MATHEMATICAL ANALYSIS OF THE PHOTO-ACOUSTIC IMAGING MODALITY USING RESONATING DIELECTRIC NANO-PARTICLES: THE 2D TM-MODEL

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Abstract. We deal with the photoacoustic imaging modality using dielectric nanoparticles as contrast agents. Exciting the heterogeneous tissue, localized in a bounded domain \( \Omega \), with an electromagnetic wave, at a given incident frequency, creates heat in its surrounding which in turn generates an acoustic pressure wave (or fluctuations). The acoustic pressure can be measured in the accessible region \( \partial \Omega \) surrounding the tissue of interest. The goal is then to extract information about the optical properties (i.e. the permittivity and conductivity) of this tissue from these measurements. We describe two scenarios. In the first one, we inject single nanoparticles while in the second one we inject couples of closely spaced nanoparticles (i.e. dimers). From the acoustic pressure measured, before and after injecting the nanaparticles (for each scenario), at two single points \( x_1 \) and \( x_2 \) of \( \partial \Omega \) and two single times \( t_1 \neq t_2 \) such that \( t_1, t_2 > \text{diam}(\Omega) \),

1. we localize the center point \( z \) of the single nanoparticles and reconstruct the phaseless total field \( |u_0| \) on that point \( z \) (where \( u_0 \) is the total field in the absence of the nanoparticles). Hence, we transform the photoacoustic problem into the inversion of phaseless internal electric fields.

2. we localize the centers \( z_1 \) and \( z_2 \) of the injected dimers and reconstruct both the permittivity and the conductivity of the tissue on those points.

This can be done using dielectric nanoparticles enjoying high contrasts of both its electric permittivity and conductivity.

These results are possible using frequencies of incidence close to the resonances of the used dielectric nanoparticles. These particular frequencies are computable. This allows us to solve the photoacoustic inverse problem with direct approximation formulas linking the measured pressure to the optical properties of the tissue. The error of approximations are given in terms of the scales and the contrasts of the dielectric nanoparticles. The results are justified in the 2D TM-model.

1. Introduction and statement of the results

1.1. Motivation and the mathematical models. Imaging using small scaled contrast agents has known in the recent years a considerable attention, see for instance [7,8,24]. To motivate it, let us recall that conventional imaging techniques, as microwave imaging, are known to be potentially capable of extracting features in breast cancer, for instance, in case of the relatively high contrast of the permittivity, and conductivity, between healthy tissues and malignant ones, [10]. However, it is observed that in case of benign tissue, the variation of the permittivity is quite low so that such conventional imaging modalities are limited to be used for early detection of such diseases. In these cases, creating such missing contrast is highly desirable. One way to do it is to use micro or nano scaled particles as contrast agents, [7,8]. There are several imaging modalities using contrast agents as acoustic imaging using gas microbubbles, optical imaging and photoacoustic using dielectric or magnetic nanoparticles [7,10,20]. The first two modalities are single wave based methods. In this work, we deal with the last imaging modality.
Photoacoustic imaging is a hybrid imaging method which is based on coupling electromagnetic waves with acoustic waves to achieve high-resolution imaging of optical properties of biological tissues, \[16,19\]. Precisely, exciting the heterogeneous tissue with an electromagnetic wave, at a certain frequency related to the used small scale particles, creates heat in its surrounding which in turn generates an acoustic pressure wave (or fluctuations). The acoustic wave can be measured in a region surrounding the tissue of interest. The goal is then to extract information about the optical properties of this tissue from these measurements, \[16,19\].

A main reason why such a modality is promising is that injecting nanoparticles, see \[7,8\] for information on its feasibility, with appropriate scales between their sizes and optical properties, in the targeted tissue will create localized contrasts in the tissue and hence amplify the local electromagnetic energy around its location. This amplification can be more pronounced if the used incident electromagnetic wave is sent at frequencies close to resonances. In particular, dielectric or magnetic nanoparticles (as gold nanoparticles \[16\]) can exhibit such resonances when its inner electric permittivity or magnetic permeability is tuned appropriately, see below. Our target here is to mathematically analyze this imaging technique when injecting such nanoparticles.

To give more insight to this, let us briefly recall the photoacoustic model, see \[9,13,17,23,25,26\] for extensive studies and different motivations of this model and related topics. We assume the time harmonic (TM) approximation for the electromagnetic model\[1\] then the third component of the electric field, that we denote by \(u\), satisfies

\[
\Delta u + \omega^2 \varepsilon \mu_0 u = 0, \quad \text{in } \mathbb{R}^2
\]

with \(u := u^i + u^s\) where \(u^i := u^i(x, d, \omega) := e^{i \omega d \cdot x}\), is the incident plane wave, sent at a frequency \(\omega\) and direction \(d\), \(|d| = 1\), and \(u^s := u^s(x, \omega)\) is the corresponding scattered wave selected according to the outgoing Sommerfeld radiation conditions (S.R.C) at infinity. Here, \(\mu_0\) is the magnetic permeability of the vacuum, which we assume to be a positive real constant, and \(\varepsilon := \varepsilon(x)\) is defined as

\[
\varepsilon(x) := \begin{cases} 
\varepsilon_0, & \text{in } \mathbb{R}^2 \setminus \Omega, \\
\varepsilon(x), & \text{in } \Omega \setminus \bigcup_{m=1}^M D_m, \\
\varepsilon_m, & \text{in } D_m,
\end{cases}
\]

where \(\varepsilon_0\) is the positive constant permittivity of the vacuum and \(\varepsilon := \varepsilon_r + i \frac{\sigma}{\omega} \varepsilon_0\) with \(\varepsilon_r\) as the permittivity and \(\sigma\) the conductivity of the heterogeneous tissue (i.e. variable functions). The quantity \(\varepsilon_m\) is the permittivity constant of the particle \(D_m\), of radius \(a << 1\), which is taken to be complex valued, i.e. \(\varepsilon_m := \varepsilon_m, r + i \frac{\sigma_m}{\omega} \varepsilon_0\) where \(\varepsilon_m, r\) is its actual electric permittivity and \(\sigma_m\) its conductivity. The bounded domain \(\Omega\) models the region of the tissue of interest. We take the nanoparticle of dielectric type, meaning that \(\frac{\sigma_m}{\varepsilon_0} >> 1\) when \(a << 1\), and hence its relative speed of propagation is very large as well. Under particular rates of the ratio \(\frac{\sigma_m}{\varepsilon_0} >> 1\), resonances can occur, as the dielectric (or Mie-electric) resonances. These regimes will be of particular interest to us. Here, we take the \(D_m\)’s of the form \(D_m := z_m + a B_m\) where \(z_m\) models its location, \(a\) its radius and \(B_m\) as a smooth domain of radius \(1\) containing the origin.

As said above, exciting the tissue with such electromagnetic waves will generate a heat \(T\) which in turn generates acoustic pressure. Under some appropriate conditions, see \[5,26\] for instance, this process is modeled by the following system:

\[
\left\{ 
\begin{array}{l}
\rho_0 c_p \frac{\partial T}{\partial t} - \nabla \cdot \kappa \nabla T = \omega \text{Im} \left( \varepsilon \right) |u|^2 \delta_0(t), \\
\frac{1}{\varepsilon^2} \frac{\partial^2 p}{\partial t^2} - \Delta p = \rho_0 \frac{\partial^2 \delta(T)}{\partial t^2}
\end{array}
\right.
\]

\[1\] Here, we describe the photoacoustic model assuming the TM-approximation of the electromagnetic field. The more realistic model is of course the full Maxwell system.
where \( \rho_0 \) is the mass density, \( c_p \) the heat capacity, \( \kappa \) is the heat conductivity, \( c \) is the wave speed and \( \beta_0 \) the thermal expansion coefficient. To these two equations, we supplement the homogeneous initial conditions:

\[
T = p = \frac{\partial p}{\partial t} = 0, \text{ at } t = 0.
\]

Under additional assumptions on the smallness of the heat conductivity \( \kappa \), one can neglect the term \( \nabla \cdot \kappa \nabla T \) and hence, we end up with the photoacoustic model linking the electromagnetic field to the acoustic pressure:

\[
\begin{cases}
\frac{\partial^2 p}{\partial t^2} - c_s^2 \Delta p = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\
p(x, 0) = \frac{\omega \beta_0}{c_p} \text{Im}(\varepsilon)|u|^2 & \text{in } \mathbb{R}^2, \\
\frac{\partial p}{\partial t}(x, 0) = 0, & \text{in } \mathbb{R}^2.
\end{cases}
\]

The imaging problem we wish to focus on is stated in the following terms:

**Problem.** Reconstruct the coefficient \( \varepsilon \) from the given pressure \( p(x, t) \) measured for \( (x, t) \in \partial \Omega \times (0, T) \), with some positive time length \( T \),

1. after injecting single nanoparticles located in a sample of points in \( \Omega \),

or/and

2. after injecting couples of nanoparticles two by two closely spaced (i.e. dimers) and located in a sample of points in \( \Omega \).

It is natural to split this problem into two steps. The first step concerns the acoustic inversion, namely to reconstruct the source term \( \text{Im}(\varepsilon)|u|^2 \), \( x \in \Omega \), from the pressure \( p(x, t) \) for \( (x, t) \in \partial \Omega \times (0, L) \). The second step concerns the electromagnetic inversion, namely to reconstruct \( \varepsilon \) from the internal data \( \text{Im}(\varepsilon)|u|^2 \).

1.2. **The acoustic inversion.** We start by recalling the main results related to the model (1.3). More informations about this part can be found in [1] and [14].

For this inversion, there are two cases to distinguish:

**Case 1:** The speed of propagation \( c_s \) is constant everywhere in \( \mathbb{R}^2 \) and \( \Omega \) is a disc. The solution of the problem (1.3) is given by the Poisson formula

\[
p(x, t) = \frac{\omega \beta_0}{2\pi c_sc_p} \partial_t \left( \int_{|x-y|<c_s t} \frac{\text{Im}(\varepsilon)(y)|u|^2(y)}{\sqrt{c_s^2 t^2 - |x-y|^2}} dy \right).
\]

We denote by \( M(f) \) the circular means of \( f \)

\[
M(f)(x, r) := \frac{1}{2\pi} \int_{|\xi|=1} f(x + r\xi) d\sigma(\xi).
\]

The equation (1.4) takes the following form

\[
p(x, t) = \frac{\omega \beta_0}{c_sc_p} \partial_t \left( \int_0^{c_s t} \frac{r}{\sqrt{c_s^2 t^2 - r^2}} M(\text{Im}(\varepsilon)|u|^2)(x, r) dr \right).
\]

The recovery of \( \text{Im}(\varepsilon)|u|^2 \) from \( p(x, t), (x, t) \in \partial \Omega \times [0, T] \), is done in two steps. First, as \( \partial \Omega \) is a circle, the circular means can be recovered from the pressure as follows

\[
M(\text{Im}(\varepsilon)|u|^2)(x, r) = \frac{2\omega \beta_0}{c_p \pi} \int_0^{c_s r} \frac{p(x, t)}{\sqrt{r^2 - t^2}} dt, \quad x \in \partial \Omega.
\]

\[\text{We stated the model in the whole plan } \mathbb{R}^2. \text{ However, we could also state it in a bounded domain supplemented with Dirichlet or Neumann boundary conditions.}\]
Second, if $\text{Im}(\varepsilon)|u|^2 \in C^\infty(\mathbb{R}^2)$ with $\text{supp}(\text{Im}(\varepsilon)|u|^2) \subset \overline{\Omega}$, then, for $x \in \Omega$,

$$
\text{Im}(\varepsilon)(x)|u|^2(x) = \frac{1}{2\pi R_0} \int_{\partial \Omega} \int_{2R_0}^R (\partial_r \partial_r M(\text{Im}(\varepsilon)|u|^2))(p, r) \log(|r^2 - |x - p|^2|) \, dr \, d\sigma(p).
$$

We can find in [18] and [11] the justification of (1.5) and (1.6) respectively.

Case 2: The speed of propagation is variable in $\Omega$ and constant in $\R^2 \setminus \Omega$, with $\Omega$ not necessarily a disc. However, the following assumptions are needed, namely (1). $\text{supp}(\text{Im}(\varepsilon)|u|^2)$ is compact in $\Omega$, (2). $c(x) > c > 0$ and $\text{supp}(c(x) - 1)$ is compact in $\Omega$ and (3). the non trapping condition is verified. In $L^2(\Omega; c_n^{-2}(x)dx)$, we consider the operator given by the differential expression

$$
A = -c_n^{-2}(x)\Delta
$$

and the Dirichlet boundary condition on $\partial \Omega$. This operator is positive self-adjoint operator, and has discrete spectrum $\{s_k^2\}_{k \geq 1}$ with a basis set of eigenfunctions $\{\psi_k\}_{k \geq 1}$ in $L^2(\Omega; c_n^{-2}(x)dx)$. Then, the function $\text{Im}(\varepsilon)(x)|u|^2(x)$ can be reconstructed inside $\Omega$ from the data $p$, as the following $L^2(\Omega)$ convergent series

$$
\text{Im}(\varepsilon)(x)|u|^2(x) = \frac{c_p}{\omega \mu_0} \sum_k (\text{Im}(\varepsilon)(x)|u|^2)_k \psi_k(x),
$$

where the Fourier coefficients $(\text{Im}(\varepsilon)(x)|u|^2)_k$ can be recovered as:

$$(\text{Im}(\varepsilon)(x)|u|^2)_k = s_k^{-2} s_k^{-3} \int_0^\infty \sin(s_k t) p_k''(t) dt,$$

with

$$
p_k(t) := \int_{\partial \Omega} p(x, t) \frac{\partial \psi_k}{\partial \nu}(x) \, dx.
$$

More details can be found in [1].

In our work, we address the following two situations regarding the types of the used dielectric nanoparticles.

1. Only the permittivity $\varepsilon_m, r$ of the nanoparticle is contrasting. For this case, we use the results above on the acoustic inversion to obtain $\text{Im}(\varepsilon)(x)|u|^2(x), x \in \Omega$ and hence $|u|, x \in D_m$, as $\text{Im} \varepsilon = \frac{\varepsilon_m}{\omega}$ on $D_m$ which is known. With this information, we perform the electromagnetic inversion to reconstruct $\varepsilon_r$ and $\sigma_\Omega$.

2. Both the permittivity $\varepsilon_m, r$ and the conductivity $\sigma_m$ of the nanoparticle are contrasting. In this case, we do not rely on the acoustic inversion results above. Instead, we propose direct approximating formulas to link the measured data $p(x, t)$ for $x \in \partial \Omega$ and $t \in (0, T)$, to $|u|(x), x \in D_m$. Actually, we need only to measure $p(x, t)$ on two single points on $\partial \Omega$ for two distinct times $t_1$ and $t_2$. Then, we perform the electromagnetic inversion.

1.3. The electromagnetic inversion and motivation of using nearly resonant incident frequencies. We start from the model

$$
(\Delta + \omega^2 n^2) u = 0 \quad \text{in} \quad \mathbb{R}^2
$$

where, taking $M = 1$ in (1.2),

$$
n := \begin{cases}
n_p & \text{in } D \\ n_0 & \text{in } \mathbb{R}^2 \setminus D.
\end{cases}
$$

We set $\varepsilon_p - \varepsilon_0 = \tau, \, \tau >> 1$. Then, we obtain

$$
n^2 - n_0^2 = \begin{cases}
\mu_0 \tau & \text{in } D \\ 0 & \text{in } \mathbb{R}^2 \setminus D.
\end{cases}
$$
We call the dielectric (or *Mie-electric*) resonances the possible eigenvalues of (1.7), i.e. the possible solutions \((\omega, u^*)\) of (1.7) when \(u^* = 0\). It is known from the scattering theory, precisely Rellich’s lemma, that those eigenvalues belong to the lower complex plane \(\mathbb{C}_-\). However, as \(\tau \gg 1\), and \(a << 1\), their imaginary parts tend to zero, see [4] for instance. Using the Lippmann-Schwinger equation (L.S.E), such eigenvalues are also characterized by the equation

\[
(1.8) \quad u(x) = -\omega^2 \int_D (n_0^2 - n^2) G_k(x, y) u(y) dy, \quad x \in \mathbb{R}^2,
\]

where \(G_k\) is the Green’s function satisfying \((\Delta + \omega^2 n^2) G_k = -\delta\) with the S.R.C, and \(k := \omega n\) is the wave number. As \(\epsilon_p\) is constant in \(D\), and assuming \(\epsilon\) to be constant in \(\Omega\) for simplicity of the exposition here, we get from (1.8)

\[
(1.9) \quad u(x) \frac{1}{\omega^2 \mu_0 \tau} = \int_D G_k(x, y) u(y) dy, \quad x \in \mathbb{R}^2.
\]

To solve (1.9), it is enough to find and compute eigenvalues \(w_n(k)\) of the volumetric potential operator \(A_k\) defined as

\[
(1.10) \quad A_k(u)(x) := \int_D \Phi_k(x, y) u(y) dy, \quad u \in L^2(D).
\]

Then combining (1.9) and (1.10), we can write \(A_k(u) = \frac{1}{\omega^2 \mu_0 \tau} u\) and then solve in \(\omega\), and recalling that \(k = \omega n\), the dispersion equation

\[
(1.11) \quad w_n(k) = \frac{1}{\omega^2 \mu_0 \tau}.
\]

Let us now recall that the operator \(LP\), called the Logarithmic Potential operator, defined by

\[
LP(u)(\eta) := \int_B \frac{1}{2\pi} \log |\eta - \xi| u(\xi) d\xi, \quad u \in L^2(B), \eta \in B,
\]

has a countable sequence of eigenvalues with the corresponding eigenfunctions as a basis of \(L^2(B)\). For more details see [12] and [6]. Correspondingly, we define \(A_0\) to be

\[
(1.12) \quad A_0(u)(x) := \int_D \frac{1}{2\pi} \log |x - y| u(y) dy, \quad u \in L^2(D), x \in D.
\]

Rescaling, we have \(A_0(u)(x) = a^2 LP \tilde{u}(\xi) - \frac{a^2 \log(a)}{2\pi} \int_B \tilde{u}(\xi) d\xi, \quad \xi := \frac{x-a}{a}.\) Hence the eigenvalue problem \(A_0(u) = \lambda_n u, \) on \(D\), becomes

\[
LP \tilde{u} - \frac{\log(a)}{2\pi} \int_B \tilde{u}(\eta) d\eta = \frac{\lambda_n}{a^2} \tilde{u}, \quad \text{on } B.
\]

We observe that the spectrum \(\text{Spect}(A_0|_{L^2(B)})\) of \(A_0\), restricted to \(L^2_0(D) := \{v \in L^2(D), \int_D v(x) dx = 0\}\), is characterized by \(\text{Spect}(A_0|_{L^2(B)}) = a^{-2} \text{Spect}(LP|_{L^2(B)})\). However, as we see it later, the important eigenvalues are those for which the corresponding eigenfunctions are not average-zero. Therefore, we need to handle the other part of the spectrum of \(A_0\) as well. As \(L^2_0(D)\) is not invariant under the action of \(A_0\), the natural decomposition \(L^2(D) = L^2_0(D) \oplus 1\) does not decompose it.

The following properties are needed in the sequel and we state them as hypotheses to keep a higher generality.

**Hypotheses 1.** The particles \(D\), of radius \(a\), \(a \ll 1\), are taken such that the spectral problems \(A_0 u = \lambda u, \) in \(D\), have eigenvalues \(\lambda_n\) and corresponding eigenfunctions, \(e_n\), satisfying the following properties:

1. \(\int_D e_n(x) dx \neq 0, \forall a \ll 1.\)
2. \(\lambda_n \sim a^2 |\log(a)|, \forall a \ll 1.\)
In the appendix, see section 5, we show that for particles of general shapes, the first eigenvalue and the corresponding eigenfunction satisfy Hypotheses. In addition, we characterize the properties of the eigenvalues for the case when $D$ is a disc.

Since, the dominant part of the operator $A_k$ defined in (1.10) is $A_0$, we can write:

$$w_n(k) = \lambda_n + \mathcal{O}(a^3).$$

Combining (1.13), (5.1) and (1.11), we get $\lambda_n = \frac{1}{\omega^2 \mu_0 \tau a^2} + \mathcal{O}(1)$ or $\omega^2 = \frac{1}{\mu_0 \tau \lambda_n} + \mathcal{O}(\log(a)^{-1})$.

This means that (1.7) has a sequence of eigenvalues that can be approximated by:

$$\frac{1}{\mu_0 \tau \lambda_n} + \mathcal{O}(\log(a)^{-1}).$$

The dominating term is finite if the contrast of the used nanoparticle’s permittivity behaves as $\tau \sim \lambda_n^{-1} \sim a^{-2} \log(a)^{-1}$ for $a << 1$.

