Projective modules over discrete Hodge algebras

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1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let $A$ be a ring. In ([6], Theorem 1.1), Vorst proved that if all projective modules over polynomial extensions of $A$ are extended from $A$, then all projective modules over discrete Hodge $A$-algebras are extended from $A$ (An $A$-algebra $R$ is a discrete Hodge $A$-algebra if $R = A[X_0, \ldots, X_n]/I$, where $I$ is an ideal generated by monomials). In this note, we extend the above result of Vorst by proving the following result.

**Theorem 1.1** Let $A$ be a ring and $r > 0$ be an integer. Assume that all projective modules of rank $r$ over polynomial extensions of $A$ are extended from $A$. Then all projective modules of rank $r$ over discrete Hodge $A$-algebras are extended from $A$.

We note that Lindel gave another proof of Vorst’s result ([1], Theorem 1.5) and a proof of ([1.1]) is implicit in Lindel’s proof. But the idea of our proof is different from Lindel’s and it also gives other results which we describe below.

Let $A$ be a ring of dimension $d$ and let $r > d/2$. Assume that $A$ is of finite characteristic prime to $r!$. In ([5], Theorem 5), Roitman proved that if $P$ is a projective module of rank $r$ over $R = A[X_1, \ldots, X_n]$ such that $P \oplus R$ is extended from $A$, then $P$ is extended from $A$. In particular, if $A$ is a local ring of dimension $d$, characteristic of $A$ is positive and prime to $d!$, then all stably free modules of rank $> d/2$ over polynomial extensions of $A$ are free.

We will prove the following analogue of Roitman’s result for discrete Hodge $A$-algebras.

**Theorem 1.2** Let $A$ be a ring of dimension $d$. Assume $A$ is of finite characteristic prime to $r!$. Let $R$ be a discrete Hodge $A$-algebra and let $P$ be a projective $R$-module of rank $r > d/2$. If $P \oplus R$ is extended from $A$, then $P$ is extended from $A$.

As a corollary to the above result, if $A$ is a local ring of dimension $d$, characteristic of $A$ is finite and prime to $d!$, then all stably free modules of rank $> d/2$ over discrete Hodge $A$-algebras are free.

Now, we will describe our last result. Let $A$ be a ring of dimension $d$ and let $R = A[X_1, \ldots, X_n]$. In ([7], Section 4), Wiemers asked the following question: Is the natural map $U_m(R) \rightarrow U_m(R/(X_1X_2\ldots X_k))$ surjective for all $r$ and $1 \leq k \leq n$?

Wiemers ([7], Proposition 4.1) answered the above question in affirmative when $r \geq d + 2$ or $r = d + 1$ and $1/d! \in A$. We will prove the following result which gives a partial answer to Wiemers question in affirmative.
Theorem 1.3 Let $A$ be a ring of dimension $d$. Assume characteristic of $A$ is positive and prime to $d!$. Let $R = A[X_1, \ldots, X_n]$ and let $I \subset J$ be two ideals of $R$ generated by square free monomials. Then the map $\text{Um}_r(R/I) \to \text{Um}_r(R/J)$ is surjective for $r \geq \frac{d}{2} + 2$.

2 Preliminaries

Given a cartesian diagram of rings

$$
\begin{array}{ccc}
A & \longrightarrow & A_1 \\
\downarrow & & \downarrow j_1 \\
A_2 & \longrightarrow & A_0 \\
\downarrow j_2 & & \downarrow j_0
\end{array}
$$

where $j_2$ is a surjective map. If $P$ is a projective $A$-module, then the above diagram induces a cartesian diagram ([2], Section 2)

$$
\begin{array}{ccc}
P & \longrightarrow & P_1 \\
\downarrow & & \downarrow \\
P_2 & \longrightarrow & P_0
\end{array}
$$

where $P_i = P \otimes A_i$ for $i = 0, 1, 2$.

We begin by stating the following two results of A. Wiemers ([7], Proposition 2.1 and Theorem 2.3) respectively.

Proposition 2.1 Given a cartesian square of rings with $j_2$ surjective and a projective $A$-module $P$. Then

(i) If $\text{Aut}_{A_2}(P_2) \to \text{Aut}_{A_0}(P_0)$ is surjective, then so is $\text{Aut}_{A}(P) \to \text{Aut}_{A_1}(P_1)$.

