DEFORMATION CLASSIFICATION OF TYPICAL CONFIGURATIONS OF 7 POINTS IN THE REAL PROJECTIVE PLANE

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Abstract. A configuration of 7 points in $\mathbb{R}P^2$ is called typical if it has no collinear triples and no conic sextuples of points. We show that there exist 14 deformation classes of such configurations. This yields classification of real Aronhold sets.

“This is one of the ways in which the magical number seven has persecuted me.”

George A. Miller, The magical number seven, plus or minus two: some limits of our capacity for processing information

1. Introduction

1.1. Simple configurations of $n \leq 7$ points. By a simple $n$-configuration we mean a set of $n$ points in $\mathbb{R}P^2$ in which no triple of points is collinear. The dual object is a simple $n$-arrangement, that is a set of $n$ real lines containing no concurrent triples. The topological description that characterizes possible mutual position of the partition polygons of simple $n$-arrangements for $n \leq 7$ was given in [C] and [W]. After N. Mnëv in the beginning of 1980s constructed examples of topologically equivalent simple configurations which cannot be connected by a deformation (his initial example with $n \geq 19$ was a bit later improved by P. Suvorov [S]), S. Finashin has shown [F1] that for $n \leq 7$ the deformation classification still coincides with the topological one (see also [F2]).

The fact that real simple 5-configurations form a single deformation (i.e., path connected) component, denoted $LC^5$, trivially follows from existence of a non-singular conic containing all the points of such a configuration. For $n = 6$ there are

![Adjacency graphs $\Gamma_P$ of 5- and 6-configurations (cyclic, bicomponent, tricomponent and icosahedral)](image)

Figure 1. Adjacency graphs $\Gamma_P$ of 5- and 6-configurations (cyclic, bicomponent, tricomponent and icosahedral)

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On this Figure, we sketched configurations $P$ and joined pairs of points, $p, q \in P$, by an edge (a line segment) if and only if this edge is not crossed by any of the lines connecting pairs of the remaining $n - 2$ points of $P$. The graph, $\Gamma_P$, that we obtain for a given configuration $P$ is called the adjacency graph of $P$. For $n = 6$, the number of its connected components, 1, 2, 3, or 6, characterizes $P$ up to deformation. The deformation classes of 6-configurations with $i$ components are denoted $LC^6_i$, $i = 1, 2, 3, 6$, and the configurations of these four classes are called respectively cyclic, bicomponent, tricomponent, and icosahedral 6-configurations.

Given a simple 7-configuration $P$, we label a point $p \in P$ with a number $\delta = \delta(p) \in \{1, 2, 3, 6\}$ if $P \setminus \{p\} \in LC^6_\delta$. Count of the labels gives a quadruple $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_6)$, where $\sigma_k \geq 0$ is the number of points $p \in P$ with $\delta(p) = k$. We call $\sigma = \sigma(P)$ the derivative code of $P$. There exist 11 deformation classes of simple 7-configurations that are shown on Figure 2, together with their adjacency graphs and labels $\delta(p)$. It is trivial to notice that if $p, q \in P$ are adjacent vertices in graph $\Gamma_P$, then $\delta(p) = \delta(q)$, so, on Figure 2 we label whole components of $\Gamma_P$ rather than its vertices. The derivative codes happen to distinguish the deformation classes,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Deformation_classes_of_simple_7-configurations.png}
\caption{Deformation classes of simple 7-configurations}
\end{figure}

and we denote by $LC^7_\sigma$ the class formed by simple 7-configurations $P$ with the derivative code $\sigma$.

1.2. Typical configurations. Our interest to the real Aronhold sets of bitangents to quartics leaded to a refinement of the notion of simple configuration. Namely, an $n$-configuration is called typical, if it is simple and in addition does not contain
coconic sextuples of points. A deformation of n-configuration $\mathcal{P}_0$ is a continuous family $\mathcal{P}_t$, $t \in [0, 1]$, formed by $n$-configurations. We call it $L$-deformation if $\mathcal{P}_t$ are simple configurations, and $Q$-deformation if $\mathcal{P}_t$ are typical ones.

It is not difficult to observe (see Section 2) that for 6-configurations the two classifications coincide: typical 6-configurations can be connected by an L-deformation if and only if they can be connected by a Q-deformation. However, for $n > 6$, one L-deformation class may contain several Q-deformation classes, and our main goal is to find their number in the case of $n = 7$, for each of the 11 L-deformation classes shown on Figure 2.

1.2.1. **Theorem.** Typical 7-configurations split into 14 Q-deformation classes. Among these classes, two are contained in the class $LC_{(3,4,0,0)}$, three in the class $LC_{(2,2,3,0)}$, and each of the remaining 9 L-deformation classes of simple 7-configurations contains just one Q-deformation class.

1.3. **Structure of the paper.** In Section 2, we recall the scheme of L-deformation classification from [F1] and give Q-deformation classification of typical 6-configurations treating in turn also three trivial cases of 7-configurations. Sections 3–5 are devoted to Q-deformation classification for the three types of 7-configurations: heptagonal, hexagonal, and pentagonal. In the last Section, we discuss some applications including a description of the 14 real Aronhold sets (Figure 18).

1.4. **Acknowledgements.** This paper is essentially based on [Z], which was partially motivated by our attempts to understand and develop the results of [T1] concerning real nodal cubics passing through the points of heptagonal configurations. Recently, work [T1] had a continuation in [T2] and [T3], where in particular, the 14 types of typical 7-configurations were analyzed, although the classification there is topological and, so, is weaker than ours.

2. **Preliminaries**

2.1. **The monodromy group of a configuration.** By the $L$-deformation monodromy group of a simple $n$-configuration $\mathcal{P}$ we mean the subgroup, $Aut_L(\mathcal{P})$, of the permutation group $S(\mathcal{P})$ realized by L-deformations, that is the image in $S(\mathcal{P})$ of the fundamental group of the L-deformation component of $\mathcal{P}$ (using some fixed numeration of points of $\mathcal{P}$, we can and will identify $S(\mathcal{P})$ with the symmetric group $S_n$). For a typical $n$-configuration $\mathcal{P}$, we similarly define the $Q$-deformation monodromy group $Aut_Q(\mathcal{P}) \subset S(\mathcal{P}) \cong S_n$ formed by the permutations realized by Q-deformations.

In the case $n = 4$, any permutation can be realized by a deformation (and in fact, by a projective transformation), so we have $Aut_L(\mathcal{P}) = Aut_Q(\mathcal{P}) = S_4$. For $n = 5$, we obtain the dihedral group $Aut_L(\mathcal{P}) = Aut_Q(\mathcal{P}) = D_5$ associated to the pentagon $\Gamma_{\mathcal{P}}$ (as it was noted the adjacency graph is an L-deformation invariant).

