Partially coherent matter wave soliton solutions:
Multimode approach

Jun Chen\(^1\), Qiang Lin\(^2\)

\(^1\)College of Optical and Electronic Technology, China Jiliang University, Hangzhou 310018, China
\(^2\)Institute of Optics, Department of Physics, Zhejiang University, Hangzhou 310027, China

E-mail: \texttt{\textasciitilde chenj
un.sun@gmail.com}

Abstract. We introduce a multimode approach to describe the partially coherent matter wave (PCMW), and construct a set of localized soliton solutions of the one-dimensional multimode nonlinear Schrödinger equation with a periodic lattice potential and a periodic interaction potential. Based on our analytical expression of the multimode PCMW, we investigate the density distribution and coherence property of a two-mode case.

1. Introduction

The similarity between the physics of the interacting Bose gas and that of light in nonlinear medium has inspired many interesting findings in the coherent matter wave regime, such as four wave mixing\(^1\), matter wave amplification\(^2\), transverse atomic laser modes\(^3\), and matter wave soliton\(^4\). In a realistic system the cold Bose gas is not fully coherent, and hence can be considered as partially coherent matter wave (PCMW)\(^5, 6, 7\). To analyze the dynamics of PCMW, many theoretic models have been presented. Most of them rely on the assumption that the zero momentum state is macroscopically populated and atoms in all other states are small perturbations\(^8\). These models work well when the temperature is well below the critical temperature \(T_\text{c}\), but if the temperature increases to around or slightly above \(T_\text{c}\), it is not such appropriate to use the above assumption since the influences from other states can not be regarded as small perturbations. We remind that in optics, multimode theory has been extensively used to analyze the dynamics of partially coherent light propagating in a nonlinear medium\(^9, 10\). Inspired by the analogies between atom optics and traditional optics, we explore the matter wave counterpart of this approach using a similarity transformation method described by Belmonte-Beitia et.al\(^11, 12\). A difference with respect to this earlier work is that here the matter wave field is partially coherent and the evolution equation is a multimode nonlinear Schrödinger equation. We find analytical soliton solutions of the PCMW with a space modulated periodic potential, which is composed of a lattice potential and a space-modulated periodic interaction potential. We also analyze the density distribution and the coherence property of a two-mode PCMW.

2. Description

Our starting point is the dimensionless Heisenberg equation for the Bose field operator \(\hat{\psi}(x, t)\)

\[
i \frac{\partial}{\partial t} \hat{\psi}(x, t) = \left( -\frac{\partial^2}{\partial x^2} + V(x) + \alpha(x) \langle \hat{\psi}^\dagger(x, t) \hat{\psi}(x, t) \rangle \right) \hat{\psi}(x, t),
\]  

(1)
where the mean-field approximation have been used, i.e. \( \hat{\psi}^*(x,t)\hat{\psi}(\hat{x},t) = \langle \hat{\psi}^*(x,t)\hat{\psi}(\hat{x},t) \rangle \), \( V(x) \) is a lattice potential, and \( \alpha(x) \) is a nonlinear coefficient related to the interatomic interactions. Both \( V(x) \) and \( \alpha(x) \) are periodic potentials, whose periods are required to be equal, say \( L \), i.e. \( V(x) = V(x + L) \) and \( \alpha(x) = \alpha(x + L) \). The explicit forms of the potential \( V(x) \) and the nonlinearity \( \alpha(x) \) are left to be determined later.

Just as it is useful in Optics to introduce a mode expansion of the electric field operator, we can expand the matter wave field operator \( \psi(x,t) \) in a basis of orthogonal eigenfunctions \( \{\phi_m(x)\}\)[13]

\[
\hat{\psi}(x,t) = \sum_m \phi_m(x)\hat{c}_m(t),
\]

