Detecting Time Crystal and Classifying Quantum Phases with Time Order

Tie-Cheng Guo 1, ∗ and Li You 1, 2, †

1 State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China
2 Frontier Science Center for Quantum Information, Beijing, China

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Understanding phases of matter is of fundamental importance. Prior to the widespread appreciation and acceptance of topological order, the paradigm of spontaneous symmetry breaking, formulated along the Landau-Ginzburg-Wilson (LGW) dogma, has been central to the understanding of phases as well as phase transitions between order parameters with distinct symmetries. This Letter proposes to classify ground state phases of quantum matter based on temporal properties in terms of time order. More specifically, we define time order with twisted vector: the ground state acted on by a symmetry order operator, whose two-time auto-correlation function detects possible existence of nontrivial temporal structure. A (symmetry protected) time ordered phase thus implicates the presence and essence of continuous time crystal (CTC). As an example, time order phase diagram for a spin-1 atomic Bose-Einstein condensate (BEC) is presented.

A consistent theme for studying many-body system, particularly in condensed matter physics, concerns the classification of phases and their associated phase transitions [1–3]. In the celebrated Landau-Ginzburg-Wilson (LGW) paradigm [4, 5], spontaneous symmetry breaking plays a central role with distinct order parameters characterizing different phases of matter with respective symmetries. Some phases as well as their associated transitions are, however, beyond the Landau-Ginzburg-Wilson paradigm, which is by now well accepted since first uncovered decades ago [6–8]. For example, topological order, which classifies gapped quantum many-body system constitutes a topical research direction [7–10]. Current understanding of gapped system supports gapped liquid phase [11] and gapped non-liquid phase, with the former broadly including phases of topological order [7, 8], symmetry enriched topological order [12–15], and symmetry protected trivial order [16–18], while the recently discussed fracton phase [19–21] belongs to the latter of gapped non-liquid phase.

Properties of phases along temporal dimension are also worthy of pursuit as exemplified by many recent studies [22–24]. For instance, time crystal (TC) or ground state perpetual temporal dependence for a system that breaks spontaneously time translation symmetry (TTS), is a relatively new phenomenon. First proposed by Shapere and Wilczek [25] for classical systems and by Wilczek [24] for quantum systems in 2012, TC in their original sense is unfortunately ruled out by Bruno’s no-go theorem the following year [26, 27]. Watanabe and Oshikawa (WO) reformulate the idea in terms of quantum TC (QTC) [28], and prove a refined no-go theorem for any many-body system without long-range interaction [28]. More recent efforts are directed at discrete/Floquet TC at non-equilibrium breaking discrete TTS [29–34], in systems with disorder that facilitate many-body localization [30, 33], or otherwise in clean systems [35–38]. Continued studies are further extended to open systems with Floquet driving in the presence of dissipation [39–43], where experimental investigations are reported for a variety of systems [44–50].

This Letter concerns continuous TC (CTC), a ground state with time dependence breaking continuous TTS, as proposed originally [24, 25] in direct analogy to space crystal. We adopt the definition of WO first outlined in the now famous no-go theorem paper, that establishes a general and rigorous subtype of CTC: the WO CTC based on two-time auto-correlation function of an operator [28]. Recently, Kozin and Kyriienko claim to have realized such a genuine ground state CTC in a long-range interacting multi-spin model [51], although continued disputes call for more concrete implementation in realistic systems [52, 53].

Time order.—We show that temporal properties of the ground state for a quantum many-body system can be employed to characterize or classify its phases. Hence, the concept of time order can be raised analogous to order parameter, by bestowing it in the non-trivial temporal structures. After reformulating CTC and categorizing nontrivial temporal structures, we shall present an operational definition for time order and accordingly an exclusive list of all possible phases. Based on the WO proposal [28], a proper witness to CTC is the following two-time (or unequal time) auto-correlation function

$$\lim_{V \to \infty} \langle \hat{\Phi}(t) \hat{\Phi}(0) \rangle / V^2 \equiv f(t),$$

(1)

where $\hat{\Phi}(t) \equiv \int_V d^Dx \hat{\phi}(\vec{x}, t)$ is an integrated order parameter (over $D$-spatial-dimension). $\hat{\phi}(\vec{x}, t)$ is the corresponding local order parameter density $\phi \equiv \hat{\Phi}/V$ averaged (with respect to ground state) over volume,

$$f(t) = \lim_{V \to \infty} \langle \hat{\phi}(t) \hat{\phi}(0) \rangle.$$  

(2)

If $f(t)$ exhibits a nontrivial time periodic oscillation, the system is in a state of CTC. This can be refor-
mulated into an explicit operational protocol by introducing twisted vector. For a quantum many-body system with energy eigen-state $|\psi_i\rangle$, if there exists a coarse-grained Hermitian order parameter $\hat{\phi}$, $\langle \hat{\phi}|\psi_i\rangle$ is called the eigen-state twisted vector; More generally, if $\hat{\phi}$ is non-Hermitian, $\langle \hat{\phi}|\psi_i\rangle$ will be called the right eigen-state twisted vector and $\hat{\phi}^\dagger|\psi_i\rangle$ the left eigen-state twisted vector.

