Largest reduced neighborhood clique cover number revisited

Farhad Shahrokhi
Department of Computer Science and Engineering, UNT
farhad@cs.unt.edu

Abstract

Let $G$ be a graph and $t \geq 0$. The largest reduced neighborhood clique cover number of $G$, denoted by $\hat{\beta}_t(G)$, is the largest, overall $t$-shallow minors $H$ of $G$, of the smallest number of cliques that can cover any closed neighborhood of a vertex in $H$. It is known that $\hat{\beta}_t(G) \leq s_t$, where $G$ is an incomparability graph and $s_t$ is the number of leaves in a largest $t$-shallow minor which is isomorphic to an induced star on $s_t$ leaves. In this paper we give an overview of the properties of $\hat{\beta}_t(G)$ including the connections to the greatest reduced average density of $G$, or $\nabla_t(G)$, introduce the class of graphs with bounded neighborhood clique cover number, and derive a simple lower and an upper bound for this important graph parameter. We announce two conjectures, one for the value of $\hat{\beta}_t(G)$, and another for a separator theorem (with respect to a certain measure) for an interesting class of graphs, namely the class of incomparability graphs which we suspect to have a polynomial bounded neighborhood clique cover number, when the size of a largest induced star is bounded.

1 Introduction

This paper is a sequel to our paper [12]. We assume the reader is familiar with standard graph theory. Throughout this paper $G = (V, E)$ denotes an undirected graph. Recall that a graph $G$ is $k$-degenerate ($k \geq 0$), if every induced subgraph of $G$ has a vertex of degree at most $k$. Degeneracy of $G$ is the smallest integer $k$ so that $G$ is $k$-degenerate. Graphs with small degeneracy have nice structural and algorithm properties. Nešetřil and Ossona de Mendez introduced an important graph parameter which is a generalization of degeneracy. In simple words they introduced the notion of the maximum edge density of a graph taken overall $t$-shallow minors.

A $t$-shallow minor, or a $t$-minor of $G$ in short, is a minor of $G$ which is obtained by contracting connected subgraphs of radius at most $t$, and delet-
ing vertices (but not edges). Nesetril and Ossona de Mendez introduced the greatest reduced average density of $G$ (grad of $G$ in short), or $\nabla_t(G)$, to be the maximum edge density of any $t$-minor in $G$. It is easily seen that $
abla_t(G) \leq \delta(G)$, where $\delta(G)$ is the degeneracy of $G$. They define $G$ to have bounded expansion, if $\nabla_t(G)$ is finite for every $t \geq 0$. They explored very nice structural and algorithmic properties of the class of bounded expansion graphs that contains many traditionally known “sparse” graphs, including the class of $H$-minor free graphs [9, 10, 8, 7].

We introduced the largest reduced neighborhood clique cover number of $G$, in [12]. Informally, consider the minimum number of disjoint cliques that covers the closed neighborhood of any vertex in a graph; Now take the maximum value of such a minimum overall $t$-minors of the graph. Formally, for a graph $H$, let $\beta(H)$ denote the clique cover number of $H$, that is, the minimum number of disjoint cliques that partitions $V(H)$. Now for any $x \in V(H)$, let $H_x$ denote the the closed neighborhood of $x$ in $H$, and note that $\beta(H_x) \leq \text{deg}_H(x)$, where $\text{deg}_H(x)$ is the degree of $x$ in $H$. Next for any graph $G$ and $t \geq 0$ define largest reduced neighborhood clique cover number of $G$, denoted by $\hat{\beta}_t(G)$ to be the largest value of $\beta(H_x)$ for any $t$-minor $H$ of $G$. We say $G$ has a bounded neighborhood clique cover number if $\hat{\beta}_t(G)$ has a finite value for each $t \geq 0$. Note that $\hat{\beta}_t(K_n) = 1$ for any $t \geq 0$, nonetheless $\nabla_t(K_n) = \frac{n-1}{2}$. Furthermore, one can construct non trivial classes of graphs so that for every $G$ in the class $\hat{\beta}_t(G)$ is small, that is bounded by a constant, whereas, $\nabla_t(G)$ is arbitrary large. For instance, let $G = (V,E)$ be a connected graph which is the complement of a bipartite graph, where each partite class has $n$ vertices. Then $\hat{\beta}_t(G) \leq 2$, whereas, $\nabla_t(G) = \frac{|E|}{|V|} \geq \frac{n-1}{2}$, for any $t \geq 0$. Additionally, for any chordal graph $G$, $\hat{\beta}_t(G) = 1$ [12], but of course one can construct very dense non trivial chordal graphs $G$ for which $\hat{\beta}_t(G)$ is unbounded.

