The Effect of Cosmic Inhomogeneities On The Average Cosmological Dynamics

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Abstract
It is generally assumed that on sufficiently large scales the Universe is well-described as a homogeneous, isotropic FRW cosmology with a dark energy. Does the formation of nonlinear cosmic inhomogeneities produce a significant effect on the average large-scale FLRW dynamics? As an answer, we suggest that if the length scale at which homogeneity sets in is much smaller than the Hubble length scale, the back-reaction due to averaging over inhomogeneities is negligible. This result is supported by more than one approach to study of averaging in cosmology. Even if no single approach is sufficiently rigorous and compelling, they are all in agreement that the effect of averaging in the real Universe is small. On the other hand, it is perhaps fair to say that there is no definitive observational evidence yet that there indeed is a homogeneity scale which is much smaller than the Hubble scale, or for that matter, if today’s Universe is indeed homogeneous on large scales. If the Copernican principle can be observationally established to hold, or is theoretically assumed to be valid, this provides strong evidence for homogeneity on large scales. However, even this by itself does not say what the scale of homogeneity is. If that scale is today comparable to the Hubble radius, only a fully non-perturbative analysis can establish or rule out the importance of cosmological back-reaction. This brief elementary report summarizes some recent theoretical developments on which the above inferences are based.

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1 Introduction

The Universe that we see around us is lumpy - it has stars, galaxies, clusters of galaxies, superclusters, sheets, filaments and voids. We do not precisely know from observations what the size of the largest structures is; the size beyond which there are no larger structures. On the other hand the early Universe is very well-described as a homogeneous, isotropic FRW cosmology [Big-Bang nucleosynthesis and the relic CMB are evidence of success] and the present Universe is well-described as an FRW cosmology with dark energy. How does one reconcile a universe which is observed to be inhomogeneous and anisotropic on smaller scales, with a universe that is assumed to be homogeneous and isotropic on large scales? Clearly, some way of averaging the matter distribution and the related Einstein equations has to be invoked. What is the right way? The true metric of the universe is the one produced by the inhomogeneous matter distribution. On large scales, one assumes the average of the true metric to be FLRW, constructs the Einstein tensor for it, and uses it on the left hand side of the Friedmann equations wherein the matter content on the right hand side is a perfect fluid. Because Einstein equations are nonlinear, the Einstein tensor constructed from the average metric tensor will in general not be the same as the average of the Einstein tensor of the true metric:

\[ \langle G_{\mu\nu}(g_{\mu\nu}) \rangle = \langle T_{\mu\nu} \rangle \neq G_{\mu\nu}(\langle g_{\mu\nu} \rangle) \] (1)

Here \( \langle ... \rangle \) denotes the averaging operation [whose correct definition, for tensors on a curved spacetime, is itself a major challenge]; and \( \langle g_{\mu\nu} \rangle = g_{\mu\nu}|_{FLRW} \). The correct averaged Einstein equations are of course given by the first pair (the equality) in the above set, whereas in cosmology we assume the correct equations to be those given by the second pair (by assuming the inequality to actually be an equality). This is obviously done because the latter option is infinitely simpler - it is straightforward to write the Einstein tensor for the Robertson-Walker metric, but it is impossible to find the true metric of the inhomogeneous universe and then average its corresponding Einstein tensor. Since the first and the third terms in (1) could differ significantly, we might be working with the wrong averaged Einstein equation on cosmological scales. This is the averaging problem: how to correctly average Einstein equations, and to find out if the neglected terms (the so-called back-reaction) can become important in the late stages of an evolving universe, when nonlinear structures such as galaxies and clusters form. In particular, can the back-reaction mimic a dark energy, and explain the observed cosmic acceleration?

The problem of averaging of Einstein equations has a long history, and has recently been reviewed in an important article by Ellis \[1\]. Important contributions to the study of averaging have been made in recent times, amongst others, by Buchert \[2\], Coley \[3\], Wald \[4\], Zalaletdinov \[5\] and their collaborators. Specific applications have been developed by Kolb \[6\], Marra \[7\], Rasanen \[8\], Sussman \[9\], Wiltshire \[10\] and others. Much of this work, as well as earlier developments, are reviewed by Ellis, and we will not enter into details here, except in the context of specific arguments developed here.

If we want to find out the back-reaction on an FLRW universe, it certainly means we are taking an FLRW geometry as given on large scales. It is hence necessary to first know what the observational evidence for large scale homogeneity and isotropy is, and what is the length scale at which homogeneity sets in.
2 Evidence for large scale homogeneity

Careful discussions of this issue have recently been given by Clarkson and Maartens [11], Maartens [12] and Ellis [1]. What we have to say below is a summary from these earlier works, and is reported here because of its significance for the discussion on averaging in the next section.

It is well known that homogeneity on spatial surfaces can not be established by direct observations, because all our observations are on the past light-cone. Hence tests of homogeneity have to exploit the following route: isotropy around us along with the Copernican Principle [CP] implies homogeneity, and hence a FLRW Universe. So one needs to test for isotropy and for CP independently.

The observational evidence for spacetime isotropy around our world-line can be investigated from examining the isotropy of the CMB and the galaxy distribution. For a perfectly isotropic CMB, all multipoles of the distribution function higher than the monopole, as well their time derivatives, vanish. However without CP one cannot deduce the vanishing of the spatial derivatives of the higher multipoles, and hence spacetime isotropy about our world-line cannot be deduced without CP. As for baryonic matter (along with certain assumptions for the distribution of CDM and dark energy) it can be shown that isotropic distribution of the following four matter observables on the light-cone implies an isotropic spacetime geometry: angular diameter distances, galaxy number counts, bulk velocities and lensing (details and references to original work can be found in [12]). As pointed out by Maartens, it is not known whether almost-isotropy of observations leads to almost-isotropy of spacetime geometry.

Next, one considers what can be inferred about spatial homogeneity, if one assumes CP, and considers the following three cases: isotropic matter distribution, isotropic CMB, almost-isotropic CMB. If all fundamental observers measure the same isotropic distribution of the four matter observables mentioned above, this implies homogeneity, and the Universe is FLRW. It can be proved that exact isotropy of the CMB for all observers also implies an FLRW universe. Almost-isotropy of the CMB can be shown, via a non-perturbative analysis, to imply an almost-FLRW universe, provided some of the time and spatial derivatives of the multipoles are sufficiently small.

Thus it is clear that the case for an almost-FLRW universe will be strong if observational tests support the Copernican Principle. These tests can be carried out by testing the standard consistency relations in FLRW geometry. The FLRW curvature parameter which can be inferred from geometric measurements is independent of redshift, and a detection of redshift dependence of this parameter will indicate departure from homogeneity. A second test is the time drift of cosmological redshift, and a third test is a significant difference between the radial and transverse BAO scales. None of these tests have yet been carried out, but their eventual execution will play a crucial role in confirming large-scale homogeneity. The CP can also be tested by looking for a large thermal or kinetic Sunyaev-Zeldovich effect temepature distortion of the CMB. Also, a large SZ effect induced CMB polarization could indicate a violation of CP and hence of homogeneity.

As of now, there is no evidence against CP, but neither is there clinching evidence for large-scale homogeneity. Also, it is not quite clear at what scale homogeneity sets in. If we assume that there is homogeneity, and that too at a scale much less than the Hubble scale, say at around 100 MPc, then it can be shown (as discussed next) that the cosmological back-reaction is negligible. And the $\Lambda$CDM model is then a good description of the present day Universe. On the other hand if there are much larger nonlinear structures in the Universe - their formation can then no longer be described perturbatively on an FLRW background, and the back-reaction problem will have to be examined afresh.
3 Averaging in Cosmology and Calculation of Back-Reaction

Assuming that the scale at which homogeneity sets in is much smaller than the Hubble scale, we give three explanations as to why the back-reaction will be small: (i) a simple argument due to Peebles [13]; (ii) our own work [14], [15], [16], [17] which builds on Zalaletdinov’s Macroscopic Theory of Gravity [MG] [18], [19], [20], [21]; and (iii) the work of Wald and collaborators [22], [4]. Similar results obtained by a few other researchers, which support the present inference, are briefly reviewed in Paranjape’s thesis [23].

3.1 An argument due to Peebles ([13] and references therein)

For nonlinear structures such as galaxies, the Newtonian gravitational potential is of the order of the square of the velocity dispersion [about 300 km/sec], i.e. $\phi \sim 10^{-6}$. Hence the galaxy distribution can be described as a perturbation over an FLRW universe. The metric can be written as a perturbed FLRW Universe and the Einstein equations can be split into an evolution equation for the background scale-factor and the Poisson equation for the perturbed Newtonian potential determined by the density contrast (assumed to be provided by non-interacting dark matter).

In order to find the effect of averaging on the FLRW equations, spatial averages of Einstein equations need to be computed to order $\phi^2$, in particular for the dominant term which is proportional to $\nabla \phi, \nabla \phi$. When this is done, one finds corrections due to back-reaction in both the Friedmann equations - corrections in the form of a kinetic energy coming from the mean square velocity dispersion, and the averaged gravitational potential energy determined by the density contrast of the formed nonlinear structures. Both these correction terms are of the order of a part in a million, and hence much smaller than the magnitude of the observed dark energy.

