The $\text{AdS}_5 \times S^5$ superstrings
in the generalized light-cone gauge

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Abstract

The $\kappa$-symmetry-fixed Green-Schwarz action in the $\text{AdS}_5 \times S^5$ background is treated canonically in a version of the light-cone gauge. After reviewing the generalized light-cone gauge for a bosonic sigma model, we present the Hamiltonian dynamics of the Green-Schwarz action by using the transverse degrees of freedom. The remaining fermionic constraints are all second class, which we treat by the Dirac bracket. Upon quantization, all of the transverse coordinates are inevitably non-commutative.

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1 Introduction

Since the proposal of AdS/CFT correspondence, it has become an important issue to quantize the type IIB Green-Schwarz superstrings [1, 2] in the $AdS_5 \times S^5$ background [3]. One of the difficulties in quantizing the Green-Schwarz superstrings stems from the existence of the local $\kappa$ symmetry, which halves the fermionic degrees of freedom. In the canonical Hamiltonian formalism, the local $\kappa$ symmetry yields fermionic constraints. The half of these are first-class and the remaining half are second-class constraints. Covariant separation of the first and the second class constraints is a difficult task [1, 2, 4].

In the flat Minkowski target space, there was an attempt to quantize the action covariantly by introducing an infinite number of ghosts (see for example [5]). Other direction for covariant quantization is to add extra degrees of freedom in order to replace the second-class constraints with the first-class ones [6, 7, 8, 9, 10, 11, 12].

A less ambitious way to quantize the Green-Schwarz action is to abandon the covariance and to go to a non-covariant gauge. In flat target space, the Green-Schwarz action in the light-cone gauge becomes extremely simple [1, 2]. Light-cone quantization of quantum field theory was first recognised and developed in connection with the current algebra in the infinite momentum frame [13, 14, 15]. Light-cone quantization of (super)-strings played important roles in the development of string theory in seventies [16, 17, 18, 19] and that of superstring theory in eighties [1, 2]. Various gauges for the $AdS_5 \times S^5$ superstrings have been proposed [20, 21, 22, 23, 24, 25].

Recently, the Hofman-Maldacena limit [26] has attracted much attention [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. It is a limit which takes the energy $E$ and one of the angular momenta $J$ infinite while keeping $E - J$ finite. $E$ and $J$ are eigenvalues of Cartan generators of $SO(2, 4)$ and $SO(6)$ group respectively. In the Hofman-Maldacena limit, both the string and the dual gauge theory describe excitations called giant magnons and their generalizations. Good agreement in some physical quantities is found.

One way to take the Hofman-Maldacena limit is to employ a version of light-cone gauge in which $E - J$ appears as the light-cone energy. A sizable amount of literature has been accumulated which are devoted to the discussion of this gauge [53, 54, 55, 56, 25, 57, 49]. This gauge is sometimes referred to as the uniform light-cone gauge and is a generalization of that of [17] in the flat Minkowski space to the AdS background. The light-cone direction $X^\pm$ is chosen such that $X^\pm = (1/\sqrt{2})(t \pm \varphi)$, where $t$ is the global time direction of $AdS_5$ and $\varphi$ is a certain angle of $S^5$. The vectors $\partial/\partial X^\pm$ are Killing vectors of the
target space geometry. The transverse direction manifestly keeps the covariance under a $SO(4) \times SO(4)$ subgroup of the local Lorentz group $SO(1, 4) \times SO(5)$ [57]. The treatment of the fermionic second class constraints remains to be investigated however. In order to treat these remaining constraints, it is necessary to introduce the Dirac bracket.

In this paper, we study the $AdS_5 \times S^5$ superstring in the generalized light-cone gauge as a constrained Hamiltonian system. In section 2, we review the generalized light-cone gauge for the bosonic sigma models, emphasizing the central object, the light-cone Hamiltonian. In section 3, we study the case of the Green-Schwarz superstring. The fermionic second-class constraints lead to the highly non-trivial Dirac bracket among the transverse degrees of freedom. If $i\hbar$ times the Dirac bracket is replaced with the graded commutator, all of the transverse coordinates are inevitably non-commutative. Since the Dirac brackets are not $c$-number, several subtleties such as operator ordering remain to be investigated. In the Appendix, we give some details on the induced vielbeins.

2 Bosonic sigma models in the generalized light-cone gauge

2.1 Bosonic sigma model

In order to explain the generalized light-cone gauge, let us consider the following bosonic sigma model:

$$S = \frac{1}{2\pi} \int d^2 \xi \mathcal{L},$$

(2.1)

where the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2} \sqrt{-h} h^{ij} G_{mn}(X) \partial_i X^m \partial_j X^n.$$  

(2.2)

We assume that the target space is $D$-dimensional: $X^m = X^m(\xi)$, $(m = 0, 1, \ldots, D - 1)$, and $G_{mn}(X)$ is the metric of the target space. Here

$$(\xi^0, \xi^1) = (\tau, \sigma), \quad h^{ij} = \sqrt{-g} g^{ij}, \quad i, j = 0, 1,$$

(2.3)

and $\lambda$ is the coupling constant. $h^{ij}$ is the Weyl-invariant combination of the world-sheet metric $g_{ij}$. Since $\det h^{ij} = -1$, we choose $h^{00}$ and $h^{01}$ as the independent Lagrange multipliers. We also use the following notation for the Lagrange multipliers: $e^0 = 1/(2h^{00})$, $e^1 = (h^{01}/h^{00})$. 

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Equations of motion for this model are given by
\[
\partial_i (h^{ij} G_{m,n} \partial_j X^n) = \frac{1}{2} h^{ij} (\partial_m G_{k,l}) \partial_i X^k \partial_j X^l.
\] (2.4)

Here \( \partial_m = \partial / \partial X^m \).

Let us introduce the conjugate momenta by \( P_m := \partial L / \partial \dot{X}^m \).

\[
P_m = -\sqrt{\lambda} G_{m,n} h^{00} \partial_i X^n.
\] (2.5)

Let \( G_{m,n} \) be the inverse of the metric \( G_{m,n} \). The equations of motion (2.4) and the definition of the conjugate momenta (2.5) can be converted into the equations of motion in the first order form:

\[
\dot{X}^m = -\frac{1}{\sqrt{\lambda} h^{00}} G_{m,n} P_n - \left( \frac{h^{01}}{h^{00}} \right) \partial_1 X^m,
\]
\[
\dot{P}_m = \partial_1 \left[ -\left( \frac{h^{01}}{h^{00}} \right) P_m - \sqrt{\lambda} G_{m,n} \partial_1 X^n \right]
\]
\[
+ \frac{\sqrt{\lambda}}{2h^{00}} \left[ \frac{1}{\lambda} (\partial_m G_{k,l}) P_k P_l + (\partial_m G_{k,l}) \partial_1 X^k \partial_1 X^l \right].
\] (2.6)

The Hamiltonian density is given by
\[
\mathcal{H} = P_m \dot{X}^m - \mathcal{L} = -e^0 \Phi_0 - e^1 \Phi_1,
\] (2.7)

where
\[
\Phi_0 := \frac{1}{\sqrt{\lambda}} G_{m,n} P_m P_n + \sqrt{\lambda} G_{m,n} \partial_1 X^m \partial_1 X^n, \quad \Phi_1 := P_m \partial_1 X^m.
\] (2.8)

The Virasoro constraints are given by \( \Phi_0 \approx 0, \Phi_1 \approx 0 \). The Hamiltonian density vanishes weakly: \( \mathcal{H} \approx 0 \). Using (2.6), we can check that the Virasoro constraints are consistent with the time evolution
\[
\dot{\Phi}_0 = -2(\partial_1 e^1) \Phi_0 - 8(\partial_1 e^0) \Phi_1 - \partial_1 \Phi_0 - 2\partial_1 \Phi_1,
\]
\[
\dot{\Phi}_1 = -2(\partial_1 e^0) \Phi_0 - 2(\partial_1 e^1) \Phi_1 - e^0 \partial_1 \Phi_0 - e^1 \partial_1 \Phi_1.
\] (2.9)

There is no secondary constraint.

