Simple proofs for duality of generalized minimum poset weights and weight distributions of (Near-) MDS poset codes

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Abstract—In 1991, Wei introduced generalized minimum Hamming weights for linear codes and showed their monotonicity and duality. Recently, several authors extended these results to the case of generalized minimum poset weights by using different methods. Here, we would like to prove the duality by using matroid theory. This gives yet another and very simple proof of it. In particular, our argument will make it clear that the duality follows from the well-known relation between the rank function and the corank function of a matroid. In addition, we derive the weight distributions of linear MDS and Near-MDS poset codes in the same spirit.

Index Terms—duality, generalized minimum poset weight, weight distribution, MDS poset code, Near-MDS poset code, matroid.

I. INTRODUCTION

In 1991, Wei introduced the notion of generalized minimum Hamming weights for linear codes [12] and showed their monotonicity and duality, motivated by its application to cryptography [9]. Actually, similar properties were considered earlier for irreducible cyclic codes by Helleseth, Kőve and Mykkelveit in [6].

Poset codes were first introduced in [2]. They are just nonempty subsets in \( F_q^n \), equipped with any poset weight instead of the usual Hamming weight. By using different methods, the duality and monotonicity results were extended to the case of generalized minimum poset weights for linear poset codes independently by Barg and Purkayastha [1] and de Oliveira Moura and Firer [8]. Later, Choi and Kim [3] also showed the duality for generalized minimum poset weights by exploiting yet another method.

Here, we would like to explain very briefly how the duality result is proved in each case of [1], [3], and [8]. Barg and Purkayastha in [1], as in the case of Wei’s original proof in [12], do not adopt the matroid theory and exploit instead parity check and generator matrices for linear codes. The authors in [8] adopt the geometric formulation of the generalized minimum Hamming weights for projective systems in [11] and use multi-set techniques, originated from [5] and [10], in order to extend the proofs in [11, Theorem 4.1] to the case of generalized minimum poset weights. So their proof is far different from the original proof of Wei in [12]. Choi and Kim in [3] define \( P(C) \) and \( RP(C) \) for linear codes \( C \), and show the duality by using these. In doing so, they obtain more information than just the duality result.

The aim of this paper is to present simple proofs for the duality of the generalized minimum poset weights and the weight distributions of linear MDS and Near-MDS poset codes by using only very basic facts of matroid theory [13].

In more detail, Theorem 5 is fundamental in proving the duality in Theorem 6 and an analogue of the corresponding Theorem 2 in [12]. One remark here is that while the description involving inequality only is given in [12], that involving both inequality and equality is stated in our case(cf. (2), (3)). We emphasize here that in showing Theorems 5 and 6 we only need the facts in Lemma 2 all of which are trivial except perhaps (g). It is a special case of (f) applied to the matroid \( M_C \) of the linear code \( C \), and hence we may say that the duality really follows from the well-known relation between the rank function and the corank function of a matroid. The weight distributions of linear Near-MDS poset codes were investigated in [1, Theorem 4.1] by using orthogonal array. Here we deduce them in the same spirit as showing the duality theorem. Our proof depends on the formula in (4) and needs information about the values of the rank (or corank) function of the associated matroid of linear MDS and Near-MDS poset codes. For Near-MDS poset codes, we need again the relation between the rank function and the corank function of a matroid in order to have that information.

II. PRELIMINARIES

The following notations will be used throughout this paper.

- \( \mathbb{F}_q \), the finite field with \( q \) elements
- \( [n] = \{1, \ldots, n\} \)
- For \( J \subseteq [n], \mathcal{J} = [n] \setminus J \)
- \( \text{supp}(u) = \{i: u_i \neq 0\}, u_i \in \mathbb{F}_q \)
- \( \text{wt}_H(u) = |\text{supp}(u)| \) the Hamming weight of \( u \)
- \( \supp(D) = \cup_{u \in D} \text{supp}(u) \), for a subset \( D \subseteq \mathbb{F}_q^n \)
- \( \mathbb{P} = ([n], \leq_{\mathbb{P}}) \) a fixed poset on \( [n] \)
- \( \mathbb{P}_j = ([n], \leq_{\mathbb{P}}) \) the dual poset of \( \mathbb{P} \) on \( [n] \) (i.e., \( i \leq_{\mathbb{P}} j \iff j \leq_{\mathbb{P}} i \))
- A subset \( J \subseteq [n] \) is an ideal in \( \mathbb{P} \) if \( j \in J \) and \( i \leq_{\mathbb{P}} j \Rightarrow i \in J \)
- For any \( J \subseteq [n], \langle J \rangle_{\mathbb{P}} \) denotes the smallest ideal containing \( J \) (i.e., \( \langle J \rangle_{\mathbb{P}} = \{ i : i \leq_{\mathbb{P}} j \text{ for some } j \in J \} \))
- \( \text{wt}_P(u) = |\langle \text{supp}(u) \rangle_{\mathbb{P}}| \)
• $wt_p(D) = |(\text{supp}(D))_p|$
• For $J \subseteq [n]$, $u = (u_1, \ldots, u_n) \in \mathbb{P}^n$, and $D \subseteq \mathbb{P}^n$,
  
