A CONDITION THAT PREVENTS GROUPS FROM ACTING FIXED POINT FREE ON CUBE COMPLEXES

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Abstract. We describe a group theoretic condition which ensures that any strong simplicial action of a group satisfying this condition on a CAT(0) cube complex has a global fixed point. In particular, we show that this fixed point criterion is satisfied by Aut($F_n$), the automorphism group of a free group of rank $n$. For SAut($F_n$), the unique subgroup of index two in Aut($F_n$), we obtain a similar result.

1. Introduction

In the mathematical world, this article is located in the area of geometric group theory, a field at the intersection of algebra, geometry and topology. Geometric group theory studies the interaction between algebraic and geometric properties of groups. One is interested to understand on which 'nice' geometric spaces a given group can act in a reasonable way and how geometric properties of these spaces are reflected in the algebraic structure of the group. Here, the spaces will be CAT(0) cube complexes, while the groups will be Aut($F_n$) and SAut($F_n$). The questions we shall investigate are concerned with fixed point properties of these groups.

More precisely, let $Z^n$ be the free abelian group and $F_n$ the free group of rank $n$. One goal for a group theorist is to understand the structure of their automorphism groups, $GL_n(Z)$ resp. Aut($F_n$). The abelianization map $F_n \rightarrow Z^n$ gives a natural epimorphism Aut($F_n$) $\rightarrow$ $GL_n(Z)$. The special automorphism group of $F_n$, which we will denote by SAut($F_n$), is defined as the preimage of $SL_n(Z)$ under this map. Much of the work on Aut($F_n$) and SAut($F_n$) is motivated by the idea that $GL_n(Z)$ and Aut($F_n$) resp. $SL_n(Z)$ and SAut($F_n$) should have many properties in common. Here we follow this idea and present analogies between these groups with respect to fixed point properties.

Let $X$ be a class of metric spaces. A group $G$ is said to have property F$X$ if any action of $G$ by isometries on any member of $X$ has a fixed point.

The starting point for our investigation is the study of group actions on simplicial trees which was initiated by Serre, [Ser73], [Ser77]. Let $A$ be the class of simplicial trees. He proved that $GL_n(Z)$ and $SL_n(Z)$ have property F$A$ for $n \geq 3$. Regarding Aut($F_n$) and SAut($F_n$), Bogopolski was the first to prove that these groups also have property F$A$, see [Bog87].

A slight generalization of the class of simplicial trees is given by the class of metric trees, which we will denote by $R$. Different methods were developed by Culler and Vogtmann and later by Bridson to prove that Aut($F_n$) and SAut($F_n$) have property F$R$. [Bri10], [CV96]. We obtain the fixed point property of Aut($F_n$) and SAut($F_n$) for a much larger class of higher dimensional CAT(0) spaces, namely for the class of CAT(0) cube complexes, which we will denote by $C_\ast$. Our arguments are very similar to the arguments of Bridson.

Roughly speaking, a cube complex is a union of cubes of any dimension which are glued together along isometric faces. We say that a group $G$ has property F$\ast C_\ast$ if any strong simplicially action (meaning that the stabilizer group of any cube fixes that cube pointwise) of $G$ on any member of $C_\ast$ has a fixed point.

Date: September 2, 2015.
Key words and phrases. Aut($F_n$), group actions, global fixed point property, CAT(0) cube complexes.
We give a group theoretic condition which implies that a group satisfying this condition has property $F^{ss}C_s$.

**Fixed Point Criterion.** Let $G$ be a group and $Y$ a finite generating set of $G$. If each pair of elements in $Y$ generates a finite subgroup, then $G$ has property $F^{ss}C_s$.

Our proof of this Fixed Point Criterion is based on a suitable version of Helly’s Theorem, one important result of convexity theory.

There exist several variations of Helly’s Theorem in the literature, e.g. for finite families of convex open resp. closed subsets of a CAT(0) metric space, see [Deb70, Far09] and [Kle99]. Indeed, it was Farb who discovered the connection between Helly’s Theorem and the combinatorics of generating sets.

Our investigations on property $F^{ss}C_s$ are motivated by the following question which was formulated by Bridson and Vogtmann.

