Theory of the superconductivity of $\text{UGe}_2$ revisited.

Naoum Karchev

Department of Physics, University of Sofia, 1164 Sofia, Bulgaria

We present a unified theory of magnetism and superconductivity of $\text{UGe}_2$. To this end, we consider part of $5f$ uranium electrons as mostly itinerant and other ones as mostly localized. The main feature that distinguishes the localized from the itinerant electrons is the effect of the pressure on them. The pressure strongly screens the itinerant electrons while the localized ones are almost unaffected. The screening of itinerant electrons leads to decreasing of their Coulomb repulsion, therefore to formation of doubly occupied and empty states. These states are spin-singlet and the effective spin of itinerant electrons, the zero-temperature magnetization in units of Bohr magneton, decreases. We obtain an effective two-spin Heisenberg model, which explains the magnetization-temperature diagram of $\text{UGe}_2$. It is shown that the experimentally observed characteristic temperature $T_x$, is a partial order transition temperature. Below the Curie temperature ($T_x < T_c$) the system undergoes a transition from high temperature phase, were only localized electrons contribute the magnetization, to the low temperature one, were both itinerant and localized electrons contribute the magnetization. The characteristic temperature decreases when pressure increases. At the quantum partial order point $T_x = 0$, the Zeeman splitting of the itinerant electrons is zero. This permits formation of Cooper pairs and an onset of superconductivity induced by the transversal fluctuations of the localized electrons. Small deviation from the quantum partial ordered state leads to suppression of superconductivity. This explains the dome form of the superconducting transition temperature. The very low superconducting critical temperature is a consequence of the Ising ferromagnetism.

PACS numbers: 75.50.Cc,74.20.Mn,74.20.Rp

I. INTRODUCTION

Twenty years ago, in the paper [1] authors reported on the coexistence of ferromagnetism and superconductivity in $\text{UGe}_2$ under pressure. This triggered a very intense experimental and theoretical study of the phenomenon. New materials $\text{URhGe}_2$ [2] and $\text{UCoGe}_2$ [3] that possess the coexistence of ferromagnetism and superconductivity even at ambient pressure were prepared. The experimental results [4-8] and theoretical ideas forwarded with the aim to explain the unusual magnetic and transport properties of the samples [9-13] arrived at the common conclusion that $5f$ electrons of uranium are responsible for the ferromagnetism and superconductivity [4-18].

In the present paper we focus on the $\text{UGe}_2$ compound. It is a ferromagnetic metal with Curie temperature at ambient pressure $T_C = 52K$ and saturated magnetic order $\mu_s = 1.48 \mu_B/U$ along the $a$-axis [14]. Under pressure, the ferromagnetic state splits into two different ferromagnetic phases, which are separated by a characteristic temperature $T_x$. The magnetization shows an anomalous enhancement below $T_x$ [3,8]. The first theoretical attempts to explain the phenomenon are based on the exists of coupled CDW and SDW fluctuations [1,4,8]. The measurements of the resistivity bring important clarity to the nature of the two phases. The resistivity shows a downturn around $T_x$ [4]. The anomaly is best seen in terms of broad maximum in the derivative of resistivity $\frac{d\rho}{dT}$ [20]. It is known that the onset of magnetism in the itinerant systems is accompanied with strong anomaly in resistivity [21]. This permits to conclude that at $T_x$ itinerant $5f$ electrons start to form magnetic order. An additional argument in favor of the existence of itinerant $5f$ electrons is the fact, that $\text{UGe}_2$ is a very good metal with residual resistivity below $1 \mu\Omega cm$ [15]. On the other hand, a hallmark of uranium $\text{UGe}_2$ is the larger nearest-neighbor uranium separation, $d_{U-U} = 3.85\AA$. This increase leads to a stronger localization of the $5f$ electrons [4]. The experimental results tell that itinerant and localized $5f$ electrons coexist in $\text{UGe}_2$. The increase of the magnetic order below $T_x m < 1/2$, along with the fact that magnetic order of $\text{UGe}_2$ at zero temperature is $\mu_s = s + m$, where $s = 1$ [8] suggest that the ferromagnetism of $\text{UGe}_2$ is determined by spin $1/2$ fermion and spin $s = 1$ collective spin operator of localized electrons per uranium atom.

