CONNECTING TWO TYPES OF REPRESENTATIONS
OF A PERMUTATION OF $\mathbb{F}_q$

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Abstract. In this paper, we connect two types of representations of a permutation $\sigma$ of the finite field $\mathbb{F}_q$. One type is algebraic, in which the permutation is represented as the composition of degree-one polynomials and $k$ copies of $x^{q-2}$, for some prescribed value of $k$. The other type is combinatorial, in which the permutation is represented as the composition of a degree-one rational function followed by the product of $k$ 2-cycles on $\mathbb{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$, where each 2-cycle moves $\infty$. We show that, after modding out by obvious equivalences amongst the algebraic representations, then for each $k$ there is a bijection between the algebraic representations of $\sigma$ and the combinatorial representations of $\sigma$. We also prove analogous results for permutations of $\mathbb{P}^1(\mathbb{F}_q)$. One consequence is a new characterization of the notion of Carlitz rank of a permutation on $\mathbb{F}_q$, which we use elsewhere to provide an explicit formula for the Carlitz rank. Another consequence involves a classical theorem of Carlitz, which says that if $q > 2$ then the group of permutations of $\mathbb{F}_q$ is generated by the permutations induced by degree-one polynomials and $x^{q-2}$. Our bijection provides a new perspective from which the two proofs of this result in the literature can be seen to arise naturally, without requiring the clever tricks that previously appeared to be needed in order to discover those proofs.

1. Introduction

Throughout, we assume that $q > 2$ is a prime power and $\mathbb{F}_q$ is the finite field with $q$ elements. Note that permutations of $\mathbb{F}_q$ can be viewed as permutations of $\mathbb{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$ which fixes $\infty$. We will study two completely different types of representations of permutations on...
the second type of representations is essentially combinatorial. More precisely, fixing a positive integer $k$ and a permutation $\sigma$ of $\mathbb{P}^1(\mathbb{F}_q)$, we denote by $A_{\sigma,k}$ the set of all representations of $\sigma$ of the form

$$ \sigma = \mu(x) \circ x^{q-2} \circ (x-a_k) \circ x^{q-2} \circ (x-a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x-a_1) $$

with $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$ and $\mu(x) \in \mathbb{F}_q(x)$ of degree one, and similarly we denote by $C_{\sigma,k}$ the set of all representations of $\sigma$ of the form

$$ \sigma = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) $$

with $b_1, b_2, \ldots, b_k \in \mathbb{F}_q$ and $\nu(x) \in \mathbb{F}_q(x)$ of degree one.

The presentations in the set $A_{\sigma,k}$ are algebraically nice, since they are compositions of a degree-one rational function $\mu(x)$ with the monomial $x^{q-2}$ and monic degree-one polynomials $x-a_i$ with $1 \leq i \leq k$. The presentations in the set $C_{\sigma,k}$ are combinatorially nice, since they are compositions of a degree-one rational function $\nu(x)$ with $2$-cycles on $\mathbb{P}^1(\mathbb{F}_q)$ of the form $(b_i, \infty)$ with $1 \leq i \leq k$.

Although $A_{\sigma,k}$ and $C_{\sigma,k}$ are completely different in nature, Theorem 7 gives a recipe to turn any representation in $A_{\sigma,k}$ into a representation in $C_{\sigma,k}$, and Theorem 8 gives a recipe to turn any representation in $C_{\sigma,k}$ into a representation in $A_{\sigma,k}$. Moreover, due to Theorem 6 the maps $F: A_{\sigma,k} \to C_{\sigma,k}$ and $G: C_{\sigma,k} \to A_{\sigma,k}$ induced by the above two recipes respectively are inverses to one another. In other words, there exist naturally two inverse bijections between $A_{\sigma,k}$ and $C_{\sigma,k}$, which are induced by the recipes as illustrated in Theorem 7 and Theorem 8 respectively, as stated in the following:

**Theorem 1.** For any positive integer $k$ and any permutation $\sigma$ of $\mathbb{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$ with $q > 2$, there are two natural inverse bijections between the set $A_{\sigma,k}$ of all representations of $\sigma$ of the form

$$ \sigma = \mu(x) \circ x^{q-2} \circ (x-a_k) \circ x^{q-2} \circ (x-a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x-a_1) $$

with $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$ and $\mu(x) \in \mathbb{F}_q(x)$ of degree one and the set $C_{\sigma,k}$ of all representations of $\sigma$ of the form

$$ \sigma = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) $$

with $b_1, b_2, \ldots, b_k \in \mathbb{F}_q$ and $\nu(x) \in \mathbb{F}_q(x)$ of degree one. In particular, the finite sets $A_{\sigma,k}$ and $C_{\sigma,k}$ have the same cardinality.

There are two easy consequences of our results. The first consequence is a new characterization of the notion of Carlitz rank of a permutation on $\mathbb{F}_q$, based on which an explicit formula about Carlitz rank has been obtained in [11]. The second consequence involves a classical theorem of Carlitz, which says that if $q > 2$ then the group of permutations of $\mathbb{F}_q$
is generated by the permutations induced by degree-one polynomials and \( x^{q^2-2} \). Our bijection in Theorem 1 provides a new perspective from which the two proofs of this result in the literature can be seen to arise naturally, without requiring the clever tricks that previously appeared to be needed in order to discover those proofs.

