A numerical scheme for a diffusion equation with nonlocal nonlinear boundary condition

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Abstract
In this article, a numerical scheme to find approximate solutions to the McKendrick–Von Foerster equation with diffusion (M-V-D) is presented. This is a nonlinear equation for continuously structured population models. The main difficulty in employing the standard analysis to study the properties of this scheme is due to the presence of nonlinear and nonlocal term in the Robin boundary condition in the M-V-D. To overcome this, we use the abstract theory of discretizations based on the notion of stability threshold to analyze the scheme. Stability, and convergence of the proposed numerical scheme are established. Finally, some numerical experiments are illustrated.

Keywords Finite difference method · Nonlocal boundary condition · The McKendrick–Von Foerster equation · Stability threshold · Convergent numerical scheme · Structured population model

Mathematics Subject Classification 65M12 · 92D25

1 Introduction

The McKendrick–Von Foerster equation arises naturally in many areas of mathematical biology such as cell proliferation, and demography modeling (see Abia et al. 2010; Diekmann and Heesterbeek 2000; Murray 2002, 2003; Perthame 2007, 2015; Thieme 2003). In particular, the McKendrick–Von Foerster equation is one amongst the important models whenever age structure is a vital feature in the modeling (see Abia et al. 2018; Halder and Tumuluri 2023).
In the recent years, the McKendrick–Von Foerster equation with diffusion (M-V-D) has attracted interest of many engineers as well as mathematicians due to its applications in the modeling of thermoelasticity, neuronal networks, etc. (see Day 1982, 1985; Kakumani and Tumuluri 2016, 2017; Michel and Touaoula 2013; Michel and Tumuluri 2020). The main difficulty in the study of the M-V-D is due to the nonlocal nature of the partial differential equation (PDE) and the boundary condition. The qualitative properties of the M-V-D have been developed by many authors. Though, numerical study of nonlocal equations got considerable focus, relatively less attention was paid to problems with the Robin boundary condition.

In this paper, our objective is to propose and analyze a numerical scheme to find approximate solutions to the following nonlinear diffusion equation:

$$\begin{cases}
\frac{u_t(x, t)}{t} + u_x(x, t) + d(x, s_1(t))u(x, t) = u_{xx}(x, t), & x \in (0, a_{\tau}), \ t > 0 \\
u(0, t) - u_x(0, t) = \int_0^{a_{\tau}} B(x, s_2(t))u(x, t)dx, & t \geq 0, \\
u(a_{\tau}, t) = 0, & t \geq 0, \\
u(x, 0) = u_0(x), & x \in (0, a_{\tau}), \\
\psi_v(x) = \int_0^{a_{\tau}} \psi_v(x)u(x, t)dx, & t \geq 0, \ v = 1, 2, 
\end{cases}$$

(1)

where $a_{\tau} > 0$. In the given model, the unknown function $u(x, t)$ represents the age-specific density of individuals of age $x$ at time $t$. The function $d$ represents the death rate and it depends on $x$ and the environmental factor $s_1$. Similarly, the fertility rate $B$ depends on the age $x$ and the environmental factor $s_2$. Both the functions $\psi_1$ and $\psi_2$ are called the competition weights. Moreover, the functions $d$ and $B$ are assumed to be non-negative. Without loss of generality, we take the diffusion rate is equal to one.

In Kakumani and Tumuluri (2016), the authors considered the M-V-D with nonlinear nonlocal Robin boundary condition and studied the existence and uniqueness of the solution. The authors of Kakumani and Tumuluri (2018) proposed a convergent numerical scheme to the M-V-D. On the other hand, the existence of a global solution to the M-V-D in a bounded domain with nonlinear nonlocal Robin boundary condition was proved when $d = d(x)$ in Kakumani and Tumuluri (2017). Moreover, the authors of Breda et al. (2021, 2020) designed numerical schemes to compute the basic reproduction number $R_0$ for general continuously structured population models, in particular for models with boundary conditions of Robin type. Regarding the basic reproduction number for diffusion equation (1), taking the approach developed in Barril et al. (2021), one can get $R_0 = \frac{2}{\sqrt{1 + 4d_0}} \int_0^{\infty} B(x, 0) e^{(1-\sqrt{1+4d_0})r} dx$ under the assumptions that the mortality rate is a constant $d_0$ and $a_{\tau} = \infty$. Recently in Halder and Tumuluri (2023), an implicit finite difference scheme was introduced to approximate the solution to the M-V-D in a bounded domain with nonlinear nonlocal Robin boundary condition at both the boundary points. Moreover, the well-posedness and the stability of the numerical scheme were proved using the method of upper and lower solution with the aid of the discrete maximum principle.

The author of López Marcos (1991) presented an upwind scheme for a nonlinear hyperbolic integro-differential equation with nonlocal boundary condition. The analysis was carried out employing the general analytic framework developed in López-Marcos and Sanz-Serna (1988), López Marcos and Sanz-Serna (1988), Sanz-Serna (1985). The notion of ‘stability with threshold’ and a result due to Stetter (see Stetter 1973, Lemma 1.2.1) were the most important tools for the analysis.
The above mentioned results inspired us to propose and analyze an explicit finite difference numerical scheme to (1). The main difficulty in the analysis of the proposed numerical scheme is due to the nonlinearity and the Robin boundary condition that are presented in (1). The objective of this paper is to establish the stability and the convergence of our numerical method. Since the scheme is of the form \( U_i^{n+1} = F(U_0^n, \ldots, U_M^n) \), where \( F \) is a nonlinear function, the standard techniques of proving stability (for instance, the Lax theory, etc.) cannot be used. Instead, the notion of nonlinear stability (with threshold) is used to arrive at the convergence result.

This article is organized as follows. In Sect. 2, we present a finite difference scheme and define the required norms to use the general discretization framework. Moreover, we introduce the notion of stability with \( h \)-dependent thresholds. We prove consistency, stability and convergence results in Sect. 3. In Sect. 4, numerical schemes to (1) with other types of boundary conditions are discussed. Finally, numerical examples are provided in Sect. 5 to justify the convergence results that are proved.

2 The numerical scheme

Let \( h, k \) be the spatial and temporal step sizes. Denote by \((x_i, t^n)\) a typical grid point, where \( x_i = ih \), and \( t^n = nk \). Moreover, we fix \( T > 0 \), assume that \( a_i = 2(M' + 3)h \) for some \( M' \in \mathbb{N} \) and \( T = Nk \) for some \( N \in \mathbb{N} \). To simplify the notations, we write \( M = 2(M' + 3) \).

