Component Actions from Curved Superspace: Normal Coordinates and Ectoplasm

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ABSTRACT

We give efficient superspace methods for deriving component actions for supergravity coupled to matter. One method uses normal coordinates to covariantly expand the superfield action, and can be applied straightforwardly to any superspace. The other interprets the component lagrangian as a differential form on a bosonic hypersurface in superspace, and gives a simple derivation for pertinent cases such as chiral superspace.

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1. Introduction

Superspace methods have many advantages over component approaches to supersymmetry (see, e.g., [1]). They allow us to write manifestly supersymmetric actions, and they facilitate quantum calculations. However, component expansions of superfields and superspace actions can be useful for comparison to nonsupersymmetric theories, or for applying supersymmetry to nonsupersymmetric theories.

Although component expansions of superspace actions are easy and straightforward for globally supersymmetric theories, the same has not been true for locally supersymmetric ones. The two main complications are: (1) the complexity of supergravity in comparison to super Yang-Mills theory, as reflected in the commutation relations of the covariant derivatives needed for component expansions, and (2) the lack of a simple way to expand the integration measure, which is absent in the global case. Consequently, the construction of locally supersymmetric component actions, either by starting from superspace, or directly by requiring component local supersymmetry, has always been a somewhat awkward marriage of various ad hoc techniques.

This letter gives a brief overview of two recent papers [2,3] that have tackled the problem anew, using two complementary approaches. One approach is based on component expansions with respect to fermionic Riemann normal coordinates [4,5], equivalent to covariant Wess-Zumino gauge [6]. This approach is completely straightforward, and independent of the details of the particular superspace under consideration. The observation of [2] is that standard methods in Riemannian geometry for constructing such coordinate systems [7] can be applied in order to obtain component actions from superspace actions much more efficiently than older methods. (However, this method is not as efficient as one might hope, since it requires evaluation of the entire vielbein for the purpose of determining just the measure — the vielbein superdeterminant.)

On the other hand, the “ectoplasmic” method of [3] requires detailed knowledge of the properties of the relevant superspace. For example, for the standard case of simple supersymmetry in four space-time dimensions, the existence and properties of chiral superspace must be understood. However, since the method works directly with densities (as differential forms), knowledge of the explicit measure is avoided. Also, the strong dependence of this method on the existence of “subsuperspaces” like chiral superspace makes it useful for studying them in the less-known cases of extended supersymmetry. (The use of such spaces simplifies calculations for the normal coordinate method as well.)

In this letter we apply both methods to the example of four-dimensional N=1 supersymmetry, obtaining locally supersymmetric component lagrangians for old minimal supergravity. (Similar results can be obtained for other versions of supergravity.) Its torsions and curvatures are given by (we use the notation $a \equiv \alpha \dot{a}$ and in
expressions such as $\psi_a^\alpha$ that we encounter, summation over $\alpha$ is to be understood

$$[\nabla_\alpha, \nabla_\beta] = -2 R \mathcal{M}_{\alpha\beta},$$

$$[\nabla_\alpha, \nabla_\gamma] = -i C_{\alpha\beta}(\nabla_\alpha R) \delta^\gamma_\beta + i C_{\alpha\beta}(\nabla_\alpha G_\gamma^\delta) \mathcal{M}_{\beta\delta} + \text{h. c.}$$

We note for future reference that a covariantly chiral scalar can be expressed in terms of a general scalar:

$$\nabla_\alpha \Phi = 0 \Rightarrow \Phi = \left(\nabla^2 + R\right) \Psi.$$ 

Interestingly, the relevance of chiral lagrangians for four-dimensional, $N = 1$ superspace follows from the ectoplasmic approach, while the above relation between chiral and general scalars follows from the normal coordinate approach, showing their complementarity.

2. Fermionic Normal Coordinates

Normal coordinates $y^A$ around an “origin” $z^M$ can be defined by Taylor expanding the metric or vielbein (and other gauge fields) with respect to (some of) the coordinates, such that the coefficients are field strengths (and their derivatives). This covariant expansion is achieved by performing finite parallel transport from the arbitrary point $z^M$. An infinitesimal transport with parameter $y^A$ is exponentiated to yield the finite one. Taylor expansion of the exponential yields an explicit algorithm for the desired coordinate expansion.

