Discrete index transformations with squares of Bessel functions

S. Yakubovich

Department of Mathematics, Faculty of Sciences, University of Porto, Porto, Portugal

ABSTRACT

Discrete analogues of the index transforms with squares of Bessel functions of the first and second kind \( J_\nu(z) \), \( Y_\nu(z) \) are introduced and investigated. The corresponding inversion theorems for suitable classes of functions and sequences are established.

ARTICLE HISTORY

Received 24 March 2021
Accepted 9 April 2021

KEYWORDS

Bessel functions; modified Bessel functions; Struve functions; Lommel functions; Fourier series; index transforms

AMS SUBJECT CLASSIFICATIONS

45A05; 44A15; 42A16; 33C10

1. Introduction and preliminary results

In this paper, we continue to investigate mapping properties of discrete index transforms, introducing the following transformations between suitable sequences \( \{a_n\}_{n\geq1} \) and functions \( f \) in terms of the series and integrals which are associated with squares of Bessel functions of the first and second kind (cf. [1, Ch. 10]), namely,

\[
f(x) = \sum_{n=1}^{\infty} a_n \left[ J_{in/2}^2(x) + Y_{in/2}^2(x) \right], \quad x > 0,
\]

\[
a_n = \int_0^\infty \left[ J_{in/2}^2(x) + Y_{in/2}^2(x) \right] f(x) \, dx, \quad n \in \mathbb{N},
\]

\[
f(x) = \sum_{n=1}^{\infty} \frac{a_n}{\cosh(\pi n/2)} \text{Re} \left[ J_{in/2}^2(x) \right], \quad x > 0,
\]

\[
a_n = \frac{1}{\cosh(\pi n/2)} \int_0^\infty \text{Re} \left[ J_{in/2}^2(x) \right] f(x) \, dx, \quad n \in \mathbb{N},
\]

\[
f(x) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh(\pi n/2)} \text{Im} \left[ J_{in/2}^2(x) \right], \quad x > 0,
\]

CONTACT S. Yakubovich syakubov@fc.up.pt Department of Mathematics, Faculty of Sciences, University of Porto, Campo Alegre str., 687, Porto 4169-007, Portugal

© 2021 Informa UK Limited, trading as Taylor & Francis Group
We note that, for instance, continuum analogues of transformations (1.3) and (1.4) were considered by the author in [2]. Here \( i \) is the imaginary unit and \( \text{Re}, \text{Im} \) denote the real and imaginary parts of a complex-valued function. Bessel functions \( J_v(z), Y_v(z), z, v \in \mathbb{C} \), of the first and second kind, respectively, are solutions of the Bessel differential equation

\[
\frac{z^2 d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - v^2) u = 0.
\]  

These functions have the following asymptotic behaviour at infinity and near the origin

\[
J_v(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi}{4} (2v + 1) \right) [1 + O(1/z)], \quad z \to \infty, \quad |\arg z| < \pi,
\]

\[
J_v(z) = O(z^v), \quad z \to 0,
\]

\[
Y_v(z) = \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{\pi}{4} (2v + 1) \right) [1 + O(1/z)], \quad z \to \infty, \quad |\arg z| < \pi,
\]

\[
Y_v(z) = O(\log(|z|)), \quad z \to 0.
\]

The sum of squares \( J_v^2(z) + Y_v^2(z) \) is called the Nicholson function, and it is represented by the Nicholson integral (cf. [1, Entry 10.9.30])

\[
J_v^2(z) + Y_v^2(z) = \frac{8}{\pi^2} \int_0^\infty K_0(2z \sin(t)) \cosh(2vt) dt, \quad |\arg z| < \frac{\pi}{2},
\]

where \( K_v(z) \) is the modified Bessel function. Moreover, we mention the following Mellin–Barnes integral representations of the kernels of transformations (1.1)–(1.6) (see [3, Vol. III, Entries 8.4.19.17, 8.4.19.18, 8.4.20.19, 8.4.20.35])

\[
\frac{2\sqrt{\pi}}{\cosh(\pi n/2)} \text{Re} \left[ J_{in/2}^2(x) \right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma((s + in)/2)\Gamma((s - in)/2)\Gamma((1 - s)/2)}{\Gamma(s/2)[\Gamma(1 - s/2)]^2} x^{-s} \, ds,
\]

where \( x > 0, 0 < \gamma < 1, n \in \mathbb{N} \) and \( \Gamma(z) \) is the Euler gamma function,

\[
\frac{2\sqrt{\pi}}{\sinh(\pi n/2)} \text{Im} \left[ J_{in/2}^2(x) \right] = \sqrt{\pi} \left[ J_{in/2}(x)Y_{-in/2}(x) + J_{-in/2}(x)Y_{in/2}(x) \right]
\]

\[
= -\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma((s + in)/2)\Gamma((s - in)/2)}{\Gamma((s + 1)/2)\Gamma(1 - s/2)} x^{-s} \, ds,
\]

where \( x > 0, 0 < \gamma < 3/2, n \in \mathbb{N} \),

\[
\frac{\pi^{5/2}}{\cosh(\pi n/2)} \left[ J_{in/2}^2(x) + Y_{in/2}^2(x) \right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s + in}{2} \right)
\]

\[
\times \Gamma \left( \frac{s - in}{2} \right) \Gamma \left( \frac{1 - s}{2} \right) x^{-s} \, ds,
\]
where \( x > 0, 0 < \gamma < 1, n \in \mathbb{N} \). On the other hand, the Nicholson function can be represented in terms of the modified Bessel function via the integral (see [3, Vol. II, Entry 2.16.3.12])

\[
J_{in/2}^2(x) + Y_{in/2}^2(x) = \frac{8}{\pi^2} \cosh \left( \frac{\pi n}{2} \right) \int_0^\infty \frac{K_{in}(t) \, dt}{(t^2 + 4x^2)^{1/2}}, \quad x > 0.
\] (1.17)

