Semi-analytical RWA formalism to solve Schrödinger equations for multi-qudit systems with resonator couplings

H.W.L. Naus
TNO, P.O. Box 96864, 2509 JG The Hague, The Netherlands and QuTech, Delft University of Technology, P.O. Box 5046, 2600 GA Delft, The Netherlands

R. Versluis
TNO, P.O. Box 155, 2600 AD Delft, The Netherlands and QuTech, Delft University of Technology, P.O. Box 5046, 2600 GA Delft, The Netherlands

(Dated: November 27, 2018)

In this study, we develop a semi-analytical framework to solve generalized Jaynes-Tavis-Cummings Hamiltonians describing multi-qudit systems coupled via EM resonators. Besides the multi-level generalization we allow for an arbitrary number of resonators and/or modes, with nonidentical couplings to the qudits. Our method is based on generic excitation-number operators which commute with the respective Hamiltonians in the rotating wave approximation (RWA). The validity of the RWA is assessed explicitly. The formalism enables the study of eigenstates, eigenenergies and corresponding time evolutions of such coupled multi-qudit systems. The technique can be applied in cavity quantum electrodynamics and circuit quantum electrodynamics. It is also applicable to atomic physics, describing the coupling of a single-mode photon to an atom. As an example, we solve the Schrödinger equation for a two-qubit-one-resonator system, in principle to arbitrary high excitations. We also solve the Tavis-Cummings Hamiltonian in the one-excitation subspace for an arbitrary number of identical qubits resonantly coupled to one resonator. As a final example, we calculate of the low-excitation spectrum of a coupled two-transmon system.

I. INTRODUCTION

Scalable quantum processing eventually using a large number of qubits [1] and enabling fault-tolerant quantum computing is a topical subject; see [2–5] and references therein. In order to perform the required two-qubit gates or multi-qubit gates to reach computational universality [6, 7] the qubits need to interact, e.g. using electromagnetic (EM) resonators. Examples include transmon, Xmon and fluxmon qubits with fixed [8–11] or tuneable [3] couplings as well as spin qubits connected to resonators [12–14]. To determine the dynamic behavior of multi-qubit systems, the Schrödinger equation governed by the corresponding Hamiltonian needs to be solved. In the simple case of a qubit interacting with a single resonator this is the well-known Jaynes-Cummings (JC) Hamiltonian [15] with known exact solutions. A more complex example is the collective interaction of multiple identical qubits equally coupled to one single-mode resonator as described by the Tavis-Cummings (TC) Hamiltonian. Exact and approximate solutions have been derived for the TC model [16–19].

In this study, we develop a calculational framework solving Schrödinger equations for multi-qudit systems which are coupled via EM resonators. Besides the multi-level generalization we allow for an arbitrary number of resonators and/or modes, with non-identical couplings to the qudits. The semi-analytical formalism also incorporates qudits with different energy levels. The commonly used rotating wave approximation (RWA) is also adopted here [20]. We assess its applicability in perturbation theory [21]. Given the RWA, the methods are exact and semi-analytical in the sense that only finite-dimensional matrices are eventually numerically diagonalized, in principle limited only by the available computing and memory resources. The technique can be applied in cavity quantum electrodynamics and circuit quantum electrodynamics. It is also applicable to atomic physics, describing the coupling of a single-mode photon to an atom.

Some caveats are appropriate to include at the onset. First, it is tacitly assumed that the dimensions of the described physical systems are small compared to the wavelengths of the considered radiation modes. Secondly, we neither address finite temperature effects nor imperfect cavities. Only ideal, closed systems are considered. Thirdly, the qubits/qudits and the EM modes are not too far from resonance and their coupling should not be ultrastrong - otherwise the RWA would be a priori invalid. Finally, throughout this paper we use the concept of multiple single-mode resonators. Our framework covers multiple EM modes in one cavity equally well; care should be taken in the limit of an infinite number of modes [22].

The outline of this paper is as follows. In order to fix our notation and to introduce the method we start by re-analysing the generic JC Hamiltonian. Next we explicitly assess the validity of the RWA in the JC model. In section [V] we extend the formalism based on the excitation-number to Hamiltonians describing multiple qubits and resonators with arbitrary couplings. The technique is applied to a system of two qubits and one resonator in section [V]. For this specific problem we show that the applicability extends to arbitrary high excitations.
Section \[\text{VII}\] addresses an arbitrary number of identical qubits resonantly coupled to one resonator. We explicitly solve the Schrödinger equation for the one-excitation subspace. The framework is generalized further to multilevel systems called qudits in section \[\text{VIII}\]. The concrete example in section \[\text{VIII}\] presents the calculation of the low-excitation spectrum of the Hamiltonian modeling a system consisting of two qutrits coupled to a single resonator. Finally, in section \[\text{IX}\] we give the formal expression for the time evolution operator of coupled multiqutrit systems. The paper is concluded with a summary.

II. PRELUDE: JAYNES-CUMMINGS PHYSICS

We first present the generic Jaynes-Cummings (JC) Hamiltonian \[\text{(1)}\] describing the coupling of a single-mode photon to an atom in the two-level approximation

\[
H = -\frac{1}{2}\hbar \omega' \sigma_z + \hbar \omega (a^\dagger a + \frac{1}{2}) + \hbar g (a^\dagger \sigma_+ + a \sigma_-),
\]

where \(g\) is the coupling strength. The photon mode, frequency \(\omega\), is described by creation and annihilation operators \(a, a^\dagger\) which satisfy the canonical commutation relation \([a, a^\dagger] = 1\) of a harmonic oscillator. We use standard Pauli matrices with spin raising and lowering operators \(\sigma_\pm = \frac{1}{2}(\sigma_x \pm i \sigma_y)\). Note that the RWA has already been included. This problem can be solved exactly \[\text{(2)}\] \[\text{23–25}\]. In order to further fix our notation and to introduce the method which is generalized below, we present the derivation of the solution. The standard qubit states satisfy

\[
\begin{align*}
\sigma_- |0\rangle &= |0\rangle, & \sigma_+ |0\rangle &= 0, & \sigma_+ |1\rangle &= |0\rangle, \\
\sigma_- |1\rangle &= -|1\rangle, & \sigma_- |0\rangle &= |1\rangle, & \sigma_- |1\rangle &= 0.
\end{align*}
\]

Unperturbed oscillator states are denoted by \(|n\rangle\). The excitation-number operator is defined as

\[
\mathcal{N} = a^\dagger a - \frac{1}{2} \sigma_z,
\]

which, as can be verified by explicit calculation, commutes with the JC Hamiltonian \[\text{(1)}\] \([\mathcal{N}, H] = 0\). It implies that a common set of eigenstates of \(\mathcal{N}\) and \(H\) exists. The eigenstates of \(\mathcal{N}\) are the product states

\[
\begin{align*}
\mathcal{N} |n, 0\rangle &= (n - \frac{1}{2}) |n, 0\rangle, \\
\mathcal{N} |n, 1\rangle &= (n + \frac{1}{2}) |n, 1\rangle.
\end{align*}
\]

