For every integer \( d \geq 10 \), we construct infinite families \( \{ G_n \}_{n \in \mathbb{N}} \) of \((d+1)\)-regular graphs which have a large girth \( \geq \log_d |G_n| \), and for \( d \) large enough \( \geq 1.33 \cdot \log_d |G_n| \). These are Cayley graphs on \( PGL_2(\mathbb{F}_q) \) for a special set of \( d+1 \) generators whose choice is related to the arithmetic of integral quaternions. These graphs are inspired by the Ramanujan graphs of Lubotzky-Philips-Sarnak and Margulis, with which they coincide when \( d \) is a prime.

When \( d \) is not equal to the power of an odd prime, this improves the previous construction of Imrich in 1984 where he obtained infinite families \( \{ I_n \}_{n \in \mathbb{N}} \) of \((d+1)\)-regular graphs, realized as Cayley graphs on \( SL_2(\mathbb{F}_q) \), and which are displaying a girth \( \geq 0.48 \cdot \log_d |I_n| \).

And when \( d \) is equal to a power of 2, this improves a construction by Morgenstern in 1994 where certain families \( \{ M_n \}_{n \in \mathbb{N}} \) of \( 2^k + 1 \)-regular graphs were shown to have girth \( \geq 2/3 \cdot \log_{2^k} |M_n| \).

1. Introduction

The “Moore bound” follows from a simple counting argument, and permits to show that a \( d \)-regular graph \( G \) of order \( |G| \), admits the following upper bound on its girth (see [1, Ch. III, Theorem 1.2]):

\[
girth(G) \leq \begin{cases} 
2 \log_{d-1} |G| + 1 & \text{if } \text{girth}(G) \text{ is odd}, \\
2 \log_{d-1} |G| + 2 - 2 \log_{d-1} 2 & \text{if } \text{girth}(G) \text{ is even}.
\end{cases}
\]
This implies that for $d \geq 5$,

$$
(2) \quad \text{girth}(G) \leq \left( 2 + \frac{2}{\log_{d-1} |G|} \right) \log_{d-1} |G|.
$$

It is not known if this bound is tight. A convenient way to formulate what is meant by “tight”, is to consider large graphs, and even better, infinite family of constant degree regular graphs. Let us recall the following definition: a family of $d$-regular graphs $\{G_n\}_{n \in \mathbb{N}}$ is said to have *large girth* if there exists a constant $c > 0$ independent of $n$ (but possibly dependent on $d$), such that:

$$
\text{girth}(G_n) \geq (c + o_n(1)) \log_{d-1} |G_n|.
$$

The property of large girth, besides its own theoretical interest, can be applied to LDPC codes. This approach was pioneered by Margulis in [12], where he gave the first constructive example of a family of LDPC codes of unbounded minimum distance by providing explicit families of regular graphs of large girth. Another important application of large girth graphs to LDPC codes can be found in [17].

Given an infinite family of $d$-regular graphs, let us define:

$$
\gamma(\{G_n\}) = \liminf_{n \to \infty} \frac{\text{girth}(G_n)}{\log_{d-1} |G_n|}, \quad \text{and} \quad \gamma_d := \sup_{\{G_n\} \text{ family of } d\text{-regular graphs}} \gamma(\{G_n\}).
$$

What the bound (2) says is that $\gamma_d \leq 2$, for any $d \geq 3$. As for lower bound, it was proved that $\gamma_d \geq 1$ by Erdős and Sachs [5] for any $d \geq 3$. Their proof, of probabilistic nature, did not provide explicit families $\{G_n\}_n$. Currently, the best lower bounds for $\gamma_d$ that are deduced from *explicit* examples of family of graphs, are:

1. $\gamma_d \geq \frac{4}{3}$ for $d = p^k + 1$, $p$ an odd prime and $k \in \mathbb{N}^*$, (for $d = p + 1$ where $p$ is an odd prime; this was first achieved by Lubotzky-Philips-Sarnak [11] and independently by Margulis [13], then later also by Lazebnik-Ustimenko [10] with a different construction. Finally, Morgenstern [14] treated the case $d-1$ equal to any prime power).
2. $\gamma_d \geq \frac{2}{3}$ for $d = 2^k + 1$ with $k \in \mathbb{N}^*$. This is also due to Morgenstern [3, Theorem 5.13-3].
3. $\gamma_d \geq 0.48$ for other values of $d$ (this is due to Imrich [9], extending the method of Margulis [12] where it was proved that $\gamma_d \geq \frac{4}{9}$ for odd $d$).

These are the best results we are aware of. This paper presents improvements on the lower bounds on $\gamma_d$ in the cases 2 and 3, that is, when $d-1$ is not a prime power. For other values of $d$, the lower bounds that would be obtained
do not improve the best ones shown in the case 1. That is why we focus only on the cases where \( d - 1 \) is not the power of a prime, and henceforth consider only \( d \geq 10 \) (lower values are either prime powers or non manageable by our method).

**Theorem 1.1.** For any integer \( d \geq 10 \), which is not a prime power, there is an explicit infinite family \( \{G_n\}_n \) of \((d+1)\)-regular graphs, bipartite and connected, as well as having large girth. Precisely:

\[
girth(G_n) \geq c(d) \log_d |G_n| - \log_d 4,
\]

where \( c(d) \) is a constant independent of \( n \), such that \( c(d) \leq \frac{4}{3} \) and:

\[
\text{case } d \text{ odd} \begin{cases} 
  \text{if } d \geq 1335, & c(d) \geq 1.33 \\
  \text{if } 35 \leq d \leq 1331 & c(d) \geq 1.3 \\
  \text{if } 15 \leq d \leq 31 & c(d) \geq 1.27
\end{cases}
\]

\[
\text{case } d \text{ even} \begin{cases} 
  \text{if } d \geq 4826, & c(d) \geq 1.33 \\
  \text{if } 184 \leq d \leq 4824 & c(d) \geq 1.3 \\
  \text{if } 44 \leq d \leq 182 & c(d) \geq 1.25 \\
  \text{if } 22 \leq d \leq 42 & c(d) \geq 1.1 \\
  c(10) \geq 1.28 & c(12) \geq 1.12 & c(14) \geq 1.19 & c(18) \geq 1.3 & c(20) \geq 1.061
\end{cases}
\]

Related to the families \( \{G_n\}_n \), there are also explicit families of \((d+1)\)-regular graphs \( \{H_n\}_n \), connected and non-bipartite, for which the girth verifies:

\[
girth(H_n) \geq \frac{c(d)}{2} \log_d |H_n|.
\]

The family \( \{G_n\}_n \) will be \( \mathcal{X}_d \) and \( \{H_n\}_n \) will be \( \mathcal{Y}_d \) introduced in Definition 1.3.

The values in the theorem are indicative, having been chosen for their readability. More precise values of \( c(d) \) for each \( d \) can be obtained, but they are of limited interest. More interesting is to mention that \( c(d) \to \frac{4}{3} \) when \( d \) becomes large. These results on \( c(d) \) provide significantly better lower bounds for \( \gamma_{d+1} \) that was previously known \( \gamma_{d+1} \geq c(d) \) improving upon \( \gamma_{d+1} \geq 0.48 \) in the case 2, and improving upon \( \gamma_{d+1} \geq \frac{2}{3} \) in the case 3. The fact that \( c(d) \leq \frac{4}{3} \) shows that no further improvement can be expected from the trick introduced in the present paper.
Furthermore, these explicit families of graphs do even better than what the probabilistic method [5] is able to achieve, namely a $\gamma_d \geq 1$. When dealing with Cayley graphs on $PGL_2(\mathbb{F}_q)$, it was proved in Theorem 9 of [6] that random Cayley graphs$^1$ have a girth $\geq \left(\frac{1}{3} - o(1)\right) \log_d |PGL_2(\mathbb{F}_q)|$ for $q$ sufficiently large. The exact value is not known, but the new graphs of the present paper have most likely much larger girth than the one for the corresponding random Cayley graph.