  $$u | J = (u_i)_{i \in J}, \quad D | J = \{u | J : u \in D\}$$

• $C$ an $[n, k]$ code over $\mathbb{P}_q$, with a generator matrix $G$ (an $k \times n$ matrix with rank $k$) and a parity check matrix $H$ (an $(n-k) \times n$ matrix with rank $n-k$), and with $\rho$ and $\rho^\perp$ respectively the rank function and the corank function of the matroid $\mathcal{M}_C$ of $C$. Such a $C$ can be viewed as a linear $\mathbb{P}$-code (i.e., we regard it as a subspace of the $\mathbb{P}$-space $(\mathbb{P}_q, wt_p)$ and the dual $\mathcal{C}^\perp$ of $C$ as a linear $\overline{\mathbb{P}}$-code
• $C_J = C|J$ the puncturing of $C$ with respect to $J$
• $C^J = \{u | J : u \in C, supp(u) \subseteq J\}$ the shortening of $C$ with respect to $J$. Hereafter we will identify $C^J$ with the space

$$I\{u \in C : supp(u) \subseteq J\}$$

• $\Phi_r(C)$ the set of all $r$-dimensional subspaces of $C$, for $0 \leq r \leq \dim(C)$
• $\Omega(\mathbb{P})$ the set of ideals in $\mathbb{P}$
• $\Lambda^r(\mathbb{P})$ the set of ideals in $\mathbb{P}$ of size $r$
• $S_I = \{x \in \mathbb{P}^n : (\text{supp}(x))_p = I\}$, for $I \in \Omega(\mathbb{P})$
• $M(I)$ the set of maximal elements in $I$, for $I \in \Omega(\mathbb{P})$
• $I_M = I \setminus M(I)$, for $I \in \Omega(\mathbb{P})$
• $\Lambda(I) = \{J \in \Omega(\mathbb{P}) : I_M \subseteq J \subseteq I\}$, for $I \in \Omega(\mathbb{P})$
• $\{\Lambda_r(\mathbb{P})\}_{r=0}^\infty$ the $\mathbb{P}$-weight distribution of $C$ with $\Lambda_r(\mathbb{P}) = \{I \in \Omega(\mathbb{P}) : \dim(C | I) = r\}$

A matroid $\mathcal{M}$ on $S$ is a finite set $S$ together with a function (called the rank function of $\mathcal{M}$) $\rho : 2^S \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the following three properties: for $A, B \subseteq S$,

1. $0 \leq \rho(A) \leq |A|$
2. $A \subseteq B \Rightarrow \rho(A) \leq |A|$
3. $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$

A corank function $\rho^\perp$ is the rank function of the dual matroid $\mathcal{M}^*$ of $\mathcal{M}$. It is well-known that, for a matroid $\mathcal{M}$ with the rank function $\rho$ and the corank function $\rho^\perp$, we have the following: for $A \subseteq S$,

$$\rho^\perp(S \setminus A) = |S| - |A| - \rho(S) + \rho(A),$$

or equivalently

$$\rho^\perp(A) = |A| - \rho(S) + \rho(S \setminus A). \quad (1)$$

For $A \subseteq [n]$, let $G|A$ and $H|A$ be respectively the submatrices of $G$ and $H$ consisting of the columns indexed by $A$. Then we observe that

$$\rho(A) = \text{rank}(G|A) = \dim(C|A),$$

$$\rho^\perp(A) = \text{rank}(H|A) = \dim(C^\perp|A).$$

**Definition 1.** Let $C$ be an $[n, k]$ linear code. For $r(1 \leq r \leq k)$, the $r$-th generalized minimum poset weight($\mathbb{P}$-weight, if the reference to $\mathbb{P}$ is needed) is defined by

$$d_r^P(C) = \min\{wt_p(D) : D \in \Phi_r(C)\};$$

for $s(1 \leq s \leq n-k)$,

$$d_s^P(C^\perp) = \min\{wt_p(D) : D \in \Phi_s(C^\perp)\}.$$

The following lemma contains all the stuffs that are needed in proving Theorems 2 and 3. Here, all the statements are trivial except perhaps (g), which is just (1) applied to the matroid $\mathcal{M}_C$ of $C$.