A quantum system described by Hamiltonian $\hat{H}$ possesses an orthonormal set of eigen-wavefunctions $|\psi_i\rangle$ ($i = 0, 1, 2, \cdots$) in increasing eigen-energies $\epsilon_i$ with $i = 0$ denoting the ground state. When the coarse-grained order parameter $\hat{\phi}$ is Hermitian, the ground state twisted vector $|\psi(0)\rangle$ can be expanded $|\psi(0)\rangle = \sum_{i=0}^{\infty} a_i|\psi_i\rangle$ into the eigen-basis. With the help of the Schrödinger equation $i\partial\psi(t)/\partial t = \hat{H}|\psi(t)\rangle$ ($h = 1$ assumed throughout) for the system wave function $|\psi(t)\rangle$, we obtain

$$f(t) = \lim_{V \to \infty} \langle \psi_0|e^{iHt}\hat{\phi}(0)e^{-iHt}\hat{\phi}(0)|\psi_0\rangle = \lim_{V \to \infty} e^{\epsilon_0 t} \langle \psi|e^{-iHt}|\psi\rangle = \lim_{V \to \infty} \sum_{j=0}^{\infty} c_j e^{-i(\epsilon_j - \epsilon_0)t},$$

(3)

where $c_j \equiv |a_j|^2$ denotes the weight of the ground state twisted vector, $c_0$ the corresponding ground state weight, and $c_j$ (with $j > 0$) the excited state weight.

When the coarse-grained order parameter $\hat{\phi}$ is non-Hermitian, we use $|\psi^{(l)}\rangle$ and $|\psi^{(r)}\rangle$ to denote respectively the left and right ground state twisted vectors and expand them analogously in the eigen-basis to arrive at $|\psi^{(l)}\rangle \equiv \hat{\phi}^\dagger|\psi_0\rangle = \sum_{i=0}^{\infty} b_i|\psi_i\rangle$ and $|\psi^{(r)}\rangle \equiv \hat{\phi}|\psi_0\rangle = \sum_{i=0}^{\infty} a_i|\psi_i\rangle$. In this case, we find

$$f(t) = \lim_{V \to \infty} \langle \psi_0|e^{iHt}\hat{\phi}(0)e^{-iHt}\hat{\phi}(0)|\psi_0\rangle = \lim_{V \to \infty} e^{\epsilon_0 t} \langle \psi^{(l)}|e^{-iHt}|\psi^{(r)}\rangle = \lim_{V \to \infty} \sum_{j=0}^{\infty} c_j e^{-i(\epsilon_j - \epsilon_0)t},$$

(4)

with $c_j \equiv b_j^* a_j$ the weight of the ground state twisted vector instead. Similarly $c_0$ and $c_j$ (with $j > 0$) respectively denote the ground and excited state weights.

Given an order parameter $\hat{\phi}$, we see quite generally $f(t)$ will be a superposition of many harmonic functions with amplitudes $c_j$ and frequencies $\omega_j \equiv \epsilon_j - \epsilon_0$. Nontrivial time dependence of the two-time auto-correlation function is thus imbedded in the energy spectrum of Hamiltonian $\hat{H}$ as well as the weight distributions of twisted vector. For a CTC order to exist, one of the excited state weight must be non-vanishing. In rare cases, $f(t)$ can include harmonics of commensurate frequencies.

If $f(t)$ is a constant, the time dependence will be trivial. However, a subtlety appears when $f(t)$ vanishes. Since what we are after is the system’s explicit temporal behavior or time dependence, which is easily washed out to $f(t) = 0$ by accident from a zero norm of twisted vector. Such a difficulty can be mitigated by multiplying the system volume $V$, i.e., using the twisted vector $|\psi| \to V|\psi|$ to check if $F(t) = V^2f(t) = 0$ vanishes or not. If $f(t) = 0$ but $F(t)$ is a periodic function, the system is considered a generalized CTC [54].

The above analysis can be extended to non-Hermitian systems in a straightforward manner as long as a plausible “ground state” for the non-Hermitian system can be identified, e.g. by requiring its eigen-energy to possess the largest imaginary part or the smallest norm. In this case, if energy eigen-value $E_i$ possesses a non-vanishing imaginary part, a term $\propto e^{i\Im(E_i)t}$ will contribute to two-time auto-correlation function, leading to unusual time functional order. When ground state twisted vector is replaced by excited ones, our formulation above directly generalizes to TC of excited eigen-states [55]. Hence, phases for a quantum many-body system can be classified according to time order, which is defined based on WO CTC [28] as follows. If $f(t)$ exhibits nontrivial time dependence, time order exists. If $f(t) = 0$, but $F(t)$ displays nontrivial time dependence instead, generalized time order exists.

