GLOBAL EXPONENTIAL ATTRACTION FOR MULTI-VALUED SEMIDYNAMICAL SYSTEMS WITH APPLICATION TO DELAY DIFFERENTIAL EQUATIONS WITHOUT UNIQUENESS

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Dedicated to the memory of professor V.S. Mel’nik
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ABSTRACT. We first prove the existence of a compact positively invariant set which exponentially attracts any bounded set for abstract multi-valued semi-dynamical systems. Then, we apply the abstract theory to handle retarded ordinary differential equations and lattice dynamical systems, as well as reaction-diffusion equations with infinite delays. We do not assume any Lipschitz condition on the nonlinear term, just a continuity assumption together with growth and dissipative conditions, so that uniqueness of the Cauchy problem fails to be true.

1. Introduction. The long-time behavior of multi-valued semidynamical systems has been extensively investigated over the last one and a half decades, see, e.g., [1, 3, 9, 20, 21, 25, 32] and the references therein. The intention of this article is to show the existence of a compact positively invariant set $\mathcal{A}$ which exponentially attracts every bounded set for multi-valued semidynamical systems. It is worth mentioning that here we do not consider the finite dimensionality of the set $\mathcal{A}$, since it is difficult to show that multi-valued systems possess some kind of smoothing property, which was used in the construction of the exponential attractors for the single-valued case, see, e.g., [2, 12, 13, 14, 15, 24, 34], and in fact, many single-valued semigroups have infinite dimensional global attractors, see, e.g., [33, 36].

Very recently, the concept of global exponential $\kappa$-dissipativity was introduced in [35], and under the assumption that the single-valued semigroup $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set, authors have proven that $\{S(t)\}_{t \geq 0}$ is globally exponentially $\kappa$-dissipative if and only if $\{S(t)\}_{t \geq 0}$ possesses a compact positively invariant set which exponentially attracts every bounded set. In the present paper, we first extend the above theory result to the multi-valued case, and then we apply this
general theory to retarded ordinary differential equations and lattice dynamical systems, as well as reaction-diffusion equations with infinite delays. The existence of exponential attractors for delay differential equations was proved recently in [10, 16, 19, 28] and the robustness of exponential attractors for such systems was treated in [7, 15, 17]. Here since we do not assume any Lipschitz condition on the nonlinear term, in order to obtain the existence of compact positively invariant sets which exponentially attract every bounded set for multi-valued delay systems, we need to use the theory framework of multi-valued dynamical systems.

The paper is organized as follows. In Section 2, we recall some preliminary results and definitions. In Section 3, we present the sufficient conditions for the existence of a compact positively invariant set which exponentially attracts every bounded set for multi-valued semidynamical systems. Sections 4-6 are devoted to considering retarded ordinary differential equations and lattice systems as well as reaction-diffusion equations with infinite delays.

2. Preliminaries. We recall now some standard definitions for multi-valued semidynamical systems and some results ensuring the existence of a global attractor for these systems.

Let $X$ be a Banach space with norm $\| \cdot \|_X$, and denote by $C(X)$ and $K(X)$ the sets of all nonempty closed and nonempty compact subsets of $X$, respectively. Let also denote by $\text{dist}(A, B)$ the Hausdorff semimetric, i.e., for given subsets $A$ and $B$ we have

$$\text{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$ 

Finally, denote by $O(A, r)$ the open neighborhood $\{y \in X \mid \text{dist}_X(y, A) < r\}$ of radius $r > 0$ of a subset $A$ of $X$.

**Definition 1.** A family of multi-valued mappings $S(t) : X \rightarrow C(X), t \in \mathbb{R}^+$, is said to be an (autonomous) multi-valued semidynamical system (MVSS in short) if:

1. $S(0)x = \{x\}, \forall x \in X$;
2. $S(t+s)x = S(t)S(s)x, \forall t, s \in \mathbb{R}^+, \forall x \in X,$

where $S(t)B = \bigcup_{x \in B} S(t)x, B \subset X$.

**Remark 2.** It should be pointed out that Kapustyan and Mel’nik used more common definition of multi-valued semidynamical system which involves only inclusion $\subset$ in the semigroup property [22, 23].

**Definition 3.** Let $\{S(t)\}_{t \geq 0}$ be a multi-valued semidynamical system on $X$.

1. A bounded set $\mathcal{U} \subset X$ is said to be an absorbing set for $\{S(t)\}_{t \geq 0}$ if for every bounded set $B \subset X$, there exists a $T_0 = T_0(B) \in \mathbb{R}^+$ such that $S(t)B \subset \mathcal{U}, \forall t \geq T_0$.
2. $\{S(t)\}_{t \geq 0}$ is said to be $\omega$-limit compact if for any bounded subset $B$ of $X$ and $\varepsilon > 0$, there exists a $T_1 = T_1(B, \varepsilon) \in \mathbb{R}^+$ such that

$$\kappa\left(\bigcup_{t \geq T_1} S(t)B\right) \leq \varepsilon,$$

where $\kappa$ is the measure of noncompactness.

**Definition 4.** A nonempty compact subset $\mathcal{A}$ of $X$ is called to be a global attractor for the multi-valued semidynamical system $\{S(t)\}_{t \geq 0}$, if it satisfies
(1) \( A \) is an invariant set, i.e.,
\[
S(t)A = A, \quad \forall t \in \mathbb{R}^+;
\]
(2) \( A \) attracts each bounded subset \( B \) of \( X \), i.e.,
\[
\lim_{t \to +\infty} \text{dist}_X(S(t)B, A) = 0.
\]

Now we recall a general result on the existence and uniqueness of global attractors associated to multi-valued semidynamical systems, which was proved in \([8, 25, 30]\).

**Theorem 5.** Let \( \{S(t)\}_{t \geq 0} \) be a multi-valued semidynamical system on \( X \). Suppose that \( S(t) : X \to C(X) \) is upper semicontinuous for any \( t \in \mathbb{R}^+ \). Then \( \{S(t)\}_{t \geq 0} \) has a unique global attractor \( A \) given by
\[
A = \omega(U) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)U
\]
if and only if \( \{S(t)\}_{t \geq 0} \) is \( \omega \)-limit compact and \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( U \subset X \).

Let us recapitulate now the standard definition of the measure of noncompactness and its basic properties; see \([11]\) ect. for more details.

**Definition 6.** Let \((M, d)\) be a complete metric space and \( B \) be a bounded subset of \( M \). The measure of noncompactness \( \kappa(B) \) of \( B \) is defined by
\[
\kappa(B) = \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets whose diameter } \leq \delta\}.
\]

**Lemma 7.** Let \((M, d)\) be a complete metric space and \( \kappa \) be the measure of noncompactness. Then

1. \( \kappa(B) = 0 \) if and only if \( \overline{B} \) is compact, where \( \overline{B} \) is the closure of \( B \);
2. \( \kappa(B) = \kappa(\overline{B}) \);
3. if \( B_1 \subset B_2 \), then \( \kappa(B_1) \leq \kappa(B_2) \);
4. \( \kappa(B_1 \cup B_2) \leq \max\{\kappa(B_1), \kappa(B_2)\} \);
5. if \( B_t \) is a family of nonempty, closed and bounded sets defined for \( t > r \) that satisfy \( B_s \subset B_t \) whenever \( s \leq t \), and \( \kappa(B_t) \to 0 \) as \( t \to +\infty \), then \( \bigcap_{t > r} B_t \) is a nonempty, compact set in \( M \).
6. If, in addition, \( M \) is a Banach space, then
   \[ \kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2) \]
7. \( \kappa(\overline{coB}) = \kappa(B) \), where \( \overline{coB} \) is the closed convex hull of \( B \);
8. let \( M \) has the following decomposition:
   \[ M = M_1 \oplus M_2, \quad \text{with } \dim M_1 < +\infty, \]
   \[ P : M \to M_1, \quad Q : M \to M_2 \text{ be the canonical projectors, and } B \text{ be a bounded subset of } M. \text{ If the diameter of } QB \text{ is less than } \varepsilon, \text{ then } \kappa(B) < \varepsilon. \]

3. Construction of global exponential attracting sets for MVSSs. In this section, we will construct a compact positively invariant set which exponentially attracts each bounded set for abstract multi-valued semidynamical systems.

**Theorem 8.** Let \( \{S(t)\}_{t \geq 0} \) be a multi-valued semidynamical system on \( X \), and let \( S(t) : X \to K(X) \) be upper semicontinuous for any \( t \in \mathbb{R}^+ \). Assume that \( \{S(t)\}_{t \geq 0} \)
has a bounded absorbing set $U \subset X$, and \( \{ S(t) \}_{t \geq 0} \) is globally exponentially $\kappa$-dissipative, i.e., for every bounded set $B \subset X$, there exist positive constants $C_0$ and $\alpha_0$ such that
\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) \leq C_0 e^{-\alpha_0 s}, \quad \forall s \geq 0.
\] (3.1)

Then there exists a compact set $A^* \subset X$ for the multi-valued semidynamical system $\{ S(t) \}_{t \geq 0}$ such that
(1) $A^*$ is positively invariant, i.e.,
\[
S(t)A^* \subset A^*, \quad \forall t \in \mathbb{R}^+;
\]
(2) $A^*$ exponentially attracts each bounded subset $B$ of $X$.

Proof. By the assumption (3.1), we find that the multi-valued semidynamical system $\{ S(t) \}_{t \geq 0}$ is $\omega$-limit compact. Then it follows from Theorem 5 that $\{ S(t) \}_{t \geq 0}$ has a unique global attractor $A$ given by
\[
A = \omega(U) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)U.
\]
Since $\{ S(t) \}_{t \geq 0}$ is globally exponentially $\kappa$-dissipative, there exist positive constants $C_0$ and $\alpha_0$ such that
\[
\kappa \left( \bigcup_{t \geq s} S(t)U \right) \leq C_0 e^{-\alpha_0 s}, \quad \forall s \geq 0.
\]

Taking $s_1 = 1$, we get
\[
\kappa \left( \bigcup_{t \geq s_1} S(t)U \right) \leq C_0 e^{-\alpha_0 s_1}.
\]

By the definition of the measure of noncompactness, there exist finite points \( \{ y_i^{(1)} \}_{i=1}^{k_1} \subset \bigcup_{t \geq s_1} S(t)U \) such that
\[
\bigcup_{t \geq s_1} S(t)U \subset \bigcup_{i=1}^{k_1} \mathcal{O} \left( y_i^{(1)}, C_0 e^{-\alpha_0 s_1} \right).
\]
Noticing that for each $i$, $y_i^{(1)} \in \bigcup_{t \geq s_1} S(t)U$. Hence there exist $x_i^{(1)} \in U$ and $t_i^{(1)} \geq s_1$ such that $y_i^{(1)} \in S(t_i^{(1)})x_i^{(1)}$, and consequently,
\[
\bigcup_{t \geq s_1} S(t)U \subset \bigcup_{i=1}^{k_1} \mathcal{O}(S(t)S(s_1)x_i^{(1)}, \varepsilon_1),
\]
where $\varepsilon_1 := C_0 e^{-\alpha_0 s_1}$. This implies that
\[
\text{dist}_X \left( \bigcup_{t \geq s_1} S(t)U, \bigcup_{i=1}^{k_1} \bigcup_{t \geq 0} \mathcal{O}(S(t)S(s_1)x_i^{(1)}, \varepsilon_1) \right) \leq \varepsilon_1.
\]

Proceeding inductively and taking $s_m = m$, then there exist $x_i^{(m)} \in U$ and $t_i^{(m)} \geq s_m$ such that
Let 
\[
A^* := A \bigcup \left( \bigcup_{m=1}^{+\infty} \bigcup_{i=1}^{k_m} S(t)S(s_m)x_i^{(m)} \right).
\] (3.3)

Note that \(A\) is invariant, hence for any \(t \geq 0\),
\[
S(t)A^* = S(t)A \bigcup S(t) \left( \bigcup_{m=1}^{+\infty} \bigcup_{i=1}^{k_m} S(\tau)S(s_m)x_i^{(m)} \right)
\]
\[
= A \bigcup \left( \bigcup_{m=1}^{+\infty} \bigcup_{i=1}^{k_m} S(t+\tau)S(s_m)x_i^{(m)} \right)
\]
\[
\subset A \bigcup \left( \bigcup_{m=1}^{+\infty} \bigcup_{i=1}^{k_m} S(t)S(s_m)x_i^{(m)} \right) = A^*.
\]

Therefore, \(A^*\) is positively invariant. On the other hand, arguing as in the proof of Theorem 4.1 in [35], we obtain that \(A^*\) exponentially attracts each bounded set \(B \subset X\).

