Ground State and Quasiparticle Spectrum of a Two Component Bose-Einstein Condensate

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Abstract

We consider a dilute homogenous atomic Bose-Einstein condensate with two non-degenerate internal energy levels. We discuss the case in which the two components achieve a state of chemical equilibrium in the presence of an external radiation field which couples the two states. The presence of the radiation field can result in new ground states for the condensate as a consequence of the lowering of the condensate energy due to the interaction energy with the field. We analyze the ground state energy as a function of the coupling constants for the two-body interactions, the Rabi frequency of the radiation field, and the detuning of the field. We also give explicit expressions for the quasiparticle excitation spectrum.

I. INTRODUCTION

The recent experimental realization of trapped Bose-Einstein condensates (BEC) with internal degrees of freedom [1] has sparked much theoretical and experimental study of the properties of multicomponent condensates. Of fundamental importance is the structure of the ground state and the energy spectrum of collective excitations above the condensate ground state. This has been explored extensively for the case of spinor condensates [2] in which the internal degrees of freedom correspond to the different Zeeman states of a particular hyperfine manifold such as the $F = 1$ manifold in $^{23}Na$. The two-body interaction Hamiltonian for a spinor condensate is invariant under rotations in spin space, since for two-body collisions, the s-wave scattering length can depend only on the total spin angular momentum of the two atoms owing to the rotational symmetry of the collision. In contrast, for two component condensates, such as $^{87}Rb$, in which the two states correspond to non-degenerate internal states of the atoms (i.e. states which differ in either their principal quantum number or total angular momentum quantum numbers), the two-body interaction is not symmetric with respect to pseudo-spin $SU(2)$ transformations since the s-wave scatter-
ing length depends on the internal eigenstates of the two atoms. If an external field couples the two internal states, one would expect effects that are not present in spinor condensates.

The ground state and energy spectrum of the quasiparticle excitations of a two component homogenous condensate in the presence of a coupling field has been calculated in Ref. [3]. The coupling field allows atoms to make transitions between the two internal states. In that article, as well as in the calculations to be presented below, it is assumed that the two components of the condensate are in a state of chemical equilibrium (i.e. the chemical potential for the two components are equal [4]). This condition is equivalent to assuming that the total number of atoms in the condensate is fixed rather than the number of atoms in each of the two components. For nonzero temperature, the relative number of atoms in the two components is determined by the condition that the atoms are in thermal equilibrium [5].

However, for zero temperature (which is the case considered in Ref. [3]), the chemical potential may be identified with the energy of a single atom in the condensate. In this case, chemical equilibrium corresponds to the condensate being in a stationary state of the system. The stationary states of a free atom interacting with an external radiation field are known as dressed states in quantum optics [6]. Consequently, chemical equilibrium for a two-component condensate corresponds to the direct generalization of the atomic dressed states [7].

Under the condition of chemical equilibrium, Goldstein and Meystre [3] found that the quasiparticle spectrum could contain states having imaginary energies, indicating the onset of instabilities in the condensate. This is a rather surprising and unexpected result; the energy spectrum for quasiparticle excitations above the ground state of an interacting many body system are expected to be real. Given these results, we have calculated the ground state and quasiparticle spectrum for the model of a two component condensate discussed in [3]. We obtain an expression for the ground state energy density as a function of the relative concentration of the two components of the condensate, given a fixed total number of atoms in the condensate. For a coupling field of finite strength, the energy density can exhibit
both maxima and minima as a function of the number of atoms in one of the components. The minima correspond to points of stable equilibrium. As one might expect, the excitation spectra calculated around these new minima are real for all momenta. For values of the relative concentration other than those corresponding to the minima, in the presence of relaxation, the system will be driven to a point of stable equilibrium. The instabilities found in Ref. [3] can be traced to an expansion about a point of unstable equilibrium.

In section II, the Hamiltonian for a two component condensate, expressed in a form which emphasizes the broken $SU(2)$ symmetry, is used to study the ground state energy as a function of the relative fraction of atoms in each of the two states. In section III, the quasiparticle spectrum is derived.

### II. HAMILTONIAN AND GROUND STATE ENERGY

The Hamiltonian operator for a two component BEC may be written as

$$\hat{H} = \hat{H}_1 + \hat{H}_2$$

where $\hat{H}_1$ is the single body Hamiltonian given by

$$\hat{H}_1 = \int d^3r \left\{ \hat{\Psi}_a^\dagger(r) \left[ -\frac{\hbar^2\nabla^2}{2m} + V_a(r) + \frac{\hbar\delta}{2} \right] \hat{\Psi}_a(r) + \hat{\Psi}_b^\dagger(r) \left[ -\frac{\hbar^2\nabla^2}{2m} + V_b(r) - \frac{\hbar\delta}{2} \right] \hat{\Psi}_b(r) \right. + \\
\left. \frac{\hbar R}{2} \left[ \hat{\Psi}_a^\dagger(r)\hat{\Psi}_b(r) + \hat{\Psi}_b^\dagger(r)\hat{\Psi}_a(r) \right] \right\}$$