The discussion by Peebles is patterned in part on the nice work of Siegel and Fry [24]. It seems to us that there is room for improvement in this argument: one should not fix the background, but allow for the possibility that as perturbations grow, the background around which back-reaction should be calculated may itself be changing, because of feedback from the perturbations. One has to ascertain that a runaway process leading to breakdown of perturbation theory does not take place. This is the study we attempted by applying Zalaletdinov’s averaging theory [Macroscopic Gravity] to cosmology. Before we summarize our work on applying MG, it will be useful to briefly review Buchert’s averaging scheme. We do this because the Buchert approach provides simple averaged equations, while being less ambitious than MG. Remarkably, the averaged equations that arise from MG are very similar to Buchert’s averaging equations, enforcing a certain high degree of reliability of both approaches, despite their conceptual differences.

3.2 Buchert’s averaging scheme for a dust spacetime

For a general spacetime containing irrotational dust, the metric can be written as

$$ds^2 = -dt^2 + h_{ij}(\vec{x}, t)dx^i dx^j. \quad (2)$$

The expansion tensor $\Theta^i_j$ is given by $\Theta^i_j \equiv (1/2)h^{ik}h_{kj}$ where the dot refers to a derivative with respect to time $t$. The traceless symmetric shear tensor is defined as $\sigma^i_j \equiv \Theta^i_j - (\Theta/3)\delta^i_j$ where $\Theta = \Theta^i_i$ is the expansion scalar. The Einstein equations can be split into a set of scalar equations and a set of vector and traceless tensor equations. The scalar equations are the Hamiltonian constraint (3a).
and the evolution equation for $\Theta$ (3b),

\[ (3) R + \frac{2}{3} \Theta^2 - 2\sigma^2 = 16\pi G \rho \]  

(3a)

\[ (3) R + \dot{\Theta} + \Theta^2 = 12\pi G \rho \]  

(3b)

where the dot denotes derivative with respect to time $t$, $(3)R$ is the Ricci scalar of the 3-dimensional hypersurface of constant $t$ and $\sigma^2$ is the rate of shear defined by $\sigma^2 \equiv (1/2)\sigma^i_j\sigma_j^i$. Eqns. (3a) and (3b) can be combined to give Raychaudhuri’s equation

\[ \dot{\Theta} + \frac{1}{3} \Theta^2 + 2\sigma^2 + 4\pi G \rho = 0. \]  

(4)

The continuity equation $\dot{\rho} = -\Theta \rho$ which gives the evolution of $\rho$, is consistent with Eqns. (3a), (3b). We only consider the scalar equations, since the spatial average of a scalar quantity can be defined in a gauge covariant manner within a given foliation of space-time. For the space-time described by (2), the spatial average of a scalar $\Psi(t, \vec{x})$ over a comoving domain $D$ at time $t$ is defined by

\[ \langle \Psi \rangle = \frac{1}{V_D} \int_D d^3x \sqrt{h} \Psi \]  

(5)

where $h$ is the determinant of the 3-metric $h_{ij}$ and $V_D$ is the volume of the comoving domain given by $V_D = \int_D d^3x \sqrt{h}$.

Spatial averaging is, by definition, not generally covariant. Thus the choice of foliation is relevant, and should be motivated on physical grounds. In the context of cosmology, averaging over freely-falling observers is a natural choice, especially when one intends to compare the results with standard FRW cosmology. Following the definition (5), the following commutation relation then holds [2]

\[ \langle \Psi \rangle - \langle \dot{\Psi} \rangle = \langle \Psi \Theta \rangle - \langle \Psi \rangle \langle \Theta \rangle \]  

(6)

which yields for the expansion scalar $\Theta$

\[ \langle \Theta \rangle - \langle \dot{\Theta} \rangle = \langle \Theta^2 \rangle - \langle \Theta \rangle^2. \]  

(7)

Introducing the dimensionless scale factor $a_D \equiv (V_D/V_{D_{in}})^{1/3}$ normalized by the volume of the domain $D$ at some initial time $t_{in}$, we can average the scalar Einstein equations (3a), (3b) and the continuity equation to obtain

\[ \langle \Theta \rangle = 3 \frac{\dot{a}_D}{a_D}, \]  

(8a)

\[ 3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi G \langle \rho \rangle + \frac{1}{2} \langle R \rangle = -\frac{Q_D}{2}, \]  

(8b)

\[ 3 \left( \frac{\dot{a}_D}{a_D} \right) + 4\pi G \langle \rho \rangle = Q_D, \]  

(8c)

\[ \langle \rho \rangle = -\langle \Theta \rangle \langle \rho \rangle = -3 \frac{\dot{a}_D}{a_D} \langle \rho \rangle. \]  

(8d)
Here $\langle \mathcal{R} \rangle$, the average of the spatial Ricci scalar $^{(3)}\mathcal{R}$, is a domain dependent spatial constant. The ‘backreaction’ $Q_D$ is given by

$$Q_D \equiv \frac{2}{3} \left( \langle \Theta^2 \rangle - \langle \Theta \rangle^2 \right) - 2 \langle \sigma^2 \rangle$$

and is also a spatial constant. The last equation (8d) simply reflects the fact that the mass contained in a comoving domain is constant by construction: the local continuity equation $\dot{\rho} = -\Theta \rho$ can be solved to give $\rho \sqrt{h} = \rho_0 \sqrt{h_0}$ where the subscript 0 refers to some arbitrary reference time $t_0$. The mass $M_D$ contained in a comoving domain $D$ is then

$$M_D = \int_D \rho \sqrt{h} d^3x = \int_D \rho_0 \sqrt{h_0} d^3x = \text{constant}.$$  

Hence

$$\langle \rho \rangle = \frac{M_D}{V_{\text{lin}} a_D^3} (10)$$

which is precisely what is implied by Eqn. (8d).

This averaging procedure can only be applied for spatial scalars, and hence only a subset of the Einstein equations can be smoothed out. As a result it may appear that the outcome of such an approach is severely restricted, and essentially incomplete due to the impossibility to analyze the full set of equations. However one should note that the cosmological parameters of interest are scalars, and the averaging of the exact scalar part of Einstein equations provides the requisite needed information. A more general strategy would be to consider the smoothing of tensors, which is beyond the scalar approach that certainly provides useful information, albeit not the full information.

The dynamical equations above can be cast in a form which is immediately comparable with the standard FRW equations [2]. Namely,

$$\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} \left( \rho_{\text{eff}} + 3 P_{\text{eff}} \right) (11a)$$

$$\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} \rho_{\text{eff}} (11b)$$

with $\rho_{\text{eff}}$ and $P_{\text{eff}}$ defined as

$$\rho_{\text{eff}} = \langle \rho \rangle - \frac{Q_D}{16\pi G} - \frac{\langle \mathcal{R} \rangle}{16\pi G}; \quad P_{\text{eff}} = -\frac{Q_D}{16\pi G} + \frac{\langle \mathcal{R} \rangle}{48\pi G}. (12)$$

A necessary condition for (11a) to integrate to (11b) takes the form of the following differential equation involving $Q_D$ and $\langle \mathcal{R} \rangle$

$$\dot{Q}_D + 6 \frac{\dot{a}_D}{a_D} Q_D + \langle \mathcal{R} \rangle \cdot + 2 \frac{\dot{a}_D}{a_D} \langle \mathcal{R} \rangle = 0$$

and the criterion to be met in order for the effective scale factor $a_D$ to accelerate, is

$$Q_D > 4\pi G \langle \rho \rangle. (14)$$

The Buchert scheme has been applied extensively, and in particular can be used to show that there indeed are toy cosmological models which when averaged over inhomogeneities can produce an apparent acceleration. However, not all Einstein equations are averaged, and one does not have an averaged metric here [which we would like to be the FLRW metric]. Macroscopic Gravity can achieve that, while reproducing modified Friedmann equations analogous to the Buchert equations, when applied to cosmology.
4 Macroscopic Gravity

This theory is developed comprehensively in the works of Zalaletdinov; briefly introduced in [14], and reviewed in Paranjape’s thesis [23]. For detailed discussions and the primary interpretation of MG, the reader is referred to Zalaletdinov’s original papers cited in this article.

For the purpose of averaging of tensors the key new element which is introduced is a bivector $\mathcal{W}_{ab}(x', x)$ which transforms as a vector at event $x'$ and as a co-vector at event $x$. The bivector is used to define the “bilocal extension” of a general tensorial object

$$\tilde{P}^a(x', x) = \mathcal{W}_{ab}^a(x', x)P^b(x')$$ (15)

The “average” of $P^a(x)$ over a 4-dimensional spacetime region $\Sigma$ with a supporting point $x$ is

$$\bar{P}^a(x) = \langle \tilde{P}^a \rangle_{ST} = \frac{1}{V_\Sigma} \int_\Sigma d^4x' \sqrt{-g} \tilde{P}^a(x', x)$$ (16)

and this averaging operation preserves tensorial properties.

