised via the string parametrisation in direction \( i \) (see \([4, \S 2]\), \([17]\) and \([29, \S 2]\)), which we denote by \( c_i : B^* \rightarrow C_i \), where \( C_i \subseteq \mathbb{N}^N \) is known as the string cone corresponding to \( i \).

\[\text{Date: December 27, 2020.}\]
Our main result is that the set of string parameters (in direction i) of a certain twisting of the standard adapted subalgebra of the dual canonical basis corresponding to i coincides with the Lusztig cone corresponding to i. The twisting is done with the help of the Schützenberger involution. This gives a realisation of all Lusztig cones in terms of the dual canonical basis. It also implies that all Lusztig cones are simplicial (generalising results of Bedard [1] and the second author [2]), and enables us to give an explicit description of their spanning vectors; see Theorem 8.10.

The paper is organised as follows. Sections 1 and 2 give preliminary results on quantum groups and the canonical basis, including its parametrisations associated to a reduced word, and adapted algebras. In Section 4, we introduce the Schützenberger involution \( \phi \) and its action on the dual canonical basis. We also recall some facts on geometric lifting of the canonical basis in order to give a formula which describes \( \phi \) in terms of the parametrisation of the dual canonical basis. A remarkable property is that, with a good choice of parametrisations, the action of the Schützenberger involution on the dual canonical basis is given by an affine map.

In Sections 6 and 7, we apply the results from previous sections to describe explicitly the twisted standard adapted subalgebra associated to a reduced word \( i \), in terms of \( i \)-string parametrisation. By the multiplicative property of the adapted subalgebra and the "affine map" property, this can be provided with a good choice of parametrisations, the action of the Schützenberger involution on the dual canonical basis resulting from this decomposition. The set of positive roots is denoted by \( \Delta^+ \).

Let \( P \) be the weight lattice generated by the fundamental weights \( \varpi_i \), \( 1 \leq i \leq n \). Set \( P^+ := \sum_i \mathbb{Z}_{\geq 0} \varpi_i \), endowed with the ordering \( \sum_i \lambda_i \varpi_i \leq \sum_i \mu_i \varpi_i \iff \lambda_i \leq \mu_i \). The Weyl group \( W \) is generated by the reflections \( s_i \) corresponding to the simple roots. We denote by \( <,> \) the \( W \)-invariant form on \( P \); we have \( a_{ij} = < \alpha_j, \alpha_i^\vee > \) for all \( i, j \).

For \( n \) a nonnegative integer and \( \alpha \) a positive root, we set \( q_\alpha = q^{<\alpha, \alpha>/2} \), \( [n]\alpha = q^n - q^{-n} \), \( [n]_\alpha! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha \).

2.1. Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be the Cartan matrix of a finite dimensional semi-simple Lie algebra \( g \) over \( \mathbb{C} \). Let \( g = n^- \oplus h \oplus n \) be a triangular decomposition, where \( h \) is a Cartan subalgebra and where \( n^-, n \) are opposite maximal nilpotent subalgebras of \( g \). Let \( \{\alpha_i\}_i \) be the set of simple roots of the root system \( \Delta \) resulting from this decomposition. The set of positive roots is denoted by \( \Delta^+ \).

Let \( P \) be the weight lattice generated by the fundamental weights \( \varpi_i \), \( 1 \leq i \leq n \). Set \( P^+ := \sum_i \mathbb{Z}_{\geq 0} \varpi_i \), endowed with the ordering \( \sum_i \lambda_i \varpi_i \leq \sum_i \mu_i \varpi_i \iff \lambda_i \leq \mu_i \). The Weyl group \( W \) is generated by the reflections \( s_i \) corresponding to the simple roots. We denote by \( <,> \) the \( W \)-invariant form on \( P \); we have \( a_{ij} = < \alpha_j, \alpha_i^\vee > \) for all \( i, j \).

For \( n \) a nonnegative integer and \( \alpha \) a positive root, we set \( q_\alpha = q^{<\alpha, \alpha>/2} \), \( [n]\alpha = q^n - q^{-n} \), \( [n]_\alpha! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha \).

2.2. Let \( W \) be the Weyl group of \( g \), with Coxeter generators \( s_1, s_2, \ldots, s_n \) and corresponding length function. An expression \( s_{i_1} s_{i_2} \cdots s_{i_m} \) for an element of \( w \) is called reduced if it is of minimal length; we identify such an expression with the tuple \( i = (i_1, i_2, \ldots, i_m) \). Set \( N := \dim n \). It is known that \( N \) is the length of the longest element \( w_0 \) of the Weyl group. Let \( R \) be the set of reduced expressions for \( w_0 \).

Fix \( i \) in \( R \). Let \( \mathcal{L}_i \) be the set of points \( (c_1, \ldots, c_N) \in \mathbb{Z}_{\geq 0}^N \) with the following property: for any two indices \( p < p' \) in \( \{1, \ldots, N\} \) such that \( i_p = i_{p'} \) and \( i_q \neq i \) whenever \( p < q < p' \), we have

\[
c_p + c_{p'} + \sum_{p < q < p'} a_{i_p, i_q} c_q \leq 0.
\]

The cone \( \mathcal{L}_i \) is the so-called Lusztig cone associated to the reduced expression \( i \). This is defined in [19 §16] for the simply-laced case. We use here a natural generalisation to the general case which also appears implicitly (for type \( B_2 \)) in [20].

2.3. Let \( d \) be an integer such that \( < P, P > \subset (2/d)\mathbb{Z} \). Let \( q \) be an indeterminate and set \( K = \mathbb{C}(q^{1/d}) \).

We define the simply connected quantised enveloping \( K \)-algebra \( U_q(g) \) as in [12]. Set \( d_i = < \alpha_i, \alpha_i > /2 \) and \( q_i = q^{d_i} \) for all \( i \). Let \( U_q(n) \), resp. \( U_q(n^-) \), be the subalgebra generated by the canonical generators \( E_i := E_{\alpha_i}, \) resp. \( F_i := F_{\alpha_i}, \) of positive, resp. negative, weights, subject to the quantum Serre relations.
For all $\lambda$ in $P$, let $K_\lambda$ be the corresponding element in the algebra $U_q^0 = \mathbb{K}[P]$ of the torus of $U_q(\mathfrak{g})$ and $K_i := K_{\alpha_i}$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. Set

\[ U_q(b) = U_q(n) \otimes U_q^0, \quad U_q(b^-) = U_q(n^-) \otimes U_q^0. \]

The algebra $U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication $\Delta$, antipode $S$ and augmentation $\varepsilon$ given by

\[ \Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_\lambda = K_\lambda \otimes K_\lambda, \]

\[ S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_\lambda) = K_{-\lambda}, \]

\[ \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1. \]

Let $(\cdot, \cdot)$ be the Hopf bilinear form, $[26]$, on $U_q(b) \times U_q(b^-)$, uniquely defined by

\[ (E_i, F_j) = \delta_{ij}(1-q_i^2)^{-1}, \quad 1 \leq i, j \leq n, \]

\[ (X K_\lambda, Y K_\mu) = q^{(\lambda, \mu)} (X, Y), \quad X \in U_q(n), \quad Y \in U_q(n^-). \]

2.4. In this section, we define automorphisms of the quantised enveloping algebra and the Poincaré-Birkhoff-Witt basis. The automorphisms $T_i$, $1 \leq i \leq n$, as in $[15]$, are given by:

\[ T_i(E_i) = -K_i^{-1} F_i, \quad T_i(E_j) = \sum_{k+l=-a_{ij}} (-1)^k \frac{q_{a_{ij}}}{[k][l]} E_i^k E_j^l E_i, \quad 1 \leq i, j \leq n, \quad i \neq j, \]

\[ T_i(F_i) = -E_i K_i, \quad T_i(F_j) = \sum_{k+l=-a_{ij}} \frac{q_{a_{ij}}}{[l][k]} F_i^l F_j^k F_i, \quad 1 \leq i, j \leq n, \quad i \neq j, \]

\[ T_i(K_{\alpha_j}) = K_{\alpha_j}, \quad 1 \leq i, j \leq n. \]

It is known that the $T_i$’s define a braid action on $U_q(\mathfrak{g})$. Fix $i = (i_1, \ldots, i_N)$ in $\mathcal{R}$. For each $k$, $1 \leq k \leq N$, set $\beta_k := s_{i_k} \cdots s_{i_{k-1}}(\alpha_k)$. It is well known that \{ $\beta_k$, $1 \leq k \leq N$ \} is the set of positive roots and that

\[ \beta_1 < \beta_2 < \ldots < \beta_N \]

defines a so-called convex ordering on $\Delta^+$. This ordering identifies the semigroup $\mathbb{Z}_{\geq 0}^N$ with the semigroup $\mathbb{Z}_{\geq 0}$. In the sequel, we denote by \{ $e_k$, $1 \leq k \leq N$ \} the natural basis of this semigroup.

