UNIFORMLY CONVEX-TRANSITIVE FUNCTION SPACES

FERNANDO RAMBLA-BARRENO AND JARNO TALPONEN

Abstract. We introduce a property of Banach spaces, called uniform convex-transitivity, which falls between almost transitivity and convex-transitivity. We will provide examples of uniformly convex-transitive spaces. This property behaves nicely in connection with some vector-valued function spaces. As a consequence, we obtain some new examples of convex-transitive Banach spaces.

1. Introduction

In this paper we study the symmetries of some well-known, in fact, almost classical Banach spaces. We denote the closed unit ball of a Banach space $X$ by $B_X$ and the unit sphere of $X$ by $S_X$. A Banach space $X$ is called transitive if for each $x \in S_X$ the orbit $G_X(x) = \{T(x)\mid T: X \to X$ is an isometric automorphism\} = $S_X$. If $G_X(x) = S_X$ (resp. $\text{conv}(G_X(x)) = B_X$) for all $x \in S_X$, then $X$ is called almost transitive (resp. convex-transitive). These concepts are motivated by the Banach-Mazur rotation problem appearing in [2, p.242], which remains unsolved. We refer to [3] and [6] for a survey and discussion on the matter.

The known concrete examples of convex-transitive spaces are scarce, and the ultimate aim of this paper is to provide more examples by establishing the convex-transitivity of some vector-valued function spaces and other natural spaces. It was first reported by Pelczynski and Rolewicz [15] in 1962 that the space $L^p$ is almost transitive for $p \in [1, \infty)$ and convex-transitive for $p = \infty$ (see also [17]). Later, Wood [20] characterized the spaces $C_0^0(L)$ whose norm is convex-transitive (see Preliminaries). Greim, Jamison and Kaminska [11] proved that if $X$ is almost transitive and $1 \leq p < \infty$, then the Lebesgue-Bochner space $L^p(X)$ is also almost transitive. Recently, an analogous study of the spaces $C_0^0(L, X)$ was done by Aizpuru and Rambla [1], and some related spaces were studied by Talponen [19]. For some other relevant contemporary results, see [5], [13] and [16].

We will extend these investigations into the vector-valued convex-transitive setting, which differs considerably in many respects from the scalar-valued almost transitive one. For this purpose we will introduce a new concept which is (formally) stronger than convex-transitivity and weaker than almost transitivity, called uniform convex-transitivity. With the aid of this class of Banach spaces we produce new natural examples of convex-transitive vector-valued function spaces. The main results of this paper are the following:

- Characterization of locally compact Hausdorff spaces $L$ such that $C_0^0(L)$ is uniformly convex-transitive.
- If $X$ is a uniformly convex-transitive Banach space, then so is $L^\infty_X(X)$.
- If $X$ and $C_0^0(L)$ are uniformly convex-transitive, then so is $C_0^0(L, X)$.

Date: May 6, 2009.

2000 Mathematics Subject Classification. Primary 46B04, 46B20; Secondary 46B25.
Preliminaries. The scalar field of a Banach space $X$ is denoted by $\mathbb{K}$ and whenever there are several Banach spaces under discussion, then $\mathbb{K}$ is the scalar field of the space denoted by $X$. The open unit ball of $X$ is denoted by $U_X$. The group of rotations $G_X$ of $X$ consists of isometric automorphisms $T: X \to X$, the group operation being the composition of the maps and the neutral element being the identity map $I: X \to X$. We will always consider $G_X$ equipped with the strong operator topology (SOT). An element $x \in S_X$ is called a big point if $\text{conv} Y(x) = B_X$. Thus $X$ is convex-transitive if and only if each $x \in S_X$ is a big point.

Recall that a topological space is totally disconnected if each connected component of the space is a singleton. In what follows $L$ is a locally compact Hausdorff space and $K$ is a compact Hausdorff space, unless otherwise stated. In [20] Wood characterized convex-transitive $C^0_0(L)$ spaces. Namely, $C^0_0(L)$ is convex-transitive if and only if $L$ is totally disconnected and for every regular probability measure $\mu$ on $L$ and $t \in L$ there exists a net $\{\gamma_\alpha\}_\alpha$ of homeomorphisms on $L$ such that the net $\{\mu \circ \gamma_\alpha\}_\alpha$ is $\omega^t$-convergent to the Dirac measure $\delta_t$. The above mapping $\mu \circ \gamma_\alpha$ is given by $\mu \circ \gamma_\alpha(A) = \mu(\gamma_\alpha(A))$ for Borel sets $A \subset L$.

We refer to [14] for background information on measure algebras and isometries of $L^p$-spaces and to [8] for a suitable source to Banach spaces in general. In what follows $\Sigma$ is the completed $\sigma$-algebra of Lebesgue measurable sets on $[0,1]$ and we denote by $m: \Sigma \to \mathbb{R}$ the Lebesgue measure. Define an equivalence relation $\sim$ on $\Sigma$ by setting $A \sim B$ if $m((A \cup B) \setminus (A \cap B)) = 0$.

Recall that a rotation $R$ on the space $C^0_0(L, X)$ is said to be of the Banach-Stone type, if $R$ can be written as

$$R(f)(t) = \sigma(t)(f \circ \phi(t)), \quad f \in C^0_0(L, X),$$

where $\phi: L \to L$ is a homeomorphism and $\sigma: L \to G_X$ is a continuous map. A Banach space $Y$ is said to be contained almost isometrically in a Banach space $X$ if for each $\varepsilon > 0$ there is a linear map $\psi: Y \to X$ such that

$$||y||_Y \leq ||\psi(y)||_X \leq (1 + \varepsilon)||y||_Y \quad \text{for } y \in Y.$$

2. Uniform convex-transitivity

Provided that the space $X$ under discussion is understood, we denote

$$C_n(x) = \left\{ \sum_{i=1}^{n} a_i T_i(x) | T_1, \ldots, T_n \in G_X, \ a_1, \ldots, a_n \in [0, 1], \ \sum_{i=1}^{n} a_i = 1 \right\}$$

for $n \in \mathbb{N}$ and $x \in S_X$. We call a Banach space $X$ uniformly convex-transitive if for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ satisfying the following condition: For all $x \in S_X$ and $y \in B_X$ it holds that $\text{dist}(y, C_n(x)) \leq \varepsilon$, that is

$$\lim_{n \to \infty} \sup_{x \in S_X, y \in B_X} \text{dist}(y, C_n(x)) = 0.$$

We denote by $K_\varepsilon$ the least integer $n$, which satisfies the above inequality involving $\varepsilon$ and such $K_\varepsilon$ is called the constant of uniform convex transitivity of $X$ associated to $\varepsilon$. We call $x \in S_X$ a uniformly big point if

$$\lim_{n \to \infty} \sup_{y \in B_X} \text{dist}(y, C_n(x)) = 0.$$

Clearly almost transitive spaces are uniformly convex-transitive, and uniformly convex-transitive spaces are convex-transitive. It is well-known that $C^0(S^1)$ is a
convex-transitive, non-almost transitive space, and it is easy to see (see e.g. the subsequent Theorem 2.4) that it is even uniformly convex-transitive. Unfortunately, we have not been able so far to find an example of a convex-transitive space which is not uniformly convex-transitive. However, we suspect that such examples exist and we note that the absence of such a complicated space would make some proofs regarding convex-transitive spaces much more simple. Observe that the canonical unit vectors $e_k \in \ell^1$ are far from being uniformly big points:

$$\lim_{n \to \infty} \sup_{y \in B_1} \text{dist}(y, C_n(e_k)) = 1,$$

even though they are big points, i.e. $\text{conv}(G_{\ell^1}(e_k)) = B_{\ell^1}$ for $k \in \mathbb{N}$. In any case, we will provide examples of uniformly convex-transitive spaces, most of which are not previously known to be even convex-transitive.