The ferromagnetism and superconductivity in $\text{UGe}_2$ coexist. The superconductivity appears in narrow pressure range near the ferromagnetic quantum critical point [1]. There is a very close relationship between the characteristic temperature $T_x$ and the superconductivity. Under pressure $T_x$ decreases and it becomes zero at $p = p_x$, where the superconducting critical temperature is highest. As mentioned above at $T_x$ starts the formation of the magnetic order of the itinerant electrons. $T_x = 0$ means that the magnetic order of the itinerant electrons is zero which in turn means that the Zeeman splitting of the itinerant electrons is zero. Small deviation from $p_x$ leads to suppression of the superconductivity which is evident from the dome form of the experimental curve of the superconducting critical temperature [3]. The effect of Zeeman splitting on superconductivity is decisive for determining Cooper pairs symmetry. The spin singlet Cooper pairs are not possible because transversal fluctuations of localized electrons suppress them. It is commonly accepted that spin-triplet, spin-parallel pair-
ing is realized in $\text{UGe}_2$, but this pairing is not so sen-
tive to Zeeman splitting, as experiment shows. Most ade-
quate to describe the dome form of the supercon-
ducting transition temperature is the spin-triplet spin-
antiparallel pairing of 5f electrons. Superconductivity
being entirely in ferromagnetic phase is expected to be
induced by transversal fluctuations of localized 5f ele-
trons. The paramagnon induced Cooper pairs is the most
popular idea, but the disappearance of superconductivity
in the paramagnetic phase casts a shadow over it. This
make magnon induced superconductivity most appropri-
ate for $\text{UGe}_2$. In the present paper we consider a spin-
fermion model in terms of spin-1 transversal fluctuations
of the localized 5f electrons and spin-1/2 fermions de-
scribing the itinerant ones. Under the pressure Coulomb
repulsion of itinerant electrons decrease and part of them
form doubly occupied and empty states. They are spin-
singlet and the effective spin of itinerant electrons, the
zero-temperature magnetization in units of Bohr magneton,
decreases. We calculate the zero temperature mag-
etization of itinerant electrons $m$ as a function of $2J/U$,
where $U$ is Coulomb repulsion and $J$ is the spin exchange
of itinerant and localized electrons. The result, depicted
in Fig.1, shows that increasing $0<2J/U \leq 1$ the mag-
etization decreases and at $2J/U = 1$ it is zero. Hence the
parameter $2J/U$ is the theoretical analog of the pressure
$p/p_x$. We calculate the magnetization of localized and
itinerant electrons as a function of temperature $T/J$ for
different values of $2J/U$. The results, depicted in Fig.2,
show that when the parameter $2J/U = 0.1$ is small, at
ambient pressure, the onset of magnetic order of local-
ized and itinerant electrons is at Curie temperature, but
if the parameter increases $2J/U = 0.4$ the magnetization
of itinerant and localized electrons decreases when
temperature increases but magnetization of the itinerant
electrons becomes zero at $T_x$ while the magnetization of
localized electrons is still nonzero. This demonstrates
that temperature $T_x$ marks the onset of itinerant elec-
trons' magnetic order, i.e. $T_x$ is partial order transition
temperature.

The feature of the partial order transition is that only
part of electrons form magnetic order above a character-
istic temperature, while all electrons contribute the mag-
etization below this temperature. There are exact results
for the partially ordered systems $^{[22][24]}$. The partial
order transition has been predicted in frustrated anti-
ferromagnetic $^{[23]}$, ferrimagnetic $^{[24]}$ systems and ferromag-
etic spin-fermion compounds $^{[25]}$. Experimentally the
partial order has been observed in $\text{Gd}_2\text{T}2\text{O}_7$ $^{[28]}$.

The temperature $T_x$ is equal to zero at $2J/U = 1$, i.e
$p = p_x$, and Zeeman splitting of itinerant electrons is zero.
This allows formation of Cooper pairs and an on-
set of superconductivity induced by the transversal fluc-
tuations of the localized electrons. We consider spin-
triplet spin anti-parallel order parameter and calculate
superconducting critical temperature as a function of
$2J/U$. The result, depicted in Fig.4 reproduces the ex-
perimentally known form of the curve with maximum at
$2J/U = 1$, i.e. $p = p_x$.

