Rotating Fluids in General Relativity

T. Papakostas
Department of Natural Resources and Environment, Technological Educational Institute of Crete, 3 Romanou Str, Chania 731-33 GREECE

E-mail: taxiar@chania.teicrete.gr

Abstract. We define the notion of a surface of revolution in the curved spaces of General Relativity and we present the corresponding Einstein’s equations in the case of an anisotropic fluid with bulk and shear viscosity and heat conduction. We indicate a method of integration and some particular solutions.

1. Introduction

The study of equilibrium structures of rotating fluid masses is a central problem in Newtonian Gravity and mostly in General Relativity where the solution of this problem constitutes an alternative to the existence of black holes. A complete review for the results in Newtonian Gravity can be found in [1].

In General Relativity the problem of finding an equilibrium structure for a fluid, bounded by a surface of zero pressure and matched across this surface to an asymptotically flat vacuum solution is very important: it represents an alternative to the implicitly admitted conjecture that the Kerr metric might have no other source but a black hole.

In the literature the results concerning solutions of Einstein’s equations are focusing on the following issues:

(I) Approximate solutions of Einstein’s equations for the interior configuration, matched to the Kerr metric up to a certain order in the perturbative expansion.

(II) Exact solutions of Einstein’s equations for the interior configuration that might be matched to the Kerr metric across the zero pressure surface.

In group (I), Florides [2] pursuing further the method developed with Synge [3], matched an interior solution for a rotating fluid sphere, with anisotropic pressures to the Kerr metric up to fifth order of a small parameter. These results suggest that the interior configuration consists of an anisotropic fluid rather than a perfect one.

In group (II), the only rigidly rotating perfect fluid solution, containing Kerr as limiting vacuum case is the Wahlquist metric, which has an unphysical equation of state \( e + 3p = const \) and it exhibits a prolate ellipsoid rather than oblate as surface of zero pressure, in the slow rotation approximation [4], [5], therefore this solution can not be matched to Kerr solution. The other solutions of group (II) represent anisotropic fluids [6], [7], [8] but either they fail to satisfy the energy conditions [6], [7] or either the hydrostatic pressure can not be isolated from the other stresses [8], in order to fix the bounding surface of zero pressure.

Finally Krasinski [9] defines the notion of an ellipsoid in curved spaces following the principles of General Relativity. This approach enables him to investigate the analogue to Newtonian Gravity, where the solutions are obtained by imposing the ellipsoid as bounding surface of the fluid configuration. Curiously this approach was not pursued farther, perhaps due to the fact that no new solution was obtained.

We consider that the choice of an ellipsoidal bounding surface is restrictive and quite unsatisfactory for the description of Physical Reality. It would be more reasonable to deduce the shape of the surface of revolution from the equations governing the motion of the fluid; however this was never achieved until now.

Following Krasinski, we propose a generalization or the bounding surface of the fluid; we look for the most general surface compatible with the symmetries of our problem: stationarity and axial
symmetry. Obviously these surfaces are the surfaces of revolution, which are obtained by revolving a
plane curve C about a line L in its plane. The line L will be identified with the axis of rotation of the
fluid configuration. Then the imposition of Einstein’s equations will permit to characterize the curve C
and study it using well known techniques of Differential Geometry.

The most general problem in this context is the search of stationary axisymmetric spaces of General
Relativity, possessing a surface of revolution and satisfying Einstein’s equations. The energy –
momentum tensor will be that of an anisotropic fluid, with bulk and shear viscosity and heat
conduction.

However this general problem leads to complicated differential equations, so we are forced to
consider a special family of stationary, axisymmetric spaces: the Carter’s family [A] of solutions.

In 2 we present the Carter’s family [A] of solutions and in 3 we introduce the notion of a space-time
possessing a surface of revolution, following the approach of Krasinski, then in 4 we present the
energy-momentum tensor of an anisotropic fluid, imposing that the velocity of the comoving with the
fluid observer defines a quotient space representing a surface of revolution. Then we deduce the
necessary and sufficient conditions of such a tensor in the case of Carter’s family [A] of solutions.

