Isometric multipliers of $L^p(G, X)$

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MS received 21 January 2004

Abstract. Let $G$ be a locally compact group with a fixed right Haar measure and $X$ a separable Banach space. Let $L^p(G, X)$ be the space of $X$-valued measurable functions whose norm-functions are in the usual $L^p$. A left multiplier of $L^p(G, X)$ is a bounded linear operator on $L^p(G, X)$ which commutes with all left translations. We use the characterization of isometries of $L^p(G, X)$ onto itself to characterize the isometric, invertible, left multipliers of $L^p(G, X)$ for $1 \leq p < \infty$, $p \neq 2$, under the assumption that $X$ is not the $\ell^p$-direct sum of two non-zero subspaces. In fact we prove that if $T$ is an isometric left multiplier of $L^p(G, X)$ onto itself then there exists a $y \in G$ and an isometry $U$ of $X$ onto itself such that $Tf(x) = U(R_yf)(x)$. As an application, we determine the isometric left multipliers of $L^1 \cap L^p(G, X)$ and $L^1 \cap C_0(G, X)$ where $G$ is non-compact and $X$ is not the $\ell^p$-direct sum of two non-zero subspaces. If $G$ is a locally compact abelian group and $H$ is a separable Hilbert space, we define $A^p(G, H) = \{ f \in L^1(G, H) : \hat{f} \in L^p(\Gamma, H) \}$ where $\Gamma$ is the dual group of $G$. We characterize the isometric, invertible, left multipliers of $A^p(G, H)$, provided $G$ is non-compact. Finally, we use the characterization of isometries of $C(G, X)$ for $G$ compact to determine the isometric left multipliers of $C(G, X)$ provided $X^*$ is strictly convex.

Keywords. Locally compact group; Haar measure; Banach space-valued measurable functions; isometric multipliers.

1. Introduction

Let $G$ be a locally compact group with right Haar measure $\mu$. Suppose $X$ is a separable Banach space. If $1 \leq p < \infty$, let $L^p(G, X)$ be the space of $X$-valued measurable functions $F$ such that $\int_G \| F(x) \|^p \, d\mu < \infty$. The $p$-norm of $F$ is defined by $\left( \int_G \| F(x) \|^p \, d\mu \right)^{1/p}$. In case $X$ is a one-dimensional complex Banach space, $L^p(G, X)$ is denoted by $L^p(G)$.

The left and right translation operators $L_g$ and $R_g$ are defined by $(L_gF)(x) = F(gx)$ and $(R_gF)(x) = F(xg)$. A left multiplier of $L^p(G, X)$ is a bounded linear operator on $L^p(G, X)$ which commutes with all left translations. The main result of this paper gives a characterization of the isometric, invertible, left multipliers of $L^p(G, X)$ for $1 \leq p < \infty$, $p \neq 2$, under the assumption that $X$ is not the $\ell^p$-direct sum of two non-zero subspaces. More precisely we shall prove the following theorem.

**Theorem 1.** Let $G$ be a locally compact group and $T$ an isometric, invertible, left multiplier on $L^p(G, X)$ for $1 \leq p < \infty$, $p \neq 2$. Suppose that $X$ is not the $\ell^p$-direct sum of two non-zero subspaces. Then there exists an isometry $U$ of $X$ onto itself and $y \in G$ such that $T$ is of the form

$$(TF)(x) = UR_yF(x).$$
Wendel [7] proved this result for $L^1(G)$ in 1952. Later Strichartz [6] and Parrot [4] proved it for $L^p(G)$ if $1 \leq p < \infty$, $p \neq 2$.

Let $G$ be a non-compact locally compact group. If $f \in L^1 \cap L^p(G, X)$, we define $\|f\| = \|f\|_1 + \|f\|_p$. Then $L^1 \cap L^p(G, X)$ is a Banach space with this norm. Similarly for $f \in L^1 \cap C_0(G, X)$, we define $\|f\| = \|f\|_1 + \|f\|_\infty$. Then $L^1 \cap C_0(G, X)$ is a Banach space. In both cases, we shall show that if $T$ is an isometric, invertible, left multiplier, then $T$ is of the form

