The total edge product cordial labeling of graph with pendant vertex

R M Prihandini\textsuperscript{1,4}, I H Agustin\textsuperscript{1,3}, Dafik\textsuperscript{1,2}, E R Albirri\textsuperscript{1,2}, R Adawiyah\textsuperscript{1,2}, and R Alfarisi\textsuperscript{1,4}

\textsuperscript{1}CGANT University of Jember, Indonesia
\textsuperscript{2}Mathematics Edu. Dept. University of Jember
\textsuperscript{3}Mathematics Depart. University of Jember, Indonesia
\textsuperscript{4}Elementary School Teacher Edu, University of Jember, Indonesia

E-mail: rafiantikap.fkip@unej.ac.id, ikahestiagustin@gmail.com, d.dafik@unej.ac.id

Abstract. One of the topics in graph theory is labeling. The object of the study is a graph generally represented by vertex, edge and sets of natural numbers called label. For a graph $G$, the function of vertex labeling $g : V(G) \rightarrow \{0, 1\}$ induces an edge labeling function $g^* : E(G) \rightarrow \{0, 1\}$ defined as $g^*(uv) = g(u)g(v)$. The function $g^*$ is called total product cordial labeling of $G$ if $|v_g(0) + e_g(0) - v_g(1) - e_g(1)| \leq 1$ with $v_g(0), v_g(1), e_g(0)$, and $e_g(1)$ respectively are the number of vertex which has label zero, the number of vertex which has label one, the number of edge which has label zero and the number of edge which has label one. All graphs used in this paper are simple and connected graphs. In this paper, we will prove that some graphs with pendant vertex admit total edge product cordial labeling.

1. Introduction
Graph can be interpreted as a collection of vertices which is connected to each other through an edges. Let $G$ be a graph with a set of vertex $V$ and the edge $E$, thus the number of vertex in graph $G$ is $|V(G)| = p$ and the number of edge in $G$ is $|E(G)| = q$. One of the emerging topics in graph theory is graph labeling. Labeling has many types such as graceful labeling, harmonic labeling, magic labeling, anti-magical labeling, cordial labeling and others. Some studies which is related to labeling of graph can be seen in [2, 3, 4, 10]. One of the simple application of binary labeling in everyday life is a contradictory expressions. Suppose between the words yes and no. The word yes is indicated or labeled with one and no is indicated or labeled with zero.

In this paper, we concern about the total edge product cordial labeling. The total edge product cordial labeling is a development of the cordial labeling. Some important
definitions which related to this research are as follows:

**Definition 1.** A mapping $g : V(G) \to 0, 1$ is called binary vertex labeling of $G$ and $g(v)$ is called the label of vertex $v$ of $G$ under $g$.

**Definition 2.** A binary vertex labeling of graph $G$ is called cordial labeling if \[ |(v_{g}(0) + e_{g}(0)) - (v_{g}(1) + e_{g}(1))| \leq 1. \]

Some related studies with the topic of cordial labeling are as follows: (1) Vaidya et al in [7] have explained that some special graphs namely cycle with one chord, friendship graph, cycle with twin chords, and middle graph of $P_n$ are product cordial labeling of graphs; (2) Vaidya et al in [8] have investigated that the graph obtained by joining two copies of cycle $C_n$ by a path $P_k$ of arbitrary length is a product cordial graph and path union of $k$ copies of cycle $C_n$ is a product cordial graph except odd $k$ and even $n$; and (3) its development, Ponraj et al in [9] have proved that stars and bistar are a $k$-cordial labeling and some simple graphs namely complete graphs, combs, paths are a $k$-cordial labeling with $k = 4$. Another related results can be seen at [1-5, 6, 12].

To determine a graph is a total edge product cordial labeling is not easy. It needs a special technique and specific method in labeling the vertex and edge thus the reduction of the number of vertex and edge that have label zero with the number of vertex and edge that have label one is less than equal to one.

Based on some research above, author want to investigate the total edge product cordial labeling. Vaidya et al in [11] have proved that some graph admit total edge cordial labeling. Author interest in some graph which have pendant vertex namely caterpillar, firecracker, kite, helm, and sun graph. A graph with pendant vertex is graph which has vertex with degree one.

### 2. Results

In this section, we will prove that a graph with pendant vertex namely caterpillar, firecracker, kite, helm, and sun graph admit a total edge product cordial labeling. In this section, author also prove that $Kite(n, 1)$ for $n$ odd is not a total edge product cordial labeling.