In general, we expand (choose a normal gauge) with respect to a subset of the coordinates: We divide them into sets $(z^i, z^a)$ and $(y^i, y^a)$. After using the algorithm we set $z^a = y^i = 0$ (more generally the $z^a$ can be set equal to some arbitrary constants) and use $(z^i, y^a)$ as new coordinates. The surviving $y^i$’s are the normal-gauge-fixed coordinates, while the remaining $z$’s are still arbitrary coordinates. Such gauges are useful, e.g., for “compactification”, where expansions are made in some of the coordinates, while coordinate invariance is still desired in the remaining coordinates. A familiar case is that of Riemann normal coordinates, where we set all the $z$’s to vanish, and keep all the $y$’s as our new coordinates. A more relevant example
is Gaussian normal coordinates, where we choose $y$ to be a single timelike coordinate, and $z$ the spacelike coordinates, by setting $z^0 = y^i = 0$. This construction then gives the timelike gauge $g_{m0} = \eta_{m0}$, fixing the time coordinate while leaving the space coordinates arbitrary. For covariant component expansions in supersymmetry, the idea is to fix the fermionic coordinates, while maintaining coordinate invariance in the bosonic coordinates.

Once the fermionic expansions have been obtained, they can be applied to the superspace action. Consider, for example, the derivation of a covariant component action from a superspace action of the form

$$S = \int d^4 x \ d^4 \theta \ E^{-1} L$$

where $E^{-1} = sdet \ E_M^A$, and $E_M^A$ is the vielbein. We first apply the algorithm to $E^{-1} L$ with respect to $z^M = (z^m, z^\mu)$ and $y^A = (y^a, y^\alpha)$, and then set $y^a = z^\mu = 0$, identifying $y^a$ as the fermionic coordinates $\theta^\alpha$ in the Wess-Zumino gauge and $z^m$ as the bosonic coordinates $x^m$, still arbitrary with respect to spacetime coordinate transformations. (Generically we use lower-case Greek letters for both dotted and undotted spinor indices, wherever no distinction needs to be made.) Integration over $\theta$ can then be performed as in flat superspace, by picking out the highest-order terms in $E^{-1} L$. The expansion of $E^{-1}$ automatically gives the usual factor of $e^{-1} = det \ e_m^a$. As mentioned earlier, $z^\mu$ can be set to an arbitrary constant (distinct from the integration variable $\theta^\alpha = y^\alpha$); the result for the action is independent of it.

However, superspace integrations are often performed over “subsuperspaces” parametrized by the usual spacetime coordinates plus a subset of fermionic coordinates, for example (anti)chiral superspace for D=4, N=1. (In fact, the use of such subsuperspaces can be avoided only in degenerate cases for D<4. For massless theories, only one quarter of the off-shell supersymmetries are physical, except when there are fewer than four to start with.) In such cases normal coordinate expansions can be used for two purposes: (1) reducing a full superspace action, e.g. $S = \int d^4 x \ d^4 \theta \ E^{-1} L$ to a subsuperspace action, e.g. $S = \int d^4 x \ d^2 \theta \ E^{-1} L_{ch}$, and (2) deriving the component expansion of the subsuperspace action, e.g. $S = \int d^4 x e^{-1} L$.

In fact, these are the two steps we use in practice to evaluate the component expansion of a full superspace action. We can interpret $S$ as being obtained from $S$, where we have expanded in (and integrated over) only with respect to the $\tilde{\theta}^\alpha$. (Of course, in any situation coordinates can be integrated out one at a time, but the result will not always be simple. Reduction to subsuperspaces produces a manifestly covariant result only when scalars can be defined on such spaces: For example, chiral scalars exist in curved 4D, N=1 superspace.) The procedure in both steps is the same; the only difference is the choices of the various sets of coordinates ($z$’s and $y$’s). For our chiral superspace example, which we will discuss in more detail below, the first step parallel transports with respect to $\tilde{\theta}^\alpha$, dividing up the superspace coordinates as $(z^m, z^\mu; \tilde{y}^\alpha)$, while the second step transports with respect
to $\theta^\alpha$, dividing up the coordinates as $(z^m, \bar{z}^i; y^\alpha)$ where we now explicitly distinguish dotted and undotted indices.