$$(Tf)(x) = U R_y f(x).$$

If $G$ is a locally compact abelian group and $H$ is a separable Hilbert space, we define $A^p(G, H) = \{f \in L^1(G, H) : f \in L^p(\Gamma, H)\}$ where $\Gamma$ is the dual group of $G$. For $f \in A^p(G, H)$, we define $\|f\| = \|f\|_1 + \|\hat{f}\|_p$. $A^p(G, H)$ is a Banach space with this norm. We will prove that if $T$ is an isometric, invertible, left multiplier, then $T$ is of the form

$$(Tf)(x) = U R_y f(x).$$

Let $G$ be a compact group and $X$ be a separable Banach space. $C(G, X)$ denotes the Banach space of continuous $X$-valued functions. Using the characterization of isometries of $C(G, X)$, we will prove that if $T$ is an isometric, invertible, left multiplier, then $T$ is of the form

$$(T F)(x) = U R_y F(x).$$

provided $X^*$ is strictly convex.

2. Preliminaries

Let $(\Omega, \Sigma, \mu)$ be a measure space. Suppose $\Sigma'$ is the $\sigma$-ring generated by the sets of $\sigma$-finite measure. A mapping $\Phi$ of $\Sigma'$ onto itself, defined modulo null sets, is said to be a regular set isomorphism if

1. $\Phi(A \setminus A') = \Phi(A) \setminus \Phi(A')$ for $A, A' \in \Sigma'$.
2. $\Phi(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty \Phi(A_n)$, where $\{A_n\}$ is a sequence of disjoint sets in $\Sigma'$.
3. $\mu(\Phi(A)) = 0$ iff $\mu(A) = 0$.

A regular set isomorphism induces a linear map on $X$-valued measurable functions. If $A \in \Sigma'$ and $x \in X$, define $\Phi(\chi_A)(x) = \chi_{\Phi(A)}(x)$ where $\chi_A$ is the characteristic function of $A$. This extends linearly to simple functions. Let $f$ be an $X$-valued measurable function. Then there exists a sequence $\{f_n\}$ of simple functions converging to $f$ in measure. Then $\{\Phi(f_n)\}$ is a Cauchy sequence in measure and hence converges to a measurable function $\Phi(f)$. It is easy to show that $\Phi(f)$ depends only on $f$ and not on the particular sequence $\{f_n\}$.

We also note that any $\Sigma'$-measurable function is also $\Sigma$-measurable and any $\Sigma'$-measurable function with $\sigma$-finite support is $\Sigma'$-measurable. Thus the spaces of $\Sigma'$ and $\Sigma$ measurable functions with $\sigma$-finite support coincide.

If $\Phi$ is a regular set isomorphism, define a measure $\nu$ by $\nu(A) = \mu(\Phi^{-1}(A))$. The measure $\nu$ is absolutely continuous with respect to $\mu$. Let $h = ((\nu)/(d\mu))^{1/p}$. It is easy to see that $h$ is a function on $\Omega$ whose restriction to any measurable set of $\sigma$-finite measure is measurable. Further, if $f \in L^p(\Omega, X)$, then $h \Phi(f) \in L^p(\Omega, X)$ and $\|h \Phi(f)\|_p = \|f\|_p$. 

We say that a Banach space $X$ is the $\ell^p$-direct sum of two Banach spaces $X_1$ and $X_2$ if $X$ is isometrically isomorphic to $X_1 \oplus X_2$ where the norm on the direct sum is given by $\|x_1 \oplus x_2\| = (\|x_1\|^p + \|x_2\|^p)^{1/p}$.

Our main tool for the proof of the main result is a theorem of Sourour [5]. We state it in a form slightly different from that of [5], but virtually no modification of the proof given there is necessary. The assumption that $\Omega$ is $\sigma$-finite is not needed for our conclusion because every function in $L^p(\Omega, X)$ has $\sigma$-finite support.

**Theorem S.** Let $(\Omega, \Sigma, \mu)$ be a measure space and $T$ be an isometry of $L^p(\Omega, X)$ onto itself. Suppose $X$ is not the $\ell^p$-direct sum of two non-zero Banach spaces. Then there exists a regular set isomorphism $\Phi$ of $\Sigma'$ onto itself, a measurable function $h$ on $\Omega$ and a strongly measurable map $S$ of $\Omega$ into the Banach space of bounded linear maps of $X$ into $X$ with $S(t)$ a surjective isometry of $X$ for almost all $t \in \Omega$, such that

$$T(F(t)) = S(t)h(t)\Phi(F)(t)$$

for $F \in L^p(\Omega, X)$ and almost all $t \in \Omega$.

