Approximation by Shifts of Compositions of Dirichlet $L$-Functions with the Gram Function

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Abstract: In this paper, a joint approximation of analytic functions by shifts of Dirichlet $L$-functions $L(s + ia_1 t_r, \chi_1), \ldots, L(s + ia_r t_r, \chi_r)$, where $a_1, \ldots, a_r$ are non-zero real algebraic numbers linearly independent over the field $\mathbb{Q}$ and $t_r$ is the Gram function, is considered. It is proved that the set of their shifts has a positive lower density.

Keywords: Dirichlet $L$-function; Gram function; joint universality

1. Introduction

Let $\chi : \mathbb{N} \to \mathbb{C}$ be a Dirichlet character modulo $q \in \mathbb{N}$. Note that $\chi(m)$ is periodic with period $q$, completely multiplicative (i.e., $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$ and $\chi(1) = 1$), $\chi(m) = 0$ for $(m, q) = 1$ and $\chi(m) \neq 0$ for $(m, q) = 1$. Let $s = \sigma + it$. In [1], L. Dirichlet introduced a function

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad (\sigma > 1),$$

which is now called the Dirichlet $L$-function. In virtue of the complete multiplicativity of $\chi(m)$, the function (1) can be written as an Euler product

$$L(s, \chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers and has a meromorphic continuation to the whole complex plane with a unique simple pole at the point $s = 1$ (if $\chi$ is the principal character modulo $q$) with residue $\prod_{p\nmid q}(1 - 1/p)$. Since then, the function (1) has become a subject of intensive investigation. See, for instance, References [2–4] for some very recent papers on its zeros and moments. For $q = 1$, the function $L(s, \chi)$ becomes the Riemann zeta-function $\zeta(s)$.

In Reference [5], S. M. Voronin established the universality of Dirichlet $L$-functions. He proved that if $f(s)$ is a continuous non-vanishing function on the disc $|s| \leq r$ with any fixed $r$, $0 < r < 1/4$, and analytic in the interior of that disc, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leq r} |L(s + 3/4 + i\tau, \chi) - f(s)| < \varepsilon.$$

The Voronin theorem was extended to more general compact sets independently in References [6–8]. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements, and by $H_0(\mathcal{K})$, where $K \in \mathcal{K}$, the class of continuous non-vanishing functions on $K$ that
are analytic in the interior of $K$. Then the modern version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,
\[
\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} \left| L(s + i\tau, \chi) - f(s) \right| < \varepsilon \right\} > 0,
\]
where $\operatorname{meas} A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ (see, for example, Reference [9]). The latter inequality shows that there are infinitely many shifts $L(s + i\tau, \chi)$ approximating a given function from the class $H_0(K)$.

In Reference [10], Voronin considered the joint functional independence of Dirichlet $L$-functions using the joint universality. We recall that two Dirichlet characters are called non-equivalent if they are not generated by the same primitive character. Thus, the following statement is valid [10,11]; see also References [9,12,13].

**Theorem 1.** Let $\chi_1, \ldots, \chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,
\[
\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r, s \in K_j} \left| L(s + i\tau, \chi_j) - f_j(s) \right| < \varepsilon \right\} > 0.
\]

The non-equivalence of the characters $\chi_1, \ldots, \chi_r$ ensures a certain independence of the functions $L(s, \chi_1), \ldots, L(s, \chi_r)$ which is necessary for a simultaneous approximation of the collection $f_1(s), \ldots, f_r(s)$. Later, it turned out that, in place of non-equivalent characters, different shifts can be used. This was observed by Nakamura [14]. More precisely, he proved the following theorem.

**Theorem 2.** Let $a_1 = 1, a_2, \ldots, a_r$ be real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$ and $\chi_1, \ldots, \chi_r$ be arbitrary Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $a \in \mathbb{R} \setminus \{0\}$,
\[
\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r, s \in K_j} \left| L(s + i\tau, \chi_j) - f_j(s) \right| < \varepsilon \right\} > 0.
\]

In Reference [15], Pańkowski obtained the joint universality of Dirichlet $L$-functions using the shifts $L(s + ia_j \log \Gamma(s/2), \chi_j), j = 1, \ldots, r$, where $a_1, \ldots, a_r \in \mathbb{R}, a_1, \ldots, a_r \in \mathbb{R}^+$ are distinct, $b_1, \ldots, b_r$ are distinct and satisfy
\[
b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{N}, \\ (-\infty, 0) \cup (1+\infty) & \text{if } a_j \in \mathbb{N}. \end{cases}
\]