This paper is organized as follows. In the next section, we prove some results about permutations on \( \mathbb{P}^1(\mathbb{F}_q) \) which will be used in our treatment. In section 3 we give a proof for Theorem 1 together with the explicit recipe, which turns an algebraically nice representation of a permutation of \( \mathbb{P}^1(\mathbb{F}_q) \) into a combinatorially nice representation of the same permutation and vice versa. We conclude in section 4 by illustrating the above mentioned two consequences of our results.

2. Basic facts

Let us begin with the following basic observation:

**Lemma 2.** We have \((0, \infty) = x^{-1} \circ x^{q^2-2}\) as permutations of \( \mathbb{P}^1(\mathbb{F}_q) \).

*Proof.* Note that \( x^{q^2-2} = x^{-1} \) for any nonzero element \( x \in \mathbb{F}_q \), so the map \( x^{-1} \circ x^{q^2-2} \) fixes each element in \( \mathbb{F}_q^* \). It is not hard to verify directly that \( x^{-1} \circ x^{q^2-2} \) exchanges the remaining two points 0 and \( \infty \). \[\square\]

Next, we need to know the conjugations of a given 2-cycle \((b, \infty)\) with \( b \in \mathbb{F}_q \) by a degree-one polynomial \( x-a \) and by the monomial \( x^{q^2-2} \) respectively. For this purpose let us give first the following fact:

**Lemma 3.** The relation \( f \circ (u,v) = (f(u), f(v)) \circ f \) holds for any injective map \( f \) of sets from \( X \) to \( Y \) and any two distinct \( u,v \in X \).

*Proof.* Both \( f \circ (u,v) \) and \( (f(u), f(v)) \circ f \) send \( x \) to \( f(x) \) for any element \( x \in X \setminus \{u,v\} \), while both of them send \( u \) to \( f(v) \) and send \( v \) to \( f(u) \). \[\square\]

**Corollary 4.** The following identities hold as permutations of \( \mathbb{P}^1(\mathbb{F}_q) \):

- \((x-a) \circ (b, \infty) = (b-a, \infty) \circ (x-a)\) for any \( a,b \in \mathbb{F}_q \),
- \(x^{q^2-2} \circ (b, \infty) = (b^{q^2-2}, \infty) \circ x^{q^2-2}\) for any \( b \in \mathbb{F}_q \).

*Proof.* The results follows from Lemma 3 by taking \( X = Y := \mathbb{P}^1(\mathbb{F}_q) \), \( u := b, v := \infty \), letting \( f(x) := x-a \) and \( f(x) := x^{q^2-2} \) respectively. \[\square\]

Corollary 4 implies particularly that \( x^{q^2-2} \circ (0, \infty) = (0, \infty) \circ x^{q^2-2}, \) which is just equal to \( x^{-1} \) by Lemma 2. Moreover, it is interesting to note that the composition of any two of the three permutations \( x^{-1}, x^{q^2-2}, (0, \infty) \) of \( \mathbb{P}^1(\mathbb{F}_q) \) is equal to the third one. In other words, the subgroup of permutations of \( \mathbb{P}^1(\mathbb{F}_q) \) with \( q > 2 \) generated by \( x^{-1}, x^{q^2-2}, \) and \( (0, \infty) \) is exactly the Klein four-group.
Proof.

By applications of Corollary 4 and Lemma 2, we have
\[
\text{Corollary 5. } (a, \infty) = (x^{-1} + a) \circ x^{q-2} \circ (x - a) \text{ holds for any } a \in \mathbb{F}_q.
\]

\[x\]

which is equal to \((x^{-1} + a) \circ x^{q-2} \circ (x - a),\)

which is equal to \((x^{-1} + a) \circ x^{q-2} \circ (x - a)\) obviously. \qed

\[\]

3. Proof of Theorem 1

For a fixed positive integer \(k\), let us define maps \(F : \mathbb{F}_q^k \to \mathbb{F}_q^k\) and \(G : \mathbb{F}_q^k \to \mathbb{F}_q^k\) as follows, which will be used to illustrate the recipes in Theorem 7 and Theorem 8 respectively.

Define \(F : \mathbb{F}_q^k \to \mathbb{F}_q^k\) by sending \((a_1, a_2, \ldots, a_k)\) to \((b_1, b_2, \ldots, b_k)\), where \(b_i := c_{i,i}\) for \(1 \leq i \leq k\), and for any fixed \(1 \leq i \leq k\) we define \(c_{i,j}\) with \(0 \leq j \leq i\) inductively by \(c_{i,0} := 0\) and \(c_{i,j} := e_i^q - 1 + a_{i-j+1}\).

In order to define the map \(G : \mathbb{F}_q^k \to \mathbb{F}_q^k\), we need to use the map \(\Phi_{i} : \mathbb{F}_q^{i} \to \mathbb{F}_q^{i-1}\) for each \(2 \leq i \leq k\), which is defined to send \((e_1, e_2, \ldots, e_{i})\) to \(((e_2 - e_1)^{q-2}, (e_3 - e_1)^{q-2}, \ldots, (e_{i} - e_1)^{q-2})\). For ease of notation, we will drop the subscript \(i\) and write \(\Phi\) for \(\Phi_{i}\) when the value of \(i\) is clear from context.