More generally, we can consider a class of simple $n$-gonal $n$-configurations, $\mathcal{P}$, that are defined as ones forming a convex $n$-gon in the complement $\mathbb{R}^2 = \mathbb{R}P^2 \setminus \ell$, of some line $\ell \subset \mathbb{R}P^2 \setminus \mathcal{P}$. For $n \geq 5$ this $n$-gon (that coincides with the adjacency graph $\Gamma_{\mathcal{P}}$) is preserved by the monodromy group action, and it is easy to conclude that $Aut_L(\mathcal{P}) = D_n$. In particular, $Aut_L(\mathcal{P}) = D_6$ for $\mathcal{P} \in LC_6^1$.

2.1.1. **Remark.** It is also not difficult to show (see [F1]) that for 6-configurations $\mathcal{P}$ from the components $LC_5^2$, $LC_6^3$, and $LC_6^6$, groups $Aut_L(\mathcal{P})$ are respectively $\mathbb{Z}/4$, $D_3$, and the icosahedral group.
2.2. Aut(\(P\))-action on L-polygons. The \(\binom{n}{2}\) lines passing through the pairs of points of a simple \(n\)-configuration \(P\) divide \(\mathbb{R}^2\) into polygons that we call L-polygons associated to \(P\). Group \(\text{Aut}_L(P)\) acts naturally on the L-polygons that cannot be collapsed in a process of L-deformation.

For \(n = 5\) none of the L-polygons can be collapsed, and we obtain an action of \(\text{Aut}_L(P) = D_5\), which divide the set of 31 L-polygons into 6 orbits: three internal orbits formed by L-polygons lying inside pentagon \(\Gamma_P\) and three external ones, placed outside \(\Gamma_P\) (see Figure 3).

![Figure 3. \(D_5\)-orbits of L-polygons for \(P \in LC^5\). Labels \(i = 1, 2, 3, 6\) represent the corresponding class \(LC^5_i \ni P\)](image)

By adding a point \(p\) in one of the L-polygons associated to \(P\) we obtain a simple 6-configuration, \(P' = P \cup \{p\}\). The L-deformation class of \(P'\) depends obviously only on the \(\text{Aut}_L(P)\)-orbit of the L-polygon containing \(p\). Figure 3 shows the correspondence between the 6 orbits and the four classes \(LC^5_i\). Note that for \(i\) equal to 1 and 6 class \(LC^5_i\) is represented by one orbit, while for 2 and 3 such class is represented by two orbits. This is because \(\text{Aut}_L(P)\) acts transitively on the points of \(P\) for \(i = 1, 6\), while for \(i = 2, 3\) the vertices of \(P\) split into two \(\text{Aut}_L(P)\)-orbits.

2.3. The dual viewpoint. In the dual projective plane \(\hat{\mathbb{R}}^2\), consider the arrangement of lines \(\hat{P} = \{\hat{p}_1, \ldots, \hat{p}_n\}\) which is dual to a given \(n\)-configuration \(\hat{P} = \{\hat{p}_1, \ldots, \hat{p}_n\}\). Lines \(\hat{p}_i\) divide \(\hat{\mathbb{R}}^2\) into polygons that we call subdivision polygons of \(\hat{P}\). We define the polygonal spectrum of an \(n\)-configuration \(P\) as the \((n-2)\)-tuple \(f = (f_3, f_4, \ldots, f_n)\) where \(f_k\) is the number of \(k\)-gonal subdivision polygons of \(\hat{P}\). Euler’s formula easily implies that \(\sum_{k=3}^{n}(k-4)f_k = -4\) for simple \(n\)-configurations. It is easy to see also that \(f_3 \geq 5\) if \(n \geq 5\) (in fact, it is known that \(f_3 \geq n\)), which implies that for \(n \geq 5\), at least one subdivision polygon has 5 or more sides.

If \(\ell \subset \mathbb{R}^2 \setminus P\) is a line, then point \(\hat{\ell} \in \hat{\mathbb{R}}^2\) that is dual to \(\ell\) should belong to one of the subdivision polygons, say \(F\). Then \(F\) is an \(m\)-gon if and only if the convex hull, \(H\), of \(P\) in the affine plane \(\mathbb{R}^2 \setminus \ell\) is an \(m\)-gon. The \(m\)-gons \(F\) and \(H\) are dual: points \(p \in H\) are dual to lines \(\hat{p}\) disjoint from \(F\) (and vice versa).

This gives two options for a simple 6-configuration \(P\). The first option is \(f_6 > 0\) that implies \(P \in LC^6_6\). The second option is \(f_5 = 0, f_6 > 0\), which means that the convex hull of \(P\) is a pentagon in some affine chart \(\mathbb{R}^2 \setminus \ell\).
A simple 7-configurations is called *heptagonal* if \( f_7 > 0 \), *hexagonal* if \( f_7 = 0 \) and \( f_6 > 0 \), and *pentagonal* if \( f_6 = f_7 = 0 \) and \( f_5 > 0 \). Note that \( f_5 + f_6 + f_7 > 0 \) for any simple 7-configuration, and so, one of these three conditions is satisfied. In terms of the affine chart \( \mathbb{R}^2 = \mathbb{P}^2 \setminus \ell \), these three cases give (if point \( \hat{l} \) is chosen inside a subdivision polygon with the maximal number of sides): 7 points forming a convex heptagon, 6 points forming a convex hexagon plus a point inside it, and 5 points forming a convex pentagon plus two points inside it (see Table below).

### Table 1. Derivative codes and polygonal spectra

| \( \mathcal{P} \in \text{LC}' \) | \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) | \( f = (f_5, f_6, f_7, f_{\hat{f}}) \) |
|-----------------|------------------|------------------|
| Hexagonal       | (7, 14, 0, 0)    | (7, 13, 1, 0)    |
| (3, 4, 0, 0)    | (2, 2, 3, 0)     | (8, 11, 0, 0)    |
| (1, 2, 2, 2)    | (1, 0, 6, 0)     | (9, 3, 1, 0)     |
| Pentagonal with \( \sigma_1 = 1 \) | (1, 6, 0, 0) | (7, 12, 3, 0) |
| (1, 4, 2, 0)    | (1, 2, 4, 0)     | (8, 10, 4, 0)    |
| Pentagonal with \( \sigma_1 = 0 \) | (0, 4, 3, 0) | (8, 10, 4, 0) |
| (0, 6, 1, 0)    | (0, 3, 3, 1)     | (7, 13, 3, 0)    |
| (0, 3, 3, 1)    | (0, 6, 6, 0)     | (10, 6, 6, 0)    |

2.4. Simple 7-configurations with \( \sigma_1 > 0 \). Following [F1], we outline here the L-deformation classification in the most essential for us case of simple 7-configurations with \( \sigma_1 > 0 \). By definition, such configurations can be presented as \( \mathcal{P}' = \mathcal{P} \cup \{p\} \), where \( \mathcal{P} \) is its cyclic 6-subconfiguration and \( p \) is an additional point (there are precisely \( \sigma_1 \) ways to choose such a decomposition of \( \mathcal{P}' \)).

