Thermal Fluctuation Driven Topological Transitions of Magnons in Collinear Antiferromagnets

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Thermal fluctuation plays a central role in driving thermodynamical phase transitions, such as the transition between magnetic ordered and paramagnetic phases in magnets. Here we show that it could also drive topological transitions for magnons in two-dimensional collinear antiferromagnets with sublattice asymmetries by increasing temperatures, attributed to the magnon-magnon interactions. We present the phase diagram and showcase that the topological trivial magnon bands at low temperatures will experience topological transitions at a critical temperature and exhibit Chern insulating phase at higher temperatures. These temperature driven topological transitions originate from bandgap closing and reopening at Γ or K points, accompanied by a magnon chirality switch, which can be probed experimentally via magneto-Raman spectroscopy, polarized neutral scattering or polarization-selective spectroscopy.

The last twenty years have witnessed the extraordinary developments of topological insulators and semimetals in the field of condensed matter physics [1–3]. And vast works reported the realization of topological phases of electrons by changing various experimental parameters [1–14]. In analog to electronic systems, the topological phases have also been extended to the bosonic systems, such as the photonic [15] and acoustic systems [16]. Magnons, quantized spin excitations in magnets, another boson, can also host nontrivial topological phases, which can be realized from Dzyaloshinskii-Moriya interactions (DMIs), dipolar interactions or in artificially designed structures [17–29], holding great potential applications in the field of spintronics [30] with low dissipation and power consumption.

One of the key features in magnetic systems is the presence of magnon-magnon interactions (MMIs) and thermal fluctuations. Their interplay would renormalize the magnon modes with a softening effect due to the self-energy corrections from MMIs, and induce thermodynamical phase transitions between ordered and paramagnetic phases by increasing temperatures [31–38]. However, the studies on how such interplay between thermal fluctuations and MMIs affects the topological phases of magnons are elusive, an interesting and open problem is: can thermal fluctuation drive the trivial magnons to topological phases through increasing the temperature?

In this Letter, we theoretically demonstrate that by simply breaking sublattice symmetries in collinear antiferromagnets (Fig. 1), the thermal fluctuation can drive topological transitions for magnons by increasing temperatures. At zero temperature, the Chern insulating phase exists only in an interval for the sublattice easy-axis anisotropy. At finite temperatures, this topology always survives. Surprisingly, out of the interval, the separated trivial magnon bands will be dragged closer. The band gap at Γ or K points will be closed and reopen above critical temperature $T_c$, thus inducing topological transitions. The magnon bands are topologically trivial at low temperatures but exhibit Chern insulating phase above $T_c$. We find these topological transitions are accompanied by a magnon chirality switch at Γ or K points for the acoustic branch. The magneto-Raman spectroscopy, polarized neutral scattering or polarization-selective spectroscopy are promising experimental techniques to probe these topological transitions. Our proposal and conclusion are quite universal, and can be extended to the case of ferrimagnets. The van der Waals antiferromagnets MnPS$_3$ and MnPSe$_3$ are promising candidate platforms.

The temperature tunable behavior is quite important for designing topological quantum devices operating at easily achievable higher temperatures, and have already been demonstrated in certain electronic materials with electron-phonon interaction [10–14]. In magnonic systems, recent two works [28, 29] addressed that gapped Dirac magnons can host opposite Chern numbers at different temperatures. On the contrary, we proposed that the trivial magnons can be driven to nontrivial topological phase with increasing thermal fluctuations. Mean-

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FIG. 1: Illustration of the honeycomb antiferromagnet with sublattice asymmetry. The numbers denote the bond index. The two sublattices NNN interactions are denoted as $J_2^a$ and $J_2^b$, respectively.
while, the two works [28, 29] studied the honeycomb ferromagnets with DMI. Differently, we here consider the collinear antiferromagnets, proposed to be superior in the spintronics [39] due to the ultrafast spin dynamics and additional chirality degree of freedom.

