A dynamical stability criterion for inhomogeneous quasi-stationary states in long-range systems

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Abstract. We derive a necessary and sufficient condition for linear dynamical stability for inhomogeneous Vlasov stationary states of the Hamiltonian mean field (HMF) model. The condition is expressed by an explicit inequality that has to be satisfied by the stationary state, and it generalizes the known inequality for homogeneous stationary states. In addition, we derive analogous inequalities that express necessary and sufficient conditions for formal stability for the stationary states. Their usefulness, from the point of view of linear dynamical stability, is that they are simpler, although they provide only sufficient criteria for linear stability. We show that for homogeneous stationary states the relations become equal, and therefore linear dynamical stability and formal stability become equivalent.

Keywords: exact results, stationary states
1. Introduction

There are numerous distinctive features that characterize the behaviour of many-body systems with long-range interactions, features that are not present in systems with short-range interactions. These peculiarities concern both the equilibrium properties, such as the inequivalence of ensembles and negative specific heats in the microcanonical ensemble, and the out-of-equilibrium dynamical behaviour, such as the existence of long-lived quasi-stationary states and of out-of-equilibrium phase transitions. The study of these properties is interesting in its own right, but it is also justified by the many different physical systems in which long-range interactions play the prominent role, e.g., self-gravitating systems [1]–[3], unscreened Coulomb systems [4], some models in plasma physics [5] and in hydrodynamics [6], and trapped charged particles [7]. Recent reviews give the state of the art for the subject [8, 9].

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In this paper, we treat the subject of the long-lived quasi-stationary states (QSS). They are out-of-equilibrium states in which the distribution functions are non-Boltzmannian, and their lifetime increases with the size of the system as given by the number $N$ of degrees of freedom; this increase generally scales as a power law in $N$ but it can also be exponential. It has to be emphasized that the QSSs are not related to the usual metastable states that are found also in short-range systems. The latter are realized by local extrema of the thermodynamical potential (e.g., they are local maxima of the entropy or local minima of the free energy, if these quantities are computed as a function of an order parameter of the system), in which the system is trapped until it is driven away by some perturbation, and then heads towards the global extremum, i.e., the equilibrium state. Global and local extrema, i.e., equilibrium and metastable states, are obtained on the basis of the usual Boltzmann–Gibbs statistics. The evaluation of these states can be done following different routes; e.g., one can compute the partition function of the $N$-body system or work at the level of the one-particle distribution function, but only with a direct relation to the Boltzmann–Gibbs statistics, that governs equilibrium.

On the other hand, QSSs in principle have nothing to do with some sort of equilibrium state, global or local, of the system; nevertheless the system can be trapped for macroscopic times in these states. We want to underline that this does not imply at all any failure of the Boltzmann–Gibbs description of the equilibrium states. It simply means that the approach to equilibrium, that both in long-range and short-range systems should be described, at least approximately, by a kinetic equation, happens often in a manner, when long-range interactions are present, in which the system resides for macroscopic times in dynamical states that are very far from the equilibrium states.

From this it should be clear that the ultimate reason for the existence of the QSSs should be looked for in the dynamics, i.e., in the representation of the dynamics via a kinetic equation. It turns out that the dynamics of many-body long-range systems can be described to a very good approximation, in a certain time range, by the Vlasov equation for the one-particle distribution function. This equation is also sometimes called the collisionless Boltzmann equation, since it represents the interactions between the particles (whatever they might be; they can even be stars or galaxies in astrophysical problems) by a mean field term, i.e., the two-body interaction potential averaged over the whole system. The theoretical justification for this fact can be looked upon at different levels of mathematical rigour. It is not our task here to give this justification, but we think it useful to give a flavour of the reason from a physical point of view.

If a system is initially prepared in a state away from equilibrium, it will evolve towards the equilibrium state because of the interactions between the particles. In short-range systems each particle interacts only with the nearby particles, and therefore the dynamical evolution is determined by the ‘collisions’ of any given particle with the few others surrounding it. Since the positions of nearby particles are strongly correlated, it is completely useless to approximate the field acting on a particle by an averaged field, but it is necessary to find a way to describe, in a kinetic equation, the collisions between close particles. In long-range systems, even in the cases where the field at close distances is strong, the field acting on a particular particle is determined by all the others, and it looks quite plausible that an averaged field can be a good
approximation. Obviously, at the end, the ‘collisional’ regime will take its toll and the Boltzmann–Gibbs equilibrium will be realized, but before that point, the dynamics, to a high degree of approximation, will follow the evolution determined by the Vlasov equation.

It is then not surprising that a considerable amount of work has been dedicated to the properties of the Vlasov equation in relation to important systems like self-gravitating and plasma systems. It is worth noting a difference between these two classes of systems. In fact, globally neutral plasmas have a spatially homogeneous equilibrium state, and it is in reference to this case that the theory of the Vlasov linear stability of small perturbations of the homogeneous equilibrium state has been developed [10]–[12]. In contrast, the necessarily inhomogeneous states of large but finite self-gravitating systems have motivated research into the stability properties of inhomogeneous stationary states of the Vlasov equation [1], [13]–[17].

The Hamiltonian mean field (HMF) model [18,19], a simple 1D toy model of systems with long-range interactions, has been very useful for studying the various statistical and dynamical properties of long-range systems. Also, a good amount of work has been dedicated recently to the stability of Vlasov stationary states. The comments of the previous paragraph suggest that one should be interested in the stability of both homogeneous and inhomogeneous states. However, it is mainly homogeneous states that have been considered (both theoretically and numerically; see, e.g., [19]–[26]), and there exist only few theoretical results for inhomogeneous states (see sections 3.3 and 4.4 of [24], appendix F of [26], and [27] for polytropes). In this paper, we present a criterion for the Vlasov stability of inhomogeneous stationary states of the HMF model that generalizes the known criterion valid for homogeneous states. The results presented are based on the stability conditions for Vlasov stationary states that have been derived in the (essentially astrophysical) literature. We here rederive the results, and extend them, leaving the interaction potential unspecified. We will introduce the HMF interaction potential when the aforementioned results are used to obtain explicit necessary and sufficient conditions on the stationary distribution function.

We emphasize that the stability conditions already in the literature are in the form of relations that have to be satisfied by the perturbations to the stationary states. To our knowledge, such conditions have not been yet transformed into explicit conditions that have to be obeyed by the stationary distribution itself. Therefore, such explicit conditions are the core of this paper.

It turns out that the necessary and sufficient conditions for linear stability can be simplified at the price of obtaining conditions that are only sufficient. This can be done in the framework of the formal stability of the Vlasov stationary states. Therefore, for more completeness and for a useful comparison, we find it useful to treat also the problem of formal stability. The stationary point of a general dynamical system is said to be formally stable [28] if a conserved quantity has an extremum at the stationary point and if the second variation about the stationary point is either positive definite or negative definite.

Section 2 introduces the stationary states in the framework of the Vlasov equation. The sections from 3 to 6 derive results later employed for the study of linear stability and formal stability. This is done, for the HMF model, in sections 7 and 8. Section 9 contains the conclusions.

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2. The Vlasov equation and the quasi-stationary states

This paper will use the HMF model as a benchmark for our analysis, and therefore it is convenient to introduce the Vlasov equation from the beginning in this framework. Consider $N$ particles of unit mass moving on a circle, with the Hamiltonian of the system given by

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2N} \sum_{i \neq j} V(\theta_i - \theta_j),$$

where $\theta_i \in [0, 2\pi]$ is the angle giving the position of a particle on the circle and $-\infty < p_i < \infty$ is its linear momentum (equal to the velocity since the mass is unitary). The $1/N$ normalization of the interaction potential $V(\theta_i - \theta_j)$ is the usual one introduced in order to have an extensive energy; it is equivalent to a system size dependent rescaling of time, and it does not affect the study of the properties of the system. The Vlasov equation associated with this system, governing the evolution of the one-particle distribution function $f(\theta, p, t)$, is

$$\frac{\partial f(\theta, p, t)}{\partial t} + p \frac{\partial f(\theta, p, t)}{\partial \theta} - \frac{\partial \Phi(\theta, t; f)}{\partial \theta} \frac{\partial f(\theta, p, t)}{\partial p} = 0,$$

where $\Phi(\theta, t; f)$ is the mean field potential:

$$\Phi(\theta, t; f) = \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' V(\theta - \theta') f(\theta', p', t).$$

The last equation shows that the Vlasov equation (2) is a nonlinear integrodifferential equation. It is immediately seen that it conserves the normalization of $f(\theta, p, t)$:

$$\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta f(\theta, p, t) = 1,$$

and the total energy, given by

$$E = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left( \frac{p^2}{2} + \frac{1}{2} \Phi(\theta, t; f) \right) f(\theta, p, t).$$

The HMF model is obtained when $V(\theta - \theta')$ is a cosine potential. In the following, we will present the criteria that determine the dynamical stability of stationary states of the Vlasov equation (2) for a general potential $V(\theta - \theta')$. These criteria can be trivially generalized to arbitrary space dimension. Afterwards, we will specialize to the HMF model, when the relations will be transformed into explicit conditions for the stationary states.

As we have described in section 1, there is a time regime in which the dynamics of the $N$-body system is, to a high degree of accuracy, represented by the time evolution of the one-particle distribution function as governed by the Vlasov equation. In this framework, it is natural to expect that the QSSs will be associated with stationary states (i.e., time independent states) of this equation. Of course, these stationary states should also be stable, i.e., a small perturbation should not drive the system away from the stationary state. Then, it is natural to be interested in the dynamical stability of the
stationary states of the Vlasov equation. It is easy to see that any function of the form $f(\theta, p) = f((p^2/2) + \Phi(\theta; f))$ is stationary; therefore, in principle one is interested in determining the stability of any such function.

The function $f((p^2/2) + \Phi(\theta; f))$ is by definition linearly stable if it is possible to choose the norm of the perturbation $\delta f(\theta, p, t)$ at time $t = 0$ such that this norm remains smaller than any (small) positive number, provided that the dynamics of the perturbation is governed by the linearized Vlasov equation, with the linearization made around the stationary state. Introducing the individual energy:

$$\epsilon(\theta, p) \equiv \frac{p^2}{2} + \Phi(\theta; f),$$  \hspace{1cm} (6)

the linearized Vlasov equation for $\delta f(\theta, p, t)$ is easily obtained as

$$\frac{\partial \delta f(\theta, p, t)}{\partial t} + p \frac{\partial \delta f(\theta, p, t)}{\partial \theta} = \frac{d\Phi(\theta; f)}{d\theta} \frac{\partial \delta f(\theta, p, t)}{\partial p} - \frac{\partial \Phi(\theta, t; \delta f)}{\partial \theta} p f'(\epsilon(\theta, p)) = 0, \hspace{1cm} (7)$$

where the potential $\Phi(\theta; f)$ is constant in time (and thus the partial derivative with respect to $\theta$ has become a total derivative) and where in the last term the functional dependence of $f$ on $(\theta, p)$ only through $\epsilon(\theta, p)$ has been exploited. The problem of the linear stability associated with this equation has been treated long ago by Antonov [13] in astrophysics. In order to have a self-contained presentation, in section 4 we will reproduce, although with a somewhat different formulation and scope, the Antonov results, that later will be used to derive the stability conditions on the stationary state $f((p^2/2) + \Phi(\theta; f))$. These will be necessary and sufficient conditions for linear stability.