Finally, we show that \(A^*\) is compact. Let us consider an arbitrary sequence \(\{y_n\}\) in \(A^*\), it suffices to prove that \(\{y_n\}\) has a convergent subsequence in \(A^*\). Note that \(A\) is a compact set, if \(\{y_n\}\) has a subsequence in \(A\), then it is clear that \(\{y_n\}\) is precompact. Hence, without loss of generality, we assume that
\[
\{y_n\} \subset \bigcup_{m=1}^{+\infty} \bigcup_{i=1}^{k_m} S(t)S(s_m)x_i^{(m)}.
\]

Then for each \(n \in \mathbb{N}\), there exist \(m_n, t_n\) and \(1 \leq i_n \leq k_{m_n}\) such that
\[
y_n \in S(t_n)S(s_{m_n})x_{i_n}^{(m_n)} = S(t_n + s_{m_n})x_{i_n}^{(m_n)}.
\]

Two cases may occur.

**Case 1.** \(t_n \to +\infty\) or \(s_{m_n} \to +\infty (n \to \infty)\). Recall that \(A\) attracts \(U\), hence
\[
\lim_{n \to \infty} \text{dist}_X \left( S(t_n + s_{m_n})x_{i_n}^{(m_n)}, A \right) = 0,
\]
which implies that
\[
\lim_{n \to \infty} \text{dist}_X (y_n, A) = 0.
\]

Since \(A\) is compact, \(\{y_n\}\) has a subsequence which converges in \(X\).

**Case 2.** \(\{t_n\}\) and \(\{s_{m_n}\}\) are bounded. In this case, there exists a \(N_0 \in \mathbb{N}\) such that \(t_n \leq N_0\) and \(m_n \leq N_0\) for all \(n \in \mathbb{N}\). Observing that \(s_{m_n} = m_n\) and
\[
\{y_n\} \subset \bigcup_{m_n=1}^{N_0} \bigcup_{i_n=1}^{k_{m_n}} \bigcup_{t_n=1}^{N_0} S(t_n + m_n)x_{i_n}^{(m_n)}.
\]

Since \(S(t)x\) is compact for any \(t \in \mathbb{R}^+\) and \(x \in X\), in view of property (4) for the measure of noncompactness in Lemma 7, we deduce that \(\{y_n\}\) is compact. Therefore, the proof of this theorem is finished. □
Remark 9. It is worth mentioning that similar to the arguments of Theorem 3.2 in [35], we see that if there exists a compact set \( \mathcal{A}^* \subset X \) for the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) such that \( \mathcal{A}^* \) exponentially attracts each bounded set \( B \subset X \), i.e., there exist positive constants \( C_0' \) and \( \alpha_0' \) such that
\[
\text{dist}_X (S(t)B, \mathcal{A}^*) \leq C_0' e^{-\alpha_0't}, \quad \forall t \geq 0,
\]
then \( \{S(t)\}_{t \geq 0} \) is globally exponentially \( \kappa \)-dissipative.

3.1. Multi-valued semidynamical systems with delays. In this subsection, several sufficient conditions to obtain the global exponential \( \kappa \)-dissipativity of the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) will be addressed.

Since the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) satisfies the property (2) of Definition 1, we have the following result from Theorem 3.1 in [35].

**Theorem 10.** Let \( \{S(t)\}_{t \geq 0} \) be a multi-valued semidynamical system on \( X \). Suppose that \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( \mathcal{U} \subset X \) and is uniformly \( \kappa \)-contracting, i.e., there exist \( t_0 > 0 \) and \( \beta_0 \in (0, 1) \) such that for each bounded set \( B \subset X \),
\[
\kappa (S(t_0)B) \leq \beta_0 \kappa (B).
\]
Then \( \{S(t)\}_{t \geq 0} \) is globally exponentially \( \kappa \)-dissipative.

Now we present a method for checking the global exponential \( \kappa \)-dissipativity of the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) with finite delays. Let \( C_Y \) denote the Banach space \( C([-h, 0]; Y) \) endowed with the norm
\[
\| \varphi \|_{C_Y} := \sup_{\theta \in [-h, 0]} \| \varphi(\theta) \|_Y,
\]
where \( Y \) is a Banach space with norm \( \| \cdot \|_Y \), and \( h \) is a given positive number, which will denote the delay time.

**Theorem 11.** Let \( \{S(t)\}_{t \geq 0} \) be a multi-valued semidynamical system on \( C_Y \). Suppose that for any bounded set \( B \subset C_Y \), there exist positive constants \( t_1, C_1 \) and \( \alpha_1 \) so that for any \( \varepsilon > 0 \), there exist a finite-dimensional subspace \( Y_\varepsilon \) of \( Y \) and a \( \delta_1 > 0 \) such that
\begin{enumerate}
\item for each fixed \( \theta \in [-h, 0] \),
\[
\left\| \bigcup_{t \geq t_1} \bigcup \{ Pu(t + \theta) \} \right\|_Y \text{ is bounded ;}
\]
\item for all \( t \geq t_1 \), \( u(\cdot) \in S(t)B \), \( \theta_1, \theta_2 \in [-h, 0] \) with \( \theta_2 - \theta_1 < \delta_1 \),
\[
\| P(u(t + \theta_1) - u(t + \theta_2)) \|_Y < \varepsilon;
\]
\item for all \( t \geq t_1 \), \( u(\cdot) \in S(t)B \),
\[
\sup_{\theta \in [-h, 0]} \| (I - P)u(t + \theta) \|_Y < C_1 e^{-\alpha_1 t} + \varepsilon,
\]
\end{enumerate}
where \( P : Y \to Y_\varepsilon \) is the canonical projector, and \( u(\cdot) \) is defined for \( \theta \in [-h, 0] \) as \( u(\theta) = u(t + \theta) \). Then \( \{S(t)\}_{t \geq 0} \) is globally exponentially \( \kappa \)-dissipative.

**Proof.** Let a bounded set \( B \subset C_Y \), \( s \geq t_1 \) and \( \varepsilon > 0 \) be given arbitrarily. We set
\[
\tilde{B} = \{ u(\cdot) : u(\cdot) \in \bigcup_{t \geq s} S(t)B \}.
\]
Then it follows from assumptions (1) and (2) that
\(P \tilde{B}\) is relatively compact in \(C_Y\). Hence by Lemma 7 and assumption (3), we deduce that

\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) = \kappa \left( \bigcup_{t \geq s} S(t)B \right) = \kappa(\tilde{B}) \leq \kappa(P \tilde{B}) + \kappa((I - P)\tilde{B}) \leq 2C_1e^{-\alpha_1s} + 2\varepsilon,
\]

where \(\kappa\) is the measure of noncompactness on \(C_Y\). Then the global exponential \(\kappa\)-dissipativity of \(\{S(t)\}_{t \geq 0}\) on \(C_Y\) follows from letting \(\varepsilon \to 0\).

\(\square\)

Finally, we are ready to consider the multi-valued semidynamical system \(\{S(t)\}_{t \geq 0}\) with infinite delays. In particular, a useful method for checking the global exponential dissipativity of infinite delay systems is given. For given \(\gamma > 0\) we denote \(C_{\gamma,Y}\) the space

\[
C_{\gamma,Y} = \left\{ \varphi \in C((-\infty, 0]; Y) \mid \lim_{s \to -\infty} \varphi(s)e^{\gamma s} \text{ exists} \right\},
\]

and set

\[
\|\varphi\|_{C_{\gamma,Y}} := \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta}\|\varphi(\theta)\|_Y < \infty.
\]

This is a separable Banach space.

**Theorem 12.** Let \(\{S(t)\}_{t \geq 0}\) be a multi-valued semidynamical system on \(C_{\gamma,Y}\). Suppose that for any bounded set \(B \subset C_{\gamma,Y}\), there exist positive constants \(t_2\), \(C_2\) and \(\alpha_2\) so that for any \(\varepsilon > 0\), there exist \(T = T(B, \varepsilon) > 0\), a finite-dimensional subspace \(Y_\varepsilon\) of \(Y\) and a \(\delta_2 > 0\) such that

1. for all \(t \geq t_2\), \(u_\varepsilon(\cdot) \in S(t)B\),

\[
\sup_{\theta \in (-\infty, -T]} e^{\gamma\theta}\|u(t + \theta)\|_Y < \varepsilon;
\]

2. for each fixed \(\theta \in [-T, 0]\),

\[
\left\| \bigcup_{t \geq t_2, u_\varepsilon(\cdot) \in S(t)B} Pu(t + \theta) \right\|_Y \text{ is bounded};
\]

3. for all \(t \geq t_2\), \(u_\varepsilon(\cdot) \in S(t)B\), \(\theta_1, \theta_2 \in [-T, 0]\) with \(|\theta_2 - \theta_1| < \delta_2\),

\[
\|P(u(t + \theta_1) - u(t + \theta_2))\|_Y < \varepsilon;
\]

4. for all \(t \geq t_2\), \(u_\varepsilon(\cdot) \in S(t)B\),

\[
\sup_{\theta \in [-T, 0]} \|(I - P)u(t + \theta)\|_Y < C_2e^{-\alpha_2t} + \varepsilon,
\]

where \(P : Y \to Y_\varepsilon\) is the canonical projector, and \(u_\varepsilon(\cdot)\) is defined for \(\theta \in (-\infty, 0]\) as \(u_\varepsilon(\theta) = u(t + \theta)\). Then \(\{S(t)\}_{t \geq 0}\) is globally exponentially \(\kappa\)-dissipative.

