Robust Online Algorithms for Dynamic Problems

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Abstract

Online algorithms that allow a small amount of migration or recourse have been intensively studied in the last years. They are essential in the design of competitive algorithms for dynamic problems, where objects can also depart from the instance. In this work, we give a general framework to obtain so called robust online algorithms for these dynamic problems: these online algorithms achieve an asymptotic competitive ratio of $\gamma + \epsilon$ with migration $O(1/\epsilon)$, where $\gamma$ is the best known offline asymptotic approximation ratio. In order to use our framework, one only needs to construct a suitable online algorithm for the static online case, where items never depart. To show the usefulness of our approach, we improve upon the best known robust algorithms for the dynamic versions of generalizations of Strip Packing and Bin Packing, including the first robust algorithm for general $d$-dimensional Bin Packing and Vector Packing.

2012 ACM Subject Classification Theory of computation → Online algorithms

Keywords and phrases online algorithms, dynamic algorithms, competitive ratio, recourse, packing problems

Funding Sebastian Berndt: Supported by DFG Project, "Robuste Online-Algorithmen für Scheduling- und Packungsprobleme", JA 612 /19-1  
Kilian Grage: Supported by by GIF-Project "Polynomial Migration for Online Scheduling".

1 Introduction

Online algorithms are a very natural way to deal with uncertain inputs and the need for a sequence of good solutions throughout the evolution of an instance. For a surprisingly large number of problems, one can obtain online algorithms that produce solutions within a constant ratio of the optimal solutions. The worst-case ratio between an optimal solution and a solution produced by the online algorithm is called the competitiveness of the algorithm. For most algorithms with constant competitiveness, the algorithms rely heavily on the fact that the instances involve in a monotone way: every object that becomes part of the instance will stay part of the instance forever. This monotonicity property is often not present in real-world applications, where the objects might be removed from the instance later on. These
objects might be used at a different place, they might expire, or they are no longer relevant. Hence, in order to still give a performance guarantee, the online algorithms need to be able to modify parts of an existing solutions. Clearly, such a modification might be costly and should thus be minimized. If no such boundary on the modification is given, one could easily use an offline approximation algorithm. A natural way to measure the amount of modification needed is the migration factor: it compares the total size of modified objects with the size of the newly inserted or removed object. An algorithm with a bounded migration factor roughly translates to the fact that the insertion or departure of a small object can only lead to small changes in the structure of the current solution. On the other hand, if a large (and thus impactful) object is inserted or removed, we are allowed to modify a larger part of the solution.

It is easy to see that online algorithms can in general not achieve the same solution quality as offline algorithms. The question how much these two values differ is one of the central questions in the field of online and approximation algorithms. Clearly, an online algorithm that is allowed a certain amount of migration is able to gap between these extremes. If the amount of migration needed directly corresponds to the improvement of the solution guarantee, we call such an algorithm robust. As an example, consider the well-studied Bin Packing problem. No online algorithm (without migration) can achieve a competitiveness smaller than \(\frac{3}{2}\) \[25\], while an offline algorithm with asymptotic approximation ratio of \(1 + \epsilon\) can be obtained in polynomial time \[20\]. A robust online algorithm would then have competitiveness \(1 + \epsilon\) and migration factor \(f(1/\epsilon)\) for some function. In other words, we only need to increase the migration if we want to obtain a better solution. Such robust online algorithms thus have a continuous behavior between the performance of the best offline algorithm (\(\epsilon \rightarrow 0\)) and the performance of the best online algorithm (\(\epsilon \rightarrow \infty\)).

In this paper, we describe a general framework to construct robust online algorithms for dynamic online problems. The only ingredients needed for our framework are (i) a good offline approximation algorithm and a (ii) suitably designed online algorithm for the static case. Our framework is applicable to many problems, especially in the field of geometric packing problems. To show the versatility of our approach, we use it to improve upon existing robust algorithms and to construct new robust algorithms.

### 1.1 Related Works

The migration factor model was introduced by Sanders, Sivadasan and Skutella \[22\]. They studied the classical problem of minimizing the makespan on parallel identical machines and made use of sensitivity results of integer programming to obtain a robust \(1 + \epsilon\)-competitive algorithm. This work spanned a lot of follow up works for a wide range of different problems:

- Skutella and Verschae \[23\] also studied the same problem and were able to obtain a robust \(1 + \epsilon\)-competitive algorithm for the dynamic case, where jobs may depart. They also studied the dual version of makespan minimization problem, where the minimum load should be maximized. This problem (called Machine Covering or Santa Claus) also admits a robust \(1 + \epsilon\)-competitive algorithm even in the dynamic case. Skutella and Verschae also proved that for all \(\epsilon > 0\) there is no online algorithm for the Machine Covering problem that achieves competitive ratio \(20/19 - \epsilon\) with worst-case migration \(f(1/\epsilon)\) for any function \(f\). Gálvez et al. \[15\] gave two simple competitive robust algorithms (one that is \(1.7 + \epsilon\)-competitive and one that is \(4/3 + \epsilon\)-competitive) with polynomial migration factor for the static version of the problem of makespan minimization.

- For the Bin Packing problem, Epstein and Levin \[11\] gave the first robust \(1 + \epsilon\)-competitive
algorithm for Bin Packing based on the same sensitivity results for integer programming. Jansen and Klein [17] designed new techniques in order to obtain a migration factor polynomial in $1/\epsilon$. These techniques were improved by Berndt et al. [5] to also handle the dynamic version of the Bin Packing problem where items can also depart. In all of these works, the migration factor is worst-case and can thus not be saved up for later use. In contrast, Feldkord et al. [14] presented a simple robust $1 + \epsilon$-competitive algorithm with amortized migration factor that also works for the dynamic case. Berndt et al. also showed that a worst-case migration factor of $\Omega(1/\epsilon)$ is needed [5]. This lower bound was shown to also hold for amortized migration by Feldkord et al. [14]. Feldkord et al. also studied a problem variant, where the migration needed for an item does not correspond to its size.

For the makespan problem, where jobs can be scheduled preemptively (i.e., they can be split, but parts are not allowed to run simultaneously), Epstein and Levin [13] designed an optimal online algorithm with migration factor $1 - 1/m$, where $m$ is the number of machines. Again, the migration factor used here is worst-case. They also showed that exact algorithms for the makespan minimization problem on uniform machines and for identical machines in the restricted assignment have worst-case migration factor at least $\Omega(m)$.

The offline variants of the geometric packing problems have also been studied intensively. See e.g. the survey of Christensen et al. [8] and the references therein.

1.2 Our Results

We present a very general framework that allows to construct robust online algorithms that have optimal amortized migration factor of $O(1/\epsilon)$ and achieve the same competitive ratio as the best known offline approximation algorithm (up to an additive error of $\epsilon$). This framework is suitable for many different optimization problems, especially geometric packing problems. The algorithms created by our framework can all deal with the dynamic versions of these problems, where items can also depart from the instance. To present a general framework, we introduce the notion of flexible online algorithms and show that such algorithms can be combined with offline approximation algorithms under certain circumstances. We then show the versatility of our approach by looking at several well-studied problems including generalizations of Strip Packing and Bin Packing. We give robust algorithms for the multi-dimensional variants of these problems, where we can also handle both the departure and the rotation of items. This improves and generalizes several known results and gives the first robust algorithms for all of the other problems. Additionally, our compact and clean framework gives much easier algorithms compared with the previously known algorithms.

2 Online Algorithms for Dynamic Problems: A Framework

2.1 Preliminaries

While the techniques presented in this paper also work for maximization problems, we will focus on minimization problems to improve the accessibility of our results.

Definition 1. Let $\text{ITEMS} \subseteq \{0,1\}^*$ be a prefix-free set describing some items. A minimization problem $\Pi = (\mathcal{I}, \text{sol}, \text{costs})$ consist of a set of instances $\mathcal{I} \subseteq \text{ITEMS}$, a mapping $\text{sol}$ that maps an instance $I \in \mathcal{I}$ to a non-empty set of feasible solution $\text{sol}(I)$, and a mapping $\text{costs}$ that maps a solution $S \in \text{sol}(I)$ to its costs $\text{costs}(S) \in \mathbb{Q}_{\geq 0}$. A solution
For the sake of simplicity, we also sometimes treat Π and I interchangeably and also write I ∈ Π to denote that I is an instance.