There is a certain degree of non-uniqueness in the choice of the coordination bi-vector - the freedom coming from the presence of undetermined structure constants in the commutation relations for a vector basis in terms of which one can solve for the coordination bivector. The simplest choice is to set these structure constants to zero. When that is done, then in a volume preserving coordinate system $\phi^m$, [VPC], i.e. one with $g(\phi^m) = \text{constant}$, the coordination bivector takes its most simple form, namely

$$\mathcal{W}_{j}^a(x', x) \mid_{\text{proper}} = \delta_{j}^a.$$ (17)

The effect of this non-uniqueness on the physical results for averaging in cosmology remains to be estimated. Nonetheless, it is noteworthy that the averaged Friedmann equations to be derived from this approach are similar to Buchert’s and the physical results about the magnitude of the back-reaction is identical to the one due to Peebles. This gives confidence in the robustness of the results obtained, even though there is freedom in the choice of the coordination bivector. It is also useful to note that this bi-vector is different from the Synge bi-tensor which leaves the metric invariant upon averaging, and hence cannot really be used to average an inhomogeneous geometry.

Averaged Geometry: the key idea of Macroscopic Gravity is that the average connection $\bar{\Omega}^a_b(x)$

$$\bar{\Omega}^a_b \equiv \langle \Omega^a_b \rangle,$$ (18)

is defined as the connection 1-form on a new, averaged manifold $\bar{\mathcal{M}}$. Next one defines a correlation 2-form

$$Z_{b}^a = \langle \Omega^a_b \wedge \Omega_i^j \rangle_{ST} - \bar{\Omega}_b^a \wedge \bar{\Omega}_j^i.$$ (19)

Denoting $R^a_b \equiv \langle \tilde{r}_b^a \rangle_{ST}$, where $r^a_b$ is the curvature 2-form of the inhomogeneous geometry, and the curvature 2-form on the averaged manifold $\bar{\mathcal{M}}$ as $M^a_b$ can be shown to give

$$M^a_b = R^a_b - Z_{c}^a_c b.$$ (20)

The inhomogeneous Einstein equations

$$g^{ab} r_{kb} - \frac{1}{2} \delta^a_b g^{ij} r_{ij} = -\kappa l^a_{(\text{mic})},$$ (21)
average out to

\[ E^a_b = -\kappa T^a_b + C^a_b, \quad (22) \]

\[ C^a_b = \left( Z^a_{ijb} - \frac{1}{2} \delta^a_b Z^m_{ijm} \right) G^{ij}. \quad (23) \]

\( G^{ij} \) is the metric on the averaged geometry. The correlation 2-form is assumed to satisfy certain differential conditions which amount to closure conditions for the above system of averaged equations.

These averaged equations of Macroscopic Gravity carry, in a covariant and non-perturbative manner, information about the effect of the underlying inhomogeneities on the averaged geometry.

### 5 Application of Macroscopic Gravity to Cosmology

In order to apply MG to cosmology we start with the assumption that Einstein’s equations are to be imposed on length scales where stars are pointlike objects (we denote such a scale as \( L_{\text{inhom}} \)). The averaging we perform will be directly at a length scale \( L_{\text{FLRW}} \) larger than about \( 100h^{-1}\text{Mpc} \) or so. This averaging scale is assumed to satisfy \( L_{\text{inhom}} \ll L_{\text{FLRW}} \ll L_{\text{Hubble}} \) where \( L_{\text{Hubble}} \) is the length scale of the observable universe. The averaging will be assumed to yield a geometry which has homogeneous and isotropic spatial sections. In other words, we will assume that the averaged manifold \( \hat{\mathcal{M}} \) admits a preferred, hypersurface-orthogonal unit timelike vector field \( \hat{v}^a \), which defines 3-dimensional spacelike hypersurfaces of constant curvature, and that \( \hat{v}^a \) is tangent to the trajectories of observers who see an isotropic Cosmic Background Radiation. (These “observers” are defined in the averaged manifold – we will clarify below what they correspond to in the inhomogeneous manifold.) Throughout the rest of this article, for simplicity, we will work with the special case where the spatial sections on \( \hat{\mathcal{M}} \) defined by \( \hat{v}^a \) are flat. (In principle the entire calculation can be repeated for non-flat spatial sections as well.) One can then choose coordinates \((t, x^A), A = 1, 2, 3, \) on \( \hat{\mathcal{M}} \) such that the spatial line element takes the form

\[ \langle \mathcal{M} \rangle ds_{\text{spatial}}^2 = a^2(t)\delta_{AB} dx^A dx^B, \quad (24) \]

where \( \delta_{AB} = 1 \) for \( A = B, \) and 0 otherwise, and we have \( \hat{v}^a = (\hat{v}^t, 0, 0, 0) \) so that the spatial coordinates are comoving with the preferred observers. The vector field \( \hat{v}^a \) also defines a proper time (the cosmic time) \( \tau \) such that \( \partial_t = \hat{v}^a \partial_a = \hat{v}^t \partial_t. \) We will further assume that the averaged energy-momentum tensor \( T^a_b \) can be written in the form of a perfect fluid, as

\[ T^a_b = \rho \hat{v}^a \hat{v}_b + p \pi^a_b, \quad (25) \]

where the projection operator \( \pi^a_b \) is defined as

\[ \pi^a_b = \delta^a_b + \hat{v}^a \hat{v}_b, \quad (26) \]

and \( \rho \) and \( p \) are the homogeneous energy density and pressure respectively, as measured by observers moving on trajectories (in \( \hat{\mathcal{M}} \)) with the tangent vector field \( \hat{v}^a, \)

\[ \rho \equiv T^a_b \hat{v}^b \hat{v}_a ; \quad p \equiv \frac{1}{3} \pi^a_b T^a_b. \quad (27) \]
\( \rho \) and \( p \) are observationally relevant quantities, since all measurements of the matter energy density, especially those from studies of Large Scale Structure, interpret observations in the context of the averaged geometry. An important consequence of the above assumptions is that the correlation tensor \( C^a_b \), when expressed in terms of the natural coordinates adapted to the spatial sections defined by the vector field \( \bar{v}^a \), is spatially homogeneous. This is clear when the modified Einstein equations are written in these natural coordinates.

The existence of the vector field \( \bar{v}^a \) with the attendant assumptions described above, allows us to separate out the nontrivial components of the (FLRW) Einstein tensor \( E^a_b \) on \( \mathcal{M} \) in a coordinate independent fashion – the Einstein tensor can be written as

\[
E^a_b = j_1(x)\bar{v}^a\bar{v}_b + j_2(x)\pi^a_b
\]

\[
j_1(x) \equiv E^a_b\bar{v}^b\bar{v}_a \quad ; \quad j_2(x) \equiv \frac{1}{3}(\pi^a_aE^a_b),
\]

(28)

where \( j_1(x) \) and \( j_2(x) \) are scalar functions whose form depends upon the coordinates used. The remaining components given by \( \pi^a_kE^b_k\bar{v}_a \) and the traceless part of \( \pi^a_k\pi^b_kE^a_b \), vanish identically. Since the energy-momentum tensor \( T^a_b \) in Eqn. (25) also has an identical structure, this structure is therefore also imposed on the correlation tensor \( C^a_b \). Namely, \( \pi^b_kC^a_k\bar{v}_a \) and the traceless part of \( \pi^a_k\pi^b_kC^b_k \) must vanish. This is a condition on the underlying inhomogeneous geometry, irrespective of the coordinates used on either \( \mathcal{M} \) or \( \mathcal{M} \), and is clearly a consequence of demanding that the averaged geometry have the symmetries of the FLRW spacetime.

This leads us to the crucial question of the choice of gauge for the underlying geometry: namely, what choice of spatial sections for the inhomogeneous geometry, will lead to the spatial sections of the FLRW metric in the comoving coordinates defined in Eqn. (24)? Since the matter distribution at scale \( L_{inhom} \) need not be pressure-free (or, indeed, even of the perfect fluid form), there is clearly no natural choice of gauge available, although locally, a synchronous reference frame can always be chosen. We note that there must be at least one choice of gauge in which the averaged metric has spatial sections in the form (24) – this is simply a refinement of the Cosmological Principle, and of the Weyl postulate, according to which the Universe is homogeneous and isotropic on large scales, and individual galaxies are considered as the “observers” travelling on trajectories with tangent \( \bar{v}^a \).

In the averaging approach, it makes more sense to replace “individual galaxies” with the averaging domains considered as physically infinitesimal cells – the “points” of the averaged manifold \( \mathcal{M} \). This is physically reasonable since we know after all, that individual galaxies exhibit peculiar motions, undergo mergers and so on. This idea is also more in keeping with the notion that the Universe is homogeneous and isotropic only on the largest scales, which are much larger than the scale of individual galaxies.

