MATRICES ASSOCIATED WITH MOVING LEAST-SQUARES APPROXIMATION AND CORRESPONDING INEQUALITIES

SVETOSLAV NENOV AND TSVETELIN TSVETKOV

Abstract. In this article, some properties of matrices of moving least-squares approximation have been proven. The used technique is based on singular-value decomposition and inequalities for singular-values. Some inequalities for the norm of coefficients-vector of the linear approximation have been proven.

1. Statement

Let us remind the definition of moving least-squares approximation and a basic result.

Let:

1. $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$.
2. $x_i \in \mathcal{D}$, $i = 1, \ldots, m$; $x_i \neq x_j$, if $i \neq j$.
3. $f: \mathcal{D} \rightarrow \mathbb{R}$ be a continuous function.
4. $p_i: \mathcal{D} \rightarrow \mathbb{R}$ be continuous functions, $i = 1, \ldots, l$. The functions $\{p_1, \ldots, p_l\}$ are linearly independent in $\mathcal{D}$ and let $\mathcal{P}_l$ be their linear span.
5. $W: (0, \infty) \rightarrow (0, \infty)$ be a strong positive function.

Usually the basis in $\mathcal{P}_l$ is constructed by monomials. For example:

$$p_l(x) = x_{k_1} \ldots x_{k_d},$$

where $x = (x_1, \ldots, x_d)$, $k_1, \ldots, k_d \in \mathbb{N}$, $k_1 + \cdots + k_d \leq l - 1$. In the case $d = 1$, the standard basis is $\{1, x, \ldots, x^{l-1}\}$.

Following [1], [10], [11], [12], we will use the following definition. The moving least-squares approximation of order $l$ at a fixed point $x$ is the value of $p^*(x)$, where $p^* \in \mathcal{P}_l$ is minimizing the least-squares error

$$\sum_{i=1}^{m} W(\|x - x_i\|) (p(x) - f(x_i))^2$$

among all $p \in \mathcal{P}_l$.

The approximation is “local” if weight function $W$ is fast decreasing as its argument tends to infinity and interpolation is achieved if $W(0) =$
∞. So, we define additional function \( w : [0, \infty) \to [0, \infty) \), such that:

\[
  w(r) = \begin{cases} 
    \frac{1}{W(r)}, & \text{if } (r > 0) \text{ or } (r = 0 \text{ and } W(0) < \infty), \\
    0, & \text{if } (r = 0 \text{ and } W(0) = \infty). 
  \end{cases}
\]

Some examples of \( W(r) \) and \( w(r) \), \( r \geq 0 \):

\[
  W(r) = e^{-\alpha^2 r^2} \quad \text{exp-weight}, \\
  W(r) = r^{-\alpha^2} \quad \text{Shepard weights}, \\
  w(x, x_i) = r^2 e^{-\alpha^2 r^2} \quad \text{McLain weight}, \\
  w(x, x_i) = e^{\alpha^2 r^2} - 1 \quad \text{see Levin’s works}.
\]

Here and below: \( \| \cdot \| = \| \cdot \|_2 \) is 2-norm, \( \| \cdot \|_1 \) is 1-norm in \( \mathbb{R}^d \); the superscript \( ^t \) denotes transpose of real matrix; \( I \) is the identity matrix.

We introduce the notations:

\[
  E = \begin{pmatrix} 
    p_1(x_1) & p_2(x_1) & \cdots & p_l(x_1) \\
    p_1(x_2) & p_2(x_2) & \cdots & p_l(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_1(x_m) & p_2(x_m) & \cdots & p_l(x_m) 
  \end{pmatrix}, \quad 
  a = \begin{pmatrix} 
    a_1 \\
    a_2 \\
    \vdots \\
    a_m 
  \end{pmatrix}, \\
  D = 2 \begin{pmatrix} 
    w(x, x_1) & 0 & \cdots & 0 \\
    0 & w(x, x_2) & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & w(x, x_m) 
  \end{pmatrix}, \quad 
  c = \begin{pmatrix} 
    p_1(x) \\
    p_2(x) \\
    \vdots \\
    p_l(x) 
  \end{pmatrix}.
\]

Through the article, we assume the following conditions (H1):

(H1.1) \( 1 \in \mathcal{P}_l \).

