SELBERG’S CENTRAL LIMIT THEOREM FOR DIRICHLET L-FUNCTIONS
OF q-ASPECT

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Abstract. We give a simple proof of Selberg’s central limit theorem for Dirichlet L-functions for q-aspect by following the method established by Radziwiłł and Soundararajan.

1. Introduction

Selberg observed that [4,5] the logarithm of the Riemann zeta function behaves like a Gaussian random variable with mean 0 and variance \( \frac{1}{2} \log \log |t| \). Later in 2017, Radziwiłł and Soundararajan [1] gave a new and elegant proof of this result known as Selberg’s central limit theorem. Later P. Hsu and P. Wong [3] proved this result for the Dirichlet L-functions of level aspect by following the method of [1]. We have proved [2, 6] this result for automorphic L-functions of level aspect for L-functions of degree 2 and 3 associated with the primitive holomorphic cusp forms and Hecke-Maass cusp forms respectively.

In this paper, we give a simple proof of Selberg’s central limit theorem for the Dirichlet L-function of q-aspect by using the technique of [1]. Let \( L(s, \chi) \) be a Dirichlet L-function given by

\[
L(s, \chi) = \sum_n \frac{\chi(n)}{n^s}
\]

where \( \chi \) is a primitive Dirichlet character modulo \( q \), for \( \Re(s) > 1 \).

Let \( q \) be the conductor of \( L(s, \chi) \). Since the subconvexity bound [7] for the Dirichlet L-functions has already been proven we can get all the necessary tools we need to prove our result. We need to keep in mind that we will be handling the character sums so orthogonality relations of the Dirichlet character is a very useful tool here.

Theorem 1. Let \( V \) be a fixed real number. Let \( \chi \) be a primitive Dirichlet character modulo \( q \) and \( q \asymp Q \). Then for all \( t \),

\[
\frac{1}{\phi(q)} \text{meas} \left\{ \chi(\mod q) : \log |L(\frac{1}{2} + it, \chi)| \geq V \sqrt{\frac{1}{2} \log \log Q} \right\} \sim \frac{1}{\sqrt{2\pi}} \int_V^\infty e^{-u^2/2} du.
\]

We start with the setup of the proof then we will get into the details. Notice that we are proving this result for the q-aspect so in this case \( q \asymp Q \) and \( t \) is fixed. In the first step, let us take away the problem from the critical line by proving that for a suitable choice of \( \sigma > \frac{1}{2} \) \( L(\frac{1}{2} + it, \chi) \) and \( L(\sigma + it, \chi) \) are typically close to each other.

Date: May 2021.
2010 Mathematics Subject Classification. 11M06.
Proposition 1. Let $q$ be the conductor and $q \asymp Q$. Then for any $\sigma > \frac{1}{2}$ we have

$$\sum_{\chi(\text{mod } q)} \left| \log |L(\frac{1}{2} + it, \chi)| - \log |L(\sigma + it, \chi)| \right| \ll \left( \sigma - \frac{1}{2} \right) \log Q.$$ 

Fix the parameters

$$W = (\log \log \log q)^4, \quad X = q^{(\log \log q)^2}, \quad Y = q^{(\log \log q)^2}, \quad \sigma_0 = \frac{1}{2} + \frac{W}{\log q},$$

where $q \asymp Q$ and sufficiently large so $W \geq 3$.

In the next step we consider the auxiliary series given by

$$\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; X) = \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi(n)}{n^s \log n},$$

where $\Lambda(n)$ is the Von-Mangoldt function. We determine the distribution of the auxiliary series $\mathcal{P}(f, s)$ by computing its moments.

Proposition 2. As $q \asymp Q$ the distribution of $\Re(\mathcal{P}(\sigma_0 + it))$ is approximately normal with mean 0 and variance $\frac{1}{2} \log \log Q$.

It is now obvious that if we can connect the $L$-function with the auxiliary series $\mathcal{P}(s, \chi)$ that will prove Theorem 1.