Throughout this paper, we make the following two natural assumptions that hold for all problems considered in this work:

**Assumption 1.**
1. For every instance I, every instance I′ ⊆ I, and every solution S ∈ SOL(I), the solution S↾I, induced by the items in I′, is also feasible, i.e., S↾I ∈ SOL(I′). Hence, removing some items from a feasible solution results in a feasible solution of the remaining items.
2. For every instance I, and every instance I′ ⊆ I, we have OPT(I′) ≤ OPT(I). Hence, removing items from an instance can only decrease the optimum.

In the classical offline version of Π = (I, SOL, COSTS), we are given the complete instance I ∈ I all in one. An α-approximation for Π is a polynomial-time algorithm A such that for all instances I ∈ I, we have A(I) ≤ α OPT(I) + c for some constant c not depending on the instance. Here, A(I) is the value of the solution produced by A.

In the online version of a minimization problem Π = (I, SOL, COSTS), an online instance I is a sequence of instances I1, I2, . . . , I|I| ∈ I such that |It \ It+1| = 1 for all t = 1, . . . , |I| − 1, where \( \triangle \) is the symmetric difference of two sets. This means that we insert a new item \( i^* \) (if \( I_{t+1} \setminus I_t = \{i^*\} \)) or an item \( i^* \) departs (if \( I_t \setminus I_{t+1} = \{i^*\} \)). The set of online instances of Π is denoted as \( I^{\text{on}} \). An online algorithm A maintains a sequence of solutions \( S_1, \ldots, S_t \), where \( S_t ∈ SOL(I_t) \) and furthermore, \( S_{t+1}(i) = S_t(i) \) for all \( i ∈ I_{t+1} \cap I_t \). In the static online version, we only have insertions and thus \( I_t ⊆ I_{t+1} \). A non-static problem is called dynamic. We say that an online algorithm A producing such a sequence of solutions \( S_1, S_2, \ldots \) is \( \beta \)-competitive, if

\[
\text{COSTS}(S_t) \leq \beta \cdot \text{OPT}(I_t) + c
\]

for all \( t = 1, \ldots, |I| \) and some constant c not depending on the instance. This notion of competitiveness is sometimes also called asymptotic competitiveness in contrast to the notion of absolute competitiveness, where no additional additive term c is allowed. Whenever we talk about competitiveness, this is with regard to the notion of asymptotic competitiveness.

**Migration**

Achieving bounded competitiveness is usually impossible for dynamic problems, even for very simple problems such as Bin Packing where every item has the same size. This is simply due to the fact that an online algorithm is not allowed to change the placement of already placed items. This very strict restriction thus comes with a high cost with regard to the competitiveness. As many real-world applications are not only static, but allow the departure of items, one must thus make more flexible. We will thus allow a small amount of repacking to be able to handle dynamic problems. The model of repacking we use is called the amortized migration factor model. In this setting, every item \( i ∈ \text{ITEMS} \) comes with a size \( v_i ∈ \mathbb{Q}_{≥ 0} \). Usually this size will be the space needed for an item, like its total area or volume or some identical criteria like the side length of a hypercube. For a set of items \( I ⊆ \text{ITEMS} \), we denote by vol(I) = \( \sum_{i ∈ I} v_i \) the complete volume of I. If I ∈ \( I^{\text{on}} \) is an online instance and \( \vec{S} = (S_1, \ldots, S_{|I|}) \) is a sequence of solutions with \( S_j ∈ SOL(I_j) \), the migrated items \( M_t(\vec{S}) \) at time t are defined as \( M_t(\vec{S}) = \{i ∈ I_{t-1} \cap I_t \mid S_{t-1}(i) ≠ S_t(i)\} \). The total migration \( μ(\vec{S}, t) \)
used until time $t$ is defined as $\mu(\vec{S}, t) := \sum_{j=1}^{t} \sum_{i \in M_j(\vec{S})} v_i$, i.e. as the sum of the sizes of the migrated items. Inserting an item $i$ builds up a migration potential of $v_i$, i.e. the sum of the sizes of the migrated items. More formally, for $t = 1, \ldots, |I|$, let $A_t = \bigcup_{j \in \{0, 1, \ldots, t-1\}} (I_{j+1} \setminus I_j)$ be the set of items that were inserted until time $t$ and $D_t = \bigcup_{j \in \{0, 1, \ldots, t-1\}} (I_j \setminus I_{j+1})$ be the set of items that departed until time $t$. Let $A$ be an online algorithm that is allowed to migrate items. We say that $A$ has migration factor $\beta$, if

$$\mu(\vec{S}, t) \leq \beta[\text{VOL}(A_t) + \text{VOL}(D_t)].$$

for all $t = 1, \ldots, |I|$ and all $I \in \mathcal{T}^\infty$. Here $\vec{S}$ is the sequence of solutions produced by $A$. Note that our definition of migration is amortized, i.e. we can build a potential to use later on. This will essentially allow us to repack the complete instance from time to time. In contrast, in the notion of worst-case migration, the total size of all repacked items is at most $\beta \cdot v_i$ at each time $t$, i.e. one cannot save up migration for later use.

If $A$ is $(\gamma + \epsilon)$-competitive for some constant $\gamma$ and all $\epsilon > 0$, and additionally has migration factor bounded by $f(1/\epsilon)$ for some function $f$, we say that $A$ is robust. This is due to the fact that the migration needed only depends on the desired quality of the solution.

### 2.2 The General Framework

Before we start looking at the specific problems, we will introduce our very general framework. The simple algorithm presented by Feldkord et al. [14] can be seen as a special case of our framework. By using the concept of amortized migration, we can save up repacking potential, to be used at a later time. Now the design of an algorithm boils down to three basic questions. How do we pack arriving items? How do we repack arriving items? And at what time do we repack items? The main idea behind our framework for packing problems is to use general algorithms for the first two problems: an online algorithm to pack arriving items and an offline algorithm for the repacking. The third problem is then solved by a generic combination of the two algorithms. In order for this approach to work, we need different criteria for both algorithms and the packing problem that we are trying to solve. A simpler version of this framework was also used Feldkord et al. [14] for the Bin Packing problem.

▶ **Definition 2.** Let $\Pi$ be an minimization problem with sizes $v_i$. We call $\Pi$ space related, if $\text{OPT}(I) \geq \text{VOL}(I)$ instances $I \in \Pi$. Here $\text{VOL}(I) := \sum_{i \in I} v_i$ is the total size of $I$.

Intuitively, this definition captures the fact that $\Pi$ is a packing problem that needs to pack items of a certain volume in some non-overlapping way. All problems in this paper will be space related. To make use of this relation, we also need online algorithms such that competitiveness not only holds for the optimum $\text{OPT}(I)$, but also for the volume $\text{VOL}(I)$. We therefore also formally introduce this necessity.

▶ **Definition 3.** Let $\Pi$ be an online minimization problem with sizes $v_i$ and $A$ be an online algorithm for $\Pi$. We say that $A$ is space related with ratio $\beta$, if

$$\text{costs}(S_t) \leq \beta \cdot \text{VOL}(I_t) + c$$

for all online instances $I \in \mathcal{T}^\infty$ and all time points $t = 1, 2, \ldots, |I|$. Here, $S_t$ denotes the solution produced by $A$ at time $t$.

Trivially, a space related algorithm with ratio $\beta$ for a space related problem implies $\beta$-competitiveness, like Next-Fit for the Bin Packing problem. As indicated above, we will
combine such an online algorithm with another offline algorithm. To be able to efficiently combine these two algorithms the online algorithms need to be able to build flexibly on top of the solution of the offline algorithm.

More formally, we require that our online algorithm takes another optional argument \( S \in \text{SOL}(I_t) \) describing an existing solution to the previous instance \( I_t \) to build upon.

**Definition 4.** Let \( \Pi \) be an online minimization problem with sizes \( v_i \), and \( A \) be a space related online algorithm for \( \Pi \) with ratio \( \beta \). Furthermore let \( I \in T^\text{on} \) be a static online instance (i.e. it contains no departures), \( t < t' \leq |I| \) two time points, and \( S \) be a solution of \( I_t \). We say that \( A \) is flexible, if it also accepts \( S \) as another parameter and produces upon input \( I_{t'} \) and \( S \) a solution \( S' \) with \( S'(i) = S(i) \) for all \( i \in I_t \cap I_{t'} \).