(H1.2) \( 1 \leq l \leq m \).

(H1.3) rank(\( E^t \)) = l.

(H1.4) \( w \) is smooth function.

**Theorem 1.1** (see [10]). Let the conditions (H1) hold true. Then:

1. The matrix \( E^t D^{-1} E \) is non-singular.
2. The approximation defined by the moving least-squares method is

\[
  \hat{L}(f) = \sum_{i=1}^{m} a_i f(x_i),
\]

where

\[
  a = A_0 c \quad \text{and} \quad A_0 = D^{-1} E (E^t D^{-1} E)^{-1}.
\]
If \( w(\|x_i - x_i\|) = 0 \) for all \( i = 1, \ldots, m \), then the approximation is interpolatory.

For the approximation order of moving least-squares approximation (see [10] and [5]) it is not difficult to receive (for convenience we suppose \( d = 1 \) and standard polynomial basis, see [5]):

\[
\left| f(x) - \hat{L}(f)(x) \right| \leq \|f(x) - p^*(x)\|_\infty \left[ 1 + \sum_{i=1}^{m} |a_i| \right],
\]

and moreover (\( C=\text{const.} \))

\[
\|f(x) - p^*(x)\|_\infty \leq C h^{l+1} \max \{ \|f^{(l+1)}(x)\| : x \in \overline{D} \}.
\]

It follows from (3) and (4) that the error of moving least-squares approximation is upper-bounded from the 2-norm of coefficients of approximation (\( \|a\|_1 \leq \sqrt{m}\|a\|_2 \)). That is why, the goal in this short note, is to discuss a method for majorization in the form

\[
\|a\|_2 \leq M \exp (N\|x - x_i\|),
\]

Here the constants \( M \) and \( N \) depends on singular values of matrix \( E^t \), and numbers \( m \) and \( l \) (see Section 3). In Section 2 some properties of matrices associated with approximation (symmetry, positive semi-definiteness, and norm majorization by \( \sigma_{\min}(E^t) \) and \( \sigma_{\max}(E^t) \)) are proven.

The main result in Section 3 is formulated in the case of exp-moving least-squares approximation, but it is not hard to receive analogous results in the different cases: Backus-Gilbert weight functions, McLain weight functions, etc.

2. Some Auxiliary Lemmas

**Definition 2.1.** We will call the matrices

\[
A_1 = A_0 E^t = D^{-1} E (E^t D^{-1} E)^{-1} E^t \quad \text{and} \quad A_2 = A_1 - I
\]

\( A_1 \)-matrix and \( A_2 \)-matrix of the approximation \( \hat{L} \), respectively.

**Lemma 2.1.** Let the conditions (H1) hold true. Then, the matrices \( A_1 D^{-1} \) and \( A_2 D^{-1} \) are symmetric.

**Proof.** Direct calculation of the corresponding transpose matrices. \( \square \)

**Lemma 2.2.** Let the conditions (H1) hold true. Then:
(1) All eigenvalues of $A_1$ are 1 and 0 with geometric multiplicity $l$ and $m - l$, respectively.
(2) All eigenvalues of $A_2$ are 0 and -1 with geometric multiplicity $l$ and $m - l$, respectively.

Proof. Part 1. We will prove that the dimension of the null-space $\dim(\ker(A_2))$ is at least $l$.

Using the definition of $A_2 = D^{-1}E (E^t D^{-1} E)^{-1} E^t - I$, we receive
$$E^t A_2 = (E^t D^{-1} E)(E^t D^{-1} E)^{-1} E^t - E^t = 0.$$\(\text{Hence} \im(A_2) \subseteq \ker(E^t).\)

Using (H1.3), $E^t$ is $(l \times m)$-matrix with maximal rank $l$ ($l < m$). Therefore $\dim(\ker(E^t)) = m - l$. Moreover $\dim(\im(A_2)) = m - \dim(\ker(A_2))$. That is why $m - \dim(\ker(A_2)) \leq m - l$ or $l \leq m - \dim(\ker(A_2))$.

Part 2. We will prove that $-1$ is eigenvalue of $A_2$ with geometric multiplicity $m - l$, or the system
$$A_2 \eta = -\eta \iff A_1 \eta = 0$$
has $m - l$ linearly independent solutions.