II. MAGNETIC ORDER OF $\text{UGe}_2$

We consider spin-fermion system with Hamiltonian
\[
\hat{h} = -t \sum_{\langle ij \rangle} \left( c_{iσ}^+ c_{jσ} + h.c. \right) + U \sum_i n_{i↑} n_{i↓} - \mu \sum_i n_i - J' \sum_{\langle ij \rangle} \left( S_i^+ \cdot S_j^+ + \kappa S_i^z S_j^z \right) - J \sum_i S_i^t \cdot S_i^t, \tag{1}
\]
where $t > 0$ is the hopping parameter, $S_i^t$ is the spin
of the itinerant electrons with components $(S_i^t)^{σσ'} = \frac{1}{2} \sum_{σσ'} \tau_{σσ'}^t \gamma_{σσ'}^t$, and $(τ^x, τ^y, τ^z)$ are the Pauli matrices,
$S_i^t$ is the spin operator of the localized electrons, $μ$ is the
chemical potential, $n_{iσ} = c_{iσ}^+ c_{iσ}$ and $n_i = n_{i↑} + n_{i↓}$. The
sums are over all sites of a cubic lattice, $(i,j)$ denotes the
sum over the nearest neighbors. The spin exchange
constants are ferromagnetic ($J' > 0$, $J > 0$) and $κ > 0$ is
Ising parameter. The term with the constant $U > 0$ is the
Coulomb repulsion.

We follow the technique of calculation developed for
ferromagnetic spin-fermion systems $^{[27][29]}$, and repre-
sent the Fermi operators, the spin of the itinerant elec-
trons and the density operators of itinerant electrons in
terms of the Schwinger-bosons $(φ_{i,σ}, φ_{i,σ}^+)$ and slave
fermions $(h_i, h_i^+, d_i, d_i^+)$, which satisfy the condition
\[
φ_{i1} φ_{i1}^+ + φ_{i2} φ_{i2}^+ + d_i^+ d_i + h_i^+ h_i = 1. \tag{2}
\]
To solve the constraint one introduces new Bose doublet
$ζ_{iσ}$ and $ζ_{iσ}^+$ $^{[30]}$, where the new fields satisfy the constraint
$ζ_{iσ}^+ ζ_{iσ} = 1$. In terms of these fields the components of the spin
vectors of the itinerant electrons have the form
\[
S_i^t = \frac{1}{2} \sum_{σσ'} ζ_{iσ}^+ τ_{σσ'}^t ζ_{iσ} \left[ 1 - h_i^+ h_i - d_i^+ d_i \right], \tag{3}
\]
where the unit vector $n_i^σ = \sum_{σσ'} ζ_{iσ}^+ τ_{σσ'}^t ζ_{iσ}$ ($n_i^2 = 1$)
identifies the local orientation of the spin of the itinerant
electron.

The Hamiltonian for the free $d$ and $h$ Fermions reads
\[
h_0 = \sum_{k \in B_2} \left( ε_k^d d_k^+ d_k + ε_k^h h_k^+ h_k \right). \tag{4}
\]
At half-filling $n_i^t = 1$, one sets the chemical potential
$μ = U/2$ and the dispersions adopt the form
\[
ε_k^d = -2t ε_k + \frac{U}{2} - Js^l,
ε_k^h = 2t ε_k + \frac{U}{2} - Js^l, \tag{5}
\]
where $s^l = 1$ is the spin of localized electrons. It is
important to emphasize that the minus sign in front of $J$
constant in equation (5) is because the exchange between the spins of localized and itinerant electrons is ferromagnetic. This fact has important consequences, which are discussed later on.

Let us average the spin of electrons in the subspace of the free Fermions \(\{d_i^+, d_i, h_i\}\) (to integrate the Fermions out in the path integral approach). One obtains

\[
<s_i^> = mn_i = M_i \quad M_i^2 = m^2 \quad (6)
\]

\[
m = \frac{1}{2} \left(1 - <h_i^+h_i>_f - <d_i^+d_i>_f\right),
\]

where \(<......>_f\) means an average in the subspace of the free Fermions. Hence, the amplitude of the spin vector \(m\) is an effective spin of the itinerant electrons accounting for the fact that some sites, in the ground state, are doubly occupied or empty.

