Dynamics and stability conditions of semiconductor lasers under external optical feedback from both sides of the laser cavity

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Abstract

To increase the spectral efficiency of coherent communication systems, lasers with ever-narrower linewidths are required as they enable higher-order modulation formats with lower bit-error rates. In particular, semiconductor lasers are a key component due to their compactness, low power consumption, and potential for mass production. In field-testing scenarios their output is coupled to a fiber, making them susceptible to external optical feedback (EOF). This has a detrimental effect on its stability, thus it is traditionally countered by employing, for example, optical isolators and angled output waveguides. In this work, EOF is explored in a novel way with the aim to reduce and stabilize the laser linewidth. EOF has been traditionally studied in the case where it is applied to only one side of the laser cavity. In contrast, this work gives a generalization to the case of feedback on both sides. It is implemented using photonic components available via generic foundry platforms, thus creating a path towards devices with high technology-readiness level. Numerical results shows an improvement in performance of the double-feedback case with respect to the single-feedback case. In particular, by appropriately selecting the phase of the feedback from both sides, a broad stability regime is discovered. This work paves the way towards low-cost, integrated and stable narrow-linewidth integrated lasers.

1 Introduction

The effect of external optical feedback (EOF) on diode laser dynamics has been extensively studied for the past half century [1–13]. EOF has been proven to affect laser performance, showing regimes that can aid in linewidth reduction [14–17], as well as others responsible for highly unstable behavior, from mode hopping to the case of coherence collapse [18–23]. Methods to improve laser stability thus need to take EOF into account, as even weak feedback can be detrimental. A traditional approach to mitigate its effects is to include an off-chip isolator at the laser output. Yet, this component negatively impacts the dimensions of packaged devices as well as fabrication times and costs. As such, research is ongoing to develop an integrated solution that can minimize the negative effects of EOF. Efforts include adjusting the feedback phase to tune into line-narrowing regimes [24, 25], using unidirectional phase modulators [26, 27], reducing the linewidth enhancement factor e.g. using quantum dots [28–30], employing electromagnetic effects [31, 32], harnessing the mode propagation properties of ring lasers [33, 34], or the extended cavity approach [35–38].
The established line of thought relies on the key assumption that feedback is introduced from only one side of the laser cavity. Current integration technologies make this assumption obsolete, as they allow for arbitrarily complex design geometries with a variety of functionalities, such as tunability and modulation, while maintaining narrow-linewidth performance [39–43]. Consequently, this work aims to extend the theoretical foundations of EOF to the case of feedback coupling into the laser cavity from both sides. This system is studied to obtain and analyze its dynamic rate equations. Furthermore, the frequency noise power spectral density and subsequently the intrinsic linewidth’s dependence on feedback is computed. The Lang-Kobayashi approach is used [44], where an additional term to account for the extra feedback term is included. In a similar fashion, to obtain analytical solutions both small-signal and weak feedback conditions are assumed. The obtained equations are then numerically solved. Results show the existence of a feedback-insensitive regime achieved by tuning the feedback parameters. This regime is feasible using mature components of photonic integrated circuits available in generic foundry platforms [45, 46], thus creating a path towards devices with high technology-readiness level while maintaining low cost and size.

2 Rate equations model

This section includes a comprehensive derivation the dynamic equations of a laser cavity with EOF under the revised assumption that feedback couples into the laser cavity from both sides. Starting from the Lang-Kobayashi model [44], the lasing frequency and threshold gain shifts, as well as the change of intrinsic linewidth due to feedback are obtained.

In contrast with previous literature, this work proposes a revised laser system, as shown in Fig. 1. The laser cavity of length $L$ is delimited by two mirrors with complex reflection coefficients $\rho_1$ and $\rho_2$ respectively. Assuming two interfaces at each side of the main cavity, two additional back-reflections ($\rho_{1,ext}$ and $\rho_{2,ext}$) have to be considered, which can be accounted for by calculating effective reflection coefficients. To simplify the problem, this analysis does not take into account multiple reflections in the external cavities. This is justified for:

$$|\rho_{j,ext}\rho_j| \ll 1,$$

which includes both weak feedback $|\rho_{j,ext}| \ll |\rho_j|$ and strong feedback $|\rho_{j,ext}| \gg |\rho_j|$.

The following parameters are introduced:

$$\kappa_j \equiv \frac{1 - |\rho_j|^2}{\rho_j} t_{cav},$$

$$t_{cav} = \frac{2L}{v_g},$$

$$\phi_j \equiv \omega_{FB} t_j + \phi_{mj},$$

$$\delta\omega \equiv \omega_{FB} - \omega_{ref},$$

with $j = 1, 2$, where $\kappa_j$ is the coupling coefficient; $t_{cav}$ is the cavity roundtrip time, with the group velocity $v_g$; $\phi_j$ is the phase delay due to the external cavities determined by the external roundtrip time $t_j$, the lasing frequency in the presence of feedback $\omega_{FB}$, and a phase shift at the external mirrors $\phi_{mj}$; $\omega_{ref}$ is the free running laser frequency; $\beta$ is the propagation constant; $n$ the effective refractive index of the lasing mode; and $c$ the speed of light in vacuum. The first step is extracting the lasing conditions of the proposed laser system.
2.1 Lasing conditions

By analyzing $A$, the slowly-varying amplitude of the complex electric field drawn in Fig. 1, the effective reflection coefficients ($\rho_j^{eff}$) are obtained in Appendix A:

$$\frac{\rho_j^{eff}}{\rho_j} \approx 1 + \kappa_j t_{cav} e^{\pm i\phi_j} = 1 + \kappa_j t_{cav} \cos(\phi_j) \pm i\kappa_j t_{cav} \sin(\phi_j),$$

(2.3)

where the plus and minus signs correspond to $j = 1$ (left mirror) and $j = 2$ (right mirror), respectively. The additional reflection influences the lasing condition, as shown in Appendix B:

$$\left(\frac{\rho_2}{\rho_1}\right)^2 \approx \rho_2 \rho_1 \left[1 + \kappa_2 t_{cav} \cos(\phi_2)\right]\left[1 + \kappa_1 t_{cav} \cos(\phi_1)\right] e^{(\Gamma g_{FB} - \alpha)L},$$

(2.4)

and

$$2\pi m \approx 2\beta L + \kappa_2 t_{cav} \sin(\phi_2) - \kappa_1 t_{cav} \sin(\phi_1), \quad m \in \mathbb{Z},$$

(2.5)

where $\Gamma$ is the confinement factor, $\alpha$ is the attenuation coefficient, and $g_{FB}$ is the threshold gain coefficient with feedback. The interplay between the feedback parameters $\kappa_1$, $\kappa_2$, $\phi_1$ and $\phi_2$ determines the dynamics and stability of the system. The calculations to obtain the threshold gain reduction and lasing frequency shift due to EOF are shown in Appendix C. The following definitions are convenient:

$$G_{FB} \equiv \Gamma g_{FB} v_g$$

(2.6a)

$$\gamma_H \equiv \sqrt{1 + \alpha_H^2}$$

(2.6d)

$$G_{th} \equiv \Gamma g_{th} v_g$$

(2.6b)

$$\theta_H \equiv \arctan(\alpha_H)$$

(2.6e)

$$\delta G \equiv G_{FB} - \frac{1}{\tau_{ph}}$$

(2.6c)

where $\tau_{ph}$ is the photon decay time, which accounts for cavity and mirror losses; $g_{th}$ is the threshold gain without feedback; and $\alpha_H$ is the linewidth enhancement factor [47]. Thus, from Appendix C, it is possible to obtain:

$$\delta G \equiv G_{FB} - G_{th} \approx -2\kappa_2 \cos(\phi_2) - 2\kappa_1 \cos(\phi_1)$$

(2.7a)

and

$$\delta \omega \approx -\gamma_H \left[\kappa_2 \sin(\phi_2 + \theta_H) + \kappa_1 \sin(\theta_H - \phi_1)\right],$$

(2.7b)

where both $\phi_j$ and $\delta \omega$ depend on the lasing frequency. The relation between the right-hand terms determines the shift in threshold gain and lasing frequency with feedback. Given the
transcendental form of Eq. (2.7b), a numerical analysis under different feedback conditions is studied in Sec. 3. Nevertheless, an analytical solution for lasing frequency stability can be found for the condition:

$$\delta \omega = 0 \Rightarrow \omega_{FB} = \omega_{ref}. \quad (2.8)$$

Under this condition, Eq. (2.7b) can be rewritten as:

$$\kappa_2 \sin(\phi_2 + \theta_H) = -\kappa_1 \sin(\theta_H - \phi_1), \quad (2.9)$$

which is determined by the feedback parameters $\kappa_j$ and $\phi_j$, the latter being dependent on the time delay $t_j$ as well as the lasing frequency. Finding a stable solution that does not depend on the lasing frequency is of particular interest, as it can be advantageous for tunable laser and their numerous applications. With the following assumption:

$$\kappa_2 = \kappa_1 = \kappa, \quad (2.10)$$

Eq. (2.9) can be rewritten as:

$$\sin(\phi_2 + \theta_H) = -\sin(\theta_H - \phi_1) \quad (E.5)$$

$$\Rightarrow \phi_2 + \theta_H = \phi_1 - \theta_H + 2m\pi. \quad (2.11)$$

Without loss of generality, the parameter $m$ is set to $m = 0$, thus:

$$2\theta_H = \phi_1 - \phi_2 \quad (2.2b)$$

$$\Rightarrow 2\theta_H - (\phi_{m1} - \phi_{m2}) = \omega_{FB}(t_1 - t_2). \quad (2.12)$$

Choosing an equal time delay (i.e. length) in both external cavities:

$$t_2 = t_1 = t_{ext}, \quad (2.13)$$

sets the left hand term of Eq. (2.12) equal to zero, so that:

$$\Delta \phi_m = \phi_{m1} - \phi_{m2} \quad (2.14)$$

$$2\theta_H = \Delta \phi_m. \quad (2.15)$$

This result shows that by tuning the phase in the external cavities, so that condition (2.15) is met, it is possible to obtain a feedback-insensitive lasing frequency. An active method to tune the phase is however required as $\alpha_H$ is dependent on laser parameters, such as carrier density and wavelength [48]. This can be managed by e.g. phase shifters, which are mature and widely used components that can be included on-chip in a laser.

