On regularity of primal and dual dynamic value functions related to investment problem

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Abstract. We study regularity properties of the dynamic value functions of primal and dual problems of optimal investing for utility functions defined on the whole real line. Relations between decomposition terms of value processes of primal and dual problems and between optimal solutions of basic and conditional utility maximization problems are established. These properties are used to show that the value function satisfies a corresponding backward stochastic partial differential equation. In the case of complete markets we give conditions on the utility function when this equation admits a solution.

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1 Introduction

We consider a financial market model, where the dynamics of asset prices is described by the continuous semimartingale $S$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with continuous filtration $F = (F_t, t \in [0, T])$, where $\mathcal{F} = F_T$ and $T < \infty$. We work with discounted terms, i.e. the bond is assumed to be a constant.
Denote by $\mathcal{M}^e$ (resp. $\mathcal{M}^a$) the set of probability measures $Q$ equivalent (resp. absolutely continuous with respect) to $P$ such that $S$ is a local martingale under $Q$.

Throughout the paper we assume that the filtration $F$ is continuous (i.e. all $F$-local martingales are continuous) and

$$\mathcal{M}^e \neq \emptyset. \quad (1)$$

The continuity of $F$ and the existence of an equivalent martingale measure imply that the structure condition is satisfied, i.e. $S$ admits the decomposition

$$S_t = M_t + \int_0^t \lambda_s \langle M \rangle_s, \quad \int_0^t \lambda_s^2 d\langle M \rangle_s < \infty$$

for all $t$ $P$-a.s., where $M$ is a continuous local martingale and $\lambda$ is a predictable process.

Let $U = U(x) : R \to R$ be a utility function taking finite values at all points of real line $R$ such that $U$ is continuously differentiable, increasing, strictly concave and satisfies the Inada conditions

$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0, \quad U'(-\infty) = \lim_{x \to -\infty} U'(x) = \infty. \quad (2)$$

We also assume that $U$ satisfies the condition of reasonable asymptotic elasticity (see [6] and [13]), i.e.

$$\limsup_{x \to \infty} \frac{x U'(x)}{U(x)} < 1, \quad \liminf_{x \to -\infty} \frac{x U'(x)}{U(x)} > 1. \quad (3)$$

We consider the utility maximization problem, i.e. the problem of finding a trading strategy $(\pi_t, t \in [0, T])$ such that the expected utility of terminal wealth $X_T^{x, \pi}$ becomes maximal. The wealth process, determined by a self-financing trading strategy $\pi$ and initial capital $x$, is defined as a stochastic integral

$$X_t^{x, \pi} = x + \int_0^t \pi_u dS_u, \quad 0 \leq t \leq T.$$ 

The predictable, $S$-integrable process $\pi$ we call admissible if the stochastic integral ($\int_0^t \pi_u dS_u, t \in [0, T]$) is uniformly bounded from below.

The value function $V$ associated to the problem is given by

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E} \left[ U \left( x + \int_0^T \pi_u dS_u \right) \right], \quad (4)$$

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where $\Pi$ is the class of admissible strategies.

For the utility function $U$ we denote by $\tilde{U}$ its convex conjugate

$$\tilde{U}(y) = \sup_x (U(x) - xy), \quad y > 0.$$  (5)

The dual problem to (4) is

$$\tilde{V}(y) = \inf_{Q \in \mathcal{M}^e} E[\tilde{U}(y \rho^Q_T)], \quad y > 0,$$  (6)

where $\rho^Q_t = dQ_t/dP_t$ is the density process of the measure $Q \in \mathcal{M}^e$ relative to the basic measure $P$.

Let $\tau$ be a stopping time valued in $[0, T]$. Denote by $\Pi_\tau$ the class of admissible processes, such that $\pi = \pi 1_{[\tau, T]}$. Define $Z_{\tau, y} = \{Y : Y = y \rho^\tau_T, \rho_T = \frac{dQ}{dP}, \ Q \in \mathcal{M}^e(S)\}$.

The dynamic value functions of primal and dual problems are defined as

$$V(\tau, x) = \text{ess sup}_{\pi \in \Pi_\tau} E\left[U\left(x + \int^T_\tau \pi_u dS_u\right) \mid F_\tau\right],$$  (7)

$$\tilde{V}(\tau, y) = \text{ess inf}_{Y \in Z_{\tau, y}} E\left[\tilde{U}(Y) \mid F_\tau\right], \quad y > 0.$$  (8)

For $V(0, x)$ and $\tilde{V}(0, y)$ we use the notations $V(x)$ and $\tilde{V}(y)$ respectively. Following [13] we make

Asumption 1. For each $y > 0$ the dual value function $\tilde{V}(y)$ is finite and the minimizer $Q^*(y) \in \mathcal{M}^e$ (called the minimax martingale measure) exists.

Let $\Pi_x$ be the class of predictable $S$ integrable processes $\pi$ such that $U(x + (\pi \cdot S)_T) \in L^1(P)$ and $\pi \cdot S$ is a supermartingale under each $Q \in \mathcal{M}^a$ with finite $\tilde{U}$-expectation $E\tilde{U}(\frac{dQ}{dP})$, where the notation $\pi \cdot S$ stands for the stochastic integral.

Denote $Q(x) = Q^*(y) = Q^*(V'(x))$.

It was proved in [12] that optimal strategy $\pi(x) \in \Pi_x$ of problem [4] exists, is unique and $V(x) = EU(X_T(x))$, where the optimal wealth $X_T(x) = x + \int^T_0 \pi_u(x) dS_u$ is a uniformly integrable $Q(x)$-martingale.

Besides, the following duality relations hold true almost surely

$$U'(X_T(x)) = Z_T(y), \quad y = V'(x),$$  (9)
\[ V'(t, x + \int_0^t \pi_u(x) \, dS_u) = Z_t(y), \quad t \in [0, T], \quad (10) \]

where \( y = V'(x) \) (see [13] and Proposition A3 from [11] for the dynamic version). Hereafter we shall use these results without further comments.

Our goal is to study the properties of the dynamic value function \( V(t, x) \) and the optimal wealth process \( X_t(x) \). It is well known (see e.g., [10]) that for any \( x \in \mathbb{R} \) the process \((V(t, x), t \in [0, T])\) is a supermartingale admitting an RCLL (right-continuous with left limits) modification.

Therefore, using the Galchouk–Kunita–Watanabe (GKW) decomposition, the value function is represented as

\[ V(t, x) = V(0, x) - A(t, x) + \int_0^t \psi(s, x) \, dM_s + L(t, x), \quad (11) \]

where for any \( x \in \mathbb{R} \) the process \( A(t, x) \) is increasing and \( L(t, x) \) is a local martingale orthogonal to \( M \).

Let consider the following assumptions:

a) \( V(t, x) \) is two-times continuously differentiable at \( x \) \text{ P-} a.s. for any \( t \in [0, T] \),

b) for any \( x \in \mathbb{R} \) the process \( V(t, x) \) is a special semimartingale with bounded variation part absolutely continuous with respect to \( \langle M \rangle \), i.e.

\[ A(t, x) = \int_0^t a(s, x) \, d\langle M \rangle_s, \]

for some real-valued function \( a(s, x) \) which is predictable and \( \langle M \rangle \)-integrable for any \( x \in \mathbb{R} \),

c) for any \( x \in \mathbb{R} \) the process \( V'(t, x) \) is a special semimartingale with the decomposition

\[ V'(t, x) = V'(0, x) - \int_0^t a'(s, x) \, d\langle M \rangle_s + \int_0^t \psi'(s, x) \, dM_s + L'(t, x), \]

where \( V', a', \psi' \) and \( L' \) are partial derivatives at \( x \) of \( V, a, \psi \) and \( L \), respectively.

We shall say that \((V(t, x), t \in [0, T])\) is a regular family of semimartingales if for \( V \) conditions a), b) and c) are satisfied.