We distinguish two cases as related to our imaging problem.

1. Injecting one nanoparticle and then sending incident plane waves at real frequencies $\omega$ close to the real values

$$\omega_n := (\mu_0 \tau \lambda_n)^{-1/2},$$

we can excite, approximately, the sequence of eigenvalues described above. As a consequence, see the justification later, if we excite with incident frequencies near $\omega_n$, we get $u_n = \frac{1}{\omega^2 \mu_0 \tau a^2} + \mathcal{O}(1)$ or $\omega^2 = \frac{1}{\mu_0 \tau \lambda_n} + \mathcal{O}(\log(a)^{-1})$.

As $\omega_n$ and $D$ are in principle known, then we can recover the total field $|u_0(z)|$. Taking a sampling of points $z$ in $\Omega$, we get at hand the phaseless internal total field $|u_0(z)|$, $z \in \Omega$.

2. Now, we inject a dimer, meaning a couple of close nanoparticles, instead of only single particles, with prescribed high contrasts of the relative permittivity or/and conductivity. Sending incident plane wave at frequencies close to the dielectric resonances, we recover also the amplitude of the field generated by the first interaction of the two nano-particles. Indeed, based on point-approximation expansions, this field can be approximated by the Foldy-Lax field. This field describes the one due to multiple interactions between the nanoparticles. We show that the acoustic inversion approximately reconstruct the first multiple interaction field (i.e. the Neumann series cut at the first, and not the zero, order term). From this last field, we recover the value of $|\epsilon_0(z)|$, $z \in \Omega$.

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3More exactly, using the expansion and the scales of the fundamental solution, we show that an eigenvalue of $A_k$ can be written as

$$a^2 \left( \lambda_n + \frac{1}{2} \log(a) \left( \int_B \epsilon_n(\xi) d\xi \right)^2 - \frac{1}{2} \log(k)(z + \pi \Gamma) \right) + \mathcal{O}(a^3).$$
Both steps are justified using the incident frequencies close to the dielectric resonance of the nanoparticles. This wouldn’t be possible using incident frequencies away from these resonances.

Recall that $\epsilon_0 = \epsilon_r + \frac{i\sigma_\Omega}{\omega}$, then $|\epsilon_0|^2 = \epsilon_r^2 + \frac{\sigma_\Omega^2}{\omega^2}$. Hence using two different dielectric resonances, we can reconstruct both the permittivity $\epsilon_r$ and the conductivity $\sigma_\Omega$.

1.4. Statement of the results. We recall that the mathematical model of the photoacoustic imaging modality is $\{1.1\}, \{1.2\}$ and $\{1.3\}$.

Next, we set $u := u_j, j = 0, 1, 2$, the solution of $\{1.1\}$ and $\{1.2\}$ when there is no nanoparticle injected, there is one or two nanoparticles, respectively (i.e take $M = 0, 1$ or 2 in $\{1.2\}$).

To keep the technicality at the minimum, we deal only with the case when the electromagnetic properties of the injected particles are the same i.e, $\epsilon_1 = \cdots = \epsilon_M$.

1.4.1. Imaging using dielectric nanoparticles with permittivity contrast only. Let the permittivity $\epsilon$, of the medium, be $W^{1,\infty}$—smooth in $\Omega$ and the permeability $\mu_0$ to be constant and positive. Let also the injected nanoparticles $D$ satisfy Hypotheses $\{1\}$. We assume these nanoparticles to be characterized with moderate magnetic permeability and their permittivity and conductivity are such that $\epsilon_{m,r} \sim a^{-2} |\log(a)|^{-1}$ while $\sigma_m \sim 1$ as $a << 1$. The frequency of the incidence $\omega$ is chosen close to the dielectric resonance $\omega_{\nu_0}$

$$\omega_{\nu_0}^2 := (\mu_0 \tau \lambda_{\nu_0})^{-1},$$

as follows

$$\omega^2 = \omega_{\nu_0}^2 (1 \pm |\log(a)|^{-h}), \quad 0 < h < 1. \quad 4$$

Theorem 1.1. We assume that the acoustic inversion is already performed using one of the methods given in section $\{1.2\}$. Hence, we have at hands

$$|u_j|(x), x \in D, j = 1, 2.$$

1. (Injecting one nanoparticle. In this case, we use the data $|u_1|(x), x \in D$. We have the following approximation

$$\int_D |u_1|^2(x)dx = \frac{|u_0|^2(z)(\int_D e_{\nu_0}(x)dx)^2}{1 - \omega^2 \mu_0 \tau \lambda_{\nu_0}} + O(a^2). \quad (1.15)$$

2. (Injecting two closely spaced nanoparticles. These two particles are distant to each other as

$$|z_1 - z_2| \geq \exp(-|\log(a)|^{1-h}), a << 1,$$

where $z_1$ and $z_2$ are the location points of the particles. In this case, we use as data $|u_j|(x), x \in D, j = 1, 2$, where $D$ is any one of the two particles. The following expansion is valid

$$\log(|k|)(z) = 2\pi \gamma - \frac{A_1 - (1 - C\Phi_0)^2}{A_1 - 2(1 - C\Phi_0)} + O(|\log(a)|^{\max(h-1,1-2h)}), \quad \text{where } \gamma \text{ is the Euler constant,}$$

$$A_1 := \int_D |u_2|^2(x)dx \int_D |u_1|^2(x)dx, \quad \Phi_0 := \frac{-1}{2\pi} \log |z_1 - z_2|$$

and

$$C := \int_D \frac{1}{\omega^2 \mu_0 \tau - A_0}(-1)(x)dx = \frac{\omega^2 \mu_0 \tau}{1 - \omega^2 \mu_0 \tau \lambda_{\nu_0}} \left(\int_D e_{\nu_0}(x)dx\right)^2 + O(|\log(a)|^{-1}).$$

$4$Choosing + or - does not make a difference for the results in Theorem 1.1.
From the formula (1.15), we can derive an estimate of the total field in the absence of the nanoparticles, i.e. \(|u_0|\)(x), \(x \in \Omega\), by repeating the same experiment scanning the targeted tissue located in \(\Omega\) by injecting single nanoparticles. Hence, we transform the photoacoustic inverse problem to the reconstruction of \(\epsilon_0\) in the equation \((\Delta + \omega^2 \mu_0 \epsilon_0)u_0 = 0\), in \(\mathbb{R}^2\), from the phaseless internal data \(|u_0|(x), x \in \Omega\).

From the formula (1.16), we can reconstruct \(|k|(z)\) using the data \(|u_1|(x)\) and \(|u_2|(x)\), with \(x \in D\). Indeed,

\[|k|(z) = \omega^2 |\epsilon_0| \mu_0 = \omega^2 \left(|\epsilon_r|^2 + \frac{|\epsilon_\Omega|^2}{\omega^2}\right)^{1/2} \mu_0,\]

then using two different resonances \(\omega_{n_0}\) and \(\omega_{n_1}\), we can reconstruct both the permittivity \(\epsilon_0(z)\) and the conductivity \(\sigma_0(z)\).

1.4.2. Imaging using dielectric nanoparticles with both permittivity and conductivity contrasts. As in section 1.4.1, let the permittivity \(\epsilon\), of the medium, be \(W^{1,\infty}\)-smooth in \(\Omega\) and the permeability \(\mu_0\) to be constant and positive. Let also the injected nanoparticles \(D\) satisfy Hypotheses 1. Here, we assume that \(\epsilon_{m,x} \sim a^{-2} |\log(a)|^{-1}\) and \(\sigma_{m} \sim a^{-2} |\log(a)|^{-1-h-s}\), \(s \geq 0\). The frequency of the incidence \(\omega\) is chosen close to the dielectric resonance \(\omega_{n_0}\)

\[\omega_{n_0}^2 := \left(\mu_0 \lambda_{n_0}\right)^{-1}\]

as follows

(1.17) \[\omega^2 = \omega_{n_0}^2 (1 \pm |\log(a)|^{-h}), \quad 0 < h < 1.\]

**Theorem 1.2.** Let \(x \in \partial\Omega\) and \(t \geq \text{diam}(\Omega)\). We have the following expansions of the pressure:

1. Injecting one nanoparticle. In this case, we have the expansion

(1.18) \[
(p^+ + p^- - 2p_0)(t, x) = \frac{-t \omega \beta_0}{c_p (t^2 - |x - z|^2)^{3/2}} \frac{2 \text{Im}(\tau)|u_0(z)|^2}{|1 - \omega^2 \mu_0 \lambda_{n_0} \tau|^2} \left(\int_D \epsilon_{n_0} dx\right)^2 + O(|\log(a)|^{\max(-1, 2h - 2)}),
\]

under the condition \(0 \leq s < \max\{h, 1 - h\}\), where \(p^+\) and \(p^-\) correspond to the pressure after injecting the nanoparticles and exciting with frequencies of incidence (1.17) while \(p_0\) is the pressure in the absence of the nanoparticles.

2. Injecting two close dielectric nanoparticles. These two particles are distant to each other as

\[|z_1 - z_2| \geq \exp(-|\log(a)|^{1-h}), a << 1,\]

where \(z_1\) and \(z_2\) are the location points of the particles. We set

\[\hat{p}(t, x) := (p^+ - p_0)(t, x) + \frac{1 - \omega_{n_0}^2}{1 + \omega_{n_0}^2} (p^- - p_0)(t, x),\]

then we have the following expansion\(^5\)

(1.19) \[
\hat{p}(t, x) = \frac{-t \omega \beta_0}{c_p (t^2 - |x - z_2|^2)^{3/2}} \frac{4 \text{Im}(\tau)}{1 + \omega_{n_0}^2} \left(\int_D u_2(x)\epsilon_{n_0}(x) dx\right)^2 + O(|\log(a)|^{\max(-1, 2h - 2)}),
\]

where \(D\) is any one of the two nanoparticles.

The formula (1.18) means that if we measure before and after injecting one nanoparticle, then we can reconstruct the phaseless data \(|u_0|(x), x \in \Omega\). Hence, we transform the photoacoustic inverse problem to the inverse scattering using phaseless internal data.

\(^5\)Since \(z_1\) and \(z_2\) are sufficiently close, we make in (1.19) an arbitrary choice of one of them, i.e. (1.19) does not distinguish between \(z_1\) and \(z_2\).
The formula (1.19) can be expressed using \( u_0 \) instead of \( u_2 \) under the condition \( 0 \leq s < \min\{h, 1 - h\} \) as for (1.18). The formula (1.19) means that if we measure before and after injecting two closely spaced nanoparticles, then we can reconstruct \( \int_D u_2(x)e_{n_0}(x)dx \) and hence \( \int_D |u_2(x)|^2dx \). In addition, a slightly different form of formula (1.18), see (3.15),

\[
(p^+ + p^- - 2p_0)(t, x) = -2 t \text{Im}(\tau) \left\{ \int_D u_1(x)e_{n_0}(x)dx \right\}^2 \left( t^2 - |x - z|^2 \right)^{3/2} + O\left( |\log(a)|^{\max(-1.2h - 2)} \right),
\]

shows that if we measure before and after injecting one nanoparticle, we can reconstruct \( \int_D |u_1(x)|^2dx \). Using these two last data, i.e. \( \int_D |u_1(x)|^2dx \) and \( \int_D |u_2(x)|^2dx \), we can reconstruct, via (1.16), \( |e_0| \).

Hence, using two different resonances, we reconstruct both the permittivity \( \epsilon \) and the conductivity \( \sigma_0 \).

Let us show how we can use (1.18) to localize the position \( z \) of the injected nanoparticles and estimate \( |u_0(z)| \). The corresponding results can also be shown using (1.19). For this, we use the notations

\[
\hat{p} := (p^+ + p^- - 2p_0), \quad A := \frac{2 t \text{Im}(\tau) |u_0(z)|^2 \left( \int_D e_{n_0}dx \right)^2}{1 - \omega^2 \mu_0 \lambda_{n_0} \tau^2} \quad \text{and} \quad \text{Err} := O\left( |\log(a)|^{\max(-1.2h - 2)} \right).
\]

Let \( t_1 \neq t_2 \) then we have

\[
\frac{\hat{p}(t_1, x)}{\hat{p}(t_2, x)} = A \left( \frac{t_1}{t_1^2 - |x - z|^2} \right)^{3/2} + \text{Err} = A \left( \frac{t_2}{t_2^2 - |x - z|^2} \right)^{3/2} + \text{Err} = \frac{t_1}{t_2} \left( \frac{t_1^2 - |x - z|^2}{t_2^2 - |x - z|^2} \right)^{3/2} + O\left( |\log(a)|^{s + \max(-h, h - 1)} \right),
\]

where

\[
0 \leq s < \min\{h; 1 - h\}.
\]

From (1.20) we derive the formula

\[
|x - z| = \left[ \left( t_1^2 (t_2 \hat{p}(t_1, x))^2 - t_2^2 (t_3 \hat{p}(t_2, x))^2 \right) / \left( t_2 \hat{p}(t_1, x)^2 - (t_1 \hat{p}(t_2, x))^2 \right) \right]^{1/2} + O\left( |\log(a)|^{s + \max(-h, h - 1)} \right).
\]

The expression (1.22) tells that, for \( x \in \partial\Omega \), the point \( z \) is in the arc given by the intersection of \( \Omega \) and the circle \( S \) with center \( x \) and radius computable as

\[
\frac{t_1^2 (t_2 \hat{p}(t_1, x))^2 - t_2^2 (t_1 \hat{p}(t_2, x))^2}{(t_2 \hat{p}(t_1, x))^2 - (t_1 \hat{p}(t_2, x))^2}.
\]

Then in order to localise \( z \), we repeat the same experience with another point \( x_\star \neq x \), and take the intersection of two arcs, see Figure 1.

Assume that \( z \) is obtained, then from the equation (1.18), we get

\[
|u_0(z)|^2 = - \frac{1 - \omega^2 \mu_0 \lambda_{n_0} \tau^2}{2 t \text{Im}(\tau)} \left( \int_D e_{n_0} \right)^2 + O\left( |\log(a)|^{s + \max(-h, h - 1)} \right),
\]

with

\[
0 \leq s < \min\{h; 1 - h\}.
\]

Let us finish this introduction by comparing our findings with the previous results. To our knowledge, the only work published to analyse the photo-acoustic imaging modality using contrast agents is the recent work [26]. The authors propose to use plasmonic resonances instead of dielectric ones. Assuming the acoustic inversion to be known and done, as described in section 1.2, they perform the electromagnetic inversion. They state the 2D-electromagnetic model where the magnetic fields satisfy a divergence form equation. Performing asymptotic expansions, close to these resonances they derive the dominant part of
the magnetic field and reconstruct the permittivity by an optimization step applied on this dominating term. This result could be compared to Theorem 1.1, i.e formula (1.15).

The rest of the paper is organized as follows. In section 2 and section 3, we prove Theorem 1.1 and Theorem 1.2, respectively. In section 4, we derive the needed estimates on the electric fields, used in section 2 and section 3 in terms of the contrast of the permittivity, conductivity and for frequencies close to the dielectric resonances. Finally, in section 5, we discuss the validity of the conditions in Hypotheses 1.

Notations: Only $L^2$-norms on domains are involved in the text. Therefore, unless indicated, we use $\| \cdot \|$ without specifying the domain. In addition, we use $\langle \cdot, \cdot \rangle$ for the corresponding scalar product.

For a given function $f$ defined on $\bigcup_{j=1}^{M} D_j$, we denote by $f_j := f_{|D_j}$; $j = 1, \cdots, M$.

The eigenfunctions $(e^{(i)}_n)_{n \in \mathbb{N}}$ of the Newtonian operator stated on $D_i$ depend, of course, on $D_i$. Nevertheless, unless specified, we use the notation $(e_n)$ even when dealing with multiple particles located in different positions.

2. Proof of Theorem 1.1

We split the proof into two subsections. In the first one, we derive the Foldy-Lax algebraic system, see (2.7) in proposition 2.4, as an approximation of the continuous L.S.E satisfied by the electric field. In the second subsection, we invert the algebraic system and extract the needed formulas, see (2.27).

2.1. Approximation of the L.S.E. In the following, we notice by $G_k$ the Green kernel for Helmholtz equation in dimension two. This means that $G_k$ is a solution of:

$$(\Delta + \omega^2 n_0^2(\cdot))G_k(\cdot, \cdot) = -\delta(\cdot), \quad \text{in} \quad \mathbb{R}^2$$

satisfying the S.R.C.
Lemma 2.1. The Green kernel \( G_k \) admits the following asymptotic expansion
\[
G_k(|x - y|) = \frac{-1}{2\pi} \log(|x - y|) - \frac{1}{2\pi} \log(k(y)) + \frac{i}{4} + \frac{1}{2\pi} \lim_{p \to +\infty} \left( \sum_{m=1}^{p} \frac{1}{m} - \log(p) \right)
\]
(2.1)
\[+ \quad \mathcal{O}(|x - y| \log(|x - y|)), \quad x \text{ near } y.\]

Proof. Following the same steps as in [2], pages 10-12, and taking into account the logarithmic singularity of the fundamental solution of 2D Helmholtz equation we can deduce the expansion (2.1). \( \square \)

Definition 2.2. We define
\[
a := \frac{1}{21 \leq m \leq M} \text{diam}(D_m), \quad d_{mj} := \text{dist}(D_m, D_j), \quad d := \min_{1 \leq m, j \leq M} d_{mj},
\]
where \( D_m = z_m + a B \) with \( B \) is a bounded domain containing the origin.

The unique solution of the problem (1.7), with \( D := \bigcup_{j=1}^{M} D_j \), satisfies the L.S.E
\[
(\Delta + \omega^2 n_0(x)) u = 0 \quad \text{in } \mathbb{R}^2
\]
(2.2)
\[u(x) - \omega^2 \mu_0 \int_{D} G_k(x, y) (\varepsilon_p - \varepsilon_0)(y) u(y) \, dy = u_0(x) \quad \text{in } D.
\]
We set \( v_m := u|_{D_m} \), \( m = 1, \ldots, M \). Then (2.2), for \( x \in D_m \), rewrites as
\[
v_m(x) - \omega^2 \mu_0 \int_{D_m} G_k(x, y) (\varepsilon_p - \varepsilon_0(y)) v_m(y) \, dy - \omega^2 \mu_0 \sum_{j \neq m} \int_{D_j} G_k(x, y) (\varepsilon_p - \varepsilon_0(y)) v_j(y) \, dy = u_0(x).
\]
We set: \( \tau_j = (\varepsilon_p - \varepsilon_0(z_j)) \). Assuming \( \varepsilon_0 \) to be \( W^{1,\infty}(\Omega) \), the solution \( u \) of the scattering problem
\[
(\Delta + \omega^2 \mu_0) u = 0 \quad \text{in } \mathbb{R}^2
\]
(2.3)
\[\Phi_0(x, y) := \frac{-1}{2\pi} \log(|x - y|) \quad \text{and } \Gamma := \frac{i}{4} + \frac{1}{2\pi} \lim_{p \to +\infty} \left( \sum_{m=1}^{p} \frac{1}{m} - \log(p) \right).
\]
Expanding \( (\varepsilon_p - \varepsilon_0,) \), \( u_0 \) and \( G_k(\cdot, \cdot) \) near the center \( z_m \), we obtain
\[
v_m(x) - \omega^2 \mu_0 \tau_m \int_{D_m} (\Phi_0(x, y) - \frac{1}{2\pi} \log(k(y)) + \Gamma) v_m(y) \, dy
\]
\[- \omega^2 \mu_0 \tau_m \int_{D_m} |x - y| \log |x - y| v_m(y) \, dy
\]
\[+ \quad \omega^2 \mu_0 \sum_{j \neq m} \tau_j \int_{D_j} \left[ G_k(z_m; z_j) + \int_0^{1} \nabla G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) \, dt \right] v_j(y) \, dy
\]
\[+ \quad \int_0^{1} \nabla y G_k(z_m; z_j + t(y - z_j)) \cdot (y - z_j) \, dt
\]
\[+ \quad \int_0^{1} \nabla x G_k(z_m + t(x - z_m); z_j + t(y - z_j)) \cdot (x - z_m) \, dt \cdot (x - z_m) \, dt \] v_j(y) \, dy
\]
\[6 \text{We use the notation } v_m := u|_{D_m}, \text{ instead of } u_m := u|_{D_m}, \text{ to avoid confusion with } u_0, u_1 \text{ and } u_2 \text{ we defined before concerning the electric fields in the absence or the presence of one or two particles.}
\]
\[7 \text{The constant } \Gamma \text{ will be written as } \Gamma := \frac{i}{4} + \gamma \text{ where } \gamma \text{ is the Euler constant.} \]
\[
\omega^2 \mu_0 \sum_{j \neq m} \int_{D_j} G_k(x, y) \int_0^1 (y - z_j) \cdot \nabla \varepsilon_0(z_j + t(y - z_j)) \, dt \, v_j(y) \, dy \\
= u_0(z_m) + \int_0^1 \nabla u_0(z_m + t(x - z_m)) \cdot (x - z_m) \, dt.
\]