2. Stationary, axisymmetric spaces and Carter’s family [A] of solutions

An important advance in the study of stationary axially symmetric systems in General Relativity was
made by Papapetrou [10] when he showed, in the vacuum case, that any connected region containing
the axis of symmetry, the orbits of the two –parameter Abelian group admit orthogonal two-surfaces.
The work of Papapetrou has been extended by Carter, who introduced the concepts of orthogonal
transitivity and invertibility of its isometry group. We recall that an isometry group in an n-dimensional
pseudo-Riemannian space is said to be orthogonal transitive, if the p-dimensional orbits of the group are
orthogonal to a family of (n-p)-dimensional surfaces. The group is said to be invertible if the isotropy
subgroup contains an element of order two (an involution) which inverts the sense of p independent
directions in the surface of transitivity at a point P but leaves unaltered the sense of directions
orthogonal to the surface of transitivity at P. It was noticed by Kundt and Trumper [11] and by Carter
[12], that Papapetrou theorem could be extended to the case when matter is present provided that the
Ricci tensor is invertible in the group.

Hypothesis 2.1

We suppose that the space admits an isometry Abelian group invertible with non null surface of
transitivity [13]. This implies that there is a local coordinate chart in which the metric can be written as
follows:

\[ ds^2 = g_{tt} dt^2 + 2 g_{tz} dt dz + g_{zz} dz^2 - g_{xx} dx^2 - g_{yy} dy^2 \]  

(1)

The group \( G_2 \) is generated by \( \frac{\partial}{\partial t} \) (time-like Killing vector) and \( \frac{\partial}{\partial z} \) (space –like Killing vector).

The invertibility of the group is obvious for the fact that the transformation:

\[ (t, z) = (-t, -z) \]  

(2)

Is an isometry. The form (1) is identical with that used by Krasinski, but we present an alternative form
which has been very useful for the calculations, the symmetric tetrad in the Newman-Penrose (NP)
coefficients are equal by pairs:

\[ ds^2 = (L dt + M dz)^2 - (N dt + P dz)^2 - S^2 dx^2 - R^2 dy^2 \]  

(3)
Where L, M, N, P, S, R are real functions of x and y. This form has been used by Carter [13] in the discovery of his spaces, by Debever [14] and exploited by Znajek [15], Carter and McLenaghan [16] and by us [17].

The Carter’s spaces are characterized by the fact that the Hamilton-Jacobi (HJ) equation for the geodesics is solvable by separation of variables. The separation of variables takes place in a particular way and give rise to a fourth constant of the motion, for particle orbits [13],[18]. This quadratic in the velocities integral is due to the existence of a second rank Killing tensor and the relation between separability of HJ equation and Killing tensors is studied in [19]. The special kind of separation of variables imposed by Carter is translated by a particular canonical form of the corresponding Killing tensor which admits two double eigenvalues \( \lambda_1 \) and \( \lambda_2 \) [20],[17]. This Killing tensor can be written in the following form in the NP formalism:

\[
K_{ij} = \lambda_i(n_i l_j + l_i n_j) + \lambda_j(m_i m_j + m_i \overline{m}_j)
\]

(4)

Then we can state the second hypothesis:

**Hypothesis 2.2**

We suppose that the space admits a second order Killing tensor, with two double eigenvalues, having the form (4) in the NP formalism.

Then we can prove [13], [17] that the spaces satisfying hypothesis 2.1 and 2.2 are the Carter’s family [A] of solutions and the corresponding metric is:

\[
ds^2 = (x^2 + y^2) \left\{ \frac{E^2(y)}{(x^2 + y^2)^2} (dt - x^2 dz)^2 - \frac{H^2(x)}{(x^2 + y^2)^2} (dt + y^2 dz)^2 - \frac{x^2}{F^2(x)} dx^2 - \frac{y^2}{G^2(y)} dy^2 \right\}
\]

(5)

We can choose the one forms of the null tetrad, of the NP formalism in the Cahen, Debever, Defrise complex vectorial representation [18], [21], [22], to be those of the symmetric null tetrad in which case the spin coefficients are equal by pairs and the null tetrad components for the Weyl tensor and the Ricci traceless tensor satisfy the following relations:

\[
\Psi_0 = \Psi_4 = 0 \quad \Psi_1 = \Psi_3 \quad \Psi_2 \neq 0
\]