3. Isometric multipliers of $L^p(G, X)$

In this section we characterize the isometric, invertible, left multipliers of $L^p(G, X)$.

**Proof of Theorem 1.** Let $T$ be an isometric, invertible, left multiplier of $L^p(G, X)$. It follows from Theorem S that

$$TF(t) = h(t)S(t)\Phi(F)(t) \quad \text{a.e.}$$

for every $F \in L^p(G, X)$.

Let $A(t) = h(t)S(t) \forall t \in G$. Fix $s \in G$. We will show that $L_sA(t) = A(t)$. If this is not true, then there exists a set $E$ of positive finite measure such that $A(st) \neq A(t) \forall t \in E$.

The sets $s\Phi^{-1}(E)$ and $\Phi^{-1}(sE)$ may be of $\sigma$-finite measure. But by choosing a suitable subset $E$ still of positive finite measure, we can assume that $s\Phi^{-1}(E)$ and $\Phi^{-1}(sE)$ are of positive finite measure. Having done this, let $F = s\Phi^{-1}(E) \cup \Phi^{-1}(sE)$. Then $\forall t \in E$, $st \in sE \subseteq \Phi(F)$ and $E \subseteq \Phi(s^{-1}F)$. Now for $t \in E$ and $x \in X$,

$$L_s(T\chi_Fx)(t) = T(\chi_Fx)(st) = \chi_{\Phi(F)}(st)A(st)(x) = A(st)(x).$$

Also,

$$T(L_s\chi_Fx)(t) = T(\chi_{s^{-1}F}x)(t) = \chi_{\Phi(s^{-1}F)}(t)A(t)(x) = A(t)(x).$$

Since $L_sT = TL_s$, it follows that $A(st)(x) = A(t)(x)$ for almost all $t \in E$. By choosing a countable dense set $\{x_n\}_{n=1}^{\infty}$ in $X$, we conclude that

$$A(st)(x) = A(t)(x)$$
for almost all \( t \in E \) and all \( x \in X \). But this is a contradiction. Hence

\[ A(st) = A(t) \]

for almost all \( t \in G \). Therefore for each \( x \in X \),

\[ h(t)S(t)(x) = h(st)S(st)(x) \]

for almost all \( t \in G \). Since \( S(t) \) is an isometry of \( X \) onto itself and \( h(t) \geq 0 \), we have

\[ h(st) = h(t) \]

for almost all \( t \in G \). This implies that \( h \) is a constant, say \( k \). It also follows that

\[ S(st) = S(t) \]

for almost all \( t \in G \). Hence \( S \) is also a constant operator, say \( V \). Therefore, \( T \) is an isometric multiplier of \( L^p(G, X) \) onto itself for all \( p \), in particular for \( p = 1 \). Now fix \( x \in X \) such that \( \|x\| = 1 \). Then for \( f \in L^1(G) \),

\[ L_s T(fx) = L_s k V \Phi(f) x = L_s \Phi(f) k V(x) \]

and

\[TL_s(f x) = k V \Phi(L_s f) x = \Phi(L_s f) k V(x) . \]

Hence \( L_s \Phi(f) = \Phi(L_s(f)) \). This implies that the map \( f \rightarrow k \Phi(f) \) is an isometric multiplier of \( L^1(G) \) onto itself. Hence by Wendel’s characterization there exists an \( s \in G \) and a scalar \( c \) such that \( |c| = 1 \) for which we have

\[ k \Phi(f)(t) = cf(ts) . \]

Let \( U = k V \). Then \( U \) is an isometry of \( X \) onto itself such that \( T = U \circ R_s \) and

\[(TF)(t) = UF(ts) \]

for almost all \( t \in G \) and all \( F \in L^p(G, X) \). This completes the proof of the theorem. \( \square \)

We shall now show that the condition that \( X \) is not an \( \ell^p \)-direct sum is a necessary (as well as a sufficient) condition for the conclusion of Theorem 1 to hold. In fact, we prove the following theorem.