Consider any 3 + 1 spacetime splitting in the form of a lapse function \( N(t,x^i) \), a shift vector \( N^A(t,x^i) \), and a metric for the 3-geometry \( h_{AB}(t,x^i) \), so that the line element on \( \mathcal{M} \) can be written as

\[
^{(\mathcal{M})}ds^2 = -(N^2 - N_A N^A) dt^2 + 2N_B dx^B dt + h_{AB} dx^A dx^B,
\]

(29)

where \( N_A = h_{AB}N^B \). At first sight, it might seem reasonable to leave the choice of gauge arbitrary. However the analysis is then complicated. On the other hand, if we make the assumption that the spatial sections on \( \mathcal{M} \) leading to the spatial metric (24) on \( \mathcal{M} \), are spatial sections in a volume preserving gauge, then the correlation terms simplify greatly. This is not surprising since the MG formalism is nicely adapted to the choice of volume preserving coordinates. The case when the gauge is left unspecified is dealt with in our original papers.
To begin our calculation, we perform a coordinate transformation and shift to the gauge wherein the new lapse function \( N \) is given by \( N = 1/\sqrt{h} \) where \( h \) is the determinant of the new 3-metric \( h_{AB} \). In general, one will now be left with a non-zero shift vector \( N^A \); however, the condition \( N\sqrt{h} = 1 \) ensures that the coordinates we are now using are volume preserving, since the metric determinant is given by \( g_{\bar{t}\bar{t}} = -N^2 = -1/\sqrt{\det{h}} \). For simplicity, we make the added assumption that \( N^A = 0 \) in the inhomogenous geometry, so that \( g_{\bar{t}\bar{t}} = -N^2 = -1/\sqrt{\det{h}} \) and \( g_{\bar{t}A} = 0 \). The line element for the inhomogenous manifold \( \mathcal{M} \) becomes

\[
(\mathcal{M}) ds^2 = -\frac{dt^2}{h(\bar{t}, \mathbf{x})} + h_{AB}(\bar{t}, \mathbf{x})d\bar{x}^A d\bar{x}^B. \tag{30}
\]

Note that in this gauge, the average takes on a particularly simple form: for a tensor \( p^{ij}(x) \), with a spacetime averaging domain given by the “cuboid” \( \Sigma \) defined by

\[
\Sigma = \{ (\bar{t}, x, y, z) | -T/2 < \bar{t} < T/2, -L/2 < x, y, z < L/2 \}, \tag{31}
\]

where \( T \) and \( L \) are averaging time and length scales respectively, the average is given by

\[
\langle p^{ij}_{ST}(\bar{t}, \mathbf{x}) \rangle = \frac{1}{TL^3} \int_{-T/2}^{T/2} dt' \int_{-L/2}^{+L/2} dx' dy' dz' \left[ p^{ij}_{\bar{t}\bar{t}}(t', x', y', z') \right], \tag{32}
\]

where the limits on the spatial integral are understood to hold for all three spatial coordinates. We define the “spatial averaging limit” as the limit \( T \to 0 \) (or \( T \ll L_{\text{Hubble}} \)) which is interpreted as providing a definition of the average on a spatial domain corresponding to a “thin” time slice, the averaging operation now being given by

\[
\langle p^{ij}(\bar{t}, \mathbf{x}) \rangle = \frac{1}{L^3} \int_{-L/2}^{+L/2} dx' dy' dz' \left[ p^{ij}_{\bar{t}\bar{t}}(\bar{t}, x', y', z') \right] + \mathcal{O}(TL^{-1}_{\text{Hubble}}). \tag{33}
\]

(Note the time dependence of the integrand.) Henceforth, averaging will refer to spatial averaging, and will be denoted by \( \langle ... \rangle \), in contrast to the spacetime averaging considered thus far (denoted by \( \langle ... \rangle_{ST} \)). The choice of a cube with sides of length \( L \) as the spatial averaging domain was arbitrary, and is in fact not essential for any of the calculations to follow. In particular, all calculations can be performed with a spatial domain of arbitrary shape. We will only use the cube for definiteness and simplicity in displaying equations. The significance of introducing a spatial averaging in this manner is that the construction of spatial averaging is not isolated from spacetime averaging, but is a special limiting case of the latter and is, in fact, still a fully covariant operation.

For the volume preserving gauge, we have

\[
G_{\bar{t}\bar{t}} = \langle g_{\bar{t}\bar{t}} \rangle = \langle \frac{-1}{h} \rangle = -f^2(\bar{t}) \;
G_{AB} = \langle h_{AB} \rangle = \tilde{a}^2(\bar{t}) \delta_{AB}, \tag{34}
\]

where \( \tilde{a} \) and \( f \) are some functions of the time coordinate alone. A few remarks are in order on this particular choice of assumptions. Apart from the fact that the spacetime averaging operation takes
on its simplest possible form \(^{(32)}\) in this gauge and allows a transparent definition of the spatial averaging limit, it can also be shown that the assumptions in Eqn. \(^{(34)}\) are sufficient to establish the following relations:

\[
f^2(\bar{t}) = \left\langle \frac{1}{\bar{h}} \right\rangle = \frac{1}{\bar{a}} = \frac{1}{\bar{a}^6}.
\]

(35)

Here the second equality arises from the condition \(\bar{g}^{ij} = G^{ij}\) which can be assumed whenever the averaged metric is of the FLRW form. The last equality follows on considering the conditions \(\langle \bar{\Gamma}_{bc} \rangle = (\text{FLRW}) \Gamma_{bc}\) in obvious notation, (the basic assumption of the MG averaging scheme). Eqn. \(^{(35)}\) reduces the line element on \(\bar{M}\) to the form

\[
(ds)^2 = -\frac{d\bar{t}^2}{\bar{a}^6(\bar{t})} + \bar{a}^2(\bar{t})\delta_{AB}dx^A dx^B.
\]

(36)

The line element in Eqn. \(^{(36)}\) clearly corresponds to the FLRW metric in a volume preserving gauge.

In other words, the (spatial) average of the inhomogeneous geometry in the volume preserving gauge leads to a geometry with homogeneous and isotropic spatial sections, also in a volume preserving gauge. Note that the gauge in Eqn. \(^{(36)}\) for the FLRW spacetime differs from the standard synchronous and comoving gauge, only by a redefinition of the time coordinate. The vector field \(\bar{v}^a\) introduced at the beginning of this section and which defines the FLRW spatial sections, is now given by

\[
\bar{v}^a = (\bar{a}^3, 0, 0, 0) ; \quad \bar{v}_a = G_{ab} \bar{v}^b = \left( -\frac{1}{\bar{a}^3}, 0, 0, 0, \right).
\]

(37)

Before proceeding to the calculation of the correlation terms and the averaged Einstein equations, we briefly describe why it is important to consider the spatial averaging limit of the MG averaging operation. The key idea to emphasize is that an average of the homogeneous and isotropic FLRW geometry, should give back the same geometry. Since the FLRW geometry has a preferred set of spatial sections, it is important therefore to perform the averaging over these sections. Further, since the FLRW metric adapted to its preferred spatial sections depends on the time coordinate, it is also essential that the spacetime average should involve a time range that is short compared to the scale over which say the scale factor changes significantly. Clearly then, averaging the FLRW metric (denoted \(^{(\text{FLRW})}g_{ab}\)) given in Eqn. \(^{(36)}\) (which is in volume preserving gauge) will strictly yield the same metric only in the limit \(T \to 0\). Namely, for the cuboid \(\Sigma\) defined in Eqn. \(^{(31)}\)

\[
\langle (\text{FLRW})\tilde{g}_{ab} \rangle = \lim_{T \to 0} \frac{1}{TL^3} \int_\Sigma dt' d^3x' (\text{FLRW}) g_{ab}(t', x')
\]

\[
= (\text{FLRW}) g_{ab},
\]

(38)

which should be clear from the definition of the metric. The result \(\langle (\text{FLRW})\tilde{g}_{ab} \rangle = (\text{FLRW}) g_{ab}\) in the spatial averaging limit can also be shown to hold for the FLRW metric in synchronous gauge, where the coordination bivector \(\mathcal{V}^{ab}_{\tau} \) can be easily computed using the transformation from the VPCs \((\bar{t}, x^A)\) to the synchronous coordinates \((\tau, y^A)\) given by

\[
\tau = \int_{\bar{t}}^{\bar{t}} \frac{dt}{\bar{a}^3(t)} ; \quad y^A = x^A.
\]

(39)

The transformation \(^{(39)}\) will also later allow us to write the averaged equations in the synchronous gauge for the averaged geometry.

