Hierarchical Fractal Weyl Laws for Chaotic Resonance States in Open Mixed Systems

M. J. Körber,1,2 M. Michler,1 A. Bäcker,1,2 and R. Ketzmerick1,2

1Technische Universität Dresden, Institut für Theoretische Physik and Center for Dynamics, 01062 Dresden, Germany
2Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany
(Dated: September 16, 2013)

In open chaotic systems the number of long-lived resonance states obeys a fractal Weyl law, which depends on the fractal dimension of the chaotic saddle. We study the generic case of a mixed phase space with regular and chaotic dynamics. We find a hierarchy of fractal Weyl laws, one for each region of the hierarchical decomposition of the chaotic phase-space component. This is based on our observation of hierarchical resonance states localizing on these regions. Numerically this is verified for the standard map and a hierarchical model system.

PACS numbers: 05.45.Mt, 03.65.Sq, 05.45.Df

It is just a century ago that Hermann Weyl published his celebrated theorem on the asymptotic distribution of eigenmodes of the Helmholtz equation in a bounded domain [1] which has found fundamental applications in the context of acoustics, optical cavities and quantum billiards [2–4]. For a quantum billiard with a d dimensional phase space the number \( N(k) \) of eigenmodes with a wave number below \( k \) is on average and in the limit of large \( k \) given by \( N(k) \sim k^{d/2} \) up to corrections of higher order [5–9]. Only recently, this fundamental question has been addressed for open scattering systems, where for the case of fully chaotic systems a fractal Weyl law was found [10–23]. Due to the opening of the system one classically obtains a fractal chaotic saddle (sometimes also called repeller), which is the invariant set of points in phase space that do not escape, neither in the future nor in the past [24, 25]. Its fractal dimension \( \delta \) plays an important role quantum mechanically: The number \( N \) of long-lived resonance states is given by a fractal Weyl law,

\[
N(h) \sim h^{-\delta/2},
\]

which here is stated for open chaotic maps, where the \( k \) dependence is replaced by the dependence on the effective size of Planck’s cell \( h \).

Generic Hamiltonian systems exhibit a mixed phase space where regular and chaotic motion coexist [26], see Fig. 1(a). Regular resonance states of the open system obey a standard Weyl law, while for chaotic resonance states one would naively expect that their number follows the fractal Weyl law, Eq. (1). This ignores, however, that the dynamics in the chaotic region of generic two-dimensional maps is dominated by partial transport barriers, see Fig. 1(a). A partial barrier is a curve which decomposes phase space into two almost invariant regions. The small area enclosed by the partial barrier and its preimage (dotted line in Fig. 1(a), magnification) consists of two parts of size \( \Phi \) on opposite sides of the partial barrier, which are mapped to the other side in one iteration of the map. This flux \( \Phi \) is the characteristic property of a partial barrier. There are infinitely many partial barriers which are hierarchically organized with decreasing fluxes towards the regular regions [27,31]. The partial barriers strongly impact the system’s classical [27,33] and quantum mechanical [34,44] properties, and lead to, e.g., the localization of eigenstates in phase space [34,36,40,44] and fractal conductance fluctuations [37,38,42].

Classically, the chaotic saddle, see Fig. 1(b), in generic two-dimensional open maps gives rise to an individual fractal dimension for each region of the hierarchical decomposition of phase space [22]. It is important to stress that these are effective fractal dimensions, which are constant over sev-

