Optical waveguide Hamiltonians leading to step-2 difference equations

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Abstract. We examine the evolution of an \(N\)-point signal produced and sensed at finite arrays of points transverse to a planar waveguide, within the framework of the finite quantization of geometric optics. In contradistinction to the common mechanical Hamiltonians (kinetic plus potential energy terms), the classical waveguide Hamiltonian is the square root of a difference of squares of the refractive index profile minus the optical momentum. The finitely quantized model requires the solution of the square eigenvalue and eigenfunction problem, which leads to a step-two difference equation that contains two solutions and two signs of energy. We find the proper linear combinations to fit the Kravchuk functions of the finite oscillator model.

1. Introduction

Most studies in symmetry, supersymmetry, and separation of variables, pertain Hamiltonian systems of mechanical nature, i.e., whose Hamiltonian functions separate kinetic from potential energy terms,

\[
H^{\text{mech}}(\vec{x}, \vec{p}) = |\vec{p}|^2/2M + V(\vec{x}),
\]

where \(M\) is the mass. Schrödinger quantization of this classical Hamiltonian provides the well-known Schrödinger second-order differential equation, whose eigenvalues are the energies and the stationary states are the eigenvectors of such systems. Waveguides with finite sensor arrays provide Hamiltonian functions which are of a different structure, and whose quantization is proposed to be distinct from the Schrödinger one, where now the geometric optical system is associated to a model where the observables of position, momentum and ‘energy’ have a discrete, finite set of eigenvalues. This we call ‘so(3)’ or ‘finite quantization.’ This is tailored for parallel analysis of finite signals and the processing of pixellated images.

We set up the Hamilton equations of geometric optics starting from the two similar right triangles in Figure 1. Let light rays be abstracted as lines in 3-space \(\vec{x}(s), s \in \mathbb{R}\); and let their tangent vectors be \(\vec{p}(s)\). In [1, Chap. 1] we prove succinctly that, postulating the change in ray direction to be in linear response to the gradient of the refractive index \(n(\vec{x})\), as \(d\vec{p} = \nabla n(\vec{x})\, ds\), leads to the restriction \(|\vec{p}| = n(\vec{x})\). Then, writing

\[
\vec{p} = \begin{pmatrix} P \\ p_z \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ z \end{pmatrix},
\]
the similarity of the triangles in Figure 1 implies that
\[ \frac{dx}{dz} = \frac{p}{p_z}, \quad \frac{dp}{dz} = \frac{dp}{ds} = n \nabla n/p_z, \quad \frac{dp_z^2}{dz} = \partial n^2/\partial z. \] (3)

These equations fit into the Hamiltonian form
\[ \frac{dx}{dz} = \frac{\partial h}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial h}{\partial x}, \quad \frac{dp_z}{dz} = -\frac{\partial h}{\partial z}, \] (4)

with the Hamiltonian function
\[ h(x, p, z) := -\sigma \sqrt{n(x, z)^2 - |p|^2} = -p_z = -n(\bar{x}) \cos \theta, \] (5)

where \( \sigma = + \) applies to forward rays (in the \( +z \) direction), \( \sigma = - \) to backward rays (in the \( -z \) direction), and \( \theta \) is the angle at the screen \( z = 0 \) between the ray and the optical axis \( +z \). (The sign assignment can be seen easily for free propagation in a medium \( n = \text{constant} \).) The square-root form of the Hamiltonian (5) is characteristic of optics, and we shall particularly devote our attention to the case when \( n \) is independent of \( z \)—namely, when the optical medium is a waveguide, so its refractive index is \( n(x) \).

We point to the resemblance of (5) with the total relativistic energy of a freely moving mass point,
\[ E = c\sqrt{M^2c^2 + |p|^2} = Mc^2 + \frac{1}{2M}|p|^2 - \frac{1}{8M^3c^2}|p|^4 + \cdots. \] (6)

The difference between both is of course the sign of the momentum term \( |p|^2 \) in the square root: in (6) the evolution parameter is time \( t \), and the metric of \( (x, t) \) and \( (p, E) \) is Lorentzian, while in (5) the metric of \( (x, z) \) and \( (p, p_z) \) is Euclidean. The Schrödinger quantization of the square of (6) leads to the Klein-Gordon equation as a relativistic quantum mechanical model; it is the Dirac quantization however, which posits a linear matrix equation that is most productive to describe Nature. Similarly, while the wavization of the square of (5) produces the Helmholtz wave equation, we feel that its matrix square-root form can yield interesting results provided with a proper interpretation as a model of finite optics.

