PARAMETRIC ESTIMATION FOR CONVOLUTIONALLY OBSERVED DIFFUSION PROCESSES

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ABSTRACT. We propose a new statistical observation scheme of diffusion processes named convolutional observation, where it is possible to deal with smoother observation than ordinary diffusion processes by considering convolution of diffusion processes and some kernel functions with respect to time parameter. We discuss the estimation and test theories for the parameter determining the smoothness of the observation, as well as the least-square-type estimation for the parameters in the diffusion coefficient and the drift one of the latent diffusion process. In addition to the theoretical discussion, we also examine the performance of the estimation and the test with computational simulation, and show an example of real data analysis for one EEG data whose observation can be regarded as smoother one than ordinary diffusion processes with statistical significance.

1. Introduction

We consider a d-dimensional diffusion process defined by the following stochastic differential equation,

$$\mathrm{d}X_t = b(X_t, \beta) \mathrm{d}t + a(X_t, \alpha) \mathrm{d}w_t, \quad X_{-\lambda} = x_{-\lambda},$$

where \( \lambda > 0 \), \( \{w_t\}_{t \geq -\lambda} \) is a standard \( r \)-dimensional Wiener process, \( x_{-\lambda} \) is an \( \mathbf{R}^d \)-valued random variable independent of \( \{w_t\}_{t \geq -\lambda} \), \( \alpha \in \Theta_1 \) and \( \beta \in \Theta_2 \) are unknown parameters, \( \Theta_1 \subset \mathbf{R}^{m_1} \) and \( \Theta_2 \subset \mathbf{R}^{m_2} \) are compact and convex parameter spaces, \( a: \mathbf{R}^d \times \Theta_1 \to \mathbf{R}^{d} \otimes \mathbf{R}^r \) and \( b: \mathbf{R}^d \times \Theta_2 \to \mathbf{R}^d \) are known functions. Our concern is statistical estimation for \( \alpha \) and \( \beta \) from observation. \( \theta^\star = (\alpha^\star, \beta^\star) \) denotes the true value of \( \theta := (\alpha, \beta) \).

We denote the observation as the discretised process \( \{X_{t,n}, n: i = 0, \ldots, n\} \) with discretisation step \( h_n > 0 \) such that \( h_n \to 0 \) and \( T_n := nh_n \to \infty \), where the convoluted process \( \{X_{t,n}\}_{t \geq 0} \) is defined as

$$X_{t,n} := \int_{t-\rho h_n}^t V_{h_n}(t-s) X_s \mathrm{d}s = \int_{\mathbf{R}} V_{h_n}(t-s) X_s \mathrm{d}s = (V_{h_n} * X)(t),$$

where \( V_{h_n} \) is an \( \mathbf{R}^d \otimes \mathbf{R}^d \)-valued kernel function whose support is a subset of \([0,\rho h_n]\), and \( \rho > 0 \) such that \( \sup_n \rho h_n \leq \lambda \). In this paper, we specify \( V_{h_n} = V_{\rho,h_n} \) which is a parametric kernel function whose support is a subset of \([0,\rho h_n]\) defined as

$$V_{\rho,h_n}^{(i,j)}(t) := \begin{cases} (\rho(i)h_n)^{-1} 1_{[0,\rho(i)h_n]}(t) & \text{if } i = j \text{ and } \rho(i) > 0, \\ \delta(t) & \text{if } i = j \text{ and } \rho(i) = 0, \\ 0 & \text{if } i \neq j, \end{cases}$$

\( \delta(t) \) is the Dirac-delta function, \( \rho = [\rho(1), \ldots, \rho(d)]^T \in \Theta_{\rho} := [0,\rho]^d \) is the smoothing parameter determining the smoothness of observation. That is to say, the observed process

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is defined as follows:

\[
X^{(\ell)}_{ih_n, n} = \begin{cases} 
(\rho^{(\ell)} h_n)^{-1} \int_{(1-\rho^{(\ell)}) h_n}^{ih_n} X^{(\ell)}_{s} \, ds & \text{if } \rho^{(\ell)} > 0, \\
X^{(\ell)}_{ih_n} & \text{if } \rho^{(\ell)} = 0,
\end{cases}
\]

for all \( \ell = 1, \ldots, d \). Let us consider both the problems that (i) \( \rho \) is a known parameter, and (ii) \( \rho \) is an unknown one and this is estimated by observation \( \{X^{(\ell)}_{ih_n, n}\} \), and the parameter space is denoted as \( \Xi := \Theta_\rho \times \Theta \).

When assuming \( \rho \) as a known parameter, we can find researches for parametric estimation for \( \alpha \) and/or \( \beta \) based on observation schemes which can be represented as special cases for some specific \( \rho \). If \( \rho = 0 \), our scheme is simply equivalent to parametric inference based on discretely observed diffusion processes \( \{X^{(\ell)}_{ih_n} : i = 0, \ldots, n\} \) studied in Florens-Zmirou (1989); Yoshida (1992); Bibby and Sørensen (1995); Kessler (1997); Kessler and Sørensen (1999); Yoshida (2011); Uchida and Yoshida (2012, 2014) and references therein. If \( \rho = [1, \ldots, 1]^T \), we can regard the problem as parametric estimation for integrated diffusion processes discussed in Gloter (2000); Ditlevsen and Sørensen (2004); Gloter (2006); Gloter and Gobet (2008); Sørensen (2011). Even for the case \( \rho = [0, \ldots, 0, 1, \ldots, 1]^T \) where some axes correspond to direct observation and the others do to integrated observation, we give consistent estimators for \( \alpha \) and \( \beta \) by considering the scheme of convolutionally observed diffusion processes and this is one of the contributions of our study.

What is more, our contribution is to consider the scheme where \( \rho \) is unknown and succeed in representation of the microstructure noise which makes the observation smoother than the latent diffusion process itself. As Zhang et al. (2005) studies, the existence of microstructure noise in financial data affects realised volatilities to increase as the subsampling frequency gets higher (for instance, see Figure 7.1 in Aït-Sahalia and Jacod, 2014). However, realised volatilities of some biological data such as EEG decrease as subsampling frequency increases: for instance, some time series data for the 2nd participant in the dataset named Two class motor imagery (002-2014) of BNCI Horizon 2020 (2014) show clear tendency of decreasing realised volatilities as subsampling frequency increases. Figure 1 shows the path of the 2nd axis of the data S02E.mat of BNCI Horizon 2020 (2014) for all 222 seconds (the observation frequency is 512Hz, and hence the entire data size is 113664) and that for the first one second; it seems to perturb like a diffusion process. We define realised volatilities with subsampling as for a one dimensional

![Figure 1. The path of the second column of S02E.mat of BNCI Horizon 2020 (2014) for all 222 seconds (left) and the first one second (right).](image-url)
observation \( \{Y_i\}_{i=0,...,n} \),

\[
RV (k) = \sum_{1 \leq i \leq [n/k]} (Y_{ik} - Y_{(i-1)k})^2,
\]

where \( k = 1, \ldots, 100 \) is the subsampling frequency parameter, and provide a plot of the realised volatilities the 2nd axis of the data S02E.mat in Figure 2: the altitudes of the graph represented in the \( y \)-axis correspond to the values of the realised volatilities \( RV (k) \) with subsampling at every \( k \) observation represented in the \( x \)-axis. It is observable that the increasing subsampling frequency results in decreasing realised volatilities, which cannot be explained by the existent major microstructure noises (e.g., see Jacod et al., 2009, 2010; Bibinger et al., 2014; Koike, 2016; Ogihara, 2018). To explain this phenomenon, we consider the smoother process than the latent one though ordinarily microstructure noises make the observation rougher than the latent process, because quadratic variation of a sufficiently smooth function is zero. One way to deal with smoother observation than the latent state is convolutional observation. As a concrete example, we show a convolutionally observed diffusion process and its characteristics in realised volatilities: let us consider the following 1-dimensional stochastic differential equation defining an Ornstein-Uhlenbeck (OU) process:

\[
dX_t = -20X_tdt + 10dw_t, \quad X_{-\lambda} = 0,
\]

where \( \lambda = 10^{-2/5} \). We simulate the stochastic differential equation by Euler-Maruyama method (see Iacus, 2008) with parameters \( n = 10^7, h_n = 10^{-5} \), and \( T_n = 10^2 \) and its convolution approximated by summation with the smoothing parameter \( \rho = 10 \) (for details, see Section 5). Figure 3 shows the latent diffusion process and the convoluted observation on \([0, 1]\), and we can see that the observation is indeed smoothed compared to the latent state. In Figure 4, we also give the plot of realised volatilities of the convolutionally ob-
Figure 3. The left figure is the plot of the latent diffusion process, and the right one is that of the convolutionally observed process on $[0, 1]$ respectively.

Figure 4. The realised volatilities of the convolutionally observed diffusion process with subsampling.

served process with subsampling as Figure 2. It is seen that the convolutional observation of a diffusion process also has the characteristics of decreasing realised volatilities as subsampling frequency increases, which can be seen in some biological data such as BNCT Horizon 2020 (2014). Of course, graphically comparing characteristics of simulation and real data is insufficient to verify the convolutional observation with smoothing parameter $\rho > 0$ in 1-dimensional case; therefore, we propose statistical estimation method for unknown $\rho$ and hypothesis test with the null hypothesis $H_0 : \rho = 0$ and the alternative one $H_1 : \rho \neq 0$ from convolutional observation in Section 3. Moreover, in Section 6, we examine the real EEG data plotted in Figure 2 by the statistical hypothesis testing we propose, and see it is more appropriate to consider the data as a convolutional observation of a latent diffusion process with $\rho \neq 0$ rather than direct observation of the latent process, which indicates the validity to deal with the problem of the convolutional observation scheme with unknown $\rho$. 
The paper is composed of the following sections: Section 2 gives the notations and assumptions used in this paper; Section 3 discusses the estimation and test for smoothing parameter $\rho$, and the discussion provides us with the tools to examine whether we should consider the convolutional observation scheme; Section 4 proposes the quasi-likelihood functions for the parameter of diffusion processes $\alpha$ and $\beta$, and corresponding estimators with consistency; Section 5 is for the computational simulation to examine the theoretical results in the previous sections; and Section 6 shows an application of the methods we propose in real data analysis.

2. Notations and assumptions

2.1. Notations. First of all, we set $A(x, \alpha) := a(x, \alpha)^{\otimes 2}$, $a(x) := a(x, \alpha_*)$, $A(x) := A(x, \alpha_*)$ and $b(x) := b(x, \beta_*)$. We also give the notation for a matrix-valued function $\mathbb{G}(x, \alpha|\rho)$ such that $\mathbb{G}^{(i,j)}(x, \alpha|\rho) := A^{(i,j)}(x, \alpha) f_G(\rho^{(i)}, \rho^{(j)})$, where

$$f_G(\rho^{(i)}, \rho^{(j)}) = \begin{cases} 1 & \text{if } \rho^{(i)} = \rho^{(j)} = 0, \\ 1 - \frac{\rho^{(i)}}{2} & \text{if } \rho^{(i)} = 0, \rho^{(j)} \in (0, 1], \\ \frac{1}{2\rho^{(i)}} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(j)} = 0, \\ \frac{1}{2\rho^{(j)}} - 3(\rho^{(i)})^2\rho^{(j)} + 3(\rho^{(i)})^2 + 6(\rho^{(i)})\rho^{(j)} - 2(\rho^{(j)})^3 & \text{if } \rho^{(i)}, \rho^{(j)} \in (0, 1], \rho^{(i)} > \rho^{(j)}, \\ \frac{3(\rho^{(i)})^2\rho^{(j)} - 3(\rho^{(i)})^2 - 6(\rho^{(i)})\rho^{(j)} - 2(\rho^{(j)})^3}{6(\rho^{(i)})^2} & \text{if } \rho^{(i)}, \rho^{(j)} \in (0, 1], \rho^{(i)} \leq \rho^{(j)}, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (1, \bar{\rho}], \rho^{(j)} \in (0, 1], \rho^{(i)} > \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (1, \bar{\rho}], \rho^{(i)} \leq \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} > \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} \leq \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (1, \bar{\rho}], \rho^{(i)} \leq \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} > \rho^{(j)} + 1, \\ \frac{6(\rho^{(i)})^2}{6(\rho^{(i)})^2 - 1} & \text{if } \rho^{(i)} \in (1, \bar{\rho}], \rho^{(i)} > \rho^{(j)} + 1. 
\end{cases}$$

The continuity of these functions is examined in Lemma 15 and Lemma 16. In addition, we also give the notation used throughout this paper.

- For every matrix $A$, $A^T$ is the transpose of $A$, and $A^{\otimes 2} := AA^T$.
- For every set of matrices $A$ and $B$ of the same size, $A[B] := \text{tr} \left(AB^T \right)$. Moreover, for any $m \in \mathbb{N}$, $A \in \mathbb{R}^{m} \otimes \mathbb{R}^{m}$ and $u, v \in \mathbb{R}^{m}$, $A[u, v] := v^T A u$.
- $v^T$ and $A^{(\ell_1, \ell_2)}$ denote the $\ell$-th element of a vector $v$ and the $(\ell_1, \ell_2)$-th one of a matrix $A$, respectively.
- For any vector $v$ and any matrix $A$, $|v| := \sqrt{\text{tr} \left(v^T v \right)}$ and $\|A\| := \sqrt{\text{tr} \left(A^T A \right)}$.
- $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$ denotes the stochastic basis, where $\mathcal{F}_t := \sigma \left(x_{-\lambda}, w_s : s \leq t \right)$.

2.2. Assumptions. With respect to $X_t$, we assume the following conditions.
For these functions, let us assume the following identifiability conditions hold.

(i) For a constant $C$, for all $x_1, x_2 \in \mathbb{R}^d$,
\[
\sup_{\alpha \in \Theta_1} \|a(x_1, \alpha) - a(x_2, \alpha)\| + \sup_{\beta \in \Theta_2} |b(x_1, \beta) - b(x_2, \beta)| \leq C |x_1 - x_2|.
\]

(ii) For all $p \geq 0$, $\sup_{t \geq -\lambda} E_{\theta_*} [|X_t|^p] < \infty$.

(iii) There exists a unique invariant measure $\nu_0$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and for all $p \geq 1$ and $f \in L^p(\nu_0)$ with polynomial growth,
\[
\frac{1}{T} \int_{-\lambda}^{T} f(X_t) \, dt \to P \int_{\mathbb{R}^d} f(x) \, \nu_0(\, dx).\]

[A2] There exists $C > 0$ such that $a: \mathbb{R}^d \times \Theta_1 \to \mathbb{R}^d \otimes \mathbb{R}^r$ and $b: \mathbb{R}^d \times \Theta_2 \to \mathbb{R}^d$ have continuous derivatives satisfying
\[
\sup_{\alpha \in \Theta_1} |\partial_2^i \partial_\alpha^j a(x, \alpha)| \leq C (1 + |x|)^C, \quad 0 \leq i \leq 2, \quad 0 \leq j \leq 2,
\]
\[
\sup_{\beta \in \Theta_2} |\partial_2^i \partial_\beta^j b(x, \beta)| \leq C (1 + |x|)^C, \quad 0 \leq i \leq 2, \quad 0 \leq j \leq 2.
\]

With the invariant measure $\nu_0$, we define
\[
\mathbb{V}_1 (\alpha|\xi_*): = -\int_{\mathbb{R}^d} \|G(x, \alpha|\rho_*) - G(x, \alpha_*|\rho_*)\|^2 \, \nu_0(\, dx),
\]
\[
\mathbb{V}_2 (\beta|\xi_*): = -\int_{\mathbb{R}^d} |b(x, \beta) - b(x, \beta_*)|^2 \, \nu_0(\, dx).
\]

For these functions, let us assume the following identifiability conditions hold.

[A3] There exist $\chi_1(\alpha_*) > 0$ and $\chi_1'(\beta_*) > 0$ such that for all $\alpha \in \Theta_1$ and $\beta \in \Theta_2$,
\[
\mathbb{V}_1 (\alpha|\xi_*) \leq -\chi_1(\alpha_*) |\alpha - \alpha_*|^2 \quad \text{and} \quad \mathbb{V}_2 (\beta|\xi_*) \leq -\chi_1'(\beta_*) |\beta - \beta_*|^2.
\]

### 3. Estimation and Test of the Smoothing Parameter

In this section, we discuss the case where the smoothing parameter $\rho$ of the kernel function $V_{\rho, h_0}$ is unknown. The estimation is significant for estimation of $\alpha$ and $\beta$ since we utilise the estimate for $\rho$ in quasi-likelihood functions of $\alpha$ and $\beta$. The test problem for hypotheses $H_0 : \rho = 0$ and $H_1 : \rho \neq 0$ is also important to examine whether our framework of convolutional observation is meaningful.

3.1. **Estimation of the smoothing parameter.** For simplicity of notation, let us consider the case $\overline{\rho} > 2$; otherwise the discussion is quite parallel. We should note that for all $i = 1, \ldots, d$,
\[
G^{(i,i)}(x|\rho) = \begin{cases} 
A^{(i,i)}(x) & \text{if } \rho^{(i)} = 0, \\
A^{(i,i)}(x) \left(1 - \frac{\rho^{(i)}}{3}\right) & \text{if } \rho^{(i)} \in (0, 1], \\
A^{(i,i)}(x) \left(\frac{1}{\rho^{(i)}} - \frac{1}{3(\rho^{(i)})^2}\right) & \text{if } \rho^{(i)} \in (1, \overline{\rho}].
\end{cases}
\]
Let us consider the estimation of $\rho^{(i)}$ with using the next statistics: the full quadratic variation

$$\frac{1}{nh_n} \sum_{k=1}^{n} \left( X_{kh_n,n}^{(i)} - X_{(k-1)h_n,n}^{(i)} \right)^2 \rightarrow^P \begin{cases} 
\nu_0 \left(A^{(i)}(\cdot)\right) & \text{if } \rho_*^{(i)} = 0, \\
\nu_0 \left( A^{(i)}(\cdot) \right) \left( 1 - \frac{\rho_*^{(i)}}{3} \right) & \text{if } \rho_*^{(i)} \in (0, 1], \\
\nu_0 \left( A^{(i)}(\cdot) \right) \left( \frac{1}{\rho_*^{(i)}} - \frac{1}{3(\rho_*^{(i)})^2} \right) & \text{if } \rho_*^{(i)} \in (1, \overline{p}], 
\end{cases}$$

because of Proposition 13 in Appendix A, and the reduced quadratic variation defined as $\frac{1}{nh_n} \sum_{2 \leq 2k \leq n} \left( X_{2kh_n,n}^{(i)} - X_{(2k-2)h_n,n}^{(i)} \right)^2$ converges in probability as follows.

**Lemma 1.** Under $[A1]$, we have the convergence in probability such that

$$\frac{1}{nh_n} \sum_{2 \leq 2k \leq n} \left( X_{2kh_n,n}^{(i)} - X_{(2k-2)h_n,n}^{(i)} \right)^2 \rightarrow^P \begin{cases} 
\nu_0 \left(A^{(i)}(\cdot)\right) & \text{if } \rho_*^{(i)} = 0, \\
\nu_0 \left( A^{(i)}(\cdot) \right) \left( 1 - \frac{\rho_*^{(i)}}{6} \right) & \text{if } \rho_*^{(i)} \in (0, 2], \\
\nu_0 \left( A^{(i)}(\cdot) \right) \left( \frac{2}{\rho_*^{(i)}} - \frac{4}{3(\rho_*^{(i)})^2} \right) & \text{if } \rho_*^{(i)} \in (2, \overline{p}]. 
\end{cases}$$

Then we define the ratio of those statistics such that

$$R_{n}^{(i)} := \left( \frac{1}{nh_n} \sum_{k=1}^{n} \left( X_{kh_n,n}^{(i)} - X_{(k-1)h_n,n}^{(i)} \right)^2 \right)^{-1} \left( \frac{1}{nh_n} \sum_{2 \leq 2k \leq n} \left( X_{2kh_n,n}^{(i)} - X_{(2k-2)h_n,n}^{(i)} \right)^2 \right)^{-1}$$

$$= \begin{cases} 
1 & \text{if } \rho_*^{(i)} = 0, \\
\left( 1 - \frac{\rho_*^{(i)}}{3} \right) \left( 1 - \frac{\rho_*^{(i)}}{6} \right)^{-1} & \text{if } \rho_*^{(i)} \in (0, 1], \\
\left( \frac{1}{\rho_*^{(i)}} - \frac{1}{3(\rho_*^{(i)})^2} \right) \left( 1 - \frac{\rho_*^{(i)}}{6} \right)^{-1} & \text{if } \rho_*^{(i)} \in (1, 2], \\
\left( \frac{1}{\rho_*^{(i)}} - \frac{1}{3(\rho_*^{(i)})^2} \right) \left( \frac{2}{\rho_*^{(i)}} - \frac{4}{3(\rho_*^{(i)})^2} \right)^{-1} & \text{if } \rho_*^{(i)} \in (2, \overline{p}]. 
\end{cases}$$

$$=: R_{n}^{(i)}$$

where $R$ has the next property.

**Lemma 2.** $R$ is a $[(3\overline{p} - 1)(6\overline{p} - 4)^{-1}, 1]$-valued monotonically decreasing continuous function, and has a continuous inverse $R^{-1} : [(3\overline{p} - 1)(6\overline{p} - 4)^{-1}, 1] \rightarrow [0, \overline{p}]$. 
We define \( \hat{\rho}_n \) such that
\[
\hat{\rho}_n^{(i)} := \begin{cases} 
0 & \text{if } R_n^{(i)} > 1, \\
R_n^{-1}(R_n^{(i)}) & \text{if } R_n^{(i)} \in [(3\overline{p} - 1)(6\overline{p} - 4)^{-1}, 1], \\
\overline{p} & \text{if } R_n^{(i)} < (3\overline{p} - 1)(6\overline{p} - 4)^{-1},
\end{cases}
\]
and then continuous mapping theorem for convergence in probability verifies the next result.

