Families of relatively exact Lagrangians, free loop spaces and generalised homology

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Abstract

We prove that (under appropriate orientation conditions, depending on $R$) a Hamiltonian isotopy $\psi^1$ of a symplectic manifold $(M, \omega)$ fixing a relatively exact Lagrangian $L$ setwise must act trivially on $R_*(L)$, where $R_*$ is some generalised homology theory. We use a strategy inspired by that of Hu, Lalonde and Leclercq ([HLL11]), who proved an analogous result over $\mathbb{Z}/2$ and over $\mathbb{Z}$ under stronger orientation assumptions. However the differences in our approaches let us deduce that if $L$ is a homotopy sphere, $\psi^1|_{L}$ is homotopic to the identity. Our technical set-up differs from both theirs and that of Cohen, Jones and Segal ([CJS95, Coh09]).

We also prove (under similar conditions) that $\psi^1|_{L}$ acts trivially on $R_*(\mathcal{L}L)$, where $\mathcal{L}L$ is the free loop space of $L$. From this we deduce that when $L$ is a surface or a $K(\pi, 1)$, $\psi^1|_{L}$ is homotopic to the identity.

Using methods of [LM03], we also show that given a family of Lagrangians all of which are Hamiltonian isotopic to $L$ over a sphere or a torus, the associated fibre bundle cohomologically splits over $\mathbb{Z}/2$.

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1 Introduction

1.1 Background

Let $(M^{2n}, \omega)$ be a symplectic manifold of dimension $2n$, and $L^n \subseteq M$ a Lagrangian submanifold. Let $\psi^t$ be a Hamiltonian isotopy of $M$ such that its time-1 flow preserves $L$ setwise, $\psi^1(L) = L$. We can consider the Hamiltonian monodromy group $G_L \subseteq \text{Diff}(L)$ of diffeomorphisms of $L$ arising in this way. A natural question is:

Question 1.1. What is $G_L$?

An elementary argument using the Weinstein neighbourhood theorem shows that if $f$ and $g$ are isotopic diffeomorphisms of $L$ and $g$ lies in $G_L$, then $f$ is too. This implies that $G_L$ is a union of connected components in $\text{Diff}(L)$, and hence to study $G_L$, it suffices to study its image in the mapping class group $\pi_0 \text{Diff}(L)$.

The subgroup $G_L$ was first studied by Yau in [Yau09], who proved the following theorem.

Theorem 1.2 ([Yau09]). Let $T$ and $T'$ be the standard monotone Clifford and Chekanov tori in $\mathbb{C}^2$ with the same monotonicity constant. Then in both cases, $\pi_0 G_L \cong \mathbb{Z}/2$. 

2 Index Bundles and Their Orientations

3 The Moduli Spaces $\mathcal{P}$ and $\mathcal{W}$

4 Monodromy in the Looped Case

5 Families Over Other Bases

A de Silva’s Theorem

B An Example in $K$-Theory

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However, there is no isomorphism $H_1(T) \cong H_1(T')$ which respects the $\mathbb{Z}/2$ action on both and also preserves the Maslov index homomorphism.

This provides a new proof that these two Lagrangians are not Hamiltonian isotopic.

Question 1.1 has been studied in few other places: by Hu, Lalonde and Leclercq in [HLL11], by Mangolte and Welschinger in [MW12], by Ono in [Ono15], by Keating in [Kea21], and by Augustynowicz, Smith and Wornbard in [ASW22]. Out of these, only Hu, Lalonde and Leclercq focus on the relatively exact setting, which is where we will focus.

We now assume (for the rest of the paper) the following:

**Assumption 1.3.**
1. $M$ is a product of symplectic manifolds which are either compact or Liouville.
2. $L$ is compact, connected and relatively exact, i.e. $\omega$ vanishes on the relative homotopy group $\pi_2(M, L)$.
3. $\psi^t$ is compactly supported for all $t$ and constant in $t$ near 0 and 1.

Note that given $\psi^t$ not satisfying (3), we can always deform it so that it does, without changing $\psi^1|_L$.

$G_L$ was studied by Hu, Lalonde and Leclercq in this more restrictive setting in [HLL11], where they proved the following:

**Theorem 1.4 (HLL11).**
1. $\psi^1|_L$ acts as the identity on $H_*(L; \mathbb{Z}/2)$.
2. If $L$ admits a relative spin structure in $M$ which $\psi^1$ preserves, $\psi^1|_L$ acts as the identity on $H_*(L; \mathbb{Z})$.

It follows from Yau’s results that relative exactness is a necessary condition here.

We will recover this as a special case of Theorem 1.8. Note that in the case that $L$ is a homotopy sphere, this theorem does not give any information on the homotopy class of $\psi^1|_L$: if $\psi^1|_L$ were not homotopic to the identity, it would not preserve any orientation of $L$ and hence would not preserve any relative spin structure.

Our goals will be to weaken the orientation assumptions, extend this result to some other generalised cohomology theories, and extend this result to the free loop space of $L$. From this, we will fully compute $G_L$ in the case that $n = 2$. Our technical set-up will be very different to theirs but our general approach is inspired by theirs.

**Remark 1.5.** Hu, Lalonde and Leclercq prove this using Morse and quantum cohomology (as in [BC09]), and one possible approach to extending this theorem to other generalised cohomology theories would be to recreate their proof using the methods of Cohen, Jones and Segal ([CJS95, Coh09]). However one may need to require stronger orientation hypotheses than Assumption 1.6.
1.2 Extensions to Generalised Homology Theories

We will define a space $D_0$ which is roughly the space of smooth maps $D^2 \to M$ sending 1 to $L$, with moving Lagrangian boundary conditions on the rest of the boundary. This admits a map $\pi : D_0 \to L$ by evaluating at 1. We will construct a virtual vector bundle called the index bundle, Ind, on any finite CW complex in $D_0$ (compatible with restriction). After picking a (generic) almost complex structure $J$ on $M$ and some specific choices of moving Lagrangian boundary conditions, the tangent bundle of the moduli space of $J$-holomorphic discs with these boundary conditions will be stably isomorphic to Ind.

We fix some ring spectrum $R$, and will recall definitions of ring spectra and $R$-orientability in Section 2. We will make the following assumption, and in Section 4.2 find conditions under which it holds.

**Assumption 1.6.** The virtual vector bundle $\text{Ind} - \pi^*TL$ is $R$-orientable.

We will show:

**Theorem 1.7.** Under Assumption 1.6, the map 
$$\psi^1|_L : \Sigma^\infty +_R L \to \Sigma^\infty +_R L$$

is homotopic to the identity as a map of $R$-modules.

**Remark 1.8.** Our proof of this will use a minimal amount of technical machinery: we will only use standard Gromov compactness results and standard transversality results, and we will not need any form of gluing. In fact, it is possible to prove this without invoking any transversality results, and instead only use the fact that certain operators are Fredholm, using ideas in [Hof88].

From Corollaries 2.21 and 2.20, we deduce:

**Corollary 1.9.** Under Assumption 1.6

1. $\psi^1|_L$ induces the identity map on $R^*(L)$.
2. If $TL$ is $R$-orientable, $\psi^1|_L$ induces the identity map on $R^*(L)$.

In Section 4 we will define a moduli space $P$ of holomorphic discs in $M$, with moving boundary conditions, lying naturally inside $D_0$. Its tangent space $TP$ will by stably isomorphic to Ind. This will satisfy the following:

**Lemma 1.10.** $P$ is a closed smooth manifold of dimension $n$.

and

**Lemma 1.11.** The following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\psi^1} & L \\
\pi \downarrow & & \downarrow \pi \\
L & & L
\end{array}
$$
Furthermore, by varying the boundary conditions in a 1-parameter family, we will construct a cobordism $\pi : W \to L$ from $Id : L \to L$ to $\pi : P \to L$ over $L$, such that $TW - \pi^* TL$ will be $R$-orientable under Assumption 1.6. A standard application of the Pontryagin-Thom construction (see Section 3) will prove the following lemma:

**Lemma 1.12.** The induced map $\pi : \Sigma^\infty P \wedge R \to \Sigma^\infty L \wedge R$ admits a section of $R$-module maps (up to homotopy of $R$-module maps).

**Proof.** This follows from the construction of $W$ in Section 4, along with Lemma 2.19.

From these three lemmas, Theorem 1.7 follows. In Section 4.2, we will show the following:

**Proposition 1.13.** Assumption 1.6 holds when:

1. $R = \mathbb{H}_Z/2$ representing mod-2 singular homology.
2. $R = \mathbb{H}_Z$ representing integral singular homology, and $w_2(L)(x) = 0$ for $x$ in the image of the composition
   \[ \pi_3(M, L) \to \pi_2(L) \to H_2(L; \mathbb{Z}) \]
3. $R = KU$ representing complex $K$-theory, and $L$ admits a spin structure.
4. $R = KO$ representing real $K$-theory, and $TL$ admits a stable trivialisation over a 3-skeleton of $L$ which extends (after applying $\cdot \otimes \mathbb{C}$) to a stable trivialisation of $TM$ over a 4-skeleton of $M$.
5. $R$ is a complex-orientable generalised cohomology theory and $L$ is stably parallelisable.
6. $R = S$ is the sphere spectrum and there is a stable trivialisation of $TL$ which (after applying $\cdot \otimes \mathbb{C}$) extends to a stable trivialisation of $TM$.

**Remark 1.14.** Our methods can be used to show that when $R = \mathbb{H}_Z$, Assumption 1.6 holds if $L$ is spin. We will deduce the stronger statement (2) from [Geo13, Theorem 1.1].

**Corollary 1.15.** When $L$ is a homotopy sphere, $\psi^1|_L$ is homotopic to the identity.

**Proof.** $L$ admits a spin structure, so $\psi^1$ acts as the identity on integral homology and so is homotopic to the identity.

It is nontrivial to find examples of self-diffeomorphisms of spin manifolds which act trivially on integral cohomology but non-trivially on complex $K$-theory, so we provide an example in Appendix B following a suggestion of Randal-Williams [RW].
1.3 Extensions to the Free Loop Space

Fix a ring spectrum $R$. We will prove an analogue of Theorem 1.7 for the free loop space $LL$ of $L$, the space of maps $S^1 \to L$.

Theorem 1.16. Assume Assumption 5.9 holds. Then the induced map

$$\psi^1|_L : \Sigma^\infty_+ LL \wedge R \to \Sigma^\infty_+ LL \wedge R$$

is homotopic to the identity.

Idea of proof. To prove this, we will define $L_1$ to be the space of free loops in the mapping torus $L_{\psi^1}$ of $\psi^1|_L$, which have winding number one over $S^1$, and whose basepoint lies over $L$. $\psi$ determines an automorphism $\Psi^1 : L_1 \to L_1$, which one should think of as parallel transporting once around this mapping torus.

We will construct a map

$$p \circ ([N] \cdot) : \bigvee_j \Sigma^\infty_+ jL_1 \wedge R \to \Sigma^\infty_+ LL \wedge R$$

This will use a moduli space of holomorphic discs with moving boundary conditions $N$, as well as Cohen and Jones’ version of the Chas-Sullivan product, constructed similarly to [CJ02]. We will then show:

1. $p \circ ([N] \cdot)$ intertwines the actions of $\Psi^1$ and $\psi^1|_L$.
2. $\Psi^1$ acts as the identity on $L_1$ up to homotopy.
3. $p \circ ([N] \cdot)$ admits a section.

Together, these will imply Theorem 1.16. □

Proposition 1.17. Assumption 5.9 holds when

1. $R = H\mathbb{Z}/2$.
2. $R = H\mathbb{Z}$ and $L$ admits a spin structure which is preserved by $\psi^1$.
3. $R = KU$ and $L$ admits a homotopy class of stable trivialisations over a 2-skeleton which is preserved under $\psi^1|_L$.
4. $R$ is any complex-orientable generalised cohomology theory and $TL$ admits a homotopy class of stable trivialisations which is preserved under $\psi^1|_L$.

Corollary 1.18. Assume Assumption 5.9 holds. Then the map

$$(\psi^1|_L)_* : R_*(LL) \to R_*(LL)$$

is the identity.
From this, we can obtain strong restrictions on $G_L$ for certain $L$ that we could not get from Theorem 1.7.

**Corollary 1.19.**
1. If $L$ is a $K(\pi_1(L), 1)$, $\psi^1|_L$ is homotopic to the identity.
2. If $n = 2$, $\psi^1|_L$ is isotopic to the identity, and hence $G_L$ is trivial.

**Proof.** Recall that a diffeomorphism of a closed surface is homotopic to the identity if it is isotopic to the identity.

It follows from Corollary 1.15 that if $L = S^2$, then $\psi^1|_L$ is isotopic to the identity. Furthermore, since the mapping class group of $\mathbb{R}P^2$ is trivial, we can assume that $L$ is a $K(\pi_1(L), 1)$.

Note that $H_0(\mathcal{L}L; \mathbb{Z}/2)$ is the free $\mathbb{Z}/2$-module generated by homotopy classes of free loops in $L$. Therefore by Theorem 1.16, $\psi^1|_L$ acts as the identity on this set of generators. Then by [CV20, Theorem 2.4], $\psi^1|_L$ must be homotopic to the identity.

### 1.4 Extensions to Other Bases

In this subsection and Section 6, we restrict to the case $R = \mathbb{H}\mathbb{Z}/2$.

**Definition 1.20.** Given a fibre bundle $E \rightarrow B$, we say it c-splits if the inclusion of a fibre into $E$ induces an injection on mod-2 singular homology, for any fibre of this bundle.

We define $\text{Lag}_L$ to be the space of (unparametrised) Lagrangian submanifolds of $M$ which are Hamiltonian isotopic to $L$. There is a natural fibre bundle $\mathcal{E}$ over $\text{Lag}_L$, where the fibre over $K$ in $\text{Lag}_L$ is $K$. This is naturally a subbundle of $\text{Lag}_L \times M$. Given a map $\gamma : S^1 \rightarrow \text{Lag}_L$, Theorem 1.4 implies that the pullback bundle $\gamma^*\mathcal{E}$ c-splits. We will use techniques in [LM03] to deduce the following from Theorem 1.4.

**Theorem 1.21.**
(i). Given a map $\gamma : (S^1)^i \rightarrow \text{Lag}_L$, the pullback bundle $\gamma^*\mathcal{E} \rightarrow (S^1)^i$ c-splits.