For \(n = 0\) we get the lowest eigenvalue of \(\mathcal{N}\), which equals \(-\frac{1}{2}\). The concomitant eigenstate also satisfies

\[
H |0, 0\rangle = -\frac{1}{2} \hbar \Delta |0, 0\rangle,
\]

where \(\Delta = \omega' - \omega\), cf. \[\text{22}\], and equals the ground state of the Hamiltonian. Other eigenvalues of \(\mathcal{N}\) are \(n - \frac{1}{2}, n \geq 1\) with degenerate eigenstates. In order to find the energy eigenvalues and the excited states of the JC Hamiltonian we therefore make the Ansatz

\[
|E\rangle = |\alpha_n n, 0\rangle + |\beta_n n - 1, 1\rangle.
\]

For each \(n \geq 1\), we obtain a two-dimensional subspace, called RWA strip in \[\text{24}\], in which we have to diagonalize the corresponding \(2 \times 2\) matrix, yielding the energy eigenvalues

\[
E_n^\pm = n \hbar \omega \pm \frac{1}{2} \hbar \sqrt{4g^2 n + \Delta^2}
\]

and the exact orthonormal solutions

\[
\begin{align*}
|E_n^+\rangle &= |\alpha_n^+ |n, 0\rangle + |\beta_n^+ |n - 1, 1\rangle, \\
|E_n^-\rangle &= |\alpha_n^- |n, 0\rangle + |\beta_n^- |n - 1, 1\rangle.
\end{align*}
\]

The coefficients are given by

\[
\begin{align*}
\alpha_n^+ &= \rho_n^{-1} \left(\sqrt{g^2 n + \frac{1}{4} \Delta^2} - \frac{\Delta}{2}\right) = \beta_n^-, \\
\beta_n^+ &= \rho_n^{-1} g \sqrt{n} = -\alpha_n^-,
\end{align*}
\]

where \(\rho_n^2 = 2g^2 n + \Delta^2 - \sqrt{g^2 n + \frac{1}{4} \Delta^2}\) are normalization factors. Equivalently, the coefficients are expressed in terms of angles \[\text{13–23}\] \[\text{24}\] :

\[
\alpha_n^+ = \cos \theta_n, \quad \beta_n^+ = \sin \theta_n \quad \text{with}
\tan (2\theta_n) = -\frac{2g \sqrt{n}}{\Delta}.
\]

The splitting in the \(\pm\) energy levels \[\text{(7)}\] corresponds to the Rabi frequencies of the two-level subsystems \[\text{11} [\text{27}]. The eigenstates \(|E_n^\pm\rangle\) are known as dressed states in the literature \[\text{23–25}\].

III. VALIDITY OF THE RWA

The validity of the RWA is assessed by treating the omitted interaction terms in perturbation theory. To this end, we straightforwardly calculate the resulting shifts in eigenenergies. The in the RWA omitted terms correspond to the interaction Hamiltonian

\[
H_p = -\hbar g (\sigma_- a^\dagger + a \sigma_+).
\]

They are associated with high frequencies and are therefore usually neglected \[\text{24}\]. Alternatively, it is proposed to control these interactions \[\text{27}\] or, for high photon numbers, go beyond the RWA \[\text{24}\]. Here \(H_p\) will be treated as a perturbation. If we denote the exact energies by \(E_0\) and \(E_n^\pm\), we do not obtain any modification up to first order \(E_0 \simeq E_0, \quad E_n^\pm \simeq E_n^\pm\). Note, however, that the eigenstates are modified in first order. Though easily calculable, we omit the explicit expressions.
The energy eigenvalues including second order corrections are given by

\[ \tilde{E}_0 \simeq E_0 + \hbar^2 g^2 \left( \frac{(\beta_2^+)^2}{E_0 - E_0^+} + \frac{(\beta_2^-)^2}{E_0 - E_0^-} \right), \]

\[ \tilde{E}_1^\pm \simeq E_1^\pm + 2\hbar^2 g^2 \left( \frac{(\alpha_1^\pm \beta_3^+)^2}{E_1^\pm - E_3^+} + \frac{(\alpha_1^\pm \beta_3^-)^2}{E_1^\pm - E_3^-} \right), \]

and for \( n \geq 3 \)

\[ \tilde{E}_n^\pm \simeq E_n^\pm + (n+1)\hbar^2 g^2 \left( \frac{(\alpha_n^\pm \beta_{n+2}^+)^2}{E_n^\pm - E_{n+2}^+} + \frac{(\alpha_n^\pm \beta_{n+2}^-)^2}{E_n^\pm - E_{n+2}^-} \right) + (n-1)\hbar^2 g^2 \left( \frac{(\alpha_n^\pm \beta_{n-2}^+)^2}{E_n^\pm - E_{n-2}^+} + \frac{(\alpha_n^\pm \beta_{n-2}^-)^2}{E_n^\pm - E_{n-2}^-} \right). \]

Figures 1 and 2 depict some typical results of the perturbative approach for the four lowest excitation levels. The qubit frequency is taken as \( f' = 6 \text{ GHz} \) and the resonator is tuned at \( f = 7 \text{ GHz} \). In order to visualize the effects, the range of the coupling is extended to 10 GHz. For couplings well below the qubit and resonator frequencies, in practice below 1 GHz, the corrections due to omitted terms are small: less than \(< 0.4\%\) for the ground state and less than \(< 0.04\%\) for \( n = 1, 2, 3 \). These results \textit{a priori} justify the RWA in such ranges.

![Figure 1](image1.png)

**FIG. 1.** Energy levels \( E/\hbar \) of the ground state (l.h.s.) and first excitation (r.h.s.) of a single qubit coupled to a single resonator as function of the coupling \( g \) without and with second order RWA correction.

![Figure 2](image2.png)

**FIG. 2.** Energy levels \( E/\hbar \) of the second excitation (l.h.s.) and third excitation (r.h.s.) of a single qubit coupled to a single resonator as function of the coupling \( g \) without and with second order RWA correction.

