On Hilbert’s 8th Problem

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Abstract

A Hadamard factorisation of the Riemann ξ-function is constructed to characterize the zeros of the zeta function.

Keywords: RH, GGC, Zeta function, Xi function

1 Introduction

Riemann [4] defines the zeta function, ζ(s), as the analytic continuation of \( \sum_{n=1}^{\infty} n^{-s} \) on \( \text{Re}(s) > 1 \) and the ξ-function by

\[
\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).
\]  

(1)

The Riemann Hypothesis (RH) states that all the non-trivial zeroes of ζ(s) lie on the critical line \( \text{Re}(s) = \frac{1}{2} \), or equivalently, those of ξ(\( \frac{1}{2} + is \)) = ξ(\( \frac{1}{2} - is \)) lie on the real axis.

The ξ-function is an entire function of order one and hence admits a Hadamard factorisation. Titchmarsh ([6], 2.12.5) shows Hadamard’s factorization theorem gives, for all values of \( s \), with \( b_0 = \frac{1}{2} \log(4\pi) - 1 - \frac{1}{2}\gamma \) and \( \xi(0) = -\zeta(0) = \frac{1}{2} \), such that

\[
\xi(s) = \xi(0)e^{b_0s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}
\]  

(2)

The zeros, \( \rho \), of \( \xi \) correspond to the non-trivial zeros of \( \zeta \).

The argument to prove RH proceeds in three steps.

(a) RH is equivalent to the existence of a generalised gamma convolution (GGC) random variable (Bondesson [1, p.124], Roynette and Yor [5]), denoted by \( H^{\xi}_{\frac{1}{2}} \), whose

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Laplace transform expresses the reciprocal $\xi$-function as

$$\frac{\xi\left(\frac{1}{2}\right)}{\xi\left(\frac{1}{2} + \sqrt{s}\right)} = E(\exp(-sH_{\xi}^{\frac{1}{2}})).$$ \hfill (3)

(b) Theorem 1 constructs a GGC random variable $H_{\alpha}^{\xi}$ for $\alpha > 1$, such that

$$\frac{\xi(\alpha)}{\xi(\alpha + \sqrt{s})} e^{-b_\alpha \sqrt{s}} = E(\exp(-sH_{\alpha}^{\xi})), \ s > 0 \ \text{where} \ b_\alpha = \frac{\xi' (\alpha)}{\xi (\alpha)} + \frac{1}{\alpha - 1}. \hfill (4)$$

(c) Theorem 2 then extends (b) to the case $\alpha = \frac{1}{2}$.

There is an intermediate Lemma 1 which helps identify the GGC random variable $H_{\alpha}^{\xi}$. Propositions 1 and 2 provide similar GGC representations for the zeta function and $(\alpha - 1)/((\alpha - 1) + s)$, respectively.

To show (a), first assume that RH is true. Then the zeroes of $\xi$ are of the form $\rho = \frac{1}{2} \pm i\tau$ as $\xi(s) = \xi(1 - s)$. The Hadamard factorisation then becomes

$$\xi(s) = \xi(0) \prod_{\tau > 0} \left(1 - \frac{s}{\frac{1}{2} + i\tau}\right) \left(1 - \frac{s}{\frac{1}{2} - i\tau}\right) \hfill (5)$$

and $\xi\left(\frac{1}{2}\right) = \xi(0) \prod_{\tau > 0} \tau^{2}/\left(\frac{1}{4} + \tau^{2}\right)$ as

$$\xi(s) = \xi(0) \prod_{\tau > 0} \frac{(s - \frac{1}{2})^{2} + \tau^{2}}{\frac{1}{4} + \tau^{2}}. \hfill (6)$$

From (6), the reciprocal $\xi$-function then satisfies

$$\frac{\xi\left(\frac{1}{2}\right)}{\xi\left(\frac{1}{2} + s\right)} = \prod_{\tau > 0} \frac{\tau^{2}}{\tau^{2} + s^{2}}. \hfill (7)$$

