Leading quantum gravitational corrections to QED

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We consider the leading post-Newtonian and quantum corrections to the non-relativistic scattering amplitude of charged spin-\frac{1}{2} fermions in the combined theory of general relativity and QED. The coupled Dirac-Einstein system is treated as an effective field theory. This allows for a consistent quantization of the gravitational field. The appropriate vertex rules are extracted from the action, and the non-analytic contributions to the 1-loop scattering matrix are calculated in the non-relativistic limit. The non-analytical parts of the scattering amplitude are known to give the long range, low energy, leading quantum corrections, are used to construct the leading post-Newtonian and quantum corrections to the two-particle non-relativistic scattering matrix potential for two massive fermions with electric charge.

I. INTRODUCTION

On the search for a theory of quantum gravity, Donoghue \cite{1} proposed 12 years ago an interesting new way to look at general relativity. He suggested that when treating general relativity as an effective field theory \cite{2}, reliable quantum predictions at the low energies could be made, in the same way as chiral perturbation theory is used as the low energy approximation of QCD, being the effective field theory of QCD. It is well known that a field theory need not be strictly renormalizable in order to be able to yield quantum predictions at low energies. A fundamental quantum theory of gravity does not appear in this way, but it is possible to calculate quantum corrections order by order in a momentum expansion.

Having laid the foundations for this new approach, Donoghue and collaborators turned their attention to the practical applications of this idea. A number of interesting calculations has been made involving quantum gravitational corrections to various quantities \cite{3,4,5,6,7,8}.

Prior to the effective field theoretical description of general relativity, attempts had been made to find a quantum theory for gravity. In particular many proved that general relativity was indeed not renormalizable, be it pure general relativity or general relativity coupled to bosonic or fermionic matter, see e.g. \cite{9,10,11,12}. Of course it is a well known fact that general relativity is a non-renormalizable theory per se, and these authors succeeded in exactly confirming that gravity indeed is explicitly non-renormalizable, with or without matter. However, when looked at in the framework of an effective field theory, these theories do become order by order renormalizable in the low energy limit. All possible counter-terms, also those not present in the initial Lagrangian, are generated. When general relativity is treated as an effective theory, renormalizability simply fails to be an issue. The ultraviolet divergences arising e.g. at the 1-loop level are dealt with by renormalizing the parameters of higher derivative terms in the action. Many interesting results have been found from this procedure. Most interesting for the point of view of this paper is the bosonic quantum corrections to the Newtonian/Coulomb potentials \cite{4,8}.

When approaching general relativity in this manner, it is convenient to use the background field method \cite{4,5}. Divergent terms are absorbed away into phenomenological constants which characterize the effective action of the theory. The price paid is the introduction of a set of never-ending higher order derivative couplings into the theory. General relativity thus turns into a minimal theory in the myriad of higher order terms in the action, however still remaining a valid theory for gravitational interactions at low energies. The effective action contains all terms consistent with the underlying symmetries of the theory. Perturbatively certain terms of the action play leading roles at certain energy scales, hence only a finite number of terms are required to be accounted for at each loop order. Here we consider the low energy limit of the effective theory of quantum gravity. Because we work to lowest order, our results are free of the new additional terms that must be appended to the Einstein action and which manifest themselves as components of the high energy couplings of the effective field theory.

In ref. \cite{8} the post-Newtonian as well as the quantum corrections, that were generated to the Newtonian and Coulomb potentials were explicitly found. We wish to repeat this calculation, but now in terms of couplings to fermions. We wish in particular to see explicitly if the post-Newtonian as well as the quantum gravitational corrections generated are identical to \cite{8} or not.

We will more or less follow a similar procedure as in \cite{8} mostly to avoid confusion about conventions and to make it easy to compare the results at the end. However, it is by no means a straightforward task to complete a similar calculation in terms of fermions. Some obstacles have to be overcome compared to bosonic matter in curved space, e.g. the issue of introducing fermionic matter into curved space-time. Luckily, this issue has been dealt with before \cite{9,10,11,12,13,14,15,16}. The additional formal-
ism required is the introduction of the vierbein formalism and deriving a proper covariant derivative for the spinor fields.

Donoghue devised a particular elegant way to extract relevant information in terms of analytical and non-analytical contributions to the scattering matrix. This was realized through the integrals occurring in the calculations, and propagation of massless particles. Because the post-Newtonian and the quantum corrections are fully determined by the non-analytical pieces of the 1-loop amplitude generated by the lowest order Einstein action, it becomes possible to perform this calculation completely. We will also only consider 1-loop effects in this paper. We will extract the non-analytical parts of the full set of 1-loop diagrams needed for the 1-loop scattering matrix in the combined quantum theory of general relativity and QED. As we will see in this paper, and as can also seen in [1, 2, 6, 7, 8], the non-analytical contributions correspond exactly to the long range corrections of the potential.

The mostly minus Minkowski metric convention \(\eta_{\mu\nu}=\begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}\) will be used and the natural units are \((\hbar = c = 1)\) when nothing else is stated.

In section II we will shortly review the concept of vierbein fields and how to introduce a proper covariant derivative for the spinors when working with fermions. We will also quantize the metric and vierbeins using the background field method. We will explain the correspondence between the metric and vierbein formulation of our theory. In section III we will see how to combine QED with general relativity by using the vierbein formalism and moreover introduce the ghost fields. Next in section IV we will focus first on the distinction between non-analytical and analytical contributions to the scattering matrix amplitude, where after we will define the potential. Finally, in the succeeding section V we will evaluate the Feynman diagrams contributing non-analytically to the scattering matrix, in order to construct the leading corrections to the non-relativistic Newtonian and Coulomb potential. We will end this paper with a discussion in relation to II. In the appendix, the vertex rules are presented.

II. INTRODUCING THE VIERBEIN FIELDS

At every space-time point \(x_0\) it is possible to erect a set of coordinates \(\xi^a_{x_0}\) locally inertial at the given point in question, in accordance with the theory of special relativity. This in turn implies that the erected set of coordinates \(\xi^a_{x_0}\) vary from point to point, and that the information about the gravitational field is in fact contained in the change of the local inertial coordinate systems from point to point. Hence it is possible to express \(\xi^a_{x_0}\) as a local function of any non-inertial coordinates (i.e. in a general coordinate system) \(x^\mu\) i.e.

\[
d\xi^a = e^a_\mu dx^\mu \hspace{2cm} (1a)
\]
\[
dx^\mu = e_a^\mu d\xi^a \hspace{2cm} (1b)
\]
evaluating the derivatives at the point of interest, where the inverse operation is also given at the point in question. The transformation matrix relating the local inertial frames to the arbitrary coordinate system is the called vierbein field, it is denoted \(e^a_\mu(x)\) and is a function of \(x^\mu\).

It is seen that due to the transformations \(x \to \xi\) and \(\xi \to x\) being nonsingular, \(e_a^\mu(x)\) is the inverse transformation matrix.