**Theorem 1.** Caterpillar $C_{n,m}$ admits a total edge product cordial labeling.

**Proof.** Caterpillar is obtained by connecting the central vertex $x$ of star graph consecutive. Let $C_{n,m}$ be the caterpillar. Let $x_1, x_2, x_3, \ldots, x_n$ be the central vertex of star graph and $x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{n,m}$ be the pendant vertex of star. The number of vertices $C_{n,m}$ is $mn$ and the number of edges of $C_{n,m}$ is $mn + m - 1$. Thus, we can obtain the number of element of elements (vertex and edge) of $C_{n,m}$ is $2mn + m - 1$. In order to determine the labeling of the vertex and edge of the caterpillar, then there are two cases as follows:

**Case 1.**

When $n$ is odd and $m$ is even. The vertex labeling of graph $C_{n,m}$, $g_1 : V(G) \to \{0, 1\}$ is:

$$g_1(x_i) = 0; \text{ for } i = [1, n]$$

$$g_1(x_{i,j}) = \begin{cases} 1; & i = [1, n]; j = [1, m]; j \text{ odd} \\ 0; & i = [1, n]; j = [1, m]; j \text{ even} \end{cases}$$
The edge labeling of graph $C_{n,m}$, $g_1 : E(G) \rightarrow \{0,1\}$ is:

$$g_1(x_i x_{i+1}) = \begin{cases} 1: i = [1,n]; j = [1,m]; j \text{ odd} \\ 0: i = [1,n]; j = [1,m]; j \text{ even} \end{cases}$$

By the vertex and edge labels which is defined by function $g_1$, it can be seen that the number of vertex and edge which has label one and zero has a certain pattern. So, the values can be obtained as follows:

- Vertex labeling:
  $$v_{g_1}(0) = m + \frac{mn-m-2}{2}$$
  $$v_{g_1}(1) = \frac{mn+m}{2}$$

- Edge labeling:
  $$e_{g_1}(0) = m - 1 + m\frac{n-1}{2}$$
  $$e_{g_1}(1) = \frac{mn+m}{2}$$

So, we get the following calculation:

$$|(v_{g_1}(0) + e_{g_1}(0)) - (v_{g_1}(1) + e_{g_1}(1))| = |(m + \frac{mn-m-2}{2}) + \frac{mn+m-2}{2} - (mn+m - \frac{mn+m}{2})|$$

$$= |m + mn - 1 - m - mn|$$

$$= |-1| \leq 1$$

**Case 2.**

When $n$ even and $m$ odd, the vertex labeling of graph $C_{n,m}$, $g_1^* : V(G) \rightarrow \{0,1\}$ is:

$$g_1^*(x_j) = 0; i = [1,n]$$

$$g_1^*(x_{i,j}) = \begin{cases} 1: i = [1,n]; j = [1,m]; j \text{ odd} \\ 0: i = [1,n]; j = [1,m]; j \text{ even} \end{cases}$$

The edge labeling of graph $C_{n,m}$, $g_1^* : E(G) \rightarrow \{0,1\}$ is:

$$g_1^*(x_i x_{i+1}) = \begin{cases} 1: i = [1,n]; j = [1,m]; j \text{ odd} \\ 0: i = [1,n]; j = [1,m]; j \text{ even} \end{cases}$$

By the vertex and edge labels which is defined by function $g_1^*$, it can be seen that the number of vertex and edge which has label one and zero has a certain pattern. So, the
Figure 1. The total edge cordial labeling of graph $C_{3,3}$

Figure 2. The total edge cordial labeling of graph $C_{4,4}$

values can be obtained as follows:

\[ v_{g_1^*(0)} = m + m\left(\frac{n}{2}\right) \]
\[ v_{g_1^*(1)} = \frac{nm}{2} \]
\[ e_{g_1^*(0)} = \frac{mn}{2} \]
\[ e_{g_1^*(1)} = \frac{2m - 2 + mn}{2} \]

So, we get the following calculation:

\[ |(v_{g_1^*(0)} + e_{g_1^*(0)}) - (v_{g_1^*(1)} + e_{g_1^*(1)})| = \left( m + m\left(\frac{n}{2}\right) + \frac{nm}{2} \right) - \left( \frac{mn}{2} + \frac{2m - 2 + mn}{2} \right) \]
\[ = |n + nm - nm - n + 1| \]
\[ = |1| \leq 1 \]

The total edge product cordial labeling of caterpillar for case 1 is depicted in figure 1 and for case 2 is depicted in figure 2. Based on two cases above, it is complete the proof that the caterpillar graph $C_{n,m}$ admit a total edge product cordial graph. □

