Abstract

We characterize the behavior of integral operators associated with multi-layer perceptrons in two eigenvalue decay regimes. We obtain as a result sharper risk bounds for multi-layer perceptrons highlighting their behavior in high dimensions. Doing so, we also improve on previous results on integral operators related to power series kernels on spheres, with sharper eigenvalue decay estimates in a wider range of eigenvalue decay regimes.

1 Introduction

Multi-layer perceptrons have been an important family of machine learning methods, whose history alternates between periods of wide popularity and periods of fading interest [21, 18, 11]. Described in a mathematical language, a multi-layer perceptron is the iterated composition of parameterized affine maps and nonlinear maps, ultimately composed with a prediction map either for the purpose of predicting a scalar value as in curve fitting or regression, or a binary label as in supervised classification. As the number of layers or network depth grows, the parameterized map resulting from this iterated composition may change in terms of smoothness properties.

However the impact of network depth on the regularity of the resulting deep network function is, quite surprisingly, still not fully understood, in particular for multi-layer perceptrons. Consider the setting, where the number of datapoints is finite, the number of hidden layers is finite, and the number of weights per layers is finite. We would like to understand, in the case of least-squares regression through with regularized empirical risk minimization for instance [17, 5, 29, 8, 24], how fast does the learned function converge to the target function as the network depth grows and the sample size grows.

Understanding the relative scaling of the sample size and the network depth can be related to the detailed study of the spectrum of the integral operator associated with a reproducing kernel related to the network architecture. This reproducing kernel may or may not be universal, hence Bayes-consistency may or may not be satisfied. Yet, as we show in this paper, the universality and Bayes-consistency is guaranteed for wide array of situations.

The integral operator is compact, self-adjoint and admits a spectral decomposition. The rate of eigenvalue decay directly controls the convergence rate of the learned function. In this work, we obtain a tight control of the rate of eigenvalue decay for several regimes depending on the nonlinear maps involved in the construction of the network. From this control, we establish high-probability bounds on the learned function and provide guidance on the effect of the depth for such networks, depending on the sample size. Moreover the results obtained extends previous results related to the Mercer decomposition of dot product kernels on spheres [1].

Reproducing kernels are powerful tools which have been used to analyze deep networks in various ways. Recent works used reproducing kernels to build equivalent instantaneous approximations of deep networks during the training process, blending algorithmic and statistical considerations in the analysis. Although remarkable progress has been made in recent years, the understanding of training processes of deep networks
is still limited. Many results are qualitative in spirit and various seemingly incompatible interpretations have been proposed and subject to discussion [16 26 9].

We adopt here a nonparametric learning approach [14 29], with the modest goal to uncover the interplay of regularity controlled by the network depth and the sample size, with the expectation to obtain high-probability bounds on the learned function explicitly involving the eigenvalue decay of the integral operator. We shall not assume that the target function belongs to the function spaces used for learning.

We believe such results are important steps to shed light on the role of network depth on statistical performance. We present here results of this kind for multi-layer perceptrons. In Sections 2-3, we present general convergence rates for regularized least-squares regression with kernel-based methods, improving upon known convergence rates in the near geometric regime of eigenvalue decay of integral operators. In Section 4, we consider deep networks of multi-layer perceptron type and show how to approach them through reproducing kernel Hilbert spaces and associated integral operators. In Section 5, we obtain convergence rates in two regimes of eigenvalue decay of the integral operators, geometric and super-geometric, matching known minimax optimal rates in nonparametric least-squares regression, and uncovering interesting approximation-estimation trade-offs. We round off the paper with illustrations of the relationship between the eigenvalue decay and the network depth.

2 Regularized Least-Squares

We consider the standard nonparametric learning framework, where the goal is to learn, from independent and identically distributed examples \( z = \{(x_1, y_1), \ldots, (x_T, y_T)\} \) from an unknown distribution \( \rho \), a functional dependency \( f_z: \mathcal{X} \to \mathcal{Y} \) between input \( x \in \mathcal{X} \) and output \( y \in \mathcal{Y} \) [14 29]. The joint distribution \( \rho(x, y) \), the marginal distribution \( \rho_X \), and the conditional distribution \( \rho(\cdot|x) \), are related through \( \rho(x, y) = \rho_X(x) \rho(y|x) \).

We call the \( f_z \) the learning method or the estimator and the learning algorithm is the procedure that, for any sample size \( T \in \mathbb{N} \) and training set \( z \in Z^T \) yields the learned function or estimator \( f_z \). If the output space \( \mathcal{Y} \subset \mathbb{R} \), given a function \( f: \mathcal{X} \to \mathcal{Y} \), the ability of \( f \) to describe the distribution \( \rho \) is measured by its expected risk

\[
R(f) := \int_{\mathcal{X} \times \mathcal{Y}} (f(x) - y)^2 \, d\rho(x, y).
\]

The minimizer over the space of measurable \( \mathcal{Y} \)-valued functions on \( \mathcal{X} \) is the function

\[
f_\rho(x) := \int_{\mathcal{Y}} y d\rho(y|x).
\]

The final aim of learning theory is to find an algorithm such that \( R(f_z) \) is close to \( R(f_\rho) \) with high probability. Let us now introduce the regularized least-squares algorithm. Consider as hypothesis space a Hilbert space \( H \) of functions \( f: \mathcal{X} \to \mathcal{Y} \). For any regularization parameter \( \lambda > 0 \) and training set \( z \in Z^T \), the Regularized Least-Square (RLS) estimator \( f_{H,z,\lambda} \) is the solution of

\[
\min_{f \in H} \left\{ \frac{1}{T} \sum_{i=1}^T (f(x_i) - y_i)^2 + \lambda \| f \|_H^2 \right\}.
\]

Let us recall basic definitions yet important for the exposition of the convergence rates. The goal is to establish bounds, either in expectation or in probability, on \( \| f_{H,z,\lambda} - f_\rho \|_\rho \) where the norm \( \| \cdot \|_\rho \) is the \( L^2_\rho \)-norm and to obtain the convergence rate. Let \( \mathcal{F} \) a class of Borel probability distributions on \( \mathcal{X} \times \mathcal{Y} \) satisfying general assumptions.

