The Complexity of Propositional Implication

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Abstract
The question whether a set of formulae $\Gamma$ implies a formula $\varphi$ is fundamental. The present paper studies the complexity of the above implication problem for propositional formulae that are built from a systematically restricted set of Boolean connectives. We give a complete complexity-theoretic classification for all sets of Boolean functions in the meaning of Post’s lattice and show that the implication problem is efficiently solvable only if the connectives are definable using the constants $\{0, 1\}$ and only one of $\{\land, \lor, \oplus\}$. The problem remains coNP-complete in all other cases. We also consider the restriction of $\Gamma$ to singletons which makes the problem strictly easier in some cases.

Key words: Computational complexity, propositional implication, Post’s lattice

1. Introduction
SAT, the satisfiability problem for propositional formulae, is the most fundamental and historically the first NP-complete problem (proven by S. Cook and L. Levin [6, 7]). A natural question, posed by H. Lewis in 1979, is what the sources of hardness in the Cook-Lewis Theorem are. More precisely, Lewis systematically restricted the language of propositional formulae and determined the computational complexity of the satisfiability problem depending on the set of allowed connectives. E.g., if only logical “and” ($\land$) and “or” ($\lor$) are allowed, we deal with monotone formulae for which the satisfiability problem obviously is easy to solve (in polynomial time). Lewis proved that SAT is NP-complete iff the negation of implication, $x \land \neg y$, is among the allowed connectives or can be simulated by the allowed connectives [8]. To simulate a logical connective $f$ by a set of logical connectives (or, in other words, a set of Boolean functions) $B$ formally means that $f$ can be obtained from functions from $B$ by superposition, i.e., general composition of functions. Equivalently, we can express this fact by saying that $f$ is a member of the clone generated by $B$, in symbols $f \in [B]$.

This brings us into the realm of Post’s lattice, the lattice of all Boolean clones [11]. In this framework, Lewis’ result can be restated as follows. Let $\text{SAT}(B)$ denote the satisfiability problem for propositional formulae with connectives restricted to the set $B$ of Boolean functions. Then $\text{SAT}(B)$ is NP-complete iff $S_1 \subseteq [B]$; otherwise the problem is polynomial-time solvable. Note that the 2-ary Boolean function $x \land \neg y$ forms a basis for $S_1$.

Since then, many problems related to propositional formulae and Boolean circuits have been studied for restricted sets of connectives or gates, and their computational complexity has been classified, depending on a parameter $B$, as just explained for SAT. These include, e.g., the equivalence problem [12], the circuit value problem [13], the quantified Boolean formulae problem QBF [13], but also more recent questions related to non-classical logics like LTL [1], CTL [9], or default logic [?]. An important part of the proof of the classification of different reasoning tasks for default logic in the latter paper [?] was the identification of the coNP-complete and polynomial-time solvable fragments of the propositional implication problem. Though implication is without doubt a very fundamental and natural problem, its computational complexity has not yet been fully classified. This is the purpose of the present note.

We study the problem, given a set $\Gamma$ of propositional formulae and a formula $\varphi$, to decide whether $\varphi$ is implied by $\Gamma$. Depending on the set of allowed connectives in the occurring formulae, we determine the computational complexity of this problem as coNP-complete, $\oplus L$-complete, in $AC^0[2]$, or in $AC^0$.

The type of reduction we use are constant-depth reductions [5] and the weaker $AC^0$ many-one reductions. For both reductions, $AC^0$ forms the $\emptyset$-degree. We also consider the case of the problem restricted to singleton sets $\Gamma$, the singleton-premise implication problem. Interestingly, the complexity of the previously $\oplus L$-complete cases now drops down to the class $AC^0[2]$; in all other cases the complexity remains the same as for the unrestricted problem. Finally, as an easy consequence our results give a refinement of Reith’s previous classification of the equivalence problem for propositional formulae [12]. While Reith only considered the dichotomy between the coNP-complete and logspace-solvable cases, we show that under constant-depth reductions, three complexity degrees occur: coNP-complete, membership in $AC^0[2]$, and membership in $AC^0$. 

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2. Preliminaries

In this paper we make use of standard notions of complexity theory. The arising complexity degrees encompass the classes \( AC^0, AC^0[2], \oplus L, \Pi, \) and \( \text{coNP} \) (cf. [10, 15] for background information).

\( AC^0 \) forms the class of languages recognizable by logtime-uniform Boolean circuits of constant depth and polynomial size over \( \{\land, \lor, \neg\} \), where the fan-in of gates of the first two types is not bounded. The class \( AC^0[2] \) is defined similarly as \( AC^0 \), but in addition to \( \{\land, \lor, \neg\} \) we also allow \( \oplus \)-gates of unbounded fan-in. The class \( \oplus L \) is defined as the class of languages \( L \) for which there exists a nondeterministic logspace Turing machine \( M \) such that for all \( x, x \in L \) iff \( M(x) \) has an odd number of accepting paths.