**The Main Inequality.** This paragraph presents the main intermediate result (4), and the next paragraph will show how to deduce from it the bounds of Theorem 1.1. A few more notations are necessary:

**Definition 1.2.** Given an integer $d$, $p$ denotes any prime number $p \geq d$, with the additional condition $p \equiv 3 \mod 8$ when $d$ is even. Let $\kappa := \log_p d \geq 1$, so that $p = d^\kappa$. Define

$$Q_d(p) := \max\{p^8, 120^\kappa p\}.$$ 

Given another prime $q > Q_d(p)$, there is a symmetric$^2$ subset $D_{p,q}$ of $PGL_2(\mathbb{F}_q)$ of cardinality $d+1$, such that if we define:

$$G_{d,p,q} := \begin{cases} 
\text{Cay}(PGL_2(\mathbb{F}_q), D_{p,q}) & \text{if } \left(\frac{p}{q}\right) = -1 \\
\text{Cay}(PSL_2(\mathbb{F}_q), D_{p,q}) & \text{if } \left(\frac{p}{q}\right) = 1
\end{cases}$$

(See Definition 1.3 for more details on $G_{d,p,q}$). Then:

- $G_{d,p,q}$ is a $(d+1)$-regular graph of size $|PGL_2(\mathbb{F}_q)| = q^3 - q$ or $|PSL_2(\mathbb{F}_q)| = \frac{1}{2}(q^3 - q)$ according to the sign of the Legendre symbol $\left(\frac{p}{q}\right)$.
- $G_{d,p,q}$ is connected, bipartite if $\left(\frac{p}{q}\right) = -1$, and not bipartite if $\left(\frac{p}{q}\right) = 1$.
- the girth of $G_{d,p,q}$ satisfies the **Main Inequality**:

$$\text{girth}(G_{d,p,q}) \geq \begin{cases} 
\frac{2}{3\kappa} \log_d |G_{d,p,q}| & \text{if } \left(\frac{p}{q}\right) = 1 \\
\frac{4}{3\kappa} \log_d |G_{d,p,q}| - \log_p 4 & \text{if } \left(\frac{p}{q}\right) = -1
\end{cases}$$

Let us point out here that $\text{girth}(G_{d,p,q}) \leq \frac{4}{3} \log_d |G_{d,p,q}| + 1$ or $\text{girth}(G_{d,p,q}) \leq \frac{2}{3} \log_d |G_{d,p,q}| + 1$, for any $d$. Indeed, these lower bounds already occur for the Ramanujan graphs [13, Last proposition], from which the graphs $G_{d,p,q}$ are derived. This is why $c(d) \leq \frac{4}{3}$ in Theorem 1.1.

---

$^1$ the model of random Cayley graphs is described p. 2 of [6]  
$^2$ that is if $x \in D_{p,q}$, then $x^{-1} \in D_{p,q}$ as well
Fixing $p$ and $d$, we can consider the following two kinds of infinite families of graphs, indexed by $q$:

(5) \[ \mathcal{X}_{d,p} := \{ G_{d,p,q} \}_{q \text{ prime}, q > Q_d(p), \left( \frac{p}{q} \right) = -1} \]

and

(6) \[ \mathcal{Y}_{d,p} := \{ G_{d,p,q} \}_{q \text{ prime}, q > Q_d(p), \left( \frac{p}{q} \right) = 1} \]

From Main Inequality (4) above, we infer: $\gamma(\mathcal{X}_{d,p}) \geq 4 \frac{3}{\kappa}$ and $\gamma(\mathcal{Y}_{d,p}) \geq 2 \frac{3}{\kappa}$, where $\kappa = \log_d p$.

**Main Inequality implies Theorem 1.1.** It is quite easy to recover the bounds on $c(d)$ of Theorem 1.1 from Main Inequality (4). The lower bound on the girth in (4) is indeed the largest when $\kappa$ is the smallest. To minimize $\kappa$, let us first introduce some notations:

**Definition 1.3.** Given an integer $u > 5$, let

\[ p(u) := \min\{ p \geq u : p \text{ prime} \} \]

and

\[ p_3(u) := \min\{ p \geq u : p \text{ prime} \equiv 3 \mod 8 \} \]

Then, for each $d \geq 10$, we consider two families of graphs $\mathcal{X}_d$ and $\mathcal{Y}_d$ as:

if $d$ is even: \[ \mathcal{X}_d := \mathcal{X}_{d,p(d)} \]

and if $d$ is odd: \[ \mathcal{X}_d := \mathcal{X}_{d,p_3(d)} \]

The real number $\kappa$ of Definition 1.2 verifies then $\kappa = \log_d p(d)$ if $d$ is odd and, $\kappa = \log_d p_3(d)$ if $d$ is even.

Then, minimizing $\kappa$ brings in the question: Given $u$ odd, how big is the smallest prime $p(u)$ larger than $u$? Similarly, if $u$ is even, how big can $p_3(u)$ be?

Considering the worst case where $u$ is equal to a prime plus one, this is related to the problem of gap between primes [7, pp.10-12]. Bertrand’s postulate affirms that $p(u) < 2u$, Cramér’s conjecture suggests that $p(u) < \log(u)^2$ for some reasonably large $u$, in between various upper bounds on the gap between two primes have appeared, most being valid only for “large enough” values of $u$. For us, small values of $u$ must be taken into account and therefore we use the unconditional estimate $p(u) < u \left( 1 + \frac{1}{2(\log u)^2} \right)$ valid for $u \geq 3275$ (see [4, Sec. 4]). Sharper estimates would yield (tiny) better
estimates on the girth only for large degrees of regularity $d$, when $1.333 < c(d) < 4/3$.

It implies that: $\kappa \leq \log_u \left( u \left( 1 + \frac{1}{2 \log_u u} \right) \right)$ for $u \geq 3275$, and proves that $c(d) = \frac{4}{3\kappa} \geq 1.33$ for $d \geq 3275$. For smaller values of $d$, I used a computer and found the following. The smallest integer $d_1$ for which $d \geq d_1 \Rightarrow 4/3 \kappa \geq 1.33$ with $\kappa = \log_{d_1} p(d_1)$ is 1335, and then $p(1335) = 1361$. The smallest integer $d_2$ for which $d \geq d_2 \Rightarrow 4/3 \kappa \geq 1.33$ with $\kappa = \log_{d_1} p(d_1)$ is 35, and then $p(35) = 37$. Between 15 and 31, it is easy to check that $4/3 \kappa \geq 1.27$. There is no integer smaller than 15 and greater than 10 which is not a prime power. This achieves the proof of the bound on $c(d)$ in Theorem 1.1, when $d$ is odd.

