Preface

I wrote these lecture notes for a graduate topics course I taught at Cornell University in Fall 2011 (Math 7770). The ostensible primary goal of the course was for the students to learn some of the fundamental results and techniques in the study of probability on infinite-dimensional spaces, particularly Gaussian measures on Banach spaces (also known as abstract Wiener spaces). As others who have taught such courses will understand, a nontrivial secondary goal of the course was for the instructor (i.e., me) to do the same. These notes only scratch the very surface of the subject, but I tried to use them to work through some of the basics and see how they fit together into a bigger picture. In addition to theorems and proofs, I’ve left in some more informal discussions that attempt to develop intuition.

Most of the material here comes from the books [14, 16, 2], and the lecture notes prepared by Bruce Driver for the 2010 Cornell Probability Summer School [4, 5]. If you are looking to learn more, these are great places to look.

Any text marked Question N is something that I found myself wondering while writing this, but didn’t ever resolve. I’m not proposing them as open problems; the answers could be well-known, just not by me. If you know the answer to any of them, I’d be happy to hear about it! There are also still a few places where proofs are rough or have some gaps that I never got around to filling in.

On the other hand, something marked Exercise N is really meant as an exercise.

I would like to take this opportunity to thank the graduate students who attended the course. These notes were much improved by their questions and contributions. I’d also like to thank several colleagues who sat in on the course or otherwise contributed to these notes, particularly Clinton Conley, Bruce Driver, Leonard Gross, Ambar Sengupta, and Benjamin Steinhurst. Obviously, the many deficiencies in these notes are my responsibility and not theirs.

Questions and comments on these notes are most welcome. I am now at the University of Northern Colorado, and you can email me at neldredge@unco.edu.

\footnote{In reading these notes in conjunction with [16], you should identify Nualart’s abstract probability space \((\Omega, F, P)\) with our Banach space \((W, B, \mu)\). His “Gaussian process” \(h \mapsto W(h)\) should be viewed as corresponding to our map \(T\) defined in Section 4.3, his indexing Hilbert space \(H\) may be identified with the Cameron–Martin space, and his \(W(h)\) is the random variable, defined on the Banach space, that we have denoted by \(Th\) or \(\langle h, \cdot \rangle\). There’s a general principle in this area that all the “action” takes place on the Cameron–Martin space, so one doesn’t really lose much by dropping the Banach space structure on the space \(W\) and replacing it with a generic \(\Omega\) (and moreover generality is gained). Nonetheless, I found it helpful in building intuition to work on a concrete space \(W\); this also gives one the opportunity to explore how the topologies of \(W\) and \(H\) interact.}
1 Introduction

1.1 Why analysis and probability on $\mathbb{R}^n$ is nice

Classically, real analysis is usually based on the study of real-valued functions on finite-dimensional Euclidean space $\mathbb{R}^n$, and operations on those functions involving limits, differentiation, and integration. Why is $\mathbb{R}^n$ such a nice space for this theory?

- $\mathbb{R}^n$ is a nice topological space, so limits behave well. Specifically, it is a complete separable metric space, and it’s locally compact.
- $\mathbb{R}^n$ has a nice algebraic structure: it’s a vector space, so translation and scaling make sense. This is where differentiation comes in: the derivative of a function just measures how it changes under infinitesimal translation.
- $\mathbb{R}^n$ has a natural measure space structure; namely, Lebesgue measure $m$ on the Borel $\sigma$-algebra. The most important property of Lebesgue measure is that it is invariant under translation. This leads to nice interactions between differentiation and integration, such as integration by parts, and it gives nice functional-analytic properties to differentiation operators: for instance, the Laplacian $\Delta$ is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^n, m)$.

Of course, a lot of analysis only involves local properties, and so it can be done on spaces that are locally like $\mathbb{R}^n$: e.g. manifolds. Let’s set this idea aside for now.

1.2 Why infinite-dimensional spaces might be less nice

The fundamental idea in this course will be: how can we do analysis when we replace $\mathbb{R}^n$ by an infinite dimensional space? First we should ask: what sort of space should we use? Separable Banach spaces seem to be nice. They have a nice topology (complete separable metric spaces) and are vector spaces. But what’s missing is Lebesgue measure. Specifically:

**Theorem 1.1.** “There is no infinite-dimensional Lebesgue measure.” Let $W$ be an infinite-dimensional separable Banach space. There does not exist a translation-invariant Borel measure on $W$ which assigns positive finite measure to open balls. In fact, any translation-invariant Borel measure $m$ on $W$ is either the zero measure or assigns infinite measure to every open set.

**Proof.** Essentially, the problem is that inside any ball $B(x,r)$, one can find infinitely many disjoint balls $B(x_i,s)$ of some fixed smaller radius $s$. By translation invariance, all the $B(x_i,s)$ have the same measure. If that measure is positive, then $m(B(x,r)) = \infty$. If that measure is zero, then we observe that $W$ can be covered by countably many balls of radius $s$ (by separability) and so $m$ is the zero measure.

The first sentence is essentially Riesz’s lemma: given any proper closed subspace $E$, one can find a point $x$ with $||x|| \leq 1$ and $d(x, E) > 1/2$. (Start by picking any $y \notin E$, so that $d(y, E) > 0$; then by definition there is an $z \in E$ with $d(y, z) < 2d(y, E)$. Now look at $y - z$ and rescale as needed.) Now let’s look at $B(0,2)$ for concreteness. Construct $x_1, x_2, \ldots$ inductively by letting $E_n = \text{span}\{x_1, \ldots, x_n\}$ (which is closed) and choosing $x_{n+1}$ as in Riesz’s lemma with $||x_{n+1}|| \leq 1$ and $d(x_{n+1}, E_n) > 1/2$. In particular, $d(x_{n+1}, x_i) > 1/2$ for $i \leq n$. Since our space is infinite dimensional, the finite-dimensional subspaces $E_n$ are always proper and the induction can continue, producing a sequence $\{x_i\}$ with $d(x_i, x_j) > 1/2$ for $i \neq j$, and thus the balls $B(x_i, 1/4)$ are pairwise disjoint. 


Exercise 1.2. Prove the above theorem for \( W \) an infinite-dimensional Hausdorff topological vector space. (Do we need separability?)

A more intuitive idea why infinite-dimensional Lebesgue measure can’t exist comes from considering the effect of scaling. In \( \mathbb{R}^n \), the measure of \( B(0, 2) \) is \( 2^n \) times larger than \( B(0, 1) \). When \( n = \infty \) this suggests that one cannot get sensible numbers for the measures of balls.

There are nontrivial translation-invariant Borel measures on infinite-dimensional spaces: for instance, counting measure. But these measures are useless for analysis since they cannot say anything helpful about open sets.

So we are going to have to give up on translation invariance, at least for now. Later, as it turns out, we will study some measures that recover a little bit of this: they are quasi-invariant under some translations. This will be explained in due course.

1.3 Probability measures in infinite dimensions

If we just wanted to think about Borel measures on infinite-dimensional topological vector spaces, we actually have lots of examples from probability, that we deal with every day.

Example 1.3. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(X_1, X_2, \ldots\) a sequence of random variables. Consider the infinite product space \( \mathbb{R}^\infty \), thought of as the space of all sequences \( \{x(i)\}_{i=1}^\infty \) of real numbers. This is a topological vector space when equipped with its product topology. We can equip \( \mathbb{R}^\infty \) with its Borel \( \sigma \)-algebra, which is the same as the product Borel \( \sigma \)-algebra (verify). Then the map from \( \Omega \) to \( \mathbb{R}^\infty \) which sends \( \omega \) to the sequence \( x(i) = X_i(\omega) \) is measurable. The pushforward of \( \mathbb{P} \) under this map gives a Borel probability measure \( \mu \) on \( \mathbb{R}^\infty \).

The Kolmogorov extension theorem guarantees lots of choices for the joint distribution of the \( X_i \), and hence lots of probability measures \( \mu \). Perhaps the simplest interesting case is when the \( X_i \) are iid with distribution \( \nu \), in which case \( \mu \) is the infinite product measure \( \mu = \prod_{i=1}^\infty \nu \). Note that in general one can only take the infinite product of probability measures (essentially because the only number \( a \) with \( 0 < \prod_{i=1}^\infty a < \infty \) is \( a = 1 \)).

Example 1.4. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and \( \{X_t : 0 \leq t \leq 1\} \) be any stochastic process. We could play the same game as before, getting a probability measure on \( \mathbb{R}^{[0,1]} \) (with its product \( \sigma \)-algebra). This case is not as pleasant because nothing is countable. In particular, the Borel \( \sigma \)-algebra generated by the product topology is not the same as the product \( \sigma \)-algebra (exercise: verify this, perhaps by showing that the latter does not contain singleton sets.) Also, the product topology on \( \mathbb{R}^{[0,1]} \) is rather nasty; for example it is not first countable. (In contrast, \( \mathbb{R}^\infty \) with its product topology is actually a Polish space.) So we will avoid examples like this one.

Example 1.5. As before, but now assume \( \{X_t : 0 \leq t \leq 1\} \) is a continuous stochastic process. We can then map \( \Omega \) into the Banach space \( C([0,1]) \) in the natural way, by sending \( \omega \) to the continuous function \( X.(\omega) \). One can check that this map is measurable when \( C([0,1]) \) is equipped with its Borel \( \sigma \)-algebra. (Hint: \( ||x|| \leq 1 \) if and only if \( |x(t)| \leq 1 \) for all \( t \) in a countable dense subset of \( [0,1] \).) So by pushing forward \( \mathbb{P} \) we get a Borel probability measure on \( C([0,1]) \). For example, if \( X_t \) is Brownian motion, this is the classical Wiener measure.

So probability measures seem more promising. We are going to concentrate on Gaussian probability measures. Let’s start by looking at them in finite dimensions.
1.4 Gaussian measures in finite dimensions

In one dimension everyone knows what Gaussian means. We are going to require our measures / random variables to be centered (mean zero) to have fewer letters floating around. However we are going to include the degenerate case of zero variance.

**Definition 1.6.** A Borel probability measure $\mu$ on $\mathbb{R}$ is **Gaussian** with variance $\sigma^2$ iff

$$\mu(B) = \int_B \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2} \, dx$$

for all Borel sets $B \subset \mathbb{R}$. We also want to allow the case $\sigma = 0$, which corresponds to $\mu = \delta_0$ being a Dirac mass at 0.

We could also specify $\mu$ in terms of its Fourier transform (or characteristic function). The above condition is equivalent to having

$$\int_{\mathbb{R}} e^{i\lambda x} \mu(dx) = e^{-\sigma^2\lambda^2/2}$$

for all $\lambda \in \mathbb{R}$. (Note $\sigma = 0$ is naturally included in this formulation.)

A random variable $X$ on some probability space $(\Omega, \mathcal{F}, P)$ is Gaussian with variance $\sigma^2$ if its distribution measure is Gaussian, i.e.

$$P(X \in B) = \int_B \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2} \, dx.$$

for all Borel sets $B$. For $\sigma = 0$ we have the constant random variable $X = 0$. Equivalently

$$E[e^{i\lambda X}] = e^{-\sigma^2\lambda^2/2}$$

for all $\lambda \in \mathbb{R}$.

Let’s make a trivial observation: $\mu$ is not translation invariant. However, translation doesn’t mess it up completely.

**Notation 1.7.** Let $\mu$ be a measure on a vector space $W$. For $y \in W$, we denote by $\mu_y$ the translated measure defined by $\mu_y(A) = \mu(A - y)$. In other words, $\int_W f(x) \mu_y(dx) = \int_W f(x + y)$.

**Exercise 1.8.** Check that I didn’t screw up the signs in the previous paragraph.

**Definition 1.9.** A measure $\mu$ on a vector space $W$ is said to be **quasi-invariant** under translation by $y \in W$ if the measures $\mu, \mu_y$ are mutually absolutely continuous (or equivalent); that is, if $\mu(A) = 0 \iff \mu_y(A) = 0$ for measurable sets $A \subset W$.

Intuitively, quasi-invariance means that translation can change the measure of a set, but it doesn’t change whether or not the measure is zero.

One way I like to think about equivalent measures is with the following baby example. Suppose I have two dice which look identical on the surface, but one of them is fair, and the other produces numbers according to the distribution $(0.1, 0.1, 0.1, 0.1, 0.1, 0.5)$ (i.e. it comes up 6 half the time). (Note that they induce equivalent measures on $\{1, 2, 3, 4, 5, 6\}$: in both cases the only set of measure zero is the empty set.) I pick one of the dice and ask you to determine which one it is. If you roll a lot of 6s, you will have a strong suspicion that it’s the unfair die, but you can’t absolutely rule out the possibility that it’s the fair die and you are just unlucky.
On the other hand, suppose one of my dice always comes up even, and the other always comes up odd. In this case the induced measures are mutually singular: there is a set (namely \{1, 3, 5\}) to which one gives measure 0 and the other gives measure 1. If I give you one of these dice, then all you have to do is roll it once and see whether the number is even or odd, and you can be (almost) sure which die you have.

For Gaussian measures on \(\mathbb{R}\), note that if \(\sigma \neq 0\), then \(\mu\) is quasi-invariant under translation by any \(y \in \mathbb{R}\). This is a trivial fact: both \(\mu\) and \(\mu_y\) have positive densities with respect to Lebesgue measure \(m\), so \(\mu(A) = 0\) iff \(\mu_y(A) = 0\) iff \(m(A) = 0\). We'll also note that, as absolutely continuous measures, they have a Radon-Nikodym derivative, which we can compute just by dividing the densities:

\[
\frac{d\mu_y}{d\mu}(x) = \frac{\frac{1}{\sqrt{2\pi\sigma}} e^{-(x-y)^2/2\sigma}}{\frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2} + \frac{xy}{\sigma^2}}.
\]

Just pay attention to the form of this expression, as we will see it again later.

On the other hand, in the degenerate case \(\sigma = 0\), then \(\mu = \delta_0\) and \(\mu_y = \delta_y\) are mutually singular.

Now let's look at the \(n\)-dimensional case.

**Definition 1.10.** An \(n\)-dimensional random vector \(X = (X_1, \ldots, X_n)\) is **Gaussian** if and only if \(\lambda \cdot X := \sum \lambda_i X_i\) is a Gaussian random variable for all \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\).

Or in terms of measures:

**Definition 1.11.** Let \(\mu\) be a Borel probability measure on \(\mathbb{R}^n\). For each \(\lambda \in \mathbb{R}^n\), we can think of the map \(\mathbb{R}^n \ni x \mapsto \lambda \cdot x \in \mathbb{R}\) as a random variable on the probability space \((\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \mu)\). \(\mu\) is **Gaussian** if and only if this random variable is Gaussian for each \(\lambda\).

Of course, we know that the distribution of \(X\) is uniquely determined by its \(n \times n\) covariance matrix \(\Sigma\), where \(\Sigma_{ij} = \text{Cov}(X_i, X_j)\). Note that \(\Sigma\) is clearly symmetric and positive semidefinite. Furthermore, any symmetric, positive semidefinite matrix \(\Sigma\) can arise as a covariance matrix: let \(Z = (Z_1, \ldots, Z_n)\) where \(Z_i\) are iid Gaussian with variance 1, and set \(X = \Sigma^{1/2}Z\).

A consequence of this is that if \((X_1, \ldots, X_n)\) has a joint Gaussian distribution, then the \(X_i\) are independent if and only if they are uncorrelated (i.e. \(\text{Cov}(X_i, X_j) = 0\) for \(i \neq j\), i.e. \(\Sigma\) is diagonal). Note that this fails if all we know is that each \(X_i\) has a Gaussian distribution.

**Proposition 1.12.** \(X\) is Gaussian if and only if it has characteristic function

\[
\mathbb{E}[e^{i\lambda \cdot X}] = e^{-\frac{1}{2} \lambda \cdot \Sigma \lambda}
\]

where \(\Sigma\) is the covariance matrix of \(X\).

Or, in terms of measures:

**Proposition 1.13.** A probability measure \(\mu\) on \(\mathbb{R}^n\) is Gaussian if and only if

\[
\int_{\mathbb{R}^n} e^{i\lambda \cdot x} \mu(dx) = e^{-\frac{1}{2} \lambda \cdot \Sigma \lambda}
\]

for some \(n \times n\) matrix \(\Sigma\), which is necessarily positive semidefinite and can be chosen symmetric.
We would like to work more abstractly and basis-free, in preparation for the move to infinite dimensions. The map \( x \mapsto \lambda \cdot x \) is really just a linear functional on \( \mathbb{R}^n \). So let’s write:

**Definition 1.14.** Let \( \mu \) be a Borel probability measure on a finite-dimensional topological vector space \( W \). Then each \( f \in W^* \) can be seen as a random variable on the probability space \( (W, \mathcal{B}_W, \mu) \). \( \mu \) is **Gaussian** if and only if, for each \( f \in W^* \), this random variable is Gaussian.

Equivalently, \( \mu \) is Gaussian iff the pushforward \( \mu \circ f^{-1} \) is a Gaussian measure on \( \mathbb{R} \) for each \( f \in W^* \).

Of course we have not done anything here because a finite-dimensional topological vector space \( W \) is just some \( \mathbb{R}^n \) with its usual topology. Every linear functional \( f \in W^* \) is of the form \( f(x) = \lambda \cdot x \), and all such linear functionals are continuous, hence measurable.

If \( f, g \in W^* \) are thought of as Gaussian random variables on \( (W, \mu) \), then \( q(f, g) = \text{Cov}(f, g) \) is a symmetric, positive semidefinite, bilinear form on \( W^* \). We’ll also write \( q(f) = q(f, f) = \text{Var}(f) \).

Of course we have not done anything here because a finite-dimensional topological vector space \( W \) is just some \( \mathbb{R}^n \) with its usual topology. Every linear functional \( f \in W^* \) is of the form \( f(x) = \lambda \cdot x \), and all such linear functionals are continuous, hence measurable.

Another way to think about this is that since each \( f \in W^* \) is Gaussian, it is certainly square-integrable, i.e. \( \int_W |f(x)|^2 \mu(dx) = E|f|^2 = \text{Var}(f) < \infty \). So \( V^* \) can be thought of as a subspace of \( L^2(V, \mu) \). Then \( q \) is nothing but the restriction of the \( L^2 \) inner product to the subspace \( W^* \).

(technically, \( q \) may be degenerate, in which case it is actually the quotient of \( W^* \) by the kernel of \( q \) that we identify as a subspace of \( L^2(W, \mu) \).

**Exercise 1.15.** The support of the measure \( \mu \) is given by

\[
\text{supp} \mu = \bigcap_{q(f,f)=0} \ker f.
\]

One could write \( \text{supp} \mu = (\ker q)^\perp \). In particular, if \( q \) is positive definite, then \( \mu \) has full support. (Recall that the support of a measure \( \mu \) is defined as the smallest closed set with full measure.)

**Exercise 1.16.** The restriction of \( \mu \) to its support is a nondegenerate Gaussian measure (i.e. the covariance form is positive definite).

**Exercise 1.17.** \( \mu \) is quasi-invariant under translation by \( y \) if and only if \( y \in \text{supp} \mu \). If \( y \notin \text{supp} \mu \), then \( \mu \) and \( \mu_y \) are mutually singular. (We’ll see that in infinite dimensions, the situation is more complex.)

In terms of characteristic functions, then, we have

**Proposition 1.18.** A Borel probability measure \( \mu \) on a finite-dimensional topological vector space \( W \) is Gaussian if and only if, for each \( f \in W^* \), we have

\[
\int_W e^{if(x)} \mu(dx) = e^{-\frac{1}{2}q(f,f)}
\]

where \( q \) is some positive semidefinite symmetric bilinear form on \( W^* \).
2 Infinite-dimensional Gaussian measures

Definition 1.14 will generalize pretty immediately to infinite-dimensional topological vector spaces. There is just one problem. An arbitrary linear functional on a topological vector space can be nasty; in particular, it need not be Borel measurable, in which case it doesn’t represent a random variable. But continuous linear functionals are much nicer, and are Borel measurable for sure, so we’ll restrict our attention to them.

As usual, $W^*$ will denote the continuous dual of $W$.

**Definition 2.1.** Let $W$ be a topological vector space, and $\mu$ a Borel probability measure on $W$. $\mu$ is **Gaussian** iff, for each continuous linear functional $f \in W^*$, the pushforward $\mu \circ f^{-1}$ is a Gaussian measure on $\mathbb{R}$, i.e. $f$ is a Gaussian random variable on $(W, B_W, \mu)$.

As before, we get a covariance form $q$ on $W^*$ where $q(f, g) = \text{Cov}(f, g)$. Again, $W^*$ can be identified as a subspace of $L^2(W, \mu)$, and $q$ is the restriction of the $L^2$ inner product.

**Proposition 2.2.** A Borel probability measure $\mu$ on a topological vector space $W$ is Gaussian if and only if, for each $f \in W^*$, we have

$$\int_W e^{if(x)} \mu(dx) = e^{-\frac{1}{2} q(f, f)}$$

where $q$ is some positive semidefinite symmetric bilinear form on $W^*$.

**Exercise 2.3.** If $f_1, \ldots, f_n$ in $W^*$, then $(f_1, \ldots, f_n)$ has a joint Gaussian distribution.

**Exercise 2.4.** Any $q$-orthogonal subset of $W^*$ is an independent set of random variables on $(W, \mu)$.

For the most part, we will concentrate on the case that $W$ is a separable Banach space. But as motivation, we first want to look at a single case where it isn’t. If you want more detail on the general theory for topological vector spaces, see Bogachev [2].

3 Motivating example: $\mathbb{R}^\infty$ with product Gaussian measure

As in Example 1.3, let’s take $W = \mathbb{R}^\infty$ with its product topology. Let’s record some basic facts about this topological vector space.

**Exercise 3.1.** $W$ is a Fréchet space (its topology is generated by a countable family of seminorms).

**Exercise 3.2.** $W$ is a Polish space. (So we are justified in doing all our topological arguments with sequences.)

**Exercise 3.3.** The topology of $W$ does not come from any norm.

**Exercise 3.4.** The Borel $\sigma$-algebra of $W$ is the same as the product $\sigma$-algebra.

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2In the case of separable Banach spaces, or more generally Polish topological vector spaces, this sufficient condition is also necessary: a linear functional is Borel measurable if and only if it is continuous. Even the weaker assumption of so-called Baire measurability is sufficient, in fact. See 9.C of [13]. So we are not giving up anything by requiring continuity. Thanks to Clinton Conley for explaining this to me and providing a reference. This sort of goes to show that a linear functional on a separable Banach space is either continuous or really really nasty.
Exercise 3.5. Every continuous linear functional $f \in W^*$ is of the form
\[ f(x) = \sum_{i=1}^{n} a_i x(i) \]
for some $a_1, \ldots, a_n \in \mathbb{R}$. Thus $W^*$ can be identified with $c_{00}$, the set of all real sequences which are eventually zero.

Let’s write $e_i$ for the element of $W$ with $e_i(j) = \delta_{ij}$, and $\pi_i$ for the projection onto the $i$’th coordinate $\pi_i(x) = x(i)$. (Note $\pi_i \in W^*$; indeed they form a basis.)

As in Example 1.3, we choose $\mu$ to be an infinite product of Gaussian measures with variance 1. Equivalently, $\mu$ is the distribution of an iid sequence of standard Gaussian random variables. So the random variables $\pi_i$ are iid standard Gaussian.

Exercise 3.6. $\mu$ is a Gaussian measure. The covariance form of $\mu$ is given by
\[ q(f, g) = \sum_{i=1}^{\infty} f(e_i)g(e_i). \]
(Note that the sum is actually finite.)

Exercise 3.7. $\mu$ has full support.

$q$ is actually positive definite: the only $f \in W^*$ with $q(f, f) = 0$ is $f = 0$. So $W^*$ is an honest subspace of $L^2(W, \mu)$. It is not a closed subspace, though; that is, $W^*$ is not complete in the $q$ inner product. Let $K$ denote the $L^2(W, \mu)$-closure of $W^*$.

Exercise 3.8. Show that $K$ consists of all functions $f : W \to \mathbb{R}$ of the form
\[ f(x) = \sum_{i=1}^{\infty} a_i x(i) \quad (3.1) \]
where $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. This formula requires some explanation. For an arbitrary $x \in W$, the sum in (3.1) may not converge. However, show that it does converge for $\mu$-a.e. $x \in W$. (Hint: Sums of independent random variables converge a.s. as soon as they converge in $L^2$; see Theorem 2.5.3 of Durrett [7].) Note well that the measure-1 set on which (3.1) converges depends on $f$, and there will not be a single measure-1 set where convergence holds for every $f$. Moreover, show each $f \in K$ is a Gaussian random variable.

($K$ is isomorphic to $\ell^2$; this should make sense, since it is the completion of $W^* = c_{00}$ in the $q$ inner product, which is really the $\ell^2$ inner product.)

Now let’s think about how $\mu$ behaves under translation. A first guess, by analogy with the case of product Gaussian measure on $\mathbb{R}^n$, is that it is quasi-invariant under all translations. But let’s look closer at the finite-dimensional case. If $\nu$ is standard Gaussian measure on $\mathbb{R}^n$, i.e.
\[ d\nu = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx \]
then a simple calculation shows
\[ \frac{d\nu_y}{d\nu}(x) = e^{-\frac{1}{2}|y|^2 + x \cdot y}. \quad (3.2) \]
Note that the Euclidean norm of $y$ appears. Sending $n \to \infty$, the Euclidean norm becomes the $\ell^2$ norm. This suggests that $\ell^2$ should play a special role. In particular, translation by $h$ is not going to produce a reasonable positive Radon-Nikodym derivative if $\|y\|_{\ell^2} = \infty$.

Let’s denote $\ell^2$, considered as a subset of $W$, by $H$. $H$ has a Hilbert space structure coming from the $\ell^2$ norm, which we’ll denote by $\| \cdot \|_H$. We note that, as shown in Exercise 3.8, that for fixed $h \in H$, $(h, x)_H$ makes sense not only for $x \in H$ but for $\mu$-a.e. $x \in W$, and $(h, \cdot)_H$ is a Gaussian random variable on $(W, \mu)$ with variance $\|h\|_H^2$.

**Theorem 3.9** (Special case of the Cameron-Martin theorem). *If $h \in H$, then $\mu$ is quasi-invariant under translation by $h$, and

$$
\frac{d\mu_h}{d\mu}(x) = e^{-\frac{1}{2}\|h\|_H^2 + (h, x)_H}. \tag{3.3}
$$

Conversely, if $y \not\in H$, then $\mu, \mu_y$ are mutually singular.*

**Proof.** We are trying to show that

$$
\mu_h(B) = \int_B e^{-\frac{1}{2}\|h\|_H^2 + (h, x)_H} \mu(dx) \tag{3.4}
$$

for all Borel sets $B \subset W$. It is sufficient to consider the case where $B$ is a “cylinder set” of the form $B = B_1 \times \cdots \times B_n \times \mathbb{R} \times \ldots$, since the collection of all cylinder sets is a $\pi$-system which generates the Borel $\sigma$-algebra. But this effectively takes us back to the $n$-dimensional setting, and unwinding notation will show that in this case (3.3) is the same as (3.2).

This is a bit messy to write out; here is an attempt. If you don’t like it I encourage you to try to just work it out yourself.

Since $W$ is a product space, let us decompose it as $W = \mathbb{R}^n \times \mathbb{R}^\infty$, writing $x \in W$ as $(x_n, x_\infty)$ with $x_n = (x(1), \ldots, x(n))$ and $x_\infty = (x(n+1), \ldots)$. Then $\mu$ factors as $\mu^n \times \mu^\infty$, where $\mu^n$ is standard Gaussian measure on $\mathbb{R}^n$ and $\mu^\infty$ is again product Gaussian measure on $\mathbb{R}^\infty$. $\mu_h$ factors as $\mu^n_{h_n} \times \mu^\infty_{h_\infty}$. Also, the integrand in (3.4) factors as

$$
e^{-\frac{1}{2}|h_n|^2 + h_n \cdot x_n} e^{-\frac{1}{2}|h_\infty|^2_\infty + (h_\infty, x_\infty)_{\ell^2}}.
$$

So by Tonelli’s theorem the right side of (3.4) equals

$$
\int_{B_1 \times \cdots \times B_n} e^{-\frac{1}{2}|h_n|^2 + h_n \cdot x_n} \mu^n(dx_n) \int_{\mathbb{R}^\infty} e^{-\frac{1}{2}|h_\infty|^2_\infty + (h_\infty, x_\infty)_{\ell^2}} \mu^\infty(dx_\infty).
$$

The first factor is equal to $\mu^n_{h_n}(B_1 \times \cdots \times B_n)$ as shown in (3.2). Since $(h_\infty, \cdot)_{\ell^2}$ is a Gaussian random variable on $(\mathbb{R}^\infty, \mu^\infty)$ with variance $\|h_\infty\|_{\ell^2}$, the second factor is of the form $E[e^{X - \sigma^2/2}]$ for $X \sim N(0, \sigma^2)$, which is easily computed to be 1. Since $\mu^\infty_{h_\infty}(\mathbb{R}^\infty) = 1$ also (it is a probability measure), we are done with the forward direction.

For the converse direction, suppose $h \not\in H$. Then, by the contrapositive of Lemma A.1, there exists $g \in \ell^2$ such that $\sum h(i)g(i)$ diverges. Let $A = \{x \in W : \sum x(i)g(i) \text{ converges}\}$; this set is clearly Borel, and we know $\mu(A) = 1$ by Exercise 3.8. But if $\sum x(i)g(i)$ converges, then $\sum(x - h)(i)g(i)$ diverges, so $A - h$ is disjoint from $A$ and $\mu_h(A) = \mu(A - h) = 0$.

We call $H$ the **Cameron–Martin space** associated to $(W, \mu)$. 9
Exercise 3.10. \( H \) is dense in \( W \), and the inclusion map \( H \hookrightarrow W \) is continuous (with respect to the \( \ell^2 \) topology on \( H \) and the product topology on \( W \)).

Although \( H \) is dense, there are several senses in which it is small.

Proposition 3.11. \( \mu(H) = 0 \).

Proof. For \( x \in W, x \in H \) iff \( \sum_i |\pi_i(x)|^2 < \infty \). Note that the \( \pi_i \) are iid \( N(0,1) \) random variables on \((W,\mu)\). So by the strong law of large numbers, for \( \mu \)-a.e. \( x \in W \) we have \( \frac{1}{n} \sum_{i=1}^n |\pi_i(x)|^2 \to 1 \); in particular \( \sum_i |\pi_i(x)|^2 = \infty \). \( \square \)

Exercise 3.12. Any bounded subset of \( H \) is precompact and nowhere dense in \( W \). In particular, \( H \) is meager in \( W \).

So \( \mu \) is quasi-invariant only under translation by elements of the small subset \( H \).

4 Abstract Wiener space

Much of this section comes from Bruce Driver’s notes \cite{driver} and from Kuo’s book \cite{kuo}.

Definition 4.1. An abstract Wiener space is a pair \((W,\mu)\) consisting of a separable Banach space \( W \) and a Gaussian measure \( \mu \) on \( W \).

Later we will write an abstract Wiener space as \((W,H,\mu)\) where \( H \) is the Cameron–Martin space. Technically this is redundant because \( H \) will be completely determined by \( W \) and \( \mu \). Len Gross’s original development \cite{gross1,gross2} went the other way, starting with \( H \) and choosing a \((W,\mu)\) to match it, and this choice is not unique. We’ll discuss this more later.

Definition 4.2. \((W,\mu)\) is non-degenerate if the covariance form \( q \) on \( W^* \) is positive definite.

Exercise 4.3. If \( \mu \) has full support (i.e. \( \mu(U) > 0 \) for every nonempty open \( U \)) then \((W,\mu)\) is non-degenerate. (For the converse, see Exercise 4.25 below.)