One can find other relations between the vierbein fields, e.g.

\[
e^a_\mu e_b^\mu = \delta^a_b \hspace{2cm} (2a)
\]
\[
e^a_\mu e^\nu = \delta^\mu_\nu \hspace{2cm} (2b)
\]
The metric expressed in general coordinates can be found by looking at the proper time in general coordinates

\[
d\tau^2 = \eta_{ab}d\xi^a(x)d\xi^b(x) \hspace{2cm} (3)
\]
where the metric in general coordinates is

\[
g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab} \hspace{2cm} (4)
\]
where \(a, b\ldots\) are Lorentz indices and \(\mu, \nu\ldots\) are the general coordinate indices. The inverse of the metric is

\[
g^{\mu\nu} = e_{a\mu}(x)e_{b\nu}(x)\eta^{ab} \hspace{2cm} (5)
\]
from which a familiar result is obtained

\[
g_{\mu\nu}g^{\nu\sigma} = (e^a_\mu e^b_\nu \eta_{ab})(e_{c\nu} e_{d\sigma} \eta^{cd}) = e_{a\mu}e_{a\sigma} = \delta^\sigma_\mu \hspace{2cm} (6)
\]

Thus the \(e^a_\mu(x)\) fields relate the Lorentz axes to the coordinate axes at each point in space-time, explicitly

\[
e^a_\mu(x = x_0) = \left(\frac{\partial \xi^a_{x_0}(x)}{\partial x^\mu}\right)_{x=x_0} \hspace{2cm} (7a)
\]

When changing the general coordinates from \(x^\mu \to x'^\mu\), these fields transform as covariant vectors and under local Lorentz transformations they transform as Lorentz vectors

\[
e^a_\mu(x) \to e'^a_\mu(x') = \frac{\partial x'^\nu}{\partial x^\mu} e_{a\nu}(x) \hspace{2cm} (7a)
\]
\[
e^a_\mu \to e'^a_\mu = \Lambda^a_b e^b_\mu \hspace{2cm} (7b)
\]
The vierbein fields are the only index changing objects in this theory. For given covariant/contravariant vector fields or also a tensor field, we can refer their components at \(x\) to the locally inertial coordinate system \(\xi^a_{x_0}(x)\) at \(x_0\) by using the vierbein

\[
e^a_\mu A^\mu = A^a \hspace{2cm} (8)
\]
$A^a$ transforms as a collection of four scalars under general coordinate transformations \([\mathbf{10}]\), and under the local Lorentz transformations \([\mathbf{11}]\) it behaves as a vector. It is thus possible to use the vierbeins, to convert general tensors into local, Lorentz-transforming tensors, whereby shifting the additional space-time dependence into the vierbeins.

\section*{A. Gauge transformation of the fermion fields and the spin connection}

Using the generators of the Lorentz group $\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b]$ for spinors belonging to a given representation $S(\Lambda) = e^{\frac{i}{2} \lambda^{ab} \sigma_{ab}}$ for small $\lambda$ in infinitesimal form we get for spinor transformations

$$\psi' \rightarrow (1 + \frac{1}{2} \sigma_{ab} \lambda^{ab}(x)) \psi$$  \hspace{0.5cm} (9)

Since partial derivatives on fields almost always occur in the Lagrangian density functions, it would be worthwhile to subject \([\mathbf{9}]\) to a partial differentiation, and see if it transforms covariantly or not. We immediately see from the transformation

$$\partial_{\mu} \psi' \rightarrow (1 + \frac{1}{2} \sigma_{ab} \lambda^{ab}(x)) \partial_{\mu} \psi + \frac{1}{2} \sigma_{ab} \partial_{\mu} \lambda^{ab}(x) \psi$$  \hspace{0.5cm} (10)

that the fermion field does not transform as a proper Lorentz spinor under this operation, even though the ordinary derivative $\partial_{\mu} \psi$ is a covariant vector since the fermion fields are defined to be scalar objects under coordinate transformations. In order to get a feasible theory \([\mathbf{10}]\) must transform covariantly, hence we must invent a “covariant derivative” $D_{\mu}$ for fields that transform in this manner, such that in the end result the second term in \([\mathbf{10}]\) gets canceled.

We can find this object by studying the behavior of the vierbein fields under the transformations \([\mathbf{10}]\) and \([\mathbf{11}]\) and comparing it with the transformation properties of a general four vector \([\mathbf{11}]\).

The rule for covariant derivative for an object mixed in both Lorentz and coordinate indices $A_{\mu}^a$ is \([\mathbf{24}]\)

$$D_{\mu} A_{\nu}^a = \partial_{\mu} A_{\nu}^a - \Gamma_{\mu\nu}^\kappa A_{\kappa}^a + \omega_{\mu}^{ab} A_{\nu}^b$$  \hspace{0.5cm} (11)

where $\omega_{\mu}^{ab}$ is called the spin connection. To determine the structure of the spin connection one can require that the covariant differentiation should commute with the operation of index changing, not only just index lowering/raising \([\mathbf{3}]\), thus using the condition \([\mathbf{24}]\)

$$D_{\nu} e_{\mu}^a = \partial_{\nu} e_{\mu}^a - \Gamma_{\mu\nu}^\kappa e_{\kappa}^a + \omega_{\nu}^{ab} e_{\mu}^b = 0$$  \hspace{0.5cm} (12)

one finds that the following object does the job

$$\omega_{\mu}^{ab} = \frac{1}{2} \left( e_{\nu}^{[ab} \partial_{\mu]} e_{\nu]}^c + e_{\nu}^{[ab} \partial_{\mu} e_{\nu]}^c - e_{\nu}^{[ab} \partial_{\mu} e_{\nu]}^c - e_{\nu}^{[ab} \partial_{\mu] e_{\nu]}^c - e_{\nu}^{[ab} \partial_{\mu} e_{\nu]}^c - e_{\nu}^{[ab} \partial_{\mu]} e_{\nu]}^c \right)$$  \hspace{0.5cm} (13)

Note that the indices commute only with alike (space-time-)indices, i.e. Latin characters with Latins and Greek characters with Greek.

This object has exactly the transformation property that we were searching for

$$\omega_{\mu}^{ab} \rightarrow \omega_{\mu}^{ab} - \partial_{\mu} \lambda^{ab}$$  \hspace{0.5cm} (14)

where the primed object shows transformation as a tensor. The covariant derivative for spinor fields now becomes

$$D_{\mu} \psi \equiv (\partial_{\mu} + \frac{1}{2} \sigma_{ab} \omega_{\mu}^{ab}) \psi$$  \hspace{0.5cm} (15)

This will transform as a proper Lorentz spinor and a covariant vector under the mentioned gauge transformations.

\section*{B. Quantization of the vierbein and metric fields}

To quantize the combined theory of QED and gravity we will have to quantize the vierbein fields in the same way as it is done when quantizing the metric.