### 2.2 Generalized light-cone gauge

Let us decompose the target space index \( m \) into \( m = (a, m), a = \pm, m = 1, 2, \ldots, D - 2 \).

We assume that the target space metric takes the form
\[
G_{m,n} dX^m dX^n = G_{ab} dX^a dX^b + G_{mn} dX^m dX^n,
\] (2.10)
and $\partial/\partial X^\pm$ are Killing vectors.

We first recall the procedure of the light-cone gauge fixing in the flat target space. In this case, the world-sheet diffeomorphism is fixed by setting the world-sheet metric conformally flat. The residual symmetry is used to set $X^+ = \kappa \tau$.

But in a certain curved target space such as AdS space-time, there is an obstacle in making the world-sheet metric be conformally flat and obey the light-cone gauge condition \[25\]. Instead, in the generalized light-cone approach, the world-sheet diffeomorphism is fixed by imposing the following two conditions

\[ X^+ = \kappa \tau, \quad \dot{P}_- = 0. \] (2.11)

Equation of motion (2.6) for $X^+$

\[ \dot{X}^+ = \kappa = -\frac{1}{\sqrt{\lambda \kappa}} (G^{++} P_+ + G^{+-} P_-) \] (2.12)

determines the Lagrange multiplier $h^{00}$ as

\[ h^{00} = -\frac{1}{\sqrt{\lambda \kappa}} (G^{++} P_+ + G^{+-} P_-). \] (2.13)

Equation of motion for $P_-$

\[ 0 = \dot{P}_- = \partial_1 \left[ -\left( \frac{h^{01}}{h^{00}} \right) P_- - \frac{\sqrt{\lambda}}{h^{00}} G_{--} \partial_1 X^- \right] \] (2.14)

fixes $h^{01}$ up to an arbitrary function of $\tau$:

\[ h^{01} = -\frac{\sqrt{\lambda}}{P_-} G_{--} \partial_1 X^- - \frac{f(\tau)}{P_-} h^{00}. \] (2.15)

The function $f(\tau)$ arises from the residual symmetry. The residual symmetry is fixed by setting $f(\tau) = 0$. Solving the Virasoro constraint $\Phi_1 = 0$ gives the relation $\partial_1 X^- = -(1/P_-) P_m \partial_1 X^m$.

Therefore, the worldsheet metric is fixed as

\[ h^{00} = -\frac{1}{\sqrt{\lambda \kappa}} (G^{++} P_+ + G^{+-} P_-), \quad h^{01} = \frac{\sqrt{\lambda}}{P_-^2} G_{--} P_m \partial_1 X^m. \] (2.16)

The Virasoro constraint $\Phi_0 = 0$ gives a quadratic equation for $P_+$:

\[ G^{++} P_+^2 + 2P_- G^{+-} P_+ + P_-^2 G^{--} + G^{mn} P_m P_n P_+ + \frac{\lambda}{P_-^2} G_{--} (P_m \partial_1 X^m)^2 + \lambda G_{mn} \partial_1 X^m \partial_1 X^n = 0. \] (2.17)

\[ ^1\text{Here for simplicity we consider the sector with vanishing winding number.} \]
The equations of motion for the dynamical variables in the reduced phase space are given by

\[
\dot{X}^m = -\frac{\sqrt{\lambda}}{\hbar^{00}} \left[ \frac{1}{\lambda} G^{mn} p_n + \frac{G_{--}(p_n \partial_1 X^n) \partial_1 X^m}{p_2} \right],
\]

\[
\dot{P}_m = \partial_1 \left[ -\left( \frac{h^{01}}{h^{00}} \right) P_m - \frac{\sqrt{\lambda}}{h^{00}} G_{mn} \partial_1 X^n \right] + \frac{1}{2\sqrt{\lambda} h^{00}} \left[ (\partial_m G^{++}) P_+^2 + 2(\partial_m G^{+-}) P_+ P_- + (\partial_m G^{--}) P_-^2 + (\partial_m G^{kl}) P_k P_l \right.
\]
\[
\left. + \frac{\lambda}{P_2^2} (\partial_m G_{--}) (P_n \partial_1 X^n)^2 + \lambda (\partial_m G_{kl}) \partial_1 X^k \partial_1 X^l \right].
\]

Using the Poisson bracket \(\{X^m(\tau, \sigma), P_n(\tau, \sigma')\}_\text{P.B.} = 2\pi \delta_n^m \delta(\sigma - \sigma')\), the equations of motion can be rewritten as

\[
\dot{X}^m(\tau, \sigma) = \{X^m(\tau, \sigma), H_{\text{LC}}\}_\text{P.B.}, \quad \dot{P}_m(\tau, \sigma) = \{P_m(\tau, \sigma), H_{\text{LC}}\}_\text{P.B.}.
\]

The light cone Hamiltonian is found to be\(^2\)

\[
H_{\text{LC}} := -\frac{\kappa}{2\pi} \int_{-\pi}^{\pi} d\sigma P_+.
\]

Here \(P_+\) is a solution to the quadratic equation (2.17).

In the Hamilton formalism, the light-cone Hamiltonian \(H_{\text{LC}}\) can be understood by using a canonical transformation \((X^m, P_\mu) \rightarrow (\tilde{X}^m, \tilde{P}_\mu)\)

\[
\tilde{X}^+ = X^+ - \kappa \tau, \quad \tilde{X}^- = X^-, \quad \tilde{X}^m = X^m, \quad \tilde{P}_\mu = P_\mu.
\]

whose generating functional is given by

\[
W(X, \tilde{P}, \tau) = \int \frac{d\sigma}{2\pi} \left[ (X^+(\tau, \sigma) - \kappa \tau) \tilde{P}_+(\tau, \sigma) + X^-(\tau, \sigma) \tilde{P}_-(\tau, \sigma) + X^m(\tau, \sigma) \tilde{P}_m(\tau, \sigma) \right].
\]

The transformed Hamiltonian is given by

\[
\tilde{H} = H + \frac{\partial W}{\partial \tau} = H_{\text{LC}}.
\]

\(^2\) In addition we must examine the open string and the closed string boundary conditions and the level-matching condition. We will not dwell upon these in this paper.
2.3 $AdS_5 \times S^5$ case

The bosonic part of the Green-Schwarz model for the $AdS_5 \times S^5$ background is a special case of the sigma model (2.2). The coordinates for $D = 10$-dimensional target space is chosen as

$$X^m = (X^+, X^-, X^a, X^{4+s}), \quad X^a = z^a, \; a = 1, 2, 3, 4, \quad X^{4+s} = y^s, \; s = 1, 2, 3, 4,$$

