MODULAR LATTICE FOR $C_0$-OPERATORS.

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Abstract. We study modularity of the lattice $\text{Lat}(T)$ of closed invariant subspaces for a $C_0$-operator $T$ and find a condition such that $\text{Lat}(T)$ is a modular. Furthermore, we provide a quasiaffinity preserving modularity.

Introduction

A partially ordered set is said to be a lattice if any two elements $M$ and $N$ of it have a least upper bound or supremum denoted by $M \lor N$ and a greatest lower bound or infimum denoted by $M \land N$. For a Hilbert space $H$, $L(H)$ denotes the set of all bounded linear operators from $H$ into $H$. For an operator $T$ in $L(H)$, the set $\text{Lat}(T)$ of all closed invariant subspaces for $T$ is a lattice. For $L, M,$ and $N$ in $\text{Lat}(T)$ such that $N \subset L$, if following identity is satisfied:

$$L \cap (M \lor N) = (L \cap M) \lor N,$$

then $\text{Lat}(T)$ is called modular. We study $\text{Lat}(T)$ where $T$ is a $C_0$-operator which were first studied in detail by B.Sz.-Nagy and C. Foias [4]. In this paper $D$ denotes the open unit disk in the complex plane.

This paper is organized as follows. Section 1 contains preliminaries about operators of class $C_0$ and the Jordan model of $C_0$-operators.

For operators $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if $X \in \{A \in L(H) : AT_1 = T_2 A\}$, then we define a function $X_* : \text{Lat}(T_1) \to \text{Lat}(T_2)$ as following:

$$X_*(M) = (XM)^-.$$

In Theorem 2.14 we provide a quasiaffinity $Y$ such that $Y_*$ preserves modularity. Furthermore, in section 2, we provide a definition and prove some fundamental results of property $(P)$ which was introduced by H. Bercovici [2].

In Theorem 3.5 we prove that if $T \in L(H)$ is an operator of class $C_0$ with property $(P)$, then $\text{Lat}(T)$ is a modular lattice.

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1. **C₀-Operators Relative to D**

1.1. **A Functional Calculus.** It is well-known that for every linear operator \( A \) on a finite dimensional vector space \( V \) over the field \( F \), there is a minimal polynomial for \( A \) which is the (unique) monic generator of the ideal of polynomials over \( F \) which annihilate \( A \). If the dimension of \( F \) is not finite, then generally there is no such a polynomial. However, to provide a function similar to a minimal polynomial, B. Sz.-Nagy and C. Foias focused on a contraction \( T \in L(H) \) which is called to be completely nonunitary, i.e. there is no invariant subspace \( M \) for \( T \) such that the restriction \( T|_M \) of \( T \) to the space \( M \) is a unitary operator.

Let \( H \) be a subspace of a Hilbert space \( K \) and \( P_H \) be the orthogonal projection from \( K \) onto \( H \). We recall that if \( A \in L(K) \), and \( T \in L(H) \), then \( A \) is said to be a dilation of \( T \) provided that for \( n = 1, 2, \ldots \),

\[
T^n = P_H A^n | H.
\]

If \( A \) is an isometry (unitary operator) then \( A \) will be called an isometric (unitary) dilation of \( T \). An isometric (unitary) dilation \( A \) of \( T \) is said to be minimal if no restriction of \( A \) to an invariant subspace is an isometric (unitary) dilation of \( T \). B. Sz.-Nagy proved the following interesting result:

**Proposition 1.1.** \([4]\) Every contraction has a unitary dilation.

Let \( T \in L(H) \) be a completely nonunitary contraction with minimal unitary dilation \( U \in L(K) \). For every polynomial \( p(z) = \sum_{j=0}^{n} a_j z^j \) we have

\[
p(T) = P_H p(U) | H,
\]

and so this formula suggests that the functional calculus \( p \to p(T) \) might be extended to more general functions \( p \). Since the mapping \( p \to p(T) \) is a homomorphism from the algebra of polynomials to the algebra of operators, we will extend it to a mapping which is also a homomorphism from an algebra to the algebra of operators. By Spectral Theorem, since \( U \in L(H) \) is a normal operator, there is a unique spectral measure \( E \) on the Borel subsets of the spectrum of \( U \) denoted as usual by \( \sigma(U) \) such that

\[
U = \int_{\sigma(U)} zdE(z).
\]

