The socle tableau as a dual version of the Littlewood–Richardson tableau

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Dedicated to the memory of Professor Andrzej Skowroński

Abstract
Like the LR-tableau, a socle tableau is given as a skew diagram with certain entries. Unlike in the LR-tableau, the entries in the socle tableau are weakly increasing in each row, strictly increasing in each column and satisfy a modified lattice permutation property. In the study of embeddings of a subgroup in a finite abelian $p$-group, socle tableaux occur as isomorphism invariants, they are given by the socle series of the subgroup. We show that each socle tableau can be realized by some embedding. Moreover, the socle tableau of an embedding and the LR-tableau of the dual embedding determine each other — a correspondence which appears to be related to the tableau switching studied by Benkart, Sottile and Stroomer. We illustrate how the entries in the socle tableau position the object within the Auslander–Reiten quiver. Like the LR-tableau, the socle tableau determines the irreducible component of the object in representation space.

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1 | INTRODUCTION

1.1 | Combinatorics

The Littlewood–Richardson (LR) coefficient occurs in a meaningful way in a variety of areas in algebra: for example, as the structure constant of the product of Schur polynomials in the ring of...
symmetric functions; as the multiplicity of a given irreducible representation of the symmetric
group in a decomposition of the tensor product of two others; or as the number of irreducible
components of invariant subspace varieties [2, 4–7].

Combinatorially, the LR-coefficient \( c_{\alpha, \beta}^{\gamma} \) can be computed as the number of LR-tableaux of
shape \((\alpha, \beta, \gamma)\), where \(\alpha, \beta, \gamma\) are given partitions. In this paper we introduce a new kind of tableau,
which we call the socle tableau. Here, the entries in the tableau are weakly decreasing in each row,
strictly decreasing in each column, and satisfy a third condition which corresponds to the lattice
permutation property.

The number of socle tableaux of shape \((\alpha, \beta, \gamma)\) is equal to the number of LR-tableaux of the
same shape. However to the knowledge of the authors, there is no ‘natural’ one-to-one correspon-
dence between LR-tableaux and socle tableaux of the same shape.

It is well known that the number \( c_{\alpha, \gamma}^{\beta} \) of LR-tableaux of shape \((\alpha, \beta, \gamma)\) equals the number \( c_{\gamma, \alpha}^{\beta} \)
of LR-tableaux of shape \((\gamma, \beta, \alpha)\). In Section 4.3, we present an explicit combinatorial one-to-one
correspondence between the set of socle tableaux of shape \((\alpha, \beta, \gamma)\) and the set of LR-tableaux of
shape \((\gamma, \beta, \alpha)\). For example,

\[
\begin{array}{ccc}
4 & 2 & 1 \\
3 & 1 \\
1
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 \\
1 & 3
\end{array}
\]

\(\Sigma_2: \) corresponds to \(\Gamma_2: \).

As a consequence, we can interpret the LR coefficient \( c_{\alpha, \gamma}^{\beta} \) as the number of socle tableaux of
shape \((\alpha, \beta, \gamma)\) (Corollary 4.8).

1.2 Algebra

Both LR- and socle tableaux occur as isomorphism invariants of embeddings of the form \((A \subset B)\)
where \(B\) is a finite length module over a discrete valuation ring \(\Lambda\) and \(A\) is a submodule of \(B\).
The shape parameters \(\alpha, \beta, \gamma\) are the partitions which describe the isomorphism types of the
\(\Lambda\)-modules \(A, B, B/A\). While the LR-tableau encodes the isomorphism types of the quotients
\(B/\text{rad}^\ell A\), for integers \(\ell \geq 0\), given by the radical series of \(A\), the socle tableau encodes the iso-
morphism types of the quotients \(B/\text{soc}^\ell A\) given by the socle series of \(A\). Hence the name.

We present examples to illustrate that the LR-tableau of an embedding does not determine the
socle tableau of that embedding, and conversely.

Certain embeddings of \(\Lambda\)-modules, the pickets, play a key role. For natural numbers \(1 \leq m, 0 \leq \ell \leq m\),
denote by \(P^m_\ell\) the embedding \((A \subset B)\) where \(B = \Lambda/\text{rad}^m \Lambda\) is the cyclic \(\Lambda\)-module
of (composition) length \(m\) and \(A = \text{soc}^\ell B\) the unique submodule of \(B\) of length \(\ell\). It turns out
that either the socle tableau or the dual LR-tableau of an embedding \(X\) determines and is deter-
mined by the Hom-matrix \(H = (h^m_\ell)\), where \(h^m_\ell\) measures the length of the homomorphism space
\(\text{Hom}(P^m_\ell, X)\) (Theorem 4.1). In Subsection 4.3, we will give an explicit combinatorial construction
how to obtain the socle tableau from the corresponding dual LR-tableau, and conversely.

1.3 Contents of the sections

In Section 2, we define the socle tableau and discuss the modified lattice permutation prop-
erty. We introduce embeddings of modules over a discrete valuation domain and the associated
LR- and socle tableaux. Examples show that the LR-tableau of an embedding does not determine the socle tableau, and conversely.

In Section 3, we verify that the socle tableau of an embedding does indeed satisfy the properties of a socle tableau. Our main result is the version of the Green–Klein Theorem ([3, 7]) for socle tableaux: For a given socle tableau $\Sigma$, we explicitly construct an embedding with this socle tableau (Theorem 3.9).

The Hom-matrix $(h^m_{\ell, m})$ of an embedding $X$ encodes the lengths of the homomorphism spaces $\text{Hom}(P^m_{\ell}, X)$ between pickets and the given embedding. In Section 4, we show that socle tableau, hom-matrix and the LR-tableau of the dual embedding determine each other.

This correspondence between socle tableau and dual LR-tableau appears to be related to the tableau switching studied by Benkart, Sotille and Stroomer. We give an example and state the conjecture. In the remaining parts of Section 5, we describe how each entry in a certain row in the socle tableau specifies the position of the object in the Auslander–Reiten quiver; we also illustrate how the socle tableau, like the LR-tableau, determines the irreducible component of an object within the representation space.

2 | NOTATION AND EXAMPLES

2.1 | The LR-tableau and the socle tableau

Let $\alpha, \beta, \gamma$ be partitions. For LR- and socle tableaux of shape $(\alpha, \beta, \gamma)$ to exist, there are two necessary conditions. First, $\gamma \leq \beta$ in the sense that the Young diagram for $\gamma$ (which has columns of length $\gamma_1, \gamma_2, ...$) fits into the Young diagram for $\beta$ (that is, $\gamma_i \leq \beta_i$ holds for all $i$), so that we can consider the skew diagram $\beta \setminus \gamma$. The second condition is that $|\alpha| + |\gamma| = |\beta|$ so that we can fill the skew diagram $\beta \setminus \gamma$ with $\alpha'_1$ entries 1, $\alpha'_2$ entries 2, etc. Here, $\alpha'$ is the transpose of the partition $\alpha$, so $\alpha'_i$ is the length of the $i$th row in the Young diagram for $\alpha$.

**Definition.** A **Littlewood–Richardson (LR-) tableau** of shape $(\alpha, \beta, \gamma)$ is a filling of the skew diagram $\beta \setminus \gamma$ with $\alpha'_1$ boxes $\Box_1$, $\alpha'_2$ boxes $\Box_2$, etc., such that:

- (LR-1) in each row, the entries are weakly increasing;
- (LR-2) in each column, the entries are strictly increasing;
- (LR-3) (lattice permutation property) for each vertical line, and for each natural number $\ell$, there are at most as many entries $\ell + 1$ on the right-hand side of the line as there are entries $\ell$.

Socle tableaux are defined correspondingly:

**Definition.** A **socle tableau** of shape $(\alpha, \beta, \gamma)$ is a filling of the skew diagram $\beta \setminus \gamma$ with $\alpha'_1$ boxes $\Box_1$, $\alpha'_2$ boxes $\Box_2$, etc., such that

- (ST-1) in each row, the entries are weakly decreasing;
- (ST-2) in each column, the entries are strictly decreasing;
- (ST-3) for each vertical line, and each natural number $\ell$, there are at most as many entries $\ell + 1$ on the left hand side of the line as there are entries $\ell$. 
Example. There are two LR-tableaux and two socle tableaux of shape \( (42, 532, 31) \) (for the notation, see below):

\[
\begin{array}{ccc}
\Gamma_1: & 1 & 4 \\
\ & 1 & 2 \\
\ & 3 & 2 \\
\ & 4 & 3 \\
\end{array},
\begin{array}{ccc}
\Gamma_3: & 2 & 1 \\
\ & 3 & 2 \\
\ & 2 & 4 \\
\ & 1 & 1 \\
\end{array},
\begin{array}{ccc}
\Sigma_1: & 2 & 2 \\
\ & 3 & 3 \\
\end{array},
\begin{array}{ccc}
\Sigma_3: & 4 & 1 \\
\ & 2 & 1 \\
\end{array}
\]

In all diagrams, we number rows from top to bottom and columns from left to right.