For all $k$, define $E_{\beta_k} = E_{\beta_k} = T_{i_k} \cdots T_{i_{k-1}}(E_{\alpha_k})$. For all $t = (t_i) \in \mathbb{Z}_{\geq 0}^N$, set $E^t(E) := E_{\beta_{t_N}} \cdots E_{\beta_{t_1}}$. It is known that \{ $E(t)$, $t \in \mathbb{Z}_{\geq 0}^N$ \} is a basis of $U_q(n)$ called the Poincaré-Birkhoff-Witt basis, in short PBW-basis, associated to the reduced expression $i$. In the same way, we can define the PBW-basis \{ $F(t)$, $t \in \mathbb{Z}_{\geq 0}^N$ \} of $U_q(n^-)$. We define now the automorphisms $\omega$, $\omega$, and the antiautomorphism $\sigma$ of $U_q(\mathfrak{g})$ by

\[ \tilde{E}_i = E_i, \quad \tilde{K}_i = K_i^{-1}, \quad \tilde{F}_i = F_i, \quad \tilde{q} = q^{-1}, \quad 1 \leq i \leq n, \]

\[ \omega(E_i) = F_i, \quad \omega(K_i) = K_i^{-1}, \quad \omega(F_i) = E_i, \quad \omega(q) = q, \quad 1 \leq i \leq n, \]

\[ \sigma(E_i) = E_i, \quad \sigma(K_i) = K_i^{-1}, \quad \sigma(F_i) = F_i, \quad \sigma(q) = q, \quad 1 \leq i \leq n. \]

Note that $\omega$ is a coalgebra antiautomorphism.
2.5. For any (left) $U_q(\mathfrak{g})$-module $M$, and any weight $\mu$ in $P$, let $M_\mu = \{ m \in M : K_\lambda m = q^{<\lambda,\mu>} m \}$ be the subspace of $M$ of weight $\mu$. For all $\lambda$ in $P^+$, let $V(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight $\lambda$ and highest weight vector $v_\lambda$. The module $V(\lambda)$ satisfies the Weyl character formula. For $\lambda$ in $P^+$, $V_q(\lambda)^*$ is naturally endowed with a structure of right $U_q(\mathfrak{g})$-module. Let $\eta_\lambda$ be its weight element such that $\eta_\lambda(v_\lambda) = 1$. For $\lambda, \mu$ in $P^+$, the module $V_q(\lambda) \otimes V_q(\mu)$, endowed with the diagonal action of $U_q(\mathfrak{g})$, has a unique component of type $V_q(\lambda + \mu)^*$. Hence the restriction map provides a map $r_{\lambda,\mu} : V_q(\lambda)^* \otimes V_q(\mu)^* \rightarrow V_q(\lambda + \mu)^*$. We can suppose that the $v_\lambda$ are normalised such that $r_{\lambda,\mu}(\eta_\lambda \otimes \eta_\mu) = \eta_{\lambda+\mu}$, $\lambda, \mu \in P^+$.

Set $R^+ := \bigoplus_{\lambda \in P^+} V_q(\lambda)^* \otimes v_\lambda$. The space $R^+$ can be equipped with a structure of algebra by the multiplication rule : $(\xi \otimes v_\lambda)(\xi' \otimes v_\lambda') = r_{\lambda,\lambda'}(\xi \otimes \xi') \otimes v_{\lambda+\lambda'}, \xi \in V_q(\lambda)^*, \xi' \in V_q(\lambda')^*$.

The map $R^+ \rightarrow U_q(b^-)^*$, $\xi \otimes v_\lambda \rightarrow \xi(?v_\lambda)$, $\xi \in V_q(\lambda)^*$, provides an embedding of algebras.

Define the map $\beta : U_q(b) \rightarrow U_q(b^-)^*$, $\beta(u)(v) = (u, v)$.

Then we have:

**Theorem 2.1.** The map $\beta$ is an injective antihomomorphism of algebras which maps $K_\lambda$ to $\eta_\lambda \otimes v_\lambda \in R^+ \subseteq U_q(b^-)^*$. There exists a unique subspace $E_\lambda$ of $U_q(n)$ such that $\beta(E_\lambda K_\lambda) = V_q(\lambda)^* \otimes v_\lambda$.

3. Recollection on canonical bases and adapted algebras.

The recollection on canonical bases is from §. Results on adapted algebras can be found in § and §.

3.1. The following theorem defines the canonical basis and its Lusztig parametrisation.

**Theorem 3.1.** Fix $i$ in $R$. For all $t$ in $\mathbb{Z}_{\geq 0}$, there exists a unique element $t = b_i(t)$ in $U_q(n^-)$ such that $b = b$ and $b - F_i(t) \in q^{-1} \sum_{l < 1} \mathbb{Z}[q^{-1}] F_i(t')$. The map $t \mapsto b_i(t)$ defines a bijection from $\mathbb{Z}_{\geq 0}$ to a basis $\mathcal{B}$ of $U_q(n^-)$ which does not depend on the choice of $i$.

The basis $\mathcal{B}$ is called the canonical (or global) basis and the map $b_i : t \mapsto b_i(t)$ is the Lusztig parametrisation of $\mathcal{B}$ associated to the reduced expression $i$. We can define the action of the Kashiwara operators on the canonical basis as follows : for $1 \leq i \leq n$, there exists a unique injective map $\tilde{e}_i : \mathcal{B} \rightarrow \mathcal{B}$, such that for all $i$ with $i_1 = i$, we have

$$\tilde{e}_i(b_i(t_1, t_2, \ldots, t_N)) = b_i(t_1 + 1, t_2, \ldots, t_N).$$

For $1 \leq i \leq n$, let $\tilde{c}_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{0\}$ be such that $\tilde{c}_i(b) = b'$ if there exists $b'$ such that $\tilde{e}_i(b') = b$ and $\tilde{e}_i(b) = 0$ if not. Let $\varepsilon_i(b) = \max \{k \geq 0 : \tilde{c}_i(b) \neq 0\}$ and let $\mathcal{E} : \mathcal{B} \rightarrow P^+$, $b \mapsto \sum_{i} \varepsilon_i(b) \sigma_i$. Now, the basis $\mathcal{B}$ is stable under $\sigma$. For all $\lambda$ in $P^+$, set $\mathcal{B}(\lambda) = \{ b \in \mathcal{B} : \varepsilon_i(b) \leq \lambda \}$. A nice theorem of compatibility of the canonical basis with the Weyl modules $V_q(\lambda)$ can be stated as follows :

**Theorem 3.2.** Fix $\lambda$ in $P^+$. Then, for $b$ in $\mathcal{B}$, we have $b \cdot v_\lambda \neq 0$ if and only if $b \in \mathcal{B}(\lambda)$. Moreover, $\mathcal{B}(\lambda) \cdot v_\lambda$ is a basis of $V_q(\lambda)$.

In the sequel, we will identify $\mathcal{B}(\lambda)$ with its image in $V_q(\lambda)$.

3.2. We now introduce the string parametrisation of the canonical basis and the various transition maps.

Fix a reduced expression $i$ in $R$ and $b$ in $\mathcal{B}$. The string of $b$ in the direction $i$ is the sequence of integers $c_i(b) := (t_1, \ldots, t_N)$ defined recursively by :

$$t_1 = \varepsilon_i(b), \quad t_2 = \varepsilon_i(c_{i2}(b)), \ldots, \quad t_N = \varepsilon_i(c_{iN-1}(b)) \cdots c_{i1}(b)).$$

The map $c_i$ defines a bijection from $\mathcal{B}$ onto the set of integral points of a rational convex polyhedral cone $\mathcal{C}_i$ in $\mathbb{R}^N$.

We can now define :

$$R_i' = (b_i)^{-1} \circ b_i : \mathbb{Z}_0^N \rightarrow \mathbb{Z}_0^N.$$
3.3. Let $B^* \subset U_q(n)$ be the basis dual to $B$ with respect to the form $(\ , \ )$ on $U_q(n) \times U_q(n^-)$. We call it the dual canonical basis. For $b$ in $B$, we denote by $b^*$ the corresponding element in $B^*$. Since we work with the dual canonical basis, in the sequel we shall regard $b_i$ as a map from $\mathbb{Z}_{\geq 0}^N$ to $B^*$ (rather than $B$), using the identification $b_i \leftrightarrow b^*_i$. Similarly we shall regard $c_i$ as a map from $B^*$ to $\mathbb{Z}_{\geq 0}^N$.

