ON THE ALGEBRA OF NONLOCAL SYMMETRIES FOR THE 4D MARTÍNEZ ALONSO-SHABAT EQUATION

I.S. KRASIL’SHCHIK AND P. VOJČÁK

Abstract. We consider the 4D Martínez Alonso-Shabat equation $E_{ty} = u_z u_{xy} - u_y u_{xz}$ (also referred to as the universal hierarchy equation) and using its known Lax pair construct two infinite-dimensional differential coverings over $E$. In these coverings, we give a complete description of the Lie algebras of nonlocal symmetries. In particular, our results generalize the ones obtained in [12] and contain the constructed there infinite hierarchy of commuting symmetries as a subalgebra in a much bigger Lie algebra.

Contents

Introduction 1
1. Preliminaries 2
2. The equation and its coverings 3
3. Algebras of nonlocal symmetries 7
Acknowledgments 13
References 13

Introduction

To the best of our knowledge, the equation
\[ u_{ty} = u_z u_{xy} - u_y u_{xz} \] (1)
was introduced in the work [9] by L. Martínez Alonso, A.B. Shabat, where the authors studied multi-dimensional systems whose reductions lead to the known $(1+1)$-integrable equations (see also [8] for additional motivations). By this reason we call Equation (1) the Martínez Alonso-Shabat equation, or shortly the 4D MASH equation. The equation arises also in classification of integrable 4D systems, see [4].

A differential covering (Lax pair) with a non-removable parameter was constructed in [11], as well as a recursion operator for symmetries of the 4D MASH equation. Using this covering, the authors of [12] found a hierarchy of nonlocal symmetries and proved its commutativity.

We study Equation (1) using the approach successfully applied to integrable linearly degenerate 3D systems in [1] and [5]. Expanding defining equations of the Lax pair in formal series of the spectral parameter, we construct two differential coverings (which we call the negative and positive ones) and describe the algebras of nonlocal symmetries in these coverings. As the reader will see, the structure of these algebras is quite complicated. The commutative hierarchy found in [12] appears as a subalgebra in one of them. We also analyze the action of the recursion operator from [11] on our symmetries.

The structure of the paper is as follows: in Section 1 we present very briefly necessary facts from the geometrical theory of PDEs [3] and differential coverings [7]. Section 2 contains the construction of the positive and negative coverings and defining equations for symmetries in them. In Section 3 the symmetry algebras are described.

2010 Mathematics Subject Classification. 35B06.
Key words and phrases. 4D Martínez Alonso-Shabat equation, universal hierarchy equation, Lax pairs, differential coverings, nonlocal symmetries.
1. Preliminaries

Let us very shortly recall the necessary theoretical background. All the details may be found, e.g., in [3] and [7]. A particular implementation of all the general constructions will be presented in Section 2.

Equations. From the geometrical viewpoint, a differential equation is a submanifold in a jet space. More precisely, this means the following. Let \( \pi: E \to M \) be a locally trivial vector bundle over a smooth manifold, \( \dim M = n \), rank \( \pi = m \), and \( \pi_\infty: J^\infty(\pi) \to M \) be the corresponding bundle of infinite jets. For us, a differential equation (imposed on sections of \( \pi \)) is a submanifold \( \mathcal{E} \subset J^\infty(\pi) \) obtained by the prolongation procedure from a submanifold in the space of finite jets. We use the same notation \( \pi_\infty \mid E \) for the restriction \( \pi_\infty|_E: \mathcal{E} \to M \).

The structure of equation on \( \mathcal{E} \) is defined by the Cartan connection \( \mathcal{C} \), which takes vector fields \( X \in D(M) \) to vector fields \( \mathcal{C}_X \in D(\mathcal{E}) \) on \( \mathcal{E} \). The connection is flat, i.e., \( \mathcal{C} = [\mathcal{C}_X, \mathcal{C}_Y] \). The corresponding integrable \( \pi_\infty \)-horizontal distribution is called the Cartan distribution on \( \mathcal{E} \) and its maximal \((n\text{-dimensional})\) integral manifolds are identified with solutions of \( \mathcal{E} \).

Local symmetries. An (infinitesimal higher local) symmetry of \( \mathcal{E} \) is a \( \pi_\infty \)-vertical vector field \( S \in D(\mathcal{E}) \) on \( \mathcal{E} \) such that the commutator \( [S, \mathcal{C}_X] \) lies in the Cartan distribution for any \( X \in D(M) \). Symmetries form a Lie \( \mathbb{R} \)-algebra denoted by \( \text{sym} \mathcal{E} \).

To describe \( \text{sym} \mathcal{E} \), consider another vector bundle \( \xi: G \to M \), rank \( \xi = r \), and assume that \( \mathcal{E} = \{ F = 0 \} \) is given as the set of zeros of some section \( F \in P = \Gamma(\pi_\infty^* (\xi)) \), where \( \Gamma(\cdot) \) denotes the \( C^\infty(M) \)-module of sections. Consider also the module \( \mathcal{E} = \Gamma(\pi_\infty^* (\pi)) \) and the linearization operator

\[ \ell_\xi = \ell_F|_\xi : \mathcal{E} \to P. \]

Then one has

\[ \text{sym} \mathcal{E} = \ker \ell_\xi. \]

Thus, to any symmetry \( S \in \text{sym} \mathcal{E} \) there corresponds a section \( \varphi \in \mathcal{E} \), its generating section, or characteristic, and we use the notation \( S = E_\varphi \) in this case. The commutator of symmetries generates a bracket in the \( \mathbb{R} \)-space of generating sections defined by \( [E_\varphi, E_\psi] = E_{\{\varphi, \psi\}} \). The bracket \( \{\cdot, \cdot\} \) is called the (higher) Jacobi bracket.

Differential coverings. Let \( \tau: \tilde{\mathcal{E}} \to \mathcal{E} \) be a locally trivial bundle. It is called a (differential) covering over \( \mathcal{E} \) if there exists a flat connection \( \tilde{\mathcal{C}} \) in the bundle \( \tilde{\pi}_\infty = \pi_\infty \circ \tau: \tilde{\mathcal{E}} \to M \) such that \( \tau_* (\tilde{\mathcal{C}}_X) = \mathcal{C}_X \) for any field \( X \in D(M) \). The manifold \( \tilde{\mathcal{E}} \), locally at least, is always an equation in some bundle over \( M \) and is called the covering equation. The number \( s = \text{rank} \tau \) is the covering dimension and it may be infinite. Coordinates in fibers of \( \tau \) are called nonlocal variables.

Let \( \tau_1: \tilde{\mathcal{E}}_1 \to \mathcal{E}, \ i = 1, 2 \), be two coverings. Then their Whitney product \( \tau_1 \oplus \tau_2 \) carries a natural structure of a covering called the Whitney product of \( \tau_1 \) and \( \tau_2 \) and all the arrows in the diagram

\[
\begin{array}{c}
\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\tau_2} \mathcal{E}_1 \\
\mathcal{E}_1 \mathcal{E}_2 \xrightarrow{\tau_1} \mathcal{E}_1 \\
\mathcal{E}_2 \end{array}
\]

are coverings.