The aim of this paper is to introduce new shifts of Dirichlet $L$-functions that approximate collections of analytic functions from the class $H_0(K)$. Let, as usual, $\Gamma(s)$ be the Euler gamma-function. For $t > 0$, denote the increment $\theta(t)$ of the argument of the function $\pi^{-s/2} \Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. Then it is known (see, for example, Reference [16] [Lemma 1.1]) that, for $\tau \geq 0$, the equation
\[
\theta(t) = (\tau - 1)\pi
\]
has the unique solution $t_\tau$ satisfying $\theta'(t_\tau) > 0$. For $n \in \mathbb{N}$, the numbers $t_n$ are called the Gram points. They were introduced and studied in Reference [17]. Therefore, we call $t_n$ the Gram function. A very interesting property of the Gram points is the relation $t_n \sim \gamma_n$ as $n \to \infty$, where $\gamma_n > 0$ are imaginary parts of non-trivial zeros of the Riemann zeta-function. In the paper, we will consider the
joint approximation of analytic functions by shifts of Dirichlet $L$-functions involving the Gram function. More precisely, we will prove the following joint universality theorem.

**Theorem 3.** Suppose that $a_1, \ldots, a_r$ are real non-zero algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1, \ldots, \chi_r$ are arbitrary Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$
\liminf_{T \to \infty} \frac{1}{T - 2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.
$$

Moreover, the limit

$$
\lim_{T \to \infty} \frac{1}{T - 2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t, \chi_j) - f_j(s)| < \varepsilon \right\} > 0
$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 3, we will use the probabilistic approach based on weakly convergent probability measures in the space of analytic functions.

2. Lemmas

We start with a lemma on the functional properties of the function $t_\tau$. (Its proof can be found in Reference [16] [Lemma 1.1].)

**Lemma 1.** Suppose that $\tau \to \infty$. Then

$$
t_\tau = \frac{2\pi \tau}{\log \tau} \left( 1 + \frac{\log \log \tau}{\log \tau} (1 + o(1)) \right),
$$

$$
t'_\tau = \frac{2\pi}{\log \tau} \left( 1 + \frac{\log \log \tau}{\log \tau} (1 + o(1)) \right)
$$

and

$$
t''_\tau = -\frac{\pi}{\tau (\log \tau)^2} \left( 1 + \frac{\log \log \tau}{\log \tau} (2 + o(1)) \right).
$$

The next lemma provides an estimate for certain trigonometric integral.

**Lemma 2.** Suppose that $F(x)$ is a real differentiable function, the derivative $F'(x)$ is monotonic and $F'(x) \geq \lambda > 0$ or $F'(x) \leq -\lambda < 0$ on the interval $(a, b)$. Then

$$
\left| \int_a^b \exp \{iF(x)\} \, dx \right| \leq \frac{4}{\lambda}.
$$

The proof of the lemma is given, for example, in Reference [11].

We will also use Baker’s theorem on linear forms in logarithms of algebraic numbers (see, for example, Reference [18]).

**Lemma 3.** Suppose that $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}$ are such that their logarithms $\log \lambda_1, \ldots, \log \lambda_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \ldots, \beta_r$, not all zero, we have

$$
|\beta_0 + \beta_1 \log \lambda_1 + \cdots + \beta_r \log \lambda_r| > H^{-C},
$$

where $H$ is the maximum of the heights of $\beta_0, \beta_1, \ldots, \beta_r$, and $C$ is an effectively computable constant depending on $r, \lambda_1, \ldots, \lambda_r$ and the maximum of the degrees of $\beta_0, \beta_1, \ldots, \beta_r$.
Let \( \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \), and \( \Omega = \prod_{p \in \mathbb{P}} \gamma_p \)
where \( \gamma_p = \gamma \) for all \( p \in \mathbb{P} \). With the product topology and pointwise multiplication, the infinite-dimensional torus \( \Omega \) is a compact topological Abelian group. Define
\[
\Omega' = \Omega_1 \times \cdots \times \Omega_r,
\]
where \( \Omega_j = \Omega \) for \( j = 1, \ldots, r \). Then \( \Omega' \) is also a compact topological Abelian group. Therefore, denoting by \( B(\mathcal{X}) \) the Borel \( \sigma \)-field of the space \( \mathcal{X} \), we see that, on \( (\Omega', B(\Omega')) \), the probability Haar measure \( m'_H \) exists. This gives the probability space \( (\Omega', B(\Omega'), m'_H) \).

For \( A \in B(\Omega') \), define
\[
Q_T(A) = \frac{1}{T - 2} \text{meas}\{ \tau \in [2, T] : (p^{-ia_1 t \tau} : p \in \mathbb{P}), \ldots, (p^{-ia_r t \tau} : p \in \mathbb{P}) \} \in A \}.
\]
Then the following limit theorem holds.

**Lemma 4.** Under hypotheses of Theorem 2 on the numbers \( a_1, \ldots, a_r \), \( Q_T \) converges weakly to the Haar measure \( m'_H \) as \( T \to \infty \).

