We formulate soliton equations on a lattice in terms of the reduced Moyal algebra that includes one parameter. The limit in which parameter vanishes leads to continuous soliton equations.

It is well known that the soliton equations can be expressed as a zero-curvature equation emerging as a compatibility condition of the scattering problem equation and the time evolution equation in the inverse scattering problem. The potentials are \( sl(N, C) \) valued. Many of the familiar soliton equations fall into the \( N = 2 \) category. In this paper we consider a formulation of the soliton equation by using the Moyal algebra \(^1\) which is identical to \( su(N) \) algebra for some value of parameter it contains. \(^2\)–\(^4\) In order to realize this, we replace the commutation relation in the zero-curvature equation by the Moyal algebra. This introduces two new variables into the equations. We assume that the potentials are expanded in powers of \( e^p \), where the variable \( p \) is one of the newly introduced variables. Specification of the expansion determines the type of soliton equation. Substituting the potentials into the zero-curvature equation, we obtain various soliton equations. Taking out coefficients at each order of the power of \( e^p \) in the zero-curvature equation, we obtain soliton equations on a lattice, one of which is the well-known Toda lattice equation. In general, the equations thus obtained include one parameter originating from the Moyal algebra. It is interesting to note that the parameter of the Moyal bracket has the physical meaning in the soliton equation as the spacing between particles on the lattice. \(^5\) We then naturally obtain the continuous correspondence of the discrete soliton equations by taking the limit in which this parameter vanishes.

We first recapitulate the notation of the Moyal algebra following Strachan. \(^6\)

Define the star product by

\[
    f \star g = \exp \left[ \kappa \epsilon^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} \right] f(x)g(\tilde{x}) |_{x=\tilde{x}},
\]

and the brackets by

\[
    \{ f, g \}_\pm = \frac{f \star g \pm g \star f}{\kappa},
\]

where \( x = (x^0, x^1) \) and \( \tilde{x} = (\tilde{x}^0, \tilde{x}^1) \). Here the “−” bracket is the Moyal bracket. We also introduce the Hirota operator \( D_x \) by

\[
    D_x f \bullet g = \frac{df}{dx} g - f \frac{dg}{dx}.
\]

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We have not yet assumed any dependence of the functions in the Moyal algebra. Here we assume that these functions are products of a function of $x$ and the power of $e^p$. It is interesting to note that the star product for those functions is related to the Hirota operator according to

$$ [e^p f(x)] \star [e^p f(x)] = e^{2p} \exp(\kappa D_x) f \cdot \tilde{f}. \quad (4) $$

In more general cases with other $e^p$ dependences, we have

$$ [e^{mp} f(x)] \star [e^{np} g(x)] = \exp\{ (m+n)p \} \left[ \exp \left( n\kappa \frac{d}{dx} f(x) \right) \right] \left[ \exp \left( -m\kappa \frac{d}{dx} g(x) \right) \right] = \exp\{ (m+n)p \} f(x+np)g(x-mn), \quad (5) $$

where $m$ and $n$ are integers. The brackets are explicitly obtained as

$$ \{ e^{mp} f, e^{np} g \}_\pm = \frac{1}{\kappa} \exp\{ (m+n)p \} \left[ f(x+np)g(x-mn) \pm g(x+mn)f(x-nm) \right]. \quad (6) $$

We can naturally obtain the $\kappa \to \infty$ limit of the brackets:

$$ \lim_{\kappa \to 0} \{ e^{mp} f, e^{np} g \}_- = 2 \left( \frac{df}{dx} g - m \frac{dg}{dx} f \right) \exp\{ (m+n)p \}, \quad (7) $$

$$ \lim_{\kappa \to 0} \kappa \{ e^{mp} f, e^{np} g \}_+ = 2fg. \quad (8) $$

We next consider the soliton equation. Of the various formulations of soliton theory, we consider the inverse scattering method, where the soliton equation is obtained as a compatibility condition of the scattering problem equation and the time evolution equation. The soliton equation has geometrical meaning as the zero-curvature equation:

$$ \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + \{ A_\mu, A_\nu \} = 0, \quad (9) $$

where the potentials $A_\mu = A_\mu(x_0, x_1)(\mu = 0, 1)$ are $sl(N, C)$ valued fields and include a spectral parameter in the soliton theory. The expansion of potentials in terms of the spectral parameter produces the equations at each order of the parameter, which lead to various soliton equations.

