Boundary Hölder Regularity for Elliptic Equations on Reifenberg Flat Domains

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Abstract. In this paper, we investigate the boundary Hölder regularity for elliptic equations (precisely, the Poisson equation, linear equations in divergence form and non-divergence form, the p-Laplace equations and fully nonlinear elliptic equations) on Reifenberg flat domains. We prove that for any $0 < \alpha < 1$, there exists $\delta > 0$ such that the solution is $C^\alpha$ at $x_0 \in \partial \Omega$ provided that $\Omega$ is $\delta$-Reifenberg flat at $x_0$ (see Definition 1.1). In particular, for any $0 < \alpha < 1$, if $\partial \Omega$ is $C^1$ and $u = g$ on $\partial \Omega$ with $g \in C^\alpha(x_0)$, then $u \in C^\alpha(x_0)$. A similar result for the Poisson equation has been proved by Lemenant and Sire [Lemenant and Sire(2013)], where the Alt-Caffarelli-Friedman’s monotonicity formula is used.

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1. Introduction

In the regularity theory of partial differential equations, the Hölder continuity is a kind of quantitative estimate. It is usually the first smooth regularity for solutions and the beginning for higher regularity. Take the uniformly elliptic equations in divergence form or non-divergence form for example. With respect to the interior Hölder regularity, De Giorgi [De Giorgi(1957)], Nash [Nash(1958)] and Moser [Moser(1960)] proved it for elliptic equations in divergence form, and it was extended to elliptic equations in non-divergence form by Krylov and Safonov [Krylov and Safonov(1979)]. In particular, the

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fully nonlinear elliptic equations can also be treated (see [Caffarelli(1989)] and [Caffarelli and Cabré(1995)]). With regard to the boundary Hölder continuity, a well-known result is that if \( \Omega \) satisfies the exterior cone condition at \( x_0 \in \partial \Omega \), the solution is Hölder continuous at \( x_0 \) (see [Miller(1967)] and [Gilbarg and Trudinger(2001), Theorem 8.29 and Corollary 9.28]).

All results above mentioned show that there exists some \( 0 < \alpha < 1 \) such that \( u \in C^{\alpha} \). It is interesting to ask whether the exponent \( \alpha \) could be bigger? Based only on the uniform ellipticity condition, it is impossible to obtain a higher interior Hölder regularity (see [Safonov(1987)]). On the other hand, for the boundary Hölder regularity, we can do expect a bigger exponent.

For the Poisson equation, Lemenant and Sire [Lemenant and Sire(2013)] proved that for any \( 0 < \alpha < 1 \), there exists \( \delta > 0 \) such that the solution is \( C^{\alpha} \) at \( x_0 \in \partial \Omega \) provided that \( \Omega \) is \( \delta \)-Reifenberg flat at \( x_0 \). In this paper, we prove analogue results for different types of elliptic equations and our approach is simple. Moreover, even for the Poisson equation, we relax the requirements imposed in [Lemenant and Sire(2013)]. The main idea in our method is that a Reifenberg flat domain can be regarded as a perturbation of half balls in different scales.

First, we introduce some notions.

**Definition 1.1 (Reifenberg flat domain).** We say that \( \Omega \) is \( \delta \)-Reifenberg flat from the exterior at \( x_0 \in \partial \Omega \) if there exists \( r_0 > 0 \) such that the following holds: for any \( 0 < r < r_0 \), there exists a coordinate system \( \{y_1, ..., y_n\} \) (isometric to the original coordinate system) such that 
\[
B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}.
\]
(1.1)

Next, we introduce a pointwise characterization for a function.

**Definition 1.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded set (may be not a domain) and \( f : \Omega \to \mathbb{R} \) be a function. We say that \( f \) is \( C^{\alpha} \) \( (0 < \alpha \leq 1) \) at \( x_0 \in \Omega \) or \( f \in C^{\alpha}(x_0) \) if there exist \( r_0 > 0 \) and \( K > 0 \) such that 
\[
|f(x) - f(x_0)| \leq K|x-x_0|^{\alpha}, \forall x \in \Omega \cap B_{r_0}(x_0).
\]
(1.2)

Then, define 
\[
[f]_{C^{\alpha}(x_0)} = \min \{K |(1.2) \text{ holds with } K]\}
\]
and 
\[
\|f\|_{C^{\alpha}(x_0)} = |f(x_0)| + [f]_{C^{\alpha}(x_0)}.
\]

We say that \( f \) is \( C^{p,\alpha}_{q} \) at \( x_0 \) or \( f \in C^{p,\alpha}_{q}(x_0) \) \( (p > 0, q \geq 1, \alpha > 0) \) if there exist \( r_0 > 0 \) and \( K > 0 \) such that 
\[
\|f\|_{L^q(\Omega\cap B_r(x_0))} \leq Kr^{\alpha-p+n/q}, \forall 0 < r < r_0.
\]
(1.3)

Similarly, we set 
\[
\|f\|_{C^{p,\alpha}_{q}(x_0)} = \min \{K |(1.3) \text{ holds with } K\}.
\]

If \( f \in C^{p,\alpha}_{q}(x) \) for any \( x \in \Omega \) with the same \( r_0 \) and 
\[
\|f\|_{C^{p,\alpha}_{q}(\Omega)} := \sup_{x \in \Omega} \|f\|_{C^{p,\alpha}_{q}(x)} < +\infty,
\]
we say that $f \in C_{q}^{-p,\alpha}(\Omega)$.

Remark 1.3. Since we study the boundary pointwise regularity, throughout this paper, we always assume that $0 \in \partial \Omega$ and $r_0 = 1$ in Definition 1.1 and Definition 1.2 without loss of generality.