For the hardness results we use constant-depth and \( AC^0 \) many-one reductions, defined as follows: A language \( A \) is constant-depth reducible to a language \( B = (A \leq_{cd} B) \) if there exists a logtime-uniform \( AC^0 \)-circuit family \( \{C_n\}_{n \geq 0} \) with \( \{\land, \lor, \neg\} \)-gates and oracle gates for \( B \) such that for all \( x, C_n(x) = 1 \) iff \( x \in A \). A language \( A \) is \( AC^0 \) many-one reducible to a language \( B = (A \leq_{m1} B) \) if there exists a function \( f \) computable by a logtime-uniform \( AC^0 \)-circuit family such that \( x \in A \) iff \( f(x) \in B \).

For both reductions, the class \( AC^0 \) forms the \( 0 \)-degree. Furthermore, it is easy to see that

\[
Mod_2 := \{w \in \{0, 1\}^* \mid |w|_1 \equiv 1 \pmod{2}\},
\]

where \( |w|_1 = |i \mid 1 \leq i \leq n, w_i = 1| \), is complete for \( AC^0[2] \) under \( \leq_{cd} \)-reductions, for \( AC^0[2] \) merely extends \( AC^0 \) with oracle gates for \( \text{MOD}_2 \).

We assume familiarity with propositional logic. The set of all propositional formulae is denoted by \( \mathcal{L} \). For \( \Gamma \subseteq \mathcal{L} \) and \( \varphi \in \mathcal{L} \), we write \( \Gamma \vdash \varphi \) iff all assignments satisfying all formulae in \( \Gamma \) also satisfy \( \varphi \).

3. Boolean Clones

In order to completely classify the complexity of the implication problem for all possible sets \( B \) of Boolean functions, one has to consider an infinite number of parameterized problems. We introduce the notion of a clone to reduce the number of problems to be considered to a finite set.

A propositional formula using only connectives from a finite set \( B \) of Boolean functions is called a \( B \)-formula. The set of all \( B \)-formulae is denoted by \( \mathcal{L}(B) \). A clone is a set \( B \) of Boolean functions that is closed under superposition, i.e., \( B \) contains all projections and is closed under arbitrary composition. We denote by \( [B] \) the smallest clone containing \( B \) and call \( B \) a base for \( [B] \). In [11] Post classified the lattice of all clones and found a finite base for each clone, see Fig. 1. In order to introduce the clones relevant to this paper, we define the following notions for \( n \)-ary Boolean functions \( f \):

- \( f \) is \( c \)-reproducing if \( f(\cdot, \ldots, c) = c \), \( c \in \{0, 1\} \).
- \( f \) is monotone if \( a_1 \leq b_1, \ldots, a_n \leq b_n \) implies \( f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n) \).
- \( f \) is \( c \)-separating if there exists an index \( i \in \{1, \ldots, n\} \) such that \( f(a_1, \ldots, a_n) = c \) implies \( a_i = c, c \in \{0, 1\} \).
- \( f \) is self-dual if \( f \equiv \text{dual}(f) \), where \( \text{dual}(f)(x_1, \ldots, x_n) := \neg f(\neg x_1, \ldots, \neg x_n) \).
- \( f \) is linear if \( f \equiv c_0 + c_1 x_1 + \cdots + c_n x_n + c \) for constants \( c_i \in \{0, 1\} \), \( 0 \leq i \leq n \), and variables \( x_1, \ldots, x_n \).

The clones relevant to this paper are listed in Table 1. The definition of all Boolean clones can be found, e.g., in [3].

4. The Complexity of the Implication Problem

Let \( B \) be a finite set of Boolean functions. The implication problem for \( B \)-formulae is defined as

**Problem:** \( \text{IMP}(B) \)

**Instance:** A finite set \( \Gamma \) of \( B \)-formulae and a \( B \)-formula \( \varphi \).

**Question:** Does \( \Gamma \vdash \varphi \) hold?

In the general case \( [B] = \text{BF} \), verifying an instance \( (\Gamma, \varphi) \in \text{IMP}(B) \) amounts to verifying that the formula \( \Gamma \vdash \varphi \) is tautological. We hence obtain a \( \text{coNP} \) upper bound. The following theorem classifies the complexity of the implication problem for all possible sets \( B \).