To establish this connection we use the mollification technique. Consider the Dirichlet polynomial $M(s, \chi)$ given by

$$M(s, \chi) = \sum_n \frac{\mu(n)a(n)\chi(n)}{n^s}$$

where $a(n)$ is given by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is composed only primes below } X \text{ and has at most } 100 \log \log Q \text{ primes below } Y, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $a(n)$ it takes the value 0 except when $n \leq Y^{100 \log \log q}X^{100 \log \log \log q} < Q^\epsilon$. It is now evident that $M(s, \chi)$ is short Dirichlet polynomial. In the next step, we establish the connection between $M(s, \chi)$ and $\mathcal{P}(s, \chi)$.

Proposition 3. For the primitive Dirichlet character $\chi(\text{mod } q)$

$$M(\sigma_0 + it) = (1 + o(1)) \exp(-\mathcal{P}(\sigma_0 + it)).$$

It remains to prove that the mollifier and the $L$-function are inverse to each other.

Proposition 4. Let $\chi$ be a primitive Dirichlet character modulo $q$.

$$\sum_{\chi(\text{mod } q)} |1 - L(\sigma_0 + it, \chi)M(\sigma_0 + it, \chi)|^2 = o(1)$$

so as $q \asymp Q$ we have

$$L(\sigma_0 + it, \chi)M(\sigma_0 + it, \chi) = 1 + o(1).$$
Now we prove our main theorem. We prove the propositions in later sections.

**Proof of Theorem 1.** Recalling Proposition 4, it typically says that \( L(\sigma_0 + it, \chi) \approx M(\sigma_0 + it, \chi)^{-1} \). By Proposition 3 we know that \( M(\sigma_0 + it, \chi) \) is normally distributed with mean 0 and variance \( \frac{1}{2} \log \log Q \). Finally with the help of Proposition 1 we deduce that \( \log |L(\sigma_0 + it, \chi)| \) and \( \log |L(\frac{1}{2} + it, \chi)| \) have the same distribution if we set \( \sigma = \sigma_0 \) under the estimate \( W^2 = o(\sqrt{\log \log q}) \). This completes the proof. \( \square \)

2. **Proof of Proposition 1**

Let \( \chi \) be the primitive Dirichlet character and \( q \) be the conductor with \( q \approx Q \). We set

\[
G(s, \chi) = \left( \frac{\pi}{q} \right)^{-(s+a)/2} \Gamma \left( \frac{s+a}{2} \right)
\]

where

\[
a = a(\chi) = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
1 & \text{if } \chi(-1) = -1.
\end{cases}
\]

Consider the Stirling approximation \( \Gamma(s+it) \sim e^{\frac{1}{2} \pi |t|} |t|^{s-\frac{1}{2}} \sqrt{2\pi} \).}

\[
\log \left| \frac{G(s+it, \chi)}{G(\frac{1}{2}+it, \chi)} \right| = \log \left( \frac{\left( \frac{\pi}{q} \right)^{-(s+it+a)/2} \Gamma \left( \frac{s+it+a}{2} \right)}{\left( \frac{\pi}{q} \right)^{-(\frac{1}{2}+it+a)/2} \Gamma \left( \frac{\frac{1}{2}+it+a}{2} \right)} \right)
\]

\[
\ll \left( \sigma - \frac{1}{2} \right) \log q.
\]

Note that we are working with the \( q \)-aspect here so we can fix \( t \) and ignore the term inside the logarithm. By putting the Stirling’s approximation we conclude the above equation. To prove proposition consider the complete Dirichlet \( L \)-function \( \xi(s, \chi) = G(s, \chi)L(s, \chi) \). Recall Hadamard’s factorization formula, there exist constant \( A \) and \( B \) with \( \Re(B(\chi)) = -\sum_{\rho \in Z_{\chi}} \frac{1}{\rho} \) such that

\[
\xi(s, \chi) = e^{A + Bs} \prod_{1 - \frac{1}{2} e^{s/\rho}}
\]

where we denote the set of non-trivial zeros by \( Z_{\chi} \). Thus we have

\[
\log \left| \frac{\xi(\frac{1}{2} + it, \chi)}{\xi(s + it, \chi)} \right| = \sum_{\rho \in Z_{\chi}} \log \left| \frac{\frac{1}{2} + it - \rho}{s + it - \rho} \right|
\]

which implies that
Suppose $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$, if $|t - \gamma| \geq 2$ we have (as argued in [1])

$$
\sum_{\rho \in \mathbb{Z}_\chi} \log \left| \frac{\xi \left( \frac{1}{2} + it, \chi \right)}{\xi \left( \sigma + it, \chi \right)} \right| \ll \frac{\sigma - \frac{1}{2}}{1 + (t - \gamma)^2}.
$$