Our main results are listed in the following. First, we consider the Laplace operator:

**Theorem 1.4.** Let $0 < \alpha < 1$ and $u$ be a weak solution of

$$\begin{cases}
\Delta u = f & \text{in } \Omega \cap B_1; \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}$$

(1.4)

where $g \in C^\alpha(0)$ and $f \in C_{p}^{-2,\alpha}(0)$ for some $p > n/2$. Suppose that $\Omega$ is $\delta$-Reifenberg flat from the exterior at $0$, where $\delta > 0$ depends only on $n$ and $\alpha$.

Then $u$ is $C^\alpha$ at $0$ and

$$|u(x) - u(0)| \leq C|x|^\alpha \left( \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C_{p}^{-2,\alpha}(x_0)} + [g]_{C^\alpha(0)} \right), \forall x \in \Omega \cap B_1,$$

where $C$ depends only on $n, \alpha$ and $p$.

Remark 1.5. If $f \in L^q$ with $q > n/(2 - \alpha)$, then $f \in C_{p}^{-2,\alpha}(0)$ for some $p > n/2$. In particular, $f \in L^n$ is enough for any $0 < \alpha < 1$.

Remark 1.6. Theorem 1.4 is more general by comparing with the result of [Lemenant and Sire(2013)].

For linear elliptic equations in divergence form, we have

**Theorem 1.7.** Let $0 < \alpha < 1$ and $u$ be a weak solution of

$$\begin{cases}
(a^{ij}u_i)_j = f & \text{in } \Omega \cap B_1; \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}$$

(1.5)

where $a^{ij}$ is uniformly elliptic with ellipticity constants $\lambda$ and $\Lambda$, $g \in C^\alpha(0)$ and $f \in C_{p}^{-2,\alpha}(0)$ for some $p > n/2$. Suppose that $|a^{ij} - \delta^{ij}| \leq \delta$ and $\Omega$ is $\delta$-Reifenberg flat from the exterior at $0$, where $\delta > 0$ depends only on $n, \lambda, \Lambda$ and $\alpha$.

Then $u$ is $C^\alpha$ at $0$ and

$$|u(x) - u(0)| \leq C|x|^\alpha \left( \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C_{p}^{-2,\alpha}(0)} + [g]_{C^\alpha(0)} \right), \forall x \in \Omega \cap B_1,$$

where $C$ depends only on $n, \lambda, \Lambda, \alpha$ and $p$.

Remark 1.8. In (1.5), the Einstein summation convention is used (similarly hereinafter), i.e., repeated indices means summation. The symbol $\delta^{ij}$ denotes the unit matrix. In fact, $\delta^{ij}$ can be replaced by any constant symmetric matrix with eigenvalues lying in $[\lambda, \Lambda]$. 

Remark 1.9. The smallness condition $|a_{ij} - \delta_{ij}| \leq \delta$ is necessary, which is different from the boundary regularity for equations in non-divergence form (see Theorem 1.10 below). In fact, based only on the uniform ellipticity, we can’t expect any higher boundary Hölder regularity for elliptic equations in divergence form (see [Byun and Wang(2004), (1.4)] and [Meyers(1963), Section 5]).

For linear equations in non-divergence form, the corresponding boundary Hölder regularity holds. More generally, it holds for fully nonlinear equations in non-divergence form:

**Theorem 1.10.** Let $0 < \alpha < 1$ and $u$ be a viscosity solution of
\[
\begin{cases}
u \in S(\lambda, \Lambda, f) & \text{in } \Omega \cap B_1; \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}
\]
where $g \in C^\alpha(0)$ and $f \in L^n(\Omega \cap B_1)$. Suppose that $\Omega$ is $\delta$-Reifenberg flat from the exterior at 0, where $\delta > 0$ depends only on $n, \lambda, \Lambda$ and $\alpha$.

Then $u$ is $C^\alpha$ at 0 and
\[
|u(x) - u(0)| \leq C|x|\alpha \left( \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{L^n(\Omega \cap B_1)} + [g]_{C^\alpha(0)} \right), \forall x \in \Omega \cap B_1,
\]
where $C$ depends only on $n, \lambda, \Lambda$ and $\alpha$.

**Remark 1.11.** In (1.6), $S(\lambda, \Lambda, f)$ denotes the Pucci class with ellipticity constants $\lambda, \Lambda$ and righthand $f$. The Pucci class is a natural generalization of linear uniformly elliptic equations in non-divergence form. For the notion of viscosity solutions and basic properties of the Pucci class, we refer to [Caffarelli and Cabré(1995)], [Caffarelli et al.(1996)] and [Crandall et al.(1992)]

**Remark 1.12.** We can relax $f \in L^n$ to $f \in L^{n-\varepsilon_0}$ for some $\varepsilon_0 > 0$ depending only on $n, \lambda, \Lambda$ and $\alpha$ (see [Escauriaza(1993)]) since the proof mainly relies on the A-B-P maximum principle.

For the $p$-Laplace equations, we have

**Theorem 1.13.** Let $0 < \alpha < 1$, $1 < p < \infty$ and $u$ be a weak solution of
\[
\begin{cases}
\operatorname{div}(|Du|^{p-2}Du) = 0 & \text{in } \Omega \cap B_1; \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}
\]
where $g \in C^\alpha(0)$. Suppose that $\Omega$ is $\delta$-Reifenberg flat from the exterior at 0, where $\delta > 0$ depends only on $n, \alpha$ and $p$.