Since the spectral measure \( E \) of \( U \) is absolutely continuous with respect to Lebesgue measure on \( \partial D \), for \( g \in L^\infty(\sigma(U), E) \), \( g(U) \) can be defined as follows:

\[
g(U) = \int_{\sigma(U)} g(z)dE(z).
\]

It is clear that if \( g \) is a polynomial, then this definition agrees with the preceding one. Since the spectral measure of \( U \) is absolutely continuous with respect to Lebesgue measure on \( \partial D \), the expression \( g(U) \) makes sense for every \( g \in L^\infty = L^\infty(\partial D) \). We generalize formula \((1.2)\), and so for \( g \in L^\infty \), define \( g(T) \) by

\[
g(T) = P_H g(U) | H.
\]

While the mapping \( g \to g(T) \) is obviously linear, it is not generally multiplicative, i.e. it is not a homomorphism. Evidently it is convenient to find a subalgebra in \( L^\infty \) on which the functional calculus is multiplicative. Recall that \( H^\infty \) is the Banach
space of all (complex-valued) bounded analytic functions on the open unit disk $D$ with supremum norm $\| \cdot \|$. It turns out that $H^\infty$ is the unique maximal algebra making the map a homomorphism between algebras. We know that $H^\infty$ can be regarded as a subalgebra of $L^\infty(\partial D)$ \[1\].

We note that the functional calculus with $H^\infty$ functions can be defined in terms of independent of the minimal unitary dilation. Indeed, if $u(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H^\infty$, then

\[ u(T) = \lim_{r \to 1} u(rT) = \lim_{r \to 1} \sum_{n=0}^{\infty} a_n r^n T^n, \tag{1.6} \]

where the limit exists in the strong operator topology.

B. Sz.-Nagy and C. Foias introduced this important functional calculus for completely nonunitary contractions.

**Proposition 1.2.** Let $T \in L(H)$ be a completely nonunitary contraction. Then there is a unique algebra representation $\Phi_T$ from $H^\infty$ into $L(H)$ such that:

(i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;

(ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in D$;

(iii) $\Phi_T$ is continuous when $H^\infty$ and $L(H)$ are given the weak*-topology.

(iv) $\Phi_T$ is contractive, i.e., $\|\Phi_T(u)\| \leq \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B. Sz.-Nagy and C. Foias \[4\] defined the class $C_0$ consisting of completely nonunitary contractions $T$ on $H$ such that the kernel of $\Phi_T$ is not trivial. If $T \in L(H)$ is an operator of class $C_0$, then

$$\ker \Phi_T = \{ u \in H^\infty : u(T) = 0 \}$$

is a weak*-closed ideal of $H^\infty$, and hence there is an inner function generating ker $\Phi_T$. The minimal function $m_T$ of an operator of class $C_0$ is the generator of ker $\Phi_T$, and it seems as a substitute for the minimal polynomial. Also, $m_T$ is uniquely determined up to a constant scalar factor of absolute value one \[1\]. The theory of class $C_0$ relative to the open unit disk has been developed by B. Sz.-Nagy, C. Foias \[4\] and H. Bercovici \[1\].

1.2. Jordan Operator. We know that every $n \times n$ matrix over an algebraically closed field $F$ is similar to a unique Jordan canonical form. To extend that theory to the $C_0$ operator $T \in L(H)$, B. Sz.-Nagy and C. Foias \[4\] introduced a weaker notion of equivalence. They defined a quasiaffine transform of $T$ which is bounded operator $T'$ defined on a Hilbert space $H'$ such that there exists an injective operator $X \in L(H, H')$ with dense range in $H'$ satisfying $T'X = XT$. We write $T \prec T'$ if $T$ is a quasiaffine transform of $T'$. Instead of similarity, they introduced quasisimilarity of two operators, namely, $T$ and $T'$ are quasisimilar, denoted by $T \sim T'$, if $T \prec T'$ and $T' \prec T$.