Combine this with equation (2) and the fact that the number of zeros of $L(s, \chi)$ in the box $k \leq |t - \gamma| \leq k + 1$ concludes the proposition. \qed

3. Proof of Proposition 2

We prove this proposition by restricting the sum to primes and then compute moments. We can restrict the sums to primes because for the primes $p$, $p^k$ terms contribute $O(1)$ for $k \geq 3$. For the terms involving $p^2$ we can ignore them as argued in [3].

Consider the auxiliary series

$$
P_0(\sigma_0 + it) = P_0(\sigma_0 + it; \chi; X) = \sum_{p \leq X} \frac{\chi(p)}{p^{\sigma_0 + it}}
$$

We study the moments by obtaining a mean value estimate to prove Proposition 2.

Lemma 1. Suppose that $k$ and $l$ are non-negative integers with $X^{k+l} \ll q \ll Q$. Then if $k \neq l$

$$
\sum_{\chi( \mod q)} |P_0(\sigma_0 + it)|^k \frac{\overline{P_0(\sigma_0 + it)}}{k^{1+\epsilon}} \ll \phi(q)Q
$$

If $k = l$, we have

$$
\sum_{\chi( \mod q)} |P_0(\sigma_0 + it)|^{2k} = k!\phi(q)(\log \log Q)^k + O_k(\phi(q)(\log \log Q)^{k-1+\epsilon})
$$

Proof. Write $P_0(s)^k = \sum_n \frac{a_k(n)\chi(n)}{n^s}$ where

$$
a_k(n) = \begin{cases} 
\frac{k!}{\alpha_1! \cdots \alpha_r!} & \text{if } n = \prod_{j=1}^{r} p_j^{\alpha_j}, p_1 < \ldots < p_r < X, \sum_{j=1}^{r} \alpha_j = k. \\
0 & \text{otherwise}.
\end{cases}
$$

Therefore,
We conclude the first part of the lemma.

For $k = l$ we have the diagonal term

$$\sum_{\chi(\mod q)} |P_0(\sigma + it)|^{2k} dt = \sum_{\chi(\mod q)} \sum_{n} \frac{a_k(n)^2 |\chi(n)|^2}{n^{2\sigma_0}} + O \left( \sum_{\chi(\mod q)} \sum_{n \neq m, (n, q) = 1} \frac{a_k(n)a_k(m)\chi(n)\chi(m)}{(mn)^{\sigma_0}} \right)$$

For $n$ not being square free the diagonal term contribute

$$k! \sum_{p_1, \ldots, p_k \leq X, (p_1 \cdots p_k)^{2\sigma_0} = n} \frac{\chi(p)^2}{(p_1 \cdots p_k)^{2\sigma_0}} = k! \left( \sum_{p \leq X} \frac{1}{p^{2\sigma_0}} \right)^k = k!(\log \log q)^k + O_k((\log \log q)^k-1)$$

Recalling the definition of $X$ we conclude the proof. \qed

Proof of Proposition 4 \ By Lemma 1 for any odd $k$ we have

$$\sum_{\chi(\mod q)} (\Re(P_0(\sigma_0 + it)))^k = \frac{1}{2^k} \sum_{\chi(\mod q)} \left( P_0(\sigma_0 + it) + P_0(\sigma_0 + it) \right)^k$$

$$= \frac{1}{2^k} \sum_{l=0}^k \binom{k}{l} \sum_{\chi(\mod q)} (P_0(\sigma_0 + it)P_0(\sigma_0 + it))^{k-l} \ll \phi(q)Q$$
Observe that it is impossible to have $l = k = l$ for any odd $k$, we have for all even $k, l = k - l = k/2$ and again with the help of Lemma 1 we obtain,

$$
\frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} \left(\Re(P_0(\sigma_0 + it))\right)^k = \frac{1}{2^k} \left(\frac{k}{2}\right) \left(\frac{k}{2}\right)! \log \log Q + O\left((\log \log Q)^{\frac{k}{2} - 1 + \epsilon}\right)
$$

The above equation matches with the moments of Gaussian random variable with mean 0 and variance $\frac{1}{2} \log \log Q$.