The Hamiltonian of the system is quadratic with respect to slave fermions and one can integrate them out. As a result, we obtain an effective theory of two vectors \(S_i^l\) and \(M_i\) with Hamiltonian

\[
h_{eff} = -J^l \sum_{(ij)} [S_i^l \cdot S_j^l + \kappa S_i^l[S_j^l]_z] \quad (7)
\]

\[
- J^l \sum_{(ij)} M_i \cdot M_j - J \sum_i S_i^l \cdot M_i.
\]

The exchange constant \(J^l\) can be calculated, but the result depends on many parameters, so we consider all three exchange constants \((J, J^l, J^f)\) as free parameters to be determined so that the theoretical results are close to the experimental one.

We calculate the effective spin \(m\) of itinerant electrons at zero temperature using equation (6). It follows from equation (5) that if \(U - 2J - 12t > 0\), \(\varepsilon_k^l\) and \(\varepsilon_k^h\) are positive for all values of wave vectors \(k\), therefore \(<h_i^+h_i>_f=<d_i^+d_i>_f=0\) and \(m = 1/2\). The system is spin-3/2 ferromagnetic insulator. In opposite case \(U - 2J - 12t < 0\) the Fermi system has two different, due to Zeeman splitting, Fermi surfaces and \(m < 1/2\). In the special case \(U = 2J\), one obtains \(<h_i^+h_i>_f=<d_i^+d_i>_f=1/2\) and \(m = 0\). This means that itinerant electrons do not contribute the magnetization of the system. The result is independent of the hopping parameter \(t\), wherefore we fix it \(2t = J\) throughout the paper. We calculate \(m\) as a function of \(2J/U\).

The result is depicted in figure (1). At \(2J/U = 0.1\), zero temperature magnetization of itinerant electrons is \(m = 1/2\). When \(2J/U\) increases the magnetization decreases and at \(2J/U = 1\), \(m = 0\). The experimental measurements demonstrate that below \(T_C\) there is enhancement of magnetization so that at zero temperature the magnetization is \(s^l + m\), where \(s^l = 1\) is the collective spin of localized electrons and \(m\) is zero temperature value of the enhancement [8]. The experiment shows that the enhancement varies between 0.5 and zero, and decreases when pressure increases. The comparison with the present theoretical results permits to conclude that the enhancement is due to the onset of magnetism of itinerant electrons. It is evident that parameter \(2J/U\) can be thought as theoretical analog of the pressure, and the theoretical state \(2J/U = 1\) is the experimental one \(p = p_x\). Because of this one studies the ferromagnetism of the system as a function of \(2J/U\).

We calculate the magnetization of localized and itinerant electrons as a function of temperature for two different values of \(2J/U\). The results are depicted in figure (2).

The figure (2a) shows that at low value of the parameter \(2J/U = 0.1\), corresponding to the ambient pressure, the magnetic order of localized and itinerant electrons occurs at the same temperature \(T_C\). At higher value of the parameter \(2J/U = 0.4\) the temperature dependence of magnetic order of localized \(M^l\) and itinerant \(M^i\) electrons is depicted in figure (2b). Both magne-
tization of itinerant and localized electrons decrease as the temperature rises, but at $T_x/J$ the magnetization of itinerant electrons $M^I$ becomes equal to zero while $M^l$ is still nonzero. This reveals that $T_x$ is the partial order transition temperature, from low-temperature phase where both itinerant and localized electrons contribute the magnetization to high-temperature one where only localized electrons form the magnetic order of the system.

With this in mind we calculate and draw in figure 3, three representative magnetization-temperature curves. The upper one is the magnetization as function of

$$
M = - t \sum_{<ij>}(c_{i\sigma}^+ c_{j\sigma} + h.c.) + Z \sum_{l=\sigma}^N c_{i\sigma}^* \tau_{3\sigma\sigma'}^l c_{i\sigma'} 
$$

$$
- J' \sum_{(ij)} [S_i^l \cdot S_j^l + \kappa S_i^l \cdot S_j^l] 
$$

$$
- \sqrt{\frac{J}{2}} \sum_i \left( c_{i\uparrow}^+ c_{i\downarrow} + c_{i\downarrow}^+ c_{i\uparrow} \right),
$$

where $Z = U/2 - J$ is the Zeeman splitting, $s^l = 1$ and $\tau_{3\sigma\sigma'}^l$ is the third Pauli matrix. The Bose operators $a_i, a_i^+$ are from the Holstein-Primakoff representation of spin operators of localized electrons $S_i^l$. In the spin-fermion interaction they are accounted for in linear approximation.