We assumed that all nano-particles have the same electromagnetic properties, then \( \tau_j \) is the same for every \( j \) and let us denote it by \( \tau \). Define

\[
w = \omega^2 \mu_0 \tau \left[ I - \omega^2 \mu_0 \tau A_0 \right]^{-1} (1) = \left[ \frac{1}{\omega^2 \mu_0 \tau} I - A_0 \right]^{-1} (1),
\]

and set the following notations

\[
C_m = \int_{D_m} w \, dx \quad \text{and} \quad C_m^* = C_m \left[ 1 - \left( -\frac{1}{2\pi} \log(k(z_m) + 1) \right) \right] \quad \text{and} \quad Q_m = \omega^2 \mu_0 \tau (C_m^*)^{-1} \int_{D_m} v_m \, dx.
\]

Using the definition of \( w \), and integrate \( y \) over \( D_m \), the self-adjointness of the operator \( (\lambda I - A_0) \) and we multiplying both sides of this equation by \( \omega^2 \mu_0 \tau C_m^{-1} \), we obtain

\[
Q_m - \sum_{j \neq m} G_k(z_m; z_j) \, C_j^* \, Q_j = u_0(z_m) \, \omega^2 \mu_0 \tau \, C_m^{-1} \left[ \\
+ \int_{D_m} w \int_{D_m} |x - y| \log |x - y| \, v_m(y) \, dy \, dx \\
- \tau \int_{D_m} w \int_{D_m} G_k(x, y) \int_0^1 (y - z_m) \cdot \nabla \varepsilon_0(z_m + t(y - z_m)) \, dt \, v_m(y) \, dy \, dx \\
+ \sum_{j \neq m} \int_{D_m} w \int_0^1 \nabla G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) \, dt \, dx \int_{D_j} v_j \, dy \\
+ C_m \sum_{j \neq m} \int_{D_j} \int_0^1 \nabla_y G_k(z_m; z_j + t(y - z_j)) \cdot (y - z_j) \, dt \, v_j(y) \, dy \\
+ \sum_{j \neq m} \int_{D_m} w \int_{D_j} \int_0^1 \nabla G_k(z_m + t(x - z_m); z_j + t(y - z_j)) \cdot (y - z_j) \, dt \, dx \int_{D_j} v_j \, dy \, dy dx \\
+ \tau^{-1} \sum_{j \neq m} \int_{D_m} w \int_{D_j} G_k(x, y) \int_0^1 (y - z_j) \cdot \nabla \varepsilon_0(z_j + t(y - z_j)) \, dt \, v_j(y) \, dy \, dx \\
+ (\omega^2 \mu_0 \tau)^{-1} \int_{D_m} w \int_0^1 (x - z_m) \cdot \nabla u_0(z_m + t(x - z_m)) \, dt \, dx \int_{D_m} v_m \, dx \right].
\]

For the right side, we keep \( u_0(z_m) \) as a dominant term and estimate the other terms as an error. To achieve this goal, we need the following proposition.

**Proposition 2.3.** We have:

\[
\|u\|_{L^2(D)} \leq |\log(a)|^h \|u_0\|_{L^2(D)};
\]

and

\[
C_m = O(|\log(a)|^{h-1}).
\]

**Proof.** See Section 4 \( \square \)

As the incident wave is smooth and independent on \( a \), thanks to (2.6), we get

\[
\|w\|_{L^2(D)} \leq a^{-1} |\log(a)|^{h-1}.
\]
We recall that
\[ \tau \sim 1/a^2 \log(a). \]
The error part contains eight terms. Next we define and estimate every term, then we sum them up. More precisely, we have

- **Estimation of** \( S_1 := \tau C_m^{-1} \int_{D_m} w \int_{D_m} |x-y| \log(|x-y|) v_m(y) dy \) \( dx \)

\[
|S_1| \lesssim a^{-2} |\log(a)|^{-1} |\log(a)|^{1-h} \|w\| \left[ \int_{D_m} \left( \int_{D_m} |x-y| \log(|x-y|) v_m(y) dy \right)^2 \right]^\frac{1}{2} \]

\[
\lesssim a^{-2} |\log(a)|^{-h} a^{-1} |\log(a)|^{h-1} \left[ \int_{D_m} \left( \int_{D_m} |v_m(y) dy \right)^2 \right]^\frac{1}{2} a |\log(a)| \]

\[ = O\left( a^{-2} |\log(a)|^{-1} a \|v_m\| a |\log(a)| \right), \]

and then

\[ S_1 = O\left( a |\log(a)|^h M \right). \]

- **Estimation of** \( S_2 := C_m^{-1} \int_{D_m} w(x) \int_{D_m} G_k(x, y) \int_0^1 (y - z_m) \cdot \nabla \varepsilon_0(y - z_m) dt v_m(y) dy \) \( dx \)

\[
|S_2| \lesssim a^{-1} \left[ \int_{D_m} \left( \int_{D_m} |G_k(x, y)| \int_0^1 (y - z_m) \cdot \nabla \varepsilon_0(y - z_m) dt \right. \left. |v_m(y) dy \right)^2 \right]^\frac{1}{2} . \]

The smoothness of \( \varepsilon_0 \) implies

\[ |S_2| \lesssim \|v_m\| \left[ \int_{D_m} \int_{D_m} |G_k|^2(x, y) dy \right] \lesssim \|u\| a^2 |\log(a)|, \]

and then

\[ S_2 = O\left( a^3 |\log(a)|^{1+h} M \right). \]

- **Estimation of** \( S_3 := \tau C_m^{-1} \sum_{j \neq m} \int_{D_m} w \int_0^1 \nabla_x G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) dt dx \int_{D_j} v_j dy \)

\[
|S_3| \lesssim \frac{1}{a |\log(a)|^h} \sum_{j \neq m} \|w\| \|v_j\| \left[ \int_{D_m} \left( \int_0^1 \nabla_x G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) dt \right)^2 \right]^\frac{1}{2} . \]

Without difficulties, we can check that

\[ \left[ \int_{D_m} \left( \int_0^1 \nabla_x G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) dt \right)^2 \right]^\frac{1}{2} \lesssim \frac{a^2}{d_m^2}, \]

then we plug this on the previous equation and use Cauchy-Schwarz inequality, to get

\[ |S_3| \lesssim |\log(a)|^{-1} \|v\| \left( \sum_{j \neq m} \frac{1}{d_m^2} \right)^\frac{1}{2} \lesssim |\log(a)|^{h-1} a M d^{-1}. \]

Set

\[ S_4 := \tau \sum_{j \neq m} \int_{D_j} \int_0^1 \nabla_y G_k(z_m; z_j + t(y - z_j)) \cdot (y - z_j) dt v_j(y) dy, \]

and remark that \( S_4 \) has a similar expression as \( S_3 \), then we obtain:

\[ S_3 = O(|\log(a)|^{h-1} a M d^{-1}) \quad \text{and} \quad S_4 = O(|\log(a)|^{h-1} a M d^{-1}). \]
• Estimation of

\[ S_5 := \frac{\tau}{C_m} \sum_{j \neq m} \int_{D_m} w \int_{D_j} \int_{0}^{1} \int_{0}^{1} \nabla G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt v_j(y) dy dx \]

\[ |S_5| \lesssim \frac{\|u\|}{a^2 |\log(a)|^h} \sum_{j \neq m} \left| \int_{D_m} \int_{D_j} \int_{0}^{1} \int_{0}^{1} \nabla G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt v_j(y) dy \right| \]

\[ \lesssim \frac{a^{-3}}{|\log(a)|} \sum_{j \neq m} \|v_j\| \left( \int_{D_m} \int_{D_j} \int_{0}^{1} \nabla G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt dy dx \right)^{\frac{1}{2}} \]

we have

\[ \int_{D_m} \int_{D_j} \int_{0}^{1} \nabla G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt dy dx \lesssim O\left(\frac{a^8}{d_{mj}}\right), \]

hence

\[ |S_5| \lesssim a |\log(a)|^{-1} \|u\| \left( \sum_{j \neq m} \frac{1}{d_{mj}} \right)^{\frac{1}{2}} \lesssim a^2 |\log(a)|^{-1} M d^{-2}, \]

then

\[ S_5 = O(a^2 |\log(a)|^{-1} M d^{-2}). \]

• Estimation of \( S_6 := C_m^{-1} \sum_{j \neq m} \int_{D_m} w \int_{D_j} G_k(x, y) \int_{0}^{1} (y - z_j) \cdot \nabla z_0(z_j + l(y - z_j)) \cdot v_j(y) dy dx \]

\[ |S_6| \lesssim \sum_{j \neq m} \|v_j\| \left( \int_{D_m} \int_{D_j} |G_k|^2(x; y) dy dx \right)^{\frac{1}{2}} \lesssim a^2 \|v\| M^{\frac{1}{2}} \lesssim a^3 |\log(a)|^{h} M. \]

Then

\[ S_6 = O\left(\frac{a^3}{|\log(a)|^h} M\right). \]

• Estimation of \( S_7 := C_m^{-1} \int_{D_m} w \int_{0}^{1} (x - z_m) \cdot \nabla u_0(\bar{z}_m + t(x - z_m)) dt dx \]

\[ |S_7| \lesssim |\log(a)|^{-h} \|u\| \left( \int_{D_m} \int_{0}^{1} (x - z_m) \cdot \nabla u_0(\bar{z}_m + t(x - z_m)) dt dy dx \right)^{\frac{1}{2}}. \]

As \( u_0 \) is smooth, we have

\[ \left( \int_{D_m} \int_{0}^{1} (x - z_m) \cdot \nabla u_0(\bar{z}_m + t(x - z_m)) dt dy dx \right)^{\frac{1}{2}} = O(a). \]

Hence

\[ S_7 = O(a). \]

• Estimation of \( S_8 := a \tau \int_{D_m} v_m \]

\[ |S_8| \leq \tau a \|v\| \|v_m\| = O(a |\log(a)|^{-h-1}). \]

Finally, the error part is

\[ Error = S_1 + \cdots + S_8 = O(a d^{-1} |\log(a)|^{-1} M). \]

**Proposition 2.4.** The vector \( Q_m \) satisfy the following algebraic system

(2.7) \[ Q_m - \sum_{j \neq m} G_k(z_m; \bar{z}_j) C_j^* Q_j = u_0(z_m) + O(a d^{-1} |\log(a)|^{-1} M). \]
The algebraic system can be written, in a matrix form, as
\[
(I - B) Q = V + Err
\]
with \( B = \left( B_{mj} \right)_{m,j=1}^{M} \) such that \( B_{mj} := G_k(z_m; z_j) \) and \( V := (u_0(z_1), \cdots, u_0(z_M))^T \).

In the next proposition, we give conditions under which the linear system \((2.8)\) is invertible.

**Lemma 2.5.** The algebraic system \((2.8)\) is invertible if
\[
d > \exp \left( - \left| \log(a) \right|^{1-k} \right),
\]
where \( d \) is the minimal distance between the particles.

**Proof. of Lemma 2.5.** Let us evaluate the norm of \( B \). For this we have:
\[
\|B\| = \max_m \sum_{j \neq m} |B_{mj}| = \max_m \sum_{j \neq m} \left| G_k(z_m; z_j) \left[ C_j^{-1} - \left( -\frac{1}{2\pi} \log(k)(z_j) + \Gamma \right) \right]^{-1} \right|
\]
\[
\leq \left| \log(a) \right|^{h-1} \sum_{j \neq m} \log \left( \frac{1}{d_{mj}} \right).
\]
We need the following lemma

**Lemma 2.6.** We have
\[
\sum_{j \neq m} \log(1/d_{mj}) = \log(1/d).
\]

**Proof. of Lemma 2.6**
We set \( \log(1/d_{mj}) = 1/l_{mj} \) and \( l = \min l_{mj} \). Then
\[
\sum_{j \neq m} \log \left( \frac{1}{d_{mj}} \right) = \sum_{j \neq m} \frac{1}{l_{mj}} \overset{(*)}{=} \frac{1}{l},
\]
At first we assume that (*) is checked. Then we have
\[
l = \min_{j \neq m} \frac{1}{\log(1/d_{mj})} = \frac{1}{\log(\max(1/d_{mj}))} = \frac{1}{\log(1/(\min d_{mj}))} = \frac{1}{\log(1/d)}.
\]
Then \((2.11)\) combined with \((2.12)\) give a justification of \((2.10)\).
Now, in order to prove (*) we modify to the two dimensional case the proof, done for three dimensional case, given in (2, page 13). We get
\[
\sum_{i=1}^{M} \frac{1}{l_{ik}} = \begin{cases} 
\mathcal{O}(l^{-k}) + \mathcal{O}(l^{-2\alpha}) & \text{if } k < 2 \\
\mathcal{O}(l^{-2}) + \mathcal{O}(l^{-2\alpha} \log(l)) & \text{if } k = 2 \\
\mathcal{O}(l^{-k}) + \mathcal{O}(l^{-\alpha k}) & \text{if } k > 2.
\end{cases}
\]

Based on lemma \(2.6\) the condition \( \|B\| < 1 \), is fulfilled if
\[
\log(1/d) < \left| \log(a) \right|^{1-h} \quad \text{or} \quad d > \exp \left( - \left| \log(a) \right|^{1-h} \right).
\]
2.2. Inversion of the derived Foldy-Lax algebraic system \((2.7)\). Here, we deal with the case of two particles, i.e. \(M = 2\). In the equation \((2.7)\) we use the condition \(d \sim a^{\log(a)-h}\), then we get
\[
\begin{cases}
Q_1 - G_k(z_1; z_2) C_2^* Q_2 = u_0(z_1) + O(a^{1-\log(a)-h} |\log(a)|^{-1}), \\
Q_2 - G_k(z_2; z_1) C_1^* Q_1 = u_0(z_2) + O(a^{1-\log(a)-h} |\log(a)|^{-1}).
\end{cases}
\]
We check that the condition \(d \sim a^{\log(a)-h}\) is sufficient for the invertibility of the last system. For this, we have from \((2.13)\)
\[
d > \exp(-|\log(a)|^{-1}) = \left(\frac{1}{a^{\log(a)}}\right)^{-1-h} = a^{\log(a)-h}.
\]
Now, we assume that \({\mathbf{C}_1 = C_2 = C}\) and use the expansion of \(G_k(z_m; z_j)\), see \((2.1)\), to obtain
\[
\begin{cases}
Q_1 - \left[\Phi_0(z_1; z_2) - \frac{1}{2\pi} \log(k)(z_1) + \Gamma\right] C_2^* Q_2 = u_0(z_1) + O(a^{1-\log(a)-h}) + O(d |\log(d)| C_2^* Q_2), \\
Q_2 - \left[\Phi_0(z_2; z_1) - \frac{1}{2\pi} \log(k)(z_2) + \Gamma\right] C_1^* Q_1 = u_0(z_2) + O(a^{1-\log(a)-h}) + O(d |\log(d)| C_1^* Q_1).
\end{cases}
\]
We can estimate
\[
d |\log(d)| C_i^* Q_i = O(a^{\log(a)-h}), \quad \text{for} \quad i = 1, 2,
\]
because, by the definition of \(Q_i\), see \((2.5)\), we have
\[
d |\log(d)| C_i^* Q_i = d |\log(d)| C_i^* \omega^2 \mu_0 \tau (C_i^*)^{-1} \int_{D_i} v \, dx \lesssim d |\log(d)| |\tau| |1| |\|u||.
\]
The value of \(d\) and \(|\tau|\) are known, and we have an a priori estimate about \(|u||\) given by \((2.6)\), then
\[
O(d |\log(d)|) C_i^* Q_i \lesssim a^{\log(a)-h} |\log(a)|^{-1-h} a^{-2} |\log(a)|^{-1} a |\log(a)|^{h} a = a^{\log(a)-h}.
\]
This proves \((2.14)\).

With these estimations the last system can be written as
\[
\begin{cases}
Q_1 - \left[\Phi_0(z_1; z_2) - \frac{1}{2\pi} \log(k)(z_1) + \Gamma\right] C_2^* Q_2 = u_0(z_1) + O(d), \\
Q_2 - \left[\Phi_0(z_2; z_1) - \frac{1}{2\pi} \log(k)(z_2) + \Gamma\right] C_1^* Q_1 = u_0(z_2) + O(d).
\end{cases}
\]
We need the following lemma to simplify the last system.

**Lemma 2.7.** Since \(k\) is \(C^1\)-smooth and \(z_1\) is close to \(z_2\) at a distance \(d\), we obtain
\[
|\log(k)(z_2)| = |\log(k)(z_1)| + O(d) \quad \text{and} \quad C_2^* = C_1^* + O(d C^2).
\]

**Proof.** Use Taylor expansion of the function \(k\) to get the first equality. Now the first one is proved, we use the definition of \(C_1^*\) and the fact that \(C_1 = C_2\) to obtain the second equality. \(\square\)

We use the last lemma to write the system \((2.15)\) as
\[
\begin{cases}
Q_1 - \left[\Phi_0(z_1; z_2) - \frac{1}{2\pi} \log(k)(z_1) + \Gamma\right] C_1^* Q_2 = u_0(z_1) + O(d), \\
Q_2 - \left[\Phi_0(z_2; z_1) - \frac{1}{2\pi} \log(k)(z_2) + \Gamma\right] C_1^* Q_1 = u_0(z_2) + O(d).
\end{cases}
\]

**Remark 2.8.** To simplify notations, we write \(\Phi_0\) (respectively \(\frac{1}{2\pi} \log(k)\), \(C^*\)) instead of \(\Phi_0(z_1; z_2)\) (respectively \(\frac{1}{2\pi} \log(k(z_1))\), \(C_1^*\)).

\(^{8}\text{Remember that we assumed that all nano-particles have the same electromagnetic properties.}\)
After resolution of this algebraic system, we obtain

\[
\begin{align*}
Q_1 &= \frac{u_0(z_1)}{1 - \left[\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right] C^*} + O(d), \\
Q_2 &= \frac{u_0(z_2)}{1 - \left[\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right] C^*} + O(d).
\end{align*}
\]

We use the definition of \(Q_{1,2}\), see (2.15), to get

\[
\int_{D_1} v \, dx = \frac{u_0(z_1)}{\omega^2 \mu_0 \tau \left[\left(C^*\right)^{-1} - \left(\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right)\right]} + O(d a^2 |\log(a)|^h).
\]

Then

\[
\int_{D_1} v \, dx = \frac{u_0(z_2)}{\omega^2 \mu_0 \tau \left[\left(C^*\right)^{-1} - \left(\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right)\right]} + O(d a^2 |\log(a)|^h),
\]

we estimate the term

\[
\frac{u_0(z_1) - u_0(z_2)}{\omega^2 \mu_0 \tau \left[\left(C^*\right)^{-1} - \left(\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right)\right]}
\]

as \(O(d a^2 |\log(a)|^h)\), and use this to obtain

\[
\int_{D_1} v \, dx = \frac{u_0(z_2)}{\omega^2 \mu_0 \tau \left[\left(C^*\right)^{-1} - \left(\Phi_0 + \left(\frac{1}{2\pi \log(k)} + \Gamma\right)\right)\right]} + O(d a^2 |\log(a)|^h)
\]

\[
= \int_{D_2} v \, dx + O(d a^2 |\log(a)|^h),
\]

and finally

(2.16) \[
\int_{D_1} v \, dx = \int_{D_2} v \, dx + O(d a^2 |\log(a)|^h).
\]

By adding the two equations of system (2.15), we get

(2.17) \[
C^{-1} - \left(\Phi_0 + 2(- \log(k)/2\pi + \Gamma)\right) = \frac{u_0(z_1) + u_0(z_2)}{\omega^2 \mu_0 \tau \left[\int_{D_1} v \, dy + \int_{D_2} v \, dy\right]} + \frac{O(d \tau^{-1})}{\int_{D_1} v \, dy + \int_{D_2} v \, dy}.
\]

We use equation (2.16) to rewrite the denominator as

\[
\omega^2 \mu_0 \tau \left[\int_{D_1} v \, dy + \int_{D_2} v \, dy\right] = \omega^2 \mu_0 \tau \left[2 \int_{D_2} v \, dy + O(d a^2 |\log(a)|^h)\right]
\]

\[
= 2 \omega^2 \mu_0 \tau \int_{D_2} v \, dy \left[1 + \frac{O(d a^2 |\log(a)|^h)}{\int_{D_2} v \, dy}\right]
\]

\[
= 2 \omega^2 \mu_0 \tau \int_{D_2} v \, dy \left[1 + O(d)\right],
\]

then equation (2.17) takes the form

(2.17) \[
C^{-1} - \left(\Phi_0 + 2(- \log(k)/2\pi + \Gamma)\right) = \frac{u_0(z_1) + u_0(z_2)}{2 \omega^2 \mu_0 \tau \left[\int_{D_2} v \, dy\right] \left[1 + O(d)\right]} + \frac{O(d \tau^{-1})}{\int_{D_2} v \, dy \left[1 + O(d)\right]},
\]
We manage the errors

\[
C^{-1} \left( \Phi_0 + 2(-\log(k)/2\pi + \Gamma) \right) = \frac{u_0(z_1) + u_0(z_2)}{2\omega^2 \mu_0 \tau \int_{D_2} v \, dy} + \mathcal{O}(d |\log(a)|^{1-h})
\]

\[
= \frac{2u_0(z_2)}{2\omega^2 \mu_0 \tau \int_{D_2} v \, dy} + \frac{\int_{0}^{1}(z_1 - z_2) \cdot \nabla u_0(z_2 + t(z_1 - z_2))dt}{2\omega^2 \mu_0 \tau \int_{D_2} v \, dy} + \mathcal{O}(d |\log(a)|^{1-h})
\]

\[
= \frac{u_0(z_2)}{\omega^2 \mu_0 \tau \int_{D_2} v \, dy} + \mathcal{O}(d |\log(a)|^{1-h}),
\]

and take the modulus, we derive the identity:

\[
(2.20) \quad \left| C^{-1} \left( \Phi_0 + 2(-\log(k)/2\pi + \Gamma) \right) \right|^2 = \frac{|u_0(z_2)|^2}{|\omega^2 \mu_0 \tau|^2 \left| \int_{D_2} v \, dy \right|^2} + \mathcal{O}(d |\log(a)|^{2(1-h)}).
\]

Unfortunately, from the acoustic inversion, we get only data of the form \( \int_{D_{1,2}} |v|^2 dx \) and in the last equation we deal with \( |\int_{D_{1,2}} v \, dx|^2 \). The next lemma makes a link between these two quantities.