**Question.** ([BV06]) If $n \geq 4$, can $\text{Aut}(F_n)$ act without a global fixed point on a finite dimensional CAT(0) cube complex?

Under the additional assumption that the action is strong simplicial we show the answer to this question to be negative. More precisely, we show that the Fixed Point Criterion is satisfied by the groups $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$. We hence obtain

**Theorem A.**

(i) For $n \geq 3$ the group $\text{Aut}(F_n)$ has property $F^{ss}C_s$.

(ii) For $n \geq 4$ the group $\text{SAut}(F_n)$ has property $F^{ss}C_s$.

We use these results to show that actions of $\text{SAut}(F_n)$ on certain CAT(0) cube complexes are automatically trivial.

**Theorem B.** Let $n \geq 4$ and $X$ be a CAT(0) cube complex such that every vertex has at most $m$ neighbours. Let $\Phi : \text{SAut}(F_n) \to \text{Isom}(X)$ be a strong simplicial action. If $m < n$, then $\Phi$ is trivial.

**Remark on Kazhdan’s property (T).** Another fixed point property of interest is property $FH$ where $H$ denotes the class of Hilbert spaces. This property was first introduced by Kazhdan and is known as Kazhdan’s property (T). Groups with property (T) play an important role in many areas of mathematics and even in computer science.

It is a fundamental open question in geometric group theory whether the groups $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$ for $n \geq 4$ have property (T) or not. This fixed point property is related to the ones studied here by the following result of Niblo and Reeves.

**Theorem.** ([NR97]) If $G$ is a finitely generated group satisfying Kazhdan’s property (T) and $X$ is a CAT(0) cube complex on which $G$ acts simplicially, then the action has a global fixed point. In other words, $G$ has property $F^sC_s$.

A first step in answering this question would be to show that the group $\text{Aut}(F_n)$ has property $F^sC_s$.

Note that property $F^sC_s$ trivially implies $F^{ss}C_s$. The converse is not true, see for example [NR03].

2. **Preliminaries**

2.1. **Some facts about CAT(0) spaces.** In this section we briefly present the main definitions and properties concerning CAT(0) metric spaces. A detailed description of these spaces and their geometry can be found in [BH99].

We start by reviewing the concept of geodesic spaces. Let $(X, d)$ be a metric space and $x, y \in X$. A **geodesic** joining $x$ and $y$ is a map $c_{xy} : [0, l] \to X$, such that $c_{xy}(0) = x$, $c_{xy}(l) = y$ and $d(c_{xy}(t), c_{xy}(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image of $c_{xy}$, denoted by $[x, y]$, is called a
geodesic segment. A metric space \((X,d)\) is said to be a geodesic space if every two points in \(X\) are joined by a geodesic. We say that \(X\) is uniquely geodesic if there is exactly one geodesic joining \(x\) and \(y\) for all \(x,y \in X\).

A geodesic triangle in \(X\) consists of three points \(p_1,p_2,p_3\) in \(X\) and a choice of three geodesic segments \([p_1,p_2], [p_2,p_3], [p_3,p_1]\). Such a geodesic triangle will be denoted by \(\Delta(p_1,p_2,p_3)\). A triangle \(\overline{\Delta(p_1,p_2,p_3)}\) in Euclidian space \(\mathbb{R}^2\) is called a comparison triangle for \(\Delta(p_1,p_2,p_3)\) if it is a geodesic triangle in \(\mathbb{R}^2\) and if \(d(p_i,p_j) = d(\overline{p_i},\overline{p_j})\) for \(i,j = 1,2,3\). A point \(\overline{p}\) in \([\overline{p}_i,\overline{p}_j]\) is called a comparison point for \(x \in [p_i,p_j]\) if \(d(x,p_i) = d(\overline{x},\overline{p}_i)\) and \(d(x,p_j) = d(\overline{x},\overline{p}_j)\). A geodesic triangle in \(X\) is said to satisfy the CAT(0) inequality if for all \(x\) and \(y\) in the geodesic triangle and all comparison points \(\overline{x}\) and \(\overline{y}\), the inequality \(d(x,y) \leq d(\overline{x},\overline{y})\) holds.

**Definition 2.1.** A metric space \(X\) is called a CAT(0) space if \(X\) is a geodesic space and all of its geodesic triangles satisfy the CAT(0) inequality.