(2.24)

and the $AdS_5 \times S^5$ metric is given by

$$G_{mn}(X) dX^m dX^n = G_{ab} dX^a dX^b + G_z \sum_{a=1}^{4} (dz^a)^2 + G_y \sum_{s=1}^{4} (dy^s)^2,$$

(2.25)

where

$$G_{++} = G_{--} = -\frac{1}{2} \left( 1 + \frac{z^2}{4} \right) \frac{1}{\left( 1 - \frac{z^2}{4} \right)} + \frac{1}{2} \left( \frac{1 - y^2}{1 + y^2} \right) \frac{1}{\left( 1 + \frac{y^2}{4} \right)},$$

(2.26)

$$G_{+-} = G_{-+} = -\frac{1}{2} \left( 1 + \frac{z^2}{4} \right) - \frac{1}{2} \left( \frac{1 - y^2}{1 + y^2} \right) \frac{1}{\left( 1 + \frac{y^2}{4} \right)},$$

(2.27)

$$G_z = \frac{1}{(1 - \frac{z^2}{4})^2}, \quad G_y = \frac{1}{(1 + \frac{y^2}{4})^2}.$$

(2.28)

Here

$$z^2 = \sum_{a=1}^{4} (z^a)^2, \quad y^2 = \sum_{s=1}^{4} (y^s)^2.$$

(2.29)

The coupling constant $\lambda$ is related to the radius $R$ of the $AdS_5$ and $S^5$ as follows: $\sqrt{\lambda} = R^2 / \alpha'$. In the generalized light-cone gauge, $P_+$ is determined by the following equation

$$G^{++} P_+^2 + 2BP_+ + C = 0,$$

(2.30)

where $B = G^{+-} P_-$,

$$C = G^{--} P_-^2 + \frac{1}{G_z} \sum_{a=1}^{4} P_a^2 + \frac{1}{G_y} \sum_{s=1}^{4} P_{4+s}^2$$

$$+ \frac{\lambda}{P_-} G_{--} (P_a \partial_1 z^a + P_{4+s} \partial_1 y^s)^2 + \lambda G_z \sum_{a=1}^{4} (\partial_1 z^a)^2 + \lambda G_y \sum_{s=1}^{4} (\partial_1 y^s)^2.$$

(2.31)

For $AdS_5 \times S^5$, we can take the flat Minkowski limit $R \to \infty$. In this case,

$$G^{++} = 0 + O(R^{-2}), \quad G^{+-} = -1 + O(R^{-2}).$$

(2.32)
Therefore, in order to have a finite Minkowski limit, the sign for $P_+$ must be chosen as

$$P_+ = \frac{1}{G^{++}}(-B + \epsilon_B \sqrt{B^2 - G^{++} C}),$$

where $\epsilon_B$ is 1 for $B > 0$ and $-1$ for $B < 0$.

3 The $AdS_5 \times S^5$ Green-Schwarz superstring in the generalized light-cone gauge

3.1 The Green-Schwarz action in the $AdS_5 \times S^5$ background

The Green-Schwarz superstring in the flat target space was proposed in [1, 2]. Generalization to the action for the curved supergravity background was done in [58].

More explicit Green-Schwarz action in the $AdS_5 \times S^5$ background was constructed in [3] based on the coset superspace $PSU(2,2|4)/(SO(1,4) \times SO(5))$. (See also [59, 60]). Originally, the Wess-Zumino term is written in the three-dimensional form. The manifestly two-dimensional form of the Wess-Zumino term was presented in [61, 62].

The Green-Schwarz action for the $AdS_5 \times S^5$ is given by

$$S_{GS} = \frac{1}{2\pi} \int d^2 \xi L_{GS},$$

$$L_{GS} = -\frac{1}{2} \sqrt{h} h^{ij} \eta_{ab} E_i^a E_j^b + \sqrt{h} e^{ij} \left( E_i^a \eta_{ab} E_j^b - E_i^a \eta_{ab} E_j^b \right).$$

Here $E_i^A$ is the induced vielbein for the type IIB superspace:

$$E_i^A = E^A_M \partial_i Z^M = E_i^a \partial_i X^a + E_i^\theta \partial_i \theta^\alpha + \bar{E}_i^\bar{\alpha} \partial_i \bar{\theta}^{\bar{\alpha}}.$$ (3.3)

The local Lorentz index $A = (a, \alpha, \bar{\alpha})$ take values in the following way: $a = (a, a, 4 + s)$, $a = \pm, a = 1, 2, 3, 4, s = 1, 2, 3, 4, \alpha = 1, 2, \ldots, 16$ and $\bar{\alpha} = \bar{1}, \bar{2}, \ldots, \bar{16}$. We use the 16-component notation for Weyl spinors. The constant matrix $\eta$ in the Wess-Zumino term is given by

$$C \Gamma^{01234} = \begin{pmatrix} \eta_\alpha^\beta & 0 \\ 0 & \eta^{\bar{\alpha}}_\bar{\beta} \end{pmatrix}, \quad \Gamma^a = \begin{pmatrix} 0 & (\gamma^a)^\alpha_\beta \\ (\gamma^a)^\bar{\alpha}_{\bar{\beta}} & 0 \end{pmatrix}.$$ (3.4)

It is related to the existence of the self-dual Ramond-Ramond 5-form flux.

In the large radius limit, (3.3) goes to the Lagrangian in the flat Minkowski space up to (divergent) surface terms.

\[ ^{3}\text{The surface terms purely come from the Wess-Zumino term.} \]
Let us decompose each of the two 16-component Weyl spinors into two 8-component \(SO(4) \times SO(4)\) spinors:

\[
\theta^\alpha = \begin{pmatrix} \theta^{+\alpha} \\ \theta^{-\dot{\alpha}} \end{pmatrix}, \quad \bar{\theta}^{\dot{\alpha}} = \begin{pmatrix} \bar{\theta}^{+\dot{\alpha}} \\ \bar{\theta}^{-\alpha} \end{pmatrix},
\]

(3.5)

where \(\alpha = 1, 2, \ldots, 8, \dot{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{8}\), \(\bar{\alpha} = \bar{1}, \bar{2}, \ldots, \bar{8}\) and \(\dot{\bar{\alpha}} = \hat{\bar{1}}, \hat{\bar{2}}, \ldots, \hat{\bar{8}}\).

We first fix the \(\kappa\)-symmetry by setting \(\theta^{-\dot{\alpha}} = \bar{\theta}^{-\alpha} = 0\). In the 32-component notation, these conditions are equivalent to the condition \(\Gamma^+ \Theta = 0\). In the large radius limit, it directly goes to the \(\kappa\)-symmetry fixing condition for the flat Minkowski target space.

