Hidden symmetry of the quantum Calogero-Moser system

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Abstract
Hidden symmetry of the quantum Calogero-Moser system with the inverse-square potential is explicitly demonstrated in algebraic sense. We find the underlying algebra explaining the super-integrability phenomenon for this system. Applications to related multi-variable Bessel functions are also discussed.

\textsuperscript{1}Supported by the grant from Forskerakademiet through a Guest Professorship.
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1 Introduction

It is well-known \[1\] that the rational \(N\)-particle Calogero-Moser problem in classical mechanics is superintegrable. It means that to each of the integrals of motion \(H_k, k = 1, \ldots, N\), in involution w.r.t. the standard Poisson bracket one can find a set of \(N - 1\) additional algebraic, functionally independent integrals \(B_r^{(k)}, r = 1, \ldots, N - 1\), which are all in involution with the \(H_k\). So, generally, one has a \(N \times N\) table of the form:

\[
\begin{pmatrix}
  H_1 & B_1^{(1)} & \cdots & B_1^{(N-1)} \\
  H_2 & B_2^{(2)} & \cdots & B_2^{(N-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  H_N & B_N^{(N)} & \cdots & B_N^{(N-1)}
\end{pmatrix}.
\]

The entries of the first column in this table are distinguished by the fact that each of them defines a superintegrable system, i.e. the \(k\)th entry \(H_k\) is in involution with those in the first column and in the \(k\)th row. Each entry of the rest of the table, so anyone of the \(B_r^{(k)}\)'s, defines an integrable system with the integrals of motion being some functions of the \(H_k\)'s.

In the present paper we show how to construct the same kind of table for the quantum Calogero-Moser system with the inverse square potential. The 'quantum table' will have the same properties as the 'classical table', i.e. the first column will give us the partial differential operators (PDOs) of the \(k\)th order \((k = 1, \ldots, N)\) with the superintegrability property, while \(B_r^{(k)}\)'s will be the PDOs of the following orders: \(\text{order}(B_r^{(k)}) = k + r - 1\). We derive the non-linear algebraic relations for all these operators.

2 Operators of Dunkl’s type

Let \(P_{ij}\) be the permutation operators acting on the indices \(i\) and \(j\), i.e. the generators of the permutation group \(S_N\) of \(N\) numbers \((i, j = 1, \ldots, N)\):

\[
P_{ij} = P_{ji}, \quad (P_{ij})^2 = 1, \quad P_{ij}P_{jl} = P_{ji}P_{li} = P_{li}P_{ij}.
\]

Hereafter all indices run from 1 to \(N\), unless otherwise stated.

Introduce the operators

\[
\Delta_i = \sum_{j \neq i} \frac{1}{x_{ij}} (1 - P_{ij}), \quad i = 1, \ldots, N,
\]

where we use the notation

\[
x_{ij} = x_i - x_j,
\]

and have the following algebra for the \(P_{ij}\) and \(x_k\):

\[
[x_i, x_j] = 0, \quad P_{ij}x_k = x_k P_{ij} \quad \text{if} \quad \{k\} \cap \{i, j\} = \emptyset, \quad (2.3)
\]

Then it is straightforward to verify the following relations for the operators \(P_{ij}, \Delta_k, \text{and } x_l\).
Proposition 2.1

\[ (a) \quad P_{ij} \Delta_k = \Delta_k P_{ij} \quad \text{if} \quad \{k\} \cap \{i, j\} = \emptyset, \]
\[ P_{ij} \Delta_j = \Delta_i P_{ij}, \quad (2.4) \]
\[ (b) \quad [\Delta_i, P_{ik}] = 0 \quad \Leftrightarrow \quad [\Delta_i, \Delta_k] = 0, \quad (2.5) \]
\[ (c) \quad [x_i, \Delta_j] = (1 - \delta_{ij}) P_{ij} - \delta_{ij} \sum_{k \neq i} P_{ik}. \quad (2.6) \]

Introduce the differential operators:

\[ \partial_i = \frac{\partial}{\partial x_i}, \quad [\partial_i, x_j] = \delta_{ij}, \]

which are closed together with \( P_{ij} \) to the same kind of algebra as in (2.3):

\[ [\partial_i, \partial_j] = 0, \]
\[ P_{ij} \partial_k = \partial_k P_{ij} \quad \text{if} \quad \{k\} \cap \{i, j\} = \emptyset, \quad (2.7) \]
\[ P_{ij} \partial_j = \partial_i P_{ij}. \]

It is not difficult to verify the following algebraic relations for the operators \( \partial_i, \Delta_j, \) and \( P_{kl}. \)

Proposition 2.2

\[ (a) \quad [\partial_i, \Delta_j] = \frac{1}{x_{ij}} (1 - P_{ij}) + \frac{1}{x_{ij}} (\partial_i - \partial_j) P_{ij}, \quad i \neq j, \]
\[ (b) \quad [[\partial_i, \Delta_j], P_{ij}] = 0 \quad \Leftrightarrow \quad [\partial_i, \Delta_j] = [\partial_j, \Delta_i], \quad i \neq j. \quad (2.8) \]

The final operators that will be introduced in this Section are the Dunkl’s type operators

\[ D_i = \partial_i + g \Delta_i, \quad g \in \mathbb{R}, \quad (2.9) \]

which are the differential-permutation operators acting on the function space \( f(x_1, x_2, \ldots, x_N) \in C^\infty(\mathbb{R}^N). \) Let us now prove the following Theorem about the algebraic relations for the operators \( P_{ij}, D_k, \) and \( x_l. \)

Theorem 2.3

\[ (a) \quad P_{ij} D_k = D_k P_{ij} \quad \text{if} \quad \{k\} \cap \{i, j\} = \emptyset, \]
\[ P_{ij} D_j = D_i P_{ij}, \quad (2.10) \]
\[ (b) \quad [D_i, D_j] = 0, \quad (2.11) \]
\[ (c) \quad [D_i, x_j] = \delta_{ij} (1 + g \sum_{k \neq i} P_{ik}) - (1 - \delta_{ij}) g P_{ij}. \quad (2.12) \]

Proof  The statement (a) follows from (2.9), (2.7), (2.4). The commutativity of the operators \( D_i \) results from (2.9), (2.8), (2.7), (2.3). The last commutator (c) is easy to derive from the relation (2.6).

