Voronoi-Dickson Hypothesis on Perfect Forms and L-types

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Short Version

Abstract

George Voronoi (1908, 1909) introduced two important reduction methods for positive quadratic forms: the reduction with perfect forms, and the reduction with $L$-type domains, often called domains of Delaunay type. The first method is important in studies of dense lattice packings of spheres. The second method provides the key tools for finding the least dense lattice coverings with equal spheres in lower dimensions. In his investigations Voronoi heavily relied on that in dimensions less than 6 the partition of the cone of positive quadratic forms into $L$-types refines the partition of this cone into perfect domains. Voronoi conjectured implicitly and Dickson (1972) explicitly that the $L$-partition is always a refinement of the partition into perfect domains. This was proved for $n \leq 5$ (Voronoi, Delaunay, Ryshkov, Baranovskii). We show that Voronoi-Dickson conjecture fails already in dimension 6.

Keywords: positive quadratic form, perfect form, point lattice, Delaunay tiling ($L$-partition), $L$-type, repartitioning complex, lattice packings and coverings, lattices $E_6$ and $E_6^*$, Voronoi reduction, integral representations of groups $D_4$, $E_6$, $E_6^*$, Gosset polytope $2_{21}$

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1 Introduction and main result

Positive quadratic forms (referred to as PQFs) in $n$ indeterminate form a closed cone $\mathcal{P}(n)$ of dimension $N = \frac{n(n+1)}{2}$ in $\mathbb{R}^N$, and this cone is the main object of study in our paper. The interior of $\mathcal{P}(n)$ consists of positive definite forms of rank $n$. We abbreviate positive definite forms as PDQFs. PDQFs serve as algebraic representations of point lattices. There is a one-to-one correspondence between isometry classes of $n$-lattices and integral equivalence classes (i.e. with respect to $GL(n, \mathbb{Z})$-conjugation).
of PDQFs in $n$ indeterminates. For basic results of the theory of lattices and PQFs and their applications see Ryshkov and Baranovskii (1978), Gruber and Lekkerkerker (1987), Erdős, Gruber and Hammer (1989), Conway and Sloane (1999).

$GL(n,\mathbb{Z})$ acts pointwise on the space of quadratic forms $\text{Sym}(n,\mathbb{R}) \cong \mathbb{R}^N$. A *polyhedral reduction partition* $R$ of $\mathfrak{P}(n)$ is a partition of this cone into open convex polyhedral cones such that:

**Definition 1.1**
1. it is invariant with respect to $GL(n,\mathbb{Z})$;
2. there are finitely many inequivalent cones in this partition;
3. for each cone $C$ of $R$ and any PQF $f$ in $n$ indeterminates, $f$ can be $GL(n,\mathbb{Z})$-equivalent to at most finitely many forms lying in $C$.

The *partition into perfect cones* and the *$L$-type partition* (also referred to as the Voronoi partition of the 2nd kind, or the partition into Voronoi reduction domains) are important polyhedral reduction partitions of $\mathfrak{P}(n)$. (Our usage of term *domain* is lax; it should be clear from the context whether we mean the whole arithmetic class, or just one element of this class.) These partitions have been intensively studied in geometry of numbers since times of Korkin, Zolotareff (1873) and Voronoi (1908), and more recently in combinatorics (e.g. Deza et al. (1997)), and algebraic geometry (e.g. Alexeev (1999a,b)). *In most previous works* (e.g. Voronoi (1908,1909), Ryshkov, Baranovskii (1976), Dickson (1972)) *the $L$-type partition of* $P(n)$, or *sub-cones of* $P(n)$, *was constructed by refining the perfect partition*. It is not an exaggeration to say that in almost any systematic study, except for Engels’ computational investigations, $L$-types were approached via perfect forms. For example, Voronoi started classifying 4-dimensional $L$-types by analyzing the Delaunay ($L$-)tilings of forms lying in the 1st ($A_n$) and 2nd ($D_n$) perfect domains. The same route was followed by Ryshkov and Baranovskii (1976). It was widely believed that the $L$-type partition is the refinement of the perfect partition, i.e. each convex cone of the perfect partition is the union of finitely many convex cones from the $L$-type partition. This conjecture is implicit in Voronoi’s memoirs (1908-1909), and explicit in Dickson (1972), where he showed that the first perfect domain is the only perfect domain which coincided with an $L$-type domain.

In our paper we present the results of our study of the relationship between these two partitions for $n = 6$. We have proved that for $n = 6$ the $L$-type partition is NOT a refinement of the partition into perfect cones, as was believed before. This note contains only a sketch of the proof of this result, which will be later published in a longer journal paper (about 35-40pp).

## 2 Perfect and $L$-type partitions
2.1 L-types

Definition 2.1 Let \( L \) be a lattice in \( \mathbb{R}^n \). A convex polyhedron \( P \) in \( \mathbb{R}^n \) is called a Delaunay cell of \( L \) with respect to a positive quadratic form \( f(x, x) \) if:

1. for each face \( F \) of \( P \) we have \( \text{conv}(L \cap F) = F \);
2. there is a quadric circumscribed about \( P \), called the empty ellipsoid of \( P \) when \( f(x, x) \) is positive definite, whose quadratic form is \( f(x, x) \) (in case rank \( f < n \), this quadric is an elliptic cilinder);
3. no points of \( L \) lie inside the quadric circumscribed about \( P \).

When \( f = \sum_{i=1}^{n} x_i^2 \), our definition coincides with the classical definition of Delaunay cell in \( \mathbb{R}^n \). Delaunay cells form a convex face-to-face tiling of \( L \) that is uniquely defined by \( L \) (Delaunay, 1937). Two Delaunay cells are called homologous if they can be mapped to each other with a composition of a lattice translation and a central inversion with respect to a lattice point.

Definition 2.2 PQFs \( f_1 \) and \( f_2 \) belong to the same convex \( L \)-domain if the Delaunay tilings of \( \mathbb{Z}^n \) with respect to \( f_1 \) and \( f_2 \) are identical. \( f_1 \) and \( f_2 \) belong to the same \( L \)-type if these tilings are equivalent with respect to \( \text{GL}(n, \mathbb{Z}) \).

The following proposition establishes the equivalence between the Delaunay’s definition of \( L \)-equivalence for lattices and the notion of \( L \)-equivalence for arbitrary PQFs, which is introduced above.

Proposition 2.3 Positive definite forms \( f \) and \( g \) belong to the same \( L \)-type if the corresponding lattices belong to the same \( L \)-type with respect to the form \( \sum_{i=1}^{n} x_i^2 \).

Theorem 2.4 (Voronoi) The partition of \( \mathcal{P}(n) \) into \( L \)-types is a reduction partition. Moreover, it is face-to-face.

The notions of Delaunay tiling and \( L \)-type are important in the study of extremal and group-theoretic properties of lattices. For example, the analysis of Delaunay cells in the Leech lattice conducted by Conway, Sloane (e.g. see (1999)) and Borcherds showed that 23 ”deep holes” (Delaunay cells of radius equal to the covering radius of the lattice) in the Leech lattice correspond to 23 even unimodular 24-dimensional lattices (Niemier’s list) that, in turn, give rise to 23 ”gluing” constructions of the Leech lattice from root lattices. Barnes and Dickson (1967, 1968) and, later, in a geometric form, Delaunay et al. (1969, 1970) proved the following

Theorem 2.5 The closure of any \( N \)-dimensional convex \( L \)-domain contains at most one local minimum of the sphere covering density. The group of \( \text{GL}(n, \mathbb{Z}) \)-automorphisms of the domain maps this form to itself.
Using this approach, Delaunay, Ryshkov and Baranovskii (1963, 1976) found the best lattice coverings in $\mathbb{E}^4$ and $\mathbb{E}^5$. The theory of $L$-types also has numerous connections to combinatorics and, in particular, to cuts, hypermetrics, and regular graphs (see Deza et al. (1997)). Recently, V. Alexeev (1999a,b) found exciting connections between compactifications of moduli spaces of principally polarized abelian varieties and $L$-types.

2.2 Perfect cones

The $L$-type partition of $\mathcal{P}(n)$ is closely related to the theory of perfect forms originated by Korkine and Zolotareff (1873). Let $f(x,x)$ be a PDQF. The arithmetic minimum of $f(x,x)$ is the minimum of this form on $\mathbb{Z}^n$. The integral vectors on which this minimum is attained are called the representations of the minimum, or the minimal vectors of $f(x,x)$: these vectors have the minimal length among all vectors of $\mathbb{Z}^n$ when $f(x,x)$ is used as the metrical form. Form $f(x,x)$ is called perfect if it can be reconstructed up to scale from all representations of its arithmetic minimum. In other words, a form $f(x,x)$ with the arithmetic minimum $m$ and the set of minimal vectors $\{v_k | k = 1, ..., 2s\}$ is perfect if the system

$$\sum_{i,j=1}^{n} a_{ij} v_i^k v_j^k = m,$$

where $k = 1, ..., 2s$, has a unique solution $(a_{ij})$ in $\text{Sym}(n, \mathbb{R}) \cong \mathbb{R}^N$ (indeed, uniqueness requires at least $n(n+1)$ minimal vectors).

Definition 2.6 PQFs $f_1$ and $f_2$ belong to the same cone of the perfect partition if they both can be written as strictly positive linear combinations of some subset of minimal vectors of a perfect form $\phi$. $f_1$ and $f_2$ belong to the same perfect type if there is $f_1'$, equivalent to $f_1$, such that $f_1'$ and $f_2$ belong to the same cone of the perfect partition.

Theorem 2.7 (Voronoi) The partition of $\mathcal{P}(n)$ into perfect domains is a reduction partition. Moreover, it is face-to-face. Each 1-dimensional cone of this partition lies on $\partial \mathcal{P}(n)$.

Perfect forms play an important role in lattice sphere packings. Voronoi’s theorem (1908) says that if a form is extreme—i.e., a maximum of the packing density—it must also be perfect (see Coxeter (1951), Conway, Sloane (1988) for the proof). The notion of eutactic form arises in the study of the dense lattice sphere packings and is directly related to the notion of perfect form. The reciprocal of $f(x,x)$ is a form whose Gramm matrix is the inverse of the Gramm matrix of $f(x,x)$. The dual form is normally denoted by $f^*(x,x)$. A form $f(x,x)$ is called eutactic if the dual form $f^*(x,x)$ can be written as $\sum_{k=1}^{s} \alpha_k (v_k \cdot x)^2$, where $\{v_k | k = 1, ..., s\}$ is the set of mutually non-collinear minimal vectors of $f(x,x)$, and $\alpha_k > 0$.

Theorem 2.8 (Voronoi) A form $f(x,x)$ is a maximum of the sphere packing density if and only if $f(x,x)$ is perfect and eutactic.
Voronoi gave an algorithm finding all perfect domains for given \( n \). This algorithm is known as Voronoi’s reduction with perfect forms. For the computational analysis of his algorithm and its improvements see Martinet (1996). The perfect forms and the incidence graphs of perfect partitions of \( \mathfrak{P}(n) \) have have been completely described for \( n \leq 7 \).