The function \( K_{in}(t) \) is the kernel of the discrete Kontorovich–Lebedev transform recently investigated by the author [4]. It can be estimated by virtue of the Lebedev inequality (see [5, p. 219])

\[
|K_{in}(t)| \leq A t^{-1/4} \sqrt{\sinh(\pi n)}, \quad (1.18)
\]

where \( A > 0 \) is an absolute constant. Therefore, we get from (1.17)

\[
\left| J_{in/2}^2(x) + Y_{in/2}^2(x) \right| \leq \frac{8A \cosh (\pi n/2)}{\pi^2} \sqrt{\sinh(\pi n)} \int_0^\infty \frac{t^{-1/4} \, dt}{(t^2 + 4x^2)^{1/2}}
\]

\[
= \left( \frac{2}{\pi^2} \right)^{5/4} A \Gamma \left( \frac{3}{8} \right) \Gamma \left( \frac{1}{8} \right) x^{-1/4} \coth^{1/2} \left( \frac{\pi n}{2} \right)
\]

\[
\leq \left( \frac{2}{\pi^2} \right)^{5/4} A \Gamma \left( \frac{3}{8} \right) \Gamma \left( \frac{1}{8} \right) \coth^{1/2} \left( \frac{\pi}{2} \right) x^{-1/4},
\]

i.e.

\[
\left| J_{in/2}^2(x) + Y_{in/2}^2(x) \right| \leq B x^{-1/4}, \quad (1.19)
\]

where \( x > 0, B \) is a positive constant. For the modulus of the Bessel function, we have the inequality from [2]

\[
|J_{it}(x)| \leq e^x \sqrt{\frac{\sinh(\pi \tau)}{\pi \tau}}, \quad x \geq 0, \quad \tau \in \mathbb{R} \setminus \{0\}.
\] (1.20)

Hence, we find for \( x \geq 0, n \in \mathbb{N} \),

\[
\left| \Re \left[ \frac{J_{in/2}^2(x)}{\cosh(\pi n/2)} \right] \right| \leq \frac{2 e^{2x}}{\pi n} \tanh \left( \frac{\pi n}{2} \right) \leq \frac{2 e^{2x}}{\pi n},
\] (1.21)

\[
\left| \Im \left[ \frac{J_{in/2}^2(x)}{\sinh(\pi n/2)} \right] \right| \leq \frac{2 e^{2x}}{\pi n}.
\] (1.22)

Finally, we mention Struve functions \( H_{\nu}(z), K_{\nu}(z) \), which are related with the Bessel function of the second kind \( Y_{\nu}(z) \) via the equality (see [1, Entry 11.2.5])

\[
H_{\nu}(z) = K_{\nu}(z) + Y_{\nu}(z).
\] (1.23)

The Struve function \( K_{\nu}(z) \) has the integral representation (cf. [1, Entry 11.5.2])

\[
K_{\nu}(z) = \frac{2(z/2)^{\nu}}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^\infty e^{-zt} (1 + t^2)^{\nu-1/2} \, dt, \quad \Re(z) > 0,
\] (1.24)
and behaves at infinity as \( K_\nu(z) = O(z^{-1}), \ z \to \infty \) (cf. [1, Entry 11.6.1]). The Mellin–Barnes representation for this function is given in [3, Vol. III, Entry 8.4.25.3]

\[
K_\nu(x) = \frac{\cos(\pi \nu)}{4\pi^3 i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right) \times \Gamma\left(\frac{s+\nu+1}{2}\right) \Gamma\left(\frac{1-\nu-s}{2}\right) \left(\frac{x}{2}\right)^{-s} ds, \quad |\text{Re}\nu| < \gamma < 1 - \text{Re}\nu. \tag{1.25}
\]

We note that these preliminary results will be applied in the sequel to prove inversion theorems for discrete transformations (1.1)–(1.6).

2. Inversion theorems

We begin with

**Theorem 2.1:** Let a sequence \( a = \{a_n\}_{n \in \mathbb{N}} \in l_1 \), i.e.

\[
\|a\|_{l_1} = \sum_{n=1}^{\infty} |a_n| < \infty. \tag{2.1}
\]

Then discrete transformation (1.1) can be inverted by the formula

\[
a_n = \sinh\left(\frac{\pi n}{2}\right) \int_0^\infty \Phi_n(x)f(x) \, dx, \quad n \in \mathbb{N}, \tag{2.2}
\]

where the kernel \( \Phi_n(x) \) is defined by

\[
\Phi_n(x) = x \int_0^\pi J_0(2x \cosh(u)) \sinh(2u) \sin(nu) \, du, \quad x > 0, \quad n \in \mathbb{N}, \tag{2.3}
\]

and integral (2.2) converges in the improper sense.

