Limits of Ordered Graphs and Images

Omri Ben-Eliezer∗  Eldar Fischer†  Amit Levi‡  Yuichi Yoshida§

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Abstract

The emerging theory of graph limits exhibits an interesting analytic perspective on graphs, showing that many important concepts and tools in graph theory and its applications can be described naturally in analytic language. We extend the theory of graph limits to the ordered setting, presenting a limit object for dense vertex-ordered graphs, which we call an orderon. Images are an example of dense ordered bipartite graphs, where the rows and the columns constitute the vertices, and pixel colors are represented by row-column edges; thus, as a special case, we obtain a limit object for images.

Along the way, we devise an ordered locality-preserving variant of the cut distance between ordered graphs, showing that two graphs are close with respect to this distance if and only if they are similar in terms of their ordered subgraph frequencies. We show that the space of orderons is compact with respect to this distance notion, which is key to a successful analysis of combinatorial objects through their limits. For the proof we combine techniques used in the unordered setting with several new techniques specifically designed to overcome the challenges arising in the ordered setting. We derive several results related to sampling and property testing on ordered graphs and images; For example, we describe how one can use the analytic machinery to obtain a new proof of the ordered graph removal lemma [Alon et al., FOCS 2017].

∗Tel Aviv University, Israel. Email: omrib@mail.tau.ac.il.
†Technion - Israel Institute of Technology, Israel. Email: eldar@cs.technion.ac.il.
‡University of Waterloo, Canada. Email: amit.levi@uwaterloo.ca. Research supported by the David R. Cheriton Graduate Scholarship. Part of this work was done while the author was visiting the Technion.
§National Institute of Informatics (NII), Japan. Email: yyoshida@nii.ac.jp. Research supported by JSPS KAKENHI Grant Number JP17H04676.
1 Introduction

Large graphs appear in many applications across all scientific areas. Naturally, it is interesting to try to understand their structure and behavior: When can we say that two graphs are similar (even if they do not have the same size)? How can the convergence of graph sequences be defined? What properties of a large graph can we capture by taking a small sample from it?

The theory of graph limits addresses such questions from an analytic point of view. The investigation of convergent sequences of dense graphs was started to address three seemingly unrelated questions asked in different fields: statistical physics, theory of networks and the Internet, and quasi-randomness. A comprehensive series of papers [BCL+06a, BCL+06b, LS06, FLS07, LS07, BCL+08, BCL10, BCL+12] laid the infrastructure for a rigorous study of the theory of dense graph limits, demonstrating various applications in many areas of mathematics and computer science. The book of Lovász on graph limits [Lov12] presents these results in a unified form.

A sequence \( \{G_n\}_{n=1}^\infty \) of finite graphs, whose number of vertices tends to infinity as \( n \to \infty \), is considered convergent\(^1\) if the frequency\(^2\) of any fixed graph \( F \) as a subgraph in \( G_n \) converges as \( n \to \infty \). The limit object of a convergent sequence of (unordered) graphs in the dense setting, called a graphon, is a measurable symmetric function \( W : [0,1]^2 \to [0,1] \), and it was proved in [LS06] that, indeed, for any convergent sequence \( \{G_n\} \) of graphs there exists a graphon serving as the limit of \( G_n \) in terms of subgraph frequencies. Apart from their role in the theory of graph limits, graphons are useful in probability theory, as they give rise to exchangeable random graph models; see e.g. [DJ08, OR15]. An analytic theory of convergence has been established for other types of discrete structures. These include sparse graphs, for which many different (and sometimes incomparable) notions of limits exist – see e.g. [BC17, BCG17] for two recent papers citing and discussing many of the works in this field; permutations, first developed in [HKM+13] and further investigated in several other works; partial orders [Jan11]; and high dimensional functions over finite fields [Yos16]. The limit theory of dense graphs has also been extended to hypergraphs, see [Zha15, ES12] and the references within.

In this work we extend the theory of dense graph limits to the ordered setting, establishing a limit theory for vertex-ordered graphs in the dense setting, and as a by-product, for images, which can be viewed as ordered graph-like structures that are inherently dense; see Subsection 1.4 for a discussion regarding images. An ordered graph is a symmetric function \( G : [n]^2 \to \{0,1\} \). \( G \) is simple if \( G(x,x) = 0 \) for any \( x \). A weighted ordered graph is a symmetric function \( F : [n]^2 \to \{0,1\} \). Unlike the unordered setting, where \( G, G' : [n]^2 \to \Sigma \) are considered isomorphic if there is a permutation \( \pi \) over \( [n] \) so that \( G(x,y) = G'\pi(x), \pi(y) \) for any \( x \neq y \in [n] \), in the ordered setting, the automorphism group of a graph \( G \) is trivial: \( G \) is only isomorphic to itself through the identity function.

For simplicity, we consider in the following only graphs (without edge colors). All results here can be generalized in a relatively straightforward manner to edge-colored graph-like ordered structures, in which pairs of vertices may have one of \( r \geq 2 \) colors (the definition above corresponds to the case \( r = 2 \)). This is done by replacing the range \( [0,1] \) with the \( (r-1) \)-dimensional simplex (which corresponds to the set of all possible distributions over \( [r] \)). As we shall see in Subsection 1.2, the

\(^1\)In unordered graphs, this is also called convergence from the left; see the discussion on [BCL+08].

\(^2\)The frequency of \( F \) in \( G \) is roughly defined as the ratio of induced subgraphs of \( G \) isomorphic to \( F \) among all induced subgraphs of \( G \) on \( |F| \) vertices.
main results proved in this paper are, in a sense, natural extensions of results in the unordered setting. However, proving them requires machinery that is heavier than that used in the unordered setting: the tools used in the unordered setting are not rich enough to overcome the subtleties materializing in the ordered setting. In particular, the limit object we use in the ordered setting – which we call an orderon – has a 4-dimensional structure that is more complicated than the analogous 2-dimensional structure of the graphon, the limit object for the unordered setting. The tools required to establish the ordered theory are described next.

1.1 Main ingredients

Let us start by considering a simple yet elusive sequence of ordered graphs, which has the makings of convergence. The odd-clique ordered graph $H_n$ on $2n$ vertices is defined by setting $H_n(i,j) = 1$ – i.e., having an edge between vertices $i$ and $j$ – if and only if $i \neq j$ and $i,j$ are both odd, and otherwise setting $H_n(i,j) = 0$. In this subsection we closely inspect this sequence to demonstrate the challenges arising while trying to establish a theory for ordered graphs, and the solutions we propose for them. First, let us define the notions of subgraph frequency and convergence.

The (induced) frequency $t_F(G)$ of a simple ordered graph $F$ on $k$ vertices in an ordered graph $G$ with $n$ vertices is the probability that, if one picks $k$ vertices of $G$ uniformly and independently (repetitions are allowed) and reorders them as $x_1 \leq \cdots \leq x_k$, $F$ is isomorphic to the induced ordered subgraph of $G$ over $x_1, \ldots, x_k$. (The latter is defined as the ordered graph $H$ on $k$ vertices satisfying $H(i,j) = G(x_i, x_j)$ for any $i,j \in [k]$.) A sequence $\{G_n\}_{n=1}^\infty$ of ordered graphs is convergent if $|V(G_n)| \to \infty$ as $n \to \infty$, and the frequency $t(F,G_n)$ of any simple ordered graph $F$ converges as $n \to \infty$. Observe that the odd-clique sequence $\{H_n\}$ is indeed convergent: The frequency of the empty $k$-vertex graph in $H_n$ tends to $(k+1)2^{-k}$ as $n \to \infty$, the frequency of any non-empty $k$-vertex ordered graph containing only a clique and a (possibly empty) set of isolated vertices tends to $2^{-k}$, and the frequency of any other graph in $H_n$ is $0$.

In light of previous works on the unordered theory of convergence, we look for a limit object for ordered graphs that has the following features.

**Representation of finite ordered graphs** The limit object should have a natural and consistent representation for finite ordered graphs. As is the situation with graphons, we allow graphs $H$ and $G$ to have the same representation when one is a blowup\(^3\) of the other.

**Usable distance notion** Working directly with the definition of convergence in terms of subgraph frequencies is difficult. The limit object we seek should be endowed with a metric, like the cut distance for unordered graphs (see discussion below), that should be easier to work with and must have the following property: A sequence of ordered graph is convergent (in terms of frequencies) if and only if it is Cauchy in the metric.

**Completeness and compactness** The space of limit objects must be complete with respect to the metric: Cauchy sequences should converge in this metric space. Combined with the previous requirements, this will ensure that any convergent sequence of ordered graphs has a limit

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\(^3\)A graph $G$ on $nt$ vertices is an ordered $t$-blowup of $H$ on $n$ vertices if $G(x,y) = H([x/t], [y/t])$ for any $x$ and $y$. 

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(in terms of ordered frequencies), as desired. It is even better if the space is compact, as compactness is essentially an “ultimately strong” version of Szemerédi’s regularity lemma [Sze76], and will help to develop applications of the theory in other areas.

Additionally, we would like the limit object to be as simple as possible, without unnecessary over-representation. In the unordered setting, the metric used is the cut distance, introduced by Frieze and Kannan [FK96, FK99] and defined as follows. First, we define the cut norm \( \|W\|_\square \) of a function \( W: [0,1]^2 \to \mathbb{R} \) as the supremum of \( |\int_{S \times T} W(x,y)dx\,dy| \) over all measurable subsets \( S, T \subseteq [0,1] \). The cut distance between graphons \( W \) and \( W' \) is the infimum of \( \|W\phi - W'\|_\square \) over all measure-preserving bijections \( \phi: [0,1] \to [0,1] \), where \( W\phi(x,y) \overset{\text{def}}{=} W(\phi(x),\phi(y)) \).

For the ordered setting, we look for a similar metric; the cut distance itself does not suit us, as measure-preserving bijections do not preserve ordered subgraph frequencies in general. A first intuition is then to try graphons as the limit object, endowed with the metric \( d_\square(W,W') \overset{\text{def}}{=} \|W - W'\|_\square \). However, this metric does not satisfy the second requirement: the odd-clique sequence is convergent, yet it is not Cauchy in \( d_\square \), since \( d_\square(H_n,H_{2n}) = 1/2 \) for any \( n \). Seeing that \( d_\square \) seems “too strict” as a metric and does not capture the similarities between large odd-clique graphs well, it might make sense to use a slightly more “flexible” metric, which allows for measure-preserving bijections, as long as they do not move any of the points too far from its original location. In view of this, we define the cut-shift distance between two graphons \( W, W' \) as

\[
d_\Delta(W,W') \overset{\text{def}}{=} \inf_f \left( \text{Shift}(f) + \|Wf - W'f\|_\square \right),
\]

where \( f: [0,1] \to [0,1] \) is a measure-preserving bijection, \( \text{Shift}(f) = \sup_{x \in [0,1]} |f(x) - x| \), and \( Wf(x,y) = W(f(x),f(y)) \) for any \( x,y \in [0,1] \). As we show in this paper (Theorem 1.2 below), the cut-shift distance settles the second requirement: a sequence of ordered graphs is convergent if and only if it is Cauchy in the cut-shift distance.

Consider now graphons as a limit object, coupled with the cut-shift distance as a metric. Do graphons satisfy the third requirement? In particular, does there exist a graphon whose ordered subgraph frequencies are equal to the limit frequencies for the odd-clique sequence? The answers to both of these questions are negative: it can be shown that such a graphon cannot exist in view of Lebesgue’s density theorem, which states that there is no measurable subset of \( [0,1] \) whose density in every interval \( (a,b) \) is \((b - a)/2\) (see e.g. Theorem 2.5.1 in the book of Franks on Lebesgue measure [Fra09]). Thus, we need a somewhat richer ordered limit object that will allow us to “bypass” the consequences of Lebesgue’s density theorem. Consider for a moment the graphon representations of the odd clique graphs. In these graphons, the domain \( [0,1] \) can be partitioned into increasingly narrow intervals that alternately represent odd and even vertices. Intuitively, it seems that our limit object needs to be able to contain infinitesimal odd and even intervals at any given location, leading us to the following limit object candidate, which we call an orderon.

An orderon is a symmetric measurable function \( W: ([0,1]^2)^\square \to [0,1] \) viewed, intuitively and loosely speaking, as follows. In each point \( (x,a) \in [0,1]^2 \), corresponding to an infinitesimal “vertex” of the orderon, the first coordinate, \( x \), represents a location in the linear order of \([0,1] \). Each set \( \{x\} \times [0,1] \) can thus be viewed as an infinitesimal probability space of vertices that have the same location in the linear order. The role of the second coordinate is to allow “variability” (in terms of probability) of the infinitesimal “vertex” occupying this point in the order. The definition of the
frequency \( t(F, W) \) of a simple ordered graph \( F = ([k], E) \) in an orderon \( W \) is a natural extension of frequency in graphs. First, define the random variable \( G(k, W) \) as follows: Pick \( k \) points in \([0, 1]^2\) uniformly and independently, order them according to the first coordinate as \((x_1, a_1), \ldots, (x_k, a_k)\) with \( x_1 \leq \cdots \leq x_k \), and then return a \( k \)-vertex graph \( G \), in which the edge between each pair of vertices \( i \) and \( j \) exists with probability \( W((x_i, a_i), (x_j, a_j)) \), independently of other edges. The frequency \( t(F, W) \) is defined as the probability that the graph generated according to \( G(k, W) \) is isomorphic to \( F \).

Consider the orderon \( W \) satisfying \( W((x, a), (y, b)) = 1 \) if and only if \( a, b \leq 1/2 \), and otherwise \( W((x, a), (y, b)) = 0 \). \( W \) now emerges as a natural limit object for the odd-clique sequence: one can verify that the subgraph frequencies in it are as desired.

The cut-shift distance for orderons is defined similarly to (1), except that \( f \) is now a measure-preserving bijection from \([0, 1]^2\) to \([0, 1]^2\) and \( \text{Shift}(f) = \sup_{(x,a) \in [0,1]^2} |\pi_1(f(x,a)) - x| \), where \( \pi_1(y, b) \overset{\text{def}}{=} y \) is the projection to the first coordinate.

### 1.2 Main results

Let \( \mathcal{W} \) denote the space of orderons endowed with the cut-shift distance. In view of Lemma 2.8 below, \( d_\Delta \) is a pseudo-metric for \( \mathcal{W} \). By identifying \( W, U \in \mathcal{W} \) whenever \( d_\Delta(W, U) = 0 \), we get a metric space \( \tilde{\mathcal{W}} \). The following result is the main component for the viability of our limit object, settling the third requirement above.

**Theorem 1.1.** The space \( \tilde{\mathcal{W}} \) is compact with respect to \( d_\Delta \).

The proof of Theorem 1.1 is significantly more involved than the proof of its unordered analogue. While at a very high level, the roadmap of the proof is similar to that of the unordered one, our setting induces several new challenges, and to handle them we develop new shape approximation techniques. These are presented along the proof of the theorem in Section 4.

The next result shows that convergence in terms of frequencies is equivalent to being Cauchy in \( d_\Delta \). This settles the second requirement.

**Theorem 1.2.** Let \( \{W_n\}_{n=1}^\infty \) be a sequence of orderons. Then \( \{W_n\} \) is Cauchy in \( d_\Delta \) if and only if \( t(F, W_n) \) converges for any fixed simple ordered graph \( F \).

As a corollary of the last two results, we get the following.

**Corollary 1.3.** For every convergent sequence of ordered graphs \( \{G_n\}_{n \in \mathbb{N}} \), there exists an orderon \( W \in \mathcal{W} \) such that \( t(F, G_n) \to t(F, W) \) for every ordered graph \( F \).

The next main result is a sampling theorem, stating that a large enough sample from an orderon is almost always close to it in cut-shift distance. For this, we define the orderon representation \( W_G \) of an \( n \)-vertex ordered graph \( G \) by setting \( W_G((x, a), (y, b)) = G(Q_n(x), Q_n(y)) \) for any \( x, a, y, b \), where we define \( Q_n(x) = \lfloor nx \rfloor \) for \( x > 0 \) and \( Q_n(0) = 1 \). This addresses the first requirement.
**Theorem 1.4.** Let $k$ be a positive integer and let $W \in \mathcal{W}$ be an orderon. Let $G \sim G(k, W)$. Then,

$$d_\Delta(W, W_G) \leq C \left( \frac{\log \log k}{\log k} \right)^{1/3}$$

holds with probability at least $1 - C \exp(-\sqrt{k}/C)$ for some constant $C > 0$.

Theorem 1.4 implies, in particular, that ordered graphs are a dense subset in $\mathcal{W}$.

**Corollary 1.5.** For every orderon $W$ and every $\varepsilon > 0$, there exists a simple ordered graph $G$ on at most $2^{\varepsilon^{-3+o(1)}}$ vertices such that $d_\Delta(W, W_G) \leq \varepsilon$.

**Applications**

We finish by mentioning two applications of the ordered limit theory to illustrate the use of our theory. We provide a full proof for the first one and a detailed sketch for the other. The first application is concerned with naturally estimable ordered graph parameters, defined as follows.

**Definition 1.6 (Naturally Estimable Parameter).** An ordered graph parameter $f$ is *naturally estimable* if for every $\varepsilon > 0$ and $\delta > 0$ there is a positive integer $k = k(\varepsilon, \delta) > 0$ satisfying the following. If $G$ is an ordered graph with at least $k$ nodes and $G|_k$ is the subgraph induced by a uniformly random ordered set of exactly $k$ nodes of $G$, then

$$\Pr_{G|_k}[|f(G) - f(G|_k)| > \varepsilon] < \delta.$$

The following result provides an analytic characterization of ordered natural estimability.