The commutative operators \( D_i \) were first introduced and studied in [2].
3 Quantum Calogero-Moser system

Let us define a quantum integrable system by fixing the complete set of the symmetric polynomials 
\[ I_k, \quad k = 1, \ldots, N, \] 
onumber
on \( N \) Dunkl’s operators \( D_i \) as corresponding mutually commuting integrals of motion
\[ I_k = \sum_i D_i^k, \quad k = 1, \ldots, N. \quad (3.1) \]

The first two integrals have the form
\[ I_1 = \sum_k \partial_k, \]
\[ I_2 = \sum_k \partial_k^2 + 2g \sum_{i<j} \frac{1}{x_{ij}} \left[ \partial_i - \partial_j - \frac{1}{x_{ij}} (1 - P_{ij}) \right]. \]

Introduce now the operation \( \text{Res} \) which acts on operators sending symmetric functions to symmetric ones (i.e. operators leaving invariant the sub-space of symmetric functions) and means the restriction of these operators on the sub-space of symmetric functions. Then, for instance,
\[ H_2 \equiv \text{Res}(I_2) = \sum_k \partial_k^2 + 2g \sum_{i<j} \frac{1}{x_{ij}} (\partial_i - \partial_j), \quad (3.2) \]

since the operator \( 1 - P_{ij} \) vanishes on symmetric functions. Define a set of the PDOs \( H_k \) of orders from 1 to \( N \) by the rule:
\[ H_k = \text{Res}(I_k), \quad k = 1, \ldots, N. \quad (3.3) \]

Notice that all these operators (after applying the operation \( \text{Res} \)) become purely differential operators, i.e. do not contain any \( P_{ij} \)’s parts.

**Proposition 3.1** The operator \( H_k \) has the order \( k \) and all of them are mutually commuting
\[ [H_i, H_j] = 0. \]

**Proof** follows from the relations (3.3), (3.1), (2.11), (2.10).

The set of operators
\[ \tilde{H}_i = w \circ H_i \circ w^{-1}, \quad w = \prod_{i<j} (x_i - x_j)^g \]
gives the integrals of motion of the quantum Calogero-Moser system with the second integral (Hamiltonian) having the form
\[ \tilde{H}_2 = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} - 2g(g-1) \sum_{i<j} \frac{1}{(x_i - x_j)^2}. \]

We refer here to the lectures [3, 4] where the Dunkl’s type operators were used for proving the quantum complete integrability of the (trigonometric) Calogero-Sutherland model and of its generalisation to the other classical root systems.

4 Superstructure

In this Section we introduce the superintegrability structure for the quantum Calogero-Moser system which explains the degeneration of this model. The crucial role in the whole construction is played by the operators \( S_i, \quad i = 0, 1, 2, \ldots \), introduced below.
Proposition 4.1 Let \( n = 1, 2, 3, \ldots \). Then

\[
[D_i^n, x_j] = \delta_{ij} \left( nD_i^{n-1} + \sum_{k \neq i} R_{n-1}(D_i, D_k) gP_{ik} \right) - (1 - \delta_{ij})R_{n-1}(D_i, D_j) gP_{ij},
\]

(4.1)

where \( R_{n-1}(x, y) \) is the symmetric polynomial of the order \( n - 1 \) of the form

\[
R_{n-1}(x, y) = \frac{x^n - y^n}{x - y}.
\]

Proof is done by induction on \( n \) with help of the statement (c) from Theorem 2.3.

As a corollary of this Proposition we have the following

Proposition 4.2 Let \( n = 1, 2, 3, \ldots \), \( l = 0, 1, 2, \ldots \). Then

\[
[x_iD_i^n, x_jD_j^l] = \delta_{ij} \left( nx_iD_i^{l+n-1} + \sum_{k \neq i} x_iD_k^l R_{n-1}(D_i, D_k) gP_{ik} \right)
- (1 - \delta_{ij})x_iD_i^l R_{n-1}(D_i, D_j) gP_{ij},
\]

(4.2)

Introduce now the additional operators \( S_i \) which, together with the integrals \( I_k \), constitute the 'superstructure' of the quantum Calogero-Moser system

\[
S_k = \sum_{i=1}^N x_iD_i^k, \quad k = 0, 1, 2, \ldots.
\]

(4.3)

It is quite straightforward calculation to derive the following algebra for the operators \( S_i, I_n \) on the basis of the Propositions 4.1 and 4.2.

Theorem 4.3 The operators \( S_i = \sum x_iD_i^l, I_n = \sum D_i^n \) are closed to the algebra

\[(i) \quad [I_l, I_n] = 0,
(ii) \quad [S_l, I_n] = -nI_{l+n-1},
(iii) \quad [S_l, S_n] = (l - n)S_{l+n-1}.
\]

(4.4)  (4.5)  (4.6)

Definition 4.4 Consider \( k \) operators \( X_1, \ldots, X_k \). Fix an ordering \( X_1 \prec X_2 \prec \ldots \prec X_k \). The operators \( X_1, \ldots, X_k \) are algebraically independent if \( p \equiv 0 \) is the only polynomial such that

\[p(X_1, X_2, \ldots, X_k) = 0.\]

For fixed \( N \) there are only \( N \) additional, algebraically independent operators \( S_i \) because of the easily verified identity

\[
S_{N+k} = \sum_{i=1}^N (-1)^{i+1} S_{N+k-i} M_i(D), \quad k = 0, 1, 2, \ldots,
\]

(4.7)

where \( M_i(D) \) are elementary monomial symmetric functions

\[
M_i(D) = \sum_{k_1 < \ldots < k_i} D_{k_1} \cdots D_{k_i}.
\]

(4.8)
Let us think always of the first \( N \) lower order operators
\[
S_0, \; S_1, \; \ldots, \; S_{N-1}
\]
as \( N \) chosen algebraically independent operators additional to the \( I_k, \; k = 1, \ldots, N \).

It is easy to observe that the new operators \( S_i \) commute with all the permutations
\[
[S_i, P_{jk}] = 0,
\]
(as well as the \( I_n \) do!). Hence, they send symmetric functions to symmetric ones and one can restrict the relations (4.4)–(4.6) on the sub-space of the symmetric functions. So, we get the following algebra
\[
(\text{Res}(I_n) = H_n):
\]
\[
[H_l, H_n] = 0, \tag{4.9}
\]
\[
[\text{Res}(S_l), H_n] = -nH_{l+n-1}, \tag{4.10}
\]
\[
[\text{Res}(S_l), \text{Res}(S_n)] = (l-n)\text{Res}(S_{l+n-1}). \tag{4.11}
\]
The operators \( \text{Res}(S_l) \) and \( H_n \) are now purely differential operators of orders \( l \) and \( n \), respectively.

Let us construct the action of the operators \( \text{Res}(S_l) \) on the common eigenfunction \( \Psi_{\vec{m}}(\vec{x}) \) of the quantum integrals of motion
\[
H_k \Psi_{\vec{m}}(\vec{x}) = \sum_{i=1}^{N} m_k^i \Psi_{\vec{m}}(\vec{x}), \quad m_i \in \mathbb{R}. \tag{4.12}
\]
The eigenfunction \( \Psi_{\vec{m}}(\vec{x}) \in C^\infty(\mathbb{R}^{2N}) \) depends on \( N \) variables \( \vec{x} = (x_1, \ldots, x_N) \) and on \( N \) real spectral parameters \( \vec{m} = (m_1, \ldots, m_N) \).