We note that if $X$ is convex-transitive and there exists a uniformly big point $x \in S_X$ and the value of $\varepsilon$ is very large, for example, $\varepsilon = 1/2$, then $X$ should be uniformly convex-transitive. Next we give an equivalent condition to uniform convex transitivity, which is more applicable in calculations than the condition introduced above.

**Proposition 2.1.** Let $X$ be a Banach space. The following condition of $X$ is equivalent to $X$ being uniformly convex-transitive: For each $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that for each $x \in S_X$ and $y \in B_X$ there are $T_1, \ldots, T_{N_\varepsilon} \in \mathcal{G}_X$ such that

$$\left\| y - \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} T_i(x) \right\| \leq \varepsilon. \tag{2.1}$$

**Proof.** It is clear that (2.1) implies uniform convex transitivity, even for the value $K_\varepsilon = N_\varepsilon$ for each $\varepsilon > 0$. Towards the other direction, let $X$ be a uniformly convex-transitive Banach space, $\varepsilon > 0$ and $x \in S_X$, $y \in B_X$. Let $K$ be the constant of uniform convex-transitivity of $X$ associated to $\varepsilon$. Then there are $a_1, \ldots, a_K \in [0, 1]$, $\sum_i a_i = 1$ and $T_1, \ldots, T_K \in \mathcal{G}_X$ such that

$$\left\| y - \sum_{i=1}^{K} a_i T_i(x) \right\| \leq \varepsilon.$$

Put $m = \left\lceil \frac{4K}{\varepsilon} \right\rceil \in \mathbb{N}$, so that $K \cdot \frac{1}{m} \leq \frac{\varepsilon}{4}$. Next we define an $m$-uple $(S_1, \ldots, S_m) \subset \mathcal{G}_X$ as follows: For each $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, K\}$ we put $S_j = T_i$ if $\left[m \sum_{n<i} a_n\right] < j \leq \left[m \sum_{n<i} a_n\right] + \left[m \sum_{n<i} a_n\right]$. (Here $\sum_{n} a_n = 0$.) By applying the triangle inequality several times, we obtain that

$$\left\| y - \frac{1}{m} \sum_{j=1}^{m} S_j(x) \right\| \leq \varepsilon.$$

Hence it suffices to put $N_\varepsilon = m = \left\lceil \frac{4K}{\varepsilon} \right\rceil$, where $K$ depends only on the Banach space $X$ and the value of $\varepsilon$. \hfill \square

In what follows, we will apply the constant $N_\varepsilon$ freely without explicit reference to the previous proposition, and if there is no danger of confusion, also without mentioning explicitly $X$ and $\varepsilon$, either.

The following condition on the locally compact space $L$ turns out to be closely related to the uniform convex-transitivity of $C^0_u(L)$:

\begin{itemize}
  \item[(*)] For each $\varepsilon > 0$ there is $M_\varepsilon \in \mathbb{N}$ such that for every non-empty open subset $U \subset L$ and compact $K \subset L$ there are homeomorphisms $\phi_1, \ldots, \phi_{M_\varepsilon} : L \to L$
with
\[
\frac{1}{M_\varepsilon} \sum_{i=1}^{M_\varepsilon} \chi_{\phi^{-1}_i}(t) \geq 1 - \varepsilon \quad \text{for } t \in K.
\]

This condition should be compared with the conditions found by Cabello (see [3] p.110-113), especially condition (g)), which characterize the convex transitivity of \(C_0(L)\). Next we will give this characterization the uniformly convex-transitive counterpart. If \(L\) is a locally compact Hausdorff space, by \(\alpha L\) we denote its one-point compactification and if \(L\) is noncompact, we denote such point by \(\infty\). Prior to the theorem we need the following two lemmas.

**Lemma 2.2.** (16 Thm. 3.1) Let \(T\) be a normal topological space with \(\dim T \leq 1\). If \(F \subseteq T\) is a closed subset and \(f : F \to S_C\) is a continuous map, then \(f\) admits a continuous extension \(g : T \to S_C\).

**Lemma 2.3.** Let \(L\) be a locally compact, Hausdorff, 0-dimensional space. Then for every \(g \in B_{C_0^0}(L)\) and \(k \in \mathbb{N}\) there exist disjoint clopen sets \(C_1, C_2, \ldots, C_{2k-1}\) such that the function \(h \in B_{C_0^0}(L)\) defined by \(h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}\) satisfies \(\|h - g\| \leq \frac{3}{2k}\).

**Proof.** We regard \(g\) as defined in \(\alpha L\). Consider \(i \in \{-k, -k+1, \ldots, k-1\}\) and let \(K_i = g^{-1}[\frac{i-1}{k}, \frac{i+1}{k}]\). Every \(x \in K_i\) has a clopen neighbourhood \(A_x\) such that \(g(A_x) \subseteq [\frac{2i-1}{2k}, \frac{2i+1}{2k}]\). By compactness there exist \(x_1, \ldots, x_n\) such that \(K_i \subseteq \bigcup_{j=1}^{n} A_{x_j} = B_i\). Finally, define \(C_0 = B_0, C_1 = B_{-k} \setminus C_0, \ldots, C_k = B_{-1} \setminus (C_0 \cup \cdots \cup C_{k-1}), C_{k+1} = B_1 \setminus (C_0 \cup \cdots \cup C_k), \ldots, C_{2k-1} = B_{k-1} \setminus (C_0 \cup \cdots \cup C_{2k-2})\). Note that the \(C_i\)'s are a partition of \(\alpha L\).

Now take \(h : L \to \mathbb{R}\) given by \(h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}\). It is easy to check that \(\|h - g\| \leq \frac{3}{2k}\) and \(h \in B_{C_0^0}(L)\).

**Theorem 2.4.** Let \(L\) be a locally compact Hausdorff space. The space \(C_0^0(L)\) is uniformly convex-transitive if and only if \(L\) is totally disconnected and satisfies (*). If the space \(C_0^0(L)\) is uniformly convex-transitive, then \(L\) satisfies (*). Moreover, if \(\dim(\alpha L) \leq 1\), then also the converse implication holds.

Before the proof we comment on the above assumptions.

**Remark 2.5.** The spaces \(C^R(S^1, \mathbb{R}^2)\) and \(C^C(S^1, \mathbb{C})\) are uniformly convex-transitive, their rotations are of the Banach-Stone type, and clearly \(S^1, \mathbb{G}_{\mathbb{R}^2}\) and \(\mathbb{G}_{\mathbb{C}}\) are not totally disconnected.