As for $p_3(u)$, I used results of [15]. This requires to introduce the classical arithmetic function

$$\theta(x; k, \ell) := \sum_{\substack{p \equiv \ell \mod k \leq x}} \ln(x), \quad \text{where } p \text{ denotes a prime number.}$$

Indeed, there is a prime number equal to 3 modulo 8 in the interval $[a; b]$ if $\theta(b; 8, 3) - \theta(a; 8, 3) > 0$. The estimate of [15, Theorem 1] shows:

$$\max_{1 \leq y \leq x} \left| \theta(y; 8, 3) - \frac{y}{4} \right| \leq 0.002811 \frac{x}{4}, \quad \text{for } x \geq 10^{10}.$$

Setting $\epsilon = 0.002811$, for $x \geq 10^{10}$ and any $y$, it comes:

$$\frac{y}{4} - \epsilon \frac{x}{4} \leq \theta(y; 8, 3) \leq \epsilon \frac{x}{4} + \frac{y}{4}.$$

It follows that for all $b > a \geq 10^{10}$,

$$\theta(b; 8, 3) - \theta(a; 8, 3) \geq \frac{b}{4}(1 - 2\epsilon) - \frac{a}{4}.$$  

This insures that for $a \geq 10^{10}$ there is a prime equal to 3 modulo 8 in each interval $[a; \frac{a}{1 - 2\epsilon}]$. For $d \geq 10^{10}$, this clearly proves that $4/3 \kappa \geq 1.33$, since then $\kappa = \log_d p_3(d) \leq \log_d \frac{d}{1 - 2\epsilon}$. For values $d \leq 10^{10}$, a laptop computer may not be powerful enough to check what the maximal values of $\log_d p_3(d)$ are. Again, from [15, Theorem 2], in this case:

$$\max_{1 \leq y \leq x} \left| \theta(y; 8, 3) - \frac{y}{4} \right| \leq 1.82 \sqrt{x}, \quad \text{for } 1 \leq x \leq 10^{10}.$$

It follows that $\theta(b; 8, 3) - \theta(a; 8, 3) \geq \frac{b-a}{4} - 2 \cdot 1.82 \sqrt{b}$ for $b > a$. This shows that in the interval $[a; a \left( 1 + \frac{8 \cdot 1.82}{\sqrt{a} - 8 \cdot 1.82} \right)]$ there is a prime equal to 3 modulo
8. Hence, \( \kappa = \log_d p_3(d) \leq 1 + \log_d \left( 1 + \frac{8 \cdot 1.82}{\sqrt{d} - 8 \cdot 1.82} \right) \), showing that \( \frac{4}{3\kappa} \geq 1.33 \) if \( d \geq 228050 \).

The other values of \( c(d) \) of Theorem 1.1 in the case \( d \) even, for \( d \leq 228050 \) are easily obtained with the help of a computer. This concludes the proof of Theorem 1.1 assuming the Main Inequality (4).

2. Proof of the Main Inequality

It remains to show that Main Inequality (4) holds. All the necessary material is contained in the monograph [3]. To make this section a minimum self-contained, many results appearing therein are recalled.

2.1. Unique factorization of quaternions and regular trees

The construction of Ramanujan graphs by Lubotzky-Philips-Sarnak is achieved by taking finite quotients of a “mother graph”, which is a regular tree. They used simply the factorization of quaternions to build these regular trees.

We briefly recall this here, referring to Ch. 2.6 of the aforementioned monograph [3] for the details.

**Quaternions.** For \( R \) a commutative ring, let \( \mathbb{H}(R) \) denotes the Hamilton quaternion algebra over \( R \):

\[
\mathbb{H}(R) := R + Ri + Rj + Rk, \quad i^2 = j^2 = k^2 = -1, \quad k = ij = -ji.
\]

The *conjugate* of an element \( \alpha = a_0 + a_1i + a_2j + a_3k \) is \( \overline{\alpha} := 2a_0 - \alpha = a_0 - a_1i - a_2j - a_3k \), and the *norm* of \( \alpha \) is \( N(\alpha) = \alpha \overline{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \). The multiplication of quaternions makes the norm multiplicative: \( N(\alpha \beta) = N(\alpha)N(\beta) \). Given a quaternion \( \alpha = a_0 + a_1i + a_2j + a_3k \) the non-negative integer \( \gcd(a_0, a_1, a_2, a_3) \) is called the *content* of \( \alpha \) and is denoted \( c(\alpha) \). If \( c(\alpha) = 1 \), then \( \alpha \) is *primitive*.

Let us set \( R = \mathbb{Z} \). We introduce a property of unique factorization for integral quaternions \( \mathbb{H}(\mathbb{Z}) \), yet in a special easy case that is sufficient for the purpose of this article. This restriction is to consider only quaternions whose norm is a power of an odd prime \( p \) (instead of considering any quaternion in \( \mathbb{H}(\mathbb{Z}) \)).

Given an odd prime \( p \), and a primitive quaternion \( \alpha \in \mathbb{H}(\mathbb{Z}) \) of norm \( p^k \), then there exist *prime quaternions* \( \pi_1, \ldots, \pi_k \) (prime means that if \( \pi = \gamma \delta \), then either \( \gamma \) or \( \delta \) is a unit in \( \mathbb{H}(\mathbb{Z}) \)) such that \( \alpha = \pi_1 \cdots \pi_k \). In a word, this
follows from the possibility to perform a Euclidean division in \( \mathbb{H}(\mathbb{Z}) \) of two such quaternions whose norm is a power of \( p \); a non-commutative Euclidean algorithm (one “on the right”, one “on the left”) is deduced, in order to compute left and right gcds. This permits to show that prime quaternions are precisely those whose norm is a prime number. Then the existence of a factorization follows easily by induction on the exponent \( k \) of the norm \( p^k = N(\alpha) \).

The default of uniqueness is completely related to the units of \( \mathbb{H}(\mathbb{Z}) \) (which are \( \pm 1, \pm i, \pm j, \pm k \)). What this means is that two distinct factorizations \( \pi_1 \cdots \pi_k \) and \( \mu_1 \cdots \mu_k \) of \( \alpha \) verify: \( \pi_i = \epsilon_i \mu_i \), for some \( \epsilon_i \in \mathbb{H}(\mathbb{Z})^* \) and for \( 1 \leq i \leq k \). The group of 8 units \( \mathbb{H}(\mathbb{Z})^* \) acts on the set of quaternions of norm \( p \). By isolating one quaternion per orbit, uniqueness can be recovered. Since the number of quaternions of norm \( p \) is \( 8(p+1) \) by a famous theorem of Jacobi (indeed, such quaternions \( x_0 + x_1 i + x_2 j + x_3 k \) give a solution in \( \mathbb{Z}^4 \) of \( f(x) = p \), where \( x = (x_0, x_1, x_2, x_3) \) and \( f(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2 \)). As perfectly explained in p. 67-68 of [3], a quite natural way to isolate one quaternion per orbit is to introduce:

\[
\mathcal{P}(p) = \{ \pi \in \mathbb{H}(\mathbb{Z}) \text{ primitive: } N(\pi) = p, \pi_0 > 0, \pi_1 - 1 \in 2\mathbb{H}(\mathbb{Z}) \} \quad \text{if } p \equiv 1 \mod 4,
\]

\[
\mathcal{P}(p) = \{ \pi \in \mathbb{H}(\mathbb{Z}) \text{ primitive: } N(\pi) = p, \pi_0 > 0 \text{ if } \pi_0 \neq 0, \text{ and } \pi_1 > 0 \text{ else, } \pi - i - j - k \in 2\mathbb{H}(\mathbb{Z}) \} \quad \text{if } p \equiv 3 \mod 4
\]

The fact that \( \pi_0 \neq 0 \), or \( \pi_1 \neq 0 \) if \( \pi_0 = 0 \) is made clear by the explanations coming hereafter.

**Remark 2.1.** Some general remarks about this set:

(a) if \( \alpha \in \mathcal{P}(p) \), then \( \epsilon \alpha \) and \( \alpha \epsilon \) are not in \( \mathcal{P}(p) \), for any unit \( \epsilon \in \mathbb{H}(\mathbb{Z})^* \) different from 1.

(b) Similarly, given \( \beta \in \mathbb{H}(\mathbb{Z}) \), \( N(\beta) = p \), there are exactly two units \( \epsilon, \epsilon' \in \mathbb{H}(\mathbb{Z})^* \) that yield \( \epsilon \beta \in \mathcal{P}(p) \) and \( \beta \epsilon' \in \mathcal{P}(p) \).