**Theorem 3.** Under \([A1]\), \( \hat{\rho}_n \) has consistency, i.e., \( \hat{\rho}_n \to^P \rho_* \).

**Remark 1.** We can compute \( y = R^{-1}(x), \ x \in [(3\overline{p} - 1)(6\overline{p} - 4)^{-1}, 1] \) by solving the following equations:

(i) \( x = (6 - 2y)(6 - y)^{-1} \) if \( x \in (4/5, 1) \),

(ii) \( x = (6 - 2y)(6y^2 - y^3)^{-1} \) if \( x \in (5/8, 4/5) \),

(iii) \( x = (3y - 1)(6y - 4)^{-1} \) if \( x \in [(3\overline{p} - 1)(6\overline{p} - 4)^{-1}, 5/8] \).

3.2. **Test for smoothed observation.** For all \( i = 1, \ldots, d \), we consider the next hypothesis testing:

\[
H_0 : \rho^{(i)} = 0, \ H_1 : \rho^{(i)} > 0.
\]

Let us consider the following test statistic:
\[
T_{i,n} := \sqrt{\frac{2}{3nh_n^2} \sum_{k=1}^n (\overline{X}_{kh_n,n}^{(i)} - \overline{X}_{(k-1)h_n,n}^{(i)})^4}
\times \left( \frac{1}{n} \sum_{k=1}^n (\overline{X}_{kh_n,n}^{(i)} - \overline{X}_{(k-1)h_n,n}^{(i)})^2 - \frac{1}{n} \sum_{2 \leq 2k \leq n} (\overline{X}_{2kh_n,n}^{(i)} - \overline{X}_{(2k-2)h_n,n}^{(i)})^2 \right)
= \sqrt{\frac{3/2}{\sum_{k=1}^n (\overline{X}_{kh_n,n}^{(i)} - \overline{X}_{(k-1)h_n,n}^{(i)})^4}}
\times \left( \sum_{k=1}^n (\overline{X}_{kh_n,n}^{(i)} - \overline{X}_{(k-1)h_n,n}^{(i)})^2 - \sum_{2 \leq 2k \leq n} (\overline{X}_{2kh_n,n}^{(i)} - \overline{X}_{(2k-2)h_n,n}^{(i)})^2 \right),
\]
and we abbreviate \( T_{i,n} \) to \( T_n \) if \( d = 1 \). Under \( H_0 \), we have the next result.

**Theorem 4.** Under \( H_0 \) and \([A1]\), we have the convergence in law such that
\[
T_{i,n} \to^L N(0, 1).
\]

We also obtain the result to support the consistency of the test.

**Theorem 5.** Under \( H_1 \) and \([A1]\), we have the divergence in probability such that for any \( c \in \mathbb{R} \),
\[
P(T_{i,n} < c) \to 1.
\]

Hence when we set the significance level \( \alpha_{\text{sig}} \in (0, 1) \), then we have the rejection region
\[
T_{i,n} < \Phi^{-1}(\alpha_{\text{sig}})
\]
where $\Phi$ is the distribution function of the standard Gaussian distribution. Theorem 5 supports the consistency of the test.

This test is essential in terms of examining the validity to consider the scheme of convolutional observation: if $\rho = 0$, then the ordinary observation scheme can be applied, but if $\rho \neq 0$, then we have the motivation to consider the convolutional observation scheme.

4. LEAST SQUARE ESTIMATION OF THE DIFFUSION AND DRIFT PARAMETERS

Let us set the least-square quasi-likelihood functions such that

\[
H_{1,n} (\alpha | \rho) := - \sum_{k=1}^{n} \left\| \frac{1}{h_n} (\overline{X}_{kh_n,n} - \overline{X}_{(k-1)h_n,n}) \otimes 2 - G (\overline{X}_{(k-1)h_n,n}, \alpha | \rho) \right\|^2,
\]

\[
H_{2,n} (\beta | \rho) := - \sum_{k=\max, \rho(\beta)}^{n} \frac{1}{h_n} |\overline{X}_{kh_n,n} - \overline{X}_{(k-1)h_n,n} - h_n b \left( \overline{X}_{(k-2-\max, \rho(\beta))}h_n, \beta \right) |^2,
\]

and the least-square estimators $\hat{\alpha}_n$ and $\hat{\beta}_n$ satisfying

\[
H_{1,n} (\hat{\alpha}_n | \rho_*) = \sup_{\alpha \in \Theta_1} H_{1,n} (\alpha | \rho_*) , \quad H_{2,n} (\hat{\beta}_n | \rho_*) = \sup_{\beta \in \Theta_2} H_{2,n} (\beta | \rho_*) .
\]

when $\rho_*$ is known, and

\[
H_{1,n} (\hat{\alpha}_n | \hat{\rho}_n) = \sup_{\alpha \in \Theta_1} H_{1,n} (\alpha | \hat{\rho}_n) , \quad H_{2,n} (\hat{\beta}_n | \hat{\rho}_n) = \sup_{\beta \in \Theta_2} H_{2,n} (\beta | \hat{\rho}_n) .
\]

when $\rho_*$ is unknown.

**Theorem 6.** Under $[A1]$-$[A3]$, $\hat{\alpha}_n$ and $\hat{\beta}_n$ are consistent, i.e., $\hat{\alpha}_n \to^P \alpha_*$ and $\hat{\beta}_n \to^P \beta_*$.

5. SIMULATIONS

In this simulation section, we only consider the case where $\rho$ is unknown and should be estimated by data with the method proposed in Section 3.

5.1. 1-Dimensional simulation. We examine the following 1-dimensional stochastic differential equation whose solution is a 1-dimensional Ornstein-Uhlenbeck (OU) process:

\[
dX_t = (\beta^{(1)} X_t + \beta^{(2)}) dt + \alpha dw_t , \quad X_{-\lambda} = 0,
\]

$\alpha \in \Theta_1 := [0.01, 10]$, $\beta \in \Theta_2 := [-10, -0.01] \times [-10, 10]$, and $\lambda = 10^{-7/3}$. The procedure of the simulation is as follows: in the first place we iterate an approximated OU process by Euler-Maruyama scheme (for example, see Iacus, 2008) with simulation parameters $n_{\text{sim}} = 10^{5+m}$, $h_{\text{sim}} = 10^{-10/3-m}$, $T_{\text{sim}} = 10^{5/3}$ where $m \in \mathbb{N}$ is a parameter to determine the precision of approximation; secondly, we give the approximation of convolution by summation such that

\[
\overline{X}_{ih_n,n} \approx \left\{ \begin{array}{ll}
\frac{1}{[10^m \rho]} \sum_{k=0}^{10^m-1} X_{ih_n-kh_{\text{sim}}} & \text{if } [10^m \rho] \geq 1, \\
X_{ih_n} & \text{if } [10^m \rho] < 0,
\end{array} \right.
\]

where $i = 0, \ldots, n$, the sampling frequency $h_n = 10^{-10/3}$ and $n = 10^{5/3}$. In this Section 5.1, we fix the true value of $\alpha$ and $\beta$ as $\alpha_* = 3$ and $\beta_* = [-2, 1]^T$, and change the true value of $\rho \in \Theta_\rho := [0, 100]$ to see the corresponding changes of performance of estimation for $\xi$, and test for $\rho$ in comparison to estimation by an existent method called local Gaussian approximation (LGA) for parametric estimation of discretely observed diffusion process.
processes (e.g., see Kessler, 1997) which does not concern convolutional observation. All the numbers of iterations for different $\rho$'s are 1000.

In the first place, we see the estimation and test with small values of $\rho$, such that $\rho = 0, 0.1, 0.2, \ldots, 1$ to observe how the performance of statistics changes by difference in $\rho$. Table 1 summarises the results of simulation of $\hat{\rho}_n$ for $\rho$'s with respective empirical means and root mean square error (RMSE). We can see the proposed estimator $\hat{\rho}_n$ works well for small $\rho$. With respect to the performance of the test statistic $T_n$ proposed in Section 3.2, Table 2 shows the empirical ratio of the number of iterations whose $T_n$ is lower than some typical critical values where $\Phi$ indicates the distribution function of 1-dimensional standard Gaussian distribution as well as the maximum value of $T_n$ in 1000 iterations. Even for $\rho = 0.1$, the simulation result supports the theoretical discussion of the

| $\rho$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 |
|-------|-----|-----|-----|-----|-----|
| mean  | 0.00990 | 0.0971 | 0.198 | 0.298 | 0.398 |
| RMSE  | (0.0182) | (0.0256) | (0.0235) | (0.0215) | (0.0197) |

| $\rho$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-------|-----|-----|-----|-----|-----|-----|
| mean  | 0.498 | 0.598 | 0.699 | 0.799 | 0.899 | 0.999 |
| RMSE  | (0.0180) | (0.0164) | (0.0150) | (0.0135) | (0.0123) | (0.0110) |

Table 1. Estimation performance of $\rho$ with small $\rho$.

| $\rho$ | $\Phi^{-1}(0.10)$ | $\Phi^{-1}(0.05)$ | $\Phi^{-1}(0.025)$ | $\Phi^{-1}(0.01)$ | $\Phi^{-1}(0.001)$ | max. of $T_n$ |
|-------|------------------|------------------|------------------|------------------|------------------|----------------|
| $\rho = 0.0$ | 0.101 | 0.053 | 0.025 | 0.005 | 0.000 | 3.060 |
| $\rho = 0.1$ | 0.989 | 0.980 | 0.966 | 0.914 | 0.759 | −0.710 |
| $\rho = 0.2$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −4.593 |
| $\rho = 0.3$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −9.341 |
| $\rho = 0.4$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −13.985 |
| $\rho = 0.5$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −19.152 |
| $\rho = 0.6$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −24.816 |
| $\rho = 0.7$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −30.848 |
| $\rho = 0.8$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −37.557 |
| $\rho = 0.9$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −44.829 |
| $\rho = 1.0$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | −52.759 |

Table 2. Empirical ratio of test statistic $T_n$ less than some critical values, and the maximum value of $T_n$ in 1000 iterations.

test with consistency. Because $\Phi(10^{-16}) = −8.222$, all the iterations with $\rho \geq 0.3$ result in rejection of $H_0$ with substantially significance level $10^{-16}$. Let us see the estimation for $\alpha$ and $\beta$ by our proposal method and that by the LGA in Table 3. Note that the biases of the estimation by LGA increase as the true value of $\rho$ gets larger, while the estimation by our proposal method is not influenced by the true value of $\rho$. This result of the simulation supports the theoretical discussion in Section 4 stating the consistency of $\hat{\theta}_n$, and necessity to consider the convolutional observation scheme where the LGA method does not work properly.
Secondly, we consider the estimation and test with large \( \rho \), such that \( \rho_\star = 10, 15, 20 \) to see if our proposal methods work even for large \( \rho \). We note that the maximum values of \( T_n \) for \( \rho = 10, 15, 20 \) in 1000 iterations are \(-55.091, -68.462\) and \(-79.105\), and hence we can see that the smoothed observation easily. Table 4 shows the empirical means and RMSEs of \( \hat{\rho}_n \) for \( \rho = 10, 15, 20 \) and we can see that the RMSEs increase as \( \rho \)'s increase; it indicates the difficulty to estimate \( \rho \) accurately when \( \rho_\star \) is large. Table 5 summarises the estimation for \( \theta \) by means and RMSE, and tells us that although the large RMSE of \( \hat{\rho}_n \) results in increase of RMSE of \( \hat{\alpha}_n \), estimation by our method is substantially better than that by LGA of course.

### 5.2. 2-dimensional simulation

We consider the following 2-dimensional stochastic differential equation whose solution is a 2-dimensional OU process:

\[
d \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \end{bmatrix} = \begin{bmatrix} \beta^{(1)} & \beta^{(3)} \\ \beta^{(4)} & \beta^{(6)} \end{bmatrix} \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \end{bmatrix} + \begin{bmatrix} \alpha^{(1)} \\ \alpha^{(2)} \end{bmatrix} \ d \omega_t, X_{-\lambda} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
$\lambda = 10^{-7/3}$. The simulation is conducted with the settings as follows: firstly, we iterate the OU process by Euler-Maruyama scheme with the simulation sample size $n_{\text{sim}} = 10^{5+m}$, $T_{\text{sim}} = 10^{5/3}$ and discretisation step $h_{\text{sim}} = 10^{-10/3-m}$, where $m = 2$ is the precision parameter for approximation of convolution; in the second place, we approximate the convoluted process with summation such that

$$X_{ih_n,m}^{(j)} \approx \begin{cases} \frac{1}{[10^m \rho^{(j)}]} \sum_{k=0}^{10^m-1} X_{ih_n-kh_{\text{sim}}}^{(j)} & \text{if } [10^m \rho^{(j)}] \geq 1, \\ X_{ih_n}^{(j)} & \text{if } [10^m \rho^{(j)}] < 0, \end{cases}$$

where $i = 0, \ldots, n$, $j = 1, 2$, the sampling scheme for inference is defined as $n = 10^5$ and $h = 10^{-10/3}$; the true value of $\rho$, $\alpha$ and $\beta$ are set as $\rho_* = [2, 4]^T$, $\alpha_* = [2, 0, 3]^T$, $\beta_* = [-2, -0.4, 0, 0.1, -3, 5]^T$; the parameter spaces are defined as $\Theta = [0, 10]^2$, $\Theta_1 = [1 + 10^{-8}, 10] \times [-1 + 10^{-8}, 1 - 10^{-8}] \times [1 + 10^{-8}, 10]$, and $\Theta_2 = [-10, 10]^6$; the total iteration number is set to 1000.

Table 6 summarises the estimation for $\rho$ with the method proposed in Section 3 (the inverse of $r$ is computationally obtained) with empirical means and empirical RMSEs of $\hat{\rho}_n$ in 1000 iterations. We can see that $\hat{\rho}_n$ is sufficiently precise to estimate the true value of $\rho$ indeed in this result, which is significant to estimate the other parameters $\alpha$ and $\beta$.

We also note that the maximum values of the test statistics for smoothed observation proposed in Section 3.2 in 1000 iterations are $-17.947$ and $-33.159$ for each axis. The $p$-value for them are smaller than $10^{-16}$; therefore, we can conclude that it is possible to detect the smoothed observation with the proposed test statistic in the case $\rho^{(i)} = 2.0$ if $d = 2$ from this result.

With respect to the estimation for $\alpha$ and $\beta$, we compare the estimates by our proposal method with that by LGA which does not concern convolutional observation. Table 7 is the summary for $\alpha$ estimate by both the methods: we can see that the estimation precision for $\alpha$ by our proposal outperforms those by LGA. This results support validity of our estimation method if we have convolutional observation for diffusion processes. Regarding $\beta$, the simulation result is summarised in Table 8: though the estimation

| $\rho = 10$ | mean \(\alpha\) & \(\beta^{(1)}\) & \(\beta^{(2)}\) | LGA | \(\alpha\) & \(\beta^{(1)}\) & \(\beta^{(2)}\) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| mean \(\rho = 10\) | 2.989 & -2.101 & 1.030 | \(0.0347\) & \(0.323\) & \(0.496\) | 0.933 & -0.204 & 0.0811 |
| RMSE | \(0.0347\) & \(0.323\) & \(0.496\) | \(2.067\) & \(1.796\) & \(0.920\) |
| $\rho = 15$ | mean \(\rho = 15\) | 2.996 & -2.095 & 1.027 | \(0.0475\) & \(0.321\) & \(0.495\) | 0.765 & -0.138 & 0.0473 |
| RMSE | \(0.0475\) & \(0.321\) & \(0.495\) | \(2.235\) & \(1.862\) & \(0.953\) |
| $\rho = 20$ | mean \(\rho = 20\) | 2.977 & -2.090 & 1.024 | \(0.0526\) & \(0.319\) & \(0.493\) | 0.664 & -0.104 & 0.0302 |
| RMSE | \(0.0526\) & \(0.319\) & \(0.493\) | \(2.336\) & \(1.896\) & \(0.970\) |

Table 5. Estimation of $\theta$ by the proposed method with large $\rho$
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\[ \alpha \]

true value 2.0 0.0 3.0
Our proposal mean 1.993 0.000256 2.992
RMSE (0.0115) (0.00739) (0.0213)
LGA mean 1.295 −0.00320 1.442
RMSE (0.705) (0.0154) (1.558)

Table 7. summary for \( \alpha \) estimate

for \( \beta^{(3)} \) by our method has the smaller bias in comparison to that by LGA, the RMSE of our method is larger than that of LGA; in the estimation for other parameters, our proposal method outperforms the method by LGA. We can conclude that our proposal for estimation of \( \alpha \) and \( \beta \) concerning convolutional observation performs better than that not considering this observation scheme.

6. Real data analysis

In this section, we analyse the EEG dataset named S02E.mat provided in “2. Two class motor imagery (002-2014)” of BNCI Horizon 2020 (2014). The datasets including S02E.mat are also studied by Steyrl et al. (2016).

6.1. Estimation and test for the smoothing parameters. In the first place, we pick up the first 15 axes of the dataset and compute \( \hat{\rho}_n \) and \( T_n \) proposed in Section 3.1 and 3.2 respectively. The results are shown in Table 9. We can observe that all the 15 time series data have the smoothing parameter \( \rho > 0 \) with statistical significance when we assume ordinary significance levels. These results motivate us to use our methods for parametric

| true value | \( \beta^{(1)} \) | \( \beta^{(2)} \) | \( \beta^{(3)} \) | \( \beta^{(4)} \) | \( \beta^{(5)} \) | \( \beta^{(6)} \) |
|------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Our proposal | mean | −2.137 | −0.408 | −0.0439 | 0.0788 | −3.103 | 5.091 |
| | RMSE | (0.362) | (0.252) | (0.540) | (0.473) | (0.399) | (0.777) |
| LGA | mean | −0.917 | 0.340 | −0.326 | −0.696 | 0.221 | 1.243 |
| | RMSE | (1.093) | (0.802) | (0.386) | (0.804) | (3.242) | (3.765) |

Table 8. summary for \( \beta \) estimate

| | 1st axis | 2nd axis | 3rd axis | 4th axis | 5th axis |
|-----------------|----------|----------|----------|----------|----------|
| \( \hat{\rho}_n \) | 0.449 | 1.037 | 0.894 | 0.736 | 0.937 |
| \( T_n \) | −20.398 | −58.631 | −46.649 | −35.201 | −49.741 |
| 6th axis | 7th axis | 8th axis | 9th axis | 10th axis |
| \( \hat{\rho}_n \) | 0.951 | 0.971 | 1.017 | 0.958 | 0.967 |
| \( T_n \) | −51.392 | −52.607 | −55.455 | −51.221 | −51.797 |
| 11th axis | 12th axis | 13th axis | 14th axis | 15th axis |
| \( \hat{\rho}_n \) | 0.949 | 0.649 | 0.952 | 0.977 | 0.932 |
| \( T_n \) | −50.457 | −30.094 | −50.633 | −50.978 | −48.842 |

Table 9. The values of \( \hat{\rho}_n \) and \( T_n \) for the first 15 axes of S02.mat by BNCI Horizon 2020 (2014).
estimation proposed in Section 4 when we fit stochastic differential equations for these data.

6.2. Parametric estimation for a diffusion process. We fit a 1-dimensional OU process for the time series data in the 2nd column of the data file S02E.mat with 512Hz observation for 222 seconds (the plot of the path can be seen in Figure 1), whose $\hat{\rho}_n = 1.037$ is the largest among those for the 15 axes and it is larger than 0 with statistical significance. According to the simulation result shown in Section 5.1, this size of the smoothing parameter gives critical biases when we estimate $\alpha$ and $\beta$ with LGA method not concerning convolutional observation scheme.

The stochastic differential equation with parameters $\alpha \in \Theta_1 := [0.01, 200]$ and $\beta \in \Theta_2 := [-100, -0.01] \times [-100, 100]$ is defined as follows:

$$dX_t = (\beta^{(1)} X_t + \beta^{(2)}) \, dt + \alpha dw_t, \ X_{-\lambda} = x_{-\lambda}.$$ 

We set 5 seconds as the time unit: hence $n = 113664$ and $h_n = 1/(5 \times 512)$. If we fit the OU process with the LGA method, i.e., we do not concern convolutional observation scheme, we obtain the fitting result such that

$$dX_t = ((-17.378) X_t + (-1.091)) \, dt + (122.892) \, dw_t, \ X_{-\lambda} = x_{-\lambda}.$$ 

In the next place, we fit $\alpha$ and $\beta$ with the least square method proposed in Section 4, and then we have the next fitting result:

$$dX_t = ((-2.146) X_t + (0.552)) \, dt + (151.919) \, dw_t, \ X_{-\lambda} = x_{-\lambda}.$$ 

It is worth noting that this fitting result is substantially different to that by LGA as shown above: hence these results indicate the significance to examine if the observation is convoluted with the smoothing parameter $\rho > 0$ and otherwise the estimation is strongly biased.

7. Summary

We have discussed the convolutional observation scheme which deals with the smoothness of observation in comparison to ordinary diffusion processes. The first contribution is to propose this new observation scheme with the statistical test to confirm whether this scheme is valid in real data. The second one is to prove consistency of the estimator $\hat{\rho}_n$ for the smoothing parameter $\rho$, those for parameters in diffusion and drift coefficient, i.e., $\alpha$ and $\beta$, of the latent diffusion process $\{X_t\}$. Thirdly, we have examined the performance of those estimators and the test statistics in computational simulation, and verified these statistics work well in realistic settings. In the fourth place, we have shown a real example of observation where $\rho \neq 0$ holds with statistical significance.