**Proof:** The key ingredient to prove (2.2) will be the following relatively convergent integral (see [3, Vol. II, Entry 2.13.25.9])

\[
\int_0^\infty xJ_0(2x \cosh(u)) \left[ J_{\ln/2}^2(x) + Y_{\ln/2}^2(x) \right] \, dx = \frac{2 \sin(nu)}{\pi \sin(2u) \sinh(\pi n/2)}. \tag{2.4}
\]

In fact, substituting \( f \) by formula (1.1) and \( \Phi_n(x) \) by (2.3) on the right-hand side of (2.2), we change the order of the proper integration and summation to obtain

\[
\sinh\left(\frac{\pi n}{2}\right) \int_0^\infty \Phi_n(x)f(x) \, dx = \sinh\left(\frac{\pi n}{2}\right) \lim_{T \to \infty} \sum_{m=1}^\infty a_m \int_0^\pi \sinh(2u) \sin(nu) \times \int_0^T xJ_0(2x \cosh(u)) \left[ J_{\ln/2}^2(x) + Y_{\ln/2}^2(x) \right] \, dx \, du, \tag{2.5}
\]
where the interchange follows immediately from assumption (2.1) and the estimate for each fixed \( T > 0 \) by virtue of (1.8) and (1.19)

\[
\sum_{m=1}^{\infty} |a_m| \int_0^\pi \sinh(2u) |\sin(nu)| \sinh(2u) \left[ J_{im/2}^2(x) + Y_{im/2}^2(x) \right] dx du \\
\leq C \|a\|_{l1} \int_0^T x^{1/4} dx \int_0^\pi \sinh(u) \cosh^{1/2}(u) du = \frac{8}{15} C \frac{T^{5/4}}{\|a\|_{l1}} \left[ \cosh^{3/2}(\pi) - 1 \right],
\]

where \( C > 0 \) is an absolute constant. The problem is to justify the passage to the limit in (2.5) under the summation sign when \( T \to \infty \). To do this, we appeal to asymptotic behaviour (1.8) of Bessel function \( J_0 \), the modified Bessel function \( K_0 \) and integral representation (1.13) of the Nicholson function. Then for a big enough positive \( T \), we have via integration by parts and Entry 2.16.2.2 in [3, Vol. II]

\[
\left| \int_T^\infty xJ_0(2x \cosh(u)) \left[ J_{im/2}^2(x) + Y_{im/2}^2(x) \right] dx \right| \\
= \frac{8 \sqrt{2}}{\pi^{5/2}} \left| \int_T^\infty \sqrt{x} \cos \left( 2x \cosh(u) - \frac{\pi}{4} \right) \left[ 1 + O \left( \frac{1}{x} \right) \right] \\
\times \int_0^\infty K_0(2x \sinh(t)) \cos(mt) dt dx \right| \\
\leq \frac{8 \sqrt{2}}{\pi^{5/2}} \left| - \frac{\sqrt{T}}{2 \cosh(u)} \sin \left( 2T \cosh(u) - \frac{\pi}{4} \right) \\
\times \int_0^\infty K_0(2T \sinh(t)) \cos(mt) dt \\
- \frac{1}{4 \cosh(u)} \int_T^\infty \frac{1}{\sqrt{x}} \sin \left( 2x \cosh(u) - \frac{\pi}{4} \right) \sinh(t) \cos(mt) dt dx \\
+ \frac{1}{\cosh(u)} \int_T^\infty \sqrt{x} \sin \left( 2x \cosh(u) - \frac{\pi}{4} \right) \\
\times \int_0^\infty K_1(2x \sinh(t)) \sinh(t) \cos(mt) dt dx \right| \\
+ O \left( \int_T^\infty \frac{1}{\sqrt{x}} \int_0^\infty K_0(2xt) dt dx \right) \\
\leq \frac{8 \sqrt{2}}{\pi^{5/2}} \left[ \frac{1}{4 \cosh(u) \sqrt{T}} \int_0^\infty K_0(t) dt + \frac{1}{8 \cosh(u)} \int_T^\infty \frac{dx}{x^{3/2}} \int_0^\infty K_0(t) dt \\
- \frac{1}{4 \cosh(u)} \int_T^\infty \frac{1}{x^{3/2}} \int_0^\infty K_0(t) \cosh \left( \frac{t}{2x} \right) dt dx + O \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
\leq \frac{8\sqrt{2}}{\pi^{5/2}} \left[ \frac{\pi}{4 \cosh(u)\sqrt{T}} + \frac{1}{2 \cosh(u)\sqrt{T}} \int_0^\infty K_0(t) \cosh \left( \frac{t}{2} \right) \, dt + O \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
= O \left( \frac{1}{\sqrt{T}} \right), \quad T \to \infty,
\]
(2.6)
where the estimate is uniform with respect to \( m \in \mathbb{N} \) and \( u \in [0, \pi] \). Consequently, returning to (2.5), we pass to the limit when \( T \to \infty \) under the summation sign and use (2.4) to derive
\[
\sinh \left( \frac{\pi n}{2} \right) \int_0^\infty \Phi_n(x)f(x) \, dx = \frac{2}{\pi} \sinh \left( \frac{\pi n}{2} \right) \times \sum_{m=1}^\infty \frac{a_m}{\sinh(\pi m/2)} \int_0^\pi \sin(nu) \sin(mu) \, du = a_n,
\]
completing the proof of Theorem 2.1.

Concerning an inversion formula for the transformation (1.2), we have the following result.

**Theorem 2.2:** Let \( f \) be a complex-valued function on \( \mathbb{R}_+ \) which is represented by the integral
\[
f(x) = x \int_{-\pi}^\pi J_0(x \cosh(u)) \varphi(u) \, du, \quad x > 0,
\]
(2.7)
where \( \varphi(u) = \psi(u) \sinh(2u) \) and \( \psi \) is a \( 2\pi \)-periodic function, satisfying the Lipschitz condition on \([-\pi, \pi]\), i.e.
\[
|\psi(u) - \psi(v)| \leq C|u - v|, \quad \forall \, u, v \in [-\pi, \pi],
\]
(2.8)
where \( C > 0 \) is an absolute constant. Then for all \( x > 0 \) the following inversion formula for transformation (1.2) holds
\[
f(x) = \sum_{n=1}^\infty \sinh \left( \frac{\pi n}{2} \right) \Phi_n(x)a_n,
\]
(2.9)
where \( \Phi_n \) is defined by (2.3).