**Proof.** Let a bounded set \(D \subset C_{\gamma,Y}\), \(s \geq t_2\) and \(\varepsilon > 0\) be given arbitrarily. We set \(\tilde{D} = \left\{u(\cdot) : u(\cdot) \in \bigcup_{t \geq s} S(t)D \right\}\), and we define the operators \(\Pi_T\) and \(\Pi_T'\) by setting

\[
(\Pi_T(u))(\theta) := \begin{cases} u(\theta), & \theta \in [-T, 0], \\ 0, & \theta \in (-\infty, -T). \end{cases}
\]
Ordinary differential equations with delays.

4.

and

\[(\Pi'_T(u)) \theta := \begin{cases} 0, & \theta \in [-T, 0], \\ u(\theta), & \theta \in (-\infty, -T), \end{cases}\]

respectively. Thanks to Theorem 11, assumptions (2)-(4) imply that

\[\kappa \left( \Pi_T \left( \bigcup_{t \geq s} S(t)D \right) \right) = \kappa \left( \Pi_T(D) \right) \leq 2C_2e^{-\alpha s} + 2\varepsilon.\]

Then by Lemma 7 and assumption (1), we conclude that

\[\kappa \left( \bigcup_{t \geq s} S(t)D \right) = \kappa \left( \bigcup_{t \geq s} S(t) \right) \leq \kappa(D) \leq \kappa \left( \Pi_T(D) \right) + \kappa \left( \Pi'_T(D) \right) \leq 2C_2e^{-\alpha s} + 4\varepsilon,\]

where \(\kappa\) is the measure of noncompactness on \(C_{\gamma,Y}\). Let \(\varepsilon \to 0\), hence \(\{S(t)\}_{t \geq 0}\) on \(C_{\gamma,Y}\) is globally exponentially \(\kappa\)-dissipative. \(\square\)

4. Ordinary differential equations with delays. Let \(h > 0\) be a given positive number, which will denote the delay time. Consider the equation

\[x'(t) = F_0(x(t)) + F_1(x(t - \rho)) + \int_{-h}^{0} b(s, x(t + s)) ds = f(x_t),\]  

\[x_0 = \varphi \in C_{\mathbb{R}^n},\]

where \(F_0, F_1 \in C(\mathbb{R}^n; \mathbb{R}^n), b \in C([-h, 0] \times \mathbb{R}^n; \mathbb{R}^n), \rho \in [0, h]\) and the initial condition \(\varphi\) is specified in \(C_{\mathbb{R}^n} = C([-h, 0]; \mathbb{R}^n)\), the space of continuous functions from \([-h, 0]\) into \(\mathbb{R}^n\) endowed with the norm \(\|\varphi\|_{C_{\mathbb{R}^n}} = \sup_{\theta \in [-h, 0]} |\varphi(\theta)|\). Given a function \(x \in C([-h, T]; \mathbb{R}^n)\), for each \(t \in [0, T]\) the notation \(x_t\) denotes the function in \(C_{\mathbb{R}^n}\) given by

\[x_t(\theta) = x(t + \theta) \text{ for all } \theta \in [-h, 0].\]

Denote by \(\langle \cdot, \cdot \rangle\) and \(|\cdot|\) the scalar product and norm in \(\mathbb{R}^n\), respectively.

Let us now introduce the following conditions:

(H1) There exist positive scalar functions \(m_0, m_1 \in L^1([-h, 0])\) such that

\[|b(s, x)| \leq m_0(s) + m_1(s)|x|.\]  

\[\text{(H2) There exist positive constants } k_1, k_2, \alpha \text{ and } \beta \text{ such that}\]

\[\langle x, F_0(x) \rangle \leq -\alpha|x|^2 + \beta, \quad \forall x \in \mathbb{R}^n,\]  

\[|F_1(x)|^2 \leq k_1^2 + k_2^2|x|^2, \quad \forall x \in \mathbb{R}^n.\]

Denote \(m_i = \int_{-h}^{0} m_i(s) ds, i = 0, 1\). In the sequel \(C\) denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

It is known (Hale [18]) that the assumptions (H1) and (H2) ensure the local existence of solutions for problem (4.1). By slightly modifying the proof of Theorem 35 in [4], we have

Lemma 13. Let conditions (H1)-(H2) hold. Also, assume that

\[2m_1eh < 1,\]  

\[k_2^2 < e^{-1}\alpha(\alpha - \lambda^*),\]  

and
where \( \lambda^* \in (\lambda_0, \lambda_1) \), being \( \lambda_0 < \lambda_1 \) the solutions of the equation \( \lambda e^{-\lambda h} = 2m_1 \), and let

\[
\lambda^* < \alpha.
\] (4.7)

Then, every solution \( x(\cdot) \) with initial data \( \varphi \in C_{\mathbb{R}^n} \) satisfies

\[
\|x_t\|_{C_{\mathbb{R}^n}} \leq C\|\varphi\|_{C_{\mathbb{R}^n}}^2 e^{\lambda^* t} + C\|\varphi\|_{C_{\mathbb{R}^n}}^2 e^{-(\lambda^* - L)|t|} + C, \quad \forall t \in [0, T^*),
\] (4.8)

where \( L = 2m_1 e^{\lambda^* h} \) and \( T^* \) is the maximal time of existence.

Assume that the conditions of Lemma 13 hold. Then it follows from Lemma 13 that every local solution of Eq. (4.1) can be defined globally, and thus we define a family of multi-valued mappings \( S(t) : C_{\mathbb{R}^n} \rightarrow C_{\mathbb{R}^n} \) by setting

\[
S(t)\varphi = \{x_t(\cdot; \varphi) \mid x(\cdot) \text{ is a solution of Eq. (4.1) with initial data } \varphi \in C_{\mathbb{R}^n}\}.
\]

By a standard way in [4] (Lemma 8), we see that \( \{S(t)_{t \geq 0}\} \) satisfies (1) and (2) of Definition 1.

The Ascoli-Arzelà Theorem allows us to prove the following result which will be used to check the global exponential \( \kappa \)-dissipativity of the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \).

**Lemma 14.** Assume that the conditions of Lemma 13 hold. Then the next properties hold:

1. For any bounded set \( B \subset C_{\mathbb{R}^n} \) and \( t \geq 0, \overline{S(t)B} \) is a compact subset of \( C_{\mathbb{R}^n} \).
2. For any \( t \geq 0 \), the map \( S(t) \) is upper semicontinuous and has compact values.

**Proof.** (1) Let \( t \geq 0 \), a bounded set \( B \subset C_{\mathbb{R}^n} \) and a sequence of points \( \psi^m \in S(t)B \) be given arbitrarily. Then we will show that \( \{\psi^m\} \) is precompact in \( C_{\mathbb{R}^n} \).

Observe that there exists a sequence of solutions of (4.1), \( x^m : [-h, t] \rightarrow \mathbb{R}^n \), with \( x^m_t = \psi^m \), and by Lemma 13 there exists a constant \( R_0 > 0 \) such that

\[
\|x^m_s\|_{C_{\mathbb{R}^n}} \leq R_0 \quad \text{for all } s \geq 0 \text{ and } m \in \mathbb{N}.
\] (4.9)

On the other hand, by (H1)-(H2) and (4.9), we can deduce that for all \( m \in \mathbb{N} \) and \( \theta_1, \theta_2 \in [-h, 0] \) with \( \theta_1 < \theta_2 \),

\[
|x^m(t + \theta_1) - x^m(t + \theta_2)| \leq \int_{t + \theta_1}^{t + \theta_2} \left| \frac{dx^m(s)}{ds} \right| ds
\]

\[
\leq \int_{t + \theta_1}^{t + \theta_2} \left( |F_0(x^m(s))| + |F_1(x^m(s - \rho))| + \left| \int_{-h}^{0} b(r, x^m(s + r)) dr \right| \right) ds
\]

\[
\leq C \int_{t + \theta_1}^{t + \theta_2} \left( |F_0(x^m(s))|^2 + |F_1(x^m(s - \rho))|^2 + \left| \int_{-h}^{0} b(r, x^m(s + r)) dr \right|^2 \right) ds
\]

\[
+ C(\theta_2 - \theta_1)
\]

\[
\leq C(\theta_2 - \theta_1) + C \int_{t + \theta_1}^{t + \theta_2} |x^m(s - \rho)|^2 ds + C \int_{t + \theta_1}^{t + \theta_2} \|x^m_s\|_{C_{\mathbb{R}^n}}^2 ds
\]

\[
\leq C(\theta_2 - \theta_1).
\] (4.10)

(4.9) and (4.10) allow us to apply the Ascoli-Arzelà Theorem and conclude the precompactness of \( \{\psi^m\} \).

(2) Since the compactness of the values of \( S(t) \) follows easily from the conclusion (1) and the upper semicontinuity of the map \( S(t) \), now we only need to prove that the map \( \varphi \mapsto S(t)\varphi \) is upper semicontinuous. Suppose not. Then there exist some
\[ \phi \in C_{R^n}, \text{ a neighborhood } \mathcal{O} \text{ of } S(t)\phi \text{ and sequences } \phi^m \to \phi, \xi^m \in S(t)\phi^m \text{ such that } \xi^m \notin \mathcal{O} \text{ for all } m \in \mathbb{N}. \text{ Clearly, there exists a sequence of solutions of } (4.1), \]
\[ x^m : [-h, t] \to \mathbb{R}^n, \text{ with } x^m_0 = \phi^m \text{ and } x^m_t = \xi^m. \text{ Note that } \phi^m \to \phi, \text{ hence by Lemma 13 there exists a constant } R'_0 > 0 \text{ such that} \]
\[ \|x^m_s\|_{C_{R^n}} \leq R'_0 \quad \text{for all } s \geq 0 \text{ and } m \in \mathbb{N}. \tag{4.11} \]

Similar to the argument in (4.10), we conclude that for all \( m \in \mathbb{N} \) and \( s_1, s_2 \in [0, t] \) with \( s_1 < s_2, \)
\[ |x^m(s_1) - x^m(s_2)| \leq C(s_2 - s_1). \tag{4.12} \]

Therefore, the Ascoli-Arzelà Theorem implies the existence of a converging subsequence \( x^\mu \) to a function \( x : [0, t] \to \mathbb{R}^n \) (the convergence is uniform on \([0, t])\).

Using
\[ x^m(t) = \phi^m(0) + \int_0^t f(x^\mu_s)ds, \]
by the Lebesgue Theorem, we pass to the limit and obtain that
\[ \bar{x}(r) = \begin{cases} \phi(r), & r \in [-h, 0], \\ x(r), & r \in [0, t], \end{cases} \]
solves (4.1) with initial data \( \phi \). Recall that \( x^\mu(\cdot) \to \bar{x}(\cdot) \) in \( C([-h, t]; \mathbb{R}^n) \) and \( x^\mu_t = \xi^\mu, \) hence \( \xi^\mu \to \bar{x}_t \) in \( C_{R^n} \) and \( \bar{x}_t \in S(t)\phi, \) which is a contradiction. \( \square \)

Then, we have the following result.