Next step is to integrate out the Bose fields. The result is an effective four fermion theory with Hamiltonian

$$
h_{fs} = \sum_k \varepsilon_k^c c_k^+ c_k^\sigma 
$$

$$
- \frac{1}{N} \sum_{k,p} \delta(k_1 - k_2 - p_1 + p_2)V_{k_1} - k_2 c_{k_1\uparrow}^+ c_{k_2\uparrow} c_{p_2\downarrow}^+ c_{p_1\downarrow},
$$

where

$$
\varepsilon_k^c = -2t(\cos k_x + \cos k_y + \cos k_z) + \frac{U}{2} - J
$$

$$
\varepsilon_k^l = -2t(\cos k_x + \cos k_y + \cos k_z) - \frac{U}{2} + J
$$

the wave vectors $k$ and $p$ run over the Brillouin zone of a simple cubic lattice, and the potential $V_k$ reads

$$
V_k = \frac{J^2}{8J'}(3 - \cos k_x - \cos k_y - \cos k_z + 3\kappa).
$$

In the introduction we discussed that the spin-triplet spin-antiparallel pairing is most appropriate for $UGe_2$. In weak coupling approximation we use the Hartree-Fock approximation of the Hamiltonian

$$
h_{eff}^{HF} = \sum_k \left[ \varepsilon_{k\sigma} c_{k\sigma}^+ c_{k\sigma} + \Delta_k c_{k\downarrow}^+ c_{k\uparrow} + \Delta_k^* c_{k\uparrow} c_{k\downarrow} \right],
$$

with gap function

$$
\Delta_k = \frac{1}{N} \sum_p <c_p^+ c_{-p}> V_{p-k}.
$$

Following standard procedure we can rewrite the gap equation in the form

$$
\Delta_k = \frac{1}{N} \sum_p V_{k-p} \frac{\Delta_p}{\sqrt{(\varepsilon_{p}^c + \varepsilon_{p}^l)^2 + 4|\Delta_p|^2}} 
$$

$$
\times \left( \frac{1}{e^{E_p^c/T} + 1} - \frac{1}{e^{E_p^l/T} + 1} \right),
$$

III. SUPERCONDUCTIVITY OF $UGe_2$

The superconducting phase of $UGe_2$ is located entirely in the ferromagnetic phase. There is a narrow relationship between $UGe_2$ superconductivity and critical temperature $T_x$. In the previous section, was shown that $T_x$ is a partial order transition temperature, and that reducing the temperature means reducing Zeeman’s splitting of itinerant electrons. At pressure $p = p_x$ ($2J = U$) the temperature $T_x$ is zero and the itinerant electrons do not form transversal spin fluctuations. One can represent them by means of creation and annihilation operators in equation (1). The spin-fermion Hamiltonian of $UGe_2$ near $2J/U = 1$ point is

$$
h_{fs} = - t \sum_{<ij>}(c_{i\sigma}^+ c_{j\sigma} + h.c.) + Z \sum_{l=\sigma}^N c_{i\sigma}^* \tau_{3\sigma\sigma'}^l c_{i\sigma'} 
$$

$$
- J' \sum_{(ij)} [S_i^l \cdot S_j^l + \kappa S_i^l \cdot S_j^l] 
$$

$$
- \sqrt{\frac{J}{2}} \sum_i \left( c_{i\uparrow}^+ c_{i\downarrow} + c_{i\downarrow}^+ c_{i\uparrow} \right),
$$