**Lemma 2.9.** We have

\[
(2.19) \quad \left| \int_{D_1} v \, dy \right|^2 = a^2 \left( \int_B \tau_{n_0} \, d\eta \right)^2 \int_{D_1} |v|^2 \, dy + \mathcal{O}(a^4 |\log(a)|^h), \quad i = 1, 2.
\]

**Proof.** We split the proof into two steps.

Step 1: Estimation of \( |\int_{D_1} v \, dy|^2 \).

We use the same techniques as in the proof of the a priori estimation i.e proposition 2.3. We have

\[
\int_{D_1} v \, dy = <v; e^{(1)}_{n_0}> \int_D e_{n_0} \, dx + a^2 \sum_{n \neq n_0} <\hat{v}; \tau_n> \int_B \tau_n \, d\eta
\]

\[
<\hat{v}; e^{(1)}_{n_0}> \int_D e_{n_0} \, dx + \mathcal{O}(a^2) \sum_{n \neq n_0} \left[ \frac{<\hat{u}_0; \tau_n>}{(1 - \omega^2 \mu_0 \tau \lambda_n)} + O(|\log(a)|^{-h}) <1, \tau_n>< \int_B \tau_n \, d\eta \right].
\]

When the used frequency is not close to the resonance the following estimation holds

\[
\sum_{n \neq n_0} \left[ \frac{<\hat{u}_0; \tau_n>}{(1 - \omega^2 \mu_0 \tau \lambda_n)} + O(|\log(a)|^{-h}) <1, \tau_n>< \int_B \tau_n \, d\eta \right] \int_B \tau_n \, d\eta \sim O(1),
\]

and plug this in the previous equation to obtain

\[
\int_{D_1} v \, dy \quad a^2 \left[ \frac{<\hat{u}_0; e^{(1)}_{n_0}>}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) - \omega^2 \mu_0 \tau a^2 \Phi_0 (\int_B \tau_{n_0})^2} + O(1) \right] \int_B \tau_{n_0} \, d\eta + O(a^2).
\]

Then

\[
(2.20) \quad |\int_{D_1} v \, dy|^2 = a^4 \frac{|<\hat{u}_0; e^{(1)}_{n_0}>|^2}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) - \omega^2 \mu_0 \tau a^2 \Phi_0 (\int_B \tau_{n_0})^2} \left( \int_B \tau_{n_0} \, d\eta \right)^2 + O(a^4 |\log(a)|^h).
\]
Step 2: Estimation of $\int_{D_1} |v|^2 \, dy$.

We have

$$\int_{D_1} |v|^2 \, dx = \sum_n |<v, e_n^{(1)}>|^2 = a^2 \left( |<\tilde{v}^1, \tilde{\tau}_{n_0}>|^2 + \sum_{n \neq n_0} |<\tilde{v}^1, \tilde{\tau}_n>|^2 \right) = a^2 |<\tilde{v}^1, \tilde{\tau}_{n_0}>|^2 + \mathcal{O}(a^2)\quad(2.20)$$

We continue with equation (2.20) and (2.21), we obtain

$$\left| \int_{D_1} v \, dy \right|^2 = a^4 \left( |<\tilde{\omega}_0;\tilde{\tau}_n^{(1)}>|^2 \right) \left( 1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau a^2 \Phi_0(z_1, z_2) \left( \int_B \tilde{\tau}_{n_0} \, d\eta \right)^2 \right)^2 + \mathcal{O}(a^4 |\log(a)|^h)$$

Combining (2.20) and (2.21), we obtain

$$\left| \int_{D_1} v \, dy \right|^2 = a^4 \left( |<\tilde{\omega}_0;\tilde{\tau}_n^{(1)}>|^2 \right) \left( 1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau a^2 \Phi_0(z_1, z_2) \left( \int_B \tilde{\tau}_{n_0} \, d\eta \right)^2 \right)^2 + \mathcal{O}(a^4 |\log(a)|^h)$$

which proves the formula (2.19).

We continue with equation (2.18), then

$$\left| C^{-1} - \left( \Phi_0 + 2(-\log(k)/2\pi + \Gamma) \right) \right|^2 = \frac{|u_0(z_2)|^2}{|\omega^2 \mu_0 \tau|^2} \left( \int_{D_2} v \, dy \right)^2 + \mathcal{O}(d |\log(a)|^{2(1-h)})$$

Combining (2.18) and (2.19), we obtain

$$\left| C^{-1} - \left( \Phi_0 + 2(-\log(k)/2\pi + \Gamma) \right) \right|^2 = \frac{|u_0(z_2)|^2}{|\omega^2 \mu_0 \tau|^2} \left( \int_{D_2} v \, dy \right)^2 + \mathcal{O}(d |\log(a)|^{2(1-h)})$$

In the following proposition, we write an estimation of $|u_0(z_2)|$ in the case of one particle inside the domain.

**Proposition 2.10.** We have

$$|u_0(z_2)|^2 = \frac{1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D \tilde{e}_{n_0} \right)^2}{\left( \int_D \tilde{e}_{n_0} \right)^2} \left( \int_D |u_1|^2 \, dx + \mathcal{O}(|\log(a)|^{\max(-2h,-1)}) \right).$$
Proof. To fix notations recall L.S.E for one particle
\[ u_1(x) - \omega^2 \mu_0 \int_D G_k(x, y) (\varepsilon_p - \varepsilon_0)(y) u_1(y) \, dy = u_0(x), \quad x \in D. \]
With this notation the equation (4.14) takes the following form
\[
< u_1; e_{n_0} > = \frac{< u_0; e_{n_0} >}{1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} \right)^2} + O(a | \log(a)|^{h-1}).
\]
Next,
\[
\int_D |u_1|^2 \, dx = |< u_1; e_{n_0} >|^2 + a^2 \sum_{n \neq n_0} |< \hat{u}_1; \tau_n >|^2
\]
\[
\int_D |u_1|^2 \, dx = \left[ |u_0(z_2)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(a^3) \right] \quad 1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} \right)^2
\]
\[
= \frac{|u_0(z_2)|^2 \left( \int_D e_{n_0} \, dx \right)^2}{1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} \right)^2} + O(a^2 \log(a)^{\max(0,2h-1)}).
\]
We develop \( u_0 \) near the point \( z \) to obtain
\[
\int_D |u_1|^2 \, dx = \left| 1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} \right)^2 \right|^2.
\]
This proves (2.22). \( \square \)

In (2.22), we use the following notation
\[
\Psi := \left| 1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} \right)^2 \right|^2.
\]
With this, we get
\[
\left| C^{-1} - \left( \Phi_0 + 2 \left( \frac{-1}{2\pi} \log(k) + \Gamma \right) \right) \right|^2 = \frac{|u_0(z_2)|^2}{\omega^2 \mu_0 \tau^2 \left( \int_D e_{n_0} \, dy \right)^2 \int_{D_2} |v|^2 \, dy} \quad 2.22
\]
\[
\Psi \int_D \frac{|u_1|^2 \, dx}{\omega^2 \mu_0 \tau^2 \left( \int_D e_{n_0} \, dy \right)^4 \int_{D_2} |v|^2 \, dy} + O(|\log(a)|^{2-3h}).
\]
We set
\[
(2.25) \quad B := \int_D |u_1|^2 \, dx \quad \frac{\omega^2 \mu_0 \tau^2 \left( \int_D e_{n_0} \, dy \right)^4 \int_{D_2} |v|^2 \, dy}{\omega^2 \mu_0 \tau^2 \left( \int_D e_{n_0} \, dy \right)^4 \int_{D_2} |v|^2 \, dy}
\]
Referring to (2.23), we set \( \Gamma := \gamma + i/4 \). We develop the left side of the last equation as
\[
\left| C^{-1} - \left( \Phi_0 + 2 \left( \frac{-1}{2\pi} \log(k) + \Gamma \right) \right) \right|^2 = \left( C^{-1} - \Phi_0 \right)^2 - 4 \left( C^{-1} - \Phi_0 \right) \left( \frac{-1}{2\pi} \log |k| + \gamma \right)
\]
Using (2.25), we have

\[
(C^{-1} - \Phi_0)^2 - 4 (C^{-1} - \Phi_0) \left( \frac{-1}{2\pi} \log |k| + \gamma \right) + 4 \left( \frac{-1}{2\pi} \log |k| + \gamma \right)^2 + 4 \left( \frac{-1}{2\pi} \text{Arg}(k) + \frac{1}{4} \right)^2
\]

(2.26)

then, we have

\[
\left( C^{-1} - \Phi_0 \right)^2 - 4 \left( C^{-1} - \Phi_0 \right) \left( \frac{-1}{2\pi} \log |k| + \gamma \right) + 4 \left( \frac{-1}{2\pi} \log |k| + \gamma \right)^2 + 4 \left( \frac{-1}{2\pi} \text{Arg}(k) + \frac{1}{4} \right)^2
= \Psi B + \mathcal{O}(\log(a)^{2-3k}).
\]

Remark that \( \Psi \) can be written as

\[
\Psi = \left| 1 - \omega^2 \mu_0 \tau \lambda_n_0 \right|^2 + (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 \left[ \left( \frac{-1}{2\pi} \log |k| + \gamma \right)^2 + \left( \frac{-1}{2\pi} \text{Arg}(k) + \frac{1}{4} \right)^2 \right]
- 2\omega^2 \mu_0 \left( \int_D e_{no} \right)^2 \left( \frac{-1}{2\pi} \log |k| + \gamma \right) \text{Re} \left[ \tau (1 - \omega^2 \mu_0 \tau \lambda_n_0) \right] + \mathcal{O}(a^2).
\]

Hence using (4.25), we have

\[
\Psi = C^{-2} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 + (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 \left[ \left( \frac{-1}{2\pi} \log |k| + \gamma \right)^2 + \left( \frac{-1}{2\pi} \text{Arg}(k) + \frac{1}{4} \right)^2 \right]
- 2C^{-1} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 \left( \frac{-1}{2\pi} \log |k| + \gamma \right) + \mathcal{O}(\log(a)^{-3k}).
\]

Replace \( \Psi \) in (2.26) and use the fact that \( B = \mathcal{O}(\log(a)^2) \) to cancel all the terms of order \( \mathcal{O}(1) \). The formula (2.26) will be

\[
(C^{-1} - \Phi_0)^2 - 4 (C^{-1} - \Phi_0) \left( \frac{-1}{2\pi} \log |k| + \gamma \right) = -2C^{-1} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 \left( \frac{-1}{2\pi} \log |k| + \gamma \right) B
+ C^{-2} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 B + \mathcal{O}(\log(a)^{\max(0,2-3k)}).
\]

Then

\[
\left( \frac{-1}{2\pi} \log |k| + \gamma \right) \left[ -4 \left( C^{-1} - \Phi_0 \right) + 2C^{-1} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 B \right] = C^{-2} (\omega^2 \mu_0)^2 |\tau|^2 \left( \int_D e_{no} \right)^4 B
- \left( C^{-1} - \Phi_0 \right)^2 + \mathcal{O}(\log(a)^{\max(0,2-3k)}).
\]

Using (2.25), we get an explicit expression

\[
\log |k| = 2\pi \gamma - \frac{\pi}{C} \left( \frac{\int_D |u_1|^2 dx}{\int_{D_2} |v|^2 dx} - (1 - C \Phi_0)^2 \right)
- \frac{\pi}{C} \left( \frac{\int_D |u_1|^2 dx}{\int_{D_2} |v|^2 dx} - 2(1 - C \Phi_0) \right)
+ \mathcal{O}(\log(a)^{\max(h-1,1-2h)}).
\]
Remark 2.11. To justify that (2.27) is well defined, we use (2.24), (2.21) and (4.25) to obtain the following relation

\[
\int_D |u_1|^2 \, dx = \frac{1 - C \Phi_0}{1 - C \left( \frac{1}{2\pi} \log(k) + \Gamma \right)^2} + O\left( |\log(a)|^{\max(h-1, -h)} \right).
\]

Hence,

\[
\frac{\int_D |u_1|^2 \, dx}{\int_{D_2} |v|^2 \, dx} \quad \frac{(1 - C \Phi_0)^2}{\int_{D_2} |v|^2 \, dx} = \left( 1 - C \Phi_0 \right) C \left\{ 2 \Re \left[ \frac{1}{2\pi} \log(k) + \Gamma \right] - C \left| \frac{1}{2\pi} \log(k) + \Gamma \right|^2 \right\} \sim O(C).
\]

Therefore the error term in (2.27) is indeed negligible as soon as \( \frac{1}{2} < h < 1 \).

Taking the exponential in both side of (2.27) and using the smallness of \( O\left( |\log(a)|^{\max(1-2h, h-1)} \right) \), we write

\[
|k| = \exp \left\{ 2\pi \gamma - \frac{\pi}{C} \frac{\int_D |u_1|^2 \, dx - (1 - C \Phi_0)^2}{\int_{D_2} |v|^2 \, dx} - 2 \frac{(1 - C \Phi_0)}{\int_{D_2} |v|^2 \, dx} \right\} + O\left( |\log(a)|^{\max(h-1,1-2h)} \right).
\]

3. Proof of Theorem 1.2

We recall the model problem for photo-acoustic imaging:

\[
\begin{align*}
\partial_t^2 p(x, t) - \Delta_x p(x, t) &= 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\
p(x, 0) &= \frac{\omega \beta_0}{\varepsilon_0} \Im(\varepsilon)(x) |u|^2(x) \chi_{\Omega}, \quad \text{in } \mathbb{R}^2, \\
\partial_t p(x, 0) &= 0 \quad \text{in } \mathbb{R}^2.
\end{align*}
\]

Remark 3.1. Next, when we solve the equation (3.1), we omit the multiplicative term \( \frac{\omega \beta_0}{2\pi \varepsilon_0} \).

3.1. Photo-acoustic imaging using one particle. Proof of (1.18).

The next lemma gives an estimation of the total field for \( x \in \Omega \setminus D \).

Lemma 3.2. The total field behaves as

\[
|u_1(x)|^2 = O(1) + O(\max(h-1,1-2h)|\log(dist)|) \quad \text{in } \Omega \setminus D,
\]

where \( dist = dist(x, D) \).

---

9The constant \( 2\pi \) in the denominator comes from the Poisson formula.
This proves (3.2).

Proof. We use L.S.E

(3.3) \( u_1(x) = u_0(x) + \omega^2 \mu_0 \int_D (\varepsilon_p - \varepsilon_0)(y)G_k(y, x)u_1(y)dy, \quad x \in \mathbb{R}^2. \)

Now, for \( x \) away from \( D \)

\[
|u_1(x)| \leq |u_0(x)| + \mathcal{O}\left( \frac{1}{a^2 \log(a)} \left| \int_D |G_0|(y, x) |u_1(y)| dy \right| \right) = \mathcal{O}(1) + \mathcal{O}\left( \frac{1}{a^2 \log(a)} ||u_1|| \left( \int_D |G_0|^2(y, x) dy \right)^{1/2} \right) = \mathcal{O}(1) + \mathcal{O}\left( |\log(a)|^{2h-1} |\log(\text{dist})| \right)
\]

This proves (3.2). \( \square \)

Let us recall from proposition (4.1), the following relation

(3.4) \( <u_1, e_{n_0}> = \frac{1}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} <u_0, e_{n_0}> + \mathcal{O}(a|\log(a)|^{2h-1}). \)

We use Poisson’s formula to solve the system (3.1), see ([21], Chapter 9), to represent the pressure as follows

\[
p(t, x) = \partial_t \int_{|x-y|<t} \frac{(\text{Im} \varepsilon_p) |u_1|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{D, dy} + \partial_t \int_{|x-y|<t} \frac{(\text{Im} \varepsilon_0) |u_1|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{\Omega, dy}
\]

Let \( t > \text{diam}(\Omega) \). For \( x \in \partial \Omega \), the representation above reduces to:

\[
p(t, x) = \int_{D(z, a)} \partial_t \frac{(\text{Im} \varepsilon_p - \varepsilon_0) |u_1|^2(y)}{\sqrt{t^2 - |x-y|^2}} dy + \int_{\Omega} \partial_t \frac{(\text{Im} \varepsilon_0) |u_1|^2(y)}{\sqrt{t^2 - |x-y|^2}} dy.
\]

Set \( T_4 \) to be

\[
T_4 := \int_{\Omega} \partial_t \frac{(\text{Im} \varepsilon_0) |u_1|^2(y)}{\sqrt{t^2 - |x-y|^2}} dy.
\]

Recalling that \( \tau := \varepsilon_p - \varepsilon_0(z) \), we have

\[
p(t, x) = -t \text{Im} \tau \int_{D(z, a)} \frac{|u_1|^2(y)}{(t^2 - |x-y|^2)^{3/2}} dy + T_4 + \int_{D(z, a)} |u_1|^2(y) \partial_t \frac{\text{Im} \int_0^1 (y-z) \cdot \nabla \varepsilon_0(z + s(y-z)) ds}{\sqrt{t^2 - |x-y|^2}} dy.
\]

We estimate the remainder term as follows

(3.5) \[
\left| \int_{D(z, a)} |u_1|^2(y) \partial_t \frac{\text{Im} \int_0^1 (y-z) \cdot \nabla \varepsilon_0(z + s(y-z)) ds}{\sqrt{t^2 - |x-y|^2}} dy \right| \leq a ||u_1||_{L^2(D)} = \mathcal{O}(a^3 |\log(a)|^{2h}),
\]

then

\[
p(t, x) = -t \text{Im} \tau \int_{D(z, a)} \frac{|u_1|^2(y)}{(t^2 - |x-y|^2)^{3/2}} dy + T_4 + \mathcal{O}(a^3 |\log(a)|^{2h}).
\]

By Taylor expansion of the function \( (t^2 - |x-z|^2)^{-3/2} \) near \( z \), we have

\[
p(t, x) = -t \text{Im} \tau \int_{D(z, a)} \frac{|u_1|^2(y)}{(t^2 - |x-z|^2)^{3/2}} dy + T_4 + \mathcal{O}(a^3 |\log(a)|^{2h}) + \mathcal{O} \left( \text{Im} \tau \int_{D(z, a)} (|y-z|^2 + 2 < x-z; z-y >) |u_1|^2(y) dy \right).
\]
We estimate the remainder term as

\[ \text{Im}(\tau) \int_D (|y-z|^2 + 2 < x-z; z-y>)|u_1|^2(y)dy \leq \text{Im}(\tau) a \|u_1\|^2 = \mathcal{O}(\text{Im}(\tau)a^3 \log(a)^{2h}), \]

and then

\[ p(t, x) = \frac{-t \text{Im}(\tau)}{(t^2 - |x-z|^2)^{3/2}} \int_{D(z,a)} |u_1|^2(y)dy + T_4 + \mathcal{O}(a^3 \text{Im}(\tau) |\log(a)|^{2h}). \]