(c) this implies that \( |\mathcal{P}(p)| = p + 1 \) (according to Jacobi’s theorem on the sum of four squares).

(d) given \( \pi \in \mathcal{P}(p) \), if \( \pi_0 \neq 0 \) then \( \overline{\pi} \in \mathcal{P}(p) \) (easy to check). If \( \pi \) is such that \( \pi_0 = 0 \), as it may happen when \( p \equiv 3 \mod 4 \) (actually when \( p \equiv 3 \mod 8 \) after Proposition 2.3), then \( \overline{\pi} = -\pi \notin \mathcal{P}(p) \), in conformity with the two points (a) and (b) above.

Remark that the first point (a) allows a form of uniqueness of the factorization of quaternions [3, 2.6.13 Theorem].
Theorem 2.2. Given $\alpha$ of norm $p^k$, and of content $c(\alpha) = p^\ell$, then there exist unique $\pi_1, \ldots, \pi_{k-2\ell} \in \mathcal{P}(p)$ and a unique unit $\epsilon \in \mathbb{H}(\mathbb{Z})^*$ such that:

$$\alpha = c(\alpha)\epsilon \pi_1 \cdots \pi_{k-2\ell},$$

with $\pi_i \neq \pi_{i-1}$ if $\pi_i \in \mathcal{P}(p)$, and with $\pi_i \neq \pi_{i-1}$ else.

Let us stress that under these conditions, the quaternion $\pi_1 \cdots \pi_{k-2\ell}$ is primitive (motivating the definition of irreducible product in Definition (2.6)).

We focus on the case $\pi \in \mathcal{P}(p)$ and $\pi \notin \mathcal{P}(p)$, which may happen when $p \equiv 3 \mod 4$ as mentioned in (d) above.

Proposition 2.3. There is an element $\pi = \pi_0 + \pi_1i + \pi_2j + \pi_3k \in \mathcal{P}(p)$ for which $\pi_0 = 0$ (equivalently $\pi = \pi$, or $\pi \notin \mathcal{P}(p)$) if and only if $p \equiv 3 \mod 8$.

Proof. By definition of $\mathcal{P}(p)$ this can only happen if $p \equiv 3 \mod 4$, since otherwise $\pi_0 \equiv 1 \mod 2$. For such a $\pi$, $N(\pi) = \pi_1^2 + \pi_2^2 + \pi_3^2$ and consequently $p$ is a sum of 3 squares. Reciprocally, a sum of 3 squares $x_1^2 + x_2^2 + x_3^2$ equal to $p$ gives a quaternion $x = x_1i + x_2j + x_3k \in \mathbb{H}(\mathbb{Z})$ of norm $p$, which is also necessarily primitive (because $p$ is prime). Since $p \equiv 3 \mod 4$, $p$ is not the sum of 2 squares. Hence, necessarily, $x_1 \equiv x_2 \equiv x_3 \equiv 1 \mod 4$, implying $x \in \mathcal{P}(p)$.

We have proved that such a $\pi$ exists in $\mathcal{P}(p)$ if and only if $p \equiv 3 \mod 4$ and $p$ is the sum of 3 squares. This is true if and only if $p \equiv 3 \mod 8$, as the Gauss theorem [16, Ch. IV, Appendice] on sum of 3 squares shows:

Theorem 2.4 (Gauss). An integer $n$ is the sum of 3 squares if and only if $n$ is not equal to $4^k(8\ell + 7)$ whatsoever are $k, \ell \in \mathbb{N}$.

Hence, $p \equiv 3 \mod 4$ is the sum of 3 squares if and only if $p \neq 4^k(8\ell + 7)$. Suppose $p = 4^k(8\ell + 7)$, then $p \equiv 3 \mod 4$ gives $k = 0$, and $p = 8\ell + 7$, implying $p \equiv 3 \mod 8$. This proves that $p \equiv 3 \mod 4$ is sum of 3 squares if and only if $p \equiv 3 \mod 8$, achieving the proof of Proposition 2.3.

In the case $p \equiv 3 \mod 8$, we denote a quaternion in $\mathcal{P}(p)$ of the form shown in Proposition 2.3 by the letter $\nu$, and the others by the letter $\mu$. One has:

$$\text{if } p \equiv 3 \mod 8, \quad \mathcal{P}(p) = \{\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_t\},$$

with $s + t = p + 1$ and $t > 0$.

Note that $s$ is even because each $\mu_i$ comes along with its conjugate, that is there is an $i' \neq i$ such that $\overline{\mu_i} = \mu_{i'}$. That is, $t = p + 1 - s$ is also even.
Trees built on quaternions. The unique factorization Theorem 2.2 permits to build infinite regular trees of arbitrary degree \( d \). As in Definition 1.2, let \( p \geq d \) be a prime number, ordinary if \( d \) is odd, and equal to 3 modulo 8 if \( d \) is even.

**Lemma 2.5.** We can choose a subset \( \mathcal{D}(d) \subset \mathcal{P}(p) \) of cardinality \( d + 1 \) such that, given \( \pi \in \mathcal{D}(d) \), one has: \( \pi \in \mathcal{D}(d) \) if and only if \( \pi \in \mathcal{P}(p) \).

In particular, if \( d \) is even then \( \mathcal{D}(d) \) contains at least one \( \pi \) such that \( \pi \not\in \mathcal{D}(d) \) (this latter case happens only if \( p \equiv 3 \mod 8 \) according to Proposition 2.3).

**Proof.** If \( d \) is odd, then Property (7) in the definition (8) of \( \mathcal{P}(p) \) when \( p \not\equiv 3 \mod 8 \) makes it clear: it suffices to choose \( \frac{d+1}{2} \) elements pairwise not conjugate, as well as their \( \frac{d+1}{2} \) conjugates (that are also in \( \mathcal{P}(p) \) in this case). For the case \( p \equiv 3 \mod 8 \), let us use the two even integers \( s \) and \( t \) defined in (9). We first choose \( k_1 := \max \left\{ \frac{d+1}{2}, \frac{s}{2} \right\} \) couple of conjugates in \( \mathcal{P}(p) \), and, if necessary, \( d + 1 - 2k_1 \) elements \( \pi \) such that \( \pi \not\in \mathcal{P}(p) \).

If \( d \) is even, then \( p \equiv 3 \mod 8 \) by Definition 1.2. A way to choose the set \( \mathcal{D}(d) \) is as follows. First choose \( k_1 := \max \left\{ \frac{d}{2}, \frac{s}{2} \right\} \) couples of conjugates, completed with \( d + 1 - 2k_1 \) elements \( \pi \) such that \( \pi \not\in \mathcal{P}(p) \).

Notice that in general, there are several other possible ways of choosing \( \mathcal{D}(d) \) inside \( \mathcal{P}(p) \).

**Definition 2.6.** An irreducible product of length \( \ell \) over \( \mathcal{D}(d) \) is the product of \( \ell \) elements \( \alpha_1, \ldots, \alpha_\ell \) in \( \mathcal{D}(d) \) where two consecutive elements:
- are not conjugate, \( \alpha_i \neq \alpha_{i+1} \), if \( \alpha_i \in \mathcal{P}(p) \)
- are not equal, \( \alpha_i \neq \alpha_{i+1} \), if \( \alpha_i \not\in \mathcal{P}(p) \).

The set of all irreducible products over \( \mathcal{D}(d) \) is denoted \( \Lambda_\mathcal{D} \).

The motivation of this terminology comes from the following fact, resulting of the unique factorization (Theorem 2.2): the product of a sequence of elements in \( \mathcal{D}(d) \) that does not verify the conditions mentioned in the definition can be reduced, yielding a non primitive quaternion.