and $\hat{H}_2$ is the two-body interaction,

$$\hat{H}_2 = \frac{1}{2} \int d^3rd^3r' \left\{ \hat{\Psi}_a^\dagger(r)\hat{\Psi}_a^\dagger(r')U_a(r - r')\hat{\Psi}_a(r) + \hat{\Psi}_b^\dagger(r)\hat{\Psi}_b^\dagger(r')U_b(r - r')\hat{\Psi}_b(r') + \\
2\hat{\Psi}_a^\dagger(r)\hat{\Psi}_b^\dagger(r')U_a(r - r')\hat{\Psi}_b(r')\hat{\Psi}_a(r) \right\}.$$
written in a field interaction representation which is rotating at the frequency of the external field, \( \omega_e \). Consequently, there appears in Eq. (2) the detuning, \( \delta = \omega_o - \omega_e \), where \( \hbar \omega_o \) is the energy difference between the two internal states. In Eq. (2), the \( V_i(r) \) are external potentials which we take to be zero since we wish to consider only the case of a homogenous condensate.

For the two-body interaction, we use a contact potential, \( U_i(r - r') = V_i \delta(r - r') \). The coupling constants, \( V_i \), are expressed in terms of the s-wave scattering lengths by

\[
V_i = \frac{4\pi \hbar^2 a_i}{m}
\]

where \( a_a \) and \( a_b \) are the scattering lengths for collisions between two atoms in states \( a \) and \( b \), respectively while \( a_x \) is the scattering length for collisions between atoms in different internal states. In accordance with [3], \( \hat{H}_2 \) can be simplified by assuming that \( V_a \approx V_b = V_s \neq V_x \).

It is possible to rewrite \( \hat{H}_2 \) in a simplified form that emphasizes the lack of \( SU(2) \) symmetry. To do this one expresses the field operators as two-component spinors,

\[
\Psi(r) = \begin{pmatrix} \Psi_a(r) \\ \Psi_b(r) \end{pmatrix}.
\]

The two-body Hamiltonian may then be expressed as

\[
\hat{H}_2 = \frac{1}{2} \int d^3 r d^3 r' \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r') V_I(r, r') \hat{\Psi}(r) \hat{\Psi}(r');
\]

\[
V_I(r, r') = \frac{1}{2} (V_s + V_x) I(r) I(r') \delta(r - r') + \frac{1}{2} (V_s - V_x) \sigma_z(r) \sigma_z(r');
\]

where \( I(r) \) is the identity matrix and \( \sigma_z(r) \) is the Pauli spin matrix which act on the spinor at \( r \). The single body Hamiltonian may also be expressed in the compact form,

\[
\hat{H}_1 = \int d^3 r \hat{\Psi}^\dagger(r) \left\{ -\frac{\hbar^2 \nabla^2}{2m} + \frac{\hbar \delta}{2} \sigma_z + \frac{\hbar R}{2} \sigma_x \right\} \hat{\Psi}(r).
\]

The previously mentioned lack of \( SU(2) \) symmetry in \( \hat{H}_2 \) is now obvious. If we carry out an active rotation that diagonalizes \( \hat{H}_1 \), \( \hat{H}_2 \) will not be invariant since the interactions are proportional to \( \sigma_z(r) \sigma_z(r') \) instead of \( \sigma(r) \cdot \sigma(r') \).

One can now write a c-number function for the ground state energy. This is accomplished by rewriting the field operators as \( \Psi_i(r) = \phi_i(r) + \delta \hat{\Psi}_i(r) \) where \( \phi_i(r) \) is the condensate wave
function which is defined as $\phi_i(r) \equiv \langle \hat{\Psi}_i(r) \rangle = \sqrt{\rho_i} e^{iS_i}$ and the expectation value is taken with respect to the condensate state $|\Psi\rangle$. Here $\rho(r) = \rho_a(r) + \rho_b(r)$ is the total density which is conserved and $S_{ab} = S_a - S_b$ is the relative phase between the two components. By making the replacement $\hat{\Psi}_i(r) \rightarrow \phi_i(r)$ in Eq. (1), one obtains an energy functional for the condensate. For a homogenous stationary system, Bose-Einstein condensation occurs in the state with zero momentum and as such, the $\phi_i(r)$ are independent of $r$. The energy density for the homogenous condensate with volume $V$ is,

$$
\frac{E_o}{V} = \frac{\hbar\delta}{2} (\rho_a - \rho_b) + \hbar R \sqrt{\rho_a \rho_b} \cos(S_{ab}) + \frac{1}{4} (V_s - V_x)(\rho_a - \rho_b)^2 + \frac{1}{4} (V_s + V_x)\rho^2
$$

(8a)

$$
= \frac{\hbar\delta}{2} (2\rho_a - \rho) + \hbar R \sqrt{\rho_a(\rho - \rho_a)} \cos(S_{ab}) + \frac{1}{4} (V_s - V_x)(2\rho_a - \rho)^2 + \frac{1}{4} (V_s + V_x)\rho^2
$$

(8b)

Notice that Eq. (8a) may also be derived from $\langle \hat{H} \rangle = E_o - \frac{1}{2\rho} (V_s (\rho_a^2 + \rho_b^2) + 2V_x \rho_a \rho_b)$ where the expectation value is taken with respect to the wave function

$$
|C\rangle = \frac{1}{\sqrt{N_o!}} \left( \sqrt{\frac{N_{ao}}{N_o}} e^{iS_a} \hat{a}_{a0}^\dagger + \sqrt{\frac{N_{bo}}{N_o}} e^{iS_b} \hat{a}_{b0}^\dagger \right)^{N_o} |0\rangle
$$