**Proof:** Plugging the right-hand side of representation (2.7) in (1.2), we change the order of integration and employ (2.4) to obtain
\[
a_n = \frac{2}{\pi \sinh(\pi n/2)} \int_{-\pi}^\pi \varphi(u) \sin(nu) \sinh(2u) \, du.
\]
(2.10)
This interchange is valid by virtue of (2.6), which guarantees the uniform convergence of integral (2.4) with respect to \( u \in [-\pi, \pi] \). Hence, following ideas which are elaborated in
[4], we substitute $a_n$ by formula (2.10) and $\Phi_n$ by (2.3) into the partial sum of the series (2.9) $S_N(x)$, and it becomes

$$S_N(x) = \frac{x}{\pi} \sum_{n=1}^{N} \int_{-\pi}^{\pi} J_0(x \cosh(t)) \sinh(2t) \sin(nt) \, dt \int_{-\pi}^{\pi} \frac{\varphi(u)}{\sinh(2u)} \sin(nu) \, du. \quad (2.11)$$

Hence, calculating the sum via the known identity

$$\sum_{n=1}^{N} \sin(nt) \sin(nu) = \frac{1}{4} \left[ \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} - \frac{\sin((2N + 1)(u + t)/2)}{\sin((u + t)/2)} \right], \quad (2.12)$$

and invoking the definition of $\varphi$, equality (2.11) turns to be as follows:

$$S_N(x) = \frac{x}{4\pi} \int_{-\pi}^{\pi} J_0(x \cosh(t)) \sinh(2t) \times \int_{-\pi}^{\pi} \frac{\varphi(u) + \varphi(-u)}{\sinh(2u)} \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du \, dt$$

$$= \frac{x}{4\pi} \int_{-\pi}^{\pi} J_0(x \cosh(t)) \sinh(2t) \times \int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du \, dt. \quad (2.13)$$

Since $\psi$ is $2\pi$-periodic, we treat the latter integral with respect to $u$ as follows:

$$\int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du$$

$$= \int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du$$

$$= \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.$$

Moreover,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du - [\psi(t) - \psi(-t)]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.$$

When $u + t > \pi$ or $u + t < -\pi$, then we interpret the value $\psi(u + t) - \psi(t)$ by formulas

$$\psi(u + t) - \psi(t) = \psi(u + t - 2\pi) - \psi(t - 2\pi),$$

$$\psi(u + t) - \psi(t) = \psi(u + t + 2\pi) - \psi(t + 2\pi),$$
respectively. Analogously, the value \( \psi(-u - t) - \psi(-t) \) can be treated. Then due to Lipschitz condition (2.8), we have the uniform estimate for any \( t \in [-\pi, \pi] \)

\[
\frac{|\psi(u + t) - \psi(t) + \psi(-u - t) - \psi(-u - t)|}{|\sin(u/2)|} \leq 2C \left| \frac{u}{\sin(u/2)} \right|.
\]

Therefore, owing to the Riemann–Lebesgue lemma

\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du = 0
\]

for all \( t \in [-\pi, \pi] \). Besides, returning to (2.13), we estimate the iterated integral (see (1.8))

\[
\int_{-\pi}^{\pi} |J_0(x \cosh(t)) \sinh(2t)| \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du \, dt \\
\times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du \, dt \leq \frac{4C_1}{\sqrt{x}} \int_{0}^{\pi} \frac{\sinh(2t)}{\cosh^{1/2}(t)} \, dt \int_{-\pi}^{\pi} \left| \frac{u}{\sin(u/2)} \right| \, du < \infty, \quad x > 0,
\]

where \( C_1 > 0 \) is constant. Consequently, via the dominated convergence theorem, it is possible to pass to the limit when \( N \to \infty \) under the integral sign, and recalling (2.14), we derive

\[
\lim_{N \to \infty} \frac{x}{4\pi} \int_{-\pi}^{\pi} J_0(x \cosh(t)) \sinh(2t) \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du \, dt = 0.
\]

Hence, combining with (2.13), we obtain by virtue of the definition of \( \varphi \) and \( f \)

\[
\lim_{N \to \infty} S_N(x) = \frac{x}{2} \int_{-\pi}^{\pi} J_0(x \cosh(t)) \left[ \varphi(t) + \varphi(-t) \right] \, dt = f(x),
\]

where integral (2.7) converges since \( \varphi \in C[-\pi, \pi] \). Thus we established (2.9), completing the proof of Theorem 2.2. \( \blacksquare \)

In order to invert discrete transformation (1.3), we will need the following lemmas.

**Lemma 2.1:** Let \( u \in [-\pi, \pi] \), \( n \in \mathbb{N} \). Then, the following formula takes place

\[
\int_{0}^{\infty} xS_{-1,0}(2x \cosh(u)) \text{Re} \left[ j_{in/2}^2(x) \right] \, dx = \frac{\pi \sin(nu)}{\sinh(2u) \sinh(\pi n/2)},
\]

where \( S_{\mu,v}(z) \) is the Lommel function (cf. [1, Sec. 11.9]) and integral (2.15) converges absolutely.