Writing \( u_1 \) as a Fourier series over the basis \( (e_n)_{n \in \mathbb{N}} \), we obtain

\[ p(t, x) = \frac{-t \text{Im}(\tau)}{(t^2 - |x-z|^2)^{3/2}} \sum_{n \neq n_0} | < u_1; e_n > |^2 + T_4 + \mathcal{O}(a^3 \text{Im}(\tau) |\log(a)|^{2h}), \]

since \( n \neq n_0 \) we estimate the series as

\[ \mathcal{O}(\text{Im}(\tau) \sum_{n \neq n_0} | < u_1; e_n > |^2) = \mathcal{O}(\text{Im}(\tau) a^2). \]

Next,

\[ p(t, x) = \frac{-t \text{Im}(\tau)}{(t^2 - |x-z|^2)^{3/2}} \left[ | < u_0; e_{n_0} > |^2 + \mathcal{O}(a^2 |\log(a)|^{3h-1}) \right] + T_4 + \mathcal{O}(\text{Im}(\tau) a^2), \]

hence

\[ p(t, x) = \frac{-t \text{Im}(\tau)}{(t^2 - |x-z|^2)^{3/2}} \left[ | < u_0; e_{n_0} > |^2 + T_4 + \mathcal{O}(\text{Im}(\tau) a^2) + \mathcal{O}(\text{Im}(\tau) a^2 |\log(a)|^{3h-1}) \right]. \]

In order to calculate the term \( T_4 \), we use L.S.E

\[ u_1(x) - \omega^2 \mu_0 \int_D (\varepsilon_p - \varepsilon_0(\eta))G_k(x, \eta) u_1(\eta)d\eta = u_0(x) \quad \text{in} \quad \Omega, \]

and define

\[ p_0(t, x) := \int_\Omega \partial_t \frac{1}{\sqrt{t^2 - |x-y|^2}} \text{Im}(\varepsilon_0)(y) |u_0|^2(y)dy. \]

Observe that \( p_0(t, x) \) is the measured pressure at point \( x \in \partial \Omega \) and time \( t \) when no particle is inside \( \Omega \). We set

\[ f := \partial_t \frac{1}{\sqrt{t^2 - |x-y|^2}} \text{Im}(\varepsilon_0)(y). \]

With this, we get

\[ T_4 = \int_\Omega \partial_t \frac{1}{\sqrt{t^2 - |x-y|^2}} \text{Im}(\varepsilon_0)(y) |u_1|^2(y)dy = \int_{\Omega \setminus D} f |u_1|^2(y)dy + \int_D f |u_1|^2(y)dy. \]

If we compare (3.12) to (3.2) we deduce that the term \( (*) := \int_D f |u_1|^2(y)dy \) is less dominant than the one given on \( \Omega \setminus D \). Now, since \( f \) is smooth we can estimate \( (*) \), with help of a priori estimation, as

\[ |(*)| = \left| \int_D f |u_1|^2(y)dy \right| \leq \|u_1\|^2 = \mathcal{O}(a^2 |\log(a)|^{2h}), \]

and, from L.S.E, see for instance (3.3), we can rewrite \( T_4 \) as

\[ T_4 = p_0(t, x) - \int_D f |u_0|^2(y)dy + (\omega^2 \mu_0)^2 \int_{\Omega \setminus D} f \left| \int_D (\varepsilon_p - \varepsilon_0(\eta))G_k(\eta, y) u_1(\eta)d\eta \right|^2 dy \]

\[ + 2 \omega^2 \mu_0 \text{Re} \left[ \int_{\Omega \setminus D} \frac{1}{\text{Im}(\varepsilon_0)} f \int_D (\varepsilon_p - \varepsilon_0(\eta))G_k(\eta, y) u_1(\eta)d\eta dy \right] + \mathcal{O}(a^2 |\log(a)|^{2h}). \]

The smoothness of \( u_0 \) is enough to justify the following estimation

\[ \left| \int_D f |u_0|^2(y)dy \right| \sim \mathcal{O}(a^2). \]
To finish the estimation of \( T_4 \) we still have to deal with two terms. More exactly we set
\[
S_3 := \int_{\Omega \setminus D} f \left( \int_D (\varepsilon_p - \varepsilon_0(\eta)) G_k(\eta, y) u_1(\eta) d\eta \right)^2 dy.
\]

Expanding \((\varepsilon_p - \varepsilon_0(.)\)) near \( z \), we obtain
\[
|S_3| \leq |\tau|^2 \int_{\Omega \setminus D} |f| \left( \int_D |G_k(\eta, y) u_1(\eta)| d\eta \right)^2 dy
+ \int_{\Omega \setminus D} |f| \left( \int_D \int_0^1 (z - \eta) \cdot \nabla \varepsilon_0(z + s(\eta - z)) ds G_k(\eta, y) u_1(\eta) d\eta \right)^2 dy
+ 2 \int_{\Omega \setminus D} |f| \left| \text{Re} \left[ \int_D \int_D \int_D (z - \eta) \cdot \nabla \varepsilon_0(z + s(\eta - z)) ds G_k(\eta, y) u_1(\eta) d\eta \right] \right| dy,
\]

then apply Cauchy Schwartz inequality and exchange the integration variables to obtain
\[
|S_3| \leq |\tau|^2 \|u_1\|^2 \int_D J(\eta) d\eta + O(a^2) \|u_1\|^2 \int_D J(\eta) d\eta + O(a^2) \|u_1\|^2 \int_D J(\eta) d\eta \lesssim |\tau|^2 \|u_1\|^2 \int_D J(\eta) d\eta,
\]

where \( J(\eta) := \int_{\Omega \setminus D} |f| \left| G_k(\eta, y) \right|^2 dy \).

Remark that \( J \) is a smooth function because \( f \) is a smooth and \( \eta \) and \( y \) are in two disjoint domains. Then
\[
S_3 = O(|\log(a)|^{2h-2}).
\]

The last term to estimate, that we set as \( S_4 \), is more delicate. We split it as:
\[
S_4 := 2 \omega^2 \mu_0 \text{Re} \left[ \int_{\Omega \setminus D} f \left( \varepsilon_p - \varepsilon_0(\eta) \right) G_k(\eta, y) u_1(\eta) d\eta \right] dy
= 2 \omega^2 \mu_0 \text{Re} \left[ \sum_n < u_1; e_n > \tau \int_{\Omega \setminus D} f \left( \varepsilon_p - \varepsilon_0(\eta) \right) G_k(\eta, y) e_n(\eta) d\eta \right] dy
- 2 \omega^2 \mu_0 \text{Re} \left[ \int_{\Omega \setminus D} f \left( \varepsilon_p - \varepsilon_0(\eta) \right) \int_D G_k(\eta, y) e_n(\eta) d\eta \right] dy.
\]

The same techniques, as previously, allows to estimate the second term of \( S_4 \) as \( O\left(a^4 |\log(a)|^h\right) \). Then
\[
S_4 = 2 \omega^2 \mu_0 \text{Re} \left[ < u_1; e_n > \tau \int_{\Omega \setminus D} f \left( \varepsilon_p - \varepsilon_0(\eta) \right) G_k(\eta, y) e_n(\eta) d\eta \right] dy
+ O \left( \sum_{n \neq n_0} \text{Re} \left[ < u_1; e_n > \tau \int_{\Omega \setminus D} f \left( \varepsilon_p - \varepsilon_0(\eta) \right) G_k(\eta, y) e_n(\eta) d\eta \right] \right) + O\left(a^4 |\log(a)|^h\right).
\]

We keep the term with index \( n_0 \) and estimate the series as
\[
|O(\cdots)| \lesssim |\tau| \int_{\Omega \setminus D} |f| \left| \varepsilon_0(\eta) \right| \sum_{n \neq n_0} < u_1; e_n > \left| \int_D G_k(\eta, y) e_n(\eta) d\eta \right| dy
\]
\[
\lesssim |\tau| \|u_1\| \|f \varepsilon_0(\eta)\|_{L^2(\Omega \setminus D)} \left( \int_D \int_{\Omega \setminus D} \left| G_k(\eta, y) \right|^2 dy d\eta \right)^\frac{1}{2} = O\left(|\log(a)|^{-1}\right).
\]

Plug this in the last equation to obtain
\[
S_4 \leq 2 \omega^2 \mu_0 \text{Re} \left[ < u_0; e_n > \tau \int_{\Omega \setminus D} f \varepsilon_0(\eta) \int_D G_k(\eta, y) e_n(\eta) d\eta \right] dy
\]
The equation (3.13) allows us to deduce that

\[ a \log(a)^{2h-1} \text{Re} \left[ \tau \int_{\Omega \setminus D} f \, \varpi_0(y) \int_{D} G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right] = O\left( |\log(a)|^{-1} \right), \]

The last step is to use Taylor expansion to write \( <u_0; e_{n_0}> \) on function of the center \( z \). We have

\[
S_4 = 2 \omega^2 \mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ \frac{u_0(z)}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} f \, \varpi_0(y) \int_D G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right]
+ O \left( |\log(a)|^2 \right) + O \left( |\log(a)|^{-1} \right),
\]

then we compute an estimation of the remainder term from Taylor expansion. More precisely, we have

\[
|\mathcal{O}(\cdots)| \lesssim a \log(a)^h \int_D e_{n_0} \, dx \left| \int_{\Omega \setminus D} f \, \varpi_0(y) \int_D G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right|
\leq |\log(a)|^{h-1} \int_D \int_{\Omega \setminus D} f \, \varpi_0(y) G_k(\eta, y) dy |e_{n_0}(\eta)| \, d\eta = O\left( a \log(a)^{h-1} \right).
\]

Finally,

\[
S_4 = 2 \omega^2 \mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ \frac{u_0(z)}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im}(\varepsilon_0)(y) \, \varpi_0(y)}{\sqrt{t^2 - |x - y|^2}} \int_D G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right]
+ O\left( |\log(a)|^{-1} \right) + O\left( |\log(a)|^{2h-2} \right).
\]

Hence

\[
T_4 = p_0(t, x) + S_3 + S_4
\]

\[
= p_0(t, x) + 2 \omega^2 \mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ \frac{u_0(z)}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im}(\varepsilon_0)(y) \, \varpi_0(y)}{\sqrt{t^2 - |x - y|^2}} \int_D G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right]
+ O\left( |\log(a)|^{-1} \right) + O\left( |\log(a)|^{2h-2} \right).
\]

The equation (3.8) takes the form

\[
(p - p_0)(t, x) = 2 \omega^2 \mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ \frac{u_0(z)}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im}(\varepsilon_0)(y) \, \varpi_0(y)}{\sqrt{t^2 - |x - y|^2}} \int_D G_k(\eta, y) e_{n_0}(\eta) \, d\eta \, dy \right]
- \frac{-t \text{Im} \left( \tau \right)}{(t^2 - |x - z|^2)^{3/2}} |<u_0; e_{n_0}>|^2 + O(\log(a)^{\max(2h-2, 1)} - 1) + O(\log(a)^{\max(0, 3h-1)}).
\]

Recall that we take,

\[
\text{Im} \left( \tau \right) = \frac{1}{a^2 |\log(a)|^{1+h+s}}
\]

with

\[
0 \leq s < \min(h, 1 - h).
\]
With this choice, the error part of (3.12) will be of order $O(|\log(a)|^{\max(-1.2h^{-2})})$. Hence
\[
(p - p_0)(t, x) = 2\omega^2\mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ -t \right. \\
\left. \frac{u_0(z)}{1 - \omega^2\mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im} (\varepsilon_0) (y) \overline{\mu_0 (y)} - 1}{\sqrt{t^2 - |x - y|^2}} \int_D G_k (\eta, y) e_{n_0} (\eta) \, dy \, dy \right] \\
+ \frac{-t \text{Im} (\tau)}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{1}{1 - \omega^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0, e_{n_0} > |^2 + O(|\log(a)|^{\max(-1.2h^{-2})}).
\]
Using again the estimate
\[
| < u_0, e_{n_0} > |^2 = |u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(a^3),
\]
we get
\[
(p - p_0)(t, x) = 2\omega^2\mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ -t \right. \\
\left. \frac{u_0(z)}{1 - \omega^2\mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im} (\varepsilon_0) (y) \overline{\mu_0 (y)} - 1}{\sqrt{t^2 - |x - y|^2}} \int_D G_k (\eta, y) e_{n_0} (\eta) \, dy \, dy \right] \\
+ \frac{-t \text{Im} (\tau)}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{1}{1 - \omega^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(|\log(a)|^{\max(-1.2h^{-2})}).
\]
Now, if we take two frequencies $\omega_{\pm}^2$, such that $\omega_{\pm}^2 = \omega_0^2 \pm |\log(a)|^{-h}$, we obtain
\[
(p^+ - p_0)(t, x) = 2\omega_+^2\mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ -t \right. \\
\left. \frac{u_0(z)}{1 - \omega_+^2\mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im} (\varepsilon_0) (y) \overline{\mu_0 (y)} - 1}{\sqrt{t^2 - |x - y|^2}} \int_D G_k (\eta, y) e_{n_0} (\eta) \, dy \, dy \right] \\
+ \frac{-t \text{Im} (\tau)}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{1}{1 - \omega_+^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(|\log(a)|^{\max(-1.2h^{-2})}).
\]
We use $1 - \omega_{\pm}^2\mu_0 \lambda_{n_0} \tau = 0$ to deduce that $|1 - \omega_{\pm}^2\mu_0 \lambda_{n_0} \tau| = O(|\log(a)|^{-h})$. After some simplifications we get
\[
(p^+ - p_0)(t, x) = 2\omega_\pm^2\mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ -t \right. \\
\left. \frac{u_0(z)}{1 - \omega_\pm^2\mu_0 \tau \lambda_{n_0}} \int_{\Omega \setminus D} \partial_t \frac{\text{Im} (\varepsilon_0) (y) \overline{\mu_0 (y)} - 1}{\sqrt{t^2 - |x - y|^2}} \int_D G_k (\eta, y) e_{n_0} (\eta) \, dy \, dy \right] \\
+ \frac{-t \text{Im} (\tau)}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{1}{1 - \omega_\pm^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(|\log(a)|^{\max(-1.2h^{-2})}).
\]
Next,
\[
(p^+ + p^- - 2p_0)(t, x) = 4\omega_{n_0}^2\mu_0 \int_D e_{n_0} \, dx \text{Re} \left[ -t \right. \\
\left. \frac{u_0(z) \tau (1 - \omega_{n_0}^2\mu_0 \lambda_{n_0} \tau)}{(1 - \omega_+^2\mu_0 \tau \lambda_{n_0}) (1 - \omega_-^2\mu_0 \tau \lambda_{n_0})} \int_{\Omega \setminus D} \partial_t \frac{\text{Im} (\varepsilon_0) (y) \overline{\mu_0 (y)} - 1}{\sqrt{t^2 - |x - y|^2}} \int_D G_k (\eta, y) e_{n_0} (\eta) \, dy \, dy \right] \\
+ \frac{-2t \text{Im} (\tau)}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{1}{1 - \omega_{n_0}^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(|\log(a)|^{\max(-1.2h^{-2})}),
\]
thanks to (4.8), we know that $(1 - \omega_{n_0}^2\mu_0 \lambda_{n_0} \tau) = 0$, then the right term of this equation will be reduced to only the dominant term. Finally, we obtain
\[
(p^+ + p^- - 2p_0)(t, x) = \frac{-t}{(t^2 - |x - z|^2)^{3/2}} \left. \frac{2 \text{Im} (\tau)}{1 - \omega_{n_0}^2 \mu_0 \lambda_{n_0} \tau^2} \right| u_0(z)|^2 \left( \int_D e_{n_0} \, dx \right)^2 + O(|\log(a)|^{\max(-1.2h^{-2})}),
\]
or, with help of (3.4),
\[
(p^+ + p^- - 2p_0)(t, x) = \frac{-2t \text{Im} (\tau) | < u_1; e_{n_0} > |^2}{(t^2 - |x - z|^2)^{3/2}} + O(|\log(a)|^{\max(-1.2h^{-2})}).
\]
3.2. Photo-acoustic imaging using two close particles (Dimers). Proof of (1.19)

To avoid using, in the proof, more notations we keep the same ones as in the case of one particle whenever this is possible.

**Lemma 3.3.** We have

\( u_2(x) = \mathcal{O}(1) + \mathcal{O}(\log(a)^{h-1} \log(\text{dist}(x, D_1 \cup D_2))) \), \quad x \notin D_1 \cup D_2. 

**Proof.** We skip the proof since it is similar to that of one particle (see the proof of Lemma 3.2). \( \square \)

Now, from Poisson’s formula, the solution can be written as

\[
p(t, x) = \sum_{i=1}^{2} \partial_i \int_{|x-y|<t} \frac{\text{Im} (\varepsilon_p) |u_2(y)|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{D_i} dy + \partial_i \int_{|x-y|<t} \frac{\text{Im} (\varepsilon_0) |u_2(y)|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{\Omega \setminus D} dy 
\]

For \( t > \text{diam}(\Omega) \) we have

\[
p(t, x) = \sum_{i=1}^{2} \partial_i \int_{D_i} \frac{\text{Im} (\varepsilon_p - \varepsilon_0) |u_2(y)|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{D_i} dy + \partial_i \int_{\Omega} \frac{\text{Im} (\varepsilon_0) |u_2(y)|^2(y)}{\sqrt{t^2 - |x-y|^2}} \chi_{\Omega} dy.
\]

Next, we assume that \( \tau_1 = \tau_2 = \tau \) and we use Taylor expansion of \( (\varepsilon_p - \varepsilon_0)(\cdot) \) and \( (t^2 - |x-y|^2)^{-3/2} \) near \( z_{1,2} \) to obtain

\[
p(t, x) = -t \text{Im} (\tau) \sum_{i=1}^{2} \int_{D_i} \frac{|u_2(y)|^2(y)}{(t^2 - |x-y|^2)^{3/2}} dy + T_4^* + \mathcal{O}\left( \sum_{i=1}^{2} \int_{D_i} \int_0^1 (y - z_i) \cdot \nabla \varepsilon_0(z_i + t(y - z_i)) dt |u_2(y)|^2(y) dy \right) .
\]

The remainder term, as done in (3.5), is of order \( \mathcal{O}(a^3 \log(a)^{2h}) \). Then, as in the case of one particle, we have

\[
p(t, x) = -t \text{Im} (\tau) \sum_{i=1}^{2} \int_{D_i} \frac{|u_2(y)|^2(y)}{(t^2 - |x-y|^2)^{3/2}} dy + T_4^* 
+ \mathcal{O}\left( \sum_{i=1}^{2} \text{Im} (\tau) \int_{D_i} (|y - z_i|^2 + 2 < x - z_i; z_i - y >) |u_2(y)|^2(y) dy \right) + \mathcal{O}(a^3 \log(a)^{2h}).
\]

We deduce as in (3.6) that the remainder term can be estimated as \( \mathcal{O}(\text{Im} (\tau) a^4 \log(a)^{2h}) \). Next, we develop \( u_2 \) over the basis and we use (3.7) to estimate the remainder term to obtain

\[
p(t, x) = -t \text{Im} (\tau) \sum_{i=1}^{2} \frac{1}{(t^2 - |x-z_i|^2)^{3/2}} | < u_2; e_n^{(i)} > |^2 + T_4^* + \mathcal{O}(\text{Im} (\tau) a^3 \log(a)^{2h}).
\]

Then we get

\[
p(t, x) = -t \text{Im} (\tau) \sum_{i=1}^{2} \frac{1}{(t^2 - |x-z_i|^2)^{3/2}} | < u_2; e_n^{(i)} > |^2 + T_4^* + \mathcal{O}(\text{Im} (\tau) a^2).
\]

Set \( \Omega_{1,2} := \Omega \setminus (D_1 \cup D_2) \) and write \( T_4^* \) as:

\[
T_4^* = \int_{\Omega} \frac{\text{Im} (\varepsilon_0) |u_2(y)|^2(y)}{\sqrt{t^2 - |x-y|^2}} dy = \int_{\Omega_{1,2}} |u_2(y)|^2 f(y) dy + \int_{D_1 \cup D_2} |u_2(y)|^2 f(y) dy.
\]
From the a priori estimate, see (4.17), and lemma (3.3) we deduce that the first integral dominates the second one. Now, since $f$ is smooth, the a priori estimate allows to estimate the integral over $D_1 \cup D_2$ as follows

$$\left| \int_{D_1 \cup D_2} |u_2|^2 f \, dy \right| \lesssim \|u_2\|^2 = \mathcal{O}(a^2 \log(a)^{2h}).$$