**Theorem 1.7.** Let $f$ be a bounded simple ordered graph parameter. Then, the following are equivalent:

1. $f$ is naturally estimable.

2. For every convergent sequence $\{G_n\}_{n \in \mathbb{N}}$ of ordered simple graphs with $|V(G_n)| \to \infty$, the sequence of numbers $\{f(G_n)\}_{n \in \mathbb{N}}$ is convergent.

3. There exists a functional $\hat{f}(W)$ over $\mathcal{W}$ that satisfies the following:
   
   (a) $\hat{f}(W)$ is continuous with respect to $d_\Delta$.
   
   (b) For every $\varepsilon > 0$, there is $k = k(\varepsilon)$ such that for every ordered graph $G$ with $|V(G)| \geq k$, it holds that $\left|\hat{f}(W_G) - f(G)\right| \leq \varepsilon$.

Our second application is a new analytic proof of the ordered graph removal lemma of [ABEF17], implying that every hereditary property of ordered graphs (and images over a fixed alphabet) is testable, with one-sided error, using a constant number of queries. (For the relevant definitions, see [ABEF17] and Definition 1.6 here.)
Theorem 1.8 ([ABEF17]). Let $\mathcal{P}$ be an hereditary property of ordered graphs, and fix $\varepsilon, c > 0$. Then there exists $k = k(\mathcal{P}, \varepsilon, c)$ satisfying the following: For every ordered graph $G$ on $n \geq k$ vertices that is $\varepsilon$-far from $\mathcal{P}$, the probability that $G|_k$ does not satisfy $\mathcal{P}$ is at least $1 - c$.

The proof is rather long and involved (but somewhat cleaner than the combinatorial proof in [ABEF17]), and here we only provide a detailed sketch for it. The complete proof will appear in the full version of this paper, and will contain a proof (via Theorem 1.7) that the distance from any given hereditary property of ordered graphs is a naturally estimable graph parameter.

1.3 Related work

The theory of graph limits has strong ties to the area of property testing, especially in the dense setting. Regularity lemmas for graphs, starting with the well-known regularity lemma of Szeméredi [Sze76], later to be joined by the weaker (but more efficient) versions of Frieze and Kannan [FK96, FK99] and the stronger variants of Alon et al. [AFKS00], among others, have been very influential in the development of property testing. For example, regularity was used to establish the testability of all hereditary properties in graphs [AS08], the relationship between the testability and estimability of graph parameters [FN07], and combinatorial characterizations of testability [AFNS09]. The analytic theory of convergence, built using the cut distance and its relation to the weak regularity lemma, has proved to be an interesting alternative perspective on these results. Indeed, the aforementioned results have equivalent analytic formulations, in which both the statement and the proof seem cleaner and more natural. A recent line of work has shown that many of the classical results in property testing of dense graphs can be extended to dense ordered graph-like structures, including vertex-ordered graphs and images. In [ABEF17], it was shown that the testability of hereditary properties extends to the ordered setting (see Theorem 1.8 above). Shortly after, in [BEF18] it was proved that characterizations of testability in unordered graphs can be partially extended to similar characterizations in ordered graph-like structures, provided that the property at stake is sufficiently “well-behaved” in terms of order.

Graphons and their sparse analogues have various applications in different areas of mathematics, computer science, and even social sciences. The connections between graph limits and real-world large networks have been very actively investigated; see the survey of Borgs and Chayes [BC17]. Graph limits have applications in probability and data analysis [OR15]. Graphons were used to provide new analytic proofs of results in extremal graph theory; see Chapter 16 in [Lov12]. Through the notion of free energy, graphons were also shown to be closely connected to the field of statistical physics [BCL+12]. We refer the reader to [Lov12] for more details.

1.4 Limits of images

An interesting direction for future investigation is to establish a theory of convergence for images, suitable for practical applications. A two-dimensional image is one of the most widely investigated structures in computer science, being the main object of interest in computer vision. Nowadays, this field is largely dominated by deep learning based methods (see the recent survey [VDDP18]), that are usually very effective, but the mathematical theory behind them is not yet sufficiently established. The use of analytic models to represent images, like those naturally arising when studying theories
of convergence, might be an intriguing approach in which meaningful mathematical results can be proved.

The ordered limit theory presented here can be easily adapted to binary images, i.e., ordered bipartite graphs $I : [m] \times [n] \rightarrow \{0, 1\}$, as long as $m = \Theta(n)$. We believe that the results can be generalized to images with range $\Sigma = [0, 1]^r$ for fixed $r > 0$ (including greyscale images and RGB (red-green-blue) images, corresponding to the cases $r = 1$ and $r = 3$) through a suitable generalization of the relevant definitions, as was done for unordered graphs; see Chapter 5 of Lovász’s book [Lov12] for more details. The limit object for images can be viewed as a bipartite variant of an orderon: This is a measurable (and not necessarily symmetric) function $W : ([0, 1]^2)^2 \rightarrow \Sigma$, where sets of the form $\{x\} \times [0, 1]$ in the first and second coordinate of $W$ correspond to infinitesimal rows and columns, respectively. Convergence is in terms of (non-consecutive) sub-image frequencies, calculated by picking $s$ points in infinitesimal rows and $t$ points in infinitesimal columns uniformly at random, and inspecting the value of $W$ on their $s \times t$ intersection.

While our type of limit object is a natural extension of the unordered one and has applications in other areas, it seems that for practical applications in computer vision, one has to design more specifically tailored types of limit objects for images. Let us present two possible use-cases. First, the problem of understanding a continuous scene from a (possibly sparse) series of still images is one of the central tasks in computer vision, and when describing each of the images as an analytic object in some suitable space, the continuous scene should naturally correspond to a smooth curve in this space. To design a useful limit object for problems of this type, one has to consider analytic objects that are relatively robust against geometric transformations typically occurring due to movements of the camera and the objects in the scene, temporary occlusions of elements in the scene, slight changes in the amount of light, and other challenging phenomena typically occurring in continuous scenes. On the other hand, the analytic representation should be “sufficiently far” for images that do not look similar to the human eye. Another possible application is template matching, where the task is to find an approximate instance of a given pattern in an image, possibly rotated, shadowed, and partly occluded. Local sampling-based methods were shown to be useful for template matching [KRTA17], raising hopes that a local type of limit object might be helpful here. A recent line of works [BEKR17, BE19] studies local properties of images and pattern matching from the perspective of property testing, and might be useful in the development of such a limit object.

2 Preliminaries

In this section we formally describe some of the basic ingredients of our theory, including the limit object – the orderon, and several distance notions including the cut-norm for orderons (both unordered and ordered variants are presented), and the cut-shift distance. We then show that the latter is a pseudo-metric for the space of orderons. This will later allow us to view the space of orderons as a metric space, by identifying orderons of cut-shift distance 0.

The measure used here is the Lebesgue measure, denoted by $\lambda$. We start with the formal definition of an orderon.

**Definition 2.1 (Orderon).** An orderon is a measurable function $W : ([0, 1]^2)^2 \rightarrow [0, 1]$ that is symmetric in the sense that $W((x, a), (y, b)) = W((y, b), (x, a))$ for all $(x, a), (y, b) \in [0, 1]^2$. For the
sake of brevity, we also denote \( W((x, a), (y, b)) \) by \( W(v_1, v_2) \) for \( v_1, v_2 \in [0, 1]^2 \).

We denote the set of all orderons by \( W \).

**Definition 2.2** (measure-preserving bijection). A map \( g : [0, 1]^2 \to [0, 1]^2 \) is measure preserving if the pre-image \( g^{-1}(X) \) is measurable for every measurable set \( X \) and \( \lambda(g^{-1}(X)) = \lambda(X) \). A measure preserving bijection is a measure preserving map whose inverse map exists (and is also measure preserving).

Let \( F \) denote the collection of all measure preserving bijections from \([0, 1]^2 \) to itself. Given an orderon \( W \in W \) and \( f \in F \), we define \( Wf \) as the unique orderon satisfying \( Wf((x, a), (y, b)) = W(f(x, a), f(y, b)) \) for any \( x, a, y, b \in [0, 1] \). Additionally, denote by \( \pi_1 : [0, 1]^2 \to [0, 1] \) the projection to the first coordinate, that is, \( \pi_1(x, a) = x \) for any \( (x, a) \in [0, 1]^2 \).

### 2.1 Cut-norm and ordered cut-norm

The definition of the (unordered) cut-norm for orderons is analogous to the corresponding definition for graphons.

**Definition 2.3** (cut-norm). Given a symmetric measurable function \( W : ([0, 1]^2)^2 \to \mathbb{R} \), we define the cut-norm of \( W \) as

\[
\|W\|_\square = \sup_{S,T \subseteq [0,1]^2} \left| \int_{(x,a) \in S, (y,b) \in T} W((x, a), (y, b)) \, dx \, dy \, db \right|.
\]

As we are working with ordered objects, the following definition of ordered cut-norm will sometimes be of use (in particular, see Section 6). Given \( v_1, v_2 \in [0, 1]^2 \), we write \( v_1 \leq v_2 \) to denote that \( \pi_1(v_1) \leq \pi_1(v_2) \). Let \( 1_E \) be the indicator function for the event \( E \).

**Definition 2.4** (Ordered cut-norm). Let \( W : ([0, 1]^2)^2 \to \mathbb{R} \) be a symmetric measurable function. The ordered cut norm of \( W \) is defined as

\[
\|W\|_{\square'} = \sup_{S,T \subseteq [0,1]^2} \left| \int_{(v_1,v_2) \in S \times T} W(v_1, v_2) 1_{v_1 \leq v_2} \, dv_1 \, dv_2 \right|.
\]

We mention two important properties of the ordered-cut norm. The first is a standard smoothing lemma, and the second is a relation between the ordered cut-norm and the unordered cut-norm.

**Lemma 2.5.** Let \( W \in W \) and \( \mu, \nu : [0, 1]^2 \to [0, 1] \). Then,

\[
\left| \int_{v_1,v_2} \mu(v_1) \nu(v_2) W(v_1, v_2) 1_{v_1 \leq v_2} \, dv_1 \, dv_2 \right| \leq \|W\|_{\square'}.
\]

**Proof:** Fix partitions \( S = \{S_i\} \) and \( T = \{T_j\} \) of \([0, 1]^2 \). We show below that the claim holds when \( \mu \) and \( \nu \) are step functions on \( S \) and \( T \), respectively. Then, the proof is complete by the fact that all integrable functions are approximable in \( L_1(([0, 1]^2)^2) \) by step functions.
Since $\mu$ and $\nu$ are step functions, we can write $\mu = \sum_i a_i 1_{S_i}$ and $\nu = \sum_j b_j 1_{T_j}$ for some vectors $a \in [0, 1]^{|S|}$ and $b \in [0, 1]^{|T|}$. We define

$$f(a, b) \stackrel{\text{def}}{=} \int_{v_1, v_2} \mu(v_1) \nu(v_2) W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2.$$  

When $a \in \{0, 1\}^{|S|}$ and $b \in \{0, 1\}^{|T|}$, we have

$$|f(a, b)| = \left| \int \sum_i \sum_j a_i b_j 1_{S_i}(v_1) 1_{T_j}(v_2) W(v_1, v_2) dv_1 dv_2 \right| = \left| \int \bigcup_{i:a_i = 1} S_i \int \bigcup_{j:b_j = 1} T_j W(v_1, v_2) dv_1 dv_2 \right| \leq \|W\|_{\mathcal{C}^r},$$

where the last inequality follows from the definition of the ordered cut-norm. As $f(a, b)$ is bilinear in $a$ and $b$, and $|f(a, b)| \leq \|W\|_{\mathcal{C}^r}$ for any $a \in \{0, 1\}^{|S|}$ and $b \in \{0, 1\}^{|T|}$, we have $|f(a, b)| \leq \|W\|_{\mathcal{C}^r}$ for any $a \in \{0, 1\}^{|S|}$ and $b \in \{0, 1\}^{|T|}$.

**Lemma 2.6.** Let $W : ([0, 1]^2) \rightarrow [-1, 1]$ be a symmetric measurable function. Then,

$$\frac{\|W\|_{\mathcal{C}^r}^2}{4} \leq \|W\|_{\mathcal{C}^0} \leq 2\|W\|_{\mathcal{C}^r}.$$

**Proof:** The inequality $\|W\|_{\mathcal{C}^0} \leq 2\|W\|_{\mathcal{C}^r}$ follows immediately from the fact that $W$ is symmetric. For the other inequality, let $\xi = \|W\|_{\mathcal{C}^r}$, fix $\gamma > 0$, and let $S, T \subseteq [0, 1]^2$ be a pair of sets satisfying

$$\left| \int_{S \times T} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right| \geq \xi - \gamma.$$

We partition $[0, 1]^2$ into strips $I = \{I_1, \ldots, I_{2/\xi}\}$, such that for every $j \in [2/\xi]$, $I_j = \left[\frac{j-1}{2}, \frac{j}{2}\right] \times [0, 1]$. For every $j \in [2/\xi]$, let $I^{(j)} = \bigcup_{i<j} I_i$ (where $I^{(1)} = \emptyset$). Then,

$$\xi - \gamma \leq \left| \int_{S \times T} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right| \leq \sum_{i \in [2/\xi]} \left| \int_{(S \cap I_i) \times (T \cap I_i)} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right| + \sum_{j \in [2/\xi]} \left| \int_{(S \cap I^{(j)}) \times (T \cap I_j)} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right|$$

Note that by the fact that $|W(v_1, v_2)| \leq 1$ for all $v_1, v_2 \in [0, 1]^2$,

$$\sum_{i \in [2/\xi]} \left| \int_{(S \cap I_i) \times (T \cap I_i)} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right| \leq \sum_{i \in [2/\xi]} \lambda(I_i \times I_i) \leq \xi/2,$$

and therefore,

$$\sum_{j \in [2/\xi]} \left| \int_{(S \cap I^{(j)}) \times (T \cap I_j)} W(v_1, v_2) 1_{v_1 \leq v_2} dv_1 dv_2 \right| \geq \xi/2 - \gamma.$$
On the other hand, the above implies that there exists \( j \in \lfloor 2/\xi \rfloor \) such that
\[
\left| \int_{(S \cap I^{<j}) \times (T \cap I_j)} W(v_1, v_2) \mathbf{1}_{v_1 \leq v_2} dv_1 dv_2 \right| \geq \xi^2/4 - \xi \gamma/2,
\]
Note that for every \((v_1, v_2) \in (S \cap I^{<j}) \times (T \cap I_j)\), we have that \( \mathbf{1}_{v_1 \leq v_2} = 1 \), and thus
\[
\|W\|_\square \geq \left| \int_{(S \cap I^{<j}) \times (T \cap I_j)} W(v_1, v_2) dv_1 dv_2 \right| \geq \frac{\xi^2}{4} - \frac{\gamma \xi}{2}.
\]
Since the choice of \( \gamma \) is arbitrary, the lemma follows.

\[\blacksquare\]

### 2.2 The cut and shift distance

The next notion of distance is a central building block in this work. It can be viewed as a locality preserving variant of the unordered cut distance, which accounts for order changes resulting from applying a measure preserving function.

**Definition 2.7.** Given two orderons \( W, U \in W \) we define the **CS-distance** (cut-norm+shift distance) as:
\[
d_{\triangle}(W, U) \overset{\text{def}}{=} \inf_{f \in \mathcal{F}} \left( \text{Shift}(f) + \|W - U^f\|_\square \right),
\]
where \( \text{Shift}(f) = \sup_{x,a \in [0,1]} |x - \pi_1(f(x,a))| \).

**Lemma 2.8.** \( d_{\triangle} \) is a pseudo-metric on the space of orderons.

**Proof:** First note that non-negativity follows trivially from the definition. In addition, it is easy to see that \( d_{\triangle}(W, W) = 0 \) for any orderon \( W \). For symmetry,
\[
d_{\triangle}(W, U) = \inf_{g \in \mathcal{F}} (\text{Shift}(g) + \|W - U^g\|_\square) = \inf_{g \in \mathcal{F}} (\text{Shift}(g^{-1}) + \|W - U^g\|_\square)
\]
\[
= \inf_{g^{-1} \in \mathcal{F}} (\text{Shift}(g^{-1}) + \|W^{-1} - U\|_\square) = \inf_{f \in \mathcal{F}} (\text{Shift}(f) + \|U - W^f\|_\square)
\]
\[
= d_{\triangle}(U, W).
\]

Where we used the fact that \( g \) is a measure preserving bijection and that \( \text{Shift}(g^{-1}) = \text{Shift}(g) \) for any \( g \in \mathcal{F} \).

Consider three orderons \( W, U, Z \). We now show that \( d_{\triangle}(W, U) \leq d_{\triangle}(W, Z) + d_{\triangle}(Z, U) \).
\[
d_{\triangle}(W, U) = \inf_{f, g \in \mathcal{F}} \left( \text{Shift}(g^{-1} \circ f) + \|W - Ug^{-1} \circ f\|_\square \right)
\]
\[
\leq \inf_{f, g \in \mathcal{F}} \left( \text{Shift}(f) + \text{Shift}(g^{-1}) + \|W^g - U^f\|_\square \right)
\]
\[
\leq \inf_{f \in \mathcal{F}} \left( \text{Shift}(f) + \|Z - U^f\|_\square \right) + \inf_{g \in \mathcal{F}} \left( \text{Shift}(g) + \|W^g - Z\|_\square \right)
\]
\[
= d_{\triangle}(W, Z) + d_{\triangle}(Z, U),
\]
where the first equality holds since \( g^{-1} \circ f \) is a measure preserving bijection, and the last inequality follows from the triangle inequality; note that \( \text{Shift}(g^{-1}) = \text{Shift}(g) \) for any \( g \in \mathcal{F} \).
3 Block orderons and their density in $W$

In this section we show that weighted ordered graphs are dense in the space of orderons coupled with the cut-shift distance. To start, we have to define the orderon representation of a weighted ordered graph, called a naive block orderon. A naive $n$-block orderon is defined as follows.