A Bäcklund transformation between equations \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) is a diagram of the form

\[
\begin{array}{c}
\tilde{\mathcal{E}} \xrightarrow{\tau_1} \mathcal{E}_1 \\
\tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2.
\end{array}
\]
where \( \tau_1 \) and \( \tau_2 \) are coverings. When \( \mathcal{E}_1 = \mathcal{E}_2 \) then we speak about Bäcklund auto-transformation. If the equations \( \mathcal{E}_i, \ i = 1, 2, \) are given by the systems \( \{ F_i(u_i) = 0 \} \) then the system \( \{ \tilde{F}(u_1, u_2) = 0 \} \) that corresponds to \( \tilde{\mathcal{E}} \) possesses the following property: if \( u_1 \) is a solution of \( \mathcal{E}_1 \) and \( (u_1, u_2) \) is a solution of \( \tilde{\mathcal{E}} \) then \( u_2 \) solves \( \mathcal{E}_2 \) and vice versa.

**Nonlocal symmetries and shadows.** A nonlocal symmetry of \( \mathcal{E} \) in the covering \( \tau \) is a symmetry of \( \tilde{\mathcal{E}} \). These symmetries form the algebra \( \text{sym}_r \mathcal{E} = \text{sym} \tilde{\mathcal{E}} \). Thus, to find nonlocal symmetries, we need to solve the equation \( \ell_{\mathcal{E}}(\tilde{\phi}) = 0 \).

Denote by \( \mathcal{F} \) and by \( \tilde{\mathcal{F}} \) the algebras of smooth functions on \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \), respectively. The projection \( \tau \) leads to the embedding \( \tau^*: \mathcal{F} \to \tilde{\mathcal{F}} \). We say that an \( \mathbb{R} \)-linear derivation \( Y: \mathcal{F} \to \tilde{\mathcal{F}} \) is a shadow in \( \tau \) if the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{Y} & \mathcal{F} \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
\tilde{\mathcal{F}} & \xrightarrow{\tilde{Y}} & \tilde{\mathcal{F}}
\end{array}
\]

is commutative for any \( X \in D(M) \). In particular, for any nonlocal symmetry \( \tilde{S}: \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \) the restriction \( \tilde{S}|_{\mathcal{F}}: \mathcal{F} \to \tilde{\mathcal{F}} \) is a shadow. We say that \( \tilde{S} \) is invisible if its shadow vanishes. Note that any local symmetry \( S \) may be regarded as a \( \tau \)-shadow if one takes the composition \( \tau^* \circ S \). A nonlocal symmetry \( \tilde{S} \) is a lift of a shadow \( Y \) if \( \tilde{S}|_{\mathcal{F}} = Y \). A lift, if it exists, is defined up to invisible symmetries. The defining equation for shadows is

\[
\tilde{\ell}_{\mathcal{E}}(\tilde{\phi}) = 0,
\]

where \( \tilde{\ell}_{\mathcal{E}} \) is the natural extension of the linearization operator from \( \mathcal{E} \) to \( \tilde{\mathcal{E}} \).

**Recursion operators (see \([6, 10]\)).** Let an equation \( \mathcal{E} \) be given by \( \{ F(u) = 0 \} \). Then its tangent equation is

\[
\mathcal{T}\mathcal{E}: \quad F(u) = 0, \quad \ell_{\mathcal{F}}(p) = 0,
\]

where \( p = (p^1, \ldots, p^m) \) is a new unknown of the same dimension as \( u \). The projection \( t: \mathcal{T}\mathcal{E} \to \mathcal{E}, \ (u, p) \mapsto u \), is called the tangent covering of \( \mathcal{E} \). Properties of \( \mathcal{T}\mathcal{E} \) are closely related with symmetries of \( \mathcal{E} \): sections of \( t \) which take the Cartan distribution on \( \mathcal{E} \) to that on \( \mathcal{T}\mathcal{E} \) are in one-to-one correspondence with symmetries.

Let \( \mathcal{R} \) be a Bäcklund auto-transformation of \( \mathcal{T}\mathcal{E} \). Then it relates shadows of symmetries of \( \mathcal{E} \) with each other, i.e., may be understood as recursion operator.

## 2. The equation and its coverings

Here we present the necessary formulas for the computations to be done in Section 3.

**Internal coordinates and the total derivatives.** The manifold \( \mathcal{E} \) corresponding to Equation (1) lies in \( J^\infty(\pi) \), where \( \pi: \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4 \) is the trivial bundle. We denote the coordinates in the base by \( x, y, z, t \), while \( u \) denotes a coordinate in the fiber. Then internal coordinates

\[
u_{x^i z^j}, \ u_{x^i z^j y^k}, \ u_{x^i z^j t^l}, \quad i, j \geq 0, \ k, l > 0,
\]
on \( \mathcal{E} \) arise. Then the Cartan connection is completely determined by its values on the basis vector fields \( \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial t \). The result is the corresponding total derivatives on \( \mathcal{E} \):

\[
\begin{align*}
D_x &= \frac{\partial}{\partial x} + \sum_{i,j \geq 0, k,l > 0} \left( u_{x^{i+1} z^j} \frac{\partial}{\partial u_{x^i z^j}} + u_{x^i z^{j+1} y^k} \frac{\partial}{\partial u_{x^i z^j y^k}} + u_{x^i z^{j+1} t^l} \frac{\partial}{\partial u_{x^i z^j t^l}} \right), \\
D_y &= \frac{\partial}{\partial y} + \sum_{i,j \geq 0, k,l > 0} \left( u_{x^i z^j y^k} \frac{\partial}{\partial u_{x^i z^j y^k}} + u_{x^i z^j t^{l+1}} \frac{\partial}{\partial u_{x^i z^j y^k}} + D_x D_y D_t^{l-1} \left( u_{x^i z^j y^k} - u_y u_{x^i z^j y^k} \right) \frac{\partial}{\partial u_{x^i z^j t^l}} \right),
\end{align*}
\]
\[ D_z = \frac{\partial}{\partial z} + \sum_{i,j \geq 0, k, l > 0} \left( u_{x^iz^j} \frac{\partial}{\partial u_{x^iz^j}} + u_{x^iz^j+y^k} \frac{\partial}{\partial u_{x^iz^j+y^k}} + u_{x^iz^j+l} \frac{\partial}{\partial u_{x^iz^j+l}} \right) , \]
\[ D_t = \frac{\partial}{\partial t} + \sum_{i,j \geq 0, k, l > 0} \left( u_{x^iz^j+l} \frac{\partial}{\partial u_{x^iz^j+l}} + D_y^l D_y^{k-1} \left( u_{u_{x^iy} - u_y u_{x^iz^j}} \frac{\partial}{\partial u_{x^iz^j+y^k}} + u_{x^iz^j+l+1} \frac{\partial}{\partial u_{x^iz^j+l+1}} \right) \right) . \]

The Cartan distribution on \( E \) is spanned by these fields.

**The defining equations for local symmetries.** The linearization of Equation (1) has the form
\[ D_y D_t (\varphi) = u_{xy} D_z (\varphi) - u_{xz} D_y (\varphi) + u_x D_x D_y (\varphi) - u_y D_x D_z (\varphi), \] (2)
where \( \varphi \) is a function that depends on a finite number of internal coordinates. The vector field on \( E \) that corresponds to a solution \( \varphi \) is
\[ E_\varphi = \sum_{i,j \geq 0, k, l > 0} \left( D_y^l D_y^{k-1} \left( \varphi \frac{\partial}{\partial u_{x^iz^j+y^k}} + D_x D_z \left( \varphi \frac{\partial}{\partial u_{x^iz^j+l}} \right) \right) \right) , \] (3)
but we shall mainly deal with the generating functions \( \varphi \) rather than with the fields \( E_\varphi \) themselves.