**Definition 2.1.** (Upper rate of convergence) A sequence \( (\alpha_t)_{t \geq 1} \) of positive numbers is called upper rate of convergence in \( L^2_\rho \) norm over the model \( \mathcal{F} \), for the sequence of estimated solutions \( (f_{z,\lambda_t})_{t \geq 1} \) using regularization parameters \( (\lambda_t)_{t \geq 0} \) if

\[
\lim_{\tau \to +\infty} \limsup_{T \to \infty} \sup_{\rho \in \mathcal{F}} \rho^T \left( z : \| f_{z,\lambda_t} - f_\rho \|_{\rho}^2 > \tau \alpha_t \right) = 0.
\]
Definition 2.2. (Minimax Lower Rate of Convergence) A sequence \((w_\ell)_{\ell \geq 1}\) of positive numbers is called minimax lower rate of convergence in \(L^2_2\) norm over the model \(\mathcal{F}\) if

\[
\lim_{\tau \to 0^+} \lim_{\ell \to \infty} \inf_{f_\ast \in \mathcal{F}} \sup_{\rho \in \mathcal{I}} \rho^\ell (z : \|f_\ast - f_\rho\|_\rho^2 > \tau w_\ell) = 1
\]

where the infimum is taken over all measurable learning methods with respect to \(\mathcal{F}\).

Such sequences \((w_\ell)_{\ell \geq 1}\) are called minimax lower rates. Every sequence \((\hat{w}_\ell)_{\ell \geq 1}\) decreasing at least with the same rate as \((w_\ell)_{\ell \geq 1}\) is a lower rate for this set of probability measures and at least the same lower rate holds on any larger set of probability measures. When the rate of the sequence of learned functions coincides with the minimax lower rates, it is then said to be optimal in the minimax sense.

### 3 General Convergence Rates

**Approach.** We first present results for regularized least-squares under general assumptions. The convergence rates depend on the eigenvalue decay of the integral operator associated with the kernel defining the reproducing kernel Hilbert space.

Using these results, we shall be able to treat in Section 5 the case of multi-layer perceptrons (MLPs) and obtain convergence rates depending on the eigenvalue decay of an appropriately defined integral operator associated with an appropriately defined kernel. For MLPs, the different eigenvalue decay regimes result from different choices of nonlinear activation functions.

**Setting.** Let \((\mathcal{X}, \mathcal{B})\) a measurable space, \(\mathcal{Y} = \mathbb{R}\) and \(\rho(x, y) = \rho_X(x)\rho(y|x)\) an unknown distribution on \(Z := \mathcal{X} \times \mathcal{Y}\). We assume that \((\mathcal{X}, \mathcal{B})\) is \(\rho_X\)-complete. Let \(H\) be separable reproducing kernel Hilbert space on \(\mathcal{X}\) with respect to a measurable and bounded kernel \(k\). Define the integral operator on \(L^2_2(\mathcal{X})\) associated with \(k\) as

\[
T_\rho : L^2_2(\mathcal{X}) \to L^2_2(\mathcal{X})
\]

\[
f \to \int_\mathcal{X} k(x,.) f(x) d\rho_X(x)
\]

Since \(k\) is bounded, \(T_\rho\) is self-adjoint, positive semi-definite and trace-class. Moreover we have that \(\text{Im}(T_\rho) \subset H \subset L^2_2(\mathcal{X})\) and we denote \(C_\rho\), the restriction of \(T_\rho\) to \(H\), the associated covariance operator.

The spectral theorem for compact operators tells us for an for at most countable index set \(I\), we have a positive, decreasing sequence \((\mu_i)_{i \in I} \in \ell_1(I)\) and a family \((e_i)_{i \in I} \subset H\), such that \((\mu_i \frac{1}{2} e_i)_{i \in I}\) is an orthonormal system in \(H\) while \((e_i)_{i \in I}\) is an orthonormal system in \(L^2_2(\mathcal{X})\) with

\[
C_\rho = \sum_{i \in I} \mu_i \langle \cdot, \mu_i^{1/2} e_i \rangle_H \mu_i^{1/2} e_i
\]

\[
T_\rho = \sum_{i \in I} \mu_i \langle \cdot, e_i \rangle_{L^2_2(\mathcal{X})} e_i
\]

We omit the dependency of \(I\) with \(\rho\) to simplify the notations. We shall therefore assume in the following that \(I = \mathbb{N}\). When \(I\) is finite, results can be found for example in [8].

We work here under general assumptions on the set of probability measures \(\rho\) on \(\mathcal{X} \times \mathcal{Y}\).

**Assumption [Probability measures on \(\mathcal{X}\)].** Let \(H\) be an infinite dimensional separable reproducing kernel Hilbert space (RKHS) on \(\mathcal{X}\) with respect to a bounded and measurable kernel \(k\). Furthermore, let \(C_0, \gamma > 0\) be some constants and \(\alpha > 0\) be a parameter. By \(\mathcal{P}_{H, C_0, \gamma, \alpha}\) we denote the set of all probability measures \(\nu\) on \(\mathcal{X}\) with the following.

- The measurable space \((\mathcal{X}, \mathcal{B})\) is \(\nu\)-complete.
- The eigenvalues fulfill the following upper bound \(\mu_i \leq C_0 e^{-\gamma i^{1/\alpha}}\) for all \(i \in I\).
Let us introduce for a constant $c > 0$ and a parameter $q \geq \gamma > 0$ the subset $P_{H,C_0,\gamma,\alpha,c,q} \subset P_{H,C_0,\gamma,\alpha}$ of probability measures $\mu$ on $\mathcal{X}$ which additionally have the following property.