**Model and methodology**—We consider a honeycomb collinear antiferromagnet, illustrated in Fig. 1. We here assume a nearest-neighboring exchange anisotropy, sublattice asymmetric easy-axis anisotropy and next-nearest-neighboring (NNN) interactions, which could be induced and tuned by the substrate, adatoms or in heterostructures [40–42]. The spin Hamiltonian is given by

\[
H = \sum_{ij} \left[ \frac{J_{ij}}{2} (S_i^+ S_j^- + S_i^- S_j^+) + J'_{ij} S_i^z S_j^z \right] + \frac{J_3}{2} \sum_{(ij)} \left[ \gamma_{ij} S_i^+ S_j^+ + \gamma_{ij} S_i^- S_j^- \right] - \sum_{i} K_i (S_i^z)^2.
\]

The first term denotes the Heisenberg exchange interactions up to NNN, \( J_{ij}^2 > J_1 = J > 0 \) for NN, \( J_2 \neq J_3 \) and no exchange anisotropy for NNN. For negative and quite small positive \( J_2^2 \) and \( J_3 \), the ground state stays in collinear AFM phase [43, 44]. The second term is the bond-dependent interactions [45], arising from spin-orbit coupling. \( \gamma_{ij} = e^{2\pi n_{ij}/3} \) with \( n = 0, 1, 2 \) the bond index, illustrated in Fig. 1. The last term is the sublattice asymmetric easy-axis anisotropy characterized by \( K_a \) and \( K_b \) for the two sublattices, respectively.

We apply the Holstein-Primakoff transformation, \( S^\pm = S - a^\dagger a, S^\dagger = a^\dagger \sqrt{2S - a^\dagger a}, S^- = a^\dagger \sqrt{2S - a^\dagger a} \) for A-sublattice, and \( S^\pm = -S + b^\dagger b, S^\dagger = b^\dagger \sqrt{2S - b^\dagger b}, S^- = \sqrt{2S - b^\dagger b} b \) for B-sublattice. The Hamiltonian in Eq. (1) can be expanded as \( H = \sum_{p=0}^{\infty} H_{2p} \), where \( 2p \) denotes the number of bosonic operators. We here keep the terms up to quartic order and neglect the ground state energy term. With a Fourier transformation, the two-particle term can be written in the form \( H_2 = \frac{1}{2} \sum_{k} \Psi_k^\dagger H_k \Psi_k \), where \( \Psi_k = (a_k, b_k, a_k^\dagger, b_k^\dagger)^T \), the uppercase denote the transpose.

\[
H_k = \begin{pmatrix}
 h_k & \Delta_k \\
\Delta_k^\dagger & h_{T_k}
\end{pmatrix},
\]

where \( h_k = h_0 \mathbb{1}_2 + h \cdot \sigma \), \( \Delta_k = \delta \cdot \sigma \). \( \sigma \) denotes the sublattice index. \( h_0 = (h_A^2 + h_B^2)/2, h_x = J_a S \text{Re}(g_k), h_y = -J_a S \text{Im}(g_k), h_z = (h_A^2 - h_B^2)/2, \delta_x = J_1 \text{Re}(f_k), \delta_y = -J_1 \text{Im}(f_k), h_A^k = K_a (2S - 1) + 3J_1 S - J_2 (6 - d_k), h_B^k = K_b (2S - 1) + 3J_1 S - J_2 (6 - d_k), \Delta_k = 1 + 2e^{i k_\lambda/2} \cos(\frac{2\pi n_{\lambda}}{3}), f_k = 1 - e^{i k_\lambda/2} \cos(\frac{2\pi n_{\lambda}}{3}), d_k = \sum_{\lambda=1}^6 \cos(\lambda \cdot a_k), \) with \( a_k \) the NNN lattice vectors. By diagonalizing Eq. (2), \( \Lambda_k^T H_k \Lambda_k = \text{diag} \{ E_{k_1}, E_{k_2}, E_{k_3} \} \), we have \( H_k = \sum_{k} \{ E_k^a \alpha_k^a \beta_k + E_k^b \beta_k^a \beta_k \} \) respect to a constant. In the discussions below, we always denote the mode with lower (higher) energy the acoustic (optical) branch. The paraunitary eigenvectors satisfy \( \Lambda_k^T \tau_2 \Lambda_k = \tau_2 \) and \( \tau_2 \) is the Pauli matrix acting on the particle-hole space [18]. Note that the diagonalization gives us the relation \( \Psi_k = \Lambda_k \Phi_k \) with \( \Phi_k = (\alpha_k, \beta_k, \alpha_k^\dagger, \beta_k^\dagger)^T \).