(ii). Given a map $\gamma : S^1 \rightarrow \text{Lag}_L$, the pullback bundle $\gamma^*\mathcal{E} \rightarrow S^1$ c-splits.

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2 Spectra and Pontryagin-Thom Maps

We assume all spaces that we work with are Hausdorff, paracompact and homotopy equivalent to CW complexes. For an unbased space $X$, we will write $X_+$ for $X$ with a disjoint basepoint added.

We will use the category of spectra described in [Rud98], whose construction and properties we will sketch here.

2.1 Spectra

Definition 2.1. A spectrum $X$ is a sequence $\{X_n\}_{n \in \mathbb{N}}$ of based spaces along with structure maps $\Sigma X_n \to X_{n+1}$.

A function $f$ from $X$ to $Y$ is a family of based maps of spaces $f_n : X_n \to Y_n$ such that the square

\[
\begin{array}{ccc}
\Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\
\downarrow & & \downarrow \\
X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}
\end{array}
\]

commutes.

We will not define a morphism of spectra (see [Rud98, §II.1] for details), but any function of spectra is a morphism of spectra, and all morphisms of spectra that we will consider will arise in this way.

Remark 2.2. Rudyak actually defines a spectrum in a slightly different way but shows the two are equivalent in [Rud98, Lemma II.1.19].

Definition 2.3. Given a spectrum $X$ and a based space $Z$, there is a spectrum $X \wedge Z$ with $(X \wedge Z)_n = X_n \wedge Z$.

A homotopy between two functions $f, g : X \to Y$ is a function

\[ H : X \wedge [0, 1]_+ \to Y \]

restricting to $f$ over $\{0\}_+$ and to $g$ over $\{1\}_+$.

One can extend this definition to homotopies of morphisms of spectra, as in [Rud98, Definition II.1.9], and for homotopic morphisms $f$ and $g$ we write $f \simeq g$. Then for spectra $X$ and $Y$, we define $[X, Y]$ to be the set of homotopy classes of morphisms $X \to Y$. There is a natural structure of an abelian group on this set.

From this definition of a homotopy we can define a notion of (homotopy) equivalence as with spaces, which we denote by $\simeq$.

There is a natural functor $\Sigma^\infty$ from based spaces to $Sp$ sending a space $X$ to the spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$, with structure maps the identity. We write $\Sigma_+^\infty$ for the functor from unbased spaces to $Sp$ sending $X$ to $\Sigma^\infty X_+$. These both send homotopies to homotopies. We will denote $\Sigma_+^\infty \{\ast\}$ by $\mathbb{S}$, and call this the sphere spectrum.
For any \( N \) in \( \mathbb{Z} \), there is an endofunctor \( \Sigma^N \) of \( Sp \) which sends \( X \) to the spectrum \( \Sigma^N X \) with

\[
(\Sigma^N X)_n = \begin{cases} 
X_{n+N} & \text{if } n + N \geq 0 \\
\ast & \text{Otherwise}
\end{cases}
\]

This satisfies \( \Sigma^N \Sigma^M X \simeq \Sigma^{N+M} X \) for all \( X \), and \([X,Y] = [\Sigma^N X, \Sigma^N Y]\) for all \( X,Y \). We write \( \Sigma^\infty \Sigma^N X \) for \( \Sigma^N \Sigma^\infty X \).

**Definition 2.4.** Given a family \( \{X_\lambda\}_{\lambda \in \Lambda} \) of spectra, we can define their wedge product to be the spectrum \( \bigwedge_{\lambda} X_\lambda \) with

\[
(\bigwedge_{\lambda} X_\lambda)_n = \bigwedge_{\lambda} X_n
\]

**Lemma 2.5 ([Rud98, Proposition II.1.16]).** For any spectrum \( Y \), there are natural isomorphisms

\[
\left[ \bigwedge_{\lambda} X_\lambda, Y \right] \cong \prod_{\lambda} [X_\lambda, Y]
\]

and

\[
\left[ Y, \bigwedge_{\lambda} X_\lambda \right] \cong \bigoplus_{\lambda} [Y, X_\lambda]
\]

We can extend the smash product \( \wedge \) to two spectra, functorially in each argument, as in [Ada74]. We will not need explicit construction but will use some of its properties:

**Theorem 2.6 ([Rud98, Theorem II.2.1]).** There are equivalences

1. \( (X \wedge Y) \wedge Z \simeq X \wedge (Y \wedge Z) \)
2. \( X \wedge Y \simeq Y \wedge X \)
3. \( X \wedge S \simeq X \)
4. \( \Sigma X \wedge Y \simeq \Sigma(X \wedge Y) \)

such that all natural diagrams made up of these equivalences commute up to homotopy.

**Definition 2.7.** A ring spectrum \( R \) is a spectrum equipped with a unit morphism \( \eta : S \to R \) and product morphism \( \mu : R \wedge R \to R \), satisfying appropriate associativity and unitality conditions up to homotopy.

A module over a ring spectrum is defined similarly.

A map of \( R \)-modules \( X \to Y \) is a map of spectra \( X \to Y \) which commutes with the actions of \( R \) on \( X \) and \( Y \) up to homotopy.

A homotopy of \( R \)-module maps between \( f \) and \( g : X \to Y \) is a homotopy of spectra \( X \wedge [0,1]_+ \to Y \) between \( f \) and \( g \), which is also a map of \( R \)-modules.
Example 2.8. The sphere spectrum $S$ is naturally a ring spectrum, and every spectrum is naturally a module spectrum over it.

Example 2.9. For a ring spectrum $R$, there is a functor from spectra to $R$-module spectra, given by $\cdot \wedge R$.

Definition 2.10. We define the stable homotopy groups of $X$ to be

$$\pi_i X := [\Sigma^i S, X]$$

This is covariantly functorial in $X$.

Given a spectrum $R$, we define $R_i(X)$ to be $\pi_i(X \wedge R)$ and $R^i(X)$ to be $[X, \Sigma^i R]$. These are functorial in $X$, covariantly and contravariantly respectively.

Given a space $Z$, we define $R_*(Z)$ to be $R_*(\Sigma^\infty_+ Z)$ and $R^*(Z)$ to be $R^*(\Sigma^\infty_+ Z)$. These are functorial in $Z$, covariantly and contravariantly respectively.

By Brown’s representability theorem ([Hat02, Theorem 4E.1]), for an abelian group $G$ there is a (unique up to homotopy equivalence) spectrum $HG$ such that $HG_*$ and $HG^*$ are homology and cohomology with co-efficients in $G$ respectively, and when $G$ is a ring these are in fact ring spectra. Similarly there are (unique up to homotopy equivalence) ring spectra $KO$ and $KU$ such that $KO^*$ and $KU^*$ are real and complex $K$-theory respectively.

2.2 Thom Spectra

Let $\xi : E \to X$ be a vector bundle of rank $n$ over a space $X$. We let $DE$ and $SE$ denote the unit disc and unit sphere bundles of $E$ respectively, with respect to some choice of metric.

Definition 2.11. We define the Thom space of $E$, denoted $X^E_u$, to be the quotient $DE/SE$. This is a based space with basepoint given by the image of $SE$.

We define the Thom spectrum of $E$, denoted $X^E$, to be the spectrum $\Sigma^\infty X^E_u$.

If $E'$ is a virtual vector bundle which can be written as $E' \cong E - \mathbb{R}^m$ for some $m$ (in particular, this includes any virtual vector bundle pulled back from one over a compact space), we define the Thom spectrum of $E'$ to be $\Sigma^{\infty-m} X^E_u$.

Remark 2.12. None of these depend on the choice of metric up to homotopy equivalence, and furthermore the Thom spectrum only depends on the stable isomorphism class of the virtual vector bundle by [Rud98, Lemma IV.5.14].

Now let $\xi : E \to X$ be a virtual vector bundle.

Definition 2.13. $E$ is orientable with respect to $R$ if there is a morphism $U : X^E \to \Sigma^n R$, called the Thom class, whose restriction to a fibre (which is a copy of $\Sigma^n S$) represents plus or minus the unit in $R$.

An orientation is a homotopy class of such morphisms.
It follows from Remark 2.12 that being \( R \)-orientable only depends on the stable isomorphism class of \( E \).

**Lemma 2.14** ([Rud98 Proposition V.1.10 and Examples V.1.23]).

1. Any virtual vector bundle is canonically oriented with respect to \( H\mathbb{Z}/2 \).

2. A virtual vector bundle is orientable with respect to \( H\mathbb{Z} \) iff it is orientable in the usual sense, and there is a natural bijection between \( H\mathbb{Z} \)-orientations and orientations in the usual sense.

3. A trivial virtual vector bundle is orientable with respect to any \( R \).

4. If \( E \) is oriented with respect to \( R \) and \( f : Y \to X \) is any map, the pullback bundle \( f^*E \) admits a natural \( R \)-orientation.

5. If \( F \) is another virtual vector bundle over \( X \) and any two of \( E, F \) and \( E \oplus F \) are \( R \)-oriented, then the third admits a natural \( R \)-orientation.

We will need a stable version of the Thom isomorphism theorem:

**Theorem 2.15** ([Rud98 Theorem V.1.15 and Exercise V.1.28]). An \( R \)-orientation of \( E \) induces an equivalence of \( R \)-module spectra

\[
\Sigma^\infty_+X \wedge R \simeq X^E \wedge R
\]

More generally, if \( F \to X \) is another virtual vector bundle, then an \( R \)-orientation of \( E \) induces an equivalence of \( R \)-module spectra

\[
\Sigma^nX^F \wedge R \simeq X^{E \oplus F} \wedge R
\]

### 2.3 Pontryagin-Thom Collapse Maps

Let \( i : X \to Y \) be a smooth embedding of manifolds with \( X \) compact, such that \( i(\partial X) \subseteq \partial Y \). Let \( \nu \) be its normal bundle, with unit disc bundle \( D\nu \) and unit sphere bundle \( S\nu \) (with respect to some choice of metric). Furthermore we fix a tubular neighbourhood of \( X \), i.e. we pick an embedding \( j : D\nu \to Y \) extending \( i \), and not touching \( \partial Y \) apart from over \( \partial X \).

We will denote the one-point compactification of \( Y \) by \( Y_\infty \), viewed as a based space with basepoint at infinity. If \( Y \) is already compact then this is \( Y_+ \).

**Definition 2.16.** We define the Pontryagin-Thom collapse map \( i_{t,u} : Y_\infty \to X_u^\nu \) by

\[
i_{t,u}(x) = \begin{cases} j^{-1}(x) & \text{if } x \in j(D\nu) \\ S\nu & \text{otherwise} \end{cases}
\]

We denote the stabilisation \( \Sigma^\infty i_{t,u} \) by \( i_t \).

The map \( i_{t,u} \) is continuous, and both the space \( X_u^\nu \) and the map \( i_{t,u} \) are independent of the choices of metric and tubular neighbourhood up to homotopy.

Now let \( f : X^m \to Y^n \) be any smooth map of manifolds.
Definition 2.17. We define the stable normal bundle of \( f, \nu_f \), to be the virtual vector bundle \( f^*TY - TX \).

When \( f \) is an embedding, this is stably isomorphic to the normal bundle of the embedding, defined in the usual sense.

Now assume \( X \) and \( Y \) are closed, and choose a smooth embedding

\[
i : X \hookrightarrow \mathbb{R}^N
\]

for some \( N \). Now \( \nu_f \oplus \mathbb{R}^N \) is stably isomorphic to the normal bundle \( \nu_{f \times i} \) of the embedding

\[
f \times i : X \hookrightarrow Y \times \mathbb{R}^N
\]

We can consider the map

\[
(f \times i)_! : \Sigma^N \Sigma^\infty Y_\infty \simeq \Sigma^\infty \left( Y \times \mathbb{R}^N \right)_\infty \to X^\nu_{f \times i} \simeq \Sigma^N X^\nu_f
\]

Lemma 2.18. Up to homotopy the morphism \( \Sigma^\infty + N Y_\infty \to \Sigma^N X^\nu_f \) does not depend on the choice of \( i \) or \( N \).

Proof. First, observe that if we compose \( i \) with the standard embedding \( \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1} \), the resulting map is the same up to suspension.

Assume \( i, i' \) are two choices of embedding \( X \hookrightarrow \mathbb{R}^N \). Pick an embedding

\[
I : X \times [0, 1] \hookrightarrow \mathbb{R}^N \times [0, 1]
\]

over \([0, 1]\) restricting to \( i \) and \( i' \) over 0 and 1 respectively, increasing \( N \) if necessary as above. Then \((f \times I)_!, u \) provides a homotopy between \((f \times i)_!, u \) and \((f \times i')_!, u \).

Now that we know the desuspended map \( \Sigma^\infty Y_\infty \to X^\nu_f \) doesn’t depend on these choices (up to homotopy), we denote it by \( f_! \).

We fix a ring spectrum \( R \). Given an \( R \)-orientation of \( \nu_f \), we consider the following composition, which is a map of \( R \)-modules:

\[
p_f : \Sigma^\infty Y_\infty \wedge R \xrightarrow{f_!} X^\nu_f \wedge R \xrightarrow{\nu_f} \Sigma^\infty + d X \wedge R \xrightarrow{\nu_f} \Sigma^\infty + d Y_\infty \wedge R
\]

where \( d = n - m \). By construction, this factors through \( \Sigma^\infty + d X \wedge R \).

Now suppose that \( n = m \) (so \( d = 0 \)), \( Y \) is compact (so \( Y_\infty = Y_+ \)), and there is a cobordism \( W \) from \( X \) to \( Y \) along with a map \( F : W \to Y \) restricting to the identity on \( Y \) and to \( f \) on \( X \), as in the following diagram:

\[
\begin{array}{c}
X \\
\downarrow f \\
W \\
\downarrow p \\
Y \\
\downarrow \text{id}
\end{array}
\]

We assume that \( \nu_F \) admits an orientation with respect to \( R \), which, by restriction, induces one for \( \nu_f \).
Lemma 2.19. After applying $\Sigma^\infty_+ \cdot \wedge R$, $f$ admits a section of $R$-module maps up to homotopy (of $R$-module maps).

Proof. We will show that the map $p_f$ constructed above is an equivalence, then $f_t \circ p_f^{-1}$ will be the desired section.