Define \(G : \mathbb{F}_q^k \to \mathbb{F}_q^k\) by sending \((b_1, b_2, \ldots, b_k)\) to \((a_1, a_2, \ldots, a_k)\), where for \(1 \leq i \leq k\) we denote by \(a_i\) the first entry of \(\Phi^{-1}(b_1, b_2, \ldots, b_k)\), in which the map \(\Phi^{-1} : \mathbb{F}_q^k \to \mathbb{F}_q^{k-1}\) is really the composition \(\Phi_{k-i+2} \circ \Phi_{k-i+3} \circ \cdots \circ \Phi_k\) with the convention that \(\Phi^0\) is the identity map on \(\mathbb{F}_q^k\). More precisely, if we write

\[
\Phi^{-1}(b_1, b_2, \ldots, b_k) = (d_{1,i-1}, d_{2,i-1}, \ldots, d_{k-i+1,i-1})
\]

for \(1 \leq i \leq k\), then by definition we have

\[
(a_1, a_2, \ldots, a_k) := (d_{1,0}, d_{1,1}, \ldots, d_{1,k-1}).
\]

\[\]

Theorem 6. The maps \(F : \mathbb{F}_q^k \to \mathbb{F}_q^k\) and \(G : \mathbb{F}_q^k \to \mathbb{F}_q^k\) are inverses of one another. In particular, both \(F\) and \(G\) are bijections on \(\mathbb{F}_q^k\).

\[\]

Proof. It suffices to show that \(G \circ F\) is the identity map on \(\mathbb{F}_q^k\), since then \(F\) is injective and \(G\) is surjective, but since each of \(F\) and \(G\) is a map on \(\mathbb{F}_q^k\), it follows that \(F\) and \(G\) are bijective, whence \(F\) and \(G\) are inverse bijections since \(G \circ F\) is the identity map on \(\mathbb{F}_q^k\) again.

Now, let us show that \(G \circ F\) is the identity map on \(\mathbb{F}_q^k\). Given \(a_1, a_2, \ldots, a_k\) in \(\mathbb{F}_q\), recall that for \(1 \leq i \leq k\) the elements \(c_{i,j}\) with
0 \leq j \leq i$ are defined inductively by $c_{i,0} = 0$ and $c_{i,j} = c_{i,j-1}^{q-2} + a_{i-j+1}$.
Put $b_j = c_{j,j}$ for $1 \leq j \leq k$, so that by the definition of $F$ we get
\[ F(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k). \]
For $1 \leq i \leq k$ let us write
\[ \Phi^{i-1}(b_1, b_2, \ldots, b_k) = (d_{1,i-1}, d_{2,i-1}, \ldots, d_{k-i+1,i-1}), \]
then by the definition of $G$ we have
\[ G(b_1, b_2, \ldots, b_k) = (d_{1,0}, d_{1,1}, \ldots, d_{1,k-1}). \]
We will show by induction on $j$ that $d_{i,j} = c_{i+j,i}$ holds for all $i, j$ with $0 \leq j \leq k-1$ and $1 \leq i \leq k-j$. The base case $j = 0$ says that $d_{i,0} = c_{i,i}$ for $1 \leq i \leq k$, which is true since both sides equal $b_i$. Inductively, if $1 \leq j \leq k-1$, then for each $1 \leq i \leq k-j$ we have\[
d_{i,j} = (d_{i+1,j-1} - d_{1,j-1})^{q-2} = (c_{i+j,i+1} - c_{j,1})^{q-2} = (c_{i+j,i+1} - a_j)^{q-2} = (c_{i+j,i})^{q-2} = c_{i+j,i},\]
which concludes the induction. Thus
\[ G \circ F(a_1, a_2, \ldots, a_k) = G(b_1, b_2, \ldots, b_k) \]
\[ =(d_{1,0}, d_{1,1}, \ldots, d_{1,k-1}) = (c_{1,1}, c_{2,1}, \ldots, c_{k,1}) = (a_1, a_2, \ldots, a_k). \]
So $G \circ F$ is the identity map on $\mathbb{F}_q^k$, whence as explained above it follows that $F$ and $G$ are inverse bijections on $\mathbb{F}_q^k$.

\textbf{Theorem 7.} For any $a_1, a_2, \ldots, a_k$ in $\mathbb{F}_q$, denote
\[ (b_1, b_2, \ldots, b_k) := F(a_1, a_2, \ldots, a_k), \]
then
\[ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1) = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty), \]
where
\[ \nu(x) := x^{-1} \circ (x - a_k) \circ x^{-1} \circ (x - a_{k-1}) \circ \cdots \circ x^{-1} \circ (x - a_1) \]
is a degree-one rational function in $\mathbb{F}_q(x)$.