**Definition 3.1** (Naive block orderon). Let $m \in \mathbb{N}$ be an integer. For $z \in (0,1]$, we denote $Q_n(z) = \lfloor nz \rfloor$; we also set $Q_n(0) = 1$. An $m$-block naive orderon is a function $W: ([0,1]^2)^2 \to [0,1]$ that can be written as

$$W((x,a),(y,b)) = G(Q_n(x), Q_n(y)),$$

for some weighted ordered graph $G$ on $n$ vertices.

Following the above definition, we denote by $W_G$ the naive block orderon defined using $G$, and view $W_G$ as the orderon “representing” $G$ in $W$. Similarly to the unordered setting, this representation is slightly ambiguous (but this will not affect us). Indeed, it is not hard to verify that two weighted ordered graphs $F$ and $G$ satisfy $W_F = W_G$ if and only if both $F$ and $G$ are blowups of some weighted ordered graph $H$. Here, a weighted ordered graph $G$ on $nt$ vertices is a $t$-blowup of a weighted ordered graph $H$ on $n$ vertices if $G(x,y) = H([x/nt], [y/nt])$ for any $x, y \in [nt]$.

We call an orderon $U \in W$ a step function with at most $k$ steps if there is a partition $P = \{S_1, \ldots, S_k\}$ of $[0,1]^2$ such that $U$ is constant on every $S_i \times S_j$.

**Remark** (The name choices). The definition of a step function in the space of orderons is the natural extension of a step function in graphons. Note that a naive block orderon is a special case of a step function, where the steps $S_i$ are rectangular (this is why we call these “block orderons”). The “naive” prefix refers to the fact that we do not make use of the second coordinate in the partition.

For every $W \in W$ and every partition $P = \{S_1, \ldots, S_k\}$ of $[0,1]^2$ into measurable sets, let $W_P: ([0,1]^2)^2 \to [0,1]$ denote the step function obtained from $W$ by replacing its value at $((x,a),(y,b)) \in S_i \times S_j$ by the average of $W$ on $S_i \times S_j$. That is,

$$W_P((x,a),(y,b)) = \frac{1}{\lambda(S_i) \lambda(S_j)} \int_{S_i \times S_j} W((x',a'),(y',b')) dx' da' dy' db'.$$

Where $i$ and $j$ are the unique indices such that $(x,a) \in S_i$ and $(y,b) \in S_j$, respectively.

The next lemma is an extension of the regularity lemma to the setting of Hilbert spaces.

**Lemma 3.2** ([LS07] Lemma 4.1). Let $\{K_i\}_i$ be arbitrary non-empty subsets of a Hilbert space $H$. Then, for every $\varepsilon > 0$ and $f \in H$ there is an $m \leq \lceil 1/\varepsilon^2 \rceil$ and there are $f_i \in K_i$ ($1 \leq i \leq k$) and $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ such that for every $g \in K_{k+1}$

$$|\langle g, f - (\gamma_1 f_1 + \cdots + \gamma_k f_k) \rangle| \leq \varepsilon \|f\| \|g\|.$$

The next lemma is a direct consequence of Lemma 3.2.

**Lemma 3.3.** For every $W \in W$ and $\varepsilon > 0$ there is a step function $U \in W$ with at most $\lfloor 2^8/\varepsilon^2 \rfloor$ steps such that

$$\|W - U\|_\square \leq \varepsilon.$$
Proof: We apply Lemma 3.2 to the case where the Hilbert space is $L^2([0,1]^4)$, and each $K_i$ is the set of indicator functions of product sets $S \times S$, where $S \subseteq [0,1]^2$ is a measurable subset. Then for $f \in \mathcal{W}$, there is an $f' = \sum_{j=1}^k \gamma_j f_j$, which is a step function with at most $2^k$ steps. Therefore, we get a step function $U \in \mathcal{W}$ with at most $2^{[1/\varepsilon^2]}$ steps such that for every measurable set $S \subseteq [0,1]^2$

$$\left| \int_{v_1,v_2 \in S \times S} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 \right| \leq \varepsilon .$$

By the above and the fact that

$$\left| \int_{v_1,v_2 \in (S \cup T) \times (S \cup T)} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 \right| = \int_{v_1,v_2 \in S \times S} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 + 2 \cdot \int_{v_1,v_2 \in S \times T} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 + \int_{v_1,v_2 \in T \times T} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 \leq \varepsilon ,$$

we get that for any two measurable sets $S, T \subseteq [0,1]^2$,

$$\left| \int_{v_1,v_2 \in S \times T} (W(v_1,v_2) - U(v_1,v_2))dv_1dv_2 \right| \leq 2\varepsilon ,$$

which implies the lemma.

Similarly to the graphon case, the step function $U$ might not be a stepping of $W$. However, it can be shown that these stepplings are almost optimal.

Claim 3.4. Let $W \in \mathcal{W}$, let $U$ be a step function, and let $\mathcal{P}$ denote the partition of $[0,1]^2$ into the steps of $U$. Then $\|W - W_{\mathcal{P}}\| \leq 2\|W - U\|$.  

Proof: The proof follows from the fact that $U = U_{\mathcal{P}}$ and the fact that the stepping operator is contractive with respect to the cut norm. More explicitly,

$$\|W - W_{\mathcal{P}}\| \leq \|W - U\| + \|U - W_{\mathcal{P}}\| = \|W - U\| + \|U_{\mathcal{P}} - W_{\mathcal{P}}\| \leq 2\|W - U\| .$$

Using Lemma 3.3 and Claim 3.4 we can obtain the following lemma.

Lemma 3.5. For every function $W \in \mathcal{W}$ and every $\varepsilon > 0$, there is a partition $\mathcal{P}$ of $[0,1]^2$ into at most $2^{[32/\varepsilon^2]}$ sets with positive measure such that $\|W - W_{\mathcal{P}}\| \leq \varepsilon$.

Using the above lemma, we can impose stronger requirements on our partition. In particular, we can show that there exists a partition of $[0,1]^2$ to sets of the same measure. Such a partition is referred to as an equipartition. Also, we say that a partition $\mathcal{P}$ refines $\mathcal{P}'$, if $\mathcal{P}$ can be obtained from $\mathcal{P}'$ by splitting each $P_j \in \mathcal{P}'$ into a finite number of sets (up to sets of measure 0).

Lemma 3.6. Fix some $\varepsilon > 0$. Let $\mathcal{P}$ be an equipartition of $[0,1]^2$ into $k$ sets, and fix $q \geq 2k^2 \cdot 2^{162/\varepsilon^2}$ such that $k$ divides $q$. Then, for any $W \in \mathcal{W}$, there exists an equipartition $\mathcal{Q}$ that refines $\mathcal{P}$ with $q$ sets, such that $\|W - W_{\mathcal{Q}}\| \leq \frac{8\varepsilon}{q} + \frac{2}{k}$.
**Theorem 3.8.** For every orderon $W \in \mathcal{W}$ and every $\varepsilon > 0$, there exist a naive $\frac{c}{\varepsilon} 2^{162/\varepsilon^2}$-block orderon $W'$ (for some constant $c > 0$) such that

$$d_\Delta(W, W') \leq \varepsilon.$$
Proof: Fix $\varepsilon > 0$ and $\gamma = \gamma(\varepsilon) > 0$. We consider an interval equipartition $J = \{J_1, \ldots, J_{1/\gamma}\}$ of $[0, 1]$ (namely, for each $j \in [\frac{1}{\gamma} - 1]$, $J_j = [(j - 1) \cdot \gamma, j \cdot \gamma)$, and for $j = 1/\gamma$, $J_j = [(j - 1) \cdot \gamma, j \cdot \gamma))$. In addition, let $\mathcal{P} = (J_i \times J_j | i, j \in [1/\gamma])$ be an equipartition of $[0, 1]^2$. By Lemma 3.6, there exists an equipartition $\mathcal{Q}$ of $[0, 1]^2$ of size $q = \frac{2}{\gamma^2} 102/\varepsilon^2$ that refines $\mathcal{P}$, such that

$$\|W - W_\mathcal{Q}\| \leq \frac{8\varepsilon}{9} + 2\gamma^2.$$

Next we construct a small shift measure preserving function $f$ as follows. For every $i \in [1/\gamma]$, consider the collection of sets $\{Q^k_i | k \in [\gamma q]\}$ in $\mathcal{Q}$ such that $(J_i \times [0, 1]) \cap \mathcal{Q} = \{Q^k_i | k \in [\gamma q]\}$.

For each $k \in [\gamma q]$, the function $f$ maps $Q^k_i$ to a rectangular set

$$\left[\left(\frac{(i - 1)\gamma + \frac{(k - 1)}{q}}{q}, \frac{(i - 1)\gamma + \frac{k}{q}}{q}\right) \times [0, 1]\right].$$

Finally, for every $i, j \in [q]$ and every $(x, a), (y, b) \in Q_i \times Q_j$, we define $W'(f(x, a), f(y, b)) = W_\mathcal{Q}((x, a), (y, b))$.

Note that the resulting function $W'$ obeys the definition of a naive $q$-block orderon and $\text{Shift}(f) \leq \gamma$.

Therefore, setting $\gamma = \varepsilon/100$,

$$d_\Delta(W, W') \leq \gamma + \frac{8\varepsilon}{9} + 2\gamma^2 \leq \varepsilon/100 + 8\varepsilon/9 + 2\varepsilon^2/100^2 \leq \varepsilon.$$ 

\[\blacksquare\]

4 Compactness of the space of orderons

In this section we prove Theorem 1.1. We construct a metric space $\tilde{W}$ from $W$ with respect to $d_\Delta$, by identifying $W, U \in W$ with $d_\Delta(W, U) = 0$. Let $\tilde{W}$ be the image of $W$ under this identification. On $\tilde{W}$ the function $d_\Delta$ is a distance function.

We start with some definitions and notations. Let $(\Omega, \mathcal{M}, \lambda)$ be some probability space, $\mathcal{P}_\ell = \{P^{(\ell)}_i\}_i$ a partition of $\Omega$, and let $\beta(\mathcal{P}_\ell : \cdot): \mathcal{P}_\ell \rightarrow [0, 1]$ be a function. For $v \in \Omega$, we slightly abuse notation and write $\beta(\mathcal{P}_\ell : v)$ to denote $\beta(\mathcal{P}_\ell : i)$ for $v \in P^{(\ell)}_i$. With this notation, observe that for every $\ell$

$$\int_{v \in \Omega} \beta(\mathcal{P}_\ell : v)dv = \sum_{i \in [\mathcal{P}_\ell]} \lambda(P^{(\ell)}_i) \beta(\mathcal{P}_\ell : i).$$

(2)

The following two results serve as useful tools to prove convergence. The first result is known as the martingale convergence theorem, see e.g. Theorem A.12 in [Lov12]. The second result is an application of the martingale convergence theorem, useful for our purposes.
The next lemma states that for any order on $\ell$.

**Lemma 4.2.** Let $\{P_\ell\}_\ell$ be a sequence of partitions of $\Omega$ such that for every $\ell$, $P_{\ell+1}$ refines $P_\ell$. Assume that for every $\ell$ and $j \in [\|P_\ell\|]$, the functions $\beta(P_\ell : \cdot)$ satisfy

$$
\lambda^\ell(P_j^\ell) \beta(P_\ell : j) = \sum_{i \in [\|P_{\ell+1}\|]} \lambda^\ell(P_j^\ell \cap P_i^{\ell+1}) \beta(P_{\ell+1} : i).
$$

(3)

Then, there is a measurable function $\beta : \Omega \to [0,1]$ such that $\beta(v) = \lim_{\ell \to \infty} \beta(P_\ell : v)$ for almost all $v \in \Omega$.

**Proof:** Fix some $\ell \in \mathbb{N}$. Let $X$ be a uniformly distributed random variable in $\Omega$. Let $\psi_\ell : \Omega \to [\|P_\ell\|]$ be the function mapping each $v \in \Omega$ to its corresponding part in $P_\ell$ and let $Z_\ell = \beta(P_\ell : X)$. We now show that the sequence $(Z_1, Z_2, \ldots)$ is a martingale. That is, $E_{X \sim \Omega}[Z_{\ell+1} | Z_1, \ldots, Z_\ell] = Z_\ell$, for every $\ell \in \mathbb{N}$. Note that by the fact that $P_{\ell+1}$ refines $P_\ell$, $\psi_\ell(X)$ determines $\psi_i(X)$ for every $i < \ell$. By definition, the value $\beta(P_\ell : X)$ is completely determined by $\psi_\ell(X)$, and so it suffices to prove that $Z_\ell = E_{X \sim \Omega}[Z_{\ell+1} | \psi_\ell(X)]$. By the fact that for every $j \in [\|P_\ell\|]$ Equation (3) holds (and in particular holds for $\psi_\ell(X)$), we can conclude that the sequence $(Z_1, Z_2, \ldots)$ is a martingale.

Since $Z_\ell$ is bounded, we can invoke the martingale convergence theorem (Theorem 4.1) and conclude that $\lim_{\ell \to \infty} Z_\ell$ exists with probability 1. That is, $\beta(v) = \lim_{\ell \to \infty} \beta(P_\ell : v)$ exists for almost all $v \in \Omega$. ■

**Definition 4.3.** Fix some $d \in \mathbb{N}$ and define $I_d = \left\{ I_1^{(d)}, \ldots, I_2^d \right\}$ so that for every $t \in [2^d]$, $I_t^{(d)} = \left[ \frac{t-1}{2^d}, \frac{t}{2^d} \right] \times [0,1]$. We refer to this partition as the strip partition of order $d$.

The next lemma states that for any order on $W$ we can get a sequence of partitions $\{P_\ell\}_\ell$, with several properties that will be useful later on.

**Lemma 4.4.** For any order on $W \in \mathcal{W}$ and $\ell \in \mathbb{N}$, there is a sequence of partitions $\{P_\ell\}_\ell$ of $[0,1]^2$ with the following properties.

1. $P_\ell$ has $g(\ell)$ many sets (for some monotone increasing $g : \mathbb{N} \to \mathbb{N}$).
2. For every $\ell$, $\Gamma_\ell \overset{\text{def}}{=} \frac{g(\ell)}{g(\ell-1)} \in \mathbb{N}$.
3. For every $\ell' \geq \ell$, the partition $P_{\ell'}$ refines both $P_\ell$ and the strip partition $I_{\ell'}$. In particular, for every $j \in [g(\ell-1)]$,

$$
P_j^{\ell-1} = \bigcup_{j'=(j-1)\Gamma_\ell+1}^{j\Gamma_\ell} P_{j'}^\ell.
$$

4. $W_\ell = (W)_{P_\ell}$ satisfies $\|W - W_\ell\|_\square \leq \frac{4}{g(\ell-1)2^\ell}$.

**Proof:** We invoke Lemma 3.7 with the trivial partition $\{[0,1]^2\}$ and the strip partition $I_1$, to get a partition $P_{n,1}$ with $g(1)$ many sets such that $P_{n,1}$ refines $I_1$ and $\|W_n - W_{n,1}\|_\square \leq 1$. For $\ell > 1$, we
invoke Lemma 3.7 with \( I_\ell \) and \( P_{n,\ell - 1} \) to get a partition \( P_{n,\ell} \) of size \( g(\ell) = (g(\ell - 1) \cdot 2^\ell) \cdot 2^O(g(\ell - 1)^2) \) which refines both \( I_\ell \) and \( P_{n,\ell - 1} \) such that \( \|W_n - W_{n,\ell}\|_r \leq \frac{4}{g(\ell - 1)^2} \). In order to take care of divisibility, we add empty (zero measure) sets in order to satisfy items (2) and (3).

Consider a sequence of orderons \( \{W_n\}_{n \in \mathbb{N}} \). For every \( n \in \mathbb{N} \), we use Lemma 4.4 to construct a sequence of functions \( \{W_n,\ell\}_{\ell \in \mathbb{N}} \) such that \( \|W_n - W_{n,\ell}\|_r \) is small. For each \( \ell \), we would like to approximate the shape of the limit partition resulting from taking \( n \to \infty \). Inside each strip \( I^{(\ell)}_t \), we consider the relative measure of the intersection of each set contained in \( I^{(\ell)}_t \), with a finer strip partition \( I^{(\ell)}_t \).

**Definition 4.5** (Shape function). For fixed \( n \in \mathbb{N} \), let \( \{P_{n,\ell}\}_{\ell} \) be partitions of \([0,1]^2\) with the properties listed in Lemma 4.4. For every \( \ell > \ell' \) and \( I^{(\ell')}_t \in I^{(\ell)}_t \), we define \( \alpha^{(n,\ell)}_{j}(I^{(\ell')}_t : t') \) as \( 2^{\ell'} \cdot \lambda \left( P^{(n,\ell)}_j \cap I^{(\ell')}_t \right) \) to be the relative volume of the set \( P^{(n,\ell)}_j \) in \( I^{(\ell')}_t \).