2.3 The relationship in low dimensions. The case of \( n = 6 \).

Voronoi (1908-09) proved that for \( n = 2, 3 \) the \( L \)-partition and the perfect partition of \( \mathfrak{P}(n) \) coincide. The perfect facet \( D_4 \) (the 2nd perfect form in 4 variables) exemplifies a new pattern in the relation of these partitions. Namely, the facet \( D_4 \) is decomposed into a number of simplicial \( L \)-type domains like a pie: this decomposition consists of the cones with apex at the affine center of this facet over the \((N-2)\)-faces. These simplexes are \( L \)-type domains of two arithmetic types: type I is adjacent to the the perfect/\( L \)-type domain of \( A_4 \), type II is adjacent to an arithmetically equivalent \( L \)-type domain (also type II, indeed) from the \( L \)-subdivision of the adjacent \( D_4 \) domain (for details see Delaunay et al. (1963, 1968)).

Voronoi also proved that for \( n = 4 \) the tiling of \( \mathfrak{P}(n) \) with \( L \)-type domains refines the partition of this cone into perfect domains. Ryshkov and Baranovskii (1975) proved the refinement hypothesis for \( n = 5 \). In his paper of 1972 T.J. Dickson proved that the perfect domain of \( A_n \), also called the first perfect form after Korkine and Zolotareff (1873), is the only perfect domain that is also an \( L \)-type domain; he was also first to explicitly mention the common believe in Voronoi’s refinement hypothesis.

3 Metrical forms for the lattices \( E_6 \) and \( E_6^* \)

Consider the following symmetric sets of vectors in \( \mathbb{Z}^6 \):

\[
\mathcal{P}_1 = \{\pm[-3;2^5], \pm[2;-2;-1^4], \pm[1;0;-1^4]\}
\]

\[
\mathcal{P}_2 = \{\pm[2;-1;-2,-1^3]\times 4, \pm[1;-1;0,-1^3]\times 4, \pm[1;0^2,-1^3]\times 10, \pm[0;1,0^4]\times 5, \pm[2;-1^5]\}
\]

\[
\mathcal{P}_3 = \{\pm[0;0;1,-1,0^2]\times 6, \pm[1;0;-1^2,0^2]\times 6\}
\]

Here we use a short-hand notation for families of vectors obtained from some \( n \)-vector by all circular permutations of selected subsets of its components: (1) \( m^k \) stands for \( k \) consecutive \( m \)'s, (2) square brackets \( [a_1...a_n] \) are used to denote all vectors that can be obtained from vector \((a_1...a_n)\) by circular permutations in strings of symbols that are separated by commas and bordered on the sides by semicolons and/or brackets, (3) numbers between semicolons and/or brackets are not permuted.

The two sets of perfect vectors \( \mathcal{P}_{E_6^*} = \mathcal{P}_1 \cup \mathcal{P}_2 \) and \( \mathcal{P}_{E_6} = \mathcal{P}_2 \cup \mathcal{P}_3 \) are the minimal vectors for the metrical forms \( \pi_{E_6^*} \) and \( \pi_{E_6} \), with arithmetic minimum \( m \) and
coefficient matrices

\[ \mathbf{P}_{E_6} = \frac{m}{2} \begin{bmatrix} 8 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 & 1 & 1 \\ 3 & 0 & 1 & 2 & 1 & 1 \\ 3 & 0 & 1 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{P}_{*E_6} = \frac{m}{4} \begin{bmatrix} 16 & 5 & 5 & 5 & 5 & 5 \\ 5 & 4 & 1 & 1 & 1 & 1 \\ 5 & 1 & 4 & 1 & 1 & 1 \\ 5 & 1 & 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 & 4 & 1 \\ 5 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}. \]

The metrical forms \( \pi_{E_6}^* \) and \( \pi_{E_6} \) are integrally equivalent to forms \( \phi_4 \) and \( \phi_2 \) from Barnes’s table (1957). That \( \pi_{E_6}^* \) and \( \pi_{E_6} \) are forms for the root lattices \( E_6^* \) and \( E_6 \) is established by the numbers of perfect vectors, which are given by \( |\mathcal{P}_{E_6}| = 54 \) and \( |\mathcal{P}_{E_6}| = 72 \). The. As shown below, if this scale parameter is set equal to \( \sqrt{8/3} \), then, the geometric lattices corresponding to \( \pi_{E_6}^* \) and \( \pi_{E_6} \) are dual lattices.

**Perfect vectors and perfect domains.** Let \( V \subset \mathbb{Z}^d \) be centrally symmetric. Then define the domain of \( V \) to be given by

\[ \Phi(V) = \{ \varphi(x) = \sum_{v \in V^+} \omega_v(v \cdot x)^2 | \omega_v \geq 0 \}; \]

as indicated by the superscript +, this summation is over an oriented subset of vectors − one vector from each pair of opposites in the symmetric set \( V \). The domains \( \Phi(\mathcal{P}_{E_6}), \Phi(\mathcal{P}_{E_6}^*) \) are perfect cones of types \( E_6 \) and \( E_6^* \), respectively; they are defined on the perfect vectors for the perfect forms \( \pi_{E_6} \) and \( \pi_{E_6}^* \).

**Proposition 3.1** \( \Phi(\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^*) \) is a facet of both \( \Phi(\mathcal{P}_{E_6}) \) and \( \Phi(\mathcal{P}_{E_6}^*) \).

**Proof.** If forms \( \pi, \varphi \) have coefficient matrices \( \mathbf{P}, \mathbf{F} \), define the scalar product \( \langle \pi, \varphi \rangle := \text{trace}(\mathbf{P} \mathbf{F}) \). Consider the form \( \pi_{E_6}^* - \pi_{E_6} \) and the rank one form \( \varphi_p(x) = (p \cdot x)^2 \). Then, \( \langle \pi_{E_6}^* - \pi_{E_6}, \varphi_p \rangle = \text{trace}(\mathbf{P}_{E_6}^* - \mathbf{P}_{E_6})pp^T = p^T(\mathbf{P}_{E_6}^* - \mathbf{P}_{E_6})p = \pi_{E_6}^*(p) - \pi_{E_6}(p) \) if and only if \( p \in \mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^* \); if \( p \in \mathcal{P}_{E_6} \), then \( \pi_{E_6}^*(p) - \pi_{E_6}(p) = m - \pi_{E_6}(p) \leq 0 \), with equality if and only if \( p \in \mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^* \). It follows that the hyperplane with equation \( \langle \pi_{E_6}^* - \pi_{E_6}, \varphi \rangle = 0 \) separates \( \Phi(\mathcal{P}_{E_6}) \) and \( \Phi(\mathcal{P}_{E_6}^*) \), and that \( \Phi(\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^*) \) is a face of both these perfect domains.

That \( \Phi(\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^*) \) is a facet follows by showing that the linear span of the forms \( \varphi_p(x) = (p \cdot x)^2 \), \( p \in \mathcal{P}_{E_6} \cap \mathcal{P}_{E_6}^* \) has co-dimension one in the linear space of metrical forms. (We omit this argument.) ■

**Eutactic forms.** The forms

\[ \varphi_{E_6}(x) = \frac{m}{12} \sum_{p \in \mathcal{P}_{E_6}^+} (p \cdot x)^2, \quad \varphi_{E_6}^*(x) = \frac{m}{16} \sum_{p \in \mathcal{P}_{E_6}^+} (p \cdot x)^2, \]
lie on the central rays of the perfect cones $\Phi_{E_6^*}, \Phi_{E_6}$, and have coefficient matrices given by

$$
F_{E_6} = \frac{m}{2} \begin{bmatrix}
8 & -5 & -5 & -5 & -5 & -5 \\
-5 & 4 & 3 & 3 & 3 & 3 \\
-5 & 3 & 4 & 3 & 3 & 3 \\
-5 & 3 & 3 & 4 & 3 & 3 \\
-5 & 3 & 3 & 3 & 4 & 3 \\
-5 & 3 & 3 & 3 & 3 & 4 
\end{bmatrix},
$$

$$
F_{E_6^*} = \frac{m}{4} \begin{bmatrix}
10 & -5 & -6 & -6 & -6 & -6 \\
-5 & 4 & 3 & 3 & 3 & 3 \\
-6 & 3 & 6 & 3 & 3 & 3 \\
-6 & 3 & 3 & 6 & 3 & 3 \\
-6 & 3 & 3 & 3 & 6 & 3 \\
-6 & 3 & 3 & 3 & 3 & 6 
\end{bmatrix}.
$$

These forms are related to the original $\pi_{E_6}, \pi_{E_6^*}$ by the formulas $\varphi_{E_6}(x) = \pi_{E_6}(U_*x)$, $\varphi_{E_6^*}(x) = \pi_{E_6^*}(U_*x)$, where $U_* \in GL(6, \mathbb{Z})$ is given by

$$
U_* = \begin{bmatrix}
2 & -2 & 0 & -1 & -1 & -1 \\
-2 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 1 
\end{bmatrix},
$$

$$
U_*^{-1} = \begin{bmatrix}
0 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 \\
-1 & 0 & 0 & -1 & 0 & -1 \\
-1 & 0 & 0 & -1 & -1 & 0 
\end{bmatrix}.
$$

Being arithmetically equivalent, $\varphi_{E_6}, \pi_{E_6}$ are alternate metrical forms for the same geometric lattice, and similarly for $\varphi_{E_6^*}, \pi_{E_6^*}$. We will refer to the minimal vectors of these forms as short. The short vectors for $\varphi_{E_6}$ and $\varphi_{E_6^*}$ are related to the perfect vectors for $\pi_{E_6}$ and $\pi_{E_6^*}$ by the formulas: $S_{E_6} = S_2 \cup S_3, S_{E_6^*} = S_1 \cup S_2$, where $S_1 = U_*^{-1}(P_1), S_2 = U_*^{-1}(P_2), S_3 = U_*^{-1}(P_3)$. Explicit form for the sets $S_1, S_2, S_3$ can be obtained by a direct calculation.