![FIG. 1.](image-url) (Color online) (a) Phase space of the standard map at \( \kappa = 2.9 \) with regular (solid gray lines) and chaotic (gray points) orbits, three partial barriers (solid colored lines) and the preimage of the outermost partial barrier (dotted magenta line). (b) Chaotic saddle of the opened map (gray-shaded absorbing stripes) colored according to the regions (A0: green, A1: blue). (c) Rescaled hierarchical fractal Weyl laws \( \tilde{N}_j(h) \) vs. \( h^{-1} \) (filled symbols) counting hierarchical resonance states in the outer (A0, triangles) and inner (A1, circles) chaotic regions (with corresponding typical Husimi representations for \( h = 1/1000 \)). Their power-law scaling is compared to the rescaled box-counting scaling \( \tilde{N}_j^{bc}(\varepsilon) \) vs. \( \varepsilon^{-2} \) (open symbols) with fractal dimension \( \delta_j \) in region \( A_j \) of the chaotic saddle.
eral orders, while in the limit of arbitrarily small scales, they approach two. \textsuperscript{12,43} Quantum mechanically, fractal Weyl laws for open systems with a mixed phase space have been investigated in Refs. \textsuperscript{46–49}, but the influence of the hierarchical phase-space structure remains open. In particular, the individual effective fractal dimensions of the chaotic saddle have not been taken into account, so far.

In this Letter we propose a generalization of the Weyl law to open systems with a mixed phase space. We obtain hierarchical fractal Weyl laws,

\[
N_j(h) \sim h^{-\delta_j/2},
\]

one for each phase-space region \(A_j\) of the hierarchical decomposition of the chaotic component in a generic two-dimensional phase space. Here, \(\delta_j\) denotes the effective fractal dimension of the chaotic saddle in each region. Quantum mechanically, this result is based on our observation of hierarchical resonance states, which predominantly localize on one of the regions \(A_j\). Their number \(N_j\) follows the hierarchical fractal Weyl laws, Eq. (2). This holds over ranges of \(h\) where on the corresponding classical scale the effective fractal dimension \(\delta_j\) is constant. In the semiclassical limit we expect a scaling \(h^{-1}\). Equation (2) is confirmed for the generic standard map and a hierarchical model system.

Classical properties: We first review the classical properties of the chaotic saddle in a generic mixed system and illustrate them for the prototypical example of the Chirikov standard map \textsuperscript{50}. It is obtained from the kicked rotor Hamiltonian \(H(q, p, t) = T(p) + V(q) \sum_{n \in \mathbb{Z}} \delta(t - n)\) with kinetic energy \(T(p) = p^2/2\) and kick potential \(V(q) = \frac{\pi}{2} \cos(2\pi q)\).

At integer times \(t\) it leads to the symmetrized map \(q_{t+1} = q_t + T'(p^*)\), \(p_{t+1} = p_t - \frac{1}{2}V'(q_{t+1})\) with \(p^* = p_t - \frac{1}{2}V'(q_t)\) on the torus \(0,1) \times (-\frac{1}{2}, \frac{1}{2})\). We open the system by defining absorbing stripes of width 0.05 on the left and right, see Fig. (1b). This leads to a chaotic saddle \(\Gamma\), for which a finite-time approximation is shown in Fig. (1b) for \(\kappa = 2.9\). The chaotic saddle \(\Gamma\) of the open system is strongly structured by the presence of partial barriers. They originate from Cantor or stable/unstable manifolds of hyperbolic periodic orbits \textsuperscript{31}. Partial barriers provide a hierarchical tree-like decomposition \textsuperscript{30} of the chaotic component of phase space into regions \(A_j\): A typical orbit explores a region \(A_j\) before it enters a neighboring region \(A_k\). The transition rate is appropriately given by the ratio \(\Phi/A_j\), where \(\Phi\) is the flux across the partial barrier separating \(A_j\) and \(A_k\). The route of escape from region \(A_j\) to the opening is determined by the tree-like decomposition of phase space. It traverses the sequence of neighboring regions connecting \(A_j\) with the opening in \(A_0\). The escape rate from a region \(A_j\) is dominated by the first transition rate, as subsequent transition rates are much larger. Figure (1a) shows the outermost partial barrier separating the largest two regions \(A_0\) and \(A_1\) (which are quantum mechanically accessible), as well as its preimage illustrating its flux \(\Phi\). In addition one can see the two partial barriers separating region \(A_1\) from the chaotic region near the central island and near the period-four regular island chain. All three partial barriers are constructed from stable/unstable manifolds of a period 4 and a period 28 orbit.