In the background field method the quantum corrections to general relativity are described by quantum vibrations of the metric tensor, making it possible to expand the metric and vierbein fields into two separate contributions, a classical background field and a quantum field

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa h_{\mu\nu}$$  \hspace{0.5cm} (16)

$$e_{\mu}^a = \tilde{e}_{\mu}^a + \kappa e_{\mu}^a$$  \hspace{0.5cm} (17)

where $\kappa^2 = 32\pi G$ and the background fields are denoted as $\tilde{g}_{\mu\nu}$ and $\tilde{e}_{\mu}^a$. The quantum part - the graviton field - is denoted by $h_{\mu\nu}$ and $e_{\mu}^a$, the sum of these being the full metric and vierbein respectively. We will soon see how the metric and vierbein quantum variables for the gravitons are related. Furthermore we find the following inverses and other relations

$$e_{a\mu} = \tilde{e}_{a\mu} + \kappa e_{a\mu} + \ldots$$

$$e_{a\mu}^a = \tilde{e}_{a\mu}^a - \kappa e_{a\mu} + \ldots$$  \hspace{0.5cm} (18)

for the vierbeins

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa h_{\mu\nu}$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa (e_{a\mu} e_{a\mu}) + \kappa^2 e_{a\mu} e_{a\nu}$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} - \kappa e_{a\mu} e_{a\nu} + \ldots$$  \hspace{0.5cm} (19)

$$h_{\mu\nu} = e_{\mu\nu} + \kappa c_{a\mu} e_{a\nu} = s_{\mu\nu} + \kappa c_{a\mu} e_{a\nu}$$  \hspace{0.5cm} (20)
showing us that the quantized metric field is equal the quantized symmetric vierbein field to first order in the quantum fields, i.e. $h_{\mu\nu} = c_{\mu\nu} + c_{\mu\nu} = s_{\mu\nu}$. It is easy to deduce the determinants of the vierbein and metric fields

$$ e = \det[e^a_\mu] \approx \hat{e} \left(1 + \kappa e^a_\mu + \ldots\right) $$

(21)

with $\hat{e} \equiv \det \bar{e}^a_\mu$, as well as the square-root of the metric tensor $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$

$$ \sqrt{-g} \approx \sqrt{-\hat{g}} \left(1 + \frac{\kappa}{2} h^a_\mu + \ldots\right) $$

(22)

with $\hat{g} \equiv \det \bar{g}_{\mu\nu}$. Finally it will be necessary to expand the spin connection in terms of the vierbein fields as well, the leading order terms appearing in the calculations are (in flat background field)

$$ w^\text{Background}_{\muab} = 0 $$

(23a)

$$ w^\text{First order}_{\muab} = \frac{1}{2} \partial_\mu a_{ab} + \frac{1}{2} \partial_\mu s_{ab} - \frac{1}{2} \partial_a s_{\mu b} $$

(23b)

where we have defined a new field, an antisymmetric field $a_{\mu\nu} = c_{\mu\nu} - c_{\nu\mu}$.

III. THE DIRAC-EINSTEIN SYSTEM AS A COMBINED EFFECTIVE FIELD THEORY

The combined theory of QED in a gravitational field is given by the sum of the QED and Einstein Lagrangian densities

$$ \mathcal{L} = \mathcal{L}_{\text{Gravity}} + \mathcal{L}_{\text{QED}} $$

(24)

The interacting field theory for Quantum Electrodynamics is well known, with the Dirac equation minimally coupled to the electromagnetic field

$$ \mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} $$

$$ \mathcal{L}_{\text{QED}} = \frac{\bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} g^{\mu\nu} g^{\beta\nu} F_{\alpha\mu} F_{\mu\beta}}{\kappa^2} $$

(25)

where $m$ is the mass, $e_q$ is the electron charge with $e_q = |e_q|$ and finally $D_\mu = \partial_\mu - ie_q A_\mu(x)$ is the covariant derivative.

To make the action of the Dirac Lagrangian density invariant under general coordinate transformations, we follow the general procedure, i.e. multiply it with $\sqrt{-\hat{g}}$ and at the same time introduce our new covariant derivative

$$ \mathcal{L}_{\text{Dirac}} = \sqrt{-\hat{g}} \bar{\psi} (i\gamma^\mu D_\mu - m) \psi = \sqrt{-\hat{g}} \left( ie_q A_\mu D_\mu - m \right) \psi $$

(26)

now $D_\mu = \partial_\mu - ie_q A_\mu + \frac{1}{2} \sigma^{ab} w_{\mu ab}$ and we have used $\sqrt{-\hat{g}} = \det(e^a_\mu) \equiv e$ i.e. the determinant of the vierbein is the matrix square-root of the metric. Finally $\gamma^\mu = \gamma^\mu e^a_\mu$.

The full generally covariant Lagrangian density including the fermionic degrees of freedom may collectively be written as

$$ \mathcal{L} = e \frac{2}{\kappa^2} R + e \sqrt{-\hat{g}} \bar{\psi} e_q A_\mu D_\mu \psi - \bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi} \gamma^\mu \gamma^\nu D_\mu \psi $$

(27)

This will account for our full theory. The Lagrangian density is to be expanded in powers of $c_{\mu\nu}$ (where we choose $c_{\mu\nu}$ to be linearly symmetrically equal to $h_{\mu\nu}$ as seen earlier) in the case of the Dirac field and only $h_{\mu\nu}$ in the case of the Maxwell fields, specifically we expand the Lagrangian density as follows

$$ \mathcal{L} = \mathcal{L}_{\text{Background}} + \mathcal{L}_{\text{Linear order}} + \ldots $$

(28)

where the ellipses denote second and higher order terms that will not contribute at the 1-loop level calculations.

The Lagrangian density for the photon field can now be expanded in powers of $h_{\mu\nu}$, explicitly

$$ \mathcal{L}_{\text{Maxwell}} = -\frac{\kappa}{4} h (\partial_\mu A_\nu \partial^\mu A_\nu - \partial_\nu A_\mu \partial^\mu A_\nu) + \frac{\kappa}{2} h^{\rho\sigma} (\partial_\mu A_\nu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\rho A_\sigma - \partial_\sigma A_\rho \partial_\mu A_\nu - \partial_\sigma A_\rho \partial_\nu A_\mu) $$

(29)

where the trace of $h \equiv h^a_\alpha = h^a_\alpha = h$. And likewise for the fermionic part

$$ \mathcal{L}_{\text{Dirac}} = e_q \bar{\psi} \gamma^\mu A_\mu \psi + \frac{ie_q}{2} \bar{\psi} \gamma^a (I_a^\mu \beta - \delta_\mu^\alpha e_q A_\beta) A_\mu \psi + \frac{\kappa}{4} h (\bar{\psi} \gamma^\mu \partial_\mu \psi - \bar{\psi} m \psi) - \frac{\kappa}{2} \bar{\psi} \gamma^\mu h^{\nu\rho} \partial_\nu \psi + \frac{\kappa}{2} \bar{\psi} \gamma^\mu \sigma^{ab} \partial_\nu h_{\mu\nu} \psi $$

(30)

where the symmetric identity $I_a^\mu \beta = \frac{1}{2} h_a^a (\delta \eta^a)^{\mu\nu}$.

All the necessary lowest order interaction vertices of fermions, gravitons and photons can be found for the theory from the Linear order expansions as stated above in equations (29) and (30). A summary of these rules is presented in the appendix.
In principle we should also expand

\[ \mathcal{L} = \frac{2}{\kappa^2} R \]  

(31)

however it is known that the metric and vierbein formulations are equivalent for fields with only covariant vector indices. The coupling to the vierbein field only occur as symmetric combinations of vierbein fields, the symmetric combination is as we have seen to linear order equal to the metric tensor. No new aspects of the traditional quantization of the pure gravitational action are introduced. We will therefore use the known vertices and propagators for the bosonic and gravitational fields.