Theorem 2. Firecracker $F_{n,m}$ is total edge cordial labeling
Proof. Firecracker is obtained by connecting one pendant vertex $x_1$ of star graph consecutive. Let $F_{n,m}$ be the firecracker. Let $x_1, x_2, x_3, \ldots , x_m$ be the central vertex of star graph and $x_{1,1}, x_{1,2}, x_{1,3}, \ldots , x_{n,m}$ be the pendant vertex of star. To determine the labeling of the vertex and edge of the firecracker, there are two cases as follows:

Case 1.
When $n$ is odd. The vertex labeling of graph $F_{n,m}$, $g_2 : V(G) \rightarrow \{0, 1\}$ is:

$$g_2(x_j) = 0; j = [1, m]$$

The edge labeling of graph $F_{n,m}$, $g_2 : E(G) \rightarrow \{0, 1\}$ is:

$$g_2(x_{i,j}x_{i,j+1}) = 1; j = [1, m-1]$$

Based on the case above, we have:

$$v_{g_2}(0) = m + m(m - 1)$$

$$v_{g_2}(1) = \frac{mm - m}{2}$$

$$e_{g_2}(0) = \frac{mm - m}{2}$$

$$e_{g_2}(1) = \frac{mm + m - 2}{2}$$

So,

$$|(v_{g_2}(0) + e_{g_2}(0)) - (v_{g_2}(1) + e_{g_2}(1))| = |(m + \frac{mm - m}{2} + \frac{mm - m}{2})$$

$$-\left(\frac{mm - m}{2} + \frac{mm + m - 2}{2}\right)|$$

$$= |m - m + 1|$$

$$= 1 \leq 1$$

Case 2.
When $n$ is even. The vertex labeling of graph $F_{n,m}$, $g_2^*: V(G) \rightarrow \{0, 1\}$ is:

$$g_2^*(x_j) = 0; j = [1, m]$$

The edge labeling of graph $F_{n,m}$, $g_2^*: E(G) \rightarrow \{0, 1\}$ is:

$$g_2^*(x_{i,j}x_{i,j+1}) = 1; j = [1, m-1]$$

$$g_2^*(x_{i,j}x_{j}) = \begin{cases} 
1; i = [1, n-1]; j = [1, m]; i \text{ odd, } j \text{ odd} \\
0; i = [1, n-1]; j = [1, m]; i \text{ even, } j \text{ even}
\end{cases}$$
Figure 3. The total edge cordial labeling of graph $F_{5,3}$

Based on the case above, we have:

$$v_{g^*}(0) = \frac{1}{2}(mn + m)$$
$$v_{g^*}(1) = \frac{1}{2}(mn - m)$$
$$e_{g^*}(0) = \frac{1}{2}(mn - m)$$
$$e_{g^*}(1) = \frac{1}{2}(mn + m - 2)$$

So,

$$|(v_{g^*}(0) + e_{g^*}(0)) - (v_{g^*}(1) + e_{g^*}(1))| = |\left(\frac{mn + m}{2}\right) - \left(\frac{mn - m}{2}\right)|$$

$$= |m - m + 1|$$
$$= 1$$

The total edge product cordial labeling of firecracker is depicted in figure 3. Based on two cases above, it is complete the proof that the firecracker graph $F_{n,m}$ is a total edge product cordial graph.

Theorem 3. **Kite $Kite(n, m)$, $n = m$ is total edge cordial labeling**

**Proof.** Kite graph consists of cycle $C_n$ and path $P_m$ which is added at one vertex on the cycle. Let $\{x_1, x_2, \ldots, x_n\}$ be the vertex of the cycle and let $\{y_1, y_2, \ldots, y_m\}$ be the vertex of the path. Therefore, the number of the vertices is $m + n$ and the number of edges is $m + n$. Thus, we can define the elements in $Kite(n, m)$ is $2(m + n)$. The vertex labeling of graph $Kite(n, m)$, $g_3 : V(G) \rightarrow \{0, 1\}$ is:

$$g_3(x_i) = 0; i = [1, n]$$
$$g_3(y_i) = 1; i = [1, m]$$

□
Figure 4. The total edge cordial labeling of graph $K(4,4)$

The edge labeling of graph $K_{n,m}$, $g_3 : E(G) \rightarrow \{0, 1\}$ is:

\[
\begin{align*}
g_3(x_ix_{i+1}) & = 0; i = [1, n - 1] \\
g_3(x_1x_n) & = 0 \\
g_3(x_1y_1) & = 1 \\
g_3(y_iy_{i+1}) & = 1; i = [1, m - 1]
\end{align*}
\]

