Glauber-Sudarshan P function for a single-emitter laser

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Abstract. We consider the model of a single-emitter laser: an incoherently pumped single two-level system interacting with a single cavity mode of finite finesse. In the stationary regime using normally-ordered diagonal representation of the density matrix over the coherent states we derive equation for the phase-averaged Glauber-Sudarshan P function. In the strong-coupling regime between atom and cavity mode the asymptotic solution of this equation is obtained. This solution is a linear combination of two basis solutions. We analyze one of them which was ignored in previous work without any discussions.

1. Introduction
The master equation written in terms of coherent states is often used by different authors for investigation of a single-emitter laser problem (see e.g. [1, 2, 3, 4]). In the stationary regime from this equation one can obtain the system of differential equations for the Glauber-Sudarshan P-function (hereinafter just P-function) and some additional quasi-probability or equivalent closed equation only for the P-function. The latter is the second order differential equation with the polynomial coefficients – one of the simplest equations which is nonintegrable in explicit form. This equation have been obtained in [4], but its derivation was not so clearly: it was made incorrect assumption, that two dimensional quasi-probability current density \( \vec{J}(z, z^*) \) defined in this work is equal to zero when its divergence is vanished (see the text after Eqs.(36) in [4]). Also, without any discussions it was ignored the second basis solution in the asymptotic solution of the equation for the P-function in the case of strong-coupling regime.

Thus in our paper at first we shall perform accurate derivation of the stationary equation for the P function and show that latter equation can be done only for the phase-averaged P function. After we shall discuss the second asymptotic basis solution of this equation which arises in the strong coupling regime.

2. Stationary equation for the phase-averaged P function
The single-emitter laser consists of the incoherently pumped two-level atom which interacts with the single cavity mode of finite finesse. Just four constants characterize this laser: \( \kappa/2 \) – the decay rate of the cavity mode, \( \gamma/2 \) – the atomic polarization decay rate, \( \Gamma \) – the pumping rate from the low level \( |1\rangle \) to the excited level \( |2\rangle \) and \( g \) – the coupling constant between atom and cavity mode. Following the lead of [4] let us write the master equation for the normally-ordered
diagonal representation of the density matrix of our laser $\hat{\rho}(z, z^*)$ over the coherent states $|z\rangle$ of the field and over the projections on the atomic states (Eqs.(36) in [4])

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial z} J(z, z^*) + \frac{\partial}{\partial z^*} J^*(z, z^*) = \text{div} \vec{J}(z, z^*),$$

$$\frac{\partial D}{\partial t} = \langle (\Gamma - \gamma) P - (\Gamma + \gamma) D + \frac{\partial}{\partial z} \left( \frac{\kappa}{2} z D + g \rho_{12} \right) \rangle + \frac{\partial}{\partial z^*} \left( \frac{\kappa}{2} z^* D + g \rho_{21} \right) - 2g |z^* \rho_{21} + z \rho_{12}|,$$

$$\frac{\partial \rho_{21}}{\partial t} = \frac{(\Gamma + \gamma)}{2} \rho_{21} + \frac{\partial}{\partial z} \left( \frac{\kappa}{2} z \rho_{21} \right) + \frac{\partial}{\partial z^*} \left( \frac{\kappa}{2} z^* \rho_{21} \right) + g \left[ z D - \frac{1}{2} \frac{\partial}{\partial z^*} (P + D) \right],$$  \hspace{0.5cm} (1)

where was introduced the following quasi-probabilities: $\rho_{ik}(z, z^*) = \langle i | \hat{\rho}(z, z^*) | k \rangle$ for $i, k = 1, 2$, the difference $D(z, z^*) = \rho_{22}(z, z^*) - \rho_{11}(z, z^*)$, and the P-function $P(z, z^*) = \rho_{11}(z, z^*) + \rho_{22}(z, z^*)$ was included in the vector of quasi-probability current density $\vec{J}(z, z^*) = (J(z, z^*), J^*(z, z^*))$ as $J(z, z^*) = \frac{\kappa}{2} z P(z, z^*) - g \rho_{21}(z, z^*)$.

The physical meaning of the additional quasi-probabilities $D(z, z^*)$ and $\rho_{21}(z, z^*)$ are expressed through the following integrals $\langle \hat{D} \rangle = \int D(z, z^*) d^2z$ and $\langle \hat{\sigma} \rangle = \int \rho_{21}(z, z^*) d^2z$, where $\langle \hat{D} \rangle$ and $\langle \hat{\sigma} \rangle$ are the mean values of the atomic inversion and polarization correspondingly.

Let us show that in the stationary regime we can equate to zero only the quantity $\langle \hat{\sigma} \rangle$ which is a result of averaging of $\vec{J}(z, z^*)$ all over phase. For that we rewrite the right-hand member of the first equation in (1) in terms of polar coordinates $I$ and $\varphi$ as $z = \sqrt{I} \exp(i\varphi)$ and average it all over $\varphi$. The result is

$$\langle \text{div} \vec{J}(z, z^*) \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i\varphi} \left( \sqrt{I} \frac{\partial}{\partial I} - \frac{i}{2\sqrt{I}} \frac{\partial}{\partial \varphi} \right) J(I, \varphi)$$

$$+ e^{i\varphi} \left( \sqrt{I} \frac{\partial}{\partial I} + \frac{i}{2\sqrt{I}} \frac{\partial}{\partial \varphi} \right) J^*(I, \varphi) d\varphi = \sqrt{I} \frac{\partial J(I)}{\partial I} + \frac{1}{2\sqrt{I}} j(I),$$  \hspace{0.5cm} (2)