We have excluded the antisymmetric fields in all our expressions. This is due to the fact that the antisymmetric fields have propagators that go as \( \sim \kappa^2 \). This can be seen when we fix the gauges in our quantization scheme. As pointed out earlier, our theory (i.e. fermions including gravitational effects) has two types of invariances. One is the general coordinate transformations (1a), under which the fermions behave as scalars (since they are defined with respect to the local Lorentz frame). The other is the local Lorentz transformations (1b), under which the fermions transform as spinors. If Einstein action is included, then the coordinate gauge can be fixed by choosing the harmonic (de Donder) gauge

\[ \mathcal{L}^C = -\frac{1}{2} \sqrt{-g} (h_{\mu\nu,\nu} - \frac{1}{2} h_{\nu,\nu})^2 \]  

(32)

whereas the local Lorentz invariance is broken by choosing the sum of the squares of the antisymmetric vierbein components

\[ \mathcal{L}^L = -\frac{1}{2} \epsilon \kappa^{-2} a_{\mu\nu}^2 \]  

(33)

Gauge fixing of both these fields will result in an introduction of two sets of ghost fields. We do not need to be concerned about the ghost introduced due to the antisymmetric field. In a vierbein description of pure gravity, the ghosts are never external, furthermore neither the antisymmetric vierbein fields nor its ghosts propagate (they cancel each other (2)), thus we will not need to calculate vertices for the external ghosts fields. This is very reassuring since the pure gravity theory in vierbein formulation can be covariantly quantized and is equivalent to the quantized metric approach. That is we could in principle describe the theory without introducing these variables. But if we do not have pure gravity and include fermions, the antisymmetric fields become coupled to the vertices. We need only consider the symmetric fields of the interactions. This is due to the fact that we will only be interested in the long range corrections to the background field, and the antisymmetric fields do not produce non-analytic terms to the order at which we are working, due to the proportionality factor of its propagator \( \sim \kappa^2 \).

In a diagram consisting of at least an antisymmetric field and a graviton vertex will at least go as \( \sim \kappa^3 \) which is an order higher than \( \sim \kappa^2 \). However in a full treatment of gravitational interaction between fermionic matter these fields will have important contributions. They will most likely contribute to higher order calculations.

**IV. THE SCATTERING MATRIX AND THE POTENTIAL**

It will be fruitful to make a distinction between non-analytical and analytical contributions from the diagrams, in order to compute the leading long range, low energy quantum corrections to this theory.

This distinction originates from the impossibility of expanding a massless propagator \( \sim \frac{1}{q} \) while on the other hand we have

\[ \frac{1}{q^2 - m^2} = -\frac{1}{m^2} \left(1 + \frac{q^2}{m^2} + \ldots\right) \]  

(34)

expansion for the massive propagator. No \( \sim \frac{1}{q} \) terms are generated by the above expansion of the massive propagator. We see that the non-analytical contributions are inherently non-local effects which cannot be expanded in a power series in momentum. Thus the non-analytical effects appear from the propagation of massless particle modes, in our case the gravitons and photons. These non-analytical contributions will be governed to leading order only by the minimally coupled Lagrangian.

The non-analytical effects are e.g. terms in the S-matrix which go as \( \sim \ln(-q^2) \) or \( \sim \frac{1}{\sqrt{-q^2}} \). General example of an analytical effect is a power series in momentum. Thus the non-analytical contributions are governed to leading order only by the minimally coupled Lagrangian.

As in (1), the high energy renormalization of the theory will also be of no concern for us, since we are also only interested in finding the leading finite non-analytical momentum contributions for the 1-loop diagrams in the low energy scale of the theory. The singular analytical momentum parts which are absorbed into coefficients of the higher derivative couplings, will have no part to play here, they will ultimately not be manifested in this energy regime of the theory.

**1. Defining the Potential**

The S-matrix is defined as the scattering matrix between incoming and outgoing particles. The invariant matrix element \( i\mathcal{M} \) originating from the diagrams is

\[ \langle k_1 k_2 \ldots | iT | k_A k_B \rangle = (2\pi)^3 \delta^4(k_A + k_B - \Sigma \delta_{k\text{final}}) \left( i\mathcal{M} \right) \]  

(35)

here we have two incoming particles. If we Fourier transform the earlier mentioned non-analytic terms to real
or rather potential transformed in momentum space. We should be inventions, thus we divide with these to obtain the non-

\[ \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} = \frac{1}{|\vec{q}|} \tag{36} \]

\[ \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} = \frac{1}{2\pi^2 r^2} \tag{37} \]

\[ \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \ln(q^2) = -\frac{1}{2\pi^3} \tag{38} \]

obviously these terms indeed do contribute to the long-range corrections. When we calculate the tree diagrams, we explicitly see that the non-analytic contribution of the type \( \delta \), will correspond to the Coulomb and Newtonian part of the potentials and the higher power of \( q \) will generate the leading order and classical corrections to the Coulomb and Newtonian potentials. Explicitly the invariant matrix element will look like

\[
\mathcal{M} = \left( A + Bq^2 + (\alpha_1 k^2 + \alpha_2 e^2) \frac{1}{q^2} \right. \\
+ \beta_1 k^2 e^2 q^2 \ln(-q^2) + \beta_2 k^2 e^2 q^2 \frac{m}{\sqrt{-q^2}} \ldots \right) \tag{39}
\]

where \( A, B, \ldots \) correspond to the analytical, local and short-ranged interactions, these terms will only dominate in the high energy regime of the effective field theory, whereas \( \alpha_1, \alpha_2, \ldots \) and \( \beta_1, \beta_2, \ldots \) correspond to the leading non-analytic, non-local, long-range contributions to the amplitude. Many diagrams will yield pure analytic contributions to the S-matrix, such diagrams will not be necessary in our calculations, we will only consider the non-analytic contributions from the 1-loop diagrams. The diagrams which will yield non-analytic contributions to the S-matrix amplitude are those containing two or more massless propagating particles.

Relating the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, we get in terms of \( iT \)

\[ \langle k_1 k_2 \ldots | iT | k_A k_B \rangle = -i\tilde{V}(q)(2\pi)\delta(E - E') \tag{40} \]

where \( q = p' - p \) and \( \tilde{V}(q) \) is the non-relativistic potential transformed in momentum space. We should be careful when comparing with \( i\mathcal{M} \), in \( (i\mathcal{M}) \) factors of \((2m_1 \times 2m_2)\) arise due to relativistic normalization conventions, thus we divide with these to obtain the non-relativistic limit. Equating the two we deduce

\[ -i\tilde{V}(q)(2\pi)\delta(E - E') \sim (2\pi)^4 \delta^4(k_A + k_B - \Sigma k_{\text{final}})(i\mathcal{M}) \tag{41} \]

or rather

\[ \tilde{V}(q) = -\frac{1}{2m_1 2m_2} \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 \delta^3(k_A + k_B - \Sigma k_{\text{final}})(\mathcal{M}) \tag{42} \]

Momentum integration yields the non-relativistic potential

\[ \tilde{V}(q) = -\frac{1}{2m_1 2m_2} M \tag{43} \]

In our calculations, \( M \) will only contain the non-analytic contributions of the amplitude of the scattering process to 1-loop order, and we will not compute the full amplitude of the S-matrix, only the long-range corrections will be of our interest. In order to obtain their contribution to the potential, only a subclass of scattering matrix diagrams will be required. If we wanted to find the full total non-relativistic potential, we would merely have to include the remaining 1-loop diagrams. This type of calculation has e.g. been performed in \( \cite{21} \) (who also have used the same definition of the potential as us) where the full amplitude is considered. Their choice of potential included all 1-loop diagrams, hence they obtained a gauge invariant definition of the potential. This choice of the potential makes good physical sense since it is gauge invariant, but other choices are also possible. The most convenient choice could depend on the physical situation at hand or how the total energy is defined. The gauge invariant choice is also equivalent to the suggestion in \( \cite{21} \), where it is suggested that one should use the full set of diagrams constituting the scattering matrix, from which one can decide the non-relativistic potential from the total sum of the 1-loop diagrams. However, it is worthwhile to note that we consider all the non-analytic corrections to 1-loop order, thus if we had the full amplitude to 1-loop order we would still need to extract the non-analytical parts! We will continue using this definition of the potential.