The shown stable solution thus requires meeting the conditions (2.10), (2.13) and (2.15), which constrain the feedback parameters of one side of the cavity with respect to those of the other side, but do not restrict their absolute value. Nevertheless, if conditions (2.10) and (2.13) are not met, solutions for stable performance become frequency dependent. This case would thus only be satisfied for certain lasing frequency values for a given set of feedback parameters, which can potentially yield unstable solutions for other frequencies.

Finally, looking at the (Eq. 2.7a), under conditions (2.10) and (2.13) Eq. (2.7a) becomes:

$$G_{FB} - G_{th} = -2\kappa \left[ \cos(\omega_{ref}t_{ext} + \phi_{m1}) + \cos(\omega_{ref}t_{ext} + \phi_{m2}) \right], \quad (2.16)$$
which becomes zero if:
\[
\cos(\omega \text{ref} t_{\text{ext}} + \phi_{m_1}) = -\cos(\omega \text{ref} t_{\text{ext}} + \phi_{m_2})
\]
\[
\cos(\omega \text{ref} t_{\text{ext}} + \phi_{m_1}) = \cos(\omega \text{ref} t_{\text{ext}} + \phi_{m_2} - \pi)
\]
\[
\Rightarrow \omega \text{ref} t_{\text{ext}} + \phi_{m_1} = \omega \text{ref} t_{\text{ext}} + \phi_{m_2} - \pi + 2m\pi \quad m \in \mathbb{N}
\]
\[
m=0 \Rightarrow \Delta \phi_m = \pi. \quad (2.17)
\]
This condition, while different than condition (2.15), also yields stability regardless of lasing frequency in this case for the threshold gain. Both cases are studied numerically in Sec. 3.

2.2 Rate equations

In order to obtain the frequency noise (FN) power spectral density (PSD), and from it the laser linewidth, the laser rate equations for the intensity and phase, as well as one for the carrier density need to be studied. The former two can be extracted from the dynamic equations for the field inside the laser cavity, following the Lang-Kobayashi [44] approach. Its full derivation is shown in Appendix D. Furthermore, Langevin noise terms are included to account for shot noise fluctuations. The following definitions are useful to simplify notation:

\[
A(t) = \sqrt{S(t)} e^{-i\phi(t)}, \quad (2.18a)
\]
\[
\Delta \Phi^+_1 = \phi(t) - \phi_{t_1}^+ - \phi_1 \quad (2.18d)
\]
\[
\Delta \Phi^-_2 = \phi(t) - \phi_{t_2}^- + \phi_2 \quad (2.18e)
\]
\[
\Delta \Phi^+_2 = \phi(t + t_j) \quad (2.18c)
\]

with \(j = 1, 2\) relating to the EOF components from the right and left respectively. Assuming that the field amplitude \(A\) is slowly varying, where \(S\) is the photon number inside the laser cavity and \(\phi\) is the phase of the field, the rate equations of the system can be written as:

\[
\dot{S} = S \Delta G + 2S_2 \sqrt{S} \cos(\Delta \Phi^-_2) + 2S_1 \sqrt{S} \cos(\Delta \Phi^+_1) + R_{sp} + F_S \quad (2.19a)
\]
\[
\dot{\phi} = \frac{\alpha_n \Delta G}{2} - \delta \omega - \frac{S^-}{\sqrt{S}} \sin(\Delta \Phi^-_2) - \frac{S^+}{\sqrt{S}} \sin(\Delta \Phi^+_1) + F_\phi \quad (2.19b)
\]
\[
\dot{N} = I - GS(t) - N \tau_{sp}^{-1} + F_N, \quad (2.19c)
\]

where \(I\) is the effective rate of injected current (in electrons), \(\tau_{sp}\) is the carrier lifetime, and \(R_{sp}\) is the spontaneous recombination rate. The Langevin noise sources \(F_S(t), F_\phi(t)\) and \(F_N(t)\) satisfy [49]:

\[
\langle F_i(t) \rangle = 0 \quad (2.20a)
\]
\[
\langle F_i(t_1) F_j(t_2) \rangle = 2D_{ij} \delta(t_1 - t_2) \quad \text{with}\ i, j = S, \phi \text{or} N, \quad (2.20b)
\]
where:

\[
D_{SS} = R_{sp} S ; \quad D_{\phi\phi} = \frac{R_{sp}}{4S} ; \quad D_{NN} = R_{sp} S + N \tau_{sp}^{-1} ; \quad D_{SN} = -R_{sp} S, \quad (2.21)
\]
are standard diffusion coefficients. Using the Fourier transform:

\[
\hat{f}(\Omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\Omega t} \, dt,
\]

(2.22)

Eq. (2.20b) is rewritten the frequency domain as:

\[
\langle \hat{F}_i(\Omega_1)\hat{F}_j^*(\Omega_2) \rangle \equiv 2D_{ij}\delta(\Omega_1 - \Omega_2).
\]

(2.23)

A usual approach for solving the system from Eq. (2.19) involves small-signal analysis. Small deviations from a steady-state value are assumed:

\[
S \simeq S_0 + S_{\Delta} = S_0 + \int_{-\infty}^{\infty} e^{i\Omega t} S_{0p}(\Omega')d\Omega' \quad \text{with} \quad S_0 \gg S_{\Delta}
\]

(2.24a)

\[
\phi \simeq \phi_{\Delta} = \int_{-\infty}^{\infty} e^{i\Omega t} \phi_{0p}(\Omega')d\Omega'
\]

(2.24b)

\[
N \simeq N_0 + N_{\Delta} = N_0 + \int_{-\infty}^{\infty} e^{i\Omega t} N_{0p}(\Omega')d\Omega' \quad \text{with} \quad N_0 \gg N_{\Delta},
\]

(2.24c)

where the steady state value of the phase is assumed to be zero. The full linearization of the rate equations is shown in Appendix E, which uses the following definitions:

\[
\kappa^c_j \equiv \kappa_j t_j \cos(\phi_j) \quad \text{(2.25a)}
\]

\[
\kappa^s_j \equiv \kappa_j t_j \sin(\phi_j) \quad \text{(2.25b)}
\]

\[
K_s \equiv \kappa^s_2 + \kappa^s_1 \quad \text{(2.25c)}
\]

\[
K_c \equiv 1 + \kappa^c_2 - \kappa^c_1 \quad \text{(2.25d)}
\]

\[
\zeta_s \equiv R_{sp}/S_0 \quad \text{(2.25e)}
\]

\[
a_g = \Gamma v_g a \quad \text{(2.25f)}
\]

\[
G_i \approx a_g(N_i - N_{tr}) \quad \text{(2.25g)}
\]

\[
\tau_e^{-1} \equiv a_g S_0 + \tau_{sp}^{-1} \quad \text{(2.25h)}
\]

where a linear approximation for the gain has been introduced, with \(a\) the differential gain coefficient and \(N_{tr}\) the number of electrons at transparency. Applying the Fourier transform from Eq. (2.22) to Eqs. (E.10a), (E.10b) and (E.10c), the following system of equations is obtained in the frequency domain:

\[
i\Omega K_c S_{0p} \overset{\text{(E.10a)}}{=} a_g S_0 N_{0p} - \zeta_s S_{0p} - 2i\Omega S_0 K_s \phi_{0p} + \hat{F}_S
\]

(2.26a)

\[
2i\Omega K_c \phi_{0p} \overset{\text{(E.10b)}}{=} \alpha_\Omega a_g N_{0p} + i\Omega \frac{K_s}{S_0} S_{0p} + 2\hat{F}_\phi
\]

(2.26b)

\[
i\Omega N_{0p} \overset{\text{(E.10c)}}{=} -\tau_e^{-1} N_{0p} - G_0 S_{0p} + \hat{F}_N,
\]

(2.26c)

where the unknowns \(S_{0p}, \phi_{0p}, N_{0p}\), and \(\hat{F}_S, \hat{F}_\phi\) and \(\hat{F}_N\) depend on the fourier frequency \(\Omega\). These equations are the first step to obtain the FN PSD.
2.3 Power spectral density and laser intrinsic linewidth

The next step is to find an expression for $\phi_{0p}$ from which the FN PSD, and thus the laser intrinsic linewidth, can be computed. Defining:

$$A_\phi \equiv (i\Omega + \tau_e^{-1}) \left( i\Omega K_c + \frac{a_g G_0}{i\Omega + \tau_e^{-1} S_0} + \zeta_s \right)$$  (2.27a)

$$2A_S \equiv i\Omega K_s \frac{i\Omega + \tau_e^{-1}}{S_0} - a_h a_g G_0$$  (2.27b)

$$2A_N \equiv \frac{\alpha_h A_\phi + 2S_0 A_S}{i\Omega + \tau_e^{-1}} a_g$$  (2.27c)

$$B_{\phi} \equiv K_c A_\phi - a_h a_g G_0 S_0 K_s + i\Omega K_s^2 (i\Omega + \tau_e^{-1})$$  (2.27d)

the expression for $\phi_{0p}$, as shown in Appendix F, can be written as:

$$\phi_{0p} (F.3) = A_N \hat{F}_N + A_S \hat{F}_S + A_\phi \hat{F}_\phi$$  (2.28)