Given an inner function $\theta \in H^\infty$, the Jordan block $S(\theta)$ is the operator acting on $H(\theta) = H^2 \ominus \theta H^2$, which means the orthogonal complement of $\theta H^2$ in the Hardy space $H^2$, as follows:

\[ S(\theta) = P_{H(\theta)}^* S|H(\theta) \tag{1.7} \]
where \( S \in L(H^2) \) is the unilateral shift operator defined by 
\[
(Sf)(z) = zf(z)
\]
and \( P_{H(\theta)} \in L(H^2) \) denotes the orthogonal projection of \( H^2 \) onto \( H(\theta) \).

**Proposition 1.3.** \([1]\)** For every inner function \( \theta \) in \( H^\infty \), the operator \( S(\theta) \) is of class \( C_0 \) and its minimal function is \( \theta \).

Let \( \theta \) and \( \theta' \) be two inner functions in \( H^\infty \). We say that \( \theta \) divides \( \theta' \) (or \( \theta \mid \theta' \)) if \( \theta' \) can be written as \( \theta' = \theta \phi \) for some \( \phi \in H^\infty \). It is clear that \( \phi \in H^\infty \) is also inner. We will use the notation \( \theta \equiv \theta' \) if \( \theta \mid \theta' \) and \( \theta' \mid \theta \).

**Proposition 1.4.** \([1]\)** Let \( T_1 \in L(H) \) and \( T_2 \in L(H) \) be two completely nonunitary contractions of class \( C_0 \). If \( T_1 \) and \( T_2 \) are quasisimilar, then \( m_{T_1} \equiv m_{T_2} \).

From Proposition 1.3 and Proposition 1.4, we can easily see that for every inner functions \( \theta_1 \) and \( \theta_2 \) in \( H^\infty \), if \( S(\theta_1) \) and \( S(\theta_2) \) are quasisimilar, then \( \theta_1 \equiv \theta_2 \).

**Conversely,**

**Proposition 1.5.** \([1]\)** Let \( \theta_1 \) and \( \theta_2 \) be inner functions in \( H^\infty \). If \( \theta_1 \equiv \theta_2 \), then \( S(\theta_1) \) and \( S(\theta_2) \) are quasisimilar.

Let \( \gamma \) be a cardinal number and
\[
\Theta = \left \{ \theta_\alpha \in H^\infty : \alpha < \gamma \right \}
\]
be a family of inner functions. Then \( \Theta \) is called a model function if \( \theta_\alpha \mid \theta_\beta \) whenever \( \text{card}(\beta) \leq \text{card}(\alpha) < \gamma \). The Jordan operator \( S(\Theta) \) determined by the model function \( \Theta \) is the \( C_0 \) operator defined as
\[
S(\Theta) = \bigoplus_{\alpha < \gamma'} S(\theta_\alpha)
\]
where \( \gamma' = \min \{ \beta : \theta_\beta \equiv 1 \} \).

We will call \( S(\Theta) \) the Jordan model of the operator \( T \) if
\[
S(\Theta) \sim T,
\]
and in the sequel \( \bigoplus_{i < \gamma'} S(\theta_i) \) always means a Jordan operator determined by a model function.

By using Jordan blocks, \( C_0 \)-operators relative to the open unit disk \( D \) can be classified \([1]\) Theorem 5.1):

**Theorem 1.6.** Any \( C_0 \)-operator \( T \) relative to the open unit disk \( D \) acting on a Hilbert space is quasisimilar to a unique Jordan operator.

**Theorem 1.7.** If \( \Theta \) and \( \Theta' \) are two model functions and \( S(\Theta) \prec S(\Theta') \), then \( \Theta \equiv \Theta' \) and hence \( S(\Theta) = S(\Theta') \).

From Theorem 1.6 and Theorem 1.7, we can conclude that \( \prec \) is an equivalence relation on the set of \( C_0 \)-operators.

