Original Article

Weighted fractional Sobolev spaces as interpolation spaces in bounded domains

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Funding information
Fondo para la Investigación Científica y Tecnológica, Grant/Award Numbers: PICT-2018-00583, PICT-2018-03017; Universidad de Buenos Aires, Grant/Award Number: 20020160100144BA

1 INTRODUCTION

The aim of this paper is to contribute to the study of weighted fractional Sobolev spaces in arbitrary bounded domains in $\mathbb{R}^n$, when the weights are positive powers of the distance to the boundary, by characterizing them as real interpolation spaces. The results we obtain are new even in the case of smooth domains.

To be more precise, let us introduce some notations. For any bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $d(x) = d(x, \partial \Omega)$ the distance from $x$ to the boundary of $\Omega$. We will consider the weighted Sobolev spaces

$$W^{1,p}(\Omega, d^\alpha, d^\beta) = \{ f \in L^p(\Omega, d^\alpha) : \| \nabla f \|_{L^p(\Omega, d^\beta)} < \infty \},$$

where $\alpha, \beta \geq 0$ and $\| f \|_{L^p(\Omega, d^\alpha)} = \| f d^{\alpha} \|_{L^p(\Omega)}$, and their fractional counterpart

$$\tilde{W}^{s,p}(\Omega, d^\alpha, d^\beta) = \{ f \in L^p(\Omega, d^\alpha) : \| f \|_{\tilde{W}^{s,p}(\Omega, d^\beta)} < \infty \},$$

where $0 < s < 1$ and

$$\| f \|_{\tilde{W}^{s,p}(\Omega, d^\beta)} = \int_{\Omega} \int_{|x-y|^s < \frac{d(x)}{2}} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, d(x)^\beta \, dx. \quad (1.1)$$
In the unweighted case, we will simply write $W^{1,p}(\Omega) = W^{1,p}(\Omega, 1, 1)$, $\widetilde{W}^{s,p}(\Omega) = \widetilde{W}^{s,p}(\Omega, 1, 1)$, and $\|\cdot\|_{\widetilde{W}^{s,p}(\Omega)} = 1 \cdot \|\cdot\|_{\widetilde{W}^{s,p}(\Omega)}$.

Observe that, due to the fact that the region of integration of its inner integral is restricted to $|x - y| < \frac{d(x)}{2}$, the seminorm (1.1) is equivalent to

$$
\int \int_{|x-y|<\frac{d(x)}{2}} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \delta^2(x,y) \, dy \, dx,
$$

(1.2)

where $\delta(x,y) = \min\{d(x), d(y)\}$. Moreover, replacing $\frac{1}{2}d(x)$ by $\tau d(x)$ for any fixed $\tau \in (0, 1)$ in Equations (1.1) and (1.2), the norms of the associated spaces are equivalent (see Remark 2.2).

The seminorm $\|f\|_{\widetilde{W}^{s,p}(\Omega)}$ was introduced in the context of Poincaré and Sobolev–Poincaré inequalities in John domains in [9] and further results on its relevance for these inequalities were obtained in [2, 6, 7, 11]. It is equivalent to the usual Gagliardo seminorm given by

$$
\int \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \delta^2(x,y) \, dy \, dx,
$$

when $\Omega$ is a Lipschitz domain (see [5, Equation (13)]) or, more generally, a uniform domain (see [12, Corollary 4.5]). More importantly, it is a replacement for the non-inclusion $W^{1,p}(\Omega) \not\subset W^{s,p}(\Omega)$ when $\Omega$ is an irregular domain (see [3, Example 2.1] for such an example). In fact, the embedding $W^{1,p}(\Omega) \subset \widetilde{W}^{s,p}(\Omega)$ holds for any bounded domain (see [4, Lemma 2.2]) and one also has the Bourgain–Brézis–Mironescu limit property, namely that for any $1 < p < \infty$ and $f \in L^p(\Omega)$,

$$
\lim_{s \to 1-} (1-s) \|f\|^p_{\widetilde{W}^{s,p}(\Omega)} = K_{s,p} \|\nabla f\|^p_{L^p(\Omega)},
$$

where $K_{s,p}$ is an explicit constant and the right-hand side of the inequality is understood to be infinity if $f \notin W^{1,p}(\Omega)$ (see [4, Theorem 1.1]). A deeper result in [3, Theorem 3.2] is that for a general class of irregular domains one has $(L^p(\Omega), W^{1,p}(\Omega))_{s,p} = \widetilde{W}^{s,p}(\Omega)$ which is a generalization of the classical result $(L^p(\Omega), W^{1,p}(\Omega))_{s,p} = W^{s,p}(\Omega)$ for smooth domains (see [10, Section 1.1.2, Exercise 7] or [15, Lemmas 35.2 and 36.1]). Here, $(L^p(\Omega), W^{1,p}(\Omega))_{s,p}$ stands for the real interpolation space (see Section 2 for its definition).

While for irregular domains it may happen that $\widetilde{W}^{s,p}(\Omega) \neq W^{s,p}(\Omega)$, it turns out that, for every $\alpha \geq 0$, $\widetilde{W}^{s,p}(\Omega, d^\alpha, d^{(\alpha+s)p}) = W^{s,p}(\Omega, d^\alpha, d^{(\alpha+s)p})$ with equivalence of norms, where

$$
W^{s,p}(\Omega, d^\alpha, d^{(\alpha+s)p}) = \{f \in L^p(\Omega, d^\alpha) : \|f\|_{W^{s,p}(\Omega, \delta^{(\alpha+s)p})} < \infty\},
$$

and

$$
\|f\|^p_{W^{s,p}(\Omega, \delta^2)} = \int \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \delta(x,y)^2 \, dy \, dx
$$

(1.3)

(see Lemma 2.1).

