SYMMETRY BREAKING AND MORSE INDEX OF SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS IN THE PLANE

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Abstract. In this paper we study the problem

\[
\begin{aligned}
-\Delta u &= \left(\frac{2+\alpha}{2}\right)^2 |x|^\alpha f(\lambda, u), \quad \text{in } B_1 \\
u &> 0, \quad \text{in } B_1 \\
u &= 0, \quad \text{on } \partial B_1
\end{aligned}
\]

where \(B_1\) is the unit ball of \(\mathbb{R}^2\), \(f\) is a smooth nonlinearity and \(\alpha, \lambda\) are real numbers with \(\alpha > 0\). From a careful study of the linearized operator we compute the Morse index of some radial solutions to \((P)\). Moreover, using the bifurcation theory, we prove the existence of branches of non-radial solutions for suitable values of the positive parameter \(\lambda\). The case \(f(\lambda, u) = \lambda e^u\) provides more detailed information.

1. Introduction and main results

In this paper we study the problem

\[
\begin{aligned}
-\Delta u &= \left(\frac{2+\alpha}{2}\right)^2 |x|^\alpha f(\lambda, u), \quad \text{in } B_1 \\
u &> 0, \quad \text{in } B_1 \\
u &= 0, \quad \text{on } \partial B_1
\end{aligned}
\]  

(1.1)

where \(\alpha, \lambda\) are real numbers, \(\alpha > 0\) and \(B_1\) is the unit ball of \(\mathbb{R}^2\). The nonlinearity \(f(t, s)\) satisfies

\[
f : (a, b) \times \mathbb{R}^+ \to \mathbb{R}^+, \quad a, b \in \mathbb{R}, \quad f \in C^{1,\gamma}_{loc}((a, b) \times \mathbb{R}^+)  
\]  

(1.2)

for some \(\gamma \in (0, 1]\).

Problem (1.1) models different type of equations. When \(f(\lambda, s) = s^\lambda\) with \(\lambda > 1\) it is known as Hénon problem and arises in the study of stellar clusters. Other interesting examples are given by \(f(\lambda, s) = \lambda(1 + s)^p\) with \(p > 1, \lambda > 0\) and \(f(\lambda, s) = \lambda e^s\). In this last case problem (1.1) is sometimes known as Liouville equation and arises in the study of vortices of Euler flows in the gauge field theory, when the equation involves singular sources. Observe that the presence of the term \(|x|^\alpha\) allows the existence of nonradial solutions for (1.1) and the constant \(\left(\frac{2+\alpha}{2}\right)^2\) in (1.1), in many cases, can be merged into the equation.

The first two authors are supported by PRIN-2009-WRJ3W7 grant, S. Neves has been partially supported by FAPESP.
In the particular case of \( f(\lambda, s) = \lambda e^s \) problem (1.1) has been studied by many authors. Del Pino, Kowalczyk and Musso in [10] proved the existence of solutions concentrating at one or more points when \( \alpha \) is fixed and \( \lambda \) is close to zero (see also [6] for a generalization of this result). We quote also the results in [1], [2] and [4] where the behavior of the solutions as \( \lambda \to 0 \) is considered.

In this paper we are interested in studying existence of nonradial solutions to (1.1). The main tool to get our results will be the bifurcation theory. We want to stress that the results we obtain, both for the general problem (1.1) that for the special case of \( f(\lambda, s) = \lambda e^s \), allow \( \alpha \) and \( \lambda \) to vary in all the range of existence of solutions and not only in a neighborhood of some specific value (as \( \lambda = 0 \)).

To our knowledge one of the few results where \( \lambda \) varies in all its range is due to Suzuki [18], where he proved the nondegeneracy of the solution in simply connected domains when \( \alpha = 0, f(\lambda, s) = \lambda e^s, \lambda \int_\Omega e^u < 8\pi \) and \( \lambda \in (0, \lambda^*) \). Here \( \lambda^* \) is the maximal value of \( \lambda \) such that (1.1) has a solution. In a similar spirit we will obtain some existence results of solutions to (1.1).

A crucial role in our analysis is given by the autonomous problem associated to (1.1),

\[
\begin{cases}
-\Delta v = f(\lambda, v), & \text{in } B_1 \\
v > 0, & \text{in } B_1 \\
v = 0, & \text{on } \partial B_1.
\end{cases}
\]  

(1.3)

The well known result by Gidas, Ni and Nirenberg ([11]) tell us that all solutions to (1.3) are radial (this is no longer true for (1.1)).

Let us remark that, if we restrict ourselves to considering only radial solutions, then problems (1.1) and (1.3) are equivalent. This can be easily seen using the map \( r \mapsto 2 + 2\alpha \) as pointed out in [5] (see also [13] and the proof of Proposition 1.1).

Let us now turn to illustrate the strategy of our paper: first of all observe that, in many situations, it is possible to establish the existence of radial solutions \( v_\lambda \) to (1.3) for some suitable values \( \lambda \in (a, b) \).

Then we get that

\[ u_{\lambda, \alpha}(r) = v_\lambda(r^{\frac{2+\alpha}{2}}) \]  

(1.4)

are radial solutions to (1.1) for any \( \alpha > 0 \) and \( \lambda \in (a, b) \). If we look for nonradial solutions which bifurcate from \( u_{\lambda, \alpha} \), by the implicit function theorem, the values of \( \alpha \) and \( \lambda \) must satisfy the degeneracy condition,

\[
\begin{cases}
-\Delta w - \left(\frac{2+\alpha}{2}\right)^2 |x|^\alpha f'(\lambda, u_{\lambda, \alpha}) w = 0, & \text{in } B_1 \\
w = 0, & \text{on } \partial B_1
\end{cases}
\]  

(1.5)

for some nontrivial \( w \in H^1_0(B_1) \), where \( f'(\lambda, u) = \frac{\partial f}{\partial u}(\lambda, u) \).

In general the computation of the values \( \alpha \) and \( \lambda \) for which \( u_{\lambda, \alpha} \) is degenerate is a very difficult problem. However, in the case where the solution \( v_\lambda \)
of (1.3) has Morse index 1, we will be able to characterize them. This is our first result.

**Proposition 1.1.** Assume (1.2) and that problem (1.3) has a solution \( v_\lambda \) for any \( \lambda \in (a,b) \). Then problem (1.1) has a radial solution \( u_{\lambda,\alpha}(r) = v_\lambda \left( r^{\frac{2+\alpha}{2}} \right) \) for any \( \lambda \in (a,b) \) and \( \alpha > 0 \).

Moreover if \( v_\lambda \) is radially nondegenerate and it has Morse index 1, setting

\[
\nu_1(\lambda) := \inf_{\eta \in H^1((0,1),rdr) \atop \eta \neq 0, \eta(1)=0} \frac{\int_0^1 r(\eta')^2 \, dr - \int_0^1 r f'(\lambda, v) \eta^2 \, dr}{\int_0^1 r^{-1} \eta^2 \, dr} \tag{1.6}
\]

we have that \( u_{\lambda,\alpha} \) is degenerate if and only if \( \lambda \) and \( \alpha \) satisfy

\[
\nu_1(\lambda) = -\frac{4k^2}{(2+\alpha)^2} \tag{1.7}
\]

for some integer \( k \geq 1 \). The solutions of (1.5) corresponding to the values of \( \lambda \) and \( \alpha \) which satisfy (1.7) are given by, in polar coordinates,

\[
\psi_{\lambda,\alpha}(r, \theta) = \tilde{\psi}_{1,\lambda} \left( r^{\frac{2+\alpha}{2}} \right) \left[ A \sin \left( \frac{2 + \alpha}{2} \sqrt{-\nu_1(\lambda)} \right) \theta + B \cos \left( \frac{2 + \alpha}{2} \sqrt{-\nu_1(\lambda)} \right) \theta \right] \tag{1.8}
\]

for any real constant \( A \) and \( B \) where \( \tilde{\psi}_{1,\lambda} \) is the function which achieves (1.6).

Else if \( v_\lambda \) is degenerate for some \( \hat{\lambda} \in (a,b) \), with eigenfunction \( \tilde{\psi}_{\hat{\lambda}} \), then \( u_{\hat{\lambda},\alpha} \) is radially degenerate for any \( \alpha > 0 \) with eigenfunction \( \psi_{\hat{\lambda},\alpha}(r) = \tilde{\psi}_{\hat{\lambda}} \left( r^{\frac{2+\alpha}{2}} \right) \).