**Proof of Theorem 2.4.** Let us first consider the only if directions. Since uniformly convex-transitive spaces are convex-transitive, we may apply Wood’s characterization for convex-transitive \(C_0^R(L)\) spaces, and thus we obtain that \(L\) must be totally disconnected. Let \(C_0^R(L), \mathbb{R} \in \{\mathbb{R}, \mathbb{C}\}\), be uniformly convex-transitive. Next we aim to check that \(L\) satisfies (*), so let \(U \subseteq L\) be a non-empty open subset and \(K \subseteq L\) a compact subset. Fix \(x_0 \in U\). Since \(\alpha L\) is normal, there exist continuous functions \(f, g : \alpha L \to [-1, 1]\) satisfying \(f(\alpha L \setminus U) = \{0\}\), \(f(x_0) = 1\), \(g(K) = \{1\}\) and \(g(\infty) = 0\). Since both functions vanish at infinity, we can consider that \(f, g \in S_{C_0^R(L)}\).

Fix \(\varepsilon > 0\) appearing in condition (*). Let \(N_\varepsilon\) be the associated constant provided by the uniform convex-transitivity and condition (2.1). Then by the definition of \(N_\varepsilon\) and the Banach-Stone characterization of rotations of \(C_0^R(L)\) we obtain that there are continuous functions \(\sigma_1, \ldots, \sigma_{N_\varepsilon} : L \to \mathbb{R}\) and homeomorphisms...
\(\phi_1, \ldots, \phi_{N_\varepsilon} : L \to L\) such that

\[
(2.2) \quad \left\| g - \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} \sigma_i(f \circ \phi_i) \right\| \leq \varepsilon.
\]

In particular, this yields for each \(t \in K\) that

\[
\varepsilon \geq \left| 1 - \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} \sigma_i f(\phi_i(t)) \right| = \left| \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} 1 - \sigma_i f(\phi_i(t)) \right| \geq \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} \epsilon_{L \setminus \phi_i^{-1}(U)}(t),
\]

where we applied the fact that \(f\) vanishes outside \(U\). This justifies (*) for \(M_\varepsilon = N_\varepsilon\).

Let us see the if direction for \(C_0^R(L)\). Let \(k \in \mathbb{N}, f \in S_{C_0^R(L)}\) and \(g \in B_{C_0^R(L)}\). We may assume \(\max f = 1\). Take \(h\) as in Lemma 2.3 i.e. \(h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}\) with each \(C_i\) clopen and \(\|h - g\| \leq \frac{1}{2k}\).

Note that \(K = \bigcup_{i=1}^{2k-1} C_i\) is compact and apply (*) to this \(K\), the subset \(U = \{ t \in L : f(t) > 1 - k^{-1} \}\) and \(\varepsilon = \frac{1}{2k}\). Write \(M = M_\varepsilon\). There exist homeomorphisms \(\phi_1, \ldots, \phi_M\) such that if \(t \in K\) then \(\frac{1}{M} \sum_{i=1}^{M} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - k^{-1}\). For each \(j \in \{1, \ldots, 2k - 1\}\), define \(B_j = \bigcup_{i=j}^{2k-1} C_i\) and let \(T_j\) be the rotation on \(C_0^R(L)\) given by \(T_j x = (\chi_{B_j} - \chi_{B_j}^c) \cdot x\) if \(j \leq k\) and \(T_j x = (\chi_{B_j} - \chi_{B_j}^c + 2 \chi_{L \setminus k}^c) \cdot x\) if \(j > k\). Now only a few calculations are needed to see that

\[
\left\| g - \frac{1}{M} \sum_{j=1}^{2k-1} \sum_{i=1}^{M} T_j (f \circ \phi_i) \right\| \leq 6k^{-1}
\]

and thus \(C_0^R(L)\) is uniformly convex transitive.

In order to justify the last claim it is required to verify that if \(L\) satisfies \(\dim(o(L)) \leq 1\) and \((*)\), then \(C_0^R(L)\) is uniformly convex-transitive. Let \(k \in \mathbb{N}\) and let \(M_k\) be the corresponding constant in condition \((*)\) associated to value \(k^{-1}\). Fix \(f \in S_{C_0^R(L)}\) and \(g \in B_{C_0^R(L)}\). We may assume without loss of generality, possibly by multiplying \(f\) with a suitable complex number of modulus 1, that \(f(t_0) = 1\) for a suitable \(t_0 \in L\). Let \(U = \{ t \in L : |1 - f(t)| < k^{-1} \}\) and \(K = \{ t \in L : |g(t)| \geq k^{-1} \}\). Let \(\phi_1, \ldots, \phi_{M_k} : L \to L\) be homeomorphisms such that \(\frac{1}{M_k} \sum_{i=1}^{M_k} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - k^{-1}\) for \(t \in K\). This means that the average

\[
(2.3) \quad F = \frac{1}{M_k} \sum_{i=1}^{M_k} f \circ \phi_i \in B_{C_0^R(L)}
\]
satisfies \(|1 - F(t)| \leq 3k^{-1}\) for each \(t \in K\).

Next we define some auxiliary mappings. Put \(\alpha : S_{\mathbb{C}} \times [0,1] \to S_{\mathbb{C}}\): \(\alpha(z, s) = -i^{2s} z\). Note that this is a continuous map, and \(\alpha(z, 0) = -z, \alpha(z, 1) = z\) for \(z \in S_{\mathbb{C}}\). Taking into account Lemma 2.2 with \(T = \alpha L\), let \(\beta_L : L \to S_{\mathbb{C}}\) be a continuous extension of the function \(\frac{g(1)}{|g(1)|}\) defined on \(K\).

For \(j \in \{1, \ldots, k\}\) we define rotations on \(C_0^R(L)\) by putting \(e_{ja}(x)(t) = \beta_L(t) \cdot x(t)\) and \(e_{jb}(x)(t) = \alpha(\beta_L(t), \min(1, \max(0, k |g(t)| - j))) \cdot x(t)\). The main point above is that \((e_{ja} + e_{jb})(F)(t) = 0\) for \((j, t) \in \{1, \ldots, k\} \times L\) such that \(|g(t)| \leq \frac{1}{k}\) and \((e_{ja} + e_{jb})(F)(t) = 2F(t) \beta_L(t)\) for \((j, t) \in \{1, \ldots, k\} \times L\) such that \(g(t) \geq \frac{4k + 1}{k}\). Thus, by using (2.3) we obtain that

\[
\left\| g(t) |\beta_L(t) - \frac{1}{2k} \sum_{j=1}^{k} (e_{ja} + e_{jb})(F)(t) \right\| \leq 2k^{-1}\quad \text{for}\quad t \in L.
\]

Here \(\|g(\cdot) - g(\cdot)|\beta_L(\cdot)|\leq k^{-1}\), so that \(C_0^R(L)\) is uniformly convex-transitive. \(\square\)
Note that Theorem [2.4] yields the fact that if $C^0_0(L)$ is uniformly convex-transitive, then so is $C^0_0(L)$. By the above reasoning one can also see that if $C^0_0(L)$ is convex-transitive and $|L| > 1$, then $L$ contains no isolated points and thus it follows that each non-empty open subset of $L$ is uncountable. Cabello pointed out [5, Cor. 1] that locally compact spaces $L$ having a basis of clopen sets $C$ such that $L \setminus C$ is homeomorphic to $C$, have the property that $C^0_0(L)$ is convex-transitive. Consequently, this provides a route to the fact that the spaces $L^\infty$, $F^\infty/c_0$ and $C(\Delta)$ over $\mathbb{R}$, where $\Delta$ is the Cantor set, are convex-transitive. By applying Theorem [2.4] and following Cabello’s argument with slight modifications, one arrives at the conclusion that these spaces are in fact uniformly convex-transitive. When studying [6] it is helpful to observe that each occurrence of ‘basically disconnected’ in the paper must be read as totally disconnected. (1).