These contributions, especially the third one, will cultivate the motivation to study statistical approaches for convolutionally observed diffusion processes furthermore, such as estimation of kernel function $V$ appearing in the convoluted diffusion $\tilde{X}_t := (V * X)(t)$, test theory for parameters $\alpha$ and $\beta$ as likelihood-ratio-type test statistics (for example, see Kitagawa and Uchida, 2014; Nakakita and Uchida, 2019b), large deviation inequalities for quasi-likelihood functions and associated discussion of Bayes-type estimators (e.g., Yoshida, 2011; Ogihara and Yoshida, 2011; Clinet and Yoshida, 2017; Nakakita and Uchida, 2018). By these future works, it is expected that the applicability of stochastic differential equations in real data analysis and contributions to the areas with high frequency observation of phenomena such as EEG will be enhanced.
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**Appendix A. Proofs for preliminary lemmas and main results**

**Nonasymptotic results.** We assume $\Delta \leq \lambda$, $k \in \mathbb{N}$, $M > 0$, and consider a class of $\mathbb{R}^k \otimes \mathbb{R}^d$-valued kernel functions on $\mathbb{R}$ denoted as $\mathcal{K}(\Delta, k, M)$ such that for all $\Phi_{\Delta} \in \mathcal{K}(\Delta, k, M)$, it holds:

(i) $\text{supp} \Phi_{\Delta} \subset [0, \Delta]$,

(ii) for all $f : [0, \Delta] \times \Omega \to \mathbb{R}^k, \omega \in \Omega$, $\left| \int_0^{\Delta} \Phi_{\Delta} (\Delta - s) f (s, \omega) \mathrm{d}s \right| \leq M \sup_{s \in [0, \Delta]} |f (s, \omega)|$

(iii) for all $t_0 \geq -\lambda, f : \mathbb{R}^d \to \mathbb{R}$ which is continuous and at most polynomial growth,

$$
\mathbb{E} \left[ \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) f (X_s) \mathrm{d}s \bigg| \mathcal{F}_{t_0} \right] = \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) \mathbb{E} [f (X_s)] \mathcal{F}_{t_0} \mathrm{d}s.
$$

**Remark 2.** Note one sufficient condition for $\Phi_{\Delta} \in \mathcal{K}(\Delta, k, M)$ is (i) $\Phi_{\Delta} : \mathbb{R} \to \mathbb{R}^k \otimes \mathbb{R}^d$,

(ii) $\text{supp} \Phi_{\Delta} \subset [0, \Delta]$, (iii) $\int_0^{\Delta} \| \Phi_{\Delta} (\Delta - s) \| \mathrm{d}s \leq M$ and (iv) $\mathcal{B} ([0, \Delta])-\text{measurable}$ since

$$
\left| \int_0^{\Delta} \Phi_{\Delta} (\Delta - s) f (s, \omega) \mathrm{d}s \right| \leq \int_0^{\Delta} \| \Phi_{\Delta} (\Delta - s) \| |f (s, \omega)| \mathrm{d}s \leq M \sup_{s \in [0, \Delta]} |f (s, \omega)|
$$

for Cauchy-Schwarz inequality, and Fubini’s theorem.

It is easily checked that $V_{\rho,h_n} \in \mathcal{K}(\max_{i=1,...,d} \rho_i h_n, d, d)$. 

The next theorem is a generalization of Proposition 2.2 of Gloter (2000), Theorem 1 of Gloter (2006) and Corollary 1 of Nakakita and Uchida (2019a).

**Theorem 7.** Set \( t \geq 0, \Delta \in (0, \lambda], k \in \mathbb{N}, M > 0, \Phi_\Delta, \Psi_\Delta \in \mathcal{K}(\Delta, k, M), \) and assume \([A1]\). Then we have

\[
\begin{align*}
\int_{t-\Delta}^{t} \Phi_\Delta(t-s) X_s ds &= \left( \int_{0}^{\Delta} \Phi_\Delta(\Delta-s) ds \right) X_{t-\Delta} + \left( \int_{0}^{\Delta} \Phi_\Delta(\Delta-s) s ds \right) b(X_{t-\Delta}) \\
&\quad + \int_{t-\Delta}^{t} \Phi_\Delta(t-s) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) dw_{s_2} \right) ds_1 + e_{t-\Delta},
\end{align*}
\]

where \( e_{t-\Delta} \) is an \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-measurable random variable such that

\[
\begin{align*}
(i) \quad |E[|e_{t-\Delta}|]| &\leq C(M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)}, \\
(ii) \quad \text{for all } m > 0, \quad E[|e_{t-\Delta}|^m] &\leq C(m, M) \Delta^m (1 + |X_{t-\Delta}|)^{C(m,M)}, \\
(iii) \quad \left| E \left[ e_{t-\Delta} \left| \int_{t-\Delta}^{t} \Psi_\Delta(t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) dw_{s_2} \right) ds_1 \right] \mathcal{F}_t \right| &\leq C(M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)}. \quad (3)
\end{align*}
\]

The following corollaries correspond more directly to Proposition 2.2 and Theorem 2.3 of Gloter (2000), or Proposition 1 and Theorem 1 of Gloter (2006).

**Corollary 8.** Set \( t \geq 0, \Delta \in (0, \lambda], M > 0 \) and \( \Phi_\Delta \in \mathcal{K}(\Delta, d, M) \). We assume \([A1]\) and \( \int_{0}^{\Delta} \Phi_\Delta(\Delta-s) ds = I_d \). Then we obtain

\[
\begin{align*}
\int_{t-\Delta}^{t} \Phi_\Delta(t-s) X_s ds &= X_{t-\Delta} + \int_{t-\Delta}^{t} \Phi_\Delta(t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) dw_{s_2} \right) ds_1 + e_{t-\Delta},
\end{align*}
\]

where \( e_{t-\Delta} \) is an \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-measurable random variable such that

\[
\begin{align*}
(i) \quad |E[|e_{t-\Delta}|]| &\leq C(M) \Delta (1 + |X_{t-\Delta}|)^{C(M)}, \\
(ii) \quad \text{for all } m > 0, \quad E[|e_{t-\Delta}|^m] &\leq C(m, M) \Delta^m (1 + |X_{t-\Delta}|)^{C(m,M)}. \quad (4)
\end{align*}
\]

**Remark 3.** This corollary leads to the same result as Proposition 2.2 of Gloter (2000) by setting \( \Phi_\Delta(s) = I_d(1_{[0,\Delta]}(s) / \Delta) \); we can see \( \Phi_\Delta \in \mathcal{K}(\Delta, d, \sqrt{d}) \) because \( \int_{0}^{\Delta} \|\Phi_\Delta(s)\| ds = \sqrt{d} \). We have the following equalities

\[
\begin{align*}
\int_{0}^{\Delta} \Phi_\Delta(s) ds &= I_d, \\
\int_{t-\Delta}^{t} \Phi_\Delta(t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) dw_{s_2} \right) ds_1 &= \int_{t-\Delta}^{t} \left( \int_{s_1}^{t} \Phi_\Delta(t-s_2) ds_2 \right) a(X_{t-\Delta}) dw_{s_1} \\
&= \frac{1}{\Delta} a(X_{t-\Delta}) \int_{t-\Delta}^{t} (t-s_1) dw_{s_1}.
\end{align*}
\]

Then we can see that this result is identical to that of Gloter (2000).

**Corollary 9.** Set \( t \geq 0, \Delta \in (0, \lambda], M > 0, \Phi_\Delta \in \mathcal{K}(\Delta, d, M), \) and \( f(x, \xi) \) is a real-valued function such that \( f : \mathbb{R}^d \times \Xi \to \mathbb{R} \), and \( f, \partial_x f \) and \( \partial_x^2 f \) are polynomial growth
Corollary 10. It is obvious that $\Phi_{\text{ph}} \xi_1 \in \mathcal{S} H \text{NAKAKITA AND UCHIDA (2019a)}$; let us set $p$ for all $4$

Remark 4. We see that this result generalises Proposition 4 of Nakakita and Uchida (2019a); let us set $p \in \mathbb{N}$ and $h > 0$ such that $\text{ph} \leq \lambda$, and $\Phi_{\text{ph}}$ as follows:

$$\Phi_{\text{ph}} (s) = \frac{1}{p} \sum_{i=0}^{p-1} \delta (ih - s) I_d.$$ 

It is obvious that $\Phi_{\text{ph}} \in \mathcal{K} (\text{ph}, d, d)$ and $\int_0^{\text{ph}} \Phi_{\text{ph}} (\text{ph} - s) \, ds = I_d.$

Corollary 10. Set $t \geq 0, \Delta \in (0, \lambda], k \in \mathbb{N}, M > 0$ and $\Phi_{\Delta} \in \mathcal{K} (\Delta, k, M)$. We assume that $\int_0^{\Delta} \Phi_{\Delta} (\Delta - s) \, ds = O$. Then we obtain

$$\int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) \, X_s \, ds = \left( \int_0^{\Delta} \Phi_{\Delta} (\Delta - s) \, sds \right) b (X_{t-\Delta})$$

$$+ \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s_1) \left( \int_{t-\Delta}^{s_1} a (X_{t-\Delta}) \, dw_{s_2} \right) \, ds_1 + e_{t-\Delta,\Delta},$$

where $e_{t-\Delta,\Delta}$ is an $\mathbb{R}^k$-valued $\mathcal{F}_t$-measurable random variable such that

$$(i) \quad |\mathbb{E} [e_{t-\Delta,\Delta} | \mathcal{F}_{t-\Delta}]| \leq C (M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)}, \quad (6)$$

(ii) for all $m > 0$, $\mathbb{E} [|e_{t-\Delta,\Delta}|^m | \mathcal{F}_{t-\Delta}] \leq C (m, M) \Delta^m (1 + |X_{t-\Delta}|)^{C(m,M)}, \quad (7)$

(iii) $|\mathbb{E} [e_{t-\Delta,\Delta} \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s_1) \left( \int_{t-\Delta}^{s_1} a (X_{t-\Delta}) \, dw_{s_2} \right) \, ds_1 \right) | \mathcal{F}_{t-\Delta}]|^m$ 

$$\leq C (M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)}. \quad (8)$$

Remark 5. We can have a result similar to Theorem 3.2 of Gloter (2000) if we assume that $\Delta' = \Delta / 2$ and $\Phi_{\Delta} (s) = I_d \left( \mathbb{1}_{[0, \Delta']} (s) - \mathbb{1}_{[\Delta', \Delta]} (s) \right) / \Delta'$ where $\Delta \leq \lambda$ and $k = d$. $\Phi_{\Delta} (s) \in \mathcal{K} (\Delta, d, 2\sqrt{d})$ because $\int_0^{\Delta} |\Phi_{\Delta} (\Delta - s)| \, ds = 2\sqrt{d}$. We have the following
equalities
\[ \int_0^\Delta \Phi_\Delta (\Delta - s) \, ds = O, \]
\[ \int_0^\Delta \Phi_\Delta (\Delta - s) \, s \, ds = I_d \left( \frac{1}{\Delta} \int_0^\Delta s \, ds - \frac{1}{\Delta} \int_0^{\Delta'} s \, ds \right) = I_d \left( \Delta - \frac{\Delta'}{2} - \frac{\Delta'}{2} \right) = \Delta' I_d, \]
\[ \int_{t-\Delta}^t \Phi_\Delta (t - s_1) \left( \int_{s_1}^{s_1} a (X_{t-\Delta}) \, dw_{s_2} \right) \, ds_1 \]
\[ = \int_{t-\Delta}^t \left( \int_{s_1}^{t} \Phi_\Delta (t - s_2) \, ds_2 \right) a (X_{t-\Delta}) \, dw_{s_1} \]
\[ = \frac{1}{\Delta} a (X_{t-\Delta}) \int_{t-\Delta}^t \left( \int_{s_1}^{t} (1_{[0,\Delta']} (t - s_2) - 1_{[\Delta',\Delta]} (t - s_2)) \, ds_2 \right) \, dw_{s_1} \]
\[ = \frac{1}{\Delta} a (X_{t-\Delta}) \int_{t-\Delta}^t \left( (t - s_1) 1_{[0,\Delta']} (t - s_1) \right) \, dw_{s_1} \]
\[ = \frac{1}{\Delta} a (X_{t-\Delta}) \int_{t-\Delta}^t \left( t - \Delta' - s_1 \right) 1_{[0,\Delta']} \, dw_{s_1} \]
\[ = \frac{1}{\Delta} a (X_{t-\Delta}) \int_{t-\Delta}^t \left( \int_{s_1}^{t-\Delta'} (s_1 - (t - \Delta)) \, ds_1 + \int_{t-\Delta'}^t (t - s_1) \, ds_1 \right) \]

because of Lemma 14 in Appendix B. Hence this result and Corollary 9 give the same evaluation as Gloter (2000).

Remark 6. This corollary also generalises Corollary 1 of Nakakita and Uchida (2019a) when we ignore noise term; let us set \( p \in \mathbb{N} \) and \( h > 0 \) such that \( 2ph \leq \lambda \), and \( \Phi_{2ph} \) as follows:

\[ \Phi_{2ph} (s) = \frac{1}{p} \sum_{i=0}^{p-1} \delta (s - ih) I_d - \frac{1}{p} \sum_{i=0}^{p-1} \delta (s - (p + i) h) I_d. \]

It is obvious that \( \Phi_{2ph} \in \mathbb{K} (2ph, d, 2d) \) and \( \int_0^{2ph} \Phi_{2ph} (2ph - s) \, ds = O \). We can evaluate

\[ \int_0^{2ph} \Phi_{2ph} (2ph - s) \, ds = \frac{1}{p} \sum_{i=0}^{p-1} (2ph - ih) I_d - \frac{1}{p} \sum_{i=0}^{p-1} (2ph - (p + i) h) I_d = ph I_d, \]
\[ \int_{t-2ph}^t \Phi_{2ph} (t - s_1) \left( \int_{s_1}^{s_1} a (X_{t-2ph}) \, dw_{s_2} \right) \, ds_1 \]
\[ = a (X_{t-2ph}) \left( \frac{1}{p} \sum_{i=0}^{p-1} \int_{t-2ph}^{t-ih} dw_s - \frac{1}{p} \sum_{i=0}^{p-1} \int_{t-2ph}^{t-(p+i)h} dw_s \right) \]
\[ = a (X_{t-2ph}) \left( \frac{1}{p} \sum_{i=0}^{p-1} (i + 1) \int_{t-(2p-i)h}^{t-(p-i)h} dw_s + \frac{1}{p} \sum_{i=0}^{p-1} (p - 1 - i) \int_{t-(p-i)h}^{t-(p-i)h} dw_s \right). \]
Proof of Theorem 7. We have
\[
\int_{t-\Delta}^{t} \Phi_{\Delta} (t-s) X_s \, ds
= \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( X_{t-\Delta} + \int_{t-\Delta}^{s_1} b(X_{s_2}) \, ds_2 + \int_{t-\Delta}^{s_1} a(X_{s_2}) \, dw_{s_2} \right) \, ds_1
= \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \, ds \right) X_{t-\Delta} + \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} b(X_{t-\Delta}) \, ds_2 \right) \, ds_1
+ \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) \, dw_{s_2} \right) \, ds_1
+ \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) \, ds_1
+ \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (a(X_{s_2}) - a(X_{t-\Delta})) \, dw_{s_2} \right) \, ds_1.
\]

Set \( e_{t-\Delta, \Delta} := e_{t-\Delta, \Delta, 1} + e_{t-\Delta, \Delta, 2} \) where
\[
e_{t-\Delta, \Delta, 1} := \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) \, ds_1,
\]
\[
e_{t-\Delta, \Delta, 2} := \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (a(X_{s_2}) - a(X_{t-\Delta})) \, dw_{s_2} \right) \, ds_1.
\]

We examine that the properties (i)-(iii) hold for this \( e_{t-\Delta, \Delta} \). The assumption of \( \Phi_{\Delta} \) and martingale property of stochastic integral verify
\[
\mathbb{E} [e_{t-\Delta, \Delta, 2} | \mathcal{F}_{t-\Delta}] = 0.
\]

Then, in order to show (i) and (ii), it is sufficient to prove the following inequalities.
\[
\mathbb{E} [e_{t-\Delta, \Delta, 1}] \leq C(M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)},
\]
\[
\mathbb{E} [e_{t-\Delta, \Delta, 1}^m | \mathcal{F}_{t-\Delta}] \leq C(m, M) \Delta^{2m/2} (1 + |X_{t-\Delta}|)^{C(m, M)},
\]
\[
\mathbb{E} [e_{t-\Delta, \Delta, 2}^m | \mathcal{F}_{t-\Delta}] \leq C(m, M) \Delta^m (1 + |X_{t-\Delta}|)^{C(m, M)}.
\]

In the first place, we have
\[
\mathbb{E} \left[ \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) \, ds_1 | \mathcal{F}_{t-\Delta} \right]
= \int_{t-\Delta}^{t} \mathbb{E} \left[ \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) | \mathcal{F}_{t-\Delta} \right] \, ds_1
= \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \mathbb{E} \left[ \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) | \mathcal{F}_{t-\Delta} \right] \, ds_1
= \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \mathbb{E} \left[ \left( \int_{t-\Delta}^{s_1} \left( b(X_{s_2}) - b(X_{t-\Delta}) \right) \, ds_2 \right) | \mathcal{F}_{t-\Delta} \right] \, ds_1
\leq C(M) \sup_{s_1 \in [t-\Delta, t]} \int_{t-\Delta}^{s_1} \mathbb{E} \left[ \left( b(X_{s_2}) - b(X_{t-\Delta}) \right) | \mathcal{F}_{t-\Delta} \right] \, ds_2.
\]
because of Proposition A of Gloter (2000). Secondly, we obtain

\[ \leq C(M) \Delta \sup_{s \in [t-\Delta, t]} |\mathbb{E}[b(X_s) - b(X_{t-\Delta})|\mathcal{F}_{t-\Delta}]| \]

\[ \leq C(M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)} \]

by \( \Phi_{\Delta} \in K(\Delta, k, M) \); therefore, we obtain

\[ \mathbb{E} \left[ \left| \int_{t-\Delta}^t \Phi_{\Delta}(t-s_1) \left( \int_{t-\Delta}^{s_1} (b(X_{s_2}) - b(X_{t-\Delta})) \, ds_2 \right) \, ds_1 \right|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^m \mathbb{E} \left[ \sup_{s \in [t-\Delta, t]} |b(X_s) - b(X_{t-\Delta})|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^{3m/2} (1 + |X_{t-\Delta}|)^{C(m,M)} \]

due to Proposition 5.1 of Gloter (2000). Thirdly, by Hölder’s inequality and the Burkholder-Davis-Gundy one, and Proposition A of Gloter (2000),

\[ \mathbb{E} \left[ \left| \int_{t-\Delta}^t \Phi_{\Delta}(t-s_1) \left( \int_{t-\Delta}^{s_1} (a(X_{s_2}) - a(X_{t-\Delta})) \, dw_{s_2} \right) \, ds_1 \right|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \mathbb{E} \left[ \sup_{s \in [t-\Delta, t]} \int_{t-\Delta}^{s_1} (a(X_{s_2}) - a(X_{t-\Delta})) \, dw_{s_2} \right|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^{m/2} \mathbb{E} \left[ \|a(X_s) - a(X_{t-\Delta})\|^2 |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^{m/2} \mathbb{E} \left[ \|a(X_s) - a(X_{t-\Delta})\|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^m (1 + |X_{t-\Delta}|)^{C(m,M)} . \]

To show (iii), it is obvious that

\[ \left| \mathbb{E} \left[ e_{t-\Delta, t} \left( \int_{t-\Delta}^t \Phi_{\Delta}(t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{s_2}) \, dw_{s_2} \right) \, ds_1 \right) \right] \right| \]

\[ \leq C(M) \Delta^2 (1 + |X_{t-\Delta}|)^{C(M)} \]

due to Hölder’s inequality and the evaluation analogous to \( \mathbb{E} [\|e_{t-\Delta, t}\|^m |\mathcal{F}_{t-\Delta}] \) such that

\[ \mathbb{E} \left[ \left| \int_{t-\Delta}^t \Phi_{\Delta}(t-s_1) \left( \int_{t-\Delta}^{s_1} a(X_{t-\Delta}) \, dw_{s_2} \right) \, ds_1 \right|^m |\mathcal{F}_{t-\Delta} \right] \]

\[ \leq C(m, M) \Delta^{m/2} (1 + |X_{t-\Delta}|)^{C(m,M)} . \]
Hence it is sufficient to show
\[
\left| \mathbb{E} \left[ e_{t-\Delta,2} \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a \left( X_{t-\Delta} \right) \, dw_{s_2} \right) \, ds_1 \right) \right] \right| \leq C (M) \Delta^2 \left( 1 + |X_{t-\Delta}| \right)^{C(M)} .
\]

We can evaluate the left hand side such that
\[
\left| \mathbb{E} \left[ \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \, dw_{s_2} \right) \, ds_1 \right] \right| = \left| \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \, dw_{s_2} \right) \, ds_1 \right| 
\leq C (M) \sup_{s_1 \in [t-\Delta,t]} \left| \mathbb{E} \left[ \int_{t-\Delta}^{s_1} a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \, dw_{s_2} \right] \right| \left| \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \, dw_{s_2} \right) \, ds_1 \right|
\leq C (M) \Delta \sup_{s \in [t-\Delta,t]} \left| \mathbb{E} \left[ \left( a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \right) \left( a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \right) \, ds_2 \right] \right| \left| \int_{t-\Delta}^{t} \Phi_{\Delta} (t-s_1) \left( \int_{t-\Delta}^{s_1} a \left( X_{s_2} \right) - a \left( X_{t-\Delta} \right) \, dw_{s_2} \right) \, ds_1 \right|
\leq C (M) \Delta^2 \left( 1 + |X_{t-\Delta}| \right)^{C(M)} .
\]

Hence we obtain the proof of (iii).
**Proof of Corollary 9.** By Taylor’s expansion,

\[
f \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds, \xi \right) - f (X_{t-\Delta}, \xi) = \partial_{x} f (X_{t-\Delta}, \xi) \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right) \\
+ \int_{0}^{1} (1 - u) \partial_{x}^{2} f \left( X_{t-\Delta} + u \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right), \xi \right) du \\
\left[ \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right) \right]^{\otimes 2}.
\]

It is obvious that

\[
\sup_{\xi \in \Xi} \left| \mathbb{E} \left[ \partial_{x} f (X_{t-\Delta}, \xi) \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right) \right] \right| \\
\leq \sup_{\xi \in \Xi} \left| \partial_{x} f (X_{t-\Delta}, \xi) \right| \left| \mathbb{E} \left[ \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right] \right| \\
\leq C (M) (1 + |X_{t-\Delta}|) \mathbb{E} (1 + |X_{t-\Delta}|) \mathbb{E} \left[ |e_{t-\Delta}| \right] \\
\leq C (M) \Delta (1 + |X_{t-\Delta}|) \mathbb{E} \left[ |e_{t-\Delta}| \right]
\]

by Corollary 8. We also have

\[
\sup_{\xi \in \Xi} \left| \mathbb{E} \left[ \int_{0}^{1} (1 - u) \partial_{x}^{2} f \left( X_{t-\Delta} + u \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right), \xi \right) du \right] \right| \\
\leq \sup_{\xi \in \Xi} \mathbb{E} \left[ \left| \int_{0}^{1} (1 - u) \partial_{x}^{2} f \left( X_{t-\Delta} + u \left( \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right), \xi \right) du \right| \right] \\
\leq \mathbb{E} \left[ \left| \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right|^{2} \right] \\
\leq \mathbb{E} \left[ \left| \int_{t-\Delta}^{t} \Phi_{\Delta} (t - s) X_{s} ds - X_{t-\Delta} \right|^{2} \right] \\
\leq C (M) \Delta (1 + |X_{t-\Delta}|) \mathbb{E} \left[ |e_{t-\Delta}| \right]
\]

because of Corollary 8 and Proposition 5.1 of Gloter (2000). Here we obtain the first evaluation. With respect to the second one, we can have the following evaluation as
We define \( G \) a function on a set \( \{ \sum \{ \left. \Phi \right| |F| \leq n \} \} \) and there exist a matrix \( (\Phi, B) \) such that \( \Phi^2 + (1 + |x|)^2 \neq 0 \), a set \( L \subset \{ 0, \ldots, p \} \) such that there exist functions \( D_\ell : \mathbb{R}^d \to \mathbb{R}^d \) for \( \ell \in L \) such that

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) sds - h_n B \right) \right] \leq C h_n^2 (1 + |x|)^2,
\]

a function \( G : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \right]
\]

\[
\leq C h_n^2 (1 + |x|)^C,
\]

a function \( G : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \right]
\]

\[
\leq C h_n^2 (1 + |x|)^C.
\]

We define

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \right]
\]

\[
\leq C h_n^2 (1 + |x|)^C.
\]

Proofs of the results for some laws of large numbers.