We first give the algorithm for finite parallel transport by exponentiating an infinitesimal transformation. We apply repeatedly the rules for variation of the occurring quantities (see [2] for details):

$$\delta y^A = 0, \quad \delta T = y \cdot \nabla T,$$

$$\delta E^A = D y^A + y^C E^B T_{BC}^A, \quad \delta (D y^A) = y^B y^C E^D R_{DCB}^A,$$

where $T$ is any tensor (including $T$ and $R$), which is a function of just $z^M$, and

$$E^A \equiv dz^M E_M^A(z), \quad D y^A \equiv E^B \nabla_B y^A = dy^A - y^B E^C \omega_{CB}^A(z).$$

The transformation of the tensor identifies $y^A$ as the translation parameter, while $\delta y^A = 0$ is the geodesic condition. (The other transformations follow from that of the tensor, since then $\delta \nabla = [y \cdot \nabla, \nabla].$)

Applying the above algebra mechanically, we evaluate the transformed vielbein, breaking it up into a part proportional to $E^B(z)$ and a part proportional to $D y^B$:

$$E'{}^A(z; y) \equiv \sum_{n=0}^\infty \frac{1}{n!} \delta^n E^A = E^B(z) F_B^A + (D y^B) G_B^A,$$

where $F$ and $G$ depend explicitly only on tensors $T(z)$ ($T$ and $R$ and their covariant derivatives), and on $y^A$. For the present purpose we need only the terms of order $n = 0, 1$ (given above), 2:

$$\delta^2 E^A = y^B y^C E^D R_{DCB}^A + y^C (D y^B + y^E E^D T_{DE}^B) T_{BC}^A + y^C E^B y^D \nabla_D T_{BC}^A.$$

and the part of $\delta^3 E^A$ proportional to $D y^B$.

The first step in applying the algorithm is to divide up the $z^M$ and $y^A$ coordinates into complementary sets. The same procedure applies for expansion about any subset of the coordinates; only the ranges of the indices change. To simplify notation, we will use indices corresponding to the special case of expansion of the full superspace over all the fermionic coordinates, with the understanding that appropriate modifications can be made for other cases. We thus evaluate at

$$z^\mu = 0, \quad y^a = 0,$$

(and similarly for $dz^\mu$ and $dy^a$): This is the expansion in the fermionic normal coordinates $y^a$ about the bosonic hypersurface coordinatized by $z^m$. The left-hand-side of (2.4) then gives the complete vielbein in all superspace

$$E^A(z^m, y^\alpha) = E'^A(z^m, 0, y^\alpha),$$

while on the right-hand-side we have only $y^\alpha$ (and $dy^\alpha$), $\mathcal{T}(z^m, 0)$ (and $\omega(z^m, 0)$, but the connection cancels out in any Lorentz invariant quantity and need not be
considered in the expansion of the action), and only the nontrivial part of the vielbein (since \(dz^\mu = 0\)):

\[
E^A(z^m,0) = dz^n E^n_A(z^m,0) = E^b(z^m,0) \tilde{E}_b^A(z^m,0),
\]

\[
\tilde{E}_a^B(z^m,0) = (\delta_a^b, -\psi_a^\beta)
\]

(2.8)

(In the first equation we have factored out the component vielbein; the second equation is consistent with the standard definition of the gravitino. We do not need to explicitly fix a gauge and use \(E_\mu^A\).) We thus have

\[
E^A(z,y) = E^b(z,0) \tilde{E}_b^A(z,y) + (Dy^\beta) \hat{E}_\beta^A(z,y);
\]

\[
\hat{E}_b^A = \hat{E}_b^C(z,0) F_{CA}^A(z,y), \quad \hat{E}_\beta^A = G_{\beta}^A(z,y)
\]

(2.9)

From the normal coordinate expansion, we read off the vielbein components, and evaluate the superdeterminants

\[
E^{-1}(z,y) \equiv sdet \ E_M^A(z,y) = \tilde{E}^{-1} \hat{E}^{-1};
\]

\[
\hat{E}^{-1} \equiv sdet \ E_m^a(z,0), \quad \tilde{E}^{-1} \equiv sdet \ \hat{E}_A^B(z,y)
\]

(2.10)