**Theorem 15.** Assume that the conditions of Lemma 13 hold. Then the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) associated with problem (4.1) possesses a compact positively invariant set \( A^* \subset C_{R^n} \) which exponentially attracts each bounded subset of \( C_{R^n}. \)

**Proof.** Using point (1) of Lemma 14 and property (2) of Lemma 7, we obtain that for any bounded set \( B \subset C_{R^n} \) and \( t \geq 0 \)
\[ \kappa \left(S(t)B \right) = \kappa (S(t)B) = 0. \]

This implies that \( \{S(t)\}_{t \geq 0} \) is uniformly \( \kappa \)-contracting. Thanks to Theorem 10 and Lemma 13, we obtain that \( \{S(t)\}_{t \geq 0} \) is globally exponentially \( \kappa \)-dissipative. Then we complete the proof by using Theorem 8 and Lemma 14. \( \square \)

5. **Lattice differential equations with delays.** Let us consider a first order lattice differential equation with finite delays
\[ \dot{u}_i(t) + (-1)^p \Delta^p u_i(t) + \lambda_i u_i(t) = F_{0,i}(u_i(t)) + F_{1,i}(u_i(t - \rho)) + \int_{-h}^0 G_i(r, u_i(t + r))dr + g_i, \tag{5.1} \]
with the initial condition
\[ u_i(t) = \phi_i(t), \quad t \in [-h, 0], \quad i \in \mathbb{Z}, \tag{5.2} \]
de where \( u = (u_i)_{i \in \mathbb{Z}} \in \ell^2, \quad p \geq 2, \quad \rho \in [0, h], \quad \lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \ell^2, \quad g = (g_i)_{i \in \mathbb{Z}} \in \ell^2, \quad h > 0 \)
denotes the delay time, \( F_{0,i}, F_{1,i} \) and \( G_i \) are continuous functions satisfying the assumptions below. In (5.1), \( p \) is any positive integer and \( \Delta^p = \Delta \cdots \Delta, p \) times, where \( \Delta \) denotes the discrete one-dimensional Laplace operator, which is defined by \( \Delta u_i = u_{i+1} + u_{i-1} - 2u_i. \) We also write \( \partial^+ u_i = u_{i+1} - u_i, \partial^- u_i = u_i - u_{i-1} \) and
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$$D^p := \begin{cases} \Delta^\frac{p}{2}, & p \text{ even,} \\ \partial^+ \Delta^\frac{p-1}{2}, & p \text{ odd.} \end{cases}$$

It is easy to see by induction that

$$\sum_{i \in \mathbb{Z}} |D^p u_i|^2 \leq 4^p \|u\|^2_{l^2}, \quad (5.3)$$

for all \(u = (u_i)_{i \in \mathbb{Z}} \in l^2\).

We consider the following assumptions on the nonlinearities:

\(G1\) There exist two positive constants \(\lambda_0\) and \(\lambda_0\) such that

$$0 < \lambda_0 \leq \lambda_i \leq \lambda_0, \quad \forall i \in \mathbb{Z}.$$ \(G2\) \(F_{0,i}\) are continuous and satisfy that \(F_{0,i}(x) \leq -\alpha_1 |x|^q + \psi_{1,i}\), for all \(x \in \mathbb{R}\), where \(\alpha_1 > 0\), \(q \geq 2\) and \(\psi_1 = (\psi_{1,i})_{i \in \mathbb{Z}} \in l^1\).

\(G3\) \(|F_{0,i}(x)| \leq |\alpha_2|x|^{q-1} + |\psi_{2,i}|\), for all \(x \in \mathbb{R}\), where \(\alpha_2 > 0\), \(q \geq 2\) and \(\psi_2 = (\psi_{2,i})_{i \in \mathbb{Z}} \in l^2\).

\(G4\) \(F_{1,i}\) are continuous and verify that \(|F_{1,i}(x)|^2 \leq |k_{1,i}|^2 + k_2^2 |x|^2\), for all \(x \in \mathbb{R}\), where \(k_1 = (k_{1,i})_{i \in \mathbb{Z}} \in l^1\), \(k_2 > 0\).

\(G5\) \(|G_i(r,x)| \leq m_{0,i}(r) + m_{1,i}(r) |x|\), for all \(x \in \mathbb{R}\) and a.a. \(r \in (-h,0)\), where \(G_i\) are Caratheodory, that is, measurable in \(r\) and continuous in \(x\).

Also, \(m_{0,i}()\), \(m_{1,i}() \in l^1((-h,0), l^0)\), \(m_{0,i}(r), m_{1,i}(r) \geq 0\) and defining

$$M_{j,i} = \int_{-h}^r m_{j,i}(r)dr.$$ We also assume that \(M_j^2 := \sum_{i \in \mathbb{Z}} M_{j,i}^2 < \infty, \quad j = 0, 1.\)

Let

$$l^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}}, u_i \in \mathbb{R} : \sum_{i \in \mathbb{Z}} u_i^2 < +\infty \right\},$$

and equip it with the inner product and norm as

$$(u,v) = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \|u\|^2_{l^2} = (u,u), \quad \forall u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in l^2.$$ The norm of \(l^2\) is usually written as \(\| \cdot \|_{l^2}\). Denote by \(C_{l^2}\) the Banach space \(C([-h,0]; l^2)\) endowed with the norm \(\|\varphi\|_{C_{l^2}} = \sup_{t \in [-h,0]} \|\varphi(t)\|_{l^2}\). By \(u_t\) we will denote the element in \(C_{l^2}\) given by \(u_t(s) = u(t+s)\) for all \(s \in [-h,0]\).

Following the arguments in [6, P. 63], and thanks to Theorem 10 and Corollary 13 in [6], we conclude that there exists at least one solution \(u_t(\cdot) \in C^1([0,a]; l^2)\) in a maximal interval \([0,a)\). Arguing as in the proofs of Proposition 17 in [6] and Lemma 12 in [31], we obtain some estimates of solutions.

**Lemma 16.** In addition to the assumptions (G1)-(G5) and \(g \in l^2\), assume that there exists a positive constant \(\eta\) satisfying

$$\eta > \frac{4M_1^2 e^{\eta h}}{\lambda_0}, \quad (5.4)$$

and

$$\eta + \frac{2k_2^2 e^{\eta h}}{\lambda_0} < \lambda_0. \quad (5.5)$$

Then for any initial data \(\varphi \in C_{l^2}\), every solution \(u_t\) of Eqs. (5.1) and (5.2) satisfies

$$\|u_t\|^2_{C_{l^2}} \leq Ce^{-(\eta-L')t} \|\varphi\|^2_{C_{l^2}} + C, \quad \forall t \in [0,a), \quad (5.6)$$
where \( L' = \frac{4M_2 \epsilon^{\gamma h}}{\lambda_0} \) and \( a \) is the maximal time of existence.

Thanks to Lemma 16, we define a family of multi-valued mappings \( S(t) : C_{l^2} \to C_{l^2} \) by setting

\[
S(t) \varphi = \{ u_t(\cdot ; \varphi) \mid u(\cdot) \text{ is a solution of Eq. (5.1)} \text{ with the initial data } \varphi \in C_{l^2} \}.
\]

By a standard way (see [26, Lemma 13]) one can prove that \( \{ S(t) \}_{t \geq 0} \) satisfies (1) and (2) of Definition 1.

In order to check the global exponential \( \kappa \)-dissipativity of \( \{ S(t) \}_{t \geq 0} \), we need to use some estimates of the solutions.

**Lemma 17.** Assume the conditions of Lemma 16. Let \( B \) be a bounded subset of \( C_{l^2} \). Then there exists \( T_1 > 0 \) such that for any positive integer \( M \), any solution \( u \in S(t)B \) satisfies

\[
\sup_{\theta \in [-h,0]} \sum_{i \geq 2M} |u_i(t+\theta)|^2 \leq Ce^{-\frac{(\eta-L')t}{2}} + K(M), \quad \forall t \geq T_1, \tag{5.7}
\]

where \( \eta \) and \( L' \) are given in Lemma 16, and \( K(M) \) is a positive constant depending on \( M \) with \( \lim_{M \to +\infty} K(M) = 0 \), which will be fixed later on.

**Proof.** Choose a smooth function \( \chi \) such that \( 0 \leq \chi(s) \leq 1 \) for \( s \in \mathbb{R}^+ \), and

\[
\chi(s) = 0 \quad \text{for } 0 \leq s \leq 1, \quad \chi(s) = 1 \quad \text{for } s \geq 2.
\]

Then there exists a positive constant \( R_1 \) such that \( |\chi'(s)| \leq R_1 \) for \( s \in \mathbb{R}^+ \). Let \( M \) be an arbitrary positive integer, and set \( v(t) = (v_i(t))_{i \in \mathbb{Z}} \) with

\[
v_i(t) = \chi \left( \frac{|i|}{M} \right) u_i(t), \quad i \in \mathbb{Z}.
\]

By Lemma 3.6 in [27] and Lemma 14 in [31], we conclude that

\[
(-1)^p \sum_{i \in \mathbb{Z}} \Delta_i^p u_i \chi \left( \frac{|i|}{M} \right) u_i = \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) \sum_{i \in \mathbb{Z}} |D_i^p u_i|^2 + \sum_{i \in \mathbb{Z}} \left( \sum_{i \in \mathbb{Z}} \theta^+ \chi \left( \frac{|i|}{M} \right) \right) z_p,i, \tag{5.8}
\]

where \( \sum_{i \in \mathbb{Z}} |z_p,i| \leq C(p) \| u \|_{l^2}^2 \), for some positive constant \( C(p) \) depending on \( p \).

Taking the inner product of Eq. (5.1) with \( v(t) = (v_i(t))_{i \in \mathbb{Z}} \), in view of (5.8) and (G1), we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i|^2 + \lambda_0 \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i|^2 + \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |D_i^p u_i|^2 \\
+ \sum_{i \in \mathbb{Z}} \left( \sum_{i \in \mathbb{Z}} \theta^+ \chi \left( \frac{|i|}{M} \right) \right) z_p,i \leq \\
\sum_{i \in \mathbb{Z}} F_{0,i}(u_i) \chi \left( \frac{|i|}{M} \right) u_i + \sum_{i \in \mathbb{Z}} F_{1,i}(u_i(t-\rho)) \chi \left( \frac{|i|}{M} \right) u_i \\
+ \sum_{i \in \mathbb{Z}} \int_-^0 G_i(r,u_i(t+r)) \chi \left( \frac{|i|}{M} \right) u_i dr + \sum_{i \in \mathbb{Z}} g_i \chi \left( \frac{|i|}{M} \right) u_i.
\end{align*}
\]