$$
S_1 = \{ \pm[2, -1, 1^4], \pm[-2, 0, -1^4], \pm[0, 1, 0^4] \}
$$

$$
S_2 = \{ \pm[0, 1, -1, 0^3] \times 4, \pm[2, 0, 0, 1^2] \times 4, \pm[1, 1^2, 0^3] \times 10, \pm[1, 1, 0^4] \times 5, \pm[3, 1^5] \}
$$

$$
S_3 = \{ \pm[0, 0, 1, -1, 0^2] \times 6, \pm[2, 1, 1^2, 0^2] \times 6 \}
$$

The four coefficient matrices satisfy the relations $F_{E_6}P_{E_6} = F_{E_6^*}P_{E_6} = \frac{3}{8}m^2I$, so that when $m = \sqrt{8/3}$, $F_{E_6} = (P_{E_6})^{-1}, F_{E_6^*} = (P_{E_6})^{-1}$. Under these circumstances the pair of forms $\varphi_{E_6}, \pi_{E_6}$, and the pair $\varphi_{E_6^*}, \pi_{E_6}$, are in duality: $\varphi_{E_6} = \pi_{E_6^*}, \pi_{E_6} = \varphi_{E_6^*}, \pi_{E_6^*} = \varphi_{E_6}, \pi_{E_6} = \varphi_{E_6^*}$. Dual forms correspond to dual lattices, so the single geometric lattice corresponding to the forms $\varphi_{E_6}, \pi_{E_6}$ is dual to the geometric lattice corresponding to $\varphi_{E_6^*}, \pi_{E_6^*}$. This pair of geometric lattices is the pair of root lattices $E_6$ and $E_6^*$. The above representations for $\varphi_{E_6}, \varphi_{E_6^*}$ can equally well be considered as representations for the dual forms $\pi_{E_6^*} = \varphi_{E_6}, \pi_{E_6} = \varphi_{E_6^*}$, which shows that $\pi_{E_6^*}, \pi_{E_6}$ lie interior to the domains $\Phi_{E_6^*}, \Phi_{E_6}$ determined, respectively, by $\pi_{E_6^*}, \pi_{E_6}$. Since
\[\pi_{E_6} = \varphi_{E_6}^\circ, \quad \pi_{E_6}^* = \varphi_{E_6}^\circ,\] the original forms have eutaxy (see Section 2) representations identic to those for \(\varphi_{E_6}, \varphi_{E_6}^\circ\):

\[\pi_{E_6}(x) = \frac{m}{12} \sum_{s \in S_{E_6}^+} (s \cdot x)^2, \quad \pi_{E_6}^*(x) = \frac{m}{16} \sum_{s \in S_{E_6}^*} (s \cdot x)^2;\]

the summation again is over an oriented subset of minimal vectors for the reciprocal forms, respectively, the forms \(\varphi_{E_6}^\circ, \varphi_{E_6}^*\).

The symmetrical arrangement provided by the domains, \(\Phi_{E_6}, \Phi_{E_6}^\circ\), is remarkable – it is a rare occurrence that forms on the central rays of adjacent domains correspond to dual geometric lattices. While the lattices are dual, the forms \(\varphi_{E_6}, \varphi_{E_6}^\circ\) are not. These forms fit into the following intricate pattern: \(\varphi_{E_6}\) is dual to \(\pi_{E_6}^*\), which in turn is arithmetically equivalent to \(\varphi_{E_6}^\circ\) and, \(\varphi_{E_6}^\circ\) is dual to \(\pi_{E_6}\), which in turn is arithmetically equivalent to \(\varphi_{E_6}\). This pattern is known since the times of Korkine and Zolotareff (1873). See also Coxeter (1951). Barnes (1957) showed that there is only one arithmetic type of wall between \(E_6\) and \(E_6^*\).

### 3.0.1 The invariance groups for the forms \(\varphi_{E_6}\) and \(\varphi_{E_6}^\circ\).

The point group for the root lattice \(E_6\) is the product of the two element group generated by central inversion and the reflection group \(E_6\), and has order \(2^83^45^1\).

There is a representation \(G_{E_6} \subset GL(6, \mathbb{Z})\) that is the invariance group of the perfect form \(\varphi_{E_6}\), with the following characterization: \(g \in G_{E_6}\) if and only if \(\varphi_{E_6}(gx) = \varphi_{E_6}(x)\).

There is a second representation \(G_{E_6}^\circ\) that is the invariance group of the form \(\varphi_{E_6}^\circ\).

The dual group (representation) \(G_{E_6}^\circ\) is the invariance group of the perfect form \(\pi_{E_6}^\circ\), and the dual group (representation) \(G_{E_6}^\circ\) is the invariance group of the form \(\pi_{E_6}\).

The subgroup \(G_{E_6}^\circ\) is defined by

\[G_{E_6}^\circ = \{(g^{-1})^T | g \in G_{E_6}\},\]

where \(g^\circ = (g^{-1})^T\) is the matrix for the dual transformation; there is a similar definition for the subgroup \(G_{E_6}^\circ\). From the arithmetic equivalences \(P_{E_6}^* = U_s S_{E_6}^\circ\) and \(P_{E_6} = U_s S_{E_6}\), it follows that \(G_{E_6}^\circ = U_s G_{E_6} U_s^{-1}\) and \(G_{E_6}^\circ = U_s G_{E_6} U_s^{-1}\), where \(U_s \in GL(6, \mathbb{Z})\) was described above.

**Proposition 3.2** The subgroups \(G_{E_6}, G_{E_6}^\circ, G_{E_6}^\circ, G_{E_6}^\circ \subset GL(6, \mathbb{Z})\) are the full invariance groups of the respective sets of vectors \(S_{E_6}, S_{E_6}^\circ, P_{E_6}^*, P_{E_6}\), and act transitively on these sets.

The remaining sections will be devoted to the proof of the main theorem that provides for a counterexample to the refinement hypothesis:

**Theorem 3.3** (main) The segment \(t\varphi_{E_6} + (1-t)\varphi_{E_6}^\circ\), \(0 \leq t \leq 1\) has forms of 5 \(L\)-types: \(t = 0, 0 < t < 1/2, t = 1/2, 1/2 < t < 1, t = 1\). The wall between the perfect domains \(\Phi(P_{E_6})\) and \(\Phi(P_{E_6}^\circ)\) crosses this segment at point \(\frac{2}{5}\varphi_{E_6} + \frac{3}{5}\varphi_{E_6}^\circ\).
4 L-types

4.1 Commensurate Delaunay Tilings

A convex polyhedron $P$ in $\mathbb{R}^n$ is called a $\mathbb{Z}^n$-polyhedron if any face $F$ of $P$ is $\text{conv}(\text{aff } F \cap \mathbb{Z}^n)$. A $\mathbb{Z}^n$-tiling is a face-to-face tiling of $\mathbb{R}^n$ by convex $\mathbb{Z}^n$-polyhedra. If $T_1, T_2$ are two convex tilings of $\mathbb{R}^n$, one can define the intersection tiling of $T_1$ and $T_2$: the (open) tiles of this tiling are all non-empty intersections of (open) tiles of $T_1$ and $T_2$. The intersection tiling of two $\mathbb{Z}^n$-tilings is not always a $\mathbb{Z}^n$-tiling; it is a $\mathbb{Z}^n$-tiling if and only if the vertex set of the intersection tiling is $\mathbb{Z}^n$.

**Definition 4.1** Let $P_1$ and $P_2$ be two $\mathbb{Z}^n$-polyhedra. They are called commensurate if all the faces of $P_1 \cap P_2$ are $\mathbb{Z}^n$-polyhedra.

In particular, two $\mathbb{Z}^n$-polytopes $P_1$ and $P_2$ are commensurate if the vertex set of $P_1 \cap P_2$ belongs to $\mathbb{Z}^n$.

**Definition 4.2** Let $T_1, T_2$ be two $\mathbb{Z}^n$-tilings. They are called commensurate if their intersection tiling consist of $\mathbb{Z}^n$-polyhedra only.

In particular, the vertex set of the intersection tiling of $T_1$ and $T_2$ must be a subset of $\mathbb{Z}^n$ (possibly empty).

**Lemma 4.3** Assume that lattice tilings $\mathcal{D}(\varphi_1), \mathcal{D}(\varphi_2)$ of $\mathbb{Z}^n$ are Delaunay with respect to the PQFs $\varphi_1$ and $\varphi_2$. Also assume that $C_1, C_2$ are closed Delaunay cells for $\mathcal{D}(\varphi_1)$ and $\mathcal{D}(\varphi_2)$. If $C_1 \cap C_2 \cap \mathbb{Z}^d \neq \emptyset$, then $C = \text{conv}(C_1 \cap C_2 \cap \mathbb{Z}^d)$ is a closed Delaunay cell for all intermediate forms $\varphi_t = (1 - t)\varphi_1 + t\varphi_2$, where $0 < t < 1$.

**Proof.** Since $C_1$ is Delaunay, there is a scalars $c_1$ and vector $p_1$ so that $f_1(x) = c_1 + p_1 \cdot x + \varphi_1(x)$ is zero-valued on the vertices of $C_1$, but has positive values on all other elements of $\mathbb{Z}^d$. Therefore, the equation $f_1(x) = 0$ is for an empty ellipsoid circumscribing $C_1$. There is a similar function for $C_2$, so that $f_2(x) = c_2 + p_2 \cdot x + \varphi_2(x) = 0$ is the equation of an empty ellipsoid circumscribing $C_2$. For $0 < t < 1$, the function $f_t(x) = (1 - t)f_1(x) + tf_2(x)$ is zero-valued on the vertices of $C$, and positive on all other elements of $\mathbb{Z}^d$. Therefore, $C$ is circumscribed by the empty ellipsoid with equation $f_t(x) = 0$, and is Delaunay with respect to the form $\varphi_t = (1 - t)\varphi_1 + t\varphi_2$. ■

**Lemma 4.4** If Delaunay tilings $\mathcal{D}(f)$ and $\mathcal{D}(g)$ of $\mathbb{Z}^n$ with respect to PQFs $f$ and $g$ are commensurate, then their intersection $\mathbb{Z}^n$-tiling is Delaunay with respect to $\alpha f + \beta g$ for any $\alpha, \beta > 0$. Conversely, if for all $0 < t < 1$ the Delaunay tiling of $\mathbb{Z}^n$ with respect to $tf + (1 - t)g$ is the same, then Delaunay tilings $\mathcal{D}(f)$ and $\mathcal{D}(g)$ of $\mathbb{Z}^n$ are commensurate.
Proof. \[ \implies \) Since \( \mathcal{D}(f) \) and \( \mathcal{D}(g) \) are commensurate, any cell \( C \) of the intersection tiling of \( \mathcal{D}(f) \) and \( \mathcal{D}(g) \) can be written as \( C = \text{conv}(C_1 \cap C_2 \cap \mathbb{Z}^d) \), where \( C_1, C_2 \) are closed Delaunay cells for \( \mathcal{D}(\varphi_1) \) and \( \mathcal{D}(\varphi_2) \). By the above lemma, \( C \) is Delaunay for \( \alpha f + \beta g \) for any \( \alpha, \beta > 0 \).

\[ \iff \) If \( C \) is a Delaunay cell with respect to \( tf + (1 - t)g \), then, by a standard continuity argument, there are \( \mathbb{Z}^n \)-polyhedra \( C_f, C_g \) which are Delaunay relative to \( f \) and \( g \) respectively, such that \( C \subseteq C_f \) and \( C \subseteq C_g \). Therefore, \( C = C_f \cap C_g = \text{conv}(C_f \cap C_g \cap \mathbb{Z}^d) \). Thus, all cells of the Delaunay tiling defined by \( tf + (1 - t)g \) are the intersections of the Delaunay cells of \( \mathcal{D}(f) \) and \( \mathcal{D}(g) \). This implies that \( \mathcal{D}(f) \) and \( \mathcal{D}(g) \) are commensurate. \( \blacksquare \)

The following Proposition directly follows from Lemma 4.4.

**Proposition 4.5** Two forms \( f \) and \( g \) belong to the same \( L \)-cone (not type!) if and only if their Delaunay tilings \( \mathcal{D}(f) \) and \( \mathcal{D}(g) \) are commensurate.