Using the box-counting method \textsuperscript{51} one can associate a fractal dimension \(\delta_j\) to the intersection \(\Gamma \cap A_j\) of the chaotic saddle \(\Gamma\) with each of the regions \(A_j\). The number \(N_j^{bc}(\varepsilon)\) of occupied boxes of side length \(\varepsilon\) scales like \(N_j^{bc}(\varepsilon) \sim \varepsilon^{-\delta_j}\), see Fig. (1c), with \(\delta_0 = 1.68\) and \(\delta_1 = 1.86\). To emphasize the difference between such dimensions close to two, the ordinate is rescaled by \(\varepsilon^2\), yielding the rescaled counting function \(N_j^{bc}(\varepsilon) = N_j^{bc}(\varepsilon) \cdot \varepsilon^2\). The increase of \(\delta_j\) towards two when going deeper into the hierarchy can be qualitatively understood by adapting the Kantz–Grassberger relation \textsuperscript{52} from fully chaotic systems.

Hierarchical resonance states: We now present the essential quantum effect that resonance states localize predominantly on one of the regions \(A_j\). The closed quantum system is described by the time-evolution operator \(U = \exp\{-\frac{\pi}{\hbar} V(q)\} \exp\{-\frac{\pi}{\hbar} T(p)\} \exp\{-\frac{\pi}{\hbar} V(q)\}\). The corresponding open quantum system is given by \(U_{open} = PU\), where \(P\) is a projector on all positions not in the absorbing regions. The resonance states \(\psi\) are given by \(U_{open} \psi = \exp[-i(\varphi - i\gamma/2)] \psi\). Regular resonance states are predominantly located in the regular region. Chaotic resonance states are predominantly located in either of the hierarchical regions \(A_j\), see Fig. (1c). Hence, we will call them hierarchical resonance states (of region \(A_j\)). Such a localization of chaotic eigenstates on different sides of a partial barrier is well known for closed quantum systems \textsuperscript{27,36,44}. Chaotic eigenstates localized in the hierarchical region of a mixed phase space were termed hierarchical states \textsuperscript{40}. They require that the classical flux \(\Phi\) across a partial barrier is small compared to the size \(h\) of a Planck cell, i.e. \(\Phi \ll h\). In the opposite case, eigenstates would be equidistributed ignoring the partial barrier \textsuperscript{27,36,44}. Quite surprisingly, in open quantum systems we find that this condition from closed systems is irrelevant for hierarchical resonance states. In the standard map at \(\kappa = 2.9\) we have \(\Phi \approx 1/80\), and for \(h = 1/1000\), such that the condition \(\Phi \ll h\) is violated, typical resonance states still predominantly localize in one of the regions \(A_j\), as shown in Fig. (1c). This is still the case for \(h = 1/12800\), see Fig. (2). This crucial phenomenon for our study highlights the strong impact of the opening.

One can qualitatively understand this localization of hierarchical resonance states in the following way: The localization on an almost invariant region \(A_j\) seems plausible in view of the semiclassical eigenfunction hypothesis for invariant regions \textsuperscript{53–55}. However, eigenstates localized on neighboring regions hybridize, if their coupling due to the flux \(\Phi\) is larger than their energy spacing. In closed systems this happens for \(\Phi > h\) \textsuperscript{27,36,44}. In open systems, though, the distance of resonance energies in the complex plane is larger due to their imaginary part. In fact, it is much larger due to the different decay rates of resonance states of neighboring regions \(A_j\) corresponding to their different classical escape rates. Therefore the localization of resonance states on regions \(A_j\) is possible in open systems, even if the criterion for the closed system,
Hierarchical fractal Weyl laws: For each region $A_j$ of the hierarchical phase space we now relate the number $N_j$ of hierarchical resonance states of that region to the fractal dimension $\delta_j$ of the chaotic saddle in that region. To this end we use the fractal Weyl law of fully chaotic systems \([12, 13]\), Eq. (1), individually for each region $A_j$. This gives our main result that in open systems with a mixed phase space one obtains a hierarchy of fractal Weyl laws, one for each phase-space region $A_j$, Eq. (2). We stress that this result is based on the surprising existence of hierarchical resonance states. Note that as a consequence of Eq. (2) the total number of long-lived hierarchical resonance states is a superposition of power laws with different exponents and not a single power law.