- The eigenvalues fulfill the following lower bound $\mu_i \geq e^{-q^{1/\alpha}}$ for all $i \in I$.

We shall denote $P_{H,\alpha} := P_{H,C_0,\gamma,\alpha}$ and $P_{H,\alpha,q} := P_{H,C_0,\gamma,\alpha,c,q}$.

**Assumption [Probability measures on $\mathcal{X} \times \mathcal{Y}$].** Let $H$ be a separable RKHS on $\mathcal{X}$ with respect to a bounded and measurable kernel $k$ and $\mathcal{P}$ a set of probability measures on $\mathcal{X}$. Furthermore, let $B, B_0, L, \sigma > 0$ be some constants and $0 < \beta \leq 2$ a parameter. Then we denote by $F_{H,B,B_0,L,\sigma,\beta}(\mathcal{P})$ the set of all probability measures $\rho$ on $\mathcal{X} \times \mathcal{Y}$ with the following properties:

- There exists $g \in L_2^{dx}(\mathcal{X})$ such that $f_0 = T_0^{\beta/2} g$ and $\|g\|_\rho^2 \leq B$.
- There exist $\sigma > 0$ and $L > 0$ such that $\int_{\mathcal{X}} |y - f_0(x)|^m \rho(y|x) \leq \frac{1}{2} m! L^{m-2}$.

A sufficient condition for the last assumption is that $\rho$ is concentrated on $\mathcal{X} \times [-M, M]$ for some constant $M > 0$. We shall denote $F_{H,\alpha,\beta} := F_{H,B,B_0,L,\sigma,\beta}(P_{H,\alpha})$ and $F_{H,\alpha,q,\beta} := F_{H,B,B_0,L,\sigma,\beta}(P_{H,\alpha,q})$.

### 3.1 Upper rate of convergence

We state a theorem establishing $L_2^{dx}$-convergence rates. See Appendix A.1 for the proof.

**Theorem 3.1.** Let $H$ be a separable RKHS on $\mathcal{X}$ with respect to a bounded and measurable kernel $k$, $\alpha > 0$ and $0 < \beta \leq 2$. Then for any $\rho \in F_{H,\alpha,\beta}$ and $\tau \geq 1$ we have:

- If $\beta > 1$, then for $\lambda_\ell = \frac{1}{\ell \tau^\beta}$ and
  $$\ell \geq \max \left( e^{\beta}, \left( \frac{N}{\tau^{\alpha}} \right)^{\frac{2\beta}{\alpha}} \frac{2\beta}{\alpha} \log(\ell) \frac{\alpha}{\tau^{\alpha/\beta}} \right),$$
  with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds
  $$\|f_{H,\mathbf{x},\lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^\alpha}{\ell^\beta}$$

- If $\beta = 1$, then for $\lambda_\ell = \frac{\log(\ell)^\mu}{\ell}$, $\mu > \alpha > 0$ and $\ell \geq \max \left( \exp \left( (N\tau)^{\frac{1}{\alpha}} \right), e^{1 \log(\ell)\mu} \right)$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds
  $$\|f_{H,\mathbf{x},\lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^\mu}{\ell^\beta}$$

- If $\beta < 1$, then for $\lambda_\ell = \frac{\log(\ell)^{\frac{2}{\beta}}}{\ell}$ and
  $$\ell \geq \max \left( \exp \left( (N\tau)^{\frac{1}{\alpha}} \right), e^{1 \log(\ell)^{\frac{2}{\beta}}} \right),$$
  with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds
  $$\|f_{H,\mathbf{x},\lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^{\alpha}}{\ell^\beta}$$

where $N$ and $C$ are constants independent of $\alpha$ and $\beta$. See Appendix A.1 for definitions.

From the above theorem, we obtain the upper rate of convergence, which holds

$$\lim_{\tau \to +\infty} \limsup_{\ell \to \infty} \sup_{\rho \in F_{H,\alpha,\beta}^{\rho}} \rho^\ell \left( z : \|f_{H,\mathbf{x},\lambda_\ell} - f_\rho\|_\rho^2 > \tau a_\ell \right) = 0$$

if one of the following conditions hold
\[ \beta > 1, \lambda_\ell = \frac{1}{\beta^{\ell-1}} \text{ and } a_\ell = \frac{\log(\ell)^\alpha}{\ell} \]
\[ \beta = 1, \lambda_\ell = \frac{\log(\ell)^\alpha}{\ell} \text{ and } a_\ell = \frac{\log(\ell)^\mu}{\ell} \text{ for } \mu > \alpha > 0 \]
\[ \beta < 1, \lambda_\ell = \frac{\log(\ell)^{\beta}}{\ell} \text{ and } a_\ell = \frac{\log(\ell)^\alpha}{\ell} \]

3.2 Lower rate of convergence

In order to investigate the optimality of the convergence rates, let us take a look at the lower rates. See Appendix A.2 for the proof.

**Theorem 3.2.** Let \( H \) be a separable RKHS on \( X \) with respect to a bounded and measurable kernel \( k \), \( q \geq \gamma > 0, \alpha > 0, 0 < \beta \leq 2 \) such that \( \mathcal{P}_{H,\alpha,q} \) is not empty. Then we have
\[
\lim_{\tau \to 0^+} \lim_{\ell \to \infty} \inf_{f \in \mathcal{F}_{H,\alpha,q,\beta}} \sup_{\rho \in \mathcal{F}_{H,\alpha,q,\beta}} \rho^\ell (z : \|f_x - f_\rho\|_p^2 > \tau w_\ell) = 1
\]
where \( w_\ell = \log(\ell)^\alpha / \ell \). The infimum is taken over all measurable learning methods in \( \mathcal{F}_{H,\alpha,q,\beta} \).