For every grid point \((x_i, t^n)\), we denote the numerical solution by \( U_i^n \), and set

\[
\Psi_{v,i} = \psi_v(x_i),
\]

\[
\Psi_v = (\Psi_{v,1}, \Psi_{v,2}, \ldots, \Psi_{v,M-1}), \quad v = 1, 2,
\]

\[
B(\cdot) = (B(x_1, \cdot), B(x_2, \cdot), \ldots, B(x_{M-1}, \cdot)),
\]

\[
d(\cdot) = (d(x_1, \cdot), d(x_2, \cdot), \ldots, d(x_{M-1}, \cdot)),
\]

\[
U^n = (U_1^n, U_2^n, \ldots, U_{M-1}^n).
\]

To approximate the integrals in (1), we use the following quadrature rule which is a combination of the composite Simpson–\(\frac{1}{3}\) and Minle’s rules. For \( V = (V_1, \ldots, V_{M-1}) \in \mathbb{R}^{M-1} \), we define the quadrature formula

\[
Q_h(V) = \frac{4h}{3}(2V_1 - V_2 + 2V_3) + \frac{h}{3} \sum_{i=2}^{M'} (V_{2i} + 4V_{2i+1} + V_{2i+2})
\]

\[
+ \frac{4h}{3}(2V_{2M'+3} - V_{2M'+4} + 2V_{2M'+5}).
\]

If \( V = (V_1, \ldots, V_{M-1}), W = (W_1, \ldots, W_{M-1}) \) are in \( \mathbb{R}^{M-1} \), then \( V \cdot W \) denotes the vector in \( \mathbb{R}^{M-1} \) which is obtained by the element wise multiplication of \( V \) and \( W \), i.e.,

\[
V \cdot W = (V_1 W_1, \ldots, V_{M-1} W_{M-1}).
\]

With the notation introduced so far, we propose the following scheme for (1) using the forward difference approximation for \( u_t \), the backward difference for \( u_x \), and the central
difference for $u_{xx}$:

\[
\begin{aligned}
U^n_i &- U^{n-1}_i + \frac{U^{n-1}_{i} - U^{n-1}_{i-1}}{k} + d(x_i, Q_h(\Psi_1 \cdot U^{n-1})) U^{n-1}_i \\
&= \frac{U^{n-1}_{i+1} + U^{n-1}_{i-1} - 2U^{n-1}_i}{h^2}, \quad 1 \leq i \leq M - 1, 1 \leq n \leq N,
\end{aligned}
\]

(2)

To carry out the analysis within an abstract theory of discretizations, we introduce the general discretization framework. For, we define the spaces

\[X_h = Y_h = \mathbb{R}^{N+1} \times \left(\mathbb{R}^{M-1}\right)^{N+1} \times \mathbb{R}^{N+1}.\]

We also introduce the operator $\Phi_h : X_h \to Y_h$, defined through the formulae

\[
\Phi_h\left(V_0, V^0, V^1, \ldots, V^N, V_M\right) = \left(P_0, P^0, P^1, \ldots, P^N, P_M\right),
\]

where

\[
\begin{aligned}
P_0 & = \left(P^0_0, P^1_0, \ldots, P^N_0\right), \\
P^n_0 & = \left(1 + \frac{1}{h}\right) V^n_0 - \frac{1}{h} V^n_1 - Q_h\left(B\left( Q_h(\Psi_2 \cdot V^n) \right) \cdot V^n\right), \quad 0 \leq n \leq N, \\
P_M & = \left(P^0_M, P^1_M, \ldots, P^N_M\right), \\
P^n_M & = \frac{V^n_M}{h}, \quad 0 \leq n \leq N, \\
P^n & = \left(P^n_1, P^n_2, \ldots, P^n_{M-1}\right), \quad 0 \leq n \leq N, \\
P^n_i & = V^n_i - U^n_i, \quad 1 \leq i \leq M - 1, \\
P^n_i & = \frac{V^n_i - V^{n-1}_i}{k} + \frac{V^{n-1}_i - V^{n-1}_{i-1}}{h} + d(x_i, Q_h(\Psi_1 \cdot V^{n-1})) V^{n-1}_i \\
&\quad - \frac{V^{n-1}_{i+1} + V^{n-1}_{i-1} - 2V^{n-1}_i}{h^2}, \quad 1 \leq n \leq N, 1 \leq i \leq M - 1.
\end{aligned}
\]

(3)

Now $U_h = (U_0, U^0, U^1, \ldots, U^N) \in X_h$ is a solution to (2) if and only if it is a solution of the discrete problem

\[
\Phi_h(U_h) = 0 \in Y_h.
\]

(4)

To investigate how close $U_h$ is to $u$, we first need to choose an element $u_h \in X_h$, which is a suitable discrete representation of $u$. In particular, our choice is the set of nodal values of the theoretical solution $u$, namely

\[
u_h = (u_0, u^0, \ldots, u^N, u_M) \in X_h.
\]

(5)
Then the global discretization error is defined to be the vector
\[ e_h = u_h - U_h \in X_h, \]
and the local discretization error is given by
\[ I_h = \Phi_h(u_h) \in Y_h. \]

To measure the magnitude of errors, we define the following norms in the spaces \( X_h \) and \( Y_h \):
\[
\| (V_0, V^1, \ldots, V^N, V_M) \|_{X_h} = h(\|V_0\|_* + \|V_M\|_*) + \max\{\|V^0\|, \|V^1\|, \ldots, \|V^N\|\},
\]
\[
\| (P_0, P^1, \ldots, P^N, P_M) \|_{Y_h} = \left( \|P_0\|_*^2 + \|P^0\|_*^2 + h\|P_M\|_*^2 + \sum_{n=1}^{N} k\|P^n\|_*^2 \right)^{1/2},
\]
where \( \|V^n\|_*^2 = \sum_{i=1}^{M-1} h|V^n_i|^2 \) and \( \|V_0\|_*^2 = \sum_{n=0}^{N} k|V^n_0|^2 \).

For \( V, W \in \mathbb{R}^{M-1} \) and \( Z \in \mathbb{R}^{N+1} \), we define
\[
\langle V, W \rangle = \sum_{i=1}^{M-1} h V_i W_i,
\]
\[
\|V\|_* = \max_{1 \leq j \leq M-1} |V_j|, \quad \|Z\|_* = \max_{0 \leq n \leq N} |Z^n|. \]

Throughout the article, we use \( C \) to denote the generic positive constant which does not depend on the step sizes, grid points and it need not be the same constant as in the preceding calculations.

For the sake of completeness, we give the following standard definitions (see López Marcos 1991).

**Definition 1** (Consistency) Discretization (4) is said to be consistent with (1) if
\[
\lim_{h \to 0} \|\Phi_h(u_h)\|_{Y_h} = \lim_{h \to 0} \|I_h\|_{Y_h} = 0.
\]
Moreover, if \( \|I_h\|_{Y_h} = \mathcal{O}(h^p) + \mathcal{O}(k^q) \) then we say that \((p, q)\) is the order of the consistency.