In ref. [2] we have given a number of examples of the procedure. Here we summarize the case of four-dimensional N=1 minimal supergravity. We consider evaluation of a 4D N=1 superspace integral over \(d^4\theta = d^2\theta d^2\bar{\theta}\). This would require expansion up to fourth (and partly fifth) order in all the fermionic coordinates and therefore we consider instead the two-step procedure described above. In the first step we do only the \(\bar{\theta}\) integration, by expanding with respect to \(\bar{y}^\dot{\alpha}\) and evaluating at

\[
\bar{z}^\dot{\alpha} = 0, \quad (y^\alpha, y^\alpha) = 0
\]

(2.11)

(where we now explicitly distinguish between dotted and undotted spinors). We now need to expand \(E^A\) only to second (and partly third) order. We find

\[
\hat{E}_B^A = \begin{pmatrix}
\delta^\alpha_b - i\bar{y}^\dot{\alpha} \psi_b^\alpha - \frac{1}{2} \delta^\alpha_b \bar{y}^2 R & -\psi^\alpha_b + i\bar{y}^\dot{\alpha} C_{\gamma\beta}, \delta^\alpha_b R + \cdots -\psi^\alpha_b + \cdots \\
-i\bar{y}^\dot{\alpha} \delta^\alpha_b + \cdots & \delta^\alpha_b + \bar{y}^2 \delta^\alpha_b R & \cdots \\
0 & 0 & \delta^\alpha_\beta + \frac{1}{2} \delta^\alpha_\beta \bar{y}^2 R
\end{pmatrix}
\]

(2.12)

We have written only the relevant terms. Evaluating the superdeterminant we find

\[
E^{-1} = \tilde{E}^{-1} (1 - \bar{y}^2 R)
\]

(2.13)

where \(\bar{y}^2 = \frac{1}{2} \bar{y}^\dot{\alpha} \bar{y}_{\dot{\alpha}}\); and

\[
\tilde{E}^{-1} = sdet \begin{pmatrix}
E_m^a & E_m^\alpha \\
E_\mu^a & E_\mu^\alpha
\end{pmatrix}
\]

(2.14)
is the measure of the chiral subspace. (Explicit evaluation by the usual methods \[1\] shows it equals the usual measure \[\phi \] in terms of the chiral compensator \[4\]). We also expand the scalar lagrangian

\[ L(\bar{y}) = \left( 1 + \bar{y}^2 \nabla^2 - \bar{y}^2 \nabla^2 \right) L(0) \]  

Performing the \( \bar{y} \) integration, we find

\[ \int d^4x d^4\theta E^{-1} L = \int d^4x d^2\theta \tilde{E}^{-1}(\nabla^2 + R) L \]  

The normal coordinate method thus gives the solution to the chirality condition as well as the chiral integration measure. Note that the \( \psi \) terms cancelled: This expresses the covariance of the chiral subsuperspace (i.e., of chiral scalar superfields).

The same procedure can be applied to the next step. The coordinate restriction is now the antichiral one

\[ z^\mu = 0, (y^a, \bar{y}^\dot{a}) = 0 \]  

We first need to evaluate \( \tilde{E} \) of (2.14), now treated as a function of the new \( z^M \) and \( y^A \). We can use the result of the previous calculation by: (1) replacing (2.12) with the hermitian conjugate (effectively just switching dotted and undotted indices), and (2) deleting the second row and column before taking the superdeterminant, so we get the contribution to the smaller superdeterminant of (2.14). The result is

\[ \tilde{E}^{-1} = e^{-1} \left[ 1 - i \bar{y}^\alpha \bar{\psi}^\dot{\alpha} \right] - y^2 \left( 3 \bar{R} + \frac{1}{2} C^{\alpha\beta} \bar{\psi}^\dot{\alpha} \bar{\psi}^\dot{\beta} \right) \]  

where \( e^{-1} = \det e^m_\alpha \) is the new \( \tilde{E}^{-1} \) part of this subdeterminant, and the rest is the new \( \tilde{E}^{-1} \). The expansion of the lagrangian is similar to the previous case (just switching \( \bar{y} \rightarrow y \)), giving the final result

\[ S = \int d^4x d^4\theta E^{-1} L = \int d^4x e^{-1} \mathcal{D}^2(\nabla^2 + R) L = \int d^4x e^{-1} \mathcal{D}^1 L \]  

where we have defined a superdifferential operator, the “chiral density projector” \( \mathcal{D}^2 \), which (for the present case of old minimal supergravity) takes the form

\[ \mathcal{D}^2 \equiv \nabla^2 + i \bar{\psi}^\dot{\alpha} \nabla_\alpha + 3 \bar{R} + \frac{1}{2} C^{\alpha\beta} \bar{\psi}^\dot{\alpha} \bar{\psi}^\dot{\beta} \]  

and the general density projector \( \mathcal{D}^4 \equiv \mathcal{D}^2(\nabla^2 + R) \). We could use instead the complex conjugate \( \mathcal{D}^4 \equiv \mathcal{D}^2(\bar{\nabla}^2 + \bar{R}) \), which differs only by a total space-time derivative.