2. Lattice of subspaces

2.1. Modular Lattice. Let \( H \) be a Hilbert space. If \( F_i (i \in I) \) is a subset of \( H \), then the closed linear span of \( \bigcup_i F_i \) will be denoted by \( \bigvee_i F_i \). The collection of all subspaces of a Hilbert space is a lattice. This means that the collection is partially ordered (by inclusion), and that any two elements \( M \) and \( N \) of it have a least
upper bound or supremum (namely the span $\mathbf{M} \vee \mathbf{N}$) and a greatest lower bound or infimum (namely the intersection $\mathbf{M} \cap \mathbf{N}$). A lattice is called *distributive* if
\begin{equation}
\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee (\mathbf{L} \cap \mathbf{N})
\end{equation}
for any element $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ in the lattice.

In the equation (2.1), if $\mathbf{N} \subseteq \mathbf{L}$, then $\mathbf{L} \cap \mathbf{N} = \mathbf{N}$, and so the identity becomes
\begin{equation}
\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee \mathbf{N}
\end{equation}
If the identity (2.2) is satisfied whenever $\mathbf{N} \subseteq \mathbf{L}$, then the lattice is called *modular*.

For an arbitrary operator $T \in L(H)$, $\text{Lat}(T)$ denotes the collection of all closed invariant subspaces for $T$. The following fact is well-known [3].

**Proposition 2.1.** The lattice of subspaces of a Hilbert space $H$ is modular if and only if $\dim H$ is finite.

We will think about $\text{Lat}(T)$ for a $C_0$-operator $T$.

**Definition 2.2.** The *cyclic multiplicity* $\mu_T$ of an operator $T \in L(H)$ is the smallest cardinal of a subset $A \subseteq H$ with the property that $\bigvee_{n=0}^\infty T^n A = H$. The operator $T$ is said to be *multiplicity-free* if $\mu_T = 1$.

Thus $\mu_T$ is the smallest number of cyclic subspaces for $T$ that are needed to generate $H$, and $T$ is multiplicity-free if and only if it has a cyclic vector.

2.2. **Property** ($P$). Let $H$ be a Hilbert space and for an operator $T \in L(H)$, $T^*$ denote the adjoint of $T$. It is well known that $H$ is finite-dimensional if and only if every operator $X \in L(H)$, with the property $\ker(X) = \{0\}$, also satisfies $\ker(X^*) = \{0\}$. The following definition is a natural extension of finite dimensionality.

**Definition 2.3.** An operator $T \in L(H)$ is said to have property ($P$) if every operator $X \in \{T\}'$ with the property that $\ker(X) = \{0\}$ is a quasiaffinity, i.e., $\ker(X^*) = \ker(X) = \{0\}$.

From the fact that the commutant $\{0\}'$ of zero operator on $H$ coincides with $L(H)$, we can see that $H$ is finite-dimensional if and only if the zero operator on $H$ has property ($P$).

Let $T_1$ and $T_2$ be operators in $L(H)$. Suppose that
\[ X \in \{ A \in L(H) : AT_1 = T_2 A \}. \]
If $M$ is in $\text{Lat}(T_1)$, then $(XM)^-$ is in $\text{Lat}(T_2)$. By using these facts, we define a function $X_*$ from $\text{Lat}(T_1)$ to $\text{Lat}(T_2)$ as following :
\begin{equation}
X_*(M) = (XM)^-.
\end{equation}
The operator $X$ is said to be a $(T_1, T_2)$-*lattice-isomorphism* if $X_*$ is a bijection of $\text{Lat}(T_1)$ onto $\text{Lat}(T_2)$. We will use the name lattice-isomorphism instead of $(T_1, T_2)$-lattice-isomorphism if no confusion may arise.

If $X \in \{ A \in L(H) : AT_1 = T_2 A \}$, then $X^* T_2^* = T_1^* X^*$. Thus $(X^*)_*: \text{Lat}(T_2^*) \to \text{Lat}(T_1^*)$ is well-defined by
\[ (X^*)_*(M') = (X^* M')^- \]

**Proposition 2.4.** [1] (Theorem 7.1.9) Suppose that $T \in L(H)$ is an operator of class $C_0$ with Jordan model $\bigoplus_{j=0}^\infty S(\theta_j)$. Then $T$ has property ($P$) if and only if
\[ \bigwedge_{j<\omega} \theta_j \equiv 1. \]
Thus, if $T$ has property $(P)$, then $H$ is separable and $T^*$ also has property $(P)$.