**Definition 2** (Stability) Discretization (4) is said to be stable restricted to the thresholds \( R_h \) if there exist two positive constants \( h_0 \) and \( S \) such that
\[
\|V_h - W_h\|_{X_h} \leq S\|\Phi_h(V_h) - \Phi_h(W_h)\|_{Y_h},
\]
whenever \( h \in (0, h_0], V_h, W_h \in B(u_h, R_h) \), where \( B(u_h, R_h) = \{z \in X_h \mid \|z - u_h\|_{X_h} < R_h\} \).

**Definition 3** (Convergence) Discretization (4) is said to be convergent if there exists \( h_0 > 0 \) such that, for all \( 0 < h \leq h_0, (4) \) has a solution \( U_h \) for which
\[
\lim_{h \to 0} \|u_h - U_h\|_{X_h} = \lim_{h \to 0} \|e_h\|_{X_h} = 0.
\]
The following theorem which is established in López Marcos and Sanz-Serna (1988) is based on a result due to Stetter (see Stetter 1973), and plays an important role in the proof of convergence of (2).

**Theorem 1** (Cf. López Marcos and Sanz-Serna 1988) Assume that (4) is consistent and stable with thresholds $M_h$. If $\Phi_h$ is continuous in $B(u_h, M_h)$ and $\|I_h\| = O(M_h)$ as $h \to 0$, then the following hold.

(i) For sufficiently small $h > 0$, discrete equation (4) admits a unique solution in $B(u_h, M_h)$.

(ii) The solutions to (4) converge to the solution to (2) as $h \to 0$. Furthermore, the order of convergence is not smaller than the order of consistency.

### 3 Consistency, stability and convergence

In this section, we prove that numerical scheme (2) is consistent and stable. To obtain the stability result, we first need to prove an elementary inequality. Next, with the help of Theorem 1, we establish the convergence result. We begin with the consistency result in the following theorem.

**Theorem 2** (Consistency) Assume that $d, B, \psi_i, i = 1, 2$, are sufficiently smooth such that the solution $u$ to (1) is four times continuously differentiable with bounded derivatives. Moreover, we assume that there exists $L > 0$ such that for every $0 \leq x \leq a^\dagger, s_1, s_2 > 0$,

$$|d(x, s_1) - d(x, s_2)| \leq L|s_1 - s_2|,$$

and

$$|B(x, s_1) - B(x, s_2)| \leq L|s_1 - s_2|.$$

Then the local discretization error satisfies

$$\|\Phi_h(u_h)\|_{Y_h} = \left\{ \|U^0 - u^0\|^2 + O(h^2) + O(k^2) \right\}^{1/2}, \text{ as } h \to 0.$$

**Proof** Using the notation introduced in (6), it is standard to verify that

$$\sup_{i, n} \left| \frac{u_{i}^{n} - u_{i}^{n-1}}{k} - u_t(x_i, t^{n-1}) \right| = O(k), \text{ as } k \to 0,$$

and

$$\sup_{i, n} \left| \frac{u_{i}^{n-1} - u_{i-1}^{n-1}}{h} - u_x(x_i, t^{n-1}) \right| = O(h), \text{ as } h \to 0,$$

and

$$\sup_{i, n} \left| \frac{u_{i+1}^{n} + u_{i-1}^{n} - 2u_{i}^{n-1}}{h^2} - u_{xx}(x_i, t^{n-1}) \right| = O(h^2), \text{ as } h \to 0.$$

On the other hand, it is well known that if $f \in C^4[0, a^\dagger]$, then

$$|\int_{0}^{a^\dagger} f(x)dx - Q_h(f)| \leq Ch^4,$$

where $C > 0$ is independent of $h$.

Lipschitz continuity of $d$ on compact sets readily implies
From the boundary condition, it follows that

\[ \text{Lemma 1} \]

If \( x, y, a, b \) and \( h \) are positive real numbers such that

\[ \text{Using (7)–(9), (11) and (13), one can easily conclude the proof of the required result.} \]

\[ \therefore \text{This completes the proof of the lemma.} \]

\[ \text{Proof} \]

Consider

\[
\left(1 + \frac{1}{h} \right) x^2 - \frac{1}{h} y^2 \leq \left(1 + \frac{1}{h} \right) \left( \frac{ah + bh + y}{h+1} \right)^2 - \frac{1}{h} y^2 \\
= \frac{a^2 h^2 + b^2 h^2 + 2ab h^2 + y^2 + 2yh(a+b)}{h(h+1)} - \frac{1}{h} y^2 \\
\leq \frac{2h(h+1)(a^2 + b^2) + (h+1)y^2}{h(h+1)} - \frac{1}{h} y^2 \\
= 2(a^2 + b^2). \]

This completes the proof of the lemma.

\[ \therefore \text{Now, we are ready to establish the following stability theorem.} \]

\[ \text{Theorem 3 (Stability)} \]

Assume the hypotheses of Theorem 2. Let \( r \) and \( \lambda \) be such that \( k = rh^2 = \lambda h \), and \( \lambda + 2r \leq 1 \). Then discretization (4) is stable with thresholds \( R_h = Rh \), where \( R \) is a fixed positive constant independent of \( h \).
Therefore, there exists $\leq$ for $1$ such that

This readily implies $\phi_i$. On the other hand, from the definition of $\phi_i$, we obtain