Then $u$ is $C^\alpha$ at 0 and
\[
|u(x) - u(0)| \leq C|x|\alpha \left( \|u\|_{L^\infty(\Omega \cap B_1)} + [g]_{C^\alpha(0)} \right), \forall x \in \Omega \cap B_1,
\]
where $C$ depends only on $n, \alpha$ and $p$.

For $2 \leq p < \infty$, we have the boundary Hölder regularity for the Poisson equations:
Theorem 1.14. Let $0 < \alpha < 1$, $2 \leq p < \infty$ and $u$ be a weak solution of
\[
\begin{cases}
\text{div}(|Du|^{p-2}Du) = f & \text{in } \Omega \cap B_1; \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}
\]
where $g \in C^\alpha(0)$ and $f \in C_q^{-p,(p-1)\alpha}(0)$ for some $q > n/p$ and $1/p + 1/q \leq 1$. Suppose that $\Omega$ is $\delta$-Reifenberg flat from the exterior at 0, where $\delta > 0$ depends only on $n, \alpha$ and $p$.

Then $u$ is $C^\alpha$ at 0 and
\[
|u(x) - u(0)| \leq C|x|^\alpha \left( \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C_q^{-p,(p-1)\alpha}(0)} + [g]_{C^\alpha(0)} \right), \quad \forall x \in \Omega \cap B_1,
\]
where $C$ depends only on $n, \alpha, p$ and $q$.

We say that $\Omega$ satisfies the exterior cone condition with slope $\delta$ at $x_0 \in \partial \Omega$ if there exist $r_0 > 0$ and a unit vector $\vec{n}$ such that
\[
\{x \in B(x_0, r) : (x - x_0) \cdot \vec{n} < -\delta|x - x_0|\} \subset \Omega^c.
\]
Clearly, if $\Omega$ satisfies the exterior cone condition with slope $\delta$, it is $\delta$-Reifenberg flat from the exterior. Hence, we have

Corollary 1.15. In Theorems 1.4 to 1.14, the condition that $\Omega$ is $\delta$-Reifenberg flat from the exterior at 0 can be replaced by that $\Omega$ satisfies the exterior cone condition at 0 with slope $\delta$. In particular, if $\partial \Omega \in C^1$, Theorems 1.4 to 1.14 hold.

Notation 1.16.

1. $\{e_i\}_{i=1}^n$: the standard basis of $\mathbb{R}^n$, i.e., $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.
2. $x^t = (x_1, x_2, \ldots, x_{n-1})$ and $x = (x_1, \ldots, x_n) = (x', x_n)$.
3. $S^n$: the set of $n \times n$ symmetric matrices and $\|A\| = \text{the spectral radius of } A$ for any $A \in S^n$.
4. $R^n_+ = \{x \in \mathbb{R}^n | x_n > 0\}$.
5. $B_r(x_0) = \{x \in \mathbb{R}^n | |x - x_0| < r\}$, $B_r = B_r(0)$, $B_r^+(x_0) = B_r(x_0) \cap R^n_+$ and $B_r^+ = B_r^+(0)$.
6. $T_r(x_0) = \{(x', 0) \in \mathbb{R}^n | |x' - x_0'| < r\}$ and $T_r = T_r(0)$.
7. $A^c$: the complement of $A$ and $\bar{A}$: the closure of $A$, $\forall A \subset \mathbb{R}^n$.
8. $\Omega_r = \bar{\Omega} \cap B_r$ and $(\partial \Omega)_r = \partial \Omega \cap B_r$.
9. $\varphi_i = D_i \varphi = \partial \varphi / \partial x_i$ and $D \varphi = (\varphi_1, \ldots, \varphi_n)$. Similarly, $\varphi_{ij} = D_{ij} \varphi = \partial^2 \varphi / \partial x_i \partial x_j$ and $D^2 \varphi = (\varphi_{ij})_{n \times n}$.

2. Boundary Hölder regularity

In this section, we give the detailed proofs of Theorems 1.4 to 1.14. The main idea is the following. Since we consider the continuity of the solution up to the boundary, we only need to control the solution from both the above and the below. This is usually carried out by constructing a proper barrier. Here, we construct a sequence of “barrier-like” functions and adopt the scaling
argument to prove the continuity up to the boundary. Although we can’t obtain the Hölder regularity at once as done by the method of constructing a barrier, the method is more flexible.

We regard the boundary of a Reifenberg flat domain as a perturbation of a hyperplane in different scales. Hence, we only need to construct “barrier-like” functions with respect to a flat boundary. Thus, they are easy to construct. In fact, a function like

$$v_0(x) = \left(\frac{1}{2}\right)^{-\beta} - x + \left(\delta + \frac{1}{2}\right) e_n$$

is enough for Theorems 1.4 to 1.14 by proper rescaling and choosing a sufficient large $\beta$. From above observation, this kind of boundary regularity is the most easily to prove. One can compare it with the boundary Hölder regularity for “some” rather than “any” $0 < \alpha < 1$ (see [Lian et al.(2020b)Lian, Zhang, Li and Hong], the boundary Lipschitz regularity (see [Safonov(2008)] and [Lian and Zhang(2018)]), the boundary differentiability (see [Li and Wang(2006), Li and Wang(2009), Li and Zhang(2013)]) and the boundary $C^{k,\alpha}$ regularity for $k \geq 1$ (see [Lian and Zhang(2020)] and [Lian et al.(2020a)Lian, Wang and Zhang]).

In the following, we give the detailed proofs of our main results.