As we have previously underlined, we are also interested in less refined stability criteria, that provide only sufficient, but simpler, conditions for linear stability. With that purpose, we can use the notion of formal stability of a stationary point of a general dynamical system, a notion that we have defined at the end of section 1. It can be shown [28] that formal stability implies linear stability; therefore, proving that a stationary point is formally stable gives a sufficient condition for linear stability. Formal stability is also a prerequisite for nonlinear stability, although formal stability does not imply nonlinear stability for infinite dimensional systems. With a slight extension of the definition, we will consider the formal stability of a stationary point also for the cases in which both the extremization of the conserved quantity and the sign of its second variation are studied under some constraints. The problem will be related, as will be clear, by the fact that if one finds that the second variation has a definite sign for the unconstrained case, this is sufficient for having the same definite sign also for the constrained case. However, the converse is wrong and this is similar to the notion of ensemble inequivalence in statistical physics [29].

To study the formal stability problem, we use a general result obtained in [30]. Again for a self-contained presentation, we find it useful to briefly reproduce it here. This is done in the following section, in slightly more general terms than those necessary for the linear Vlasov equation.

3. The maximization of a class of functionals

Let $\mathcal{L}$ be the functional space of real differentiable functions $f(\theta, p)$ defined for $0 \leq \theta \leq 2\pi$ and $p \in \mathcal{R}$. We also assume that the functions decay sufficiently fast for $|p| \to \infty$ that
the integral of \( p^2 f \) is finite. The scalar product in this space is naturally defined by

\[
\langle g, f \rangle = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta g(\theta, p) f(\theta, p).
\] (8)

We consider here the problem \([30]\) of finding the constrained maximum of the functional

\[
S[f] = -\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta C(f(\theta, p)),
\] (9)

with the function \( C(x) \) at least twice differentiable and strictly convex, i.e., with the second derivative strictly positive. The constraints are given by two functionals. The first is a linear–quadratic expression:

\[
E = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta a(\theta, p) f(\theta, p) + \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' f(\theta, p) b(\theta, p, \theta', p') f(\theta', p'),
\] (10)

that we can call the ‘total energy’ (in analogy with equation (5)), and that is constrained to have a given value \( E_0 \). In this expression, \( a \) and \( b \) are two given functions, with \( b(\theta', p', \theta, p) = b(\theta, p, \theta', p') \). The second constraint is the normalization

\[
I = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta f(\theta, p) = 1.
\] (11)

By using Lagrange multipliers \( \beta \) and \( \mu \) the extremum is given by equating the first-order variation to zero:

\[
\delta S - \beta \delta E - \mu \delta I = 0.
\] (12)

We thus have

\[
-C'(f(\theta, p)) - \beta \left[ a(\theta, p) + \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' b(\theta, p, \theta', p') f(\theta', p') \right] - \mu = 0.
\] (13)

If we denote the ‘individual energy’ (in analogy with equation (6)) by

\[
\varepsilon(\theta, p) \equiv a(\theta, p) + \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' b(\theta, p, \theta', p') f(\theta', p'),
\] (14)

then the extremum relation can be written as

\[
-C'(f(\theta, p)) = \beta \varepsilon(\theta, p) + \mu.
\] (15)

From the convexity property of \( C(x) \) it follows that this relation can be inverted to give

\[
f(\theta, p) = F(\beta \varepsilon + \mu) \equiv f(\varepsilon),
\] (16)

where \( F \) is the inverse function of \(-C'\). Inserting this function in equations (10) and (11) we obtain the values of the Lagrange multipliers. It is clear that equation (14) is also a consistency equation. We note that in order to interpret \( f(\theta, p) \) as a distribution function, the only acceptable functions \( C(f) \) in equation (9) are those that, through equation (16),...
provide a positive definite function. We also note the identity
\[
f'(\varepsilon(\theta, p)) = -\frac{\beta}{C''(f(\theta, p))},
\]
that is obtained by differentiating equation (15). From the convexity property of \(C(x)\) it follows that \((1/\beta)f'(\varepsilon(\theta, p))\) is negative definite. On the other hand, since we have \(\partial f / \partial p = pf'(\varepsilon(\theta, p))\), the integrability in \(p\) requires that, if \(f'(\varepsilon(\theta, p))\) has a definite sign, this sign must be negative. Therefore, \(\beta\) is restricted to positive values. We therefore conclude that the extremization of \(S\) at fixed \(E\) and \(I\) determines distribution functions of the form \(f = f(\varepsilon)\) with \(f'(\varepsilon) < 0\).

The extremum so obtained will be a maximum only if the second-order variation of \(S[f]\), for all the allowed displacements \(\delta f(\theta, p)\), is negative definite. The variation of the functional \(S[f]\) is given, up to second order, by
\[
\delta S = -\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \left[ C'(f(\theta, p)) \delta f(\theta, p) + \frac{1}{2} C''(f(\theta, p)) (\delta f(\theta, p))^2 \right],
\]
with the derivatives computed at the extremal point. Then, from equations (15)–(17) we obtain
\[
\delta S = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \left[ (\beta \varepsilon(\theta, p) + \mu) \delta f(\theta, p) + \frac{1}{2} \beta \delta f(\theta, p) - \frac{\beta}{f'(\varepsilon(\theta, p))} (\delta f(\theta, p))^2 \right].
\]
We can transform this expression by using the fact that the variations of \(E\) and \(I\) must identically vanish for the allowed displacements \(\delta f(\theta, p)\). These variations are given by
\[
\delta E = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \varepsilon(\theta, p) \delta f(\theta, p)
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \delta f(\theta, p) b(\theta, p, \theta', p') \delta f(\theta', p') \equiv 0
\]
and
\[
\delta I = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \delta f(\theta, p) \equiv 0
\]
respectively. Adding to equation (19) the zero-valued expression \(-\beta \delta E - \mu \delta I\) we have
\[
\delta S = \frac{1}{2} \beta \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \left( \frac{1}{f'(\varepsilon(\theta, p))} \right) (\delta f(\theta, p))^2
- \frac{1}{2} \beta \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \delta f(\theta, p) b(\theta, p, \theta', p') \delta f(\theta', p').
\]
The right-hand side of this expression must be negative definite for all allowed displacements \(\delta f(\theta, p)\), i.e., for all those that at first order do not change \(E\) and \(I\).

The problem of maximizing \(S[f]\) at constant \(E\) and \(I\) can be shown to be equivalent to that of minimizing the energy \(E\) at constant \(S\) and \(I\). In fact, using Lagrange multipliers \(1/\beta\) and \(-\mu/\beta\), the equation of the first-order variation is now
\[
\delta E - \frac{1}{\beta} \delta S + \frac{\mu}{\beta} \delta I = 0,
\]
which is the same as equation (12); then the solution is again given by equation (13). Now, we have to study the variation of $E$, that up to second order is given by the left-hand side of equation (20), i.e.

$$
\delta E = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \varepsilon(\theta, p) \delta f(\theta, p) \\
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \delta f(\theta, p) b(\theta, p, \theta', p') \delta f(\theta', p').
$$

(24)

The variations of $S$ and $I$ must identically vanish; the latter is expressed by equation (21), while the former is given by

$$
\delta S = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left[ (\beta \varepsilon(\theta, p) + \mu) \delta f(\theta, p) + \frac{1}{2} \frac{1}{f'(\varepsilon(\theta, p))} (\delta f(\theta, p))^2 \right] = 0.
$$

(25)

Adding to equation (24) the zero-valued expression $-(1/\beta) \delta S + (\mu/\beta) \delta I$ we arrive at

$$
\delta E = -\frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{f'(\varepsilon(\theta, p))} (\delta f(\theta, p))^2 \\
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \delta f(\theta, p) b(\theta, p, \theta', p') \delta f(\theta', p').
$$

(26)

This is the same as equation (22) divided by $-\beta < 0$. Therefore, equation (26) is positive definite, i.e., $E$ is minimum at the stationary state, if equation (22) is negative definite. To complete the proof of the equivalence between the maximization of $S$ at constant $E$ and $I$ and the minimization of $E$ at constant $S$ and $I$ we have to see that in both cases the allowed displacements $\delta f(\theta, p)$ are the same. This can be deduced in the following way. For equation (22) the allowed displacements are all those at first order give $\delta E = \delta I = 0$. By equation (12), they also at first order give $\delta S = 0$; then, they are also allowed for equation (26). In turn, the allowed displacements for equation (26) are all those that at first order give $\delta S = \delta I = 0$. By equation (23), they also at first order give $\delta E = 0$; then, they are also allowed for equation (22). This concludes the proof.

After treating the problem of the linear stability of the stationary states of the Vlasov equation, we will use the results of this section to study their formal stability.

4. The linear stability of Vlasov stationary states

The energy functional in the previous section, equation (10), was more general than the one associated with the Vlasov equation, equation (5). The former reduces to the latter when the functions $a(\theta, p)$ and $b(\theta, p, \theta', p')$ are related to the kinetic energy and to the potential energy of the system, respectively; namely, when $a = p^2/2$ and $b = V(\theta - \theta')$. For convenience, we rewrite here the relevant expressions. We have the individual energy

$$
\varepsilon(\theta, p) = \frac{p^2}{2} + \Phi(\theta; f),
$$

(27)

with the mean field potential

$$
\Phi(\theta; f) = \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' V(\theta - \theta') f(\theta', p').
$$

(28)
and the total energy
\[ E[f] = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left( \frac{p^2}{2} + \frac{1}{2} \Phi(\theta, t; f) \right) f(\theta, p, t). \] (29)

We will consider stationary states that are associated with the extremization of functionals of the form (9). We have seen that in this case \( f(\theta, p) = f(\epsilon(\theta, p)) \) with \( f'(\epsilon(\theta, p)) < 0 \). For more compactness, we will use the notation \( \gamma(\theta, p) \equiv f'(\epsilon(\theta, p)) \). As remarked above, we follow and complete the treatment of Antonov [13].