For any \( \ell' \geq \ell \) and \( I^{(\ell')}_t \in I^{(\ell)}_t \), by the compactness of \([0,1]\), we can select a subsequence of \( \{W_n\}_{n \in \mathbb{N}} \) such that \( \alpha^{(n,\ell)}_{j}(I^{(\ell')}_t : t') \) converges for all \( j \in [g(\ell)] \) as \( n \to \infty \). Let

\[
\alpha^{(\ell)}_{j}(I^{(\ell')}_t : t') \overset{\text{def}}{=} \lim_{n \to \infty} \alpha^{(n,\ell)}_{j}(I^{(\ell')}_t : t').
\]

Next we define the limit density function.

**Definition 4.6** (Density function). For fixed \( n \in \mathbb{N} \), let \( \{P_{n,\ell}\}_{\ell} \) be partitions of \([0,1]^2\) with the properties listed in Lemma 4.4. We let \( \delta^{(n,\ell)}(P_{n,\ell} \times P_{n,\ell} : i,j) \) as \( W_{n,\ell}(x,a)\), \( (y,b) \) for \( (x,a) \in P_{i}^{(n,\ell)} \) and \( (y,b) \in P_{j}^{(n,\ell)} \).

By the compactness of \([0,1]\), we can select a subsequence of \( \{W_n\}_{n \in \mathbb{N}} \) such that \( \delta^{(n,\ell)}(P_{n,\ell} \times P_{n,\ell} : i,j) \) converge for all \( i,j \in [g(\ell)] \) as \( n \to \infty \). Let

\[
\delta^{(\ell)}(i,j) \overset{\text{def}}{=} \lim_{n \to \infty} \delta^{(n,\ell)}(P_{n,\ell} \times P_{n,\ell} : i,j).
\]

The following lemma states that by taking increasingly refined strip partitions \( I^{(\ell)}_t \), we obtain a limit shape function for each set contained in any strip of \( I^{(\ell)}_t \).

**Lemma 4.7.** For fixed \( \ell \) and \( j \in [g(\ell)] \), there is a measurable function \( \alpha^{(\ell)}_{j} : [0,1] \to [0,1] \) such that \( \alpha^{(\ell)}_{j}(x) = \lim_{\ell' \to \infty} \alpha^{(\ell')}_{j}(I^{(\ell')}_t : x) \) for almost all \( x \in [0,1] \).

**Proof:** Fix \( n,\ell \) and \( \ell' > \ell \). For every \( j \in [g(\ell)] \), by the definition of \( \alpha^{(n,\ell)}_{j}(I^{(\ell')}_t : t') \) and the strip partition \( I^{(\ell')}_t \)

\[
\lambda \left( I^{(\ell')}_t \right) \cdot \alpha^{(n,\ell)}_{j}(I^{(\ell')}_t : t') = \lambda \left( P^{(n,\ell)}_j \cap I^{(\ell')}_t \right) \forall t' \in [2^\ell].
\]

On the other hand, since \( I^{(\ell+1)}_t \) refines \( I^{(\ell)}_t \),

\[
\lambda \left( I^{(n,\ell)}_j \cap I^{(\ell')}_t \right) = \lambda \left( P^{(n,\ell)}_j \cap I^{(\ell')}_t \right) + \lambda \left( I^{(n,\ell)}_j \cap I^{(\ell' + 1)}_t \right)
\]

\[
= \lambda \left( I^{(\ell + 1)}_j \right) \cdot \alpha^{(n,\ell)}_{j}(I^{(\ell+1)}_t : 2t' - 1) + \lambda \left( I^{(\ell + 1)}_j \right) \cdot \alpha^{(n,\ell)}_{j}(I^{(\ell+1)}_t : 2t').
\]

The following lemma states that by taking increasingly refined strip partitions \( I^{(\ell)}_t \), we obtain a limit shape function for each set contained in any strip of \( I^{(\ell)}_t \).
Therefore, when \( n \to \infty \) we get that,
\[
\lambda \left( I_{t'}^{(e')} \right) \cdot \alpha_j^{(e)}(I_{t'} : t') = \lambda \left( I_{2t' - 1}^{(e'+1)} \right) \cdot \alpha_j^{(e)}(I_{e+1} : 2t' - 1) + \lambda \left( I_{2t' - 1}^{(e'+1)} \right) \cdot \alpha_j^{(e)}(I_{e+1} : 2t'),
\]
which is exactly the condition in Equation (3). By applying Lemma 4.2 with the sequence of strip partitions \( \{I_{t'}\}_t \) on \( \alpha_j^{(e)} \) the lemma follows.

The next lemma asserts that the limit shape functions behave consistently.

**Lemma 4.8.** For every \( \ell \) and \( j \in [g(\ell - 1)] \),
\[
\alpha_j^{(\ell-1)}(x) = \sum_{j'=(j-1)^+\Gamma_{\ell+1}}^{j\Gamma_{\ell}} \alpha_j^{(\ell)}(x),
\]
for almost all \( x \in [0,1] \).

**Proof:** Fix some \( n, \ell \) and \( \ell' > \ell \). By the additivity of the Lebesgue measure,
\[
\alpha_j^{(n,\ell-1)}(I_{t'} : x) = \sum_{j'=(j-1)^+\Gamma_{\ell+1}}^{j\Gamma_{\ell}} \alpha_j^{(n,\ell)}(I_{t'} : x) \quad \forall x \in [0,1].
\]

By the fact that for every \( j \in [g(\ell - 1)] \) and \( x \in [0,1] \) the sequence \( \{\alpha_j^{(n,\ell-1)}(I_{t'} : x)\}_n \) converges to \( \alpha_j^{(\ell-1)}(I_{t'} : x) \) as \( n \to \infty \), we get that
\[
\alpha_j^{(\ell-1)}(I_{t'} : x) = \sum_{j'=(j-1)^+\Gamma_{\ell+1}}^{j\Gamma_{\ell}} \alpha_j^{(\ell)}(I_{t'} : x) \quad \forall x \in [0,1].
\]

By applying Lemma 4.7 on each \( j' \in [g(\ell)] \), where \( \ell' \to \infty \), we get that
\[
\alpha_j^{(\ell-1)}(x) = \sum_{j'=(j-1)^+\Gamma_{\ell+1}}^{j\Gamma_{\ell}} \alpha_j^{(\ell)}(x),
\]
for almost all \( x \in [0,1] \).

Using the sequence of \( \{\alpha_j^{(\ell)}\}_j \) we define a limit partition \( \mathcal{A}_\ell = \{A_1^{(\ell)}, \ldots, A_{g(\ell)}^{(\ell)}\} \) of \([0,1]^2\) as follows.

**Definition 4.9** (Limit partition). For every \( \ell \in \mathbb{N} \), let \( \mathcal{A}_\ell = \{A_1^{(\ell)}, \ldots, A_{g(\ell)}^{(\ell)}\} \) be a partition of \([0,1]^2\) such that,
\[
A_j^{(\ell)} = \left\{(x, a) : \sum_{i<j} a_i^{(\ell)}(x) \leq a < \sum_{i\leq j} a_i^{(\ell)}(x) \right\} \quad \forall j \in [g(\ell)].
\]

**Lemma 4.10.** For any \( \ell \), the partition \( \mathcal{A}_\ell \) has the following properties

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1. $A_\ell$ refines the strip partition $\mathcal{I}_\ell$.

2. The partition $A_\ell$ refines $A_{\ell-1}$.

3. For every $j \in [g(\ell)]$, \( \lambda \left( A_j^{(\ell)} \right) = \lim_{n \to \infty} \lambda \left( P_j^{(n,\ell)} \right) \).

**Proof:** The first item follows by the fact that each $\alpha_j^{(\ell)}$ is non-zero inside only one strip.

By the definition of the sets $A_j^{(\ell)}$ and Lemma 4.8 it follows that for each $j \in [g(\ell - 1)]$,

\[ A_j^{(\ell)} \subset A_j^{(\ell-1)} \quad \text{for all} \quad (j-1) \cdot \Gamma_\ell + 1 \leq j' \leq j \cdot \Gamma_\ell, \]

and therefore,

\[ A_j^{(\ell-1)} = \bigcup_{j'=(j-1)\Gamma_\ell+1}^{j\Gamma_\ell} A_{j'}^{(\ell)}, \]

which shows the second item. To prove the third item of the lemma, note that for every $n, \ell$ and $\ell' > \ell$,

\[ \lim_{n \to \infty} \lambda \left( P_j^{(n,\ell)} \right) = \lim_{n \to \infty} \sum_{t' \in [2^{\ell'}]} 2^{-\ell'} \cdot \alpha_j^{(n,\ell)} (\mathcal{I}_{\ell'} : t') = \sum_{t' \in [2^{\ell'}]} 2^{-\ell'} \cdot \alpha_j^{(\ell)} (\mathcal{I}_{\ell'} : t') = \int_x \alpha_j^{(\ell)} (\mathcal{I}_{\ell'} : x) \, dx, \]

where the last equality follows from Equation (2). Finally, by taking $\ell' \to \infty$ and using Lemma 4.7, we get

\[ \lim_{n \to \infty} \lambda \left( P_j^{(n,\ell)} \right) = \int_x \alpha_j^{(\ell)} (x) \, dx = \lambda \left( A_j^{(\ell)} \right). \]

Using the definition of $\delta^{(\ell)}$ and $A_\ell$, we define a density function on the limit partition. For $(x,a) \in A_i^{(\ell)}$ and $(y,b) \in A_j^{(\ell)}$, let

\[ \delta (A_\ell \times A_\ell : (x,a), (y,b)) \overset{\text{def}}{=} \delta^{(\ell)}(i,j). \]

**Lemma 4.11.** For each $\ell \in \mathbb{N}$ and $i, j \in [g(\ell-1)]$,

\[ \sum_{i'=(i-1)\Gamma_\ell+1}^{i\Gamma_\ell} \sum_{j'=(j-1)\Gamma_\ell+1}^{j\Gamma_\ell} \lambda \left( A_{i'}^{(\ell)} \right) \cdot \lambda \left( A_{j'}^{(\ell)} \right) \delta \left( A_\ell \times A_\ell : i', j' \right) \]

\[ = \lambda \left( A_i^{(\ell-1)} \right) \cdot \lambda \left( A_j^{(\ell-1)} \right) \delta \left( A_{\ell-1} \times A_{\ell-1} : i, j \right). \]

**Proof:** Fix $n, \ell$ and $i, j \in [g(\ell-1)]$. By the definition of the partitions $\mathcal{P}_{n,\ell}$, $\mathcal{P}_{n,\ell-1}$ and the density functions $\delta^{(n,\ell)}$, $\delta^{(n,\ell-1)}$

\[ \sum_{i'=(i-1)\Gamma_\ell+1}^{i\Gamma_\ell} \sum_{j'=(j-1)\Gamma_\ell+1}^{j\Gamma_\ell} \lambda \left( P_{i'}^{(n,\ell)} \right) \cdot \lambda \left( P_{j'}^{(n,\ell)} \right) \delta^{(n,\ell)} (\mathcal{P}_{n,\ell} \times \mathcal{P}_{n,\ell} : i', j') \]

\[ = \lambda \left( P_i^{(n,\ell-1)} \right) \cdot \lambda \left( P_j^{(n,\ell-1)} \right) \delta \left( \mathcal{P}_{n,\ell-1} \times \mathcal{P}_{n,\ell-1} : i, j \right). \]
By taking the limit as \( n \to \infty \) and using the third item of Lemma 4.10,

\[
\sum_{i'=(i-1)\Gamma_{\ell}+1}^{i\Gamma_{\ell}} \sum_{j'=(j-1)\Gamma_{\ell}+1}^{j\Gamma_{\ell}} \lambda (A_{i'}^{(\ell)}) \cdot \lambda (A_{j'}^{(\ell)}) \delta (A_{\ell} \times A_{\ell} : i', j') = \lambda (A_{i}^{(\ell-1)}) \cdot \lambda (A_{j}^{(\ell-1)}) \delta (A_{\ell-1} \times A_{\ell-1} : i, j).
\]

The next Lemma asserts that the natural density function of the limit partition is measurable. It follows directly from the combination of Lemma 4.2 and Lemma 4.11.

**Lemma 4.12.** There exists a measurable function \( \delta : ([0,1]^2)^2 \to [0,1] \) such that \( \delta((x,a),(y,a)) = \lim_{\ell \to \infty} \delta(A_{\ell} \times A_{\ell} : (x,a),(y,a)) \) for almost all \((x,a),(y,b) \in ([0,1]^2)^2\).

Finally, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1:** We start by giving a high-level overview of the proof. Let \( \{W_n\}_{n \in \mathbb{N}} \) be a sequence of functions in \( \mathcal{W} \). We show that there exists a subsequence that has a limit in \( \mathcal{W} \).

For every \( n \in \mathbb{N} \), we use Lemma 4.4 to construct a sequence of functions \( \{W_{n,\ell}\}_{\ell} \) such that \( \|W_n - W_{n,\ell}\|_\square \leq \frac{4}{g(\ell)^{1/2}} \). Then, for every fixed \( \ell \in \mathbb{N} \), we find a subsequence of \( \{W_{n,\ell}\} \) such that their corresponding \( a_j^{(n,\ell)} \) and \( \delta^{(n,\ell)}(i,j) \) converge for all \( i,j \in [g(\ell)] \) as \( n \to \infty \). For every \( \ell \), we consider the partition \( A_{\ell} \) (which by Definition 4.9, is determined by \( \{a_j^{(\ell)}\}_{j} \) and \( \delta^{(\ell)} \)). Using \( A_{\ell} \) and \( \delta^{(\ell)} \), we can define the function \( U_{\ell} \), such that \( W_{n,\ell} \to U_{\ell} \) almost everywhere as \( n \to \infty \).

Given the sequence of functions \( \{U_{\ell}\}_{\ell} \), we use Lemma 4.12 to show that \( \{U_{\ell}\}_{\ell} \) converges to some \( U \) almost everywhere as \( \ell \to \infty \) (where \( U \) is defined according to the limit density function \( \delta \)). Finally we show that for any fixed \( \varepsilon > 0 \), there is \( n_0(\varepsilon) \) such that for any \( n > n_0(\varepsilon) \), \( d_\triangle(W_n, U) \leq \varepsilon \).

Fix some \( \varepsilon > 0 \) and \( \xi(\varepsilon) > 0 \) which will be determined later. Consider the sequence \( \{U_{\ell}\}_{\ell} \) which is defined by the partition \( A_{\ell} \) and the density function \( \delta^{(\ell)} \). By Lemma 4.12, the sequence \( \{U_{\ell}\}_{\ell} \) converges (as \( \ell \to \infty \)) almost everywhere to \( U \), which is defined by the limit density function \( \delta \). Therefore, we can find some \( \ell > 1/\xi \) such that \( \|U_{\ell} - U\|_1 \leq \xi \).

Fixing this \( \ell \), we show that there is \( n_0(\ell) \) such that \( d_\triangle(W_{n,\ell}, U_{\ell}) \leq 2^{-\ell} + 3\xi \) for all \( n > n_0(\ell) \). We shall do it in two steps by defining an interim function \( W'_{n,\ell} \) and using the triangle inequality.

Recall that the function \( W_{n,\ell} \) is defined according to the partition \( P_{n,\ell} \) and the density function \( \delta^{(n,\ell)} \). Let \( W'_{n,\ell} \) be the function defined according to the partition \( A_{\ell} \) and the density function \( \delta^{(n,\ell)} \). That is, for every \((x,a) \in A_{i}^{(\ell)} \) and \((y,b) \in A_{j}^{(\ell)} \), \( W'_{n,\ell}((x,a), (y,b)) \) \( \overset{\text{def}}{=} \delta^{(n,\ell)}(P_{n,\ell} \times P_{n,\ell} : i,j) \). By the third item of Lemma 4.10, for every \( j \in [g(\ell)] \), \( \lambda (A_{j}^{(\ell)}) = \lim_{n \to \infty} \lambda (P_{j}^{(n,\ell)}) \). Then, we can find \( n_0(\ell) \) such that for all \( n > n_0(\ell) \),

\[
\max \left( \lambda (A_{j}^{(\ell)}) , \lambda (P_{j}^{(n,\ell)}) \right) \leq \min \left( \lambda (A_{j}^{(\ell)}) , \lambda (P_{j}^{(n,\ell)}) \right) \leq \frac{\xi}{g(\ell)} \quad \forall j \in [g(\ell)].
\]

We define a measure preserving map \( f \) from \( W_{n,\ell} \) to \( W'_{n,\ell} \) as follows. For every strip \( I_{t}^{(\ell)} \in \mathcal{I}_{\ell} \), we consider all the sets \( \{P_{j}^{(n,\ell)} \} \) in \( P_{n,\ell} \) such that \( \bigcup_{j'=j_t} P_{j'}^{(n,\ell)} = I_{t}^{(\ell)} \). Similarly, consider all
the sets \( \{A^{(\ell)}_{j_1}, \ldots, A^{(\ell)}_{j_t}\} \) in \( \mathcal{A}_\ell \) such that \( \bigcup_{j' = j_t} A^{(\ell)}_{j'} = I^{(\ell)}_t \). For every \( j' \in \{j_1, \ldots, j_t\} \), we map an arbitrary subset \( S^{(n,\ell)}_{j'} \subseteq P^{(n,\ell)}_{j'} \) of measure \( \min \left( \lambda(A^{(\ell)}_{j'}), \lambda(P^{(n,\ell)}_{j'}) \right) \) to an arbitrary subset (with the same measure) of \( A^{(\ell)}_{j'} \). Next, we map \( I^{(\ell)}_t \setminus \bigcup_{j' = j_t} S^{(n,\ell)}_{j'} \) to \( I^{(\ell)}_t \setminus \bigcup_{j' = j_t} f(S^{(n,\ell)}_{j'}) \). Note that by (4) and the fact that \( W_{n,\ell} \) and \( W'_{n,\ell} \) have the same density function \( \delta^{(n,\ell)} \), the functions \( W_{n,\ell} \) and \( W'_{n,\ell} \) disagree on a set of measure at most \( 2\xi \). Note that for every \( I^{(\ell)}_t \in \mathcal{I}_\ell \), the function \( f \) maps sets from \( \mathcal{P}_{n,\ell} \) that are contained in \( I^{(\ell)}_t \) to sets in \( \mathcal{A}_\ell \) that are contained in \( I^{(\ell)}_t \), and thus, \( \text{Shift}(f) \leq 2^{-\ell} \). Therefore, for \( n > n'_0 \), we get that \( d_\Delta(W_{n,\ell}, W'_{n,\ell}) \leq 2^{-\ell} + 2\xi \), and the first step is complete.

