SOME IMPLICATIONS BETWEEN GROTHENDIECK’S ANABELIAN CONJECTURES

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ABSTRACT. The main result of this paper is to establish two implications between Grothendieck’s anabelian conjectures. We define anabelian Deligne-Mumford stacks as those DM stacks that satisfy the section conjecture in a strong sense: for schemes, this amounts to the section conjecture plus triviality of the centralizers of sections. We prove that anabelian DM stacks satisfy the hom conjecture (which characterizes morphisms from a smooth scheme $T$ to an anabelian variety, or DM stack $X$). As a corollary, for hyperbolic curves the section conjecture implies the hom conjecture. In doing this, we actually show that for $T$ normality is enough. We show that if a positive dimensional DM stack is anabelian then its geometric fundamental group cannot have abelian finite index subgroups. We enlarge the class of elementary anabelian varieties to elementary anabelian DM stacks. We prove that they satisfy the injectivity part of the conjecture, and that the section conjecture for hyperbolic curves implies that elementary anabelian DM stacks satisfy the section conjecture too.

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1. INTRODUCTION

Throughout this paper, we fix a base field $k$ of characteristic 0 finitely generated over $Q$ (except in the appendix, where characteristic 0 is enough).
1.1. Grothendieck’s anabelian conjectures. In his letter to Faltings [Gro97], Grothendieck proposed a series of conjectures about “anabelian” varieties, varieties whose fundamental group is highly non abelian. In dimension one, anabelian varieties are hyperbolic curves, i.e. those with negative Euler characteristic. In higher dimension the picture is not so clear, since Grothendieck gave no precise definition. He said that they should contain the so called elementary anabelian varieties, i.e. those varieties that can be obtained by subsequent fibrations by hyperbolic curves, and moduli stacks of curves.

The idea of Grothendieck’s anabelian philosophy is that for an anabelian variety $X$, if the base field $k$ is finitely generated over $\mathbb{Q}$, it should be possible to recover the geometry of the variety from the étale fundamental group $\pi_1(X)$ with its projection to the absolute Galois group $\text{Gal}(\bar{k}/k)$. Grothendieck’s conjectures predict more precisely how this happens.

Disclaimer: Throughout the paper, we restrict our attention to $X$ proper. The reason is that, thanks to an idea of Borne and Emsalem (see [BE14, §2.2.3] and [Bre18, §7]), if one states the anabelian conjectures for DM stacks rather than schemes, then the non-proper case can be seen as a projective limit of the proper case for stacks. The upside is that conjectures are much easier to state and manipulate in the proper case, the downside is that one has to embrace the formalism of stacks (which, however, is very appropriate for the anabelian world).

Let us fix a base field $k$ finitely generated over $\mathbb{Q}$, and write $G_k$ for $\text{Gal}(\bar{k}/k)$.

If $X$ is geometrically connected and $x$ is a geometric point, there is a short exact sequence of étale fundamental groups

$$0 \rightarrow \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, x) \rightarrow G_k \rightarrow 0$$

Let $T$ be another geometrically connected scheme with a geometric point $t$. Write $\text{Hom-ext}_{G_k}(\pi_1(T, t), \pi_1(X, x))$ for the set of continuous homomorphisms $\pi_1(T, t) \rightarrow \pi_1(X, x)$ which commute with the projections to $G_k$, considered up to conjugation by elements of $\pi_1(\bar{X}, \bar{x})$. Functoriality of $\pi_1$ gives a natural map

$$\text{Hom}_k(T, X) \rightarrow \text{Hom-ext}_{G_k}(\pi_1(T, t), \pi_1(X, x)).$$

For proper varieties, Grothendieck gave two forms of his “main conjecture”.

**Conjecture 1.1 (Hom conjecture).** If $T$ is smooth and $X$ is smooth, proper and anabelian, then

$$\text{Hom}_k(T, X) \rightarrow \text{Hom-ext}_{G_k}(\pi_1(T, t), \pi_1(X, x))$$

is a bijection.

We call this the hom conjecture. There is a form of the hom conjecture which restricts the attention to dominant morphisms, and this has been proved by Mochizuki for hyperbolic curves. We discuss briefly Mochizuki’s result and its connection to the hom conjecture in § 1.3. The second form of the main conjecture is the so called section conjecture.

**Conjecture 1.2 (Section conjecture).** If $X$ is smooth, proper and anabelian, then

$$X(k) \rightarrow \text{Hom-ext}_{G_k}(G_k, \pi_1(X, x))$$

is a bijection.
Grothendieck stated that the two forms are equivalent. One implication is obvious, in fact the section conjecture is just the hom conjecture for $T = \text{Spec} \ k$. For the other one, Grothendieck stated that the hom conjecture follows from the section conjecture by looking at the generic point of $T$, but I don’t know of any proof of this in the literature. He also said that these conjectures for hyperbolic curves imply the analogous statements for elementary anabelian varieties. Again, I don’t know of any proof of this in the literature.

We can now say which are the main results of this article:

- In § 2.1 we give a form of the conjecture well suited for proper DM stacks. This is a stronger statement than the usual section conjecture, but in the case of hyperbolic curves it turns out to be equivalent, see Lemma 2.4.
- In § 2.3 we explain why we should not consider Artin stacks, and show that an Artin stack satisfying our version of the section conjecture is actually a DM stack.
- In Proposition 2.6 we prove that DM stacks satisfying the conjecture have a strong topological feature, i.e. they have a finite étale cover by an algebraic space.
- In Theorem 3.3 we prove that, as stated by Grothendieck, the section conjecture implies the hom conjecture. Actually, our form of the hom conjecture is stronger in some ways: most notably, we allow $T$ to be normal, while Grothendieck assumed smooth.
- In Theorem 3.6 we prove that the geometric fundamental group of a positive dimensional DM stack satisfying the conjecture cannot have finite index abelian subgroups.
- In § 4 we generalize the class of elementary anabelian varieties to elementary anabelian DM stacks.
- In Theorem 4.10 we prove that, as stated by Grothendieck for varieties, the section conjecture for hyperbolic curves implies the section conjecture for elementary anabelian DM stacks.

All of these are more conveniently stated and proved in terms of étale fundamental gerbes.

1.2. Étale fundamental gerbes. In [BV15] Borne and Vistoli developed the theory of fundamental gerbes, which can be thought as a more convenient way to repackage the theory of various types of fundamental groups. In particular, the étale fundamental gerbe repackages the content of the étale fundamental group, along with its natural projection to $G_k$. This construction builds on the theory of gerbes, which are a particular type of stack "concentrated on one point". By analogy with the topological theory of the fundamental group $\pi_1$, gerbes allow to construct in algebraic geometry the analogous of the classifying space $B\pi_1$ of the fundamental group, rather than the fundamental group itself.

More precisely, if $X$ is a geometrically connected fibered category over $k$ (for example a geometrically connected scheme, or algebraic stack, see [Bre18, §8] for a general definition) then the étale fundamental gerbe is a pro-étale stack $\Pi_{X/k}$ with a natural morphism

$$X \to \Pi_{X/k}$$

universal among morphisms to étale stacks. If the base field is algebraically closed and $x \in X$ is a geometric point, then $\text{Aut}_{\Pi_{X/k}}(x) = \pi_1(X, x)$. 

This is convenient for a number of reasons, among which the fact that fundamental gerbes do not depend on a base point and that, like classifying spaces in topology, they suggest naturally a lot of geometric constructions which are more complicated from the point of view of fundamental groups. Moreover, they allow a very nice interpretation of the anabelian conjectures.