FIG. 3: The magnetization-temperature curves calculated at: upper(black) $2J/U = 0.1$; middle(red) $2J/U = 0.5$; bottom(green)$2J/U = 1$. The upper and bottom curves have not anomalous temperature dependence, while the middle one shows anomalous increasing of magnetization at $T = T_x$ due to onset of magnetic order of itinerant electrons

temperature at $2J/U = 0.1$ and $m = 0.5$. It is without anomaly and resembles the result for spin-3/2 system. The bottom curve calculated at $2J/U = 1$ and $m = 0$, shows an anomaly free magnetization close to spin-1 system. Between them is the magnetization-temperature curve at $2J/U = 0.5$ and $m = 0.32$, with anomalous enhancement of magnetization at $T_x$ which is due to onset of magnetic order of itinerant electrons, as is evident from figure (2). The partial order transition temperature as a function of the parameter $2J/U$ is depicted in figure (1).
where
\[ E_{p}^{s} = \frac{U}{2} - J \pm J \sqrt{\frac{f^{2} \epsilon_{p}^{2} + |\Delta p|^{2}}{2}} \] (15)

with \( \epsilon_{p} \) from equation (3) and \( T \) is the temperature.

We consider superconductivity on the simple cubic lattice, and restrict our analysis to nearest neighbor exchange. Following the classifications for spin-triplet gap functions \( \Delta_{-k} = -\Delta_{k} \) [31], we present \( \Delta_{k} \) with \( T_{1u} \) configuration in the form
\[ \Delta_{k} = \Delta (\sin k_{x} + \sin k_{y} + \sin k_{z}) \] (16)

The equation for the gap parameter \( \Delta \) is
\[
3 \Delta = \Delta \int \frac{d^{3}k}{(2\pi)^{3}} \frac{d^{3}p}{(2\pi)^{3}} V_{k-p} \\
\times \frac{\sin p_{x} + \sin p_{y} + \sin p_{z}}{\sqrt{J^{2} \epsilon_{p}^{2} + \Delta^{2} (\sin p_{x} + \sin p_{y} + \sin p_{z})^{2}}} \\
\times \left( \frac{1}{e^{E_{p}^{s}/T} + 1} - \frac{1}{e^{E_{p}^{s}/T + 1}} \right) .
\]

It is a basis for obtaining the equation for the superconducting critical temperature \( T_{sc} \).

\[
24 \frac{J^{f}}{J} = \int \frac{d^{3}k}{(2\pi)^{3}} \frac{d^{3}p}{(2\pi)^{3}} \left( \frac{1}{e^{E_{p}^{s}/T_{sc}} + 1} - \frac{1}{e^{E_{p}^{s}/T_{sc} + 1}} \right) \\
\times \cos p_{x} + \cos p_{y} + \cos p_{z} \right) \\
\times \left( \sin p_{x} + \sin p_{y} + \sin p_{z} \right)^{2} \\
\times \left( \cos p_{x} + \cos p_{y} + \cos p_{z} \right)^{2} \right)
\]

where
\[ E_{p}^{s} = \frac{U}{2} - J \pm J \sqrt{\left( \cos p_{x} + \cos p_{y} + \cos p_{z} \right)^{2}} \] (19)

The solution of the equation \( T_{sc}/J \) as a function of \( 2J/U \) is depicted in figure 4. The curves show that theoretically calculated relationship between \( T_{x} \) and \( T_{sc} \) is in agreement with the experimentally obtained one.

Very important is the dependence of the superconducting critical temperature on \( 2J/U \). Having in mind that \( Z = U/2 - J \) is the Zeeman splitting of itinerant electrons, we see that superconductivity onsets when the parameter \( 2J/U \) is large i.e. the Zeeman splitting is small enough. The abrupt increase of the superconducting critical temperature up to the maximum, where the Zeeman splitting is zero, is in agreement with the experiment and supports the assumption for the spin-triplet spin-antiparallel pairing. The parameter \( J^{f}/J \) in equation (18) affects the critical temperature and the speed at which it reaches its maximum. In the present paper we set \( J^{f}/J = 0.07 \).

**IV. CONCLUSION**

In the present paper we have addressed the superconductivity of \( UGe_{2} \). One can not understand the superconductivity without detail knowledge of the magnetism of \( UGe_{2} \). A hallmark of the magnetism of these materials is that below characteristic temperature \( T_{x} \) the ferromagnetic moment abruptly increases. The most popular suggestion is that coupled charge and spin density wave fluctuations are origin of \( T_{x} \) anomaly. Instead, we consider \( T_{x} \) as a critical temperature of partial order transition induced by different interaction of the transversal fluctuations of the total magnetization with itinerant and localized electrons. \( T_{x} \) transition is a transition between two spin-ordered phases in contrast to the transition from the spin-ordered state to the disordered state at Curie temperature. There is no additional symmetry breaking and the Goldstone boson has a ferromagnetic dispersion in both phases.