Note that we define flexibility only with regard to static online instances where no items depart. One advantage of our framework is that we only need to design such online algorithms, but our combined algorithm will also be able to deal with departures. Remember that our online algorithm \( A \) is given some solution \( S \) to the previous items \( I_t \). In order to be able to ignore the departure of items in the combined approach, we need to guarantee that \( A \) only introduces an error of \( \beta [\text{vol}(I_{t'}) - \text{vol}(I_t)] \) when it packs instance \( I_{t'} \).

**Definition 5.** We say that \( A \) is flexible with ratio \( \beta \), if \( A \) is flexible and

\[
A(I_{t'}, S) \leq \text{costs}(S) + \beta [\text{vol}(I_{t'}) - \text{vol}(I_t)] + c
\]

for some constant \( c \) for all instances \( I \in T^\text{on} \), all \( t \in \{1, \ldots, |I|\} \), and all \( t' \in \{t+1, \ldots, |I|\} \).

For the problems we will address later, Bin Packing and Strip Packing, we can simply solve the instance containing the newly arriving jobs separately and pack the new partial solution on top of the old one. Therefore this will be a property easily fulfilled by all online algorithms that we consider later. Note that a flexible algorithm with ratio \( \beta \) is also space related with ratio \( \beta \), as we can simply choose \( t = 1 \) and a trivial solution \( S \) for this instance with a single item.

### Combining the Algorithms

In order to obtain a robust PTAS or a robust online algorithm that is \( \gamma + \epsilon \)-competitive, we will need a flexible online algorithm with a constant ratio. For the offline algorithm we will need an offline \( \gamma + \epsilon \)-approximation (in the case of \( \gamma = 1 \), this is simply a PTAS). Basically, our final algorithm will have the same ratios as the offline algorithm and migration factor \( O(1/\epsilon) \). In order to achieve a bounded competitiveness, we need to migrate items at some special time points. These time points will be determined by the total volume of items that arrived or departed since the last such time point. At these special time points, we will use the offline algorithm to rebuild the solution completely. In between these points, we will only apply the online algorithm for the static case. We will therefore define phases such that during a phase we only apply the online algorithm.

**Definition 6.** Let \( \Pi \) be an online minimization packing problem with sizes \( v_i \), and \( I \in T^\text{on} \) be an online instance.

We partition \( I_1, I_2, \ldots \) into phases as follows: The start time of the first phase is 1. If \( \tau \) is the start time of the current phase, and \( t \geq \tau \) is some time point, we define the following values: (i) the complete volume of the instance at time \( \tau \) is denoted by \( V_\tau = \text{vol}(I_\tau) \), (ii) the items inserted since \( \tau \) are defined as \( \text{Ins}_\tau = \bigcup_{i=\tau}^{t-1} (I_i \setminus I_\tau) \) and its volume is \( A_t = \text{vol}(\text{Ins}_\tau) \), and (iii) the items departed since \( \tau \) are defined as \( \text{Dep}_t = \bigcup_{i=\tau}^{t-1} (I_i \setminus I_{i+1}) \) and its total volume
Theorem 7. Migration factor: Let $(\Pi, \alpha, \beta)\text{-approximation algorithm for the offline version of } \Pi$ for some constant $\gamma \in O(1)$, let $1/2 \geq \epsilon > 0$, and let $\mathcal{A}_{\text{off}}$ be a flexible algorithm with ratio $\beta$. Then the combination of these two algorithms, denoted with $\text{ALG}(\mathcal{A}_{\text{on}}, \mathcal{A}_{\text{off}})$, is an $(\gamma + O(1)\beta\epsilon)$-competitive robust algorithm for $\Pi$ with amortized migration factor $O(\frac{1}{\epsilon})$.

The running time of $\text{ALG}$ at time point $t$ is at most $T_{\text{off}}(t) + T_{\text{on}}(t)$, where $T_{\text{off}}(t)$ (resp. $T_{\text{on}}(t)$) is the worst-case running time of $\mathcal{A}_{\text{off}}$ (resp. $\mathcal{A}_{\text{on}}$) on an instance of $t$ items.

Proof. Migration factor: Let $I_\tau$ be the instance at the start of a phase with volume $V_\tau = \text{Vol}(I_\tau)$ and let $\tau'$ be the ending time of this phase. As the phase ends at time $\tau'$, we have $A_{\tau'} + R_{\tau'} > \epsilon V_\tau$ (where $A_{\tau'}$ and $R_{\tau'}$ are defined as in Definition 6). As we only ever migrate items at the end of a phase, we only need to consider the amortized migration factor at these time points. We assign the volume of all items inserted and departed during the phase that ends at time $\tau'$ to this phase. Hence, a total migration potential of $A_{\tau'} + R_{\tau'}$ was build up in this phase. The total volume of the instance at this point is at most $V_\tau + A_{\tau'}$ and our amortized migration factor is thus

$$\frac{V_\tau + A_{\tau'}}{A_{\tau'} + R_{\tau'}} \leq \frac{V_\tau}{A_{\tau'} + R_{\tau'}} + 1 < \frac{V_\tau}{\epsilon V_\tau} + 1 = O(1/\epsilon).$$

As the phases are disjoint, the total amortized migration factor is thus also bounded by $O(1/\epsilon)$.
Competitiveness: To show the competitiveness of ALG, assume that

\[ A_{\text{off}}(I_t) \leq (\gamma + \epsilon) \cdot \text{OPT}(I_t) + c_{\text{off}} \quad \text{(*off)} \]

for some constant \( c_{\text{off}} \) and furthermore

\[ A_{\text{on}}(I_{\tau}, S) \leq \text{COSTS}(S) + \beta[\text{VOL}(I_{\tau}) - \text{VOL}(I_t)] + c_{\text{on}} \quad \text{(*on)} \]

for some constant \( c_{\text{on}} \).

Let \( I \in I_{\text{on}} \) be some online instance and \( t \in \{1, \ldots, |I|\} \). We distinguish whether \( t \) is the end of a phase or in the middle of a phase. If \( t = \tau \) is the end of a phase, we use the offline algorithm \( A_{\text{off}} \) and thus have

\[ \text{ALG}(I_t) = A_{\text{off}}(I_t) \leq (\gamma + \epsilon) \cdot \text{OPT}(I_t) + c_{\text{off}}, \]

where the inequality follows from Equation (*off). Now consider any point of time \( t \) during the phase starting at \( \tau \). Like above, let \( A_t, R_t \) denote the total volumes of arrived and removed items in this phase up time \( t \). Note that this time \( A_t + R_t \leq \epsilon V_\tau \), since otherwise we would repack.

\[ \triangleright \text{Claim 8}. \quad \text{We have } \text{OPT}(I_\tau) \leq \text{OPT}(I_t) + \beta \epsilon V_\tau + c_{\text{on}}. \]

**Proof.** By [Assumption 1] the value \( \text{OPT}(I_t) \) is minimal if only departures happened. We can thus assume w.l.o.g. that up till time \( t \) some items were removed and no new items arrived. Consider an optimal solution \( S_t \) for the instance \( I_t \), where items departed and let \( D_t = I_\tau \setminus I_t \) be the set of departed items. Let \( d_1, \ldots, d_k \) be some total ordering of \( D_t \). We now construct a new online instance \( I' \) that somehow reverses the removal of \( D_t \). The instance \( I' \) is of length \( t + k \) with \( I'_i = I_i \) for \( i \leq t \). For \( i > t \), we define \( I'_i = I_{i-1} \cup \{d_i\} \). We now use the online algorithm \( A_{\text{on}} \) on this instance \( I' \) with solution \( S_t \) for instance \( I_t \). At the end of instance \( I' \), Equation (*on) implies that \( A_{\text{on}} \) gives a feasible solution to \( I'_{t+k} = I_t \cup D_t = I_\tau \) with

\[ A_{\text{on}}(I'_{t+k}, S_t) \leq \text{COSTS}(S_t) + \beta[\text{VOL}(I_{t+k}) - \text{VOL}(I_t)] + c_{\text{on}} = \]

\[ \text{OPT}(I_t) + \beta[\text{VOL}(I_\tau) - \text{VOL}(I_t)] + c_{\text{on}} = \]

\[ \text{OPT}(I_t) + \beta \epsilon V_\tau + c_{\text{on}} \leq \]

\[ \text{OPT}(I_t) + \beta \epsilon V_\tau + c_{\text{on}} \]

The last inequality follows from the fact that \( R_t = \text{VOL}(D_t) \) by definition and \( R_t \leq \epsilon V_\tau \) by assumption. As \( A_{\text{on}} \) produces a feasible solution for \( I'_{t+k} = I_\tau \), we clearly have \( \text{OPT}(I_\tau) \leq A_{\text{on}}(I'_{t+k}, S_t) \) thus proving the claim. \( \blacktriangle \)

Since the problem II is space related, we can conclude that

\[ \text{OPT}(I_t) \geq \text{VOL}(I_t) = V_\tau + A_t - R_t \geq V_\tau - R_t \geq V_\tau - \epsilon V_\tau \geq \frac{V_\tau}{2}, \quad (*) \]

Hence \( V_\tau \leq 2 \text{OPT}(I_t) \).