**Theorem 4.1.** Let \( B \) be a finite set of Boolean functions. Then the implication problem for propositional \( B \)-formulae, \( \text{IMP}(B) \), is

1. \( \text{coNP-complete under } \leq_{AC^0} \)-reductions if \( S_{00} \subseteq [B] \) or \( S_{10} \subseteq [B] \) or \( D_2 \subseteq [B] \).
2. \( \oplus L \)-complete under \( \leq_{AC^0} \)-reductions if \( L_2 \subseteq [B] \subseteq L \).
3. In \( AC^0[2] \) and \( \text{MOD}_2 \leq_{m1} \text{IMP}(B) \) if \( N_2 \subseteq [B] \subseteq N \), and
4. in \( AC^0 \) for all other cases.

In contrast to the first two cases, we do not state a completeness result for the third case, where \( N_2 \subseteq [B] \subseteq N \). Under \( \leq_{AC^0} \)-reductions, however, \( \text{IMP}(B) \) is \( AC^0[2] \)-complete in this case. For \( \leq_{m1} \)-reductions, the existence of a complete problem \( A \) would state that any \( AC^0[2] \)-circuit is equivalent to an \( AC^0 \)-computation followed by a single oracle call to \( A \). To date, there is no such problem known.

We split the proof of Theorem 4.1 into several lemmas.

**Lemma 4.2.** Let \( B \) be a finite set of Boolean functions such that \( S_{00} \subseteq [B] \) or \( S_{10} \subseteq [B] \). Then \( \text{IMP}(B) \) is \( \text{coNP-complete under } \leq_{m1} \)-reductions.

**Proof.** Membership in \( \text{coNP} \) is apparent, because given \( \Gamma \) and \( \varphi \), we just have to check that for all assignments \( \sigma \) to the variables of \( \Gamma \) and \( \varphi \), either \( \varphi \not\vdash \Gamma \) or \( \sigma \not\vdash \varphi \).

The hardness proof is inspired by [12]. Observe that \( \text{IMP}(B) \equiv_{cd} \text{IMP}(B \cup \{1\}) \) if \( \land \in [B] \), and that \( \text{IMP}(B) \equiv_{cd} \text{IMP}(B \cup \{0\}) \) if \( \lor \in [B] \) (because \( \varphi \equiv \psi \iff \varphi_{t/1} \land t \equiv \psi_{t/1} \) and \( \varphi \equiv \psi \iff \varphi_{t/0} \cup \psi_{t/0} \cup f \) where \( t, f \) are new variables). It hence suffices to show that \( \text{IMP}(B) \) is \( \text{coNP} \)-hard for \( M_0 = [S_{00} \cup \{0\}] = \{\land, \lor, 0\} \) and \( M_1 = [S_{10} \cup \{1\}] = \{\land, \lor, 1\} \).
| Name | Definition | Base |
|------|------------|------|
| BF   | All Boolean functions | $\{\land, \neg\}$ |
| $M_2$ | $(f : f$ is monotone and 0- and 1-reproducing$)$ | $\{\lor, \land\}$ |
| $S_{00}$ | $(f : f$ is 0-separating$) \cap M_2$ | $\{x \lor (y \land z)\}$ |
| $S_{10}$ | $(f : f$ is 1-separating$) \cap M_2$ | $\{x \land (y \lor z)\}$ |
| $D_2$ | $(f : f$ is monotone and self-dual$)$ | $(\{x \land y\} \lor (y \land z) \lor (x \land z))$ |
| $L$ | $(f : f$ is linear$)$ | $\{\oplus, 1\}$ |
| $L_2$ | $(f : f$ is linear and 0- and 1-reproducing$)$ | $\{x \oplus y \oplus z\}$ |
| $V$ | $(f : f \equiv c_0 \lor \bigvee_{i=1}^{n} c_i x_i \text{ for } c_i \in \{0, 1\}, 1 \leq i \leq n\}$ | $\{\lor, 0, 1\}$ |
| $E$ | $(f : f \equiv c_0 \land \bigwedge_{i=1}^{n} c_i x_i \text{ for } c_i \in \{0, 1\}, 1 \leq i \leq n\}$ | $\{\land, 0, 1\}$ |
| $I$ | $(f : f$ depends on at most one variable$)$ | $\{\neg, 1\}$ |
| $N_2$ | $(f : f$ is the negation or a projection$)$ | $\{\neg\}$ |

Table 1: A list of Boolean clones with definitions and bases.

Figure 1: Post’s lattice. Colors indicate the complexity of $\text{IMP}(B)$, the implication problem for $B$-formulae.
We will show that $\text{IMP}(B)$ is coNP-hard for each base $B$ with $M_2 = \{(x, \lor, 0) \leq \{B\}$. To prove this claim, we will provide a reduction from $\text{TAUT}_{\text{DNF}}$ to $\text{IMP}(B)$, where $\text{TAUT}_{\text{DNF}}$ is the coNP-complete problem to decide, whether a given propositional formula in disjunctive normal form is a tautology.