Like in the case of 6-configurations considered above, the monodromy group \( \text{Aut}_L(\mathcal{P}) = \mathbb{D}_6 \) of \( \mathcal{P} \in \text{LC}'_6 \) acts on the L-polygons associated to \( \mathcal{P} \) (see Figure 4), and the L-deformation class of \( \mathcal{P}' \) depends only on the orbit \( (\mathcal{A}, \mathcal{B}, \ldots) \) of the L-polygon that contains point \( p \). An additional attention is required to the four L-polygons that can be collapsed, namely, the “central” triangle \( E \) in Figure 4 and three other triangles, one of which is marked as \( F \) on this Figure: it is bounded by a principal diagonal and the two sides of the hexagon that have no common vertices with this diagonal. We skip here the arguments from [F1] (but after passing to the Q-deformation classification in Propositions 4.4.1 and 5.2.2 we will provide in fact a more subtle version of this proof).

Note that \( \mathbb{D}_6 \)-action is well-defined on the contractible L-polygons as well as on the non-contactable ones: this action preserves \( E \) invariant and naturally permute the three polygons of type \( F \) so that they form a single orbit, \( [F] \).

It is easy to see that \( f_6 + f_7 \leq 1 \) for any simple 7-configuration, and that our assumption on \( \mathcal{P}' \) admits three options. The first option is \( f_7 = 1 \), that is to say, \( \mathcal{P}' \) is a heptagonal configuration. This correspond to location of \( p \) inside one of the six L-polygons from the \( \mathbb{D}_6 \)-orbit \( \mathcal{A} \) of a triangle \( A \). The second option is location of \( p \) inside hexagon \( \Gamma \mathcal{P} \), in one of the *internal L-polygons* from the orbits \( \mathcal{B}, \mathcal{C}, \mathcal{D}, \) or \( \mathcal{E} \) (see Figure 4). In this case, \( f_7 = 0 \) and \( f_6(\mathcal{P}') = 1 \), that is to say, configuration \( \mathcal{P}' \) is hexagonal. In the remaining case, \( p \) lies outside \( \Gamma \mathcal{P} \), but not inside one of the six triangles on type \( A \). Then \( \mathcal{P}' \) may be either hexagonal with \( \sigma_1 \geq 2 \), or pentagonal with \( \sigma_1 \geq 1 \).
Totally, we enumerated 8 L-deformation classes out of 11. The remaining 3 pentagonal classes with $\sigma_1 = 0$ can be described by placing two points inside a convex affine 5-configuration, as it can be understood from Figure 2 (for details see [F1]).

2.5. Coloring of graphs $\Gamma_P$ for typical 6-configurations. Given a typical 6-configuration $P$, we say that its point $p \in P$ is dominant (subdominant) if it lies inside (respectively, outside of) conic $Q_p$ that passes through the remaining 5 points of $P$. Here, by points inside (outside of) $Q_p$ we mean points lying in the component of $\mathbb{R}P^2 \setminus Q_p$ homeomorphic to a disc, (respectively, in the other component). We color the vertices of adjacency graph $\Gamma_P$: the dominant points of $P$ in black and subdominant ones in white, see Figure 5 for the result.

Graphs $\Gamma_P$ are bipartite, i.e., adjacent vertices have different colors.

2.5.1. Lemma. For a typical 6-configuration $P$, every edge of $\Gamma_P$ connects a dominant and a subdominant points.

Proof. It follows from analysis of the pencil of conics passing through 4 points of $P$: a singular conic from this pencil cannot intersect an edge of $\Gamma_P$ connecting the remaining two points. $\square$
2.6. Q-deformation components of typical 6-configurations. Let $Q\Delta^n \subset LC^n$ denote the subset formed by simple $n$-configurations, which have a subconfiguration of six points lying on a conic. Then $QC^n = LC^n \setminus Q\Delta^n$ is the set of typical $n$-configurations.

Note that $Q\Delta^6 \subset LC^6_1$, so, $LC^6_i$ for $i = 2, 3, 6$ are formed entirely by typical configurations and thus, give three Q-deformation components $QC^6_i = LC^6_i \setminus Q\Delta^6$, $i = 2, 3, 6$, of typical 6-configurations. It follows immediately also that $Aut_Q(P) = Aut_L(P)$ for $P$ from these three Q-deformation components.

To complete the classification it is left to observe connectedness of $QC^6_1 = LC^6_1 \setminus Q\Delta^6$, which implies that $QC^6_1$ is the remaining Q-deformation component in $QC^6$.

Connectedness follows immediately from the next Lemma and the fact that any 6-configuration $P \in QC^6_1$ has a dominant point (in fact, it has exactly three such points, see Figure 5).

2.6.1. Lemma. Consider two hexagonal 6-configurations $P^i \in QC^6_1$, with marked dominant points $p^i \in P^i$, $i = 0, 1$. Then, there is a Q-deformation $P^t$, $t \in [0, 1]$ that takes $p^0$ to $p^1$.

Proof. The same idea as in Subsection 2.2 is applied: the triangular L-polygons marked by 1 on Figure 3 are divided into pairs of Q-regions by the conic passing through the vertices of a pentagon. Placing of the sixth point outside (inside) the conic gives a dominant (respectively, subdominant) point. The monodromy group $D_5$ acts transitively on the Q-regions of the same kind (in our case, on the parts of triangles marked by 1 that lie outside the conic), and these regions cannot be contracted in the process of L-deformation of the pentagon. Therefore, an L-deformation between $P^0 \setminus \{p^0\}$ and $P^1 \setminus \{p^1\}$ that brings the Q-region containing $p^0$ into the one containing $p^1$ can be extended to a required Q-deformation $P^t$ (see Figure 6).

![Figure 6. Continuous family of triangles of type “1”](image)

It follows easily that $Aut_Q(P) \cong D_3$, for $P \in QC^6_1$, namely, it is a subgroup of $Aut_L(P) \cong D_6$ that preserves the colors of vertices of graph $\Gamma_P$ on Figure 4.