The set $B^*$ is stable under $\sigma$ up to a power of $q^n$. To be more precise, for $b$ in $B^*$, there exists an integer $m$ such that $\sigma(b^*) = q^n \sigma(b)^*$. For $\lambda \in P^+$ and $b$ in $B(\lambda)$, let $\pi_\lambda(b^*)$ be the element of $V_q(\lambda)^*$ such that $\pi_\lambda(b^*)(b', v_\lambda) = \delta_{b, b'}$, where $\delta$ is the Kronecker symbol. It is easily seen from the definitions that:

**Lemma 3.3.** For all $\lambda$ in $P^+$ and $b$ in $B(\lambda)$, we have $\beta(b^* K_\lambda) = \pi_\lambda(b^*) \otimes v_\lambda$.

In the notation, we will sometimes omit $\pi_\lambda$.

The lemma implies that the spaces $E_\lambda$ defined by Theorem 2.1 are compatible with the dual canonical basis. By the Weyl character formula, for all $w$ in $W$ and $\lambda$ in $P^+$, there is a unique element of $B^* \cap E_\lambda$ with weight $\lambda - w\lambda$. We denote this element by $b^*_{w, \lambda}$ and the corresponding element in the canonical basis by $b_{w, \lambda}$.

In the sequel, $\{\pi_\lambda(b^*) \otimes v_\lambda, b \in B(\lambda), \lambda \in P^+\}$ will be called dual canonical basis of $R^+$. By a misuse of language, its Lusztig, resp. string, parametrisation will be the Lusztig, resp. string, parametrisation of the corresponding element $b$ in $B(\lambda)$.

3.4. Two elements of the dual canonical basis are called multiplicative if their product is an element of the dual canonical basis, up to a power of $q$. By [24], multiplicative elements $q$-commute. We start with the definition of adapted algebras.

**Definition 3.4.** A subalgebra $A$ of $U_q(n)$, resp. $R^+$, is called adapted if:

1) the intersection of $A$ and the dual canonical basis is a basis of $A$, called adapted basis,
2) the elements of this basis are pairwise multiplicative.

We define standard adapted subalgebras of $R^+$, associated to a fixed reduced expression $i$ for $w_0$; see [8].

Set $y_i := \eta_i \otimes v_\lambda$. Let $A_i$ be the subalgebra of $R^+$ generated by $y_i := y_{\omega_i}, 1 \leq i \leq n$, and $c^k_i := b^*_{s_{i_1} \ldots s_{i_k} \omega_i} \otimes v_{\omega_i}, 1 \leq k \leq N$. (This notation is standard, but shouldn’t be confused with $c_i$ which is used for the string parametrisation). Using the antihomomorphism $\beta$, we obtain from [3] the following proposition:

**Proposition 3.5.** Fix $i \in \mathbb{R}$. The algebra $A_i$ is an adapted subalgebra of $R^+$ and the adapted basis is given by monomials in the $y_i$ and the $c^k_i$, up to a power of $q$.

As $\sigma$ is an antiautomorphism which preserves the dual canonical basis up to a power of $q$, we can now define another family of adapted algebras by twisting the standard ones.

Note that $\sigma(b_{s_{i_1} \ldots s_{i_k} \omega_i}) \in B(\mu_k)$, where $\mu_k := E(b_{s_{i_1} \ldots s_{i_k} \omega_i})$. Let $A^\sigma_i$ be the subalgebra generated by the $y_i, 1 \leq i \leq n$, and the $c^k_i := \sigma(b_{s_{i_1} \ldots s_{i_k} \omega_i}) \otimes v_{\mu_k}, 1 \leq k \leq N$. These elements $q$-commute by [24].

Moreover, by [3] and [8, 2.2], we have

**Proposition 3.6.** Fix $i \in \mathbb{R}$. Then,

(i) the algebra $A^\sigma_i$ is an adapted subalgebra of $R^+$,
(ii) the adapted basis is given by monomials in the $y_i$ and the $c^k_i$, up to a power of $q$,
(iii) the Lusztig parametrisation of $y_i$ is zero,
(iv) the Lusztig parametrisation of $c^k_i$ is $\sum c_l$ where $l$ runs over $\{l \leq k, i_l = i_k\}$.
3.5. In this section, we are concerned with the lowest weight vectors in $R^+$. For all $\lambda$ in $P^+$, set $\lambda^* := -w_0\lambda$. Let $v_{w_0\lambda}$ be the unique element of weight $w_0\lambda$ in the canonical basis of $V_q(\lambda)$. Then, \( z_\lambda := v_{w_0\lambda}^* \otimes v_\lambda \) is an element of the dual canonical basis of $R^+$. It is known (see [8]) that these elements belong to all of the standard adapted algebras defined in the section above.

**Lemma 3.7.** The elements $z_\lambda$, $\lambda \in P^+$ satisfy $z_\lambda z_\mu = z_{\lambda + \mu}$, $\lambda$, $\mu \in P^+$.

**Proof.** Let $b$, $b'$ be two elements of the canonical basis of $U_q(n^-)$, with $b \in B(\lambda)$. By [8] Proposition 3.1, we have $b''b' \in q^{<\lambda, \nu'>}B^*$, where $\nu'$ is the weight of $b''$.

Applying this formula when $b''$, resp. $b''$, is the element of $B(\lambda)^*$, resp. $B(\mu)^*$, of weight $\lambda + \lambda^*$, resp. $\mu + \mu^*$, and using Theorem 2.1, we obtain that

\[
z_\mu z_\lambda = q^{-<\lambda, \mu^*}> q^{<\lambda, \mu + \mu^*}> z_{\lambda + \mu} = z_{\lambda + \mu}.
\]

This proves the lemma. \( \square \)

In the sequel, we set $z_i = z_{\omega_i}$.

4. The Schützenberger involution.

We can now define an involution $\phi$ of $R^+$ by twisting the dual Weyl modules by the automorphism $\omega$. This involution generalises the Schützenberger involution, [27], up to a diagram automorphism.

4.1. Let $M$ be a $U_q(\mathfrak{g})$-module. With the help of the automorphism $\omega$, we define a twisted $U_q(\mathfrak{g})$-module structure on $M^*$ via

\[ x.\xi(m) = \xi(\omega(x)m), \xi \in M^*, \quad x \in U_q(\mathfrak{g}), \quad m \in M. \]

Let $M_{\omega}^*$ be the space $M^*$ endowed with this $U_q(\mathfrak{g})$-module structure.

Fix $\lambda$ in $P^+$. Then, $V_q(\lambda)_{\omega}^*$ is a simple right module. By dualising [13] Chap XXI, we obtain

**Proposition 4.1.** The module $V_q(\lambda)_{\omega}^*$ is isomorphic to $V_q(\lambda^*)$. There exists a unique right $U_q(\mathfrak{g})$-module isomorphism $\phi_\lambda$ which sends the dual canonical basis of $V_q(\lambda)_{\omega}^*$ to the dual canonical basis of $V_q(\lambda^*)$. It sends highest weight vectors to lowest weight vectors and conversely.

4.2. We define a map $\phi : R^+ \to R^+$, such that

\[ \phi(\xi \otimes v_\lambda) = \phi_\lambda(\xi) \otimes v_{\lambda^*}, \quad \xi \in V_q(\lambda)^*, \quad \lambda \in P^+, \]

where $\phi_\lambda$ is as above. Then,

**Proposition 4.2.** The map $\phi$ is an involutive antiisomorphism of the algebra $R^+$ which preserves the dual canonical basis of $R^+$.

**Proof.** Recall that $\omega$ is involutive. So, by Proposition 4.1, $\phi_\lambda \phi_\lambda$ is the identity and this implies that $\phi$ is involutive.

Now, let $R^+_\lambda$ be the $\lambda$-component of $R^+$, which is isomorphic to $V_q(\lambda)^*$ as a right $U_q(\mathfrak{g})$-module. As noted in [27] $\omega$ is a coalgebra antiisomorphism. Hence, the map $m : R^+_\lambda \otimes R^+_\mu \to R^+_{\lambda+\mu}$,

\[ m(a \otimes b) = \phi^{-1}(\phi(b)\phi(a)) \]

is a morphism of right $U_q(\mathfrak{g})$-modules, where $R^+_\lambda \otimes R^+_\mu$ is endowed with the diagonal action. Using Lemma 3.7, we obtain

\[ m(y_\lambda \otimes y_\mu) = \phi^{-1}(z_{\lambda+\mu}) = y_{\lambda+\mu} = y_\lambda y_\mu. \]

As $\dim \text{Hom}_{U_q(\mathfrak{g})}(R^+_\lambda \otimes R^+_\mu, R^+_{\lambda+\mu}) = 1$, this proves that $m$ is the multiplication of $R^+$ and thus that $\phi$ is an algebra antiisomorphism. The last assertion of the proposition is clear by Proposition 4.1. \( \square \)

**Corollary 4.3.** Fix a reduced expression $i$ in $\mathcal{R}$. Then, $\phi(A_i^\mathfrak{g})$ is an adapted subalgebra of $R^+$.