**Proof:** The absolute converges of integral (2.15) follows immediately from (1.8) and asymptotic behaviour of the Lommel function at infinity [1, Entry 11.9.9] \( S_{\mu,v}(z) = \)
\(O(z^{\mu-1}), \ z \to \infty\). Hence, recalling (1.14), Entry 8.4.27.3 in [3, Vol. III] and Stirling’s asymptotic formula for the gamma function [3], which gives for \(0 < \gamma < 1/2\)
\[
\frac{\Gamma((s + in)/2)\Gamma((s - in)/2)\Gamma((1 - s)/2)}{\Gamma(s/2)[\Gamma(1 - s/2)]^2} = O\left(|s|^{\gamma - 3/2}\right), \ |s| \to \infty,
\]
we appeal to the Parseval equality for the Mellin transform [3, Vol. III] to derive
\[
\int_0^\infty xS_{-1,0}(2x \cosh(u)) \Re \left[ J_{in/2}(x) \right] dx
= \frac{\cosh(\pi n/2)}{8\pi^{3/2}i \cosh^2(u)} \int_{-\infty}^{\gamma + i\infty} (1 - s/2) \Gamma\left(\frac{s + in}{2}\right) \Gamma\left(\frac{s - in}{2}\right) \Gamma\left(\frac{1 - s}{2}\right) (\cosh(u))^s ds.
\]
But the latter Mellin–Barnes integral in (2.16) can be calculated in terms of the Gauss hypergeometric function [5] via Entry 8.4.49.21 in [3, Vol. III]. Indeed, we obtain
\[
\int_{\gamma + i\infty}^{\gamma - i\infty} (1 - s/2) \Gamma\left(\frac{s + in}{2}\right) \Gamma\left(\frac{s - in}{2}\right) \Gamma\left(\frac{1 - s}{2}\right) (\cosh(u))^s ds
= \frac{\pi n}{2 \cosh(u) \sinh(\pi n/2)} 2F_1\left(\frac{1 + in}{2}, \frac{1 - in}{2}; \frac{3}{2}; -\sinh^2(u)\right). \tag{2.17}
\]
Finally, the Gauss hypergeometric function in (2.17) can be simplified due to Entry 7.3.1.91 in [3, Vol. III], and it becomes
\[
\frac{\pi n}{2 \cosh(u) \sinh(\pi n/2)} 2F_1\left(\frac{1 + in}{2}, \frac{1 - in}{2}; \frac{3}{2}; -\sinh^2(u)\right) = \frac{\pi \sin(nu)}{\sin(2u) \sinh(\pi n/2)}.
\]
Thus, combining with (2.16), we end up with (2.15), completing the proof of Lemma 2.1. \(\blacksquare\)

**Lemma 2.2:** Let \(x > 0, n \in \mathbb{N}\). Then
\[
\Re \left[ \frac{J_{in/2}(x)}{\cosh(\pi n/2)} \right] = \frac{2}{\pi} \int_0^\infty \cos(nt)H_0(2x \cosh(t)) \, dt, \tag{2.18}
\]
where \(H_0\) is the Struve function (1.23) of zero index and integral (2.18) converges absolutely.

**Proof:** Since we find from (1.24)
\[
K_0(x) = \frac{2}{\pi} \int_0^\infty e^{-x \sinh(t)} \, dt, \tag{2.19}
\]
and \(\sqrt{x} Y_0(x), \ x > 0,\) is bounded (see (1.10), (1.12)), we have via (1.23) and Entry 2.4.4.4, 2.4.18.4 in [3, Vol. I]
\[
\int_0^\infty |\cos(nt)H_0(2x \cosh(t))| \, dt \leq \frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-2x \cosh(t) \sinh(y)} \, dy \, dt
\]
where $K_0$ is the modified Bessel function of zero index, we establish the absolute convergence of integral (2.18). Then, appealing to (1.13) and the following integrals (cf. [3, Vol. II, Entries 2.13.5.2, 2.16.3.12 (cf. (1.17))])

$$
\frac{4}{\pi} \int_0^\infty \cos(nt) Y_0(2x \cosh(t)) \, dt = |J_{in}(x)|^2 - |Y_{in}(x)|^2,
$$

(2.21)

$$
\int_0^\infty K_{in}(2x \sinh(t)) \, dt = \frac{\pi^2}{8 \cosh(\pi n/2)} \left[ J_{in/2}^2(x) + Y_{in/2}^2(x) \right],
$$

(2.22)

we obtain

$$
\frac{4}{\pi} \int_0^\infty \cos(nt) H_0(2x \cosh(t)) \, dt = \frac{1}{\cosh(\pi n/2)} \left[ J_{in/2}^2(x) + Y_{in/2}^2(x) \right] + |J_{in/2}(x)|^2 - |Y_{in/2}(x)|^2.
$$

Hence the final result follows immediately from the relation between Bessel functions of the first and the second kind (see [1, Entry 10.2.3])

$$
Y_\nu(z) = \frac{1}{\sin(\pi \nu)} \left[ J_\nu(z) \cos(\pi \nu) - J_{-\nu}(z) \right].
$$

(2.23)

From (2.18) and (2.20) we arrive at

**Corollary 2.1:** The following inequality holds

$$
\left| \frac{\text{Re} \left[ J_{in/2}^2(x) \right]}{\cosh(\pi n/2)} \right| \leq \frac{1}{2} \left[ J_0^2(x) + Y_0^2(x) + \frac{\Gamma^2(1/4)}{\pi \sqrt{\pi x}} \sup_{u>0} \sqrt{u} \, Y_0(u) \right].
$$

(2.24)

**Theorem 2.3:** Let a sequence $a = \{a_n\}_{n \in \mathbb{N}} \in l_1$, i.e. satisfy condition (2.1). Then, discrete transformation (1.3) can be inverted by the formula

$$
a_n = \frac{1}{\pi^2} \sinh(\pi n) \int_0^\infty \Psi_n(x)f(x) \, dx, \quad n \in \mathbb{N},
$$

(2.25)

where the kernel $\Psi_n(x)$ is defined by

$$
\Psi_n(x) = x \int_0^\pi S_{-1,0}(2x \cosh(u)) \sinh(2u) \sin(nu) \, du, \quad x > 0, \quad n \in \mathbb{N},
$$

