Anti-continuum approach on nonlinear localized modes in dipolar Bose-Einstein condensates in optical lattice

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Abstract. This paper aims to analyze the stability of nonlinear localized modes in dipolar Bose-Einstein condensates in optical lattice by anti-continuum approach. It is relatively convenient to employ the anti-continuum method for solving discrete nonlinear Schrödinger equation. This method has shown the ability to produce similar result of analysis of the system stability as produced by other methods. The dynamical stability of the system has been investigated within certain permissible range of parameters.

1. Introduction
The nonlinear localized modes in dipolar Bose-Einstein condensates (BEC) in optical lattices model has been investigated in [1] by mainly using modulational instability (MI) method. This paper however revisits the same problem but using anti-continuum (AC) approach. This approach has been applied in numerous discrete nonlinear Schrödinger problems for example, recently in [2,3]. Generally, AC approach starts from anti-continuous limit where coupling parameter that quantifying the tunneling between adjacent sites is initially set to zero and then followed by continuation procedure by slowly increases the value of the coupling constant. It has the advantage to be able to provide high precision solutions for small positive coupling constant while maintaining the localization of the solution. Hence the stability of the system can be analyzed concurrently during the process [4-8].

The organization of this paper is as follows: we first introduce the discrete nonlinear Schrödinger equation (DNLSE) model of dipolar BEC in optical lattice system in section 2; in section 3 we discuss the AC approach for solving the DNLS system; while in section 4, the potentially initial wave profiles are presented and the existence of the solutions are investigated; in section 5, the results and analysis for the stability of the system are demonstrated; and finally it is completed with some conclusions in section 6.

2. The model
The model of nonlinear localized modes in dipolar BEC in optical lattice is fully based on [1], where originally it is the combination of basic DNLS with nonlinear Tamm states generalization. In brief, this equation is derived from the original equation of 1D Gross-Pitaevskii equation (GPE) with
nonlocal interaction terms. This combination produces a DNLSE with additional local (contact interactions) on-site and nonlocal (long-range interaction) cubic terms as expressed below

\[ iu_n(t) = -\kappa (u_{n+1} + u_{n-1}) - q|u_n|^2 u_n - g(|u_{n+1}|^2 + |u_{n-1}|^2)u_n \]  

(1)

where \( \kappa \) is the coupling constant while \( q \) and \( g \) are the nonlinear local and nonlocal coefficients respectively. In this study, energy conservation of the system is quantified by two means; i.e. the norms, \( N \) and Hamiltonian, \( H \) respectively given by

\[ N = \sum_{n=1}^{M} |u_n|^2 \]  

(2)

and

\[ H = -\sum_{n=1}^{M} \left[ \kappa u_{n+1} u_n^* + \frac{q}{4} |u_n|^4 + \frac{g}{2} |u_{n+1}|^2 |u_n|^2 + c.c \right] \]  

(3)

where \( n \) is the number of lattice and \( M \) is the total number of the lattice.

3. The anti-continuum (AC) approach

Basically, we follow the AC approach as applied in reference [4]. Starting by transformation of the equation (1) into stationary state equation by substitution of ansatz \( u_n = A_n e^{i \omega t} \) yields

\[ \omega A_n = \kappa (A_{n+1} + A_{n-1}) + q |A_n|^2 A_n + g (|A_{n+1}|^2 + |A_{n-1}|^2) A_n \]  

(4)

where \( \omega \) denotes the frequency. The existence of solutions can be found analytically by imposing some assumptions and later it can be verified numerically for example by using Newton method. Assuming \( A_n \) real numbers, equation (4) can be rewritten as

\[ \omega A_n = \kappa (A_{n+1} + A_{n-1}) + q A_n^3 + g (A_{n+1}^2 + A_{n-1}^2) A_n \]  

(5)