Equation (1.7) characterizes all the degeneracy points of the radial solutions \( u_{\lambda,\alpha} \). It seems important to emphasize that the map \( r \mapsto r^{\frac{2+\alpha}{2}} \), that links the radial solutions of (1.1) and (1.3), allows us to say more: it identifies the degeneracy points in terms of \( \nu_1(\lambda) \) that is not directly related with problem (1.1) but just with problem (1.3).

In addition, the quantity \( \nu_1(\lambda) \), in some specific cases, can be explicitly computed as we shall see, for example, in Proposition 1.4.

It seems to us that this link has never been highlighted before and can bring useful information such as the calculation of the Morse index.

The degeneracy result of Proposition 1.1 can be generalized also to solutions \( v_\lambda \) with Morse index greater than 1. This case, however, goes beyond this work and we do not treat it.

A first consequence of Proposition 1.1 is the computation of the Morse index of the solution \( u_{\lambda,\alpha} \).

**Theorem 1.2.** Let \( v_\lambda \) be a solution of (1.3) with Morse index 1 and \( u_{\lambda,\alpha} = v_\lambda \left( r^{\frac{2+\alpha}{2}} \right) \). Then the Morse index \( m(\lambda, \alpha) \) of the radial solution \( u_{\lambda,\alpha} \) to (1.1) (see (1.4)) is equal to

\[
m(\lambda, \alpha) = \begin{cases} 
1 + 2 \left( \frac{\alpha + 2}{2} \sqrt{-\nu_1(\lambda)} \right) & \text{if } \frac{\alpha + 2}{2} \sqrt{-\nu_1(\lambda)} \not\in \mathbb{N} \\
(\alpha + 2) \sqrt{-\nu_1(\lambda)} - 1 & \text{if } \frac{\alpha + 2}{2} \sqrt{-\nu_1(\lambda)} \in \mathbb{N} 
\end{cases} \tag{1.9}
\]
where \([x]\) denotes the greatest integer less than or equal to \(x\).

Moreover \(m(\lambda, \alpha) \to +\infty\) as \(\alpha \to +\infty\).

Observe that starting from a solution \(v_\lambda\) of (1.3) of Morse index 1 we get a radial solution \(u_{\lambda, \alpha}\) of (1.1) which is degenerate in a certain number of points and whose Morse index increases with \(\alpha\).

Next result shows that, under some additional assumptions, the condition (1.7) is also sufficient to get solutions which bifurcate from the radial one.

**Theorem 1.3.** Suppose that \(f\) satisfies (1.2), \(f(\lambda, 0) \geq 0\) and assume that problem (1.3) has a solution \(v_\lambda\) for any \(\lambda \in (a, b)\) which is radially nondegenerate with Morse index 1.

Let \(\alpha > 0\) be fixed and set

\[
F_k(\lambda) = \nu_1(\lambda) + \frac{4k^2}{(2 + \alpha)^2}
\]

for any integer \(k \geq 1\). If

i) there exists, for some \(k \geq 1\) a real value \(\lambda = \lambda(k) \in (a, b)\) at which the function \(F_k(\lambda)\) changes sign in a neighborhood of \(\lambda = \lambda(k)\);

ii) for any \(k \geq 1\) the zeros of \(F_k(\lambda)\) in a neighborhood of \(\lambda(k)\) are isolated,

then there exists a branch of nonradial solutions of (1.1) bifurcating from \((\lambda(k), u_{\lambda(k), \alpha})\).

Moreover branches of bifurcating solutions related to different values of \(k\) are separated and one of the following alternative holds

- they are unbounded in \((a, b) \times C_0^1(\bar{B}_1)\)

- they intersect the curve of radial solutions in another bifurcation point related to the same value of \(k\),

- they meet the boundary of \((a, b) \times C_0^1(\bar{B}_1)\).

In the previous theorem a crucial role is played by the curves

\[
\gamma_k = \left\{ (\lambda, \alpha) \in (a, b) \times (0, +\infty) \text{ such that } \nu_1(\lambda) + \frac{4k^2}{(2 + \alpha)^2} = 0 \right\}. \quad (1.10)
\]

For any integer \(k\) greater than 1 we have that each \(\gamma_k\) is a smooth curve of \(\mathbb{R}^2\).

Theorem 1.3 says that there exists a set \(\mathcal{U}_k\) containing \(\gamma_k\) in \((a, b) \times (0, +\infty)\) such that if \((\lambda, \alpha) \in \mathcal{U}_k\) then the problem (1.1) has at least one nonradial solution (see figure 1).

We observe that in Theorem 1.3 we can obtain a similar result by fixing \(\lambda\) in \((a, b)\) and using \(\alpha\) as a parameter. Repeating the proof we get new bifurcation branches of solutions to (1.1) related to \(\lambda\) and \(\alpha\) but we cannot claim that the new set of parameters \(\mathcal{U}_k\) is larger (see Remark 2.3).

Next we consider a special case where to apply the results of Theorem 1.3, namely \(f(\lambda, s) = \lambda e^s\). Here we will get more accurate information than those provided in Theorems 1.1-1.3.
First of all, we recall that all radial solutions to (1.3) are given by, for \( \lambda \in (0, 2) \),

\[
v_\lambda(r) = \log \left( \frac{8\delta_\lambda}{\lambda (\delta_\lambda + r^2)^2} \right) \quad \text{where} \quad \delta_\lambda = \delta^+_\lambda = \frac{4 - \lambda \pm \sqrt{16 - 8\lambda}}{\lambda}. \quad (1.11)
\]

The solution corresponding to \( \delta^+_\lambda \) is the minimal one while that corresponding to \( \delta^-_\lambda \) has Morse index 1 and they give rise to radial solutions to (1.1) (see Theorem 3.1). As before we set \( u_{\lambda, \alpha}(r) = v_\lambda(r^{2\alpha}) \) where \( v_\lambda \) is the solution in (1.11) corresponding to \( \delta^-_\lambda \) which has Morse index 1.

Our next result deals with the function \( \nu_1(\lambda) \) in (1.6), that we are able to compute explicitly.

**Proposition 1.4.** Set \( f(\lambda, s) = \lambda e^s \) and let \( v_\lambda \) be the unique radial solution of (1.3) with Morse index 1. Then

\[
\nu_1(\lambda) = \frac{\lambda - 2}{2}
\]

and the function which achieves \( \nu_1(\lambda) \) is given by

\[
\tilde{\psi}_{1, \lambda}(r) = r^\frac{\lambda - 2}{2} \frac{2(1 - r^4) + (1 - r^2)^2 \sqrt{4 - 2\lambda}}{\lambda (1 - r^2)^2 + 8r^2}. \quad (1.13)
\]

The computation of (1.12) will be done using the generalized Legendre equation which turns out to play a natural role in solving the linearized equation to (1.3) at \( v_\lambda \) (this was pointed out in [1] and [12]).

Using this result we can calculate exactly the Morse index of the radial solution \( u_{\lambda, \alpha} \) and also the values of \( \lambda \) where the bifurcation occurs applying...
Theorems 1.2-1.3 (see Theorems 3.3-3.4).
Let us state the bifurcation result for the case of the exponential nonlinearity, where the constant \((\alpha^2 + 2)^2\) is merged into the equation, i.e.

\[
\begin{cases}
-\Delta u = \mu |x|^\alpha e^u, & \text{in } B_1 \\
u > 0, & \text{in } B_1 \\
u = 0, & \text{on } \partial B_1
\end{cases}
\]

(1.14)

where \(\mu \in (0, \frac{(2+\alpha)^2}{2})\). It is worth noting that using some \(L^\infty\) estimates proved in [3] and [2] we give a more detailed description of the branches of nonradial solutions. This leads to the following

**Theorem 1.5.** Let \(\alpha > 0\) be fixed and let \(u_{\mu,\alpha}\) be the radial solution of (1.14) which is not minimal (see (3.24)). There are \(j\) values

\[
\mu_k = \frac{(2 + \alpha)^2}{2} - 2k^2 \quad \text{for } k = 1, \ldots, j,
\]

with \(j = \begin{cases} 1 + \left\lfloor \frac{\alpha}{2} \right\rfloor & \text{if } \frac{\alpha}{2} \notin \mathbb{N} \\ \frac{\alpha}{2} & \text{if } \frac{\alpha}{2} \in \mathbb{N} \end{cases}\)

(1.15)

such that there exists a branch of nonradial solutions of (1.14) bifurcating from \((\mu_k, u_{\mu_k,\alpha})\). The bifurcation is global, the branches are separated and solutions blow up at \(\mu = 0\).