General results. Let \( p \) denote an integer such that \( \sup_{n \in \mathbb{N}} ph_n \leq \lambda, \Delta_n := ph_n \). We set the sequence of the kernels \( \{ \Phi_{\Delta_n,n} \}_{n \in \mathbb{N}} \) such that \( \Phi_{\Delta_n,n} \in K (\Delta_n, m, d) \) for some \( M > 0 \),

\[
\int_0^{\Delta_n} \Phi_{\Delta_n,n} ds = I_d \text{ and there exist a matrix } B \in \mathbb{R}^d \otimes \mathbb{R}^d \text{ such that}
\]

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) sds - h_n B \right) \right] \leq C h_n^2 (1 + |x|)^2,
\]

a set \( L \subset \{ 0, \ldots, p \} \) such that there exist functions \( D_\ell : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) for \( \ell \in L \) such that

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \right]
\]

\[
\leq C h_n^2 (1 + |x|)^C,
\]

a function \( G : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\mathbb{E} \left[ \left( \int_0^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s) - \Phi_{\Delta_n,n}(\Delta_n-s)) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \right]
\]

\[
\leq C h_n^2 (1 + |x|)^C.
\]

Hence the proof is complete. \( \square \)
and the following random quantities such that
\[ \nu_n(f(\cdot, \xi)) := \frac{1}{n} \sum_{i=1}^{n} f(X_{ih_n, n}, \xi); \]
\[ \bar{T}_{\ell, n}(v(\cdot, \xi)) := \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v(X_{(i-1-\ell)h_n, n}, \xi) [X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b(X_{(i-1-\ell)h_n, n})], \]
\[ \bar{Q}_n(M(\cdot, \xi)) := \frac{1}{nh_n} \sum_{i=1}^{n} M(X_{(i-1)h_n, n}, \xi) [(X_{ih_n, n} - X_{(i-1)h_n, n})^2], \]
where \( f: \mathbb{R}^d \times \Xi \to \mathbb{R}, v: \mathbb{R}^d \times \Xi \to \mathbb{R}^d, M: \mathbb{R}^d \times \Xi \to \mathbb{R}^d \otimes \mathbb{R}^d \) are in \( C^2 \)-class, and their first and second derivatives and themselves are at most polynomial growth uniformly in \( \xi \in \Xi \).

**Proposition 11.** Under \([A1] \), \( \nu_n(f(\cdot, \xi)) \to^P \nu_0(f(\cdot, \xi)) \) uniformly in \( \xi \in \Xi \).

**Proposition 12.** If \( \ell \in \mathbb{L} \) and \([A1] \) hold, \( \bar{T}_{\ell, n}(v(\cdot, \xi)) \to^P \nu_0(\partial_x v[D_T^\ell](\cdot, \xi)) \) uniformly in \( \xi \in \Xi \).

**Remark 7.** When \( \Phi_{\Delta, n}(s) = \frac{1}{h_n} 1_{[0, h_n]}(s) I_d, p = 1, \; M = 1 \), then
\[ \int_0^{2h_n}(\Phi_{\Delta, n}(2h_n - s) - \Phi_{\Delta, n}(h_n - s)) sds = \frac{1}{h_n} I_d \int_{h_n}^{2h_n} sds - \frac{1}{h_n} I_d \int_0^{h_n} sds \]
\[ = \frac{1}{h_n} I_d \left[ \frac{(2h_n)^2 - 2(h_n)^2}{2} \right] \]
\[ = h_n I_d \]
and hence we obtain \( B = I_d \); we also can evaluate \( D_0(x) = \frac{1}{h} A(x) \), which coincides with that of Gloter (2006).

**Proposition 13.** Under \([A1] \), \( \bar{Q}_n(M(\cdot, \xi)) \to^P \nu_0(M[G](\cdot, \xi)) \) uniformly in \( \xi \in \Xi \).

**Remark 8.** As the previous remark, we can obtain \( G(x) = \frac{2}{3} A(x) \) as shown in Gloter (2006).

**Proof of Proposition 11.** It is obvious by Corollary 9 and the assumption for \( f \). \( \square \)

**Proof of Proposition 12.** We decompose the summation as follows:
\[ \bar{T}_{\ell, n}(v(\cdot, \xi)) \]
\[ = \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v(X_{(i-1-\ell)h_n, n}, \xi) [X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b(X_{(i-1-\ell)h_n, n})] \]
\[ = \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v(X_{(i-1-\ell)h_n, n}, \xi) [X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b(X_{(i-2p-1)h_n})] \]
\[ + \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v(X_{(i-1-\ell)h_n, n}, \xi) [(h_n B) b(X_{(i-2p-1)h_n}) - (h_n B) b(X_{(i-1-\ell)h_n})] \]
\[ = \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v(X_{(i-2p-1)h_n}, \xi) [X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b(X_{(i-2p-1)h_n})] \]
\[
+ \frac{1}{nh_n} \sum_{i=1+\ell}^{n} \partial_{x}v \left( X_{(i-2p-1)h_n}, \xi \right) \left[ (X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n})) \right] \\
\left( X_{(i-1)h_n, n} - X_{(i-2p-1)h_n} \right)^T \\
+ \frac{1}{nh_n} \sum_{i=1+\ell}^{n} \sum_{j_1=1}^{d} \sum_{j_2=1}^{d} \\
\int_{0}^{1} (1-s) \partial_{x(j_1)} \partial_{x(j_2)} v \left( X_{(i-p-1)h_n} + s (X_{(i-1)h_n, n} - X_{(i-p-1)h_n}), \xi \right) ds \\
\left( X_{(i-1-\ell)h_n, n} - X_{(i-2p-1)h_n} \right)^{(j_1)} \left( X_{(i-1)h_n, n} - X_{(i-2p-1)h_n} \right)^{(j_2)} \\
\left[ X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n}) \right] \\
+ \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v \left( X_{(i-1-\ell)h_n, n}, \xi \right) \left[ (h_n B) b (X_{(i-2p-1)h_n}) - (h_n B) b (X_{(i-1)h_n, n}) \right].
\]

Because of the evaluation such that
\[
\frac{1}{nh_n} \sum_{1+\ell \leq (2p+1) \leq n} \left| \mathbb{E} \left[ v \left( X_{(2p+1)(i-1)h_n}, \xi \right) \left[ X_{(2p+1)h_n, n} - X_{((2p+1)i-1)h_n, n} - (h_n B) b (X_{(2p+1)(i-1)h_n}) \right] \mathcal{F}_{(2p+1)(i-1)h_n} \right] \right| \xrightarrow{P} 0,
\]
\[
\frac{1}{n^2h_n^2} \sum_{1+\ell \leq (2p+1) \leq n} \mathbb{E} \left[ \left| v \left( X_{(2p+1)(i-1)h_n}, \xi \right) \left[ X_{(2p+1)h_n, n} - X_{((2p+1)i-1)h_n, n} - (h_n B) b (X_{(2p+1)(i-1)h_n}) \right] \right|^2 \mathcal{F}_{(2p+1)(i-1)h_n} \right] \xrightarrow{P} 0
\]
for all \(\xi \in \Xi\) and Lemma 9 of Genon-Catalot and Jacod (1993), we have
\[
\frac{1}{nh_n} \sum_{i=1+\ell}^{n} v \left( X_{(i-2p-1)h_n}, \xi \right) \left[ X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n}) \right] \xrightarrow{P} 0
\]
for all \(\xi \in \Xi\). To verify the uniform convergence in probability of this summation, we show that the following inequalities hold (Ibragimov and Has’minskii, 1981): there exist \(C > 0\) and \(k > \dim \Xi\) such that for all \(n \in \mathbb{N}\) and \(\xi, \xi' \in \Xi\),
\[
\mathbb{E} \left[ \left| \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v \left( X_{(i-2p-1)h_n}, \xi \right) \left[ X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n}) \right] \right|^k \right] \leq C,
\]
\[
\mathbb{E} \left[ \left| \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v \left( X_{(i-2p-1)h_n}, \xi \right) \left[ X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n}) \right] \right|^k \right] \\
- \frac{1}{nh_n} \sum_{i=1+\ell}^{n} v \left( X_{(i-2p-1)h_n}, \xi' \right) \left[ X_{ih_n, n} - X_{(i-1)h_n, n} - (h_n B) b (X_{(i-2p-1)h_n}) \right] \right|^k \right] \leq C |\xi - \xi'|^k.
\]
These evaluations can be led by the assumption of $\Phi_\Delta$ and Burkholder’s inequality in a similar way to Nakakita and Uchida (2019a).

With respect to the second summation, we can easily have the evaluation such that

$$
\frac{1}{nh_n} \sum_{1 + \ell \leq (2p+1)i \leq n} \partial_x v \left( X_{(2p+1)(i-1)h_n}, \xi \right) \\
\left[ (\overline{X}_{(2p+1)ih_n, n} - \overline{X}_{((1p+1)i-1)h_n, n} - (h_nB) b \left( X_{(2p+1)(i-1)h_n} \right)) \right. \\
\left. - \left( \int_0^{\Delta_n} \Phi_{\Delta_n,n} (\Delta_n - s) sds \right) b \left( X_{(p+1)(i-1)} \right) \right]^T
$$

$$
\xrightarrow{P} \frac{1}{2p+1} \nu_0 \left( \partial_x v \left[ D^T \right] (\cdot, \xi) \right) \text { uniformly in } \xi \in \Xi
$$

by an analogous manner to Gloter (2006). Hence we obtain

$$
\frac{1}{nh_n} \sum_{i=1+\ell}^n \partial_x v \left( X_{(i-2p-1)h_n}, \xi \right) \left[ (\overline{X}_{ih_n, n} - \overline{X}_{(i-1)h_n, n} - (h_nB) b \left( X_{(i-2p-1)h_n} \right)) \right. \\
\left. - \left( \int_0^{\Delta_n} \Phi_{\Delta_n,n} (\Delta_n - s) sds \right) b \left( X_{(i-1)h_n} \right) \right]^T \\
\xrightarrow{P} \nu_0 \left( \partial_x v \left[ D^T \right] (\cdot, \xi) \right) \text { uniformly in } \xi \in \Xi.
$$

For the residual terms, it is obvious that they converge to zero in probability uniformly in $\xi \in \Xi$. Hence we complete the proof. $\square$

**Proof of Proposition 13.** Because of the fact

$$
\frac{1}{nh_n} \sum_{i=1}^n M \left( \overline{X}_{(i-1)h_n, n}, \xi \right) \left[ (\overline{X}_{ih_n, n} - \overline{X}_{(i-1)h_n, n}) \otimes^2 \right] \\
- \frac{1}{nh_n} \sum_{i=1}^n M \left( X_{(i-p-1)h_n}, \xi \right) \left[ (\overline{X}_{ih_n, n} - \overline{X}_{(i-1)h_n, n}) \otimes^2 \right] \\
\xrightarrow{P} 0 \text { uniformly in } \xi \in \Xi
$$

which can be easily obtained, it is sufficient to evaluate

$$
\overline{Q}_n \left( M (\cdot, \xi) \right) = \frac{1}{nh_n} \sum_{i=1}^n M \left( X_{(i-p-1)h_n}, \xi \right) \left[ (\overline{X}_{ih_n, n} - \overline{X}_{(i-1)h_n, n}) \otimes^2 \right],
$$

and we can have an analogous result to Gloter (2006) such that

$$
\overline{Q}_n' \left( M (\cdot, \xi) \right) \xrightarrow{P} \nu_0 \left( M [G] (\cdot, \xi) \right) \text { uniformly in } \xi \in \Xi
$$

and hence obtain the proof. $\square$
Some specific evaluation. We set $p = [\bar{p}] + 1$, $\Delta_n = ph_n$ and show the evaluation of $B$, $D_\ell$ and $G$ when setting our kernel $\{\Phi_{\Delta,n}\} = \{\Phi_{p,h_n}\}$ as follows: we have $\Delta_n = ph_n$, $B = I_d$, $D_0 (x) = D_0 (x|\rho)|_{\rho = p_n}$, where $D_0 \Phi \bigl( x|\rho \bigr) = A^{(i,j)} (x) f_{D_0} \bigl( \rho^{(i)}, \rho^{(j)} \bigr)$.

\[
D_\ell = O \text{ for } \ell \geq \left[ \max_{i = 1, \ldots, d} \rho^{(d)} \right] + 1 \text{ because of independent increments of the Wiener process, and } G \bigl( x \bigr) = G \bigl( x|\rho \bigr)|_{\rho = p_n} \text{ where } G \bigl( x|\rho \bigr) = G \bigl( x, \alpha|\rho \bigr)|_{\alpha = \alpha_n}.
\]

Remark 9. Note that $D_0 (x|\rho)$ and $G \bigl( x|\rho \bigr)$ is continuous w.r.t. $\rho$ for all fixed $x$ by Lemma 15 and Lemma 16 in Appendix B.

For all $i = 1, \ldots, d$, if $\rho^{(i)} = 0$, then

\[
\left[ \int_0^{\Delta_n+h_n} (V_{p,h_n} ((\Delta_n + h_n) - s) - V_{p,h_n} (\Delta_n - s)) \, ds \right]^{(i,i)} = \int_0^{(p+1)h_n} (\delta ((p + 1) h_n - s) - \delta (ph_n - s)) \, ds
\]

\[
= (p + 1) h_n - ph_n
\]

\[
h_n,
\]

and if $\rho^{(i)} \in (0, 1]$, then

\[
\left[ \int_0^{\Delta_n+h_n} (V_{p,h_n} ((\Delta_n + h_n) - s) - V_{p,h_n} (\Delta_n - s)) \, ds \right]^{(i,i)}
\]

\[
= \int_0^{(p+1)h_n} (\rho^{(i)} h_n)^{-1} \left( 1_{[0,\rho^{(i)} h_n]} ((p + 1) h_n - s) - 1_{[0,\rho^{(i)} h_n]} (ph_n - s) \right) \, ds
\]

\[
= \int_{(p+1-\rho^{(i)})h_n}^{ph_n} (\rho^{(i)} h_n)^{-1} \, ds - \int_{(p-\rho^{(i)})h_n}^{ph_n} (\rho^{(i)} h_n)^{-1} \, ds
\]

\[
= h_n \left[ (p + 1)^2 - (p + 1 - \rho^{(i)})^2 - p^2 + (p - \rho^{(i)})^2 \right]
\]

\[= h_n,
\]
and if \( \rho^{(i)} \in (1, \bar{p}] \), then

\[
\frac{1}{h_n} \left[ \int_0^{\Delta_n+h_n} \left( V_{\rho,h_n} \left( (\Delta_n + h_n) - s \right) - V_{\rho,h_n} \left( \Delta_n - s \right) \right) s ds \right]^{(i,i)} \\
= \int_0^{(p+1)h_n} \rho^{(i)} h_n^{-1} s ds - \int_{(p-\rho^{(i)})h_n}^{ph_n} \rho^{(i)} h_n^{-1} s ds \\
= \int_0^{(p+1)h_n} \rho^{(i)} h_n^{-1} s ds - \int_{(p-\rho^{(i)})h_n}^{(p+1-\rho^{(i)})h_n} \rho^{(i)} h_n^{-1} s ds \\
= \frac{h_n}{2\rho^{(i)}} \left[ (p+1)^2 - p^2 - (p + 1 - \rho^{(i)})^2 + (p - \rho^{(i)})^2 \right] \\
= h_n.
\]

It is obvious that if \( i, j = 1, \ldots d \) and \( i \neq j \),

\[
\left[ \int_0^{\Delta_n+h_n} \left( V_{\rho,h_n} \left( (\Delta_n + h_n) - s \right) - V_{\rho,h_n} \left( \Delta_n - s \right) \right) s ds \right]^{(i,j)} = 0;
\]

therefore it holds that \( B = I_d \). Regarding to \( D_0(x) \), for all \( i, j = 1, \ldots, d \), let us define

\[
D_0^{(i,j)}(x|\rho) = \frac{1}{h_n} E \left[ \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right) \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( (p+1)h_n - s_1 \right) - V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right)^T \right]^{(i,j)} \\
= \frac{1}{h_n} E \left[ \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right)^{(i)} \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right)^{(j)} \left( \int_0^{(p+1)h_n} V_{\rho,h_n} \left( (p+1)h_n - s_1 \right) - V_{\rho,h_n} \left( ph_n - s_1 \right) \left( \int_0^{s_1} a(x) dw_{s_2} \right) ds_1 \right)^{(j)} \right] \\
= \frac{1}{h_n} E \left[ \left( \int_0^{(p+1)h_n} V_{\rho,h_n}^{(i,i)} \left( ph_n - s \right) \left( a(x) w_s \right) \left( a(x) w_s \right)^{(i)} ds \right) \left( \int_0^{(p+1)h_n} V_{\rho,h_n}^{(j,j)} \left( ph_n - s' \right) \left( a(x) w_{s'} \right) \left( a(x) w_{s'} \right)^{(j)} ds' \right) \right] \\
= \frac{1}{h_n} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} E \left[ \left( a(x) w_s \right)^{(i)} \left( a(x) w_{s'} \right)^{(j)} \right] \left( V_{\rho,h_n}^{(i,i)} \left( ph_n - s \right) \left( V_{\rho,h_n}^{(j,j)} \left( ph_n - s' \right) - V_{\rho,h_n}^{(j,j)} \left( (p+1)h_n - s' \right) \right) \left( a(x) w_{s'} \right)^{(j)} ds' \right] ds \\
= \frac{1}{h_n} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{s, s'\} \left( V_{\rho,h_n}^{(i,i)} \left( ph_n - s \right) \left( V_{\rho,h_n}^{(j,j)} \left( (p+1)h_n - s' \right) - V_{\rho,h_n}^{(j,j)} \left( ph_n - s' \right) \right) \right) ds' ds,
\]
and if $\rho^{(i)} = \rho^{(j)} = 0$,

$$
\begin{align*}
\mathbf{D}^{(i,j)}_0 (x | \rho) &= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{ s, s' \} \\
&\quad \left( V^{(i,i)}_{\rho,h_n} (ph_n - s) \left( V^{(j,j)}_{\rho,h_n} ((p + 1) h_n - s') - V^{(j,j)}_{\rho,h_n} (ph_n - s') \right) \right) \ ds' \ ds \\
&= 0,
\end{align*}
$$