Obviously the systems
$$A_1 \eta = D^{-1}E (E^t D^{-1} E)^{-1} E^t \eta = 0 \quad (5)$$
and
$$E^t \eta = 0 \quad (6)$$
are equivalent. Indeed, if $\eta_0$ is a solution of (5), then
$$D^{-1}E (E^t D^{-1} E)^{-1} E^t \eta_0 = 0 \implies E^t D^{-1} E (E^t D^{-1} E)^{-1} E^t \eta_0 = 0 \implies E^t \eta_0 = 0,$$
i.e. $\eta_0$ is solution of (6).

On the other hand, if $\eta_0$ is a solution of (6), then
$$(D^{-1}E (E^t D^{-1} E)^{-1} E^t) \eta_0 = (D^{-1}E (E^t D^{-1} E)^{-1}) (E^t \eta_0) = 0,$$
i.e. $\eta_0$ is solution of (5). Therefore
$$\dim(\im(A_1)) = \dim(\im(E^t)) = m - l.$$\(\text{Part 3. It follows from parts 1 and 2 of the proof that 0 is an eigenvalue of A_2 with multiplicity exactly l and -1 is an eigenvalue of A_2 with multiplicity exactly m - l.}\)
It remains to prove that 1 is eigenvalue of $A_1$ with multiplicity at least $l$, but this is analogous to the proven part 1 or it follows directly from the definition of $A_1 = A_2 + I$. □

The following two results are proven in [13].

**Theorem 2.1** (see [13], Theorem 2.2). Suppose $U, V$ are $(m \times m)$ Hermitian matrices and either $U$ or $V$ is positive semi-definite. Let

$$\lambda_1(U) \geq \cdots \geq \lambda_m(U), \quad \lambda_1(V) \geq \cdots \geq \lambda_m(V)$$

denote the eigenvalues of $U$ and $V$, respectively.

Let:

1. $\pi(U)$ is the number of positive eigenvalues of $U$;
2. $\nu(U)$ is the number of negative eigenvalues of $U$;
3. $\xi(U)$ is the number of zero eigenvalues of $U$.

Then:

1. If $1 \leq k \leq \pi(U)$, then
   \[\min_{1 \leq i \leq k} \{\lambda_i(U)\lambda_{k+1-i}(V)\} \geq \lambda_k(VU) \geq \max_{k \leq i \leq m} \{\lambda_i(U)\lambda_{m+k-i}(V)\}.\]
2. If $\pi(U) < k \leq m - \nu(U)$, then
   \[\lambda_k(VU) = 0.\]
3. If $m - \nu(U) < k \leq m$, then
   \[\min_{1 \leq i \leq k} \{\lambda_i(U)\lambda_{m+i-k}(V)\} \geq \lambda_k(VU) \geq \max_{k \leq i \leq m} \{\lambda_i(U)\lambda_{i+1-k}(V)\}.\]

**Corollary 2.1** (see [13], Corollary 2.4). Suppose $U, V$ are $(m \times m)$ Hermitian positive definite matrices.

Then for any $1 \leq k \leq m$

$$\lambda_1(U)\lambda_1(V) \geq \lambda_k(VU) \geq \lambda_m(U)\lambda_m(V).$$

As a result of Lemma 2.1, Lemma 2.2 and Theorem 2.1, we may prove the following lemma.

**Lemma 2.3.** Let the conditions (H1) hold true.

1. Then $A_1 D^{-1}$ and $-A_2 D^{-1}$ are symmetric positive semi-definite matrices.
2. The following inequality holds true

$$\lambda_{\max}(A_1 D^{-1}) \leq \frac{1}{\lambda_{\min}(D)}.$$
Proof. (1) We apply Theorem 2.1, where
\[ U = D, \quad V = A_1D^{-1}. \]

Obviously, \( U \) is a symmetric positive definite matrix (in fact it is a diagonal matrix). Moreover \( \pi(U) = m, \mu(U) = \xi(U) = 0 \), if \( x \neq x_i, i = 1, \ldots, m \).

The matrix \( V \) is symmetric, see Lemma 2.1.

