On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space

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Abstract

In this paper, we proved decay properties of solutions to the Stokes equations with surface tension and gravity in the half space \( \mathbb{R}^+ = \{ (x', x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N > 0 \} \) \((N \geq 2)\). In order to prove the decay properties, we first show that the zero points \( \lambda_{\pm} \) of Lopatinski determinant for some resolvent problem associated with the Stokes equations have the asymptotics: \( \lambda_{\pm} = \pm i c_\varphi \sqrt{2} |\xi'|^{1/2} - 2 |\xi'|^2 + O(|\xi'|^{5/2}) \) as \( |\xi'| \to 0 \), where \( c_\varphi > 0 \) is the gravitational acceleration and \( \xi' \in \mathbb{R}^{N-1} \) is the tangential variable in the Fourier space. We next shift the integral path in the representation formula of the Stokes semi-group to the complex left half-plane by Cauchy’s integral theorem, and then it is decomposed into closed curves enclosing \( \lambda_{\pm} \) and the remainder part. We finally see, by the residue theorem, that the low frequency part of the solution to the Stokes equations behaves like the convolution of the \((N-1)\)-dimensional heat kernel and \( F^{-1}_N [ e^{\pm i c_\varphi \sqrt{2} |\xi'|^{1/2}} ](x') \) formally, where \( F^{-1}_N \) is the inverse Fourier transform with respect to \( \xi' \). However, main task in our approach is to show that the remainder part in the above decomposition decay faster than the residue part.

1 Introduction and main results

Let \( \mathbb{R}^+_N \) and \( \mathbb{R}^0_N \) \((N \geq 2)\) be the half space and its boundary, that is,

\[
\mathbb{R}^+_N = \{ (x', x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N > 0 \}, \quad \mathbb{R}^0_N = \{ (x', x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N = 0 \}.
\]

In this paper, we consider the following Stokes equations with the surface tension and gravity in the half space \( \mathbb{R}^+_N \):

\[
\begin{cases}
\partial_t U - \text{Div} S(U, \Theta) = 0, & \text{div} U = 0 \quad \text{in} \ \mathbb{R}^+_N, \ t > 0, \\
\partial_t H + U_N = 0 & \text{on} \ \mathbb{R}^0_N, \ t > 0, \\
S(U, \Theta) \nu + (c_\varphi - c_\sigma \Delta) H \nu = 0 & \text{on} \ \mathbb{R}^0_N, \ t > 0, \\
U|_{t=0} = f & \text{in} \ \mathbb{R}^+_N, \ H|_{t=0} = d & \text{on} \ \mathbb{R}^0_N.
\end{cases}
\tag{1.1}
\]

Here the unknowns \( U = (U_1(x, t), \ldots, U_N(x, t))^{T} \) and \( \Theta = \Theta(x, t) \) are the velocity field and the pressure at \( (x, t) \in \mathbb{R}^+_N \times (0, \infty) \), respectively, and also \( H = H(x', t) \) is the height function at \( (x', t) \in \mathbb{R}^0_N \times \mathbb{R}^0_N \).
The operators $\text{div}$ and $\Delta'$ are defined by
\[
\text{div} \, U = \sum_{j=1}^N D_j U_j, \quad \Delta' H = \sum_{j=1}^{N-1} D_j^2 H \quad (D_j = \frac{\partial}{\partial x_j})
\]
for any $N$-component vector function $U$ and scalar function $H$. $S(U, \Theta) = -\Theta I + D(U)$ is the stress tensor, where $I$ is the $N \times N$ identity matrix and $D(U)$ is the doubled strain tensor whose $(i,j)$ component is $D_{ij}(U) = D_i U_j + D_j U_i$. Moreover, $\text{Div} \, S(U, \Theta)$ is the $N$-component vector function with the $i$th component:
\[
\sum_{j=1}^N D_j(D_j U_i + D_i U_j - \delta_{ij} \Theta) = \Delta U_i + D_i \text{div} \, U - D_i \Theta.
\]

Let $\nu = (0, \ldots, 0, -1)^T$ be the unit outer normal to $\mathbb{R}_N^N$, and then
\[
\text{ith component of } S(U, \Theta)\nu = \begin{cases} 
-D_N U_i + D_i U_N & (i = 1, \ldots, N-1), \\
-2D_N U_N + \Theta & (i = N).
\end{cases}
\]

The parameters $c_\sigma > 0$ and $c_\tau > 0$ describe the gravitational acceleration and the surface tension coefficient, respectively, and the functions $f = (f_1(x), \ldots, f_N(x))^T$ and $d = d(x')$ are given initial data.

The equations (1.1) arise in the study of a free boundary problem for the incompressible Navier-Stokes equations. The free boundary problem is mathematically to find a $N$-component vector function $u = (u_1(x,t), \ldots, u_N(x,t))^T$, a scalar function $\theta = \theta(x,t)$, and a free boundary $\Gamma(t) = \{(x',x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N = h(x',t)\}$ satisfying the following Navier-Stokes equations:

\[
\begin{cases}
\rho(\partial_t u + u \cdot \nabla u) - \text{Div} \, S(u, \theta) = -pc_\sigma \nabla x_N, & \text{div} \, u = 0 \quad \text{in } \Omega(t), \ t > 0, \\
\partial_t h + u' \cdot \nabla' h - u_N = 0 & \text{on } \Gamma(t), \ t > 0, \\
\partial_t \mu_t = c_\sigma \nu_t & \text{on } \Gamma(t), \ t > 0, \\
w_{|t=0} = u_0 & \text{in } \Omega(0), \\
h_{|t=0} = h_0 & \text{on } \mathbb{R}^{N-1}.
\end{cases}
\]

Here $\Omega(t) = \{(x',x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N < h(x',t)\}$, and $\Omega(0)$ is a given initial domain; $\rho$ is a positive constant describing the density of the fluid; $\kappa = \kappa(x,t)$ is the mean curvature of $\Gamma(t)$, and $\nu_t$ is the unit outer normal to $\Gamma(t)$; $u \cdot \nabla u = \sum_{j=1}^N u_j D_j u$ and $u' \cdot \nabla' h = \sum_{j=1}^{N-1} u_j D_j h$.

A problem is called the finite depth one if the equations (1.2) is considered in $\Omega(t) = \{(x',x_N) \mid x' \in \mathbb{R}^{N-1}, \ -b < x_N < h(x',t)\}$ for some constant $b > 0$ with Dirichlet boundary condition on the lower boundary: $\Gamma_b = \{(x',x_N) \mid x' \in \mathbb{R}^{N-1}, \ x_N = -b\}$. There are several results for the finite depth problem. In fact, Beale [4] proved the local well-posedness in the case of $c_\sigma > 0$ and $c_\tau > 0$, and also [5] proved the global well-posedness for small initial data when $c_\sigma > 0$ and $c_\tau > 0$. Beale and Nishida [6] proved decay properties of the solution obtained in [5], but the paper is just survey. We can find the detailed proof in Hataya [7]. Tani and Taniaka [8] also treated both case of $c_\sigma > 0$ and $c_\tau > 0$ under the condition $c_\sigma > 0$. Along with these results, we refer to Allain [9], Hataya and Kawashima [10], and Bae [11]. Note that they treated the problem in the $L_2-L_2$ framework, that is, their classes of solutions are contained in the space-time $L_2$ space, and their methods are based on the Hilbert space structure. Thus, their methods do not work in general Banach spaces. From this viewpoint, we need completely different techniques since our aim is to treat (1.2) in the $L^2-L^2$ framework.

The study of free boundary problems with surface tension and gravity in the $L_p$-$L_q$ maximal regularity class were started by Shibata and Shimizu [12]. We especially note that Abels [13] proved the local well-posedness of the finite depth problem with $p = q > N$, $c_\sigma = 0$, and $c_\tau > 0$. In the case of the $L_p$-$L_q$ framework, Shibata [14] proved the local well-posedness of free boundary problems for the Navier-Stokes equations with $c_\sigma = c_\tau = 0$ in general unbounded domains containing the finite depth problem, where $p$ and $q$ are exponents satisfying the conditions: $1 < p, q < \infty$ and $2/p + N/q < 1$.

Concerning (1.2), under some smallness condition of initial data, Prüss and Simonett [15] showed the local well-posedness of the two-phase problem containing (1.2) with $c_\sigma > 0$ and $c_\tau = 0$, and also [16] and [17] proved the local well-posedness of the case where $c_\sigma > 0$ and $c_\tau > 0$. Recently, there are two papers due to Shibata and Shimizu [18, 19], which treat the linearized problem of (1.2) and some
where $1/q < \infty$, non-integer $s > 0$, and $m \in \mathbb{N}$, $W^s_q(\mathbb{R}^m)$ denotes the Sobolev space of order $s$ defined by

\[ W^s_q(\mathbb{R}^m) = \{ u \in L^q(\mathbb{R}^m) \mid \| u \|_{W^s_q(\mathbb{R}^m)} < \infty \}, \]

\[ \| u \|_{W^s_q(\mathbb{R}^m)} = \| u \|_{W^s_q(\mathbb{R}^m)} + \sum_{|\alpha| = s} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|D^s u(x) - D^s u(y)|^q}{|x-y|^{m+(s-|\alpha|)q}} \, dx \, dy \right)^{1/q}, \]

where $[s]$ is the largest integer lower than $s$. For any vector function $u = (u_1, \ldots, u_N)^T$ and $v = (v_1, \ldots, v_N)^T$ defined on $\mathbb{R}^N_+$, we set

\[ (u, v)_{\mathbb{R}^N_+} = \int_{\mathbb{R}^N_+} u(x) \cdot v(x) \, dx = \sum_{j=1}^N \int_{\mathbb{R}^N_+} u_j(x) v_j(x) \, dx. \]

The letter $C$ denotes a generic constant and $C(a, b, c, \ldots)$ a generic constant depending on the quantities $a, b, c, \ldots$. The value of $C$ and $C(a, b, c, \ldots)$ may change from line to line.

Let $\tilde{W}^{1}_{q,0}(\mathbb{R}^N_+)$ be the homogeneous spaces of order 1 defined by $\tilde{W}^{1}_{q,0}(\mathbb{R}^N_+) = \{ \theta \in L_q(\mathbb{R}^N_+ \mid \nabla \theta \in L_q(\mathbb{R}^N_+) \}$. In addition, we set $\tilde{W}^{1}_{q,0}(\mathbb{R}^N_+) = \{ \theta = \tilde{W}^{1}_{q,0}(\mathbb{R}^N_+) \mid \theta \in \mathbb{R} \}$ and $W^{1}_{q,0}(\mathbb{R}^N_+) = \{ \theta = \tilde{W}^{1}_{q,0}(\mathbb{R}^N_+) \mid L_q(\mathbb{R}^N_+) \}$. As was seen in [33 Theorem A.3], $W^{1}_{q,0}(\mathbb{R}^N_+)$ is dense in $\tilde{W}^{1}_{q,0}(\mathbb{R}^N_+)$ with the gradient norm $\| \nabla \cdot \|_{L_q(\mathbb{R}^N_+)}$.

Then the second solenoidal space $J^0_q(\mathbb{R}^N_+)$ is defined by

\[ J^0_q(\mathbb{R}^N_+) = \{ f \in L_q(\mathbb{R}^N_+) \mid (f, \nabla \varphi)_{\mathbb{R}^N_+} = 0 \text{ for any } \varphi \in \tilde{W}^{1}_{q,0}(\mathbb{R}^N_+) \}, \]

where $1/q + 1/q' = 1$. For simplicity, we set

\[ X_q = J^0_q(\mathbb{R}^N_+) \times W^{2-1/q}_{q}(\mathbb{R}^{N-1}) \quad Y_q = L_q(\mathbb{R}^N_+) \times L_q(\mathbb{R}^{N-1}), \]

\[
X_q^i = L_q(\mathbb{R}^N_+) \times W^{i-1/q}_{q}(\mathbb{R}^{N-1}) \quad (i = 1, 2),
\]

(1.3)
and let $\mathcal{E}H$ be the harmonic extension of $H$, that is,
\[
\begin{cases}
\Delta \mathcal{E}H = 0 & \text{in } \mathbb{R}^N_p, \\
\mathcal{E}H = H & \text{on } \mathbb{R}^N_0.
\end{cases}
\]

The main results of this paper then is stated as follows:

**Theorem 1.1.** Let $1 < p < \infty$, $c_g > 0$, and $c_\sigma > 0$.

1. For every $t > 0$ there exists operators
   \[
   S(t) \in \mathcal{L}(X^p_1, W^2_p(\mathbb{R}^N_1)), \quad \Pi(t) \in \mathcal{L}(X^p_1, \tilde{W}^1_p(\mathbb{R}^N_1)), \quad T(t) \in \mathcal{L}(X^p_1, W^{3-1/p}_p(\mathbb{R}^{N-1}));
   \]
   such that for $F = (f, d) \in X^p_1$,
   \[
   S(t)F \in C^1((0, \infty), J_p(R^N_1)) \cap C^0((0, \infty), W^2_p(R^N_1)), \\
   \Pi(t)F \in C^0((0, \infty), \tilde{W}^1_p(R^N_1)), \\
   T(t)F \in C^1((0, \infty), W^{2-1/p}_p(R^{N-1})) \cap C^0((0, \infty), W^{3-1/p}_p(R^{N-1})),
   \]
   and that $(U, \Theta, H) = (S(t)F, \Pi(t)F, T(t)F)$ solves uniquely (1.1) with
   \[
   \lim_{t \to 0} \|(U(t), H(t)) - (f, d)\|_{X^p_0} = 0.
   \]

2. Let $1 \leq r \leq 2 \leq q \leq \infty$ and $F = (f, d) \in X^r \cap X^q_1$. The operators, obtained in (1), then are decomposed into
   \[
   S(t)F = S_0(t)F + S_\infty(t)F + R(t)f, \\
   \Pi(t)F = \Pi_0(t)F + \Pi_\infty(t)F + P(t)f, \\
   T(t)F = T_0(t)F + T_\infty(t)F;
   \]
   which satisfy the estimates as follows: For $k = 1, 2$, $\ell = 0, 1, 2$, and $t \geq 1$
   \[
   \|S_0(t)F, \partial_\ell \mathcal{E}(T_0(t)F)\|_{L^q(R^N_1)} \leq C(t + 1)^{-m(q, r)}\|F\|_{X^q_0} \quad \text{if } (q, r) \neq (2, 2),
   \]
   \[
   \|\nabla^k S_0(t)F\|_{L^q(R^N_1)} \leq C(t + 1)^{-n(q, r) - k/8}\|F\|_{X^q_0},
   \]
   \[
   \|\partial_\ell S_0(t)F, \nabla \Pi_0(t)F\|_{L^q(R^N_1)} \leq C(t + 1)^{-m(q, r) - 1/4}\|F\|_{X^q_0},
   \]
   \[
   \|\nabla^k \partial_\ell \mathcal{E}(T_0(t)F)\|_{L^q(R^N_1)} \leq C(t + 1)^{-m(q, r) - k/2}\|F\|_{X^q_0},
   \]
   \[
   \|\nabla^{1+\ell} \mathcal{E}(T_0(t)F)\|_{L^q(R^N_1)} \leq C(t + 1)^{-m(q, r) - 1/4 - \ell/2}\|F\|_{X^q_0}
   \]
   (1.6)
   with some positive constant $C$, where we have set
   \[
   m(q, r) = \frac{N - 1}{2} \left( \frac{1}{r} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right),
   \]
   \[
   n(q, r) = \frac{N - 1}{2} \left( \frac{1}{r} - \frac{1}{q} \right) + \min \left\{ \frac{1}{8} \left( \frac{1}{r} - \frac{1}{q} \right), \frac{1}{8} \left( 2 - \frac{1}{q} \right) \right\}.
   \]
   In addition, there exist positive constants $\delta$ and $C$ such that for $t \geq 1$
   \[
   \|\partial_\ell S_\infty(t)F, \nabla \Pi_\infty(t)F\|_{L^p(R^N_1)} + \|(S_\infty(t)F, \partial_\ell \mathcal{E}(T_\infty(t)F), \nabla \mathcal{E}(T_\infty(t)F))\|_{W^2_p(R^N_1)} \leq C e^{-\delta t}\|F\|_{X^q_0}.
   \]

Finally, for $t \geq 1$ and $\ell = 0, 1, 2$,
   \[
   \|\nabla^\ell R(t)f\|_{L^p(R^N_1)} \leq C(t + 1)^{-\ell/2}\|f\|_{L^p(R^N_1)},
   \]
   \[
   \|\partial_\ell R(t)f, \nabla P(t)f\|_{L^p(R^N_1)} \leq C(t + 1)^{-1}\|f\|_{L^p(R^N_1)}.
   \]

This paper consist of five sections. In the next section, we introduce some symbols and lemmas, and also consider some resolvent problem associated with $\mathbb{L}^1$ with $c_g = c_\sigma = 0$. In Section 3, we construct the operators $S(t), \Pi(t), \text{and } T(t)$, and also give the decompositions (1.5). Finally, Theorem 1.1 (2) is proved in Section 4 and Section 5.
2 Preliminaries

We first give some symbols used throughout this paper. Set
\[ \Sigma_\varepsilon = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| < \pi - \varepsilon, \lambda \neq 0 \}, \quad \Sigma_{\varepsilon, \lambda_0} = \{ \lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0 \} \]
for any \( 0 < \varepsilon < \pi/2 \) and \( \lambda_0 > 0 \). We then define
\[
A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2} \quad (\text{Re} B \geq 0), \quad \mathcal{M}(a) = \frac{e^{-Ba} - e^{-Aa}}{B - A},
\]
\[
D(A, B) = B^3 + AB^2 + 3A^2 B - A^5,
\]
\[
L(A, B) = (B - A)D(A, B) + A(c_\sigma + c_\sigma A^2)
\]
for \( \xi' = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N, \lambda \in \Sigma_\varepsilon, \) and \( a > 0 \). Especially, we have, for \( \ell = 1, 2, \)
\[
\frac{\partial^\ell}{\partial a^\ell} \mathcal{M}(a) = (-1)^\ell \left((B + A)^{\ell-1}e^{-Ba} + A^\ell \mathcal{M}(a)\right),
\]
\[
\mathcal{M}(a) = -a \int_0^1 e^{-(B\theta + A(1-\theta))a} d\theta.
\]

The following lemma was proved in [14] Lemma 5.2, Lemma 5.3, Lemma 7.2.