An important characteristic of the system, not discussed in the paper, is the temperature dependence of the specific heat. In the case of \( UGe_{2} \) it has been measured only at ambient pressure [1]. The convex form of the curve is interpreted as an indication of contribution by phonons. The ferrimagnets are popular systems with partial order transition [32, 33]. The specific heat measurements, as a function of temperature [34, 35], show anomalous peak at partial order transition temperature. We think of the convex form of the specific heat curve as a precursor to well-separated peak known in ferrimagnets. The peaks, measured at higher pressure will give us information for the nature of the two ferromagnetic phases in \( UGe_{2} \), and can be used to determine the \( T_{x} \) temperature of the partial order transition.
An important step toward understanding the relationship between the ferromagnetism and the superconductivity is the recognition of the $2J/U$ parameter as a theoretical analog of the pressure. At ambient pressure it is small ($2J/U < 1$) and increases when the pressure increases. The point $2J/U = 1$ corresponds to the pressure $p = p_x$ at which the superconducting transition temperature is maximal. To understand this is enough to mention that the Zeeman splitting of the itinerant electrons is $Z = U/2 - J$. The increase of $2J/U$ decreases the zero temperature magnetic order of the system which leads to anomalous behavior of the magnetization dependence on temperature, decreases the transition temperature $T_x$ and the $Z$ splitting which in turn allows the formation of Cooper pairs and the onset of superconductivity.

Very recently was published a paper [39] focused on the ferromagnetic superconductivity. The authors have considered Ruderman-Kittel-Kasuya-Yosida interaction of conducting electrons and localized spins. The model does not account for Coulomb repulsion, which does not permit the introduction of "theoretical pressure" and respectively does not reproduce the dome form of the superconducting transition temperature. The most important point is the conclusion that spin-wave exchange mechanism results in equal spin-triplet pairing described by two-component order parameter. In the present paper we have presented strict arguments that superconducting order parameter of $UGe_2$ is spinantiparallel component of spin-triplet induced by magnons of the localized electrons. The magnon mediated four fermion interaction [40] shows that it is not possible to realize non-zero $< c_k c_{-k}>$ and $< c_k c_{-k}^\dagger >$ in Hartree-Fock approximation. It is possible if there is a paramagnon exchange interaction [40].