We discuss the Moyal version of the above equation. Our discussion is based on the fact that the Moyal algebra and the $su(N)$ algebra are identical with some assumption regarding the parameter in the Moyal algebra. This means that we can replace the commutation relation by the Moyal algebra. The equation reads

$$ \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + \{ A_\mu, A_\nu \} = 0. \quad (10) $$

In general, the replacement of the commutation relation by the Moyal bracket introduces two new variables, say, $x$ and $p$, and thus transfers the original equation to the 2 dimensional higher equation in compensation of the loss of matrix form. Here we
assume that each function appearing in the above equation (10) can be written as the product of function of the induced variable $x$ and a function of another induced variable $p$ as
\begin{align}
A_0(x_0, x_1; x, p) &= \sum_k e^{kp}a_k(x_0, x_1; x), \\
A_1(x_0, x_1; x, p) &= \sum_l e^{lp}b_l(x_0, x_1; x).
\end{align}

Here $k$ and $l$ run over positive and/or non-positive integers whose choice determines the soliton equation. The specification of the range of $k$ and $l$ together with the coefficients leads to various soliton equations. The expansions are to be compared with the inverse scattering method, where the potentials are expanded in terms of the spectral parameter. Here $e^p$ plays the role of the spectral parameter. The zero-curvature equation (10) implies a set of equations taken out at each order of the power of $e^p$. Therefore, in the transition to the Moyal version of the soliton equation, we do not add two dimensions but, rather, one dimension to the original equation.

In a previous paper,\(^5\) we obtained an algebra composed of brackets between operators which are the exponentiation of the operators satisfying conformal algebra. The brackets are found to be identical to those derived from the Moyal algebra by the reduction assuming that functions of $x$ and $p$ are expanded in $e^0$ and $e^{\pm p}$. The restriction of functions to such $e^p$ dependences implies the restriction on the potentials that they are allowed to take values only on the Cartan and simple algebra. The following examples (i) and (ii) are of this type. But we can allow for other $e^p$ dependences. Such an example is given by (iii).

We assume that $A_\mu = A_\mu(r; x, p)$ or $A_\mu = A_\mu(t; x, p)$, where $x_0 = 0$ and $x_1 = r$, just to identify our examples with well-known equations. These lead to
\begin{align}
\frac{dA_0}{dr} &= \{A_0, A_1\}_-, \\
\frac{dA_1}{dt} &= \{A_1, A_0\}_-,
\end{align}

respectively. We consider several examples below.

(i) The Toda lattice equation
Assume that
\begin{align}
A_1(t; x, p) &= b(t; x) + \frac{1}{2}(e^p + e^{-p})a(t; x), \\
A_0(t; x, p) &= \frac{1}{2}(-e^p + e^{-p})a(t; x).
\end{align}

Then Eq. (14) implies
\begin{align}
\frac{db(t; x)}{dt} &= \frac{1}{2}\{e^p a, e^{-p}a\}_-, \\
e^{\pm p}\frac{da(t; x)}{dt} &= \pm\{e^{\pm p}a, b\}_-.
\end{align}
By using (6), these are
\[ \frac{d}{dt} b(t; x) = -\frac{1}{2\kappa} \left[ a^2(t; x + \kappa) - a^2(t; x - \kappa) \right] , \tag{19} \]
\[ \frac{d}{dt} a(t; x) = \frac{1}{\kappa} \left[ b(t; x + \kappa) - b(t; x - \kappa) \right] a(t; x) . \tag{20} \]
These equations lead to
\[ \frac{d^2}{dt^2} \rho(t; x) = \frac{1}{\kappa^2} \left[ e^{-\rho(t; x+2\kappa)} - 2e^{-\rho(t; x)} + 2e^{-\rho(t; x-2\kappa)} \right] , \tag{21} \]
where
\[ \rho(t; x) = -2 \log a(t; x) . \tag{22} \]
When we impose the condition \( x = n\kappa, (n \in \mathbb{Z}) \) on the above equation, we find that the equation is to be regarded as a Toda lattice equation, except at endpoints where the equations differ from the above form. This demonstrates an important aspect of the Moyal version of soliton equation. The Toda lattice equation can be written in terms of the Cartan matrix, which reflects the algebra we choose. The Toda lattice equation for finite size particles corresponds to the choice of a finite dimensional algebra \( A_n \). The present type of equation is either the Toda lattice equation with an infinite number of particles or that on a one-dimensional link lattice. Mathematically, these are derived assuming \( A_\infty \) or the Kac-Moody algebra denoted by \( A_{n,1} \), respectively. On the other hand, it is known that the \( \kappa \to 0 \) limit in the Moyal algebra leads to the Poisson bracket. In identifying the Moyal algebra with \( su(N) \) algebra one imposes the relation \( \kappa \propto \frac{1}{N} \), which implies that finite \( \kappa \) corresponds to finite \( N \), i.e., a finite number of particles. This shows that the Moyal version of the soliton equation leads to the soliton equation with periodic boundary condition whose algebraic specification is \( A_{n,1} \). We can take the \( \kappa \to 0 \) limit of Eqs. (17) and (18) by using Eq. (7):
\[ \frac{\partial}{\partial t} b(t; x) = -\frac{\partial a^2}{\partial x} , \tag{23} \]
\[ e^{\pm p} \frac{\partial}{\partial t} a(t; x) = 2e^{\pm p} a \frac{\partial b}{\partial x} . \tag{24} \]
We thus obtain the continuous counterpart of the Toda lattice equation quite naturally:
\[ \frac{\partial^2}{\partial t^2} \rho(t; x) = 4 \frac{\partial^2}{\partial x^2} e^{-\rho(t; x)} . \tag{25} \]
(ii) The Bogomolny equation\(^7\)-\(^9\)
We assume that
\[ A_0(r; x, p) = \psi(r) + \frac{1}{2} (e^p f(r; x) + e^{-p} f^*(r; x)) , \tag{26} \]
\[ A_1(r; x, p) = \frac{1}{2} (e^{-p} f^*(r; x) - e^p f(r; x)) . \tag{27} \]
Then Eq. (13) implies\(^6\)