(ii) If $\text{Aut}_{A_2}(P_2) \to \text{Aut}_{A_0}(P_0)$ is surjective and $Q \otimes A_i \xrightarrow{\sim} P_i$, $i = 1, 2$ for another projective $A$-module $Q$, then $P \xrightarrow{\sim} Q$. In particular, if $P_1$ and $P_2$ have the cancellation property, then so does $P$.

(iii) Let, in addition, $j_1$ be surjective. If $\text{Um}(P_2) \to \text{Um}(P_0)$ is surjective, then so is $\text{Um}(P) \to \text{Um}(P_1)$.

Theorem 2.2 Let $A$ be a ring and let $J$ be an ideal of $R = A[X_1, \ldots, X_n]$ generated by square free monomials. Then the natural map $\text{GL}_r(R) \to \text{GL}_r(R/J)$ is surjective.

Given a simplicial subcomplex $\Sigma$ of $\Delta_n$ and a ring $A$, let $I(\Sigma)$ be the ideal of $A[X_0, \ldots, X_n]$ generated by all square free monomials $X_{i_1}X_{i_2}\cdots X_{i_k}$ with $0 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $\{i_1, \ldots, i_k\}$ is not a face of $\Sigma$. By $A(\Sigma)$, we denote the discrete Hodge $A$-algebra $A[X_0, \ldots, X_n]/I(\Sigma)$.

The following result is due to Vorst ([6], Lemma 3.4) and is very crucial for the proof of our results.

Proposition 2.3 Let $\Sigma$ be a simplicial subcomplex of $\Delta_n$ which is not a simplex. Then there exists an $i \in \{0, 1, \ldots, n\}$ and simplicial subcomplexes $\Sigma_2 \subset \Sigma_1 \subset \Sigma$ such that we have a cartesian square of
Hence, if $Q$ is not a vertex and $C(\Sigma_2)$ is the cone on $\Sigma_2$ with vertex $i$. Note that $j_2$ is a split surjection and $A(C(\Sigma_2)) = A(\Sigma_2)[X_i]$.

We end this section by stating two results of Wiemers (\cite{7}, Theorem 3.6) and (\cite{8}, Theorem 4.3) respectively which will be used in section 4.

**Theorem 2.4** Let $A$ be a ring of dimension $d$. Let $I \subset J$ be ideals in $R = A[X_1, \ldots, X_n]$ generated by square free monomials. Let $P$ be a projective module over $R/I$. If either rank $P \geq d + 1$ or rank $P \geq d$ and $1/d! \in A$, then the natural map $\text{Aut}_{R/J}(P) \to \text{Aut}_{R/J}(P/JP)$ with $J = J/I$ is surjective.

**Theorem 2.5** Let $A$ be a ring of dimension $d$ with $1/d! \in A$ and $B = A[X_1, \ldots, X_n]$. Let $P$ and $P_1$ be projective $B$-modules of rank $\geq d$. Assume $P \oplus B \xrightarrow{\sim} P_1 \oplus B$. If $P/(X_1, \ldots, X_n)P \xrightarrow{\sim} P_1/(X_1, \ldots, X_n)P_1$, then $P \xrightarrow{\sim} P_1$.

In other words, if the projective $A$-module $P/(X_1, \ldots, X_n)P$ is cancellative, then $P$ is cancellative.

### 3 Main Theorem

In this section we prove our main results mentioned in the introduction.

**Proof of Theorem 1.1**: Let $B = A[X_0, \ldots, X_n]/I$ be a discrete Hodge $A$-algebra and let $P$ be a projective $B$-module of rank $r$ (here $I$ is a monomial ideal). It is enough to assume that $I$ is a square free monomial ideal. Then $I = I(\Sigma)$ for some simplicial subcomplex $\Sigma$ of $\Delta_n$ and $B = A(\Sigma)$. We will use induction on $n$.