**Discussion.** When \( \beta > 1 \) the convergence rates of RLS stated in Theorem 3.1 coincide with the minimax lower rates from Theorem 3.2, i.e. the rates are optimal in the minimax sense in this context. Note that, to the best of our knowledge, previous results in nonparametric learning assumed a polynomial eigenvalue decay of the integral operator; see e.g. recent works [5, 13] and references therein. We show here that optimal rates for regularized least-squares still hold when the eigenvalue decay is geometric. We shall show now that, in fact, an MLP can be embedded in an appropriately defined RKHS, whose corresponding integral operator eigenvalue decay we can control to obtain convergence rates.

4 Multi-Layer Perceptron

We consider a fully-connected deep neural network. We shall refer to it as a multi-layer perceptron (MLP). Let \( X \subset \mathbb{R}^d \), \( N \) be the number of hidden layers, \( \sigma_1, \ldots, \sigma_N \) real-valued functions defined on \( \mathbb{R} \) be the activation functions at each layer and \( m_1, \ldots, m_N \) be the width of each layer. Set \( m_0 := d \) and \( m_{N+1} := 1 \). Then, any function \( \mathcal{N} \) defined by an MLP, parameterized by weight matrices \((W^k)_{k=1}^{N+1}\) where each \( W^k \in \mathbb{R}^{m_k \times m_{k-1}} \), can be recovered as follows. Let \( x \in X \), define \( \mathcal{N}^0(x) := x \) and for \( k \in \{1, \ldots, N\} \), denote \( W^k := (w_{1}^k, \ldots, w_{m_k}^k) \) where for all \( j \in \{1, m_k\} \), \( w_j \in \mathbb{R}^{m_{k-1}} \). Then, for all \( k \in \{1, \ldots, N\} \), define the \( k^{th} \) layer as
\[
\mathcal{N}^k(x) := (\sigma_k(\mathcal{N}^{k-1}(x), w_1^k), \ldots, \sigma_k(\mathcal{N}^{k-1}(x), w_{m_k}^k))
\]
We obtain \( \mathcal{N}(x) = \langle \mathcal{N}(x), W^{N+1} \rangle_{\mathbb{R}^m} \). We shall denote \( \mathcal{F}_{X,(\sigma_i)_{i=1}^N}^{N_1, (m_i)_{i=1}^N} \) the function space defined by all functions \( \mathcal{N} \) defined as above on \( X \) for any choice of \((W^k)_{k=1}^{N+1}\). We shall also consider the union space
\[
\mathcal{F}_{X,(\sigma_i)_{i=1}^N}^{N_1, (m_i)_{i=1}^N} := \bigcup_{(m_1, \ldots, m_N) \in \mathbb{N}^N} \mathcal{F}_{X,(\sigma_i)_{i=1}^N}^{N_1, (m_i)_{i=1}^N}
\]

4.1 Multilayer Perceptron Function Space

We now show that there exists an RKHS containing the function space \( \mathcal{F}_{X,(\sigma_i)_{i=1}^N}^{N_1, (m_i)_{i=1}^N} \) for any activation function, \((\sigma_i)_{i=1}^N\), admitting a Taylor decomposition on \( \mathbb{R} \). Moreover, for well chosen activation maps, the kernel is a cc-universal kernel on \( X \) in \( C(X) \) sense. See Appendix A.1.

**Definition 4.1** (cc-universal kernel). A continuous positive semi-definite kernel \( k \) on a Hausdorff space \( X \) is said to be cc-universal if the RKHS, \( H \) induced by \( k \) is dense in \( C(X) \) endowed with the topology of compact convergence.
Lemma 4.1. Let $X$ be any subspace of $\mathbb{R}^d$, $N \geq 1$, $(\sigma_i)_{i=1}^N$ functions which admits a Taylor decomposition on $\mathbb{R}$. Moreover let $(f_i)_{i=1}^N$ be the sequence of functions such that for every $i \in \{1, ..., N\}$:

$$f_i(x) = \sum_{n \geq 0} \frac{|\sigma_i^{(n)}(0)|}{n!} x^n$$

Then the RKHS $H_N$ of the kernel, $K_N$ defined on $X \times X$ by:

$$K_N(x,x') := f_N \circ ... \circ f_1((x,x')_{\mathbb{R}^d})$$

contains the function space $\mathcal{F}_{X,(\sigma_i)_{i=1}^N}$. If we assume in addition that for every $i \in \{1, ..., N\}$ and $n \in \mathbb{N}$, $\sigma_i^{(n)}(0) \neq 0$, then the kernel $K_N$ is $cc$-universal.

The result above extend previous results on the universality of kernels associated with deep neural networks [4, 31, 32], in which where the number of weights was assumed fixed and activation functions were the same at each layer.

Remark 1. Arbitrary Width Networks. It is worth emphasizing that, for any number of weights $(m_i)_{i=1}^N$, the function space $\mathcal{F}_{X,(\sigma_i)_{i=1}^N,(m_i)_{i=1}^N}$ belongs to the RKHS we have just defined. In other words, the RKHS we define does not require that all layers of the network are infinitely large as is commonly assumed in several previous works.

For suitably chosen activation functions, the kernel $K_N$ defined above is $cc$-universal. Therefore the RKHS $H_N$ obtained approximate the Bayes risk of a large class of loss, in particular the least squares loss. See Corollary 5.29 [29].

$$\inf_{f \in H} \mathbb{E}[(f(X) - Y)^2] = R^*$$

where $R^*$ is the Bayes risk. For instance, if at each layer the nonlinear function is $\sigma_{\text{exp}}(x) = \exp x$, then the RKHS becomes universal. There are other examples of activation functions satisfying the assumptions of the Lemma [1] such as the square activation $\sigma_2(x) = x^2$, the smooth hinge activation $\sigma_{\text{sh}}$, close to the rectifier linear unit (ReLU) activation, or a sigmoid-like function such as $\sigma_{\text{erf}}$, similar to the sigmoid function. See [31].

$$\sigma_{\text{erf}}(x) = 1 \sqrt{\pi} \int_{-x}^{x} e^{-t^2} dt \quad \sigma_{\text{sh}}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} xe^{-t^2} dt + \exp(-\sqrt{\pi x^2})$$

Thanks to Lemma 4.1, we can have a direct control on the RKHS norm of the network through the spectral norm of its weights.