**Lemma 2.** Let $J \subseteq [n]$. Then we have the following.

1. $\supp(C^J) \subseteq J$.
2. For any subset $D \subseteq C$, $\supp(D) \subseteq J \Rightarrow D \subseteq C^J$.
3. If $J$ is an ideal in $\mathbb{P}$, then $\langle J \rangle_p = J$.
4. $J$ is an ideal in $\mathbb{P} \Leftrightarrow \overline{J}$ is an ideal in $\overline{\mathbb{P}}$.
5. $\dim(C^J) = dim(C) - \rho(J)$.
6. If $supp(D) \subseteq J$, for some $D \in \Phi_r(C)$, then $\rho(D^J) \leq dim(C^J) - r$.
7. $\rho^{-1}(J) = dim(C) - \rho(\overline{J}) = dim(C^J)$.

**Proof:** (a), (b), (c), (d) Clear. (e) Let $\psi : C \rightarrow \overline{C}$ be the linear map given by $u \mapsto u \langle J \rangle$. Then the kernel of this map is $C^r$. (f) As $D \subseteq C^J$ by (b), dim($C^J$) $\geq r$. The result now follows from (e). (g) This follows from (f) and (e).

**III. PROOF OF DUALITY**

We do not provide the proof of the following theorem. One refers its proof to [1].

**Theorem 3.** Let $C$ be an $[n, k]$ linear code. Then

$$1 \leq d_1^p(C) < d_2^p(C) < \cdots < d_k^p(C) \leq n,$$

and

$$1 \leq d_1^p(C^\perp) < d_2^p(C^\perp) < \cdots < d_{n-k}^p(C^\perp) \leq n.$$

**Corollary 4.** For $1 \leq r \leq k$,

$$r \leq d_r^p(C) \leq n - k + r.$$

For $1 \leq s \leq n - k$,

$$s \leq d_s^p(C^\perp) \leq k + s.$$

**Theorem 5.** Let $C$ be an $[n, k]$ linear code. For $1 \leq r \leq k$,

$$d_r^p(C) = \min\{|J| : |J| - \rho^+(J) \geq r\} \quad (2),$$

$$\min\{|J| : |J| - \rho^{-1}(J) = r\}. \quad (3)$$

For $1 \leq s \leq n - k$,

$$d_s^p(C^\perp) = \min\{|J| : |J| - \rho^+(J) \geq s\}$$

$$= \min\{|J| : |J| - \rho^{-1}(J) = s\}. \quad (4)$$

**Proof:** Firstly, we show that $d_r^p(C) \leq \min\{|J| : |J| - \rho^+(J) = r\}$. Let $d$ denote the right hand side of this. Let $J$ be such that $|J| - \rho^+(J) = r$, $|J|_p = d$. Then, by Lemma 2 (g), dim($C^J$) $\leq |J|_p = d$. So, $d_r^p(C) \leq wt_p(C^J) \leq |J|_p = d$, by Lemma 2 (a). Secondly, we show that $\min\{|J| : |J| - \rho^+(J) \geq r\}$ $\leq d_r^p(C)$. Let $e$ denote the left hand side of this. To show this, let $wt_p(D) = d_r^p(C)$, for some $D \in \Phi_r(C)$. Set $J = (\supp(D))_p$, Then $D \subseteq C^J$, by Lemma 2 (b) and dim($C^J$) $= |J| - \rho^+(J)$ $\geq r$ (cf. Lemma 2 (g)). So, by Lemma 2 (e), $e \leq |J|_p$. Thus, $d_r^p(C)$ $\leq d_r^p(C^J)$. Lastly, it is enough to see that $d \leq e$. Let $e = |J|_p$, with $|J| - \rho^+(J) \geq r$. Then we claim that $|J| - \rho^+(J) = r$. Assume on the contrary that $|J| - \rho^+(J) = r' > r$. Then, by the first and second steps,
\( d_r^2(C) \leq \min\{|(I)\varphi| : |I| - \rho(I) = r'\} \leq |(J)\varphi| = e \leq d_r^2(C) \), a contradiction to Theorem 3.

**Theorem 6.** Let \( C \) be an \([n, k, d]\) linear code and \( A = \{d_r^2(C) : 1 \leq r \leq k\} \). Then \( A \) and \( B = \{n + 1 - d_r^2(C) : 1 \leq s \leq n - k\} \), and, by Theorem 5 and Lemma 2, a contradiction to Theorem 3.