More specifically, if $f(t) = \text{const.} \neq 0$, the system exhibits time trivial order. The same applies when $f(t) = 0$ but $F(t) = \text{const.} \neq 0$. For all other situations, nontrivial time order exhibits in a system. A complete ground state classification for a quantum many-body system is shown in Table I, categorizing different phases based on temporal behaviors of its two-time auto-correlation function $f(t)$ or $F(t)$. As shown in the supplementary material [56], the above discussions on time order can be extended to finite temperature as well.

The approach above presents an explicit procedure for detecting time order, albeit with reference to an order parameter. Perhaps more appropriately, it should be called order parameter assisted time order or symmetry-based time order, to emphasize its reference to symmetry (based on an order parameter) of a quantum many-body system when classification of ground state phases with their temporal properties is carried out. The operational procedure for the classification is straightforward with the introduced twisted vector, which easily distinguishes between different time order phases from time trivial ones. It is reasonable to expect that transitions between different time order phases can occur, reminiscent of phase transitions in the LGW spontaneous symmetry breaking paradigm.

According to the WO no-go theorem [28], the ground state or the Gibbs ensemble for a general many-body Hamiltonian with not-too-long range interactions belongs to time trivial order. At first sight, this seems to sweep many important models of condensed matter physics into the same boring class of time trivial order phase. How-
TABLE I. Classification of the ground state phases for a quantum many-body system

| Phase                                | Property of two-time auto-correlator |
|--------------------------------------|--------------------------------------|
| Time trivial order                   |                                       |
| Time crystalline order               | \( f(t) = \text{const.} \neq 0 \) or \( f(t) = 0, F(t) = \text{const.} \neq 0 \) |
| Time quasi-crystalline order         | \( f(t) \) is periodic and nonvanishing |
| Time functional order                | \( f(t) \) is aperiodic with beats from two incommensurate frequencies |
| Generalized time crystalline order   | \( f(t) = 0, F(t) \) is periodic and nonvanishing |
| Generalized time quasi-crystalline order | \( f(t) = 0, F(t) \) contains beats from two incommensurate frequencies |
| Generalized time functional order    | \( f(t) = 0, F(t) \) is aperiodic     |

ever, such a blunt conclusion should not be drawn so directly as we still don’t understand whether different quantum phases of matter can be further identified by the values of their two-time auto-correlation function \( f(t) \) or not? Inspired by the recent study of CTC [51], we believe more time order phases will be uncovered and further understanding will be gained with time.

**Time order in a spin-1 atomic condensate.**—A spin-1 atomic Bose-Einstein condensate (BEC) under single spatial mode approximation (SMA) [57–59] is described by the following Hamiltonian

\[
\hat{H} = \frac{c_2}{2N} \left( 2\hat{N}_0 - 1 \right) \left( \hat{N}_1 + \hat{N}_{-1} \right) + 2 \left( \hat{a}_1^{\dagger} \hat{a}_{-1} \hat{a}_0 + \text{h.c.} \right) - p \left( \hat{N}_1 - \hat{N}_{-1} \right) + q \left( \hat{N}_1 + \hat{N}_{-1} \right), \tag{5}
\]

where \( \hat{a}_{m_F} \) (\( m_F = 0, \pm 1 \)) \( (\hat{a}_{m_F}^\dagger) \) denotes the annihilation (creation) operator for atom in the ground state manifold \( |F = 1, m_F \rangle \) with corresponding number operator \( \hat{N}_{m_F} = \hat{a}_{m_F}^\dagger \hat{a}_{m_F} \). The total atom number \( \hat{N} = \hat{N}_1 + \hat{N}_0 + \hat{N}_{-1} \) is conserved. \( p \) and \( q \) are linear and quadratic Zeeman shifts that can be tuned independently [60], while \( c_2 \) describes the strength of spin exchange interaction. Unless otherwise noted, we will take \( |c_2| = 1 \) as unit of energy in the following.