Furthermore, Theorem 2.2 also tells that two different irreducible products yields two different quaternions. This allows to define a \( (d+1) \)-regular tree \( T_d \) in the following way:

- the vertex set \( V(T_d) \) is identified with the irreducible products of \( \Lambda_\mathcal{D} \) over \( \mathcal{D}(d) \subset \mathcal{P}(p) \)
- the root is identified with the void product; given another vertex identified with the irreducible product \( \alpha_1 \cdots \alpha_s \), we define \( d \) adjacent vertices
whose irreducible products are:

\[ \alpha_1 \cdots \alpha_s \alpha_{s+1}, \quad \alpha_{s+1} \in \mathcal{D}(d) \quad \text{where} \quad \begin{cases} \alpha_{s+1} \neq \alpha_s & \text{if } \alpha_s \in \mathcal{P}(p) \\ \alpha_{s+1} \neq \alpha_s & \text{if } \alpha_s \notin \mathcal{P}(p) \end{cases} \]

- and the last adjacent vertex is the irreducible product \( \alpha_1 \cdots \alpha_{s-1} \).

### 2.2. Algebraic construction of the tree and definition of the graphs \( G_{d,p,q} \)

It is necessary to give an interpretation of the tree \( T_d \) constructed above more algebraically. Indeed, the graphs \( G_{d,p,q} \) are naturally defined algebraically.

#### Algebraic construction of the trees \( T_d \). It consists in seeing the trees \( T_d \) as Cayley graphs on free groups if \( d \) is even, or on groups with involutions and unique factorization property in terms of the generating set if \( d \) is odd.

**Proposition 2.7.** The set \( \Lambda_\mathcal{D} \) of all irreducible products over \( \mathcal{D}(d) \) can be endowed with the structure of group. If \( d \) is odd, then \( \Lambda_\mathcal{D} \) is free over \( \mathcal{D}(d) \).

If \( d \) is even, \( \mathcal{D}(d) \) contains at least one involution – hence, \( \Lambda_\mathcal{D} \) is not free on \( \mathcal{D}(d) \) – but each element of \( \Lambda_\mathcal{D} \) can be uniquely written as a product of elements in \( \mathcal{D}(d) \).

**Proof.** Given two irreducible products \( \alpha := \alpha_1 \cdots \alpha_n \), and \( \beta := \beta_1 \cdots \beta_m \) in \( \Lambda_\mathcal{D} \), we associate an irreducible product \( \alpha \times \beta \) as follows.

- there is no integer \( i \geq 0 \) such that \( \alpha_{n-i} \neq \beta_{i+1} \) if \( \beta_{i+1} \in \mathcal{P}(p) \), or \( \alpha_{n-i} \neq \beta_{i+1} \) if \( \beta_{i+1} \notin \mathcal{P}(p) \). Then we define \( \alpha \times \beta = 1 \).

- else, let \( \ell \geq 0 \) be the largest such integer \( i \). Then the content of \( \alpha \beta \) is \( c(\alpha \beta) = p^\ell \), and \( \frac{\alpha \beta}{p^\ell} \) is primitive. Its unique factorization is given by: \( \frac{\alpha \beta}{p^\ell} = \pm \alpha_1 \cdots \alpha_{n-\ell} \beta_{\ell+1} \cdots \beta_m \). This allows to define,

\[ \alpha \times \beta := \alpha_1 \cdots \alpha_{n-\ell} \beta_{\ell+1} \cdots \beta_m. \]

Note that this is an irreducible product in \( \Lambda_\mathcal{D} \).

It is easy to check that \( \times \) defines an associative operations on \( \Lambda_\mathcal{D} \) with unit element 1 (the void irreducible product). The inverse of an irreducible product \( \alpha := \alpha_1 \cdots \alpha_n \) is \( \beta := \alpha_n \cdots \alpha_1 \) where \( \alpha_i = \alpha_i \in \mathcal{D}(d) \) if \( \alpha_i \in \mathcal{P}(p) \), and \( \alpha_i = -\alpha_i \) if \( \alpha_i \notin \mathcal{P}(p) \). The content of \( \alpha \beta \) is then \( p^n \), hence \( \alpha \times \beta = 1 \).

It remains to show that each element of the group \( (\Lambda_\mathcal{D}, \times) \) can be uniquely written as a product of elements in \( \mathcal{D}(d) \). This follows by the definition 2.6 of irreducible products on \( \mathcal{D}(d) \), that yields different quaternions by the unique factorization theorem 2.2.
Remark 2.8. Using the notations in (9), \( \mathcal{D}(d) \) consists of elements \( \mu_1, \ldots, \mu_u, \nu_1, \ldots, \nu_v \) with \( u \leq s \) and \( v \leq t \), such that \( \overline{\nu_i} \notin \mathcal{D}(d) \) and \( \overline{\mu_i} = \mu_i \in \mathcal{D}(d) \). Let \( (K, \times) \) be the subgroup of \( (\Lambda_{\mathcal{G}}, \times) \) generated by \( \mu_1, \ldots, \mu_u \). This is a free group for \( \times \), and we have:

\[
(A_{\mathcal{G}}, \times) \simeq (K, \times) \ast \langle \nu_1 \rangle \ast \cdots \ast \langle \nu_v \rangle,
\]

where \( \langle \nu_i \rangle \) is the subgroup of order 2 of \( (A_{\mathcal{G}}, \times) \) generated by \( \nu_i \) and \( \ast \) is the free product on subgroups of \( (A_{\mathcal{G}}, \times) \).

The combinatorial definition of the tree \( T_d \) given at the end of Section 2.2 is the Cayley graph of the group \( \Lambda_{\mathcal{G}} \) with generating set \( \mathcal{D}(d) \).

\[
T_d \simeq Cay(\Lambda_{\mathcal{G}}, \mathcal{D}(d)).
\]

Graphs \( G_{d,p,q} \) as finite quotients of the tree \( T_d \). As above, we let \( d \) be an integer greater than 10, and \( p \) a prime greater than \( d \), equal to 3 modulo 8 if \( d \) is even (and without condition if \( d \) is odd). Now we let \( q > Q_d(p) \) where \( Q_d(p) \) is the constant introduced in Definition 1.2.

The next step consists in taking finite quotients of the tree \( T_d \). Let

\[
\tau_q: \mathbb{H}(\mathbb{Z}) \to \mathbb{H}(\mathbb{F}_q)
\]

the reduction map modulo \( q \). When restricted to \( \Lambda_{\mathcal{G}} \), we observe the following:

- \( \tau_q(\Lambda_{\mathcal{G}}) \subset \mathbb{H}(\mathbb{F}_q)^* \)
- \( \tau_q(\alpha \beta) \) and \( \tau_q(\alpha \ast \beta) \) differ multiplicatively by \( \tau_q(p^\ell) \), where \( p^\ell \) is the content of \( \alpha \beta \), which is in the center \( \mathcal{Z} \) of the group \( \mathbb{H}(\mathbb{F}_q)^* \).