Let \( S_\tau \) be the solution produced by \( A_{\text{off}} \) at time \( \tau \) that the online algorithm \( A_{\text{on}} \) is building upon. Note that we do only remove the departed item at a time point where the offline algorithm is used. Hence, at time \( t \), the online algorithm does not produce a solution for the instance \( I_t \), where the departed items are already removed. The algorithm rather
works on the instance that still contains all items that have departed since time \( \tau \). This instance is denoted by \( I'_r \). We thus have

\[
\text{ALG}(I_r) = A_{\text{on}}(I'_r, S_r) \leq \quad \text{// Equation (1)}
\]

\[
\text{COSTS}(S_r) + \beta[\text{vol}(I'_r) - \text{vol}(I_r)] + c_{\text{on}} = \quad \text{// S_r produced by } A_{\text{off}}
\]

\[
A_{\text{off}}(I_r) + \beta[\text{vol}(I'_r) - \text{vol}(I_r)] + c_{\text{on}} \leq \quad \text{// Equation (\ref{eq:claim8})}
\]

\[
(\gamma + \epsilon) \cdot \text{OPT}(I_r) + c_{\text{off}} + \beta[\text{vol}(I'_r) - \text{vol}(I_r)] + c_{\text{on}} \leq \quad \text{// Claim 8}
\]

\[
(\gamma + \epsilon) \cdot [\text{OPT}(I_r) + \beta\epsilon V_r + c_{\text{on}}] + c_{\text{off}} + \beta[\text{vol}(I'_r) - \text{vol}(I_r)] + c_{\text{on}} = \quad \text{\hspace{1cm}}
\]

\[
(\gamma + \epsilon) \cdot \text{OPT}(I_r) + \beta\epsilon V_r + c_{\text{on}} \quad + c_{\text{off}} + \beta A_t + c_{\text{on}} \leq \quad \text{// A_t + R_t \leq \epsilon V_r}
\]

\[
(\gamma + \epsilon) \cdot \text{OPT}(I_r) + (\gamma + 1)\beta\epsilon V_r + (\gamma + \epsilon + 1)c_{\text{on}} + c_{\text{off}} \leq \quad \text{// Equation (\ref{eq:claim8})}
\]

\[
(\gamma + \epsilon) \cdot \text{OPT}(I_r) + (\gamma + 1)\beta\epsilon \text{OPT}(I_r) + (\gamma + 1)c_{\text{on}} + c_{\text{off}} = \quad \text{\hspace{1cm}}
\]

\[
(\gamma + \epsilon + 2(\gamma + 1)\beta\epsilon) \text{OPT}(I_t) + (\gamma + 1)c_{\text{on}} + c_{\text{off}}.
\]

As \( \gamma \), \( c_{\text{on}} \), and \( c_{\text{off}} \) are constants, the last term can be written as

\[
(\gamma + \epsilon + 2(\gamma + 1)\beta\epsilon) \text{OPT}(I_t) + (\gamma + 1)c_{\text{on}} + c_{\text{off}} \leq \quad \text{\hspace{1cm}}
\]

\[
(\gamma + O(1) \cdot \beta\epsilon) \text{OPT}(I_t) + O(1).
\]

The running time bound follows easily from the fact that we essentially only use \( A_{\text{on}} \) or \( A_{\text{off}} \) at any given time.

\section{2-Dimensional Strip Packing}

In the online Strip Packing problem, we are given a two-dimensional strip of width 1 and infinite height. At time \( t \), either a rectangle \( r_t \) with width \( w(r_t) \leq 1 \) and height \( h(r_t) \leq 1 \) is inserted and needs to be packed into this strip or a rectangle \( r_t \) is removed from the strip. A packing is valid if no two rectangles intersect. The size \( v(r) \) of a rectangle \( r \) is defined as \( v(r) = h(r) \cdot w(r) \). We first focus on the case that rectangles are not allowed to be rotated. A simple adaption of our algorithm also handles the case, where rotations by 90 degree are allowed. In both cases, the goal is to minimize the height of the produced packing. This problem has been studied intensively in the online setting (see for example the works cited in \cite{11}). Jansen et al. \cite{13} studied the static case in the migration scenario, where rectangles can only arrive.

To use our framework, we need the following ingredients:

i) We need to show that the Strip Packing problem is space related;

ii) we need to construct a flexible online algorithm with ratio \( \beta \);

iii) We need to construct an offline approximation algorithm.

Concerning the first point, it is well-known that \( \text{OPT}(I_t) \geq \text{vol}(I_t) \), as the rectangles are not allowed to intersect and the width of the strip is exactly 1.

\begin{remark}
The Strip Packing problem is space related.
\end{remark}

We will now present a flexible online algorithm with ratio \( \beta = 4 \). This algorithm is a simple adaption of the shelf algorithms presented by Baker and Schwarz \cite{1}. In the notion of Csisrik and Woeginger \cite{10}, this algorithm would be denoted as SHELF(FirstFit, 1/2). For the sake of completeness, we give a self-contained description and analysis. We first define several types of containers. A container \( c \) of type \( \gamma_0 \) has width 1 and height \( h(c) = 1 \). For
Lemma 10. Let $t$ be any time point, $i \in \mathbb{Z}_{\geq 0}$, and $c_1, \ldots, c_k$ be the containers of type $\gamma_i$ at time $t$. We have

$$
\sum_{j=1}^{k} \sum_{r \in c_j} v(r) \geq \frac{1}{4} \left( \sum_{j=1}^{k} h(c_j) \right) - 2^{-i+1}.
$$

Proof. If $k = 1$, only one container of type $\gamma_i$ exists. As $h(c_j) \leq 2^{-i+1}$, the inequality holds. The left hand side is strictly larger than 0 and the right hand side is strictly smaller than 0. If $k > 1$, we can argue about the contained volume. If $i = 0$, we have opened container $j > 1$, as $\left[ \sum_{r \in c_{j-1}} h(r) \right] + \left[ \sum_{r \in c_j} h(r) \right] > 1$. As all of these rectangles were placed in a container of type $\gamma_0$, they have width at least $1/2$. Summing up, as $h(c_j) = 1$, this gives

$$
\sum_{j=1}^{k} \sum_{r \in c_j} v(r) \geq \sum_{j=0}^{[k/2]-1} \left[ \sum_{r \in c_{j+1}} h(r) \cdot w(r) \right] + \sum_{r \in c_{j+2}} h(r) \cdot w(r) \geq (1/2) \sum_{j=0}^{[k/2]-1} h(r) \cdot w(r) = (1/2) \sum_{r \in c_{j+1}} h(r) + (1/2) \sum_{r \in c_{j+2}} h(r) \geq (1/2)[[k/2]] 
$$

$$
(1/2) - (1/2)(k/2 - 1) = (1/4)(k - (1/2)) = (1/4) \sum_{j=1}^{k} h(c_j) - (1/2) > (1/4) \sum_{j=1}^{k} h(c_j) - \frac{2}{2^{2^{0+1}}}
$$

If $i > 0$, we have opened container $c_j$ for $j > 1$, as $\left[ \sum_{r \in c_{j-1}} w(r) \right] + \left[ \sum_{r \in c_j} w(r) \right] > 1$. As all of these rectangles were placed in a container of type $\gamma_i$, they have height at least $2^{-i}$. 