The validity for the above model is well established based on extensive studies of spinor BEC over the years [61–64], supported by a rich variety of experimental investigations [60, 65–67]. These studies [64, 66, 68, 69] often employ \( \hat{n}_{\text{sum}} \equiv N_{\text{sum}} / N \) with \( N_{\text{sum}} = \hat{N}_1 + \hat{N}_{-1} = N - \hat{N}_0 \) as a suitable order parameter, which measures the fractional population in the spin states \( |1, 1 \rangle \) and \( |1, -1 \rangle \), with \( N \) assuming the role of system size. The ground state twisted vector then becomes \( |v\rangle \equiv \hat{n}_{\text{sum}} |\psi_0\rangle \), and

\[
f(t) = \lim_{N \to \infty} \langle \hat{n}_{\text{sum}}(t) \hat{n}_{\text{sum}}(0) \rangle, \tag{6}
\]

\[
F(t) = \lim_{N \to \infty} \left\langle \hat{N}_{\text{sum}}(t) \hat{N}_{\text{sum}}(0) \right\rangle. \tag{7}
\]

We will concentrate on the zero magnetization \( F_z = 0 \) subspace and employ exact diagonalization (ED) to calculate eigen-states. Since \( F_z \) is conserved, we can set \( p = 0 \). The overall time order phase diagram we find is shown in Fig. 1. With ferromagnetic interaction \( c_2 < 0 \) as for \(^{87}\text{Rb} \) atoms, the critical quadratic Zeeman energy \( q/|c_2| = 2 \) splits the whole region into time trivial order (TT) phase for smaller \( q \) that observes TTS, and generalized time crystalline (gTC) order phase for \( q/|c_2| > 2 \) where TTS is spontaneously broken. The latter (gTC) phase is found to coincide with the polar phase [64]. Limited by computation resource used for ED calculation, the finite system sizes we explored prevents us from mapping out the finer details in the immediate neighborhood of \( q = 2 \), where time order properties need to be further elaborated. On the other hand, with antiferromagnetic interaction \( c_2 > 0 \) for \(^{23}\text{Na} \) atoms, we find \( q = 0 \) separates TT phase from gTC order.

![FIG. 1. Time order phase diagram for spin-1 atomic BEC, where TT and gTC respectively denote time trivial and generalized time crystalline order. The (hashed) line segment region of \( c_2 = 0 \) for noninteracting system is to be excluded. More detailed discussions, including the dependence on system size and possible approaches for detecting the various time ordered phase is presented in the SM [56].](image-url)

Compared to the recent progress and understanding of topological order [9, 10], one could try to develop a framework for entanglement-based time order rather than symmetry-based time order we introduce here in this study. Quantum entanglement in a many-body system is responsible for topological order, whose origin lies at the tensor product structure of the quantum many-body Hilbert space \( \mathcal{H}_{\text{tot}} = \bigotimes_i \mathcal{H}_i \) with \( \mathcal{H}_i \) the finite-dimensional Hilbert space for site-\( i \). An entanglement-based time or-
der therefore calls for a combined investigation to exploit quantum entanglement and temporal properties of a quantum many-body system in the future.

While ground state phases of a quantum many-body system are mostly classified with its Hamiltonian based on two paradigms: LGW symmetry breaking order parameter and topological order, this Letter proposes to study phases from the time dimension with time order, which hopefully could also be developed into a paradigm. For every quantum many-body Hamiltonian $\hat{H}$, there accompanies its evolution operator $e^{-i\hat{H}t}$. Such a dual establishes a solid foundation for time order and provides further information concerning time domain quantum phases. Through time order, one focuses on temporal structure of the evolution operator $e^{-i\hat{H}t}$. The symmetry-based time order therefore unifies LGW paradigm with the concept of time order; while an entanglement-based time order could amalgamate topological order paradigm (or entanglement beyond that) with time order. A more basic definition for time order will extend in-depth exploration in this direction.

In conclusion, understanding phases of matter constitutes a corner stone of contemporary physics. This study shows that information from time dimension can be employed to provide a new angle to further this understanding. Based on a generalization of the WO definition for CTC, we introduce time order, provide an operational definition, and present a complete classification of all time order phases. Besides the time crystalline order which has already attracted some attention, other phases we identify, e.g. time quasi-crystalline order and time functional order, represents exciting new possibilities.

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Supplementary Material for “Detecting Time Crystal and Classifying Quantum Phases with Time Order”

Tie-Cheng Guo and Li You
1State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China
2Frontier Science Center for Quantum Information, Beijing, China

This supplementary provides supporting material and related details for the presentation of the main text. It is organized as follows: in Sec. I we extend the discussion of time order to finite temperatures; in Sec. II, we present calculation details related to the spin-1 atomic Bose-Einstein condensate (BEC) example considered. Finally, as a more straightforward approach to understand numerical results, we present a variational approach for treating the polar ground state of a spin-1 BEC in Sec. III.