To give an intuitive understanding of the hierarchical fractal Weyl laws, let us recall the interpretation of the fractal Weyl law \([12]\), and apply it in the presence of a hierarchical phase space. The number of quantum states localizing on a particular phase-space region is given by the number of Planck cells necessary to cover the chaotic saddle in that region. Using the scaling, $N_j(\varepsilon) \sim \varepsilon^{-\delta_j}$, of the number of boxes $N_j$ to cover the chaotic saddle in region $A_j$ and the identification of the box area $\varepsilon^2$ with the Planck cell area $h$ directly leads to Eq. (2). This holds for values of $h$ not too small, such that on the corresponding classical scale the effective fractal dimension $\delta_j$ still remains constant. Asymptotically ($\varepsilon \to 0$), all $\delta_j$ approach two \([12, 45]\). Therefore, in the semiclassical limit ($h \to 0$), we expect an individual resonance state to extend over all regions $A_j$ and that their number scales as $h^{-1}$.

Standard map: The numerical investigation of the standard map supports the existence of hierarchical fractal Weyl laws, as we now show. By the classification of resonance states we are able to determine the number $N_j(h)$ of long-lived hierarchical resonance states associated with a particular region $A_j$ depending on $h$. We restrict ourselves to the consideration of small $h$ such that $\Phi/h \gtrsim 10$ where quantum mechanics can very well mimic classical transport in phase space \([44]\). Short-lived states are discarded by defining an arbitrary cut-off rate $\gamma_c = 1$, as usual for the fractal Weyl law \([13]\). In globally chaotic systems the particular choice of $\gamma_c$ (within a reasonable range) does not influence the power-law exponent of the fractal Weyl law but its prefactor only \([14]\). Here this merely affects resonance states of the outermost region $A_0$. We obtain distinct behavior for each rescaled counting function $N_j(h) = N_j(h) \cdot h \cdot f_j$, see Fig. (1c), corresponding to the previous classical rescaling. We fitted prefactors $f_j$ to the quantum results to better demonstrate their scaling with power laws in agreement with the classical counterparts (both prefactors $f_j$ are of order one: $f_0 = 2.6, f_1 = 0.85$). Apart from the smallest values of $1/h$, one observes the power-law scaling of Eq. (2) and good agreement with the box-counting results for the fractal dimensions $\delta_j$ of $\Gamma \cap A_j$. Note that the deviations between corresponding classical and quantum power-law exponents are much smaller than the differences between the exponents associated with different regions $A_j$ of the hierarchy. Figure (1c) confirms for two regions $A_j$ of the standard map that they give rise to hierarchical fractal Weyl laws. Note that the shape and position of the absorbing region modifies the fractal dimension of the chaotic saddle and the power-law exponent of the fractal Weyl law, but their relation remains valid (not shown).

Hierarchical model system: To verify the hierarchical fractal Weyl laws for more than two regions, we suggest the following system that models the hierarchical structure of partial barriers in a generic mixed phase space, similar in spirit to a one-dimensional model \([32]\) and a Markov chain \([29]\). The numerics for the corresponding quantum model allows for studying three regions.