(11)

\[
\Phi_{02} = \Phi_{20} \quad \Phi_{01} = \Phi_{21}
\]

(12)

In the case of vacuum Einstein’s equations, we obtain the following expressions for the functions of the metric (5):

\[
G^2 = y^2 E^2 \quad F^2 = x^2 H^2 \quad E^2 = \frac{1}{2} ay^2 + by + c \quad H^2 = -\frac{1}{2} ax^2 + dx + c
\]

(13)

Where a, b, c, d are constants of integration. The Kerr metric is obtained if we set:
\[ a = 2 , \quad d = 0 , \quad b = -2M , \quad c = \alpha^2 , \quad y = r , \quad x = \alpha \cos \theta \]

(14)

M is the mass, \( \alpha \) the angular momentum per unit mass and \( r, \theta \) the Boyer-Lindquist coordinates. If we redefine \( t \) and \( z \) as follows:

\[
dz = \alpha d\phi \quad , \quad dt = d\tilde{t} + d\phi
\]

(15)

We get the Kerr metric in the canonical form of Boyer-Lindquist:

\[
ds^2 = \frac{\Delta}{\rho^2} \left[ dt - \alpha (\sin \theta)^2 d\phi \right]^2 - \frac{\sin \theta}{\rho^2} \left[ (r^2 + \alpha^2) d\phi - \alpha dt \right]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2
\]

(16)

Where

\[
\Delta = r^2 - 2Mr + \alpha^2 \quad , \quad \rho^2 = r^2 + \alpha^2 (\cos \theta)^2
\]

(17)

The Einstein’s equations in the presence of a perfect fluid reduce to two equations [23]:

\[
\Phi_{00} \Phi_{02} = \Phi_{01}^2 \quad , \quad 2\Phi_{11} = \Phi_{00} + \Phi_{02}
\]

(18)

These equations imply that the energy-momentum tensor admits one simple eigenvalue (the energy-mass density) and a triple eigenvalue (the isotropic hydrostatic pressure). The solution of the first of equations (22) permit to define \( G^2 \) and \( F^2 \) as functions of \( E^2 \) and \( H^2 \) respectively:

\[
W^2 = \frac{G^2}{E^2} = k_4 y^4 + k_2 y^2 - k_0 \quad , \quad Z = \frac{F^2}{H^2} = -k_4 x^4 + k_2 x^2 + k_0
\]

(19)

\( k_0, k_2, k_4 \) are constants of integration and \( k_4 \) has to be negative in order to preserve the Lorentzian signature. Now if we put:

\[
k_4 = -q^2 \quad , \quad l^2 = k_2 - 4k_0 q^2
\]

(20)

And if we define new coordinates \( \xi \) and \( \zeta \) (Wahlquist coordinates) by the relations:

\[
x^2 = l\xi^2 - \frac{k_2 - l}{2q^2} \quad , \quad y^2 = l\zeta^2 + \frac{k_2 - l}{2q^2}
\]

(21)
We can solve the second of equations (22) and we get a generalization of Wahlquist solution [23].

3. Stationary axisymmetric spaces admitting surfaces of revolution

We consider a three dimensional Euclidean space filled with a congruence of surfaces of revolution with common centre and common axis of symmetry. Each surface is obtained by revolving a plane curve C about a line L in its plane, this line coincides with the axis of symmetry of the fluid configuration. In a Cartesian coordinates system \((x_1, x_2, x_3)\), the curve C lies on the plane \(Ox_1 x_2\) and \(Ox_3\) is the axis of revolution. In this coordinate system, a parametric representation for the curve C is defined as follows:

\[ x_1 = h_1(t), \quad x_3 = h_2(t) \]  

(22)

Where \(t\) is the corresponding parameter. The elimination of the parameter between equations (22), permit to write the Cartesian equation of the curve:

\[ x_i = f(x_3) \]  

(23)

And the Cartesian equation of the surface will be:

\[ f(x_3) = \sqrt{x_1^2 + x_2^2} \]  

(24)