Furthermore, for any \(\mu \in (0, \mu_j)\) problem (1.14) has at least \(j\) nonradial solutions.

Finally, for any \(\mu \in (\mu_{k+1}, \mu_k)\) problem (1.14) has at least \(k\) nonradial solutions.

As before let us introduce the curves

\[
\gamma_k = \left\{(\mu, \alpha) \in \left(0, \frac{(2 + \alpha)^2}{2}\right) \times (0, +\infty) \mid 2\mu = (2 + \alpha)^2 - 4k^2\right\}
\]

(1.16)

**Remark 1.6.** As a consequence of Theorem 1.5 we get, for any \(k \geq 1\) and for any fixed \(\alpha\), the existence of a branch starting from \(\gamma_k\) that reaches \(\mu = 0\). Then varying \(\alpha\) we recover the region \((\mu, \alpha) \in \left(0, \frac{(2 + \alpha)^2}{2}\right) \times (0, +\infty)\) such that \((\mu, \alpha)\) lies to the left of \(\gamma_k\). In this region there exist at least \(k\) nonradial solutions to (1.14) (see figure 2).

We end comparing Theorem 1.6 with other known existence results. To do this we need the following estimate, which is, in our opinion, interesting in itself.

**Proposition 1.7.** Any smooth solution \(u\) of (1.14) must satisfy

\[
2\pi \left(2 + \alpha - \sqrt{(2 + \alpha)^2 - 2\mu}\right) \leq \mu \int_{B_1} |x|^{\alpha} e^u \leq 2\pi \left(2 + \alpha + \sqrt{(2 + \alpha)^2 - 2\mu}\right).
\]

(1.17)
The bound on $\mu \int_{B_1} |x|^{\alpha} e^u$ in (1.17) provides some interesting information on solutions to (1.14). In fact the bounds in (1.17) correspond to $\mu \int_{B_1} |x|^{\alpha} e^u$ when $u$ are the radial solutions to (1.14). The first inequality is trivial, being $u$ the minimal solution. The second one highlights that the other radial solution “maximizes” the quantity $\mu \int_{B_1} |x|^{\alpha} e^u$ (recall that there exist no solution to (1.14) which stays above any other solution).

Moreover, passing to the limit as $\mu \to 0$ in (1.17) we get

$$\limsup_{\mu \to 0} \mu \int_{B_1} |x|^{\alpha} e^u \leq 8 \pi \left(1 + \frac{\alpha}{2}\right).$$

(1.18)

In [10] the authors proved the existence of a solution $u_{\mu, \alpha}$ concentrating as $\mu \to 0$ at $j$ points with $1 < j < 1 + \left\lceil \frac{\alpha}{2} \right\rceil$. Each concentration point carries a “quantized mass” $\mu \int_{B_1} |x|^{\alpha} e^u = 8 \pi$. Then, by (1.18) we derive that $j = 1 + \left\lceil \frac{\alpha}{2} \right\rceil$ is the maximum number of peaks for which such a solution exists. So the result in [10] in the unit ball is sharp when $\alpha$ is not an even integer.

Moreover, the pair $(\mu, \alpha)$ in [10] such that there exist solutions to (1.14) identify a narrow strip close to $\mu = 0$ in the plane $(\mu, \alpha)$. Our result shows that this region can be extended till to the curve $\gamma_k$ (see figure 3). We think that the nonradial solutions in Theorem 1.5 are the same found in [10].

The paper is organized as follows: in Section 2 we consider the general problem (1.1) and we prove Proposition 1.1 and Theorems 1.2-1.3. In Section 3 we turn to the exponential case and we prove Propositions 1.4, 1.7 and Theorem 1.5.
Finally in Section 4, using the transformation $r \mapsto r^{\frac{2+\alpha}{2}}$, we retrieve some results, partly known and partly new, for the problem
\[
\begin{cases}
-\Delta u = |x|^\alpha e^u, & \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} |x|^\alpha e^u < +\infty.
\end{cases}
\]
while in the Appendix we give the details of some technical known facts.

2. The abstract existence result

This section is devoted to study the general problem (1.1). Our first result studies the nondegeneracy of solutions to (1.1).

Proof of Proposition 1.1: Let $v_\lambda$ be a solution of (1.3). From the symmetry results of Gidas, Ni, Nirenberg [11] we get that $v_\lambda$ is radial. Setting $u_{\lambda,\alpha}(r) = v_\lambda \left( r^{\frac{2+\alpha}{2}} \right)$, where $r = |x|$, a straightforward computation shows that $u_{\lambda,\alpha}$ is a radial solution of (1.1) for any $\lambda \in (a, b)$ and $\alpha > 0$.

Let us consider the linearized operator of (1.1) at the radial solution $u_{\lambda,\alpha}$, i.e. the problem (1.5). Decomposing (1.5) using the spherical harmonic functions $Y_k(\theta)$, we get that
\[
w(r, \theta) = \sum_{k=0}^{+\infty} w_k(r) Y_k(\theta)
\]
is a solution of (1.5) if and only if $w_k(r) := \int_{S^1} w(r, \theta) Y_k(\theta) \, d\theta$ is a solution of
\[
\begin{cases}
-w''_k - \frac{1}{r} w'_k + \frac{k^2}{r^2} w_k = \left( \frac{2+\alpha}{2} \right)^2 r^\alpha f'(\lambda, u_{\lambda,\alpha}) w_k, & \text{in } (0, 1) \\
w_k'(0) = 0 = w_k(1) & \text{if } k = 0, \quad \text{and } w_k(0) = 0 = w_k(1) & \text{if } k \geq 1
\end{cases}
\] (2.1)
Note that, in order to \( w \) be a smooth solution to \((1.5)\), we must have \( w'(0) = 0 \) and \( w_k(0) = 0 \) for \( k \geq 1 \). Letting \( \eta_k(r) = w_k(r^{\frac{2}{2+\alpha}}) \), we have that \( \eta_k \) solves

\[
\begin{aligned}
-\eta''_k - \frac{1}{r} \eta'_k - f'(\lambda, v_\lambda) \eta_k &= -\frac{4k^2}{(2+\alpha)^2} \eta_k, \quad \text{in } (0,1) \\
\eta'_k(0) &= 0 = \eta_k(1) \quad \text{if } k = 0, \quad \text{and } \eta_k(0) = 0 = \eta_k(1) \quad \text{if } k \geq 1.
\end{aligned}
\]  

(2.2)

In order to study this problem we consider the more general eigenvalue problem,

\[
\begin{aligned}
-\eta'' - \frac{1}{r} \eta' - f'(\lambda, v_\lambda) \eta &= \nu \eta, \quad r \in (0,1) \\
\eta \in L^\infty(0,1), \quad \eta(1) = 0.
\end{aligned}
\]  

(2.3)

It is known that (2.3) is a Sturm-Liouville eigenvalue problem and it has a sequence of eigenvalues \( \nu_1(\lambda) < \nu_2(\lambda) < \ldots \) which are all simple.

If \( v_\lambda \) is radially nondegenerate and has Morse index 1, by [16, Proposition 3.3], the radial operator \(-\eta'' - \frac{1}{r} \eta' - f'(\lambda, v_\lambda)\eta\) has one negative eigenvalue and the same is true for the eigenvalue problem with weight \( \frac{1}{\nu^2} \) in (2.3). This is a standard result, see Lemma 2.1 of [14] for a proof. So we have that the first eigenvalue \( \nu_1(\lambda) \) of (2.3), defined in (1.6), is negative and the second one \( \nu_2(\lambda) \) is positive.

Then the first eigenfunction \( \hat{\psi}_{1,\lambda} \) of (2.3) satisfies \( \hat{\psi}_{1,\lambda}(0) = 0 \) (see the Appendix) and it is a solution of (2.2) related to some \( k \geq 1 \) if and only if (1.7) is satisfied. Using the reversed map \( r \mapsto r^{\frac{2+\alpha}{2}} \) we get that \( \psi_{\lambda,\alpha}(r) = \hat{\psi}_{1,\lambda} \left( r^{\frac{2+\alpha}{2}} \right) \) is a solution to (2.1) so that (1.8) holds.

If, else \( v_\lambda \) is degenerate then, from [16], the linearized operator at \( v_\lambda \) has a unique solution \( \hat{\psi}_\lambda \) which is radial. Then, letting \( \psi_{\lambda,\alpha}(r) = \hat{\psi}_\lambda \left( r^{\frac{2+\alpha}{2}} \right) \), we get that \( \psi_{\lambda,\alpha} \) is a radial solution of the linearized operator (1.5) for any \( \alpha > 0 \).