Lemma 2.1. Let \( 0 < \varepsilon < \pi/2, s \in \mathbb{R}, a > 0, \) and \( \alpha' \in \mathbb{N}_0^{N-1}. \)

1. There holds the estimate
\[
b_c(|\lambda|^{\frac{1}{2}} + A) \leq |\text{Re} B| \leq |\lambda|^{\frac{1}{2}} + A)
\]
for any \( (\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\varepsilon \) with \( b_c = (1/\sqrt{2}) \sin(\varepsilon/2))^{3/2}. \)

2. There exist a positive constant \( C = C(\varepsilon, s, \alpha') \) such that for any \( (\xi', \lambda) \in (\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_\varepsilon \)
\[
|D_{\xi'}^\alpha A^s| \leq CA^{s-|\alpha'|}, \quad |D_{\xi'}^\alpha e^{-Aa}| \leq CA^{-|\alpha'|}e^{-((A/2)a)}, \quad |D_{\xi'}^\alpha B^s| \leq C(|\lambda|^{\frac{1}{2}} + A)^s-|\alpha'|,
\]
\[
|D_{\xi'}^\alpha e^{-Ba}| \leq C(|\lambda| + A)^{-|\alpha'|}e^{-(b_\alpha/8)(|\lambda|^{1/2} + A)a}, \quad |D_{\xi'}^\alpha D(A, B)^s| \leq C(|\lambda|^{\frac{1}{2}} + A)^{3s}A^{-|\alpha'|}
\]
\[
|D_{\xi'}^\alpha \mathcal{M}(a)| \leq CA^{-|\alpha'|}e^{-(b_\alpha/8)Aa}, \quad |D_{\xi'}^\alpha \mathcal{M}(a)| \leq C|\lambda|^{-\frac{1}{2}}A^{-|\alpha'|}e^{-(b_\alpha/8)Aa}.
\]

3. There exist positive constants \( \lambda_0 = \lambda_0(\varepsilon) \geq 1 \) and \( C = C(\varepsilon, \lambda_0, \alpha') \) such that for any \( (\xi', \lambda) \in (\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon, \lambda_0} \)
\[
|D_{\xi'}^\alpha L(A, B)^{-1}| \leq C(|\lambda|(|\lambda|^{\frac{1}{2}} + A)^2 + A(c_\sigma + c_\sigma A^2))^{-1}A^{-|\alpha'|}.
\]

Let \( f(x) \) and \( g(\xi) \) be functions defined on \( \mathbb{R}^N \), and then the Fourier transform of \( f(x) \) and the inverse Fourier transform of \( g(\xi) \) are defined by
\[
\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\xi} f(x) \, dx, \quad \mathcal{F}^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\xi} g(\xi) \, d\xi.
\]

We also define the partial Fourier transform of \( f(x) \) and the inverse partial Fourier transform of \( g(\xi) \) with respect to tangential variables \( x' = (x_1, \ldots, x_{N-1}) \) and its dual variable \( \xi' = (\xi_1, \ldots, \xi_{N-1}) \) by
\[
\mathcal{F}[f](\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\xi'} f(x', x_N) \, dx',
\]
\[
\mathcal{F}^{-1}[g](x', \xi_N) = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\xi'} g(\xi', \xi_N) \, d\xi'.
\]

Next we consider the following resolvent problem:
\[
\begin{cases}
\lambda w - \text{Div} S(w, p) = f, & \text{div} w = 0 \quad \text{in } \mathbb{R}_+^N, \\
S(w, p)\nu = 0 & \text{on } \mathbb{R}_0^N.
\end{cases}
\]
Lemma 2.2. Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $\lambda \in \Sigma_\varepsilon$, and $f \in L_q(\mathbb{R}_+^N)$. Then the equations (2.3) admits a unique solution $(w, p) \in W^1_q(\mathbb{R}_+^N) \times W^1_q(\mathbb{R}_+^N)$ possessing the estimate:

$$
\| (\lambda w, \lambda^{1/2} \nabla w, \nabla^2 w, \nabla p) \|_{L_q(\mathbb{R}_+^N)} \leq C \| f \|_{L_q(\mathbb{R}_+^N)}
$$

with some positive constant $C = C(\varepsilon, q, N)$. In addition, $\hat{w}_N(\xi', 0, \lambda)$ is given by

$$
\hat{w}_N(\xi', 0, \lambda) = \sum_{k=1}^{N-1} \frac{i \xi_k (B - A)}{D(A, B)} \int_0^\infty e^{-B y_n} \hat{f}_k(\xi', y_N) \, dy_N
$$

$$
+ \frac{A(B + A)}{D(A, B)} \int_0^\infty e^{-B y_n} \hat{f}_N(\xi', y_N) \, dy_N
$$

$$
- \sum_{k=1}^{N-1} \frac{i \xi_k (B^2 + A^2)}{D(A, B)} \int_0^\infty \mathcal{M}(y_N) \hat{f}_k(\xi', y_N) \, dy_N
$$

$$
- \frac{A(B^2 + A^2)}{D(A, B)} \int_0^\infty \mathcal{M}(y_N) \hat{f}_N(\xi', y_N) \, dy_N. \tag{2.4}
$$

$$
= \sum_{k=1}^{N-1} \frac{i \xi_k (B - A)}{D(A, B)} \int_0^\infty e^{-A y_n} \hat{f}_k(\xi', y_N) \, dy_N
$$

$$
+ \frac{A(B + A)}{D(A, B)} \int_0^\infty e^{-A y_n} \hat{f}_N(\xi', y_N) \, dy_N
$$

$$
- \sum_{k=1}^{N-1} 2i \xi_k AB \frac{A^2}{D(A, B)} \int_0^\infty \mathcal{M}(y_N) \hat{f}_k(\xi', y_N) \, dy_N
$$

$$
- \frac{2A^3}{D(A, B)} \int_0^\infty \mathcal{M}(y_N) \hat{f}_N(\xi', y_N) \, dy_N. \tag{2.5}
$$

Proof. The lemma was proved by Shibata and Shimizu \cite{14} Theorem 4.1 except for (2.4) and (2.5), so that we prove (2.4) and (2.5) here.

Given functions $g(x)$ defined on $\mathbb{R}_+^N$, we set their even extensions $g^e(x)$ and odd extensions $g^o(x)$ as

$$
g^e(x) = \begin{cases} g(x', x_N) & \text{in } \mathbb{R}_+^N, \\
g(x', -x_N) & \text{in } \mathbb{R}_-^N, \end{cases} \quad g^o(x) = \begin{cases} g(x', x_N) & \text{in } \mathbb{R}_+^N, \\
-g(x', -x_N) & \text{in } \mathbb{R}_-^N, \end{cases} \tag{2.6}
$$

where $\mathbb{R}_+^N = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1}, x_N < 0\}$. In addition, given the right member $f = (f_1, \ldots, f_N)^T$ of (2.3), we set $Ef = (f_1^1, \ldots, f_{N-1}^1, f_N^3)^T$. Let $(w^1, p^1)$ be the solution to the following resolvent problem:

$$
\lambda w^1 - \text{Div} S(w^1, p^1) = Ef, \quad \text{div} w^1 = 0 \quad \text{in } \mathbb{R}_+^N.
$$

We then have the following solution formulas (cf. \cite{17} Section 3):

$$
w^1_j(x, \lambda) = \mathbf{F}^{-1}_\xi \left[ \frac{(Ef)_j(\xi)}{\lambda + |\xi|^2} \right] (x)
$$

$$
- \sum_{k=1}^{N} \mathbf{F}^{-1}_\xi \left[ \frac{\xi_j \xi_k}{|\xi|^2 (\lambda + |\xi|^2)} (Ef)_k(\xi) \right] (x) \quad (j = 1, \ldots, N),
$$

$$
p^1(x, \lambda) = - \mathbf{F}^{-1}_\xi \left[ \frac{i \xi_j}{|\xi|^2} \cdot Ef(\xi) \right] (x). \tag{2.7}
$$

As was seen in \cite{14} Section 4], we have, by the definition of the extension $E$,

$$
D_N w^1_k(x', 0, \lambda) = 0, \quad p^1_j(x', 0, \lambda) = 0. \tag{2.8}
$$

Next we give the exact formulas of $\hat{w}^1_N(\xi', 0, \lambda)$ and $\hat{D}_N w^1_j(\xi', 0, \lambda)$ for $j = 1, \ldots, N - 1$. To this end, we use the following lemma which is proved by the residue theorem.
Lemma 2.3. Let $a \in \mathbb{R} \setminus \{0\}$, and let $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$. Then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ia_0 \xi_N} d\xi_N = e^{-A|a|}, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi_N e^{ia_0 \xi_N} d\xi_N = -\text{sign}(a) e^{-A|a|}/2,$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ia_0 \xi_N} d\xi_N = e^{-B|a|}, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_N e^{ia_0 \xi_N} d\xi_N = \text{sign}(a) e^{-B|a|},$$

where $\text{sign}(a)$ defined by the formula: $\text{sign}(a) = 1$ when $a > 0$ and $\text{sign}(a) = -1$ when $a < 0$.

In order to obtain

$$\hat{w}_N^1(\xi', 0, \lambda) = \sum_{k=1}^{N} \frac{i\xi_k}{\lambda} \int_0^{\infty} \left( e^{-Ay_N} - e^{-B\lambda_N} \right) \hat{f}_k(\xi', y_N) dy_N$$

$$+ \int_0^{\infty} \frac{e^{-B\lambda_N}}{B} \hat{f}_N(\xi', y_N) dy_N$$

$$+ \frac{1}{\lambda} \int_0^{\infty} \left( Ae^{-Ay_N} - Be^{-B\lambda_N} \right) \hat{f}_N(\xi', y_N) dy_N,$$

$$\widehat{D_Nw}_j^1(\xi', 0, \lambda) = - \sum_{k=1}^{N} \frac{\xi_k \xi_j}{\lambda} \int_0^{\infty} \left( e^{-Ay_N} - e^{-B\lambda_N} \right) \hat{f}_k(\xi', y_N) dy_N$$

$$+ \int_0^{\infty} e^{-B\lambda_N} \hat{f}_j(\xi', y_N) dy_N$$

$$+ \frac{i\xi_j}{\lambda} \int_0^{\infty} \left( Ae^{-Ay_N} - Be^{-B\lambda_N} \right) \hat{f}_N(\xi', y_N) dy_N,$$  \hspace{1cm} (2.9)

we apply the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ to (2.7), insert the identities in Lemma 2.3 into the resultant formula, and use the formulas:

$$\mathcal{F}[f_{yN}^0](\xi) = \int_0^{\infty} \left( e^{-y_N \xi_N} - e^{y_N \xi_N} \right) \hat{f}_j(\xi', y_N) dy_N \quad (j = 1, \ldots, N - 1),$$

$$\mathcal{F}[f_{yN}^0](\xi) = \int_0^{\infty} \left( e^{-y_N \xi_N} + e^{y_N \xi_N} \right) \hat{f}_N(\xi', y_N) dy_N.$$  

Here and in the following, $j$ runs from 1 through $N - 1$. By (2.3) and the fact that $\lambda = B^2 - A^2$ and $e^{-B\lambda_N} - e^{-A\lambda_N} = (B - A)\mathcal{M}(y_N)$, we have

$$\hat{w}_N^1(\xi', 0, \lambda) = \frac{A}{B(B + A)} \int_0^{\infty} e^{-B\lambda_N} \hat{f}_N(\xi', y_N) dy_N$$

$$- \sum_{k=1}^{N-1} \frac{i\xi_k}{B + A} \int_0^{\infty} \mathcal{M}(y_N) \hat{f}_k(\xi', y_N) dy_N$$

$$- \frac{A}{B + A} \int_0^{\infty} \mathcal{M}(y_N) \hat{f}_N(\xi', y_N) dy_N,$$

$$\widehat{D_Nw}_j^1(\xi', 0, \lambda) = \int_0^{\infty} e^{-B\lambda_N} \hat{f}_j(\xi', y_N) dy_N - \frac{i\xi_j}{B + A} \int_0^{\infty} e^{-B\lambda_N} \hat{f}_N(\xi', y_N) dy_N$$

$$+ \sum_{k=1}^{N-1} \frac{\xi_k \xi_j}{B + A} \int_0^{\infty} \mathcal{M}(y_N) \hat{f}_k(\xi', y_N) dy_N$$

$$- \frac{i\xi_j A}{B + A} \int_0^{\infty} \mathcal{M}(y_N) \hat{f}_N(\xi', y_N) dy_N.$$  \hspace{1cm} (2.10)
Next we give the exact formula of $\hat{w}_N^2(\zeta', 0, \lambda)$. Setting $w = w^1 + w^2$ and $p = p^1 + p^2$ in (2.3) and noting (2.8), we achieve the equations:

$$\begin{align*}
\begin{cases}
\lambda w - \text{Div} S(w, p) = f & \text{div} w = 0 \quad \text{in } \mathbb{R}^N_+, \\
\lambda h + u_N = d & \text{on } \mathbb{R}^N_0, \\
S(u, \theta) \nu + (c_g - c_\sigma \Delta) h \nu = 0 & \text{on } \mathbb{R}^N_0.
\end{cases}
\end{align*}$$

(3.1)

Let $(w, p)$ be the solution to (2.3) and $(v, \pi, h)$ the solution to the equations:

$$\begin{align*}
\begin{cases}
\lambda v - \nabla v + \nabla \pi = 0 & \text{div } v = 0 \quad \text{in } \mathbb{R}^N_+, \\
\lambda h + v_N = -w_N + d & \text{on } \mathbb{R}^N_0, \\
S(v, \pi) \nu + (c_g - c_\sigma \Delta) h \nu = 0 & \text{on } \mathbb{R}^N_0.
\end{cases}
\end{align*}$$

(3.2)

Then, $u = v + w$, $\theta = \pi + p$, and $h$ solve (3.1). Let $j$ and $k$ run from 1 through $N - 1$ and $J$ from 1 through $N$, respectively, in the present section. The exact formulas of $(v, \pi, h)$ are given by

$$\begin{align*}
v_j(x, \lambda) &= \mathcal{F}^{-1}_{\xi'}[\hat{v}_j(\zeta', x_N, \lambda)](x'), \\
\pi(x, \lambda) &= \mathcal{F}^{-1}_{\xi'}[\hat{\pi}(\zeta', x_N, \lambda)](x') \\
h(x', \lambda) &= \mathcal{F}^{-1}_{\xi'} \left[ \frac{D(A, B)}{(B + A)L(A, B)} \left( -\hat{w}_N(\zeta', 0, \lambda) + \hat{d}(\zeta') \right) \right](x').
\end{align*}$$

(3.11)

By (2.10) and (3.11),

$$\begin{align*}
\hat{w}_N^2(x', x_N, \lambda) &= \mathcal{F}^{-1}_{\xi'}[\hat{\omega}_N^2(\zeta', x_N, \lambda)](x'), \\
\hat{w}_N^2(\zeta', x_N, \lambda) &= \left( \frac{B - A}{D(A, B)} e^{-Bx_N} + \frac{2AB}{D(A, B)} M(x_N) \right) \sum_{j=1}^{N-1} i \xi_j \hat{h}_j(\zeta', 0, \lambda).
\end{align*}$$

(2.11)

which combined with (2.10) furnishes (2.3), because $\hat{w}_N(\zeta', 0, \lambda) = \hat{w}_N^1(\zeta', 0, \lambda) + \hat{w}_N^2(\zeta', 0, \lambda)$.

Finally, using the relation: $e^{-Bx_N} = e^{-\lambda y_N} + (B - A) M(y_N)$ in (2.4), we have (2.8). This completes the proof of the lemma.

\[\Box\]

### 3 Decompositions of operators

In this section, we construct the operators $S(t), \Pi(t)$, and $T(t)$ in Theorem 1.1, and also show the decompositions (1.5). For this purpose, we first give the exact formulas of the solution $(u, \theta, h)$ to

$$\begin{align*}
\begin{cases}
\lambda u - \text{Div} S(u, \theta) = f & \text{div} u = 0 \quad \text{in } \mathbb{R}^N_+, \\
\lambda h + u_N = d & \text{on } \mathbb{R}^N_0, \\
S(u, \theta) \nu + (c_g - c_\sigma \Delta) h \nu = 0 & \text{on } \mathbb{R}^N_0.
\end{cases}
\end{align*}$$

(3.1)

Let $(w, p)$ be the solution to (2.3) and $(v, \pi, h)$ the solution to the equations:

$$\begin{align*}
\begin{cases}
\lambda v - \nabla v + \nabla \pi = 0 & \text{div } v = 0 \quad \text{in } \mathbb{R}^N_+, \\
\lambda h + v_N = -w_N + d & \text{on } \mathbb{R}^N_0, \\
S(v, \pi) \nu + (c_g - c_\sigma \Delta) h \nu = 0 & \text{on } \mathbb{R}^N_0.
\end{cases}
\end{align*}$$

(3.2)

Then, $u = v + w$, $\theta = \pi + p$, and $h$ solve (3.1). Let $j$ and $k$ run from 1 through $N - 1$ and $J$ from 1 through $N$, respectively, in the present section. The exact formulas of $(v, \pi, h)$ are given by

$$\begin{align*}
v_j(x, \lambda) &= \mathcal{F}^{-1}_{\xi'}[\hat{v}_j(\zeta', x_N, \lambda)](x'), \\
\pi(x, \lambda) &= \mathcal{F}^{-1}_{\xi'}[\hat{\pi}(\zeta', x_N, \lambda)](x') \\
h(x', \lambda) &= \mathcal{F}^{-1}_{\xi'} \left[ \frac{D(A, B)}{(B + A)L(A, B)} \left( -\hat{w}_N(\zeta', 0, \lambda) + \hat{d}(\zeta') \right) \right](x').
\end{align*}$$

(3.11)
where we have set

\[
\begin{align*}
\nu_{jk}^{BB}(\zeta', x, \lambda) &= -\frac{i\xi_k(B - A)^2}{(B + A)D(A, B)}, \\
\nu_{jk}^{BB}(\zeta', x, \lambda) &= -\frac{i\xi_k(A - B)}{D(A, B)}, \\
\nu_{jk}^{BM}(\zeta', x, \lambda) &= \frac{\xi_k(B - A)(B^2 + A^2)}{(B + A)D(A, B)}, \\
\nu_{jk}^{BM}(\zeta', x, \lambda) &= \frac{i\xi_kA(B^2 + A^2)}{D(A, B)}, \\
\nu_{jk}^{MB}(\zeta', x, \lambda) &= \frac{\xi_k(B - A)(B^2 + A^2)}{(B + A)D(A, B)}, \\
\nu_{jk}^{MB}(\zeta', x, \lambda) &= \frac{i\xi_kA(B^2 + A^2)}{D(A, B)}, \\
\nu_{jk}^{MM}(\zeta', x, \lambda) &= \frac{\xi_k(B^2 + A^2)^2}{(B + A)D(A, B)}, \\
\nu_{jk}^{MM}(\zeta', x, \lambda) &= \frac{i\xi_kA(B^2 + A^2)^2}{(B + A)D(A, B)}.
\end{align*}
\]

}\]
\[
\mathcal{V}^{\joint}(\xi', \lambda) = -\frac{i\kappa (B^2 + A^2)^2}{(B + A)D(A, B)}, \\
\mathcal{P}^{AA}(\xi', \lambda) = -\frac{i\kappa (B - A)(B^2 + A^2)}{D(A, B)}, \\
\mathcal{P}^{AM}(\xi', \lambda) = \frac{2i\kappa AB(B^2 + A^2)}{D(A, B)}, \\
\mathcal{P}^{AN}(\xi', \lambda) = \frac{2A^3(B^2 + A^2)}{D(A, B)}.
\]

In addition, we see, by inserting (2.5) into \(\hat{h}(\xi', \lambda)\), that \(\hat{h}(\xi', \lambda) = \hat{h}^f(\xi', \lambda) + \hat{h}^d(\xi', \lambda)\) with

\[
\hat{h}^f(\xi', \lambda) = -\sum_{k=1}^{N-1} \frac{i\kappa (B - A)}{(B + A)L(A, B)} \int_0^\infty e^{-A\gamma_N} \hat{f}_k(\xi', y_N) dy_N \\
- \frac{A}{L(A, B)} \int_0^\infty e^{-A\gamma_N} \hat{f}_N(\xi', y_N) dy_N \\
+ \sum_{k=1}^{N-1} \frac{2i\kappa AB}{(B + A)L(A, B)} \int_0^\infty M(y_N) \hat{f}_k(\xi', y_N) dy_N \\
+ \frac{2A^3}{(B + A)L(A, B)} \int_0^\infty M(y_N) \hat{f}_N(\xi', y_N) dy_N,
\]

\[
\hat{h}^d(\xi', \lambda) = \frac{D(A, B)}{(B + A)L(A, B)} \hat{d}(\xi').
\]

Next we shall construct cut-off functions. Let \(\varphi \in C_0^\infty(\mathbb{R}^{N-1})\) be a function such that \(0 \leq \varphi(\xi') \leq 1\), \(\varphi(\xi') = 1\) for \(|\xi'| \leq 1/3\), and \(\varphi(\xi') = 0\) for \(|\xi'| \geq 2/3\). Let \(A_0\) be a number in \((0, 1)\), which is determined in Section 4 below. We then define \(\varphi_0\) and \(\varphi_\infty\) by

\[
\varphi_0(\xi') = \varphi(\xi'/A_0), \quad \varphi_\infty(\xi') = 1 - \varphi(\xi'/A_0),
\]

and also set, for \(a \in \{0, \infty\}, g \in \{f, d\}\), and \(F = (f, d)\),

\[
S^g_a(t; A_0) = \frac{1}{2\pi i} \int_{\Gamma(a)} e^{\lambda t} F^{-1}[\varphi_a(\xi') \hat{\varphi}^g(\xi', x_N, \lambda)](x') \, d\lambda, \\
\Pi^g_a(t; A_0) = \frac{1}{2\pi i} \int_{\Gamma(a)} e^{\lambda t} F^{-1}[\varphi_a(\xi') \hat{\varphi}^g(\xi', x_N, \lambda)](x') \, d\lambda, \\
T^g_a(t; A_0) = \frac{1}{2\pi i} \int_{\Gamma(a)} e^{\lambda t} F^{-1}[\varphi_a(\xi') \hat{\varphi}^g(\xi', x_N, \lambda)](x') \, d\lambda, \\
R(t)f = \frac{1}{2\pi i} \int_{\Gamma(a)} e^{\lambda t} F^{-1}[\hat{\varphi}(\xi', x_N, \lambda)](x') \, d\lambda, \\
P(t)f = \frac{1}{2\pi i} \int_{\Gamma(a)} e^{\lambda t} F^{-1}[\hat{\varphi}(\xi', x_N, \lambda)](x') \, d\lambda \quad (t > 0)
\]

with \(\hat{\varphi}^g(\xi', x_N, \lambda) = (\hat{\varphi}_1^g(\xi', x_N, \lambda), \ldots, \hat{\varphi}_N^g(\xi', x_N, \lambda))^T\). Here we have taken the integral path \(\Gamma(\varepsilon)\) as follows:

\[
\Gamma(\varepsilon) = \Gamma^+(\varepsilon) \cup \Gamma^-(\varepsilon), \quad \Gamma^\pm(\varepsilon) = \{\lambda \in \mathbb{C} \mid \lambda = \tilde{\lambda}_0(\varepsilon) + se^{\pm i(\pi - \varepsilon)}, \ s \in (0, \infty)\}
\]

for \(\tilde{\lambda}_0(\varepsilon) = 2\lambda_0(\varepsilon)/\sin \varepsilon\) with \(\varepsilon \in (0, \pi/2)\), where \(\lambda_0(\varepsilon)\) is the same number as in Lemma 2.1 (3).