\[
\frac{d\psi}{dr} = \frac{1}{2} \{ e^p f, e^{-p} f^* \}_-, \tag{28}
\]

\[
e^p \frac{df}{dr} = -\{ \psi, e^p f \}_-, \tag{29}
\]

\[
e^{-p} \frac{df^*}{dr} = \{ \psi, e^{-p} f^* \}_-. \tag{30}
\]

Equivalently, these can be written as

\[
\frac{d\psi(r; x)}{dr} = -\frac{1}{2\kappa} \left[ |f(x + \kappa)|^2 - |f(x - \kappa)|^2 \right], \tag{31}
\]

\[
\frac{df(r; x)}{dr} = -\frac{1}{\kappa} \left[ \psi(r; x + \kappa) - \psi(r; x - \kappa) \right], \tag{32}
\]

\[
\frac{df^*(r; x)}{dr} = -\frac{1}{\kappa} \left[ \psi(r; x + \kappa) - \psi(r; x - \kappa) \right]. \tag{33}
\]

These lead to

\[
\frac{d^2}{dr^2} \rho(r; x) = -\frac{1}{\kappa^2} \left[ e^{-\rho(r; x + 2\kappa)} - 2e^{-\rho(r; x)} + 2e^{-\rho(r; x - 2\kappa)} \right], \tag{34}
\]

where

\[
\rho(r; x) = -\log |f(r; x)|^2. \tag{35}
\]

By taking the \(\kappa \to 0\) limit in the Moyal brackets, we can rewrite Eqs. (28)–(30) by using the derivative operators instead of the differences. We thus obtain the continuous limit of the Bogomolny equation\(^{5,10,11}\)

\[
\frac{\partial^2}{\partial r^2} \rho(r; x) = -4 \frac{\partial^2}{\partial x^2} e^{-\rho(r; x)}, \tag{36}
\]

which can be interpreted as the \(su(\infty)\) Bogomolny equation.

(iii) The KM equation

This example corresponds to the case in which the expansion of potentials include \(e^{\pm 2p}\) terms in addition to \(e^{\pm p}\) terms, which means that this expansion does not have an algebra counterpart taking values only on the Cartan sub-algebra and simple roots, in contrast to the previous examples. Assuming that

\[
A_1(t; x) = (e^p + e^{-p})a(t; x), \tag{37}
\]

\[
A_0(t; x) = \kappa \left[ -\{ e^p a(t; x), e^p a(t; x) \}_+ + \{ e^{-p} a(t; x), e^{-p} a(t; x) \}_+ \right], \tag{38}
\]

we obtain from (14)

\[
e^{\pm p} \frac{da(t; x)}{dt} = \{ e^{\pm p} a, \kappa \{ e^{\pm p} a, e^{\pm p} a \}_+ \}_-, \tag{39}
\]