**Proof.** We apply the Fourier transform method. It is well known that the dual group of \( \Omega' \) is isomorphic to the group
\[
\bigoplus_{j=1}^r \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{kp_j},
\]
where \( \mathbb{Z}_{kp_j} = \mathbb{Z} \) for all \( j = 1, \ldots, r, p \in \mathbb{P} \). Hence it follows that characters of the group \( \Omega' \) are of the form
\[
\prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_j^{k_{jp}}(p),
\]
where \( \omega_j(p) \) is the \( p \)th component of an element \( \omega_j \in \Omega_j, \) \( j = 1, \ldots, r \), and the sign "*" means that only a finite number of integers \( k_{jp} \) are distinct from zero. Therefore
\[
\int_{\Omega'} \left( \prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_j^{k_{jp}}(p) \right) d\mu
\]
is the Fourier transform of a measure \( \mu \) on \( (\Omega', B(\Omega')) \).

Let \( S_{Q_T}(\mathbf{k}), \mathbf{k} = (k_1, \ldots, k_r), \mathbf{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P}), j = 1, \ldots, r, \) be the Fourier transform of \( Q_T \). In view of (2) we have
\[
S_{Q_T}(\mathbf{k}) = \int_{\Omega'} \left( \prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_j^{k_{jp}}(p) \right) dQ_T.
\]
Thus, by the definition of \( Q_T \),
\[
S_{Q_T}(\mathbf{k}) = \frac{1}{T - 2} \int_2^T \int_{\mathbb{P}} \prod_{j=1}^r \prod_{p \in \mathbb{P}} p^{-ik_{jp} t \tau} d\tau
\]
\[
= \frac{1}{T - 2} \int_2^T \exp \left\{ -it \sum_{j=1}^r \sum_{p \in \mathbb{P}} a_j k_{jp} \log p \right\} d\tau.
\]
Obviously, if $k = (0, \ldots, 0)$, then
\[ g_{\mathbb{Q}_r}(k) = 1. \] (4)

Now suppose that $k = (k_1, \ldots, k_r) \neq (0, \ldots, 0)$. Note that
\[
A_k \overset{\text{def}}{=} \sum_{j=1}^{r} \sum_{p \in \mathbb{P}} a_j k_j p \log p = \sum_{p \in \mathbb{P}} \log p \sum_{j=1}^{r} a_j k_j p.
\]

Since $k_j \neq 0$ for some $j \in \{1, 2, \ldots, r\}$, there is a prime number $p$ such that $k_j p \neq 0$. For this $p$, the sum $\beta_p \overset{\text{def}}{=} \sum_{j=1}^{r} \omega_j(p) k_j p \log p$ is non-zero, because the numbers $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3,
\[
A_k = \sum_{p \in \mathbb{P}} \beta_p \log p \neq 0.
\] (5)

Now, (3) and Lemmas 1 and 2 show that, in the case $k \neq (0, \ldots, 0)$,
\[ g_{\mathbb{Q}_r}(k) \ll \log TA_k. \]

This together with (4) and (5) give
\[
\lim_{T \to \infty} g_{\mathbb{Q}_r}(k) = \begin{cases} 1 & \text{if } k = (0, \ldots, 0), \\ 0 & \text{if } k \neq (0, \ldots, 0). \end{cases}
\]

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure $m_{H'}$, the lemma follows by a continuity theorem for probability measures on compact groups.

$H(D)$ denotes the space of analytic functions on the strip $D$ endowed with the topology of uniform convergence on compacta. Lemma 4 implies a limit theorem for probability measures on $(H(D), B(H(D)))$ defined by means of absolutely convergent Dirichlet series.

For a fixed number $\theta > 1/2$ and $m, n \in \mathbb{N}$, set
\[
v_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^\theta \right\}.
\] (6)

Then we define the series
\[
L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s},
\]
and
\[
L_n(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) \omega_j(m) v_n(m)}{m^s},
\]
j = 1, \ldots, r, where the functions $\omega_j(p)$ are extended to the set $\mathbb{N}$ by the formula
\[
\omega_j(m) = \prod_{p \mid m \atop p^{l+1} \mid m} \omega_j(p), \quad m \in \mathbb{N}.
\]

Denote the elements of $\Omega'$ by $\omega = (\omega_1, \ldots, \omega_r)$. Put $\chi = (\chi_1, \ldots, \chi_r)$, and set
\[
L_n(s, \chi) = (L_n(s, \chi_1), \ldots, L_n(s, \chi_r))
\] (7)
and
\[
L_n(s, \omega, \chi) = (L_n(s, \omega_1, \chi_1), \ldots, L_n(s, \omega_r, \chi_r)).
\]
Moreover, let \( u_n : \Omega' \to H'(D) \) be given by the formula
\[
    u_n(\omega) = L_n(s, \omega, \chi).
\]
The absolute convergence of the series for \( L_n(s, \omega_j, \chi_j) \) implies the continuity of the mapping \( u_n \). Let \( V_n = m_H' u_n^{-1} \), where, for \( A \in \mathcal{B}(H'(D)) \),
\[
    V_n(A) = m_H' u_n^{-1}(A) = m_H'(u_n^{-1} A). \tag{8}
\]
In view of (7) and (8) we conclude that Lemma 4, the continuity of \( u_n \) and the well-known property on preservation of weak convergence under mapping lead to the following statement.