Using Frullani’s identity, $\log(z/z + s^{2}) = \int_{0}^{\infty} (1 - e^{-s^{2}t})e^{-tz} dt/t$, write

$$\frac{\xi\left(\frac{1}{2}\right)}{\xi\left(\frac{1}{2} + s\right)} = \prod_{\tau > 0} \frac{\tau^{2}}{\tau^{2} + s^{2}} = \exp\left\{ \int_{0}^{\infty} \log\left(\frac{z}{z + s^{2}}\right) U(dz) \right\}$$

$$= \exp\left(-\int_{0}^{\infty} (1 - e^{-s^{2}t})g_{\frac{1}{2}}(t) \frac{dt}{t}\right) \hfill (8)$$

$$= E(\exp(-s^{2}H_{\xi}^{\frac{1}{2}})). \hfill (9)$$

Here $g_{\frac{1}{2}}(t) = \int_{0}^{\infty} e^{-t\tau} U_{\frac{1}{2}}(d\tau)$ and $U_{\frac{1}{2}}(d\tau) = \sum_{\tau > 0} \delta_{\tau^{2}}(d\tau)$ is the Thorin measure where $\delta$ is a Dirac measure.
The GGC random variable $H^\xi D \overset{D}{=} \sum_{\tau > 0} Y_\tau$ where $Y_\tau \sim \text{Exp}(\tau^2)$ satisfies

$$\prod_{\tau > 0} \frac{\tau^2}{\tau^2 + s^2} = E \left( \exp(-s^2 H^\xi \frac{1}{2}) \right).$$  \hspace{1cm} (11)

Conversely, if $\xi(\frac{1}{2}) / \xi(\frac{1}{2} + \sqrt{s}) = E(\exp(-sH^\xi \frac{1}{2}))$ then $\xi(\frac{1}{2} + s)$ has no zeros. Then $\xi(s)$ has no zeros for $Re(s) > \frac{1}{2}$ and $\xi(s) = \xi(1 - s)$, implies no zeroes for $Re(s) < \frac{1}{2}$ either.

2 Reciprocal $\xi$-function and GGC representation

To show (b), consider the following theorem

**Theorem 1.** The reciprocal $\xi$-function satisfies for $\alpha > 1$ and $s > 0$,

$$\frac{\xi(\alpha)}{\xi(\alpha + s)} e^{-b_\alpha s} = \exp \left( - \int_0^\infty (1 - e^{-\frac{1}{2} s^2 t}) g^\xi_\alpha(t) \frac{dt}{t} \right)$$  \hspace{1cm} (12)

where

$$b_\alpha = -\frac{\xi'(\alpha)}{\xi(\alpha)} + \frac{1}{\alpha - 1}$$  \hspace{1cm} (13)

and $g^\xi_\alpha$ is completely monotone

$$g^\xi_\alpha(t) = \int_0^\infty e^{-tz} U^\xi_\alpha(dz).$$  \hspace{1cm} (14)

**Proof.** To derive $\xi(\alpha)/\xi(\alpha + s)$, use the definitions,

$$\xi(\alpha) = (\alpha - 1)\pi^{-\frac{1}{2}\alpha} \Gamma(1 + \frac{1}{2}\alpha) \zeta(\alpha)$$  \hspace{1cm} (15)

$$\xi(\alpha + s) = (\alpha - 1 + s)\pi^{-\frac{1}{2}(\alpha + s)} \Gamma(1 + \frac{1}{2}(\alpha + s)) \zeta(\alpha + s).$$  \hspace{1cm} (16)

Now, $\xi(0) = \xi(1) = \frac{1}{2}$ and $\xi(\frac{1}{2}) \neq 0$ and $\xi(s) = \xi(1 - s)$, so $\xi'(\frac{1}{2}) = 0$ and $\xi'(\frac{1}{2})/\xi(\frac{1}{2}) = 0$.