We pick an embedding

$$I : W \hookrightarrow \mathbb{R}^N \times [0, 1]$$

for some $N$, such that $I^{-1} (\mathbb{R}^N \times \{0\}) = X$ and $I^{-1} (\mathbb{R}^N \times \{1\}) = Y$. Then choosing an orientation of $\nu_F$ and performing the above construction to $F$ gives us a homotopy from $p_f$ to $p_{Id}$, where $p_{Id}$ is constructed similarly to $p_f$ but for $Id : Y \to Y$, using some choice of trivialisation on the trivial vector bundle over $Y$. But $p_{Id}$ is a composition of three equivalences and hence $p_f$ is an equivalence.

All the maps and homotopies are constructed either by applying $\cdot \wedge R$ to a map of spectra or by applying the Thom isomorphism theorem over $R$, so they are all $R$-module maps.

Corollary 2.20. The map $f_* : R_* X \to R_* Y$ is surjective.

It follows from [Rud98, Lemma II.2.4 and Theorem V.2.3] that

Corollary 2.21. If $W$ is $R$-orientable (and hence $X$ and $Y$ are too), the map $f^* : R^* Y \to R^* X$ is injective.

We can also define Pontryagin-Thom collapse maps without smoothness assumptions. Let $i : X \hookrightarrow Y$ be an embedding of spaces.

Definition 2.22. We say $X$ admits a tubular neighbourhood in $Y$ with normal bundle $\nu$ if there is a vector bundle $\nu$ over $X$ and an open neighbourhood $U$ of $X$ in $Y$, such that there is a homeomorphism $\phi : U \to DE$ between $U$ and the unit disc bundle $DE$ of $\nu$ (for some choice of fibrewise metric on $\nu$), sending $X$ to the zero section.

Definition 2.23. Let $E \to Y$ be another vector bundle. We define the Pontryagin-Thom collapse map $i_{t,u} : Y_{u}^E \to X_{u}^{\nu \oplus E}$ by

$$(x, v) \mapsto \begin{cases} (\phi(x), v) & \text{if } x \in U \\ S(\nu \oplus E) & \text{otherwise} \end{cases}$$

We write $i_t$ for the induced map on Thom spectra.

This definition extends to the case when $E$ is a virtual vector bundle of the form $E' - \mathbb{R}^N$, as we can apply this construction to $E'$ and desuspend. Similarly to before, up to homotopy $i_t := \Sigma^\infty i_{t,u}$ then only depends on the virtual vector bundle up to stable isomorphism.

Example 2.24. When $X \hookrightarrow Y$ is a proper embedding of smooth manifolds, $X$ always admits a tubular neighbourhood in $Y$, and we recover the earlier construction.
2.4 Fundamental Classes

Let $X^n$ be a closed manifold of dimension $n$. Choose an embedding

$$i : X \hookrightarrow \mathbb{R}^N \hookrightarrow S^N$$

for some $N$, with normal bundle $\nu$. Then as virtual vector bundles, there is a natural isomorphism $\nu - \mathbb{R}^N \cong -TX$. Consider the map $i: S^N \to X^\nu$. Stabilising and desuspending gives us a well-defined (up to homotopy) map of spectra

$$[X] : \mathbb{S} \to X^{-TX}$$

Given a ring spectrum $R$ and an $R$-orientation of $TX$ (which induces one on $-TX$), the Thom isomorphism theorem gives us a well-defined (up to homotopy) map of spectra

$$[X] : \mathbb{S} \to \Sigma^\infty_{+} X \wedge R$$

representing an element $[X]$ in $R_\ast(X)$. More generally, given a space $Z$, a map of spaces $f : X \to Z$, a vector bundle $E \to Z$, and an $R$-orientation of $TX - f^*E$ (which we assume to have rank $d$), we can use the Thom isomorphism theorem to obtain a well-defined (up to homotopy) map of spectra

$$\mathbb{S} \to X^{-TX} \wedge R \cong \Sigma^{-d} X^{-f^*E} \wedge R \to \Sigma^{-d} Z^{-E} \wedge R$$

which we also denote by $[X]$.

We call all of these maps $[X]$ fundamental classes.

**Lemma 2.25.** Let $Y$ be another closed manifold of dimension $n$. Suppose $f : X \to Z$ and $g : Y \to Z$ are two maps, such that there is a cobordism $W$ from $X$ to $Y$, and a map $F : W \to Z$ extending $f$ and $g$, as shown below.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow{W} & \nearrow{f} & \nearrow{g} \\
Y & \xrightarrow{g} & W
\end{array}
\]

Let $E \to Z$ be a vector bundle and assume there is an $R$-orientation of $TW - F^*E$, which we assume to have rank $d + 1$. Then the two fundamental classes $[X], [Y]$ in $R_d(Z^{-E})$ (defined using the orientations of $TX - f^*E$ and $TY - g^*E$ given by restricting the orientation of $TW - F^*E$) agree.

**Proof.** Pick an embedding

$$I : W \to \mathbb{R}^N \times [0, 1]$$

for some $N$, such that $W^{-1}(\mathbb{R}^N \times \{0\}) = X$ and $W^{-1}(\mathbb{R}^N \times \{1\}) = Y$. Then the composition

$$\mathbb{S} \wedge [0, 1]_+ \to W^{R-TW} \wedge R \cong \Sigma^{-d} W^{-F^*E} \to \Sigma^{-d} Y^{-E}$$

defines a homotopy between $[X]$ and $[Y]$. \qed
Remark 2.26. If \( X = \bigsqcup_{i \geq 0} X^i \) is a compact manifold consisting of components \( X^i \) of dimension \( i \), we can consider the fundamental classes of these independently and obtain a fundamental class \([X]\) in \( \bigoplus_i R_i (X^{-TX})\). We will use this later in Section 3.

3 Index Bundles and Their Orientations

3.1 Cauchy-Riemann Operators

Let \( B \) be a space and \( A \subseteq B \) some subspace.

Definition 3.1. A bundle pair over the pair \((B, A)\) is a complex vector bundle \( E \) over \( B \) along with a totally real subbundle \( F \) of \( E|_A \). This is written as

\[
(E, F) \to (B, A)
\]

We can perform certain operations on bundle pairs, analogously to vector bundles.

- Given a real vector bundle \( G \) over \( B \), we can define \( (E, F) \otimes G \) to be the bundle pair

\[
(E \otimes_R G, F \otimes_R G|_A)
\]

- Given \( (E', F') \to (B, A) \) another bundle pair, we can define \( (E, F) \oplus (E', F') \) to be the bundle pair

\[
(E \oplus E', F \oplus F')
\]

- A virtual bundle pair is a formal difference \( (E, F) - (E', F') \) of two bundle pairs, and we say two virtual bundle pairs \( (E, F) - (E', F') \) and \( (G, H) - (G', H') \) are stably isomorphic if there is an isomorphism of bundle pairs

\[
(E, F) \oplus (G', H') \oplus (\mathbb{C}^N, \mathbb{R}^N) \cong (E', F') \oplus (G, H) \oplus (\mathbb{C}^N, \mathbb{R}^N)
\]

for some \( N \).

- The rank of a bundle pair \( (E, F) \) is \( \text{Rank}_\mathbb{C} E \), and the (virtual) rank of a virtual bundle pair \( (E, F) - (E', F') \) is \( \text{Rank}_\mathbb{C} E - \text{Rank}_\mathbb{C} E' \).

- Given a bundle pair \( (E, F) \to (B, A) \) and a map of pairs \( f : (B', A') \to (B, A) \), we define the bundle pair \( f^* (E, F) \to (B', A') \) to be the bundle pair \( (f^* E, f^* F) \).

- We define a section of \( (E, F) \) to be a section of \( E \) whose restriction to \( A \) lies in \( F \). We write \( \Gamma(E, F) \) for the space of sections.
Similarly to the case of real or complex vector bundles, when \( B \) is a finite CW complex and \( A \) a subcomplex, every virtual bundle pair over \((B, A)\) is stably isomorphic to \((E, F) \to (\mathbb{C}^N, \mathbb{R}^N)\) for some \((E, F)\) and some \(N\). Similarly the pullbacks of a bundle pair under homotopic maps are isomorphic.

We will usually take \((B, A)\) to be \((D^2, \partial D^2) \times X\) for some space \(X\), and we will assume this to be the case for the rest of Section 3.

For the rest of Section 3.1 we will assume that \(X\) is a point, so \((E, F)\) is a bundle pair over \((D^2, \partial D^2)\).

**Definition 3.2.** A (real) Cauchy-Riemann operator on \((E, F)\) is an \(R\)-linear first order differential operator

\[
D : \Gamma(E, F) \to \Omega^{0,1}(E)
\]

satisfying the Leibniz rule:

\[
D(f \eta) = (\bar{\partial} f) \eta + f D \eta
\]

for \(f \in C^\infty(D^2)\) and \(\eta \in \Gamma(E, F)\).

**Lemma 3.3** ([MS12, Remark C.1.2]). The space of Cauchy-Riemann operators on \((E, F)\) is contractible (and in fact, convex).

For a choice of Hermitian metric on \(E\) and a real number \(q > 2\) (which we fix), a Cauchy-Riemann operator induces an operator

\[
\hat{D} : W^{1,q}(E, F) \to W^q \left( \Lambda^{0,1}T^*D^2 \otimes E \right)
\]

where these spaces are appropriate Sobolev completions of the above spaces of smooth sections. By [MS12, Theorem C.1.10], this operator \(\hat{D}\) is Fredholm, and in fact \(\text{Ker} \hat{D} = \text{Ker} D\).

### 3.2 Index Bundles

Let \(X\) be a finite CW complex with basepoint \(x_0\), and let \((E, F) \to (D^2, \partial D^2) \times X\) be a bundle pair.

By Lemma 3.3 we can choose a Cauchy-Riemann operator \(D_x\) on

\[
\left( E|_{D^2 \times \{x\}}, F|_{\partial D^2 \times \{x\}} \right)
\]

for each \(x\), varying continuously in \(x\). The space of such choices is contractible.

**Definition 3.4.** Assume \(\hat{D}_x\) is surjective for all \(x\). The index bundle of \((E, F)\) is the vector bundle \(\text{Ind}(E, F)\) over \(X\), with fibre at a point \(x\) given by \(\text{Ker} \hat{D}_x\).

If \(\hat{D}_x\) is not always surjective, we can still define the index bundle, following [Ati68]. Since \(X\) is compact, we can find, for some finite \(N\), a continuous family of linear maps

\[
\phi_x : \mathbb{R}^N \to W^q \left( \Lambda^{0,1}T^*D^2 \otimes E|_{D^2 \times \{x\}} \right)
\]
for \( x \) in \( X \), such that the stabilised operator

\[ T_x := \dot{D}_x + \phi_x : W^{1,q} \left( E|_{D^2 \times \{x\}}, F|_{\partial D^2 \times \{x\}} \right) \oplus \mathbb{R}^N \to W^q \left( \Lambda^{0,1} T^* D^2 \otimes E|_{D^2 \times \{x\}} \right) \]

is surjective. In this situation we call \( \phi \) a stabilisation of rank \( N \). We then define \( \text{Ind}(E,F) + \mathbb{R}^N \) to be the vector bundle with fibre over \( x \) given by \( \text{Ker} T_x \).

**Lemma 3.5.** The vector bundle \( \text{Ind}(E,F) \) is well-defined up to (unique up to weakly contractible choice) stable isomorphism.

**Proof.** Firstly we note that stabilising \( \phi \) by adding another copy of \( \mathbb{R} \) which is sent to 0 does not change the index bundle up to canonical isomorphism.

We choose a metric on \( E \), and note that the space of such choices is contractible. This defines a \( W^{0,2} \) inner product on \( W^{1,q}(E,F) \).

Given two choices of family of Cauchy-Riemann operators and stabilisation \( (D, \phi) \) and \( (D', \phi') \), by the above we can assume that both stabilisations are of the same rank. If \( (D, \phi) \) and \( (D', \phi') \) have distance less than 1 with respect to the operator norm, orthogonal projection defines an isomorphism between their kernels. Note that since the space of such operators is convex, this open ball is contractible.

In general, we can pick a path in the space of such \( (D, \phi) \), and iterate this process along the path. Because the spaces of such pairs \( (D, \phi) \) (up to stabilisation) are weakly contractible, the uniqueness result follows from a similar argument.

We will require some elementary properties of index bundles.

**Lemma 3.6.** If \( X' \) is another finite CW complex and \( f : X' \to X \) is any map, then there is a natural isomorphism of vector bundles

\[ f^* \text{Ind}(E,F) \cong \text{Ind}f^*(E,F) \]

where we write \( f^*(E,F) \) for \( (\text{Id}_{D^2} \times f)^*(E,F) \).

**Proof.** All of the choices made to define the index bundle are compatible under pullbacks.

We will require some elementary properties of index bundles.

**Lemma 3.7.** If \( (E,F) \) and \( (E',F') \) are bundle pairs on \( (D^2, \partial D^2) \times X \), then there is a natural isomorphism of virtual vector bundles

\[ \text{Ind}(E,F) \oplus \text{Ind}(E',F') \cong \text{Ind}(E \oplus E', F \oplus F') \]

**Proof.** All of the choices made to define the index bundle are compatible under direct sums.

**Corollary 3.8.** If \( (E,F) \) is stably isomorphic to \( (E',F') \), then \( \text{Ind}(E,F) \) is stably isomorphic to \( \text{Ind}(E',F') \).
Lemma 3.9. Let $G \to X$ be a real virtual vector bundle over $X$. Then there is a stable isomorphism

$$(\text{Ind}(E, F)) \otimes G \cong \text{Ind} \left( (E, F) \otimes G \right)$$

Proof. We assume $G$ is an actual vector bundle, and observe that the general case follows from Lemma 3.7.

Let $D$ be a family of Cauchy-Riemann operators on $(E, F)$, and $\phi$ a stabilisation of rank $N$, corresponding to the family of stabilised operators $T$. We pick another vector bundle $G'$ over $X$, along with an isomorphism $G \oplus G' \cong \mathbb{R}^M$ for some $M$.

For each $x$ in $X$, we consider the Cauchy-Riemann operator

$$D_x \otimes \text{Id}_{G_x} : \Gamma \left( \left( E|_{D^2 \times \{x\}}, F|_{\partial D^2 \times \{x\}} \right) \otimes G_x \right) \to \Omega^{0,1} \left( E|_{D^2 \times \{x\}} \otimes G_x \right)$$

along with the stabilisation of rank $NM$

$$\theta_x : (\mathbb{R}^N \otimes G_x) \oplus (\mathbb{R}^N \otimes G'_x) \to W^q \left( \Lambda^{0,1} T^* D^2 \otimes E|_{D^2 \times \{x\}} \otimes G_x \right)$$

sending $(a \otimes u, b \otimes v)$ to $\phi_x(a) \otimes u$.