and if $\rho^{(i)} = 0$, $\rho^{(j)} \in (0, 1]$,

$$
\begin{align*}
\mathbf{D}^{(i,j)}_0 (x | \rho) &= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{ s, s' \} \\
&\quad \left( V^{(i,i)}_{\rho,h_n} (ph_n - s) \left( V^{(j,j)}_{\rho,h_n} ((p + 1) h_n - s') - V^{(j,j)}_{\rho,h_n} (ph_n - s') \right) \right) \ ds' \ ds \\
&= \int_0^{(p+1)h_n} A^{(i,j)} (x) \ min \{ ph_n, s' \} \\
&\quad \left( \rho^{(j)} h_n \right)^{-1} \left( \mathbf{1}_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - \mathbf{1}_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) \ ds' \\
&= \int_{ph_n}^{\rho h_n} A^{(i,j)} (x) ph_n \\
&\quad \left( \rho^{(j)} h_n \right)^{-1} \left( \mathbf{1}_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - \mathbf{1}_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) \ ds' \\
&\quad + \int_0^{ph_n} A^{(i,j)} (x) s' \\
&\quad \left( \rho^{(j)} h_n \right)^{-1} \left( \mathbf{1}_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - \mathbf{1}_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) \ ds' \\
&= A^{(i,j)} (x) \left( ph_n - \frac{p^2 h_n^2 - (p - \rho^{(j)})^2 h_n^2}{2 \rho^{(j)} h_n} \right) \\
&= \frac{\rho^{(j)} h_n A^{(i,j)} (x)}{2}.
\end{align*}
$$

and if $\rho^{(i)} = 0$, $\rho^{(j)} \in (1, \rho]$,

$$
\begin{align*}
\mathbf{D}^{(i,j)}_0 (x | \rho) &= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{ s, s' \} \\
&\quad \left( V^{(i,i)}_{\rho,h_n} (ph_n - s) \left( V^{(j,j)}_{\rho,h_n} ((p + 1) h_n - s') - V^{(j,j)}_{\rho,h_n} (ph_n - s') \right) \right) \ ds' \ ds \\
&= \int_{ph_n}^{\rho h_n} A^{(i,j)} (x) ph_n \\
&\quad \left( \rho^{(j)} h_n \right)^{-1} \left( \mathbf{1}_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - \mathbf{1}_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) \ ds' \\
&\quad + \int_0^{ph_n} A^{(i,j)} (x) s' \\
&\quad \left( \rho^{(j)} h_n \right)^{-1} \left( \mathbf{1}_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - \mathbf{1}_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) \ ds'
\end{align*}
$$
\[
A^{(i,j)}(x) = \left( \frac{p h_n}{\rho^{(j)}} + \frac{p^2 h_n^2 - (p + 1 - \rho^{(j)})^2 h_n^2 - p^2 h_n^2 + (p - \rho^{(j)})^2 h_n^2}{2\rho^{(j)} h_n} \right)
\]

\[
= A^{(i,j)}(x) \left( \frac{p h_n}{\rho^{(j)}} + \frac{-2 (p - \rho^{(j)}) h_n^2 - h_n^2}{2\rho^{(j)} h_n} \right)
\]

\[
= \frac{(2\rho^{(j)} - 1) h_n A^{(i,j)}(x)}{2\rho^{(j)}},
\]

and if \( \rho^{(i)} > 0, \rho^{(j)} = 0 \),

\[
h_n D_0^{(i,j)}(x|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \}
\]

\[
\left( V_{\rho, h_n}^{(i,j)}(ph_n - s) \left( V_{\rho, h_n}^{(j,j)} ((p + 1) h_n - s') - V_{\rho, h_n}^{(j,j)} (ph_n - s') \right) \right) ds' ds
\]

\[
= \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, (p + 1) h_n \} \left( \rho^{(i)} h_n \right)^{-1} 1_{[0, \rho^{(i)} h_n]} (ph_n - s) ds
\]

\[
- \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, ph_n \} \left( \rho^{(i)} h_n \right)^{-1} 1_{[0, \rho^{(i)} h_n]} (ph_n - s) ds
\]

\[
= \int_0^{(p+1)h_n} A^{(i,j)}(x) s \left( \rho^{(i)} h_n \right)^{-1} 1_{[0, \rho^{(i)} h_n]} (ph_n - s) ds
\]

\[
- \int_0^{(p+1)h_n} A^{(i,j)}(x) ph_n \left( \rho^{(i)} h_n \right)^{-1} 1_{[0, \rho^{(i)} h_n]} (ph_n - s) ds
\]

\[
- \int_{ph_n}^{p h_n} A^{(i,j)}(x) s \left( \rho^{(i)} h_n \right)^{-1} 1_{[0, \rho^{(i)} h_n]} (ph_n - s) ds
\]

\[
= 0,
\]

and if \( \rho^{(i)} > 0, \rho^{(j)} > 0 \),

\[
h_n D_0^{(i,j)}(x|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \}
\]

\[
\left( V_{\rho, h_n}^{(i,j)}(ph_n - s) \left( V_{\rho, h_n}^{(j,j)} ((p + 1) h_n - s') - V_{\rho, h_n}^{(j,j)} (ph_n - s') \right) \right) ds' ds
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)} \rho^{(j)} h_n^2} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} \min \{ s, s' \} 1_{[0, \rho^{(i)} h_n]} (ph_n - s)
\]

\[
\left( 1_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - 1_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) ds' ds
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)} \rho^{(j)} h_n^2} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} s 1_{[0, \rho^{(i)} h_n]} (ph_n - s)
\]

\[
\left( 1_{[0, \rho^{(j)} h_n]} ((p + 1) h_n - s') - 1_{[0, \rho^{(j)} h_n]} (ph_n - s') \right) ds' ds
\]

\[
+ \frac{A^{(i,j)}(x)}{\rho^{(i)} \rho^{(j)} h_n^2} \int_0^{(p+1)h_n} \int_0^{s} s' 1_{[0, \rho^{(i)} h_n]} (ph_n - s)
\]
\[
A^{(i,j)} (x) \frac{p^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} s \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \ ds' \ ds \\
+ A^{(i,j)} (x) \frac{p^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \ ds' \ ds \\
+ A^{(i,j)} (x) \frac{\rho^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \ ds' \ ds \\
= A^{(i,j)} (x) \frac{p^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} s \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \ ds \\
- A^{(i,j)} (x) \frac{p^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} s \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \max \{ s, (p + 1 - \rho^{(j)}) h_n \} \ ds \\
- A^{(i,j)} (x) \frac{\rho^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} s \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[0, p h_n]} (s) \ ds \\
+ A^{(i,j)} (x) \frac{\rho^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} s \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[0, p h_n]} (s) \max \{ s, (p - \rho^{(j)}) h_n \} \ ds \\
+ A^{(i,j)} (x) \frac{2 \rho^{(i)} p^{(j)}}{h_n^{(i) h_n^{(j)}}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[(p+1-\rho^{(j)}) h_n, (p+1) h_n]} (s) \ s^2 \ ds \\
- A^{(i,j)} (x) \frac{(p + 1 - \rho^{(j)})^2}{2 \rho^{(i)} p^{(j)}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[(p+1-\rho^{(j)}) h_n, (p+1) h_n]} (s) \ ds \\
- A^{(i,j)} (x) \frac{(p - \rho^{(j)})^2}{2 \rho^{(i)} p^{(j)}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[(p-\rho^{(j)}) h_n, (p+1) h_n]} (s) \ min \{ s^2, p^2 h_n^2 \} \ ds \\
+ A^{(i,j)} (x) \frac{(p - \rho^{(j)})^2}{2 \rho^{(i)} p^{(j)}} \int_0^{(p+1) h_n^{(i) h_n^{(j)}}} \mathbf{1}_{[0, \rho^{(i)} h_n]} (p h_n - s) \mathbf{1}_{[(p-\rho^{(j)}) h_n, (p+1) h_n]} (s) \ ds \\
= A^{(i,j)} (x) \frac{(p + 1)}{\rho^{(i)} p^{(j)}} \int_{(p-\rho^{(i)}) h_n}^{p h_n} s \ ds 
\]
CONVOLUTIONALLY OBSERVED DIFFUSION PROCESSES

\[
\begin{align*}
- A^{(i,j)}(x) & \int_{(p+1)^2_{(\rho^{(i)})}}^{(p+1)^2_{(\rho^{(j)})}} s^2 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds \\
- \frac{A^{(i,j)}(x)(p + 1 - \rho^{(j)})}{\rho^{(i)}\rho^{(j)}/h_n^2} & \int_{0}^{(p+1)^2_{(\rho^{(j)})}} s 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds \\
- \frac{A^{(i,j)}(x)p}{\rho^{(i)}\rho^{(j)}/h_n} & \int_{(p-\rho^{(i)})}^{ph_n} sds \\
+ A^{(i,j)}(x) & \int_{(p+1)^2_{(\rho^{(j)})}}^{(p+1)^2_{(\rho^{(i)})}} s^2 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds \\
+ \frac{A^{(i,j)}(x)(p - \rho^{(j)})}{\rho^{(i)}\rho^{(j)}/h_n^2} & \int_{0}^{(p-\rho^{(i)})} s 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds \\
+ \frac{A^{(i,j)}(x)}{2\rho^{(i)}\rho^{(j)/h_n^2}} & \int_{(p-\rho^{(i)})}^{ph_n} 1_{[p+1,\rho^{(i)}]}(s) s^2 \, ds \\
- \frac{A^{(i,j)}(x)(p + 1 - \rho^{(j)})^2}{2\rho^{(i)}\rho^{(j)}} & \int_{(p-\rho^{(i)})}^{ph_n} 1_{[p+1,\rho^{(i)}]}(s) \, ds \\
- \frac{A^{(i,j)}(x)(p - \rho^{(j)})^2}{2\rho^{(i)}\rho^{(j)}} & \int_{(p-\rho^{(i)})}^{ph_n} 1_{[p+1,\rho^{(i)}]}(s) \, ds \\
= \frac{A^{(i,j)}(x)(p + 1)}{2\rho^{(i)}\rho^{(j)}} & \left(\rho^{2}h_n^2 - (\rho - \rho^{(i)})^2 h_n^2\right) \\
- \frac{A^{(i,j)}(x)1_{[1,\rho]}(\rho^{(j)})}{\rho^{(i)}\rho^{(j)}/h_n^2} & \int_{\max\{p,(p+1-\rho^{(j)})\}}^{p} s^2 \, ds \\
- \frac{A^{(i,j)}(x)(p + 1 - \rho^{(j)})}{\rho^{(i)}\rho^{(j)/h_n}} & \int_{(p-\rho^{(i)})}^{ph_n} 1_{[0,\rho^{(i)}]}(\rho^{(j)}) \, ds \\
+ A^{(i,j)}(x)(p - \rho^{(j)}) & \int_{(p-\rho^{(i)})}^{(p-\rho^{(j)})} s 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds \\
+ \frac{A^{(i,j)}(x)1_{[1,\rho]}(\rho^{(j)})}{2\rho^{(i)}\rho^{(j)/h_n^2}} & \int_{\max\{p,(p+1-\rho^{(j)})\}}^{ph_n} s^2 \, ds \\
- \frac{A^{(i,j)}(x)(p + 1 - \rho^{(j)})^2}{2\rho^{(i)}\rho^{(j)}} & \int_{\max\{p,(p+1-\rho^{(j)})\}}^{ph_n} s 1_{[0,\rho^{(i)}]}(ph_n - s) \, ds
\end{align*}
\]
\[- \frac{A^{(i,j)}(x)}{2 \rho^{(i)} \rho^{(j)} h_n^2} \int_{\max \{\rho^{(i)}, \rho^{(j)}\} h_n}^{p h_n} s^2 \, ds + \frac{A^{(i,j)}(x) (p - \rho^{(j)})^2}{2 \rho^{(i)} \rho^{(j)}} \int_{\max \{\rho^{(i)}, \rho^{(j)}\} h_n}^{p h_n} ds \]
\[= \frac{A^{(i,j)}(x) (p + 1) h_n}{2 \rho^{(i)} \rho^{(j)}} (p^2 - (p - \rho^{(j)})^2) \]
\[- \frac{A^{(i,j)}(x) 1_{[1, \infty]}(\rho^{(j)}) h_n}{6 \rho^{(i)} \rho^{(j)}} (p^3 - \max \{(p - \rho^{(i)})^3, (p + 1 - \rho^{(j)})^3\}) \]
\[- \frac{A^{(i,j)}(x) (p + 1 - \rho^{(j)})^2 1_{[0, \rho^{(i)} + 1]}(\rho^{(j)}) h_n}{2 \rho^{(i)} \rho^{(j)}} (p - \max \{(p - \rho^{(i)}, (p + 1 - \rho^{(j)})\}) \]
\[- \frac{A^{(i,j)}(x) h_n}{6 \rho^{(i)} \rho^{(j)}} (p^3 - \max \{(p - \rho^{(i)})^3, (p - \rho^{(j)})^3\}) \]
\[+ \frac{A^{(i,j)}(x) (p - \rho^{(j)})^2 h_n}{2 \rho^{(i)} \rho^{(j)}} (p - \max \{(p - \rho^{(i)}, (p - \rho^{(j)})\}) \]
\[= \frac{A^{(i,j)}(x) h_n}{2 \rho^{(i)} \rho^{(j)}} (p^2 - (p - \rho^{(i)})^2) \]
\[- \frac{A^{(i,j)}(x) 1_{[1, \infty]}(\rho^{(j)}) h_n}{6 \rho^{(i)} \rho^{(j)}} (p^3 - \max \{(p - \rho^{(i)})^3, (p + 1 - \rho^{(j)})^3\}) \]
\[- \frac{A^{(i,j)}(x) (p + 1 - \rho^{(j)})^2 1_{[0, \rho^{(i)} + 1]}(\rho^{(j)}) h_n}{2 \rho^{(i)} \rho^{(j)}} (p - \max \{(p - \rho^{(i)}, (p + 1 - \rho^{(j)})\}) \]
\[+ \frac{A^{(i,j)}(x) h_n}{6 \rho^{(i)} \rho^{(j)}} (p^3 - \max \{(p - \rho^{(i)})^3, (p - \rho^{(j)})^3\}) \]
\[\frac{A^{(i,j)}(x) (p - \rho^{(j)})^2 1_{[0, \rho^{(i)}]}(\rho^{(j)}) h_n}{2 \rho^{(i)} \rho^{(j)}} ((p - \rho^{(j)})^2 - (p - \rho^{(i)})^2) \]
\[+ \frac{A^{(i,j)}(x) h_n}{6 \rho^{(i)} \rho^{(j)}} (p^3 - \max \{(p - \rho^{(i)})^3, (p - \rho^{(j)})^3\}) \]
and then we should consider five cases as follows: (i) \( \rho^{(i)} + 1 < \rho^{(j)} \); (ii) \( \rho^{(j)} > 1 \), \( \rho^{(i)} < \rho^{(j)} \leq \rho^{(i)} + 1 \); (iii) \( \rho^{(j)} > 1 \), \( \rho^{(i)} \geq \rho^{(j)} \); (iv) \( \rho^{(j)} \leq 1 \), \( \rho^{(i)} < \rho^{(j)} \); (v) \( \rho^{(j)} \leq 1 \), \( \rho^{(i)} \geq \rho^{(j)} \), and for the case (i), we have

\[
\begin{align*}
&+ \frac{A^{(i,j)}(x) \left(p - \rho^{(j)}\right)^2 h_n}{2\rho^{(i)}\rho^{(j)}} (p - \max \{ (p - \rho^{(i)}), (p - \rho^{(j)}) \}) \\
&\text{and for the case (ii),}
\end{align*}
\]

\[
\begin{align*}
 h_n D_0^{(i,j)}(x|\rho) \\
= & \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \\
&\left( V_{\rho,h_n}(ph_n - s) \left( \rho^{(j)} \left((p + 1)h_n - s'\right) - V_{\rho,h_n}(ph_n - s') \right) \right) ds' ds \\
&= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) - \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left( p^3 - (p - \rho^{(i)})^3 \right) \\
&+ \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left( p^3 - (p - \rho^{(i)})^3 \right) - \frac{A^{(i,j)}(x) \left(p + 1 - \rho^{(j)}\right) h_n}{2\rho^{(i)}\rho^{(j)}} (p - (p - \rho^{(i)})) \\
&+ \frac{A^{(i,j)}(x) \left(p - \rho^{(j)}\right)^2 h_n}{2\rho^{(i)}\rho^{(j)}} (p - (p - \rho^{(i)})) \\
&= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) \\
&- \frac{A^{(i,j)}(x) \left(p + 1 - \rho^{(j)}\right)^2 \rho^{(i)} h_n}{2\rho^{(i)}\rho^{(j)}} + \frac{A^{(i,j)}(x) \left(p - \rho^{(j)}\right)^2 \rho^{(i)} h_n}{2\rho^{(i)}\rho^{(j)}} \\
&= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 - 2(p - \rho^{(j)}) \rho^{(i)} - \rho^{(i)} \right) \\
&= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - p^2 + 2p\rho^{(i)} - (\rho^{(i)})^2 - 2p\rho^{(i)} + 2\rho^{(i)}\rho^{(j)} - \rho^{(i)} \right) \\
&= \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left( 6\rho^{(i)}\rho^{(j)} - 3(\rho^{(i)})^2 - 3\rho^{(i)} \right),
\end{align*}
\]
and for the case (iii),

\[
\begin{align*}
&h_n \mathbf{D}^{(i,j)}_0 (x|\rho) \\
&= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{s, s'\} \\
&\quad \left( V_{p,h_n} (ph_n - s) \left( V^{(j)}_{p,h_n} ((p + 1) h_n - s') - V^{(j)}_{p,h_n} (ph_n - s') \right) \right) ds' ds \\
&= \frac{A^{(i,j)} (x) h_n}{2 \rho^{(i)} \rho^{(j)}} \left( p^2 - (p - \rho^{(j)})^2 \right) - \frac{A^{(i,j)} (x) h_n}{6 \rho^{(i)} \rho^{(j)}} \left( p^3 - (p + 1 - \rho^{(j)})^3 \right) \\
&\quad - \frac{A^{(i,j)} (x) (p + 1 - \rho^{(j)}) h_n}{2 \rho^{(i)} \rho^{(j)}} \left( (p + 1 - \rho^{(j)})^2 - (p - \rho^{(j)})^2 \right) \\
&\quad + \frac{A^{(i,j)} (x) h_n}{6 \rho^{(i)} \rho^{(j)}} \left( p^3 - (p - \rho^{(j)})^3 \right) \\
&\quad + \frac{A^{(i,j)} (x) (p - \rho^{(j)}) h_n}{2 \rho^{(i)} \rho^{(j)}} \left( (p - \rho^{(j)})^2 - (p - \rho^{(j)})^2 \right) \\
&\quad - \frac{A^{(i,j)} (x) (p + 1 - \rho^{(j)})^2 h_n}{2 \rho^{(i)} \rho^{(j)}} (p - (p + 1 - \rho^{(j)}) \right) \\
&\quad + \frac{A^{(i,j)} (x) (p - \rho^{(j)})^2 h_n}{2 \rho^{(i)} \rho^{(j)}} (p - (p - \rho^{(j)})\right) \\
&= \frac{A^{(i,j)} (x) h_n}{6 \rho^{(i)} \rho^{(j)}} \left[ 3 \left( p^2 - (p - \rho^{(i)})^2 \right) - \left( p^3 - (p + 1 - \rho^{(j)})^3 \right) \\
&\quad - 3 (p + 1 - \rho^{(j)}) \left( (p + 1 - \rho^{(j)})^2 - (p - \rho^{(i)})^2 \right) \\
&\quad + \left( p^3 - (p - \rho^{(j)})^3 \right) + 3 (p - \rho^{(j)}) \left( (p - \rho^{(j)})^2 - (p - \rho^{(j)})^2 \right) \\
&\quad - 3 (p + 1 - \rho^{(j)})^2 (p - (p + 1 - \rho^{(j)}) \right) + 3 (p - \rho^{(j)})^2 \rho^{(j)}\right],
\end{align*}
\]
\[ h_n(D_0^{(i,j)}(x)|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min\{s, s'\} \left( V_{\rho, h_n}(ph_n - s) \left( V_{\rho, h_n}((p + 1)h_n - s') - V_{\rho, h_n}(ph_n - s') \right) \right) ds' ds \\
= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) \\
- \frac{A^{(i,j)}(x) (p + 1 - \rho^{(j)}) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) \\
+ \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left( p^3 - (p - \rho^{(i)})^3 \right) \\
+ \frac{A^{(i,j)}(x) (p - \rho^{(j)})^2 h_n}{2\rho^{(i)}\rho^{(j)}} (p - (p - \rho^{(i)})^2) \\
= \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left[ (\rho^{(i)} - \rho^{(j)})^3 + (\rho^{(j)})^3 \right] \]

and for the case (v),

\[ h_n(D_0^{(i,j)}(x)|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min\{s, s'\} \left( V_{\rho, h_n}(ph_n - s) \left( V_{\rho, h_n}((p + 1)h_n - s') - V_{\rho, h_n}(ph_n - s') \right) \right) ds' ds \\
= \frac{A^{(i,j)}(x) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) \\
- \frac{A^{(i,j)}(x) (p + 1 - \rho^{(j)}) h_n}{2\rho^{(i)}\rho^{(j)}} \left( p^2 - (p - \rho^{(i)})^2 \right) \\
+ \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left( p^3 - (p - \rho^{(i)})^3 \right) \\
+ \frac{A^{(i,j)}(x) (p - \rho^{(j)})^2 h_n}{2\rho^{(i)}\rho^{(j)}} (p - (p - \rho^{(i)})^2) \\
= \frac{A^{(i,j)}(x) h_n}{6\rho^{(i)}\rho^{(j)}} \left[ 3 \left( p^2 - (p - \rho^{(i)})^2 \right) \\
- 3(p + 1 - \rho^{(j)}) \left( p^2 - (p - \rho^{(i)})^2 \right) + (p^3 - (p - \rho^{(i)})^3) \right] \]
Hence, it follows from (i)-(v) that \( D_{38} \left[ \frac{S H NAKAKITA AND M UCHIDA}{\text{K}} \right] = \max \left[ (\int_{h_{\rho}} (p + 1 + \ell) h_n - s_1) - V_{h_{\rho}} ((p + \ell) h_n - s_1) \right] \times \left( \int_{0}^{s_1} a(x) \, dw_s \right) \right] = 0 \]