Then we use L.S.E to obtain

$$T_4^* = \int_{\Omega} |u_0|^2 dy - \int_{D_1 \cup D_2} f |u_0|^2 dy + 2 \omega^2 \mu_0 \sum_{i=1}^2 \text{Re} \left[ \int_{\Omega_{i,2}} f \bar{u}_0(y) \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right]$$

$$+ \left( \omega^2 \mu_0 \right)^2 \sum_{i=1}^2 \int_{\Omega_{i,2}} f \left| \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \right|^2 dy.$$ 

Clearly, by the smoothness of $f$ and $|u_0|$, we have

$$\left| \int_{D_1 \cup D_2} f |u_0|^2 dy \right| = \mathcal{O}(a^2).$$

Then, we obtain

$$T_4^* = p_0(t, x) + 2 \omega^2 \mu_0 \sum_{i=1}^2 \text{Re} \left[ \int_{\Omega_{i,2}} f \bar{u}_0(y) \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right]$$

$$+ 2 \text{Re} \left[ \int_{\Omega_{i,2}} f \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \int_{D_2} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right]$$

$$+ \left( \omega^2 \mu_0 \right)^2 \sum_{i=1}^2 \int_{\Omega_{i,2}} f \left| \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \right|^2 dy + \mathcal{O}(a^2 \log(a)^{2h}).$$

We remark that

$$\left( \omega^2 \mu_0 \right)^2 \int_{\Omega_{i,2}} f \left| \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \right|^2 dy \text{ for } i = 1, 2$$

have the same expression as $S_i$ given in section (3.1) (more exactly see (3.10)). Then we estimate it as $\mathcal{O}(\log(a)^{2h-2})$. Similarly, regardless of whether the position of $y$ is in $D_1$ or $D_2$, the same estimation holds for

$$2 \text{Re} \left[ \int_{\Omega_{i,2}} f \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \int_{D_2} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right]$$

We synthesize the above to get

$$T_4^* = p_0(t, x) + 2 \omega^2 \mu_0 \sum_{i=1}^2 \text{Re} \left[ \int_{\Omega_{i,2}} f \bar{u}_0(y) \int_{D_i} (\varepsilon_p - \varepsilon_0)(\eta) G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right] + \mathcal{O}(\log(a)^{2h-2}).$$

Next, we develop $u_2$ over the basis and use the Taylor expansion of $(\varepsilon_p - \varepsilon)(\cdot)$ to obtain

$$T_4^* = p_0(t, x) + 2 \omega^2 \mu_0 \sum_{i=1}^2 \text{Re} \left[ \tau < u_2; \epsilon_{\eta_0}^{(i)} > \int_{\Omega_{i,2}} f \bar{u}_0(y) \int_{D_i} G_k(\eta, y) \epsilon_{\eta_0}^{(i)}(\eta) \, d\eta \, dy \right]$$

$$- 2 \omega^2 \mu_0 \sum_{i=1}^2 \text{Re} \left[ \int_{\Omega_{i,2}} f \bar{u}_0(y) \int_{D_i} \int_{0}^{1} (z_i - \eta) \cdot \nabla \epsilon_0(z_i + s(\eta - z_i)) \, ds \, G_k(\eta, y) u_2(\eta) \, d\eta \, dy \right].$$

10For the definition of $p_0(t, x)$, see (3.9).
To precise the value of the error we need to estimate
\[ \sum_{i=1}^{2} \text{Re} \left[ \int_{\Omega_{1,2}} f \, p_0(y) \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy \right] \]
\[
\leq a \sum_{i=1}^{2} \int_{\Omega_{1,2}} \left| f \, p_0(y) \right| \left| \int_{D_i} G_k(\eta, y) u_2(\eta) \, d\eta \right| \, dy \leq a \sum_{i=1}^{2} \left( \int_{\Omega_{1,2}} \left| \int_{D_i} G_k(\eta, y) u_2(\eta) \, d\eta \right|^2 \, dy \right)^{\frac{1}{2}}
\leq a \|u_2\| \left( \int_{D} \int_{\Omega_{1,2}} |G_k|^2(\eta, y) \, dy \, d\eta \right)^{\frac{1}{2}} = O(a^3 |\log(a)|^4)
\]
and
\[
\sum_{i=1}^{2} \text{Re} \left[ \tau < u_2; e_n^{(i)} > \int_{\Omega_{1,2}} f \, p_0(y) \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy \right] \]
\[
\leq |\tau| \left( \sum_{i=0}^{2} |< u_2; e_n^{(i)} >|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega_{1,2}} \left| f \, p_0(y) \right|^2 \, dy \right)^{\frac{1}{2}} \left( \int_{\Omega_{1,2}} \sum_{i=1}^{2} \left| \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \right|^2 \, dy \right)^{\frac{1}{2}}
\leq |\tau| \|u_0\| \left( \int_{D} \int_{\Omega_{1,2}} |G_k|^2(\eta, y) \, dy \, d\eta \right)^{\frac{1}{2}} = O(|\log(a)|^{-1})
\]
We keep the dominant term and sum the others as an error to obtain
\[ T_4^* = p_0(t, x) + 2\omega^2 \mu_0 \sum_{i=1}^{2} \text{Re} \left[ \tau < u_2; e_n^{(i)} > \int_{\Omega_{1,2}} f \, p_0(y) \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy \right] + O(|\log(a)|^{\max(-1,2h-2)}). \]

Use (4.19) to obtain
\[ T_4^* = 2\omega^2 \mu_0 \sum_{i=1}^{2} \text{Re} \left[ \frac{\tau < u_0; e_n^{(i)} >}{\text{det}^*} \int_{\Omega_{1,2}} f \, p_0(y) \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy \right] + p_0(t, x) \]

(3.18) + \[ \text{d} \left( \tau a^2 \phi(z_1; z_2) \left( \int_{D} \tau \, e_n^{(i)}(\eta) \, d\eta \right)^2 \right), \]
where
\[ \text{det}^* := (1 - \omega^2 \mu_0 \tau \lambda_0) - \omega^2 \mu_0 \tau a^2 \phi(z_1; z_2) \left( \int_{D} \tau \, e_n^{(i)}(\eta) \, d\eta \right)^2, \]
and
\[ \tau a \sum_{i=1}^{2} \int_{\Omega_{1,2}} f \, p_0(y) \int_{D_i} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy = \tau a \sum_{i=1}^{2} \int_{\Omega_{1,2}} f \, p_0(y) G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta = O(|\log(a)|^{-1}). \]

The last equality is justified by the fact that we integrate a smooth function over \( \Omega_{1,2} \) and we know that the integral over \( D \) of an eigenfunction is of the order \( a \).
Also we can write (3.18) as
\[ T_4^* = 2\omega^2 \mu_0 \int_{D} e_n^{(i)} \, dx \sum_{i=1}^{2} \text{Re} \left[ \frac{\tau u_0(z_i)}{\text{det}^*} \int_{\Omega_{1,2}} \frac{\text{Im}(\bar{\eta})}{\sqrt{1 - |x - y|^2}} \int_{D} G_k(\eta, y) e_n^{(i)}(\eta) \, d\eta \, dy \right] + p_0(t, x) \]
We use the next lemma to simplify the expression of \(a/det\) since, if we compare it with the error term given in equation (3.19),

\[
\text{Lemma 3.4.}
\]

We have

\[
\int_\Omega \frac{\tau^2}{det^*} f \frac{\bar{\mu}_0(\eta)}{\tau} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy = \mathcal{O}(|\log(a)|^{\max(-1,2h-2)}).
\]

We have

\[
\int_\Omega \frac{\tau^2}{det^*} f \frac{\bar{\mu}_0(\eta)}{\tau} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy = \mathcal{O}(a^2 |\log(a)|^{h-1})
\]

since, if we compare it with the error term given in equation (3.18) we deduce that they are different by a term of order \(a/det^*\). Finally:

\[
T^*_i = 2\omega^2 \mu_0 \int_D e_n^0 dx \sum_{i=1}^2 \Re \left[ \frac{\tau}{det^*} \int_{\Omega_{1,2}} \partial_\tau \frac{\Im(\bar{\mu}_0(\eta))}{\sqrt{1 - |\eta|^2}} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy \right] + p_0(t, x)
\]

\[
+ \mathcal{O}(1 |\log(a)|^{\max(-1,2h-2)}).
\]

We set \(I_i\) to be

\[
I_i := \int_{\Omega_{1,2}} \partial_\tau \frac{\Im(\bar{\mu}_0(\eta))}{\sqrt{1 - |\eta|^2}} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy,
\]

and use the estimation of \(T^*_i\) in the equation (3.17) to obtain:

\[
(p - p_0)(t, x) = 2\omega^2 \mu_0 \left( \int_D e_n^0 dx \right) \sum_{i=1}^2 \Re \left[ \frac{\tau}{det^*} \int_{\Omega_{1,2}} \partial_\tau \frac{\Im(\bar{\mu}_0(\eta))}{\sqrt{1 - |\eta|^2}} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy \right] - t \Im(\tau) \sum_{i=1}^2 \frac{| < u_2; e_n^{(i)} > |^2}{(1 - |x - z_i|^2)^{3/2}}
\]

\[
+ \mathcal{O}(1 |\log(a)|^{\max(-1,2h-2)}).
\]

We use the next lemma to simplify the expression of \(p(t, x)\)

**Lemma 3.4.** We have

\[
< u_2; e_n^{(1)} > = < u_2; e_n^{(2)} > + \mathcal{O}(a).
\]

**Proof.** Remember, from (2.16), that we have:

\[
\int_{D_1} u_2 dx = \int_{D_2} u_2 dx + \mathcal{O}(a^2 |\log(a)|^h).
\]

Write each integral over the basis:

\[
< u_2; e_n^{(1)} > \int_{D_1} e_n^0 dx = < u_2; e_n^{(2)} > \int_{D_1} e_n^0 dx + \mathcal{O}(a^2 |\log(a)|^h)
\]

Clearly, by Holder inequality, we have

\[
\left| \sum_{i=1}^2 \frac{| < u_2; e_n^{(i)} > |^2}{(1 - |x - z_i|^2)^{3/2}} \right| \leq \| u_0 \| \| 1 \| = \mathcal{O}(a^2)
\]

and it follows that

\[
< u_2; e_n^{(1)} > = < u_2; e_n^{(2)} > + \mathcal{O}(a).
\]

From (3.19) we deduce:

\[
| < u_2; e_n^{(1)} > |^2 = | < u_2; e_n^{(2)} > |^2 + \mathcal{O}(a^2 |\log(a)|^h).
\]

By lemma 3.4 we have

\[
(p - p_0)(t, x) = 2\omega^2 \mu_0 \left( \int_{D_1} e_n^0 dx \right) \sum_{i=1}^2 \Re \left[ \frac{\tau}{det^*} \int_{\Omega_{1,2}} \partial_\tau \frac{\Im(\bar{\mu}_0(\eta))}{\sqrt{1 - |\eta|^2}} G_k(\eta, y) e_n^{(i)}(\eta) d\eta dy \right] - t \Im(\tau) \sum_{i=1}^2 \frac{| < u_2; e_n^{(i)} > |^2}{(1 - |x - z_i|^2)^{3/2}}
\]

\[
+ \mathcal{O}(1 |\log(a)|^{\max(-1,2h-2)}).
\]
We have also:
\[
\frac{1}{(t^2 - |x - z_1|^2)^{3/2}} = \frac{1}{(t^2 - |x - z_2|^2)^{3/2}} \left(1 + \mathcal{O}(d)\right).
\]

Then
\[
(p-p_0)(t, x) = 2 \omega_n^2 \mu_0 \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{\text{det}^* I_i} \right] - 2t \text{Im} \left( \tau \right) \frac{|u_2; e_{n_0}^{(2)}|^2}{(t^2 - |x - z_2|^2)^{3/2}} \right] + \mathcal{O} \left( \log(a) \right)^{\max(-1, 2h - 2)}.
\]

Next, we use the same technique as before by taking two frequencies \( \omega_n^2 = \omega_n^2 \pm |\log(a)|^{-h} \), we get
\[
(p^\pm - p_0)(t, x) = 2 \omega_n^2 \mu_0 \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{(1 - \omega_n^2 \mu_0 \tau \lambda_{n_0}) - \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} - \frac{\tau u_0(z_i)}{1 + \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \right] \right] + \mathcal{O} \left( \log(a) \right)^{\max(-1, 2h - 2)}.
\]

We estimate the error part as
\[
|\log(a)|^{-h} \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{(1 - \omega_n^2 \mu_0 \tau \lambda_{n_0}) - \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \right] \right] \sim \mathcal{O} \left( \log(a) \right)^{-1}.
\]

Define \( \tilde{p}(t, x) \) as
\[
\tilde{p}(t, x) := (p^+ - p_0)(t, x) + \frac{1 - \omega_n^2}{1 + \omega_n^2} (p^- - p_0)(t, x),
\]
hence
\[
\tilde{p}(t, x) = \frac{-4t \text{Im} \left( \tau \right) \frac{|u_2; e_{n_0}^{(2)}|^2}{(t^2 - |x - z_2|^2)^{3/2}} + 2 \omega_n^2 \mu_0 \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{(1 - \omega_n^2 \mu_0 \tau \lambda_{n_0}) - \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \right] \right]}{1 + \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} + \mathcal{O} \left( \log(a) \right)^{\max(-1, 2h - 2)}.
\]

We compute the following quantity
\[
J := \frac{1}{(1 - \omega_n^2 \mu_0 \tau \lambda_{n_0}) - \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} + \frac{1}{1 + \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \left[ \frac{1}{1 + \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \right] \left[ \frac{1}{1 + \omega_n^2 \mu_0 \tau a^2 \Phi_0(\int_B \bar{\tau}_{n_0})^2} \right]
\]

hence \( J = \mathcal{O}(1) \). Going back to the formula of \( \tilde{p}(t, x) \), we obtain:
\[
\tilde{p}(t, x) = \frac{-4t \text{Im} \left( \tau \right) \frac{|u_2; e_{n_0}^{(2)}|^2}{(t^2 - |x - z_2|^2)^{3/2}} + 2 \omega_n^2 \mu_0 \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{I_i J} \right] \right] + \mathcal{O} \left( \log(a) \right)^{\max(-1, 2h - 2)},
\]

and
\[
\left| 2 \omega_n^2 \mu_0 \left[ \int_{D_1} e_{n_0} \sum_{i=1}^2 \text{Re} \left[ \frac{\tau u_0(z_i)}{I_i J} \right] \right] \right| \leq a |\tau| |\tau| = \mathcal{O} \left( \log(a) \right)^{-1}.
\]
Finally, we have the desired approximation formula
\[
\tilde{p}(t, x) = \frac{-4}{1 + \omega^2} t \text{Im} (\tau) < u_2, e_n(2^2) > |^2 \frac{1}{(t^2 - |x - z_2|^2)^{3/2}} + O(|\log(a)|^{\max(-1, 2 - 2)}).
\]

4. A priori estimations

4.1. A priori estimates on the electric field.

Proof. of Proposition 2.3

In order to prove the a priori estimation (2.6), we proceed in two steps. First we do it for one single particle and then for multiple particles.

Step 1/ Case of one particle:

Remember that the eigenvalues and eigenfunctions of the logarithmic operator satisfy
\[
\int_D \Phi_0(x, y) e_n(y) dy = \lambda_n e_n(x) \quad \text{in} \quad D,
\]
and after scaling we get, with \( \tilde{e}_n(\cdot) := e_n \left( \frac{\cdot - x}{\varepsilon} \right) \),
\[
\lambda_n \tilde{e}_n(\eta) = \int_B \Phi_0(\eta, \xi) \tilde{e}_n(\xi) d\xi - \frac{1}{2\pi} \log(a) \int_B \tilde{e}_n(\xi) d\xi = \int_B \tilde{e}_n(\eta) d\eta.
\]
Integrating the equation (4.1) over \( B \) we obtain
\[
\int_B \int_B \Phi_0(\eta, \xi) \tilde{e}_n(\xi) d\eta d\xi = \left[ \frac{1}{2\pi} \log(a)|B| + \frac{\lambda_n}{a^2} \right] \int_B \tilde{e}_n d\eta.
\]
Multiplying (4.1) by \( e_m \) and integrating over \( B \) we get:
\[
\int_B \int_B \Phi_0(\eta, \xi) \tilde{e}_n(\xi) e_m(\eta) d\eta d\xi - \frac{1}{2\pi} \log(a) \int_B \tilde{e}_n(\xi) d\xi \int_B e_m(\eta) d\eta = \frac{\lambda_n}{a^2} \int_B \tilde{e}_n e_m d\eta.
\]
Remark that when \( m \neq n \), thanks to the fact that \( \{ e_n \}_{n \in \mathbb{N}} \) forms an orthogonal basis in \( L^2(B) \), we obtain
\[
\int_B \int_B \Phi_0(\eta, \xi) \tilde{e}_n(\xi) e_m(\eta) d\eta d\xi = \frac{1}{2\pi} \log(a) \int_B \tilde{e}_n d\xi \int_B e_m d\xi
\]
and when \( m = n \), we get
\[
\int_B \int_B \Phi_0(\eta, \xi) \tilde{e}_n(\xi) e_n(\eta) d\eta d\xi - \frac{1}{2\pi} \log(a) \left( \int_B \tilde{e}_n d\xi \right)^2 = \frac{\lambda_n}{a^2} ||e_n||^2.
\]
After normalization
\[
\int_B \int_B \Phi_0(\eta, \xi) \frac{\tilde{e}_n(\xi)}{||e_n||} \frac{\tilde{e}_n(\eta)}{||e_n||} d\eta d\xi = \frac{1}{2\pi} \log(a) \left( \int_B \tilde{e}_n d\xi \right)^2 + \frac{\lambda_n}{a^2} ||e_n||^2.
\]
We denote \( \tau_n := \tilde{e}_n / ||e_n|| \) the orthonormalized basis in \( L^2(B) \), and we set
\[
\tilde{\lambda}_n := \int_B \int_B \Phi_0(\eta, \xi) \tau_n(\eta) \tau_n(\xi) d\eta d\xi,
\]
from (4.4) and (4.5) we deduce that
\[
\tilde{\lambda}_n = \frac{\lambda_n}{a^2} + \frac{1}{2\pi} \log(a) \left( \int_B \tau_n d\xi \right)^2.
\]
Thanks to L.S.E and Green kernel expansion (2.1), we have
\[
    u_1(x) = \omega^2 \mu_0 \tau \int_D \left( \Phi_0(x, y) - \frac{1}{2\pi} \log(k(y) + \Gamma) \right) u_1(y) \, dy \\
    = u_0(x) + \omega^2 \mu_0 \tau \Theta \left( \int_D |x - y| \log(|x - y|) u_1(y) \, dy \right) \quad \text{in } D.
\]

After Taylor expansion of the function \( \log(k) \) near the point \( z \), we obtain
\[
    u_1(x) = \omega^2 \mu_0 \tau \int_D \left( \Phi_0(x, y) - \frac{1}{2\pi} \log(k(z) + \Gamma) \right) u_1(y) \, dy \\
    = u_0(x) + \Theta \left( \tau \int_D |x - y| \log(|x - y|) u_1(y) \, dy \right) \\
    + O \left( \tau \int_D \int_D (y - z) \cdot \nabla \log(k(z + t(y - z))) dt \, u_1(y) \, dy \right). 
\]

Now scaling, we have
\[
    \tilde{u}_1(\eta) = \omega^2 \mu_0 \tau a^2 \int_B \left( \Phi_0(\eta, \xi) - \frac{1}{2\pi} \log(k(z) + \Gamma) \right) \tilde{u}_1(\xi) \, d\xi + \omega^2 \mu_0 \tau \frac{1}{2\pi} \log(a) \int_B \tilde{u}_1(\xi) \, d\xi \\
    = \tilde{u}_0(\eta) + \Theta \left( \tau a^3 \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \right) \\
    + O \left( \tau a^3 \int_B \tilde{u}_1(\xi) \, d\xi \right). 
\]

Using the basis, we obtain
\[
    < \tilde{u}_1; \varpi_{n_0} > = \left[ 1 - \omega^2 \mu_0 \tau \int_B \Phi_0(\eta, \xi) \varpi_{n_0}(\xi) \, d\xi \varpi_{n_0}(\eta) \, d\eta + \omega^2 \mu_0 \tau \frac{1}{2\pi} \log(a) \int_B \varpi_{n_0} \, d\xi \right]^2 \\
    = < \tilde{u}_0; \varpi_{n_0} > + \Theta \left( \tau a^3 \int_B \varpi_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \\
    + O \left( \tau a^3 \log(a) \int_B \varpi_{n_0}(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \\
    - \omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) < 1; \varpi_{n_0} > \sum_{n \neq n_0} < \tilde{u}_1; \varpi_n > \int_B \varpi_n \, d\xi \\
    + \omega^2 \mu_0 \tau a^2 \left( - \frac{1}{2\pi} \log(k(z) + \Gamma) \right) \int_B \tilde{u}_1 \, d\xi \int_B \varpi_{n_0} \, d\eta \\
    + \omega^2 \mu_0 \tau a^2 \sum_{n \neq n_0} < \tilde{u}_1; \varpi_n > \int_B \int_B \Phi_0(\eta, \xi) \varpi_n(\xi) \, d\xi \varpi_{n_0}(\eta) \, d\eta + O \left( \tau a^3 \int_B \tilde{u}_1 \, d\xi \right). 
\]