One can easily verify from the definition of a CAT(0) space that these spaces are uniquely geodesic, therefore we use the notation \([x,y]\) for the geodesic segment between \(x\) and \(y\) in the CAT(0) space \(X\). A subset \(Y\) of a CAT(0) space \(X\) is called convex if for all \(x\) and \(y\) in \(Y\) the geodesic segment \([x,y]\) is contained in \(Y\). Indeed, convex subspaces of a CAT(0) space are obviously CAT(0) spaces.

The class of CAT(0) spaces is large. Perhaps the easiest examples of CAT(0) spaces besides \(d\)-dimensional Euclidean spaces \(\mathbb{R}^d\) are metric trees, in particular simplicial trees by metrizing each edge of a simplicial tree as interval with length one.

### 2.2. CAT(0) cube complexes

The purpose of this subsection is to introduce a higher dimensional analogues of simplicial trees, namely CAT(0) cube complexes. Roughly speaking, a cube complex is a space which one obtains by taking a union of unit cubes of possibly different dimensions and gluing them along isometric faces.

Let us give the formal definition of a cube complex. We denote by \(I^d\) the unit \(d\)-dimensional cube \([0,1]^d\). By convention, \(I^0\) is a point. A face of \(I^d\) is a subset \(F\) of \(I^d\) which is a product \(F^1 \times F^2 \times \ldots \times F^d\) where each \(F^i\) is either \(\{0\}\), \(\{1\}\) or \([0,1]\). Let \(C_1,C_2\) be two cubes with faces \(F_1 \subset C_1, F_2 \subset C_2\). A gluing of \(C_1\) and \(C_2\) along \(F_1,F_2\) is a bijective isometry \(\psi_{C_1,C_2}: F_1 \to F_2\).

**Definition 2.2.** Let \(\mathcal{C}\) be a family of cubes and \(\mathcal{F}\) be a family of glueings of elements of \(\mathcal{C}\) with the properties that no cube is glued with itself and that for all cubes \(C_1\) and \(C_2\) there is at most one gluing \(\psi_{C_1,C_2}\).

A cube complex \(K\) is the quotient of the disjoint union of cubes \(X = \bigsqcup \mathcal{C}\) by the gluing equivalence relation which is generated by

\[
\{x \sim \psi_{C_1,C_2}(x) \mid \psi_{C_1,C_2} \in \mathcal{F}, x \in \text{domain}(\psi_{C_1,C_2})\}.
\]

A subset \(L\) of \(K\) is a cube subcomplex of \(K\) if there exist a subset \(\mathcal{C}'\) of \(\mathcal{C}\) such that \(L\) is equal to the quotient of a disjoint union of cubes \(Y = \bigsqcup \mathcal{C}'\) by a gluing equivalence relation which is generated by

\[
\{x \sim \psi_{C_1,C_2}(x) \mid C_1,C_2 \in \mathcal{C}', \psi_{C_1,C_2} \in \mathcal{F}, x \in \text{domain}(\psi_{C_1,C_2})\}.
\]

The cube complex is CAT(0) if the cube complex with the length metric is a CAT(0) space. We note that the CAT(0) inequality condition for a cube complex can be expressed by a combinatorial condition on the cells, see [BH99].

For example, the \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) is a CAT(0) cube complex in the obvious way with \(\mathbb{Z}^d\) as a set of vertices.

The following version of the Bruhat-Tits Fixed Point Theorem valid for non-complete CAT(0) cube complexes was proven by GERASIMOV in [Ger98], (see [Cor13] 5.18 for details). For the sake of completeness, we give here the complete argument for finite groups.
Proposition 2.3. Let $G$ be a finite group acting simplicially on a CAT(0) cube complex. Then $G$ has a global fixed point.

Proof. The group $G$ is finite, therefore any orbit of vertices is finite. The convex hull of a finite orbit is a finite dimensional CAT(0) cube subcomplex. In particular, this subcomplex is complete and invariant under the action of $G$. We obtain by the classical Bruhat-Tits Fixed Point Theorem [BH99, II 2.8] a fixed point in this subcomplex. □

3. Fixed Point Criterion

3.1. Helly’s Theorem for cube complexes. In this subsection we prove a version of one important result of convexity theory, Helly’s Theorem (restated below for convenience), for the class of CAT(0) cube complexes.