**Proposition 1.3.** Let \( X/k \) be a quasi-compact, quasi-separated and geometrically connected algebraic stack with a geometric point \( \bar{x} : \text{Spec} \Omega \to X \), and \( T \) any geometrically connected scheme with a geometric point \( \bar{t} : \text{Spec} \Omega \to X \). There is a (non canonical) equivalence of categories
\[
\Pi_{X/k}(T) \to \text{Hom-ext}_{G_k}(\pi_1(T, \bar{t}), \pi_1(X, \bar{x}))
\]
that composed with the canonical functor \( X(T) \to \Pi_{X/k}(T) \) is a lifting of the natural map
\[
\text{Hom}_k(T, X) \to \text{Hom-ext}_{G_k}(\pi_1(T, \bar{t}), \pi_1(X, \bar{x})).
\]

This is a straightforward generalization of [BV15, Proposition 9.3] of Borne and Vistoli, we don’t repeat the proof. Proposition 1.3 gives a very natural environment for the anabelian conjectures:

\[
\left\{ \begin{array}{c}
\text{morphisms in the} \\
\text{"classifying space" of } \pi_1
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\text{hom. of fundamental} \\
\text{groups as extensions of } G_k
\end{array} \right\}/\sim
\]

1.3. **The anabelian conjecture for dominant morphisms.** Finally, even if it is not directly tied with our work, for the sake of completeness we shall say some words about the anabelian conjecture for dominant morphisms and Mochizuki’s theorem.

If a morphism \( T \to X \) is dominant, the induced homomorphism \( \pi_1(T, t) \to \pi_1(X, x) \) has finite index, i.e. it is open. If we denote by \( \text{Hom}^{\text{dom}}(T, X) \) the set of dominant homomorphisms and \( \text{Hom}^{\text{op}}_{G_k}(\pi_1(T, \bar{t}), \pi_1(X, \bar{x})) \) the set of open outer homomorphisms, one gets a map
\[
\text{Hom}^{\text{dom}}(T, X) \to \text{Hom}^{\text{op}}_{G_k}(\pi_1(T, \bar{t}), \pi_1(X, \bar{x})).
\]

In [Gro97], Grothendieck conjectured that this map is a bijection as long as \( X \) is elementary anabelian (his \( Y \) correspond to our \( X \)).

In the case where \( Y \) is replaced by an elementary anabelian variety, the bijectivity of (6) is valid, as long as one restricts oneself to dominant homomorphisms on the left-hand side, keeping the same restriction (finite index image) on the right-hand side.

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\(^1\)In the case where \( Y \) is replaced by an elementary anabelian variety, the bijectivity of (6) is valid, as long as one restricts oneself to dominant homomorphisms on the left-hand side, keeping the same restriction (finite index image) on the right-hand side.
elementary anabelian variety with \( \dim X = 2 \), hence it is a fibration \( X \to C \) in hyperbolic curves with \( C \) an hyperbolic curve, then \( T \) is an hyperbolic curve too, since it is either a fiber or it dominates \( C \). Hence the conjecture for dominant morphisms cannot hold even if we ask \( T \) to be anabelian.

However, in [Moc99, Theorem B] Mochizuki proved a birational version of Grothendieck’s conjecture for dominant morphisms in arbitrary dimension, and just before the quoted line Grothendieck is talking about the case of non proper curves (he is correcting the hom conjecture in the non proper case in view of the existence of cuspidal sections):

\[
\text{[...]} \text{ im Fall wo } Y \text{ nicht eigentlich ist, ist es notwendig, sich im ersten}
\text{ Glied von (6) auf nicht-konstante Homomorphismen zu beschränken, und}
\text{ im zweiten auf Homomorphismen } \pi_1(X) \to \pi_1(Y), \text{ deren Bild von}
\text{ endlichem Index (nämlich offen) ist.}\]

Asking that the elementary anabelian variety is not proper is clearly still not enough, since we can take the product of an affine hyperbolic curve with a proper elementary anabelian variety and adapt our counterexample to this case. It is likely that Grothendieck had in mind good Artin neighbourhoods, see [SGA4-III, Exposé XI, Definition 3.2]: the definition is almost the same as elementary anabelian varieties, but with the additional condition that the fibrations defining them must have affine fibers.

Returning to the case where \( X \) is a proper hyperbolic curve and \( T \) is smooth and connected, since morphisms \( T \to X \) are either constant on a closed point or dominant, the hom conjecture implies immediately both the section conjecture and the dominant version of the conjecture, i.e. Mochizuki’s theorem. The converse seems to hold on the nose, too, but this is not correct. In fact, the hom conjecture implies also the following statement.

**Consequence of the hom conjecture.** If \( X \) is an hyperbolic curve and \( T \) is smooth and connected, for every \( \varphi \in \text{Hom-ext}_{G_k}(\pi_1(T,t),\pi_1(X,x)) \) exactly one of the following is true:

- \( \varphi \) is open, or
- \( \varphi \) factorizes as \( \pi_1(T,t) \to \text{Gal}(\bar{k}/k') \to \pi_1(X,x) \), where \( k'/k \) is a finite extension.

For curves, the hom conjecture for curves is equivalent to the sum of Mochizuki’s theorem, the section conjecture and this last statement. In view of Theorem 3.3, the section conjecture for curves actually implies the hom conjecture for curves, hence Mochizuki’s theorem can be seen as a consequence of the section conjecture.

### 1.4. Acknowledgements

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\[\text{\[\ldots\] in the case where } Y \text{ is not proper, it is necessary to restrict oneself, on the left-hand side of (6), to non-constant homomorphisms, and on the right-hand side to homomorphisms } \pi_1(X) \to \pi_1(Y), \text{ whose images are of finite index (i.e. open).}\]
2. Anabelian DM stacks

2.1. Toward the definition. We want now to understand what the section conjecture for DM stacks should look like. Clearly, one can just directly translate Grothendieck’s section conjecture to DM stacks. Here we hope to show that the right thing to conjecture in general is slightly stronger (but equivalent in the case of hyperbolic curves).

Proposition 2.1. Let $X$ be a proper, smooth, geometrically connected Deligne-Mumford stack over $k$. The following are equivalent:

1. for every finitely generated extension $k'/k$ and for every finite étale connected cover $Y \to X_{k'}$,
   $$Y(k') \to \text{Hom-Ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$$
   is bijective (resp. injective) on isomorphism classes,
2. the natural map
   $$X(k') \to \Pi_{X/k}(k')$$
   is an equivalence of categories (resp. fully faithful) for every finitely generated extension $k'/k$.

Proof. Suppose that $X(k') \to \Pi_{X/k}(k')$ is an equivalence (resp. fully faithful). Then by Proposition 5.3 $X_{k'}(k') \to \Pi_{X_{k'}/k}(k')$ is an equivalence (resp. fully faithful), too, and hence $Y(k') \to \text{Hom-Ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$ is bijective (resp. injective) thanks to the going up theorem [Bre18, Proposition 4.2].

Suppose now that (1) holds, let $k'/k$ be a finitely generated extension and $x \in X(k')$ a point and $\pi(x) \in \Pi_{X/k}(k')$. Since by hypothesis $X(k') \to \Pi_{X/k}(k')$ is bijective (resp. injective) on isomorphism classes, then we only have to show that

$$\text{Aut}_X(x) \to \text{Aut}_{\Pi_{X/k}}(\pi(x))$$

induces a bijection on $k'$-rational points. Thanks to Proposition 5.3, we may suppose $k' = k$.