Note that it can be easily shown that Equation (1) admits point symmetries only, i.e., solutions of (2) may depend only on the variables \( x, y, z, t, u, u_x, u_y, u_z, \) and \( u_t \).

**The \( \tau^+ \) - and \( \tau^- \) coverings** All our subsequent nonlocal constructions are based on the covering
\[ w_t = u_z w_x - \lambda^{-1} w_z, \quad w_y = \lambda u_y w_x, \] (4)
where \( 0 \neq \lambda \in \mathbb{R} \) and \( w \) is the nonlocal variable, see [11]. It is readily checked that the compatibility conditions for the overdetermined system (4) amount to Equation (1). We denote the covering (4) by \( \tau_\lambda \).

**Remark 1.** At first glance, the covering \( \tau_\lambda \) is one-dimensional. This is not the case, actually, because \( x \)- and \( z \)-derivatives of \( w \) are not defined in (4). To make the definition complete, we must introduce infinite number of nonlocal variables \( w_{\alpha, \beta}^0, \alpha, \beta = 0, 1, 2, \ldots, \) \( w_{\alpha, \beta}^0 = w \) and set
\[ w_{\alpha, \beta}^0 = w^{\alpha+1, \beta}, \quad w_{\alpha, \beta}^0 = w^{\alpha, \beta+1} \]
\[ w_{\alpha, \beta}^0 = (u_z w_x - \lambda^{-1} w_z)_x^{\alpha, \beta}, \quad w_{\alpha, \beta}^0 = (\lambda u_y w_x)_x^{\alpha, \beta}. \]
So, (4) defines an infinite-dimensional covering.

Assume now that \( w = w(\lambda) \) and consider the expansion \( w = \sum_{i \in \mathbb{Z}} \lambda^i w_i \). Substituting the latter into (4), we get
\[ w_{i,t} = u_z w_{i,x} - w_{i+1,z}, \quad w_{i,y} = u_y w_{i-1,x}, \quad i \in \mathbb{Z}. \] (5)
Thus, we obtain an infinite-dimensional covering over \( E \), but the problem is that this is ‘bad infinity’ which has ‘neither beginning nor end’. To overcome this inconvenience, we divide (5) in two parts assuming that \( w_i = 0 \) for \( i > 0 \) in one case and \( w_i = 0 \) for \( i < 0 \) in the other. In this way, we obtain two different coverings that we call the negative (\( \tau^- \)) and positive (\( \tau^+ \)) ones, respectively. After suitable relabellings, the defining equations for these coverings acquire the form
\[ \tau^- : E^- \to E \quad \begin{align*}
  r_0 &= y, \\
  r_{i,t} &= u_z u_y^{-1} r_{i-1,y} - r_{i-1,z}, \\
  r_{i,x} &= u_y^{-1} r_{i-1,y}, \quad i \geq 1,
\end{align*} \] (6)
and
\[ \tau^+ : E^+ \to E \quad \begin{align*}
  q_{-1} &= x, \quad q_0 = u, \\
  q_{i,y} &= u_y q_{i-1,x}, \\
  q_{i,z} &= u_z q_{i-1,x} - q_{i-1,t}, \quad i \geq 1.
\end{align*} \] (7)
and $q_1, q_2 \ldots$ are the nonlocal variables in $\tau^+$ and $r_1, r_2, \ldots$ are those in $\tau^-$. 

**Remark 2.** Strictly speaking, we must enrich (6) with infinite number of formal variables that would define $y$- and $z$-derivatives of $r$. In a similar way, additional variables that define $x$- and $t$-derivatives of $q$ are needed (cf. Remark 1). To be more precise, in $\tau^-$, we consider the variables $r_i^{\alpha, \beta}$, $\alpha, \beta = 0, 1, \ldots$, such that $r_i^{0,0} = r_i$ and

$$
\begin{align*}
\alpha^{t, y} &= r_i^{\alpha+1, \beta}, & \alpha^{t, z} &= r_i^{\alpha, \beta+1}, \\
\alpha^{t, \tau} &= (u_z u_y^{-1} r_{i-1, y} - r_{i-1, z}) y^\alpha z^\beta, & \alpha^{t, x} &= (u_y^{-1} r_{i-1, y}) y^\alpha z^\beta.
\end{align*}
$$

Similarly, we introduce $q_i^{\alpha, \beta}$ in $\tau^+$ and set $q_i^{0,0} = q_i$,

$$
\begin{align*}
\alpha^{x, y} &= q_i^{\alpha+1, \beta}, & \alpha^{x, t} &= q_i^{\alpha, \beta+1}, \\
\alpha^{x, \tau} &= (u_y q_{i-1, x}) x^\alpha t^\beta, & \alpha^{x, z} &= (u_z q_{i-1, x} - q_{i-1, t}) x^\alpha t^\beta.
\end{align*}
$$

But, as we shall see below, this formalization does not influence the subsequent computations.

**The defining equations for nonlocal symmetries.** Let us begin with writing down the total derivatives in the negative and positive coverings. In $\tau^-$, due to (6) and Remark 2 one has

$$
D_x^- = D_x + X^-, \quad D_y^- = D_y + Y^-, \quad D_z^- = D_z + Z^-, \quad D_t^- = D_t + T^-,
$$

where

$$
\begin{align*}
X^- &= D_x + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} (u_y^{-1} r_{i-1, y}) y^\alpha z^\beta \frac{\partial}{\partial r_i^{\alpha, \beta}}, \\
Y^- &= D_y + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} r_i^{\alpha+1, \beta} \frac{\partial}{\partial r_i^{\alpha, \beta}}, \\
Z^- &= D_z + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} r_i^{\alpha, \beta+1} \frac{\partial}{\partial r_i^{\alpha, \beta}}, \\
T^- &= D_t + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} (u_z u_y^{-1} r_{i-1, y} - r_{i-1, z}) \frac{\partial}{\partial q_i^{\alpha, \beta}}.
\end{align*}
$$

The total derivatives in $\tau^+$ are

$$
D_x^+ = D_x + X^+, \quad D_y^+ = D_y + Y^+, \quad D_z^+ = D_z + Z^+, \quad D_t^+ = D_t + T^+,
$$

where

$$
\begin{align*}
X^+ &= D_x + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} q_i^{\alpha+1, \beta} \frac{\partial}{\partial q_i^{\alpha, \beta}}, \\
Y^+ &= D_y + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} (u_y q_{i-1, x}) x^\alpha t^\beta \frac{\partial}{\partial q_i^{\alpha, \beta}}, \\
Z^+ &= D_z + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} (u_z q_{i-1, x} - q_{i-1, t}) x^\alpha t^\beta \frac{\partial}{\partial q_i^{\alpha, \beta}}, \\
T^+ &= D_t + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} q_i^{\alpha, \beta+1} \frac{\partial}{\partial q_i^{\alpha, \beta}}.
\end{align*}
$$

Finally, the total derivatives in the Whitney product $\tau^\pm = \tau^- \oplus \tau^+$ of $\tau^-$ and $\tau^+$ read

$$
D_x^\pm = D_x + X^- + X^+, \quad D_y^\pm = D_y + Y^- + Y^+, \quad D_z^\pm = D_z + Z^- + Z^+, \quad D_t^\pm = D_t + T^- + T^+
$$

and the lift of $\ell_\xi$ to $\tau^\pm$ will be denoted by $\ell_\xi^\pm$ with the obvious meaning of the notation.
To proceed, let us agree on notation. Denote by $E^\pm_\varphi$ the field on $\tau^\pm$ obtained from the field $E_\varphi$ presented in [3] by changing the total derivatives $D_\bullet$ to $D^\pm_\bullet$, where $\bullet$ denotes $x, y, z$ or $t$. We also obtain operators $\ell^\pm_\xi$ from $\ell_\xi$ in the same way.