Corollary 4.1. For $(W^k)_{k=1}^{N+1}$ a sequence such that for all $k \in [1, N+1]$, $W^k \in \mathbb{R}^{m_{k-1} \times m_k}$ and $N$ the function in $\mathcal{F}_{X,(\sigma_i)_{i=1}^N}$ associated, we have

$$\|N\|_{H_N}^2 \leq \|W^{N+1}\|^2 f_N(\|W\|^2 f_{N-1}(\|W^1\|^2 ...))$$

where $\|\cdot\|$ is the spectral norm.

We can then obtain statistical bounds for deep networks such as as the ones in [31, 32, 3, 4]. Indeed the RKHS norm of the network allows one to get for instance generalization bounds involving Rademacher complexities as in [7].
To be more precise, if $X$ is compact and $Y \subset [-M, M]$, then there exists constants $G, B$ such that for any $\delta > 0$ with a probability $\rho^n$ not less than $1 - \delta$

$$R(N) \leq \frac{1}{\ell} \sum_{i=1}^{\ell} (N(x_i) - y_i)^2 + \frac{2G\|N\|_{H_N}}{\gamma n} \sqrt{\sum_{i=1}^{n} K_N(x_i, x_i)} + \frac{2M^2}{\sqrt{n}} + 3B \sqrt{\frac{\log\left(\frac{\ell}{n}\right)}{2n}},$$

where the RKHS norm of the network is controlled by

$$\|N\|_{H_N} \leq \|W^{N+1}\|^2 f_N(\|W^N\|^2 f_{N-1}(...f_1(\|W^1\|^2)...)).$$

We present in the next section bounds that are, if compared on an equal footing, tighter in principle, owing to a better control of the complexity through the eigenvalue decay of the integral operator. Depending on the choice of the nonlinear functions involved in the construction of the network, we show that for suitably chosen regularization parameter sequences, we get an estimator satisfying with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ either

$$\|f_{H_N, \omega, \lambda} - f_{\rho}\|^2_{\rho} \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{\ell},$$

or

$$\|f_{H_N, \omega, \lambda} - f_{\rho}\|^2_{\rho} \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{\log(\log(\ell))^{d-1}}.$$

## 5 Regularized Least-Squares for Multi-layer Perceptrons

We can now get the statistical bounds for regularized least-squares with a multi-layer perceptron (MLP). The desiderata for the statistical bounds we wish to obtain is to uncover the effect of the network depth i.e. the length of the chain composition defining the network. There are two regimes that arise: i) geometric case; ii) super-geometric case.

We assume that the input space is the unit sphere $S^{d-1}$ of $\mathbb{R}^d$ where $d \geq 2$. This is a common assumption in the literature, motivated by common input pre-processing procedures in signal processing and computer vision [2, 26, 12]. We show now that a multi-layer perceptron can be concisely described thanks to the results presented earlier.

**Setting.** Given a dataset $z = (x_i, y_i)_{i=1}^{\ell}$ independently sampled from an unknown distribution $\rho(x, y)$ on $Z := S^{d-1} \times Y$ the goal least-squares regression is to estimate the conditional mean function $f_{\rho} : S^{d-1} \to \mathbb{R}$ given by $f_{\rho}(x) := E(Y | X = x)$. Here the hypothesis space considered is the Hilbert space associated with the kernel defined in Lemma 4.1. More precisely, let $N \geq 1$, $d \geq 2$ and $(\ell_i)_{i=1}^{N}$ be a sequence of $N$ real value functions such that $f_N \circ ... \circ f_1$ admits a Taylor decomposition on $[-1, 1]$ with non negative coefficients $(b_m)_{m \geq 0}$:

$$f_N \circ ... \circ f_1(x) := \sum_{m \geq 0} b_m x^m$$

and let $H_N$ be the RKHS associated with the kernel $K_N$ defined on $S^{d-1}$ by:

$$K_N(x, x') := f_N \circ ... \circ f_1(\langle x, x' \rangle_{S^{d-1}}).$$

**Assumptions.** For $\omega \geq 1$, we denote by $W_\omega$ the set of all probability measures $\nu$ on $S^{d-1}$ which satisfy $\frac{d\nu}{d\sigma_{d-1}} < \omega$ where $d\sigma_{d-1}$ is the induced Lebesgue measure on $S^{d-1}$. Furthermore, we introduce for a constant $\omega \geq 1 > h > 0$, $W_{\omega, h} \subset W_\omega$ the set of probability measures $\nu$ on $S^{d-1}$ which additionally satisfy $\frac{d\nu}{d\sigma_{d-1}} > h$. In the following we denote $G_{\omega, \beta} := F_{H_N, B, B_{\omega, h}, L, \omega, \beta}(W_\omega)$ and $G_{\omega, h, \beta} := F_{H_N, B, B_{\omega, h}, L, \omega, \beta}(W_{\omega, h})$. 

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5.1 Geometric Case

We first consider the case of multi-layer perceptrons such that the composition of nonlinear functions involved in the network admits a Taylor decomposition with coefficients decreasing almost geometrically. In the following theorem, we provide our result about generalization error analysis for the empirical risk minimizer. See proof in Appendix C.2.