A parametric representation of this surface is given by the following relations:

\[ x_1 = h_1(t) \cos \Phi, \quad x_2 = h_1(t) \sin \Phi, \quad x_3 = h_2(t) \]  

(25)

Now we can define a system of coordinates adapted to the surface of revolution, it is clear that one of the coordinates will be \(\Phi\), the azimuthal angle measured around the symmetry axis. The other two coordinates being \(r\) and \(\theta\), a generalization of spherical coordinates. This system \((r, \theta, \Phi)\) suggests that the parameter \(t\) in (29) can be chosen to be the angle \(\theta\):

\[ x_1 = h_1(r, \theta) \cos \Phi, \quad x_2 = h_1(r, \theta) \sin \Phi, \quad x_3 = h_2(r, \theta) \]  

(26)

The equation of the surface of revolution is, in this system of coordinates, of the form:

\[ r = r(\theta) \]  

(27)

The metric of the Euclidean space assumes now the expression:

\[ dx^2 + dy^2 + dz^2 = (h_{rr}^2 + h_{r\theta}^2) dr^2 + (h_{\theta\theta}^2 + h_{\theta\phi}^2) d\theta^2 + h_1 d\Phi^2 + 2(h_{r\theta} h_{\theta\phi} + h_{r\phi} h_{\theta\phi}) dr d\theta \]  

(28)

where
\[ h_r = \frac{\partial h_r}{\partial r}, \quad h_\theta = \frac{\partial h_\theta}{\partial \theta} \quad \ldots \ldots \]

The new coordinates are orthogonal only when:

\[ h_r, h_\theta, h_2, h_\phi = 0 \quad (29) \]

We restrict our research to orthogonal coordinates for reasons of simplicity and geometrical intuition. In fact in the case of ellipsoids of revolution, the condition (29) implies that the ellipsoids are confocal, we think that the surfaces of revolution have to share this property. Then the metric of a surface of revolution reduces to:

\[ dx^2 + dy^2 + dz^2 = (h_r^2 + h_\phi^2)dr^2 + (h_\theta^2 + h_\phi^2)d\theta^2 + h_r^2d\Phi^2 \quad (30) \]

An ellipsoidal space is characterized by:

\[ h_r(r, \theta) = g(r)\sin \theta \quad h_\phi(r, \theta) = r\cos \theta \quad (31) \]

The condition (29) reduces to:

\[ gg_r - r = 0 \quad (32) \]

The integration of this equation yields:

\[ g^2 = r^2 \pm \alpha^2 \quad (33) \]

And equation (24) is that of an ellipsoid of revolution:

\[ x_1^2 + x_2^2 + x_3^2 \pm \alpha^2 = 1 \quad (34) \]

The ellipsoid is oblate if \( g^2 = r^2 + \alpha^2 \) and prolate if \( g^2 = r^2 - \alpha^2 \) and the corresponding metric is that obtained by Krasinski [9]:

\[ dx^2 + dy^2 + dz^2 = \frac{(r^2 \pm \alpha^2 \cos^2 \theta)}{(r^2 \pm \alpha^2)}dr^2 + (r^2 \pm \alpha^2 \cos^2 \theta)d\theta^2 + (r^2 \pm \alpha^2 \sin^2 \theta)d\Phi^2 \quad (35) \]

The next step is to generalize the metric (30) to a three dimensional axisymmetric curved space, supposing that the metric can be written as follows:

\[ ds^2 = \frac{(h_r^2 + h_\phi^2)}{f^2(r, \theta)}dr^2 + (h_\theta^2 + h_\phi^2)d\theta^2 + h_r^2d\Phi^2 \quad (36) \]

Then if \( f^2(r, \theta) = 1 \) we have the corresponding Euclidean space, if \( f^2(r, \theta) \) is not constant the space is a three dimensional axisymmetric curved space. We introduce finally a coordinate system which is
of great utility for the integration of Einstein’s equations. Instead of \( r, \theta \) we will use \( x \) and \( y \) that appear in metric (5) and they are related by relations (14). Then the metric is given by:

\[
ds^2 = \frac{(h_{1y}^2 + h_{2y}^2)}{f^2(x,y)}dy^2 + (h_{1x}^2 + h_{2x}^2)dx^2 + h_1^2d\Phi^2
\]