Suppose that $\varphi \in \text{Aut}_{\Pi_{X/k}}(\pi(x))(k)$ is not in the image. Then, since $\Pi_{X/k} = B\text{Aut}_{\Pi_{X/k}}(\pi(x))$ is profinite, there exists a finite index subgroup $H \subseteq \text{Aut}_{\Pi_{X/k}}(\pi(x))$ such that $\varphi \notin H$. Consider the fiber product

$$
\begin{array}{ccc}
Y & \longrightarrow & BH \\
\downarrow & & \downarrow \\
X & \longrightarrow & B\text{Aut}_{\Pi_{X/k}}(\pi(p)) = \Pi_{X/k}
\end{array}
$$

where $BH$ identifies naturally with $\Pi_{Y/k}$ thanks to [Bre18, Proposition 8.16] (we have a natural map $\Pi_{Y/k} \to BH$, and $\Pi_{Y/k} \to \Pi_{X/k}$. $BH \to \Pi_{X/k}$ are subgerbes with the same finite index). Then id, $\varphi$ define two non isomorphic rational points $q, q' \in Y(k)$ over $p \in X(k)$ with the same image in $BH(k) = \Pi_{Y/k}(k)$, but this is absurd since by hypothesis $Y(k') \to \text{Hom-Ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$ is injective.

We want now to prove that $\text{Aut}_X(x) \to \text{Aut}_{\Pi_{X/k}}(\pi(x))$ is injective. Since $\text{Aut}_X(x)$ is finite étale, up to enlarging the base field we may suppose that $\text{Aut}_X(x)$ is discrete, and we can consider a finite index subgroup $H \subseteq \text{Aut}_{\Pi_{X/k}}(\pi(x))$ such that

$$H \cap \text{im} \left( \text{Aut}_X(x) \to \text{Aut}_{\Pi_{X/k}}(\pi(x)) \right) = \{ \text{id} \}.$$
By passing to the fiber product \( Y = X \times_{\Pi X/k} BH \) as above, we find \( y \in Y(k) \) over \( x \in X(k) \) such that \( \text{Aut}_Y(y) \subseteq \text{Aut}_X(x) \) is the kernel \( \pi : \text{Aut}_X(x) \to \text{Aut}_{\Pi X/k}(\pi(x)) \): in fact, \( \text{Aut}_Y(y) = \pi^{-1}(H) = \pi^{-1}(id) \subseteq \text{Aut}_X(x) \).

Hence, we want to prove that \( \text{Aut}_Y(y) \) is trivial.

Now, since \( Y(k') \to \Pi Y/k \) is by hypothesis injective on isomorphism classes for every finitely generated extension \( k' / k \) and \( \text{Aut}_Y(y) \to \text{Aut}_{\Pi Y/k}(\pi(y)) \) factorizes through the identity, we get that

\[
B \text{Aut}_Y(y)(k') \subseteq \pi^{-1}(\pi(y))(k') = \{y\}
\]

has only one isomorphism class for every finitely generated \( k' / k \), i.e. \( \text{Aut}_Y(y) \) is special. But an étale special group is trivial, as desired. \( \square \)

We define now anabelian DM stacks as those DM stacks satisfying the equivalent conditions of Proposition 2.1.

**Definition 2.2.** Let \( X \) be a smooth, proper, geometrically connected Deligne-Mumford stack. We say that \( X \) is anabelian (resp. fundamentally fully faithful, or fff) if the natural morphism

\[
X(k') \to \Pi X/k(k')
\]

is an equivalence of categories (resp. fully faithful) for every finitely generated extension \( k' / k \).

**Remark 2.3.** While the core of the definition of anabelian DM stacks is due to Grothendieck, there are some remarks to be made.

- Extending the definition to Deligne-Mumford stacks seems natural for at least two reasons. One is that moduli stacks of curves are expected to be anabelian, the second is that hyperbolic orbicurves are anabelian if and only if hyperbolic curves are anabelian, see [Bre18, Theorem 6.3]. We address the question “why not Artin stacks?” in § 2.3.
- Classical conjectures and theorems in anabelian geometry are stated in terms of isomorphisms classes, rather than of equivalence of categories. However, both the points of a Deligne-Mumford stack and the étale fundamental gerbe have a natural structure of a category whose morphism are invertible rather than that of a set, hence asking an equivalence of categories seems more natural, particularly in view of Proposition 2.1.
- The étale fundamental gerbe behaves well with respect to base change of the base field, i.e. \( \Pi X_{k'/k'} = \Pi X_k \times_k k' \), see Proposition 5.3. Hence if \( X \) is anabelian then \( X_{k'} \) is anabelian too, for every finitely generated extension \( k'/k \).
- In terms of my preceding paper [Bre18], \( X \) is anabelian if \( X_{k'} \) satisfies the stacky section conjecture [Bre18, Conjecture 1.3] for every finitely generated extension \( k'/k \).

In the following, we show what it means for a scheme to be anabelian in the classical terms of the section conjecture and of centralizers of sections, see [Sti13, §3.3].
Lemma 2.4. If $X$ is a smooth, proper, geometrically connected scheme, then $X$ is anabelian if and only if $X_{k'}$ satisfies the section conjecture for every finitely generated extension $k'/k$ and every section $\varphi \in \text{Hom-ext}_{G_{k'}}(G_{k'}, \pi_1(X))$ has trivial centralizer.

In particular, hyperbolic curves are anabelian if and only if they satisfy the section conjecture over every finitely generated extension of the base field.

Proof. The automorphism groups of the points of the fundamental gerbe correspond to the centralizers of sections of the étale fundamental group (see [Bre18, Remark 4.1]), hence if $X$ is a scheme asking an equivalence of categories corresponds to asking a bijection on isomorphism classes plus the triviality of centralizers.

For hyperbolic curves, centralizers of sections are trivial, thanks either to [Sti13, Proposition 36, Proposition 104] or to the full faithfulness part of Proposition 2.1.

Proposition 2.5. Let $Y$, $X$ be smooth, proper, geometrically connected DM stacks, and $Y \to X$ a finite étale covering. Then $Y$ is anabelian (resp. fff) if and only if $X$ is anabelian (resp. fff).

Proof. This is a straightforward application of the going up and down theorems [Bre18, Proposition 4.2, Proposition 4.6].

2.2. Étale covers by algebraic spaces. It turns out that anabelian DM stacks (actually, fff is enough) must have a non obvious topological feature, i.e. they have a finite étale cover by an algebraic space.

Proposition 2.6. Let $X$ be a geometrically connected DM stack locally of finite type over $k$, and suppose that the natural morphism

$$X(k') \to \Pi_{X/k}(k')$$

is fully faithful for every finitely generated $k'/k$.

Then $X \to \Pi_{X/k}$ is representable by algebraic spaces and there exists a profinite étale cover $\tilde{X} \to X$ with $\tilde{X}$ an algebraic space.

If moreover $X$ is of finite type and separated, there exists a finite gerbe $\Phi$ with a representable morphism $X \to \Phi$, and a finite étale cover $E \to X$ with $E$ an algebraic space.

Proof. Let $\text{Spec} \Omega \to X$ be any geometric point. Since $X$ is locally of finite type, $x$ is defined over a finitely generated extension $k'/k$, $x \in X(k')$. Let $\pi(x) \in \Pi_{X/k}(L)$ its image. Up to extending $k'$, we may suppose that the finite étale group scheme $\text{Aut}_X(x)$ over $k'$ is discrete. Since $X(k') \to \Pi_{X/k}(k')$ is fully faithful, the map

$$\text{Aut}_X(x)(k') \to \text{Aut}_{\Pi_{X/k}}(\pi(x))(k')$$

is injective. In particular, since $\text{Aut}_X(x)$ is discrete, the homomorphism of group schemes

$$\text{Aut}_X(x) \to \text{Aut}_{\Pi_{X/k}}(x)$$

is injective, hence $X \to \Pi_{X/k}$ is representable.