We get the number of vertex and edge that have labels one and zero as follows:

\[
\begin{align*}
v_{g_3}(0) & = n \\
v_{g_3}(1) & = m \\
e_{g_3}(0) & = n \\
e_{g_3}(1) & = m
\end{align*}
\]

Therefore,

\[
|v_{g_3}(0) + e_{g_3}(0)| - |v_{g_3}(1) + e_{g_3}(1)| = |n + n| - |m + m| = 0 \leq 1
\]

Since, $n = m$ hence

\[
|v_{g_3}(0) + e_{g_3}(0)| - |v_{g_3}(1) + e_{g_3}(1)| = 0 - 0 = 0 \leq 1
\]

Hence, the kite graph $K_{n,m}$ is a total edge cordial labeling. $\square$

As an illustration of kite graph and its total edge cordial labeling can be seen in figure 4.

**Theorem 4.** $K_{n,1}$ for $n$ odd is not a total edge cordial labeling

**Proof.** $K_{n,1}$ is graph with $n$ vertex on the cycle $C_n$ and one vertex which connected to a vertex on the cycle. Let the number of vertex and edge in cycle that have label one and zero be balanced. Hence, $|v_g(1) + e_g(1)| = n$ and $|v_g(0) + e_g(0)| = n$. Furthermore, we will investigate the label of the pendant vertex. The pendant vertex has one degree, thus $f(e) = f(v)$. If the pendant edge is placed at the vertex which has label 0, then $|v_g(1) + e_g(1)| = n + 2$ or $|v_g(0) + e_g(0)| = n + 2$. Hence, $|(v_g(0) + e_g(0)) - (v_g(1) + e_g(1))| = |n + 2 - n| = 2 \not\leq 1$. If the pendant edge that

\[
\]

7
has label 1 is placed at the vertex which has label 1, then $|v_g(1) + e_g(1)| = n + 2$ and $|v_g(0) + e_g(0)| = n$. Hence, $|(v_g(0) + e_g(0)) - (v_g(1) + e_g(1))| = |n + 2 - n| = 2 < 1$. If the pendant edge which has label 0 is placed at the vertex which has label 1, then $|v_g(1) + e_g(1)| = n - 1$ and $|v_g(0) + e_g(0)| = n + 3$. Hence, $|(v_g(0) + e_g(0)) - (v_g(1) + e_g(1))| = |n - 1 - n - 3| = 4 < 1$. Based on the some cases above, it is conclude the proof.

**Theorem 5.** Sun graph $SG_n$, is total edge cordial labeling

**Proof.** The sun graph denoted by $SG_n$ is cycle with an edge terminating in a pendant vertex attached to each vertex. Let $x_1, x_2, \ldots, x_n$ be the vertex on the cycle and $y_1, y_2, \ldots, y_n$ be the pendant vertex. The order of sun graph is $2n$. The vertex labeling of graph $SG_n$, $g_4 : V(G) \rightarrow \{0, 1\}$ is:

$$
g_4(x_i) = 0; i = [1, n] \quad \text{and} \quad g_4(y_i) = 1; i = [1, n]
$$

The edge labeling of graph $SG_n$, $g_4 : E(G) \rightarrow \{0, 1\}$ is:

$$
g_4(x_i x_{i+1}) = 0; i = [1, n] \\
g_4(x_i y_i) = 1; i = [1, n]
$$

We have:

$$
v_{g_4}(0) = n \\
v_{g_4}(1) = n \\
e_{g_4}(0) = n \\
e_{g_4}(1) = n
$$

So,

$$
|(v_{g_4}(0) + e_{g_4}(0)) - (v_{g_4}(1) + e_{g_4}(1))| = |(n + n) - (n + n)| = 0 \leq 1
$$

It is complete the proof that the sun graph $SG_n$ admit total edge cordial labeling. See figure 5 as an illustration.