It is quite easy to verify that if $L_1, \ldots, L_n$, where $n \in \mathbb{N}$, are totally disconnected locally compact Hausdorff spaces satisfying (+), then so is the product $L_1 \times \cdots \times L_n$. It follows that the space $C^0_0(L_1 \times \cdots \times L_n)$ (also known as the injective tensor product $C^0_0(L_1) \otimes \cdots \otimes C^0_0(L_n)$, up to isometry) is uniformly convex-transitive.

3. Uniform convex-transitivity of Banach-valued function spaces

With a proof similar to that of Lemma 2.3 we obtain the following:

**Lemma 3.1.** Let $L$ be a locally compact, Hausdorff, 0-dimensional space and $X$ a Banach space over $\mathbb{K}$. Given $g \in B_{C^0_0(L,X)}$ and $j \in \mathbb{N}$, there exist nonzero $x_1, \ldots, x_n \in B_X$ and disjoint clopen sets $C_1, C_2, \ldots, C_n \in L$ such that the function $h \in B_{C^0_0(L,X)}$ defined by $h(t) = \sum_{i=1}^n \chi_{C_i}(t)x_i$ satisfies $||h - g|| < \frac{1}{j}$.

**Theorem 3.2.** Let $L$ be a locally compact Hausdorff space and $X$ a Banach space over $\mathbb{K}$. Consider the following conditions:

1. $L$ is totally disconnected and satisfies (+), i.e. $C^0_0(L)$ is uniformly convex-transitive.
2. $X$ is uniformly convex-transitive.
3. $C^0_0(L,X)$ is uniformly convex-transitive.

We have the implication $(1) + (2) \implies (3)$. If the rotations of $C^0_0(L,X)$ are of the Banach-Stone type and $\dim_{\mathbb{K}}(X) \geq 1$, then $(3) \implies (+) + (2)$. If additionally $\mathbb{K} = \mathbb{R}$ and $G_X$ is totally disconnected, then $(3) \implies (1) + (2)$.

Recall Remark 2.3 related to the last claim above.

**Proof of Theorem 3.2.** We begin by proving the implication $(1) + (2) \implies (3)$. Fix $k \in \mathbb{N}$. Then condition (+) provides us with an integer $N_k$ associated to $\frac{1}{4k}$. Let $f \in S_{C^0_0(L,X)}$ and $g \in B_{C^0_0(L,X)}$. Take $h$ and $C_1, \ldots, C_n \in L$ as in Lemma 3.1 with $j = 2k$.

Let $B = \bigcup_{i=1}^n C_i$ and $K = \{t \in L : \|g(t)\| \geq k^{-1}\}$. Note that $B$ is a compact clopen set and $K \subset B$. There are $y \in S_X$ and $T_1, \ldots, T_{N_k} \in G_{C^0_0(L,X)}$ such that

\[
\|y - \left(\frac{1}{N_k} \sum_{i=1}^{N_k} T_i f\right)(t)\| < \frac{1}{k}, \quad \text{for } t \in B.
\]

Indeed, observe that the continuous map $L \to \mathbb{R}$: $t \mapsto ||f(t)||$ attains its supremum, the value 1. Thus, let $t_0 \in L$ be such that $||f(t_0)|| = 1$ and let $y = f(t_0) \in S_X$. Write $V = \{t \in L : \|f(t) - y\| < \frac{1}{4k}\}$. By using (+) there are homeomorphisms
\[ \sigma_1, \ldots, \sigma_{N_k} : L \to L \text{ such that} \]
\[ (3.2) \quad \frac{1}{N_k} \sum_{i=1}^{N_k} \chi_V(\sigma_i(t)) \geq 1 - \frac{1}{N_k}, \quad \text{for } t \in B. \]

Let \( T_i \in \mathcal{G}_{C_0^k(L, X)} \) be given by \((T_iF)(t) = F(\sigma_i(t))\) for \(1 \leq i \leq N_k\) and \(F \in C_{0}^k(L, X)\). Thus, for all \(t \in B\) we obtain by (3.2) and the definition of \(V\) that
\[ \left\| y - \left( \frac{1}{N_k} \sum_{i=1}^{N_k} T_i f \right)(t) \right\| = \left\| \frac{1}{N_k} \sum_{i=1}^{N_k} y - \frac{1}{N_k} \sum_{i=1}^{N_k} f(\sigma_i(t)) \right\| \leq \frac{1}{N_k} \sum_{i=1}^{N_k} \| y - f(\sigma_i(t)) \| < (1 - \frac{1}{N_k}) \cdot \frac{1}{N_k} + \frac{1}{N_k} \cdot 2 < \frac{1}{k}. \]

Since \(X\) is uniformly convex-transitive, there is an integer \(2M = N_k\) satisfying (2.1) for the value \(\varepsilon = k^{-1}\). Let \(S_1^{(i)}, \ldots, S_{2M}^{(i)} \in \mathcal{G}_X\) for \(1 \leq i \leq n\) such that
\[ (3.3) \quad \left\| x_i - \frac{1}{2M} \sum_{i=1}^{2M} S_i^{(i)}(y) \right\| < k^{-1} \quad \text{for } 1 \leq i \leq n. \]

Then for each \(1 \leq l \leq 2M\) we define a rotation on \(C_{0}^k(L, X)\) by
\[ R_l(F)(t) = \chi_{L \setminus B}(t)(-1)^l F(t) + \sum_{i=1}^{n} \chi_C S_i^{(i)}(F(t)), \quad F \in C_{0}^k(L, X), \quad t \in L. \]

Indeed, this defines rotations, since the sets \(L \setminus B\) and \(C_i\) are clopen. It is easy to see by combining (3.1) and (3.3) that
\[ \left\| g - \frac{1}{2M} \sum_{i=1}^{2M} R_l \frac{1}{N_k} \sum_{i=1}^{N_k} T_i f \right\| < 2k^{-1}. \]

This verifies the first implication.

Next we will prove the implication (3) \(\implies\) (\(\ast\)) \(\implies\) (2) under the assumption that the rotations are of the Banach-Stone type. In fact, the verification of claim (3) \(\implies\) (\(\ast\)) reduces to the analogous scalar-valued case, which was treated in the proof of Theorem 2.4. Moreover, by using the Banach-Stone representation of rotations and functions of type \(f \otimes x, g \otimes y \in S_{C_{0}^k(L, X)}\) it is easy to verify that the uniform convex-transitivity of \(C_{0}^k(L, X)\) implies that of \(X\).