**Remark 3.1.** (1) If we set

\[
S(t)F = \sum_{a \in \{0, \infty\}} \sum_{g \in \{f, d\}} S^g_a(t; A_0)F + R(t)f, \\
\Pi(t)F = \sum_{a \in \{0, \infty\}} \sum_{g \in \{f, d\}} \Pi^g_a(t; A_0)F + P(t)f,
\]

Figure 1 is reprinted from Ph.D. thesis of the first author.
Applying the partial Fourier transform with respect to tangential variable \( f \) with Lemma 3.2. Let

\[
T(t)F = \sum_{a \in \{0, \infty\}} \sum_{g \in \{f, d\}} T^a_g(t; A_0)F,
\]

then \( S(t)F, \Pi(t)F, \) and \( T(t)F \) are the requirements in Theorem (1.1) (1). Especially, let \( S(t) : F \mapsto (S(t)F, T(t)F) \) and \( 1 < p < \infty, \) and then \( \{S(t)\}_{t \geq 0} \) is an analytic semi-group on \( X_p, \) defined in (1.3), as was seen in [18]. On the other hand, by Lemma 2.2 \( \{R(t)\}_{t \geq 0} \) is an analytic semi-group on \( J_p(R_N^\varepsilon), \) and also \( R(t) \) and \( P(t) \) satisfy

\[
\|\nabla^\ell R(t)f\|_{L_p(R_N^\varepsilon)} \leq Ct^{-\ell/2}\|f\|_{L_p(R_N^\varepsilon)},
\]

\[
\|\partial_t R(t)f, \nabla P(t)f\|_{L_p(R_N^\varepsilon)} \leq Ct^{-1}\|f\|_{L_p(R_N^\varepsilon)}
\]

for \( f \in L_p(R_N^\varepsilon)^N, \ell = 0, 1, 2, \) and \( \ell > 0. \) These estimates imply that (1.8) holds.

(2) For \( a \in \{0, \infty\} \) and \( g \in \{f, d\}, \) the extension \( \mathcal{E}(T^a_g(t; A_0)F) \) defined as (1.4) is decomposed into

\[
\mathcal{E}(T^a_g(t; A_0)F) = \frac{1}{2\pi i} \int_{\Gamma(\varepsilon)} e^{\lambda t} F^-\ell^{-1}[\varphi(a)(\xi) e^{-Ax_N\tilde{h}_\lambda(\xi')}](x') d\lambda.
\]

(3) In the following sections, we show, for \( a \in \{0, \infty\}, \)

\[
S_a(t)F = \sum_{g \in \{f, d\}} S^g_a(t; A_0)F, \quad \Pi_a(t)F = \sum_{g \in \{f, d\}} \Pi^g_a(t; A_0)F;
\]

\[
T_a(t)F = \sum_{g \in \{f, d\}} T^g_a(t; A_0)F
\]

satisfy the estimates (1.0) and (1.7), respectively.

We devote the last part of this section to the proof of the following lemma.

**Lemma 3.2.** Let \( \xi' \in R^{N-1} \setminus \{0\} \) and \( \lambda \in \{z \in \mathbb{C} \mid \text{Re}z \geq 0\}. \) Then \( L(A,B) \neq 0. \)

**Proof.** Applying the partial Fourier transform with respect to tangential variable \( x' \) to the equations (3.1) with \( f = 0 \) and \( d = 0 \) yields that

\[
\lambda \tilde{u}_j(x_N) - \sum_{k=1}^{N-1} i\xi_k(i\xi_k \tilde{u}_k(x_N) + i\xi_k \tilde{u}_j(x_N)) = -D_N(D_N \tilde{u}_j(x_N) + i\xi_j \tilde{u}_N(x_N)) + i\xi_j \tilde{\theta}(x_N) = 0,
\]
\[
\begin{align*}
\lambda \tilde{u}_N(x_N) - \sum_{k=1}^{N-1} i\xi_k (D_N \tilde{u}_k(x_N) + i\xi_k \tilde{u}_N(x_N)) - 2D_N^2 \tilde{u}_N(x_N) + D_N \tilde{\theta}(x_N) &= 0, \\
\sum_{k=1}^{N-1} i\xi_k \tilde{u}_k(x_N) + D_N \tilde{u}_N(x_N) &= 0, \\
D_N \tilde{u}_j(0) + i\xi_j \tilde{u}_N(0) &= 0, \quad -\tilde{\theta}(0) + 2D_N \tilde{u}_N(0) + (c_g + c_\sigma A^2) \tilde{h} &= 0
\end{align*}
\]  
(3.10)

for \(x_N > 0\), where we have used the symbols:

\[
\tilde{u}_j(x_N) = \tilde{u}_j(\xi', x_N), \quad \tilde{\theta}(x_N) = \tilde{\theta}(\xi', x_N), \quad \tilde{h} = \tilde{h}(\xi').
\]

We here set \(\tilde{u}(x_N) = (\tilde{u}_1(x_N), \ldots, \tilde{u}_N(x_N))^T\), \(\|f\|^2 = \int_0^\infty f(x_N)f(x_N) \, dx_N\), and show that \(L(A, B) \neq 0\) by contradiction. Suppose that \(L(A, B) = 0\). We know that (3.10) admits a solution \((\tilde{u}(x_N), \tilde{\theta}(x_N), \tilde{h}) \neq 0\) that decays exponentially when \(x_N \to \infty\) (see e.g. [17, Section 4]). To obtain

\[
0 = \lambda \|\tilde{u}\|^2 + 2\|D_N \tilde{u}_N\|^2 + \sum_{j=1}^{N-1} \|i\xi_k \tilde{u}_j\|^2
\]

\[
+ \sum_{j=1}^{N-1} \|D_N \tilde{u}_j + i\xi_j \tilde{u}_N\|^2 + \sum_{j=1}^{N-1} \|D_N \tilde{u}_j + i\xi_j \tilde{u}_N\|^2 + 2(c_g + c_\sigma A^2) \|\tilde{h}\|^2,
\]

(3.11)

we multiply the first equation of (3.10) by \(\tilde{u}_j(x_N)\) and the second equation by \(\tilde{u}_N(x_N)\), and integrate the resultant formulas with respect to \(x_N \in (0, \infty)\), and furthermore, after integration by parts, we use the third to sixth equations of (3.10). Taking the real part of (3.11), we have

\[
D_N \tilde{u}_N(x_N) = 0, \quad D_N \tilde{u}_j(x_N) + i\xi_j \tilde{u}_N = 0 \quad \text{for Re}\, \lambda \geq 0.
\]

In particular, \(\tilde{u}_N\) is a constant, but \(\tilde{u}_N = 0\) since \(\lim_{x_N \to \infty} \tilde{u}_N = 0\). We thus have \(D_N \tilde{u}_j = 0\), which implies that \(\tilde{u}_j = 0\) since \(\lim_{x_N \to \infty} \tilde{u}_j = 0\). Combining \(\tilde{u}_j = 0\) and the first equation of (3.10) yields that \(i\xi_j \tilde{\theta} = 0\). This implies that \(\tilde{\theta} = 0\) because \(\xi' \neq 0\). In addition, by the sixth equation of (3.10), we have \(\langle c_g + c_\sigma A^2 \rangle \tilde{h} = 0\). Since \(c_g + c_\sigma A^2 \neq 0\), we see that \(\tilde{h} = 0\). We thus have \(\tilde{u} = 0\), \(\tilde{\theta} = 0\), and \(\tilde{h} = 0\), which leads to a contradiction. This completes the proof of Lemma 3.2.

4 Analysis of low frequency parts

In this section, we show the estimates (1.6) in Theorem 1.1 (2). If we consider the Lopatinskii determinant \(L(A, B)\) defined in (2.1) as a polynomial with respect to \(B\), then it has four roots \(B_j^\pm (j = 1, 2)\), which have the following asymptotics:

\[
B_j^\pm = e^{\pm i(j-1)(\pi/4)c_g^{1/4}} A^{1/4} - \frac{A^{7/4}}{2e^{\pm i(j-1)(\pi/4)c_g^{1/4}}} - \frac{c_\sigma A^{9/4}}{e^{\pm i(j-1)(3\pi/4)c_g^{3/4}}} + O(A^{10/4})
\]

as \(A \to 0\). Set \(\lambda_\pm = (B_j^\pm)^2 - A^2\), and then

\[
\lambda_\pm = \pm ic_g^{1/2} A^{1/2} - 2A^2 + \frac{2c_\sigma}{ic_g^{1/2}} A^{10/4} + O(A^{11/4}) \quad \text{as} \, A \to 0.
\]

(4.2)

Remark 4.1. For \(\lambda \in \Sigma_c\), we choose a brunch such that \(\text{Re} B = \text{Re} \sqrt{\lambda + A^2} > 0\). Note that \(\lambda_\pm \in \Sigma_c\) and \(\text{Re} (\lambda_\pm + A^2) < 0\).
We define a positive number $\varepsilon_0$ by $\varepsilon_0 = \tan^{-1}\{ (A^2/8)/A^2 \} = \tan^{-1}(1/8)$, and furthermore, we set

\[
\Gamma_0^\pm = \{ \lambda \in \mathbb{C} \mid \lambda = \lambda_\pm + \imath \left( e^{1/2}/4 \right) A^{1/2} e^{\pm i \varepsilon_0}, \ u : 0 \to 2\pi \},
\]

\[
\Gamma_1^\pm = \{ \lambda \in \mathbb{C} \mid \lambda = -A^2 + (A^2/4) e^{\pm i \varepsilon_0}, \ u : 0 \to \pi/2 \},
\]

\[
\Gamma_2^\pm = \{ \lambda \in \mathbb{C} \mid \lambda = -(A^2(1-u) + \gamma_0 u) \pm i((A^2/4)(1-u) + \gamma_0 u), \ u : 0 \to 1 \},
\]

\[
\Gamma_3^\pm = \{ \lambda \in \mathbb{C} \mid \lambda = -\gamma_0 \pm i\varepsilon_0 + u e^{\pm i(\pi - \varepsilon_0)}, \ u : 0 \to \infty \}
\]

with $\gamma_0 = \lambda_0(\varepsilon_0)$ given by Lemma 2.1 (3) and

\[
\tilde{\gamma}_0 = \frac{1}{8} \left( \lambda_0(\varepsilon_0) + \tilde{\lambda}_0(\varepsilon_0) \right) = \frac{1}{8} \left( 1 + \frac{2}{\sin \varepsilon_0} \right) \lambda_0(\varepsilon_0) = \frac{(1 + 2\sqrt{65})\gamma_0}{8}, \tag{4.3}
\]

where $\tilde{\lambda}_0(\varepsilon_0)$ is the same constant as in 3.8 with $\varepsilon = \varepsilon_0$.

![Figure 2: $\Gamma^+_{\sigma} (\sigma = 0, 1, 2, 3)$](image)

Then, by Cauchy’s integral theorem, we decompose $S_0^\sigma(t;A_0)F$, $\Pi_0^\sigma(t;A_0)F$, and $E(T_0^\sigma(t;A_0)F)$ given by (5.7) and (5.9) as follows: For $g \in \{ f, d \}$

\[
S_0^\sigma(t;A_0)F = \sum_{\sigma=0}^{3} S_0^{\sigma,\sigma}(t;A_0)F, \quad \Pi_0^\sigma(t;A_0)F = \sum_{\sigma=0}^{3} \Pi_0^{\sigma,\sigma}(t;A_0)F,
\]

\[
E(T_0^\sigma(t;A_0)F) = \sum_{\sigma=0}^{3} E(T_0^{\sigma,\sigma}(t;A_0)F) \tag{4.4}
\]

with

\[
S_0^{\sigma,\sigma}(t;A_0)F = \mathcal{F}_{\xi}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma_1^\pm \cup \Gamma_2^\pm} e^{\imath \xi \varphi_0(\xi')(\xi', x_N, \lambda)} d\lambda \right] (x'),
\]

\[
\Pi_0^{\sigma,\sigma}(t;A_0)F = \mathcal{F}_{\xi}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma_1^\pm \cup \Gamma_2^\pm} e^{\imath \xi \varphi_0(\xi')(\xi', x_N, \lambda)} d\lambda \right] (x'),
\]

\[
E(T_0^{\sigma,\sigma}(t;A_0)F) = \mathcal{F}_{\xi}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma_1^\pm \cup \Gamma_2^\pm} e^{\imath \xi \varphi_0(\xi')(\xi', x_N, \lambda)} e^{-Ax_N \hat{\Phi}_0^{\sigma}(\xi', \lambda)} d\lambda \right] (x'), \tag{4.5}
\]

where $\varphi_0(\xi')$ is the cut-off function given in (3.3). In order to estimate each term in (4.5), we here introduce operators $K_0^{\sigma,\sigma}(t;A_0)$ and $L_0^{\sigma,\sigma}(t;A_0)$ defined by

\[
[K_0^{\sigma,\sigma}(t;A_0)f](x) = \int_{\Gamma_1^\pm} \mathcal{F}_{\xi}^{-1} \left[ \int_{\Gamma_1^\pm} e^{\imath \xi \varphi_0(\xi')(\xi', x_N, \lambda)} \hat{h}_0(\xi', y_N) d\lambda \hat{f}(\xi', y_N) \right] (x') dy_N,
\]

\[
[L_0^{\sigma,\sigma}(t;A_0)d](x) = \mathcal{F}_{\xi}^{-1} \left[ \int_{\Gamma_1^\pm} e^{\imath \xi \varphi_0(\xi')(\xi', x_N, \lambda)} \hat{\gamma}_0(\xi', y_N) d\lambda \hat{d}(\xi) \right] (x') \quad (\sigma = 0, 1, 2, 3) \tag{4.6}
\]

Figure 2 is reprinted from Ph.D. thesis of the first author.
with some multipliers $k_n(\xi', \lambda)$ and $\ell_n(\xi', \lambda)$, where $X_n(x_N, y_N)$ and $Y_n(x_N)$ are given by

$$X_n(x_N, y_N) = \begin{cases} e^{-A(x_N+y_N)} & (n = 1), \\ e^{-2A_N}M(y_N) & (n = 2), \\ e^{-B(x_N+y_N)} & (n = 3), \\ e^{-B_NN}M(y_N) & (n = 4), \\ M(x_N)e^{-BN_N} & (n = 5), \\ M(x_N)M(y_N) & (n = 6), \end{cases}$$

$$Y_n(x_N) = \begin{cases} e^{-Ax_N} & (n = 1), \\ e^{-B_N} & (n = 2), \\ M(x_N) & (n = 3). \end{cases}$$

### 4.1 Analysis on $\Gamma^\pm_0$

Our aim here is to show the following theorem for the operators given in (4.5) with Theorem 4.2.

**Theorem 4.2.** Let $1 \leq r \leq 2 \leq q \leq \infty$ and $F = (f, d) \in L_r(R^n)^N \times L_r(R^{N-1})$. Then there exists an $A_0 \in (0, 1)$ such that the following assertions hold:

1. Let $k = 0, 1, \ell = 0, 1, 2$, and $\alpha' \in N_0^{N-1}$. Then there exist a positive constant $C = C(\alpha')$ such that for any $t > 0$

   $$\|D^\alpha D^\ell d_{N}^{(0)}(t; A_0)F\|_{L_q(R^n)} \leq C(t + 1)^{-\frac{n-1}{2}} \frac{1}{2} \|f\|_{L_r(R^n)}.$$

2. There exists a positive constant $C$ such that for any $t > 0$

   $$\|\nabla \Pi^0_{d0}(t; A_0)F\|_{L_q(R^n)} \leq C(t + 1)^{-\frac{n-1}{2}} \frac{1}{2} \|f\|_{L_r(R^n)}.$$

3. Let $\alpha \in N_0^N$. Then there exists a positive constant $C = C(\alpha)$ such that for any $t > 0$

   $$\|D^\alpha d_{N}^{(0)}(t; A_0)F\|_{L_q(R^n)} \leq C(t + 1)^{-\frac{n-1}{2}} \frac{1}{2} \|f\|_{L_r(R^n)}.$$

We here introduce some fundamental lemmas to show Theorem 4.2.

**Lemma 4.3.** Let $s_i \geq 0$ ($i = 0, 1, 2, 3$). Then there exists a positive constant $C = C(s_0, s_1, s_2, s_3)$ such that for any $t > 0, \alpha > 0,$ and $Z \geq 0$

$$e^{-s_0Z^2}e^{-s_2Z^3} \leq C(t^{s_1/2} + a^{s_1/s_3})^{-1}.$$

**Lemma 4.4.** Let $1 \leq q, r \leq \infty$, $a > 0$, $b_1 > 0$, and $b_2 > 0$.

1. Set $g(x_N, \tau) = (\tau^n + (x_N)^{b_1})^{-1}$ for $x_N > 0$ and $\tau > 0$. Then there exists a positive constant $C$ such that for any $\tau > 0$

   $$\|g(\tau)\|_{L_q((0, \infty))} \leq C \tau^{-a(1-\frac{1}{n+1})},$$

   provided that $b_1q > 1.$
(2) Let \( f \in L_r((0, \infty)) \), and set, for \( x_N > 0 \) and \( \tau > 0 \),
\[
g(x_N, \tau) = \int_0^\infty \frac{f(y_N)}{\tau^{\alpha} + (x_N)^{\alpha} + (y_N)^{\alpha}} \, dy_N.
\]
Then there exists a positive constant \( C \) such that for any \( \tau > 0 \)
\[
\|g(\tau)\|_{L_2((0, \infty))} \leq C \tau^{-\alpha} \left(1 - \frac{1}{\tau + \frac{1}{r}}\right) \frac{1}{\|f\|_{L_r((0, \infty))}},
\]
provided that for \( r' = r/(r - 1) \)
\[
b_1 q > 1, \quad b_2 \left(1 - \frac{1}{b_1 q}\right) r' > 1.
\]

By using Lemma 4.3 and Lemma 4.3, we obtain the following lemma.

**Lemma 4.5.** Let \( 1 \leq r \leq 2 \leq q \leq \infty \), and let \( f \in L_r(\mathbb{R}^N) \) and \( d \in L_r(\mathbb{R}^{N-1}) \). For multipliers \( \kappa_n(\xi', \lambda) \) and \( m_n(\xi', \lambda) \) given below, we set, in (4.0),
\[
k_n(\xi', \lambda) = \frac{\kappa_n(\xi', \lambda)}{L(A, B)}, \quad \ell_n(\xi', \lambda) = \frac{m_n(\xi', \lambda)}{L(A, B)}.
\]

(1) Let \( s \geq 0 \) and suppose that there exist constants \( A_1 \in (0, 1) \) and \( C = C(s) > 0 \) such that for any \( A \in (0, A_1) \)
\[
|\kappa_1(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |\kappa_2(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |\kappa_3(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad
|\kappa_4(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |\kappa_5(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |\kappa_6(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}.
\]
Then there exist constants \( A_0 \in (0, A_1) \) and \( C = C(s) > 0 \) such that for any \( t > 0 \)
\[
\|K_n^{\pm, 0}(t; A_0) f\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{\varphi}{2} + \frac{q}{2}} ||f||_{L_q(\mathbb{R}^N)} \quad (n = 1, 2, 6),
\]
\[
\|K_n^{\pm, 0}(t; A_0) f\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{q}{2} - \frac{q}{2}} ||f||_{L_q(\mathbb{R}^N)},
\]
\[
\|K_n^{\pm, 0}(t; A_0) f\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{q}{2} - \frac{q}{2}} ||f||_{L_q(\mathbb{R}^N)},
\]
\[
\|K_n^{\pm, 0}(t; A_0) f\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{q}{2} - \frac{q}{2}} ||f||_{L_q(\mathbb{R}^N)}.
\]

(2) Let \( s \geq 0 \) and suppose that there exist constants \( A_1 \in (0, 1) \) and \( C = C(s) > 0 \) such that for any \( A \in (0, A_1) \)
\[
|m_1(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |m_2(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}, \quad |m_3(\xi', \lambda_\pm)| \leq CA^{\frac{q}{2} + s}.
\]
Then there exist constants \( A_0 \in (0, A_1) \) and \( C = C(s) > 0 \) such that for any \( t > 0 \)
\[
\|L_n^{\pm, 0}(t; A_0) d / \|L_q(\mathbb{R}^N) \leq C(t + 1)^{-\frac{\varphi}{2} + \frac{q}{2}} ||d||_{L_q(\mathbb{R}^{N-1})} \quad (n = 1, 3),
\]
\[
\|L_n^{\pm, 0}(t; A_0) d / \|L_q(\mathbb{R}^N) \leq C(t + 1)^{-\frac{q}{2} - \frac{q}{2}} ||d||_{L_q(\mathbb{R}^{N-1})},
\]

**Proof.** We use the abbreviations: \( \|\cdot\|_2 = \|\cdot\|_{L_2(\mathbb{R}^{N-1})} \), \( \hat{f}(y_N) = \hat{f}(\xi', y_N) \), and \( \tilde{t} = t + 1 \) for \( t > 0 \) in this proof, and consider only the estimates on \( \Gamma_{\alpha}^{\pm} \) since the estimates on \( \Gamma_{\alpha} \) can be shown similarly.