To simplify expressions, we combine the remaining fermionic coordinates into \(\Psi^{\hat{\alpha}}\):

\[
(\Psi^{\hat{\alpha}}) = \begin{pmatrix} \theta^{+\alpha} \\ \bar{\theta}^{+\dot{\alpha}} \end{pmatrix}, \quad \hat{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{16}.
\]

(3.6)

The coordinates for the reduced type IIB superspace is given by \(Z^M = (X^m, \Psi^{\hat{\alpha}}) = (X^+, X^-, X^m, \theta^{+\alpha}, \bar{\theta}^{\dot{\alpha}})\). We further decompose \(X^m = (X^a, X^4) = (z^a, y^a)\) and choose a representative of the coset superspace as follows

\[
G(Z) = \exp \left( X^+ \hat{P}_+ + X^- \hat{P}_- \right) \exp \left( \theta^{+\alpha} \hat{Q}_\alpha^+ + \bar{\theta}^{\dot{\alpha}} \hat{Q}_{\dot{\alpha}}^+ \right) g_z g_y.
\]

(3.7)

Here \(\hat{P}_\pm\), \(\hat{Q}^+_{\alpha}\) and \(\hat{Q}^+_{\dot{\alpha}}\) belong to \(psu(2,2|4)\) generators. The vielbeins \(E^A_M\) can be read from the Cartan one-form \(G^{-1}dG\). See Appendix for details.

The \(\kappa\)-symmetry fixed action for \(AdS_5 \times S^5\) can be written as

\[
\mathcal{L}_{GS} = -\frac{1}{2} \sqrt{h_{ij}} G_{m,n}(X) D_i X^m D_j X^n + \frac{1}{2} \sqrt{\epsilon_{ij}} B_{\hat{\alpha} \hat{\beta}} D_i \Psi^{\hat{\alpha}} D_j \Psi^{\hat{\beta}}.
\]

(3.8)

The target space metric \(G_{m,n}\) is the same as the bosonic one \((2.25)\), \(B_{\hat{\alpha} \hat{\beta}} = B_{\hat{\alpha} \hat{\beta}}(Z)\), and \(\Lambda\)'s are introduced through

\[
D_i X^+ = \partial_i X^+, \\
D_i X^- = \partial_i X^- + \Lambda^-_{\hat{\alpha}} \partial_i \Psi^{\hat{\alpha}}, \\
D_i X^m = \partial_i X^m + (\Lambda^m_{n\hat{\alpha}} \partial_i \Psi^{\hat{\alpha}}) X^n, \\
D_i \Psi^{\hat{\alpha}} = \partial_i \Psi^{\hat{\alpha}} + (\Lambda^{\hat{\alpha}}_{\hat{\beta}} \partial_i X^+) \Psi^{\hat{\beta}}.
\]

(3.9)

Here \(\Lambda^-_{\hat{\alpha}}\) and \(\Lambda^m_{n\hat{\alpha}}\) depend only on fermionic variables \(\Psi^{\hat{\gamma}}\) and \(\Lambda^{\hat{\alpha}}_{\hat{\beta}}\) is a constant. See Appendix for details.
The conjugate momenta are given by
\[ P_+ = -\sqrt{\lambda} h^{i\alpha} G_{+,a} D_i X^a + \mathcal{P}_a \Lambda_{\bar{\alpha} \beta} \Psi^\beta, \]
\[ P_- = -\sqrt{\lambda} h^{i\alpha} G_{-,a} D_i X^a, \]
\[ P_m = -\sqrt{\lambda} h^{i\alpha} G_{mn} D_i X^n, \]
\[ P_\alpha = -\sqrt{\lambda} B_{\bar{\alpha} \beta} D_1 \Psi^\beta + P_\Lambda - \bar{\alpha} + P_m \Lambda_{m n} \bar{\alpha} X^n. \] (3.10)

We have fermionic primary constraints:
\[ \Phi_\alpha = P_\alpha + \sqrt{\lambda} B_{\bar{\alpha} \beta} D_1 \Psi^\beta - P_\Lambda - \bar{\alpha} - P_m \Lambda_{m n} \bar{\alpha} X^n \approx 0. \] (3.11)

The Hamiltonian density is given by
\[ \mathcal{H} = P_m \dot{X}^m + P_\alpha \dot{\Psi}^\alpha - \mathcal{L} = -e^0 \Phi_0 - e^1 \Phi_1, \] (3.12)
where
\[ \Phi_0 = \frac{1}{\sqrt{\lambda}} G^{ab} \Pi_a \Pi_b + \sqrt{\lambda} G_{ab} D_1 X^a D_1 X^b + \frac{1}{\sqrt{\lambda}} G^{mn} P_m P_n + \sqrt{\lambda} G_{mn} D_1 X^m D_1 X^n, \]
\[ \Phi_1 = \Pi_a D_1 X^a + P_m D_1 X^m. \] (3.13)

Here \( \Pi_+ := P_+ - P_\alpha \Lambda_{\bar{\alpha} \beta} \Psi^\beta \) and \( \Pi_- := P_- \).

Since the action (3.8) is a singular system, it is necessary to introduce fermionic Lagrange multipliers \( \chi^{\hat{\alpha}} \) for (3.11). The Hamilton form of the equations of motion are given by
\[ \dot{Z}^M (\tau, \sigma) = \{ Z^M (\tau, \sigma), H \}_{\text{P.B.}} + \frac{1}{2\pi} \int d\sigma' \{ Z^M (\tau, \sigma), \Phi_\alpha (\tau, \sigma') \}_{\text{P.B.}} \chi^{\hat{\alpha}} (\tau, \sigma'), \] (3.15)
\[ \dot{P}_M (\tau, \sigma) = \{ P_M (\tau, \sigma), H \}_{\text{P.B.}} + \frac{1}{2\pi} \int d\sigma' \{ P_M (\tau, \sigma), \Phi_\alpha (\tau, \sigma') \}_{\text{P.B.}} \chi^{\hat{\alpha}} (\tau, \sigma'), \] (3.16)
where
\[ H = \frac{1}{2\pi} \int d\sigma \mathcal{H}. \] (3.17)

The singularity of the action comes from the fact that \( \dot{\Psi}^{\hat{\alpha}} \) or equivalently \( D_0 \Psi^{\hat{\alpha}} \) does not appear in (3.10); \( \dot{\Psi}^{\hat{\alpha}} \) can not be expressed by the phase space variables. The equations of motion (3.15) for \( Z^M = \Psi^{\hat{\alpha}} \) can be rewritten as \( D_0 \Psi^{\hat{\alpha}} = \chi^{\hat{\alpha}} \). Therefore, the introduction of the fermionic Lagrange multipliers \( \chi^{\hat{\alpha}} \) is eventually equivalent to converting \( D_0 \Psi^{\hat{\alpha}} \) into \( \chi^{\hat{\alpha}} \).
3.2 Generalized light-cone gauge

We first reduce the phase space from \( \Gamma = \{(\mathbf{x}^m, \mathbf{p}_m, \psi^\alpha, \mathbf{p}_\alpha)\} \) to \( \Gamma^* = \{(\mathbf{x}^m, \mathbf{p}_m, \psi^\alpha, \mathbf{p}_\alpha)\} \) by taking the generalized light-cone gauge and by solving the Virasoro constraints \( \Phi_0 = 0 \) and \( \Phi_1 = 0 \).