As previously announced, our aim is to characterize the above fractional weighted spaces as real interpolation spaces. More precisely, we prove:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $0 < s < 1$ and $\alpha \geq 0$, then we have

$$(L^p(\Omega, d^\alpha), W^{1,p}(\Omega, d^\alpha, d^{(\alpha+1)p}))_{s,p} = W^{s,p}(\Omega, d^\alpha, d^{(\alpha+s)p}) = W^{s,p}(\Omega, d^\alpha, d^{(\alpha+s)p})$$

with equivalence of norms.

The weighted space $W^{s,2}(\Omega, 1, \delta^2)$ was used in Lipschitz domains in [1] to establish regularity results and to develop efficient numerical methods for nonlocal equations involving the fractional Laplace operator, while the spaces $\widetilde{W}^{s,p}(\Omega, d^\alpha, d^2)$ have been proved useful for Poincaré and Sobolev–Poincaré inequalities in John domains [2, 11]. But, to the best of our knowledge, this is the first result where they are studied in the context of interpolation spaces. Related interpolation results, with applications to elliptic eigenvalue problems, were studied in [13] but for Lipschitz domains only, and with somewhat different characterizations.
The rest of the paper is as follows. We will first prove the equivalence $W^{s,p}(Ω, d^{αp}, d^{(α+s)p}) = W^{s,p}(Ω, d^{αp}, d^{(α+s)p})$ and quickly review some necessary preliminaries and notations. Then, for simplicity, we will write the proof of the above theorem in full detail using the norm of the space $\tilde{W}^{s,p}(Ω, 1, d^{sp})$ for the special case $α = 0$, as all the ideas and technical difficulties are already present there. Finally, in the last section of this paper we will indicate how the proof for $α = 0$ can be modified to obtain the proof of Theorem 1.1 for any $α > 0$.

2 NOTATION AND PRELIMINARY RESULTS

Throughout this paper, $1 ≤ p < ∞$ and $p'$ is its conjugate exponent, $\frac{1}{p'} + \frac{1}{p} = 1$, while $C$ represents a positive constant that might change from line to line.

Lemma 2.1. Let $Ω ⊂ \mathbb{R}^n$ be an open-bounded domain, $0 < s < 1$ and $α ≥ 0$. Then,

$$W^{s,p}(Ω, d^{αp}, d^{(α+s)p}) = W^{s,p}(Ω, d^{αp}, d^{(α+s)p})$$

with equivalence of norms.

Proof. Clearly, $W^{s,p}(Ω, d^{αp}, d^{(α+s)p}) \subset \tilde{W}^{s,p}(Ω, d^{αp}, d^{(α+s)p})$. For the opposite embedding, write

$$|f|^p_{W^{s,p}(Ω, d^{(α+s)p})} = |f|^p_{\tilde{W}^{s,p}(Ω, d^{(α+s)p})} + \intΩ \int |x-y| ≥ \frac{d(x)}{2} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \delta(x, y)^{(α+s)p} dy dx$$

$$\leq |f|^p_{\tilde{W}^{s,p}(Ω, d^{(α+s)p})} + 2^p \intΩ \int |x-y| ≥ \frac{d(x)}{2} \frac{|f(x)|^p + |f(y)|^p}{|x-y|^{n+sp}} \delta(x, y)^{(α+s)p} dy dx \quad (2.1)$$

Observe that

$$\intΩ \int |x-y| ≥ \frac{d(x)}{2} \frac{|f(x)|^p}{|x-y|^{n+sp}} \delta(x, y)^{(α+s)p} dy dx \leq \intΩ |f(x)|^p d(x)^{αp} \left\{ \int |x-y| ≥ \frac{d(x)}{2} \frac{1}{|x-y|^{n+sp}} dy \right\} dx$$

$$\leq C \intΩ |f(x)|^p d(x)^{αp} dx \quad (2.2)$$

Also, when $|x - y| ≥ \frac{d(x)}{2}$ we have $d(y) ≤ |x - y| + d(x) ≤ 3|x - y|$ and so

$$\intΩ \int |x-y| ≥ \frac{d(x)}{2} \frac{|f(y)|^p}{|x-y|^{n+sp}} \delta(x, y)^{(α+s)p} dy dx \leq \intΩ |f(y)|^p d(y)^{αp} \left\{ \int |x-y| ≥ \frac{d(y)}{3} \frac{1}{|x-y|^{n+sp}} dx \right\} dy$$

$$\leq C \intΩ |f(y)|^p d(y)^{αp} dy \quad (2.3)$$

Plugging Equations (2.2) and (2.3) into Equation (2.1) we obtain the desired result. □

Remark 2.2. It is easily seen that the previous proof can be modified replacing $\frac{1}{2}d(x)$ by $rd(x)$ for any $0 < r < 1$ in the definition of the $\tilde{W}^{s,p}(Ω, d^{αp}, d^{(α+s)p})$ seminorm, thus proving that the norms of the associated spaces are equivalent.