I. TIME ORDER AT FINITE TEMPERATURE

At finite temperature $T$, excited states will be populated, which can be taken into account with the Gibbs ensemble $\rho \equiv e^{-\beta H}/Z$, where $Z \equiv \text{Tr} e^{-\beta H}$ denotes the partition function and $\beta \equiv 1/T$ the inverse temperature. We then find

$$f(t) \rightarrow \lim_{V \rightarrow \infty} \text{Tr} \left( e^{i\hat{H}t} \hat{\phi}(0)e^{-i\hat{H}t} \hat{\phi}(0)\rho \right)$$

$$= \lim_{V \rightarrow \infty} \sum_{k=0}^{\infty} \langle \psi_k | e^{i\hat{H}t} \hat{\phi}(0)e^{-i\hat{H}t} \hat{\phi}(0) \frac{e^{-\beta H}}{Z} | \psi_k \rangle$$

$$= \lim_{V \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{Z} e^{i\epsilon_k t - \beta \epsilon_k} \langle v_k | e^{-i\hat{H}t} | v_k \rangle$$

$$= \lim_{V \rightarrow \infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{Z} c_{jk} e^{-\beta \epsilon_k} e^{-i(\epsilon_j - \epsilon_k)t},$$

(1)

where $|v_k\rangle$ is the eigen-state twisted vector for $|\psi_k\rangle$, and $c_{jk}$ its associated weight. Analogously, for the non-Hermitian case, we find

$$f(t) = \lim_{V \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{Z} e^{i\epsilon_k t - \beta \epsilon_k} \langle v_k^{(l)} | e^{-i\hat{H}t} | v_k^{(r)} \rangle$$

$$= \lim_{V \rightarrow \infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{Z} c_{jk} e^{-\beta \epsilon_k} e^{-i(\epsilon_j - \epsilon_k)t},$$

(2)

where $|v_k^{(l)}\rangle$ and $|v_k^{(r)}\rangle$ are the left and right twisted vectors for eigen-state $|\psi_k\rangle$, $c_{jk}$ is the corresponding weight.

It is easily noted that $f(t)$ at finite temperature contains contributions from all eigen-states of the quantum many-body system $\hat{H}$, with a temperature dependent weight factor for different energy level, but $f(t)$ remains to include contributions from different periodic functions. Hence the quantum phase classification task essentially remains the same (including its possible reference to $F(t)$) as is shown in the Letter for the ground state. At finite temperature, due to thermal excitations to ground state, the temporal behavior will be more complex thus opening up for more interesting possibilities, e.g., to control time order phases and to study crossover or driven phase transitions between different time order phases.

II. TIME ORDER IN A SPIN-1 ATOMIC BEC

For typical interaction parameters of a spin-1 BEC (e.g., of ground state $^{87}$Rb or $^{23}$Na atoms) in a tight trap, spin domain formation is energetically suppressed when the atom number is not too large as spin-dependent interaction strength is much weaker than spin-independent interaction [1–4]. This facilitates a single-spatial-mode approximation (SMA) by assuming all spin states share the same spatial wave function $\phi(r)$, which effectively decouples the spatial
degrees of freedom from the spin and results in the following Hamiltonian [2, 5]
\[
\hat{H} = \frac{c_2}{2N} \left[ (2\hat{N}_0 - 1) \left( \hat{N}_1 + \hat{N}_-1 \right) + 2 \left( \hat{a}_1 \hat{a}_-1 \hat{a}_0 \hat{a}_0 + \text{h.c.} \right) \right] - p \left( \hat{N}_1 - \hat{N}_-1 \right) + q \left( \hat{N}_1 + \hat{N}_-1 \right),
\]
(3)
for the model many body system, where \(\hat{a}_{m_F}(m_F = 0, \pm 1)\) is the annihilation operator of the ground manifold state \(|F = 1, m_F\rangle\) with corresponding number operator \(\hat{N}_{m_F} = \hat{a}_{m_F}^\dagger \hat{a}_{m_F}\). \(p\) and \(q\) are linear and quadratic Zeeman shifts which could be tuned independently in experiments [6], while \(c_2\) denotes spin exchange interaction strength. The total particle number operator \(\hat{N} = \hat{N}_1 + \hat{N}_0 + \hat{N}_-1\) as well as the longitudinal magnetization operator \(\hat{F}_z = \hat{N}_1 - \hat{N}_-1\) are both conserved. Thus, linear Zeeman shift can be set to \(p = 0\) effectively.