Let $\varphi$ be a formula in disjunctive normal form over the propositions $X = \{x_1, \ldots, x_k\}$. Then $\varphi = \bigvee_{i=1}^m \bigwedge_{j=1}^{l_i} \neg x_j$, where $l_i$ are literals over $X$. Take new variables $Y = \{y_1, \ldots, y_k\}$ and replace in $\varphi$ each negative literal $\neg x_j$ by $y_j$. Define the resulting formula as $\varphi_2$ and let $\varphi_1 := \bigwedge_{i=1}^k (x_i \lor y_i)$. We claim that $\varphi \in \text{TAUT}_{\text{DNF}} \implies \varphi_1 \equiv \varphi_2$.

Let us first assume $\varphi \in \text{TAUT}_{\text{DNF}}$ and let $\sigma: X \cup Y \rightarrow \{0, 1\}$ be an assignment such that $\sigma \equiv \varphi$. As $\varphi$ is a tautology, $\sigma \equiv \top$. But also $\sigma \equiv \varphi_2$, as we simply replaced the negated variables in $\varphi$ by positive ones and $\sigma_2$ is monotone. It follows that $\varphi_1 \equiv \varphi_2$, since $\sigma$ was arbitrarily chosen.

For the opposite direction, let $\varphi \notin \text{TAUT}_{\text{DNF}}$. Then there exists an assignment $\sigma: X \rightarrow \{0, 1\}$ such that $\sigma \not\equiv \varphi$. We extend $\sigma$ to an assignment $\sigma': X \cup Y \rightarrow \{0, 1\}$ by setting $\sigma'(y_i) = 1 - x_i$ for $i = 1, \ldots, k$. Then $\sigma'(x_i) = 0$ iff $\sigma'(y_i) = 1$, and consequently $\sigma'$ simulates $\sigma$ on $\varphi'$. As a result, $\sigma' \not\equiv \varphi_2$. Yet, either $\sigma'(x_i) = 1$ or $\sigma'(y_i) = 1$ for $i = 1, \ldots, k$. Thus $\sigma' \equiv \varphi_1$, yielding $\varphi_1 \not\equiv \varphi_2$.

**Lemma 4.3.** Let $B$ be a finite set of Boolean functions such that $D_2 \subseteq \{B\}$. Then $\text{IMP}(B)$ is coNP-complete under $\leq^m_{\text{SN}}$ reductions.

**Proof.** Again we just have to argue for coNP-hardness of $\text{IMP}(B)$. We give a reduction from $\text{TAUT}_{\text{DNF}}$ to $\text{IMP}(B)$ for $D_2 \subseteq \{B\}$ by modifying the reduction given in the proof of Lemma 4.2.

Given a formula $\varphi$ in disjunctive normal form, we define the formulae $\varphi_1$ and $\varphi_2$ as above. As $D_2 \subseteq \{B\}$, we know that $g(x, y, z) := (x \land y) \lor (y \land z) \lor (x \land z) \in \{B\}$. Clearly, $g(x, y, 0) \equiv x \land y$ and $g(x, y, 1) \equiv x \lor y$. Denote by $\psi^B_i(t, f)$, $i \in \{1, 2\}$, the formula $\psi$ with all occurrences of $x \land y$ and $x \lor y$ replaced by a $B$-representation of $g(x, y, f)$ and $g(x, y, t)$, respectively, where $t$ and $f$ are new propositional variables. Then $\psi^B_1(1, 0) \equiv \psi_1$ and $\psi^B_2(0, 1) \equiv \psi_2$. The variables $x$ and $y$ occur several times in $g$, hence $\psi^B_i(t, f)$ and $\psi^B_2(t, f)$ might be exponential in the length of $\varphi$ (recall that $\varphi_2$ is $\varphi$ with all negative literals replaced by new variables). That is not the case follows from the associativity of $\land$ and $\lor$: we insert parentheses in such a way that $\psi^B_i$ can be transformed into a tree of logarithmic depth.

We now map a pair $(\varphi_1, \varphi_2)$ to $(\varphi'_1, \varphi'_2)$ where

$\varphi'_1 := g(\psi^B_1(t, f), t, f)$ and $\varphi'_2 := g(g(\psi^B_2(t, f), \psi^B_2(t, f), t, f), t, f)$.

We claim that $\varphi_1 \equiv \varphi_2 \iff \varphi'_1 \equiv \varphi'_2$. To verify this claim, let $\sigma$ be an arbitrary assignment for the variables in $\varphi$. Then $\sigma$ may be extended to $(t, f)$ in the following ways:

$\sigma(t) = 1$ and $\sigma(f) = 0$: This is the intended interpretation. In this case, $g(\psi^B_1(0, 1), 0, 1) \equiv \psi_1 \lor 1 \equiv \psi_1$ and $g(g(\psi^B_1(0, 1), 0, 1), 0, 1) \equiv (\psi_1 \lor \psi_2) \lor 1 \equiv \psi_1 \lor \psi_2$. Then $\varphi_1 \equiv \varphi_2$ if $\varphi'_1 \equiv \psi_1 \lor \psi_2$.