The aim of the remaining sections is to prove that the Lusztig cone $\mathcal{L}_i$ is the $i$-string parameterisation of the adapted basis of $\phi(A_i^\mathfrak{g})$. We note that this is given by monomials in the $z_i$ and the $\phi(c_k^{\mathfrak{g}})$ up to a power of $q$. 
5. Geometric Lifting and Parametrisation.

Let’s fix a dominant weight \( \lambda \). In this section, we study the geometric lifting of the morphism \( \phi_\lambda \). By using results of \([5]\), we obtain an explicit formula for the morphism \( b_i^{-1} \phi_\lambda c_i^{-1} \) which gives the Lusztig parametrisation \( t' = (t_1', \ldots, t_N') \) of the element \( \phi_\lambda(b) \) in terms of the string \( t = (t_1, \ldots, t_N) \) of \( b \), where \( b \) is in the dual canonical basis of \( R^+ \).

5.1. We give here notation and recollection of \([5]\). Let \( G \) be the semisimple simply connected complex Lie group with Lie algebra \( \mathfrak{g} \). For all \( i, 1 \leq i \leq n \), we denote by \( \varphi_i : SL_2 \hookrightarrow G \) the canonical embedding corresponding to the simple root \( \alpha_i \). Consider the one parameter subgroups of \( G \) defined by

\[
x_i(t) = \varphi_i \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right), \quad y_i(t) = \varphi_i \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right), \quad t \in \mathbb{C},
\]

and

\[
t^{\alpha_i} = \varphi_i \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right), \quad t \in \mathbb{C}^*.
\]

The \( x_i(t) \), (resp. \( y_i(t) \), \( t^{\alpha_i} \)) generate subgroups \( N_i \), (resp. \( N_i^-, H \)). We have the following commutation relations:

\[
(5.1) \quad t^{\alpha_i} x_j(t') = x_j(t'^{\alpha_i})t^{\alpha_i}, \quad t^{\alpha_i} y_j(t') = y_j(t'^{-\alpha_i})t^{\alpha_i}.
\]

We define two involutive antiautomorphisms of \( G \), \( x \mapsto x^T \) and \( x \mapsto x' \), by:

\[
x_i(t)^T = y_i(t), \quad y_i(t)^T = x_i(t), \quad (t^{\alpha_i})^T = t^{\alpha_i},
\]

\[
x_i(t)' = x_i(t), \quad y_i(t)' = y_i(t), \quad (t^{\alpha_i})' = t^{-\alpha_i}.
\]

The first one is called transposition and the second one is called inversion.

Let \( G_0 := N^- H N \) be the set of elements in \( G \) which have a (unique) gaussian decomposition; we write \( x = [x] - [x]_0 [x]_+ \) for the gaussian decomposition of \( x \) in \( G_0 \).

For all reduced expressions \( i = (i_1, \ldots, i_N) \) and all \( N \)-tuples \( t = (t_1, \ldots, t_N) \) in \( \mathbb{C}^N \), we set:

\[
x_i(t) := x_i(t_1) \cdots x_i(t_N), \quad \text{and} \quad x_{-i}(t) := y_i(t_1)^{-\alpha_i} \cdots y_i(t_N)^{-\alpha_i}.
\]

The \( x_i \) and the \( x_{-i} \) parametrise subvarieties of \( G \).

**Theorem 5.1.** \([5]\) There exists a subvariety of \( G \), \( L^{w_0, e} \), resp. \( L^{w_0, c} \), such that for all \( i \) in \( \mathcal{R} \), the map \( x_i \), resp. \( x_{-i} \), is a bijection from \( \mathbb{R}^N \) to \( L^{w_0, e} \), resp. \( L^{w_0, c} \).

We denote by \( \tilde{R}_i := x_i^{-1} \circ x_i \) and \( \tilde{R}_{-i} := x_{-i} \circ x_{-i} \) the transition maps. A remarkable result of \([5]\) asserts that \( \tilde{R}_i \) (respectively, \( \tilde{R}_{-i} \)) is a geometric lifting of the map \( R_i \) (respectively, \( R_{-i} \)), which was defined in the first section. Let’s be more precise.

By using results on semifields (see \([3]\), the authors define the so-called tropicalisation, denoted by \([.]_{\text{Trop}} \). The map \([.]_{\text{Trop}} \) is a map from the semifield \( \mathbb{Q}_{>0}(t_1, \ldots, t_N) \) to the set of maps \( \mathbb{Z}^N \rightarrow \mathbb{Z} \). The elements of \( \mathbb{Q}_{>0}(t_1, \ldots, t_N) \) are called subtraction-free rational expressions in the \( t_1, \ldots, t_N \). Tropicalising a subtraction-free expression means replacing the multiplication by the operation \( a \odot b := a + b \) and the sum by the operation \( a \oplus b := \min(a, b) \). Let’s give an example from \([3]\).

**Example 5.2.** Let \( x, y \) be two indeterminates and set \( f = x^2 - xy + y^2 \). Then, \( f \) is a subtraction-free expression because \( f = \frac{x^2 + y^2}{x + y} \). We have:

\[
[f]_{\text{Trop}} : \mathbb{Z}^2 \rightarrow \mathbb{Z}, \quad [f]_{\text{Trop}}(m, n) = \min(3m, 3n) - \min(m, n) = \min(2m, 2n).
\]

A geometric lifting is an element of the inverse image of this map. We can see in the example above that it is in general not unique.

The following theorem is a result of \([5]\). In the theorem, the notation \((.)^\vee \) means that we consider the analogue formula in the Langlands dual of \( G \), and the notation \([.]_{\text{Trop}} \) is the componentwise tropicalisation.
Proposition 5.4. Let

Theorem 5.3. \( \zeta \)

Theorem 5.5. \( \phi \)

5.2. Let \( \zeta : L_{> 0}^{w_0, e} \rightarrow L_{> 0}^{c, w_0} \) be the map defined by

By 5.4, \( \zeta \) is well defined and \( \zeta \) is affine. This is only true in the case \( i' = i \). In general, the tropical term in the right hand side of 5.5 is piecewise linear.

Remark 5.7. It is remarkable that this formula is affine. This is only true in the case \( i = i' \). In general, the tropical term in the right hand side of 5.5 is piecewise linear.

Sketch of the proof. By Proposition 5.3, \( (x_i^{-1} \circ \zeta \circ x_{-i})' \) is a subtraction-free expression. The first assertion of the theorem is obtained by composing with \( \tilde{R}_i^i \).

Let \( \phi_{i'} : \tilde{C}_i \rightarrow \mathbb{Z}^N \) be a family of maps labelled by two reduced expressions \( i \) and \( i' \) such that the three following conditions are satisfied:

1. \( \phi_{i'}(0, \cdots, 0) = b_i^{-1}\phi_{\lambda}(\eta_{\lambda}) \),
2. \( \phi_{i'} = R_i^i \circ \phi_{i'}, \)
3. \( \phi_{i'}(t_1, \cdots, t_N) = (t'_1, \cdots, t'_N), t'_k + t_1 \) and \( t'_k, k \neq 1, \) depend only on \( t_2, \cdots, t_N \).

The theorem follows from the proposition:

Proposition 5.8. [22] We have,

(i) \( (\phi_{i'}) \) is a family satisfying (1), (2), (3) if and only if

\[ \phi_{i'} = b_i^{-1}\phi_{i'}^{-1}. \]

(ii) The family \( (\phi_{i'}) \) defined by

\[ \phi_{i'}(t) = [(x_i^{-1} \circ \zeta \circ x_{-i})'](t)|_{\text{Trop}} + b_i^{-1}\phi_{\lambda}(\eta_{\lambda}) \]

satisfies the conditions (1), (2), (3).
6. From the PBW-parametrisation to the string parametrisation

Let \( i = (i_1, \cdots, i_N) \) be a reduced expression for \( w_0 \). Our aim in this section is to describe the map \( u \mapsto c_3 \phi_3 b_i(u) \), using Theorem 5.9.