In the following we will need the usual extension of the scalar product defined in equation (8) to complex valued functions:
\[ \langle g, f \rangle = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta g^*(\theta, p) f(\theta, p), \] (30)

where the asterisk denotes complex conjugation.

It is not difficult to see that the linearized Vlasov equation (7), determining the dynamics of \( \delta f(\theta, p) \), can be cast in the form
\[ \frac{\partial \delta f}{\partial t}(\theta, p, t) = -\gamma(\theta, p)(DK\delta f)(\theta, p, t), \] (31)

where \( D \) is the antisymmetric linear differential operator (advective operator):
\[ (Dg)(\theta, p) = p \frac{\partial}{\partial \theta} g(\theta, p) - \frac{d\Phi}{d\theta} \frac{\partial}{\partial p} g(\theta, p), \] (32)

while \( K \) is the linear integral operator:
\[ (Kg)(\theta, p) = \frac{1}{\gamma(\theta, p)} g(\theta, p) - \Phi(\theta; g). \] (33)

This can be obtained by seeing that \( D\epsilon = 0 \); then, the action of \( D \) on any function of \( \epsilon \) gives zero; in particular \( D\gamma = 0 \). Finally, it can be easily checked that the operator \( B \equiv DKD \), needed shortly, is Hermitian.

The stationary point of the dynamics \( \delta f(\theta, p) \equiv 0 \) will be linearly stable iff: (i) all nonsecular solutions of the type \( \delta f(\theta, p, t) = \delta f(\theta, p, 0) \exp(\lambda t) \) have eigenvalues \( \lambda \) with nonpositive real part; (ii) in the presence of secular terms (i.e., if there are eigenvalues with an algebraic multiplicity larger than the geometric multiplicity), the associated eigenvalue must have a negative real part.

Before proceeding further, we need to provide evidence for the properties of the operators \( D \) and \( K \) when acting on functions that are either symmetric or antisymmetric in \( p \). From the definition of \( D \) in equation (32) it is clear that \( D \) transforms symmetric functions into antisymmetric functions, and vice versa. As regards \( K \), we see from its definition in equation (33) that it maintains the symmetry of the functions; however, since for antisymmetric functions \( g_a(\theta, p) \) we have \( \Phi(\theta; g_a) \equiv 0 \), the action of \( K \) in this case simplifies to
\[ (Kg_a)(\theta, p) = \frac{1}{\gamma(\theta, p)} g_a(\theta, p). \] (34)
We now suppose that $\delta f(\theta, p; \lambda)$ is the eigenfunction associated with the eigenvalue $\lambda$. We then have, from equation (31),
\[ \lambda \delta f(\theta, p; \lambda) = -\gamma(DK\delta f)(\theta, p; \lambda), \tag{35} \]
where for simplicity we have dropped the dependence of $\gamma$ on the coordinates. We now separate $\delta f$ into the symmetric and antisymmetric parts: $\delta f = \delta f_s + \delta f_a$. Taking into account the aforementioned properties of the operators $D$ and $K$ we obtain
\[ \lambda \delta f_s(\theta, p; \lambda) = -\gamma(D\delta f_a)(\theta, p; \lambda) \tag{36} \]
and
\[ \lambda \delta f_a(\theta, p; \lambda) = -\gamma(K\delta f_s)(\theta, p; \lambda). \tag{37} \]
If we multiply the second of these equations by $\lambda$ and substitute in $\lambda \delta f_s$ from the first we have
\[ \lambda^2 \delta f_a(\theta, p; \lambda) = \gamma(DK\delta f_a)(\theta, p; \lambda) = \gamma(B\delta f_a)(\theta, p; \lambda). \tag{38} \]
If $\lambda \neq 0$, equations (36) and (37) prove two properties. The first is that an eigenfunction cannot be either symmetric or antisymmetric, but both components must be nonvanishing. The second is that, if $\delta f_s + \delta f_a$ is associated with the eigenvalue $\lambda$, then $\delta f_s - \delta f_a$ is associated with the eigenvalue $-\lambda$.

From equation (38) we have
\[ \lambda^2 \frac{\delta f_a(\theta, p; \lambda)}{\gamma(\theta, p)} = (B\delta f_a)(\theta, p; \lambda). \tag{39} \]

The scalar product of the two sides of this expression with $\delta f_a$ gives
\[ \lambda^2 \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{|\delta f_a(\theta, p; \lambda)|^2}{\gamma(\theta, p)} = \langle \delta f_a, B\delta f_a \rangle, \tag{40} \]
while the scalar product of its complex conjugate with $\delta f_a^*$ gives, exploiting the hermiticity of $B$,
\[ (\lambda^*)^2 \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{|\delta f_a(\theta, p; \lambda)|^2}{\gamma(\theta, p)} = \langle \delta f_a, B\delta f_a \rangle. \tag{41} \]
Since $\gamma$ is negative definite, equations (40) and (41) imply that, if $\lambda \neq 0$, $\lambda^2$ is necessarily real, and therefore $\lambda$ is either real or pure imaginary. In much the same way, it can be shown that, if $\delta f(\theta, p; \lambda_1)$ and $\delta f(\theta, p; \lambda_2)$ correspond to two different nonzero eigenvalues, then
\[ \langle \delta f_a(\lambda_1), B\delta f_a(\lambda_2) \rangle = 0. \tag{42} \]

The case $\lambda = 0$ will be considered later.

Let us now assume that an eigenvalue $\lambda$ has an algebraic multiplicity larger than its geometric multiplicity. Then, if $\delta f(\theta, p; \lambda)$ is an eigenvector associated with this eigenvalue, there will also exist a solution of equation (31) given by $[\delta f(\theta, p; \lambda) + \delta f^{(1)}(\theta, p; \lambda)]\exp(\lambda t)$. Substituting in equation (31) and using equation (35) we obtain
\[ \delta f(\theta, p; \lambda) + \lambda \delta f^{(1)}(\theta, p; \lambda) = -\gamma(DK\delta f^{(1)})(\theta, p; \lambda). \tag{43} \]
Separating both $\delta f$ and $\delta f^{(1)}$ into their symmetric and antisymmetric parts we obtain

$$\delta f_s(\theta, p; \lambda) + \lambda \delta f_s^{(1)}(\theta, p; \lambda) = -(D\delta f_a^{(1)})(\theta, p; \lambda)$$

(44)

and

$$\delta f_a(\theta, p; \lambda) + \lambda \delta f_a^{(1)}(\theta, p; \lambda) = -\gamma (DK\delta f_s^{(1)})(\theta, p; \lambda).$$

(45)

If we multiply the second of these equations by $\lambda$ and we substitute in $\lambda \delta f_s^{(1)}$ from the first we have

$$\lambda \delta f_a(\theta, p; \lambda) + \lambda^2 \delta f_a^{(1)}(\theta, p; \lambda) = \gamma (B\delta f_a^{(1)})(\theta, p; \lambda) + \gamma (DK\delta f_s)(\theta, p; \lambda).$$

(46)

Using equation (37) we arrive at

$$\frac{2\lambda \delta f_a(\theta, p; \lambda) + \lambda^2 \delta f_a^{(1)}(\theta, p; \lambda)}{\gamma(\theta, p)} = (B\delta f_a^{(1)})(\theta, p; \lambda).$$

(47)

The scalar product of this expression with $\delta f_a$ gives

$$\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{2 \lambda |\delta f_a(\theta, p; \lambda)|^2 + \lambda^2 |\delta f_a^{(1)}(\theta, p; \lambda)|^2}{\gamma(\theta, p)} = \langle \delta f_a, B\delta f_a^{(1)} \rangle.$$  

(48)

Subtracting from this equation the one that is obtained by forming the scalar product of the complex conjugate of equation (39) with $\delta f_a^{(1)*}$ we have

$$\lambda \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{|\delta f_a(\theta, p; \lambda)|^2}{\gamma(\theta, p)} = 0.$$  

(49)

If $\lambda \neq 0$ we deduce that $\delta f_a(\theta, p; \lambda) \equiv 0$, since $\gamma$ is negative definite; then also $\delta f_a(\theta, p; \lambda) \equiv 0$ and thus $\delta f(\theta, p; \lambda) \equiv 0$. This shows that no eigenvalue $\lambda$ different from 0 has an algebraic multiplicity larger than its geometric multiplicity.

We now consider the case $\lambda = 0$. We note that the presence of a zero eigenvalue with an algebraic multiplicity larger than the geometrical multiplicity would imply the presence of a solution of the equation of motion (31) of the form $[\delta f(\theta, p; 0) + \delta f^{(1)}(\theta, p; 0)]$, and therefore the stationary point $\delta f(\theta, p) \equiv 0$ would be linearly unstable. In the following we need to consider also the possibility that the difference between the algebraic and the geometric multiplicities of the zero eigenvalue is such that solutions with higher powers of the time $t$ exist.

For $\lambda = 0$, equations (36) and (37) become

$$(D\delta f_a)(\theta, p; 0) = 0$$

(50)

and

$$(DK\delta f_s)(\theta, p; 0) = 0.$$  

(51)

We note that equation (50) implies $(B\delta f_a)(\theta, p; 0) = 0$.

In the case where the eigenvalue $\lambda = 0$ has an algebraic multiplicity larger than the geometric multiplicity, we obtain, from equations (44) and (45),

$$\delta f_s(\theta, p; 0) = -(D\delta f_a^{(1)})(\theta, p; 0)$$

(52)
and
\[ \delta f_a(\theta, p; 0) = -\gamma(DK\delta f_s^{(1)})(\theta, p; 0). \] (53)

Substitution of equation (52) in (51) gives \((B\delta f_a^{(1)})(\theta, p; 0) = 0\). Dividing both sides of equation (53) by \(\gamma\) and forming the scalar product with \(\delta f_a\) we get
\[ \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \frac{|\delta f_a(\theta, p; 0)|^2}{\gamma(\theta, p)} = -\langle \delta f_a, DK\delta f_s^{(1)} \rangle = \langle D\delta f_a, K\delta f_s^{(1)} \rangle = 0, \] (54)
where in the last step we have used equation (50). It follows that \(\delta f_a(\theta, p; 0) \equiv 0\), and then \(\delta f_s(\theta, p; 0) \neq 0\) in order to have a nontrivial solution. Then, from equation (52) we obtain that \(\delta f_s^{(1)}(\theta, p; 0) \neq 0\) and \((D\delta f_s^{(1)}(\theta, p; 0) = -\delta f_s(\theta, p; 0) \neq 0\).