$i \in \mathbb{N}_{\geq 1}$, a container of type $\gamma_i$ has width 1 and height $h(c) = 2^{-i+1}$. For each $i \in \mathbb{Z}_{\geq 0}$, we will have at most one active container of type $\gamma_i$. For all other containers of this type – which we call closed – we will guarantee that at least $1/4$ of their volume is used by items.

We perform the following operation whenever a new rectangle $r_i$ arrives:

- If $w(r_i) \geq 1/2$, check whether a container of type $\gamma_0$ exists. If not, open a new active container of type $\gamma_0$ and place $r_i$ into it. If such a container $c$ already exists and $h(r_i) + \sum_{r \in c} h(r) > 1$, declare $c$ as closed and open a new active container of type $\gamma_0$. Otherwise $(h(r_i) + \sum_{r \in c} h(r) \leq 1)$, put $r_i$ on top of the top item in $c$.

- If $w(r_i) \leq 1/2$ and $h(r_i) \in (2^{-i}, 2^{-i+1}]$, check whether a container of type $\gamma_i$ exist. If not, open a new active container of type $\gamma_i$ and place $r_i$ into it. If such a container $c$ already exists and $w(r_i) + \sum_{r \in c} w(r) > 1$, declare $c$ as closed and open a new active container of type $\gamma_i$. Otherwise $(w(r_i) + \sum_{r \in c} w(r) \leq 1)$, put $r_i$ right to the right-most item in $c$.

We have at most one active container of type $\gamma_i$ for each $i \in \mathbb{Z}_{\geq 0}$: we only open a new active container if we simultaneously declare another container of the same type as closed.

We also have the following simple lemma showing that the volume of closed containers can be bounded.
Theorem 11. The presented algorithm $A_{SP}$ is a flexible online algorithm for Strip Packing with ratio 4.

Proof. The flexibility of $A_{SP}$ follows directly due to the fact that we only build upon the existing packing. We will now show that $A_{SP}$ has ratio 4. Let $t$ be any time point, $S$ be the previous packing we built upon, and $c_1, \ldots, c_k$ be the containers constructed at time $t$. The set of rectangles contained in the containers is denoted as $J$. As noted above, the current packing has height $h(S) + \sum_{i=1}^{k} h(c_j)$. For $i \in \mathbb{Z}_{\geq 0}$, let $C_i \subseteq \{c_1, \ldots, c_k\}$ be the set of containers of type $\gamma_i$. We then have $\sum_{j=1}^{k} h(c_j) = \sum_{i \geq 0} \sum_{c \in C_i} h(c)$. Using Lemma 10 gives

$$\sum_{i \geq 0} \sum_{c \in C_i} h(c) \leq \sum_{i \geq 0} [\sum_{c \in C_i} h(c)] - 2^{-i+3} + 2^{-i+3} = (\sum_{i \geq 0} [\sum_{c \in C_i} h(c)] - 2^{-i+3}) + \sum_{i \geq 0} 2^{-i+3} \leq (\sum_{i \geq 0} [\sum_{c \in C_i} h(c)] - 2^{-i+3}) + 16 \leq 4(\sum_{i \geq 0} [\sum_{c \in C_i} (1/4)h(c)] - 2^{-i+1}) + 16 \leq 4 \sum_{i \geq 0} \sum_{c \in C_i} v(r) + 16 = 4 \text{vol}(J) + 16.$$ 

We have now shown the first two ingredients for our framework: the problem is space related and we gave a suitable online algorithm. The final piece – an offline approximation algorithm – is given by the asymptotic fully polynomial time approximation scheme (AFPTAS) of Kenyon and Rémiia [21], which is an $1 + \epsilon$-approximation. We can thus use Theorem 7 with $\gamma = 1$ and $\beta = 4$ to conclude the following theorem.

Theorem 12. There is a robust online algorithm for the dynamic Strip Packing problem that is $1 + \epsilon$-competitive and has amortized migration factor $O(1/\epsilon)$.

Rotations

If rotations by 90 degree are allowed, the resulting problem is called Strip Packing With Rotations. For an instance $I$, we denote the height of a corresponding optimal packing by
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OPT\(_R(I)\). As the volume of a rotated rectangle does not change, we have OPT\(_R(I) \geq\) VOL\(_I\). Similarly, the volume bound of Lemma 10 also remains true. We can thus conclude the following adaption of Theorem 11.

**Theorem 13.** The presented algorithm A\(_{SP}\) is a flexible online algorithm for Strip Packing With Rotations with ratio 4.

Instead of using the classical AFPTAS by Kenyon and Rémi [21], we use the AFPTAS of Jansen and van Stee [19] for the case that rotations are allowed. Using Theorem 7 with \(\gamma = 1\) and \(\beta = 4\) gives the following theorem.

**Theorem 14.** There is a robust online algorithm for the dynamic Strip Packing With Rotations problem that is \(1 + \epsilon\)-competitive and has amortized migration factor \(O(1/\epsilon)\).

The best known online algorithm with migration known is due to Jansen et al. [18]. It also is \(1 + \epsilon\)-competitive, but an amortized migration factor of \(O(1/\epsilon^9 \log^2(1/\epsilon))\), only works for the static case (no rectangles are removed), and cannot handle rotations.

4 Bin Packing

4.1 2-Dimensional Bin Packing

In 2-D Bin Packing, each item \(i\) is given by its height \(h_i \leq 1\) and its width \(w_i \leq 1\). The goal is to pack these items non-overlapping into as few unit-sized squares (called bins) as possible. As above, we will show the following:

i) We need to show that the 2-D Bin Packing problem is space related;
ii) We need to construct a flexible online algorithm with ratio \(\beta\);
iii) We need to construct an offline approximation algorithm.

As the rectangles are not allowed to overlap and each bin has a total volume of 1, 2-D Bin Packing is space related.

**Remark 15.** The 2-D Bin Packing problem is space related.

We will now present a flexible online algorithm. This algorithm is a simple extension of the classical algorithm presented by Coppersmith and Raghavan [9]. For the sake of completeness, we give a self-contained description and analysis. We categorize items as follows: We call an item *vertical* if \(w_i \leq h_i\) and *horizontal* if \(w_i > h_i\). Note that squares with \(h_i = w_i\) will be considered as vertical. Without loss of generality we will explain in the following how to pack vertical items. Horizontal items can and will be placed into separate bins with the same strategy, just altered for horizontal items. Note that we later also need to account for these horizontal bins.

We further assign each item \(i\) a *size class*. Item \(i\) is in size class \(j \in \mathbb{N}_{\geq 1}\) if \(1/2^{j-1} > h_i > 1/2^j\). Further we say an item \(i\) is *square-like*, if \(i\) is in size class \(j\) and furthermore \(w_i > 1/2^j\).

The general idea is that for every arriving item in size class \(j\), we will assign a square slot of size \(1/2^{j-1}\) in some bin. A square slot of this size is called a *slot of class \(j\)*. Note that an item of size class \(j\) always fits into a slot of class \(j\), as we only handle vertical items with \(w_i \leq h_i\) here. Our goal is to fill all opened slots with items until \(1/4\) of the total area of the slot is covered. Square-like items will immediately fill such a slot to this extent. For the other items, we will reserve such slots for a size class \(j\) and stack items from left to right until the total width of all items in that slot exceeds \(1/2^j\). Since the height of items assigned to this slot exceed \(1/2^j\) as well, \(1/4\) of the total slot will be covered at that point. A slot can
have three states: it is either (i) empty and thus contains no item, (ii) reserved for class $j$ and thus only contains items of class $j$ or (iii) closed if at least $\frac{1}{4}$ of its total volume is filled with items.