With respect to \( G(x) \), for all \( i, j = 1, \ldots, d \), let us define \( G^{(i,j)}(x|\rho) \) such that

\[
G^{(i,j)}(x|\rho) := \frac{1}{h_n} E \left[ \left( \int_{0}^{\Delta_n+h_n} (\Phi_{\Delta_n,n}((\Delta_n+h_n)-s_1)) \left( \int_{0}^{s_1} a(x) \, dw_s \right) \right)^{(i,j)} \right] \]

\[
= \frac{1}{h_n} \int_{0}^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,\rho_n}^{(i,i)}((p+1) h_n - s) - V_{\rho,\rho_n}^{(i,i)}(p h_n - s) \right) \times \left( \int_{0}^{s_1} a(x) \, dw_s \right) \right] \]

and \( K^{(i,j)}(x|\rho) := G^{(i,j)}(x|\rho) + D_{0}^{(i,j)}(x|\rho) \). If \( \rho^{(i)} = 0 \), as evaluation of \( B \),

\[
h_n K^{(i,j)}(x|\rho) \]

\[
= \int_{0}^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,\rho_n}^{(i,i)}((p+1) h_n - s) \right) \times \left( \int_{0}^{s_1} a(x) \, dw_s \right) \]

\[
= A^{(i,j)}(x) \int_{0}^{(p+1)h_n} \left( V_{\rho,\rho_n}^{(i,j)}((p+1) h_n - s') - V_{\rho,\rho_n}^{(i,j)}(p h_n - s') \right) \, ds'ds' \]

\[
= h_n A^{(i,j)}(x),
\]
and if $\rho^{(i)} \in (0, 1]$ and $\rho^{(j)} = 0$,

$$h_n K^{(i,j)}(x | \rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right)$$

$$\left( V_{\rho,h_n}^{(j,j)} ((p + 1) h_n - s') - V_{\rho,h_n}^{(j,j)} (p h_n - s') \right) ds'ds$$

$$= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right)$$

$$\times V_{\rho,h_n}^{(j,j)} ((p + 1) h_n - s') ds'ds$$

$$- \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right)$$

$$\times V_{\rho,h_n}^{(j,j)} (p h_n - s') ds'ds$$

$$= A^{(i,j)}(x) \int_0^{(p+1)h_n} s \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right) ds$$

$$- A^{(i,j)}(x) \int_0^{(p+1)h_n} \min \{ s, ph_n \} \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right) ds$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} sds - \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} \min \{ s, ph_n \} ds \right)$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} sds - \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} ph_n ds \right)$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \frac{1}{2} (p + 1)^2 h_n^2 - \frac{1}{2} (p + 1 - \rho^{(i)})^2 h_n^2 - p (p + 1) h_n^2 + p (p + 1 - \rho^{(i)}) h_n^2 \right)$$

$$= h_n A^{(i,j)}(x) \left( 1 - \frac{\rho^{(i)}}{2} \right)$$

and if $\rho^{(i)} \in (1, \overline{\rho})$ and $\rho^{(j)} = 0$,

$$h_n K^{(i,j)}(x | \rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{ s, s' \} \left( V_{\rho,h_n}^{(i,i)} ((p + 1) h_n - s) \right)$$

$$\left( V_{\rho,h_n}^{(j,j)} ((p + 1) h_n - s') - V_{\rho,h_n}^{(j,j)} (p h_n - s') \right) ds'ds$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} sds - \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} \min \{ s, ph_n \} ds \right)$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} sds - \int_{(p+1-\rho^{(i)})h_n}^{(p+1)h_n} ph_n ds \right)$$

$$= \frac{A^{(i,j)}(x)}{\rho^{(i)} h_n} \left( \frac{1}{2} (p + 1)^2 h_n^2 - \frac{1}{2} (p + 1 - \rho^{(i)})^2 h_n^2 - p (p + 1) h_n^2 + p (p + 1 - \rho^{(i)}) h_n^2 \right)$$

$$+ p^2 h_n^2 - \frac{1}{2} p^2 h_n^2 + \frac{1}{2} (p + 1 - \rho^{(i)})^2 h_n^2$$
\[ h_n A^{(i,j)}(x) = \frac{h_n A^{(i,j)}(x)}{2\rho^{(i)}} \]

and if \( \rho^{(i)} > 0 \) and \( \rho^{(j)} > 0 \),

\[
 h_n K^{(i,j)}(x|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min\{s, s'\} \left( V_{p,h_n}^{(i,j)}((p+1)h_n - s) \right) \\
\left( V_{p,h_n}^{(j)}((p+1)h_n - s') - V_{p,h_n}^{(j)}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} \min\{s, s'\} \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') - 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} s \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') - 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') - 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') - 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
- \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
+ \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s' \left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') \right) \text{d}s'\text{d}s
\]

\[
- \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s' \left( 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
= \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}((p+1)h_n - s') - 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]

\[
- \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} s \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n - s) \right) \\
\left( 1_{[0,\rho^{(j)}h_n]}(ph_n - s') \right) \text{d}s'\text{d}s
\]
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\[
\int_{\max\{s,(p-\rho^{(j)})h_n\}}^{ph_n} \left( 1_{[0,ph_n]}(s) \right) ds' ds \\
+ \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n-s) \right) \left( 1_{[(p+1-\rho^{(j)})h_n,(p+1)h_n]}(s) \right) ds' ds \\
- \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \int_0^{(p+1)h_n} \left( 1_{[0,\rho^{(i)}h_n]}((p+1)h_n-s) \right) \left( 1_{[\rho^{(j)}h_n,(p+1)h_n]}(s) \right) ds' ds
\]
and we consider the following cases: (i) \(\rho^{(i)} > 1\) and \(\rho^{(j)} > \rho^{(i)} + 1\); (ii) \(\rho^{(i)} > 1\) and \(\rho^{(j)} < \rho^{(i)} \leq \rho^{(j)} + 1\); (iii) \(\rho^{(i)} > 1\) and \(\rho^{(i)} \leq \rho^{(j)}\); (iv) \(\rho^{(i)} \leq 1\) and \(\rho^{(j)} < \rho^{(i)}\); (v) \(\rho^{(i)} \leq 1\) and \(\rho^{(j)} \leq \rho^{(i)}\), and for the case (i),

\[
h_n \mathbf{K}^{(i,j)} (x|\rho) = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{s, s'\} \left( V_{\rho, h_n} ((p+1)h_n - s) - V_{\rho, h_n} ((p+1)h_n - s') \right) ds' ds
\]

...
\[
\begin{align*}
&= A^{(i,j)}(x) \left( \frac{(p + 1)^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) (p + 1) h_n \\
&- A^{(i,j)}(x) \left( \frac{(p + 1)^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{3} \right) \\
&+ A^{(i,j)}(x) \mathbf{1}_{(\rho^{(j)}, \bar{\rho})} \left( \rho^{(i)} \right) \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) \left( \rho^{(j)} h_n \right) \\
&- A^{(i,j)}(x) \mathbf{1}_{(\rho^{(j)} + 1, \bar{\rho})} \left( \rho^{(i)} \right) \left( \frac{(p - \rho^{(j)})^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) \left( \rho^{(j)} h_n \right) \\
&+ A^{(i,j)}(x) \left( \frac{(p + 1)^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{6} \right) \\
&- A^{(i,j)}(x) \left( \rho^{(j)} h_n \right) \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \\
&- A^{(i,j)}(x) \mathbf{1}_{(\rho^{(j)}, \bar{\rho})} \left( \rho^{(i)} \right) \frac{p^2 h_n^2}{2} - \frac{(p - \rho^{(j)})^2 h_n^2}{2} h_n \\
&- A^{(i,j)}(x) \mathbf{1}_{(\rho^{(j)} + 1, \bar{\rho})} \left( \rho^{(i)} \right) \frac{p^3 h_n^3}{6} - \frac{(p - \rho^{(j)})^3 h_n^3}{6} \\
&+ A^{(i,j)}(x) \mathbf{1}_{(\rho^{(j)}), \bar{\rho}} \left( \rho^{(i)} \right) \left( \rho^{(j)} h_n \right) \frac{(p - \rho^{(j)})^2 h_n^2}{2} \\
&= \frac{h_n A^{(i,j)}(x)}{\rho^{(i)} \rho^{(j)}} \left( \frac{(\rho^{(j)})^2}{2} + \frac{\rho^{(j)}}{2} \right),
\end{align*}
\]

and for the case (ii),

\[
\begin{align*}
&= \frac{h_n K^{(i,j)}(x)}{\rho^{(i)} \rho^{(j)}} \\
&= \int_0^{(p + 1)h_n} \int_0^{(p + 1)h_n} A^{(i,j)}(x) \min \{s, s'\} \left( V^{(i,j)}_{\rho, h_n} ((p + 1) h_n - s) \right) \\
&\quad \left( V^{(j)}_{\rho, h_n} ((p + 1) h_n - s') - V^{(j)}_{\rho, h_n} (p h_n - s') \right) ds' ds \\
&= A^{(i,j)}(x) \left( \frac{(p + 1)^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) ((p + 1) h_n) \\
&- A^{(i,j)}(x) \left( \frac{(p + 1)^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{3} \right) \\
\end{align*}
\]
\begin{align*}
&+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(\rho^{(i)}, \rho^{(j)})} \left( \rho^{(i)} \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) (\rho^{(j)} h_n) \right) \\
&- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) (p h_n) \right) \\
&+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{3} \right) \right) \\
&+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \frac{(p + 1)^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{6} \right) \\
&- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \rho^{(j)} h_n \right) \left( \rho^{(i)} \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) \right) \\
&- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^2 h_n^2}{2} - \frac{(p - \rho^{(i)})^2 h_n^2}{2} \right) h_n \right) \\
&- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{6} \right) \right) \\
&+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( (\rho^{(i)} - 1) h_n \right) \left( \rho^{(j)} \left( \frac{(p - \rho^{(j)})^2 h_n^2}{2} \right) \right) \\
&= h_n A^{(i,j)} (x) \\
&\times \left( (\rho^{(i)})^3 - 3 (\rho^{(i)})^2 \rho^{(j)} - 3 (\rho^{(i)})^2 + 3 \rho^{(i)} (\rho^{(j)})^2 + 6 \rho^{(i)} \rho^{(j)} + 3 \rho^{(i)} - (\rho^{(j)})^3 - 1 \right)
\end{align*}

and for the case (iii),

\begin{align*}
h_n K^{(i,j)} (x|\rho) \\
= \int_0^{(p + 1) h_n} \int_0^{(p + 1) h_n} A^{(i,j)} (x) \min \{s, s'\} \left( V^{(i,i)}_{\rho, h_n} ((p + 1) h_n - s) \\
- \left( V^{(i,j)}_{\rho, h_n} ((p + 1) h_n - s') - V^{(j,j)}_{\rho, h_n} (p h_n - s') \right) \right) ds' ds \\
= \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \frac{(p + 1)^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) ((p + 1) h_n) \\
- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \frac{(p + 1)^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{3} \right) \\
- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) (p h_n) \right) \\
+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} 1_{(1, \rho^{(i)})} \left( \rho^{(i)} \left( \frac{p^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{3} \right) \right)
\end{align*}
\[ + \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \left( \frac{(p + 1)^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{6} \right) \]

\[ - \frac{A^{(i,j)}(x)}{\rho^{(i)}\rho^{(j)}h_n^2} \left( \frac{h_n^2}{2} \right) \]

\[ - A^{(i,j)}(x) \left( \rho^{(i)} \right) \left( \frac{p^2 h_n^2}{2} - \frac{(p - \rho^{(j)})^2 h_n^2}{2} \right) h_n \]

\[ - A^{(i,j)}(x) \left( \rho^{(i)} \right) \left( \frac{p^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{6} \right) \]

\[ + A^{(i,j)}(x) \left( \rho^{(i)} \right) \left( (\rho^{(i)} - 1) h_n \right) \left( \frac{(p - \rho^{(j)})^2 h_n^2}{2} \right) \]

\[ = h_n A^{(i,j)}(x) \left( -\frac{(\rho^{(i)})^2}{2} + \rho^{(i)}\rho^{(j)} + \frac{\rho^{(i)}}{2} - \frac{1}{6} \right) , \]

and for the case (iv),

\[ h_n K^{(i,j)}(x|\rho) \]

\[ = \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)}(x) \min \{s, s'\} \left( V^{(i,j)} ((p + 1) h_n - s) - \frac{1}{\rho^{(i)}h_n} h_n \right) ds'ds \]

\[ = A^{(i,j)}(x) \left( \frac{(p + 1)^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) \left( (p + 1) h_n \right) \]

\[ - A^{(i,j)}(x) \left( \frac{(p + 1)^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{3} \right) \]

\[ + A^{(i,j)}(x) \left( \rho^{(i)} \right) \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} - \frac{(p + 1 - \rho^{(i)})^2 h_n^2}{2} \right) \left( \rho^{(j)} h_n \right) \]

\[ + A^{(i,j)}(x) \left( \frac{(p + 1)^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(j)})^3 h_n^3}{6} \right) \]

\[ - A^{(i,j)}(x) \left( \rho^{(i)} h_n \right) \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) \]

\[ - A^{(i,j)}(x) \left( \rho^{(i)} \right) \left( \rho^{(j)} h_n \right) \left( \frac{p^2 h_n^2}{2} - \frac{(p - \rho^{(j)})^2 h_n^2}{2} \right) \]

\[ = h_n A^{(i,j)}(x) \left( -\frac{(\rho^{(i)})^2}{2} + \rho^{(i)}\rho^{(j)} + \frac{\rho^{(i)}}{2} - \frac{1}{6} \right) , \]

and for the case (v),

\[ h_n K^{(i,j)}(x|\rho) \]
= \int_0^{(p+1)h_n} \int_0^{(p+1)h_n} A^{(i,j)} (x) \min \{s, s'\} \left( V^{(i,i)}_{\rho,h_n} ((p + 1) h_n - s) \right. \\
\left. \left( V^{(j,j)}_{\rho,h_n} ((p + 1) h_n - s') - V^{(j,j)}_{\rho,h_n} (p h_n - s') \right) ds' ds \right)
\\
= \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( (p + 1)^2 h_n^2 - \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right) \left( (p + 1) h_n \right)
\\
- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \frac{(p + 1)^3 h_n^3}{3} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{3} \right)
\\
+ \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \frac{(p + 1)^3 h_n^3}{6} - \frac{(p + 1 - \rho^{(i)})^3 h_n^3}{6} \right)
\\
- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \rho^{(i)} h_n \right) \left( \frac{(p + 1 - \rho^{(j)})^2 h_n^2}{2} \right)
\\
- \frac{A^{(i,j)} (x)}{\rho^{(i)} \rho^{(j)} h_n^2} \left( \rho^{(i)} h_n \right) \left( \frac{p^2 h_n^2}{2} - \frac{(p - \rho^{(j)})^2 h_n^2}{2} \right) \left( \rho^{(i)} h_n \right)
\\
= H_n A^{(i,j)} (x) \left( \rho^{(i)} \rho^{(j)} - \frac{(\rho^{(i)})^3}{6} \right);
\\
therefore, we obtain
\\
G^{(i,j)} (x|\rho) + D^{(i,j)}_0 (x|\rho)
\\
= \begin{cases} 
A^{(i,j)} (x) & \text{if } \rho^{(i)} = 0, \\
A^{(i,j)} (x) \left( 1 - \frac{\rho^{(j)}}{2} \right) & \text{if } \rho^{(i)} \in (0, 1], \rho^{(j)} = 0, \\
A^{(i,j)} (x) \frac{1}{2 \rho^{(i)} h_n} & \text{if } \rho^{(i)} \in (1, \overline{\rho}], \rho^{(j)} = 0, \\
A^{(i,j)} (x) \left( \rho^{(i)} \rho^{(j)} + \rho^{(i)} \right) & \text{if } \rho^{(i)} \in (\rho^{(j)} + 1, \overline{\rho}], \rho^{(j)} > 0, \\
A^{(i,j)} (x) \left( \rho^{(i)} \rho^{(j)} - \rho^{(i)} \right)^2 - 3(\rho^{(j)})^2 + 6\rho^{(i)} \rho^{(j)} + 3\rho^{(i)} - 1 \left( 6\rho^{(i)} \rho^{(j)} \right) & \text{if } \rho^{(i)} \in (1, \overline{\rho}], \rho^{(i)} \leq \rho^{(j)}, \\
A^{(i,j)} (x) - 3(\rho^{(i)})^2 + 6\rho^{(i)} \rho^{(j)} + 3\rho^{(i)} - 1 \left( 6\rho^{(i)} \rho^{(j)} \right) & \text{if } \rho^{(i)} \in (1, \overline{\rho}], \rho^{(i)} \leq \rho^{(j)}, \\
A^{(i,j)} (x) - 3(\rho^{(i)})^2 + 3\rho^{(i)} \rho^{(j)} - (\rho^{(i)})^3 \left( 6\rho^{(i)} \rho^{(j)} \right) & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} > \rho^{(j)}, \\
A^{(i,j)} (x) \frac{6\rho^{(i)} \rho^{(j)} - (\rho^{(i)})^3}{6\rho^{(i)} \rho^{(j)}} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} \leq \rho^{(j)}, 
\end{cases}
\\
and G (x|\rho) = G_c (x|\rho) = G (x, \alpha|\rho)|_{\alpha = \alpha^*}.
\\
Proof of the results in Section 3.1.
\\
Proof of Lemma 1. By following the proof of the Proposition 13, it is sufficient to evaluate
\\
= \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} A^{(i,i)} (x) \min \{s, s'\} \left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s) - V^{(i,i)}_{\rho,h_n} (p h_n - s') \right)
\\
\left. \left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s') - V^{(i,i)}_{\rho,h_n} (p h_n - s') \right) ds' ds \right)
for the asymptotic behaviour of the reduced quadratic variation. If $\rho^{(i)} = 0$,

$$
\int_0^{(p+2)h_n} \int_0^{(p+2)h_n} A^{(i,i)} (x) \min \{s, s'\} \left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s) - V^{(i,i)}_{\rho,h_n} (ph_n - s) \right) \\
\left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s') - V^{(i,i)}_{\rho,h_n} (ph_n - s') \right) \, ds' \, ds \\
= A^{(i,i)} (x) \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \min \{s, s'\} \left( \delta ((p + 2) h_n - s) - \delta (ph_n - s) \right) \\
\left( \delta ((p + 2) h_n - s') - \delta (ph_n - s') \right) \, ds' \, ds \\
= A^{(i,i)} (x) ((p + 2) h_n - 2ph_n + ph_n) \\
= 2h_n A^{(i,i)} (x),
$$

and if $\rho^{(i)} \in (0, \bar{\rho}]$,

$$
\int_0^{(p+2)h_n} \int_0^{(p+2)h_n} A^{(i,i)} (x) \min \{s, s'\} \left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s) - V^{(i,i)}_{\rho,h_n} (ph_n - s) \right) \\
\left( V^{(i,i)}_{\rho,h_n} ((p + 2) h_n - s') - V^{(i,i)}_{\rho,h_n} (ph_n - s') \right) \, ds' \, ds \\
= A^{(i,i)} (x) \left( \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \min \{s, s'\} \right) \\
\frac{1}{(\rho^{(i)} h_n)^2} \left( \int_{p+2-\rho^{(i)}h_n}^{(p+2)h_n} \int_{p+2-\rho^{(i)}h_n}^{(p+2)h_n} \min \{s, s'\} \right) \\
- 2A^{(i,i)} (x) \int_0^{(p+2)h_n} \int_0^{(p+2)h_n-\rho^{(i)}h_n} \min \{s, s'\} \\
+ \frac{1}{(\rho^{(i)} h_n)^2} \left( \int_{p-\rho^{(i)}h_n}^{ph_n} \int_{p-\rho^{(i)}h_n}^{ph_n} \min \{s, s'\} \right) \\
= A^{(i,i)} (x) \left( \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \left( \int_s^{p+2-\rho^{(i)}h_n} sds' + \int_s^{p+2-\rho^{(i)}h_n} s' ds' \right) \right) \, ds \\
- 2A^{(i,i)} (x) \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \left( \int_{p-\rho^{(i)}h_n}^{ph_n} s' ds' \right) \, ds \\
- 2A^{(i,i)} (x) \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \left( \int_{p-\rho^{(i)}h_n}^{ph_n} s' ds' \right) \, ds \\
- 2A^{(i,i)} (x) \int_0^{(p+2)h_n} \int_0^{(p+2)h_n} \left( \int_{p-\rho^{(i)}h_n}^{ph_n} s' ds' \right) \, ds \\
+ \frac{1}{(\rho^{(i)} h_n)^2} \left( \int_{p-\rho^{(i)}h_n}^{ph_n} \int_{p-\rho^{(i)}h_n}^{ph_n} \left( \int_s^{p+2-\rho^{(i)}h_n} sds' + \int_s^{p+2-\rho^{(i)}h_n} s' ds' \right) \right) \, ds.
\[
A(\mathbf{x}_n) = A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
+ A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
= A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
+ A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
= A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
+ A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
= A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
- 2A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
+ A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]

\[
= A(\mathbf{x}_n) \int_{h_n}^{(p+2)h_n} \left( (p + 2) h_n s - s^2 \right) + \left( \frac{s^2}{2} - \left( s + \rho(\mathbf{d}) \right)^2 h_n^2 \right) \, ds 
\]
and hence it is sufficient to show that

\[
\begin{align*}
- \frac{A(i,i)(x)}{(\rho(i) h_n)^2} & \mathbf{1}_{(0,2)} (\rho(i)) \left( p^2 \rho(i) h_n^3 - (p - \rho(i))^2 \rho(i) h_n^3 \right) \\
- \frac{A(i,i)(x)}{(\rho(i) h_n)^2} & \left( \frac{p^3 h_n^3}{6} - \frac{(p - \rho(i))^3 h_n^3}{6} \right) \\
+ \frac{A(i,i)(x)}{(\rho(i) h_n)^2} & \left( \frac{p^2 h_n^2}{2} - \frac{(p - \rho(i))^2 h_n^2}{2} \right) \rho h_n \\
- \frac{A(i,i)(x)}{(\rho(i) h_n)^2} & \frac{(p - \rho(i))^2 \rho(i) h_n^3}{2} \\
= & \begin{cases} 
2 h_n A(i,i)(x) \left( 1 - \frac{\rho(i)}{6} \right) & \text{if } \rho(i) \in (0, 2], \\
2 h_n A(i,i)(x) \left( \frac{2 \rho(i)}{\rho(i)^2} - \frac{4}{3(\rho(i))^2} \right) & \text{if } \rho(i) \in (2, \overline{p}] . 
\end{cases}
\end{align*}
\]