We first define a composed symplectic map $C \circ M$ on the
We obtain the fractal dimensions $\delta_j = 0.69, \delta_1 = 1.94$, and $\delta_2 = 1.99$. Quantum mechanically, we again find hierarchical resonance states predominantly localizing on one of the regions $A_j$, even though $h \ll \Phi_1, \Phi_2$. Their number follows the proposed hierarchical fractal Weyl laws according to Eq. (2). For the rescaled numbers $\tilde{N}_j$ in Fig. 3 we use prefactors $f_0 = 1.75, f_1 = 1.55$, and $f_2 = 0.8$, which are of order one.

An experimental verification of the hierarchical fractal Weyl laws should be feasible using microwave cavities as in a recent study on chaotic resonance states [59]. A future challenge is the study of fractal Weyl laws in higher dimensional systems with a generic phase space.

We are grateful to E. G. Altmann, H. Kantz, H. Schomerus, A. Shudo, and T. Tél for helpful comments and stimulating discussions, and acknowledge financial support through the DFG Forschergruppe 760 “Scattering systems with complex dynamics.”

[1] H. Weyl, Math. Ann. 71, 441 (1912).
[2] M. Kac, Amer. Math. Mon. 73, 1 (1966).
[3] H.-J. Stöckmann, Quantum Chaos: An Introduction (Cambridge University Press, Cambridge, 1999).
[4] W. Arendt et al., in Mathematical Chaos: An Introduction, Information and Complexity, edited by W. Arendt and W.-P. Schleich (Wiley-VCH, Weinheim, 2009) pp. 1–71.
[5] R. Balian and C. Bloch, Ann. Phys. 60, 401 (1970).
[6] H. P. Baltes and E. R. Hilf, Spectra of Finite Systems (B. I. Wissenschaftsverlag, Mannheim, 1976).
[7] M. V. Berry and C. J. Howls, Proc. R. Soc. Lond. A 447, 527 (1994).
[8] V. Ivrii, Microlocal Analysis and Precise Spectral Asymptotics, Springer Monographs in Mathematics (Springer, Berlin, 2010).
[9] A. Bäcker et al., Europhys. Lett. 94, 30004 (2011).
[10] J. Sjöstrand, Duke Math. J. 60, 1 (1990).
[11] K. K. Lin, J. Comp. Phys. 176, 295 (2002).
[12] W. T. Lu, S. Sridhar, and M. Zworski, Phys. Rev. Lett. 91, 154101 (2003).
[13] H. Schomerus and J. Tworzydlo, Phys. Rev. Lett. 93, 154102 (2004).
[14] S. Nonnenmacher and M. Zworski, J. Phys. A 38, 10683 (2005).
[15] J. P. Keating, M. Novaes, S. D. Prado, and M. Sieber, Phys. Rev. Lett. 97, 150406 (2006).
[16] S. Nonnenmacher and M. Rubin, Nonlinearity 20, 1387 (2007).
[17] J. Wiersig and J. Main, Phys. Rev. E 77, 036205 (2008).
[18] D. L. Shepelyansky, Phys. Rev. E 77, 015202(R) (2008).
[19] J. A. Ramilowski, S. D. Prado, F. Borondo, and D. Farrelly, Phys. Rev. E 80, 055201(R) (2009).
[20] A. Eberspächer, J. Main, and G. Wunner, Phys. Rev. E 82, 046201 (2010).
[21] L. Ermann and D. L. Shepelyansky, Eur. Phys. J. B 75, 299 (2010).
[22] A. Potzuweit et al., Phys. Rev. E 86, 066205 (2012).
[23] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, Ann. of Math. in press (2013) arXiv:1105.3128.