**Theorem 5.1.** Let us assume that there exists $1 > r > 0$, $0 < c_1$ some constants such that the sequence $(b_m)_{m \geq 0}$ satisfies for all $m \geq 0$:

$$b_m \leq c_{1r}^m$$  \hspace{1cm} (5)

Let also $w \geq 1$ and $0 < \beta \leq 2$. Then there exists $A, C > 0$ some constants independent of $\beta$ (see Appendix A.1 for their definitions) such that for any $\rho\in\mathcal{G}_{w,\beta}$ and $\tau \geq 1$ we have:

- If $\beta > 1$, then for $\lambda_\ell = \frac{1}{\rho \ell^{\beta}}$ and $\ell \geq \max \left( e^\beta, \left( \frac{4}{\beta^2} \right) \frac{2\beta^2}{\beta^2 - 1} \log(\ell)^{\frac{d-1}{\beta}} \right)$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

$$\|f_{H_N,\lambda,\ell} - f_\rho\|^2_\rho \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{\ell^\beta}$$

- If $\beta = 1$, then for $\lambda_\ell = \frac{\log(\ell)^{d-1}}{\ell^\beta}$, $\mu > d - 1 > 0$ and $\ell \geq \max \left( \exp \left( (A\tau)^{\frac{1}{\beta}} \right), e\rho \log(\ell)^\mu \right)$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

$$\|f_{H_N,\lambda,\ell} - f_\rho\|^2_\rho \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{\ell^\beta}$$

- If $\beta < 1$, then for $\lambda_\ell = \frac{\log(\ell)^{d-1}}{\ell^\beta}$ and $\ell \geq \max \left( \exp \left( (A\tau)^{\frac{1}{\beta}} \right), e\rho \log(\ell)^\mu \right)$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

$$\|f_{H_N,\lambda,\ell} - f_\rho\|^2_\rho \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{\ell^\beta}$$

Moreover the following theorem investigates the optimality of our learning rates

**Theorem 5.2.** See proof in Appendix C.2. Let us assume that there exists $0 < c_2 < c_1$ such that for all $m \geq 0$:

$$c_{2r}^m \leq b_m \leq c_{1r}^m$$

Then for any $w \geq 1 > h > 0$ such that $\mathcal{W}_{w,h}$ is not empty it holds

$$\lim_{\tau \to 0^+} \lim_{\ell \to \infty} \inf_{\rho} \sup_{z \in \mathcal{G}_{w,h,\beta}} \rho^\ell(z : \|f_{\rho} - f_z\|^2_\rho > \tau b_\ell) = 1$$

where $b_\ell = \frac{\log(\ell)^{d-1}}{\ell^\beta}$. The infimum is taken over all measurable learning methods with respect to $\mathcal{G}_{w,h,\beta}$.

Therefore for such networks, when $\beta > 1$ the learning rates of the regularized least-squares estimator stated in theorem 5.1 coincide with the minimax lower rates from theorem 5.2 and therefore are optimal in the minimax sense.
Remark 2. Instead of the regularized least-squares algorithm, let us consider for any sequence \((m_i)_{i=1}^N \in \mathbb{N}_*^N\) the following estimator:

\[
\hat{f}_{(m_i)_{i=1}^N} = \arg\min_{f \in \mathcal{F}_{N,\tau}(m_i)_{i=1}^N} \left\{ \frac{1}{\ell} \sum_{i=1}^{\ell} (f(x_i) - y_i)^2 \right\}
\]

Then the results of theorems 5.1 and 5.2 suggests that for any choice of \((m_i)_{i=1}^N\), the generalization performance of such estimator cannot have better asymptotic rates than the one obtained by regularized least square algorithm in this setting.

Proof. (Sketch) The main challenge here is to control the rate of eigenvalue decay associated to the integral operator \(T_\rho\). In fact, in this regime, we show that the rate of eigenvalue decay is almost geometric (see Appendix. Propositions C.2 and C.1), therefore such RKHS satisfy the assumptions of section 3 from which upper and lower rates are obtained. \(\square\)

Example. Consider the case where \(f_1\) is the inverse polynomial function \(f_1(x) = 1/(2 - x)\), while the other layers are polynomial functions. Then, for all \(k \geq 2\), \(f_\ell \circ ... \circ f_1\) satisfy also Assumption 5. Thus Theorem 5.1 can be applied. On the one hand, the convergence rate remains the same asymptotically as the network gets deeper. On the other hand, the space in which the target function is assumed to live is allowed to grow as the network gets deeper.

5.2 Super-geometric Case

Another regime of interest is the one where the composition of nonlinear functions involved in the network admits a Taylor decomposition with coefficients decreasing super-geometrically. As in the geometric case, the goal is to obtain a control of the eigenvalues associated with the integral operator \(T_{K_N}\). See Appendix C.3.

Theorem 5.3. Assume that there exists \(1 > \delta > 0\) such that

\[
\left| \frac{b_m}{b_{m-1}} \right| \in O(m^{-\delta}).
\] (6)

Let also \(0 < \beta \leq 2\) and \(\omega > 0\). Then there exist a constant \(C\) independent of \(\beta\) such that for any \(\rho \in \mathcal{G}_{\omega,\beta}\) and \(\tau \geq 1\) we have:

- If \(\beta > 1\), then there exists \(\ell_\tau > 0\) such that for all \(\ell \geq \ell_\tau\) and \(\lambda_\ell = \frac{1}{e^{\tau \ell}}, \rho = \frac{1}{e^{\tau \ell}}, \tau \geq 1\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{H_N, x, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{[\log(\log(\ell))]^{d-1} \ell^\omega}
\]

- If \(\beta = 1\), then there exists \(\ell_\omega > 0\) such that for all \(\ell \geq \ell_\omega\), \(\lambda_\ell = \frac{\log(\ell)^\mu}{[\log(\log(\ell))]^d \ell^\omega}\) and \(\mu \geq d - 1\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{H_N, x, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^\mu}{[\log(\log(\ell))]^{d-1} \ell^\omega}
\]

- If \(\beta < 1\), then there exists \(\ell_\tau > 0\) such that for all \(\ell \geq \ell_\tau\) and \(\lambda_\ell = \frac{\log(\ell)^{d-1}}{[\log(\log(\ell))]^{d-1} \ell^\omega}\) with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{H_N, x, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C\tau^2 \frac{\log(\ell)^{d-1}}{[\log(\log(\ell))]^{d-1} \ell^\omega}
\]