Hence, by taking the quotient group \( \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z} \) the following map:

\[
\mu_q: \Lambda_{\mathcal{G}} \to \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z},
\]

is a group homomorphism. Next, we identify the image of this group homomorphism. Recall that since \( p \neq 2 \), the quaternion algebra over \( \mathbb{F}_q \) as defined in Section 2.1 is isomorphic to the algebra of 2-by-2 matrices over \( \mathbb{F}_q \). Indeed, in \( \mathbb{F}_q \) there are two elements \( x \) and \( y \) such that \( x^2 + y^2 + 1 = 0 \) (see Prop. 2.5.2 and 2.5.3 in [3]). The following map is an isomorphism of \( \mathbb{F}_q \)-algebra:

\[
\phi: \mathbb{H}(\mathbb{F}_q) \to M_2(\mathbb{F}_q),
\]

\[
\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 x + \alpha_3 y & -\alpha_1 y + \alpha_2 + \alpha_3 x \\ -\alpha_1 y - \alpha_2 + \alpha_3 x & \alpha_0 - \alpha_1 x - \alpha_3 y \end{pmatrix}.
\]
Moreover, $N(\alpha) = \det \phi(\alpha)$. We deduce the following group isomorphism $\psi$ from $\phi$:

$$\psi: \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z} \rightarrow PGL_2(\mathbb{F}_q),$$

and we let:

$$\overline{\mu}_q := \psi \mu_q, \text{ and } \ker \overline{\mu}_q := A_\phi(q), \text{ so that } A_\phi/A_\phi(q) \hookrightarrow PGL_2(\mathbb{F}_q).$$

**Lemma 2.9.** If $q$ is such that $p$ is a quadratic residue modulo $q$, then $\overline{\mu}_q(\mathcal{D}(d)) \subset PSL_2(\mathbb{F}_q)$. Else, $\overline{\mu}_q(\mathcal{D}(d)) \subset PGL_2(\mathbb{F}_q) - PSL_2(\mathbb{F}_q)$.

**Proof.** The group homomorphism $\epsilon: \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z} \rightarrow \{-1,1\}$, $x \mapsto \left( \frac{N(\alpha)}{q} \right)$ takes the same value on each class modulo the center $\mathcal{Z}$. The factor map $\overline{\epsilon}: \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z} \rightarrow \{-1,1\}$, $x\mathcal{Z} \mapsto \epsilon(x)$, is well-defined. The set of quaternions in $\mathbb{H}(\mathbb{F}_q)^*$ of norm 1, denoted $\mathbb{H}_1$, is sent to 1 by $\epsilon$, and hence, $\ker \overline{\epsilon} \supset \mathbb{H}_1/(\mathcal{Z} \cap \mathbb{H}_1)$. Now, given $\pi \in \mathcal{D}(d)$, $\left( \frac{p}{q} \right)$ and $\epsilon(\mu_q(\pi))$ are equal. This shows that if $\left( \frac{p}{q} \right) = 1$, then $\mu_q(\mathcal{D}(d)) \subset \ker \overline{\epsilon}$, and if $\left( \frac{p}{q} \right) = -1$, then $\mu_q(\mathcal{D}(d)) \subset \mathbb{H}(\mathbb{F}_q)^*/\mathcal{Z} - \ker \overline{\epsilon}$. Using the isomorphism $\psi$, we obtain $\overline{\mu}_q(\mathcal{D}(d)) \subset PSL_2(\mathbb{F}_q)$ if $\left( \frac{p}{q} \right) = 1$, and $\overline{\mu}_q(\mathcal{D}(d)) \subset PGL_2(\mathbb{F}_q) - PSL_2(\mathbb{F}_q)$ else.

By the above discussion, comes:

$$A_\phi/A_\phi(q) \hookrightarrow \begin{cases} PSL_2(\mathbb{F}_q) & \text{if } \left( \frac{p}{q} \right) = 1 \\ PGL_2(\mathbb{F}_q) & \text{if } \left( \frac{p}{q} \right) = -1 \end{cases}$$

**Lemma 2.10.** Let $\mathcal{D}_{p,q} := \overline{\mu}_q(\mathcal{D}(d))$. One has $|\mathcal{D}_{p,q}| = |\mathcal{D}(d)| = d + 1$.

**Proof.** The map $\psi$ (2.2) being an isomorphism it suffices to show that $|\mathcal{D}(d)| = |\mu_q(\mathcal{D}(d))|$. Since $\mathcal{D}(d) \subset \mathcal{P}(p)$, this will certainly follow from $|\mathcal{P}(p)| = |\mu_q(\mathcal{P}(p))|$. The later is (easily) proved in [3, 4.2.1 Lemma], under the assumption that $q > 2\sqrt{2}$, verified because $q > Q_d(p) \geq p^8$.

Already mentioned in the Introduction, we can now give a precise definition of the graph $G_{d,p,q}$:

**Definition 2.11.** Given the three integers $d$, $p$ and $q$ as defined above, the graph $G_{d,p,q}$ is:

$$G_{d,p,q} := \begin{cases} \text{Cay}(PGL_2(\mathbb{F}_q), \mathcal{D}_{p,q}) & \text{if } \left( \frac{p}{q} \right) = -1 \\ \text{Cay}(PSL_2(\mathbb{F}_q), \mathcal{D}_{p,q}) & \text{if } \left( \frac{p}{q} \right) = 1 \end{cases}$$
By Lemma 2.10, the graphs $G_{d,p,q}$ are $(d+1)$-regular. Moreover:

**Lemma 2.12.** The graphs $G_{d,p,q}$ are bipartite when $\left(\frac{p}{q}\right) = -1$.

Moreover, assuming that $G_{d,p,q}$ is connected when $\left(\frac{p}{q}\right) = 1$, $G_{d,p,q}$ is non-bipartite.

**Proof.** In the first case, a bipartition $A \cup B$ of the set of vertices $V(G_{d,p,q})$ is given by $A := PSL_2(\mathbb{F}_q)$, and $B := PGL_2(\mathbb{F}_q) - PSL_2(\mathbb{F}_q)$. Indeed, the index of $PSL_2(\mathbb{F}_q)$ in $PGL_2(\mathbb{F}_q)$ is 2, and Lemma 2.9 shows that the generating set $D_{p,q}$ lies in the non-trivial coset $\subseteq B$.

As for the case $\left(\frac{p}{q}\right) = 1$, saying that $G_{d,p,q}$ is connected is equivalent to saying that $D_{p,q}$ generates $PSL_2(\mathbb{F}_q)$. Then a bipartition would imply a non-trivial group homomorphism $PSL_2(\mathbb{F}_q) \rightarrow \{-1,1\}$, whose kernel would be a proper normal subgroup of $PSL_2(\mathbb{F}_q)$, excluded since $PSL_2(\mathbb{F}_q)$ is simple [3, 3.2.2 Theorem].

To end this subsection, let us mention that all these Cayley graphs are connected (this is Proposition 2.15, in particular, $G_{d,p,q}$ is non-bipartite when $\left(\frac{p}{q}\right) = 1$ by the lemma just above). This point is important for estimating the girth, and is not trivial. In [11] the authors resort to a deep and technical result of Malyshev on the number of integer solutions of quadratic definite positive forms; the construction of Margulis [13] differs slightly from the one of [11], where a density argument (strong approximation theorem) was used. In our modified construction of graphs, the connectedness is also crucial, but none of these two proofs would work. Fortunately, later appeared in [3] (see discussion p. 6) a simple proof of the connectedness, based on the properties of the subgroups of $PSL_2(\mathbb{F}_q)$, observed by Frobenius. It will be instrumental in the present work.

### 2.3. Connectedness and final proof

Following the method of Ch. 4.3 in [3], this is achieved by showing logarithmic girth.

Let $X$ denote the connected component of $G_{d,p,q}$ containing the identity.

**Lemma 2.13.** Let $\mathcal{D}'(d)$ denotes the image of $\mathcal{D}(d) \subseteq \Lambda_\mathcal{D}$ through the group homomorphism: $\Lambda_\mathcal{D} \rightarrow \Lambda_\mathcal{D}/\Lambda_\mathcal{D}(q)$. The following isomorphism of graphs holds: $X \cong Cay(\Lambda_\mathcal{D}/\Lambda_\mathcal{D}(q), \mathcal{D}'(d))$. 

