RESEARCHPAPER

Numerical Methods for Solving the System of Volterra-Fredholm Integro-Differential Equations.

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ABSTRACT:
In this paper, we defined the system of Volterra-Fredholm integro-differential equations of the second kind with the initial conditions. The solution for this equation is introduced by using the modified decomposition method. An algorithm is applied to get that solution. Moreover, we generated some examples and discussed to illustrate the employment of the method.

KEYWORDS: System of integro-differential equations, Linear Volterra-Fredholm, Modified decomposition method.
DOI: http://dx.doi.org/10.21271/ZJPAS.31.2.4
ZJPAS (2019), 31(2):25-30.

INTRODUCTION:

In science and engineering, sometimes there are vital problems that can be minimized to a system of integral and integro-differential equations. The latter equation has pulled in much consideration of math scholars to work on and has been a subject of interest for them to solve.

In recent years several researchers have adopted different techniques for solving the systems of Volterra and Fredholm integro-differential equations. The decomposition method has been applied to obtain formal solutions to wide class of problems in many interesting mathematics and physics areas. Recently, (Darania and Ivaz, 2008) used Taylor expansion for nonlinear Volterra–Fredholm integro-differential equations, also (Hasan and Suleiman, 2018b) used Trigonometric Functions and Laguerre Polynomials to solve mixed Volterra-Fredholm integral equation. (Fariborzi Araghi and Behzadi, 2011) used Homotopy Analysis Method for the numerical solution of nonlinear Volterra-Fredholm integro-differential equations. (Hasan and Suleiman, 2018a) shows numerical solution of Mixed Volterra-Fredholm integral equations by using Linear Programming Problem, and (Hassan T.I., Sulaiman N.A., 2017) studied Aitken method to solve Volterra-Fredholm integral equations of the second kind with Homotopy perturbation method. (Akyüz-Daşcioğlu and Sezer, 2005) used Chebyshev collocation method to solve the systems of higher-order linear Fredholm–Volterra integro-differential equations.

If we focus on the Adomian decomposition method (ADM), (Wazwaz, 2006, 2011) used Modified ADM for solving nonlinear Fredholm
and Volterra integral equations of the second kind. (Rabbani and Zarali, 2012) used MADM to solve the system of linear Fredholm integro-differential equations. (Bakodah, 2012) used some modifications of Adomian decomposition method apply for solving the system of nonlinear Fredholm integral equations of the second kind, also (Bakodah, H O, Al-Mazmumy, M Almuhalbedi, 2017), He presented an efficient modification of ADM for solving the system of nonlinear Volterra and Fredholm integro-differential equations.

In this paper we introduce a solution of the system of linear Volterra-Fredholm integro-differential equations of the second kind of the following form:

\[
\begin{cases}
y_1^{(n)}(x) = f_1(x) + \int_a^x k_{11}(x,t)y_1(t)dt + \int_a^b k_{12}(x,t)y_2(t)dt \\
y_2^{(n)}(x) = f_2(x) + \int_a^x k_{21}(x,t)y_1(t)dt + \int_a^b k_{22}(x,t)y_2(t)dt
\end{cases}
\]  

(1)

With initial conditions

\[
y_1^{(0)}(x_0), y_1^{(1)}(x_0), \ldots, y_1^{(n-1)}(x_0) \quad \text{and} \quad y_2^{(0)}(x_0), y_2^{(1)}(x_0), \ldots, y_2^{(n-1)}(x_0)
\]

The unknown functions \(y_1(x)\), and \(y_2(x)\) that will be determined, occur inside the integral sign whereas the derivatives of \(y_1(x)\), and \(y_2(x)\) appear mostly outside the integral sign. The kernels \(k_{ij}(x,t)\), and the function \(f_i(x)\) for \(i, j = 1, 2\) are known real-valued functions.

2. MODIFIED ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method (Adomian, 1988) is the process of fragmenting the unknown function \(y(x)\) of an equation into the summation of an infinite terms of components defined by the decomposition series:

\[
y(x) = \sum_{n=0}^{\infty} y_n(x)
\]  

(2)

in which the parts \(y_n(x), n \geq 0,\) are to be set in a reoccurring way. This method concerns itself with discovering the components \(y_0, y_1, y_2, \ldots\) separately.

