MAL’CEV CLASSES OF LEFT-QUASIGROUPS AND QUANDLES

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Abstract

In this paper we investigate some Mal’cev classes of varieties of left-quasigroups. We prove that the weakest Mal’cev condition for a variety of left-quasigroup is having a Mal’cev term. Then we specialize to the setting of quandles for which we prove that the meet semidistributive varieties are those which have no finite models.

Introduction

A variety is an equational class of algebras of the same type, i.e. a class of algebras defined by a set of identities in a given language. Equivalently, varieties are classes closed under taking homomorphic images, subalgebras and direct products.

Varieties are usually grouped by certain properties they have, called Mal’cev classes of varieties. A strong Mal’cev condition is a set of identities in a given language. A class of varieties is called a strong Mal’cev class if it is defined by a strong Mal’cev condition. A class is said to be a weak Mal’cev class, if it is a numerable union of strong Mal’cev classes.

Mal’cev conditions turned out to be extremely useful to characterize property of varieties, in particular concerning properties of congruence lattices. The seminal result in this direction is the classical theorem of Mal’cev, which shows that congruence permutable varieties are defined by a strong Mal’cev condition.

Theorem. [Mc54] Let \( V \) be a variety. The following are equivalent:

(i) \( V \) is congruence-permutable.
(ii) \( V \) has a Mal’cev term, i.e. a ternary term \( m \) such that

\[
m(x, y, y) \equiv x \equiv m(y, y, x).
\]

For a variety, satisfying a non-trivial idempotent Mal’cev condition (i.e. any Mal’cev condition which is not satisfied by any projection algebra) was known to be a weak Mal’cev class. Indeed such varieties were characterized by the existence of a Taylor term, namely an \( n \)-ary term satisfying a set of identity respecting a given pattern. Recently such class was proven to be strong in [Olfrm–7], i.e. there exists the weakest strong Mal’cev condition.

Theorem. [Tay77] Let \( V \) be a variety. The following are equivalent:

(i) \( V \) has a Taylor term.
(ii) The variety \( V \) does not contain any projection algebra.

A variety \( V \) is meet-semidistributive if the congruence lattice of every algebra in \( V \) is meet-semidistributive. Namely, the implication

\[
\alpha \wedge \beta = \alpha \wedge \gamma \implies \alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma),
\]

holds for every triple of congruences of any algebra in \( V \). It is still unknown if meet-semidistributivity is defined by a strong Mal’cev condition, nevertheless it can be characterized in several different ways in [Olfrm–8]. On the other hand, we are going to use the characterization of meet-semidistributive varieties in terms of commutator of congruences as defined in [PM87].

Theorem 0.1. [KK13, Theorem 8.1] Let \( V \) be a variety. The following are equivalent:

(i) \( V \) is a congruence meet-semidistributive variety.
(ii) No member of \( V \) has a non-trivial abelian congruence.
(iii) \( [\alpha, \beta] = \alpha \wedge \beta \) for every \( \alpha, \beta \in \text{Com}(A) \) and every \( A \in V \).

In particular, an idempotent variety \( V \) is meet-semidistributive if and only if it does not contain an abelian algebra.

An algebra \( A \) is said to be:
(i) **coherent** if every subalgebra of $A$ which contains a block of a congruence $\alpha \in \text{Con}(A)$ is a union of blocks of $\alpha$.

(ii) **Congruence regular** if whenever $[a]_{\alpha} = [a]_{\beta}$ for some $a \in A$ and $\alpha, \beta \in \text{Con}(Q)$ then $\alpha = \beta$.

(iii) **Congruence uniform** if the blocks of every congruence $\alpha \in \text{Con}(A)$ have all the same cardinality.

A variety $V$ is coherent (resp. **congruence uniform**, **congruence regular**) if all the algebras in $V$ are coherent (resp. congruence uniform, congruence regular). Congruence uniform varieties are congruence regular. Congruence regularity and coherency are weak Mal’cev classes (see [Csa70] and [Gei74]). On the other hand, it is known that congruence uniformity is not defined by a Mal’cev condition [Tay74].

A lattice containing some of the most studied Mal’cev classes of varieties is displayed in figure 1. We refer the reader to [Ber12] for further informations about such classes.

![Figure 1](image1.png)

**Figure 1.** Mal’cev classes: $T =$ Taylor term, $\text{wDF} = $ weak difference term, $\text{CE} = $ non trivial congruence equation, $\text{DF} = $ difference term, $\text{CM} = $ congruence modularity, $\text{Ed} = $ edge term, $\text{CP} = $ congruence permutability, $M = $ Mal’cev term, $\text{CO} = $ congruence coherency, $\text{SD}(\land) = $ meet semidistributivity, $\text{SD}(\lor) = $ join semidistributivity, $\text{CD} = $ congruence distributivity, $\text{NU} = \text{CD} \cap \text{Ed} = $ near unanimity term, $\text{CA} = \text{CD} \cap M = $ congruence arithmeticity.

The main goal of this paper is to investigate Mal’cev conditions for racks and quandles. In particular, this paper is concerned with certain Mal’cev classes of varieties, namely, the varieties having a Taylor term, a Mal’cev term and meet semi-distributive congruence lattices.

In the first part of the paper we investigate left-quasigroups, which are rather combinatorial objects. Nevertheless, Mal’cev classes of varieties of left-quasigroups behave in a pretty rigid way. Indeed, in Theorem 2.2 we show that several Mal’cev conditions are equivalent for varieties of left-quasigroups. In particular, all the classes in the interval between the class of Taylor varieties and coherent varieties in figure 1 collapse into the strong Mal’cev class of varieties with a Mal’cev term. So the weakest Mal’cev condition for left-quasigroups is having a Mal’cev term, and all such varieties are congruence uniform. Therefore, the lattice of Mal’cev classes of left-quasigroups in figure 1 coincides with the chain showed in figure 2.

![Figure 2](image2.png)

**Figure 2.** Mal’cev classes of varieties of left-quasigroups.
A characterization of a Mal’cev classes of left-quasigroups and quandles is provided in Theorem 1.1. They are the varieties for which every left-quasigroup is connected, (a left-quasigroup is connected if the action of its left multiplication group is transitive). We characterize finite Mal’cev idempotent left-quasigroups, thanks to a general result in [BKLS15] as the superconnected idempotent left-quasigroups (i.e. left-quasigroups such that all the subalgebras are connected, Corollary 1.1).

Then we turn our attention to quandles, i.e. idempotent left distributive left-quasigroups. Quandles has been studied because they provide knot invariants [Joy82, Mat82]. The class of quandles used for such topological applications is the class of connected quandles. According to the characterization of Mal’cev varieties of idempotent left-quasigroups, connectedness is actually a relevant property also algebraically.

We investigate the class of superconnected quandles and we show that several results on latin quandles (i.e. left distributive quasigroups) can be extended to this class e.g. the properties of the commutator of congruences (see Proposition 3.8). We also show that some class of superconnected quandles are latin, as the nilpotent ones in Proposition 3.10 and the involutory ones in Corollary 3.10.

We introduce the displacement metric on connected quandles, which provides a necessary condition for connectedness of an arbitrary direct product of connected quandles (see Proposition 3.11).

In Proposition 1.1 we prove that the quandles in a Mal’cev varieties are uniformly bounded in terms of this metric. Examples of non-trivial Mal’cev varieties of quandles (which members are not just left-quasigroup reducts of quasigroups) are provided in table 1.

We also investigate meet semidistributive varieties of quandles. In Theorem 4.4 we prove that a variety of quandles is meet semidistributive if and only if it has not finite models, making use of the characterization of strictly simple and simple abelian quandles [Hon20]. We also prove that there is no meet semidistributive variety of involutory quandles and that the displacement group of quandles meet semidistributive varieties are perfect. The problem of finding an example of such variety is still open.