Taking derivatives at $s = \alpha$ of

$$\log \xi(s) = \log(s - 1) - \frac{1}{2} s \log \pi + \log \Gamma(1 + \frac{1}{2}s) + \log \zeta(s)$$  \hspace{1cm} (17)

with $\psi(s) = \Gamma'(s)/\Gamma(s)$, gives

$$\frac{\xi'(\alpha)}{\xi(\alpha)} = \frac{1}{\alpha - 1} - \frac{1}{2} \log \pi + \frac{\zeta'(\alpha)}{\zeta(\alpha)} + \frac{1}{2} \psi(1 + \frac{1}{2}\alpha).$$  \hspace{1cm} (18)

In particular,

$$0 = \frac{\xi'(\frac{1}{2})}{\xi(\frac{1}{2})} = -2 - \frac{1}{2} \log \pi + \frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} + \frac{1}{2} \psi(\frac{1}{2}).$$  \hspace{1cm} (19)
Theorem 1 now follows from the decomposition, for $\alpha > 1$,

$$\frac{\xi(\alpha)}{\xi(\alpha + s)} \exp \left\{ s \left( \frac{\xi'(\alpha)}{\xi(\alpha)} - \frac{1}{\alpha - 1} \right) \right\} = \frac{(\alpha - 1)}{s + (\alpha - 1)} \cdot \frac{\Gamma(1 + \frac{1}{2}\alpha)}{\Gamma(1 + \frac{1}{2}(\alpha + s))} \cdot \frac{\exp \left\{ s \left( \frac{\xi'(\alpha)}{\xi(\alpha)} - 1 \right) \right\}}{\xi(\alpha + s)}$$

(20)

where

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}} \right) dt.$$  

(21)

Each term on the r.h.s. of (20) will be treated separately.

Euler’s product formula, for $\alpha > 1$, gives

$$\zeta(\alpha + s) = \prod_{p \text{ prime}} (1 - p^{-\alpha - s})^{-1} = \prod_{p \text{ prime}} \zeta_p(\alpha + s) \text{ where } \zeta_p(s) := \frac{p^s}{p^s - 1}$$

(22)

and $\zeta(\alpha) = \prod_p \zeta_p(\alpha)$, yields

$$\log \frac{\zeta(\alpha + s)}{\zeta(\alpha)} = \sum_p \log \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - s}} = \sum_{r=1}^\infty \frac{1}{r} p^{-\alpha r} (e^{-sr \log p} - 1)$$

(23)

$$= \int_0^\infty (e^{-sx} - 1)e^{-\alpha x} \frac{\mu(dx)}{x} \text{ where } \mu(dx) = \sum_{r=1}^\infty \sum_p \left( \log p \right) \delta_{e^{-sr \log p}}(dx).$$

(24)

The following lemma identifies the Thorin measure of a GGC distribution.

**Lemma 1.** Suppose that the function $f$ satisfies

$$\log \left( \frac{f(\alpha + s)}{f(\alpha)} \right) - s f'(\alpha) = \int_0^\infty (e^{-sx} - 1 + sx)e^{-\alpha x} \frac{\mu(dx)}{x}$$

(25)

for some arbitrary $\mu(dx)$. This can then be re-expressed as

$$\log \left( \frac{f(\alpha)}{f(\alpha + s)} \right) + s f'(\alpha) = - \int_0^\infty (1 - e^{-\frac{1}{2}x^2 t}) \frac{\nu_\alpha(t)}{t} dt$$

(26)

where $\nu_\alpha(t)$ is the completely monotone function

$$\nu_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tz} \left( \int_0^\infty 2 \sin^2(x \sqrt{z/2}) e^{-\alpha x} \mu(dx) \right) \frac{dz}{\sqrt{\pi z}}.$$  

(27)

This follows from the identities, valid for $s > 0$ and $x > 0$,

$$e^{-sx} + sx - 1 = \int_0^\infty (1 - e^{-\frac{1}{2}x^2 t})(1 - e^{-\frac{1}{2}x^2 / t}) \frac{x}{\sqrt{2\pi t^3}} dt,$$

(28)

$$1 - e^{-\frac{1}{2}x^2 / t} = \int_0^\infty e^{-tz} 2 \sin^2(x \sqrt{z/2}) \frac{dz}{\sqrt{\pi z}}.$$  

(29)
In particular, when \( f \) is the gamma or zeta function, this yields

1. When \( f(s) = \Gamma(1 + \frac{1}{2}s) \) then

\[
\mu^\Gamma(dx) = \frac{dx}{e^{2x} - 1}. \tag{30}
\]