The modified decomposition method (Wazwaz, 2011) depends on dividing the function \(f(x)\) into two parts, so it can't be used if the function \(f(x)\) consists of only one term.

To give a whole description of the method consider a general functional equation

\[
Ly + Ry + Ny = f(x),
\]  

(3)

Where \(y(x)\) is the unknown function, and the linear terms are decomposed into \(L + R\) and \(Ny\) denote the nonlinear terms. Since \(L\) is easily invertible and \(R\) is the remainder of the nonlinear operator and \(f(x)\) is the source term. From Eq. (3)

\[
Ly = f(x) - R - Ny,
\]  

(4)

Applying \(L^{-1}\) on both sides of the above equation and using the given conditions, we get

\[
y(x) = g(x) - L^{-1}(Ry) - L^{-1}(Ny),
\]  

(5)

Where

\[
L^{-1}(f(x)) = g(x)
\]

applying the Adomian method the series solution \(y(x)\) as defined in Eq. (2)

\[
y(x) = \sum_{n=0}^{\infty} y_n(x)
\]

The components \(y_0, y_1, y_2, \ldots\) can easily be determine recursively from the following relations:

\[
y_0(x) = g(x),
\]

\[
y_{n+1}(x) = -L^{-1}(Ry_n) - L^{-1}(Ny_n), \quad n \geq 0
\]  

(6)

typically the decomposition method assign the zeroth components \(y_0(x)\) as the function \(g(x)\). But modified decomposition method propose that the function \(g(x)\) defined above in Eq. (5) can be decomposed into two parts namely \(g_0(x)\) and \(g_1(x)\) i.e.,

\[
g(x) = g_0(x) + g_1(x)
\]  

(7)

In the above equation proper choice of \(g_0(x)\) and \(g_1(x)\) is essential and depends mainly on the trail basis. Thus the following recursive relations
for the modified decomposition method are formed as:

\[
\begin{align*}
    y_0(x) &= g_0(x) \\
    y_1(x) &= g_1(x) - L^{-1}(Ry_0) - L^{-1}(Ny_0) \\
    y_{n+1}(x) &= -L^{-1}(Ry_n) - L^{-1}(Ny_n)
\end{align*}
\]

(8)

3. APPLYING MODIFIED (ADM) FOR SOLVING SYSTEM OF LINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND KIND

The Adomian decomposition method gives the solution in an infinite series of components that can be repetitively specified. The acquired series may give the exact solution if such a solution exists. Otherwise, the series gives a parataxis for the solution that gives high accuracy level.

Recall system (1) of linear Volterra-Fredholm integro-differential equations of the second kind:

\[
\begin{align*}
    y_1^{(n)}(x) &= f_1(x) + \int_a^x k_{11}(x,t)y_1(t)dt + \int_a^b k_{12}(x,t)dy_2(t)dt \\
    y_2^{(n)}(x) &= f_2(x) + \int_a^x k_{21}(x,t)y_1(t)dt + \int_a^b k_{22}(x,t)dy_2(t)dt
\end{align*}
\]

With initial conditions

\[
\begin{align*}
    y_1^{(0)}(x), y_1^{(1)}(x), \ldots, y_1^{(n-1)}(x) \\
    y_2^{(0)}(x), y_2^{(1)}(x), \ldots, y_2^{(n-1)}(x)
\end{align*}
\]

Integrating both sides of (1) from 0 to \( x \) \( n \) times leads to

\[
\begin{align*}
    y_1(x) &= g_1(x) + L^{-1} \int_a^x k_{11}(x,t)y_1(t)dt + \int_a^b k_{12}(x,t)y_1(t)dt \\
    y_2(x) &= g_2(x) + L^{-1} \int_a^x k_{21}(x,t)y_1(t)dt + \int_a^b k_{22}(x,t)y_2(t)dt
\end{align*}
\]

(9)

Substituting (10) in (9), we get

\[
\begin{align*}
    \sum_{n=0}^{\infty} y_1^{(n)} &= g_1(x) + L^{-1} \int_a^x k_{11}(x,t)\sum_{n=0}^{\infty} y_1^{(n)}(t)dt + \int_a^b k_{12}(x,t)\sum_{n=0}^{\infty} y_2^{(n)}(t)dt \\
    L^{-1} \int_a^b k_{12}(x,t)\sum_{n=0}^{\infty} y_2^{(n)}(t)dt
\end{align*}
\]