We now proceed to calculating the correlation 2-form \(Z^{a}_{b\ i\ j}\) and thereby the averaged Einstein equations.
6 The averaged cosmological field equations

We start by defining (in any gauge with \( N^A = 0 \)) the expansion tensor \( \Theta^A_B \) by

\[
\Theta^A_B \equiv \frac{1}{2N} h^{AC} \dot{h}_{CB},
\]

where the dot will always refer to a derivative with respect to the VPC time \( \bar{t} \), and \( h^{AB} \) is the inverse of the 3-metric \( h_{AB} \). (This also gives the symmetric tensor \( \Theta_{AB} = (1/2N) \dot{h}_{AB} \), which is the negative of the extrinsic curvature tensor.) The traceless symmetric shear tensor \( \sigma_{AB} \) and the shear scalar \( \sigma^2 \) are defined by

\[
\sigma_{AB} \equiv \Theta_{AB} - (\Theta/3) \delta^A_B; \quad \sigma^2 \equiv \frac{1}{2} \sigma^A_B \sigma_B^A,
\]

where \( \Theta \equiv \Theta^A_A \approx (1/N) \partial_t \ln \sqrt{h} \) is the expansion scalar.

The connection 1-forms \( \omega^i_j = \Gamma^i_jk d^k \) can be easily calculated in terms of the expansion tensor, for an arbitrary lapse function \( N \). Specializing to the volume preserving gauge (\( N = h^{-1/2} \)), the bilocal extensions \( \Omega^i_j \) of the connection 1-forms are trivial and are simply given by

\[
\Omega^i_j(x',x) = \Gamma^i_jk(x') dx^k.
\]

Since \( G_{ab} = g_{ab} \), the connection 1-forms \( \Omega^i_j \) for the averaged manifold \( \bar{M} \) are constructed using the FLRW metric in volume preserving gauge given in Eqn. (36), and can also be easily evaluated.

We can now construct the correlation 2-form \( Z^i_{ab} \) and from there the correlation tensor:

\[
C^a_b = \left( Z^a_{ijb} - \frac{1}{2} \delta^a_b Z^{jm}_{ijm} \right) G^{ij}.
\]

Now, the components of the Einstein tensor \( E^a_b \) for the averaged spacetime with metric (36) are given by

\[
\begin{align*}
E^t_t &= 3 \ddot{a} H^2; \quad E^t_A = 0 = E^A_t, \\
E^A_B &= \ddot{a}^6 \delta^A_B \left[ 2 \left( \frac{\ddot{a}}{a} + 3H^2 \right) + H^2 \right],
\end{align*}
\]

where the peculiar splitting of terms in the last equation is for later convenience. Recall that the overdot denotes a derivative with respect to the VPC time \( \bar{t} \), not synchronous time. In terms of the coordinate independent objects introduced in Eqn. (28), we have

\[
\begin{align*}
\dot{j}_1(x) &= -3 \ddot{a} H^2; \quad \dot{j}_2(x) = \ddot{a}^6 \left[ 2 \left( \frac{\ddot{a}}{a} + 3H^2 \right) + H^2 \right].
\end{align*}
\]

From the averaged Einstein equations we next construct the scalar equations which in the standard case would correspond to the Friedmann equation and the Raychaudhuri equation. These correspond to the Einstein tensor components,

\[
E^a_b v^b \bar{v}_a = j_1(x); \quad \pi^b_a E^a_b + E^a_b v^b \bar{v}_a = 3 j_2(x) + j_1(x),
\]

where \( \pi^b_a \) is the spatial pressure.
and are given by

\[ 3\bar{a}^6 H^2 = (\kappa T^a_b - C^a_b) \bar{v}_a \bar{v}^b \]
\[ = \kappa \bar{\rho} - \frac{1}{2} [Q^{(1)} + S^{(1)}] , \]  

(47a)

\[ 6\bar{a}^6 \left( \frac{\ddot{a}}{a} + 3H^2 \right) = (-\kappa T^a_b + C^a_b) \left( \bar{v}_a \bar{v}^b + \pi_a^b \right) \]
\[ = -\kappa (\bar{\rho} + 3\bar{\rho}) + 2 [Q^{(1)} + Q^{(2)} + S^{(2)}] . \]  

(47b)

Here Eqn. (47a) is the modified Friedmann equation and Eqn. (47b) the modified Raychaudhuri equation (in the volume preserving gauge on \( \mathcal{M} \)). We have used Eqn. (27), with the overbar on \( \rho \) and \( p \) reminding us that they are expressed in terms of the nonsynchronous time \( \bar{t} \), and we have defined the correlation terms

\[ Q^{(1)} = \bar{a}^6 \left[ \frac{2}{3} \left( \frac{1}{\bar{h}} \Theta^2 - \frac{1}{\bar{a}^6} (F \Theta^2) \right) - 2 \left( \frac{1}{\bar{h}} \sigma^2 \right) \right] ; \]
\[ \frac{1}{\bar{a}^6} (F \Theta^2) = (3H)^2 , \]  

(48a)

\[ S^{(1)} = \frac{1}{\bar{a}^2} \delta^{AB} \left[ \langle (3)^J \Gamma^J_{AC} (3)^J \Gamma^C_{BJ} \rangle \right. \]
\[ - \langle \partial_A (\ln \sqrt{\bar{h}}) \partial_B (\ln \sqrt{\bar{h}}) \rangle \]  

,  

(48b)

\[ Q^{(2)} = \bar{a}^6 \left( \frac{1}{\bar{h}} \Theta^A \Theta^B \right) - \frac{1}{\bar{a}^2} \delta^{AB} \langle \Theta^A_J \Theta^B_J \rangle , \]  

(48c)

\[ S^{(2)} = \bar{a}^6 \left( \frac{1}{\bar{h}} h^{AB} \partial_A (\ln \sqrt{\bar{h}}) \partial_B (\ln \sqrt{\bar{h}}) \right) \]
\[ - \frac{1}{\bar{a}^2} \delta^{AB} \langle \partial_A (\ln \sqrt{\bar{h}}) \partial_B (\ln \sqrt{\bar{h}}) \rangle . \]  

(48d)

In defining \( Q^{(1)} \) we have used the relation \( \Theta^2 - \Theta^A_B \Theta^B_A = (2/3) \Theta^2 - 2\sigma^2 \). \( Q^{(1)} \) and \( Q^{(2)} \) are correlations of the extrinsic curvature, whereas \( S^{(1)} \) and \( S^{(2)} \) are correlations restricted to the intrinsic 3-geometry of the spatial slices of \( \mathcal{M} \). Since the components of \( C^a_b \) are not explicitly constrained we can treat the combinations \( (1/2)(Q^{(1)} + S^{(1)}) = -C^a_0 \) and \( 2(Q^{(1)} + Q^{(2)} + S^{(2)}) = (C^a_A - C^a_0) \) as independent, subject only to the differential constraints which we will come to below.

As discussed in the beginning of Section 5 the remaining components of \( C^a_b \) must be set to zero, giving constraints on the underlying inhomogeneous geometry. In coordinate independent language, these constraints read

\[ \pi^i_k C^a_k \bar{v}_a = 0 = \pi^i_k C^a_k \bar{v}^b ; \]
\[ \pi^i_a \pi^k_a C^a_k - \frac{1}{3} \pi^i_k \left( \pi^a_a C^a_k \right) = 0 . \]  

(49)
Eqns. (49) reduce to the following for our specific choice of volume preserving coordinates,

\[ C^0_A = 0 \; ; \; C^0_C = 0 \; ; \; C^A_B - \frac{1}{3} \delta^A_B (C^j_j) = 0, \tag{50} \]

It can be shown that the VPC assumption \( N = h^{-1/2} \) reduces the correlations \( Q^{(2)} \) and \( S^{(2)} \) defined in Eqns. (48c) and (48d), as well as several terms in the explicit expansion of Eqn. (50), to the form

\[
\frac{1}{\langle g_{00} \rangle} \langle g_{00} g^{AB} \Gamma_{b_1 c_1}^{a_1} \Gamma_{j_1 k_1}^{i_1} \rangle - \langle g^{AB} \rangle \langle \Gamma_{b_2 c_2}^{a_2} \Gamma_{j_2 k_2}^{i_2} \rangle = 0. \tag{51}\]

It can be shown that

\[
\langle g_{00} g^{AB} \Gamma_{b c}^{a} \Gamma_{j k}^{i} \rangle = \langle g_{00} g^{AB} \rangle \langle \Gamma_{b c}^{a} \Gamma_{j k}^{i} \rangle = -\langle h^{AB} / h \rangle \langle \Gamma_{b c}^{a} \Gamma_{j k}^{i} \rangle. \tag{52}\]

An interesting point is that the VPC assumption \( N = h^{-1/2} \) further allows us to assume \( \langle h^{AB} / h \rangle = \langle h^{AB} \rangle (1/h) \) consistently with the formalism. Using Eqn. (55) this gives us

\[
\langle h^{AB} / h \rangle = \frac{1}{a^6} \langle h^{AB} \rangle. \tag{53}\]

This shows that the correlation terms \( Q^{(2)} \) and \( S^{(2)} \) in fact vanish,

\[
Q^{(2)} = 0 = S^{(2)}, \tag{54}\]

and leads to some remarkable cancellations in Eqns. (50), which simplify to give

\[
\delta^{JK} \left[ \sqrt{h} \Theta^{(3)AB}_{JB} \Gamma_{AK}^{(3)} - \sqrt{h} \Theta^{(3)AB}_{JK} \Gamma_{AB}^{(3)} \right] = 0, \tag{55a}\]

\[
\delta^{JK} \left( \frac{1}{\sqrt{h}} \Theta^{B}_{KJ} \Gamma_{JB}^{(3)} - \frac{1}{\sqrt{h}} \Theta^{K}_{BJ} \Gamma_{AB}^{(3)} \right) = 0, \tag{55b}\]

\[
\delta^{JK} \left( \Gamma_{JC}^{A} \Gamma_{KB}^{(3)} - \delta^{AJ} \Gamma_{JC}^{(3)} \Gamma_{KB}^{(3)} \right) = \frac{1}{3} \delta^A_B \left( a^2 S^{(1)} \right). \tag{55c}\]

These simplifications are solely a consequence of assuming that the inhomogeneous metric in the volume preserving gauge averages out to give the FLRW metric in standard form. In general, these simplifications will not occur when the standard FLRW metric arises from an arbitrary choice of gauge for the inhomogeneous metric.