(1) We first show the inequality for \( K_n^{\pm, 0}(t; A_0) \). Noting that \( B^2 - (B_1^+)^2 = \lambda - \lambda_+ \), by the residue theorem, we have
\[
[K_n^{\pm, 0}(t; A_0) f](x) = \int_0^\infty \int_{\Gamma_{\alpha}^{\pm}} e^{\lambda t} \varphi_0(\xi') \kappa_1(\xi', \lambda)(B + B_1^+) e^{-A(x_N + y_N)} d\lambda \hat{f}(y_N) \frac{d\Lambda}{d\hat{y}_N} (x') \, dy_N.
\]
\[
= 4 \pi \int_0^\infty \int_{\Gamma_{\alpha}^{\pm}} e^{\lambda t} \varphi_0(\xi') \kappa_1(\xi', \lambda)(B_1^+)^2 e^{-A(x_N + y_N)} \hat{f}(y_N) \frac{d\Lambda}{d\hat{y}_N} (x') \, dy_N.
\]
In view of (4.1) and (4.2), we can choose \( A_0 \in (0, A_1) \) in such a way that

\[
|e^{\lambda t}| \leq Ce^{-\lambda t}, \quad |B_1^+ - A_0^i| \geq CA^i, \quad |B_1^+ - B_1^+| \geq CA^i, \quad |B_1^+ - B_1^-| \geq CA^i
\]

(4.8)

for any \( A \in (0, A_0) \) and \( t > 0 \) with some constant \( C \). Thus, by \( L_2 \)-estimates of the \((N - 1)\)-dimensional heat kernel and Parseval’s theorem, we have

\[
\|K_i^{+,0}(t; A_0)f(\cdot, xN)\|_{L_2(\mathbb{R}^{N-1})} \leq C_i \int_0^{\infty} \left\| e^{-(A^2/2)i}A^i e^{-A(xN + yN)} \hat{f}(yN) \right\|_2 dyN
\]

where we have used Lemma 4.3 with \( s_0 = 1/8, s_1 = 1 \) (\( i = 1, 2, 3 \)), \( a = xN + yN \), and \( Z = A \). If \( q > 2 \), then applying Lemma 4.4(2) with \( a = 1/2 \) and \( b_1 = b_2 = 1 \) to (1.2) furnishes that

\[
\|K_i^{+,0}(t; A_0)f\|_{L_q(\mathbb{R}^{N})} \leq C_i \int_0^{\infty} \left\| \mathcal{F}^{-1} \left[ e^{-(A^2/2)i}A^i e^{-A(xN + yN)} \hat{f}(yN) \right] \right\|_2 dyN,
\]

(4.9)

In the case of \((q, r) = (2, 2)\), by (1.2)

\[
\|K_i^{+,0}(t; A_0)f(\cdot, xN)\|_2 \leq C_i \int_0^{\infty} \left\| \mathcal{F}^{-1} \left[ e^{-(A^2/2)i}A^i e^{-A(xN + yN)} \hat{f}(yN) \right] \right\|_2 dyN,
\]

and then it follows from 4.4. Lemma 5.4) that

\[
\|K_i^{+,0}(t; A_0)f\|_{L_2(\mathbb{R}^{N})} \leq C_i \|f\|_{L_2(\mathbb{R}^{N})}.
\]

On the other hand, in the case of \( 1 \leq r < 2 \) and \( q = 2 \), by the second inequality of (4.1), Lemma 1.3 and Hölder’s inequality

\[
\|K_i^{+,0}(t; A_0)f\|_{L_2(\mathbb{R}^{N})} \leq C_i \int_0^{\infty} \left\| e^{-(A^2/2)i}A^i e^{-A(xN + yN)} \hat{f}(yN) \right\|_2 dyN
\]

\[
\leq C_i \int_0^{\infty} \left\| \mathcal{F}^{-1} \left[ e^{-(A^2/2)i}A^i e^{-A(xN + yN)} \hat{f}(yN) \right] \right\|_2 dyN
\]

\[
\leq C_i \left\| f(\cdot, yN) \right\|_{L_q(\mathbb{R}^{N-1})} \|f\|_{L_r(\mathbb{R}^{N})}.
\]

which implies that the required inequality for \( K_i^{+,0}(t; A_0) \) holds. Summing up the arguments above, we see that the following lemma holds.

**Lemma 4.6.** Let \( 1 \leq r \leq 2 \leq q \leq \infty \), \( \tau > 0 \), and \( s_i > 0 \) \((i = 1, 2)\). For \( xN > 0 \) and \( f \in L_r(\mathbb{R}^{N}) \), we set

\[
F(xN, \tau) = \int_0^{\infty} \left\| e^{-s_1 A^2 \tau} A e^{-s_2 A(xN + yN)} \hat{f}(\xi', yN) \right\|_{L_2(\mathbb{R}^{N-1})} dyN.
\]

Then there exists a positive constant \( C \) such that for any \( \tau > 0 \)

\[
\|F(\tau)\|_{L_1([0, \infty))} \leq C_\tau^{-1} \left\| e^{-s_1 A^2 \tau} A e^{-s_2 A(xN + yN)} \hat{f}(\xi', yN) \right\|_{L_2(\mathbb{R}^{N-1})}.
\]

Secondly we show the inequality for \( K_i^{+,0}(t; A_0) \). We here set

\[
\mathcal{M}_\pm(a) = \frac{e^{-B_1^+ a} - e^{-A_0}}{B_1^+ - A_0} \quad \text{for} \ a > 0.
\]
In view of (4.1) and (4.2), we can choose $A_0 \in (0, A_1)$ in such a way that for any $A \in (0, A_0)$ and $a > 0$
\begin{align}
|\mathcal{M}_\pm(a)| &= \left| e^{-B^\pm_1 a} - e^{-A a} \right| 
\leq CA^{-1/4} e^{-A a}
\end{align}
with some constant $C$. Thus, by the same calculations as in (4.7) and (4.9), we obtain
\begin{align}
\| [K^+_n(t; A_0) f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} 
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \int_0^\infty \left\| e^{-(A^2/2)t} A e^{-A(x_N + y_N) \hat{f}(y_N)} \right\|_2 \, dy_N,
\end{align}
which furnishes the required inequality of $K^+_n(t; A_0)$ by Lemma 4.6.

Thirdly we show the inequality for $K^+_3(t; A_0)$ in view of (4.1) and (4.2), we can choose $A_0 \in (0, A_1)$ such that
\begin{align}
|e^{-B^+_1(x_N + y_N)}| 
\leq e^{-CA^{1/4}(x_N + y_N)}
\end{align}
for any $A \in (0, A_0)$ with some constant $C$, so that we easily see that by Lemma 4.3
\begin{align}
\| [K^+_3(t; A_0) f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} 
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \int_0^\infty \left\| e^{-(A^2/2)t} A e^{-A(x_N + y_N) \hat{f}(y_N)} \right\|_2 \, dy_N.
\end{align}
Combining the inequality above with Lemma 4.4 (2) with $a = 1/2$ and $b_1 = b_2 = 4$, we obtain the required inequality of $K^+_n(t; A_0)$.

Finally we show the inequalities for $K^+_n(t; A_0)$ ($n = 4, 5, 6$). Using similar argumentations to the above cases, we have for $n = 4, 5$
\begin{align}
\| [K^+_n(t; A_0) f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} 
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \int_0^\infty \left\| e^{-(A^2/2)t} A e^{-A(x_N + y_N) \hat{f}(y_N)} \right\|_2 \, dy_N,
\end{align}
which, combined with Lemma 4.4 (2), furnishes the required inequalities of $K^+_n(t; A_0)$ ($n = 4, 5$). In addition, for $n = 6$, we have
\begin{align}
\| [K^+_6(t; A_0) f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} 
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \int_0^\infty \left\| e^{-(A^2/2)t} A e^{-A(x_N + y_N) \hat{f}(y_N)} \right\|_2 \, dy_N
\end{align}
with a positive constant $C$, which yields the required inequality of $K^+_6(t; A_0)$ by Lemma 4.5 (2) We consider the case of $n = 1, 3$. Noting that $B^2 - (B^\pm_1)^2 = \lambda - \lambda_\pm$, by the residue theorem, we have
\begin{align}
[L^+_n(t; A_0) d](x) = 4\pi i F_{\nu}^{-1} \left[ \frac{e^{\nu t} \varphi_0(\xi') m_n(\xi', \lambda_\pm) B^+_1}{(B^+_1 - B^-_1)(B^+_1 - B^-_2)(B^+_2 - B^-_1)} \right] (x').
\end{align}
Thus, by (4.3, 4.10), Lemma 4.3 $L_q-L_r$ estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem, we have
\begin{align}
\| [L^+_n(t; A_0) d](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} 
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \left\| e^{-(A^2/2)t} A^{1/2} e^{-A x_N \tilde{d}(\xi')} \right\|_2
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \left\| e^{-(A^2/4) t} \tilde{d}(\xi') \right\|_2 / (t^{1/4} + (x_N)^{1/2})
\leq C T^{\frac{N-1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \left\| d \right\|_{L_q(\mathbb{R}^{N-1})} / (t^{1/4} + (x_N)^{1/2}).
\end{align}
If \( q > 2 \), then by Lemma 4.3 (1) we obtain the required inequality of \( L^+_{\alpha}(t; A_0) \) (\( n = 1, 3 \)). In the case of \( q = 2 \), we see that by the first inequality of (4.11),

\[
\|L^+_{\alpha}(t; A_0)d\|_{L_2(R^2)} \leq C\|\nu\|_{e^{-(\lambda^2/2)}\partial^2} \|\xi\|_{L_2(R^{N-1})} \leq C\|\frac{\partial \nu}{\partial \lambda} \|_{L_2(R^{N-1})}.
\]

Analogously, we can obtain the required inequality of \( L^+_{\alpha}(t; A_0) \), which complete the proof of the lemma.

Noting that for some \( A_2 \in (0, 1) \) and \( C > 0 \) there holds \( |D(A, B^\pm)| \geq CA^{3/4} \) for \( A \in (0, A_2) \), we see that there exist positive numbers \( A_1 \in (0, A_2) \) and \( C \) such that for any \( A \in (0, A_1) \) and \( k, j, t > 0 \),

\[
|V_{jk}^{AB}(\xi, \lambda) \leq CA^\frac{3}{4}, \quad |V_{jk}^{AB}(\xi, \lambda) \leq CA^\frac{3}{4}, \quad |V_{jk}^{AB}(\xi, \lambda) \leq CA^\frac{3}{4},
\]

Therefore, recalling the formulas (4.3), (4.4), (4.5), and (4.6) with \( \sigma = 0 \) and using (2.2), we obtain the required inequalities of Theorem 4.2 by Lemma 4.5.

4.2 Analysis on \( \Gamma^+_1 \)

Our aim here is to show the following theorem for the operators defined in (4.5) with \( \sigma = 1 \).

**Theorem 4.7.** Let \( 1 \leq r \leq 2 \leq q \leq \infty \) and \( F = (f, d) \in L_r(R^N) \times L_r(R^{N-1}) \). Then, there exists an \( A_0 \in (0, 1) \) such that we have the following assertions:

(1) Let \( k = 0, 1, \ell = 0, 1, 2, \) and \( \alpha' \in N^N \). Then there exists a positive constant \( C = C(\alpha') \) such that for any \( t > 0 \),

\[
\|\partial_{\alpha} L^0_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{\alpha}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|f\|_{L_2(R^N)},
\]

\[
\|\partial_{\alpha} L^1_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{k+\alpha+1}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|d\|_{L_2(R^{N-1})}.
\]

(2) There exists a positive constant \( C \) such that for any \( t > 0 \),

\[
\|\nabla L^0_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{\alpha}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|f\|_{L_2(R^N)},
\]

\[
\|\nabla L^1_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{k+\alpha+1}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|d\|_{L_2(R^{N-1})}.
\]

(3) Let \( \alpha \in N^N \). Then there exists a positive constant \( C = C(\alpha) \) such that for any \( t > 0 \),

\[
\|D_{\alpha} L^0_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{\alpha}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|f\|_{L_2(R^N)},
\]

\[
\|D_{\alpha} L^1_{\alpha}(t; A_0)F\|_{L_2(R^N)} \leq C(t + 1)^{-\frac{k+\alpha+1}{2}(t+\frac{1}{2})} - \frac{2\alpha+\alpha^2}{2}\|d\|_{L_2(R^{N-1})}.
\]

We start with the following lemmas in order to show Theorem 4.7.

**Lemma 4.8.** Let \( f(z) = 2z^2 + 2z + 12z - 8 \). Then \( f(z) \neq 0 \) for \( z \in \{ \omega \in C \mid \Re \omega \geq 0 \} \setminus (0, 1) \).

**Proof.** We note that \( f(z) \) has only one real root \( \alpha \) because \( f(0) = -8 \), \( f(1) = 7 \) and \( f'(z) = 3z^2 + 4z + 12 > 0 \) for \( z \in R \), and it is clear that \( \alpha \) is in \( (0, 1) \). Let \( \beta \) and \( \beta' \) be the other roots of \( f(z) \). Since \( \alpha + \beta + \beta' = -2 \), we have \( 2\Re \beta = -2 \). This completes the proof.

**Lemma 4.9.** Let \( \lambda \in \Gamma^+_1 \) and \( \xi' \in R^{N-1} \). Then

\[
\frac{A}{4} \leq \Re B \leq \frac{A}{2}, \quad |D(A, B)| \geq CA^3
\]

for some positive constant \( C \) independent of \( \xi' \) and \( \lambda \). In addition, there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that \( |L(A, B)| \geq CA \) for any \( A \in (0, A_1) \).
Proof. We first show the inequalities for $B$ and $D(A, B)$. Note that

$$B = \sqrt{\lambda + A^2} = (A/2)e^{\pm i(u/2)}$$

(4.12)

since $\lambda = -A^2 + (A^2/4)e^{\pm iu}$ for $u \in [0, \pi/2]$ on $\Gamma_1^\pm$. Therefore, it is clear that the required inequalities of $B$ hold. We insert the identity (4.12) into $D(A, B)$ to obtain

$$D(A, B) = \frac{A^3}{8} \left( (e^{\pm i(u/2)})^3 + 2(e^{\pm i(u/2)})^2 + 12(e^{\pm i(u/2)}) - 8 \right),$$

which, combined with Lemma 4.8, furnishes that $|D(A, B)| \geq CA^3$ for some positive constant $C$ independent of $\xi'$ and $\lambda$.

We finally show the last inequality. By (4.12)

$$B^2 - (B_1^\pm)^2 = \mp \frac{1}{2} e^{\pm i(u/2)} A^1/2 + A^2 \left( 1 + \frac{e^{\pm iu}}{4} \right) + O(1)$$

as $A \to 0,$

so that there exist positive constants $A_1 \in (0, 1)$ and $C$ such that

$$|B^2 - (B_1^\pm)^2| \geq CA^{1/2} \quad \text{for any } A \in (0, A_1).$$

(4.13)

On the other hand, we have $|B + B_1^\pm| \leq CA^{1/4}$ on $\Gamma_1^\pm$ when $A$ is sufficiently small, which, combined with (4.13), furnishes that

$$|B - B_1^\pm| = \frac{|B^2 - (B_1^\pm)^2|}{|B + B_1^\pm|} \geq CA^{1/4} \quad \text{for any } A \in (0, A_1).$$

Since $|B - B_1^\pm| \leq |B - B_2^\pm|$ as follows from Re$B \geq 0$ and (4.11), we thus obtain

$$|L(A, B)| = |(B - B_1^\pm)(B - B_2^\pm)(B - B_2^-)| \geq CA$$

for any $A \in (0, A_1), \lambda \in \Gamma_1^\pm$, and a positive constant $C$ independent of $\xi'$ and $\lambda$. \hfill \Box

Next, we show some multiplier theorem on $\Gamma_1^\pm$.

**Lemma 4.10.** Let $1 \leq r \leq 2 \leq q \leq 6$, and let $f \in L_r(\mathbb{R}_r^N)$ and $d \in L_r(\mathbb{R}^{N-1})$. We use the symbols defined in (4.16).

1. Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0, 1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_1^\pm$ and $A \in (0, A_1)

$$|k_n(\xi', \lambda)| \leq CA^{-1+s} \quad (n = 1, 3), \quad |k_n(\xi', \lambda)| \leq CA^s \quad (n = 2, 4, 5),$$

$$|k_0(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ we have the estimates:

$$\|K_n^{\pm,1}(t; A_0) f\|_{L_r(\mathbb{R}_r^N)} \leq C(t + 1)^{-\frac{1}{t'} - \frac{n}{t} - \frac{1}{t'}} \|f\|_{L_r(\mathbb{R}_r^N)} \quad (n = 1, 2, 3, 4, 5, 6).$$

2. Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0, 1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_1^\pm$ and $A \in (0, A_1)

$$|\ell_0(\xi', \lambda)| \leq CA^s \quad (n = 1, 2), \quad |\ell_3(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ we have the estimates:

$$\|L_n^{\pm,1}(t; A_0) d\|_{L_r(\mathbb{R}_r^N)} \leq C(t + 1)^{-\frac{1}{t'} + \frac{n}{t} - \frac{1}{t'}} \|d\|_{L_r(\mathbb{R}^{N-1})} \quad (n = 1, 2, 3).$$

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Proof. We use the abbreviations: \( \| \cdot \|_2 = \| \cdot \|_{L^2(\mathbb{R}^{N-1})}, \hat{f}(y_N) = \hat{f}(\xi', y_N) \), and \( \tilde{t} = t + 1 \) for \( t > 0 \) in this proof, and consider only the estimates on \( \Gamma_1^+ \) since the estimates on \( \Gamma_1^{-} \) can be shown similarly.

(1) Since \( \lambda = -A^2 + (A^2/4)e^{iu} \) for \( u \in [0, \pi/2] \) on \( \Gamma_1^+ \), we have

\[
[K_{11}^{-1}(t; A_0) f](x) = \int_0^\infty F_{\xi'}^{-1} \left[ \int_0^\frac{\pi}{2} e^{-A^2 + (A^2/4)e^{iu}} \varphi_0(\xi') k_1(\xi', \lambda) e^{-A(x_N + y_N)} \frac{iA^2}{4} e^{iu} du \hat{f}(y_N) \right] (x') dy_N.
\]

Noting that \( |e^{-A^2 + (A^2/4)e^{iu}}| \leq Ce^{-(3/4)A^2} \) for some positive constant \( C \) independent of \( \xi' \), \( u \), and \( t \), we see that by Lemma 4.3, \( L_q-L_r \) estimates of the \((N-1)\)-dimensional heat kernel, and Parseval’s theorem

\[
\| [K_{11}^{-1}(t; A_0)f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} \leq Ct^{-\frac{N-1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \int_0^\infty \left\| e^{-(A^2/2)\tilde{t}} A e^{-A(x_N + y_N)} f(y_N) \right\|_2 dy_N,
\]

and furthermore, for \( n = 2, 3, 4, 5, 6 \)

\[
\| [K_{11}^{-1}(t; A_0)f](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} \leq Ct^{-\frac{N-1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \int_0^\infty \left\| e^{-(A^2/2)\tilde{t}} A e^{-CA(x_N + y_N)} f(y_N) \right\|_2 dy_N
\]

with some positive constant \( C \) analogously, where we have used the fact that for \( a > 0 \) and \( \lambda \in \Gamma_1^{\pm} \)

\[
|\mathcal{M}(a)| \leq a \int_0^1 |e^{-(B\theta + A(1-\theta))y_N}| d\theta \leq ae^{-(A/4)a} \leq 8A^{-1} e^{-(A/8)a} \quad (4.14).
\]

We thus obtain the required inequality for \( K_{11}^{-1}(t; A_0) \) \((n = 1, \ldots, 6)\) by Lemma 4.6.

(2) Since \( \lambda = -A^2 + (A^2/4)e^{iu} \) for \( u \in [0, \pi/2] \) on \( \Gamma_1^+ \), we have

\[
[L_{11}^{-1}(t; A_0) d](x) = F_{\xi'}^{-1} \left[ \int_0^\frac{\pi}{2} e^{-A^2 + (A^2/4)e^{iu}} \varphi_0(\xi') k_1(\xi', \lambda) e^{-A(x_N + y_N)} \frac{iA^2}{4} e^{iu} du \tilde{d}(\xi') \right] (x').
\]

By calculations similar to the case of \( K_{11}^{-1}(t) \) and Lemma 4.3

\[
\| [L_{11}^{-1}(t; A_0)d](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} \leq C t^{-\frac{N-1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} + \frac{\pi}{2} \left\| e^{-(A^2/2)\tilde{t}} A e^{-A(x_N + y_N)} \tilde{d}(\xi') \right\|_2 du
\]

\[
\leq C t^{-\frac{N-1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \| d \|_{L_q(\mathbb{R}^{N-1})/((\tilde{t}/2) + x_N)}
\]

and also for \( n = 2, 3 \) we have by (4.14)

\[
\| [L_{n1}^{-1}(t; A_0)d](\cdot, x_N) \|_{L_q(\mathbb{R}^{N-1})} \leq C t^{-\frac{N-1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \| d \|_{L_q(\mathbb{R}^{N-1})/((\tilde{t}/2) + x_N)}.
\]

We thus obtain the required inequality for \( L_{n1}^{-1}(t; A_0) \) \((n = 1, 2, 3)\) by Lemma 4.4 (1).