We shall consider also the conditions:
d) the conditional optimization problem (11) admits a solution, i.e., for any \( t \in [0, T] \) and \( x \in R \) there exists a strategy \( \pi(t, x) \) such that

\[
V(t, x) = E\left(U(x + \int_t^T \pi_u(t, x)dS_u)|F_t\right),
\]

(12)

e) for each \( s \in [t, T] \) the function \( X_s(t, x) = x + \int_t^s \pi_u(t, x)dS_u, s \geq t \) is continuous at \((t, x)\) \( P\text{-}a.s. \).

It was shown in [8, 9, 10] that if the value function satisfies conditions a)-e) then it solves the following backward stochastic partial differential equation (BSPDE)

\[
V(t, x) = V(0, x) + \frac{1}{2} \int_0^t \left( \frac{\varphi'(s, x) + \lambda(s)V'(s, x))^2}{V''(s, x)} \right) d\langle M \rangle_s \nonumber
\]

\[
+ \int_0^t \varphi(s, x) dM_s + L(t, x), \quad V(T, x) = U(x).
\]

(13)

Our aim is to study conditions on the basic objects (on the asset price model and on the objective function \( U \)) which will guaranty that the value function \( V(t, x) \) is a regular family of semimartingales and conditions d) and e) are also satisfied, in order to show that the solution of equation (13) exists. In Theorem 3 of section 5 we provide such type conditions in the case of complete markets.

The main example, where all conditions a)-e) are satisfied is the case of exponential utility function \( U(x) = -e^{-\gamma x} \) with risk aversion parameter \( \gamma \in (0, \infty) \). In this case the corresponding value function is of the form \( V(t, x) = -e^{-\gamma x}V_t \), where \( V_t \) is a special semimartingale. Besides, \( \tilde{U}(y) = \frac{y}{\gamma} (\ln \frac{y}{\gamma} - 1) \) and Assumption 1 is equivalent to the existence of \( Q \in \mathcal{M}^e \) with finite relative entropy \( EZ_T^Q \ln Z_T^Q \) (see e.g. [11]).

We first investigate whether Assumption 1 implies an existence of an optimal strategy to the conditional maximization problem (7) and how is this strategy related to the optimal strategy of the basic problem (4).

It was shown in [13] that if we start at time \( \tau \) with the optimal wealth \( X_{\tau}(x) \) then the optimal value in (7) is attained by \( \pi(\tau, x) = \pi(0, x)I_{[\tau, T]} \), i.e.,

\[
E[U(X_T(x))|F_{\tau}] \geq E[U(X_{\tau}(x) + \int_{\tau}^T \pi_u dS_u) | F_{\tau}], \quad \pi \in \Pi_{\tau},
\]

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which is well understood from the Bellman Principle.

Under additional conditions we shall show (see Theorem 1) that if we start at time $\tau$ with the wealth equal to arbitrary amount $x$, then the optimal strategy $\pi(\tau, x)$ of (7) is expressed in terms of the optimal strategy $\pi(x) = \pi(0, x)$ and the optimal wealth $X_{\tau}(x) = X_{\tau}(0, x)$ of (11) at time $\tau$ by the equality

$$
\pi_t(\tau, x) = \pi_t(X_{-1}^{-1}(x)), \quad t \geq \tau \quad \mu^{(S)} - a.e.,
$$

where $X_{-1}^{-1}(x)$ is the inverse of the optimal wealth $X_t(x)$.

In section 3 we establish relation between decomposition terms of the value process $V(t, x)$ (11) with corresponding terms of the dual value process $\tilde{V}(t, y)$.

The problem related with condition a) was studied in [5] for utility functions defined on the positive real line for value functions at time 0 and in [11] for dynamic value function $V(t, x)$ corresponding to utility functions defined on the whole real line.

The problems related with conditions b) and c) we connect with an existence of the inverse flow $X_{-1}^{-1}(x)$ of the optimal wealth. In [11] conditions are given when for any $t$ the optimal wealth is an increasing function of $x$ $P$-a.s. and that an adapted inverse of $X_t(x)$ exists. In Proposition 2 of section 4 we derive a stochastic differential equation for the inverse of the optimal wealth $\psi_t(x) = X_{-1}^{-1}(x)$ and based on this result we give in Proposition 3 sufficient conditions when b) and c) are fulfilled.

Finally in Section 5 in the case of complete markets we give conditions on utility function for which all conditions a)-e) are fulfilled and the value function $V(t, x)$ satisfies $BSPDE$ (13).

In the paper [3] a new approach was developed, where the solution of the problem (4) was reduced to the solvability of a system of Forward-Backward equations which is also a heavy task. Note that they showed that in case of complete markets this system admits a solution under conditions similar to condition r1) of section 5.

2 The relation between the basic and conditional utility maximization problems

In this section we study basic and conditional utility maximization problem in incomplete markets for utility functions defined on the whole real line and
establish relations between optimal strategies of these problems.

To this end we first give some definitions and auxiliary assertions.

We shall say that an adapted stochastic process \((X_t, t \in [\tau, T])\) is a

1) \(E(|X_t|/F_\tau) < \infty\), for any \(t \in [\tau, T]\)

2) \(E(X_t/F_t) = X_t\) (resp. \(\leq X_t\)) for any \(t' \leq t\), where \(t', t \in [\tau, T]\)

(see the definition of generalized conditional expectations and of generalized supermartingales for discrete time in [14])

**Definition.** A predictable \(S\)-integrable process \(\pi\) is in \(\Pi_{x,\tau}\), if

\[
E(U(x + \int_\tau^T \pi_u dS_u)/F_\tau) \quad \text{is finite and } \quad ((\pi \cdot S)_t, t \geq \tau) \quad \text{is a generalized supermartingale}
\]

under each \(Q \in \mathcal{M}^a\) with finite \(\tilde{U}\)-expectation \(E\tilde{U}(\frac{dQ}{dP})\).

We shall also need two complementary assumptions

**Assumption 2.** The filtration \(F\) is continuous and \(\liminf_{y \to \infty} Z_T(y)/y > 0\) for the

process

\[
Z_T(y) = y^\theta \rho^\gamma \frac{dQ}{dP}
\]

**Assumption 3.** The utility function \(U\) is two times differentiable and there are constants \(c_1 > 0\) and \(c_2 > 0\) such that

\[
c_1 < -\frac{U''(x)}{U'(x)} < c_2, \quad x \in R.
\]

The last condition is similar to the condition on relative risk-aversion introduced in [3]. Note that for exponential utility function the risk-aversion coefficient \(-\frac{U''(x)}{U'(x)} = \gamma\) is a constant and condition (14) is also satisfied for linear combinations of exponential utility functions with different risk-aversion parameters.

The proof of the following assertion follows from Theorem 4.1 and Proposition 3.1 of [11].

**Proposition 1.** Let Assumptions 1-3 be satisfied.

Then for any \(t \in [0, T]\) there exists a modification of the optimal wealth process \((X_t(x), x \in R)\) (resp. of \(Z_t(y)\)) almost all paths of which are strictly increasing and absolutely continuous with respect to \(dx\) (resp. \(dy\)). Besides

\[
X_t'(x) > 0, \quad E^Q(x)(X_t'(x))^2 \leq C, \quad \text{(15)}
\]

\[
\lim_{x \to \infty} X_t(x) = \infty, \quad \lim_{x \to -\infty} X_t(x) = -\infty \quad \text{(16)}
\]

\(P\)-a.s. for any \(t \in [0, T]\) and the adapted inverse \(X_t^{-1}(x)\) (resp. \(Z_t^{-1}(y)\)) of the optimal wealth process exists.
We shall need also the continuity properties of the square characteristics \( \langle X(x) - X(y) \rangle \) which can be deduced from Proposition 1.

**Lemma 1.** Let conditions of Proposition 1 be satisfied. Then, for any \( t \in [0, T] \) the random field \( \langle X(x) - X(y) \rangle_t, x, y \in R \) admits a continuous modification.