Next we compute the Morse index of the radial solution to (1.1) which proves Theorem 1.2.

**Proof of Theorem 1.2:** As we pointed out in the previous proposition, the Morse index of the radial solution \( u_{\lambda,\alpha} \) coincides with the number of negative eigenvalues \( \Lambda \) of the problem, counted with their multiplicity,

\[
\begin{aligned}
-\Delta w - \frac{(2+\alpha)^2}{2} |x|^\alpha f'(\lambda, u_{\lambda,\alpha}) w &= \frac{\Lambda}{|x|^\alpha} w \quad \text{in } B_1 \\
w &= 0, \quad \text{on } \partial B_1.
\end{aligned}
\]  

(2.4)

Denoting by \( w_i \) the eigenfunction of (2.4) related to \( \Lambda_i < 0 \) and arguing as in the previous proposition setting \( w_{i,k}(r) := \int_{S^1} w_i(r, \theta) Y_k(\theta) d\theta \) and then \( \eta_{i,k}(r) = w_{i,k}(r^{\frac{2}{2+\alpha}}) \), we get that \( \eta_{i,k} \) satisfies

\[
\begin{aligned}
-\eta''_{i,k} - \frac{1}{r} \eta'_{i,k} - f'(\lambda, v_\lambda) \eta_{i,k} &= 4 \frac{\Lambda_i - k^2}{(2+\alpha)^2} \eta_{i,k}, \quad \text{in } (0,1) \\
\eta'_{i,k}(0) &= 0 = \eta_{i,k}(1) \quad \text{if } k = 0, \quad \text{and } \eta_{i,k}(0) = 0 = \eta_{i,k}(1) \quad \text{if } k \geq 1.
\end{aligned}
\]  

(2.5)
for some value of $k$ and $\Lambda_i < 0$. Since problem (2.3) admits only one negative eigenvalue $\nu_1(\lambda)$, arguing as before, from (2.5) we derive that
\[
\nu_1(\lambda) = 4 \frac{\Lambda_i - k^2}{(2 + \alpha)^2}.
\] (2.6)

So we have that the modes $k$ which contribute to the Morse index of the solution $u_{\lambda, \alpha}$ verify
\[
4k^2 (2 + \alpha^2) + \nu_1(\lambda) \leq 0, \quad i.e. \quad k \leq \frac{2 + \alpha}{2} \sqrt{-\nu_1(\lambda)}.
\] Finally recalling that the dimension of the eigenspace of the Laplace-Beltrami operator on $\mathbb{S}^1$ is 2 for any $k \geq 1$ then (1.9) follows.

We end this section proving the bifurcation result from the radial solution $u_{\lambda, \alpha}$.

Proof of Theorem 1.3: Let $\alpha$ be fixed and consider the operator $T(\lambda, v) : (a, b) \times C^{1, \gamma}_0(\bar{B}_1) \to C^{1, \gamma}_0(\bar{B}_1)$ defined by
\[
T(\lambda, v) := (-\Delta)^{-1} \left( \frac{(2 + \alpha)}{2} \alpha f(\lambda, v) \right).
\]
$T$ is a compact operator for every fixed $\lambda$ and it is continuous with respect to $\lambda$. Let us define $S(\lambda, v) : (a, b) \times C^{1, \gamma}_0(\bar{B}_1) \to C^{1, \gamma}_0(\bar{B}_1)$ as
\[
S(\lambda, v) := v - T(\lambda, v).
\] A function $v \in C^{1, \gamma}_0(\bar{B}_1)$ is a solution of (1.1) related to $\lambda$ if and only if $(\lambda, v)$ is in the kernel of $S$ and $v > 0$ in $B_1$.

Using assumption ii) and (1.7) we can find $\varepsilon > 0$ such that the interval $(\lambda(k) - \varepsilon, \lambda(k) + \varepsilon)$ does not contain degeneracy points of (1.1) other than $\lambda(k)$. Moreover the Morse index of the radial solution $u_{\lambda, \alpha}$ changes at $\lambda(k)$ because $\nu_1(\lambda) + \frac{4k^2}{(2 + \alpha)^2}$ changes sign at $\lambda(k)$. Since the eigenspace of the Laplace Beltrami operator on $\mathbb{S}^1$ related to $k$ is spanned by $\{\cos k \theta, \sin k \theta\}$, we have that, for $k \geq 1$, the change in the Morse index is exactly 2. However, if we restrict to the space
\[
X = \left\{ v \in C^{1, \gamma}_0(\bar{B}_1) \text{ such that } v = v(r, \theta) \text{ and } v \text{ is even in } \theta \right\}
\] (2.7)
then only the spherical harmonic $\cos k \theta$ contributes to the Morse index and the change in the Morse index of $u_{\lambda, \alpha}$ in $X$ at the point $\lambda(k)$ is exactly one.

To prove the bifurcation result we use the cones
\[
C^k := \left\{ v \in X, \text{ such that } v \geq 0 \text{ in } B_1, \right. \left. v(r, \theta) = v \left( r, \theta + \frac{2\pi}{k} \right), \text{ for } r \in [0, 1], \right. \right. \nonumber \\
\left. \theta \in [0, 2\pi], v(r, \theta) \text{ non increasing in } \theta \text{ for } 0 \leq \theta \leq \frac{\pi}{k}, 0 \leq r \leq 1 \right\}
\] introduced by Dancer in [8].

The operator $S(\lambda, v)$ then maps $(a, b) \times C^k \to C^k$ (see [8, Lemma 1]) and the compactness of $T$ allows us to compute the Leray-Schauder degree of $S$ in a suitable neighborhood of the radial solution $(\lambda(k), u_{\lambda(k), \alpha})$. The
odd change in the Morse index of $u_{\lambda(k),\alpha}$ in $\lambda(k)$ causes a change in the Leray-Schauder degree along the curve of radial solutions $(\lambda, u_{\lambda,\alpha})$ going from $(\lambda(k) - \epsilon, u_{\lambda(k) - \epsilon,\alpha})$ to $(\lambda(k) + \epsilon, u_{\lambda(k) + \epsilon,\alpha})$ so that the bifurcation occurs. Finally, since it is not difficult to see that $C^k \cap \mathcal{C}^h$ contains only radial solutions ([7]), then branches of nonradial solutions related to different values of $k$ are separated. The Rabinowitz alternative theorem holds (see [7, Theorem 1]) and the bifurcation is indeed global and this proves the final part of the Theorem. □

Remark 2.1. In some particular cases it is possible to improve the statement of Theorem 1.3. For example, if $f(s) = \lambda e^s$ it will be showed in the next section that the function $F_k(\lambda)$ in (1.3) is strictly increasing in $\lambda$ for any $k$. This allow us to compute exactly the number of nonradial bifurcation points.

Remark 2.2. Hypothesis ii) in Theorem 1.3 is satisfied if the first eigenvalue $\nu_1(\lambda)$ is analytic in $\lambda$. This is the case, for example, if $f(\lambda, s)$ is analytic in $\lambda$, see [15]. Anyway, the analyticity of $f(\lambda, s)$ is not a necessary condition for ii) to hold, and in some cases a strict monotonicity property of $\nu_1(\lambda)$ can be proved directly (see Proposition 1.4).

Remark 2.3. As we did in Theorem 1.3 we can state a bifurcation result with respect to the parameter $\alpha$, getting the following result:

Let $\lambda \in (a, b)$ be fixed; then if

$$\alpha = \alpha_k = \frac{2k}{\sqrt{-\nu_1(\lambda)}} - 2 > 0 \text{ with } k \in \mathbb{N} \quad (2.8)$$

there exists a branch of solutions bifurcating from $(\alpha_k, u_{\lambda,\alpha_k})$ in $(0, +\infty) \times C^1_0(\bar{B}_1)$. Moreover, branches of bifurcating solutions related to different values of $k$ are separated and either they are unbounded in the space $(0, +\infty) \times C^1_0(\bar{B}_1)$ or they meet $\{0\} \times C^1_0(\bar{B}_1)$. These branches of solutions are obviously different from those obtained in Theorem 1.3, but do not allow to derive if there are other pairs $(\lambda, \alpha)$ other than those previously found. However, we think that the set $V_k$ obtained in this way coincides with the set $U_k$ of the Theorem 1.3.