In Section 2 we introduce the specific family of waveguides to be treated here, whose refractive index profile is elliptic, arriving at their symmetry expressed in algebraic terms through Poisson brackets. Then in Section 3 we quantize the classical system to a finite system given by a Hamiltonian matrix defined by the square root of a self-adjoint matrix. The signs of the
eigenvalues and eigenvector problem that stems from a step-two difference equation are analyzed and resolved in Section 4. The concluding Section 5 refers to results on the evolution of finite Kravchuk coherent states in such waveguides, and the possible use of aberration expansions in modeling nonlinear optical media.

2. Waveguide Hamiltonians

In a two-dimensional (planar) geometric optical system $\vec{x} = (x, z)$ where the index of refraction is $n(x, z)$, the $x$-coordinate measures the screen $z = 0$, and its $z$-coordinate is the independent evolution parameter along the optical axis of the device. In this paper we shall consider media that are invariant under $z$-translations, i.e. waveguides, where $n \equiv n(x)$; and moreover, for reasons of parametric proximity, we propose waveguides whose refractive index profile be elliptic, namely

$$n^\nu(x) := +\sqrt{\nu^2 - \mu^2 x^2}, \quad \nu \geq 1, \quad \mu \geq 0, \quad |x| \leq \nu/\mu.$$  \hspace{1cm} (7)

At the center of the guide $n(0) = \nu$, while $1/\mu$ estimates the width of the guide. Although physically $n \geq 1$ this is not necessary for the analysis, so we consider the guide to extend between $\pm \nu/\mu$.

The Hamiltonian (5) of the waveguide medium (7) is

$$h^\nu,\mu(x, p) = -\sigma \sqrt{\nu^2 - (p^2 + \nu^2 x^2)}$$

$$= -\sigma \nu + \sigma \frac{p^2 + \nu^2 x^2}{2\nu} + \sigma \frac{(p^2 + \nu^2 x^2)^2}{8\nu^4} + \cdots, \hspace{1cm} (8)$$

where we have expanded the square root into a series of powers of the oscillator Hamiltonian $k_{\nu} := \frac{1}{2}(p^2 + \nu^2 x^2)$. In the waveguide (8), the evolution of phase space along $z$ will be that of an incompressible fluid, flowing along the ellipses $p^2 + \nu^2 x^2 = \text{constant} < \nu^2$. The trajectory of lowest energy $h = -\nu$ is a ray along the $+z$ axis — at the center of phase space, and that of highest energy $\nu$ is along the $-z$ axis. Two particular cases of waveguide with $\nu = 1$ will serve as reference:

- circular profile $\mu = 1, \ k_1 = \frac{1}{2}(p^2 + x^2)$,  \hspace{1cm} (10)
- free particle $\mu = 0, \ k_0 = \frac{1}{2}p^2$.  \hspace{1cm} (11)

Written with Poisson brackets, the Hamilton equations are

$$\{k_{\mu}, x\} = -p, \quad \{k_{\mu}, p\} = \mu^2 x, \quad \{x, p\} = 1.$$  \hspace{1cm} (12)

The four quantities $\{k_{\mu}, x, p, 1\}$ form, for $\mu = 1$, the oscillator Lie algebra $\text{osc}$, while for $\mu = 0$ they form the nilpotent algebra $\text{free}$.

3. Finite $so(3)$-quantization

Finite quantization involves a deformation (or ‘pre-contraction’) of the Lie algebra that contains the classical system [2, 3, 4]. Poisson brackets between the classical phase space observables of position and momentum $(x, p)$ are replaced by $i$ times the commutator Lie brackets between operators $(\mathcal{X}, \mathcal{P})$, closing with a pseudo-Hamiltonian $\mathcal{K}_{\mu}$ — for which we assume no specific form — that obey the same geometry and dynamics. Corresponding to (12), the two Hamilton equations are:

$$[\mathcal{K}_{\mu}, \mathcal{X}] = -i \mathcal{P}, \quad [\mathcal{K}_{\mu}, \mathcal{P}] = i \mu^2 \mathcal{X}.$$  \hspace{1cm} (13)

The deformation is contained in the third — non-standard — commutator

$$[\mathcal{X}, \mathcal{P}] = -i \mathcal{K}_{\mu}.$$  \hspace{1cm} (14)
Thus we represent the classical quantities by the self-adjoint matrices that belong to the Euclidean algebra so(2).