From now on, we will assume \((W,\mu)\) is non-degenerate unless otherwise specified. (This assumption is really harmless, as will be justified in Remark 4.26 below.) So \( W^* \) is honestly (injectively) embedded into \( L^2(\mu) \), and \( q \) is the restriction to \( W^* \) of the \( L^2(\mu) \) inner product. As before, we let \( K \) denote the closure of \( W^* \) in \( L^2(\mu) \).

Note that we now have two different topologies on \( W^* \): the operator norm topology (under which it is complete), and the topology induced by the \( q \) or \( L^2 \) inner product (under which, as we shall see, it is not complete). The interplay between them will be a big part of what we do here.

4.1 Measure-theoretic technicalities

The main point of this subsection is that the continuous linear functionals \( f \in W^* \), and other functions you can easily build from them, are the only functions on \( W \) that you really have to care about.

Let \( \mathcal{B} \) denote the Borel \( \sigma \)-algebra on \( W \).

Lemma 4.4. Let \( \sigma(W^*) \) be the \( \sigma \)-field on \( W \) generated by \( W^* \), i.e. the smallest \( \sigma \)-field that makes every \( f \in W^* \) measurable. Then \( \sigma(W^*) = \mathcal{B} \).
Note that the topology generated by $W^*$ is not the same as the original topology on $W$; instead it’s the weak topology.

**Proof.** Since each $f \in B^*$ is Borel measurable, $\sigma(W^*) \subset \mathcal{B}$ is automatic.

Let $B$ be the closed unit ball of $W$; we will show $B \in \sigma(W^*)$. Let $\{x_n\}$ be a countable dense subset of $W$. By the Hahn-Banach theorem, for each $x_n$ there exists $f_n \in W^*$ with $||f_n||_{W^*} = 1$ and $f_n(x_n) = ||x_n||$. I claim that

$$B = \bigcap_{n=1}^{\infty} \{x : ||f_n(x)|| \leq 1\}. \quad (4.1)$$

The $\subset$ direction is clear because for $x \in B$, $||f_n(x)|| \leq ||f_n|| - ||x|| = ||x|| \leq 1$. For the reverse direction, suppose $||f_n(x)|| \leq 1$ for all $n$. Choose a sequence $x_{n_k} \to x$; in particular $f_{n_k}(x_{n_k}) = ||x_{n_k}|| \to ||x||$. But since $||f_{n_k}|| = 1$, we have $||f_{n_k}(x_{n_k}) - f_{n_k}(x)|| \leq ||x_{n_k} - x|| \to 0$, so $||x|| = \lim f_{n_k}(x_{n_k}) = \lim f_{n_k}(x) \leq 1$. We have thus shown $B \in \sigma(W^*)$, since the right side of (4.1) is a countable intersection of sets from $W^*$.

If you want to show $B(y, r) \in \sigma(W^*)$, we have

$$B(y, r) = \bigcap_{n=1}^{\infty} \{x : ||f_n(x) - f_n(y)|| < r\}.$$ 

Now note that any open subset $U$ of $W$ is a countable union of closed balls (by separability) so $U \in \sigma(W^*)$ also. Thus $\mathcal{B} \subset \sigma(W^*)$ and we are done. \hfill $\square$

Note that we used the separability of $W$, but we did not assume that $W^*$ is separable.

**Exercise 4.5.** If $W$ is not separable, Lemma 4.4 may be false. For a counterexample, consider $W = \ell^2(E)$ for some uncountable set $E$. One can show that $\sigma(W^*)$ consists only of sets that depend on countably many coordinates. More precisely, for $A \subset E$ let $\pi_A : \ell^2(E) \to \ell^2(A)$ be the restriction map. Show that $\sigma(W^*)$ is exactly the set of all $\pi_A^{-1}(B)$, where $A$ is countable and $B \subset \ell^2(A)$ is Borel. In particular, $\sigma(W^*)$ doesn’t contain any singletons (in fact, every nonempty subset of $\sigma(W^*)$ is non-separable).

**Question 1.** Is Lemma 4.4 always false for non-separable $W$?

Functionals $f \in W^*$ are good for approximation in several senses. We are just going to quote the following results. The proofs can be found in [2], and are mostly along the same lines that you prove approximation theorems in basic measure theory. For this subsection, assume $\mu$ is a Borel probability measure on $W$ (not necessarily Gaussian).

**Notation 4.6.** Let $\mathcal{FC}_c^\infty(W)$ denote the “smooth cylinder functions” on $W$: those functions $F : W \to \mathbb{R}$ of the form $F(x) = \varphi(f_1(x), \ldots, f_n(x))$ for some $f_1, \ldots, f_n \in W^*$ and $\varphi \in C_c^\infty(\mathbb{R})$. (Note despite the notation that $F$ is not compactly supported; in fact there are no nontrivial continuous functions on $W$ with compact support.)

**Notation 4.7.** Let $\mathcal{T}$ be the “trigonometric polynomials” on $W$: those functions $F : W \to \mathbb{R}$ of the form $F(x) = a_1 e^{if_1(x)} + \cdots + a_n e^{if_n(x)}$ for $a_1, \ldots, a_n \in \mathbb{R}$, $f_1, \ldots, f_n \in W^*$.

**Theorem 4.8.** $\mathcal{FC}_c^\infty$ and $\mathcal{T}$ are each dense in $L^p(W, \mu)$ for any $1 \leq p < \infty.$
A nice way to prove this is via Dynkin’s multiplicative system theorem (a functional version of the \( \pi \)-\( \lambda \) theorem).

**Theorem 4.9** (Uniqueness of the Fourier transform). Let \( \mu, \nu \) be two Borel probability measures on \( W \). If \( \int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx) \) for all \( f \in W^* \), then \( \mu = \nu \).

We could think of the Fourier transform of \( \mu \) as the map \( \hat{\mu} : W^* \to \mathbb{R} \) defined by \( \hat{\mu}(f) = \int e^{if(x)} \mu(dx) \). The previous theorem says that \( \hat{\mu} \) completely determines \( \mu \).

### 4.2 Fernique’s theorem

The first result we want to prove is Fernique’s theorem [8], which in some sense says that a Gaussian measure has Gaussian tails: the probability of a randomly sampled point being at least a distance \( t \) from the origin decays like \( e^{-t^2} \). In one dimension this is easy to prove: if \( \mu \) is a Gaussian measure on \( \mathbb{R} \) with, say, variance 1, we have

\[
\mu(\{|x| > t\}) = 2 \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx
\]

\[\leq \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2} \, dx\]

\[= \frac{2}{t\sqrt{2\pi}} e^{-t^2/2}\]  \hspace{1cm} (4.2)

where the second line uses the fact that \( \frac{x}{t} \geq 1 \) for \( x \geq t \), and the third line computes the integral directly.

This is sort of like a heat kernel estimate.

**Theorem 4.10** (Fernique [8]). Let \((W, \mu)\) be an abstract Wiener space. There exist \( \epsilon > 0, C > 0 \) such that for all \( t \),

\[
\mu(\{|x|_W \geq t\}) \leq Ce^{-\epsilon t^2}.
\]

The proof is surprisingly elementary and quite ingenious.

Let’s prove Fernique’s theorem. We follow Driver’s proof [5, Section 43.1]. Some of the details will be sketched; refer to Driver to see them filled in.

The key idea is that products of Gaussian measures are “rotation-invariant.”

**Lemma 4.11.** Let \((W, \mu)\) be an abstract Wiener space. Then the product measure \( \mu^2 = \mu \times \mu \) is a Gaussian measure on \( W^2 \).

If you’re worried about technicalities, you can check the following: \( W^2 \) is a Banach space under the norm \( \|(x, y)\|_{W^2} := \|x\|_W + \|y\|_W \); the norm topology is the same as the product topology; the Borel \( \sigma \)-field on \( W^2 \) is the same as the product of the Borel \( \sigma \)-fields on \( W \).

**Proof.** Let \( F \in (W^2)^* \). If we set \( f(x) = F(x, 0), \ g(y) = F(0, y) \), we see that \( f, g \in W^* \) and \( F(x, y) = f(x) + g(y) \). Now when we compute the Fourier transform of \( F \), we find

\[
\int_{W^2} e^{i\lambda F(x,y)} \mu^2(dx, dy) = \int_{W} e^{i\lambda f(x)} \mu(dx) \int_{W} e^{i\lambda g(y)} \mu(dy)
\]

\[= e^{-\frac{1}{2} \lambda^2 (q(f,f) + q(g,g))}.
\]

\[\square\]
Proposition 4.12. For $\theta \in \mathbb{R}$, define the “rotation” $R_\theta$ on $W^2$ by

$$R_\theta(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta).$$

(We are actually only going to use $R_{\pi/4}(x, y) = \frac{1}{\sqrt{2}}(x - y, x + y)$. ) If $\mu$ is Gaussian, $\mu^2$ is invariant under $R_\theta$.

Invariance of $\mu^2$ under $R_{\pi/4}$ is the only hypothesis we need in order to prove Fernique’s theorem. You might think this is a very weak hypothesis, and hence Fernique’s theorem should apply to many other classes of measures. However, it can actually be shown that any measure $\mu$ satisfying this condition must in fact be Gaussian, so no generality is really gained.

Proof. Let $\nu = \mu^2 \circ R_\theta^{-1}$; we must show that $\nu = \mu^2$. It is enough to compare their Fourier transforms. Let $F \in (W^2)^*$, so $W(x, y) = f(x) + g(y)$, and then

$$\int_{W^2} e^{i(f(x) + g(y))} \nu(dx, dy) = \int_{W^2} e^{i(f(Tx) + g(Ty))} \mu^2(dx, dy)$$

$$= \int_{W^2} e^{i(\cos \theta f(x) - \sin \theta g(y) + \sin \theta f(x) + \cos \theta g(y))} \mu^2(dx, dy)$$

$$= \int_{W} e^{i(\cos \theta f(x) + \sin \theta g(x))} \mu(dx) \int_{W} e^{i(- \sin \theta f(y) + \cos \theta g(y))} \mu(dy)$$

$$= e^{-\frac{1}{2}(\sin^2 \theta + \cos^2 \theta) q(f, f) + (\sin^2 \theta + \cos^2 \theta) q(g, g)}$$

$$= \int_{W} e^{i(f(x) + g(y))} \mu^2(dx, dy).$$

We can now really prove Fernique’s theorem.

Proof. In this proof we shall write $\mu(\|x\| \leq t)$ as shorthand for $\mu(\{x : \|x\| \leq t\})$, etc.

Let $0 \leq s \leq t$, and consider

$$\mu(\|x\| \leq s) \mu(\|x\| \geq t) = \mu^2(\{(x, y) : \|x\| \leq s, \|y\| \geq t\})$$

$$= \mu^2 \left( \left\| \frac{1}{\sqrt{2}}(x - y) \right\| \leq s, \left\| \frac{1}{\sqrt{2}}(x + y) \right\| \geq t \right)$$

by $R_{\pi/4}$ invariance of $\mu^2$. Now some gymnastics with the triangle inequality shows that if we have $\left\| \frac{1}{\sqrt{2}}(x - y) \right\| \leq s$ and $\left\| \frac{1}{\sqrt{2}}(x + y) \right\| \geq t$, then $\|x\|, \|y\| \geq \frac{t - s}{\sqrt{2}}$. So we have

$$\mu(\|x\| \leq s) \mu(\|x\| \geq t) \leq \mu^2 \left( \|x\| \geq \frac{t - s}{\sqrt{2}}, \|y\| \geq \frac{t - s}{\sqrt{2}} \right)$$

$$= \left( \mu \left( \|x\| \geq \frac{t - s}{\sqrt{2}} \right) \right)^2.$$

If we rearrange and let $a(t) = \frac{\mu(\|x\| \geq t)}{\mu(\|x\| \leq s)}$, this gives

$$a(t) \leq a \left( \frac{t - s}{\sqrt{2}} \right)^2. \quad (4.3)$$

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Once and for all, fix an $s$ large enough that $\mu(\|x\| \geq s) < \mu(\|x\| \leq s)$ (so that $a(s) < 1$). Now we’ll iterate (4.3). Set $t_0 = s$, $t_{n+1} = \sqrt{2}(t_n + s)$, so that (4.3) reads $a(t_{n+1}) \leq a(t_n)^2$, which by iteration implies $a(t_n) \leq a(s)^{2^n}$.

Since $t_n \uparrow \infty$, for any $r \geq s$ we have $t_n \leq r \leq t_{n+1}$ for some $n$. Note that

$$t_{n+1} = s \sum_{k=0}^{n+1} 2^{k/2} \leq C2^{n/2}$$

since the largest term dominates. $a$ is decreasing so we have

$$a(r) \leq a(t_n) \leq a(s)^{2^n} \leq a(s)^{r^2/C^2}$$

so that $a(r) \leq e^{-cr^2}$, taking $\epsilon = -\log(a(s))/C^2$. Since $a(r) = \mu(\|x\| \leq s)\mu(\|x\| \geq r)$ we are done.

\[ \square \]

**Corollary 4.13.** If $\epsilon$ is as provided by Fernique’s theorem, for $\epsilon' < \epsilon$ we have $\int W e^{\epsilon' \|x\|^2} \mu(dx) < \infty$.

\[ \text{Proof.} \text{ Standard trick: for a nonnegative random variable } X, EX = \int_0^\infty P(X > t) \, dt. \text{ So}
\]

$$\int_W e^{\epsilon' \|x\|^2} \mu(dx) = \int_0^\infty \mu(\{x : e^{\epsilon' \|x\|^2} > t\}) \, dt$$

$$= \int_0^\infty \mu \left( \left\{ x : \|x\| > \sqrt{\frac{\log t}{\epsilon'}} \right\} \right) \, dt$$

$$\leq \int_0^\infty t^{-\epsilon'/\epsilon'} \, dt < \infty.$$ 

\[ \square \]

The following corollary is very convenient for dominated convergence arguments.

**Corollary 4.14.** For any $p > 0$, $\int_W \|x\|^p \mu(dx) < \infty$.

\[ \text{Proof.} \text{ } t^p \text{ grows more slowly than } e^{ct^2}. \]

\[ \square \]

**Corollary 4.15.** The inclusion $W^* \hookrightarrow L^2(\mu)$ is bounded. In particular, the $L^2$ norm on $W^*$ is weaker than the operator norm.

\[ \text{Proof.} \text{ For } f \in W^*, \|f\|^2_{L^2} = \int_W |f(x)|^2 \mu(dx) \leq \|f\|^2_{W^*} \int_W \|x\|^2 \mu(dx) \leq C\|f\|^2_{W^*} \text{ by the previous corollary.} \]

\[ \square \]

(This would be a good time to look at Exercises C.1–C.6 to get some practice working with different topologies on a set.)

Actually we can say more than the previous corollary. Recall that an operator $T : X \rightarrow Y$ on normed spaces is said to be **compact** if it maps bounded sets to precompact sets, or equivalently if for every bounded sequence $\{x_n\} \subset X$, the sequence $\{Tx_n\} \subset Y$ has a convergent subsequence.

**Proposition 4.16.** The inclusion $W^* \hookrightarrow L^2(\mu)$ is compact.
Proof. Suppose \( \{f_n\} \) is a bounded sequence in \( W^* \); say \( \|f_n\|_{W^*} \leq 1 \) for all \( n \). By Alaoglu's theorem there is a weak-* convergent subsequence \( f_{n_k} \), which is to say that \( f_{n_k} \) converges pointwise to some \( f \in W^* \). Note also that \( |f_{n_k}(x)| \leq \|x\|_W \) for all \( k \), and \( \int_W \|x\|_W^2 \mu(dx) < \infty \) as we showed. So by dominated convergence, \( f_{n_k} \to f \) in \( L^2(W, \mu) \), and we found an \( L^2 \)-convergent subsequence. □

This fact is rather significant: since compact maps on infinite-dimensional spaces can't have continuous inverses, this shows that the \( W^* \) and \( L^2 \) topologies on \( W^* \) must be quite different. In particular:

**Corollary 4.17.** \( W^* \) is not complete in the \( q \) inner product (i.e. in the \( L^2(\mu) \) inner product), except in the trivial case that \( W \) is finite dimensional.

**Proof.** We've shown the identity map \( (W^*, \|\cdot\|_{W^*}) \to (W^*, q) \) is continuous and bijective. If \( (W^*, q) \) is complete, then by the open mapping theorem, this identity map is a homeomorphism, i.e. the \( W^* \) and \( q \) norms are equivalent. But the identity map is also compact, which means that bounded sets, such as the unit ball, are precompact (under either topology). This means that \( W^* \) is locally compact and hence finite dimensional. □

So the closure \( K \) of \( W^* \) in \( L^2(W, \mu) \) is a proper superset of \( W^* \).

### 4.3 The Cameron–Martin space

Our goal is to find a Hilbert space \( H \subset W \) which will play the same role that \( \ell^2 \) played for \( \mathbb{R}^\infty \). The key is that, for \( h \in H \), the map \( W^* \ni f \mapsto f(h) \) should be continuous with respect to the \( q \) inner product on \( W^* \).

As before, let \( K \) be the closure of \( W^* \) in \( L^2(W, \mu) \). We'll continue to denote the covariance form on \( K \) (and on \( W^* \)) by \( q \). We'll also use \( m \) to denote the inclusion map \( m : W^* \to K \). Recall that we previously argued that \( m \) is compact.

**Lemma 4.18.** Every \( k \in K \) is a Gaussian random variable on \( (W, \mu) \).

**Proof.** Since \( W^* \) is dense in \( K \), there is a sequence \( f_n \in W^* \) converging to \( k \) in \( L^2(W, \mu) \). In particular, they converge in distribution. By Lemma [A.3] \( k \) is Gaussian. □

**Definition 4.19.** The **Cameron–Martin space** \( H \) of \( (W, \mu) \) consists of those \( h \in W \) such that the evaluation functional \( f \mapsto f(h) \) on \( W^* \) is continuous with respect to the \( q \) inner product.

\( H \) is obviously a vector space.

For \( h \in H \), the map \( W^* \ni f \mapsto f(h) \) extends by continuity to a continuous linear functional on \( K \). Since \( K \) is a Hilbert space this may be identified with an element of \( K \) itself. Thus we have a mapping \( T : H \to K \) such that for \( f \in W^* \),

\[
q(Th, f) = f(h).
\]

A natural norm on \( H \) is defined by

\[
\|h\|_H = \sup \left\{ \frac{|f(h)|}{\sqrt{q(f,f)}} : f \in W^*, f \neq 0 \right\}.
\]

This makes \( T \) into an isometry, so \( \|\cdot\|_H \) is in fact induced by an inner product \( \langle \cdot, \cdot \rangle_H \) on \( H \).
Next, we note that \( H \) is continuously embedded into \( W \). We have previously shown (using Fernique) that the embedding of \( W^* \) into \( K \) is continuous, i.e. \( q(f, f) \leq C^2 \|f\|_{W^*}^2 \). So for \( h \in H \) and \( f \in W^* \), we have
\[
\frac{|f(h)|}{\|f\|_{W^*}} \leq C \frac{|f(h)|}{\sqrt{q(f, f)}}
\]
When we take the supremum over all nonzero \( f \in W^* \), the left side becomes \( \|h\|_{W} \) (by Hahn–Banach) and the right side becomes \( C \|h\|_{H} \). So we have \( \|h\|_{W} \leq C \|h\|_{H} \) and the inclusion \( i : H \hookrightarrow W \) is continuous.

(Redundant given the next paragraph.) Next, we check that \( (H, \|\cdot\|_{H}) \) is complete. Suppose \( h_n \) is Cauchy in \( H \)-norm. In particular, it is bounded in \( H \) norm, so say \( \|h_n\|_{H} \leq M \) for all \( n \). Since the inclusion of \( H \) into \( W \) is bounded, \( h_n \) is also Cauchy in \( W \)-norm, hence converges in \( W \)-norm to some \( x \in W \). Now fix \( \epsilon > 0 \), and choose \( n \) so large that \( \|h_n - h_m\|_{H} \leq \epsilon \) for all \( n \geq m \). Given a nonzero \( f \in W^* \), we can choose \( m \geq n \) so large that \( |f(h_m - x)| \leq \epsilon \sqrt{q(f, f)} \). Then
\[
\frac{f(h_n - x)}{\sqrt{q(f, f)}} \leq \frac{|f(h_n - h_m)|}{\sqrt{q(f, f)}} + \frac{|f(h_m - x)|}{\sqrt{q(f, f)}} \leq \|h_n - h_m\|_{H} + \epsilon \leq 2\epsilon.
\]
We can then take the supremum over \( f \) to find that \( \|h_n - x\|_{H} < 2\epsilon \), so \( h_n \to x \) in \( H \)-norm.

Next, we claim the inverse of \( T \) is given by
\[
Jk = \int_{W} xk(x)\mu(dx)
\]
where the integral is in the sense of Bochner. (To see that the integral exists, note that by Fernique \( \|\cdot\|_{L^2(W, \mu)} \) is continuous, exists for all \( k \in K \), we have
\[
|f(Jk)| = \left| \int_{W} f(x)k(x)\mu(dx) \right| = |q(f, k)| \leq \sqrt{q(f, f)}q(k, k)
\]
whence \( \|\int_{W} xk(x)\mu(dx)\|_{H} \leq \sqrt{q(k, k)} \). So \( J \) is a continuous operator from \( K \) to \( H \). Next, for \( f \in W^* \) we have
\[
q(TJk, f) = f(Jk) = q(k, f)
\]
as we just argued. Since \( W^* \) is dense in \( K \), we have \( TJk = k \). In particular, \( T \) is surjective, and hence unitary.

**Question 2. Could we have done this without the Bochner integral?**

We previously showed that the inclusion map \( i : H \to W \) is continuous, and it’s clearly 1-1. It has an adjoint operator \( i^* : W^* \to H \). We note that for \( f \in W^* \) and \( h \in H \), we have
\[
q(f, Th) = f(h) = \langle i^*f, h \rangle_{H} = q(Ti^*f, Th).
\]
Since \( T \) is surjective we have \( q(f, k) = q(Ti^*f, k) \) for all \( k \in K \); thus \( Ti^* \) is precisely the inclusion map \( m : W^* \to K \). Since \( m \) is compact and 1-1 and \( T \) is unitary, it follows that \( i^* \) is compact and 1-1.

Since \( i^* \) is 1-1, it follows that \( H \) is dense in \( W \): if \( f \in W^* \) vanishes on \( H \), it means that for all \( h \in H \), \( 0 = f(h) = \langle i^*f, h \rangle_{H} \), so \( i^*f = 0 \) and \( f = 0 \). The Hahn–Banach theorem then implies \( H \) is dense in \( W \). Moreover, Schauder’s theorem from functional analysis (see for example [3, Theorem VI.3.4]) states that an operator between Banach spaces is compact if its adjoint is compact, so \( i \) is compact as well. In particular, \( H \) is not equal to \( W \), and is not complete in the \( W \) norm.

We can sum up all these results with a diagram.
**Theorem 4.20.** The following diagram commutes.

\[
\begin{array}{ccc}
W^* & \longrightarrow & K \\
\downarrow & & \downarrow \\
H & \longrightarrow & W \\
\end{array}
\]

(4.4)

All spaces are complete in their own norms. All dotted arrows are compact, 1-1, and have dense image. All solid arrows are unitary.

Sometimes it’s convenient to work things out with a basis.

**Proposition 4.21.** There exists a sequence \( \{e_k\}_{k=1}^\infty \subset W^* \) which is an orthonormal basis for \( K \). \( e_k \) are iid \( N(0,1) \) random variables under \( \mu \). For \( h \in H \), we have \( \|h\|_H^2 = \sum_{k=1}^\infty |e_k(h)|^2 \), and the sum is infinite for \( h \in W \setminus H \).

Proof. The existence of \( \{e_k\} \) is proved in Lemma A.2. They are jointly Gaussian random variables since \( \mu \) is a Gaussian measure. Orthonormality means they each have variance 1 and are uncorrelated, so are iid.

If \( h \in H \), then \( \sum_k |e_k(h)|^2 = \sum_k |q(e_k, Th)|^2 = \|Th\|_K^2 = \|h\|_H^2 \) since \( T \) is an isometry. Conversely, suppose \( x \in W \) and \( M := \sum_k |e_k(x)|^2 < \infty \). Let \( E \subset X^* \) be the linear span of \( \{e_k\} \), i.e. the set of all \( f \in W^* \) of the form \( f = \sum_{k=1}^n a_k e_k \). For such \( f \) we have

\[
|f(x)|^2 = \left| \sum_{k=1}^n a_k e_k(x) \right|^2 \\
\leq \left( \sum_{k=1}^n |a_k|^2 \right) \left( \sum_{k=1}^n |e_k(x)|^2 \right) \quad \text{(Cauchy–Schwarz)}
\]

Thus \( x \mapsto f(x) \) is a bounded linear functional on \( (E, q) \). \( (E, q) \) is dense in \( (W^*, q) \) so the same bound holds for all \( f \in X^* \). Thus by definition we have \( x \in H \).

**Proposition 4.22.** \( \mu(H) = 0 \).

Proof. For \( h \in H \) we have \( \sum_k |e_k(h)|^2 < \infty \). But since \( e_k \) are iid, by the strong law of large numbers we have that \( \sum |e_k(x)|^2 = +\infty \) for \( \mu \)-a.e. \( x \).

**Notation 4.23.** Fix \( h \in H \). Then \( \langle h, x \rangle_H \) is unambiguous for all \( x \in H \). If we interpret \( \langle h, x \rangle_H \) as \( (Th)(x) \), it is also well-defined for almost every \( x \), and so \( \langle h, \cdot \rangle_H \) is a Gaussian random variable on \( (W, \mu) \) with variance \( \|h\|_H^2 \).

**Theorem 4.24** (Cameron–Martin). For \( h \in H \), \( \mu_h \) is absolutely continuous with respect to \( \mu \), and

\[
\frac{d\mu_h}{d\mu}(x) = e^{-\frac{1}{2}\|h\|_H^2 + \langle h, x \rangle_H}.
\]

For \( x \in W \setminus H \), \( \mu_x \) and \( \mu \) are singular.
Proof. Suppose $h \in H$. We have to show $\mu_h(dx) = e^{-\frac{1}{2}\|h\|^2_H + \langle h, x \rangle} \mu(dx)$. It is enough to show their Fourier transforms are the same (Theorem 4.9). For $f \in W^*$ we have

$$\int_W e^{if(x)} \mu_h(dx) = \int_W e^{if(x+h)} \mu(dx) = e^{i\langle f, h \rangle} \int_W e^{i\langle f, h \rangle} \mu(dx).$$

On the other hand,

$$\int_W e^{if(x)} e^{-\frac{1}{2}\|h\|^2_H + \langle h, x \rangle} \mu(dx) = e^{-\frac{1}{2}\|h\|^2_H} \int_W e^{i\langle f - iTh, h \rangle} \mu(dx) = e^{-\frac{1}{2}\|h\|^2_H} e^{-\frac{1}{2}q(f - iTh, f - iTh)}$$

since $f - iTh$ is a complex Gaussian random variable (we will let the reader check that everything works fine with complex numbers here). But we have

$$q(f - iTh, f - iTh) = q(f, f) - 2iz(q(f, Th) - q(Th, Th)) = q(f, f) - 2iz(f, h) - \|h\|^2_H$$

by properties of $T$, and so in fact the Fourier transforms are equal.

Conversely, if $x \in W \setminus H$, by Lemma 4.4.21 we have $\sum_k |e_k(x)|^2 = \infty$. By Lemma A.4 there exists $a \in l^2$ such that $\sum a_k e_k(x)$ diverges. Set $A = \{ y \in W : \sum a_k e_k(y) \text{ converges} \}$. We know that $\sum a_k e_k$ converges in $L^2(W, \mu)$, and is a sum of independent random variables (under $\mu$), hence it converges $\mu$-a.s. Thus $\mu(A) = 1$. However, if $y \in A$, then $\sum a_k e_k(y - x)$ diverges, so $A - x$ is disjoint from $A$, and thus $\mu_x(A) = \mu(A - x) = 0$. \qed

Exercise 4.25. $\mu$ has full support, i.e. for any nonempty open set $U$, $\mu(U) > 0$. This is the converse of Exercise 4.3. (Hint: First show this for $U \ni 0$. Then note any nonempty open $U$ contains a neighborhood of some $h \in H$. Translate.) (Question: Can we prove this without needing the Cameron–Martin hammer? I think yes, look for references.)

Remark 4.26. There really isn’t any generality lost by assuming that $(W, \mu)$ is non-degenerate. If you want to study the degenerate case, let $F = \{ f \in W^* : q(f, f) = 0 \}$ be the kernel of $q$, and consider the closed subspace

$$W_0 := \bigcap_{f \in F} \ker f \subset W.$$  

We claim that $\mu(W_0) = 1$. For each $f \in F$, the condition $q(f, f) = \int f^2 \ d\mu = 0$ implies that $f = 0 \mu$-almost everywhere, so $\mu(\ker f) = 1$, but as written, $W_0$ is an uncountable intersection of such sets. To fix that, note that since $W$ is separable, the unit ball $B^*$ of $W^*$ is weak-* compact metrizable, hence weak-* separable metrizable, hence so is its subset $F \cap B^*$. So we can choose a countable weak-* sequence $\{f_n\} \subset F \cap B^*$. Then I claim

$$W_0 = \bigcap_n \ker f_n.$$  

The inclusion is obvious. To see the other direction, suppose $x \in \bigcap_n \ker f_n$ and $f \in F$; we will show $f(x) = 0$. By rescaling, we can assume without loss of generality that $f \in B^*$. Now choose a subsequence $f_{n_k}$ converging weak-* to $f$; since $f_{n_k}(x) = 0$ by assumption, we have $f(x) = 0$ also. Now $W_0$ is written as a countable intersection of measure-1 subsets, so $\mu(W_0) = 1$.  

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We can now work on the abstract Wiener space \((W_0, \mu|_{W_0})\). Note that the covariance form \(q_0\) defined on \(W_0^*\) by \(q_0(f_0, f_0) = \int_{W_0} f_0^2 d\mu\) agrees with \(q\), since given any extension \(f \in W^*\) of \(f_0\) will satisfy
\[
q(f, f) = \int_W f^2 d\mu = \int_{W_0} f^2 d\mu = \int_{W_0} f_0^2 d\mu = q_0(f_0, f_0).
\]

This makes it easy to see that that \(q_0\) is positive definite on \(W_0^*\). Suppose \(q_0(f_0, f_0) = 0\) and use Hahn–Banach to choose an extension \(f \in W^*\) of \(f_0\). Then \(q(f, f) = 0\), so by definition of \(W_0\), we have \(W_0 \subset \ker f\); that is, \(f\) vanishes on \(W_0\), so the restriction \(f_0 = f|_{W_0}\) is the zero functional.

It now follows, from the previous exercise, that the support of \(\mu\) is precisely \(W_0\). So \((W_0, \mu|_{W_0})\) is a non-degenerate abstract Wiener space, and we can do all our work on this smaller space.

I’d like to thank Philipp Wacker for suggesting this remark and sorting out some of the details.

### 4.4 Example: Gaussian processes

Recall that a one-dimensional stochastic process \(X_t, 0 \leq t \leq 1\) is said to be **Gaussian** if, for any \(t_1, \ldots, t_n \geq 0\), the random vector \((X_{t_1}, \ldots, X_{t_n})\) has a joint Gaussian distribution. If the process is continuous, its distribution gives a probability measure \(\mu\) on \(W = C([0, 1])\). If there is any good in the world, this ought to be an example of a Gaussian measure.

By the Riesz representation theorem, we know exactly what \(W^*\) is: it’s the set of all finite signed Borel measures \(\nu\) on \([0, 1]\). We don’t yet know that all of these measures represent Gaussian random variables, but we know that some of them do. Let \(\delta_t\) denote the measure putting unit mass at \(t\), so \(\delta_t(\omega) = \int_0^1 \omega(t) d\delta_t = \omega(t)\). We know that \(\{\delta_t\}_{t \in [0, 1]}\) are jointly Gaussian. If we let \(E \subset W^*\) be their linear span, i.e. the set of all finitely supported signed measures, i.e. the set of measures \(\nu = \sum_{i=1}^n a_i \delta_{t_i}\), then all measures in \(E\) are Gaussian random variables.

