One sided conformal collars and the reflection principle

V. Liontou and V. Nestoridis

November 5, 2018

Abstract

If a Jordan curve $\sigma$ has a one-sided conformal collar with "good" properties, then, using the Reflection principle, we show that any other conformal collar of $\sigma$ from the same side has the same "good" properties. A particular use of this fact concerns analytic Jordan curves, but in general the Jordan arcs we consider do not have to be analytic. We show that if an one-sided conformal collar bounded by $\sigma$ is of class $A^p$, then any other collar bounded by $\sigma$ and from the same side of $\sigma$ is of class $A^p$.

A.M.S classification no: primary 30C99, 30E25 secondary 30H10

Key words and phrases: Reflection principle, one sided conformal collar, analytic curves, $A^p$ spaces.

1 Introduction

In this paper, for two conformal mappings $\Phi$ and $F$, we write $F = \Phi \circ (\Phi^{-1} \circ F)$ whenever this has a meaning, as explained later. Using our assumptions, the function $\Phi^{-1} \circ F$ maps a circular arc $\tau$ on another circular arc. Thus, by Reflection $h = \Phi^{-1} \circ F$ extends to an injective holomorphic function on a neighbourhood $V$ of a part of $\tau$. Thus, $h'$ extends continuously on $\tau$ and does not vanish at any point of $\tau$. Since $F = \Phi \circ h$ and $F' = (\Phi' \circ h)h'$, if $\Phi$ or $\Phi'$ has some good properties we can transfer them to $F$ and $F'$. The first application of this fact is described below.

Let $J$ be a Jordan curve in $\mathbb{C}$, that is a homeomorphic image of the unit
circle $T$ defined by $\gamma : T \to J$. The Jordan curve $J$ is called analytic if there is a two sided conformal collar around it. That is, if there exists an annulus $D(0, r_1, r_2) = \{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$, $0 < r_1 < 1 < r_2$ and an injective holomorphic mapping $\phi : D(0, r_1, r_2) \to \mathbb{C}$ such that $\phi(e^{i\theta}) = \gamma(e^{i\theta})$ for all $e^{i\theta} \in T$, $\theta \in \mathbb{R}$. A consequence of the existence of the above two sided conformal collar is that, if $\Omega$ is the Jordan region bounded by $J$ and $\Phi : D \to \Omega$ is a Riemann mapping from the open disk $D$ onto $\Omega$, then $\Phi$ has a conformal extension on a larger disc $D(0, r) = \{ z \in \mathbb{C} : |z| < r \}$, $r > 1$. In particular, the function $\Phi$ extends continuously on the unit circle $T$ and the same holds for the derivative $\Phi'$ and $\Phi'(e^{i\theta}) \neq 0$ for all $e^{i\theta} \in T$.

In the present paper we investigate which conclusions still hold if we do not assume the existence of a two-sided conformal collar around the Jordan curve $J$ but we assume the existence of a one sided conformal collar bounded by $J$.

Thus, we assume the existence of an injective holomorphic function $\phi : D(0, r, 1) \to \mathbb{C}$, $D(0, r, 1) = \{ z \in \mathbb{C} : r < |z| < 1 \}$ which extends continuously on $D(0, r, 1)$ and let us call $\phi$ the extension as well. We also assume that $\phi$ is one to one on $D(0, r, 1) \cup T$ where $T = \{ z \in \mathbb{C} : |z| = 1 \}$ and that $\phi|_T : T \to J$ is a homeomorphism. Furthermore, we assume that $\phi'$ has a continuous extension on $\overline{D(0, r, 1)}$ and that $\phi'(e^{i\theta}) \neq 0$ for all $\theta \in \mathbb{R}$. Then we show that the Riemann map $\Phi : D \to \Omega$, onto the Jordan region $\Omega$ bounded by $J$ has similar properties; that is, $\Phi$ and $\Phi'$ extend continuously on $\overline{D}$ and $\Phi'(e^{i\theta}) \neq 0$ for all $\theta \in \mathbb{R}$. Certainly, the fact that $\Phi$ extends continuously on $\overline{D}$ is a consequence of the Ozgood-Caratheodory theorem [6]. The rest is a consequence of the Reflection Principle ([1]). We also can extend the above fact for every finite number $p + 1$ of derivatives, provided that the one-sided conformal collar is of class $A^p$([3],[5]). Furthermore, instead of considering the unit circle we can replace it by one-sided free boundary arcs ([1]) which are analytic arcs and obtain similar results (see Th.2.2 below).