\textbf{Proof.} We prove it by induction on $k$. The base case $k = 1$ says
\[ x^{q-2} \circ (x - a_1) = x^{-1} \circ (x - a_1) \circ (b_1, \infty) \]
where $b_1 = F(a_1) = a_1$. This identity follows directly from Corollary 5. Inductively, suppose $k > 1$ and $a_1, a_2, \ldots, a_k$ are in $\mathbb{F}_q$. By definition, for $1 \leq i \leq k$ we have $c_{i,0} = 0$ and $c_{i,j} = c_{i,j-1}^{q-2} + a_{i-j+1}$ with $1 \leq j \leq i$.
Write $b_i = c_{i,i}$ for $1 \leq i \leq k$, so that
\[ (b_1, b_2, \ldots, b_k) = F(a_1, a_2, \ldots, a_k). \]
For ease of expression, let us denote
$$\lambda_i := x^{-1} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ x^{q-2} \circ \cdots \circ x^{q-2} \circ (x - a_i)$$
for $1 \leq i \leq k$, and denote
$$\rho_i := x^{q-2} \circ (x - a_i) \circ x^{q-2} \circ (x - a_{i-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1)$$
for $0 \leq i \leq k$ where by convention $\rho_0 = x$. Write
$$\sigma := x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1).$$

Hence we have
$$\infty = \rho_k = x^{q-2} \circ (x - a_k) \circ \rho_{k-1}$$
$$= x^{-1} \circ (0, \infty) \circ (x - a_k) \circ \rho_{k-1}$$
$$= x^{-1} \circ (x - a_k) \circ (a_k, \infty) \circ \rho_{k-1}$$
$$= \lambda_k \circ (c, \infty) \circ \rho_{k-1}.$$

We claim that
$$\lambda_{i+1} \circ (c, \infty) \circ \rho_i = \lambda_i \circ (c, \infty) \circ \rho_{i-1}$$
for $1 \leq i \leq k - 1$. Indeed, for each $i$ with $1 \leq i \leq k - 1$, we have
$$(c, \infty) \circ \rho_i$$
$$= (c, \infty) \circ x^{q-2} \circ (x - a_i) \circ \rho_{i-1}$$
$$= x^{q-2} \circ (c, \infty) \circ (x - a_i) \circ \rho_{i-1}$$
$$= x^{q-2} \circ (x - a_i) \circ (c, \infty) \circ \rho_{i-1}.$$

Thus by prepending $\lambda_{i+1}$ to these permutations we get
$$\lambda_{i+1} \circ (c, \infty) \circ \rho_i$$
$$= \lambda_{i+1} \circ x^{q-2} \circ (x - a_i) \circ (c, \infty) \circ \rho_{i-1}$$
$$= \lambda_i \circ (c, \infty) \circ \rho_{i-1},$$

which concludes the proof of the claim. So we have
$$\sigma = \lambda_k \circ (c, \infty) \circ \rho_{k-1} = \lambda_{k-1} \circ (c, \infty) \circ \rho_{k-2} = \cdots = \lambda_1 \circ (c, \infty) \circ \rho_0,$$
which by definition is equal to
$$x^{-1} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ x^{q-2} \circ \cdots \circ x^{q-2} \circ (x - a_1) \circ (b, \infty).$$

By inductive hypothesis, we have
$$x^{q-2} \circ (x - a_{k-1}) \circ x^{q-2} \circ (x - a_{k-2}) \circ \cdots \circ x^{q-2} \circ (x - a_1)$$
$$= x^{-1} \circ (x - a_{k-1}) \circ x^{-1} \circ (x - a_{k-2}) \circ \cdots \circ x^{-1} \circ (x - a_1) \circ (b_1, \infty) \circ (b, \infty) \circ \cdots \circ (b_{k-1}, \infty).$$
Therefore, we get
\[
\sigma = x^{-1} \circ (x - a_k) \circ x^{-1} \circ (x - a_{k-1}) \circ \cdots \circ x^{-1} \circ (x - a_1) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty),
\]
which concludes the induction and hence concludes the proof. □

**Theorem 8.** For any \(b_1, b_2, \ldots, b_k\) in \(\mathbb{F}_q\), denote
\[
(a_1, a_2, \ldots, a_k) := G(b_1, b_2, \ldots, b_k),
\]
then
\[
(b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1),
\]
where
\[
\mu(x) := (x + a_1) \circ x^{-1} \circ (x + a_2) \circ x^{-1} \circ \cdots \circ (x + a_k) \circ x^{-1}
\]
is a degree-one rational function in \(\mathbb{F}_q(x)\).