In this notation, any $\tau^-$-nonlocal symmetry is of the form

$$ S = E^-_\varphi + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} \varphi^i_{\alpha, \beta} \frac{\partial}{\partial q^i_{\alpha, \beta}}, $$

where $\varphi, \varphi^i_{\alpha, \beta}$ are functions on $\tau^-$. Then $\varphi^i_{\alpha, \beta} = (D^-_\gamma)^\alpha (D^-_\zeta^\beta) (\varphi^i), \varphi^i = \varphi^i_{1, \beta}$, and

$$ \ell^-_\xi (\varphi) = 0, $$

$$ D^-_t (\varphi^i) = u_y^{-2} (u_y D^-_y (\varphi) - u_z D^-_y (\varphi)) r_{i-1, y} + u_z u_y^{-1} D^-_y (\varphi^{i-1}) - D^-_z (\varphi^{i-1}), $$

$$ D^-_z (\varphi^i) = -u_y^{-2} D^-_y (\varphi) r_{i-1, y} + u_z u_y^{-1} D^-_y (\varphi^{i-1}). $$

Hence, any such a symmetry $S = S_\Phi$ is completely determined by the vector-function $\Phi = (\varphi, \varphi^1, \ldots)$ and the formula $[S_\Phi, S_\Psi] = S_{\{\Phi, \Psi\}}$ defines a bracket on the space of these functions. Nonlocal shadows are just the functions $\varphi$ that satisfy the first of Equations [5], while invisible symmetries are $\Phi = (0, \varphi^1, \ldots)$ with

$$ D^-_t (\varphi^i) = u_z u_y^{-1} D^-_y (\varphi^{i-1}) - D^-_z (\varphi^{i-1}), $$

$$ D^-_z (\varphi^i) = u_y^{-1} D^-_y (\varphi^{i-1}). $$

(9)

Of course, the scheme is almost the same in $\tau^+$. Any symmetry is

$$ S = E^+_\varphi + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} \varphi^i_{\alpha, \beta} \frac{\partial}{\partial q^i_{\alpha, \beta}}, $$

where $\varphi, \varphi^i_{\alpha, \beta}$ are functions on $\tau^+$ and $\varphi^i_{\alpha, \beta} = (D^+_x)^\alpha (D^+_t^\beta) (\varphi^i), \varphi^i = \varphi^i_{1, \beta}$. The defining equations for $\varphi$ and $\varphi^i$ are

$$ \ell^+_\xi (\varphi) = 0, $$

$$ D^+_y (\varphi^i) = D^+_y (\varphi) q_{i-1, x} + u_y D^+_x (\varphi^{i-1}), $$

$$ D^+_z (\varphi^i) = D^+_z (\varphi) q_{i-1, x} + u_z D^+_x (\varphi^{i-1}) - D^+_t (\varphi^{i-1}). $$

(10)

As above, we introduce generating vector-functions $\Phi = (\varphi, \varphi^1, \ldots)$ and using the notation $S = S_\Phi$ define the bracket between these functions. Nonlocal shadows in the positive covering are identified with solutions of $\ell^+_\xi (\varphi) = 0$, while invisible symmetries $\Phi = (0, \varphi^1, \ldots)$, where $\varphi^i$ satisfy the system

$$ D^+_y (\varphi^i) = u_y D^+_x (\varphi^{i-1}), $$

$$ D^+_t (\varphi^i) = u_z D^+_x (\varphi^{i-1}) - D^+_t (\varphi^{i-1}). $$

(11)

Symmetries in the Whitney product are vector fields

$$ S = E^\pm_\varphi + \sum_{i=1}^{\infty} \sum_{\alpha, \beta=0}^{\infty} \left( D^\alpha_y D^\beta_z (\varphi^-) \frac{\partial}{\partial q^i_{\alpha, \beta}} + D^\alpha_y D^\beta_t (\varphi^+_{\alpha, \beta}) \frac{\partial}{\partial q^i_{\alpha, \beta}} \right), $$

where the functions $\varphi^\pm, \varphi^-_i, \varphi^+_i \in C^\infty (E^- \times \xi, E^+)$ enjoy the relations

$$ \ell^\pm_\xi (\varphi^\pm) = 0, $$

$$ D^\pm_t (\varphi^i) = u_y^{-2} (u_y D^\pm_y (\varphi) - u_z D^\pm_y (\varphi)) r_{i-1, y} + u_z u_y^{-1} D^\pm_y (\varphi^{i-1}) - D^\pm_z (\varphi^{i-1}), $$

$$ D^\pm_z (\varphi^i) = -u_y^{-2} D^\pm_y (\varphi) r_{i-1, y} + u_z u_y^{-1} D^\pm_y (\varphi^{i-1}), $$

(12)

$$ D^\pm_y (\varphi^i) = D^\pm_y (\varphi) q_{i-1, x} + u_y D^\pm_x (\varphi^{i-1}), $$

$$ D^\pm_x (\varphi^i) = D^\pm_x (\varphi) q_{i-1, x} + u_z D^\pm_x (\varphi^{i-1}) - D^\pm_t (\varphi^{i-1}). $$
Remark 3. A useful instrument in analysis of Lie algebra structures is the weights (gradings) that may be assigned to all the variables in the coverings under consideration and all polynomial functions in these variables. Namely, if we set the weights on independent variables to be
\[ x \mapsto |x|, \quad y \mapsto |y|, \quad z \mapsto |z|, \quad t \mapsto |t|, \]
then from Equations (11), (6) and (7) it follows that
\[ |u| = |x| + |z| - |t|, \quad |r_i| = |y| + i(|t| - |z|), \quad |q_i| = |x| + (i + 1)(|z| - |t|). \]
To a vector field \( A\partial/\partial a \) we assign the weight \(|A| - |a|\). Then for any two fields one has \(|[A, B]| = |A| + |B|\). Thus, Lie algebras spanned by homogeneous fields become graded.

So, we have four independent way to introduce weights reflects existence of four independent scaling symmetries in \( \text{sym} \mathcal{E} \) (see Section 3). Weights of differential polynomials are computed in an obvious way. In what follows, it will be convenient to use the following choice:
\[ |x| = -1, \quad |t| = |y| = |u| = 0, \quad |z| = 1, \]
and thus
\[ |r_i| = -i, \quad |q_i| = i. \]

To conclude the discussion of structures inherent to the equation under study, we mention the recursion operator found in [11]. The tangent equation corresponding to (1) is of the form
\[ D_y^i(\varphi) = u_y D_x^i(\varphi') - u_{xy} \varphi', \quad D_z^i(\varphi) = -D_t^i(\varphi') + u_B D_z(\varphi') - u_{xz} \varphi'. \]
If \( \varphi \) is a solution of Equation (2) then \( \varphi' \) also solves it and vice versa. The correspondence \( \varphi \mapsto \varphi' \) defined by relations (13) will be denoted by \( \underline{\mathcal{R}} \) and the opposite one by \( \overline{\mathcal{R}} \). The operator \( \underline{\mathcal{R}} \) changes the weight by +1, while \( \overline{\mathcal{R}} \) changes it by -1.