**Proposition 2.5.** An operator $T$ of class $C_0$ fails to have property $(P)$ if and only if $T$ is quasisimilar to $T|N$, where $N$ is a proper invariant subspace for $T$.

**Proposition 2.6.** (Lemma 7.1.20) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. If the mapping $X_*$ is onto $\text{Lat}(T_2)$ if and only if $(X^*)_*$ is one-to-one on $L(T_2^*)$.

**Corollary 2.7.** Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. The mapping $X_*$ is one-to-one on $\text{Lat}(T_1)$ if and only if $(X^*)_*$ is onto $\text{Lat}(T_1^*)$.

**Proof.** Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. By Proposition 2.8 $(X^*)_*$ is onto $\text{Lat}(T_1^*)$ if and only if $(X^{**})_* = X_*$ is one-to-one on $\text{Lat}(T_1)$. □

From Proposition 2.9 and Corollary 2.7, we obtain the following result.

**Corollary 2.8.** If $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$, then $X$ is a lattice-isomorphism if and only if $X^*$ is a lattice-isomorphism.

**Proposition 2.9.** (Proposition 7.1.21) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class $C_0$, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ is an injection. If $T_1$ has property $(P)$, then $X$ is a lattice-isomorphism.

Recall that if $T$ is an operator on a Hilbert space, then $\ker T = (\text{ran } T^*)^\perp$ and $\ker T^* = (\text{ran } T)^\perp$.

**Corollary 2.10.** Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class $C_0$, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ has dense range. If $T_2$ has property $(P)$, then $X$ is a lattice-isomorphism.

**Proof.** Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. Let $Y = X^*$ and so

$$YT_2^* = T_1^*Y.$$  \hspace{1cm} (2.4)

From the fact that $\ker Y = \ker(X^*) = (\text{ran } X)^\perp = \{0\}$, we conclude that $Y$ is injective. Since $T_2$ has property $(P)$, so does $T_2^*$ by Proposition 2.4. By Proposition 2.9 and equation (2.4), $Y = X^*$ is a lattice-isomorphism. From Corollary 2.8 it is proven that $X$ is a lattice-isomorphism. □

**Corollary 2.11.** Suppose that $T_i \in L(H_i)\{i = 1, 2\}$ is a $C_0$-operator and $T_1$ has property $(P)$. If $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ and $X$ is an injection, then $X$ is a lattice-isomorphism.

**Proof.** Define $Y : H_1 \to (XH_1)^-$ by

$$Yh = Xh$$ for any $h \in H_1$.

Since $X$ is an injection, so is $Y$. Clearly, $Y$ has dense range. Note that $(XH_1)^-$ is invariant for $T_2$. By definition of $Y$,

$$YT_1 = (T_2|(XH_1)^-)Y.$$  \hspace{1cm} (2.5)

It follows that $T_1 < (T_2|(XH_1)^-)Y$ and so $T_1 \sim (T_2|(XH_1)^-)$. By Proposition 2.10 it is proven.
Corollary 2.12. Suppose that $T_i \in L(H_i) (i = 1, 2)$ is a $C_0$-operator and $T_2$ has property (P). If $X \in \{ A \in L(H_1, H_2): AT_1 = T_2A \}$ and $X$ has a dense range, then $X$ is a lattice-isomorphism.

Proof. By assumption, $X^*T_2^* = T_1^*X^*$. Since $T_2$ has property (P), by Proposition 2.11 so does $T_2^*$.

Because $X$ has dense range, $X^*: H_2 \to H_1$ is an injection. By Corollary 2.11, $X^*$ is a lattice isomorphism. From Corollary 2.12, $X$ is also a lattice isomorphism. □

2.3. Quasi-Affinity and Modular Lattice. For operators $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if $Y \in \{ B \in L(H_1, H_2): BT_1 = T_2B \}$, then we define a function $Y_* : \text{Lat}(T_1) \to \text{Lat}(T_2)$ the same way as equation (2.3). For any $N \in \text{Lat}(T_2)$, if $M = Y^{-1}(N)$, then $YT_1(M) = T_2Y(M) \subset T_2N \subset N$ and so $T_1(M) \subset M$. It follows that $M = Y^{-1}(N) \in \text{Lat}(T_1)$ for any $N \in \text{Lat}(T_2)$. If $Y$ is invertible, that is, $T_1$ and $T_2$ are similar, and $\text{Lat}(T_1)$ is modular, then clearly, $\text{Lat}(T_2)$ is also modular. In this section, we consider when $T_1$ and $T_2$ are quasi-similar instead of similar, and find an assumption in Theorem 2.14 such that $\text{Lat}(T_2)$ is modular, whenever $\text{Lat}(T_1)$ is modular.