**Lemma 4.27.** \(E\) is weak-* dense in \(W^*\), and dense in \(K\).

**Proof.** Suppose \(\nu \in W^*\). Given a partition \(\mathcal{P} = \{0 = t_0 < \cdots < t_n = 1\}\) of \([0, 1]\), set \(\nu_\mathcal{P} = \sum_{j=1}^n \int_{[t_{j-1}, t_j]} d\nu \delta_{t_j}\). Then for each \(\omega \in C([0, 1])\), \(\int \omega d\nu_\mathcal{P} = \int \omega d\nu\), where
\[
\omega_\mathcal{P} = \sum_{j=1}^n \omega(t_j) \mathbf{1}_{[t_{j-1}, t_j]}.
\]

But by uniform continuity, as the mesh size of \(\mathcal{P}\) goes to 0, we have \(\omega_\mathcal{P} \to \omega\) uniformly, and so \(\int \omega_\mathcal{P} d\nu \to \int \omega d\nu\). Thus \(\nu_\mathcal{P} \to \nu\) weakly-*.

**Corollary 4.28.** \(\mu\) is a Gaussian measure.

**Proof.** Every \(\nu \in W^*\) is a pointwise limit of a sequence of Gaussian random variables, hence Gaussian.

**Lemma 4.29.** \(E\) is dense in \(K\).

**Proof.** \(\{\nu_\mathcal{P}\}\) is bounded in total variation (in fact \(\|\nu_\mathcal{P}\| \leq \|\nu\|\)). So by Fernique’s theorem and dominated convergence, \(\nu_\mathcal{P} \to \nu\) in \(L^2(X, \mu)\). Thus \(E\) is \(L^2\)-dense in \(W^*\). Since \(W^*\) is \(L^2\)-dense in \(K\), \(E\) is dense in \(K\).
Note that in order to get \( \mu \) to be non-degenerate, it may be necessary to replace \( W \) by a smaller space. For example, if \( X_t \) is Brownian motion started at 0, the linear functional \( \omega \mapsto \omega(0) \) is a.s. zero. So we should take \( W = \{ \omega \in C([0,1]) : \omega(0) = 0 \} \). One might write this as \( C_0((0,1]) \).

Recall that a Gaussian process is determined by its covariance function \( a(s,t) = E[X_sX_t] = \langle \delta_s, \delta_t \rangle \). Some examples:

1. Standard Brownian motion \( X_t = B_t \) started at 0: \( a(s,t) = s \) for \( s < t \). Markov, martingale, independent increments, stationary increments.
2. Ornstein–Uhlenbeck process defined by \( dX_t = -X_t \, dt + \sigma \, dB_t \): \( a(s,t) = \frac{\sigma^2}{2}(e^{-(t-s)} - e^{-(t+s)}) \), \( s < t \). Markov, not a martingale.
3. Fractional Brownian motion with Hurst parameter \( H \in (0,1) \): \( a(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - (t-s)^{2H}) \), \( s < t \). \( H \) is non-degenerate, it may be necessary to replace \( W \) by a smaller space with the candidate norm \( H \).
4. Brownian bridge \( X_t = B_t - tB_1 \): \( a(s,t) = s(1-t) \), \( s < t \). (Here \( W \) should be taken as \( \{ \omega \in C([0,1]) : \omega(0) = \omega(1) = 0 \} = C_0((0,1)) \), the so-called pinned loop space.)

**Lemma 4.30.** The covariance form \( q \) for a Gaussian process is defined by

\[
q(\nu_1, \nu_2) = \int_0^1 \int_0^1 a(s,t) \nu_1(ds) \nu_2(dt)
\]

for \( \nu_1, \nu_2 \in W^* \).

**Proof.** Fubini’s theorem, justified with the help of Fernique. \( \square \)

**Lemma 4.31.** \( J : K \to H \) is defined by \( Jk(t) = q(k, \delta_t) \). For \( k = \nu \in W^* \) this gives \( J\nu(t) = i^*\nu(t) = \int_0^1 a(s,t)\nu(ds) \). In particular \( J\delta_s(t) = a(s,t) \).

**Proof.** \( Jk(t) = \delta_t(Jk) = q(k, \delta_t) \).

Observe that \( a \) plays the role of a reproducing kernel in \( H \): we have

\[
\langle h, a(s, \cdot) \rangle_H = \langle h, J\delta_s \rangle = \delta_s(h) = h(s).
\]

This is why \( H \) is sometimes called the “reproducing kernel Hilbert space” or “RKHS” for \( W \).

### 4.5 Classical Wiener space

Let \( \mu \) be Wiener measure on classical Wiener space \( W \), so \( a(s,t) = s \land t \).

**Theorem 4.32.** The Cameron–Martin space \( H \subset W \) is given by the set of all \( h \in W \) which are absolutely continuous and have \( \dot{h} \in L^2([0,1],m) \). The Cameron-Martin inner product is given by

\[
\langle h_1, h_2 \rangle_H = \int_0^1 \dot{h}_1(t)\dot{h}_2(t) \, dt.
\]

**Proof.** Let \( \tilde{H} \) be the candidate space with the candidate norm \( \| \cdot \|_{\tilde{H}} \). It’s easy to see that \( \tilde{H} \) is a Hilbert space.

Note that \( J\delta_s(t) = s \land t \in \tilde{H} \), so by linearity \( J \) maps \( E \) into \( \tilde{H} \). Note \( J\delta_s = 1_{[0,s]} \). Moreover,

\[
\langle J\delta_s, J\delta_r \rangle_{\tilde{H}} = \int_0^1 1_{[0,s]}1_{[0,r]} \, dm = s \land r = q(\delta_s, \delta_r)
\]

so \( J \) is an isometry from \( (E,q) \) to \( \tilde{H} \). Hence it extends to an isometry of \( K \) to \( \tilde{H} \). Since \( J \) is already an isometry from \( K \) to \( H \) we have \( H = \tilde{H} \) isometrically. \( \square \)
Now what can we say about $T$? It’s a map that takes a continuous function from $H$ and returns a random variable. Working informally, we would say that

$$Th(\omega) = \langle h, \omega \rangle_H = \int_0^1 h(t) \dot{\omega}(t) \, dt. \quad (4.5)$$

This formula is absurd because $\dot{\omega}$ is nonexistent for $\mu$-a.e. $\omega$ (Brownian motion sample paths are nowhere differentiable). However, it is actually the right answer if interpreted correctly.

Let’s suppose that $h$ is piecewise linear: then its derivative is a step function $\dot{h} = \sum_{i=1}^n b_i 1_{[c_i, d_i]}$.

Note that the reproducing kernel $a(s, \cdot)$ has as its derivative the step function $1_{[0, s]}$. So by integrating, we see that we can write

$$h(t) = \sum_{i=1}^n b_i (a(d_i, t) - a(c_i, t)).$$

Now we know that $T[a(s, \cdot)] = \delta_s$, i.e. the random variable $B_s$. So we have

$$Th = \sum_{i=1}^n b_i (B_{d_i} - B_{c_i}).$$

We can recognize this as the stochastic integral of the step function $\dot{h} = \sum_{i=1}^n b_i 1_{[c_i, d_i]}$:

$$Th = \int_0^1 \dot{h}(t) \, dB_t. \quad (4.6)$$

Moreover, by the Itô isometry we know that

$$\left\| \int_0^1 \dot{h}(t) \, dB_t \right\|_{L^2(W, \mu)}^2 = \| \dot{h} \|_{L^2([0, 1])}^2 = \| h \|_H^2.$$

Thus both sides of (4.6) are isometries on $H$, and they are equal for all piecewise linear $H$. Since the step functions are dense in $L^2([0, 1])$, the piecewise linear functions are dense in $H$ (take derivatives), so in fact (4.6) holds for all $h \in H$. We have rediscovered the stochastic integral, at least for deterministic integrands. This is sometimes called the Wiener integral. Of course, the Itô integral also works for stochastic integrands, as long as they are adapted to the filtration of the Brownian motion. Later we shall use our machinery to produce the Skorohod integral, which will generalize the Itô integral to integrands which need not be adapted, giving us an “anticipating stochastic calculus.”

Exercise 4.33. For the Ornstein–Uhlenbeck process, show that $H$ is again the set of absolutely continuous functions $h$ with $\dot{h} \in L^2([0, 1])$, and the Cameron–Martin inner product is given by

$$\langle h_1, h_2 \rangle_H = \frac{1}{\sigma^2} \int_0^1 \dot{h}_1(t) \dot{h}_2(t) + h_1(t)h_2(t) \, dt.$$

Exercise 4.34. For the Brownian bridge, show that $H$ is again the set of absolutely continuous functions $h$ with $\dot{h} \in L^2([0, 1])$, and the Cameron–Martin inner product is given by $\langle h_1, h_2 \rangle_H = \int_0^1 \dot{h}_1(t) \dot{h}_2(t) \, dt$, where

$$\dot{h}(t) = \dot{h}(t) + \frac{h(t)}{1-t}.$$
Perhaps later when we look at some stochastic differential equations, we will see where these formulas come from.

Note that in this case the Cameron–Martin theorem is a special case of Girsanov’s theorem: it says that a Brownian motion with a “smooth” drift becomes a Brownian motion without drift under an equivalent measure. Indeed, suppose $h \in H$. If we write $B_t(\omega) = \omega(t)$, so that $\{B_t\}$ is a Brownian motion on $(W, \mu)$, then $B_t + h(t)$ is certainly a Brownian motion (without drift!) on $(W, \mu_h)$. The Cameron-Martin theorem says that $\mu_h$ is an equivalent measure to $\mu$. Anything that $B_t$ can’t do, $B_t + h(t)$ can’t do either (since the $\mu$-null and $\mu_h$-null sets are the same). This fact has many useful applications. For example, in mathematical finance, one might model the price of an asset by a geometric Brownian motion with a drift indicating its average rate of return (as in the Black–Scholes model). The Cameron–Martin/Girsanov theorem provides an equivalent measure under which this process is a martingale, which makes it possible to compute the arbitrage-free price for options involving the asset. The equivalence of the measures is important because it guarantees that changing the measure didn’t allow arbitrage opportunities to creep in.

4.6 Construction of $(W, \mu)$ from $H$

This section originates in [10] via Bruce Driver’s notes [5].

When $W = \mathbb{R}^n$ is finite-dimensional and $\mu$ is non-degenerate, the Cameron–Martin space $H$ is all of $W$ (since $H$ is known to be dense in $W$), and one can check that the Cameron–Martin norm is

$$\langle x, y \rangle_H = x \cdot \Sigma^{-1} y$$

(4.7)

where $\Sigma$ is the covariance matrix. We also know that $\mu$ has a density with respect to Lebesgue measure $dx$, which we can write as

$$\mu(dx) = \frac{1}{Z} e^{-\frac{1}{2} \|x\|^2_H} dx$$

(4.8)

where $Z = \int_{\mathbb{R}^n} e^{-\frac{1}{2} \|x\|^2_H} dx$ is a normalizing constant chosen to make $\mu$ a probability measure. Informally, we can think of $\mu$ as being given by a similar formula in infinite dimensions:

$$\mu(dx) = \frac{1}{Z} e^{-\frac{1}{2} \|x\|^2_H} Dx$$

(4.9)

where $Z$ is an appropriate normalizing constant, and $Dx$ is infinite-dimensional Lebesgue measure. Of course this is nonsense in at least three different ways, but that doesn’t stop physicists, for instance.

For classical Wiener measure this reads

$$\mu(dx) = \frac{1}{Z} e^{-\frac{1}{2} \int_0^1 |\dot{\omega}(t)|^2 dt} D\omega.$$ 

(4.10)

Since the only meaningful object appearing on the right side of (4.9) is $\|\cdot\|_H$, it is reasonable to ask if we can start with a Hilbert space $H$ and produce an abstract Wiener space $(W, \mu)$ for which $H$ is the Cameron–Martin space.
4.6.1 Cylinder sets

Let \((H, \| \cdot \|_H)\) be a separable Hilbert space.

**Definition 4.35.** A cylinder set is a subset \(C \subset H\) of the form

\[
C = \{ h \in H : (\langle h, k_1 \rangle_H, \ldots, \langle h, k_n \rangle_H) \in A \}
\]  

(4.11)

for some \(n \geq 1\), orthonormal \(k_1, \ldots, k_n\), and \(A \subset \mathbb{R}^n\) Borel.

**Exercise 4.36.** Let \(\mathcal{R}\) denote the collection of all cylinder sets in \(H\). \(\mathcal{R}\) is an algebra: we have \(\emptyset \in \mathcal{R}\) and \(\mathcal{R}\) is closed under complements and finite unions (and intersections). However, if \(H\) is infinite dimensional then \(\mathcal{R}\) is not a \(\sigma\)-algebra.

Note by Lemma 4.4 that \(\sigma(\mathcal{R}) = B_H\), the Borel \(\sigma\)-algebra.

We are going to try to construct a Gaussian measure \(\tilde{\mu}\) on \(H\) with covariance form given by \(\langle \cdot, \cdot \rangle_H\). Obviously we can only get so far, since we know of several obstructions to completing the task. At some point we will have to do something different. But by analogy with finite dimensions, we know what value \(\tilde{\mu}\) should give to a cylinder set of the form (4.11): since \(k_1, \ldots, k_n\) are orthonormal, they should be iid standard normal with respect to \(\tilde{\mu}\), so we should have

\[
\tilde{\mu}(C) = \mu_n(A)
\]  

(4.12)

where \(d\mu_n = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx\) is standard Gaussian measure on \(\mathbb{R}^n\).

**Proposition 4.37.** The expression for \(\tilde{\mu}(C)\) in (4.12) is well-defined, and \(\tilde{\mu}\) is a finitely additive probability measure on \(\mathcal{R}\).

**Proof.** To check that \(\tilde{\mu}(C)\) is well-defined, suppose that

\[
C = \{ h \in H : (\langle h, k_1 \rangle_H, \ldots, \langle h, k_n \rangle_H) \in A \subset \mathbb{R}^n \} = \{ h \in H : (\langle h, k'_1 \rangle_H, \ldots, \langle h, k'_m \rangle_H) \in A' \subset \mathbb{R}^m \}.
\]  

(4.13)

Let \(E\) be the span in \(H\) of \(\{k_1, \ldots, k_n, k'_1, \ldots, k'_m\}\), and let \(m = \dim E\). Since \(\{k_1, \ldots, k_n\}\) is orthonormal in \(E\), we can extend it to an orthonormal basis \(\{k_1, \ldots, k_m\}\) for \(E\), and then we have

\[
C = \{ h \in H : (\langle h, k_1 \rangle_H, \ldots, \langle h, k_m \rangle_H) \in A \times \mathbb{R}^{m-n} \}.
\]

Since \(\mu_m\) is a product measure, we have \(\mu_m(A \times \mathbb{R}^{m-n}) = \mu_n(A)\). So by playing the same game for \(\{k'_1, \ldots, k'_m\}\), there is no loss of generality in assuming that in (4.13) we have \(n = n' = m\), and that \(\{k_1, \ldots, k_m\}\) and \(\{k'_1, \ldots, k'_m\}\) are two orthonormal bases for the same \(E \subset H\). We then have to show that \(\mu_m(A) = \mu_m(A')\).

We have two orthonormal bases for \(E\), so there is a unitary \(T : E \to E\) such that \(Tk_i = k'_i\). Let \(P : H \to E\) be orthogonal projection, and define \(S : E \to \mathbb{R}^m\) by \(Sx = (\langle x, k_1 \rangle, \ldots, \langle x, k_m \rangle)\). Then \(S\) is unitary. If we define \(S'\) analogously, then \(S' = ST^* = ST\), and we have

\[
C = P^{-1}S^{-1}A = P^{-1}S'^{-1}A' = P^{-1}T^{-1}S^{-1}A'.
\]

Since \(P : H \to E\) is surjective, we must have \(S^{-1}A = T^{-1}S'^{-1}A'\); since \(S, T\) are bijective this says \(A' = STS^{-1}A\), so \(A'\) is the image of \(A\) under a unitary map. But standard Gaussian measure
on $\mathbb{R}^m$ is invariant under unitary transformations, so indeed $\mu_m(A) = \mu_m(A')$, and the expression \((4.12)\) is well defined.

It is obvious that $\tilde{\mu}(\emptyset) = 0$ and $\tilde{\mu}(H) = 1$. For finite additivity, suppose $C_1, \ldots, C_n \in \mathcal{R}$ are disjoint. By playing the same game as above, we can write $C_i = P^{-1}(A_i)$ for some common $P : H \to \mathbb{R}^m$, where the $A_i \subset \mathbb{R}^m$ are necessarily disjoint, and then $\tilde{\mu}(C_i) = \mu_m(A_i)$. Since $\bigcup_i C_i = P^{-1}\left(\bigcup_i A_i\right)$, the additivity of $\mu_m$ gives us that $\tilde{\mu}\left(\bigcup_i C_i\right) = \sum_i \tilde{\mu}(C_i)$. \[\Box\]

We will call $\tilde{\mu}$ the canonical Gaussian measure on $H$. As we see in the next proposition, we’re using the term “measure” loosely.

**Proposition 4.38.** If $H$ is infinite dimensional, $\tilde{\mu}$ is not countably additive on $\mathcal{R}$. In particular, it does not extend to a countably additive measure on $\sigma(\mathcal{R}) = \mathcal{B}_H$.

**Proof.** Fix an orthonormal sequence $\{e_i\}$ in $H$. Let

$$A_{n,k} = \{x \in H : |\langle x, e_i \rangle| \leq k, i = 1, \ldots, n\}.$$ 

$A_{n,k}$ is a cylinder set, and we have $B(0, k) \subset A_{n,k}$ for any $n$. Also, we have $\tilde{\mu}(A_{n,k}) = \mu_n([-k, k]^n) = \mu_1([-k, k])^n$ since $\mu_n$ is a product measure. Since $\mu_1([-k, k]) < 1$, for each $k$ we can choose an $n_k$ so large that $\tilde{\mu}(A_{n_k,k}) = \mu_1([-k, k])^{n_k} < 2^{-k}$. Thus $\sum_{k=1}^\infty \tilde{\mu}(A_{n_k,k}) < 1$, but since $B(0, k) \subset A_{n_k,k}$ we have $\bigcup_{k=1}^\infty A_{n_k,k} = H$ and $\tilde{\mu}(H) = 1$. So countable additivity does not hold. \[\Box\]

Of course we already knew that this construction cannot produce a genuine Gaussian measure on $H$, since any Gaussian measure has to assign measure 0 to its Cameron–Martin space. The genuine measure has to live on some larger space $W$, so we have to find a way to produce $W$. We’ll produce it by producing a new norm $\|\cdot\|_W$ on $H$ which is not complete, and set $W$ to be the completion of $H$ under $\|\cdot\|_W$. Then we will be able to extend $\tilde{\mu}$, in a certain sense, to an honest Borel measure $\mu$ on $W$.

It’s common to make an analogy here with Lebesgue measure. Suppose we were trying to construct Lebesgue measure $m$ on $\mathbb{Q}$. We could define the measure of an interval $(a, b) \subset \mathbb{Q}$ to be $b - a$, and this would give a finitely additive measure on the algebra of sets generated by such intervals. But it could not be countably additive. If we want a countably additive measure, it has to live on $\mathbb{R}$, which we can obtain as the completion of $\mathbb{Q}$ under the Euclidean metric.

### 4.6.2 Measurable norms

**Definition 4.39.** A finite rank projection means a map $P : H \to H$ which is orthogonal projection onto its image $PH$ with $PH$ finite dimensional. We will sometimes abuse notation and identify $P$ with the finite-dimensional subspace $PH$, since they are in 1-1 correspondence. We will write things like $P_1 \perp P_2$, $P_1 \subset P_2$, etc.

We are going to obtain $W$ as the completion of $H$ under some norm $\|\cdot\|_W$. Here is the condition that this norm has to satisfy.

**Definition 4.40.** A norm $\|\cdot\|_W$ on $H$ is said to be measurable if for every $\epsilon > 0$ there exists a finite rank projection $P_0$ such that

$$\tilde{\mu}\{h : \|Ph\|_W > \epsilon\} < \epsilon$$

for all $P \perp P_0$ of finite rank \((4.14)\)

where $\tilde{\mu}$ is the canonical Gaussian “measure” on $H$. (Note that $\{x : \|Ph\|_W > \epsilon\}$ is a cylinder set.)
A quick remark: if $P_0$ satisfies (4.14) for some $\varepsilon$, and $P_0 \subset P_0'$, then $P_0'$ also satisfies (4.14) for the same $\varepsilon$. This is because any $P \perp P_0'$ also has $P \perp P_0$.

In words, this definition requires that $\tilde{\mu}$ puts most of its mass in “tubular neighborhoods” of $P_0H$. Saying $\|Ph\|_W > \varepsilon$ means that $x$ is more than distance $\varepsilon$ (in $W$-norm) from $P_0H$ along one of the directions from $PH$.

As usual, doing the simplest possible thing doesn’t work.

**Lemma 4.41.** $\|\cdot\|_H$ is not a measurable norm on $H$.

**Proof.** For any finite-rank projection $P$ of some rank $n$, we can find an orthonormal basis $\{h_1, \ldots, h_n\}$ for $PH$. Then it’s clear that $Ph = \sum_{i=1}^n \langle h, h_i \rangle H h_i$, so $\{h : \|Ph\|_H > \varepsilon\} = P^{-1}(B_{PH}(0, \varepsilon)^c)$, where $B_{PH}(0, \varepsilon)$ is a ball in $PH$. By definition of $\tilde{\mu}$ we can see that

$$\tilde{\mu}(\{h : \|Ph\|_H > \varepsilon\}) = \mu_n(B_{\mathbb{R}^n}(0, \varepsilon)^c) \geq \mu_n([-\varepsilon, \varepsilon]^n)^c)$$

since the ball is contained in the cube

$$= 1 - \mu_1([-\varepsilon, \varepsilon]^n)^n.$$ 

Thus for any $\varepsilon > 0$ and any finite-rank projection $P_0$, if we choose $n$ so large that $1 - \mu_1([-\varepsilon, \varepsilon]^n)^n > \varepsilon$, then for any projection $P$ of rank $n$ which is orthogonal to $P_0$ (of which there are lots), we have $\tilde{\mu}(\{h : \|Ph\|_H > \varepsilon\}) > \varepsilon$. So $\|\cdot\|_H$ is not measurable.

As a diversion, let’s explicitly verify this for the classical example.

**Proposition 4.42.** Let $H$ be the classical Cameron–Martin space of Theorem 4.32. The supremum norm $\|h\|_W = \sup_{t \in [0, 1]} h(t)$ is a measurable norm on $H$.

Together with Gross’s theorem (Theorem 4.44 below), this proposition constitutes a construction of Brownian motion: the completion $W$ of $H$ under $\|\cdot\|_W$ is precisely $C([0, 1])$ (since $H$ is dense in $C([0, 1])$), and the measure $\mu$ on $W$ is Wiener measure (having $H$ as its Cameron–Martin space, we can check that its covariance function is $a(s, t) = s \wedge t$ as it ought to be).

With the proof we will give, however, it will not be an essentially new construction. Indeed, we are going to steal the key ideas from a construction which is apparently due to Lévy and can be found in [12, Section 2.3], which one might benefit from reading in conjunction with this proof. In some sense, Gross’s theorem is simply an abstract version of an essential step of that construction.

**Proof.** Observe up front that by Cauchy–Schwarz

$$|h(t)| = \left| \int_0^t \dot{h}(t) \, dt \right| \leq t \|h\|_H$$

so taking the supremum over $t \in [0, 1]$, we have $\|h\|_W \leq \|h\|_H$.

We want to choose a good orthonormal basis for $H$. We use the so-called “Schauder functions” which correspond to the “Haar functions” in $L^2([0, 1])$. The Haar functions are given by

$$f^n_k(t) := \begin{cases} 
2^{(n-1)/2}, & k \frac{1}{2^n} \leq t < \frac{k+1}{2^n} \\
-2^{(n-1)/2}, & \frac{k}{2^n} \leq t \leq \frac{k+1}{2^n} \\
0, & \text{else}
\end{cases}$$

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where \( f_0^1 = 1 \). Here \( k \) should be taken to range over the set \( I(n) \) consisting of all odd integers between 0 and \( 2^n \). (This somewhat peculiar indexing is from Karatzas and Shreve’s proof. It may or may not be optimal.) We note that for \( n \geq 1 \), we have \( \int_0^1 f_k^n(t) \, dt = 0 \); that for \( n > m \), \( f_j^m \) is constant on the support of \( f_k^n \), and that for fixed \( n \), \( \{f_k^n : k \in I(n)\} \) have disjoint supports. From this it is not hard to check that \( \{f_k^n : k \in I(n), n \geq 0\} \) are an orthonormal set in \( L^2(0,1) \). Indeed, the set forms an orthonormal basis.

The Schauder functions are defined by \( h_k^n(t) := \int_t^1 f_k^n(s) \, ds \); since \( f \mapsto \int_0^1 f \, dt \) is an isometric isomorphism from \( L^2([0,1]) \) to \( H \), we have that \( \{h_k^n : k \in I(n), n \geq 0\} \) is an orthonormal basis for \( H \). (We can check easily that it is a basis: if \( \langle h, h_k^n \rangle_H = 0 \) then \( \langle h, (2^n)^{-1} h \rangle = h(k+1) \). If this holds for all \( n, k \), then \( h(t) = 0 \) for all dyadic rationals \( t \), whence by continuity \( h = 0 \).) Stealing Karatzas and Shreve’s phrase, \( h_k^n \) is a “little tent” of height \( 2^{-(n+1)/2} \) supported in \([k-1, k+1] \); in particular, for each \( n \), \( \{h_k^n : k \in I(n)\} \) have disjoint supports.

Let \( P_m \) be orthogonal projection onto the span of \( \{h_k^n : k \in I(n), n \leq m\} \), and suppose \( P \) is a projection of finite rank which is orthogonal to \( P_m \). Then for any \( h \in H \), we can write \( Ph \) in terms of the Schauder functions

\[
Ph = \sum_{n=m}^{\infty} \sum_{k \in I(n)} h_k^n \langle Ph, h_k^n \rangle_H.
\]

where the sum converges in \( H \) and hence also in \( W \)-norm, i.e. uniformly. Since for fixed \( n \) the \( h_k^n \) have disjoint supports, we can say

\[
\|Ph\|_W \leq \sum_{n=m}^{\infty} \left\| \sum_{k \in I(n)} h_k^n \langle Ph, h_k^n \rangle_H \right\|_W \tag{Triangle inequality}
\]

\[
= \sum_{n=m}^{\infty} \max_{k \in I(n)} \|h_k^n\|_W |\langle Ph, h_k^n \rangle_H| \tag{since \( h_k^n \) have disjoint support}
\]

\[
= \sum_{n=m}^{\infty} 2^{-(n+1)/2} \max_{k \in I(n)} |\langle Ph, h_k^n \rangle_H|.
\]

To forestall any nervousness, let us point out that all the following appearances of \( \bar{\mu} \) will be to measure sets of the form \( P^{-1}B \) for our single, fixed \( P \), and on such sets \( \bar{\mu} \) is an honest, countably additive measure (since it is just standard Gaussian measure on the finite-dimensional Hilbert space \( PH \)). Under \( \bar{\mu} \), each \( \langle Ph, h_k^n \rangle_H \) is a centered Gaussian random variable of variance \( \|Ph_k^n\|_H^2 \leq 1 \) (note that \( \langle Ph, h_k^n \rangle_H = \langle Ph, Ph_k^n \rangle_H \), and that \( P \) is a contraction). These random variables will be correlated in some way, but that will not bother us since we are not going to use anything fancier than union bounds.

We recall the standard Gaussian tail estimate: if \( N \) is a Gaussian random variable with variance \( \sigma^2 \leq 1 \), then \( P(|N| \geq t) \leq Ce^{-t^2/2} \) for some universal constant \( C \). (See \textit{L.E.} or for overkill, Fernique’s theorem.) Thus we have for each \( n, k \)

\[
\bar{\mu}(\{h : |\langle Ph, h_k^n \rangle_H| \geq n\}) \leq Ce^{-n^2/2}
\]

and so by union bound

\[
\bar{\mu} \left( \left\{ h : \max_{k \in I(n)} |\langle Ph, h_k^n \rangle_H| \geq n \right\} \right) = \bar{\mu} \left( \bigcup_{k \in I(n)} \{h : |\langle Ph, h_k^n \rangle_H| \geq n\} \right) \leq C2^n e^{-n^2/2}
\]

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Lemma 4.43. If \( |I(n)| \leq 2^n \). By another union bound,
\[
\hat{\mu} \left( \bigcup_{n=m}^{\infty} \left\{ h : \max_{k \in I(n)} |\langle Ph, h_k^n \rangle_H| \geq n \right\} \right) \leq C \sum_{n=m}^{\infty} 2^n e^{-n^2/2}.
\]
On the complement of this event, we have \( \max_{k \in I(n)} |\langle Ph, h_k^n \rangle_H| < n \) for every \( n \), and so using \((4.15)\) we have \( \|Ph\|_W < \sum_{n=m}^{\infty} n2^{-(n+1)/2} \). Thus we have shown
\[
\hat{\mu} \left( \left\{ h : \|Ph\|_W \geq \sum_{n=m}^{\infty} n2^{-(n+1)/2} \right\} \right) \leq C \sum_{n=m}^{\infty} 2^n e^{-n^2/2}. \tag{4.16}
\]
Since \( \sum_{n=m}^{\infty} n2^{-(n+1)/2} \) and \( \sum_{n=m}^{\infty} 2^n e^{-n^2/2} \) both converge, for any given \( \epsilon > 0 \) we may choose \( m \) so large that \( \sum_{n=m}^{\infty} n2^{-(n+1)/2} < \epsilon \) and \( C \sum_{n=m}^{\infty} 2^n e^{-n^2/2} < \epsilon \). Then for any finite-rank projection \( P \) orthogonal to \( P_m \), we have
\[
\hat{\mu}(\{ h : \|Ph\|_W > \epsilon \}) \leq \epsilon \tag{4.17}
\]
which is to say that \( \|\cdot\|_W \) is a measurable norm.

The name “measurable” is perhaps a bit misleading on its face: we are not talking about whether \( h \mapsto \|h\|_W \) is a measurable function on \( H \). It just means that \( \|\cdot\|_W \) interacts nicely with the “measure” \( \hat{\mu} \). However, \( \|\cdot\|_W \) is in fact a measurable function on \( H \), in fact a continuous function, so that it is a weaker norm than \( \|\cdot\|_H \).

**Lemma 4.43.** If \( \|\cdot\|_W \) is a measurable norm on \( H \), then \( \|h\|_W \leq C \|h\|_H \) for some constant \( C \).

