NIL HAPPENS. WHAT ABOUT SOL?

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Abstract. We construct complete, finite volume, 4-dimensional manifolds with sectional curvature $-1 < K < 0$ with cusp cross sections compact solvmanifolds.

1. Introduction

Let $M$ be a connected, noncompact, complete, finite volume manifold with sectional curvature $-1 < K(M) < 0$. By [4], $M$ is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary $\partial \overline{M}$. It is known that if $-1 \leq K(M) \leq -a^2 < 0$, then $\partial \overline{M}$ is an infra-nil manifold. For example, cross sections of cusps of finite volume, complex hyperbolic 4-dimensional manifolds are closed 3-manifolds with Nil geometry. In fact, compact flat or infra-nil manifolds can be realized as cusp cross sections of finite volume manifolds with pinched negative curvature by [5, Corollary 6] and [1]. Fujiwara ([3]) constructed examples of $M$ for which $\partial M$ is a circle bundle over a compact hyperbolic manifold (thus is not infra-nil) for each dimension $\geq 4$.

While there are closed 3-manifolds with Sol geometry realized as cusp cross sections of finite volume, nonpositively curved manifolds, e.g. Hilbert modular surfaces, examples of finite volume manifolds $M$ with $-1 < K(M) < 0$ whose cusp cross sections are diffeomorphic to a Sol manifold have not shown up (or it is unknown to the author). It might not be surprising that negatively curved cusps with Sol cross section exist because of the Fujiwara examples. We give a construction of such manifolds $M$ with Sol cusp cross sections in this paper.

2. The construction

The procedure of the construction is that first we construct a negatively curved cusp with cross section a 3-manifold with Sol geometry. Then we use Ontaneda’s procedure of pinched smooth hyperbolization [5] to glue a “thick part” to the cusp keeping the metric smooth and negatively curved.

Lemma 1. Let $C$ be a compact 3-manifold with Sol geometry. There is a complete, finite volume, negatively curved, Riemannian metric on $C \times \mathbb{R}$ with sectional curvature $-1 < K < 0$ on $(-R^2, \infty)$ for some $R^2 > 0$. Moreover, this metric can be chosen so that it is a warped product metric on $C \times (0, 1)$.

Proof. The Sol metric on $C$ is

$$g_{\text{Sol}} = dz^2 + e^{-2z} dx^2 + e^{2z} dy^2.$$ 

Consider the following metric on $C \times (0, \infty)$.

$$g = dt^2 + f(t)^2 dz^2 + e^{-2t} (e^{-2z} dx^2 + e^{2z} dy^2)$$
for some function $f: (0, \infty) \longrightarrow \mathbb{R}^+$. The nonzero components (up to symmetry) of Riemann curvature tensor are

$$R_{1212} = \frac{e^{-4t}(1 - f(t)^2)}{f(t)^2}, \quad R_{1313} = e^{-2t - 2z}(f(t)f'(t) - 1), \quad R_{1414} = -e^{-2t - 2z},$$

$$R_{2323} = e^{-2t + 2z}(f(t)f'(t) - 1), \quad R_{2424} = -e^{-2t + 2z}, \quad R_{3434} = -f(t)f''(t),$$

$$R_{1431} = e^{-2t - 2z} \left(1 + \frac{f'(t)}{f(t)}\right), \quad R_{2432} = -e^{-2t + 2z} \left(1 + \frac{f'(t)}{f(t)}\right).$$

The metric $g$ has negative curvature if the following four conditions are satisfied.

a) $f(t) > 1$ for all $t$,
b) $f'(t) < 0$ for all $t$,
c) $f''(t) > 0$ for all $t$, and

d) $1 - f(t)f'(t) > \left(1 + \frac{f'(t)}{f(t)}\right)^2$ for all $t$.

We see that $f(t) = e^{-t}$ satisfies the above four conditions for $t < 0$. Also, $f(t) = 1 + e^{-t}$ satisfies the above four conditions for all $t \in \mathbb{R}$, and the sectional curvature in this case is bounded from below. We pick the function $f(t)$ to be an interpolation between these two functions, i.e. $f(t) = e^{-t}$ for $t << 0$, and $f(t) = 1 + e^{-t}$ for $t \geq 0$, in such a way that the above four conditions are satisfied during the interpolating process. It is an exercise for the reader that such a function exists.

We see that for $t$ negative enough the metric $g$ (with the chosen function $f$) is a warped product

$$g = dt^2 + e^{-2t}(dz^2 + e^{-2z}dx^2 + e^{2z}dy^2).$$

It is clear that this metric is complete. Finiteness of volume follows since there is the warping factor $e^{-2t}$ decays fast enough, i.e. $\int_0^\infty e^{-2t}dt < \infty$. \hfill \Box

By the same procedure as in [5, Section 11], e.g. apply pinched smooth hyperbolization to the suspension $\Sigma C$ of $C$, we get a finite volume, manifold with sectional curvature $-1 < K(M) < 0$ with two cusps, each with cross section $C$.

**Acknowledgement.** The author would like to thank Benson Farb for asking her the question in this paper. She should like thank the PCMI Summer School in 2012 for providing a good air-conditioned working environment during which this paper was written up. Finally she would like to thank God and the communist party of Vietnam for their great guidance.

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