Abstract

The twist and writhe numbers and magnetic energy of an orthogonally perturbed vortex filaments are obtained from the computation of the magnetic helicity of geodesic and abnormal magnetohydrodynamical (MHD) vortex filament solutions. Twist is computed from a formula recently derived by Berger and Prior [J. Phys. A 39 (2006) 8321] and finally writhe is computed from the theorem that the helicity is proportional to the sum of twist and writhe. The writhe number is proportional to the total torsion and to integrals of the vector potential. The magnetic energy is computed in terms of the integral of the torsion squared which allows us to place a bound to energy in in the case of helical filaments due to a theorem by Fukumoto [J. Phys. Soc. Japan,(1987)] in fluid vortex filaments. It is also shown that filament torsion coincides with the magnetic twist in the case under consideration, where a small orthogonal magnetic field exist along the thin filament. A new Aharonov-Bohm (AB) phase term is obtained in the writhe number expression which is not present in the Moffatt-Ricca (Proc Roy Soc London A,1992) computation. PACS numbers: 02.40
I Introduction

The topology and geometry of fluid vortex filaments [1] have immensely contributed to our comprehension of the [2] dynamics and kinematics of magnetic filamentary structures with applications in solar and plasma physics [3, 4]. In particular twisted filamentary structure have helped solar physicists to better understand the mechanism which allow the highly twisted coronal magnetic flux tube to emerge from the solar surface and produce those beautiful solar flares and loops. Recently Berger and Prior [5] have presented a detailed discussion of twist and writhe of open and closed curves presenting a formula for the twist of magnetic filaments in terms of parallel electric currents. In the present paper we apply Berger-Prior formula and the theorem of the sum of writhe and twist numbers investigated previously by Moffat and Ricca [6] and Berger and Field [7] to obtain an expression for the writhe number for the magnetically perturbed vortex filamentary twisted structure. To obtain the writhe number we make use of the helicity local expression for the magnetic field proportional to the magnetic field itself and decompose the magnetic vector field along the Serret-Frenet frame in 3D dimensions. This allows us to solve scalar MHD equations to obtain constraints on the global helicity expression which will allow us to compute the writhe number. The knowledge of tilt, twist and writhe of solar filaments for example has recently helped solar physicists [8] to work out data obtained from the vector magnetograms placed in solar satellites. This is already an strong motivation to go on investigating topological properties of these filamentary twisted magnetic structures. Earlier the Yokkoh solar mission has shown that the sigmoids which are nonplanar solar filaments are obtained due to the action of electric currents along these filaments which was used recently [9] as motivation to investigate current-carrying torsioned twisted magnetic curves Throughout the paper we use the notation of a previously paper on vortex filaments in MHD. Mathematical notation is used based on the book by C. Rogers and W. Schief [10] on the geometry of solitons. The paper is organized as follows: In section 2 we decompose MHD equations on a Frenet frame along the twisted thin filament and solve the scalar equations obtained. In section 3 we compute the twist, writhe and energy of the magnetic filament. In section 4 we present the conclusions.
Let us now start by considering the MHD field equations

\[ \nabla \cdot \vec{B} = 0 \]  
\[ \nabla \times \vec{B} = \alpha \vec{B} \]  

where \( \alpha \) is the magnetic twist and the magnetic field \( \vec{B} \) along the filament is defined by the expression \( \vec{B} = B_s \vec{t} + B_n \vec{n} \) and \( B_n \) is the magnetic field perturbation orthogonal to the filament all along its extension, and \( B_s \) is the component along the arc length \( s \) of the filament. The vectors \( \vec{t} \) and \( \vec{n} \) along with binormal vector \( \vec{b} \) together form the Frenet frame which obeys the Frenet-Serret equations

\[ \vec{t}' = \kappa \vec{n} \]  
\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \]  
\[ \vec{b}' = -\tau \vec{n} \]  

the dash represents the ordinary derivation with respect to coordinate \( s \), and \( \kappa(s, t) \) is the curvature of the curve where \( \kappa = R^{-1} \). Here \( \tau \) represents the Frenet torsion. We follow the assumption that the Frenet frame may depend on other degrees of freedom such as that the gradient operator becomes

\[ \nabla = \vec{t} \frac{\partial}{\partial s} + \vec{n} \frac{\partial}{\partial n} + \vec{b} \frac{\partial}{\partial b} \]  