5 G-topes and G*-topes

In this section we consider the Delaunay tilings for \( \mathbb{Z}^6 \) relative to the forms \( \varphi_{E_6} \) and \( \varphi_{E_6}^* \). More specifically, we study the Delaunay tilings of forms lying on the segment joining \( \varphi_{E_6} \) and \( \varphi_{E_6}^* \) of \( \mathfrak{P}(n) \). These are affinely equivalent to the Delaunay tilings for the root lattice \( E_6 \) and the dual lattice \( E_6^* \). Two Delaunay cells are said to belong to the same homology class if one of them can be obtained from the other by the composition of a lattice translation and inversions.

Our discussion is mostly descriptive and adapted specifically for studying the change of \( L \)-type along the segment joining forms \( \varphi_{E_6} \) and \( \varphi_{E_6}^* \). More information about these tilings can be found in Coxeter (1995), and Baranovski (1991).

**Tiling with G-topes.** Consider the 45 \( \mathcal{L}_{E_6} \)-triangles:

\[
\Delta_1^G = \text{conv}\{[0; 0^5], [0; 1, 0^4], [2; 0, 1^4]\} \times 5, \\
\Delta_2^G = \text{conv}\{[3; 1^5], [-1; -1, 0^4], [0; 1, 0^4]\} \times 5, \\
\Delta_3^G = \text{conv}\{[1; 0, 1^2, 0^2], [1; 0^3, 1^2], [0; 1, 0^4]\} \times 15, \\
\Delta_4^G = \text{conv}\{[1; 1^2, 0^3], [-1; 0, -1, 0^3], [2; 0, 1^4]\} \times 20
\]

have a common centroid \( \mathbf{c}_G = \frac{1}{3}[2; 1^5] \). In these triples of points admissible positions of non-zero components can be determined from that the affine centroid is always at \( \mathbf{c}_G = \frac{1}{3}[2; 1^5] \). The convex hull is a polytope with 27 vertices, the Gosset polytope \( G \) (usually denoted by \( 2_{21} \)), which we call the reference \( G \)-tope. There are 54 homologous copies of \( G \) that fit together facet-to-facet around the origin to form the star at the origin. This arrangement can be extended to a tiling by \( G \)-topes, which is the Delaunay tiling \( \mathcal{D}_{E_6} \) for \( \mathbb{Z}^6 \) determined by \( \varphi_{E_6} \) (e.g. Baranovski (1991)).
These 45 triangles inscribed in $G$ are $\mathcal{L}_{E_6}$-triangles because the edge vectors:

$$E_G^1 = \{\pm[2; -1,1^4], \pm[-2; 0,-1^4], \pm[0; 1,0^4]\} \times 5,$$
$$E_G^2 = \{\pm[-4; -2,-1^4], \pm[1;2,0^4], \pm[3;0,1^4]\} \times 5,$$
$$E_G^3 = \{\pm[0;0,-1^2,1^2], \pm[-1;1,0^2,-1^2], \pm[1;1,1^2,0^2]\} \times 15,$$
$$E_G^4 = \{\pm[-2;-1,-2,0^3], \pm[3;0,2,1^3], \pm[-1;1,0,-1^3]\} \times 20,$$

are subsets of $\mathcal{L}_{E_6}$. This is the set of 270 long vectors for $\varphi_{E_6}$, and is defined by

$$\mathcal{L}_{E_6} = \{z \in \mathbb{Z}^6 | \varphi_{E_6}(z) = 2m\};$$

$2m$ is the second minimum for $\varphi_{E_6}$, which is the minimal value assumed on the non-zero elements of $\mathbb{Z}^6$ not belonging to $\mathcal{S}_{E_6}$.

As is indicated by the notation, these inscribed triangles belong to four $\mathcal{A}_5$-classes of size 5, 5, 15, and 20, where $\mathcal{A}_5$ is the subgroup of permutations of the last five coordinates. Each long vector appears only once in the list, so the 45 edge sets account for the $270 = 45 \times 6$ long vectors for $\varphi_{E_6}$.

The group $\mathcal{G}_{E_6}$ has the following useful characterization: $\mathcal{G}_{E_6} \subset GL(6, \mathbb{Z})$ is the stability group of the set of short vectors $\mathcal{S}_{E_6}$. The following theorem gives information on $\mathcal{G}_{E_6}$ (see e.g. Coxeter (1995) or Baranovskii (1991)).

**Theorem 5.1 (Coxeter)** The group $\mathcal{G}_{E_6}$ acts transitively on the 72 elements of $\mathcal{S}_{E_6}$, the 270 elements of $\mathcal{L}_{E_6}$, and, the 54 $G$-topes in the star at the origin.

### 5.0.1 The inscribed long triangles.

The triangle $\Delta_5^1$ has a vertex at the origin and is inscribed in $G$. There are five such $\mathcal{L}_{E_6}$-triangles that belong to a single $\mathcal{A}_5$ class and together have ten vertices at distance $2m$ from the origin. These vertices, which are given by $\{r_1, r_2, ..., r_5\} = \{[0;1,0^4] \times 5\}, \{b_1, b_2, ..., b_5\} = \{[2;0,1^4] \times 5\}$, are the vertices of a five-dimensional cross-polytope. The five diagonals intersect at $\frac{1}{2}[2;1^5]$, and the ten diagonal vectors are given by

$$\mathcal{L}_1 = \{\pm(b_1 - r_1), \pm(b_2 - r_2), ..., \pm(b_5 - r_5)\} = \{\pm[2; -1,1^4] \times 5\},$$

and belong to a single parity class. If $g \in \mathcal{G}_{E_6}(G)$, which is the stability group for $G$, then $g$ must fix the center $\frac{1}{2}[2;1^5]$ and therefore belong to the stability group of the cross-polytope.

The group of linear transformations that map the cross-polytope onto itself has order $2^5 \times 5!$, and can be written as the product $\mathcal{I}_5 \times \mathcal{A}_5$; the subgroup $\mathcal{I}_5$ inverts arbitrary numbers of axes, and $\mathcal{A}_5$ permutes the red vertices $\{r_1, r_2, ..., r_5\}$ by permuting the last five co-ordinates. The $\mathcal{A}_5$-action clearly stabilizes the set $\mathcal{S}_{E_6}$, and therefore $\mathcal{A}_5 \subset \mathcal{G}_{E_6}(G)$. 

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The transformations corresponding to

\[
i_1 = \frac{1}{2} \begin{bmatrix}
  0 & 4 & 0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 2 & 2 & 0 & 0 & 0 \\
 -1 & 2 & 0 & 2 & 0 & 0 \\
 -1 & 2 & 0 & 0 & 2 & 0 \\
 -1 & 2 & 0 & 0 & 0 & 2
\end{bmatrix},
\]

\[
i_{12} = \begin{bmatrix}
 -1 & 2 & 2 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
 -1 & 1 & 1 & 1 & 0 & 0 \\
 -1 & 1 & 1 & 0 & 1 & 0 \\
 -1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix},
\]

belong to \(\mathcal{I}_5\). The first inverts the first axis of the cross-polytope so that

\[i_1[r_1, r_2, ..., r_5; b_1, b_2, ..., b_5] = [b_1, r_2, ..., r_5; r_1, b_2, ..., b_5],\]

but is non-integral and therefore not an element of \(G_{E_6}(G)\). The second transformation inverts the first two axes so that

\[i_{12}[r_1, r_2, ..., r_5; b_1, b_2, ..., b_5] = [b_1, b_2, r_3, ..., r_5; r_1, r_2, b_3, ..., b_5].\]

A direct calculation shows this transformation leaves \(S_{E_6}\) invariant, so \(i_{12} \in G_{E_6}(G)\).

Proposition 5.2 \(G_{E_6}(G) = \mathcal{I}_5^+ \times A_5\), where the \(A_5\)-action permutes the last four co-ordinates. \(|G_{E_6}(\Delta_G^1)| = 2^4 \times 4!\).

Proof. This follows immediately from the preceding discussion. ■

Corollary 5.3 \(G_{E_6}\) acts transitively on the 54 \(G\)-topes in the star of \(G\)-topes at the origin.

Proof. \(|G_{E_6}|/|G_{E_6}(G)| = |G_{E_6}|/|\mathcal{I}_5^+ \times A_5| = 2^8 \times 3^4 \times 5/2^4 \times 4! = 54. ■

The \(G_{E_6}\)-action maps the cross-polytope centered at \(\frac{1}{2}[2; 1]^5\) to 53 others, making 54 it total. Since cross-polytopes are centrally symmetric, opposite \(G\)-topes have cross-polytopes that are translates. In particular, the cross-polytopes for \(G\) and \(-G\) are homologous. Consequently, there are 27 sets of diagonal vectors, each with 10 long vectors belonging to the same parity class. These sets of are given by:

\[
\mathcal{L}_1 = \{\pm[2; -1, 1^4] \times 5\} \times 1; \quad \mathcal{L}_4 = \{\pm[1; 2, 0^4] \times 5\} \times 1;
\]

\[
\mathcal{L}_2 = \{\pm[-2; 0 - 1^4], \pm[4; 2, 1^4], \pm[0; 0, -1^2, 1^2], \pm[0; 0, -1, 1, -1, 1], \pm[0; 0, -1, 1^2, -1]\} \times 5;
\]

\[
\mathcal{L}_3 = \{\pm[0; 1, 0^4], \pm[2; 1; 2, 0^2], \pm[2; 1, 0, 2, 0^2], \pm[2; 1, 0^2; 2, 0], \pm[2; 1, 0^3; 2]\} \times 5;
\]

\[
\mathcal{L}_5 = \{\pm[3; 0, 1^4], \pm[-1; 0, 1, -1^3], \pm[-1; 0, -1, 1, -1^2], \pm[-1; 0, 1^2, -1, 1], \pm[-1; 0, -1^3, 1]\} \times 5;
\]

\[
\mathcal{L}_6 = \{\pm[-3; -2, 0, -1^3], \pm[-3; 0, -2, -1^3], \pm[1; 0^2; -1, 1^2], \pm[1; 0^2, 1, -1, 1], \pm[1; 0^2, 1^2, -1]\} \times 10.
\]
These parity classes are grouped into six $A_5$-classes, as indicated by the notation. This gives a second accounting for the 270 elements of $L_{E_6}$ in geometric terms.

More directly related to our main line of argument are the following results on $\Delta^1_G \subset G$.

**Corollary 5.4** $G_{E_6}(\Delta^1_G) = \mathcal{T}_0^+ \times A_4$, where $A_4$ is permutation of the last four coordinates.

**Proof.** This statement follows from the geometric description of $\Delta^1_G$ above, and the observation that $G_{E_6}(\Delta^1_G) \subset G_{E_6}(G)$.

**Corollary 5.5** There are 270 $L_{E_6}$-triangles $G_{E_6}$-equivalent to $\Delta^1_G$, which are inscribed in $G$-topes in the star at the origin.