Notation and terminology. An algebra is a set with an arbitrary set of operations, called the language or the type of the algebra. A term $t(x_1, \ldots, x_n)$ for an algebra is a well-formed formal expression using the variables $x_1, \ldots, x_n$ and the basic operations of such algebra. We say that $A$ satisfies the identity $t_2(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ where $t_1$ and $t_2$ are terms if $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ for every $a_i \in A$.

Two algebras are said to be term equivalent if they have the same terms.

A projection algebra is an algebra whose all terms are projection (i.e. $t(x_1, \ldots, x_n) = x_i$) for some $1 \leq i \leq n$.

The projection algebra of size $n$ is denoted by $P_n$.

We denote by $H(A)$, $S(A)$ and $P(A)$ respectively the set of isomorphism classes of factors, subalgebras and powers of the algebra $A$. The subalgebra generated by a subset $X \subseteq A$ is the smallest subalgebra of $A$ containing $X$ and we denote it $Sg(X)$. The congruence lattice of $A$ is denoted by $Con(A)$, the block of $a \in A$ with respect to a congruence $\alpha$ is denoted by $[a]_\alpha$ (or simply by $[a]$) and the factor algebra by $A/\alpha$.

See [Her12] for further standard notions from universal algebra.

Through all the paper, concrete examples of left-quasigroups are computed using the software Mace4 [McC10] and examples of quandles are taken from the RIG library of connected quandles on GAP.

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1. Preliminary results

1.1. Left-quasigroups. A left-quasigroup is a binary algebra $(Q, \ast, \backslash)$ such that the identities

\[ x \ast (y \ast x) = y = x \ast (x \backslash y) \]

hold, i.e. the left multiplications $L_a : b \mapsto a \ast b$ are bijective for every $a \in Q$. The left multiplication group of $Q$ is $L(Q) = \{ (L_a : a \in Q) \}$. If $\alpha$ is a congruence of $Q$, the mapping

\[ \pi_\alpha : L(Q) \rightarrow L(Q/\alpha), \quad L_a \mapsto L_{[a]_\alpha}, \]

is a well defined surjective homomorphism of groups (see [AG03] Lemma 1.8 for racks and [BS19b] for left-quasigroups). As a particular case we have the following: let $(Q_i : i \in I)$ be a set of left quasigroups and let $Q = \Pi_{i \in I} Q_i$. Then the mapping

\[ \pi_j : L(Q) \rightarrow L(Q_j), \quad L_{(a_i)_{i \in I}} \mapsto L_{a_j} \]

(1)

can be extended to a surjective morphism of groups for every $j \in I$. 

The displacement group relative to a congruence \( \alpha \) is the smallest normal subgroup of \( \text{LMLt}(Q) \) containing \( \{L_nL_n^{-1} : a \alpha b \} \) (see [1ST19a] Section 3.1)) i.e.

\[
\text{Dis}_\alpha = \langle hL_nL_n^{-1}h^{-1}, a \alpha b, h \in \text{LMLt}(Q) \rangle.
\]

For \( \alpha = 1_Q \) we denote the relative displacement group as \( \text{Dis}(Q) \) and we call it the displacement group of \( Q \).

**Lemma 1.1.** Let \( Q \) be a left quasigroup. Then

\[
\text{Dis}(Q) = \{ L_k^{k_1} \cdots L_k^{k_n} : \sum_{i=1}^n k_i = 0 \}
\]

and in particular \( \text{LMLt}(Q) = \text{Dis}(Q)(L_a) \) for every \( a \in Q \).

**Proof.** The set on the right-hand side of (2) is a normal subgroup of \( \text{LMLt}(Q) \) and it contains the generators of \( \text{Dis}(Q) \). Therefore \( \text{Dis}(Q) \) is contained in it. Let \( h = L_k^{k_1} \cdots L_k^{k_n} \) with \( \sum_{i=1}^n k_i = 0 \). If \( n = 2 \) we get one of the generators of \( \text{Dis}(Q) \). Since \( \sum_{i=1}^n k_i = 0 \) then there exists a a proper interval \([l,j]\) of \([1,n]\) such that \( \sum_{i=1}^n k_i = 0 \) then conjugating by \( w = L_{a_1}^{k_1} \cdots L_{a_{n-1}}^{k_{n-1}} \) we obtain

\[
w^{-1}hw = L_{a_1}^{k_1} \cdots L_{a_{n-1}}^{k_{n-1}} L_{a_1}^{j-1} \cdots L_{a_{n-1}}^{j-1} L_{a_1}^{k_1} \cdots L_{a_{n-1}}^{k_{n-1}}.
\]

The sum of the exponents of \( w^{-1}hw \) and in \( f \) is zero, then also the sum of the exponent of \( g \) is zero. Then by induction on the length, \( f, g \in \text{Dis}(Q) \). Hence \( h = wfgw^{-1} \in \text{Dis}(Q) \) since \( \text{Dis}(Q) \) is normal in \( \text{LMLt}(Q) \).

In particular, if \( h = L_k^{k_1} \cdots L_k^{k_n} \in \text{LMLt}(Q) \) then

\[
h = L_k^{k_1} \cdots L_k^{k_n} L_{a_1}^{\sum_{i=1}^n k_1} L_{a_1}^{\sum_{i=1}^n k_i}
\]

\[\text{Dis}(Q) \]

for every \( a \in Q \), so \( \text{LMLt}(Q) = \text{Dis}(Q)(L_a) \). \( \square \)

If \( \alpha \leq \beta \) the image of \( \text{Dis}_\beta \) under \( \pi_\alpha \) is \( \text{Dis}_{\beta/\alpha} \) and then in particular the restriction of \( \pi_\alpha \) to \( \text{Dis}(Q) \) gives a surjective homomorphism \( \text{Dis}(Q) \to \text{Dis}(Q/\alpha) \). The kernels of \( \pi_\alpha \) and of its restriction will be denoted respectively by \( \text{LMLt}^\alpha \) and \( \text{Dis}^\alpha \). The set-wise block stabilizers in \( \text{LMLt}(Q) \) is the subgroup \( \text{LMLt}(Q)_{[\alpha]} = \{ h \in \text{LMLt}(Q) : h(\{[a]_\alpha\}) = [a]_\alpha \} \) (and similarly \( \text{Dis}(Q)_{[\alpha]} = \{ h \in \text{Dis}(Q) : h([a]_\alpha) = [a]_\alpha \} \)). Note that both \( \text{LMLt}(Q)_{[\alpha]} \) and \( \text{Dis}^\alpha \) are contained in \( \text{LMLt}(Q)_{[\alpha]} \) (and the same is true for \( \text{Dis}(Q)_\alpha \), \( \text{Dis}^\alpha \) and \( \text{Dis}(Q)_{[\alpha]} \)).

The Cayley kernel is the equivalence relation \( \lambda_Q \) defined as

\[
a \lambda_Q b \quad \text{if and only if} \quad a \lambda = b.
\]

If \( \lambda_Q = 0_Q \) then \( Q \) is called faithful. In particular, if \( Q/\alpha \) is faithful, then \( \lambda_Q \leq \alpha \). If \( \lambda_Q = 1_Q \), i.e. \( *b = f(b) \) for every \( a, b \in Q \) where \( f \in \text{Sym}(Q) \), then \( Q \) is called *permutation left-quasigroup* and denoted by \( (Q, f) \). If \( f \) is the identity mapping then \( a * b = b \) for every \( a, b \in Q \). \( Q \) is a projection left-quasigroup. We call trivial left-quasigroup the one-element projection left-quasigroup.