$\sigma(t) = 0$ and $\sigma(f) = 1$: In this case, we obtain that $g(\psi^B_1(0, 1), 0, 1) \equiv \varphi_1 \lor 0 \equiv \varphi_1$ and $g(g(\psi^B_2(0, 1), 0, 1), 1, 0) \equiv (\varphi_1 \lor \varphi_2) \lor 0 \equiv \varphi_1 \lor \varphi_2$. As $\varphi_1 \equiv \varphi_2$, we conclude that $\varphi'_1 \equiv \varphi'_2$ in this case.

$\sigma(t) = \sigma(f) = c$ with $c \in \{0, 1\}$: Then both $\varphi'_1$ and $\varphi'_2$ are equivalent to $c$. Thus, as in the previous case, $\varphi'_1 \equiv \varphi'_2$.

From this analysis, it follows that $\varphi_1 \equiv \varphi_2$ if $\varphi'_1 \equiv \varphi'_2$. Hence, $\text{TAUT}_{\text{DNF}} \leq^m_{\text{SN}} \text{IMP}(B)$ via the reduction $\varphi \mapsto (\varphi'_1, \varphi'_2)$.

**Lemma 4.4.** Let $B$ be a finite set of Boolean functions such that $L_2 \subseteq \{B\} \subseteq L$. Then $\text{IMP}(B)$ is $\oplus L$-complete under $\leq^m_{\text{AC}}$ reductions.

**Proof.** Observe that $L \equiv \varphi$ iff $\Gamma \cup \{\varphi \circ r, t\}$ is inconsistent, where $r$ is a new variable. Let $L'$ denote $\Gamma \cup \{\varphi \circ r, t\}$ rewritten such that for all $\varphi \in L'$, $\psi = c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$, where $c_0, \ldots, c_n \in \{0, 1\}$. $L'$ is logspace constructible, since $c_0 = 1$ if $\psi(0, \ldots, 0) = 1$, and for $1 \leq i \leq n, c_i = 1$ if $\psi(0, \ldots, 0) \neq 0, \ldots, 0, 1, 0, \ldots, 0$.

$\Gamma'$ can now be transformed into a system of linear equations $S$ via

$c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n \mapsto c_0 + c_1 x_1 + \cdots + c_n x_n = 1$ (mod 2).

Clearly, the resulting system of linear equations has a solution iff $\Gamma'$ is consistent. The equations are furthermore defined over the field $\mathbb{Z}_2$, hence existence of a solution can be decided in $\oplus L$ [4].

For the $\oplus L$-hardness, note that solving a system of linear equations over $\mathbb{Z}_2$ is indeed $\oplus L$-complete, under $\leq^m_{\text{AC}}$ reductions: let $\text{MOD-GAP}_2$ denote the $\oplus L$-complete problem to decide whether a given directed acyclic graph $G$ with nodes $s$ and $t$ has an odd number of distinct paths leading from $s$ to $t$. Buntrock et al. [4] give an $\text{NC}^1$-reduction from $\text{MOD-GAP}_2$ to the problem whether a given matrix over $\mathbb{Z}_2$ is non-singular. The given reduction is actually an $\text{AC}^0$ many-one reduction. We reduce the latter problem to the complement of $\text{IMP}(x \oplus y \oplus z)$ and then generalize the result to arbitrary finite sets $B$ such that $\{B\} = L_2$. The lower bound then follows from $\oplus L$ being closed under complement.

First map the system $S$ of linear equations into a set of linear formulae $\Gamma$ via

$c_1 x_1 + \cdots + c_n x_n = c \mod 2 \mapsto c' \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$, where $c' = 1$ if $c = 0$, and $c' = 0$ otherwise. Next replace the constant 1 with a fresh variable $t$, pad all formulae having an even number of non-fictive variables with another fresh variable $f$, and let $\Gamma \equiv \Gamma \cup \{t\}$. We claim that $S$ has a solution iff $\Gamma \not\equiv f$.

Suppose that $S$ has no solutions. If $\Gamma$ is inconsistent, then $\Gamma' \not\equiv f$. Otherwise, $\Gamma'$ has a satisfying assignment $\sigma$. Clearly, $\sigma(t) = 1$. If $\sigma(f) = 0$, then $\Gamma[\{t/1, f/0\}]$ is equivalent to $\Gamma'$; hence the transformation of $\Gamma[\{t/1, f/0\}]$ yields a system of linear equations $S'$ that is equivalent to $S$ and that has a solution.
corresponding to $\sigma$ — a contradiction to our assumption. Thus $\sigma(f) = 1$ and, consequently, $\Gamma \models f$.