As discussed in the main text, a suitable order parameter for this model system is \(\tilde{n}_{\text{sum}} \equiv \hat{n}_{\text{sum}}/N\) \((\hat{n}_{\text{sum}} = \hat{N}_1 + \hat{N}_-1 = N - \hat{N}_0)\), which measures the fractional atomic population in the states \(|1, 1\rangle\) and \(|1, -1\rangle\) and \(N\) assumes the role of system size. Following our formulation and denoting the system energy eigen-state by \(|\psi_i\rangle\) \((i = 0, 1, 2, \cdots)\) with increasing eigen-energy \(\epsilon_i\), the ground state twisted vector becomes \(|v\rangle \equiv |\tilde{n}_{\text{sum}}|\psi_0\rangle = \sum_{i=0}^{\infty} a_i|\psi_i\rangle\), with \(a_i = \langle \psi_i | v \rangle\) its expansion coefficient on the eigen-state \(|\psi_i\rangle\). We find
\[
f(t) = \lim_{N \to \infty} \langle \tilde{n}_{\text{sum}}(t)\tilde{n}_{\text{sum}}(0) \rangle = \lim_{N \to \infty} \sum_{j=0}^{\infty} b_j e^{-i(\epsilon_j - \epsilon_0)t},
\]
(4)
where \(b_j \equiv |a_j|^2\) is the weight of the ground state twisted vector, \(b \equiv \sum_{j=0}^{\infty} b_j\) the total weight, and
\[
F(t) = \lim_{N \to \infty} \langle \hat{N}_{\text{sum}}(t)\hat{N}_{\text{sum}}(0) \rangle = \lim_{N \to \infty} \sum_{j=0}^{\infty} B_j e^{-i(\epsilon_j - \epsilon_0)t},
\]
(5)
where \(A_i = N\langle \psi_i | v \rangle\), \(B_j \equiv |A_j|^2\) is the weight of the enlarged ground state twisted vector, and \(B \equiv \sum_{j=0}^{\infty} B_j\) the total weight.

Our study below is for the zero magnetization \(\hat{F}_z = 0\) subspace and employs exact diagonalization (ED) to calculate eigen-states as well as eigen-energies. The overall time order phase diagram for spin-1 BEC is shown in the main Letter. For ferromagnetic interaction \(c_2 < 0\), the critical quadratic Zeeman energy \(q/|c_2| = 2\) splits the whole region into time trivial order (TT) phase for smaller \(q\) that observes TTS, and the generalized time crystalline (gTC) order phase for \(q/|c_2| > 2\) where TTS is spontaneously broken. The latter (gTC phase) is found to coincide with the ground state polar phase. The available computation resource limits the calculation to a finite system size, which prevents us from mapping out the exact details in the immediate neighborhood of \(q = 2\), where further elaboration is need for its time order properties. On the other hand, for antiferromagnetic interactions, we find \(q = 0\) separates TT phase from and gTC order.

In Figure 1, the weights for the ground state as well as for the low-lying excited states are shown as functions of \(q\) for a typical system size of \(N = 10000\). Only the ground state weight \(b_0\) is nonvanishing in the \(q < 2\) \((q < 0)\) region for ferromagnetic (antiferromagnetic) interactions, but the total weight \(b\) is zero in the \(q > 2\) \((q > 0)\) region for ferromagnetic (antiferromagnetic) interaction, which prompts us to examine further the enlarged weights \(B_i\) corresponding to the bulk order parameter. For ground and the first excited states, the volume enlarged weights \(B_{0,1}\) are found to be nonvanishing, although both decrease as \(q\) increases and grow with \(N\) as \(q\) approaches \(q = 2\) \((q = 0)\) for ferromagnetic (antiferromagnetic) interaction. However, as mentioned above, limited to a system size of \(N = 10000\) by computation resource in the ED calculation, we cannot exactly map out the behavior near \(q = 2\) \((q = 0)\) for ferromagnetic (antiferromagnetic) interaction. This consequently leaves empty for \(q\) in region \([2.0, 2.02] ([0.0, 0.01])\) for ferromagnetic (antiferromagnetic) interaction.

The dependence on system size \(N\) is clearly revealed by Fig. 2, with the enlarged weights in the gTC regime attain fixed values as system approaches thermodynamic limit \((N \to \infty)\). In regions away from \(q = 2\) \((q = 0)\) for ferromagnetic (antiferromagnetic) interaction, ED numerics can always approach thermodynamic limit except for the immediate neighbourhood near \(q = 2\) \((q = 0)\), where we infer with confidence the tendencies to divergence of the weights \(B_{0,1}\) as \(q\) approaches \(q = 2\) \((q = 0)\).

The time evolution of two-time auto-correlation function \(F(t)\) is plotted in Fig. 3 (a) for ferromagnetic and (c) for antiferromagnetic interactions, while Figs. 3 (b) and (d) display energy gaps between ground and the first excited states as a function of \(q\) for ferromagnetic and antiferromagnetic interactions respectively at a system size of \(N = 5000\). The behavior of \(F(t)\) is quantitatively consistent with that of the weights \(B_i(q)\) \((i = 0, 1)\) shown in Fig. 1 and the energy gap \(\epsilon_1 - \epsilon_0\) shown in Figs. 3 (b) and (d).