Then $\text{Ind} \left( (E, F) \otimes G \right)_x \oplus \mathbb{R}^{NM}$ is equal to $\text{Ker} S_x$ by definition, where $S_x$ is the stabilised operator

$$\left( D_x \otimes \text{Id}_G \right) \oplus \theta_x$$

But $\text{Ker} S_x$ is isomorphic to

$$\left( (\text{Ker} T_x) \otimes G_x \right) \oplus \left( \mathbb{R}^N \otimes G'_x \right)$$

which is isomorphic to

$$\left( (\text{Ind}(E, F))_x \otimes G_x \right) \oplus \mathbb{R}^{N,M}$$

There is a standard bundle pair over a point, $H = (\mathbb{C}, \delta) \to (D^2, \partial D^2)$, where $\delta(z) = \sqrt{-1} \mathbb{R}$ for $z$ in $\partial D^2$.

We can pick a trivialisation $E \cong \mathbb{C}^n$ over $D^2 \times \{x\}$ for a point $x$ in $X$. Then $F|_{\partial D^2 \times \{x\}}$ determines a loop in the space of totally real subspaces of $\mathbb{C}^n$, which is isomorphic to $U(n)/O(n)$. There are isomorphisms

$$\pi_1 U(n)/O(n) \cong \mathbb{Z}$$

for all $n$, compatibly with stabilisation in $n$. There are two such choices of isomorphism, and we fix the one such that $H \oplus (\mathbb{C}^{n-1}, \mathbb{R}^{n-1})$ is sent to 1.

Definition 3.10. We define the Maslov index of $(E, F)$, denoted $\mu(E, F)$, to be the image of $F$ under the isomorphism $\pi_1 U(n)/O(n) \to \mathbb{Z}$ after choosing a trivialisation of $E$ over a point. This is well-defined on each connected component of $X$. 

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Using the index theorem, one can compute the virtual rank of the index bundle.

**Lemma 3.11** ([MSS12, Theorem C.1.10]). The virtual rank of $\text{Ind}(E, F)$ is

$$n + \mu(E, F)$$

If $(E, F)$ is a virtual bundle pair over $(D^2, \partial D^2) \times Y$ for some (possibly non-compact) space $Y$, for any map from a finite CW complex $X$ to $Y$, $\text{Ind}(E, F)|_X$ is well-defined, and furthermore this is compatible with restriction.

**Definition 3.12.** We say that $\text{Ind}(E, F)$ is $R$-orientable if $\text{Ind}(E, F)|_X$ is $R$-orientable for all such $X$.

An $R$-orientation on $\text{Ind}(E, F)$ is a choice of $R$-orientation on $\text{Ind}(E, F)|_X$ for all $X$, compatible with restriction.

### 3.3 Orientations

Our goal in this section will be to establish the following two propositions, the first of which is clear from Corollary 3.8. These will be used later to find conditions under which the tangent bundles to the moduli spaces constructed in Sections 4.1 and 5.3 are $R$-orientable for various ring spectra $R$.

**Proposition 3.13.** A stable trivialisation of $(E, F)$ induces a stable trivialisation of $\text{Ind}(E, F)$, and hence an $R$-orientation for any ring spectrum $R$, compatibly with direct sums and pullbacks.

*Proof.* By Corollary 3.8 it suffices to prove the result for $(E, F) = (\mathbb{C}^n, \mathbb{R}^n)$. Over every point $x$ in the base $X$, we pick the standard Cauchy-Riemann operator

$$\bar{\partial} : \Gamma (\mathbb{C}^n, \mathbb{R}^n) \to \Gamma (\mathbb{C}^n)$$

on sections over the disc $D^2 \times \{x\}$.

Evaluation at any fixed point $z$ in $\partial D^2$ defines a map $\text{Ker} \bar{\partial} \to \mathbb{R}^n$, which is an isomorphism (and which does not depend on $z$). Then because the index of this Cauchy-Riemann operator is $n$, this Cauchy-Riemann operator must be surjective. This gives us the required trivialisation of $\text{Ind}(E, F)$.

The following proposition seems well-known to experts, but to the author’s knowledge a proof has not yet appeared in the literature.

**Proposition 3.14.** A stable trivialisation of $F$ induces a stable complex structure of $\text{Ind}(E, F)$, compatibly with direct sums and pullbacks.

*Proof.* Assume $X$ is connected. Recall that a based vector bundle (or bundle pair) over $X$ is a vector bundle (or bundle pair) over $X$ (or $(D^2, \partial D^2) \times X$) with a fixed trivialisation over the basepoint of $X$.

Let $\mu = \mu(E, F)$ be the Maslov index. Then since $F$ is stably trivial, $\mu$ must be even. Therefore the real part of $H^{\oplus \mu}$ is also stably trivial. Furthermore,
Ind($H^n$) is canonically trivial and hence admits a canonical stable complex structure.

Note that the virtual bundle pair $(E, F) - H^\otimes \mu$ has Maslov index 0. Therefore it suffices to assume $\mu = 0$, which we now do.

The stable trivialisation of $F$ induces a stable trivialisation of $E$ which restricts to that of $F$ over $\{1\} \times X$ (but not necessarily over the rest of $\partial D^2 \times X$). Furthermore because $\mu = 0$ we can pick this stable trivialisation to extend $F$ over $\partial D^2 \times \{x_0\}$ too. Thus $F$ determines a based map $\Sigma X \rightarrow U(n)/O(n)$ and, stabilising, a based map

$$\psi : \Sigma X \rightarrow U/O$$

whose based homotopy class, along with the rank $n$, classifies $(E, F)$ up to stable, based isomorphism.

The stable trivialisation of $F$ determines a nullhomotopy of the composition

$$\Sigma X \rightarrow U/O \xrightarrow{\zeta} BO$$

where the second map $\zeta$ is the forgetful map, sending a totally real subspace $V$ in $\mathbb{C}^N$, to $V$ viewed as in $\mathbb{R}^{2N}$. After applying $[\Sigma X, \cdot]_*$, this sends a bundle pair to its real part. Taking adjoints determines a nullhomotopy of the adjoint map

$$X \rightarrow \Omega BO$$

We define a map

$$[\Sigma X, U/O]_* \rightarrow [X, BO]_*$$

sending a based bundle pair $(E, F)$ of rank $n$ to the classifying map of Ind$(E, F) - \mathbb{R}^n$. This was proved to be a bijection by de Silva in [dS98], and we recap his proof in Appendix A, de Silva constructed an explicit inverse to this map, which we call

$$\beta : [X, BO]_* \rightarrow [\Sigma X, U/O]_*$$

sending a based vector bundle $E$ of rank $n$ to the map $\Sigma X \rightarrow U/O$ corresponding to the based virtual bundle pair of rank $n$:

$$H \otimes E - (\mathbb{C}, \mathbb{R}) \otimes E - H \otimes \mathbb{R}^n$$

We define a map

$$\varepsilon : [X, BO]_* \rightarrow [\Sigma X, BO]_*$$

to send a based vector bundle $E$ of rank $n$ to

$$\lambda \otimes E - \mathbb{R} \otimes E - \lambda \otimes \mathbb{R}^n$$

where $\lambda \rightarrow S^1$ is the Möbius line bundle. Then it is clear that the following diagram commutes:

$$
\begin{array}{ccc}
[X, BO]_* \ & \xrightarrow{\varepsilon} & \ [\Sigma X, BO]_* \\
\downarrow \beta & & \downarrow = \\
[\Sigma X, U/O]_* \ & \xrightarrow{\zeta} & \ [\Sigma X, BO]_* = [X, \Omega BO]_*
\end{array}
$$
Consider the fibration sequence

\[ BU \to BO \to \Omega BO \]

where the first arrow corresponds to the realification of a vector bundle and the second to \( \zeta \). Then the given contraction of our map \( X \to \Omega BO \) induces a lift to a map \( X \to BU \), which is the stable complex structure we desired.

It it clear from construction that everything is compatible with direct sums and pullbacks.

4 The Moduli Spaces \( \mathcal{P} \) and \( \mathcal{W} \)

4.1 Construction of the Moduli Spaces

Let \( C \) be a convex domain in the upper half plane in \( \mathbb{C} \), with smooth boundary \( \partial C \) containing 0. Let \( f : \partial C \to [0, 1] \) be a smooth map sending 0 to 0, and let \( J \) be an \( \omega \)-tame almost complex structure on \( M \) which is convex at infinity.

**Definition 4.1.** We define \( D_{f,C} \) to be the space of smooth maps \( u : C \to M \) such that for all \( z \) in \( \partial C \), \( u(z) \) lies in \( \psi f(z)(L) \).

We define \( \pi \) to be the natural evaluation map \( \pi : D_{f,C}(J) \to L \) sending \( u \) to \( u(0) \).

**Definition 4.2.** We define \( U_{f,C}(J) \) to be the space of maps \( u \) in \( D_{f,C} \) which are \( J \)-holomorphic, namely

\[ J \circ du = du \circ j \]

where \( j \) is the complex structure on \( C \subseteq \mathbb{C} \).

We call the triple \( (C, f, J) \) the moduli data.

Let \( G \) be a fixed convex domain in \( \mathbb{C} \) with smooth boundary, such that both \( (-\eta, \eta) \) and \( i + (-\eta, \eta) \) are contained in \( \partial G \) for some \( \eta > 0 \). Let \( G_{\pm} \) be \( G \cap \mathbb{C}_{\pm \text{Re} \geq 0} \).

For \( l \geq 0 \), define \( Z_l \) to be \([0, 1)i + [-l, l]\), and \( G_l \) to be

\[ Z_l \cup (G_+ + l) \cup (G_- - l) \]

as shown below.
We now define a 1-parameter family of moduli data \((C_l, f_l, J_l)\) as follows.

\[
C_l := \begin{cases} 
  G_{l-1} & \text{if } l \geq 1 \\
  G_0 & \text{if } l \leq 1
\end{cases}
\]

\[
f_l(z) := \begin{cases} 
  \text{Im } z & \text{if } l \geq 1 \\
  l\text{Im } z & \text{if } l \leq 1
\end{cases}
\]

As a 1-parameter family of domains in \(\mathbb{C}\) and maps to \([0, 1]\), both of these are continuous everywhere in \(l\) and vary smoothly in \(l\) except at 1, so we pick a small smoothing of these families supported near \(l = 1\).

Using relative exactness of \(L\) in \(M\), the proof of [Hof88, Lemma 2] shows that

**Lemma 4.3.** There exists some \(A\) in \(\mathbb{R}\) such that for any \(l \geq 0\) and any smooth map \(u\) in \(D_{f_l, C_l}\), the topological energy \(\int u^* \omega\) is bounded above by \(A\).

We now pick a Riemannian metric on \(L\), with injectivity radius \(\varepsilon\) and distance function \(d\). Using this energy bound, [Hof88, Proposition 3] then directly implies the following result, which should be viewed as a form of Gromov compactness. Roughly, he shows that for large \(l\), these moduli spaces, when restricted to \(Z_l\), live close to holomorphic strips with boundary on \(L\) and \(\psi^1(L)\), which we know are constant since \(L = \psi^1(L)\) is relatively exact.

**Lemma 4.4.** There is some \(c > 1\) such that for all \(l \geq c\) and \(u\) in \(U_{f_l, C_l}(J_l)\),

\[
d(u(i), u(0)) < \varepsilon
\]

Standard techniques in [MS12] show that for a generic (\(l\)-dependent) perturbation of \(J_l\), \(W' := \bigcup_{l \geq 0} U_{f_l, C_l}(J_l)\) is a smooth manifold with boundary \(U_{f_0, C_0}(J_0)\), which consists only of constant maps and hence \(\pi : U_{f_0, C_0}(J_0) \to L\) is a diffeomorphism. If we choose the perturbation small enough then Lemma 4.4 still holds, by Lemma 4.4 along with Gromov compactness.

Let \(U_l = U_{f_l, C_l}(J_l)\), and let \(p : W' \to \mathbb{R}\) send \(U_l\) to \(l\).
Lemma 4.5. Each $U_l$ is compact and $p$ is a proper map.

Proof. We will show each $U_l$ is compact, a similar argument shows that $p$ is a proper map.

Let $u_i$ be a sequence in $U_l$. Gromov compactness as in [FZ15] implies that this contains a convergent subsequence, but only after precomposing with a sequence of holomorphic automorphisms of the domain.

Let $\tilde{M} = M \times \mathbb{C}$, with almost complex structure $J_l$ times the standard complex structure on $\mathbb{C}$. Let $\tau : \tilde{M} \to \mathbb{C}$ be the projection map onto the second factor. Let $\tilde{L}$ be

$$\bigcup_{z \in \partial C_l} \psi^{h(z)}(L) \times \{z\}$$

a totally real submanifold of $\tilde{M}$.

We let $U_l$ be the moduli space of holomorphic discs in $\tilde{M}$ with boundary in $\tilde{L}$. There is an embedding $\theta : U_l \hookrightarrow \tilde{U}_l$ sending $u$ to the map $\tilde{u}$ which sends $z$ to $(u(z), z)$. This clearly is a bijection with the subset of $\tilde{U}_l$ consisting of sections of $\tau|_{C_l}$.

For $u$ in $U_l$, the energy of $\tilde{u}$ is equal to the energy of $u$ plus the area of $C_l$, so in particular by Lemma 4.3 the energies of $\tilde{u}_i$ are uniformly bounded. Therefore by Gromov compactness, there is a subsequence (which we will continue to call $\tilde{u}_i$, by abuse of notation) and a sequence of holomorphic automorphisms $\phi_i$ of $C_l$, such that $\tilde{u}_i \circ \phi_i$ lives in $\tilde{U}_l$ for all $i$, and $\tilde{u}_i \circ \phi_i$ converges to some $v$ in $\tilde{U}_l$.

$\tau \circ \tilde{u}_i \circ \phi_i = \phi_i$ also converges to $\phi := \tau \circ v$ which is a holomorphic automorphism of $C_l$, so $\tilde{u}_i$ converges to $v \circ \phi^{-1}$, which lies in the image of $\theta$ and so $v = \tilde{u}$ for some $u$ in $U_l$. Therefore $u_i$ converges to $u$ in $U_l$. \vspace{4mm}

We pick $c' > c$ (where $c$ is from Lemma 4.4) a regular value of $p$. Then we define $W$ to be the path component of $U_0 = L$ in $p^{-1}[0, c']$, and $P$ to be

$$W \cap P_{c'}$$

Then, by construction, $\pi : W \to L$ is a cobordism from $Id : L \to L$ to $\pi : P \to L$ over $L$. Furthermore whilst $W$ may have had different path components of different dimensions, $W$ is a manifold (with boundary) of dimension $n + 1$.