If there exists a solution of the equation of motion (31) of the form \([t^2\delta f(\theta, p; 0) + t\delta f^{(1)}(\theta, p; 0) + \delta f^{(2)}(\theta, p; 0)]\), then the evaluation of (31) at \(t = 0\) gives
\[ \delta f^{(1)}(\theta, p; 0) = -\gamma(DK\delta f^{(2)})(\theta, p; 0). \] (55)

The usual separation into the symmetric and antisymmetric parts gives
\[ \delta f_s^{(1)}(\theta, p; 0) = -(D\delta f_a^{(2)})(\theta, p; 0) \] (56)
and
\[ \delta f_a^{(1)}(\theta, p; 0) = -\gamma(DK\delta f_s^{(2)})(\theta, p; 0). \] (57)

The substitution of equation (56) in (52), taking into account that \(\delta f_a(\theta, p; 0) = 0\), gives \((B\delta f_a^{(2)})(\theta, p; 0) = 0\). Finally, it can be shown that a solution of the form \([t^2\delta f(\theta, p; 0) + t\delta f^{(1)}(\theta, p; 0) + \delta f^{(2)}(\theta, p; 0) + \delta f^{(3)}(\theta, p; 0)]\) cannot exist. In fact, the evaluation of (31) at \(t = 0\) gives, after separation into the symmetric and antisymmetric parts,
\[ \delta f_s^{(2)}(\theta, p; 0) = -(D\delta f_a^{(3)})(\theta, p; 0). \] (58)

Substitution in equation (57) gives
\[ \delta f_a^{(1)}(\theta, p; 0) = \gamma(B\delta f_a^{(3)})(\theta, p; 0). \] (59)

Dividing both sides by \(\gamma\) and forming the scalar product with \(\delta f_a^{(1)}\) we obtain
\[ \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \frac{|\delta f_a^{(1)}(\theta, p; 0)|^2}{\gamma(\theta, p)} = \langle \delta f_a^{(1)}, B\delta f_a^{(3)} \rangle = \langle B\delta f_a^{(1)}, \delta f_a^{(3)} \rangle = 0, \] (60)

since \((B\delta f_a^{(1)})(\theta, p; 0) = 0\). This implies that \(\delta f_a^{(1)}(\theta, p; 0) = 0\), that in turn gives, from equation (52), \(\delta f_s(\theta, p; 0) = 0\). This is not acceptable, since we already have \(\delta f_a(\theta, p; 0) = 0\).

Summarizing all the results, we have that the initial value of \(\delta f(\theta, p)\) can be decomposed in general as
\[ \delta f(\theta, p) = \sum_{\lambda \neq 0} c(\lambda)\delta f(\theta, p; \lambda) + \sum_{j=1}^{r} \left[ c_j \delta f_j(\theta, p; 0) + c_j^{(1)} \delta f_j^{(1)}(\theta, p; 0) + c_j^{(2)} \delta f_j^{(2)}(\theta, p; 0) \right], \] (61)
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where the sum over the nonzero eigenvalues stands also for an integral if the eigenvalues are continuously distributed. The sum over \( r \) different contributions coming from the zero eigenvalue takes into account possible separation into disjoint eigenspaces; some of the corresponding functions might have only \( \delta f_j(\theta, p; 0) \) different from zero, and some only \( \delta f_{j_0}(\theta, p; 0) \) and \( \delta f_{j_1}^{(1)}(\theta, p; 0) \). The fact that \((B\delta f_{j,a})(\theta, p; 0) = (B\delta f_{j,a}^{(1)})(\theta, p; 0) = (B\delta f_{j,a}^{(2)})(\theta, p; 0) = 0\) implies that if we take the antisymmetric part of \( \delta f(\theta, p) \) in equation (61) and form the scalar product \( \langle \delta f_a, B\delta f_a \rangle \), only the eigenfunctions corresponding to the eigenvalues \( \lambda \) different from zero contribute. To be precise, using the orthogonality property (42) we have

\[
\langle \delta f_a, B\delta f_a \rangle = \sum_{\lambda \neq 0} |c(\lambda)|^2 \langle \delta f_a(\lambda), B\delta f_a(\lambda) \rangle. 
\]  

(62)

With the use of equation (40) we get

\[
\langle \delta f_a, B\delta f_a \rangle = \sum_{\lambda \neq 0} |c(\lambda)|^2 \lambda^2 \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \left| \frac{\delta f_a(\theta, p; \lambda)}{\gamma(\theta, p)} \right|^2. 
\]  

(63)

This is the first important expression of this section. From it we deduce that, if the stationary point \( \delta f(\theta, p) = 0 \) is linearly stable, then necessarily the last expression is positive for all cases in which not all of the coefficients \( c(\lambda) \) are zero. In fact, the linear stability requires that all eigenvalues different from zero are pure imaginary (we recall that if a real negative eigenvalue exists, also its opposite exists and leads to instability; we also recall that the antisymmetric part of any eigenfunction corresponding to an eigenvalue different from zero is nonvanishing). From this and from the fact that \( \gamma \) is negative definite, our statement follows.

The scalar product (63) can be zero for a nonvanishing \( \delta f \) if one or some of the coefficients \( c_j, c_j^{(1)}, c_j^{(2)} \) are nonzero. We know that, if one of the eigenspaces, e.g. the one corresponding to \( j = j_0 \) in equation (61), has an algebraic multiplicity larger than its geometric multiplicity, the stationary point \( \delta f(\theta, p) = 0 \) is linearly unstable. In this case, taking \( \delta f(\theta, p) = \delta f_{j_0}^{(1)}(\theta, p) \) we have \( \langle \delta f_a, B\delta f_a \rangle = 0 \) with, according to what was proved just after equation (54), \( (D\delta f_a)(\theta, p) \neq 0 \). On the other hand, if all the eigenspaces corresponding to \( \lambda = 0 \) have equal algebraic and geometric multiplicities, then only the terms with \( c_j \) appear in equation (61) as the contribution from the zero eigenvalue. But in this case, it follows from equation (50) that the scalar product (63) can be zero only for a function \( \delta f(\theta, p) \) such that \( (D\delta f_a)(\theta, p) = 0 \).

In conclusion, the necessary and sufficient condition for the linear stability is that the scalar product (63) is nonnegative, and it is zero only for functions \( \delta f(\theta, p) \) such that \( (D\delta f_a)(\theta, p) = 0 \).

We now rewrite the scalar product \( \langle \delta f_a, B\delta f_a \rangle \) in another way. To be precise,

\[
\langle \delta f_a, B\delta f_a \rangle = \langle \delta f_a, DKD\delta f_a \rangle = -\langle D\delta f_a, KD\delta f_a \rangle = -\left( D\delta f_a, \frac{1}{\gamma} D\delta f_a \right) + \langle D\delta f_a, \Phi(D\delta f_a) \rangle, 
\]  

(64)

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where use has been made of the definition of the operator $K$ in equation (33). Using the definition of $\Phi$ in equation (3) we finally obtain

$$
\langle \delta f_a, B \delta f_a \rangle = -\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} \langle (D\delta f_a)(\theta, p) \rangle^2.
$$

$$
+ \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \langle (D\delta f_a)^*(\theta, p) V(\theta - \theta')(D\delta f_a)(\theta', p') \rangle.
$$

(65)

We should note two things. Firstly, since our equation of motion is real, it is always possible to choose $\delta f(\theta, p)$ real. Secondly, it is not difficult to check, from the last expression, that if we consider the whole function $\delta f(\theta, p)$ instead of its antisymmetric part $\delta f_a(\theta, p)$, we will always have $\langle \delta f, B \delta f \rangle \geq \langle \delta f_a, B \delta f_a \rangle$. We then arrive at the following necessary and sufficient condition for linear stability:

$$
-\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} \langle (D\delta f)(\theta, p) \rangle^2
$$

$$
+ \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \langle (D\delta f)(\theta, p) V(\theta - \theta')(D\delta f)(\theta', p') \rangle \geq 0
$$

(66)

for any $\delta f(\theta, p)$, with the equality holding only when $(D\delta f)(\theta, p) = 0$. This is the main expression of this section. In treating the particular case of the HMF model, it will be the basis for obtaining a condition on the stationary state $f(\varepsilon(\theta, p))$.

5. The energy principle and the most refined formal stability criterion

The stability of stationary states of the Vlasov equation has been studied by Kandrup [16, 17], introducing a Hamiltonian formulation for this equation, obtaining a sufficient condition for linear stability. We give here few details, mainly to show that, for stationary states of the form $f(\varepsilon(\theta, p))$, with $f'(\varepsilon(\theta, p)) < 0$, this condition becomes identical to the one given in equation (66), thus becoming also necessary.

This approach is based on a Hamiltonian formulation of the Vlasov equation, and on the observation that the Vlasov dynamics admits an infinite number of conserved quantities, called Casimir invariants, given by

$$
C_A[f] = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta A(f(\theta, p)),
$$

(67)

for any function $A(x)$. Note that functionals of the form (9) are particular Casimirs. The Hamiltonian formulation of the Vlasov equation (2) for $f(\theta, p)$ is realized by casting it in the form

$$
\frac{\partial f}{\partial t} + \{f, \varepsilon\} = 0,
$$

(68)

where $\varepsilon$ is the individual energy (27) and the curly brackets denote the Poisson bracket

$$
\{a, b\} = \frac{\partial a}{\partial \theta} \frac{\partial b}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial b}{\partial \theta}.
$$

(69)
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It can be shown that, if \( f(\theta, p) \) is a stationary state of the Vlasov equation of a general form, then a sufficient condition for its linear stability is the following [17]: the difference between the total energy (29) computed at the perturbed state \( f(\theta, p) + \delta f(\theta, p) \) and the total energy computed at the stationary state \( f(\theta, p) \) is positive for all \( \delta f(\theta, p) \) that conserve all the Casimirs. In other words, \( f(\theta, p) \) is linearly stable if it is a local minimum of energy with respect to perturbations that conserve all the Casimirs. This forms the most refined formal stability criterion. These so-called ‘phase-preserving’ or symplectic perturbations can be expressed in the form

\[
\delta E = e^{(a, f)} f(\theta, p),
\]

for some ‘small’ generating function \( a(\theta, p) \). They amount to a rearrangement of phase levels by a mere advection in phase space. Expanding to second order in \( a \) we have

\[
f(\theta, p) + \delta f(\theta, p) = f(\theta, p) + \{a, f\} + \frac{1}{2} \{a, \{a, f\}\}.
\]

The corresponding expansion of the total energy (29) is easily obtained as

\[
E[f + \delta f] - E[f] = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \varepsilon(\theta, p) \{a, f\} (\theta, p)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \varepsilon(\theta, p) \{a, \{a, f\}\} (\theta, p)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \{a, f\} (\theta, p) V(\theta - \theta') \{a, f\} (\theta', p'),
\]

where the individual energy (27) for the stationary distribution has been used (and for clarity the explicit dependence of the Poisson brackets has been written out). Now, it is possible to exploit the identity

\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta c_1 \{c_2, c_3\} = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta c_2 \{c_3, c_1\}
\]

(73)

for any three functions \( c_1, c_2, c_3 \). Using in addition that for a stationary state \( \{f, \varepsilon\} = 0 \), we have that the first line of the right-hand side of equation (72), i.e., the first-order variation of the total energy, vanishes. This shows that the total energy for a stationary distribution is an extremum (\( \delta E = 0 \)) with respect to symplectic perturbations. However, this does not guarantee that it is an extremum with respect to all perturbations. Using again the identity (73), the second-order variations of energy deduced from equation (72) are

\[
\delta^2 E = \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \{\varepsilon, a\} (\theta, p) \{a, f\} (\theta, p)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \{a, f\} (\theta, p) V(\theta - \theta') \{a, f\} (\theta', p').
\]

(74)

The positive definiteness of this expression is a sufficient condition for linear stability.