(2.26)

and integral (2.25) converges absolutely.
**Proof:** In the same manner as in the proof of Theorem 2.1, we employ (1.3), (1.8), (1.10), (2.25), (2.26) and Corollary 2.1 to write the equality

\[
\frac{1}{\pi^2} \sinh (\pi n) \int_{0}^{\infty} \Psi_n(x)f(x) \, dx = \frac{1}{\pi^2} \sinh (\pi n) \sum_{m=1}^{\infty} \frac{a_m}{\cosh(\pi m/2)} \times \int_{0}^{\pi} \sinh(2u) \sin(nu) \int_{0}^{\infty} x S_{-1,0}(2x \cosh(u)) \text{Re} \left[ J_{m/2}^2(x) \right] \, dx \, du,
\]

(2.27)

where the interchange of the order of integration and summation is permitted by virtue of Fubini’s theorem, owing to the estimate

\[
\sum_{m=1}^{\infty} |a_m| \int_{0}^{\pi} \sinh(2u) |\sin(nu)| \int_{0}^{\infty} x |S_{-1,0}(2x \cosh(u)) \text{Re} \left[ J_{m/2}^2(x) \right]| \, dx \, du \leq [\cosh(\pi) - 1] \|a\|_1 \int_{0}^{\infty} x |S_{-1,0}(2x)| \left[ J_{0}^2 \left( \frac{x}{\cosh(\pi)} \right) + Y_{0}^2 \left( \frac{x}{\cosh(\pi)} \right) \right] \sup_{t>0} \sqrt{t} Y_0(t) \, dx < \infty.
\]

Hence, recalling equality (2.15), we establish (2.25) from (2.27), completing the proof of Theorem 2.3.

Further, let us consider discrete transformation (1.4). We have

**Theorem 2.4:** Let \( f \) be a complex-valued function on \( \mathbb{R}_+ \) which is represented by the integral

\[
f(x) = x \int_{-\pi}^{\pi} S_{-1,0}(x \cosh(u)) \varphi(u) \, du, \quad x > 0,
\]

(2.28)

where \( \varphi(u) = \psi(u) \sinh(2u) \) and \( \psi \) is a \( 2\pi \)-periodic function, satisfying Lipschitz condition (2.8) on \([-\pi, \pi]\). Then for all \( x > 0 \), the inversion formula for transformation (1.4) takes place

\[
f(x) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \sinh(\pi n) \Psi_n(x) a_n,
\]

(2.29)

where \( \Psi_n \) is defined by (2.26).

**Proof:** Following the same scheme as in the proof of Theorem 2.2, we substitute the right-hand side of representation (2.28) in (1.4), we interchange the order of integration and use (2.15) to get

\[
a_n = \frac{\pi}{\sinh(\pi n)} \int_{-\pi}^{\pi} \frac{\varphi(u) \sin(nu)}{\sinh(2u)} \, du.
\]

(2.30)

The interchange is allowed by virtue of Lemma 2.1, where the absolute and uniform convergence of integral (2.15) with respect to \( u \in [-\pi, \pi] \) is shown. Hence, we plug \( a_n \) by...
formula (2.30) and \( \Psi_n \) by (2.26) into the partial sum of series (2.29) to obtain

\[
S_N(x) = \frac{x}{\pi} \sum_{n=1}^{N} \int_{-\pi}^{\pi} S_{-1,0}(x \cosh(t)) \sinh(2t) \sin(nt) \, dt \int_{-\pi}^{\pi} \frac{\varphi(u)}{\sinh(2u)} \sin(nu) \, du.
\]

(2.31)

Hence, making use of (2.12) and the definition of \( \varphi \), we find

\[
S_N(x) = \frac{x}{4\pi} \int_{-\pi}^{\pi} S_{-1,0}(x \cosh(t)) \sinh(2t) \int_{-\pi}^{\pi} [\psi(u) - \psi(-u)]
\times \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du \, dt.
\]

(2.32)

Hence as in the proof of Theorem 2.2, we take into account (2.28) and the definition of \( \varphi \) to deduce

\[
\lim_{N \to \infty} S_N(x) = \frac{x}{2} \int_{-\pi}^{\pi} S_{-1,0}(x \cosh(t)) [\varphi(t) + \varphi(-t)] \, dt = f(x).
\]

Theorem 2.4 is proved. \[\blacksquare\]

In order to invert discrete transformation (1.5), we will prove the following lemma.

**Lemma 2.3:** Let \( u \in [-\pi, \pi] \), \( n \in \mathbb{N} \). Then, the formula

\[
\int_{0}^{\infty} \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \Im \left[ J_{\infty/2}^2(x) \right] \, dx = \frac{\sin(nu)}{\pi \sinh(2u) \cosh(\pi n/2)}
\]

(2.33)

holds valid, and integral (2.33) converges absolutely.