In order to come as close as possible to the standard approach in Cosmology, we will now rewrite the scalar equations (47) (which are the cosmologically relevant ones) after performing the transformation given in Eqn. (39) in order to get the FLRW metric to the form

\[
^{(M)} ds^2 = -dt^2 + a^2(t) dy^A dy^B \; ; \; a(t) = \bar{a}(\bar{t}(t)). \tag{56}\]

Since Eqns. (47) are scalar equations, this transformation only has the effect of reexpressing all the terms as functions of the synchronous time \( \tau \). Although the transformation will change the explicit form of the coordination bivector \( W^{a'}_2 \), this change involves only the time coordinate, and
in the spatial averaging limit there is no difference between averages computed in the VPCs and those computed after the time redefinition. This again emphasizes the importance of the spatial averaging limit of spacetime averaging, if we are to succeed operationally in explicitly displaying the correlations as corrections to the standard cosmological equations. The correlation terms in Eqns. (48) are therefore still interpreted with respect to the volume preserving gauge, but are treated as functions of $\tau$. For the scale factor on the other hand, we have

$$a^3 H = \frac{1}{a} \frac{da}{d\tau} \equiv H_{\text{FLRW}} ; \quad \bar{a}^6 \left( \frac{\ddot{a}}{a} + 3H^2 \right) = \frac{1}{a} \frac{d^2 a}{d\tau^2}. \quad (57)$$

Further writing

$$\rho(\tau) = \bar{\rho}(\bar{t}(\tau)) ; \quad p(\tau) = \bar{p}(\bar{t}(\tau)), \quad (58)$$
équations (47) become

$$H_{\text{FLRW}}^2 = \frac{8\pi G N}{3} \rho - \frac{1}{6} \left[ Q^{(1)} + S^{(1)} \right], \quad (59a)$$

$$\frac{1}{a} \frac{d^2 a}{d\tau^2} = -\frac{4\pi G N}{3} (\rho + 3p) + \frac{1}{3} Q^{(1)}. \quad (59b)$$

We emphasize that the quantities $Q^{(1)}$ and $S^{(1)}$, defined in Eqns. (48a) and (48b) as correlations in the volume preserving gauge, are to be thought of as functions of the synchronous time $\tau$, where the coordinate $\tau$ itself was defined after the spatial averaging. Such an identification is justified since we are dealing with scalar combinations of these quantities. Note that $Q^{(1)}$ and $S^{(1)}$ can be treated independently, apart from the constraints imposed by conservation conditions, which we turn to next. These conservation conditions can be decomposed into a scalar part and a 3-vector part, given respectively by

$$\ddot{a} - ru^b C^a_{b\alpha} = 0 ; \quad \dot{x}_k C^a_{b\alpha} = 0. \quad (60)$$

In the synchronous gauge (56) for the FLRW metric, the scalar equation reads

$$(\partial_\tau Q^{(1)} + 6H_{\text{FLRW}} Q^{(1)}) + (\partial_\tau S^{(1)} + 2H_{\text{FLRW}} S^{(1)}) = 0. \quad (61)$$

We recall that this equation is a consequence of setting the correlation 3-form and the correlation 4-form to zero, and it relates the evolution of $Q^{(1)}$ and $S^{(1)}$. The 3-vector equation (on imposing the first set of conditions in Eqn. (19)) simply gives $\partial_\tau C^a_A = 0$, so that $C^a_A = 0 = \text{constant}$, which also implies that $C^A_\tau = 0 = \text{constant}$ and hence this equation gives nothing new. (We have used the relations $C^0_0 = C^\tau_\tau, C^0_A = \bar{a}^3 C^A_A$ and $C^A_0 = (1/\bar{a}^3)C^A_\tau$ where 0 denotes the nonsynchronous time coordinate $\bar{t}$.)

The cosmological equations (59), along with the constraint equations (55) and (61) are the key results of this section. Subject to the acceptance of the volume preserving gauge on the underlying manifold $\mathcal{M}$ they can in principle be used to study the role of the correction terms resulting from spatial averaging.

### 6.1 A comparison with the averaging formalism of Buchert

The averaging formalism developed by Buchert is based exclusively on the manifold $\mathcal{M}$, and there is no analog of the averaged manifold $\bar{\mathcal{M}}$ in this scheme. Given an inhomogeneous metric on $\mathcal{M}$ one
takes the trace of the Einstein equations in the *inhomogeneous* geometry, and carries out a spatial averaging of the inhomogeneous scalar equations.

For ease of comparison, we again recall in brief Buchert’s construction, by first writing down the averaged equations for the simplest case of pressureless and irrotational inhomogeneous dust. The metric can be written in synchronous and comoving gauge as

$$ds^2 = -dt^2 + b_{AB}(x,t)dx^A dx^B .$$  \hfill (62)

The Einstein equations can be split into a set of scalar equations and a set of vector and traceless tensor equations. The scalar equations are the Hamiltonian constraint \((63a)\) and the evolution equation for \(\Theta \) \((63b)\),

\[
\mathcal{R} + \frac{2}{3} \Theta^2 - 2\sigma^2 = 16\pi G \rho ,
\]

\[
\mathcal{R} + \partial_t \Theta + \Theta^2 = 12\pi G \rho ,
\]

where \(\mathcal{R}\) is the Ricci scalar of the 3-dimensional hypersurface of constant \(t\), \(\Theta\) and \(\sigma^2\) are the expansion scalar and the shear scalar defined earlier and \(\rho\) is the inhomogeneous matter density of the dust. Note that all quantities in Eqns. \((63)\) generically depend on both position \(x\) and time \(t\). Eqns. \((63a)\) and \((63b)\) can be combined to give Raychaudhuri’s equation

\[
\partial_t \Theta + \frac{1}{3} \Theta^2 + 2\sigma^2 + 4\pi G \rho = 0 .
\]

\hfill (64)

The continuity equation \(\partial_t \rho = -\Theta \rho\) which gives the evolution of \(\rho\), is consistent with Eqns. \((63a)\), \((63b)\). Only scalar Einstein equations are considered, since the spatial average of a scalar quantity can be defined in a gauge covariant manner, within a given foliation of space-time. We return to this point below. For the space-time described by \((62)\), the spatial average of a scalar \(\Psi(x,t)\) over a *comoving* domain \(D\) at time \(t\) is defined by

\[
\langle \Psi \rangle_D = \frac{1}{V_D} \int_D d^3x \sqrt{b} \Psi ,
\]

\hfill (65)

where \(b\) is the determinant of the 3-metric \(b_{AB}\) and \(V_D\) is the volume of the comoving domain given by \(V_D = \int_D d^3x \sqrt{b}\). Spatial averaging is, by definition, not generally covariant. Thus the choice of foliation is relevant, and should be motivated on physical grounds. In the context of cosmology, averaging over freely-falling observers is a natural choice, especially when one intends to compare the results with standard FLRW cosmology. Following the definition \((65)\) the following commutation relation then holds

\[
\partial_t \langle \Psi \rangle_D - \langle \partial_t \Psi \rangle_D = \langle \Psi \Theta \rangle_D - \langle \Psi \rangle_D \langle \Theta \rangle_D ,
\]

\hfill (66)

which yields for the expansion scalar \(\Theta\)

\[
\partial_t \langle \Theta \rangle_D - \langle \partial_t \Theta \rangle_D = \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 .
\]

\hfill (67)

Introducing the dimensionless scale factor \(a_D \equiv (V_D/V_D_i)^{1/3}\) normalized by the volume of the domain \(D\) at some initial time \(t_i\), we can average the scalar Einstein equations \((63a)\), \((63b)\) and the continuity equation to obtain

\[
\partial_t \langle \rho \rangle_D = -\langle \Theta \rangle_D \langle \rho \rangle_D ; \quad \langle \Theta \rangle_D = 3 \frac{\partial_t a_D}{a_D} ,
\]

\hfill (68a)
\[
\left( \frac{\partial_t a_D}{a_D} \right)^2 = \frac{8\pi G}{3} \langle \rho \rangle_D - \frac{1}{6} (Q_D + \langle R \rangle_D), \quad (68b)
\]

\[
\left( \frac{\partial^2 a_D}{a_D} \right) = -\frac{4\pi G}{3} \langle \rho \rangle_D + \frac{1}{3} Q_D. \quad (68c)
\]

Here, the ‘kinematical backreaction’ \( Q_D \) is given by

\[
Q_D = \frac{2}{3} \left( \langle (\Theta^2)_D \rangle - \langle \Theta \rangle^2_D \right) - 2 \langle \sigma^2 \rangle_D \quad (69)
\]

and is a spatial constant over the domain \( D \).