In general, the equivalence \( \lambda_Q \) is not a congruence. A variety of left-quasigroups \( \mathcal{V} \) is a called a Cayley variety if \( \lambda_Q \) is a congruence for every \( Q \in \mathcal{V} \). For instance the variety of racks and the variety of cycle set defined in [Hm85] as a special class of solutions of Yang-Baxter equation are Cayley varieties. According to [ saint ] \( \lambda_Q \) is the biggest strongly abelian congruence in the sense of [IB05] for every Cayley left-quasigroup \( Q \).

A quasigroup is an algebra \( (Q, *, \backslash, /) \) such that \( (Q, *, \backslash) \) is a left-quasigroup (the left-quasigroup reduct of \( Q \)) and the identity

\[
(x * y)/y \equiv x \equiv (x/y) * y
\]

holds, i.e. also the right multiplications \( R_a : b \mapsto b * a \) are bijective for every \( a \in Q \). Note that terms, subalgebras and congruences of a quasigroup and of its left-quasigroup reduct might be different. Nevertheless they are the same in the finite case, since they are term equivalent. If a left-quasigroup is a reduct of a quasigroup we call it Latin (in the finite case its multiplication table is a Latin square) when we consider it as a left quasigroup.

A left quasigroups \( Q \) is said to be idempotent if \( x * x \equiv x \) holds and involutory if \( x * (x * y) \equiv y \) holds.
1.2. Connected left-quasigroup. In this section we introduce the classes of connected and superconnected left-quasigroups.

Definition 1.2. A left-quasigroup $Q$ is said to be:

(i) connected if $\text{LMlt}(Q)$ acts transitively on $Q$.

(ii) Superconnected if every subalgebra of $Q$ is connected.

The orbit decomposition $O_Q$ defined by the action of $\text{LMlt}(Q)$ (as $a O_Q b$ if and only if $a$ and $b$ are in the same orbit with respect to the action of $\text{LMlt}(Q)$) is a congruence on $Q$ and $Q/O_Q$ is a projection left-quasigroup. Indeed, if $a$ and $b$ are in the same orbit, then so they are $L^m_2(c)$ and $L^m_2(c)$ and $L^m_2(c)$ and $L^m_2(c)$.

Proposition 1.3. Let $Q$ be a left-quasigroup and $\alpha \in \text{Con}(Q)$. Then $Q/\alpha$ is a projection left-quasigroup if and only if $O_Q \leq \alpha$. In particular, $Q$ is connected if and only if $P_2 \in H(Q)$.

Proof. If $O_Q \leq \alpha$, then $Q/\alpha \cong (Q/O_Q)/(\alpha/O_Q)$. Therefore, $Q/\alpha$ is a projection left-quasigroup. On the other hand, if $Q/\alpha$ is a projection left-quasigroup, then $[b * a]_\alpha = [0][a]_\alpha = [a]_\alpha$ for every $a, b \in Q$. Hence, $O_Q \leq \alpha$.

A left-quasigroup is connected if and only if $Q/O_Q$ is trivial, i.e. no factor of $Q$ is projection.

Corollary 1.4. A left-quasigroup $Q$ is superconnected if and only if $P_2 \in HS(Q)$.

The class of connected left-quasigroups is closed under $H$. It is not a closed under $S$ (for instance it is easy to find connected left-quasigroups with projection subalgebras). The class of superconnected left-quasigroups is closed under $S$ and $H$. On the other hand it is not closed under $P$ (e.g. the permutation left-quasigroup $Q = (\mathbb{Z}_m, +1)$ is superconnected, but $Q^2$ is not even connected).

The following is a criterion for connectedness for left-quasigroups. The proof is the same for the analog criterion for racks stated in [BB19 Proposition 1.3].

Lemma 1.5. Let $Q$ be left-quasigroup and $\alpha \in \text{Con}(Q)$. Then $Q$ is connected if and only if $Q/\alpha$ is connected and $\text{LMlt}(Q)[a]_\alpha$ is transitive on $[a]_\alpha$ for every $a \in Q$.

The property of being superconnected is determined by the connectedness of the two-generated subalgebras.

Lemma 1.6. Let $Q$ be a left-quasigroup. The following are equivalent:

(i) $Q$ is superconnected.

(ii) $Sg(a, b)$ is connected for every $a, b \in Q$.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Let $M$ be a subalgebra of $Q$, let $a, b \in M$ and $Sg(a, b)$ the subalgebra generated by $a$ and $b$. Then the subgroup $\in \text{LMlt}(Sg(a, b))$ is transitive on $Sg(a, b)$ and then so in particular there exists $h \in \{L_+, c \in Sg(a, b)\} \leq \text{LMlt}(M)$ such that $h(a) = b$. Therefore $M$ is connected.

The property of being (super)connected is also reflected by the properties of congruences.

Lemma 1.7. Connected left-quasigroup are congruence uniform and congruence regular.

Proof. Let $Q$ be a connected left-quasigroup and assume that $[a]_\alpha = [a]_\beta$ for some $a \in Q$. For every $b \in Q$ there exists $h \in \text{LMlt}(Q)$ with $b = h(a)$. The blocks of congruences are blocks with respect to the action of $\text{LMlt}(Q)$, then $[b]_\alpha = [h(a)]_\alpha = h([a]_\alpha) = h([a]_\beta) = [b]_\beta$, and so $\alpha = \beta$. In particular, the mapping $h$ is a bijection between $[a]_\alpha$ and $[b]_\alpha$ for every $a \in \text{Con}(Q)$.

Lemma 1.8. Superconnected left-quasigroup are coherent.

Proof. Let $Q$ be a superconnected left-quasigroup, $M \in S(Q)$ and $\alpha \in \text{Con}(Q)$ with $[a]_\alpha \subseteq M$ for some $a \in M$. For every $b \in M$ there exists $h \in \text{LMlt}(M)$ such that $b = h(a)$. The blocks of $\alpha$ are blocks with respect to the action of $\text{LMlt}(Q)$ and $M$ is a subalgebra, then $h([a]_\alpha) = [b]_\alpha \subseteq M$. Therefore, $M = \bigcup_{a \in M} [b]_\alpha$.

Example 1.9. Latin left-quasigroup are connected. If a quasigroup and its quasigroup reduct are term equivalent then the left-quasigroup reduct is superconnected. Hence, any finite latin left-quasigroup is superconnected.

Example 1.10. The following is a superconnected left-quasigroup which is not latin:
1.3. **Idempotent left-quasigroups.** The blocks of congruences of idempotent left-quasigroups are subalgebras, and in particular, the classes of $\lambda_Q$ are projection subalgebras. The orbits of $\text{LMlt}(Q)$ and of $\text{Dis}(Q)$ coincide, because of the structure of $\text{LMlt}(Q)$ given in Lemma 2. We extend [Bon20, Proposition 1.4] to the setting of idempotent left-quasigroups.

**Lemma 1.11.** Let $Q$ be an idempotent left-quasigroup. The following are equivalent:

(i) $P_2 \notin S(Q)$.

(ii) All the subalgebras of $Q$ are faithful.

In particular, if $Q$ is superconnected then $Q$ is faithful.

**Proof.** (i) $\Rightarrow$ (ii) Let $M$ be a subalgebra of $Q$. The classes of $\lambda_M$ are projection subalgebras, therefore they are trivial.

(ii) $\Rightarrow$ (i) The subalgebra $P_2$ is not faithful.

Note that the class of idempotent left-quasigroup with no projection subalgebras is closed under $S$ and $P$. Therefore the class of idempotent left-quasigroup with no projection subalgebras is a quasi-variety axiomatized by the quasi-identity:

$$x \ast y = y \Rightarrow x = y.$$  

A pseudovariety of idempotent left-quasigroup $V$ satisfies some non-trivial quasi-identity if and only if $P_2 \notin S(Q)$ for every $Q \in V$.