**Theorem 6.** Helm graph $H_n$, is total edge cordial labeling

**Proof.** The helm $H_n$ is the graph obtained from a wheel graph by attaching a pendant edge at each vertex of the $n$ cycle. Let $x$ be the central vertex of wheel, let $x_i$ be the vertex of the cycle, and let $y_i$ be the pendant vertex that is attached at each vertex of the cycle. The vertex labeling of graph $H_n$, $g_5 : V(G) \rightarrow \{0, 1\}$ is:

$$
g_5(x) = 0 \\
g_5(x_i) = 0; i = [1, n] \\
g_5(y_i) = 1; i = [1, n]
$$

8
Figure 5. The total edge cordial labeling of $SG_8$

The vertex labeling of graph $H_n$, $g_5 : E(G) \rightarrow \{0, 1\}$ is:

- $g_5(x_i x_{i+1}) = 0; i = [1, n]$
- $g_5(x_i y_i) = 1; i = [1, n]$
- $g_5(xx_i) = 1; i = [1, n]; i$ odd
- $g_5(xx_i) = 0; i = [1, n]; i$ even

We have:

- $v_{g_5}(0) = n + 1$
- $v_{g_5}(1) = n$
- $e_{g_5}(0) = n + \left\lceil \frac{n}{2} \right\rceil$
- $e_{g_5}(1) = n + \left\lfloor \frac{n}{2} \right\rfloor$

For the the number of edge which has label zero $e_{g_5}(0)$ since $n$ odd, $\left\lceil \frac{n}{2} \right\rceil = \frac{n-1}{2}$, and since $n$ even $\left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$. For the the number of edge which has label one $e_{g_5}(1)$ since $n$ odd, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n+1}{2}$ and since $n$ even $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$. So,

$$|v_{g_5}(0) + e_{g_5}(0)) - (v_{g_5}(1) + e_{g_5}(1))| = |(n + 1 + n + \left\lceil \frac{n}{2} \right\rceil) - (n + n + \left\lfloor \frac{n}{2} \right\rfloor)|$$

$$= |2n + 1 + n + \left\lceil \frac{n}{2} \right\rceil - 2n - n - \left\lfloor \frac{n}{2} \right\rfloor| = 1 \leq 1$$
Figure 6. The total edge product cordial labeling of graph $H_8$

It is conclude the proof. □

See figure 6 as an illustration of the total edge product cordial labeling of graph $H_8$.

**Conjecture 1.** Kite $K(n,1)$ for $n$ even is not a total edge cordial labeling

**3. Conclusion**

The study about the total edge cordial labeling is very potential for further research. We have proved that some graphs with pendant vertex, namely caterpillar, firecracker, kite, helm, and sun graph admit the total edge cordial labeling. Other researches related to this subject is still open. So we are giving one of the open problem in example:

**Open Problem 1.** Does kite graph for $n \neq m$ admit the total edge cordial labeling?

**Open Problem 2.** Does every graph with pendant vertex admit the total edge cordial labeling?

**Acknowledgement**

We gratefully acknowledge the support from CGANT - University of Jember of year 2018.

**References**

[1] Babujee J B and Shobana L 2010 Prime and Prime Cordial Labeling for Some Special Graphs *Int. J. Contemp. Math. Sciences* **5** (47) 2347-2356.

[2] Dafik, Slamin, Tanna D, Semanico-Fenovcikova A and Baca M 2017 On super (a, d)-edge-antimagic total labeling of disconnected graphs *Ars Combinatoria* **133** 233-245.

[3] Dafik, Hasan M, Azizah Y N and Agustin I H 2017 A generalized shackle of any graph $H$ admits a super $H$-antimagic total labeling *Journal of Physics: Conference Series* **893**(1) 012042.
[4] Dafik, Agustin IH, Nurvitaningrum A I, and Prihandini R M. 2017 On super Hantimagicness of an edge comb product of graphs with subgraph as a terminal of its amalgamation Journal of Physics: Conference Series 855 (1) 012010.

[5] Kuo D, Chang GJ and Kwong YH 1997 Cordial Labeling of mKn Discrete Mathematics 169(1-3) 121-131.

[6] Youssef M Z 2009 On k- Cordial Labeling Australas. J. Combin 43 31-37.

[7] Vaidya S K and Barasara C M 2011 Product Cordial Labeling for Some New Graphs journal of mathematics Research 3 (2) 206

[8] Vaidya S K and Kanani K K 2010 Some Cycle Related Product Cordial Graphs Int. J. of Algorithms, Comp. and Math 3 (1) 109-116.

[9] Ponraj R, Sivakumar M and Sundaram M 2012 k-Product Cordial Labeling of Graphs Int. J. Contemp. Math. Sciences 7(15) 733-742.

[10] Prihandini R M, Agustin I H, and Dafik 2018 Journal of Physics: Conference Series, 1008 (1) 012039.

[11] Vaidya S K and Barasara C M 2013 Total Edge Product Cordial Labeling of Graphs Malaya Journal of Matematik 3 (1) 55-63.

[12] Murugan A N and Suganya V B 2014 Cordial Labeling of Path Related Splitted Graphs Indian Journal of Applied Research 1-8.