At finite temperature, excited states come into play by also contributing to the correlation function. We find the gTC order hosted in the polar phase persists for both ferromagnetic and antiferromagnetic interactions. The corresponding time evolution and Fourier transform of \(F(t)\) are shown in Fig. 4, calculated for \(N = 500\) at a temperature of
FIG. 1. Weights of ground state twisted vector in the ground and low-lying excited states as functions of $q$ at system size $N = 10000$. The upper panel is for ferromagnetic interaction, where weights $b_i$ for $q < 2$ are shown in (a) while weights $B_i$ for $q > 2$ are shown in (b). The lower panel is for antiferromagnetic interaction, where weights $b_i$ for $q < 0$ are shown in (c) while weights $B_i$ for $q > 0$ are shown in (d).

FIG. 2. Weights of ground state twisted vector in the ground and low-lying excited states as functions of system size $N$ at $q = 2.1$ for ferromagnetic interaction (a) and at $q = 0.2$ for antiferromagnetic interaction (b).

$\beta = 1/T = 1$. The Fourier transform is performed for $\text{Re}(F)$ over $t = [0, 1000]$ with the zero frequency (DC) component subtracted or for $\text{Im}(F)$. The upper (lower) panel corresponds to ferromagnetic (antiferromagnetic) interaction at $q = 3$ ($q = 2$). For ferromagnetic interaction, two distinct frequency components are clearly identified for $q = 3$, associated with the two different energy level gaps. Thus, the gTC phase remains at finite temperature. Moreover, we also find a generalized time quasicrystalline order phase assuming the two frequencies are incommensurate, by fine tuning their corresponding energy gaps such that the relation $\Delta_1/\Delta_2 = m_1/m_2$ with $m_1$ and $m_2$ being co-primes is not satisfied. The gTC phase at finite temperature here is robust which is in contrast to the melting behavior of CTC shown in Ref. [7].

Finally, we hope to address the critical question about how could this time order, sort of a perpetual time dependence, can be observed. We note the bulk two-time auto-correlation function introduced $F(t) = \lim_{N \to \infty} \langle \hat{N}_{\text{sum}}(t)\hat{N}_{\text{sum}}(0) \rangle$ denotes nothing but the ground state (averaged) conditional outcome of measuring
FIG. 3. $F(t)$ for different $q$ as a function of time $t$. The solid and dotted lines correspond to Re($F$) and Im($F$) respectively. The red, green, and blue lines correspond to $q = 2.5$, $q = 3$, and $q = 5$ respectively for ferromagnetic interaction (a). The red, green, and blue lines correspond to $q = 0.7$, $q = 1$, and $q = 3$ respectively for antiferromagnetic interaction (b). The energy gap between ground and the first excited state $\epsilon_1 - \epsilon_0$ as a function of $q$ for ferromagnetic (b) and antiferromagnetic interactions (d), at system size $N = 5000$.

FIG. 4. $F(t)$ as a function of time $t$ at $q = 3$ for ferromagnetic interaction (a) and $q = 2$ for antiferromagnetic interaction (c). The red and blue solid lines respectively correspond to Re($F$) and Im($F$). The Fourier transform spectrum $\tilde{F}(\nu)$ of Re($F$) or Im($F$) with $\nu = 1/T$ the frequency, $T$ the period, for ferromagnetic (b) and antiferromagnetic (d) interactions, at temperature $\beta = 1$ and system size $N = 500$. 
$N_{\text{sum}}(t)$ at $t$ after starting with $N_{\text{sum}}(0)$ initially. The dynamics of $F(t)$ follows that of $N_{\text{sum}}(t)$ as in quantum regression theorem. Given the system is well controlled, highly reproducible, one can simply detect $F(t)$ by measuring $N_{\text{sum}}(t)$, although for each measurement at an instant $t$, a condensate is destroyed, and a follow up one will have to be prepared as closely as possible in every respects (through selection and post-selection) and be measured at a different $t' > t$. Thus, a plausible way to detecting the ground state time dependence will require reconstructing the time dependence of $F(t)/N_{\text{sum}}(0)$. As along as the oscillation amplitude is more than a few percent, it will be easily observable with not too much difficulty, although such a reconstruction will still be difficult as $N_{\text{sum}}(0)$ can be rather small compared to $N_0 \sim N$ in the polar state. Alternatively, one can perhaps start from a twin-Fock state, i.e., by preparing an initial state with $N_{\text{sum}}(0) \sim N$.

In Figure 5(a) we show the behavior of oscillation amplitude for $F(t)/N_{\text{sum}}(0)$. The time dependence of $F(t)/N_{\text{sum}}(0)$ at $q = 2.5$ for ferromagnetic interaction is shown in Fig. 5(b).