Let \( S \) be the complex rational map extending

\[
(t_1, t_2, \ldots, t_N) \mapsto (u_1, u_2, \ldots, u_N) = \left( x_1^{-1} \circ \zeta \circ x_1^{-1}(t_1, t_2, \ldots, t_N) \right)
\]

(see Proposition 5.4). Then Theorem 5.5 states that

(6.1)

\[
S^{-1}(u_1, u_2, \ldots, u_N) = (t_1, t_2, \ldots, t_N) \in \mathbb{C}^N,
\]

where, for \( 1 \leq k \leq N \), we have

(6.2)

\[
t_k = u_k^{-1} \prod_{j > k} u_j^{< s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_j}> }.
\]

Proof. We note that, by Proposition 5.2.4 if \( (t_1, t_2, \ldots, t_N) \in \mathbb{C}^N \) and \( S(t_1, t_2, \ldots, t_N) = (u_1, u_2, \ldots, u_N) \in \mathbb{C}^N \), then, for \( 1 \leq k \leq N \), we have:

(6.3)

\[
u_k = t_k^{-1} \prod_{j > k} t_j^{a_{i_j} / k}.
\]

Since this map is a monomial transformation of \( \mathbb{C}^N \), it is sufficient to show that substituting the expression (6.2) for \( u_k \) in terms of the \( t_j \) into right hand side of equation (6.1) reduces to the left hand side. We obtain:

\[
\left( t_k^{-1} \prod_{l > k} t_l^{-a_{i_l} / i_k} \right) \prod_{j > k} \left( t_j^{-1} \prod_{l > j} t_l^{-a_{i_l} / i_j} \right)^{< s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_j}> }.
\]

It is clear that the exponent of \( t_k \) in this expression is 1, and that, if \( l < k \), then the exponent of \( t_l \) is zero. So we consider the case where \( l > k \). The exponent of \( t_l \) is given by

\[
a_{i_l} \alpha_{i_l} + \sum_{l > j > k} -a_{i_l, i_j} < s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_j}> > a_{i_l} \alpha_{i_l} < s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_l}>.
\]

Since \( s_{i_j} (\alpha_{i_l}^\vee) = \alpha_{i_l}^\vee - a_{i_l, i_j} \alpha_{i_j}^\vee \), this is equal to:

\[
< \alpha_{i_k} \alpha_{i_l}^\vee > + \sum_{l > j > k} < s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_j} \alpha_{i_l}^\vee > - a_{i_l, i_j} < s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k} \alpha_{i_l}^\vee > = 0,
\]

and we are done.

\[
\text{Proposition 6.2.} \quad \text{Fix a reduced expression } i \text{ for } w_0, \text{ and let } u = (u_1, u_2, \ldots, u_N) \in \mathbb{Z}^N_{\geq 0}. \text{ Then}
\]

\[
c_3 \phi_3 b_i(u) = [(S^{-1})(u)]_{\text{Trop}} + c_3 \phi_3 (\eta_{\lambda}),
\]

where \( S^{-1}(u) \) is as in Lemma 6.1.
Proof. By Theorem 5.5, we have that

\[(6.3)\]
\[b^{-1}_i \phi_i c_i^{-1}(t) = [S(t)]_{\text{trop}} + b^{-1}_i \phi_i (\eta).\]

It follows that

\[c_i \phi_i b_i(u) = [S^{-1}(u)]_{\text{trop}} - [S^{-1}(b^{-1}_i \phi_i (\eta))]_{\text{trop}}.\]

noting that \(\phi_i, \phi_i\) is the identity map. We note that \(S\) is an invertible monomial map, so its tropicalisation is linear. Hence

\[c_i \phi_i b_i(u) = [S^{-1}(u)]_{\text{trop}} + c_i \phi_i (\eta).\]

Substituting \(t = c_i \phi_i (\eta)\) into (6.3), we obtain

\[0 = b^{-1}_i \eta = [S(c_i \phi_i (\eta))]_{\text{trop}} + b^{-1}_i \phi_i (\eta).\]

Hence

\[[S^{-1}(b^{-1}_i \phi_i (\eta))]_{\text{trop}} = -c_i \phi_i (\eta).\]

Hence we have:

\[c_i \phi_i b_i(u) = [S^{-1}(u)]_{\text{trop}} + c_i \phi_i (\eta),\]

giving the required result (since \(\lambda \mapsto \lambda^*\) is an involution).

We can compute the constant term in this formula as follows:

**Lemma 6.3.** Let \(i\) be a reduced expression for \(w_0\). Then \(c_i (\phi_i \eta) = (v_1, v_2, \ldots, v_N)\), where, for \(1 \leq k \leq N\),

\[v_k = \langle s_{i_{k-1}} \cdots s_{i_1} \lambda, \alpha_{i_k}^\vee \rangle.\]

Proof. This follows from [15, 28.1.4], since we are computing the string of the lowest weight vector in \(V_q(\lambda^*)^*\).

We remark that \(\phi_i (\eta) = z_{\lambda^*}\), so this lemma is computing the string \(c_i z_{\lambda^*}\).

7. **The string parametrisation of a twisted standard adapted subalgebra**

Recall (see Corollary 5.3) that the elements \(z_1, z_2, \ldots, z_n\), together with the elements \(\phi(e_k^{\sigma})\), \(k = 1, 2, \ldots, N\), form an adapted basis for the twisted standard adapted subalgebra \(\phi A_1^p\) of \(R^+\). Our aim is to compute the string parameters of these elements. We therefore use Proposition 5.3 to apply the map \(c_i \mu_k b_i\) to the element \(b^{-1}_i (e_k^{\sigma})\), for each \(1 \leq k \leq N\); see 5.3. These last vectors were described in Proposition 3.6(iv).

7.1. For convenience, we define a matrix \(V\) with columns given by these vectors.

**Definition 7.1.** Let \(M_N(\mathbb{Z})\) denote the ring of \(N \times N\) matrices with integer entries. Let \(V = (V_{jk}) \in M_N(\mathbb{Z})\) be defined by:

\[V_{jk} = \begin{cases} 1 & \text{if } i_j = i_k, \\ 0 & \text{otherwise.} \end{cases}\]

Let \(v_k\) denote the \(k\)th column of \(V\); this is \(b^{-1}_i (e_k^{\sigma})\) by Proposition 5.3(iv).
7.2. In order to apply Proposition 6.2, to compute $c_i\phi_{\mu_k} b_i(v_k)$, for each $1 \leq k \leq N$, we first need to apply $(S^{-1})_{\text{Trop}}$ to $v_k$.
Let $S$ be the matrix defining the linear map $S_{\text{Trop}}$. Then the matrix of $(S^{-1})_{\text{Trop}}$ is $S^{-1}$. By Lemma 7.3, we have that $S^{-1} = T = (T_{jk})$, where

$$
T_{jk} = \begin{cases} 
-1 & j = k, \\
< s_{i_{j+1}} \cdots s_{i_k}, \alpha_{i_j}, \alpha_{i_j}^\vee > & j < k, \\
0 & \text{otherwise}. 
\end{cases}
$$

**Definition 7.2.** Let $C = (C_{jk}) \in M_N(\mathbb{Z})$ be the matrix given by:

$$
C_{jk} = \begin{cases} 
< s_{i_{j+1}} \cdots s_{i_k}, \varpi_{i_k}, \alpha_{i_j}^\vee > & j \leq k, \\
0 & \text{otherwise}. 
\end{cases}
$$

We have:

**Lemma 7.3.** Let $S, V$ and $C$ be the matrices defined above. Then we have $S^{-1}V = -C$.

*Proof.* We have that $(S^{-1})_{jl} = \sum_{j \leq k \leq l} T_{jk} V_{kl}$, and is zero if $j > l$. If $j \leq l$ then, using that $<, >$ is $W$-invariant,

$$
(S^{-1})_{jl} = T_{jj}V_{jl} + \sum_{j < k \leq l, i_k = i_j} < s_{i_{k-1}} \cdots s_{i_{j+1}}, \alpha_{i_j}, \alpha_{i_j}^\vee > .
$$

Remark that $T_{jj}V_{jl} = -\delta_{i_j, i_l}$. Now, by calculating explicitly the coefficient of $\alpha_{i_j}^\vee$ in $s_{i_j} \cdots s_{i_l} \alpha_{i_j}^\vee$, we find that it is equal to the right hand side of (7.1) Hence,

$$(S^{-1})_{jl} = < \varpi_{i_j}, s_{i_l} \cdots s_{i_j} \alpha_{i_j}^\vee > = - < s_{i_{j+1}} \cdots s_{i_l} \varpi_{i_l}, \alpha_{i_j}^\vee > ,$$

which is equal to $-C_{jl}$ as required. $\square$

We have the following Corollary:

**Corollary 7.4.** The entries of $C$ are nonnegative.

*Proof.* In proof of Lemma 7.3, we noted that $C_{jl}$ is equal to the negative of the coefficient of $\alpha_{i_j}^\vee$ in the negative coroot $s_{i_j} \cdots s_{i_l} \alpha_{i_j}^\vee$. $\square$

7.3. We now would like to compute $c_i\phi_{\mu_k} b_i(v_k)$ for each $k$ (note that the $v_k$ are the columns of $V$). In the following lemma we set $\varepsilon_l(c_k^i) = \varepsilon_l(b_{s_{i_1} \cdots s_{i_k}} \varpi_{i_k})$ by a misuse of notation.