Finally, let us prove the total disconnectedness of \(L\) in the case when \(\mathcal{G}_X\) is totally disconnected and \(K = \mathbb{R}\). Assume to the contrary that \(L\) contains a connected subset \(C\), which is not a singleton. Pick \(t, s \in C\), \(t \neq s\), and \(x \in S_X\). Let \(x^* \in S_{\mathcal{G}_X}\) with \(x^*(x) = 1\). Let \(f, g \in S_{C_{0}^k(L)}\) be functions with disjoint supports and such that \(f(t) = g(s) = 1\). Consider \(f \otimes x, f \otimes x - g \otimes x \in S_{C_{0}^k(L, X)}\). Since \(C_{0}^k(L, X)\) is convex-transitive we obtain that \(f \otimes x - g \otimes x \in \mathcal{G}_{C_{0}^k(L, X)}(f \otimes x)\).

It follows easily by using the Banach-Stone representation of rotations of \(C_{0}^k(L, X)\) that there exists a continuous map \(\sigma : L \to \mathcal{G}_X\) such that
\[ x^*(\sigma(t)(x)), x^*(-\sigma(s)(x)) > 0. \]

By using the facts that \(\sigma(t) \neq \sigma(s)\) and that \(\mathcal{G}_X\) is totally disconnected we obtain that \(\sigma(C)\) is not connected. However, we have a contradiction, since \(\sigma(C)\) is a continuous image of a connected set. This contradiction shows that \(L\) must be totally disconnected.

By following the argument in the previous proof with slight modifications one obtains an analogous result in the convex-transitive setting.
Theorem 3.3. If $C^p_0(L)$ is convex-transitive and $X$ is a convex-transitive space over $K$, then $C^p_0(L, X)$ is convex-transitive.

Proof. The proof of Theorem 3.2 has the convex-transitive counterpart with convex combinations of rotations in place of averages of rotations. Indeed, in the equation (3.1) one uses the convex-transitivity of $C^p_0(L)$ and the corresponding Banach-Stone type rotations applied on $C^p_0(L, X)$. After equation (3.1), the argument proceeds similarly. Note that in the convex-transitive setting there does not exist, a priori, an upper bound $M$ depending only on $\varepsilon$. □

Recall that the Lebesgue-Bochner space $L^p(X)$ consists of strongly measurable maps $f : [0, 1] \to X$ endowed with the norm
\[ ||f||_{L^p(X)} = \int_0^1 ||f(t)||_X^p \, dt, \quad \text{for } p \in [1, \infty) \]
and
\[ ||f||_{L^\infty(X)} = \text{ess sup}_{t \in [0, 1]} ||f(t)||_X. \]
We refer to [7] for precise definitions and background information regarding the Banach-valued function spaces appearing here.

Recall that $L^\infty$ is convex-transitive (see [15] and [17]). Greim, Jamison and Kaminska proved that $L^p(X)$ is almost transitive if $X$ is almost transitive and $1 \leq p < \infty$, see [11] Thm. 2.1. We will present the analogous result for uniformly convex-transitive spaces, that is, if $X$ is uniformly convex-transitive, then $L^p(X)$ are also uniformly convex-transitive for $1 \leq p \leq \infty$.

Theorem 3.4. Let $X$ be a uniformly convex-transitive space over $K$. Then the Bochner space $L^p_\infty(X)$ is uniformly convex-transitive for $1 \leq p \leq \infty$.

We will make some preparations before giving the proof. Suppose that $(A_n)_{n \in \mathbb{N}}$ is a countable measurable partition of the unit interval and $(x_n)_{n \in \mathbb{N}} \subset X$. We will use the short-hand notation $F = \sum_n \chi_{A_n} x_n$ for the function $F \in L^\infty(X)$ defined by $F(t) = x_n$ for a.e. $t \in A_n$ for each $n \in \mathbb{N}$. The following two auxiliary observations are obtained immediately from the fact that the countably valued functions are dense in $L^\infty(X)$ and the triangle inequality, respectively.

Fact 3.5. Consider $F = \sum_n \chi_{A_n} x_n$, where $(A_n)$ is a measurable partition of $[0, 1]$ and $(x_n) \subset B_X$. Functions $F$ of such type are dense in $B_{\ell^\infty(X)}$.

Fact 3.6. Let $X$ be a Banach space and $T_1, \ldots, T_n \in \mathcal{G}_X$, $n \in \mathbb{N}$. Assume that $x, y, z \in X$ satisfy $||y - \frac{1}{n} \sum_i T_i(x)|| = \varepsilon \geq 0$ and $||x - z|| = \delta \geq 0$. Then $||y - \frac{1}{n} \sum_i T_i(z)|| \leq \varepsilon + \delta$.

Proof of Theorem 3.4. We mainly concentrate on the case $p = \infty$. Fix $k \in \mathbb{N}$, $x \in S_X$, $(x_n)_{n \in \mathbb{N}} \subset B_X$ and measurable partitions $(A_n)$ and $(B_n)$ of the unit interval. Let $N_k$ be the integer provided by the uniform convex transitivity of $X$ associated to the value $\varepsilon = \frac{1}{k}$. Write
\[ F = \sum_n \chi_{A_n} x_n \text{ and } G = \sum_n \chi_{B_n} y_n. \]
We assume additionally that $||F|| = 1$.

For each $n \in \mathbb{N}$ there are isometries $\{T_i^{(n)}\}_{i \leq N_k} \subset \mathcal{G}_X$ such that
\[ ||\frac{1}{N_k} \sum_{i=1}^{N_k} T_i^{(n)} (x) - y_n|| < \frac{1}{k} \quad \text{for } n \in \mathbb{N}. \]
Observe that one obtains rotations on $L^\infty(X)$ by putting
\[ R_t(f)(t) = \sum_n \chi_{B_n} T_i^{(n)}(f(t)) \]
for a.e. $t \in [0,1]$, where $f \in L^\infty(X)$, $i \leq N_k$, and the above summation is understood in the sense of pointwise convergence almost everywhere. We define a convex combination of elements of $G_{L^\infty(X)}$ by
\[ A_1(f) = \frac{1}{N_k} \sum_{i=1}^{N_k} R_i(f), \quad f \in L^\infty(X). \]
Condition (3.5) implies that
\[ \|G - A_1(\chi_{[0,1]}x)\| \leq \frac{1}{k}. \]
By the definition of $F$ one can find $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$ and
\[ \|x_{n_0}\|_X \geq 1 - \frac{1}{k}. \]
Put $\Delta_n = [1 - 2^{-n}, 1 - 2^{-(n+1)}]$ for $n \leq k$. By composing suitable bijective transformations one can construct measurable mappings $g_n: [0, 1] \to [0, 1]$ and $\hat{g}_n: [0, 1] \to [0, 1]$ such that
\[ g_n(A_{n_0}) \overset{m}{\to} [0, 1] \setminus \Delta_n \text{ and } g_n([0, 1] \setminus A_{n_0}) \overset{m}{\sim} \Delta_n, \]
the measure $\mu_n(\cdot) \overset{m}{=} m(g_n(\cdot)): \Sigma \to \mathbb{R}$ is equivalent to $m$ and
\[ \hat{g}_n \circ g_n(t) = t \quad \text{for a.e. } t \in [0, 1] \]
for each $n \leq k$.