**Proof of Theorem 1.4.** Without loss of generality, we assume that $g(0) = 0$. Let $M = \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C^{-2,\alpha}(x_0)} + |g|_{C^{\alpha}(0)}$ and $\Omega_r = \Omega \cap B_r$. To prove that $u$ is $C^\alpha$ at 0, we only need to show the following:

there exist constants $0 < \delta < 1/4$, $0 < \eta < 1$ (depending only on $n$ and $\alpha$) and $\hat{C} \geq 1$ (depending only on $n, \alpha$ and $p$) such that for all $k \geq 0$,

$$\|u\|_{L^\infty(\Omega_{\eta^k})} \leq \hat{C} M \eta^{k\alpha}. \quad (2.2)$$

We prove (2.2) by induction. For $k = 0$, it holds clearly. Suppose that it holds for $k$. We need to prove that it holds for $k + 1$.

Let $r = \eta^{k+1}/2$ and there exists a new coordinate system denoted by $\{x_1, ..., x_n\}$ again such that

$$B_r \cap \Omega \subset B_r \cap \{x_n > -\delta r\}. \quad (2.3)$$

Let $\tilde{B}_r^+ = B_r^+ - \delta re_n$, $\tilde{T}_r = T_r - \delta re_n$ and $\tilde{\Omega}_r = \Omega \cap \tilde{B}_r^+$ where $e_n = (0,0,...,0,1)$. Then $\Omega_{r/2} \subset \tilde{\Omega}_r \subset \Omega_{\eta^k}$.

Take (note that $v_0$ is defined in (2.1))

$$v(x) = \hat{C} M \eta^{k\alpha} v_0(x/r)$$

and by choosing $\beta$ large enough, $v$ satisfies

$$\begin{align*}
\Delta v &\leq 0 \quad \text{in } \tilde{B}_r^+; \\
v &\geq 0 \quad \text{in } \tilde{B}_r^+; \\
v &\geq \hat{C} M \eta^{k\alpha} \quad \text{on } \partial \tilde{B}_r^+ \setminus \tilde{T}_r; \\
v(−\delta re_n) &= 0.
\end{align*}$$
Set \( w = u - v \) and we have (note that \( v \geq 0 \))

\[
\begin{align*}
\Delta w & \geq f \quad \text{in } \tilde{\Omega}_r; \\
\Delta w & = g - v \leq g \quad \text{on } \partial \Omega \cap \tilde{B}_{r}^+; \\
\Delta w & \leq 0 \quad \text{on } \partial \tilde{B}_{r}^+ \cap \tilde{\Omega}.
\end{align*}
\]

Let \( \delta = \eta \).

For \( v \), it can be calculated directly that

\[
v \leq C_1 \eta \cdot \hat{C} M \eta^{(k+1)\alpha} \quad \text{on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\},
\]

where \( C_1 \) depends only on \( n \). For \( w \), by the A-B-P maximum principle, we have for \( x \in \tilde{\Omega}_r \),

\[
w(x) \leq \|g\|_{L^\infty(\partial \Omega \cap \tilde{B}_{r}^+)} + C r^{2-n/p} \|f\|_{L^p(\tilde{\Omega}_r)} \\
\leq M(2r)^{\alpha} + C M(2r)^{\alpha} \\
= \frac{C_2}{\hat{C} \eta^{\alpha}} \cdot \hat{C} M \eta^{(k+1)\alpha},
\]

where \( C_2 \) depends only on \( n \) and \( p \).

Take \( \eta \leq 1/4 \) small enough such that

\[
C_1 \eta^{1-\alpha} \leq 1/2.
\]

Next, take \( \hat{C} \) large enough such that

\[
\frac{C_2}{\hat{C} \eta^{\alpha}} \leq 1/2.
\]

Then by combining with (2.4) and (2.5), we have

\[
u = v + w \leq \hat{C} M \eta^{(k+1)\alpha} \quad \text{on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\}.
\]

By translating \( v \) to proper positions (with \( v(x', -\delta r e_n) = 0 \) for some \( x' \in B_{\eta^{k+1}} \) and similar arguments, we obtain

\[
\sup_{\Omega_{\eta^{k+1}}} u \leq \hat{C} M \eta^{(k+1)\alpha}.
\]

The proof for

\[
\inf_{\Omega_{\eta^{k+1}}} u \geq -\hat{C} M \eta^{(k+1)\alpha}
\]

is similar and we omit it here. Therefore,

\[
\|u\|_{L^\infty(\Omega_{\eta^{k+1}})} \leq \hat{C} M \eta^{(k+1)\alpha}.
\]

By induction, the proof is completed. \( \square \)

**Remark 2.1.** From (2.6), we infer an explicit relation between \( \delta \) and \( \alpha \):

\[
\delta^{1-\alpha} = c_0
\]

for some \( 0 < c_0 < 1 \) depending only on \( n \).
Proof of Theorem 1.7. The proof is almost the same as that of Theorem 1.4 since (1.5) can regarded as a perturbation of the Laplace equation. As before, we assume that \( g(0) = 0 \). Let \( M = \|u\|_{L^\infty(\Omega \cap B_r)} + \|f\|_{C^2_p} + [g]C^\alpha(0) \) and \( \Omega_r = \Omega \cap B_r \). To prove that \( u \) is \( C^\alpha \) at 0, we only need to show the following:

there exist constants \( 0 < \delta < 1/4, 0 < \eta < 1 \) (depending only on \( n, \lambda, \Lambda \) and \( \alpha \)) and \( \hat{C} \geq 1 \) (depending only on \( n, \lambda, \Lambda, \alpha \) and \( p \)) such that for all \( k \geq 0 \),

\[
\|u\|_{L^\infty(\Omega_{\eta^k})} \leq \hat{C} M \eta^{k \alpha}. \tag{2.7}
\]

We prove (2.7) by induction. For \( k = 0 \), it holds clearly. Suppose that it holds for \( k \). We need to prove that it holds for \( k + 1 \).