From the cited theorem, for any index \( k \) \((k = 1, \ldots, m = \pi(U))\) we have
\[ \lambda_k(A_1) = \lambda_k(A_1D^{-1}D) = \lambda_k(VU) \leq \min_{1 \leq i \leq k} \{\lambda_i(U)\lambda_{m+i-k}(V)\}. \]

In particular, if \( k = m \):
\[ \lambda_m(A_1) \leq \min_{1 \leq i \leq m} \{\lambda_i(U)\lambda_i(V)\}. \]  \( \text{(7)} \)

Let us suppose that there exists index \( i_0 \) \((i_0 = 1, \ldots, m - 1)\) such that
\[ \lambda_1(V) \geq \cdots \geq \lambda_{i_0}(V) \geq 0 > \lambda_{i_0+1}(V) \geq \cdots \geq \lambda_m(V). \]  \( \text{(8)} \)

It follows from (8) and positive definiteness of \( U \), that
\[ \min_{1 \leq i \leq m} \{\lambda_i(U)\lambda_i(V)\} \leq \lambda_{i_0+1}(U)\lambda_{i_0+1}(V) < 0. \]

Therefore (see (7)) \( \lambda_m(A_1) < 0 \). This contradiction (see Lemma 2.2) proves that the matrix \( A_1D^{-1} \) is positive semi-definite.

If we set \( U = D, V = -A_2D^{-1} \) then by analogical arguments, we see that the matrix \( -A_2D^{-1} \) is positive semi-definite.

(2) From the first statement of Lemma 2.3 \( V = A_1D^{-1} \) is positive semi-definite. Therefore (see Corollary 2.1 and Lemma 2.2):
\[ 1 \geq \lambda_k(A_1) = \lambda_k(VU) \geq \max \{\lambda_m(U)\lambda_k(V), \lambda_m(V)\lambda_k(U)\} \]
for all \( k = 1, \ldots, m \). Moreover all numbers \( \lambda_k(U), \lambda_k(V) \) are non-negative and
\[ \lambda_{\max}(D) = \lambda_1(U) \geq \cdots \geq \lambda_m(U) = \lambda_{\min}(D), \quad \lambda_1(V) \geq \cdots \geq \lambda_m(V). \]

Therefore
\[ 1 \geq \max \{\lambda_m(U)\lambda_1(V), \lambda_m(V)\lambda_1(U)\}, \]
or
\[ \lambda_{\max}(A_1D^{-1}) = \lambda_1(V) \leq \frac{1}{\lambda_m(U)} = \frac{1}{\lambda_{\min}(D)}. \]

In the following, we will need some results related to inequalities for singular values. So, we will list some necessary inequalities in the next lemma.
Lemma 2.4 (see [19], [8]). Let $U$ be an $(d_1 \times d_2)$-matrix, $V$ be an $(d_3 \times d_4)$-matrix.

Then:

\[ \sigma_{\text{max}}(UV) \leq \sigma_{\text{max}}(U)\sigma_{\text{max}}(V), \] (9)
\[ \sigma_{\text{max}}(U^{-1}) = \frac{1}{\sigma_{\text{min}}(U)}, \quad \text{if } d_1 = d_2, \ det \ U \neq 0, \] (10)
\[ \sigma_{\text{max}}(V)\sigma_{\text{min}}(U) \leq \sigma_{\text{max}}(UV), \quad \text{if } d_1 \geq d_2 = d_3, \] (11)
\[ \sigma_{\text{max}}(U)\sigma_{\text{min}}(V) \leq \sigma_{\text{max}}(UV), \quad \text{if } d_4 \geq d_3 = d_2, \] (12)

If $d_1 = d_2$ and $U$ is Hermitian matrix, then $\|U\| = \sigma_{\text{max}}(U)$, $\sigma_i(U) = |\lambda_i(U)|$, $i = 1, \ldots, d_1$.

Lemma 2.5. Let the conditions (H1) hold true and let $x \neq x_i$, $i = 1, \ldots, m$.

