\( \mathcal{N} = 8 \) supersymmetric mechanics on special Kähler manifolds

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Abstract

We propose the Hamiltonian model of \( \mathcal{N} = 8 \) supersymmetric mechanics on \( n \)-dimensional special Kähler manifolds (of the rigid type).

Introduction

One of the most studied theories since its introduction has been the concept of supersymmetric mechanics [1]. In spite of its simplicity this system inherits the qualitative properties of supersymmetric field/string theories. Moreover, it is relevant to the specific problems of condensed matter, quantum optics and mathematical physics. The quantum-mechanical studies of this system were focussed mainly on the simplest case of standard \( \mathcal{N} = 2 \) supersymmetry\(^1\) (see the review in [2] and refs therein). The systems with \( \mathcal{N} = 4 \) supersymmetry also received enough attention, but mostly in the classical context, concentrated on building the appropriate Lagrangian and Hamiltonian models of supersymmetric mechanics (see, e.g., [3, 4, 5, 6, 7, 8] and refs therein).

For many reasons, the most interesting case seems to be the \( \mathcal{N} = 8 \) supersymmetric mechanics. First of all, due to general considerations developed in [9] this is the highest \( \mathcal{N} \) case of minimal \( \mathcal{N} \)-extended supersymmetric mechanics admitting realization on \( \mathcal{N} \) bosons (physical and auxiliary) and \( \mathcal{N} \) fermions. Secondly, systems with eight supercharges are the highest \( \mathcal{N} \) ones among extended supersymmetric systems which still possess a non-trivial geometry in the bosonic sector [10]. When the number of supercharges exceeds 8, the target spaces are restricted to be symmetric spaces. Finally, \( \mathcal{N} = 8 \) supersymmetric mechanics should be related via a proper dimensional reduction with four-dimensional \( \mathcal{N} = 2 \) supersymmetric field theories. So, one may hope that some interesting properties of the latter will survive after reduction. Nevertheless, the study of \( \mathcal{N} = 8 \) supersymmetric mechanics remains in the initial stage, though there were a few interesting investigations on this subject [11, 12, 13, 14]. Particularly, the \( \mathcal{N} = 8 \) supersymmetric mechanics constructed by the use of the vector supermultiplet reduced to one dimension at the prepotential level has been considered in [11]. However, for the reduced (5,8,3) supermultiplet involving five physical bosons, supersymmetric mechanics is quite different from the original four-dimensional \( \mathcal{N} = 2 \) super Yang-Mills theory. In [14] the \( \mathcal{N} = 8 \) supersymmetric mechanics was considered using the (2,8,6) supermultiplet constructed by the dimensional reduction of four-dimensional \( \mathcal{N} = 2 \) super Yang-Mills theory at the level of the field strength.

In this note we propose the Hamiltonian mechanics with \( \mathcal{N} = 8 \) supersymmetry algebra\(^2\)

\[
\{Q_{\alpha i}, Q_{\beta j}\} = \{\overline{Q}_{\alpha i}, \overline{Q}_{\beta j}\} = 0, \quad \{Q_{\alpha i}, \overline{Q}_{\beta j}\} = \epsilon_{\alpha \beta} \epsilon_{ij} \mathcal{H}_{\text{SUSY}}, \quad i, j = 1, 2; \alpha, \beta = 1, 2.
\]

We require the configuration space of the underlying bosonic mechanics to be a Kähler manifold (\( M_0, g_{ab} dz^a d\bar{z}^b \)), as in the \( \mathcal{N} = 4 \) mechanics considered in Ref. [8]. Our model has the \((2n,4n)_{\mathcal{N}}\)–dimensional phase space, while in its \( \mathcal{N} = 4 \) counterpart the phase space is the \((2n,2n)_{\mathcal{N}}\)–dimensional one. In the former system the supercharges have a cubic dependence on the fermionic variables, while in the latter one they have a linear dependence. There is no doubt, that our model is the Hamiltonian counterpart of the supersymmetric mechanics, which could be constructed at the Lagrangian level within the superfield approach by the use of the \((2,8,6)\) supermultiplet suggested in [14].