We want now to show that $X$ has a profinite étale cover by an algebraic space. Up to a finite separable extension of the base field, we may suppose that $X$ has a
rational point $x_0 \in X(k)$, let $\pi(x_0) \in \Pi_{X/k}$ be its image. Then just take the fiber product
\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \text{Spec } k \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Pi_{X/k}
\end{array}
\]

Since $X \rightarrow \Pi_{X/k}$ is representable, $\tilde{X}$ is an algebraic space.

Suppose now that $X$ is of finite type and separated. Let $\xi_1, \ldots, \xi_n$ be the generic points of the irreducible components of $X$. Since $\text{Aut}_{X}(\xi_i)$ is finite for every $i$, thanks to the hypothesis there exists a finite gerbe $\Phi_1$ with a morphism $X \rightarrow \Phi_1$ such that $B\text{Aut}_{X}(\xi_i) \rightarrow \Phi_1$ is representable for every $i$. Hence, there exists a dense open subset $U_1 \subseteq X$ such that $U_1 \rightarrow \Phi_1$ is representable: $U_1$ is open since it is the locus where the relative inertia $I_{X/\Phi_1} \rightarrow X$ is an isomorphism. Now take the generic points of the irreducible components of $X \setminus U_1$, and repeat the argument in order to find $X \rightarrow \Phi_2 \rightarrow \Phi_1$ and $U_2 \supseteq U_1$ with $U_2 \rightarrow \Phi_2$ representable. Since $X$ is of finite type, the process ends.

In order to find $E$, since $\Phi$ is finite there exists a finite, separable extension $k'/k$ and a section $\text{Spec } k' \rightarrow \Phi$. Take $E = \text{Spec } k' \times_{\Phi} X$. \hfill $\Box$

**Remark 2.7.** In [Bre18], to show that the section conjecture for orbicurves is equivalent to the section conjecture for curves we used the non obvious fact that hyperbolic orbicurves are covered by hyperbolic curves: it is quite remarkable that it was not only a useful feature, but a necessary one, and it happened to be true.

### 2.3. Artin stacks vs Deligne-Mumford stacks

One may wonder: why DM stacks and not Artin stacks? The answer is based on one’s taste. DM stacks seem more natural, since $\Pi_{X/k}$ is pro-étale and Proposition 2.1 fails for Artin stacks. For example, if $G$ is a connected algebraic group, then condition (1) of Proposition 2.1 holds for $BG$ if and only if $G$ is special, while condition (2) if and only if $G$ is trivial. Hence, it makes a difference if we choose condition (1) or (2) as definition of anabelianity for Artin stacks.

If we choose (1), we should for instance consider $BGL_n$ as anabelian even if $BGL_n \rightarrow \Pi_{BGL_n} = \text{Spec } k$ is not an equivalence of categories on rational points, and this seems not very pleasant. On the other hand, if we choose (2), the following proposition shows that we get back to DM stack.

**Proposition 2.8.** Let $X$ be a geometrically connected Artin stack. Suppose that $X(k') \rightarrow \Pi_{X/k}(k')$ is fully faithful for every finitely generated extension $k'/k$. Then $X$ is a Deligne-Mumford stack.

**Proof.** Let $x : \text{Spec } \Omega \rightarrow X$ be any geometric point, we want to show that $\text{Aut}_{X}(x)$ is finite étale. Since $X$ is of finite type, we may suppose that $x$ is defined over a finitely generated extension $k'/k$. Thanks to Proposition 5.3, we may suppose $k' = k$, i.e. $x \in X(k)$ is a rational point.

Let $\pi(x) \in \Pi_{X/k}(k)$ be the image of $x$, we have an homomorphism of group schemes
\[
\text{Aut}_{X}(x) \xrightarrow{\pi} \text{Aut}_{\Pi_{X/k}(\pi(x))}.
\]
This homomorphism is injective: in fact, if the kernel $\ker(\pi) \subseteq \text{Aut}_X(x)$ is non-trivial, since it is of finite type up to enlarging the base field we may suppose that there exists a rational point $\varphi \in \ker(\pi)(k)$ different from the identity. But $\text{Aut}_X(x) \to \text{Aut}_{\Pi/X/k}(\pi(x))$ is injective on rational points by hypothesis, hence $\ker(\pi)$ is trivial and

$$\text{Aut}_X(x) \subseteq \text{Aut}_{\Pi/X/k}(\pi(x))$$

is a subgroup scheme. Now, $\text{Aut}_X(x)$ is of finite type and $\text{Aut}_{\Pi/X/k}(\pi(x))$ is pro-étaile, hence $\text{Aut}_{\Pi/X/k}(\pi(x))$ is finite étale, as desired. \qed

3. FROM THE SECTON CONJECTURE TO THE HOM CONJECTURE

If $X$ is anabelian, we expect the functor

$$X(T) \to \Pi/X/k(T)$$

to be an equivalence for a much larger class than finitely generated extensions of $k$. At least, we should have smooth schemes: we actually show that normality plus a finiteness condition on local rings is enough.

Recall that a $k$-algebra is essentially of finite type if it is the localization of a $k$-algebra of finite type.

**Definition 3.1.** Let $k$ be a field. A $k$-scheme $T$ is left over $k$ (short for locally essentially of finite type) if $O_{T,p}$ is essentially of finite type over $k$ for every $p \in T$.

**Remark 3.2.** This condition on local rings may seem strange at first glance, but it is really everything that we need: there is no need of conditions on open neighbourhoods. Observe that this definition is somewhat similar to Mochizuki’s smooth pro-varieties [Moc99, Definition 16.4]. Imposing that the local rings are essentially of finite type ensures both the fact that residue fields are finitely generated over $k$ and that local rings are noetherian. Being left is a quite general finiteness condition: it contains schemes locally of finite type, finitely generated extensions of $k$, curves with an arbitrary set of closed points removed. For example, 

$$\lim_{n} A_{k}^{1} \setminus \{1, \ldots, n\} = \text{Spec} k \left[ x, \frac{1}{x + 1}, \frac{1}{x + 2}, \ldots \right]$$

is left.

**Theorem 3.3** (Pointwise criterion). Let $X$ be a DM stack and $T$ an integral, normal left scheme over $k$. If $X$ is fff, then $X(T) \to \Pi/X(T)$ is fully faithful. If $X$ is anabelian, then $X(T) \to \Pi/X(T)$ is an equivalence of categories.

**Proof.** **Full faithfulness.** Let $t_1, t_2 : T \to X$ be two morphisms, $\pi(t_1), \pi(t_2)$ their images in $\Pi/X(k)(T)$ and $(t_1, t_2) \in X \times X(T)$. Then $\text{Isom}_X(t_1, t_2)$ is proper, unramified and hence finite over $T$ (because $X$ is separated and DM, hence it has proper and unramified diagonal), while $\text{Isom}_{\Pi/X/k}(\pi(t_1), \pi(t_2))$ is pro-étale over $T$ (because $\Pi/X/k$ is a pro-étale gerbe, hence it has pro-étale diagonal).