Let \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be positive parameters to be fixed later on. Multiplying (5.9) by \( e^{|t|} \), and using Young’s inequality, (G2) and (G4), we find that
Integrating (5.10) over \((0, t)\), in view of \((G5)\) and Young’s inequality, we obtain that
\[
\int_0^t e^{nt} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(t - \rho)|^2 ds \\
\leq e^{nh} \int_{-h}^t e^{ns} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s)|^2 ds \\
\leq \frac{e^{nh}}{\eta} \|\varphi\|^2_{C_{i_2}} + e^{nh} \int_0^t e^{ns} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s)|^2 ds,
\]
and
\[
\left| 2 \int_0^t e^{ns} \left( \sum_{i \in \mathbb{Z}} \int_{-h}^0 G_i(r, u_i(s + r)) \chi \left( \frac{|i|}{M} \right) u_i(s) dr \right) ds \right|
\leq 2 \int_0^t e^{ns} \left( \sum_{i \in \mathbb{Z}} \int_{-h}^0 m_{0,i}(r) \chi \left( \frac{|i|}{M} \right) |u_i(s)| dr \\
+ \sum_{i \in \mathbb{Z}} \int_{-h}^0 m_{1,i}(r) |u_i(s + r)| \chi \left( \frac{|i|}{M} \right) |u_i(s)| dr \right) ds \\
\leq \varepsilon_3 \int_0^t e^{ns} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s)|^2 ds + \frac{2e^{nt}}{\eta \varepsilon_3} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) M_0^2, \\
+ \frac{2M_i^2}{\varepsilon_3} \int_0^t e^{ns} \sup_{r \in [-h, 0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s + r)|^2 ds.
\]
Therefore,
\[
e^{nt} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(0)|^2 \\
\leq \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(0)|^2 + \frac{k_2 e^{nh}}{2 \varepsilon_1 \eta} \|\varphi\|^2_{C_{i_2}} \\
+ \left( \eta - 2\lambda_0 + 2\varepsilon_1 + \varepsilon_2 + 3 + \frac{k_2 e^{nh}}{2 \varepsilon_1} \right) \int_0^t e^{ns} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s)|^2 ds
\]
(5.13)
where

Then, it follows from (5.14) and (5.15) that

\[ \epsilon_1 \leq C \left( \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |\psi_{1,i}| + \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |k_{1,i}|^2 + \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |M_{0,i}|^2 \right) \]

\[ + \frac{e^{nt}}{\eta} \left( 2 \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |\psi_{1,i}| + \frac{1}{2 \varepsilon_1} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |k_{1,i}|^2 + \frac{2}{\varepsilon_2} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |M_{0,i}|^2 \right) \]

\[ + \frac{e^{nt}}{\varepsilon_2 \eta} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |g_i|^2 + \frac{2 M^2}{\varepsilon_3} \int_{0}^{t} e^{\eta s} \sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s + \theta)|^2 ds \]

\[ - 2 \int_{0}^{t} e^{\eta s} \sum_{i \in \mathbb{Z}} \left( \partial^+ \chi \left( \frac{|i|}{M} \right) \right) z_{p,i}(s) ds. \]

Let \( \varepsilon_1 = \frac{\lambda_0}{\eta} \) and \( \varepsilon_3 = \frac{\lambda_0}{\eta} \). By assumption (5.5), we can neglect the fourth term in (5.13) if \( \varepsilon_2 \) is chosen small enough. Setting now \( t + \theta \) instead of \( t \) (where \( \theta \in [-h,0] \)), multiplying by \( e^{-\eta(t+\theta)} \), it yields

\[ \sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(t + \theta)|^2 \leq C e^{-\eta t} \| \varphi \|_{C_t^2}^2 \]

\[ + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |\psi_{1,i}| + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |k_{1,i}|^2 + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |M_{0,i}|^2 \]

\[ + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |g_i|^2 + \frac{4 M^2 e^{\eta h}}{\lambda_0} \int_{0}^{t} e^{\eta s} \sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s + \theta)|^2 ds \]

\[ - Ce^{-\eta t} \int_{0}^{t} e^{\eta s} \sum_{i \in \mathbb{Z}} \left( \partial^+ \chi \left( \frac{|i|}{M} \right) \right) z_{p,i}(s) ds. \]

(5.14)

Note that \( \sum_{i \in \mathbb{Z}} |z_{p,i}| \leq C(p) \| u \|_{L^2}^2 \). By (5.6) we have

\[ Ce^{-\eta t} \int_{0}^{t} e^{\eta s} \sum_{i \in \mathbb{Z}} \left( \partial^+ \chi \left( \frac{|i|}{M} \right) \right) z_{p,i}(s) ds \leq \frac{C}{M} e^{-\eta t} \int_{0}^{t} e^{\eta s} \sum_{i \in \mathbb{Z}} |z_{p,i}(s)| ds \]

\[ \leq \frac{CC(p)}{M} e^{-\eta t} \int_{0}^{t} e^{\eta s} \| u(s) \|_{L^2}^2 ds \leq \frac{CC(p)}{M} e^{-\eta t} \int_{0}^{t} e^{\eta s} \left( e^{-(n-L')s} \| \varphi \|_{C_t^2}^2 + 1 \right) ds \]

\[ \leq \frac{CC(p)}{M} \| \varphi \|_{C_t^2}^2 e^{-(n-L')t} + \frac{CC(p)}{M} \leq C \| \varphi \|_{C_t^2}^2 e^{-(n-L')t} + \frac{C}{M}. \]

(5.15)

Then, it follows from (5.14) and (5.15) that

\[ e^{nt} \sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(t + \theta)|^2 \leq C e^{L't} \| \varphi \|_{C_t^2}^2 + \bar{C}(M) e^{nt} \]

\[ + L' \int_{0}^{t} e^{\eta s} \sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(s + \theta)|^2 ds, \]

where \( L' := \frac{4 M^2 e^{\eta h}}{\lambda_0} \) and

\[ \bar{C}(M) := C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |\psi_{1,i}| + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |k_{1,i}|^2 + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |M_{0,i}|^2 \]

\[ + C \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |g_i|^2 + \frac{C}{M}. \]

(5.17)
Applying Gronwall’s inequality and by (5.4), we deduce that
\[
\sup_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) |u_i(t + \theta)|^2 \leq C e^{-(\eta - L')t} \|\varphi\|_{C^2}^2 \tag{5.18}
\]
\[+ C e^{-(\eta - L')t} \|\varphi\|_{C^2}^2 + L C(M) + \frac{L'C(M)}{\eta - L'}.
\]
Since \(\psi_1 = (\psi_{1,i})_{i \in \mathbb{Z}} \in l^1, k_1 = (k_{1,i})_{i \in \mathbb{Z}} \in l^2, M_0 = (M_{0,i})_{i \in \mathbb{Z}} \in l^2\) and \(g = (g_i)_{i \in \mathbb{Z}} \in l^2\), we have \(\lim_{M \to \infty} C(M) = 0\). Thus the conclusion (5.7) follows immediately from (5.18). The proof of this lemma is complete. \(\square\)

**Lemma 18.** Assume the conditions of Lemma 16. Let \(B\) be a bounded subset of \(C^2\). Then for any \(\theta > 0\), there exists a \(\delta_1 > 0\) such that for any positive integer \(M\) and any solution \(u_t \in S(t)B\) with \(\theta_1, \theta_2 \in [-h,0]\) and \(|\theta_1 - \theta_2| < \delta_1\),
\[
\left\| (u_t(t+\theta_1) - u_t(t+\theta_2)) \right\|_{l^2, i \leq 2M} < \varepsilon, \quad \forall t \geq h.
\] \(\tag{5.19}\)

**Proof.** Let \(B\) be a bounded subset of \(C^2\), and let \(M\) be an arbitrary positive integer. Without loss of generality, we assume that \(\theta_1, \theta_2 \in [-h,0]\) with \(0 < \theta_1 - \theta_2 < 1\). Then for any fixed \(t \geq h\) and any \(u_t \in S(t)B\),
\[
\left\| (u_t(t+\theta_1) - u_t(t+\theta_2)) \right\|_{l^2, i \leq 2M} \leq \int_{t+\theta_2}^{t+\theta_1} \left\| \left( \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) u_i(s) \right) \right\|_{l^2, i \leq 2M} ds
\]
\[\leq C \int_{t+\theta_2}^{t+\theta_1} \left\| \left( \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) u_i(s) \right) \right\|_{l^2, i \leq 2M} ds + \int_{t+\theta_2}^{t+\theta_1} \left\| (\Delta u_i(s)) \right\|_{l^2, i \leq 2M} ds
\]
\[+ \int_{t+\theta_2}^{t+\theta_1} \left\| \left( \int_{-h}^0 G_i(r, u_i(s + r)) \right) ds \right\|_{l^2, i \leq 2M} ds
\]
\[+ \int_{t+\theta_2}^{t+\theta_1} \left\| (g_i) \right\|_{l^2, i \leq 2M} ds.
\] \(\tag{5.20}\)

Since \(\|(-1)^p \Delta^p u\|_{l^2}^2 \leq 4^p \|u\|_{l^2}^2\) for all \(u \in l^2\), by (5.6) we have
\[
\int_{t+\theta_2}^{t+\theta_1} \left\| \left( \sum_{i \in \mathbb{Z}} \chi \left( \frac{|i|}{M} \right) u_i(s) \right) \right\|_{l^2, i \leq 2M} ds + \int_{t+\theta_2}^{t+\theta_1} \left\| (\Delta u_i(s)) \right\|_{l^2, i \leq 2M} ds
\]
\[\leq C \int_{t+\theta_2}^{t+\theta_1} \left\| u(s) \right\|_{l^2} ds \leq C \int_{t+\theta_2}^{t+\theta_1} \left\| u(s) \right\|_{l^2}^2 ds + C(\theta_1 - \theta_2) \] \(\tag{5.21}\)
\[\leq C \left( e^{-(\eta - L')\theta_2} - e^{-(\eta - L')\theta_1} \right) + C(\theta_1 - \theta_2).
\]
Using assumptions (G3)-(G5) and (5.6), we deduce that

\[
\int_{t+\theta_1}^{t+\theta_2} \left\| \left( F_{0,i} \left( u_i(s) \right) \right) \right\|_{|i| \leq 2M} \left\|_{\mathbb{R}^{4M+1}} \right. \, ds \\
\leq \int_{t+\theta_1}^{t+\theta_2} \left( \left\| \left( F_{0,i} \left( u_i(s) \right) \right) \right\|_{|i| \leq 2M} \right)^2 + C \right) \, ds \\
\leq \int_{t+\theta_1}^{t+\theta_2} \left( \sum_{i \in \mathbb{Z}} \left( 2\alpha_2^2 \left\| u_{s,i} \right\|^2 \right) + \right) \, ds + C(\theta_1 - \theta_2) \tag{5.22}
\]

\[
\leq \int_{t+\theta_1}^{t+\theta_2} \left( \sum_{i \in \mathbb{Z}} \left( 2\alpha_2^2 \left\| u_{s,i} \right\|^2 \right) \, ds + C(\theta_1 - \theta_2) \leq C \left( e^{-(q-1)(n-L')\theta_2} - e^{-(q-1)(n-L')\theta_1} \right) + C(\theta_1 - \theta_2), \tag{5.23}
\]

and

\[
\int_{t+\theta_2}^{t+\theta_1} \left( \left\| \left( F_{1,i} \left( u_i(s - \rho) \right) \right) \right\|_{|i| \leq 2M} \right\|_{\mathbb{R}^{4M+1}} \right. \, ds \\
\leq \int_{t+\theta_2}^{t+\theta_1} \left( \left\| \left( F_{1,i} \left( u_i(s - \rho) \right) \right) \right\|_{|i| \leq 2M} \right)^2 + C \right) \, ds \\
\leq \int_{t+\theta_2}^{t+\theta_1} \left( \sum_{i \in \mathbb{Z}} \left( 2\alpha_2^2 \left\| u_{s,i} \right\|^2 \right) \, ds + C(\theta_1 - \theta_2) \right) \tag{5.24}
\]

\[
\leq 2M_1^2 \int_{t+\theta_2}^{t+\theta_1} \left\| u_{s,i} \right\|^2 \, ds + C(\theta_1 - \theta_2) \\
\leq C \left( e^{-(q-1)(n-L')\theta_2} - e^{-(q-1)(n-L')\theta_1} \right) + C(\theta_1 - \theta_2).
\]

It follows from \( g \in \ell^2 \) that

\[
\int_{t+\theta_2}^{t+\theta_1} \left\| \left( g_1 \right) \right\|_{|i| \leq 2M} \left\|_{\mathbb{R}^{4M+1}} \right. \, ds \leq \int_{t+\theta_2}^{t+\theta_1} \left( \left\| \left( g_1 \right) \right\|_{|i| \leq 2M} \right)^2 + C \right) \, ds \leq C(\theta_1 - \theta_2). \tag{5.25}
\]

Thus, the conclusion (5.19) follows from (5.20)-(5.25).