Then:

\[ \|A_1 D^{-1}\| \leq \frac{1}{\lambda_{\text{min}}(D)}, \] (13)
\[ \sigma_{\text{max}}(A_1)\sigma_{\text{min}}(D^{-1}) \leq \sigma_{\text{max}}(A_1 D^{-1}), \] (14)
\[ 1 \leq \|A_1\| \leq \sqrt{\frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)}}, \] (15)

Proof. The matrix $A_1 D^{-1}$ is symmetric and positive semi-definite (see Lemma 2.3(1)). Using the second statement of Lemma 2.3 and Lemma 2.4 we receive

\[ \|A_1 D^{-1}\| = \sigma_{\text{max}}(A_1 D^{-1}) = \lambda_{\text{max}}(A_1 D^{-1}) \leq \frac{1}{\lambda_{\text{min}}(D)}. \]

The inequality (13) follows from (12) ($d_4 = d_3 = m$).

From (14) and (10), we receive

\[ \sigma_{\text{max}}(A_1) \leq \frac{\sigma_{\text{max}}(A_1 D^{-1})}{\sigma_{\text{min}}(D^{-1})} = \frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)}. \]

Therefore the equality $\|A_1\| = \sqrt{\sigma_{\text{max}}(A_1)}$ implies the right inequality in (15).

Using $E^t = E^t A_1$ and inequality (9), we receive

\[ \sigma_{\text{max}}(E^t) \leq \sigma_{\text{max}}(E^t)\sigma_{\text{max}}(A_1), \]

or $1 \leq \sigma_{\text{max}}(A_1) = \|A_1\|^2$, i.e. the left inequality in (15).

The lemma has been proved. \qed
3. An Inequality for the Norm of Approximation Coefficients

We will use the following hypotheses:
H2.1. The hypotheses (H1) hold true.
H2.2. $d = 1$, $x_1 < \cdots < x_m$.
H2.3. The map $c$ is $C^1$-smooth in $[x_1, x_m]$.
H2.4. $w(|x - x_i|) = \exp(\alpha (x - x_i)^2)$, $i = 1, \ldots, m$.

**Theorem 3.1.** Let the following conditions hold true:
1. Hypotheses (H2).
2. Let $x \in [x_1, x_m]$ be a fixed point.
3. The index $k_0 \in \{1, \ldots, m\}$ is chosen such that

$$|x - x_{k_0}| = \min\{|x - x_i| : i = 1, \ldots, m\}.$$

Then, there exist constants $M_1, M_2 > 0$ such that

$$\|a(x)\| \leq \left( \|a(x_{k_0})\| + M_1|x - x_{k_0}| \right) \exp(M_2|x - x_{k_0}|).$$

**Proof.** Part 1. Let

$$H = \begin{pmatrix} 2\alpha(x - x_1) & 0 & \cdots & 0 \\ 0 & 2\alpha(x - x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\alpha(x - x_m) \end{pmatrix},$$

then

$$\frac{dD}{dx} = HD, \quad \frac{dD^{-1}}{dx} = -HD^{-1}.$$

We have (obviously $D = D(x)$, $H = H(x)$, and $c = c(x)$)

$$\frac{da(x)}{dx} = \frac{d}{dx} \left( D^{-1} E \left( E^t D^{-1} E \right)^{-1} c \right)$$

$$= \left( \frac{d}{dx} D^{-1} \right) E \left( E^t D^{-1} E \right)^{-1} c + D^{-1} E \left( \frac{d}{dx} \left( E^t D^{-1} E \right)^{-1} \right) c$$

$$+ D^{-1} \left( E^t D^{-1} E \right)^{-1} \frac{d}{dx} c$$

$$= -HD^{-1} E \left( E^t D^{-1} E \right)^{-1} c$$

$$+ D^{-1} \left( - \left( E^t D^{-1} E \right)^{-1} \left( \frac{d}{d\alpha} E^t D^{-1} E \right) \left( E^t D^{-1} E \right)^{-1} \right) c$$

$$+ D^{-1} E \left( E^t D^{-1} E \right)^{-1} \frac{d}{dx} c.$$
\[= - H a + D^{-1} E (E^t D^{-1} E)^{-1} (E^t H D^{-1} E) (E^t D^{-1} E)^{-1} c + D^{-1} E (E^t D^{-1} E)^{-1} \frac{d}{dx} c \]

\[= \left( D^{-1} E (E^t D^{-1} E)^{-1} E^t - I \right) H a + D^{-1} E (E^t D^{-1} E)^{-1} \frac{d}{dx} c \]

\[= A_2 H a + A_0 \frac{d}{dx} c. \]