or

\[
\frac{da(t; x)}{dt} = -\frac{2}{\kappa} \left[ a^2(t; x + 2\kappa) - a^2(t; x - 2\kappa) \right] a(t; x). \tag{40}
\]
This can be written as
\[
\frac{d}{dt} u(t; x) = \frac{4}{\kappa} \left[ e^{-u(t; x+2\kappa)} - e^{-u(t; x-2\kappa)} \right],
\]
where
\[
u(t; x) = -2 \log a(t; x).
\]
This is the equation discussed by Kac and van Moerbeke (KM).\(^{12}\) We can also obtain the continuous limit,
\[
\frac{\partial}{\partial t} u(t; x) = 16 \frac{\partial}{\partial x} e^{-u(t; x)}.
\]

We have considered three examples of the Moyal formulation of the soliton equations. One of the most intriguing features of the formulation is that it includes a parameter which interpolates between the discrete and continuous soliton theories. In the discrete theory, which is obtained by fixing the parameter, the Lax pair formulation is well known and is used to show the complete integrability. The Toda equation and the KM equation have been extensively studied by this formulation, which makes clear their completely integrable structure. On the other hand, the continuous equations have been less thoroughly investigated. In the remainder of this paper we discuss the completely integrable feature of the equations by giving an infinite number of conservation laws. In Ref. 10 we give an infinite number of conservation laws relevant to the Bogomolny Eq. (36) whose discussion is based on the Hamiltonian formulation of the theory. Following the discussion there, we give an infinite number of conservation laws relevant to Eqs. (25) and (43). We introduce a new field convenient for the physical interpretation as a dynamical system. Note that these equations can be rewritten as
\[
\frac{\partial^2}{\partial t^2} \phi(t; x) = \pm \frac{\partial}{\partial x} e^{-\frac{\partial}{\partial x} \phi(t; x)}.
\]
Here the equation with the plus sign is obtained from Eq. (25), and that with the minus sign is obtained from Eq. (43) by introducing \(\phi(t; x)\) by
\[
\rho(t; x) = \frac{\partial}{\partial x} \phi(t; x),
\]
\[
u(t; x) = \frac{\partial}{\partial x} \phi(t; x),
\]
in Eq. (25) and Eq. (43), respectively, together with an adjustment of the coefficient by scaling the variable \(t\). In deriving Eq. (44) from Eq. (43), we have set the integration constants to zero. Define the charge densities by
\[
\sigma^{(n)}(t; x) = \sum_{k=0}^{[n/2]} b^{(n)}_k \left( \frac{\partial \phi}{\partial t} \right)^{n-2k} \left( e^{-\frac{\partial \phi}{\partial x}} \right)^k,
\]
where \([n/2]\) is the maximum integer that does not exceed \(n/2\), and the constants \(b^{(n)}_k\) are defined by
\[
b^{(n)}_k = \frac{1}{(k!)^2(n-2k)!}.
\]
and
\[ b_k^{(n)} = \frac{(-1)^k}{(k!)^2(n-2k)!}, \]
for Eq. (44) with plus sign and minus sign, respectively. It is straightforward to show that the charges defined by
\[ Q^{(n)} = \int_{-\infty}^{\infty} \sigma^{(n)}(t; x) dx \]
are constants by using Eq. (44). In the proof we have assumed that
\[ \frac{\partial \phi}{\partial t} \to 0, \quad \frac{\partial \phi}{\partial x} \to 0, \quad \text{as} \quad |x| \to \infty. \]

We have thus found an infinite number of conservation laws, where the conserved densities are expressed by the derivatives with respect to $t$ and $x$ derivatives of the field constituting Eq. (44). Although we can also find an infinite number of conservation laws with respect to Eqs. (25) and (43) by use of Eqs. (45) and (46), their conserved densities include integrals of fields constituting the equations.

In this paper we have formulated soliton equations in terms of the Moyal algebra and have used well-known equations as examples. The specification of how the potentials are expanded in terms of $e^p$ leads to the various soliton equations on a one-dimensional link lattice whose algebraic specification is $A_{n,1}$. We found that, in the present formulation, the role of $e^p$ is the same as the spectral parameter in the soliton equation in the inverse scattering method, and the parameter $\kappa$ in the Moyal algebra has the physical meaning of the spacing on the lattice. In the KM equation we have an anti-commutation bracket expression for one of the potentials, which might be related to the super-symmetric formulation of the Moyal version of the zero-curvature equation. We leave investigation of this point as a future work.

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