We are now ready to state and prove the main result of this section.
Theorem 19. Assume the conditions of Lemma 16. Then the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) associated with problem (5.1) possesses a compact positively invariant set \( \mathcal{A}^* \subset C_l \) which exponentially attracts each bounded subset of \( C_l \).

Proof. Arguing as in the proof of Corollary 23 in [6], we obtain that the multi-valued map \( \varphi \mapsto S(t)\varphi \) is upper semicontinuous and has compact values for any \( t \in \mathbb{R}^+ \). Invoking Theorems 8 and 11, in view of Lemmas 16-18, the result of this theorem can be deduced and thus the proof is complete.

6. Reaction-diffusion equations with infinite delays. Consider the following reaction-diffusion equation with infinite delays

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \lambda u = F(x, u(t - \rho)) + \int_{-\infty}^{0} G(x, r, u(t + r))dr \\
\quad + g(x), & \text{in } (0, +\infty) \times \Omega, \\
u = 0, & \text{on } (0, +\infty) \times \partial\Omega, \\
\quad u(t, x) = \psi(t, x), & t \in (-\infty, 0], x \in \Omega,
\end{cases}
\]

(6.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary, \( \lambda \) is a positive constant, \( \rho \in [0, \infty) \), \( g \in L^2(\Omega) \) and \( \psi \) is the initial datum in the interval of time \( (-\infty, 0] \), the other symbols satisfy the following conditions:

(H3) There exist positive constant \( \tilde{k}_2 \) and a function \( \tilde{k}_1 \in L^2(\Omega) \) such that the function \( F \in C(\Omega \times \mathbb{R}; \mathbb{R}) \) satisfies

\[
|F(x, v)|^2 \leq \tilde{k}_1(x)^2 + \tilde{k}_2^2 e^{-2\gamma \rho} |v|^2, \quad \forall x \in \Omega, v \in \mathbb{R},
\]

(6.2)

where \( \gamma \) is the positive parameter which will be determined later on.

(H4) there exist a positive scalar function \( e^{-2\gamma r} \tilde{m}_1(\cdot) \in L^1((-\infty, 0]; \mathbb{R}) \) and a function \( \tilde{m}_0 \in L^1((-\infty, 0]; L^2(\Omega)) \) such that the function \( G \in C(\Omega \times \mathbb{R} \times \mathbb{R}; \mathbb{R}) \) satisfies

\[
|G(x, r, v)| \leq |\tilde{m}_0(r, x)| + \tilde{m}_1(r)|v|, \quad \forall x \in \Omega, r, v \in \mathbb{R},
\]

(6.3)

and we will denote

\[
\tilde{m}_0 = \int_{-\infty}^{0} \|\tilde{m}_0(r, \cdot)\|_{L^2(\Omega)}dr \quad \text{and} \quad \tilde{m}_1 = \int_{-\infty}^{0} e^{-2\gamma r} \tilde{m}_1(r)dr.
\]

For convenience, let \( H = L^2(\Omega) \) with the norm \( \| \cdot \|_2 \) and the inner product \( \langle \cdot, \cdot \rangle \), and let \( V = H^1_0(\Omega) \) with the norm \( \| \cdot \| \) and the inner product \( \langle \cdot, \cdot \rangle \).

Let \( Au = -\Delta u \) for any \( u \in D(A) \) where \( D(A) = \{ u \in V : Au \in H \} = H^1_0(\Omega) \cap H^2(\Omega) \). Also, we will use the Banach space \( C_{\gamma, H} \) and \( C_{\gamma, V} \) in the following discussions with the norms

\[
\|\psi\|_{C_{\gamma, H}} := \sup_{\theta \in (-\infty, 0]} e^{\gamma \theta} \|\psi(\theta)\|_2 < \infty,
\]

and

\[
\|\psi\|_{C_{\gamma, V}} := \sup_{\theta \in (-\infty, 0]} e^{\gamma \theta} \|\psi(\theta)\| < \infty,
\]

respectively.

By the standard Galerkin method, we have the following existence result of solutions:

Theorem 20. Suppose that (H3)-(H4) hold true and \( g \in H \). Then
(1) for any \( \psi \in C_{\gamma,H} \), there exists a solution \( u(t) \) to problem (6.1), and \( u(t) \) satisfies
\[
u \in L^2(0,T; V) \cap L^\infty(-\infty,T; H) \cap C((-\infty,T]; H), \quad \forall T > 0,
\]
(2) for any \( \psi \in C_{\gamma,V} \), problem (6.1) admits a strong solution
\[
u \in L^2(0,T; D(A)) \cap L^\infty(-\infty,T; V) \cap C((-\infty,T]; V), \quad \forall T > 0.
\]

We first establish suitable estimates of the solutions in \( C_{\gamma,H} \) and \( C_{\gamma,V} \) with the purpose of proving the existence of a bounded absorbing set.

**Lemma 21.** In addition to the assumptions (H3)-(H4), suppose that \( g \in H \) and there exists \( \bar{\eta} > 0 \) such that
\[
\lambda - \bar{\eta} > 0, \tag{6.4}
\]
and
\[
\frac{k_2^2}{\lambda} + \bar{m} < \bar{\eta} \leq 2\gamma. \tag{6.5}
\]

Then for any initial data \( \psi \in C_{\gamma,H} \), any solution \( u_t \) of problem (6.1) satisfies
\[
\|u_t\|_{C_{\gamma,H}}^2 \leq 4\|\psi\|^2_{C_{\gamma,H}} e^{\left(\frac{\bar{\eta}^2 + \frac{k_2^2}{\lambda} - \bar{\eta}}{\lambda}\right)t} + C, \quad \forall t \in [0,T^*],
\]
where \( T^* \) is the maximal time of existence.

**Proof.** Taking the scalar product of (6.1) with \( u(t) \) in \( H \), we obtain that
\[
\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\|u(t)\|^2 + 2\lambda\|u(t)\|_{L^2}^2 = 2(F(x,u(t-\rho)),u(t)) + 2(g(x),u(t)).
\]
Let positive parameters \( \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3 \) to be fixed later on. By making use of (H3) and Young’s inequality, we have
\[
\frac{d}{dt}(e^{\bar{\eta}t}\|u(t)\|_{L^2}^2) = \bar{\eta}e^{\bar{\eta}t}\|u(t)\|_{L^2}^2 + e^{\bar{\eta}t}\frac{d}{dt}\|u(t)\|_{L^2}^2
\]
\[
\leq -2e^{\bar{\eta}t}\|u(t)\|^2 - (2\lambda - \bar{\eta} - \bar{\varepsilon}_1 - \bar{\varepsilon}_2)e^{\bar{\eta}t}\|u(t)\|_{L^2}^2 + \frac{1}{\bar{\varepsilon}_1}e^{\bar{\eta}t}\|u(t)\|_{L^2}^2 + \frac{k_2^2}{\bar{\varepsilon}_1}e^{\bar{\eta}t}\|u(t)\|_{C_{\gamma,H}}^2,
\]
\[
+ 2e^{\bar{\eta}t}\left(\int_{-\infty}^{0} G(x,r,u(t+r))dr,u(t)\right) + \frac{1}{\bar{\varepsilon}_2}e^{\bar{\eta}t}\|g\|_{L^2}^2.
\] (6.6)

By Hölder’s inequality, Young’s inequality and the trivial bound \( |u(s)| \leq \|u_s\|_{C_{\gamma,H}} \), we get from (H4) that
\[
2\int_{0}^{t} \bar{\eta}e^{\bar{\eta}s}\int_{\Omega}\int_{-\infty}^{0} G(x,r,u(s+r))u(s)drdxds
\]
\[
\leq 2\int_{0}^{t} \bar{\eta}e^{\bar{\eta}s}\int_{\Omega}\int_{-\infty}^{0} (|\bar{m}_0(r,x)| + \bar{m}_1(r)|u(s+r)||u(s)|)drdxds \tag{6.7}
\]
\[
\leq 2\int_{0}^{t} \bar{\eta}e^{\bar{\eta}s}\bar{m}_0|u(s)|_{L^2}^2 + \int_{0}^{t} \bar{\eta}e^{\bar{\eta}s}\int_{\Omega}\int_{-\infty}^{0} \bar{m}_1(r)|u(s+r)|^2 drdxds
\]
and thus we complete the proof of this lemma. 

By the assumption (6.5), we have

\[ \tilde{\gamma} \]

Let \( \tilde{\varepsilon}_1 = \lambda \). Then we can neglect the second term on the right-hand side of inequality (6.8), since by (6.4)

\[ \lambda - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3 - \tilde{\eta} - \tilde{m}_1 > 0 \]

if \( \tilde{\varepsilon}_2 \) and \( \tilde{\varepsilon}_3 \) are chosen small enough. Thus,

\[
\begin{align*}
\tilde{u}(t) &\leq |u(0)|^2 + Ce^{-\tilde{\eta}t} + \left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} \right) \int_0^t e^{\tilde{\eta}s} \| u_s \|_{C_{\gamma,H}}^2 ds.
\end{align*}
\]

By the assumption (6.5), we have \( \tilde{\eta} \leq 2\gamma \) and so \( e^{(2\gamma - \tilde{\eta})\theta} \leq 1 \) for \( \theta \in (-\infty, 0] \). Multiplying (6.9) by \( e^{2\gamma\theta}e^{-2\gamma\theta} \) and replacing \( t \) by \( t + \theta \), it yields

\[
\sup_{\theta \in [-t, 0]} e^{2\gamma\theta} |u(t + \theta)|_2 \leq |u(0)|_2 e^{-\tilde{\eta}t} + C + \left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} \right) e^{-\tilde{\eta}t} \int_0^t e^{\tilde{\eta}s} \| u_s \|_{C_{\gamma,H}}^2 ds.
\]

Note that

\[
\begin{align*}
e^{2\gamma\theta} |u(t + \theta)|_2^2 &= e^{2\gamma\theta} |\psi(t + \theta)|_2^2 = e^{-2\gamma t} e^{2\gamma(t + \theta)} |\psi(t + \theta)|_2^2 \\
&\leq e^{-2\gamma t} \| \psi \|_{C_{\gamma,H}}^2 \leq e^{-\tilde{\eta}t} \| \psi \|_{C_{\gamma,H}}^2, \quad \forall \theta \in (-\infty, -t].
\end{align*}
\]