A necessary condition for (68c) to integrate to (68b) takes the form of the following differential equation involving \( Q_D \) and \( \langle R \rangle_D \),

\[
\partial_t Q_D + 6 \frac{\partial_t a_D}{a_D} Q_D + \partial_t \langle R \rangle_D + 2 \frac{\partial_t a_D}{a_D} \langle R \rangle_D = 0. \quad (70)
\]

The equations above describe the essence of Buchert’s averaging formalism, for the dust case. We note that the remaining eight Einstein equations for the inhomogeneous geometry, which are not scalar equations, are not averaged. These are the five evolution equations for the trace-free part of the shear,

\[
\partial_t \left( \sigma^A_B \right) = -\Theta \sigma^A_B - R^A_B + \frac{2}{3} \delta^A_B \left( \sigma^2 - \frac{1}{3} \Theta^2 + 8\pi G \rho \right), \quad (71)
\]

and the three equations relating the spatial variation of the shear and the expansion,

\[
\sigma^A_{B||A} = \frac{2}{3} \Theta_{||B}. \quad (72)
\]

Here, \( R^A_B \) is the spatial Ricci tensor and, in Buchert’s notation, a || denotes covariant derivative with respect to the 3-metric.

In analogy with the dust case, Buchert’s averaging formalism can be applied to the case of a perfect fluid, by starting from the metric

\[
ds^2 = -N^2 dt^2 + b_{AB} dx^A dx^B. \quad (73)
\]

The averaged scalar Einstein equations for the scale factor \( a_D \) are

\[
3 \frac{\partial^2 a_D}{a_D} + 4\pi G \left( N^2 (\rho + 3p) \right)_D = \bar{Q}_D + \bar{P}_D, \quad (74)
\]

\[
6H^2_D - 16\pi G \left( N^2 \rho \right)_D = -\bar{Q}_D - \left( N^2 R \right)_D; \quad H_D = \frac{\partial_t a_D}{a_D}, \quad (75)
\]

where the kinematical backreaction \( \bar{Q}_D \) is given by

\[
\bar{Q}_D = \frac{2}{3} \left( \langle (N\Theta)^2 \rangle_D - \langle N\Theta \rangle^2_D \right) - 2 \langle N^2 \sigma^2 \rangle_D, \quad (76)
\]

and the dynamical backreaction \( \bar{P}_D \) is given by

\[
\bar{P}_D = \langle N^2 \dot{A} \rangle_D + \langle \Theta \partial_t N \rangle_D, \quad (77)
\]

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where \( \mathcal{A} = \nabla_j (u^i \nabla_i u^j) \) is the 4-divergence of the 4-acceleration of the fluid. Eqn. (75) follows as an integral from Eqn. (74) if and only if the relation

\[
\partial_t Q_D + 6H_D Q_D + \partial_t \langle N^2 \mathcal{R} \rangle_D + 2H_D \langle N^2 \mathcal{R} \rangle_D + 4H_D \bar{P}_D - 16\pi G \left[ \partial_t \langle N^2 \rho \rangle_D + 3H_D \langle N^2 (\rho + p) \rangle_D \right] = 0 ,
\]

is satisfied. There are also the unaveraged equations (which we do not display here) for the shear, analogous to the shear equations (71) and (72) for dust.

Buchert’s approach is the only other approach, apart from Zalaletdinov’s MG, which is capable of treating inhomogeneities in a nonperturbative manner, although it is limited to using only scalar quantities within a chosen 3 + 1 splitting of spacetime. Buchert takes the trace of the Einstein equations in the inhomogeneous geometry, and averages these inhomogeneous scalar equations. In the context of Zalaletdinov’s MG however, we have used the existence of the vector field \( \bar{v}^a \) in the FLRW spacetime to construct scalar equations after averaging the full Einstein equations. As far as observations are concerned, it has been noted by Buchert and Carfora that the spatially averaged matter density \( \langle \rho \rangle_D \) defined by Buchert is not the appropriate observationally relevant quantity – the “observed” matter density (and pressure) is actually defined in a homogeneous space. Since we have done precisely this in Eqn. (27), we are directly dealing with the appropriate observationally relevant quantity in the MG framework.

Another important difference between the two approaches is the averaging operation itself. Buchert’s spatial average, defined for scalar quantities, is given (for some scalar \( \Psi(t, x^A) \)) by (65) above. On the other hand the averaging operation we have been using (given by Eqn. (33) using the volume preserving gauge) is a limit of a spacetime averaging defined using the coordination bivector \( W^a_{j} \), and is different from the one in Eqn. (65).

Most importantly though, Buchert’s averaging scheme by itself does not incorporate the concept of an averaged manifold \( \mathcal{M} \) (although the work of Buchert and Carfora [2] does deal with 3-spaces of constant curvature). In a recent paper we had argued that Buchert’s “effective scale factor” \( a_D(t) \equiv (V_D(t)/V_D(t_{in}))^{1/3} \) must be the scale factor for the metric of the averaged manifold, upto some corrections arising due to such effects as calculated by Buchert and Carfora. In the present work however, it is clear that such a suggestion is necessarily incomplete due to the presence of Eqns. constraining the underlying geometry. These constraints are in general nontrivial and hence indicate that it is not sufficient to assume that the metric of the inhomogeneous manifold averages out to the FLRW form – there are additional conditions which the correlations must satisfy.

To our understanding, Buchert’s averaging formalism is a valid approach, even though it is based on a spatial averaging. A central difference from the MG approach is the issue of closure: not all the Einstein equations have been averaged in Buchert’s approach, but only the scalar ones. This puts a constraint on the allowed solutions considered for the averaged equations: (68) for the dust case, and (74) and (75) for the fluid case. Solutions to these equations must necessarily be checked for consistency with the unaveraged equations for the shear. Further, averaging over successively larger scales can bring in additional corrections to the averaged equations, as discussed by Buchert and Carfora. Also, if one does not wish to identify \( a_D \) with the scale factor in FLRW cosmology, one is compelled to develop a whole new set of ideas in order to try and compare theory with observation. On the other hand, if one does identify \( a_D \) with the scale factor, comparison with standard cosmology becomes more convenient, but this brings in additional constraints on the underlying inhomogeneous geometry. Thus our conclusion is that the Buchert formalism is a correct and tractable averaging scheme, provided all the caveats pointed out in this paragraph are
taken care of. Also, when these caveats have been taken care of correctly, the Buchert formalism is expected to give the same physical results as the MG approach. We recall that in the covariant MG approach also, once a spacetime geometry has been identified for the averaged manifold $\mathcal{M}$, a gauge must be selected for the geometry on the underlying manifold, in order to explicitly compute the correction scalars for comparison with observation.

The advantage of the MG approach is that it accomplishes in a neat package what the Buchert approach, with its attendant caveats, sets out to do. In the MG approach, there are no unaveraged shear equations, because the trace of the Einstein equations has been taken after performing the averaging on the underlying geometry. Since the averaged geometry is FLRW, the shear is zero by definition. There is a natural metric on the averaged manifold by construction, the FLRW metric. The correlations satisfy additional constraints, given by Eqns. (55). Thus, once a gauge has been chosen and if one can overcome the computational complexity of the averaging operation, the cosmological equations derived by us in the MG approach are complete and ready for application, without any further caveats.

In spite of these differences, our equations (59) and (61) for the volume preserving gauge are strikingly similar to Buchert’s effective FLRW equations and their integrability condition in the dust case; and in the case of general $N$, the role of Buchert’s dynamical backreaction $\bar{P}_D$ in Eqns. (74) and (78) is identical to that of our combination of $(\bar{S}^{(2)} + \tilde{S}^{(2)})$. Concentrating on the volume preserving case, the structure of the correlation $Q^{(1)}$ is identical to Buchert’s kinematical backreaction $Q_{D}$ (or $\bar{Q}_{D}$ in the general case). The correlation $\bar{S}^{(1)}$ appears in place of the averaged 3-Ricci scalar $\langle R \rangle_D$ in Buchert’s dust equations. This is not unreasonable since Buchert’s $\langle R \rangle_D$ can be thought of as $\langle R \rangle_D = 6k_D/a_D^2 + \text{corrections}$, where $6k_D/a_D^2$ represents the 3-Ricci scalar on the averaged manifold which in our case is zero, and hence $S^{(1)}$ represents the corrections due to averaging. Further, these similarities are in spite of the fact that our correlations were defined assuming that a volume preserving gauge averages out to the FLRW 3-metric in standard form, whereas Buchert’s averaging is most naturally adapted to beginning with a synchronous gauge. This remarkable feature, at least to our understanding, does not seem to have any deeper meaning – it simply seems to arise from the structure of the Einstein equations themselves, together with our assumption $D_{ab}Z_{ij} = 0$. In the absence of this latter condition, one would have to consider the correlation 3- and 4-forms mentioned earlier, and the structure of the correlation terms and their “conservation” equations would be far more complicated.