Let \( \alpha, \beta, \gamma \) be partitions and put \( \alpha_1 = s \). Formally, an LR-tableau \( \Gamma \) can be given by an increasing sequence \( (\gamma(0), \gamma(1), \ldots, \gamma(s)) \) of partitions where the Young diagram for \( \gamma(i) \) consists of all empty boxes and boxes with entries at most \( i \). In particular, \( \gamma(0) = \gamma \) and \( \gamma(s) = \beta \).

Similarly, a socle tableau \( \Sigma \) is given by a decreasing sequence \( (\sigma(0), \sigma(1), \ldots, \sigma(s)) \) of partitions where the Young diagram for \( \sigma(i) \) consists of all empty boxes and boxes with entries strictly greater than \( i \). In particular, \( \sigma(0) = \beta \) and \( \sigma(s) = \gamma \).

**Lemma 2.1.** In the definition of the socle tableau, assuming conditions (ST-1) and (ST-2), the property (ST-3) is equivalent to each of the following conditions.

(ST-3′) For each row, and each natural number \( \ell \), there are at most as many entries \( \ell + 1 \) in this row and underneath as there are entries \( \ell \) strictly underneath the row.

(ST-3″) For each natural number \( \ell \), there exists a one-to-one map \( \varphi_{\ell} \) from the set of boxes with entry \( \ell + 1 \) to the set of boxes with entry \( \ell \) such that \( \varphi_{\ell}(b) \) is in the same column as \( b \) provided this column contains a box with entry \( \ell \), and in a column to the left of the column of \( b \) otherwise.

**Proof.** It is clear that conditions (ST-3) and (ST-3″) are equivalent. Let \( \Sigma \) be a socle tableau. For a vertical line \( L \) and a row \( R \), write

\[
\begin{align*}
L_{\ell} &= \# \{ \square \text{ in } \Sigma \text{ on the left of } L \}, \\
R_{\ell} &= \# \{ \square \text{ in } \Sigma \text{ strictly underneath } R \}, \\
R'_{\ell} &= \# \{ \square \text{ in } \Sigma \text{ in row } R \text{ or underneath} \}.
\end{align*}
\]

(ST-3) \( \Rightarrow \) (ST-3′): Given a row \( R \) and a natural number \( \ell \), let \( L \) be the vertical line through \( \Sigma \) which separates the entries greater than \( \ell \) in row \( R \) from those less than or equal to \( \ell \). Then

\[
R'_{\ell+1} = L_{\ell+1} \leq L_\ell = R_\ell.
\]

(ST-3′) \( \Rightarrow \) (ST-3): Given a vertical line \( L \) and a natural number \( \ell \), let \( R \) be the first row which contains an entry \( \ell + 1 \) on the left of \( L \). (If there is no such entry then there is nothing to show.) Let \( u \) be the number of entries \( \ell + 1 \) in row \( R \) on the right of \( L \). Then

\[
L_{\ell+1} = R'_{\ell+1} - u \leq R_\ell - u \leq L_\ell
\]

where the last inequality holds since there are at most \( u \) entries \( \ell \) on the right of \( L \) and underneath row \( R \).

\( \square \)

### 2.2 \( \Lambda \)-modules and embeddings

Let \( \Lambda \) be a (commutative) discrete valuation ring with maximal ideal generator \( p \) and radical factor field \( k = \Lambda/(p) \). In this paper, we assume that all \( \Lambda \)-modules have finite length. Examples of
discrete valuation rings include the localization $\Lambda = \mathbb{Z}(p)$ of the ring of integers at the prime ideal $(p)$, then the $\Lambda$-modules are the finite abelian $p$-groups; and the power series ring $\Lambda = k[[T]]$ with coefficients in a field $k$, then the $\Lambda$-modules are the finite-dimensional modules over the polynomial ring $k[T]$ which are annihilated by some power of $T$.

It is well known there is a one-to-one correspondence between the set of isomorphism classes of $\Lambda$-modules and the set of partitions: For a partition $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1 \geq \cdots \geq \alpha_n \geq 1$ are natural numbers, we denote by $N_\alpha$ the $\Lambda$-module

$$N_\alpha = \Lambda/(p^{\alpha_1}) \oplus \cdots \oplus \Lambda/(p^{\alpha_n})$$

and write $\alpha = \text{type}(A)$ if $A \cong N_\alpha$. We denote by $\text{len} A$ the (composition) length of the $\Lambda$-module $A$; in particular, the length of $N_\alpha$ as a $\Lambda$-module equals the length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ of $\alpha$ as a partition.

An embedding $(A \subset B)$ consists of a $\Lambda$-module $B$ and a submodule $A$ of $B$. By $S = S(\Lambda)$, we denote the category of all embeddings, with homomorphisms given by commutative diagrams.

For a $\Lambda$-module $B$, the multiplication by the radical generator $p \in \Lambda$ gives rise to two maps $\text{Sub}_B \to \text{Sub}_B$ where $\text{Sub}_B$ is the set of $\Lambda$-submodules of $B$.

$$p_B : \text{Sub}_B \to \text{Sub}_B, \quad A \mapsto \{pa : a \in A\}$$

$$p_B^{-1} : \text{Sub}_B \to \text{Sub}_B, \quad A \mapsto \{b \in B : pb \in A\}$$

In particular, if $A \subset B$ is an embedding, then the layers of the radical series and the socle series of $A$ are the submodules of $B$ given by

$$\text{rad}^m A = p_B^m(A), \quad \text{soc}^\ell A = p_A^{-\ell}(0).$$

For two $\Lambda$-modules $B, C$, and two embeddings $X, Y \in S(\Lambda)$, the homomorphism groups are $\Lambda$-modules and we write

$$\text{hom}_\Lambda(B, C) = \text{len} \text{Hom}_\Lambda(B, C), \quad \text{hom}_S(X, Y) = \text{len} \text{Hom}_S(X, Y).$$

### 2.3 Tableaux given by an embedding

Let $(A \subset B)$ be an embedding, and denote by $\alpha, \beta,$ and $\gamma$ the partition type of the $\Lambda$-module $A, B,$ and $B/A$, respectively. Let $s = \alpha_1$, so $p^s A = 0$.

The radical sequence for $A$,

$$0 = \text{rad}^s A \subset \text{rad}^{s-1} A \subset \cdots \subset \text{rad} A \subset A,$$

gives rise to a sequence of epimorphisms

$$B = B/\text{rad}^s A \to B/\text{rad}^{s-1} A \to \cdots \to B/\text{rad} A \to B/A,$$
and hence to an increasing sequence of partitions,
\[
\beta = \gamma^{(s)} \geq \gamma^{(s-1)} \geq \cdots \geq \gamma^{(1)} \geq \gamma^{(0)} = \gamma,
\]
where \(\gamma^{(i)}\) is the partition type of \(B/\text{rad}^i A\). The corresponding tableau \(\Gamma = (\gamma^{(i)})\) is an LR-tableau, according to the Theorem by Green and Klein, see Section 3. We call \(\Gamma\) the LR-tableau of the embedding \((A \subseteq B)\).

Dually, the socle sequence for \(A\),
\[
0 \subset \text{soc} A \subset \cdots \subset \text{soc}^{s-1} A \subset \text{soc}^s A = A,
\]
gives rise to a sequence of epimorphisms,
\[
B \to B/\text{soc} A \to \cdots \to B/\text{soc}^{s-1} A \to B/\text{soc}^s A = B/A,
\]
and hence to a decreasing sequence of partitions,
\[
\beta = \sigma^{(0)} \geq \sigma^{(1)} \geq \cdots \geq \sigma^{(s-1)} \geq \sigma^{(s)} = \gamma,
\]
where \(\sigma^{(i)}\) is the partition type of \(B/\text{soc}^i A\). We will see in the next section that \(\Sigma = (\sigma^{(i)})\) is a socle tableau. We call \(\Sigma\) the socle tableau of the embedding \((A \subseteq B)\).