**Lemma 5.** Under hypothesis of Theorem 3 on the numbers \( a = (a_1, \ldots, a_r) \), we have
\[
    P_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T - 2}\text{meas}\left\{ \tau \in [2, T] : L_n(s + i\tau, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)),
\]
converges weakly to the measure \( V_n \) as \( T \to \infty \).

The probability measure \( V_n \) is very important for the proof of Theorem 3. Let
\[
    L(s, \omega, \chi) = (L(s, \omega_1, \chi_1), \ldots, L(s, \omega_r, \chi_r)),
\]
where
\[
    L(s, \omega, \chi) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega_j(p) \chi_j(p)}{p^s} \right)^{-1}, \quad j = 1, \ldots, r. \tag{9}
\]
Note that the latter products are uniformly convergent on compact subsets of the strip \( D \) for almost all \( \omega_j \in \Omega_j \), and define the \( H(D) \)-valued random elements on the probability space \((\Omega_j, \mathcal{B}(\Omega_j), m_{jH})\), where \( m_{jH} \) is the probability Haar measure on \((\Omega_j, \mathcal{B}(\Omega_j))\). Therefore, \( L(s, \omega, \chi) \) is the \( H'(D) \)-valued random element on \((\Omega', \mathcal{B}(\Omega'), m_H')\). Denote by \( P_L \) the distribution of the random element \( L(s, \omega, \chi) \), that is,
\[
    P_L(A) = m_H' \left\{ \omega \in \Omega' : L(s, \omega, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)).
\]
We recall that the support of a probability measure \( P \) on \((X, \mathcal{B}(X))\), where the space \( X \) is separable, is a minimal closed set \( S_P \subset X \) such that \( P(S_P) = 1 \). The set \( S_P \) consists of all elements \( x \in X \) such that, for every open neighbourhood \( G \) of \( x \), the inequality \( P(G) > 0 \) is satisfied.

The measure \( V_n \) is independent on any hypothesis. Therefore, from Reference [19] it follows that:

**Lemma 6.** The measure \( V_n \) converges weakly to \( P_L \) as \( n \to \infty \). Moreover, the support of \( P_L \) is the set \( S' \), where
\[
    S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

**Proof.** To be precise, in Reference [19] it was proved that a certain measure \( P_N \) converges weakly to a certain probability measure \( P \) on \((H'(D), \mathcal{B}(H'(D)))\) (as \( N \to \infty \)), and the measure \( P \) is the limit measure of \( V_n \) as \( n \to \infty \). Moreover, it was proved that \( P = P_L \).

It remains to prove that the support of \( P_L \) is the set \( S' \). It is well known that the support of the random element
\[
    \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega(p) \chi(p)}{p^s} \right)^{-1}, \quad \omega \in \Omega, \tag{10}
\]
is the set $S$ for every Dirichlet character $\chi$. Since the space $H'(D)$ is separable, we have

$$\mathcal{B}(H'(D)) = \mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))$$

(see [20]). Therefore, it suffices to consider the measure $P_L$ on the sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \ldots, A_r \in \mathcal{B}(H(D)).$$

Since the Haar measure $m'_H$ is the product of the Haar measures $m_{jH}$ on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \ldots, r$, we deduce that

$$m'_H(\omega \in \Omega' : L(s, \omega, \chi) \in A) = \prod_{j=1}^{r} m_{jH}(\omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j).$$

This equality and the minimality of the support together with remark on the support of the element (10) show that the support of $P_L$ is the set $S'$. \( \square \)

3. Mean Square Estimates

Define

$$L(s, \chi) = (L(s, \chi_1), \ldots, L(s, \chi_r)). \quad (11)$$

To pass from $L_n(s + iat, \chi)$ (defined by (7)) to $L(s + iat, \chi)$, certain mean square estimates for Dirichlet $L$-functions are necessary. Let $\chi$ be an arbitrary character modulo $q$.