**Lemma 7.5.** For $1 \leq l \leq n$ and $1 \leq k \leq N$, we have:

$$
\varepsilon_l(c_k^i) = \begin{cases} 
- < s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee > & \text{if } < s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee > \leq 0, \\
0 & \text{otherwise}. 
\end{cases}
$$

In particular,

$$
\mu_k = \sum_{1 \leq l \leq n, < s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee > \leq 0} - < s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee > \varpi_l .
$$

*Proof.* This follows from the definition of the $c_k^i$ and [18, 28.1.4] (and $sl_2$-representation theory). $\square$

Let $c_k$ be the $k$th column of $C = -S^{-1}V$, and let $p_k = c_i(\phi_{\mu_k} \eta_{\mu_k})$. Let $P$ be the matrix with columns $p_k$, $k = 1, 2, \ldots, N$.

**Lemma 7.6.** For $1 \leq k \leq N$, we have

$$
c_i\phi_{\mu_k} b_i(v_k) = -c_k + p_k.
$$

In particular, the entries of $-C + P$ are nonnegative.

*Proof.* This follows immediately from Proposition 6.2 and Lemma 7.3. $\square$

As a consequence, we have:
Proposition 7.7. Let $i$ be any reduced expression for $w_0$. Then $c_i \rho(A^\tau)$ coincides with the nonnegative integer span of the columns of $-C + P$ and the strings $c_i(z_i)$, $1 \leq i \leq n$.

Proof. By Proposition 6.6, $c_i \rho(A^\tau)$ is the nonnegative integer span of the $c_i \rho_{\mu_k} b_i(v_k)$ together with the strings $c_i(z_i)$, $1 \leq i \leq n$. Since $c_i(b^* b^*) = c_i(b^*) + c_i(b^*)$ when the elements $b^*$ and $b^*$ of the dual canonical basis are multiplicative \cite[Cor. 3.3]{C-M}, the proposition follows from Lemma 7.4 and Corollary 4.3. \hfill \Box

7.4. We also note the following formula for the entries of $P$:

Lemma 7.8. For $1 \leq j, k \leq N$, we have

$$P_{jk} = \sum_{1 \leq l \leq n, <s_{i_1} \cdots s_{i_k} \varpi_{ik}, \alpha_l^\vee > \leq 0} < s_{i_1} \cdots s_{i_k} \varpi_{ik}, \alpha_l^\vee > < s_{i_{j-1}} \cdots s_{i_1} \varpi_l, \alpha_{i_j}^\vee > .$$

Proof. By definition $P_{jk}$ is the $j$th entry of $c_i(\rho_{\mu_k} \eta_{ik})$. Hence, the lemma results from Lemma 6.5 and Lemma 6.3. \hfill \Box

7.5. We will next show that some of the columns of $-C + P$ are entirely zero, and therefore can be neglected in Proposition 7.7.

Definition 7.9. Given $k \in \{1, 2, \ldots, N\}$, we set $k(1) = \min\{j : j > k, i_j = i_k\}$, i.e. the first occurrence of $i_k$ to the right of $i_k$ in $i$. If there is no such occurrence, we set $k(1) = N + 1$.

Lemma 7.10. Suppose that $k(1) = N + 1$. Then, for $j = 1, 2, \ldots, N$, we have $P_{jk} = C_{jk}$, i.e. the $k$th column of $P$ coincides with the $k$th column of $C$.

Proof. We have

$$C_{jk} = \begin{cases} < s_{i_{j+1}} \cdots s_{i_k} \varpi_{ik}, \alpha_{i_j}^\vee > & j \leq k, \\ 0 & j > k. \end{cases}$$

Since $k(1) = N + 1$, we have

$$C_{jk} = < s_{i_{j+1}} \cdots s_{i_N} \varpi_{ik}, \alpha_{i_j}^\vee > ,$$

(in either case). By Lemma 6.6

$$\mu_k = \sum_{1 \leq l \leq n, <s_{i_1} \cdots s_{i_k} \varpi_{ik}, \alpha_l^\vee > \leq 0} < s_{i_1} \cdots s_{i_k} \varpi_{ik}, \alpha_l^\vee > \varpi_l = \sum_{1 \leq l \leq n, <s_{i_1} \cdots s_{i_N} \varpi_{ik}, \alpha_l^\vee > \leq 0} < s_{i_1} \cdots s_{i_N} \varpi_{ik}, \alpha_l^\vee > \varpi_l = \sum_{1 \leq l \leq n, <w_0 \varpi_{ik}, \alpha_l^\vee > \leq 0} < w_0 \varpi_{ik}, \alpha_l^\vee > \varpi_l = -w_0 \varpi_{ik}.$$

Hence, by Lemma 6.3

$$P_{jk} = < s_{i_{j-1}} \cdots s_{i_1} w_0 \varpi_{ik}, \alpha_{i_j}^\vee > = < s_{i_j} \cdots s_{i_N} \varpi_{ik}, \alpha_{i_j}^\vee > = C_{jk}.$$ 

\hfill \Box
7.6. We replace the zero columns in $-C + P$ with vectors which we will see are the strings of the elements $z_1, z_2, \ldots, z_n$, in order to obtain a matrix whose nonnegative integer span is the set of string parameters of the twisted standard adapted subalgebra of $R^+$ corresponding to $i$. We call this matrix $X$:

**Definition 7.11.** Let $X = (X_{jk}) \in M_N(\mathbb{Z})$ be the matrix defined as follows:

$$X_{jk} = \begin{cases} < s_{i_j} \cdots s_{i_1} \omega_{i_k}, \alpha_i^\vee >, & k(1) = N + 1, \\ -C_{jk} + P_{jk}, & \text{otherwise.} \end{cases}$$

We note that the $P_{jk}$ are given by Lemma 7.8, and that the $C_{jk}$ are given in Definition 7.2. Also, it follows from Lemma 6.3 that if $k(1) = N + 1$ then the $k$th column of $X$ is the string $c_{i_k} z_{\omega_{i_k}^\vee}$.

We have:

**Proposition 7.12.** Let $i$ be any reduced expression for $w_0$. Then $c_i \phi(A_i^\tau)$ coincides with the nonnegative integer span of the columns of $X$. In particular, the entries of $X$ are nonnegative.

**Proof.** We note that the matrix $X$ is the same as $-C + P$, except that if $1 \leq k \leq N$ and $k(1) = N + 1$, then the $k$th column (which is zero by Lemma 7.10) is replaced by the string of $z_{\omega_{i_k}^\vee}$. The result now follows from Proposition 7.7. $\square$

8. **Lusztig cones and twisted standard adapted subalgebras**

In this section, we will show that the cone of string parameters of $c_i \phi(A_i^\tau)$, given by the nonnegative integer span of the columns of $X$, coincides with the Lusztig cone corresponding to $i$. At the same time we will show that the Lusztig cones are simplicial.

8.1. We first of all define a matrix $\tilde{L}$ whose rows include the defining inequalities of the Lusztig cone corresponding to $i$ as a subset of $\mathbb{N}^N$. This matrix will later be modified to a matrix defining the Lusztig cone as a subset of $\mathbb{Z}^N$.

**Definition 8.1.** Let $\tilde{L} \in M_N(\mathbb{Z})$ be the matrix defined by:

$$\tilde{L}_{jk} = \begin{cases} -1 & k = j \text{ or } k = j(1) \\ -a_{i_j,i_k} & j < k < j(1), \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 8.2.** Let $\tilde{r}_j$ denote the $j$th row of $\tilde{L}$. The defining inequalities of the Lusztig cone $L_i$ (as a subset of $\mathbb{Z}_{\geq 0}^N$) are those inequalities of the form $\tilde{r}_j \cdot c \geq 0$ for those $j$ such that $j(1) \leq N$.