We find that \( \mathcal{N} = 8 \) supersymmetry yields a strong restriction on the geometry of configuration space: it has to be a special Kähler manifold (of the rigid type) (see, e.g., [15]). In a distinguished coordinate frame its Kähler potential is given by the expression

\[
K(z, \bar{z}) = i \left( \frac{\partial f(z)}{\partial z^a} \bar{z}^a - \frac{\partial f(\bar{z})}{\partial \bar{z}^a} z^a \right).
\]

\(^1\)Hereafter \( \mathcal{N} \) denotes the number of real supercharges. Often one considers supersymmetric mechanics with \( \mathcal{N} \) even, dividing the supercharges into complex pairs \( (Q, \bar{Q}) \).
\(^2\)We use the following convention for the skew-symmetric tensor: \( \epsilon_{ijk} = \delta_{ij}^k, \epsilon_{12} = \epsilon^{21} = 1 \).
During the last decade such a manifold was paid much attention to, due to its relevance for the phenomenon
of electric-magnetic duality of four-dimensional $\mathcal{N} = 2$ super Yang-Mills theories \[16\]. The proposed system
inherits, besides the special Kähler geometry, the duality symmetry of super Yang-Mills theory, which makes
the constructed $\mathcal{N} = 8$ supersymmetric mechanics a good \textit{“probe”} to analyze some subtle properties of
$\mathcal{N} = 2, d = 4$ superYang-Mills theory. Finally, much like the four-dimensional $\mathcal{N} = 2$ super Yang-Mills case \[17\] the action of our supersymmetric mechanics can be modified by introducing Fayet-Iliopoulos terms which
give rise to potential terms.

The model

In order to construct $\mathcal{N} = 8$ supersymmetric mechanics, let us define the (2n,4n)-dimensional symplectic
structure

$$\Omega = d\mathcal{A} = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - R_{abcd}\eta^c_{\bar{\imath}a} \eta^d_{\bar{\jmath}a} dz^a \wedge d\bar{z}^b + g_{ab} D\eta^a_{\bar{\imath}a} \wedge D\eta^b_{\bar{\jmath}a},$$

where

$$\mathcal{A} = \pi_a dz^a + \bar{\pi}_a d\bar{z}^a + \frac{1}{2} \eta^a_{\bar{\imath}a} g_{ab} D\eta^b_{\bar{\jmath}a} + \frac{1}{2} \eta^b_{\bar{\jmath}a} g_{ab} D\eta^a_{\bar{\imath}a}, \quad D\eta^a_{\bar{\imath}a} = d\eta^a_{\bar{\imath}a} + \Gamma^b_{\bar{\imath}a} \eta^b_{\bar{\jmath}a} dz^c,$$

and $\Gamma^a_{\bar{\imath}a} = g^{da} g_{bd,e}, R_{abcd} = -g_{eb} (\Gamma^c_{\bar{\imath}a}, \bar{d})$ are, respectively, the components of the connection and curvature of
the Kähler structure. The corresponding Poisson brackets are given by the following non-zero relations (and
their complex-conjugates):

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \bar{\pi}_b\} = -\Gamma^b_{\bar{\imath}a} \eta^c_{\bar{\jmath}a}, \quad \{\pi_a, \bar{\pi}_b\} = R_{abcd} \eta^c_{\bar{\imath}a} \bar{\eta}^d_{\bar{\jmath}a}, \quad \{\eta^a_{\bar{\imath}a}, \bar{\eta}^b_{\bar{\jmath}a}\} = g^{ab} \delta^\imath_\jmath \delta^\alpha_\beta.$$

It is clear that the symplectic structure is covariant under following holomorphic transformations:

$$z^a = z^a(z), \quad \bar{\eta}^a_{\bar{\imath}a} = \frac{\partial z^a(z)}{\partial \bar{z}^b} \eta^b_{\bar{\jmath}a}, \quad \bar{\pi}_a = \frac{\partial z^b}{\partial \bar{z}^a} \pi_b.$$