**Lemma 7.** Suppose that $\sigma$, $1/2 < \sigma < 1$, and $a \in \mathbb{R} \setminus \{0\}$ are fixed. Then, for $t \in \mathbb{R}$,

$$\int_{\frac{T}{2}}^{T} \left| L(\sigma + it + iat, \chi) \right|^2 \, d\tau \ll T(1 + |t|).$$

**Proof.** It is well known that, for fixed $\sigma > 1/2$,

$$\int_{\frac{T}{2}}^{T} \left| L(\sigma + it, \chi) \right|^2 \, dt \ll_{\sigma} T.$$

Therefore, in view of Lemma 1, for $1/2 < \sigma < 1$,

$$\int_{\frac{T}{2}}^{T} \left| L(\sigma + it + iat, \chi) \right|^2 \, d\tau = \frac{1}{a} \int_{\frac{T}{2}}^{T} \frac{1}{t} \left| L(\sigma + it + iat, \chi) \right|^2 \, d(at)$$

$$= \frac{1}{a} \int_{\frac{T}{2}}^{T} \frac{1}{t} \left( \int_{\frac{T}{2}}^{t + at} \left| L(\sigma + iu, \chi) \right|^2 \, du \right)$$

$$\ll \log T \int_{\frac{T}{2}}^{t + at} \left| L(\sigma + iu, \chi) \right|^2 \, du$$

$$\ll_{\sigma, a} \log T \left( \frac{T}{\log T} \right) \ll_{\sigma, a} T(1 + |t|),$$

which is the required estimate. \( \square \)

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \sup_{s \in K_l} \left| g_1(s) - g_2(s) \right| \frac{1}{1 + \sup_{s \in K_l} \left| g_1(s) - g_2(s) \right|}.$$  \( (12) \)
where \( \{K_l\} \subset D \) is a sequence of compact subsets such that
\[
D = \bigcup_{l=1}^\infty K_l.
\]

Then, in view of (16),
\[
\text{Let } \rho \text{ be a metric in the space } H(D) \text{ inducing the topology of uniform convergence on compacta. Now, putting, for } g_1 = (g_{11}, \ldots, g_{1r}), g_2 = (g_{21}, \ldots, g_{2r}) \in H(D),
\[
\rho(g_1, g_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})
\]
gives a metric in \( H(D) \) inducing the product topology. The next lemma provides a certain approximation of \( L(s, \chi) \) (see definition (11)) by \( L_n(s, \chi) \).

**Lemma 8.** Suppose that \( a \neq (0, \ldots, 0) \). Then
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T-2} \int_2^T \rho \left( L(s + it \tau, \chi), L_n(s + it \tau, \chi) \right) \, d\tau = 0.
\]

**Proof.** From the definition (13) of the metric \( \rho \), it follows that it suffices to prove that, for \( a \neq 0, \)
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T-2} \int_2^T \rho \left( L(s + it \tau, \chi), L_n(s + it \tau, \chi) \right) \, d\tau = 0
\]
for every \( j = 1, \ldots, r \). We will prove the above equality for the character \( \chi \) modulo \( q \).

Let \( \theta \) be from the definition (6) of \( v_n(m) \), and
\[
l_n(s) = \frac{s}{\theta} \Gamma \left( \frac{s}{\theta} \right) n^s.
\]

Then the representation
\[
L_n(s, \chi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s + z, \chi)n(z) \frac{dz}{z},
\]
is true. Its proof is the same as in Section 5.4 of [21] for the Riemann zeta-function. Hence, taking \( \theta_1 > 0 \), by the residue theorem, we obtain
\[
L_n(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} L(s + z, \chi)n(z) \frac{dz}{z} + R_n(s, \chi),
\]
where
\[
R_n(s, \chi) = \begin{cases}
0 & \text{if } \chi \text{ is a non-principal character}, \\
\prod_{p|q} \left( 1 - \frac{1}{p} \right) \frac{l_n(1-s)}{1-s} & \text{otherwise}.
\end{cases}
\]

Let \( K \subset D \) be an arbitrary compact set. Denote by \( s = \sigma + iv \) the points of \( K \), and suppose that
\[
1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon \text{ with fixed } \varepsilon > 0 \text{ for } s \in K.
\]
More precisely, we select \( \theta_1 = \sigma - \varepsilon - 1/2 \geq \varepsilon > 0 \). Then, in view of (16),
\[
|L_n(s + it \tau, \chi) - L(s + it \tau, \chi)| \ll \int_{-\infty}^\infty |L(s + it \tau - \theta_1 + it, \chi)| \frac{|l_n(-\theta_1 + it)|}{|\theta_1 + it|} \, dt + |R_n(s + it \tau, \chi)|.
\]
Now, taking $t$ in place of $t + \nu$, we get that, for $s \in K$,

\[
|L_n(s + i\alpha t, \chi) - L(s + i\alpha t, \chi)| \
\ll \int_{-\infty}^{\infty} |L(1/2 + \epsilon + i(t + \alpha t), \chi)| \left| \frac{l_n(1/2 + \epsilon - s + it)}{1/2 + \epsilon - s + it} \right| dt \
+ |R_n(s + i\alpha t, \chi)|.
\]