(10)

By using the modified decomposition method, we decompose the function \( g_i(x) \) as follows:

\[
    g_i(x) = g_{i0}(x) + g_{i1}(x)
\]

We set

\[
    y_{i0}(x) = g_{i0}(x)
\]

and

\[
\begin{align*}
    y_{i1}(x) &= g_{i1}(x) + L^{-1} \int_a^x k_{11}(x,t)y_{i0}(t)dt + \\
    &+ L^{-1} \int_a^b k_{12}(x,t)y_{i0}(t)dt \\
    y_{i2}(x) &= g_{i2}(x) + L^{-1} \int_a^x k_{21}(x,t)y_{i0}(t)dt + \\
    &+ L^{-1} \int_a^b k_{22}(x,t)y_{i0}(t)dt
\end{align*}
\]

(11)

And also we take

Where \( g_i(x) \) includes \( L^{-1}(f_i(x)) ; i = 1,2 \) and the initial conditions, and \( L^{-1} \) is an \( n \)-fold integral operator.

Now decomposing the unknown functions \( y_i(x), i = 1,2 \) of any equation into a sum of an infinite number of terms defined by the decomposition series as
\begin{align*}
    y_{1,j+1}(x) &= L^{-1} \int_{a}^{x} k_{11}(x,t)y_{1,j}(t)dt + \\
    &+ L^{-1} \int_{a}^{b} k_{12}(x,t)y_{2,j}(t)dt \\
    y_{2,j+1}(x) &= L^{-1} \int_{a}^{x} k_{21}(x,t)y_{1,j}(t)dt + \\
    &+ L^{-1} \int_{a}^{b} k_{22}(x,t)y_{2,j}(t)dt
\end{align*}

(12)

Algorithm:

Step (1): Integrate both sides the system \( n \) times.
Step (2): We set the function \( g_i(x) = g_{i0}(x) + g_{i1}(x) \) for \( i = 1,2 \).
Step (3): assigned to the zeroth component \( y_{i0}(x) = g_{i0}(x) \) for \( i = 1,2 \).
Step (4): substitute \( y_{i0}(x) = g_{i0}(x) \) for \( i = 1,2 \) in equation (11) to get \( y_{i1}(x) \).
Step (5): substitute \( y_{i1}(x) \) for \( i = 1,2 \) in equation (12) to get \( y_{i1}(x) \).

4. NUMERICAL EXAMPLES

Example 1. Consider the linear system of Volterra-Fredholm integro differential equations

\[ y_1^{(2)}(x) = x - 2\cos x - \int_{0}^{x} (x - t)y_1(t)dt - \int_{0}^{\pi / 2} (x - t)y_2(t)dt \]
\[ y_2^{(2)}(x) = \sin x - x - 1 + \int_{0}^{x} (x - t)^2y_1(t)dt - \int_{0}^{\pi / 2} (x - t)y_2(t)dt \]

With initial conditions \( y_1(0) = y_1'(0) = 1 \) and \( y_2(0) = y_2'(0) = 0 \).

The above equations in an operator form equations become

\[ L_1y_1 = x - 2\cos x - \int_{0}^{x} (x - t)y_1(t)dt - \int_{0}^{\pi / 2} (x - t)y_2(t)dt \]
\[ L_2y_2 = \sin x - x - 1 + \int_{0}^{x} (x - t)^2y_1(t)dt - \int_{0}^{\pi / 2} (x - t)y_2(t)dt \]

Applying \( L^{-1} \), a two-fold integral operator, on both sides we have

\[ y_1(x) = \frac{x^3}{6} - 1 + 2\cos x \]
\[ y_2(x) = 2x - \frac{x^2}{2} - \frac{x^3}{3} - \sin x \]

Now splitting \( g_1(x) \) into two parts \( g_{10}(x) = \cos x, g_{11}(x) = \cos x - 1 + \frac{x^3}{6} \) and also splitting \( g_2(x) \) into two parts \( g_{20}(x) = \sin x, g_{21}(x) = 2x - \frac{x^2}{2} - \frac{x^3}{3} - 2\sin x \)