**Proof.** Let us prove it by induction on \(k\). The base case \(k = 1\) says
\[
(b_1, \infty) = (x + a_1) \circ x^{-1} \circ x^{q-2} \circ (x - a_1)
\]
where \(a_1 = G(b_1) = b_1\), which is true by Corollary 5. For the inductive step, assume that \(k > 1\) and \(b_1, b_2, \ldots, b_k\) are in \(\mathbb{F}_q\). Write
\[
\Phi^{i-1}(b_1, b_2, \ldots, b_k) = (d_{1,i-1}, d_{2,i-1}, \ldots, d_{k-i+1,i-1})
\]
for \(1 \leq i \leq k\), so that
\[
(a_1, a_2, \ldots, a_k) = G(b_1, b_2, \ldots, b_k) = (d_{1,0}, d_{1,1}, \ldots, d_{1,k-1}).
\]
For ease of expression, let us define
\[
\gamma_i := (x + b_1) \circ x^{-1} \circ (d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{i-1,1}, \infty) \circ x^{q-2} \circ (x - b_1)
\]
for \(1 \leq i \leq k\), and define
\[
\theta_i := (b_{i+1}, \infty) \circ (b_{i+2}, \infty) \circ \cdots \circ (b_k, \infty)
\]
for \(0 \leq i \leq k\), where by convention \(\gamma_1 = (x + b_1) \circ x^{-1} \circ x^{q-2} \circ (x - b_1)\) and \(\theta_k = x\). Denote
\[
\tau := (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty).
\]
Hence by Corollary 5 we have
\[
\tau = \theta_0 = (b_1, \infty) \circ \theta_1 = (x + b_1) \circ x^{-1} \circ x^{q-2} \circ (x - b_1) \circ \theta_1 = \gamma_1 \circ \theta_1.
\]
We claim that $\gamma_i \circ \theta_i = \gamma_{i+1} \circ \theta_{i+1}$ for $1 \leq i \leq k-1$. Indeed, for each $i$ with $1 \leq i \leq k-1$, we have

$$x^{q-2} \circ (x - b_1) \circ \theta_i$$

$$= x^{q-2} \circ (x - b_1) \circ (b_{i+1}, \infty) \circ \theta_{i+1}$$

$$= x^{q-2} \circ (b_{i+1} - b_1, \infty) \circ (x - b_1) \circ \theta_{i+1}$$

$$= (d_{i+1}, \infty) \circ x^{q-2} \circ (x - b_1) \circ \theta_{i+1}.$$ 

Thus by prepending

$$(x + b_1) \circ x^{-1} \circ (d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{k-1,1}, \infty)$$

to these permutations, we get

$$\gamma_i \circ \theta_i = (x + b_1) \circ x^{-1} \circ (d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{i-1,1}, \infty) \circ x^{q-2} \circ (x - b_1) \circ \theta_i$$

$$= (x + b_1) \circ x^{-1} \circ (d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{i-1,1}, \infty) \circ (d_{i,1}, \infty) \circ x^{q-2} \circ (x - b_1) \circ \theta_{i+1}$$

$$= \gamma_{i+1} \circ \theta_{i+1},$$

which concludes the proof of the claim. So we have

$$\tau = \gamma_1 \circ \theta_1 = \gamma_2 \circ \theta_2 = \cdots = \gamma_k \circ \theta_k$$

which by definition is equal to

$$(x + b_1) \circ x^{-1} \circ (d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{k-1,1}, \infty) \circ x^{q-2} \circ (x - b_1).$$

By inductive hypothesis, if we denote

$$(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{k-1}) := G(d_{1,1}, d_{2,1}, \ldots, d_{k-1,1}),$$

then

$$(d_{1,1}, \infty) \circ (d_{2,1}, \infty) \circ \cdots \circ (d_{k-1,1}, \infty)$$

$$= (x + \hat{a}_1) \circ x^{-1} \circ (x + \hat{a}_2) \circ x^{-1} \circ \cdots \circ (x + \hat{a}_{k-1}) \circ x^{-1} \circ x^{q-2} \circ (x - \hat{a}_{k-2}) \circ \cdots \circ x^{q-2} \circ (x - \hat{a}_1).$$

Note that for $1 \leq i \leq k-1$ we have $\hat{a}_i = a_{i+1}$ since

$$\Phi^{-1}(d_{1,1}, d_{2,1}, \ldots, d_{k-1,1}) = \Phi(b_1, b_2, \ldots, b_k).$$

Therefore, we have

$$\tau = (x + b_1) \circ x^{-1} \circ (x + a_2) \circ x^{-1} \circ (x + a_3) \circ x^{-1} \circ \cdots \circ (x + a_k) \circ x^{-1} \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_2) \circ x^{q-2} \circ (x - a_1),$$

which concludes the induction and the proof since $b_1 = a_1$. \qed
We remark that Theorem 7 gives a recipe for turning an algebraical presentation of a permutation on $\mathbb{P}^1(\mathbb{F}_q)$ of the form 
\[ \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1) \]
into a combinatorial presentation of the same permutation of the form 
\[ \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty), \]
and conversely Theorem 8 gives a recipe for turning a combinatorial presentation of a permutation on $\mathbb{P}^1(\mathbb{F}_q)$ of the form 
\[ \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) \]
into an algebraical presentation of the same permutation of the form 
\[ \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1), \]
where $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ are elements in $\mathbb{F}_q$ and $\mu(x), \nu(x)$ are degree-one rational functions in $\mathbb{F}_q(x)$.