### 2.4 Examples

Let \(m\) be a natural number and \(0 \leq \ell \leq m\). The picket \(P^m_\ell\) is given by the embedding \((\text{soc}^\ell P^m \subset P^m)\) or \((\text{rad}^{m-\ell} P^m \subset P^m)\) where \(P^m = \Lambda/(p^m)\). We picture the picket as a column of \(m\) boxes (representing the \(\Lambda\)-module \(P^m\)) and put a dot in the \((m - \ell)\)-th box from the top to represent the generator of the submodule: The top box stands for the element \(1 + (p^m)\) in \(P^m\), the second box for \(p + (p^m)\), etc. The partition type for \(P^m_\ell\) is easily computed as \(\beta = (\text{len} P^m_\ell) = (m), \alpha = (\text{len} \text{soc}^\ell P^m) = (\ell), \) or \(\alpha = ()\) if \(\ell = 0; \gamma = \text{len} P^m_\ell/\text{soc}^\ell P^m = (m - \ell), \) or \(\gamma = ()\) if \(\ell = m.\)

The modules in the radical series of the submodule have length \(\text{len} \text{rad}^i (\text{soc}^\ell P^m) = \ell - i\) for \(0 \leq i \leq \ell\) hence the partitions defining the LR-tableau are \(\gamma^{(i)} = \text{type} P^m/\text{rad}^i (\text{soc}^\ell P^m) = (m - \ell + i).\) Similarly, the modules in the socle series of the submodule have length \(\text{soc}^i (\text{soc}^\ell P^m) = i\) for \(0 \leq i \leq \ell\), so the partitions in the socle tableau are \(\sigma^{(i)} = \text{type} P^m/\text{soc}^i (\text{soc}^\ell P^m) = (m - i).\)

Here we picture the embedding \(P^5_4\) together with its LR-tableau \(\Gamma^5_4\) and its socle tableau \(\Sigma^5_4\).

Here are some more examples. \(M_1\) is the direct sum \(P^5_4 \oplus P^3_0 \oplus P^2_2\). In \(M_2\) and \(M_3\), the ambient space is \(P^5 \oplus P^3 \oplus P^2\), say generated by \(b, b',\) and \(b''\); in \(M_2\), the subspace generators are \(pb + b''\) and \(pb'\); in \(M_3\), the submodule is generated by \(pb + b'\) and \(pb' + pb''.\)
Each of the embeddings $M_i$ has partition type $\alpha = (42), \beta = (532), \gamma = (31)$.

The LR- and socle tableaux are as follows.

\[
\begin{align*}
\Sigma_1 : & \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}, & \Gamma_1 = \Gamma_2 : & \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}, & \Sigma_2 = \Sigma_3 : & \begin{array}{c}
1 \\
2 \\
3 \\
2 \\
1 \\
\end{array}, & \Gamma_3 : & \begin{array}{c}
1 \\
2 \\
3 \\
1 \\
\end{array}
\end{align*}
\]

We observe that embeddings with the same LR-tableau may have different socle tableaux, and conversely.

## 2.5 Duality

**Definition.** Let $E$ be the injective envelope of the simple $\Lambda$-module $\Lambda/(p)$. We write $DB = \text{Hom}_\Lambda(B, E)$ for the dual of the $\Lambda$-module $B$. For an embedding $X = (A \subset B) \in S(\Lambda)$, the dual embedding $X^*$ is given by the inclusion $\text{Hom}_\Lambda(\pi, E) : DC \to DB$ where $C = B/A$ and $\pi : B \to C$ is the canonical map.

Note that if the embedding $(A \subset B)$ has partition type $(\alpha, \beta, \gamma)$, then the dual embedding has type $(\gamma, \beta, \alpha)$. We define the dual LR-tableau of the embedding $(A \subset B)$ as the LR-tableau of the dual of the embedding, $\Gamma_{X^*}^* = \Gamma_{X^*}$, it is a tableau of shape $(\gamma, \beta, \alpha)$.

**Example.** We present the dual embeddings for $M_1, M_2, M_3$ from the previous subsection.

\[
\begin{align*}
M_1^* : & \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}, & M_2^* : & \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}, & M_3^* : & \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}
\end{align*}
\]

They have the following LR- and socle tableaux.

\[
\begin{align*}
\Gamma_1^* : & \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}, & \Sigma_1^* = \Sigma_2^* : & \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}, & \Gamma_2^* = \Gamma_3^* : & \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}, & \Sigma_3^* : & \begin{array}{c}
1 \\
3 \\
2 \\
\end{array}
\end{align*}
\]

We have seen in the previous subsection that the socle tableau of an embedding does not determine its LR-tableau or conversely. However, the socle tableau of an embedding and the dual LR-tableau do determine each other. In Section 4.3, we describe how to obtain one tableau from the other combinatorially.
3 | THE GREEN–KLEIN THEOREM REVISITED

This section is inspired and motivated by the Theorem of Green and Klein, which we present in the version for $\Lambda$-modules.

**Theorem 3.1** [3, 7]. Given partitions $\alpha$, $\beta$, $\gamma$, there exists a short exact sequence of $\Lambda$-modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

where $A$, $B$, $C$ have partition type $\alpha$, $\beta$, $\gamma$, respectively, if and only if there exists an LR-tableau of shape $(\alpha, \beta, \gamma)$.

Our aim is to show the corresponding result for socle tableaux.

**Theorem 3.2.** Given partitions $\alpha$, $\beta$, $\gamma$, there exists a short exact sequence of $\Lambda$-modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

where $A$, $B$, $C$ have partition type $\alpha$, $\beta$, $\gamma$, respectively, if and only if there exists a socle tableau of shape $(\alpha, \beta, \gamma)$.

In the first subsection, we show that the socle tableau of the embedding $(A \subset B)$ as defined in Subsection 2.3 does satisfy conditions (ST-1) through (ST-3). In the second subsection, we show that any such tableau can be realized as the socle tableau of an embedding.

### 3.1 Conditions (ST-1) to (ST-3) are satisfied

Let $\Sigma$ be the socle tableau of an embedding $(A \subset B)$, so there are partitions $\alpha = \text{type } A$, $\beta = \text{type } B$, $\gamma = \text{type } B/A$ such that $\Sigma$ is the skew diagram of shape $\beta \setminus \gamma$. If the socle tableau $\Sigma$ is given by partition sequence $(\sigma(i))$ where $\sigma(i) = \text{type } B/\text{soc }^i A$, then for each natural number $\ell$, the entries $[\ell]$ occur in the skew diagram $\sigma(\ell - 1) \setminus \sigma(\ell)$.

We adapt to our situation the proof of the analogous fact for LR-tableaux, see [7, II.3.4]. Let $s = \alpha_1$.

Obviously we have $\sigma(s) = \gamma$ and $\sigma(0) = \beta$, because $\text{soc }^s A = A$ and $\text{soc }^0 A = 0$. Moreover, there is a short exact sequence

$$0 \rightarrow \text{soc }^\ell A/\text{soc }^{\ell - 1} A \rightarrow B/\text{soc }^{\ell - 1} A \rightarrow B/\text{soc }^\ell A \rightarrow 0$$

and therefore $\sigma(\ell) \subseteq \sigma(\ell - 1)$ for all $\ell$, see [7, II.3.1]. This shows (ST-1) and the fact that there are $\alpha'_\ell = \text{len}(\text{soc }^\ell A/\text{soc }^{\ell - 1} A)$ boxes $[\ell]$ in $\Sigma$.

Since $\text{soc }^\ell A/\text{soc }^{\ell - 1} A$ is a semisimple module, $\sigma(\ell - 1) \setminus \sigma(\ell)$ is a horizontal strip see [7, II.3.3]. Thus condition (ST-2) is also satisfied.

It remains to show property (ST-3), or equivalently, (ST-3’) (Lemma 2.1).
Lemma 3.4. For $\ell \geq 2$, the map

$$\mu_p : \frac{\text{soc } \ell A}{\text{soc } (\ell-1) A} \to \frac{\text{soc } (\ell-1) A}{\text{soc } (\ell-2) A}$$

given by multiplication by $p$ is a monomorphism.