From Eq. (2.28) it is possible to calculate an expression for the PSD [50]:

$$S_f^{(1)}(\Omega) = \frac{\Omega^2}{2\pi^2} \langle |\phi_{0p}(\Omega)|^2 \rangle,$$  (2.29)

which, using the following definitions:

$$F_0 \equiv \zeta_s^2 + K_c^2 \tau_e^{-2} - 2K_c a_g G_0 S_0,$$  (2.30a)

$$F_2 \equiv K_c a_h + K_s$$  (2.30c)

$$F_1 \equiv K_s^2 + K_c^2$$  (2.30b)

$$F_3 \equiv K_c - a_h K_s,$$  (2.30d)

and:

$$A_4 \equiv 4F_1 D_{\phi \phi}$$  (2.30e)

$$A_2 \equiv a_h^2 F_2^2 D_{NN} + 4D_{\phi \phi} (K_c^2 \tau_e^{-2} + F_0) - \frac{2K_c}{S_0} D_{SS} a_g \left( \tau_e^{-1} F_2 - \zeta_s \alpha_h + \alpha_h G_0 \right)$$  (2.30f)

$$A_0 \equiv a_h^2 \left[ D_{SS} \left( \zeta_s^2 + G_0^2 + 2\zeta_s G_0 \right) + \zeta_s \tau_e^{-1} \tau_{sp} + \left( \tau_e^{-1} \tau_s + a_g G_0 S_0 \right)^2 \right] 4D_{\phi \phi}$$  (2.30g)

can be written as:

$$2\pi^2 S_f^{(1)} (2.30)(F.15) = \frac{\Lambda_4 \Omega^4 + \Lambda_2 \Omega^2 + \Lambda_0}{2|B_\phi|^2}.$$  (2.31)

Finally, from the following expression: [6]

$$S_f^{(1)} (f \to 0) = 2\pi \Delta f,$$  (2.32)

which is valid for a Lorentzian lineshape, the intrinsic linewidth can be obtained. As shown in Appendix G:

$$F \equiv \frac{\Delta f}{\Delta f_0 (1 + \alpha_H^2)} \overset{(G.3)}{=} \left[ 1 + \gamma_h \kappa_2 t_2 \cos (\phi_2 + \theta_H) - \gamma_h \kappa_1 t_1 \cos (\phi_1 - \theta_H) \right]^{-2},$$  (2.33)
where $\Delta f_0$ is the Schawlow–Townes linewidth [51]. The expression found for the intrinsic linewidth has two feedback terms that account for one contribution from each side, with a sign that depends on $\phi_1$ and $\phi_2$. Recalling from Eq. (2.2b) that these quantities are a function of $t_j$ and $\phi_m$, a proper design of the laser can yield linewidth stability or a reduction of the intrinsic linewidth with respect to the case of one-sided feedback. This is further explored using a numerical analysis in Sec. 3. Additionally, it is possible to find an analytical expression for Eq. (2.33) under the conditions for frequency stability, namely conditions (2.10), (2.13) and (2.15). Using this assumptions in Eq. (2.33):

$$F = \left\{1 + \gamma_H \kappa t_2 \left[ \cos (\phi_2 + \theta_H) - \cos (\phi_1 - \theta_H) \right] \right\}^{-2}. \quad (2.34)$$

Taking a closer look at the feedback terms yields:

$$\cos (\phi_2 + \theta_H) = \omega_{ref} t_2 + \phi_m - 2\theta_H + \theta_H = \cos (\phi_1 - \theta_H). \quad (2.35)$$

Using Eq. (2.35) in Eq. (2.34) yields a value of $F = 1$ which indicates that, under the assumed conditions, the intrinsic linewidth is insensitive to feedback. This result is significant as under the same condition the frequency is also feedback-insensitive, as shown in Sec. 2.1, regardless of lasing frequency. It is worth noting however that weak feedback was assumed in this analysis, with which the upper bound for feedback strength under which these equation are valid is not established. Nevertheless, achieving stability in the full frequency domain even under this condition is an improvement with respect to the single feedback case.

3 Numerical study

Laser stability is studied by numerically evaluating the equations for the shift in lasing frequency, threshold gain and intrinsic linewidth under the revised EOF conditions, namely Eqs. (2.7) and (2.33), under different feedback parameters. Particular attention is given to the previously analyzed case under conditions (2.10), (2.13) and (2.15), which shows stable solutions for the lasing frequency and intrinsic linewidth. System tolerances to each of these conditions are explored by varying each while keeping the other two fixed. The simulated equations are plotted as a function of the unperturbed laser frequency multiplied by $t_2$ which, given the periodicity of the functions, is thus kept between 0 and 1 (i.e. $\omega_{ref} t_2 \in (0, 2\pi)$). Additionally, simulations assume $\alpha_H = 3$. This value is compatible with measurements for semiconductor lasers [52], and thus meeting condition (2.15) requires that $\Delta \phi_m = 2\theta_H \approx 2.5$.

Results are compared with the case with feedback from a single side, in which parameter:

$$\kappa_1 = 0. \quad (3.1)$$

In this case, as shown in [18], as feedback strength increases, solutions for the lasing frequency become multi-valued. This gives rise to instabilities in the system such as mode hopping or coherence collapse regimes. The separation between single-valued and multi-valued solutions is related to the coefficient:

$$C = \gamma_H \kappa_2 t_2, \quad (3.2)$$

where $C = 1$ is the critical value that separates both behaviours.

Simulation results of the proposed system under conditions (2.10) and (2.13) are thus compared to the single feedback case for three cases:

Case 1: $C = 0.5$. This represents the single-feedback case with a single solution, and results for various values of $\Delta \phi_m$ are shown in Fig. 2. Column A, B and C show the variations in lasing frequency, threshold gain and intrinsic linewidth respectively, and the blue and orange plots represent the full system and the single-feedback case respectively. These labels are maintained throughout the document. The upper row shows the case where $\Delta \phi_m = 0$, i.e. there is no additional phase difference between the external cavities. Linewidth narrowing for
Figure 2: Simulations for $C = 0.5$ under conditions (2.10) and (2.13) for different $\Delta \phi_m$. Full solution shown in blue, single feedback case shown in orange. Column A shows the lasing frequency shift results. Column B shows the threshold gain shift. Column C shows the intrinsic linewidth variations.

a wide range of frequencies can be observed, evidenced by negative values, whose magnitude is higher than in the single feedback case. Additionally, the signal is singled valued in the full domain, yet it is close to the critical point where multi-valued solutions arise. As $\Delta \phi_m$ increases, the amplitude of the lasing frequency shift is reduced until becoming zero for all values when condition (2.15) is met, as expected from previous analysis. Under this condition the intrinsic linewidth does not experience fluctuations either, and the amplitude of the threshold gain fluctuations is lower than in the single feedback case, indicating better stability across the three analyzed parameters with respect to the single feedback case. For the case of $\Delta \phi_m = \pi$ the threshold gain shows no fluctuations as predicted by Eq. (2.17), and while the lasing frequency and intrinsic linewidth fluctuations are no longer zero, they are less pronounced than in the single feedback case. Further increases in $\Delta \phi_m$ show an increase in the fluctuations across all functions, and for $\phi_m > 6$ multi-valued solutions arise.

Case 2: $C = 1$. This represents the limiting case between single and multi-valued solutions in the single-feedback case. Results for various values of $\Delta \phi_m$ are shown in Fig. 3. As expected, the threshold gain is stable for $\Delta \phi_m = \pi$, and meeting condition (2.15) results in a stable lasing frequency and intrinsic linewidth. As $\Delta \phi_m$ deviates from these optimal points in either direction, the amplitude of fluctuations increase until reaching multi-valued solutions for $\Delta \phi_m < 1.5$ and $\Delta \phi_m > 3.5$. Comparing these results with the previous case shows that as feedback increases, the single valued solutions become more sensitive to the value of $\Delta \phi_m$.

Case 3: $C = 1.3$. This represents the single-feedback case with multi-valued solutions.
Results for various values of $\Delta \phi_m$ are shown in Fig. 4. In the single feedback case, multi-valued solutions are present in a given frequency range, and this span increases with increasing feedback strength. The multi-valued characteristics are evidenced experimentally with unstable regimes characterized by mode hopping and eventually coherence collapse for sufficiently high feedback. In contrast, the system proposed in this work shows that by tuning the value of $\Delta \phi_m$ to meet condition (2.15), even with increasing feedback it is possible to achieve stable performance regardless of frequency. In the case shown in Fig. 4 for $C = 1.3$, single valued solutions can be found for $\Delta \phi_m \in (1.5, 3.3)$ which is equivalent to a phase variation of more than 90°. Still, comparing with previous cases it is possible to see that as feedback increases, the single valued solutions tolerance with respect to $\Delta \phi_m$ decreases. Nevertheless, it is an improvement with respect to the single feedback case which shows no single value solutions across all frequencies for $C > 1$.

Taking all cases into account, it can be seen that linewidth narrowing regions are present for all cases of $\Delta \phi_m$ analyzed. The only exception are the stable cases when meeting the three conditions (2.10), (2.13) and (2.15), yet the feedback-insensitivity provided by this case is also beneficial. Selecting an appropriate $\Delta \phi_m$ can thus be used to harness linewidth narrowing properties at a desired frequency.