**Proof.** It follows from the equality $|G_{E_6}|/|G_{E_6}(\Delta^1_G)| = |\mathcal{T}_0^+ \times A_4| = 2^6 3^4 5^2 2^4 4! = 720$ that there are 270 $L_{E_6}$-triangles $G_{E_6}$-equivalent to $\Delta^1_G$. That these are inscribed in $G$-topes in the star follows from **Corollary 5.3**.

**Tiling with $G^*$-topes.** The three $L_{E_6}$-triangles

\[
\Delta^1_T = \text{conv}\{[0,0], [0^4, 1, 0], [0^5, 1]\},
\]

\[
\Delta^2_T = \text{conv}\{[2,0,1], [-1,0^2,-1,0^2], [-1,0,-1,0^3]\},
\]

\[
\Delta^3_T = \text{conv}\{[0,1,0^4], [1,0^3,1^2], [-1^2,0^4]\},
\]

have a common centroid $c_T = \frac{1}{3}[^40,1^2]$, and belong to complementary 2-spaces. Their convex hull is a lattice polytope with nine vertices, which we denote by $G^*$. This reference $G^*$-tile is a tile in the Delaunay tiling $D_{E_6}$ determined by $\varphi_{E_6}$. All others are isometrically equivalent to $G^*$ relative to $\varphi_{E_6}$. In total, there are 40 homology classes of $G^*$-topes, accounting for 720 = 2 × 9 × 40 $G^*$-topes in the star at the origin.

These inscribed triangles are $L_{E_6}$-triangles because the edge vectors

\[
E^1_T = \{\pm[0^4,-1,1], \pm[0^5,-1], \pm[0^4,1,0]\},
\]

\[
E^2_T = \{\pm[0^3,1,-1,0^2], \pm[3,0,-2,-1^3], \pm[3,0,1,2,1^2]\},
\]

\[
E^3_T = \{\pm[2,1,0^2,1^2], \pm[1,-2,0^4], \pm[-1,1,0^2,-1^2]\},
\]

belong to $L_{E_6}$, the set of 72 long vectors for $\varphi_{E_6}$. The long vectors for $\varphi_{E_6}^*$ are defined by

\[
L_{E_6} = \{z \in \mathbb{Z}^6 | \varphi_{E_6}^*(z) = \frac{3}{2}m\},
\]

where $\frac{3}{2}m$ is the second minimum for $\varphi_{E_6}^*$.

There is a second representation $G_{E_6}^* \subset GL(6, \mathbb{Z})$ for the point group for $E_6$, which can be characterized as either the invariance group for the form $\varphi_{E_6}^*$ or the stabilizer of $S_{E_6}$. The following theorem gives the essential information we will need on this action (see e.g. Baranovskii (1991)).
Theorem 5.6 The group $G_{E_6^*}$ acts transitively on the 54 elements of $S_{E_6^*}$, the 72 elements of $L_{E_6^*}$, and the 270 $G^*$-topes in the star at the origin.

From the arithmetic equivalences $P_{E_6^*} = U_s S_{E_6^*}$ and $P_{E_6} = U_s S_{E_6}$, it follows that $G_{E_6} = U_s G_{E_6^*} U_s^{-1}$ and $G_{E_6}^* = U_s G_{E_6^*} U_s^{-1}$, where $U_s \in GL(6, \mathbb{Z})$ was described above.

The dual groups $G_{E_6}^* = \{g^o | g \in G_{E_6}\}$ and $G_{E_6}^* = \{g^o | g \in G_{E_6}\}$, where $g^o = (g^{-1})^T$ is the matrix for the dual transformation.

The proof of following corollary will be included into the full-length version of this paper.

Corollary 5.7 There are 270 $P_{E_6^*}$-triangles with a vertex at the origin. These are $G_{E_6}^*$-equivalent.

The $S_{E_6^*}$-triangle

$$\Delta^4_T = \text{conv}\{[0; 0], [0; 1, 0^4], [2; 0, 1^4]\}$$

is a 2-face of $G^*$, and is equal to the $L_{E_6^*}$-triangle $\Delta^1_G$. The edge set $E^1_T = E^1_G = L_{E_6} \cap S_{E_6}$.  

Corollary 5.8 There are 720 $S_{E_6^*}$-triangles with a vertex at the origin. These are 2-faces of $G^*$-topes, and are $G_{E_6}^*$-equivalent.

Proof. This is an immediate consequence of the arithmetic equivalences $S_{E_6^*} = U_s^{-1} P_{E_6^*}$, $G_{E_6}^* = U_s^{-1} G_{E_6^*} U_s$, and Corollary 5.7. □

Long triangles inscribed in $G^*$-topes. The 72 long vectors for $\varphi_{E_6^*}$ are given by $L_{E_6} = L_b \cup L_c$, where $L_c = S_3$, and

$$L_b = \{\pm[3, 0, 1^4], \pm[1, 2, 0^4], \pm[0^2; 1, 0^3], \pm[2, 1; 0, 1^3],$$

$$\pm[1, -1; 0, 1^3], \pm[3, 0; 2, 1^3], \pm[1, -1; 1^2, 0^2]\}$$

The proof of the following lemma will be included into the full-length exposition of our counterexample.

Lemma 5.9 There are 120 homology classes of $L_{E_6^*}$-triangles.

Proposition 5.10 The lattice polytope $T$ is a $G^*$-tope if and only if it is the convex hull of three $L_{E_6}$-triangles with a common centroid.

Proof. Each $G^*$-tope is the convex hull of three $L_{E_6}$-triangles with a common centroid. Since there are 40 homology classes of $G^*$-topes, there are 120 homology classes of $L_{E_6}$-triangles inscribed in $G^*$-topes. By Lemma 5.9, this accounts for all the homology classes of $L_{E_6}$-triangles, from which the proof of the Proposition immediately follows. □
6 An embedded copy of $D_4$

Since $\mathcal{P}_1 = \{\pm p_1, \pm p_2, \pm p_3\} = \{\pm[-3;2^5], \pm[1;0,-1^4], \pm[2;-2,-1^4]\}$ is the edge set for a $\mathcal{P}_{E_6}$-triangle, this set spans a two-dimensional subspace. Let $\Lambda_{D_4}$ be the four-dimensional sublattice defined by $\mathbb{Z}^6 \cap \mathcal{P}_1^\perp$, and let $\varphi_{D_4}$ be the restriction of $\varphi_{E_6}$ to $\Lambda_{D_4}$. The minimal vectors for $\varphi_{D_4}$ are given by $\mathcal{S}_{D_4} = \mathcal{S}_{E_6} \cap \mathcal{P}_1^\perp = S_3$. These minimal vectors determine $\varphi_{D_4}$, and $|\mathcal{S}_{D_4}| = 24$. This is sufficient to identify $\varphi_{D_4}$ as a metrical form for the root lattice $D_4$.

The set of long vectors for $\varphi_{D_4}$ is given by

$$\mathcal{L}_{D_4} = \mathcal{L}_{E_6} \cap \mathcal{P}_1^\perp = (\mathcal{L}_1 \cap \mathcal{P}_1^\perp) \cup (\mathcal{L}_2 \cap \mathcal{P}_1^\perp) \cup (\mathcal{L}_3 \cap \mathcal{P}_1^\perp),$$

which is the union of three parity classes of eight vectors each. These classes are the diagonal vectors for three orientations of cross polytopes, which tile $\Lambda_{D_4}$ by translates. This is the Delaunay tiling $\mathcal{D}_{D_4}$ of $\Lambda_{D_4}$ relative to the form $\varphi_{D_4}$. This Delaunay tiling can also be realized as the four-dimensional section $\mathcal{D}_{E_6} \cap \mathcal{P}_1^\perp$.

The group $G_{D_4}$. Let $G_{D_4} = G_{E_6} \cap G_{E_7}$. Since by Proposition 5.2 $G_{E_6}$ is the stabilizer of $\mathcal{S}_{E_6} = S_2 \cup S_3$, and $G_{E_7}$ is the stabilizer of $\mathcal{S}_{E_7} = S_1 \cup S_2$, $G_{D_4}$ can be characterized as the subgroup of $GL(6, \mathbb{Z})$ that stabilizes each of the sets $S_1, S_2, S_3$.

The short and long vectors for $\varphi_{D_4}$ relate to the sets $S_1$ and $S_3$ in the following way

$$\mathcal{S}_{D_4} = S_3; \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 = S_1 \cup \mathcal{L}_{D_4},$$

where the last union is disjoint. The set $S_1 = \{\pm l_1, \pm l_2, \pm l_3\} = \{\pm[2;-1,1^4], \pm[0;1,0^4], \pm[-2;0,-1^4]\}$ is the edge set $E_1^4$ for the triangle $\Delta_1^4$, which is inscribed in $G$.

The proof of the following proposition will be included in to a full-length paper.

**Proposition 6.1** $G_{D_4} = G_{E_6}(E_1^4) = G_{E_7}(E_1^4)$.

There are six $\mathcal{L}_{E_6}$-triangles with a vertex at the origin, and with edge set $E_1^4$. These fit around the origin, edge-to-edge, to form a hexagon. The triangle $\Delta_1^4$ is one of the tiles in this hexagon, and the other five are triangles homologous to it. Since $G_{E_6}(\Delta_1^4) \subset G_{E_6}(E_1^4) = G_{D_4}$, it follows from Proposition 5.2 and Corollary 5.3 that $|G_{D_4}| = |G_{E_6}(E_1^4)| = 6 |G_{E_6}(\Delta_1^4)| = 2^8 3^2$. However, the point group for the lattice $D_4$ has order $2^7 3^2$, which is less than $|G_{D_4}|$ by a factor of two. This discrepancy is explained by the following Corollary.

**Corollary 6.2** The restriction of $G_{D_4}$ to $\Lambda_{D_4}$ is a representation of the point group for the root lattice $D_4$.  

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Proof. Since the point group for $D_4$ is the only lattice point group for a four-dimensional lattice of order $2^73^2$, we need only show the kernel of the restriction map is a two element group. The kernel $K \subset G_{D_4}$ is the subgroup that fixes each point in $\Lambda_{D_4}$.

The four diagonals $(b_2 - r_2), \ldots, (b_5 - r_5)$, of the cross-polytope in the long layer of $G$ form a basis for $\mathcal{P}_1^+$, and therefore an arbitrary element $k \in K$ must fix these diagonals. It follows that $k$ must then fix the parity class $L_1$. These two conditions can hold only if $k$ maps $(b_1 - r_1)$ to $\pm (b_1 - r_1)$. By the geometric discussion of the group $G_{E_6}(\Delta^1_G)$, it follows that $k \in \{e, i_0\} \times I_5^+ \subset G_{D_4}$, where $i_0$ is central inversion and $e$ is the identity. Let $k = k_1k_2$, where $k_1 \in \{e, i_0\}$, $k_2 \in I_5^+$. If $k_1 = e$, then $k_2$ must also equal $e$, and $k = e$. If $k_1 = i_0$, then $k_2$ must invert the four diagonals $(b_2 - r_2), \ldots, (b_5 - r_5)$, which uniquely determines the element $k_2$. One of the ways to write $k_2$ in this case is as the product $i_{23}i_{34}$, where $i_{23}$ inverts the second and third axes of the cross-polytope, and $i_{45}$ inverts the fourth and fifth. Therefore, when $k_1 = i_0$, $k = k_0 = i_0i_{23}i_{34}$, and the matrix for $k_0$ is given by $\mathcal{S}_{D_4}$

$$k_0 = \begin{bmatrix} 3 & 0 & -2 & -2 & -2 & -2 \\ 2 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}.$$ 

It follows that $K = \{e, k_0\}$, and is a two element group. ■

The faces of $D_{D_4}$. The origin $0 \in \Lambda_{D_4}$, and the star of Delaunay tiles at the origin has 24 cross-polytopes, eight belonging to each orientation. The lower dimensional cells are $S_{D_4}$—simplexes.