We now discuss the effect of magnon-magnon interactions, i.e. the four-particle term \( H_4 \), we have

\[
H_4 = -\frac{1}{N} \sum_{\{k\}} \left[ \frac{J_4}{4} \left( f_{k_1} a_{k_1}^\dagger a_{k_2} a_{k_3} a_{k_4} + f_{k_1} a_{k_1} a_{k_2}^\dagger b_{k_3} b_{k_4} + f_{k_1} a_{k_1}^\dagger b_{k_2} b_{k_3} + f_{k_1} a_{k_2} b_{k_3}^\dagger a_{k_4} + a_{k_1}^\dagger b_{k_2} a_{k_3} b_{k_4} + b_{k_1}^\dagger a_{k_2} b_{k_3} a_{k_4} + b_{k_1} a_{k_2}^\dagger b_{k_3}^\dagger a_{k_4} + b_{k_1} a_{k_2} b_{k_3}^\dagger a_{k_4} \right) + H.c. \right] + \frac{J_4}{4} \left( g_{k_1} a_{k_1}^\dagger a_{k_2} a_{k_3} a_{k_4} + g_{k_1} a_{k_1}^\dagger b_{k_2} b_{k_3} a_{k_4} + H.c. \right) \right) + \left( \epsilon_{a} a_{k_1}^\dagger a_{k_2} a_{k_3} a_{k_4} + \epsilon_{b} a_{k_1}^\dagger b_{k_2} b_{k_3} a_{k_4} \right) \delta^2_{\{k\}},
\]

where \( \delta^2_{\{k\}} = \delta_{k_1 + k_2, k_3 + k_4}, \delta^2_{\{k\}} = \delta_{k_1 + k_2, k_3 + k_4}, \epsilon_{a,b} = J_4^{a,b} d_{\{k\}} + K_{a,b} \) and \( d_{\{k\}} = \frac{a_{k_1} + a_{k_2} - 2d_{k_3} + d_{k_4}}{2} \). To take many-body effect into account and consider the temperature effect, we employ the Green function method and define a matrix Green function as \( \tilde{G}(k, \tau) = \frac{1}{\left( T \right)} \sum_{\nu} \tilde{G}(k, \omega_n) e^{-i \omega_n \tau} \) with \( \beta = 1/T, T \) the temperature and \( \omega_n \) the bosonic Matsubara frequency, we can get the Dyson’s equation \( \tilde{G}^{-1}(k, \omega_n) = i\omega_n \tau_2 - H_k^{\text{eff}} \) and the effective Hamiltonian

\[
H_k^{\text{eff}} = H_k + \Sigma_k,
\]

where \( \Sigma_k \) is the first-order self-energy with the formalism presented in the Supplementary Material (SM) [47]. Using the relation above \( \Psi_k = \Lambda_k \Phi_k \), the matrix element of self-energy can be expressed as

\[
\Sigma_{ij} = \frac{1}{N} \sum_{\lambda,\alpha,\beta} \left[ T_{ij}^\lambda (k, \mathbf{q}) n_{q,\lambda}(T) + Q_{ij}^\lambda (k, \mathbf{q}) \right],
\]

\( n_{q,\lambda} \) is the Bose-Einstein distribution function \( n_{q,\lambda} = (e^{\omega_a/T} - 1)^{-1} \) with a zero chemical potential. The right two terms correspond to the thermal and quantum corrections, respectively. The calculation method of them are presented in SM [47]. Notice that the self-energy correction does not vanish even at zero temperature due to the quantum corrections. The Eq. (2), (4), (5) and the diagonalizing relation above form the self-consistent relations to determine the effect of MMs. We numerically calculate the band structures and corresponding Chern numbers at zero and finite temperatures. The results on the topological phase at zero temperature and temperature induced topological transitions are presented below.

**Topological transitions and phase diagram**—We first discuss the topological phases at zero temperature \( T = 0 \) to
get an intuitive picture of the system. The two magnon branches are not degenerate due to the sublattice asymmetries. When \( J_a = 0 \) and setting \( J_{a}^{2} > J_{b}^{2} \), we find in the region \( K_{a} < K_{b} < K_{c} \), the two magnon bands would show a ring-like band intersection, as shown in Fig. 2 (a) by the solid lines. We here do not give the explicit expression for \( K_{a} \) due to the quantum correction. While at \( K_{a} = K_{b} (K_{a} = K_{c}) \), the two band show a point touching at \( \Gamma (K) \) point in the Brillouin zone (BZ). Outside the above region, the two bands are always separated and topologically trivial even for \( J_a \neq 0 \).