**Theorem 2.** Let \( X \) be a separable Banach space which is \( \ell^p \)-direct sum of two non-zero subspaces of \( X \). Then there exists an isometric, invertible, left multiplier \( T \) of \( L^p(G, X) \) which is not of the form \( U \circ R_y \) for any isometry \( U \) of \( X \) and \( y \in G \).

**Proof.** Suppose \( X = X_1 \oplus_p X_2 \). Then

\[ L^p(G, X) = L^p(G, X_1) \oplus_p L^p(G, X_2) . \]

Choose \( z \in G \) where \( z \) is not the identity element of \( G \). Define \( T \) by

\[ T(f_1 \oplus f_2) = f_1 \oplus R_z f_2 . \]

Then it is easy to verify that \( T \) is an isometric, invertible, left multiplier of \( L^p(G, X) \) which is not of the form \( U \circ R_y \) for any isometry \( U \) of \( X \) and \( y \in G \). \( \square \)
4. Isometric multipliers of \( L^1 \cap L^p(G, X) \) and \( L^1 \cap C_0(G, X) \)

In this section we assume that \( G \) is non-compact and \( X \) is not an \( \ell^p \)-direct sum of two non-zero subspaces of \( X \). We will prove that if \( T \) is an isometric, invertible, left multiplier of \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \) then \( T \) is of the form \( U \circ R_y \) for some isometry \( U \) of \( X \) and \( y \in G \).

The proof of the following proposition is similar to the proof of Theorems 3.5.1 and 3.5.2 in [2] and hence omitted.

**Proposition 3**

Suppose \( G \) is non-compact. If \( T \) is a left multiplier of \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \) then \( T \) has a unique extension to \( L^1(G, X) \) as a left multiplier such that \( \|Tf\|_1 \leq \|T\| \|f\|_1 \), where \( \|T\| \) is the norm of \( T \) as an operator on \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \).

We now prove the characterization of an isometric, invertible, left multiplier of \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \).

**Theorem 4.** Suppose \( G \) is non-compact and \( X \) is not \( \ell^p \)-direct sum of two non-zero subspaces of \( X \). If \( T \) is an isometric, invertible, left multiplier of \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \) then \( T \) is of the form \( U \circ R_y \) for some isometry \( U \) of \( X \) and \( y \in G \).

**Proof.** Since \( T \) and \( T^{-1} \) are both isometric multipliers of \( L^1 \cap L^p(G, X) \) or \( L^1 \cap C_0(G, X) \), it follows from Proposition 3 that \( T \) extends to \( L^1(G, X) \) as an isometric left multiplier. Therefore by Theorem 1, there exists an isometry of \( X \) onto itself and \( y \in G \) such that \( T = U \circ R_y \). \( \square \)

5. Isometric multipliers of \( A^p(G, H) \)

Let \( G \) be a locally compact Abelian group and \( H \) be a separable Hilbert space. We define the Fourier transform of \( f \in L^1(G, H) \) by

\[
\hat{f}(\gamma) = \int_G \overline{\gamma(x)} f(x) dx,
\]

where \( \gamma \in \Gamma \), the dual group of \( G \). Given a Haar measure on \( G \) there exists a unique Haar measure on \( \Gamma \) such that the map \( f \rightarrow \hat{f} \) is an isometry of \( L^1 \cap L^2(G, H) \) into \( L^2(\Gamma, H) \) and extends to an isometry of \( L^2(G, H) \) onto \( L^2(\Gamma, H) \). The Fourier–Plancherel formula \( \|\hat{f}\|_2 = \|f\|_2 \) holds for \( f \in L^2(G, H) \), see [1].

For \( f \in A^p(G, H) \), we define \( \|f\| = \|f\|_1 + \|\hat{f}\|_p \). Then \( A^p(G, H) \) is a Banach space. We note that left and right translates mean the same for Abelian groups. Suppose \( G \) is non-compact. We will prove that if \( T \) is an isometric and invertible multiplier of \( A^p(G, H) \) then \( T = U \circ R_y \), where \( U \) is an isometry of \( H \) onto itself and \( y \in G \).