An entirely different outlook towards his approach has been emphasized to us by Buchert. According to Buchert, the absence of an averaged manifold $\bar{M}$ is not to be thought of as a ‘caveat’, but as a feature deliberately retained ‘on purpose’. The actual inhomogeneous Universe is regarded by Buchert as the only fundamental entity, and the introduction of an averaged Universe is in fact regarded as an unphysical and unnecessary approximation. As we mentioned earlier, this is probably the most important difference between MG and Buchert’s approach. In the latter, contact with observations is to be made by constructing averaged quantities, such as the scalars defined earlier in this section, and by introducing the expansion factor $a_D$. The assertion here is that the averaging of geometry, as discussed in MG or in the Renormalization Group approach of Buchert and Carfora [2], is not an indispensable step in comparing the inhomogeneous Universe with actual observations. The need for averaging of geometry is to be physically separated from simply looking at effective properties (such as the constructed scalars) which can be defined for any inhomogeneous metric. Averaging of geometry becomes relevant if (i) an observer insists on interpreting the data in a FLRW template model, so that (s)he needs a mapping from the actual inhomogeneous slice and its average properties to the corresponding properties in this template, or (ii) one desires a
mock metric, to sort of have a thermodynamic effective metric to approximate the real one. In this context it should perhaps also be mentioned that the importance of a thin time-slice approximation of spacetime averaging (as opposed to a strict spatial averaging) has been stressed also by Buchert.

7 Perturbation theory, structure formation, and backreaction

We have in hand the machinery to ask the following question: Is cosmological perturbation theory stable against growth of backreaction? The answer must be found iteratively. Assume a background with perturbations on it, calculate the back-reaction, feed it in the right hand of the modified Friedmann equations to find the new background, and so on:

\[ a^{(0)} \rightarrow \phi^{(0)} \rightarrow C^{(0)} \rightarrow a^{(1)} \rightarrow \phi^{(1)} \rightarrow \ldots \] (79)

Let the perturbed FLRW metric be

\[ ds^2 = a^2 \left[ -(1 + 2\phi) d\eta^2 + 2\omega_A dx^A d\eta + ((1 - 2\psi) \gamma_{AB} + \chi_{AB}) dx^A dx^B \right]. \] (80)

We work with a VPC which has no residual degrees of freedom. Further, this VPC is constructed by starting from the conformal Newtonian gauge, and by making a steady coordinate transformation. This ensures that all averaged quantities are gauge invariant. We evaluate the correlation scalars for a given initial power spectrum - standard CDM.

\[ \frac{k^3 P_{\phi^2}(k)}{2\pi^2} = A(k/H_0)^{n_s - 1}, \] (81)

the back reaction is

\[ \frac{S^{(1)}}{H_0^2} \sim -\frac{1}{a^2} (10^{-4}). \] (82)

The smallness of backreaction holds also for the exact sCDM model thus demonstrating the stability of perturbation theory against the growth of back-reaction.

This analysis ignores contribution of scales that have become fully nonlinear in matter density at late times and it is important to ask if structure formation can significantly modify large scale dynamics.

We studied backreaction in a toy model of spherical collapse, using the LTB solution. The initial density is chosen to be

\[ \rho(t_i, r) = \rho_{bi} \begin{cases} 
(1 + \delta_*), & r < r_* \\
(1 - \delta_v), & r_* < r < r_v \\
1, & r > r_v, 
\end{cases} \] (83)

We match the initial velocity and coordinate scaling to the global background solution, by requiring

\[ R(t_i, r) = a_i r, \] (84)
\[ \dot{R}(t_i, r) = a_i H_i r, \] (85)
For the FLRW background we consider an Einstein-deSitter (EdS) solution with scale factor and Hubble parameter given by
\[ a(t) = \left( \frac{t}{t_0} \right)^{2/3} ; \quad t_0 = \frac{2}{(3H_0)} , \]
\[ H(t) \equiv \frac{\dot{a}}{a} = \frac{2}{(3t)} , \]
with \( t_0 \) denoting the present epoch. \( a_i \) fixes the initial time as
\[ t_i = \frac{2}{(3H_0)} a_i^{3/2} . \]
We use \( a_i = 10^{-3}, \) so that the initial conditions are being set around the CMB last scattering epoch.

The mass function \( M(r) \) and curvature function \( k(r) \) in this LTB solution are given by
\[ GM(r) = \frac{1}{2} H_0^2 r^3 \begin{cases} 
1 + \delta_s , & 0 < r < r_s \\
1 + \delta_v \left((\delta_v/r)^3 - 1\right) , & r_s < r < r_v \\
1 + (\delta_v/r^3) (r_c^3 - r_v^3) , & r > r_v , 
\end{cases} \]
where we have defined a “critical” radius \( r_c \) by the equation
\[ \left( \frac{r_c}{r_s} \right)^3 = 1 + \frac{\delta_s}{\delta_v} . \]
The significance of \( r_c \) is brought out by \( k(r) : \)
\[ k(r) = \frac{H_0^2}{a_i} \begin{cases} 
\delta_s , & r < r_s \\
\delta_v \left((\delta_v/r)^3 - 1\right) , & r_s < r < r_v \\
(\delta_v/r^3) (r_c^3 - r_v^3) , & r > r_v . 
\end{cases} \]
Since \( \delta_s, \delta_v > 0, \) we have \( r_c > r_s \) by definition. The following possibilities arise:

If \( r_c > r_v, \) then \( k(r) > 0 \) for all \( r, \) and every shell will ultimately collapse, including the “void” region \( r_s < r < r_v \). If \( r_c < r_v, \) then \( k(r) > 0 \) for \( r < r_c \) and changes sign at \( r = r_c \). Hence, the region \( r_s < r < r_c \) will collapse even though it is underdense, while the region \( r > r_c \) will expand forever. If \( r_c = r_v, \) then the “void” exactly compensates for the overdensity, and the universe is exactly EdS for \( r > r_v. \)

Transforming to the perturbed FLRW form : We want a coordinate transformation \((t, r) \rightarrow (\tau, \tilde{r})\) such that the metric in the new coordinates is
\[ ds^2 = -(1 + 2\phi)d\tau^2 + \alpha^2(\tau)(1 - 2\psi) \left( d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \right) , \]
with at least the conditions
\[ | \phi | \ll 1 ; \quad | \psi | \ll 1 , \]
being satisfied. Since \( t \) is the proper time of each matter shell, the quantity \( \partial \tilde{r} / \partial \tau \) is simply the velocity of matter in the \((\tau, \tilde{r})\) frame (which is comoving with the Hubble flow) :
\[ \tilde{v} \equiv \frac{\partial \tilde{r}}{\partial \tau} , \]
is the radial comoving peculiar velocity of the matter shells in the \((\tau, \tilde{r})\) frame. We showed that the required transformation exists, provided matter peculiar velocities remain small, which is consistent with what has been shown by other authors, and is true for the observed Universe.
In the cosmological equations derived from Macroscopic Gravity we already have in place the formalism for calculating the backreaction when the metric is of the perturbed FLRW form. From there it follows that the backreaction is very small, in the nonlinear structure formation regime, provided matter peculiar velocities are small. It can be argued that this result is independent of the assumption of spherical symmetry in the toy model. The situation could be very different though, if there are dominant nonlinear structures in today’s Universe, comparable to the Hubble radius.

7.1 Perturbation theory around a background - the shortwave approximation

Green and Wald [22] have recently given an analysis of the growth of metric perturbations, assuming that the metric is always close to a given background, although matter perturbations can be arbitrarily large. No averaging of an underlying spacetime geometry is done, and it is assumed that there is a homogeneity length scale at around 100 Mpc, much smaller than the Hubble radius. It is shown that if the small-scale motions of matter inhomogeneities are non-relativistic, the deviations from the background metric are small, and well-described by Newtonian gravity. This result tallies with what has been found by others before, including us. It is further shown that subject to the matter satisfying weak energy condition, the effect of small scale inhomogeneities on large scale dynamics is to produce an effective trace-free stress energy tensor. One might ask if this traceless nature of the correction has to do with no averaging over finite volumes being carried out.

Thus the assumption of non-relativistic peculiar velocities along with the assumption of a homogeneity scale much smaller than the Hubble radius strongly suggest a negligible effect of small scale inhomogeneities on the average large-scale dynamics. The first of these two assumptions is well supported by observations. There is no observational evidence against the second assumption, but nor is it firmly established by observations. If this assumption is correct, either a small cosmological constant, or a modification of general relativity on large scales, is indicated by the observed cosmic acceleration. If this assumption turns out to be not correct, the effect of inhomogeneities could be significant, and remains an important question for further investigation.

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