2.7. Q-deformation components of typical 7-configurations: trivial cases. Let $QC^7_\sigma = LC^7_\sigma \setminus Q\Delta^7$, where $\sigma$ is one of the 11 derivative codes of 7-configurations (see Figure 2 or Table 1). Like in the case of 6-configurations, some of L-deformation components $LC^7_\sigma$, namely, the ones with $\sigma_1 = 0$ are disjoint from $Q\Delta^7$, therefore in these cases $QC^7_\sigma = LC^7_\sigma$ are Q-deformation components. From Table 1 this holds for $\sigma$ being $(0, 4, 3, 0)$, $(0, 6, 1, 0)$, and $(0, 3, 3, 1)$. 
3. Heptagonal 7-configurations

3.1. Dominance indices. We shall prove in Subsection 3.3 connectedness of space $QC_7^{(7,0,0,0)}$ formed by heptagonal typical configurations, and begin here with a key observation: for any $\mathcal{P} \in QC_7^{(7,0,0,0)}$ there exists a point $p \in \mathcal{P}$ that lies outside the six conics $Q_{p,q}$ passing through the 5-subconfigurations $\mathcal{P} \setminus \{p, q\}$, where $q \in \mathcal{P} \setminus \{p\}$. For $p \in \mathcal{P} \in QC_7^{(7,0,0,0)}$, we count the number of points $q \in \mathcal{P} \setminus \{p\}$ for which $p$ lies outside of the conic $Q_{p,q}$, denote this number $d(p)$ and call it the dominance index of $p$.

Among the 14 ways to numerate cyclically the vertices of heptagon $\Gamma$ (starting from any vertex, one can go around in two possible directions), one can distinguish a particular one that we call the canonical cyclic numeration.

3.1.1. Proposition. For any $\mathcal{P} \in QC_7^{(7,0,0,0)}$, there exists a canonical cyclic numeration of its points, $\mathcal{P} = \{p_0, \ldots, p_6\}$, such that $d(p_k)$ is $k$ for odd $k$ and $6 - k$ for even. In the other words, the sequence of $d(k)$ is 6, 1, 4, 3, 2, 5, 0.

The first step of the proof is the following observation.

3.1.2. Lemma. For any $\mathcal{P} \in QC_7^{(7,0,0,0)}$, there exists at most one point $p \in \mathcal{P}$ with the dominance index $d(p) = 6$ and at most one with $d(p) = 0$.

Proof. Assume that by contrary, $d(p) = d(q) = 6$ for $p, q \in \mathcal{P}$. Then $p$ and $q$ are dominant points in the 6-configuration $\mathcal{P}_r = \mathcal{P} \setminus \{r\}$ for any $r \in \mathcal{P} \setminus \{p, q\}$. But dominant and subdominant points in the hexagon $\Gamma_{P_r}$ are alternating (see Figure 5), and so, the parity of the dominant points is the same in a cyclic numeration of the hexagon vertices. By an appropriate choice of $r$ this parity however can be made different, contradiction. In the case $d(p) = d(q) = 0$ a proof is similar. □

3.2. Position of the vertices and edges of $\Gamma_\mathcal{P}$ with respect to conics $Q_{i,j}$. Let us fix any cyclic numeration $p_0, \ldots, p_6$ of points of $\mathcal{P} \in QC_7^{(7,0,0,0)}$. We denote by $Q_{i,j}$ the conic passing through the points of $\mathcal{P}$ different from $p_i$ and $p_j$ and put $d_{i,j} = 0$ if $p_i$ lies inside conic $Q_{i,j}$ and $d_{i,j} = 1$ if outside, $0 \leq i, j \leq 6$, $i \neq j$. By definition, we have

$$d(p_i) = \sum_{0 \leq j \leq 6, j \neq i} d_{i,j}, \quad i = 0, \ldots, 6.$$

In what follows we apply “modulo 7” index convention in notation for $p_i$, $Q_{i,j}$ and $d_{i,j}$, that is put $p_{i+1} = p_0$ if $i = 6$, $p_{i-1} = p_6$ if $i = 0$, etc.

3.2.1. Lemma. Assume that $0 \leq i \leq 6$. Then

(a) $d_{i,j} + d_{i+1,j} = 1$ for all $0 \leq j \leq 6$, $j \neq i, i + 1$.

(b) $d_{i,i+1} = d_{i+1,i}$ and $d_{i-1,i} = d_{i,i-1}$ provided $d(p_i) \neq 0, 6$.

(c) $d_{i-i} \neq d_{i+1} + 1$ provided $d(p_i) \neq 0, 6$.

Proof. (a) follows from Lemma 2.5.1 applied to $\mathcal{P} \setminus \{p_j\}$. Assume that (b) does not hold, say, $d_{i,i+1} = 1$ and $d_{i+1,i} = 0$ (the other case is analogous). This means that $p_i$ lies outside of conic $Q_{i,i+1}$ and $p_{i+1}$ lies inside. Since $d(p_i) \neq 6$, there is another conic, $Q_{i,j}$ containing $p_i$ inside. This contradicts to the Bezout theorem, since $Q_{i+1}$ and $Q_{i,j}$ have 4 common points $p_k$, $0 \leq k \leq 6$, $k \neq i, i + 1, j$, and in addition one more point as it is shown on Figure 7. For proving (c) we apply Lemma 2.5.1 to the cyclic 6-configuration $\mathcal{P} \setminus \{p_i\}$, in which points $p_{i-1}$ and $p_{i+1}$
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Figure 7. An extra intersection point of conics $Q_{i,i+1}$ and $Q_{i,j}$

become consecutive, and thus, one and only one of them is dominant, say $p_{i-1}$ (the other case is analogous). Then $d_{i-1,i} = 1$ and $d_{i+1,i} = 0$, and thus, $d_{i,i+1} = 0$ as it follows from (b).

We say that an edge $[p_i, p_{i+1}]$ of heptagon $\Gamma_P$ is internal (respectively, external) if its both endpoints lie inside (respectively, outside of) conic $Q_{i,i+1}$, or in the other words, if $d_{i,i+1} = d_{i+1,i} = 0$ (respectively, if $d_{i,i+1} = d_{i+1,i} = 1$). If one endpoint lies inside and the other outside, we say that this edge is special (see Figure 8a–c).

3.2.2. Corollary. A special edge of the heptagon $\Gamma_P$ should connect a vertex of dominance index 0 with a vertex of index 6, and in particular, such an edge is unique if exists. The internal and external edges are consecutively alternating. In particular, a special edge must exist (since the number of edges is odd).

We sketch the internal and external edges of $\Gamma(P)$ respectively thin and thick. The special edge is shown dotted and directed from the vertex of dominance index 0 to the one of dominance index 6. Corollary 3.2.2 means that graph $\Gamma(P)$ decorated this way should look like is shown on Figure 8d.

3.2.3. Lemma. The sum $d(p_i) + d(p_{i+1})$ is 5 if edge $[p_i, p_{i+1}]$ is internal, and is 7 if external.

Proof. By 3.2.1, we have

$$d(p_i) + d(p_{i+1}) = (d_{i,i+1} + d_{i+1,i}) + \sum_{0 \leq j \leq 6, j \neq i,i+1} (d_{i,j} + d_{i+1,j}),$$

where by Lemma 3.2.1(b), $(d_{i,i+1} + d_{i+1,i}) = 2d_{i,i+1}$ is 0 if edge $[p_i, p_{i+1}]$ is internal and 2 if external. By Lemma 3.2.1(a), the remaining sum is 5.