This implies the estimate

\[
\frac{1}{T - 2} \int_{2}^{T} \sup_{s \in K} |L(s + i\alpha t, \chi) - L_n(s + i\alpha t, \chi)| \, d\tau 
\ll \frac{1}{T - 2} \int_{2}^{T} \int_{-\infty}^{\infty} |L(1/2 + \epsilon + i(t + \alpha t), \chi)| \sup_{s \in K} \left| \frac{l_n(1/2 + \epsilon - s + it)}{1/2 + \epsilon - s + it} \right| \, dt \, d\tau 
+ \frac{1}{T - 2} \int_{2}^{T} \sup_{s \in K} |R_n(s + i\alpha t, \chi)| \, d\tau 
\ll J_1 + J_2,
\]

where

\[
J_1 = \int_{-\infty}^{\infty} \frac{1}{T - 2} \int_{2}^{T} \left( |L(1/2 + \epsilon + i(t + \alpha t), \chi)| \, d\tau \right) \sup_{s \in K} \left| \frac{l_n(1/2 + \epsilon - s + it)}{1/2 + \epsilon - s + it} \right| \, dt
\]

and

\[
J_2 = \frac{1}{T - 2} \int_{2}^{T} \sup_{s \in K} |R_n(s + i\alpha t, \chi)| \, d\tau.
\]

It is well known that uniformly in $\sigma, 0 \leq \sigma \leq \sigma_2$, with arbitrary $\sigma_1 < \sigma_2$,

\[
\Gamma(\sigma + i\alpha) \ll \exp\{ -c|\alpha|\}, \quad c > 0.
\]

Therefore, by the definition (15) of the function $l_n(s)$, we find that, for $s \in K$,

\[
\left| \frac{l_n(1/2 + \epsilon - s + it)}{1/2 + \epsilon - s + it} \right| = n^{1/2 + \epsilon - \sigma} \left| \Gamma\left( \frac{1/2 + \epsilon - \sigma}{\theta} + \frac{i(t - \nu)}{\theta} \right) \right| 
\ll \Theta_{\theta, K} n^{-\epsilon} \exp \left\{ -\frac{c_1}{\theta} |t| \right\}, \quad c_1 > 0.
\]

In the same way, for $s \in K$, we obtain

\[
R_n(s + i\alpha t, \chi) \ll_{\theta, \alpha, K} n^{1-\epsilon} \exp \left\{ -\frac{c_2}{\theta} |a| t \right\}.
\]

Suppose that $\theta = 1/2 + \epsilon$. Then (17), (19) and Lemma 7 lead to the bound

\[
J_1 \ll_{c, K} n^{-\epsilon} \int_{-\infty}^{\infty} (1 + |t|) \exp\{ -c_3|t| \} \, dt 
\ll_{c, K} n^{-\epsilon}, \quad c_3 > 0.
\]

Moreover, by (18), Lemma 1 and (20),
\[ J_2 \ll_k n^{1/2 - 2\varepsilon} \frac{1}{T - 2} \int_2^T \exp \left\{ -c_4 |a| \frac{T}{\log T} \right\} d\tau \]
\[ \ll_k n^{1/2 - 2\varepsilon} \frac{\log T}{T - 2} + n^\varepsilon \frac{T^{1/2 - 2\varepsilon}}{T - 2} \int_{\log T}^T \exp \left\{ -c_4 |a| \frac{T}{\log T} \right\} d\tau \]
\[ \ll_k n^{1/2 - 2\varepsilon} \frac{\log T}{T - 2} . \]

Thus, in view of (17) and (21),
\[ \frac{1}{T - 2} \int_2^T \sup_{s \in K} |L(s + iat, \chi) - L_n(s + iat, \chi)| d\tau \ll_k n^{-\varepsilon} + n^{1/2 - 2\varepsilon} \frac{\log T}{T - 2} \]

From this, it follows that
\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - 2} \int_2^T \sup_{s \in K} |L(s + iat, \chi) - L_n(s + iat, \chi)| d\tau = 0. \tag{22} \]

Now, the definition (12) of the metric \( \rho \) implies (14), which completes the proof of Lemma 8. \( \square \)

4. A Limit Theorem

For \( A \in B(H'(D)) \), define
\[ P_T(A) = \frac{1}{T - 2} \text{meas} \left\{ \tau \in [2, T] : L(s + iat, \chi) \in A \right\} . \tag{23} \]

In this section, we will prove the following statement.

**Theorem 4.** Suppose that \( a_1, \ldots, a_r \) are non-zero real algebraic numbers linearly independent over \( \mathbb{Q} \), and \( \chi_1, \ldots, \chi_r \) are arbitrary Dirichlet characters. Then \( P_T \) converges weakly to \( P_L \) as \( T \to \infty \). The support of \( P_L \) is the set \( S' \).

First we recall a useful property of convergence in distribution \( (D) \) (see Theorem 4.2 in Reference [20]).