Next we will apply some observations which appear e.g. in [10] and [9]. Denote by $\Sigma_m$ the quotient $\sigma$-algebra of Lebesgue measurable sets on $[0, 1]$ formed by identifying the $m$-null sets with $\emptyset$. Note that (3.8) gives in particular that the map $\phi_n: \Sigma_m \to \Sigma_m$ determined by $\phi_n(A) \overset{m}{=} \hat{g}_n(A)$ for $A \in \Sigma$ is a Boolean isomorphism for each $n \leq k$. Observe that $\hat{g}_n(A) \overset{m}{=} \phi_n^{-1}(A)$ for $A \in \Sigma$ and $n \leq k$.

By (3.9) there are rotations $\{T_i\}_{i \leq N_k} \subset G_X$ such that
\[ \frac{1}{N_k} \sum_{i=1}^{N_k} T_i(x_{n_0}) \overset{\|X\|_X}{=} \frac{2}{k}. \]
According to (3.9) we may define mappings $S_i: L^\infty(X) \to L^\infty(X)$ for $n \leq k$ and $i \leq N_k$ by putting
\[ S_i^{(n)}(F)(t) = T_i(F(\hat{g}_n(t))) \quad \text{for a.e. } t \in [0, 1], \quad F \in L^\infty(X). \]
By (3.8) we get that $S_i^{(n)} \in G_{L^\infty(X)}$ (see also [7] p.467-468).

The function $\chi_{[0,1]}x$ can be approximated by convex combinations as follows:
\[ \chi_{[0,1]}x - \frac{1}{N_k} \sum_{n=1}^{k} \sum_{i=1}^{N_k} \phi_i^{(n)}(F) \overset{\|L^\infty(X)\|_X}{=} \frac{1}{k(2 + \sum_{i=1}^{k-1} 2k-1)}. \]
Indeed, for $n \leq k$ and a.e. $t \in [0, 1] \setminus \Delta_n$ it holds by (3.10) that
\[ \frac{1}{N_k} \sum_{i=1}^{N_k} \phi_i^{(n)}(F)(t) \overset{\|X\|_X}{=} \frac{1}{N_k} \sum_{i=1}^{N_k} T_i^{(n)}(x_{n_0}) \overset{\|X\|_X}{=} \frac{2}{k}. \]
On the other hand, $\|x - \frac{1}{N_k} \sum_{i=1}^{N_k} S_i^{(n)}(F)(t)\|_X \overset{\|X\|_X}{=} 2$ for a.e. $t \in \Delta_n$. In (3.11) we apply the fact that $\Delta_n$ are pairwise essentially disjoint.
Denote \( A_2 = \frac{1}{k} \sum_{i=1}^{k} \frac{1}{N_i} \sum_{n=1}^{N_i} S_i^{(n)} \in \text{conv}(\mathcal{G}_{L^\infty(X)}) \). By combining the estimates \((3.13)\) and \((3.11)\) we obtain by Fact 3.6 that
\[ \|G - A_1 A_2(F)\| < \frac{5}{k}. \]

Observe that \( A_1 A_2 \) is an average of \( N_i N_k \) many rotations on \( L^\infty(X) \). We conclude by Fact 5.5 that \( L^\infty(X) \) is uniformly convex-transitive.

The case \( 1 \leq p < \infty \) is a straightforward modification of the proof of [11, Thm. 2.1], where one replaces \( U_i x_i \) by suitable averages belonging to \( \text{conv}(\mathcal{G}_X(x_i)) \) for each \( i \).

In fact it is not difficult to check the following fact: If the rotations of \( L^\infty(X) \) are of the Banach-Stone type, then \( L^\infty(X) \) is convex-transitive if and only if each \( x \in S_X \) is a uniformly big point.

We already mentioned that \( \ell^\infty/c_0 \) is uniformly convex-transitive as a real space. Next we generalize this result to the vector-valued setting.

**Theorem 3.7.** Let \( X \) be a uniformly convex-transitive Banach space over \( \mathbb{K} \). Then \( \ell^\infty(X)/c_0(X) \) (over \( \mathbb{K} \)) is uniformly convex-transitive.

**Proof.** Observe that the formula
\[ (3.12) \quad T((x_n)_n) = (S_n x_{\pi(n)})_n, \]
where \( \pi : N \to N \) is a bijection and \( S_n \in \mathcal{G}_X \), \( n \in N \), defines a rotation on \( \ell^\infty(X) \).

Also note that such an isometry \( T \) restricted to \( c_0(X) \) is a member of \( \mathcal{G}_{c_0(X)} \).

If \( T \in \mathcal{G}_{\ell^\infty(X)} \) is as in \((3.12)\), then \( \hat{T} : x + c_0(X) \to T(x) + c_0(X) \), for \( x \in \ell^\infty(X) \), defines a rotation \( \ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X) \). Indeed, it is clear that \( \hat{T} : \ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X) \) is a linear bijection. Moreover,
\[ \inf_{z \in c_0(X)} ||x - z|| = \inf_{z \in c_0(X)} ||T(x) - T(z)|| = \inf_{z \in c_0(X)} ||T(x) - z||, \]
so that \( \hat{T} : \ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X) \) is an isometry.

Fix \( u, v \in S_{\ell^\infty(X)/c_0(X)} \). If \( x, y \in \ell^\infty(X) \) are such that \( u = x + c_0(X) \) and \( v = y + c_0(X) \), then
\[ (3.13) \quad \text{dist}(x, c_0(X)) = \limsup_{n \to \infty} ||x_n|| = 1 = \text{dist}(y, c_0(X)) = \limsup_{n \to \infty} ||y_n||, \]
since \( u, v \in S_{\ell^\infty(X)/c_0(X)} \). Hence we may pick \( x, y \in S_{\ell^\infty(X)} \) such that \( u = x + c_0(X) \) and \( v = y + c_0(X) \).

Fix \( k \in \mathbb{N} \), \( e \in S_X \) and let \( A = \{n \in \mathbb{N} : ||x_n|| \geq 1 - \frac{1}{2^k}\} \). Observe that \( A \) is an infinite set by \((3.13)\). Since \( X \) is uniformly convex-transitive, there exists \( N_k \in \mathbb{N} \) such that for each \( n \in A \) there are \( T_{l_1}^{(n)}, \ldots, T_{N_k}^{(n)} \in \mathcal{G}_X \) such that
\[ (3.14) \quad \left| e - \frac{1}{N_k} \sum_{l=1}^{N_k} T_l^{(n)} x_n \right| < \frac{1}{k}. \]

Fix \( j_{(k)} \in \mathbb{N} \) such that
\[ (3.15) \quad \frac{1}{j_{(k)}} (2 + (j_{(k)} - 1)(\frac{1}{k})) < \frac{2}{k}. \]