3. The case of the exponential nonlinearity

In this section we apply the results of Section 2 to the exponential nonlinearity $f(\lambda, s) = \lambda e^s$, i.e. to the problem

$$\begin{cases}
-\Delta u = \left(\frac{2 + \alpha}{2}\right)^2 \lambda |x|^\alpha e^u, & \text{in } B_1 \\
u > 0, & \text{in } B_1 \\
u = 0, & \text{on } \partial B_1
\end{cases} \quad (3.1)$$

for $\lambda > 0$. Let us start by considering radial solutions to the problem (3.1). In this case there exists a maximal value of $\lambda$ that separates the threshold between existence and nonexistence of solutions of (3.1).
Theorem 3.1. Let us consider the problem (3.1). We have that, 

i) if \( \lambda \in (0, 2) \) there exist exactly two radial solutions \( u_+ \) and \( u_- \) given by 
\[
    u_{\pm}(x) = \log \frac{8\delta_{\lambda}^\pm}{\lambda(\delta_{\lambda}^\pm + |x|^{2+\alpha})^2} \quad (3.2)
\]
with  
\[
    \delta_{\lambda}^\pm = \frac{4 - \lambda \pm 2\sqrt{4 - 2\lambda}}{\lambda}. \quad (3.3)
\]
The solution \( u_+ \) is the minimal one and \( u_- \) blows up at the origin as \( \lambda \to 0^+ \).

ii) If \( \lambda = 2 \) there is only the solution 
\[
    u(x) = \log \frac{4}{(1 + |x|^{2+\alpha})^2}. \quad (3.4)
\]

iii) There is no solution if \( \lambda > 2 \).

Proof. We set \( v(r) = u(r^{\frac{2}{1+\alpha}}) \), where \( r = |x| \). In this way, we are led to the problem
\[
\begin{cases}
    -v'' - \frac{1}{r}v' = \lambda e^v, & \text{in } 0 < r < 1 \\
    v > 0, & \text{in } 0 < r < 1 \\
    v'(0) = 0 = v(1). \quad (3.5)
\end{cases}
\]
It is well known that the above problem admits solutions only if \( 0 < \lambda \leq 2 \), and all solutions are given by 
\[
    v(r) = \log \left( \frac{8\delta_{\lambda}}{(\delta_{\lambda} + r^{2})^2} \right) \quad \text{where} \quad \delta_{\lambda} = \delta_{\lambda}^\pm = \frac{4 - \lambda \pm \sqrt{16 - 8\lambda}}{\lambda}.
\]
The solution with 
\[
    \delta_{\lambda}^+ = \frac{4 - \lambda + \sqrt{16 - 8\lambda}}{\lambda}
\]
is the minimal solution which goes to zero uniformly as \( \lambda \to 0^+ \), while the solution with 
\[
    \delta_{\lambda}^- = \frac{4 - \lambda - \sqrt{16 - 8\lambda}}{\lambda}
\]
blows up at the origin as \( \lambda \to 0^+ \). Turning back to (3.1) by inverting the transformation \( v(r) = u(r^{\frac{2}{1+\alpha}}) \) we get i) and (3.4). Finally, reasoning exactly as in the paper of Mignot and Puel [17, Theorem 1], we can prove that problem (3.1) has a unique solution for \( \lambda = 2 \) and no solutions for \( \lambda > 2 \) concluding the proof. \( \square \)

Proposition 3.2. Any smooth solution \( u \) of (3.1) must satisfy
\[
    \lambda \int_{B_1} |x|^\alpha e^{u_+} \, dx \leq \lambda \int_{B_1} |x|^\alpha e^{u} \, dx \leq \lambda \int_{B_1} |x|^\alpha e^{u_-} \, dx \quad (3.6)
\]
Proof. Note that the first inequality is trivial, being \( u_+ \), the minimal solution. In order to prove the other inequality we use the well known Pohozaev identity. We have

\[
\frac{(2 + \alpha)^3}{4} \lambda \int_{B_1} |x|^\alpha e^{u} \, dx - \frac{(2 + \alpha)^2}{2} \pi \lambda = \frac{1}{2} \int_{\partial B_1} \left( \frac{\partial u}{\partial \nu} \right)^2 \, ds. \tag{3.7}
\]

Using the Schwartz inequality we get

\[
\left( \int_{\partial B_1} \frac{\partial u}{\partial \nu} \, ds \right)^2 \leq 2\pi \int_{\partial B_1} \left( \frac{\partial u}{\partial \nu} \right)^2 \, ds
\]

which turns in an equality if and only if \( u \) is radial, so that \( \frac{\partial u}{\partial \nu} \) is constant on \( \partial B_1 \). Now we integrate equation (3.1) in \( B_1 \), getting

\[
1 \geq \frac{1}{4\pi} \left( \frac{(2 + \alpha)^2}{4} \lambda \int_{B_1} |x|^\alpha e^{u} \, dx \right)^2.
\]

Inserting (3.8) into (3.7) then we obtain that the following inequality holds

\[
4\pi (2 + \alpha) \lambda \int_{B_1} |x|^\alpha e^{u} \, dx \geq 8\pi^2 \lambda \geq \frac{(2 + \alpha)^2}{4} \left( \lambda \int_{B_1} |x|^\alpha e^{u} \, dx \right)^2. \tag{3.9}
\]

Observing that the inequality (3.9) becomes an equality if \( u \) coincides with the radial solutions \( u_\pm \) of the previous theorem, by direct computation we get that

\[
\lambda \int_{B_1} |x|^\alpha e^{u_+} \, dx \leq \lambda \int_{B_1} |x|^\alpha e^{u} \, dx \leq \lambda \int_{B_1} |x|^\alpha e^{u_-} \, dx,
\]

and the claim follows.

In the following we refer to \( u_{\lambda,\alpha} \) as the solution which blows up as \( \lambda \to 0^+ \), i.e.

\[
u_{\lambda}(r) = \log \left( \frac{8\delta_\lambda}{\lambda(\delta_\lambda + r^{2+\alpha})^2} \right) \quad \text{where} \quad \delta_\lambda = \frac{4 - \lambda - \sqrt{16 - 8\lambda}}{\lambda}.
\]

Using the map \( r \mapsto r^{2+\alpha} \) we have that (3.10) becomes

\[
v_{\lambda}(r) = \log \frac{8\delta_\lambda}{\lambda(\delta_\lambda + r^2)^2}
\]

which is a nondegenerate solution of

\[
\begin{align*}
-\Delta v &= \lambda e^{v}, & & \text{in } B_1(0) \\
v &= 0 & & \text{on } \partial B_1(0)
\end{align*}
\]
with Morse index 1 for $0 < \lambda < 2$. So we can define the quantity $\nu_1(\lambda)$ as in (1.6). Since in this case we know explicitly the solution $v_\lambda$ we have,

$$\nu_1(\lambda) := \inf_{\eta \in H^1((0,1), r \, dr), \eta \neq 0, \eta(1) = 0} \int_0^1 r(\eta')^2 \, dr - \int_0^1 \frac{8\lambda}{(\delta_\lambda + r^2)^2} r \eta^2 \, dr. \quad (3.13)$$

We are now in position to prove Proposition 1.4,

Proof of Proposition 1.4: Let $\tilde{\psi}_{1,\lambda}$ be the first eigenfunction corresponding to the first eigenvalue $\nu_1(\lambda)$. It solves

$$\begin{cases}
-\tilde{\psi}_{1,\lambda} - \frac{1}{r} \tilde{\psi}_{1,\lambda}' - \frac{8\lambda}{(\delta_\lambda + r^2)^2} \tilde{\psi}_{1,\lambda} = \frac{\nu_1(\lambda)}{r^2} \tilde{\psi}_{1,\lambda} & \text{for } r \in (0, 1) \\
\tilde{\psi}_{1,\lambda}(1) = 0.
\end{cases} \quad (3.14)$$

We let $\xi := \frac{\delta_\lambda - r^2}{\delta_\lambda + r^2}$ and $R(\xi) := \tilde{\psi}_{1,\lambda}(r)$. Then $R(\xi)$ solves

$$\begin{cases}
(1 - \xi^2)R'' - 2\xi R' + \frac{\nu_1(\lambda)}{1 - \xi^2} R + 2R = 0 & \text{for } \xi \in \left(\frac{\delta_\lambda - 1}{\delta_\lambda + 1}, 1\right) \\
R\left(\frac{\delta_\lambda - 1}{\delta_\lambda + 1}\right) = 0.
\end{cases} \quad (3.15)$$

Equation (3.15) is the classical Legendre equation and it has

$$R(\xi) = \left(\frac{1 + \xi}{1 - \xi}\right)^{\frac{3}{2}} (\xi - \gamma)$$

with $\gamma^2 = -\nu_1(\lambda)$ as a solution. Moreover, for $\gamma = \frac{\delta_\lambda - 1}{\delta_\lambda + 1} < 0$, $R(\xi)$ is strictly positive in $\left(\frac{\delta_\lambda - 1}{\delta_\lambda + 1}, 1\right)$ and satisfies the boundary condition. This means that $\nu_1(\lambda) = -\left(\frac{\delta_\lambda - 1}{\delta_\lambda + 1}\right)^2$ is the first eigenvalue of (3.14). Using the value of $\delta_\lambda$ in (3.10) we have that (1.12) follows straightforward. Inverting the transformation we get that

$$\tilde{\psi}_{1,\lambda}(r) = r^{-\gamma} \left(\frac{\delta_\lambda - r^2}{\delta_\lambda + r^2} - \gamma\right).$$

and since $\gamma = \frac{\delta_\lambda - 1}{\delta_\lambda + 1}$ then (1.13) follows by straightforward computations. $\square$

Once we know the explicit value of $\nu_1(\lambda)$, not only we can apply the results of Section 2, but we have even more accurate results.