Proposition 2.13. Let $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$. Suppose that $Y \in \{ B \in L(H_1, H_2): BT_1 = T_2B \}$ and for any $N \in \text{Lat}(T_2)$, the condition $M = Y^{-1}(N)$ implies that $Y_*(M) = N$.

Then for any $M_i = Y^{-1}(N_i)$ with $N_i \in \text{Lat}(T_2)$ ($i = 1, 2$),

$Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2)$.

Proof. Assume that $N_i \in \text{Lat}(T_2)$ and $M_i = Y^{-1}(N_i)$ for $i = 1, 2$. Then by assumption, we obtain

(2.6) $Y_*(M_i) = N_i$.

Since $Y^{-1}(N_1 \cap N_2) = Y^{-1}(N_1) \cap Y^{-1}(N_2) = M_1 \cap M_2$, by assumption,

$Y_*(M_1 \cap M_2) = N_1 \cap N_2$

which proves that $Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2)$ by equation (2.6). □

Theorem 2.14. Let $T_1 \in L(H_1)$ be a quasiaffine transform of $T_2 \in L(H_2)$ and $Y \in \{ B \in L(H_1, H_2): BT_1 = T_2B \}$ be a quasiaffinity.

If $Y_* : \text{Lat}(T_1) \to \text{Lat}(T_2)$ is onto and $\text{Lat}(T_1)$ is modular, then $\text{Lat}(T_2)$ is also modular.

Proof. Suppose that $\text{Lat}(T_2)$ is not modular. Then there are invariant subspaces $N_i (i = 1, 2, 3)$ for $T_2$ such that

(2.7) $N_3 \subset N_1$,

and

$(N_1 \cap N_2) \vee N_3 \neq N_1 \cap (N_2 \vee N_3)$.

Let

(2.8) $M_i = Y^{-1}(N_i)$,

for $i = 1, 2, 3$. Since $YT_1 = T_2Y$, definition (2.5) of $M_i$ implies that for $i = 1, 2, 3$,
YT_1(M_i) = T_2Y(M_i) \subset T_2N_i \subset N_i.

It follows that \( T_1M_i \subset Y^{-1}(N_i) = M_i \) for \( i = 1, 2, 3 \). Thus \( M_i \) is a closed invariant subspace for \( T_1 \). Condition (2.7) implies that

\[ M_3 \subset M_1. \]

Since \( Y(M_i) \subset N_i \), for \( i = 1, 2, 3 \),

\[ (2.9) \quad Y_*(M_i) = (Y(M_i))^\sim \subset N_i. \]

Since \( Y_* \) is onto, there is a function \( \phi : \text{Lat}(T_2) \to \text{Lat}(T_1) \) such that \( Y_* \circ \phi \) is the identity mapping on \( \text{Lat}(T_2) \). Hence for \( i = 1, 2, 3 \),

\[ Y_*(\phi(N_i)) = Y(\phi(N_i))^\sim = N_i. \]

It follows that for \( i = 1, 2, 3 \),

\[ (2.10) \quad \phi(N_i) \subset M_i. \]

Since \( Y_* \circ \phi \) is the identity mapping on \( \text{Lat}(T_2) \), (2.10) implies that for \( i = 1, 2, 3 \),

\[ (2.11) \quad N_i = Y_*(\phi(N_i)) \subset Y_*(M_i). \]

By (2.9) and (2.11), we get

\[ (2.12) \quad Y_*(M_i) = N_i, \]

for \( i = 1, 2, 3 \). Hence we can easily see that function \( Y \) satisfies the assumptions of Proposition 2.13.