(37)

Where \( h_1(x,y) \) and \( h_2(x,y) \) satisfy:

\[
h_{1x}h_{1y} + h_{2x}h_{2y} = 0
\]

(38)

In General Relativity the shape of a surface will depend on the observer performing its description, who takes also in account of the deformation due to relative motions. Obviously the bounding surface of a fluid will be characterized as surface of revolution only by a particular class of observers. An observer moving along the flow lines of the fluid, which are trajectories of the symmetry group, is defined by his four-velocity vector field \( u \). This vector field will be necessarily a linear combination of the Killing vector fields \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial z} \), so we can state that the vector field of a comoving with the fluid observer is given by:

\[
u = U(x,y)\frac{\partial}{\partial t} + V(x,y)\frac{\partial}{\partial z}
\]

(39)

And since \( g_{ij}u^iu^j = 1 \) must hold:

\[
g_{\alpha\beta}U^2 + 2g_{\alpha\gamma}UV + g_{\gamma\gamma}V^2 = 1
\]

(40)

Where the metric coefficients appeared in (40) are those of metric (5). The appropriate three space to consider the surfaces of revolution is the union of local rest spaces of observers, comoving with the fluid, this is nothing else but the quotient space of the comoving observers:

\[
dS^2 = h_0dx^i dx^j = g_{ij}dx^i dx^j - u_{ij} dx^i dx^j
\]

(41)

Where the \( g_{ij} \) are the metric coefficients of Carter’s spaces (5) and \( h_0 \) is the projection tensor. Then we can state the following definition:

**Definition 3.1**

The Carter’s family \([A]\) of spaces are said to admit a family of surfaces of revolution, if the quotient space of the comoving observers (39) is identified with the spaces (37).

If we apply this definition we can state that:

**Theorem 3.1**

The quotient space of the comoving with the fluid, observer in the case of Carter’s class \([A]\) of spaces, is that of a surface of revolution, if the velocity field of the observer can be written in the null tetrad:
\[ u_i = \frac{\sqrt{2}}{2} \left\{ \Pi_1(n_i + l_i) + \Pi_2(m_i + m_i) \right\} \]

Where:
\[ \Pi_1 = \frac{E(U - x^2V)}{(x^2 + y^2)^{\frac{1}{2}}} \]
\[ \Pi_2 = \frac{H(U + y^2V)}{(x^2 + y^2)^{\frac{1}{2}}} \]

\[ V = \pm \frac{h_1}{EH} \]
\[ U = \frac{1}{(E^2 - H^2)} \left\{ \pm (x^2E^2 + y^2H^2) \frac{h_1}{EH} + \left[ (E^2 - H^2)(x^2 + y^2) + h_1(x^2 + y^2)^2 \right]^{\frac{1}{2}} \right\} \]

And \( h_1, h_2 \) satisfy:
\[ \frac{(h_1^2 + h_2^2)}{f^2} = \frac{(x^2 + y^2)y^2}{G^2}, \quad h_1^2 + h_2^2 = \frac{(x^2 + y^2)x^2}{F^2}, \quad h_1h_1 + h_2h_2 = 0 \]

The equations of theorem 3.1 can be solved and the integrability conditions permit to prove that:

A. \[ f(x, y) = \frac{f_1(y)}{f_2(x)} \]

B. \[ G = f_1 \sqrt{p_4y^4 + p_2 + p_0} \quad F = f_2 \sqrt{-p_4x^4 + p_2x^2 - p_0} \quad \text{(42)} \]

\[ f_2 = \left\{ 2\sqrt{-p_4} \sqrt{-p_4x^4 + p_2x^2 - p_0} - 2p_4x^2 + p_2 \right\}^{\frac{k}{2}} \]

Where \( p \)'s and \( k \) are constants of integration.