In the second step we bound \( d_\Delta(W'_{n,\ell}, U_\ell) \). The two functions \( W'_{n,\ell} \) and \( U_\ell \) are defined on the same partition \( \mathcal{A}_\ell \), however, their values are determined by the density functions \( \delta^{(n,\ell)} \) and \( \delta^{(\ell)} \) respectively. By the fact that \( \delta^{(n,\ell)} \) converges to \( \delta^{(\ell)} \) (as \( n \to \infty \)), we can find \( n'_0(\ell) \) such that for all \( n > n'_0 \),

\[
\left| \delta^{(n,\ell)}(i, j) - \delta^{(\ell)}(i, j) \right| \leq \frac{\xi}{g(\ell)^2} \quad \forall i, j \in [g(\ell)].
\]

Thus, for every \( n > n'_0 \), it holds that \( d_\Delta(W'_{n,\ell}, U_\ell) \leq ||W'_{n,\ell} - U_\ell||_1 \leq \xi \). By choosing \( n_0 = \max(n'_0, n''_0) \) we get that

\[
d_\Delta(W_{n,\ell}, U_\ell) \leq d_\Delta(W_{n,\ell}, W'_{n,\ell}) + d_\Delta(W'_{n,\ell}, U_\ell) \leq 2^{-\ell} + 3\xi.
\]

By putting everything together we get that for every \( n > n_0 \)

\[
d_\Delta(W, U) \leq d_\Delta(W, W_{n,\ell}) + d_\Delta(W_{n,\ell}, U_\ell) + d_\Delta(U_\ell, U)
\leq ||W_{n,\ell} - U||_2 + d_\Delta(W_{n,\ell}, U_\ell) + ||U_\ell - U||_1
\leq O \left( \frac{1}{g(\ell - 1)^2} \right) + 2^{-\ell} + 3\xi + \xi.
\]

By our choice of \( \ell > 1/\xi \) we get that

\[
d_\Delta(W, U) \leq 6\xi.
\]

By choosing \( \xi = \varepsilon/6 \) the theorem follows. \( \blacksquare \)

### 5 Sampling theorem

In this section we prove Theorem 1.4. We start by defining two models of random graphs which are constructed using orderons.

**Definition 5.1** (Ordered \( W \)-random graphs). Given a function \( W \in \mathcal{W} \) and an integer \( n > 0 \), we generate an edge-weighted ordered random graph \( \mathbf{H}(n, W) \) and an ordered random graph \( \mathbf{G}(n, W) \), both on nodes \([n] \), as follows. We generate \( Z_1, \ldots, Z_n \) i.i.d random variables, distributed uniformly in \([0, 1]\), and let \( X_1 \leq \cdots \leq X_n \) be the result of sorting \( Z_1, \ldots, Z_n \). In addition, we generate \( Y_1, \ldots, Y_n \) i.i.d random variables, distributed uniformly in \([0, 1]\). Then, for all \( i, j \in [n] \), we set the edge weight of \((i, j)\) in \( \mathbf{H}(n, W) \) to be \( W((X_i, Y_i), (X_j, Y_j)) \). Also, for all \( i, j \in [n] \), we set \((i, j)\) to be an edge in \( \mathbf{G}(n, W) \) with probability \( W((X_i, Y_i), (X_j, Y_j)) \) independently.
The proof consists of two main parts. The first (and simpler) part states that large enough samples from a naive block orderon are typically close to it in cut-shift distance. The second and main part shows that samples from orderons that are close with respect to cut-shift distance are typically close as well. We start with the proof of the first part, regarding sampling from naive block orderons.

**Lemma 5.2.** Let $k$ be a positive integer and $W \in \mathcal{W}$ be a naive $m$-block orderon. For any $\varepsilon > 0$, we have

$$
\Pr \left[ d_\triangle(W, W_{G(k,W)}) > \frac{2m^{3/2}}{\sqrt{k}} + \frac{\varepsilon}{\sqrt{k}} \right] \leq \exp \left( -C\varepsilon^2 k \right).
$$

for some constant $C > 0$.

**Proof:** We first show that $d_\triangle(W, W_{H(k,W)})$ is small with high probability and then discuss how it derives a concentration bound for $d_\triangle(W, W_{G(k,W)})$.

First, we show that the expectation of $d_\triangle(W, W_{H(k,W)})$ is at most $2m^{3/2}/k$. Let $P = \{ P_i \mid i \in [m] \}$ be the block partition of $W$. That is, $P_i = [(i-1)/m, i/m]$ for every $i \in [m]$. Note that for any $i \in [m]$, $\lambda(P_i) = 1/m$. Let $Z_1, \ldots, Z_k$ be independent uniformly random variables in $[0, 1]$ used to construct $H(k,W)$. For every $i \in [m]$, let $A_i$ be the number of samples $Z_\ell$ falling into $P_i$. By the fact that the variables are uniform, for every $i \in [m]$,

$$
\mathbb{E}_{Z_1, \ldots, Z_k}[A_i] = \frac{k}{m} \quad \text{and} \quad \text{Var}[A_i] = \frac{1}{m} \left(1 - \frac{1}{m}\right) k < \frac{k}{m}.
$$

We construct a partition $P' = \{ P'_i \mid i \in [m] \}$ of $[0, 1]$ using the values $A_i$. For every $i \in [m]$, we define

$$
P'_i = \left[ \frac{\sum_{i' < i} A_{i'}}{k}, \frac{\sum_{i' \leq i} A_{i'}}{k} \right].
$$

We construct an orderon $W_{H(k,W)} \in \mathcal{W}$ so that the value of $W_{H(k,W)}$ on $(P'_i \times [0, 1]) \times (P'_j \times [0, 1])$ is the same as the value of $W$ on $(P_i \times [0, 1]) \times (P_j \times [0, 1])$. Therefore, $W_{H(k,W)}$ agrees with $W$ on a set

$$
Q = \bigcup_{i,j \in [m]} \left( (P_i \cap P'_i) \times [0, 1] \right) \times \left( (P_j \cap P'_j) \times [0, 1] \right).
$$

We note that $\lambda(P'_i) = A_i/k$ and

$$
\lambda\left( \bigcup_{i \in [m]} (P_i \cap P'_i) \right) \geq 1 - \sum_{i \in [m]} \left| \sum_{i' \leq i} \frac{A_{i'}}{k} - \frac{i}{m} \right| \geq 1 - \sum_{i \in [m]} \sum_{i' \leq i} \left| \frac{A_{i'}}{k} - \frac{1}{m} \right| \geq 1 - m \sum_{i \in [m]} \left| \frac{A_i}{k} - \frac{1}{m} \right|.
$$

Then,

$$
d_\triangle(W, W_{H(k,W)}) \leq \|W - W_{H(k,W)}\|_\triangle \leq 1 - \lambda(Q) = 1 - \left( 1 - m \sum_{i \in [m]} \left| \frac{A_i}{k} - \frac{1}{m} \right| \right)^2 \leq 2m \sum_{i \in [m]} \left| \frac{A_i}{k} - \frac{1}{m} \right| \leq 2m \left( m \sum_{i \in [m]} \left( \frac{1}{m} - \frac{A_i}{k} \right)^2 \right)^{1/2} = \left( \frac{4m^3}{k^2} \sum_{i \in [m]} \left( \frac{k}{m} - A_i \right)^2 \right)^{1/2}.
$$
Therefore, by taking expectation
\[
\mathbb{E} \left[ d_{\Delta}(W, W_{H(k,W)})^2 \right] \leq \frac{4m^3}{k^2} \sum_{i \in [m]} \text{Var}(A_i) < \frac{4m^3}{k},
\]
and \( \mathbb{E} \left[ d_{\Delta}(W, W_{H(k,W)}) \right] < 2m^{3/2}/\sqrt{k} \) holds by Jensen’s inequality.

By applying Azuma’s inequality (see [Lov12, Corollary A.15]), noting that a single change in \( Z_\ell \) changes the value of \( d_{\Delta}(W, W_{H(k,W)}) \) by at most \( O(1/k) \), we have for any \( \varepsilon > 0 \),
\[
\Pr \left[ d_{\Delta}(W, W_{H(k,W)}) > \frac{2m^{3/2}}{\sqrt{k}} + \frac{\varepsilon}{k} \right] \leq \exp \left( -C'\varepsilon^2 k \right). \tag{5}
\]
for some constant \( C' > 0 \).

For an edge-weighted ordered graph \( H \) on nodes \([k]\), we define \( G(H) \) to be the ordered graph obtained by, for all \( i, j \in [n] \), setting \((i, j)\) to be an edge in \( G(H) \) with probability being the weight of \((i, j)\) in \( H \) independently. By [Lov12, Lemma 10.11], we have for any edge-weighted ordered graph \( H \) and \( \varepsilon > 0 \)
\[
\Pr \left[ d_{\Delta}(W_{G(H)}, W_H) > \frac{\varepsilon}{\sqrt{k}} \right] \leq \Pr \left[ \|W_{G(H)} - W_H\|_\square > \frac{\varepsilon}{\sqrt{k}} \right] \leq \exp \left( -\varepsilon^2 k / 100 \right). \tag{6}
\]
The desired concentration bound is obtained by (5), (6) and a union bound.

Before proceeding to the next lemma, we first recall the notion of a coupling of distributions.

**Definition 5.3 (Couplings).** Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be distributions over domains \( \Omega_1 \) and \( \Omega_2 \), respectively. Then, a coupling of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) is a distribution \( \mathcal{D} \) over \( \Omega_1 \times \Omega_2 \) such that the marginal distributions of \( \mathcal{D} \) on \( \Omega_1 \) and \( \Omega_2 \) are the same distributions as \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), respectively.

Let \( W, W' \in \mathcal{W} \) be orderons. The following lemma says that the random ordered graphs \( G(k,W) \) and \( G(k,W') \) can be coupled (note that \( G(k,W) \) and \( G(k,W') \) can be regarded as distributions over ordered graphs of \( k \) vertices) so that, when \( d_{\Delta}(W,W') \) is small, \( d_{\Delta}(W_{G(k,W)}, W_{G(k,W')}) \) is also small with high probability.

Roughly speaking, the main idea is as follows. The “cut part” follows from results in the unordered setting, specifically Corollary 10.12 in [Lov12]. For the “shift part”, we show that there is a coupling between samples \( G \sim G(k,W) \) and samples \( G' \sim G(k,W') \), so that for most of the pairs \((G, G')\), one can turn \( G \) into \( G' \) by only making local reordering of vertices, where no vertex is moved more than \( O(k \cdot \text{Shift}(f)) \) steps away from its original location. This follows from the fact that if one takes a sample from an orderon, then the location of the \( i \)-th order statistic – the vertex that is the \( i \)-th smallest in the first coordinate – is typically close to its mean.

**Lemma 5.4.** Let \( W, W' \in \mathcal{W} \) be orderons and \( k \) be a positive integer. Then, the random ordered graphs \( G(k,W) \) and \( G(k,W') \) can be coupled so that
\[
d_{\Delta}(W_{G(k,W)}, W_{G(k,W')}) \leq 9d_{\Delta}(W,W') + \frac{10}{k^{1/4}}
\]
holds with probability at least \( 1 - k(\varepsilon/4)^{4kd_{\Delta}(W,W')} - 5\sqrt{k}/10 \).
Proof: For any $\delta > 0$, there exists $f \in \mathcal{F}$ such that $\text{Shift}(f) + \|W^f - W'\|_\square < d_\triangle(W, W') + \delta$. Here, we choose $\delta = d_\triangle(W, W')$ and fix $f$ for this choice. To define the desired coupling between $G \overset{\text{def}}{=} G(k, W)$ and $G' \overset{\text{def}}{=} G(k, W')$, we first define a coupling between $G$ and $G' \overset{\text{def}}{=} G(k, W^f)$ so that $d_\triangle(W, G')$ is small with high probability, and then define a coupling between $G'$ and $G'$ so that $d_\triangle(G', G')$ is small with high probability. We obtain the desired bound by chaining the couplings and a union bound.

Recall that, in the construction of $G(k, W)$, we used two sequences of independent random variables $Z = (Z_i)_{i \in [k]}$ and $Y = (Y_i)_{i \in [k]}$. To look at the construction more in detail, it is convenient to introduce another sequence of independent random variables $R = (R_{ij})_{i, j \in [k], i < j}$, where each $R_{ij}$ is uniform over $[0, 1]$. After defining $X = (X_i)_{i \in [k]}$ as in Definition 5.1, we obtain $G(k, W)$ by setting $(i, j)$ to be an edge if $W((X_i, Y_i), (X_j, Y_j)) \geq R_{ij}$ for each $i, j \in [k]$ with $i < j$. To make the dependence on these random variables more explicit, we write $G(Z, Y, R; W)$ to denote the ordered graph obtained from $W$ by using $Z$, $Y$, and $R$.

Let $(Z = (Z_i)_{i \in [k]}, Y = (Y_i)_{i \in [k]}, R = (R_{ij})_{i, j \in [k], i < j})$ be uniform over $[0, 1]^{k^2 + \binom{k}{2}}$. Then, we define $(Z' = (Z'_i)_{i \in [k]}, Y' = (Y'_i)_{i \in [k]}, R' = (R'_{ij})_{i, j \in [k], i < j})$ so that $(Z'_i, Y'_i) = f^{-1}(Z_i, Y_i)$ for every $i \in [k]$ and $R'_{ij} = R_{ij}$ for every $i, j \in [k]$ with $i < j$. Note that the marginal $(Z', Y', R')$ is uniform over $[0, 1]^{k^2 + \binom{k}{2}}$, and hence the distribution of $G(Z', Y', R'; W^f)$ is exactly same as that of $G(k, W^f)$. Now, we couple $G(Z, Y, R; W)$ with $G(Z', Y', R'; W^f)$.

We can naturally define a measure preserving function $g \in \mathcal{F}$ from $W_G$ to $W_G^f$ with $W_G^f = W_G$ as follows ($G$ and $G'$ are coupled as in the last paragraph). Let $\pi: [k] \to [k]$ be a permutation of $[k]$ such that $Z_{\pi^{-1}(1)} \leq Z_{\pi^{-1}(2)} \leq \cdots \leq Z_{\pi^{-1}(k)}$. Then, $\pi(i)$ is the position of $Z_i$ in this sorted sequence. Similarly, we define a permutation $\pi': [k] \to [k]$ using $Z'$. Then, we arbitrarily choose $g$ so that the part corresponding to $\pi(i)$ is mapped to the part corresponding to $\pi'(i)$, that is, $(\{g(v) \mid v \in [(\pi(i) - 1)/k, \pi(i)/k] \times [0, 1]\} = [(\pi'(i) - 1)/k, \pi'(i)/k] \times [0, 1])$ for every $i \in [k]$.

We now show a concentration bound for $\text{Shift}(g)$. For each $i \in [k]$, we consider a segment $A_i = [Z_i - 2d_\triangle(W, W'), Z_i + 2d_\triangle(W, W')]$ in a circular domain $[0, 1]$, where we identify $-x$ with $1 - x$. Letting $M$ be the maximum number of overlaps of the segments at a point $x$ over $x$ in the circular domain, we can upper bound $\text{Shift}(g)$ by $(M - 1)/k$ because $\text{Shift}(f) \leq 2d_\triangle(W, W')$ and the overlap of two segments may cause a shift of $1/k$ in $g$ to map the vertex corresponding to one segment to the vertex corresponding to the other. Let $\mu = 4kd_\triangle(W, W')$ be the average overlap at a point. As the segments $A_1, \ldots, A_k$ are independently and uniformly distributed, by [SV03, Theorem 3.1] (to apply this theorem, we considered the circular domain instead of the interval $[0, 1]$), we have

$$\Pr[M \geq 2\mu + 1] \leq k\left(\frac{e}{4}\right)^\mu.$$  \hspace{1cm} (7)

Hence, we have

$$\text{Shift}(g) \leq 8d_\triangle(W, W')$$

with probability at least $1 - k(e/4)^{4kd_\triangle(W, W')}$. 

Next, we couple $G^f$ with $G'$ by coupling $G(Z', Y', R'; W^f)$ with $G(Z', Y', R'; W')$. By [Lov12, Corollary 10.12], we have

$$\|W_{G^f} - W_G\|_\square = \|W_{G(Z', Y', R', W^f)} - W_{G(Z', Y', R', W')}\|_\square \leq \|W^f - W'\|_\square + \frac{10}{k^{1/4}}$$

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with probability at least $1 - 5e^{-\sqrt{k}/10}$. Note that we can apply the corollary because the sorting process according to $Z'$ during the constructions of $G^f$ and $G'$ does not affect the cut norm.