**Proposition 4.5** The multiplication operators
\[
\mathcal{H}_n \Psi_{\vec{m}} = \sum_{i=1}^{N} m_i^n \Psi_{\vec{m}} \tag{4.13}
\]
and the following first-order differential operators (in \( m_i \)’s!)
\[
S_l \Psi_{\vec{m}} = -\sum_{i=1}^{N} m_l^i \frac{\partial}{\partial m_i} \Psi_{\vec{m}} \tag{4.14}
\]
give a representation of the algebra (4.9)–(4.11), i.e.
\[
[H_l, H_n] = 0, \tag{4.15}
\]
\[
[S_l, H_n] = -nH_{l+n-1}, \tag{4.16}
\]
\[
[S_l, S_n] = (l-n)S_{l+n-1}. \tag{4.17}
\]

**Proof** can be done by straightforward computation.

It is well-known [3, 4] that the so-called multi-variable Bessel function \( J_{\vec{m}}^{(g)}(\vec{x}) \) solves the spectral problem
\[
H_k J_{\vec{m}}^{(g)}(\vec{x}) = \sum_{i=1}^{N} m_k^i J_{\vec{m}}^{(g)}(\vec{x}), \quad m_i \in \mathbb{R}. \tag{4.18}
\]
Hence, as the corollary of the relations (4.13)–(4.17) we have the following identities for such functions:
\[
\text{Res} \left[ \sum_{i=1}^{N} x_i \left( \frac{\partial}{\partial x_i} + g \sum_{j \neq i} \frac{1}{x_i - x_j} (1 - P_{ij}) \right) \right] J_{\vec{m}}^{(g)}(\vec{x}) = \sum_{i=1}^{N} m_i^l \frac{\partial}{\partial m_i} J_{\vec{m}}^{(g)}(\vec{x}). \tag{4.19}
\]
Here in the l.h.s. we have the order $l$ differential operator in $x_i$'s while in the r.h.s. we have the first-order differential operator in the spectral parameters $m_i$'s. In the limit $g = 0$ the multivariable Bessel function $J_{m_i}^{(g)}(\vec{x})$ turns into the symmetrised exponential $\sum_{\sigma_x \in S_N} \sigma_x(\exp(\vec{m}\vec{x}))$ and the relations (1.19) look like $(l = 0, 1, 2, \ldots)$

$$\sum_{i=1}^{N} x_i \frac{\partial^l}{\partial x_i^l} \sum_{\sigma_x \in S_N} \sigma_x(\exp(\vec{m}\vec{x})) = \sum_{i=1}^{N} m_i \frac{\partial}{\partial m_i} \sum_{\sigma_x \in S_N} \sigma_x(\exp(\vec{m}\vec{x})).$$  \hspace{1cm} (4.20)

5 Quadratic algebra

Introduce the additional operators

$$A_{ij}^{(k)} = S_i I_{j+k-1} - S_j I_{i+k-1},$$

where $k = 1, 2, 3, \ldots, i, j = 0, 1, 2, \ldots$. By definition we have the following properties:

$$A_{ji}^{(k)} = 0, \quad A_{ij}^{(k)} = -A_{ji}^{(k)},$$

so that we assume that $i < j$. The order of the PDO which is the restriction of the operator $A_{ij}^{(k)}$ is

$$\text{order}[\text{Res}(A_{ij}^{(k)})] = i + j + k - 1.$$  \hspace{1cm} (5.2)

Proposition 5.1

(a) \ \ [A_{ij}^{(k)}, S_n] = iA_{i+n-1,j}^{(k)} + jA_{i,j+n-1}^{(k)} + (k - 1)A_{i,j}^{(k+n-1)} - nA_{i+1,j+n-1}^{(k+1-n)},

(b) \ \ [A_{ij}^{(k)}, I_k] = 0,

(c) \ \ [\text{Res}(A_{ij}^{(k)}), H_k] = 0.  \hspace{1cm} (5.3) \hspace{1cm} (5.4) \hspace{1cm} (5.5)

Proof follows from the definition (5.1) and from Theorem 4.3.

Notice that from the statement (c) it follows that we have constructed a big supply of the PDOs $\text{Res}(A_{ij}^{(k)})$, $i, j = 0, 1, 2, \ldots$, all commuting to the integral $H_k$. These additional operators of the order $i + j + k - 1$ describe the hidden symmetry (or the super-integrability) of the quantum integrals of motion for the Calogero-Moser system.

We are now in a position to derive the global quadratic algebra for the integrals $I_n$ and the ‘hidden symmetry generators’ $A_{ij}^{(k)}$.

Theorem 5.2 The operators $I_n$ and $A_{ij}^{(k)}$ are closed to the quadratic algebra

(a) \ \ [I_i, I_j] = 0,

(b) \ \ [A_{ij}^{(k)}, I_n] = -n(I_{i+n-1} - I_{j+n-1} - I_{i+j+n-1}),

(c) \ \ [A_{ij}^{(k)}, A_{ij'}^{(k')} = i(A_{i,j}^{(k)}I_{j',k'-1} - A_{i,j}^{(k')}I_{i+k'-1}) + i'(A_{i,j}^{(k')}I_{j+k'-1} - A_{i,j}^{(k)}I_{j+k'-1}) + j(A_{i,j}^{(k)}I_{j',k'-1} - A_{i,j}^{(k')}I_{j'+k'-1}) + j'(A_{i,j}^{(k')}I_{j'+k'-1} - A_{i,j}^{(k)}I_{j+k'-1}) + (k - 1)(A_{i,j}^{(k)}I_{k'-1,j'} - A_{i,j}^{(k')}I_{k'-1+j'}),

+ (k' - 1)(A_{i,j}^{(k)}I_{k'-1+j'} - A_{i,j}^{(k')}I_{k'-1+j'}).  \hspace{1cm} (5.6) \hspace{1cm} (5.7) \hspace{1cm} (5.8)
Not all of the operators $A_{ij}^{(k)}$ are algebraically independent. Actually, for each given $k$ (and a fixed $N$) we can supply only $N-1$ operators $A_{ij}^{(k)}$ which all commute to the $I_k$ and all of them plus the integrals of motion are algebraically independent. In this way we get a $N \times N$ table putting, for instance, $i = 0$ to diminish the order of the generators:

$$
\begin{bmatrix}
I_1 & A_{01}^{(1)} & \cdots & A_{0,N-1}^{(1)} \\
I_2 & A_{01}^{(2)} & \cdots & A_{0,N-1}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
I_N & A_{01}^{(N)} & \cdots & A_{0,N-1}^{(N)}
\end{bmatrix}.
$$

If we take the restriction of all operators in the table to the invariant sub-space of the symmetric functions and denote $B_j^{(k)} = \text{Res}(A_{0j}^{(k)})$ then we have the same table that was announced in the Introduction Section:

$$
\begin{bmatrix}
H_1 & B_1^{(1)} & \cdots & B_{N-1}^{(1)} \\
H_2 & B_1^{(2)} & \cdots & B_{N-1}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
H_N & B_1^{(N)} & \cdots & B_{N-1}^{(N)}
\end{bmatrix} . \tag{5.9}
$$

It is easy now to derive the quadratic algebra between the entries of this table.