Helly’s classical Theorem. ([Hel23]) Let $\mathcal{S}$ be a finite family of non-empty closed convex subspaces of $\mathbb{R}^d$. If the intersection of each $(d + 1)$-elements of $\mathcal{S}$ is non-empty, then $\bigcap \mathcal{S}$ is non-empty.

We start by giving the following definition and result about median graphs.

Definition 3.1. Let $\Gamma$ be a graph. The interval $I(u,v)$ between two vertices $u$ and $v$ consists of all vertices on a shortest paths between $u$ and $v$, i.e.

$$I(u,v) := \{ x \in V \mid d(u,x) + d(x,v) = d(u,v) \}.$$

A graph $\Gamma$ is called **median** if for each triple $x,y,z$ of vertices the interval intersection consists of exactly one vertex, denoted by $m(x,y,z)$, i.e.

$$I(x,y) \cap I(y,z) \cap I(x,z) = \{m(x,y,z)\}.$$

The relation between CAT(0) cube complexes and median graphs is as follows.

Proposition 3.2. ([Rol98, §10]) Let $X$ be a CAT(0) cube complex and $X^{(1)}$ be the 1-skeleton of $X$. Then $X^{(1)}$ is a median graph.

With the help of above proposition we can now prove a suitable version of Helly’s Theorem.

Helly’s Theorem for CAT(0) cube complexes. ([Rol98, 2.2]) Let $X$ be a CAT(0) cube complex and $\mathcal{S}$ a finite family of non-empty convex subcomplexes of $X$. If the intersection of each two elements of $\mathcal{S}$ is non-empty, then $\bigcap \mathcal{S}$ is non-empty.

Proof. We argue by induction on $m := | \mathcal{S} |$. For $m = 2$ there is nothing to prove. Let $m \geq 3$ and

$$Y := X_3 \cap X_4 \cap \ldots \cap X_m.$$

By induction hypothesis $X_1 \cap Y$, $X_2 \cap Y$ and $X_1 \cap X_2$ are non-empty. Choose vertices $P \in X_1 \cap Y$, $Q \in X_1 \cap Y$ and $R \in X_1 \cap X_2$. Since $Y$ is a convex subcomplex and $P, Q \in Y$, we have $I(P,Q) \subseteq Y$ and therefore also $m(P,Q,R) \in Y$. For the same reason $P, R \in X_1$ gives $m(P,Q,R) \in X_1$ and $Q, R \in X_2$ implies $m(P,Q,R) \in X_2$. We have shown that $m(P,Q,R) \in X_1 \cap X_2 \cap Y$. This finishes the proof. □

We need the following definitions.

Definition 3.3. (i) A simplicial action on a CAT(0) cube complex is called **strong simplicial** if the stabilizer group of any cube fixes that cube pointwise.

(ii) A group $G$ is said to have property $\Gamma^\approx \mathcal{C}_*$ if any strong simplicial action of $G$ on any member of $\mathcal{C}_*$ has a fixed point (in the geometric realization).

Using Helly’s Theorem for CAT(0) cube complexes we obtain the following group theoretic condition for the fixed point property.
Proposition 3.4. Let $G = \langle g_1, \ldots, g_k \rangle$ be a finitely generated group and $X$ a CAT(0) cube complex. If $\Phi : G \to \text{Isom}(X)$ is a strong simplicial action such that the fixed point sets $\text{Fix}(\langle g_i, g_j \rangle)$ are non-empty for all $i, j = 1, \ldots, k$, then the fixed point set $\text{Fix}(G)$ is non-empty as well.

Proof. Since the action is strong simplicial, the fixed point sets of the generators are convex subcomplexes. By assumption we have that $\text{Fix}(\langle g_i, g_j \rangle) = \text{Fix}(g_i) \cap \text{Fix}(g_j)$ is non-empty for all $i, j = 1, \ldots, k$. It hence follows from Helly’s Theorem for CAT(0) cube complexes that $\text{Fix}(g_1) \cap \ldots \cap \text{Fix}(g_k) = \text{Fix}(G)$ is non-empty. \hfill \Box

3.2. Fixed Point Criterion. Combining the above proposition with Proposition 2.3, we immediately find

Fixed Point Criterion. Let $G$ be a group and $Y$ a finite generating set of $G$. If each pair of elements in $Y$ generates a finite subgroup, then $G$ has property $\text{F}_{\text{fix}}C_\ast$.