Proof of Proposition 3.1.1. Lemma 3.2.3 together with Corollary 3.2.2 let us recover the whole sequence $d(p_i)$, $i = 0, \ldots, 6$, from any value different from 0 and 6 (which exists by Lemma 3.1.2), see Figure 8e.
3.3. **Connectedness of QC\(_7^{(7,0,0,0)}\)**. This proof is similar to the proof of connectedness of QC\(_6^0\) in Subsection 2.3. Given \(\mathcal{P} \in \text{QC}\(_7^{(7,0,0,0)}\)) assume that its points \(p_0, \ldots, p_6\) have the canonical cyclic numeration, and consider subconfiguration \(\mathcal{P}_0 = \mathcal{P} \setminus \{p_0\} \in \text{QC}\(_6^0\)). As we observed in Subsection 2.4 point \(p_0\) lies in a triangular L-polygon \(A\) associated with \(\mathcal{P}_0\) (see Figure 4).

Such a triangle \(A\) is subdivided into 6 or 7 Q-regions \(A_i\) by the conics \(Q_i = Q_{0,i}\), \(i = 1, \ldots, 6\), that connect quintuples of points of \(\mathcal{P}_0\), see Figure 5. The only one of these Q-regions that can be collapsed is \(A_6\), so, the monodromy group \(\text{Aut}_Q(\mathcal{P}_0) \cong \mathbb{D}_3\) act on the Q-regions of types \(A_i\), \(0 \leq i < 6\) and clearly, form 6 orbits, denoted respectively \([A_i]\). Our choice of \(p_0\) with \(d(p_0) = 6\) means that it lies inside region \(A_0\). Transitivity of \(\mathbb{D}_3\)-action on the Q-regions of type \(A_0\) and impossibility for such a region to be collapsed in the process of a Q-deformation implies connectedness of QC\(_7^{(7,0,0,0)}\).

3.3.1. **Remark**. Placing point \(p_0\) in a Q-region \(A_i\), \(0 < i \leq 6\), instead of \(A_0\) lead to a new canonical cyclic ordering of the points of \(\mathcal{P}\) (different from \(p_0, \ldots, p_6\)). To recover that order, it is sufficient to know the two points, of dominance index 0 and 6. Table 2 shows how this pair of points depend on the region \(A_i\).

4. **Q-deformation classification of hexagonal 7-configurations**

4.1. **General scheme of arguments: subdivision of L-polygons into Q-regions**. In all the cases we follow the same scheme of Q-deformation classification.
Table 2. The indices $d(p_j)$ in case of $p_0 \in A_i$

| Location of $p_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|-------------------|-------|-------|-------|-------|-------|-------|-------|
| The point of $\mathcal{P}$ with $d = 6$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ |
| The point of $\mathcal{P}$ with $d = 0$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ |

as for heptagonal configurations in Subsection 3.3. Namely, we consider a typical 7-configuration with a marked point $\mathcal{P} = \mathcal{P}_0 \cup \{p_0\}$, so that $\mathcal{P}_0 \in QC_1^6$ (a cyclic 6-subconfiguration). In this section we assume that $p_0$ lies inside hexagon $\Gamma_{\mathcal{P}_0}$, which corresponds to the case of hexagonal configuration $\mathcal{P}$ (see Subsection 2.3). For a given $\mathcal{P}$ such a choice of marked point $p_0$ is unique, because $f_6(\mathcal{P}) \leq 1$. In the next Subsection we consider $p_0$ lying outside $\Gamma_{\mathcal{P}_0}$ in one of L-polygons that correspond to pentagonal configurations (so, we exclude previously considered cases of heptagonal and hexagonal 7-configurations).

Conics $Q_i$ passing through the points of 5-subconfigurations $\mathcal{P}_0 \setminus \{p_i\}$, $i = 1, \ldots, 6$, can subdivide an L-polygon into several $Q$-regions like in Subsection 3.3 and our aim is to analyze which of these regions cannot be contracted in a process of $Q$-deformation, and how the monodromy group $\text{Aut}_Q(\mathcal{P}_0)$ does act on them.

We always choose a cyclic order of points $p_1, \ldots, p_6 \in \mathcal{P}_0$ so that $p_1$ is dominant (then $p_3$ and $p_5$ are dominant too, whereas $p_2, p_4$, and $p_6$ are subdominant). Then conics $Q_2, Q_4$, and $Q_6$ contain hexagon $\Gamma_{\mathcal{P}_0}$ inside, whereas $Q_1, Q_3$, and $Q_5$ intersect the internal L-polygons of $\mathcal{P}_0$, see Figure 10.

4.1.1. **Remark.** On Figure 10 conics $Q_1, Q_3, Q_5$ do not intersect the sides of the hexagon. If they do intersect, then the shape of $Q$-regions of types $B_1$ and $B_2$ may change, which changes Figure 10 a bit, but not essentially for our arguments.

4.2. **$D_3$-orbits.** The internal L-polygons of types $D$ and $E$ are obviously contained inside these conics and only L-polygons of types $B$ and $C$ are actually subdivided into $Q$-regions. Namely, the latter L-polygons are subdivided into $Q$-regions $B_1, B_2$ and respectively $C_1, C_2, C_3$ as it is shown. Next, we can easily see that the monodromy group $\text{Aut}_Q(\mathcal{P}_0) \cong D_3$ acts transitively on the $Q$-regions of each type, which gives 7 $D_3$-orbits: $B_1, B_2, C_1, C_2, C_3, D$ and $E$. 

![Figure 10](image-url)
4.3. Deformation classification in the cases of non-collapsible Q-regions. Note that triangle $E$ is the only internal Q-region of $P_0$ that can be collapsed by a Q-deformation. Thus, any pair of hexagonal 7-configurations $P'$ and $P''$ whose marked points, $p'_0$ and $p''_0$ belong to the same $D_3$-orbit of the internal Q-regions can be connected by a Q-deformation. Namely, we start with a Q-deformation between $P'_0 = P' \setminus \{p'_0\}$ and $P''_0 = P'' \setminus \{p''_0\}$ that transforms the Q-region containing $p'_0$ to the one containing $p''_0$ and extend this deformation to the seventh points using non-contractibility of the given type of Q-regions.

This yields Q-deformation classes $QC^7(3,4,0,0)_1$, $QC^7(3,4,0,0)_2$, $QC^7(2,3,3,0)_1$, $QC^7(2,3,3,0)_2$, $QC^7(2,3,3,0)_3$, and $QC^7(1,2,2,2)$ that correspond respectively to the $D_3$-orbits of types $B_1$, $B_2$, $C_1$, $C_2$, $C_3$, and $D$, see Figure 11.