By Lemma 4.9 we see that there exist constants \( A_1 \in (0, 1) \) and \( C > 0 \) such that for any \( \lambda \in \Gamma_1^{\pm} \), \( A \in (0, A_1) \), and \( j, k = 1, \ldots, N \)

\[
|\mathcal{V}^{BB}_{jk}(\xi', \lambda)/L(A, B)| \leq CA^{-1}, \quad |\mathcal{V}^{BM}_{jk}(\xi', \lambda)/L(A, B)| \leq C,
\]

\[
|\mathcal{V}^{MB}_{jk}(\xi', \lambda)/L(A, B)| \leq C, \quad |\mathcal{V}^{MM}_{jk}(\xi', \lambda)/L(A, B)| \leq CA,
\]

\[
|\mathcal{P}^{AB}_{jk}(\xi', \lambda)/L(A, B)| \leq C, \quad |\mathcal{P}^{AM}_{jk}(\xi', \lambda)/L(A, B)| \leq CA.
\]

Therefore, recalling (3.3)–(3.5) and (4.5) with \( \sigma = 1 \) and using Lemma 4.10 we have Theorem 4.7.
4.3 Analysis on $\Gamma^\pm_2$

Our aim here is to show the following theorem for the operators defined in (4.5) with $\sigma = 2$.

**Theorem 4.11.** Let $1 \leq r \leq 2 \leq q \leq \infty$ and $F = (f, d) \in L_r(R^N) \times L_r(R^{N-1})$. Then there exists an $A_0 \in (0, 1)$ such that the following assertions hold:

1. Let $k = 0, 1, \ell = 0, 1, 2$, and $\alpha' \in \mathbb{N}^{N-1}$. Then there exists a positive constant $C = C(\alpha')$ such that for any $t > 0$
   
   \[
   \|\partial_t^{k} D_0^{\alpha'} D_N^{\ell} S_0^{t^2} (t; A_0) F\|_{L_q(R^N)} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-\alpha'+|\alpha'|}{2}} \|f\|_{L_r(R^N)},
   \]
   
   \[
   \|\partial_t^{k} D_0^{\alpha'} D_N^{\ell} S_0^{t^2} (t; A_0) F\|_{L_q(R^{N-1})} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-\alpha'+|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|d\|_{L_r(R^{N-1})},
   \]
   
   provided that $k + \ell + |\alpha'| \neq 0$. In addition, if $(q, r) \neq (2, 2)$, then
   
   \[
   \|\partial_t^{k} D_0^{\alpha'} D_N^{\ell} S_0^{t^2} (t; A_0) F\|_{L_q(R^N)} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-\alpha'+|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|f\|_{L_r(R^N)},
   \]
   
   \[
   \|\partial_t^{k} D_0^{\alpha'} D_N^{\ell} S_0^{t^2} (t; A_0) F\|_{L_q(R^{N-1})} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-\alpha'+|\alpha'|}{2} - \frac{|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|d\|_{L_r(R^{N-1})}.
   \]

2. There exists a positive constant $C$ such that for any $t > 0$
   
   \[
   \|\nabla_0^{t^2} (t; A_0) F\|_{L_q(R^N)} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2}} \|f\|_{L_r(R^N)},
   \]
   
   \[
   \|\nabla_0^{t^2} (t; A_0) F\|_{L_q(R^{N-1})} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|d\|_{L_r(R^{N-1})},
   \]

3. Let $\alpha \in \mathbb{N}^N$. Then there exists a positive constant $C = C(\alpha)$ such that for any $t > 0$
   
   \[
   \|D_0^{\alpha'} \nabla F_0^{t^2} (t; A_0) F\|_{L_q(R^N)} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2}} \|f\|_{L_r(R^N)},
   \]
   
   \[
   \|D_0^{\alpha'} \nabla F_0^{t^2} (t; A_0) F\|_{L_q(R^{N-1})} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|d\|_{L_r(R^{N-1})},
   \]
   
   provided that $|\alpha| \neq 0$, if $(q, r) \neq (2, 2)$, then
   
   \[
   \|\partial_t^{\alpha} (t; A_0) F\|_{L_q(R^N)} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2} - \frac{|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|f\|_{L_r(R^N)},
   \]
   
   In addition, if $(q, r) \neq (2, 2)$, then
   
   \[
   \|\partial_t^{\alpha} (t; A_0) F\|_{L_q(R^{N-1})} \leq C(t + 1)^{-\frac{2}{1} - \frac{1}{4} \frac{1-|\alpha'|}{2} - \frac{|\alpha'|}{2} - \frac{|\alpha'|}{2} - \frac{|\alpha'|}{2}} \|d\|_{L_r(R^{N-1})}.
   \]

We start with the following lemma in order to show Theorem 4.11.

**Lemma 4.12.** There exist positive constants $A_1 \in (0, 1)$, $b_0 \geq 1$, and $C$ such that for any $\lambda \in \Gamma^\pm_2$ and $A \in (0, A_1)$

\[
 b_0^{-1}(A \sqrt{1 - u} + \sqrt{u} + A) \leq |B| \leq b_0(A \sqrt{1 - u} + \sqrt{u} + A),
\]

\[
 |D(A, B)| \geq C(A \sqrt{1 - u} + \sqrt{u} + A)^3,
\]

\[
 |L(A, B)| \geq C(A \sqrt{1 - u} + \sqrt{u} + A^{1/4})^4.
\]

**Proof.** We first show the inequalities for $B$. Set $\sigma = \lambda + A^2$ and $\theta = \arg \sigma$. Noting that

\[
 \lambda = -(A^2(1 - u) + \gamma_0 u) + \pm i((A^2/4)(1 - u) + \tilde{\gamma}_0 u)
\]

for $u \in [0, 1]$ on $\Gamma^\pm_2$, we have

\[
 |\sigma| + A^2(1 - u) + \gamma_0 u - A^2 \leq 2(A^2(1 - u) + \gamma_0 u + A^2) + \frac{A^2}{4}(1 - u) + \tilde{\gamma}_0 u
\]

\[
 \leq 3 \max(\gamma_0, \tilde{\gamma}_0)(A^2(1 - u) + u + A^2) \leq 3 \max(\gamma_0, \tilde{\gamma}_0)(A \sqrt{1 - u} + \sqrt{u} + A)^2,
\]
which is used to obtain
\[
\Re B = |\sigma|^\frac{\theta}{2} \cos \frac{\theta}{2} = \frac{|\sigma|}{|\sigma|} (1 + \cos \theta) = \frac{|\sigma|}{|\sigma|} (|\sigma| + \Re \sigma) = \frac{1}{\sqrt{2}} \left( |\sigma|^2 - (\Re \sigma)^2 \right)^{\frac{1}{2}}
\]
\[
\geq \frac{(A^2/4)(1-u) + \gamma_0 u + (A^2/8) - (A^2/8)(1-u) - (A^2/8)u}{\sqrt{2}(|\sigma| + A^2(1-u) + \gamma_0 u - A^2)^{1/2}}
\]
\[
\geq \frac{(A^2/8)(1-u) + \gamma_0 u + (A^2/8) - (A^2/8)u}{\sqrt{6} \max(\gamma_0^2, \gamma_0^2)(A\sqrt{1-u} + \sqrt{u} + A)}
\]
\[
\geq \frac{(1/8)|A^2(1-u) + \gamma_0 u + A^2|}{\sqrt{6} \max(\gamma_0^2, \gamma_0^2)(A\sqrt{1-u} + \sqrt{u} + A)} \geq \frac{A\sqrt{1-u} + u + A}{24 \sqrt{6} \max(\gamma_0^2, \gamma_0^2)}
\]
for any $A \in (0, A_1)$ provided that $A_1^2 \leq 7\gamma_0$. It is clear that the other inequalities concerning $B$ hold.

Next we consider $D(A, B)$. Noting that $\lambda \in \Gamma_2^s \subset \Sigma_{\epsilon_0}$ and using Lemma \[2.1\] \[2\], we obtain
\[
|D(A, B)| \geq C(\epsilon_0)(|\lambda|^\frac{\theta}{2} + A)^3 \geq C(\epsilon_0)(A\sqrt{1-u} + \sqrt{u} + A)^3.
\]
Finally, we show the inequality for $L(A, B)$. By \[4.2\]
\[
B^2 - (\gamma_1^s)^2 = -(A^2(1-u) + \gamma_0 u) \pm i \left( \frac{A^2}{4}(1-u) + \gamma_0 u - c_{1/2} A^{1/2} \right) + 2A^2 + O(A^{10/4})
\]
as $A \to 0$, and also we have
\[
\left| -(A^2(1-u) + \gamma_0 u) \pm i \left( \frac{A^2}{4}(1-u) + \gamma_0 u - c_{1/2} A^{1/2} \right) \right|^2
\]
\[
= (A^2(1-u) + \gamma_0 u)^2 + \left( \frac{A^2}{4}(1-u) + \gamma_0 u \right)^2 + c_g A - 2c_{1/2} A^{1/2} \left( \frac{A^2}{4}(1-u) + \gamma_0 u \right)
\]
\[
\geq \left( A^2(1-u) + \gamma_0 u \right)^2 + \frac{1}{11} c_g A - \frac{1}{10} \left( \frac{A^2}{4}(1-u) + \gamma_0 u \right)^2
\]
\[
\geq \frac{1}{90} (A^2(1-u) + \gamma_0 u)^2 + \frac{1}{11} c_g A \geq C \left( A\sqrt{1-u} + \sqrt{u} + A^{1/4} \right)^4.
\]
We thus see that there exist positive constants $A_1$ and $\lambda$ such that for any $A \in (0, A_1)$ and $\lambda \in \Gamma_2^s$
\[
|B - \gamma_1^s| = \frac{|B^2 - (\gamma_1^s)^2|}{|B + \gamma_1^s|} \geq \frac{C \left( A\sqrt{1-u} + \sqrt{u} + A^{1/4} \right)^2}{b_0(A\sqrt{1-u} + \sqrt{u} + A) + c_{1/4} A^{1/4}} \geq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4}).
\]
Since $|B - \gamma_1^s| \leq |B - \gamma_2^s|$ as follows from $\Re B \geq 0$ and \[4.1\], we have the required inequality for $L(A, B)$, which completes the proof of Lemma \[4.1\].

Lemma 4.13. Let $1 \leq r \leq 2 \leq q \leq \infty$, and let $f \in L_r(R_+^N)$ and $d \in L_r(R^{N-1})$. We use the symbols defined in \[4.9\].

(1) Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0, 1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_2^s$ and $A \in (0, A_1)$
\[
|k_n(\xi, \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A |B|^s \quad (n = 1, 3),
\]
\[
|k_2(\xi, \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A |B|^s,
\]
\[
|k_n(\xi, \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A^2 |B|^s \quad (n = 4, 5),
\]
\[
|k_0(\xi, \lambda)| \leq CA |B|^s.
\]

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Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ and $n = 1, \ldots, 6$ we have the estimates:

$$\|K_n^{\pm,2}(t; A_0)f\|_{L_1^s(R^N_+)} \leq (t + 1)^{-\frac{N}{2} + \frac{1}{2} - \frac{n}{2}} \|f\|_{L_1^s(R^N_+)}$$

provided that $s > 0$. In the case of $s = 0$, we have

$$\|K_n^{\pm,2}(t; A_0)f\|_{L_1^s(R^N_+)} \leq (t + 1)^{-\frac{N}{2} + \frac{1}{2} - \frac{n}{2}} \|f\|_{L_1^s(R^N_+)} \quad \text{if } (q,r) \neq (2,2).$$

(2) Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0, 1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma^+_2$ and $A \in (0, A_1)$

$$|\ell_n(\xi', \lambda)| \leq C(A\sqrt{1 - u} + \sqrt{u} + A^{1/4})^{-4}A|B|^s \quad (n = 1, 2),$$

Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ we have the estimates:

$$\|L_n^{\pm,2}(t; A_0)d\|_{L_1^s(R^N_+)} \leq (t + 1)^{-\frac{N}{2} + \frac{1}{2} - \frac{n}{2}} \|d\|_{L_1^s(R^N_+)} \quad (n = 1, 3, s > 0),$$

$$\|L_2^{\pm,2}(t; A_0)d\|_{L_1^s(R^N_+)} \leq (t + 1)^{-\frac{N}{2} + \frac{1}{2} - \frac{n}{2}} \|d\|_{L_1^s(R^N_+)} \quad (s \geq 0).$$

In the case of $s = 0$, we have

$$\|L_n^{\pm,2}(t; A_0)d\|_{L_1^s(R^N_+)} \leq (t + 1)^{-\frac{N}{2} + \frac{1}{2} - \frac{n}{2}} \|d\|_{L_1^s(R^N_+)} \quad \text{if } (q,r) \neq (2,2).$$

Proof. We use the abbreviations: $\| \cdot \|_2 = \| \cdot \|_{L_2(R^N_+)}$, $\hat{f}(y_N) = \hat{f}(\xi', y_N)$ and $\tilde{t} = t + 1$ for $t > 0$ in this proof, and consider only the estimates on $\Gamma^+_2$ since the estimates on $\Gamma^+_2$ can be shown similarly.

(1) We first show the inequality for $K_n^{\pm,2}(t; A_0)$. Recalling that $\lambda = -(A^2(1 - u) + \gamma_0 u) + i((A^2/4)(1 - u) + \gamma_0 u)$ for $u \in [0, 1]$ on $\Gamma^+_2$, we have, by \ref{4.16},

$$[K_1^{\pm,2}(t; A_0)f](x) = \int_0^\infty \hat{f}(x_\xi) \left[ \int_0^{\infty} e^{-\left(A^2(1-u)+\gamma_0 u\right)t+i((A^2/4)(1-u)+\gamma_0 u)t} \varphi_0(\xi') k_1(\xi', \lambda)e^{-A(x_N+y_N)} \right] \, dy_N.$$ 

Since it follows from Lemma \ref{4.12} that

$$|e^{-\left(A^2(1-u)+\gamma_0 u\right)t+i((A^2/4)(1-u)+\gamma_0 u)t}| \leq e^{-\frac{1}{2}A^2 t}e^{-\frac{1}{2}(A^2(1-u)+\gamma_0 u)t} \leq Ce^{-\frac{1}{2}A^2 t}e^{-C|B|^2 t}$$

with some positive constant $C$, independent of $\xi'$, $\lambda$, and $t$, for any $A \in (0, A_0)$ by choosing suitable $A_0 \in (0, A_1)$, we have, by $L_q$-estimates of the $(N - 1)$-dimensional heat kernel and Parseval’s theorem,

$$\|\|K_1^{\pm,2}(t; A_0)f|_{L_q(R^N_+)}\|_{L_q(R^N_+)} \leq C \int_0^{\frac{N-1}{2} + \frac{1}{2}} \int_0^1 e^{-\frac{1}{2}A^2 t}e^{-\frac{1}{2}(A^2(1-u)+\gamma_0 u)t} \, dy_N$$

for a sufficiently small $\delta > 0$. If $s > 0$, then we have, by Lemma \ref{4.12} and Lemma \ref{4.13} with $Z = |B|$ and $a = 0$,

$$\int_0^1 e^{-C|B|^2 t}e^{-\delta \varphi_0(\xi')} \, du \leq C \int_0^1 e^{-\delta \varphi_0(\xi')} \, du \leq C \int_0^1 e^{-\delta \varphi_0(\xi')} \, du \leq CT^{-\frac{1}{2}}.$$
We thus obtain
\[ \| [K_n^{+2}(t; A_0) f] (\cdot, x_N) \|_{L_q(R^{N-1})} \]
\[ \leq C \int^{\infty}_{0} \left\| \int_{0}^{1} e^{-\frac{C |B|^2 \varphi_0(\zeta)}{(\sqrt{u})^{2-\delta}}} \, du \, e^{-\frac{4 \alpha^2}{\delta} A^{1-\delta} e^{-A(x_N+y_N)} f(y_N)} \right\|_{2} \, dy_N, \]
which furnishes the required inequality by Lemma 4.6. In the case of \( s = 0 \), by Lemma 4.3 and (4.16)
\[ \| K_n^{+2}(t; A_0) f \|_{L_q(R^{N-1})} \]
\[ \leq C \int^{\infty}_{0} \left\| \int_{0}^{1} e^{-\frac{C |B|^2 \varphi_0(\zeta)}{(\sqrt{u})^{2-\delta}}} \, du \, e^{-\frac{4 \alpha^2}{\delta} A^{1-\delta} e^{-A(x_N+y_N)} f(y_N)} \right\|_{2} \, dy_N, \]
which implies that the required inequality holds by Lemma 4.4 (2) if we choose a sufficiently small \( \delta > 0 \) and \( (q, r) \neq (2, 2) \). Analogously, for \( K_2^{+2}(t; A_0) \), we see that the required inequality holds, noting that there holds, by (2.22) and Lemma 4.12
\[ |M(a)| \leq a \int^{1}_{0} e^{-\int_{0}^{1} \left( (\text{Re}B)\theta + A(1-\theta) \right) d\theta} \leq a e^{-B_{0}^{-1} A_{0} A} \leq 2b_{0} A_{0}^{-1} e^{-\left(B_{0}^{-1}/2 \right) A_a} \]
for \( a > 0 \) and any \( A \in (0, A_{0}) \) by choosing suitable \( A_{0} \in (0, A_{1}) \). Then, by (4.15) and Lemma 4.12 there holds
\[ \| K_n^{+2}(t; A_0) f \|_{L_q(R^{N-1})} \]
\[ \leq C \int^{\infty}_{0} \left\| \int_{0}^{1} e^{-\frac{C |B|^2 \varphi_0(\zeta)}{(\sqrt{u})^{2-\delta}}} \, du \, e^{-\frac{4 \alpha^2}{\delta} A^{1-\delta} e^{-A(x_N+y_N)} f(y_N)} \right\|_{2} \, dy_N, \]
which furnishes the required inequalities of \( K_n^{+2}(t; A_0) \) \( (n = 3, 4, 5) \) hold in the same manner as we have obtained the inequality of \( K_n^{+2}(t; A_0) \) from (4.16).
Finally, we consider \( K_0^{+2}(t; A_0) \). By (4.17) and (4.18), we have for \( s > 0 \)
\[ \| K_0^{+2}(t; A_0) f \|_{L_q(R^{N-1})} \]
\[ \leq C \int^{\infty}_{0} \left\| \int_{0}^{1} e^{-\frac{C |B|^2 \varphi_0(\zeta)}{(\sqrt{u})^{2-\delta}}} \, du \, e^{-\frac{4 \alpha^2}{\delta} A^{1-\delta} e^{-A(x_N+y_N)} f(y_N)} \right\|_{2} \, dy_N, \]
by choosing sufficiently small \( \delta > 0 \). We thus obtain the required inequality of \( K_0^{+2}(t; A_0) \) by Lemma 4.6 if \( s > 0 \). In the case of \( s = 0 \), since it follows from Lemma 4.12 that
\[ |M(a)| \leq a \int^{1}_{0} e^{-\int_{0}^{1} \left( (\text{Re}B)\theta + A(1-\theta) \right) d\theta} \leq a \int^{1}_{0} e^{-\int_{0}^{1} \left( (b_{0}^{-1} \sqrt{\text{Re}B} + A(1-\theta) \right) x_N d\theta} \]
\[ \leq a e^{-b_{0}^{-1} A a} \int^{1}_{0} e^{-b_{0}^{-1} \sqrt{a_{0}} d\theta} \quad (a > 0, \lambda \in \Gamma_{2}^{\frac{1}{2}}) \]
for any $A \in (0, A_0)$ by choosing some $A_0 \in (0, A_1)$, we easily obtain by Lemma 4.3

$$\|K_0^{+,2}(t; A_0)f(\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} \leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty \left\| \int_0^1 e^{-\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} e^{-C |B|^{\frac{1}{2}}} \varphi_0(\xi') A M(x_N) M(y_N) d\xi(y_N) \right\|_2 \text{d}y_N$$

$$\leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty x_t y_N \left\| e^{-\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} A e^{-C A(x_N+y_N)} f(y_N) \right\|_2 \times \int \int_0^1 \int_0^1 e^{-C A e^{-C A(x_N+y_N)}} \text{d}u \text{d}v \text{d}w$$

$$\leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty x_t y_N \left\| f(\cdot, y_N) \right\|_{L_q(\mathbb{R}^{N-1})} \int_0^1 \int_0^1 \frac{d\varphi d\psi}{t^{1/2} + x_N + y_N} \text{d}y_N$$

for some positive constant $C$. The change of variable: $\psi y_N = \{t + (\varphi x_N)^2\}^{1/2} \ell$ yields that

$$\int_0^1 \frac{d\varphi}{t + (\varphi x_N)^2 + (\psi y_N)^2} \leq \frac{1}{t + (\varphi x_N)^2} \int_0^\infty \frac{1}{1 + \ell^2} \frac{\{t + (\varphi x_N)^2\}^{1/2} \ell}{y_N} \text{d}\ell \leq \frac{C}{y_N(t^{1/2} + \varphi x_N)}$$

for a positive constant $C$, so that

$$\|K_0^{+,2}(t; A_0)f(\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} \leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty x_t y_N \left\| f(\cdot, y_N) \right\|_{L_q(\mathbb{R}^{N-1})} \int_0^1 \int_0^1 \frac{d\varphi}{t^{1/2} + x_N + y_N} \text{d}y_N$$

$$\leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty x_t y_N \left\| f(\cdot, y_N) \right\|_{L_q(\mathbb{R}^{N-1})} \int_0^1 \int_0^1 \frac{d\varphi d\psi}{t^{1/2} + x_N + y_N} \text{d}y_N$$

for any $0 < \delta < 1$. By the change of variable: $\varphi x_N = t^{1/2} \ell$, we then have

$$\int_0^1 \frac{d\varphi}{(t^{1/2} + x_N + y_N)\delta(t^{1/2} + \varphi x_N)} \leq \int_0^1 \frac{d\varphi}{(t^{1/2} + x_N + y_N)^\delta} \leq C \int_0^1 \int_0^{t^{1/2} + (\varphi x_N)^{1+\delta}} \frac{d\varphi d\ell}{t^{1/2} + x_N + y_N} \leq C \frac{t^{1/2}}{t^{1+\delta}} \int_0^1 \int_0^{t^{1/2}} \frac{d\ell}{x_N^{1+\delta}}$$

with a positive constant $C$, which furnishes that

$$\|K_0^{+,2}(t; A_0)f(\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} \leq C \int_0^{\frac{\alpha}{A_0+1} \left( \frac{1}{4} - \frac{1}{4} \right)} \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{t^{1/2} + x_N + y_N} \text{d}y_N.$$

We therefore obtain the required inequality by Lemma 4.4 (2) by choosing a sufficiently small $\delta > 0$ when $(q, r) \neq (2, 2)$.