Furthermore, system tolerances to conditions (2.10) and (2.13) are studied, while maintaining condition (2.15). Results for 20% deviation are shown in Figs. (6), (7) and (8) for cases 1, 2 and 3 respectively, where single valued solutions are found in all cases. Results show a
Figure 4: Simulations for $C = 1.3$ under conditions (2.10) and (2.13) for different $\Delta \phi_m$. Full solution shown in blue, single feedback case shown in orange. Column A shows the lasing frequency shift results. Column B shows the threshold gain shift. Column C shows the intrinsic linewidth variations.

high tolerance with respect to feedback strength. Looking at case 3, single valued solutions are obtained for $\kappa_1/\kappa_2 \in (0.2, 1.7)$. While the system is no longer feedback-insensitive, results evidence single-valued solutions that are robust with respect to condition (2.10). This stable case is more sensitive with respect to (2.13), with single valued solutions achieved within $t_1/t_2 \in (0.47, 1.2)$. Nevertheless, tolerances become once again stricter as feedback increases for both parameters, thus laser design is of paramount importance and should focus on meeting the discussed stability conditions. In particular, choosing equal lengths for both external cavities should be enough to meet condition (2.13). Fabrication tolerances in foundry processes are the main limiting factor for time delay accuracy. To meet condition (2.10), a possible approach is to merge the output of the two external cavities into a single one using a coupler, which can be included in the laser on-chip. Finally, as mentioned before, condition (2.15) can be met by using a phase shifter, which is a mature component in active platforms.

All in all, results demonstrate that, with proper design of the laser cavity the conditions (2.10), (2.13) and (2.15) can be met, with which it is possible to obtain lasing frequency and intrinsic linewidth insensitivity to feedback.
4 Discussion

The existing mature photonic integration fabrication processes are very flexible with respect to device geometry. They are however limited by a lack of commercially available on-chip isolators, and thus new approaches are required to minimize the effect feedback has on laser stability. The current work proposes a theoretical extension of laser dynamics under EOF by considering two external reflections, one from each side of the cavity, instead of the single feedback approach explored in previous literature. The proposed analysis yields new laser dynamic equations. These are numerically solved, which show the existence of a stable regime in terms of lasing frequency and intrinsic linewidth, with high feedback tolerances. Moreover, feedback-insensitivity is achievable under conditions that can be met with current laser fabrication processes and components, such as phase shifters for meeting condition (2.15) and couplers for condition (2.10). It has previously been shown, for the single feedback case, that tuning the feedback phase can result in linewidth narrowing [24], however this still requires low feedback levels and a precise phase shift which can suffer variations due to external parameters, such as temperature or driving currents. In the proposed approach, the stability conditions do not require specific values, instead relating the feedback parameters from one side to those from the other side. For example, condition (2.13) only implies equal roundtrip times at the external cavities, regardless of value. This allows for additional flexibility in the feedback parameters and gives versatility to the device. Furthermore, this method allows for feedback-insensitivity across the full spectra, which is not seen in the single feedback case. This is of particular importance for tunable lasers, as all lasing frequencies are thus equally affected. Additionally, it relaxes the need for an isolator, reducing the cost and size of packaging processes. Another significant improvement of the proposed method with respect to the single feedback case is the increase in feedback tolerance: higher levels of feedback strength are allowed without seeing multi-valued solutions, which result in mode-hopping seen experimentally. As a weak feedback approximation is used, the upper bound for feedback tolerance cannot be extracted from this analysis. Despite this, even under this approximation, tolerances are higher than that of the single feedback case. Furthermore, this system has a high tolerance to deviations from the optimal stability conditions as analyzed in the previous section. Linewidth narrowing can be achieved in these cases, for certain frequency values which can be tuned by selecting appropriate feedback parameters, as was the case for single feedback conditions, while maintaining stable solutions.

Finally, while the dynamics under consideration are complex, the laser system itself involves a straight-forward configuration using widely used components, which are available in generic foundry platforms. Previously studied methods to reduce feedback sensitivity include resourceful yet intricate designs. The proposed system is, in contrast, potentially easier to design, fabricate and characterize. An experimental study of this laser system is essential to validate the obtained results, and more importantly to explore the limitations of the model, and is the next step for a more comprehensive understanding of the proposed system.

5 Conclusions

This work explores an extension of the theoretical background of EOF. By assuming that feedback couples into the laser cavity from both sides, new dynamic equations are found for the lasing frequency, the threshold gain and the intrinsic linewidth. These are numerically evaluated to analyze laser stability. Results show the existence of a stable solution, with feedback-insensitive lasing frequency and intrinsic linewidth, regardless of the lasing frequency. This case is obtained by tuning the phase of the feedback field, for external cavities with equal lengths and coupling factors. Furthermore, the feedback-insensitive case exists regardless of the feedback strength, within a weak feedback approximation, which is an major improvement with respect to the single feedback case. Furthermore, the stability conditions shows good
tolerances with respect to all feedback parameters, albeit they become stricter as the feedback strength increases. Additionally, solutions with linewidth reduction are observed. In particular, choosing feedback parameters close to the feedback-insensitive conditions ensure stable solutions that are feedback tolerant. Finally, the proposed system relies on few components in straightforward configurations, and the stable conditions can be met with mature components available in generic foundry platforms. This enables close to market, low cost, feedback tolerant semiconductor lasers and has direct applications in multiple fields which rely on stable laser sources, such as coherent communications and spectroscopy.

A EFFECTIVE REFLECTION COEFFICIENTS

Consider the slowly-varying amplitudes $A$ shown in Fig. 1, where a slowly varying electric field is assumed:

$$E(t) = A(t)e^{-i\omega_{FB}t}, \quad (A.1)$$

in which the lasing frequency with feedback is $f_{FB} = \frac{\omega_{FB}}{2\pi}$. The relation between $A_{1}^{\pm}$ and $A_{1}^{\pm}$ is:

$$A_{1}^{-}e^{-i\phi_{2}/2} = A_{1}^{-} \quad (A.2a) \quad A_{1}^{+} = A_{1}^{+}e^{-i\phi_{2}/2}. \quad (A.2b)$$

Using the transmission coefficient:

$$\tau_{j}^{2} = 1 - |\rho_{j}|^{2}, \quad (A.3)$$

where $j = 1, 2$, the equations that describe the fields propagation are obtained by considering the reflections and transmissions at the interfaces:

$$A_{0}^{-} = -\rho_{2}A_{0}^{+} + \tau_{2}A_{1}^{-} \quad (A.4a) \quad A_{1}^{-} = -\rho_{2,ext}A_{1}^{+}$$

$$A_{1}^{+} = -\rho_{2}A_{1}^{+} + \tau_{2}A_{0}^{+} \quad (A.4b) \quad \Rightarrow A_{1}^{-}e^{i\phi_{2}/2} = \rho_{2,ext}A_{1}^{+}e^{-i\phi_{2}/2}. \quad (A.4c)$$

Inserting Eq. (A.4c) into Eq. (A.4a) and (A.4b):

$$A_{0}^{-} = -\rho_{2}A_{0}^{+} - \tau_{2}\rho_{2,ext}e^{-i\phi_{2}}A_{1}^{+} \quad (A.5a)$$

$$A_{1}^{+} = \rho_{2}\rho_{2,ext}e^{-i\phi_{2}}A_{1}^{+} + \tau_{2}A_{0}^{+} \quad (A.5b) \quad \Rightarrow A_{1}^{+} = \frac{\tau_{2}}{1 - \rho_{2}\rho_{2,ext}e^{-i\phi_{2}}}A_{0}^{+}. \quad (A.5c)$$

and replacing Eq. (A.5c) in Eq. (A.5a):

$$A_{0}^{-} = -\rho_{2}A_{0}^{+} - \frac{\tau_{2}^{2}\rho_{2,ext}e^{-i\phi_{2}/2}}{1 - \rho_{2}\rho_{2,ext}e^{-i\phi_{2}/2}} e^{-i\phi_{2}/2}A_{0}^{+}$$

$$\stackrel{(2.1)}{=} A_{0}^{-}/A_{0}^{+} = \rho_{2}^{\text{eff}} (A.3)(2.2a) = \rho_{2} \left(1 + \kappa_{2}\text{cav}e^{-i\phi_{2}} \right). \quad (A.6)$$

An analogous analysis can be done for mirror 1. The relations from Eq. (A.4) remain the same, as the equations for field transmission and reflection are valid in either case. However, the phase induced by the external cavity, represented by Eq. (A.2), has an opposite sign given that the fields travel in opposite directions:
and thus:

\[ \rho_{\text{eff}}^{1} = \rho_{1} \left( 1 + \kappa_{1} t_{\text{cav}} e^{i\phi_{1}} \right). \]  

\[ \text{(A.8)} \]

Eqs. (A.6) and (A.8) can be summarized as:

\[ \rho^{j} = \rho_{j}^{\text{eff}} \left( 1 + \kappa_{j} t_{\text{cav}} e^{\pm i\phi_{j}} \right) = 1 + \kappa_{j} t_{\text{cav}} \cos(\phi_{j}) \pm i\kappa_{j} t_{\text{cav}} \sin(\phi_{j}). \]  

\[ \text{(A.9)} \]

where the plus and minus signs correspond to \( j = 1 \) (left mirror) and \( j = 2 \) (right mirror), respectively. Both effective reflection coefficients are considered in the analysis to obtain the laser dynamic equations.