**Example 1.12.** Subalgebras of idempotent latin left-quasigroups have no projection subalgebras, but they might not be superconnected. Let $Q = (Q, \ast, \backslash)$ where $a \ast b = 2a - b$. Then $Q$ is a latin left-quasigroup, but $(\mathbb{Z}, \ast, \backslash)$ is a non-connected subalgebra of $Q$.

A class of idempotent left-quasigroups $K$ is said to be closed under extensions if, whenever $Q/\alpha$ and $[a]_\alpha$ belongs to $K$ for every $a \in Q$ then also $Q$ belongs to $K$.

**Proposition 1.13.** The class of faithful (resp. connected) idempotent left-quasigroups is closed under extensions.

**Proof.** If $L_a = L_b$ then $L_{[a]} = L_{[b]}$ and so $[a] = [b]$ since $Q/\alpha$ is faithful. Therefore $L_a|_{[a]} = L_b|_{[a]}$ which implies $a = b$ since $[a]$ is faithful.

Let $Q/\alpha$ and $[a]$ be connected. Let $a \in Q$ and the group $D = \langle \{L_a L_c^{-1} : b, c \in [a]\} \rangle$. Since $[a]$ is a subalgebra, then $D \leq \text{LMlt}(Q)_{[a]}$. If $[a]$ is connected, $D$ is transitive on $[a]$ and so is $\text{LMlt}(Q)_{[a]}$. Therefore $Q$ is connected by virtue of Lemma 1.5.

**Corollary 1.14.** The class of idempotent left-quasigroups with no projection subalgebra (resp. superconnected) is closed under extensions.

**Proof.** It is enough to apply Proposition 1.13 to the subalgebras of $Q$. Indeed if $M$ is a subalgebra of $Q$, then its factor $M/\alpha$ is faithful (resp. connected) and $M \cap [a]_\alpha \subseteq [a]$ is faithful (resp. connected).

**Remark 1.15.** Let $Q$ be a left quasigroup and assume that $Q/\alpha$ is idempotent. The blocks of $\alpha$ are subalgebras of $Q$. If $Q/\alpha$ and the blocks of $\alpha$ are superconnected then the same argument of Corollary 1.14 we have that $Q$ is superconnected.

The class of latin left-quasigroup is not closed under extensions. For instance the following superconnected idempotent left-quasigroup has a congruence with a factor of size 3 and blocks of size 3 which are latin, but it is not latin itself:

$$Q = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix}.$$
2. Mal’cev classes of left-quasigroups

A term \( t(x_1, \ldots, x_n) \) in the language of left-quasigroups is a well-formed formal expression using the variables \( x_1, \ldots, x_n \) and the operations \( \{\ast, \setminus\} \). So the term \( t(x_1, \ldots, x_n) \) can be written as

\[
 t(x_1, \ldots, x_n) = s_1(x_1, \ldots, x_n) \ast \cdots \ast s_m(x_1, \ldots, x_n) = t
\]

where \( \ast, \setminus \in \{\ast, \setminus\} \) and \( s_1, \ldots, s_m \) are suitable subterms. Somehow less formally, \( t \) can be understood as

\[
 t(x_1, \ldots, x_n) = L_{k_1}^{s_1(x_1, \ldots, x_n)} \cdots L_{k_m}^{s_m(x_1, \ldots, x_n)}(x_t)
\]

where \( k_i = \pm 1 \) for \( 1 \leq i \leq m \) and some \( 1 \leq r \leq n \). Every identity \( t_1 = t_2 \) can therefore be written in the form

\[
 L_{s_1(x_1, \ldots, x_n)}^{k_1} \cdots L_{s_m(x_1, \ldots, x_n)}^{k_m}(x_i) = L_{u_1(y_1, \ldots, y_l)}^{u_1} \cdots L_{u_t(y_1, \ldots, y_l)}^{u_t}(y_j),
\]

or equivalently as

\[
 L_{r_1(y_1, \ldots, y_l)}^{u_1} \cdots L_{r_t(y_1, \ldots, y_l)}^{u_t} L_{x_1, \ldots, x_n}^{k_1} \cdots L_{x_1, \ldots, x_n}^{k_m}(x_i) = y_j.
\]

The projection left-quasigroup \( P_2 \) satisfies (3) if and only if \( x_i = y_j \). So a variety of left-quasigroups \( V \) has a Taylor term if and only if it satisfies an identity as in (3) with different right-most variable of the following form

\[
 t(x_1, \ldots, x_n) = L_{k_1}^{s_1(x_1, \ldots, x_n)} \cdots L_{k_m}^{s_m(x_1, \ldots, x_n)}(y) = x.
\]

Note that, such an identity might have just the trivial model. For instance if \( V \) is a variety of idempotent left-quasigroups satisfying (3), if the variable \( x \) does not appear in the left handside then the variety is trivial. Indeed, identifying all the variables \( x_1, \ldots, x_n, y \) we have \( L_y^{k_1+\cdots+k_m}(y) = y = x \).

**Example 2.1.** The varieties axiomatized by some identities as in (1) might be made up of latin left-quasigroups. For instance, the left-quasigroups satisfying the identity

\[
 x \ast (y \setminus z) = z \ast (y \setminus x)
\]

are latin. Indeed (5) is satisfied by a left-quasigroup \( Q \) if and only if \( R_{y \setminus z} = L_z L_y \) for every \( y, z \in Q \). So (5) implies that \( R_y = L_y \setminus \setminus L_y \) is bijective for every \( y \in Q \).

On the other hand, the identity

\[
 x \ast (y \setminus (x \ast y)) = y \ast (x \setminus (y \ast x))
\]

has non-latin models as

\[
 Q = \begin{bmatrix}
 1 & 3 & 2 & 7 & 8 & 9 & 4 & 5 & 6 \\
 3 & 2 & 1 & 7 & 8 & 9 & 4 & 5 & 6 \\
 2 & 1 & 3 & 7 & 8 & 9 & 4 & 5 & 6 \\
 7 & 8 & 9 & 4 & 6 & 5 & 1 & 2 & 3 \\
 7 & 8 & 9 & 6 & 5 & 1 & 1 & 2 & 3 \\
 7 & 8 & 9 & 5 & 4 & 6 & 1 & 2 & 3 \\
 4 & 5 & 6 & 1 & 2 & 3 & 7 & 9 & 8 \\
 4 & 5 & 6 & 1 & 2 & 3 & 9 & 8 & 7 \\
 4 & 5 & 6 & 1 & 2 & 3 & 8 & 7 & 9
\end{bmatrix}
\]

For left-quasigroups, the interval of Mal’cev classes between the class of Taylor varieties and the class of coherent varieties collapse into the class of Mal’cev varieties.

**Theorem 2.2.** Let \( V \) be a variety of left-quasigroups. The following are equivalent:

(i) \( V \) has a Taylor term.
(ii) \( V \) has a Mal’cev term.
(iii) \( V \) is coherent.
(iv) Every algebra in \( V \) is (super)connected.

In particular, every Mal’cev variety of left-quasigroup is congruence uniform.
Proof. The implications (iii) ⇒ (ii) ⇒ (i) are true in general.

(i) ⇒ (iv) The variety $V$ is Taylor, then $P_2 \not\in V$. By Lemma 1.10 every left-quasigroup in $V$ is connected and then superconnected.

(iv) ⇒ (iii) Every left-quasigroup in $V$ is superconnected. By Lemma 1.18 every superconnected is coherent, i.e. $V$ is coherent.