2. When \( f(s) = \zeta_p(s) \) then

\[
\mu^\zeta(dx) = \sum_{\kappa \geq 1} (\log p) \delta_{\kappa \log p}(dx). \tag{31}
\]

Using (31) and Lemma 1, the \( \zeta \)-function satisfies

**Proposition 1.** For \( \alpha > 1 \) and \( s > 0 \),

\[
\frac{\zeta(\alpha + s)}{\zeta(\alpha)} e^{-s \zeta'(\alpha)} = \exp \left( \int_0^\infty (e^{-sx} + sx - 1)e^{-\alpha x} \mu^\zeta(dx) \right) \tag{32}
\]

with

\[
\frac{\mu^\zeta(dx)}{x} = \sum_p \frac{\mu_p^\zeta(dx)}{x} = \sum_{n \geq 2} \frac{\Lambda(n)}{\log n} \delta_{\log n}(dx) \tag{33}
\]

where \( \Lambda(n) \) is the von Mangoldt function, \( \Lambda(n) = \log p \) if \( n = p^r \) where \( p \) is a prime. The reciprocal zeta function then satisfies, for \( \alpha > 1 \),

\[
\frac{\zeta(\alpha)}{\zeta(\alpha) + s} e^{s \zeta'(\alpha)} = \exp \left( - \int_0^\infty (1 - e^{-\frac{1}{2}t^2}) \nu^\zeta_\alpha(t) dt \right) \quad \text{where} \quad \nu^\zeta_\alpha(t) = \sum_{p \ \text{prime}} \nu^\zeta_{p,\alpha}(t) \tag{34}
\]

with \( \int_0^\infty \nu^\zeta_\alpha(t) dt < \infty \), and

\[
\nu^\zeta_\alpha(t) = \frac{1}{\sqrt{2\pi t}} \sum_{n \geq 2} \frac{\Lambda(n)}{n^\alpha} \left( 1 - e^{-(\log n)^2/2t} \right) \tag{35}
\]

\[
= \frac{1}{\sqrt{2\pi t}} \sum_{p \ \text{prime}} \log p \left\{ \sum_{r \geq 1} \frac{1}{r^\alpha} \left( 1 - e^{-(r^2 \log p)^2/2t} \right) \right\}. \tag{36}
\]

The term \( (\alpha - 1)/(s + (\alpha - 1)) \) in the product (20) follows from

**Proposition 2.** Let \( a = \alpha - 1 > 0 \), then \( a/(s + a) \) has representation

\[
\frac{a}{s + a} = \int_0^\infty e^{-sx} ae^{-ax} dx = \int_0^\infty e^{-\frac{1}{2}t^2} \left\{ \int_0^\infty x e^{-\frac{x^2}{2t^3}} ae^{-ax} dx \right\} dt \tag{37}
\]

\[
= \exp \left\{ - \int_0^\infty (1 - e^{-\frac{1}{2}t^2}) \frac{\nu^\zeta_\alpha(t)}{t} dt \right\} \tag{38}
\]
with completely monotone function

\[ \nu^{(0)}_\alpha(t) = \frac{1}{2} e^{\frac{1}{2} a^2 t} \text{erfc}(a \sqrt{\frac{1}{2}} t) = E(e^{-tZ^{(0)}_\alpha}) \]  

(39)

where \( Z^{(0)}_\alpha \) has density \( 2a/\pi \sqrt{2x(a^2 + 2x)} \) for \( x > 0 \).

The Hadamard factorization (20), for \( \alpha > 1 \), is then

\[ \frac{\xi(\alpha)}{\xi(\alpha + s)} = e^{b_\alpha s} \exp \left( - \int_0^\infty (1 - e^{-\frac{1}{2}s^2 t}) \frac{g^{(\xi)}_\alpha(t)}{t} dt \right) \]  

(40)

with \( b_\alpha = -\frac{e^{\xi}}{\xi}(\alpha) + \frac{1}{\alpha - 1} \) and completely monotone function

\[ g^{(\xi)}_\alpha(t) = \nu^{(0)}_\alpha(t) + \nu^{(\xi)}_\alpha(t) + \nu^{(\xi)}_\alpha(t). \]  

(41)

This defines the Thorin measure, \( U^{(\xi)}_{\alpha} \), of \( H^{(\xi)}_{\alpha} \) via \( g^{(\xi)}_\alpha(t) = \int_0^\infty e^{-tZ} U^{(\xi)}_{\alpha}(dz) \).