We now suppose that the stationary distribution function is a function of \( \varepsilon(\theta, p) \). In this case \( \{a, f\} = f'(\varepsilon)\{a, \varepsilon\} \). Furthermore, we also have \( \{a, \varepsilon\} = Da \), where \( D \) is the
linear differential operator defined in equation (32). Using that \( D\varepsilon = 0 \), and therefore \( Df'(\varepsilon) = 0 \), equation (74) becomes in this case

\[
\delta^2 E = -\frac{1}{2} \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \frac{1}{f'(\varepsilon(\theta, p))} ((D\tilde{a})(\theta, p))^2 \\
+ \frac{1}{2} \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' (D\tilde{a})(\theta, p)V(\theta - \theta')(D\tilde{a})(\theta', p'),
\]

where \( \tilde{a} \equiv f'(\varepsilon)a \). The positive definiteness of this expression is exactly the necessary and sufficient condition for linear stability (66), recalling the definition of \( \gamma(\theta, p) \).

It is convenient to introduce the notation \( \delta f(\theta, p) \equiv D\tilde{a}(\theta, p) \). Then, the necessary and sufficient condition for linear stability can be written as

\[
-\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} (\delta f(\theta, p))^2 \\
+ \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \delta f(\theta, p)V(\theta - \theta')\delta f(\theta', p') \geq 0,
\]

for any perturbation of the form \( \delta f(\theta, p) \equiv D\tilde{a}(\theta, p) \) where \( \tilde{a}(\theta, p) \) is any function. These perturbations correspond to a mere displacement (by the advective operator \( D \)) of the phase levels, i.e. to dynamically accessible perturbations. It is straightforward to check by a direct calculation that these perturbations conserve energy and all the Casimirs at first order (this is of course obvious for symplectic perturbations). Indeed, using \( \delta f(\theta, p) = \{a, f\} \) and identity (73), we get

\[
\delta E = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \delta f(\theta, p)\varepsilon = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \{a, f\} \varepsilon \\
= \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \{f, \varepsilon\} a = 0,
\]

and

\[
\delta C_A = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \delta f(\theta, p)A'(f) = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \{a, f\} A'(f) \\
= \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \{f, A'(f)\} a = 0.
\]

### 6. Less refined formal stability criteria: sufficient conditions for stability

As we have seen above, the minimization of \( E \) with respect to symplectic perturbations, i.e. dynamically accessible perturbations that conserve all the Casimirs, is a necessary and sufficient condition for linear stability. It is also the most refined criterion for formal stability since all the constraints of the Vlasov equation are taken into account individually. Less refined formal stability criteria that provide only sufficient (albeit simpler) conditions for linear stability can be obtained by relaxing some constraints.
6.1. The 'microcanonical' formal stability

As we have seen in section 3, a stationary state of the form \( f(\theta, p) = f(\varepsilon(\theta, p)) \) with \( f'(\varepsilon) < 0 \) is obtained by extremizing a functional \( S \) of the form (9), with the function \( C(x) \) given by equation (16), under the constraints given by the normalization \( I \), equation (11), and the total energy \( E \), equation (29); but also by extremizing the total energy \( E \) at constant \( S \) and \( I \). Furthermore, we have proven that a maximum of \( S \) at fixed \( E \) and \( I \) is a minimum of \( E \) at fixed \( S \) and \( I \) and vice versa. Since we have seen that for distributions of the form \( f(\varepsilon(\theta, p)) \) the minimization of \( E \) with respect to symplectic perturbations (i.e. perturbations that conserve all the Casimirs) is a necessary and sufficient condition for linear stability, it is clear that the minimization of \( E \) with respect to all perturbations that conserve \( S \) and \( I \) (i.e., two particular Casimirs) gives a sufficient condition for linear stability. In turn, this means that the maximization of \( S \) at constant \( E \) and \( I \) gives the same sufficient condition.

Specializing equation (26) to our case, we immediately obtain our sufficient condition:

\[
\begin{align*}
- \int_{-\infty}^{\infty} dp & \int_0^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} (\delta f(\theta, p))^2 \\
+ \int_{-\infty}^{\infty} dp & \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \delta f(\theta, p)V(\theta - \theta') \delta f(\theta', p') \geq 0
\end{align*}
\]

(79)

for all \( \delta f(\theta, p) \) that at first order give \( \delta S = \delta I = 0 \), or equivalently \( \delta E = \delta I = 0 \).

Another way to see that equation (79) gives a sufficient condition if equation (76) gives a necessary and sufficient condition is the following. Considering that \( D\varepsilon = 0 \), we have that all functions of the type \( \delta f = D\tilde{a} \) give \( \delta E = \delta I = 0 \) at first order. Therefore, we arrive at the conclusion that the condition for the maximization of \( S \) at constant \( E \) and \( I \), or for the minimization of \( E \) at constant \( S \) and \( I \), is stronger than the condition for linear dynamical stability. To put this differently, if inequality (79) is satisfied for all perturbations \( \delta f(\theta, p) \) that conserve \( E \) and \( I \) at first order, it is a fortiori satisfied for all perturbations that conserve \( E \), \( I \) and all the Casimirs at first order. However, the reciprocal is wrong. Therefore, equation (79) gives only a sufficient condition for linear stability.

At this stage, it is interesting to note some analogies with thermodynamics. In particular, the formal stability obtained by maximizing \( S \) at constant \( E \) and \( I \) can be interpreted as a 'microcanonical' formal stability problem if we regard \( S \) as a 'pseudo-entropy'. Less refined stability properties can be found by relaxing one or both constraints (see below).

On the other hand, taking \( S \) as the Boltzmann entropy, we note that thermodynamical stability (in the usual sense) implies Vlasov linear dynamical stability. However, the converse may not be true in the general case, i.e. the Maxwell–Boltzmann distribution could be linearly stable according to (76) without being a maximum of Boltzmann entropy at fixed energy and normalization (i.e. a thermodynamical state).

6.2. The 'canonical' formal stability

Following the usual procedure of thermodynamics, we pass from the ‘microcanonical’ problem of maximizing \( S \), equation (9), at constant \( E \) and \( I \), to the ‘canonical’ problem...
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of maximizing the ‘pseudo-free energy’ $S - \beta E$ (equivalent to minimizing $E - (1/\beta)S$) at constant $I$. Introducing the Lagrange multiplier $\mu$, we obtain again the first-order variational problem

$$\delta S - \beta \delta E - \mu \delta I = 0,$$

(80)
equivalent to equation (12), and therefore the same extremizing stationary state. Without repeating again the computations made in section 3, it is now clear that the condition for a maximum (i.e., for formal stability) is given by the relation

$$- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} (\delta f(\theta, p))^2 + \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \delta f(\theta, p)V(\theta - \theta')\delta f(\theta', p') \geq 0$$

(81)

for all $\delta f(\theta, p)$ that at first order give $\delta I = 0$.

6.3. The ‘grand-canonical’ formal stability

Relaxing also the constraint of normalization is associated with the passage to the ‘grand-canonical’ problem. Namely, we look for the maximum of the ‘pseudo-grand potential’ $S - \beta E - \mu I$ (or the minimum of $E - (1/\beta)S + (\mu/\beta)I$) without any constraint. The first-order variational problem will be again given by equation (80), thus obtaining the same stationary state, while the condition for formal stability is given by the relation

$$- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta, p)} (\delta f(\theta, p))^2 + \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \delta f(\theta, p)V(\theta - \theta')\delta f(\theta', p') \geq 0$$

(82)

for all $\delta f(\theta, p)$.

This unconstrained problem corresponds to the usual energy Casimir method [28].

6.4. Summary of stability problems

We have found that the necessary and sufficient condition for linear dynamical stability for a stationary state $f(\varepsilon(\theta, p))$ of the Vlasov equation is given by equation (66). We have proven that this is equivalent to the fact that the stationary distribution function $f$ satisfies (locally) the problem

$$\min_{f} \{ E[f] \mid \text{all Casimirs} \}.$$  

(83)

This is the most refined criterion for formal stability as it takes into account an infinity of constraints. By relaxing some constraints, we have then found that progressively less refined, sufficient conditions for linear stability are given by the following problems. First, the ‘microcanonical’ stability problem

$$\max_{f} \{ S \mid E, I \},$$

(84)
equivalent to
\[ \min_f \{ E \mid S, I \}. \]  (85)

Then, the ‘canonical’ stability problem
\[ \max_f \{ S - \beta E \mid I \}. \]  (86)

Finally, the ‘grand-canonical’ stability problem
\[ \max_f \{ S - \beta E - \mu I \}. \]  (87)

The solution of an optimization problem is always a solution of a more constrained dual problem [29]. Therefore, a distribution function that satisfies the ‘grand-canonical’ stability problem (no constraint) will satisfy the ‘canonical’ stability problem, a distribution that satisfies the ‘canonical’ stability problem (one constraint) will satisfy the ‘microcanonical’ stability problem, and a distribution that satisfies the ‘microcanonical’ stability problem (two constraints) will satisfy the infinitely constrained stability problem (83). This is the analogue of what happens in the study of the stability of macrostates in thermodynamics. Of course, the converse of these statements is wrong and this is similar to the notion of ensemble inequivalence in thermodynamics. We have the chain of implications
\[ (87) \Rightarrow (86) \Rightarrow (85) \Leftrightarrow (84) \Rightarrow (83). \]  (88)

The usefulness of these less refined conditions for linear stability will be clear after the stability conditions, that now appear as conditions to be satisfied by the perturbation to the stationary distribution function, are transformed, in the application to the HMF model, into explicit conditions on the stationary distribution function itself. It will be shown that the less refined conditions for stability are associated with simpler expressions, and therefore, in a concrete calculation, one might use the simpler expressions if the more refined ones appear to be practically unfeasible. The procedure is to start by the simplest problem and progressively consider more and more refined stability problems so as to prove (if necessary) the stability of a larger and larger class of distributions. Of course, if we can prove the stability of all the distribution functions with a particular criterion (see, e.g., the appendix), it is not necessary to consider more refined criteria.