According to Lemma 1.24 connected left-quasigroups are congruence uniform, therefore so it is any Mal’cev variety of left-quasigroup. \[\square\]

According to Theorem 2.2 for left-quasigroups the lattice of Mal’cev classes in figure 1 turns into the chain in figure 2.

Corollary 2.3. Let $V$ be a variety of idempotent left-quasigroups. The following are equivalent:

(i) $V$ has a Mal’cev term.

(ii) Every algebra in $V$ is faithful and (super)connected.

In particular, if $Q$ is a finite idempotent left-quasigroup then $V(Q)$ has a Mal’cev term if and only if $Q$ is superconnected.

Proof. (i) ⇔ (ii) We can just apply Theorem 2.2 taking into account that superconnected idempotent left-quasigroups are faithful, according to Lemma 1.11.

Let $Q$ be a finite idempotent left-quasigroup. According to [BKST15, Theorem 1.1], $V(Q)$ has Taylor term if and only if $P_2 \not\in \text{HS}(Q)$, i.e. $Q$ is superconnected by Corollary 1.4. \[\square\]

For Cayley varieties we have the following:

Proposition 2.4. Let $V$ be a Cayley variety of left-quasigroups. The following are equivalent:

(i) $V$ has a Mal’cev term.

(ii) Every algebra in $V$ is (super)connected and faithful.

Proof. (i) ⇒ (ii) According to [KK13, Theorem 3.13] a variety with a Taylor term does not contain any strongly abelian congruence. Strongly abelian congruences of left-quasigroups are contained in the Cayley kernel [BS19a, Proposition 5.1], therefore every left-quasigroup in $V$ is faithful.

(ii) ⇒ (iii) It follows by Theorem 2.2 \[\square\]

Quasigroups have a Mal’cev term. If every quasigroup of a given variety is term equivalent to its left-quasigroup reduct, then the left-quasigroup reducts of such quasigroups is a Mal’cev variety of left-quasigroup. For instance:

(i) the variety of commutative quasigroups, i.e. the quasigroups satisfying

$$x \ast y = y \ast x.$$

(ii) The variety of quasigroups satisfying the identity

$$((x \ast y) \ast y) \ast y \approx x$$

for every $n \in \mathbb{N}$.

(iii) The variety of quasigroups satisfying the identity

$$(x \ast y) \ast x = y \approx x \ast (y \ast x).$$

On the other hand, Mal’cev varieties of left-quasigroups are not limited to this kind of examples, as witnessed by the following example.

Example 2.5. Let $V_n$ be the variety of left-quasigroups satisfying $L_n^+(x) = L_n^-(y)$ where $n \in \mathbb{Z}$ then

$$(6) \quad m(x, y, z) = L_n^+ L_n^- (z)$$

is a Mal’cev term and the unique idempotent algebra in $V_n$ is the trivial one.

A construction for such left-quasigroups over a set $Q$ and for $n \in \mathbb{Z}$ which are not latin is the following: let $e$ be a fixed element in $Q$, we define $L_e = 1$ and $L_a$ to be any cycle $(a, \ldots, e)$ of length $n$ for every $a \in Q$ if $n > 0$. If $n < 0$ we define $L_a^{-1}$ in the same way (see Example 1.10 for $n = 1$ and $|Q| = 4$).
3. Racks and Quandles

3.1. Superconnected racks. A rack is a left distributive left-quasigroup, i.e. a left-quasigroup satisfying the identity

\[ x * (y * z) = (x * y) * (x * z). \]

An idempotent rack is a quandle. Left-distributivity implies that for a quandle \( Q, hL_a h^{-1} = L_{h(a)} \) for every \( h \in \text{LMlt}(Q) \) and \( a \in Q \). In particular, the displacement group is simply given by

\[ \text{Dis}(Q) = \{ L_a L_a^{-1}, a, b \in Q \}. \]

Note that latin quandles are faithful and that faithful racks are quandles.

Example 3.1. Every permutation left-quasigroup is a rack.

Example 3.2. Let \( G \) be a group, \( f \in \text{Aut}(G) \) and \( H \leq \text{Fix}(f) = \{ a \in G : f(a) = a \} \). Let \( G/H \) be the set of left cosets of \( H \) and the multiplication defined by

\[ aH * bH = af(a^{-1}b)H. \]

Then \( \mathcal{Q}_{\text{Hom}}(G,H,f) = (G/H,*,\setminus) \) is a quandle, called a coset quandle. A coset quandle \( \mathcal{Q}_{\text{Hom}}(G,H,f) \) is called principal if \( H = 1 \) and is such case it is denoted by \( \mathcal{Q}_{\text{Hom}}(G,f) \). A principal quandle is called affine if \( G \) is abelian and in such case it is denoted by \( \text{Aff}(G,f) \).

The construction in Example 3.2 is very important. Indeed, connected quandles can be represented as coset quandles over their displacement group.

Proposition 3.3. [HSV16 Theorem 4.1] A quandle \( Q \) is connected then \( Q \cong \mathcal{Q}(\text{Dis}(Q), \text{Dis}(Q)_a, \overline{L}_a) \) for every \( a \in Q \).

Let \( Q \) be a rack, \( A \) an abelian group, \( \psi \in \text{Aut}(A) \) and a map \( \theta : Q \times Q \to A \). If the mapping \( \theta \) and the automorphism \( \psi \) satisfy some suitable conditions (in order to guarantee left distributivity, see [BS19b, Section 7]) the left-quasigroup \( E = Q \times_{\psi,\theta} A = (Q \times A, *) \) where

\[ (a, s) * (b, t) = (a * b, (1 - \psi)(s) + \psi(t) + \theta_{a,b}) \]

for every \( a, b \in Q \) and \( s, t \in A \) is a rack and it is called central extension of \( Q \) by \( A \). The projection onto \( Q \) is a rack morphism and if \( \psi = 1 \), then its kernel is contained in the congruence \( \lambda_E \). In this case, following [BS19a] and [BS19b] in such case we say that \( E \) is an abelian covering of \( Q \).

A permutation rack \( Q = (Q,f) \) is connected if and only if \( Q = \{ f^j(a) : j \in \mathbb{Z} \} \) for every \( a \in Q \). The rack \( Q \) is isomorphic to \( (A, +1) \) where \( A \) is a cyclic group and \( a * b = b + 1 \) for every \( a, b \in Q \). In particular, \( Q \) is generated by any of its elements and then it is superconnected.

Recall that for a rack \( Q \) the equivalence relation \( \mathbf{ip}_Q \) which blocks are \( [a]_{\mathbf{ip}_Q} = Sg(a) \) is a congruence of \( Q \) and it is contained in \( \lambda_0 \). [BS19a, Proposition 7.1].

Proposition 3.4. Let \( Q \) be a rack. The following are equivalent:

(i) \( Q \) is superconnected.
(ii) \( \mathbf{ip}_Q \) is superconnected.

If particular, if (i) holds then \( \lambda_Q = \mathbf{ip}_Q \) and \( Q \) is an abelian covering of the superconnected quandle \( Q/\mathbf{ip}_Q \).

Proof. The blocks of \( \mathbf{ip}_Q \) are subracks since \( \mathbf{ip}_Q \) is idempotent. The block \( [a]_{\mathbf{ip}_Q} \) is the subrack generated by \( a \) and so it is superconnected (it has no proper subrack and it is connected). So, the equivalence between (i) and (ii) follows by Remark 1.15.