V. RESULTS FOR THE FEYNMAN DIAGRAMS

A. Diagrams contributing to the non-analytic parts of the scattering matrix potential

In this section we shall extract the non-analytical parts of a limited set of 1-loop diagrams needed for the 1-loop scattering matrix in the combined quantum theory of QED and general relativity (however, it is a practically complete set of diagrams in terms of non-analytical contributions to the scattering matrix). We will explicitly see that the non-analytic contributions indeed correspond to the long range corrections of the potential. This will become obvious when the amplitudes are Fourier transformed to produce the scattering potential, whence all the analytic pieces are disregarded. The resulting non-analytic piece of the scattering amplitude will then be used to construct the leading corrections to the non-relativistic gravitational potential.
B. Classical physics

Here we will look at the tree diagrams. The fermion-fermion scattering process at tree level should of course reproduce the results of classical physics both for gravitational interactions as well as for electromagnetic interactions.

1. Tree diagrams

Given in figure 1, we have depicted the scattering process, where the (incoming/outgoing) momenta for particle one are \((k/k')\) with the (mass/charge) being \((m_1/e_1)\), and similarly for the second particle with the (incoming/outgoing) momenta \((p,p')\) and \((m_2/e_2)\) being the (mass/charge). This is assigned for all the other diagrams as well. The formal expression for the diagram depicted in figure 1(a), the scattering process involving a photon exchange, is

\[
iM_{1(a)} = \bar{u}(p')[\tau^\alpha]u(p)\bar{u}(k')[\tau^\beta]u(k)\left[-\frac{\eta_{\alpha\beta}}{q^2}\right]
\]

and a graviton exchange, figure 1(b)

\[
iM_{1(b)} = \bar{u}(p')[\tau^{\mu\nu}]u(p)\bar{u}(k')[\tau^{\alpha\beta}]u(k)\left[iP_{\mu\nu\alpha\beta}\right]
\]

yielding the well known classical results, namely the Coulomb

\[
V_{1(a)}(r) = \frac{e_1e_2}{4\pi r}
\]

and Newtonian potentials.

\[
V_{1(b)}(r) = -\frac{Gm_1m_2}{r}
\]

It is worthwhile to note that already at this stage the level of difficulty is not obvious. There is virtually no problem in working out the Coulomb term for the interaction, technically and mathematically it is straightforward. However, in comparison, the Newtonian term is much more sophisticated to work out. This is mainly due to the many \(\gamma\)-matrices involved (since we are working with fermions) and the long vertex rules theories get when coupled to gravity. This difference will play a much bigger role when more complicated diagrams are involved. Indeed, the next set of diagrams were perhaps the most challenging of them all, the box and crossed box diagrams.

C. The 1PI diagrams

We will calculate all the relevant 1PI diagrams necessary to find the long range corrections to the potentials. We will start with the box and crossed box diagrams and continue with the set of triangular diagrams. Lastly we will work out the circular loop diagram.

1. The box and crossed box diagrams

There are in all four distinct diagrams. Two box and two crossed box diagrams, these are depicted in figure 2. We will not treat all of these diagrams here. We will only look at one of these, the rest can be treated in the same manner. Explicitly diagram 2(a) is defined by

\[
iM_{2(a)} = \int \frac{d^4\ell}{(2\pi)^4} \left[-\frac{\eta_{\beta\gamma}}{\ell^2}\right] \left[iP_{\mu\nu\rho\sigma}\right] \times \bar{u}(p')[\tau^{\rho\sigma}(p-\ell,p')D_F(p-\ell)\tau^\delta(p,p-\ell)]u(p) \\
\times \bar{u}(k')[\tau^{\mu\nu}(k+\ell,k')D_F(k+\ell)\tau^\gamma(k,k+\ell)]u(k)
\]

The methods and techniques to work these diagrams are identical to those shown in [6], we will repeat them shortly here.

The only difference lies in the fact that these diagrams require four different integrals that were not worked out previously. We have worked them out, and the coefficients can be obtained by contacting us, they are too
tedious to be written down in the appendix. Other than these integrals, these diagrams simply had to be worked out even though they involved enormous amounts of calculations. The level of difficulty is much larger than in the previous case, due to the reasons mentioned earlier. All the box diagrams have been calculated by using symbolic manipulation on a computer. These have partly been checked by hand.

---

On the mass shell we will have the following type of identities

\[ \ell \cdot q = \frac{1}{2}((\ell + q)^2 - q^2 - \ell^2), \quad \ell \cdot k = \frac{1}{2}((\ell + k)^2 - m_1^2 - \ell^2), \quad \ell \cdot p = -\frac{1}{2}((\ell - p)^2 - m_2^2 - \ell^2) \]  

(50)

so

\[ q_{\mu} K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{((\ell \cdot q)^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} \rightarrow -\frac{q^2}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} = -\frac{q^2}{2} K^{\nu} \]  

(51)

since the terms with \((\ell + q)^2\) and \(\ell^2\) simply do not contribute with non-analytical results.

A more important reduction of the integrals is with contraction of the sources momenta rather than the exchange momentum

\[ k_{\mu} K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell \cdot k)^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} \rightarrow \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} = \frac{1}{2} I_{\nu}^{\nu} \]  

(52)

or in similar manner

\[ p_{\mu} K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell \cdot p)^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} \rightarrow -\frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^{\nu}}{\ell^2((\ell + k)^2 - m_1^2)((\ell - k)^2 - m_2^2)} = -\frac{1}{2} I_{\nu}^{\nu} \]  

(53)

where the subscripts \(k\) and \(p\) on the \(I\)'s are written to indicate that the propagators left in the integrals are either from the particle with momentum \(k\) or \(p\). Thus a loop momentum contracted with a source momentum simplifies our integrals considerably.

The kinematics are (on the mass shell)

\[ k \cdot q = p' \cdot q = \frac{q^2}{2} \quad k' \cdot q = p \cdot q = -\frac{q^2}{2} \]  

(54)

\[ k \cdot k' = m_1^2 - \frac{q^2}{2} \quad p \cdot p' = m_2^2 - \frac{q^2}{2} \]  

(55)

The potential contribution from these diagrams are found to be

\[ V(r)_{2(a)+2(b)+2(c)+2(d)} = \frac{3}{4} \frac{e_1 e_2 (m_1 + m_2) G}{\pi r^2} - \frac{118}{48} \frac{G e_1 e_2}{\pi^2 r^3} \]  

(56)

These diagrams yield both a classical contribution - the \(\sim \frac{\pi}{r}\) - and a quantum correction - the \(\sim \frac{1}{r^2}\). In the scalar QED calculations the box diagrams only generated quantum corrections, it is interesting to see that the Feynman diagrams do not necessarily generate identical results for the individual diagrams. This will become more clear in the other diagrams.