Remark. The connection between the optimization problems (83)–(87) was first discussed in relation to the Vlasov equation in [31], in section 8.4 of [26], and in section 3.1 of [32]. Similar results are obtained in 2D fluid mechanics for the Euler–Poisson system [33]. Criterion (83) is equivalent to the so-called Kelvin–Arnol’d energy principle, criterion (87) is equivalent to the standard Casimir energy method introduced by Arnol’d [34] and criterion (84) is equivalent to the refined stability criterion given by Ellis et al [35].
7. The linear dynamical stability of Vlasov stationary states of the HMF model

For the HMF model we have \( V(\theta - \theta') = -\cos(\theta - \theta') \). The extremization of a functional of the type (9) leads to a function of the type (see equation (16))

\[
f(\theta, p) = F \left[ \beta \left( \frac{p^2}{2} - M_x(f) \cos \theta - M_y(f) \sin \theta \right) + \mu \right],
\]

with \( \beta > 0 \), and with \( M_x(f) \) and \( M_y(f) \) given by self-consistency equations

\[
M_x(f) = \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \cos \theta' f(\theta', p'),
\]

and

\[
M_y(f) = \int_{-\infty}^{\infty} dp' \int_0^{2\pi} d\theta' \sin \theta' f(\theta', p').
\]

These are the two components of the magnetization. In this case the mean field potential \( \Phi(\theta; f) \) is

\[
\Phi(\theta; f) = -M_x(f) \cos \theta - M_y(f) \sin \theta,
\]

and therefore the individual energy is given by

\[
\varepsilon(\theta, p) \equiv \frac{p^2}{2} + \Phi(\theta; f) = \frac{p^2}{2} - M_x(f) \cos \theta - M_y(f) \sin \theta.
\]

Substituting equations (92), (90) and (91) in equation (29), we obtain the total energy

\[
E = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \frac{p^2}{2} f(\theta, p) - \frac{1}{2}(M_x^2(f) + M_y^2(f)).
\]

The Vlasov equation for the HMF model reads

\[
\frac{\partial f(\theta, p)}{\partial t} + p \frac{\partial f(\theta, p)}{\partial \theta} - (M_x(f) \sin \theta - M_y(f) \cos \theta) \frac{\partial f(\theta, p)}{\partial p} = 0.
\]

For the HMF model, we are interested in studying the stability of stationary solutions of equation (95) given by functions of the form \( f(\theta, p) = f(\frac{p^2}{2} - M_x(f) \cos \theta - M_y(f) \sin \theta) \).

Without loss of generality we can suppose that \( M_y(f) = 0 \) and therefore that \( f(\theta, p) = F \left[ \beta \left( \frac{p^2}{2} - M \cos \theta \right) + \mu \right] \),

where for simplicity we have defined \( M \equiv M_x(f) \) (dropping the explicit dependence on \( f \)). In this case we have

\[
\Phi(\theta; f) = -M \cos \theta.
\]

The individual energy is

\[
\varepsilon(\theta, p) \equiv \frac{p^2}{2} + \Phi(\theta; f) = \frac{p^2}{2} - M \cos \theta,
\]
while the total energy is
\[ E = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{p^2}{2} f(\theta, p) - \frac{1}{2} M^2. \] (99)

The linearized Vlasov equation, governing the linear dynamics of \( \delta f(\theta, p, t) \) around the stationary distribution \( f(\theta, p) \) of the form (96), is obtained by linearizing equation (95) (in our case with \( M_y(f) = 0 \) and \( M_x(f) = M \)), and it is given by
\[ \frac{\partial}{\partial t} \delta f = -p \frac{\partial}{\partial \theta} \delta f + M \sin \theta \frac{\partial}{\partial p} \delta f + \frac{p f'(\varepsilon(\theta, p))}{\gamma(\theta, p)} \frac{\partial}{\partial \theta} \Phi(\theta; \delta f), \] (100)
where \( \Phi(\theta; \delta f) \) must contain also the contribution of \( \delta f \) to a magnetization in the \( y \) direction:
\[ \Phi(\theta; \delta f) = -\int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \cos(\theta - \theta') \delta f(\theta', p'). \] (101)

Equation (100) is in the form of equation (31), with the linear operator \( D \) now taking the form
\[ (Dg)(\theta, p) = -\gamma(\theta, p)g(\theta, p). \] (102)

This operator has the property that \( D\varepsilon = 0 \) and \( D\gamma = 0 \), with \( \varepsilon \) given in equation (98) and where we again use for simplicity the notation \( \gamma(\theta, p) \) for the negative definite function \( f'(\varepsilon) \). The linearized Vlasov equation can then be written, similarly to equation (31), as
\[ \frac{\partial}{\partial t} \delta f(\theta, p, t) = -\gamma(\theta, p)(DK\delta f)(\theta, p, t), \] (103)
where for the HMF model the operator \( K \) is defined by
\[ (Kg)(\theta, p) = \frac{1}{\gamma(\theta, p)}g(\theta, p) - \int_{-\infty}^{\infty} dp' \int_{0}^{2\pi} d\theta' \cos(\theta - \theta')g(\theta', p'). \] (104)

We are now in the position to follow the general results presented in section 4. Exploiting the particularly simple expression for the interaction potential in the HMF model we have, from equation (66), that the necessary and sufficient condition for the linear stability of \( f(\theta, p) \) is
\[ -\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta, p)}((D\delta f)(\theta, p))^2 - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta (D\delta f)(\theta, p) \right)^2 \]
\[ - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \sin \theta (D\delta f)(\theta, p) \right) \geq 0. \] (105)

We note that in equation (105) the first term in the left-hand side is positive definite, while the second and third terms are negative definite.

We have to find in which case the condition in equation (105) is satisfied. We can exploit the antisymmetry of the operator \( D \), that implies that the functional subspace orthogonal to the kernel of the operator is transformed into itself. In fact, if \( g_1 \) belongs to the kernel of \( D \), and \( g_2 \) is orthogonal to \( g_1 \), then
\[ \langle g_1, Dg_2 \rangle = -\langle Dg_1, g_2 \rangle = 0. \] (106)
The kernel is made of the functions which depend on \((\theta,p)\) through \(((p^2/2) - M \cos \theta)\). We may therefore transform the problem (105) into the problem of satisfying the relation

\[
- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta,p)} (\delta f(\theta,p))^2 - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta \delta f(\theta,p) \right)^2
\]

subject to the conditions:

\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left( \frac{p^2}{2} - M \cos \theta \right)^s \delta f(\theta,p) = 0 \quad s = 0, 1, \ldots
\] (108)

However, we should take into account the case in which the stationary distribution function \(f\) has a power law decay for large \(p\). We therefore replace the previous conditions with

\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left[ \frac{p^2}{2} - M \cos \theta \right]^{s} \delta f(\theta,p) = 0 \quad s = 0, 1, \ldots
\] (109)

where \(h\) is a function that ensures integrability; it may be chosen, e.g., equal to \(\exp[-((p^2/2) - M \cos \theta)]\).

Since the function \(\gamma(\theta,p)\) is even in \(\theta\), it is useful to separate \(\delta f\) into its even and odd parts in \(\theta\), i.e., \(\delta f(\theta,p) = \delta f_e(\theta,p) + \delta f_o(\theta,p)\). In this way, our problem of satisfying the relation in equation (107) subject to the conditions given in equation (109) is separated into a pair of separate problems. To be precise, for the even part we have to satisfy

\[
- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta,p)} (\delta f_e(\theta,p))^2 - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta \delta f_e(\theta,p) \right)^2 \geq 0
\] (110)

for all \(\delta f_e(\theta,p)\) such that the relations

\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left[ \frac{p^2}{2} - M \cos \theta \right]^{s} \delta f_e(\theta,p) = 0 \quad s = 0, 1, \ldots
\] (111)

are verified. For the odd part we have to satisfy

\[
- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \frac{1}{\gamma(\theta,p)} (\delta f_o(\theta,p))^2 - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \sin \theta \delta f_o(\theta,p) \right)^2 \geq 0
\] (112)

without any condition.

At this point, we note the following things. Firstly, any \(\delta f_e\) such that \(\cos \theta \delta f_e\) has a vanishing integral will trivially give a positive value for the left-hand side of equation (110), and similarly any \(\delta f_o\) such that \(\sin \theta \delta f_o\) has a vanishing integral will trivially give a positive value for the left-hand side of equation (112). Therefore the problems can only come from functions \(\delta f_e\) and \(\delta f_o\) that do not have the aforementioned properties. Secondly, since both (110) and (112) are quadratic functions, of \(\delta f_e\) and \(\delta f_o\), respectively, the sign of the expression is not changed by the multiplication of \(\delta f_e\) or \(\delta f_o\) by any number. We can therefore study the sign of the left-hand sides of (110) and (112) also by imposing a linear condition on \(\delta f_e\) and \(\delta f_o\).
Therefore we proceed in the following way. We look for the extremum of the left-hand side of equation (110), constrained by the conditions (111), with the further convenient constraint
\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta \delta f_e(\theta, p) = 1.
\] (113)

Similarly, we look for the extremum of the left-hand side of equation (112) under the constraint
\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \sin \theta \delta f_o(\theta, p) = 1.
\] (114)

It is useful at this point to introduce the following definitions:
\[
\alpha_s^{(h)}(\theta) \equiv \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta h \left[ \frac{p^2}{2} - M \cos \theta \right] \gamma(\theta, p) \left( \frac{p^2}{2} - M \cos \theta \right)^s
\]
\[
= \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta h \left[ \frac{p^2}{2} - M \cos \theta \right] \gamma(\theta, p) (\varepsilon(\theta, p; f))^s
\] (115)
and
\[
\eta_s^{(h)}(\theta) \equiv \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta h \left[ \frac{p^2}{2} - M \cos \theta \right] \gamma(\theta, p) \cos \theta \left( \frac{p^2}{2} - M \cos \theta \right)^s
\]
\[
= \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta h \left[ \frac{p^2}{2} - M \cos \theta \right] \gamma(\theta, p) \cos \theta (\varepsilon(\theta, p; f))^s
\] (116)
where the dependence on the function \( h \) is explicitly indicated.