Below we recall the essential definitions of the real interpolation method. We refer the reader to [10, Chap. 1] for more details.
Definition 2.3. For $0 < s < 1$, the real interpolation space between $L^p(\Omega, d^{\alpha p})$ and $W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})$ is given by

$$(L^p(\Omega, d^{\alpha p}), W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p}))_{s,p} = \{ f \in L^p(\Omega, d^{\alpha p}) : \| f \|_{(L^p(\Omega, d^{\alpha p}), W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p}))_{s,p}} < \infty \}$$

where

\[
\| f \|_{(L^p(\Omega, d^{\alpha p}), W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p}))_{s,p}} = \left( \int_0^\infty \lambda^{-sp} K(\lambda, f)^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}}
\]

and the $K$ functional is given by

$$K(\lambda, f) = \inf\{\| g \|_{L^p(\Omega, d^{\alpha p})} + \lambda \| h \|_{W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})} : f = g + h\}.$$  

3 PROOF OF THE EMBEDDING $(L^p(\Omega), W^{1,p}(\Omega, 1, d^p))_{s,p} \subset \widetilde{W}^{s,p}(\Omega, 1, d^p)$

We begin this section with two lemmas.

Lemma 3.1. Let $\phi \in L^1(\Omega)$, $\phi \geq 0$, $w \in \mathbb{R}^n$ such that $|w| < \frac{1}{2}$, and $0 \leq t \leq 1$. Then,

$$\int_{\Omega} \phi(x + t d(x) w) \, dx \leq 2^n \int_{\Omega} \phi(x) \, dx. \quad (3.1)$$

Proof. Consider the change of variables $y = F(x) := x + t d(x) w$. First of all, observe that $F(\Omega) \subset \Omega$ and that $F$ is injective, since $F(x) = F(\bar{x})$ implies that

$$|x - \bar{x}| = t |w| |d(\bar{x}) - d(x)| \leq \frac{1}{2} |x - \bar{x}|$$

and therefore $x = \bar{x}$. Consequently,

$$\int_{\Omega} \phi(x + t d(x) w) \, dx = \int_{F(\Omega)} \phi(y) |J^{-1}(y)| \, dy \quad (3.2)$$

where $J = \det DF$. But $DF = I - tB$ with $B = -w(\nabla d)^T$, and

$$\|B\| = \max_{|v|=1} |Bv| = \max_{|v|=1} |w(\nabla d)^T v| \leq \max_{|v|=1} |w||(\nabla d)^T v| \leq |w| |\nabla d| \leq \frac{1}{2}$$

Then, we have that

$$|DF^{-1}| = |(I - tB)^{-1}| = \left| \sum_{j=0}^{\infty} t^j B^j \right| \leq 2^n$$

and so $|J^{-1}| \leq 2^n$, which together with Equation (3.2) concludes the proof.  

Lemma 3.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 \leq p \leq \infty$, and $\alpha \geq 0$. Then, $C^\infty(\Omega) \cap W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})$ is dense in $W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})$.

Proof. Since $W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p}) \subset W^{1,p}_{loc}(\Omega)$, by [8, Theorem 1] for any $f \in W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})$ and $\varepsilon > 0$ there exists $g \in C^\infty(\Omega)$ such that $f - g \in W^{1,p}_0(\Omega)$ and $\| f - g \|_{W^{1,p}(\Omega)} < \varepsilon$. Hence, $\| f - g \|_{W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})} \leq C \| f - g \|_{W^{1,p}(\Omega)} < C \varepsilon$ and $g \in W^{1,p}(\Omega, d^{\alpha p}, d^{(\alpha+1)p})$ by the triangle inequality. 

\[\Box\]
Now, we are ready to show that \((L^p(\Omega), W^{1,p}(\Omega, 1, d^p))_{s,p} \subset \tilde{W}^{s,p}(\Omega, 1, d^{sp})\). Observe that, since one always has the continuous embedding \((X, Y)_{s,p} \subset X + Y\) and \(L^p(\Omega) + W^{1,p}(\Omega, 1, d^p) = L^p(\Omega)\), it follows that \(\|f\|_{L^p} < C\|f\|_{(L^p(\Omega), W^{1,p}(\Omega, 1, d^p))_{s,p}}\). Hence, it is enough to prove that

\[
|f|^p_{\tilde{W}^{s,p}(\Omega, d^{sp})} \leq C \int_0^{\infty} r^{-sp} K(r, f)^p \frac{dr}{r}. \tag{3.3}
\]

We begin by making the change of variables \(w = \frac{y - x}{d(x)}\) in definition (1.1) (for \(\beta = sp\)) to obtain

\[
|f|^p_{\tilde{W}^{s,p}(\Omega, d^{sp})} = \int_{\Omega} \int_{|w| < \frac{1}{2}} \left| \frac{|f(y + d(x)w) - f(x)|^p}{|w|^{n+sp}} \right| dw \, dx. \tag{3.4}
\]

Now, for each \(w\), calling \(r = |w|\), we consider a decomposition \(f = g_r + h_r\) such that

\[
\|g_r\|_{L^p(\Omega)} + \|\nabla h_r\|_{L^p(\Omega, d^{sp})} \leq 2K(r, f).
\]

By Lemma 3.2, we may also assume that \(h_r\) is smooth.