where the modal label \( m \) stands for the complete set of quantum numbers necessary to characterize that mode, and \( N \) is the amount of the modes included in the field. \( \hat{c}_m \) (\( \hat{c}^\dagger_m \)) are bosonic annihilation operators (creation operators) for a particle in mode \( m \), and are uncorrelated with one another, that is \( \langle \hat{c}^\dagger_m \hat{c}_n \rangle = \delta_{mn} \). The eigenfunctions \( \{\phi_m(x)\} \) are orthogonal, i.e., \( \int \phi_n^*(x)\phi_m(x)dx = \lambda_m\delta_{mn} \) with \( \lambda_m \) being the modal occupancy, and satisfy a set of stationary multimode nonlinear Schrödinger (NLS) equations

\[
E_m\phi_m(x) = \left( \frac{\partial^2}{\partial x^2} + V_m(x) + \alpha(x)\sum_n |\phi_n(x)|^2 \right)\phi(x),
\]

where \( E_m \) is the eigenenergy of mode \( m \), and \( V_m(x) \) represents the linear potential acting on the \( m \)-mode matter wave. For the case of simplicity, we assume that the nonlinearity \( \alpha(x) \) is mode-independent, and the periodic linear potential has a zero mean value, \( \langle V_m(x) \rangle = \frac{1}{L} \int_0^L V_m(x)dx = 0 \).

In order to find an analytical solution to the above space-modulated nonlinear problem of PCMW, we would look for a transformation reducing Eq.(3) to a form

\[
E_m\Phi_m(X) = \left( -\frac{\partial^2}{\partial X^2} + A\sum_n |\Phi_n(X)|^2 \right)\Phi_m(X),
\]

where \( X \equiv X(x) \) is a new spatial variable, \( E_m \) and \( A \) are constants. In order to connect Eq.(3) and Eq.(4), we follow the idea of J. Belmote-Beitia et al.[11, 12], and introduce a transformation

\[
\phi_m(x) = \rho_m(x)\Phi_m(X)
\]

where \( \rho_m(x) \) is a periodic function with the same period as the lattice potential \( V(x) \), i.e. \( \rho_m(x) = \rho_m(x + L) \), and \( \Phi(X) \) is undetermined. Using the relation that \( \{\phi_m(x)\} \) are solutions of Eq.(3) and \( \{\Phi_m(X)\} \) satisfy Eq.(4), we can obtain a set of equations and find a relation that \( X' = \frac{1}{\rho''_m(x)} \) where the prime represents the derivative with respect to \( x \). Knowing that \( X \) is the independent variable of Eq.(4), in other words, \( X \) should not be affected by mode numbering, we can deduce that \( \rho_m(x) \) are independent of mode numbering, i.e. \( \rho_m(x) = \rho_n(x) = \rho(x) \) \((m \neq n)\). The parameters, \( X(x) \), \( V_m(x) \), \( \alpha(x) \), and \( E_m \), can be expressed through \( \rho(x) \) as

\[
X(x) = \int_0^x \frac{1}{\rho^2(s)}ds,
\]

\[
V_m(x) = E_m + \rho''(x)/\rho(x) - E_m/\rho^4(x),
\]

\[
\alpha(x) = A/\rho^4(x),
\]

\[
E_m = E_m < 1/\rho^4(x) > - < \rho''(x)/\rho(x) > .
\]

where \( X(0) = 0 \) and \( \lim_{x \to \pm \infty} X(x) = \pm \infty \).
According to the above results, Eqs.(6-9), given the explicit expression of \( \rho(x) \), we can construct a solvable nonlinear lattice system Eq.(3) with \( A \) and \( E_m \)’s. However, there is still one function undetermined, \( \Phi_m(X) \). Note that \( \{ \Phi_m(X) \} \) satisfy Eq.(4), and this set of equations is integrable for an arbitrary set of real nondegenerate \( \{ E_m \} \) under the condition that the nonlinearity is attractive \( (A < 0) \). A standard multisolution expression of \( \{ \Phi_m(X) \} \) can be found \( (A = -2) \)\cite{14, 15}.

\[
\Phi = D^{-1}e
\]  