We begin with the problem related to \( \delta f_e \). Introducing the Lagrange multipliers \( 2\mu_s^{(h)} \) for the constraints (111) and \( 2\nu \) for the constraint (113), the conditioned extremum of the left-hand side of equation (110) is given by the equation
\[
-\frac{1}{\gamma(\theta, p)} \delta f_e(\theta, p) - (1 + \nu) \cos \theta - \sum_{s=0}^{\infty} \mu_s^{(h)} h \left[ \frac{p^2}{2} - M \cos \theta \right] \left( \frac{p^2}{2} - M \cos \theta \right)^s = 0.
\] (117)

It is clear that this extremum is a minimum, since the second variation is simply \(-1/\gamma > 0\). Therefore the necessary and sufficient condition is that the inequality (110) is satisfied for the extremal \( \delta f_e(\theta, p) \) determined by equation (117). Defining furthermore \( \xi = -(1 + \nu) \), equation (117) gives
\[
\delta f_e(\theta, p) = \xi \gamma(\theta, p) \cos \theta - \sum_{s=0}^{\infty} \mu_s^{(h)} \gamma(\theta, p) h (\varepsilon(\theta, p; f)) \left( \varepsilon(\theta, p; f) \right)^s.
\] (118)

Substituting this expression into equations (111), we obtain the system of equations
\[
\sum_{s'=0}^{\infty} \mu_{s'}^{(h)} \alpha_{s+s'}^{(h)} = \xi \eta_s^{(h)} \quad s = 0, 1, \ldots,
\] (119)
while substitution in equation (113) gives
\[
\xi \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \gamma(\theta, p) \cos^2 \theta - \sum_{s=0}^{\infty} \mu_s^{(h)} \eta_s^{(h)} = 1.
\] (120)
Stability of inhomogeneous quasi-stationary states

From the system (119), we may obtain the multipliers $\mu_s^{(h)}$ as a function of the multiplier $\xi$. We see in particular that the multipliers $\mu_s^{(h)}$ are proportional to $\xi$. We therefore introduce the ‘normalized’ multipliers $\tilde{\mu}_s^{(h)}$, given by the solution of the system of equations

$$\sum_{s=0}^{\infty} \tilde{\mu}_s^{(h)} \alpha_{s+s'} = \eta_s^{(h)}$$

We have that $\mu_s^{(h)} = \xi \tilde{\mu}_s^{(h)}$; substituting in equation (120) we obtain

$$\xi = \frac{1}{\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta,p) \cos^2 \theta - \sum_{s=0}^{\infty} \tilde{\mu}_s^{(h)} \eta_s^{(h)}}$$

The relation (110) for $\delta f_e$ equal to the extremal function given by equation (118) can now be easily obtained, taking into account equations (113), (121) and (122). Introducing the further shorthand notation

$$\sum_{s=0}^{\infty} \tilde{\mu}_s^{(h)} \eta_s^{(h)} \equiv z(\gamma)$$

(where we provide evidence for the dependence on the stationary distribution function through $\gamma$), we have

$$1 + \pi \int_{-\infty}^{\infty} dp \gamma(p) \geq 0.$$  

We have thus obtained a relation involving only the stationary distribution function. This is the main expression of this paper.

The relation valid in the case of the linear dynamical stability of homogeneous (i.e., with $M=0$) stationary distribution functions is easily obtained. In fact, in that case $\eta_s^{(h)} = 0$; therefore $\tilde{\mu}_s^{(h)} = 0$ and thus $z(\gamma) = 0$. Then, taking into account that the integral of $\cos^2 \theta$ is equal to $\pi$, equation (124) becomes in this case

$$1 + \pi \int_{-\infty}^{\infty} dp \gamma(p) \geq 0.$$  

This is identical to the expression generally found in the literature for the linear stability of homogeneous distribution functions in the HMF model (see, e.g., [20], [22]–[24], [26]); for this comparison we have to consider that for homogeneous distribution functions, $\gamma(p) = f'(p)/p$. We will show that for homogeneous distribution functions all extremal problems lead to the same result. This explains why the same expression is obtained from the ‘canonical’ formal stability problem [23] (in principle, this approach only gives a sufficient condition for linear stability, but our present results show that it is in fact sufficient and necessary).

The result just obtained for the problem associated with the even part $\delta f_e(\theta, p)$ can be immediately transformed into that for the problem associated with the odd part $\delta f_o(\theta, p)$. In this case, the only constraint is equation (114), so no multipliers $\mu_s^{(h)}$ are present. Then, we easily obtain the further condition, analogous to equation (124), that has to be
satisfied by \( \gamma \), namely,

\[
1 + \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \, \gamma(\theta, p) \sin^2 \theta \geq 0. \tag{126}
\]

For homogeneous distribution functions this relation becomes equal to equation (125). However, it can easily be shown that for inhomogeneous distribution functions of the form given in equation (96), i.e., when \( M \) is strictly positive, equation (126) is satisfied as an equality, independently of the particular form of the function and of the value of its parameters. In fact,

\[
\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \, \gamma(\theta, p) \sin^2 \theta \equiv \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \sin^2 \theta \frac{\partial F}{\partial \varepsilon} = \frac{1}{M} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \, \sin^2 \theta \frac{\partial F}{\partial \varepsilon} = -\frac{1}{M} \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta F = -1. \tag{127}
\]

This equality is clearly associated with a \( \delta f_o(\theta, p) \) that simply rotates, at first order, the distribution function (96). This shows that, for inhomogeneous distribution functions (96), any odd \( \delta f_o(\theta, p) \) will satisfy equation (112).

Summarizing the results of this section, the distribution function (96) is linearly dynamically stable iff the relation (124) is satisfied. This relation reduces to the simpler form given in equation (125) for a homogeneous (i.e., with \( M = 0 \)) distribution function.

As we have shown in the general case, the most refined formal stability criterion (83) leads to the same necessary and sufficient condition.

8. The formal stability of Vlasov stationary states of the HMF model: sufficient conditions for stability

8.1. The ‘microcanonical’ formal stability for the HMF model

The problem related to the ‘microcanonical’ formal stability is obtained by specializing to the HMF potential the expressions given in section 6.1. We thus obtain

\[
- \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \, \frac{1}{\gamma(\theta, p)} (\delta f(\theta, p))^2 - \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \cos \theta \delta f(\theta, p) \right)^2
- \left( \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \sin \theta \delta f(\theta, p) \right)^2 \geq 0 \tag{128}
\]

for all \( \delta f(\theta, p) \) such that the constraints of normalization and of total energy are satisfied at first order. Using the expression for the total energy for the HMF model, we have that the allowed \( \delta f(\theta, p) \) have to satisfy

\[
\delta E = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \left( \frac{p^2}{2} - M \cos \theta \right) \delta f(\theta, p) = 0, \tag{129}
\]

and

\[
\delta I = \int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta \, \delta f(\theta, p) = 0. \tag{130}
\]
We immediately see that this is analogous to the linear stability problem, with the difference that now we have only the constraints associated with \( s = 0 \) and \( s = 1 \), and the function \( h \) is the constant unitary function. In fact, the constraints (130) and (129) are nothing more than the constraints (111) for \( s = 0 \) and \( s = 1 \), respectively, and for the case \( h = 1 \). Therefore, the only thing we need before writing down the result for this case is to adapt the definition of the parameters \( \alpha_s \) and \( \eta_s \), and of the multipliers \( \tilde{\mu}_s \), to the present situation. We thus define

\[
\alpha_s \equiv \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \, \gamma(\theta, p) \, (\varepsilon(\theta, p; f))^s.
\] (131)

and

\[
\eta_s \equiv \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \, \gamma(\theta, p) \, \cos \theta \, (\varepsilon(\theta, p; f))^s.
\] (132)

We therefore introduce the ‘normalized’ multipliers \( \tilde{\mu}_s \), given by the solution of the system of equations

\[
\sum_{s' = 0}^{1} \tilde{\mu}_{s'} \alpha_{s+s'} = \eta_s \quad s = 0, 1.
\] (133)

The condition on \( \gamma(\theta, p) \) analogous to equation (124) can now be immediately written down. It is given by

\[
\frac{1}{w(\gamma) - \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \, \gamma(\theta, p) \cos^2 \theta} \geq 1,
\] (134)

where now the shorthand notation \( w(\gamma) \) stands for

\[
\sum_{s = 0}^{1} \tilde{\mu}_s \eta_s \equiv w(\gamma).
\] (135)

Summarizing, the distribution function (96) is formally stable with respect to the ‘microcanonical’ criterion iff the relation (134) is satisfied. This relation reduces to the simpler form given in equation (125) for a homogeneous (i.e., with \( M = 0 \)) distribution function, since in that case \( \tilde{\mu}_s = 0 \). Thus, dynamical linear stability and formal stability lead to identical conditions for stationary homogeneous distribution functions.

Although it is not evident from the two expressions (134) and (124), we know that if the former relation is satisfied, so is the latter, since we had found that the necessary and sufficient condition for the formal stability is also a sufficient condition for the linear dynamical stability.

8.2. The ‘canonical’ formal stability for the HMF model

At this point, it is straightforward to derive the relation for the less refined formal stability problems. In particular, the ‘canonical’ formal stability condition is completely analogous to the ‘microcanonical’ formal stability condition, with the difference that the allowed \( \delta f(\theta, p) \) in equation (128) have to satisfy only the normalization constraint, i.e., equation (130). This corresponds to taking only the constraint associated with \( s = 0 \). In
particular, the system (133) reduces to the single equation
\[ \tilde{\mu}_0 \alpha_0 = \eta_0. \]  
(136)

Then, the condition on \( \gamma(\theta, p) \) now becomes
\[ \frac{1}{(\eta_0^2 / \alpha_0) - \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p) \cos^2 \theta} \geq 1, \]  
(137)
or, more explicitly,
\[ \frac{1}{((\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p) \cos \theta)^2 / \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p)) - \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p) \cos^2 \theta} \geq 1. \]  
(138)

Again, for \( M = 0 \) this condition becomes identical to equation (125), since in that case \( \eta_0 = 0 \).

For consistency, it is interesting to note that the equality in equation (138) corresponds to the condition for marginal stability found by another method in appendix F of [26].

8.3. The ‘grand-canonical’ formal stability for the HMF model

Finally, the ‘grand-canonical’ formal stability condition has no constraint at all. Therefore, the condition on \( \gamma(\theta, p) \) is simply
\[ - \frac{1}{\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p) \cos^2 \theta} \geq 1. \]  
(139)

Again, for \( M = 0 \) this condition becomes identical to equation (125).