(9)

where $\hat{a}_{i0}^\dagger$ is the creation operator for an atom in state $i$ with zero momentum (see next section), $N_{oi} = \rho_i V$, and $N_o = N_{ao} + N_{ob}$. Note that $\frac{1}{2\rho} (V_s (\rho_a^2 + \rho_b^2) + 2V_x \rho_a \rho_b)$ is an intensive quantity whereas the total energy, $E_o$, is extensive and consequently $\frac{1}{2\rho} (V_s (\rho_a^2 + \rho_b^2) + 2V_x \rho_a \rho_b)$ is negligible in the thermodynamic limit. Finally, it is clear from Eq. (9) that $S_{ab} = 0$ and $\pi$ correspond to symmetric and antisymmetric superpositions of the internal atomic states.

In the remainder of the paper, we consider the case in which there is a nonvanishing coupling between the two components which allows the atoms to make transitions between states $a$ and $b$. In other words, $R$ is never identically zero although it may be infinitesimally small, $R \sim 0$. If $R \equiv 0$, then the number of atoms in states $a$ and $b$ would be separately conserved and the condensate energy would be given Eq. (8a) for fixed $\rho_a$ and $\rho_b$. However, when Bose-Einstein condensation occurs in the presence of a coupling field, the condensate will form in that state which minimizes Eq. (8a) for fixed total number of atoms ($\rho =$ const.);
this is the limit we consider. For non-interacting atoms, this would correspond to the lowest energy dressed state.

Consequently, the ground state of the condensate will be a function of \( \rho, \delta, R, S_{ab}, \) and \( V_s - V_x \). Note that Steel and Collett have carried out a fully quantum mechanical calculation of the ground state for small condensates (a few hundred atoms) under the same conditions \[10\]. Similarly, the ground state of a one dimensional inhomogeneous condensate has been studied under these conditions by Blakie \textit{et al.} using numerical solutions of the time independent coupled Gross-Pitaevskii equations \[7\]. In both these cases, however, the relative roles played by the mean-field interactions and interaction with the external field in determining the ground state of the condensate is not as physically clear as it is for the case of a homogeneous condensate.

In the following two subsections we classify the extrema of Eq. (8a) for the two cases of \( R \sim 0 \) and \( \delta = 0 \). The key point is that nonzero \( (V_s - V_x) \) can significantly modify the ground state structure from what one would expect based on the single body Hamiltonian. However, before proceeding it is helpful to make a few definitions. We define the polarization of the condensate to be \( \xi = |\rho_a - \rho_b|/\rho \). Consequently, a polarized condensate corresponds to \( \xi = 1 \) and an unpolarized condensate would correspond to \( \xi = 0 \) while \( 0 < \xi < 1 \) represents a state of partial polarization.

\textbf{A.} \( R \sim 0 \)

In this case we set \( R \) equal to zero in Eq. (8a). When \( \delta = 0 \), the ground state will either be unpolarized for \( V_s > V_x \) or polarized for \( V_s < V_x \). When \( \delta > 0 \) and \( V_s < V_x \), the ground state corresponding to the minimum of \( E_o/V \) occurs at \( \rho_b = \rho \), while \( E_o/V \) is a maximum at \( \rho_a = \frac{-\hbar \delta}{2(V_s - V_x)} + \rho/2 \). For the case \( V_s > V_x \) (again with \( \delta > 0 \)), \( \rho_a = \frac{-\hbar \delta}{2(V_s - V_x)} + \rho/2 \) now corresponds to the minimum for \( E_o/V \). Notice that when \( R \sim 0 \), the relative phase, \( S_{ab} \), is arbitrary. For \( \delta < 0 \), the results are the same if one interchanges states \( a \) and \( b \). Note that for binary condensates, where \( N_a \) and \( N_b \) are separately conserved, \( V_s < V_x \) would lead
to a phase separation of the condensates into two components that occupy nonoverlapping regions in space [13] [14] [15].

**B. $\delta = 0$**

When $\mathcal{R}$ is finite, the ground state can exhibit interesting new structure. It is easy to see from Eq. (8a), that the energy is an extremum only if $S_{ab} = 0, \pi$. By requiring that $E_o/V$ be an extremum with respect to $\rho_a$, one finds from $\partial (E_o/V) / \partial \rho_a$, that the extrema are located at $\rho_a = \rho/2$ and $\rho_a = \rho_a^{(\pm)}$ where

$$\rho_a^{(\pm)} \equiv \frac{\rho}{2} \left( 1 \pm \sqrt{1 - 1/(2\rho\alpha)^2} \right)$$

and

$$\alpha = \frac{V_s - V_x}{2h\mathcal{R}}. \tag{11}$$

Note that the extrema at $\rho_a^{(\pm)}$ are degenerate and occur only when $|\rho\alpha| > 1/2$ and $\alpha/\cos(S_{ab}) > 0$. There are four cases to consider:

(i) $\rho\alpha < 1/2$ and $S_{ab} = 0$. In this case the only extremum is at $\rho_a = \rho/2$ and this is a global maximum of $E_o/V$.