Let \( r = \eta^k \/ 2 \) and there exists a new coordinate system denoted by \( \{x_1, ..., x_n\} \) again such that

\[ B_r \cap \Omega \subset B_r \cap \{x_n \geq -\delta r\}. \]

Let \( \tilde{B}_r^+ = B_r^+ - \delta r e_n \), \( \tilde{T}_r = T_r - \delta r e_n \) and \( \tilde{\Omega}_r = \Omega \cap \tilde{B}_r^+ \). Then \( \Omega_{r/2} \subset \tilde{\Omega}_r \subset \Omega_{\eta^{k+1}} \).

Define

\[ v(x) = \hat{C} M \eta^{k \alpha} v_0(x/r) \]

and by choosing \( \beta \) large enough, \( v \) satisfies

\[
\begin{cases}
\Delta v \leq 0 & \text{in } \tilde{B}_r^+; \\
v \geq 0 & \text{in } \tilde{B}_r^+; \\
v \geq \hat{C} M \eta^{k \alpha} & \text{on } \partial \tilde{B}_r^+ \setminus \tilde{T}_r; \\
v(-\delta r e_n) = 0.
\end{cases}
\]

Set \( w = u - v \) and we have

\[
\begin{cases}
(a^{ij} w_i)_j = f + ((\delta^{ij} - a^{ij}) v_i)_j & \text{in } \tilde{\Omega}_r; \\
w \leq g & \text{on } \partial \Omega \cap \tilde{B}_r^+; \\
w \leq 0 & \text{on } \partial \tilde{B}_r^+ \cap \tilde{\Omega}_r.
\end{cases}
\]

Take \( \delta = \eta \). As before, for \( v \), it can be calculated directly that

\[
v \leq C_1 \eta^{1-\alpha} \cdot \hat{C} M \eta^{(k+1)\alpha} \quad \text{on} \quad \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2n \delta r\}, \tag{2.8}
\]

where \( C_1 \) depends only on \( n \). For \( w \), by the A-B-P maximum principle, we have for \( x \in \tilde{\Omega}_r \),

\[
w(x) \leq \|g\|_{L^\infty(\partial \tilde{\Omega} \cap \tilde{B}_r^+)} + C r^{2-n/p} \|f\|_{L^p(\tilde{\Omega}_r)} + C \eta r \|Dv\|_{L^\infty(\tilde{\Omega}_r)} \\
\leq M(2r)^\alpha + C M(2r)^\alpha + C \eta \cdot \hat{C} M \eta^{k \alpha} \\
= \left( \frac{C_2}{C \eta^{\alpha}} + C_3 \eta^{1-\alpha} \right) \cdot \hat{C} M \eta^{(k+1)\alpha}, \tag{2.9}
\]

where \( C_2 \) depends only on \( n, \lambda, \Lambda \) and \( p \), and \( C_3 \) depends only on \( n, \lambda \) and \( \Lambda \).

Take \( \eta \leq 1/4 \) small enough such that

\[
(C_1 + C_3) \eta^{1-\alpha} \leq 1/4.
\]
Next, take $\hat{C}$ large enough such that
\[
\frac{C_2}{C\eta^\alpha} \leq 1/4.
\]

Then by combining with (2.8) and (2.9), we have
\[
u = v + w \leq \hat{C}M^\eta^{(k+1)\alpha} \text{ on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\}.
\]

By translating $\nu$ to proper positions and similar arguments, we obtain
\[
sup_{\Omega_{\eta,k+1}} \nu \leq \hat{C}M^\eta^{(k+1)\alpha}.
\]

The proof for
\[
inf_{\Omega_{\eta,k+1}} \nu \geq -\hat{C}M^\eta^{(k+1)\alpha}
\]
is similar and we omit here. Therefore,
\[
\|u\|_{L^\infty(\Omega_{\eta,k+1})} \leq \hat{C}M^\eta^{(k+1)\alpha}.
\]

By induction, the proof is completed. $\square$

**Proof of Theorem 1.10.** Without loss of generality, we assume that $g(0) = 0$. Let $M = \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{L^\infty(\Omega \cap B_1)} + [g]_{C^\alpha(0)}$ and $\Omega_r = \Omega \cap B_r$. To prove that $u$ is $C^\alpha$ at 0, we only need to show the following:

there exist constants $0 < \delta < 1/4, 0 < \eta < 1$ and $\hat{C} \geq 1$ depending only on $n, \lambda, \Lambda$ and $\alpha$ such that for all $k \geq 0$,

\[
\|u\|_{L^\infty(\Omega_{\eta,k})} \leq \hat{C}M^\eta^{k\alpha}. \tag{2.10}
\]

We prove (2.18) by induction. For $k = 0$, it holds clearly. Suppose that it holds for $k$. We need to prove that it holds for $k + 1$.

Let $r = \eta^k/2$ and there exists a new coordinate system denoted by $\{x_1, ..., x_n\}$ again such that

\[
B_r \cap \Omega \subset B_r \cap \{x_n > -\delta r\}.
\]

Let $\tilde{B}_r^+ = B_r^+ - \delta r e_n, \tilde{T}_r = T_r - \delta r e_n$ and $\tilde{\Omega}_r = \Omega \cap \tilde{B}_r^+$. Then $\Omega_{r/2} \subset \tilde{\Omega}_r \subset \Omega_{\eta^k}$.