4. Carter’s class [A] of spaces, admitting a family of surfaces of revolution, in the presence of an anisotropic fluid.

The energy-momentum for a fluid which satisfies the equations of viscous relativistic hydrodynamics (corresponding to the Navier-Stokes equations) is given by [23]:
\[ T_{ij} = (e + p)u_iu_j - pg_{ij} + 2\eta \sigma_{ij} + \zeta \partial h_{ij} + q_iu_j + q_ju_i \quad \text{(43)} \]

In this expression \( e \) is the rest-energy density of the fluid \( p \) is the hydrostatic pressure and \( \eta, \zeta \) are the coefficients of viscosity which are in general non negative. The fluid moves along the integral
curves of the fluid four velocity (39) as specified by theorem 3.1, the heat flow is described by the space-like vector field \( q_i \) which is then orthogonal to \( u_i \):

\[
q_i u^i = 0
\]

Finally \( \sigma^i, \theta, a, h_i \) are the shear tensor, the expansion, the acceleration and the projection tensor corresponding to \( u_i \).

The Einstein’s equations:

\[
R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}
\]

permit to prove the following theorem:

**Theorem 4.1**

The necessary and sufficient conditions that the Carter’s class of solutions admitting a family of surfaces of rotations, obeying equations (44), in the NP formalism are:

\[
2\Phi_{11} - \Phi_{00} - \Phi_{02} = 0
\]

\[
e = 3(\Phi_{00} - \Phi_{02}) + 6\Lambda
\]

\[
p = \Phi_{00} - \Phi_{02} - 6\Lambda
\]

And four expressions for the components of the vector field \( q_i \), also \( \Lambda = -\frac{R}{24} \), where \( R \) is the scalar curvature.

Equation (45) is an ODE for the functions \( E, G, H, \) and \( F \) which can be solved by separation of variables to give:

\[
\frac{G^2}{E^2} = \gamma E^2 + k_1 y^4 + k_2 y^2 + k_0
\]

\[
\frac{Z^2}{H^2} = \gamma H^2 + l_1 x^4 + l_2 x^2 + l_0
\]

And two Ode’s one for \( E(y) \) and one \( H(x) \), solving thus completely the problem! However the next step is to make a systematic study of the metric, including singularities, energy conditions, and regularity of the symmetry axis, thermodynamics and finally the problem of the junction conditions for the procedure of the matching with Kerr solution.

5. References

[1] Chandrasekhar S 1987 Ellipsoidal Figures of equilibrium Dover.
[2] Florides P 1973 Nuovo Cimento B13 1.
[3] Florides P and Synge J 1964 Proc.Roy.Soc. A280 459.
[4] Wahlquist H 1968 Phys.Rev. 172, 5 1291.
[5] Wahlquist H 1992 J.Math.Phys. 33(1) 304.
[6] Herrera L and Jimenez L 1982 J.Math.Phys. 23(12) 2339.
[7] Gurses M and Gursey F 1979 J.Math.Phys 16(12) 2385.
[8] Papakostas T 2001 Int.J.Mod.Phys. D 10 869.
[9] Krasinski A 1978 Ann.of Phys.112 22.
[10] Papapetrou A 1966 Ann.Inst. H Poincare A-IV 83.
[11] Kundt W and Trumper M 1966 Z.Phys. 192 419.
[12] Carter B 1964 J.Math.Phys. 10 70.
[13] Carter B 1968 Commun. Math. Phys. 10 280.
[14] Debever R 1971 Bull. Soc. Math.Belgique XXIII 360.
[15] Znajek R 1977 Mon.Not.R.Astr.Soc. 179 457.
[16] Carter B and McLenaghan R 1979 Phys.Rev. D 19 1093.
[17] Papakostas T 1983 Bull.Sci.Acad.R.Belgique LXIX 495.
[18] Debever R and McLenaghan R 1981 J.Math.Phys. 22 1771.
[19] Woodhouse N 1975 Commun.Math.Phys. 44 9.
[20] Hauser H and Mahliot R 1976 J.Math.Phys.17 1306.
[21] Debever R 1969 Cah . Phys.18 303.
[22] Debever R,McLenaghan R,Tariq N 1979 Gen.Rel.Grav. 10 853.
[23] Papakostas T 1998 Int.J.Mod.Phys. D 7 927.
[24] Weinberg S 1972 Gravitation and Cosmology John Wiley and Sons.