Now, we combine by chaining the couplings $(G, G')$ and $(G^f, G')$. By a union bound, we have

$$d_\Delta(W_G, W_{G'}) \leq \text{Shift}(g) + \|W_{G'} - W_G\|_{\square}$$

$$\leq 8d_\Delta(W, W') + \|W^f - W'\|_{\square} + \frac{10}{k^{1/4}} \leq 9d_\Delta(W, W') + \frac{10}{k^{1/4}}$$

with probability at least $1 - k(e/4)^{4k_d}(W; W') - 5e^{-\sqrt{k}/10}$.

The proof of Theorem 1.4 now follows easily from the above two lemmas and Theorem 3.8. Indeed, from Theorem 3.8 we know that any orderon $W$ has an arbitrarily close naive block orderon $W'$, and from Lemma 5.4 we conclude that the cut-shift distance between samples from $W$ and samples from $W'$ is typically not much larger than $d_\Delta(W, W')$. Finally, Lemma 5.2 implies that $W'$ is typically close in $d_\Delta$ to large samples taken from it.

**Proof of Theorem 1.4:** Let $W'$ be the naive $m$-block orderon obtained by applying Theorem 3.8 on $W$. Let $(G, G')$ be the coupling obtained by applying Lemma 5.4 on $W$ and $W'$. By the triangle inequality, we have

$$d_\Delta(W_G, W_{G'}) \leq d_\Delta(W, W') + d_\Delta(W', W_{G'}) + d_\Delta(W_{G'}, W_G).$$

By Lemmas 5.2 and 5.4 and the union bound, we have for any $\varepsilon' > 0$

$$d_\Delta(W, W') \leq d_\Delta(W, W') + \frac{2m^{3/2}}{k} + \varepsilon' + 9d_\Delta(W', W) + \frac{10}{k^{1/4}}$$

$$= 10d_\Delta(W, W') + \frac{2m^{3/2}}{k} + \varepsilon' + \frac{10}{k^{1/4}}$$

$$= 10\varepsilon + \frac{\varepsilon'^2}{\varepsilon k} + \frac{10}{k^{1/4}}$$

with probability at least $1 - k(e/4)^{4k} - 5e^{-\sqrt{k}/10} - \exp(C(\varepsilon')^2 k)$. By setting $\varepsilon = \Theta(\log \log k / \log k)^{1/3}$ and $\varepsilon' = \Theta(\sqrt{k} / \log k)$, we have the desired bound.

6 Subgraph statistics

For $k \in \mathbb{N}$, let $\Omega = (\{0, 1\}^2)^k$. We define $\Omega_o \subset \Omega$ such that $v \in \Omega_o$ if and only if $v$ is ordered according to the entries of the first coordinate in the tuple (in such case, we write $v_1 \leq \cdots \leq v_k$). Namely, $v = ((x_1, y_1), \ldots, (x_k, y_k)) \in \Omega_o$ if and only if for every $i \leq j$, $x_i \leq x_j$. In addition, given a set of pairs $E \subseteq \binom{[k]}{2}$ we let $E_\prec = \{(i, j) \in E \mid i < j\}$. Let us restate the definition of homomorphism density of an ordered graph in an orderon from Subsection 1.1 in a slightly different but equivalent form.

**Definition 6.1 (Homomorphism Density).** Let $F = ([k], E)$ be a simple ordered graph and let $W \in \mathcal{W}$ be an orderon. We define the (induced) homomorphism density of $F$ in $W$ as

$$t(F, W) \overset{\text{def}}{=} k! \cdot \int_{v \in \Omega_o} \prod_{(i, j) \in E_\prec} W(v_i, v_j) \cdot \prod_{(i, j) \in E_\prec} (1 - W(v_i, v_j)) \, dv,$$
or equivalently,

\[ t(F, W) = k! \cdot \int_{v \in \Omega} \left( \prod_{(i,j) \in E} W(v_i, v_j) \cdot \prod_{(i,j) \in E, i<j} (1 - W(v_i, v_j)) \cdot \prod_{i<j} 1_{v_i \leq v_j} \right) \, dv. \]

Recall the definition of \( t(F, G) \) where \( G \) is an ordered graph, presented in Subsection 1.1. Clearly, \( t(F, G) = t(F, W_G) \) always holds. Our first main result of the section proves the “only if” direction of Theorem 1.2, showing that if a sequence of orderons \( W_n \) is Cauchy in \( d_\Delta \), then it is convergent in terms of subgraph frequencies.

**Lemma 6.2.** Let \( W, U \in W \). Then, for every simple ordered graph \( F = ([k], E) \)

\[ |t(F, W) - t(F, U)| \leq 6k! \left( \frac{k}{2} \right) \sqrt{d_\Delta(W, U)}. \]

In order to prove the above, we introduce the following two lemmas. Lemma 6.3 considers pairs of orderons that are close in cut-norm, whereas Lemma 6.4 describes the effect of shifts.

**Lemma 6.3.** For any \( W, U \in W \) and every ordered graph \( F = ([k], E) \),

\[ |t(F, W) - t(F, U)| \leq 2k! \left( \frac{k}{2} \right) \sqrt{\|W - U\|_\square}. \]

**Proof:** Fix some arbitrary ordering on \( \binom{k}{2} \subset \). For every \( (i, j) \in \binom{k}{2} \subset \) we define the function

\[ \gamma_W(v_i, v_j) = \begin{cases} W(v_i, v_j) \cdot 1_{v_i \leq v_j}, & \text{if } (i, j) \in E(F) \\ (1 - W(v_i, v_j)) \cdot 1_{v_i \leq v_j}, & \text{if } (i, j) \notin E(F) \end{cases} \]

and define \( \gamma_U \) similarly.

\[ t(F, W) - t(F, U) = k! \int_{v \in \Omega} \left( \prod_{(i,j) \in \binom{k}{2} \subset } \gamma_W(v_i, v_j) - \prod_{(i,j) \in \binom{k}{2} \subset } \gamma_U(v_i, v_j) \right) \, dv \]

By identifying each \( e_r \in \binom{k}{2} \subset \) with \( (i_r, j_r) \), the integrand can be written as

\[ \sum_{s=1}^{k} \left( \prod_{r<s} \gamma_W(v_{i_s}, v_{j_s}) \cdot \prod_{r>s} \gamma_U(v_{i_s}, v_{j_s}) \cdot \left( \gamma_W(v_{i_s}, v_{j_s}) - \gamma_U(v_{i_s}, v_{j_s}) \right) \right). \]

To estimate the integral of a given term, we fix all variables except \( v_{i_s} \) and \( v_{j_s} \). Then, the integral is of the form

\[ \int_{v_{i_s}, v_{j_s}} g(v_{i_s})h(v_{j_s}) \left( \gamma_W(v_{i_s}, v_{j_s}) - \gamma_U(v_{i_s}, v_{j_s}) \right) dv_{i_s} dv_{j_s}, \]
where \( g, h : [0, 1]^2 \rightarrow [0, 1] \) are some functions. By applying Lemma 2.5, it suffices to provide an upper bound on
\[
\sup_{S,T \subseteq [0,1]^2} \left| \int_{S \times T} \left( \gamma_W(v_{i_s}, v_{j}) - \gamma_U(v_{i_s}, v_{j}) \right) dv_{i_s} dv_{j} \right|.
\]
By using Lemma 2.6, we get that
\[
\sup_{S,T \subseteq [0,1]^2} \left| \int_{S \times T} \left( \gamma_W(v_{i_s}, v_{j}) - \gamma_U(v_{i_s}, v_{j}) \right) dv_{i_s} dv_{j} \right| = \sup_{S,T \subseteq [0,1]^2} \left| \int_{S \times T} \left( W(v_{i_s}, v_{j}) - U(v_{i_s}, v_{j}) \right) 1_{v_{i_s} \leq v_{j}} dv_{i_s} dv_{j} \right| = \| W - U \|_{\infty} \leq 2 \sqrt{\| W - U \|}.
\]
By summing up over all \( \binom{k}{2} \) pairs of vertices, the lemma follows.

**Lemma 6.4.** Let \( U \in \mathcal{W} \) and let \( \phi : [0, 1]^2 \rightarrow [0, 1]^2 \) be a measure preserving function. Then, for every ordered graph \( F = ([k], E) \)
\[
\left| t(F, U) - t(F, U^\phi) \right| \leq 4k! \binom{k}{2} \cdot \text{Shift}(\phi).
\]

**Proof:** The proof is similar to Lemma 6.3. However, we shall slightly change notation. Let \( \gamma_W \) be defined as follows.
\[
\gamma_W(v_i, v_j) = \begin{cases} W(v_i, v_j), & \text{if } (i, j) \in E(F) \\ 1 - W(v_i, v_j), & \text{if } (i, j) \notin E(F). \end{cases}
\]
Then, by changing the integration variables in \( t(F, U^\phi) \) from \( v_i, v_j \) to \( \phi^{-1}(v_i), \phi^{-1}(v_j) \), we have
\[
t(F, U) - t(F, U^\phi) = k! \int_{\Omega} \left( \prod_{(i,j) \in \binom{[k]}{2}_<} \gamma_U(v_i, v_j) 1_{v_i \leq v_j} - \prod_{(i,j) \in \binom{[k]}{2}_<} \gamma_U(\phi^{-1}(v_i), \phi^{-1}(v_j)) 1_{\phi^{-1}(v_i) \leq \phi^{-1}(v_j)} \right) dv
\]
\[
= k! \int_{\Omega} \left( \prod_{(i,j) \in \binom{[k]}{2}_<} \gamma_U(v_i, v_j) 1_{v_i \leq v_j} - \prod_{(i,j) \in \binom{[k]}{2}_<} \gamma_U(v_i, v_j) 1_{\phi^{-1}(v_i) \leq \phi^{-1}(v_j)} \right) dv
\]
\[
= k! \int_{\Omega} \prod_{(i,j) \in \binom{[k]}{2}_<} \gamma_U(v_i, v_j) \left( \prod_{(i,j) \in \binom{[k]}{2}_<} 1_{v_i \leq v_j} - \prod_{(i,j) \in \binom{[k]}{2}_<} 1_{\phi^{-1}(v_i) \leq \phi^{-1}(v_j)} \right) dv.
\]
Hence,
\[
\left| t(F, U) - t(F, U^\phi) \right| \leq k! \int_{\Omega} \left( \prod_{(i,j) \in \binom{[k]}{2}_<} 1_{v_i \leq v_j} - \prod_{(i,j) \in \binom{[k]}{2}_<} 1_{\phi^{-1}(v_i) \leq \phi^{-1}(v_j)} \right) dv
\]
\[
= k! \int_{\Omega} \sum_{s=1}^{\binom{k}{2}} \left( \prod_{r<s} 1_{v_r \leq v_j} \right) \left( \prod_{r>s} 1_{\phi^{-1}(v_r) \leq \phi^{-1}(v_j)} \right) \left( 1_{v_s \leq v_j} - 1_{\phi^{-1}(v_s) \leq \phi^{-1}(v_j)} \right) dv.
\]
Similarly to Lemma 6.3, we fix all the variables except \( v_{i_s} \) and \( v_{j_s} \). Then, by using Lemma 2.5, it suffices to estimate
\[
\left| \int_{v_{i_s},v_{j_s}} \left( 1_{v_{i_s} \leq v_{j_s}} - 1_{\phi^{-1}(v_{i_s}) \leq \phi^{-1}(v_{j_s})} \right) dv_{i_s} dv_{j_s} \right| \leq \left| \int_{v_{i_s},v_{j_s}} \left( 1_{v_{i_s} \leq v_{j_s}} - 1_{\phi^{-1}(v_{i_s}) \leq \phi^{-1}(v_{j_s})} \right) dv_{i_s} dv_{j_s} \right|.
\]
Note that whenever the intersection between \([\pi_1(v_{i_s}), \pi_1(\phi^{-1}(v_{i_s}))]\) and \([\pi_1(v_{j_s}), \pi_1(\phi^{-1}(v_{j_s}))]\) is empty, the difference between the indicators is zero. Therefore,
\[
\left| \int_{v_{i_s},v_{j_s}} \left( 1_{v_{i_s} \leq v_{j_s}} - 1_{\phi^{-1}(v_{i_s}) \leq \phi^{-1}(v_{j_s})} \right) dv_{i_s} dv_{j_s} \right| \leq \int_{v_{i_s},v_{j_s}} 1_{\pi_1(v_{i_s}), \pi_1(\phi^{-1}(v_{i_s})) \cap \pi_1(v_{j_s}, \pi_1(\phi^{-1}(v_{j_s})) \neq \emptyset} dv_{i_s} dv_{j_s} \leq 4\text{Shift}(\phi). \tag{9}
\]
By summing up over all \( \binom{n}{2} \) pairs of vertices, the lemma follows.

Using the above two lemmas, it is straightforward to prove Lemma 6.2.

**Proof of Lemma 6.2:** For any \( \gamma > 0 \) let \( \phi : [0,1]^2 \to [0,1]^2 \) be the measure preserving function such that \( \text{Shift}(\phi) + \|W - U^\phi\|_\square \leq d_\square(W, U) + \gamma \). For this specific \( \phi \) we have that
\[
\|W - U^\phi\|_\square = \sup_{S, T \subseteq [0,1]^2} \left| \int_{S \times T} W(v_1, v_2) - U(\phi(v_1), \phi(v_2)) dv_1 dv_2 \right| \leq d_\square(W, U) + \gamma,
\]
and \( \text{Shift}(\phi) \leq d_\square(W, U) + \gamma \). Then, by assuming \( d_\square(W, U) > 0 \) (note that the case \( d_\square(W, U) = 0 \) is covered by considering what happens when \( d_\square(W, U) \to 0 \)), and using the triangle inequality combined with Lemma 6.3 and Lemma 6.4, we get
\[
|t(F, W) - t(F, U)| \leq |t(F, W) - t(F, U^\phi)| + |t(F, U^\phi) - t(F, U)| \leq 2k! \left( \begin{array}{c} k \end{array} \right) \sqrt{d_\square(W, U)} + k! \left( \begin{array}{c} k \end{array} \right) \frac{\gamma}{\sqrt{d_\square(W, U)}} + 4k! \left( \begin{array}{c} k \end{array} \right) (d_\square(W, U) + \gamma) \leq 6k! \left( \begin{array}{c} k \end{array} \right) \sqrt{d_\square(W, U)} + 4k! \left( \begin{array}{c} k \end{array} \right) \frac{\gamma}{\sqrt{d_\square(W, U)}} + 4k! \left( \begin{array}{c} k \end{array} \right) \sqrt{\gamma}.
\]
As the choice of \( \gamma \) is arbitrary, the lemma follows.

Next we prove a converse statement, showing that if all frequencies of \( k \)-vertex graphs in a pair of orderons \( W \) and \( U \) are very similar, than \( d_\square(W, U) \) is small. This establishes the “if” component of Theorem 1.2.

**Lemma 6.5.** Let \( k \in \mathbb{N} \) and \( W, U \in \mathcal{W} \). Assume that for every ordered graph \( F \) on \( k \) vertices,
\[
|t(F, W) - t(F, U)| \leq 2^{-k^2}.
\]
Then, \( d_\square(W, U) \leq 2C \left( \frac{\log \log k}{\log k} \right)^{1/3} \) for some constant \( C > 0 \).

**Proof:** We start by showing that if for some \( k \geq 2 \), the total variation distance between the distribution \( G(k, W) \) and \( G(k, U) \) is small then they are close in CS-distance.
Assume that for $U, W \in W$ and some $k \geq 2$ it holds that

$$d_{TV}(G(k, W), G(k, U)) < 1 - \exp\left(-\frac{k}{2 \log k}\right).$$

This assumption implies that there exists a joint distribution $(G(k, W), G(k, U))$ so that $G(k, W) = G(k, U)$ with probability larger than $\exp\left(-\frac{k}{2 \log k}\right)$. By Theorem 1.4, with probability at least $1 - C \exp(-\sqrt{k}/C)$, we have that $d_{\Delta}(W, G(k, W)) \leq C \left(\frac{\log \log k}{\log k}\right)^{1/3}$ for some constant $C > 0$.

Let $E_1, E_2, E_3$ denote the events that $G(k, W) = G(k, U)$, $d_{\Delta}(W, G(k, W)) \leq C \left(\frac{\log \log k}{\log k}\right)^{1/3}$ and $d_{\Delta}(U, G(k, U)) \leq C \left(\frac{\log \log k}{\log k}\right)^{1/3}$, respectively.

Therefore, by using a union bound, $\Pr[E_1 \lor E_2 \lor E_3] \leq 2C \exp(-\sqrt{k}/C) + \exp(-k/2 \log k) < 1$. Hence, there is a positive probability for all the three events to occur, implying that

$$d_{\Delta}(W, U) \leq d_{\Delta}(W, G(k, W)) + d_{\Delta}(U, G(k, U)) + d_{\Delta}(G(k, U), G(k, W)) \leq 2C \left(\frac{\log \log k}{\log k}\right)^{1/3}.$$

The lemma follows by noting that,

$$|t(F, W) - t(F, U)| = \left|\Pr_{G \sim G(k, W)}[G = F] - \Pr_{G \sim G(k, U)}[G = F]\right| \leq 2^{-k^2},$$

and hence,

$$d_{TV}(G(k, W), G(k, U)) = \sum_{F} \left|\Pr_{G \sim G(k, W)}[G = F] - \Pr_{G \sim G(k, U)}[G = F]\right| \leq 2^{\binom{k}{2}} \cdot 2^{-k^2} \leq 2^{-k} < 1 - \exp\left(-\frac{k}{2 \log k}\right).$$

\[\square\]

### 7 Parameter estimation

In this section we prove Theorem 1.7. Recall the definition of natural estimability from Subsection 1.2. First, observe that the random graph distributions $G|_k$ (no vertex repetitions) and $G(k, W_G)$ (allowing vertex repetitions) are very close in terms of variation distance.