Then from the definition of the norm in $X_h$, we find that

Assume that $\phi_i$. Proof

Now consider

\begin{align*}
\| d(Q_h(\Psi_1 \cdot V^{n-1})) V^{n-1} - d(Q_h(\Psi_1 \cdot W^{n-1})) W^{n-1} \|
\leq & \| d(Q_h(\Psi_1 \cdot V^{n-1})) \|_\infty \| V^{n-1} - W^{n-1} \|
+ \| W^{n-1} \|_\infty \| d(Q_h(\Psi_1 \cdot V^{n-1})) - d(Q_h(\Psi_1 \cdot W^{n-1})) \|
\leq C \| V^{n-1} - W^{n-1} \|. \tag{17}
\end{align*}
for some \( C > 0 \) independent of \( h, k \). Thus (16), (17) together give
\[
(1 - 2k) \| V^n - W^n \|^2 \leq (1 + Ck) \| V^{n-1} - W_0^{n-1} \|^2 + k \| P^n - R^n \|^2
+ \frac{1}{h} \| V_0^{n-1} - W_0^{n-1} \|^2 - \frac{1}{h} \| V_1^{n-1} - W_1^{n-1} \|^2.
\]
Using (14) and the left boundary condition, we can write
\[
0 \leq \left( 1 + \frac{1}{h} \right) |V_0^n - W_0^n| - \frac{1}{h} |V_1^n - W_1^n|,
\]
\[
\leq |P_0^n - R_0^n| + |Q_h \left( B \left( Q_h (\Psi_2 \cdot V^n) \right) \cdot V^n \right) - Q_h \left( B \left( Q_h (\Psi_2 \cdot W^n) \right) \cdot W^n \right)|
\]
\[
\leq |P_0^n - R_0^n| + |Q_h \left( B \left( Q_h (\Psi_2 \cdot V^n) \right) - B \left( Q_h (\Psi_2 \cdot W^n) \right) \right) \cdot W^n |
\]
\[
\leq |P_0^n - R_0^n| + \| B \|_{\infty} |Q_h (V^n - W^n)|
+ La't \| \Psi_2 \|_{\infty} |Q_h (V^n - W^n)| \| W^n \|_{\infty}
\]
\[
\leq |P_0^n - R_0^n| + C \| V^n - W^n \|,
\]
for some \( C > 0 \) independent of mesh sizes \( h \) and \( k \). From (19) and Lemma 1, we deduce that
\[
\left( 1 + \frac{1}{h} \right) \| V_0^{n-1} - W_0^{n-1} \|^2 - \frac{1}{h} \| V_1^{n-1} - W_1^{n-1} \|^2
\]
\[
\leq C \left( \| V^{n-1} - W^{n-1} \|^2 + \| P_0^{n-1} - R_0^{n-1} \|^2 \right). \tag{20}
\]
On substituting this bound in (18), we obtain
\[
\| V^n - W^n \|^2 \leq \frac{1 + Ck}{1 - 2k} \| V^{n-1} - W^{n-1} \|^2 + \frac{Ck}{1 - 2k} \left( \| P^n - R^n \|^2 \right)
+ \frac{1}{h} \left( \| P_0^{n-1} - R_0^{n-1} \|^2 + h \| P_M^{n-1} - R_M^{n-1} \|^2 \right). \tag{21}
\]
From the discrete Gronwall lemma, there exists \( C_T \) depending solely on \( T \) such that
\[
\| V^n - W^n \|^2 \leq C_T \left\{ \| V^0 - W^0 \|^2 + \frac{Ck}{1 - 2k} \sum_{m=1}^{\infty} \left( \| P^m - R^m \|^2 \right)
+ |P_0^{m-1} - R_0^{m-1}|^2 + h |P_M^{m-1} - R_M^{m-1}|^2 \right\}. \tag{22}
\]
Thus, for \( k \) sufficiently small, this immediately gives
\[
\| V^n - W^n \|^2 \leq C_T \left\{ \| P^0 - R^0 \|^2 + C \left( \sum_{m=1}^{\infty} k \| P^m - R^m \|^2 \right)
+ C \| P_0 - R_0 \|^2 + h \| P_M - R_M \|^2 \right\}^{\frac{1}{2}}. \tag{23}
\]
Again, from (19), we have

$$(1 + h)|V^n_0 - W^n_0| - |V^n_1 - W^n_1| \leq h \left( C\|V^n - W^n\| + |P^n_0 - R^n_0| \right).$$

On multiplying both sides with $|V^n_0 - W^n_0|$ and using the AM-GM inequality, we get

$$|V^n_0 - W^n_0|^2 \leq |V^n_1 - W^n_1|^2 + h \left( C\|V^n - W^n\|^2 + |P^n_0 - R^n_0|^2 \right).\quad (24)$$

On multiplying both sides by $hk$, taking summation on $n$, we find that

$$h\|V_0 - W_0\|^2_* \leq \sum_{n=0}^{N} hk|V^n_1 - W^n_1|^2 + \sum_{n=0}^{N} kh^2 \left( C\|V^n - W^n\|^2 + |P^n_0 - R^n_0|^2 \right)$$

$$\leq (1 + Ch^2) \sum_{n=0}^{N} k\|V^n - W^n\|^2 + h^2\|P_0 - R_0\|^2_*.$$

The second boundary condition immediately gives

$$\|V_M - W_M\|_* = h\|P_M - R_M\|_*.$$

From (23), (25) and (26), we observe that

$$h \left( \|V_0 - W_0\|_* + \|V_M - W_M\|_* \right)$$

$$+ \max \left\{ \|V_0 - W_0\|, \|V^1 - W^1\|, \ldots, \|V^N - W^N\| \right\}$$

$$\leq K \left( \|P_0 - R_0\|^2_* + \|P_0 - R_0\|^2 + h\|P_M - R_M\|^2_* + \sum_{m=1}^{N} k\|P^m - R^m\|^2 \right)^{\frac{1}{2}},$$

where $K$ is a constant. This completes the proof.

In the following result, we establish that (2) is indeed a convergent scheme.

**Theorem 4** (Convergence) Assume the hypotheses of Theorem 3. If

$$\|U^0 - u^0\|_{X_h} = O(h), \text{ as } h \to 0,$$

then discretization (4) is convergent.

**Proof** The proof is an immediate consequence of Theorems 1–3.

**4 Other types of boundary conditions**

In this section, we discuss the M-V-D with two other boundary conditions. In particular, we study (1) when the right boundary condition is non-homogeneous instead of homogeneous. On the other hand, in Sect. 4.2, we consider Robin boundary condition at both the end points.
4.1 Non-homogeneous boundary condition at \( x = a \)

In this subsection, we consider (1) with non-homogeneous Dirichlet boundary condition, i.e.,

\[
\begin{align*}
\psi(x) & = \psi_0(x), \\
\psi'(x) & = \psi_1(x), \\
\psi''(x) & = \psi_2(x)
\end{align*}
\]

Moreover, we consider the operator \( X_h \) as before, to carry out the analysis, we use the spaces introduced in Sect. 2. Let \( h, k, T, M \) be as in Sect. 2 and \( U^n_i \) denote the approximate solution to (27) at the grid point \((x_i, t^n)\). Moreover, we define \( g^n = g(t^n), 0 \leq n \leq N \).