Proof. From the description of the socle as $\text{soc } \ell A = p_{\ell}^{-\ell}(0)$, we obtain for $a \in \text{soc } \ell A$ that $pa \in \text{soc } (\ell-1) A$, hence the map $\mu_p$ is defined. Moreover, if $pa \in \text{soc } (\ell-2) A = p_{\ell}^{-2}(\ell-2)(0)$, then $a \in p_{\ell}^{-\ell}(\ell-1) = \text{soc } (\ell-1) A$; so $\mu_p$ is a monomorphism. \hfill $\square$

We will use the following easy generalization.

Corollary 3.5. For an embedding $A \subset B$ and natural numbers $\ell \geq 2$, $s \geq 1$, the map

$$\mu_p : \frac{\text{soc } \ell A \cap \text{rad }^{s-1} B}{\text{soc } (\ell-1) A \cap \text{rad }^{s-1} B} \to \frac{\text{soc } (\ell-1) A \cap \text{rad }^{s} B}{\text{soc } (\ell-2) A \cap \text{rad }^{s} B}$$

given by multiplication by $p$ is a monomorphism.

The factors in the corollary describe the numbers in (ST-3').

Lemma 3.6. For an embedding $A \subset B$ with socle tableau $\Sigma$ and for natural numbers $\ell', r \geq 1$,

$$\text{len}\left(\frac{\text{soc } \ell A \cap \text{rad }^{r-1} B}{\text{soc } (\ell-1) A \cap \text{rad }^{r-1} B}\right) = \# \{\text{entries } \ell \text{ with row numbers } \geq r \text{ in } \Sigma\}.$$ 

Proof. For $S_{\ell'} = B / \text{soc } \ell A$, note that $\text{len } S_{\ell'}$ counts the number of boxes in $\Sigma$ which are either empty or contain an entry $> \ell$. Hence, $\text{len } \text{rad }^{r-1} S_{\ell'}$ counts the number of boxes in $\Sigma$ which occur in rows $\geq r$ and which are either empty or contain an entry $> \ell$. Thus in the lemma, the number on the right-hand side is

$$\text{len } \text{rad }^{r-1} S_{\ell'-1} - \text{len } \text{rad }^{r-1} S_{\ell'}.$$

Note that

$$\text{rad }^{r-1} S_{\ell'} = \text{rad }^{r-1}\left(\frac{B}{\text{soc } \ell A}\right) = \frac{\text{rad }^{r-1} B + \text{soc } \ell A}{\text{soc } \ell A \cap \text{rad }^{r-1} B} \approx \frac{\text{rad }^{r-1} B}{\text{soc } \ell A \cap \text{rad }^{r-1} B}.$$ 

The formula implies

$$\text{len } \text{rad }^{r-1} S_{\ell'-1} - \text{len } \text{rad }^{r-1} S_{\ell'} = \text{len}\left(\frac{\text{soc } \ell A \cap \text{rad }^{r-1} B}{\text{soc } (\ell-1) A \cap \text{rad }^{r-1} B}\right).$$ 

This finishes the proof of the lemma. \hfill $\square$

Proposition 3.7. The socle tableau $\Sigma$ of an embedding $A \subset B$ satisfies conditions (ST-1) through (ST-3).
Proof. We have seen that conditions (ST-1) and (ST-2) hold for $\Sigma$. Here we verify property (ST-3') which is equivalent to (ST-3) by Lemma 2.1.

The map $\mu_p$ in Corollary 3.5 is a monomorphism, hence

\[
\text{len}\left(\frac{\text{soc}^\ell A \cap \text{rad}^{s-1} B}{\text{soc}^{\ell-1} A \cap \text{rad}^{s-1} B}\right) \leq \text{len}\left(\frac{\text{soc}^{\ell-1} A \cap \text{rad}^{s} B}{\text{soc}^{\ell-2} A \cap \text{rad}^{s} B}\right).
\]

The claim follows from Lemma 3.6. □

3.2 Every socle tableau is the socle tableau of an embedding

We will use the following tool to construct, for a given socle tableau $\Sigma$, a corresponding embedding $(A \subset B)$.

Lemma 3.8. Let

\[ B = C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots C_{s-1} \xrightarrow{f_s} C_s = C \]

be a sequence of surjective $\Lambda$-homomorphisms with semisimple kernels satisfying the following condition

\[ (\ast) \quad \text{soc}(\text{Ker} f_{\ell+1} f_{\ell}) = \text{Ker} f_{\ell} \text{ for all } \ell = 1, \ldots, s-1. \]

Then $A = \text{Ker} f$ where $f = f_s \cdots f_1$ has partition type $\alpha$ given by $\alpha'_\ell = \text{len} \text{Ker} f_{\ell}$ for all $\ell$. Moreover, $B/\text{soc}^\ell A \cong C^\ell$ holds for each $0 \leq \ell \leq s$.

Proof. We first show by induction on $j$ that for each $\ell$ the equality $\text{soc}(\text{Ker} f_{\ell+j} f_{\ell+j-1} \cdots f_{\ell}) = \text{Ker} f_{\ell}$ holds. The case where $j = 0, 1$ is satisfied by assumption. Let $j > 1$. The inclusion $\text{Ker} f_{\ell} \subset \text{soc}(\text{Ker} f_{\ell+j} \cdots f_{\ell})$ is clear, so let $x \in \text{soc}(\text{Ker} f_{\ell+j} \cdots f_{\ell})$. If $x \in \text{Ker} f_{\ell+j-1} \cdots f_{\ell}$, we are finished by induction. If $f_{\ell+j-1} \cdots f_{\ell}(x) \neq 0$, then $f_{\ell}(x) \in \text{soc}(\text{Ker} f_{\ell+j} \cdots f_{\ell+1})$. By induction, $f_{\ell}(x) \in \text{Ker} f_{\ell+1}$ and we have a contradiction to the assumption that $f_{\ell+j-1} \cdots f_{\ell}(x) \neq 0$. Hence $x \in \text{Ker} f_{\ell}$.

Next we observe that for any $\ell = 1, \ldots, s-1$ there is the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\text{Ker} f_{\ell} & \xrightarrow{f_{\ell}} & \text{Ker} f_{\ell} \\
\downarrow & & \downarrow \\
0 & \to & \text{Ker} f_s \cdots f_{\ell} & \xrightarrow{f_s \cdots f_{\ell}} & C_{\ell-1} & \xrightarrow{f_s \cdots f_{\ell}} & C_{\ell} & \xrightarrow{f_s \cdots f_{\ell}} & C_s & \to & 0 \\
\downarrow & & \downarrow f_{\ell} & & \downarrow & & \downarrow f_{\ell} & & \downarrow & & \downarrow \\
0 & \to & \text{Ker} f_s \cdots f_{\ell+1} & \xrightarrow{f_s \cdots f_{\ell+1}} & C_{\ell} & \xrightarrow{f_s \cdots f_{\ell+1}} & C_s & \to & 0 \\
\end{array}
\]
In the first part, we have seen that \( \ker f_{\ell} = \soc \ker f_s \cdots f_{\ell} \); using the exactness of the left column in the diagram, we obtain

\[
\frac{\ker f_s \cdots f_{\ell}}{\soc(\ker f_s \cdots f_{\ell})} \cong \ker f_s \cdots f_{\ell+1}.
\]

Thus the Young diagram for \( \ker f_s \cdots f_{\ell} \) is obtained from the Young diagram for \( \ker f_s \cdots f_{\ell+1} \) by adding a new top row of length

\[
\text{len } \soc(\ker f_s \cdots f_{\ell}) = \text{len } \ker f_{\ell}.
\]

Suppose that \( A = \ker f_s \cdots f_1 \) has partition type \( \alpha \). Then it follows that \( \alpha'_{\ell} = \text{len } \ker f_{\ell} \).

Next we show by induction on \( \ell \) that \( B/\soc \ell A \cong C_{\ell} \). More precisely, we show in each step that \( \soc \ell A = \ker f_{\ell} \cdots f_1 \). Since the epimorphism

\[
\pi_{\ell} : B \to C_{\ell}, \quad b \mapsto f_{\ell} \cdots f_1(b),
\]

maps \( A \) onto \( \ker f_s \cdots f_{\ell+1} \), it has kernel \( \soc \ell A \) and hence induces the desired isomorphism \( B/\soc \ell A \cong C_{\ell} \).