As expected, in that regime, we obtain faster rates compared to the geometric case. Indeed the eigenvalues decrease faster. Moreover as in the geometric case, we can deduce asymptotic rates. See Appendix Corollary C.1.
Example 1. Let consider the case where \( N = 2 \) and \( f_2(t) = f_1(t) = \exp(t) \). Thanks to Lemma 7 from Appendix (D), we have an explicit expression of the sequence \((b_m)_{m \geq 0}\) with \( b_0 = e^1 \)

\[
b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \frac{b_k}{(m-k)!}
\]

We can compute the coefficients \((b_m)_{m \geq 0}\) thanks to this recurrence relation. In Figure 1, we plot the rate of convergence of \((|b_m/b_{m-1}|)_{m \geq 1}\) (in blue) and compare it to \((1/m^\delta)_{m \geq 1}\) where \( \delta = 1/3 \) (in green) and \( \delta = 1/5 \) (in red): assumption (6) hold for \( \delta = 1/5 < 1/2 \) and Theorem 5.3 hold.

![Figure 1: Decay of \((|b_m/b_{m-1}|)_{m \geq 1}\).]

Example 2. Consider the case where \( f_1 \) is the exponential function, while the other layers are polynomial functions. Then \( f_N \circ ... \circ f_1 \) satisfies also Assumption 6. Then Theorem 5.3 can be applied.

Proof. (Sketch) As in the geometric case, the main goal here is to control the rate of eigenvalue decay associated to the integral operator \( T_\rho \) in order to control the degrees-of-freedom defined as

\[
df(\lambda) := \text{Tr} \left( (C_\rho + \lambda)^{-1} C_\rho \right)
\]

To do so, we first study the eigenvalue decay of the integral operator \( T_{K_N} \) defined as

\[
L_2^{d\sigma_{d-1}} (S^{d-1}) \rightarrow \int_{S^{d-1}} K_N(x,.) f(x) d\sigma_{d-1}(x)
\]

In [1], the author present a control of distinct eigenvalues \((\omega_m)_{m \geq 0}\) of \( T_{K_N} \) only for \( \delta \geq 1/2 \).

\[
\omega_m \in O \left( \frac{b_m}{2^{m+1} m^{(d-2)/2}} \right)
\]

We extend the above result for any \( \frac{1}{2} > \delta > 0 \). See Appendix for proof of Proposition C.3. For \( \frac{1}{2} > \delta > 0 \), with \( \alpha := 1/(1-2\delta) \), we obtain

\[
\omega_m \in O \left( \frac{m^{\alpha} b_m}{2^{m+1} m^{(d-2)/2}} \right)
\]
From this control, we are able to obtain the rate of convergence of the positive eigenvalues of the integral operator $T_{K_N}$ associated to the kernel $K_N$ ranked in a non-increasing order with their multiplicities. See Appendix for proof of Proposition C.4. Finally we obtain a tight control of the degrees-of-freedom from which an analogue proof of the theorem 3.1 can be derived to obtain rates in this regime.

$$\text{df}(\lambda) \in \mathcal{O} \left( \log(\lambda^{-1})^{d-1} / \left( \log((\lambda^{-1})) \right)^{d-1} \right)$$

6 Experiments

(a) Accuracy vs Depth.

(b) Decay vs. Depth.

First, we investigate the relations between network depth and prediction accuracy on MNIST \(^1\). We construct fully connected neural networks from 1 to 3 layers. We model large-width networks by setting number of weights per layer to be constant and sufficiently large. The number of weights can be either 128, 256 or 512 at each layer. Moreover we normalize the input data so that the input datapoints lie on a subspace of the unit sphere.

We see in Figure 2(a) that, in each situation, as the number of layers increases, the prediction accuracy gets better. This means that the networks get better at approximating the target function as the network depth increases.

Second, we investigate the relation between the network depth and the eigenvalue decay, in the supergeometric setting. We proceed similarly, including regarding input normalization. The first nonlinear activation is the exponential function, while the others are quadratic activation functions for which assumption (6) is indeed satisfied. We obtain the spectral decomposition of the kernel associated with the network by computing the eigenvalues associated with the last hidden layer as a feature map of the kernel. Figure 2(b) shows the result after 15 epochs and for a number of weights per layer equal to 512. See Appendix \(^{11}\) for 128 and 256. We observe that, as the number of layers increases, the eigenvalue decay is slower, which means that the function space generated by such structures is larger. Therefore, as the number of layers increases, the network gets better at approximating the target function in the asymptotic regime.

\(^{1}\)http://yann.lecun.com/exdb/mnist/
7 Related works

An extensive survey of the abundant literature on theoretical analysis of deep networks is beyond the scope of this paper. We focus here on relevant recent works to contrast our work with.

**Eigenvalue decay regimes.** Recent works such as [5] [13] establish convergence rates for regularized least-squares in reproducing kernel Hilbert spaces; see [29] for an earlier survey of this literature. In these papers and previous ones referenced therein, the analysis focuses on the case of polynomial eigenvalue decay of the integral operator, while we extend the analysis to geometric and super-geometric decays.

In [2], the author studies also single-hidden layer neural networks with input data on the sphere. An RKHS is built allowing to get an explicit control of the eigenvalue decay of the integral operator. The eigenvalue considered in [2] is \( m^{-1/2d} \), that is a polynomial eigenvalue decay. Our work considers multiple hidden layer neural networks and cover a wider range of eigenvalue decay regimes.

**Embedding an MLP in a function space.** The papers [23] [19] focus on the realizable case, i.e. when the target function \( f_\rho \) lives in the function space used to learn, and characterize the norm required to realize such functions using a network, with a single-hidden layer, with an infinite width, and a ReLU activation. Their viewpoint is diametrically opposed to ours, the proof techniques as well, since we do not assume that the target function is realizable, nor do we assume that the layers have infinite width, and consider deep networks instead of a single-hidden layer ones. The two viewpoints can therefore be seen as complementary.