**Proof:** Indeed, recalling (1.15), we write the representation of the transformation kernel in (1.5), using Entry 2.12.7.4 in [3, Vol. II], in terms of the integral

\[
\frac{\Im \left[ J_{\infty/2}^2(x) \right]}{\sinh(\pi n/2)} = -\frac{2}{\pi} \int_{0}^{\infty} \cos(nt) J_0(2x \cosh(t)) \, dt.
\]

(2.34)

Then, we have

\[
\frac{1}{\sinh(\pi n/2)} \int_{0}^{\infty} xK_0(2x \cosh(u)) \Im \left[ J_{\infty/2}^2(x) \right] \, dx
\]

\[
= -\frac{2}{\pi} \lim_{T \to \infty} \int_{0}^{T} \cos(nt) \int_{0}^{T} xK_0(2x \cosh(u)) J_0(2x \cosh(t)) \, dx \, dt
\]

(2.35)

where the interchange of the order of integration is guaranteed by the estimate (see (1.8) and (2.19))

\[
\int_{0}^{\infty} |\cos(nt)| \int_{0}^{T} xK_0(2x \cosh(u)) |J_0(2x \cosh(t))| \, dx \, dt
\]
\[ \leq \frac{1}{\pi \sqrt{2 \cosh(u)}} \sup_{y > 0} |\sqrt{y}J_0(y)| \int_0^\infty \frac{dt}{\cosh^{1/2}(t)} \int_0^T dx \sqrt{x} \]
\[ = \frac{\sqrt{T} \Gamma^2(1/4)}{(2\pi)^{3/2} \cosh(u)} \sup_{y > 0} |\sqrt{y}J_0(y)|. \]

Then using the asymptotic behaviour of the Bessel and Struve functions and integration by parts, we motivate the passage to the limit under the integral sign in (2.35), since for a big enough \( T \)
\[ \left| \int_T^\infty xK_0(2x \cosh(u))J_0(2x \cosh(t)) \, dx \right| \]
\[ = \frac{1}{\pi \sqrt{T} \cosh(u) \cosh^{1/2}(t)} \times \left| \int_T^\infty \frac{1}{\sqrt{x}} \cos \left( 2x \cosh(t) - \frac{\pi}{4} \right) \, dx + O \left( \frac{1}{\sqrt{T}} \right) \right| \]
\[ = O \left( \frac{1}{(T \cosh(t))^{1/2}} \right), \quad T \to \infty. \]

Meanwhile, the integral with respect to \( x \) in (2.35) over \((0, \infty)\) is calculated in [3, Vol. III, Entry 2.7.16.3], and we find
\[ \int_0^\infty xK_0(2x \cosh(u))J_0(2x \cosh(t)) \, dx = [2\pi \cosh(t)(\cosh(t) + \cosh(u))]^{-1}. \quad (2.36) \]

Therefore, combining with (2.35), we obtain (see [3, Vol. I, Entry 2.5.46.6, Vol. II, Entry 2.16.6.1])
\[ \frac{1}{\sinh(\pi n/2)} \int_0^\infty xK_0(2x \cosh(u)) \Im \left[ J_{in/2}^2(x) \right] \, dx \]
\[ = -\frac{1}{\pi^2} \int_0^\infty \frac{\cos(nt) \, dt}{\cosh(t)(\cosh(t) + \cosh(u))} \]
\[ = \frac{1}{\pi^2 \cosh(u)} \int_0^\infty \frac{\cos(nt) \int_0^\infty e^{-y(\cosh(u) + \cosh(t))} \, dy \, dt}{\cosh(t)} \]
\[ - \frac{1}{\pi^2 \cosh(u)} \int_0^\infty \frac{\cos(nt) \, dt}{\cosh(t)} \]
\[ = \frac{2 \sin(nu)}{\pi \sin(\pi n) \sin(2u)} - \frac{1}{2\pi \cosh(\pi n/2) \cosh(u)}. \quad (2.37) \]

Finally, from (1.15) and the reciprocal Mellin transform
\[ \frac{1}{\sinh(\pi n/2)} \int_0^\infty \Im \left[ J_{in/2}^2(x) \right] \, dx = -\frac{1}{2 \cosh(\pi n/2)}, \]
and this yields (2.33). The absolute convergence of the integral follows, in turn, from (1.8) and (1.9) and the asymptotic behaviour of the Struve function (cf. [1, Entry 11.6.1]) since
\[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \sim -\frac{1}{4\pi x^2 \cosh^3(u)}, \quad x \to \infty. \quad (2.38) \]
Lemma 2.3 is proved. ■

Recalling integral representation (2.34), we find

\[ \text{Corollary 2.2: The following inequality holds} \]

\[ \left| \Im \left[ J_{\text{im}}^{2/3}(x) \right] \right| \leq \frac{\Gamma^2(1/4)}{2\pi \sqrt{\pi x}} \sup_{u > 0} \sqrt{u} J_0(u). \]  \hspace{1cm} (2.39)

Now we are ready to prove

\[ \text{Theorem 2.5: Let a sequence } a = \{a_n\}_{n \in \mathbb{N}} \in l_1. \text{ Then discrete transformation} \ (1.5) \text{ can be inverted by the formula} \]

\[ a_n = \sinh (\pi n) \int_0^\infty \Omega_n(x)f(x) \, dx, \quad n \in \mathbb{N}, \] \hspace{1cm} (2.40)

where the kernel \( \Omega_n(x) \) is defined by

\[ \Omega_n(x) = \int_0^\pi \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \sinh(2u) \sin(nu) \, du, \quad x > 0, \quad n \in \mathbb{N}, \] \hspace{1cm} (2.41)

and integral (2.40) converges absolutely.