### IV. MULTIPLE QUBITS AND RESONATORS

Next we consider a system of \( K \) qubits and \( P \) resonators. We allow the coupling of each resonator to each qubit. The generalized JC Hamiltonian of such a system
is in RWA given by

\[ H = \sum_{k=1}^{K} H_0^{[k]} + \sum_{p=1}^{P} H_{\text{res}}^{[p]} + \sum_{i=1}^{P} \sum_{j=1}^{K} H_{\text{int}}^{ij}. \]  

(13)

We have introduced qubit, resonator and interaction Hamiltonians respectively as

\[ H_0^{[k]} = -\frac{1}{2} \hbar \omega_k \sigma_z^{[k]}, \]
\[ H_{\text{res}}^{[p]} = \hbar \omega_p (a_{\text{p}}^\dagger a_{\text{p}} + \frac{1}{2}), \]
\[ H_{\text{int}}^{ij} = \hbar g_{ij} (a_1^\dagger \sigma_+^{[j]} + a_1 \sigma_-^{[j]}). \]  

(14)

In realistic systems, many couplings out of the set \( g_{ij} \) may vanish. The corresponding excitation-number operator \( N \) reads in this case

\[ N = \sum_{q=1}^{P} a_q^\dagger a_q - \frac{1}{2} \sum_{s=1}^{K} \sigma_z^{[s]}. \]  

(15)

In order not to overload the notation we use the same symbol for the various excitation-number operators. Obviously \( N \) commutes with the ‘free’ Hamiltonians \([N, H_0^{[k]}] = 0\), \([N, H_{\text{res}}^{[p]}] = 0\). Next we calculate the commutator of the excitation-number operator and the interaction Hamiltonians

\[ [N, H_{\text{int}}^{ij}] = \hbar g_{ij} \left( \sum_{q=1}^{P} \delta_{qj} (a_1^\dagger \sigma_+^{[j]} - a_1 \sigma_-^{[j]}) \right. \]
\[ \left. - \frac{1}{2} \sum_{s=1}^{K} \delta_{sj} (2a_1^\dagger \sigma_+^{[j]} - 2a_1 \sigma_-^{[j]}) \right) \]
\[ = \hbar g_{ij} \left( a_1^\dagger \sigma_+^{[j]} - a_1 \sigma_-^{[j]} - a_1^\dagger \sigma_+^{[j]} + a_1 \sigma_-^{[j]} \right) = 0. \]  

(16)

As a consequence we obtain \([N, H] = 0\). The excitation-number operator and the Hamiltonian are therefore simultaneously diagonalizable. To that end, we note that the product states \(|n_1, \ldots, n_p, b_1, \ldots, b_K\rangle\) are eigenstates of \( N \) with eigenvalues \( N \). We indicate the photon state by means of \( n_p = 0, 1, 2 \cdots \) and the qubits by \( b_k = 0, 1 \). The eigenvalues \( N \) follow as

\[ N = \sum_{p=1}^{P} n_p + \sum_{k=1}^{K} (b_k - \frac{1}{2}) = \sum_{p=1}^{P} n_p + \sum_{k=1}^{K} b_k - \frac{1}{2} K. \]  

(17)

With the exception of the state corresponding to \( N = -\frac{1}{2} K \), expectedly the ground state of the Hamiltonian, the eigenstates are (highly) degenerate. The degree of degeneracy \( L \) depends on \( N \), i.e., \( L = L(N) \); where irrelevant we do not explicitly indicate this. Labelling the degenerate states with the index \( l \) leads to the alternative notation \(|n_1, \ldots, n_p, b_1, \ldots, b_K\rangle = |N; l\rangle\). Of course, we expect that the Hamiltonian does not couple the various subspaces corresponding to different values of \( N \). We check this by explicitly calculating its matrix elements.

The matrix elements of the ‘free’ Hamiltonians are

\[ \langle N'; l' | \sum_{k=1}^{K} H_0^{[k]} |N; l\rangle = -\frac{1}{2} \hbar \delta_{N'N} \delta_{l'l} \sum_{k=1}^{K} (-1)^{b_k} \omega_k \]  

(18)

and

\[ \langle N'; l' | \sum_{p=1}^{P} H_{\text{res}}^{[p]} |N; l\rangle = \hbar \delta_{N'N} \delta_{l'l} \sum_{p=1}^{P} \omega_p (n_p + \frac{1}{2}). \]  

(19)

For the interaction terms of the Hamiltonian we get

\[ \langle N'; l' | \sum_{i=1}^{P} \sum_{j=1}^{K} H_{\text{int}}^{ij} |N; l\rangle = \hbar \sum_{i=1}^{P} \sum_{j=1}^{K} g_{ij} \delta_{n'_i, n_i} \delta_{b'_j, b_j} \left( \delta_{n_i, n_{i+1}} \delta_{b_j, b_{j+1}} \sqrt{n_i + 1} \right) \]
\[ + \delta_{n'_i, n_{i+1}} \delta_{b'_j, b_{j+1}} \sqrt{n_j}, \]  

(20)

where \( \delta_{n'_i, n_i} = 1 \) if \( n'_i = n_i \) for all \( k \neq i; \) \( \delta_{b'_j, b_j} \) is defined analogously. Using the Kronecker deltas it follows that \( N = N' + n_i - n'_i + b_j - b'_j = N' \), i.e., nonzero matrix elements have equal excitation number. It is also clear that the Hamiltonians in the various subspaces are symmetric. Consequently, these matrices have real eigenvalues and - within their particular subspace- a complete set of eigenvectors. In other words, they are diagonalizable by the standard Jacobi procedure. In this sense, the general multi qubit-resonator problem is solved.

V. TWO QUBITS COUPLED BY A CAVITY

As an explicit example we first address the simple multi-qubit problem of two qubits coupled by one electromagnetic resonator. Thus we derive the solution of the time-independent Schrödinger equation for the Hamiltonian

\[ H = -\frac{1}{2} \hbar \omega_1 \sigma_+^{[1]} - \frac{1}{2} \hbar \omega_2 \sigma_+^{[2]} + \hbar \omega (a_1^\dagger a_1 + \frac{1}{2}) \]
\[ + \hbar g_1 \left( a_1^\dagger \sigma_+^{[1]} + a_2 \sigma_-^{[1]} \right) + \hbar g_2 \left( a_1^\dagger \sigma_+^{[2]} + a_2 \sigma_-^{[2]} \right). \]  

(21)

Recall that the RWA has been made.

A. Derivation of subspace Hamiltonians

Analogously to the one-qubit problem, we define the operator

\[ N = a_1^\dagger a_1 - \frac{1}{2} \sigma_+^{[1]} - \frac{1}{2} \sigma_-^{[1]}. \]  

(22)

Once more, a brief calculation shows that

\[ [N, H] = 0. \]  

(23)

The vanishing commutator guarantees a common set of eigenstates of \( N \) and \( H \). The eigenstates of \( N \) are given by

\[ N |j, 0, 0\rangle = (j - 1) |j, 0, 0\rangle \]
\[ N |k, 1, 0\rangle = k |k, 1, 0\rangle \]
\[ N |l, 0, 1\rangle = l |l, 0, 1\rangle \]
\[ N |m, 1, 1\rangle = (m + 1) |m, 1, 1\rangle. \]  

(24)
The lowest eigenvalue of $\mathcal{N}$ is $-1$ and the unique eigenstate is $|0, 0, 0\rangle$. It is also an eigenstate of the Hamiltonian $H$.

$$H|0, 0, 0\rangle = E_{-1}|0, 0, 0\rangle = \frac{i\hbar}{2}(\omega - \omega_1 - \omega_2)|0, 0, 0\rangle,$$  \quad (25)

expectedly the ground state.