Furthermore, linear stability analysis is carried out by linearization using perturbed ansatz \( u_n = (A_n + \delta E_n) e^{i \omega t} \) through \( O(\delta) \) [9], hence it could be reduced into an eigenvalue problem (EVP) of

\[ \lambda \left( \begin{array}{c} a_k \\ b_k^* \end{array} \right) = \tilde{J} \left( \begin{array}{c} a_k \\ b_k^* \end{array} \right) \]  

(6)

where \( \tilde{J} \) is a \( 2n \times 2n \) Jacobian (stability) matrix of the form

\[ \tilde{J} = \left( \begin{array}{cc} \frac{\partial F_i}{\partial u_j} & \frac{\partial F_i}{\partial u_j} \\ -\frac{\partial F_i}{\partial u_j} & -\frac{\partial F_i}{\partial u_j} \end{array} \right) \]  

(7)

where

\[ F_i = \kappa (A_{i+1} + A_{i-1}) + q A_i^2 A_i^* + g (A_{i+1} A_i^* + A_{i-1} A_i^*) A_i - \omega A_i. \]  

(8)

Note that the elements of matrix \( \tilde{J} \) only become nonzero when \( j = i, j = i - 1 \) and \( j = i + 1 \). Thus, matrix \( \tilde{J} \) could be divided into four parts of tridiagonal \( n \times n \) matrix as shown in equation (7). Here, the instability is determined by the existence of imaginary part from the eigenvalues \( \lambda \) in the equation (6), i.e. when \( \lambda^2 < 0 \), otherwise the system is linearly stable. Consequently, equation (1) can be solved
by using fourth order Runge-Kutta (RK) method where the solutions obtained from previous stability analysis are used as initial condition for the RK method.

4. Existence of solutions

The scheme for identifying strongly localized mode (SLM) to prove the existence of solutions when \( \kappa \ll 1 \) has been described in [10]. The similar analytical estimations are presented here however, for simplicity only two types of matter wave profiles are considered; on-site and inter-site profiles.

4.1. On-site profile

We consider ansatz \( A_n = A(..., \alpha_1, \alpha_2, 0, 0, ..., A, ...), \) where \( A \) could be an arbitrary number and \( \alpha_2 = 1 \). For \( \kappa \ll 1 \), we expect a very small increment of value \( \alpha_1 \) such that \( \alpha_1 \ll 1 \). By substituting this ansatz into equation (5) and after neglecting all higher order small terms, we get

\[
A \approx \frac{\omega}{\sqrt{q}}, \tag{10}
\]

\[
\alpha_1 \approx \frac{\kappa}{A^2 (q-g)}, \tag{11}
\]

and

\[
\omega \approx qA^2 + \frac{2k^2}{A^2 (q-g)} \left( 1 + \frac{g}{(q-g)} \right). \tag{12}
\]

Consequently, in order to obtain the condition of \( \alpha_1 \ll 1 \), the condition of \( |q - g| \gg \kappa \) must be satisfied where \( \kappa \ll 1 \). For AC limit case, we simply substitute \( \kappa = 0 \) into every term in \( A_n \), producing \( A_n = (...0,0,A,0,0,...) \) where \( A \) as stated before in equation (10).

4.2. Inter-site profile

Similar to the on-site profile, we consider ansatz \( A_n = B(..., 0, \beta_1, \beta_0, \beta_1, 0, 0, ..., \) where \( B \) could be an arbitrary number and \( \beta_2 = 1 \). For \( \kappa \ll 1 \), we expect a very small increment of value \( \beta_1 \) such that \( \beta_1 \ll 1 \). Again, by substituting this ansatz into equation (5) and after neglecting all higher order small terms, yields

\[
B \approx \frac{\omega}{\sqrt{q+g}}, \tag{13}
\]

\[
\beta_1 \approx \frac{\kappa}{q^2 \beta_1}, \tag{14}
\]

and

\[
\omega \approx \kappa + B^2 (q + g) + \frac{\omega^2}{q^2 \beta_1} \left( 1 + \frac{1}{q} \right). \tag{15}
\]

In order to obtain the condition of \( \beta_1 \ll 1 \), the condition of \( q \gg \kappa \) must be satisfied where \( \kappa \ll 1 \). In AC limit case, \( \kappa = 0 \) thus \( A_n = (...0,0,B,B,0,0,...) \).