Theorem 3.3. Let $v_\lambda$ be the unique radial solution of (1.3) with Morse index 1 and $u_{\lambda,\alpha}(r) = v_\lambda(r^{\frac{2 + \alpha}{2}})$. Then $u_{\lambda,\alpha}$ is degenerate if and only if $\lambda$ and $\alpha$ satisfy

$$\frac{2 - \lambda}{2} = \frac{4k^2}{(2 + \alpha)^2} \quad (3.16)$$
for some integer \( k \geq 1 \). The solutions of the linearized equation at the values of \( \alpha \) and \( \lambda \) that satisfy (3.16) are given by, in polar coordinates,

\[
\psi_k(r, \theta) = r^k \frac{2(2 + \alpha) (1 - r^2(2+\alpha)) + 4k(1 - r^2(2+\alpha))^2}{(2 + \alpha)^2 - 8k^2(1 - r^2(2+\alpha))^2 + 8(2 + \alpha)^2 r^2(2+\alpha)} (A \sin k\theta + B \cos k\theta)
\]

(3.17)

for any constants \( A, B \in \mathbb{R} \). Finally the Morse index of \( u_{\lambda, \alpha} \) is equal to

\[
m(\lambda, \alpha) = \begin{cases} 
1 + 2 \left[ \frac{\alpha + 2}{2} \sqrt{\frac{2-\lambda}{2}} \right] & \text{if } \frac{\alpha + 2}{2} \sqrt{\frac{2-\lambda}{2}} \notin \mathbb{N} \\
(\alpha + 2) \sqrt{\frac{2-\lambda}{2}} - 1 & \text{if } \frac{\alpha + 2}{2} \sqrt{\frac{2-\lambda}{2}} \in \mathbb{N}
\end{cases}
\]

(3.18)

and \( m(\lambda, \alpha) \to +\infty \) as \( \alpha \to +\infty \).

\begin{proof}
By (1.7) of Proposition 1.1 and (1.12) we have that \( u_{\lambda, \alpha} \) is degenerate if and only if (3.16) holds. Moreover, as said in Proposition 1.1, the solutions of the linearized equation (1.5) at the degeneracy points (3.16), are given by \( \psi_k(r) = \tilde{\psi}_{1, \lambda}(r^{\frac{2+\alpha}{2}}) \) multiplied by the \( k \)-th spherical harmonic, so that (3.17) follows.

Finally, inserting (1.12) in (1.9) of Theorem 1.2 we get (3.18).
\end{proof}

Our next step is to apply Theorem 1.3 to (3.1) getting the bifurcation result.

**Theorem 3.4.** Let \( \alpha > 0 \) be fixed and let \( u_{\lambda, \alpha} \) be as defined above. There exist \( j \) values

\[
\lambda_k = 2 - \frac{8k^2}{(\alpha + 2)^2} \quad \text{for } k = 1, \ldots, j,
\]

with \( j = \left\lfloor \alpha \right\rfloor \) if \( \frac{\alpha}{2} \notin \mathbb{N} \) and \( j = \left\lfloor \frac{\alpha}{2} \right\rfloor \) if \( \frac{\alpha}{2} \in \mathbb{N} \).

(3.19)

such that \((\lambda_k, u_{\lambda_k, \alpha})\) is a nonradial bifurcation point for the curve of radial solutions \( u_{\lambda, \alpha} \) of (3.1). The bifurcation is global, and the branches are separated and unbounded in \((0, 2) \times C_0^{1, \gamma}(\bar{B}_1)\).

\begin{proof}
All the assumptions of Theorem 1.3 are verified and since the equation (1.7) can be explicitly solved we get the values \( \lambda_k \) given in (3.19). Observe that at each of the degeneracy values \( \lambda_k \), the Morse index of the radial solution \( u_{\lambda, \alpha} \) changes. Then Theorem 1.3 implies that the bifurcation occurs at \((\lambda_k, u_{\lambda_k, \alpha})\). Moreover, setting \( C(\lambda_k) \) as the branch of nonradial solutions bifurcating from \( \lambda_k \), we have that it satisfies the alternative in Theorem 1.3. Observe that from (3.16) we have a unique bifurcation point corresponding to each value of \( k \geq 1 \). This implies that each branch of bifurcating solutions does not intersect the curve of radial solutions again.

Let us show that the branches are unbounded in \( C_0^{1, \gamma}(\bar{B}_1) \). From what we said, we only need to show that the branch \( C(\lambda_k) \) can not stay bounded and intersect \\{0\} \times \( C_0^{1, \gamma}(\bar{B}_1) \) or \\{2\} \times \( C_0^{1, \gamma}(\bar{B}_1) \). The case where \( C(\lambda_k) \) is bounded and meets \\{0\} \times \( C_0^{1, \gamma}(\bar{B}_1) \) cannot happen since problem (3.1) has,
at $\lambda = 0$ only the trivial solution, which is nondegenerate and isolated. On the other hand, the case where $C(\lambda_k)$ is bounded and meets $\{2\} \times C^{1,\gamma}_0(B_1)$ cannot happen since problem (3.1) has, when $\lambda = 2$, a unique solution which is radially degenerate. Then each branch $C(\lambda_k)$ for $k = 1, \ldots, j$ is unbounded.

Using the result of the previous theorem we can prove a multiplicity result for problem (3.1) (see Corollary 3.6). To do this we need an $L^\infty$-estimate for solutions of (3.1) when $\lambda$ is bounded and bounded away from zero. This estimate can be proved collecting the results of [3] and [2] and reads as follows,

**Theorem 3.5.** Let $u$ be a solution of (3.1) with $\alpha$ fixed, such that $0 < c_1 \leq \lambda \leq c_2$ for some positive constants $c_1, c_2$. Then there exists $C = C(c_1, c_2) > 0$ such that

$$
\|u\|_{L^\infty(B_1)} \leq C.
$$

**Proof.** Using the assumption on $\lambda$ and estimate (1.17) we can apply Theorem 1.3 of [2] with $V(x) = (2 + \alpha)^2 \lambda > 0$ getting that $\sup_K u(x) \leq C$ for any compact set $K \subset B_1$. The $L^\infty$-estimate near the boundary of $B_1$ follows exactly as in Theorem 1 of [3] without assuming their assumption (i). Indeed, that hypothesis can be replaced by the integral estimate (1.17). This provides the boundary estimate and ends the proof. \qed

Using Theorem 3.5 then we have

**Corollary 3.6.** Let $\alpha > 0$ be fixed and let $\lambda_k$ and $j$ be as defined in (3.19). For any $\lambda \in (0, \lambda_j)$ problem (3.1) has at least $j$ nonradial solutions. Moreover, for any $\lambda \in (\lambda_{k+1}, \lambda_k)$ problem (3.1) has at least $k$ nonradial solutions.
Proof. From the Theorem 3.4 we know that there are $j$ branches that bifurcate from the radial solutions that are unbounded in $C_{0}^{1,\gamma}(B_1)$. By standard regularity theory they are also unbounded in $L^\infty(B_1)$. By Theorem 3.5 the solutions of (3.1) can blow up only as $\lambda \to 0^+$. This implies that the branch bifurcating from the value $\lambda_k$ exists in the interval $(0,\lambda_k)$. □

Now we give the results for solutions of problem (1.14). Let us set

$$\mu = \lambda \left(\frac{2 + \alpha}{2}\right)^2$$

for $\mu > 0$. Of course all the previous results follow substituting $\mu = \lambda \left(\frac{2 + \alpha}{2}\right)^2$. First, Theorem 3.1 becomes

**Theorem 3.7.** Let us consider the problem (1.14). We have that,

i) if $\mu \in \left(0, \left(\frac{2 + \alpha}{2}\right)^2\right)$ there exist exactly two radial solutions $u_+$ and $u_-$ given by

$$u_{\pm}(x) = \log \left(\frac{2\delta_{\mu}^+(2 + \alpha)^2}{\mu(\delta_{\mu}^+ + |x|^{2+\alpha})^2}\right)$$

with

$$\delta_{\mu}^\pm = \frac{(2 + \alpha)^2 - \mu \pm (2 + \alpha)\sqrt{(2 + \alpha)^2 - 2\mu}}{\mu}.$$  \hspace{1cm} (3.22)

The solution $u_+$ is the minimal one and $u_-$ blows up as $\mu \to 0^+$.

ii) If $\mu = \frac{(2 + \alpha)^2}{2}$ there is only the solution

$$u(x) = \log \left(\frac{4}{(1 + |x|^{2+\alpha})^2}\right).$$  \hspace{1cm} (3.23)

iii) There is no solution if $\mu > \frac{(2 + \alpha)^2}{2}$.