The proof of the following Proposition is similar to the argument in the proof of Theorem 6.3.1 in [2] where it is shown that if \( T \) is a multiplier of \( A^p(G) \) then \( \|Tf\|_1 \leq \|T\| \|f\|_1 \) for \( f \in A^p(G) \), where \( \|T\| \) denotes the operator norm of \( T \). The necessary modifications are easy and hence we omit the details.

**Proposition 5**

Let \( G \) be a non-compact locally compact Abelian group and \( 1 \leq p < \infty \). Suppose \( T \) is a multiplier of \( A^p(G, H) \) then \( \|Tf\|_1 \leq \|T\| \|f\|_1 \) for \( f \in A^p(G, H) \).
We now prove the characterization of isometric multipliers of $A^p(G, H)$.

**Theorem 6.** Let $G$ be a non-compact locally compact Abelian group and $1 \leq p < \infty$. Suppose $T$ is an isometric multiplier of $A^p(G, H)$. Then there exists a unique $y \in G$ and an isometry $U$ of $H$ onto itself such that $T = U \circ R_y$.

**Proof.** Let $T$ be an isometric multiplier of $A^p(G, H)$. Then $T^{-1}$ is also an isometric multiplier and we conclude from Proposition 5 that $\|Tf\|_1 = \|f\|_1$ for every $f \in A^p(G, H)$. It follows that $T$ extends to $L^1(G, H)$ as an isometric multiplier of $L^1(G, H)$. Hence, by Theorem 1, there exists an isometry $U$ of $H$ onto itself and $y \in G$ such that $T = U \circ R_y$.

6. **Isometric multipliers of $C(G, X)$**

In this section we describe the isometric, invertible, left multipliers of $C(G, X)$ where $G$ is a compact group and $X^*$ is strictly convex. The space $C(G, X)$ consists of all continuous $X$-valued function and is a Banach space under the supremum norm. The norm of $f \in C(G, X)$ will be denoted by $\|f\|_\infty$. For the space $X$, we denote the set of isometries of $X$ onto itself by $I(X)$. The isometries of $C(G, X)$ were characterized by Lau [3]. He has shown that if $T$ is an isometry of $C(G, X)$ onto itself, then there exists a homeomorphism $\phi$ of $G$ onto itself and a continuous map $\lambda: X \to I(X)$ (with the strong operator topology) such that

$$Tf(s) = \lambda(s)f(\phi(s)) \quad \forall s \in G.$$  

Using this characterization of isometries of $C(G, X)$, we prove the following:

**Theorem.** Let $T$ be an isometric, invertible, left multiplier of $C(G, X)$. Then there exists an isometry $U$ of $X$ onto itself and $y \in G$ such that $T = U \circ R_y$.

**Proof.** Since $T$ is an isometry of $C(G, X)$, there exists a continuous map $\lambda: X \to I(X)$ and a homeomorphism $\phi$ of $G$ onto itself such that

$$Tf(s) = \lambda(s)f(\phi(s)) \quad \forall s \in G.$$

Fix $x \in X$ and let $f(s) = x \forall s \in G$. Then

$$TL_t f(s) = \lambda(s)f(t(\phi(s))) \quad (1)$$

and

$$L_t Tf(s) = \lambda(ts)f(\phi(ts)). \quad (2)$$

Since $TL_t = L_t T$, it follows that $\lambda(s)(x) = \lambda(ts)(x)$. Since $x \in X$ is arbitrary, we conclude that $\lambda(ts) = \lambda(s) \forall s, t \in G$. Hence there exists an isometry $U$ of $X$ onto itself such that $\lambda(s) = U \forall s \in G$. Therefore

$$Tf(s) = Uf(\phi(s)) \quad \forall f \in C(G, X).$$

Let $g \in C(G)$ and $x \in X$. Define $f$ by $f(s) = g(s)x \forall s \in G$. Then (1) and (2) imply that

$$g(t\phi(s)) = g(\phi(ts)) \quad \forall g \in C(G).$$

Since $C(G)$ separates points, we conclude that $t\phi(s) = \phi(ts) \forall s, t \in G$. Let $s$ be the identity element of $G$. Then $\phi(t) = t\phi(e)$. Let us denote $\phi(e)$ by $y$. Then we have $Tf(s) = Uf(sy) \forall f \in C(G, X)$ and $s \in G$. Therefore we have $T = U \circ R_y$. 

\[\square\]
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