3. ALGEBRAS OF NONLOCAL SYMMETRIES

We accomplish the construction of the desired algebra in several steps that are:
- explicit computation of basic shadows and their lifts to \( \tau^- \), \( \tau^+ \), and \( \tau^\pm \) (Proposition 1);
- construction of hierarchies by means of commutators of the basic symmetries;
- construction of new hierarchies by somewhat artificial trick (Theorems 1 and 2);
- computation of the Lie algebra structure (Theorem 3).

Notation. In what follows, \( A = A(y, z) \) and \( B = B(x, t) \) are arbitrary smooth functions. Notation \( S_i^j \) for a symmetry indicates its weight \( i \) (and the position in a hierarchy), while the superscript \( j \) (if any) enumerates the hierarchies. If a symmetry contains a function \( A \), we compute its weight assuming \( A = y \); if it contains \( B \), the assumption is \( B = x \).

The coefficient of \( S_i^j \) at \( \partial/\partial u \) (the shadow) will be denoted by \( s_{i,j}^\alpha \), while its coefficients at \( \partial/\partial r_\alpha \) and \( \partial/\partial q_\alpha \) will be \( s_{i,k}^{j-} \) and \( s_{i,k}^{j+} \), respectively. Thus, any symmetry is presented by its generating vector-function
\[ S_i^j \sim \left[ s_{i,0}^j, s_{i,1}^{j-}, s_{i,1}^{j+}, \ldots, s_{i,0}^{j-}, s_{i,0}^{j+}, \ldots \right], \]
where \( s_{i,0}^{j-}, s_{i,0}^{j+}, s_{i,1}^{j-} \) are smooth functions on \( \mathcal{E}^- \otimes \mathcal{E}^+ \).

The basic shadows. The following shadows are found by direct computations:
\[ \psi_{1,0}^0 = -u_z, \quad \psi_{0,0}^0 = u_t, \quad \psi_{1,0}^0 = q_{1,t} - u_t u_x, \]
\[ 1 \text{For the convenience of the subsequent exposition, we present it a slightly different from [11] form, which is of course equivalent to the original one.} \]
\[ \omega_{2,0}^0 = u_y(2r_2 + zr_{2,z} - r_{1,y}(r_1 + zr_{1,z})), \quad \omega_{1,0}^0 = u_y(r_1 + zr_{1,z}), \quad \omega_{0,0}^0 = u - zu_z, \]
\[ \omega_{1,0}^0 = 2q_1 - uu_x + zu_t, \quad \omega_{2,0}^0 = 3q_2 - 2u_xq_1 - uq_{1,x} + zu^2 - zu_tu_x, \]
\[ \xi_{-1,0}(A) = u_y(Ar_{1,y} - Ayr_1 + Az_t), \quad \xi_{0,0}(A) = -Au_y, \quad \xi_{1,0}(A) = 0, \]
\[ v_{-1,0}(B) = B, \quad v_{0,0}(B) = -Bux + Bx, u - Bz, \]
\[ v_{1,0}(B) = B(u_x^2 - q_{1,x}) + Bz(q_1 - uu_x) + Bzu_x + \frac{1}{2}B_{xx}u^2 + \frac{1}{2}B_{zz}z^2 - B_{xz}zu. \]

**Proposition 1.** All the above listed shadows admit lifts to \( \tau^\pm \).

**Proof.** The lifts of the shadows \( \psi_{i,0}^j \) and \( \omega_{i,0}^j \) are described explicitly. Namely, we set

\[
\psi_{i,0}^{0,-} = -r_{\alpha,z}, \quad \psi_{i,0}^{0,+} = q_{\alpha,z}, \quad \psi_{i,0}^{0,-} = r_{\alpha,t}, \quad \psi_{i,0}^{0,+} = q_{\alpha,t}, \quad \psi_{i,0}^{0,-} = r_{\alpha-1,t} - ur_{\alpha,x}, \quad \psi_{i,0}^{0,+} = q_{\alpha+1,t} - utq_{\alpha,x},
\]
and

\[
\omega_{-1,0}^0 = -(\alpha + 2)r_{\alpha+2} - zr_{\alpha+2,z} + (r_1 + zr_{1,z})r_{\alpha+1,y} + (2r_2 + zr_{2,z} - (r_1 + zr_{1,z})r_{1,y})r_{\alpha,y},
\]
\[
\omega_{2,0}^0 = s_{\alpha-3,t} - zq_{\alpha-3} + (\alpha - 1)q_{\alpha-2} + u_y(r_1 + zr_{1,z})q_{\alpha-2} - x
\]
\[ + u_y(2r_2 + zr_{2,z} - (r_1 + zr_{1,z})r_{1,y})q_{\alpha-1,x}, \]
\[ \omega_{1,0}^0 = -(\alpha + 1)r_{\alpha+1} - zr_{\alpha+1,z} + (r_1 + zr_{1,z})r_{\alpha,y}, \]
\[ \omega_{1,0}^0 = zq_{\alpha-2} - zq_{\alpha-2,x} + \alpha q_{\alpha-1} + u_y(r_1 + zr_{1,z})q_{\alpha-1,x}, \]
\[ \omega_{0,0}^0 = -\alpha r_{\alpha} - zr_{\alpha,z}, \quad \omega_{0,0}^0 = (\alpha + 1)q_{\alpha} - zq_{\alpha,x}, \]
\[ \omega_{1,0}^0 = -(\alpha - 1)r_{\alpha-1} - zr_{\alpha-1,z} + (zr_{1,t} - ur_{1,x})r_{\alpha-1,y}, \]
\[ \omega_{1,0}^0 = (\alpha + 2)q_{\alpha+1} - uq_{\alpha+1,x} + zq_{\alpha+1,t}, \]
\[ \omega_{2,0}^0 = -(\alpha - 2)r_{\alpha-2} - zr_{\alpha-2,z} - (u - zu_z)r_{\alpha-1,x} - (2q_1 - uu_x + zu_t)r_{\alpha,x}, \]
\[ \omega_{2,0}^0 = +(\alpha + 3)q_{\alpha+2} - uq_{\alpha+1,x} + zq_{\alpha+1,t} - (2q_1 - uu_x + zu_t)q_{\alpha,x}. \]

To lift the shadows \( \xi_{i,0}(A) \) and \( v_{i,0}(B) \), let us introduce the operators

\[ y = -\frac{\partial}{\partial z} + \sum_{i=0}^{\infty} (i + 1)r_{i+1} \frac{\partial}{\partial r_{i+1}}, \quad \chi = -z \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} (i + 2)q_{i+1} \frac{\partial}{\partial q_{i+1}} \]
(recall that \( r_0 = y \) and \( q_0 = u \)) and the quantities \( P_{\alpha}(A) \), \( Q_{\alpha}(B) \), \( j = 0, 1, 2, \ldots \), defined by induction as follows:

\[ P_0(A) = A, \quad P_{\alpha}(A) = \frac{1}{\alpha}y(P_{\alpha-1}(A)), \quad Q_0(B) = B, \quad Q_{\alpha}(B) = \frac{1}{\alpha} \chi(Q_{\alpha-1}(B)), \quad \alpha \geq 1. \]

We also tacitly assume that \( P_{\alpha}(A) \) and \( Q_{\alpha}(B) \) vanish if \( \alpha \) is negative. Then

\[ \xi_{-1,0}^-(A) = Ar_{\alpha,y} - r_{\alpha+1,y} - Ayr_{\alpha,y} + Az_tr_{\alpha,y} + P_{\alpha+1}(A), \]
\[ \xi_{-1,0}^+(A) = u_y(Ar_{1,y}q_{\alpha-1,x} - q_{\alpha-2,x}) - Ayr_{1,y}q_{\alpha-1,x} + Az_tq_{\alpha-1,x}, \]
\[ \xi_{0,0}^-(A) = -Ar_{\alpha,y} + P_{\alpha}(A), \]
\[ \xi_{0,0}^+(A) = -Aq_{\alpha,y}, \]
\[ \xi_{1,0}^-(A) = P_{\alpha-1}(A), \]
\[ \xi_{1,0}^+(A) = 0. \]
Let \( j \in \{ \pm \} \) and \( \Omega \).