Among the realizations of so(3) we choose the self-adjoint irreducible representation matrices in the eigenspaces of the Casimir operator [5, 6],

\[ C := \mathcal{A}^2 + \mathcal{P}^2 + \mathcal{K}^2 = j(j+1) I, \]

whose eigenvalues (numbered by \( j \)) determine the vector space to be of dimension \( N = 2j+1 \). Thus we represent the classical quantities by the self-adjoint matrices that belong to the representation \( j \) of so(3): \( x \sim \mathbf{X} = \|X_{m,m'}\| \) will be diagonal, \( p \sim \mathbf{P} = \|P_{m,m'}\| \) skew-symmetric, and \( h + \text{const.} \sim \mathbf{K} = \|K_{m,m'}\| \) symmetric, with elements [6]

\[ X_{m,m'} = m \delta_{m,m'}, \quad m, m' \in \{-j, -j+1, \ldots, j\}, \]
\[ P_{m,m'} = -i \frac{1}{2} \sqrt{(j-m)(j+m+1)} \delta_{m+1,m'} + i \frac{1}{2} \sqrt{(j+m)(j-m+1)} \delta_{m-1,m'}, \]
\[ K_{m,m'} = \frac{1}{2} \sqrt{(j-m)(j+m+1)} \delta_{m+1,m'} + \frac{i}{2} \sqrt{(j+m)(j-m+1)} \delta_{m-1,m'}, \]

and their sum of squares (15) is \( \mathbf{C} := \mathbf{X}^2 + \mathbf{P}^2 + \mathbf{K}^2 = j(j+1) I \). The spectrum of the position operator \( \mathbf{X} \) is thus \( x_m \equiv m \in \{-j, -j+1, \ldots, j\} \); the eigenvalues \( \{\omega_r\}_{r=-j}^j \) of the momentum \( \mathbf{P} \), and \( \{\kappa_n\}_{n=-j}^j \) of the pseudo-energy operator \( \mathbf{K} \), take values in the same range. Since we consider important to have sensor arrays that include one point \( x_0 = 0 \) at the center, we consider only integer \( j \)'s, to the exclusion of the su(2) spin representations.

The elliptic-profile waveguide Hamiltonian \( \mathbf{h}^{\nu,\mu}(x,p) \) in (8) will be thus finitely-quantized to an \( N \times N \) hermitian matrix which, taking (15) into account, provides the equation for its eigenvalues and eigenvectors:

\[ \mathbf{H}^{\nu,\mu} := -\sqrt{[(\nu^2-1)\mathbf{C} - (\mu^2-1)\mathbf{X}^2] + \mathbf{K}^2}, \]
\[ \mathbf{H}^{\nu,\mu} \Psi^{\nu,\mu}_\eta(m) = \eta^{\nu,\mu} \Psi^{\nu,\mu}_\eta(m), \quad m \in \{-j, -j+1, \ldots, j\}, \]

where the matrix between the square brackets in (19) is diagonal. We shall call \( \eta^{\nu,\mu} \) the energy of the state \( \Psi^{\nu,\mu}_\eta(m) \), corresponding to the value of the classical Hamiltonian (8). In the special case \( \nu = 1 = \mu \) we have \( \mathbf{H}^{1,1} = -\sqrt{\mathbf{K}^2} \), and we refer to this as a Kravchuk guide; since then the eigenvalues are \( \eta^{1,1}_n = \pm \kappa_n \in \{-j, -j+1, \ldots, j\} \), and the eigenfunctions \( \Psi^{1,1}_n(m) \) are the Kravchuk functions studied in Refs. [4, 7, 8, 9], which are finite deformations of the harmonic oscillator (Hermit-Gauss) wavefunctions.