The next eigenvalue of $\mathcal{N}$ is equal to $0$ and we see that there is a three-fold degeneracy since the three states $|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle$ correspond to this eigenvalue. In this subspace we therefore make the Ansatz

$$|E_0\rangle = \alpha_0|1, 0, 0\rangle + \beta_0|0, 1, 0\rangle + \gamma_0|0, 0, 1\rangle.$$  \quad (26)

The time-independent Schrödinger equation in this subspace gives

$$-\hbar \alpha_0 \left\{ \left(-\frac{1}{2}\omega_+'' + \frac{3}{2}\omega_+\right) |1, 0, 0\rangle + g_1 |0, 1, 0\rangle + g_2 |0, 0, 1\rangle \right\} + \hbar \beta_0 \left\{ \left(\frac{1}{2}\omega_-' + \frac{1}{2}\omega_\gamma\right) |0, 1, 0\rangle + g_1 |1, 0, 0\rangle \right\} + \hbar \gamma_0 \left\{ \left(-\frac{1}{2}\omega_-' + \frac{1}{2}\omega_\gamma\right) |0, 0, 1\rangle + g_1 |1, 0, 0\rangle \right\} = E_0 \left\{ \alpha_0 |1, 0, 0\rangle + \beta_0 |0, 1, 0\rangle + \gamma_0 |0, 0, 1\rangle \right\},$$  \quad (27)

with $\omega_+' = \omega_1 + \omega_2$. We take the inner product with the basis states of this subspace and obtain the matrix eigenvalue equation

$$\mathcal{H}_0 \tilde{\eta}_0 = E_0 \tilde{\eta}_0, \quad \text{where} \quad \tilde{\eta}_0 = \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}.$$  \quad (28)

and

$$\mathcal{H}_0 = \hbar \begin{pmatrix} -\frac{1}{2}\omega_+' + \frac{3}{2} \omega_+ & g_1 & g_2 \\ g_1 & \frac{1}{2}\omega_-' + \frac{1}{2} \omega_\gamma & 0 \\ g_2 & 0 & -\frac{1}{2}\omega_-' + \frac{1}{2} \omega_\gamma \end{pmatrix}.$$  \quad (29)

This real symmetric matrix can be diagonalized. Three real eigenvalues $E_{0\xi}, \xi = 1, 2, 3$ follow from the characteristic equation

$$\det (\mathcal{H}_0 - E_0 I) = 0.$$  \quad (30)

The concomitant orthonormal eigenvectors $\tilde{\eta}_{0\xi}$ can then be calculated. It results in the eigenstates in the $n = 0$ subspace

$$|E_{00}\rangle = \alpha_{00}|1, 0, 0\rangle + \beta_{00}|0, 1, 0\rangle + \gamma_{00}|0, 0, 1\rangle.$$  \quad (31)

The higher eigenvalues of $\mathcal{N}$ are equal to $n, n \geq 1$. Here we encounter a four-fold degeneracy; the corresponding states are $|n+1, 0, 0\rangle, |n, 1, 0\rangle, |n, 0, 1\rangle, |n-1, 1, 1\rangle$. Consequently, we make the Ansatz for the eigenstates $|E_n\rangle$

$$|E_n\rangle = \alpha_n|n+1, 0, 0\rangle + \beta_n|n, 1, 0\rangle + \gamma_n|n, 0, 1\rangle + \zeta_n|n-1, 1, 1\rangle.$$  \quad (32)

The eigenvalue equation $H|E_n\rangle = E_n|E_n\rangle$ explicitly yields for $n \geq 1$

$$E_n \left\{ \alpha_n |n+1, 0, 0\rangle + \beta_n |n, 1, 0\rangle + \gamma_n |n, 0, 1\rangle + \zeta_n |n-1, 1, 1\rangle \right\} =$$

$$\hbar \alpha_n \left\{ \left(-\frac{1}{2}\omega_+' + \omega(n + \frac{3}{2})\right) |n+1, 0, 0\rangle + g_1 \sqrt{n+1} |n, 1, 0\rangle + g_2 \sqrt{n+1} |n, 0, 1\rangle \right\} +$$

$$\hbar \beta_n \left\{ \left(\frac{1}{2}\omega_-' + \omega(n + \frac{1}{2})\right) |n, 1, 0\rangle + g_1 \sqrt{n+1} |n+1, 0, 0\rangle + g_2 \sqrt{n+1} |n-1, 1, 1\rangle \right\} +$$

$$\hbar \gamma_n \left\{ \left(-\frac{1}{2}\omega_-' + \omega(n + \frac{1}{2})\right) |n, 0, 1\rangle + g_1 \sqrt{n+1} |n-1, 1, 1\rangle + g_2 \sqrt{n+1} |n+1, 0, 0\rangle \right\} +$$

$$\hbar \zeta_n \left\{ \left(\frac{1}{2}\omega_+' + \omega(n - \frac{1}{2})\right) |n-1, 1, 1\rangle + g_1 \sqrt{n+1} |n, 0, 1\rangle + g_2 \sqrt{n+1} |n, 1, 0\rangle \right\}.$$  \quad (33)

Taking the inner product with the basis states of $\mathcal{N}$ leads to the matrix eigenvalue equations

$$\mathcal{H}_n \tilde{\eta}_n = E_n \tilde{\eta}_n, \quad \text{where} \quad \tilde{\eta}_n = \begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \zeta_n \end{pmatrix}.$$  \quad (34)

and matrices

$$\mathcal{H}_n = \hbar \begin{pmatrix} -\frac{1}{2}\omega_+' + \omega(n + \frac{3}{2}) & g_1 \sqrt{n+1} & g_2 \sqrt{n+1} & 0 \\ g_1 \sqrt{n+1} & \frac{1}{2}\omega_-' + \omega(n + \frac{1}{2}) & 0 & g_2 \sqrt{n+1} \\ g_2 \sqrt{n+1} & 0 & -\frac{1}{2}\omega_-' + \omega(n - \frac{1}{2}) & g_1 \sqrt{n+1} \\ 0 & g_2 \sqrt{n+1} & g_1 \sqrt{n+1} & \frac{1}{2}\omega_+' + \omega(n - \frac{1}{2}) \end{pmatrix}.$$  \quad (35)

These real symmetric matrices can again be diagonalized and four energy eigenvalues $E_{n\nu}, \nu = 1, 2, 3, 4$ are obtained for each $n$ by solving the characteristic equations

$$\det (\mathcal{H}_n - E_n I) = 0.$$  \quad (36)
As above the corresponding orthonormal eigenvectors $\eta_{n\nu}$ can be computed. In this way, we get the eigenstates the subspaces defined by the value of $n$

$$|E_{n\nu}\rangle = \alpha_{n\nu}|n + 1, 0, 0\rangle + \beta_{n\nu}|n, 1, 0\rangle + \gamma_{n\nu}|n, 0, 1\rangle + \zeta_{n\nu}|n - 1, 1, 1\rangle.$$  \hspace{1cm} (37)

Note that, besides the RWA, no further approximations have been made in solving the Schrödinger equation for the two-qubit JC Hamiltonian \[^{(22)}\].