(ii) $\rho\alpha > 1/2$ and $S_{ab} = 0$. In this case $\rho_a = \rho/2$ is now a global minimum of $E_o/V$. The extrema at $\rho_a = \rho_a^{(\pm)}$ are global maxima of $E_o/V$.

(iii) $\rho\alpha > -1/2$ and $S_{ab} = \pi$. Again, $\rho_a = \rho/2$ is a global minimum of $E_o/V$.

(iv) $\rho\alpha < -1/2$ and $S_{ab} = \pi$. The extremum at $\rho_a = \rho/2$ has now become the global maximum of $E_o/V$. The extrema at $\rho_a = \rho_a^{(\pm)}$ are the global minima in the ground state energy. Note that when $\rho\alpha \to \infty$, the minima are located at $\rho_a = 0$ or $\rho_a = \rho$.

The nature of these extrema may be understood by a consideration of the physical meaning of the parameter $\rho\alpha$. Since the $(V_s + V_x)$ term in $E_o/V$ simply gives an overall constant energy, it may neglected and, as such, the only relevant mean field interaction energy for determining the ground state is the $\frac{1}{4}(V_s - V_x)(\rho_a - \rho_b)^2$ term. Therefore, $\rho\alpha$
is the ratio of the mean-field energy per atom, \( \sim (V_s - V_x) \rho \), to the atom-field interaction energy, \( \sim \hbar \mathcal{R} \). Consequently, for \( |\rho \alpha| > 1/2 \) the mean-field interactions dominate the ground state energy while for \( |\rho \alpha| < 1/2 \) the energy of the ground state is dominated by the atom-field interaction energy. For \( \rho \alpha < -1/2 \) the extremum at \( \rho_a = \rho/2 \) is a maximum, regardless of \( S_{ab} \), since the mean-field interactions favor a polarized ground state \( (\xi = 1) \) in this limit. Similarly, for \( \rho \alpha > 1/2 \), \( \rho_a = \rho/2 \) is always a minimum since the mean-field interactions favor an unpolarized state \( (\xi = 0) \).

On the other hand, for \(-1/2 < \rho \alpha < 1/2\), the single body atom-field interaction dominates over the mean-field interactions. In this limit, the minimum of \( E_o/N = \rho^{-1}(E_o/V) \) coincides with the lowest energy dressed state of the free atoms. For \( \delta = 0 \), the lowest energy dressed is the antisymmetric state \( \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle) \) with energy \(-\hbar \mathcal{R}/2\) which corresponds to \( \rho_a = \rho/2 \) and \( S_{ab} = \pi \). Consequently, the state \( \rho_a = \rho/2 \) and \( S_{ab} = \pi \) is the global minimum of \( E_o/V \) in the interval \(-1/2 < \rho \alpha < 1/2\). The other dressed state for \( \delta = 0 \) is the symmetric state \( \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) \) with energy \( \hbar \mathcal{R}/2 \) which is not the ground state of the system. The symmetric state corresponds to \( \rho_a = \rho/2 \) and \( S_{ab} = 0 \) which is a global maximum in the interval \(-1/2 < \rho \alpha < 1/2\).

For case (ii) and (iv), the energy difference between the extremum at \( \rho/2 \) and \( \rho_a^{(\pm)} \) is given by

\[
\Delta E/V = \frac{1}{V} \left( E(\rho_a = \rho/2) - E(\rho_a = \rho_a^{(\pm)}) \right) = \frac{\rho \hbar \mathcal{R}}{2} \left[ -\frac{1}{4\rho \alpha} - \rho \alpha + \cos(S_{ab}) \right].
\] (12)

The energy difference helps to elucidate the transition of the central extremum at \( |\rho \alpha| = 1/2 \). One can see that as \( |\rho \alpha| \to 1/2 \) from above, \( \rho_a^{(\pm)} \to \rho/2 \) and \( \Delta E/V \to 0 \) so that the three extrema merge at \( |\rho \alpha| = 1/2 \) and for \( |\rho \alpha| < 1/2 \), there is a single extremum at \( \rho_a = \rho/2 \). This is illustrated in Figures 1 and 2.

One should note that \( E_o/V \) always possess at least one global minimum for finite \( \mathcal{R} \) but this minimum does not necessarily correspond to the state with equal population in the two components [3]. It is interesting to note that for \( \rho \alpha < -1/2 \) the condensate state exhibits another broken symmetry in addition to the usual broken U(1) gauge symmetry.
since condensation occurs at either $\rho_a^{(+)}$ or $\rho_a^{(-)}$.

III. QUASIPARTICLE SPECTRUM

In this section the Bogoliubov prescription is used to linearize the Hamiltonian around the ground states discussed in the previous section. A canonical transformation is then used to diagonalize the Hamiltonian and find the spectrum of elementary excitations above the condensate.