**Lemma 9.** Suppose that the space \( (X, d) \) is separable, the random elements \( X_{kn} \) and \( Y_n \), \( k \in \mathbb{N}, n \in \mathbb{N} \), are defined on the same probability space with measure \( \mu \),
\[ X_{kn} \xrightarrow{D} X_k, \]
for every \( k \in \mathbb{N}, \)
\[ X_k \xrightarrow{D} X, \]
and, for every \( \varepsilon > 0, \)
\[ \lim_{k \to \infty} \limsup_{n \to \infty} \mu \{ d(X_{kn}, Y_n) \geq \varepsilon \} = 0. \]

Then \( Y_n \xrightarrow{D} X \).

In the theory of weak convergence of probability measures, the notions of relative compactness and tightness of families of probability measures are very useful. We recall that the family \( \{ P \} \) of probability measures on \( (X, B(X)) \) is called relatively compact if every sequence \( \{ P_n \} \subset \{ P \} \) contains a weakly convergent subsequence to a certain measure on \( (X, B(X)) \), and this family is called tight, if for every \( \varepsilon > 0 \), there exists a compact set \( K = K(\varepsilon) \subset X \) such that
\[ P(K) > 1 - \varepsilon \]
for all $P \in \{P\}$. By the direct Prokhorov theorem (see Theorem 5.1 in Billingsley [20]), every tight family $\{P\}$ is relatively compact. We apply the above remarks to the sequence $\{V_n: n \in \mathbb{N}\}$, where $V_n$ (defined by (8)) is the limit measure in Lemma 5.

**Lemma 10.** The sequence $\{V_n\}$ is relatively compact.

**Proof.** By the above mentioned Prokhorov theorem, it suffices to prove that the sequence $\{V_n\}$ is tight.

Suppose $\theta_T$ is a random variable defined on a certain probability space with measure $\mu$ and uniformly distributed on $[2, T]$. Define the $H'(D)$-valued random element

$$X_{T,n} = X_{T,n}(s) = (X_{T,n,1}(s), \ldots, X_{T,n,r}(s)) = L_n(s + i\theta_T, X).$$

Moreover, let

$$X_n = X_n(s) = (X_{n,1}(s), \ldots, X_{n,r}(s))$$

be the $H'(D)$-valued random element with the distribution $V_n$. Then Lemma 5 implies the relation

$$X_{T,n} \xrightarrow{D} X_n \text{ as } T \to \infty. \tag{25}$$

By Lemma 7 with $t = 0$, we have, for $1/2 < \sigma < 1$,

$$\int_{\sigma}^{T} |L(\sigma + i\tau, \chi_j)|^2 \, d\tau \ll_{\sigma,\beta} T, \quad j = 1, \ldots, r. \tag{26}$$

Let $K_j$ be a compact set from the definition of the metric $\rho$. Then (26) together with the Cauchy integral formula show that

$$\int_{\sigma}^{T} \sup_{s \in K_j} |L(s + i\tau, \chi_j)| \, d\tau \ll_{\sigma,\beta} T, \quad j = 1, \ldots, r.$$

This combined with (22) implies the inequality

$$\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T - 2} \int_{\sigma}^{T} \sup_{s \in K_j} |L_n(s + i\tau, \chi_j)| \, d\tau \ll_{\sigma,\beta} R_{ij}, \quad j = 1, \ldots, r. \tag{27}$$

Fix $\varepsilon > 0$, and define $M_{ij} = M_{ij}(s) = 2^j r R_{ij} \varepsilon^{-1}$. Then, in view of (27), we find that, for each $n \in \mathbb{N}$,

$$\limsup_{T \to \infty} \mu \left\{ \exists j : \sup_{s \in K_j} |X_{T,n,j}(s)| > M_{ij} \right\} \leq \sum_{j=1}^{r} \limsup_{T \to \infty} \mu \left\{ \sup_{s \in K_j} |X_{T,n,j}(s)| > M_{ij} \right\} \leq \sum_{j=1}^{r} \frac{R_{ij}}{M_{ij}} = \frac{\varepsilon}{2^j}.$$  

This together with (25) shows that, for all $l, n \in \mathbb{N}$,

$$\mu \left\{ \exists j : \sup_{s \in K_l} |X_{n,j}(s)| > M_{ij} \right\} \leq \frac{\varepsilon}{2^j}. \tag{28}$$
Define the set
\[ K_j = K_j(s) = \left\{ g \in H(D) : \sup_{s \in K_j} |g(s)| \leq M_{lj}, \ l \in \mathbb{N} \right\}. \]

Then \( K_j \) is a compact set in \( H(D) \), and, in virtue of (24) and (28),
\[ \mu \{ X_n \in K \} \geq 1 - \varepsilon \]
for all \( n \in \mathbb{N} \). In other words, we have
\[ V_n(K) \geq 1 - \varepsilon \]
for all \( n \in \mathbb{N} \). Thus, the sequence \( \{ V_n : n \in \mathbb{N} \} \) is tight. \( \Box \)