We also mention that our result relates to the considerations of ([2]).

Finally, a second application of the method of the present paper can be found in [7], see proof of Th. 4.2 where it is proven the following. If $\Omega$ is a domain in $\mathbb{C}$, whose boundary contains an analytic arc $J$ and the boundary values of a holomorphic function $f \in H(\Omega)$ define a function in $C^p(J)$, then the derivatives $f^{(l)}(z), z \in \Omega, l = 0, \ldots, p$ extend continuously on $J$. 

2
2 One sided conformal collar

We start with the following definition.

Definition 2.1. A complex function \( \phi(t) \) of a real variable \( t \) defined on an interval \( a < t < b \), defines an analytic curve if, for every \( t_0 \) in the \( (a, b) \), the derivative \( \phi'(t_0) \) exists and it is non-zero and \( \phi \) has a representation

\[
\phi(t) = \sum_{n=0}^{\infty} \alpha_n(t_0)(t - t_0)^n, \alpha_n(t_0) \in \mathbb{C},
\]

where the power series converges in some interval \( (t_0 - p, t_0 + p) \), \( p = p(t_0) > 0 \), and coincides with \( \phi \) in this interval.

But if this is so, then from Abel’s lemma, the series is also convergent for complex values of \( t \), as long as \( |t - t_0| < p \) and represents an analytic function on that disk. In overlapping disks the functions are the same, because they coincide on a segment of the real axis. Thus the function \( \phi(t) \) can be extended as an analytic and injective function in a neighbourhood of \( (a, b) \) in \( \mathbb{C} \). \[1\]

Theorem 2.2. Let \( W_1 \) and \( W_2 \) be regions and \( J_1 \) and \( J_2 \) be analytic arcs of the boundaries of the regions \( W_1, W_2 \) respectively. We also assume that there are neighbourhoods \( U_1, U_2 \) of \( J_1, J_2 \) respectively for which \( U_1 \cap W_1 \) and \( U_2 \cap W_2 \) are connected regions. For the analytic arcs \( J_1 \) and \( J_2 \) we assume that:

1. For every point of \( J_1 \) or \( J_2 \) there exists a neighbourhood whose intersection with the whole boundary of \( W_1 \) or \( W_2 \) respectively is the same with its intersection with \( J_1 \) or \( J_2 \).

2. For every point \( z \) in \( J_i \), \( i = 1, 2 \) and for every disk \( D(z, \epsilon) \) such that \( D(z, \epsilon) \cap \partial W_i = D(z, \epsilon) \cap J_i \), the arc \( J_i \) separates \( D(z, \epsilon) \) in two subsets, one completely inside \( W_i \) and the other completely outside \( W_i \).

3. The arcs \( J_i, i = 1, 2 \) are simple analytic and the injective maps \( \phi_1, \phi_2 \) with \( J_1 := \phi_1 : (a_1, b_1) \rightarrow \mathbb{C} \) and \( J_2 := \phi_2 : (a_2, b_2) \rightarrow \mathbb{C} \) satisfy that \( \phi'_1 \) and \( \phi'_2 \) exist and are non-zero for every point of the segments \( (a_1, b_1) \) and \( (a_2, b_2) \) respectively.