Since $\text{Isom}_X(t_1, t_2) \to T$ is finite, $\text{Isom}_{\Pi/X/k}(\pi(t_1), \pi(t_2)) \to T$ is pro-étale and $T$ is normal, we have that

$$\text{Isom}_X(t_1, t_2)(T) = \text{Isom}_X(t_1, t_2)(k(T)),$$

$$\text{Isom}_{\Pi/X/k}(\pi(t_1), \pi(t_2))(T) = \text{Isom}_{\Pi/X/k}(\pi(t_1), \pi(t_2))(k(T)),$$
and hence
\[ \text{Isom}_X(t_1, t_2)(T) \cong \text{Isom}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(T) \]
because by hypothesis
\[ \text{Isom}_X(t_1, t_2)(k(T)) \cong \text{Isom}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(k(T)). \]

**Essential surjectivity.** Let \( T \) be an integral, normal left scheme over \( k \) with a morphism \( T \to \Pi_{X/k} \). Thanks to Proposition 2.6, there exists a finite gerbe \( \Phi \) and a representable morphism \( X \to \Phi \). Hence we have an induced morphism \( T \to \Pi_{X/k} \to \Phi \), and by hypothesis we have a generic section \( \text{Spec}(T) \to X \) which induces a section \( \text{Spec}(T) \to X' = X \times_{\Phi} T \). Since \( X \to \Phi \) is representable, \( X' \) is an algebraic space smooth and proper over \( T \), and thus \( Z \subseteq X' \) the closure of \( \text{Spec}(T) \to X' \). Finally, let \( \tilde{X} \) be \( X \times_{\Pi_{X/k}} T \), we also have a generic section \( \text{Spec}(T) \to \tilde{X} \). The situation is illustrated in the following diagram.

\[
\begin{array}{ccc}
\tilde{X} = X \times_{\Pi_{X/k}} T & \xrightarrow{\phi} & Z \\
\downarrow & & \downarrow \\
\text{Spec}(T) & \xrightarrow{\exists ?} & T
\end{array}
\]

If we can show that \( Z \to T \) is an isomorphism, we have a section \( T \to X' \) which by composition gives us a section \( T \to X \) generically isomorphic to the morphism \( \text{Spec}(T) \to X \) we started with. As we have shown in the preceding part about full faithfulness, this implies that \( T \to X \) lifts the initial morphism \( T \to \Pi_{X/k} \).

Observe that the fact that \( Z \to T \) is an isomorphism makes sense, but is tricky. In fact, if the thesis is true we have a section \( T \to X \to \Pi_{X/k} \), thus \( Z \cong T \). But how do we prove this? On one hand, \( X' \) is an algebraic space smooth and proper over \( T \), but that’s almost everything we know about it, and it is clearly not enough. On the other hand, \( \tilde{X} \to T \) induces an equivalence \( \tilde{X}(k') \to T(k') \) for every finitely generated extension \( k'/k \) (because it is a base change of \( X \to \Pi_{X/k} \)), but \( \tilde{X} \to T \) is not even of finite type. In order to prove the claim, we have to find a way to use the information on both \( X' \) and \( \tilde{X} \).

Since \( X' \to T \) is proper, \( Z \to T \) is surjective (\( Z \) is closed and its image contains the generic point of \( T \)), we want to show that it is injective too. Since we have a section \( \text{Spec}(T) \to Z \), the image of the generic point of \( T \) is closed in the generic fiber \( X'_{k(T') \to T} \) and hence there is only one point in the fiber of \( Z \to T \). Now take any point \( t \in T \) different from the generic one, we want to show that the fiber \( Z_t \) has only one point. We already know that \( Z \to T \) is surjective, take any point \( z \) in the fiber \( Z_t \).

**Claim:** any \( z \in Z_t \) is the image through \( \tilde{X} \to X' \) of the unique lifting \( \text{Spec}(k(t)) \to \tilde{X} \) of \( \text{Spec}(k(t)) \to T \), hence the fiber \( Z_t \) has only one point. Let us show why. We can find a germ of a curve on \( Z \) passing through \( z \) and such that the image in \( T \) is non-constant: we can do this by using Chow’s lemma [Stacks, Tag 088U] and Bertini’s theorem. Observe that in order to use Chow’s lemma we would need \( T \) to be noetherian, but this may be false under our hypotheses: we can bypass this problem by reasoning on \( \text{Spec} \mathcal{O}_{T,t} \) rather than on \( T \), since \( \mathcal{O}_{T,t} \) is noetherian (it is essentially of finite type by hypothesis).
Hence, we have a DVR $R$ which is the localization of a $k$-algebra of finite type and a morphism $\text{Spec} R \to Z$ (not necessarily an embedding) such that the closed point of $R$ maps to $z$ and the composition $\text{Spec} R \to T$ is not constant, i.e. the open point of $\text{Spec} R$ does not map to $t$, call $t_0$ the image of the open point. By hypothesis, the morphism $\text{Spec} k(R) \to \text{Spec} k(t_0) \to T$ lifts to a unique morphism $\text{Spec} k(R) \to \tilde{X}$. But then, thanks to the valuative criterion, we may lift $\text{Spec} R \to T$ to a morphism $\text{Spec} R \to \tilde{X}$. Here we are using the valuative criterion of universal closedness [Stacks, Tag 0A3X]: in order to do it, we don’t need finite type hypotheses, but just the fact that $\tilde{X} \to T$ is universally closed and separated.

This is true, since $\tilde{X} = X \times_{\text{Proj} X/k} T = X \times T$ is representable by integral morphisms of schemes (it is obtained by base change from the diagonal of $\text{Proj} X/k$), and hence both separated and universally closed, while $X \times T \to T$ is proper since it is the base change of $X \to \text{Spec} k$. If one wants to avoid this general valuative criterion, we can also use the fact that $\tilde{X}$ is a projective limit of algebraic spaces of finite type.

Now, recall that $\tilde{X} \to T$ induces an equivalence $\tilde{X}(k') \to T(k')$ for every finitely generated extension $k'/k$. Since the residue field $R/m_R$ of $R$ is finitely generated over $k$, we get that the closed point of $\text{Spec} R/m_R \to \text{Spec} R$ maps to the unique point $\text{Spec} R/m_R \to \text{Spec} R/m_R \to \text{Spec} k(t) \to T$. This means that $z \in Z_t$ is uniquely determined, since it is the image of the composition $\text{Spec} k(t) \to \tilde{X} \to X_t$.

Hence, we know that $Z \to T$ is a 1 : 1 proper map. Since $Z$ is an algebraic space quasi-finite over a scheme, $Z$ is a scheme, too. Moreover, $Z$ is integral by construction (it is the closure of $\text{Spec} k(T) \to X'$), and by Zariski’s main theorem [Stacks, Tag 05K0] $Z \to T$ is a 1 : 1 birational finite morphism. Since $T$ is normal, we get that $Z \to T$ is an isomorphism too.

Actually, we have cheated, since in order to apply Zariski’s main theorem we need $T$ to be quasi-compact and quasi-separated and this is not a consequence of our hypotheses, but this is easily fixed. Cover $T$ by open affine schemes $T_i$, for each $i$ the argument above works since $T_i$ is quasi-compact and quasi-separated, hence we have a section $T_i \to X$ of $T_i \to T \to \text{Proj} X/k$. We already know the fact that $X(T_i \cap T_j) \to \text{Proj} X/k(T_i \cap T_j)$ is an equivalence, hence the gluing data on $T_i \cap T_j \Rightarrow \text{Proj} X/k$ gives us gluing data on $T_i \cap T_j \Rightarrow X$, and thus finally we get a global section $T \to X$.  