**Proof of Theorem 4.** By Lemma 10, there exists a subsequence \( \{ V_{n_k} \} \) of the sequence \( \{ V_n \} \) that is weakly convergent to a certain probability measure \( P \) on \( (H'(D), B(H'(D))) \) as \( k \to \infty \). This can be written as
\[ X_{n_k} \xrightarrow{D} P. \quad (29) \]

Define one more \( H'(D) \)-valued random element
\[ X_T = X_T(s) = L(s + i\theta_T, \chi). \]

Then Lemma 8 implies that, for every \( \varepsilon > 0 \),
\[ \lim_{n \to \infty} \limsup_{T \to \infty} \mu \left\{ \rho(X_T, X_{T,n}) \geq \varepsilon \right\} \leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{(T - 2)^2} \int_2^T \rho \left( L(s + i\theta_T, \chi), L_n(s + i\theta_T, \chi) \right) d\tau = 0. \]

The latter equality together with (25), (29), and Lemma 9 shows that
\[ X_T \xrightarrow{T \to \infty} P, \quad (30) \]
or, in other words, \( P_T \) converges weakly to \( P \) as \( T \to \infty \). Moreover, by the relation (30), the measure \( P \) is independent of the subsequence \( \{ V_{n_k} \} \). Thus, we deduce that
\[ X_n \xrightarrow{n \to \infty} P, \]
or \( V_n \) converges weakly to \( P \) as \( n \to \infty \). Therefore, the theorem follows by Lemma 6. \( \Box \)

5. **Proof of Universality**

The proof of Theorem 3 is based on Mergelyan’s theorem on the approximation of analytic functions by polynomials [22], Theorem 4, and the properties of weak convergence. For convenience, we state them as lemmas.

**Lemma 11 (Mergelyan theorem).** Suppose that \( K \subset \mathbb{C} \) is a compact set with connected complement, and \( f(s) \) be a continuous function on \( K \) and analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \), there exists a polynomial \( p(s) \) such that
\[ \sup_{s \in K} |f(s) - p(s)| < \varepsilon. \]

We recall that \( A \in B(\mathbb{X}) \) is called a continuity set of the measure \( P \) on \( (\mathbb{X}, B(\mathbb{X})) \) if \( P(\partial A) = 0 \), where \( \partial A \) is a boundary of \( A \).
Lemma 12. Let $P_n$, $n \in \mathbb{N}$, and $P$ be probability measures on $(X, \mathcal{B}(X))$. Then the following statements are equivalent:

1° $P_n$ converges weakly to $P$ as $n \to \infty$;

2° For every open set $G \subset X$,
$$\liminf_{n \to \infty} P_n(G) \geq P(G);$$

3° For every continuity set $A$ of $P$,
$$\lim_{n \to \infty} P_n(A) = P(A).$$

The above lemma is a part of Theorem 2.1 from Reference [20]. Now, we can give the proof of Theorem 3.

Proof of Theorem 3. First part. In view of Lemma 11, there exist polynomials $p_1(s), \ldots, p_r(s)$ such that
$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| f_j(s) - e^{p_j(s)} \right| < \frac{\varepsilon}{2}. \quad (31)$$

The set
$$G'_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H'(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| g_j(s) - e^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\} \quad (32)$$
is an open neighbourhood of the element \( e^{p_1(s)}, \ldots, e^{p_r(s)} \) $\in S'$. Thus, by Theorem 4, $P_L(G'_\varepsilon) > 0$, where the distribution $P_L$ is defined by (9). Hence, from Theorem 4 again and Lemma 12,
$$\liminf_{T \to \infty} P_T(G'_\varepsilon) \geq P_L(G'_\varepsilon) > 0,$$
and the definitions (23) and (32) of $P_T$ and $G'_\varepsilon$ together with (31) prove the first part of the theorem.

Second part. Introduce one more set
$$A_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H'(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}. \quad (33)$$

Then the boundary of $A_\varepsilon$ lies in the set
$$\left\{ (g_1, \ldots, g_r) \in H'(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| = \varepsilon \right\},$$
thus, $\partial A_{\varepsilon_1} \cap \partial A_{\varepsilon_2} = \emptyset$ for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. This shows that the set $A_\varepsilon$ is a continuity set of the measure $P_L$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemma 12,
$$\lim_{T \to \infty} P_T(A_\varepsilon) = P_L(A_\varepsilon) \quad (34)$$
for all but at most countably many $\varepsilon > 0$. Moreover, (31) shows the inclusion $G'_\varepsilon \subset A_\varepsilon$. This, (34) and the definitions (23) and (33) of $P_T$ and $A_\varepsilon$ prove the second assertion of the theorem. □

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