(2) First, we show the inequality for $L_t^{+,2}(t; A_0)$. Noting that $\lambda = -(A^2(1-u) + \gamma_0u) + i((A^2/4)(1-u) + \gamma_0u)$ for $u \in [0, 1]$ on $\Gamma_1^+$, we have, by 4.6,

$$[L_t^{+,2}(t; A_0)d](x) = \mathcal{F}_t^{-1} \left[ \int_0^1 e^{-((A^2(1-u) + \gamma_0u) + i((A^2/4)(1-u) + \gamma_0u))t} \varphi_0(\xi') \ell_1(\xi', \lambda) e^{-A(x_N+y_N)} \times \left\{ -(\gamma_0 - A^2) + i \left( \gamma_0 - \frac{A^2}{4} \right) \right\} \tilde{d}(\xi') \right](x').$$
In a similar way to the case of $K_{1}^{+2}(t; A_0)$, we have by (4.17) and Lemma 4.3
\[
\|L_{1}^{+2}(t; A_0)d(\cdot, x_N)\|_{L_{4}(R^{n-1})}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \int_{0}^{1} e^{-C|B|^{2}\hat{t}} e^{-C|B|^{2\frac{\varphi_{0}(\xi)}{A^{1/2}}} B^{\alpha} e^{-A_{\hat{t}}} A^{1/2}} e^{-\varphi_{1}(\xi)} du \widehat{\theta}(\xi) \right\|_{2}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \int_{0}^{1} e^{-C|B|^{2}\hat{t}} e^{-C|B|^{2\frac{\varphi_{0}(\xi)}{A^{1/2}}} B^{\alpha} e^{-A_{\hat{t}}}} du \widehat{\theta}(\xi) \right\|_{2}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| d\right\|_{L_{4}(R^{n-1})/(\hat{t}^{1/4} + (x_N)^{1/2})}.
\] (4.19)

We thus obtain the required inequality by Lemma 4.11 (1) if $s > 0$ and $q > 2$. In the case of $s = 0$ and $q = 2$, by (4.20) and using (4.17) again, we have
\[
\|L_{1}^{+2}(t; A_0)d(\cdot, x_N)\|_{L_{4}(R^{n-1})}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \int_{0}^{1} e^{-C|B|^{2}\hat{t}} e^{-C|B|^{2\frac{\varphi_{0}(\xi)}{A^{1/2}}} B^{\alpha} e^{-A_{\hat{t}}}} du \widehat{\theta}(\xi) \right\|_{2}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| d\right\|_{L_{4}(R^{n-1})}.
\] (4.20)

If $s = 0$, then we have by Lemma 4.3 and Lemma 4.12
\[
\|L_{1}^{+2}(t; A_0)d(\cdot, x_N)\|_{L_{4}(R^{n-1})}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \int_{0}^{1} e^{-C|B|^{2}\hat{t}} e^{-C|B|^{2\frac{\varphi_{0}(\xi)}{A^{1/2}}} B^{\alpha} e^{-A_{\hat{t}}}} du \widehat{\theta}(\xi) \right\|_{2}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \int_{0}^{1} e^{-C|B|^{2}\hat{t}} e^{-C|B|^{2\frac{\varphi_{0}(\xi)}{A^{1/2}}} B^{\alpha} e^{-A_{\hat{t}}}} du \widehat{\theta}(\xi) \right\|_{2}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| d\right\|_{L_{4}(R^{n-1})/(\hat{t}^{1/4} - \delta/8) + (x_N)^{1/2} - (\delta/4)}.
\] (4.21)

which, combined with Lemma 4.4 (1), furnishes that the required inequality holds for $q > 2$ by choosing a sufficiently small $\delta > 0$. In the case of $s = 0$ and $q = 2$, by (4.21) and Young's inequality with $1 + (1/2) = (1/p) + (1/r)$ for $1 \leq r < 2$, we have
\[
\|L_{1}^{+2}(t; A_0)d\|_{L_{4}(R^{n})}
\leq C\hat{t}^{-\frac{n-1}{2}} \left\| \|F_{\xi}^{-1}[e^{-(A^{2}/2\hat{t}) A^{-\delta/4}}]\|_{L_{p}(R^{n-1})}\right\|_{L_{4}(R^{n})}d\|_{L_{4}(R^{n-1})}.
\] (4.22)

We use the following proposition proved by [15, Theorem 2.3] to calculate the right-hand side of (4.22).

**Proposition 4.14.** Let $X$ be a Banach space and $\| \| X$ its norm. Suppose that $L$ and $n$ be a non-negative integer and positive integer, respectively. Let $0 < \sigma \leq 1$ and $s = L + \sigma - n$. Let $f(\xi)$ be a $C^{\infty}$-function, defined on $R^{n} \setminus \{0\}$ with value $X$, which satisfies the following two conditions:

1. $D_{\xi}^{\alpha}f \in L_{1}(R^{n}, X)$ for any multi-index $\alpha \in N_{n}^{0}$ with $|\alpha| \leq L$.

2. For any multi-index $\alpha \in N_{n}^{0}$, there exists a positive constant $C(\alpha)$ such that
\[
\|D_{\xi}^{\alpha}f(\xi)\|_{X} \leq C(\alpha)|\xi|^{\alpha}
\] (\xi \in R^{n} \setminus \{0\}).

Then there exists a positive constant $C(n, s)$ such that
\[
\|F_{\xi}^{-1}[f](x)\|_{X} \leq C(n, s) \max_{|\alpha| \leq L+2} C(\alpha) |x|^{-(n+s)}
\] (x \in R^{n} \setminus \{0\}).

By Proposition 4.14 with $n = N - 1$, $L = N - 2$, and $\sigma = 1 - \delta/4$, we have
\[
\|F_{\xi}^{-1}[e^{-(A^{2}/2\hat{t}) A^{-\delta/4}}](x')\| \leq C|x'|^{-(N-1-\delta/4)}
\] for a positive constant $C$, and furthermore, by direct calculations
\[
\|F_{\xi}^{-1}[e^{-(A^{2}/2\hat{t}) A^{-\delta/4}}](x')\| \leq C\hat{t}^{-1/2}(N-1-\delta/4).
\]
We thus obtain

$$\|F_x^{-1}[e^{-(A^2/2)\tilde{r}A^{-\delta/4}}](x')\| \leq \frac{C}{t^{(1/2)(N-1-\delta/4)}} |x'|(N-1-\delta/4)$$

for some positive constant $C$. Therefore, by choosing a sufficiently small $\delta > 0$, we see that

$$\|F_x^{-1}[e^{-(A^2/2)\tilde{r}A^{-\delta/4}}]|\mathcal{L}_p(\mathbb{R}^N)| \leq C\tilde{r}^{-\frac{N-1}{(1-\delta/4)}}(1-\frac{\delta}{4}) + \frac{\delta}{4}$$

since $p > 1$ by $1 \leq r < 2$, which, combined with (4.22), furnishes that the required inequality holds.

Summing up in the case of $s = 0$, we have obtained

$$\|L_1^{+2}(t; A_0)\|_{\mathcal{L}_r(\mathbb{R}^N)} \leq C\tilde{r}^{-\frac{N-1}{(1-\delta/4)}}(1-\frac{\delta}{4}) \frac{\|d\|_{\mathcal{L}_s(\mathbb{R}^N)}}{t}$$

for some positive constant $C$ and $1 \leq r \leq 2 \leq q < \infty$ when $(q, r) \neq (2, 2)$.

Concerning $L_3^{+2}(t; A_0)$, we see, by Lemma 4.3 that

$$\|L_3^{+2}(t; A_0)d(\cdot, x_N)\|_{\mathcal{L}_s(\mathbb{R}^N)} \leq C\tilde{r}^{-\frac{N-1}{(1-\delta/4)}} \frac{1}{t} \int_0^1 e^{-(A^2/2)\tilde{r}e^{-C|B|^2}\tilde{r} \varphi_0(\xi')e^{-\mathbb{R}Bx_N}} du \tilde{d}(\xi')$$

which, combined with Lemma 4.3 (1), furnishes the required inequality for $L_3^{+2}(t; A_0)$.

Finally, we show the inequality for $L_3^{+2}(t; A_0)$. We easily have by (4.18) and Lemma 4.12

$$\|L_3^{+2}(t; A_0)d(\cdot, x_N)\|_{\mathcal{L}_s(\mathbb{R}^N)} \leq C\tilde{r}^{-\frac{N-1}{(1-\delta/4)}} \frac{1}{t} \int_0^1 e^{-(A^2/2)\tilde{r}e^{-C|B|^2}\tilde{r} \varphi_0(\xi')A^{1/2}|B|^2e^{-C\mathbb{R}Bx_N}} du \tilde{d}(\xi')$$

for a positive constant $C$. We thus obtain the required inequality in the same manner as we have obtained the inequality of $L_1^{+2}(t; A_0)$ from (4.19).

**Corollary 4.15.** Let $1 \leq r \leq 2 \leq q < \infty$, and let $f \in \mathcal{L}_r(\mathbb{R}^N)$ and $d \in \mathcal{L}_r(\mathbb{R}^N)$. We use the symbols defined in (4.1).

1. Let $\alpha \in \mathbb{N}_0^N$ and we assume that there exist positive constants $A_1 \in (0, 1)$ and $C$ such that for any $\lambda \in \Gamma_{\frac{\alpha}{2}}^N$ and $A \in (0, A_1)$

$$|k_1(\xi', \lambda)| \leq C(A\sqrt{1-u + \sqrt{u} + A^{1/4}})^{-4}A,$$

$$|k_2(\xi', \lambda)| \leq C(A\sqrt{1-u + \sqrt{u} + A^{1/4}})^{-4}A^2,$$

$$|\ell_1(\xi', \lambda)| \leq C(A\sqrt{1-u + \sqrt{u} + A^{1/4}})^{-4}|B|^2.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(\alpha)$ such that for any $t > 0$ and $n = 1, 2$

$$\|D_\alpha^2 \nabla K_{n}^{\pm, 2}(t; A_0)f\|_{\mathcal{L}_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N-1}{(1-\delta/4)}} \frac{\|f\|_{\mathcal{L}_r(\mathbb{R}^N)}}{t}$$

if $|\alpha| \neq 0$, and

$$\|D_\alpha^2 \nabla L_{n}^{+2}(t; A_0)d\|_{\mathcal{L}_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N-1}{(1-\delta/4)}} \frac{\|d\|_{\mathcal{L}_s(\mathbb{R}^N)}}{t}.$$

In addition, if $(q, r) \neq (2, 2)$, then we have for any $t > 0$ and $n = 1, 2$

$$\|\partial_t K_{n}^{\pm, 2}(t; A_0)f\|_{\mathcal{L}_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N-1}{(1-\delta/4)}} \|f\|_{\mathcal{L}_r(\mathbb{R}^N)}.$$

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(2) Let $k = 0, 1, \ell = 0, 1, 2$, and $\alpha' \in \mathbb{N}^{N-1}_0$. We assume that there exist positive constants $A_1 \in (0, 1)$ and $C$ such that for any $\lambda \in \Gamma^\perp$ and $A \in (0, A_1)$

$$|k_3(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2}A,$$

$$|k_0(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2}A|B| \quad (n = 4, 5),$$

$$|k_0(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2}A|B|^2,$$

$$|\ell_2(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4}A,$$

$$|\ell_3(A, B)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3}A.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(\alpha')$ such that for any $t > 0$

$$|\partial^k_t D_{x'}^\alpha D_{x''}^\beta K_n^{\pm,2}(t; A_0)f|_{L_4(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})^{-k - \frac{|\alpha'|}{2}}\|f\|_{L_4(R^N_x)} \quad (n = 3, 4, 5, 6),$$

$$|\partial^k_t D_{x'}^\alpha D_{x''}^\beta L_n^{\pm,2}(t; A_0)d|_{L_4(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})^{-k - \frac{|\alpha'|}{2}}\|d\|_{L_4(R^{N-1})} \quad (n = 2, 3).$$

provided that $k + \ell + |\alpha'| \neq 0$. In addition, if $(q, r) \neq (2, 2)$, then there hold for any $t > 0$

$$|K_n^{\pm,2}(t; A_0)f|_{L_q(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|f\|_{L_q(R^N_x)} \quad (n = 3, 4, 5, 6),$$

$$|L_n^{\pm,2}(t; A_0)d|_{L_q(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|d\|_{L_q(R^{N-1})} \quad (n = 2, 3).$$

**Proof.** We only show the inequalities for $K_n^{\pm,2}(t)$, $K_n^{\pm,2}(t)$, and $L_n^{\pm,2}(t)$. The other inequalities can be proved by Lemma [4.13] directly. By (4.6)

$$\partial^k_t D_{x'}^\alpha [K_n^{\pm,2}(t; A_0)f](x) = \int_0^{\infty} \mathcal{F}_x^{-1} \left[ \int_{\Gamma^\perp} e^{\lambda t} \varphi_0(\xi') \lambda^k(i\xi')^\alpha k_n(\xi', \lambda) \chi_n(xN, yN) d\lambda \hat{f}(\xi', yN) \right](x'),$$

$$\partial^k_t D_{x'}^\alpha [L_n^{\pm,2}(t; A_0)d](x) = \mathcal{F}_x^{-1} \left[ \int_{\Gamma^\perp} e^{\lambda t} \varphi_0(\xi') \lambda^k(i\xi')^\alpha \ell_3(\xi', \lambda) \chi(xN) d\lambda \hat{d}(\xi') \right](x')$$

for $n = 5, 6$. Since by Lemma [4.12]

$$|\lambda^k(i\xi')^\alpha k_n(\xi', \lambda)| \leq C \begin{cases} (A\sqrt{1-u} + \sqrt{u} + A)^{-4}A|B|^{2k+|\alpha'|} \quad (n = 5), \\
A|B|^{2k+|\alpha'|} \quad (n = 6), \end{cases}$$

$$|\lambda^k(i\xi')^\alpha \ell_3(\xi', \lambda)| \leq (A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3}A|B|^{2k+|\alpha'|}$$

for $\lambda \in \Gamma^\perp$ and $A \in (0, A_0)$ by choosing some $A_0 \in (0, A_1)$, we obtain by Lemma [4.13]

$$\|\partial^k_t D_{x'}^\alpha K_n^{\pm,2}(t)f\|_{L_4(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|f\|_{L_4(R^N_x)} \quad (n = 5, 6),$$

$$\|\partial^k_t D_{x'}^\alpha L_n^{\pm,2}(t)d\|_{L_4(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|d\|_{L_4(R^{N-1})} \quad (n = 2, 3).$$

for any $t > 0$, provided that $k + |\alpha'| \neq 0$. In the case of $k + |\alpha'| = 0$, we have by Lemma [4.13]

$$\|K_n^{\pm,2}(t)f\|_{L_q(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|f\|_{L_q(R^N_x)} \quad (n = 5, 6),$$

$$\|L_n^{\pm,2}(t)d\|_{L_q(R^N_x)} \leq C(t + 1)^{-\frac{3}{2}}(\frac{1}{t} - \frac{1}{t})\|d\|_{L_q(R^{N-1})}$$

(4.24)
when \((q, r) \neq (2, 2)\). On the other hand, by (2.2)

\[
\begin{aligned}
\partial^\ell_x D_{x'}^\ell D_N^\pm [K_5^{\pm, 2}(t)f](x) &= (-1)^\ell \\
\int_0^\infty \mathcal{F}^{-1}_{\ell} \left[ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} (B + A) \lambda^{\ell-1} k_0(\xi', \lambda) e^{-B(x + y) + \lambda N} d\lambda \tilde{f}(\xi', y_n) \right] (x') dy_n \\
&+ \int_0^\infty \mathcal{F}^{-1}_{\ell} \left[ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} A' k_0(\xi', \lambda) M(x_n) e^{-B y_n} d\lambda \tilde{f}(\xi', y_n) \right] (x') dy_n,
\end{aligned}
\]

\[
\partial^\ell_x D_{x'}^\ell D_N^\pm [K_6^{\pm, 2}(t)f](x) = (-1)^\ell \\
\int_0^\infty \mathcal{F}^{-1}_{\ell} \left[ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} (B + A)^{\ell-1} k_0(\xi', \lambda) e^{-B x_n} M(y_n) d\lambda \tilde{f}(\xi', y_n) \right] (x') dy_n \\
+ \int_0^\infty \mathcal{F}^{-1}_{\ell} \left[ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} A' k_0(\xi', \lambda) M(x_n) M(y_n) d\lambda \tilde{f}(\xi', y_n) \right] (x') dy_n,
\]

\[
\partial^\ell_x D_{x'}^\ell D_N^\pm [L_{3, 2}^{\pm, 2}(t)d](x) = (-1)^\ell \left\{ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} (B + A)^{\ell-1} \lambda_3(\xi', \lambda) e^{-B x_n} d\lambda \tilde{f}(\xi', y_n) \right\} (x')
\]

\[
+ \int_{\Gamma_{\ell}} e^{\lambda t} \phi_0(\xi') \lambda^k (i \xi')^{\alpha'} A' \lambda_3(\xi', \lambda) M(x_n) d\lambda \tilde{f}(\xi', y_n) \right\} (x')
\]

for \(\ell = 1, 2\). Since by Lemma 4.12

\[
|\lambda^k (i \xi')^{\alpha'} (B + A)^{\ell-1} k_0(\xi', \lambda)| \leq C (A \sqrt{1 - u} + \sqrt{u} + A)^{-2} |A| B^{2k+|\alpha'|+\ell}
\]

\[
|\lambda^k (i \xi')^{\alpha'} A' k_0(\xi', \lambda)| \leq C (A \sqrt{1 - u} + \sqrt{u} + A)^{-1} |A| B^{2k+|\alpha'|+\ell},
\]

\[
|\lambda^k (i \xi')^{\alpha'} (B + A)^{\ell-1} \lambda_3(\xi', \lambda)| \leq C (A \sqrt{1 - u} + \sqrt{u} + A)^{-1} |A| B^{2k+|\alpha'|+\ell},
\]

\[
|\lambda^k (i \xi')^{\alpha'} A' \lambda_3(\xi', \lambda)| \leq C A |B|^{2k+|\alpha'|+\ell}
\]

for any \(\lambda \in \Gamma_{\frac{3}{2}}^+\) and \(A \in (0, A_0)\) by choosing suitable \(A_0 \in (0, A_1)\), we have by Lemma 4.13

\[
||\partial^\ell_x D_{x'}^\ell D_N^\pm K_n^{\pm, 2}(t)f||_{L_n(\mathbb{R}^n_+)} \leq C (t + 1)^{-1/2} (\ell + 1/4)^{-k - |\alpha'|/2} ||f||_{L_n(\mathbb{R}^n_+)} \quad (n = 5, 6)
\]

(4.25) for \(\ell = 1, 2\). In addition,

\[
|\lambda^k (i \xi')^{\alpha'} (B + A)^{\ell-1} \lambda_3(\xi', \lambda)| \leq C (A \sqrt{1 - u} + \sqrt{u} + A)^{-4} |A| B^{2k+|\alpha'|+\ell}
\]

\[
|\lambda^k (i \xi')^{\alpha'} A' \lambda_3(\xi', \lambda)| \leq C (A \sqrt{1 - u} + \sqrt{u} + A)^{-3} |A| B^{2k+|\alpha'|+\ell}
\]

for any \(\lambda \in \Gamma_{\frac{3}{2}}^+\) and \(A \in (0, A_0)\), and therefore by Lemma 4.13

\[
||\partial^\ell_x D_{x'}^\ell D_N^\pm L_3^{\pm, 2}(t)d||_{L_n(\mathbb{R}^n_+)} \leq C (t + 1)^{-1/2} (\ell + 1/4)^{-k - |\alpha'|/2} ||d||_{L_n(\mathbb{R}^{n-1})}
\]

for \(\ell = 1, 2\), which, combined with (4.23), (4.24), and (4.25), furnishes the required estimates for \(K_5^{\pm, 2}(t)\), \(K_6^{\pm, 2}(t)\), and \(L_3^{\pm, 2}(t)\). This completes the proof of Corollary 4.13. \(\square\)

By Lemma 4.12 there exist a positive number \(A_1 \in (0, 1)\) and a positive constant \(C\) such that for
Lemma 4.17. This completes the proof of Theorem 4.11.

\[ \frac{\mathcal{V}_{jk}^{A,B}(\xi', \lambda)}{L(A,B)} \leq \frac{CA}{(\sqrt{1 - u} + \sqrt{u} + A)^2}, \]
\[ \frac{\mathcal{V}_{jk}^{A,M}(\xi', \lambda)}{L(A,B)} \leq \frac{CA|B|}{(\sqrt{1 - u} + \sqrt{u} + A)^2}, \]
\[ \frac{\mathcal{P}_{jk}^{A}(\xi', \lambda)}{L(A,B)} \leq \frac{CA}{(\sqrt{1 - u} + \sqrt{u} + A^{1/4})^4}, \]
\[ \frac{\mathcal{P}_{jk}^{A,M}(\xi', \lambda)}{L(A,B)} \leq \frac{CA|B|^2}{(\sqrt{1 - u} + \sqrt{u} + A)^2}, \]

and furthermore,
\[ |A/L(A,B)| \leq C(A\sqrt{1 - u} + \sqrt{u} + A^{1/4})^{-4}A, \]
\[ |\{A(B^2 + A^2)\}/\{(B + A)L(A,B)| \leq C(A\sqrt{1 - u} + \sqrt{u} + A^{1/4})^{-3}A, \]
\[ |D(A/B)/\{(B + A)L(A,B)| \leq C(A\sqrt{1 - u} + \sqrt{u} + A^{1/4})^{-4}|B|^2. \]

Therefore, remembering (3.3)-(3.5), and (4.3) with \( \sigma = 2 \), and using Corollary 4.15, we have Theorem 4.11. This completes the proof of Theorem 4.11.