When \( \mu = 0 \), the free Hamiltonian (19) has eigenvalues \( \eta^{\nu,0}_n = \sqrt{\nu^2(j+1) - \omega_r^2} \), with \( \omega_r = r \) having the same range as \( \kappa_n \). We restrict the waveguide parameters to \( 0 \leq \mu \leq \nu \geq 1 \) to keep the eigenvalues of the radicand positive. Since the square root of a matrix is not uniquely defined, we propose to use its square and solve (20) through

\[ (\mathbf{H}^{\nu,\mu})^2 \Psi^{\nu,\mu}_\eta(m) = \lambda^{\nu,\mu}_m \Psi^{\nu,\mu}_\eta(m), \quad \lambda^{\nu,\mu} := (\eta^{\nu,\mu})^2, \]

knowing that the eigenvectors of \( \mathbf{H} \) and of its square are the same. We thus diagonalize (21) by its eigenvector matrix,

\[ \Psi^{\dagger} (\mathbf{H}^{\nu,\mu})^2 \Psi = \Lambda^{\nu,\mu}, \quad \Psi = \|\Psi^{\nu,\mu}_\eta(m)\| \]
\[ \Lambda^{\nu,\mu} = \text{diag}(\lambda^{\nu,\mu}_{m_{\text{max}}}, \lambda^{\nu,\mu}_{m_{\text{max}-1}}, \ldots, \lambda^{\nu,\mu}_{m_{\text{min}+1}}, \lambda^{\nu,\mu}_{m_{\text{min}}}), \]

where the \( N = 2j+1 \) eigenvalues \( \{\lambda^{\nu,\mu}_m\} \) are ordered from maximum to minimum, observing that away from \( \mu = 0, 1 \), they are non-degenerate, although \( j \) pairs are very close in various ranges of \( \mu \). In Figure 2 (left) we show the computed eigenvalues \( \{\lambda^{\nu,\mu}_n\} \) for \( \nu = 1 \) and \( \nu = \sqrt{2} \), in guides with \( \mu = \nu \), and for free \( \mu = 0 \).
4. Eigenvalue signs and eigenvectors

We now proceed to analyze the sign of the square roots $\sqrt{(\eta^{\nu,\mu})^2}$. Notice that the ground state of a waveguide has the lowest energy eigenvalue $\eta_{gr}^{\nu,\mu} < 0$ of $H^{\nu,\mu}$, and this appears as the highest $\lambda_{\text{max}}^{\nu,\mu} = (\eta_{gr}^{\nu,\mu})^2 > 0$ of $(H^{\nu,\mu})^2$. In the Kravchuk case this eigenvalue is $\eta_{-j}^{1,1} = -j$, whose square $j^2$ is degenerate with that of the highest energy that the waveguide can carry, $\eta_j^{1,1} = j$. In the other extreme case $\mu = 0$, the ground state has zero transverse momentum $\omega_0 = 0$, so $\eta_{gr,0}^{\nu,0} = -\nu/j(j+1)$; this state also has the highest eigenvalue $\lambda_{\text{max}}^{\nu,\mu} = (\eta_{gr,0}^{\nu,0})^2 > 0$ under $(H^{\nu,\mu})^2$, and is nondegenerate. Since eigenvalues do not cross, we conclude that for all $0 \leq \mu < 1$ the highest eigenvalue of $(H^{\nu,\mu})^2$ corresponds to the ground state. We thus take the square root of this eigenvalue with a negative sign: $\eta_{gr}^{\nu,\mu} := -\sqrt{\lambda_{\text{max}}^{\nu,\mu}}$. On the other hand, the state of highest energy that the waveguide can carry, $\eta_{\text{top}}^{\nu,\mu}$, is positive; and this corresponds with the next-highest square eigenvalue, so we should opt for the positive sign: $\eta_{\text{top}}^{\nu,\mu} := +\sqrt{\lambda_{\text{max}}^{\nu,\mu} - 1}$. Again, since eigenvalues do not cross, $\lambda_{\text{max}}^{\nu,\mu} < \lambda_{\text{max}}^{\nu,\mu}$. Continuing in this way with the $2j+1$ eigenvalues $\{\lambda_n^{\nu,\mu}\}$ of $(H^{\nu,\mu})^2$ from top to bottom, we conclude that $\eta_{gr,n}^{\nu,\mu} = -\sqrt{\lambda_{\text{max}}^{\nu,\mu} - 2n}$ and $\eta_{\text{top},n}^{\nu,\mu} = +\sqrt{\lambda_{\text{max}}^{\nu,\mu} - 2n - 1}$, for $n \in \{0,1,\ldots,j\}$. In Figure 2 (right) we show the $\eta$'s with this alternating sign selection.