Then it follows from (6.10) and (6.11) that

\[
\begin{align*}
\tilde{u}(t) &\leq 2 \| \psi \|_{C_{\gamma,H}}^2 + Ce^{\tilde{\eta}t} + \left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} \right) \int_0^t e^{\tilde{\eta}s} \| u_s \|_{C_{\gamma,H}}^2 ds.
\end{align*}
\]

Using Gronwall's lemma and (6.5), we find that

\[
\begin{align*}
|u(t)|_{C_{\gamma,H}}^2 &\leq 2 \| \psi \|_{C_{\gamma,H}}^2 e^{-\tilde{\eta}t} + 2 \| \psi \|_{C_{\gamma,H}}^2 e^{\left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} - \tilde{\eta} \right)t + C} \\
&\leq 4 \| \psi \|_{C_{\gamma,H}}^2 e^{\left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} - \tilde{\eta} \right)t + C},
\end{align*}
\]

and thus we complete the proof of this lemma. \( \square \)

**Lemma 22.** Under the same assumptions of Lemma 21, if

\[
\tilde{\eta} < 2\lambda \leq 2\gamma
\]

holds true, then for any initial data \( \psi \in C_{\gamma,V} \), any solution \( u_t \) of problem (6.1) satisfies

\[
\begin{align*}
|u(t)|_{C_{\gamma,V}}^2 &\leq 2 \| \psi \|_{C_{\gamma,V}}^2 e^{-\tilde{\eta}t} + C \| \psi \|_{C_{\gamma,H}}^2 e^{\left( \tilde{m}_1 + \frac{\tilde{k}_2}{\lambda} - \tilde{\eta} \right)t + C}, \quad \forall t \in [0, T^*),
\end{align*}
\]

where \( T^* \) is the maximal time of existence.
Proof. Taking the scalar product of (6.1) with $-\Delta u(t)$, using Young’s inequality and $(H3)$, we have

$$
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 + \lambda \|u(t)\|^2 = (F(x, u(t - \rho)), -\Delta u(t))
$$

$$
+ \left( \int_{-\infty}^{0} G(x, r, u(t + r))dr, -\Delta u(t) \right) + (g(x), -\Delta u(t))
$$

$$
\leq \frac{1}{2} |\Delta u(t)|^2 + \bar{k}_1 \|u(t)\|_{C_{\gamma, h}}^2 + |g|_2^2 + \left( \int_{-\infty}^{0} G(x, r, u(t + r))dr, -\Delta u(t) \right).
$$

(6.16)

Arguing as in (6.7), the last term in (6.16) is estimated by

$$
\left| \int_{-\infty}^{0} G(x, r, u(t + r))dr, -\Delta u(t) \right|
$$

$$
\leq \int_{\Omega} \int_{-\infty}^{0} (|\tilde{m}_0(r, x)| + \bar{m}_1(r)|u(t + r)|) |\Delta u(t)| dr dx
$$

(6.17)

$$
\leq \int_{-\infty}^{0} |\tilde{m}_0(r)|_2 |\Delta u(t)|_2 dr + \int_{-\infty}^{0} \bar{m}_1(r)|u(t + r)|_2 |\Delta u(t)|_2 dr
$$

$$
\leq \frac{1}{2} |\Delta u(t)|^2 + \bar{m}_0^2 + \bar{m}_1^2 \|u_t\|_{C_{\gamma, h}}^2,
$$

and thus,

$$
\frac{d}{dt} (e^{\bar{\eta} t} \|u(t)\|^2) = \bar{\eta} e^{\bar{\eta} t} \|u(t)\|^2 + e^{\bar{\eta} t} \frac{d}{dt} \|u(t)\|^2
$$

$$
\leq (\bar{\eta} - 2\lambda) e^{\bar{\eta} t} \|u(t)\|^2 + (2 \bar{k}_1 \|g\|^2 + 2 \bar{m}_0 e^{\bar{\eta} t} + (2 \bar{k}_2^2 + 2 \bar{m}_1^2)) e^{\bar{\eta} t} \|u_t\|_{C_{\gamma, h}}^2.
$$

By Gronwall’s inequality, in view of (6.13), we have

$$
e^{\bar{\eta} t} \|u(t)\|^2 \leq \|u(0)\|^2 + C \int_{0}^{t} e^{(\bar{\eta} - 2\lambda)(t-s)} e^{\bar{\eta} s} \left( 1 + \|u_s\|_{C_{\gamma, h}}^2 \right) ds
$$

(6.18)

$$
\leq \|u(0)\|^2 + C \int_{0}^{t} e^{(\bar{\eta} - 2\lambda)(t-s)} e^{\bar{\eta} s} \left( 1 + \|\psi\|_{C_{\gamma, h}}^2 e^{\left( \bar{m}_1 + \frac{\bar{k}_2^2}{2} \right) t} \right) ds
$$

Then in a similar way as in (6.10)-(6.11), the conclusion (6.15) follows immediately.

Thanks to Theorem 20 and Lemma 22, we define a family of multi-valued mappings $S(t) : C_{\gamma, V} \to C_{\gamma, V}$ by setting

$$
S(t) \psi = \{ u_t(\cdot ; \psi) \mid u(\cdot) \text{ is a solution of Eq. (6.1) with the initial data } \psi \in C_{\gamma, V} \}.
$$

By a standard way in [5] (Lemma 5.1), we see that $\{ S(t) \}_{t \geq 0}$ satisfies (1) and (2) of Definition 1.

In order to prove the existence of a compact positively invariant set which exponentially attracts any bounded set for the multi-valued semidynamical system $\{ S(t) \}_{t \geq 0}$, we need the following lemma.

**Lemma 23.** Suppose $(H3)-(H4)$, $g \in H$, (6.4)-(6.5) and (6.14) hold true. Then the multi-valued semidynamical system $\{ S(t) \}_{t \geq 0}$ associated with problem (6.1) is globally exponentially $\kappa$-dissipative.
Proof. Invoking Theorem 12 and Lemma 22, it only remains to check (1) and (3)-(4) in Theorem 12. Let $V_m = \text{span}\{e_1, \ldots, e_m\}$ in $V$ and $P_m : V \to V_m$ is an orthogonal projector, and let $u = u_1 + u_2$ where $u_1 = P_m u$ and $u_2 = (I - P_m) u$. We decompose Eq. (6.1) as follows:

$$\begin{align*}
\begin{cases}
\frac{\partial u_2(t)}{\partial t} - \Delta u_2(t) + \lambda u_2(t) &= F(x, u(t - \rho)) - P_m F(x, u_1(t - \rho)) \\
+ \int_{-\infty}^{0} G(x, r, u(t + r)) dr - \int_{-\infty}^{0} P_m G(x, r, u_1(t + r)) dr + (I - P_m) g(x),
\end{cases}
\end{align*}$$

and

$$\begin{align*}
\begin{cases}
\frac{\partial u_1(t)}{\partial t} - \Delta u_1(t) + \lambda u_1(t) &= P_m F(x, u_1(t - \rho)) \\
+ \int_{-\infty}^{0} P_m G(x, r, u_1(t + r)) dr + P_m g(x),
\end{cases}
\end{align*}$$

We divide the proof into three steps.  

**Step 1.** By (6.5) and (6.18), we deduce that for all $t \geq 0$ and $\theta$ with $-t \leq \theta \leq 0$,

$$e^{2\gamma \theta} \|u(t + \theta)\|^2 \leq e^{(2\gamma - \tilde{\gamma}) \theta} \|u(0)\|^2 + C e^{2\gamma \theta} + C e^{\left(\tilde{m}_1 + \frac{\tilde{m}_2^2}{2} + 2\gamma - \tilde{\gamma}\right) \theta} \|\psi\|^2_{C_{\gamma, H}}.$$

Note that for all $\theta \leq -t$,

$$e^{2\gamma \theta} \|u(t + \theta)\|^2 = e^{-2\gamma t} e^{2\gamma (t+\theta)} \|u(t + \theta)\|^2 \leq e^{-2\gamma t} \|\psi\|^2_{C_{\gamma, V}}.$$

Hence for all $t \geq 0$ and $\theta \in (-\infty, 0]$,

$$e^{2\gamma \theta} \|u(t + \theta)\|^2 \leq e^{-2\gamma t} \|\psi\|^2_{C_{\gamma, V}} + e^{(2\gamma - \tilde{\gamma}) \theta} \|u(0)\|^2 + C e^{2\gamma \theta} + C e^{\left(\tilde{m}_1 + \frac{\tilde{m}_2^2}{2} + 2\gamma - \tilde{\gamma}\right) \theta} \|\psi\|^2_{C_{\gamma, H}},$$

which implies that for any bounded set $B \subset C_{\gamma, V}$ and $\varepsilon > 0$, there exists a $T > 0$ and we can choose $T' > 2T$ such that for all $t \geq T'$ and $u(t) \in S(t) B$,

$$\sup_{\theta \in (-\infty, -T']} e^{2\gamma \theta} \|u(t + \theta)\|^2 < \varepsilon.$$

(6.21)

Thus, the conclusion (1) in Theorem 12 holds.  

**Step 2.** Taking the inner product in $H$ of (6.19) with $-\Delta u_2 = -\Delta (I - P_m) u$, we obtain

$$\frac{d}{dt} \|u_2(t)\|^2 + 2\|\Delta u_2(t)\|^2 + 2\lambda \|u_2(t)\|^2 = 2 (F(x, u(t - \rho)), -\Delta u_2(t))$$

$$- 2 (P_m F(x, u_1(t - \rho)), -\Delta u_2(t)) + 2 \left( \int_{-\infty}^{0} G(x, r, u(t + r)) dr, -\Delta u_2(t) \right)$$

$$- 2 \left( \int_{-\infty}^{0} P_m G(x, r, u_1(t + r)) dr, -\Delta u_2(t) \right) + 2 ((I - P_m) g(x), -\Delta u_2(t)).$$