**Theorem 5.3** The operators $H_n$ and $B_j^{(k)} = \text{Res}(A_{0j}^{(k)})$ are closed to the quadratic algebra

\begin{align}
(a) \quad [H_i, H_j] &= 0, \quad \tag{5.10} \\
(b) \quad [B_j^{(k)}, H_n] &= -n(H_{n-1}H_{j+k-1} - H_{k-1}H_{j+n-1}), \quad \tag{5.11} \\
(c) \quad [B_j^{(k)}, B_{j'}^{(k')} &= j(-B_{j+j'-1}^{(k)}H_{k'-1} + B_j^{(k)}H_{j'+k'-1}) \\
&\quad + j'(-B_{j'+j'-1}^{(k')}H_{k'-1} + B_{j'}^{(k')}H_{j+k'-1}) \\
&\quad + (k-1)(B_j^{(k-1)}H_{k'-1} - B_j^{(k)}H_{k'-1}) \\
&\quad + (k'-1)(B_{j'}^{(k'-1)}H_{k-1} - B_{j'}^{(k')}H_{k-1}) . \quad \tag{5.12}
\end{align}

Recall that the superintegrability of the classical Calogero-Moser system means that each $H_n$ of the integrals of motion is in involution with another $2N-2$ quantities. In the quantum case we have shown (cf. the statements (a) and (b) in the Theorem 5.3) that the $H_n$ commutes with $2N-2$ operators: $H_1, \ldots, H_{n-1}, H_{n+1}, \ldots, H_N, B_1^{(n)}, \ldots, B_{N-1}^{(n)}$. This is the fact of the quantum superintegrability or degeneration of the quantum integrable system.

**Remark** For the table (5.9) one gets in the r.h.s.’s of the commutators (5.11)–(5.12) some $H_k$’s and $B_j^{(k)}$’s which are out of the table. The simple analysis shows that all these operators are not algebraically independent from those in the table, i.e. they can be expressed as some polynomials (linear in $B_j^{(k)}$’s) on the operators inside the table.

Let us consider in details two first examples: $N = 2$ and $N = 3$.

### 6 $N = 2$ case

We have the following $2 \times 2$ table in the case of $N = 2$:

$$
\begin{bmatrix}
H_1 & B_1^{(1)} \\
H_2 & B_1^{(2)}
\end{bmatrix} .
$$
and the quadratic algebra of the form

\[
[H_1, H_2] = [H_1, B_1^{(1)}] = [H_2, B_1^{(2)}] = 0,
\]

\[
[B_1^{(1)}, H_2] = -2(H_1^2 - 2H_2),
\]

\[
[B_1^{(2)}, H_1] = H_1^2 - 2H_2,
\]

\[
[B_1^{(1)}, B_1^{(2)}] = 4B_1^{(2)} - 2B_1^{(1)}H_1.
\]

Let us denote the restrictions of the operators \(S_i\) by the small letters:

\[
s_i = \text{Res}(S_i).
\]

Then the additional operators \(s_0, s_1, s_2\) satisfy the following relations:

\[
[s_0, H_1] = -2 \cdot 1, \quad [s_0, B_1^{(1)}] = 0,
\]

\[
[s_0, H_2] = -2H_1, \quad [s_0, B_1^{(2)}] = -B_1^{(1)},
\]

\[
[s_1, H_1] = -H_1, \quad [s_1, B_1^{(1)}] = 0,
\]

\[
[s_1, H_2] = -2H_2, \quad [s_1, B_1^{(2)}] = -B_1^{(2)}, \quad [s_0, s_1] = -s_0,
\]

\[
s_2 = s_1H_1 + \frac{1}{2}s_0(H_2 - H_1^2),
\]

\[
[s_2, H_1] = -H_2, \quad [s_2, B_1^{(1)}] = B_1^{(1)}H_1 - 2B_1^{(2)},
\]

\[
[s_2, H_2] = H_1^3 - 3H_1H_2, \quad [s_2, B_1^{(2)}] = \frac{3}{2}B_1^{(1)}(H_1^2 - H_2) - 2B_1^{(2)}H_1.
\]

The explicit form of all these operators is as follows:

\[
H_1 = \partial_1 + \partial_2,
\]

\[
H_2 = \partial_1^2 + \partial_2^2 + \frac{2g}{x_1 - x_2}(\partial_1 - \partial_2),
\]

\[
B_1^{(1)} = (x_1 - x_2)(\partial_2 - \partial_1),
\]

\[
B_1^{(2)} = x_2\partial_1^2 + x_1\partial_2^2 - (x_1 + x_2)\partial_1\partial_2 + 2g\frac{x_1 + x_2}{x_1 - x_2}(\partial_1 - \partial_2),
\]

\[
s_0 = x_1 + x_2,
\]

\[
s_1 = x_1\partial_1 + x_2\partial_2,
\]

\[
s_2 = x_1\partial_1^2 + x_2\partial_2^2 + g\frac{x_1 + x_2}{x_1 - x_2}(\partial_1 - \partial_2).
\]

**Theorem 6.1** The spectral problem

\[
H_1 J_{m_1m_2}^{(g)}(x_1, x_2) = (m_1 + m_2) J_{m_1m_2}^{(g)}(x_1, x_2), \quad m_1, m_2 \in \mathbb{R},
\]

\[
H_2 J_{m_1m_2}^{(g)}(x_1, x_2) = (m_1^2 + m_2^2) J_{m_1m_2}^{(g)}(x_1, x_2)
\]

has the following symmetric function as a solution

\[
J_{m_1m_2}^{(g)}(x_1, x_2) = e^{m_+x_+} (m_-x_-)^{\frac{1}{2} - g} I_{g - \frac{1}{2}} \left( \frac{m_-}{2} x_- \right),
\]

\[
x_\pm = x_1 \pm x_2, \quad m_\pm = m_1 \pm m_2,
\]

\[
2 \cdot 1.
\]
where

\[ I_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\nu}}{k! \Gamma(k+\nu+1)} \]  

(6.21)

is the modified Bessel function of the first kind \([4, 7.2(12)]\). Moreover

\[ J_{m_1m_2}^{(0)}(x_1, x_2) = \frac{1}{\sqrt{\pi}} (e^{m_1 x_1 + m_2 x_2} + e^{m_1 x_2 + m_2 x_1}). \]  

(6.22)

Notice that if \( \nu \in \mathbb{R} \) and \( z > 0 \) then \( I_\nu(z) \in \mathbb{R} \).