4.4. The case of Q-region $E$. Consider a subset $\tilde{LC}^6_1 \subset LC^6_1$ formed by typical hexagonal 6-configurations $P = \{p_1, \ldots, p_6\}$ whose principal diagonals $p_1p_4$, $p_2p_5$, and $p_3p_6$ are not concurrent, or in the other words, whose L-polygon $E$ is not collapsed. Connectedness of space $QC^7_{(1,0,6,0)}$ formed by 7-configurations $P = P_0 \cup \{p_0\}$ with $p_0$ placed in the Q-region $E$ would follow from connectedness of $LC^6_1$.

4.4.1. Proposition. Space $\tilde{LC}^6_1$ is connected.

Proof. Given a pair of configurations, $P^i \in QC^6_{C_1}$, $i = 0, 1$, we need to connect them by some deformation $P^t \in QC^6_{C_1}$, $t \in [0, 1]$. Let us choose a cyclic numbering of points, $p'_1, \ldots, p'_6 \in P^i$, $i = 0, 1$, so that $p'_1$ are dominant. At the first step, we can achieve that the triangular Q-regions “E” of the both configurations coincide, so that the dominant and subdominant points in $P^i$ go in the same order, as it is shown on Figure 11(a), that is, points $p'_1 \in P^i$, $i = 0, 1$, lie on the ray that is the extension of side $XY$ (then the other rays extending the sides if triangle $XYZ$ are also of the same color).

![Figure 11](image)

**Figure 11.** (a) Similar mutual position of dominant and subdominant points on the common principal diagonals of $P^0$ and $P^1$, (b) conic $Q$ containing triangle $XYZ$ inside and a point $p$ outside.

This can be done by a projective transformation sending the diagonals $p'_1p'_2$, $p'_2p'_3$, $p'_3p'_4$, and the infinity line, $L_\infty$, (that pass in the complement of $\Gamma_{p_0}$) to the corresponding diagonals and the “infinity line” for configuration $P^1$ (existence of a deformation is due to connectedness of $PGL(3, \mathbb{R})$). If the mutual positions of the dominant and subdominant points on the lines in $P^0$ and $P^1$ will differ, then it can be made like on Figure 11(a) by a projective transformation that permutes
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the three diagonals while preserving $L_\infty$. (One can also use flexibility of the initial numeration of vertices in $P^1$).

Fixing a triangle $XYZ \subset \mathbb{R}^2 = \mathbb{R}P^2 \setminus L_\infty$, let us denote by $QC^6_{E,XYZ}$ the subspace of $QC^6_1$ consisting of hexagonal 6-configurations whose dominant points lie on the affine rays that are continuations of sides $XY$, $YZ$, and $ZX$, and subdominant points lie on the continuations of $YX$, $ZY$, and $XZ$, as it is shown on Figure 11(a).

The final step of the proof is connectedness of $QC^6_{E,XYZ}$.

4.4.2. Lemma. For a fixed triangle $XYZ \subset \mathbb{R}^2 = \mathbb{R}P^2 \setminus L_\infty$, the configuration space $QC^6_{E,XYZ}$ is connected.

Proof. Consider $P \in QC^6_{E,XYZ}$. $P = \{p_1, \ldots, p_6\}$ where $p_1$ lies on the continuation of $XY$ (and so, is dominant). With respect to the conic $Q$ passing trough $p_2, \ldots, p_6$, triangle $XYZ$ lies inside and $p_1$ outside. This gives a one-to-one correspondence between $QC^6_{E,XYZ}$ and the space of pairs $(Q, p)$, where $Q$ is an ellipse containing $XYZ$ inside and $p$ is a point on the continuation of $XY$ lying outside $Q$ (see Figure 11(b)). The space of such ellipses is connected (and in fact, contractible), and the projection of $QC^6_{E,XYZ}$ to this space is fibration with a contractible fiber. Thus, $QC^6_{E,XYZ}$ is connected (and in fact, is contractible). □

4.5. Decoration of the adjacency graphs for typical 7-configurations. Figure 12 shows the adjacency graphs of typical hexagonal configurations $P$ endowed additionally with the vertex coloring for $P_0 = P \setminus \{p_0\}$ as in Subsection 2.5 (black

![Diagram](image)

Figure 12. $Q$-deformation classes of hexagonal 7-configurations
for dominant and white for subdominant points) and with the edge decoration for the connected component of $\Gamma_P$ labeled by 1. Namely, such an edge $[p_i,p_j]$ is thin if $p_i$ and $p_j$ lie inside conic $Q_{ij}$, thick if they lie outside, or dotted and directed from $p_i$ to $p_j$ if $p_i$ lies inside and $p_j$ lies outside (like is shown on Figure 8). Such decoration lets us distinguish Q-deformation types of hexagonal configurations.

We endow graphs $\Gamma_P$ with a vertex coloring also in the case of pentagonal typical 7-configuration $P$ with $\sigma_1 > 0$. It can be done using that for pentagonal configuration $\sigma_1 \leq 1$ (see Table 1) and thus, $P$ has a unique vertex $p_0$ such that $P_0 = P \setminus \{p_0\}$ is cyclic, which gives a canonical way to color its vertices.

5. Pentagonal 7-configurations, the case of $\sigma_1 > 0$

5.1. $\mathbb{D}_3$-orbits of types $G$ and $H$. A pentagonal typical 7-configurations with $\sigma_1 > 0$ can be presented like in the previous section as $P = P_0 \cup \{p_0\}$, where $P_0 \in QC_7^{(1,4,2,0)}$. The difference is that now point $p_0$ lies outside hexagon $\Gamma_{P_0}$. More precisely, $p_0$ should lie in an L-polygon of type $F$, or $G$, or $H$, since the other types of L-poligons correspond either to a heptagonal (the case of L-polygon of type $A$) or a hexagonal (the case of types $I$ and $J$) configurations $P$ that were analyzed before.

The first crucial observation is that none of the conics $Q_i$, $i = 1, \ldots, 6$ can intersect these three types of external polygons, and therefore, such L-poligons are not subdivided into Q-regions like in the case of types $A$, $B$ and $C$. It is clear from Figure 13 conics $Q_i$ should lie in the shaded part that is formed by L-poligons of types $A$, $I$ and $J$.

![Figure 13. External Q-regions representing pentagonal 7-configurations with $\sigma_1 = 1$. $\mathbb{D}_3$-action is clear on the left, and the shape of L-poligons $G$, $H$, $F$ on the right.]

The second observation is that L-poligons of each type, $F$, $G$, or $H$, form a single orbit with respect to the action of monodromy group $\text{Aut}_Q(P_0) = \mathbb{D}_3$. The third evident observation is that L-poligons of types $G$ and $H$ cannot be contracted by a Q-deformation (as they cannot be contracted even by an L-deformation). Together these observations imply that the corresponding to L-polygon types $G$ and $H$ (see Table 1) configuration spaces $QC_7^{(1,4,2,0)}$ and $QC_7^{(1,2,4,0)}$ are connected, and thus, are Q-deformation components.
5.2. The case of L-polygons of type $F$.