8.2. We will next show how this matrix is related to the matrices $S$ and $V$ already considered, i.e. that $V^{-1} S = \tilde{L}$. This will have the consequence that $\tilde{L} S^{-1} V = I$, in particular showing that the columns of $S^{-1} V$ satisfy the defining inequalities of the Lusztig cone corresponding to $i$. We recall from Proposition 5.4 that $S_{\text{rhop}}$ is defined by the matrix $S = (S_{jk}) \in M_N(\mathbb{Z})$ where

$$S_{jk} = \begin{cases} -1 & j = k \\ -a_{i_j,i_k} & j < k, \\ 0 & \text{otherwise.} \end{cases}$$

We next need to compute the inverse of the matrix $V$.

**Lemma 8.3.** Let $W = (W_{jk}) \in M_N(\mathbb{Z})$ be the matrix defined as follows:

$$W_{jk} = \begin{cases} 1 & j = k \\ -1 & j < k, k = j(1), \\ 0 & \text{otherwise.} \end{cases}$$

Then $W = V^{-1}$. 

Proof. We show that $WV = I$, the identity matrix. The $j,l$-entry of $WV$ is given by $Z_{jl} = \sum_{k=1}^{N} W_{jk} V_{kl}$.

For this to be non-zero, we must have $j \leq k \leq l$ and $k = j$ or $k = j(1)$. There are 5 cases:

Case (a): If $l < j$, then clearly $Z_{jl} = 0$.

Case (b): If $l = j$, then $Z_{jj} = W_{jj} V_{jj} = 1 \cdot 1 = 1$.

Case (c): If $j < l < j(1)$, then $Z_{jl} = W_{jj} V_{jl} = 1 \cdot 0 = 0$.

Case (d): If $l = j(1)$, then $Z_{jl} = W_{jj} V_{j(1)j(1)} + W_{j(1)j(1)} V_{j(1)j(1)} = 1 \cdot 1 + (-1) \cdot 1 = 0$.

Case (e): If $l > j(1)$, then $Z_{jl} = W_{jj} V_{jl} + W_{j(1)j(1)} V_{j(1)l} = V_{jl} - V_{j(1)l} = 0$ since $V_{jl} = V_{j(1)l}$. □

Lemma 8.4. Let $V$, $S$ and $\bar{L}$ be the matrices as defined above. Then $V^{-1}S = \bar{L}$.

Proof. The $j,l$-entry of $V^{-1}S$ is given by $Y_{jl} = \sum_{k=1}^{N} W_{jk} S_{kl}$. To be non-zero, we must have $j \leq k \leq l$ and $k = j$ or $k = j(1)$. As before, we have the 5 cases:

Case (a): If $l < j$ then clearly $Y_{jl} = 0$.

Case (b): If $l = j$, then $Y_{jj} = Y_{jj} = W_{jj} S_{jj} = 1 \cdot (-1) = -1$.

Case (c): If $j < l < j(1)$, then $Y_{jl} = W_{jj} S_{jl} = S_{jl} = -a_{ij, i}$.

Case (d): If $l = j(1)$, then $Y_{jl} = Y_{j(1)j(1)} = W_{j(1)j(1)} S_{j(1)j(1)} = 1 \cdot (-2) + (-1) \cdot (-1) = 1$.

Case (e): If $l > j(1)$, then $Y_{jl} = W_{jj} S_{jl} + W_{j(1)j(1)} S_{j(1)l} = S_{jl} - S_{j(1)l} = -a_{ij, i} + a_{i(j), i} = -a_{ij, i} + a_{i(j), i} = 0$.

We see that $Y_{jl} = \bar{L}_{jl}$ in every case. □

8.3 We now define a slightly altered version of the matrix $\bar{L}$, whose rows will eventually be seen to be the defining inequations of the Lusztig cone as a subset of $\mathbb{Z}^{N}$. We will also see that this matrix is the inverse of the matrix $X$.

Definition 8.5. Let $L = (L_{jk}) \in M_{N}(\mathbb{Z})$ be the matrix defined as follows:

$$L_{jk} = \begin{cases} 1 & j = k \text{ or } k = j(1), \\ -a_{ij, i} & j < k < j(1), \\ 1 & j(1) = N + 1, \\ 0 & \text{otherwise}. \end{cases}$$

Firstly, we show how some rows of $L$ are related to the strings of lowest weight vectors:

Lemma 8.6. Let

$$v = c_{1}(\phi_{1},\eta_{1}) = (v_{1}, v_{2}, \ldots, v_{N}),$$

as in Lemma 6.3. Suppose that $1 \leq j \leq j(1) \leq N$. Let $r_{j}$ be the $j$th row of $L$. Then we have $r_{j} \cdot v = 0$.

Proof. Recall that for $k = 1, 2, \ldots, N$, we have

$$v_{k} = s_{i_{k-1}} \cdots s_{i_{1}} \lambda, a_{i_{k}}^{\vee},$$

by Lemma 6.3. We have

$$r_{j} \cdot v = -v_{j} - v_{j(1)} - \sum_{j < k < j(1)} a_{ij, i} v_{k}$$

$$= - s_{i_{j-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > - s_{i_{j(1)-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > - \sum_{j < k < j(1)} a_{ij, i} < s_{i_{k-1}} \cdots s_{i_{1}} \lambda, a_{i_{k}}^{\vee} >.$$

We note that

$$s_{i_{k}}(a_{i_{j}}^{\vee}) = a_{i_{j}}^{\vee} - < a_{i_{k}}^{\vee} \alpha_{i_{j}}^{\vee} = a_{i_{j}}^{\vee} - a_{ij, i} a_{i_{k}}^{\vee},$$

so

$$r_{j} \cdot v = - s_{i_{j-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > - s_{i_{j(1)-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > + \sum_{j < k < j(1)} s_{i_{k-1}} \cdots s_{i_{1}} \lambda, s_{i_{k}}(a_{i_{j}}^{\vee}) = a_{i_{j}}^{\vee} >$$

$$= - s_{i_{j-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > - s_{i_{j(1)-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > + s_{i_{j(1)-1}} \cdots s_{i_{1}} \lambda, a_{i_{j}}^{\vee} > = 0,$$
the sum telescoping.

Next, we show that some entries of $-C + P = S^{-1}V + P$ are zero:

**Lemma 8.7.** Suppose that $1 \leq j \leq N$ and that $s_{i_1}s_{i_2} \cdots s_{i_{j-1}}(\alpha_j^\vee) = \alpha_r^\vee$ is a simple coroot. Then

\[ P_{jk} = C_{jk}. \]

**Proof.** We note that, by Lemma 8.6

\[ P_{jk} = \begin{cases} -s_{i_1} \cdots s_{i_k} \alpha_r^\vee > & s_{i_1} \cdots s_{i_k} \alpha_r^\vee > \leq 0, \\ 0 & s_{i_1} \cdots s_{i_k} \alpha_r^\vee >> 0, \end{cases} \]

using the fact that $< \alpha_\ell, \alpha_r^\vee > = \delta_{\ell r}$. Suppose first that $j \leq k$. Then:

\[
< s_{i_1} \cdots s_{i_k} \alpha_r^\vee > = < s_{i_{j+1}} \cdots s_{i_k} \alpha_r^\vee > = < s_{i_{j+1}} \cdots s_{i_k} \alpha_r^\vee > = < s_{i_{j+1}} \cdots s_{i_k} \alpha_i^\vee > = -C_{jk} \leq 0,
\]

by Corollary 8.4. So $P_{jk} = C_{jk}$. If $j > k$ then

\[
< s_{i_1} \cdots s_{i_k} \alpha_r^\vee > = < s_{i_k} \cdots s_{i_1} \alpha_r^\vee > = < s_{i_{j-1}} \cdots s_{i_k} \alpha_i^\vee > = < s_{i_{j-1}} \cdots s_{i_k} \alpha_i^\vee > = < s_{i_{j+1}} \cdots s_{i_k} \alpha_i^\vee > \geq 0,
\]

since $s_{i_{j+1}} \cdots s_{i_k} \alpha_i^\vee$ is a positive coroot. It follows that $P_{jk} = 0 = C_{jk}$. □

**Remark 8.8.** We note that the condition

\[ s_{i_1} \cdots s_{i_k} (\alpha_i^\vee) = \alpha_r^\vee \]

is equivalent to the condition

\[ s_{i_1} \cdots s_{i_k} (\alpha_i) = \alpha_r. \]

Secondly, this result shows that $(S^{-1}V + P)_{jk} = 0$ under this assumption, by Lemma 7.4.