Therefore, the function \( a(x) \) satisfies the differential equation

\[ \frac{d a(x)}{dx} = A_2 H a + A_0 \frac{d}{dx} c. \quad (16) \]

Part 2. Obviously

\[ \| A_2 H \| = \| (A_1 - I) H \| \leq (\| A_1 \| + 1) \| H \|. \]

It follows from (15) that

\[ \| A_1 \| \leq \sqrt{\frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)}}. \]

Here \( \sigma_{\text{max}}(D) \leq 2 \exp(\alpha r^2) \), \( r = x_m - x_1 \), and \( \sigma_{\text{min}}(D) \geq 2 \). Hence

\[ \| A_1 \| \leq \sqrt{\exp(\alpha r^2)}. \]

For the norm of diagonal matrix \( H \), we receive

\[ \| H \| \leq 2 \alpha r. \]

Therefore \( \| A_2 H \| \leq M_2 \), where

\[ M_2 = 2 \alpha r \left( 1 + \sqrt{\exp(\alpha r^2)} \right). \]

We will use Lemma 2.4 to obtain the norm of \( A_0 \).

Obviously \( A_0 E^t = A_1 \). Therefore by (12) \( (m = d_4 \geq d_3 = l) \) we have

\[ \sigma_{\text{max}}(A_0) \sigma_{\text{min}}(E^t) \leq \sigma_{\text{max}}(A_1), \]

i.e.

\[ \| A_0 \| \leq \frac{1}{\sigma_{\text{min}}(E^t)} \sqrt{\frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)}}. \]

Therefore, if we set \( M_{11} = \frac{M_2}{\sigma_{\text{min}}(E^t)} \), then \( \| A_0 \| \leq M_1 \).
Let the constant $M_{12}$ is choosen such that
\[
\left\| \frac{d}{dx} c(x) \right\| \leq M_{12}, \quad x \in [x_1, x_m]
\]
and let $M_1 = M_11 M_{12}$.

Part 3. On the end, we have only to apply Lemma 4.1 form \cite{7} to the equation (16):
\[
\|a(x)\| \leq \left( \|a(x_{k_0})\| + \int_{x_{k_0}}^{x} \left| A_0 \frac{d}{dx} c \right| dx \right) \exp \int_{x_{k_0}}^{x} \|A_2 H\| dx \leq (\|a(x_{k_0})\| + M_1 |x - x_{k_0}|) \exp (M_2 |x - x_{k_0}|). \tag*{\square}
\]

Remark 3.1. Let the hypotheses (H2) hold true and let moreover
\[
p_1(x) = 1, \quad p_2(x) = x, \ldots, \quad p_l(x) = x^{l-1}, \quad l \geq 1.
\]
In such a case, we may replace the differentiation of vector-fuction
\[
c(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_l(x) \end{pmatrix}
\]
by left-multiplication:
\[
\frac{dc(x)}{dx} = \begin{pmatrix} 0 & 1 & 2x & 3x^2 & \cdots & (l-2)x^{l-3} & (l-1)x^{l-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & l-2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & l-1 & 0 \end{pmatrix}
\]
\[
= \partial c(x).
\]

The singular values of the matrix $\partial$ are: $0, 1, \ldots, l-1$. Therefore $\|\partial\| = \sqrt{l-1}$.

That is why, we may chose
\[
M_{22} = \sqrt{(l-1)} \max_{1 \leq i \leq l} \left\{ \max_{x_1 < \cdots < x_m} |p_i(x)| \right\}.
\]

Additionally, if we supose $|x_1| \leq |x_m|$, then
\[
\max_{x_1 < x < x_m} |p_i(x)| = |p_i(x_m)|, \quad i = 1, \ldots, l.
\]
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Therefore, in such a case:

$$M_{22} = \sqrt{(l - 1) \max_{1 \leq i \leq l} \{|p_i(x_m)|\}}.$$  

If we suppose $-1 \leq x_1 \leq x \leq x_m \leq 1$, then obviously, we may set

$$M_{22} = \sqrt{l - 1}.$$  

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E-mail address, Corresponding author: nenov@uctm.edu

E-mail address: tstsvetkov@uctm.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHEMICAL TECHNOLOGY AND METALLURGY, SOFIA 1756, BULGARIA