### B. Numerical solution of the eigenvalue problems

The eigenvalue problems defined by equations \[^{(28, 29, 34, 35)}\] will be solved numerically for given values of the parameters. To this end, we use the Fortran 95 subroutine \textit{ jacobi }\[^{(31)}\] which does the job for real symmetric $n \times n$ matrices. It is based on Jacobi rotations, cf. \[^{(32)}\]. The lowest energy levels for some typical parameter values are shown in Fig. 3. Three energy scales are visible in the energy levels. The energy differences between states corresponding to a different eigenvalue of $N$ are $\hbar \omega$. The larger differences for equal eigenvalues of $N$ are $\hbar(\omega - \frac{1}{2}(\omega_1^2 + \omega_2^2))$ whereas the smallest ones are $\hbar(\omega_1 - \omega_2)$. These relations are approximately valid for small couplings.

![Energy levels](image)

\textbf{FIG. 3.} Energy levels $E/\hbar$ of the two-qubit Hamiltonian \[^{(22)}\] for parameter values in GHz; $g_1/\hbar = 0.1, g_2/\hbar = 0.12$; l.h.s.: $f_1' = 6.0, f_2' = 6.3, f = 7.0$; r.h.s.: $f_1' = 6.2, f_2' = 6.7, f = 8.0$.

### VI. RESONANT COUPLING OF $K$ QUBITS

The system described above is extended to $K$ qubits with identical transition frequency $\omega'$. It is the original Tavis-Cummings model for a single-mode quantized radiation field interacting with $K$ molecules \[^{(11)}\]. For resonant interactions a collective interaction which scales as $\sqrt{K}$ has been expected and observed \[^{(23)}\]. Exploiting our formalism, we analyze such systems for $\omega = \omega'$ and in the one-excitation subspace.

For the two-qubit system considered above, the results in the one-excitation subspace yield the eigen energies

$$E = \frac{\hbar}{2} \omega \quad \text{and} \quad E_\pm = \frac{\hbar}{2} \omega \pm \hbar \sqrt{2g_{12}},$$  \hspace{1cm} (38)

with mean coupling $g_{12} = \sqrt{\frac{1}{2}(g_1^2 + g_2^2)}$. The corresponding eigenstates are respectively given by

$$|E\rangle = \frac{1}{\sqrt{2g_{12}}} (g_2|0, 1, 0\rangle - g_1|0, 0, 1\rangle)$$  \hspace{1cm} (39)

$$|E_\pm\rangle = \pm \frac{1}{2} \sqrt{2}|1, 0, 0\rangle + \frac{1}{2g_{12}} (g_1|0, 1, 0\rangle + g_2|0, 0, 1\rangle).$$

For obvious reasons $|E\rangle$ is referred to as dark state whereas $|E_\pm\rangle$ are called bright states in \[^{(22)}\].

We immediately proceed to the $K$-qubit case. In the one-excitation subspace the $(K+1) \times (K+1)$ Hamiltonian is explicitly given by

$$\mathcal{H}_1 = \hbar \begin{pmatrix} \Omega_K & g_1 & g_2 & \cdots & g_{K-1} & g_K \\ g_1 & \Omega_K & 0 & \cdots & \cdots & 0 \\ g_2 & 0 & \Omega_K & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ g_{K-1} & 0 & \cdots & \cdots & \ddots & 0 \\ g_K & 0 & \cdots & \cdots & 0 & \Omega_K \end{pmatrix},$$  \hspace{1cm} (40)
The \textquote{spin} raising/lowering operators 

\[ S = \pm \hbar \sqrt{2} g_{12-K} \]

with a \((K - 1)\)-fold degenerate dark subspace 

\[ |E\rangle = \alpha_1 |0, 1, 0, \ldots, 0\rangle + \alpha_2 |0, 0, 1, \ldots, 0\rangle + \cdots + \alpha_K |0, 0, 0, \ldots, 1\rangle \]  

\textbf{VII. COUPLED QU DIT SYSTEMS}

The formalism developed thus far can be generalized to qudits, coupled by electromagnetic resonators. A qudit is a finite-dimensional quantum system, obviously the generalization of qubits and qutrits.

\textbf{A. Formalism}

We consider \(K\) qudits and allow qudits of different dimensionality \(D_k - 1 = 2M_k\), with \(M_k = 1/2, 1, 3/2, \ldots\). The Hamiltonian of the coupled system is given in the RWA by

\[ H = \sum_{k=1}^{K} H_0^{[k]} + \sum_{p=1}^{P} H^{[p]}_{\text{res}} + \sum_{p=1}^{P} \sum_{k=1}^{K} H^{[pk]}_{\text{int}}. \]

The qudit, resonator and interaction Hamiltonians are respectively given by

\[ H_0^{[k]} = \hbar \sum_{m=0}^{2M_k} \omega_{mk} |m\rangle_k \langle m|_k, \]

\[ H^{[p]}_{\text{res}} = \hbar \omega_p (a_p^+ a_p + \frac{1}{2}), \]

\[ H^{[pk]}_{\text{int}} = \hbar \left( a_p^+ S^{[pk]}_+ + a_p S^{[pk]}_- \right). \]

The \textquote{spin} raising/lowering operators \(S^{[pk]}_\pm\) are explicitly defined as

\[ S^{[pk]}_+ = \sum_{l=0}^{2M_k - 1} g_{l+1}^{[pk]} |l+1\rangle_k \langle l|_k, \]

\[ S^{[pk]}_- = \sum_{l=0}^{2M_k - 1} g_{l+1}^{[pk]} |l+1\rangle_k \langle l|_k. \]

where \(\Omega_K = (\frac{\hbar}{2g_{12-K}}) \omega\). The mean coupling is defined as \(g_{12-K} = \sqrt{\frac{1}{K}(g_1^2 + g_2^2 + \cdots + g_K^2)}\) and the coefficients \(\alpha_k\) satisfy \(\sum_{k=1}^{K} \alpha_k g_k = 0\). These results can be verified by straightforward calculation of \(H_1|E\rangle\) and \(H_1|E_\pm\rangle\). In this way we have confirmed the strength of the collective interaction to be \(g_{12-K} \sqrt{\Omega}\) as experimentally demonstrated for two and three qubits [33].