$\hat{\beta}_t(G)$, is an effective tool to study the properties of those graphs that are not “sufficiently sparse”, to have a bounded expansion, but yet there is need to explore their properties. For instance, another interesting class of graphs for which $\hat{\beta}(G)$ is bounded, but grad of $G$ can be arbitrary large is the intersection graph of fact objects (spheres, cubes, boxes with bounded aspect ratio) [2] when geometric dimension is bounded. Specifically, see [12] for the following Theorem.

**Theorem 1.1** Let $G$ be the intersection graph of fat objects in $\mathbb{R}^d$ (spheres, cubes, boxes with bounded aspect ratio), then, $\hat{\beta}_t(G) = O(b^{d^2t^2})$, where $b$ is a constant that depends on the shape of the object.

Section two contains a simple lower bound and an upper bound on $\hat{\beta}_t(G)$ in terms of the clique cover width of $G$, and some constructions that
measures the ratio of the upper bound to the lower bound. Section three contains two conjectures related to incomparability graphs that arise from our studies here.

2 Bounds on $\hat{\beta}_t(G)$

It is interesting to observe that $\hat{\beta}_0(K_{n,n}) = n$, therefore, $\hat{\beta}(K_{n,n})$ is not bounded. In fact, the following observation is easy to prove.

**Observation 2.1** Let $p$ be the largest integer so that a $t-$shallow minor of $G$ is isomorphic to $K_{p,p}$, then $\hat{\beta}(G) \geq p$.

For a clique cover $C$ in $G$, the **clique cover graph** of $C$ is the graph obtained by contracting the vertices of each clique in $C$ into a single vertex. The **clique cover width** of $G$, denoted by $CCW(G)$, is the minimum value of the bandwidth of all clique cover graphs in $G[15, 13, 14]$. In this paper when we write $C = \{C_1, C_2, ..., C_K\}$, we mean $C$ is an ordered set. Let $ab$ be an edge width $a \in C_i$ and $b \in C_j, j > i$, and let $W(e) = j - i$. We call $W(e)$ the **width** of $e$. An important application of the clique cover width is in the derivation of separation theorems in dense graphs, where separation can be defined for other types measures [15], instead of just the number of vertices. Recall that according to the planar separation theorem, any $n$ vertex planar graph can be separated into two subgraphs, each having at most $2n/3$ vertices, by removing $O(\sqrt{n})$ vertices. Any $G$ can be separated with respect to an optimal (or feasible) set $C$ of cliques (utilizing $CCW(G)$): There is partition of $\{A, S, B\}$ of $V(G)$ so that (i) there are no edges between $A$ and $B$, (ii) $S$ can be covered with at most $CCW(G)$ many cliques from $C$, and (iii) $A$ and $B$ are each covered with at most $2|C|/3$ cliques from $C[15, 14]$.

**Theorem 2.1** For any graph $G$, $\hat{\beta}(G) \leq k + 1$, where $k$ is the largest clique cover width of any $t-$shallow minor of $G$.

**Proof.** Let $\{C_1, C_2, ..., C_K\}$ be a clique cover of a graph $H$. Let $e_a = ab, a \in C_1, b \in C_i$ be an edge of largest width incident to $a$. Let $e^*$ be an edge having an end point in $C_1$ with $W(e^*) = \min\{W(e_a) | a \in C_1\}$. By definition of $e^*$, $H_a$ can be covered with $W(e^*) + 1$ cliques, and hence $\hat{\beta}(H) \leq W(e^*) + 1$. Therefore $\hat{\beta}(H) \leq CCW(H) + 1$, since $CCW(H) \geq W(e^*)$. To finish the proof take $H$ to be a $t$-minor of $G$. $\square$

**Corollary 2.1** Let $k$ denote the largest clique cover width of any $t-$shallow minor of $G$, and $p$ be largest integer so that any $t-$shallow minor of $G$ is isomorphic to $K_{p,p}$. Then, $p \leq \hat{\beta}(G) \leq k + 1$. 

It is easy to verify that $CCW(H) \leq CCW(G)$, for any induced subgraph $H$ of $G$. Nonetheless, for a $t$-minor $H$ of $G$, $CCW(H)$, or $k$ in Corollary, 2.1 may be much larger than $CCW(G)$. Generally speaking, it would nice to know how large the ratio $k/p$ may be.

**Observation 2.2** For any $t \geq 0$, and $n > t$, there is an $n$ vertex graph $G$, with $CCW(G) = 1$, so that for a $t$-minor $H$ of $G$, $t \geq CCW(H) \geq t/2$. Moreover, in this case, neither $G$, nor $H$ contain $K_{2,2}$ as an induced subgraph.