**Proof.** It follows from Proposition 1 that \( X_t(b) - X_t(a) = \int_a^b X'_t(x) dx \) and

\[
\int_a^b E^Q(x) \langle X'(x) \rangle_T dx = \int_a^b E^Q(x) X_T^2(x) dx < \infty
\]

and by the Fubini theorem \( \int_a^b V'(x) \langle X'(x) \rangle_T dx < \infty, \ P - a.s. \) Thus by continuity of \( \frac{V'(x)}{U'(x_T(x))} \) we obtain

\[
\int_a^b \langle X'(x) \rangle_T dx \leq \max_{x \in [a,b]} \frac{V'(x)}{U'(x_T(x))} \int_a^b \frac{U'(X_T(x))}{V'(x)} \langle X'(x) \rangle_T dx < \infty, \ P - a.s.
\]

Therefore, using the Kunita-Watanabe and Hölder's inequalities we have

\[
\langle X(b) - X(a) \rangle_t = \int_a^b \int_a^b \langle X'(x), X'(y) \rangle_t dx dy \\
\leq \int_a^b \int_a^b \langle X'(x) \rangle_t^{1/2} \langle X'(y) \rangle_t^{1/2} dx dy = \left( \int_a^b \langle X'(x) \rangle_t^{1/2} dx \right)^2 \\
\leq (b - a) \int_a^b \langle X'(x) \rangle_t dx < \infty, \ P - a.s.
\]

and it follows from inequality

\[
\langle X(b') - X(a') \rangle_t - \langle X(b) - X(a) \rangle_t \\
\leq \langle X(b') - X(b) \rangle_t^{1/2} \langle X(b') - X(a') + X(b) - X(a) \rangle_t^{1/2} \\
+ \langle X(a') - X(a) \rangle_t^{1/2} \langle X(b') - X(a') + X(b) - X(a) \rangle_t^{1/2}
\]

that \( \langle X(b_n) - X(a_n) \rangle_t \to \langle X(b) - X(a) \rangle_t, \ P - a.s. \) when \( b_n \to b, \ a_n \to a. \) Thus the stochastic field defined by

\[
\langle X(x) - X(y) \rangle_t = \begin{cases} 
\lim_{r \to a, r' \to b} \langle X(r) - X(r') \rangle_t, & r, r' \text{ are rational,} \\
0, & \text{if the limit does not exists}
\end{cases}
\]

is continuous and stochastically equivalent to \( \langle X(x) - X(y) \rangle_t. \)
**Theorem 1.** Let Assumptions 1-3 be satisfied. Then there exist the maximizer of (7) and the minimizer of (8) in the classes \( \Pi_{\tau,x} \) and \( Z_{\tau,y} \) respectively and equalities

\[ X_T(\tau,x) = X_T(X_{\tau}^{-1}(x)), \quad \pi_t(\tau,x) = \pi_t(X_{\tau}^{-1}(x)), t \geq \tau, \]

\[ Y(\tau,y) = Z_T(Z_{\tau}^{-1}(y)), \quad \rho_T^Q(\tau,y) = \rho_T^Q(y) \frac{Z_T(Z_{\tau}^{-1}(y))}{y} \]

are satisfied.

Moreover \( P \)-a.s.

\[ V(\tau,x) = E\left[U \left( x + \int_{\tau}^{T} \pi_u(X_{\tau}^{-1}(x))dS_u \right) \mid F_\tau \right], \]

\[ \tilde{V}(\tau,y) = E\left[\tilde{U}(Z_T(Z_{\tau}^{-1}(y))) \mid F_\tau \right], \]

the following duality relation holds

\[ U' \left( x + \int_{\tau}^{T} \pi_u(X_{\tau}^{-1}(x))dS_u \right) = Z_T(Z_{\tau}^{-1}(y)), \quad y = V'(\tau,x) \]

and the process

\[ Z_t(Z_{\tau}^{-1}(y))X_t(X_{\tau}^{-1}(x)), \quad t \in [\tau,T], \text{ where } y = V'(\tau,x), \]

is a generalized martingale.

**Proof.** By the optimality principle (see, e.g. [10]) \( V(t,X_t(x)) \) is a martingale and since \( V(T,x) = U(x) \) we have that for any \( x \in R \)

\[ V(\tau,X_\tau(x)) = E(U(X_T(x))/F_\tau) \quad P \text{- a.s.} \]  

Since for any \( \tau \) the functions \( V(\tau,x) \) and \( X_\tau(x) \) are continuous for almost all \( \omega \in \Omega \), the equality \( [22] \) holds \( P \)-a.s. for all \( x \in R \) and substituting \( X_{\tau}^{-1}(x) \) in this equality we obtain that

\[ V(\tau,x) = E(U(X_T(X_{\tau}^{-1}(x)))/F_\tau) \quad P \text{- a.s.}, \]

which means the maximality of \( X_T(X_{\tau}^{-1}(x)) \). Let us show that \( X_T(X_{\tau}^{-1}(x)) \) is equal to the stochastic integral

\[ X_T(X_{\tau}^{-1}(x)) = x + \int_{\tau}^{T} \pi_u(X_{\tau}^{-1}(x))dS_u \]

(23)
and that \( \pi(X_{\tau}^{-1}(x)) \) belongs to the class \( \Pi_{\tau,x} \).

In order to show equality (23) it is enough to show that
\[
\int_{\tau}^{T} \pi_u(x) dS_u \big|_{x=\xi} = \int_{\tau}^{T} \pi_u(\xi) dS_u,
\]
for \( \xi = X_{\tau}^{-1}(x) \).

Let us consider the sequence of simple random variables \( \xi_n = \sum_{k=-\infty}^{\infty} c_k 1_{A_k} \), where \( A_k = (\frac{k}{n} \leq \xi < \frac{k+1}{n}) \), \( c_k = \frac{k}{n} \). We have \( \xi_n \to \xi \) uniformly and
\[
\int_{\tau}^{T} \pi_u(\xi_n) dS_u = \sum_{k=-\infty}^{\infty} \int_{\tau}^{T} \pi_u(c_k) 1_{A_k} dS_u = \\
= \sum_{k=-\infty}^{\infty} 1_{A_k} \int_{\tau}^{T} \pi_u(c_k) dS_u = \int_{\tau}^{T} \pi_u(x) dS_u |_{x=\xi_n}.
\]

On the other hand
\[
\int_{\tau}^{T} \pi_u(x) dS_u |_{x=\xi} - \int_{\tau}^{T} \pi_u(x) dS_u |_{x=\xi} = \\
= X_T(\xi_n) - X_T(\xi_n) - (X_T(\xi) - X_T(\xi)) \to 0,
\]
as \( n \to \infty \), since \( X_t(x) \) is continuous and
\[
\int_{\tau}^{T} (\pi_u(\xi_n) - \pi_u(\xi))^2 d(S)_u = \\
= (X(x) - X(y))_T - (X(x) - X(y))_T |_{x=\xi_n, y=\xi} \to 0, P - a.s.
\]
as \( n \to \infty \), by continuity of \((X(x) - X(y))_T \). Hence \( \int_{\tau}^{T} \pi_u(\xi_n) dS_u \to \int_{\tau}^{T} \pi_u(\xi) dS_u \) in probability and \( \int_{\tau}^{T} \pi_u(x) dS_u |_{x=\xi} = \int_{\tau}^{T} \pi_u(\xi) dS_u - P.a.s. \).