The other equations for the other legs of the Frenet frame are

\[ \frac{\partial}{\partial n} \vec{t} = \theta_{ns} \vec{n} + [\Omega_b + \tau] \vec{b} \]  
\[ \frac{\partial}{\partial n} \vec{n} = -\theta_{ns} \vec{t} - (\text{div} \vec{b}) \vec{b} \]  
\[ \frac{\partial}{\partial n} \vec{b} = -[\Omega_b + \tau] \vec{t} - (\text{div} \vec{n}) \vec{n} \]  
\[ \frac{\partial}{\partial b} \vec{t} = \theta_{bs} \vec{b} - [\Omega_n + \tau] \vec{n} \]  
\[ \frac{\partial}{\partial b} \vec{n} = [\Omega_n + \tau] \vec{t} - \kappa + (\text{div} \vec{n}) \vec{b} \]
\[
\frac{\partial \vec{b}}{\partial b} = -\theta_{bs} \vec{t} - \left[ \kappa + (\text{div}\vec{n}) \right] \vec{n}
\]  

(II.12)

Substitution of these equations into the magnetic helicity equation reads

\[
\nabla \times \vec{B} = \vec{t}[B_n(\text{div}\vec{b}) - B_s(\Omega_s + \tau)] + \vec{n}\tau B_n + \vec{b}[\kappa B_s - B_n(1 + \theta_{ns})]
\]

(II.13)

while its RHS is

\[
\alpha \vec{B} = \alpha[B_n\vec{n} + B_s\vec{t}]
\]

(II.14)

Comparing both sides component by component one obtains the three scalar equations

\[
\tau B_n = \alpha B_n
\]

(II.15)

This result seemed to be noticed earlier by Parker [11] which used to call the parameter \(\alpha\) torsion. Throughout this derivation we consider that the parameter \(\alpha\) is constant, which is not very usual in other works on magnetic helicity. The other scalar equations are

\[
\frac{B_n}{B_s} = \frac{\kappa}{(1 + \theta_{ns})}
\]

(II.16)

which is also usual when one deals with magnetic flux tubes [12] and finally

\[
\alpha B_s = B_n(\text{div}\vec{b}) - B_s(\Omega_s + \tau)
\]

(II.17)

The equation \(\nabla \cdot \vec{B} = 0\) becomes

\[
\partial_s B_s + [\theta_{bs} + \text{div}\vec{b}] B_s = 0
\]

(II.18)

A simple solution of this equation can be obtained if one considers that \(\text{div}\vec{b}\) does not vary appreciably on a short distance \(ds\) along the filament. The solution is

\[
B_s = B_0[1 - \text{div}\vec{b} \int \kappa ds]
\]

(II.19)

where \(B_0\) is an integration constant. Substitution into equation (II.17) assuming that the flow is geodesic along the filament and abnormality relation \(\Omega_s = 0\) along with \(\theta_{bs} = \theta_{ns} = 0\) one obtains

\[
B_n = B_0[(\text{div}\vec{b})^{-1} - \int \kappa ds]
\]

(II.20)

which allows us to say that we obtain a solution for the magnetic filament in terms of the vortex filament invariant \(\int \kappa ds\). In the next section we shall compute the magnetic energy in terms of the magnetic vector potential \(\vec{A}\), which allows us to compute the helicity integral and the writhe number in terms of the AB phase.
III Magnetic energy, Twist and Writhe

To be able to compute the magnetic energy, twist and writhe we now compute the vector potential assuming that it obeys the Coulomb gauge $\nabla \cdot \vec{A} = 0$ and the definition $\vec{B} = \nabla \times \vec{A}$. The Coulomb gauge becomes

$$\partial_s A_s + \text{div} \vec{n} A_n = 0 \quad (\text{III.21})$$

together with the equations for the definition of $\vec{B}$, taking $\vec{A} = A_s \vec{t} + A_n \vec{n} + A_b \vec{b}$ yields

$$\frac{A_s}{A_b} = -\frac{\tau}{\kappa} \quad (\text{III.22})$$

which was obtained from the constraint $B_s = 0$ and

$$B_n = -\tau A_n \quad (\text{III.23})$$

$$B_s = -[2\tau A_s + \text{div} \vec{b} A_n + (\kappa + \text{div} \vec{n}) A_b] \quad (\text{III.24})$$