Thus by Proposition 2.13 and equation (2.12),

\[ (2.13) \quad Y_*[M_1 \cap (M_2 \lor M_3)] = Y_*(M_1) \cap Y_*(M_2 \lor M_3) = N_1 \cap (N_2 \lor N_3). \]

Since \( M_1 \cap M_2 = Y^{-1}(N_1) \cap Y^{-1}(N_2) = Y^{-1}(N_1 \cap N_2) \), by the same way as above, we obtain

\[ (2.14) \quad Y_*(M_1 \cap M_2) = N_1 \cap N_2. \]

By equations (2.12) and (2.14), we obtain

\[ (2.15) \quad Y_*[(M_1 \cap M_2) \lor M_3] = (N_1 \cap N_2) \lor N_3. \]

Since \( (N_1 \cap N_2) \lor N_3 \neq N_1 \cap (N_2 \lor N_3) \), from equations (2.13) and (2.15), we can conclude that

\[ (M_1 \cap M_2) \lor M_3 \neq M_1 \cap (M_2 \lor M_3). \]

Therefore \( \text{Lat}(T_1) \) is not modular.
Proposition 3.1. [1] Let $\theta$ be a nonconstant inner function in $H^\infty$. Then every invariant subspace $M$ of $S(\theta)$ has the form
\[ \phi H^2 \oplus \theta H^2 \]
for some inner divisor $\phi$ of $\theta$.

We can easily check that if $M_1 = \theta_1 H^2 \oplus \theta H^2$ and $M_2 = \theta_2 H^2 \oplus \theta H^2$ where $\theta_i$ ($i = 1, 2$) is an inner divisor of $\theta$, then
\begin{equation}
M_1 \cap M_2 = (\theta_1 \lor \theta_2)H^2 \oplus \theta H^2 \tag{3.1}
\end{equation}
and
\begin{equation}
M_1 \lor M_2 = (\theta_1 \land \theta_2)H^2 \oplus \theta H^2 \tag{3.2}
\end{equation}
where $\theta_1 \land \theta_2$ and $\theta_1 \lor \theta_2$ denote the greatest common inner divisor and least common inner multiple of $\theta_1$ and $\theta_2$, respectively. Note that if $M_1 \subset M_2$, then
\begin{equation}
\theta_2 | \theta_1. \tag{3.3}
\end{equation}

Lemma 3.2. If $\theta$ is an inner function in $H^\infty$, then $\text{Lat}(S(\theta))$ is distributive.

Proof. Let $M_1$, $M_2$, and $M_3$ be invariant subspaces for $S(\theta)$. Then by Proposition 3.1, there are nonconstant inner functions $\theta_1$, $\theta_2$, and $\theta_3$ in $H^\infty$ such that
\[ M_i = \theta_i H^2 \oplus \theta H^2 \text{ for } i = 1, 2, 3. \]

From equations (3.1) and (3.2), we obtain that
\begin{equation}
M_1 \cap (M_2 \lor M_3) = (\theta_1 \lor (\theta_2 \land \theta_3))H^2 \oplus \theta H^2, \tag{3.4}
\end{equation}
and
\begin{equation}
(M_1 \cap M_2) \lor (M_1 \cap M_3) = ((\theta_1 \lor \theta_2) \land (\theta_1 \lor \theta_3))H^2 \oplus \theta H^2. \tag{3.5}
\end{equation}
Since $\theta_1 \lor (\theta_2 \land \theta_3) = (\theta_1 \lor \theta_2) \land (\theta_1 \lor \theta_3)$, by equations (3.4) and (3.5), this lemma is proven.

In this section, we will consider a sufficient condition for $\text{Lat}(T)$ of a $C_0$-operator $T$ to be modular.

Proposition 3.3. [2] (Proposition 2.4.3) Let $T \in L(H)$ be a completely nonunitary contraction, and $M$ be an invariant subspace for $T$. If
\begin{equation}
T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{3.6}
\end{equation}
is the triangularization of $T$ with respect to the decomposition $H = M \oplus (H \ominus M)$, then $T$ is of class $C_0$ if and only if $T_1$ and $T_2$ are operators of class $C_0$.