Define

\[
v(x) = \hat{C}M^\eta^{k\alpha}v_0(x/r)
\]

and by choosing $\beta$ large enough, $v$ satisfies

\[
\begin{cases}
M^+(D^2v, \lambda, \Lambda) \leq 0 & \text{in } \tilde{B}_r^+; \\
v \geq 0 & \text{in } \tilde{B}_r^+; \\
v \geq \hat{C}M^\eta^{k\alpha} & \text{on } \partial \tilde{B}_r^+ \setminus \tilde{T}_r; \\
v(-\delta r e_n) = 0.
\end{cases}
\]
Set $w = u - v$ and we have
\[
\begin{cases}
    w \in S(\lambda, \Lambda, f) \quad \text{in } \tilde{\Omega}_r; \\
    w \leq g \quad \text{on } \partial \Omega \cap \tilde{B}_r^+; \\
    w \leq 0 \quad \text{on } \partial \tilde{B}_r^+ \cap \tilde{\Omega}.
\end{cases}
\]

Take $\delta = \eta$. For $v$, it can be calculated directly that
\[
v \leq C_1 \eta^{1-\alpha} \cdot \hat{C} M \eta^{(k+1)\alpha} \quad \text{on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\},
\]
where $C_1$ depends only on $n, \lambda$ and $\Lambda$. For $w$, by the A-B-P maximum principle, we have for $x \in \tilde{\Omega}_r$,
\[
w(x) \leq \|g\|_{L^\infty(\partial \Omega \cap \tilde{B}_r^+)} + Cr\|f\|_{L^n(\tilde{\Omega}_r)} \leq M(2r)^\alpha + CMr = \frac{C_2}{\hat{C} \eta^\alpha} \cdot \hat{C} M \eta^{(k+1)\alpha},
\]
where $C_2$ depends only on $n, \lambda$ and $\Lambda$.

Take $\eta \leq 1/4$ small enough such that
\[C_1 \eta^{1-\alpha} \leq 1/2.\]

Next, take $\hat{C}$ large enough such that
\[\frac{C_2}{\hat{C} \eta^\alpha} \leq 1/2.\]

Then by combining with (2.19) and (2.20), we have
\[
u + w \leq \hat{C} M \eta^{(k+1)\alpha} \quad \text{on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\}.
\]

By translating $v$ to proper positions and similar arguments, we obtain
\[\sup_{\Omega_{\eta^{k+1}}} u \leq \hat{C} M \eta^{(k+1)\alpha}.\]

The proof for
\[\inf_{\Omega_{\eta^{k+1}}} u \geq -\hat{C} M \eta^{(k+1)\alpha}\]
is similar and we omit here. Therefore,
\[\|u\|_{L^\infty(\Omega_{\eta^{k+1}})} \leq \hat{C} M \eta^{(k+1)\alpha}.
\]

By induction, the proof is completed. \qed

For the $p$-Laplace equation, we can prove the boundary Hölder regularity by the same argument since it can be rewritten as a uniformly elliptic equation in nondivergence form if the gradient of the solution is nonvanishing.

**Proof of Theorem 1.13.** Without loss of generality, we assume that $g(0) = 0$. Let $M = \|u\|_{L^\infty(\Omega \cap B_1)} + [g]_{C^0(0)}$ and $\Omega_r = \Omega \cap B_r$. To prove that $u$ is $C^\alpha$ at $0$, we only need to show the following:

there exist constants $0 < \delta < 1/4$, $0 < \eta < 1$ and $\hat{C} \geq 1$ depending only on $n, \alpha$ and $p$ such that for all $k \geq 0$,
We prove (2.13) by induction. For $k = 0$, it holds clearly. Suppose that it holds for $k$. We need to prove that it holds for $k + 1$.

Let $r = \eta^k / 2$. Then there exists a new coordinate system denoted by \{ $x_1, ..., x_n$ \} again such that

$$B_r \cap \Omega \subset B_r \cap \{ x_n > -\delta r \}.$$ 

Let $\tilde{B}_r^+ = B_r^+ - \delta r e_n$, $\tilde{T}_r = T_r - \delta r e_n$ and $\tilde{\Omega}_r = \Omega \cap \tilde{B}_r^+$. Then $\Omega_r / 2 \subset \tilde{\Omega}_r \subset \Omega_{\eta^k}$.

Define

$$v(x) = \hat{C} M \eta^{k\alpha} v_0 (x/r)$$

and by choosing $\beta$ large enough, $v$ satisfies

$$\begin{cases}
\text{div}(|Dv|^{p-2} Dv) \leq 0 & \text{in } \tilde{B}_r^+; \\
v \geq 0 & \text{in } \tilde{B}_r^+; \\
v \geq \hat{C} M \eta^{k\alpha} & \text{on } \partial \tilde{B}_r^+ \setminus \tilde{T}_r; \\
v(-\delta r e_n) = 0.
\end{cases}$$

Take $\delta = \eta$. As before,

$$v \leq C\eta^{1-\alpha} \cdot \hat{C} M \eta^{(k+1)\alpha} \text{ on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\},$$

where $C$ depends only on $n$ and $p$. From the comparison principle, we have for $x \in \tilde{\Omega}_r$,

$$u(x) \leq v(x) + \|g\|_{L^\infty(\partial \Omega \cap \tilde{B}_r^+)} \leq \left( C\eta^{1-\alpha} + \frac{C}{C\eta^{\alpha}} \right) \cdot \hat{C} M \eta^{(k+1)\alpha},$$

where $C$ depends only on $n$ and $p$.