Let us take the generalized light-cone gauge:

\[
X^+ = \kappa \tau, \quad \hat{P}_- = 0.
\] (3.18)

The Virasoro constraint \( \Phi_1 = 0 \) is solved by setting

\[
D_1 X^+ = -\frac{1}{P_-} G^{++} P_+ + G^{+-} P_- + G_{mn} D_1 X^m D_1 X^n = 0,
\] (3.20)

which gives a solution

\[
P_+ = P_\alpha \Lambda^\alpha \hat{\psi}^\beta + \Pi_+^{(sol)}.
\] (3.21)

Here

\[
\Pi_+^{(sol)} = \frac{1}{G^{++}} \left(-B + \epsilon_B \sqrt{B^2 - G^{++} \tilde{C}}\right),
\] (3.22)

with \( B = G^{+-} P_- \), \( \epsilon_B = \text{sign}(B) \),

\[
\tilde{C} = G^{--} P_-^2 + \frac{1}{G_z} \sum_{a=1}^4 P_a^2 + \frac{1}{G_y} \sum_{s=1}^4 P_{4+s}^2 + \frac{\lambda}{P_-^2} G_{--} (P_d D_1 z^a + P_{4+s} D_1 y^s)^2 + \lambda G_{zz} \sum_{a=1}^4 (D_1 z^a)^2 + \lambda G_{yy} \sum_{s=1}^4 (D_1 y^s)^2.
\] (3.23)

The time evolution for the reduced phase variables is given by

\[
\hat{F}(\tau, \sigma) = \{F(\tau, \sigma), H_{LC}\}^*_{P, B} + \frac{1}{2\pi} \int d\sigma' \{F(\tau, \sigma), \Phi_\alpha(\tau, \sigma')\}^*_{P, B} \chi^{\hat{\alpha}}(\tau, \sigma').
\] (3.24)

Here \( \{F, G\}^*_{P, B} \) is the Poisson bracket in the reduced phase space \( \Gamma^* \). The light-cone Hamiltonian is given by

\[
H_{LC} = -\frac{\kappa}{2\pi} \int_{-\pi}^\pi d\sigma P_+,
\] (3.25)
and $\Phi_\dot{\alpha}$ is the fermionic constraints in the reduced phase space:

$$\Phi_\dot{\alpha} = P_\dot{\alpha} + \sqrt{\lambda} B_{\dot{\alpha}\dot{\beta}} \partial_\tau \bar{\Psi}^{\dot{\beta}} - P_- \Lambda^{-\dot{\alpha}} - P_m \Lambda^m_{n\dot{\alpha}} X^n. \quad (3.26)$$

As in the bosonic case, the light-cone Hamiltonian can be understood by canonical transformation. It can be also explained by using the first order form of the action:

$$S = \frac{1}{2\pi} \int d^2 \xi \left( P_+ \dot{X}^+ + P_- \dot{X}^- + P_m \dot{X}^m + P_\dot{\alpha} \dot{\Psi}^{\dot{\alpha}} - \mathcal{H} - \Phi_\dot{\alpha} \bar{\chi}^{\dot{\alpha}} \right). \quad (3.27)$$

By taking the generalized light-cone gauge and by substituting the solutions of the Virasoro constraints into the action, we have

$$S = \frac{1}{2\pi} \int d^2 \xi \left( P_m \dot{X}^m + P_\dot{\alpha} \dot{\Psi}^{\dot{\alpha}} - \mathcal{H}_{LC} - \Phi_\dot{\alpha} \bar{\chi}^{\dot{\alpha}} \right). \quad (3.28)$$

Here $\mathcal{H}_{LC} = -\kappa P_+$ and we have dropped the total $\tau$-derivative term $P_- \dot{X}^-$. We can see that the remaining fermionic constraints $\Phi_\dot{\alpha} \approx 0$ are second class:

$$\{ \Phi_\dot{\alpha}(\tau, \sigma), \Phi_\dot{\beta}(\tau, \sigma') \}^*_{P.B.} = -2\pi C_{\dot{\alpha}\dot{\beta}}(\tau, \sigma) \delta(\sigma - \sigma'), \quad (3.29)$$

where

$$C_{\dot{\alpha}\dot{\beta}} = P_- (\partial \Lambda^{-\dot{\alpha}}/\partial \bar{\Psi}^{\dot{\beta}}) + P_- (\partial \Lambda^{-\dot{\beta}}/\partial \bar{\Psi}^{\dot{\alpha}})$$

$$- P_m X^n \left( \Lambda^m_{\dot{\alpha}\dot{\beta}} \Lambda^k_{n\dot{\gamma}} + \Lambda^m_{\dot{\beta}\dot{\gamma}} \Lambda^k_{n\dot{\alpha}} - (\partial \Lambda^m_{n\dot{\alpha}}/\partial \bar{\Psi}^{\dot{\beta}}) - (\partial \Lambda^m_{n\dot{\beta}}/\partial \bar{\Psi}^{\dot{\alpha}}) \right) + \sqrt{\lambda} (\Lambda_1 B_{\dot{\alpha}\dot{\beta}})$$

$$- \sqrt{\lambda} \left[ (\partial_\tau B_{\dot{\alpha}\dot{\gamma}}) \Lambda^m_{\dot{\beta}\dot{\gamma}} X^k + (\partial_\tau B_{\dot{\beta}\dot{\gamma}}) \Lambda^m_{\dot{\alpha}\dot{\gamma}} X^k - (\partial B_{\dot{\alpha}\dot{\gamma}}/\partial \bar{\Psi}^{\dot{\beta}}) - (\partial B_{\dot{\beta}\dot{\gamma}}/\partial \bar{\Psi}^{\dot{\alpha}}) \right] \partial_\tau \bar{\Psi}^{\dot{\gamma}}. \quad (3.30)$$

We assume that $\mathcal{C}$ is invertible. For $AdS_5 \times S^5$, this is indeed the case since the terms in the first line of the above equation start with an invertible matrix:

$$P_- (\partial \Lambda^{-\dot{\alpha}}/\partial \bar{\Psi}^{\dot{\beta}}) + P_- (\partial \Lambda^{-\dot{\beta}}/\partial \bar{\Psi}^{\dot{\alpha}}) = 2\sqrt{2} i P_- (\gamma^+)_{\dot{\alpha}\dot{\beta}} + \mathcal{O}(\bar{\Psi}^2), \quad (3.31)$$

$$\gamma^+_{\dot{\alpha}\dot{\beta}} = \left( \begin{array}{cc} 0 & (\gamma^+)_{\dot{\alpha}\dot{\beta}} \\ (\gamma^+)_{\dot{\alpha}\dot{\beta}} & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1_8 \\ 1_8 & 0 \end{array} \right). \quad (3.32)$$

The consistency of the time evolution of the fermionic constraints ($\dot{\Phi}_\dot{\alpha} = 0$) determines the fermionic Lagrange multipliers as follows:

$$\chi^{\dot{\alpha}}(\tau, \sigma) = (\mathcal{C}^{-1}(\tau, \sigma))^{\dot{\alpha}\dot{\beta}} \chi^{\dot{\beta}}(\tau, \sigma). \quad (3.33)$$

Here $\chi^{\dot{\beta}}(\tau, \sigma) = \{ \Phi_\dot{\beta}(\tau, \sigma), H_{LC} \}_{P.B.}$. Since the explicit form of $\chi^{\dot{\beta}}$ is rather lengthy and is not necessary here, we do not write it in this paper.
The (equal \( \tau \)) Dirac bracket is given by

\[
\{ F, G \}_{\text{D.B.}} = \{ F, G \}_{\text{P.B.}}^* + \frac{1}{2\pi} \int \text{d}\sigma \{ F, \Phi_\alpha(\tau, \sigma) \}_{\text{P.B.}}^* \left( C^{-1}(\tau, \sigma) \right)^{\hat{\alpha}\hat{\beta}} \{ \Phi_{\hat{\beta}}(\tau, \sigma), G \}_{\text{P.B.}}^*.
\] (3.33)