![Figure 5](image_url)

**FIG. 5.** (a) $B_i/N_{\text{sum}}(0)$ as a function of $q$ for ferromagnetic interaction. (b) $F(t)/N_{\text{sum}}(0)$ as a function of time $t$ at $q = 2.5$ for ferromagnetic interaction. The red and blue solid lines respectively correspond to the real and imaginary part of $F(t)/N_{\text{sum}}(0)$.

### III. A VARIATIONAL POLAR STATE FOR FERROMAGNETIC SPIN-1 BEC

One might naively expect that nothing particularly interesting could happen in the polar phase of a ferromagnetic spin-1 BEC, where essentially all atoms reside in the single particle state $|1,0\rangle$. Nevertheless, due to the competition between spin exchange interaction $c_2$ and quadratic Zeeman shift $q$, the ground state of our system differs from $|N_1 = 0, N_0 = N, N_{-1} = 0\rangle$, which can be affirmed based on a simple variational analytical calculation given in this section.

We use the number state basis $|N_1, N_0, N_{-1}\rangle \equiv |[N], M, k\rangle$, where $N_m$ denotes occupation number of the $m_F$ magnetic state, $M \equiv N_1 - N_{-1}$, and $k \equiv N_{-1}$. We take the following ground state variational ansatz $|\psi_0\rangle = \frac{1}{\sqrt{1+r^2}} ((0, N, 0) + a|1, N - 2, 1\rangle)$ for the polar state of ferromagnetic spin-1 BEC, where $a = r e^{i\phi}$ is a (complex) variational parameter with $r$ and $\phi$ real. From Eq. (3) and (assumed) $p = 0$, the ground state energy follows from

$$E = \frac{\langle \psi_0 | H | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}$$

$$= \frac{1}{1+a^*a} \left[ \left( \frac{c_2(2N-5)}{N} + 2q \right) a^*a + c_2 \sqrt{\frac{N-1}{N}} (a^* + a) \right]$$

$$= \frac{1}{1+r^2} \left[ \left( \frac{c_2(2N-5)}{N} + 2q \right) r^2 + 2c_2 \sqrt{\frac{N-1}{N}} r \cos(\phi) \right].$$

(6)

We see the extreme value (the minimum) of $E$ is reached when $\cos(\phi) = \pm 1$, i.e., for a real variational parameter $a$, which will be assumed from now on. This gives

$$E = \frac{x_1 a^2 + x_2 a}{1 + a^2},$$

(7)
with \( x_1 = \frac{c_2(2N-5)}{N} + 2q \) and \( x_2 = 2c_2\sqrt{\frac{N-1}{N}} \). The derivative of the energy function \( E(a) \) is

\[
E'(a) = -\frac{x_2a^2 + 2x_1a + x_2}{(1+a^2)^2},
\]

which determines the locations for the extreme values

\[
a_\pm = \frac{1}{2c_2\sqrt{N(N-1)}} \left[ c_2(2N-5) + 2Nq \pm N\sqrt{\frac{c_2^2(8N^2 - 24N + 25)}{N^2} + \frac{4c_2q(2N - 5)}{N} + 4q^2} \right],
\]

and the corresponding extreme values are

\[
E_\pm = c_2 + q \pm \frac{1}{2}\sqrt{4q^2 + 8c_2q + 8c_2^2 + \frac{-24c_2^2 - 10c_2q}{N} + \frac{25c_2^2}{N^2} - \frac{5c_2}{2N}}.
\]

In the thermodynamic limit \( N \to \infty \), they reduce respectively to \( a_\pm = 1 + \frac{q}{c_2} \pm \sqrt{\frac{2c_2^2 + 2c_2q + q^2}{c_2^2}} \) and \( E_\pm = c_2 + q \pm \sqrt{2c_2^2 + 2c_2q + q^2} \). The left and right asymptotic value for the energy function \( E(a) \) is therefore

\[
E(a) = c_2(2 - \frac{5}{N}) + 2q, \quad \text{when } a \to \pm \infty.
\]

For ferromagnetic interaction \((c_2 < 0)\), \( E_- \) assumes the minimum, which corresponds to the ground state \( |\psi_0\rangle = \frac{1}{\sqrt{1 + a_-^2}} (|0, N, 0\rangle + a_-|1, N - 2, 1\rangle) \) with \( N_{\text{sum}} = 2a_-^2/(1 + a_-^2) \), and \( a_- = 1 + \frac{q}{c_2} - \sqrt{\frac{2c_2^2 + 2c_2q + q^2}{c_2^2}} \) in the thermodynamic limit \( N \to \infty \).

Despite of the vanishing order parameter \( n_{\text{sum}} \) in the polar phase (here the gTC order phase from the time order perspective), the enlarged quantity \( N_{\text{sum}} \) retains a finite value. Hence, the physics we present here clearly belongs to the realm of quantum effects, beyond the reach of mean-field theory.

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