**Proof.** Choose a \( P_0 \) such that \((4.14)\) holds with \( \epsilon = 1/2 \). Pick some vector \( k \in (P_0 H)^\perp \) with \( \|k\|_H = 1 \). Then \( Ph = \langle h, k \rangle k \) is a (rank-one) projection orthogonal to \( P_0 \), so
\[
\frac{1}{2} > \hat{\mu}(\{ h : \|Ph\|_W > \frac{1}{2} \}) = \hat{\mu}(\{ h : |\langle h, k \rangle| > \frac{1}{2\|k\|_W} \}) = \mu_1 \left( \left[ -\frac{1}{2\|k\|_W}, \frac{1}{2\|k\|_W} \right]^C \right).
\]
Since \( \mu_1([-t, t]^C) = 1 - \mu_1([-t, t]) \) is a decreasing function in \( t \), the last line is an increasing function in \( \|k\|_W \), so it follows that \( \|k\|_W \leq M \) for some \( M \). \( k \in (P_0 H)^\perp \) was arbitrary, and so by scaling we have that \( \|k\|_W \leq M \|k\|_H \) for all \( k \in (P_0 H)^\perp \). On the other hand, \( P_0 H \) is finite-dimensional, so by equivalence of norms we also have \( \|k\|_W \leq M \|k\|_H \) for all \( k \in P_0 H \), taking \( M \) larger if needed. Then for any \( k \in H \), we can decompose \( k \) orthogonally as \( (k - P_0 k) + P_0 k \) and obtain
\[
\|k\|_W^2 = \|(k - P_0 k) + P_0 k\|_W^2 \leq (\|k - P_0 k\|_W + \|P_0 k\|_W)^2 \tag{Triangle inequality}
\]
\[
\leq 2(\|k - P_0 k\|_W^2 + \|P_0 k\|_W^2) \tag{since \( (a + b)^2 \leq 2(a^2 + b^2) \), follows from AM-GM}
\]
\[
\leq 2M^2(\|k - P_0 k\|_H^2 + \|P_0 k\|_H^2)
\]
\[
= 2M^2 \|k\|_H^2 \tag{Pythagorean theorem}
\]
and so the desired statement holds with \( C = \sqrt{2}M \). 

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Theorem 4.44 (Gross [10]). Suppose $H$ is a separable Hilbert space and $\|\cdot\|_W$ is a measurable norm on $H$. Let $W$ be the completion of $H$ under $\|\cdot\|_W$. There exists a Gaussian measure $\mu$ on $(W, \|\cdot\|_W)$ whose Cameron-Martin space is $(H, \|\cdot\|_H)$.

Proof. We start by constructing a sequence of finite-rank projections $P_n$ inductively. First, pick a countable dense sequence $\{v_n\}$ of $H$. Let $P_0 = 0$. Then suppose that $P_{n-1}$ has been given. By the measurability of $\|\cdot\|_W$, for each $n$ we can find a finite-rank projection $P_n$ such that for all finite-rank projections $P \perp P_n$, we have

$$\tilde{\mu}(\{h \in H : \|P h\|_W > 2^{-n}\}) < 2^{-n}.$$ 

As we remarked earlier, we can always choose $P_n$ to be larger, so we can also assume that $P_{n-1} \subset P_n$ and also $v_n \in P_n H$. The latter condition ensures that $\bigcup_n P_n H$ is dense in $H$, from which it follows that $P_n h \to h$ in $H$-norm for all $h \in H$, i.e. $P_n \uparrow I$ strongly. Let us also note that $R_n := P_n - P_{n-1}$ is a finite-rank projection which is orthogonal to $P_{n-1}$ (in fact, it is projection onto the orthogonal complement of $P_{n-1}$ in $P_n$), and in fact we have the orthogonal decomposition $H = \bigoplus_{n=1}^{\infty} R_n H$.

Given an orthonormal basis for $P_n H$, we can extend it to an orthonormal basis for $P_{n+1} H$. Repeating this process, we can find a sequence $\{h_j\}_{j=1}^\infty$ such that $\{h_1, \ldots, h_{kn}\}$ is an orthonormal basis for $P_n H$. Since $\bigcup P_n H$ is dense in $H$, it follows that the entire sequence $\{h_j\}$ is an orthonormal basis for $H$.

Let $\{X_n\}$ be a sequence of iid standard normal random variables defined on some unrelated probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the $W$-valued random variable

$$S_n = \sum_{j=1}^{k_n} X_j h_j.$$ 

Note that $S_n - S_{n-1}$ has a standard normal distribution on the finite-dimensional Hilbert space $R_n H$, so by definition of $\tilde{\mu}$ we have

$$\mathbb{P}(\|S_n - S_{n-1}\|_W > 2^{-n}) = \tilde{\mu}(\{h \in H : \|R_n h\|_W > 2^{-n}\}) < 2^{-n}.$$ 

Thus $\sum_n \mathbb{P}(\|S_n - S_{n-1}\|_W > 2^{-n}) < \infty$, and by Borel–Cantelli, we have that, $\mathbb{P}$-almost surely, $\|S_n - S_{n-1}\|_W \leq 2^{-n}$ for all but finitely many $n$. In particular, $\mathbb{P}$-a.s., $S_n$ is Cauchy in $W$-norm, and hence convergent to some $W$-valued random variable $S$.

Let $\mu = \mathbb{P} \circ S^{-1}$ be the law of $S$; $\mu$ is a Borel measure on $W$. If $f \in W^*$, then $f(S) = \lim_{n \to \infty} f(S_n) = \lim_{n \to \infty} \sum_{j=1}^{k_n} f(h_j) X_j$ is a limit of Gaussian random variables. Hence by Lemma A.3 $f(S)$ is Gaussian, and moreover we have

$$\infty > \text{Var}(f(S)) = \lim_{n \to \infty} \text{Var}(f(S_n)) = \lim_{n \to \infty} \sum_{j=1}^{k_n} |f(h_j)|^2 = \sum_{j=1}^{\infty} |f(h_j)|^2.$$ 

Pushing forward, we have that $f$ is a Gaussian random variable on $(W, \mu)$ with variance $q(f, f) = \sum_{j=1}^{\infty} |f(h_j)|^2 < \infty$. So $\mu$ is a Gaussian measure and $q$ is its covariance form. Let $H_\mu$ be the Cameron–Martin space associated to $(W, \mu)$. We want to show that $H = H_\mu$ isometrically. This is basically just another diagram chase.
Let \( i \) denote the inclusion map \( i : H \hookrightarrow W \). We know by Lemma 4.43 that \( i \) is 1-1, continuous and has dense range, so its adjoint \( i^* : W^* \to H \) is also 1-1 and continuous with dense range (Exercise C.7). Also, we have

\[
\|i^* f\|^2_H = \sum_{j=1}^{\infty} |\langle i^* f, h_j \rangle_H|^2 = \sum_{j=1}^{\infty} |f(h_j)|^2 = q(f, f)
\]

so that \( i^* : (W^*, q) \to H \) is an isometry.

Next, for any \( h \in H \) and any \( f \in W^* \), Cauchy–Schwarz gives

\[
|f(h)|^2 = |\langle i^* f, h \rangle_H|^2 \leq \|i^* f\|^2_H \|h\|^2_H = q(f, f) \|h\|^2_H
\]

so that \( f \mapsto f(h) \) is a continuous linear functional on \((W^*, q)\). That is, \( h \in H_\mu \), and rearranging and taking the supremum over \( f \) shows \( \|h\|_{H_\mu} \leq \|h\|_H \). On the other hand, if \( f_n \in W^* \) with \( i^* f_n \to h \) in \( H \), we have by definition

\[
\frac{|f_n(h)|}{\sqrt{q(f_n, f_n)}} \leq \|h\|_{H_\mu}.
\]

As \( n \to \infty \), \( f_n(h) = \langle i^* f_n, h \rangle_H \to \|h\|^2_H \), and since \( i^* \) is an isometry, \( q(f_n, f_n) = \|i^* f_n\|^2_H \to \|h\|^2_H \), so the left side tends to \( \|h\|_H \). Thus \( \|h\|_H = \|h\|_{H_\mu} \).

We have shown \( H \subset H_\mu \) isometrically; we want equality. Note that \( H \) is closed in \( H_\mu \), since \( H \) is complete in \( H \)-norm and hence also in \( H_\mu \)-norm. So it suffices to show \( H \) is dense in \( H_\mu \).

Suppose there exists \( q \in H_\mu \) with \( \langle g, h \rangle_{H_\mu} = 0 \) for all \( h \in H \). If \( i_\mu : H_\mu \hookrightarrow W \) is the inclusion map, we know that \( i_\mu^* : (W^*, q) \to H_\mu \) has dense image and is an isometry. So choose \( f_n \in W^* \) with \( i_\mu^* f_n \to g \) in \( H_\mu \). Then \( f_n \) is \( q \)-Cauchy, and so \( i_\mu^* f_n \) converges in \( H \)-norm to some \( k \in H \). But for \( h \in H \),

\[
\langle k, h \rangle_H = \lim \langle i_\mu^* f_n, h \rangle_H = \lim f_n(h) = \lim \langle i_\mu^* f_n, h \rangle_{H_\mu} = \langle g, h \rangle_{H_\mu} = 0
\]

so that \( k = 0 \). Then

\[
\|g\|^2_{H_\mu} = \lim \|i_\mu^* f_n\|^2_{H_\mu} = \lim q(f_n, f_n) = \lim \|i^* f_n\|^2_H = 0
\]

so \( g = 0 \) and we are done. \( \square \)

Here is one way to describe what is going on here. If \( h_j \) is an orthonormal basis for \( H \), then \( S = \sum X_j h_j \) should be a random variable with law \( \mu \). However, this sum diverges in \( H \) almost surely (since \( \sum |X_j|^2 = \infty \) a.s.). So if we want it to converge, we have to choose a weaker norm.

The condition of measurability is not only sufficient but also necessary.

**Theorem 4.45.** Let \((W, \mu)\) be an abstract Wiener space with Cameron–Martin space \( H \). Then \( \|\cdot\|_W \) is a measurable norm on \( H \).

The first proof of this statement, in this generality, seems to have appeared in [1]. For a nice proof due to Daniel Stroock, see Bruce Driver’s notes [2].

**Remark 4.46.** The mere existence of a measurable norm on a given Hilbert space \( H \) is trivial. Indeed, since all infinite-dimensional separable Hilbert spaces are isomorphic, as soon as we have found a measurable norm for one Hilbert space, we have found one for any Hilbert space.
One might wonder if the completion \( W \) has any restrictions on its structure. Equivalently, which separable Banach spaces \( W \) admit Gaussian measures? This is a reasonable question, since Banach spaces can have strange “geometry.” However, the answer is that there are no restrictions.

**Theorem 4.47** (Gross [10, Remark 2]). If \( W \) is any separable Banach space, there exists a separable Hilbert space densely embedded in \( W \), on which the \( W \)-norm is measurable. Equivalently, there exists a non-degenerate Gaussian measure on \( W \).

**Proof.** The finite-dimensional case is trivial, so we suppose \( W \) is infinite dimensional. We start with the case of Hilbert spaces.

First, there exists a separable (infinite-dimensional) Hilbert space \( W \) with a densely embedded separable Hilbert space \( H \) on which the \( W \)-norm is measurable. Proposition 4.59 tells us that, given any separable Hilbert space \( H \), we can construct a measurable norm \( \| \cdot \|_W \) on \( H \) by letting \( \| h \|_W = \| Ah \|_H \), where \( A \) is a Hilbert–Schmidt operator on \( H \). Note that \( \| \cdot \|_W \) is induced by the inner product \( \langle h, k \rangle_W = \langle Ah, Ak \rangle_H \), so if we let \( W \) be the completion of \( H \) under \( \| \cdot \|_W \), then \( W \) is a separable Hilbert space with \( H \) densely embedded. (We should take \( A \) to be injective. An example of such an operator is given by taking an orthonormal basis \( \{ e_n \} \) and letting \( Ae_n = \frac{1}{n} e_n \).)

Now, since all infinite-dimensional separable Hilbert spaces are isomorphic, this shows that the theorem holds for any separable Hilbert space \( W \).

Suppose now that \( W \) is a separable Banach space. By the following lemma, there exists a separable Hilbert space \( H_1 \) densely embedded in \( W \). In turn, there is a separable Hilbert space \( H \) densely embedded in \( H_1 \), on which the \( H_1 \)-norm is measurable. The \( W \)-norm on \( H \) is weaker than the \( H_1 \)-norm, so it is measurable as well. (Exercise: check the details.)

Alternatively, there exists a non-degenerate Gaussian measure \( \mu_1 \) on \( H_1 \). Push it forward under the inclusion map. As an exercise, verify that this gives a non-degenerate Gaussian measure on \( W \).

**Lemma 4.48** (Gross). If \( W \) is a separable Banach space, there exists a separable Hilbert space \( H \) densely embedded in \( W \).

**Proof.** We repeat a construction of Gross [10]. Since \( W \) is separable, we may find a countable set \( \{ z_i \} \subset W \) whose linear span is dense in \( W \); without loss of generality, we can take \( \{ z_n \} \) to be linearly independent. We will construct an inner product \( \langle \cdot, \cdot \rangle_K \) on \( K = \text{span}\{ z_i \} \) such that \( \| x \|_W \leq \| x \|_K \) for all \( x \in K \); thus \( K \) will be an inner product space densely embedded in \( W \).

We inductively construct a sequence \( \{ a_n \} \) such that \( a_i \neq 0 \) for any real numbers \( b_1, \ldots, b_n \) with \( \sum_{i=1}^n b_i^2 \leq 1 \), we have \( \| \sum_{i=1}^n a_i b_i z_i \|_W < 1 \). To begin, choose \( a_1 \) with \( 0 < |a_1| < \| z_1 \|^{-1}_W \). Suppose now that \( a_1, \ldots, a_{n-1} \) have been appropriately chosen. Let \( D^n = \{(b_1, \ldots, b_n) : \sum_{i=1}^n b_i^2 \leq 1 \} \) be the closed Euclidean unit disk of \( \mathbb{R}^n \) and consider the map \( f : D^n \times \mathbb{R} \to W \) defined by

\[
f(b_1, \ldots, b_n, a) = \sum_{i=1}^{n-1} a_i b_i z_i + a b_n z_n.
\]

Now \( f \) is obviously continuous, and by the induction hypothesis we have \( f(D^n \times \{0\}) \subset S \), where \( S \) is the open unit ball of \( W \). So by continuity, \( f^{-1}(S) \) is an open set containing \( D^n \times \{0\} \); hence

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\(^{\ast}\)A particularly bizarre example was given recently in [3]: a separable Banach space \( X \) such that every bounded operator \( T \) on \( X \) is of the form \( T = \lambda I + K \), where \( K \) is a compact operator. In some sense, \( X \) has almost the minimum possible number of bounded operators.
\( f^{-1}(S) \) contains some set of the form \( D^n \times (-\epsilon, \epsilon) \). Thus if we choose any \( a_n \) with \( 0 \leq |a_n| < \epsilon \), we have the desired property for \( a_1, \ldots, a_n \).

Set \( y_i = a_i z_i \); since the \( a_i \) are nonzero, the \( y_i \) span \( K \) and are linearly independent. Let \( \langle \cdot, \cdot \rangle_K \) be the inner product on \( K \) which makes the \( y_i \) orthonormal; then we have \( \| \sum_{i=1}^n b_i y_i \|_K^2 = \sum_{i=1}^n b_i^2 \). By our construction, we have that any \( x \in K \) with \( |x|^2_K \leq 1 \) has \( \|x\|_W < 1 \) as well, so \( K \) is continuously and densely embedded in \( W \). That is to say, the inclusion map \( i : (K, \| \cdot \|_K) \to (W, \| \cdot \|_W) \) is continuous.

Let \( \bar{K} \) be the abstract completion of \( K \), so \( \bar{K} \) is a Hilbert space. Since \( W \) is Banach, the continuous map \( i : K \to W \) extends to a continuous map \( \bar{i} : \bar{K} \to W \) whose image is dense (as it contains \( K \)). It is possible that \( \bar{i} \) is not injective, so let \( H = (\ker \bar{i})^\perp \) be the orthogonal complement in \( \bar{K} \) of its kernel. \( H \) is a closed subspace of \( \bar{K} \), hence a Hilbert space in its own right, and the restriction \( \bar{i}|_H : H \to W \) is continuous and injective, and its range is the same as that of \( \bar{i} \), hence still dense in \( W \).

\[ \square \]

\textbf{Remark 4.49.} The final step of the previous proof (passing to \( (\ker \bar{i})^\perp \)) is missing from Gross’s original proof, as was noticed by Ambar Sengupta, who asked if it is actually necessary. Here is an example to show that it is.

Let \( W \) be a separable Hilbert space with orthonormal basis \( \{e_n\}_{n=1}^\infty \), and let \( S \) be the left shift operator defined by \( Se_1 = 0, Se_n = e_{n-1} \) for \( n \geq 2 \). Note that the kernel of \( S \) is one-dimensional and spanned by \( e_1 \). Let \( E \) be the subspace of \( W \) spanned by the vectors \( h_n = e_n - e_{n+1}, n = 1, 2, \ldots \). It is easy to check that \( E \) is closed in \( W \); for suppose \( x \in E^\perp \). Since \( \langle x, h_n \rangle_W = 0 \), we have \( \langle x, e_n \rangle_W = \langle x, e_{n+1} \rangle_W \), so in fact there is a constant \( c \) with \( \langle x, e_n \rangle_W = c \) for all \( n \). But Parseval’s identity says \( \sum_{n=1}^\infty \|x, e_n\|^2 = \|x\|^2_W < \infty \) so we must have \( c = 0 \) and thus \( x = 0 \).

We also remark that \( SE \) is also dense in \( W \); since \( Sh_n = h_{n-1} \), we actually have \( E \subset SE \).

So we have a separable inner product space \( E \), a separable Hilbert space \( W \), and a continuous injective map \( S|_E : E \to W \) with dense image, such that the continuous extension of \( S|_E \) to the completion of \( E \) (namely \( W \)) is not injective (since the extension is just \( S \) again).

To make this look more like Gross’s construction, we just rename things. Set \( K = SE \) and define an inner product on \( K \) by \( \langle Sx, Sy \rangle_K = \langle x, y \rangle_W \) (this is well defined because \( S \) is injective on \( E \)). Now \( K \) is an inner product space, continuously and densely embedded in \( W \), but the completion of \( K \) does not embed in \( W \) (the continuous extension of the inclusion map is not injective, since it is really \( S \) in disguise).

The inner product space \( (K, \langle \cdot, \cdot \rangle_K) \) could actually be produced by Gross’s construction. By applying the Gram–Schmidt algorithm to \( \{h_n\} \), we get an orthonormal set \( \{s_n\} \) (with respect to \( \langle \cdot, \cdot \rangle_W \)) which still spans \( E \). (In fact, \( \{s_n\} \) is also an orthonormal basis for \( W \).) Take \( z_n = Ss_n \); the \( z_n \)'s are linearly independent and span \( K \), which is dense in \( W \). If \( \sum_{i=1}^n b_i^2 \leq 1 \), then \( \sum_{i=1}^n b_i z_i \) is a contraction and the \( g_i \) are orthonormal. So we can take \( a_i = 1 \) in the induction step.\footnote{Technically, since we were supposed to have \( \sum a_i b_i z_i \) with a strict inequality, we should take \( a_i = c < 1 \), and this argument will actually produce \( c \| \cdot \|_K \) instead of \( \| \cdot \|_K \), which of course makes no difference.}

Then of course the inner product which makes the \( z_n \) orthonormal is just \( \langle \cdot, \cdot \rangle_K \).

We can make the issue even more explicit: consider the series \( \sum_{n=1}^\infty \langle g_n, e_1 \rangle W z_n \). Under \( \| \cdot \|_K \), this series is Cauchy, since \( z_n \) is orthonormal and \( \sum_{n=1}^\infty | \langle g_n, e_1 \rangle W |^2 = \| e_1 \|_W^2 = 1 \); and its limit is not zero, since there must be some \( g_k \) with \( \langle g_k, e_1 \rangle W \neq 0 \), and then we have \( \langle z_k, \sum_{n=1}^m \langle g_n, e_1 \rangle W z_n \rangle = \langle g_k, e_1 \rangle W \) for all \( m \geq k \). So the series corresponds to some nonzero element of the completion \( \bar{K} \).

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However, under $\|\cdot\|_H$, the series converges to zero, since $\sum_{n=1}^{\infty} \langle g_n, e_1 \rangle_{W} z_n = S \sum_{n=1}^{\infty} \langle g_n, e_1 \rangle_{W} g_n \to S e_1 = 0$, using the continuity of $S$ and the fact that $g_n$ is an orthonormal basis for $W$.

The following theorem points out that measurable norms are far from unique.

**Theorem 4.50.** Suppose $\|\cdot\|_W$ is a measurable norm on a Hilbert space $(H, \|\cdot\|_H)$. Then there exists another measurable norm $\|\cdot\|_{W'}$ which is stronger than $\|\cdot\|_W$, and if we write $W, W'$ for the corresponding completions, the inclusions $H \hookrightarrow W' \hookrightarrow W$ are compact.

*Proof.* See [14, Lemma 4.5].

### 4.7 Gaussian measures on Hilbert spaces

We have been discussing Gaussian measures on separable Banach spaces $W$. This includes the possibility that $W$ is a separable Hilbert space. In this case, there is more that can be said about the relationship between $W$ and its Cameron–Martin space $H$.

Let $H, K$ be separable Hilbert spaces.

**Exercise 4.51.** Let $A : H \to K$ be a bounded operator, $A^*$ its adjoint. Let $\{h_n\}, \{k_m\}$ be orthonormal bases for $H, K$ respectively. Then

$$\sum_{n=1}^{\infty} \|Ah_n\|^2_K = \sum_{m=1}^{\infty} \|A^*k_m\|^2_H.$$

**Definition 4.52.** A bounded operator $A : H \to K$ is said to be **Hilbert–Schmidt** if

$$\|A\|^2_{HS} = \sum_{i=1}^{\infty} \|Ae_n\|^2_K < \infty$$

for some orthonormal basis $\{e_n\}$ of $H$. By the previous exercise, this does not depend on the choice of basis, and $\|A\|_{HS} = \|A^*\|_{HS}$.

**Exercise 4.53.** If $\|A\|_{L(H,K)}$ denotes the operator norm of $A$, then $\|A\|_{L(H)} \leq \|A\|_{HS}$.

**Exercise 4.54.** $\|\cdot\|_{HS}$ is induced by the inner product $\langle A, B \rangle_{HS} = \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle_K$ and makes the set of all Hilbert–Schmidt operators from $H$ to $K$ into a Hilbert space.

**Exercise 4.55.** Every Hilbert–Schmidt operator is compact. In particular, Hilbert–Schmidt operators do not have bounded inverses if $H$ is infinite-dimensional. The identity operator is not Hilbert–Schmidt.

**Exercise 4.56.** If $A$ is Hilbert–Schmidt and $B$ is bounded, then $BA$ and $AB$ are Hilbert–Schmidt. So the Hilbert–Schmidt operators form a two-sided ideal inside the ring of bounded operators.

**Exercise 4.57.** If $A$ is a bounded operator on $H$, $H_0$ is a closed subspace of $H$, and $A|_{H_0}$ is the restriction of $A$ to $H_0$, then $\|A|_{H_0}\|_{HS} \leq \|A\|_{HS}$.

**Lemma 4.58.** If $H$ is a finite-dimensional Hilbert space, $A : H \to K$ is linear, and $X$ has a standard normal distribution on $H$, then $E \|AX\|^2_K = \|A\|^2_{HS}$.
Proof. If \( Z \) has a normal distribution on \( \mathbb{R}^n \) with covariance matrix \( \Sigma \), then clearly \( E|Z|^2 = \text{tr} \Sigma \). The covariance matrix of \( AX \) is \( A^*A \), and \( \text{tr}(A^*A) = \|A\|_{HS}^2 \).

**Proposition 4.59.** Let \( A \) be a Hilbert–Schmidt operator on \( H \). Then \( \|h\|_W = \|Ah\|_H \) is a measurable norm on \( H \).

Proof. Fix an orthonormal basis \( \{e_n\} \) for \( H \), and suppose \( \epsilon > 0 \). Since \( \sum_{n=1}^\infty \|A e_n\|^2_H < \infty \), we can choose \( N \) so large that \( \sum_{n=N}^\infty \|A e_n\|^2_H < \epsilon^3 \). Let \( P_0 \) be orthogonal projection onto the span of \( \{e_1, \ldots, e_{N-1}\} \). Note in particular that \( \|A\|_{(P_0 H)^\perp}^2 < \epsilon^3 \). Now suppose \( P \perp P_0 \) is a finite rank projection. Then

\[
\tilde{\mu}(\{h : \|P h\|_W > \epsilon\}) = \tilde{\mu}(\{h : \|A P h\|_H > \epsilon\}) = \mathbb{P}(\|AX\|_H > \epsilon)
\]

where \( X \) has a standard normal distribution on \( PH \). By the previous lemma,

\[
\mathbb{E} \|AX\|_H^2 = \|A|P H\|_{HS}^2 \leq \|A|_{(P_0 H)^\perp}^2 < \epsilon^3
\]

so Chebyshev’s inequality gives \( \mathbb{P}(\|AX\|_H > \epsilon) < \epsilon \) as desired.

Since the norm \( \|h\|_W = \|Ah\|_H \) is induced by an inner product (namely \( \langle h, k \rangle_W = \langle Ah, Ak \rangle_H \)), the completion \( W \) is a Hilbert space. Actually, this is the only way to get \( W \) to be a Hilbert space. Here is a more general result, due to Kuo.

**Theorem 4.60** ([14, Corollary 4.4]). Let \( W \) be a separable Banach space with Gaussian measure \( \mu \) and Cameron–Martin space \( H \), \( i : H \to W \) the inclusion map, and let \( Y \) be some other separable Hilbert space. Suppose \( A : W \to Y \) is a bounded operator. Then \( Ai : H \to Y \) (i.e. the restriction of \( A \) to \( H \subset W \)) is Hilbert–Schmidt, and \( \|Ai\|_H \leq C \|A\|_{L(W,Y)} \) for some constant \( C \) depending only on \((W, \mu)\).

Proof. We consider instead the adjoint \( (Ai)^* = i^*A^* : Y \to H \). Note \( A^* : Y \to W^* \) is bounded, and \( \|i^*A^* y\|_H^2 = q(A^*y, A^*y) \). So if we fix an orthonormal basis \( \{e_n\} \) for \( Y \), we have

\[
\|i^*A^*\|_{HS}^2 = \sum_{n=1}^\infty q(A^*e_n, A^*e_n)
\]

\[
= \sum_{n=1}^\infty \int_W |(A^*e_n)(x)|^2 \mu(dx)
\]

\[
= \int_W \sum_{n=1}^\infty |(A^*e_n)(x)|^2 \mu(dx)
\]

\[
= \int_W \sum_{n=1}^\infty |(A^*e_n)(x)|^2 \mu(dx)
\]

\[
= \int_W \|Ax\|_Y^2 \mu(dx)
\]

\[
\leq \|A\|_{L(W,Y)}^2 \int_W \|x\|_W^2 \mu(dx).
\]

By Fernique’s theorem we are done.
Corollary 4.61. If $W$ is a separable Hilbert space with a Gaussian measure $\mu$ and Cameron–Martin space $H$, then the inclusion $i : H \to W$ is Hilbert–Schmidt, as is the inclusion $m : W^* \to K$.

Proof. Take $Y = W$ and $A = I$ in the above lemma to see that $i$ is Hilbert–Schmidt. To see $m$ is, chase the diagram. \hfill \square

Corollary 4.62. Let $\|\cdot\|_W$ be a norm on a separable Hilbert space $H$. Then the following are equivalent:

1. $\|\cdot\|_W$ is measurable and induced by an inner product $\langle \cdot , \cdot \rangle_W$;
2. $\|h\|_W = \|Ah\|_H$ for some Hermitian, positive definite, Hilbert–Schmidt operator $A$ on $H$.

Proof. Suppose 1 holds. Then by Gross’s theorem (Theorem 4.44) the completion $\tilde{W}$, which is a Hilbert space, admits a Gaussian measure with Cameron–Martin space $H$. Let $i : H \to W$ be the inclusion; by Corollary 4.61 $i$ is Hilbert–Schmidt, and so is its adjoint $i^* : W \to H$. Then $i^* i : H \to H$ is continuous, Hermitian, and positive semidefinite. It is also positive definite because $i$ and $i^*$ are both injective. Take $A = (i^* i)^{1/2}$. $A$ is also continuous, Hermitian, and positive definite, and we have $\|Ah\|_H^2 = \langle i^*ih, h \rangle_H = \langle ih, ih \rangle_W = \|h\|_W^2$. $A$ is also Hilbert–Schmidt since $\sum \|Ae_n\|_H^2 = \sum \|ie_n\|_W^2$ and $i$ is Hilbert–Schmidt.

The converse is Lemma 4.59. \hfill \square

5 Brownian motion on abstract Wiener space

Let $(W, H, \mu)$ be an abstract Wiener space.

Notation 5.1. For $t \geq 0$, let $\mu_t$ be the rescaled measure $\mu_t(A) = \mu(t^{-1/2}A)$ (with $\mu_0 = \delta_0$). It is easy to check that $\mu_t$ is a Gaussian measure on $W$ with covariance form $q_t(f, g) = tq(f, g)$. For short, we could call $\mu_t$ Gaussian measure with variance $t$.

Exercise 5.2. If $W$ is finite dimensional, then $\mu_s \sim \mu_t$ for all $s, t$. If $W$ is infinite dimensional, then $\mu_s \perp \mu_t$ for $s \neq t$.

Lemma 5.3. $\mu_s * \mu_t = \mu_{s+t}$, where $*$ denotes convolution: $\mu * \nu(E) = \int_{W^2} 1_E(x + y) \mu(dx) \nu(dy)$. In other words, $\{\mu_t : t \geq 0\}$ is a convolution semigroup.

Proof. Compute Fourier transforms: if $f \in W^*$, then

$$\hat{\mu_s * \mu_t}(f) = \int_W \int_W e^{if(x+y)} \mu_s(dx) \mu_t(dy)$$

$$= \int_W e^{if(x)} \mu_s(dx) \int_W e^{if(y)} \mu_t(dy)$$

$$= e^{-\frac{1}{2}q_s(f,f)} e^{-\frac{1}{2}tq(f,f)}$$

$$= e^{-\frac{1}{2}(s+t)q(f,f)}$$

$$= \hat{\mu_{s+t}}(f).$$

\hfill \square
Theorem 5.4. There exists a stochastic process \( \{B_t, t \geq 0\} \) with values in \( W \) which is a.s. continuous in \( t \) (with respect to the norm topology on \( W \)), has independent increments, and for \( t > s \) has \( B_t - B_s \sim \mu_{t-s} \), with \( B_0 = 0 \) a.s. \( B_t \) is called standard Brownian motion on \( (W, \mu) \).

Proof. Your favorite proof of the existence of one-dimensional Brownian motion should work. For instance, one can use the Kolmogorov extension theorem to construct a countable set of \( W \)-valued random variables \( \{B_t : t \in E\} \), indexed by the dyadic rationals \( E \), with independent increments and \( B_t - B_s \sim \mu_{t-s} \). (The consistency of the relevant family of measures comes from the property \( \mu_t * \mu_s = \mu_{t+s} \), just as in the one-dimensional case.) If you are worried that you only know the Kolmogorov extension theorem for \( \mathbb{R} \)-valued random variables, you can use the fact that any Polish space can be measurably embedded into \([0, 1]\). Then the Kolmogorov continuity theorem (replacing absolute values with \( \|\cdot\|_W \)) can be used to show that, almost surely, \( B_t \) is Hölder continuous as a function between the metric spaces \( E \) and \( W \). Use the fact that

\[
\mathbb{E}\|B_t - B_s\|_W^2 = \int_W \|x\|_W^2 \mu_{t-s}(dx) = (t-s)^{3/2} \int_W \|x\|_W^2 \mu(dx) \leq C(t-s)^{3/2}
\]

by Fernique. In particular \( B_t \) is, almost surely, uniformly continuous and so extends to a continuous function on \([0, \infty)\). \( \Box \)

Exercise 5.5. For any \( f \in W^* \), \( f(B_t) \) is a one-dimensional Brownian motion with variance \( q(f, f) \). If \( f_1, f_2, \ldots \) are \( q \)-orthogonal, then the Brownian motions \( f_1(B_t), f_2(B_t), \ldots \) are independent.