**Proof:** In fact, we have from (1.5) and (2.40)

\[ \sinh (\pi n) \int_0^\infty \Omega_n(x)f(x) \, dx \]

\[ = \sinh (\pi n) \sum_{m=1}^\infty \frac{a_m}{\sinh(\pi m/2)} \]

\[ \times \int_0^\pi \sinh(2u) \sin(nu) \int_0^\infty \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \Im \left[ J_{\text{im}}^{2/3}(x) \right] \, dx \, du, \] \hspace{1cm} (2.42)

where the interchange of the order of integration and summation is valid by virtue of Fubini’s theorem via the asymptotic behaviour of Struve function (2.38), Corollary 2.2, owing to the estimate

\[ \sum_{m=1}^\infty \frac{|a_m|}{\sinh(\pi m/2)} \int_0^\pi \sinh(2u) |\sin(nu)| \]

\[ \times \int_0^\infty \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \Im \left[ J_{\text{im}}^{2/3}(x) \right] \, dx \, du \]

\[ \leq \frac{\Gamma^2(1/4) \|a\|_{l_1}}{2\pi^{3/2}} \sup_{t > 0} \sqrt{t} J_0(t) \int_0^\pi \sinh(2u) \]
\[
\times \int_0^\infty \left| xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right| \frac{dx \, du}{\sqrt{x}} = 2 \Gamma^2(1/4) \frac{\|a\|_1}{\pi^{3/2}} \left[ \cosh^{1/2}(\pi) - 1 \right] \sup_{t > 0} \left| \sqrt{t} J_0(t) \right| \times \int_0^\infty \left| xK_0(2x) - \frac{1}{\pi} \right| \frac{dx}{\sqrt{x}} < \infty.
\]

Consequently, returning to (2.42) and appealing to Lemma 2.3, we calculate the integral with respect to \(x\) by formula (2.33) and then, due to the orthogonality of trigonometric functions, end up with inversion formula (2.40). Theorem 2.5 is proved.

Finally, we will proceed with a proof of the inversion theorem for transformation (1.6).

**Theorem 2.6:** Let \(f\) be a complex-valued function on \(\mathbb{R}_+\) which is represented by the integral

\[
f(x) = \int_{-\pi}^\pi \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \varphi(u) \, du, \quad x > 0,
\]

where \(\varphi(u) = \psi(u) \sinh(2u)\) and \(\psi\) is a 2\(\pi\)-periodic function, satisfying Lipschitz condition (2.8) on \([-\pi, \pi]\). Then for all \(x > 0\), the inversion formula for transformation (1.6) holds

\[
f(x) = \frac{1}{2} \sum_{n=1}^{\infty} \sinh(\pi n) \Omega_n(x) a_n,
\]

where \(\Omega_n\) is defined by (2.41).

**Proof:** Substituting the right-hand side of (2.44) in (1.6), we interchange the order of integration and use (2.33) to derive

\[
a_n = \frac{2}{\pi \sinh(\pi n)} \int_{-\pi}^\pi \varphi(u) \sin(nu) \frac{1}{\sinh(2u)} \, du.
\]

This interchange is verified via the estimate

\[
\frac{1}{\sinh(\pi n/2)} \int_0^\infty \left| \text{Im} \left[ f_{in/2}^2(x) \right] \right| \int_{-\pi}^\pi \left[ xK_0(2x \cosh(u)) - \frac{1}{\pi \cosh(u)} \right] \varphi(u) \, du \, dx
\]

\[
\leq \frac{\Gamma^2(1/4)}{2\sqrt{\pi}} \sup_{t > 0} \left| \sqrt{t} J_0(t) \right| \max_{t \in [-\pi, \pi]} |\varphi(t)| \int_{-\pi}^\pi \left| xK_0(2x) - \frac{1}{\pi} \right| \frac{dx}{\sqrt{x}} \int_{-\pi}^\pi \cos^{-3/2}(u) \, du
\]

\[
\leq \frac{\Gamma^2(1/4)}{\sqrt{\pi}} \sup_{t > 0} \left| \sqrt{t} J_0(t) \right| \max_{t \in [-\pi, \pi]} |\varphi(t)| \int_0^\infty \left| xK_0(2x) - \frac{1}{\pi} \right| \frac{dx}{\sqrt{x}} < \infty,
\]

which is an immediate consequence of (2.38) and (2.39). Therefore, in the same manner as above, we find the following representation for the partial sum of series (2.44)

\[
S_N(x) = \frac{1}{\pi} \sum_{n=1}^N \int_{-\pi}^\pi \left[ xK_0(2x \cosh(t)) - \frac{1}{\pi \cosh(t)} \right] \sinh(2t) \sin(nt) \, dt
\]
\[ \times \int_{-\pi}^{\pi} \frac{\varphi(u)}{\sinh(2u)} \sin(nu) \, du \]

\[ = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[ xK_0(2x \cosh(t)) - \frac{1}{\pi \cosh(t)} \right] \sinh(2t) \int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \]

\[ \times \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du \, dt. \]

Hence as in the proof of Theorem 2.2, we take into account (2.43) and the definition of \( \varphi \) to deduce

\[ \lim_{N \to \infty} S_N(x) = \frac{1}{2} \int_{-\pi}^{\pi} \left[ xK_0(2x \cosh(t)) - \frac{1}{\pi \cosh(t)} \right] [\varphi(t) + \varphi(-t)] \, dt = f(x). \]

Theorem 2.6 is proved.  

\[ \square \]

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

The work was partially supported by CMUP, which was financed by national funds through FCT (Portugal) under the project with reference UIDB/00144/2020.

**References**

[1] Olver FWJ, Olde Daalhuis AB, Lozier DW, et al. NIST digital library of mathematical functions. Release 1.0.17 of 2017-12-22. Available from: http://dlmf.nist.gov/

[2] Yakubovich S. Index transforms with the squares of Bessel functions. Integral Transforms Spec Funct. 2016;27(12):981–994.

[3] Prudnikov AP, Brychkov YA, Marichev OI. Integrals and series: Vol. I: Elementary functions. New York (NY): Gordon and Breach; 1986; Vol. II: Special functions. New York (NY): Gordon and Breach; 1986; Vol. III: More special functions. New York (NY): Gordon and Breach; 1990.

[4] Yakubovich S. Discrete Kontorovich-Lebedev transforms. Ramanujan J. DOI: 10.1007/s11139-020-00313-7

[5] Yakubovich S. Index transforms. Singapore: World Scientific Publishing Company; 1996.