4. Proof of Proposition 3

As we have decomposed $\mathcal{P}(s)$ in [2] we are going to use the same technique here. So we have

$$
\mathcal{P}_1(s, \chi) = \sum_{2 \leq n \leq Y} \frac{\Lambda(n)\chi(n)}{n^s \log n}
$$

$$
\mathcal{P}_2(s, \chi) = \sum_{Y < n \leq X} \frac{\Lambda(n)\chi(n)}{n^s \log n}
$$

We further set

$$
\mathcal{M}_1(s, \chi) = \sum_{0 \leq k \leq 100 \log \log Q} \frac{(-1)^k}{k!} \mathcal{P}_1(s)^k
$$

$$
\mathcal{M}_2(s, \chi) = \sum_{0 \leq k \leq 100 \log \log Q} \frac{(-1)^k}{k!} \mathcal{P}_1(s)^k
$$

Lemma 2. For the primitive Dirichlet character $\chi$ modulo $q$

$$
|\mathcal{P}_1(\sigma_0 + it)| \leq \log \log Q
\quad (3)

|\mathcal{P}_2(\sigma_0 + it)| \leq \log \log Q
\quad (3)

Moreover,

$$
\mathcal{M}_1(\sigma_0 + it) = \exp(-\mathcal{P}_1(\sigma_0 + it))(1 + O(\log Q)^{-99})
\quad (4)

\mathcal{M}_2(\sigma_0 + it) = \exp(-\mathcal{P}_2(\sigma_0 + it))(1 + O(\log \log Q)^{-99})
\quad (4)

Proof. For the first assertion of the lemma,

$$
\sum_{\chi(\text{mod } q)} |\mathcal{P}_1(\sigma_0 + it)|^2
$$

$$
= \sum_{\chi(\text{mod } q)} \left( \sum_{2 \leq n_1 = n_2 \leq Y} \frac{\Lambda(n_1)\Lambda(n_2)\chi(n_1)\chi(n_2)}{(n_1n_2)^s \log n_1 \log n_2} \right) + O \left( \sum_{\chi(\text{mod } q)} \sum_{2 \leq n_1 \neq n_2 \leq Y} \frac{\Lambda(n_1)\Lambda(n_2)\chi(n_1)\chi(n_2)}{(n_1n_2)^s \log n_1 \log n_2} \right)
$$

$$
= \sum_{\chi(\text{mod } q)} \left( \sum_{2 \leq n_1 = n_2 \leq Y} \frac{\Lambda(n_1)\Lambda(n_2)}{(n_1n_2)^s \log n_1 \log n_2} \right) + O \left( \phi(q) \sum_{2 \leq n_1 \neq n_2 \leq Y} \frac{\Lambda(n_1)\Lambda(n_2)}{(n_1n_2)^s \log n_1 \log n_2} \right)
\ll \phi(q) \log \log Q
$$
Since we know that \( q \asymp Q \) by recalling the definition of \( Y \) we conclude the above equation. Similarly, we have

\[
\sum_{\chi(\text{mod } q)} |\mathcal{P}_2(\sigma_0 + it)|^2 \ll \phi(q) \log \log Q
\]

and the first part of the lemma follows. For the second part we simply apply the Stirling formula.

If \( |z| \leq k \) then by Stirling formula

\[
\left| e^z - \sum_{0 \leq k \leq 100k} \frac{z^k}{k!} \right| \leq e^{-99k}
\]

For the second part we simply apply the Stirling formula by taking \( z = -\mathcal{P}_1(\sigma_0 + it) \) and \( k = \log \log q \) and \( \Box \) holds.