Question 3. If \( h_j \) is an orthonormal basis for \( H \), and \( B_t^j \) is an iid sequence of one-dimensional standard Brownian motions, does \( \sum_{j=1}^\infty B_t^j h_j \) converge uniformly in \( W \), almost surely, to a Brownian motion on \( W \) ? That would be an even easier construction.

Exercise 5.6. Let \( W' = C([0, 1], W) \) (which is a separable Banach space) and consider the measure \( \mu' \) on \( W' \) induced by \( \{B_t, 0 \leq t \leq 1\} \). Show that \( \mu' \) is a Gaussian measure. For extra credit, find a nice way to write its covariance form.

Exercise 5.7. Suppose \( W = C([0, 1]) \) and \( \mu \) is the law of a one-dimensional continuous Gaussian process \( X \), with covariance function \( a(s_1, s_2) \). Let \( Y_{s,t} = B_t(s) \) be the corresponding two-parameter process (note \( B_t \) is a random element of \( C([0, 1]) \) so \( B_t(s) \) is a random variable). Show \( Y_{s,t} \) is a continuous Gaussian process whose covariance function is

\[
\mathbb{E}[Y_{s_1,t_1}Y_{s_2,t_2}] = (t_1 \wedge t_2)a(s_1, s_2).
\]

If \( X \) is one-dimensional Brownian motion, then \( Y_{s,t} \) is called the Brownian sheet.

\( B_t \) has essentially all the properties you would expect a Brownian motion to have. You can open your favorite textbook on Brownian motion and pick most any theorem that applies to \( d \)-dimensional Brownian motion, and the proof should go through with minimal changes. We note a few important properties here.

Proposition 5.8. \( B_t \) is a Markov process, with transition probabilities \( \mathbb{P}^x(B_t \in A) = \mu_t^x(A) := \mu(t^{-1/2}(A - x)) \).

Proof. The Markov property is immediate, because \( B_t \) has independent increments. Computing the transition probabilities is also very simple. \( \Box \)
**Proposition 5.9.** \(B_t\) is a martingale.

*Proof.* Obvious, because it has independent mean-zero increments. 

**Proposition 5.10.** \(B_t\) obeys the Blumenthal 0-1 law: let \(\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)\) and \(\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s\). Then \(\mathcal{F}_0^+\) is \(\mathbb{P}^x\)-almost trivial, i.e. for any \(A \in \mathcal{F}_0^+\) and any \(x \in W\), \(\mathbb{P}^x(A) = 0\) or 1.

*Proof.* This holds for any continuous Markov process. 

The transition semigroup of \(B_t\) is

\[
P_t F(x) = E_x F(B_t) = \int F \mu_t^x = \int F(x + t^{1/2} y) \mu(dy)
\]

which makes sense for any bounded measurable function. Clearly \(P_t\) is Markovian (positivity-preserving and a contraction with respect to the uniform norm).

**Notation 5.11.** Let \(C_b(W)\) denote the space of bounded continuous functions \(F : W \to \mathbb{R}\). Let \(C_u(W)\) denote the subspace of bounded *uniformly continuous* functions.

**Exercise 5.12.** \(C_b(W)\) and \(C_u(W)\) are Banach spaces.

**Proposition 5.13.** \(P_t\) is a Feller semigroup: if \(F\) is continuous, so is \(P_tF\).

*Proof.* Fix \(x \in W\), and \(\epsilon > 0\). Since \(\mu_t\) is Radon (see Section [B]), there exists a compact \(K\) with \(\mu_t(K^C) < \epsilon\).

For any \(z \in K\), the function \(F(\cdot + z)\) is continuous at \(x\), so there exists \(\delta_z\) such that for any \(u\) with \(\|u\|_W < \delta_z\), we have \(|F(x + z) - F(x + u + z)| < \epsilon\). The balls \(B(z, \delta_z/2)\) cover \(K\) so we can take a finite subcover \(B(z_i, \delta_i/2)\). Now suppose \(\|x - y\|_W < \min \delta_i/2\). For any \(z \in K\), we may choose \(z_i\) with \(z \in B(z_i, \delta_i/2)\). We then have

\[
|F(x + z) - F(y + z)| \leq |F(x + z) - F(x + z_i)| + |F(x + z_i) - F(y + z)|
\]

\[
= |F((x + (z - z_i)) + z_i) - F(x + z_i)|
\]

\[
+ |F(x + z_i) - F((x + (y - x) + (z - z_i)) + z_i)|
\]

Each term is of the form \(|F(x + z_i) - F(x + u + z_i)|\) for some \(u\) with \(\|u\|_W < \delta_i\), and hence is bounded by \(\epsilon\).

Now we have

\[
|P_tF(x) - P_tF(y)| \leq \int_W |F(x + z) - F(y + z)| \mu_t(dz)
\]

\[
= \int_K |F(x + z) - F(y + z)| \mu_t(dz) + \int_{K^C} |F(x + z) - F(y + z)| \mu_t(dz)
\]

\[
\leq \epsilon + 2\epsilon \|F\|_\infty.
\]
Remark 5.14. This shows that $P_t$ is a contraction semigroup on $C_b(W)$. We would really like to have a strongly continuous contraction semigroup. However, $P_t$ is not in general strongly continuous on $C_b(W)$. Indeed, take the one-dimensional case $W = \mathbb{R}$, and let $f(x) = \cos(x^2)$, so that $f$ is continuous and bounded but not uniformly continuous. One can check that $P_t f$ vanishes at infinity, for any $t$. (For instance, take Fourier transforms, so the convolution in $P_t f$ becomes multiplication. The Fourier transform $\hat{f}$ is just a scaled and shifted version of $f$; in particular it is bounded, so $P_t \hat{f}$ is integrable. Then the Riemann–Lebesgue lemma implies that $P_t f \in C_0(\mathbb{R})$. One can also compute directly, perhaps by writing $f$ as the real part of $e^{ix^2}$.) Thus if $P_t f$ were to converge uniformly as $t \to 0$, the limit would also vanish at infinity, and so could not be $f$. (In fact, $P_t f \to f$ pointwise as $t \to 0$, so $P_t f$ does not converge uniformly.)

Remark 5.15. One should note that the term “Feller semigroup” has several different and incompatible definitions in the literature, so caution is required when invoking results from other sources. One other common definition assumes the state space $X$ is locally compact, and requires that $P_t$ be a strongly continuous contraction semigroup on $C_0(X)$, the space of continuous functions “vanishing at infinity”, i.e. the uniform closure of the continuous functions with compact support. In our non-locally-compact setting this condition is meaningless, since $C_0(W) = 0$.

Proposition 5.16. $B_t$ has the strong Markov property.

Proof. This should hold for any Feller process. The proof in Durrett looks like it would work. $\square$

Theorem 5.17. $P_t$ is a strongly continuous contraction semigroup on $C_u(W)$.

Proof. Let $F \in C_u(W)$. We first check that $P_t F \in C_u(W)$. It is clear that $P_t F$ is bounded; indeed, $\|P_t F\|_\infty \leq \|F\|_\infty$. Fix $\epsilon > 0$. There exists $\delta > 0$ such that $|F(x) - F(y)| < \epsilon$ whenever $\|x - y\|_W < \delta$. For such $x, y$ we have

$$|P_t F(x) - P_t F(y)| \leq \int |F(x + z) - F(y + z)| \mu_t(dz) \leq \epsilon.$$ 

Thus $P_t F$ is uniformly continuous.

Next, we have

$$|P_t F(x) - F(x)| \leq \int |F(x + t^{1/2} y) - F(x)| \mu(dy)$$

$$= \int_{\|t^{1/2} y\|_W < \delta} |F(x + t^{1/2} y) - F(x)| \mu(dy) + \int_{\|t^{1/2} y\|_W \geq \delta} |F(x + t^{1/2} y) - F(x)| \mu(dy)$$

$$\leq \epsilon + 2 \|F\|_\infty \mu(\{y : \|t^{1/2} y\|_W \geq \delta\}).$$

But $\mu(\{y : \|t^{1/2} y\|_W \geq \delta\}) = \mu(B(0, t^{-1/2} C)) \to 0$ as $t \to 0$. So for small enough $t$ we have $|P_t F(x) - F(x)| \leq 2\epsilon$ independent of $x$. $\square$

Remark 5.18. $C_u(X)$ is not the nicest Banach space to work with here; in particular it is not separable, and its dual space is hard to describe. However, the usual nice choices that work in finite dimensions don’t help us here. In $\mathbb{R}^n$, $P_t$ is a strongly continuous semigroup on $C_0(\mathbb{R}^n)$; but in infinite dimensions $C_0(W) = 0$.

In $\mathbb{R}^n$, $P_t$ is also a strongly continuous symmetric semigroup on $L^2(\mathbb{R}^n, m)$, where $m$ is Lebesgue measure. In infinite dimensions we don’t have Lebesgue measure, but we might wonder whether $\mu$
could stand in: is \( P_t \) a reasonable semigroup on \( L^2(W, \mu) \)? The answer is emphatically no; it is not even a well-defined operator. First note that \( P_1 = 1 \) for all \( t \). Let \( e_i \in W^* \) be \( q \)-orthonormal, so that under \( \mu \) the \( e_i \) are iid \( N(0,1) \). Under \( \mu_t \) they are iid \( N(0,t) \). Set \( s_n(x) = \frac{1}{n} \sum_{i=1}^n |e_i(x)|^2 \); by the strong law of large numbers, \( s_n \to \mu_t \), \( \mu_t \)-a.e. Let \( A = \{ x : s_n(x) \to 1 \} \), so \( 1_A = 1 \) \( \mu_t \)-a.e. On the other hand, for any \( t > 0 \),

\[
\int_W P_1 A(x) \mu(dx) = \int_W \int_W 1_A(x+y) \mu_t(dy) \mu(dx) = (\mu_t * \mu)(A) = \mu_{1+t}(A) = 0.
\]

Since \( P_1 A \geq 0 \), it must be that \( P_1 A = 0 \), \( \mu_t \)-a.e. Thus \( 1 \) and \( 1_A \) are the same element of \( L^2(W, \mu) \), but \( P_1 \) and \( P_1 A \) are not.

We knew that \( H \) was “thin” in \( W \) in the sense that \( \mu(H) = 0 \). In particular, this means that for any \( t \), \( \mathbb{P}(B_t \in H) = 0 \). Actually more is true.

**Proposition 5.19.** Let \( \sigma_H = \inf \{ t > 0 : B_t \in H \} \). Then for any \( x \in W \), \( \mathbb{P}^x(\sigma_H = \infty) = 1 \). That is, from any starting point, with probability one, \( B_t \) never hits \( H \). In other words, \( H \) is polar for \( B_t \).

**Proof.** Fix \( 0 < t_1 < t_2 < \infty \). If \( b_t \) is a standard one-dimensional Brownian motion started at \( 0 \), let \( c = P(\inf \{ b_t : t_1 \leq t \leq t_2 \} \geq 1) \), i.e. the probability that \( b_t \) is above \( 1 \) for all times between \( t_1 \) and \( t_2 \). Clearly \( c > 0 \). By the strong Markov property it is clear that for \( x_0 > 0 \), \( P(\inf \{ b_t + x_0 : t_1 \leq t \leq t_2 \} \geq 1) \) \( > c \), and by symmetry \( P(\sup \{ b_t - x_0 : t_1 \leq t \leq t_2 \} \leq -1) \) \( > c \) also. So for any \( x_0 \in \mathbb{R} \), we have

\[
P(\inf \{ |b_t + x_0|^2 : t_1 \leq t \leq t_2 \} \geq 1) > c.
\]

Fix a \( q \)-orthonormal basis \( \{ e_k \} \subset W^* \) as in Lemma 4.21 so that \( B_t \in H \) iff \( \| B_t \|_H^2 = \sum_k |e_k(B_t)|^2 < \infty \). Under \( \mathbb{P}^x \), \( e_k(B_t) \) are independent one-dimensional Brownian motions with variance \( 1 \) and starting points \( e_k(x) \). So if we let \( A_k \) be the event \( A_k = \{ \inf \{ |e_k(B_t)|^2 : t_1 \leq t \leq t_2 \} \geq 1 \} \), by the above computation we have \( \mathbb{P}^x(A_k) > c \). Since the \( A_k \) are independent we have \( \mathbb{P}^x(\bigcap_{k \in \mathbb{N}} A_k) = 1 \). But on the event \( \{ A_k \} \) we have \( \| B_t \|_H^2 = \sum_{k=1}^\infty |e_k(B_t)|^2 = \infty \) for all \( t \in [t_1, t_2] \). Thus \( \mathbb{P}^x \)-a.s. \( B_t \) does not hit \( H \) between times \( t_1 \) and \( t_2 \). Now let \( t_1 \downarrow 0 \) and \( t_2 \uparrow \infty \) along sequences to get the conclusion. \( \square \)

**Remark 5.20.** A priori it is not obvious that \( \sigma_H : \Omega \to [0, \infty) \) is even measurable (its measurability it is defined by an uncountable infimum, and there is no apparent way to reduce it to a countable infimum), or that \( \{ \sigma_H = \infty \} \) is a measurable subset of \( \Omega \). What we really showed is that there is a (measurable) event \( A = \{ A_k \text{ i.o.} \} \) with \( \mathbb{P}^x(A) = 1 \) and \( \sigma_H = \infty \) on \( A \). If we complete the measurable space \( (\Omega, \mathcal{F}) \) by throwing in all the sets which are \( \mathbb{P}^x \)-null for every \( x \), then \( \{ \sigma_H = \infty \} \) will be measurable and so will \( \sigma_H \).

In the general theory of Markov processes one shows that under some mild assumptions, including the above completion technique, \( \sigma_B \) is indeed measurable for any Borel (or even analytic) set \( B \).

### 6 Calculus on abstract Wiener space

The strongly continuous semigroup \( P_t \) on \( C_u(W) \) has a generator \( L \), defined by

\[
Lf = \lim_{t \downarrow 0} \frac{1}{t} (P_t f - f).
\]

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This is an unbounded operator on $C_u(W)$ whose domain $D(L)$ is the set of all $f$ for which the limit converges in $C_u(W)$. It is a general fact that $L$ is densely defined and closed.

In the classical setting where $W = \mathbb{R}^n$ and $\mu$ is standard Gaussian measure (i.e. $q = \langle \cdot , \cdot \rangle_{\mathbb{R}^n}$), so that $B_t$ is standard Brownian motion, we know that $L = -\frac{1}{2} \Delta$ is the Laplace operator, which sums the second partial derivatives in all orthogonal directions. Note that “orthogonal” is with respect to the Euclidean inner product, which is also the Cameron–Martin inner product in this case.

We should expect that in the setting of abstract Wiener space, $L$ should again be a second-order differential operator that should play the same role as the Laplacian. So we need to investigate differentiation on $W$.

**Definition 6.1.** Let $W$ be a Banach space, and $F : W \to \mathbb{R}$ a function. We say $F$ is **Fréchet differentiable** at $x \in W$ if there exists $g_x \in W^*$ such that, for any sequence $W \ni y_n \to 0$ in $W$-norm,

$$\frac{F(x + y_n) - F(x) - g_x(y_n)}{\|y_n\|_W} \to 0.$$  

$g_x$ is the Fréchet derivative of $F$ at $x$. One could write $F'(x) = g_x$. It may be helpful to think in terms of directional derivatives and write $\partial_y F(x) = F'(x)y = g_x(y)$.

**Example 6.2.** Suppose $F(x) = \phi(f_1(x), \ldots, f_n(x))$ is a cylinder function. Then

$$F'(x) = \sum_{i=1}^n \partial_i \phi(f_1(x), \ldots, f_n(x)) f_i.$$  

As it turns out, we will be most interested in differentiating in directions $h \in H$, since in some sense that is really what the usual Laplacian does. Also, Fréchet differentiability seems to be too much to ask for; according to references in Kuo \cite{14}, the set of continuously Fréchet differentiable functions is not dense in $C_u(W)$.

**Definition 6.3.** $F : W \to \mathbb{R}$ is **$H$-differentiable** at $x \in W$ if there exists $g_x \in H$ such that for any sequence $H \ni h_n \to 0$ in $H$-norm,

$$\frac{F(x + h_n) - F(x) - \langle g_x, h_n \rangle H}{\|h_n\|_H} \to 0.$$  

We will denote the element $g_x$ by $DF(x)$, and we have $\partial_h F(x) = \langle DF(x), h \rangle_H$. $DF : W \to H$ is sometimes called the **Malliavin derivative** or **gradient** of $F$.

**Example 6.4.** For a cylinder function $F(x) = \phi(f_1(x), \ldots, f_n(x))$, we have

$$\langle DF(x), h \rangle_H = \sum_{i=1}^n \partial_i \phi(f_1(x), \ldots, f_n(x)) f_i(h)$$  

or alternatively

$$DF(x) = \sum_{i=1}^n \partial_i \phi(f_1(x), \ldots, f_n(x)) J f_i \quad (6.1)$$

We know that for $F \in C_u(W)$, $P_t F$ should belong to the domain of the generator $L$, for any $t > 0$. The next proposition shows that we are on the right track with $H$-differentiability.
Proposition 6.5. For $F \in C_a(W)$ and $t > 0$, $P_tF$ is $H$-differentiable, and
\[
\langle DP_tF(x), h \rangle_H = \frac{1}{t} \int_W F(x + y) \langle h, y \rangle_H \mu_t(dy).
\]

Proof. It is sufficient to show that
\[
P_tF(x + h) - P_tF(x) = \frac{1}{t} \int_W F(x + y) \langle h, y \rangle_H \mu_t(dy) + o(\|h\|_H)
\]
since the first term on the right side is a bounded linear functional of $h$ (by Fernique’s theorem).

The Cameron–Martin theorem gives us
\[
P_tF(x + h) = \int_W F(x + y) \mu_t(dy) = \int_W F(x + y) J_t(h, y) \mu_t(dy)
\]
where
\[
J_t(h, y) = \exp \left( -\frac{1}{2t} \|h\|_H^2 + \frac{1}{t} \langle h, y \rangle_H \right)
\]
is the Radon–Nikodym derivative, or in other words the “Jacobian determinant.” Then we have
\[
P_tF(x + h) - P_tF(x) = \int_W F(x + y) (J_t(h, y) - 1) \mu_t(dy).
\]

Since $J_t(0, y) = 1$, we can write $J_t(h, y) - 1 = \int_0^1 \frac{d}{ds} J_t(sh, y) \, ds$ by the fundamental theorem of calculus. Now we can easily compute that $\frac{d}{ds} J_t(sh, y) = \frac{1}{t} (\langle h, y \rangle_H - s \|h\|_H^2) J_t(sh, y)$, so we have
\[
P_tF(x + h) - P_tF(x) = \frac{1}{t} \int_W F(x + y) \int_0^1 \langle (h, y)_H - s \|h\|_H^2 \rangle J_t(sh, y) \, ds \, \mu_t(dy)
\]
\[
= \frac{1}{t} \int_W F(x + y) (h, y)_H \mu_t(dy)
\]
\[
+ \frac{1}{t} \int_W F(x + y) \int_0^1 \langle (h, y)_H \rangle J_t(sh, y) \, ds \, \mu_t(dy)
\]
\[
+ \frac{1}{t} \int_W F(x + y) \int_0^1 s \|h\|_H^2 J_t(sh, y) \, ds \, \mu_t(dy)
\]
\[
(\alpha)
\]
\[
(\beta).
\]

So it remains to show that the remainder terms $\alpha, \beta$ are $o(\|h\|_H)$.

To estimate $\alpha$, we crash through with absolute values and use Tonelli’s theorem and Cauchy–Schwarz to obtain
\[
|\alpha| \leq \frac{\|F\|_\infty}{t} \int_0^1 \int_W |\langle h, y \rangle_H| |J_t(sh, y) - 1| \, \mu_t(dy) \, ds
\]
\[
\leq \frac{\|F\|_\infty}{t} \int_0^1 \sqrt{\int_W |\langle h, y \rangle_H|^2 \, \mu_t(dy)} \int_W |J_t(sh, y) - 1|^2 \, \mu_t(dy) \, ds.
\]

But $\int_W |\langle h, y \rangle_H|^2 \, \mu_t(dy) = t \|h\|_H^2$ (since under $\mu_t$, $\langle h, \cdot \rangle_H \sim N(0, t \|h\|_H^2)$). Thus
\[
|\alpha| \leq \frac{\|F\|_\infty}{\sqrt{t}} \|h\|_H \int_0^1 \sqrt{\int_W |J_t(sh, y) - 1|^2 \, \mu_t(dy)} \, ds.
\]
Now, a quick computation shows

$$|J_t(sh, y) - 1|^2 = e^{s^2\|h\|^2_H/t} J_t(2sh, y) - 2J_t(sh, y) + 1.$$  

But $J_t(g, y)\mu_t(dy) = \mu_t^g(dy)$ is a probability measure for any $g \in H$, so integrating with respect to $\mu_t(dy)$ gives

$$\int_W |J_t(sh, y) - 1|^2 \mu_t(dy) = e^{s^2\|h\|^2_H/t} - 1.$$  

So we have

$$\int_0^1 \sqrt{\int_W |J_t(sh, y) - 1|^2 \mu_t(dy)} ds = \int_0^1 \sqrt{e^{s^2\|h\|^2_H/t}} - 1 ds = o(1)$$  

as $\|h\|_H \to 0$, by dominated convergence. Thus we have shown $\alpha = o(\|h\|_H)$.

The $\beta$ term is easier: crashing through with absolute values and using Tonelli’s theorem (and the fact that $J_t \geq 0$), we have

$$|\beta| \leq \frac{1}{t} \|F\|_\infty \|h\|_H^2 \int_0^1 s \int_W J_t(sh, y) \mu_t(dy) ds$$

$$= \frac{1}{t} \|F\|_\infty \|h\|_H^2 \int_0^1 s \int_W \mu_t^g(dy) ds$$

$$= \frac{1}{2t} \|F\|_\infty \|h\|_H^2 = o(\|h\|_H).$$

With more work it can be shown that $P_tF$ is in fact infinitely $H$-differentiable.

**Question 4.** Kuo claims that the second derivative of $P_tF$ is given by

$$\langle D^2 P_tFh, k \rangle_H = \frac{1}{t} \int \int W F(x + y) \left( \frac{\langle h, y \rangle_H \langle k, y \rangle_H}{t} - \langle h, k \rangle \right) \mu_t(dy).$$

If the generator $L$ is really the Laplacian $\Delta$ defined below, then in particular $D^2 P_tF$ should be trace class. But this doesn’t seem to be obvious from this formula. In particular, if we let $h = k = h_n$ and sum over an orthonormal basis $h_n$, the obvious approach of interchanging the integral and sum doesn’t work, because we get an integrand of the form $\sum_n (\xi_n^2 - 1)$ where $\xi_n$ are iid $N(0, 1)$, which diverges almost surely.

**Corollary 6.6.** The (infinitely) $H$-differentiable functions are dense in $C_u(W)$.

**Proof.** $P_tF$ is a strongly continuous semigroup on $C_u(W)$, so for any $F \in C_u(W)$ and any sequence $t_n \downarrow 0$, we have $P_{t_n}F \to F$ uniformly, and we just showed that $P_{t_n}F$ is $H$-differentiable.

Now, on the premise that the $H$ inner product should play the same role as the Euclidean inner product on $\mathbb{R}^n$, we define the Laplacian as follows.
Definition 6.7. The Laplacian of a function $F : W \rightarrow \mathbb{R}$ is

$$\Delta F(x) = \sum_{k=1}^{\infty} \partial_{h_k} \partial_{h_k} F(x)$$

if it exists, where $\{h_k\}$ is an orthonormal basis for $H$. (This assumes that $F$ is $H$-differentiable, as well as each $\partial_{h} F$.)

Example 6.8. If $F : W \rightarrow \mathbb{R}$ is a cylinder function as above, then

$$\Delta F(x) = \sum_{i,j=1}^{n} \partial_{i} \partial_{j} \phi(f_1(x), \ldots, f_n(x))q(f_i, f_j).$$

Theorem 6.9. If $F$ is a cylinder function, then $F \in D(L)$, and $LF = -\frac{1}{2} \Delta F$.

Proof. We have to show that

$$\frac{p_t F(x) - F(x)}{t} \rightarrow \frac{1}{2} \Delta F(x)$$

uniformly in $x \in W$. As shorthand, write

$$G_i(x) = \partial_{i} \phi(f_1(x), \ldots, f_n(x))$$
$$G_{ij}(x) = \partial_{i} \partial_{j} \phi(f_1(x), \ldots, f_n(x))$$

so that $\Delta F(x) = \sum_{i,j=1}^{n} G_{ij}(x)q(f_i, f_j)$. Note that $G_{ij}$ is Lipschitz.

First note the following identity for $\alpha \in C^2([0,1])$, which is easily checked by integration by parts:

$$\alpha(1) - \alpha(0) = \alpha'(0) + \int_{0}^{1} (1-s)\alpha''(s) \, ds \quad (6.2)$$

Using $\alpha(s) = F(x + sy)$, we have

$$F(x + y) - F(x) = \sum_{i=1}^{n} G_i(x)f_i(y) + \int_{0}^{1} (1-s) \sum_{i,j=1}^{n} G_{ij}(x + sy)f_i(y)f_j(y) \, ds.$$

Integrating with respect to $\mu_t(dy)$, we have

$$P_t F(x) - F(x) = \sum_{i=1}^{n} G_i(x) \int_{W} f_i(y) \mu_t(dy) + \sum_{i,j=1}^{n} \int_{W} f_i(y)f_j(y) \int_{0}^{1} (1-s)G_{ij}(x + sy) \, ds \, \mu_t(dy)$$

$$= t \sum_{i,j=1}^{n} \int_{W} f_i(y)f_j(y) \int_{0}^{1} (1-s)G_{ij}(x + st^{1/2}y) \, ds \, \mu(dy)$$

rewriting the $\mu_t$ integral in terms of $\mu$ and using the linearity of $f_i, f_j$. Now if we add and subtract $G_{ij}(x)$ from $G_{ij}(x + st^{1/2}y)$, we have

$$\int_{W} f_i(y)f_j(y)G_{ij}(x) \, \mu(dy) \int_{0}^{1} (1-s) \, ds = \frac{1}{2} G_{ij}(x) \int_{W} f_i(y)f_j(y) \mu(dy) = \frac{1}{2} G_{ij}(x)q(f_i, f_j)$$
and, if $C$ is the Lipschitz constant of $G_{ij}$,

$$\left| \int_W f_i(y) f_j(y) \int_0^1 (1-s)(G_{ij}(x + st^{1/2}y) - G_{ij}(x)) \, ds \, dy \right|$$

$$\leq \|f_i\|_{W^*} \|f_j\|_{W^*} \int_W \|y\|^2 \int_0^1 (1-s)(Cst^{1/2} \|y\|_W) \, ds \, dy$$

$$\leq Ct^{1/2} \|f_i\|_{W^*} \|f_j\|_{W^*} \int_0^1 (s - s^2) \, ds \int_0^1 \|y\|^3 \, dy.$$

The $\mu$ integral is finite by Fernique’s theorem, so this goes to 0 as $t \to 0$, independent of $x$. Thus we have shown

$$P_t F(x) - F(x) = \frac{t}{2} \left( \sum_{i,j=1}^n G_{ij}(x)q(f_i, f_j) + o(1) \right)$$

uniformly in $x$, which is what we wanted. \qed

**Question 5.** Kuo [14] proves the stronger statement that this holds for $F$ which are (more or less) Fréchet-$C^2$. Even this is not quite satisfactory, because, as claimed by Kuo’s references, these functions are not dense in $C_u(W)$. In particular, they are not a core for $L$. Can we produce a Laplacian-like formula for $L$ which makes sense and holds on a core of $L$?

### 6.1 Some $L^p$ theory

Here we will follow Nualart [16] for a while.

It is worth mentioning that Nualart’s approach (and notation) are a bit different from ours. His setting is a probability space $(\Omega, \mathcal{F}, P)$ and a “process” $W$, i.e. a family $\{W(h) : h \in H\}$ of jointly Gaussian random variables indexed by a Hilbert space $H$, with the property that $E[W(h)W(k)] = \langle h, k \rangle_H$. This includes our setting: take an abstract Wiener space $(B, H, \mu)$, set $\Omega = B$, $P = \mu$, and $W(h) = \langle h, \cdot \rangle_H$. We will stick to our notation.

We want to study the properties of the Malliavin derivative $D$ as an unbounded operator on $L^p(W, \mu)$, $p \geq 1$. If we take the domain of $D$ to be the smooth cylinder functions $\mathcal{F}C_c^\infty(W)$, we have a densely defined unbounded operator from $L^p(W, \mu)$ into the vector-valued space $L^p(W, \mu; H)$. (Note that $\|DF(x)\|_H$ is bounded as a function of $x$, so there is no question that $DF \in L^p(W; H)$.)

**Lemma 6.10** (Integration by parts). Let $F \in \mathcal{F}C_c^\infty(W)$ be a cylinder function, and $h \in H$. Then

$$\int_W \langle DF(x), h \rangle_H \, dx = \int_W F(x) \langle h, x \rangle_H \, dx. \tag{6.3}$$

**Proof.** It is easy to see that both sides of (6.3) are bounded linear functionals with respect to $h$, so it suffices to show that (6.3) holds for all $h$ in a dense subset: hence suppose $h = i^* f$ for some $f \in W^*$.

Now basically the proof is to reduce to the finite dimensional case. By adjusting $\phi$ as needed, there is no loss of generality in writing $F(x) = \phi(f_1(x), \ldots, f_n(x))$ where $f_1 = f$. We can also apply Gram–Schmidt and assume that all the $f_i$ are $q$-orthonormal. Then $\langle DF(x), h \rangle_H = \int_W \phi(ty) \, dy$, which is finite by the Paley–Wiener theorem. Thus we have shown (6.3) holds for all $(f_i) \in H^n$. The general case now follows by linearity, as desired.
\[ \partial \phi(f_1(x), \ldots, f_n(x)) \]. The \( f_i \) are i.i.d \( N(0, 1) \) random variables under \( \mu \), so we have

\[
\int_W \langle DF(x), h \rangle_H \, \mu(dx) = \int_{\mathbb{R}^n} \partial \phi(x_1, \ldots, x_n) \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} \, dx
\]

\[
= \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) x_1 \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} \, dx
\]

\[
= \int_W \phi(f_1(x), \ldots, f_n(x)) f_1(x) \, \mu(dx)
\]

\[
= \int_W F(x) \langle h, x \rangle_H \, \mu(dx)
\]

so we are done. \( \square \)

We also observe that \( D \) obeys the product rule: \( D(F \cdot G) = F \cdot DG + DF \cdot G \). (This is easy to check by assuming, without loss of generality, that we have written \( F, G \) in terms of the same functionals \( f_1, \ldots, f_n \in W^* \).) Applying the previous lemma to this identity gives:

\[
\int_W G(x) \langle DF(x), h \rangle_H \, \mu(dx) = \int_W (-F(x) \langle DG(x), h \rangle_H + F(x) G(x) \langle h, x \rangle_H) \mu(dx).
\]  

(6.4)

We can use this to prove:

**Proposition 6.11.** The operator \( D : L^p(W, \mu) \to L^p(W, \mu; H) \) is closable.

**Proof.** Suppose that \( F_n \in \mathcal{F}C_c^{\infty}(W) \) are converging to 0 in \( L^p(W, \mu) \), and that \( DF_n \to \eta \) in \( L^p(W; H) \). We have to show \( \eta = 0 \). It is sufficient to show that \( \int_W \langle \eta(x), h \rangle G(x) \, m(u(dx)) = 0 \) for all \( h \in H \) and \( G \in \mathcal{F}C_c^{\infty}(W) \). (Why?) Now applying (6.4) we have

\[
\int_W G(x) \langle DF_n(x), h \rangle_H \, \mu(dx) = -\int_W F_n(x) \langle DG(x), h \rangle_H + \int_W F_n(x) G(x) \langle h, x \rangle_H \, \mu(dx).
\]

As \( n \to \infty \), the left side goes to \( \int_W G(x) \langle \eta(x), h \rangle_H \, \mu(dx) \). The first term on the right side goes to 0 (since \( DG \in L^p(W; H) \) and hence \( \langle DG, h \rangle \in L^p(W) \)) as does the second term (since \( G \) is bounded and \( \langle h, \cdot \rangle \in L^q(W) \) because it is a Gaussian random variable). \( \square \)

Now we can define the Sobolev space \( \mathbb{D}^{1,p} \) as the completion of \( \mathcal{F}C_c^{\infty}(W) \) under the norm

\[
\|F\|_{\mathbb{D}^{1,p}}^p = \int_W |F(x)|^p + \|DF\|_H^p \, \mu(dx).
\]

Since \( D \) was closable, \( \mathbb{D}^{1,p} \subset L^p(W, \mu) \). We can also iterate this process to define higher derivatives \( D^k \) and higher order Sobolev spaces.