**Observation 7.1.** Let $G$ be an ordered graph on $n$ vertices. For every fixed $k$ and a large enough $n$ the distribution $G|_k$ is arbitrarily close (in variation distance) to the distribution $G(k, W_G)$.

We now turn to the proof of the theorem. First we prove that natural estimability of $f$ implies convergence of $f(G_n)$ for any convergent $\{G_n\}$; then we prove that the latter condition implies the existence of a continuous functional on orderons that satisfies both items of the last condition of Theorem 1.7; and finally, we prove the other direction of both statements.
(1) $\implies$ (2): Let $\{G_n\}_{n \in \mathbb{N}}$ be a convergent sequence with $|V(G_n)| \to \infty$. Given $\varepsilon > 0$, let $k$ be such that for every ordered graph $G$ on at least $k$ nodes, $|f(G) - \mathbf{E}_{G|k}[f(G|k)]| \leq 2\varepsilon$ (this can be done by setting $\delta = \varepsilon/M$ where $M$ is an upper bound on the values $f$). By the fact that $\{G_n\}_{n \in \mathbb{N}}$ is convergent, $t(F, W_{G_n})$ tends to a limit for all ordered graphs $F$ on $k$ vertices, which by Observation 7.1 implies that $\lim_{n \to \infty} \Pr_{G_n|k}[G_n|k = F] = \lim_{n \to \infty} t(F, W_{G_n}) = t(F, W)$. Therefore,

$$r_k \overset{\text{def}}{=} \lim_{n \to \infty} \mathbf{E}_{G_n|k}[f(G_n|k)] = \sum_{F} \lim_{n \to \infty} \Pr_{G_n|k}[G_n|k = F] \cdot f(F) = \sum_{F} t(F, W) \cdot f(F).$$

Thus, for all sufficiently large $n$,

$$|f(G_n) - r_k| \leq |f(G_n) - \mathbf{E}_{G_n|k}[f(G_n|k)]| + \varepsilon \leq 3\varepsilon,$$

which implies that $\{f(G_n)\}_n$ is convergent.

(2) $\implies$ (3): For a sequence $\{G_n\}_{n \in \mathbb{N}}$ converging to $W$, let $\hat{f}(W) \overset{\text{def}}{=} \lim_{n \to \infty} f(G_n)$. Note that this quantity is well-defined: Given two ordered graph sequences $\{G_n\}_{n \in \mathbb{N}}$ and $\{H_n\}_{n \in \mathbb{N}}$ converging to $W$, we can construct a new sequence $\{S_n\}_{n \in \mathbb{N}}$ such that $S_{2n} = G_n$ and $S_{2n-1} = H_n$. By definition, the sequence $\{S_n\}_{n \in \mathbb{N}}$ also converges to $W$, and hence $\lim_{n \to \infty} f(H_n) = \lim_{n \to \infty} f(G_n) = \lim_{n \to \infty} f(S_n) = \hat{f}(W)$.

To prove (3a), assume that $\{W_n\}_{n \in \mathbb{N}} \in W$ converges to $W$. For every $n$, we can apply Theorem 1.4 and obtain a sequence $\{G_{n,k}\}_k$ such that $\lim_{k \to \infty} W_{G_{n,k}} = W_n$. In addition, we can pick a subsequence of $\{G_{n,k}\}_k$ such that

$$d_{\triangle}(W_n, W_{G_{n,k}}) \leq 2^{-k} \text{ and } \left| \hat{f}(W_n) - f(G_{n,k}) \right| \leq 2^{-k}.$$

Now, since for every $n$ the sequence $G_{n,k}$ converges uniformly to $W_n$ (as $k \to \infty$) and $W_n$ converges to $W$, we have that the diagonal sequence $G_{n,n}$ converges to $W$ as well. Therefore, by the fact that $\hat{f}$ is well defined, we have that $\lim_{n \to \infty} f(G_{n,n}) = \hat{f}(W)$. Therefore, for every $\varepsilon > 0$, we can find $N$ such that for all $n > N$, $\left| f(G_{n,n}) - \hat{f}(W) \right| \leq \varepsilon/2$. Then, for every $\varepsilon > 0$ and all $n \geq \max(N, \log(2/\varepsilon))$ we have

$$\left| \hat{f}(W_n) - \hat{f}(W) \right| \leq \left| \hat{f}(W_n) - f(G_{n,n}) \right| + \left| f(G_{n,n}) - \hat{f}(W) \right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which concludes the proof for (3a).

To prove (3b): Assume towards a contradiction that (3b) does not hold. Let $\{G_n\}_n$ be an ordered graph sequence with $|V(G_n)| \to \infty$ such that $|f(G_n) - \hat{f}(W_{G_n})| > \varepsilon$. For each $n$, consider the sequence $\{G^{(j)}_n\}_j$ where $G^{(j)}_n$ is the $j$ blow-up of $G_n$. Note that by the fact that for every $j$ we have that $W_{G^{(j)}_n} = W_{G_n}$, the sequence $\{G^{(j)}_n\}_j$ converges to $W_{G_n}$. Therefore, by the above construction of $\hat{f}$, $\{f(G^{(j)}_n)\}$ converges to $\hat{f}(W_{G_n})$. Thus, for each $n$ we can find $j_n \in \mathbb{N}$ such that $|f(G^{(j_n)}_n) - \hat{f}(W_{G_n})| \leq \varepsilon/2$. Combined with the triangle inequality, this implies that $|f(G_n) - f(G^{(j_n)}_n)| > \varepsilon/2$. By compactness we can assume that $\{G'_n\}_n$ has a subsequence $\{G''_n\}_n$ that converges to $W$. Construct a sequence $\{H_n\}_n$ such that $H_{2n-1} = G'_n$ and $H_{2n} = G''_n$. Note that the sequence
\{H_n\}_n converges to \(W\) as well (since \(d_\Delta(W_G, W_{G_n \cup j_n}) = 0\)), but \(\{f(H_n)\}_n\) does not converge, as it is not Cauchy, which is a contradiction to (2).

(3) \implies (2): Consider any convergent ordered graph sequence \(\{G_n\}_{n \in \mathbb{N}}\) such that \(|V(G_n)| \to \infty\), and let \(W \in \mathcal{W}\) be its limit. Then, \(d_\triangle(W_G, W) \to 0\), and by the continuity of \(\hat{f}\), we have that \(\hat{f}(W_G) - \hat{f}(W) \to 0\). Namely, for every \(\varepsilon > 0\), we can find \(N\) such that for all \(n \geq N\),\n
\[
|\hat{f}(W_G) - \hat{f}(W)| \leq \varepsilon/2.
\]

By assumption, we also have that for every \(\varepsilon > 0\) there is a \(k(\varepsilon)\) such that for all \(|G_n| \geq k\), \(|f(G_n) - \hat{f}(W_G)| \leq \varepsilon/2\). This implies that for a large enough \(n\), \(|f(G_n) - \hat{f}(W)| \leq \varepsilon\), concluding the proof.

(2) \implies (1): Assume towards a contradiction that (1) does not hold. Namely, that there are \(\varepsilon > 0\) and \(\delta > 0\) such that for all \(k\), there is \(G\) on at least \(k\) vertices such that \(|f(G) - f(G|_k)| > \varepsilon\) with probability at least \(\delta\). Suppose we have a sequence \(\{G_k\}_k\) where \(|V(G_k)| = n(k) \to \infty\), and \(|f(G_k) - f(G_k|_k)| > \varepsilon\) with probability at least \(\delta\). By the compactness of \(\mathcal{W}\), we can select a subsequence \(\{G'_k\}_k\) of \(\{G_k\}_k\) such that \(\{G'_k\}_k\) converges to some \(W \in \mathcal{W}\). Using Theorem 1.4 (along with a union bound on the confidence probabilities) and the assumption of (2), for every \(k\), let \(H_k = G'_k|_k\) be some specific induced subgraph such that \(d_\triangle(W_{G'_k}, W_{H_k}) = o_k(1)\) and \(|f(H_k) - f(G'_k)| > \varepsilon\). Note that by the triangle inequality, the sequence \(\{H_k\}_k\) converges to \(W\). Let \(\{S_\ell\}_\ell\) be the sequence where \(S_{2\ell} = H_\ell\) and \(S_{2\ell-1} = G'_\ell\). Since both \(\{H_\ell\}_\ell\) and \(\{G'_\ell\}_\ell\) converge to \(W\), the sequence \(\{S_\ell\}_\ell\) also converges to \(W\). However, the sequence \(\{f(S_\ell)\}_\ell\) does not converge (as it is not Cauchy), which is a contradiction to (2).

8 Testability of hereditary ordered graph properties

This section contains a relatively detailed sketch of the proof of Theorem 1.8; The full proof will be provided in the full version of this paper.

Let \(\mathcal{H}\) be a hereditary ordered graph property. Recall that a property is hereditary if it satisfies the following: if \(G \in \mathcal{H}\) then every induced subgraph of \(G\) (vertex repetitions are not allowed) is also in \(\mathcal{H}\).

Let \(\overline{\mathcal{H}}\) be the set of all functions \(W \in \mathcal{W}\) such that \(\Pr[G(k, W) \in \mathcal{H}] = 1\) for every \(k\). Our main technical lemma is the following.

**Lemma 8.1.** For every \(U \in \overline{\mathcal{H}}\) and every \(\varepsilon > 0\) there exists \(\delta(U, \varepsilon)\), so that if \(W\) is such that \(d_\Delta(W, U) \leq \delta(U, \varepsilon)\), then \(d_1(W, \overline{\mathcal{H}}) \leq \varepsilon\).

Note that the parameter \(\delta\) depends on the object \(U\). In order to eliminate this dependency, we first show that \(\overline{\mathcal{H}}\) is closed with respect to \(d_\Delta\), and then use a compactness argument to conclude the proof.

**Lemma 8.2.** The set \(\overline{\mathcal{H}}\) is closed in \(\mathcal{W}\) with respect to \(d_\Delta\).

**Proof:** Suppose that \(\{W_n\}_n \in \overline{\mathcal{H}}\) converges to \(W\) with respect to \(d_\Delta\). Then, for every ordered graph \(F\), \(t(F, W_n)\) converges to \(t(F, W)\) (as \(n \to \infty\)). Hence, if \(t(F, W_n) = 0\) for every \(F \notin \mathcal{H}\), then also \(t(F, W) = 0\), which implies that \(W \in \overline{\mathcal{H}}\). \(\blacksquare\)
The above lemma, together with the fact that \((\overline{W}, d_{\Delta})\) is compact, implies that \((\overline{F}, d_{\Delta})\) is compact, and we can prove the following.

**Lemma 8.3.** For every \(\varepsilon > 0\), there exists \(\delta(\varepsilon)\) such that if \(d_{\Delta}(W, \overline{H}) \leq \delta\), then \(d_1(W, \overline{H}) \leq \varepsilon\).

**Proof:** Fix \(\varepsilon > 0\). Lemma 8.1 implies that for every \(U \in \overline{H}\) there exists \(\delta(U, \varepsilon)\) such that if \(d_{\Delta}(W, U) \leq \delta(U, \varepsilon)\) then \(d_1(W, \overline{H}) \leq \varepsilon\).

Assume \(U' \in \overline{H}\) and \(d_{\Delta}(U', U) \leq \eta\). Clearly \(\delta(U', \varepsilon) \geq \delta(U, \varepsilon) - \eta\) by the triangle inequality. So for a fixed \(\varepsilon\) and every \(U\) there is a ball around it with a guaranteed lower bound of \(\delta(U, \varepsilon)/2\) on \(\delta(U', \varepsilon)\) for any \(U'\) in that ball. By compactness, we can cover \(\overline{H}\) with a finite subset of this set of balls, obtaining a positive universal lower bound on \(\delta(U, \varepsilon)\) for every \(U \in \overline{H}\).

Before proving Lemma 8.1, we describe how to derive from it the proof of Theorem 1.8. For this we need the following two lemmas.

**Lemma 8.4.** The limit of any convergent sequence of graphs from \(\mathcal{H}\) is in \(\overline{\mathcal{H}}\).

**Proof:** Let \(\{H_n\}_n\) be such a sequence. For any fixed \(k\), we know that any \(k\)-vertex induced subgraph of \(H_n\) (with no vertex repetitions) is in \(\mathcal{H}\), and using Observation 7.1 we get that for any \(F \notin \mathcal{H}\), \(t(F, H_n) \to 0\) as \(n \to \infty\). Thus, \(H_n\) converges to an order on \(W\) satisfying \(t(F, W) = 0\) for any \(F \notin \mathcal{H}\), as desired.

**Lemma 8.5.** For every \(\varepsilon > 0\) there is a finite number of graphs \(H \in \mathcal{H}\), such that \(d_1(W, \overline{H}) > \varepsilon\).

**Proof:** Assume there is a sequence of ordered graphs \(\{G_n\}_{n \in \mathbb{N}}\) in \(\mathcal{H}\) such that \(d_1(W, \overline{G_n}) \geq \varepsilon\). By compactness, we can assume that the sequence converges to some \(W\). By Lemma 8.4, it follows that \(W \in \overline{\mathcal{H}}\). Combined with Theorem 1.2, we get that \(d_{\Delta}(W, \overline{G_n})\) converges to zero. By combining this with Lemma 8.3, we get that \(d_1(W, \overline{G_n})\) tends to zero as well, which is a contradiction.

Let us now turn to the proof of Theorem 1.8. Fix \(\varepsilon, c > 0\). By Lemma 8.3, there exists \(\delta > 0\) (depending only on \(\varepsilon\), so that any order \(W\) with \(d_1(W, \overline{H}) \geq 2\varepsilon/3\) satisfies \(d_{\Delta}(W, \overline{H}) \geq \delta\).

Apply Lemma 8.5 with parameter \(\varepsilon/3\) and Let \(N_1\) denote the maximum number of vertices of a graph \(H \in \mathcal{H}\) with \(d_1(W, \overline{H}) > \varepsilon/3\). Suppose now that \(G\) is a graph on \(n > N_1\) vertices that is \(\varepsilon\)-far from \(\mathcal{H}\) (in Hamming distance). In other words, \(d_1(W, \overline{H}) \geq \varepsilon\) for any \(H \in \mathcal{H}\) on \(n\) vertices. However, for any such \(H\) we have \(d_1(W, \overline{H}) \leq \varepsilon/3\), and so \(d_1(W, \overline{H}) \geq 2\varepsilon/3\), implying that \(d_{\Delta}(W, \overline{H}) \geq \delta\).

By Lemma 8.4 and Theorem 1.1, there is only a finite number of graphs \(H \in \mathcal{H}\) for which \(d_{\Delta}(W, \overline{H}) > \delta/3\). Let \(N_2\) denote the maximum number of vertices in such a graph.

By Theorem 1.4 and Observation 7.1, there exists integers \(N_3 \geq s > \max\{N_1, N_2\}\) satisfying the following. For any graph \(G\) on \(n \geq N_3\) vertices, with probability at least \(1 - c\) it holds that \(d_{\Delta}(W, \overline{G}) \leq \delta/3\).

The proof of Theorem 1.8 now follows by taking \(k(\mathcal{H}, \varepsilon, c) = N_3\) in the statement of the theorem. Indeed, suppose that \(G\) on \(n \geq N_3\) vertices is \(\varepsilon\)-far from \(\mathcal{H}\) in Hamming distance. Since \(s > N_1\), we know that \(d_{\Delta}(W, \overline{G}) \geq \delta\). By our choice of \(s\) and \(N_3\), with probability at least \(1 - c\) we have \(d_{\Delta}(W, \overline{G}) \leq \delta/3\). If the latter holds, then \(d_{\Delta}(W, \overline{G}) \geq 2\delta/3\). However, since \(s > N_2\), we
know that any graph on $s$ vertices whose distance from $\overline{H}$ is bigger than $\delta/3$ does not satisfy $\mathcal{H}$. Thus, $G|_{s} \notin \mathcal{H}$ with probability at least $1 - c$, implying that $G|_{N_3} \notin \mathcal{H}$ with probability at least $1 - c$, as desired (we moved from $s$ to $N_3$ so that graphs with less than $N_3$ vertices will be read in their entirety by the test).

### 8.1 Proof sketch of Lemma 8.1

In this subsection we sketch our proof of Lemma 8.1. We need to show that for every $U \in \overline{H}$ and every $\varepsilon > 0$ there exists $\delta(U, \varepsilon) > 0$, so that if $W$ is such that $d_\Delta(W, U) \leq \delta(U, \varepsilon)$, then $d_1(W, \overline{H}) \leq \varepsilon$.

Fix $\varepsilon > 0$, and let $U \in \overline{H}$. We would like to modify $U$ slightly, so that the resulting object has the structure of a \textit{layered strip orderon}, defined as follows.