At this point it is advantageous to introduce the grand canonical Hamiltonian, $\hat{K} = \hat{H} - \mu \hat{N}$. Here, $\hat{N} = \hat{N}_a + \hat{N}_b = \int d^3r \left( \hat{\Psi}_a^\dagger(\mathbf{r}) \hat{\Psi}_a(\mathbf{r}) + \hat{\Psi}_b^\dagger(\mathbf{r}) \hat{\Psi}_b(\mathbf{r}) \right)$ is the total number operator and $\mu$ is the chemical potential. We have assumed that the system is in a state of chemical equilibrium so that $\mu = \mu_a = \mu_b$ where $\mu_i = \frac{\partial E}{\partial N_i}$ is the chemical potential of the two components. The chemical potential insures that the ground state expectation value, $\langle \hat{K} \rangle$, is a minimum with respect to the total number of atoms. The chemical potential may also be interpreted as a Lagrange multiplier which insures that $\langle \hat{N} \rangle$ is conserved [11]. A pair of equations for $\mu$ may be derived by requiring that $E_o[\phi_a, \phi_b] - \mu \int d^3r \left( |\phi_a|^2 + |\phi_b|^2 \right)$ be an extremum with respect to the variations $\delta \phi_a^* \delta \phi_b^*$.

$$
\mu \phi_a = \frac{\hbar \delta}{2} \phi_a + \frac{\hbar \mathcal{R}}{2} \phi_b + \left( V_s |\phi_a|^2 + V_x |\phi_b|^2 \right) \phi_a; \\
\mu \phi_b = -\frac{\hbar \delta}{2} \phi_b + \frac{\hbar \mathcal{R}}{2} \phi_a + \left( V_s |\phi_b|^2 + V_x |\phi_a|^2 \right) \phi_b. 
$$

(13a) (13b)

It should be noted that Eqs. (13a)-(13b) are equivalent to the condition that $\langle \hat{K} \rangle$ is an extremum with respect to the total number of atoms which is found by varying $N_{oa}$ while keeping $N_{ob}$ fixed and vice versa. In contrast, the extrema for $E_o$ found in section II corresponded to finding $\left( \frac{\partial E_o}{\partial N_{oa}} \right)_{N_o} = 0$, i.e. the extrema for variations in $N_{oa}$ for a fixed total number of atoms in the condensate. As such, Eqs. (13a) (13b) serve to define the chemical potential, but not the relative fraction of $\phi_a$ and $\phi_b$, which is determined by minimizing Eq. (8a).

For a homogenous condensate we can expand $\hat{\Psi}_i(\mathbf{r})$ in a basis of plane wave states,
\[ \hat{\psi}_i(r) = \frac{1}{\sqrt{V}} \sum_p \hat{a}_i p e^{ip \cdot r/\hbar}, \]  

with \( \phi_i = \frac{1}{\sqrt{V}} \langle \hat{a}_i o \rangle \) and \( [\hat{a}_i p, \hat{a}_j^\dagger p'] = \delta_{ij} \delta_{p,p'}. \) This gives the following expression for \( \hat{K} \),

\[
\hat{K} = \sum_p \left\{ \left( \frac{p^2}{2m} + \frac{\hbar \delta}{2} - \mu \right) \hat{a}_i p \hat{a}_i^\dagger p + \left( \frac{p^2}{2m} - \frac{\hbar \delta}{2} - \mu \right) \hat{a}_i^\dagger p \hat{a}_i p + \frac{\hbar R}{2} \left( \hat{a}_i p \hat{a}_p + \hat{a}_i^\dagger p \hat{a}_p^\dagger \right) \right\} 
\]

\[
+ \frac{1}{2V} \sum_{p_1+p_2=p_3+p_4} \left\{ V_s \left( \hat{a}_i p_1 \hat{a}_i^\dagger p_2 \hat{a}_i p_3 \hat{a}_i^\dagger p_4 + \hat{a}_i^\dagger p_2 \hat{a}_i p_3 \hat{a}_i^\dagger p_4 + 2V_x \hat{a}_i p_1 \hat{a}_i^\dagger p_2 \hat{a}_i p_3 \hat{a}_i^\dagger p_4 \right) \right\}; \quad (14)
\]

which may be linearized around the ground state solutions found in section II by making the replacement \( \hat{a}_i o \rightarrow \langle \hat{a}_i o \rangle \) and keeping only the lowest order quadratic terms in the operators for \( p \neq 0 \). By utilizing Eqs. (13a-13b), the resulting expression for \( \hat{K} \) is

\[
\hat{K} = E_o - \mu N_o + \sum_{p \neq 0} \left\{ \left( \frac{p^2}{2m} + \rho_a V_s - \frac{hR}{2} \sqrt{\frac{\rho_a}{\rho_b} \cos S_{ab}} \right) \hat{a}_i p \hat{a}_i^\dagger p + \left( \frac{p^2}{2m} + \rho_b V_s - \frac{hR}{2} \sqrt{\frac{\rho_a}{\rho_b} \cos S_{ab}} \right) \hat{a}_i^\dagger p \hat{a}_i p \right\} 
\]

\[
+ \left( \frac{hR}{2} + V_x \sqrt{\rho_a \rho_b \cos S_{ab}} \right) \left( \hat{a}_i p \hat{a}_i^\dagger p + \hat{a}_i^\dagger p \hat{a}_i p \right) + \frac{1}{2} \rho_a V_s \left( \hat{a}_i p \hat{a}_i^\dagger p_{a-p} + \hat{a}_i^\dagger p \hat{a}_i p_{a-p} \right) 
\]

\[
+ \frac{1}{2} \rho_b V_s \left( \hat{a}_i p \hat{a}_i^\dagger p_{b-p} + \hat{a}_i^\dagger p \hat{a}_i p_{b-p} \right) + V_x \sqrt{\rho_a \rho_b \cos S_{ab}} \left( \hat{a}_i p \hat{a}_i^\dagger p_{b-p} + \hat{a}_i^\dagger p \hat{a}_i p_{b-p} \right) \right\}; \quad (15)
\]