Since \( E|U(X_T(x))| < \infty \) and \( E^Q|X_t(x)| < \infty, t \in [0,T] \) for any \( Q \in \mathcal{M}^a \) and \( X_{\tau}^{-1}(x) \) is \( F_{\tau} \)-measurable we have that
\[
E|U(X_T(X_{\tau}^{-1}(x)))| | F_{\tau} < \infty, \quad E^Q(|X_t(X_{\tau}^{-1}(x))|/F_{\tau}) < \infty \quad P-a.s., t \geq \tau.
\]

On the other hand, since for any \( t \in [0,T] \) the function \((X_t(x), x \in R)\) is continuous and increasing, the supermartingale inequality
\[
E^Q(X_t(x)/F_{t'}) \leq X_{t'}(x), \quad t' \leq t \leq T
\]
is valid for any \( Q \in \mathcal{M}^a \), hence \( \pi(\tau, x) = \pi(X_{\tau}^{-1}(x)) \) belongs to the class \( \Pi_{\tau,x} \) and the equality (19) holds. Similarly one can show the minimality of \( Z_{\tau}(Z_{\tau}^{-1}(y)) \),
so conditional density of the minimax martingale measure to the problem (11) is
\( Z_T^{-1}(y) \).

Since for any \( t \in [0,T] \) the functions \( V'(t, x), x \in R \) and \( Z_t(y), y > 0 \) are
continuous and the inverse of \( Z_t(y) \) exists, from (10) we have that \( P \)-a.s.
\[
Z_t^{-1}(V'(\tau, x)) = V'(X^{-1}_\tau(x))
\] (24)
which, together with (9) implies the conditional duality relation (20).

Note also that since \( Z_t(y)X_t(x) \) is a martingale (see Theorem 1 from [13]),
by continuity of \( X(x) \) and \( Z(y) \) the process \( (Z_t(V'(X^{-1}_\tau(x)))X_t(X^{-1}_\tau(x)), t \geq \tau) \)
will be a generalized martingale and by equality (24) this is equivalent to
(21).

3 Relations between decomposition terms of
the value processes of primal and dual problems

In this section additionally to the continuity of the filtration \( F \) we assume
that any orthogonal to \( M \) local martingale \( L \) is represented as a stochastic
integral with respect to the given continuous local martingale \( M^\perp \). Therefore,
the value process \( V(t, x) \) admits the decomposition
\[
V(t, x) = V(0, x) - A(t, x) + \int_0^t \phi(s, x)dM_s + \int_0^t \phi^\perp(s, x)dM^\perp_s,
\]
where \( A(t, x) \) is an increasing process for any \( x \in R \), \( \phi \) and \( \phi^\perp \) are \( M \) and \( M^\perp \)
integrable predictable processes respectively. Since the value process \( \tilde{V}(t, y) \)
of the dual problem is a submartingale for each \( y > 0 \) it is decomposable as
\[
\tilde{V}(t, y) = \tilde{V}(0, y) + \tilde{A}(t, y) + \int_0^t \tilde{\phi}(s, y)dM_s + \int_0^t \tilde{\phi}^\perp(s, y)dM^\perp_s,
\] (25)
with \( M \) and \( M^\perp \) integrable predictable processes \( \tilde{\phi} \) and \( \tilde{\phi}^\perp \) and an increasing
process \( \tilde{A}(t, y) \).

It is known that the value processes of the primal and dual problems are
related by the equality
\[
V(t, -\tilde{V}'(t, y)) = \tilde{V}(t, y) - y\tilde{V}'(t, y).
\] (26)
We are interested how are related the decomposition terms $A, \varphi$ and $\varphi_\perp$ with $\tilde{A}, \tilde{\varphi}$ and $\tilde{\varphi}_\perp$ respectively.

**Theorem 2.** Assume that the filtration $F$ is continuous and any orthogonal to $M$ local martingale $L$ is represented as a stochastic integral with respect to a local martingale $M^\perp$. Assume that $V(t, x)$ is a regular family of semimartingales (i.e., satisfies conditions a)-c) of introduction) and that $\tilde{V}'(t, y)$ is a semimartingale with the decomposition

$$\tilde{V}'(t, y) = \tilde{V}'(0, y) + \tilde{B}(t, y) + \int_0^t \tilde{\varphi}'(s, y) dM_s + \int_0^t \tilde{\varphi}'_\perp(s, y) dM^\perp_s,$$  \hspace{1cm} (27)

where $\tilde{B}(t, y)$ is the process of finite variation for any $y$.

Then $(\tilde{V}(t, y), y > 0)$ is a regular family of semimartingales and

$$\tilde{\varphi}(s, y) = \varphi(s, -\tilde{V}'(s, y)), \quad \mu^{(M)} \text{ a.e.,} \hspace{1cm} (28)$$

$$\tilde{\varphi}_\perp(s, y) = \varphi_\perp(s, -\tilde{V}'(s, y)), \quad \mu^{(M)} \text{ a.e.,} \hspace{1cm} (29)$$

$$\tilde{A}(t, y) = \int_0^t a(s, -\tilde{V}'(s, y)) d\langle M \rangle_s - \frac{1}{2} \int_0^t \frac{\varphi'(s, -\tilde{V}'(s, y))^2}{V''(s, -\tilde{V}'(s, y))} d\langle M \rangle_s - \frac{1}{2} \int_0^t \frac{\varphi'_\perp(s, -\tilde{V}'(s, y))^2}{V'(s, -\tilde{V}'(s, y))} d\langle M^\perp \rangle_s. \hspace{1cm} (30)$$

Besides $\tilde{V}(t, y)$ satisfies the BSPDE

$$\tilde{V}(t, y) = \tilde{V}(0, y) + \int_0^t \varphi(s, -\tilde{V}'(s, y)) dM_s + \int_0^t \varphi_\perp(s, -\tilde{V}'(s, y)) dM^\perp_s, \hspace{1cm} \tilde{V}(T, y) = \tilde{U}(y). \hspace{1cm} (31)$$

**Proof.** Using the duality relation (26) and the Itô-Ventzel formula (see, e.g., [7] or [15]) we have

$$V(t, -\tilde{V}'(t, y)) =$$

$$= V(0, -\tilde{V}'(0, y)) + \int_0^t \varphi(s, -\tilde{V}'(s, y)) dM_s + \int_0^t \varphi_\perp(s, -\tilde{V}'(s, y)) dM^\perp_s -$$

$$- \int_0^t V'(s, -\tilde{V}'(s, y)) \tilde{\varphi}'(s, y) dM_s - \int_0^t V'(s, -\tilde{V}'(s, y)) \tilde{\varphi}'_\perp(s, y) dM^\perp_s -$$

$$+ \frac{1}{2} \int_0^t \frac{\varphi'(s, -\tilde{V}'(s, y))^2}{V''(s, -\tilde{V}'(s, y))} d\langle M \rangle_s + \int_0^t \varphi(s, -\tilde{V}'(s, y)) dM_s + \int_0^t \varphi_\perp(s, -\tilde{V}'(s, y)) dM^\perp_s,$$
\[ + \int_0^t a(s, -\ddot{V}'(s, y))d\langle M\rangle_s - \int_0^t V'(s, -\ddot{V}'(s, y))d\ddot{B}(s, y) + \]
\[ + \frac{1}{2} \int_0^t V''(s, -\ddot{V}'(s, y))\dddot{\varphi}(s, y)^2 d\langle M\rangle_s + \]
\[ + \frac{1}{2} \int_0^t V''(s, -\ddot{V}'(s, y))\dddot{\varphi}_\perp(s, y)^2 d\langle M^\perp\rangle_s - \]
\[ - \int_0^t \varphi'(s, -\ddot{V}'(s, y))\dddot{\varphi}(s, y) d\langle M\rangle_s - \]
\[ - \int_0^t \varphi'_\perp(s, -\ddot{V}'(s, y))\dddot{\varphi}_\perp(s, y) d\langle M^\perp\rangle_s = \]
\[ = \dddot{A}(t, y) + \int_0^t \dddot{\varphi}(s, y) dM_s + \int_0^t \dddot{\varphi}_\perp(s, y) dM^\perp_s - \]
\[ - y\dddot{B}(t, y) - y \int_0^t \dddot{\varphi}(s, y) dM_s - y \int_0^t \dddot{\varphi}_\perp(s, y) dM^\perp_s. \quad (32) \]