Proof. By definition of Cayley graphs $G_{d,p,q}$, we see that

$$X = \text{Cay}(\langle \mathcal{D}_{p,q} \rangle, \mathcal{D}_{p,q})$$

where $\langle \mathcal{D}_{p,q} \rangle$ denotes the subgroup of $PGL_2(\mathbb{F}_q)$ generated by $\mathcal{D}_{p,q}$. On the other hand, since $\mathcal{D}(d)$ generates $\Lambda_{\mathcal{G}}$, $\mathcal{D}'(d)$ generates $\Lambda_{\mathcal{G}} / \Lambda_{\mathcal{G}}(d)$. The embedding (11) shows that $\Lambda_{\mathcal{G}} / \Lambda_{\mathcal{G}}(q)$ is isomorphic to a subgroup of $PGL_2(\mathbb{F}_q)$, which is precisely $\langle \mathcal{D}_{p,q} \rangle$. This induces the graph isomorphism

$$\text{Cay}(\Lambda_{\mathcal{G}} / \Lambda_{\mathcal{G}}(d), \mathcal{D}'(d)) \simeq \text{Cay}(\langle \mathcal{D}_{p,q} \rangle, \mathcal{D}_{p,q})$$

concluding the proof. \hfill \Box

By vertex-transitivity of a Cayley graph on a group, the closed paths of length $\ell$ (starting and ending) at a vertex $x$ and the ones (starting and ending) at a vertex $y$ are in one-to-one correspondence. In particular a closed path of minimal length in the graph is found at each vertex, including the vertex 1. Thanks to Lemma 2.13, a closed path starting at the identity of $\Gamma$ has minimal length of all cycles, is an even.

Recall that $X$ is the connected component of $G_{d,p,q}$ containing 1. Its cardinality verifies $|X| \leq |PGL_2(\mathbb{F}_q)| = q^3 - q$, and even $|X| \leq |PSL_2(\mathbb{F}_q)| = \frac{1}{2}(q^3 - q)$ when $\left(\frac{p}{q}\right) = 1$. The definition (1.2) of $\kappa$ along with the above show that

$$\frac{2}{3} \log p |X| = \frac{2}{3 \kappa} \log d |X| \leq \frac{2}{3 \kappa} \log d q^3 \leq \text{girth}(X)$$

if $\left(\frac{p}{q}\right) = 1$. And similarly,

$$\frac{4}{3 \kappa} \log d |X| - \log p 4 \leq \text{girth}(X)$$

if $\left(\frac{p}{q}\right) = -1$. 

The computations that follow are classical. They already appeared in [11]. Note that $x = x_0 + x_1 i + x_2 j + x_3 k \in \Lambda_{\mathcal{G}}(q)$ implies that $q | x_i$ for $i = 1, 2, 3$. If we write $x_i = q y_i$, then $N(x) = x_0^2 + q^2 (y_1^2 + y_2^2 + y_3^2) = p^t$. At least one $y_i \neq 0$ among the values of $i = 1, 2, 3$, else $x \not\in \Lambda_{\mathcal{G}}$. Hence, $t \geq 2 \log_p q = \frac{2}{3} \log_q q^3$. 

In the case where $\left(\frac{p}{q}\right) = -1$, the graphs $G_{d,p,q}$ are bipartite by Lemma 2.12 and the girth, as is minimum length of all cycles, is an even. Hereafter, the girth is equal to $2t$. A basic refinement is possible in this case: as before, we get $p^{2t} = x_0^2 + q^2 (y_1^2 + y_2^2 + y_3^2)$, with at least one $y_i \neq 0$ among $y_1, y_2, y_3$. Hence, $p^{2t} \equiv x_0^2 \mod q^2$. This is equivalent to $p^t \equiv \pm x_0 \mod q^2$, the group $(\mathbb{Z}/q^2 \mathbb{Z})^*$ being cyclic. Therefore, $p^t = \pm x_0 + m q^2$ for a positive integer $m$. A simple calculation yields $2p^t - m q^2 > 0$, from which $t \geq \log_p q - \log_p 2$ follows. The girth in this case satisfies

$$\text{girth}(X) \geq \frac{4}{3} \log_p q^3 - 2 \log_p 2$$

Recall that $X$ is the connected component of $G_{d,p,q}$ containing 1. Its cardinality verifies $|X| \leq |PGL_2(\mathbb{F}_q)| = q^3 - q$, and even $|X| \leq |PSL_2(\mathbb{F}_q)| = \frac{1}{2}(q^3 - q)$ when $\left(\frac{p}{q}\right) = 1$. The definition (1.2) of $\kappa$ along with the above show that

$$\frac{2}{3} \log p |X| = \frac{2}{3 \kappa} \log d |X| \leq \frac{2}{3 \kappa} \log d q^3 \leq \text{girth}(X)$$

if $\left(\frac{p}{q}\right) = 1$. And similarly,

$$\frac{4}{3 \kappa} \log d |X| - \log p 4 \leq \text{girth}(X)$$

if $\left(\frac{p}{q}\right) = -1$. 
The graph $X$ has logarithmic girth. A trick that first appeared in [3, 3.3.4 Theorem] proves that it implies connectedness. We recall this theorem resulting from the properties of subgroups of $SL_2(F_q)$ due to Frobenius; a group is said to be metabelian if it admits a normal subgroup $N$ such that both $N$ and $H/N$ are abelian. It is easy to see that $H$ is metabelian if and only if for any four elements $h_1, h_2, h_3, h_4 \in H$ one has
\[ [[h_1, h_2], [h_3, h_4]] = 1, \quad \text{(where } [a, b] = aba^{-1}b^{-1}). \]

**Theorem 2.14 ([3], 3.3.4 Theorem).** Let $q$ be a prime. Let $H$ be a proper subgroup of $PSL_2(F_q)$, such that $|H| > 60$. Then $H$ is metabelian. \hfill \qed

Hence, to prove that $H = PSL_2(F_q)$, it suffices to prove that $|H| > 60$ and that $H$ is not metabelian.

**Proposition 2.15.** Since $d \geq 10$ and
\[ q > \max\{d^{8\kappa}, (120d)^\kappa\} = \max\{p^8, 120^\kappa p\}, \]
one has that the graph $G_{d,p,q}$ is connected.

**Proof.** It amounts to show that $X = G_{d,p,q}$. Thanks to Lemma 2.13, it suffices to show that the embedding (11) is onto, that is:
\[ \Lambda_q / \Lambda_q(q) \simeq \begin{cases} PSL_2(F_q) & \text{if } \left(\frac{p}{q}\right) = 1 \\ PGL_2(F_q) & \text{if } \left(\frac{p}{q}\right) = -1 \end{cases} \]
This is equivalent to show that $\overline{\mu}_q(\Lambda_q) = PSL_2(F_q)$ or $PGL_2(F_q)$. Since $PSL_2(F_q)$ is an index 2 normal subgroup of $PGL_2(F_q)$ and that $\overline{\mu}_q(\Lambda_q) \not\subset PSL_2(F_q)$ if $\left(\frac{p}{q}\right) = -1$, it suffices to show that $\overline{\mu}_q(\Lambda_q) \cap PSL_2(F_q) = PSL_2(F_q)$.

Let $L := \overline{\mu}_q(\Lambda_q) \cap PSL_2(F_q)$. First, we have $|L| > 60$. Indeed, by Equation (1) and the bound on the girth of $X$ obtained above,
\[ 2 \log_p q \leq \text{girth}(X) < 2 \log_d |X| + 2, \]
from which follows $\log_p q - 1 < \log_d |X|$, then $|X| > d^{\log_p q - 1} = p^{\frac{1}{\kappa}(\log_p q - 1)}$ and finally $|X| > \left(\frac{q}{p}\right)^{\frac{1}{\kappa}}$.

Next, $|X| \leq 2|L|$. The equality may occur if $G_{d,p,q}$ is connected, i.e., $X = G_{d,p,q}$, and if $\left(\frac{p}{q}\right) = -1$. It follows that $|L| > \frac{1}{2} \left(\frac{q}{p}\right)^{\frac{1}{\kappa}}$. Since, $q \geq 120^\kappa p$, this implies $|L| > 60$. 