**Proof** can be done through the separation of variables \( x_+ \) and \( x_- \) in the spectral problem (6.18)–(6.19).

**Theorem 6.2** The action of the operators (6.1)–(6.17) on the eigenfunction (6.20) is as follows:

\[
\begin{align*}
H_1 J_{m_1m_2}^{(g)}(x_1, x_2) &= (m_1 + m_2) J_{m_1m_2}^{(g)}(x_1, x_2), \\
H_2 J_{m_1m_2}^{(g)}(x_1, x_2) &= (m_1^2 + m_2^2) J_{m_1m_2}^{(g)}(x_1, x_2), \\
B^{(1)}_1 J_{m_1m_2}^{(g)}(x_1, x_2) &= (m_2 - m_1) \left( \partial_{m_1} - \partial_{m_2} \right) J_{m_1m_2}^{(g)}(x_1, x_2), \\
B^{(2)}_1 J_{m_1m_2}^{(g)}(x_1, x_2) &= (m_2 - m_1) \left( m_2 \partial_{m_1} - m_1 \partial_{m_2} \right) J_{m_1m_2}^{(g)}(x_1, x_2), \\
B^{(3)}_1 J_{m_1m_2}^{(g)}(x_1, x_2) &= (\partial_{m_1} + \partial_{m_2}) J_{m_1m_2}^{(g)}(x_1, x_2), \\
B^{(1)}_2 J_{m_1m_2}^{(g)}(x_1, x_2) &= m_1 \partial_{m_1} + m_2 \partial_{m_2} J_{m_1m_2}^{(g)}(x_1, x_2), \\
B^{(2)}_2 J_{m_1m_2}^{(g)}(x_1, x_2) &= m_2 \partial_{m_1} + m_1 \partial_{m_2} J_{m_1m_2}^{(g)}(x_1, x_2).
\end{align*}
\]

(6.23–6.29)

**Proof** follows from the Proposition 4.5 (cf. formula (4.13)) and the definition of \( A^{(k)}_{ij} \) (5.1) (recall also that \( B^{(k)}_j = \text{Res}(A^{(k)}_{ij}) \)).

7 \( N = 3 \) case

We have the following 3 \( \times \) 3 table in the case of \( N = 3 \):

\[
\begin{bmatrix}
H_1 & B^{(1)}_1 & B^{(1)}_2 \\
H_2 & B^{(2)}_1 & B^{(2)}_2 \\
H_3 & B^{(3)}_1 & B^{(3)}_2
\end{bmatrix},
\]

and the non-linear algebra of the form

\[
[H_i, H_j] = [H_i, B^{(i)}_1] = [H_i, B^{(i)}_2] = 0,
\]

(7.1)

\[
[H_1, B^{(2)}_1] = 3H_2 - H_1^2,
\]

(7.2)

\[
[H_1, B^{(2)}_2] = 3H_3 - H_1 H_2,
\]

(7.3)

\[
[H_1, B^{(3)}_1] = 3H_3 - H_1 H_2,
\]

(7.4)

\[
[H_1, B^{(3)}_2] = 4H_3 H_1 + \frac{1}{2} H_2^2 - 3H_2 H_1^2 + \frac{1}{2} H_1^4,
\]

(7.5)
\[ [H_2, B_1^{(1)}] = -2(3H_2 - H_1^2), \quad (7.6) \]
\[ [H_2, B_2^{(1)}] = -2(3H_3 - H_1H_2), \quad (7.7) \]
\[ [H_2, B_3^{(1)}] = -2(H_2^2 - H_1H_3), \quad (7.8) \]
\[ [H_2, B_2^{(3)}] = -2H_3H_2 + \frac{8}{3}H_3H_1^2 + H_2^2H_1 - 2H_2H_1^3 + \frac{1}{3}H_1^5, \quad (7.9) \]
\[ [H_3, B_1^{(1)}] = -3(3H_3 - H_1H_2), \quad (7.10) \]
\[ [H_3, B_2^{(1)}] = -12H_3H_1 - \frac{3}{2}H_2^2 + 9H_2H_1^2 - \frac{3}{2}H_1^3, \quad (7.11) \]
\[ [H_3, B_1^{(2)}] = 3(H_2^2 - H_1H_3), \quad (7.12) \]
\[ [H_3, B_2^{(2)}] = 3H_3H_2 - 4H_3H_1^2 - \frac{3}{2}H_2^2H_1 + 3H_2H_1^3 - \frac{1}{2}H_1^5, \quad (7.13) \]

\[ [B_1^{(1)}, B_2^{(1)}] = 3B_2^{(1)} - 2B_1^{(1)} H_1, \quad (7.14) \]
\[ [B_1^{(1)}, B_2^{(2)}] = 6B_2^{(2)} - 2B_1^{(1)} H_1, \quad (7.15) \]
\[ [B_1^{(1)}, B_2^{(2)}] = 9B_2^{(2)} - 2(B_2^{(1)} + B_2^{(2)}) H_1, \quad (7.16) \]
\[ [B_1^{(1)}, B_3^{(1)}] = 9B_3^{(1)} - 2B_1^{(2)} H_1 - B_1^{(1)} H_2, \quad (7.17) \]
\[ [B_1^{(1)}, B_2^{(3)}] = 12B_2^{(3)} - 2(B_1^{(3)} + B_2^{(2)}) H_1 - B_2^{(1)} H_2, \quad (7.18) \]
\[ [B_2^{(1)}, B_2^{(2)}] = 3(B_1^{(3)} + B_2^{(2)}) - 2B_1^{(1)} H_1 + B_1^{(1)} H_2, \quad (7.19) \]
\[ [B_2^{(1)}, B_2^{(2)}] = 3B_2^{(3)} + 6B_2^{(2)} H_1 + B_1^{(2)} (H_2 - 3H_1^2) \]
\[ -B_2^{(3)} (H_2 + 2H_1^2) + B_1^{(1)} (H_3^2 - H_1H_2 + 2H_3), \quad (7.20) \]
\[ [B_2^{(1)}, B_1^{(3)}] = 3B_3^{(3)} + 6B_3^{(1)} H_1 + B_2^{(2)} (H_2 - 3H_1^2) \]
\[ -2B_2^{(1)} H_2 + B_1^{(1)} (H_3^3 - 3H_1H_2 + 4H_3), \quad (7.21) \]
\[ [B_2^{(1)}, B_2^{(3)}] = 12B_3^{(3)} H_1 + (B_1^{(3)} + B_2^{(2)}) (H_2 - 3H_1^2) \]
\[ + B_2^{(1)} (H_3^3 - 5H_1H_2 + 2H_3) + B_1^{(1)} (\frac{1}{3}H_3^3 - H_1H_2 + \frac{8}{3}H_3) H_1, \quad (7.22) \]
\[ [B_2^{(2)}, B_2^{(2)}] = (2B_2^{(2)} - B_1^{(3)}) H_1 - (2B_2^{(2)} + B_1^{(1)}) H_2 + B_1^{(1)} H_3, \quad (7.23) \]
\[ [B_2^{(2)}, B_1^{(3)}] = 3B_4^{(3)} H_1 - 4B_2^{(2)} H_2 + B_1^{(1)} H_3, \quad (7.24) \]
\[ [B_2^{(2)}, B_2^{(3)}] = 4B_3^{(3)} H_1 - 3(B_1^{(3)} + B_2^{(2)}) H_2 + B_1^{(1)} (\frac{1}{6}H_1^4 - H_2H_1^2 + \frac{1}{2}H_2^2 + \frac{4}{3}H_3H_1), \quad (7.25) \]
\[ [B_2^{(2)}, B_1^{(3)}] = B_2^{(3)} H_1 + 2B_1^{(3)} H_1^2 - 3B_2^{(2)} H_2 + B_2^{(1)} (H_2 - H_1^2) H_1 \]
\[ + B_2^{(1)} H_3 + \frac{1}{3}B_1^{(1)} (2H_3 - 3H_1H_2 + H_3^3) H_1, \quad (7.26) \]
\[ [B_2^{(2)}, B_2^{(3)}] = B_2^{(3)} (4H_1^2 - H_2^2) + B_1^{(3)} (H_1H_2 - H_1^3 - 2H_3) - B_2^{(2)} (H_1H_2 + H_3^3 + 2H_3) \]
\[ + B_1^{(1)} (\frac{8}{3}H_3 - H_1H_2 + \frac{1}{3}H_1^3) H_1 + B_2^{(1)} (2H_3H_1 - 2H_2^2H_2 + \frac{1}{2}H_1^4 + \frac{1}{2}H_2^4), \quad (7.27) \]
\[ [B_2^{(3)}, B_2^{(3)}] = 3B_3^{(3)} H_2 - 2B_1^{(3)} (H_3 + H_1H_2) - 2B_2^{(2)} H_3 \]
\[ + B_1^{(2)} (\frac{1}{3}H_3^3 - H_1H_2 + \frac{8}{3}H_3H_1 - \frac{1}{3}B_1^{(1)} (H_1^3 - 3H_1H_2 + 2H_3) H_2). \quad (7.28) \]