Finite \( J_a \) is expected to gap the two bands and bring a nontrivial topology for \( K_a < K_b < K_c \). As \( g_k \) vanishes at \( \Gamma \) and \( K \) points, the band point touching at \( K_a = K_b \) and \( K_c \) will not be changed. Thus the two conditions is the phase boundary between trivial and nontrivial topological phases. In Fig. 2 (a) by the dashed lines, we can see the two bands are gapped in the region. From Fig. 2 (b), we can see the two bands always have a gap along the \( \Gamma - K \) direction also other directions. We have checked that the two bands are gapped in the whole BZ. We use the Chern integer as the topological invariant, defined as \( C_n = \frac{1}{4\pi} \int_{BZ} B^2 d^2k \) with Berry curvature \( B_{a} = \nabla \times A_{a} \) and Berry connection \( A_{a} = i \text{Tr}[\gamma^n a_{k} B^2 d^2k] \), where \( \gamma^n \) is the diagonal matrix taking +1 for n-th digonal component and zero otherwise. We find the Chern integer is 1 in the region \( K_0 < K_a < K_c \) for the acoustic branch, as presented in Fig. 2 (b), suggesting a parameter driven topological transition, similar to the most of the previous works discussing topological phases of magnons. When the sublattice asymmetric NNN exchange interaction is reversed, i.e., \( J_a^{2} < J_b^{2} \), the nontrivial topological region lies in \( K_0 < K_b < K_c \) and the Chern integer for the acoustic branch is \(-1\). In the subsequent discussions, we adopt \( J_a^{2} > J_b^{2} \) as the two settings share the same physics.

We turn to the effects of MMs on the topological phases at finite temperatures and show that the separated trivial bands at zero temperature could host topological phase at high temperatures. We present the phase diagram in the first and the order parameter is chosen as the Chern integer of the acoustic branch, calculated from the effective Hamiltonian after the self-consistent treatment at given temperatures. The self-consistent process also help us to confirm that the temperatures we choose are below Neél temperature \( T_N \). The phase diagram in the \( K_a - T \) plane is shown in Fig. 3 (a). Same to the previous works, the topological phase in the region \( K_b < K_a < K_c \) at \( T = 0 \) remains unbroken at any finite temperatures. Interestingly, in the trivial region at zero temperature, i.e. when \( K_a < K_b \) and \( K_b > K_c \), the trivial magnon bands will become nontrivial and host a non-zero Chern integer above critical temperatures. This phenomenon establishes the concept of thermal fluctuation driven topological transitions, which is the main results of this work. Note that the phase diagram is similar to the one of topological Anderson insulators [48–51], where the temperature in our system is analog to the disorder strength.

To uncover the origin of these topological transitions at finite temperatures, we plot the magnon bands at three temperature for \( K_a \approx 0.0425 < K_0 \) (K1) and \( K_a \approx 0.16 > K_0 \) (K2) in Fig. 3 (b) and (c), respec-

![FIG. 2: (a) The magnon band structures at \( T = 0 \) and \( K_{a} = 0.1 \). The other parameters are adopted as \( S = 2.5, J_{1}^{b} = 1.05, J_{1}^{a} = 1.0, J_{0}^{a} = 0.08, J_{0}^{b} = 0.065, K_{0} = 0.05 \). In subsequent calculations, we adopt the same parameters when not pointing out in particular. \( J_a = 0 \) for solid line and \( J_a = 0.02 \) for dashed lines. (b) The Chern integer for the acoustic branch and bandgap with respect to \( K_a \) at \( T = 0 \). The other parameters are the same to (a) with \( J_a = 0.02 \). The bandgap is defined as \( E_g = \min_{k}(E_{k}^{a} - E_{k}^{b}) \) along \( \Gamma - K \) direction.