Algebraic manipulation of the above equations allow us to compute the magnetic energy

$$E_B = \frac{1}{8\pi} \int B^2 dV \quad (\text{III.25})$$

which yields

$$E_B = \frac{1}{8\pi} \int (B_s^2 + B_n^2) dV \quad (\text{III.26})$$

or in terms in terms of the vector potential component $A_n$ yields

$$E_B = \frac{R^2 A_n^2}{2} \left[ \int \tau^2 ds + \frac{1}{4} \int \text{div} \vec{b} ds \right] \quad (\text{III.27})$$

which upon the assumption $\text{div} \vec{b} \ll \tau^2$ reduces to

$$E_B = \frac{R^2 A_n^2}{2} \left[ \int \tau^2 ds \right] \quad (\text{III.28})$$

In case of helical filaments one knows that $\tau = c_0 \kappa$ where $c_0$ is constant and the expression (III.27) reads

$$E_B = \frac{c_0 R^2 A_n^2}{2} \left[ \int \kappa^2 ds \right] \quad (\text{III.29})$$

Due to a theorem by Fukumoto [13]

$$\int \kappa^2 ds < 16\pi^2 L \quad (\text{III.30})$$
which clearly places a bound on the magnetic energy $E_B$. Here $L$ denotes the length of the filament. Algebraic manipulation of the same equations yields a relation between the Frenet curvature and torsion in terms of $\text{div}\vec{n}$ as

$$\kappa \pm \sqrt{2} \tau = -\frac{\text{div}\vec{n}}{2}$$  \hspace{1cm} (III.31)

In the helical case we obtain the following equation

$$\kappa[1 \pm \sqrt{2}] = -\frac{\text{div}\vec{n}}{2}$$  \hspace{1cm} (III.32)

Now let us compute the global helicity

$$H = \int \vec{A} \cdot \vec{B} dV$$  \hspace{1cm} (III.33)

which is equivalent to

$$H = \int [A_s B_s + A_n B_n] dV$$  \hspace{1cm} (III.34)

substitution of the relations above yields

$$H = -\int [2\tau A_n + \text{div}\vec{b}] A_n ds$$  \hspace{1cm} (III.35)

Due to a theorem which states

$$H = \frac{\Phi}{2\pi} [Tw + Wr]$$  \hspace{1cm} (III.36)

where $\Phi$ is the magnetic flux $\int \vec{B} \cdot d\vec{S}$. Before computing the writhe number we need to compute the twist of the magnetic filament. But this becomes quite simple thanks to a formula recently derived by Berger and Prior [5] which is given by

$$\frac{d}{ds}Tw = \frac{J||}{|B|} = \frac{\alpha B_s}{B_s} = \alpha$$  \hspace{1cm} (III.37)

where $J||$ is the electric current along the twisted filament. This result was obtained thanks to the assumption that $B_n << B_s$ since $B_n$ is nothing but a simple perturbation of $B_s$. Integration of the last expression yields

$$Tw = \int \tau ds$$  \hspace{1cm} (III.38)

Substitution of the twist and magnetic helicity $H$ into expression (III.36) yields finally the expression for the writhe number

$$Wr = -[(1 + \frac{4\pi A_n^2}{\Phi^2}) \int \tau ds + \frac{2\pi}{\Phi^2} (\text{div}\vec{b}) \int A_n ds]$$  \hspace{1cm} (III.39)

Note that the last term in this expression is proportional to the AB phase $\int A_n ds$. 
IV Conclusions

In conclusion, a new expression for the writhe of magnetic vortex twisted filament has been obtained presenting a new term representing a Berry's phase which was not present in the previous calculation of Moffatt and Ricca. A bound in the energy has been obtained thanks to a theorem by Fukumoto on the bound of the integral invariant of the square of the Frenet curvature value in the helical case. Future applications in plasma physics or in DNA [12] be appear elsewhere. Expressions for the writhe number for the elastodynamics have been obtained by Klapper and Tabor [13]. Other integral invariants in vortex fluid dynamics have been recently also computed by Maggioni and Ricca [14] which can be generalised to MHD. Yet other interesting examples of the topological bounds to energy [2] have been provided by Khesin [15].

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