Since

\[
|f|^p_{\tilde{W}^{s,p}(\Omega, d^{sp})} \leq \left( \int_{\Omega} \int_{|w| < \frac{1}{2}} \frac{|g_r(y + d(x)w) - g_r(x)|^p}{|w|^{n+sp}} \, dw \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} \int_{|w| < \frac{1}{2}} \frac{|h_r(y + d(x)w) - h_r(x)|^p}{|w|^{n+sp}} \, dw \, dx \right)^{\frac{1}{p}}
\]

it is enough to estimate each term separately. For the first term, we have

\[
\int_{\Omega} \int_{|w| < \frac{1}{2}} \frac{|g_r(y + d(x)w) - g_r(x)|^p}{|w|^{n+sp}} \, dw \, dx \leq C \int_{|w| < \frac{1}{2}} \int_{\Omega} \frac{|g_r(x)|^p}{|w|^{n+sp}} \, dx \, dw + C \int_{|w| < \frac{1}{2}} \int_{\Omega} \frac{|g_r(y + d(x)w)|^p}{|w|^{n+sp}} \, dx \, dw
\]

\[
\leq C \int_{|w| < \frac{1}{2}} \frac{\|g_r\|_{L^p(\Omega)}^p}{|w|^{n+sp}} \, dw
\]

where in the last step we have used Lemma 3.1 for \(\phi = |g_r|^p\) and \(t = 1\).

On the other hand, observe that for \(x \in \Omega, |w| < 1/2\) and \(0 \leq t \leq 1\) we have \(d(x) \leq 2d(x + td(x)w)\), and then

\[
|h_r(x + d(x)w) - h_r(x)|^p = \left| \int_0^1 \nabla h_r(x + td(x)w) \cdot d(x)w \, dt \right|^p \leq \int_0^1 |\nabla h_r(x + td(x)w)|^p d(x)^p |w|^p \, dt \leq 2^p \int_0^1 |\nabla h_r(x + td(x)w)|^p d(x + td(x)w)^p |w|^p \, dt.
\]

Therefore,

\[
\int_{\Omega} \int_{|w| < \frac{1}{2}} \frac{|h_r(x + d(x)w) - h_r(x)|^p}{|w|^{n+sp}} \, dw \, dx \leq 2^p \int_{|w| < \frac{1}{2}} \frac{|w|^p}{|w|^{n+sp}} \int_0^1 \int_{\Omega} |\nabla h_r(x + td(x)w)|^p d(x + td(x)w)^p \, dx \, dt \, dw
\]

\[
\leq C \int_{|w| < \frac{1}{2}} \frac{\|\nabla h_r\|_{L^p(\Omega)}^p}{|w|^{-n-sp+sp}} \, dw,
\]

where we have used again Lemma 3.1, now for \(\phi = |\nabla h_r|^p d^p\).
Putting together the previous estimates and integrating in polar coordinates we have

\[ |f|_{W^{s,p}((\Omega, 1, d^s \rho))}^p \leq C \left( \int_0^{\frac{1}{2}} r^{n-1} \frac{r^{p-1}}{r^{s p}} \|g_r\|_{L^p(\Omega)}^p \, dr + \int_0^{\frac{1}{2}} r^{n-1} \frac{r^{p-s}}{r^{p-1}} \|\nabla h_r\|_{L^p(\Omega, d^p)}^p \, dr \right) \]

\[ \leq C \int_0^{\frac{1}{2}} r^{-sp} (\|g_r\|_{L^p(\Omega)} + r \|\nabla h_r\|_{L^p(\Omega, d^p)})^p \, dr \]

\[ \leq C \int_0^{\frac{1}{2}} r^{-sp} K(r, f)^p \, dr \]

which proves Equation (3.3) and the claimed embedding.

4 | PROOF OF THE EMBEDDING $W^{s,p}(\Omega, 1, d^s \rho) \subset (L^p(\Omega), W^{1,p}(\Omega, 1, d^p))_{s,p}$

Observe first that, since $K(\lambda, f) \leq \|f\|_{L^p(\Omega)}$ and $W^{1,p}(\Omega, 1, d^p) \subset L^p(\Omega)$, we always have

\[ \int_1^\infty (\lambda^{-s} K(\lambda, f))^p \frac{d\lambda}{\lambda} \leq \|f\|_{L^p(\Omega)}^p \int_1^\infty \lambda^{-sp} \frac{d\lambda}{\lambda} \leq C \|f\|_{L^p(\Omega)}^p . \]

Also, for a given decomposition,

\[ \int_0^1 (\lambda^{-s} \|g\|_{L^p(\Omega)})^p \frac{d\alpha}{\alpha} \leq \int_0^1 (\lambda^{-s} \|f\|_{L^p(\Omega)})^p \frac{d\alpha}{\alpha} + \int_0^1 (\lambda^{-s} \|\nabla h\|_{L^p(\Omega, d^p)})^p \frac{d\alpha}{\alpha} . \]

\[ \leq C \left( \|f\|_{L^p(\Omega)}^p + \int_0^1 (\lambda^{-s} \|g\|_{L^p(\Omega)})^p \frac{d\alpha}{\alpha} \right) . \]

Therefore,

\[ \int_0^\infty (\lambda^{-s} K(\lambda, f))^p \frac{d\lambda}{\lambda} \leq C \left( \|f\|_{L^p(\Omega)}^p + \int_0^1 (\lambda^{-s} \|g\|_{L^p(\Omega)} + \lambda \|\nabla h\|_{L^p(\Omega, d^p)})^p \frac{d\alpha}{\alpha} \right) , \quad (4.1) \]

and to prove the claimed embedding it suffices to bound the integral on the right-hand side.