We also assume that a function \( \phi : W_1 \cup J_1 \rightarrow \mathbb{C} \) is injective, continuous on \( W_1 \cup J_1 \) and holomorphic in \( W_1 \) and we denote \( G := \phi(W_1) \). We suppose that
$\phi'$ can be extended in $W_1 \cup J_1$ continuously. Let $\gamma = \phi(J_1)$ and let a function $F : W_2 \to \mathbb{C}$ be continuous on $W_2 \cup J_2$ and holomorphic on $W_2$. Suppose that for all $z$ in $\phi(J_1)$, there exists $\epsilon > 0$ so that $D(z, \epsilon) \cap F(W_2) \subseteq G$ and if a sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in W_2$ for all $n \in \mathbb{N}$ accumulates only in $J_2$, then $F(z_n)$ accumulates only in $\gamma$ and $F(J_2) \subseteq \gamma = \phi(J_1)$.

Then:

(a) The function $F'$, can be continuously extended in $W_2 \cup J_2$.

(b) If $\phi'(z) \neq 0$ for all $z$ in $J_1$ and $F$ is injective in $W_2$ then $F'(\zeta) \neq 0$ for all $\zeta$ in $J_2$.

Proof. First we prove statement (a).

Let $z \in J_2$. There exists a $\delta_z > 0$ such that $D(z, \delta_z) \cap (\partial W_2 \setminus J_2) = \emptyset$. Therefore, the function

$$h := \phi^{-1} \circ F : D(z, \delta_z) \cap W_2 \to W_1$$

is well defined. The function $h$ is holomorphic, since $F$ is holomorphic in $W_2$. The function

$$\phi^{-1} : \phi(W_1) \to W_1$$

is holomorphic also holomorphic in $\phi(W_1)$ and if $\delta_z > 0$ is small enough, then $F(D(z, \delta_z) \cap W_2) \subseteq \phi(W_1)$, for every $z \in \gamma$.

We assume that $\{z_n, n \in \mathbb{N}\}$ is a sequence of points in $W_2$. We are going to prove that if all the accumulation points of $\{z_n\}_{n \in \mathbb{N}}$ belong in $J_2$, then all the accumulation points of sequence $b_n := h(z_n), n \in \mathbb{N}$ belong in $J_2$. From our hypothesis it follows that $F(z_n), n \in \mathbb{N}$ accumulates in $\gamma$. It suffices to prove that $\{\phi^{-1} \circ F(z_n), n \in \mathbb{N}\}$ accumulates in $J_1$. If we suppose that there is an accumulation point $w$ of the sequence $b_n := \phi^{-1} \circ F(z_n)$ that does not belong in $J_1$, then , there exists a subsequence $\{b_{k_n}, n \in \mathbb{N}\}$ that converges to $w \in W_1$. Thus, $\phi(b_{k_n})$ defines a subsequence of $F(z_n)$ in $\phi(W_1)$. Since $\phi$ is continuous on $W_1 \cup J_1$, then $\phi(z_{k_n})$ converges to $\phi(w)$. Since all accumulation points of $F(z_n)$ belong in $\gamma$, therefore $\phi(w)$ belongs in $\gamma$. Thus, there exists an $z_1 \in J_1$ such as $\phi(z_1) = \phi(w)$. This is impossible since $z_1$ is on the boundary of $W_1$ and if $\Phi$ is injective on $W_1 \cup J_1$.

Therefore if $(z_n)_n$ accumulates on $J_2$, $h(z_n)$ accumulates on $J_1$.

Finally, $h$ is holomorphic on $D(z, \delta_z) \cap W_2$ and continuous on $\overline{D(z, \delta_z) \cap W_2}$. According to the Reflection Principle for analytic arcs $\forall$, $h$ is holomorphic
in $D(z, \delta'_z)$ for some $\delta'_z, 0 < \delta'_z \leq \delta_z$. Therefore $F = \phi \circ \phi^{-1} \circ F$ is continuous in $J_2$ since $\phi$ can be continuously extended in $J_1$.