By discretizing (27) as in Sect. 2, we arrive at the following finite difference scheme (see (2))

\[
\begin{align*}
\frac{U^n_i - U^{n-1}_i}{k} & + \frac{U^{n-1}_i - U^{n-1}_{i-1}}{h} + d(x_i, Q_h(\Psi_1 \cdot U^{n-1}))U^{n-1}_i \\
& = \frac{U^{n-1}_{i+1} + U^{n-1}_{i-1} - 2U^{n-1}_i}{h^2}, 1 \leq i \leq M - 1, 1 \leq n \leq N, \\
\left(1 + \frac{1}{h}\right)U^n_0 - \frac{1}{h}U^n_1 + Q_h(B(Q_h(\Psi_2 \cdot U^n)) \cdot U^n), 0 \leq n \leq N, \\
U^n_M & = g^n, 0 \leq n \leq N, \\
U^n_i & = u_0(x_i), 1 \leq i \leq M - 1.
\end{align*}
\]

As before, to carry out the analysis, we use the spaces \( X_h \) and \( Y_h \) that are introduced in Sect. 2. Moreover, we consider the operator \( \Phi_h : X_h \rightarrow Y_h \), defined through the formulae

\[
\Phi_h \left( V_0, V^0, V^1, \ldots, V^N, V_M \right) = \left( P_0, P^0, P^1, \ldots, P^N, P_M \right),
\]

where

\[
\begin{align*}
P_0 & = \left( P^n_0, P^n_1, \ldots, P^n_M \right), \\
P^n_0 & = \left(1 + \frac{1}{h}\right)V^n_0 - \frac{1}{h}V^n_1 - Q_h(B(Q_h(\Psi_2 \cdot V^n)) \cdot V^n), 0 \leq n \leq N, \\
P^n_M & = \left( P^n_M, P^n_{M-1}, \ldots, P^n_1 \right), \\
P^n_i & = \left( P^n_i, P^n_{i-1}, \ldots, P^n_1 \right), 1 \leq i \leq M - 1, \\
P^n_i & = \left( P^n_i, P^n_{i-1}, \ldots, P^n_1 \right), 1 \leq i \leq M - 1.
\end{align*}
\]
Using the definition of $\Phi_h$, and the arguments used in Theorems 2–4, one can easily show that (28) is indeed a convergent scheme whenever the hypotheses in Theorem 4 hold.

### 4.2 Robin condition at both $x = 0$, $a_\ell$

Consider the following M-V-D with nonlinear nonlocal Robin boundary conditions:

\[
\begin{align*}
    u_t(x, t) + u_x(x, t) + d(x, s_1(t))u(x, t) &= u_{xx}(x, t), & x \in (0, a_\ell), & t > 0, \\
    u(0, t) - u_x(0, t) &= \int_0^{a_\ell} B_1(x, s_2(t))u(x, t)dx, & t \geq 0, \\
    u(a_\ell, t) + u_x(a_\ell, t) &= \int_0^{a_\ell} B_2(x, s_3(t))u(x, t)dx, & t \geq 0, \\
    u(x, 0) &= u_0(x), & x \in (0, a_\ell), \\
    s_v(t) &= \int_0^{a_\ell} \psi_v(x)u(x, t)dx, & t \geq 0 & v = 1, 2, 3.
\end{align*}
\]

In view of Barril et al. (2021), Eq. (30) can be interpreted as a model for population living in a one-dimensional habitat. In that case, $x$ represents the spatial position instead of age. In the right boundary condition, $B_2 = 0$ is an important case which represents the nonflux condition at $x = a_\ell$. The authors of Halder and Tumuluri (2023) designed a numerical scheme to (30), and studied well-posedness and long time behavior of the solution of that numerical scheme. Their numerical scheme is nonlinear and it is proved that the scheme is indeed stable. In this subsection, we propose a numerical scheme to (30) and establish its convergence.

For, we use the notation from the earlier sections. Moreover, we denote $\Psi_{v,i} = \psi_v(x_i)$, $\Psi_v = (\Psi_{v,1}, \Psi_{v,2}, \ldots, \Psi_{v,M-1})$, $v = 1, 2, 3$, $B_1(\cdot) = (B_1(x_1, \cdot), B_1(x_2, \cdot), \ldots, B_1(x_{M-1}, \cdot))$, and $B_2(\cdot) = (B_2(x_1, \cdot), B_2(x_2, \cdot), \ldots, B_2(x_{M-1}, \cdot))$.

Now we discretize (30) to get the following finite difference scheme

\[
\begin{align*}
    U_i^n - U_i^{n-1} &= \frac{k}{h} U_i^{n-1} - U_{i-1}^{n-1} + d(x_i, \Psi_h(\Psi_1 \cdot U^{n-1})) U_i^{n-1} \\
    &= U_{i+1}^{n-1} + U_{i-1}^{n-1} - 2U_i^{n-1}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \\
    \left(1 + \frac{1}{h}\right) U_0^n - \frac{1}{h} U_1^n &= \Psi_h(B_1(\Psi_1 \cdot U^n)), \quad 0 \leq n \leq N, \\
    \left(1 + \frac{1}{h}\right) U_M^n - \frac{1}{h} U_{M-1}^n &= \Psi_h(B_2(\Psi_3 \cdot U^n)), \quad 0 \leq n \leq N, \\
    U_i^0 &= u_0(x_i), \quad 1 \leq i \leq M - 1.
\end{align*}
\]

To establish the convergence of the solution of (31) to the solution to (30), we introduce the operator $\Phi_h : X_h \to Y_h$ given by

\[
  \Phi_h \left( V_0, V^0, V^1, \ldots, V^N, V_M \right) = \left( P_0, P^0, P^1, \ldots, P^N, P_M \right),
\]
where
\[
P_0 = \left( P_0^0, P_0^1, \ldots, P_0^N \right),
\]
\[
P_0^n = \left( 1 + \frac{1}{h} \right) V_0^n - \frac{1}{h} V_i^n - Q_h \left( B_1 \left( Q_h (\Psi_2 \cdot V^n) \right) \cdot V^n \right), \quad 0 \leq n \leq N,
\]
\[
P_M = \left( P_M^0, P_M^1, \ldots, P_M^N \right),
\]
\[
P_M^n = \left( 1 + \frac{1}{h} \right) V_M^n - \frac{1}{h} V_{M-1}^n - Q_h \left( B_2 \left( Q_h (\Psi_3 \cdot V^n) \right) \cdot V^n \right), \quad 0 \leq n \leq N,
\]
\[
P_n = \left( P_n^1, P_n^2, \ldots, P_n^{N-1} \right), \quad 0 \leq n \leq N,
\]
\[
P_i^n = \frac{V_i^n - V_{i-1}^{n-1}}{k} + \frac{V_{i+1}^{n-1} - V_{i-1}^{n-1}}{h} d \left( x_i, Q_h (\Psi_1 \cdot V^{n-1}) \right) V_i^{n-1}
- \frac{V_{i-1}^{n-1} + V_{i+1}^{n-1} - 2V_i^{n-1}}{h^2}, \quad 1 \leq n \leq N, 1 \leq i \leq M - 1.
\]