5. Stability analysis of bulk solutions

In this section, numerical stability analysis of bulk localized solutions for dipolar BEC in optical lattices system by AC approach is performed. The aim is to find the effect of linear and nonlinear interactions specifically on \( \kappa \) and \( g \) with regards to different values of \( \omega \) and relate the results to [1]. Here, the number of lattices is set \( M = 100 \) and the amplitude for each site is taken from
approximation in section 4, \( q = 1 \) and take various \( \omega > 2 \), representing large norm limit \([1]\). The investigation on the stability is carried out based on factor \( g \) within three different range (i.e. \( 0 \leq g < q \), \( g \rightarrow q \) and \( g < 0 \)) and analyse for \( 0 \leq \kappa \leq 1 \).

5.1. Case 1: \( 0 \leq g < q \)
When \( g = 0 \), the original DNLSE stability behavior is obtained where the on-site solutions is stable but on the other hand inter-site solutions is unstable along the range of \( \kappa \) \([11]\) and those stability behaviours remain unchanged even with increment of \( g \). Figure 1 and 2 display respectively, the on-site and inter-site profiles for some \( \omega \) and \( g \). As \( g \) increases, the peak amplitude for on-site is slightly reduced and its first nearest-neighbour site amplitudes raise. On the other hand, for inter-site, significant reduction occurs on the peak amplitude and its first nearest-neighbour site amplitudes raise a little bit compare to the former. Therefore, the profile for on-site has a wider base as \( g \) increases while the inter-site profile does not change much. During this stage, increment of \( \omega \) does not affect the stability behaviour, but the amplitude for both fundamental solutions increase as shown in the figures.

![Figure 1. On-site profile at \( \kappa = 0.1 \) and \( q = 1 \) for different \( \omega \) (10 & 20) and \( g \) (0.2 & 0.8).](image1)

![Figure 2. Inter-site profile for same parameters as in figure 1.](image2)

5.2. Case 2: \( g \rightarrow q \)

![Figure 3. Stability analysis of the system for (a) on-site and (b) inter-site at \( q = 1 \), \( \omega = 10 \) and \( g = 0.93 \) along \( 0 \leq \kappa \leq 1 \).](image3)

![Figure 4. Stability analysis for same parameters as in figure 3 except for \( \omega = 20 \). (a) For on-site. (b) For inter-site.](image4)
As $g$ is getting closer to $q$, exist a region (as $g \approx q$) where [1] reported that on-site and inter-site solution exchange their stability property. This behavior however significantly depends on $\kappa$ as shown in Figure 3. For simplicity, let denote $\tilde{\kappa}$ is the threshold value for $\kappa$ where the stability exchanges start. Furthermore, we fix $\omega$ and increase the value of $g$, consequently $\tilde{\kappa}$ becomes smaller for both fundamental solutions, indicating wider region for larger $\kappa$ to exchange the stability behavior. Despite the fact that $\tilde{\kappa}$ becomes smaller as $g$ increase, the region for smaller $\kappa$ will not be banished at all, only the region will be too small. This is corresponding to the fact that the system will only has full stability exchange at $\kappa = 0$. On the other hand, when we increase $\omega$, there will slower reduction of $\tilde{\kappa}$ by factor $g$, giving larger regions for smaller $\kappa$ for the same value of $g$, thus further increment of $\omega$, could maintain the stability behaviour as in case 1. Figure 3 and 4 demonstrate the stability behavior for different value of $\omega$ and as an example, it can be estimated from figure 3(a) $\tilde{\kappa} \approx 0.41$.