A Hamiltonian which is quadratic in bosonic operators with the general form,

\[
\hat{H} = H_o + \frac{1}{2} \sum_{i,j,p \neq 0} A_{ij}(p) \hat{a}_i^\dagger p \hat{a}_j p + \frac{1}{2} \sum_{i,j,p \neq 0} A_{ij}^*(p) \hat{a}_i p \hat{a}_j^\dagger p + \sum_{i,j,p \neq 0} B_{ij}(p) \hat{a}_i^\dagger p \hat{a}_j p. \quad (16)
\]

where \( B_{ij}(p) \) is Hermitian and \( A_{ij}(p) \) is a symmetric matrix may be diagonalized by a canonical transformation. This is done by defining quasiparticle annihilation and creation operators, \( \hat{b}_i p \) and \( \hat{b}_i^\dagger p \), respectively, given by

\[
\hat{b}_i p = \sum_i \left( u_i p^*(p) \hat{a}_i p - v_i p^*(p) \hat{a}_i^\dagger p \right) \quad (17)
\]

which satisfy bosonic commutation relations. The quasiparticle operators obey the equation of motion

\[
\left[ \hat{b}_i p, \hat{H} \right] = \varepsilon_i(p) \hat{b}_i p; \quad (18)
\]

which is consistent with \( \hat{H} = \sum_{i,p \neq 0} \varepsilon_i(p) \hat{b}_i p \hat{b}_i^\dagger p + E_{vac} \). Using Eqs. (18) and (17) it is easy to show that the quasiparticle energies satisfy the eigenvalue equation \( [12] \),
\[ \varepsilon_i(p)u_i(p) = \sum_j (B_{ij}(p)u_{ij}(p) + A_{ij}(p)v_{ij}(p)); \]
\[ -\varepsilon_i(p)v_i(p) = \sum_j (B_{ij}^*(p)v_{ij}(p) + A_{ij}^*(p)u_{ij}(p)); \quad (19a) \]

Since \( \varepsilon_i(p) \) are not the eigenvalues of a Hermitian matrix, there is no guarantee that they will be real. However, when the \( \varepsilon_i(p) \) are complex, the \( \hat{b}_p \) and \( \hat{b}_p^\dagger \) do not satisfy bosonic commutation relations and consequently, the quasiparticles may no longer be interpreted as bosons. In addition, if \((u_i(p), v_i(p))\) are a solution with eigenvalue \( \varepsilon_i(p) \), then \((v_i^*(p), u_i^*(p))\) are a solution with eigenvalue \(-\varepsilon_i(p)\). However, only \( \varepsilon_i(p) > 0 \) are physically significant since the energy of the system must be bounded from below.

Since Eq. (15) has the form of Eq. (16), one may directly apply Eqs. (19a) to calculate the energy spectrum of the quasiparticle excitations above the condensate. As in the previous section, we focus on the two cases of \( R \sim 0 \) and \( \delta = 0 \) and limit the discussion to the case \( V_x > V_s \) (\( \alpha < 0 \)).

**A. \( R \sim 0 \)**

For \( \delta > 0 \), the ground state is given by \( \rho_b = \rho \) and \( \rho_a = 0 \). In this case Eq. (15) has the form

\[ \hat{K} = E_o - \mu N_o + \sum_{p \neq 0} \left( \frac{p^2}{2m} \hat{a}_{op}^\dagger \hat{a}_{op} + \varepsilon_B(p) \hat{B}_{p}^\dagger \hat{B}_p \right) \quad (20) \]

where \( \varepsilon_B(p) = \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + 2\rho V_s \right)} \) and \( \hat{B}_p \) is the quasiparticle annihilation operator for excitations in state \( b \) and is related to \( \hat{a}_{bp} \) through the canonical transformation, \( \hat{a}_{bp} = \cosh \varphi_p \hat{B}_p - \sinh \varphi_p \hat{B}_p^\dagger \). For \( \delta < 0 \), condensation occurs in \( \rho_a = \rho \) and Eq. (20) remains valid with the interchange of \( \hat{a}_{ap} \leftrightarrow \hat{a}_{bp} \). Finally, for \( \delta = 0 \) and \( V_x > V_s \), condensation only occurs in the state \( \rho_b = \rho \) or \( \rho_a = \rho \) since the two states are degenerate. Bose condensation will not occur in the state \( \rho_b = \rho_a = \rho/2 \) as given in Ref. [3] since this represents a maximum in \( E_o/V \).
\textbf{B. } \delta = 0, \mathcal{R} > 0