Now, observe that (16) can be written as
\[
\| V^n - W^n \|^2 \leq \frac{1}{2} \| V^{n-1} - W^{n-1} \|^2 + \left( \frac{1}{2} + k \right) \| V^n - W^n \|^2
+ \frac{k}{2} \| d \left( Q_h (\Psi_1 \cdot V^{n-1}) \right) V^{n-1} - d \left( Q_h (\Psi_1 \cdot W^{n-1}) \right) W^{n-1} \|^2
+ \frac{k}{2} \| P^n - R^n \|^2 + \frac{k}{2} \left( 1 + \frac{1}{h} \right) | V_0^{n-1} - W_0^{n-1} |^2
- \frac{1}{h} | V_1^{n-1} - W_1^{n-1} |^2 - \frac{1}{h} | V_{M-1}^{n-1} - W_{M-1}^{n-1} |^2
+ \left( 1 + \frac{1}{h} \right) | V_M^{n-1} - W_M^{n-1} |^2 \right). \tag{33}
\]

Using the same argument to establish (20), we obtain
\[
\left( 1 + \frac{1}{h} \right) | V_M^{n-1} - W_M^{n-1} |^2 - \frac{1}{h} | V_{M-1}^{n-1} - W_{M-1}^{n-1} |^2
\leq C \left( \| V^{n-1} - W^{n-1} \|^2 + | P_{M-1}^{n-1} - R_{M-1}^{n-1} |^2 \right). \tag{34}
\]

Using (33), (34), the definition of \( \tilde{\Phi}_h \), and the arguments used in Theorems 2–4, it is straightforward to show that (28) is indeed a convergent scheme whenever the hypotheses in Theorem 4 hold.

## 5 Numerical simulations

In this section, we present some examples in which the numerical solutions to (1), (27) and (30) are computed using (2), (28) and (31), respectively, to validate the results in the earlier sections. If \( E_h \) denotes the magnitude of the error with step size \( h \), then the experimental
order of convergence can be computed using the standard formula

$$\text{Order} = \frac{\log(E_h) - \log(E_h^{(2)})}{\log 2}.$$ 

All the computations that are presented in this section have been performed using Matlab 8.5 (R2015a). In all the examples, we have taken $a_+ = 1$, $\psi_1(x) \equiv \psi_2(x) \equiv \psi_3(x) \equiv 1$ and $r = \frac{k}{h^2} = 0.4$.

**Example 1** To test the efficacy of the numerical scheme, we assume that $u_0$, $d$, and $B$ are given by

$$u_0(x) = e - e^x, \; d(x, s) = 1, \; B(x, s) = e, \; x \in (0, 1), \; s \geq 0.$$ 

Note that given vital rates ($d$ and $B$) are constant. Therefore, the first equation in (1) becomes linear. We now seek a solution to (1) of the form $u(x, t) = (c_1 e^{(\Lambda_1 x)} + c_2 e^{(\Lambda_2 x)}) e^{(\lambda t)}$. On substituting $u$ in (1), an easy computation gives $\Lambda_{1,2} = \frac{(1 \pm \sqrt{1 - 4(d + B)})}{2}$, where $\lambda$ is a solution of the characteristic equation

$$\det\begin{pmatrix} e^{\Lambda_1} & e^{\Lambda_2} \\ 1 - \Lambda_1 + \frac{1 - e^{\Lambda_1}}{\Lambda_1} & 1 - \Lambda_2 + \frac{1 - e^{\Lambda_2}}{\Lambda_2} \end{pmatrix} = 0.$$ 

(35)

One can easily verify that $\lambda = -1$ is a solution of (35). This readily gives us that $\Lambda_1 = 0$, $\Lambda_2 = 1$. After substituting $u(x, t) = (c_1 e + c_2 e^x) e^{-t}$ in the initial condition and the right boundary condition given in (1), we find $c_1 = 1$ and $c_2 = -1$. Hence for the given set of vital rates, $u(x, t) = (e - e^x) e^{-t}$ is the solution to (1). It is straightforward to check that $d$ and $B$ satisfy the hypotheses of Theorem 4. Hence (2) is a convergent numerical scheme.

In Fig. 1, we show the absolute difference between the exact solution and the computed solution. In Fig. 1(left), we present the exact solution $u$ to (1) and the corresponding numerical solutions using (2) with $h = 0.05$, $0.01$, $0.005$ at $t = 0.2$. From this figure, it is evident that $U_{0.05}$, $U_{0.01}$ and $U_{0.005}$ are very close to $u$ at $t = 0.2$. This phenomenon re-validates the result that is proved in Theorem 4. In Fig. 1(right), the difference between $u(x, 0.2)$ and $U_h$ at $t = 0.2$, with $h = 0.05$, $0.01$, $0.005$ are shown. From these figures, we can conclude
Table 1 The magnitude of the global discretization error and the order of convergence for different choices of \( h \) at \( t = 0.2 \) with \( d, B \) given in Example 1

| \( h \) | \( \| U_0 - u_0 \|_* \) | Order | \( \max_{1 \leq n \leq N} \{ \| U^n - u^n \| \} \) | Order | \( \| e_h \|_{X_h} \) | Order |
|------|-----------------|-------|-----------------|-------|-----------------|-------|
| 0.1  | 0.0212          | 1.1009| 0.0391          | 0.9805| 0.0412          | 1.0212|
| 0.05 | 0.0099          | 1.0509| 0.0198          | 0.9921| 0.0203          | 1.0105|
| 0.02 | 0.0037          | 1.0204| 0.0079          | 0.9972| 0.0080          | 1.0041|
| 0.01 | 0.0018          | 1.0102| 0.0039          | 0.9986| 0.0040          | 1.0020|
| 0.005| 0.0009          | 1.0051| 0.0020          | 0.9993| 0.0020          | 1.0010|

that the sequence \( U_h \) indeed converges to the solution \( u \) as \( h \) tends to zero at \( t = 0.2 \), as mentioned in Theorem 4.

In Table 1, we display the magnitude of the global discretization error and the experimental order of convergence in \([0, 1] \times [0, 0.2]\) for different choices of \( h \). In the second column of Table 1, we show the error at the boundary point \( x = 0 \), and in the fourth column the interior error, i.e., \( \max_{1 \leq n \leq N} \| U^n - u^n \| \) is shown. In the third, and fifth columns, the experimental order of convergence corresponding to the boundary \( x = 0 \), interior of the domain are given, respectively. Finally, the experimental order of convergence corresponding to the global discretization error is shown in the last column. From Table 1, one can easily observe that the order of convergence of the proposed numerical scheme is one.

Example 2 In this example, we consider the non-homogeneous case described in Sect. 4.1. In particular, we consider (27) with \( u_0, d, B \), and \( g \) are given by

\[
\begin{align*}
    u_0 &= \frac{e^{-x}}{2}, \\
    d(x, s) &= 1 + \frac{s}{1 - e^{-s}}, \quad B(x, s) = 2e^x, \quad x \in (0, 1), \quad s \geq 0, \\
    g(t) &= \frac{e^{-1}}{1 + e^{-t}}, \quad t > 0.
\end{align*}
\]

We observe that \( d \) and \( B \) satisfy hypotheses of Theorem 4. Therefore (28) is a convergent numerical scheme. On the other hand, it is easy to check that for the given set of functions, the function

\[
    u(x, t) = \frac{e^{-x}}{1 + e^{-t}}, \quad x \in (0, 1), \quad t \geq 0,
\]

is a solution to (27).