Interestingly, the threshold value for on-site, denoted as $\tilde{\kappa}_{on}$ always smaller than for the inter-site, say $\tilde{\kappa}_{\text{inter}}$ for the same fixed value of $\omega$ and $g$, indicating that the on-site will tend to change their stability behaviour faster than the inter-site as $\kappa$ increase. This situation let us to define a particular region $\Delta \tilde{\kappa}$ such that it includes all $\kappa$ in the range of $\tilde{\kappa}_{on} < \kappa < \tilde{\kappa}_{\text{inter}}$, in which both fundamental solutions simultaneously unstable which is unlikely to occur in the original DNLSE when $g = 0$ [11]. This is a special situation where exist symmetric solutions that are being stable along the range of $\Delta \tilde{\kappa}$ which is also known as intermediate solutions (IS). This phenomenon is well discussed in [1] and been observed in other systems [12-14]. Figure 5 and 6 depict the simultaneous instability (i.e. $\kappa \in \Delta \tilde{\kappa}$) for both on-site and inter-site solutions while figure 7 and 8 show a case for $\kappa \notin \Delta \tilde{\kappa}$. Further, if we fix $\omega$ and increase $g$, the $\Delta \tilde{\kappa}$ will shift towards smaller $\kappa$. Conversely, if we increase $\omega$, the $\Delta \tilde{\kappa}$ will shifts towards larger $\kappa$. This coincides with the effect of $g$ and $\omega$ on $\tilde{\kappa}$ (including $\tilde{\kappa}_{on}$ and $\tilde{\kappa}_{\text{inter}}$) as discussed previously. Additionally, the range of $\Delta \tilde{\kappa}$ changes with the change of value $g$, without specific patterns, nevertheless the increment of $\omega$ increases the size of $\Delta \tilde{\kappa}$.
Figure 7. The transient solution at $\kappa = 0.4$ for on-site (stable) by same parameters as in figure 4.

Figure 8. Similar description as for figure 7 except this figure for inter-site (unstable).

5.3. Case 3: $g < 0$

Figure 9. Stability analysis for on-site (stable), at $\omega = 50$ and $g = -0.2$ along $0 \leq \kappa \leq 1$.

Figure 10. Stability analysis for inter-site (unstable), by same parameters as in figure 9.

For this case, the stability behavior is very similar to case 1, in the sense that the on-site always stable while inter-site always unstable for $0 \leq \kappa \leq 1$ as illustrated in figure 9 and 10. The fundamental solutions profile for this case however different from case 1 by factor $g$ where the peak for inter-site profile has a faster step-up than on-site profile and the nearest-neighbor sites for on-site profile degrade more than inter-site profile as $g$ decreases. However, the profile patterns are very similar to case 1 when factor $\omega$ is considered. In other word, either increment in $\omega$ or reduction in $g$, improve localization for both fundamental solutions as illustrated in figure 11 and 12.

Figure 11. On-site profile at $\kappa = 0.1$ and $q = 1$ for different $\omega$ (10 & 20) and $g$ (-0.2 & -0.4).

Figure 12. Inter-site profile for same parameter as in figure 11.
6. Conclusion
In this work, we have analyzed the stability of DNLSE model of dipolar BECs in optical lattice system for two fundamental symmetric solutions i.e. on-site and inter-site solutions by using the AC approach. The results from analysis are comparable with the analysis of the same model by different approach, but from different views. In addition, at the small coupling constant, $\kappa$ limit ($\kappa \ll 1$), we have checked the existence for both fundamental solutions analytically. Then, the solutions are taken as initial modes profile for the AC approach, and furthermore affirmed the existence numerically. Next, the linear stability of the system was numerically studied. Essentially, we divided the works into three cases concerning difference range of $g$, i.e. $0 \leq g < q$, $g \rightarrow q$ and $g < 0$. Here, emphasize is given to the effect of $\kappa$ on the stability of the system by considering the interactions towards different value of $g$ and $\omega$. For case $g \rightarrow q$, an interesting phenomenon occurred where on-site and inter-site solutions were simultaneously unstable.

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