\section*{B Amplitude and Phase Conditions}

To obtain the revised lasing conditions resulting from the additional feedback term, extracting the amplitude and phase of the effective reflection coefficients is needed. In polar notation:

\[ \rho^{j} = \left| \rho^{j} \right| e^{i\varphi_{j}}, \]  

\[ \text{(B.1)} \]

where the magnitude is computed as:

\[ \left| \rho^{j} \right|^{2} = \left[ 1 + \kappa_{j} t_{\text{cav}} \cos(\phi_{j}) \right]^{2} + \left[ \kappa_{j} t_{\text{cav}} \sin(\phi_{j}) \right]^{2} \]  

\[ = 1 + 2\kappa_{j} t_{\text{cav}} \cos(\phi_{j}) + \kappa_{j}^{2} t_{\text{cav}}^{2}. \]  

\[ \text{(B.2)} \]

Assuming that the external feedback is weak compared to the internal one:

\[ \left| \rho_{j,\text{ext}} \right| \ll \left| \rho_{j} \right| \overset{(2.2a)}{\Rightarrow} \kappa_{j} t_{\text{cav}} \ll 1, \]  

\[ \text{(B.3)} \]

the last term on the right-hand side of Eq. (B.2) can be neglected, resulting in:

\[ \left| \rho^{j} \left( \omega_{\text{FB}} \right) \right| / \rho_{i} = \sqrt{1 + 2\kappa_{j} t_{\text{cav}} \cos(\phi_{j})} \overset{(B.3)}{\approx} 1 + \kappa_{j} t_{\text{cav}} \cos(\phi_{j}). \]  

\[ \text{(B.4)} \]

The phase of the effective reflection coefficient is extracted from:

\[ \varphi_{j} = \arctan \left[ \left( \rho^{j} \right)^{\prime} / \left( \rho^{j} \right)^{\prime\prime} \right] = \arctan \left[ \pm \kappa_{j} t_{\text{cav}} \sin(\phi_{j}) \right] \overset{(B.3)}{\approx} \arctan[\pm \kappa_{j} t_{\text{cav}} \sin(\phi_{j})] \overset{(B.3)}{\approx} \pm \kappa_{j} t_{\text{cav}} \sin(\phi_{j}). \]  

\[ \text{(B.5)} \]

The fields traveling forward and backward in the laser cavity, \( \mathcal{E}_{f} \) and \( \mathcal{E}_{b} \) shown in Fig. 5, can now be related by the effective reflection coefficients:

\[ \mathcal{E}_{f}(z = 0) = \rho_{1}^{\text{eff}} \mathcal{E}_{b}(z = 0) \quad \text{(B.6a)} \]

\[ \mathcal{E}_{b}(z = L) = \rho_{2}^{\text{eff}} \mathcal{E}_{f}(z = L). \]  

\[ \text{(B.6b)} \]

Using the propagation constant from Eq. (2.2e), the fields can be written as:
Figure 5: Schematic of the effective cavity of the laser, resulting from calculating effective reflection coefficients.

\[ E_f = A_f e^{-i\beta z + \frac{1}{2}(\Gamma g - \alpha)z} \quad \text{(B.7a)} \]
\[ E_b = A_b e^{-i\beta(L-z) + \frac{1}{2}(\Gamma g - \alpha)(L-z)} \quad \text{(B.7b)} \]

where \( g \) is the gain coefficient. Replacing Eq. (B.6) into Eq. (B.7):

\[ E_{f0}^{(B.1)} = \rho_2^{\text{eff}} e^{i\varphi_2} A_b e^{-i\beta L + (\Gamma g - \alpha)L/2} \quad \text{(B.8a)} \]
\[ E_{b0}^{(B.1)} = \rho_1^{\text{eff}} e^{i\varphi_1} A_f e^{-i\beta L + (\Gamma g - \alpha)L/2} \quad \text{(B.8b)} \]

and inserting Eq. (B.8a) into Eq. (B.8b) results in:

\[ 1 = \left| \frac{\rho_2^{\text{eff}}}{\rho_1^{\text{eff}}} \right| e^{i\varphi_2 - 2i\beta L + (\Gamma g - \alpha)L} = \left| \frac{\rho_1^{\text{eff}}}{\rho_2^{\text{eff}}} \right| e^{-i(\Gamma g - \alpha)L}. \]

Once lasing has been established, the gain assumes its threshold value:

\[ g = g_{FB}, \quad \text{(B.10)} \]

where \( g_{FB} \) is the threshold gain with feedback. Thus, Eq. (B.9) yields a lasing condition for the amplitude:

\[ 1 \approx \rho_2 \rho_1 \left[ 1 + \kappa_2 t_{cav} \cos(\phi_2) \right] \left[ 1 + \kappa_1 t_{cav} \cos(\phi_1) \right] e^{(\Gamma g_{FB} - \alpha)L}, \quad \text{(B.11)} \]

and the phase:

\[ 2\pi m = 2\beta L - \varphi_2 - \varphi_1 \quad \text{(B.12)} \]

where the influence of feedback gives rise to two terms in Eqs. (B.11) and (B.12), one from each side. The new lasing conditions result in a variation of the lasing frequency and threshold gain of the system, and thus have to be studied to determine the laser dynamics.

C THRESHOLD GAIN REDUCTION AND LASING FREQUENCY SHIFT

Under feedback conditions, new lasing conditions are found which subsequently result in a shift in the laser threshold gain and lasing frequency with respect to the no feedback case. Without feedback:

\[ \kappa_j = 0, \quad \text{(C.1)} \]
the amplitude condition from Eq. (B.11) becomes:

\[ 1 \equiv \rho_2 \rho_1 e^{(\Gamma_{gth} - \alpha)L}. \]  

(C.1)

Using the following expansion:

\[ \ln(1 + x) \simeq x, \]  

(C.3)

the threshold gain reduction due to feedback can be found by computing the ratio between Eqs. (2.4) and (C.2):

\[ 1 = \frac{\rho_2 \rho_1 [1 + \kappa_2 t_{cav} \cos(\phi_2)] [1 + \kappa_1 t_{cav} \cos(\phi_1)] e^{(\Gamma_{gFB} - \alpha)L}}{\rho_2 \rho_1 e^{\Gamma_{gth} - \alpha L}} \]

\[ = [1 + \kappa_2 t_{cav} \cos(\phi_2)] [1 + \kappa_1 t_{cav} \cos(\phi_1)] e^{\Gamma(\gamma_{FB} - g_{th})L} \]

\[ \approx -2\kappa_2 \cos(\phi_2) - 2\kappa_1 \cos(\phi_1). \]  

(C.4)

The relation between the right hand terms determines the threshold gain reduction, as discussed in Sec. 2.1.

The phase lasing condition from Eq. (B.12) yields the lasing frequency shift equation. Consider the following definitions related to the effective refractive index [53]:

\[ n \equiv n' + in'', \]  

(C.5a) \hspace{1cm} \alpha_H \equiv \Delta n'/\Delta n'' \]  

(C.5d)

\[ n_g \equiv n + \omega \frac{\partial n}{\partial \omega}, \]  

(C.5b) \hspace{1cm} \frac{\partial n}{\partial N} = \frac{\partial n''}{\partial N''} \]  

(C.5c)\hspace{1cm} \frac{\partial n''}{\partial N} \]  

(C.5e)

where \( n_g \) is the group refractive index and \( \alpha_H \) the linewidth enhancement factor [47]. To find the lasing frequency shift equation, calculating the change in \( \beta \) is first needed:

\[ e\Delta \beta \equiv \Delta(n\omega) = \omega \Delta n + n \Delta \omega = \omega \left[ \frac{\partial n}{\partial N}(N - N_{th}) + \frac{\partial n}{\partial \omega} \delta \omega \right] + n \delta \omega \]

\[ \equiv -\frac{G - G_{th}}{2\nu_g} \alpha_H c + n_g \delta \omega \]

\[ \equiv [\kappa_2 \cos(\phi_2) + \kappa_1 \cos(\phi_1)] \frac{\alpha_H c}{\nu_g} + n_g \delta \omega, \]  

(C.6)

with which:

\[ 2L \Delta \beta \equiv [\kappa_2 \cos(\phi_2) + \kappa_1 \cos(\phi_1)] \alpha_H t_{cav} + t_{cav} \delta \omega, \]  

(C.7)

Furthermore, using:

\[ \sin[\arctan(x)] = \frac{x}{\sqrt{1 + x^2}} \]  

(C.9a) \hspace{1cm} \cos(x \pm y) = \cos(x) \cos(y) \pm \sin(x) \sin(y) \]  

(C.9c)

\[ \cos[\arctan(x)] = \frac{1}{\sqrt{1 + x^2}} \]  

(C.9b) \hspace{1cm} \cos(\theta_H) \]  

(C.9d)
the following can be computed:

\[
\alpha_H \cos(\phi_j) = \gamma_H \sin(\theta_H) \cos(\phi_j)
\]

\[
\gamma_H \left[ \sin(\theta_H \pm \phi_j) \mp \cos(\theta_H) \sin(\phi_j) / \gamma_H \right]
\]

\[
\Leftrightarrow \alpha_H \cos(\phi_j) \pm \sin(\phi_j) = \gamma_H \sin(\theta_H \pm \phi_j).
\]

Finally, using the phase condition in Eq. (B.12) and assuming without generality loss that \( m = 0 \):

\[
2 \pi m = t_{\text{cav}} \delta \omega + \kappa_2 t_{\text{cav}} [\alpha_H \cos(\phi_2) + \sin(\phi_2)] + \kappa_1 t_{\text{cav}} [\alpha_H \cos(\phi_1) - \sin(\phi_1)]
\]

\[
\Leftrightarrow \delta \omega = -\kappa_2 [\alpha_H \cos(\phi_2) + \sin(\phi_2)] + \kappa_1 [\alpha_H \cos(\phi_1) - \sin(\phi_1)]
\]

\[
\delta \omega = -\gamma_H [\kappa_2 \sin(\phi_2 + \theta_H) + \kappa_1 \sin(\phi_1 - \theta_H)].
\]

which describes the lasing frequency shift as a function of the feedback parameters \( \kappa_1, \kappa_2, \phi_1 \) and \( \phi_2 \) as expected. Similarly to the threshold gain reduction, the interaction between the two right hand terms determines the lasing frequency stability as discussed in Sec. 2.1.