We display the exact solution \( u \) to (27) and the numerical solutions \( U \) using (28) in Fig. 2. In Fig. 2(left), the exact solution \( u \) to (27) and numerical solutions using (28) with \( h = 0.05, \ 0.01, \ 0.005 \) at \( t = 0.8 \) are presented. From this figure, it is evident that \( U_{0.05}, U_{0.01} \) and \( U_{0.005} \) are approaching to \( u \) at \( t = 0.8 \). In Fig. 2(right), we show the absolute difference between \( u \) and \( U_h \) at \( t = 0.8 \), with \( h = 0.05, \ 0.01, \ 0.005 \). We conclude from these figures that the sequence of numerical solutions \( U_h \) indeed converges to the solution \( u \) at \( t = 0.8 \) as \( h \) tends to 0.

In Table 2, we show computational errors and their experimental order of convergence for various choices of \( h \) at \( t = 0.8 \). In particular, we display the error at the boundary point \( x = 0 \), the maximum error in the interior of domain and the global discretization error in the second, fourth and sixth columns of the table, respectively. On the other hand, the experimental order...
Fig. 2 The exact solution to (27) and the approximate solutions using (28) at $t = 0.8$ with $d(x, s)$, $B(x, s)$, $g(t)$ given in Example 2; Left: $u(x, 0.8)$ (solid line), $U_{0.05}$ (dotted line), $U_{0.01}$ (dash-dotted line), $U_{0.005}$ (dashed line) for $0 \leq x \leq 1$, Right: $|u(x, 0.8) - U_{0.05}|$ (dotted line), $|u(x, 0.8) - U_{0.01}|$ (dash-dotted line) and $|u(x, 0.8) - U_{0.005}|$ (dashed line)

Table 2 The magnitude of the global discretization error and the order of convergence for different choices of $h$ at $t = 0.8$ with $d(x, s)$, $B(x, s)$, $g(t)$ given in Example 2

| $h$   | $\|U_0 - u_0\|_\infty$ | Order $\max_{1 \leq n \leq N} (\|U^n - u^n\|)$ | Order $\|e_h\|_{X_h}$ | Order |
|-------|------------------------|---------------------------------|------------------------|-------|
| 0.1   | 0.0451                 | 1.0404                          | 0.0416                 | 0.9425| 0.0461 |
| 0.05  | 0.0219                 | 1.0202                          | 0.0216                 | 0.9722| 0.0227 |
| 0.02  | 0.0086                 | 1.0080                          | 0.0088                 | 0.9891| 0.0090 |
| 0.01  | 0.0042                 | 1.0040                          | 0.0044                 | 0.9946| 0.0045 |
| 0.005 | 0.0021                 | 1.0020                          | 0.0022                 | 0.9973| 0.0023 |

of convergence corresponding to the error at the boundary point $x = 0$, the maximum error in the interior of domain and the global discretization error are presented in the third, fifth and seventh columns of the table, respectively. From Table 2, we observe that the experimental order of convergence of the proposed scheme is indeed one.

**Example 3** In this example, we take the nonflux boundary condition at the right boundary described in Sect. 4.2, i.e., $B_2 = 0$. Let the vital rates $d$, $B_1$, $B_2$ and the initial data $u_0$ be given by

$$u_0(x) = e^{-x}, \quad d(x, s) = 2 + 4\left(\frac{s}{1 - e^{-1}}\right)^\frac{1}{2},$$

$$B_1(x, s) = 2e^x, \quad B_2(x, s) = 0, \quad (0, 1), \quad s \geq 0.$$

Once again, using the ansatz $u(x, t) = X(s)T(t)$ and substituting it in (30), we obtain that $u(x, t) = \frac{e^{-x}}{(1+t)^3}$ is the solution to (30). Moreover, it is easy to verify that $d$, $B_1$ and $B_2$ satisfy hypotheses of Theorem 4. Therefore (31) is a convergent numerical scheme.

We compare the exact solution to (30) and the approximate solutions that are computed using (31) for different values of $h$ at $t = 0.8$ in Fig. 3. In particular, the exact solution to (30) and approximate solutions to (30) with $h = 0.05$, 0.01, 0.005 at $t = 0.8$ are shown in Fig. 3(left). Moreover, we plot the absolute difference between $u(x, 0.8)$ and $U_h$ with $h = 0.05$, 0.01, 0.005 at $t = 0.8$ in Fig. 3(right). From these graphs, it is clear that $U_h$
Fig. 3 The exact solution to (30) and the approximate solutions using (31) at $t = 0.8$ with $d(x, s)$, $B_1(x, s)$, $B_2(x, s)$ given in Example 3; Left: $u(x, 0.8)$ (solid line), $U_{0.05}$ (dotted line), $U_{0.01}$ (dash-dotted line), $U_{0.005}$ (dashed line) for $0 \leq x \leq 1$, Right: $|u(x, 0.8) - U_{0.05}|$ (dotted line), $|u(x, 0.8) - U_{0.01}|$ (dash-dotted line) and $|u(x, 0.8) - U_{0.005}|$ (dashed line)

Table 3 The magnitude of the global discretization error and the order of convergence for different choices of $h$ at $t = 0.8$ with $d(x, s)$, $B_1(x, s)$, $B_2(x, s)$ given in Example 3

| $h$   | $\|U_M - u_M\|_*$ | Order | $\max_{1 \leq n \leq N} (\|U^n - u^n\|)$ | Order | $\|e_h\|_{X_h}$ | Order |
|-------|-------------------|-------|------------------------------------------|-------|----------------|-------|
| 0.1   | 0.0076            | 1.2307| 0.0141                                   | 0.8375| 0.0169         | 0.9790|
| 0.05  | 0.0032            | 1.0394| 0.0079                                   | 0.9180| 0.0085         | 0.9788|
| 0.02  | 0.0012            | 0.9942| 0.0033                                   | 0.9671| 0.0035         | 0.9898|
| 0.01  | 0.0006            | 0.9932| 0.0017                                   | 0.9835| 0.0017         | 0.9947|
| 0.005 | 0.0003            | 0.9956| 0.0008                                   | 0.9917| 0.0008         | 0.9973|

approaches $u(x, 0.8)$ as $h$ goes to zero at $t = 0.8$. Furthermore, one can conclude that the numerical scheme (31) converges.