The additional operators \( s_0, s_1, s_2 \) satisfy the following relations:

\[ [s_0, H_1] = -3 \cdot 1, \quad [s_1, H_1] = -H_1, \quad (7.29) \]
The explicit form of all these operators is as follows:

\[ [s_0, H_2] = -2H_1, \quad [s_1, H_2] = -2H_2, \quad [s_0, H_3] = -3H_2, \quad [s_1, H_3] = -3H_3, \quad [s_2, H_1] = -H_2, \quad [s_2, H_2] = -2H_3, \quad [s_2, H_3] = -\frac{1}{2}H_1^4 + 3H_2H_1^2 - \frac{3}{2}H_2^2 - 4H_3H_1, \quad [s_0, s_1] = -s_0, \quad [s_0, s_2] = -2s_1, \quad [s_1, s_2] = -s_2, \]

\[ [s_0, B_1^{(1)}] = 0, \quad [s_0, B_2^{(1)}] = -2B_1^{(2)} - B_2^{(1)}, \quad [s_0, B_2^{(2)}] = -2B_1^{(1)}, \quad [s_0, B_2^{(3)}] = -2B_1^{(2)}, \quad [s_1, B_1^{(1)}] = 0, \quad [s_1, B_2^{(1)}] = -2B_2^{(2)}, \quad [s_1, B_2^{(2)}] = -B_2^{(1)}, \quad [s_1, B_1^{(3)}] = -2B_2^{(1)}, \quad [s_1, B_2^{(3)}] = -B_2^{(2)}, \quad [s_1, B_2^{(3)}] = -3B_2^{(3)}, \]

\[ [s_2, B_1^{(1)}] = B_2^{(1)} - 2B_2^{(2)}, \quad [s_2, B_2^{(1)}] = -2B_1^{(3)}, \quad [s_2, B_1^{(2)}] = B_2^{(2)} - 3B_1^{(3)}, \]

\[ [s_2, B_2^{(2)}] = -B_2^{(3)} - 2B_1^{(3)}(H_1 - B_1^{(2)}(H_2 - H_1^2) - \frac{1}{3}B_1^{(1)}(2H_3 - 3H_1H_2 + H_1^3)), \quad [s_2, B_1^{(3)}] = B_2^{(3)} - 4B_1^{(3)}H_1 - 2B_2^{(2)}(H_2 - H_1^2) - \frac{2}{3}B_1^{(1)}(2H_3 - 3H_1H_2 + H_1^3), \quad [s_2, B_2^{(3)}] = -2B_2^{(3)}H_1 + B_1^{(3)}H_2 - \frac{5}{3}H_1^2 - B_2^{(2)}(H_2 - H_1^2) + B_1^{(2)}(\frac{8}{3}H_3 + \frac{2}{3}H_1H_2 + \frac{2}{3}H_3^2)
\]
\[ -\frac{1}{3}B_2^{(1)}(2H_3 - 3H_1H_2 + H_1^3) - \frac{1}{3}B_1^{(1)}(H_1^2 - H_2)(H_1^2 - 2H_2). \]

The explicit form of all these operators is as follows:

\[ H_1 = \partial_1 + \partial_2 + \partial_3, \quad H_2 = \partial_1^2 + \partial_2^2 + \partial_3^2 + \frac{2g}{x_{12}}\partial_{12} + \frac{2g}{x_{13}}\partial_{13} + \frac{2g}{x_{23}}\partial_{23}, \quad H_3 = \partial_1^3 + \partial_2^3 + \partial_3^3 + 3g\left(\frac{1}{x_{12}} + \frac{1}{x_{13}}\right)\partial_1^2 + 3g\left(\frac{1}{x_{21}} + \frac{1}{x_{23}}\right)\partial_2^2 + 3g\left(\frac{1}{x_{31}} + \frac{1}{x_{32}}\right)\partial_3^2
\]
\[ + \frac{6g^2}{x_{12}x_{13}}\partial_1 + \frac{6g^2}{x_{21}x_{23}}\partial_2 + \frac{6g^2}{x_{31}x_{32}}\partial_3, \]