In order to assign items to these slots, we will keep up to two open bins. The first open bin will hold items of size class 1 and we use the complete bin as a single reserved slot. The second open bin will receive items of size class $\geq 2$. Initially, this bin is split into four empty slots of class 2. We now define our online algorithm $A_{2,D}$:

1. If a square-like item of size class 1 arrives, we open a new bin, place the item in there and close it immediately. By definition of size class 1, at least $\frac{1}{4}$ of the volume of this bin will be filled.
2. If a non-square-like item $i$ of size class 1 arrives, we place it to the right of the non-square-like items in the first open bin or on the left end, if no such items exist. As $w_i \leq \frac{1}{2}$, we can always place this item in the bin. If the sum of the widths of the items in the first bin now is at least $\frac{1}{2}$, close this bin and open a new first bin. As every item placed in the bin has height $h_i \geq \frac{1}{2}$, at least $\frac{1}{4}$ of the volume of this bin will be filled.
3. If an item $i$ of class $j \geq 2$ arrives, find a non-closed slot of class $j' \leq j$ with maximal $j'$ in the second open bin.
   a. If $j' = j$, we choose a slot reserved for $j'$ or - if no such slot exists - the top-most left-most of the slots of class $j'$. We declare this slot as reserved for $j'$ and place the item right of the items placed in this slot or on the left end, if no such items exist. As $w_i \leq h_i \leq \frac{1}{2^{\leq i}}$, we can always place $i$ in this slot. If the sum of the widths of the items placed in this slot is at least $\frac{1}{2^{\leq i}}$, we close this slot.
   b. If $j' < j$, note that we can split such a slot into four equal-sized slots of class $j' + 1$. Splitting one slot will leave us with four slots: three of the slots will remain empty and the last slot will either be split even more (if its class $j' + 1$ is smaller than $j$) or we reserve this slot for class $j$ and put the new item into it. We repeat this splitting until we have created a slot of class $j$ and put $i$ on the left end. If the sum of the widths of the items placed in this slot is at least $\frac{1}{2^{\leq i}}$ (i.e. if $i$ is square-like), we close this slot.
   c. If no slot of size $j' \leq j$ exists, we close the second bin and open a new second bin containing four empty slots of class 2. We place $i$ into this bin as above.

We will now show that every bin created by the algorithm contains a most three empty slots of a certain class.

**Lemma 16.** For each $j \geq 1$, every bin created by $A_{2,D}$ contains at most three empty slots of class $j$.

**Proof.** For $j = 1$, every bin contains at most a single slot of class 1.

For $j \geq 2$, assume that this is not the case and our algorithm may generate a slot assignment with four or more slots of the same class $j^*$. Consider some arriving item $i$ of class $j$ such that before the arrival of $i$ there are at most three empty slots of class $j^*$ and afterwards, there are four or more of these slots. Since the number of empty slots of class $j^*$ increased, our algorithm must have split a slot of class $j' < j^*$. Note that this will only happen if $j' \leq j$. Splitting this slot of class $j'$ over and over will create three slots of class $j' + 1, j' + 2, \ldots, j^* - 1$. Finally, four slots of class $j^*$ are created. One of these slots will either be directly used for $i$ (if $j' = j$) or will be split into smaller slots. Hence, only three new slots of class $j^*$ are created. By assumption, the insertion of $i$ leads to at least four slots of class $j^*$. Hence, there must have been a slot of class $j^*$ before. But, by definition of the
To make the algorithm $A_{2,D}$ flexible, we simply ignore all previous used bins and only work on the newly generated bins.

**Theorem 17.** The proposed algorithm $A_{2,D}$ for 2-D Bin Packing is a flexible online algorithm with ratio $\frac{48}{5}$.

**Proof.** The flexibility of $A_{2,D}$ follows directly from the fact that we only work on newly created bins.

To analyze the ratio of the algorithm we look at how much total area is covered when a complete bin has been closed. First we can observe that every closed slot inside a closed bin has at least an overall area of $\frac{1}{4}$ covered: we only close a slot if half of its width is covered and we only put items into a slot that cover at least half of the height.

Another easy observation is the fact that the algorithm may have at most one reserved slot for each class, since the algorithm only opens a new reserved slot for a class whenever the current reserved one is closed. We may open a second slot for a square-like item, but will also close it immediately. By neglecting the items in reserved slots, the unused space due to reserved slots in every bin therefore can be bounded by

$$\sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{1}{1 - \frac{1}{4}} - 1 = \frac{1}{3}. \quad (*)$$

By [Lemma 16](#) we know that every bin contains at most three empty slots of each size class. Note that a closed bin cannot contain an empty slot of class 2, as such a bin will only receive items of this size class or higher. With this we can now conclude that the unused space due to empty slots in a closed bin is at most

$$3 \sum_{i=2}^{\infty} \frac{1}{4^i} = 3 \left[ \frac{1}{1 - \frac{1}{4}} - 1 - \frac{1}{4} \right] = 3 \cdot \frac{1}{12} = \frac{1}{4}. \quad (***)$$

By combining Equation $(*)$ and Equation $(***)$, at least a total area of $1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$ of every closed bin is occupied by closed slots. As discussed earlier, every such closed slot has at least $\frac{1}{4}$ of its total area covered so we finally can conclude that at least a total area of $\frac{5}{48}$ of every closed bin must be covered.

The algorithm keeps at most four open bins, one for items of size class 1, one for items of other size classes, and we need two respective bins for horizontal items containing the same classes as well. Finally we get that our algorithm for an instance $I$ with total volume $\text{vol}(I)$ will use at most $\frac{48}{5} \cdot \text{vol}(I) + 4$ bins in total.

Finally, we can use the approximation algorithm of Bansal and Khan [4] that is an 1.405-approximation for 2-D Bin Packing. We can thus use [Theorem 7](#) with $\gamma = 1.405$ and $\beta = \frac{48}{5}$ to conclude the following theorem.

**Theorem 18.** There is a robust online algorithm for the dynamic 2-D Bin Packing problem that is $1.405 + \epsilon$-competitive and has amortized migration factor $O(1/\epsilon)$.

To the best of our knowledge, this is the first robust online algorithm for dynamic 2-D Bin Packing. Note that the best known lower bound for the competitiveness of any online algorithm for online 2-D Bin Packing without migration is 1.856 due to Van Vliet [24].
Rotations

As for Strip Packing, allowing rotations of the rectangles by 90 degrees gives rise to a problem called 2-D Bin Packing With Rotations. The corresponding optimal number of bins needed to pack instance $I$ is denoted by $\text{OPT}_R(I)$. Rotations are invariant with regard to the volume of a rectangle and thus $\text{OPT}_R(I) \geq \text{vol}(I)$. We can thus again use our online algorithm $A_{2,D}$ to obtain the following adaption of Theorem 17.

\textbf{Theorem 19.} The proposed algorithm $A_{2,D}$ for 2-D Bin Packing With Rotations is a flexible online algorithm with ratio $48/5$.

The approximation algorithm of Bansal and Khan [4] used above can also handle the case of rotation and thus is an $1.405$-approximation for 2-D Bin Packing With Rotations. We can thus use Theorem 7 with $\gamma = 1.405$ and $\beta = 48/5$ to conclude the following theorem.

\textbf{Theorem 20.} There is a robust online algorithm for the dynamic 2-D Bin Packing With Rotations problem that is $1.405 + \epsilon$-competitive and has amortized migration factor $O(1/\epsilon)$.

To the best of our knowledge, this is the first robust online algorithm for dynamic 2-D Bin Packing With Rotations. Note that the best known lower bound for the competitiveness of any online algorithm for 2-D Bin Packing With Rotations without migration is 1.6707 due to Heydrich and van Stee [6].

4.2 $d$-Dimensional Bin Packing

We will now look at the problem of packing $d$-dimensional hyperrectangles into as few unit-sized hypercubes as possible for higher dimensions $d > 2$. This problem is called $d$-Dimensional Hyperrectangle Packing.

To obtain an suitable flexible online algorithm for this problem, we will generalize the 2-dimensional Bin Packing algorithm $A_{2,D}$ from Section 4.1. The side length of dimension $i \in \{1, \ldots, d\}$ of a $d$-dimensional hyperrectangle $r$ is denoted as $r_i$ and its volume is $v(r) = \prod_{i=1}^{d} r_i$. Items will be classified almost the same as before in a straight-forward generalization. For a permutation $\pi : \{1, \ldots, d\} \rightarrow \{1, \ldots, d\}$, we associate a set of rectangles $r$ with $r_{\pi(1)} \leq r_{\pi(2)} \leq \ldots \leq r_{\pi(d)}$ with it. We treat each of these permutations separately and pack the corresponding rectangles in separate bins. This will only give an additive error of $d!$ in our approximation guarantee. In the following, we thus fix a permutation $\pi$. A rectangle associated with $\pi$ is in size class $j \in \mathbb{N}_{\geq 1}$, if $1/2\cdot2^{-1} \leq r_{\pi(d)} > 1/2\cdot2^{-1}$, hence the class of a rectangle depends on its largest side length. Rectangles of class $j$ are packed into slots that are hypercubes with side length $1/2\cdot2^{-1}$.