For this purpose, we will make use of the Whitney decomposition of $\Omega$, whose definition we recall below (see, e.g., [14, Chap. VI] for a proof of its existence). Given a cube $Q \subset \mathbb{R}^n$, we denote by $Q'$ the cube with the same center but expanded by a factor $9/8$. The distance from $Q$ to the boundary of $\Omega$ is denoted by $d(Q, \partial \Omega)$, while $diam(Q)$ and $\ell_Q$ are the diameter and length of the edges of $Q$, respectively.

**Definition 4.1.** Given $\Omega \subset \mathbb{R}^n$ an open bounded set, a Whitney decomposition of $\Omega$ is a family $\mathcal{W}$ of closed dyadic cubes with pairwise disjoint interiors and satisfying the following properties:

1) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$
2) $diam(Q) \leq d(Q, \partial \Omega) \leq 4 \ diam(Q)$ \quad $\forall Q \in \mathcal{W}$
3) $\frac{1}{4} \ diam(Q) \leq diam(\tilde{Q}) \leq 4 \ diam(Q)$ \quad $\forall Q, \tilde{Q} \in \mathcal{W}$ such that $Q \cap \tilde{Q} \neq \emptyset$.

Let $\mathcal{W} = \{ Q \}$ be a Whitney decomposition of $\Omega$. For a fixed $0 < \lambda \leq 1$ we denote by $\mathcal{W}^{\lambda} = \{ Q^\lambda \}$ a new dyadic decomposition obtained from $\mathcal{W}$ by dividing each $Q \in \mathcal{W}$ in such a way that $\frac{1}{4} \lambda \ell_Q \leq \ell_{Q^\lambda} \leq 4 \lambda \ell_Q$. Note that, in particular, this means that $\frac{1}{4} \lambda \ diam(Q) \leq diam(Q^\lambda) \leq \lambda \ diam(Q)$. The center of $Q^\lambda$ in this new partition will be denoted by $x^\lambda_j$, and we will write $\ell_j$ instead of $\ell_{Q^\lambda_j}$. 
For each $W^3$, we can define the covering of expanded cubes $W^{3*}=\{(Q^j)^*\}$. Observe that it satisfies $\sum_j \chi_{(Q^j)^*}(x) \leq C$ for every $x \in \Omega$, and that for $x \in (Q^j)^*$

\[
\frac{3 \text{ diam}(Q^j)}{4} \leq d(x) \leq \frac{41 \text{ diam}(Q^j)}{\lambda}.
\] (4.2)

Associated with this covering we can also introduce a smooth partition of unity $\{\psi^j\}$ such that $\text{supp}(\psi^j) \subset (Q^j)^*$, $0 \leq \psi^j \leq 1$, $\sum_j \psi^j = 1$ in $\Omega$, and $\|\nabla \psi^j\|_{\infty} \leq \frac{C}{r^j}$.

For a given (fixed) $C^\infty$ function $\varphi \geq 0$ such that $\text{supp}(\varphi) \subset B(0, \frac{1}{4})$ and $\int \varphi = 1$, and for each $t > 0$, we define $\varphi_t(x) = t^{-n} \varphi(t^{-1}x)$. Then, for a given $f \in \tilde{W}^{3,P}(\Omega, d^P)$ we can define

\[
h^j(y) = \sum_j f^j \psi^j(y),
\]

with

\[
f^j = \int_{\mathbb{R}^n} f * \varphi_{\varepsilon_j}^j(z) \varphi_{\varepsilon_j}^j(z - x_j^j) \, dz,
\]

which is a smooth approximation of $f$.

Now, we are ready to show that

\[
\int_0^1 \lambda^{-3P} \left( \|f - h^j\|_{L^P(\Omega)} + \lambda \|\nabla h^j\|_{L^P(\Omega,d^P)} \right) \frac{d\lambda}{\lambda} \leq C \|f\|_{\tilde{W}^{3,P}(\Omega,d^P)}^P
\] (4.3)

which will prove the embedding $\tilde{W}^{3,P}(\Omega, 1, d^P) \subset (L^P(\Omega), W^{1,P}(\Omega, 1, d^P))_{s,P}$.

Since the elements of $W^{3*}$ have finite overlapping, we have that

\[
\|f - h^j\|_{L^P(\Omega)}^P \leq C \sum_j \|f - f^j\|_{L^P((Q^j)^*)}^P.
\] (4.4)

Now, if $y \in (Q^j)^*$,

\[
f(y) - f^j = \int_{\mathbb{R}^n} (f(y) - f * \varphi_{\varepsilon_j}^j(z)) \varphi_{\varepsilon_j}^j(z - x_j^j) \, dz,
\]

so, if we let $u(x, t) = f * \varphi_t(x)$ and $g(t) = u(tz + (1 - t)y, t\ell_j^j)$, we have that

\[
f(y) - f * \varphi_{\varepsilon_j}^j(z) = g(0) - g(1) = -\int_0^1 g'(t) \, dt
\]

\[
= -\int_0^1 \left[ \nabla u(y + t(z - y), t\ell_j^j) \cdot (z - y) + \frac{\partial u}{\partial t}(y + t(z - y), t\ell_j) \ell_j^j \right] \, dt.
\]