Proof of Lemma 4.4. We define $\pi' : P \to L$ to send $u$ to $u(t)$. Then by Lemma 4.3 this is homotopic to $\pi$. We now write down a homotopy $H : P \times [0, 1] \to L$ from $\pi'$ to $\psi^1 \circ \pi$.

Let $\gamma : [0, 1] \to \partial C_{c'}$ be some fixed path from $0$ to $i$. For $u$ in $P$ and $t$ in $[0, 1]$, we define $H(u, t)$ to be

$$\psi^1 \left( \left( \psi^{\text{Im}}(\gamma(t)) \right)^{-1} \left( u(\gamma(t)) \right) \right)$$

\vspace{4mm}

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Remark 4.6. [Hof88, Theorem 1] shows that even without any genericity assumption on the almost complex structure, the map $\pi : \mathcal{P} \to \mathcal{L}$ induces an injective map

$$H^*(\mathcal{L}; \mathbb{Z}/2) \cong \check{H}^*(\mathcal{P}; \mathbb{Z}/2)$$

where $\check{H}^*$ denotes Čech cohomology. The statement of Lemma 4.4 still holds and hence so does Lemma 1.11. From this it follows that $\psi^1|_\mathcal{L}$ induces the identity map on $H^*(\mathcal{L}; \mathbb{Z}/2)$.

4.2 Orientations on the Moduli Space

We define $\mathcal{D}_0$ to be the space of triples $(f, C, u)$, where $C$ is a convex domain in the upper half plane in $\mathbb{C}$ with smooth boundary $\partial C$ containing $0$, $f : \partial C \to [0, 1]$ is a smooth map sending $0$ to $0$, and $u$ is an element of $\mathcal{D}_{f,C}$. Note that each inclusion $\mathcal{D}_{f,C} \hookrightarrow \mathcal{D}_0$ is a homotopy equivalence.

There is a tautological smooth fibre bundle over $\mathcal{D}_0$ with fibre over $(f, C, u)$ given by $C$. The structure group of this bundle is the group of diffeomorphisms of $(D^2, \partial D^2)$ which send $1$ to itself. This group is contractible, so we can pick a trivialisation $\Phi$ from $D^2 \times \mathcal{D}_0$ to this bundle, preserving $0$. Note the space of such choices is contractible.

There is an evaluation map

$$ev : (D^2, \partial D^2) \times \mathcal{D}_0 \rightarrow (M, L)$$

sending $(z, u)$ in $D^2 \times \mathcal{D}_{f,C}$ to

$$\left(\tilde{f}(x)\right)^{-1} \left(u(\Phi(z))\right)$$

where $\tilde{f}$ is some smooth extension of $f$ to the entirety of $C$. Since for each $(f, C)$ the space of such extensions is contractible, we can make some consistent choice of such things to define $ev$, and therefore this defines $ev$ uniquely up to a contractible space of choices.

There is a bundle pair over $(D^2, \partial D^2) \times \mathcal{D}_0$ given by $ev^* (TM, TL)$. For a map from a CW complex $g : X \rightarrow \mathcal{D}_0$, we write Ind for the index of the pullback of this bundle pair along $g$. Note that the virtual rank of Ind may be different on different components of $X$ if the evaluation map lands in different components of $\mathcal{D}_0$.

By construction, if $J$ is regular, $T\mathcal{U}_{f,C}(J)$ is stably isomorphic to the index bundle Ind, and similarly $T\mathcal{W}$ is stably isomorphic to $\text{Ind} \oplus \mathbb{R}$.

Let $X$ be a finite CW complex and $g : X \rightarrow \mathcal{D}_0$ some map. In the rest of this section we will find conditions under which Ind is $R$-orientable over $X$ for various ring spectra $R$.

Proposition 4.7. If there is a stable trivialisation $TL \cong \mathbb{R}^n$ which (after applying $\otimes \mathbb{C}$) extends to a stable trivialisation $TM \cong \mathbb{C}^n$, then $\text{Ind} - \pi^* TL$ is stably trivial and hence $R$-orientable for any ring spectrum $R$. 
Proof. Follows from Proposition 3.8.

Proposition 4.8. If there is a stable trivialisation $TL \cong \mathbb{R}^n$ over the 3-skeleton of $L$ which (after applying $\cdot \otimes \mathbb{C}$) extends to a stable trivialisation $TM \cong \mathbb{C}^n$ over the 4-skeleton of $M$, then $\text{Ind}$ and hence $\text{Ind} - \pi^* TL$ are KO-orientable.

Proof. We pick a 3-skeleton $L_3$ of $L$, a 4-skeleton $M_4$ of $M$ containing $L_3$, and a 2-skeleton $X_2$ of $X$. Then the map of pairs $ev : (D^2, \partial D^2) \times X_2 \to (M, L)$ can be homotoped to land in $(M_4, L_3)$, and we can apply Proposition 3.8 to see that $\text{Ind}$ admits a stable trivialisation over a 2-skeleton. Then by [ABS64, Theorem 12.3], $\text{Ind}$ is KO-orientable.

Lemma 4.9. If there is a stable trivialisation $TL \cong \mathbb{R}^n$ over the $i+1$-skeleton of $L$, $\text{Ind}$ admits a stable complex structure over the $i$-skeleton of $X$.

Proof. The real part of the bundle pair $(ev^* TM, ev^* TL) \to (D^2, \partial D^2) \times X$ is pulled back from a map $\partial D^2 \times X \to L$. Letting $X_i$ be some choice of $i$-skeleton of $X$, $ev : \partial D^2 \times X_i \to L$ can be homotoped to have image in the $i+1$-skeleton of $L$, so by Proposition 3.14, $\text{Ind}$ admits a stable complex structure over $X_i$.

Corollary 4.10. 1. If $L$ admits a spin structure, $\text{Ind}$ and $\text{Ind} - \pi^* TL$ are orientable.

2. If $L$ admits a spin structure, $\text{Ind}$ and $\text{Ind} - \pi^* TL$ are KU-orientable.

Proof. 1. If $L$ admits a spin structure, $TL$ is stably trivial over the 2-skeleton of $L$, and so $\text{Ind}$ admits a complex structure over the 1-skeleton of $X$ and hence is orientable.

2. A vector bundle over a finite CW complex $X$ is stably trivial over an $i$-skeleton $X_i$ iff the classifying map $X \to BO$ is nullhomotopic when restricted to $X_i$. Since $\pi_3 BO = 0$, if $X_2 \to BO$ is nullhomotopic, so is $X_3 \to BO$. So if $L$ admits a spin structure, $TL$ admits a stable trivialisation over any 3-skeleton. Therefore $\text{Ind}$ admits a spin structure, i.e. it admits a stable complex structure over any 2-skeleton. Then by [ABS64, Theorem 12.3], $\text{Ind}$ is KU-orientable.

Lemma 4.11. If $w_2(L)(x) = 0$ for $x$ in the image of the composition

$$\pi_3(M, L) \to \pi_2(L) \to H_2(L; \mathbb{Z})$$

then $\text{Ind} - \pi^* TL$ is orientable.

Proof. $w_1(\text{Ind} - \pi^* TL)$ vanishes by [Geo13, Theorem 1.1].
5 Monodromy in the Looped Case

5.1 General Set-Up

We define $L_{\psi^1}$ to be the space of pairs $(t, x)$, where $t$ lies in $S^1 = \mathbb{R}/\mathbb{Z}$ and $x$ lies in $\psi^1(L)$. We let $q : L_{\psi^1} \to S^1$ be the projection map to the first co-ordinate. This is a model for the mapping torus of $\psi^1$. We denote its vertical tangent space by $T_vL_{\psi^1}$.

The Hamiltonian flow induces a natural family of maps which we denote by $\Psi^t : L_{\psi^1} \to L_{\psi^1}$ for $t$ in $\mathbb{R}$, which live over the identity map on $S^1$ for $t$ in $\mathbb{Z}$.

More explicitly, pick $x$ in $\psi^s(L)$ and $t$ in $\mathbb{R}$, and write $t + s = m + \tau$, where $m$ is in $\mathbb{Z}$ and $\tau$ is in $[0, 1)$. Then we define $\Psi^t(x)$ to be

$$\psi^\tau \left( \left( \psi^1 \right)^m \left( (\psi^s)^{-1} (x) \right) \right)$$

Then $\Psi^0$ is the identity map and $\Psi^t \circ \Psi^s = \Psi^{t+s}$ for all $t, s$ in $\mathbb{R}$. Furthermore $\Psi^1|_{q^{-1}(0)} = \psi^1$. $\Psi^t$ should be thought of as parallel transporting by a distance $t$ around the base of the fibre bundle $L_{\psi^1} \to S^1$.

The free loop space $\mathcal{L}L_{\psi^1}$ of $L_{\psi^1}$ is naturally a fibre bundle over $L_{\psi^1}$. We define $\mathcal{L}$ to be the restriction of this to the fibre over 0 in $S^1$. Then $\mathcal{L}$ is naturally a fibre bundle over $L$ and splits into a disjoint union

$$\mathcal{L} = \bigsqcup_{j \in \mathbb{Z}} \mathcal{L}_j$$

where $\mathcal{L}_j$ is the space of loops in $\mathcal{L}$ with winding number $j$ about $S^1$.

$\Psi^1$ induces a map $\mathcal{L} \to \mathcal{L}$ which preserves winding numbers, so $\Psi^1$ induces an automorphism of $\mathcal{L}_i$ for all $i$, which we will also denote by $\Psi^1$.

We define $\mathcal{L}_j'$ to be subspace of $\mathcal{L}_j$ given by loops $\gamma$ such that the composition $S^1 \to L_{\psi^1} \to S^1$ is given by $z \mapsto jz$, and $\mathcal{L}'$ to be the disjoint union of all $\mathcal{L}_j'$. Then $\Psi^1$ sends $\mathcal{L}'$ to $\mathcal{L}'$.

**Lemma 5.1.** The inclusion $\mathcal{L}' \hookrightarrow \mathcal{L}$ is a homotopy equivalence.

**Proof.** We first define a continuous map $f : \mathcal{L}_j \to \{ \text{continuous maps } [0, 1] \to \mathbb{R} \}$. For all $\gamma$ in $\mathcal{L}_j$, $f_\gamma$ is uniquely determined by the following properties.

1. $f_\gamma(0) = 0$
2. $f_\gamma(1) = j$
3. The composition $[0, 1] \to S^1 \xrightarrow{\gamma} L_{\psi^1} \xrightarrow{q} S^1$

agrees with the composition $[0, 1] \xrightarrow{f_\gamma} \mathbb{R} \to S^1$.

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We define a map $F : \mathcal{L} \to \mathcal{L}'$ by

$$F(\gamma)(t) = \Psi^{j_t - f_t}(\gamma(t))$$

for $t$ in $[0, 1]$. Then this is a homotopy inverse to the inclusion $\mathcal{L}' \hookrightarrow \mathcal{L}$. □

By Lemma 5.1, $\mathcal{L}_0 \simeq \mathcal{L}L$ and it is clear that the action of $\Psi^1$ on $\mathcal{L}_0$ is the same as the action of $\psi^1$ on $\mathcal{L}L$.

**Lemma 5.2.** $\Psi^1$ acts as the identity on $\mathcal{L}_{\pm 1}$ up to homotopy.

**Proof.** We prove this in the case of $+1$, the case of $-1$ is similar. Note by Lemma 5.1 it suffices to prove $\Psi^1$ acts as the identity on $\mathcal{L}'_{+1}$.

We define $H : \mathcal{L}'_{+1} \times [0, 1] \to \mathcal{L}'_{+1}$ by

$$H(\gamma, s)(t) = \Psi^s(\gamma(t - s))$$

Then $H(\cdot, 0)$ is the identity and $H(\cdot, 1)$ is $\Psi^1$. □

### 5.2 A Chas-Sullivan Type Product

We first note that by [Mil59], all spaces considered so far in this section are homotopy equivalent to CW complexes, and furthermore they are all Hausdorff and paracompact.

We denote the pullback of the virtual vector bundle $-TL \to \mathcal{L}$ also by $-TL$. We will define ring and module structures on the spectra $\mathcal{L}^{-TL}$ and $\mathcal{L}$, following [CJ02].

We have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{L}_j \times_L \mathcal{L}_k & \xrightarrow{\Delta} & \mathcal{L}_j \times \mathcal{L}_k \\
\downarrow e & & \\
\mathcal{L}_{j+k} & &
\end{array}$$

where $\mathcal{L}_j \times_L \mathcal{L}_k$ is the fibre product of $\mathcal{L}_j$ and $\mathcal{L}_k$ over $L$, $\Delta$ is the natural inclusion map into $\mathcal{L}_j \times \mathcal{L}_k$, and $e$ is a concatenation map: given $(\gamma, \delta)$ in $\mathcal{L}_j \times L \mathcal{L}_k$ and $t$ in $[0, 1]$,

$$c(\gamma, \delta)(t) = \begin{cases} 
\gamma(2t) & \text{if } t \leq \frac{1}{2} \\
\delta(2t - 1) & \text{if } t \geq \frac{1}{2}
\end{cases}$$

Then $\mathcal{L}_j \times \mathcal{L}_k$ admits a tubular neighbourhood in $\mathcal{L}_j \times \mathcal{L}_k$ with normal bundle $TL$, and by applying Definition 2.23 with the virtual vector bundles $-TL \boxplus -TL$ and $-TL \boxplus 0$ over $\mathcal{L}_j \times \mathcal{L}_k$, we obtain two maps, both of which we denote by $\mu$:

$$\mu : \mathcal{L}_j^{-TL} \times \mathcal{L}_k^{-TL} \xrightarrow{\Delta_{\otimes} \otimes} (\mathcal{L}_j \times \mathcal{L}_k)^{-TL} \xrightarrow{\boxplus} \mathcal{L}_{j+k}^{-TL}$$

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which fit together to give maps

\[ \mu : \mathcal{L}^{-TL} \wedge \Sigma_+^\infty \mathcal{L}_k \overset{\Delta}{\to} \Sigma_+^\infty (\mathcal{L}_j \times \mathcal{L}_k) \overset{\mu}{\to} \Sigma_+^\infty \mathcal{L}_{j+k} \]

It follows from homotopy coassociativity of \( \Delta \) that these define a homotopy associative product on \( \mathcal{L}^{-TL} \) and a left module action of this on \( \Sigma_+^\infty \mathcal{L} \).