For \( \rho \alpha < 0 \) and \( \delta = 0 \), the only minima in \( E_0/V \) occur for \( S_{ab} = \pi \). For the extremum at \( \rho_a = \rho/2 \), the quasiparticle energies are

\begin{align*}
\varepsilon_-(p) &= \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + \rho (V_s + V_x) \right)}; \quad (21a) \\
\varepsilon_+(p) &= \sqrt{\left( \frac{p^2}{2m} + \hbar \mathcal{R} \right) \left( \frac{p^2}{2m} + \hbar \mathcal{R} + \rho (V_s - V_x) \right)}; \quad (21b)
\end{align*}

which agrees with the results obtained in Ref. \[3\]. It is possible for \( \varepsilon_+(p) \) to be imaginary and this situation corresponds to the instabilities mentioned in \[3\]. The condition that \( \varepsilon_+(p) \) be real for all momenta is \( \rho \alpha \geq -1/2 \) which is exactly the condition that \( \rho_a = \rho/2 \) be a minimum of \( E_0/V \). Consequently, Eqs. \((21a \text{-} 21b)\) can be interpreted as the quasiparticle spectrum for excitations above the condensate \textit{only} when \( \rho \alpha \geq -1/2 \). When Eqs. \((21a \text{-} 21b)\) are extended to \( \rho \alpha < -1/2 \), they no longer correspond to physically meaningful results since \( \varepsilon_\pm(p) \) then correspond to expansions of small amplitude oscillations around an energy maximum.

When \( \rho \alpha < -1/2 \), the location of the minima of \( E_0/V \) are \( \rho_a = \rho_a^{(\pm)} \). Since Eq. \((15)\) is symmetric under the interchange of \( a \) and \( b \) atoms, the quasiparticle spectrum will be the same at \( \rho_a^{(+)} \) and \( \rho_a^{(-)} \). The excitation spectrum about \( \rho_a^{(-)} \) (or \( \rho_a^{(+)} \)) is significantly more complicated than at \( \rho_a = \rho/2 \) and is given by

\begin{align*}
\varepsilon_\pm(p) &= \sqrt{\Xi_1 \pm \frac{1}{2} \sqrt{\Xi_2}}; \quad (22a) \\
\Xi_1 &= \frac{p^2}{2m} \left( \frac{p^2}{2m} + \rho V_x \right) + \frac{1}{2} (\hbar \mathcal{R})^2 \left( (2 \rho \alpha)^2 - 1 \right) \quad (22b) \\
\Xi_2 &= (\hbar \mathcal{R})^4 \left( (2 \rho \alpha)^2 - 1 \right)^2 + 4 (\hbar \mathcal{R})^2 \left( (2 \rho \alpha)^2 - 1 \right) \frac{p^2}{2m} \left[ \left( \hbar \mathcal{R}(2 \rho \alpha) + \rho V_s \right) + \left( 1 + \frac{1}{2 \rho \alpha} \right) \frac{p^2}{2m} \right] \\
&\quad + 4 \left( \frac{p^2}{2m} \right)^2 \left[ (\rho V_s)^2 + \left( 1 + \frac{1}{2 \rho \alpha} \right) (1 - 2 \rho \alpha)(\hbar \mathcal{R})(\hbar \mathcal{R} - 2 \rho V_s) \right]; \quad (22c)
\end{align*}

Note that the \( \pm \) in \( \varepsilon_\pm(p) \) correspond to the two branches of the excitation spectrum for each of \( \rho_a^{(+)} \) and \( \rho_a^{(-)} \) and not to \( \rho_a^{(+) \text{ or } (-)} \) individually. It is obvious that \( \Xi_1 \geq 0 \) and one can
prove that $\Xi_2 \geq 0$ for $2\rho \alpha \leq -1$. One may also show that $\Xi_2/4 \geq 0$ which implies that 
$\varepsilon_\pm(p)$ will be real for $2\rho \alpha \leq -1$. Thus the quasiparticle spectrum is real as expected.

When $p = 0$, $\varepsilon_-(0) = 0$ and $\varepsilon_+(0) = \sqrt{\rho^2 (V_x - V_s)^2 - (\hbar \mathcal{R})^2}$ which shows that one branch is gapless while the other branch contains a gap for $\rho \alpha < -1/2$. It is easy to show that when $\rho \alpha = -1/2$, Eqs. (22a-22c) agree with Eqs. (21a-21b) which indicates that the quasiparticle spectrum varies continuously with $\rho \alpha$. There are several important limiting cases for Eq. (22a). First one may consider the limit that $\rho \alpha \to -1/2$ while $\frac{p^2}{2m}, \rho V_s \sim \hbar \mathcal{R}$. In this case the excitation spectrum reduces to

$$\varepsilon_\pm(p) = \sqrt{\frac{\rho^2}{2m} \left( \frac{p^2}{2m} + \rho (V_x \pm V_s) \right)}.$$ (23)