Considering $F$ restricted in $D(z, \delta_z) \cap W_2$ it is analytic and

$$F'(z) = [\phi \circ (\phi^{-1} \circ F)]'(z) = \phi' \circ (\phi-1 \circ F)(z) \cdot (\phi^{-1} \circ F)'(z)$$

$\forall z \in D(z, \delta_z) \cap W_2$

Since $\phi'$ is continuous in $J_1$ and from the Reflection Principle $(\phi^{-1} \circ F)'$ is holomorphic in $D(z, \delta_z)$, therefore $F'$ is continuous on $D(z, \delta_z) \cap W_2$.

The above statements are valid for all $z \in J_2$. From analytic continuation the extensions of $h \in D(z_i, \delta'_{z_i}) \cap D(z_j, \delta'_{z_j}) \cap W_2$ match, thus they also match in $D(z_i, \delta'_{z_i}) \cap D(z_j, \delta'_{z_j})$, for any $z_1, z_2 \in J_2$.

Therefore $F'$ can be continuously extended in $J_2$.

Next we prove statement (b):

We have already that $F' = [\phi \circ (\phi^{-1} \circ F)]' = \phi' \circ (\phi^{-1} \circ F) \cdot (\phi^{-1} \circ F)'$

Let $z \in J_2$. Since for all $z_n$ accumulating in $J_2$, the sequence $F(z_n)$ accumulates in $\gamma$, it follows that $\phi^{-1} \circ F(z) \in J_1$. Assuming that $\phi' \neq 0$, for all $z$ in $J_1$, it follows that $\phi' \circ (\phi^{-1} \circ F)(z) \neq 0$ for all $z$ in $J_2$.

From the Reflection Principle, the function $h = \phi^{-1} \circ F$ can be extended in an neighbourhood of $J_2$ and the extension is injective on it; thus $(\phi^{-1} \circ F)' \neq 0$, for all $z$ in $J_2$. Therefore $F'(z) \neq 0$ for all $z \in J_2$. This completes the proof.

\[\square\]

**Remark 2.3.** Let suppose that there exists $z$ in $J_1$ such that $\phi'(z) \neq 0$ and not for all points in $J_1$. Therefore, there exists $\zeta$ in $J_2$ such that $\phi^{-1} \circ F(\zeta) = z$ in $J_1$ and $F(\zeta) \neq 0$ and reversely.

**Lemma 2.4.** Let $W \subseteq \mathbb{C}$ and $\Omega \subseteq \mathbb{C}$ open sets. Let $\phi : W \rightarrow \mathbb{C}$ and $h : \Omega \rightarrow W$ holomorphic functions. We assume that $P$ is a finite linear combination of products of the functions $\phi^{(n)} \circ h, \phi^{(n-1)} \circ h, ..., \phi \circ h$ and $h^{(n)}, ..., h$, for some $n$ in $\mathbb{N}$. Then $P$ is holomorphic on $\Omega$ and its derivative is a polynomial of the functions $\phi^{(n)} \circ h, \phi^{(n-1)} \circ h, ..., \phi \circ h$ and $h^{(n)}, ..., h$ and $\phi^{(n+1)} \circ h, h^{(n+1)}$.
Proof. We will use induction to prove this lemma. We start from

\[ P(z) = (\phi \circ h)^{k_1}h^{k_2}(z) \]

, then

\[ P'(z) = k_1(\phi' \circ h)h'h^{k_2} + k_2(\phi \circ h^{k_1})h^{k_2-1}h' \]

For the inductive step we assume

\[ P = c[\phi^{(n-1)} \circ h^{k_{n-2}} \phi \circ h^{k_{n+1}} \phi^{(n)} h^{k_n} h^{k_{n+1}}], c \in \mathbb{C} \]