![FIG. 3: (a) The phase diagram in the \( K_a - T \) plane. The Chern number is for the acoustic branch, while the Chern number for optical branch is opposite. (b) Magnon bands near \( \Gamma \) point at three temperatures. \( K_{a} = K_{1} = 0.0425 \). (c) Magnon bands near \( K \) point at the same three temperatures to (b). \( K_{a} = K_{2} = 0.16 \). At \( K_1 \) and \( K_2 \), the critical temperature of topological transition is almost the same, \( T_c \approx 2 \). (d) The magnon chirality switch at \( \Gamma \) point for \( K_a = K_1 \). (e) The magnon chirality switch at \( K \) point and DoP evolution at \( K' \) point for \( K_a = K_2 \). For all the figures, \( J_a = 0.02 \).]
tively. The critical temperature of phase transitions for the two values are almost the same, \( T_c/J \simeq 2 \). For \( K_a = K_1 \), we plot the bands near \( \Gamma \) point. Besides the magnon energy renormalization, we can see the bandgap at \( \Gamma \) point decreases as the temperature increases. The spectrum become gapless at \( T = T_c \). Further increasing temperature reopens the gap. During this process, we have checked the two bands at other points in the BZ are always gapped. From the phase diagram in Fig. 3 (a), the Chern number is 0 below \( T_c \) and 1 above \( T_c \). The other \( K_a < K_b \) value shares the same behavior with different critical temperatures. This temperature driven topological transitions at the region \( K_a < K_b \) is related to the weak ferrimagnetic phase at finite temperatures \([47]\), which arises from the imbalanced occupation number of the two magnon branches. For \( K_a = K_2 \), the magnon bands experience a similar behavior but at \( K \) point instead, as shown in Fig. 3 (c). The gap at \( K \) point decreases to zero and reopens across \( T_c \). The increasing temperature also induces a topological transition with the Chern integer jumping from 0 to 1. We can see that the both topological transitions are accompanied by a gap closing and reopening induced by the temperature.

It is well-known that the two magnon modes precess circularly with opposite chiralities when \( J_a = 0 \) \([52]\). For finite \( J_a \), the precession trajectories will become elliptical. The polarization of the magnons in antiferromagnet is much similar to the one of light. We can define the degree of polarization (DoP) for magnons in the momentum space. At a given \( k \), the eigenvector at finite \( J_a \) can be expressed as a linear combination of the polarized state at \( J_a = 0 \), \( \Lambda^+_{k,\lambda} = \chi_{k,\lambda} A^+_{k,\lambda} + \chi_{k,\lambda} A^-_{k,\lambda} \) (\( \lambda = \alpha, \beta \)), where \( \Lambda^+_{k,\lambda} \) is the eigenvectors for right- and left-handed precession modes, \( \chi_{k,\lambda} \) the expansion coefficients and satisfying \( |\chi^+_{k,\lambda}|^2 + |\chi^-_{k,\lambda}|^2 = 1 \). The DoP in momentum space is defined as \( P(k) = |\chi^+_{k,\lambda}|^2 - |\chi^-_{k,\lambda}|^2 \). Below we always focus on the acoustic branch, while the DoP for the optical branch is opposite. At zero temperature and \( J_a = 0 \), the chirality is right-handed for \( K_a < K_b \) \( (P(k) = 1) \) and left-handed for \( K_a > K_c \) \( (P(k) = -1) \). Finite \( J_a \) does not break the chirality at \( K \) and \( \Gamma \) points. Therefore, the chirality at \( K \) and \( \Gamma \) points in the left trivial region in the phase diagram of Fig. 3 (a) is right-handed, while at the right trivial region is left-handed. In the middle topologically nontrivial region, we find the the chirality is right-handed at \( K \) point, while left-handed at \( \Gamma \) point. These features indicate that the topological transitions are accompanied by a chirality switch at where the gap closes and reopens. When \( K_a < K_b \), the increasing temperature would induce a band gap closing and reopening at \( \Gamma \) point, the chirality will be switched from the right-handed to left-handed, as shown in Fig. 3 (c), opposite behavior to the above one. As reported in the experimental works \([53-57]\), the magnon chirality in antiferromagnets can be distinguished by the magneto-Raman spectroscopy due to the angular momentum conservation \([53]\), the polarized neutral scattering technique \([54]\) or very recent developed polarization-selective spectroscopy \([55-57]\). These experimental methods greatly coincide with our system. The detection of the chirality switch and the gapped bands together provide experimental proofs of the thermal fluctuation driven topological transitions, instead of the observation of the appearance of chiral edge states due to the difficulties arising from the absence of a full gap and the quite small band separation.