---

2. The triangular diagrams

Diagrammatically it is given in figure \(\mathbf{6}\) these are the only triangular diagrams that contribute with non-analytic terms. We will again only consider one instance of these diagrams, namely the first fig. \(\mathbf{6a}\). Formally it is written down as follows

\[ i\mathcal{M}_{3(a)} = \int \frac{d^4\ell}{(2\pi)^4} \bar{u}(p') [\gamma^\mu(\ell)] u(p) \times \bar{u}(k') [\gamma^\nu(-\ell - k, k') D_\ell(-\ell - k) \gamma^\gamma(k, -\ell - k)] u(k) \times \left[ \frac{\bar{u}_\ell k}{\ell^2} \right] \left[ \frac{\ell^2}{(\ell + q)^2} \right] \]  

(57)

Upon applying contractions and insertion of the relevant integrals where after Fourier transformations are performed, we end up with the potential contribution

\[ V_{3(a)+3(b)+3(c)+3(d)}(r) = -\frac{9G e_1 e_2}{4\pi r^3} \]  

(58)

which is also different from the result obtained in the \(\mathbf{6}\) for the triangular diagrams.
FIG. 3: The set of triangular diagrams contributing non-analytically to the potential.

3. The circular diagram

The circular diagram is depicted in figure 4. Formally it can be written as

\[ i M_{4(a)} = \int \frac{d^4 \ell}{(2\pi)^4} \bar{u}(p') \left[ \tau^\rho_\beta D_F(p-\ell) \tau^\alpha_\alpha \right] u(p) \left[ -\frac{i\eta^\rho_\beta}{\ell^2} \right] \]

\[ \times \left[ \frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell + q)^2} \bar{u}(k') \left[ \tau^{\mu\nu}(k, k') \right] u(k) \right] \] (59)

When doing all the contractions and rearranging the \( \gamma \)-matrices one arrives at the result that the contribution to the potential from the circular loop diagram is precisely equal to nil.

\[ V_{4(a)} = 0 \] (60)

Which indeed is very different from the result obtained in [3].

D. The vertex correction diagrams

There are several sets of 1PR diagrams. All these are presented in this section.

1. The 1PR diagram

The first set of these 1PR diagrams is given in figure 5. These diagrams are the only ones corresponding to the gravitational vertex corrections. Again we will only consider one instance of these diagrams. The matrix element originating from figure 5(a) diagram is

\[ i M_{5(a)} = \int \frac{d^4 \ell}{(2\pi)^4} \bar{u}(p') \left[ \tau^\rho_\beta D_F(p-\ell) \tau^\alpha_\alpha \right] u(p) \left[ -\frac{i\eta^\rho_\beta}{\ell^2} \right] \]

\[ \times \left[ \frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell + q)^2} \bar{u}(k') \left[ \tau^{\mu\nu}(k, k') \right] u(k) \right] \] (61)

which yields the following potential when all the diagrams are summed

\[ V_{5(a)+5(b)}(r) = \frac{G(m_2 e_1^2 + m_1 e_2^2)}{8\pi r^2} - \frac{e_1^2 m_2 + e_2^2 m_1}{3\pi^2 r^3} \] (62)

This checks with [3] where it has also been calculated.

Of the next set of the 1PR diagrams, depicted in figure 6 we will also only consider the first one. This is the first of the set of diagrams corresponding to the photonic vertex corrections. Formally diagram 6(a) is given by

\[ i M_{6(a)} = \int \frac{d^4 \ell}{(2\pi)^4} \bar{u}(p') \left[ \tau^\rho_\beta D_F(p-\ell) \tau^\alpha_\alpha \right] u(p) \left[ -\frac{i\eta^\rho_\beta}{\ell^2} \right] \]

\[ \times \left[ \frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell + q)^2} \bar{u}(k') \left[ \tau^{\mu\nu}(k, k') \right] u(k) \right] \] (63)

giving the potential

\[ V_{6(a)+6(b)+6(c)+6(d)}(r) = \frac{3G e_1 e_2}{4\pi^2 r^3} \] (64)
The second set of diagrams corresponding to the pho-
tonic vertex corrections are given in figure 7. Formally
diagram 7(a) is given by
\[
iM_{7(a)} = \int \frac{d^4\ell}{(2\pi)^4} \bar{u}(p')\tau(\gamma)u(p) \left[ -\frac{i\eta_{\gamma \delta}}{q^2} \right] \times \tau_{\sigma \rho(\delta \alpha)}(q, -\ell) \left[ -\frac{i\eta_{\alpha \delta}}{\ell^2} \right] \left[ \frac{iP_{\mu \nu \rho \sigma}}{(\ell + q)^2} \right]
\]
\[
\times \tau_{\mu \nu \sigma}(\ell) \left[ -\frac{i\eta_{\beta \sigma}}{q^2} \right] \bar{u}(k')\tau(\phi)u(k)
\]
This is the only instance in these calculations that \(\gamma\)-
matrices are not explicitly involved. Simple index con-
tractions are done and one obtains the potential contribu-
tion after going to the non relativistic limit
\[
V_{7(a)}(r) = \frac{g \epsilon_1 \epsilon_2}{6\pi^2 r^3}
\]
which is identical with the bosonic version.

VI. THE RESULTS FOR THE POTENTIAL

When adding up all the separate contributions, we end
up with
\[
V(r) = -\frac{Gm_1 m_2}{r} + \frac{\tilde{\alpha} \tilde{e}_1 \tilde{e}_2}{r}
\]
\[
+ \frac{1}{2} \left( m_2 \tilde{e}_1^2 + m_1 \tilde{e}_2^2 \right) \frac{G\tilde{\alpha}}{c^2 r^2} + 3 \frac{\tilde{e}_1 \tilde{e}_2 (m_1 + m_2) \tilde{\alpha} G}{c^2 r^2}
\]
\[
- \frac{4}{3} \frac{G\hbar h}{\pi c^3 r^3} \left( \epsilon_1^2 \frac{m_2}{m_1} + \epsilon_2^2 \frac{m_1}{m_2} \right) - 15 \frac{1}{6} \frac{G\hbar h \tilde{e}_1 \tilde{e}_2}{\pi r^3}
\]
where we have included the appropriate physical factors
of \(h, c\) and we have further rescaled everything in terms
of \(\tilde{\alpha} = \frac{\hbar c}{4\pi}\), lastly \((\tilde{e}_1, \tilde{e}_2)\) are the normalized charges in
units of elementary charge. The result is divided into
three separate parts, the first two terms represent the
Newtonian and Coulomb potentials, the next two terms
represent the classical post-Newtonian corrections to the
potential, which also can be found by pure classical treat-
ment of general relativity with the inclusion of charged
matter sources\[22\]. It is interesting to see that loop calculations also reproduce classical results, and not only quantum corrections. Finally, the last two terms are the leading 1-loop quantum corrections. We have in a greater extent reproduced the results of\[6\], except for the last quantum correction where we get the factor $(-15\frac{1}{2})$ instead of $(-8)$ as in\[2\].