Equivalently, there exists \( H^{(\xi)}_{\alpha} \) with GGC density, such that, for \( \alpha > 1 \),

\[ \frac{\xi(\alpha)}{\xi(\alpha + \sqrt{s})} e^{-b_\alpha \sqrt{s}} = E(\exp(-sH^{(\xi)}_{\alpha})), \quad s > 0. \]  

(42)

This completes the derivation of Theorem 1.

Corollary 1. Riemann’s \( \xi \)-function admits a Hadamard factorization, for \( \alpha > 1 \) and \( s > 0 \),

\[ \frac{\xi(\alpha + s)}{\xi(\alpha)} = e^{-b_\alpha s} \prod_{\tau_\alpha} \left(1 + \frac{s^2}{\tau_\alpha^2}\right). \]  

(43)

Proof. This follows by inverting Theorem 1, and expressing \( H^{(\xi)}_{\alpha} \), as a generalized convolutions of mixtures of exponentials, namely \( H^{(\xi)}_{\alpha} \overset{D}{=} \sum_{\tau_\alpha > 0} Y_{\tau_\alpha} \) where \( Y_{\tau_\alpha} \sim \text{Exp}(\tau^2_\alpha) \).

Pólya’s \( X_\xi \) has symmetric density with tails \( p(x) \sim 4\pi^2 e^{-\pi^2 - \pi e^{2x}} \) as \( x \to \infty \),

\[ p(x) = \frac{1}{\xi(\frac{3}{2})} \sum_{n=1}^\infty p_n(x) \quad \text{and} \quad p_n(x) := 2n^2 \pi (2n^2 e^{-2x} - 3) e^{-\frac{5}{2}x - n^2 \pi e^{-2x}}. \]  

(44)

Hayman-Grosswald provide a bound,

\[ \log \xi(\frac{1}{2} + k) = \frac{1}{4} k \log \left( \frac{k}{2\pi e} \right) + \frac{1}{4} k \log \left( \frac{k}{2\pi} \right) + \frac{1}{2} \log(\frac{1}{4} \pi) + o(1) \text{ as } k \to \infty. \]  

(45)

To extend (42) to \( \xi(\frac{1}{2})/\xi(\frac{1}{2} + \sqrt{s}) \) for \( s > 0 \) where \( b_{\frac{1}{2}} = 0 \), we Now calculate the reciprocal \( \xi \)-function using \( m_G(s) \) (see Grosswald [2] for existence), defined by

\[ G(x) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{\Gamma(k)} \xi(\frac{1}{2} + k) x^k \quad \text{and} \quad m_G(s) = \int_0^\infty x^{s-1} G(x) dx. \]  

(46)
Let $Y_\xi := e^{-X_\xi}$ be such that
\[
\frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{3}{2})} = E(e^{-sX_\xi}) = E(Y_\xi^s).
\] (47)

Using (42) with $1 < \alpha \leq \frac{3}{2}$ and $c_\alpha = \frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{1}{2} + \frac{\alpha}{2})}/\xi(\alpha)$, yields
\[
\frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{3}{2} + k)} = \frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{1}{2} + \alpha + \frac{1}{2})} = c_\alpha E\left(e^{bk-(k-\alpha+\frac{1}{2})^2H}\right) \text{ for } k = 1, 2, \ldots
\] (48)

where $H = H_\xi$ and $b = b_\alpha$.