For \( \ell = 0 \), there is nothing to show. Assume \( \ell > 0 \).

\[
\soc \ell A = \pi_{\ell-1}^{-1}(\soc \ker f_s \cdots f_{\ell})
\]

\[
= \pi_{\ell-1}^{-1}(\ker f_{\ell})
\]

\[
= \{ b : \pi_{\ell-1}(b) \in \ker f_{\ell} \}
\]

\[
= \ker f_{\ell} \cdots f_1
\]

\[
= \ker \pi_{\ell}.
\]

The first equality holds by induction hypothesis: \( \pi_{\ell-1} : A \to \ker f_s \cdots f_{\ell} \) is an epimorphism with kernel \( \soc \ell-1 A \). Hence by definition of the \( \ell \)th socle, \( \soc \ell A \) is the inverse image of the socle of \( \ker f_s \cdots f_{\ell} \). The equations show that \( \pi_{\ell} \) induces an isomorphism \( B/\soc \ell A \cong C_{\ell} \). \( \square \)

**Theorem 3.9.** Every socle tableau can be realized as the socle tableau of an embedding.

Before the proof, we illustrate the construction of the embedding in an example.

**Example.** Consider the socle tableau \( \Sigma_2 \) from above.

\[
\Sigma_2 : \begin{array}{ccc}
& 4 & \\
3 & 2 & \\
1 & 1 & \\
\end{array}
\]

We construct a chain of epimorphisms which satisfies the conditions of Lemma 3.8. Suppose \( \Sigma_2 \) is given by the sequence \( \sigma^{(i)} \), \( i = 0, \ldots, 4 \), of partitions. Let \( C^{(i)} \) be the \( \Lambda \) module of type \( \sigma^{(i)} \) and
$g^{(i)} : C^{(i-1)} \to C^{(i)}$ be the component-wise maps given as follows.

$$
\begin{align*}
C^{(0)} &= P^5 \oplus P^3 \oplus P^2 \\
\xrightarrow{g^{(i)}=\text{can} \oplus \text{can} \oplus 1_a} & C^{(1)} \\
C^{(1)} &= P^4 \oplus P^2 \oplus P^2 \\
\xrightarrow{g^{(i)}=\text{can} \oplus 1_b \oplus \text{can} \oplus u} & C^{(2)} \\
C^{(2)} &= P^3 \oplus P^2 \oplus P^1 \\
\xrightarrow{g^{(i)}=1 \oplus \text{can} \oplus 1_c} & C^{(3)} \\
C^{(3)} &= P^3 \oplus P^1 \oplus P^1 \\
\xrightarrow{g^{(i)}=1 \oplus 1 \oplus 0,} & C^{(4)} = P^3 \oplus P^1
\end{align*}
$$

Here $\text{can} : P^n \to P^{n-1}$ is the canonical epimorphism. Considering the lemma for the maps $g^{(i)}$ (instead of $f_i$), we see that Condition $(\ast)$ of Lemma 3.8 is violated whenever a composition of type $\text{can} \circ 1$ occurs in some component. This happens three times; we label the occurrences using subscripts $a$, $b$ and $c$.

To fix the situation, we define automorphisms $h^{(i)}$ on the target of $g^{(i)}$ and put $f_i = h^{(i)} \circ g^{(i)}$. Then Condition $(\ast)$ of Lemma 3.8 will be satisfied.

We consider the first case, marked by subscript $a$. The third column in $\Sigma_2$ contains a box $[2]$, but not a box $[1]$. Hence, on the third component, the map $g^{(1)}$ is the identity (marked $1_a$), while $g^{(2)}$ on the third component is the canonical map (marked $\text{can}_a$). We use the box $[1]$ in the second column to fix the situation. Define

$$
\begin{align*}
h^{(1)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Aut}(P^4 \oplus P^2 \oplus P^2),
\end{align*}
$$

It is straightforward to verify that the maps $g^{(1)}$ and $g^{(2)} h^{(1)} g^{(1)}$ have kernels

$$
\begin{align*}
\ker f_1 &= \ker g^{(1)} = (p^4, 0, 0) \Lambda \oplus (0, p^2, 0) \Lambda \\
\ker f_2 f_1 &= \ker (g^{(2)} h^{(1)} g^{(1)}) = (p^3, 0, 0) \Lambda \oplus (0, p, -p) \Lambda,
\end{align*}
$$

thus the Condition $(\ast)$, $\ker f_1 = \text{soc} \ker f_2 f_1$ is satisfied.

In the second case, marked by subscript $b$, the box $[3]$ in $\Sigma_2$ does not have a box $[2]$. We can use the box in the first column and define $h^{(2)}$ correspondingly:

$$
\begin{align*}
h^{(2)} &= \begin{pmatrix} 1 & \text{incl} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
h^{(3)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \\
h^{(4)} &= 1_{C^{(4)}}
\end{align*}
$$

Similarly, in the third case marked by subscript $c$, the box $[4]$ does not have a box $[3]$. We have verified the conditions of Lemma 3.8, hence the embedding $(A \subset B)$ where $A = \ker f_4 \cdots f_1$ and $B = C^{(0)}$ has socle tableau $\Sigma_2$. 

We can now show Theorem 3.9 in the general situation.

Proof. Suppose that the socle tableau \( \Sigma \) has shape \((\alpha, \beta, \gamma)\). We take \( B = N_\beta \) and \( C = N_\gamma \) and use Lemma 3.8 to construct an epimorphism \( f : B \to C \) with kernel \( A \) such that the embedding \((A \subset B)\) has the desired socle tableau.

The tableau \( \Sigma \) is given by partitions \( \sigma^{(\ell)} \), where \( 0 \leq \ell \leq s \) with \( s = \alpha_1 \). Let \( C^{(\ell)} \) be a module of type \( \sigma^{(\ell)} \), more precisely, put

\[
C^{(\ell)} = \bigoplus_j C_j^{(\ell)},
\]

where \( C_j^{(\ell)} = P_j^{\sigma^{(\ell)}} \) is the indecomposable \( \Lambda \)-module of length \( \sigma_j^{(\ell)} \). It follows from (ST-1) and (ST-2) that \( \sigma_j^{(\ell)} \subset \sigma_j^{(\ell-1)} \), hence the \( j \)th parts satisfy \( \sigma_j^{(\ell)} \leq \sigma_j^{(\ell-1)} \), so there are canonical maps

\[
g_j^{(\ell)} : C_j^{(\ell-1)} \to C_j^{(\ell)},
\]

which we can define as the diagonal map

\[
g_j^{(\ell)} = \bigoplus_j g_j^{(\ell)} : \bigoplus_j C_j^{(\ell-1)} \to \bigoplus_j C_j^{(\ell)}.
\]

(In case the partition \( \sigma^{(\ell-1)} \) has more parts than \( \sigma^{(\ell)} \), formally add parts of size 0.) Since \( \sigma^{(\ell-1)} \setminus \sigma^{(\ell)} \) is a horizontal strip of length \( \alpha'_\ell \), the kernel of \( g_\ell^{(\ell+1)} \) is a semisimple \( \Lambda \)-module of length \( \alpha'_\ell \).

For \( \ell = 1, \ldots, s-1 \), we construct automorphisms \( h^{(\ell)} : C^{(\ell)} \to C^{(\ell)} \) such that \( \text{soc}(\ker g^{(\ell+1)} h^{(\ell)} g^{(\ell)}) = \ker g^{(\ell)} \). Then the maps

\[
f_\ell = \begin{cases} h^{(\ell)} g^{(\ell)}, & \text{if } 1 \leq \ell < s \\ g^{(s)}, & \text{if } \ell = s \end{cases}
\]

satisfy Condition (*) in Lemma 3.8.

The construction for the automorphism \( h^{(\ell)} \) of \( C^{(\ell)} \) is as follows. We use condition (ST-3") on \( \Sigma \) to partition the set of columns of the Young diagram for \( \sigma^{(\ell)} \) into subsets which consist either of a single column or a pair of columns. Using the map \( \varphi^{(\ell)} \), we consider the following cases:

(a) Column \( j \) contains boxes with entries \( \ell \) and \( \ell + 1 \); by definition of \( \varphi^{(\ell)} \), the box with entry \( \ell + 1 \) is mapped to the box with entry \( \ell \). In this case, we add the singleton \( \{j\} \) to our partition of the columns and put \( h_j^{(\ell)} = 1 \), the identity map on \( C_j^{(\ell)} \).