Let \( i : L \hookrightarrow \mathcal{L} \) be the inclusion of constant loops into \( \mathcal{L}_0 \). The composition

\[ [L] : \Sigma_+^\infty \mathcal{L} \to \mathcal{L}^{-TL} \]

defines a unit for \( \mathcal{L}^{-TL} \), similarly to [CJ02]. Together, we have

**Proposition 5.3.** \( \mathcal{L}^{-TL} \) is a ring spectrum, and \( \Sigma_+^\infty \mathcal{L} \) is a left module over it. Furthermore if \( R \) is a ring spectrum, \( \mathcal{L}^{-TL} \wedge R \) is a ring spectrum, and \( \Sigma_+^\infty \mathcal{L} \wedge R \) is a left module over it.

It follows from construction that

**Lemma 5.4.** All of the maps \( \mu \) constructed in this subsection commute with the map \( \Psi^1 \) up to homotopy.

### 5.3 The Moduli Spaces \( \mathcal{M}, \mathcal{N} \) and \( \mathcal{Q} \)

We fix \( G \) a convex domain in \( \mathbb{C} \) with smooth boundary, such that \( \partial G \) contains \((-\eta, \eta) \) and \((-\eta, \eta) + i \) for some \( \eta > 0 \), and \( G \) is symmetric in the lines \( i\mathbb{R} \) and \( \frac{1}{2} + i\mathbb{R} \). We define

\[ D_+ = (\mathbb{R}_{\leq 0} + i[0,1]) \cup G \cup \{-\infty\} \]

and

\[ D_- = (\mathbb{R}_{\geq 0} + i[0,1]) \cup G \cup \{+\infty\} \]

viewed as the one-point compactifications of half-infinite strips. Both are compact Riemann surfaces with boundary, biholomorphic to a disc. We fix explicit identifications between \( D_\pm \) and the standard disc \( D^2 \): we fix biholomorphisms \( \phi_\pm : D^2 \to D_\pm \) which send \(+1\) to \(+\infty\), satisfying

\[ \phi_-(z) = i - \phi_+(z) \]

for all \( z \), as shown below.
**Definition 5.5.** We define $\mathcal{R}_\pm$ to be the space of smooth maps $w : D_\pm \to M$ such that for all $z \neq \mp \infty$ in $\partial D_\pm$, $w(z)$ lies in $\psi^{\text{im}} z(L)$.

Fix $J$ an $\omega$-tame almost complex structure on $M$ which is convex at infinity.

**Definition 5.6.** We define $\mathcal{M}$ to be the moduli space of $J$-holomorphic maps in $\mathcal{R}_-$, and $\mathcal{N}$ to be the moduli space of $J$-holomorphic maps in $\mathcal{R}_+$.

There are natural evaluation maps $\pi : \mathcal{R}_\pm \to L$ sending $u$ to $u(\mp \infty)$. We define $Q$ to be the fibre product $\mathcal{N} \times_L \mathcal{M}$ with respect to these evaluation maps.

For generic $J$ these moduli spaces are naturally smooth manifolds, and a similar argument to Section 4 shows that they are compact. Furthermore, for generic $J$ the two maps $\pi : \mathcal{M} \to L$ and $\pi : \mathcal{N} \to L$ are transverse and hence $Q$ is also a compact smooth manifold.

$\mathcal{M}$ and $\mathcal{N}$ may have different connected components which have different dimensions. We write

$$\mathcal{M} = \bigsqcup_{i \geq 0} \mathcal{M}^i$$

where $\mathcal{M}^i$ is the component of $\mathcal{M}$ lying in dimension $i$, and we write

$$\mathcal{N} = \bigsqcup_{i \geq 0} \mathcal{N}^i$$

similarly. Note that since both these moduli spaces are compact, only finitely many of these components are non-empty.

### 5.4 Bundle Pairs on the Moduli Spaces

We define $\Sigma$ to be the space

$$(D_- \cup [-1, 1] \cup D_+) / \sim$$

where $+\infty \sim -1$ and $+1 \sim -\infty$, as shown below (drawing $D_\pm$ as discs).

\begin{center}
\begin{tikzpicture}
\node (D-1) at (-2,0) {$D_-$};
\node (D+) at (2,0) {$D_+$};
\draw (D-) circle (1cm);
\draw (D+) circle (1cm);
\draw[->] (D-) -- node[above] {$+\infty$} (D+);
\draw[->] (D-) -- node[below] {$-\infty$} (D+);
\end{tikzpicture}
\end{center}

We define $\partial \Sigma \subseteq \Sigma$ to be the subspace

$$(\partial D_- \cup [-1, 1] \cup \partial D_+) / \sim$$

There are natural collapse maps

$$\xi_\pm : (\Sigma, \partial \Sigma) \to (D_\pm, \partial D_\pm)$$

collapsing $D_\mp$ and $[-1, 1]$ to $\mp \infty$. 

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We define a submanifold $\tilde{L}_{\psi} \subseteq \mathbb{C} \times M$ to be the space of pairs $(z, x)$ such that $|z|=1$ and if $z=1$ then $x$ lies in $L$, otherwise $x$ lies in $\psi \text{Im } \phi^{-1}(z)(L)$.

This is diffeomorphic to $L_{\psi}$. We let $T^* \tilde{L}_{\psi}$ be its vertical tangent bundle.

There are natural maps $ev_{\pm} : (D_{\pm}, \partial D_{\pm}) \times \mathbb{R}_{\pm} \to (\mathbb{C} \times M, \tilde{L}_{\psi})$ defined by

$ev_{\pm}(z, w) = \begin{cases} (\phi_{\pm}^{-1}(z), w(z)) & \text{if } z \in D_{\pm} \end{cases}$

and

$ev_{\pm}(z, w) = \begin{cases} (-\phi_{\pm}^{-1}(i + \bar{z}), w(z)) & \text{if } z \in \mathbb{R}_{\pm} \end{cases}$

We define a map $C : (\Sigma, \partial \Sigma) \times \mathbb{Q} \to (\mathbb{C} \times M, \tilde{L}_{\psi})$ by

$C(z, (v, u)) := \begin{cases} (ev_{-}(z, u)) & \text{if } z \in D_{-} \\ (ev_{+}(z, v)) & \text{if } z \in D_{+} \\ (1, u(+\infty)) & \text{if } z \in [-1, +1] \end{cases}$

We will consider the bundle pairs

$\xi^* C^* (TM, T^* \tilde{L}_{\psi})$

and

$\xi^* (TM, T^* \tilde{L}_{\psi})$

By construction, the fibres of the complex parts of these bundle pairs are given as follows.

$(\xi^*ev^*TM)_{(z, (v, u))} = \begin{cases} T_{u(z)}M & \text{if } z \in D_{-} \\ T_{u(+\infty)}M & \text{if } z \in D_{+} \\ T_{u(+\infty)}M & \text{if } z \in [-1, 1] \end{cases}$

$(\xi^*ev^+TM)_{(z, (v, u))} = \begin{cases} T_{u(\infty)}M & \text{if } z \in D_{-} \\ T_{v(z)}M & \text{if } z \in D_{+} \\ T_{u(+\infty)}M & \text{if } z \in [-1, 1] \end{cases}$

$(C^*TM)_{(z, (v, u))} = \begin{cases} T_{u(z)}M & \text{if } z \in D_{-} \\ T_{v(z)}M & \text{if } z \in D_{+} \\ T_{u(+\infty)}M & \text{if } z \in [-1, 1] \end{cases}$
\((\xi_+^* \pi^* TM)_{(z, (v, u))} = T_{u(+\infty)} M\)

The fibres of the real parts are given as follows, where \(z\) now lies in \(\partial \Sigma\).

\[
\left(\xi_+^* ev_+^* T^v \tilde{L}_{\psi^1}\right)_{(z, (v, u))} = \begin{cases} T_{u(z)} \psi_{\text{im}} z(L) & \text{if } z \in \partial D_- \\ T_{u(+\infty)} L & \text{if } z \in \partial D_+ \\ T_{u(+\infty)} L & \text{if } z \in [-1, 1] \end{cases}
\]

\[
\left(\xi_-^* ev_-^* T^v \tilde{L}_{\psi^1}\right)_{(z, (v, u))} = \begin{cases} T_{u(z)} \psi_{\text{im}} z(L) & \text{if } z \in \partial D_- \\ T_{v(z)} \psi_{\text{im}} z(L) & \text{if } z \in \partial D_+ \\ T_{u(+\infty)} L & \text{if } z \in [-1, 1] \end{cases}
\]

\[
\left(C^* T^v \tilde{L}_{\psi^1}\right)_{(z, (v, u))} = \begin{cases} T_{u(z)} \psi_{\text{im}} z(L) & \text{if } z \in \partial D_- \\ T_{v(z)} \psi_{\text{im}} z(L) & \text{if } z \in \partial D_+ \\ T_{u(+\infty)} L & \text{if } z \in [-1, 1] \end{cases}
\]

\[
\left(\xi_-^* \pi^* T^v \tilde{L}_{\psi^1}\right)_{(z, (v, u))} = T_{u(+\infty)} L
\]

**Lemma 5.7.** There is an isomorphism of bundle pairs \(F\) from

\[
\xi_-^* ev_-^* \left(TM, T^v \tilde{L}_{\psi^1}\right) \oplus \xi_+^* ev_+^* \left(TM, T^v \tilde{L}_{\psi^1}\right)
\]

to

\[
C^* \left(TM, T^v \tilde{L}_{\psi^1}\right) \oplus \xi_+^* \pi^* \left(TM, T^v \tilde{L}_{\psi^1}\right)
\]

**Proof.** We define \(F_{(z, (v, u))}\) explicitly as follows.

If \(z\) lies in \(D_-\), then

\[
F_{(z, (v, u))} : T_{u(z)} M \oplus T_{u(+\infty)} M \to T_{u(z)} M \oplus T_{u(+\infty)} M
\]

is given by the identity.

If \(z\) lies in \(D_+\), then

\[
F_{(z, (v, u))} : T_{u(+\infty)} M \oplus T_{v(z)} M \to T_{v(z)} M \oplus T_{u(+\infty)} M
\]

is given by the matrix

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

with respect to the decomposition above.

If \(z\) lies in \([-1, +1]\), then

\[
F_{(z, (v, u))} : T_{u(+\infty)} M \oplus T_{u(+\infty)} M \to T_{u(+\infty)} M \oplus T_{u(+\infty)} M
\]

is given by the matrix

\[
\begin{pmatrix} \cos \left(\frac{z+1}{2} \pi\right) & \sin \left(\frac{z+1}{2} \pi\right) \\ -\sin \left(\frac{z+1}{2} \pi\right) & \cos \left(\frac{z+1}{2} \pi\right) \end{pmatrix}
\]

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with respect to the decomposition above.

Note that this map \( F \) respects the totally real subbundles when \( z \) lies in \( \partial D \). \( \square \)

Note that

\[
\text{Ind} \pi^* \left( TM, T^v L_{\psi}^1 \right)
\]

is isomorphic to \( \pi^* TL \), over both \( R_\pm \).

Now pick a “pinch” map \( r : (D^2, \partial D^2) \to (\Sigma, \partial \Sigma) \) such that the compositions \( \xi_\pm \circ r \) induce maps of degree 1 from \( \partial D^2 \) to \( \partial D_\pm \). This determines \( r \) up to homotopy, but later we will make a more specific choice of \( r \).

**Lemma 5.8.** There is an isomorphism of bundle pairs \( r^* F \) from

\[
r^* \xi_\pm ev^*_\pm \left( TM, T^v L_{\psi}^1 \right) \oplus r^* \xi_\pm ev^*_\pm \left( TM, T^v L_{\psi}^1 \right)
\]

to

\[
r^* C^* \left( TM, T^v L_{\psi}^1 \right) \oplus r^* \xi_\pm \pi^* \left( TM, T^v L_{\psi}^1 \right)
\]

**5.5 Gluing and Orientations of the Moduli Spaces**

Let \( U_\infty = \mathcal{Q} \). Then as in Section 4, for a generic \((l\text{-dependent})\) perturbation of the almost complex structure, the space

\[
\mathcal{V}' := \bigsqcup_{0 \leq l < \infty} \mathcal{U}_l
\]

is a smooth manifold such that the natural map \( \mathcal{V}' \to \mathbb{R} \) is proper, and it follows from the proof of gluing in [ES16] that

\[
\mathcal{V} := \bigsqcup_{0 \leq l \leq \infty} \mathcal{U}_l
\]

is naturally a compact smooth manifold with boundary \( L \sqcup \mathcal{Q} \). This uses in an important way the fact that there is only one type of disk breaking.

The vertical tangent bundle of this cobordism restricted to \( \mathcal{Q} \) is then naturally isomorphic to

\[
\text{Ind} r^* C^* \left( TM, T^v L_{\psi}^1 \right)
\]

Therefore an \( R \)-orientation of \( \text{Ind} \) induces an \( R \)-orientation of \( T\mathcal{Q} \).

By construction, \( T\mathcal{M} \) and \( T\mathcal{N} \) are naturally stably isomorphic to

\[
\text{Ind} r^* \xi^*_\pm ev^*_\pm \left( TM, T^v L_{\psi}^1 \right)
\]

respectively. Therefore \( R \)-orientations on both of these as well as \( TL \) induce one on \( T\mathcal{Q} \), since by construction,

\[
T\mathcal{Q} \cong T\mathcal{N} + T\mathcal{M} - \pi^* TL
\]
Assumption 5.9. There are $R$-orientations of $TN, TM, \text{Ind}$ and $L$, such that the two induced $R$-orientations on $TQ$ above agree under the isomorphism $r^*F$ constructed in Lemma 5.7. Furthermore the restriction of $\text{Ind}$ to $L$ is the given orientation of $L$.

Lemma 5.10. Under Assumption 5.9, the cobordism $V$ admits an $R$-orientation, restricting to the given ones on $L$ and $Q$.

Proof. We need to check that the natural bundle isomorphism which comes from gluing

$$\text{Ind} \cong TQ \cong TN + TM - \pi^*TL$$

from above agrees with the one induced by $F$, up to homotopy of bundle isomorphisms.