\[ B_1^{(1)} = (x_{21} + x_{31})\partial_1 + (x_{12} + x_{32})\partial_2 + (x_{13} + x_{23})\partial_3, \quad B_1^{(2)} = (x_2 + x_3)(\partial_1^2 - \partial_2\partial_3) + (x_1 + x_3)(\partial_2^2 - \partial_1\partial_3) + (x_1 + x_2)(\partial_3^2 - \partial_1\partial_2)
\]
\[ + 2g(x_1 + x_2 + x_3)\left[\left(\frac{1}{x_{12}} + \frac{1}{x_{13}}\right)\partial_1 + \left(\frac{1}{x_{21}} + \frac{1}{x_{23}}\right)\partial_2 + \left(\frac{1}{x_{31}} + \frac{1}{x_{32}}\right)\partial_3\right], \quad B_2^{(1)} = (x_{21} + x_{31})\left(\partial_1^2 - \frac{g}{x_{23}}\partial_{23}\right) + (x_{12} + x_{32})\left(\partial_2^2 - \frac{g}{x_{13}}\partial_{13}\right)
\]
\[ + (x_{13} + x_{23})\left(\partial_3^2 - \frac{g}{x_{12}}\partial_{12}\right), \quad B_2^{(2)} = s_0H_3 - s_2H_1, \quad B_2^{(3)} = s_0H_3 - s_1H_2, \quad B_2^{(3)} = s_0\left(\frac{1}{6}H_1^4 - H_2H_1^2 + \frac{1}{2}H_2^2 + \frac{4}{3}H_1H_3\right) - s_2H_2, \]
where
\[
\begin{align*}
s_0 &= x_1 + x_2 + x_3, \\
s_1 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3, \\
s_2 &= x_1\partial_1^2 + x_2\partial_2^2 + x_3\partial_3^2 + g\frac{x_1 + x_2}{x_{12}}\partial_{12} + g\frac{x_1 + x_3}{x_{13}}\partial_{13} + g\frac{x_2 + x_3}{x_{23}}\partial_{23}, \\
s_3 &= s_2H_1 + \frac{1}{2}s_1(H_2 - H_1^2) + s_0\left(\frac{1}{3}H_3 - \frac{1}{2}H_1H_2 + \frac{1}{6}H_3^3\right).
\end{align*}
\] (7.54)
(7.55)
(7.56)
(7.57)

Here we use the notations \(x_{ij} = x_i - x_j\) (the same is for \(m_{ij}\) below) and \(\partial_{ij} = \partial_i - \partial_j\).

**Theorem 7.1** The action of the operators \((7.44) - (7.56)\) on the common symmetric eigenfunction \(J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3)\) of the commuting operators \(H_1, H_2, H_3\) is as follows:

\[
\begin{align*}
H_1 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (m_1 + m_2 + m_3) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
H_2 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (m_1^2 + m_2^2 + m_3^2) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
H_3 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (m_1^3 + m_2^3 + m_3^3) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_1^{(1)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_1 + m_3)\partial_{m_1} + (m_1 + m_3)\partial_{m_2} + (m_3 + m_4)\partial_{m_3}] J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_2^{(1)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_1^2 + m_2^2 - 2m_3^2)\partial_{m_1} + (m_1^2 + m_3^2 - 2m_2^2)\partial_{m_2} + (m_2^2 + m_3^2 - 2m_1^2)\partial_{m_3}] J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_1^{(2)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_1^2 - m_1m_3)\partial_{m_1} + (m_2^2 - m_1m_3)\partial_{m_2} + (m_3^2 - m_1m_3)\partial_{m_3}] J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_2^{(2)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_2^2 + m_3^2 - m_1^2)\partial_{m_1} + (m_3^2 + m_2^2 - m_1^2)\partial_{m_2} + (m_1^2 + m_3^2 - m_2^2)\partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_1^{(3)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_2^3 + m_3^3 - m_1^2)\partial_{m_1} + (m_3^3 + m_2^3 - m_1^2)\partial_{m_2} + (m_1^3 + m_2^3 - m_3^2)\partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
B_2^{(3)} J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= [(m_2^4 + m_3^4 - m_1^3)\partial_{m_1} + (m_3^4 + m_2^4 - m_1^3)\partial_{m_2} + (m_1^4 + m_2^4 - m_3^3)\partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
s_0 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (\partial_{m_1} + \partial_{m_2} + \partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
s_1 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (m_1\partial_{m_1} + m_2\partial_{m_2} + m_3\partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3), \\
s_2 J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3) &= (m_1^2\partial_{m_1} + m_2^2\partial_{m_2} + m_3^2\partial_{m_3}) J_{m_1m_2m_3}^{(g)}(x_1, x_2, x_3).
\end{align*}
\] (7.58)
(7.59)
(7.60)
(7.61)
(7.62)
(7.63)
(7.64)
(7.65)
(7.66)
(7.67)
(7.68)
(7.69)

**Proof** follows from the Proposition 4.5 (cf. formula (1.13)) and the definition of \(A_{ij}^{(k)}\) (5.1) (recall also that \(B_j^{(k)} = \text{Res}(A_{ij}^{(k)})\)).

**8 Miscellaneous results**

Recall that
\[
B_j^{(k)} = \text{Res}(S_0I_{j+k-1} - S_jI_{k-1}) = \text{Res}\left(\sum_{l<n} M_{ij}^{(k)}(D_l^j - D_l^i)\right),
\] (8.1)

where
\[
M_{ij}^{(k)} = x_iD_{j}^{k-1} - x_jD_{i}^{k-1} = -M_{ji}^{(k)}.
\] (8.2)

From the Proposition 4.1 one can derive the statement.
Proposition 8.1

\[ [I_k, M^{(k)}_{ij}] = 0. \]

This means that we could repeat all the contraction described in the previous Sections working with the operators \( M^{(k)}_{ij} \) instead of the \( S_i \) (and \( A^{(k)}_{ij} \)). The result is the same, i.e. the operators \( I_k \) commute not only to the \( D_j \) but also to the additional operators \( M^{(k)}_{ij} \) being some sort of \((g, P)\)-deformation of the operators of ‘rotations’. The only difference between the operators \( A^{(k)}_{ij} \) and \( M^{(k)}_{ij} \) is that the former ones commute with permutations but the latter ones do not.

For instance for \( k = 2 \) we have the following \((g, P)\)-deformation of the Euclidean \( e(N) \) Lie algebra of generators \( M_{ij} \equiv M^{(2)}_{ij} = x_i D_j - x_j D_i \) and \( D_k \):

\[
\begin{align*}
[M_{ij}, P_{kl}] &= 0 \quad \text{if} \quad \{i, j\} \cap \{k, l\} = \emptyset, \\
P_{ij} M_{ik} &= M_{jk} P_{ij}, \quad k \neq j, \\
P_{ij} M_{ij} &= M_{ji} P_{ij},
\end{align*}
\]

(8.3)
have the sl(2) Lie algebra commutation relations and commute to the operators

\[ [M_{ij}, M_{kl}] = (\delta_{ik} M_{lj} + \delta_{il} M_{jk}) \left( 1 + g \sum_{m \neq i} P_{im} \right) + (\delta_{jl} M_{ki} + \delta_{jk} M_{il}) \left( 1 + g \sum_{m \neq j} P_{jm} \right) \]

\[ + g M_{ij} P_{kl} (\delta_{ik} - \delta_{jl} + \delta_{jk} - \delta_{il}) - g M_{ki} P_{lj} (\delta_{ik} - \delta_{jl} - \delta_{jk} + \delta_{il}), \]

\[ [M_{ij}, D_j] = D_i \left( 1 + g \sum_{m \neq j} P_{jm} \right) + g D_j P_{ij}. \]  

Proposition 8.2

The Casimir operator of the sl(2) algebra (8.9) has the form

\[ \text{Proposition 8.2} \]

P

N

Calogero-Moser system, i.e. there are not only Euclidean space. Keeping in mind this limit we will treat the PDM and, as well as in non-deformed case, is a Casimir operator of the deformation of the Lie algebra e(\(N\)) (where \(g\) is the parameter of deformation). It is easy to see that the deformation involves the permutation operators. Operator \(\sum_i D_i^2\) is a deformed Laplace operator and, as well as in non-deformed case, is a Casimir operator of the PDM-algebra. The very fact of the existence of the PDM-algebra is an algebraic interpretation of the super-integrability of the rational Calogero-Moser system, i.e. there are not only \(N - 1\), but \(2N - 2\) independent additional integrals of motion commuting with the Hamiltonian of the rational Calogero-Moser problem.