Since \( V'(s, -\ddot{V}'(s, y)) = y \), from (32) we obtain that
\[ \int_0^t \varphi(s, -\ddot{V}'(s, y)) dM_s + \int_0^t \varphi'_\perp(s, -\ddot{V}'(s, y)) dM^\perp_s + \]
\[ + \int_0^t a(s, -\ddot{V}'(s, y)) d\langle M\rangle_s + \frac{1}{2} \int_0^t V''(s, -\ddot{V}'(s, y)) (\dddot{\varphi}(s, y))^2 d\langle M\rangle_s + \]
\[ + \frac{1}{2} \int_0^t V''(s, -\ddot{V}'(s, y)) (\dddot{\varphi}_\perp(s, y))^2 d\langle M^\perp\rangle_s - \]
\[ - \int_0^t \varphi'(s, -\ddot{V}'(s, y))\dddot{\varphi}(s, y) d\langle M\rangle_s - \]
\[ - \int_0^t \varphi'_\perp(s, -\ddot{V}'(s, y))\dddot{\varphi}_\perp(s, y) d\langle M^\perp\rangle_s = \]
\[ = \dddot{A}(t, y) + \int_0^t \dddot{\varphi}(s, y) dM_s + \int_0^t \dddot{\varphi}_\perp(s, y) dM^\perp_s. \quad (33) \]

Equalizing the martingale parts in (33) we obtain equalities (28) and (29). Since \( \dddot{V}(t, y) \) is two-times differentiable and
\[ \dddot{V}''(t, y) = -\frac{1}{V''(t, -\ddot{V}'(t, y))}. \quad (34) \]
we have that \( \tilde{\varphi}(s, y) \) and \( \tilde{\varphi}_\perp(s, y) \) are also differentiable and

\[
\tilde{\varphi}'(s, y) = \varphi'(s, -\tilde{V}'(s, y)) \tilde{V}''(t, y) = \frac{\varphi'(s, -\tilde{V}'(s, y))}{V''(t, -\tilde{V}'(s, y))}, \quad \mu(\mathcal{M}) \text{ a.e.,} \quad (35)
\]

\[
\tilde{\varphi}'_\perp(s, y) = \varphi'_\perp(s, -\tilde{V}'(s, y)) \tilde{V}''(t, y) = \frac{\varphi'_\perp(s, -\tilde{V}'(s, y))}{V''(t, -\tilde{V}'(s, y))}, \quad \mu(\mathcal{M}^\perp) \text{ a.e.} \quad (36)
\]

Therefore,

\[
\varphi'(s, -\tilde{V}''(s, y)) \tilde{\varphi}(s, y) = V''(s, -\tilde{V}'(s, y))(\tilde{\varphi}(s, y))^2, \quad \mu(\mathcal{M}) \text{ a.e.,}
\]

\[
\varphi'_\perp(s, -\tilde{V}'(s, y)) \tilde{\varphi}'_\perp(s, y) = V''(s, -\tilde{V}'(s, y))(\tilde{\varphi}'_\perp(s, y))^2, \quad \mu(\mathcal{M}^\perp) \text{ a.e.}
\]

and equalizing the finite variation parts in (33) we deduce that equality (30) holds.

Let us show now that \( \tilde{V}(t, y) \) satisfies the BSPDE (31). It follows from (13) that

\[
a(s, x) = \frac{1}{2} \left( \lambda_s V'(s, x) + \varphi'(s, x) \right)^2.
\]

Therefore, using equalities \( V'(s, -\tilde{V}'(s, y)) = y \), (31) and (35)

\[
\int_0^t a(s, -\tilde{V}'(s, y)) d\langle \mathcal{M} \rangle_s = \frac{1}{2} \int_0^t \frac{(y \lambda_s + \varphi'(s, -\tilde{V}'(s, y)))^2}{V''(s, -\tilde{V}'(s, y))} d\langle \mathcal{M} \rangle_s =
\]

\[
= \int_0^t (y \lambda_s \tilde{\varphi}'(s, y) - \frac{1}{2} y^2 \lambda_s^2 \tilde{V}''(s, y)) d\langle \mathcal{M} \rangle_s + \frac{1}{2} \int_0^t \frac{(\varphi'(s, -\tilde{V}'(s, y)))^2}{V''(s, -\tilde{V}'(s, y))} d\langle \mathcal{M} \rangle_s.
\]

which (together with (30)) implies that

\[
\tilde{A}(t, y) = \int_0^t (y \lambda_s \tilde{\varphi}'(s, y) - \frac{1}{2} y^2 \lambda_s^2 \tilde{V}''(s, y)) d\langle \mathcal{M} \rangle_s
\]

\[
+ \frac{1}{2} \int_0^t \frac{(\varphi'_\perp(s, y))^2}{V''(s, y)} d\langle \mathcal{M}^\perp \rangle_s. \quad (37)
\]

Now, (25) and (37) imply that \( \tilde{V}(t, y) \) satisfies (31).
Remark. It follows from (25), (30) and (34) that \( \tilde{V}(t, y) \) satisfies also the forward SPDE derived in [4], which takes in this case the following form

\[
\tilde{V}(t, y) = \int_0^t a(s, -\tilde{V}'(s, y))d(M)_s + \frac{1}{2} \int_0^t (\varphi'(s, -\tilde{V}'(s, y))^2\tilde{V}''(s, y)d(M)_s + \\
\frac{1}{2} \int_0^t (\varphi_1'(s, -\tilde{V}'(s, y)))^2\tilde{V}''(s, y)d(M_\perp)_s + \\
+ \int_0^t \tilde{\varphi}(s, -\tilde{V}'(s, y))dM_s + \int_0^t \tilde{\varphi}_1(s, -\tilde{V}'(s, y))dM_s^\perp
\]

4 Differential equation for the inverse flow of the optimal wealth

By Proposition 1, if the filtration \( F \) is continuous and Assumptions 1-3 are satisfied then the adapted inverse \( X_t^{-1}(x) \) of the optimal wealth process exists. Under stronger conditions we shall derive for the inverse process \( X_t^{-1}(x) \) a Stochastic Differential Equation (SDE) which will be used to show the absolutely continuity of bounded variation parts of \( V(t, x) \) and \( V'(t, x) \) with respect to square characteristic \( <S> \).

For stochastic process \( \xi_t(x) \) by \( \xi_t'(x) \) (or \( \partial_x \xi(t) \)) we denote the derivative with respect to \( x \), \( \mu^{(S)} \) denotes Dolean’s measure for \( \langle S \rangle \), i.e. the measure \( d\langle S \rangle dP \) on \([0, T] \times \Omega \). If \( F(t, x) \) is a family of semimartingales then \( \int_0^T F(ds, \xi_s) \) denotes a generalized stochastic integral (see [7]), or stochastic line integral by terminology from [2]. If \( F(t, x) = xG_t \), where \( G_t \) is a semimartingale then the generalized stochastic integral coincides with usual one denoted by \( \int_0^T \xi_s dG_s \) or \( (\xi \cdot G)_T \).

Now we shall derive an SDE for the inverse of the optimal wealth \( \psi_t(x) = X_t^{-1}(x) \) of the form

\[
d\psi_t = \sigma_t(\psi_t) dS_t + \mu_t(\psi_t) d\langle S \rangle_t, \quad \psi_0 = x, \quad (38)
\]

where \( \sigma_t(z) = -\frac{\pi_t(z)}{X_t(z)}, \quad \mu_t(z) = \frac{1}{2X_t(z)} \left( \frac{\pi_t^2(z)}{X_t^2(z)} \right)' \).

**Proposition 2.** Let \( X_t''(x), \pi_t'(x) \) exist \( \mu^{(S)} \)-a.e. and are locally Lipschitz functions with respect to \( x \) \( \mu^{(S)} \)-a.e.. Then SDE (38) or equivalently

\[
d\psi_t = -\frac{\pi_t(\psi_t)}{X_t(\psi_t)} dS_t + \frac{\pi_t'(\psi_t)\pi_t(\psi_t)}{X_t'(\psi_t)^2} d\langle S \rangle_t - \frac{1}{2} \frac{X_t''(\psi_t)\pi_t^2(\psi_t)}{X_t'(\psi_t)^3} d\langle S \rangle_t, \quad (39)
\]
admits a unique maximal solution and it coincides with $X_t^{-1}(x)$.