In Table 4, we present the absolute error $|u^n_i - U^n_i|$ and the experimental order of convergence for different choice of $h$ at $t = 0.8$. In particular, we show the error at the boundary point $x = 1$ and the maximum error in the interior of domain in the second and fourth columns, respectively. In the third and fifth columns, we display the experimental order of convergence corresponding to the boundary point $x = 1$ and the interior of domain, respectively. Moreover, the global discretization error and the corresponding experimental order of convergence are shown in the sixth and seventh columns of the table, respectively. From Table 4, one can conclude that the experimental order of convergence of the proposed numerical scheme (31) is one.

Example 4 In this example, we choose the vital rates $d$, $B_1$, $B_2$ and the initial data $u_0$ such that the solution to (30) is known in the closed form. In particular, let $u_0$, $d$, $B_1$ and $B_2$ be given by

$$u_0(x) = e^{-\frac{(2x-1)^2}{16}}, \quad d(x, s) = 1 + \frac{2x - 1}{4} + \frac{(2x - 1)^2}{16} - \frac{s}{2 \int_0^1 e^{-\frac{(2x-1)^2}{16}} dx},$$

$$B_1(x, s) = B_2(x, s) = \frac{3e^{-1/16}}{4 \int_0^1 e^{-\frac{(2x-1)^2}{16}} dx}, \quad x \in (0, 1), \quad s \geq 0.$$
Fig. 4 The exact solution to (30) and the approximate solutions using (31) at $t = 0.8$ with $d(x, s), B_1(x, s), B_2(x, s)$ given in Example 4; Left: $u(x, 0.8)$ (solid line), $U_{0.05}$ (dotted line), $U_{0.01}$ (dash-dotted line), $U_{0.005}$ (dashed line) for $0 \leq x \leq 1$. Right: $|u(x, 0.8) - U_{0.05}|$ (dotted line), $|u(x, 0.8) - U_{0.01}|$ (dash-dotted line) and $|u(x, 0.8) - U_{0.005}|$ (dashed line).

Fig. 5 The approximate solutions to (1) at $t = 0.8$ with $d(x, s), B_1(x, s), B_2(x, s)$ given in Example 5; Left: $U_{0.05}$ (dotted line), $U_{0.01}$ (dash-dotted line), $U_{0.005}$ (solid line) for $0 \leq x \leq 1$ at $t = 0.8$. Right: $|U_{0.005} - U_{0.05}|$ (dotted line) and $|U_{0.005} - U_{0.01}|$ (dash-dotted line).

Now it is straightforward to verify that $u(x, t) = \frac{2}{1+4e^{(2x-1)^2}}$ is the solution to (30). On the other hand, it is easy to check that $d, B_1$ and $B_2$ satisfy hypotheses of Theorem 4. Therefore (31) is a convergent numerical scheme.

In Fig. 4, we plot the exact solution to (30) and computed solutions using (31) for different values of $h$ at $t = 0.8$. In Fig. 4(left), the exact, and approximate solutions to (30) with $h = 0.05$, $0.01$, $0.005$ at $t = 0.8$ are presented. From these figures, it is straightforward to see that $u(x, 0.8)$ is closer to $U_{0.005}$ than $U_{0.01}$ and $U_{0.05}$ at $t = 0.8$. In Fig. 4(right), we plot the absolute difference between $u(x, 0.8)$ and $U_h$ with $h = 0.05$, $0.01$, $0.005$ at $t = 0.8$. From these graphs, one can observe that the numerical solutions $U_h$ converge.

We display various discretization errors and their experimental orders of convergence for different choice of $h$ at $t = 0.8$ in Table 4. We present the error at the boundary point $x = 0$ and the maximum error in the interior of domain in the second and fourth columns, respectively. In the third and fifth columns, we show the experimental orders of convergence corresponding to the boundary point $x = 0$ and the interior domain, respectively. Moreover, the global discretization error and the corresponding experimental order of convergence are
Table 4  The magnitude of the global discretization error and the order of convergence for different choices of \( h \) at \( t = 0.8 \) with \( d(x, s), B_1(x, s), B_2(x, s) \) given in Example 4

| \( h \)  | \( \|U_0 - u_0\|_* \) | Order | \( \max_{1 \leq n \leq N}\{|\|U^n - u^n\|\} \) | Order | \( \|e_h\|_{X_h} \) | Order |
|--------|-----------------|------|-----------------|------|----------------|------|
| 0.1    | 0.0120          |       | 1.0586          |       | 0.0114         |       |
| 0.05   | 0.0057          |       | 1.0290          |       | 0.0055         |       |
| 0.02   | 0.0022          |       | 1.0115          |       | 0.0021         |       |
| 0.01   | 0.0011          |       | 1.0057          |       | 0.0010         |       |
| 0.005  | 0.0005          |       | 1.0028          |       | 0.0005         |       |

given in the sixth and seventh columns of the table, respectively. From Table 4, it is easy to observe that the experimental order of convergence of the proposed scheme is one.

Example 5 To test our numerical scheme, we assume that \( d, B \) and \( u_0 \) are given by

\[
\begin{align*}
  u_0(x) &= e - e^x, \\
  d(x, s) &= 2 + x^2 + \frac{s^2}{2}, \\
  B(x, s) &= 2e^x + s, \\
  x &\in (0, 1), \quad s \geq 0.
\end{align*}
\]

Note that, the given set of functions \( d, B \) and \( u_0 \) satisfy hypotheses of Theorem 4. Hence, (2) is a convergent numerical scheme.

In Fig. 5(left), we present approximate solutions to (1) at \( t = 0.8 \) for \( h = 0.05, 0.01, 0.005 \). On the other hand, we display the absolute difference \( |U_h - U_{0.005}| \) at \( t = 0.8 \) for \( h = 0.05, 0.01 \) in Fig. 5(right). From this figure, it is evident that \( U_h \)'s are very close to each other as \( h \) goes to zero, and the limit of the sequence \( U_h \) indeed converges to the solution of (1) as mentioned in Theorem 4.

Conclusions

We have proposed a finite difference numerical scheme to the McKendrick–Von Foerster equation with diffusion (1) in which the Robin condition is prescribed at the boundary point \( x = 0 \), and the Dirichlet condition is given at \( x = a_\ast \). Furthermore, we have proved that the proposed numerical scheme is stable restricted to the thresholds \( R_h \). Moreover, we have established that the given scheme is indeed convergent using a result due to Stetter and the rate of convergence is one. The result is extended to the M-V-D with nonlocal nonlinear Robin boundary conditions at both the end points in a bounded domain (see (30)). Using the similar technique, one can easily obtain a convergent scheme when (1) has nonlinear, nonlocal Neumann boundary condition at \( x = 0 \). However, it is an interesting problem to design a convergent scheme for (1) in the unbounded domain \([0, \infty)\).
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Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflict of interest The authors declare no competing interests.

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