If $n = 0$, then there is nothing to prove, as $A(\Sigma) = A$ or $A[X_0]$. Let $n > 0$ and assume the result for $n - 1$. We will apply (2.3). By induction hypothesis, all projective modules of rank $r$ over $A_1 = A(\Sigma_1)$ and $A_0 = A(\Sigma_2)$ are extended from $A$. Also all projective modules of rank $r$ over $A_2 = A(C(\Sigma_2)) = A(\Sigma_2)[X_i]$ are extended from $A[X_i]$ and hence are extended from $A$.

Write $P_i = P \otimes A A_i$, $i = 0, 1, 2$. Clearly, the natural map $\text{Aut}_{A_2}(P_2) \to \text{Aut}_{A_0}(P_0)$ is surjective. Hence, if $Q = P/(X_0, \ldots, X_n)P$, then $P_1 \xrightarrow{\sim} Q \otimes A_1$ and $P_2 \xrightarrow{\sim} Q \otimes A_2$, by induction hypothesis. Hence, by (2.1(ii)), $P \xrightarrow{\sim} Q \otimes A$, i.e. $P$ is extended from $A$. This proves the result. \(\square\)

**Proof of Theorem 1.2**: Let $R = A[X_0, \ldots, X_n]/I$ be a discrete Hodge $A$-algebra and let $P$ be a projective $R$-module of rank $r$ (here $I$ is a monomial ideal). Again, it is enough to assume that $I$ is a square free monomial ideal. Then $I = I(\Sigma)$ for some simplicial subcomplex $\Sigma$ of $\Delta_n$ and $R = A(\Sigma)$. We will use induction on $n$. \(\square\)
When \( n = 0 \), there is nothing to prove as \( R = A \) or \( A[X_0] \). Let \( n > 0 \) and assume the result for \( n - 1 \). We apply \((2.3)\). Let \( A_1 = A(\Sigma_1) \), \( A_2 = A(C(\Sigma_2)) \) and \( A_0 = A(\Sigma_2) \). Write \( P_i = P \otimes_A A_i \) for \( i = 0, 1, 2 \).

Since \( R \rightarrow A_i \) are natural surjections, \( P_i \otimes A_i \) are extended from \( A_i \), \( i = 1, 2 \). Therefore, if \( Q = P/(X_0, \ldots, X_n)P \), \( P_i \xrightarrow{\sim} Q \otimes A_i \), \( i = 1, 2 \). Clearly, the natural map \( \text{Aut}_{A_2}(P_2) \rightarrow \text{Aut}_{A_0}(P_0) \) is surjective. Hence, by \((2.1(ii))\), \( P_\sim \rightarrow Q \otimes A \), i.e. \( P \) is extended from \( A \). This proves the result.

**Proof of Theorem 1.3:** It is enough to show that the natural map \( U_{mr}(R) \rightarrow U_{mr}(R/J) \) is surjective for every ideal \( J \) of \( R \) generated by square free monomials.

Let \( v \in U_{mr}(R/J) \). We have an exact sequence \( 0 \rightarrow \varphi \rightarrow (R/J)^r \rightarrow \varphi(R/J) \rightarrow 0 \).

Since \( P \otimes A/J \) is free, by \((1.2)\), \( P \) is extended from \( A \), i.e. \( P = \varphi \otimes A \), where \( \varphi = P/(X_1, \ldots, X_n)P \).

Hence, we have the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow \varphi \otimes A \rightarrow (R/J)^r \rightarrow \varphi(R/J) \rightarrow 0 \\
\downarrow \searrow \downarrow \leftarrow \downarrow \leftarrow \\
0 \rightarrow \varphi \rightarrow (R/J)^r \rightarrow \varphi(R/J) \rightarrow 0
\end{array}
\]

where \( \varphi(0) \) is the image of \( v \) in \( U_{mr}(A) \) under the map \( R/J \rightarrow A \) given by \( \overline{X_i} \rightarrow 0 \), \( i = 1, \ldots, n \).

Hence, there exists \( \sigma \in \text{GL}_{r}(R/J) \) such that \( \varphi \sigma = \varphi(0) \otimes R \). By \((2.2)\), \( \sigma \) can be lifted to \( \Delta \in \text{GL}_{r}(R) \) and \( \varphi(0) \Delta^{-1} \in U_{mr}(R) \) is a lift of \( v \). This proves the result.

\( \square \)

## 4 Some Auxiliary Results

As an application of \((2.3)\), we will give an alternative proof of the following result of Wiemers (7, Corollary 4.4).