4.2 $d$-Dimensional Bin Packing

Every closed slot inside a closed bin has at least an overall area of $2^{-d}$ covered, as in the proof of Theorem 17. We thus obtain the following generalization of Theorem 17 by not using any bins created by the previous solution $S$. 

\textbf{Lemma 21.} For each $\pi$ and each $j \geq 1$, every bin created by $A_{d,D}$ contains at most $2^d - 1$ empty slots of class $j$ associated with $\pi$. 

\textbf{Theorem 19.} The proposed algorithm $A_{2,D}$ for 2-D Bin Packing With Rotations is a flexible online algorithm with ratio $48/5$. 

The approximation algorithm of Bansal and Khan [4] used above can also handle the case of rotation and thus is an $1.405$-approximation for 2-D Bin Packing With Rotations. We can thus use Theorem 7 with $\gamma = 1.405$ and $\beta = 48/5$ to conclude the following theorem. 

\textbf{Theorem 20.} There is a robust online algorithm for the dynamic 2-D Bin Packing With Rotations problem that is $1.405 + \epsilon$-competitive and has amortized migration factor $O(1/\epsilon)$.

To the best of our knowledge, this is the first robust online algorithm for dynamic 2-D Bin Packing With Rotations. Note that the best known lower bound for the competitiveness of any online algorithm for 2-D Bin Packing With Rotations without migration is 1.6707 due to Heydrich and van Stee [6].
Theorem 22. The proposed algorithm $A_{d,D}$ for $d$-Dimensional Hyperrectangle Packing is a flexible online algorithm with ratio $\frac{2^{d-1} \cdot 2^{d+1}}{2^{2d}(2^{d-1})}$. 

Proof Sketch. By neglecting the items in reserved slots, the unused space due to reserved slots of class $j$ can be divided into $2^d$ equal sized hypercubes with side length $1/2^d$. Putting an item of class $j$ into a slot of class $j$ covers at least a fraction of $1/2^d$ of the total volume of the slot. Hence, slots are either empty or closed and do not need to be reserved. Every $d$-dimensional hypercube with side length $s$ can be divided into $2^d$ equal sized hypercubes with side length $s/2$. The slot assignment happens like above: We start with single empty active bin containing $2^d$ empty slots of class 2. An item of class 1 is simply put into its own bin. Whenever an item $i$ of class $j \geq 2$ arrives, we first try to find an empty slot of class $j$. If any such slot exists, put $i$ into it and close the slot. If no slot of class $j' < j$ exists, we close the bin and open a new bin to insert $i$ into. If a slot of class $j' < j$ exists, we split the smallest such slot that is still larger than the required slot size until an empty slot of size $j$ is created.

We call this algorithm $A_{d,\text{hyper}}$ and again make it flexible by simply opening new bins. Similar to the two-dimensional case, we can show that there can’t be too many empty slots of the same class. The proof is just an adaption of the proof of Lemma 16.

Lemma 24. For each $j \geq 1$, every bin created by $A_{d,\text{hyper}}$ contains at most $2^d - 1$ empty slots of class $j$.

Proof. For $j = 1$, every bin contains at most a single slot of class 1.
Theorem 26. The flexibility follows from the fact that we do not touch the already used bins.

Proof. The flexibility follows from the fact that we do not touch the already used bins.

For \( j \geq 2 \), assume that this is not the case and our algorithm may generate a slot assignment with \( 2^d \) or more slots of the same class \( j^* \). Consider some arriving item \( i \) of class \( j \) such that before the arrival of \( i \) there are at most \( 2^d - 1 \) empty slots of class \( j^* \) and afterwards, there are \( 2^d \) or more of these slots. Since the number of empty slots of class \( j^* \) increased, our algorithm must have split a slot of class \( j' < j^* \). Note that this will only happen if \( j^* \leq j \). Splitting this slot of class \( j' \) over and over will create \( 2^d - 1 \) slots of class \( j' + 1, j' + 2, \ldots, j^* - 1 \). Finally, \( 2^d \) slots of class \( j^* \) are created. One of this slots will either be directly used for \( i \) (if \( j^* = j \)) or will be split into smaller slots. Hence, only \( 2^d - 1 \) new slots of class \( j^* \) are created. By assumption, the insertion of \( i \) leads to at least \( 2^d \) slots of class \( j^* \). Hence, there must have been a slot of class \( j^* \) before. But, by definition of the algorithm, we would rather have chosen this slot instead of the slot of class \( j' \), as \( j' < j^* \).

Hence, this is a contradiction.

Theorem 25. The proposed algorithm \( A_{d,\text{hyper}} \) for \( d \)-Dimensional Hypercube Packing is a flexible online algorithm with ratio \( \frac{2^d}{2^d - 1} \).

Proof. The flexibility follows from the fact that we do not touch the already used bins.

For every \( d \geq 2 \), there is a robust online algorithm for the dynamic \( d \)-Dimensional Hypercube Packing problem that is \( 1 + \epsilon \)-competitive and has amortized migration factor \( O(1/\epsilon) \).

In contrast, the best known online algorithm with migration for the \( d \)-Dimensional Hypercube Packing is due to Epstein and Levin [12]. It is also \( 1 + \epsilon \)-competitive, but can only handle the static case and has migration factor \( (1/\epsilon)^{O(d)} \). Note however that they use worst-case migration, i.e. they are not allowed to repack the complete instance every once in a while but need to make slight adptions carefully throughout the run of the algorithm.


5 \textbf{d-Dimensional Strip Packing}

In the \textit{d}-dimensional version of online strip packing, called the \textit{d-Dimensional Strip Packing} problem, we are given a \textit{d}-dimensional cuboid that has size 1 in \(d-1\) dimensions and infinite size in the last dimension. Even in the general case with more dimensions we will consider this last dimension as \textit{height}. At each point of time a \textit{d}-dimensional hyperrectangle \(r\) arrives. The size of its \(i\)th dimension is denoted by \(r_i \in (0, 1)\) and its volume is \(v(r) = \prod_{i=1}^{d} r_i\). The task is again to pack these cuboids with no intersection such that the height is minimized. Like above we denote with \(I_t\) the set of rectangles present at time \(t\) with \(\text{OPT}(I_t)\) the total volume and with \(\text{OPT}(I_t)\) the optimal height. This version of Strip Packing is also space related, since the base of the packing space has side lengths of 1.

\textbf{Remark 27.} The \textit{d}-dimensional Strip Packing problem is space related.

It leaves to show that there are respective online and offline algorithms for our framework. For a flexible online algorithm we will generalize our above approach for the \textit{d}-dimensional case in a straight-forward way. A \textit{container} \(c\) of type \(\gamma_i\) is a \textit{d}-dimensional hyperrectangle with \(c_1 = c_2 = \ldots = c_{d-1} = 1\) and \(c_{d} = 2^{-i}\). For each \(i \in \mathbb{Z}_{\geq 0}\), we will have at most one \textit{active} container of type \(\gamma_i\). For all other containers of this type – which we call \textit{closed} – we will guarantee that at least a constant fraction of their volume is used by items. We assign a hyperrectangle \(r\) of height \(r_d \in (2^{-i-1}, 2^{-i}]\) to a container of type \(\gamma_i\). We then treat these hyperrectangles as \(d-1\)-dimensional hyperrectangles by projecting to its first \(d-1\) coordinates and also treat the container as a \(d-1\)-dimensional hypercube. We then pack the projected hyperrectangles into the projected hypercube with the algorithm \(A_{d-1}\) described in Section 4.2. Theorem 22 then guarantees that a fraction of \(\frac{2^{d(d-1)}(2^{d-1} - 1)}{2^{d-1} - 2^{d-1} + 1}\) of each projected container is filled. As the original, non-projected containers have height \(2^{-i}\) and every non-projected hyperrectangle has height at least \(2^{-i-1}\), we lose a factor of 2 here.