Using the Dirac bracket we can choose \((X^m, P_m, \Psi_\alpha)\) as dynamical variables and \(P_\alpha\) can be treated as the solution of the fermionic constraints:

\[
P_\alpha = -\sqrt{\lambda} B_{\alpha\hat{\beta}} \partial_1 \Psi^{\hat{\beta}} + P_\Lambda \Lambda^m_{\hat{\alpha}m} X^n.
\] (3.34)

The time evolution of the dynamical variables are now given by

\[
\dot{F} = \{ F, H_{\text{LC}} \}_{\text{D.B.}}.
\] (3.35)

Let

\[
U^m_{\hat{\alpha}} = \Lambda^m_{\hat{\alpha}m} X^n, \quad V_{m\hat{\alpha}} = \sqrt{\lambda} (\partial_m B_{\alpha\hat{\beta}}) \partial_1 \Psi^{\hat{\beta}} - P_n \Lambda^m_{n\hat{\alpha}}.
\] (3.36)

The Dirac bracket among the dynamical variables are given by

\[
\{ X^m(\tau, \sigma), X^n(\tau, \sigma') \}_{\text{D.B.}} = -2\pi U^m_{\hat{\alpha}} (C^{-1})^{\hat{\alpha}\hat{\beta}} U^n_{\hat{\beta}} \delta(\sigma - \sigma'),
\]

\[
\{ X^m(\tau, \sigma), P_n(\tau, \sigma') \}_{\text{D.B.}} = 2\pi \left( \delta_m - U^m_{\hat{\alpha}} (C^{-1})^{\hat{\alpha}\hat{\beta}} V_{n\hat{\beta}} \right) \delta(\sigma - \sigma'),
\]

\[
\{ X^m(\tau, \sigma), \Psi_{\hat{\alpha}}(\tau, \sigma') \}_{\text{D.B.}} = -2\pi U^m_{\hat{\beta}} (C^{-1})^{\hat{\alpha}\hat{\beta}} V_{n\hat{\beta}} \delta(\sigma - \sigma'),
\]

\[
\{ P_m(\tau, \sigma), P_n(\tau, \sigma') \}_{\text{D.B.}} = -2\pi V_{m\hat{\alpha}} (C^{-1})^{\hat{\alpha}\hat{\beta}} V_{n\hat{\beta}} \delta(\sigma - \sigma'),
\]

\[
\{ P_m(\tau, \sigma), \Psi_{\hat{\alpha}}(\tau, \sigma') \}_{\text{D.B.}} = -2\pi V_{m\hat{\alpha}} (C^{-1})^{\hat{\alpha}\hat{\beta}} \delta(\sigma - \sigma'),
\]

\[
\{ \Psi_{\hat{\alpha}}(\tau, \sigma), \Psi^{\hat{\beta}}(\tau, \sigma') \}_{\text{D.B.}} = 2\pi (C^{-1})^{\hat{\alpha}\hat{\beta}} \delta(\sigma - \sigma').
\] (3.37)

The quantization of these transverse degrees of freedom is then a straightforward task: to replace \( i\hbar \) times the Dirac bracket by the graded commutator. Because of the fermionic constraints, all corresponding quantum operators become non-commutative.

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A Details on the induced vielbein

The $\text{psu}(2,2|4)$ generators are given by

$$\hat{P}_a = (\hat{P}_a, \hat{P}_{a'}), \quad \hat{J}_{ab} = -\hat{J}_{b\bar{a}}, \quad \hat{J}_{a'b'} = -\hat{J}_{b'a'}, \quad \hat{Q}_\alpha, \quad \hat{\bar{Q}}_{\bar{\alpha}}, \quad \text{(A.1)}$$

where $a = 0, 1, 2, \ldots, 9$, $\dot{a}, \dot{b} = 0, 1, 2, 3, 4$, $a', b' = 5, 6, 7, 8, 9$, $\alpha = 1, 2, \ldots, 16$, $\bar{\alpha} = \bar{1}, \bar{2}, \ldots, \bar{16}$. The bosonic generators are chosen to be anti-Hermitian and $(\hat{Q}_\alpha)^\dagger = \hat{\bar{Q}}_{\bar{\alpha}}$. The non-zero commutation relations are given by

$$[\hat{P}_a, \hat{P}_b] = \hat{J}_{ab}, \quad [\hat{P}_a, \hat{J}_{\dot{b}\dot{c}}] = \eta_{\dot{a}\dot{b}} \hat{P}_{\dot{c}} - \eta_{\dot{a}\dot{c}} \hat{P}_{\dot{b}},$$

$$[\hat{P}_{a'}, \hat{J}_{b'}] = \eta_{a'b'} \hat{P}_{c'} - \eta_{a'c'} \hat{P}_{b'}, \quad [\hat{P}_{a'}, \hat{J}_{b'c'}] = \delta_{a'b'} \hat{P}_{c'} - \delta_{a'c'} \hat{P}_{b'},$$

$$[\hat{J}_{ab}, \hat{J}_{\dot{a}\dot{b}}] = \eta_{\dot{a}\dot{b}} \hat{J}_{a\dot{b}} + 3 \text{ terms}, \quad [\hat{J}_{a'b'}, \hat{J}_{\dot{b}'\dot{c}'}] = \delta_{a'b'} \hat{J}_{\dot{b}'\dot{c}'} + 3 \text{ terms},$$

$$[\hat{Q}_\alpha, \hat{P}_a] = \frac{i}{2} (\gamma_\alpha^a_\beta) \hat{Q}_\beta, \quad [\hat{Q}_\alpha, \hat{P}_{a'}] = -\frac{i}{2} (\gamma_\alpha^a_{\bar{\beta}}) \hat{\bar{Q}}_{\bar{\beta}},$$

$$[\hat{Q}_\alpha, \hat{J}_{\dot{a}\dot{b}}] = \frac{1}{2} (\gamma_{a'\beta}) \hat{Q}_\beta, \quad [\hat{\bar{Q}}_{\bar{\alpha}}, \hat{J}_{\dot{a}\dot{b}}] = \frac{1}{2} (\gamma_{\bar{a}\bar{\beta}}) \hat{\bar{Q}}_{\bar{\beta}},$$

$$\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\bar{\beta}}\} = -2i(\gamma_\alpha^a_{\beta}) \hat{P}_a + (\gamma_{\bar{a}\bar{\beta}}) \hat{\bar{Q}}_{\bar{\beta}} - (\gamma_{a'\beta}) \hat{P}_{a'}. \quad \text{(A.8)}$$

Here $\eta_{\dot{a}\dot{b}} = \text{diag}(-, +, +, +, +)$. We define

$$\hat{P}_\pm = \frac{1}{\sqrt{2}} (\hat{P}_0 \pm \hat{P}_9), \quad \gamma_{\pm} = \frac{1}{2} (\gamma_0 \pm \gamma_9). \quad \text{(A.9)}$$

$$(\gamma_+)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\gamma_-)_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{(A.10)}$$

In our notation,

$$(\gamma^a)_{\Delta\bar{\beta}} = \begin{pmatrix} 0 & (\gamma^a)_{\alpha\beta} \\ (\gamma^a)_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}, \quad (\gamma^{a+s})_{\Delta\bar{\beta}} = \begin{pmatrix} 0 & (\gamma^{a+s})_{\alpha\beta} \\ (\gamma^{a+s})_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}, \quad \text{(A.11)}$$

for $a = 1, 2, 3, 4$ and $s = 1, 2, 3, 4$.