On the other hand, if $S$ has a solution, then $\Gamma$ possesses a satisfying assignment $\sigma$ with $\sigma(t) = 1$ and $\sigma(f) = 0$. Again $\sigma \models \Gamma$ iff $\sigma \models \Gamma$. Hence, $\Gamma' \not\models f$.

It remains to show that $x \oplus y \oplus z$ can be efficiently expressed in any set $B$ such that $[B] = L_2$, that is, there exists a function $f_\emptyset \in [B]$ such that $f_\emptyset$ is equivalent to $x \oplus y \oplus z$ and each variable occurs only once in the body of $f_\emptyset$. Let $B$ be such that $[B] = L_2$ and let $g(x,y,z)$ be a function from $[B]$ depending on three variables. Such a function $g$ exists because $x \oplus y \oplus z \in [B] = L_2$. As $g$ is a linear function, replacing two occurrences of any variable with a fresh variable does not change $g$ modulo logical equivalence. Let $n$ denote the number of occurrences of $x$ in $g$ and assume that $n$ is even. Replacing all occurrences of $x$ with an arbitrary symbol yields a formula $g'(y,z) \equiv y \oplus z \notin L_2$ which gives a contradiction. Analogous arguments hold for the number of occurrences of $y$ and $z$. Hence, each of the variables $x, y,$ and $z$ occurs an odd number of times, and replacing all but one occurrence of each $x, y,$ and $z$ with a fresh variable $t$ yields a function $g''(x,y,z,t) \equiv x \oplus y \oplus z$ in which each variable occurs exactly once.

**Lemma 4.5.** Let $B$ be a finite set of Boolean functions such that $N_2 \subseteq [B] \subseteq N$. Then $IMP(B)$ is contained in $AC^0[2]$ and $MOD_2 \leq^m_{AC^0} IMP(B)$.

**Proof.** Let $B$ be a finite set of Boolean functions such that $N_2 \subseteq [B] \subseteq N$. Let $\varphi$ be a $B$-formula and $\Gamma$ be a set of $B$-formulae, both over the set of propositions $\{x_1, \ldots, x_n\}$.

We will argue on membership in $AC^0[2]$ first. For all $f \in [B]$, $f$ is equivalent to some literal or a constant. Let $L := \{l_i |$ there exists $\psi \in \Gamma: l_i \equiv \psi\}$, where $l_i = x_i$ or $l_i = \neg x_i$ for $1 \leq i \leq n$. $L$ is computable from $\Gamma$ using an $AC^0$-circuit with oracle gates for $MOD_2$: for each formula in $\Gamma$, we determine the atom and count the number of preceding negations modulo 2. In the case that $\Gamma$ is unsatisfiable, either $L = \emptyset$ or there exist $l_i, l_j \in \Gamma$ with $l_i = \neg l_j$. Both conditions can be checked in $AC^0$, hence we may w.l.o.g. assume that $\Gamma$ is satisfiable. It now holds that

$$\Gamma \models \varphi \iff \bigwedge_{l_i \in L} l_i \models \varphi \iff \text{ for some } L' \subseteq L: \varphi \equiv \bigwedge_{l_i \in L'} l_i.$$  

It remains to compute an equivalent formula of the form $\bigwedge_{l_i \in L} l_i$ from $\varphi$ and test whether $L' \subseteq L$. It is easy to see that the former task can again be performed in $AC^0[2]$, and the latter merely requires $AC^0$. Thus we conclude $IMP(B) \in AC^0[2]$.

For $MOD_2 \leq^m_{AC^0} IMP(B)$, we claim that, for $w = w_1 \cdots w_n \in \{0, 1\}^n$, $w \in MOD_2$ iff $t \models w_1w_2 \cdots w_n(\neg t)$, where $\neg t := \neg t \oplus 0 := id$ and $t$ is a variable.

First observe that $t \models w_1w_2 \cdots w_n(\neg t)$ iff all assignments $\sigma$ of $t$ to $\{0, 1\}$, $\sigma \models t$ implies $\sigma \models w_1w_2 \cdots w_n(\neg t)$. Now, if $\sigma(t) := 0$, then $t \models w_1w_2 \cdots w_n(\neg t)$ is always true, whereas, if $\sigma(t) := 1$, then $t \models w_1w_2 \cdots w_n(\neg t)$ iff $1 \models w_1w_2 \cdots w_n0$. Hence, the claim applies and $MOD_2 \leq^m_{AC^0} IMP(B)$ follows.