After simplifications and using (4.2) and (4.4) we get
\[
    < \tilde{u}_1; \varpi_{n_0} > = \frac{1}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} \left[ < \tilde{u}_0; \varpi_{n_0} > + \omega^2 \mu_0 \tau a^2 \left( - \frac{1}{2\pi} \log(k(z) + \Gamma) \right) \int_B \tilde{u}_1 \, d\xi \int_B \varpi_{n_0} \, d\eta \\
    + O \left( \tau a^3 \int_B \varpi_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \\
    + O \left( \tau a^3 \log(a) \int_B \varpi_{n_0}(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \, d\eta \right) + O \left( \tau a^3 \int_B \tilde{u}_1 \, d\xi \right) \right]. 
\]
We take\textsuperscript{11} \(\tau\) and \(\omega\) so that
\[
\tau \simeq \frac{1}{a^2 |\log(a)|} \quad \text{and} \quad \omega^2 = \frac{1 \pm |\log(a)|^{-h}}{\mu_0 \lambda_{n_0} a^{-2} |\log(a)|^{-1}}.
\]
(4.9)

With this choice we have the estimation
\[
\frac{1}{1 - \omega^2 \mu_0 \tau \lambda_{n_0}} = O(|\log(a)|^h).
\]

Then
\[
|< \tilde{u}_1; \tau_{n_0}>| \leq |\log(a)|^{h} \left[ |< \tilde{u}_0; \tau_{n_0}>| + a |\log(a)|^{-1} \left| \int_B \tau_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right| + a \left| \int_B \tau_{n_0}(\eta) \int_B |\eta - \xi| \, d\xi \, d\eta \right| + a^2 |\log(a)|^{-1} \| \tilde{u}_1 \| \right]
\]

Obviously the term \(|< \tilde{u}_0; \tau_{n_0}>|\) dominates the others, but we need to check this mathematically by estimating the error part. This last one will be subdivided into three parts. We have

**Estimation of** \(s_1 := \int_B \tau_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\eta \).\n
\[
|s_1| \leq \| \tau_{n_0} \| \left( \int_B \left| \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \right|^2 \, d\eta \right)^{\frac{1}{2}} = O(\| \tilde{u}_1 \|).
\]

**Estimation of** \(s_2 := \int_B \tau_{n_0}(\eta) \int_B |\eta - \xi| \, d\xi \, d\eta \).

\[
|s_2| \leq \| \tau_{n_0} \| \left( \int_B \left| \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \right|^2 \, d\eta \right)^{\frac{1}{2}} = O(\| \tilde{u}_1 \|).
\]

**Estimation of** \(s_3 := \int_B \tilde{u}_1 \, d\xi \int_B \tau_{n_0} \, d\eta \).

\[
|s_3| := \left| \int_B \tilde{u}_1 \, d\xi \right| \left| \int_B \tau_{n_0} \, d\eta \right| = O(\| \tilde{u}_1 \|).
\]

Then
\[
|< \tilde{u}_1; \tau_{n_0}>| \leq |\log(a)|^{h} \left[ |< \tilde{u}_0; \tau_{n_0}>| + \| \tilde{u}_1 \| \left( a |\log(a)|^{-1} + a + |\log(a)|^{-1} + a^2 |\log(a)|^{-1} \right) \right],
\]

and then
\[
(4.10) \quad |< \tilde{u}_1; \tau_{n_0}>|^2 \leq |\log(a)|^{2h} \left[ |< \tilde{u}_0; \tau_{n_0}>|^2 + |\tilde{u}_1|^2 |\log(a)|^{-2} \right].
\]

In what follows, we calculate an estimation of \(\sum_{n \neq n_0} |< \tilde{u}_1; \tau_{n}>|^2\) since the other steps are the same, we start with equation \(4.7\),

\[
< \tilde{u}_1; \tau_{n}> = \frac{1}{1 - \omega^2 \mu_0 \tau \lambda_{n}} \left[ < \tilde{u}_0; \tau_{n}> + O\left( \tau a^3 \int_B \tau_{n}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \right] + O\left( \tau a^3 \log(a) \int_B \tau_{n}(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \, d\eta \right) + \omega^2 \mu_0 \tau a^2 \left( - \frac{1}{2\pi} \log(k)(z) + \Gamma \right) \int_B \tilde{u}_1 \, d\xi \int_B \tau_{n} \, d\eta
\]

\textsuperscript{11}The dielectric-resonance that we want to excite is \(\omega_{n_0}\) given by
\[
(4.8) \quad \omega_{n_0}^2 = \frac{1}{\mu_0 \lambda_{n_0} a^{-2} |\log(a)|^{-1}}.
\]
\[ + \mathcal{O}\left( a^3 \tau \int_B \tilde{u}_1 \, d\xi \int_B \tau_n \, d\eta \right) \].

Then
\[
\sum_{n \neq n_0} | \tilde{u}_1; \tau_n > |^2 \leq C^{\text{te}} \left[ \sum_{n \neq n_0} | \tilde{u}_0; \tau_n > |^2 + a^2 | \log(a)|^{-2} \sum_{n \neq n_0} \left| \int_B \tau_n(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 \right.
\]
\[ + a^2 \sum_{n \neq n_0} \left| \int_B \tau_n(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 + | \log(a)|^{-2} \| \tilde{u}_1 \|^2 \sum_{n \neq n_0} \left| \int_B \tau_n \, d\eta \right|^2 \left. \right].
\]

On the right side, except for the first term, we need to estimate the terms containing series. For this, we have
\[
\sum_{n \neq n_0} \left| \int_B \tau_n(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 \leq \int_B \left| \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \right|^2 \, d\eta,
\]
since the function \(| \cdot | \log(| \cdot |)\) is bounded on \(B\) we get
\[
\int_B \left| \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) \, d\xi \right|^2 \, d\eta \leq \mathcal{O}(\| \tilde{u}_1 \|^2).
\]
The same argument as before allows to deduce that
\[
\sum_{n \neq n_0} \left| \int_B \tau_n(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 \leq \mathcal{O}(\| \tilde{u}_1 \|^2).
\]
Obviously we have also
\[
\sum_{n \neq n_0} \left| \int_B \tau_n \, d\eta \right|^2 \leq \| 1 \|^2.
\]

Hence
\[
(4.11) \sum_{n \neq n_0} | \tilde{u}_1; \tau_n > |^2 \leq C^{\text{te}} \left[ \sum_{n \neq n_0} | \tilde{u}_0; \tau_n > |^2 + | \log(a)|^{-2} \| \tilde{u}_1 \|^2 \right].
\]

By adding (4.10) and (4.11), we get
\[
\| \tilde{u}_1 \|^2 = | \tilde{u}_1; \tau_{n_0} > |^2 + \sum_{n \neq n_0} | \tilde{u}_1; \tau_n > |^2
\]
\[ \leq | \log(a)|^{2h} \left[ \sum_{n \neq n_0} | \tilde{u}_0; \tau_n > |^2 + \| \tilde{u}_1 \|^2 | \log(a)|^{-2} \right] + C^{\text{te}} \left[ \sum_{n \neq n_0} | \tilde{u}_0; \tau_n > |^2 + | \log(a)|^{-2} \| \tilde{u}_1 \|^2 \right]
\]
hence
\[
\| \tilde{u}_1 \|^2 \leq | \log(a)|^{2h} \| \tilde{u}_0 \|^2 + | \log(a)|^{2h-2} \| \tilde{u}_1 \|^2,
\]
\[
\| \tilde{u}_1 \|^2 (1 - | \log(a)|^{2h-2}) \leq | \log(a)|^{2h} \| \tilde{u}_0 \|^2
\]
and, as \(h < 1\),
\[
\| \tilde{u}_1 \|^2 \leq (1 - | \log(a)|^{2h-2})^{-1} | \log(a)|^{2h} \| \tilde{u}_0 \|^2 \leq | \log(a)|^{2h} \| \tilde{u}_0 \|^2,
\]
or
\begin{equation}
\|u_1\|_{L^2(D)} \leq |\log(a)|^h \|u_0\|_{L^2(D)}.
\end{equation}

The following proposition makes a link between the Fourier coefficient of the generated total field and that of the source field.

**Proposition 4.1.** We have
\begin{equation}
< u_1; e_{n_0} > = < u_0; e_{n_0} > \frac{1}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0})} + O(a |\log(a)|^{2h-1}).
\end{equation}

**Proof.** We write
\[ \int_B \tilde{u}_1 d\xi = \int_B \tilde{\tau}_{n_0} d\xi + \sum_{n \neq n_0} < \tilde{u}_1; \tilde{\tau}_n > \int_B \tilde{\tau}_n d\xi. \]
Use this representation in (4.7) and rearrange the equation to get
\begin{align*}
< \tilde{u}_1; \tilde{\tau}_{n_0} > &= \left[ \frac{1}{1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau a^2 \left( -\frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_B \tilde{\tau}_{n_0} d\eta \right)^2} \right] < u_0; \tilde{\tau}_{n_0} > \\
&\quad + \omega^2 \mu_0 \tau a^2 \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \int_B \tilde{\tau}_{n_0} d\eta \sum_{n \neq n_0} < \tilde{u}_1; \tilde{\tau}_n > \int_B \tilde{\tau}_n d\xi + O(a^3 \tau \int_B \tilde{u}_1 d\xi) \\
&\quad + \omega^2 \mu_0 \tau a^3 \int_B \tilde{\tau}_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) d\xi d\eta \\
&\quad + \omega^2 \mu_0 \tau a^3 \log(a) \int_B \tilde{\tau}_{n_0}(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) d\xi d\eta.
\end{align*}

We need to estimate the four last terms between brackets. We have
\begin{align*}
|\omega^2 \mu_0 \tau a^2 \left( \frac{1}{2\pi} \log(k) + \Gamma \right) \int_B \tilde{\tau}_{n_0} d\eta \sum_{n \neq n_0} < \tilde{u}_1; \tilde{\tau}_n > \int_B \tilde{\tau}_n d\xi| &\lesssim \tau a^2 \|u_0\| \|1\| = O(|\log(a)|^{-1}).
\end{align*}
Next, use Holder inequality and the a priori estimate to obtain
\begin{equation}
|\omega^2 \mu_0 \tau a^3 \int_B \tilde{\tau}_{n_0}(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \tilde{u}_1(\xi) d\xi d\eta| \lesssim O(a |\log(a)|^{h-1}).
\end{equation}

Remark that the following term
\[ \omega^2 \mu_0 \tau a^3 \log(a) \int_B \tilde{\tau}_{n_0}(\eta) \int_B |\eta - \xi| \tilde{u}_1(\xi) d\xi d\eta, \]
up to multiplicative constant $|\log(a)|$ behaves as $\frac{1}{4.13}$, then we estimate it as $O(a |\log(a)|^{h-1})$, and obviously we have
\[ a^3 \tau \int_B \tilde{u}_1 d\xi \sim O(a |\log(a)|^{h-1}). \]
Finally, we obtain
\begin{equation}
< u_1; e_{n_0} > = \frac{< u_0; e_{n_0} >}{1 - \omega^2 \mu_0 \tau \lambda_{n_0} - \omega^2 \mu_0 \tau \left( -\frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_D e_{n_0} d\eta \right)^2} + O(a |\log(a)|^{h-1}),
\end{equation}
or in the following form
\begin{equation}
< u_1; e_{n_0} > = \frac{< u_0; e_{n_0} >}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) \left[ 1 - \frac{\omega^2 \mu_0 \tau a^2 \left( -\frac{1}{2\pi} \log(k) + \Gamma \right) \left( \int_B \tilde{\tau}_{n_0} d\eta \right)^2}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0})} \right]} + O(a |\log(a)|^{h-1}).
\end{equation}
Step 2/ Case of multiple particles: Consider the L.S.E for multiple particles

\[
(4.15) \quad v_i(x) - \omega^2 \mu_0 \tau \int_{D_i} G_k(x;y) v_i(y) \, dy - \omega^2 \mu_0 \tau \sum_{m \neq i} \int_{D_m} G_k(x;y) v_m(y) \, dy = u_0(x), \quad x \in D_i.
\]

We use the expansion formula (2.1) of \( G_k(x;y) \) to write

\[
v_i(x) = \omega^2 \mu_0 \tau a^2 \int_B \Phi_0(x,y) v_i(y) \, dy = u_0(x) + \omega^2 \mu_0 \tau \left( -\frac{1}{2\pi} \log(k(z_i)) + \Gamma \right) \int_{D_i} v_i \, dy + O\left( \tau a \int_{D_i} v \, dy \right)
\]

\[
+ \omega^2 \mu_0 \tau \int_{D_i} |x-y| \log(|x-y|) v_i(y) \, dy + \omega^2 \mu_0 \tau \sum_{m \neq i} \int_{D_m} \Phi_0(x,y) v_m(y) \, dy + O\left( \tau a \sum_{m \neq i} \int_{D_m} v \, dy \right)
\]

\[
+ \omega^2 \mu_0 \tau \sum_{m \neq i} \left( -\frac{1}{2\pi} \log(k(z_m)) + \Gamma \right) \int_{D_m} v_m \, dy + \omega^2 \mu_0 \tau a^3 \sum_{m \neq i} \int_{D_m} |x-y| \log(|x-y|) v_m(y) \, dy.
\]

Scaling, we obtain

\[
\dot{v}_i(\eta) = \omega^2 \mu_0 \tau a^2 \int_B \Phi_0(\eta,\xi) \dot{v}_i(\xi) \, d\xi = u_0(\eta + a \eta) - \omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) \int_B \dot{v}_i \, d\xi + O\left( \tau a^3 \int_B \dot{v}_i \, d\xi \right)
\]

\[
+ \omega^2 \mu_0 \tau \left( -\frac{1}{2\pi} \log(k(z_i)) + \Gamma \right) a^2 \int_B \dot{v}_i \, d\xi + \omega^2 \mu_0 \tau a^3 \int_B |\eta - \xi| \log(|\eta - \xi|) \dot{v}_i(\xi) \, d\xi
\]

\[
+ \omega^2 \mu_0 \tau a^3 \log(a) \int_B |\eta - \xi| \dot{v}_i(\xi) \, d\xi - \frac{1}{2\pi} \omega^2 \mu_0 \tau a^2 \sum_{m \neq i} \int_B \log(|z_i - z_m| + a(\eta - \xi)) \dot{v}_m(\xi) \, d\xi
\]

\[
+ \omega^2 \mu_0 \tau a^2 \sum_{m \neq i} \left( -\frac{1}{2\pi} \log(k(z_m)) + \Gamma \right) \int_B \dot{v}_m \, d\xi + O\left( \tau a^3 \sum_{m \neq i} \int_B \dot{v}_m \, d\xi \right)
\]

\[
(4.16) + \omega^2 \mu_0 \tau a^2 \sum_{m \neq i} \int_B |z_i - z_m| + a(\eta - \xi) \log(|z_i - z_m| + a(\eta - \xi)) \dot{v}_m(\xi) \, d\xi.
\]

We recall that

\[
A_0 \, v(x) = \int_B \Phi_0(x,y) \, v(y) \, dy,
\]

and denote

\[
T \, v(x) := \int_B \, v(y) \, dy, \quad x \in B.
\]

Then:

\[
[I - \omega^2 \mu_0 \tau a^2 A_0 + \omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) T] \dot{v}_i = u_0(\eta + a \cdot) + \omega^2 \mu_0 \tau \left( -\frac{1}{2\pi} \log(k(z_i)) + \Gamma \right) a^2 \int_B \dot{v}_i \, d\xi
\]

\[
+ \omega^2 \mu_0 \tau a^3 \int_B |\eta - \xi| \log(|\eta - \xi|) \dot{v}_i(\xi) \, d\xi + \omega^2 \mu_0 \tau a^3 \log(a) \int_B |\eta - \xi| \dot{v}_i(\xi) \, d\xi
\]

\[
- \frac{1}{2\pi} \omega^2 \mu_0 \tau a^2 \sum_{m \neq i} \int_B \log(|z_i - z_m| + a(\eta - \xi)) \dot{v}_m(\xi) \, d\xi
\]
\[ + \omega^2 \mu_0 \tau a^2 \sum_{m \neq i}^M \int_B |(z_i - z_m) + a(\eta - \xi)| \log(|(z_i - z_m) + a(\eta - \xi)|) \bar{v}_m(\xi) \, d\xi \]

\[ + \omega^2 \mu_0 \tau a^2 \left( - \frac{1}{2\pi} \log(k)(z_m) + \Gamma \right) \int_B \bar{v}_m \, d\xi + \mathcal{O}\left( \tau a^3 \sum_{m=1}^M \int_B \bar{v}_m \, d\xi \right). \]

Also, we note by

\[ \mathcal{R}(A_0; T) := [I - \omega^2 \mu_0 \tau a^2 A_0 + \omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) \, T]^{-1}. \]

**Remark 4.2.** In the definition of the operator \( \mathcal{R}(A_0; T) \) we cannot neglect the operator \( T \) since it scales with the same order as \( A_0 \).

Then

\[ \bar{v}_i = \mathcal{R}(A_0; T) (u_0(z_i + a \cdot)) + \omega^2 \mu_0 \tau \left( - \frac{1}{2\pi} \log(k)(z_i) + \Gamma \right) a^2 \int B \bar{v}_i \, d\xi \mathcal{R}(A_0; T)(1) \]

\[ + \omega^2 \mu_0 \tau a^3 \mathcal{R}(A_0; T) \left( \int_B |\eta - \xi| \log(|\eta - \xi|) \bar{v}_i(\xi) \, d\xi \right) \]

\[ + \omega^2 \mu_0 \tau a^3 \log(a) \mathcal{R}(A_0; T) \left( \int_B |\eta - \xi| \bar{v}_i(\xi) \, d\xi \right) \]

\[ - \frac{1}{2\pi} \omega^2 \mu_0 \tau a^2 \sum_{m \neq i}^M \mathcal{R}(A_0; T) \left( \int_B \log(|z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right) \]

\[ + \omega^2 \mu_0 \tau a^2 \sum_{m \neq i}^M \mathcal{R}(A_0; T) \left( \int_B |z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right) \]

\[ + \mathcal{O}\left( \tau a^3 \sum_{m=1}^M \int_B \bar{v}_m \, d\xi \right) \mathcal{R}(A_0; T)(1). \]

Using the a priori estimate (4.12), we obtain

\[ \|\bar{v}_i\| \leq |\log(a)^h\|u_0(z_i + a \cdot)\| + a |\log(a)^h|^{-1}\|\bar{v}_i\| + a^2 \|\bar{v}_i\| |\log(a)^h| 1 | \]

\[ + a^3 |\log(a)^h| \left( \int_B |\eta - \xi| \log(|\eta - \xi|) \bar{v}_i(\xi) \, d\xi \right) + a^3 |\log(a)^h| \left( \int_B |\eta - \xi| \bar{v}_i(\xi) \, d\xi \right) \]

\[ + a^2 |\log(a)^h| \sum_{m \neq i}^M \left( \int_B \log(|z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right) + a^2 |\log(a)^h| 1 | \sum_{m \neq i} \|\bar{v}_m\| \]

\[ + a^2 |\log(a)^h| \sum_{m \neq i}^M \left( \int_B |z_i - z_m| + a(\eta - \xi)| \log(|z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right) \]

\[ + a |\log(a)^h|^{-1} \sum_{m=1}^M \|\bar{v}_m\|, \]

and

\[ \left\| \int_B \log(|z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right\|^2 = \int_B \left( \int_B \log(|z_i - z_m| + a(\eta - \xi)| \bar{v}_m(\xi) \, d\xi \right)^2 \, d\eta \]

\[ \leq \int_B \int_B \log(|z_i - z_m| + a(\eta - \xi)| \| \bar{v}_m\|^2 \, d\xi \, d\eta. \]
Proof. First of all, recall that
\[ \| \log(|z_i - z_m| + a(\eta - \xi)| \tilde{v}_m(\xi) d\xi \| \lesssim \log(1/d_m) \| \tilde{v}_m \|. \]

The same calculus allows to obtain
\[ \| \int_B ([z_i - z_m] + a(\eta - \xi) \log(|z_i - z_m| + a(\eta - \xi)) \tilde{v}_m(\xi) d\xi \| \lesssim d_m \log(1/d_m) \| \tilde{v}_m \|. \]

Gathering these estimates, we have
\[ \| \tilde{v}_i \| \leq \| \log(a)^h \| u_0(z_i + a \cdot) \| + \left[ \left( a^2 | \log(a)|^h + a^3 | \log(a)|^h + a^3 | \log(a)|^{1+h} + a | \log(a)|^{h-1} \right) \right] \| \tilde{v}_i \| + \left[ \left( a^2 | \log(a)|^h \log(1/d) + a^3 | \log(a)|^h \| 1 \| + a^2 | \log(a)|^h + a | \log(a)|^{h-1} \right) \right] M \| \tilde{v}_m \|. \]

Then
\[ \| \tilde{v}_i \| \leq \| \log(a)^h \| u_0(z_i + a \cdot) \| + a \| \log(a)|^{h-1} \| \tilde{v}_i \| + a | \log(a)|^{h-1} \sum_{m \neq i} \| \tilde{v}_m \|, \]

or
\[ \| \tilde{v}_i \| (1 - a | \log(a)|^{h-1}) \leq | \log(a)^h \| u_0(z_i + a \cdot) \| + a | \log(a)|^{h-1} \sum_{m \neq i} \| \tilde{v}_m \|, \]

hence
\[ \| \tilde{v}_i \|_{L^2(B)} \leq | \log(a)^h \| u_0(z_i + a \cdot) \|_{L^2(B)} + a | \log(a)|^{h-1} \sum_{m \neq i} \| \tilde{v}_m \|_{L^2(B)} \]
\[ \| \tilde{v}_i \|_{L^2(B)}^2 \leq | \log(a)^{2h} \| u_0(z_i + a \cdot) \|_{L^2(B)}^2 + a^2 | \log(a)|^{2h-2} \sum_{m \neq i} \| \tilde{v}_m \|_{L^2(B)}^2 \]
\[ \| \tilde{v}_i \|_{L^2(B)}^2 \leq | \log(a)^{2h} \| u_0(z_i + a \cdot) \|_{L^2(B)}^2 + a^2 | \log(a)|^{2h-2} \sum_{m \neq i} \| \tilde{v}_m \|_{L^2(B)}^2 \]

we sum up to \( M \), to obtain
\[ \| \tilde{u} \|_{L^2(B)} \leq | \log(a)^{2h} \| \tilde{u}_0 \|_{L^2(B)} + M^2 a^2 | \log(a)|^{2h-2} \| \tilde{u} \|_{L^2(B)}^2 \]
\[ (1 - M^2 a^2 | \log(a)|^{2h-2}) \| \tilde{u} \|_{L^2(B)}^2 \leq | \log(a)^{2h} \| \tilde{u}_0 \|_{L^2(B)}^2 \]
\[ \| \tilde{u} \|_{L^2(B)}^2 \leq | \log(a)^{2h} \| \tilde{u}_0 \|_{L^2(B)}^2 \]

We obtain after scaling back
\[ (4.17) \quad \| u \|_{L^2(B)} \leq | \log(a)^h \| u_0 \|_{L^2(B)}. \]

In the next proposition, which is analogous to proposition \[4.1\], we estimate the Fourier coefficient of the total field for dimer particles when \( n \neq n_0 \).