Corollary 6.3 For the tiling $D_{D_4}$, the star of $D_{D_4}$—cells at the origin contains 24 edges, 96 triangles, 96 3—simplexes, and 24 cross-polytopes. For each dimension, the cells in the star are $G_{D_4}$—equivalent.

Proof. By Corollary 6.2, the homorphic image of $G_{D_4}$ on $\Lambda_{D_4}$ is a group of order $2^73^2$, which acts effectively on $\Lambda_{D_4}$, and is a representation of the point group for the lattice $D_4$. The order of the full linear invariance group of a cross-polytope, attached to the origin in $\mathbb{R}^4$, is $2^3 \times 3!$. Since $2^73^2/24 = 2^3 \times 3!$, the stability group for each of the 24 cross-polytopes in the star has order $2^3 \times 3!$, and the $G_{D_4}$—orbit of each includes all 24 cross-polytopes. Since the stability group in $G_{D_4}$ of any particular cross-polytope in the star is the full invariance group, all edges, all 2—faces and 3—faces of the cross-polytope, which are attached to the origin, are $G_{D_4}$—equivalent.

The number of edges attached to the origin is equal to the number of short vectors, which is 24. Each cross-polytope has 12 2—cells attached to the origin, and each
2–cell belongs to 3 cross-polytopes in the star. Therefore, there are \((24 \times 12)/3 = 96\) 2–cells attached to the origin. Each cross-polytope has 8 facets attached to the origin, and each belongs to a pair of cross-polytopes. Therefore, there are \((24 \times 8)/2 = 96\) 3–cells attached to the origin. ■

**Equivalent facets of \(\Phi(\mathcal{P}_{E_6})\) and \(\Phi(\mathcal{P}_{E_6})\).** Since \(\mathcal{G}'_{E_6}, \mathcal{G}''_{E_6} \subset GL(6, \mathbb{Z})\) are the full invariance groups of the corresponding sets of perfect vectors \(\mathcal{P}_{E_6}\) and \(\mathcal{P}_{E_6}\), \(\Phi(\mathcal{P}_{E_6})\) is invariant with respect to the action of \(\mathcal{G}_{E_6}\), and \(\Phi(\mathcal{P}_{E_6})\) is invariant with respect to the action of \(\mathcal{G}_{E_6}^*\). The proof of the following corollary will be included in to a full-length paper.

**Corollary 6.4** The perfect domain \(\Phi(\mathcal{P}_{E_6}^*)\) has 45 facets that are \(\mathcal{G}_{E_6}^*\)–equivalent to \(\Phi(\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6})\), and the perfect domain \(\Phi(\mathcal{P}_{E_6})\) has 45 facets that are \(\mathcal{G}_{E_6}^*\)–equivalent to \(\Phi(\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6})\).

The 45 \(\mathcal{G}_{E_6}^*\)–equivalent facets of \(\Phi(\mathcal{P}_{E_6}^*)\) described in this Corollary are in one-to-one correspondence with the 45 possible edge sets for \(\mathcal{L}_{E_6}\)–triangles inscribed in \(G\)-topes, and, the 45 \(\mathcal{G}_{E_6}^*\)–equivalent facets of \(\Phi(\mathcal{P}_{E_6})\) are in one-to-one correspondence with the 45 possible edge sets for \(\mathcal{S}_{E_6}\)–triangles, which are 2–faces of \(G^*\)-topes.

### 7 Commensurate and incommensurate \(G^*\)-topes

A \(G^*\)-tope homologous to the reference \(G^*\)-tope is called a \(T\)-tope. Since \(T \subset G\) this \(G^*\)-tope is commensurate with \(G\), and therefore commensurate with the Delaunay tiling \(\mathcal{D}_{E_6}\). We will count \(G^*\)-topes below, and discover that there are 24 homology classes of such commensurate \(G^*\)-tope.

There is a second \textit{incommensurate} class of \(G^*\)-tope with representative \(Q\), which is the convex hull of the three \(\mathcal{L}_{E_6}\)–triangles

\[
\Delta^1_Q = \text{conv}\{[0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\}, \quad \Delta^2_Q = \text{conv}\{[0, 1, 0^4], [-1^2, 0^4], [0, 1^2, 1^3]\}, \\
\Delta^3_Q = \text{conv}\{[2, 0, 1^4], [-1, 0^5], [-1, 0, -1, 0^6]\},
\]

with common centroid \(c_Q = \frac{1}{9}[0^3, 1^3]\). In the counting exercise below we will find there are 16 homology classes of incommensurate \(G^*\)-topes, and will refer \(G^*\)-topes in any of these classes as a \(Q\)-tope. There are only two homology classes of \(G^*\)-tope, so the 24 commensurate classes and the 16 incommensurate classes sum to the required number of 40 homology classes of \(G^*\)-tope. The commensurate and incommensurate classes, and the 24 commensurate classes. There are sixteen homology classes of such \(G^*\)-topes, and we will refer to these as \(Q\)-topes.

An application of Proposition 5.10 shows that \(Q\) is a \(G^*\)-tope. That \(Q\) is incommensurate with \(G\) is established by considering the the simplicial 3–faces \(S_Q \subset Q\) and \(S_G \subset G\). The simplex \(S_Q = \text{conv}\{[0, 1, 0^4], [2, 0, 1^4], [-1^2, 0^4], [-1, 0^2]\} \subset Q\) is the
convex hull of an edge from each of the triangles $\Delta^2_Q$, $\Delta^3_Q$, so is a 3-face of $Q$. The centroid $\frac{1}{4}[0^2, 1^4]$ is also the centroid of the simplex $S_G = \text{conv}\{[0^2, 1, 0^3], [0^4, 1, 0^2], [0^4, 1, 0], [0^5, 0^1]\}$, which is a 3-face of $G$. These simplexes lie in complementary 3-spaces, so satisfy the condition that $S_T \cap S_G = \frac{1}{4}[0^2; 1^4]$. The relatively open cells of the intersection polytope $Q \cap G$ are the intersections of relatively open cells of $Q$ and $G$. For this reason the common centroid is a vertex of $Q \cap G$, and since it is non-integral, $Q$ is incommensurate with $G$. Therefore $Q$ is incommensurate with the Delaunay tiling $D_{E_6}$.

Another important class lattice polytope is the class of $R$-topes; the lattice polytope $R = \text{conv}(S_Q \cup S_G)$ serves as an example. This $R$-tope has full dimension and eight vertices, so can be triangulated in just two ways. By taking the convex hull of $S_Q$ with each 2-face of $S_G$, the four blue simplexes $B^1, B^2, B^3, B^4$ are formed; this collection is the star of simplexes in $R$ with the 3-face $S_Q$. These blue simplexes tile $R$. The four yellow simplexes, $Y^1, Y^2, Y^3, Y^4$, form a second star, each yellow simplex being the convex hull of $S_G$ with one of the four 2-faces of $S_Q$. This is the collection of simplexes in $R$ with the 3-face $S_G$. The yellow simplexes also tile $R$. Lattice polytopes such as $R$ are refereed to as repartitioning complexes in the literature on lattice Delaunay tilings. In the counting exercise below we will see that there are 12 homology classes of $R$-topes.

**Stars of G-topes.** In order to effectively study the geometry of $Q$-topes and $R$-topes we must first obtain information on the relationship between the perfect vectors $\mathcal{P}_{E_6}$, and the $G$-topes in the star of $G$-topes at the origin. If $\mathcal{V}(G)$ is the vertex set for $G$, then, relative to $\varphi_{E_6}$, the set vertices of $G$ that are a distance $m$ from the origin are given by $S_{E_6}(G) = \mathcal{V}(C) \cap S_{E_6}$, and the set vertices that are a distance $2m$ are given by $L_{E_6}(C) = \mathcal{V}(C) \cap L_{E_6}$. These short and long layers of vertices can also be characterized in terms of the perfect vector $p_1 = [-3, 2^5]$ using the incidence relations for $G$-topes:

$$S_{E_6}(G) = \{s \in S_{E_6}| p_1 \cdot s = 1\}; \quad L_{E_6}(G) = \{l \in L_{E_6}| p_1 \cdot l = 2\}.$$ 

The perfect vector $p_1$ determines the short and long layers of vertices, which in turn determine $G$, so we write $G = G_{p_1}$. This process can be reversed, and the incidence relations can be used to determine the perfect vector associated with $G$:

$$p_1 = \mathcal{P}_G = \{p \in \mathcal{P}_{E_6}| p \cdot s = 1, s \in S_{E_6}(G)\} \cap \{p \in \mathcal{P}_{E_6}| p \cdot l = 2, l \in L_{E_6}(G)\}.$$ 

For this case the set $\mathcal{P}_G$ contains the single elements $p_1$.

By Proposition 3.2, the incidence relations can be used to establish a one-to-one correspondence between the $G$-topes in the star at the origin, and the vectors of $\mathcal{P}_{E_6}$: if $p \in \mathcal{P}_{E_6}$, then $G_p$ is the corresponding $G$-tope in the star, and if $G'$ is in the star, then $p G'$ is the corresponding perfect vector. This association can be extended to
cover arbitrary cells $C \in \mathcal{D}_{E_6}$ with a vertex at the origin. If $S_{E_6}(C) = \mathcal{V}(C) \cap S_{E_6}$ and $\mathcal{L}_{E_6}(C) = \mathcal{V}(C) \cap \mathcal{L}_{E_6}$ are the short and long layers for $C$, define
\[ \mathcal{P}_C = \{ \mathbf{p} \in \mathcal{P}_{E_6} \mid \mathbf{p} \cdot \mathbf{s} - 1, \mathbf{s} \in S_{E_6}(C) \} \cap \{ \mathbf{p} \in \mathcal{P}_{E_6} \mid \mathbf{p} \cdot \mathbf{l} - 2, \mathbf{l} \in \mathcal{L}_{E_6}(C) \}. \]
When $\dim(C) < 6$ the set $\mathcal{P}_C$ includes several perfect vectors. The short and long layers, and therefore $C$, can be recovered from $\mathcal{P}_C$ by again invoking the incidence relations:
\[ S_{E_6}(C) = \{ \mathbf{s} \in S_{E_6} \mid \mathbf{p} \cdot \mathbf{s} = 1, \mathbf{p} \in \mathcal{P}_C \}; \mathcal{L}_{E_6}(G) = \{ \mathbf{l} \in \mathcal{L}_{E_6} \mid \mathbf{p} \cdot \mathbf{l} = 2, \mathbf{p} \in \mathcal{P}_C \}. \]

For a cell $C \in \mathcal{D}_{E_6}$, $\text{star}_G(C)$ is the collection of $G$-topes with face $C$. The following Proposition gives information on $G$-stars when $C$ has a vertex at the origin.