In the calculation above we have set the fermionic coordinates \( z^\mu \) to zero, but they could be fixed at any other value since the action is independent of the fermionic coordinates. Independence from \( \theta^\mu \) is equivalent to supersymmetry invariance; in superspace, supersymmetry transformations are formulated as coordinate transformations. This fact is the basis of the “ectoplasmic” method of superspace integration, which we will now describe.
3. Ectoplasmic Subintegration

Our goal is to use superfield methods to construct \textit{locally} supersymmetric \textit{component} actions. Component actions are written as integrals over space-time, while superfield actions are written as integrals over superspace. Since space-time is a subspace of superspace, it is natural to consider the same approach that is used for integration over subspaces of spaces with only commuting coordinates. For example, when considering integrals over three-dimensional hypersurfaces in four-dimensional space-time, we integrate the component of a vector normal to the surface. The condition that the integral be independent of the choice of hypersurface is the constraint that the vector be a conserved current. In the same fashion if we consider an integral over a space-time hypersurface in superspace, local supersymmetry invariance translates into independence of the integral from the $\theta$ variables in the integrand and imposes similar “conservation” conditions on the latter. Solving these conditions allows us to express the integrand in a manifestly locally supersymmetric component form.

This analysis can be applied to spaces without a metric. In the case under discussion it is necessary to integrate a differential three-form, or covariant third-rank antisymmetric tensor, and the hypersurface element itself is described by a contravariant third-rank antisymmetric tensor. A conserved “charge” in this case takes the form

\[
Q = \int dx^m \wedge dx^n \wedge dx^p \, J_{pnm} .
\] (3.1)

Time independence of the integral implies the conservation law; the 3-form current must be closed (curl-free) but not exact (i.e. not globally the curl of a 2-form):

\[
\frac{dQ}{dt} = 0 \Rightarrow \partial_{[m} J_{npq]} = 0 , \quad \delta J_{mnp} = \partial_{[m} \lambda_{np]} .
\] (3.2)

(In differential form notation, we write simply $Q = \int J$ satisfying $dJ = 0$, $\delta J = d\lambda$.) We assume that $J$ and $\lambda$ vanish at the boundaries.

We can also interpret the conservation law as saying that $J$ is a gauge field (with gauge parameter $\lambda$) whose field strength vanishes. However, this is not a gauge field in the usual sense of de Rham cohomology (cohomology of $d$): We consider only local functionals of the fields, not arbitrary functions of the coordinates, and the gauge invariance is just the statement that we drop total derivatives in the integrand. We can evaluate the charge $Q$ at $t = 0$ for convenience, but the conservation law was derived to make that unnecessary.

This analysis can be applied without modification to component actions, interpreted as integrals of superforms over a space-time “hypersurface” in superspace. We thus evaluate the integral in $D$ space-time dimensions:

\[
S = \int dx^{m_1} \wedge ... \wedge dx^{m_D} \, J_{m_D...m_1} = \int d^D x \, \frac{1}{D!} \epsilon^{m_1...m_D} J_{m_D...m_1} .
\] (3.3)
The integrand, $J_{m_1...m_D}$, is in general a function of superfields depending on both space-time and fermionic variables. The “conservation law” is now

$$\partial_{[M_1} J_{...M_{D+1}]} = 0, \quad \delta J_{M_1...M_D} = \partial_{[M_1} \lambda_{...M_D]}.$$  \hspace{1cm} (3.4)

Let us emphasize that the indices that appear in these equations are “holonomic” or “curved” super-vector indices. Separating out the bosonic and fermionic parts of the indices we have

$$\partial_{[m_1} J_{...m_{D+1}]} = 0, \quad \partial_{\mu} J_{m_1...m_D} - \frac{1}{(D-1)!} \partial_{[m_1} J_{\mu|m_2...m_D]} = 0, \ldots.$$  \hspace{1cm} (3.5)