We will recall a sketch of how Ekholm and Smith, in [ES16], construct their diffeomorphism from $Q$ to some $P_{l+1}$. Fix some very large $l > 0$. We define $D_{\pm,l}$ to be the subset of $D_{\pm}$ given by

$$D_{\pm} \cap \{ \pm \text{Re} \geq -(l - 1) \}$$

which we view as subsets of $G_l = C_{l+1}$ by translating by $\pm l$. We let $H$ be the closure of the complement in $G_l$ of the union of these two regions:

$$H := [-1, 1] + i[0, 1]$$

Ekholm and Smith construct a pre-gluing embedding

$$PG : Q \hookrightarrow D_{f_{l+1}, C_{l+1}}$$

sending $(v, u)$ to a map $(u#_c v) : C_{l+1} \to M$ which agrees with $u$ and $v$ on $D_{\pm,l}$ respectively, and is $C^1$-small when restricted to $H$. Roughly, this is constructed by picking a metric on $M$, picking $l$ large enough so that outside $D_{\pm,l}$ the images of both $u$ and $v$ lie inside a single geodesically convex neighbourhood of $M$, and cutting them off with a bump function.

Then there is a diffeomorphism $\rho : PG(Q) \to \mathcal{P}_{l+1}$ such that $PG$ and $\rho \circ PG$ are $C^1$ close. Therefore the derivative $d(\rho \circ PG)$ sends a pair of sections $(s_{\pm})$ of the bundle pairs $r^*C^*_\pm ev^*_{\pm} (TM, T^*L_{\psi_1})$ lying in $TQ$, to a section $s$ of the bundle pair $ev^*(TM, TL)$ over $\mathcal{P}_{l+1}$, such that $s$ is $C^1$-small on $H$ and $C^1$-close to $u$ and $v$ when restricted to $D_{\pm,l}$.

We make some choice of “pinch” map $r : C_{l+1} \to \Sigma$ which sends $\partial C_{l+1}$ to $\partial \Sigma$, and which restricts to the identity on $D_{\pm,l}$. Note that $ev \simeq C \circ r$.

By construction, we see that the map induced by $r^*F$ on sections of these bundle pairs sends a pair of sections $(s_{\pm})$, whose evaluations at $\mp \infty$ agree, to a section $s$, whose restrictions to $D_{\pm,l}$ agree with $s_{\pm}$. Consider the composition
where we weight the metric on the bundle pair to be small on $H$. Note that since $r^*F$ respects the direct sum away from $H$, if the weight is sufficiently small on $H$, any Cauchy-Riemann operator on the bundle pair in either the second or third term which respects the direct sum is of distance less than 1 to a Cauchy-Riemann operator on the other which respects the direct sum. Therefore the map $TQ \to TP_{l+1}$ coming from Lemma 3.5 agrees up to homotopy with $A$, and so under Assumption 5.9, $A$ respects the $R$-orientations.

$A$ is close to $d(\rho \circ PG)$ and so the two maps are homotopic isomorphisms of vector bundles, which is what we wanted. □

**Lemma 5.11.** If $T^v L_{\psi^1}$ admits a stable trivialisation over an $i + 1$-skeleton of $L_{\psi^1}$, then the induced stable trivialisations over an $i + 1$-skeleton of $Q$ of the bundle pairs appearing in Lemma 5.8 agree up to homotopy, under $r^*F$.

**Proof.** The two stable trivialisations are related by a map $\eta: \partial D^2 \times Q \to O$, where $O$ is the infinite orthogonal group. We will use the choice of $r$ from the proof of Lemma 5.10 for convenience, noting that this is unique up to homotopy.

From the construction of $F$, for any $y$ in $Q$ and $x$ in $r|_{\partial D^2}^{-1}\partial D_-$, $\eta(x, y)$ is given by the identity matrix. For $x$ in $r|_{\partial D^2}^{-1}\partial D_+$, $\eta(x, y)$ is given by (the stabilisation of) the block matrix

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

Then along the top part of $\partial D^2$, $\eta(\cdot, y)$ is given by some homotopy between these two matrices, and travelling in the opposite direction along the bottom produces the reverse of this homotopy. Therefore for a fixed $y$, $\eta(\cdot, y)$ is a
contractible loop. However these contractions and loops are independent of \( y \), so \( \eta \) is contractible.

Then from Proposition 3.14 it follows that

**Corollary 5.12.**

1. If \( T^u L_{\psi^1} \) admits a stable trivialisation over a 2-skeleton of \( L_{\psi^1} \), then Assumption 5.9 holds for \( R = H\mathbb{Z} \).

2. If \( T^u L_{\psi^1} \) admits a stable trivialisation over a 3-skeleton of \( L_{\psi^1} \), then Assumption 5.9 holds for \( R = KU \).

3. If \( T^u L_{\psi^1} \) admits a stable trivialisation, then Assumption 5.9 holds when \( R \) is any complex-oriented cohomology theory.

**Proof of Proposition 1.17.** We will show that if \( TL \) admits a homotopy class of stable trivialisations over an \( i \)-skeleton \( L_i \) of \( L \) which is preserved by \( \psi_1 \) and can be extended to an \( i+1 \)-skeleton \( L_{i+1} \), then \( T^u L_{\psi^1} \) admits a stable trivialisation over an \( i \)-skeleton of \( L_{\psi^1} \).

Let \( \tilde{\psi}^1 : L \to L \) be a map homotopic to \( \psi^1 \mid_L \) which sends \( L_i \) to itself. Then \( L_{\psi^1} \) is homotopy equivalent to

\[
L \times [0, 1]/\{(0, x) \sim (1, \tilde{\psi}^1)\}
\]

and an \( i+1 \)-skeleton of this is given by

\[
(L_i \times [0, 1] \cup L_{i+1})
\]

Then by assumption \( T^u L_{\psi^1} \), admits a stable trivialisation over this \( i+1 \)-skeleton.

**5.6 Composition of the Moduli Spaces**

Fix some diffeomorphism \( \tilde{L}_{\psi^1} \cong L_{\psi^1} \) covering some orientation-preserving diffeomorphism \( \partial D^2 \cong S^1 \) sending \( 1 \in \partial D^2 \) to \( 0 \in S^1 = \mathbb{R}/\mathbb{Z} \). There are natural evaluation maps

\[
S^1 \times \mathbb{R} \xrightarrow{\exp(2\pi \cdot)} \partial D^2 \times \mathbb{R} \xrightarrow{\sigma_{\pm}(\mp \cdot)} \partial D \times \mathbb{R} \xrightarrow{ev_{\pm}} \tilde{L}_{\psi^1} \cong L_{\psi^1}
\]

whose adjoints define maps

\[
\sigma_{\pm} : \mathbb{R} \to L_{\psi^1}
\]

Therefore \( Q \) admits a natural evaluation map

\[
\sigma : Q \to L_0
\]

sending \((v, u)\) to the concatenation \( c(\sigma_{-}(v), \sigma_{+}(u)) \). Then choices of \( R \)-orientations of \( L \) and all the moduli spaces allow us to use \( \sigma_{\pm} \) and \( \sigma \) to define fundamental classes \([M], [N] \) and \([Q] \) in \( \bigoplus_j R_j \left(L_{\pm 1}^{-TL} \right) \) and \( \bigoplus_j R_j \left(L_0^{-TL} \right) \) respectively.

Our goal in this subsection is to prove the following two lemmas.
Lemma 5.13.
\[ [\mathcal{N}] \cdot [\mathcal{M}] = [\mathcal{Q}] \]
in \( \bigoplus_j R_j \left( \mathcal{L}_0^{-\pi T} \right) \).

Lemma 5.14.
\[ [\mathcal{Q}] = [L] \]
in \( \bigoplus_j R_j \left( \mathcal{L}_0^{-\pi T} \right) \), for some choice of \( R \)-orientation used to define \([\mathcal{N}] \) and \([\mathcal{M}] \).

From these, we deduce

Lemma 5.15. The composition
\[
\Sigma_+^\infty \mathcal{L}_0 \wedge R \xrightarrow{[\mathcal{M}]} \bigvee_j \Sigma_+^{\infty+j} \mathcal{L}_1 \wedge R \xrightarrow{[\mathcal{N}]} \bigvee_j \Sigma_+^{\infty+j} \mathcal{L}_0 \wedge R \xrightarrow{p} \mathcal{L}_0 \wedge R
\]
is an equivalence, where \( p : \bigvee_j \Sigma_+^{\infty+j} \mathcal{L}_0 \rightarrow \Sigma_+^\infty \mathcal{L}_0 \) is the natural projection map.

Proof of Lemma 5.13. This follows from commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathbb{S} & \xrightarrow{[\mathcal{N}] \cdot [\mathcal{M}]} & [\mathcal{Q}] \\
\downarrow & & \downarrow \\
\mathcal{N}^{-TN} \wedge R \wedge \mathcal{M}^{-TM} \wedge R & \xrightarrow{\iota} & \mathcal{Q}^{-\left(TN+TM-\pi^*TL\right)} \wedge R \\
\downarrow \text{Thom} & & \downarrow \text{Thom} \\
\bigvee_j \Sigma^j \mathcal{N}^{-\pi^*TL} \wedge R \wedge \mathcal{M}^{-\pi^*TL} \wedge R & \xrightarrow{\iota'} & \bigvee_j \Sigma^j \mathcal{Q}^{-\pi^*TL} \wedge R \\
\downarrow \sigma^- \wedge \sigma_+ & & \downarrow \sigma \\
\bigvee_j \Sigma^j \mathcal{L}_{-1}^{-TL} \wedge R \wedge \mathcal{L}_{+1}^{-TL} \wedge R & \xrightarrow{\mu} & \bigvee_j \Sigma^j \mathcal{L}_0^{-TL} \wedge R
\end{array}
\]
where the two arrows labelled Thom are the isomorphisms from the Thom isomorphism theorem for our choices of \( R \)-orientations followed by inclusion into this wedge product, and \( \iota \) and \( \iota' \) are obtained by applying Definition 2.23 to the embedding
\[ Q \hookrightarrow \mathcal{N} \times \mathcal{M} \]
using the virtual vector bundles \(-TN \oplus -TM\) and \(-\pi^*TL \oplus -\pi^*TL\) respectively.

Proof of Lemma 5.14. By Lemma 5.10, there exists an \( R \)-orientable cobordism \( V \) from \( Q \) to \( L \), with respect to the given \( R \)-orientations on both ends. Furthermore, evaluating along the boundary allows us to extend the maps \( \sigma : Q \rightarrow L_0 \) and \( L \rightarrow L_0 \) to the entirety of \( V \). Therefore the result follows from Lemma 2.25. \( \square \)
5.7 Proof of Theorem 1.16

Proof of Theorem 1.16 Lemma 5.4 implies that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\Sigma_+ \infty +_0 \wedge R & \rightarrow & \Sigma_+ \infty +_j \wedge R \\
\downarrow \psi^1 & & \downarrow \psi^1 \\
\Sigma_+ \infty +_0 \wedge R & \rightarrow & \Sigma_+ \infty +_0 \wedge R \\
\end{array}
\]

By Lemma 5.2, the second vertical arrow is homotopic to the identity, and the horizontal arrows along the top are all homotopic to the corresponding horizontal arrows along the bottom. Furthermore Lemma 5.15 tells us that the composition along the top (and similarly along the bottom) is an equivalence. It follows that the map

\[\Psi^1 : \Sigma_+ \infty +_0 \wedge R \rightarrow \Sigma_+ \infty +_0 \wedge R\]

is homotopic to the identity. \[\square\]

6 Families Over Other Bases

Our goal in this section will be to prove Theorem 1.21. In this section, we only consider mod-2 singular homology. We will use the following purely topological lemma:

Lemma 6.1 ([LM03 Lemma 4.3]). Fix fibre bundles \(A \hookrightarrow B \twoheadrightarrow C\) and \(F \hookrightarrow E \twoheadrightarrow B\). Assume that the fibre bundles

1. \(A \hookrightarrow B \twoheadrightarrow C\)
2. \(E|_A \hookrightarrow E \twoheadrightarrow C\)
3. \(F \hookrightarrow E|_A \twoheadrightarrow A\)

all c-split. Then the fibre bundle \(F \hookrightarrow E \twoheadrightarrow B\) c-splits.

Proof. The inclusion of a fibre \(F \hookrightarrow E\) is the composition \(F \hookrightarrow E|_A\) and \(E|_A \hookrightarrow E\). The hypothesis of the lemma is that both these maps induce injections on mod-2 singular homology. \[\square\]

Proof of Theorem 1.21(i). The case \(i = 0\) is trivial and the case \(i = 1\) follows from Theorem 1.4 along with an application of the Mayer-Vietoris sequence. We proceed by induction, and assume we know the result for \(i - 1\), for all \(M\) and \(L\) as in the statement of Theorem 1.4.

We let \(E = \gamma^* \mathcal{E}\). Then we have two fibre bundles

\[L \hookrightarrow E \twoheadrightarrow (S^1)^i\]

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and
\[ S^1 \hookrightarrow (S^1)^i \rightarrow (S^1)^{i-1} \]
where the inclusion map is inclusion to the first factor and the projection map is projection to all the other factors.

The fibre bundle \( E|_{S^1} \hookrightarrow S^1 \) \( c \)-splits by Theorem 1.4. So by Lemma 6.1 it suffices to show the fibre bundle given by the composition
\[ E \twoheadrightarrow (S^1)^i \rightarrow (S^1)^{i-1} \]
\( c \)-splits.

For \( \tau \) in \((S^1)^{i-1}\), we let \( \gamma_\tau \) be the map \( S^1 \rightarrow \text{Lag}_L \) given by \( \gamma_\tau(t) = \gamma(t, \tau) \).

**Claim 6.2.** Possibly after perturbing \( \gamma_\tau \), there exists a smooth family of smooth maps \( H_\tau : M \times S^1 \rightarrow \mathbb{R} \) for \( \tau \) in \((S^1)^{i-1}\) such that the Hamiltonian flow of \( H_\tau \) applied to \( \gamma_\tau(0) \) is \( \gamma_\tau \).