If \( Q \) is superconnected quandle and so it is faithful. Therefore, \( \mathbf{ip}_Q = \lambda_Q \). Moreover we can apply [BS19a, Corollary 7.1(5)], i.e. \( Q \) is an abelian covering of \( Q/\lambda_Q \). \( \square \)

In [BS19a] we investigate a Galois connection between the congruence lattice of a rack and particular normal subgroups of the left multiplication group. If \( Q \) is a rack we define \( \text{Norm}(Q) = \{ N \leq \text{LMlt}(Q) : N \leq \text{Dis}(Q) \} \). The pair of mappings \( \alpha \mapsto \text{Dis}_\alpha \) and \( N \mapsto \text{con}_N = \{ (a, b) \in Q \times Q : L_a L_b^{-1} \in N \} \) provide a monotone Galois connection between \( \text{Con}(Q) \) and \( \text{Norm}(Q) \). If the mappings \( \text{Dis} \) and \( \text{con} \) are mutually inverses lattice isomorphism, we say that \( Q \) has the CDSg property (see [BS19a, Section 3.4]).

According to the commutator theory developed in [FAM87] and adapted to racks in [BS19a] we can define abelianness and centrality for congruences of general algebras and consequently nilpotence and solvability by using a special chain of congruences defined in analogy with the derived series and the lower central series of groups, using the commutator between congruences as defined in [FAM87] (we denote the commutator between two congruences \( \alpha, \beta \) by \( [\alpha, \beta] \)). The derived series of a rack \( Q \) is defined as

\[ \gamma^0(Q) = 1_Q, \quad \gamma^{n+1}(Q) = [\gamma^n(Q), \gamma^n(Q)], \]
and the lower central series as
\[ \gamma_0(Q) = 1_Q, \quad \gamma_{n+1}(Q) = [\gamma_n(Q), 1_Q], \]
for \( n \in \mathbb{N} \). A left-quasigroup is solvable (resp. nilpotent) of length \( n \) if \( \gamma^n(Q) = 0_Q \) (resp \( \gamma_n(Q) = 0_Q \)). For racks abelianess and centrality of congruences are completely determined by the properties of the relative displacement groups [BS19b, Theorem 1.1]. In particular, a quandle is nilpotent (resp. solvable) if and only if its displacement group is nilpotent (resp. solvable) [BS19b, Theorem 1.2].

Some of the results stated in [BS19b] for finite latin quandles, actually apply to the class of super-connected quandles, showing that connectedness is a relevant property also from the commutator theory viewpoint.

**Proposition 3.5.** Let \( Q \) be a superconnected quandle. Then:

(i) \([\alpha, \beta] = [\beta, \alpha] = \text{con}_Q(\alpha, \beta) = 0_Q\) for every \( \alpha, \beta \in \text{Con}(Q) \).

(ii) The mapping \( \text{Dis} \) is injective and the mapping \( \text{con} \) is surjective.

**Proof.** All factors of superconnected quandles are superconnected and then faithful. So, we can apply directly [BS19b, Proposition 5.5]. \( \square \)

Some of the contents of Section 2.4 of [Bon20] extend to principal superconnected quandles.

**Proposition 3.6.** Let \( Q = Q_{\text{Rom}}(G, f) \) be a superconnected quandle.

(i) The subquandles of \( Q \) are coset with respect to \( f \)-invariant subgroups of \( G \) and they are principal.

(ii) \( \text{Dis}_n = \text{Dis}^n = \text{Dis}(\alpha)^{(n)} \) for every \( \alpha \in \text{Con}(Q) \).

(iii) \( \text{Con}(Q) \equiv \{ N \leq G : f(N) = N \} \) and \( Q/\alpha \) is principal for every \( \alpha \in \text{Con}(Q) \).

(iv) Principal superconnected quandles have the CDSg property.

**Proof.** All subquandles of \( Q \) are connected, then we can apply [Bon20, Lemma 2.7] for (i), [Bon20, Corollary 2.11] for (ii) and (iii).

By virtue of (ii) and since every superconnected quandle is faithful, we can apply [BS19b, Proposition 3.10] and then principal superconnected quandles have the CDSg property. \( \square \)

**Problem 1.** According [Bon20, Proposition 1.8] right multiplications of principal faithful quandles are injective. So, finite superconnected principal quandles are latin. Are superconnected principal quandles latin? According to Example [17.12] the converse implication is not true (\( \text{Aff}(\mathbb{Q}, -1) \) is latin but not superconnected).

3.2. Superconnected quandles VS latin quandles. Superconnected quandles form a proper subclass of quandles with no proper projection subquandles (the quandle \( \text{Aff}(\mathbb{Z}, -1) \) has no projection subquandle but it is not connected, see Example [17.9]).

Finite latin quandles are superconnected, but the converse implication fails, although examples seem to be rare. Examples of superconnected non-latin quandles are given by locally strictly simple quandles studied in [Bon19]. The smallest such quandles are \text{SmallQuandle}(28,i) with \( i = 3, 4, 5, 6 \) in the \text{RIQ} library of GAP.

On the other hand, there exist infinite affine latin quandles which are not superconnected (e.g. \( \text{Aff}(\mathbb{Q}, -1) \)). In particular, the class of latin quandles is not a subvariety of the variety of quandles and then latin quandles have a Mal’cev term as quasigroups but they do not have a Mal’cev term as left-quasigroup.

In this section we will show that some families of superconnected quandles are indeed latin.

**Theorem 3.7.** Nilpotent superconnected quandles are latin.

**Proof.** If \( Q \) is abelian and superconnected, then it is faithful and connected and then latin [Bon20, Corollary 2.6]. Let \( Q \) be nilpotent of length \( n+1 \), i.e. \( \gamma_n(Q) \) is central. The group \( \text{Dis}_{\gamma_n(Q)} \) is transitive on each block of \( \gamma_n(Q) \) and \( Q \) is connected. Then we can apply [BS19b, Proposition 7.8] and we have that \( Q \) is a central extension of \( Q/\gamma_n(Q) \), i.e. the quandle operation of \( Q \) is defined as in (7). By induction on the nilpotency length, \( Q/\gamma_n(Q) \) is latin and the blocks of \( \gamma_n(Q) \) are abelian and therefore latin, i.e. \( 1 - \psi \) is bijective. Therefore the right multiplication \( R_{\alpha} \) has inverse \( R_{1-\alpha}^{-1}(b, t) = (R_{\alpha}^{-1}(b), (1 - \psi)^{-1}(s - \psi(t) - \theta_{\alpha, b}) \) and so \( Q \) is latin. \( \square \)

Note that the Mal’cev quandles of size 28 mentioned above are solvable but not latin, so Theorem 3.7 does not extend to the solvable case. Nevertheless finite soluble superconnected quandles have the Lagrange property (extending a known result for left distributive quasigroups [Gal81]).

**Proposition 3.8.** Finite soluble superconnected quandles have the Lagrange property.
Proof. If $Q$ is abelian, the statement is true because subquandles correspond to submodules with respect to the structure given by the affine representation \cite[Proposition 2.18]{Bor90}. Let $Q$ be solvable of length $n + 1$ i.e. $\gamma^n(Q)$ is an abelian cogroune and let $M$ be a subquandle. Then $|M| = |M/\gamma^n(Q)||a]\cap M|$. Since $[a]$ is affine, $[M\cap[a]]$ divides $[[a]]$ and since $Q/\gamma^n(Q)$ is solvable of length $n$, by induction we have that $|M/\gamma^n(Q)|$ divides $|Q/\gamma^n(Q)|$. Therefore, $|M|$ divides $|Q|/|Q/\gamma^n(Q)||[a]|$. $\square$

Recall that a quandle $Q$ such that the identity $x \ast (x \ast y) = y$ holds is called involutory.

**Proposition 3.9.** Let $Q$ be a 2-generated involutory quandle. Then $\text{Dis}(Q)$ is cyclic and if $Q$ is connected then $Q \cong \text{Aff}(\mathbb{Z}_{2^{m+1}},-1)$ for some $m \in \mathbb{N}$ and $Q$ is latin.