Now it remains to establish the connection between \( M(s, \chi) \) and \( \mathcal{P}(s, \chi) \). To do so in a similar way we decompose \( M(s, \chi) \) as well. But if we observe the definition of \( M(s, \chi) \) we see that we need to decompose \( a(n) \) first.

\[
a_1(n) = \begin{cases} 
1 & \text{if } n \text{ has at most } 100 \log \log q \text{ prime factors with all } p \leq Y \\
0 & \text{otherwise}.
\end{cases}
\]

\[
a_2(n) = \begin{cases} 
1 & \text{if } n \text{ has at most } 100 \log \log \log q \text{ prime factors with all } Y < p \leq X \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore we have

\[
M(s, \chi) = M_1(s)M_2(s)
\]

\[
M_1(s, \chi) = \sum_n \frac{\mu(n)a_1(n)\chi(n)}{n^s}
\]

\[
M_2(s, \chi) = \sum_n \frac{\mu(n)a_2(n)\chi(n)}{n^s}
\]

**Lemma 3.** For the primitive Dirichlet character \( \chi \) modulo \( q \) we have

\[
\sum_{\chi(\text{mod } q)} |\mathcal{M}_1(\sigma_0 + it) - \mathcal{M}_1(\sigma_0 + it)|^2 \ll \phi(q)(\log q)^{-60}
\]

\[
\sum_{\chi(\text{mod } q)} |\mathcal{M}_2(\sigma_0 + it) - \mathcal{M}_2(\sigma_0 + it)|^2 \ll \phi(q)(\log \log q)^{-60}
\]

**Proof.** First we define

\[
\mathcal{M}_1(s, \chi) = \sum_n \frac{b(n)\chi(n)}{n^s}
\]

where \( b(n) \) satisfies the following properties.

1. \( |b(n)| \leq 1 \) for all \( n \).
2. \( b(n) = 0 \) unless \( n \leq Y^{100 \log \log Q} \) has only prime factors below \( Y \).
3. \( b(n) = \mu(n)a_1(n) \) unless \( \Omega(n) > 100 \log \log Q \) or, \( p \leq Y \) s.t. \( p^k | n \) with \( p^k > Y \).
Set \( c(n) = (b(n) - \mu(n)a_1(n))\chi(n) \), then we have
\[
\sum_{\chi \equiv \chi(c(n))} |M_1(\sigma_0 + it) - M_1(\sigma_0 + it)|^2
\leq \phi(q) \sum_{n_1 = n_2} \left| \frac{c(n_1)c(n_2)}{(n_1n_2)^{\sigma_0}} \right| + \sum_{n_1 \neq n_2} \left| \frac{c(n_1)c(n_2)}{(n_1n_2)^{\sigma_0}} \right|
\leq \phi(q) \left( \log^2 q \right)
\]

Since we are dealing with the \( q \)-aspect here for the Dirichlet \( L \)-functions we need to normalize with \( \phi(q) \) because of the orthogonality relation of the Dirichlet characters.

Proof of Proposition 3: It follows from (4) that
\[
M_1(\sigma_0 + it) = \exp(-P_1(\sigma_0 + it))(1 + O(\log Q)^{-99})
\]
and by (3) we can write
\[
(\log Q)^{-1} \leq |M_1(\sigma_0 + it)| \leq \log Q.
\]

Combining these two equations we get
\[
M_1(\sigma_0 + it) = M_1(\sigma_0 + it) + O((\log Q)^{-25})
= \exp(-P_1(\sigma_0 + it))(1 + O(\log Q)^{-20})
\]

Similarly, we have
\[
M_2(\sigma_0 + it) = M_2(\sigma_0 + it) + O((\log \log Q)^{-25})
= \exp(-P_2(\sigma_0 + it))(1 + O(\log \log Q)^{-20})
\]

Recall the decomposition of \( M(f, s) \) and \( P(f, s) \), by multiplying these estimates we obtain
\[
M(\sigma_0 + it) = \exp(-P(\sigma_0 + it))(1 + O(\log \log Q)^{-20})
\]
completes the proof of the proposition. \( \square \)

5. Proof of Proposition 4

We do not follow the same method established in [1] to prove this proposition. The proof of this proposition relies on [8].