(6.22)
By Young’s inequality, Hölder’s inequality and the assumptions \((H3)-(H4)\), we have
\[
2\left| (F(x, u(t - \rho)) - P_m F(x, u_1(t - \rho)), -\Delta u_2(t)) \right| \leq \frac{1}{4} \|\Delta u_2(t)\|_2^2 + C|k_1|^2_2 + C\|u_t\|_{\tilde{C}_\gamma,H}^2,
\]
\[
(6.23)
\]
similar to the argument in \((6.17)\),
\[
2\left| \int_{-\infty}^0 G(x, r, u(t + r)) \, dr, -\Delta u_2(t) \right| \leq 2 \int_{-\infty}^0 \int_{-\infty}^0 \left( |\bar{m}_0(r, x)| + \bar{m}_1(r) |u(t + r)| \right) |\Delta u_2(t)| \, dr \, dx \leq 2 \int_{-\infty}^0 |\bar{m}_0(r)|_2 |\Delta u_2(t)|_2 \, dr + 2 \int_{-\infty}^0 \bar{m}_1(r) |u(t + r)|_2 |\Delta u_2(t)|_2 \, dr \leq \frac{1}{4} \|\Delta u_2(t)\|_2^2 + C\bar{m}_0^2 + C\|u_t\|_{\tilde{C}_\gamma,H}^2,
\]
\[
(6.25)
\]
and in the similar way,
\[
2\left| \int_{-\infty}^0 P_m G(x, r, u_1(t + r)) \, dr, -\Delta u_2(t) \right| \leq \frac{1}{4} \|\Delta u_2(t)\|_2^2 + C\bar{m}_0^2 + C\|u_t\|_{\tilde{C}_\gamma,H}^2.
\]
\[
(6.26)
\]
Note that \(|\Delta u_2|^2 \geq \lambda_{m+1} \|u_2\|^2\). Therefore, it follows from \((6.22)-(6.26)\) that
\[
\frac{d}{dt} \|u_2(t)\|^2 + (\lambda_{m+1} + 2\lambda) \|u_2(t)\|^2 \leq C + C\|u_t\|_{\tilde{C}_\gamma,H}^2.
\]
\[
(6.27)
\]
Applying Gronwall’s inequality, in view of \((6.4)\) and \((6.13)\), we find that
\[
\|u_2(t)\|^2 \leq e^{-(\lambda_{m+1} + 2\lambda)t} \|u_2(0)\|^2 + \frac{C}{\lambda_{m+1} + 2\lambda + \bar{m}_1 + \bar{m}_0^2 - \eta}.
\]
Replacing \(t\) by \(t + \theta\) and noticing that \(t' \geq 2T\), we deduce that for all \(t \geq t'\) and \(u_t(\cdot) \in S(t)B\),
\[
\sup_{\theta \in (-T,0)} \|u_2(t + \theta)\|^2 \leq \|\psi\|_{\tilde{C}_\gamma,H}^2 e^{-\lambda_{m+1} + 2\lambda t} \left( \frac{C}{\lambda_{m+1} + 2\lambda + \bar{m}_1 + \bar{m}_0^2 - \eta} \right) + \frac{C}{\lambda_{m+1} + 2\lambda + \bar{m}_1 + \bar{m}_0^2 - \eta}.
\]
\[
(6.28)
\]
Since \(\psi \in B\) and \(B \subset C_{1,1}\) is bounded, when \(m + 1\) is large enough, the condition \((4)\) in Theorem 12 follows immediately from \((6.28)\).

**Step 3.** Now we consider the finite-dimensional functional differential system \((6.20)\). Note that \(\|Au_1\|^2 \leq \lambda_m \|u_1\|^2 \leq \lambda_m \|u_1\|^2_2\). Without loss of generality, we
assume that $\theta_1, \theta_2 \in [-T, 0]$ with $0 < \theta_1 - \theta_2 < 1$. Then for all $t \geq t'_2$ and $u_t(\cdot) \in S(t)B$,

$$
\|u_1(t + \theta_1) - u_1(t + \theta_2)\| \\
\leq \sqrt{\lambda_m} \int_{t + \theta_2}^{t + \theta_1} \left| \frac{du_1(s)}{ds} \right| ds \\
\leq \sqrt{\lambda_m} \int_{t + \theta_2}^{t + \theta_1} \left( |\Delta u_1(s)| \right) ds$$. 

By (6.13) and the assumptions (H3)-(H4), we have

$$
\begin{align*}
\int_{t + \theta_2}^{t + \theta_1} |\Delta u_1(s)| \leq (\lambda_m + \lambda) \int_{t + \theta_2}^{t + \theta_1} |u_1(s)| ds &+ C(\theta_1 - \theta_2) \\
&\leq C \int_{t + \theta_2}^{t + \theta_1} |u_1(s)| ds + C(\theta_1 - \theta_2) \\
&\leq C \left( e^{\bar{\theta}_1 + \bar{\theta}_2} \bar{\theta} - e^{\bar{\theta}_1 + \bar{\theta}_2} \theta_1 \right) + C(\theta_1 - \theta_2).
\end{align*}
$$

By (6.13) and the assumptions (H3)-(H4), we obtain

$$
\int_{t + \theta_2}^{t + \theta_1} \left| P_m F(x, u_1(s - \rho)) \right| ds \\
\leq \int_{t + \theta_2}^{t + \theta_1} \left| F(x, u_1(s - \rho)) \right| ds + C(\theta_1 - \theta_2)
$$

and

$$
\begin{align*}
\int_{t + \theta_2}^{t + \theta_1} \left| P_m G(x, r, u_1(s + r)) \right| ds \\
&\leq \int_{t + \theta_2}^{t + \theta_1} \int_{-\infty}^{0} \left| \tilde{m}_0(r) \right| ds + C(\theta_1 - \theta_2) \\
&\leq C \left( e^{\bar{\theta}_1 + \bar{\theta}_2} \bar{\theta} - e^{\bar{\theta}_1 + \bar{\theta}_2} \theta_1 \right) + C(\theta_1 - \theta_2).
\end{align*}
$$
Then it follows from (6.29)-(6.32) that there exists a $\delta_2 > 0$ such that for all $t \geq t_2'$, $u_t \in S(t)B$, $\theta_1$, $\theta_2 \in [-T, 0]$ with $|\theta_2 - \theta_1| < \delta_2$,

$$\|u_1(t + \theta_1) - u_1(t + \theta_2)\| < \varepsilon.$$ 

This implies that the condition (3) in Theorem 12 holds true. Thus, the proof of this lemma is done. \hfill \square

Let us now prove other properties of the multi-valued semidynamical system $\{S(t)\}_{t \geq 0}$.

**Lemma 24.** Suppose (H3)-(H4), $g \in H$, (6.4)-(6.5) and (6.14) hold true. Let $\psi^n$ be a sequence converging to $\psi$ in $C_{\gamma,V}$ and fix $\bar{T} > 0$. Then for any solution $u^n_t \in S(t)\psi^n$, there exist $u_t \in S(t)\psi$ and a subsequence $u^{n_k}$ satisfying

$$u^{n_k} \to u \quad \text{in} \quad C([0, \bar{T}]; V).$$

**Proof.** Note that there exists a $C_1 > 0$ such that

$$\|\psi^n\|_{C_{\gamma,V}} \leq C_1, \quad \forall n \in \mathbb{N}. \quad (6.33)$$

Then it follows from (6.15) that

$$\|u^n_s\|_{C_{\gamma,V}} \leq 2\|\psi^n\|_{C_{\gamma,V}} e^{-\bar{s}\eta} + C\|\psi^n\|_{C_{\gamma,V}}^2 e^{\left(\tilde{m}_1 + \frac{\lambda_1^2}{2\lambda} - \bar{s}\eta\right)s_2} + C, \quad \forall s \in [0, \bar{T}]. \quad (6.34)$$

Arguing in the same way as in Lemma 23, in view of (6.33) and (6.34), we obtain that for any $\varepsilon > 0$, there exist a finite-dimensional subspace $V_m$ of $V$, a $\tilde{\delta}_2 > 0$ and a $N_2 > 0$ such that

1. for all $n \geq N_2$, $s_1, s_2 \in [0, \bar{T}]$ with $|s_2 - s_1| < \tilde{\delta}_2$,

$$\|P(u^n(s_1) - u^n(s_2))\| \leq C\left|e^{\left(\tilde{m}_1 + \frac{\lambda_1^2}{2\lambda}\right)(s_1 - s_2)} - e^{\left(\tilde{m}_1 + \frac{\lambda_1^2}{2\lambda}\right)s_1}\right| + C|s_1 - s_2| < \varepsilon; \quad (6.35)$$

2. for all $n \geq N_2$,

$$\sup_{s \in [0, \bar{T}]} \|(I - P)u^n(s)\|^2 \leq \|(I - P)\psi^n(0)\|^2 + \frac{C}{\lambda_{m+1} + 2\lambda} + \frac{1}{\lambda_{m+1} + 2\lambda + \tilde{m}_1 + \frac{\lambda_1^2}{2\lambda} - \bar{\eta}} \leq 2\|\psi^n(0) - \psi(0)\|^2 + 2\|(I - P)\psi(0)\|^2 + \frac{C}{\lambda_{m+1} + 2\lambda} \quad (6.36)$$

where $P : V \to V_m$ is the canonical projector.

Using (6.34) and (6.36), one can find that for each fixed $s \in [0, \bar{T}]$, $u^n(s)$ is pre-compact in $V$. On the other hand, by (6.35) and (6.36), we deduce that for all $s_1, s_2 \in [0, \bar{T}])$ with $|s_2 - s_1| < \tilde{\delta}_2$,

$$\|u^n(s_1) - u^n(s_2)\| \leq \|(I - P)(u^n(s_1) - u^n(s_2))\| + \|P(u^n(s_1) - u^n(s_2))\| < 3\varepsilon, \quad \forall n \geq N_2.$$
The Ascoli-Arzelà Theorem implies the existence of a subsequence \( u^{n_k} \) converging in \( C([0,T]; V) \) to some function \( u(\cdot) \). It is then easy to show that \( u \) is a solution of (6.1).

Finally, we have the following result, which shows the existence of a compact, positively invariant and exponentially attracting set for \( \{S(t)\}_{t \geq 0} \).

**Theorem 25.** Suppose (H3)-(H4), \( g \in H \), (6.4)-(6.5) and (6.14) hold true. Then the multi-valued semidynamical system \( \{S(t)\}_{t \geq 0} \) associated with problem (6.1) possesses a compact positively invariant set \( A^* \subset C_{\gamma,V} \) which exponentially attracts each bounded subset of \( C_{\gamma,V} \).

**Proof.** Lemma 24 implies that for any \( t \geq 0 \), the map \( \psi \mapsto S(t)\psi \) is upper semi-continuous. Suppose the opposite. Then there exist \( \psi, t > 0 \), a neighborhood \( \mathcal{O} \) of \( S(t)\psi \) and sequences \( \psi^n \to \psi \) in \( C_{\gamma,V} \), \( \xi^n \in S(t)\psi^n \) such that \( \xi^n \notin \mathcal{O} \). Let \( u^n_s \in S(s)\psi^n \) be such that \( u^n_s = \xi^n \) and \( u^n_0 = \psi^n \). By Lemma 24 we obtain that, up to a subsequence, \( u^n \to u \) in \( C([0,t]; V) \). Set \( \xi = u_t(\cdot) \) and

\[
\begin{align*}
  u_t(\theta) &= \begin{cases} 
    \psi(t + \theta), & \text{if } t + \theta \leq 0, \\
    u(t + \theta), & \text{if } t + \theta > 0.
  \end{cases}
\end{align*}
\]

It is clear that \( u_t(\cdot) \in S(t)\psi \). Hence,

\[
\begin{align*}
  \|\xi^n - \xi\|_{C_{\gamma,V}} &= \|u_t^n - u_t\|_{C_{\gamma,V}} \\
  &\leq e^{-\gamma t} \sup_{s \in (-\infty, 0]} e^{\gamma s} \|\psi^n(s) - \psi(s)\| + \sup_{s \in [0,t]} \|u^n(s) - u(s)\|
\end{align*}
\]

as \( n \to \infty \) and \( \xi \in S(t)\psi \), which is a contradiction. Arguing in the similar way as above, we conclude that for any \( t \geq 0 \), the map \( S(t) \cdot \) has compact values. Thanks to Lemmas 22 and 23, the conclusion of this theorem follows immediately by invoking Theorem 8. \( \square \)

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