**Justification.** Let $P_n$ be a path of $n$ vertices on vertex set $X = \{x_1, x_2, ..., x_n\}$. Now let $S$ be a an independent set of $n$ vertices. To construct $G = (V, E)$ place a perfect matching of cardinality $n$ between $S$ and $X$. It is easily verified that $CCW(G) = 1$. Now for a given $n \geq t \geq 0$, contract $x_1, x_2, ..., x_t$ into one single vertex to obtain a $t$-minor $H$. Observe that $H$ has an induced star on $t$ vertices. Thus, $CCW(H) \geq t/2$ \[13\]. Furthermore, it is not difficult to see that $G$ is an incomparability graph (a graph whose complement has a transitive orientation on edges), and so is $H$, since $H$ is obtained by contractions of edges in $G$. Since $H$ is an incomparability graph we must have $CCW(H) \leq s$, where $s$ is the number of leaves in a largest induced star \[13\]. Finally, it is easy to verify that neither $H$ or $G$ have $K_{2,2}$ as a subgraph, since $G$ is acyclic. \[\Box\]

**Observation 2.3** For any $t \geq 0$, and $n >> t$, there is a graph $G$, on $n + t(t + 1)$ vertices that excludes $K_{2,2}$ as an induced subgraph, but has a $t$-minor $H$ that contains $K_{t+1,t+1}$ as an induced subgraph. Moreover, $CCW(G) \geq n/2$.

**Justification.** Let $V(G) = A \cup \bigcup_{i=1}^{t+1} B_i$, where $A$ is a independent set of size $t + 1$, and for $i = 1, 2, ..., t$ each $B_i$ is path on $t + 1$ vertices; $B_{t+1}$ is a cycle on $n$ vertices. Now for each $i = 1, 2, ..., t + 1$ add a perfect matching of size $t$ between vertices in $A$ and vertices in $B_i$. Thus each vertex in $A$ has degree $t$, where for $i = 1, 2, ..., t$, each vertex of $B_i$ has degree at most 3. Note that $G$ does not have $K_{2,2}$ as an induced subgraph. Furthermore, since $B_{t+1}$ is a cycle of $n$ vertices, we have $CCW(G) \geq n/2$. Now for $i = 1, 2, ..., t$, contract each path $B_i$ into a single vertex. For $B_{t+1}$ contract the first $t + 1$ vertices to a vertex. Then the resulting graph $H$ has an induced subgraph isomorphic to $K_{t+1,t+1}$. \[\Box\]

### 3 Incomparability graphs

Recall that a chordal graph does not have any chord-less cycles \[3\]. An incomparability graph is a graph whose complement has a transitive orientation \[16\]. Incomparability graphs are perfect, have geometric realizations,
and have recently been subject to intense investigations, due to their intimate connections to string graphs. One wonders if there is a meaningful converse to Observation 2.1. That is, can one find a suitable upper bound on $\hat{\beta}_t(G)$ that is related to the lower bound in 2.1? It is less likely that this is the case for all graphs, nonetheless, we suspect that there is a weak converse to 2.1 when $G$ is an incomparability graph. Specifically, we have shown that if an incomparability graph $G$ does not have a $t$–shallow minor which is isomorphic to an induced star on $s_t$ leaves, then, $\hat{\beta}_t(G) \leq s_t$. Moreover, we have shown that for any incomparability graph $G$, $\frac{t}{2} \leq CCW(G) \leq s$, where $s$ is the number of leaves in a largest induced star in $G$. Hence, a natural question is how large $s_t/s$ can be?

**Conjecture 3.1** Let $G$ be an incomparability graph that does not have an induced star which is isomorphic to an induced star on $s$ leaves. Then, the size of a largest induced star $s_t$ in any $t$–shallow minor of $G$ is at most $O(t.s)$. Consequently, $\hat{\beta}_t(G) = O(t.s)$, for any $t \geq 0$.

If the above conjecture were to be true, then $\hat{\beta}_t(G) = O(t.s)$, where $t$ is the number of leaves in a largest induced star in $G$. Note that the conjecture implies that the class of incomparability graphs have a linearly bounded neighborhood clique cover number, when the size of a largest induced star is fixed.

By observation 2.1, $\hat{\beta}_t(G) \geq p_t$, where $p_t$ is the largest integer so that $K_{p_t,p_t}$ is a $t$–shallow minor of $G$. Hence to get a good estimate for $\hat{\beta}_t(G)$ (if the conjecture were to be true), one has to investigate how large $t.s/p_t$ can be.