Although they depend on the cavity index \(p\) through the couplings \(g_{l_{t+1}}^{[pk]}\), as operators they only act on the qudit labelled by \(k\). In typical applications like the transmon-cavity model and a two-level atom-cavity system \([23, 34]\), \(g_{l_{t+1}} \simeq g \sqrt{l + 1}\), with dipole strength \(g\). Here we only assume the couplings to be real and symmetric which implies hermitian Hamiltonians. Once again, we define the excitation-number operator for the system

\[ N = \sum_{p=1}^{P} a_p^+ a_p - \sum_{k=1}^{K} S^{[k]}, \]

with \textquote{spin} operators \(S^{[k]}\)

\[ S^{[k]} = \sum_{m=0}^{2M_k} (M_k - m) |m\rangle_k \langle m|_k. \]

Their eigenstates and eigenvalues are given by

\[ S^{[k]} |m\rangle_k = (M_k - m) |m\rangle_k. \]

It is also readily verified that for all \(p, k\)

\[ [S^{[k]}, S^{[pk]}] = \pm \delta_{kk'} S^{[pk]}. \]

Analogously to the qubit-resonator system discussed above, it follows that the excitation-number operator commutes with the Hamiltonian and is a conserved quantity. Therefore, excitation-number operator and Hamiltonian are again simultaneously diagonalizable. The eigenstates of \(N\) are once more product states: \(|n_1, \ldots, n_P, m_1, \ldots, m_K\rangle\). The concomitant eigenvalues \(N\) explicitly read

\[ N = \sum_{p=1}^{P} n_p + \sum_{k=1}^{K} m_k - \sum_{k=1}^{K} M_k. \]
We obtain for the interaction term
\[
\langle N'; l'| \sum_{k=1}^{K} H_{0}^{[k]} |N; l\rangle = \hbar \delta_{N',N} \delta_{l,l'} \sum_{k=1}^{K} \omega_{mk}
\] (52)
and
\[
\langle N'; l'| \sum_{p=1}^{P} H_{\text{res}}^{[p]} |N; l\rangle = \hbar \delta_{N',N} \delta_{l,l'} \sum_{p=1}^{P} \omega_{p}(n_{p} + \frac{1}{2}).
\] (53)

We obtain for the interaction term
\[
\langle N'; l'| \sum_{p=1}^{P} \sum_{k=1}^{K} H_{\text{int}}^{[p,k]} |N; l\rangle = \hbar \sum_{p=1}^{P} \sum_{k=1}^{K} \delta_{n_{p}n_{p+1}} \delta_{m_{k}m_{k+1}} \sqrt{n_{p}} + 1 \delta_{m_{k}m_{k+1}}
\]
\[
+ \delta_{n_{p}n_{p+1}} \delta_{m_{k}m_{k+1}} \sqrt{n_{p}} g_{m_{k}} \delta_{m_{k}m_{k+1}}
\].

Exploiting the various Kronecker deltas gives for nonvanishing matrix elements \(N = N' + n_{p} - n_{p'} + m_{k} - m_{k}' = N\). Thus it is shown that the Hamiltonian is diagonal with respect to \(N\). In these respective subspaces the matrices are symmetric as well. Consequently, the matrices have real eigenvalues and - within their particular subspace - a complete set of eigenvectors. In other words, they are diagonalizable. In this sense, the general multiple qudit-resonator problem is also solved.

\section{VIII. TWO COUPLED QUDITS}

A relevant example of a qudit is the transmon in the three-level approximation. It has been shown \cite{S2021} that one has to include the second excitation level to achieve a two-qubit gate for two transmons. Here we explicitly consider two transmons as qutrits which are coupled by one resonator. The Hamiltonian of this system can be written as
\[
H = \hbar \sum_{m=0}^{2} \omega_{m1}|m\rangle_{1}\langle m|_{1} + \hbar \sum_{l=0}^{2} \omega_{l2}|l\rangle_{2}\langle l|_{2}
\]
\[
+ \hbar \omega(a^\dagger a + \frac{1}{2})
\]
\[
+ \hbar \left( a^\dagger S_+^{[1]} + a S_-^{[1]} \right) + \hbar \left( a^\dagger S_+^{[2]} + a S_-^{[2]} \right). \] (55)