Note that Eq. (23) is not valid in the long wavelength phonon regime since $\frac{p^2}{2m} \sim \hbar \mathcal{R} \sim \rho (V_x - V_s)$. Another simple case is when the two minima approach the edges at $\rho_a = 0$ and $\rho_a = \rho$ which corresponds to the condition $|\rho \alpha| \gg 1/2$ while at the same time $\frac{p^2}{2m} \ll \hbar \mathcal{R}, \rho V_x, \rho V_s$ (long wavelength limit). In this case the two branches of Eq. (22a) are given by

$$\varepsilon_-(p) = \sqrt{\frac{\rho V_s}{m}} p;$$ (24a)
$$\varepsilon_+(p) = \sqrt{\frac{p^2 \rho (V_x - V_s)}{m}} + (\hbar \mathcal{R})^2 ((2\rho \alpha)^2 - 1);$$ (24b)
$$\approx \rho (V_x - V_s) + \frac{p^2}{2m}.$$ (24c)

Notice that in this case, the $\varepsilon_-(p)$ branch corresponds to phonons with a speed of sound given by $u = \sqrt{\frac{\rho V_s}{m}}$ that is independent of the interspecies scattering and that $\varepsilon_+(p)$ corresponds to single particle excitations with a mean-field shift in the energy.

Finally, one may consider the limit that $\rho \alpha \to -1/2$ and $\frac{p^2}{2m} \ll \hbar \mathcal{R}$ which corresponds to the long wavelength limit in which the two minima are displaced only slightly from $\rho/2$. In this case the excitation spectrum has the form

$$\varepsilon_\pm(p) = \sqrt{\Xi_1 \pm \frac{1}{2} \left( (\hbar \mathcal{R})^2 ((2\rho \alpha)^2 - 1) - 2\rho V_s \frac{p^2}{2m} \right)^2 + 4(\hbar \mathcal{R})^3 ((2\rho \alpha)^2 - 1) \frac{p^2}{2m}};$$ (25)
which for $\rho V_s \gg \hbar R$, $\varepsilon_{\pm}(p)$ simplifies even further to

$$\varepsilon_- (p) = \sqrt{\frac{\rho (V_x + V_s)}{2m}} p; \quad (26a)$$

$$\varepsilon_+ (p) = \sqrt{\frac{p^2 \rho (V_x - V_s)}{2m}} + \left[ \rho^2 (V_x - V_s)^2 - (\hbar R)^2 \right]. \quad (26b)$$

Again, $\varepsilon_- (p)$ corresponds to phonon excitations with a speed of sound which depends on the average of $V_s$ and $V_x$. However, the $\varepsilon_+ (p)$ branch cannot be given a simple interpretation in terms of either phonon-like collective excitations or single particle excitations since both the $p^2$ term and the term in brackets are of comparable magnitude.

### IV. DISCUSSION

In this paper we have analyzed the ground state energy of a homogeneous two-component Bose-Einstein condensate interacting with a spatially uniform radiation field which couples the two internal states of the condensates. We have argued that Bose-Einstein condensation occurs in the state which minimizes $E_0$ as a function of the density of one of the components (since the total density is fixed) and the relative phase between the two components. When the proper ground state for the condensate is chosen, the quasiparticle excitations above the ground state have real energies and no instabilities occur. The fact that the quasiparticle spectrum is real and exhibit no instabilities may not come as a surprise, but these results differ from those obtained in Ref. [3]. The instabilities found in Ref. [3] originate from an expansion of small amplitude collective excitations about a point of unstable equilibrium, represented by the global maximum of Eq. (8a).

Throughout this paper it has been assumed that the condensate is in a state of chemical equilibrium (i.e. $\mu_a = \mu_b = \mu$ where $\mu_i$ is the chemical potential for component $i$). As such the ground state energy is found by minimizing $E_0$ subject to the constraint $\rho_a + \rho_b = \rho$. An interesting extension of this work would be to consider the case when the two components are in a state of thermal equilibrium such that Bose condensation can occur, but not in a state of chemical equilibrium such as in [3]. In this case the population of the two components
is fixed by some external pumping and decay mechanism and the ground state energy of
the condensate will be given by Eq. (8a) for fixed \( \rho_a \) and \( \rho_b \). In this case one might expect
instabilities to occur in the excitation spectrum since the condensate state is not in general
the minimum of Eq. (8a). Normally, these instabilities would indicate a transition to a
phase separated state [15]. However, if the pumping and decay are spatially uniform and
time independent so that the relative densities at each point in space are fixed externally,
then it will be impossible for phase separation to occur. The dynamics of the condensate in
this case would be interesting to explore.
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Figure 1. Plot of $E_o/V$ for $S_{ab} = 0$ and $\rho\alpha > 0$ showing the transition from a central maximum at $\rho_a = \rho/2$ for $\rho\alpha < 1/2$ to a minimum for $\rho\alpha > 1/2$ along with the development of maxima at $\rho_a = \rho^{(+)}, \rho^{(-)}$. Note that $E_o/V$ is in units of $\rho h R$ and we have subtracted off the constant term $\frac{1}{4}(V_s + V_x)\rho^2$.

Figure 2. Plot of $E_o/V$ for $S_{ab} = \pi$ and $\rho\alpha < 0$ showing the transition from a central minimum at $\rho_a = \rho/2$ for $\rho\alpha > -1/2$ to a maximum for $\rho\alpha < -1/2$ along with the development of minima at $\rho_a = \rho^{(+)}, \rho^{(-)}$. Note that $E_o/V$ is in units of $\rho h R$ and we have subtracted off the constant term $\frac{1}{4}(V_s + V_x)\rho^2$.