The first equation is trivial, since there are more than $D$-valued space-time indices antisymmetrized. (Thus, there are no conditions for the nonsupersymmetric case.) After using the gauge invariance to pick a convenient gauge, the remaining equations in (3.5) can be solved to give

$$\partial_{\mu} J_{m_1...m_D} = 0 \quad (3.6)$$

Thus, in this particular gauge the integrand is in fact independent of the fermionic variables and can then be expressed as an integral (or derivative) of some object $L$ with respect to all the anticommuting coordinates.

$$J_{m_1...m_D} = \int d^{N_F} \theta \, L_{m_1...m_D} \quad .$$  \hspace{1cm} (3.7)

where $N_F$ is the number of fermionic coordinates in the superspace. In writing the solution in (3.7) we recall that the Berezinian definition of the Grassmann integral implies that it is equivalent to differentiating with respect to all of the fermionic coordinates. The original integral then is converted into the standard integral over all superspace of some superspace lagrangian.

Although this result may be sufficient for flat (super)space, it does not result in a covariant expression in the presence of supergravity. (In what follows we return to a general gauge so that (3.6) is necessarily not satisfied.) The simplest way to solve the conservation law covariantly is to convert from curved to flat indices, since the superspace of supergravity needs the tangent space for its definition. The conversion is

$$J_{m_1...m_D} = E_{m_D} A_D \ldots E_{m_1} A_1 J_{A_1...A_D} \quad .$$  \hspace{1cm} (3.8)

and similarly for $\lambda$. Note that, as for the normal coordinate method, the only parts of the vielbein that appear are the nontrivial parts, namely $e_m^a (x, \theta) \equiv E_{m}^a$ and $\psi_m^\alpha (x, \theta) \equiv -E_{m}^\alpha$ that contain at $\theta = 0$ the graviton, and the gravitino, respectively. (This is the same expansion as in (2.8).) We emphasize that the result of the integration is $\theta$-independent by construction so that the specific choice of a particular hypersurface, e.g. at $\theta = 0$, is never needed explicitly although it may be convenient.
We now need to solve the conservation constraints. We convert the indices of (3.4) by appropriately multiplying by the vielbein $E^M_A$ so that

$$\frac{1}{D!} \nabla_{\{A_1} J_{A_2...A_{D+1}} \} - \frac{1}{2(D-1)!} T^{B}_{\{A_1A_2\}B} J_{B|A_3...A_{D+1}} = 0 \ , \quad (3.9)$$

$$\delta J_{A_1...A_D} = \frac{1}{(D-1)!} \nabla_{A_1} \lambda_{A_2...A_D} - \frac{1}{2(D-2)!} T^{B}_{[A_1A_2} \lambda_{B|A_3...A_D]} \ . \quad (3.10)$$

Using the explicit form of the torsion in terms of the irreducible tensors of supergravity, and especially $T^{\alpha}_{\alpha \beta \gamma} \sim \gamma^c_{\alpha \beta \gamma}$, the pieces of $J$ with more vector indices can be recursively expressed in terms of those with more spinor indices. The form of the solution depends on the particulars of the supergravity under consideration. The general result is that a certain part of $J$ is a scalar, which may satisfy some (spinorial) differential constraint; parts of $J$ with more spinor indices vanish, those with fewer are expressed as spinorial derivatives of that scalar (with also torsion terms). The scalar is the superspace lagrangian, and its constraints (if any) define the kind of superspace.

To illustrate how all of this comes together, we examine the case of local $N = 1$ supersymmetry in the case of $D = 4$. The expanded form of (3.8) is given by

$$J_{m_1...m_4} = e_{m_1}^f \cdots e_{m_4}^k \left[ J_{f ghk} - \left( \frac{1}{3!} \psi_{[f} \alpha J_{\alpha|ghk]} + \frac{1}{4} \psi_{[f} \alpha \psi_{|gh]} \beta J_{\alpha \beta |hk]} + \text{h.c.} \right) \right.$$ 