(b) Column \( j \) contains neither a box with entry \( \ell \) nor one with entry \( \ell + 1 \). Again, column \( j \) forms a singleton, and we put \( h_j^{(\ell)} = 1 \).

(c) Column \( j \) contains a box with entry \( \ell \) which is not in the image of \( \varphi^{(\ell)} \). Also in this case, column \( j \) is a singleton and \( h_j^{(\ell)} = 1 \).

(d) The remaining columns contain either a box with entry \( \ell + 1 \), or a box with entry \( \ell \) in the image of \( \varphi^{(\ell)} \). Suppose column \( i \) contains entry \( \ell \) and column \( j \) entry \( \ell + 1 \) and \( \varphi^{(\ell)} \) maps the box in column \( j \) to the box in column \( i \). Then \( i < j \) and if \( u = \sigma_i^{(\ell)}, v = \sigma_j^{(\ell)} \) are the lengths of the two columns in the Young diagram for \( \sigma^{(\ell)} \), then \( u > v \). In this case, we add the pair \( \{i, j\} \) to our partition of the set of columns, and define

\[
h_{i,j}^{(\ell)} : C_i^{(\ell)} \oplus C_j^{(\ell)} \to C_i^{(\ell)} \oplus C_j^{(\ell)}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & \text{incl} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},
\]

where incl is the inclusion map from \( C_j^{(\ell)} = P^v \) to \( C_i^{(\ell)} = P^u \) given by multiplication by \( p^{u-v} \).
We have accounted for each column. Then \( h^{(\ell)} = \bigoplus_S h^{(\ell)}_S \) where \( S \) runs over the parts of the partition of the set of columns is an isomorphism of \( C^{(\ell)} \). Put \( g^{(\ell)}_S = \bigoplus_{j \in S} g^{(\ell)}_j \), then one can check that in each of the four cases (a)–(d),

\[
\text{soc} \left( \text{Ker} g^{(\ell+1)}_S \right) h^{(\ell)}_S g^{(\ell)}_S = \text{Ker} g^{(\ell)}_S
\]

holds (for case (d), see the example in this proof). Putting \( f_\ell = h^{(\ell)} g^{(\ell)} \) (where \( h^{(s)} = 1 \)), we see that Condition (*) in Lemma 3.8 is satisfied. Let \( A = \text{Ker} f_\ell \cdots f_1 \), then the embedding \((A \subset B)\) has socle tableau \( \Sigma \).

\[\square\]

4 | SOCLE TABLEAU, DUAL LR-TABLEAU AND HOM-MATRIX

Let \( X \) be an embedding of type \((\alpha, \beta, \gamma)\). The Hom-matrix for \( X \) describes the sizes of the homomorphism groups from the pickets into \( X \). The matrix \( H = (h^{m}_{m})_{\ell, m} \) is given by \( h^{m}_{\ell} = \text{len Hom}_{S}(P^{m}_{\ell}, X) \).

In this section, we discuss the interplay between the socle tableau for \( X \), the LR-tableau of the dual embedding \( \Gamma^{*} = \Gamma_{DX}^{*} \) and the Hom-matrix \( H \).

**Theorem 4.1.** For an embedding \( X \), the following invariants are equivalent in the sense that each one determines both of the others.

1. The socle tableau \( \Sigma = \Sigma_X \).
2. The LR-tableau of the dual embedding \( \Gamma^{*} = \Gamma_{DX}^{*} \).
3. The Hom-matrix \( H = (h^{m}_{m})_{\ell, m} \) where \( h^{m}_{\ell} = \text{hom}_{S}(P^{m}_{\ell}, X) \).

In the following three subsections, we give explicit combinatorial formulas for the conversion between the socle tableau and the Hom-matrix; the dual LR-tableau and the Hom-matrix; and the socle tableau and the dual LR-tableau.

4.1 | The socle tableau and the Hom-matrix

We denote by \( \mu_{\Sigma}(e \cdot r) \) the multiplicity of the entry \( e \) in row \( r \) in \( \Sigma \).

Our aim is to show the two formulas:

\[
\mu_{\Sigma}(e \cdot r) = h^{r+\ell-1}_{\ell} - h^{r+\ell}_{\ell-1} - h^{r+\ell-2}_{\ell-1} + h^{r+\ell-1}_{\ell-1}
\]

where we substitute \( m = r + \ell \) in the second line. Conversely, one can retrieve each entry in the Hom-matrix from the socle tableau.

\[
h^{m}_{\ell} = |\text{soc}^{\ell} A| + |\text{soc}^{m-\ell}(B/\text{soc}^{\ell} A)|
\]

\[
= \alpha'_{1} + \cdots + \alpha'_{\ell} + (\sigma^{(\ell)})'_{1} + \cdots + (\sigma^{(\ell)})'_{m-\ell}
\]
For the proof of the first formula, we use that for a \( \Lambda \)-module \( U \) of type \( \lambda \), the length of the \( r \)th row of the Young diagram for \( \lambda \) is

\[
\lambda'_r = \text{len} \ soc^r U - \text{len} \ soc^{r-1} U.
\]

We also use the following

**Lemma 4.2.** For an embedding \((A \subset B)\) and a picket \( P^m_\ell \) with \( m = \ell + r \), we have

\[
\text{hom}_S(P^m_\ell, (A \subset B)) = \text{hom}_\Lambda(P^\ell, A) + \text{hom}_\Lambda(P^r, B/\text{soc}^\ell A)
\]

**Proof.**

\[
\text{hom}_S(P^m_\ell, (A \subset B)) = \\
= \text{hom}_S(P^m_\ell, (\text{soc}^\ell A \subset B)) \\
= \text{hom}_\Lambda(P^m, p_B^r (\text{soc}^\ell A)) \\
= \text{hom}_\Lambda(P^m, \text{soc}^\ell A) + \text{hom}_\Lambda(P^m, p_B^r (\text{soc}^\ell A)/\text{soc}^\ell A) \\
= \text{hom}_\Lambda(P^\ell, A) + \text{hom}_\Lambda(P^r, B/\text{soc}^\ell A)
\]

For the first two equalities note that each \( f \in \text{Hom}_S(P^m_\ell, (A \subset B)) \) is given by an element \( b \in B \) such that \( p^r b \in A \) and \( p^r + \ell b = 0 \). Since \( p_B^r (\text{soc}^\ell A) \) is a \( \Lambda/(p^m) \)-module, the next equality follows from the exactness of the Hom-functor. For the last step note that each \( f \in \text{Hom}_\Lambda(P^\ell, C) \) is given by an element in \( p_C^{-\ell}(0) \). \( \square \)

We can now show the first equality.

\[
\mu_S(\ell^r) = (\sigma^{(\ell-1)})'_r - (\sigma^{(\ell)})'_r \\
= \text{len} \ soc^r (B/\text{soc}^{\ell-1} A) - \text{len} \ soc^{r-1} (B/\text{soc}^{\ell-1} A) \\
- \text{len} \ soc^r (B/\text{soc}^\ell A) + \text{len} \ soc^{r-1} (B/\text{soc}^\ell A) \\
= \text{hom}(P^r, B/\text{soc}^{\ell-1} A) - \text{hom}(P^{r-1}, B/\text{soc}^{\ell-1} A) \\
- \text{hom}(P^r, B/\text{soc}^\ell A) + \text{hom}(P^{r-1}, B/\text{soc}^\ell A) \\
= \text{hom}(P^r, B/\text{soc}^{\ell-1} A) + \text{hom}(P^{r-1}, A) \\
- \text{hom}(P^{r-1}, B/\text{soc}^{\ell-1} A) - \text{hom}(P^{r-1}, A) \\
- \text{hom}(P^r, B/\text{soc}^\ell A) - \text{hom}(P^r, A) \\
+ \text{hom}(P^{r-1}, B/\text{soc}^\ell A) + \text{hom}(P^r, A) \\
= \text{hom}_S(P^{m-1}_{\ell-1}, (A \subset B)) - \text{hom}_S(P^{m-2}_{\ell-1}, (A \subset B)) \\
- \text{hom}_S(P^m_{\ell-1}, (A \subset B)) + \text{hom}_S(P^{m-1}_{\ell-1}, (A \subset B)) \\
= h^{m-1}_{\ell-1} - h^{m-2}_{\ell-1} - h^m_{\ell} + h^{m-1}_{\ell}
\]
Corollary 4.3. Let \( \ell, r \) be natural numbers, \( m = \ell + r \) and \( f^m_{\ell} \) the monomorphism in the short exact sequence (where we put \( P_0^0 = 0 \)).