**Definition 8.6.** For $\ell \in \mathbb{N}$ define $I_\ell = \{I_1^\ell, \ldots, I_2^{\ell}\}$ where $I_j^\ell = [(j-1)/2^\ell, j/2^\ell] \times [0, 1]$ for any $j \in [2^\ell]$. A $2^\ell$-strip layered orderon is a step orderon whose steps refine $I_\ell$, and in addition, for each $j \in [2^\ell]$, the sets contained in $I_j^\ell$ partition it into rectangles of the form $[(j-1)/2^\ell, j/2^\ell] \times [a, b]$.

Since $U$ is a measurable function, it can be approximated in $L_1$ by a step function whose steps are rectangles, which can be viewed in turn as a layered block orderon.

**Lemma 8.7.** Let $W \in \widetilde{W}$ be an orderon. Then, there exist $\ell \in \mathbb{N}$ and a function $W^{R_\ell} \in \widetilde{W}$ which is a $2^\ell$-strip layered orderon satisfying $\|W - W^{R_\ell}\|_1 \leq \alpha_\ell(1)$.

Let us denote by $U_{R_\ell}$ the orderon resulting by applying the above lemma to $U$, where $R_\ell$ denotes the partition of $[0, 1]^2$ into the steps of $U_{R_\ell}$.

Now, we take a slight detour and consider $U_\phi$ for some measure-preserving bijection $\phi: [0, 1]^2 \to [0, 1]^2$. Consider the partition $R_\ell = \{P_1, \ldots, P_{|R_\ell|}\}$ and define the shifted partition $R_\ell^\phi$ as follows.

$$R_\ell^\phi = \{P_1^\phi, \ldots, P_{|R_\ell|}^\phi\}, \quad \text{where } P_i^\phi \overset{\text{def}}{=} \{\phi^{-1}(x, y) \in [0, 1]^2 \mid (x, y) \in P_i\} \quad \text{for any } i \in [|R_\ell|].$$

As $U_{R_\ell}$ is a good approximation of $U$ in $L_1$, we conclude that $U_{R_\ell}^\phi$ is a a good approximation of $U_\phi$ in $L_1$.

Noting that the values of $U_{R_\ell}^\phi$ are the same as the respective values in $U_{R_\ell}$, the following observation is immediate from the definition of the cut-norm.

**Observation 8.8.** Let $W \in \widetilde{W}$ be any orderon such that $\|W - U_\phi\|_\square \leq \gamma$. For every $i, j \in [|R_\ell|]$,

$$\left| \int_{v_1, v_2 \in P_i \times P_j} U(v_1, v_2) dv_1 dv_2 - \int_{v_1, v_2 \in P_i^\phi \times P_j^\phi} W(v_1, v_2) dv_1 dv_2 \right| \leq \gamma.$$

Assuming that Shift($\phi$) is small enough we can “trim” the area around the boundaries of the strips, and “stretch” (using a linear mapping) the area around the trimmed sections. By applying this procedure, we make sure that measure-preserving bijections do not cross strip boundaries. More precise details follow.
We would like to claim that there exists an orderon in $R$. Consider the partition $Q = \{Q_1, \ldots, Q_2|\mathcal{R}_\ell|\}$ of $[0,1]^2$ resulting from “trimming and stretching” $W$, where $Q_j = \{(x,a) \in [0,1]^2 \mid (\psi_{x,\eta}(x),a) \in P^\phi\}$ for each $j \in [||\mathcal{R}_\ell||]$. Note that this partition is an “adjustment” of the partition $\mathcal{R}_\ell^\phi$, to the orderon $W_{\ell,\eta}$ resulting from trimming. Two simple, but important observations:

**Observation 8.11.** If $\text{Shift}(\phi) \leq \ell$, then for every $j,t \in [||\mathcal{R}_\ell^\phi||] \times [2^\ell]$ such that $P^\phi_j$ is contained in $I^\ell_t$, the set $Q_j$ is also contained in $I^\ell_t$.

**Observation 8.12.** For an appropriate choice of $\eta$, for each $i \in [||\mathcal{R}_\ell||]$,

$$|\lambda(P^\phi_i) - \lambda(Q_i)| \leq O(1/\text{poly}(\ell)),$$

and

$$ \left| \int_{v_1,v_2 \in P^\phi_i \times P^\phi_j} W(v_1,v_2)dv_1dv_2 - \int_{v_1,v_2 \in Q_i \times Q_j} T(v_1,v_2)dv_1dv_2 \right| \leq O(1/\text{poly}(\ell)).$$

We would like to claim that there exists an orderon in $\mathcal{H}$ which has similar structure as $U_{\mathcal{R}_\ell}$ and also close to $U_{\mathcal{R}_\ell}$ in $L_1$. Later we show that this implies the existence of an orderon in $\mathcal{H}$ which is not too far from $T$.

For this we need to define a “uniform decision”.

**Definition 8.13** (Uniform Decision). For $\ell \in \mathbb{N}$, let $W \in \mathcal{W}$ be a $2^\ell$-strip layered orderon. Let $P = \{P_1, \ldots, P_{||P||}\}$ be the partition of $[0,1]^2$ according to the steps of $W$. Given a decision function $w: ||P||^2 \rightarrow \{0,1,*\}$, we define the implementation of $w_P$ on $W$, denoted $W_{=w_P}$, as follows. For every $u,v \in [0,1]^2$ such that $\pi_1(u) < \pi_1(v)$ and $u,v \in P_i \times P_j$

1. If $w_P(i,j) \neq *$, let $W_{=w_P}(u,v) = w_P(i,j)$.
2. If $w_P(i,j) = *$, let $W_{=w_P}(u,v) = W(u,v)$.


Lemma 8.14. Assume that $W$ is a $2^l$-strip layered orderon and let $P$ be the partition into its steps. In addition, let $U \in \mathcal{H}$ such that $d_1(W, U) = \varepsilon$. Then, there exists $U' \in \mathcal{H}$ which is a result of a uniform decision and $d_1(W, U') \leq \varepsilon + O(1/2^l)$.

This is the only place in our proof that uses non-analytic techniques. The existence of a decision function shall arise from the use of Ramsey theory. Specifically, we use a variant of the following “Ramsey theorem with quantified undesirable edges” from [ABEF17] (we actually use a weight function instead of a set of undesirables).

Theorem 8.15. [[See [ABEF17], Theorem 6.1] Let $G$ be a complete $k$-partite graph (possibly edge colored, with set of colors $\Sigma$), let $t$ be a positive integer, and let $\varepsilon, \alpha > 0$. Suppose that $G$ has $n \geq N(|\Sigma|, k, t, \alpha)$ vertices in each part, and that no more than an $\varepsilon$-fraction of the edges of $G$ belong to a set of undesirable edges, denoted $B$. Then there exists a $k$-partite induced subgraph $R$ of $G$ satisfying the following conditions.

- Each part of $R$ contains exactly $t$ vertices.
- For any fixed pair $V_1, V_2$ of parts in $R$, all edge colors between $V_1$ and $V_2$ are equal.
- At most a $(1 + \alpha)\varepsilon$-fraction of the edges in $R$ are undesirable (i.e., belong to $B$).

To explain the role of this theorem, first consider the following experiment: Consider sampling a point $v_1$ from each member $P_i$ of the partition $P$, looking up the values $U(v_i, v_j)$, setting $w_P(i, j) = U(v_i, v_j)$ if it is in $\{0, 1\}$, and setting $w_P(i, j) = *$ if $0 < U(v_i, v_j) < 1$.

With positive probability this would have most of the required properties of the uniform decision, but unfortunately it would fail to account for sampling graphs where some strips contain more than a single vertex.

The actual proof requires subdividing the strips into finer sub-strips before sampling, and then choosing from them through use of a variant of the above theorem.

Next, we shall show that implementing $w_{R^\mathcal{H}}$ directly on the trimmed object $W_{\mathcal{H}}$ results in a good outcome, as described below. For a partition $P$, and decision function $w_P$, we say that $w_P$ is decisive if whenever $w_P(i, j) = *$, changing the value of $w_P(i, j)$ from * (to 0 or 1) results in a different orderon when implementing it over $U_{R^\mathcal{H}}$. For example, when keeping an all-0 value for $P_i \times P_j$, the decisive function would be $w_P(i, j) = 0$ rather than $w_P(i, j) = *$.

Given a partition $P$ which respects $I_\ell$, we say that a partition $P'$ is a clone of $P$ if for each strip $I_j^{(l)}$, 

$$\left|\left\{P_i \in P \mid P_i \cap I_j^{(l)} \neq \emptyset\right\}\right| = \left|\left\{P'_i \in P' \mid P'_i \cap I_j^{(l)} \neq \emptyset\right\}\right|$$

Lemma 8.16. Let $Z \in \mathcal{W}$ be a $2^l$-strip layered orderon and let $P$ be its stepping. Let $Z' \in \mathcal{W}$ be an orderon, and let $P'$ be a clone of $P$. Then, if $w_P$ is a decisive decision function such that $Z_{\leq w_P} \in \mathcal{H}$, then $Z'_{\leq w_{P'}} \in \mathcal{H}$.

The proof of Lemma 8.16 uses a useful property of $\mathcal{H}$.
Lemma 8.17. If $U \in \overline{\mathcal{H}}$ and $U' \in \mathcal{W}$ is a function such that $U'(v_1, v_2) = U(v_1, v_2)$ whenever $U(v_1, v_2) \in \{0, 1\}$, then $U' \in \overline{\mathcal{H}}$.

Proof: By Lemma 8.2, the closure of a hereditary property can be defined by conditions of the form $t(F, U) = 0$ (for all graphs $F \not\in \mathcal{H}$). For every fixed $F \not\in \mathcal{H}$, this condition means that for almost all vectors $v \in [0, 1]^{V(F)}$, at least one of the factors in $\prod_{(i,j) \in E} U(v_i, v_j) \cdot \prod_{(i,j) \in \overline{E}} (1 - U(v_i, v_j))$ must be 0. This condition is preserved if values strictly between 0 and 1 are changed.

The next lemma states that given a strip layered orderon and a decision function $w$, the result of implementing $w$ on a different orderon with similar structural parameters will result in a similar bound on the $L_1$ difference as that resulting from the implementation of $w$ over the layered strip orderon.

Lemma 8.18. For every $\varepsilon > 0$ and $k > 0$, there exists $\delta(\varepsilon, k) > 0$ such the following holds. Let $Z \in \mathcal{W}$ be a $2^k$-strip layered orderon with stepping in $\mathcal{P}$, where $|\mathcal{P}| \leq k$. Let $Z' \in \mathcal{W}$ be an orderon, and let $\mathcal{P}'$ be a clone of $\mathcal{P}$ such that:

1. For each $i \in [|\mathcal{P}|]$, $|\lambda(P_i) - \lambda(P'_i)| \leq O(\delta/\text{poly}(\ell))$.

2. For all $i, j \in [|\mathcal{P}|]^2$,

$$\left| \int_{v_1, v_2 \in P_i \times P_j} Z(v_1, v_2)dv_1dv_2 - \int_{v_1, v_2 \in P'_i \times P'_j} Z'(v_1, v_2)dv_1dv_2 \right| \leq O(\delta/\text{poly}(\ell)).$$

Finally, let $w_\mathcal{P}$ be a decision function. Then,

$$|d_1(Z_{\ll w_\mathcal{P}}, \overline{\mathcal{H}}) - d_1(Z'_{\ll w_{\mathcal{P}'}}, \overline{\mathcal{H}})| \leq \varepsilon.$$

Observe that by an appropriate choice of a parameter bounding $\text{Shift}(\phi)$ from above and an appropriate choice of the closeness parameter of $W$ to $U^\phi$ in cut norm, we can apply Lemma 8.14 and implement a uniform decision on $U_{\mathcal{R}_\ell}$ generating some $U^* \in \overline{\mathcal{H}}$. By Observation 8.11 and Observation 8.12, $\mathcal{Q}$ is a clone of the partition $\mathcal{R}_\ell$, which satisfies both Lemma 8.16 and Lemma 8.18. Thus, we deduce that applying the same decision function on $T$ yields some orderon $\tilde{U}$ which is $L_1$-close to $W$.

References

[ABEF17] Noga Alon, Omri Ben-Eliezer, and Eldar Fischer. Testing hereditary properties of ordered graphs and matrices. In Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 848–858, 2017.

[AFKS00] Noga Alon, Eldar Fischer, Michael Krivelevich, and Mario Szegedy. Efficient testing of large graphs. Combinatorica, 20(4):451–476, 2000.

[AFNS09] Noga Alon, Eldar Fischer, Ilan Newman, and Asaf Shapira. A combinatorial characterization of the testable graph properties: It’s all about regularity. SIAM Journal on Computing, 39(1):143–167, 2009.
[AS08] Noga Alon and Asaf Shapira. A characterization of the (natural) graph properties testable with one-sided error. *SIAM Journal on Computing*, 37(6):1703–1727, 2008.

[BC17] Christian Borgs and Jennifer T. Chayes. Graphons: A nonparametric method to model, estimate, and design algorithms for massive networks. *CoRR*, abs/1706.01143, 2017.

[BCG17] Christian Borgs, Jennifer Chayes, and David Gamarnik. Convergent sequences of sparse graphs: A large deviations approach. *Random Structures & Algorithms*, 51(1):52–89, 2017.

[BCL+06a] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, Balázs Szegedy, and Katalin Vesztergombi. Graph limits and parameter testing. In *Proceedings of the 38th ACM Symposium on the Theory of Computing (STOC)*, pages 261–270, 2006.

[BCL+06b] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Counting graph homomorphisms. In *Topics in Discrete Mathematics*, pages 315–371. Springer Berlin Heidelberg, 2006.

[BCL+08] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, 219(6):1801–1851, 2008.

[BCL10] Christian Borgs, Jennifer Chayes, and László Lovász. Moments of two-variable functions and the uniqueness of graph limits. *Geometric and Functional Analysis*, 19(6):1597–1619, 2010.

[BCL12] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs II. multiway cuts and statistical physics. *Annals of Mathematics*, 176(1):151–219, 2012.

[BE19] Omri Ben-Eliezer. Testing local properties of arrays. In *Proceedings of the 10th Innovations in Theoretical Computer Science Conference (ITCS)*, 2019. To appear.

[BEF18] Omri Ben-Eliezer and Eldar Fischer. Earthmover resilience and testing in ordered structures. In *Proceedings of the 33rd Conference on Computational Complexity (CCC)*, pages 18:1–18:35, 2018.

[BEKR17] Omri Ben-Eliezer, Simon Korman, and Daniel Reichman. Deleting and testing forbidden patterns in multi-dimensional arrays. In *Proceedings of the 44th International Colloquium on Automata, Languages and Programming (ICALP)*, pages 9:1–9:14, 2017.

[DJ08] Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, 28(1):33–61, 2008.

[ES12] Gábor Elek and Balázs Szegedy. A measure-theoretic approach to the theory of dense hypergraphs. *Advances in Mathematics*, 231(3):1731–1772, 2012.
[FK96] Alan Frieze and Ravi Kannan. The regularity lemma and approximation schemes for dense problems. In *Proceedings of the 37th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 12–20, 1996.

[FK99] Alan Frieze and Ravi Kannan. Quick approximation to matrices and applications. *Combinatorica*, 19(2):175–220, 1999.

[FLS07] Michael Freedman, László Lovász, and Alexander Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *Journal of the American Mathematical Society*, 20:37–51, 2007.

[FN07] Eldar Fischer and Ilan Newman. Testing versus estimation of graph properties. *SIAM Journal on Computing*, 37(2):482–501, 2007.

[Fra09] John M. Franks. A (terse) introduction to Lebesgue integration, volume 48 of *Student Mathematical Library*. American Mathematical Society, 2009.

[HKM+13] Carlos Hoppen, Yoshiharu Kohayakawa, Carlos Gustavo Moreira, Balázs Ráth, and Rudini Menezes Sampaio. Limits of permutation sequences. *Journal of Combinatorial Theory, Series B*, 103(1):93–113, 2013.

[Jan11] Svante Janson. Poset limits and exchangeable random posets. *Combinatorica*, 31(5):529–563, 2011.

[KRTA17] Simon Korman, Daniel Reichman, Gilad Tsur, and Shai Avidan. Fast-match: Fast affine template matching. *International Journal of Computer Vision*, 121(1):111–125, 2017.

[Lov12] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Society, 2012.

[LS06] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957, 2006.

[LS07] László Lovász and Balázs Szegedy. Szemerédi’s lemma for the analyst. *Geometric And Functional Analysis*, 17(1):252–270, 2007.

[OR15] Peter Orbanz and Daniel M. Roy. Bayesian models of graphs, arrays and other exchangeable random structures. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 37(2):437–461, 2015.

[SV03] Peter Sanders and Berthold Vöcking. Tail bounds and expectations for random arc allocation and applications. *Combinatorics, Probability and Computing*, 12(3):301–318, 2003.

[Sze76] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et thorie des graphes. Colloq. Internat. CNRS*, volume 260, pages 399–401, 1976.

[VDDP18] Athanasios Voulodimos, Nikolaos Doulamis, Anastasios Doulamis, and Eftychios Protopapadakis. Deep learning for computer vision: A brief review. *Journal of Computational Intelligence and Neuroscience*, 2018. Article ID 7068349.
[Yos16] Yuichi Yoshida. Gowers norm, function limits, and parameter estimation. In *Proceedings of the 27th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1391–1406, 2016.

[Zha15] Yufei Zhao. Hypergraph limits: A regularity approach. *Random Structures & Algorithms*, 47(2):205–226, 2015.