If we decompose the fermionic generators into $\hat{Q}_\alpha = (\hat{Q}_\alpha^+, \hat{Q}_\alpha^-)$, $\hat{\bar{Q}}_{\bar{\alpha}} = (\hat{\bar{Q}}_{\bar{\alpha}}^+, \hat{\bar{Q}}_{\bar{\alpha}}^-)$, some of commutation relations can be rewritten as follows:

$$[\hat{Q}_\alpha^+, \hat{P}_+] = \frac{i}{\sqrt{2}} (\gamma_+^a)_{\alpha\beta} \hat{Q}_\beta^+, \quad [\hat{Q}_\alpha^-, \hat{P}_+] = 0, \quad \text{(A.12)}$$

$$[\hat{\bar{Q}}_{\bar{\alpha}}^+, \hat{P}_-] = 0, \quad [\hat{\bar{Q}}_{\bar{\alpha}}^-, \hat{P}_-] = \frac{i}{\sqrt{2}} (\gamma_-^a)_{\alpha\beta} \hat{\bar{Q}}_{\bar{\beta}}^-, \quad \text{(A.13)}$$
\{\hat{Q}^+_a, \hat{Q}^+_\dot{b}\} = 2\sqrt{2}i(\gamma_+)^{a\dot{b}} \hat{P}_- + (\gamma^{ab})^{a\dot{b}} \hat{J}_{ab} - (\gamma^{a'b'})^{a\dot{b}} \hat{J}_{a'b'}.

(A.14)

Here \(a, b = 1, 2, 3, 4, a', b' = 5, 6, 7, 8\).

Using the coset representative (3.7) with

\[ g_z = \exp \left( \chi^a \hat{P}_a \right), \quad \chi^a = \frac{z^a}{z} \log \left( \frac{1 + (1/2)z}{1 - (1/2)z} \right), \quad (A.15) \]

\[ g_y = \exp \left( \chi^{4+s} \hat{P}_{4+s} \right), \quad \chi^{4+s} = -i \frac{y^a}{y} \log \left( \frac{1 + (1/2)y}{1 - (1/2)y} \right), \quad (A.16) \]

we can calculate the vielbeins for the reduced type IIB superspace as follows:

\[ G^{-1}dG = E^a \hat{P}_a + E^\dot{b} \hat{Q}^\dot{b} + \bar{E}^{\dot{a}} \hat{Q}^{\dot{a}} + \text{(spin connection part)}. \quad (A.17) \]

Let us define a 16 \times 16 matrix \( \mathcal{M}^2 \) by

\[ \mathcal{M}^2 = \begin{pmatrix} (\mathcal{M}^2)^{\alpha \beta} & (\mathcal{M}^2)^{\dot{a} \dot{b}} \\ (\mathcal{M}^2)^{\dot{a} \dot{b}} & (\mathcal{M}^2)^{\beta \alpha} \end{pmatrix}, \quad (A.18) \]

where the matrix elements are defined by

\[ \text{ad}^2(\theta^+ \hat{Q}^+ + \bar{\theta}^+ \hat{Q}^+) = \hat{Q}^+_\beta (\mathcal{M}^2)^{\beta \alpha} + \hat{Q}^+_\dot{b} (\mathcal{M}^2)^{\dot{b} \dot{a}}, \quad (A.19) \]

\[ \text{ad}^2(\theta^+ \hat{Q}^+ + \bar{\theta}^+ \hat{Q}^+) = \hat{Q}^+_\beta (\mathcal{M}^2)^{\beta \dot{a}} + \hat{Q}^+_\dot{b} (\mathcal{M}^2)^{\dot{b} \beta}. \quad (A.20) \]

Explicit form of the matrix elements are given by

\[ (\mathcal{M}^2)^{\alpha \beta} = \frac{1}{2}(\theta^+ \gamma_{ab})^{\alpha}(\bar{\theta}^+ \gamma^{ab} \theta)_\beta - \frac{1}{2}(\theta^+ \gamma_{a'b'})^{\alpha}(\bar{\theta}^+ \gamma^{a'b'} \theta)_\beta, \]

\[ (\mathcal{M}^2)^{\dot{a} \dot{b}} = -\frac{1}{2}(\theta^+ \gamma_{ab})^{\dot{a}}(\bar{\theta}^+ \gamma^{ab} \theta)_\dot{b} + \frac{1}{2}(\theta^+ \gamma_{a'b'})^{\dot{a}}(\bar{\theta}^+ \gamma^{a'b'} \theta)_\dot{b}, \quad (A.21) \]

\[ (\mathcal{M}^2)^{\dot{a} \dot{b}} = \frac{1}{2}(\theta^+ \gamma_{ab})^{\dot{a}}(\bar{\theta}^+ \gamma^{ab} \theta)_\dot{b} - \frac{1}{2}(\theta^+ \gamma_{a'b'})^{\dot{a}}(\bar{\theta}^+ \gamma^{a'b'} \theta)_\dot{b}, \]

\[ (\mathcal{M}^2)^{\beta \dot{a}} = -\frac{1}{2}(\bar{\theta}^+ \gamma_{ab})^{\dot{a}}(\theta^+ \gamma^{ab} \theta)_\beta + \frac{1}{2}(\bar{\theta}^+ \gamma_{a'b'})^{\dot{a}}(\theta^+ \gamma^{a'b'} \theta)_\beta. \]

Let

\[ \frac{\cosh \mathcal{M} - 1_{16}}{\mathcal{M}^2} = \begin{pmatrix} (K_{11})^{\alpha \beta} & (K_{12})^{\alpha \dot{b}} \\ (K_{21})^{\dot{a} \dot{b}} & (K_{22})^{\dot{a} \beta} \end{pmatrix}, \quad \frac{\sinh \mathcal{M}}{\mathcal{M}} = \begin{pmatrix} (L_{11})^{\alpha \beta} & (L_{12})^{\alpha \dot{b}} \\ (L_{21})^{\dot{a} \dot{b}} & (L_{22})^{\dot{a} \beta} \end{pmatrix}. \quad (A.22) \]
The induced vielbeins are calculated as follows:

\[
E_i^\pm = e_\pm \partial_i X^\pm + e_\mp \mathcal{D}_i X^- ,
\]

\[
E_i^a = \frac{1}{1 - (z^2/4)} \mathcal{D}_i z^a ,
\]

\[
E_i^{4+s} = \frac{1}{1 + (y^2/4)} \mathcal{D}_i y^s ,
\]

\[
E_i^{\pm\alpha} = U^\alpha_\beta \left( (L_{11} \mathcal{D}_i \theta^+)^\beta + (L_{12} \mathcal{D}_i \bar{\theta}^+)^\beta \right) ,
\]

\[
E_i^{-\bar{\alpha}} = V^{\bar{\alpha}}_\beta \left( (L_{11} \mathcal{D}_i \bar{\theta}^-)^\beta + (L_{12} \mathcal{D}_i \theta^-)^\beta \right) ,
\]