D DERIVING THE RATE EQUATION FOR THE INTENSITY AND PHASE

To further inspect the laser dynamics, the laser rate equations for the intensity and phase must be studied. Considering Eqs. (2.18) and (A.2), and following the approach from Lang and Kobayashi [44], the laser field equation which considers EOF from both sides of the laser cavity can be written as:

\[
\dot{E} = \left( -i \omega_{\text{ref}} + \Delta G \frac{1 - i \alpha_H}{2} \right) E(t) + \kappa_2 E(t - t_2) + \kappa_1 E(t + t_2)
\]

\[
\Leftrightarrow \dot{A}(t) = (2.2c)\frac{d}{dt} \left[ e^{-i \omega_{\text{ref}} t} \left[ -i \omega_{\text{ref}} + \Delta G \frac{1 - i \alpha_H}{2} \right] A(t) + \kappa_2 A(t - t_2) e^{i \phi_2} + \kappa_1 A(t + t_2) e^{-i \phi_1} \right]
\]

\[
\Leftrightarrow \dot{A}(t) = (2.2c)(2.2b) \left( i \delta \omega + \Delta G \frac{1 - i \alpha_H}{2} \right) A(t) + \kappa_2 A(t - t_2) e^{i \phi_2} + \kappa_1 A(t + t_2) e^{-i \phi_1}.
\]

The last two right-hand terms appear as a result of the imposed feedback conditions, each term to account for feedback on each side of the cavity. In the case without feedback the lasing frequency become \( \omega_{\text{ref}} = \omega_{\text{ref}} \) and \( \kappa_j = 0 \), thus recovering the no-feedback field equation [47]. The slowly varying field amplitude \( A \) can be modeled as Eq. (2.18a), and therefore the rate equations for the photon number \( S \) and phase \( \phi \) can be found using:
\[
\dot{S} = \frac{d[A^*A]}{dt} = A\dot{A}^* + A^*\dot{A} \quad (\text{D.2a}) \\
\dot{\phi} = -\frac{1}{S}\Im(A^*\dot{A}) \quad (\text{D.2b})
\]

Replacing Eqs. (D.1) and (2.18a) into Eq. (D.2a) yields:
\[
\dot{S}^{(2.18\text{d})} = S\Delta G + S_2^2\sqrt{\frac{S}{S}}\left(\exp(-i\Delta \Phi_2) + \exp(i\Delta \Phi_2)\right) + S_1^+\sqrt{\frac{S}{S}}\left(\exp(-i\Delta \Phi_1^+) + \exp(i\Delta \Phi_1^+)\right),
\]
and transforming the sum of exponentials into a cosine yields the following expression for the photon rate:
\[
\dot{S} = S\Delta G + 2S_2^2\sqrt{\frac{S}{S}}\cos(\Delta \Phi_2) + 2S_1^+\sqrt{\frac{S}{S}}\cos(\Delta \Phi_1^+).
\]

In the case of the phase, its rate equation comes from replacing Eqs. (D.1) and (2.18a) into Eq. (D.2b):
\[
\dot{\phi} S^{(2.18)} = -\Im\left(\left(i\delta\omega + \Delta G\frac{1-i\alpha_H}{\alpha_H}\right)S + S_2^2\sqrt{\frac{S}{S}}e^{i\Delta \Phi_2} + S_1^+\sqrt{\frac{S}{S}}e^{i\Delta \Phi_1^+}\right)
\]
\[
\dot{\phi} = \frac{\Delta G}{2}\alpha_H - \delta\omega - \frac{S_2^2}{\sqrt{S}}\sin(\Delta \Phi_2) - \frac{S_1^+}{\sqrt{S}}\sin(\Delta \Phi_1^+).
\]

Eqs. (D.4) and (D.5) are the amplitude and phase rate equations for the laser system proposed in this work. These are the starting point to compute the frequency noise PSD, and extract the intrinsic linewidth.

E SMALL-SIGNAL ANALYSIS

To find the FN PSD, the system shown in Eq. (2.19) is to be solved. This is done using the small-signal analysis proposed in Eq. (2.24). Assuming a narrow-linewidth laser, i.e. a long coherence time with respect to the external cavity lenghts:
\[
t_{\text{ext}} < t_{\text{coh}},
\]
the following approximation is valid:
\[
\Omega_{t_{\text{ext}}} \ll 1.
\]

By linearizing the following expressions:
\[
\sqrt{\frac{S(t \pm t_j)}{S(t)}} \approx \sqrt{\frac{S \pm \dot{S} t_j}{S}} \approx \sqrt{1 \pm \frac{\dot{S}}{S} t_j} \approx \sqrt{1 \pm i\Omega S_\Delta t_j} \approx 1 \pm i\Omega S_\Delta t_j (\text{E.3a})
\]
\[
\phi - \phi(t \pm t_j) \approx \phi - \phi \mp t_j \dot{\phi} \approx t_j \Omega_\Delta \phi. (\text{E.3b})
\]
it is possible to rewrite Eqs. (2.19) as:

\[ \dot{i}ΩS_\Delta = \Delta GS + R_{sp} + 2\kappa_2 S \left( 1 - \frac{iΩS_\Delta}{2S_0} \right) \cos (iΩφ_\Delta t_2 + φ_2) \]

\[ + 2\kappa_1 S \left( 1 + \frac{iΩS_\Delta}{2S_0} \right) \cos (-iΩφ_\Delta t_1 - φ_1) + F_S \] \hspace{1cm} (E.4a)

\[ \dot{i}Ωφ_\Delta = \frac{ΔG}{2} - δω - \kappa_2 \left( 1 - \frac{iΩS_\Delta}{2S_0} \right) \sin (iΩφ_\Delta t_2 + φ_2) \]

\[ - \kappa_1 \left( 1 + \frac{iΩS_\Delta}{2S_0} \right) \sin (-iΩφ_\Delta t_1 - φ_1) + F_φ \] \hspace{1cm} (E.4b)

\[ \dot{i}ΩN_\Delta = I - G_{ph} S - Nτ^{-1}_{sp} + F_N. \] \hspace{1cm} (E.4c)

Using the following identities:

\[ \cos(x) = \cos(-x) \] \hspace{1cm} (E.5a)

\[ \sin(x) = -\sin(-x), \] \hspace{1cm} (E.5b)

the rate equations from Eqs. (E.4) can be rewritten as:

\[ \dot{i}ΩS_\Delta = S\Delta G + R_{sp} + 2\kappa_2 \frac{S}{t_{cav}} \left( 1 - \frac{iΩS_\Delta}{2S_0} \right) \cos (iΩφ_\Delta t_2 + φ_2) \]

\[ + 2\kappa_1 \left( 1 + \frac{iΩS_\Delta}{2S_0} \right) \cos (iΩφ_\Delta t_1 + φ_1) + F_S \] \hspace{1cm} (E.6a)

\[ \dot{i}Ωφ_\Delta = α_n \frac{ΔG}{2} - δω - \kappa_2 \frac{S}{t_{cav}} \left( 1 - \frac{iΩS_\Delta}{2S_0} \right) \sin (iΩφ_\Delta t_2 + φ_2) \]

\[ + \kappa_1 \left( 1 + \frac{iΩS_\Delta}{2S_0} \right) \sin (iΩφ_\Delta t_1 + φ_1) + F_φ. \] \hspace{1cm} (E.6b)

Solving Eq. (E.4) requires the steady-state solutions of Eq. (2.19). Under stationary conditions:

\[ \dot{S} = 0 \Rightarrow S(t) = S(t ± t_j) = S_0 \] \hspace{1cm} (E.7a)

\[ \dot{φ} = 0 \Rightarrow φ(t) = φ(t ± t_j) \] \hspace{1cm} (E.7b)

\[ \dot{N} = 0 \Rightarrow N(t) = N(t ± t_j) = N_0, \] \hspace{1cm} (E.7c)

with which the steady-state equations are:

\[ τ_{ph}^{-1} = G_0 + 2\kappa_2 \cos (φ_2) + 2\kappa_1 \cos (φ_1) + \frac{R_{sp}}{S_0} \] \hspace{1cm} (E.8a)

\[ δω = α_n \frac{G_0 - τ_{ph}^{-1}}{2} - \kappa_2 \sin (φ_2) + \kappa_1 \sin (φ_1) \] \hspace{1cm} (E.8b)

\[ I = G_0 S_0 + N_0τ_{sp}^{-1}, \] \hspace{1cm} (E.8c)

where the Langevin noise terms are not included as their mean value is zero. Next, using Eq. (E.8) and the following expansions:
\sin(x + \Delta) \approx \sin(x) + \Delta \cos(x) \quad \text{(E.9a)} \quad \cos(x + \Delta) \approx \cos(x) - \Delta \sin(x), \quad \text{(E.9b)}