The second step is to show that $L$ is not metabelian, that is there exist four elements $\ell_1, \ell_2, \ell_3, \ell_4$ in $L$ such that:

$$[[\ell_1, \ell_2], [\ell_3, \ell_4]] \neq 1.$$  

Let 4 elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $\mathcal{D}(d)$. The commutator $[[\alpha_1, \alpha_2], [\alpha_3, \alpha_4]]$ taken in the group $(\Lambda_\mathcal{G}, \times)$, yields an irreducible product of length smaller than 16. And it is equal to 16 if and only if $[[\alpha_1, \alpha_2], [\alpha_3, \alpha_4]]$ performed this time in $\mathbb{H}(\mathbb{Z})$ is primitive (that is no reduction occurred).

Suppose $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ verifies the latter. Let $\ell_i := \overline{m_q}(\alpha_i)\Lambda_\mathcal{G}(q) \in \Lambda_\mathcal{G}/\Lambda_\mathcal{G}(q)$. Then by construction of Cayley graphs, the commutator $[[\ell_1, \ell_2], [\ell_3, \ell_4]]$ yields a backtrackless path of length 16 in $X$. Beforehand, we have proved that girth($X$) $\geq 2\log_p q$ which is strictly greater than 16 considering that $q > p^8$. Hence, we have $[[\ell_1, \ell_2], [\ell_3, \ell_4]] \neq 1$ concluding the proof of (12), under the existence of the $\alpha_i$s in $\mathcal{D}(d)$.

It is actually always possible to find such $\alpha_i$s as soon as $|\mathcal{D}(d)| > 6$, as perfectly explained in the proof of [3] p. 120, paragraphs (a) and (b). This is the case since $d \geq 10$ by assumption.

Since $X = G_{d,p,q}$, it follows that girth($G_{p,d,q}$) $\geq 2\log_p q = \frac{2}{3\kappa} \log_d q^3 > \frac{2}{3\kappa} \log_d |G_{p,d,q}|$ if $\left(\frac{p}{q}\right) = 1$, and girth($G_{d,p,q}$) $\geq 4\log_p q - \log_p 4 > \frac{4}{3\kappa} \log_q |G_{d,p,q}| - \log_q 4$ if $\left(\frac{p}{q}\right) = -1$, achieving the proof of Main Inequality (4).

As for the non-bipartite $(d+1)$-regular graphs $H_n$ mentioned in Theorem 1.1, they correspond to the families $\mathcal{Y}_d$ of Definition 1.3. It has not be proved yet that they are not bipartite. Going back to the second point above Main Inequality (4), we must show that $G_{d,p,q}$ is non-bipartite when $\left(\frac{p}{q}\right) = 1$. It was not possible to prove it at the time of the proof of Lemma 2.12, because of the lack of knowledge of the connectedness. Granted by Proposition 2.15, this concludes the proof of Theorem 1.1.

**Concluding remarks**

**On the previous work.** By a simple modification made on the classical construction of Ramanujan graphs of [11], the lower bounds on the girth of regular graphs of degree $d \geq 10$ not a prime power were largely increased. Indeed, is obtained $\gamma_d \geq 1.06$ and even $\gamma_d \geq 1.33$ for larger values of $d$. This improves upon the 30 years old $\gamma_d \geq 0.48$ proved in [9], for $d \neq 2k + 1$. For $d = 2k + 1$, this improves upon the $\gamma_d \geq \frac{2}{3}$ of [14]. It even outperforms what the probabilistic method [5] is able to give, namely $\gamma_d \geq 1$. 
The construction of Imrich [9] is inspired by the previous work of Margulis [12]. The families that are built therein are derived from a mother graph, seen as a Cayley graph on a suitable free subgroup of $SL_2(\mathbb{Z})$. This prevents to use quaternions as done here and in [13,12,11], because the Hamilton quaternion algebra $\mathbb{H}(\mathbb{Q})$ is not split (no isomorphism with the 2-by-2 matrices). Thanks to quaternions, it is comparatively possible to do better. The lower bound obtained on the girth of the non-bipartite Cayley graphs $H_n$ on $PSL_2(\mathbb{F}_q)$ in Theorem 1.1, is $\geq 1.33 \cdot \log_d |H_n|$ for $d$ large enough. As already mentioned, this is better than for the Cayley graphs on $SL_2(\mathbb{F}_q)$ in [9], where the lower bound on the girth is worked out directly on matrices of $SL_2(\mathbb{Z})$ (see Proposition 4 of [2] for more details) and not on integral quaternions as done here.

**Expander graphs.** It should be mentioned that all the families of non-bipartite $(d + 1)$-regular graphs $\mathcal{Y}_{d,p}$ defined in (6) are expander families. This is due to their large girth property, for which the theorem of Bourgain & Gamburd [2, Theorem 3] holds. In particular, the non-bipartite graphs $G_{d,p,q}$ do not have a small chromatic number, but have a small diameter in the order of $O(\log |G_{d,p,q}|)$ (see [8, pp.455]).

**About possible generalizations.** In 1994, Morgenstern in [14] has extended the construction of families of $p + 1$-regular Ramanujan graphs by Lubotzky-Philips-Sarnak [11] and Margulis [13] coming with a construction of families of $p^k + 1$-regular graphs, $p$ any prime and $k \in \mathbb{N}^*$. The idea was to use quaternion algebras over function fields that are of class number equal to 1 (admit a unique factorization property similar to Theorem 2.2). Applying the technique developed in the present paper to those graphs raises the hope to improve further more the estimates on the girth: Indeed, given an integer $d$ the next prime power $p^k$ is always smaller than the next prime $p'$: $p^k \leq p'$ (remember that this “gap” plays an important role in the estimate of the girth). However, Dickson’s result do not hold directly for the group $PSL_2(\mathbb{F}_{p^k})$ and thus cannot guarantee the connectedness of the graphs as was done here. We have not tried, but even if connectedness can be obtained in some cases, we found out that the use of Morgenstern graphs may not be worth considering the tradeoff between simplicity and sharpness of the bounds, as explained below:

- for an even number $d$, to build a $(d + 1)$-regular tree was required some “involutions” in $\mathcal{P}(p)$, as explained in Remark 2.8. They were proved to exist only if $p \equiv 3 \mod 8$. There is no such involution in the similar special set of prime quaternions of Equality (9) of [14] (see Definitions 4.3 and 4.6 therein). Hence, to build a $(d + 1)$-regular tree we are led to consider
the prime $p=2$, and to choose $d+1$ elements in the set defined in Equality (18) and Definition 5.3 of [14] (indeed, by Corollary 5.7 they yield such involutions). But in this case, roughly because $\mathrm{PSL}_2(\mathbb{F}_{2^k})=\mathrm{PGL}_2(\mathbb{F}_{2^k})$, the Cayley graphs $\Gamma_g$ obtained are non-bipartite and only of girth $\geq \frac{2}{3} \log_q |\Gamma_g|$ (see Theorem 5.13). This does not compete with the girth of the graphs described in the present paper, even in the non-bipartite case.

- for an odd number $d$, the use of Morgenstern graphs could make sense if connectedness is proved, however, the values of $c(d)$ for $d$ odd shown in Theorem 1.1 are not too bad, becoming close to the upper limit $\frac{4}{3}$ rather quickly.

- the use of the construction of Morgenstern would induce a jump in technicality, with additional new results to address the problem of connectedness, and without a significant strengthening of the results, as shown by the two previous points.

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Xavier Dahan

*ISEE*

*Department of Informatics*

*Kyushu University, Japan*

xdahan@gmail.com