\[
\bar{E}_i^{\pm\bar{\alpha}} = \bar{U}^{\bar{\alpha}}_\beta \left( (L_{21} \mathcal{D}_i \theta^+)^\beta + (L_{22} \mathcal{D}_i \bar{\theta}^+)^\beta \right) ,
\]

\[
\bar{E}_i^{-\bar{\alpha}} = \bar{V}^{\bar{\alpha}}_\beta \left( (L_{21} \mathcal{D}_i \bar{\theta}^-)^\beta + (L_{22} \mathcal{D}_i \theta^-)^\beta \right) ,
\]

where

\[
e^\pm_+ = \frac{1}{2} \left[ \left( 1 + \frac{z^2}{4} \right) - \left( 1 - \frac{z^2}{4} \right) \right] \pm \frac{1}{2} \left[ \left( 1 - \frac{y^2}{4} \right) + \left( 1 + \frac{y^2}{4} \right) \right] ,
\]

\[
e^\pm_- = \frac{1}{2} \left[ \left( 1 + \frac{z^2}{4} \right) + \left( 1 - \frac{z^2}{4} \right) \right] \pm \frac{1}{2} \left[ \left( 1 - \frac{y^2}{4} \right) - \left( 1 + \frac{y^2}{4} \right) \right] ,
\]

\[
U^\alpha_\beta = \frac{(\delta^\alpha_\beta + (1/4) z^a y^s (\gamma_a \gamma_{4+s})^\alpha_\beta)}{(1 - (z^2/4))^{1/2} (1 + (y^2/4))^{1/2}} ,
\]

\[
\bar{U}^\bar{\alpha}_\beta = \frac{(\delta^{\bar{\alpha}}_\bar{\beta} + (1/4) z^a y^s (\gamma_a \gamma_{4+s})^{\bar{\alpha}}_{\bar{\beta}})}{(1 - (z^2/4))^{1/2} (1 + (y^2/4))^{1/2}} ,
\]

\[
V^{\bar{\alpha}}_\beta = \frac{-iz^a (\gamma_a \gamma_{4+s})^{\bar{\alpha}}_{\beta}}{2 (1 - (z^2/4))^{1/2} (1 + (y^2/4))^{1/2}} + iy^s (\gamma_{4+s} \gamma_{6+21})^{\bar{\alpha}}_{\beta} ,
\]

\[
\bar{V}^{\bar{\alpha}}_\beta = \frac{iz^a (\gamma_a \gamma_{4+s})^{\bar{\alpha}}_{\beta} - iy^s (\gamma_{4+s} \gamma_{6+21})^{\bar{\alpha}}_{\beta}}{2 (1 - (z^2/4))^{1/2} (1 + (y^2/4))^{1/2}} + iy^s (\gamma_{4+s} \gamma_{6+21})^{\bar{\alpha}}_{\beta} ,
\]

\[
\mathcal{D}_i X^- = \partial_i X^- + 2\sqrt{2} \left[ \left( \bar{\theta}^+ \gamma_+ K_{11} \right)_\alpha + \left( \theta^+ \gamma_+ K_{21} \right)_\alpha \right] \mathcal{D}_i \theta^+ \alpha
\]

\[
+ 2\sqrt{2} \left[ \left( \bar{\theta}^+ \gamma_+ K_{12} \right)_\alpha + \left( \theta^+ \gamma_+ K_{22} \right)_\alpha \right] \mathcal{D}_i \bar{\theta}^+ \bar{\alpha} ,
\]

\[
\mathcal{D}_i z^a = \partial_i z^a - 2z_b \left[ \left( \bar{\theta}^+ \gamma_+ \gamma_{ab} \gamma_{21} \right)_\alpha - \left( \theta^+ \gamma_+ \gamma_{ab} \gamma_{22} \right)_\alpha \right] \mathcal{D}_i \theta^+ \alpha
\]

\[
- 2z_b \left[ \left( \bar{\theta}^+ \gamma_+ \gamma_{ab} \gamma_{21} \right)_\alpha - \left( \theta^+ \gamma_+ \gamma_{ab} \gamma_{22} \right)_\alpha \right] \mathcal{D}_i \bar{\theta}^+ \bar{\alpha} ,
\]

\[
\mathcal{D}_i y^s = \partial_i y^s + 2y^s \left[ \left( \bar{\theta}^+ \gamma_+ \gamma_{4+s} \gamma_{a} \gamma_{21} \right)_\alpha - \left( \theta^+ \gamma_+ \gamma_{4+s} \gamma_{a} \gamma_{22} \right)_\alpha \right] \mathcal{D}_i \theta^+ \alpha
\]

\[
+ 2y^s \left[ \left( \bar{\theta}^+ \gamma_+ \gamma_{4+s} \gamma_{a} \gamma_{21} \right)_\alpha - \left( \theta^+ \gamma_+ \gamma_{4+s} \gamma_{a} \gamma_{22} \right)_\alpha \right] \mathcal{D}_i \bar{\theta}^+ \bar{\alpha} ,
\]

\[
\mathcal{D}_i \theta^+ \alpha = \partial_i \theta^+ \alpha - \frac{1}{\sqrt{2}} \left( \theta^+ \gamma_+ \gamma^a \right)^\alpha \partial_i X^+ ,
\]

\[
\mathcal{D}_i \bar{\theta}^+ \bar{\alpha} = \partial_i \bar{\theta}^+ \bar{\alpha} + \frac{1}{\sqrt{2}} \left( \bar{\theta}^+ \gamma_+ \gamma^a \right)^\bar{\alpha} \partial_i X^+ .
\]

By comparing with (339), we can read off $\Lambda^- \bar{\alpha}, \Lambda^m_{\bar{n} \bar{\alpha}}, \Lambda^{\alpha}_{\bar{\beta}}$. For example,

\[
\Lambda^- \alpha = 2\sqrt{2} i \left[ (\theta^+ \gamma_+ K_{11})_\alpha + (\theta^+ \gamma_+ K_{21})_\alpha \right] ,
\]
\[ \Lambda^{-\dot{\alpha}} = 2\sqrt{2} i \left[ (\bar{\theta}^+ \gamma_+ K_{12})_{\dot{\alpha}} + (\theta^+ \gamma_+ K_{22})_{\dot{\alpha}} \right]. \]  

(A.29)

The fields \( B_{\dot{\alpha}\dot{\beta}} \) in the Wess-Zumino term are read off from

\[ \frac{1}{2} \epsilon^{ij} B_{\dot{\alpha}\dot{\beta}}(Z) D_i \Psi_{\dot{\alpha}} D_j \Psi_{\dot{\beta}} \]

\[ = \epsilon^{ij} (E_i^{+\alpha} \theta_{\alpha\beta} E_j^{+\dot{\beta}} + E_i^{-\dot{\alpha}} \bar{\theta}_{\dot{\alpha}\dot{\beta}} E_j^{-\dot{\beta}} - \overline{E_i^{+\dot{\alpha}}} \theta_{\dot{\alpha}\beta} E_j^{+\dot{\beta}} - \overline{E_i^{-\dot{\alpha}}} \bar{\theta}_{\dot{\alpha}\dot{\beta}} E_j^{-\dot{\beta}}). \]  

(A.30)

Our convention for the Levi-Civita symbol is \( \epsilon^{01} = 1 \).

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