It is easy to observe that if $G$ is a chordal graph, then, $\hat{\beta}_t(G) = 1$ [12]. Moreover, the separation property with respect to cliques holds for any chordal graph $G$, regardless of the value the clique cover width. Particularly, given a clique tree [3] of $G$ associated with a set $C$ of maximal cliques, there is one clique $B$ in $C$, so that after removal of $B$, each the two remaining (separated) subgraph of $G$ can be covered by at most $2|C|/3$ cliques from $C$. Now let $G$ be an interval graph; Since $G$ is chordal $\hat{\beta}_t(G) = 1$, and additionally $G$ has the stated separation property. Particularly, note that $G$ is chordal and also an incomparability graph that does not have a $K_{2,2}$ as an induced subgraph. In fact, no $t$–minor of an interval graph $G$ can have $K_{2,2}$ as an induced subgraph. So one can suspect that if a incomparability graph $G$ does not have a large $K_{p,p}$ as a $t$–minor, then, $G$ has *nice* separation properties with respect to cliques.

**Conjecture 3.2** Let $p$ be fixed, and let $G$ be an incomparability graph that does not have $K_{p,p}$ as a $t$–shallow minor. Then, there is a clique cover $C$ in $G$ so that the removal of $O(\sqrt{|C|})$ cliques from $C$, separates $G$ into
two subgraphs so that each subgraph can be covered with at most 2|C|/3 cliques from C.

We remark that by a general result of Fox and Pach [4] (see also an earlier result of Bodlaender and Thilikos on $k$-chordal graphs [1]), any incomparability graph $G$ on $n$ vertices and $m$ edges has a separation $(L, S, R)$ so that $S = O(\sqrt{m})$, and $|L|, |S| \leq 2n/3$, but conjecture 3.2 does not follow from these result.

References

[1] Bodlaender H., Thilikos D., Treewidth and Small Separators for Graphs with Small Chordality, Disc. Applied Math., 79(1997), 45-61.

[2] Chan T., Polynomial-time approximation schemes for packing and piercing fat objects, Journal of Algorithms, 46(2), 178 - 189, 2003.

[3] Golumbic M. C, Algorithmic Graph Theory and Perfect Graphs (Annals of Discrete Mathematics, Vol 57, North-Holland Publishing Co., Amsterdam, The Netherlands, 2004.

[4] Fox J., Pach J., Separator theorems and Turn-type results for planar intersection graphs Advances in Mathematics 219 (3), 1070-1080.

[5] Dvorak Z., Norin S., (2015), Strongly sublinear separators and polynomial expansion, arXiv:1504.04821.

[6] Dvorak Z., Constant-factor approximation of domination number in sparse graphs, 2011 arXiv:1110.5190 [math.CO] (or arXiv:1110.5190v).

[7] Nesetril, J., Ossona de Mendez P., Grad and classes with bounded expansion II. Algorithmic aspects, European Journal of Combinatorics 29 (3), 777-791, 2008.

[8] Nesetril, J., Ossona de Mendez, P., Grad and classes with bounded expansion I. Decompositions, European Journal of Combinatorics, (29), 3, 2008, 760-776.

[9] Nesetril J., Ossona de Mendez P. (2012), "5.5 Classes with Bounded Expansion", Sparsity: Graphs, Structures, and Algorithms, Algorithms and Combinatorics 28, Springer, pp. 104107.

[10] Nesetril J., Ossona de Mendez P.; Wood, D. R. (2012), "Characterizations and examples of graph classes with bounded expansion", European Journal of Combinatorics 33 (3): 350373, arXiv:0902.3265.
[11] S. A. PLOTKIN, S. RAO, AND W. D. SMITH, Shallow excluded minors and improved graph decompositions, in Proceedings of the 5th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA94), 1994, pp. 462-470.

[12] Shahrokhi F., On the largest reduced neighborhood clique cover number of a graph, Congressus Numerantium, 226 (2016), 273-279. arXiv:1606.02370v2 [math.CO].

[13] Shahrokhi F., Unit Incomparability Dimension and Clique Cover Width in Graphs, Congressus Numerantium 213 (2012), 91-98.

[14] Shahrokhi F., On the clique cover width problem, Congressus Numerantium 205 (2010), 97-103.

[15] Shahrokhi F., A new separation theorem with geometric applications, Proceedings of 26th European Workshop on Computational Geometry, EuroCG2010, 2010, 253-2569, arXiv:1504.04938 [cs.CG].

[16] Trotter W.T., New perspectives on interval orders and interval graphs, in Surveys in Combinatorics, Cambridge Univ. Press, 1977, 237-286.