(10)

where \( \Phi = [\Phi_1 \cdots \Phi_N]^T \) and \( e = [e_1 \cdots e_N]^T \) are one-column matrices, \( D^{-1} \) is the inversion matrix of a \( N \times N \) symmetric matrix \( D \) with

\[
D_{jm}(X) = \delta_{jm} + \frac{e_j(X)e_m(X)}{k_j + k_m}, \quad e_j(X) = \sqrt{2k_j a_j} \exp[ik_jX_j].
\]  

(11)

The mode coefficient \( a_j \) is a real positive constant, and \( k_j = \sqrt{-E_j} \), \( X_j = X - X_j \) and \( X_j \) are shifts and locations for each fundamental soliton, respectively. With the knowledge of \( \rho(x) \) and \( \Phi(X) \), the exact solution of the multimode NLS equation (3) can be constructed.

To be more specific we impose the period of the lattice potential to be \( L = \pi \), and choose the periodic function \( \rho(x) \) as \( \rho(x) = 1 + b \cos 2x \), where \( b \) is a real quantity with the magnitude smaller than unity, \( |b| < 1 \). For the case of a two-mode \( (N = 2) \) PCMW field, by defining the parameters \( a_j \) \( (j = 1, 2) \), and the locations for each soliton at \( X_1, X_2 \), the solitary solutions of Eq.(3) can be obtained according to \( \phi_m(x) = \rho(x)\Phi_m(X) \). The intensity profiles of the two-mode soliton in a free space \( (b = 0) \) and a periodic lattice \( (b = 0.3) \) are shown in Fig.2(a) and Fig.2(b), respectively. It can be seen that the existence of the space modulated periodicity causes a Bloch oscillation in the envelope of the soliton profile. Decreasing the eigen energy (decreasing \( E_j \)) of the solitary wave or increasing the depth of the lattice potential(increasing \( b \)), this oscillation could become significant.

We also consider the coherence property of the two-mode PCMW solitary field. According to the mode expansion of Eq.(2), the normalized correlation function can be expressed as \( \mu(x_1, x_2) = \Gamma(x_1, x_2)/\sqrt{\Gamma(x_1)\Gamma(x_2)} \), with \( \Gamma(x_1, x_2) = \langle \hat{\psi}^\dagger(x_1, t)\hat{\psi}(x_2, t) \rangle = \sum_m \phi_m^*(x_1)\phi_m(x_2) \) and \( \Gamma(x_j) = \Gamma(x_j, x_j) \). We compare the coherence distribution of a one-mode coherent solitary matter wave and that of the two-mode PCMW soliton in a periodic lattice potential. As shown in Fig.2, the density profiles of a symmetric two-mode PCMW soliton with \( a_1 = a_2 = 3 \) and \( X_1 = X_2 = 0 \), is similar to that of the one-mode coherent wave soliton, but the coherent properties are totally different. The one-mode soliton is a purely coherent matter wave field with a uniform coherence distribution, but the two-mode solitary PCMW is a partially coherent field whose coherence distribution is curved and space modulated by the periodic lattice potential.

![Figure 1](image)

Figure 1. The density profile of an asymmetric two-mode PCMW soliton with \( a_1 = a_2 = 3, X_1 = 2, X_2 = -2 \) in (a) a free space \((b = 0)\), with \( E_1 = -1 \), and \( E_2 = -0.25 \); and (b) a lattice potential \((b = 0.3)\), with \( E_1 = 0.09 \), and \( E_2 = -0.0225 \).
3. Conclusion

In this manuscript, we expand the PCMW field operator in a basis of orthogonal coherent modes. The evolution of the mode eigenfunctions can be describe by a multimode NLS equation. Using the similarity transformation, we construct a set of analytical solutions of the multimode NLS equation with a periodic lattice potential and a periodic interaction potential. To be more specific, we take a two-mode matter wave for an example to show that the existence of the periodic lattice causes a Bloch oscillation in the envelope of the density profile of the matter wave. The depth of the lattice potential and the magnitude of the mode eigen energies would influence the significance of this oscillation. Moreover, we compare the coherence properties of a coherent matter wave and a two-mode PCMW. It is shown that although the coherence distribution are totally different, the density profiles can still be quite similar.

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