Theorem 1. These hierarchies will be used below to construct new ones.

Define the functions \( \Psi_0^0, \Psi_1^0, \Psi_{-1}^0, \Omega_{-2}^0, \Omega_{-1}^0, \Omega_0^0, \Omega_1^0, \Omega_2^0, \Xi_1(A), \Xi_0(A), \Xi_{-1}(A), \Upsilon_0(B), \Upsilon_1(B) \) and \( \Upsilon_{-1}(B) \) as shadows of symmetries. These shadows can be lifted to \( \tau^\pm \) which will serve as seeds for constructing the entire algebra of nonlocal symmetries.

Remark 4. It is worth to note that the operators \( \mathcal{X} \) and \( \mathcal{Y} \) used in the proof of Proposition 1 have a transparent geometrical interpretation. Namely, consider the system consisting of Equations (1) and (4) and let us treat the parameter \( \lambda \) as an additional independent variable with the condition \( u_\lambda = 0 \). Then the total derivative \( D\lambda \) transforms to \( \mathcal{X} \) when passing from the covering \( \lambda \) to \( \tau^+ \) and to \( \mathcal{Y} \) when passing to \( \tau^- \).

Construction of hierarchies 1. Now we use the symmetries \( \Omega_{i+1}^0 \) as hereditary ones and construct two infinite hierarchies

\[
\Psi_i^0 = \begin{cases} 
\frac{1}{i+1} \{ \Omega_{i-1}^0, \Psi_{i+1}^0 \}, & \text{if } i \leq -2, \\
\frac{1}{i-1} \{ \Omega_i^0, \Psi_{i-1}^0 \}, & \text{if } i \geq 2,
\end{cases}
\]

and

\[
\Omega_i^0 = \begin{cases} 
\frac{1}{i+2} \{ \Omega_{i-1}^0, \Omega_{i+1}^0 \}, & \text{if } i \leq -3, \\
\frac{1}{i-2} \{ \Omega_i^0, \Omega_{i-1}^0 \}, & \text{if } i \geq 3.
\end{cases}
\]

These hierarchies will be used below to construct new ones.

Construction of hierarchies 2. Define the functions

\[
\psi_{i,0}^j = \sum_{m=0}^{j} (-1)^m \binom{j}{m} t^{j-m} z^m \psi_{i-m,0}^0, \quad j \geq 1. \tag{14}
\]

Theorem 1. Formula (14) defines shadows of symmetries. These shadows can be lifted to \( \tau^\pm \) and thus define infinite number of hierarchies \( \{ \Psi_i^j \} \) of nonlocal symmetries.

Proof. The proof is accomplished in two steps: first we establish that \( \psi_{i,0}^j \) are shadows and then show that they can be lifted.

Step 1. Induction on \( j \). To this end, let us rewrite (14) recursively. Namely, we write

\[
\psi_{i,0}^j = t \psi_{i,0}^{j-1} - z \psi_{i-1,0}^{j-1}, \quad i \geq 1. \tag{15}
\]

Let \( j = 1 \). Consider the linearization operator lifted to \( \tau^\pm \)

\[
\ell^\pm_E = D_y^\pm D_t^\pm - u_{xy} D_x^\pm + u_{xz} D_y^\pm - u_z D_x^\pm D_y^\pm + u_y D_x^\pm D_z^\pm. \tag{16}
\]
Then due to (16) for \( j = 1 \) one obviously has
\[
\ell^\pm_E(\psi^j_{i,0}) = \ell^\pm_E(t\psi^0_{i,0} - z\psi^0_{i-1,0}) =
\]
\[
t\ell^\pm_E(\psi^0_{i,0}) - z\ell^\pm_E(\psi^0_{i-1,0}) + D^\pm_y(\psi^0_{i,0}) + u_x y \psi^0_{i-1,0} - u_y D^\pm_x(\psi^0_{i-1,0}) =
\]
\[
D^\pm_y(\psi^0_{i,0}) + u_x y \psi^0_{i-1,0} - u_y D^\pm_x(\psi^0_{i-1,0}),
\]
since \( \psi^0_{i,0} \) and \( \psi^0_{i-1,0} \) are shadows. But the last term in the equalities above is exactly the first equation in the formula (13) for the recursion operator. It can be checked that this operator, modulo the image of zero (see discussion in the end of the paper) connects the shadows \( \psi^0_{i,0} \) and \( \psi^0_{i-1,0} \). In particular,
\[
D^\pm_y(\psi^0_{i,0}) + u_x y \psi^0_{i-1,0} - u_y D^\pm_x(\psi^0_{i-1,0}) = 0, \tag{17}
\]
because all \( \psi^0_{\alpha,0} \) are shadows. Moreover, since (17) does not contain the total derivatives in \( z \) and \( t \), we deduce, using (15), that
\[
D^\pm_y(\psi^1_{i,0}) + u_x y \psi^1_{i-1,0} - u_y D^\pm_x(\psi^1_{i-1,0}) = 0.
\]

Let now \( j > 1 \) and assume that for all \( l < j \) and \( i \in \mathbb{Z} \) the functions \( \psi^j_{i,0} \) are shadows that enjoy the relations \( D^\pm_y(\psi^j_{i,0}) + u_x y \psi^j_{i-1,0} - u_y D^\pm_x(\psi^j_{i-1,0}) = 0 \). Then the proof of the induction step is exactly the same as the one for the case \( j = 1 \).

**Step 2.** We shall now prove that the functions
\[
\psi^j_{i,\alpha} = t\psi^0_{i,\alpha} - z\psi^0_{i-1,\alpha} \tag{18}
\]
satisfy System (12) for all \( i \in \mathbb{Z}, j \geq 0, \alpha \geq 1 \). We also use induction on \( j \) here.