Proposition 3.4. [3] (Corollary 7.1.17) Let $T \in L(H)$ is an operator of class $C_0$, $M$ be an invariant subspace for $T$, and
\begin{equation}
T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{3.7}
\end{equation}
be the triangularization of $T$ with respect to the decomposition $H = M \oplus (H \ominus M)$. Then $T$ has property (P) if and only if $T_1$ and $T_2$ have property (P).
Let $H$ and $K$ be Hilbert spaces and $H \oplus K$ denote the algebraic direct sum. Recall that $H \oplus K$ is also a Hilbert space with an inner product

\[(h_1, k_1), (h_2, k_2) = (h_1, h_2) + (k_1, k_2)\]

**Theorem 3.5.** Let $T \in L(H)$ be an operator of class $C_0$ with property $(P)$. Then $\text{Lat}(T)$ is a modular lattice.

**Proof.** Suppose that $T$ has property $(P)$ and let $M_i (i = 1, 2, 3)$ be an invariant subspace for $T$ such that $M_3 \subset M_1$. Then evidently,

\[
(M_1 \cap M_2) \vee M_3 \subset M_1 \cap (M_2 \lor M_3).
\]

Let $T_i = T|_{M_i} (i = 1, 2, 3)$. Define a linear transformation $X : M_2 \oplus M_3 \to M_2 \lor M_3$ by

\[
X(a_2 + a_3) = a_2 + a_3
\]

for $a_2 \in M_2$ and $a_3 \in M_3$.

Then for $a_2 + a_3 \in M_2 \oplus M_3$ with $\|a_2 + a_3\| \leq 1$, $\|X(a_2 + a_3)\| = \|a_2 + a_3\| \leq \|a_2\| + \|a_3\| \leq 2$. It follows that $\|X\| \leq 2$ and so $X$ is bounded.

Since $M_2 \lor M_3$ is generated by $\{a_2 + a_3 : a_2 \in M_2$ and $a_3 \in M_3\}$, $X$ has dense range. By definition of $T_i (i = 1, 2, 3)$,

\[
X(T_2 \oplus T_3)(a_2 + a_3) = Ta_2 + Ta_3
\]

and

\[
(T | M_2 \lor M_3)X(a_2 + a_3) = Ta_2 + Ta_3.
\]

Thus

\[
X(T_2 \oplus T_3) = (T | M_2 \lor M_3)X.
\]

By Proposition 3.3, $T_2 \oplus T_3$ and $T | M_2 \lor M_3$ are of class $C_0$ and since $T$ has property $(P)$, by Proposition 3.4, we conclude that $T | M_2 \lor M_3$ also has Property $(P)$. By Corollary 2.12, $X$ is a lattice-isomorphism.

Thus $X_* : \text{Lat}(T_2 \oplus T_3) \to \text{Lat}(T | M_2 \lor M_3)$ is onto. Let

\[
M = \{a_2 + a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}.
\]

Since $M = X^{-1}(M_1)$, $M$ is a closed subspace of $M_2 \oplus M_3$. Evidently, $M$ is invariant for $T_2 \oplus T_3$. From the equation (3.9), we conclude that

\[
M = (M_1 \cap M_2) \oplus M_3.
\]

Since $X^{-1}(M_1 \cap (M_2 \lor M_3)) = \{a_2 + a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1 \cap (M_2 \lor M_3)\} = \{a_2 + a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}$,

\[
X^{-1}(M_1 \cap (M_2 \lor M_3)) = M
\]

Since $X$ is a lattice-isomorphism,

\[
X_* M = (XM)^{-} = M_1 \cap (M_2 \lor M_3).
\]

By equation (3.10) and definition of $X$,

\[
X_* M = (XM)^{-} \subset (M_1 \cap M_2) \lor M_3.
\]

From (3.11) and (3.12), we conclude that

\[
M_1 \cap (M_2 \lor M_3) \subset (M_1 \cap M_2) \lor M_3.
\]
Thus if $T$ has property $(P)$, then by (3.8) and (3.13), we obtain that

$$M_1 \cap (M_2 \lor M_3) = (M_1 \cap M_2) \lor M_3.$$
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