[24] Y.-C. Lai and T. Tél, Transient Chaos: Complex Dynamics on Finite-Time Scales, Applied Mathematical Sciences (Springer, New York, 2011).
[25] E. G. Altmann, J. S. E. Portela, and T. Tél, Rev. Mod. Phys. 85, 869 (2013).
[26] L. Markus and K. R. Meyer, Generic Hamiltonian Dynamical Systems are neither Integrable nor Ergodic, Memoirs of the Mathematical Society, Vol. 144 (Amer. Math. Soc., Providence, Rhode Island, 1974).
[27] R. S. MacKay, J. D. Meiss, and I. C. Percival, Phys. Rev. Lett. 52, 697 (1984).
[28] R. S. MacKay, J. D. Meiss, and I. C. Percival, Physica 13D, 55 (1984).
[29] J. D. Hanson, J. R. Cary, and J. D. Meiss, J. Stat. Phys. 39, 327 (1985).
[30] J. D. Meiss and E. Ott, Phys. Rev. Lett. 55, 2741 (1985).
[31] J. D. Meiss, Rev. Mod. Phys. 64, 795 (1992).
[32] A. E. Motter, A. P. S. de Moura, C. Grebogi, and H. Kantz, Phys. Rev. E 71, 036215 (2005).
[33] G. Cristadoro and R. Ketzmerick, Phys. Rev. Lett. 100, 184101 (2008).
[34] R. C. Brown and R. E. Wyatt, Phys. Rev. Lett. 57, 1 (1986).
[35] T. Geisel, G. Radons, and J. Rubner, Phys. Rev. Lett. 57, 2883 (1986).
[36] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rep. 223, 43 (1993).
[37] R. Ketzmerick, Phys. Rev. B 54, 10841 (1996).
[38] A. S. Sachrajda et al., Phys. Rev. Lett. 80, 1948 (1998).
[39] N. T. Maitra and E. J. Heller, Phys. Rev. E 61, 3620 (2000).
[40] R. Ketzmerick, L. Hufnagel, F. Steinbach, and M. Weiss, Phys. Rev. Lett. 85, 1214 (2000).
[41] A. Bäcker, A. Manze, B. Huckestein, and R. Ketzmerick, Phys. Rev. E 66, 016211 (2002).
[42] A. P. Micolich et al., Phys. Rev. B 70, 085302 (2004).
[43] C.-M. Goletz, F. Grossmann, and S. Tomsovic, Phys. Rev. E 80, 031101 (2009).
[44] M. Michler, A. Bäcker, R. Ketzmerick, H.-J. Stöckmann, and S. Tomsovic, Phys. Rev. Lett. 109, 234101 (2012).
[45] Y.-T. Lau, J. M. Finn, and E. Ott, Phys. Rev. Lett. 66, 978 (1991).
[46] M. Kopp and H. Schomerus, Phys. Rev. E 81, 026208 (2010).
[47] M. E. Spina, I. García-Mata, and M. Saraceno, J. Phys. A 43, 392003 (2010).
[48] A. Ishii, A. Akaishi, A. Shudo, and H. Schomerus, Phys. Rev. E 85, 046203 (2012).
[49] C. Birchall and H. Schomerus, arXiv:1208.2259 (2012).
[50] G. Casati et al., Lecture Notes in Physics 93, 334 (1979).
[51] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, 2nd ed. (Wiley, West Sussex, 2003).
[52] H. Kantz and P. Grassberger, Physica 17D, 75 (1985).
[53] I. C. Percival, J. Phys. B 6, L229 (1973).
[54] M. V. Berry, J. Phys. A 10, 2083 (1977).
[55] A. Voros, Lecture Notes in Physics 93, 326 (1979).
[56] E. Persson, T. Gorin, and I. Rotter, Phys. Rev. E 58, 1334 (1998).
[57] E. Persson and I. Rotter, Phys. Rev. C 59, 164 (1999).
[58] E. Persson, I. Rotter, H.-J. Stöckmann, and M. Barth, Phys. Rev. Lett. 85, 2478 (2000).
[59] S. Barkhofen et al., Phys. Rev. Lett. 110, 164102 (2013).