Let us search for the supercharges among the functions

$$Q_{ia} = \pi_a \eta^a_{\bar{\imath}a} + \frac{1}{3} f_{abc} \bar{T}^{abc}_{ia} \quad \bar{Q}_{ia} = \bar{\pi}_a \bar{\eta}^a_{\bar{\imath}a} + \frac{1}{3} f_{abc} T^{abc}_{ia}.$$

Calculating mutual Poisson brackets of $Q_{ia}, \bar{Q}_{ia}$ one gets that they obey the $\mathcal{N} = 8$ supersymmetry algebra \[1\], if the following relations hold:

$$\frac{\partial}{\partial z^d} f_{abc} = 0, \quad R_{abcd} = -f_{aceg} \bar{f}^{cg} f^{e\bar{d}}.$$

The above equations guarantee, respectively, that the first and the second equations in \[1\] are fulfilled. Then,
we could immediately get the $\mathcal{N} = 8$ supersymmetric Hamiltonian

$$\mathcal{H}_{\text{SUSY}} = \pi_a g^{ab} \bar{\eta}_b + \frac{1}{3} f_{abc,d} \Lambda_{abcd} + \frac{1}{3} \bar{f}_{abc,d} \bar{\Lambda}^{abcd} + f_{abc} g^{c\bar{d}} \bar{f}^{\bar{d}e} \bar{\Lambda}^{e\bar{a}d} \Lambda^{b\bar{c}d},$$

where

$$\Lambda^{abcd} = -\frac{1}{4} \eta^a_{\bar{\imath}a} \eta^b_{\bar{\imath}b} \eta^c_{\bar{\jmath}c} \eta^d_{\bar{\jmath}d}, \quad \bar{\Lambda}^{abcd} = \frac{1}{2} (\eta^{a\bar{i}a} \eta^{b\bar{j}b} \eta^{c\bar{k}c} \eta^{d\bar{l}d} + \eta^{a\bar{j}a} \eta^{b\bar{i}b} \eta^{c\bar{l}c} \eta^{d\bar{k}d}),$$

and the covariant derivatives of the third rank symmetric tensor are defined as

$$f_{abc,d} = f_{abc,d} - \Gamma^e_{\bar{\imath}a} f_{ebc} - \Gamma^e_{\bar{\jmath}b} f_{aec} - \Gamma^e_{\bar{\imath}c} f_{abe}.$$

The equations \[8\] precisely mean that the configuration space $M_0$ is a \textit{special Kähler manifold of the rigid
type} \[15\]. Taking into account the symmetry of $f_{abc}$ over spatial indices and the explicit expression of $R_{abcd}$
via the metric $g_{ab}$, we can immediately find the local solution for the equations \[8\]

$$f_{abc} = \frac{\partial^2 f(z)}{\partial z^a \partial z^b \partial z^c}, \quad g_{ab} = e^i \eta^a_{\bar{\imath}a} \eta^d_{\bar{\jmath}a} \eta^b_{\bar{\jmath}d} + e^{-i\nu} \frac{\partial^2 f(z)}{\partial z^a \partial z^b}, \quad \nu = \text{const} \in \mathbb{R}.$$

Properly redefining the local function $f(z)$ and the odd coordinates $\eta^a_{\bar{\imath}a}$, we shall get a supersymmetric mechanics on the Kähler space with metric defined by the potential \[2\]. For sure, this local solution is
not covariant under an arbitrary holomorphic transformation, and it assumes the choice of a distinguished coordinate frame.