And,

\[ y_{11}(x) = \frac{x^3}{6} - 1 + \cos x - L^{-1} \int_{0}^{x} (x - t)y_{10}(t)dt \]
\[ y_{21}(x) = 2x - \frac{x^2}{2} - \frac{x^3}{3} - 2\sin x + L^{-1} \int_{0}^{x} (x - t)^2y_{10}(t)dt \]

Also we take

\[ y_{1,n+1}(x) = 0, \quad n \geq 1. \]
\[ y_{2,n+1}(x) = 0 \]

Thus
\begin{align*}
\begin{cases}
  y_1(x) &= \cos x \\
  y_2(x) &= \sin x \\
\end{cases}
\end{align*}

which is the exact solution.

Example 2. Consider the linear system of Volterra-Fredholm integro differential equations

\begin{align*}
\begin{cases}
  y_1^{(2)}(x) &= 6x - \frac{3x^2}{4} - \frac{x^6}{5} + \int_0^x xt y_1(t)dt + \frac{1}{0} x^2 t y_2(t)dt \\
  y_2^{(2)}(x) &= 2 - \frac{x^5}{5} - \frac{3}{2} x + \int_0^x t y_1(t)dt + \frac{1}{0} 2xt y_2(t)dt \\
\end{cases}
\end{align*}

With initial conditions \( y_1(0) = y_1'(0) = y_2(0) = 0 \) and \( y_2(0) = 1 \)

The above equations in an operator form equations become

\begin{align*}
\begin{cases}
  Ly_1 &= 6x - \frac{3x^2}{4} - \frac{x^6}{5} + \int_0^x xt y_1(t)dt + \frac{1}{0} x^2 t y_2(t)dt \\
  Ly_2 &= 2 - \frac{x^5}{5} - \frac{3}{2} x + \int_0^x t y_1(t)dt + \frac{1}{0} 2xt y_2(t)dt \\
\end{cases}
\end{align*}

Applying \( L^{-1} \), a two-fold integral operator, on both sides we have

\begin{align*}
\begin{cases}
  y_1(x) &= x^3 - \frac{x^8}{280} - \frac{x^4}{16} + L^{-1} \int_0^x xt y_1(t)dt + \frac{1}{0} x^2 t y_2(t)dt \\
  y_2(x) &= x^2 - \frac{x^7}{210} - \frac{x^3}{4} + 1 + L^{-1} \int_0^x t y_1(t)dt + \frac{1}{0} 2xt y_2(t)dt \\
\end{cases}
\end{align*}

Where \( g_1(x) = x^3 - \frac{x^8}{280} - \frac{x^4}{16} \) and \( g_2(x) = x^2 - \frac{x^7}{210} - \frac{x^3}{4} + 1 \)

Now splitting \( g_1(x) \) into two parts,

\begin{align*}
\begin{cases}
  g_{10}(x) &= x^3 \quad g_{11}(x) = -\frac{x^8}{280} - \frac{x^4}{16} \quad \text{and} \\
  g_{20}(x) &= x^2 + 1 \quad g_{21}(x) = -\frac{x^7}{210} - \frac{x^3}{4} \\
\end{cases}
\end{align*}

and,

\begin{align*}
\begin{cases}
  y_{10}(x) &= x^3 \\
  y_{20}(x) &= x^2 + 1 \\
\end{cases}
\end{align*}

Also we take

\begin{align*}
\begin{cases}
  y_{1,n+1}(x) = 0, \\
  y_{2,n+1}(x) = 0, \\
\end{cases}
\end{align*}

\( n \geq 1. \)

Thus

\begin{align*}
\begin{cases}
  y_1(x) &= x^3 \\
  y_2(x) &= x^2 + 1 \\
\end{cases}
\end{align*}

which is the exact solution.

5. CONCLUSIONS

In this paper, the modified decomposition method is used to solve the system of Volterra-Fredholm integro differential equations. This method is more proficient than its traditional one as it is less complicated, needs less time to get to the solution and most importantly the exact solution is achieved in two iterations.
The essential condition for that to succeed is that the zeroth component should include the exact solution.

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