[1] S. Saxena, P. Agarwal, K. Ahilan, F. M. Grosche, R. Haselwimmer, M. Steiner, E. Pugh, I. Walker, S. Julian, P. Monthoux, G. Lonzarich, A. Huxley, I. Sheikin, D. Braithwaite, and J. Flouquet, Nature (London) 406, 587 (2000).
[2] D. Aoki, A. Huxley, E. Ressouche, D. Braithwaite, J. Flouquet, J-P. Brison, E. Lhotel, and C. Paulsen, Nature (London) 413, 613 (2001).
[3] N. T. Huy, A. Gasparini, D. E. de Nijs, Y. Huang, J. C. P. Klaas, T. Gortonmulder, A. de Visser, A. Hamann, T. G"orlach, and H. v. Loneyes, Phys. Rev. Lett. 99, 067006 (2007).
[4] A. Huxley, I. Sheikin, E. Ressouche, N. Kernavanois, D. Braithwaite, R. Calemczuk, and J. Flouquet, Phys. Rev. B 63, 144519 (2001).
[5] N. Tateiwa, T. Kobayashi, K. Hanazono, K. Amaya, Y. Haga, R. Settai, and Y. Onuki, J. Phys.: Condens. Matter 13, L17 (2001).
[6] N. Tateiwa, K. Hanazono, T. C. Kobayashi, K. Amaya, T. Inoue, K. Kindo, Y. Koike, N. Metoki, Y. Haga, R. Settai, and Y. Onuki, J. Phys. Soc. Jpn. 70, 2876 (2001).
[7] G. Motoyama, S. Nakamura, H. Kadoya, T. Nishioka, and N. K. Sato, Phys. Rev. B 65, 020510(R) (2001).
[8] C. Pfleiderer and A. D. Huxley, Phys. Rev. Lett., 89, 147005 (2002).
[9] S. Watanabe and K. Miyake, J. Phys. Soc. Jpn. 71, 2489 (2002).
[10] G. Zwicknagl and P. Fulde, J. Phys.: Condens. Matter 15, S1911 (2003).
[11] K. V. Samokhin and V. P. Mineev, Phys. Rev. B 77, 104520 (2008).
[12] V. P. Mineev, Phys. Rev. B 83, 064515 (2011).
[13] R. Troć, Z. Gajek, and A. Pikul, Phys. Rev B86, 224403 (2012).
[14] Dai Aoki, and Jacques Flouquet, J. Phys. Soc. Jpn. 81, 010103 (2012).
[15] Dai Aoki, and Jacques Flouquet, J. Phys. Soc. Jpn. 83, 061011 (2014).
[16] V. P. Mineev, Physics-Uspekhi 60, 121 (2017).
[17] Dai Aoki, Kenji Ishida, and Jacques Flouquet, J. Phys. Soc. Jpn. 88, 022001 (2019).
[18] Christian Pfleiderer, Rev. Mod. Phys. 81, 1551 (2009).
[19] Y. Onuki, I. Ukon, S.W. Yun, I. Umehara, K. Satoh, T. Fukuhara, H. Sato, T. Takayanagi, M. Shikama, and A. Ochiai, J. Phys. Soc. Jpn. 61, 293 (1992).
[20] G. Oomi, K. Kagayama, K. Nishimura, S.W. Yun, and Y. Onuki, Physica B206, 207, 515 (1995).
[21] P. P. Craig, W. I. Goldburg, T. A. Kitchens, and J. I. Budnick, Phys. Rev. Lett., 19, 1334 (1967).
[22] V. G. Vaks, A. I. Larkin, and Y. N. Ovchinnikov, JETP Lett. 22, 820 (1966).
[23] P. Azaria, H. T. Diep, and H. Giacomini, Phys. Rev. Lett. 59, 1629 (1987).
[24] H. T. Diep (ed.), Frustrated Spin systems (World Scientific, Singapore, 2004)
[25] R. Quatu and H. T. Diep, Phys. Rev. B 55, 2975 (1997).
[26] N. Karchev, J. Magnetism and Magnetic Materials, 396, 77 (2015).
[27] Naoum Karchev, Phys. Rev. B 77, 012405 (2008).
[28] J. R. Stewart, G. Ehlers, A. S. Willis, S. T. Bramwell, and J. S. Gardner, J. Phys.: Condens. Matter 16, L321 (2004).
[29] N. Karchev, Eur.Phys. J. B 77, 47 (2010).
[30] D. Schmeltzer, Phys. Rev. B 43, 8650 (1991).
[31] S. Raghu, S. A. Kivelson, and D. J. Scalapino, Phys. Rev. B 81, 224505 (2010).
[32] K. Adachi, T. Suzuki, K. Kato, K. Osaka, M. Takata, and T. Katsufuji, Phys. Rev. Lett., 95, 197202 (2005).
[33] V. O. Garlea, R. Jin, D. Mandrus, B. Roessli, Q. Huang, M. Miller, A. J. Schultz, and S. E. Nagler, Phys. Rev. Lett., 100, 066404 (2008).
[34] S.-H. Baek, K.-Y Choi, A.P.Reyes, P.L. Kuhns, N.J.Curro, V.Ramachandran, N.S. Dalal, H. D. Zhou, and C.R. Wiebe, J. Phys.:Condens.Matter, 20, 135218 (2008).
[35] H. D. Zhou, J. Lu, and C. R. Wiebe, Phys. Rev., B 76, 174403 (2007).
[36] H. D. Zhou, J. Lu, and C. R. Wiebe, Phys. Rev. B 76, 174403 (2007).
[37] Kim Myung-Whun, J. S. Kim, T. Katsufuji, and R. K. Kremer, Phys. Rev. B 83, 024403 (2011).
[38] Q. Zhang, K. Singh, F. Guillou, C. Simon, Y. Breard,
V. Caignaert, and V. Hardy, Phys. Rev. B 85, 054405 (2012).

[39] L. Bulaevskii, R. Eneias, and A. Ferraz, Phys. Rev. B 99, 064506 (2019).

[40] Naoum Karchev, Phys. Rev. B 67, 054416 (2003).