Approximating the transmons as anharmonic oscillators gives in terms of the charging energy \(E_C\) and the Josephson energy \(E_J\)
\[
\hbar \omega_{01} = \frac{1}{2} \left( \sqrt{8E_J^{[1]} E_C^{[1]} - E_J^{[1]}} \right),
\] (56)
\[
\hbar \omega_{11} = 3\hbar \omega_{01}, \quad \hbar \omega_{21} = 5\hbar \omega_{01} + \alpha^{[1]}.
\]
The anharmonicity reads \(\alpha^{[1]} = -E_C^{[1]}\). For the second transmon analogous relations are valid. The raising/lowering operators are defined as
\[
S_+^{[1,2]} = \sum_{l=0}^{1} g_{ll'}(l+1)_{1,2} |l+1,1,2\rangle,
\]
\[
S_-^{[1,2]} = \sum_{l=0}^{1} g_{ll'}(l)_{1,2} |l,1,2\rangle
\]
and it is easily verified that
\[
S_+^{[1,2]}|0\rangle_{1,2} = 0, \quad S_-^{[1,2]}|0\rangle_{1,2} = g_{10}^{[1,2]}|1\rangle_{1,2},
\]
\[
S_+^{[1,2]}|1\rangle_{1,2} = g_{01}^{[1,2]}|0\rangle_{1,2}, \quad S_-^{[1,2]}|1\rangle_{1,2} = g_{21}^{[1,2]}|2\rangle_{1,2},
\]
\[
S_+^{[1,2]}|2\rangle_{1,2} = g_{12}^{[1,2]}|1\rangle_{1,2}, \quad S_-^{[1,2]}|2\rangle_{1,2} = 0.
\] (58)
The spin operators \(S^{[1,2]} = |0\rangle_{1,2} \langle 0|_{1,2} - |2\rangle_{1,2} \langle 2|_{1,2}\) are diagonal:
\[
S^{[1,2]}|0\rangle_{1,2} = |0\rangle_{1,2}, \quad S^{[1,2]}|1\rangle_{1,2} = 0,
\]
\[
S^{[1,2]}|2\rangle_{1,2} = -|2\rangle_{1,2}.
\] (59)
Herewith we again define the excitation number operator
\[
\mathcal{N} = a^\dagger a - S^{[1]} - S^{[2]}.
\] (60)
Its eigenstates are once more given by product states
\[
\mathcal{N}|k,0,0\rangle = (k - 2)|k,0,0\rangle, \quad \mathcal{N}|k,1,0\rangle = (k - 1)|k,1,0\rangle, \quad \mathcal{N}|k,0,1\rangle = (k - 1)|k,0,1\rangle, \quad \mathcal{N}|k,2,0\rangle = k|k,2,0\rangle, \quad \mathcal{N}|k,0,2\rangle = k|k,0,2\rangle, \quad \mathcal{N}|k,1,1\rangle = k|k,1,1\rangle.
\] (61)
The state \(|0,0,0\rangle\) has the lowest eigenvalue of \(\mathcal{N}\), i.e., \(N = -2\). Evidently it corresponds to the ground state of the Hamiltonian
\[
H|0,0,0\rangle = \hbar (\omega_{01} + \omega_{02} + \frac{1}{2} \omega) |0,0,0\rangle.
\] (62)
The first excitation subspace is three-fold degenerate with respect to the eigenvalues of \(\mathcal{N}\) because the three states \(|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\) have \(N = -1\). Hence we make the Ansatz for the eigenstates of \(H\)
\[
|E_{-1}\rangle = a|1,0,0\rangle + \beta|0,1,0\rangle + \gamma|0,0,1\rangle.
\] (63)
The time-independent Schrödinger equation explicitly yields
\[
\hbar \alpha \left\{ (\omega_{01} + \omega_{02} + \omega_{03}) |1, 0, 0\rangle + g_{10}^{[1]} |0, 1, 0\rangle + g_{10}^{[2]} |0, 0, 1\rangle \right\} + \hbar \beta \left\{ (\omega_{11} + \omega_{02} + \omega_{03}) |0, 1, 0\rangle + g_{01}^{[1]} |1, 0, 0\rangle \right\} + \hbar \gamma \left\{ (\omega_{01} + \omega_{12} + \omega_{03}) |0, 0, 1\rangle + g_{01}^{[2]} |1, 0, 0\rangle \right\} = E_{-1} \{ \alpha |1, 0, 0\rangle + \beta |0, 1, 0\rangle + \gamma |0, 0, 1\rangle \}. \tag{64}
\]
Taking the inner product with the basis states of of this subspace leads to the matrix eigenvalue equation
\[
\mathcal{H}_{-1} \vec{\eta}_{-1} = E_{-1} \vec{\eta}_{-1}, \quad \text{where} \quad \vec{\eta}_{-1} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{65}
\]
and
\[
\mathcal{H}_{-1} = \hbar \begin{pmatrix} \omega_{01} + \omega_{02} + \frac{\omega_{03}}{2} & g_{10}^{[1]} & g_{10}^{[2]} \\ g_{10}^{[1]} & \omega_{11} + \omega_{02} + \frac{\omega_{03}}{2} & 0 \\ g_{10}^{[2]} & 0 & \omega_{01} + \omega_{12} + \frac{\omega_{03}}{2} \end{pmatrix}. \tag{66}
\]

The second excitation level corresponds to eigenvalue zero of \( \mathcal{N} \). Its basis states follow analogously and lead to the Ansatz for the eigenstates of \( \mathcal{H} \):
\[
|E_0\rangle = v_1 |2, 0, 0\rangle + v_2 |1, 1, 0\rangle + v_3 |1, 0, 1\rangle + v_4 |0, 1, 1\rangle + v_5 |0, 2, 0\rangle + v_6 |0, 0, 2\rangle. \tag{67}
\]
The time-independent Schrödinger equation in the subspace can be rewritten as matrix equation for the vectors \( \vec{v} \) with components \( v_i, l = 1, \ldots, 6 \)
\[
\mathcal{H}_0 \vec{v} = E_0 \vec{v}. \tag{68}
\]
The energy eigenvalues are \( E_0 \) and the \( 6 \times 6 \) matrix \( \mathcal{H}_0 \) reads
\[
\mathcal{H}_0 = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix}, \tag{69}
\]
with the \( 3 \times 3 \) matrices
\[
\mathcal{H}_{11} = \hbar \begin{pmatrix} \omega_{01} + \omega_{02} + \frac{\omega_{03}}{2} & \sqrt{2}g_{10}^{[1]} & \sqrt{2}g_{10}^{[2]} \\ \sqrt{2}g_{10}^{[1]} & \omega_{11} + \omega_{02} + \frac{\omega_{03}}{2} & 0 \\ \sqrt{2}g_{10}^{[2]} & 0 & \omega_{01} + \omega_{12} + \frac{\omega_{03}}{2} \end{pmatrix}, \tag{70}
\]
\[
\mathcal{H}_{12} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ g_{01}^{[2]} & g_{12}^{[1]} & 0 \\ g_{01}^{[1]} & 0 & g_{12}^{[2]} \end{pmatrix}, \quad \mathcal{H}_{21} = \hbar \begin{pmatrix} 0 & g_{01}^{[2]} & g_{12}^{[1]} \\ 0 & g_{21}^{[1]} & 0 \\ 0 & 0 & g_{21}^{[2]} \end{pmatrix}. \tag{71}
\]

The program can of course be continued for higher excitation levels. The dimension of each subspace is determined by its degeneracy with respect to the eigenvalues of the excitation number operator. The eigenvalues and eigenstates follow by diagonalization of the respective subspace Hamiltonians. Here we do not pursue this further but will explicitly solve the eigenvalue problems for the first and second excitation level.

Some remarks are relevant for the numerical implementation. It is shown in [34] that one eventually can take the couplings real and symmetric. As a consequence, the matrices \( [66, 69] \) are symmetric and therefore can be diagonalized. The energy eigenvalues are real indeed, and the three/six eigenvectors are complete and orthogonal in the three/six dimensional subspaces. Of course, they can be normalized to one. Moreover, from [34] it follows that
\[
g_{12}^{[1, 2]} = \sqrt{2}g_{01}^{[1, 2]}. \tag{72}
\]
The numerical values of all parameters are taken as in [8], making a comparison of results possible.
Arguably the main result of [8] is the demonstration of a conditional phase gate. First, the one-excitation spectrum is shown as a function of the frequency of one transmon, being varied by tuning. An avoided crossing is visible in the spectrum. The resulting interaction, however, is claimed to be too small for the applications. We have re-calculated the one excitation spectrum using our formalism. The result is depicted in Fig. 4 and is consistent with Fig. 1 in [8]. The abovementioned C-phase gate relies on the two excitation spectrum, presented as Fig. 2 in [8]. Fig. 4 demonstrates that our formalism reproduces—at least qualitatively—the spectrum and especially the avoided crossing as shown in [8].