**Corollary 3.4.** If hyperbolic curves satisfy the section conjecture, then they satisfy the hom conjecture.

**Proof.** If hyperbolic curves satisfy the section conjecture, then they are anabelian thanks to Lemma 2.4. Hence, they satisfy the hom conjecture thanks to Theorem 3.3. \[□\]

Thanks to Corollary 3.4, we can also see the anabelian conjecture proved by Mochizuki as a particular case of the section conjecture, rather than a different one.

**Corollary 3.5.** For hyperbolic curves, the section conjecture implies the dominant version of the anabelian conjecture (i.e. the restriction of Mochizuki’s theorem [Moc99, Theorem A] to fields finitely generated over Q). \[□\]
Theorem 3.3 allows us to prove easily the following theorem, which is perhaps the least important of this paper. However, we think it has its own interest, since we know no other result of the form “if a variety shows anabelian behaviour, then its fundamental group is far from being abelian”: conjectures and theorems are always in the other direction.

**Theorem 3.6.** Let $X$ be an anabelian DM stack such that $\pi_1(X_{\bar{k}})$ has a finite index abelian subgroup. Then $\dim X = 0$.

**Proof.** Thanks to Proposition 2.6 and Proposition 2.5, up to a finite étale covering we may suppose that $X$ is an algebraic space. Up to another finite étale covering and a finite extension of the base field, we may suppose that $\pi_1(X_{\bar{k}})$ is abelian and $X$ has a rational point $x_0 \in X(k)$. Let $\text{Sm}_k$ be the category of smooth varieties over $k$. Since $X$ is anabelian, thanks to Theorem 3.3 $X$ and $\Pi_{X/k}$ define two naturally equivalent functors $\text{Sm}_k^{\text{op}} \to \text{Set}$ (by taking equivalence classes of $\Pi_{X/k}(T)$ for every $T \in \text{Sm}_k$). The fact that the fundamental group of $X_{\bar{k}}$ is abelian implies that the gerbe $\Pi_{X/k}$ is abelian and hence its functor is enriched in groups with identity $\pi(x_0) \in \Pi_{X/k}(x_0)$, thus the same is true for the functor defined by $X$ and $x_0$.

Now take an étale cover $U \to X$ with $U$ a scheme, and let $R = U \times_X U$. Then, since $U$ and $R$ are smooth varieties, $X(U)$ and $X(R)$ are groups with the structure inherited from $\Pi_{X/k}(U)$ and $\Pi_{X/k}(R)$, this allows us to construct the usual maps $m : X \times X \to X, i : X \to X$ giving the group structure to $X$. Hence, the functor of points of $X$ is enriched in groups over the whole category of schemes over $k$ and not just the smooth ones. This implies that $X$ is not only an algebraic space but also a scheme: the rough idea is that there exists a nonempty open subset which is a scheme, and then we can move it around with the group structure. For an actual proof, see [Art69, Theorem 4.1].

Hence, $X$ is actually a proper group scheme, i.e. an abelian variety. But it is well known that an abelian variety of positive dimension is not anabelian, see for example MathOverflow 92927 where a proof is given for elliptic curves (the proof actually works without modifications for positive dimensional abelian varieties). \(\square\)

4. **From Hyperbolic Curves to Elementary Anabelian DM Stacks**

Recall that a proper, geometrically connected variety $X$ is elementary anabelian if there exists a chain of smooth, proper morphisms

$$X = X_0 \to X_1 \to \cdots \to X_n = \text{Spec } k$$

with $X_i \to X_{i+1}$ either a finite étale morphism or a fibration whose fibers are geometrically connected hyperbolic curves. We want to extend this definition to elementary anabelian DM stacks.

4.1. **Elementary anabelian DM stacks.**

**Definition 4.1.** Let $Y \to X$ be a smooth, proper morphism representable morphism of codimension 1 with geometrically connected fibers of algebraic stacks. Let $D_1, \ldots, D_n \subseteq Y$ be distinct, reduced effective Cartier divisors étale over $X$ and $d_1, \ldots, d_n$ positive integers. Write $D = (D_1, \ldots, D_n), r = (r_1, \ldots, r_n)$. As described in [AGV08, Appendix B.2], we can construct the root stack

$$\sqrt{D/Y}$$
We call a morphism of the form $\sqrt[D]{Y} \to X$ a family of orbicurves.

Let $\sqrt[D]{Y} \to X$ be a family of orbicurves, and suppose that $X$ is connected. Let $g$ be the genus of the fibers of $Y \to X$, and $d_i$ be the degree of $D_i \to X$. Then the fibers of the family are orbicurves of rational Euler characteristic

$$2 - 2g - \sum_{i} \frac{r_i - 1}{r_i} d_i.$$  

The fibers of the family are resp. parabolic, elliptic or hyperbolic if the Euler characteristic is resp. positive, zero or negative.

**Definition 4.2.** The class of elementary anabelian DM stacks over $k$ are smooth, proper geometrically connected DM stacks defined by recursion in the following way.

1. $	ext{Spec} k$ is elementary anabelian.
2. If $Y \to X$ is a family of hyperbolic orbicurves and $X$ is elementary anabelian, then $Y$ is elementary anabelian.
3. If $Y \to X$ is finite, representable and étale, then $X$ is elementary anabelian if and only if $Y$ is elementary anabelian.
4. If $k'/k$ is an extension, then $X$ is elementary anabelian over $k$ if and only if $X_{k'}$ is elementary anabelian over $k'$.

**Remark 4.3.** Despite the name, it is obviously not known that elementary anabelian DM stacks are anabelian (with respect to our definition): this is equivalent to the section conjecture for hyperbolic curves, see Theorem 4.10.

**4.2. Étale homotopy of elementary anabelian DM stacks.** In the analytic context, hyperbolic orbicurves are $K(G,1)$ spaces (they have a covering by an hyperbolic curve, see [Bre18, Proposition 5.1], and these are $K(G,1)$). By using the long exact sequence of a fibration, it is then immediate to check that elementary anabelian DM stacks are $K(G,1)$ spaces, too. We want to show that this is true also for étale homotopy in the sense of Artin and Mazur, i.e. that the higher étale homotopy groups of elementary anabelian DM stacks are trivial. By [AM69, Theorem 6.7], it is enough to check that the topological fundamental group of elementary anabelian DM stacks is *good* in the sense of Serre, see [Ser65, §I.2.6].

Recall that a discrete group $G$ is good if the natural homomorphism

$$H^q(\hat{G},M) \to H^q(G,M)$$

is an isomorphism for every finite $G$-module $M$, where $\hat{G}$ is the profinite completion of $G$. We recall some facts about good groups.

**Facts 4.4.** [Ser65, §I.2.6 Exercises 1, 2]

1. Finite groups and finitely generated free groups are good.
2. If we have an exact sequence

$$1 \to N \to E \to G \to 1$$

with $G$ good and $N$ finitely generated, then

$$1 \to \hat{N} \to \hat{E} \to \hat{G} \to 1$$

is exact.
3. In the situation of the preceding point, if we assume that $N$ is good and and $H^q(N, M)$ is finite for every finite $E$-module $M$, then $E$ is good too.

4. If $M$ is finite and $N$ is either finite or finitely generated and free, $H^q(N, M)$ is finite. If $N$ is obtained by successive extension starting from finite groups and finitely generated free groups, by taking the long exact sequence in cohomology we see that $H^q(N, M)$ is still finite. Hence, thanks to the preceding point, all groups obtained by successive extensions starting from finite groups and finitely generated free groups are good.