$$\left. - \frac{1}{2} \psi_{[f} \alpha \overline{\psi}_{|g]} \delta J_{\alpha \beta [hk]} + \left( \frac{1}{3!} \psi_{[f} \alpha \psi_{|gh]} \beta \psi_{|h]} \gamma J_{\alpha \beta \gamma [k]} + \text{h.c.} \right) \right.$$ 

$$+ \frac{1}{2} \left( \psi_{[f} \alpha \psi_{|g]} \beta \overline{\psi}_{|h]} \delta J_{\alpha \beta \gamma [k]} + \text{h.c.} \right)$$ 

$$+ \left( \psi_{[f} \alpha J_{\alpha \beta \gamma} \delta + \text{h.c.} \right)$$ 

$$+ \left( \frac{1}{3!} \psi_{[f} \alpha \psi_{|gh]} \beta \psi_{|h]} \delta \overline{\psi}_{|k]} J_{\alpha \beta \gamma \delta} + \text{h.c.} \right)$$ 

$$+ \frac{1}{4} \psi_{[f} \alpha \psi_{|gh]} \beta \overline{\psi}_{|h]} \delta \overline{\psi}_{|k]} J_{\alpha \beta \gamma \delta} \right] \ . \quad (3.11)$$

In writing this we note the following: on the right-hand-side of the equation appear the components of the vielbein that we have denoted above as $e_{m}^f(x, \theta)$ and $-\psi_{f} \alpha(x, \theta)$. However, since we know that $J_{m_1...m_4}$ is $\theta$-independent, we can evaluate it at $\theta = 0$ and hence use the values of the vielbein and $J_{A_1...A_4}$ at $\theta = 0$ in terms of the ordinary component vielbein and gravitino.

The solution to the constraints is given simply by the supercovariantization of the rigid results [8] expressing all the nonvanishing components of the superform in terms of a covariantly chiral scalar superfield $J$. Up to total derivative terms that give no contributions to the action integral

$$J_{\alpha \beta \gamma D} = J_{\alpha \beta \gamma D} = J_{\alpha \beta cd} = 0 \ , \quad J_{\alpha \beta cd} = iC_{\gamma \delta} C_{\alpha (\gamma} C_{\delta) \beta} \tilde{J} \ , \quad \nabla_{\alpha} \tilde{J} = 0$$

$$J_{abcd} = -i \epsilon_{abcd} \nabla^a \tilde{J} \ , \quad J_{\alphaabcd} = -i \epsilon_{abcd} \nabla^a J \ ,$$

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\[
J_{abcd} = \epsilon_{abcd} \left[ (\nabla^2 + 3R)J + (\nabla^2 + 3R)\bar{J} \right], \quad (3.12)
\]

where
\[
\epsilon_{abcd} \equiv i(C_{\alpha\delta}C_{\beta\gamma}C_{\alpha\beta};C_{\gamma\delta} - C_{\alpha\beta}C_{\gamma\delta}C_{\alpha\delta};C_{\beta\gamma}) \quad . \quad (3.13)
\]

Substituting (3.11) and (3.12) into (3.3) yields the result
\[
S = \int d^4x \, e^{-1}D^2J + \text{h.c.}, \quad (3.14)
\]

with
\[
D^2 \equiv \nabla^2 + i\bar{\psi}_a^\alpha \nabla_\alpha + 3\bar{R} + \frac{1}{2}C^{\alpha\beta}_{\beta\gamma} \bar{\psi}_a^\alpha (\dot{\psi}_b^\beta) \quad (3.15)
\]

as in (2.20). Being \(\theta\)-independent, the right hand side may be evaluated at \(\theta = \bar{\theta} = 0\) without loss of generality. This result agrees with (2.19); solving the chirality condition as in (1.1), we find \(L = \Psi + \bar{\Psi}\). Even without (2.19), it is clear we can identify (3.14) with the result of integration over \(\theta\), since the chiral projector is the covariantization of the flat-space integration. (If there were an ambiguity associated with “nonminimal coupling”, it would have shown up as an ambiguity in solving the constraints.)

In conclusion, we emphasize that the “ectoplasmic” method for the construction of local density projectors is a genuinely new approach to the problem of integrating over the supermanifolds appropriate for describing locally supersymmetric theories. At no stage did we require any knowledge of the superspace measure, e.g. \((\text{sdet} E)^{-1}\). The superspace lagrangian and the form of the component action is determined solely from the existence of superforms and the solution of the constraints they satisfy.

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