8.4. We can now prove the following, as we have all the pieces we need:

**Proposition 8.9.** Let $L$ and $X$ be the matrices defined as above, so that the nonnegative integer span of the columns of $X$ is the cone of the string parameters of the twisted standard adapted subalgebra corresponding to $i$. Then $LX = I$, the identity matrix.

**Proof.** Denote by $x_k$ the $k$th column of $X$. Suppose first that $1 \leq j \leq j(1) \leq N$, and that $1 \leq k \leq k(1) \leq N$. Then $r_j$ (the $j$th row of $L$) is the same as the $j$th row of $\tilde{L}$. By Lemma 8.4 $\tilde{L}S^{-1}V = I$. Hence $r_j \cdot (-c_k) = \delta_{jk}$, since $-c_k$ is the $k$th column of $S^{-1}V$. By Lemma 8.6 $r_j \cdot p_k = 0$ (where $p_k$ is the $k$th column of $P$). It follows (from the definition of $X$) that $r_j \cdot x_k = \delta_{jk}$.

If $1 \leq j \leq j(1) \leq N$ and $k(1) = N + 1$, then $x_k = \alpha_i^\vee \alpha_i$. By Lemma 8.6 we have that $r_j \cdot x_k = 0$ (as required, noting that we must have $j \neq k$).

If $j(1) = N + 1$, then let $1 \leq l \leq N$ be defined by $s_{i_1} \cdots s_{i_l} \alpha_i = \alpha_i$. Then $L_{ji} = 1$ is the only non-zero entry in $r_j$, it follows that $r_j \cdot x_k = X_{lk}$.

Case (a) Suppose that $k(1) \leq N$. Then $X_{lk} = -C_{lk} + P_{lk} = 0$ by Lemma 8.7.

Case (b) Suppose that $k(1) = N + 1$. Then

\[
X_{lk} = < s_{i_{l-1}} \cdots s_{i_1} \alpha_i^\vee > = < s_{i_1} \cdots s_{i_{l-1}} \alpha_i^\vee > = < s_{i_1} \alpha_i^\vee > = \delta_{j,k}.
\]
as required (noting that \( j(1) = k(1) = N + 1 \)).

The proposition is proved. \( \square \)

If \( \mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{Z}^N \), we write \( \mathbf{c} \geq 0 \) to denote \( c_k \geq 0 \) for \( k = 1, 2, \ldots, N \). We have the following consequences.

**Theorem 8.10.** The Lusztig cone \( \mathcal{L}_i \) is simplicial, defined by the matrix \( L \):

\[
\mathcal{L}_i = \{ \mathbf{c} \in \mathbb{Z}^N : L \mathbf{c} \geq 0 \}.
\]

It coincides with the nonnegative integer span of the columns of the matrix \( X \) (see Definition 7.11).

**Proof.** By Proposition 7.12 the entries of \( X \) are nonnegative. By Proposition 8.9, \( X L = I \), so nonnegative integer combinations of the rows of \( L \) are of the form \((0, 0, \ldots, 0, 1, 0, \ldots, 0)\) (with a 1 in the \( k \)th position) and therefore correspond to inequalities of the form \( c_k \geq 0 \). So

\[
\{ \mathbf{c} \in \mathbb{Z}^N : L \mathbf{c} \geq 0 \} \subseteq \mathbb{N}^N.
\]

Since the inequalities corresponding to rows of \( L \) are either defining inequalities of \( \mathcal{L}_i \) or inequalities of the form \( c_k \geq 0 \), the claimed equality follows, and it is then immediate that \( \mathcal{L}_i \) is spanned by the columns of \( L^{-1} = X \). \( \square \)

**Remark 8.11.** The fact that \( \mathcal{L}_i \) is simplicial was already known for quiver-compatible reduced expressions for \( w_0 \) for \( \mathfrak{g} \) simply laced \([1]\) and for all reduced expressions for \( w_0 \) in type \( A_n \) \([21]\).

And we have:

**Theorem 8.12.** Let \( \mathbf{i} \) be any reduced expression for \( w_0 \). Let \( \mathcal{L}_i \) denote the Lusztig cone corresponding to \( \mathbf{i} \). Let \( \phi(A_i^\tau) \) denote the twisted standard adapted subalgebra corresponding to \( \mathbf{i} \). Then

\[
\mathcal{L}_i(\phi(A_i^\tau)) = \mathcal{L}_i.
\]

**Proof.** This follows from Proposition 7.12 and Theorem 8.10. \( \square \)

**Example 8.13.** Suppose that \( \mathfrak{g} = sl_4(\mathbb{C}) \) (type \( A_3 \)). Let \( \mathbf{i} = (2, 3, 2, 1, 2, 3) \), a reduced expression for \( w_0 \). Then we have:

\[
V = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
T = \begin{pmatrix}
-1 & -1 & 1 & 0 & -1 & 1 \\
0 & -1 & -1 & -1 & 0 & 1 \\
0 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
C = -TV = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
P = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The matrix \( X \) of spanning vectors of the Lusztig cone corresponding to \( \mathbf{i} \) and the defining matrix \( L \) are given by:

\[
X = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
L = X^{-1} = \begin{pmatrix}
-1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]
References

[1] R. Bedard. On the spanning vectors of Lusztig cones. *Represent. Theory*, 4:306–329, 2000.
[2] R. Bedard. On tight monomials in quantized enveloping algebras. Preprint, 2003.
[3] A. Berenstein, S. Fomin and A. Zelevinsky. Parametrization of canonical Bases and totally positive matrices *Adv. Math.*, 122, (1996), 49-149.
[4] A. Berenstein and A. Zelevinsky. String bases for quantum groups of type $A_r$. *I. M. Gelfand Seminar*, 51–89, *Adv. Soviet Math.* 16, Part 1, *Amer. Math. Soc.*, Providence, RI, 1993.
[5] A. Berenstein and A. Zelevinsky. Tensor product multiplicities, Canonical bases and Totally positive varieties. *Invent. Math.*, 143 (2001), 77–128.
[6] A. Berenstein and A. Zelevinsky. Total positivity in Schubert varieties. *Comment. Math. Helv.*, 72:128–166, 1997.
[7] P. Caldero. Sur la décomposition de certaines algèbres quantiques, *Comptes Rendus de l’Académie des Sciences*, t. 316 (1993), Série I, 327-329.
[8] P. Caldero. Adapted algebras for the Berenstein–Zelevinsky conjecture. *Transform. Groups* 8, no. 1, 37–50, 2003.
[9] P. Caldero. A multiplicative property of quantum flag minors. *Representation Theory*, 7, 164-176, 2003.
[10] R. W. Carter and R. J. Marsh. Regions of linearity, Lusztig cones and canonical basis elements for the quantized enveloping algebra of type $A_4$. *J. Algebra* 234, no.2, 545–503, 2000.
[11] S. Fomin and A. Zelevinsky. Cluster algebras I: Foundations, *J. Amer. Math. Soc.* 15, 497-529, 2002.
[12] A. Joseph. Quantum groups and their primitive ideals. *I. M. Gelfand Seminar*, 51–89, *Adv. Soviet Math.* 16, Part 1, *Amer. Math. Soc.*, Providence, RI, 1993.
[13] A. Reineke. Multiplicative properties of dual canonical bases of quantum groups. *J. Algebra* 214, no. 1, 8–21, 1999.
[14] M. Reineke. Monomials in canonical bases of quantum groups and quadratic forms. *J. Pure Appl. Algebra* 157, no. 2-3, 301–309, 2001.
[15] M. Rosso. Analogues de la forme de Killing et du théorème de Harish-Chandra pour les groupes quantiques. *Ann. Sci. Ec. Norm. Sup.*, 23, 445-467, 1990.
[16] M. P. Schützenberger. Promotion des morphismes d’ensembles ordonnés. *Discrete Math.*, 2, 73-94, 1972.
[17] N. Xi. Canonical basis for type $A_3$. *Comm. Algebra* 27, No. 11, 5703–5710, 1999.
[18] N. Xi. Canonical basis for type $B_2$. *J. Algebra* 214, no. 1, 8-21, 1999.
[19] N. Xi. Private communication, 1997.
[20] A. Zelevinsky. Connected components of real double Bruhat cells. *Internat. Math. Res. Notices*, 21: 1131-1154, 2000.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ CLAUDE BERNARD LYON I, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: caldero@igd.univ-lyon1.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD, LEICESTER LE1 7RH, ENGLAND

E-mail address: R.Marsh@mcs.le.ac.uk

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ CLAUDE BERNARD LYON I, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: morier@igd.univ-lyon1.fr