From the Proposition 4.1 one can calculate that the operators

\[ J_+ \equiv I_2 = \sum_{i=1}^{N} D_i^2, \quad J_3 = \sum_{i=1}^{N} (x_i D_i + D_i x_i), \quad J_- = \sum_{i=1}^{N} x_i^2 \]  

(8.8)

have the sl(2) Lie algebra commutation relations and commute to the operators \(M_{ij}\) and permutations \(P_{ij}\).

Proposition 8.2

\[ [J_{\pm}, J_3] = \pm 4 J_{\pm}, \quad [J_+, J_-] = 2 J_3, \quad [J_{\pm, 3}, M_{ij}] = [J_{\pm, 3}, P_{ij}] = 0. \]  

(8.9)

The Casimir operator of the sl(2) algebra (8.9) has the form

\[ C_2 = \frac{1}{2} (J_+ J_- + J_- J_+) - \frac{1}{4} J_3^2 = \sum_{i<j} M_{ij}^2 - \left( g P + \frac{N}{2} \right) \left( g P + \frac{N}{2} - 2 \right), \]  

(8.10)

where the operator \(P\) is the sum over all permutation operators:

\[ P = \sum_{i<j} P_{ij}. \]  

(8.11)

In order to get the purely differential operators from \(M_{ij}\) and \(D_k\) one needs to consider the restriction of the symmetric combinations of them like

\[ \text{Res} \left( \sum_{i,j} M_{ij}^{2r} \right) \quad \text{or} \quad \text{Res} \left( \sum_{i,j} M_{ij} D_j^r \right), \quad r = 1, 2, 3, \ldots. \]
Recall here that the additional operators $B_k^{(2)}$ look like (cf. formula (8.1))

$$
\text{Res} \left( \sum_{i,j} M_{ij} D_j^k \right).
$$

One could ask whether it is possible to find a complete set of commuting PDOs only in terms of the symmetric combinations of the operators $M_{ij}$ such that all these operators commute to the Hamiltonian $H_2$. For $N = 3$ it is possible.

**Theorem 8.3** The following second order partial differential operators are algebraically independent and mutually commuting:

\[
H_2 = \text{Res}(D_1^2 + D_2^2 + D_3^2) = \partial_1^2 + \partial_2^2 + \frac{2g}{x_{12}} \partial_{12} + \frac{2g}{x_{13}} \partial_{13} + \frac{2g}{x_{23}} \partial_{23},
\]

\[
K_1 = \text{Res}(M_{12}^2 + M_{23}^2 + M_{31}^2) = M_{12}^2 + M_{23}^2 + M_{31}^2 + 2g \left( \frac{x_1}{x_{23}} + \frac{x_2}{x_{31}} - \frac{x_1 + x_2}{x_{12}} \right) M_{12} + 2g \left( \frac{x_3}{x_{23}} + \frac{x_1}{x_{31}} - \frac{x_3 + x_1}{x_{23}} \right) M_{31},
\]

\[
K_2 = \text{Res}(\{M_{12}, M_{13}\} + \{M_{23}, M_{21}\} + \{M_{31}, M_{32}\}) = \{M_{12}, M_{13}\} + \{M_{23}, M_{21}\} + \{M_{31}, M_{32}\} + 2g \frac{x_1 - x_2 - x_3}{x_{23}} (M_{13} - M_{12}) + 2g \frac{x_3 - x_1 - x_2}{x_{12}} (M_{32} - M_{31}) - 4g \frac{x_3}{x_{23}} M_{12} - 4g \frac{x_1}{x_{23}} M_{23} - 4g \frac{x_2}{x_{31}} M_{31},
\]

\[
M_{ij} = x_i \partial_j - x_j \partial_i.
\]

Moreover, the spectral problem

\[
H_2 \Psi = h_2 \Psi, \quad K_1 \Psi = k_1 \Psi, \quad K_2 \Psi = k_2 \Psi, \quad h_2, k_1, k_2 \in \mathbb{R}
\]

can be solved through a simple separation of variables (SoV) which is some change of coordinates $x_i$'s.

**Proof** The commutativity of the operators $H_2, K_{1,2}$ follows from the Proposition 8.2 and the SoV for $\Psi$ in (8.13) is done by the same coordinate change by which Jacobi separated variables in the corresponding Hamilton-Jacobi equation for $H_2$.

The case $N = 3$ is specific, there is obviously no such situation for $N > 3$. For general $N$ (and even for $N = 3$) it is still open question to make the SoV for the symmetric solution of the spectral problem

\[
H_k \Psi_{\vec{m}}(\vec{x}) = \sum_i m_i^k \Psi_{\vec{m}}(\vec{x}), \quad m_i \in \mathbb{R}.
\]

It is quite clear that such SoV will not be a simple change of coordinates $x_i$'s but rather some integral transformation $\mathcal{K}$

\[
\mathcal{K} : \Psi_{\vec{m}}(\vec{x}) \mapsto \prod_{i=1}^N \psi_{\vec{m}}(y_i).
\]

See the work [9] where the SoV was done for the trigonometric generalisation of the 3-particle quantum Calogero-Moser system (Calogero-Sutherland model) or, respectively, for the Jack polynomials of the $A_2$ type. We hope that the results obtained there and in the present work will be helpful to work out the SoV for the spectral problem (8.16).
Concluding remarks

Apart from the well-known and well-understood super-integrable systems like $N$-dimensional harmonic oscillator and Coulomb problem, the rational Calogero-Moser system leads to the non-linear algebra of hidden symmetry.

The construction of the 'super-integrability table' showed in this paper is quite general and can be applied to various generalisations such as Calogero-Moser models associated to other classical root systems (the model considered in this paper is associated to the $A_{N-1}$ root system) and the Ruijsenaars-Schneider model which is a relativistic generalisation. This work is in progress and will be published elsewhere.

Acknowledgments The author wishes to thank Ernie Kalnins and Frank Nijhoff for valuable discussions. This work was partially supported on the earlier stage through the research grant from the Waikato University, Hamilton. The author acknowledges the hospitality of the Technical University of Denmark.

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