The special Kähler manifolds of the rigid type became widely known due to the so-called “T-duality symmetry”, which connects, in the of $\mathcal{N} = 2, d = 4$ super Yang-Mills theory, the UV and IR limits of the theory [10]. The “T-duality symmetry” is expressed as follows:

$$ (z^a, f(z)) \Rightarrow \left( u_a = \frac{\partial f(z)}{\partial z^a}, \bar{f}(u) \right), \text{ where } \frac{\partial^2 \bar{f}(u)}{\partial u_a \partial u_c \partial z^b \partial z^c} = -\delta_b^c. $$

(12)

By the use of (10), we can extend the duality transformation (12) to that of the whole phase superspace $(\pi_a, z^a, \eta^a_\alpha) \to (p^a, u_a, \psi_{a[ia]}$)

$$ u_a = \partial_a f(z), \quad p^a = \frac{\partial^2 f}{\partial z^a \partial z^b} = -\pi_b, \quad \psi_{a[ia]} = \frac{\partial^2 f}{\partial z^a \partial z^b} \eta^{b[ia}. \quad \text{(13)} $$

Now, taking into account the expression of the symplectic structure in terms of the presymplectic one-form (8), we can easily perform the Legendre transformation of the Hamiltonian to the (second-order) Lagrangian

$$ \mathcal{L} = \mathcal{A}(d/dt) - \mathcal{H}_{SU(8)} \big|_{\pi_a = g_{ab} z^b} = $$

$$ = g_{ab} \dot{z}^a \dot{z}^b + \frac{1}{2} \delta_{abcd} \frac{D\eta^{[i|a}}{dt} + \frac{1}{2} \delta_{abcd} \frac{D\eta^{[j|a}}{dt} - \frac{1}{3} f_{abcd} \Lambda^{abc} - \frac{1}{3} \bar{f}_{abcd} \bar{\Lambda}^{abc} - f_{abcd} \bar{g}^{cde} \bar{f}_{d} \bar{\Lambda}_0^{abcd}. $$

(14)

Here we denoted $d/dt = \dot{z}^a \partial/\partial z^a + \delta_{abcd} \partial/\partial \eta^a + c.c. \quad . \quad $ Clearly, the Lagrangian (11) is covariant under holomorphic transformations (9), and duality transformations as well. As we noticed in Introduction, the above Lagrangian admits the superfield description based on the $(\mathcal{N} = 8, d = 1)$ supermultiplet constructed in [14]. The basic object is the $\mathcal{N} = 8$ complex superfield $Z, \overline{Z}$ subject to the constraints

$$ D^{ia} Z = 0, \quad \overline{D}^{ia} \overline{Z} = 0, \quad \text{(15)} $$

$$ D^{(ia} D^{jb)} Z + \overline{D}^{(ia} \overline{D}^{jb)} \overline{Z} = 0, \quad \mathcal{D}^{(ia} \mathcal{D}^{jb)} Z - \overline{D}^{(ia} \overline{D}^{jb)} \overline{Z} = 0, $$

(16)

where the spinor derivatives obey the following relations:

$$ \{ \mathcal{D}^{ia}, \mathcal{D}^{jb} \} = \{ \overline{D}^{ia}, \overline{D}^{jb} \} = 0, \quad \{ \mathcal{D}^{ia}, \overline{D}^{jb} \} = 2ik^{ij} \epsilon^{ab} \partial_t . $$

(17)

The equations (15), (16) represent the direct reduction of the constraints describing the $\mathcal{N} = 2, d = 4$ vector multiplet. The most general $\mathcal{N} = 8$ superfield action mimics the action for the $\mathcal{N} = 2, d = 4$ case

$$ S = \text{Im} \int dt d^4 \theta f(Z). $$

(18)

Being integrated over $\theta, \bar{\theta}$, exploiting the constraints (15), (16) and eliminating the auxiliary fields by their equations of motion, the action (18) coincides with the one defined by the Lagrangian (13). Thus, we see that our system indeed is closely related with the action for the $\mathcal{N} = 2, d = 4$ vector supermultiplet. Clearly enough, the special Kähler geometry arises naturally from the action as in the $d = 4$ case. The analogy with $d = 4$ goes even further, since the proposed system exhibits the Seiberg-Witten duality. Moreover, one can modify the actions by adding two possible Fayet-Iliopoulos terms with some arbitrary constant vectors as it has been done in $d = 4$ [17]. For a special choice of the superpotential, we expect that the system will have also conformal symmetry. The detailed discussion of these cases goes beyond the scope of the present paper and will be carried out elsewhere.

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