Denote by \( p_1, \ldots, p_{j_{(k)}} \in \mathbb{N} \) the \( j_{(k)} \) first primes. Let \( \phi_1, \ldots, \phi_{j_{(k)}} : \mathbb{N} \to \mathbb{N} \) be permutations such that
\[ (3.16) \quad \phi_i(N \setminus A) \subset \{p_i^m \mid m \in \mathbb{N}\} \quad \text{for } i \in \{1, \ldots, j_{(k)}\}. \]
For \( l \in \{1, \ldots, N(k)\} \) put \( S_{i,n,l} = T_l^{(\phi_i^{-1}(n))} \) if \( \phi_i^{-1}(n) \in A \) and otherwise put \( S_{i,n,l} = I \). Define a convex combination of rotations on \( \ell^\infty(X) \) by letting

\[
A_1(z)_n = \frac{1}{j(k)} \sum_{i=1}^{j(k)} \frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(x_{\phi_i^{-1}(n)}),
\]

where \((z_n)_{n \in \mathbb{N}} \in \ell^\infty(X)\). Consider \( A_1 \in L(\ell^\infty(X)) \) and \( \varpi = (e, e, e, \ldots) \in \ell^\infty(X) \). We obtain that

\[
(3.17) \quad ||\varpi - A_1((x_n))||_{\ell^\infty(X)} < \frac{2}{k}.
\]

Indeed, for each \( n \in \mathbb{N} \) it holds for at least \( j(k) - 1 \) many indices \( i \) that

\[
\frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(x_{\phi_i^{-1}(n)}) = \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_l^{(\phi_i^{-1}(n))}(x_{\phi_i^{-1}(n)}),
\]

where one uses the definition of \( S_{i,n,l} \), (3.16) and the fact that the sets \( \{p_m^m | m \in \mathbb{N}\}, \{p_j^m | m \in \mathbb{N}\} \) are mutually disjoint for \( i \neq j \). Thus (3.14) and (3.15) yield that

\[
\left\| e - \frac{1}{j(k)} \sum_{i=1}^{j(k)} \frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(x_{\phi_i^{-1}(n)}) \right\| < \frac{2}{k}
\]

holds for all \( n \in \mathbb{N} \).

Next we will define another convex combination \( A_2 \) of rotations on \( \ell^\infty(X) \) as follows. By using again the uniform convex transitivity of \( X \) we obtain \( T_{n,l} \in \mathcal{G}_X \), \( 1 \leq l \leq N(k) \), \( n \in \mathbb{N} \), such that

\[
\left\| y_n - \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_{n,l} e \right\| < \frac{1}{k}
\]

holds for \( n \in \mathbb{N} \). Define

\[
A_2(z)_n = \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_{n,l} z_n.
\]

Combining the convex combinations yields

\[
||y - A_2A_1x||_{\ell^\infty(X)} < \frac{3}{k}
\]

according to Fact 3.6. Since the applied rotations induce rotations on \( \ell^\infty(X)/c_0(X) \), we may consider the corresponding convex combinations in \( L(\ell^\infty(X)/c_0(X)) \) and thus

\[
||v - A_2A_1u||_{\ell^\infty(X)/c_0(X)} < \frac{3}{k}.
\]

Tracking the formation of the convex combinations reveals that \( A_2A_1 \) can be written as an average of \( N(k)j(k)N(k) \) many rotations on \( \ell^\infty(X)/c_0(X) \).

Since \( C(\beta N \setminus N, X) \) is linearly isometric to \( \ell^\infty/c_0 \), an application of Theorem 3.2 yields that \( C(\beta N \setminus N, X) \) is uniformly convex-transitive if \( X \) is uniformly convex-transitive. However, let us recall that this space is linearly isometric to \( \ell^\infty(X)/c_0(X) \) if and only if \( X \) is finite-dimensional.

### 4. Roughness and Projections

Let \( X \) be a Banach space. For each \( x \in S_X \) we denote

\[
\eta(X, x) = \limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2}{\|h\|}.
\]

Given \( \varepsilon > 0 \), the space \( X \) is said to be \( \varepsilon \)-rough if \( \inf_{x \in S_X} \eta(X, x) \geq \varepsilon \). In addition, 2-rough spaces are usually called extremely rough.
We will denote the *coprojection constant* of $X$ by

$$\rho(X) = \sup_P \|I - P\|,$$

where the supremum is taken over all linear norm-1 projections $P: X \to Y$.

A Banach space $X$ is called *uniformly non-square* if there exists $a \in (0, 1)$ such that if $x, y \in B_X$ and $\|x - y\| \geq 2a$ then $\|x - y\| < 2a$. These spaces were introduced in [12] by R. C. James, who also proved that this property lies strictly between uniform convexity and reflexivity. Next we will illustrate how the previous concepts are related.

**Theorem 4.1.** Let $X$ be a Banach space. Then the following conditions are equivalent:

1. $X$ contains $\ell^1(2)$ almost isometrically.
2. $X$ is not uniformly non-square.
3. $\rho(X) = 2$.

Moreover, if $\sup_{x \in S_X} \eta(X, x) = 2$, then $\rho(X) = 2$.

We will require some preparations before the proof. Recall that given $x, y \in X$ the function $t \mapsto \|x + t(y + \theta x)\| - \|x\|$ is monotone in $t$ and thus the limit $\lim_{t \to 0^+} \frac{\|x + t(y + \theta x)\| - \|x\|}{t}$ exists and is finite.

**Lemma 4.2.** Let $X$ be a Banach space and $x, y \in X$, $x \neq 0$. Then

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{2t} = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{2t}$$

for $\theta \leq \lim_{t \to 0^+} \frac{\|x - ty\| - \|x + ty\|}{2t|t|x|}$.

**Proof.** Observe that for all maps $a: [0, 1] \to \mathbb{R}$ such that $\lim_{t \to 0^+} a(t) > 0$ it holds that

$$\lim_{t \to 0^+} \frac{|a(t)x + ty| - |a(t)x|}{t} = \lim_{t \to 0^+} \frac{|a(t)x + ty| - |a(t)x|}{t}$$

(4.1)

We will also apply the fact that

$$\lim_{t \to 0^+} \frac{t(\lim_{t \to 0^+} \frac{\|x - ty\| - \|x + ty\|}{2t|t|x|}) - t \frac{\|x - ty\| - \|x + ty\|}{2t|t|x|}}{t} = 0.$$
The claimed one-sided limits are calculated as follows:
\[
\lim_{t \to 0^+} \frac{\|x + t(y + \theta x)\| - \|x\|}{t} = \lim_{t \to 0^+} \frac{\|(1 + \|x-y\|\|x+t y\|)x + t y\| - \|x\|}{t} = \lim_{t \to 0^+} \frac{\|x + t y\| - \|x - t y\|}{2t}.
\]
In the first equality above we applied the fact (4.2), and in the third equality the fact (3.1). The calculations for the equation
\[
\lim_{t \to 0^+} \frac{\|x - t(y + \theta x)\| - \|x\|}{t} = \lim_{t \to 0^+} \frac{\|x + t y\| - \|x - t y\| - 2\|x\|}{2t}
\]
are similar.