**Lemma 6.12.** If \( \phi \in C^\infty(\mathbb{R}^n) \) and \( \phi \) and its first partials have polynomial growth, then \( F(x) = \phi(f_1(x), \ldots, f_n(x)) \in \mathbb{D}^{1,p} \).

**Proof.** Cutoff functions. \( \square \)

**Lemma 6.13.** The set of functions \( F(x) = p(f_1(x), \ldots, f_n(x)) \) where \( p \) is a polynomial in \( n \) variables and \( f_i \in W^* \), is dense in \( L^p(W, \mu) \).
Proof. See [5, Theorem 39.8].

Let $H_n(s)$ be the $n$'th Hermite polynomial defined by

$$H_n(s) = \frac{(-1)^n}{n!} e^{s^2/2} \frac{d^n}{ds^n} e^{-s^2/2}.$$ 

$H_n$ is a polynomial of degree $n$. Fact: $H'_n = H_{n-1}$, and $(n+1)H_{n+1}(s) = sH_n(s) - H_{n-1}(s)$. In particular, $H_n$ is an eigenfunction of the one-dimensional Ornstein–Uhlenbeck operator $Af = f'' - xf'$ with eigenvalue $n$.

Also, we have the property that if $X, Y$ are jointly Gaussian with variance 1, then

$$E[H_n(X)H_m(Y)] = \begin{cases} 0, & n \neq m \\ \frac{1}{m} E[XY]^n, & n = m. \end{cases}$$

(See Nualart for the proof, it’s simple.) This implies:

**Proposition 6.14.** If $\{e_i\} \subset W$ is a $q$-orthonormal basis, then the functions

$$F_{n_1, \ldots, n_k}(x) = \prod_i \sqrt{n_i!} H_{n_i}(e_i(x))$$

are an orthonormal basis for $L^2(W, \mu)$.

If $\mathcal{H}_n$ is the closed span of all $F_{n_1, \ldots, n_k}$ with $n_1 + \cdots + n_k = n$ (i.e. multivariable Hermite polynomials of degree $n$) then we have an orthogonal decomposition of $L^2$. Let $J_n$ be orthogonal projection onto $\mathcal{H}_n$. This decomposition is called Wiener chaos. Note $\mathcal{H}_0$ is the constants, and $\mathcal{H}_1 = K$. For $n \geq 2$, the random variables in $\mathcal{H}_n$ are not normally distributed.

We can decompose $L^2(W; H)$ in a similar way: if $h_j$ is an orthonormal basis for $H$, then $\{F_{n_1, \ldots, n_k}h_j\}$ is an orthonormal basis for $L^2(W; H)$, and if $\mathcal{H}_n(H)$ is the closed span of functions of the form $Fh$ where $F \in \mathcal{H}_n$, $h \in H$, then $L^2(W; H) = \bigoplus \mathcal{H}_n(H)$ is an orthogonal decomposition, and we again use $J_n$ to denote the orthogonal projections. We could do the same for $L^2(W; H \otimes m)$.

Note that

$$DF_{n_1, \ldots, n_k}(x) = \sum_{j=1}^k \sqrt{n_j} H_{n_j-1}(e_j(x)) \prod_{i \neq j} \sqrt{n_i!} H_{n_i}(e_i(x)) Je_j$$

$$= \sum_{j=1}^k \sqrt{n_j} F_{n_1, \ldots, n_j-1, \ldots, n_k}(x) Je_j.$$ 

We can see from this that $\|DF_{n_1, \ldots, n_k}\|_{L^2(W; H)}^2 = n_1 + \cdots + n_k$, or for short, $\|DF_{\alpha}\|_{L^2(W; H)}^2 = |\alpha|$. Also, if $\alpha \neq \beta$, $\langle DF_{\alpha}, DF_{\beta}\rangle_{L^2(W; H)} = 0$.

**Lemma 6.15.** For each $n$, $\mathcal{H}_n \subset \mathbb{D}^{1,2}$. Moreover, $\{F_\alpha : |\alpha| = n\}$ are an orthonormal basis for $\mathcal{H}_n$ with respect to the $\mathbb{D}^{1,2}$ inner product, and $\|F_\alpha\|^2 = 1 + n$. 

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Proof. Since \( \{F_\alpha : |\alpha| = n\} \) are an orthonormal basis for \( \mathcal{H}_n \), we can write \( F = \sum_{i=1}^{\infty} a_i F_{\alpha_i} \) where \( \sum a_i^2 < \infty \) and the \( x \alpha_i \) are distinct. Let \( F_m = \sum_{i=1}^{m} a_i F_{\alpha_i} \), so that \( F_m \to F \) in \( L^2(W) \). Clearly \( F_m \in \mathbb{D}^{1,2} \) and \( DF_m = \sum_{i=1}^{m} a_i DF_{\alpha_i} \). Now we have \( \langle DF_{\alpha_i}, DF_{\alpha_j} \rangle_{L^2(W; H)} = n \delta_{ij} \), so for \( k \leq m \) we have

\[
\|DF_m - DF_k\|^2_{L^2(W; H)} = \left\| \sum_{i=k}^{m} a_i DF_{\alpha_i} \right\|^2_{L^2(W; H)} = n \sum_{i=k}^{m} a_i^2
\]

which goes to 0 as \( m, k \to \infty \). So we have \( F_m \to F \) in \( L^2(W) \) and \( DF_m \) Cauchy in \( L^2(W; H) \). Since \( D \) is closed, we have \( F \in \mathbb{D}^{1,2} \).

In fact, we have shown that \( F \) is a \( \mathbb{D}^{1,2} \)-limit of elements of the span of \( \{F_\alpha\} \), and we know the \( F_\alpha \) are \( \mathbb{D}^{1,2} \)-orthogonal. \( \square \)

Note that \( \mathcal{H}_n \) are thus pairwise orthogonal closed subsets of \( \mathbb{D}^{1,2} \). Furthermore, \( D \) maps \( \mathcal{H}_n \) into \( \mathcal{H}_{n-1}(H) \).

**Proposition 6.16.** The span of all \( \mathcal{H}_n \) is dense in \( \mathbb{D}^{1,2} \), so we can write \( \mathbb{D}^{1,2} = \bigoplus \mathcal{H}_n \).

**Proof.** This proof is taken from [18].

We begin with the finite dimensional case. Let \( \mu_k \) be standard Gaussian measure on \( \mathbb{R}^k \), and let \( \phi \in C^\infty_c(\mathbb{R}^k) \). We will show that there is a sequence of polynomials \( p_m \) such that \( p_m \to \phi \) and \( \partial_i p_m \to \partial_i \phi \) in \( L^2(\mathbb{R}^k, \mu_k) \) for all \( k \).

For continuous \( \psi : \mathbb{R}^k \to \mathbb{R} \), let

\[
I_i \psi(x_1, \ldots, x_k) = \int_0^{x_i} \psi(x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_k) \, dy.
\]

By Fubini’s theorem, all operators \( I_i, 1 \leq i \leq k \) commute. If \( \psi \in L^2(\mu_k) \) is continuous, then \( I_i \psi \) is also continuous, and \( \partial_i I_i \psi = \psi \). Moreover,

\[
\int_0^\infty |I_i \psi(x_1, \ldots, x_k)|^2 e^{-x_i^2/2} \, dx_i = \int_0^\infty \left( \int_0^{x_i} \psi_1(\ldots, y, \ldots) \, dy \right)^2 e^{-x_i^2/2} \, dx_i \\
\leq \int_0^\infty \int_0^{x_i} |\psi_1(\ldots, y, \ldots)|^2 \, dy x_i e^{-x_i^2/2} \, dx_i \quad \text{Cauchy–Schwarz} \\
= \int_0^\infty |\psi(\ldots, x_i, \ldots)|^2 e^{-x_i^2/2} \, dx_i
\]

where in the last line we integrated by parts. We can make the same argument for the integral from \(-\infty \) to \( 0 \), adjusting signs as needed, so we have

\[
\int_{\mathbb{R}} |I_i \psi(x)|^2 e^{-x_i^2/2} \, dx_i \leq \int_{\mathbb{R}} |\psi(x)|^2 e^{-x_i^2/2} \, dx_i.
\]

Integrating out the remaining \( x_j \) with respect to \( e^{-x_j^2/2} \) shows

\[
\|I_i \psi\|^2_{L^2(\mu_k)} \leq \|\psi\|^2_{L^2(\mu_k)},
\]

i.e. \( I_i \) is a contraction on \( L^2(\mu_k) \).

Now for \( \phi \in C^\infty_c(\mathbb{R}^k) \), we can approximate \( \partial_1 \ldots \partial_k \phi \) in \( L^2(\mu_k) \) norm by polynomials \( q_n \) (by the finite-dimensional case of Proposition 6.13). If we let \( p_n = I_1 \ldots I_k q_n \), then \( p_n \) is again a
polynomial, and \( p_n \to I_1 \ldots I_k \partial_1 \ldots \partial_k \phi = \phi \) in \( L^2(\mu_k) \). Moreover, \( \partial_i p_n = I_1 \ldots I_{i-1} I_{i+1} \ldots I_k q_n \to I_1 \ldots I_{i-1} I_{i+1} \ldots I_k \partial_i \phi = \partial_i \phi \) in \( L^2(\mu_k) \) also.

Now back to the infinite-dimensional case. Let \( F \in \mathcal{F}C_c^\infty(W) \), so we can write \( F(x) = \phi(e_1(x), \ldots, e_k(x)) \) where \( e_i \) are \( q \)-orthonormal. Choose polynomials \( p_n \to \phi \) in \( L^2(\mathbb{R}^k, \mu_k) \) with \( \partial_i p_n \to \partial_i \phi \) in \( L^2(\mu_k) \) also, and set \( P_n(x) = p_n(e_1(x), \ldots, e_k(x)) \). Note \( P_n \in \mathcal{H}_m \) for some \( m = m_n \).

Then
\[
\int_W |F(x) - P_n(x)|^2 \mu(dx) = \int_{\mathbb{R}^k} |\phi(\mu) - p_n(\mu)|^2 \mu_k(\mu) d\mu \to 0.
\]

Exercise: write out \( \|DF - DP_n\|_{L^2(W;H)} \) and show that it goes to 0 also. \( \square \)

**Lemma 6.17.** For \( F \in \mathbb{D}^{1,2} \), \( DJ_n F = J_{n-1} DF \).

**Proof.** If \( F = F_\alpha \) where \( |\alpha| = n \), then \( J_n F_\alpha = F_\alpha \) and \( DF_\alpha \in \mathcal{H}_{n-1}(H) \) so \( J_{n-1} DF_\alpha = DF_\alpha \), so this is trivial. If \( |\alpha| \neq n \) then both sides are zero.

Now for general \( F \in \mathbb{D}^{1,2} \), by the previous proposition we can approximate \( F \) in \( \mathbb{D}^{1,2} \)-norm by functions \( F_m \) which are finite linear combinations of \( F_\alpha \). In particular, \( F_m \to F \) in \( L^2(W) \).

Since \( J_n \) is not continuous on \( L^2(W) \), \( J_n F_m \to J_n F \) in \( L^2(W) \). Also, \( DF_m \to DF \) in \( L^2(W;H) \), so \( J_{n-1} DF_m \to J_{n-1} DF \). But \( J_{n-1} DF_m = DJ_n F_m \).

We have shown \( J_n F_m \to J_n F \) and \( DJ_n F_m \to J_{n-1} DF \). By closedness of \( D \), we have \( DJ_n F = J_{n-1} DF \). \( \square \)

**Corollary 6.18.** \( J_n \) is a continuous operator on \( \mathbb{D}^{1,2} \).

**Proof.** \( F_m \to F \) in \( \mathbb{D}^{1,2} \) means \( F_m \to F \) in \( L^2(W) \) and \( DF_m \to DF \) in \( L^2(W;H) \). When this happens, we have \( J_n F_m \to J_n F \) since \( J_n \) is continuous on \( L^2(W) \), and \( DJ_n F_m = J_{n-1} DF_m \to J_{n-1} DF = DJ_n F \). \( \square \)

**Corollary 6.19.** \( J_n \) is orthogonal projection onto \( \mathcal{H}_n \) with respect to the \( \mathbb{D}^{1,2} \) inner product.

**Proof.** \( J_n \) is the identity on \( \mathcal{H}_n \), and vanishes on any \( \mathcal{H}_m \) for \( m \neq n \). Thus by continuity it vanishes on \( \bigoplus_{m \neq n} \mathcal{H}_m \) which is the \( \mathbb{D}^{1,2} \)-orthogonal complement of \( \mathcal{H}_n \). \( \square \)

**Corollary 6.20.** For \( F \in \mathbb{D}^{1,2} \), \( F = \sum_{n=0}^\infty J_n F \) where the sum converges in \( \mathbb{D}^{1,2} \).

**Corollary 6.21.** For \( F \in \mathbb{D}^{1,2} \), \( DF = \sum_{n=0}^\infty DJ_n F = \sum_{n=1}^\infty J_{n-1} DF \) where the sums converge in \( L^2(W;H) \).

**Proof.** The first equality follows from the previous corollary, since \( D : \mathbb{D}^{1,2} \to L^2(W;H) \) is continuous. The second equality is Lemma 6.17. \( \square \)

**Proposition 6.22.** \( F \in \mathbb{D}^{1,2} \) if and only if \( \sum_n n \|J_n F\|^2_{L^2(W)} < \infty \), in which case \( \|DF\|^2_{L^2(W;H)} = \sum_n n \|J_n F\|^2_{L^2(W)} \).

**Proof.** If \( f \in \mathbb{D}^{1,2} \), we have \( DF = \sum_{n=0}^\infty DJ_n F \). Since the terms of this sum are orthogonal in \( L^2(W;H) \), we have
\[
\sum_{n=0}^\infty \|DJ_n F\|^2_{L^2(W;H)} = \sum_{n=0}^\infty \|J_n F\|^2_{L^2(W)}.
\]

Conversely, if \( \sum_n n \|J_n F\|^2 = \sum_n \|DJ_n F\|^2 < \infty \) then \( J_n F \) converges to \( F \) and \( \sum DJ_n F \) converges, therefore by closedness of \( D \), \( F \in \mathbb{D}^{1,2} \). \( \square \)
Corollary 6.23. If \( F \in \mathbb{D}^{1,2} \) and \( DF = 0 \) then \( F \) is constant.

Lemma 6.24 (Chain rule). If \( \psi \in C_\infty^c(\mathbb{R}) \), \( F \in \mathbb{D}^{1,2} \), then \( \psi(F) \in \mathbb{D}^{1,2} \) and
\[
D\psi(F) = \psi'(F)DF.
\]

Proof. For \( F \in FC_\infty^c(W) \) this is just the regular chain rule. For general \( F \in \mathbb{D}^{1,2} \), choose \( F_n \in C_\infty^c(W) \) with \( F_n \to F \), \( DF_n \to DF \). Then use dominated convergence.

Actually the chain rule also holds for any \( \psi \in C^1 \) with bounded first derivative. Exercise: prove.

Proposition 6.25. If \( A \subset W \) is Borel and \( 1_A \in \mathbb{D}^{1,2} \) then \( \mu(A) \) is 0 or 1.

Proof. Let \( \psi \in C_\infty^c(\mathbb{R}) \) with \( \psi(s) = s^2 \) on \([0, 1]\). Then
\[
D1_A = D\psi(1_A) = 21_AD1_A
\]
so by considering whether \( x \in A \) or \( x \in A^c \) we have \( D1_A = 0 \) a.e. Then by an above lemma, \( 1_A \) is (a.e.) equal to a constant.

As a closed densely defined operator between Hilbert spaces \( L^2(W) \) and \( L^2(W; H) \), \( D \) has an adjoint operator \( \delta \), which is a closed densely defined operator from \( L^2(W; H) \) to \( L^2(W) \).

To get an idea what \( \delta \) does, let’s start by evaluating it on some simple functions.

Proposition 6.26. If \( u(x) = G(x)h \) where \( G \in \mathbb{D}^{1,2} \) and \( h \in H \), then \( u \in \text{dom}(\delta) \) and
\[
\delta u(x) = G(x)\langle h, x \rangle_H - \langle DG(x), h \rangle_H.
\]

Proof. If \( G \in FC_\infty^c \), use \([6.4]\). Otherwise, approximate. (Hmm, maybe we actually need \( G \in L^{2+\epsilon}(W) \) for this to work completely.)

Recall in the special case of Brownian motion, where \( W = C([0, 1]) \) and \( H = H_0^1([0, 1]) \), we had found that \( \langle h, \omega \rangle_H = \int_0^1 h(s)dB_s(\omega) \), i.e. \( \langle h, \cdot \rangle_H \) produces the Wiener integral, a special case of the Itô integral with a deterministic integrand. So it appears that \( \delta \) is also some sort of integral. We call it the Skorohod integral. In fact, the Itô integral is a special case of it!

Theorem 6.27. Let \( A(t, \omega) \) be an adapted process in \( L^2([0, 1] \times W) \). Set \( u(t, \omega) = \int_0^t A(\tau, \omega)\,d\tau \), so \( u(\cdot, \omega) \in H \) for each \( \omega \). Then \( u \in \text{dom}(\delta) \) and \( \delta u = \int_0^t A_\tau dB_\tau \).

Proof. First suppose \( A \) is of the form \( A(t, \omega) = 1_{(r,s]}(t)F(\omega) \) where \( F(\omega) = \phi(B_{t_1}(\omega), \ldots, B_{t_n}(\omega)) \) for some \( 0 \leq t_1, \ldots, t_n \leq r \) and \( \phi \in C_\infty^c(\mathbb{R}^n) \). (Recall \( B_1(\omega) = \omega(t) \) is just the evaluation map, a continuous linear functional of \( \omega \).) We have \( F \in \mathcal{F}_r \) so \( A \) is adapted. Then \( u(t, \omega) = h(t)F(\omega) \) where \( h(t) = \int_0^t 1_{(r,s]}(\tau)d\tau \). In particular \( h(t) = 0 \) for \( t \leq r \), so
\[
\langle DF(\omega), h \rangle_H = \sum_i \partial_i \phi(B_{t_1}(\omega), \ldots, B_{t_n}(\omega))h(t_i) = 0.
\]

Thus
\[
\delta u(\omega) = F(\omega)\langle h, \omega \rangle_H - \langle DF(\omega), h \rangle_H = F(\omega)(B_r(\omega) - B_s(\omega)) = \int_0^1 A_\tau dB_\tau(\omega).
\]
Now we just need to do some approximation. If $F \in L^2(W, \mathcal{F}_r)$, we can approximate $F$ in $L^2$ by cylinder functions $F_n$ of the above form. Then defining $u_n$, $u$ accordingly we have $u_n \to u$ in $L^2(W; H)$ and $\delta u_n \to F \cdot (B_r - B_s) = \int_0^1 A_r \, dB_r$ in $L^2(W)$, so the conclusion holds for $A = F(\omega)\mathbb{1}_{(r,s]}(t)$. By linearity it also holds for any linear combination of such processes. The set of such linear combinations is dense in the adapted processes in $L^2([0,1] \times W)$, so choose such $A^{(n)} \to A$. Note that $A(t, \omega) \mapsto \int_0^1 A(\tau, \omega) \, d\tau$ is an isometry of $L^2([0,1] \times W)$ into $L^2(W; H)$ so the corresponding $u_n$ converge to $u$ in $L^2(W; H)$, and by the Itô isometry, $\delta u_n = \int_0^1 A_r^{(n)} \, dB_r \to \int_0^1 A_r \, dB_r$. Since $\delta$ is closed we are done.

This is neat because defining an integral in terms of $\delta$ lets us integrate a lot more processes.

**Example 6.28.** Let’s compute $\int_0^1 B_1 \, dB_s$ (Skorohod). We have $u(t, \omega) = tB_1 = h(t)G(\omega)$ where $h(t) = t$, and $G(\omega) = \phi(f(\omega))$ where $\phi(x) = x$, and $f(\omega) = \omega(1)$. So $\delta u(\omega) = G(\omega)\langle h, \omega \rangle_H - \langle DG(\omega), h \rangle_H$. But $\langle h, \omega \rangle = \int_0^1 h(t) \, dB_t = B_1$. And $DG(\omega) = Jf$ so $\langle DG(\omega), h \rangle_H = f(h) = h(1) = 1$. So $\int_0^1 B_1 \, dB_s = B_1^2 - 1$.

Note that $B_1$, although it doesn’t depend on $t$, is not adapted. Indeed, a random variable is an adapted process iff it is in $\mathcal{F}_0$, which means it has to be constant.

All the time derivatives and integrals here are sort of a red herring; they just come from the fact that the Cameron–Martin inner product has a time derivative in it.

Another cool fact is that we can use this machinery to construct integration with respect to other continuous Gaussian processes. Again let $W = C([0,1])$ (or an appropriate subspace), $\mu$ the law of a continuous centered Gaussian process $X_t$ with covariance function $a(s, t) = E[X_sX_t]$. Since $\delta$ applies to elements of $L^2(W; H)$, and we want to integrate honest processes (such as elements of $L^2([0,1] \times W)$), we need some way to map processes to elements of $L^2(W; H)$. For Brownian motion it was $A_t \mapsto \int_0^t A_s \, ds$, an isometry of $L^2([0,1] \times W) = L^2(W; L^2([0,1])) = L^2(W) \otimes L^2([0,1])$ into $L^2(W; H)$. To construct such an map $\Phi$ in this case, we start with the idea that we want $\int_0^1 \, ds = X_1$, so we should have $\Phi 1_{[0,T]} = J\delta_s \in H$. We can extend this map linearly to $\mathcal{E}$, the set of all step functions on $[0,1]$. To make it an isometry, equip $\mathcal{E}$ with the inner product defined by

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_\mathcal{E} = \langle J\delta_s, J\delta_t \rangle_H = a(s, t)$$

again extended by bilinearity. It extends isometrically to the completion of $\mathcal{E}$ under this inner product, whatever that may be. So the processes we can integrate are we can integrate processes from $L^2(W; \mathcal{E}) = L^2(W) \otimes \mathcal{E}$ just by taking $\int_0^1 A \, dX = \delta u$ where $u(\omega) = \Phi(A(\omega))$.

Exactly what are the elements of $L^2(W; \mathcal{E})$ is a little hard to say. The elements of $\mathcal{E}$ can’t necessarily be identified as functions on $[0,1]$; they might be distributions, for instance. But we for sure know it contains step functions, so $L^2(W; \mathcal{E})$ at least contains “simple processes” of the form $\sum Y_t \mathbb{1}_{[a_i, b_i]}(t)$. In the case of fractional Brownian motion, one can show that $\mathcal{E}$ contains $L^2([0,1])$, so in particular $\Phi$ makes sense for any process in $L^2(W \times [0,1])$. Of course, there is still the question of whether $\Phi(A) \in \text{dom } \delta$.

There’s a chapter in Nualart which works out a lot of this in the context of fractional Brownian motion. Being able to integrate with respect to fBM is a big deal, because fBM is not a semimartingale and so it is not covered by any version of Itô integration.

A couple of properties of $\delta$ in terms of the Wiener chaos:

1. $\mathcal{H}_n(H) \subset \text{dom } \delta$ for each $n$. (Follows from Lemma 6.26).
2. For \( u \in \text{dom} \delta \), \( J_n \delta u = \delta J_{n-1} u \).

Using Lemma 6.17 and the fact that the \( J_n \), being orthogonal projections, are self-adjoint, we have for any \( F \in \mathbb{D}^{1,2} \),

\[
\langle J_n \delta u, F \rangle_{L^2(W)} = \langle \delta u, J_n F \rangle_{L^2(W)} = \langle u, DJ_n F \rangle_{L^2(W;H)} = \langle u, J_{n-1} D F \rangle_{L^2(W;H)} = \langle \delta J_{n-1} u, F \rangle_{L^2(W)}.
\]

\( \mathbb{D}^{1,2} \) is dense in \( L^2(W) \) so we are done.

3. \( J_0 \delta u = 0 \). For if \( F \in L^2(W) \), then \( \langle J_0 \delta u, F \rangle = \langle \delta u, J_0 F \rangle = \langle u, DJ_0 F \rangle \). But \( J_0 F \) is a constant so \( DJ_0 F = 0 \).

6.2 The Clark–Ocone formula

Until further notice, we are working on classical Wiener space, \( W = C_0([0,1]) \), with \( \mu \) being Wiener measure.

A standard result in stochastic calculus is the **Itô representation theorem**, which in its classical form says:

**Theorem 6.29.** Let \{\( B_t \)\} be a Brownian motion on \( \mathbb{R}^d \), let \{\( \mathcal{F}_t \)\} be the filtration it generates, and let \( Z \) be an \( L^2 \) random variable which is \( \mathcal{F}_1 \)-measurable (sometimes called a **Brownian functional**). Then there exists an adapted \( L^2 \) process \( Y_t \) such that

\[
Z = \mathbb{E}[Z] + \int_0^1 Y_t dB_t, \quad \text{a.s.}
\]

**Sketch.** This is claiming that the range of the Itô integral contains all the \( L^2 \) random variables with mean zero (which we’ll denote \( \mathcal{H}_0^+ \)). Since the Itô integral is an isometry, its range is automatically closed, so it suffices to show it is dense in \( \mathcal{H}_0^+ \). One can explicitly produce a dense set. \( \square \)

**Exercise 6.30.** Look up a proof.

An important application of this theorem is in finance. Suppose we have a stochastic process \( \{X_t\} \) which gives the price of a stock (call it Acme) at time \( t \). (Temporarily you can think \( X_t = B_t \) is Brownian motion, though this is not a good model and we might improve it later.) We may want to study an **option** or **contingent claim**, some contract whose ultimate value \( Z \) is determined by the behavior of the stock. For example:

- A **European call option** is a contract which gives you the right, but not the obligation, to buy one share of Acme at time 1 for a pre-agreed **strike price** \( K \). So if the price \( X_1 \) at time 1 is greater than \( K \), you will **exercise** your option, buy a share for \( K \) dollars, and then you can immediately sell it for \( X_1 \) dollars, turning a quick profit of \( Z = X_1 - K \) dollars. If \( X_1 < K \), then you should not exercise the option; it is worthless, and \( Z = 0 \). Thus we can write \( Z = (X_1 - K)^+ \).
• **A European put option** gives the right to sell one share of Acme at a price $K$. Similarly we have $Z = (K - X_t)^+$.  

• **A floating lookback put option** gives one the right, at time 1, to sell one share of Acme at the highest price it ever attained between times 0 and 1. So $Z = \sup_{t \in [0, 1]} X_t - X_1$.  

• **There are many more.**

You can’t lose money with these contracts, because you can always just not exercise it, and you could gain a profit. Conversely, your counterparty can only lose money. So you are going to have to pay your counterparty some money up front to get them to enter into such a contract. How much should you pay? A “fair” price would be $E[Z]$. But it may be that the contract would be worth more or less to you than that, depending on your appetite for risk. (Say more about this.)

Here the Itô representation theorem comes to the rescue. If $X_t = B_t$ is a Brownian motion, it says that $Z = E[Z] + \int_0^1 Y_t \, dB_t$. This represents a **hedging strategy**. Consider a trading strategy where at time $t$ we want to own $Y_t$ shares of Acme (where we can hold or borrow cash as needed to achieve this; negative shares are also okay because we can sell short). $Y_t$ is adapted, meaning the number of shares to own can be determined by what the stock has already done. A moment’s thought shows that the net value of your portfolio at time 1 is $\int_0^1 Y_t \, dB_t$. Thus, if we start with $E[Z]$ dollars in the bank and then follow the strategy $Y_t$, at the end we will have exactly $Z$ dollars, almost surely. We can **replicate** the option $Z$ for $E[Z]$ dollars (not counting transaction costs, which we assume to be negligible). So anybody that wants more than $E[Z]$ dollars is ripping us off, and we shouldn’t pay it even if we would be willing to.

So a key question is whether we can explicitly find $Y_t$.  

In Wiener space notation, $Z$ is an element of $L^2(W, \mu)$, which we had usually called $F$. Also, now $\mathcal{F}_t$ is the $\sigma$-algebra generated by the linear functionals $\{\delta_s : s \leq t\}$; since these span a weak-* dense subset of $W^*$ we have $\mathcal{F}_1 = \sigma(W^*) = B_W$, the Borel $\sigma$-algebra of $W$.

Let $L^2_a([0, 1] \times W)$ be the space of adapted processes.

**Exercise 6.31.** $L^2_a([0, 1] \times W)$ is a closed subspace of $L^2([0, 1] \times W)$.

**Exercise 6.32.** $Y_t \mapsto E[Y_t|\mathcal{F}_t]$ is orthogonal projection from $L^2([0, 1] \times W)$ onto $L^2_a([0, 1] \times W)$.

This section is tangled up a bit by some derivatives coming and going. Remember that $H$ is naturally isomorphic to $L^2([0, 1])$ via the map $\Phi : L^2([0, 1]) \to H$ given by $\Phi f(t) = \int_0^t f(s) \, ds$ (its inverse is simply $\frac{d}{dt}$). Thus $L^2(W; H)$ is naturally isomorphic to $L^2([0, 1] \times W)$. Under this identification, we can identify $D : \mathbb{D}^{1,2} \to L^2(W; H)$ with a map that takes an element $F \in \mathbb{D}^{1,2}$ to a process $D_t F \in L^2([0, 1] \times W)$; namely, $D_t F(\omega) = \frac{d}{dt} DF(\omega)(t) = \frac{d}{dt}(DF(\omega), J_{\delta_t})_H = \frac{d}{dt}(DF(\omega), \cdot \wedge t)_H$. So $D_t F = \Phi^{-1} DF$.

The Clark–Ocone theorem states:

**Theorem 6.33.** For $F \in \mathbb{D}^{1,2}$,

$$F = \int F \, d\mu + \int_0^1 E[D_t F|\mathcal{F}_t] \, dB_t. \quad (6.5)$$

To prove this, we want to reduce everything to Skorohod integrals. Let $E \subset L^2(W; H)$ be the image of $L^2_a([0, 1] \times W)$ under the isomorphism $\Phi$. Then, since the Skorohod integral extends the
Itô integral, we know that \( E \subset \text{dom}\, \delta \), and \( \delta : E \to L^2(W) \) is an isometry. Moreover, by the Itô representation theorem, the image \( \delta(E) \) is exactly \( \mathcal{H}_0^\perp \), i.e. the orthogonal complement of the constants, i.e. functions with zero mean.

Let \( P \) denote orthogonal projection onto \( E \), so that \( \mathbb{E}[\cdot | \mathcal{F}_t] = \Phi^{-1} P \Phi \).

We summarize this discussion by saying that the following diagram commutes.

\[
\begin{array}{c}
L^2([0,1] \times W) \xrightarrow{\mathbb{E}[\cdot | \mathcal{F}_t]} L^2([0,1] \times W) \\
\Phi \downarrow \Phi \downarrow \\
L^2(W; H) \xrightarrow{P} E \\
\delta \downarrow \\
L^2(W) \\
\end{array}
\]

From this diagram, we see that the Clark–Ocone theorem reads:

\[
F = \int F \, d\mu + \delta PDF. \quad (6.7)
\]

Now the proof is basically just a diagram chase.