where $j(I) = \kappa \sqrt{I} P(I) - g|\rho_{21}(I) + \rho_{12}(I)|$ and $P(I) = \left( \frac{1}{2\pi} \right) \int_{0}^{2\pi} P(I, \varphi) d\varphi$, $\rho_{21}(I) = \left( \frac{1}{2\pi} \right) \int_{0}^{2\pi} e^{i\varphi} \rho_{21}(I, \varphi) d\varphi$, $\rho_{12}(I) = \left( \frac{1}{2\pi} \right) \int_{0}^{2\pi} e^{i\varphi} \rho_{12}(I, \varphi) d\varphi$. In the stationary regime the obtained expression for $\langle \text{div} \vec{J}(z, z^*) \rangle_{\varphi}$ should be equated to zero, what yields

$$\sqrt{I} \frac{\partial j(I)}{\partial I} + \frac{1}{2\sqrt{I}} j(I) = 0. \hspace{0.5cm} (3)$$

The solution of this equation is $j(I) = j_0/\sqrt{I}$, where $j_0$ is a constant. To find the latter let us integrate the product $\pi \sqrt{I} \times j(I)$ all over $I$, the result is

$$\kappa \langle n \rangle - g(\langle \hat{a}^\dagger \hat{\sigma} \rangle + \langle \hat{\sigma}^\dagger \hat{a} \rangle) = \pi J_0 \int_{0}^{\infty} dI,$$  \hspace{0.5cm} (4)

where $\langle n \rangle = \int |z|^2 P(z, z^*) d^2z = \pi \int_{0}^{\infty} IP(I) dI$ is the photon mean value and $\langle \hat{a}^\dagger \hat{\sigma} \rangle = (\hat{\sigma}^\dagger \hat{a})^* = \int z^* \rho_{21}(z, z^*) d^2z = \pi \int_{0}^{\infty} \sqrt{I} \rho_{21}(I) dI$. From Eq. (4) it immediately follows that $j_0 = 0$, what is easy to see obtaining the same equation from the Eqs. (1). Thus we accurately demonstrated that in the stationary regime the phase-averaged current density of the quasi-probability for a single-emitter laser is equal to zero, i.e. $j(I) = 0$. 


Using the obtained equality \( j(I) = 0 \) together with the two last phase-averaged equations from Eqs. (1) one can derive the following stationary equation for the phase-averaged P-function (Eqs. (41)-(48) in [4])
\[
P''(I) + p(I)P'(I) + q(I)P(I) = 0,
\]
where prime indicates the derivative with respect to \( I \) and
\[
p(I) = a_{12} (I - I_{-1}) (I - I_{+1}) / I^2 (I - I_{00}),
\]
\[
q(I) = a_{22} (I - I_{-2}) (I - I_{+2}) / I^2 (I - I_{00}).
\]
All constants \( a_{ik} \) and \( I_{\pm1}, I_{\pm2}, I_{00} \) depend on system parameters (see Appendix in [4]).

3. Asymptotic solution in the strong coupling regime

Let us return to Eq. (5). In the strong coupling regime between atom and cavity mode \( g/\kappa >> 1 \) in Eq. (5) the small parameter arises \( \varepsilon = \kappa^2 / 4g^2 \) in front of the higher derivatives. If we apply WKB method [5, 6] to this equation we obtain the following zeroth-order approximation result which consists of two basis solutions
\[
P_0(I) = P_1(I) + P_2(I) = c_1 (I_{-1} - I)^{f_1} (I_{+1} - I)^{f_2} \exp \left( \frac{a_{22}}{a_{12}} I \right) + c_2 (I_{-1} - I)^{-(f_1+1)} (I_{+1} - I)^{-(f_2+1)} (I_{00} - I)^2 I^{1-a_{12}} \times \exp \left( -\frac{a_{22}}{a_{12}} I \right) \exp \left( \frac{cI_0^2}{2} \frac{1}{I} \right).
\]

In [4] the first solution \( P_1(I) \) was investigated in detail. One of the result of this investigation reads that the function \( P_1(I) \) is determined on the restricted domain \( 0 \leq I < I_{-1} \). Now let us analyze the second basis solution \( P_2(I) \). And the first conclusion about the latter is that the \( P_2(I) \) is not bounded at \( I = 0 \). In the same time the solution \( P_1(I) \) gives a clear physical result for some limiting cases when \( I \to 0 \) (see Eq. (52) in [4]). Thus it is natural to drop the second solution at least on the domain where \( P_1(I) \) manifests physical behaviour, i.e. on the domain \( 0 \leq I < I_{-1} \). Note that \( P_2(I) \) can probably shows some physical behaviour after the point \( I = I_{-1} \) and for the large values of the variable \( I \), because of the exponential decreasing. But for considered in [4] range of the laser parameters such behaviour failed to reveal.

4. Conclusion

In conclusion, for the single-emitter laser we have considered the derivation of the stationary equation for the phase-averaged Glauber-Sudarshan P-function \( P(I) \) and showed that the quasi-probability current density \( \tilde{J}(z, z^*) \) can be equated to zero only after averaging all over the phase. In the strong coupling regime we have analyzed the asymptotic solution of the equation for \( P(I) \). This solution consists two basis solutions (8) and we have explained the neglecting of one of them.

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