Eq. (E.6a) can be rewritten as:

\begin{align*}
i \Omega' S_{\Delta} \text{ (2.24)(E.8)(2.25)} & \approx G_0 + a_g N_{\Delta} - \left\{ G_0 + 2 [\kappa_2 \cos(\phi_2) + \kappa_1 \cos(\phi_1)] + \frac{R_{sp}}{S_0} \right\} (S_0 + S_{\Delta}) \\
& + 2 \left\{ \kappa_2 (S_0 + S_{\Delta}) \left[ \cos(\phi_2) - i \Omega' \phi t_2 \sin(\phi_2) \right] \\
& - \kappa_2 (S_0 + S_{\Delta}) \frac{i \Omega' S_{\Delta}}{2 S_0} t_2 \left[ \cos(\phi_2) - i \Omega' \phi t_2 \sin(\phi_2) \right] \\
& + \kappa_1 (S_0 + S_{\Delta}) \left[ \cos(\phi_1) - i \Omega' \phi t_1 \sin(\phi_1) \right] \\
& + \kappa_1 (S_0 + S_{\Delta}) \frac{i \Omega' S_{\Delta}}{2 S_0} t_1 \left[ \cos(\phi_1) - i \Omega' \phi t_1 \sin(\phi_1) \right] \right\} + R_{sp} + F_S.
\end{align*}

Simplifying this equation, and neglecting the quadratic terms yields:

\begin{align*}i \Omega' S_{\Delta} \text{ (2.25)} & \approx a_g N_{\Delta} S_0 - \zeta_s S_{\Delta} - i \Omega' \phi 2 K_s S_0 + i \Omega' S_{\Delta} (\kappa_1^2 - \kappa_2^2) + F_S. \quad \text{(E.10a)}
\end{align*}

In a similar way, Eqs. (E.6b) can be rewritten using Eqs. (E.8) and (2.25):

\begin{align*}i \Omega' \phi & \approx \frac{\alpha_n}{2} \left( G_0 + a_g N_{\Delta} - \tau_{ph}^{-1} \right) - \left[ \frac{G_0 - \tau_{ph}^{-1}}{2} - \kappa_2 \sin(\phi_2) + \kappa_1 \sin(\phi_1) \right] \\
& - \kappa_2 \left( 1 - \frac{i \Omega' S_{\Delta}}{2 S_0} t_2 \right) [\sin(\phi_2) + i \Omega' \phi t_2 \cos(\phi_2)] \\
& + \kappa_1 \left( 1 + \frac{i \Omega' S_{\Delta}}{2 S_0} t_1 \right) [\sin(\phi_1) + i \Omega' \phi t_1 \cos(\phi_1)] + F_\phi \\
\Leftrightarrow 2 i \Omega' \phi & \text{ (2.25a)(2.25b)} \approx \alpha_n a_g N_{\Delta} + 2 (\kappa_1^2 - \kappa_2^2) i \Omega' \phi + \frac{i \Omega' S_{\Delta}}{S_0} K_s + 2 F_\phi. \quad \text{(E.10b)}
\end{align*}

Finally, Eq. (E.4c) can be rewritten as:

\begin{align*}i \Omega' N_{\Delta} \text{ (2.24)(E.8)(2.25)} & \approx G_0 S_0 - (G_0 + a_g N_{\Delta}) S_0 - (G_0 + a_g N_{\Delta}) S_{\Delta} - \tau_{sp}^{-1} N_{\Delta} + F_N \\
\Leftrightarrow i \Omega' N_{\Delta} & \text{ (2.25b)} \approx - \tau_{sp}^{-1} N_{\Delta} - G_0 S_{\Delta} + F_N. \quad \text{(E.10c)}
\end{align*}

The linearized rate equations of the laser under study are thus Eqs. (E.10a), (E.10b) and (E.10c), from which the power spectral density, and subsequently the linewidth, can be computed.

### F POWER SPECTRAL DENSITY

Obtaining the FN PSD of the system under study requires calculating $\phi_{0p}$. To begin with, $N_{0p}$ is extracted from Eq. (2.26c):

\begin{align*}(i \Omega + \tau_{e}^{-1}) N_{0p} = -G_0 S_{0p} + \hat{F}_N \quad \Leftrightarrow \quad N_{0p} = \frac{\hat{F}_N - G_0 S_{0p}}{i \Omega + \tau_{e}^{-1}}, \quad \text{(F.1)}
\end{align*}
Next, replacing Eq. (F.1) into Eq. (2.26a) yields the expression for $S_{0p}$:

$$
[i \Omega \xi_c + \zeta_s] S_{0p} = a_g \frac{\hat{F}_N - G_0 S_{0p}}{i \Omega + \tau_e^{-1}} S_0 - i 2 \Omega S_0 \xi_s \phi_{0p} + \hat{F}_S
$$

$$
\Leftrightarrow \frac{A_{\phi}}{i \Omega + \tau_e^{-1}} S_{0p} = a_g S_0 \frac{\hat{F}_N}{i \Omega + \tau_e^{-1}} - i 2 \Omega S_0 \xi_s \phi_{0p} + \hat{F}_S
$$

$$
\Leftrightarrow S_{0p} \overset{(2.27a)}{=} \frac{a_g S_0 \hat{F}_N - i 2 S_0 \Omega \xi_s (i \Omega + \tau_e^{-1}) \phi_{0p} + (i \Omega + \tau_e^{-1}) \hat{F}_S}{A_{\phi}}. \tag{F.2}
$$

Finally, inserting Eq. (F.1) and (F.2) into Eq. (2.26b) and grouping the terms with $\phi_{0p}$, $\hat{F}_S$, $\hat{F}_{\phi}$ and $\hat{F}_N$ yields:

$$
2 i \Omega B_{\phi} \phi_{0p} \overset{(2.27b)}{=} \left[ -\alpha_H a_g G_0 S_0 + \alpha_H A_{\phi} + i \Omega \xi_s (i \Omega + \tau_e^{-1}) \right] \frac{a_g}{i \Omega + \tau_e^{-1}} \hat{F}_N + 2 A_S \hat{F}_S + 2 A_{\phi} \hat{F}_{\phi}
$$

$$
\Leftrightarrow \frac{a_g}{i \Omega + \tau_e^{-1}} \hat{F}_N + 2 A_S \hat{F}_S + 2 A_{\phi} \hat{F}_{\phi}
$$

$$
\Leftrightarrow \phi_{0p} \overset{(2.27c)}{=} \frac{A_N \hat{F}_N + A_S \hat{F}_S + A_{\phi} \hat{F}_{\phi}}{i \Omega B_{\phi}}. \tag{F.3}
$$

With Eq. (F.3) it is possible to calculate an expression for Eq. (2.29):

$$
2 \pi^2 |B_{\phi}|^2 S_f^{(1)} \overset{(2.29)}{=} \Omega^2 |B_{\phi}|^2 \langle \phi_{0p}(\Omega) \phi_{0p}^*(\Omega) \rangle
$$

$$
\overset{(F.3)}{=} \Omega^2 |B_{\phi}|^2 \left\{ A_N \hat{F}_N + A_S \hat{F}_S + A_{\phi} \hat{F}_{\phi} \left( \frac{A_N \hat{F}_N + A_S \hat{F}_S + A_{\phi} \hat{F}_{\phi}}{i \Omega B_{\phi}} \right) \right\}^* \tag{F.4}
$$

It can be seen from Eq. (2.27) that the coefficients $A_i$ are independent of time, and assuming an ergodic process they can be taken out of the average in Eq. (F.4), obtaining:

$$
2 \pi^2 |B_{\phi}|^2 S_f^{(1)} = |A_N|^2 \left\{ \hat{F}_N \hat{F}_N^* + A_S A_N^* \left( \hat{F}_S \hat{F}_N^* \right) + A_{\phi} A_N^* \left( \hat{F}_{\phi} \hat{F}_N^* \right) \right\}
$$

$$
+ A_N A_S^* \left( \hat{F}_N \hat{F}_S^* \right) + |A_S|^2 \left( \hat{F}_S \hat{F}_S^* \right) + A_{\phi} A_S^* \left( \hat{F}_{\phi} \hat{F}_S^* \right)
$$

$$
+ A_N A_{\phi}^* \left( \hat{F}_N \hat{F}_{\phi}^* \right) + A_S A_{\phi}^* \left( \hat{F}_S \hat{F}_{\phi}^* \right) + |A_{\phi}|^2 \left( \hat{F}_{\phi} \hat{F}_{\phi}^* \right), \tag{F.5}
$$

which can be rewritten as:

$$
\pi^2 |B_{\phi}|^2 S_f^{(1)} \overset{(2.23)}{=} |A_N|^2 D_{NN} + |A_S|^2 D_{SS} + |A_{\phi}|^2 D_{\phi\phi} + (A_S A_{\phi}^* + A_{\phi} A_S^*) D_{SN}. \tag{F.6a}
$$