Here we refer to $u_{\mu,\alpha}$ as the radial solution to (1.14) which blows up as $\mu \to 0^+$, i.e.

$$u_{\mu,\alpha}(r) = \log \left(\frac{-2\delta_{\mu}(2 + \alpha)^2}{\mu(\delta_{\mu} + |r|^{2+\alpha})^2}\right)$$

where $\delta_{\mu} = \frac{(2 + \alpha)^2 - \mu - (2 + \alpha)\sqrt{(2 + \alpha)^2 - 2\mu}}{\mu}$ and $\mu \in \left(0, \left(\frac{2 + \alpha}{2}\right)^2\right)$. From Theorem 3.3 then we get

**Theorem 3.8.** Let $u_{\mu,\alpha}$ be the radial solution of (1.14) defined in (3.24). Then $u_{\mu,\alpha}$ is degenerate if and only if

$$(2 + \alpha)^2 = 4k^2 + 2\mu$$

for some $\alpha > 0$, $\mu \in \left(0, \left(\frac{2 + \alpha}{2}\right)^2\right)$ and some integer $k \geq 1$.

Moreover its Morse index is given by:

$$m(\mu, \alpha) = \begin{cases} 
1 + 2 \left[\frac{1}{2}\sqrt{(2 + \alpha)^2 - 2\mu} - 1\right] & \text{if } \frac{1}{2}\sqrt{(2 + \alpha)^2 - 2\mu} \notin \mathbb{N} \\
\sqrt{(2 + \alpha)^2 - 2\mu} - 1 & \text{if } \frac{1}{2}\sqrt{(2 + \alpha)^2 - 2\mu} \in \mathbb{N}
\end{cases}$$

(3.26)
and \( m(\mu, \alpha) \to +\infty \) as \( \alpha \to +\infty \).

The proof is an easy consequence of Theorem 3.3. Finally we only have to prove Theorems 1.5 and Proposition 1.7.

**Proof of Theorem 1.5.** It follows from Theorem 3.4 and Corollary 3.6 □

**Proof of Theorem 1.7.** It follows directly from Proposition 3.2. □

### 4. Some results in \( \mathbb{R}^2 \)

In this section, using the transformation \( r \mapsto r^{2+\alpha} \), we retrieve some results, partly known and partly new, for the problem

\[
\begin{aligned}
-\Delta u &= |x|^\alpha e^u, \quad \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} |x|^\alpha e^u &< +\infty.
\end{aligned}
\] (4.1)

All radial solutions to (4.1) are given by

\[
U_{\delta, \alpha}(x) = \log \frac{2(2+\alpha)}{\delta + |x|^{2+\alpha}}.
\] (4.2)

We want to study the linearized problem to (4.1) at \( U_\alpha = U_{1, \alpha} \), i.e.

\[
\begin{aligned}
-\Delta v &= 2(\alpha + 2)^2 \frac{|x|^n}{(1 + |x|^{2+\alpha})^2} v \quad \text{in } \mathbb{R}^2 \\
v &\in L^\infty(\mathbb{R}^2).
\end{aligned}
\] (4.3)

Next theorem characterizes all solutions to (4.3). This result was already proved in [4] for \( \frac{\alpha}{2} \notin \mathbb{N} \) and in [9] if \( \alpha \in 2\mathbb{N} \). Our proof unifies the two cases and it is (according to us) shorter.

**Theorem 4.1.** The following alternative holds:

i) If \( \alpha \notin 2\mathbb{N} \) the space of solutions to (4.3) has dimension 1 and is spanned by

\[
v(x) = \frac{1 - |x|^{2+\alpha}}{1 + |x|^{2+\alpha}}.
\] (4.4)

ii) If \( \alpha = 2(k-1) \) for some integer \( k \geq 1 \) the space of solutions to (4.3) has dimension 3 and is spanned by, in polar coordinates,

\[
v(x) = \frac{1 - r^{2+\alpha}}{1 + r^{2+\alpha}}, \quad v_1(x) = \frac{r^k \cos(k\theta)}{1 + r^{2+\alpha}} \quad \text{and} \quad v_2(x) = \frac{r^k \sin(k\theta)}{1 + r^{2+\alpha}}.
\] (4.5)

**Proof.** We decompose a solution of (4.3) using the spherical harmonic functions, we get that \( v \) is a solution of (4.3) if and only if \( v_k(r) := \int_{S^1} v(r, \theta) Y_k(\theta) d\theta \) is a solution of

\[
\begin{aligned}
-\frac{1}{r} v''_k + \frac{k^2}{r^2} v_k &= 2(2+\alpha)^2 \frac{r^n}{(1 + r^{2+\alpha})^2} v_k, \quad \text{in } (0, +\infty) \\
v'_k(0) &= 0 \quad \text{if } k = 0, \quad v_k(0) = 0 \quad \text{if } k \geq 1 \quad \text{and} \quad v_k \in L^\infty(0, +\infty).
\end{aligned}
\] (4.6)
where \( Y_k(\theta) \) denotes a \( k \)-th spherical harmonic function. Letting \( \eta_k(r) = v_k(r^{\frac{2}{2+\alpha}}) \), we have that \( \eta_k \) solves
\[
\begin{aligned}
-\eta''_k - \frac{1}{r}\eta'_k + \frac{4k^2}{(2+\alpha)r^2}\eta_k &= \frac{8}{(1+r^2)^2}\eta_k, & \text{in } (0, +\infty) \\
\eta'_k(0) &= 0 \text{ if } k = 0, \eta_k(0) = 0 \text{ if } k \geq 1 \text{ and } \eta_k \in L^\infty(0, +\infty). 
\end{aligned}
\tag{4.7}
\]
Instead of considering this problem we consider the more general eigenvalue problem,
\[
\begin{aligned}
-\eta'' - \frac{1}{r}\eta' + \frac{\beta^2}{r^2}\eta &= \frac{8}{(1+r^2)^2}\eta, & \text{in } (0, +\infty) \\
\eta \in L^\infty(0, +\infty). 
\end{aligned}
\tag{4.8}
\]
For \( \beta \neq 0 \) the solutions must satisfy \( \eta(r) = O(r^\beta) \) for \( r \to 0 \) and \( \eta(r) = O(r^{-\beta}) \) for \( r \to \infty \), because we are looking for bounded solutions (see the Appendix). We also know the following explicit solutions
\[
\beta = 1, \ \eta_1(r) = \frac{r}{1 + \frac{r^2}{2+\alpha}} \text{ and } \beta = 0, \ \eta_0(r) = \frac{1 - r^2}{1 + r^2}. \tag{4.9}
\]
It follows from the Sturm comparison theorem that there are no other solutions to (4.8) besides those. Therefore, (4.7) admits a solution if, and only if, \( \frac{4k^2}{(2+\alpha)r^2} \in \{0, 1\} \), which means that we must have \( k = 0 \) or \( \alpha = 2(k-1) \).

Turning back to (4.6) we have the solutions
\[
\begin{aligned}
v_0(r) &= \frac{1 - r^2 + \alpha}{1 + r^2 + \alpha}, & \text{if } \alpha \neq 2(k-1) \ \forall \ k \in \mathbb{N} \\
v_0(r) &= \frac{1 - r^{2+\alpha}}{1 + r^{2+\alpha}}, v_k(r) = \frac{r^k}{1 + r^{2+\alpha}}, & \text{if } \alpha = 2(k-1) \ \text{for some } k \in \mathbb{N}
\end{aligned}
\]
and the proof is now complete.