**Theorem 4.1** Let \( A \) be a ring of dimension \( d \) with \( 1/d! \in A \). Let \( B = A[X_0, \ldots, X_n]/I \) be a discrete Hodge \( A \)-algebra. Let \( P \) be a projective \( B \)-module of rank \( \geq d \). If the projective \( A \)-module \( P/(X_0, \ldots, X_n)P \) is cancellative, then \( P \) is cancellative.

**Proof** If \( B \) is a polynomial ring over \( A \), then the result follows from \((2.5)\). It is enough to assume that \( I \) is generated by square free monomials. Hence \( I = I(\Sigma) \) for some simplicial subcomplex \( \Sigma \) of \( \Delta_n \). We will apply induction on \( n \).

By \((2.3)\), we have the following cartesian square

\[
\begin{array}{ccc}
A(\Sigma) & \xrightarrow{i_1} & A(\Sigma_1) \\
\downarrow i_2 & & \downarrow j_1 \\
A(C(\Sigma_2)) & \xrightarrow{j_2} & A(\Sigma_2)
\end{array}
\]
By (2.4), the natural map \( \text{Aut}_{A(C(\Sigma_2))}(P \otimes A(C(\Sigma_2))) \to \text{Aut}_{A(\Sigma_2)}(P \otimes A(\Sigma_2)) \) is surjective and by induction hypothesis on \( n \), \( P \otimes A(C(\Sigma_2)) \) and \( P \otimes A(\Sigma_1) \) are cancellative. Hence, by (2.1(ii)), \( P \) is cancellative. This proves the result.

\[ \square \]

**Theorem 4.2** Let \( A \) be a ring of dimension \( d \) with \( 1/d! \in A \) and \( R = A[X_1, \ldots, X_n] \). Let \( P \) be a projective \( R \)-module of rank \( d \) such that \( P \oplus R \) is extended from \( A \). Then \( P \) is extended from \( A \).

**Proof** By Quillen’s local-global principle ([3], Theorem 1), it is enough to assume that \( A \) is local. Then \( P \oplus R \) is free. Since \( P/(X_1, \ldots, X_n)P \) is free, by (2.5), \( P \) is free. This proves the result. \[ \square \]

**Remark 4.3** When \( P \) is stably free, the above result (4.2) is due to Ravi A Rao ([4] Corollary 2.5). More precisely, Rao proved that if \( A \) is a ring of dimension \( d \) with \( 1/d! \in A \), then every \( v \in \text{Um}_{d+1}(A[X]) \) is extended from \( A \), i.e. there exists \( \sigma \in \text{SL}_{d+1}(A[X]) \) such that \( v\sigma = v(0) \).

Following the proof of (4.2) and using (4.2), we get the following:

**Corollary 4.4** Let \( A \) be a ring of dimension \( d \) with \( 1/d! \in A \). Let \( B \) be a discrete Hodge \( A \)-algebra. Let \( P \) be a projective \( B \)-module of rank \( d \) such that \( P \oplus B \) is extended from \( A \), then \( P \) is extended from \( A \). In particular, every stably free \( B \)-module of rank \( d \) is extended from \( A \).

During CAAG VII meeting in Bangalore, Kapil H Paranjape asked if we can extend the above results ([1.1] [1.2] [1.3]) for locally discrete Hodge \( A \)-algebras (Definition: A positively graded \( A \)-algebra \( B \) is a locally discrete Hodge \( A \)-algebra if \( B_p \) is a discrete Hodge \( A_p \)-algebra for every prime ideal \( p \) of \( A \). The answer is yes and follows from the following result of Lindel ([1], Theorem 1.3) which generalises Quillen’s patching theorem ([3], Theorem 1) from polynomial rings to positively graded rings.

**Theorem 4.5** Let \( A \) be a ring and let \( M \) be a finitely presented module over a positively graded ring \( R = \oplus_{i \geq 0} R_i, R_0 = A \). Then the set \( J(A, M) \), of all \( a \in A \) for which \( M_a \) is extended from \( A_a \), is an ideal of \( A \).

In particular, if \( M_p \) is extended from \( A_p \) for all prime ideal \( p \) of \( A \), then \( M \) is extended from \( A \).

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