**Proof.** Suppose without loss of generality that \( \int F \, d\mu = 0 \), so that \( F \in \mathcal{H}_0^\perp \). Let \( u \in E \). Then

\[
\langle F, \delta u \rangle_{L^2(W)} = \langle DF, u \rangle_{L^2(W; H)} \\
= \langle DF, Pu \rangle_{L^2(W; H)} \\
= \langle PDF, u \rangle_{L^2(W; H)}
\]

(since orthogonal projections are self-adjoint)

\[
= \langle \delta PDF, \delta u \rangle_{L^2(W)}
\]

since \( PDF \in E \), \( u \in E \), and \( \delta \) is an isometry on \( E \). As \( u \) ranges over \( E \), \( \delta u \) ranges over \( \mathcal{H}_0^\perp \), so we must have \( F = \delta PDF \). \( \square \)

**Exercise 6.34.** If the stock price is Brownian motion \( (X_t = B_t) \), compute the hedging strategy \( Y_t \) for a European call option \( Z = (X_1 - K)^+ \).

**Exercise 6.35.** Again take \( X_t = B_t \). Compute the hedging strategy for a floating lookback call option \( Z = M - X_1 \), where \( M = \sup_{t \in [0,1]} X_t \). (Show that \( D_t M = 1_{\{t \leq T\}} \) where \( T = \arg \max X_t \), which is a.s. unique, by approximating \( M \) by the maximum over a finite set.)

**Exercise 6.36.** Let \( X_t \) be a **geometric Brownian motion** \( X_t = \exp \left( B_t - \frac{t}{2} \right) \). Compute the hedging strategy for a European call option \( Z = (X_1 - K)^+ \). (Note by Itô’s formula that \( dX_t = X_t dB_t \).)
7 Ornstein–Uhlenbeck process

We’ve constructed one canonical process on $W$, namely Brownian motion $B_t$, defined by having independent increments distributed according to $\mu$ (appropriately scaled). In finite dimensions, another canonical process related to Gaussian measure is the Ornstein–Uhlenbeck process. This is a Gaussian process $X_t$ which can be defined by the SDE $dX_t = \sqrt{2}dB_t - X_t dt$. Intuitively, $X_t$ tries to move like a Brownian motion, but it experiences a “restoring force” that always pulls it back toward the origin. Imagine a Brownian particle on a spring. A key relationship between the amount of “energy” contained in the distribution $\mu$ and standard Gaussian measure is that another canonical process related to Gaussian measure is the Ornstein–Uhlenbeck process, which is a Brownian motion plus a restoring force. The process often have corresponding properties for the Dirichlet form. This is great, because constructing a process is usually a lot of work, but one can often just write down a Dirichlet form. Moreover, one finds that properties of the process often have corresponding properties for the Dirichlet form.

For example, if the process $X_t$ has continuous sample paths, the form $(E, D)$ is called the Dirichlet form. Here are some basics on the subject.

7.1 Crash course on Dirichlet forms

Suppose $X_t$ is a symmetric Markov process on some topological space $X$ equipped with a Borel measure $m$. This means that its transition semigroup $T_t f(x) = E_x[f(X_t)]$ is a Hermitian operator on $L^2(X, m)$. If we add a few extra mild conditions (e.g. càdlàg, strong Markov) and make $X_t$ a Hunt process, the semigroup $T_t$ will be strongly continuous. It is also Markovian, i.e. if $0 \leq f \leq 1$, then $0 \leq T_t f \leq 1$. For example, if $X = \mathbb{R}^n$, $m$ is Lebesgue measure, and $X_t$ is Brownian motion, then $T_t f(x) = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/2t} m(dy)$ is the usual heat semigroup.

A strongly continuous contraction semigroup has an associated generator, a nonnegative-definite self-adjoint operator $(L, D(L))$ which in general is unbounded, such that $T_t = e^{-tL}$. For Brownian motion it is $L = -\Delta/2$ with $D(L) = H^2(\mathbb{R}^n)$.

Associated to a nonnegative self-adjoint operator is an unbounded bilinear symmetric form $E$ with domain $\mathbb{D}$, such that $E(f, g) = (f, Lg)$ for every $f \in \mathbb{D}$ and $g \in D(L)$. We can take $(E, \mathbb{D})$ to be a closed form, which essentially says that $E_1(f, g) = E(f, g) + (f, g)$ is a Hilbert inner product on $\mathbb{D}$. Note that $\mathbb{D}$ is generally larger than $D(L)$. For Brownian motion, $E(f, g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g dm$ and $\mathbb{D} = H^1(\mathbb{R}^n)$. Note $E_1$ is the usual Sobolev inner product on $H^1(\mathbb{R}^n)$. $E(f, f)$ can be interpreted as the amount of “energy” contained in the distribution $f dm$. Letting this distribution evolve under the process will tend to reduce the amount of energy as quickly as possible.

When $T_t$ is Markovian, $(E, \mathbb{D})$ has a corresponding property, also called Markovian. Namely, if $f \in \mathbb{D}$, let $\bar{f} = f \wedge 1 \vee 0$ be a “truncated” version of $f$. The Markovian property asserts that $\bar{f} \in \mathbb{D}$ and $E(f, \bar{f}) \leq E(f, f)$. A bilinear, symmetric, closed, Markovian form on $L^2(X, m)$ is called a Dirichlet form.

So far this is nice but not terribly interesting. What’s neat is that this game can be played backwards. Under certain conditions, one can start with a Dirichlet form and recover a Hunt process with which it is associated. This is great, because constructing a process is usually a lot of work, but one can often just write down a Dirichlet form. Moreover, one finds that properties of the process often have corresponding properties for the Dirichlet form.

For example, if the process $X_t$ has continuous sample paths, the form $(E, \mathbb{D})$ will be local: namely, if $f = 0$ on the support of $g$, then $E(f, g) = 0$. Conversely, if the form is local, then the associated process will have continuous sample paths. If additionally the process is not killed inside
X, the form is strongly local: if f is constant on the support of g, then \( \mathcal{E}(f,g) = 0 \); and the converse is also true.

So you might ask: under what conditions must a Dirichlet form be associated with a process? One sufficient condition is that \((\mathcal{E}, \mathcal{D})\) be regular: that \(\mathcal{D} \cap C_c(X)\) is \(\mathcal{E}\)-dense in \(\mathcal{D}\) and uniformly dense in \(C_c(X)\). We also have to assume that \(X\), as a topological space, is locally compact. The main purpose of this condition is to exclude the possibility that \(X\) contains "holes" that the process would have to pass through. Unfortunately, this condition is useless in infinite dimensions, since if \(X = W\) is, say, an infinite-dimensional Banach space, then \(C_c(W) = 0\).

There is a more general condition called quasi-regular, which is actually necessary and sufficient for the existence of a process. It is sufficiently complicated that I won’t describe it here; see Ma and Röckner’s book for the complete treatment.

### 7.2 The Ornstein–Uhlenbeck Dirichlet form

We are going to define the Ornstein–Uhlenbeck process via its Dirichlet form. For \(F,G \in \mathbb{D}^{1,2}\), let 
\[ \mathcal{E}(F,G) = \langle DF, DG \rangle_{L^2(W;H)}. \]
This form is obviously bilinear, symmetric, and positive semidefinite. With the domain \(\mathbb{D}^{1,2}\), \(\mathcal{E}\) is also a closed form (in fact, \(\mathcal{E}_1\) is exactly the Sobolev inner product on \(\mathbb{D}^{1,2}\), which we know is complete).

**Proposition 7.1.** \((\mathcal{E}, \mathbb{D}^{1,2})\) is Markovian.

**Proof.** Fix \(\epsilon > 0\). Let \(\varphi_n \in C^\infty(\mathbb{R})\) be a sequence of smooth functions with \(0 \leq \varphi_n \leq 1\), \(|\varphi'_n| \leq 1 + \epsilon\), and \(\varphi_n(x) \to x \wedge 1 \vee 0\) pointwise. (Draw a picture to convince yourself this is possible.) Then \(\varphi_n(F) \to F \wedge 1 \vee 0\) in \(L^2(W)\) by dominated convergence. Then, by the chain rule, for \(F \in \mathbb{D}^{1,2}\), we have 
\[ \|D\varphi_n(F)\|_{L^2(W;H)} = \|\varphi'_n(F)DF\|_{L^2(W;H)} \leq (1 + \epsilon) \|DF\|_{L^2(W;H)}. \]
It follows from Alaoglu’s theorem that \(F \wedge 1 \vee 0 \in \mathbb{D}^{1,2}\), and moreover, 
\[ \|D[F \wedge 1 \vee 0]\|_{L^2(W;H)} \leq (1 + \epsilon) \|DF\|_{L^2(W;H)}. \]
Letting \(\epsilon \to 0\) we are done. \(\square\)

**Exercise 7.2.** Fill in the details in the preceding proof.

**Theorem 7.3.** \((\mathcal{E}, \mathbb{D}^{1,2})\) is quasi-regular. Therefore, there exists a Hunt process \(X_t\), whose transition semigroup is \(T_t\), the semigroup corresponding to \((\mathcal{E}, \mathbb{D}^{1,2})\).

**Proof.** See [15, IV.4.b]. \(\square\)

**Lemma 7.4.** The operator \(D\) is local in the sense that for any \(F \in \mathbb{D}^{1,2}\), \(DF = 0 \mu\text{-a.e. on } \{F = 0\}\).

**Proof.** Let \(\varphi_n \in C^\infty(\mathbb{R})\) have \(\varphi_n(0) = 1\), \(0 \leq \varphi_n \leq 1\), and \(\varphi_n\) supported inside \([\frac{-1}{n}, \frac{1}{n}]\); note that \(\varphi_n \to 1_{\{0\}}\) pointwise and boundedly. Then as \(n \to \infty\), \(\varphi_n(F)DF \to 1_{\{F=0\}}DF\) in \(L^2(W;H)\).

Let \(\psi_n(t) = \int_0^t \varphi_n(s) \, ds\), so that \(\varphi_n = \psi'_n\); then \(\psi_n \to 0\) uniformly. By the chain rule we have \(D(\psi_n(F)) = \varphi_n(F)DF\). Now if we fix \(u \in \text{dom } \delta\), we have 
\[ \langle 1_{\{F=0\}}DF, u \rangle_{L^2(W;H)} = \lim_{n \to \infty} \langle \varphi_n(F)DF, u \rangle_{L^2(W;H)} = \lim_{n \to \infty} \langle D(\psi_n(F)), u \rangle_{L^2(W;H)} = \lim_{n \to \infty} \langle \psi_n(F), \delta u \rangle_{L^2(W)} = 0 \]
since \(\psi_n(F) \to 0\) uniformly and hence in \(L^2(W)\). Since \(\text{dom } \delta\) is dense in \(L^2(W;H)\), we have \(1_{\{F=0\}}DF = 0 \mu\text{-a.e.},\) which is the desired statement. \(\square\)
Corollary 7.5. The Ornstein–Uhlenbeck Dirichlet form \((\mathcal{E}, \mathcal{D}^{1,2})\) is strongly local.

Proof. Let \(F, G \in \mathcal{D}^{1,2}\). Suppose first that \(F = 0\) on the support of \(G\). By the previous lemma we have (up to \(\mu\)-null sets) \(\{DF = 0\} \supset \{F = 0\} \supset \{G \neq 0\} \supset \{DG \neq 0\}\). Thus, for a.e. \(x\) either \(DF(x) = 0\) or \(DG(x) = 0\). So \(\mathcal{E}(F, G) = \int_X \langle DF(x), DG(x) \rangle_H \mu(dx) = 0\).

If \(F = 1\) on the support of \(G\), write \(\mathcal{E}(F, G) = \mathcal{E}(F - 1, G) + \mathcal{E}(1, G)\). The first term vanishes by the previous step, while the second term vanishes since \(D1 = 0\).

We now want to investigate the generator \(N\) associated to \((\mathcal{E}, \mathcal{D})\).

Lemma 7.6. For \(F \in L^2(W)\), \(J_0F = \int Fd\mu\), where \(J_0\) is the orthogonal projection onto \(\mathcal{H}_0\), the constant functions in \(L^2(W)\).

Proof. This holds over any probability space. Write \(EF = \int Fd\mu\). Clearly \(E\) is continuous, \(E\) is the identity on the constants \(\mathcal{H}_0\), and if \(F \perp \mathcal{H}_0\), then we have \(EF = \langle F, 1 \rangle_{L^2(W)} = 0\) since \(1 \in \mathcal{H}_0\). So \(E\) must be orthogonal projection onto \(\mathcal{H}_0\).

Lemma 7.7. [A Poincaré inequality] For \(F \in \mathcal{D}^{1,2}\), we have

\[
\left\| F - \int Fd\mu \right\|_{L^2(W)} \leq \|DF\|_{L^2(W;H)}.
\]

Proof. Set \(G = F - \int Fd\mu\), so that \(J_0G = \int Gd\mu = 0\). Note that \(DF = DG\) since \(D1 = 0\). Then by Lemma 6.22

\[
\|DG\|_{L^2(W;H)}^2 = \sum_{n=0}^{\infty} n \|J_nG\|_{L^2(W)}^2 \geq \sum_{n=1}^{\infty} \|J_nG\|_{L^2(W)}^2 = \sum_{n=0}^{\infty} \|J_nG\|_{L^2(W)}^2 = \|G\|_{L^2(W)}^2.
\]

Note that by taking \(F(x) = f(x)\) for \(f \in W^*\), we can see that the Poincaré inequality is sharp.

Theorem 7.8. \(N = \delta D\). More precisely, if we set

\[
\text{dom } N = \text{dom } \delta D = \{F \in \mathcal{D}^{1,2} : DF \in \text{dom } \delta\}
\]

and \(NF = \delta DF\) for \(F \in \text{dom } N\), then \((N, \text{dom } N)\) is the unique self-adjoint operator satisfying \(\text{dom } N \subset \mathcal{D}^{1,2}\) and

\[
\mathcal{E}(F, G) = \langle F, NG \rangle_{L^2(W)} \text{ for all } F \in \mathcal{D}^{1,2}, G \in \text{dom } N.
\]

Proof. It is clear that \(\text{dom } N \subset \mathcal{D}^{1,2}\) and that (7.1) holds. Moreover, it is known there is a unique self-adjoint operator with this property (reference?). We have to check that \(N\) as defined above is in fact self-adjoint. (Should fill this in?)

Proposition 7.9. \(NF_\alpha = |\alpha|F_\alpha\). That is, the Hermite polynomials \(F_\alpha\) are eigenfunctions for \(N\), with eigenvalues \(|\alpha|\). So the \(\mathcal{H}_\alpha\) are eigenspaces.

Proof. Since \(F_\alpha\) is a cylinder function, it is easy to see it is in the domain of \(N\). Then \(\langle NF_\alpha, F_\beta \rangle_{L^2(W)} = \langle DF_\alpha, DF_\beta \rangle_{L^2(W;H)} = |\alpha|\delta_{\alpha\beta}\). Since the \(\{F_\beta\}\) are an orthonormal basis for \(L^2(W)\), we are done.
There is a natural identification of $\mathcal{H}_n$ with $H^\otimes n$, which gives an identification of $L^2(W)$ with Fock space $\bigoplus_n H^\otimes n$. In quantum mechanics this is the state space for a system with an arbitrary number of particles, $H^\otimes n$ corresponding to those states with exactly $n$ particles. $N$ is thus called the number operator because $\langle NF, F \rangle$ gives the (expected) number of particles in the state $F$.

**Proposition 7.10.** $NF = \sum_{n=0}^{\infty} nJ_n F$, where the sum on the right converges iff $F \in \text{dom } N$.

**Proof.** For each $m$, we have

$$N \sum_{n=0}^{m} J_n F = \sum_{n=0}^{m} N J_n F = \sum_{n=0}^{m} n J_n F.$$  

Since $\sum_{n=0}^{m} J_n F \to F$ as $m \to \infty$ and $N$ is closed, if the right side converges then $F \in \text{dom } N$ and $NF$ equals the limit of the right side.

Conversely, if $F \in \text{dom } N$, we have $\infty > \|NF\|_{L^2(W)}^2 = \sum_{n=0}^{\infty} \|J_n NF\|^2$. But, repeatedly using the self-adjointness of $J_n$ and $N$ and the relationships $J_n = J_n^2$ and $NJ_n = nJ_n$,

$$\|J_n NF\|^2 = \langle F, NJ_n NF \rangle = n \langle F, J_n NF \rangle = n^2 \langle J_n F, F \rangle = n^2 \|J_n F\|^2.$$  

Thus $\sum n^2 \|J_n F\|^2 < \infty$, so $\sum n J_n F$ converges. \qed

Let $T_t = e^{-tN}$ be the semigroup generated by $N$. Note that each $T_t$ is a contraction on $L^2(W)$, and $T_t$ is strongly continuous in $t$.

**Proposition 7.11.** For any $F \in L^2(W)$,

$$T_t F = \sum_{n=0}^{\infty} e^{-tn} J_n F.$$  

(7.2)

**Proof.** Since $J_n F$ is an eigenfunction of $N$, we must have

$$\frac{d}{dt} T_t J_n F = T_t NJ_n F = n T_t J_n f.$$  

Since $T_0 J_n F = J_n F$, the only solution of this ODE is $T_t J_n F = e^{-tn} J_n F$. Now sum over $n$. \qed

**Corollary 7.12.** $\|T_t F - \int F d\mu\|_{L^2(W)} \leq e^{-t} \|F - \int F d\mu\|$.

**Proof.** Let $G = F - \int F d\mu$; in particular $J_0 G = 0$. Then

$$\|T_t G\|^2 = \sum_{n=1}^{\infty} e^{-2tn} \|J_n G\|^2 \leq e^{-2t} \sum_{n=1}^{\infty} \|J_n G\|^2 = e^{-2t} \|G\|^2.$$  

\qed

This is also a consequence of the Poincaré inequality (Lemma 7.7) via the spectral theorem. $T_t$ is the transition semigroup of the Ornstein–Uhlenbeck process $X_t$, i.e. $T_t F(x) = \mathbb{E}_x [F(X_t)]$ for $\mu$-a.e. $x \in X$. To get a better understanding of this process, we’ll study $T_t$ and $N$ some more.

The finite-dimensional Ornstein–Uhlenbeck operator is given by

$$\tilde{N} \phi(x) = \Delta \phi(x) - x \cdot \nabla \phi(x).$$

The same formula essentially works in infinite dimensions.
Lemma 7.13. For \( F \in \mathcal{F}_c^\infty(W) \) of the form \( F(x) = \phi(e_1(x), \ldots, e_n(x)) \) with \( e_i \) \( q \)-orthonormal, we have
\[
NF(x) = (\tilde{N}\phi)(e_1(x), \ldots, e_n(x)).
\]

Proof. This follows from the formula \( N = \delta D \) and (6.1) and Proposition 6.26, and the fact that \( J : (W^*, q) \rightarrow H \) is an isometry. Note for finite dimensions, if we take \( e_1, \ldots, e_n \) to be the coordinate functions on \( \mathbb{R}^n \), this shows that \( \tilde{N} \) really is the Ornstein–Uhlenbeck operator.

Theorem 7.14. The Ornstein–Uhlenbeck semigroup \( T_t \) is given by
\[
T_tF(x) = \int_W F\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \mu(dy).
\]

Proof. Since this is mostly computation, I’ll just sketch it.

Let \( R_t \) denote the right side. We’ll show that \( R_t \) is another semigroup with the same generator.

Showing that \( R_t \) is a semigroup is easy once you remember that \( \mu_t \) is a convolution semigroup, or in other words
\[
\int_W \int_W G(ax + by) \mu(dy) \mu(dx) = \int_W G\left(\sqrt{a^2 + b^2}z\right) \mu(dz).
\]

To check the generator is right, start with the finite dimensional case. If \( \phi \) is a nice smooth function on \( \mathbb{R}^n \), and \( p(y)dy \) is standard Gaussian measure, then show that
\[
\frac{d}{dt}|_{t=0} \int_{\mathbb{R}^n} \phi\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) p(y)dy = \tilde{N}\phi(x).
\]
(First differentiate under the integral sign. For the term with the \( x \), evaluate at \( t = 0 \). For the term with \( y \), integrate by parts, remembering that \( yp(y) = -\nabla p(y) \). If in doubt, assign it as homework to a Math 2220 class.)

Now if \( F \) is a smooth cylinder function on \( W \), do the same and use the previous lemma, noting that \( (e_1, \ldots, e_n) \) have a standard normal distribution under \( \mu \).

There is probably some annoying density argument as the last step. The interested reader can work it out and let me know how it went.

This shows that at time \( t \), \( X_t \) started at \( x \) has a Gaussian distribution (derived from \( \mu \)) with mean \( e^{-t}x \) and variance \( 1 - e^{-2t} \).

Here is a general property of Markovian semigroups that we will use later:

Lemma 7.15. For bounded nonnegative functions \( F, G \), we have
\[
|T_t(FG)(x)|^2 \leq T_t(F^2)(x)T_t(G^2)(x).
\]

Proof. Note the following identity: for \( a, b \geq 0 \),
\[
ab = \frac{1}{2} \inf_{r>0} \left(ra^2 + \frac{1}{r}b^2\right).
\]
(One direction is the AM-GM inequality, and the other comes from taking \( r = b/a \).) So

\[
T_t(FG) = \frac{1}{2} T_t \left( \inf_{r>0} \left( r F^2 + \frac{1}{r} G^2 \right) \right) \\
\leq \frac{1}{2} \inf_{r>0} \left( r T_t(F^2) + \frac{1}{r} T_t(G^2) \right) \\
= \sqrt{T_t(F^2)T_t(G^2)}
\]

where in the second line we used the fact that \( T_t \) is linear and Markovian (i.e. if \( f \leq g \) then \( T_tf \leq T_tg \)).

As a special case, taking \( G = 1 \), we have \( |T_t F(x)|^2 \leq T_t(F^2)(x) \).

Alternative proof: use (7.3), or the fact that \( T_t F(x) = E_x[F(X_t)] \), and Cauchy–Schwarz.

### 7.3 Log Sobolev inequality

Recall that in finite dimensions, the classical Sobolev embedding theorem says that for \( \phi \in C_c^\infty(\mathbb{R}^n) \) (or more generally \( \phi \in W^{1,p}(\mathbb{R}^n) \)),

\[
\|\phi\|_{L^p(\mathbb{R}^n,m)} \leq C_{n,p}(\|\phi\|_{L^p(\mathbb{R}^n,m)} + \|\nabla \phi\|_{L^p(\mathbb{R}^n,m)})
\]

(7.5)

where \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \). Note everything is with respect to Lebesgue measure. In particular, this says that if \( \phi \) and \( \nabla \phi \) are both in \( L^p \), then the integrability of \( \phi \) is actually better: we have \( \phi \in L^{p^*} \). So \( W^{1,p} \subset L^{p^*} \) and the inclusion is continuous (actually, if the inclusion holds at all it has to be continuous, by the closed graph theorem).

This theorem is useless in infinite dimensions in two different ways. First, it involves Lebesgue measure, which doesn’t exist. Second, when \( n = \infty \) we get \( p^* = p \) so the conclusion is a triviality anyway.

In 1975, Len Gross discovered the logarithmic Sobolev inequality \([11]\) which fixes both of these defects by using Gaussian measure and being dimension-independent. Thus it has a chance of holding in infinite dimensions. In fact, it does.

The log-Sobolev inequality says that in an abstract Wiener space, for \( F \in D_{1,2} \) with

\[
\int |F|^2 \ln |F| \, d\mu \leq \|F\|_{L^2(W,\mu)}^2 \ln \|F\|_{L^2(W,\mu)} + \|DF\|_{L^2(W;H)}^2 \, .
\]

(7.6)

If you are worried what happens for \( F \) near 0:

**Exercise 7.16.** \( g(x) = x^2 \ln x \) is bounded below on \((0, \infty)\), and \( g(x) \to 0 \) as \( x \downarrow 0 \).

So if we define “\( 0^2 \ln 0 = 0 \)”, there is no concern about the existence of the integral on the left side (however, what is not obvious is that it is finite). What’s really of interest are the places where \( |F| \) is large, since then \( |F|^2 \ln |F| \) is bigger than \( |F|^2 \).

It’s worth noting that (7.6) also holds in finite dimensions, but there are no dimension-dependent constants appearing in it.
A concise way of stating the log Sobolev inequality is to say that
\[ D^{1,2} \subset L^2 \ln L \]
where \( L^2 \ln L \), by analogy with \( L^p \), represents the set of measurable functions \( F \) with \( \int |F|^2 \ln |F| < \infty \). This is called an Orlicz space; one can play this game to define \( \phi(L) \) spaces for a variety of reasonable functions \( \phi \).

Our proof of the log Sobolev inequality hinges on the following completely innocuous looking commutation relation.

**Lemma 7.17.** For \( F \in D^{1,2} \), \( D T_t F = e^{-t} T_t D F \).

You may object that on the right side we are applying \( T_t \), an operator on the real-valued function space \( L^2(W) \), to the \( H \)-valued function \( D F \). Okay then: we can define \( T_t \) on \( L^2(W; H) \) in any of the following ways:

1. Componentwise: \( T_t u = \sum_i (T_t \langle u(\cdot), h_i \rangle_H) h_i \) where \( h_i \) is an orthonormal basis for \( H \).
2. Via (7.3), replacing the Lebesgue integral with Bochner.
3. Via (7.2): set \( T_t u = \sum_{n=0}^\infty e^{-tn} J_n u \) where \( J_n \) is orthogonal projection onto \( \mathcal{H}_n(H) \subset L^2(W; H) \).

**Exercise 7.18.** Verify that these are all the same. Also verify the inequality
\[ \| T_t u(x) \|_H \leq T_t \| u \|_H (x). \tag{7.7} \]

It’s worth noting that for any \( F \in L^2 \), \( T_t F \in D^{1,2} \). This follows either from the spectral theorem, or from the observation that for any \( t \), the sequence \( \{ne^{-2tm}\} \) is bounded, so \( \sum_n n \| J_n T_t F \|^2 = \sum_n ne^{2tm} \| J_n F \|^2 \leq C \sum \| J_n F \|^2 \leq C \| F \|^2 \). In fact, more is true: we have \( T_t F \in \text{dom } N \), and indeed \( T_t F \in \text{dom } N^\infty \).

**Proof of Lemma 7.17.**

\[
DT_t F = D \sum_{n=0}^\infty e^{-tn} J_n F = \sum_{n=1}^\infty e^{-tn} D J_n F \quad \text{(recall } DJ_0 = 0) \\
= \sum_{n=1}^\infty e^{-tn} J_{n-1} DF \\
= \sum_{k=0}^\infty e^{-t(k+1)} J_k DF = e^{-t} T_t DF
\]

where we re-indexed by letting \( k = n-1 \). We’ve extended to \( L^2(W; H) \) some Wiener chaos identities that we only really proved for \( L^2(W) \); as an exercise you can check the details.

There’s also an infinitesimal version of this commutation:

**Lemma 7.19.** For \( F \in FC_c^\infty(W) \), \( DN F = (N + 1)DF \).
Proof. Differentiate the previous lemma at $t = 0$. Or, use Wiener chaos expansion.

Exercise 7.20. (Not necessarily very interesting) Characterize the set of $F$ for which the foregoing identity makes sense and is true.

We can now prove the log Sobolev inequality (7.6). This proof is taken from [19] which actually contains several proofs.

\[ Q := 2 \left( \int F^2 \ln F d\mu - \|F\|^2 \ln \|F\| \right) = \int G \ln G d\mu - \int G d\mu \ln \int G d\mu. \]  

(7.8)

and we want to bound this quantity $Q$ by $\|DF\|_{L^2(W;\mu)}^2$.

Note that for any $G \in L^2(W)$ we have $\lim_{t \to \infty} T_t G = J_0 G = \int G d\mu$. (Use Lemma 7.11 and monotone convergence.) So we can think of $T_t G$ as a continuous function from $[0, \infty]$ to $L^2(W;\mu)$. It is continuously differentiable on $(0, \infty)$ and hence bounded above and below, so $(T_t G) \cdot (\ln T_t G)$ is also bounded and hence in $L^2$. Then $Q = \int_W (A(0) - A(\infty)) d\mu$. Since we want to use the fundamental theorem of calculus, we use the chain rule to see that

\[ A'(t) = -(NT_t G)(1 + \ln T_t G). \]

So by the fundamental theorem of calculus, we have

\[ Q = -\int_W \int_0^\infty A'(t) \, dt \, d\mu \]

\[ = \int_W \int_0^\infty (NT_t G)(1 + \ln T_t G) \, dt \, d\mu. \]

There are two integrals in this expression, so of course we want to interchange them. To justify this, we note that $1 + \ln T_t G$ is bounded (since $0 < a^2 \leq G \leq b^2$ and $T_t$ is Markovian, we also have $a^2 \leq T_t G \leq b^2$), and so it is enough to bound

\[ \int_W \int_0^\infty |NT_t G| \, dt \, d\mu = \int_0^\infty \|NT_t G\|_{L^1(W;\mu)} \, dt \]

\[ \leq \int_0^\infty \|NT_t G\|_{L^2(W;\mu)} \, dt \]

since $\|\cdot\|_{L^1} \leq \|\cdot\|_{L^2}$ over a probability measure (Cauchy–Schwarz or Jensen). Note that $NT_t G = T_t NG$ is continuous from $[0, \infty]$ to $L^2(W;\mu)$, so $\|NT_t G\|_{L^2(W)}$ is continuous in $t$ and hence bounded on compact sets. So we only have to worry about what happens for large $t$. But Corollary 7.12 says that it decays exponentially, and so is integrable. (Note that $\int NG d\mu = \langle NG, 1 \rangle_{L^2(W)} = \langle DG, D1 \rangle_{L^2(W)} = 0$.)

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So after applying Fubini’s theorem, we get

\[ Q = \int_0^\infty \int_W (NT_tG)(1 + \ln T_tG)\,d\mu \,dt \]
\[ = \int_0^\infty \langle NT_tG, 1 + \ln T_tG \rangle_{L^2(W)}\,dt. \]

Now since \( N = \delta D \) we have, using the chain rule,

\[ \langle NT_tG, 1 + \ln T_tG \rangle_{L^2(W)} = \langle DT_tG, DT_tG + D \ln T_tG \rangle_{L^2(W;H)} \]
\[ = \langle DT_tG, DT_tG \rangle_{L^2(W;H)} \]
\[ = \int_W \frac{1}{T_tG} \| DT_tG \|_H^2 \,d\mu \]
\[ = e^{-2t} \int_W \frac{1}{T_tG} \| DT_tG \|_H^2 \,d\mu \]

where we have just used the commutation \( DT_t = e^{-t}T_tD \).

Let’s look at \( \| T_tDG \|_H^2 \). Noting that \( DG = 2FDF \), we have

\[ \| T_tDG \|_H^2 \leq (T_t \| DG \|_H^2)^2 \]
\[ = 4(T_t(F \| DF \|_H^2))^2 \]
\[ \leq 4(T_t(F^2))(T_t \| DF \|_H^2) \]

by (7.4).

Thus we have reached

\[ \int_W \frac{1}{T_tG} \| T_tDG \|_H^2 \,d\mu \leq 4 \int_W T_t \| DF \|_H^2 \,d\mu. \]

But since \( T_t \) is self-adjoint and \( T_t 1 = 1 \) (or, if you like, the fact that \( T_t \) commutes with \( J_0 \), we have

\[ \int_W T_t f \,d\mu = \int f \,d\mu \]

for any \( t \). Thus \( \int_W T_t \| DF \|_H^2 \,d\mu = \int_W \| DF \|_H^2 \,d\mu = \| DF \|_{L^2(W;H)}^2 \). So we have

\[ Q \leq \left( 4 \int_0^\infty e^{-2t} \,dt \right) \| DF \|_{L^2(W;H)}^2 \]

The parenthesized constant equals 2 (consult a Math 1120 student if in doubt). This is what we wanted.