Next corollary computes the Morse Index of the solution \( U_\alpha \), extending to the case \( \alpha \in \mathbb{N} \) the result in [4].

**Corollary 4.2.** Let \( U_\alpha \) be the solution of (4.1). Then its Morse index is equal to
\[
m(\alpha) = \begin{cases} 1 + 2 \left[ \frac{\alpha+2}{2} \right] & \text{if } \frac{\alpha+2}{2} \not\in \mathbb{N} \\ 1 + \alpha & \text{if } \frac{\alpha+2}{2} \in \mathbb{N} \end{cases} \tag{4.10}
\]
where \( [x] \) denotes the greatest integer less than or equal to \( x \). In particular we have that the Morse index of \( U_\alpha \) changes as \( \alpha \) crosses the set of even integers and also that \( m(\alpha) \to \infty \) as \( \alpha \to \infty \).

**Proof.** Let us now consider the following eigenvalue problem,
\[
\begin{aligned}
-\Delta V - 2(2 + \alpha)^2 \frac{|x|^\alpha}{(1 + |x|^{2+\alpha})^2} V &= \Lambda \frac{1}{|x|^2} V & \text{in } \mathbb{R}^2 \\
V &\in L^\infty(\mathbb{R}^2).
\end{aligned}
\tag{4.11}
\]
As in the proof of Theorem 1.2, the Morse index of \( U_\alpha \) coincides with the number of negative eigenvalues \( \Lambda \) of (4.11), counted with their multiplicity.
Let $\Lambda < 0$ be an eigenvalue of (4.11), using the spherical harmonics we are led to the equation

$$\begin{cases}
-\psi''_k(r) - \frac{1}{r} \psi'_k(r) + \frac{k^2}{r^2} \psi_k(r) - 2(2 + \alpha)^2 \frac{r^\alpha}{(1+r^2)^\alpha} \psi_k(r) = \frac{\Lambda}{r^2} \psi_k(r), & \text{in } (0, \infty) \\
\psi'_k(0) = 0 & \text{if } k = 0, \\
\psi_k(0) = 0 & \text{if } k \geq 1,
\end{cases}$$

(4.12)

and using the transformation $r \mapsto r^\frac{2}{2+\alpha}$, as in the previous theorem, we get

$$\begin{cases}
-\eta''_k(r) - \frac{1}{r} \eta'_k(r) - 8 \frac{\eta_k(r)}{(1+r^2)^2} = 4 \frac{\Lambda - k^2}{(2+\alpha)^2} \eta_k(r), & \text{in } (0, \infty) \\
\eta'_k(0) = 0 & \text{if } k = 0, \\
\eta_k(0) = 0 & \text{if } k \geq 1,
\end{cases}$$

(4.13)

Now, by (4.8) and (4.9), we must have

$$4 \Lambda - \frac{k^2}{(2+\alpha)^2} = -1$$

and since $\Lambda < 0$ necessarily $k < \frac{2+\alpha}{2}$. Conversely, for each $k < \frac{2+\alpha}{2}$ we have $\Lambda = k^2 - \left(\frac{2+\alpha}{2}\right)^2 < 0$ an eigenvalue of (4.11). Since the dimension of the eigenspace of the Laplace-Beltrami operator on $S^1$ is 2 for any $k \geq 1$, the proof is now complete. \qed

Using the transformation $r \mapsto r^\frac{2}{2+\alpha}$ we can also prove an inequality related to the classical Moser-Trudinger inequality, i.e.

$$\sup \left\{ \int_\Omega e^{4\pi u^2} dx : \int_\Omega |\nabla u|^2 dx = 1 \right\} = C(\Omega)$$

We want to prove the following refinement for radial functions in $B_1$.

**Proposition 4.3.** Let $\alpha > 0$ then

$$\sup \left\{ \int_{B_1} |x|^\alpha e^{4\pi(\frac{\alpha+2}{2})u^2} dx : u(x) = u(|x|), \int_{B_1} |\nabla u|^2 dx = 1 \right\} = C(B_1) \frac{2}{2+\alpha}$$

**Proof.** Let $u$ be a radial function such that $\int_\Omega |\nabla u|^2 dx = 1$, then

$$\int_{B_1} |x|^\alpha e^{4\pi(\frac{\alpha+2}{2})u^2(x)} dx = 2\pi \int_0^1 \rho^{\alpha+1} e^{4\pi(\frac{\alpha+2}{2})u^2(\rho)} d\rho$$

put $v(\rho) = \sqrt{\frac{\alpha+2}{2}} u(\rho^{\frac{2}{2+\alpha}})$, hence

$$2\pi \int_0^1 |v'(\rho)|^2 d\rho = 2\pi \int_0^1 |u'(s)| ds = 1.$$
Therefore
\[ 2\pi \int_0^1 \rho^{\alpha+1} e^{4\pi(\frac{\alpha+2}{2})} u^2(\rho) d\rho = 2\pi \frac{2}{2+\alpha} \int_0^1 e^{4\pi v^2(s)} s ds \leq C(B_1) \frac{2}{2+\alpha}. \]

\[ \square \]

**Appendix A.**

In this appendix we show the following result. Let us consider the problem
\[ \begin{cases} -\eta'' - \frac{1}{r} \eta' + \frac{\beta^2}{r^2} \eta = \frac{8}{(1+s^2)^2} \eta, & \text{in } (0, +\infty) \\ \eta \in L^\infty(0, +\infty). \end{cases} \tag{A.1} \]

We will prove that any solution to (A.1) with \( \beta > 0 \) necessarily satisfies \( \eta(0) = 0 \) and \( \eta(+\infty) = 0 \).

Let us observe that the function \( v(r) = r^\beta \), with \( \beta > 0 \) satisfies
\[ \begin{cases} -v'' - \frac{1}{r} v' + \frac{\beta^2}{r^2} v = 0, & \text{in } (0, +\infty) \\ v(0) = 0. \end{cases} \tag{A.2} \]

Multiplying (A.1) by \( v \), (A.2) by \( \eta \) and integrating on \((r, R)\) we get
\[ \int_r^R \frac{8s^{\beta+1}}{(1+s^2)^2} \eta = -R^{\beta+1}\eta'(R) + r^{\beta+1}\eta'(r) + \beta R^\beta \eta(R) - \beta r^\beta \eta(r). \tag{A.3} \]

Since \( \eta \) is bounded we get that \( r^\beta \eta(r) = o(1) \) as \( r \to 0 \). We claim that
\[ r^{\beta+1} \eta'(r) = o(1). \tag{A.4} \]

To prove (A.4) we integrate (A.1) from \( r \) to 1. We get
\[ -\eta'(1) + r \eta'(r) = \int_r^1 \frac{8s}{(1+s^2)^2} \eta(s) - \int_r^1 \frac{\beta^2}{s} \eta(s) \]
which implies
\[ r^{\beta+1} \eta'(r) = O\left( r^\beta \right) + r^\beta \int_r^1 \frac{8s}{(1+s^2)^2} \eta(s) - \]
\[ -r^\beta \int_r^1 \frac{\beta^2}{s} \eta(s) = o(1) \]
which proves (A.4). Hence (A.3) becomes
\[ \int_0^R \frac{8s^{\beta+1}}{(1+s^2)^2} \eta = -R^{\beta+1}\eta'(R) + \beta R^\beta \eta(R). \]
Then
\[- \left( \frac{\eta(r)}{r^\beta} \right)' \bigg|_{r=R} = \frac{1}{R^{2\beta+1}} \int_0^R \frac{8s^{\beta+1}}{(1+s^2)^2} \eta(s) ds \]
and integrating between \( t \) and 1 we deduce
\[\eta(t) - \eta(1) = \int_t^1 \frac{1}{R^{2\beta+1}} \left( \int_0^R \frac{8s^{\beta+1}}{(1+s^2)^2} \eta(s) ds \right) dR.\]
Since \( \eta \) is bounded we get
\[\eta(R) = \begin{cases} O(R^\beta + R^2) & \text{if } \beta \neq 2 \\ O(R^2 |\log R|) & \text{if } \beta = 2 \end{cases} \]
This gives the claim. Note that, repeating the proof a finite number of steps, we get the following decay near zero,
\[|\eta(R)| \leq CR^\beta \text{ for } R \text{ close to 0.}\]
In the same way one can prove that
\[|\eta(R)| \leq C \frac{1}{R^\beta} \text{ for } R \text{ large enough.}\]

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