As an immediate consequence of the above lemma, we obtain the following corollary.

**Corollary 4.6.** Let $B$ be a finite set of Boolean functions such that $N_2 \subseteq [B] \subseteq N$. Then $IMP(B)$ is $AC^0[2]$-complete under $\leq_{AC^0}$-reductions.

**Lemma 4.7.** Let $B$ be a finite set of Boolean functions such that $[B] \subseteq V$ or $[B] \subseteq E$. Then $IMP(B)$ is in $AC^0$.

**Proof.** We prove the claim for $[B] \subseteq V$ only. The case $[B] \subseteq E$ follows analogously.

Let $B$ be a finite set of Boolean functions such that $[B] \subseteq V$. Let further $\Gamma$ be a finite set of $B$-formulae and let $\varphi$ be a $B$-formula such that $\Gamma$ and $\varphi$ only use the variables $x_1, \ldots, x_n$. Let $\varphi \equiv \psi_1 \lor \psi_2 \lor \cdots \lor \psi_m$ with constants $c_i \in \{0, 1\}$ for $0 \leq i \leq n$. Equally, every formula from $\Gamma$ is equivalent to an expression of the form $c'_0 \lor c'_1 \lor \cdots \lor c'_{mn}$ with $c'_i \in \{0, 1\}$. Then, $\Gamma \models \varphi$ iff either $c_0 = 1$ or there exists a formula $\psi \equiv \psi'_0 \lor \psi'_1 \lor \cdots \lor \psi'_{mn}$ from $\Gamma$ such that $c''_i \leq c_i$ for all $0 \leq i \leq n$ and $c''_0 \in \{0, 1\}$.

The value of $c_0$ can be determined by evaluating $\varphi(0, \ldots, 0)$. Furthermore, for $1 \leq i \leq n, c_i = 0$ if $c_0 = 0$ and $\varphi(0, \ldots, 0, 1, 0, \ldots, 0) = 0$.

The values of the coefficients of formulae in $\Gamma$ can be computed analogously. Thus $IMP(B)$ can be computed in constant depth using oracle gates for $B$-formula evaluation. As $B$-formula evaluation is in $NLOGTIME$ [14] and $NLOGTIME \subseteq AC^0$, the claim follows.

**5. The Complexity of the Singleton-Premise Implication Problem**

For a finite set $B$ of Boolean functions, we define the **singleton-premise implication problem** for $B$-formulae as

**Problem:** IMP’$(B)$

**Instance:** Two $B$-formulae $\varphi$ and $\psi$.

**Question:** Does $\varphi \models \psi$ hold?

We classify the complexity of this problem as follows:

**Theorem 5.1.** Let $B$ be a finite set of Boolean functions. Then $IMP’(B) \in AC^0[2]$ and $MOD_2 \leq^m_{AC^0} IMP’(B)$ if $L_2 \subseteq [B] \subseteq L$. For all other sets $B$, the problems IMP$(B)$ and IMP’$(B)$ are equivalent.

Before we prove Theorem 5.1, let us try to give an intuitive explanation for the difference in the complexity of IMP’$(B)$ for $L_2 \subseteq [B] \subseteq L$ stems from. Deciding IMP$(B)$ is equivalent to solving a set of linear equations corresponding to the set of premises. For IMP’$(B)$, the premise is a single formula. It hence suffices to determine whether there exists an assignment satisfying the premise and setting to true an even (resp. odd) number of variables from the conclusion.