To extend this to all \( F \in \mathbb{D}^{1,2} \), we need some density arguments. Suppose now that \( F \) is a smooth cylinder function which is bounded, say \( |F| \leq M \). Fix \( \epsilon > 0 \), and for each \( n \) let \( \varphi_n \in C^\infty(\mathbb{R}) \) be a positive smooth function, such that:

1. \( \varphi_n \) is bounded away from 0;
2. \( \varphi_n \leq M \);
3. \( \varphi_n' \leq 1 + \epsilon; \)
4. \( \varphi_n(x) \to |x| \) pointwise on \([-M, M]\).
Thus \( \varphi_n(F) \) is a smooth cylinder function, bounded away from 0 and bounded above, so it satisfies the log Sobolev inequality. Since \( \varphi_n(F) \to |F| \) pointwise and boundedly, we have \( \|\varphi_n(F)\|_{L^2(W)} \to \|F\|_{L^2(W)} \) by dominated convergence. We also have, by the chain rule, \( \|D\varphi_n(F)\|_{L^2(W;H)} \leq (1 + \epsilon) \|DF\|_{L^2(W;H)} \). Thus

\[
\limsup_{n \to \infty} \int_W \varphi_n(F)^2 \ln \varphi_n(F) \, d\mu \leq \|F\|^2 \ln \|F\| + (1 + \epsilon) \|DF\|^2.
\]

Now since \( x^2 \ln x \) is continuous, we have \( \varphi_n(F)^2 \ln \varphi_n(F) \to |F|^2 \ln |F| \) pointwise. Since \( x^2 \ln x \) is bounded below, Fatou’s lemma gives

\[
\int_W |F|^2 \ln |F| \, d\mu \leq \liminf_{n \to \infty} \int_W \varphi_n(F)^2 \ln \varphi_n(F) \, d\mu
\]

and so this case is done after we send \( \epsilon \to 0 \). (Dominated convergence could also have been used, which would give equality in the last line.)

Finally, let \( F \in \mathbb{D}^{1,2} \). We can find a sequence of bounded cylinder functions \( F_n \) such that \( F_n \to F \) in \( L^2(W) \) and \( DF_n \to DF \) in \( L^2(W;H) \). Passing to a subsequence, we can also assume that \( F_n \to F \) \( \mu \)-a.e., and we use Fatou’s lemma as before to see that the log Sobolev inequality holds in the limit.

Note that we mostly just used properties that are true for any Markovian semigroup \( T_t \) that is conservative (\( T_t 1 = 1 \)). The only exception was the commutation \( DT_t = e^{-t}T_tD \). In fact, an inequality like \( \|DT_t F\|_H \leq C(t)T_t \|DF\|_H \) would have been good enough, provided that \( C(t) \) is appropriately integrable. (One of the main results in my thesis was to prove an inequality like this for a certain finite-dimensional Lie group, in order to obtain a log-Sobolev inequality by precisely this method.)

Also, you might wonder: since the statement of the log-Sobolev inequality only involved \( D \) and \( \mu \), why did we drag the Ornstein–Uhlenbeck semigroup into it? Really the only reason was the fact that \( T_\infty F = \int F \, d\mu \), which is just saying that \( T_t \) is the semigroup of a Markov process whose distribution at a certain time \( t_0 \) (we took \( t_0 = \infty \)) is the measure \( \mu \) we want to use. If we want to prove this theorem in finite dimensions, we could instead use the heat semigroup \( P_t \) (which is symmetric with respect to Lebesgue measure) and take \( t = 1 \), beak Brownian motion at time 1 also has a standard Gaussian distribution.

### 8 Absolute continuity and smoothness of distributions

This section will just hint at some of the very important applications of Malliavin calculus to proving absolute continuity results.

When presented with a random variable (or random vector) \( X \), a very basic question is “What is its distribution?”, i.e. what is \( \nu(A) := P(X \in A) \) for Borel sets \( A \)? A more basic question is “Does \( X \) has a continuous distribution?”, i.e. is \( \nu \) absolutely continuous to Lebesgue measure? If so, it has a Radon–Nikodym derivative \( f \in L^1(m) \), which is a density function for \( X \). It may happen that \( f \) is continuous or \( C^k \) or \( C^\infty \), in which case so much the better.

Given a Brownian motion \( B_t \) or similar process, one can cook up lots of complicated random variables whose distributions may be very hard to work out. For example:
\begin{itemize}
  \item $X = f(B_t)$ for some fixed $t$ (this is not so hard)
  \item $X = f(B_T)$ for some stopping time $T$
  \item $X = \sup_{t \in [0,1]} B_t$
  \item $X = \int_0^1 Y_t \, dB_t$
  \item $X = Z_t$, where $Z$ is the solution to some SDE $dZ_t = f(Z_t) dB_t$.
\end{itemize}

Malliavin calculus gives us some tools to learn something about the absolute continuity of such random variables, and the smoothness of their densities.

Let $(W,H,\mu)$ be an abstract Wiener space. A measurable function $F : W \to \mathbb{R}$ is then a random variable, and we can ask about its distribution. If we’re going to use Malliavin calculus, we’d better concentrate on $F \in \mathbb{D}^{1,p}$. An obvious obstruction to absolute continuity would be if $F$ is constant on some set $A$ of positive $\mu$-measure; in this case, as we have previously shown, $DF = 0$ on $A$. The following theorem says if we ensure that $DF$ doesn’t vanish, then $F$ must be absolutely continuous.

**Theorem 8.1.** Let $F \in \mathbb{D}^{1,1}$, and suppose that $DF$ is nonzero $\mu$-a.e. Then the law of $F$ is absolutely continuous to Lebesgue measure.

**Proof.** Let $\nu = \mu \circ F^{-1}$ be the law of $F$; our goal is to show $\nu \ll \mu$.

By replacing $F$ with something like $\arctan(F)$, we can assume that $F$ is bounded; say $0 \leq F \leq 1$. So we want to show that $\nu$ is absolutely continuous to Lebesgue measure $m$ on $[0,1]$. Let $A \subset [0,1]$ be Borel with $m(A) = 0$; we want to show $\nu(A) = 0$.

Choose a sequence $g_n \in C^\infty([0,1])$ such that $g_n \to 1_A$ $m$-a.e., and such that the $g_n$ are uniformly bounded (say $|g_n| \leq 2$). Set $\psi_n(t) = \int_0^t g_n(s) \, ds$. Then $\psi_n \in C^\infty$, $|\psi_n| \leq 2$, and $\psi_n \to 0$ pointwise (everywhere).

In particular $\psi_n(F) \to 0$ $\mu$-a.e. (in fact everywhere), and thus also in $L^1(W,\mu)$ by bounded convergence. On the other hand, by the chain rule, $D\psi_n(F) = g_n(F)DF$. Now since $g_n \to 1_A$ $\nu$-a.e., we have $g_n(F) \to 1_A(F)$ $\mu$-a.e., and boundedly. Thus $g_n(F)DF \to 1_A(F)DF$ in $L^1(W;H)$. Now $D$ is a closed operator, so we must have $1_A(F)DF = D0 = 0$. But by assumption $DF \neq 0$ $\mu$-a.e., so we have to have $1_AF = 0$ $\mu$-a.e., that is, $\nu(A) = 0$. \hfill $\Box$

So knowing that the derivative $DF$ “never” vanishes guarantees that the law of $F$ has a density. If $DF$ mostly stays away from zero in the sense that $\|DF\|^{-1}_H \in L^p(W)$ for some $p$, then this gives more smoothness (e.g. differentiability) for the density. See Nualart for precise statements.

In higher dimensions, if we have a function $F = (F^1,\ldots,F^n) : W \to \mathbb{R}^n$, the object to look at is the “Jacobian,” the matrix-valued function $\gamma_F : W \to \mathbb{R}^{n \times n}$ defined by $\gamma_F(x)_{ij} = \langle DF^i(x), DF^j(x) \rangle_H$. If $\gamma_F$ is almost everywhere nonsingular, then the law of $F$ has a density. If we have $(\det \gamma_F)^{-1} \in L^p(W)$ for some $p$, then we get more smoothness.

Here’s another interesting fact. Recall that the **support** of a Borel measure $\nu$ on a topological space $\Omega$ is by definition the set of all $x \in \Omega$ such that every neighborhood of $x$ has nonzero $\nu$ measure. This set is closed.

**Proposition 8.2.** If $F \in \mathbb{D}^{1,2}$, then the support of the law of $F$ is connected, i.e. is a closed interval in $\mathbb{R}$.  

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Proof. Let $\nu = \mu \circ F^{-1}$. Suppose $\text{supp} \nu$ is not connected. Then there exists $a \in \mathbb{R}$ such that there are points of $\text{supp} \nu$ to the left and right of $a$. Since $\text{supp} \nu$ is closed, there is an open interval $(a, b)$ in the complement of $\text{supp} \nu$. That is, we have $\mu(a < F < b) = 0$ but $0 < \mu(F \leq a) < 1$. Let $\psi \in C^\infty(\mathbb{R})$ have $\psi(t) = 1$ for $t \leq a$ and $\psi(t) = 0$ for $t \geq b$, and moreover take $\psi$ and all its derivatives to be bounded. Then $\psi(F) = 1_{(-\infty,a]}(F) = 1_{(a,\infty]}$. Since $\psi$ is smooth, $1_{(F \leq a)} = \psi(F) \in \mathbb{D}^{1,2}$ by the chain rule (Lemma [6.2.4]). By the zero-one law of Proposition [6.2.5] $\mu(F \leq a)$ is either 0 or 1, a contradiction.

As an example, let’s look at the maximum of a continuous process.

Let $(W, H, \mu)$ be an abstract Wiener space. Suppose we have a process $\{X_t : t \in [0, 1]\}$ defined on $W$, i.e. a measurable map $X : [0, 1] \times W \to \mathbb{R}$, which is a.s. continuous in $t$. (If we take $W = C([0, 1])$ and $\mu$ the law of some continuous Gaussian process $Y_t$, then $X_t = Y_t$, in other words $X_t(\omega) = \omega(t)$, would be an example. Another natural example would be to take classical Wiener space and let $X_t$ be the solution of some SDE.) Let $M = \text{sup}_{t \in [0,1]} X_t$. We will show that under certain conditions, $M$ has an absolutely continuous law.

(Note you can also index $\{X_t\}$ by any other compact metric space $S$ and the below proofs will go through just fine. If you take $S$ finite, the results are trivial. You can take $S = [0, 1]^2$ and prove things about Brownian sheet. You can even take $S$ to be Cantor space if you really want (hi Clinton!).)

Lemma 8.3. Suppose $F_n \in \mathbb{D}^{1,2}$, $F_n \to F$ in $L^2(W)$, and $\sup_n \|DF_n\|_{L^2(W;H)} < \infty$. Then $F \in \mathbb{D}^{1,2}$ and $DF_n \to DF$ weakly in $L^2(W;H)$.

Proof. This is really a general fact about closed operators on Hilbert space. Since $\{DF_n\}$ is a bounded sequence in $L^2(W;H)$, by Alaoglu’s theorem we can pass to a subsequence and assume that $DF_n$ converges weakly in $L^2(W;H)$, to some element $u$. Suppose $v \in \text{dom} \delta$. Then $\langle DF_n, v \rangle_{L^2(W;H)} = \langle F_n, \delta v \rangle_{L^2(W)}$. The left side converges to $\langle u, v \rangle_{L^2(W;H)}$ and the right side to $\langle F, \delta v \rangle_{L^2(W)}$. Since the left side is continuous in $v$, we have $F \in \text{dom} \delta^* = \text{dom} D = \mathbb{D}^{1,2}$. Moreover, since we have $\langle DF_n, v \rangle \to \langle DF, v \rangle$ for all $v$ in a dense subset of $L^2(W;H)$, and $\{DF_n\}$ is bounded, it follows from the triangle inequality that $DF_n \to DF$ weakly. Since we get the same limit no matter which weakly convergent subsequence we passed to, it must be that the original sequence $DF_n$ also converges weakly to $DF$.

Recall, as we’ve previously argued, that if $F \in \mathbb{D}^{1,2}$, then $\|F\| \in \mathbb{D}^{1,2}$ also, and $\|D[F]\|_H \leq \|DF\|_H$ a.e. (Approximate $|t|$ by smooth functions with uniformly bounded derivatives.) It follows that if $F_1, F_2 \in \mathbb{D}^{1,2}$, then $F_1 \vee F_2, F_1 \wedge F_2 \in \mathbb{D}^{1,2}$ also. $(F_1 \wedge F_2 = F_1 + F_2 - |F_1 - F_2|$, and $F_1 \vee F_2 = F_1 + F_2 + |F_1 - F_2|$.) Then by iteration, if $F_1, \ldots, F_n \in \mathbb{D}^{1,2}$, then $\min_k F_k, \max_k F_k \in \mathbb{D}^{1,2}$ as well.

Lemma 8.4. Suppose $X, M$ are as above, and:

1. $\int_W \sup_{t \in [0,1]} |X_t(\omega)|^2 \mu(d\omega) < \infty$;
2. For any $t \in [0, 1]$, $X_t \in \mathbb{D}^{1,2}$;
3. The $H$-valued process $DX_t$ has an a.s. continuous version (which we henceforth fix);
4. $\int_W \sup_{t \in [0,1]} \|DX_t(\omega)\|^2_H \mu(d\omega) < \infty$.

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Then $M \in \mathbb{D}^{1,2}$.

**Proof.** The first property guarantees $M \in L^2(W)$. Enumerate the rationals in $[0,1]$ as $\{q_n\}$. Set $M_n = \max\{X_{q_1}, \ldots, X_{q_n}\}$. Then $M_n \in \mathbb{D}^{1,2}$ (using item 2). Clearly $M_n \uparrow M$ so by monotone

convergence $M_n \to M$ in $L^2(W)$. It suffices now to show that $\sup_n \|DM_n\|_{L^2(W;H)} < \infty$. Fix $n$, and for $k = 1, \ldots, n$ let $A_k$ be the set of all $\omega$ where the maximum in $M_n$ is achieved by $X_{q_k}$, with ties going to the smaller $k$. That is,

\[
A_1 = \{\omega : X_{q_1}(\omega) = M_n(\omega)\}
\]

\[
A_2 = \{\omega : X_{q_1}(\omega) \neq M_n(\omega), X_{q_2}(\omega) = M_n(\omega)\}
\]

\[
\vdots
\]

\[
A_n = \{\omega : X_{q_1}(\omega) \neq M_n(\omega), \ldots, X_{q_{n-1}}(\omega) \neq M_n(\omega), X_{q_n}(\omega) = M_n(\omega)\}
\]

Clearly the $A_k$ are Borel and partition $W$, and $M_n = X_{q_k}$ on $A_k$. By the local property used before, for every $\omega$ we have

\[
M = \langle \omega \rangle
\]

so the above expression makes sense. Then

\[
\sup_{t \in [0,1]} \|DX_t\|_H \leq \sup_{t \in [0,1]} \|DM_t\|_H.
\]

Exercise 8.5. Let $\{X_t, t \in [0,1]\}$ be a continuous centered Gaussian process. Then we can take $W = C([0,1])$ (or a closed subspace thereof) and $\mu$ to be the law of the process, and define $X_t$ on $W$ by $X_t(\omega) = \omega(t)$. Verify that the hypotheses of Proposition 8.4 are satisfied.

**Proposition 8.6.** Suppose $X_t$ satisfies the hypotheses of the previous theorem, and moreover

\[
\mu(\{\omega : X_t(\omega) = M(\omega) \implies DX_t(\omega) \neq 0\}) = 1.
\]

(Note we are fixing continuous versions of $X_t$ and $DX_t$, so the above expression makes sense.) Then $DM \neq 0$ a.e. and $M$ has an absolutely continuous law.

**Proof.** It is enough to show

\[
\mu(\{\omega : X_t(\omega) = M(\omega) \implies DX_t(\omega) = DM(\omega)\}) = 1.
\]

Call the above set $A$. (Note that for every fixed $\omega$, $M(\omega) = X_t(\omega)$ for some $t$.)

Let $E$ be a countable dense subset of $H$. For fixed $r, s \in \mathbb{Q}$, $h \in E$, $k > 0$, let

\[
G_{r,s,h,k} = \{\omega : \sup_{t \in (r,s)} X_t(\omega) = M(\omega), \langle DX_t(\omega) - DM(\omega), h \rangle_H \geq \frac{1}{n} \text{ for all } r < t < s\}.
\]

Enumerate the rationals in $(r, s)$ as $\{q_i\}$. If we let $M' = \sup_{t \in (r,s)} X_t$, $M_n' = \max\{X_{q_1}, \ldots, X_{q_n}\}$, then as we argued before, $M_n' \to M'$ in $L^2(W)$, and $DM_n' \to DM'$ weakly in $L^2(W;H)$. On the other hand, by the local property used before, for every $\omega$ there is some $t_i$ with $DM_n' = DX_{t_i}$. Thus for $\omega \in G_{r,s,h,k}$ we have $\langle DM_n'(\omega) - DM'(\omega), h \rangle_H \geq \frac{1}{n}$ for all $r < t < s$. Integrating this inequality, we have $\langle DM_n' - DM', h \mathbb{1}_{G_{r,s,h,k}} \rangle_{L^2(W;H)} \geq \frac{1}{n} \mu(G_{r,s,h,k})$ for all $n$. The left side goes to 0 by weak convergence, so it must be that $\mu(G_{r,s,h,k}) = 0$.

However, $A^c = \bigcup G_{r,s,h,k}$ which is a countable union. (If $\omega \in A^c$, there exists $t$ such that $X_t(\omega) = M(\omega)$ but $DX_t(\omega) \neq DM(\omega)$. As such, there must exist $h \in E$ with $\langle DX_t(\omega) - DM(\omega), h \rangle_H \neq 0$; by replacing $h$ by $-h$ or something very close to it, we can assume $\langle DX_t(\omega) - DM(\omega), h \rangle_H > 0$. As $DX_t$ is assumed continuous, there exists $(r,s) \in \mathbb{Q}$ and $k > 0$ such that $\langle DX_t(\omega) - DM(\omega), h \rangle_H > \frac{1}{k}$ for all $t \in (r,s)$. So we have $\omega \in G_{r,s,h,k}$.)

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Exercise 8.7. Again let $X_t$ be a centered Gaussian process as in Exercise 8.5 above. Give an example of a process for which $M$ does not have an absolutely continuous law. However, show that if $P(M = 0) = 0$, then the hypothesis of Proposition 8.6 is satisfied. (Can we show this always holds whenever $X_t$ is strong Markov?)

A Miscellaneous lemmas

Lemma A.1. Let $y \in \mathbb{R}^\infty$, and suppose that $\sum y(i)g(i)$ converges for every $g \in \ell^2$. Then $y \in \ell^2$.

Proof. For each $n$, let $H_n \in (\ell^2)^*$ be the bounded linear functional $H_n(g) = \sum_{i=1}^n y(i)g(i)$. By assumption, for each $g \in \ell^2$, the sequence $\{H_n(g)\}$ converges; in particular $\sup_n |H_n(g)| < \infty$. So by the uniform boundedness principle, $\sup_n ||H_n||_{(\ell^2)^*} < \infty$. But $||H_n||_{(\ell^2)^*} = \sum_{i=1}^n |y(i)|^2$, so $\sum_{i=1}^\infty |y(i)|^2 = \sup_n ||H_n||_{(\ell^2)^*}^2 < \infty$ and $y \in \ell^2$.

For an elementary, constructive proof, see also [17].

Lemma A.2. Let $H$ be a separable Hilbert space and $E \subset H$ a dense subspace. There exists an orthonormal basis $\{e_i\}$ for $H$ with $\{e_i\} \subset E$.

Proof. Choose a sequence $\{x_i\} \subset E$ which is dense in $H$. (To see that this is possible, let $\{y_k\}$ be a countable dense subset of $H$, and choose one $x_i$ inside each ball $B(y_k, 1/m).$) Then apply Gram-Schmidt to $x_i$ to get an orthonormal sequence $\{e_i\} \subset E$ with $x_n \in \text{span}\{e_1, \ldots, e_n\}$. Then since $\{e_i\} \subset \text{span}\{e_i\}$, $\text{span}\{e_i\}$ is dense in $H$, so $\{e_i\}$ is an orthonormal basis for $H$.

Lemma A.3. Let $X_n \sim N(0, \sigma_n^2)$ be a sequence of mean-zero Gaussian random variables converging in distribution to a finite random variable $X$. Then $X$ is also Gaussian, with mean zero and variance $\sigma^2 = \lim \sigma_n^2$ (and the limit exists).

Proof. Suppose $\sigma_{n_k}^2$ is a subsequence of $\sigma_n^2$ converging to some $\sigma^2 \in [0, +\infty]$. (By compactness, such a subsequence must exist.) Now taking Fourier transforms, we have $e^{-\lambda^2\sigma_{n_k}^2/2} = E[e^{i\lambda X_n}] \to E[e^{i\lambda X}]$ for each $X$, so $E[e^{i\lambda X}] = e^{-\lambda^2\sigma^2/2}$. Moreover, the Fourier transform of $X$ must be continuous and equal 1 at $\lambda = 0$, which rules out the case $\sigma^2 = +\infty$. So $X \sim N(0, \sigma^2)$. Since we get the same $\sigma^2$ no matter which convergent subsequence of $\sigma_n^2$ we start with, $\sigma_n^2$ must converge to $\sigma^2$.

Lemma A.4. Let $\mu$ be any finite Borel measure on $[0, 1]$. Then $C^\infty([0, 1])$ is dense in $L^p([0, 1], \mu)$.

Proof. Use Dynkin’s multiplicative system theorem. Let $M$ consist of all $\mu$-versions of all bounded measurable functions in the closure of $C^\infty$ in $L^p(\mu)$. Then $M$ is a vector space closed under bounded convergence (since bounded convergence implies $L^p(\mu)$ convergence) and it contains $C^\infty([0, 1])$. By Dynkin’s theorem, $M$ contains all bounded $\mathcal{F}$-measurable functions, where $\mathcal{F}$ is the smallest $\sigma$-algebra that makes all functions from $C^\infty([0, 1])$ measurable. But the identity function $f(x) = x$ is in $C^\infty$. So for any Borel set $B$, we must have $B = f^{-1}(B) \in \mathcal{F}$. Thus $\mathcal{F}$ is actually the Borel $\sigma$-algebra, and $M$ contains all bounded measurable functions. Since the bounded functions are certainly dense in $L^p(\mu)$ (by dominated convergence), we are done.
B Radon measures

Definition B.1. A finite Borel measure \( \mu \) on a topological space \( W \) is said to be **Radon** if for every Borel set \( B \), we have

\[
\mu(B) = \sup\{\mu(K) : K \subset B, K \text{compact}\}
\]

(we say that such a set \( B \) is **inner regular**). Equivalently, \( \mu \) is Radon if for every Borel set \( B \) and every \( \epsilon > 0 \), there exists a compact \( K \subset B \) with \( \mu(B \setminus K) < \epsilon \).

Proposition B.2. If \( X \) is a compact metric space, every finite Borel measure on \( X \) is Radon.

Proof. Let \((X,d)\) be a compact metric space, and \( \mu \) a Borel measure. Let \( \mathcal{F} \) denote the collection of all Borel sets \( B \) such that \( B \) and \( B^C \) are both inner regular. I claim \( \mathcal{F} \) is a \( \sigma \)-algebra. Clearly \( \emptyset \in \mathcal{F} \) and \( \mathcal{F} \) is also closed under complements. If \( B_1, B_2, \ldots \in \mathcal{F} \) are disjoint, and \( B = \bigcup_n B_n \) then since \( \sum_n \mu(B_n) = \mu(B) < \infty \), there exists \( n \) so large that \( \sum_{n=1}^N \mu(B_n) < \epsilon \). For \( n = 1, \ldots, N \), choose a compact \( K_n \subset B_n \) with \( \mu(B_n \setminus K_n) < \epsilon/N \). Then if \( K = \bigcup_{n=1}^N K_n \), \( K \) is compact, \( K \subset B \), and \( \mu(B \setminus K) < 2\epsilon \). So \( B \) is inner regular.

Next, \( \mathcal{F} \) contains all open sets \( U \). For any open set \( U \) may be written as a countable union of compact sets \( K_n \). (For every \( x \in U \) there is an open ball \( B(x,r_x) \) contained in \( U \), hence \( B(x,r_x/2) \subset U \) also. Since \( X \) is second countable we can find a basic open set \( V_x \) with \( x \in V_x \subset B(x,r_x/2) \), so \( V_x \subset U \). Then \( U = \bigcup_{x \in U} V_x \). But this union actually contains only countably many distinct sets.) Thus by countable additivity, \( U \) is inner regular. \( U^C \) is compact and so obviously inner regular. Thus \( U \in \mathcal{F} \). Since \( \mathcal{F} \) is a \( \sigma \)-algebra and contains all open sets, it contains all Borel sets. \( \square \)

Proposition B.3. Every complete separable metric space \((X,d)\) is homeomorphic to a Borel subset of the compact metric space \([0,1]^\infty\).

Proof. Without loss of generality, assume \( d \leq 1 \). Fix a dense sequence \( x_1, x_2, \ldots \) in \( X \) and for each \( x \in X \), set \( F(x) = (d(x,x_1),d(x,x_2),\ldots) \in [0,1]^\infty \). It is easy to check that \( F \) is continuous. \( F \) is also injective: for any \( x \in X \) we can choose a subsequence \( x_{n_k} \to x \), so that \( d(x_{n_k},x) \to 0 \). Then if \( F(x) = F(y) \), then \( d(x_n,x) = d(x_n,y) \) for all \( n \), so \( x_{n_k} \to y \) as well, and \( x = y \). Finally, \( F \) has a continuous inverse. Suppose \( F(y_m) \to F(y) \). Choose an \( x_n \) such that \( d(x_n,y) < \epsilon \). We have \( F(y_m)_n = d(x_n,y_m) \to d(x_n,y) = F(y)_n \), so for sufficiently large \( m \), \( d(y_m,x_n) < \epsilon \), and by the triangle inequality \( d(y_m,y) < 2\epsilon \).

Lastly, we check \( F(X) \) is Borel. Well, this theorem is standard and I’m too lazy to write it out. See, e.g. Srivastava’s *A course on Borel sets*, section 2.2. \( \square \)

Corollary B.4. Any finite Borel measure \( \mu \) on a complete separable metric space \( X \) is Radon.

Proof. Let \( F \) be the above embedding of \( X \) into \([0,1]^\infty\). Then \( \mu \circ F^{-1} \) defines a Borel measure on \( F(X) \). We can extend it to a Borel measure on \([0,1]^\infty\) by setting \( \tilde{\mu}(B) = \mu(F^{-1}(B \cap F(X))) \), i.e. \( \tilde{\mu} \) assigns measure zero to all sets outside \( F(X) \). Then we know that \( \tilde{\mu} \) is Radon and hence so is \( \mu \). \( \square \)

Exercise B.5. As a corollary of this, for any Borel probability measure on a Polish space, there is a sequence of compact sets \( K_n \) such that \( \mu(\bigcup K_n) = 1 \). This is perhaps surprising because compact sets in an infinite dimensional Banach space are very thin; in particular they are nowhere dense. For classical Wiener space with Wiener measure, find explicit sets \( K_n \) with this property. (Hint: Think of some well-known sample path properties of Brownian motion.)
C Miscellaneous Exercises

Exercise C.1. Let $X$ be a set, and let $\tau_w$ and $\tau_s$ be two topologies on $X$ such that $\tau_w \subset \tau_s$. $\tau_w$ is said to be “weaker” or “coarser,” while $\tau_s$ is “stronger” or “finer.”

Fill in the following chart. Here $A \subset X$, and $Y,Z$ are some other topological spaces. All terms such as “more,” “less,” “larger,” “smaller” should be understood in the sense of implication or containment. For instance, since every set which is open in $\tau_w$ is also open in $\tau_s$, we might say $\tau_s$ has “more” open sets and $\tau_w$ has “fewer.”

| Property                   | $\tau_w$ | $\tau_s$ | Choices               |
|----------------------------|----------|----------|-----------------------|
| Open sets                  | More     | fewer    |                       |
| Closed sets                | More     | fewer    |                       |
| Dense sets                 | More     | fewer    |                       |
| Compact sets               | More     | fewer    |                       |
| Connected sets             | More     | fewer    |                       |
| Closure $A$                | Larger   | smaller  |                       |
| Interior $A^0$             | Larger   | smaller  |                       |
| Precompact sets            | More     | fewer    |                       |
| Separable sets             | More     | fewer    |                       |
| Continuous functions $X \to Y$ | More     | fewer    |                       |
| Continuous functions $Z \to X$ | More     | fewer    |                       |
| Identity map continuous    | $(X,\tau_s) \to (X,\tau_w)$ or vice versa |                      |
| Convergent sequences       | More     | fewer    |                       |

Exercise C.2. Now suppose that $X$ is a vector space, and $\tau_w \subset \tau_s$ are generated by two norms $\|\cdot\|_w, \|\cdot\|_s$. Also let $Y,Z$ be other normed spaces.

| Property                   | $\|\cdot\|_w$ | $\|\cdot\|_s$ | Choices                                |
|----------------------------|---------------|---------------|----------------------------------------|
| Size of norm               | $\|\cdot\|_s \leq C \|\cdot\|_w$ or vice versa, or neither |
| Closed (unbounded) operators $X \to Y$ | More     | fewer    |                       |
| Closed (unbounded) operators $Z \to X$ | More     | fewer    |                       |
| Cauchy sequences           | More     | fewer    |                       |

Exercise C.3. Give an example where $X$ is complete in $\|\cdot\|_s$ but not in $\|\cdot\|_w$.

Exercise C.4. Give an example where $X$ is complete in $\|\cdot\|_w$ but not in $\|\cdot\|_s$. (This exercise is “abstract nonsense,” i.e. it uses the axiom of choice.)

Exercise C.5. If $X$ is complete in both $\|\cdot\|_w$ and $\|\cdot\|_s$, show that the two norms are equivalent, i.e. $c \|\cdot\|_s \leq \|\cdot\|_w \leq C \|\cdot\|_s$ (and in particular $\tau_s = \tau_w$).

Exercise C.6. In the previous problem, the assumption that $\tau_w \subset \tau_s$ was necessary. Give an example of a vector space $X$ and complete norms $\|\cdot\|_1, \|\cdot\|_2$ which are not equivalent. (Abstract nonsense.)

Exercise C.7. Let $X,Y$ be Banach spaces with $X$ reflexive, $T : X \to Y$ a bounded operator, and $T^* : Y^* \to X^*$ its adjoint.

1. If $T$ is injective, then $T^*$ has dense range.
2. If $T$ has dense range, then $T^*$ is injective.

**Exercise C.8.** For classical Wiener space $(W, \mu)$, find an explicit sequence of compact sets $K_n \subset W$ with $\mu(\bigcup_n K_n) = 1$.

### D Questions for Nate

1. Is a Gaussian Borel measure on a separable Banach space always Radon? (Yes, a finite Borel measure on a Polish space is always Radon. See Bogachev Theorem A.3.11.)

2. Compute the Cameron-Martin space $H$ for various continuous Gaussian processes (Ornstein–Uhlenbeck, fractional Brownian motion).

3. Why should Brownian motion “live” in the space $C([0, 1])$ instead of the smaller Hölder space $C^{0, \alpha}([0, 1])$ for $\alpha < 1/2$?

4. What’s the relationship between Brownian motion on classical Wiener space and various other 2-parameter Gaussian processes (e.g. Brownian sheet)? (Compute covariances.)

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