**Proof.** According to \cite[Corollary 10.4]{Joy82}, the free 2-generated involutory quandle $F$ is isomorphic to $\text{Aff}(\mathbb{Z},-1)$ and so its displacement group is $Z$. So the surjective quandle homomorphism $F \rightarrow Q$ induces a surjective group homomorphism $Z \rightarrow \text{Dis}(Q)$ and so $\text{Dis}(Q)$ is cyclic. If $Q$ is connected, then $Q \cong \text{Aff}(\text{Dis}(Q),-1)$ and $\text{Dis}(Q)$ is finite and has odd order and $R_0 : x \mapsto 2x$ is bijective. Then $Q$ is latin. $\square$

**Corollary 3.10.** Superconnected involutive quandles are latin.

**Proof.** Let $Q$ be an involutive superconnected quandle. According to Lemma \ref{lem:finite}, every pair of elements generates a finite latin subquandle. Hence, the whole quandle is latin. Indeed, if $x * a = y * a$, then $x * a = y * a \in \text{Sg}(a, x)$, which is finite and latin and so $R_0(U) = U$. Hence, $x = y$ and right multiplications are injective. For every $a, b \in Q$ there exists $x \in \text{Sg}(a, b)$ for which $x * a = b$ and so right multiplications are surjective.

In the finite case we recover the main result of \cite{Ero03}.

**Corollary 3.11.** Locally finite involutive quandles with no projection subquandles are latin.

**Proof.** Every pair $a, b \in Q$ generates a finite subquandle of $Q$. According to Lemma \ref{lem:finite}, then $S = \text{Sg}(a, b)$ has cyclic displacement group and the orbits of $S$ are isomorphic to $\text{Aff}(\mathbb{Z}_m, -1)$. If $m$ is even then $S$ has projection subquandle and then $m$ is odd. Let $O_1, O_2 \subseteq S$ be two orbits with respect to the action of $\text{Lmt}(Q)$. Then the left multiplication of the elements of $O_1$ acts on $O_2$ which has odd size. The left multiplications have order 2 and so they have a fixed point, i.e. $P_2 \in \text{S}(Q)$. Hence $S$ is connected for every $a, b \in Q$ and so $Q$ is superconnected. Then we can conclude using Corollary \ref{cor:finite}.

### 3.3. Displacement metric for connected quandles.

For a quandle $Q$, we define the length of $h \in \text{Dis}(Q)$ as

$$l(h) = \min_{n \in \mathbb{N}} \{h = (L_{a_1} L_{b_1}^{-1})^{k_1} \cdots (L_{a_n} L_{b_n}^{-1})^{k_n}, a_i, b_i \in Q, k_i = \pm 1\}.$$ 

Note that $l(h)$ is nothing but the length of the geodesic between the identity and $h$ in the Cayley graph of $\text{Dis}(Q)$ with respect to the set of generators $\{L_{a_1} L_{b_1}^{-1} : a, b \in Q\}$. We can use the metric in such Cayley graph to define a metric on $Q$.

**Lemma 3.12.** Let $Q$ be a connected quandle. The map

$$d : Q \times Q \rightarrow \mathbb{N}, \quad (a, b) \mapsto \min_{h \in \text{Dis}(Q)} \{l(h) : a = h(b)\}$$

is a metric, i.e.:

(i) $d(a, b) = d(b, a)$;
(ii) $d(a, b) = 0$ if and only if $a = b$;
(iii) $d(a, c) \leq d(a, b) + d(b, c)$,

for every $a, b, c \in Q$.

If $Q$ is a connected quandle, we call the metric defined in \ref{def:metric} the displacement metric over $Q$ and we define the diameter of $Q$ as

$$d(Q) = \sup_{a, b \in Q} \{d(a, b)\}.$$ 

The diameter of $Q$ might be infinite.

**Lemma 3.13.** Let $Q$ be a connected quandle. Then $d(Q) = \sup_{a \in Q} \{d(a, e)\}$ for every $e \in Q$.
Proof. Since $hL_aL_b^{-1}h^{-1} = L_{h(a)}L_{h(b)}^{-1}$ for every $a, b \in Q$ and $h \in \text{Dis}(Q)$ then $h$ is an isometry of $(Q, d)$, i.e. $d(a, b) = d(h(a), h(b))$ for every $a, b \in Q$ and $h \in \text{Dis}(Q)$. The quandle $Q$ is connected, then for every $e, b \in Q$ there exists $h \in \text{Dis}(Q)$ such that $h(b) = e$. Therefore

$$d(Q) = \sup_{a, b \in Q} \{d(a, b)\} = \sup_{h \in \text{Dis}(Q)} \{d(h(a), e)\} = \sup_{a \in Q} \{d(a, e)\}$$

for every $e \in Q$. \qed

It $Q$ is connected easy to prove that:

(i) $d(Q) \leq d(Q)$ for every $a \in \text{Com}(Q)$.

(ii) $d(Q) = 1$ if and only if the right multiplications of $Q$ are surjective (e.g. if $Q$ is latin).

The direct product of a finite number of connected quandles is connected. Nevertheless, in general it is not so. In the following we give a necessary condition for the connectedness of the direct product of an infinite set of connected quandles.

Let $\{Q_i : i \in I\}$ be a set of quandles and $Q = \prod_{i \in I} Q_i$. Let us denote by $\alpha_i$ the kernel of the canonical homomorphism $\phi_i : Q \rightarrow Q_i$ and the correspondent group morphism defined as in [1] by $\pi_{\alpha_i} : \text{Dis}(Q) \rightarrow \text{Dis}(Q_i)$ for every $i \in I$. We also denote by $\mathcal{T}$ an element $(\alpha_i)_{i \in I} \in Q$.

Lemma 3.14. Let $\{Q_i : i \in I\}$ be a set of quandles and $Q = \prod_{i \in I} Q_i$. Then

$$\text{Dis}(Q) \leq \bigcup_{n \in \mathbb{N}} \{ (h_i)_{i \in I} \in \prod_{i \in I} \text{Dis}(Q_i) : l(\pi_{\alpha_i}(h)) \leq n, \text{ for every } j \in I \}.$$ 

Proof. Since $\bigwedge_{i \in I} \alpha_i = 0_Q$, then $\bigcap_{i \in I} \text{Dis}^{\alpha_i} = 1$, therefore $\text{Dis}(Q)$ embeds into $\prod_{i \in I} \text{Dis}(Q_i)$. Since every $h \in \text{Dis}(Q)$ is a product of a finite number of generators $\{L_aL_b^{-1} : \mathcal{T}, \mathcal{B} \in \mathcal{Q}\}$, then so are their images under $\pi_{\alpha_i}$ for every $i \in I$, i.e. $h \in ((h_i)_{i \in I} : l(\pi_{\alpha_i}(h)) \leq n$, for every $j \in I$) for some $n \in \mathbb{N}$.

Proposition 3.15. Let $\{Q_i : i \in I\}$ be an infinite set of connected quandles. If $Q = \prod_{i \in I} Q_i$ is connected then $\text{sup}_{\alpha_i}\{d(Q_i)\}$ is finite.