\[
0 \longrightarrow P_{\ell-1}^{m-1} \overset{f^m_{\ell}}{\longrightarrow} P_\ell^m \oplus P_{\ell-1}^{m-2} \longrightarrow P_{\ell-1}^{m-1} \longrightarrow 0
\]

If \( \Sigma \) is the socle tableau of the embedding \( X \in S(\Lambda) \), then

\[
\mu_\Sigma([\ell][r]) = \text{len CokHom}_S(f^m_{\ell}, X).
\]

Remark. In the case where \( \Lambda \) is bounded, for example, if \( \Lambda = k[T]/(T^n) \) or \( \Lambda = \mathbb{Z}/(p^n) \), and \( n = m - 1 \), then the summand \( P_\ell^m \) in the middle term of the exact sequence in Corollary 4.3 is not defined. In this case,

\[
\mu_\Sigma([\ell][r]) = \text{len CokHom}_S(f^m_{\ell}, X),
\]

where \( f^m_{\ell} \) is the canonical map \( P_{\ell-1}^{m-1} \rightarrow P_{\ell-1}^{m-2} \).

The second formula is shown by the following equations.

\[
h^m_{\ell} = \text{hom}_S(P_\ell^m, (A \subset B))
= \text{hom}(P_\ell^m, (\text{soc}^\ell A \subset B))
= \text{hom}(P_\ell^m, (\text{soc}^\ell A \subset P_B^{-m-\ell}(\text{soc}^\ell A)))
= \text{hom}_\Lambda(P_\ell^m, P_B^{-m-\ell}(\text{soc}^\ell A))
= \text{len } P_B^{-m-\ell}(\text{soc}^\ell A)
= \text{len soc}^\ell A + \text{len soc}^{m-\ell}(B/\text{soc}^\ell A)
= \alpha'_1 + \cdots + \alpha'_{\ell} + (\sigma^{(\ell)})'_1 + \cdots + (\sigma^{(\ell)})'_{m-\ell}.
\]

4.2 The dual LR-tableau and the Hom-matrix

Our aim in this subsection is to verify two formulas.

\[
\mu_\Gamma([\ell][m]) = \begin{cases} 
  h^m_{\ell} - h^{m-1}_{\ell+1} + h^{m-1}_{\ell-1} & \text{if } \ell < m \\
  h^m_{0} - h^{m-1}_{1} & \text{if } \ell = m
\end{cases}
\]

We write \( m = r + \ell \) as above. Conversely, we can obtain \( h^m_{r} \) from the tableau \( \Gamma^* = (\lambda^{(\ell)})_{\ell} \) via

\[
h^m_{r} = (\lambda^{(\ell)})'_1 + \cdots + (\lambda^{(\ell)})'_m.
\]

The first formula follows from [9, Theorem 2]:

\[
\mu_\Gamma([\ell][m]) = \text{len CokHom}_S(h^m_{r}, X),
\]
where $\tilde{h}_r^m$ is the map

$$
\tilde{h}_r^m : \begin{cases} 
P^m_0 \to P^m_1, & \text{if } r = 0 \\
P^m_r \to P^{m-1}_{r-1} \oplus P^m_{r+1}, & \text{if } 1 \leq r < m \\
P^m_m \to P^{m-1}_{m-1}, & \text{if } r = m.
\end{cases}
$$

In the case where $\ell < m$, we have $1 \leq r < m$ and the formula is obtained by applying the contravariant Hom-functor to the short exact sequence

$$
0 \to P^m_r \xrightarrow{\tilde{h}^m_r} P^{m-1}_{r-1} \oplus P^m_{r+1} \to P^{m-1}_r \to 0.
$$

If $\ell = m$, we have $r = 0$. Here we can use the short exact sequence

$$
0 \to \text{Hom}(P^m_1, X) \to \text{Hom}(P^m_0, X) \to \text{Cok}\text{Hom}(\tilde{h}_0^m, X) \to 0.
$$

For the proof of the second formula, we use

**Lemma 4.4.** Suppose the embedding $X = (A \subset B)$ has LR-tableau $\Gamma = (\gamma^{(i)})_i$ (so $\gamma^{(i)}$ is the type of $B/p^i A$). Then

$$
\text{hom}_S(Y, P^m_\ell) = (\gamma^{(\ell)})'_1 + \cdots + (\gamma^{(\ell)})'_m.
$$

**Proof.** With the notation from the Lemma, we have

$$
\text{hom}_S(X, P^m_\ell) = \text{hom}_S((A/p^\ell A \subset B/p^\ell A), P^m_\ell)
$$

$$
= \text{hom}_\Lambda(B/p^\ell A, P^m)
$$

$$
= (\gamma^{(\ell)})'_1 + \cdots + (\gamma^{(\ell)})'_m.
$$

We can now show the second formula:

**Lemma 4.5.** Suppose the embedding $X = (A \subset B)$ has dual LR-tableau $\Gamma^* = (\lambda^{(\ell)})_\ell$. Let $m$ be a natural number, $0 \leq r \leq m$ an integer and $\ell = m - r$. Then

$$
\tilde{h}_r^m = (\lambda^{(\ell)})'_1 + \cdots + (\lambda^{(\ell)})'_m.
$$

**Proof.**

$$
\tilde{h}_r^m = \text{hom}_S(P^m_r, X)
$$

$$
= \text{hom}_S(DX, DP^m_r)
$$

$$
= \text{hom}_S(DX, P^m_{m-r})
$$

$$
= (\lambda^{(m-r)})'_1 + \cdots + (\lambda^{(m-r)})'_m,
$$

where the last step follows from Lemma 4.4.
4.3 The socle tableau and the dual LR-tableau

We present two formulas which show how to obtain the socle tableau from the dual LR-tableau and conversely. Of course, this can be done via the Hom-matrix.

**Proposition 4.6.** Suppose an embedding has socle tableau $\Sigma$ and the dual embedding has LR-tableau $\Gamma^* = (\lambda(i))_i$. If $\ell, r$ are natural numbers and $m = \ell + r$, we have

$$\mu_{\Sigma}(\ell|m) = (\lambda^{(r-1)})'_m - (\lambda^{(r)})'_m.$$

**Proof.** The formula follows from the first statement in Subsection 4.1 and Lemma 4.5:

$$\mu_{\Sigma}(\ell|m) = h^m_{\ell} - h^m_{\ell-1} + h^{m-1}_{\ell-1},$$

$$= (\lambda^{(r-1)})'_m + \sum_{i=1}^{r-1} (\lambda^{(r-1)})'_m - (\lambda^{(r)})'_m,$$

$$- (\lambda^{(r-1)})'_m + \sum_{i=1}^{r-1} (\lambda^{(r)})'_m,$$

$$= (\lambda^{(r-1)})'_m - (\lambda^{(r)})'_m.$$

The second formula summarizes how to retrieve the entries in $\Gamma^*$ from $\Sigma$.

$$\mu_{\Gamma^*}(\ell|m) = \begin{cases} 0 & \text{if } m < \ell \\ \beta'_m - \sum_{j=m}^{\ell-1} \mu_{\Sigma}(1|j) & \text{if } m = \ell \\ \sum_{j=m}^{\ell} \mu_{\Sigma}(m-j) - \sum_{j=\ell+1}^{m} \mu_{\Sigma}(m-1-j) & \text{if } m > \ell \end{cases}$$

Note that the first sum in the last line counts the number of entries $r = m - \ell$ underneath the $\ell$th row, while the second sum counts the entries $m - \ell + 1$ in the $\ell$th row and underneath. We observe that by (ST-3') the difference is nonnegative.