4.4 Analysis on \( \Gamma^\pm_3 \)

Our aim here is to show the following theorem for the operators defined in (4.5) with \( \sigma = 3 \).

Theorem 4.16. Let \( 1 \leq r \leq 2 \leq q \leq \infty \), \((\alpha', \alpha) \in \mathbb{N}_0^{N-1} \times \mathbb{N}_0^{N} \), and \( F = (f, d) \in L_r(\mathbb{R}_+^{N}) \times L_r(\mathbb{R}^{N-1}) \). Then there exist positive constants \( \delta_0, A_0, \) and \( C \) such that for any \( t \geq 1 \)
\[ \|\partial_t S_{0}^{d,3}(t; A_0)F, \nabla \nabla \|_{L_4(\mathbb{R}^N_{3})} \]
\[ + \|D_{-1} S_{0}^{d,3}(t; A_0)F, D_{0}^{a} \nabla \nabla (T_{0}^{d,3}(t; A_0)F, D_{0}^{a} \nabla \nabla (T_{0}^{d,3}(t; A_0)F))\|_{L_4(\mathbb{R}^N_{3})} \]
\[ \leq Ce^{-\delta_0 t} \|f\|_{L_r(\mathbb{R}^N_{3})}, \]
\[ \|\partial_t S_{0}^{d,3}(t; A_0)F, \nabla \|_{L_4(\mathbb{R}^N_{3})} \]
\[ + \|D_{-1} S_{0}^{d,3}(t; A_0)F, D_{0}^{a} \nabla \nabla (T_{0}^{d,3}(t; A_0)F)\|_{L_4(\mathbb{R}^N_{3})} \]
\[ \leq Ce^{-\delta_0 t} \|d\|_{L_r(\mathbb{R}^{N-1})}. \]

In order to show Theorem 4.16, we start with the following lemma.

Lemma 4.17. Let \( 1 \leq r \leq 2 \leq q \leq \infty \), and let \( f \in L_r(\mathbb{R}_+^{N}) \) and \( d \in L_r(\mathbb{R}^{N-1}) \). We use the operators defined in (4.3) with the forms:
\[ k_n(\xi', \lambda) = k_n(\xi', \lambda)/L(A,B), \]
\[ \ell_n(\xi', \lambda) = m_n(\xi', \lambda)/L(A,B). \]

(1) Let \( s \geq 0 \) and suppose that there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( \lambda \in \Gamma^\pm_3 \) and \( A \in (0, A_1) \)
\[ |k_n(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^2 A^{1+s} \quad (n = 1, 2, 4, 5, 6), \]
\[ |\kappa_n(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^2 A^s. \]

Then there exist positive constants \( \delta_0, \delta_1 \in (0, A_1), \) and \( C \) such that for any \( t \geq 1 \)
\[ \|K_{n}^{d,3}(t; A_0)f\|_{L_4(\mathbb{R}^N_{3})} \leq Ce^{-\delta_0 t} \|f\|_{L_r(\mathbb{R}^N_{3})} \quad (n = 1, \ldots, 6). \]

(2) Let \( s \geq 0 \) and suppose that there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( \lambda \in \Gamma^\pm_3 \) and \( A \in (0, A_1) \)
\[ |m_n(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^2 A^{1+s}, \]
\[ |\kappa_n(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^2 A^s. \]

Then there exist positive constants \( A_0 \in (0, A_1), \delta_0, \) and \( C \) such that for any \( t \geq 1 \)
\[ \|L_{n}^{d,3}(t; A_0)d\|_{L_4(\mathbb{R}^N_{3})} \leq Ce^{-\delta_0 t} \|d\|_{L_r(\mathbb{R}^{N-1})} \quad (n = 1, 2, 3). \]
Proof. We use the abbreviations: \( \| \cdot \|_2 = \| \cdot \|_{L^2(\mathbb{R}^N)} \), \( \tilde{f}(y_N) = \tilde{f}(\xi', y_N) \), and \( \tilde{t} = t + 1 \) for \( t > 0 \) in this proof, and consider only the estimates on \( \Gamma_3^+ \), because the estimates on \( \Gamma_3^- \) can be shown similarly.

(1) First, we show the inequality for \( K_1^{+; \beta}(t) \). Noting that \( \lambda = -\gamma_0 + \tilde{r}_0 + uc(\pi - \varepsilon_0) \) for \( u \in [0, \infty) \) on \( \Gamma_3^+ \), we have, by (4.6),

\[
[K_1^{+; \beta}(t) f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \int_0^\infty e^{-(\gamma_0 + \tilde{r}_0 + uc(\pi - \varepsilon_0))t} \right.
\]
\[
\times \varphi_0(\xi') \frac{K_1(\xi', \lambda)}{L(A, B)} e^{-A(x_N + y_N)} e^{i(x - \varepsilon_0) \cdot \tilde{f}(y_N)} \left( x' \right) dy_N.
\]

Since \( e^{-(\gamma_0/2)t} e^{-A^t} \leq C e^{-A^t} \) for any \( \lambda \in (0, A_0) \) by choosing some \( A_0 \in (0, A_1) \), we obtain by Lemma 2.4 (3), \( L_q-L_r \) estimates of the \((N - 1)\) dimensional heat kernel, and Parseval’s theorem

\[
||K_1^{+; \beta}(t; A_0) f(\cdot, x_N)||_{L_q(\mathbb{R}^{N-1})} \leq C \int \frac{1}{\pi (\frac{d}{4} + t)} \frac{1}{\lambda} e^{-u(\cos \varepsilon_0) t} \frac{1}{(A + \varepsilon_0)} e^{-A(x_N + y_N)} \frac{1}{\lambda} e^{-A(x_N + y_N)} e^{i(x - \varepsilon_0) \cdot \tilde{f}(y_N)} \left( x' \right) dy_N.
\]

for any \( t \geq 1 \) with some positive constant \( C \), where we note that \( |\lambda| \geq C\varepsilon_0 \) on \( \Gamma_3^+ \) and \( A \leq C \) on \( \supp \varphi_0 \).

We thus obtain the required inequality of \( K_1^{+; \beta}(t; A_0) \) by Lemma 4.6. Analogously, we can show the case of \( n = 2, 4, 5, 6 \) by using the fact that

\[
|e^{-Ba}| \leq C e^{-Ca}, \quad |M(a)| \leq C |\lambda|^{-1} e^{-Ca} \leq C e^{-Ca}
\]

for any \( a > 0 \) and \( \lambda \in \Gamma_3^+ \) with some positive constant \( C \) by Lemma 2.1 (1) and (2.2).

We finally show the inequality for \( K_3^{+; \beta}(t; A_0) \). By Hölder’s inequality and (4.26), \( \|f\| \leq C \|f\|_{L_p(\mathbb{R}^N)} \), we easily have for \( r' = r/(r - 1) \)

\[
K_3^{+; \beta}(t; A_0) f(x_N) \leq C e^{-\gamma_0/2 t} \int_0^\infty \int_0^\infty e^{-u(\cos \varepsilon_0) t} \frac{1}{\lambda} e^{-C|\lambda|^{1/2} e^{i(x - \varepsilon_0) \cdot \tilde{f}(y_N)}} e^{-Cv' r'^{1/2} dy_N} \left( x' \right) \frac{1}{d_{\lambda} u} ||f||_{L_p(\mathbb{R}^N)}
\]

Therefore, we see that

\[
K_3^{+; \beta}(t; A_0) f(x_N) \leq C e^{-\gamma_0/2 t} \int_0^\infty \int_0^\infty e^{-u(\cos \varepsilon_0) t} \frac{1}{\lambda} e^{-C|\lambda|^{1/2} e^{i(x - \varepsilon_0) \cdot \tilde{f}(y_N)}} e^{-Cv' r'^{1/2} dy_N} \left( x' \right) \frac{1}{d_{\lambda} u} ||f||_{L_p(\mathbb{R}^N)}
\]

for any \( t \geq 1 \) with some positive constant \( C \).

(2) Employing an argument similar to (1) and using (4.26) for \( L_3^{+; \beta}(t; A_0) \), we can prove (2), so that we may omit the detailed proof of (2). This completes the proof of Lemma 4.17.

We see that by Lemma 2.4 there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( A \in (0, A_1) \) and \( \lambda \in \Gamma_3^+ \) we have

\[
|\nabla_j \bar{B}(\xi', \lambda)| \leq C, \quad |\nabla_j \bar{B}(\xi', \lambda)| \leq CA, \quad |\nabla_j \bar{B}(\xi', \lambda)| \leq CA,
\]

\[
|\bar{B}(\xi', \lambda)| \leq CA, \quad |\bar{B}(\xi', \lambda)| \leq CA,
\]

for \( j, k = 1, \ldots, N \). Therefore, remembering (3.2)–(3.5), and (4.3) with \( \sigma = 3 \), and using Lemma 4.17 we have Theorem 4.16. This completes the proof of Theorem 4.16.
We finally consider the term \( \partial_t \mathcal{E}(T_0^p(t; A_0) F) \) given by

\[
\partial_t \mathcal{E}(T_0^p(t; A_0) F) = \mathcal{F}^{-1} \left[ \frac{1}{2 \pi i} \int_{\Gamma} e^{\lambda \varphi_0(\xi')} \frac{\lambda D(A, B)}{(B + A) L(A, B)} d\lambda e^{-\lambda x N} \tilde{d}(\xi') \right] (x')
\]

\[
= \mathcal{F}^{-1} \left[ \frac{1}{2 \pi i} \int_{\Gamma} e^{\lambda \varphi_0(\xi')} e^{-\lambda x N} \tilde{d}(\xi') \right] (x')
\]

\[
- \mathcal{F}^{-1} \left[ \frac{1}{2 \pi i} \int_{\Gamma} e^{\lambda \varphi_0(\xi')} t \frac{d\lambda}{L(A, B)} e^{-\lambda x N} \tilde{d}(\xi') \right] (x'),
\]

where we have used the relations: \( D(A, B) = (B - A)^{-1} \{ L(A, B) - A(c_g + c_\sigma A^2) \} \) and \( B^2 - A^2 = \lambda \). Note that the first term vanishes by Cauchy’s integral theorem, so that it suffices to consider the second term only. Set

\[
I_0^\pm(t; A_0) = - \mathcal{F}^{-1} \left[ \frac{1}{2 \pi i} \int_{\Gamma} e^{\lambda \varphi_0(\xi')} \frac{A(c_g + c_\sigma A^2)}{L(A, B)} d\lambda e^{-\lambda x N} \tilde{d}(\xi') \right] (x'), \quad (\sigma = 0, 1, 2, 3).
\]

Since by Lemma 4.12 there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( \lambda \in \Gamma^\pm \) and \( A \in (0, A_1) \)

\[
|A(c_g + c_\sigma A^2)/L(A, B)| \leq C(A^{1/4} u + \sqrt{u} A^{1/4})^{-4} A,
\]

by Lemma 4.13, we have for \( t > 0, \alpha \in \mathbb{N}^N \) with \( |\alpha| \neq 0 \), and \( 1 \leq r \leq 2 \leq q \leq \infty \)

\[
\|D_\alpha^r I_0^\pm(t; A_0)\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N|\alpha|}{4} + \frac{3}{4} - \frac{3}{4} - \frac{1}{4}} ||d||_{L_r(\mathbb{R}^N)} \]

with some positive constant \( C \). If \( (q, r) \neq (2, 2) \), then we also have

\[
\|I_0^\pm(t; A_0)\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N|\alpha|}{4} + \frac{3}{4} - \frac{3}{4} - \frac{1}{4}} ||d||_{L_r(\mathbb{R}^N)}.
\]

In addition, by Lemma 4.15 and 4.17, we have

\[
\|D_\alpha^r I_0^\pm(t; A_0)\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N|\alpha|}{4} + \frac{3}{4} - \frac{3}{4} - \frac{1}{4}} ||d||_{L_r(\mathbb{R}^N)} \quad (n = 0, 1),
\]

\[
\|D_\alpha^r I_0^\pm(t; A_0)\|_{L_q(\mathbb{R}^N)} \leq C e^{-\delta_0 t} ||d||_{L_r(\mathbb{R}^N)}
\]

for any \( t \geq 1, \alpha \in \mathbb{N}^N \), and \( 1 \leq r \leq 2 \leq q \leq \infty \) with some positive constant \( C \). Thus, we have

\[
\|D_\alpha^r \partial_t \mathcal{E}(T_0^p(t; A_0) F)\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N|\alpha|}{4} + \frac{3}{4} - \frac{3}{4} - \frac{1}{4}} ||d||_{L_r(\mathbb{R}^N)} \quad (1 \leq r \leq 2 \leq q \leq \infty, \ |\alpha| \neq 0),
\]

\[
\|\partial_t \mathcal{E}(T_0^p(t; A_0) F)\|_{L_q(\mathbb{R}^N)} \leq C(t + 1)^{-\frac{N|\alpha|}{4} + \frac{3}{4} - \frac{3}{4} - \frac{1}{4}} ||d||_{L_r(\mathbb{R}^N)} \quad (1 \leq r \leq 2 \leq q \leq \infty \text{ and } (q, r) \neq (2, 2))
\]

for any \( t \geq 1 \) with some positive constant \( C \), which, combined with Theorem 4.2, 4.7, 4.11 and 4.16, completes the proof of (1.5) in Theorem 1.1 (2), because

\[
S_0(t) F = \sum_{g \in (f, d)} \sum_{\sigma = 0}^3 S_0^g(\sigma; t; A_0) F, \quad \Pi_0(t) F = \sum_{g \in (f, d)} \sum_{\sigma = 0}^3 \Pi_0^g(\sigma; t; A_0) F,
\]

\[
T_0(t) F = \sum_{g \in (f, d)} \sum_{\sigma = 0}^3 T_0^g(\sigma; t; A_0) F.
\]

5 Analysis of high frequency part

In this section, we show the estimate (1.7) in Theorem 1.1 (2). If we consider the Lopatinskii determinant \( L(A, B) \) defined by (2.1) as a polynomial with respect to \( B \), it has the following four roots:

\[
B_j = a_j A + \frac{c_\sigma}{4(1 - a_j - a_j^2)} + \frac{(1 + 3a_j^2)c_\sigma^2}{32(1 - a_j - a_j^2)^3} A + O \left( \frac{1}{A^2} \right) \quad \text{as } A \to \infty,
\]

(5.1)
where \(a_j (j = 1, \ldots, 4)\) are the solutions to the equation: \(x^4 + 2x^2 - 4x + 1 = 0\). We have the following informations about \(a_j\): \(a_1\) and \(a_2\) are real numbers such that \(a_1 = 1\) and \(0 < a_2 < 1/2\), and \(a_3\) and \(a_4\) are complex numbers satisfying \(\text{Re} a_j < 0\) for \(j = 3, 4\). We define \(\lambda_j\) by \(\lambda_j = B_j^2 - A^2\) for \(j = 1, 2\), and then

\[
\lambda_1 = -\frac{c_\sigma A}{2} - \frac{3}{16} \varepsilon^r + O\left(\frac{1}{A}\right), \quad \lambda_2 = -(1 - a_2^2)A^2 + \frac{a_2 c_\sigma}{2(1 - a_2 - a_2^2)} A + O(1) \quad \text{as} \quad A \to \infty. \tag{5.2}
\]

Let \(L_0 = \{\lambda \in \mathbb{C} \mid \text{Re} (A, B) = 0, \text{Re} B \geq 0, A \in \text{supp} \varphi_\infty\}\), where \(\varphi_\infty\) is defined in (3.8), and then we see, by the expansion formulas (1.2), (5.2), and Lemma 3.2, that there exist positive numbers \(0 < \varepsilon < \pi/2\) and \(\lambda_\infty > 0\) such that \(L_0 \subset \Sigma_{\varepsilon, \infty} \cap \{z \in \mathbb{C} \mid \text{Re} z < -\lambda_\infty\}\). Set \(\gamma_\infty = \min\{\lambda_\infty, 4^{-1} \times (A_0/6)^2\}\) for \(A_0\) defined in (3.6), and set, for (5.7) and \(g \in \{f, d\}\),

\[
S^g_\infty(t) = S^g_\infty(t; A_0), \quad \Pi^g_\infty(t) = \Pi^g_\infty(t; A_0), \quad T^g_\infty(t) = T^g_\infty(t; A_0).
\]

In order to estimate each term above, we use the integral paths:

\[
\Gamma^\pm = \{\lambda \in \mathbb{C} \mid \lambda = -\gamma \pm iu, \ u : 0 \to \gamma_\infty\},
\]

\[
\Gamma^\pm = \{\lambda \in \mathbb{C} \mid \lambda = -\gamma \pm i\gamma_\infty + u e^{\pm i(x - \varepsilon)}, \ u : 0 \to \gamma_\infty\},
\]

where \(\gamma_\infty = (\tan \varepsilon_0)(\lambda_0(\varepsilon_0) + \gamma_\infty)\) and \(\lambda_0(\varepsilon_0)\) is the same constant as in (3.8) with \(\varepsilon = \varepsilon_0\). Furthermore, for \(g \in \{f, d\}\), setting \(v^g_\infty(x, \lambda) = (v^g_\infty(x, \lambda), \ldots, v^g_{N, \infty}(x, \lambda))^T\) and

\[
v^g_\infty(x, \lambda) = F^{-1}_\infty[\varphi_\infty(\xi') \xi_\infty^g(\xi', x, \lambda)](x') \quad (j = 1, \ldots, N),
\]

\[
\pi^g_\infty(x, \lambda) = F^{-1}_\infty[\varphi_\infty(\xi') \xi_\infty^g(\xi', x, \lambda)](x'),
\]

\[
h^g_{A, \infty}(x, \lambda) = F^{-1}_\infty[\varphi_\infty(\xi') e^{-A x_\infty} \xi_\infty^g(\xi', \lambda)](x')
\]

by (5.3)-(5.6), we have, by Cauchy’s integral theorem, the following decompositions:

\[
S^g_\infty(t) F = \sum_{\sigma=4}^5 S^{g,\sigma}_\infty(t) F, \quad \Pi^g_\infty(t) F = \sum_{\sigma=4}^5 \Pi^{g,\sigma}_\infty(t) F, \quad E(T^g_\infty(t) F) = \sum_{\sigma=4}^5 E(T^{g,\sigma}_\infty(t) F),
\]

where the right-hand sides are given by

\[
S^{g,\sigma}_\infty(t) F = \frac{1}{2\pi i} \int_{\Gamma^\pm \cup \Gamma^\pm_x} e^{\lambda} v^{g,\sigma}_\infty(x, \lambda) d\lambda, \quad \Pi^{g,\sigma}_\infty(t) F = \frac{1}{2\pi i} \int_{\Gamma^\pm \cup \Gamma^\pm_y} e^{\lambda} \pi^{g,\sigma}_\infty(x, \lambda) d\lambda,
\]

\[
E(T^{g,\sigma}_\infty(t) F) = \frac{1}{2\pi i} \int_{\Gamma^\pm \cup \Gamma^\pm_y} e^{\lambda} h^{g,\sigma}_{A, \infty}(x, \lambda) d\lambda.
\tag{5.3}
\]