In other words, for any fixed permutation $\sigma$ of $\mathbb{P}^1(\mathbb{F}_q)$, there are two natural maps $F : \mathcal{A}_{\sigma,k} \rightarrow \mathcal{C}_{\sigma,k}$ and $G : \mathcal{C}_{\sigma,k} \rightarrow \mathcal{A}_{\sigma,k}$ induced by the recipes illustrated in Theorem 7 and Theorem 8 respectively, where $\mathcal{A}_{\sigma,k}$ is the set of all representations of $\sigma$ of the form 
\[ \sigma = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1) \]
with $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$ and $\mu(x) \in \mathbb{F}_q(x)$ of degree one, and $\mathcal{C}_{\sigma,k}$ is the set of all representations of $\sigma$ of the form 
\[ \sigma = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) \]
with $b_1, b_2, \ldots, b_k \in \mathbb{F}_q$ and $\nu(x) \in \mathbb{F}_q(x)$ of degree one.

Now we are ready to prove Theorem 1, which asserts the natural maps $F : \mathcal{A}_{\sigma,k} \rightarrow \mathcal{C}_{\sigma,k}$ and $G : \mathcal{C}_{\sigma,k} \rightarrow \mathcal{A}_{\sigma,k}$ are inverses to one another.

**Proof of Theorem 1.** It suffices to show that both $G \circ F$ and $F \circ G$ are the identity maps, where $F : \mathcal{A}_{\sigma,k} \rightarrow \mathcal{C}_{\sigma,k}$ and $G : \mathcal{C}_{\sigma,k} \rightarrow \mathcal{A}_{\sigma,k}$ are the natural maps induced by Theorem 7 and Theorem 8 respectively.

Given any algebraic representation of $\sigma$ in the set $\mathcal{A}_{\sigma,k}$ of the form 
\[ \sigma = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1) \]
with $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$ and $\mu(x) \in \mathbb{F}_q(x)$ of degree one, by definition $F$ sends it to a combinatorial representation of $\sigma$ in $\mathcal{C}_{\sigma,k}$ of the form 
\[ \sigma = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty) \]
with $b_1, b_2, \ldots, b_k \in \mathbb{F}_q$ and $\nu(x) \in \mathbb{F}_q(x)$ of degree one, in which 
\[ (b_1, b_2, \ldots, b_k) := F(a_1, a_2, \ldots, a_k), \]
\[ \nu_0(x) := x^{-1} \circ (x - a_k) \circ x^{-1} \circ (x - a_{k-1}) \circ \cdots \circ x^{-1} \circ (x - a_1), \]
and \( \nu(x) := \mu(x) \circ \nu_0(x) \). This combinatorial representation is sent by \( \mathcal{G} \) to another algebraic representation of \( \sigma \) in \( A_{\sigma,k} \) of the form

\[
\sigma = \mu^*(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1)
\]

with \( a_1, a_2, \ldots, a_k \in \mathbb{F}_q \) and \( \mu^*(x) \in \mathbb{F}_q(x) \) of degree one, in which

\[
(a_1^*, a_2^*, \ldots, a_k^*) := (x + a_1) \circ x^{-1} \circ (x + a_2) \circ x^{-1} \circ \cdots \circ (x + a_k) \circ x^{-1},
\]
and \( \mu^*(x) := \nu(x) \circ \mu_0(x) \). By Theorem 6, the \( k \)-tuple

\[
(a_1^*, a_2^*, \ldots, a_k^*) = G(b_1, b_2, \ldots, b_k) = G \circ F(a_1, a_2, \ldots, a_k)
\]
equals \((a_1, a_2, \ldots, a_k)\), i.e., we have \( a_i^* = a_i \) for any \( 1 \leq i \leq k \). Thus \( \mu_0(x) \) and \( \nu_0(x) \) are inverse to one another, which implies that

\[
\mu^*(x) = \nu(x) \circ \mu_0(x) = \mu(x) \circ \nu_0(x) \circ \mu_0(x) = \mu(x).
\]

Hence, \( \mathcal{G} \circ \mathcal{F} \) is the identity map on the set \( A_{\sigma,k} \). Similarly, we can show that \( \mathcal{F} \circ \mathcal{G} \) is the identity map on \( C_{\sigma,k} \). Therefore, the natural maps \( \mathcal{F} : A_{\sigma,k} \to C_{\sigma,k} \) and \( \mathcal{G} : C_{\sigma,k} \to A_{\sigma,k} \) induced by Theorem 7 and Theorem 8 respectively are inverse bijections.

\[ \square \]

4. Consequences

Note that any permutation of \( \mathbb{F}_q \) can be extended uniquely to a permutation of \( \mathbb{P}^1(\mathbb{F}_q) \). Hence permutations of \( \mathbb{F}_q \) can be regarded as permutations of \( \mathbb{P}^1(\mathbb{F}_q) \) in this sense. By application of Theorem 1 to permutations of \( \mathbb{F}_q \), we get the following result:

**Corollary 9.** Suppose \( f \) is a permutation of \( \mathbb{F}_q \), and then extends \( f \) to a permutation of \( \mathbb{P}^1(\mathbb{F}_q) \) which fixes \( \infty \). Then in any representation

\[
f = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1)
\]

with \( a_1, a_2, \ldots, a_k \in \mathbb{F}_q \) and \( \mu(x) \in \mathbb{F}_q(x) \) of degree one, the rational function \( \mu(x) \) must be a degree-one polynomial in \( \mathbb{F}_q[x] \). Therefore, the set of all representations of \( f \) in the form of

\[
\mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1)
\]

with \( a_1, a_2, \ldots, a_k \in \mathbb{F}_q \) and \( \mu(x) \in \mathbb{F}_q[x] \) of degree one is naturally bijective to the set of all representations of \( f \) in the form of

\[
\nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty)
\]

with \( b_1, b_2, \ldots, b_k \in \mathbb{F}_q \) and \( \nu(x) \in \mathbb{F}_q(x) \) of degree one.
Proof. Note that the degree-one rational function \( \mu(x) \in \mathbb{F}_q(x) \) in any representation of the extended permutation \( f \) on \( \mathbb{P}^1(\mathbb{F}_q) \) in the form of

\[
f = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1)
\]

fixes the point \( \infty \), since all of \( f, x^{q-2} \), and \((x - a_i)\) fix \( \infty \). Hence the first assertion follows from the fact that the degree-one rational functions which fix \( \infty \) are precisely the degree-one polynomials. The second assertion is then a direct consequence of Theorem 1. \( \square \)

The following is a classical result on permutations of \( \mathbb{F}_q \), which was posed as a question by Straus, and was first proved by Carlitz [3] and then proved again by Zieve [22]:

**Theorem 10.** If \( q > 2 \) is a prime power, then every permutation of \( \mathbb{F}_q \) is a composition of \( x^{q-2} \) and degree-one polynomials over \( \mathbb{F}_q \).

Inspired by Theorem 10, Aksoy et al. [1] introduced the notion of the **Carlitz rank** of a permutation \( f \) of \( \mathbb{F}_q \), which means the smallest value of \( n \geq 0 \) in any representation of \( f \) of the form

\[
f = \theta_0 \circ x^{q-2} \circ \theta_1 \circ x^{q-2} \circ \cdots \circ x^{q-2} \circ \theta_n
\]

in which each \( \theta_i \) is a degree-one polynomial in \( \mathbb{F}_q[x] \). For a systematic study of the general theory of Carlitz ranks and their generalizations, see [11]. Carlitz ranks have been extensively studied in the literature, see for example the papers [1, 6–9, 14–17, 19, 20]. For the history of permutation polynomials over a finite field, see the papers [2–5, 10, 12, 13, 18, 21].

As the first consequence, our results provide a new perspective on the theory of Carlitz ranks from the point of view of combinatorics, instead of from the point of view of algebra as in the literature previously.

More precisely, by Corollary 9 the Carlitz rank of a permutation \( f \) of \( \mathbb{F}_q \) is exactly the smallest integer \( n \geq 0 \) for which the uniquely extended permutation of \( \mathbb{P}^1(\mathbb{F}_q) \), which is also denoted by \( f \) by abuse of language, admits a combinatorial representation of the form

\[
f = \nu(x) \circ (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_n, \infty)
\]

for some \( b_1, b_2, \ldots, b_n \in \mathbb{F}_q \) and some \( \nu(x) \in \mathbb{F}_q(x) \) of degree one.

Based on this new characterization of Carlitz rank, the following result on the computation of Carlitz rank has been obtained in [11]:

**Theorem 11.** Let \( f = \mu \circ \sigma \) be a permutation of \( \mathbb{F}_q \), where \( \mu \) is a degree-one rational function, and \( \sigma \) is a permutation of \( \mathbb{P}^1(\mathbb{F}_q) \) which moves \( s \) points in \( \mathbb{F}_q \) and has \( t \) nontrivial orbits. Define \( n := s + t \) if \( \sigma(\infty) = \infty \) and \( n := s + t - 1 \) if \( \sigma(\infty) \neq \infty \). Then the Carlitz rank of \( f \) is at most \( n \), and it equals \( n \) if in addition \( q \geq n + s + 2 \).
As the second consequence, our results provide new perspective to Theorem 10. Indeed, Theorem 10 follows easily from Corollary 9 in various ways, since any permutation \( f \) of \( \mathbb{F}_q \) has many different ways to be represented as

\[
f = (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_k, \infty)
\]

for some \( k \geq 1 \) and some \( b_1, b_2, \ldots, b_k \in \mathbb{F}_q \), and by Corollary 9 any such combinatorial representation of \( f \) gives rise to an algebraic representation of \( f \) as a composition of \( k \) copies of \( x^{q-2} \) and degree-one polynomials over \( \mathbb{F}_q \), which concludes our proof for Theorem 10.