It is interesting to note how the different terms come about in the corrections to the potential. The classical terms, the Newtonian and Coulomb terms, of course originate from the tree diagrams alone. The first post-Newtonian term and the first quantum correction notably originate from one type of diagrams alone, namely the gravitational vertex correction, see figure\[\text{K}\]. These match\[\text{K}\] exactly. Now the second post-Newtonian correction originates from all the box diagrams and nothing else, which indeed also is in full agreement with\[\text{K}\]. However, the second quantum correction is a sum of partly the box contributions and the rest of the diagrams. This correction does not match the quantum correction in ref.\[\text{K}\]. The rest of the diagrams have been done by hand, and all the diagrams done in this paper have been checked by symbolic manipulation on a computer.

VII. DISCUSSION

We have examined the leading order quantum corrections to gravitational coupling of a spin-$\frac{1}{2}$ massive charged particle. Explicitly we have extracted the nonanalytic terms from the diagrams, which exactly manifested themselves as corrections to the long range forces, this was realized when we Fourier transformed these terms into coordinate space. These terms originated from the propagation of the massless particles, here the photons and gravitons. We have obtained similar results for most of the contributions to the corrections of the potentials, when compared with\[\text{K}\]. Only one of the leading quantum corrections differs from the bosonic calculation.

One could in similar manner do a QED - pure gravity scattering calculation as in\[\text{K}\]. However, more obstacles would be needed to overcome. First of all, one would have to find many new vertex rules involving the antisymmetric fields. Moreover we would have to derive second order Lagrangian densities both in terms of the symmetric and antisymmetric fields. Indeed, in this direction there lies a considerably interesting project ahead.

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APPENDIX A: FEYNMAN RULES

1. Propagators

The relevant propagators are presented in this section.

a. Photon propagator

The photon propagator is no stranger in QFT. In Feynman gauge the propagator becomes

$$\begin{align*}
\alpha \, \gamma \, \beta &= -\frac{i\eta_{\alpha\beta}}{q^2 + i\epsilon} \\
\mu\nu \, \gamma \, \alpha\beta &= \frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2 + i\epsilon}
\end{align*}$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

b. Graviton propagator

The graviton propagator is perhaps a stranger. However it has been worked out several places. In the Harmonic gauge we get the following for the graviton propagator

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

$$\mu\nu \, \gamma \, \alpha\beta = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

c. Fermion propagator

The fermion propagator can be found many places in literature, it is very well known

$$\begin{align*}
\gamma \, \alpha\beta &= \frac{i}{(k - m)} = \frac{i(\bar{k} + m)}{k^2 - m^2}
\end{align*}$$

2. Vertices

The vertices are presented here. They are all derived in\[\text{K}\]. For all vertices the rules of momentum conservation has been applied.

a. 1-photon-2-fermion vertex

The 1-photon-2-fermion vertex can also be looked up in literature it is worked out to be

$$\alpha \, \gamma \, \beta = \tau^\alpha(p, p')$$
with
\[ \tau^\alpha(p, p') = i\epsilon_\gamma \gamma^\alpha \]

b. 1-graviton-2-fermion vertex

The 1-graviton-2-fermion vertex is found to be
\[ \alpha\beta \quad \gamma \quad p \quad p' = \tau^{\alpha\beta}(p, p') \]

where
\[ \tau^{\alpha\beta}(p, p') = \frac{i\kappa}{2} \left[ \eta^{\alpha\beta} \left( \frac{1}{2} (\mathbf{p} + \mathbf{p'}) - m \right) - \frac{1}{4} \gamma^\gamma \{\alpha(p + p')^\beta\} \right] \]

c. 1-photon-1-graviton-2-fermion vertex

The 1-photon-1-graviton-2-fermion vertex is not known from literature, however it is found to be
\[ \alpha\beta,k \quad \gamma \quad p \quad p' = \tau^{\alpha\beta}(\gamma)(p, p') \]

where
\[ \tau^{\alpha\beta}(\gamma)(p, p') = \frac{i\kappa}{4} \gamma^\mu (2\eta^\alpha \eta^\beta - \eta^\gamma (\alpha^\gamma \beta)) \]

d. 1-graviton-2-photon vertex

We have derived the following for the 1-graviton-2-photon vertex
\[ \alpha\beta \quad \delta \quad p \quad p' = \tau^{\alpha\beta}(\delta)(p, p') \]

where
\[ \tau^{\alpha\beta}(\delta)(p, p') = ik \left[ \mathcal{P}^{\alpha\beta}(\delta)(p, p') + \frac{1}{2} (\eta^\beta p^\gamma p^\rho + \eta^\gamma p^\rho (\alpha^\gamma \beta) - \rho^\gamma p^\rho (\alpha^\rho \beta)) \right] \]

\( \mathcal{P}^{\alpha\beta}(\gamma) \) is defined as above and \{\} do not involve any symmetrization factors, they merely exchange the indices.

APPENDIX B: TABLE OF RELEVANT INTEGRALS

The following integrals are needed, note that in these integrals only the lowest order non-analytical terms are presented
\[ J = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell + q)^2} = \frac{i}{32\pi^2} \left[ -2L \right] + \ldots \] (B1)
\[ J_\mu = \int \frac{d^4\ell}{(2\pi)^4} \ell_\mu (\ell + q)^2 = \frac{i}{32\pi^2} \left[ q_\mu L \right] + \ldots \] (B2)
\[ J_{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \ell_\mu \ell_\nu (\ell + q)^2 = \frac{i}{32\pi^2} \left[ q_\mu q_\nu \left( -\frac{2}{3} L \right) - q^2 \eta_{\mu\nu} \left( -\frac{1}{6} \right) \right] + \ldots \] (B3)