Hence, we obtain the proof. \(\square\)

**Proof of Lemma 2.** Continuity is obvious, and monotonicity is obtained as follows: if \(\rho(i) \in (0, 1],\)

\[
\begin{align*}
\frac{d}{d \rho(i)} (6 - 2 \rho(i)) (6 - \rho(i))^{-1} &= (6 - \rho(i)) \frac{d}{d \rho(i)} (6 - 2 \rho(i)) (6 - \rho(i))^{-2} \\
&= (-12) (6 - \rho(i))^{-2} < 0,
\end{align*}
\]

and if \(\rho(i) \in (1, 2],\)

\[
\begin{align*}
\frac{d}{d \rho(i)} (6 \rho(i) - 2) (6 (\rho(i))^2 - (\rho(i))^3)^{-1} \\
&= \left( 6 \left( 6 (\rho(i))^2 - (\rho(i))^3 \right) - (6 \rho(i) - 2) \left( 12 \rho(i)^2 - 3 (\rho(i))^2 \right) \right) (6 (\rho(i))^2 - (\rho(i))^3)^{-2} \\
&= 6 \rho(i) (-7 \rho(i) + 4 + 2 (\rho(i))^2) (6 (\rho(i))^2 - (\rho(i))^3)^{-2} < 0,
\end{align*}
\]

and if \(\rho(i) \in (2, \overline{p}],\)

\[
\begin{align*}
\frac{d}{d \rho(i)} (3 \rho(i) - 1) (6 \rho(i) - 4)^{-1} &= (-18) (6 \rho(i) - 4)^{-2} < 0.
\end{align*}
\]

The inverse can be obtained directly. \(\square\)

**Proofs of the results in Section 3.2.**

**Proof of Theorem 4.** We can clearly prove the result by using Lemma 7 in Kessler (1997), Proposition 7 in Nakakita and Uchida (2019a), and Slutsky’s theorem. \(\square\)

**Proof of Theorem 5.** By Lemma 1, there exists a number \(\ell < 0\) such that

\[
\frac{1}{n h_n} \sum_{k=1}^{n} \left( \overline{X}_{kh_n,m}^{(i)} - \overline{X}_{(k-1)h_n,m}^{(i)} \right) \leq \frac{1}{n h_n} \sum_{2 \leq 2k \leq n} \left( \overline{X}_{2kh_n,m}^{(i)} - \overline{X}_{(2k-2)h_n,m}^{(i)} \right)^2 \rightarrow P \ell < 0,
\]

and hence it is sufficient to show that

\[
\sup_{n \in \mathbb{N}} \mathbf{E} \left[ \frac{2}{3n h_n^2} \sum_{k=1}^{n} \left( \overline{X}_{kh_n,m}^{(i)} - \overline{X}_{(k-1)h_n,m}^{(i)} \right)^4 \right] < \infty;
\]
and it is obvious that
\[
(X_{kh_n,n}^{(i)} - X_{(k-1)h_n,n}^{(i)})^4 \leq C \left( X_{kh_n,n}^{(i)} - X_{(k-1)h_n,n} \right)^4 + C \left( X_{(k-1)h_n,n} - X_{(k-1)h_n,n} \right)^4
\]
and
\[
E \left[ \left( X_{kh_n,n}^{(i)} - X_{(k-1)h_n,n} \right)^4 \left| F_{(k-1)h_n} \right. \right]
= E \left[ \left( \frac{1}{\rho_*^{(i)}} h_n \int_{(k-\rho_*^{(i)})h_n}^{kh_n} (X_s^{(i)} - X_{(k-\rho_*^{(i)})h_n}) \, ds \right)^4 \left| F_{(k-\rho_*^{(i)})h_n} \right. \right]
\leq E \left[ \left( \frac{1}{\rho_*^{(i)}} h_n \int_{(k-\rho_*^{(i)})h_n}^{kh_n} \sup_{s' \in [(k-\rho_*^{(i)})h_n,kh_n]} |X_{s'}^{(i)} - X_{(k-\rho_*^{(i)})h_n}| \, ds \right)^4 \left| F_{(k-\rho_*^{(i)})h_n} \right. \right]
\leq C h_n \left( 1 + \left| X_{(k-\rho_*^{(i)})h_n} \right| \right)^4
\]
by Proposition A in Gloter (2000), and a parallel result holds for \( \left( X_{(k-1)h_n,n}^{(i)} - X_{(k-1)h_n,n} \right)^4 \). Hence we obtain the result.

**Proof of the results in Section 4.**

**Proof of Theorem 6.** We only deal with the case where \( \rho_* \) is unknown because the discussion for the case where \( \rho_* \) is known is parallel. First of all, we prove the consistency of \( \hat{\alpha}_n \). We obtain that
\[
\left| \frac{1}{n} \mathbb{H}_{1,n} (\alpha | \hat{\rho}_n) - \frac{1}{n} \mathbb{H}_{1,n} (\alpha | \rho_*) \right|
= \left| - \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{h_n} (\hat{X}_{kh_n,n} - \hat{X}_{(k-1)h_n,n})^2 \right) \mathcal{G} (\hat{X}_{kh_n,n}, \alpha | \hat{\rho}_n) - \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \hat{\rho}_n) \right|^2
+ \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{h_n} (\hat{X}_{kh_n,n} - \hat{X}_{(k-1)h_n,n})^2 \right) \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \rho_*) \right|^2
\leq \frac{2}{nh_n} \sum_{k=1}^n \left( \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \hat{\rho}_n) - \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \rho_*) \right) \left( \hat{X}_{kh_n,n} - \hat{X}_{(k-1)h_n,n} \right) \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \rho_*) \right|^2
+ \frac{1}{n} \sum_{k=1}^n \left( \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \rho_*) \right)^2 - \left( \mathcal{G} (\hat{X}_{(k-1)h_n,n}, \alpha | \rho_*) \right)^2
\leq \frac{2}{nh_n} \sum_{k=1}^n \left( \hat{X}_{kh_n,n} - \hat{X}_{(k-1)h_n,n} \right)^2
\[
\begin{align*}
&\times \sum_{i=1}^{d} \sum_{j=1}^{d} \left| A^{(i,j)}(\mathbf{X}_{(k-1)h_n,n}, \alpha) \right| \left| f_{G}^{2}\left(\hat{\rho}^{(i)}_{n} \hat{\rho}^{(j)}_{n}\right) - f_{G}^{2}\left(\rho^{(i)}_{\ast} \rho^{(j)}_{\ast}\right)\right| \\
&\quad + \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{d} \sum_{j=1}^{d} \left| A^{(i,j)}(\mathbf{X}_{(k-1)h_n,n}, \alpha) \right|^{2} \left| f_{G}^{2}\left(\hat{\rho}^{(i)}_{n} \hat{\rho}^{(j)}_{n}\right) - f_{G}^{2}\left(\rho^{(i)}_{\ast} \rho^{(j)}_{\ast}\right)\right| \\
&\leq \frac{C}{n h_n} \sum_{k=1}^{n} \left(1 + \left| \mathbf{X}_{(k-1)h_n,n}\right|\right)^{C} \left| \mathbf{X}_{kh_n,n} - \mathbf{X}_{(k-1)h_n,n}\right|^{2} \\
&\quad \times \sum_{i=1}^{d} \sum_{j=1}^{d} \left| f_{G}^{2}\left(\hat{\rho}^{(i)}_{n} \hat{\rho}^{(j)}_{n}\right) - f_{G}^{2}\left(\rho^{(i)}_{\ast} \rho^{(j)}_{\ast}\right)\right| \\
&\quad + \frac{C}{n} \sum_{k=1}^{n} \left(1 + \left| \mathbf{X}_{(k-1)h_n,n}\right|\right)^{C} \sum_{i=1}^{d} \sum_{j=1}^{d} \left| f_{G}^{2}\left(\hat{\rho}^{(i)}_{n} \hat{\rho}^{(j)}_{n}\right) - f_{G}^{2}\left(\rho^{(i)}_{\ast} \rho^{(j)}_{\ast}\right)\right| \\
&\quad \to^{P} 0 \text{ uniformly in } \alpha,
\end{align*}
\]

because continuous mapping theorem holds. Therefore, it follows from Proposition 11 and Proposition 13 that

\[
\frac{1}{n} \mathbb{H}_{1,n} (\alpha | \hat{\rho}_{n}) - \frac{1}{n} \mathbb{H}_{1,n} (\alpha | \rho_{\ast}) = \frac{2}{n h_n} \sum_{k=1}^{n} \mathbb{G}\left(\mathbf{X}_{(k-1)h_n,n}, \alpha | \rho_{\ast}\right) \left[\left(\mathbf{X}_{kh_n,n} - \mathbf{X}_{(k-1)h_n,n}\right)^{\otimes 2}\right] \\
- \frac{1}{n} \sum_{k=1}^{n} \left\| \mathbb{G}\left(\mathbf{X}_{(k-1)h_n,n}, \alpha | \rho_{\ast}\right) \right\|^{2} \\
- \frac{2}{n h_n} \sum_{k=1}^{n} \mathbb{G}\left(\mathbf{X}_{(k-1)h_n,n}, \alpha_{\ast} | \rho_{\ast}\right) \left[\left(\mathbf{X}_{kh_n,n} - \mathbf{X}_{(k-1)h_n,n}\right)^{\otimes 2}\right] \\
+ \frac{1}{n} \sum_{k=1}^{n} \left\| \mathbb{G}\left(\mathbf{X}_{(k-1)h_n,n}, \alpha_{\ast} | \rho_{\ast}\right) \right\|^{2} \\
+ o_{P}(1) \\
\to^{P} \mathbb{V}_{1} (\alpha | \xi_{\ast}) \text{ uniformly in } \alpha
\]

where \(o_{P}(1)\) indicates the term converging in probability to zero uniformly in \(\theta\). Then we obtain that \(\hat{\alpha}_{n} \to \alpha_{\ast}\) in the same way as Kessler (1997) with Assumption [A3].

In the next place, we consider the consistency of \(\hat{\beta}_{n}\). Firstly, we consider the case \(\max_{\ell} \rho_{\ast}^{(\ell)} \in (\ell - 1, \ell)\) for an integer \(\ell \in \{1, \ldots, [\rho] + 1\}\). Then it is sufficient to show

\[
\frac{1}{n h_n} \mathbb{H}_{2,n} (\beta | \hat{\rho}_{n}) - \frac{1}{n h_n} \mathbb{H}_{2,n} (\beta_{\ast} | \rho_{\ast}) \to^{P} \mathbb{V}_{2} (\beta | \xi_{\ast}) \text{ uniformly in } \beta
\]

due to Assumption [A3]. Because the evaluation \(D_{j}(x) = O\) where \(j \geq \left[ \max_{i=1,\ldots,n} \rho_{\ast}^{(i)} \right] + 1\) using independent increments of the Wiener process, Proposition 11 and Proposition 12 verify

\[
F_{\ell}(\beta) - \frac{1}{n h_n} \mathbb{H}_{2,n} (\beta_{\ast} | \rho_{\ast}) \to^{P} \mathbb{V}_{2} (\beta | \xi_{\ast}) \text{ uniformly in } \beta,
\]
where

\[ F_j(\beta) := -\frac{1}{nh_n^2} \sum_{k=1+j}^{n} |X_{kh_n,n} - X_{(k-1)h_n,n} - h_n b(X_{(k-1)h_n,n}, \beta)|^2. \]

In addition, the exact convergences such that

\[ P\left(1_\{\ell\} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) \to 1, \quad P\left(1_{\{j\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) \to 0 \]

hold for all \( j \neq \ell \), since for all \( j = 1, \ldots, [\bar{\rho}] + 1 \),

\[ P\left(1_{\{j\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) = P\left(\max_i \hat{\rho}_n^{(i)} \in [j - 1, j)\right). \]

Therefore, for any \( \epsilon > 0 \),

\[ P\left(\sup_{\beta \in \Theta_2} \left|\frac{1}{nh_n} \mathbb{H}_{2,n}(\beta|\hat{\rho}_n) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_*|\rho_\star) - \mathbb{V}_2(\beta|\xi_\star)\right| > \epsilon\right) \]

\[ \leq \sum_{j \neq \ell} P\left(1_{\{j\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) + P\left(1_{\{\ell\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) \]

\[ \cap \left\{\sup_{\beta \in \Theta_2} \left|F_{\ell}(\beta) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_*|\rho_\star) - \mathbb{V}_2(\beta|\xi_\star)\right| > \epsilon\right\} \]

\[ \to 0. \]

For the case \( \max_i \rho_\star^{(i)} = \ell \) for an integer \( \ell = \{0, \ldots, [\bar{\rho}] + 1\} \), we similarly obtain

\[ \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta|\hat{\rho}_n) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_*|\rho_\star) \to^P \mathbb{V}_2(\beta|\xi_\star) \]

uniformly in \( \beta \) because we have

\[ F_{\ell}(\beta) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_*|\rho_\star) \to^P \mathbb{V}_2(\beta|\xi_\star) \]

uniformly in \( \beta \),

\[ F_{\ell+1}(\beta) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_*|\rho_\star) \to^P \mathbb{V}_2(\beta|\xi_\star) \]

uniformly in \( \beta \),

and

\[ P\left(1_{\{\ell\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) + 1_{\{\ell + 1\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 1\right) \to 1, \]

\[ P\left(1_{\{j\}} \left(\left[\max_i \hat{\rho}_n^{(i)}\right] + 1\right) = 0\right) \to 1, \text{ for all } j \neq \ell, \ell + 1, \]
and it holds that for any $\epsilon > 0$,
\[
P\left( \sup_{\beta \in \Theta_2} \left| \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta | \hat{\rho}_n) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_* | \rho_* \mid \xi_*) \right| > \epsilon \right)
\leq \sum_{j \neq \ell, \ell + 1} P\left( \mathbb{1}_{\{j\}} \left( \left[ \max_i \hat{\rho}^{(i)}_n \right] + 1 \right) = 1 \right)
+ P\left( \mathbb{1}_{\{\ell\}} \left( \left[ \max_i \hat{\rho}^{(i)}_n \right] + 1 \right) = 1 \right)
\cap \left\{ \sup_{\beta \in \Theta_2} \left| F_{\ell}(\beta) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_* | \rho_* \mid \xi_*) \right| > \epsilon \right\}
+ P\left( \mathbb{1}_{\{\ell + 1\}} \left( \left[ \max_i \hat{\rho}^{(i)}_n \right] + 1 \right) = 1 \right)
\cap \left\{ \sup_{\beta \in \Theta_2} \left| F_{\ell + 1}(\beta) - \frac{1}{nh_n} \mathbb{H}_{2,n}(\beta_* | \rho_* \mid \xi_*) \right| > \epsilon \right\}
\to 0.
\]

Hence it is shown that $\hat{\beta}_n \to^p \beta_*$ with Assumption [A3].

**APPENDIX B. TRIVIAL DISCUSSION**

The next lemma supports an exchangeability of integrals.

**Lemma 14.** Let us fix $C > 0$. For any $t_1, t_2$ such that $0 \leq t_1 \leq t_2$ and $t_2 - t_1 \leq C$, and $f : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^r$ such that $f \in L^1([t_1, t_2])$, the following equation holds:
\[
\int_{t_1}^{t_2} f(s) \left( \int_{t_1}^{s_1} dw_{s_2} \right) ds_1 = \int_{t_1}^{t_2} \left( \int_{s_1}^{t_2} f(s_2) ds_2 \right) dw_{s_1};
\]
and both are normally distributed with mean 0 and variance $\int_{t_1}^{t_2} \left( \int_{s_1}^{t_2} f(s_2) ds_2 \right)^2 ds_1$.

**Proof.** For $f : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^r$, we set $F(t) = \int_0^t f(s) ds$, and then it follows from Itô’s formula that
\[
F(t) w_t = \int_0^t \left( \frac{d}{dt} F(t) \right)_{t=s} w_s ds + \int_0^t F(s) dw_s
= \int_0^t f(s) w_s ds + \int_0^t F(s) dw_s,
\]
and

\[
\int_{t_1}^{t_2} f(s) \left( \int_{t_1}^{s_1} dw_{s_2} \right) \, ds_1 = \int_{t_1}^{t_2} f(s) (w_{s_1} - w_{t_1}) \, ds_1
\]

\[
= F(t_2) w_{t_2} - F(t_1) w_{t_1} - \int_{t_1}^{t_2} f(s) w_{t_1} \, ds - \int_{t_1}^{t_2} F(s) \, dw_s
\]

\[
= F(t_2) w_{t_2} - F(t_1) w_{t_1} - (F(t_2) - F(t_1)) w_{t_1} - \int_{t_1}^{t_2} F(s) \, dw_s
\]

\[
= \int_{t_1}^{t_2} (F(t_2) - F(s)) \, dw_s
\]

\[
= \int_{t_1}^{t_2} \left( \int_{s_1}^{t_2} f(s_2) \, ds_2 \right) \, dw_s,
\]

which is normally distributed because of Wiener integral obviously.

**Lemma 15.** $f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)})$ is continuous.

**Proof.** We check the continuity at (i) $\rho^{(i)} = \rho^{(j)} = 0$, (ii) $\rho^{(i)} = 0$ and $\rho^{(j)} \in (0,1)$, (iii) $\rho^{(i)} = 0$ and $\rho^{(j)} = 1$, (iv) $\rho^{(i)} = 0$ and $\rho^{(j)} \in (1, \bar{p}]$, (v) $\rho^{(j)} = 0$ and $\rho^{(i)} \in (0, \bar{p}]$, (vi) $\rho^{(j)} = \rho^{(j)} = 0$, (vii) $\rho^{(i)} = \rho^{(j)} = 1$, (viii) $\rho^{(i)} = \rho^{(j)} \in (1, \bar{p}]$, (ix) $\rho^{(i)} = 1$ and $\rho^{(j)} \in (0, \bar{p}]$, (x) $\rho^{(j)} = 1$ and $\rho^{(i)} \in (1, \bar{p}]$, (xi) $\rho^{(i)} \in (0, \bar{p}]$ and $\rho^{(i)} + 1 = \rho^{(j)}$.