Calculate $m_G(s)$ using transform $z = xy$ via
\[
\frac{\xi(s + \frac{1}{2} - s)}{\xi(s + \frac{1}{2})} m_G(s) = E(Y_\xi^s) \int_0^\infty x^{s-1} G(x) dx \tag{49}
\]
\[
= \int_0^\infty \int_0^\infty y^s q(y) x^{s-1} G(x) dxdy \tag{50}
\]
\[
= \int_0^\infty \int_0^\infty q(y) z^{s-1} G(z/y) dzdy \tag{51}
\]
\[
= \int_0^\infty z^{s-1} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k)} \xi(s + \frac{1}{2} + k) \frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{1}{2} + \alpha + \frac{1}{2})} \int_0^\infty y^{-k} q(y) dy z^k \right\} dz \tag{52}
\]
\[
= \int_0^\infty z^{s-1} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k)} \xi(s + \frac{1}{2} + k) z^k \right\} dz \tag{53}
\]
\[
= \Gamma(1 + s)\xi(s + \frac{1}{2} - s) \tag{54}
\]
using Ramanujan’s master theorem. Fubini follows from the tail of Pólya’s distribution.

Calculate $m_G(s) = \int_0^\infty x^{s-1} G(x) dx$, with $\alpha = \frac{3}{2}$ in (48), and $\xi(s + \frac{1}{2} + k)/\xi(s + \frac{1}{2}) = e^{-bk} E(e^{kX})$ from (40), with $c = \xi(s + \frac{1}{2}) e^{-b}$, to give
\[
G(x) = cE \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k)} e^{bX-(k-\alpha)^2H} x^k \right\}. \tag{55}
\]

Ramanujan’s master theorem now yields
\[
m_G(s) = c\Gamma(1 + s) E\left(e^{-sX-(s+1)^2H}\right). \tag{56}
\]

From (54) and (56), for $0 < s < \frac{1}{4}$
\[
\frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{1}{2} + \frac{3}{2})} = \frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{1}{2} - \sqrt{s})} \frac{1}{\xi(s + \frac{1}{2} - \sqrt{s})} = cE\left(e^{-\sqrt{s}X-(\sqrt{s}+1)^2H}\right) \frac{1}{\xi(s + \frac{1}{2} - \sqrt{s})}. \tag{57}
\]

The term $1/\xi(s + \frac{1}{2} - \sqrt{s})$ follows from the Hadamard factorization in (??).

The following theorem now finishes the derivation of (c).
Theorem 2. The reciprocal $\xi$-function satisfies a HCM condition, where $H_2^\xi$ is GGC,
\[
\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2} + \sqrt{s})} = E(\exp(-sH_2^\xi)).
\] (58)

Proof. The GGC property is preserved under exponential tilting. Then (57) is the LT of a GGC from the composition theorem for GGCs (Theorem 3.3.1 in [1, p.41]) as are $\phi_H(\sum_{k=1}^{N} c_k s^{a_k})$ when $c_k > 0$ and $0 < a_k \leq 1$ (Theorem 3.3.2 [1, p.41]).

As $1/\xi(\frac{1}{2} + \sqrt{s}) = E(\exp(- (\sqrt{s} + 1)^2 H_2^\xi))$, the random variable $H_2^\xi = H_2^\xi + S_H^\xi$ where $S_t$ is a stable subordinator of index $\frac{1}{2}$. The random variable $X_2^\xi$ is related to $X_2^\xi$ via (??).

Finally, the Laplace transform, $E(\exp(-sH_2^\xi))$, of a GGC distribution, is analytic in the whole complex plane cut along the negative real axis, and, in particular, it cannot have any singularities in that cut plane.

As $\xi(\frac{1}{2}) / \xi(\frac{1}{2} + \sqrt{s}) = E(\exp(-sH_2^\xi))$ for $0 < s < \frac{1}{2}$, then, by analytic continuation, this equality must hold for all values of $s$ in the cut plane, namely $\mathbb{C} \setminus (-\infty, 0)$. Hence, the denominator, $\xi(\frac{1}{2} + \sqrt{s})$ cannot have any zeros there, and $\xi(\frac{1}{2} + s)$ has no zeros for $Re(s) > 0$. Then $\xi(s)$ has no zeroes for Re(s) > $\frac{1}{2}$ and, as $\xi(s) = \xi(1-s)$, no zeroes for Re(s) < $\frac{1}{2}$ either.

Hence all non-trivial zeros of the $\zeta$ function lie on the critical line $\frac{1}{2} + is$.

3 References

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