Proof. The quandle $Q$ is connected if and only if $\text{Dis}(Q)$ is transitive. According to Lemma 3.14 the elements of $\text{Dis}(Q)$ are elements of $\prod_{i \in I} \text{Dis}(Q_i)$ for which every component can be written with a bounded number of generators of $\text{Dis}(Q_i)$. Therefore the orbit of an elements $\mathcal{T}$ is contained in $\bigcup_{n \in \mathbb{N}} \{ Q : d(\alpha_i, \alpha_i) \leq n \text{ for every } i \in I \}$. If $\text{sup}_{\alpha_i} d(Q)$ is not finite, it is witnessed by $\{\alpha_i : i \in I\}$, then $\mathcal{T}$ is not in the orbit of $\mathcal{T}$. Therefore if $Q$ is connected then $\text{sup}_{\alpha_i}\{d(Q_i)\}$ is finite. \qed

4. Mal’cev classes of racks

4.1. Mal’cev varieties of racks. A left translation term (LT term) is a term of the form

$$t(x_1, \ldots, x_n) = L_{s_1}^{k_1}(x_1) \cdots L_{s_m}^{k_m}(x_m) = s_{m+1}(x_{m+1}),$$

where $s_i$ is a unary term for each $1 \leq j \leq m+1$. Every rack term is equivalent to a LT term with $s_j(x_{i_j}) = x_{i_j}$ ([BST19b] Proposition 4.1) and then we have that a variety of racks have a Taylor term if and only if satisfies an identity of the form

$$L_{s_1}^{k_1} \cdots L_{s_n}^{k_n}(y) = x,$$

where $n \in \mathbb{N}$ and $k_i = \pm 1$ for every $1 \leq i \leq n$ and the right-most variables of the left-hand side is different from each other. Any variety of racks is a Cayley variety, so we can apply Proposition 2.4

Theorem 4.1. Let $V$ be a variety of racks. The following are equivalent:

(i) $V$ has a Taylor term.

(ii) Every algebra in $V$ is a (super)connected and faithful quandle.

(iii) $V$ has a Mal’cev term.

Proof. The implication (iii) $\Rightarrow$ (i) is clear and for (ii) $\Rightarrow$ (iii) we can apply Theorem 2.2

(i) $\Rightarrow$ (ii) We can apply Proposition 2.2 so every rack in $V$ is faithful, and then it is a quandle. \qed

The diameter of quandles in a Mal’cev variety is bounded.

Proposition 4.2. Let $Q$ be a Mal’cev variety of quandles. Then $\text{sup}_{Q \in V}\{d(Q)\}$ is finite.

Proof. Assume that $\text{sup}_{Q \in V}\{d(Q)\}$ is infinite and take $\{Q_i : i \in I\}$ be a set of witnesses. Then, according to Proposition 3.15 $\prod_{i \in I} Q_i$ is not connected, contradiction. \qed
Let $\mathcal{V}$ be a Mal’cev variety of quandle satisfying the identity

$$L_{k_1}^n \cdots L_{k_n}^n(x) \approx y.$$  

We can rewrite $h = L_{k_1}^n \cdots L_{k_n}^n$ in the left multiplication group of the free quandle $F$ on generators $\{x, y, x_1, \ldots, x_n\}$ as $h = gL_n^m$ for some $g \in \text{Dis}(F)$ and $n \in \mathbb{Z}$. Hence, we can rewrite the identity above as

$$L_{k_1}^n \cdots L_{k_n}^n(x) = gL_n^m(x) = g(x) \approx y.$$  

So, the length $l(g)$ provides an upper bound for $\sup_{Q \in \mathcal{V}} \{d(Q)\}$.

The following identities provide Mal’cev varieties of quandles. Moreover, such varieties contain the smallest examples of Mal’cev quandles which are not latin. The witnesses are taken from the RIG library of GAP:

| Identity | Witness in the RIG library |
|----------|-----------------------------|
| $L_x^2L_y^2L_xL_yL_yL_xL_y(x) \approx y$ | SmallQuandle(28,3) |
| $L_x^2L_y^2L_xL_yL_yL_xL_y(x) \approx y$ | SmallQuandle(28,4) |
| $L_x^2L_y^2L_xL_yL_yL_xL_y(x) \approx y$ | SmallQuandle(28,5) |
| $L_x^2L_y^2L_xL_yL_yL_xL_y(x) \approx y$ | SmallQuandle(28,6) |

Table 1. Examples of Mal’cev varieties of quandles

Examples of varieties of quandles which are not Mal’cev are the following:

(i) any subvariety of reductive quandles, i.e. quandles satisfying the identity

$$\left(\cdots ((x * y) * y) \cdots \right) * y \approx y$$

for every $n \in \mathbb{N}$.

(ii) Any subvariety of graphic quandles defined by the identity

$$x * (y * x) \approx y * x.$$  

Indeed, according to [Bon20] Proposition 1.4 every algebra in such varieties has projection subquandles, since every left multiplication has more than one fixed point.

Let $\mathcal{V}$ be a variety. The class of quandles in $\mathcal{V}$ with solvable (resp. nilpotent) displacement group of length less or equal to $n$ is a subvariety of $\mathcal{V}$ for every $n \in \mathbb{N}$. In particular if $\mathcal{V}$ is Mal’cev, the class of affine quandles of $\mathcal{V}$ is the subvariety of the abelian quandles of $\mathcal{V}$. The class of principal quandles of $\mathcal{V}$ is also a subvariety.

**Theorem 4.3.** Let $\mathcal{V}$ be a Mal’cev variety of quandles. The class of principal quandles of $\mathcal{V}$ is a subvariety of $\mathcal{V}$.

**Proof.** The product of principal quandles is principal [Bon20] Corollary 2.3. By virtue of Lemma 3.6(i) and (iii) subquandles and factors of principal Mal’cev quandles are principal. Hence the class of principal quandles of $\mathcal{V}$ is a subvariety.  

4.2. **Meet-semidistributive varieties of quandles.** As an immediate consequence of Theorem 0.3 we have that meet-semidistributive varieties of quandles are the varieties which have no finite model.

**Theorem 4.4.** Let $\mathcal{V}$ be a variety of quandles. The following are equivalent:

(i) $\mathcal{V}$ contains an abelian quandle.

(ii) $\mathcal{V}$ has a finite model.

In particular, $\mathcal{V}$ is meet-semidistributive if and only if it has no non-trivial finite models.

**Proof.** (i) $\Rightarrow$ (ii) If $A \in \mathcal{V}$ is an abelian quandle, then $\mathcal{V}(A) \leq \mathcal{V}$ contains a simple abelian quandle which is finite by [Bon20] Theorem 2.21.

(ii) $\Rightarrow$ (i) If $\mathcal{V}$ contains a finite quandle, then its minimal subquandles (with respect to inclusion) are abelian by [Bon20] Theorem 3.7.

According to Theorem 0.3 a variety of quandles is meet-semidistributive if and only if it does not contain an abelian quandle, i.e. it does not contain finite quandles.

As a direct consequence of Lemma 3.8 and Theorem 4.4 we have:
Corollary 4.5. There is no locally finite meet-semidistributive variety of quandles.

Corollary 4.6. There is no meet-semidistributive variety of involutory quandles.

Another consequence of Theorem [14] is reflected by the properties of the displacement groups.

Corollary 4.7. Let $V$ be a meet semidistributive variety of quandles. Then $[\text{Dis}_a, \text{Dis}_a] = \text{Dis}_a$ and $Z(\text{Dis}(Q)) = 1$ for every $a \in \text{Con}(Q)$ and every $Q \in V$. In particular, $\text{Dis}(Q)$ is perfect for every $Q \in V$.

Proof. According to Theorem [14, (iii)] and Proposition [15, (iv)], we have that $\text{Dis}_a = \text{Dis}_{[a,a]} \leq [\text{Dis}_a, \text{Dis}_a] \leq \text{Dis}_a$ for every $a \in \text{Con}(Q)$ and every $Q \in V$. The center of $Q$ is trivial. According to [15, Corollary 5.10], the center is $\text{con}(Z(\text{Dis}(Q)))$, so accordingly $Z(\text{Dis}(Q)) = 1$.

The following problem is still open.

Problem 2. Do meet semidistributive varieties of quandles exist?

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