More precisely, given any nontrivial permutation \( f \) of \( \mathbb{F}_q \), we can write \( f \) as the product of disjoint cycles of length at least two as follows:

\[
f = (b_{11}, b_{12}, \ldots, b_{1s_1})(b_{21}, b_{22}, \ldots, b_{2s_2}) \cdots (b_{t1}, b_{t2}, \ldots, b_{ts_t}),
\]

where \( t \geq 1 \) and each \( s_i \geq 2 \) with \( 1 \leq i \leq t \). Note that

\[
(b_{i1}, b_{i2}, \ldots, b_{is_i}) = (b_{i1}, \infty) \circ [(b_{is_i}, \infty) \circ \cdots \circ (b_{i2}, \infty) \circ (b_{i1}, \infty)],
\]

for any \( 1 \leq i \leq t \). Hence we get a representation of \( f \) of the form

\[
f = (b_1, \infty) \circ (b_2, \infty) \circ \cdots \circ (b_t, \infty),
\]

where \( s := \sum_{i=1}^t s_i \), \( k := s + t \), and denote by \((b_1, b_2, \ldots, b_k)\) the tuple

\[
((b_{11}, b_{1s_1}, \ldots, b_{11}), (b_{21}, b_{2s_2}, \ldots, b_{21}), \ldots, (b_{t1}, b_{ts_t}, \ldots, b_{t1})).
\]

By Theorem 8 if we denote

\[
(a_1, a_2, \ldots, a_k) := G(b_1, b_2, \ldots, b_k),
\]

then we obtain a representation of \( f \) of the form

\[
f = \mu(x) \circ x^{q-2} \circ (x - a_k) \circ x^{q-2} \circ (x - a_{k-1}) \circ \cdots \circ x^{q-2} \circ (x - a_1),
\]

where

\[
\mu(x) := (x + a_1) \circ x^{-1} \circ (x + a_2) \circ x^{-1} \circ \cdots \circ (x + a_k) \circ x^{-1} \in \mathbb{F}_q(x)
\]

is a rational function of degree one. Furthermore, \( \mu(x) \) is a degree-one polynomial over \( \mathbb{F}_q \) by Corollary 9. Therefore, we obtain an algebraic representation of the given permutation \( f \) of \( \mathbb{F}_q \) as a composition of \( k \) copies of \( x^{q-2} \) and \( k + 1 \) degree-one polynomials over \( \mathbb{F}_q \) as above. Moreover, if in addition \( q \geq 2s + t + 2 \), then this representation of \( f \) is optimal in the sense that it has the least possible copies of \( x^{q-2} \), since in this case the Carlitz rank of \( f \) is exactly \( k \) by Theorem 11.

In particular, by application of the above procedure to 2-cycles of the form \((0, a)\) with \( a \in \mathbb{F}_q^* \), we can recover uniformly both proofs for Theorem 10 in Carlitz [3] and Zieve [22], which involve some clever tricks and have not been related to each other previously.
First, let us review briefly the proofs for Theorem 10 in Carlitz 3 and Zieve 22. The starting point for both proofs is the same, i.e., it is enough to show the result in the special case that the permutation is a 2-cycle of the form \((0, a)\) with \(a \in \mathbb{F}_q^*\), since any permutation of \(\mathbb{F}_q\) can be written as a product of such 2-cycles. But Carlitz 3 and Zieve 22 give different expressions of the 2-cycle \((0, a)\) as compositions of \(x^{q-2}\) and degree-one polynomials. More precisely, Carlitz 3 relies on the mysterious observation that

\[
(0, a) = (-a^2 x) \circ x^{q-2} \circ (x - a) \circ x^{q-2} \circ (x + 1/a) \circ x^{q-2} \circ (x - a),
\]

but does not explain how it could be found; Zieve 22 observes that

\[
(0, a) = (-ax + a) \circ x^{q-2} \circ (-x + 1) \circ x^{q-2} \circ (-x + 1) \circ x^{q-2} \circ (x/a),
\]

which follows from the clever combination of Lemma 2 and the fact that the degree-one rational function \(1 - x^{-1}\) induces an order-three permutation of \(\mathbb{P}^1(\mathbb{F}_q)\) with a 3-cycle \((\infty, 1, 0)\).

Now, we apply our procedure illustrated above to 2-cycles \((0, a)\) with \(a \in \mathbb{F}_q^*\) to recover uniformly both of the above two observations in Carlitz 3 and Zieve 22 respectively. Indeed, for any \(a \in \mathbb{F}_q^*\), there are two equally natural ways to express the 2-cycle \((0, a)\) as a product of some 2-cycles of the form \((b, \infty)\) with \(b \in \mathbb{F}_q\). More precisely, by our procedure the 2-cycle \((0, a)\) can be expressed naturally as either \((0, a) = (a, \infty) \circ (0, \infty) \circ (a, \infty)\) or \((0, a) = (0, \infty) \circ (a, \infty) \circ (0, \infty)\). By some computation as explained above, the first identity leads exactly to the mysterious observation in Carlitz 3, and the second one gives

\[
(0, a) = (-a^2 x + a) \circ x^{q-2} \circ (x + a) \circ x^{q-2} \circ (x - 1/a) \circ x^{q-2} \circ x,
\]

which is the same as the observation in Zieve 22, as polynomials. To see this it is enough to normalize all except the first degree-one polynomials to be monic in the observation discovered by Zieve 22.

Finally, we emphasize that it is of essential importance to introduce the extra point \(\infty\) and to work over \(\mathbb{P}^1(\mathbb{F}_q)\) instead of \(\mathbb{F}_q\), although both Carlitz’s original Theorem 10 and the notion of Carlitz rank are only about permutations of \(\mathbb{F}_q\) and appear to have nothing to do with \(\infty\). It is clear that neither of the above two consequences can be possibly obtained by working over \(\mathbb{F}_q\) without introducing \(\infty\).

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