Proof. By Proposition 2.3 $\text{Fix}(\langle g_i, g_j \rangle)$ is non-empty for all $i, j \in \{1, \ldots, k\}$. Now apply Proposition 3.4. \hfill \Box

4. Generation $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$ by finite subgroups

We begin with the definition of the automorphism group of the free group of rank $n$. Let $F_n$ be the free group of rank $n$ with a fixed basis $X := \{x_1, \ldots, x_n\}$. We denote by $\text{Aut}(F_n)$ the automorphism group of $F_n$ and by $\text{SAut}(F_n)$ the unique subgroup of index two in $\text{Aut}(F_n)$. More precisely, the abelianization map $F_n \to \mathbb{Z}^n$ gives a natural surjection $\pi : \text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z})$. The group $\text{SAut}(F_n)$ is equal to the preimage of $\text{SL}_n(\mathbb{Z})$ under this map.

4.1. A generating set of $\text{Aut}(F_n)$. The purpose of this section is to describe a generating set of the group $\text{Aut}(F_n)$ such that each pair of its elements generates a finite subgroup.

Although it seems awkward at first glance, it is convenient and standard to work with the right action of $\text{Aut}(F_n)$ on $F_n$.

Convention 4.1. For $\alpha, \beta$ in $\text{Aut}(F_n)$ the automorphism $\alpha \beta$ is the composite where $\alpha$ acts before $\beta$.

Let us first introduce a notations for some elements of $\text{Aut}(F_n)$. We define the right Nielsen automorphism $\rho_{ij}$, involutions $(x_i, x_j)$ and $e_i$ for $i, j = 1, \ldots, n$, $i \neq j$ as follows:

$$\rho_{ij}(x_k) := \begin{cases} x_i x_j & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

$$(x_i, x_j)(x_k) := \begin{cases} x_k & \text{if } k = i, \\ x_i & \text{if } k = j, \\ x_j & \text{if } k \neq i. \end{cases}$$

$$e_i(x_k) := \begin{cases} x_i^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

It is easy to see that the image of $X = \{x_1, \ldots, x_n\}$ under any of these maps is another basis of $F_n$, therefore these elements are automorphisms. It was proven by Nielsen in [Nie24, p. 173]) that for $n \geq 3$ the group $\text{Aut}(F_n)$ is generated by the set

$$Y_1 := \{\rho_{12}, e_1, (x_1, x_2), (x_1, x_2, \ldots, x_n)\},$$

where $(x_1, x_2, \ldots, x_n)$ is equal to $(x_{n-1}, x_n)(x_{n-2}, x_{n-1}) \ldots (x_1, x_2)$.

Proposition 4.2. [Var14] Let $n \geq 3$.

(i) The group $\text{Aut}(F_n)$ is generated by

$$Y_2 := \{(x_1, x_2)e_1 e_2, (x_2, x_3)e_1, (x_i, x_{i+1}), e_2 \rho_{12}, e_n | i = 3, \ldots, n-1\}.$$

(ii) For $\alpha, \beta$ in $Y_2$ the subgroup generated by $\{\alpha, \beta\}$ is finite.

The strategy of the proof is to modify the set $Y_1$ such that each pair of elements in the new generating set generates a finite group, compare [Bri10, 1.1, 1.2].
4.2. A generating set of $\text{SAut}(F_n)$. In this subsection we give a generating set for the group $\text{SAut}(F_n)$ with the same finiteness property as the set $Y_2$ in the previous subsection.