**Proposition 7.1** If $0 \in C$, $\text{star}_G(C) = \{ G_p \mid \mathbf{p} \in \mathcal{P}_C \}$.

**Proof.** This assertion follows from the discussion on incidence relations. □

**The geometry of $R$-topes.** In the statement of the following lemma $\text{star}_G(S_G)$ is the collection of $G$-topes, which have the face $S_G$, and $\text{star}_G(S_Q)$ is the collection of $G^*$-topes, which have the face $S_Q$.

The proof of the following lemma will be included into the full-length paper.

**Lemma 7.2** $\text{star}_G(S_Q) = \{ Q^1, ..., Q^4 \}$, where $Q^1, ..., Q^4$ are $Q$-topes, $Q^1 = Q$, and the blue simplexes can be ordered so that $B^i = Q^i \cap R, i = 1, ..., 4$. Similarly, $\text{star}_G(S_G) = \{ G^1, ..., G^4 \}$, where $G^1, ..., G^4$ are $G$-topes, $G^1 = G$, and the yellow simplexes can be ordered so that $Y^i = G^i \cap R, i = 1, ..., 4$.

**Proposition 7.3** $R$ is commensurate with both $\mathcal{D}_{E_6}$ and $\mathcal{D}_{E_6}$.

**Proof.** By Lemma 7.2, the intersection $R \cap \mathcal{D}_{E_6}$ is tiled by the blue lattice simplexes, so $R$ is commensurate with $\mathcal{D}_{E_6}$. Similarly, the intersection $R \cap \mathcal{D}_{E_6}$ is tiled by the yellow lattice simplexes, so $R$ is commensurate with $\mathcal{D}_{E_6}$. □

**The geometry of $Q$-topes.** The convex hull of $\Delta^1_Q$ with one edge from each of the other triangles $\Delta^2_Q, \Delta^3_Q \subset Q$ has seven vertices, and is a lattice simplex in $Q$. Two edges can be selected in nine ways, so there are nine such simplexes, each with $\Delta^1_Q$ as a 2-face. This is the star of simplexes in $Q$ with the 2-face $\Delta^1_Q$, and this collection tiles $Q$. In this subsection we show how six of these simplexes are commensurate with both $\mathcal{D}_{E_6}$ and $\mathcal{D}_{E_6}$. We call them white. The other three are blue, and can be represented as the intersection of $Q$ with three distinct $R$-topes.

There are two other collections of simplexes that tile $Q$, the star of simplexes in $Q$ with the 2-face $\Delta^2_Q$, and the star with the 2-face $\Delta^3_Q$. However, both of these tilings play no role in our discussion. The reason for this is that $\Delta^1_Q$ is a 2-cell in $\mathcal{D}_{E_6}$, but the other two triangles $\Delta^2_Q, \Delta^3_Q$ are not.

The proof of the following lemma will be included into the full-length paper.

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Lemma 7.4 \textit{star}_G(\Delta^1_Q) = \{G^1, ..., G^6\}, where \(G^1, ..., G^6\) are \(G\)-topes and \(G^1 = G\). Moreover, for \(i = 1, ..., 6\), \(\text{conv}(G^i \cap Q \cap Z^6)\) is a six-dimensional lattice simplex with the property that \(\Delta^1_Q \subset \text{conv}(G^i \cap Q \cap Z^6)\). We will refer to these simplexes as white.

Lemma 7.5 There are three \(R\)-topes \(R^1, R^2, R^3\), such that \(R^1 = R\), and such that \(B^i = Q \cap R, \ 1 \leq i \leq 3\) are six-dimensional lattice simplexes with the 2–face \(\Delta^1_Q\). Since these simplexes are the intersections of \(R\)-topes and a \(Q\)-tope they are blue.

\textbf{Proof.} The three simplexes
\begin{align*}
S^1_Q &= \text{conv}\{[0, 1, 0^4], [2, 0, 1^4], [-1^2, 0^4], [1, 0^5], [0^5, \ 1]\},
S^2_Q &= \text{conv}\{[0, 1, 0^4], [2, 0, 1^4], [1, 0^2, 1^3], [-1, 0, -1^3]\},
S^3_Q &= \text{conv}\{[-1^2, 0^4], [-1^2, 0^5], [1, 0^2, 1^3], [-1, 0, -1^3]\},
\end{align*}
are 3–faces of \(Q\), and \(S^1_Q = S_Q\). Each is the convex hull of an edge from \(\Delta^2_Q\) and \(\Delta^3_Q\). The three simplexes
\begin{align*}
S^1_G &= \text{conv}\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},
S^2_G &= \text{conv}\{[2, 1, 0, 1^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},
S^3_G &= \text{conv}\{[-2, -1^2, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},
\end{align*}
are \(S_{D^4}\)–simplexes, and \(S^1_G = S_G\). The three centroids of the \(Q\)–simplexes \(S^1_Q, S^2_Q, S^3_Q\), are given by \(\frac{1}{2}[0^2, 1^4], \frac{1}{4}[2, 1, 0, 2^3], \frac{1}{4}[-2, -1^2, 1^3]\), and these coincide with the centroids of the corresponding \(S_{D^4}\)–simplexes \(S^1_G, S^2_G, S^3_G\). The three \(S_{D^4}\)–simplexes are translates of closed 3–cells in \(D_{D^4}\), so by Proposition 5.3 belong to \(G_{D^4}\)–equivalent homology classes. It follows that the three lattice polytopes \(R^1 = \text{conv}\{S^1_Q \cup S^1_G\}, R^2 = \text{conv}\{S^2_Q \cup S^2_G\}, R^3 = \text{conv}\{S^3_Q \cup S^3_G\}\), belong to \(G_{D^4}\)–equivalent homology classes. Since \(R^1 = \text{conv}\{S^1_Q \cup S^1_G\} = \text{conv}\{S_Q \cup S_G\} = R\), these three lattice polytopes are \(R\)-topes.

Since by Lemma 7.3 \(B^1 = Q \cap R^1 = Q \cap R\) is a six-dimensional blue lattice simplex, the intersections \(B^2 = Q \cap R^2, B^3 = Q \cap R^3\), are other six-dimensional blue lattice simplexes. By construction, \(\Delta^1_Q \subset B^1, B^2, B^3\).

Lemma 7.6 The star of simplexes in \(Q\) containing \(\Delta^1_Q\), has six white simplexes \(W^1, W^2, ..., W^6\), and three blue simplexes \(B^1, B^2, B^3\). The six white simplexes are commensurate with \(D_{E^6}\) and \(D_{E^6}\), and the three blue simplexes are commensurate with \(D_{E^6}\), but incommensurate with \(D_{E^6}\).

\textbf{Proof.} The simplexes \(W^1, W^2, ..., W^6\), are those described in the statement of Lemma 7.4. Since \(W^i \subset G^i \cap Q, i = 1, ..., 6\), these white simplexes are commensurate with both \(D_{E^6}\) and \(D_{E^6}\).

The simplexes \(B^1, B^2, B^3\), are those described in the statement of Lemma 7.4. Since \(B^i = R^i \cap Q\), the blue simplex \(B^i\) is commensurate with \(D_{E^6}\). The simplex \(S^1_Q\) is a 3–face of \(B^i\), and the simplexes \(S^1_G\) is a closed 3–cells in \(D_{E^6}\). Since the intersection \(S^1_Q \cap S^1_G\) is the centroid of each of these cells, \(S^1_Q\) and \(S^1_G\), the simplex \(B^i\) is incommensurate with \(D_{E^6}\).
**T-topes, Q-topes and R-topes.** The intersection properties of any $G^*$-tope with respect to $D_{E_6}$, are invariant with respect to the $G_{D_4}$-action. Therefore, any $G^*$-tope $G_{D_4}$-equivalent to $T$ is commensurate with $D_{E_6}$, and any $G^*$-tope $G_{D_4}$-equivalent to $Q$ is incommensurate with $D_{E_6}$. Since commensurability is a property that extends to homology classes, it is natural to make the following definitions: a *T-tope* is a lattice polytope $G_{D_4}$-equivalent to one homologous to $T$ and, a *Q-tope* is a lattice polytope $G_{D_4}$-equivalent to a lattice polytope homologous to $Q$ as a *Q-tope*. It is also natural to refer to commensurate and incommensurate homology classes of $G^*$-topes, so that homology classes of T-topes are commensurate classes and homology classes of Q-topes are incommensurate. By the definitions of T-tope and Q-tope, the homology classes of T-topes are $G_{D_4}$-equivalent and, the homology classes of Q-topes are $G_{D_4}$-equivalent.

Each of the triangles $\Delta^1_T, \Delta^2_T, \Delta^3_T$ in $T$ has two edges that are long relative to $\varphi_{E_6}$, and a single edge that is short. This is a property that is $G_{D_4}$-invariant, so each T-tope has three inscribed triangles of this type. On the other hand, the triangle $\Delta^1_Q$ in $Q$ has edges that are short relative to $\varphi_{E_6}$, and the triangles $\Delta^2_Q, \Delta^3_Q$ have edges that are long. Accordingly, each Q-tope has one short and two long inscribed triangles.

The set of R-topes is the other collection we must consider: an *R-tope* is a lattice polytope $G_{D_4}$-equivalent to a lattice polytope homologous to $R$. The homology classes of R-topes are $G_{D_4}$-equivalent, and each R-tope in any of these classes has identicle intersection properties. If $R'$ is an arbitrary R-tope, then $R' \cap D_{E_6}$ has four yellow simplexes $Y^1_{R'}, Y^2_{R'}, Y^3_{R'}, Y^4_{R'}$, which belong to distinct G-topes and fit around a 3-cell $S''_G \in D_{E_6}$; $R' \cap D_{E_6}^*$ has four blue simplexes $B^1_{R'}, B^2_{R'}, B^3_{R'}, B^4_{R'}$, which belong to distinct Q-topes and fit around a 3-cell $S''_Q \in D_{E_6}^*$. Each R-tope $R'$ is commensurate with both $D_{E_6}$ and $D_{E_6}^*$.

**Lemma 7.7** The 40 homology classes of $G^*$-topes are divided between 24 homology classes of T-topes, and 16 homology classes of Q-topes. There are 12 homology classes of R-topes.

**Proof.** will be included into the full-length paper ■

**Repackaging Q-topes.** Each Q-tope contains six white, and three blue simplexes, and since there are 16 homology classes of Q-topes, there are 96 homology classes of white, and 48 homology classes of blue simplexes. The 12 homology classes of R-topes gives a second accounting for blue simplexes, since each R-tope can be tiled by four blue simplexes. This allows a repackage of the blue simplexes contained in Q-topes, into R-topes. The portion of space tiled by the 16 classes of Q-topes, can equally well be tiled by the 96 homology classes of white simplexes, and 12 classes of R-topes. This alternate tiling has the property that each tile is commensurate with both $D_{E_6}$ and $D_{E_6}^*$. 