Consider the case \( j = 1 \). Substituting the expression \( \psi^1_{i,\alpha} = t\psi^0_{i,\alpha} - z\psi^0_{i-1,\alpha} \) to the defining equations (12), we obtain for \( \tau^- \)
\[
D^-_t(t\psi^0_{i,1} - z\psi^0_{i-1,1}) = \frac{1}{u_y^2} \left( u_y D^-_z(t\psi^0_{i,0} - z\psi^0_{i-1,0}) - u_x D^-_y(t\psi^0_{i,0} - z\psi^0_{i-1,0}) \right)
\]
in the case \( \alpha = 1 \) and
\[
D^-_t(t\psi^0_{i,\alpha} - z\psi^0_{i-1,\alpha}) = \frac{1}{u_y^2} \left( u_y D^-_z(t\psi^0_{i,0} - z\psi^0_{i-1,0}) - u_x D^-_y(t\psi^0_{i,0} - z\psi^0_{i-1,0}) \right)
\]
\[
+ \frac{1}{u_y} \left( (u_y D^-_z(t\psi^0_{i,0} - z\psi^0_{i-1,0}) - u_x D^-_y(t\psi^0_{i,0} - z\psi^0_{i-1,0})) \right),
\]
when \( \alpha > 1 \). But the functions \( \psi^0_{i,\alpha} \) are the components of the nonlocal symmetries \( \Psi^0_t \) and hence we obtain the conditions
\[
\psi^0_{i,1} = -\frac{1}{u_y} \psi^0_{i-1,0}, \quad \psi^0_{i,\alpha} = -\frac{1}{u_y} \psi^0_{i-1,0} + \psi^0_{i-1,\alpha-1}, \quad \alpha > 1. \tag{19}
\]
from the above equations.

Similar computations show that the conditions
\[
\psi^0_{i-1,1} = u_x \psi^0_{i-1,0}, \quad \psi^0_{i-1,\alpha} = q_{\alpha-1,x} \psi^0_{i-1,0} + \psi^0_{i-1,\alpha-1}, \quad \alpha > 1. \tag{20}
\]
must hold in \( \tau^+ \).

**Lemma 1.** Conditions (19) and (20) do hold for all \( \alpha > 1 \) and \( i \in \mathbb{Z} \).

**Proof of Lemma 1.** The proof comprises two inductions on \( i \) (for \( i \geq 0 \) and \( i \leq 0 \)) and consists of voluminous computations based on explicit descriptions from Proposition 1 and on the definition of the symmetries \( \Psi^0_t \). We omit the details. \( \square \)

Note now that the functions \( \psi^1_{i,\alpha} = t\psi^0_{i,\alpha} - z\psi^0_{i-1,\alpha} \) satisfy the conditions similar to (19) and (20) by linearity. This finishes the proof of the induction base. The proof of the induction step does not differ from the latter. \( \square \)
In a similar way, we define the functions
\[ \omega_{j,0}^i = \sum_{m=0}^{j} (-1)^m \binom{j}{m} t^{j-m} z^m \omega_{i-m,0}, \quad j \geq 1, \] (21)
and prove the following

**Theorem 2.** Formula (21) defines shadows of symmetries. These shadows can be lifted to \( \tau^\pm \) and thus define infinite number of hierarchies \( \{\Omega_j^i\} \) of nonlocal symmetries.

The proof almost exactly copies the one of Theorem 1. \( \square \)

**Remark 5.** As it follows from Theorems 1 and 2, the hierarchies \( \{\Psi_j^i\}, \{\Omega_j^i\}, i \in \mathbb{Z}, j \geq 0, \) exist in the Whitney product \( \tau^\pm \), but this result may be clarified. More detailed information on the \( \Psi \)-hierarchies is presented in Table 1. Note that the symmetries \( \Psi_{-1}^0, \Psi_0^0, \) and \( \Psi_1^0 \) are local.

| \( \Psi_j^i \) | \( j < i + 2 \) | \( j \geq i + 2 \) |
|-----------------|-----------------|-----------------|
| \( i \leq 0 \) | in \( \tau^-, \tau^+, \tau^\pm \) | in \( \tau^- , \tau^\pm \) |
| \( i > 0 \) | in \( \tau^+, \tau^\pm \) | in \( \tau^\pm \) only |

Table 1. Distribution of \( \Psi_j^i \) over \( \tau^-, \tau^+, \) and \( \tau^\pm \)

| \( \Omega_j^i \) | \( j < i + 1 \) | \( j \geq i + 1 \) |
|-----------------|-----------------|-----------------|
| \( i \leq 0 \) | in \( \tau^- , \tau^+ , \tau^\pm \) | in \( \tau^- , \tau^\pm \) |
| \( i > 0 \) | in \( \tau^+ , \tau^\pm \) | in \( \tau^\pm \) only |

Table 2. Distribution of \( \Omega_j^i \) over \( \tau^- , \tau^+ , \) and \( \tau^\pm \)

**Construction of hierarchies 3.** The last step is the construction of the \( (x,t) \)- and \( (y,x) \)-dependent hierarchies. To this end, we set
\[ \Xi_i(A) = \begin{cases} \frac{1}{i+1} \{ \Omega_{i-1}^0 + \Psi_{i-1}^1, \Xi_{i+1}(A) \}, & \text{if } i \leq -2, \\ \frac{1}{i-1} \{ \Omega_{i}^1 + \Psi_{i-1}^1, \Xi_{i-1}(A) \}, & \text{if } i \geq 2, \end{cases} \]
and
\[ \Upsilon_i(B) = \begin{cases} \frac{1}{i+1} \{ \Omega_{i-1}^0, \Upsilon_{i+1}(B) \}, & \text{if } i \leq -2, \\ \frac{1}{i-1} \{ \Omega_{i}^0, \Upsilon_{i-1}(B) \} & \text{if } i \geq 2 \end{cases} \]
(recall that \( A = A(y,z) \) and \( B = B(x,t) \) are arbitrary smooth functions).

**Remark 6.** As above, the structure of these hierarchies may be clarified in some respects. Namely, we have the following facts:
\[ \Xi_i(A) \text{ is a symmetry in } \begin{cases} \tau^- , \tau^+, \tau^\pm, & \text{if } i \leq -1, \\ \tau^- , \tau^+, \tau^\pm, & \text{if } i = 0, \\ \tau^\pm, & \text{if } i \geq 1. \end{cases} \]
Moreover, the symmetry \( \Xi_0(A) \) is local, while \( \Xi_i(A) \) are invisible symmetries for \( i \geq 1 \).
In a similar way,

\[ \Upsilon_i(B) \text{ is a symmetry in } \begin{cases} \tau^\pm, & \text{if } i \leq -2, \\ \tau^-, \tau^+, \tau^\pm, & \text{if } i = -1, 0, \\ \tau^+ \tau^\pm, & \text{if } i \geq 1. \end{cases} \]

The symmetries \( \Upsilon_i(B) \) are invisible for all \( i \leq -2 \) and the symmetries \( \Upsilon_{-1}(B), \Upsilon_0(B) \) are local ones.

**Lie algebra structure.** Let us now describe the structure of the Lie algebra formed by the above constructed symmetries. To this end, relabel some of them to make the results look neater. Namely, we change notation as follows:

\[ \Psi_i^j \mapsto -\Psi_i^{j+1}, \quad \Xi_i(A) \mapsto \Xi_i(A \cdot z^{-i}). \]

Then we have the following result:

**Theorem 3.** The Lie algebra \( g = \text{sym}_{\mathbb{R},+}(\mathcal{E}) \) of the \( \tau^\pm \)-nonlocal symmetries for the 4D MASH equation as an \( \mathbb{R} \)-vector space is generated by the elements

\[
\{ \{ \Psi_i^j \}_{i \in \mathbb{Z}}, \{ \Omega_i^j \}_{j \geq 0}, \{ \Upsilon_i(B) \}_{i \in \mathbb{Z}}, \{ \Xi_i(A) \}_{i \in \mathbb{Z}}, \}
\]

where \( B = B(x,t) \) and \( A = A(y,z) \) are arbitrary smooth functions. They enjoy the commutator relations presented in Table 3.

\[
\begin{array}{|c|c|c|c|}
\hline
\Psi_i^j & \Omega_i^j & \Upsilon_i(B) & \Xi_i(A) \\
\hline
(l-j)\Psi_{i+k}^{j+1} & i\Omega_{i+k}^{j+1} & \Upsilon_{i+k}(t^{j+1}Bt) & (-1)^j\Xi_{k+i-j}(z^{i+1}A_z-kz^iA) \\
(k-i)\Omega_{i+k}^{j+1} & \Upsilon_{i+k}(k^jB) & (-1)^j\Xi_{k+i-j}(z^{i+1}A_z) & 0 \\
\Upsilon_i(B) & \Xi_{i+k}(B,B) & \Xi_{i+k}(A,A) \\
\Xi_i(A) & & & \\
\hline
\end{array}
\]

Here the notation

\[ [A,\bar{A}] = AA_y - A\bar{A}_y, \quad [B,\bar{B}] = BB_x - B\bar{B}_x \]

was used.