Hence,

\[
f(y) - f^j = \int_{\mathbb{R}^n} (f(y) - f * \varphi_{\varepsilon_j}^j(z)) \varphi_{\varepsilon_j}^j(z - x_j^j) \, dz
\]

\[
= -\int_{\mathbb{R}^n} \int_0^1 \left[ \nabla u(y + t(z - y), t\ell_j^j) \cdot (z - y) + \frac{\partial u}{\partial t}(y + t(z - y), t\ell_j) \ell_j^j \right] \, dt \varphi_{\varepsilon_j}^j(z - x_j^j) \, dz.
\]
Observe that the integral vanishes unless \( z - x_j^\lambda \in \text{supp}(\varphi_{\ell^\lambda_j}) \), that is \( |z - x_j^\lambda| \leq \frac{1}{4} \ell^\lambda_j \), which means that \( z \in Q_j^\lambda \). On the other hand, \( y \in (Q_j^\lambda)^* \) and then \( |z - y| \leq 9/8 \sqrt{n} \ell^\lambda_j \). Changing variables \( x = y + t(z - y) \), we get \( |x - y| \leq 9/8 \sqrt{n} \ell^\lambda_j \) and also \( x \in (Q_j^\lambda)^* \), since \( x \) belongs to the segment joining \( y \) with \( z \). In particular, we can write

\[
 f(y) - f_j^\lambda = -\int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \left[ \nabla u \left( x, t \ell^\lambda_j \right) \cdot \left( \frac{x-y}{t} \right) + \frac{\partial u}{\partial t} \left( x, t \ell^\lambda_j \right) \ell^\lambda_j \right] \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \frac{\chi_{(Q_j^\lambda)^*}(x)}{t^n} \, dx \, dt.
\]

Since \( \int \nabla \varphi = 0 \), we have

\[
 \nabla u(x, t \ell^\lambda_j) = f * \nabla \varphi_{\ell^\lambda_j}(x) = \int_{\mathbb{R}^n} f(w) \nabla \varphi_{\ell^\lambda_j}(x - w) \, dw
\]

and, similarly, using that \( \int \frac{\partial \varphi}{\partial t} = 0 \),

\[
 \frac{\partial u}{\partial t}(x, t \ell^\lambda_j) = f * \frac{\partial \varphi_{\ell^\lambda_j}}{\partial t}(x) = \int_{\mathbb{R}^n} f(w) \frac{\partial \varphi_{\ell^\lambda_j}}{\partial t}(x - w) \, dw
\]

Therefore, we arrive at \( f(y) - f_j^\lambda = I_1 - I_2 - I_3 \), with

\[
 I_1 = \int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \int_{\mathbb{R}^n} \left( f(x) - f(w) \right) \frac{1}{(t \ell^\lambda_j)^{n+1}} \nabla \varphi \left( \frac{x-w}{t \ell^\lambda_j} \right) \cdot \left( \frac{x-y}{t} \right) \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \frac{\chi_{(Q_j^\lambda)^*}((x)}{t^n} \, dw \, dx \, dt,
\]

\[
 I_2 = \int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \int_{\mathbb{R}^n} \left( f(x) - f(w) \right) \frac{1}{(t \ell^\lambda_j)^{n+1}} \nabla \varphi \left( \frac{x-w}{t \ell^\lambda_j} \right) \cdot \left( \frac{x-y}{t} \right) \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \frac{\chi_{(Q_j^\lambda)^*}((x)}{t^n} \, dw \, dx \, dt,
\]

\[
 I_3 = \int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \int_{\mathbb{R}^n} \left( f(x) - f(w) \right) \frac{\ell^\lambda_j}{(t \ell^\lambda_j)^{n+1}} \varphi \left( \frac{x-w}{t \ell^\lambda_j} \right) \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \frac{n \chi_{(Q_j^\lambda)^*}((x)}{t^n} \, dw \, dx \, dt.
\]

Using that \( \frac{x-w}{t \ell^\lambda_j} \in \text{supp}(\varphi) \), we have that \( |x-w| < \frac{1}{4} t \ell^\lambda_j \) and we can bound

\[
 |f(y) - f_j^\lambda| \leq C \int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \int_{\mathbb{R}^n} |f(x) - f(w)| \frac{\ell^\lambda_j}{(t \ell^\lambda_j)^{n+1}} \nabla \varphi \left( \frac{x-w}{t \ell^\lambda_j} \right) \left\| \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \right\| \frac{\chi_{(Q_j^\lambda)^*}((x)}{t^n} \, dw \, dx \, dt
\]

\[
 + C \int_0^1 \int_{|x-y|<C t \ell^\lambda_j} \int_{\mathbb{R}^n} |f(x) - f(w)| \frac{\ell^\lambda_j}{(t \ell^\lambda_j)^{n+1}} \varphi \left( \frac{x-w}{t \ell^\lambda_j} \right) \left\| \varphi_{\ell^\lambda_j} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \right\| \frac{\chi_{(Q_j^\lambda)^*}((x)}{t^n} \, dw \, dx \, dt.
\]

Moreover, recalling that for \( x \in (Q_j^\lambda)^* \) it holds that \( d(x) \geq \frac{3 \ell^\lambda_j}{4} \), we also have

\[
 |x-w| < \frac{1}{4} t \ell^\lambda_j \leq \frac{1}{4} \frac{\ell^\lambda_j}{4} \leq \frac{1}{3} d(x) < \frac{d(x)}{2}.
\]
which implies
\[ |f(y) - f_j| \]
\[ \leq C \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)| \frac{\epsilon_j^\lambda}{(te_j)^{n+1}} \nabla \varphi \left( \frac{x-w}{te_j} \right) \left\| \varphi_{\epsilon_j^\lambda} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \right\| \frac{X(Q_j)\lambda(x)}{t^n} \, dw \, dx \, dt \]
\[ + C \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)| \frac{\epsilon_j^\lambda}{(te_j)^{n+1}} \varphi \left( \frac{x-w}{te_j} \right) \left\| \varphi_{\epsilon_j^\lambda} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \right\| \frac{X(Q_j)\lambda(x)}{t^n} \, dw \, dx \, dt \]
\[ = I + II. \]