**Proof:** It is enough to see that \( A \) and \( B \) are disjoint. Let \( s \) be any integer such that \( 1 \leq s \leq n - k \). Then we need to see that \( n + 1 - d_r^2(C^+) \notin A \). Firstly, let \( r = k + s - d_r^2(C^+) \), and, by Corollary 3, \( s \leq d_r^2(C^+) \leq k + s \), so that \( 0 \leq t \leq k \). Now, the first sum in (6) is

\[
\sum_{l=|I|}^{r} (-1)^{r-|I|} \sum_{J \in \Lambda(l)} (-1)^{r-|J|} q^{|M(I)|} \leq \sum_{l=|I|}^{r} (-1)^{r-|I|} \sum_{J \in \Lambda(l)} (-1)^{r-|J|} q^{|J|} - d + 1.
\]

The second sum in (6) is

\[
\sum_{s=0}^{r-d} (-1)^{r-d-s} q^{|M(I)|} (q^{r-d+s} - 1).
\]

So, for any \( I \in \Lambda(|J|, \varphi) \), from (4) we have

\[
|C \cap S_I| = \sum_{J \in \Lambda(I)} (-1)^{|J| - |I|} q^{|M(I)|}.
\]

The equation (4) in the following follows from [7, (3.1)], while the equation (5) is clear.

**Proposition 7.** Let \( I \) be an ideal in \( \mathbb{P} \).

\[
|C \cap S_I| = \sum_{J \in \Lambda(I)} (-1)^{|J| - |I|} q^{|M(I)|}.
\]

**Theorem 8.** Let \( C \) be a MDS \( \mathbb{P} \)-code with parameters \([n, k, d = n - k + 1]\). Then, for \( r \), with \( d \leq r \leq n \),

\[
A_r(C) = \sum_{l=|I|}^{r} (-1)^{r-|I|} \frac{|M(I)|}{s} (q^{|M(I)|} - 1).
\]

Recall that an \([n, k, d = n - k + 1]\) \( \mathbb{P} \)-code is called a Near-MDS \( \mathbb{P} \)-code if

\[
d = d_r^2(C) = n - k,
\]

\[
d_r^2(C) = n - k + 2.
\]
Lemma 9 ([1, Lemma 2.4 (1), (2)]). The following hold.

(a) $C$ is an $[n, k]$ Near-MDS $P$-code if and only if

1. Any $n - k - 1$ columns of the parity check matrix $H$ are linearly independent.
2. There exist $n - k$ linearly dependent columns of $H$.
3. Any $n - k + 1$ columns of $H$ have the full rank $n - k$.

(b) If $C$ is a linear Near-MDS $P$-code, then $C^\perp$ is a linear Near-MDS $P$-code.

Now, we assume that $C$ is a Near-MDS $P$-code with parameters $[n, k, d = n - k]$. Then, from Lemma 9 (a) above, we get

$$\rho^*(J) = \begin{cases} |J|, & |J| < n - k, \\ n - k, & |J| > n - k. \end{cases}$$

By invoking Lemma 2 (g) again, from (9) we have, for $J \in \Omega(P)$,

$$k - \rho(J) = \begin{cases} 0, & |J| \leq d - 1, \\ |J| - d, & |J| \geq d + 1. \end{cases}$$

We note here that (10) also follows from (9) and Lemma 9 (b). However, Lemma 9 (b) is deduced in [1] from the duality result in Theorem 6 which in turn follows from Lemma 2 (g), as we stressed in Section I.

Then, by proceeding analogously to the MDS case, we get the following result.

Theorem 10 ([1]). Let $C$ be a Near-MDS $P$-code with parameters $[n, k, d = n - k]$. Then, for $r$, with $d \leq r \leq n$,

$$A_{r,P}(C) = \sum_{I \in \Lambda(P)} \sum_{s=0}^{r-d-1} (-1)^s \binom{|M(I)|}{s} (q^{r-d-s} - 1) + (-1)^{r-d} \sum_{I \in \Lambda(P)} \sum_{J \in A(I)} A_J(C),$$

where $A_J(C) = |C \cap S_J|$.

In the case of Hamming weight (i.e., $w_H$ with $P$ the antichain on $[n]$), denoting $A_{r,P}(C)$ by $A_r(C)$ as usual, we recover the following corollary in [4].

Corollary 11 ([4]). Let $C$ be a Near-MDS code with parameters $[n, k, d = n - k]$. Then, for $r$, with $d \leq r \leq n$,

$$A_r(C) = \binom{n}{r} \sum_{s=0}^{r-d-1} (-1)^s \binom{r}{s} (q^{r-d-s} - 1) + (-1)^{r-d} \binom{n-d}{r-d} A_d(C),$$

Proof: Now, let $P$ denote the antichain. Then the second double sum in (11) is

$$\sum_{|I|=r} \sum_{u \in C} \sum_{w_H(u)=d \supp(u) \subseteq I} 1 = \binom{n-d}{r-d} A_d(C).$$

by counting $I$, with $|I| = r$, for each fixed $u \in C$, with $w_H(u) = d$. □