For $i, j = 1, \ldots, n$, $i \neq j$ we define the left Nielsen automorphism $\lambda_{ij}$ as follows:

$$\lambda_{ij}(x_k) := \begin{cases} 
x_j x_i & \text{if } k = i, \\
x_k & \text{if } k \neq i. 
\end{cases}$$

It is known that the group $\text{SAut}(F_n)$ is generated by $\{\rho_{ij}, \lambda_{ij} \mid i, j = 1, \ldots, n, \ i \neq j\}$ for $n \geq 3$, see [Get84, 2.8]. An easy calculation shows that the commutator of $\rho_{ij}$ and $\rho_{jk}$ is equal to $\rho_{ik}$ and the commutator of $\lambda_{ij}$ and $\lambda_{jk}$ is equal to $\lambda_{ik}$ for $i, j, k = 1, \ldots, n$ distinct, therefore $\text{SAut}(F_n)$ is generated by the set

$$Y_3 = \{\rho_{i(i+1)}, \rho_{n1}, \lambda_{i(i+1)}, \lambda_{n1} \mid i = 1, \ldots, n-1\}.$$

Proposition 4.3. ([Var14]) Let $n \geq 4$.

(i) The group $\text{SAut}(F_n)$ is generated by

$$Y_4 := \{(x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1, (x_i, x_{i+1})e_i, e_2e_4\rho_{12}, e_3e_4 \mid i = 3, \ldots, n-1\}.$$

(ii) For $\alpha, \beta$ in $Y_4$ the subgroup generated by $\{\alpha, \beta\}$ is finite.

The idea of the proof is to modify the set $Y_3$ such that each pair of elements in the new generating set generates a finite group.

5. PROOF OF THEOREM A

Now we have all ingredients to prove Theorem A. We show that the Fixed Point Criterion is satisfied by $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$, and therefore these groups have property $F^{ss}\mathbb{C}_s$.

Theorem A.

(i) For $n \geq 3$ the group $\text{Aut}(F_n)$ has property $F^{ss}\mathbb{C}_s$.

(ii) For $n \geq 4$ the group $\text{SAut}(F_n)$ has property $F^{ss}\mathbb{C}_s$.

Proof. The generating sets constructed in Propositions 4.2 and 4.3 show that $\text{Aut}(F_n)$ for $n \geq 3$ and $\text{SAut}(F_n)$ for $n \geq 4$ satisfy the Fixed Point Criterion. Therefore, these groups have property $F^{ss}\mathbb{C}_s$. \qed

6. TRIVIALITY FOR ACTIONS OF $\text{SAut}(F_n)$ ON CAT(0) CUBE COMPLEXES

The aim of this section is to show that $\text{SAut}(F_n)$ cannot act non-trivially on an $m$-ary CAT(0) cube complex for $m < n$. By definition, an $m$-ary cube complex is a cube complex where every vertex has at most $m$ neighbours.

For the proof we need the following variant of a result by Bridson and Vogtmann [BV11, 3.1]. For a detailed proof the reader is referred to [Var10, 1.13].

Proposition 6.1. Let $n \geq 3$, $G$ be a group and $\phi : \text{SAut}(F_n) \to G$ a group homomorphism. If there exists $\alpha \in \text{Alt}(n) - \{\text{id}_{F_n}\}$ with $\phi(\alpha) = 1$, then $\phi$ is trivial.

We finish by proving

Theorem B. Let $n \geq 4$ and $X$ be an $m$-ary CAT(0) cube complex. Let $\Phi : \text{SAut}(F_n) \to \text{Isom}(X)$ be a strong simplicial action. If $m < n$, then $\Phi$ is trivial.

Proof. The group $\text{SAut}(F_n)$ has property $F^{ss}\mathbb{C}_s$ by Theorem A, therefore $\Phi$ has a global fixed point $v \in X$. After barycentric subdivision we may assume that $v$ is a vertex. The group $\text{SAut}(F_n)$ acts on the link of $v$, i.e. the set of all neighbours of $v$, via $\text{Sym}(m)$. As $\text{SAut}(F_n)$ is perfect, we even have

$$\Phi_{\text{Alt}(m)} : \text{SAut}(F_n) \to \text{Alt}(m).$$
If \( m < n \), the restriction of this map to \( \text{Alt}(n) \) cannot be injective, therefore \( \Phi_{\text{Alt}(m)} \) is trivial by Proposition 6.1. This shows that any neighbour of \( v \) is in the fixed point set of the action, hence all vertices of \( X \) are in the fixed point set. Thus \( \text{SAut}(F_n) \) acts trivially on \( X \).

\[ \square \]

Acknowledgements. We would like to thank Yves Cornulier and Genevois Anthony for their comments concerning completeness in Proposition 2.3.

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