We briefly discuss the DoP in the full BZ. First at the \( K' \) point, the magnon will also experience of chirality switch, same to \( K \) point when \( J_a = 0 \) and \( K_a > K_c \). But for finite \( J_a \), the evolution of the DoP is shown in Fig. 3 (e), showing a negative to positive transition. There are three different regions in the phase diagram in Fig. 3 (a). The typical DoP distributions in the BZ for the three regions are shown in Fig. 4 (a) (b) (c), respectively. In the left region, the DoP is positive (Fig. 4 (a)) and the right-handed mode dominates. While in the right region, the DoP is negative (Fig. 4 (c)) and the left-handed mode dominates. In the middle topological region, \( P \) is negative near \( \Gamma \) point and positive around \( K \) point (Fig. 4 (b)). The ring-like transition zone where \( P \approx 0 \) is the band intersection region when \( J_a = 0 \). The differences indicate that the temperature would change the distribution, possibly to be tracked via the experimental techniques presented above.

Above all, we find the thermal fluctuation driven topological transitions in collinear antiferromagnets. The magnon chirality switch during the phase transitions are within the experimental approach. For the platforms, MnPX\(_3\) (X=S, Se), also other collinear antiferromagnets, are the candidate materials \([58-60]\). The parameters we adopted in all the calculations are within the approach of these materials. As our proposal is based on the sublattice asymmetries, the ferrimagnets should show similar temperature-dependent topological phases, with the
phase diagram presented in the SM [47]. Dipolar interactions can replace the role of the bond-dependent interactions. Our proposal is not limited to the honeycomb lattice configuration. Square or other lattices are expected to reach the same conclusion.

Summary—We proposed thermal fluctuation driven topological transitions in collinear antiferromagnets by breaking sublattice symmetry. Our work showcases that a nontrivial topological phases of magnons can also be created at high temperatures while at low temperature the bands are trivial. The transition between trivial and nontrivial topological phases can be probed with multiple state-of-the-art techniques via the chirality detection. Our work paves the way for the study of the interplay between topological phases, MMIs and thermal fluctuations that is beyond the linear spin wave theory.

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SUPPLEMENTARY MATERIALS

S1. The expressions for the self-energy

We here use the Green function method and random phase approximation to get the self-energy corrections. The Heisenberg equation-of-motion for the Green function is given by

$$\frac{d\hat{G}(k,\tau)}{d\tau} = -\delta(\tau)\tau_z - \langle T[H_k(\tau)\Psi^*_k(0)] \rangle = -\delta(\tau)\tau_z - \tau_z(H_k + \Sigma_k)\hat{G}(k,\tau).$$

Here the self-energy term $\Sigma_k$ is obtained from the random phase approximation. With the Fourier transformation $\hat{G}(k,\tau) = (1/\beta)\sum_n G(k,\omega_n)e^{-i\omega_n\tau}$, we can get $-\omega_n\hat{G}(k,\omega_n) = -\tau_z - \tau_z(H_k + \Sigma_k)\hat{G}(k,\omega_n)$. Thus

$$[i\omega_n - \tau_z(H_k + \Sigma_k)]\hat{G}(k,\omega_n) = \tau_z.$$

By multiply a $\tau_z$ term on both side, we can get the Dyson’s equation in the main text. The effective Hamiltonian is presented as

$$H_{eff}^i = H_k + \Sigma_k = \left( \begin{array}{cc} h_{11} & \Delta_{12}^T \\ \Delta_{21} & h_{22}^T \end{array} \right),$$

with $h_{11}^i = h_{01} + h_{12}^i + h_{22}^i$. $h_0 = (h_A^i + h_B^i)/2$, $h_{22} = J_a(\hat{S}_a + \hat{S}_b)\text{Re}(g_k)$, $h_{22}^i = -J_a(\hat{S}_a + \hat{S}_b)\text{Im}(g_k)$, $h_{22} = (h_A^i - h_B^i)/2$, where $h_A = K_a(-2S - 1 + 4\hat{S}_a) + 3J_a\hat{S}_a - J_{2s}\hat{S}_d(6 - d_k) - \hat{h}_A$, $h_B = K_b(-2S - 1 + 4\hat{S}_b) + 3J_b\hat{S}_b - J_{2s}\hat{S}_d(6 - d_k) - \hat{h}_B$, $\hat{S}_a = S - \frac{1}{N}\sum_q (a_q^a a_{-q}^a)$, $\hat{S}_b = S - \frac{1}{N}\sum_q (b_q^a b_{-q}^a)$, $\hat{S}_d = \frac{1}{N}\sum_q d_q^a d_{-q}^a - \frac{2}{N}\sum_q (b_q^a b_q^b)$, $\hat{h}_A = \frac{1}{N}\text{Re}\sum_q (a_q^a b_{q}^b)$, $\hat{h}_B = \frac{1}{N}\text{Re}\sum_q (a_q^a b_{-q}^b)$ $(\Delta_{12})^2 = [\hat{S}_a + \hat{S}_b]J_1 \hat{k}_f - \hat{J}_1 \sum_{q} \hat{k}_{f-q}(u_q^a b_{-q}^b)$.