**Proof.** For $S_{90} \subseteq [B], S_{10} \subseteq [B],$ and $D_2 \subseteq [B]$, observe that the proofs of Lemma 4.4 and Lemma 4.5 establish $coNP$-hardness of IMP’$(B)$. Analogously, for $N_2 \subseteq [B]$, MOD_2 \leq_{AC^0} IMP’$(B)$ follows by the same reduction given in the proof of
Lemma 4.5. Let $[B] \subseteq V$ and $[B] \subseteq E$, we have IMP$^m(B) \leq_{AC^0}^{m}$ IMP$(B) \in AC^0$. It thus remains to show that IMP$^m(B) \in AC^0[2]$ for $[B] \subseteq L$, and that MOD$_2 \leq_{AC^0}^{m}$ IMP$(B)$ for $L_2 \subseteq [B]$. Let $(\phi, \psi)$ be a pair of $B$-formulae over the variables $\{x_1, \ldots, x_n\}$. As $[B] \subseteq L$, $\phi$ and $\psi$ are equivalent to expressions of the form $\phi \equiv c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$ and $\psi \equiv c'_0 \oplus c'_1 x_1 \oplus \cdots \oplus c'_n x_n$, where $c_i, c'_i \in \{0, 1\}$ for $1 \leq i \leq n$. If $c_0 = \ldots = c_n = 0$, then $\phi \equiv \psi$ and the equivalence holds. Therefore, let us assume that not all coefficients $c_i$ are 0. In this situation, we claim that $\phi \equiv \psi$ is in fact equivalent to $\forall \equiv \exists$. To prove this claim observe that $\phi \equiv \psi$ iff
\[ \chi := (c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n) \land (1 \oplus c'_0 \oplus c'_1 x_1 \oplus \cdots \oplus c'_n x_n) \]
is unsatisfiable. Let us assume now $\phi \not\equiv \psi$. We will construct a satisfying assignment $\sigma$ for $\chi$. Let $I := \{i \in \{1, \ldots, n\} | c_i = c'_i\}$ and define $\sigma(x_i) := 0$ for $i \in I$. As $\phi \equiv \psi$, the set $\overline{I} := \{1, \ldots, n\} \setminus I$ is nonempty and for all $i \in \overline{I}$, $c_i = 1 \iff c'_i = 0$. Hence, there is a partition $P_1 \lor P_2 = \overline{I}$ such that
\[ \sigma \models \chi \iff \sigma \models (c_0 \oplus \bigoplus_{i \in P_1} c_i x_i) \land (1 \oplus c'_0 \oplus \bigoplus_{i \in P_2} c'_i x_i). \]
Here the subformulae $c_0 \oplus \bigoplus_{i \in P_1} c_i x_i$ and $1 \oplus c'_0 \oplus \bigoplus_{i \in P_2} c'_i x_i$ are over disjoint sets of variables. But still, both subformulae are satisfiable using an appropriate completion of $\sigma$. Consequently, $\sigma$ will also satisfy $\chi$ and hence the claim holds.

Thus $\phi \equiv \psi$ if either $c_0 = \ldots = c_n = 0$ or $\phi \equiv \psi$. Similarly to the proof of Lemma 4.2 it follows that the latter alternative holds iff $c_i = c'_i$ for all $0 \leq i \leq n$. The coefficients $c_i$ can be determined from $c_0 = \varphi(0, \ldots, 0)$ and
\[ c_i = \varphi(0, \ldots, 0, 1, 0, \ldots, 0) \oplus c_0 \]
for $1 \leq i \leq n$. The values of the $c'_i$’s can be computed analogously. As $B$-formula evaluation is equivalent to MOD$_2$ in this case, IMP$^m(B) \in AC^0[2]$.

It remains to show $MOD_2 \leq_{AC^0}^{m}$ IMP$(B)$ for $L_2 \subseteq [B]$. Consider the mapping $h : \{0, 1\}^* \rightarrow L(B)$, recursively defined by
\[ h(x) = \begin{cases} f & x = \varepsilon \\ h(y) & x = 0y \\ t \lor f \lor h(y) & x = 1y \end{cases} \]
where $\varepsilon$ denotes the empty word and $t, f$ are propositional variables. We claim that $x \mapsto (t, h(x))$ computes an $\leq_{AC^0}^{m}$-reduction from MOD$_2$ to IMP$(B)$. To verify this claim, let $x \in \{0, 1\}^*$ be an instance of MOD$_2$. Then
\[ x \in MOD_2 \Rightarrow h(x) \equiv t \Rightarrow (t, h(x)) \in IMP(B), \]
\[ x \notin MOD_2 \Rightarrow h(x) \equiv f \Rightarrow (t, h(x)) \notin IMP(B). \]
Whence, MOD$_2 \leq_{AC^0}^{m}$ IMP$(B)$ for $L_2 \subseteq [B]$.

Let EQ$(B)$ denote the equivalence problem for $B$-formulae. Obviously, $(\phi, \psi) \in EQ(B)$ iff $(\phi, \psi) \in IMP(B)$ and $(\psi, \phi) \in IMP(B)$. As AC$^0$, AC$^0[2]$, and coNP are all closed under intersection, we obtain as an immediate corollary a finer classification of the complexity of EQ than the one given by Reith.

6. Conclusion

In this paper we provided a complete classification of the complexity of the implication problem, IMP$(B)$, and the singleton-premise implication problem, IMP$^m(B)$—fundamental problems in the area of propositional logic. Though IMP$(B)$ is a restricted version of IMP$(B)$, the simplification amounts to a difference for $L_2 \subseteq [B] \subseteq L$ only: IMP$(B)$ is AC$^0[2]$-complete under constant-depth reductions, whereas IMP$(B)$ is $\oplus L$-complete under AC$^0$ many-one reductions and thus strictly harder. For all other clones, both problems have the same complexity.

Due to the close relationship between the implication and the equivalence problem, we were also able to slightly refine the classification of the complexity of the equivalence problem given in [12].

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