Take $\eta \leq 1/4$ small enough and $\hat{C}$ large enough such that

$$\left( C\eta^{1-\alpha} + \frac{C}{C\eta^{\alpha}} \right) \leq 1.$$ 

Then

$$u \leq \hat{C} M \eta^{(k+1)\alpha} \text{ on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\}.$$ 

By translating $v$ to proper positions and similar arguments, we obtain

$$\sup_{\Omega_{\eta^{k+1}}} u \leq \hat{C} M \eta^{(k+1)\alpha}.$$ 

The proof for

$$\inf_{\Omega_{\eta^{k+1}}} u \geq -\hat{C} M \eta^{(k+1)\alpha}$$

is similar and we omit here. Therefore,

$$\|u\|_{L^\infty(\Omega_{\eta^{k+1}})} \leq \hat{C} M \eta^{(k+1)\alpha}.$$ 

By induction, the proof is completed. \hfill \Box
For the corresponding Poisson equations, since the difference of two solutions is no longer a solution of some equation, we need to do a little more work. First, we introduce two lemmas, which are motivated by [Wang and Wang(2013)].

**Lemma 2.2.** Let \( \phi : [0, \infty) \to [0, \infty) \) be a nonnegative nonincreasing function. Assume that for some constants \( c > 0 \), \( \beta > 1 \) and \( \gamma > 0 \),

\[
\phi(h) \leq \frac{c}{(h-k)^\gamma} \phi^\beta(k), \quad \forall \, h > k \geq 0.
\]

Then

\[
\phi(r) = 0, \quad \text{where} \quad r = \left(c 2^{\frac{\beta \gamma}{p-1}} \phi^\beta(0) \right)^{1/\gamma}.
\]

**Proof.** Let \( k_m = r(1 - 2^{-m}) \) for \( m \geq 0 \). Obviously, it is enough to prove that for any \( m \geq 0 \),

\[
\phi(k_m) \leq \frac{\phi(0)}{2^{\mu m}}, \quad \text{where} \quad \mu = \frac{\gamma}{\beta - 1}. \tag{2.16}
\]

We prove (2.16) by induction. For \( m = 0 \), it holds clearly. Suppose that it holds for \( m \) and we need to prove that it holds for \( m + 1 \). By direct calculation,

\[
\phi(k_{m+1}) \leq \frac{c}{(2^{-m-1} r)^\gamma} \phi^\beta(k_m)
\]
\[
\leq 2^{(m+1)\gamma} 2^{-\frac{\beta \gamma}{p-1}} \phi^{1-\beta}(0) \left( \frac{\phi(0)}{2^{\mu m}} \right)^\beta
\]
\[
= \phi(0) \frac{2^{m \mu}}{2^{\mu (m+1) \gamma - \frac{\gamma}{p-1} - m \mu (\beta - 1)}}
\]
\[
= \phi(0) \frac{2^{m \mu}}{2^{m \mu}} \cdot 2^{-\frac{\gamma}{p-1}}
\]
\[
= \phi(0) \frac{2^{(m+1) \mu}}{2^{m \mu}}.
\]

By induction, the proof is completed. \( \square \)

The next lemma is a kind of A-B-P estimate for the difference of two solutions.

**Lemma 2.3.** Let \( 2 \leq p < \infty \). Suppose that \( u \) and \( v \) are weak solutions of

\[
\begin{align*}
\text{div}(|Du|^{p-2} Du) &\geq f_1 \quad \text{in} \; \Omega; \\
u &\leq g_1 \quad \text{on} \; \partial \Omega
\end{align*}
\]

and

\[
\begin{align*}
\text{div}(|Dv|^{p-2} Dv) &\leq f_2 \quad \text{in} \; \Omega; \\
v &\geq g_2 \quad \text{on} \; \partial \Omega
\end{align*}
\]

respectively, where \( f_1, f_2 \in L^q(\Omega) \) with \( q > n/p \) and \( 1/p + 1/q \leq 1 \). Then

\[
\sup_{\Omega} (u-v) \leq \|(g_1-g_2)^+\|_{L^\infty(\partial \Omega)} + C |\Omega|^{\frac{p}{(p-1)}} - \frac{1}{q(p-1)} \| (f_1-f_2)^-\|_{L^q(\Omega)}, \tag{2.17}
\]

where \( C \) depends only on \( n, p \) and \( q \).
Proof. For any \( r \geq 0 \), let \( A(r) = \{ x \in \Omega : u(x) - v(x) - \| (g_1 - g_2)^+ \|_{L^\infty(\partial \Omega)} \geq r \} \). Given \( k \geq 0 \), define

\[
wx(x) = \begin{cases}
u(x) - v(x) - \| (g_1 - g_2)^+ \|_{L^\infty(\partial \Omega)} - k, & x \in A(k); \\
0 & x \in \Omega \setminus A(k).
\end{cases}
\]

Note that \( w \geq 0 \) in \( \Omega \) and \( w = 0 \) on \( \partial \Omega \) for any \( k \geq 0 \). Then by the Poincaré inequality and basic calculation, we have

\[
|A(k)|^{-p/n} \int_{A_k} |w|^p \leq C \int_{A_k} |Dw|^p \\
= C \int_{A_k} |Du - Dv|^p \\
\leq C \int_{A_k} \langle |Du|^{p-2} Du - |Dv|^{p-2} Dv, Du - Dv \rangle \\
= C \int_{A_k} \langle |Du|^{p-2} Du - |Dv|^{p-2} Dv, Dw \rangle \\
= C \int_{A_k} (f_2 - f_1) w \\
\leq C \left( \int_{A_k} w^p \right)^{1/p} \left( \int_{A_k} \| (f_1 - f_2)^-(p/(p-1)) \right)^{(p-1)/p}.
\]

Hence,

\[
\|w\|_{L^p(A(k))}^{-p-1} \leq C |A(k)|^{p/n} \| (f_1 - f_2)^- \|_{L^{p/(p-1)}(A(k))} \\
\leq C |A(k)|^{p/n+(p-1)/p-1/q} \| (f_1 - f_2)^- \|_{L^q(A(k))}.
\]