To compute the eigenvectors $\Psi_\eta = \Psi_\eta^{\nu,\mu}$ of $(H^{\nu,\mu})^2$ in (21) we must solve a difference equation [10], which is of step-two:

$$
\begin{align*}
\frac{1}{4} \sqrt{(j-m-1)(j-m)(j+m+1)(j+m+2)} \Psi_\eta(m+2) \\
+ [j(j+1)(\nu^2 - \frac{1}{2}) - m^2(\mu^2 - \frac{1}{2}) - \eta^2] \Psi_\eta(m) \\
+ \frac{1}{4} \sqrt{(j+m-1)(j+m)(j-m+1)(j-m+2)} \Psi_\eta(m-2) &= 0,
\end{align*}
$$

(23)

where we notice that both $\Psi_\eta(m)$ and $\Psi_{-\eta}(m)$ are solutions to the same equation, as well as any linear combination of the two. Replacing $m \leftrightarrow -m$ in (23) returns the same equation for $\Psi_{\pm\eta}(-m)$, so the solutions will have definite parity. Equation (23) stands in effect for a pair of
difference equations. Assume first that \( j \) is odd; then, the values of one solution, \( \Psi_{n,j}^e(m) \), can be determined at all the even points \( x_m \equiv m \) by setting \( \Psi_{n,j}^e(\pm(j+1)) = 0 \), using the recurrence (23) to determine its values \( \Psi_{n,j}^e(m) \) for all other even \( |m| \leq j-1 \), while its values at all odd \( m \)'s are zero. A second solution, \( \Psi_{n,j}^o(m) \), nonzero at all odd points \( m \) stems from setting \( \Psi_{n,j}^o(\pm(j+2)) = 0 \) and then using the same (23) to determine its values at all odd points \( |m| \leq j \), and zero at the even \( m \)'s. (When \( j \) is even, a corresponding separation into two independent solutions is made.)

Since both \( \Psi_{n,j}^e(m) \) and \( \Psi_{n,j}^o(m) \) have zeros at alternating points, they appear as ‘porcupine’ signals, such as shown in Figure 3 (top row). A question thus arises on the relation between these two orthogonal solutions to (23) and the two solutions \( \Psi_{\pm\mu}(m) \) mentioned earlier. What we expect is to reproduce ground states with a Gaussian shape—in particular the ground Kravchuk function in a Kravchuk guide [2]; and for \( \mu = 0 \) a state resembling a free wave. In all cases, definite parity is guaranteed by (23).

For general \( \nu; \mu \) it does not seem possible to find a known discrete special function that will solve (23) ‘exactly,’ so we must resort to numerical computation to plot results. In Figure 3 (rows 2–4) the normalized sum and difference of the first three lowest energy \( (\eta < 0) \) states (24), and the three highest \( (\eta > 0) \) states (25), in the range \( 0 \leq \mu \leq \nu = 1 \). The resulting states

\[
\Psi_{\eta,j}^e(m) = \frac{1}{\sqrt{2}} [\Psi_{n,j}^e(m) + \Psi_{n,j}^o(m)],
\]

\[
\Psi_{-\eta,j}^o(m) = \frac{1}{\sqrt{2}} [\Psi_{n,j}^e(m) - \Psi_{n,j}^o(m)],
\]

have recognizable shapes and definite energy eigenvalues.