**Proof.** The SDE (38) admits unique maximal solution up to time $\tau(x) \leq T$, where $|\psi_{\tau(x)}(x)| = \infty$ if $\tau(x) < T$ (see [7]). Applying the Ito-Ventzel formula for $X_t(\psi_t) \equiv X(t, \psi_t)$ (see [7] or [15]) and using that $\psi_t$ satisfies (39) we get

\[
d\psi_t = X(dt, \psi_t) + X'(dt, \psi_t)d\psi_t + \frac{1}{2}X''(t, \psi_t)d\psi_t.
\]

Hence $X(t, \psi_t(x)) = x$ on $[0, \tau(x))$. Since $|X_t^{-1}(x)| < \infty$, we have $\tau(x) = T$ $P$-a.s. and $\psi_t(x) = X_t^{-1}(x)$.

**Remark 1.** Let $\pi_t(x) = H_t(X_t(x))$. Then

\[
d\psi_t = -\frac{H_t(X_t(\psi_t))}{X_t'(\psi_t)} X_t'(\psi_t) H_t(X_t(\psi_t)) dS_t - \frac{1}{2} \frac{X_t''(\psi_t) H_t^2(X_t(\psi_t))}{X_t'(\psi_t)^3} d\psi_t,
\]

Using equalities $X_t(\psi_t(x)) = x$, $\frac{1}{X_t'(\psi_t)} = \psi_t'(x)$, and $-\frac{X_t''(\psi_t(x))}{X_t'(\psi_t)} = \frac{\psi_t''(x)}{\psi_t'(x)^3}$, we obtain the linear Partial SDE (linear PSDE)

\[
d\psi_t(x) = -H_t(x)\psi_t'(x)dS_t + H_t'(x) H_t(x)\psi_t'(x)dS_t + \frac{1}{2} H_t^2(x)\psi_t''(x)d\psi_t,
\]

or a PSDE in the divergence form

\[
d\psi_t(x) = -H_t(x)\psi_t'(x)dS_t + \frac{1}{2} (H_t^2(x)\psi_t'(x))'d\psi_t.
\]

Let us define martingale random fields

\[
\mathcal{M}(t, x) = E[U(X_T(x)|F_t],
\]

\[
\overline{\mathcal{M}}(t, x) = E[U'(X_T(x)|F_t].
\]

**Proposition 3.** Let conditions of Proposition 2 be satisfied.
i) If $\mathcal{M}(t, x)$ is two times continuously differentiable with respect to $x$, then the finite variation part of $V(t, x) = \mathcal{M}(t, \psi_r(x))$ is absolutely continuous with respect to $\langle S \rangle$.

ii) If $\overline{\mathcal{M}}(t, x)$ is two times continuously differentiable with respect to $x$, then $V'(t, x)$ is a special semimartingale and the finite variation part of $V'(t, x) = \overline{\mathcal{M}}(t, \psi_r(x))$ is absolutely continuous with respect to $\langle S \rangle$. Besides $V'(t, x)$ admits the decomposition

$$V'(t, x) = V'(0, x) - \int_0^t a'(s, x) \, d\langle M \rangle_s + \int_0^t \psi'(s, x) \, dM_s + L'(t, x).$$  \hspace{1cm} (41)

**Proof.** i) By the optimality principle $V(t, X_t(x))$ is a martingale and since $V(T, x) = U(x)$ we have that $V(t, X_t(x)) = E[U(X_T(x))|F_t] = \mathcal{M}(t, x)$. Therefore by duality relation (9)

$$\mathcal{M}'(t, x) = V'(t, X_t(x))X_t'(x) = Z_t(y)X_t'(x)$$  \hspace{1cm} (42)

is a martingale and let

$$\mathcal{M}'(t, x) = V'(x) + \int_0^t h_t(x) \, dM_t + L_t(x), \quad L(x) \perp M$$

be the GKW decomposition of $\mathcal{M}'(t, x)$. From (39) we have

$$\left\langle \int_0^t \mathcal{M}'(dr, \psi_r(x)), \psi(x) \right\rangle = - \int_0^t h_t(\psi_r(x)) \frac{\pi_r(\psi_r(x))}{X_r'(\psi_r(x))} \, d\langle S \rangle_r.$$  \hspace{1cm} (43)

Since $V(t, x) = \mathcal{M}(t, X_t^{-1}(x))$, by the Ito-Ventzel formula we get

$$V(t, x) = V(0, x) + \int_0^t \mathcal{M}(ds, \psi_s) + \int_0^t \mathcal{M}'(s, \psi_s) \, d\psi_s$$  \hspace{1cm} (44)

$$+ \frac{1}{2} \int_0^t \mathcal{M}''(s, \psi_s) \, d\langle \psi \rangle_s + \left\langle \int_0^t \mathcal{M}'(dr, \psi_r(x)), \psi(x) \right\rangle_t$$

In view of (39) and (43) one can verify that all finite variation members of (44) are integrals with respect to $\langle S \rangle$. Namely,

$$-A(t, x)$$

$$= \int_0^t \mathcal{M}'(r, \psi_r(x)) \left( \frac{\pi_r(\psi_r(x)) \pi_r(\psi_r(x))}{X_r'(\psi_r(x))^2} - \frac{1}{2} X''_r(\psi_r(x)) \pi_r(\psi_r(x)) \right) \, d\langle S \rangle_r$$

$$+ \int_0^t \left( \frac{1}{2} \mathcal{M}''(r, \psi_r(x)) \frac{\pi_r^2(\psi_r(x))}{X_r'(\psi_r(x))^2} - h_r(\psi_r(x)) \frac{\pi_r(\psi_r(x))}{X_r'(\psi_r(x))} \right) \, d\langle S \rangle_r.$$
ii) It follows from (9) and (10) that
\[ M(t,x) = E[U'(X_T(x))|F_t] = E[Z_T(y)|F_t] = Z_t(y) = V'(t, X_t(x)), \]
which (together with (32)) implies that \( \mathcal{M} \) and \( \overline{\mathcal{M}} \) are related as
\[ \mathcal{M}'(t,x) = \overline{\mathcal{M}}(t,x) X_t'(x) \] (46)
and \( V'(t,x) = \overline{\mathcal{M}}(t, X^{-1}_t(x)) \). It follows from (45) that \( \overline{\mathcal{M}}(t,x) = Z'_t(y)V''(x) \)
is a martingale and
\[ \langle \int_0^t \overline{\mathcal{M}}(dr, \psi_r(x)), \psi(x) \rangle_t = - \int_0^t \bar{h}_r(\psi_r(x)) \frac{\pi_r(\psi_r(x))}{X'_r(\psi_r(x))} d\langle S \rangle_r, \]
(47)
where \( \overline{\mathcal{M}}(t,x) = \bar{V}''(x) + \int_0^t \bar{h}_r(x) dM_r + \bar{L}_t(x), \bar{L}(x) \perp M \) is the GKW decomposition of \( \overline{\mathcal{M}}(t,x) \). Therefore the Itô-Ventzel formula implies that \( V'(t,x) = \overline{\mathcal{M}}(t, X^{-1}_t(x)) \) is a special semimartingale and similarly to i) one can show that the finite variation part of \( V'(t,x) \) is absolutely continuous with respect to \( \langle S \rangle \). Therefore, \( V'(t,x) \) is decomposable as
\[ V'(t,x) = V'(0,x) + \int_0^t b(r,x) d\langle M \rangle_r + \int_0^t g(r,x) dM_r + N(t,x), \]
for some local martingale \( N(t,x) \) orthogonal to \( M \) for any \( x \in R \) and \( M \) and \( \langle M \rangle \) integrable processes \( g \) and \( b \) respectively. The Itô-Ventzel formula and conditions of this proposition also imply that \( b(r,x) \) and \( g(r,x) \) are continuous at \( x \). Therefore, integrating the equation (48) with respect to \( dx \) (over a finite interval) and using the stochastic Fubini theorem (taking decomposition (11) in mind), we obtain (41).