We begin by bounding $I$. Hölder's inequality in $dx \, dt$ gives us
\[ I \leq C \left( \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)| \frac{1}{(te_j)^p} \frac{X(Q_j)\lambda(x)}{t^{n+1} + \frac{n}{p}} \left\| \nabla \varphi \left( \frac{x-w}{te_j} \right) \right\| \, dw \right) \frac{1}{p} \left( \epsilon_j^\lambda \right)^{-\frac{n}{p}}, \]
where we have used that, for $\varepsilon > 0$ to be chosen below,
\[ \left( \int_0^1 \int_{\mathbb{R}^n} \frac{1}{t^{n+1-\varepsilon}} \left\| \varphi_{\epsilon_j^\lambda} \left( \frac{x-y}{t} + y - x_j^\lambda \right) \right\|^{p'} \, dt \, dx \right)^{\frac{1}{p'}} \leq C(\epsilon_j^\lambda)^{-\frac{n}{p}}. \]

A further application of Hölder's inequality in $dw$ leads to
\[ I \leq C \left( \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \frac{X(Q_j)\lambda(x)}{t^{2n+1+\varepsilon(p-1)}} \, dx \, dt \right) \frac{1}{p} \left( \epsilon_j^\lambda \right)^{-\frac{n}{p} + \frac{n}{p'} - n}, \]
where we have used that
\[ \left( \int_{\mathbb{R}^n} \left\| \nabla \varphi \left( \frac{x-w}{te_j} \right) \right\|^{p'} \, dw \right)^{\frac{1}{p'}} \leq C(\epsilon_j^\lambda)^{\frac{n}{p'}}. \]

The term $II$ can be bounded similarly observing that
\[ \left( \int_{\mathbb{R}^n} \left\| \varphi \left( \frac{x-w}{te_j} \right) \right\|^{p'} \, dw \right)^{\frac{1}{p'}} \leq C(\epsilon_j^\lambda)^{\frac{n}{p'}}. \]

Hence, for $y \in (Q_j)^*$, we arrive at
\[ |f(y) - f_j| \leq C \left( \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \frac{X(Q_j)\lambda(x)}{t^{2n+1+\varepsilon(p-1)}} \, dx \, dt \right) \frac{1}{p} \left( \epsilon_j^\lambda \right)^{-\frac{n}{p} + \frac{n}{p'} - n}. \]

Replacing this expression in Equation (4.4), we get
\[ \int_0^1 \lambda^{-sp} \| f - h_j \|^p_{L^p(Q_j)} \, d\lambda \]
\[ \leq C \int_0^1 \sum_j \int_{(Q_j)^*} \int_0^1 \int_{|x-y|<Cte_j} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \frac{X(Q_j)\lambda(x)}{t^{2n+1+\varepsilon(p-1)}} \, dx \, dt \, dy \, d\lambda \]
\[
\begin{align*}
&\leq C \int_0^1 \sum_j \int (Q^2_j)^* \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \chi_{|x-w|<\frac{1}{4}t\lambda} \, dw \, \int (Q^2_j)^* \chi_{|x-y|<\frac{1}{4}t\lambda} \, dy \\
&\quad \times \frac{\epsilon_j^{-2n \lambda^{-sp}}}{t^{2n+1+\varepsilon(p-1)}} \, dt \, dx \, d\lambda/\lambda \\
&\leq C \int_0^1 \sum_j \int (Q^2_j)^* \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \chi_{|x-w|<\frac{1}{4}t\lambda} \, dw \, \frac{(\epsilon_j^2)^{-n \lambda^{-sp}}}{t^{n+1+\varepsilon(p-1)}} \, dt \, dx \, d\lambda/\lambda \\
&\leq C \int_0^1 \sum_j \int (Q^2_j)^* \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \chi_{|x-w|<\frac{1}{4}t\lambda} \, dw \, \frac{d(x)^{-n \lambda^{-n-sp}}}{t^{n+1+\varepsilon(p-1)}} \, dt \, dx \, d\lambda/\lambda,
\end{align*}
\]

where in the last line we have used that for \( x \in (Q_j^2)^* \), \( \epsilon_j^2 \) is equivalent to \( \lambda d(x) \). Now, we add up in \( j \) and change the order of integration to obtain

\[
\begin{align*}
\int_0^1 \lambda^{-sp} \| f - h^\lambda \|_{L^p(\Omega)}^p \, d\lambda \\
&\leq C \int_0^1 \int_\Omega \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \chi_{|x-w|<\frac{1}{4}t\lambda d(x)} \, dw \, \frac{d(x)^{-n \lambda^{-n-sp}}}{t^{n+1+\varepsilon(p-1)}} \, dt \, dx \, d\lambda \\
&\leq C \int_\Omega \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|^p \, dw \, \int_0^\infty \lambda^{-n-sp-1} d\lambda \, \frac{d(x)^{-n}}{t^{n+1+\varepsilon(p-1)}} \, dt \, dx \\
&\leq C \int_\Omega \int_{|x-w|<\frac{d(x)}{2}} \left| \frac{f(x) - f(w)}{|x-w|^{n+sp}} \right| \, dw \, d(x)^p \, dx \\
&= |f|^p_{\tilde{W}^{s,p}(\Omega,1,d^p)},
\end{align*}
\]

where in the last inequality we have used that we can choose \( \varepsilon > 0 \) such that \(-sp + 1 + \varepsilon(p-1) < 1\).

