A NUMERICAL STUDY OF PENROSE-LIKE INEQUALITIES IN A FAMILY OF AXIALLY SYMMETRIC INITIAL DATA

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Abstract. Our current picture of black hole gravitational collapse relies on two assumptions: i) the resulting singularity is hidden behind an event horizon — weak cosmic censorship conjecture — and ii) space-time eventually settles down to a stationarity state. In this setting, it follows that the minimal area containing an apparent horizon is bound by the square of the total ADM mass (Penrose inequality conjecture). Following Dain et al. 2002, we construct numerically a family of axisymmetric initial data with one or several marginally trapped surfaces. Penrose and related geometric inequalities are discussed for these data. As a by-product, it is shown how Penrose inequality can be used as a diagnosis for an apparent horizon finder numerical routine.

1. Introduction and methodology. Our present goal is the study, using numerical techniques, of certain geometric inequalities conjectured to hold in asymptotically flat Cauchy slices containing an apparent horizon (AH). This is an ambitious objective, since numerical tools can offer at best a counterexample – not a general proof – and experience with these inequalities has revealed the difficulty of this task. Precisely because of this confidence in the inequalities, the line of reasoning can be reversed in appropriate settings: inequalities can be (tentatively) taken for granted and used as a diagnosis to test specific geometric and/or numerical constructions. At the end of the day, we aim at gaining geometric insight about these inequalities in regimes that are difficult to probe by standard analytic techniques. We formulate here our strategy and present some preliminary results.

1.1. Geometric inequalities. Penrose inequality is our prototype of geometric inequality. It follows from a chain of heuristic arguments (Penrose 1973) in the context of black hole gravitational collapse, in particular probing weak cosmic
censorship conjecture and the assumption of an evolution towards a final stationary state. Penrose inequality conjectures: \( A \leq 16\pi M_{\text{ADM}}^2 \), where \( M_{\text{ADM}} \) is the total ADM mass and \( A \) is the minimal area enclosing the (possibly non-connected) AH. It has been proved in the Riemannian case \( K_{ij} = 0 \) (Huisken & Ilmanen 2001, Bray 2001), an equality is conjectured to hold only for Schwarzschild. Here we focus on axisymmetric data, for which an angular momentum \( J \) can be unambiguously defined and Penrose inequality can be strengthened (Dain et al. 2002) to

\[
A \leq 8\pi \left( M_{\text{ADM}}^2 + \sqrt{M_{\text{ADM}}^4 - J^2} \right)
\]

where equality is conjectured to hold only for Kerr data. We shall refer to this latter point as the Dain’s rigidity conjecture. Rhs expression only makes sense if

\[
|J| \leq M_{\text{ADM}}^2 ,
\]

an angular momentum-mass inequality recently proved by Dain for vacuum, maximal \((K = 0)\), asymptotically flat, axisymmetric data with a connected AH (Dain 2007 and references therein). Equality holds only for extremal Kerr data. Petroff and Ansorg have proposed a quasi-local bound for \(|J|\) in terms of the area \( A \)

\[
8\pi |J| \leq A ,
\]

in the restricted setting of stationary, axisymmetric configurations of black holes surrounded by matter. This conjecture has been extended to include charges, and equality has been shown to exactly correspond to the extremal case (Ansorg & Pfister 2007). Simultaneously, it has been argued (Booth & Fairhurst 2007) the non-validity of this quasi-local inequality for generic axisymmetric data. Here we consider this latter non-stationary generic situation. We rewrite previous inequalities in terms of bounded dimensionless parameters \( \epsilon_p, \epsilon_A, \epsilon_D, \epsilon_{PA} \):

\[
\epsilon_p := \frac{A}{16\pi M_{\text{ADM}}^2} \leq 1 , \quad \epsilon_D := \frac{|J|}{M_{\text{ADM}}^2} \leq 1 , \quad \epsilon_A := \frac{A}{8\pi (M_{\text{ADM}}^2 + \sqrt{M_{\text{ADM}}^4 - J^2})} \leq 1 , \quad \epsilon_{PA} := \frac{8\pi |J|}{A} \leq 1 .
\] (0.1)

In particular, Dain’s rigidity conjecture reads: \( \epsilon_A = 1 \Leftrightarrow (\gamma_{ij}, K^{ij}) \) are Kerr data.

1.2. Axisymmetric Initial Data: deformations on Kerr. A conformal construction of vacuum, maximal and axisymmetric data — parametrized by two free functions \( q \) and \( \omega \) — is presented in Dain et al. 2002. By further restricting \( q \) and \( \omega \), we have studied: i) binary black hole data and ii) deformations of Kerr. We discuss here Kerr deformations and will present the binary case elsewhere:

1. Choice of \( q \). Fixed by a choice of conformal metric as the representative with unit determinant in the conformal class of Kerr in quasi-isotropic coordinates.

2. Choice of \( \omega \) as: \( \omega(J, M_{\text{Kerr}}, \lambda) = \omega_{BY}(J) - \lambda \cdot \omega_{A}(J, M_{\text{Kerr}}) \). Here \( \omega_{BY}(J) \) is associated with the Bowen-York extrinsic curvature in Dain et al. 2002 method, whereas \( \omega_{A}(J, M_{\text{Kerr}}, \lambda = 1) \) is such that \( \omega(J, M_{\text{Kerr}}, \lambda = 1) \) is compatible with Kerr.