(i) We have that

\[
f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)})|_{\rho^{(i)}=\rho^{(j)}=0} = 0,
\]

\[
\lim_{\rho^{(i)}=0, \rho^{(j)} \downarrow 0} f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)}=0, \rho^{(j)} \downarrow 0} \rho^{(j)}/2 = 0,
\]

\[
\lim_{\rho^{(i)}, \rho^{(j)} \downarrow 0} f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} \frac{(\rho^{(i)})^2 - 3\rho^{(i)} \rho^{(j)} + 3(\rho^{(j)})^2}{6\rho^{(j)}} = 0,
\]

\[
\lim_{\rho^{(i)}, \rho^{(j)} \downarrow 0} f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} \frac{(\rho^{(j)})^2}{6\rho^{(i)}} = 0,
\]

\[
\lim_{\rho^{(j)}=0, \rho^{(i)} \downarrow 0} f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)}) = 0.
\]

(ii) It holds that

\[
f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)})|_{\rho^{(i)}=0, \rho^{(j)} \in (0,1)} = \frac{\rho^{(j)}}{2},
\]

\[
\lim_{\rho^{(j)} \in (0,1), \rho^{(i)} \downarrow 0} f_{\mathbb{D}_0}(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(j)} \in (0,1), \rho^{(i)} \downarrow 0} \frac{(\rho^{(i)})^2 - 3\rho^{(i)} \rho^{(j)} + 3(\rho^{(j)})^2}{6\rho^{(j)}} = \frac{\rho^{(j)}}{2}.
\]
(iii) We obtain that
\[
\frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \frac{1}{2},
\]
\[
\lim_{\rho_i = 0, \rho_j \uparrow 1} \frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \frac{1}{2},
\]
\[
\lim_{\rho_i = 0, \rho_j \downarrow 1} \frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i = 0, \rho_j \downarrow 1} \frac{2\rho_j - 1}{2\rho_j} = \frac{1}{2},
\]
\[
\lim_{\rho_i \downarrow 0, \rho_j \downarrow 1} \frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i \downarrow 0, \rho_j \downarrow 1} \frac{(\rho_i)^2 - 3\rho_i \rho_j + 3 (\rho_j)^2}{6\rho_j} = \frac{1}{2},
\]
\[
\lim_{\rho_i \uparrow 1, \rho_j \uparrow 1, \rho_i < \rho_j} \frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i \uparrow 1, \rho_j \uparrow 1, \rho_i < \rho_j} \frac{(\rho_i)^3 - 3 (\rho_i)^2 \rho_j + 3\rho_i (\rho_j)^2}{6\rho_i \rho_j} = \frac{1}{2},
\]
\[
\lim_{\rho_i \downarrow 1, \rho_j \downarrow 1, \rho_i < \rho_j} \frac{\rho_i}{\rho_i + 1} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i \downarrow 1, \rho_j \downarrow 1, \rho_i < \rho_j} \frac{6\rho_i \rho_j - 3 (\rho_i)^2 - 3\rho_j}{6\rho_i \rho_j} = \frac{3}{2}.
\]

(iv) We can have that
\[
\frac{\rho_i}{\rho_j} \left( \frac{\rho_j}{\rho_j + 1} \right) \bigg|_{\rho_i = 0, \rho_j \in (1, \infty)} = \frac{2\rho_j - 1}{2\rho_j},
\]
\[
\lim_{\rho_i \in (0, \infty), \rho_j \uparrow 1} \frac{\rho_i}{\rho_j} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i \in (0, \infty), \rho_j \uparrow 1} \frac{6\rho_i \rho_j - 3 (\rho_i)^2 - 3\rho_i}{6\rho_i \rho_j} = \frac{2\rho_j - 1}{2\rho_j}.
\]

(v) It holds that
\[
\frac{\rho_i}{\rho_j} \left( \frac{\rho_j}{\rho_j + 1} \right) \bigg|_{\rho_i \in (0, \infty), \rho_j = 0} = 0,
\]
\[
\lim_{\rho_i \in (0, \infty), \rho_j \uparrow 1} \frac{\rho_i}{\rho_j} \left( \frac{\rho_j}{\rho_j + 1} \right) = \lim_{\rho_i \in (0, \infty), \rho_j \uparrow 1} \frac{(\rho_j)^3}{6\rho_i \rho_j} = 0.
\]
(vi) We have that

\[
\begin{align*}
    f_{D_0}(\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)}=\rho^{(j)} \in (0,1)} &= \frac{\rho^{(i)}}{6}, \\
    \lim_{\rho^{(i)} \to 0, \rho^{(i)} \geq \rho^{(j)}, \rho^{(i)} \in (0,1), \rho^{(i)} - \rho^{(j)} \to 0,} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \frac{\rho^{(i)}}{6}, \\
    \lim_{\rho^{(i)} \to 0, \rho^{(i)} < \rho^{(j)} \in (0,1), \rho^{(i)} - \rho^{(j)} \to 0,} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \to 0, \rho^{(i)} < \rho^{(j)} \in (0,1), \rho^{(i)} - \rho^{(j)} \to 0,} \frac{(\rho^{(i)} - \rho^{(j)})^3 + (\rho^{(j)})^3}{6\rho^{(i)}\rho^{(j)}} = \frac{\rho^{(i)}}{6}.
\end{align*}
\]

(vii) It holds that

\[
\begin{align*}
    f_{D_0}(\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)}=\rho^{(j)}=1} &= \frac{1}{6}, \\
    \lim_{\rho^{(i)} \to 1, \rho^{(i)} \downarrow} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \downarrow} \frac{(\rho^{(i)} - \rho^{(j)})^3 + (\rho^{(j)})^3}{6\rho^{(i)}\rho^{(j)}} = \frac{1}{6}, \\
    \lim_{\rho^{(i)} \to 1, \rho^{(i)} \uparrow \rho^{(j)} \downarrow} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \uparrow \rho^{(j)} \downarrow} \frac{(\rho^{(j)})^3}{6\rho^{(i)}\rho^{(j)}} = \frac{1}{6}, \\
    \lim_{\rho^{(i)} \to 1, \rho^{(i)} \uparrow \rho^{(j)} \downarrow} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \uparrow \rho^{(j)} \downarrow} \frac{(\rho^{(i)} - \rho^{(j)})^3 + 3(\rho^{(j)})^2 - 3\rho^{(j)} + 1}{6\rho^{(i)}\rho^{(j)}} = \frac{1}{6}, \\
    \lim_{\rho^{(i)} \to 1, \rho^{(i)} \downarrow \rho^{(j)} \uparrow} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \downarrow \rho^{(j)} \uparrow} \frac{3(\rho^{(j)})^2 - 3\rho^{(j)} + 1}{6\rho^{(i)}\rho^{(j)}} = \frac{1}{6}.
\end{align*}
\]

(viii) We have that

\[
\begin{align*}
    f_{D_0}(\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)}=\rho^{(j)} \in [1,\infty]} &= \frac{3(\rho^{(i)})^2 - 3\rho^{(i)} + 1}{6(\rho^{(i)})^2}, \\
    \lim_{\rho^{(i)} \in [1,\infty], \rho^{(i)} \geq \rho^{(j)} \to 0,} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \frac{3(\rho^{(i)})^2 - 3\rho^{(i)} + 1}{6(\rho^{(i)})^2}, \\
    \lim_{\rho^{(i)} \in [1,\infty], \rho^{(i)} \geq \rho^{(j)} \to 0,} f_{D_0}(\rho^{(i)}, \rho^{(j)}) &= \lim_{\rho^{(i)} \in [1,\infty], \rho^{(i)} \geq \rho^{(j)} \to 0,} \frac{(\rho^{(i)} - \rho^{(j)})^3 + 3(\rho^{(j)})^2 - 3\rho^{(j)} + 1}{6\rho^{(i)}\rho^{(j)}} = \frac{3(\rho^{(i)})^2 - 3\rho^{(i)} + 1}{6(\rho^{(i)})^2}.
\end{align*}
\]
(ix) It holds that

\[
\begin{align*}
f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) \big|_{\rho^{(i)} \in (0,1), \rho^{(j)} = 1} &= \frac{\left( \rho^{(i)} - 1 \right)^3 + 1}{6\rho^{(i)}}, \\
\lim_{\rho^{(i)} \in (0,1), \rho^{(j)} \uparrow 1} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \frac{\left( \rho^{(i)} - 1 \right)^3 + 1}{6\rho^{(i)}}, \\
\lim_{\rho^{(i)} \in (0,1), \rho^{(j)} \downarrow 1} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \lim_{\rho^{(i)} \in (0,1), \rho^{(j)} \downarrow 1} \frac{\left( \rho^{(i)} - \rho^{(j)} \right)^3 + 3 \left( \rho^{(j)} \right)^2 - 3\rho^{(j)} + 1}{6\rho^{(i)} \rho^{(j)}} \\
&= \frac{\left( \rho^{(i)} - 1 \right)^3 + 1}{6\rho^{(i)}}.
\end{align*}
\]

(x) We have that

\[
\begin{align*}
f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) \big|_{\rho^{(i)} \in (1,\rho^{(i)} = 1} &= \frac{1}{6\rho^{(i)}}, \\
\lim_{\rho^{(i)} \in (1,\rho^{(j)} \uparrow 1} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \frac{1}{6\rho^{(i)}}, \\
\lim_{\rho^{(i)} \in (1,\rho^{(j)} \downarrow 1} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \lim_{\rho^{(i)} \in (1,\rho^{(j)} \downarrow 1} \frac{3 \left( \rho^{(j)} \right)^2 - 3\rho^{(j)} + 1}{6\rho^{(i)} \rho^{(j)}} \\
&= \frac{1}{6\rho^{(i)}}.
\end{align*}
\]

(xi) It holds that

\[
\begin{align*}
f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) \big|_{\rho^{(i)} \in (0,\rho^{(i)} + 1 = \rho^{(j)} } &= \frac{1}{2}, \\
\lim_{\rho^{(i)} < \rho^{(i)} + 1 \to 0} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \frac{1}{2}, \\
\lim_{\rho^{(i)} \to 0, \rho^{(i)} + 1 - \rho^{(j)} \to 0} f_{\mathbb{D}_0} \left( \rho^{(i)}, \rho^{(j)} \right) &= \lim_{\rho^{(i)} \to 0, \rho^{(i)} + 1 - \rho^{(j)} \to 0} \frac{6\rho^{(i)} \rho^{(j)} - 3 \left( \rho^{(i)} \right)^2 - 3\rho^{(i)}}{6\rho^{(i)} \rho^{(j)}} \\
&= \lim_{\rho^{(i)} \to 0, \rho^{(i)} + 1 - \rho^{(j)} \to 0} \frac{6 \left( \rho^{(i)} + 1 \right) - 3\rho^{(i)} - 3}{6 \left( \rho^{(i)} + 1 \right)} \\
&= \frac{1}{2}.
\end{align*}
\]

Therefore we obtain the continuity of \( f_{\mathbb{D}_0} \).

\( \Box \)

**Lemma 16.** \( f_G \left( \rho^{(i)}, \rho^{(j)} \right) \) is continuous.
Proof. Since the continuity of \( f_{\mathbb{D}_0} \) is shown in Lemma 15, it is sufficient to show the continuity of \( f_G + f_{\mathbb{D}_0} \). Note that

\[
(f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \begin{cases}
1 & \text{if } \rho^{(i)} = 0, \\
1 - \frac{\delta^{(i)}}{2} & \text{if } \rho^{(i)} \in (0, 1], \rho^{(j)} = 0, \\
\frac{1}{2\rho^{(i)}} \left(\frac{\rho^{(j)} + \rho^{(i)}}{2}\right) & \text{if } \rho^{(i)} \in (1, \mathbb{P}], \rho^{(j)} = 0, \\
1 - \frac{6\rho^{(i)}\rho^{(j)}}{\rho^{(i)} - \rho^{(j)}} - \frac{3(\rho^{(i)})^2 + 6\rho^{(i)}\rho^{(j)} + 3(\rho^{(i)})^2 - 1}{6\rho^{(i)}\rho^{(j)}} & \text{if } \rho^{(i)} \in (\rho^{(j)} + 1, \mathbb{P}], \rho^{(j)} > 0, \\
-3(\rho^{(i)})^2 + 6\rho^{(i)}\rho^{(j)} + 3(\rho^{(i)})^2 - 1 & \text{if } \rho^{(i)} \in (1, \mathbb{P}], \rho^{(i)} \rho^{(j)} + 1, \\
-3(\rho^{(i)})^2 - 6\rho^{(i)}\rho^{(j)} + 3(\rho^{(i)})^2 - 1 & \text{if } \rho^{(i)} \in (1, \mathbb{P}], \rho^{(i)} \leq \rho^{(j)}, \\
-3(\rho^{(i)})^2 - 6\rho^{(i)}\rho^{(j)} + 3(\rho^{(i)})^2 - 1 & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} > \rho^{(j)}, \\
6\rho^{(i)}\rho^{(j)} - (\rho^{(i)})^3 & \text{if } \rho^{(i)} \in (0, 1], \rho^{(i)} \leq \rho^{(j)}. \\
\end{cases}
\]

We check the continuity of \( f_G + f_{\mathbb{D}_0} \) at

(i) \( \rho^{(i)} = \rho^{(j)} = 0 \),
(ii) \( \rho^{(i)} \in (0, 1) \) and \( \rho^{(j)} = 0 \),
(iii) \( \rho^{(i)} = 1 \) and \( \rho^{(j)} = 0 \),
(iv) \( \rho^{(i)} \in (1, \mathbb{P}] \) and \( \rho^{(j)} = 0 \),
(v) \( \rho^{(i)} \in (1, \mathbb{P}] \) and \( \rho^{(j)} = \rho^{(j)} + 1 \),
(vi) \( \rho^{(i)} \in (0, 1) \) and \( \rho^{(i)} = \rho^{(j)} \),
(vii) \( \rho^{(i)} = 1 \) and \( \rho^{(j)} \in (0, 1) \),
(viii) \( \rho^{(i)} = \rho^{(j)} = 1 \),
(ix) \( \rho^{(i)} = 1 \) and \( \rho^{(j)} \in (1, \mathbb{P}] \),
(x) \( \rho^{(i)} \in (1, \mathbb{P}] \) and \( \rho^{(i)} = \rho^{(j)} \),
(xi) \( \rho^{(i)} = 0, \rho^{(j)} \in (0, 1] \).

For (i), we obtain that

\[
(f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} = \rho^{(j)} = 0} = 1,
\]

\[
\lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} (f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} \left(1 - \frac{\delta^{(i)}}{2}\right) = 1,
\]

\[
\lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0, \rho^{(i)} > \rho^{(j)}} (f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} \frac{3\rho^{(i)}(\rho^{(j)})^2 - 3(\rho^{(i)})^2 \rho^{(j)} + 6\rho^{(i)}\rho^{(j)} - (\rho^{(j)})^3}{6\rho^{(i)}\rho^{(j)}} = 1,
\]

\[
\lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0, \rho^{(i)} \leq \rho^{(j)}} (f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \downarrow 0, \rho^{(j)} \downarrow 0} \frac{6\rho^{(i)}\rho^{(j)} - (\rho^{(i)})^3}{6\rho^{(i)}\rho^{(j)}} = 1,
\]

\[
\lim_{\rho^{(i)} = 0, \rho^{(j)} \downarrow 0} (f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} = 0, \rho^{(j)} \downarrow 0} 1 = 1.
\]

For (ii), one has that

\[
(f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} \in (0, 1), \rho^{(j)} = 0} = 1 - \frac{\rho^{(i)}}{2},
\]

\[
\lim_{\rho^{(i)} \in (0, 1), \rho^{(j)} \downarrow 0} (f_G + f_{\mathbb{D}_0})(\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \in (0, 1), \rho^{(j)} \downarrow 0} \frac{-3(\rho^{(i)})^2 \rho^{(j)} + 3(\rho^{(i)})^2 \rho^{(j)} + 6\rho^{(i)}\rho^{(j)} - (\rho^{(j)})^3}{6\rho^{(i)}\rho^{(j)}} = 1 - \frac{\rho^{(i)}}{2}.
\]
For (iv), we can evaluate that

\[
(f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) \bigg|_{\rho^{(i)}=1, \rho^{(j)}=0} = \frac{1}{2},
\]

\[
\lim_{\rho^{(i)} \uparrow 1, \rho^{(j)} \to 0} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) = \lim_{\rho^{(i)} \uparrow 1, \rho^{(j)} = 0} \left( 1 - \frac{\rho^{(i)}}{2} \right) = \frac{1}{2},
\]

\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (0,1), \rho^{(i)} > \rho^{(j)}} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)}))
\]

\[
= \lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} \frac{3\rho^{(i)} (\rho^{(j)})^2 - 3 (\rho^{(i)})^2 \rho^{(j)} + 6 \rho^{(i)} \rho^{(j)} - (\rho^{(j)})^3}{6 \rho^{(i)} \rho^{(j)}} = \frac{1}{2},
\]

\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)}))
\]

\[
= \lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} \frac{(\rho^{(i)})^3 - 3 (\rho^{(i)})^2 \rho^{(j)} - 3 (\rho^{(i)})^2 + 3 \rho^{(i)} - 1}{6 \rho^{(i)} \rho^{(j)}} + 1
\]

\[
= \lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} \frac{(\rho^{(i)})^3 - 3 (\rho^{(i)})^2 + 3 \rho^{(i)} - 1}{6 \rho^{(i)} \rho^{(j)}} + \frac{1}{2}
\]

\[
= \frac{1}{2},
\]

\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) = \lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} \in (\rho^{(j)},\rho^{(j)} + 1]} \frac{(\rho^{(j)})^2 + \rho^{(j)}}{2 \rho^{(i)} \rho^{(j)}} = \frac{1}{2},
\]

\[
\lim_{\rho^{(i)} \downarrow 1, \rho^{(j)} = 0} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) = \lim_{\rho^{(i)} \downarrow 1, \rho^{(j)} = 0 \atop \rho^{(i)} \in (1,\rho^{(j)}), \rho^{(i)} = \rho^{(j)} + 1} \frac{1}{2 \rho^{(i)}} = \frac{1}{2}.
\]

For (v), we obtain that

\[
(f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) \bigg|_{\rho^{(i)}=1, \rho^{(j)}=0} = \frac{1}{2 \rho^{(i)}},
\]

\[
\lim_{\rho^{(i)} \in (1,\rho^{(j}) \downarrow 0} (f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) = \lim_{\rho^{(i)} \in (1,\rho^{(j}) \downarrow 0} \frac{(\rho^{(j)})^2 + \rho^{(j)}}{2 \rho^{(i)} \rho^{(j)}} = \frac{1}{2 \rho^{(i)}}.
\]

For (v), it holds that

\[
(f_G + f_D) ((\rho^{(i)}), (\rho^{(j)})) \bigg|_{\rho^{(i)} \in (1,\rho^{(j}) = \rho^{(j)} + 1} = \frac{1}{2}.
\]
For (vi), we have that
\[
\lim_{\rho^{(i)} \in (1, \infty), \rho^{(j)} \to \rho^{(i)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \left( \rho^{(i)} \right) = \frac{1}{2},
\]
\[
\lim_{\rho^{(i)} \in (1, \infty), \rho^{(j)} \leq \rho^{(i)} \to 0} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \leq \rho^{(j)} \to 0} \frac{(\rho^{(j)})^2 + \rho^{(j)}}{2 \rho^{(i)} \rho^{(j)}} = \frac{1}{2}.
\]

For (vii), it holds that
\[
(f_G + f_{D_0}) (\rho, \rho^{(j)}) \bigg|_{\rho^{(i)} \in (0, 1), \rho^{(i)} = \rho^{(j)}} = 1 - \frac{\rho^{(j)}}{6},
\]
\[
\lim_{\rho^{(i)} \in (0, 1), \rho^{(i)} \to 0} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = 1 - \frac{\rho^{(i)}}{6},
\]
\[
\lim_{\rho^{(i)} \in (0, 1), \rho^{(i)} \to 0} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)})
\]
\[
= \lim_{\rho^{(i)} \in (0, 1), \rho^{(i)} \to 0} \frac{-3(\rho^{(i)})^2 \rho^{(j)} + 3\rho^{(i)} \rho^{(j)}^2 + 6\rho^{(i)} \rho^{(j)} - (\rho^{(j)})^3}{6\rho^{(i)} \rho^{(j)}}
\]
\[
= 1 - \frac{\rho^{(i)}}{6}.
\]

For (vii), it holds that
\[
(f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \bigg|_{\rho^{(i)} = 1, \rho^{(j)} \in (0, 1)} = \frac{1}{2} + \frac{\rho^{(j)}}{2} - \frac{(\rho^{(j)})^2}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1)} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1)} \frac{1}{2} + \frac{\rho^{(j)}}{2} - \frac{(\rho^{(j)})^2}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1), \rho^{(i)} \in [\rho^{(j)}, \rho^{(j)} + 1]} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)})
\]
\[
= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1), \rho^{(i)} \in [\rho^{(j)}, \rho^{(j)} + 1]} \frac{(\rho^{(i)} - \rho^{(j)})^3 - 3(\rho^{(i)})^2 + 6\rho^{(i)} \rho^{(j)} + 3\rho^{(i)} - 1}{6\rho^{(i)} \rho^{(j)}}
\]
\[
= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1), \rho^{(i)} \in [\rho^{(j)}, \rho^{(j)} + 1]} \frac{(\rho^{(i)} - \rho^{(j)})^3 - 1}{6\rho^{(i)} \rho^{(j)}} + 1
\]
\[
= \lim_{\rho^{(i)} \to 1, \rho^{(i)} \in (0, 1), \rho^{(i)} \in [\rho^{(j)}, \rho^{(j)} + 1]} \frac{(\rho^{(i)})^3 - 3(\rho^{(i)})^2 \rho^{(j)} + 3\rho^{(i)} \rho^{(j)}^2 - (\rho^{(j)})^3 - 1}{6\rho^{(i)} \rho^{(j)}} + 1
\]
\[
= \frac{1}{2} + \frac{\rho^{(j)}}{2} - \frac{(\rho^{(j)})^2}{6}.
\]
For (viii), we have that
\[
(f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} = \rho^{(j)} = 1} = \frac{5}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 1 \atop \rho^{(i)} \in (0,1), \rho^{(i)} \leq \rho^{(j)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{5}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 1 \atop \rho^{(i)} \in (0,1), \rho^{(i)} > \rho^{(j)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{5}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 1 \atop \rho^{(i)} \in (1, \overline{\rho}], \rho^{(i)} \in (\rho^{(j)}, \rho^{(j)} + 1]} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{5}{6},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \to 1 \atop \rho^{(i)} \in (1, \overline{\rho}], \rho^{(i)} \leq \rho^{(j)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{5}{6}.
\]

For (ix), we obtain that
\[
(f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} = 1, \rho^{(j)} \in (1, \overline{\rho}]} = 1 - \frac{1}{6\rho^{(j)}},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \in (1, \overline{\rho}] \atop \rho^{(i)} \in (0,1)} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = 1 - \frac{1}{6\rho^{(j)}},
\]
\[
\lim_{\rho^{(i)} \to 1, \rho^{(j)} \in (1, \overline{\rho}] \atop \rho^{(i)} \in (1, \overline{\rho}]} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \lim_{\rho^{(i)} \to 1, \rho^{(j)} \in (1, \overline{\rho}] \atop \rho^{(i)} \in (1, \overline{\rho}]} \frac{-3 (\rho^{(i)})^2 + 6 \rho^{(i)} \rho^{(j)} + 3 \rho^{(i)} - 1}{6 \rho^{(i)} \rho^{(j)}}
\]
\[
= 1 - \frac{1}{6\rho^{(j)}}.
\]

For (x), it holds that
\[
(f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} = \rho^{(j)} \in (1, \overline{\rho}]} = \frac{3 (\rho^{(i)})^2 + 3 \rho^{(i)} - 1}{6 (\rho^{(i)})^2},
\]
\[
\lim_{\rho^{(i)} \to \rho^{(j)} \to 0 \atop \rho^{(i)} \leq \rho^{(j)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{3 (\rho^{(i)})^2 + 3 \rho^{(i)} - 1}{6 (\rho^{(i)})^2},
\]
\[
\lim_{\rho^{(i)} \to \rho^{(j)} \to 0 \atop \rho^{(i)} > \rho^{(j)}} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = \frac{3 (\rho^{(i)})^2 + 3 \rho^{(i)} - 1}{6 (\rho^{(i)})^2}.
\]

For (xi), we obtain that
\[
(f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) \big|_{\rho^{(i)} = 0, \rho^{(j)} \in (0, \overline{\rho}]} = 1,
\]
\[
\lim_{\rho^{(i)} \to 0, \rho^{(j)} \in (0, \overline{\rho}]} (f_G + f_{D_0}) (\rho^{(i)}, \rho^{(j)}) = 1.
\]

Hence we have the continuity of $f_G$. \qed