The explicit formula for $S_f^{(1)}$ can be obtained from Eq. (F.6a), which requires calculating $|A_N|^2$, $|A_S|^2$, $|A_{\phi}|^2$ and $(A_S A_{\phi}^* + A_{\phi} A_S^*)$. For this purpose, it is helpful to separate the real and imaginary parts of these coefficients. Expanding $A_N$ yields:

$$
2 A_N \overset{(2.27c)\tag{2.30}}{=} \zeta_s \alpha_H a_g + i \Omega a_g F_2 \tag{F.7a}
$$

$$
\Leftrightarrow 4 |A_N|^2 = (\zeta_s \alpha_H a_g)^2 + \Omega^2 a_g^2 F_2^2. \tag{F.7b}
$$
In the case of $A_\phi$, the same approach yields:

$$A_\phi \equiv (2.27a) \begin{equation}
(\Omega + \tau_e^{-1}) (i\Omega \kappa + \xi_s) + a_g G_0 S_0
\end{equation}$$

$$= -\Omega^2 K_c + \tau_e^{-1} \xi_s + a_g G_0 S_0 + i\Omega (\xi_s + K_c \tau_e^{-1}) \quad (F.8a)$$

$$\Leftrightarrow |A_\phi|^2 = \left( -\Omega^2 K_c + \tau_e^{-1} \xi_s + a_g G_0 S_0 \right)^2 + \Omega^2 (\xi_s + K_c \tau_e^{-1})^2$$

$$(2.30a) \quad \Omega^4 K_c^2 + \Omega^2 F_0 + (\tau_e^{-1} \xi_s + a_g G_0 S_0)^2,$$  

and in the case for $A_S$:

$$2 A_S \equiv (2.27b) \begin{equation}
\frac{-\alpha_H a_g G_0 - \Omega^2 K_s}{S_0} + i\Omega \tau_e^{-1} \frac{K_s}{S_0}
\end{equation}$$

$$\Leftrightarrow 4 S_0^2 |A_S|^2 = \left( -\alpha_H a_g G_0 S_0 - \Omega^2 K_s \right)^2 + \Omega^2 (K_s \tau_e^{-1})^2$$

$$= \Omega^4 K_s^2 + \Omega^2 \left( [K_s \tau_e^{-1}]^2 + 2 \alpha_H a_g G_0 K_s S_0 \right) + (\alpha_H a_g G_0 S_0)^2.$$  

(F.9b)

Next, the following identity is useful for computing $(A_S A_N^* + A_N A_S^*)$:

$$XY^* + X^*Y = (a + ib)(c - id) + (a - ib)(c + id) = 2(ac + bd),$$  

(F.10)

with which:

$$4 \left( A_S A_N^* + A_N A_S^* \right) \equiv (F.7a)(F.9a) = 2 \left\{ [-\alpha_H a_g G_0 - \Omega^2 K_s / S_0] \xi_s \alpha_H a_g \right.$$  

$$+ \Omega^2 \tau_e^{-1} (K_s / S_0) a_g (K_c \alpha_H + K_s) \left\}$$

$$\Leftrightarrow 2 (A_S A_N^* + A_N A_S^*) = \Omega^2 \frac{K_s}{S_0} a_g \left[ \tau_e^{-1} (K_c \alpha_H + K_s) - \xi_s \alpha_H \right] - \alpha_H^2 a_g^2 G_0 \xi_s.$$  

(F.11)

Using Eq. (F.7b), (F.8b), (F.9b), (F.11) and (2.27d), an expression for $S_f^{(1)}$ can be found:

$$4\pi^2 |B_\phi|^2 S_f^{(1)} (F.6a)(2.30) = \left[ (\xi_s \alpha_H a_g)^2 + \Omega^2 a_g^2 F_0^2 \right] D_{NN}$$

$$+ \left[ \Omega^4 K_c^2 + \Omega^2 F_0 + (\tau_e^{-1} \xi_s + a_g G_0 S_0)^2 \right] 4 D_{\phi \phi}$$

$$+ \left\{ \Omega^2 \left[ \left( \tau_e^{-1} \frac{K_s}{S_0} \right)^2 + 2 \alpha_H a_g G_0 \frac{K_s}{S_0} \right] + \Omega^4 \left( \frac{K_s}{S_0} \right)^2 + \left[ \alpha_H a_g G_0 \right]^2 \right\} D_{SS}$$

$$- 2 \left[ -\alpha_H^2 a_g^2 G_0 \xi_s + \Omega^2 \frac{K_s}{S_0} a_g \left( \tau_e^{-1} F_2 - \xi_s \alpha_H \right) \right] D_{SS}$$

$$\Leftrightarrow 2\pi^2 S_f^{(1)} (2.30) = \frac{A_4 \Omega^4 + A_2 \Omega^2 + A_0}{2 |B_\phi|^2}.$$  

(F.12)

Furthermore, using the following definitions:

$$\Delta_1 \equiv F_1^2$$  

(F.13a)

$$\Delta_2 \equiv K_c^2 \xi_s^2 + \tau_e^{-2} F_1^2 - 2 F_1 a_g G_0 S_0 F_3$$  

(F.13b)

$$\Delta_0 \equiv (a_g G_0 S_0 F_3 + K_c \xi_s \tau_e^{-1})^2,$$  

(F.13c)
the expression for $B_\phi$ from Eq. (2.27d) can be rewritten as:

$$B_\phi = a_g G_0 S_0 F_3 - \Omega^2 F_1 + i \Omega \left( K_c \zeta_s + \tau_e^{-1} F_1 \right) + K_c \zeta_s \tau_e^{-1}$$  \hspace{1cm} (F.14a)

\[\Leftrightarrow \left| B_\phi \right|^2 = \Omega^2 \left( K_c \zeta_s + \tau_e^{-1} F_1 \right)^2 + \left( a_g G_0 S_0 F_3 - \Omega^2 F_1 + K_c \zeta_s \tau_e^{-1} \right)^2\]

\[\Leftrightarrow \Omega^4 \Delta f + \Omega^2 \left( K_c \zeta_s + \tau_e^{-1} F_1 \right)^2 - 2 F_1 a_g G_0 S_0 F_3 - 2 F_1 K_c \zeta_s \tau_e^{-1} + \Delta_0\]

\[\Leftrightarrow \Omega^4 \Delta f + \Omega^2 \Delta_2 + \Delta_0,\]  \hspace{1cm} (F.14b)

with which the expression of the FN PSD becomes:

$$4\pi^2 S_f^{(1)} \left( 2.30 \right) = \frac{4 \Lambda_4 \Omega^4 + 4 \Lambda_2 \Omega^2 + \Lambda_0}{\Delta_4 \Omega^4 + \Delta_2 \Omega^2 + \Delta_0}.$$  \hspace{1cm} (F.15)

\section{G \hspace{0.5cm} EXPRESSION FOR THE INTRINSIC LINEWIDTH}

The laser intrinsic linewidth can be found from Eq. (2.32). Defining:

$$\beta_{Ag} \equiv \frac{\tau_e^{-1} \zeta_s}{a_g G_0 S_0} ; \hspace{0.5cm} \delta_{Ag} \equiv \frac{\zeta_s}{G_0} ; \hspace{0.5cm} \Delta f_0 = \frac{R_{sp}}{4\pi S_0},$$  \hspace{1cm} (G.1)

where, following Ref. [14]:

$$\delta_{Ag} \simeq 0 \simeq \beta_{Ag},$$  \hspace{1cm} (G.2)

as $\delta_{Ag} < 10^{-2}$, which accounts for shot noise in the generation and recombination of minority carriers, and $\beta_{Ag}$ is inversely proportional to the laser power which above threshold becomes negligible. Starting from Eq. (F.15) and setting $\Omega = 0$ as required by Eq. (2.32):

$$4\pi \Delta f = \Lambda_0 / \Delta_0$$

\[\Leftrightarrow \Delta f = \frac{\left( \alpha_{Ag} \right)^2 \left( \frac{2 G_0 \zeta_s}{\zeta_s} + 2 G_0 \zeta_s + \frac{2 N \tau_{sp}^{-1}}{R_{sp} S_0} \right) + \left( \beta_{Ag} + 1 \right)^2}{\left( K_c \beta_{Ag} + F_3 \right)^2}

\Leftrightarrow \frac{\Delta f}{\Delta f_0} = \frac{\left( \frac{\alpha_{Ag}}{G_0} \right)^2 \left( \frac{2 G_0 \zeta_s}{\zeta_s} + 2 G_0 \zeta_s + \frac{2 N \tau_{sp}^{-1}}{R_{sp} S_0} \right) + \left( \beta_{Ag} + 1 \right)^2}{\left( K_c \beta_{Ag} + F_3 \right)^2}

\Leftrightarrow \frac{\Delta f}{\Delta f_0} \left( 1 + \frac{\alpha_{Ag}^2}{G_0^2} \right) \simeq F_3^{-2}

\left( 2.30 \right) \left( 2.25 \right) \left\{ 1 + \kappa_2 t_2 \left[ \cos (\phi_2) - \alpha_n \sin (\phi_2) \right] - \kappa_1 t_1 \left[ \cos (\phi_1) + \alpha_n \sin (\phi_1) \right] \right\}^{-2}

\left( C.9a \right) \left( C.9b \right) \left\{ 1 + \gamma_n \kappa_2 t_2 \cos (\phi_2 + \theta_n) - \gamma_n \kappa_1 t_1 \cos (\phi_1 - \theta_n) \right\}^{-2}.$$

The presence of EOF from both sides of the laser cavity results in two terms in the linewidth expression, one for each side, as was seen in the threshold gain reduction and lasing frequency shift due to feedback. This result is discussed in Sec. 2.3.
Figure 6: Simulations for $C = 0.5$ under condition (2.15) for a $\pm 20\%$ variation of conditions (2.10) and (2.13). Full solution shown in blue, single feedback case shown in orange. Column A shows the lasing frequency shift results. Column B shows the threshold gain shift. Column C shows the intrinsic linewidth variations.
Figure 7: Simulations for $C = 1$ under condition (2.15) for a ±20% variation of conditions (2.10) and (2.13). Full solution shown in blue, single feedback case shown in orange. Column A shows the lasing frequency shift results. Column B shows the threshold gain shift. Column C shows the intrinsic linewidth variations.
Figure 8: Simulations for $C = 1.3$ under condition (2.15) for a ±20% variation of conditions (2.10) and (2.13). Full solution shown in blue, single feedback case shown in orange. Column A shows the lasing frequency shift results. Column B shows the threshold gain shift. Column C shows the intrinsic linewidth variations.
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