**Proof of Theorem 4.1.** The equivalence of conditions (1) and (2) is well-known. The direction (1) \(\Rightarrow\) (3) is established by using the Hahn-Banach Theorem to obtain suitable rank-1 projections \(P\). Towards the implication (3) \(\Rightarrow\) (2), suppose that \(\rho(X) = 2\). Given \(\delta > 0\) there exists a projection \(P: X \to Y\), which satisfies \(|P| = 1\) and \(|1 - P| > 2 - \frac{\delta}{2}\). Choose \(x \in S_X\) such that \(|x - P(x)| > 2 - \frac{\delta}{2}\). This gives that \(|P(x)| \geq 1 - \frac{\delta}{2}\). Put \(y = \frac{P(x)}{|P(x)|}\) and note that \(y \in S_X\) and \(|y - P(x)| < \frac{\delta}{2}\).

Moreover,
\[
\|x - y\| \geq \|x - P(x)\| - \|y - P(x)\| > 2 - \delta > 2(1 - \delta)
\]
and
\[
\|x + y\| \geq \|x + P(x)\| - \|y - P(x)\| > \|x + P(x)\| - \frac{\delta}{2}
\]
\[
= \|2x + P(x) - x\| \|P\| - \frac{\delta}{2}
\]
\[
\geq \|P(2x + P(x) - x)\| - \frac{\delta}{2} = \|P(2x)\| - \frac{\delta}{2} > 2 - \delta - \frac{\delta}{2} > 2(1 - \delta).
\]
Thus \(X\) is not uniformly non-square.

To check the latter part of the claim, an application of Lemma 4.2 yields that if \(\sup_{x \in S_X} \eta(x, X) = 2\), then \(X\) is not uniformly non-square. Hence \(\rho(X) = 2\).

The extreme roughness of \(X\) is a tremendously stronger condition than \(\rho(X) = 2\). For example, if \((F_n)\) is a sequence of finite-dimensional smooth spaces such that \(\rho(F_n) \to 2\) as \(n \to \infty\), then the space
\[
X = \bigoplus_{n \in \mathbb{N}} F_n \quad \text{(summation in } \ell^2\text{-sense)}
\]
is Fréchet-smooth but \(\rho(X) = 2\).
However, for convex-transitive spaces $X$ the condition of being extremely rough is equivalent to the condition $\rho(X) = 2$. Indeed, if a convex-transitive space is not extremely rough then, by \cite[Thm. 6.8]{becerra2002transitivity}, it must be uniformly convex and thus $\rho(X) < 2$. It is unknown to us whether a convex-transitive Banach space is reflexive if it does not contain an isomorphic copy of $\ell^1$.

In the same spirit as in this section, the projection constants of $L^p$ spaces were discussed in \cite{fabian2001functional}.

5. Final Remarks: On the universality of transitivity properties

The well-known Banach-Mazur problem mentioned in the introduction asks whether every transitive, separable Banach space must be linearly isometric to a Hilbert space. It is well-known that all such (transitive+separable) spaces must be smooth; otherwise, not much is known. Even adding some properties like being a dual space or even reflexivity has not sufficed, to date, for proving that the norm is Hilbertian.

Let us make a few remarks on the universality of some spaces of continuous functions. It is well-known that $C(\Delta)$ contains $C([0,1])$ isometrically; hence, the former space is universal for the property of being uniformly convex-transitive and separable. However, it is not almost transitive.

To get a space which is universal for the property of being almost transitive and separable, just consider the almost transitive space $X = C_0^0(L)$ where $L$ is the pseudo-arc with one point removed ([13] or [10]). Since $[0,1]$ is a continuous image of $L$, every separable space is isometrically contained in $X$ (complex case) or $X_\mathbb{R}$ (real case). Finally, note that the almost transitivity of a Banach space implies that of the real underlying space.

Acknowledgements. The first author was financially supported by Junta de Andalucía grant FQM 257 and Project MTM-2006-15546-C02-01. The second author was financially supported by Väisälä Foundation and Emil Aaltonen Foundation. This paper was conceived as the second named author enjoyed the hospitality of University of Cádiz in 2008.

References

[1] A. Aizpuru, F. Rambla, Almost transitivity in $C_0$ spaces of Vector-valued functions, *Proc. Edinburgh Math. Soc.* 48 (2005), 513–520.
[2] S. Banach, *Théorie des Opérations Linéaires*, Warsaw (1932).
[3] J. Becerra, A. Rodriguez, Transitivity of the norm on Banach spaces, *Extracta Math.* 17 (2002), 1–58.
[4] Communications between F. Cabello Sanchez and J. Talponen by email in January 2009.
[5] F. Cabello, Convex transitive norms on spaces of continuous functions, *Bull. London Math. Soc.* 37 (2005), 107–118.
[6] F. Cabello, Regards sur le problème des rotations de Mazur (French), *Extracta Math.* 12 (1997), no. 2, 97–116.
[7] J. Diestel, J. Uhl Jr, *Vector measures*. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
[8] M. Fabian, P. Habala, P. Hajek, V. Montesinos Santalucia, J. Pelant, V. Zizler, *Functional Analysis and Infinite-dimensional Geometry*, CMS Books in Mathematics, Springer-Verlag 2001.
[9] P. Greim, The centralizer of Bochner $L^\infty$-Spaces, *Math. Ann.* 260 (1982), 463–468.
[10] P. Greim, Isometries and $L^p$-structure of separably valued Bochner $L^p$-spaces, in Measure Theory and its Applications, Lecture Notes in Mathematics 1033, Springer-Verlag, (1983), 209–218.
[11] P. Greim, J.E. Jamison, A. Kaminska, Almost transitivity of some functions spaces, *Math. Proc. Cambridge Phil. Soc.* **116** (1994), 475-488.

[12] R. C. James, Uniformly non-square Banach spaces. *Ann. of Math.* **80** (1964), no. 2, 542–550.

[13] K. Kawamura, On a conjecture of Wood, *Glasq. Math. J.*, **47** (2005), 1-5.

[14] H.E. Lacey, *The isometric theory of classical Banach spaces*. Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974.

[15] A. Pelczynski, S. Rolewicz, Best norms with respect to isometry groups in normed linear spaces, Short Communications on International Math. Conference in Stockholm (1962), 104.

[16] F. Rambla, A counterexample to Wood’s conjecture, *J. Math. Anal. Appl.* **317** (2006), 659–667.

[17] S. Rolewicz, *Metric Linear Spaces*, Polish Scientific Publishers, Reidel, 1985.

[18] J. Talponen, Asymptotically transitive Banach spaces, Banach Spaces and their Applications in Analysis, Eds. B.Randrianatoanina, N. Randrianantoanina, de Gruyter Proceedings in Mathematics (2007), 423–438.

[19] J. Talponen, Convex-transitivity in function spaces, *J. Math. Anal. Appl.* (to appear).

[20] G. Wood, Maximal symmetry in Banach spaces, *Proc. Roy. Irish Acad.* **82A** (1982), 177–186.

Fernando Rambla, Universidad de Cádiz, Departamento de Matemáticas, 11510, Puerto Real, Spain
*E-mail address: fernando.rambla@uca.es*

Jarno Talponen, University of Helsinki, Department of Mathematics and Statistics, Box 68, FI-00014 University of Helsinki, Finland
*E-mail address: talponen@cc.helsinki.fi*