5.2.1. **Proposition.** The configuration space $QC^7_{(1,6,0,0)}$ is connected, or equivalently, L-deformation component $LC^7_{(1,6,0,0)}$ that correspond to L-polygon of type $F$ contains a unique $Q$-deformation component.

**Proof.** A configuration $P \in QC^7_{(1,6,0,0)}$ has a unique distinguished point $p_0$, such that $P = P_0 \cup \{p_0\}$, where $P_0 \in QC^6_1$ and $p_0$ lies in the L-polygon of type $F$. Such polygon is a triangle whose vertices we denote by $X$, $Y$, and $Z$ using the following rule. By definition of $F$-type polygon, one of its supporting lines should be a principal diagonal passing through two opposite vertices of hexagon $\Gamma_{P_0}$. We can choose a cyclic numeration of points $p_1, \ldots, p_6 \in P$ so that these opposite vertices are $p_1$ and $p_4$, and $p_1$ is a dominant point (then $p_3, p_5$ are also dominant, and $p_2, p_4, p_6$ are subdominant). Two vertices of the triangle on the line $p_1p_4$ are denoted by $X$ and $Y$ in such an order that $X, Y, p_1, p_4$ go consecutively on this line, like it is shown on Figure 14(a), and the third point of the triangle is denoted by $Z$. The direction of cyclic numeration of points $p_i$ can be also chosen so that points $p_2, p_3$ lie on the line $XZ$ and $p_5, p_6$ on $YZ$ (see Figure 14(a)).

By a projective transformation we can map a triangle $XYZ$ to any other triangle on $\mathbb{RP}^2$, so, in what follows we suppose that triangle $XYZ$ is fixed and denote by $QC^6_{F,XYZ} \subset QC^6_1$ the subspace formed by typical cyclic configurations having $XYZ$ as its L-polygon of type $F$ and having a cyclic numeration of points $p_1, \ldots, p_6 \in P$ satisfying the above convention.

Then, Proposition 5.2.1 follows from connectedness of $QC^6_{F,XYZ}$.

5.2.2. **Lemma.** For a fixed triangle $XYZ$, the configuration space $QC^6_{F,XYZ}$ is connected.

**Proof.** Using the same idea as in Lemma 4.4.2, we associate with a configuration $P \in QC^6_{F,XYZ}$ a pair $(Q, p)$, where $p$ is the dominant point of $P$ on the line $XY$ (that is $p_1$ in the notation used above) and $Q$ is the conic passing through the other points of $P$. Note that $P$ can be recovered from pair $(Q, p)$ associated to it in a unique way. Position of $Q$ can be characterized by the conditions that triangle $XYZ$ lie outside $Q$ and the continuations of each side of $XYZ$ intersect conic $Q$ at two
5.3. **Proof of Theorem 1.2.1.** We have shown in Subsection 2.7 connectedness of three components $\sigma$ configurations with $P \in QC$. In Subsection 3.3 we have shown seven connected components formed by hexagonal 7-configurations. The remaining 3 cases of pentagonal configurations with $\sigma_1 = 1$ were analyzed in Subsections 5.1 and 5.2.

6. **Concluding Remarks**

6.1. **Real double sixes.** By blowing up $\mathbb{P}^2$ at the points of a typical 6-configuration $P \subset \mathbb{P}^2$ we obtain a del Pezzo surface $X_P$ of degree 3 that can be realized by anticanonical embedding as a cubic surface in $\mathbb{F}_6$. The exceptional curves of blowing up form a configuration of six skew lines $\mathcal{L}_P \subset X_P \subset \mathbb{P}^3$ that we call the skew six represented by $P$. In the real setting, for $P \subset \mathbb{R}P^2$, cubic surface $X_P$ is real and maximal, the latter means by definition that its real locus $\mathbb{R}X_P$ is homeomorphic to $\mathbb{R}P^2 \# 6\mathbb{R}P^2$. The four deformation classes of typical 6-configurations give four types of real skew sixes: cyclic, bicomponent, tricomponent and icosahedral. It was observed in [Z] that the complementary real skew six has the same type as a given one, and so, we can speak of the four types of real double sixes.

It was shown in [Maz] (see also [DV]) that there exist 10 coarse deformation classes of six skew line configurations in $\mathbb{R}P^3$: here coarse means that deformation equivalence is combined with projective (possibly orientation-reversing) equivalence, for details see [DV]. Among these 10 classes, 8 can be realized by so called join configurations, $J_\tau$, that can be presented by permutations $\tau \in S_6$ as follows. Fixing consecutive points $p_1, \ldots, p_6$ and $q_1, \ldots, q_6$ on a pair of skew lines, $L_p$ and $L_q$ respectively, we let $J_\tau = \{L_1, \ldots, L_6\}$, where line $L_i$ joins $p_i$ with $q_{\tau(i)}$, $i = 1, \ldots, 6$. We denote such a configuration (and sometimes its coarse deformation class) by $J_\tau$. The remaining two coarse deformation classes among 10 cannot be represented by join configurations $J_\tau$: these two classes are denoted in [DV] by $L$ and $M$. As it is shown in [Z], the cyclic, bicomponent, and tricomponent coarse deformation classes of real skew sixes $\mathcal{L}_P$ are realized as $J_\tau$, where $\tau$ is respectively $(12\ldots 6)$, $(123654)$, and $(214365)$, where $\tau$ is recorded as $(\tau(1) \ldots \tau(6))$ (see Figure 15). The icosahedral coarse deformation class corresponds to the class $M$ from [DV].

6.2. **Permutation Hexagrams and Pentagrams.** A change of cyclic orderings of points $p_i, q_i$ on lines $L_p$ and $L_q$ clearly does not change the coarse deformation class of $J_\tau$. In the other words, the coarse deformation class of $J_\tau$ is an invariant of the orbit $[\tau] \in S_6/\langle D_6 \times D_6 \rangle$ of $\tau \in S_6$ with respect to the left-and-right multiplication action of $D_6 \times D_6$ in $S_6$ for the dihedral subgroup $D_6 \subset S_6$. points, so that the chord of conic $Q$ that is cut by line $XY$ lies between the two other chords that are cut by $XZ$ and $YZ$ (see Figure 14(b)).

The set of conics satisfying these requirements is obviously connected (and in fact, is contractible). For each conic $Q$ like this, there is some interval on the line $XY$ (see Figure 14) formed by points $p$ such that $(Q, p)$ is associated to some $P \in QC$. Thus, the set of such pairs $(Q, p)$, or equivalently $QC_{F,XYZ}$, is also connected.