The resulting algorithm, called \(A_{d-SP}\), is thus a flexible online algorithm and we obtain the following generalization of Theorem 11.

\textbf{Theorem 28.} The presented algorithm \(A_{d-SP}\) is a flexible online algorithm for \textit{d-dimensional Strip Packing} with ratio \(O\left(\frac{2^d - 1}{2^{d-1} - 2^{d-1} + 1}\right)\).

To the best of our knowledge, there is no work that explicitly deals with the construction of approximation algorithms for the \textit{d}-dimensional Strip Packing problem. As shown above, any such result can be used in the context of our framework to obtain a robust online algorithm with corresponding competitive ratio.

5.1 \textbf{Hypercube Strip Packing}

In the following we will restrict this problem to the case where each item is a hypercube, so we have that each entry \(r_i\) is the same. In this version, that is also known as \textit{online Hypercube Strip Packing}, we will denote this value with \(s(r)\) and call it the \textit{side length} of the hypercube \(r\). The size of this hypercube is given by its volume, which is \(v(r) = s(r)^d\).

Note that in this variant allowing rotations by 90 degree makes no difference.

In fact the approach becomes a little easier due to our restriction to hypercubes. Again we define containers: For \(i \in \mathbb{N}_{\geq 1}\), a container \(c\) of type \(\gamma_i\) has size 1 in every dimension except the height and height of \(h(c) = 2^{-i+1}\). The idea is the same as before: Keep at most one \textit{open} container of each size, while we make sure at least \(2^{-d}\) of the total area of each \textit{closed} bin is covered.
Theorem 29. The presented algorithm \( A_{SP-hyper} \) is a flexible online algorithm for \( d \)-dimensional Hypercube Strip Packing restricted to hypercubes with ratio \( 2^d \).

Proof. The flexibility follows again from the properties of the Strip Packing problem, given a packing of some items \( S \), we can simply ignore that packing and put newly arriving items on top of the existing packing.

First off we show that each closed container has at least \( 2^{-d} \) of its total volume covered. Let \( c \) be a closed container of type \( \gamma_i \) with volume \( v(c) = h(c) \), since the size in every dimension except height is exactly \( 1 \). Note that we subdivided \( c \) into multiple cubic slots with total volume \( h(c)^d = 2^{d(\gamma_i+1)} \). Since \( c \) is closed each slot got assigned an item \( r \) with volume \( v(r) = s(r)^d > 2^{-i} \cdot d \cdot 2^{d(\gamma_i+1)} = 2^{-d} \cdot h(c)^d \). Since the slots divide the container completely and every slot has \( 2^{-d} \) of its total volume covered, we can conclude that \( \sum_{r \in c} v(r) \geq (2^{-d})h(c) \).

Now we will look at the height of a flexibly created solution. Let \( t \) be a point of time, \( S \) be the previous packing we extended and let \( c_1, \ldots, c_k \) be the containers created up to time \( t \). Let \( O \) be the set of open containers and \( L \) the set of closed containers. Let \( S_t \) be the result of our algorithm at time \( t \). Let \( \gamma \) be the type of the smallest open container in our current solution. We can then observe for the height of open containers that: \( \sum_{c \in O} h(c) \leq \sum_{i=1}^{t} 2^{-i+1} = \sum_{i=0}^{t} 2^{-i} \leq 2 \).

Finally we have for the total height of our solution that

\[
\begin{align*}
    h(S_t) &= h(S) + \sum_{c \in L} h(c) + \sum_{c \in O} h(c) \leq h(S) + \sum_{c \in L} 2^d \sum_{r \in c} v(r) + \sum_{c \in O} h(c) \\
    h(S) + 2^d \sum_{c \in L} \sum_{r \in c} v(r) + 2 &\leq h(S) + 2^d \text{vol}(I_t) + 2 \quad \Box
\end{align*}
\]

With this we know our problem is space-related and has an appropriate space related online algorithm. As for the offline algorithm, we will use a result from Harren, who gave an APTAS for the Hypercube Strip Packing problem \([16]\). By using Theorem 7 with \( \gamma = 1 \) and \( \beta = 2^d \) we get the following theorem.

Theorem 30. There is a robust online algorithm for the dynamic \( d \)-dimensional Hypercube Strip Packing problem that is \( 1 + \epsilon \)-competitive and has amortized migration factor \( O(1/\epsilon) \).

6 Vector Packing

In the online \( d \)-dimensional Vector Packing problem, at time \( t \) either a vector \( w_t \in (\mathbb{Q} \cap [0,1])^d \) is inserted and needs to be packed or is removed. The size \( v(w_t) \) of such a vector \( w_1 = (w[1], \ldots, w[d]) \) is defined as the average sum of its components, i.e. \( v(w_t) = \sum_{j=1}^{d} w[j]/d \).
The goal is to pack these vectors into as few as possible bins as possible. Here, a bin $B$ is a subset of vector such that $\sum_{w \in B} w[j] \leq 1$ for $j = 1, \ldots, d$.

To use our framework, we need the following ingredients:

i) We need to show that the problem is space related;
ii) we need to construct a flexible online algorithm with ratio $\beta$;
iii) We need to construct an offline approximation algorithm.

As each bin can contain items of volume at most 1, it is easy to see that the $d$-dimensional Vector Packing problem is space related.

\begin{remark}
The $d$-dimensional Vector Packing problem is space related.
\end{remark}

We will now present a flexible online algorithm with ratio $\beta = 2d$ that is a simple adaption of the well-known next fit online algorithm for bin packing. Every bin will have an index to guarantee a linear ordering. Whenever a vector $w$ arrives, we first check whether $w$ can be packed into an existing bin. If this is possible, we add $w$ to such a bin with minimal index. If no such bin exists, we open a new bin containing $w$. If we are given a previous packing $S$, we simply ignore the previous bins and do not put any vector in them.

\begin{theorem}
For every $d \geq 1$, the presented algorithm $AVP$ is a flexible online algorithm for $d$-dimensional Vector Packing with ratio $2d$.
\end{theorem}

\begin{proof}
The flexibility of $AVP$ follows directly due to the fact that we only build upon the existing packing. We will now show that $AVP$ has ratio $2d$. Let $t$ be any time point, $S$ be the previous packing we built upon, and $B_1, \ldots, B_k$ be the bins opened by $AVP$. Let $J$ be the set of vectors packed into $B_1, \ldots, B_k$. The current packing then uses $|S| + k$ bins, where $|S|$ denotes the number of bins used by the previous packing $S$. Consider two adjacent bins $B_i$ and $B_{i+1}$ for $i \in \{1, \ldots, k-1\}$. We have opened bin $B_{i+1}$, because there is some vector $w' \in B_{i+1}$ and some index $j \in \{1, \ldots, d\}$ such that $\sum_{w \in B_i} w[j] + w'[j] > 1$. Denoting this index with $j = j(i)$, we have $\sum_{w \in B_i \cup B_{i+1}} w[j(i)] > 1$ and thus $d \cdot \text{vol}(B_i \cup B_{i+1}) > 1$. Hence, we can conclude

$$
k < 1 + \sum_{i=1}^{k-1} d \cdot \text{vol}(B_i \cup B_{i+1}) = 1 + d \cdot \sum_{i=1}^{k-1} \left(\text{vol}(B_i) + \text{vol}(B_{i+1})\right) = 1 + d \cdot \left[\left(\sum_{i=1}^{k} \text{vol}(B_i)\right) + \sum_{i=2}^{k} \text{vol}(B_i)\right] < 1 + 2d \sum_{i=1}^{k} \text{vol}(B_i) = 1 + 2d \cdot \text{vol}(J).
$$

We have now shown the first two ingredients for our framework: the problem is space related and we gave a suitable online algorithm. The final piece – an offline approximation algorithm – is given by the algorithm of Bansal et. al. which is a $\ln(d+1) + 0.807 + \epsilon$-approximation. We can thus use Theorem 7 with $\gamma = \ln(d+1) + 0.807 + \epsilon$ and $\beta = 2d$ to conclude the following theorem.

\begin{theorem}
For every $d \geq 1$, there is a robust online algorithm for the dynamic $d$-dimensional Vector Packing problem that is $\ln(d+1) + 0.807 + \epsilon$-competitive and has amortized migration factor $O(1/\epsilon)$.
\end{theorem}

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