The diagonalization matrix $\Lambda_k$ reads

$$\Lambda_k = \left( \begin{array}{cccc} u_{-k,a} & u_{k,a} & v_{-k,a} & v_{k,a} \\ u_{k,b} & u_{-k,b} & v_{k,b} & v_{-k,b} \\ v_{k,a} & v_{-k,a} & u_{k,a} & u_{-k,a} \\ v_{k,b} & v_{-k,b} & u_{k,b} & u_{-k,b} \end{array} \right).$$

The above thermal average can be transformed into the basis of $\Phi_k = (\alpha_k, \beta_k, \alpha_{-k}^\dagger, \beta_{-k}^\dagger)^T$. For example,

$$\frac{1}{N}\sum_q \langle a_q^a a_q^a \rangle = \frac{1}{N}\sum_q (|u_q,a|^2 |\alpha_q^a \rangle^2 + |u_q,a|^2 |\alpha_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2 + |v_{-q,a}|^2 |\beta_{-q}^a \rangle^2).$$

The first term is the thermal correction and the second term does not vanish at zero temperature, corresponds to the quantum correction term. Repeat the above procedure for other terms in the self-energy, we can get the relation in Eq. (5) in the main text.

S2. Temperature induced weak Ferrimagnetic phase

Due to sublattice asymmetries, the band degeneracies is broken. At finite temperatures, the occupation number for the two bands is different. This will induce a weak ferrimagnetic phase. The total magnetization along z direction is defined as $< S_z > = \hat{S}_a - \hat{S}_b = \frac{1}{N}\sum_k (b_{k}^a b_{-k}^a - a_{k}^a a_{-k}^a)$. From Fig S5, we can see $< S_z >$ does not equal to zero at relatively high temperatures. When neglecting the magnon-magnon interactions, $< S_z > = 0$, the two magnon bands are always degenerate at $\Gamma$ point when $K_a = K_b$ although the sublattice NNN interactions are different. But at finite temperature, $< S_z > \neq 0$, zero band gap condition at $\Gamma$ point satisfies for some value of $K_a$ with $K_a < K_b$. 
FIG. S5: $\langle S_z \rangle$ distribution in the $K_a$-$T$ plane. The dashed gray lines are the phase boundaries adopted from Fig. 3 (a) in the main text.

FIG. S6: The phase diagram for ferrimagnets. $J^z_1 = 1.01$, $J_1 = 1.0$, $J_2 = -0.2$, $S_A = 2.5$, $S_B = 2.2$, $K_b = 0$.

S3. Phase diagram for ferrimagnets

The temperature driven topological transitions for antiferromagnets are due the sublattice asymmetries, which broken the $\mathcal{P}\mathcal{T}$ symmetry. In Ferrimagnets, the $\mathcal{P}\mathcal{T}$ symmetry is naturally absent. In recent experiment, the electron doping can drive a transition between the antiferromagnetism and ferrimagnetism in NiPS$_3$, indicating the ferrimagnets can be induced from antiferromagnets.

We here set $S_A \neq S_B$, $J_2^z = J_2^y = J_2$, $K_b = 0$. The phase diagram is presented in Fig S6. We can see the topological phase is also dependent on the temperature. But there are differences from the antiferromagnets. At zero temperature, the Chern insulating phase exists in the interval $K_a \in (K_{c1}, K_{c2})$. The trivial phase at $K_a < K_{c1}$ always stays trivial by increasing temperatures. In the interval $K_a \in (K_{c1}, K_{c2})$, the topological bands will experience phase transitions into trivial phase above critical temperatures. When $K_a > K_{c2}$, the trivial bands experience topological transitions into nontrivial phase. In the both phase transitions, the bands also experience band gap closing and reopening at $\Gamma$ or $K$ point, accompanied by a chirality switch, similar to the case of antiferromagnets.