**Remark 4.5.** In the following, we will need the long exact sequence of étale homotopy groups of a fibration. The standard reference for this is Friedlander’s paper [Fri73, Corollary 4.8], but unfortunately it covers only fibrations of schemes, not DM stacks. Since this is not the place to generalize Friedlander’s theorem, we use Facts 4.4.2 as a workaround: over $\mathbb{C}$ we can pass to the associated topological orbifold, take exact sequences in topology and then pass to profinite completions using Facts 4.4.2, since our DM stacks have no higher homotopy groups.

**Lemma 4.6.** Fix an embedding of $k$ in $\mathbb{C}$. If $X$ is an elementary anabelian DM stack over $k$, then $X_{\text{an}}$ is of type $K(G, 1)$ and its topological fundamental group is good in the sense of Serre.

*Proof.* Hyperbolic curves are $K(G, 1)$, and their topological fundamental group is obtained by successive extensions from free groups, hence it is good. Thanks to [Bre18, Proposition 5.1], every hyperbolic orbicurve has a finite étale cover which is a curve, hence we get the result for orbicurves too. We conclude by induction on dimension by taking the long exact sequence of a fibration along families of hyperbolic orbicurves. □

**Corollary 4.7.** The étale homotopy type of an elementary anabelian DM stack is of type $K(G, 1)$.

*Proof.* Just apply [AM69, Theorem 6.7] and Lemma 4.6. □

**Lemma 4.8.** If $X$ is an elementary anabelian DM stack and $f : Y \rightarrow X$ is a finite, étale gerbe, then $Y$ is an elementary anabelian DM stack.

*Proof.* Fix an embedding of $k$ in $\mathbb{C}$, since by definition of elementary anabelian DM stacks do not are invariant under base field extension we may suppose $k = \mathbb{C}$. Consider a geometric point $y \in Y(\mathbb{C})$ and its image $x \in X(\mathbb{C})$. The fiber $Y_x$ is a finite étale gerbe of the form $BG$ for some finite group $G$. Passing to the associated topological orbifolds, we may consider the topological homotopy exact sequence

$$1 \rightarrow G \rightarrow \pi_1^{\text{top}}(Y) \rightarrow \pi_1^{\text{top}}(X) \rightarrow 1,$$

where $\pi_2^{\text{top}}(X)$ is 0 by Lemma 4.6. Since $G$ is finite and $\pi_1^{\text{top}}(X)$ is good, we can pass to profinite completions

$$1 \rightarrow G \rightarrow \hat{\pi_1}^{\text{top}}(Y) = \pi_1^{\text{top}}(Y) \rightarrow \hat{\pi_1}^{\text{top}}(X) = \pi_1^{\text{top}}(X) \rightarrow 1.$$

Since $G$ is finite, there exists a connected, finite étale cover $Y' \rightarrow Y$ such that $\pi_1(Y') \cap G = \{1\} \subseteq \pi_1(Y)$. Consider now the composition $Y' \rightarrow Y \rightarrow X$: a priori, it is proper étale, but since $\pi_1(Y') \rightarrow \pi_1(X)$ is injective then we conclude that it is representable too. Hence, we have two finite étale covers $Y' \rightarrow Y$ and...
$Y' \to X$: since $X$ is an elementary anabelian DM stack, $Y'$ and $Y$ are elementary anabelian DM stacks too. □

**Corollary 4.9.** If $X$ is an elementary anabelian DM stack and $f : Y \to X$ is a proper, étale morphism, then $Y$ is an elementary anabelian DM stack.

*Proof.* We work directly on the algebraic closure of $k$. In order to reduce to Lemma 4.8, consider the Stein factorization

$$Y \to \text{Spec} \, f_*O_Y \to X.$$  

We want to show that $\text{Spec} \, f_*O_Y \to X$ is finite étale and $Y \to \text{Spec} \, f_*O_Y$ is a finite étale gerbe.

Up to taking a smooth covering of $X$ ($f_*$ commutes with flat base change), we may suppose that $X$ is a scheme of finite type over $k$. Since $f$ is proper and $X$ is locally of finite type, pushforward of coherent sheaves is coherent, see [Fal03], and hence $\text{Spec} \, f_*O_Y \to X$ is a finite morphism. Moreover, by hypothesis now the automorphism groups of geometric points of $Y$ are finite étale, hence $Y$ is a DM stack. Let $Y \to M$ be the coarse moduli space of $Y$, we have a natural morphism $M \to \text{Spec} \, f_*O_Y$. On the other hand, $M \to X$ is proper and quasi-finite, hence affine, and this gives us a natural morphism in the other direction $\text{Spec} \, f_*O_Y \to M$. These are easily checked to be inverses. In particular, we get that $Y \to \text{Spec} \, f_*O_Y$ is an homeomorphism on points.

Now take a surjective étale cover $U \to Y$ with $U$ a scheme, the composition $U \to X$ is étale. By looking at the composition

$$U \to \text{Spec} \, f_*O_Y \to X,$$

since $Y \to \text{Spec} \, f_*O_Y$ is surjective we get that $\text{Spec} \, f_*O_Y \to X$ is étale.

Finally, we have to show that since $Y$ is étale over its coarse moduli space $M$, then $Y \to M$ is a gerbe. Hence, take a scheme $S$ with a morphism $S \to M$ and two sections $S \to Y$. We have a diagram

$$
\begin{array}{ccc}
Y & \to & Y \\
\downarrow & & \downarrow \Delta \\
S & \to & Y \times_M Y
\end{array}
$$

and we want to find the dotted arrow, étale locally on $S$. But since $Y \to M$ is an étale coarse moduli space, $Y \to Y \times_M Y$ is a surjective étale morphism, hence we can find sections étale locally as desired. □

**4.3. Proof of the implication.**

**Theorem 4.10.** Elementary anabelian DM stacks are fundamentally fully faithful. If proper, hyperbolic curves satisfy the section conjecture, elementary anabelian DM stacks are anabelian too.

*Proof.* We do this by induction checking that full faithfulness and anabelianity are preserved along the elementary operations that define elementary anabelian DM stacks.

Obviously, $\text{Spec} \, k$ is anabelian since $\Pi_{\text{Spec} \, k} = \text{Spec} \, k$. If $Y \to X$ is finite étale, then by Proposition 2.5 $Y$ is anabelian if and only if $X$ is anabelian, and the same is true for full faithfulness thanks to going up and going down theorems [Bre18, Propositions 4.2, 4.6]. Both properties are also preserved along finitely generated...
extensions of the base field since the étale fundamental gerbe behaves well under extension of the base field, see Proposition 5.3. We only have to check that full faithfulness and anabelianity are preserved along families of hyperbolic orbicurves.

Let \( Y \to X \) be a family of hyperbolic orbicurves. Call \( \Pi Y/X \) the fiber product \( X \times_{\Pi X/k} \Pi Y/k \), we have a natural 2-commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & \Pi Y/X \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Pi X/k
\end{array}
\]

Fix a point \( x \in X \), and consider the fiber

\[
\Pi Y/X,x = \Pi Y/X \times_X \text{Spec} k(x) = \Pi Y/k \times_{\Pi X/k} \text{Spec} k(x).
\]

There is a natural map \( Y_x \to \Pi Y/X,x \).