For the other term of the \( K \)-functional in Equation (4.3), we write

\[
|\nabla h^\lambda(y)| = \left| \sum_j f_j^\lambda \nabla \psi_j^\lambda(y) \right| \leq \sum_j |f_j^\lambda - f(y)| |\nabla \psi_j^\lambda(y)| \leq C \sum_j |f_j^\lambda - f(y)| \frac{1}{\epsilon_j^2} \chi_{(Q_j^2)^*}(y).
\]

Therefore, using the triangle inequality and Equation (4.2),

\[
\begin{align*}
\int_0^1 \lambda^{p(1-s)} \| \nabla h^\lambda \|_{L^p(\Omega,d^p)}^p \, d\lambda / \lambda &\leq C \int_0^1 \sum_j \lambda^{p(1-s)} \| f - f_j^\lambda \chi_{(Q_j^2)^*} \|_{L^p(\Omega,d^p)}^p \, d\lambda / \lambda \\
&\leq C \int_0^1 \sum_j \lambda^{p(1-s)} \| f - f_j^\lambda \chi_{(Q_j^2)^*} \|_{L^p(\Omega,d^p)}^p \, d\lambda / \lambda \\
&\leq C \int_0^1 \sum_j \lambda^{-sp} \| f - f_j^\lambda \|_{L^p(\Omega,d^p)}^p \, d\lambda / \lambda,
\end{align*}
\]

and this expression can be bounded as before.
Summing up, by Equation (4.1) and the previous bounds, we have that

\[
\int_0^\infty \lambda^{-s_p} K(\lambda, f)^p \frac{d\lambda}{\lambda} \leq C \left\{ \|f\|_{L^p(\Omega)}^p + \int_0^1 \lambda^{-s_p} \left( \|f - h\|_{L^p(\Omega)} + \lambda \|\nabla h\|_{L^p(\Omega, d\lambda)^p} \right) \frac{d\lambda}{\lambda} \right\} 
\leq C \left\{ \|f\|_{L^p(\Omega)}^p + |f|_{\tilde{W}^{s,p}(\Omega, d\lambda)}^p \right\},
\]

and, therefore, the claimed embedding follows.

5 PROOF OF THEOREM 1.1

As said before, the proof of Theorem 1.1 for any \(\alpha > 0\) is very similar to the proof developed in detail for \(\alpha = 0\) in the previous sections, so we will only indicate briefly what are the necessary changes.

To prove the embedding \((L^p(\Omega, d\alpha^p), W^{1,p}(\Omega, d\alpha^p, d(\alpha + 1)p))_{s,p} \subset \tilde{W}^{s,p}(\Omega, d\alpha^p, d(\alpha + s)p)\) proceed as in Section 3. For the term

\[
|g_r|_{\tilde{W}(\Omega, d(\alpha + s)p)} = \int_\Omega \int_{|w|<\frac{1}{2}} \frac{|f(x + wd(x)) - f(x)|^p}{|w|^{\alpha + sp}} \, dw \, d(x)^p \, dx,
\]

split the integral as before, observe that \(d(x) \leq 2d(x + wd(x))\) and use Lemma 3.1 for \(|g_r, d^t|^p\) and \(t = 1\) to arrive at the desired bound. The part involving \(V_h\) follows exactly as in Section 3 making the necessary changes in the exponents.

To prove the embedding \(\tilde{W}^{s,p}(\Omega, d\alpha^p, d(\alpha + s)p) \subset (L^p(\Omega, d\alpha^p), W^{1,p}(\Omega, d\alpha^p, d(\alpha + 1)p))_{s,p}\), observe that the generalization of Equation (4.1) is straightforward and use the same pointwise bound as in Section 4 to arrive at

\[
\int_0^1 \lambda^{-s_p} \|f - h\|_{L^p(\Omega, d\alpha^p)}^p \frac{d\lambda}{\lambda}
\leq C \int_0^1 \sum_j \int_{(Q_j^\lambda)^p} \int_{|x-y|<\xi \ell_j^\lambda} \int_{|x-w|<\frac{\ell_j^\lambda}{2}} |f(x) - f(w)|^p \chi_{|x-w|<\frac{1}{2} \ell_j^\lambda} \, dw \, d\lambda \, dt \, dy \, dx
\]

Use that \(d(y) \leq Cd(x)\) to bound \(d(y)^{2p}\) by \(Cd(x)^{2p}\) and follow the rest of the steps in that section to finish the proof.

FUNDING INFORMATION

Supported by FONCYT, Grant numbers: PICT-2018-03017, PICT-2018-00583; by Universidad de Buenos Aires, Grant number: 20020160100144BA.

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*How to cite this article:* G. Acosta, I. Drelichman, and R. G. Durán, *Weighted fractional Sobolev spaces as interpolation spaces in bounded domains*, Math. Nachr. **296** (2023), 4374–4385.
https://doi.org/10.1002/mana.202200182