**Proof.** The proof is omitted due to its extreme length. It consists of a number of inductions with explicit computations in the bases of these inductions. \( \square \)

**Remark 7.** Denote by \( \mathfrak{h} \subset g \) the subalgebra spanned by the elements \( \Psi_i^j, \Omega_i^j, \) and \( \Upsilon_i(B), \) and let \( i(A) \subset g \) denote the ideal \( \{ \Xi_i(A) \} \). Then \( g \) is the semi-direct product \( \mathfrak{h} \ltimes i(A) \). In its turn, \( \mathfrak{h} = \mathfrak{h}_0 \ltimes i(B), \) where

\[ \mathfrak{h}_0 = \{ \Psi_i^j, \Omega_i^j \}, \quad i(B) = \{ \Upsilon_i(B) \}. \]

The structure of \( \mathfrak{h} \) is quite clear. Consider the correspondence

\[ \Psi_i^j \mapsto t^{j+1}z^i \frac{\partial}{\partial t}, \quad j \geq 1 \quad \Omega_i^j \mapsto t^j z^{j+1} \frac{\partial}{\partial z}, \quad j \geq 0 \quad \Upsilon_i(B) \mapsto z^{i}B \frac{\partial}{\partial y}, \quad i \in \mathbb{Z}. \]

Then we obtain an isomorphism between \( \mathfrak{h} \) and the Lie algebra of the corresponding vector fields. The action of \( \mathfrak{h} \) on \( i(A) \) is less conventional (see the last column of Table 3).

**Action of the recursion operator.** Let us now describe the action of the recursion operator (12) on the shadows our symmetries. First of all note that

\[ \bar{R}(0) = \xi_{0,0}(A), \quad \bar{R}(0) = \nu_{0,0}(B), \]
and thus the action is defined modulo the images of zero. Keeping this in mind we have

\[ ... \psi_{0,-2} \quad \psi_{0,-1} \quad \psi_{0,0} \quad \psi_{0,1} \quad \psi_{0,2} \quad ... \]

\[ ... \omega_{0,-2} \quad \omega_{0,-1} \quad \omega_{0,0} \quad \omega_{0,1} \quad \omega_{0,2} \quad ... \]

\[ ... \nu_{1,0}(B) \quad \nu_{0,0}(B) \quad 0 \quad \xi_{0,0}(A) \quad \xi_{1,0}(A) \quad ... \]

**Remark 8.** To conclude, recall that in [12] an infinite series of pair-wise commuting nonlocal symmetries was presented. The algebra sym\(_{\pm}(E)\) described above contains infinite number of such hierarchies. Namely, for any \(i \in \mathbb{Z}\) and \(j \in \mathbb{N}\) each of the families

\[ \Psi^j = \{\Psi^j_i\}_{i \in \mathbb{Z}} \text{ and } \Omega^i = \{\Omega^j_i\}_{j \geq 0} \]

consists of pair-wise commuting symmetries. In addition, if we fix the functions \(A(y,z)\) and \(B(x,t)\) then the families

\[ \Xi(A) = \{\Xi_i(A)\}_{i \in \mathbb{Z}} \text{ and } \Upsilon(B) = \{\Upsilon_i(B)\}_{i \in \mathbb{Z}} \]

will possess the same property.

**Acknowledgments**

Computations were supported by the Jets software, [2]. The work of I.K. was partially supported by Russian Foundation for Basic Research Grant 18-29-10013 and Simons-IUM Fellowship Grant 2020.

**References**

[1] H. Baran, I.S. Krasil’shchik, O.I. Morozov, P. Vojčák, *Nonlocal symmetries of integrable linearly degenerate equations: a comparative study*. Theoret. and Math. Phys. **196** (2018) 1089–1110 [arXiv:1902.08341v3](http://jets.math.slu.cz)

[2] H. Baran, M. Marvan, *Jets. A software for differential calculus on jet spaces and diffeoties*. Theoret. and Math. Phys. **196** (2018) 1089–1110 [arXiv:1902.08341v3](http://jets.math.slu.cz)

[3] A.V. Bocharov et al., *Symmetries of Differential Equations in Mathematical Physics and Natural Sciences*, edited by A.M. Vinogradov and I.S. Krasil’shchik). Factorial Publ. House, 1997 (in Russian). English translation: Amer. Math. Soc., 1999.

[4] B. Doubrov, E. Ferapontov, B. Kruglikov, V. Novikov, *Integrable systems in 4D associated with sextoids in Gr(4,6)*, International Math. Research Notices **21** (2019), 6585–6613 [arXiv:1705.06999](http://jets.math.slu.cz)

[5] I.S. Krasil’shchik, O.I. Morozov, P. Vojčák, *Nonlocal symmetries, conservation laws, and recursion operators of the Veronese web equation*. J. of Geom. and Phys. **146** (2019), 103519. [arXiv:1902.09341v3](http://jets.math.slu.cz)

[6] I. Krasil’chikh, A. Verbovetsky, R. Vitolo, *The symbolic computation of integrability structures for partial differential equations*. Texts & Monographs in Symbolic, Springer, 2017.

[7] I.S. Krasil’shchik, A.M. Vinogradov, *Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations*, in: Symmetries of Partial Differential Equations, Part I, Acta Appl. Math. **15** (1-2) (1989) 161–209.

[8] L. Martínez Alonso, A.B. Shabat, *Energy-dependent potentials revisited: a universal hierarchy of hydrodynamic type*. Physics Letters A **300**, Issue 1, 58–64. [arXiv:nltn/0202008v1](http://jets.math.slu.cz)

[9] L. Martínez Alonso, A.B. Shabat, *Hydrodynamic reductions and solutions of a universal hierarchy*. Theoret. Math. Phys., **104** (2004), 1073–1085 [arXiv:nltn/0312043](http://jets.math.slu.cz)

[10] M. Marvan, *Another look on recursion operators*, in: Differential Geometry and Applications, Proc. Conf. Brno, 1995 (Masaryk University, Brno, 1996) 393–402.

[11] O.I. Morozov, *The four-dimensional Martínez Alonso-Shabat equation: differential coverings and recursion operators*. J. Geom. Phys. (2014), 10.1016/j.geomphys.2014.05.022. [arXiv:1309.4993](http://jets.math.slu.cz)

[12] O.I. Morozov, A. Sergyeyev, *The four-dimensional Martínez Alonso-shabat equation: reductions and nonlocal symmetries*. J. of Geom. and Phys. **85** (2014), 40–45. [arXiv:1401.7942v2](http://jets.math.slu.cz)