Consider the approximate functional equation for Dirichlet \( L \)-function [9], we have
\[
L(\sigma_0 + it, \chi) = \sum_{\chi \equiv \chi(n) \leq q} \frac{\chi(n)}{n^{\sigma_0 + it}} + O(q^{-\frac{1}{2}})
\]

Then
\[
\sum_{\chi \equiv \chi(n) \leq q} \sum_{1 \leq m \leq q'} \frac{\mu(m)a(m)\chi(m)}{m^{\sigma_0 + it}} + O(q^{\frac{1}{2} + \epsilon})
\]

From the orthogonality relation of the Dirichlet characters and by the definition of \( a(n) \) the above equation is \( \ll \phi(q) \log q + O(q^{\frac{1}{2} + \epsilon}) \).
Expanding (1) we get
\[ \sum_{\chi \pmod{q}} |1 - L(\sigma_0 + it, \chi)M(\sigma_0 + it, \chi)|^2 \]
\begin{equation}
\sim \sum_{\chi \pmod{q}} |L(\sigma_0 + it, \chi)M(\sigma_0 + it, \chi)|^2 - \phi(q) \log q + O(q^{1/2 + \epsilon}) + o(1)
\end{equation}

In order to prove Proposition 4, we have to prove that the first term of (5) is \( \sim \phi(q) \log q \). We prove this using the argument given in [8]. Consider the integral
\[ I_\chi = \int |L(s, \chi)M(s, \chi)|^2 \Phi(t) dt \]
where \( \Phi(t) \) is smooth, \( \Phi(t) \geq 0 \) with
\[ \hat{\Phi}(1) = \int_{-\infty}^{\infty} \Phi(t) dt > 0 \]
and
\[ (1 + |t|)^j \Phi^{(j)}(t) \ll \left( 1 + \frac{|t|}{T} \right)^{-A} \]
for any \( j \geq 0 \) and any \( A \geq 0 \) the implied constant depending on \( j \) and \( A \).

Averaging over the characters as argued in [8] we have
\[ \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} I_\chi \sim c \hat{\Phi}(1) \]
As we know that there exist no perfect mollifier (in which case we have \( c = 1 \)) and \( \hat{\Phi}(1) \propto T \) where \( (\log q)^6 \leq T \leq (\log q)^A \), we can conclude that the first term of (5) is \( \sim \phi(q) \log q \), completes the proof. \( \square \)

**Remark.** In [3], the authors have set \( G(s, \chi) = L(s, \chi) + \lambda L'(s, \chi) \) for \( \lambda = 1/r \log q \) and put \( G(s, \chi) \) in the integration \( I_\chi \). Due to the mollifier we have set in our paper we can put \( L(s, \chi) \) insted of \( G(s, \chi) \) in the integration \( I_\chi \).

### 6. Conclusion

In this paper, we have proved Selberg’s central limit theorem for Dirichlet \( L \)-functions of \( q \)-aspect. Since we have the sub-convexity bound for Dirichlet \( L \)-functions of \( q \)-aspect, we have all the necessary tools we need to prove this result. Notice that to prove our main result, we heavily rely on the orthogonality relation of the Dirichlet character. The proof for \( q \)-aspect is almost similar to the proof of level aspect but we have proved the last step (or Proposition 4) differently.

One can prove this result for quadratic Dirichlet \( L \)-functions twisted by the primitive Dirichlet character \( \chi \) for \( q \)-aspect. It is the most generalized (as well as difficult) possible case to prove Selberg’s central limit theorem for \( GL(3) \) \( L \)-functions of \( q \)-aspect. We don’t have much information on the shifted convolution problem for higher degree \( L \)-functions, so the proof for these cases can not be seen easily.

Moreover, one can prove the independence of the Dirichlet \( L \)-functions for \( q \)-aspect associated with distinct primitive Dirichlet characters. The proof technique is similar to given in [3, 4].
Acknowledgement

I would like to express my gratitude to my primary supervisor, Dr Gihan Marasingha, who guided me throughout this project.

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