![Fig. 4. Two coupled transmons, energy levels E/h as function of the tuning of the frequency of one transmon. L.h.s. one-excitation subspace Hamiltonian (63, 65), cf. Fig. 1 in [8]; r.h.s two-excitation subspace Hamiltonian (69), three highest energy levels, the corresponding states have small coefficients v_1, v_2, v_3 in eq. (67), cf. Fig. 2 in [8].](image)

**IX. TIME EVOLUTION**

Finally, we address the evolution operator $U(t, t_0)$ of coupled multi-qudit systems. In order to derive formal expressions we denote the eventually obtained eigenstates of the Hamiltonian as $|E_{N\nu}\rangle$ where $N$ is the excitation number and $\nu = 1, \cdots, L(N)$; recall that $L(N)$ is the dimension of the corresponding subspace. The evolution operator follows as

$$U(t, t_0) = \sum_N \sum_\nu \sum_{l=1}^{L(N)} e^{-iE_{N\nu}(t-t_0)/\hbar} |E_{N\nu}\rangle \langle E_{N\nu}|. \quad (73)$$

cf. [27]. The energy eigenstates are linear combinations of the eigenstates of $\hat{N}$

$$|E_{N\nu}\rangle = \sum_{l=1}^{L(N)} c_{N\nu l} |N; l\rangle. \quad (74)$$

The real coefficients $c_{N1}$ and the energy eigenvalues $E_{N\nu}$ have been obtained from the diagonalization of the subspace matrices. With (73) we rewrite the evolution operator (73) in terms of these computed quantities

$$U(t, t_0) = \sum_N \sum_\nu \sum_{l,l'=1}^{L(N)} e^{-iE_{N\nu}(t-t_0)/\hbar} c_{N\nu l} c_{N\nu l'}^* |N; l\rangle \langle N; l'|. \quad (75)$$

In practice the summation over $N$ has to be truncated.

**X. SUMMARY AND OUTLOOK**

This paper has provided a computational framework for the analysis of several Jaynes-Tavis-Cummings systems in the RWA. The latter approximation is assessed first for the simplest qubit-resonator system. Generalizing the standard JC Hamiltonians, the formalism can deal with multiple qubits coupled to multiple resonators. Eventually, the formal approach is extended to qudits, i.e., multi-level systems like spins and transmons in the qutrit approximation [8]. The respective Hamiltonians are separated in decoupled subspace Hamiltonians which correspond to fixed excitation numbers. This is possible since the concomitant operator commutes with the Hamiltonian and is therefore conserved. The symmetry of the Hamiltonian is of course directly related to the RWA. In the respective subspaces, the corresponding symmetric matrices need to be diagonalized in order to obtain the eigenstates, eigenvalues and evolution operators. This is the purport of our, to the best of our knowledge new, semi-analytical formalism. We have quantitatively demonstrated the framework at the hand of a few examples. First, the system of two-qubits coupled to one resonator is analyzed completely. The problem is extended to $K$ qubits with resonant couplings. In the one-excitation subspace the $\sqrt{K}$-scaling for the collective interaction strength between bright states as well as the appearance of dark states [33] has been derived. Finally, the coupled two-transmon system in the
three-level approximation [8] has been examined. Possible consequences for developments in quantum computing, in particular for fault-tolerant quantum processors, the corresponding error corrections, stabilizers and fidelities will be discussed in a subsequent paper [35].

ACKNOWLEDGMENTS

The authors thank B. Criger and S. Poletto for useful discussions and critical readings of the manuscript. This research is supported by the Early Research Programme of the Netherlands Organisation for Applied Scientific Research (TN0). Additional support from the Top Sector High Tech Systems and Materials is highly appreciated.

[1] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK (2000).
[2] Th. K. Mavrogordatos et al., Phys. Rev. Lett. 118, 040402 (2017).
[3] Y. Chen et al., Phys. Rev. Lett. 113, 202502 (2014).
[4] T.E. O’Brien, B. Tarasinski, and L. DiCarlo, arXiv:1703.04136v1 [quant-ph] (2017).
[5] R. Versluis et al., arXiv:1612.08298v1 [quant-ph] (2016).
[6] S. Lloyd, Phys. Rev. Lett. 75, 346 (1995).
[7] D. Deutsch, Proc. R. Soc. London A 400, 97 (1989).
[8] L. DiCarlo et al., Nature 460, 240 (2009).
[9] A.D. Córcoles et al., Nature Comm. 6, 6979 (2015).
[10] R. Barends et al., Phys. Rev. Lett. 111, 080502 (2013).
[11] C.M. Quintana et al., Phys. Rev. Lett. 118, 057702 (2017).
[12] K.D. Petersson et al., Nature 490, 380 (2012).
[13] G. Burkard and A. Imamoglu, Phys. Rev. B 74, 041307 (2006).
[14] Y. Kubo et al., Phys. Rev. Lett. 105, 140502 (2010).
[15] E.T. Jaynes and F.W. Cummings, Proc. IEEE 51, 89 (1963).
[16] M. Tavis and F.W. Cummings, Phys. Rev. 170, 379 (1968).
[17] M. Tavis and F.W. Cummings, Phys. Rev. 188, 692 (1969).
[18] N.M. Bogoliubov, R.K. Bullough and J. Timonen, J. Phys. A 29, 6305 (1996).
[19] I.P. Vadeiko, G.P. Miroshnichenko, A.V. Rybin and J. Timonen, Phys. Rev. A 67, 053808 (2003).
[20] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, Third Edition, Elsevier, North-Holland Personal Library, Amsterdam (2007).
[21] C. Cohen-Tannoudji, B. Diu, F. Laloë, Quantum Mechanics, Volume 2, Chapter 9, Hermann, Paris, and Wiley-VCH (2005).
[22] M.F. Gely et al., Phys. Rev. B 95, 245115 (2017).
[23] A. Blais, R.-S. Huang, A. Wallraff, S.M. Girvin and R.J. Schoelkopf, Phys. Rev. A 69, 062320 (2004).
[24] D.I. Schuster, Circuit Quantum Electrodynamics, Dissertation, Yale University (2007).
[25] J.M. Chow, Quantum Information Processing with Superconducting Qubits, Dissertation, Yale University (2010).
[26] D. Sank et al., Phys. Rev. Lett. 117, 19503 (2016).
[27] C. Cohen-Tannoudji, B. Diu, F. Laloë, Quantum Mechanics, Volume 1, Chapter 3, Hermann, Paris, and Wiley-VCH (2005).
[28] C. Cohen-Tannoudji, Atoms in Electromagnetic Fields, World Scientific, London (1994).
[29] M.O. Scully and M.S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge (1997).
[30] J.-F. Huang, J.-Q. Liao, L. Tian, and L.-M. Kuang, arXiv:1603.08641v1 [quant-ph] (2016).
[31] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, Numerical Recipes in Fortran 90, Cambridge University Press (1999).
[32] G.H. Golub and C.F. van Loan, Matrix Computations, Third Edition, The John Hopkins University Press, Baltimore (1996).
[33] J.M. Fink et al., Phys. Rev. Lett. 103, 083601 (2009).
[34] J. Koch et al., Phys. Rev. A 76, 042319 (2007).
[35] H.W.L. Naus et al., in preparation.