□ □
| \( \mathcal{P} \in QC^6 \) | The 6-configuration \( \mathcal{L}_P \) of lines associated to \( \mathcal{P} \) | Permutation hexagram associated to \( \mathcal{L}_P \) | Permutation pentagram after dropping a line |
|---|---|---|---|
| ![Permutation pentagram](image1.png) | \( J_{(123456)} \) | (123456) | (12345) |
| ![Permutation pentagram](image2.png) | \( J_{(123654)} \) | (123654) | (12354) |
| ![Permutation pentagram](image3.png) | \( J_{(214365)} \) | (21435) | (21435) |
| ![Permutation pentagram](image4.png) | No hexagram | (13524) |

**Figure 15.** Four classes of simple 6-configurations, the corresponding real skew sixes in \( \mathbb{R}P^3 \), with their permutation hexagrams and pentagrams.

With a permutation \( \tau \in S_n \) we associate a diagram \( D_\tau \) obtained by connecting cyclically ordered vertices \( v_1, \ldots, v_n \) of a regular \( n \)-gon by diagonals \( v_{\tau(i)} v_{\tau(i+1)} \), \( i = 1, \ldots, n \), (here, \( \tau(n+1) = \tau(1) \)). Then “the shape of \( D_\tau \)” characterizes class \([\tau]\) in \( S_n/(D_n \times D_n) \), see Figure 15 for the hexagrams representing the cyclic, bicomponent, and tricomponent permutation orbits \([\tau]\), namely, \([123456]\), \([123654]\) and \([214365]\).

By dropping a line from a real skew six \( \mathcal{L} \) we obtain a real skew five, \( \mathcal{L'} \), that can be realized similarly, as a join configuration \( J_\tau \) for \( \tau \in S_5 \). It was shown in [Z] that the class \([\tau]\) in \( S_5/(D_5 \times D_5) \) does not depend on the line in \( \mathcal{L} \) that we dropped, including the case of icosahedral real double sixes, see the corresponding pentagrams \( D_{[\tau]} \) on Figure 15.

### 6.3. Real Aronhold sets.

By blowing up the points of a typical 7-configuration, \( \mathcal{P} \subset \mathbb{R}P^2 \), we obtain a non-singular real del Pezzo surface \( \mathcal{X}_P \) of degree 2 with a configuration \( \mathcal{L}_P \) of 7 disjoint real lines (the exceptional curves of blowing up). The anti-canonical linear system maps \( \mathcal{X}_P \) to a projective plane as a double covering branched along a non-singular real quartic, whose real locus has 4 connected components. Each of the 7 lines of \( \mathcal{L}_P \) is projected to a real bitangent to this quartic, and the corresponding arrangement of 7 bitangents is called an Aronhold set.

The 14 Q-deformation classes of typical 7-configurations yield 14 types of real Aronhold sets, which were described in [Z], see Figure 18.

Among various known criteria to recognize that real bitangents \( L_i, i = 1, \ldots, 7 \), to a real quartic form an Aronhold set, topologically the most practical one is
perhaps possibility to color the two line segments between the tangency points on
each $L_i$ in two colors, so that at the intersection points $L_i \cap L_j$, the corresponding
line segments of $L_i$ and $L_j$ are colored differently. Such colorings are indicated on
Figure [18]

6.4. **Real nodal cubics.** In [11], Fiedler-Le-Touzé analyzed real nodal cubics,
$C_i$, passing through the points $p_0, \ldots, p_6 \in \mathcal{P}$ of a heptagonal configuration,
$\mathcal{P} \in \mathcal{QC}_{7,0,0,0}^7$, and having a node at one of the points $p_i \in \mathcal{P}$, and described in which
order the points of $\mathcal{P}$ may follow on the real locus of $C_i$ (see Figure [10]).

![Figure 16. Cubics $C_i$, $i = 0, \ldots, 6$, passing through canonically ordered points $p_0, \ldots, p_6$ of $\mathcal{P} \in \mathcal{QC}_{7,0,0,0}^7$ and having a node at $p_i \in \mathcal{P}$](image)

Recently, a similar analysis was done for the other types of 7-configurations, see [13]. We proposed an alternative approach based on the real Aronhold set,
$L = \{L_0, \ldots, L_6\}$, corresponding to a given typical 7-configuration $\mathcal{P}$. Namely, the
order in which cubic $C_i$ passes through the points $p_j$ is the order in which bitangent
$L_i$ intersects other bitangents $L_j$. The two branches of $C_i$ at the node correspond
to the two tangency points of $L_i$.

6.4.1. **Remark.** Possibility of two shapes of cubic $C_6$ shown on Figure [16] correspond
to possibility to deform a real quartic with 4 ovals, so that bitangent $L_6$ moves away
from an oval, as it is shown on Figure [18] on the first sketch: the two tangency
points to $L_6$ on that oval are deformed into two imaginary (complex conjugate)
tangency points. Similarly, one can shift double bitangents to the same ovals in the
other of real Aronhold sets shown on Figure [18].

6.4.2. **Remark.** The two loops (finite and infinite) of a real nodal cubic $C_i$ that
correspond to the two line segments on $L_i$ bounded by the tangency points can be
distinguished by the following parity rule. Line $L_i$ contains six points of intersection
with $L_j$, $0 \leq j \leq 6$, $j \neq i$, and one more intersection point, with a line $L'_i$ obtained
by shifting $L_i$ away from the real locus of the quartic. One of the two line segments
contains even number of intersection points, and it corresponds to the “finite” loop of $C_i$, and the other line segment represents the “infinite” loop of $C_i$.

6.5. **Method of Cremona transformations.** An elementary real Cremona transfor-
mation, $\text{Cr}_{ijk} : \mathbb{RP}^2 \to \mathbb{RP}^2$, based at a triple of points $\{p_i, p_j, p_k\} \subset \mathcal{P}$ transforms
a typical 7-configuration $\mathcal{P} = \{p_0, \ldots, p_6\}$ to another typical 7-configuration
$\mathcal{P}_{ijk} = \text{Cr}_{ijk}(\mathcal{P})$. Starting with a configuration $\mathcal{P} \in \mathcal{QC}_{7,0,0,0}^7$, we can realize the
other 13 Q-deformation classes of 7-configurations as $\mathcal{P}_{ijk}$ for a suitable choice of
$i, j, k$, as it is shown on Figure [17] see [Z] for more details. This construction is
used to produce the real Aronhold sets shown on Figure [18].
Figure 17. Cremona transformations of $\mathcal{P} \in QC_{(7,0,0,0)}^7$
Figure 18. Aronhold sets representing typical planar 7-configurations. Lines having two contacts to one oval can be shifted from it by a deformation, so that the contact points become imaginary (an example is shown for the case of $QC^{7}_{(7,0,0,0)}$).
$QC_{(1,0,0)}^{7}$

$Cr_{023}$

$QC_{(1,4,2,0)}^{7}$

$Cr_{025}$
\text{Cr}_{125}

\text{Cr}_{135}

\text{QC}_{(0,6,1,0)}

\text{QC}_{(0,3,3,1)}
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