5 The case of complete markets

In this section for the case of complete markets we provide sufficient conditions on the utility function \( U \) which guarantee an existence of a solution of BSPDE (13).

Hereafter we shall assume that the market is complete, i.e.
\[ dQ = Z_T dP, \text{ where } Z_T = \mathcal{E}_T(-\lambda \cdot M) \]
is the unique martingale measure. Let
\[ R_1(x) = -\frac{U''(x)}{U'(x)}, \quad R_2(x) = -\frac{U'''(x)}{U''(x)}, \quad x \in \mathbb{R}. \]  
(49)

We shall use one of the following conditions:

r1) \( U \) is three-times differentiable, \( R_1(x) \) is bounded away from zero and infinity and \( R_2(x) \) is bounded and Lipschitz continuous.

r2) \( U \) is four-times differentiable and the density \( Z_T \) of the unique martingale measure is bounded.

**Lemma 2.** Let the market be complete and condition r1) be satisfied. Then the optimal wealth \( X_T(x) \) is two-times differentiable and the derivatives \( X'_T(x), X''_T(x) \) are bounded and Lipschitz continuous.

**Proof.** Since \( \tilde{U}(y) \) and \( U(x) \) are conjugate, \( \tilde{U}(y) \) is also three-times differentiable and
\[ \tilde{U}''(y) = -\frac{1}{U''(x)}, \quad \tilde{U}'''(y) = -\frac{U'''(x)}{(U''(x))^2}, \quad y = U'(x). \]  
(50)

Therefore the functions \( B_1(y) \) and \( B_2(y) \), where
\[ B_1(y) = y\tilde{U}''(y) = 1/R_1(x), \quad B_2(y) = y^2\tilde{U}'''(y) = R_2(x)/R_1^2(x) \]  
(51)
respectively, are also bounded. This implies that the second and the third order derivatives of \( \tilde{U}(yZ_T) \) are bounded, hence the function \( \tilde{V}(y) = E\tilde{U}(yZ_T) \) is three-times differentiable and
\[ \tilde{V}'''(y) = E^Q\tilde{U}'''(yZ_T)Z_T^2. \]

Since \( \tilde{V}(y) \) and \( V(x) \) are conjugate, \( V(x) \) is also three-times differentiable.

The duality relation (9) takes is in this case the following form
\[ U'(X_T(x)) = yZ_T, \quad X_T(x) = -\tilde{U}'(yZ_T), \quad y = V'(x). \]  
(52)

This relation implies that the function \( X_T(x) \) is two-times differentiable for all \( \omega \in \Omega' = (Z_T > 0) \) with \( P(\Omega') = 1 \) and differentiating the first equality in (52) we have that
\[ U''(X_T(x))X'_T(x) = V''(x)Z_T, \]  
(53)
\[ U'''(X_T(x))(X'_T(x))^2 + U''(X_T(x))X''_T(x) = V'''(x)Z_T. \]  

From (52) and (53) we obtain that  
\[ X'_T(x) = \frac{V''(x)}{V'(x)} \frac{U'(X_T(x))}{U''(X_T(x))}. \]

By condition r1) and Proposition 1.2 from [11] \( c_1 \leq -\frac{V''(x)}{V'(x)} \leq c_2 \). Therefore this implies that \( X'_T(x) \) is bounded, in particular  
\[ \frac{c_1}{c_2} \leq X'_T(x) \leq \frac{c_2}{c_1}, \]  
(55)

where \( c_1 \) and \( c_2 \) are constants from (14).

Comparing equations (53) and (54) we have that  
\[ X''(x) + \frac{U'''(X_T(x))}{U''(X_T(x))}(X'_T(x))^2 = \frac{V'''(x)}{V''(x)}X'_T(x). \]  
(56)

Since \( E^Q X'_T(x) = 1 \) and \( E^Q X''_T(x) = 0 \), taking expectations with respect to the measure \( Q \) in equation (56) we get  
\[ \frac{V'''(x)}{V''(x)} = E^Q \frac{U'''(X_T(x))}{U''(X_T(x))}(X'_T(x))^2, \]  
(57)

which together with (55) and condition r1) implies that \( \frac{V'''(x)}{V''(x)} \) is bounded.

Therefore, it follows from (56) that \( X''_T(x) \) is also bounded, hence \( X'_T(x) \) is Lipschitz continuous.

Since the product of bounded Lipschitz continuous functions are Lipschitz continuous, it follows from (57) that \( \frac{V'''(x)}{V''(x)} \) is Lipschitz continuous and (56) implies that \( X''_T(x) \) is also Lipschitz continuous, since all terms in (56) are bounded and Lipschitz continuous.

**Lemma 3.** Let the market be complete and condition r2) be satisfied. Then the optimal wealth \( X_T(x) \) is three-times differentiable, \( X'_T(x) \) is strictly positive and the derivatives \( X'_T(x), X''_T(x) \) and \( X'''_T(x) \) are uniformly bounded on every compact \([a, b] \in R \).

**Proof.** Since \( U(x) \) and \( \tilde{U}(y) \) are conjugate, Condition r2) implies that \( \tilde{U}(y) \) is also four times differentiable and the derivatives of \( \tilde{U}(yZ_T) \) are
bounded for any \( y \in R \), hence the function \( \tilde{V}(y) = E\tilde{U}(yZ_T) \) is four-times differentiable.

Then \( V(x) \) is also four-times differentiable, since \( V'(x) \) is the inverse of \( -\tilde{V}'(y) \). Therefore, the duality relation

\[
X_T(x) = -\tilde{U}'(V'(x)Z_T)
\]

implies that the optimal wealth \( X_T(x) \) is three-times differentiable and the derivatives \( X_T'(x), X_T''(x) \) and \( X_T'''(x) \) are bounded on every compact \([a, b] \in R \). Therefore the derivatives \( X_T'(x), X_T''(x) \) satisfy the local Lipschitz condition.

Besides,

\[
X_T'(x) = -V''(x)Z_T\tilde{U}''(V'(x)Z_T)) > 0
\]

since \( V''(x) < 0 \) and \( \tilde{U}''(y) > 0 \).

**Corollary 1.** The process \((X''_T(x), (t, x) \in [0, T] \times R)\) admits a continuous modification.

**Proof.** Since \( X''_T(x) \) is a \( Q \)-martingale, by the Doob inequality and the mean value theorem we get

\[
E^Q \sup_{t \leq T} |X''_T(x_1) - X''_T(x_2)|^2 \leq c_1 E^Q |X''_T(x_1) - X''_T(x_2)|^2 \\
\leq c_1 |x_1 - x_2| E^Q \sup_{\alpha \in [0,1]} |X''_T(\alpha x_1 + (1 - \alpha)x_2)|^2 \leq c_2 |x_1 - x_2|^2
\]

for some constants \( c_1, c_2 \). By the Kolmogorov theorem the map

\[
R \ni x \rightarrow X''_T(x) \in C[0, T]
\]

admits a continuous modification, which implies the continuity of \( X''_T(x) \) with respect to the variables \((t, x)\), \( P\)-a.s.. \( \square \)

**Proposition 4.** Assume that the market is complete and that one of the condition \( r1) \) or \( r2) \) is satisfied.

Then the optimal wealth \( X_t(x) \), the optimal strategy \( \pi_t(x) \) \((\mu^{(S)}\)-a.e.), martingale flows \( \mathcal{M}(t, x) \) and \( \mathcal{\overline{M}}(t, x) \) are two-times continuously differentiable at \( x \) for all \( t \), \( P\)-a.s. and the coefficients of equation (39) satisfy the local Lipschitz condition.