By the relation \(1 = B^2 / B^2 = (\lambda + A^2) / B^2\), we write \(v^g_\infty, \pi^g_\infty, \) and \(h^g_{A, \infty}\) as follows: For \(j = 1, \ldots, N\), \(\hat{f}_j(y_N) = \hat{f}_j(\xi', y_N), \) and \(\varphi_\infty = \varphi_\infty(\xi'),\)

\[
v^g_{j, \infty}(x, \lambda) = \sum_{k=1}^N \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty \frac{V^R_{jk}(\xi', \lambda)(c_j + c_\sigma A^2)}{AL(A, B)} A e^{-B(x_N + y_N) \hat{f}_k(y_N)} \right] (x') dy_N
\]

\[
+ \sum_{k=1}^N \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty \frac{\lambda |\lambda|^{-1/2} V_{jk}(\xi', \lambda)(c_j + c_\sigma A^2)}{AB^2 L(A, B)} A \hat{f}_k(y_N) \right] (x') dy_N
\]

\[
+ \sum_{k=1}^N \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty \frac{V^R_{j}^{MB}(\xi', \lambda)(c_j + c_\sigma A^2)}{B^2 L(A, B)} e^{-B x_N} M(y_N) \hat{f}_k(y_N) \right] (x') dy_N
\]

\[
+ \sum_{k=1}^N \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty \frac{\lambda |\lambda|^{-1/2} V_{jk}^{MB}(\xi', \lambda)(c_j + c_\sigma A^2)}{AB^2 L(A, B)} A \hat{f}_k(y_N) \right] (x') dy_N
\]

\[
+ \sum_{k=1}^N \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty \frac{V_{j}^{MB}(\xi', \lambda)(c_j + c_\sigma A^2)}{B^2 L(A, B)} M(x_N) e^{-B y_N} \hat{f}_k(y_N) \right] (x') dy_N
\]

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Moreover, using the relations:

\[
\frac{1}{A} = \int_0^\infty k \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n,
\]

\[
\sum_{k=1}^N \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n \right] \, dx_n
\]

\[
\sum_{k=1}^N \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n \right] \, dx_n,
\]

\[
\pi^d_{N}(x, \lambda) = \sum_{k=1}^N \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n \right] \, dx_n
\]

\[
\pi^d_{N}(x, \lambda) = \sum_{k=1}^N \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n \right] \, dx_n
\]

\[
\sum_{k=1}^N \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} f_k(y_n) \, dy_n \right] \, dx_n.
\]

Furthermore, using the relations:

\[
e^{-B(x_n+y_n)} g(0) = \int_0^\infty B e^{-B(x_n+y_n)} \tilde{g}(y_n) \, dy_n - \int_0^\infty e^{-B(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n,
\]

\[
\mathcal{M}(x_n) \tilde{g}(0) = \int_0^\infty \left( e^{-B(x_n+y_n)} + A \mathcal{M}(x_n + y_n) \right) \tilde{g}(y_n) \, dy_n
\]

\[
+ \int_0^\infty \mathcal{M}(x_n + y_n) D_{\nu_n} \tilde{g}(y_n) \, dy_n,
\]

where \( \tilde{g}(y_n) = \tilde{g}(\zeta', y_N) \), and using the identity: \( 1 = A^2 / A^2 = - \sum_{k=1}^{N-1} (i \xi_k)^2 / A^2 \), we write \( v^d_{N, \infty} \), \( \pi^d_{N, \infty} \), and \( h^d_{A, \infty} \) as follows: For \( j = 1, \ldots, N - 1 \),

\[
v^d_{N, \infty}(x, \lambda) = - \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n
\]

\[
+ \sum_{k=1}^{N-1} \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n
\]

\[
- \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n.
\]

\[
v^d_{N, \infty}(x, \lambda) = - \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n
\]

\[
+ \sum_{k=1}^{N-1} \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n
\]

\[
+ \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n
\]

\[
- \sum_{k=1}^{N-1} \int_0^\infty F_{\xi_k}^{-1} \left[ \varphi \int_0^\infty \rho_k(A) \mathrm{e}^{-A(x_n+y_n)} D_{\nu_n} \tilde{g}(y_n) \, dy_n \right] \, dx_n.
\]
Remark 5.1. We extend $d \in W^{2-1/p}_p(R^{N-1})$ to a function $\tilde{d}$, which is defined on $R^N$ and satisfies $\|\tilde{d}\|_{W^{2-1/p}_p(R^{N-1})} \leq C\|d\|_{W^{2-1/p}(R^{N-1})}$ for a positive constant $C$ independent of $d$ and $\tilde{d}$. For simplicity, such a $\tilde{d}$ is denoted by $d$ again in the present section.

To estimates all the terms given in (5.4) and (5.6), we introduce the following operators:

\[
\begin{align*}
[K_1(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_1(\xi',\lambda)Ae^{-A(x_N+y_N)} \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_2(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_2(\xi',\lambda)A^2e^{-Ax_N}M(y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_3(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_3(\xi',\lambda)Ae^{-B(x_N+y_N)} \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_4(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_4(\xi',\lambda)A^2e^{-Bx_N}M(y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_5(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_5(\xi',\lambda)A|\xi|^{\alpha}e^{-Bx_N}M(y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_6(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_6(\xi',\lambda)A^2M(x_N)e^{-Bx_N} \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_7(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_7(\xi',\lambda)A|\xi|^{\alpha}M(x_N)e^{-Bx_N} \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_8(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_8(\xi',\lambda)A^2M(x_N+y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_9(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_9(\xi',\lambda)A^3M(x_N)M(y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N, \\
[K_{10}(\lambda)f](x) &= \int_0^\infty F^{-1}_{\xi'} \left[ \varphi_\infty(\xi')k_{10}(\xi',\lambda)A\lambda M(x_N)M(y_N) \hat{f}(\xi',y_N) \right](x')\,dy_N.
\end{align*}
\]

We know the following proposition (cf. [17] Lemma 5.4).

Proposition 5.2. Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$, and $f \in L_\rho(R^N)$, and let $\Lambda_{\varepsilon}$ be a subset of $\Sigma_{\varepsilon}$. Suppose that for every $\alpha' \in N_0^{N-1}$ there exists a positive constant $C = C(\alpha')$ such that for any $\lambda \in \Lambda_{\varepsilon}$ and $\xi' \in R^{N-1} \setminus \{0\}$

\[
|D_{\xi'}^{\alpha'} \{ \varphi_\infty(\xi')k_n(\xi',\lambda) \}| \leq CA^{-|\alpha'|} \quad (n = 1, \ldots, 10).
\]

Then there exists a positive constant $C$ such that for any $\lambda \in \Lambda_{\varepsilon}$

\[
\|K_n(\lambda)f\|_{L_\rho(R^N)} \leq C\|f\|_{L_\rho(R^N)} \quad (n = 1, \ldots, 10).
\]

5.1 Analysis on $\Gamma_4^+$

We first show the following lemma concerning estimates of the symbols defined in (2.11).

Lemma 5.3. (1) There exists a positive constant $A_{\infty}$ such that for any $A \geq A_{\infty}$ and $\lambda \in \Gamma_4^+$

\[
2^{-1}A \leq \text{Re}B \leq |B| \leq 2A, \quad |D(A,B)| \geq A^3, \quad |L(A,B)| \geq (c_\sigma/16)(8^{-1}A)^3.
\]

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(2) There exist positive constants $C_1, C_2,$ and $C$ such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^+$,

$$C_1 A \leq \Re B \leq |B| \leq C_2 A, \quad |D(A, B)| \geq CA^3, \quad |L(A, B)| \geq CA^3,$$

where $A_\infty$ and $A_0$ are the same constants as in (1) and in (3.6), respectively.

(3) Let $\alpha' \in N_0^{N-1}, s \in \mathbb{R},$ and $a > 0$. Then there exist constants $C > 0$ and $b_\infty \geq 1$, independent of $a$, such that for any $\lambda \in \Gamma_4^+$ and $A \geq A_0/6$ with $A_0$ defined as in (3.6),

$$|D_0^{\gamma} B^4| \leq CA^{3-\alpha}, \quad |D_0^{\gamma} D(A, B^4)| \leq CA^{3-\alpha}, \quad |D_0^{\gamma} e^{-Ba}| \leq CA^{-\alpha}e^{-b\lambda}, \quad |D_0^{\gamma} L(A, B)^{-1} e^{-\gamma}| \leq CA^{-\alpha} e^{-b\lambda}.$$

Proof. (1) We first consider the estimates of $B$. For $\lambda \in \Gamma_4^+$, set $\sigma = \lambda + A^2 = -\gamma_\infty + A^2 \pm iu (u \in [0, \tilde{\gamma}_\infty])$ and $\theta = \arg \sigma$. Then we have

$$\Re B = |\sigma|^2 \cos \theta = \frac{|\sigma|^2}{\sqrt{2}} (1 + \cos \theta)^2 = \frac{1}{\sqrt{2}}(|\sigma| + A^2 - \gamma_\infty)^{\frac{1}{2}},$$

so that for any $A \geq A_\infty$

$$\Re B \geq \frac{1}{\sqrt{2}} \left(2A^2 - 2\gamma_\infty - \tilde{\gamma}_\infty \right)^{\frac{1}{2}} \geq \frac{A}{\sqrt{2}},$$

provided that $A_\infty$ satisfies $A_\infty^2 \geq 2\gamma_\infty + \tilde{\gamma}_\infty$. On the other hand, it is clear that $|B| \leq 2A$.

Next, we show the inequality for $D(A, B)$. Since

$$D(A, B)^2 = B(B^2 + 3A^2) + A(B^2 - A^2) = B(\lambda + 4A^2) + \lambda A = 4A^2B + (B + A)(-\gamma_\infty \pm iu),$$

we see, by the inequality of $B$ obtained above, that

$$|D(A, B)| \geq 4A^2|B| - |B + A| - \gamma_\infty \pm iu \geq 4A^2(\Re B) - (|B + A|)(\gamma_\infty + \tilde{\gamma}_\infty) \geq 2A^3 - 3(\gamma_\infty + \tilde{\gamma}_\infty) A \geq A^3,$$

for any $A \geq A_\infty$, provided that $A_\infty$ satisfies $A_\infty^2 \geq 3(\gamma_\infty + \tilde{\gamma}_\infty)$.

Finally, we show the inequality for $L(A, B)$. Since

$$B_1^2 - B^2 = -\frac{c_\sigma}{A} - \frac{3}{16} c_\sigma^2 (-\gamma_\infty \pm iu) + O\left(\frac{1}{A}\right) \quad \text{as} \quad A \to \infty,$$

there exist positive constants $A_\infty$ and $C$ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^+$ we have $|B_1^2 - B^2| \geq (c_\sigma/4) A$, which, combined with the inequality of $B$ obtained above and (5.11), furnishes that

$$|B_1 - B| \geq \frac{|B_1^2 - B^2|}{|B_1 + B|} \geq \frac{(c_\sigma/4) A}{4A} \geq \frac{c_\sigma}{16} (A \geq A_\infty \text{ and } \lambda \in \Gamma_4^+).$$

On the other hand, we have

$$B_2^2 - B^2 = - (1 - a_2^2) A^2 + O(A) \quad \text{as} \quad A \to \infty,$$

so that there exists a positive number $A_\infty$ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^+$ we have $|B_2^2 - B^2| \geq (A^2/2)$, from which it follows that

$$|B_2 - B| = \frac{|B_2^2 - B^2|}{|B_2 + B|} \geq \frac{(A^2/2)}{4A} = \frac{A}{8}.$$

Since $|B - B_2| \leq |B - B_j|$ $(j = 3, 4)$, we thus obtain

$$|L(A, B)| \geq (c_\sigma/16)(8^{-1} A^3) \quad (A \geq A_\infty \text{ and } \lambda \in \Gamma_4^+).$$

(2) It is sufficient to show the existence of positive constants $C_1, C_2,$ and $C$ such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^+$

$$C_1 \leq \Re B \leq |B| \leq C_2, \quad |D(A, B)| \geq C, \quad |L(A, B)| \geq C.$$
It is obvious that the inequalities for $B$ holds, so that we here consider $D(A, B)$ and $L(A, B)$ only.

First, we show the inequality for $D(A, B)$. Set

$$\tilde{A} = \frac{A}{2}, \quad \tilde{\lambda} = -\gamma_{\infty} + 3\tilde{A}^2 \pm iu \quad \text{for} \ u \in [0, \tilde{\gamma}_{\infty}],$$

and note that $B = (\tilde{\lambda} + \tilde{A}^2)^{1/2}$. We then see that

$$\{B/\tilde{A} \in \mathcal{C} \mid \lambda \in \Gamma_{4}^\pm \text{ and } A \in [A_0/6, 2A_\infty] \} \subset \{z \in \mathcal{C} \mid |1 \leq \text{Re} z\}.$$  

In fact, setting $\sigma = 1 - (\gamma_{\infty}/A^2) \pm i(u/A^2)$ and $\theta = \text{arg } \sigma$, we have

$$\text{Re} \frac{B}{A} = 2|\sigma|^{1/2} \cos \frac{\theta}{2} = 2|\sigma|^{1/2} \left(\frac{1 + \cos \theta}{2}\right)^{1/2} = \sqrt{2}|\sigma| + \text{Re} \sigma)^{1/2} \geq 2(\text{Re} \sigma)^{1/2}$$

$$= 2 \left(1 - \frac{\gamma_{\infty}}{A^2}\right)^{1/2} \geq 2 \left(1 - \frac{4^{-1} \times (A_0/6)^2}{(A_0/6)^2}\right)^{1/2} = \sqrt{3},$$

which, combined with Lemma 4.8 and the formula:

$$D(A, B) = B^3 + 2\tilde{A}B^2 + 12\tilde{A}^2B - 8\tilde{A}^3 = \tilde{A}^3 \left\{ \left(\frac{B}{A}\right)^3 + 2 \left(\frac{B}{A}\right)^2 + 12 \left(\frac{B}{A}\right) - 8 \right\},$$

furnishes the existence of a positive constant $C$ such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_{4}^\pm$ we have $|D(A, B)| \geq C$. The inequality for $L(A, B)$ follows clearly from the definition of the integral path $\Gamma_{4}^\pm$.

(3) We see, by Lemma 5.3 (1) and (2), that there exist positive constants $C_1, C_2$, and $C$ such that for any $\lambda \in \Gamma_{4}^\pm$ and $A \geq A_0/6$

$$C_1A \leq \text{Re} B \leq |B| \leq C_2A, \quad |D(A, B)| \geq CA^3, \quad |L(A, B)| \geq CA^3.$$

(5.8)

We thus obtain the required inequalities by using Leibniz’s rule and Bell’s formula, and noting

$$|D_{\xi}^\alpha D(A, B)| = |D_{\xi}^\alpha (B^3 + AB^2 + 3A^2B - A^3)| \leq CA^3,$$

$$|D_{\xi}^\alpha L(A, B)| = \left|D_{\xi}^\alpha \left(\frac{\lambda}{B + A}D(A, B) + A(e_\theta + c_A A^2)\right)\right| \leq CA^3$$

for any $\alpha' \in N_0^{N-1}$, $\lambda \in \Gamma_{4}^\pm$, and $A \geq A_0/6$ by (5.8) (cf. [17] Lemma 5.2, Lemma 5.3, Lemma 7.2). □

Now, we have a multiplier theorems on $\Gamma_{4}^\pm$.

**Lemma 5.4.** Let $1 < p < \infty$, $n = 1, \ldots, 10$, and $f \in L_p(\mathbb{R}^N)$. We use the symbols defined in [5.7] and assume that for any $\alpha' \in N_0^{N-1}$ there exists a positive constant $C = C(\alpha')$ such that $|D_{\xi}^\alpha k_n(\xi, \lambda)| \leq CA^{-|\alpha|}$ for any $\lambda \in \Gamma_{4}^\pm$ and $A \geq A_0/6$ with $A_0$ defined as in [5.6]. Then there exists a positive constant $C$ such that for any $\lambda \in \Gamma_{4}^\pm$

$$\|K_n(\lambda)f\|_{L_p(\mathbb{R}^N)} \leq C\|f\|_{L_p(\mathbb{R}^N)} \quad (n = 1, \ldots, 10).$$

**Proof.** Employing the similar argument to the proof of [17] Lemma 5.4 and using Lemma 5.3, we can prove the lemma. □

By (5.4), (5.5), (5.6), Lemma 5.3, and Lemma 5.4, we have the following lemma.

**Lemma 5.5.** Let $1 < p < \infty$, $f \in L_p(\mathbb{R}^N)^N$, and $d \in W^2_p(\mathbb{R}^N)$. Then there exists a positive constant $C$ such that for any $\lambda \in \Gamma_{4}^\pm$

$$\|f_{A,\infty}^p\|_{W^2_p(\mathbb{R}^N)} + \|f_{A,\infty}^d\|_{W^2_p(\mathbb{R}^N)} \leq C\|f\|_{L_p(\mathbb{R}^N)};$$

$$\|f_{A,\infty}^d\|_{W^2_p(\mathbb{R}^N)} + \|f_{A,\infty}^d\|_{W^2_p(\mathbb{R}^N)} \leq C\|d\|_{W^2_p(\mathbb{R}^N)}.$$
Applying Lemma 5.5 to the terms in (5.3), we have
\[
\begin{align*}
\| & (\partial_t S_{t_\infty}^{d,4}(t) F, \nabla \Pi_{t_\infty}^{d,4}(t) F)\|_{L_p(R_\infty^N)} \\
& + \| (S_{t_\infty}^{d,4}(t) F, \partial_t \mathcal{E}(T_{t_\infty}^{d,4}(t) F), \nabla \mathcal{E}(T_{t_\infty}^{d,4}(t) F))\|_{W^2_p(R_\infty^N)} \\
& \leq C e^{-\gamma_0 t} \| f \|_{L_p(R_\infty^N)},
\end{align*}
\]
for any \( t > 0 \) with some positive constant \( C \).

5.2 Analysis on \( \Gamma_5^\pm \)

By Lemma 5.4, 5.1, and Proposition 5.2, we easily see that the following lemma holds.

**Lemma 5.6.** Let \( 1 < p < \infty, f \in L_p(R_\infty^N)^N, \) and \( d \in W^2_p(R_\infty^N) \). Then there exists a positive constant \( C \) such that for any \( \lambda \in \Gamma_5^\pm \)
\[
\begin{align*}
\| & (\lambda^{1/2} S_{t_\infty}^{d,4}(t) F, \lambda^{1/2} \nabla \Pi_{t_\infty}^{d,4}(t) F)\|_{L_p(R_\infty^N)} \\
& \leq C \| f \|_{L_p(R_\infty^N)},
\end{align*}
\]
\[
\begin{align*}
\| & (\lambda^{1/2} S_{t_\infty}^{d,4}(t) F, \lambda^{1/2} \nabla \Pi_{t_\infty}^{d,4}(t) F)\|_{L_p(R_\infty^N)} \\
& \leq C \| f \|_{L_p(R_\infty^N)},
\end{align*}
\]
\[
\begin{align*}
\| & (\lambda^{1/2} S_{t_\infty}^{d,4}(t) F, \nabla \Pi_{t_\infty}^{d,4}(t) F)\|_{L_p(R_\infty^N)} \\
& \leq C \| f \|_{L_p(R_\infty^N)},
\end{align*}
\]
\[
\begin{align*}
\| & h_{A,\infty}^{d,4}(t) + h_{A,\infty}^{d,4}(t)\|_{L_p(R_\infty^N)} \\
& \leq C \| f \|_{L_p(R_\infty^N)}.
\end{align*}
\]
Applying Lemma 5.5 to the terms in (5.3), we have for \( t \geq 1 \)
\[
\begin{align*}
\| & (\partial_t S_{t_\infty}^{d,5}(t) F, \nabla \Pi_{t_\infty}^{d,5}(t) F)\|_{L_p(R_\infty^N)} \\
& + \| (S_{t_\infty}^{d,5}(t) F, \partial_t \mathcal{E}(T_{t_\infty}^{d,5}(t) F), \nabla \mathcal{E}(T_{t_\infty}^{d,5}(t) F))\|_{W^2_p(R_\infty^N)} \\
& \leq C e^{-\gamma_0 t} \| f \|_{L_p(R_\infty^N)},
\end{align*}
\]
with some positive constant \( C \).

Summing up (5.9) and (5.10), we have obtained the estimate (1.7) in Theorem 1.1 (2), since
\[
S_{t_\infty}(t) F = \sum_{g \in \{f,d\}} \sum_{\sigma=4}^5 S_{t_\infty}^{g,\sigma}(t) F, \quad \Pi_{t_\infty}(t) F = \sum_{g \in \{f,d\}} \sum_{\sigma=4}^5 \Pi_{t_\infty}^{g,\sigma}(t) F,
\]
\[
T_{t_\infty}(t) F = \sum_{g \in \{f,d\}} \sum_{\sigma=4}^5 T_{t_\infty}^{g,\sigma}(t) F.
\]

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