3. Marginally Trapped Outer Surface (MOTS) inner boundary condition at an excised sphere of coordinate radius \( r = 1 \), when solving the Hamiltonian constraint for the conformal factor. The introduced scale fixes \( M_{\text{Kerr}} \) in terms of \( J \).
In sum, we work with data $[\gamma_{ij}(J, \lambda), K^{ij}(J, \lambda)]$ parametrised by $J$ and a deformation parameter $\lambda$ — Kerr data correspond to $\lambda = 1$. Data are numerically implemented using both the spectral methods in the Meudon C++ Lorene library and the spectral methods developed by one of the authors in Ansorg et al. 2005.

1.3. Extraction of geometric information: AH-finders. The assessment of the discussed geometric inequalities primarily concerns AHS properties. First, we need to know their location. By construction, data in the considered family contain a MOTS at the inner excised sphere. In the generic case, we need an AH-finder routine to locate the outermost MOTS. In this case, we make use of the spectral 3D spectral integral-iteration AH-finder presented in Lin & Novak 2007.

2. Results. Fixing $J$ and screening different values of $\lambda$, we monitor the dimensionless quantities $\epsilon_p, \epsilon_A, \epsilon_D, \epsilon_{PA}$. First we check that, as $\lambda$ departs from $\lambda = 1$, the data indeed move away from Kerr — i.e. $\epsilon_A$ departs from 1, as they should according to Dain’s rigidity conjecture. Increasing $\lambda$, a critical $\lambda_o$ exists for each $J$ at which a second outer horizon detaches from the inner one. For sufficiently large $\lambda$, $\epsilon_A$ grows over 1 for the inner horizon, whereas the strengthened Penrose inequality still holds since the exterior horizon satisfies $\epsilon_A \leq 1$ — cf. Fig. II. The other inequalities are also satisfied, and no surprises appear.

At this point we reverse the line of reasoning. First, we use the inequalities to test the Lin & Novak AH-finder. As Fig. [II] left shows, outer $\epsilon_A$ grows with $\lambda$. Penrose inequality sets a definite geometric limit to this: $\epsilon_A \leq 1$ should hold for all $\lambda$’s if the AH-finder is properly working. Pushing $\lambda$ to very large values (bounded by numerical limits) we have checked the validity of the inequality — cf. Fig. [II] right. The AH-finder is not producing spurious solutions and is actually converging to a MOTS, otherwise there is no reason for $\epsilon_A \leq 1$ to hold. Most importantly, this asymptotic behaviour indicates the outermost character of the outer MOTS. Additional quantitative tests have shown (spectral) exponential convergence, an accuracy of $\delta A/A \sim 10^{-12}$, and the robustness of the AH-finder. Second, Fig. [II] right suggests $\lim_{\lambda \to \infty} \epsilon_A = 1$. According to Dain’s rigidity con-
jecture this limit corresponds to Kerr data. As a necessary condition for this, the AH should approach a Non-Expanding Horizon (NEH) as λ diverges, namely the outer null normal shear \( \sigma_+ \) should vanish in this limit. Fig. right confirms this vanishing asymptotic behaviour, reached after an intermediate stage in which the AH definitely departs from a NEH – similar results are obtained using a dimensionless \( \int_S |\sigma_+|^2 dA \). In this sense, our preliminary numerical results do support Dain’s conjecture.

Before concluding, we note that a mass-angular momentum inequality — milder than Dain’s — follows from Penrose and Petroff-Ansorg bounds: 

\[
8\pi |J| \leq A \leq 8\pi \left( M_{\text{ADM}}^2 + \sqrt{M_{\text{ADM}}^4 - J^2} \right).
\]

An optimal fitting among all three inequalities would be obtained with a strong Petroff-Ansorg inequality: 

\[
8\pi \left( |J| + \sqrt{M_{\text{ADM}}^4 - J^2} \right) \leq A.
\]

Our numerical experiments show its violation for \( J \) large enough, whereas standard Petroff-Ansorg inequality is not disproved. This is indeed a doubtful manner of proposing new inequalities, that could be referred to as geometric numerology.

3. Conclusions and perspectives. We have presented the elements of an approach to the numerical study of Penrose-like geometric inequalities. Potential applications to the assessment of numerics have been illustrated by performing a geometric test to the Lin & Novak AH-finder. More generally, Penrose inequality has been proposed as a practical diagnosis in the determination of the outermost character of a given MOTS. This can be useful in settings where little intuition is available about the properties of the AH — this has indeed proved to be crucial in our studies of the binary case, where employed coordinates (Ansorg 2005) strongly affect the form and location of the AH. Finally, following a strategy that resembles a canonical-conjugate version of Bray’s flow of conformal metrics approach to Penrose inequality (Bray 2001) — by employing a flow of conformal extrinsic curvatures instead — we have found support to Dain’s proposal of using \( \epsilon_A = 1 \) as a characterization of Kerr data. Given its cheap cost — evaluation of a single real number — this has a clear interest for the numerical community.

Next step consists in fully developing the outlined approach. We will pursue the study of the binary case, much richer and less understood — e.g. Dain’s mass-angular momentum inequality is not proved in the non-connected case. We will further assess Dain’s rigidity conjecture, by implementing more general deformations of Kerr, and will explore the general validity of Petroff-Ansorg inequality.

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