**Proof of Claim 6.2.** For each point \((t_0, \tau)\) in \( S^1 \times (S^1)^{i-1} \), let \( V_{t_0, \tau} \) be the space of germs near \( M \times \{0\} \) of compactly supported smooth maps \( M \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R} \) for some \( \varepsilon > 0 \) whose Hamiltonian flow \( \phi_t \) applied to \( \gamma(t_0, \tau) \) agrees with \( \gamma_\tau(t_0 + t) \). Together these define a fibre bundle over \((S^1)^i\), and the family of \( H_\tau \) we want is a choice of smooth section of this bundle. The fibres of this bundle are contractible so this exists. \( \square \)

We define a map
\[ f : E \hookrightarrow M \times T^*S^1 \times (S^1)^{i-1} \]
by
\[ f(x) = (x, t, H(x), \tau) \]
where \( x \) lies in the fibre over \( \tau \) in \((S^1)^{i-1}\), and we identify \( T^*S^1 \) with \( S^1 \times \mathbb{R} \) in the usual way. Note that this lives over \((S^1)^{i-1}\).

A direct computation shows that for each \( \tau \), this gives a relatively exact Lagrangian in \( M \times T^*S^1 \), therefore this realises \( E \rightarrow (S^1)^{i-1} \) as a family of relatively exact Lagrangians in \( M \times T^*S^1 \), and therefore by the induction hypothesis it \( c \)-splits.

Therefore by Lemma 6.1, the fibre bundle \( E \rightarrow (S^1)^i \) \( c \)-splits. \( \square \)

**Lemma 6.3** (LM03 Lemma 4.1(ii)). Let \( E \rightarrow B \) be a fibre bundle, and \( f : B' \rightarrow B \) a map which is surjective on mod-2 singular homology. Assume that the pullback bundle \( f^*E \rightarrow B' \) \( c \)-splits. Then \( E \rightarrow B \) \( c \)-splits.

**Proof of Theorem 1.21(ii).** Follows from Lemma 6.3 by considering a degree 1 map \((S^1)^i \rightarrow S^1\). \( \square \)
A de Silva’s Theorem

In the proof of Proposition 3.14, we quote a theorem of de Silva from [dS98]. We will recap a proof of this theorem since the original result was never published.

Fix a finite based connected CW complex $X$ with basepoint $x_0$. A based map

$$\phi: \Sigma X \to U/O$$

determines a based virtual bundle pair $(E, F)$ over $(D^2, \partial D^2) \times X$. Taking its index bundle gives a based virtual vector bundle Ind$(E, F)$ over $X$, classified by a based map $X \to BO$ up to homotopy. This defines a map

$$\text{Ind}: [\Sigma X, U/O]_* \to [X, BO]_*$$

Let $H$ be the bundle pair over $(D^2, \partial D^2) \times \{\ast\}$ defined as in Section 3.2. Then we see that for any based virtual vector bundle $E$ over $X$,

$$\text{Ind}((C, \mathbb{R}) \otimes E) = E$$

and

$$\text{Ind}(H \otimes E) = E \oplus E$$

**Theorem A.1** ([dS98, Theorem A]). The map Ind is a bijection.

*Proof.* We first prove surjectivity. We define a map

$$\beta: [X, BO]_* \to [\Sigma X, U/O]_*$$

to send a based virtual vector bundle $E$ of rank $n$ to the based virtual vector bundle

$$H \otimes E - (C, \mathbb{R}) \otimes E - H \otimes \mathbb{R}^n$$

Then it is clear that $\beta$ is a section of Ind and so Ind is surjective.

We now prove injectivity. We define

$$KUO\left((D^2, \partial D^2) \times X\right)$$

to be the set of virtual bundle pairs over $(D^2, \partial D^2) \times X$ up to stable isomorphism, which is an abelian group under direct sums. There is a natural forgetful map

$$g: [\Sigma X, U/O]_* \to KUO\left((D^2, \partial D^2) \times X\right)$$

whose image lies in the set of virtual bundle pairs with vanishing Maslov index. Note that by Bott periodicity, $U/O$ is a group-like monoid and so

$$[\Sigma X, U/O]_* \cong [\Sigma X, U/O]$$

From this, we see that $g$ is injective.
$g$ fits into a commutative square

$$
\begin{align*}
&\left[\Sigma X, U/O\right]_* \xrightarrow{\text{Ind}} \left[X, BO\right]_* \\
&\downarrow^g KUO\left((D^2, \partial D^2) \times X\right) \xrightarrow{\text{Ind}} \left[X, BO \times \mathbb{Z}\right]
\end{align*}
$$

Note that all the maps in this diagram are maps of $[X, BO]_*$-modules, where the module structure is induced by the tensor product.

By [Ati66, Theorem 2.1] (taking the line bundle $L$ in the statement of the theorem to be trivial), we have that

$$
KUO\left((D^2, \partial D^2) \times X\right) \cong [X, BO \times \mathbb{Z}][\theta]/(\theta^2)
$$

as $[X, BO]$-modules, where $\theta$ corresponds to $H - (\mathbb{C}, \mathbb{R})$, and the $\mathbb{Z}$ factor corresponds to the Maslov index. Furthermore

$$\text{Ind}((\mathbb{C}, \mathbb{R}) \otimes E) = E$$

and

$$\text{Ind}(\theta \otimes E) = E$$

for all $E$.

Now suppose $\psi$ in $[\Sigma X, U/O]_*$ satisfies $\text{Ind}\psi = 0$. Then $\text{Ind}g(\psi) = 0$ and so

$$g(\psi) = (\theta - (\mathbb{C}, \mathbb{R})) \otimes E$$

for some real virtual vector bundle $E$ over $X$. Because this lies in the image of $g$, by restricting to $\{1\} \times X$, we see that $E$ is trivial of some rank. But the Maslov index of $g(\psi)$ is the rank of $E$, which must therefore by 0. Therefore $g(\psi)$ and hence $\psi$ are trivial.

\begin{proof}
\end{proof}

\textbf{B An Example in $K$-Theory}

Our goal in this section is to prove the following.

\begin{proposition}
In all sufficiently high dimensions $n$, there exists a closed $n$-dimensional manifold $V$ such that:

1. $V$ is stably parallelisable (and hence spin).

2. $V$ admits a self-diffeomorphism $\theta : V \rightarrow V$ which acts as the identity on integral cohomology $H^*(V)$ but does not act as the identity on $K^*(V)$, where $K^*$ denotes complex $K$-theory.

3. If $n$ is odd, we can take $V$ to be a rational homology sphere.

\end{proposition}

\begin{corollary}
If $L$ is diffeomorphic to $V$, then $\theta$ does not lie in $G_L$.

\end{corollary}
The rest of the section will be dedicated to a proof of Proposition B.1 following a suggestion of Randal-Williams ([RW]). We will construct these in sufficiently high odd dimensions and then observe that taking a product with $S^1$ provides the desired examples in all sufficiently high even dimensions.

Let $q$ be a positive integer, and $p$ an odd prime. Let $Y$ be the homotopy mapping cone of the map $S^{2q} \to S^{2q}$ of degree $p$.

**Lemma B.3.** We have the following:

1. 
   \[ \tilde{K}^i(Y) \cong \begin{cases} 
   0 & i \text{ even} \\
   \mathbb{Z}/p & i \text{ odd}
   \end{cases} \]

2. When $i$ is odd, 
   \[ K_i(Y) \cong 0 \]

3. 
   \[ \tilde{H}^i(Y) \cong \begin{cases} 
   \mathbb{Z}/p & i = 2q + 1 \\
   0 & \text{otherwise}
   \end{cases} \]

4. 
   \[ \tilde{H}_i(Y) \cong \begin{cases} 
   \mathbb{Z}/p & i = 2q \\
   0 & \text{otherwise}
   \end{cases} \]

5. For all $i$, 
   \[ \tilde{H}^i(Y; \mathbb{Q}) = 0 \]

**Proof.** Follows from the long exact sequence of a cone. (2) also uses the decomposition 
\[ K_i(Y) \cong \tilde{K}_i(Y) \oplus K_i(\text{point}) \]

By [Ada66, Theorem 1.7], for sufficiently large $q$, there is a map 
\[ g : \Sigma^{2(p-1)}Y \to Y \]
which induces an isomorphism on $\tilde{K}^*$, but must be 0 on $\tilde{H}^*$ and $\tilde{H}_*$ for degree reasons. Furthermore we can choose $q$ to be sufficiently large that $Y$ is simply connected.

Let $Z = \Sigma^{2(p-1)}Y \vee Y$, and we define $f : Z \to Z$ to be the composition 
\[ Z = \Sigma^{2(p-1)}Y \vee Y \xrightarrow{\text{pinch} \vee \text{id}} \Sigma^{2(p-1)}Y \vee \Sigma^{2(p-1)}Y \vee Y \xrightarrow{\text{id} \vee g \vee \text{id}} \Sigma^{2(p-1)}Y \vee Y = Z \]
where the first map is the pinching map, which uses the fact that 
\[ \Sigma^{2(p-1)}Y = \Sigma \left( \Sigma^{2(p-1)-1}Y \right) \]
Lemma B.4. Let $E$ be a generalised homology or cohomology theory. Let $A$ be a connected finite CW complex, and $h : A \to A$ some map.

Then $h$ acts as the identity on $E(A)$ if and only if $h$ acts as the identity on $\tilde{E}(A)$.

Proof. There is a canonical splitting

$$E(A) \cong \tilde{E}(A) \oplus E(\text{point})$$

$h$ acts diagonally with respect to this splitting and always acts as the identity on $E(\text{point})$. \hfill \Box

Lemma B.5. Let $f : Z \to Z$ be the map constructed above.

1. $f$ does not act as the identity on $K^*(Z)$.
2. $f$ acts as the identity on $H^*(Z)$ and $H_*(Z)$. In particular, since $Z$ is simply connected, $f$ is a homotopy equivalence.

Proof. With respect to the decomposition

$$\tilde{K}^*(Z) \cong \tilde{K}^* \left( \Sigma^{2(p-1)} Y \right) \oplus \tilde{K}^*(Y)$$

we see that $f^*$ looks like

$$\begin{pmatrix}
\text{Id} & 0 \\
g^* & \text{Id}
\end{pmatrix}$$

and so $f$ does not act as the identity on $\tilde{K}^*(Z)$, and hence not on $K^*(Z)$ either, by Lemma B.5. A similar argument shows the analogous results for $H^*(Z)$ and $H_*(Z)$. \hfill \Box

Now if $Z$ were a closed manifold with the right properties and $f$ were a diffeomorphism, we would be done, but unfortunately this is not the case.

Pick an embedding $Z \hookrightarrow \mathbb{R}^N$ for some large even $N > 4(p+q)$. Let $W$ be a regular neighbourhood of $Z$ with smooth boundary such that $W$ deformation retracts to $Z$. Then $W$ is a compact manifold with boundary, has one handle for each cell of $Z$, is simply connected, and for sufficiently large $N$, $\partial W$ is also simply connected. The tangent bundle $TW$ is trivial since $W$ is a codimension 0 submanifold of $\mathbb{R}^N$, and so $T\partial W$ is also stably trivial.

Since $Z \simeq W$, $f$ induces a map $f : W \to W$, well-defined up to homotopy. If $N$ was chosen to be sufficiently large, $f$ can be homotoped to an orientation-preserving embedding $e : W \hookrightarrow W$, which we can assume has image which does not touch $\partial W$.

Lemma B.6. $e$ is homotopic to an orientation-preserving diffeomorphism $\phi : W \to W$. 

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Proof. The complement $C := W \setminus e(W \setminus \partial W)$ is a cobordism from $e(\partial W)$ to $\partial W$. All three of $C$, $e(\partial W)$ and $\partial W$ are simply connected, and since $e$ is a homotopy equivalence, by excision the inclusion $e(\partial W) \hookrightarrow C$ is a homology equivalence.

By Poincaré-Lefschetz duality, the inclusion $\partial W \hookrightarrow C$ is also a homology equivalence, and so by the $h$-cobordism theorem $C$ is a trivial cobordism, and so the result follows.

Now we let $V = \partial W$ and $\theta = \phi|_{\partial W}$.

Lemma B.7. $V$ is a rational homology sphere.

Proof. By Lemma B.3 $Y$ and hence $Z$ and $W$ are rationally homology equivalent to a point. Then the result follows from the exact sequence of a pair (using Poincaré duality):

$$H_i(W; \mathbb{Q}) \rightarrow H^{N-i}(W; \mathbb{Q}) \rightarrow H_{i-1}(V; \mathbb{Q}) \rightarrow H_{i-1}(W; \mathbb{Q}) \rightarrow H^{N-i+1}(W; \mathbb{Q})$$

The following two lemmas complete the proof of Proposition B.1.

Lemma B.8. $\theta$ acts as the identity on $H^*(V)$.

Proof. From Lemma B.3 we see that

$$\tilde{H}^i(W) \cong \begin{cases} \mathbb{Z}/p & i = 2q + 1 \text{ or } 2(p + q - 1) + 1 \\ 0 & \text{otherwise} \end{cases}$$

and since $W$ is orientable, by Poincaré duality

$$H^i(W, \partial W) \cong H_{N-i}(W) \cong \begin{cases} \mathbb{Z} & i = N \\ \mathbb{Z}/p & i = N - 2q \text{ or } N - 2(p + q - 1) \\ 0 & \text{otherwise} \end{cases}$$

Therefore by the long exact sequence of the pair $(W, \partial W)$, for all $i$ either the restriction map $\tilde{H}^i(W) \rightarrow \tilde{H}^i(\partial W)$ is an isomorphism or the boundary map $\tilde{H}^i(\partial W) \rightarrow H^{i+1}(W, \partial W)$ is an isomorphism. Both of these maps are compatible with the actions of $\phi$ and $\theta$, so by Lemma B.5 the result follows.

Lemma B.9. $\theta$ does not act as the identity on $K^*(V)$.

Proof. We will show that the restriction map $\tilde{K}^i(W) \rightarrow \tilde{K}^i(V)$ is injective for all $i$, then the result follows from Lemmas B.5 and B.4.

By the long exact sequence of a pair, the kernel of the restriction map is the image of

$$K^i(W, \partial W) \rightarrow \tilde{K}^i(W)$$
TW is trivial and hence oriented with respect to \( K^* \), so by Atiyah duality

\[
K_i(W, \partial W) \cong K_{N-i}(W)
\]

Therefore using the decomposition

\[
\tilde{K}^*(W) \cong \tilde{K}^* \left( \Sigma^2(p-1)Y \right) \oplus \tilde{K}^*(Y)
\]

and Lemma \( B3 \) we see that if \( i \) is even then \( \tilde{K}^i(W) = 0 \), and if \( i \) is odd then \( K_{N-i}(W) = 0 \). In either case this implies that the restriction map is injective in degree \( i \).

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