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Proposition 7.8 The 24 homology classes of T-topes, the 12 homology classes of \( R \)-topes, and the 96 homology classes of white simplexes fit together facet-to-facet to tile space. This lattice tiling \( \mathcal{D}_R \) is commensurate with both \( \mathcal{D}_{E_6} \) and \( \mathcal{D}_{E_6^*} \).

Proof. Since the \( T \)-topes and \( Q \)-topes tile space, the discussion immediately preceding the proposition shows that the \( T \)-topes, \( R \)-topes and white simplexes also tile space. By Lemmae 7.2, 7.6 and Proposition 7.3 this tiling is commensurate with both \( \mathcal{D}_{E_6} \) and \( \mathcal{D}_{E_6^*} \).

8 Proof of main theorem

The forms \( \varphi_{E_6} \in \Phi(\mathcal{P}_{E_6}) \) and \( \varphi_{E_6^*} \in \Phi(\mathcal{P}_{E_6}) \) lie on the central axes of their domains, and the line segment \( \varphi_t = (1 - t)\varphi_{E_6} + t\varphi_{E_6^*}, \) \( 0 \leq t \leq 1 \), runs between them. The arithmetic minimum along this segment is \( m \). At the end points the minimal vectors are respectively \( S_{E_6} \) and \( S_{E_6^*} \), but at intermediate points the minimal vectors are given by \( S_{E_6} \cap S_{E_6^*} \).

The form \( \varphi_R \). Suppose that the reference \( R \) is Delaunay with respect to the metrical form \( \varphi \). Then there is a scalar \( c \) and vector \( p \) so that \( f_R(x) = c + p \cdot x + \varphi(x) \) is non-negative on \( \mathbb{Z}^6 \), and zero valued on just the vertices of \( R \); \( f_R(x) = 0 \) is the equation of an empty ellipsoid circumscribing \( R \). The vertex sets for the component simplexes \( S_G, S_Q \) of \( R \) are given by

\[
V_G = \{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\}, \\
V_Q = \{[0, 1, 0^4], [2, 0, 1^4], [-1, 1^2, 0^4], [-1, 0^5]\},
\]

and have the property that \( \sum_{v \in V_G} v = \sum_{v \in V_Q} v = 4c_R = [0^2, 1^4] \). Since \( f_R \) is zero on each vertex of \( R \), the metrical form \( \varphi \) must satisfy the condition.

\[
0 = \sum_{v \in V_G} f_R(v) - \sum_{v \in V_Q} f_R(v) = p \cdot \left( \sum_{v \in V_G} v - \sum_{v \in V_Q} v \right) + \sum_{v \in V_G} \varphi(v) - \sum_{v \in V_Q} \varphi(v) \\
= \sum_{v \in V_G} \varphi(v) - \sum_{v \in V_Q} \varphi(v) = \text{trace}(P_R F_\varphi) = \langle \pi_R, \varphi \rangle,
\]

where \( F_\varphi \) is the matrix for \( \varphi \) and

\[
P_R = \sum_{v \in V_G} vv^T - \sum_{v \in V_Q} vv^T = -\begin{bmatrix} 6 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.
\]
The final expression $\langle \pi_R, \varphi \rangle$ is the scalar product of forms $\pi_R$ and $\varphi$, where $\pi_R$ corresponds to $P_R$; if forms $\pi$, $\varphi$ have coefficient matrices $P$, $F$, then the scalar product $\langle \pi, \varphi \rangle$ is defined to be $\text{trace}(PF)$.

The equation $\langle \pi_R, \varphi \rangle = 0$ determines a hyperplane $H_R$ in the space of metrical forms.

**Lemma 8.1** The line segment $\varphi_t = (1 - t)\varphi_{E_6} + t\varphi_{E_6}$, $0 \leq t \leq 1$, pierces the hyperplane $H_R$ at the point $\varphi_{t_R}$, where $t_R = \frac{1}{2}$. There are twelve hyperplanes of the form $H_{R'}$, where $R'$ is an $R$-tope, and the line segment $\varphi_t = (1 - t)\varphi_{E_6} + t\varphi_{E_6}$, $0 \leq t \leq 1$, pierces each of these at $\varphi_{1/2}$, which is common to all of them.

**Proof.** A direct calculation shows that

\[
\langle \pi_R, \varphi_t \rangle = \langle \pi_R, (1 - t)\varphi_{E_6} + t\varphi_{E_6} \rangle = (1 - t) \times \text{trace}(P_RF_{E_6}) + t \times \text{trace}(P_RF_{E_6}) = (1 - t)(-\frac{1}{2}m) + t(\frac{1}{2}m) = -\frac{1}{2}m + mt.
\]

Therefore, the line segment $\varphi_t, 0 \leq t \leq 1$, pierces $H_R$ when $t = t_R = \frac{1}{2}$.

Each $R$-tope $R'$ corresponds to a form $\pi_{R'}$, which in turn determines a hyperplane $H_{R'}$ with equation $\langle \pi_{R'}, \varphi \rangle = 0$. If $R' = g^{-1}R$, where $g \in G_{D_4}$, then the corresponding form is given by $\pi_{R'}(x) = \pi_R(g^*x)$ and

\[
H_{R'} = \{ \varphi(gx) \mid \varphi(x) \in H_R \};
\]

$g^*$ is the matrix dual to $g$. On the other hand, if $R'$ is homologous to $R$, then $\pi_{R'} = \pi_R$ and $H_{R'} = H_R$. There are twelve such hyperplanes, as there are twelve homology classes of $R$-topes (see Lemma 7.7). The equality $\varphi_{1/2}(gx) = \varphi_{1/2}(x)$, which holds for all $g \in G_{D_4}$, implies that the line segment $\varphi_t$ pierces all of them at the same point. ■

**Proposition 8.2** Tiling $D_R$ is Delaunay with respect to the form $\varphi_{1/2}$.

**Proof.** Recall that $D_R$ consists of 24 homology classes of $T$-topes, 96 homology classes of white simplexes, and 12 homology classes of $R$-topes (Proposition 7.8). Each $T$-tope $T$ in $D_R$ is a Delaunay cell of $D_{E_6}$ whose intersection with a $G$-tope of $D_{E_6}$ is $T$. Each white simplex in $D_R$ is the intersection of a $Q$-tope, Delaunay cell of $D_{E_6}$, and a $G$-tope, Delaunay cell of $D_{E_6}$. By Lemma 8.1 and the preceding discussion, all homology classes of $R$-topes are Delaunay for $\varphi_{1/2}$. ■

**Proof.** For $0 < t < \frac{1}{2}$, the Delaunay tiling for each form $\varphi_t = (1 - t)\varphi_{E_6} + t\varphi_{E_6}$ is constant, and obtained by replacing each $R$-tope in $D_R$ by four yellow simplexes. For $\frac{1}{2} < t < 1$, the Delaunay tiling for each form $\varphi_t$ is also constant, and obtained by replacing each $R$-tope in $D_R$ by four blue simplexes. ■
The facet $\Phi(P_{E_6} \cap P_{E_6}^*)$. Consider the form $\pi_p = \pi_{E_6}^* - \pi_{E_6}$ and the rank one form $\varphi_P(x) = (p \cdot x)^2$. Then, $\langle \pi_P, \varphi_P \rangle = \langle \pi_{E_6}^* - \pi_{E_6}^*, \varphi_P \rangle = \text{trace}(P_{E_6}^* - P_{E_6})\varphi_P = p^T(P_{E_6}^* - P_{E_6})p = \pi_{E_6}^*(p) - \pi_{E_6}(p)$. If $p \in P_{E_6}^*$, then $\pi_P(p) = \pi_{E_6}^*(p) - \pi_{E_6}(p) = m - \pi_{E_6}(p) \leq 0$, with equality if only if $p \in P_{E_6}^* \cap P_{E_6}$, and, if $p \in P_{E_6}$, then $\pi_P(p) = \pi_{E_6}^*(p) - \pi_{E_6}(p) = \pi_{E_6}(p) - m \geq 0$, with equality if and only if $p \in P_{E_6}^* \cap P_{E_6}$.

It follows that the hyperplane $H_P$, with equation $\langle \pi_P, \varphi \rangle = 0$, separates $\Phi(P_{E_6})$ and $\Phi(P_{E_6}^*)$, and contains the facet $\Phi(P_{E_6} \cap P_{E_6}^*)$.

Lemma 8.3 The line segment $\varphi_t = (1-t)\varphi_{E_6} + t\varphi_{E_6}^*$, $0 \leq t \leq 1$, pierces the facet $\Phi(P_{E_6}^* \cap P_{E_6})$ when $t = t_P = \frac{2}{5}$. Moreover, the point $\varphi_P = \varphi_{t_P}$ lies on the central axis of $\Phi(P_{E_6}^* \cap P_{E_6})$.

Proof. A direct calculation shows that

\[
\langle \pi_P, \varphi_t \rangle = \langle \pi_{E_6}^* - \pi_{E_6}, (1-t)\varphi_{E_6} + t\varphi_{E_6}^* \rangle = (1-t)\text{trace}(P_{E_6}^* F_{E_6}) + (t)\text{trace}(P_{E_6} F_{E_6}^*) = (1-t)(-\frac{2}{8}m^2) + t\left(\frac{3}{8}m^2\right) = -\frac{2}{8}m^2 + \frac{5}{8}m^2 t,
\]

where $P = P_{E_6}^* - P_{E_6}$. Therefore, the line segment $\varphi_t$, $0 \leq t \leq 1$, pierces the hyperplane $H_P$ when $t = t_P = \frac{2}{5}$.

The forms $\varphi_t$ are invariant with respect to the action of $G_{D_4} = G_{E_6} \cap G_{E_6}$. Since $G_{D_4}$ acts transitively on $P_{E_6} \cap P_{E_6}^*$, which can easily be checked, the only $G_{D_4}$-invariant positive forms on $H_P$ must lie on the central axis of $\Phi(P_{E_6} \cap P_{E_6})$. Therefore, the line segment $\varphi_t$, $0 \leq t \leq 1$, pierces the hyperplane along the central axis of $\Phi(P_{E_6} \cap P_{E_6})$.

Proof of Theorem 3.3. By Lemma 8.2, $D_R$ is the Delaunay tiling for the midpoint of the segment $\varphi_t = (1-t)\varphi_{E_6} + t\varphi_{E_6}^*$. By Proposition 7.8, $D_R$ is commensurate with both $D_{E_6}$ and $D_{E_6}^*$. Therefore by Lemma 4.4, the $L$-type does not change between $\varphi_{E_6}$ and $\varphi_{1/2}$, and between $\varphi_{E_6}^*$ and $\varphi_{1/2}$. By the results of Section 3, $\varphi_{E_6}$ and $\varphi_{E_6}^*$ are the centroids of two adjacent perfect domains and, as shown in the above lemma, the wall between them intersects the segment $\varphi_t$ at $t = \frac{2}{5}$.

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