The \( \nu \to 1 \) limit to the ‘Kravchuk’ spectrum \( \{ -j, -j+1, \ldots, j \} \) thus defines the ‘proper’ square root of the diagonal matrix \( \Lambda^{\nu;\mu} \) in (22),

\[
\tilde{H}^{\nu;\mu} := \text{diag}(\eta_1, \eta_2, \eta_3, \ldots, \eta_\nu, \eta_{\nu+1}, \ldots, \eta_{\nu\nu})
\]

The Hamiltonian matrix in (19) is finally obtained through de-diagonalizing (26) by means of the transpose of the eigenvector matrix,

\[
H^{\nu;\mu} = \Psi \tilde{H}^{\nu;\mu} \Psi^\dagger.
\]

5. Concluding remarks

The \( z \)-evolution along the waveguide is generated by the proper Hamiltonian matrix (27), exponentiated a 1-parameter group of unitary \( N \times N \) matrices

\[
U^{\nu;\mu}(z) := \exp(iz H^{\nu;\mu}) \in U(N),
\]

for \( z \in \mathbb{R} \). This is a line within the \( N^2 \)-dimensional compact manifold of \( U(N) \) that will in general not close. In Ref. [10] we computed the \( z \)-evolution of the finite Kravchuk coherent states in a Kravchuk guide, and plotted the result using the \( so(3) \) Wigner function [11] to ascertain that one reproduces the usual forward \( z > 0 \) circulation of phase space. The movement of backward rays is obtained simply by letting \( z < 0 \), which is equivalent to exchanging the signs of all energies in (26). Also, we have let the finite coherent states evolve in a non-Kravchuk guide with \( \nu = \sqrt{2} \) and \( \mu = \frac{1}{2} \sqrt{2} \) to observe the nonlinear aberration of the Wigner function that results from classical phase space points moving on concentric ellipses \( p^2 + \frac{1}{2} x^2 = \text{constant} < 2 \) at different velocities. Presently we are investigating the behavior of the coherent states in waveguides whose refractive index is different from the elliptic one in (7), such as a rectangular index profile with zeros at the endpoints, and an intermediate family of ‘smooth’ profiles.
Figure 3. Solutions $\Psi_g(m)$ to the step-two difference equation (23), for $N = 21$ points ($|m| \leq j = 10$); lines connect their values at the integer $m$'s. Top row: The ground ‘porcupine’ states, $\Psi^c_g(m)$ and $\Psi^o_g(m)$, in a ‘Kravchuk’ guide $\nu = 1 = \mu$, corresponding to the largest eigenvalue $\eta^2$ in Fig. 2, and yielding both the ground and top energy states. Second row: The sum and difference (24)–(25) of the previous porcupines produces the ground, and top states of the Kravchuk guide —indicated by a heavy line; with light lines we show the ground and top states of guides with $\mu = 0.95, 0.9, \ldots, 0.8, 0.7, \ldots, 0.0$. Third row: The same as the previous row, but for the next-to-largest eigenvalue $\eta'^2$ of Fig. 2, corresponding to the next-to-lowest ground and next-to-top states. Fourth row: Similarly, for the following lowest and highest states.

In this paper our interest was to examine optical Hamiltonian operators that are defined by the square root of a positive, self-adjoint and otherwise well known operator, which is a problem seldom addressed in mechanics. The resulting step-two difference equation separates solutions into two ‘porcupine’ components that seem to be inseparable from the sign ambiguity of the square root operation. Since several group-theoretic developments exist which (rightly or wrongly) include the square root of a positive operator —for purposes of normalization, usually—, this operation may shown to have some hidden difficulties.

As suggested by the aberration expansion (9) for the geometric optical Hamiltonian, we also considered the perturbative description of waveguide evolution as a sum of powers of harmonic oscillator Hamiltonians. The main difficulty in using this expansion for finite waveguide models is that the matrix series does not converge: in the Kravchuk $\mathfrak{so}(3)$ case the maximum element...
of $H^{osc} := P^2 + X^2 = C - K^2$ is $j(j+1) - j^2 = j$, and hence a bound for the maximum of $(H^{osc})^n$ is $j^n(2j+1)^n - 1$; meanwhile, the coefficient of this term in the $\sqrt{1-x}$ series is $(2n-3)!!/(2n)!$. Yet, the first two powers can be used to model paraxial and nonlinear Kerr media [12, 13, 14], as done in Ref. [15]. The purpose of our exploration of discrete and finite systems based on representations of compact groups such as $so(3)$ and $so(4)$, and also of discrete infinite systems [4], is to understand finite signal analysis and pixellated image processing in terms of the symmetries behind their geometric optical realizations.

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