For any \( h \geq k \),

\[
|A(h)|^{1/p} (h - k) = \| h - k \|_{L^p(A(h))} \\
\leq \| w \|_{L^p(A(h))} \\
\leq \| w \|_{L^p(A(k))} \\
\leq C |A(k)|^{p/(n(p-1)) + 1/(p-1)/(q(p-1))} \| (f_1 - f_2)^- \|_{L^q(A(k))}^{1/(p-1)}.
\]

Thus,

\[
|A(h)| \leq \frac{C \| (f_1 - f_2)^- \|_{L^p(A(k))}^{p/(p-1)}}{(h - k)^p} |A(k)|^{p^2/(n(p-1)) + 1-p/(q(p-1))}.
\]

Now, we apply Lemma 2.3 with \( \phi(r) = |A(r)|, \beta = 1 + p^2/(n(p-1)) - p/(q(p-1)) > 1 \) and \( \gamma = p \). Note that \( \phi(0) \leq |\Omega| \). Then

\[
\phi(r) = 0 \text{ for } r = C 2^{\frac{1}{p-2}} \| (f_1 - f_2)^- \|_{L^q(\Omega)}^{1/(p-1)} |\Omega|^\frac{p}{n(p-1)} - \frac{1}{q(p-1)}.
\]

That is, (2.17) holds. \( \square \)

Now, we can prove the boundary Hölder regularity for the \( p \)-Poisson equations.
Proof of Theorem 1.14. Without loss of generality, we assume that \( g(0) = 0 \). Let \( M = \|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C_{\alpha}(0)} + [g]_{C_{\alpha}(0)} \) and \( \Omega_r = \Omega \cap B_r \). To prove that \( u \) is \( C^\alpha \) at 0, we only need to show the following:

there exist constants \( 0 < \delta < 1/4 \), \( 0 < \eta < 1 \) (depending only on \( n, \alpha \) and \( p \)) and \( \hat{C} \geq 1 \) (depending only on \( n, \alpha, p \) and \( q \)) such that for all \( k \geq 0 \),

\[
\|u\|_{L^\infty(\Omega_{\eta^k})} \leq \hat{C} M \eta^{k \alpha}.
\] (2.18)

We prove (2.18) by induction. For \( k = 0 \), it holds clearly. Suppose that it holds for \( k \). We need to prove that it holds for \( k + 1 \).

Let \( r = \eta^k/2 \). Then there exists a new coordinate system denoted by \( \{x_1, \ldots, x_n\} \) again such that

\[
B_r \cap \Omega \subset B_r \cap \{x_n > -\delta r\}.
\]

Let \( \tilde{B}_r^+ = B_r^+ - \delta r e_n \), \( \tilde{T}_r = T_r - \delta r e_n \) and \( \tilde{\Omega}_r = \Omega \cap \tilde{B}_r^+ \). Then \( \Omega_{r/2} \subset \tilde{\Omega}_r \subset \Omega_{\eta^k} \).

Define

\[
v(x) = \hat{C} M \eta^{k \alpha} v_0(x/r)
\]

and by choosing \( \beta \) large enough, \( v \) satisfies

\[
\begin{cases}
\text{div}(|Dv|^{p-2}Dv) \leq 0 & \text{in } \tilde{B}_r^+; \\
v \geq 0 & \text{in } \tilde{B}_r^+; \\
v \geq \hat{C} M \eta^{k \alpha} & \text{on } \partial \tilde{B}_r^+ \setminus \tilde{T}_r; \\
v(-\delta r e_n) = 0.
\end{cases}
\]

As before, take \( \delta = \eta \) and

\[
v \leq C_1 \eta^{1-\alpha} \cdot \hat{C} M \eta^{(k+1) \alpha} \text{ on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\},
\]

where \( C_1 \) depends only on \( n \) and \( p \). In addition, by Lemma 2.3, we have for \( x \in \tilde{\Omega}_r \),

\[
u(x) - u(x) \leq \|g\|_{L^\infty(\partial\Omega \cap \tilde{B}_r^+)} + C r^{\frac{1}{p-1}(p-\frac{q}{p})} \|f\|_{L^{\frac{q}{p-1}}(\tilde{\Omega}_r)} \\
\leq M(2r)^\alpha + C M(2r)^\alpha = \frac{C_2}{C \eta^\alpha} \cdot \hat{C} M \eta^{(k+1) \alpha},
\]

where \( C_2 \) depends only on \( n, p \) and \( q \).

Take \( \eta \leq 1/4 \) small enough such that

\[
C_1 \eta^{1-\alpha} \leq 1/2.
\]

Next, take \( \hat{C} \) large enough such that

\[
\frac{C_2}{C \eta^\alpha} \leq 1/2.
\]

Then by combining with (2.19) and (2.20), we have

\[
u \leq \hat{C} M \eta^{(k+1) \alpha} \text{ on } \{(x', x_n) : x' = 0, -\delta r \leq x_n \leq 2\eta r\}.
\]
By translating $v$ to proper positions and similar arguments, we obtain
\[
\sup_{\Omega_{\eta^{k+1}}} u \leq \hat{C}M\eta^{(k+1)\alpha}.
\]
The proof for
\[
\inf_{\Omega_{\eta^{k+1}}} u \geq -\hat{C}M\eta^{(k+1)\alpha}
\]
is similar and we omit here. Therefore,
\[
\|u\|_{L^\infty(\Omega_{\eta^{k+1}})} \leq \hat{C}M\eta^{(k+1)\alpha}.
\]
By induction, the proof is completed. □

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16 Yuanyuan Lian and Kai Zhang

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