The formula follows from the rule $\mu_{\Gamma^*}(\ell|m) = (\lambda^{(\ell)})'_m - (\lambda^{(\ell-1)})'_m$ and the following

**Lemma 4.7.**

$$(\lambda^{(\ell)})'_m = \begin{cases} \beta'_m & \text{if } m \leq \ell \\ \sum_{j=\ell}^{m} \mu_{\Sigma}(m-j) & \text{if } m > \ell \end{cases}$$

**Proof.** Note that the first $\ell$ rows of $\beta$ and $\lambda^{(\ell)}$ are equal (because there is no entry bigger than $\ell$ in the first $\ell$ rows of an LR-tableau), hence $(\lambda^{(\ell)})'_m = \beta'_m$ for all $m \leq \ell$. Now, we use the first formula in this subsection repeatedly:

$$(\lambda^{(\ell)})'_m = (\lambda^{(\ell-1)})'_m - \mu_{\Sigma}(m-\ell|\ell),$$

$$= (\lambda^{(\ell-2)})'_m - \mu_{\Sigma}(m-\ell|\ell-1) - \mu_{\Sigma}(m-\ell|\ell),$$

$$= \ldots$$

$$= (\lambda^{(0)})'_m - \sum_{j=1}^{\ell} \mu_{\Sigma}(m-j),$$

$$= \alpha'_m - \sum_{j=1}^{\ell} \mu_{\Sigma}(m-j),$$

$$= \sum_{j=\ell+1}^{m} \mu_{\Sigma}(m-j).$$

□
As a consequence, we obtain the following characterisation of the LR coefficient \( c_{\alpha,\gamma}^\beta \).

**Corollary 4.8.** The number \( c_{\alpha,\gamma}^\beta \) equals the number of socle tableaux of shape \((\alpha, \beta, \gamma)\).

## 5 Applications and a Conjecture

First, given an embedding in one of the categories \( S(n) \) (see below) and its socle tableau, we determine for each entry in the tableau the possible positions of the indecomposable direct summands of the embedding within the Auslander–Reiten quiver for \( S(n) \).

Next we consider the representation space for embeddings of type \((42, 642, 42)\), it is the constructible variety of short exact sequences

\[
0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0
\]

where \( \alpha = (42), \beta = (642), \gamma = (42) \). This space contains the first occurrence of a family of pairwise nonisomorphic indecomposable embeddings. We describe how socle tableaux partition this representation space.

Finally, we note that the correspondence between the socle tableau of an embedding and the dual LR-tableau is related to the tableau switching studied in [1]. We give an example and conclude with an open problem.

### 5.1 Positioning objects in the Auslander–Reiten quiver

Given an indecomposable embedding \( M \) with socle tableau \( \Sigma \), we show that each entry \( \ell \) in a given row \( r \) in \( \Sigma \) determines a region in the Auslander–Reiten quiver \( \Gamma \) in which \( M \) does occur. This region is given by those embeddings \( X \) for which the contravariant defect \( \mu_\Sigma(\ell - r) \) in the formula in Corollary 4.3 is positive; equivalently, the region is given by those objects \( X \) for which there exists a homomorphism from \( P_\ell^{\ell + r - 1} \) to \( X \) which does not factor through \( f_\ell^{\ell + r} \).

**Example.** Consider the category \( S(5) \) of embeddings \((A \subset B)\) of a submodule in a finite length module \( B \) over \( \Lambda = k[T]/(T^5) \). Among the categories \( S(n) \), this is the largest category of finite representation type. We picture in Figure 1 the Auslander–Reiten quiver for \( S(5) \) from [8, (6.5)] which contains the 50 indecomposable objects, up to isomorphy.

For each of the objects in the Auslander–Reiten quiver, we compute the socle tableau; it is pictured at the position of the given object in Figure 2.

Let us focus on the case where \( \ell = 2 \) and \( r = 2 \), so we are interested in those embeddings \( M \) which have a 2 in the second row of their socle tableau. Consider the short exact sequence

\[
0 \rightarrow P_2^3 \rightarrow f P_2^3 \oplus P_1^2 \rightarrow P_1^3 \rightarrow 0
\]

from Corollary 4.3. We have labelled the objects \( A = P_2^3 \), \( B = P_2^4 \), \( B' = P_1^2 \) and \( C = P_1^3 \) in the diagram. The region in which \( M \) can possible occur is encircled in Figure 2.

We note that in general (and in this example), the entries in the socle tableau determine different regions in the Auslander–Reiten quiver than the entries in the LR-tableau or in the dual LR-tableau, see [9, Section 6].
FIGURE 1  The Auslander–Reiten quiver for $S(5)$

FIGURE 2  The socle tableaux for the objects in $S(5)$
Among the categories of type $S(n)$, the first category of infinite representation type is $S(6)$ (see [8]). And in $S(6)$, the smallest parametrized family of pairwise non-isomorphic indecomposable embeddings $(A \subset B)$ occurs when $A \cong N_{42}$ and $B \cong N_{642}$; in this case $B/A \cong N_{42}$.

We use the notation from [5, Example 4.5] where we have studied the representation space $V = \mathbb{V}_{42,42}^{642}$, it is the constructible variety of all $f \in \text{Hom}_k(N_{42}, N_{642})$ which give rise to a short exact sequence of $\Lambda$-modules $0 \to N_{42} \xrightarrow{f} N_{642} \to N_{42} \to 0$ where $\Lambda = k[T]/(T^6)$. This variety $V$ has three irreducible components, they are given by the three LR-tableaux of shape $(42, 642, 42)$, but also by the three socle tableaux of the same shape:

Denote by $M_x$ the embedding in Figure 3 with subscript $x$. The objects $M_3$, $M_{23}$ and $M_{123}$ have socle tableau $\Sigma_3$. In the representation space, the embeddings isomorphic to $M_3$ ($M_{23}$, $M_{123}$) form orbits of dimension 4 (3, 2), respectively. Together, the three orbits form a closed subset, hence an irreducible component, of $V$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The representation space $\mathbb{V}_{42,42}^{642}$}
\end{figure}

\section*{5.2 Positioning objects in the representation space}



The objects $M_{12}$ and $M_{2\lambda}$ have socle tableau $\Sigma_2$; each forms an orbit of dimension 3, although there is a one-parameter family of such orbits. The closure of the union of those orbits also contains $M_{23}$ and $M_{123}$. The remaining embedding $M_1$ has socle tableau $\Sigma_1$, its orbit has dimension 4 and contains in its closure $M_{12}$ and $M_{123}$.

This partition of the representation space $V$ into regions given by socle tableaux is dual to the partition given by LR-tableaux (where an object $M_x$ with subscript $x = i \ast$ belongs to the region given by the LR-tableau $\Gamma_i$). This duality is expected as socle tableaux are determined by the LR-tableaux of the dual embeddings (Theorem 4.1).

### 5.3 Socle tableaux and tableau switching

We have seen in Section 4 that there is a one-to-one correspondence between the sets of socle tableaux of shape $(\alpha, \beta, \gamma)$ and the set of LR-tableaux of shape $(\gamma, \beta, \alpha)$, given by mapping the socle tableau of an embedding to the LR-tableau of the dual embedding.

Due to an anonymous referee we observed that on the combinatorial level, this correspondence in our examples is given by tableau switching, see [1] (or [10] for an application to Schubert calculus). We illustrate the tableau switching algorithm in the example from Section 1.1.

Let $S$ be the tableau of shape $\gamma = (31)$ where each box in row $i$ has entry $i$. To be consistent with the notation in [1], let $T$ be the tableau of shape $\beta \setminus \gamma$ where the entry of a box is $5 - i$ if $i$ is in the corresponding box in the socle tableau $\Sigma_2$ (from Section 1.1).

\[
\Sigma_2 : \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad S : \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad T : \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\]

In $S \cup T$, repeat switching neighboring entries from $S$ (in bold) and $T$ such that the entries from $T$ move up or to the left, and such that after each step, the entries in $T$ and the entries in $S$ are weakly increasing in each row and strictly increasing in each column.

\[
S \cup T : \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\]

Thus, we do indeed obtain $\Gamma_2^{\ast}$ after eight steps as the union of the empty tableau of shape $\alpha = (42)$ with $S_T$.

\[
S_T : \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \Gamma_2^{\ast} : \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\]
CONJECTURE

The tableau switching algorithm always converts the socle tableau of an embedding into the LR-tableau of the dual embedding.

DEDICATION

The authors wish to dedicate this paper to the memory of Professor Andrzej Skowroński. As leader of the representation theory group in Toruń, Professor Skowroński has consistently supported both authors, in particular through conference invitations and travel support. They are grateful for Professor Skowroński’s interest in their work and his helpful feedback. Both are saddened by the untimely loss of a treasured colleague and a good friend.

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