together with
\[ I = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell + q)^2((\ell + k)^2 - m^2)} = \frac{i}{32\pi^2 m^2} \left[ -L - S \right] + \ldots \] (B4)
\[ I_\mu = \int \frac{d^4\ell}{(2\pi)^4} \ell_\mu (\ell + q)^2((\ell + k)^2 - m^2)} = \frac{i}{32\pi^2 m^2} \left[ k_\mu \left( -\frac{1}{2} \frac{q^2}{m^2} \right) L - \frac{1}{4} \frac{q^2}{m^2} S \right] + \frac{q_\mu \left( L + \frac{1}{2} S \right)}{2} + \ldots \] (B5)
\[ I_{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \ell_\mu \ell_\nu (\ell + q)^2((\ell + k)^2 - m^2)} = \frac{i}{32\pi^2 m^2} \left[ q_\mu q_\nu \left( -L - \frac{3}{8} S \right) + k_\mu k_\nu \left( -\frac{1}{2} \frac{q^2}{m^2} L - \frac{1}{8} \frac{q^2}{m^2} S \right) \right] + \ldots \] (B6)
\[
I_{\mu\nu\alpha} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell_\mu \ell_\nu \ell_\alpha}{(\ell + q)^2((\ell + k)^2 - m^2)}
\]
\[
= -\frac{i}{32\pi^2m^2}
\left[
q_\mu q_\nu q_\alpha \left(L + \frac{5}{16}S\right) + k_\mu k_\nu k_\alpha \left(-\frac{1}{6}q^2L\right)
\right.
\]
\[
\left. + (q_\mu k_\nu k_\alpha + q_\nu k_\mu k_\alpha + q_\alpha k_\mu k_\nu) \left(\frac{1}{3}q^2L + \frac{1}{16}q^2S\right)
\right]
\]
\[
+ (q_\mu q_\nu k_\alpha + q_\nu q_\alpha k_\mu + q_\alpha q_\mu k_\nu) \left(\left(-\frac{1}{3} - \frac{1}{2}q^2\right)L - \frac{5}{32}q^2S\right)
\]
\[
+ (\eta_{\mu\alpha} k_\nu + \eta_{\mu\nu} k_\alpha + \eta_{\alpha\nu} k_\mu) \left(\frac{1}{12}q^2L\right)
\]
\[
+ (\eta_{\mu\alpha} q_\nu + \eta_{\mu\nu} q_\alpha + \eta_{\alpha\nu} q_\mu) \left(-\frac{1}{6}q^2L - \frac{1}{16}q^2S\right)\right]
\] ... (B7)

where \( S = \frac{m^2}{\sqrt{-q^2}} \) and \( L = \ln(-q^2) \). The ellipses denote higher order non-analytical contributions as well as the neglected analytical terms. Please note that there seems to be a typo in \( K \), in \( I_{\mu\nu\alpha} \) the factor after \( (k_\mu k_\nu k_\alpha) \) is lacking an \( L \). Other than this typo all the integrals have been checked and are agreed upon.

The following integrals are needed to do the box diagrams.

\[
K = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell - p)^2 - m^2)} = \frac{i}{16\pi^2m_1m_2q^2} \left[ 1 - \frac{w}{3m_1m_2} \right]L + \ldots \] ... (B8)

\[
K' = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell - p')^2 - m^2)} = \frac{i}{16\pi^2m_1m_2q^2} \left[ 1 + \frac{W}{3m_1m_2} \right]L + \ldots \] ... (B9)

\[
K^\mu = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell - p)^2 - m^2)} = \alpha q^\mu + \beta k^\mu + \gamma p^\mu \] ... (B10)

\[
K'^\mu = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell + p')^2 - m^2)} = \alpha' q^\mu + \beta' k^\mu + \gamma' p'^\mu \] ... (B11)

\[
K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell - p)^2 - m^2)} = [q_\mu q_\nu a + k_\mu k_\nu b + p_\mu p_\nu c] + (q_\mu k_\nu + q_\nu k_\mu) d + (q_\mu p_\nu + q_\nu p_\mu) e + (p_\mu k_\nu + p_\nu k_\mu) f + \eta_{\mu\nu} q^2 g \] ... (B12)

\[
K'^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{(\ell + q)^2((\ell + k)^2 - m^2)((\ell + p')^2 - m^2)} = [q_\mu q_\nu a' + k_\mu k_\nu b' + p_\mu p_\nu c'] + (q_\mu k_\nu + q_\nu k_\mu) d' + (q_\mu p_\nu + q_\nu p_\mu) e' + (p_\mu k_\nu + p_\nu k_\mu) f' + \eta_{\mu\nu} q'^2 g' \] ... (B13)

Hence we have defined \( w = (k \cdot p) - m_1m_2 \) and \( W = (k \cdot p') - m_1m_2 \). From these we can deduce an important relation that becomes vital during the calculations, namely \( W = w = k \cdot (p' - p) = (k \cdot q) = \frac{-\sqrt{-q^2}}{\sqrt{-p^2}} \). The \( w \) and \( W \) are displayed here only for the \( K \) and \( K' \) integrals, for the rest of the integrals we have used \( W = (k \cdot p) - m_1m_2 - \frac{\sqrt{-q^2}}{\sqrt{-p^2}} \), see \( 23 \) for derivations. The coefficients to these are long and tedious to write down properly, however, if required, they can be obtained from us by contacting us.

For the above integrals the following constraints for the non-analytical terms can be verified directly on the mass-shell. We will use \( k I \sim \frac{1}{\sqrt{(\ell + q)^2 - m^2}} \) for \( k \) and \( \mu I \sim \frac{1}{\sqrt{(\ell - p)^2 - m^2}} \) for \( \mu \), \( \nu I \sim \frac{1}{\sqrt{(\ell + p')^2 - m^2}} \) for \( \nu \) and no particular choice is needed for contractions with \( q \)

\[
K_{\mu\nu} q^\mu = K'_{\mu\nu} q^\mu = I_{\mu\nu\alpha} q^{\mu\nu\alpha} = I_{\mu\nu} q^{\mu\nu} = J_{\mu\nu} q^{\mu\nu} = 0
\]

\[
K_{\mu\nu} q^\mu = -\frac{q^2}{2} K_{\nu}, \quad K_{\mu\nu} q^\mu = -\frac{q^2}{2} K_{\nu}, \quad K'_{\mu\nu} q^\mu = -\frac{q^2}{2} K'_{\nu}, \quad K'_{\mu\nu} q^\mu = -\frac{q^2}{2} K'_{\nu}
\]

\[
K_{\mu\nu} p^\mu = -\frac{1}{2} k_{\nu}, \quad K_{\mu\nu} p^\mu = -\frac{1}{2} k_{\nu}, \quad K_{\mu\nu} k^\mu = \frac{1}{2} p_{\nu}, \quad K_{\mu\nu} k^\mu = \frac{1}{2} p_{\nu}
\]

\[
K'_{\mu\nu} p^\mu = \frac{1}{2} k_{\nu}, \quad K'_{\mu\nu} p^\mu = \frac{1}{2} k_{\nu}, \quad K'_{\mu\nu} k^\mu = \frac{1}{2} p'_{\nu}, \quad K'_{\mu\nu} k^\mu = \frac{1}{2} p'_{\nu}
\]
\[ I_{\mu\nu\alpha\kappa\lambda}q^\alpha = -\frac{q^2}{2} I_{\mu\nu}, \quad I_{\mu\nu} q^\nu = -\frac{q^2}{2} I_{\mu}, \quad I_{\mu} q^\mu = -\frac{q^2}{2} J_{\mu}, \quad J_{\mu} q^\mu = -\frac{q^2}{2} J \]

\[ k I_{\mu\nu\alpha\kappa} = \frac{1}{2} J_{\mu\nu}, \quad k I_{\mu\nu} k^\nu = \frac{1}{2} J_{\mu}, \quad k I_{\mu} k^\mu = \frac{1}{2} J \]

\[ p I_{\mu\nu\alpha\kappa} = -\frac{1}{2} J_{\mu\nu}, \quad p I_{\mu\nu} p^\nu = -\frac{1}{2} J_{\mu}, \quad p I_{\mu} p^\mu = -\frac{1}{2} J \]

There seems to be a typo in \([6]\) the metric in \(I_{\mu\nu\alpha\kappa} q^\alpha\eta_{\mu\nu}\) is written as \(I_{\mu\nu\alpha\kappa} q^\alpha\eta^{\mu\nu}\).