Without loss of the generality, we also assume that \( k_{2n} = \cdots = k_1 = c = 1 \), hence,

\[ P' = (\phi^{(n)} \circ h)h'[\phi^{(n-1)} \circ h] \cdots (\phi^{-1} \circ F) \]

\[ + (\phi^{(n-1)} \circ h)^2h'(\phi^{(n-2)} \circ h \cdots h) \]

\[ + \cdots + [\phi' \circ h]^{(n+1)}(\phi^{(n)} \circ h) \cdots h) \]

\[ + h^{(n+1)}(\phi^{(n)} \circ h) + \cdots + h([\phi'(n) \circ h] \cdots h') \]

Therefore \( P' \) is a polynomial of \( (\phi^{(n)} \circ h)^{(k)}, h^{(k)}, k = 0, \ldots, n + 1 \)

\[ \square \]

Corollary 2.4.1. According to assumptions and notation of Theorem 2.2, if \( \phi \) is an analytic function in \( W_1 \) and continuous on \( J_1 \) and we also assume that \( \phi^{(k)} \) can be continuously extended on \( J_1 \) for \( k = 0, 1, \ldots, p \), where \( p \) is a natural number, then \( F^{(k)} \) can be continuously extended on \( W_2 \cup J_2 \) for \( k = 0, 1, \ldots, p \).

This holds because \( F^{(k)} \) is a polynomial of functions that can be continuously extended on \( J_2 \). This follows combining Theorem 2.2 with lemma 2.4. (for more details see [3])

The Theorem 2.2 implies, also, the following corollary.

Corollary 2.4.2. We suppose that \( D(0, r, 1), 0 < r < 1 \) is an open annulus and \( \phi : D(0, r, 1) \rightarrow \mathbb{C} \) is continuous, injective on \( D(0, r, 1) \) and holomorphic on \( D(0, r, 1) \). We also assume that \( \phi' \) has a continuous extension on \( D(0, r, 1) \). Furthermore, we set \( \gamma(e^{i\theta}) := \phi(e^{i\theta}) \) and \( W \) is the interior of the
Jordan curve $\gamma$. 

We, finally, assume that $\phi(D(0, r, 1)) \subseteq W$ and the function 

$$F : D(0, 1) \to W$$

is a conformal mapping of $D := D(0, 1)$ onto $W$. It is known that according to the Ozgood-Caratheodory theorem, $F$ is extended to a homeomorphism $F : D(0, 1) \to \mathbb{W}$. From the Theorem 2.2, it follows that 

1. The function $F'$ extends continuously on $\overline{D}$

2. If $\phi'(z) \neq 0$, for some $z$ with $|z| = 1$ then 
   $F'(\zeta) \neq 0$, for $\zeta$ such that $(\phi^{-1} \circ F)(\zeta) = z$

More generally, if a conformal collar of a Jordan curve has some nice properties, then the same holds for any other conformal collar of the same curve from the same side.
References

[1] Ahlfors, Complex analysis, Second edition, McGraw-Hill, New York, 1966

[2] Steven R. Bell and Steven G. Krantz, Smoothness to the boundary of conformal maps, Rocky Mountain Journal of Mathematics, Volume 17, 1987, Number 1, 23-40

[3] E. Bolkas, V. Nestoridis, C. Panagiotis and M. Papadimitrakis, One sided extendability and p-continuous analytic capacities, arxiv: 1606.05443

[4] Gauthier, P. M., Nestoridis V., Conformal extensions of functions defined on arbitrary subsets of Riemann Surfaces. Arch. Math. (Basel) 104 (2015) no 1, 61-67.

[5] Georgakopoulos N., Extensions of the Laurent Decomposition and the spaces $A^p(\Omega)$, arXiv:1605.08289

[6] Koosis P., An introduction to $H_p$ spaces, Cambridge University Press, 1998

[7] V. Mastrantonis, Relations of the spaces $A^p(\Omega)$ and $C^p(\partial \Omega)$, arXiv:1611.02971

National and Kapodistrian University of Athens,
Department of Mathematics
15784
Panepistemiopolis
Athens GREECE
e-mail: lvda20@hotmail.com
e-mail:vnestor@math.uoa.gr