Claim: \( Y_x \to \Pi Y/X,x \) is the étale fundamental gerbe of \( Y_x \). Thanks to Proposition 5.3, we may assume \( k(x) = k = \bar{k} \) is algebraically closed. Fix a base point \( y \in Y_x \). Then, since \( X \) has trivial topological second homotopy group, there is an exact sequence of étale fundamental groups

\[
0 \to \pi_1^{\text{top}}(Y_x,y) \to \pi_1^{\text{top}}(Y,y) \to \pi_1(X,x)^{\text{top}} \to 0.
\]

Since \( \pi_1^{\text{top}}(X,x) \) is good in the sense of Serre thanks to Lemma 4.6, thanks to what we have said in § 4.2 about good groups we may pass to profinite completions, i.e. étale fundamental groups:

\[
0 \to \pi_1(Y_x,y) \to \pi_1(Y,y) \to \pi_1(X,x) \to 0.
\]

Since \( \Pi Y/X,x = \Pi Y/k \times_{\Pi X/k} \text{Spec} k(x) \), there is also a short exact sequence

\[
0 \to \text{Aut}_{\Pi Y/X,x}(y) \to \text{Aut}_{\Pi Y/k}(y) \to \text{Aut}_{\Pi X/k}(x) \to 0,
\]

and there are natural identifications

\[
\pi_1(Y_x,y) = \text{Aut}_{\Pi Y/k}(y), \quad \pi_1(Y,y) = \text{Aut}_{\Pi Y/k}(y), \quad \pi_1(X,x) = \text{Aut}_{\Pi X/k}(x).
\]

These fit in a commutative diagram of short exact sequences, identifying \( Y_x \to \Pi Y/X,x \) with the étale fundamental gerbe \( Y_x \to \Pi Y/k \).

We can make another induction on dimension, hence \( X(k') \to \Pi X/k(k') \) is fully faithful and an equivalence if proper hyperbolic orbicurves satisfy the section conjecture, and the same holds for its base change \( \Pi Y/X \to \Pi Y/k \). These holds for \( Y \to \Pi Y/X \) too, since we can work fiberwise on \( Y_x \to \Pi Y/k \); in fact, thanks to [Bre18, Theorems 6.3, 6.4], since \( Y_x \) is an hyperbolic orbicurve we have that \( Y_x(k') \to \Pi Y_x/k(k') \) is fully faithful and an equivalence if proper, hyperbolic curves satisfy the section conjecture.

Finally, by composition these holds for

\[
Y \to \Pi Y/X \to \Pi Y/k.
\]
5. Appendix. Étale Fundamental Gerbe and Base Change

Suppose that we are in characteristic 0. In [BV15, Proposition 6.1], it is shown that the étale fundamental gerbe behaves well under algebraic field extensions: we want to show that, actually, it behaves well with respect to any field extension. The idea is to rephrase the theorem in terms of étale fundamental groups, and then use the fact that in characteristic 0 the étale fundamental group is invariant along extensions of algebraically closed fields, see [SGA1, Proposition 4.6].

Lemma 5.1. If $H_1, H_2$ are profinite groups and $k'/k$ is an extension of algebraically closed fields, then the natural functor

$$\text{Hom}_k(B_k H_1, B_k H_2) \to \text{Hom}_{k'}(B_{k'} H_1, B_{k'} H_2)$$

is an equivalence.

Proof. Both categories have the same description (which we skip) in purely group theoretic terms. □

Recall that a fibered category $X$ is concentrated if there exists an affine scheme $U$ over $k$ with a morphism $U \to X$ which is representable, faithfully flat, quasi-compact and quasi-separated.

Lemma 5.2. Let $k'/k$ be an extension of algebraically closed fields. Consider $S$ a quasi-compact and quasi-separated scheme over $k$, and $\Phi$ a finite étale stack over $k$. Then the natural functor

$$\text{Hom}_k(S, \Phi) \to \text{Hom}_{k'}(S_{k'}, \Phi_{k'})$$

is an equivalence of categories.

Proof. Let us prove this firstly under the additional hypothesis that $S$ is of finite type over $k$. Under this hypothesis, connected components are open, hence we may suppose that $S$ is connected and $\Phi$ is of the form $BG$ for some finite group $G$. Fix any point $s \in S(k)$. Thanks to [SGA1, Exposé XIII, Proposition 4.6], $\pi_1(S, s) = \pi_1(S_{k'}, s_{k'})$.

We have thus

$$\text{Hom}_k(S, B_k G) = \text{Hom}_k(B_k \pi_1(S, s), B_k G) = \text{Hom}_{k'}(B_{k'} \pi_1(S_{k'}, s_{k'}), B_{k'} G) = \text{Hom}_{k'}(S_{k'}, B_{k'} G).$$

In general, by noetherian approximation [TT90, Theorem C.9] we can write $S$ as an inverse limit $\lim_i S_i$ with $S_i$ of finite type over $k$. Since $\Phi$ is finite,

$$\text{Hom}_k(S, \Phi) = \lim_i \text{Hom}_k(S_i, \Phi) = \lim_i \text{Hom}_{k'}(S_{i, k'}, \Phi_{k'}) = \text{Hom}_{k'}(S_{k'}, \Phi_{k'}).$$

□

Proposition 5.3. Let $k$ be a field of characteristic 0. If $X$ is a geometrically connected, concentrated fibered category over $k$, then the natural map $\Pi_{X_{k'}/k'} \to \Pi_{X/k} \times_k k'$ is an isomorphism for every field extension $k'/k$.

Proof. In [BV15, Proposition 6.1], the result is proved for $k'/k$ algebraic (see also [Bre18, Proposition 8.14]). Hence, it is immediate to reduce to the case in which $k$ and $k'$ are both algebraically closed.

Since $X$ is concentrated, there exists a quasi-compact, quasi-separated scheme $U$ with a morphism $U \to X$ which is representable, faithfully flat, quasi-compact...
and quasi-separated. Call \( R = U \times_X U \), \( R \) is quasi-compact and quasi-separated over \( U \) and hence it is quasi-compact and quasi-separated too. We have to show that \( \Pi_{X/k} \times k' \) has the universal property of the étale fundamental gerbe of \( X \).

Since \( k' \) is algebraically closed, every finite étale stack over \( k' \) has the form \( \bigcup_i B_{k'} G_i \) for some finite number of finite groups \( G_i \). In particular, every finite étale stack over \( k' \) is isomorphic to \( \Phi_{k'} \) for some finite étale stack \( \Phi \) over \( k \), hence it is enough to show that \( \Pi_{X/k} \times k' \) has the universal property with respect to stacks of the form \( \Phi_{k'} \) with \( \Phi \) finite étale over \( k \).

Let \( \text{Hom}(R \Rightarrow U, \Phi) \) be the category of morphism \( U \rightarrow \Phi \) satisfying the usual cocycle condition on \( R \). Descent theory tells us that \( \text{Hom}(R \Rightarrow U, \Phi) \) is naturally equivalent to \( \text{Hom}(X, \Phi) \), even if \( X \) is not a stack and hence \( X \neq [U/R] \). Since \( U \) and \( R \) are quasi-compact and quasi-separated, by Lemma 5.2 we conclude that

\[
\text{Hom}_k(X_{k'}, \Phi_{k'}) = \text{Hom}_k(R_{k'} \Rightarrow U_{k'}, \Phi_{k'}) = \text{Hom}_k(R \Rightarrow U, \Phi) =
\]

\[
\text{Hom}_k(X, \Phi) = \text{Hom}_k(\Pi_{X/k}, \Phi) = \text{Hom}_k(\Pi_{X/k} \times k', \Phi_{k'}). 
\]

The last equivalence is justified by the fact that \( \Pi_{X/k} \) is concentrated, too, and hence we can repeat the argument that we have used for \( X \). To check that \( \Pi_{X/k} \) is concentrated, just observe that any section \( \text{Spec} L \rightarrow \Pi_{X/k} \) with \( L \) a field is representable, faithfully flat, quasi-compact and quasi-separated.

\[\square\]

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