**Proposition 4.3.** For \( n \neq n_0 \), we have
\[ (4.18) \quad \langle u_2; e_n^{(i)} \rangle = \frac{1}{(1 - \omega^2 \mu_0 \tau \lambda_n)} \left[ \langle u_0, e_n^{(i)} \rangle + O(| \log(a)|^{-h}) \right] \langle 1, e_n^{(i)} \rangle, i = 1, 2. \]

**Proof.** First of all, recall that \( v_m = u_{\rho m}, m = 1, 2 \) and let \( n \neq n_0 \). Take the scalar product of \[4.16\] with respect to \( \tilde{e}_n^{(i)}, i = 1, 2 \), to obtain
\[ \langle \tilde{v}_1; \tilde{e}_n \rangle = - \omega^2 \mu_0 \tau a^2 \int_B \tilde{e}_n(\eta) \int_B \Phi_0(\eta, \xi) \tilde{v}_1(\xi) d\xi d\eta \]
\[<\hat{u}_0;\tau_n> \begin{array}{c}
= -\omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) \int_B \hat{v}_1 d\xi \int_B \tau_n d\eta \\
+ \omega^2 \mu_0 \tau a^2 \left[ a \int_B \tau_n(\eta) \int_B |\eta - \xi| \log(|\eta - \xi|) \hat{v}_1(\xi) d\xi d\eta \right] \\
+ a \log(a) \int_B \tau_n(\eta) \int_B |\eta - \xi| \hat{v}_1(\xi) d\xi d\eta + \left( -\frac{1}{2\pi} \log(k) + 1 \right) \int_B (\hat{v}_1 + \hat{v}_2) d\xi \int_B \tau_n d\eta \\
- \frac{1}{2\pi} \int_B \tau_n(\eta) \int_B |z_1 - z_2| + a(\eta - \xi) |\hat{v}_2(\xi) d\xi d\eta + O\left( a \int_B (\hat{v}_1 + \hat{v}_2) d\xi \right) \\
+ \int_B \tau_n(\eta) \int_B |(z_1 - z_2) + a(\eta - \xi)| \log(|(z_1 - z_2) + a(\eta - \xi)|) \hat{v}_2(\xi) d\xi d\eta \right] .
\]

The error part, with the help of Taylor’s formula, behaves as \(O\left(|\log(a)|^{1-h}\right) = 1, \tau_n > \) and we can write

\[
<\hat{v}_1;\tau_n> = \begin{array}{c}
\frac{\lambda_n}{a^2} + \frac{1}{2\pi} \log(a) \left( \int_B \tau_n d\eta \right)^2 + \frac{1}{2\pi} \log(a) \int_B \tau_n d\eta \int_B \tau_n d\eta \int_B \Phi_0(\eta,\xi) \tau_n(\xi) d\xi d\eta
\end{array}
\]

we plug all this in the previous equation to obtain

\[
<\hat{v}_1;\tau_n> \begin{array}{c}
- \omega^2 \mu_0 \tau a^2 \left[ <\hat{v}_1;\tau_n> \int_B \tau_n(\eta) \int_B \Phi_0(\eta,\xi) \tau_n(\xi) d\xi d\eta \\
+ \sum \int_B \tau_n(\eta) \int_B \Phi_0(\eta,\xi) \tau_j(\xi) d\xi d\eta \right] = <\hat{u}_0;\tau_n>
\end{array}
\]

\[
\begin{array}{c}
- \omega^2 \mu_0 \tau a^2 \frac{1}{2\pi} \log(a) \left[ <\hat{v}_1;\tau_n> \int_B \tau_n d\eta + \sum \int_B \tau_n d\eta \int_B \tau_n d\eta <\hat{v}_1;\tau_n> \int_B \tau_j d\xi \right] \\
+ O\left(|\log(a)|^{1-h}\right) = 1, \tau_n > \)
\]

Next, we cancel the two terms given by series and those written with bold symbol and scale back the obtained formula to get \([4.18]\). The result in \([4.18]\) also applies to the case \(n = n_0\) with an error term of order \(O\left(|\log(a)|^{-h}\right)\).

The next proposition improves the error term by improving the denominator term.

**Proposition 4.4.** We have

\[
< u_2; e^{(i)}_{n_0} > = \frac{< u_0; e^{(i)}_{n_0} >}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) - \omega^2 \mu_0 \tau a^2 \Phi_0(\xi_1, \xi_2) \left( \int_B \tau_{n_0} \right)^2} + O(a), \quad i = 1, 2.
\]

**Proof.** In order to prove equality \([4.19]\) we take a scalar product with respect to \(\tau_{n_0}\) at the equation \([4.16]\), and after simplifications, we get:

\[
\begin{bmatrix}
(1 - \omega^2 \mu_0 \lambda_{n_0} \tau) & \omega^2 \mu_0 \tau a^2 \Phi_0 \left( \int_B \tau_{n_0} \right)^2 \\
-\omega^2 \mu_0 \tau a^2 \Phi_0 \left( \int_B \tau_{n_0} \right)^2 & (1 - \omega^2 \mu_0 \lambda_{n_0} \tau)
\end{bmatrix}

\begin{bmatrix}
< u_2; \tau_{n_0}^{(1)} > \\
< u_2; \tau_{n_0}^{(2)} >
\end{bmatrix} = \begin{bmatrix}
< u_0; \tau_{n_0}^{(1)} > + O(|\log(a)|^{-h}) \\
< u_0; \tau_{n_0}^{(2)} > + O(|\log(a)|^{-h})
\end{bmatrix}
\]

We denote by \(\det\) the determinant of the last matrix, i.e.

\[
\det = \left( 1 - \omega^2 \mu_0 \lambda_{n_0} \tau \right)^2 - \left( \omega^2 \mu_0 \tau a^2 \Phi_0 \left( \int_B \tau_{n_0} \right)^2 \right)^2,
\]

where \(\Phi_0 = \Phi_0(\xi_1, \xi_2)\).
Next, we check that when we are close to the resonance the determinant $det \neq 0$. For this, and by construction of $\omega^2$, we have

$$1 - \omega^2 \mu_0 \tau \lambda_{n_0} = \mp |\log(a)|^{-h},$$

and the fact that $d \sim a^{|\log(a)|^{-h}}$ implies that $\tau a^2 \Phi_0(z_1, z_2) \sim \frac{1}{2\pi} |\log(a)|^{-h}$. Plug this in (4.21) to obtain

$$det = |\log(a)|^{-2h} \left[ 1 - \left( \omega^2 \mu_0 \frac{1}{2\pi} \left( \int \tau_{n_0} \right)^2 \right)^2 \right] \sim |\log(a)|^{-2h} \left[ 1 - \frac{(1 \pm |\log(a)|^{-h}) (1 + |\log(a)|^{-1})}{1 + \frac{\lambda_{n_0} |\log(a)|^{-1}}{\# (\int \tau_{n_0})^2}} \right]$$

from Hypotheses (I) we deduce that

$$\left( \frac{\lambda_{n_0} |\log(a)|^{-1}}{\frac{1}{2\pi} \left( \int \tau_{n_0} \right)^2} \right) \sim |\log(a)|^{-1},$$

then

$$det = |\log(a)|^{-2h} \left[ 1 - (1 \pm |\log(a)|^{-h}) (1 + |\log(a)|^{-1}) \right] \sim |\log(a)|^{-3h}.$$  

Since $det \neq 0$, the algebraic system $\text{(4.20)}$ is invertible. We invert it and use the fact that

$$\langle \tilde{u}_0; \tilde{r}_{n_0}^{(2)} \rangle = \langle \tilde{u}_0; \tilde{r}_{n_0}^{(1)} \rangle > + O(d),$$

to obtain

$$\langle \tilde{u}_2; \tilde{r}_{n_0}^{(1)} \rangle = \frac{\langle \tilde{u}_0; \tilde{r}_{n_0}^{(1)} \rangle}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) - \omega^2 \mu_0 \tau a^2 \Phi_0(z_1; z_2) \left( \int_B \tau_{n_0} \right)^2} + O(1),$$

and, after scaling, we get (4.19). \hfill \square

4.2. Estimation of the scattering coefficient $C$

From (2.4) we have:

$$w = \omega^2 \mu_0 \tau \left[ I - \omega^2 \mu_0 \tau A_0 \right]^{-1} (1) \quad \text{or} \quad \frac{1}{\omega^2 \mu_0 \tau} \left[ I - \omega^2 \mu_0 \tau A_0 \right] (w) = 1.$$  

Hence

$$\langle 1, e_n \rangle = \frac{1}{\omega^2 \mu_0 \tau} < e_n; I - \omega^2 \mu_0 \tau A_0 \rangle (w) = \frac{1}{\omega^2 \mu_0 \tau} \left[ < e_n, w > - \omega^2 \mu_0 \tau \lambda_n < e_n, w > \right]$$

and then

$$\langle w, e_n \rangle = \frac{\omega^2 \mu_0 \tau}{1 - \omega^2 \mu_0 \tau \lambda_n} < 1, e_n >.$$  

The next lemma uses (4.23) to gives a precision about the value of $C$.

Lemma 4.5. The coefficient $C$ can be approximated as

$$C = \frac{\omega^2 \mu_0 \tau}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) \left( \int_D e_{n_0} \right)^2} + O(|\log(a)|^{-1}).$$

Proof. We use the definition of $C$, given by (2.5), to write

$$C := \int_D w \, dx = \sum_n < w, e_n > < 1, e_n >,$$

apply (4.23) to obtain

$$C = \omega^2 \mu_0 \tau \left[ \frac{1}{(1 - \omega^2 \mu_0 \tau \lambda_{n_0}) \left( \int_D e_{n_0} \right)^2} + \sum_{n \neq n_0} \frac{1}{(1 - \omega^2 \mu_0 \tau \lambda_n) \left( \int D e_n \right)^2},$$
and, since the frequency $\omega$ is near $\omega_n$, and hence away from the other resonances we have

$$\left| \sum_{n\neq n_0} \frac{1}{(1-\omega^2\mu_0 \tau \lambda_n)} \left( \int_D e_n \right)^2 \right| \leq \sum_{n} |<1, e_n>|^2 = \|1\|_{L^2(D)} = \mathcal{O}(a^2).$$

From \[4.24\], we see that

$$C \sim \|\log(a)\|^{h-1}.$$  \tag{4.25}

We deduce also the following formula:

$$\lambda_n = a^2 \left( \tilde{\lambda}_n + \frac{1}{2} \|\log(a)\| \left( \int_B \tilde{e}_n(\xi)d\xi \right)^2 \right),$$  \tag{5.1}

where

$$\tilde{\lambda}_n := \frac{1}{\|e_n\|_{L^2(B)}^2} \int_B L\Phi(e_n)(\eta) e_n(\eta) d\eta$$

and $\tilde{e}_n$ is the scaled of any eigenfunction $e_n$ corresponding to $\lambda_n$. Take the absolute value in \[5.1\] to obtain

$$|\lambda_n| \leq a^2 \left( |\tilde{\lambda}_n| + \frac{1}{2} \|\log(a)\| |<1; \tilde{e}_n>|^2 \right).$$

From the definition of $\tilde{\lambda}_n$, see \[4.5\], we have $|\tilde{\lambda}_n| \leq \|\Phi_0\|_{L^2(B \times B)} < \infty$ and we use the Cauchy-Schwarz inequality to obtain

$$|\lambda_n| \leq a^2 \left( \|\Phi_0\|_{L^2(B \times B)} + \frac{1}{2} \|\log(a)\| \|B\| \right) \lesssim a^2 \|\log(a)\|.$$

b) For the lower bound, the situation is less clear. Nevertheless, we have the following results:

b.1) When the shape is a disc of radius $a$, we refer to (Theorem 4.1, \[12\]) for the existence of a sequence of eigenvalues given by

$$\lambda_{k,j} = a^2 \left( \frac{\mu_j^{(k)}}{\mu_j^{(k,j)}} \right)^2,$$

and the corresponding eigenfunctions given by

$$u_{k,j}(r, \varphi) = J_k \left( \frac{\mu_j^{(k,j)}}{\mu_j^{(k,j)}} \right) \frac{r}{a} e^{i k \varphi},$$

where $J_k$ is the Bessel function of the first kind of order $k$ and $\mu_j^{(k,j)}$ are the roots of the following transcendental equation

$$k J_k \left( \frac{\mu_j^{(k,j)}}{\mu_j^{(k,j)}} \right) + \frac{\mu_j^{(k,j)}}{2} \left( J_{k-1} \left( \frac{\mu_j^{(k,j)}}{\mu_j^{(k,j)}} \right) - J_{k+1} \left( \frac{\mu_j^{(k,j)}}{\mu_j^{(k,j)}} \right) \right) = 0, \quad k = 1, 2, \cdots \tag{5.2}$$

We remark that (only) for $k = 0$, the associated eigenfunctions have a non zero average\[12\].

Next, in order to obtain a precision about the behaviour of $\{\lambda_{0,j}\}_{j \geq 1}$ with respect to $a$, we

\[12\] We can compute $\int_D u_{0,j} = \int_0^\pi \int_0^a u_{0,j}(r, \varphi) r dr d\varphi = 2\pi a^2 J_1 \left( \frac{\mu_j^{(0,j)}}{\mu_j^{(0,j)}} \right)$.\]
need to investigate the behaviour of \( \mu_{j}^{(0)} \) solutions of (5.2). For this, we use the following properties of Bessel functions
\[
J_{-1}(x) - J_{1}(x) = 2J'_{0}(x) = -2J_{1}(x),
\]
to write (5.2) as
\[
J_{0}(\mu_{j}^{(0)}) + 2 \log(a) \mu_{j}^{(0)} J_{1}(\mu_{j}^{(0)}) = 0.
\]
Set \( \Psi(x) := J_{0}(x) + 2 \log(a) x J_{1}(x) \) and use Dixon’s theorem, see [27] page 480, to deduce that the roots of \( \Psi \) are interlaced with those of \( J_{0} \), noted by \( \{x_{0,j}\}_{j \geq 1} \), and those of \( J_{1} \), noted by \( \{x_{1,j}\}_{j \geq 1} \). At this stage, we distinguish two cases

\begin{itemize}
  \item The roots of \( \Psi \) exceeding \( x_{0,1} \):
  
  For this case, a direct application of Dixon’s theorem, allows to deduce that
  \[
  \forall j \geq 2, \ x_{k,j-1} < \mu_{j}^{(0)} < x_{k,j}, \ k = 0, 1
  \]
  and
  \[
  \forall j \geq 2, \ a^{2} x_{k,j}^{-2} < \lambda_{0,j} < a^{2} x_{k,j-1}^{-2}, \ k = 0, 1,
  \]
  since \( \{x_{k,j}\}_{k \geq 1} \) are independent of \( a \) we deduce that \( \lambda_{0,j} \) behaves as \( a^{2} \).

  \begin{itemize}
    \item The root of \( \Psi \) less than \( x_{0,1} \):
      
      The analysis of this case is more delicate. First, we observe that if, for a certain \( x \), \( \Psi(x) = 0 \), then \( J_{0}(x) \neq 0 \). Otherwise, we would have also \( J_{1}(x) = 0 \) which is impossible as the zeros of \( J_{0} \) and \( J_{1} \) are disjoint, see Bourget’s Hypothesis, page 484, section 15.28 in [27]. Hence the equation \( \Psi(x) = 0 \) can be rewritten as
      \[
      \frac{1}{2 \log(a)} = -\frac{x J_{1}(x)}{J_{0}(x)} := F_{0}(x).
      \]
      Clearly, \( F_{0} \) is a smooth function on each interval not containing a zero of \( J_{0} \) and from [15], see equation 27, we deduce that it is also a decreasing function, (see figure 2, for a schematic picture).
  \end{itemize}
\end{itemize}

\textbf{Figure 2.} Graphic of \( F_{0} \).

\textbf{Figure 3.} Solving, for \( x \in (0, \nu) \),
\( F_{0}(x) = 1/(-4 \log(10)) \).

So, if we restrict our study to \( (0, \nu) \) with \( \nu < x_{0,1} \) we deduce that \( F_{0}^{-1} \) exists and is continuous, then the equation (5.3) is solvable and the solution that we obtain is also small, (see figure 3, for numerical demonstration).
Now, since $x$ is small we use the asymptotic behaviour of $F_0$, see for instance (equation 25 in [15]), $F_0(x) \sim -x^2/2$ to write (5.3) as

$$\frac{1}{2 \log(a)} \sim \frac{-x^2}{2},$$

and this implies that $x \sim \left( \log(1/a) \right)^{-1}$. Finally

$$\lambda_{0,1} \sim a^2 |\log(a)|$$

(5.4)

b.2) For the case of an arbitrary shape $D$, with $|D| = |B_a|$ where $B_a$ is the disc of radius $a$, and referring to (Theorem 2.5, [22]) we have $\|LP_D\| \leq \|LP_{B_a}\|$. From the definition of $\|LP_D\|$, we write this inequality as a Faber-Krahn type inequality

$$\frac{1}{\lambda_{0,1}(D)} = \|LP_D\| \leq \|LP_{B_a}\| = \frac{1}{\lambda_1^2(B_a)}$$

or equivalently $\lambda_1(B_a) \geq \lambda_{0,1}(D)$.

We deduce the lower bound, and hence the behaviour, of the first eigenvalue

$$\lambda_{0,1}(D) \sim a^2 |\log(a)|, \quad \forall a << 1.$$  

(5.5)

In addition from (5.1), we see that

$$\left( \int_D e_1 \right)^2 = \frac{\lambda_{0,1}}{a^2 |\log(a)|} + O(|\log(a)|^{-1})$$

and hence

$$\left( \int_D e_1 \right)^2 \sim 1 \quad \text{for} \quad a << 1.$$  

(5.6)

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