We note the following point. The problem associated with the ‘grand-canonical’ formal stability, equation (87), does not constrain the value of the normalization \( I \), that therefore can be different from 1. It is clear, however, that if we want to study the ‘grand-canonical’ formal stability of a distribution function which is an extremum also for the other stability problems, we have to restrict ourselves to just normalized distribution functions.

In the appendix, just as a useful exercise, we show that, as we should expect, the inhomogeneous Maxwell–Boltzmann distribution function is ‘canonically’ formally stable, and therefore also ‘microcanonically’ formally stable and linearly dynamically stable, but it is not ‘grand-canonically’ stable. This shows the role of the constraints and the importance of considering sufficiently refined stability criteria.

9. Discussion and conclusions

In this paper we have derived a necessary and sufficient condition for linear stability for a stationary state of the Vlasov equation. This condition is expressed by equation (124), which is the core of the paper. Less refined conditions for formal stability, which are sufficient, although not necessary, for linear dynamical stability, have also been obtained.

We should make two remarks. Firstly, it is clear that the form of the HMF interaction potential has simplified the task, and that further computations should be made for more...
complicated potentials\(^3\). Secondly, actual computations for linear dynamical stability will always require a degree of approximation, since the infinite sum implicit in the system (121) and in the definition of \(z(\gamma)\), equation (123), will have to be replaced by some finite representation. This has led us to treat also the less refined formal stability conditions. At the price of having only sufficiency, more manageable expressions are to be expected.

We would like to conclude with some comments about the relevance of the Vlasov stable stationary states from the point of view of thermodynamics.

If the system is initially in a state that is not a stationary state of the Vlasov equation, it can be argued that there will be a rather fast evolution until a stable stationary state is reached. However, some care must be exercised as regards the sense in which this statement has to be taken. Like for the Liouville theorem for the \(N\)-body distribution function of Hamiltonian systems, the time evolution of the one-body distribution function as governed by the Vlasov equation is such that its phase levels are conserved (in fact, the Vlasov equation states exactly the equality to zero of the convective derivative of \(f\)). In particular, an initial two levels \(f\), i.e.
\[ f = \text{constant in a given region of the } (\theta, p) \text{ plane and zero outside of this region}, \]
will be two levels for all the following evolution. How can we expect such a function to evolve towards a smooth stable stationary state characterized by a continuity of phase values? This can be realized only in a coarse-grained sense, when we study a sort of smeared one-body distribution function, in which the value of \(f\) at each point is substituted by the average of \(f\) taken in a small neighbourhood of the given point. If there is an efficient mixing of the dynamics, we may expect that, no matter how small the averaging neighbourhood is (provided it is not vanishing), the averaged \(f\) will evolve towards the stable stationary state of the Vlasov equation. This is exactly the framework in which the Lynden-Bell theory of violent relaxation has been proposed \[14\].

It is a typical reasoning in thermodynamics or statistical mechanics to argue that the distribution functions of a system, in particular the one-body distribution function, will evolve according to the maximization of a functional given some constraints. For example, the final Boltzmann–Gibbs state of the one-body distribution function will be given by the maximization of the Boltzmann entropy
\[
S_B[f] = -\int_{-\infty}^{\infty} dp \int_{0}^{2\pi} d\theta f(\theta,p) \ln f(\theta,p) \tag{140}
\]
subject to the constraints of normalization, equation (4), and given total energy, equation (5); the potential \(\Phi(\theta; f)\) will have to be determined self-consistently. The use of the Boltzmann–Gibbs entropy (140) is fully justified for characterizing the state reached after the ‘collisional’ regime has taken over; it will be the most mixed state given the constraints. In a collisionless regime such as the one governed by the Vlasov equation, one can make the same hypothesis about the evolution towards the most mixed state given the constraints, but one has to take into account that the mixing, even if maximally efficient, has to take place without violating the properties of the Vlasov equation, as the conservation of the phase levels of the distribution function. In this way, the Lynden-Bell expression for the entropy is obtained \[14,38\].

\(^3\) Interestingly, in stellar dynamics, using the Antonov criterion or the energy principle, it can be shown that all spherical galaxies with \(f = f(\epsilon)\) and \(f'(\epsilon) < 0\) are linearly \[36\], and even nonlinearly \[37\], stable.
The Lynden-Bell maximization problem gives a coarse-grained distribution of the form 
\( \bar{f}(\theta, p) = f(\varepsilon(\theta, p)) \) with \( \varepsilon' < 0 \), i.e. a particular steady state of the Vlasov equation. As such, it extremizes a functional of the form

\[
S[\bar{f}] = - \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta C(\bar{f}(\theta, p)),
\]

at fixed normalization and energy. It can be shown that if this distribution function is a maximum of \( S \) at fixed \( I \) and \( E \), then it is Lynden-Bell thermodynamically stable (see [33,39] for the 2D Euler equation and section V of [38] for the Vlasov equation). According to the present study, if it is a maximum of \( S \) at fixed \( I \) and \( E \), it is also guaranteed to be linearly dynamically Vlasov stable. More generally, it can be shown that the coarse-grained distribution function associated with a Lynden-Bell thermodynamical equilibrium is always dynamically stable (because it is a minimum of energy with respect to phase-preserving perturbations), even if it is not a maximum of \( S \) at fixed \( E \) and \( I \) (see section 7.8 of [33] for the 2D Euler equation).

The application of the Lynden-Bell theory to the HMF model has been studied in several works that have considered the thermodynamically stable state of the coarse-grained distribution \( \bar{f} \) predicted in the case in which this function assumes initially only two values, i.e., a constant value in a given region and zero outside (see, e.g., [40]–[42]). The numerical velocity distribution at the end of the violent relaxation, obtained with numerical simulations, is in qualitative agreement with the prediction [41].

On the other hand, since it is not guaranteed that the mixing is always completely efficient (this is referred to as ‘incomplete relaxation’), one may argue that in cases when it is not efficient the dynamics will evolve, trying to maximize, always in the coarse-grained sense, other functionals of the form (141) that are not consistent with Lynden-Bell’s theory (see [43] and section XII of [38]). This is an essentially phenomenological approach. For example, the Tsallis functional is one particular case of such functionals (that are called generalized \( H \)-functions [43]). The constraints of normalization and total energy are always present, and it is immediately seen that, if the functional to maximize is of the form (141), then the solution will always be a function of the form \( f(\theta, p) = f((p^2/2) + \Phi(\theta; f)) \). The problem at hand in this case is to show that a function of this form obtained by extremizing (141) at fixed \( I \) and \( E \) is really a maximum, i.e., the value of (141) decreases if we perturb \( f \) without changing the values of the constraints. This ‘microcanonical’ stability problem has been studied for the Tsallis distributions in [27].

We are thus led to the conclusion that functions of the form \( f(\theta, p) = f((p^2/2) + \Phi(\theta; f)) \) are relevant also in a thermodynamical sense (with respect to the collisionless dynamics). However, we have shown that, while for homogeneous states formal ‘microcanonical’ stability implies linear dynamical stability and vice versa, for inhomogeneous states only the first implication is true. Thus, there might be inhomogeneous states which are formally ‘microcanonically’ unstable (and therefore not relevant from a thermodynamical point of view) that nevertheless are dynamically linearly stable.
Appendix: The stability of the Maxwell–Boltzmann distribution function in the HMF model

As an exercise we can apply the results of section 8 to the magnetized Maxwell–Boltzmann distribution function, that realizes the Boltzmann–Gibbs global equilibrium. For this case all the quantities can be obtained analytically, basically because in this case $\gamma = -\beta f$. We can consider from the beginning the ‘canonical’ formal stability problem, given by equation (137). It is immediately obtained in this case that $a_0 = -\beta$ and $\eta_0 = -\beta M$. We also have

$$\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\theta \gamma(\theta, p) \cos^2 \theta = -\beta (M^2 + \Delta M^2),$$

where we have denoted with $\Delta M^2$ the variance of the magnetization, i.e., the expectation value $\langle (\cos \theta - M)^2 \rangle$. We then find that equation (137) reduces to

$$\beta \Delta M^2 \leq 1.$$  \hspace{1cm} (A.2)

We first note that, in the homogeneous case, the last expression reduces to the known relation $\frac{1}{2} \beta \leq 1$, i.e., $\beta \leq 2$. However, it is not difficult to show that, when $\beta > 2$ and $M > 0$, equation (A.2) is always satisfied, on the basis of the graphical construction that gives $\beta$ as a function of $M$, based on the relation

$$M = \frac{I_1(\beta M)}{I_0(\beta M)},$$

where $I_1$ and $I_0$ are the modified Bessel functions of order 1 and 0, respectively. We then have

$$\beta \Delta M^2 = \beta \frac{\partial}{\partial (\beta M)} I_1(\beta M) \frac{I_1(\beta M)}{I_0(\beta M)} = \frac{\partial}{\partial M} I_1(\beta M) I_0(\beta M) < 1,$$

where the last inequality is a consequence of the graphical solution of equation (A.3). Then, we have proven the ‘canonical’ formal stability, and hence the ‘microcanonical’ formal stability and the linear stability. In contrast, it can be seen that the ‘grand-canonical’ formal stability (139) does not hold for magnetized states. In fact, from equation (A.1), we find that this stability would require

$$\beta M^2 + \beta \Delta M^2 \leq 1,$$

that in turn, using equations (A.3) and (A.4), becomes

$$\beta \left[ \left( \frac{I_1(\beta M)}{I_0(\beta M)} \right)^2 + \frac{\partial}{\partial (\beta M)} I_1(\beta M) I_0(\beta M) \right] \leq 1.$$ \hspace{1cm} (A.6)

However, using the results in appendix B of [44], it is proven that the left-hand side of the last expression is for $M > 0$ simply equal to $\beta - 1$. Since Boltzmann–Gibbs magnetized states are realized for $\beta > 2$, the inequality cannot be satisfied. If we consider a nonnormalized Maxwell–Boltzmann distribution function, and we denote with $A$ its ‘mass’, then the left-hand sides of the last two relations are multiplied by $A$. The stability condition then becomes a relation between $A$ and $\beta$:

$$A (\beta - 1) \leq 1.$$ \hspace{1cm} (A.7)

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Finally, the reader might want to check that the ‘microcanonical’ formal stability condition (134), that must be satisfied since the ‘canonical’ one is satisfied, reduces to the expression

\[ 1 - \beta \Delta M^2 + 2 \beta^2 M^2 \Delta M^2 \geq 0. \tag{A.8} \]

Since equation (A.2) is verified, then equation (A.8) is \textit{a fortiori} verified, as it should be.

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