**Proof.** Let first assume that condition \( r1) \) is satisfied. According to Lemma 2 the optimal wealth \( X_T(x) \) is two-times differentiable and the derivatives \( X'_T(x), X''_T(x) \) are bounded and Lipschitz continuous.
To show an existence of $\pi'(x)$ we use the decomposition $X'_T(x) = 1 + \int_0^T \pi'_r(x) dS_r$ with some predictable $S$-integrable integrand $\pi^{(1)}(x)$ and inequalities

$$
E^Q \int_0^T \left( \pi^{(1)}_t(x + \varepsilon) - \pi^{(1)}_t(x) \right)^2 d\langle S \rangle_t = E^Q \langle X'_T(x + \varepsilon) - X'_T(x) \rangle_T 
$$

$$
= E^Q \left( X'_T(x + \varepsilon) - X'_T(x) \right)^2 \leq \varepsilon^2 E^Q \max_{0 \leq s \leq 1} |X''_T(x + s\varepsilon)|^2
$$

$$
\leq \varepsilon^2 \text{Const},
$$

By the Kolmogorov theorem $\pi^{(1)}(x)$ is continuous with respect to $x$ $\mu^{(S)}$-a.e.

Note that, if instead of r1) the condition r2) is satisfied, then we shall have that there exists a $\mu^{(S)}$-a.e. continuous modification of $\pi^{(1)}(x)$ on each compact of $R$ which will imply an existence of continuous modification on the whole real line.

Thus by the stochastic Fubini Theorem (see [15])

$$
x_2 - x_1 + \int_0^T (\pi_r(x_2) - \pi_r(x_1)) dS_r = X_T(x_2) - X_T(x_1)
$$

$$
= \int_{x_1}^{x_2} X'_T(x) dx = x_2 - x_1 + \int_0^T \int_{x_1}^{x_2} \pi'_r(x) dx dS_r
$$

and consequently $\pi_r(x_2) - \pi_r(x_1) = \int_{x_1}^{x_2} \pi'_r(x) dx$ $\mu^{(S)}$-a.e.. Hence $\pi^{(1)}(x) = \pi'(x)$ $\mu^{(S)}$-a.e. and

$$
X'_T(x) = 1 + \int_0^T \pi'_r(x) dS_r
$$

(58)

for all $x$ $P$-a.s.

It follows from (58) and from the Fubini theorem that

$$
X_t(x_2) - X_t(x_1) = x_2 - x_1 + \int_0^t (\pi_r(x_2) - \pi_r(x_1)) dS_r
$$

$$
= x_2 - x_1 + \int_0^t \int_{x_1}^{x_2} \pi'_r(x) dx dS_r = \int_{x_1}^{x_2} X'_t(x) dx
$$

for any $x_2 \geq x_1$ $P$-a.s. and lemma A3 from [11] implies that for each fixed $t$ there exists a modification of $(X_t(x), x \in R)$ which is absolutely continuous with respect to the Lebesgue measure $dx$. Since $(X'_t(x), t \in [0, T])$ is a $Q$-martingale

$$
|X'_t(x_2) - X'_t(x_1)| \leq E^Q(|X'_T(x_2) - X'_T(x_1)|/F_t) \leq C|x_2 - x_1|
$$

(59)
for any $x_2 \geq x_1 \ P\text{-a.s.}$ and Lemma 2 and Corollary 1 imply that there exists $\Omega' \subset \Omega, \ P(\Omega') = 1$, such that at each $\omega \in \Omega'$ the inequality (59) is fulfilled for all $(t,x)$. Since $EX'_T(x) = 0$ and the market is complete we have $X''_T(x) = \int_0^T \pi''(2)(x)dS_r$ for some predictable $S$-integrable integrand $\pi''(2)$. Similarly as above one can show that $\pi''(2)(x)$ is continuous at $x \mu^{<S>_x}$-a.e., $\pi''(2)(x) = \pi''(x) \mu^{<S>_x}$-a.e. and, hence $X''_T(x)$ admits the representation

$$X''_T(x) = \int_0^t \pi''(x)dS_r.$$ Similarly we can show that one can choose a modification of $X_T(x)$ which is two-times differentiable and such that $X''_T(x)$ is Lipschitz continuous.

In case when instead of r1) the condition r2) is fulfilled $X''_T(x)$ will satisfy the local Lipschitz condition. So, in both cases (i.e., if condition r1) or r2) is satisfied) the coefficients of equation (39) will be locally Lipschitz continuous.

Since the market is complete $\mathcal{M}(t,x) = V''(x)Z_t$ and it is evident that $\mathcal{M}(t,x)$ is two-times continuously differentiable. Besides, equality (46) implies that $\mathcal{M}(t,x)$ is also two-times continuously differentiable at $x$.

**Theorem 3.** Assume that the market is complete and that one of the condition r1) or r2) is satisfied. Then conditions a)-e) are fulfilled and the value function $V(t,x)$ satisfies BSPDE (13).

**Proof.** It is evident that boundedness of $B_1(y)$ and $B_2(y)$ (defined by (51)) implies that the dual value function $\tilde{V}(t,y) = E(\tilde{U}(y,Z_T)/F_t)$ is two-times continuously differentiable. Since

$$V''(t,x) = -\frac{1}{\tilde{V}'(t,y)}, \ y = V'(x),$$

the value function $V(t,x)$ is also two-times continuously differentiable, hence condition a) is fulfilled.

It follows from Proposition 4 that under the presence assumptions all conditions of Propositions 2 and 3 are satisfied, therefore these propositions imply that $V(t,x)$ satisfies conditions b) and c), hence $V(t,x)$ is a regular family of semimartingales.

Let us show that the condition e) is also satisfied. By optimality principle (see [10]) for any $t \in [0,T]$ the process $(V(s,X_s(t,x)), s \geq t)$ is a martingale, where $X_s(t,x) = x + \int_t^s \pi_u(t,x)dS_u$ is the solution of the conditional
optimization problem (13). This implies that $P$-a.s.

$$V(t, x) = E(V(s, X_s(t, x))/F_t). \tag{60}$$

On the other hand using again the optimality principle we have

$$V(t, X_t(x)) = E(V(s, X_s(x))/F_t),$$

and substituting in this equality the inverse of the optimal capital $X_t(x)$ we get

$$V(t, x) = E(V(s, X_s(X_t^{-1}(x))/F_t). \tag{61}$$

Since for any $t$ the function $(V(t, x), x \in R)$ is strictly convex, comparing (60) and (61) we obtain that $P$-a.s $X_s(t, x) = X_s(X_t^{-1}(x))$. By continuity at $(t, x)$ of $X_t^{-1}(x)$ as a solution of SDE (39) we obtain that condition e) is satisfied.

Thus, all conditions of Theorem 3.1 from [10] are satisfied which implies that $V(t, x)$ is a solution of the BSPDE (13).

**Corollary.** Let conditions of Theorem 3 be satisfied. Then the process

$$\tilde{V}(t, y) = E(\tilde{V}(yZ_T)/F_t), \quad t \in [0, T]$$

satisfies the BSPDE (31).

**Proof.** According to Theorem 2 it is sufficient to verify that the process

$$\tilde{V}'(t, y) = E(\frac{Z_T}{Z_t} \tilde{U}'(yZ_T)/F_t), \quad t \in [0, T],$$

is a special semimartingale.

Let $\nabla(t, y) = E(Z_T \tilde{U}'(yZ_T)/F_t)$. It is evident that $\nabla(t, y) = \frac{1}{Z_t} \nabla(t, \frac{yZ_T}{Z_t})$. But by the duality relation $\nabla(t, y) = E(Z_T \tilde{U}'(yZ_T)/F_t) = -Z_t X_t(t)$ and the martingale field $\nabla(t, y)$ is two-times differentiable by Proposition 4. Therefore the Itô-Ventzel formula implies that $\frac{1}{Z_t} \nabla(t, \frac{yZ_T}{Z_t})$ is a special semimartingale, hence so is also the process $\tilde{V}'(t, y)$.

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