The paper [30] presents generalization bounds for multi-layer perceptrons. While the paper may seem related to ours on a superficial level, the developments and resulting bounds in each paper differ substantially. We highlight below these differences. First, in this paper as well, the target function \( f_\rho \) is assumed to live in the function space used to learn. In our setting this assumption corresponds to the case where \( \beta = 1 \). The goal there is to obtain bounds for the learning method \( \hat{f}(m_\ell)_{\ell=1}^N \) with \((m_\ell)_{\ell=1}^N \in \mathbb{N}^N\) fixed defined as

\[
\hat{f}(m_\ell)_{\ell=1}^N \in \arg\min_{f \in F, \{(x_i, \sigma_i)_{i=1}^N, (m_i)_{i=1}^N\}} \left\{ \frac{1}{\ell} \sum_{i=1}^{\ell} (f(x_i) - y_i)^2 \right\}.
\]

Assuming that the number of weights per layer is of the same order as the degrees-of-freedom up to a log term, the number of samples \( \ell \) is large enough and the regularization parameter small enough, the resulting high probability bound has the form

\[
\left\| \hat{f}(m_\ell)_{\ell=1}^N - f_\rho \right\|_\rho^2 \in O \left( \frac{\log(\ell) d_f(\lambda)^2}{\ell} + \lambda \right),
\]

A major difference of our work is the explicit control of the degrees-of-freedom \( d_f(\lambda) \) through the eigenvalue decay of the integral operator. However the control of the degrees-of-freedom within the framework of [30] is unclear and deserves further investigation. Furthermore in our case, a similar bias-variance trade-off appears. Yet, when applying the above bound in each of two regimes we considered, the resulting bounds are less tight than ours in both regimes.

8 Conclusion.

We studied the complexity of the function space generated by multi-layer perceptrons in relation to the network depth. Investigating the effect of the number of weights per layer is an interesting venue for future work.

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In Section A, we prove our general results on learning rates for regularized least-squares for geometric eigenvalue decay. In Section B, we build the reproducing kernel Hilbert space (RKHS) of multi-layer perceptrons and establish its universality properties. We also show how to control the corresponding RKHS-norm. In Section C, we prove high-probability bounds in two regimes. In Sections E-F-G, we collect useful technical results and basic notions. In Section H, we illustrate the relationship between the network depth and statistical performance of a multi-layer perceptron.

A General Convergence Rates

Let us first recall the general assumptions on the set of probability measures $\rho$ on $X \times Y$ for the general setting.

Assumption [Probability measures on $X$]. Let $H$ be an infinite dimensional separable reproducing kernel Hilbert space (RKHS) on $X$ with respect to a bounded and measurable kernel $k$. Furthermore, let $C, \gamma > 0$ be some constants and $\alpha > 0$ be a parameter. By $P_{H,C,\gamma,\alpha}$ we denote the set of all probability measures $\nu$ on $X$ with the following.

• The measurable space $(X, B)$ is $\nu$-complete.

• The eigenvalues fulfill the following upper bound $\mu_i \leq C_0 e^{-\gamma i^{1/\alpha}}$ for all $i \in I$.

Let us introduce for a constant $c > 0$ and a parameter $q \geq \gamma > 0$ the subset $P_{H,C_0,\gamma,\alpha,c,q} \subset P_{H,C,\gamma,\alpha}$ of probability measures $\mu$ on $X$ which additionally have the following property.

• The eigenvalues fulfill the following lower bound $\mu_i \geq c e^{-q i^{1/\alpha}}$ for all $i \in I$.

We shall denote $P_{H,\alpha} := P_{H,C_0,\gamma,\alpha}$ and $P_{H,\alpha,q} := P_{H,C_0,\gamma,\alpha,c,q}$.

Assumption [Probability measures on $X \times Y$]. Let $H$ be a separable RKHS on $X$ with respect to a bounded and measurable kernel $k$ and $P$ a set of probability measures on $X$. Furthermore, let $B, B_\infty, L, \sigma > 0$ be some constants and $0 < \beta \leq 2$ a parameter. Then we denote by $F_{H,B,B_\infty,L,\sigma,\beta}(P)$ the set of all probability measures $\rho$ on $X \times Y$ with the following properties:

• $\rho_X \in P$, $\int_{X \times Y} y^2 d\rho(x,y) < \infty$, $\|f_\rho\|_{L^\infty_{|y|}}^2 \leq B_\infty$

• There exist $g \in L_2^{\rho_X}(X)$ such that $f_\rho = T_\rho^{\beta/2} g$ and $\|g\|_2^2 \leq B$

• There exist $\sigma > 0$ and $L > 0$ such that $\int_Y |y - f_\rho(x)|^m d\rho(y|x) \leq \frac{1}{2} m! L^{m-2}$

A sufficient condition for the last assumption is that $\rho$ is concentrated on $X \times [-M,M]$ for some constant $M > 0$. We shall denote $F_{H,\alpha,\beta} := F_{H,B,B_\infty,L,\sigma,\beta}(P_{H,\alpha})$ and $F_{H,\alpha,q,\beta} := F_{H,B,B_\infty,L,\sigma,\beta}(P_{H,\alpha,q})$.

A.1 Proof of Theorem 3.1

Proof. To show the result let us first define the degrees-of-freedom $\text{df}(\lambda) := \text{Tr} \left( (C_\rho + \lambda)^{-1} C_\rho \right) = \sum_{i \in I} \frac{\mu_i}{\mu_i + \lambda}$. Moreover for $\lambda > 0$ the minimization problem

$$\inf_{f \in H} \left\{ R(f) + \lambda \|f\|_H^2 \right\}$$

has a unique solution defined as $f_{H,\rho,\lambda} = (C_\rho + \lambda)^{-1} T_\rho f_\rho \in H$. To obtain an upper rate of convergence we first control the degrees-of-freedom with respect to the regularization parameter.
Lemma 1. Let $\alpha > 0$. If there exist $C, \gamma > 0$ such that $\mu_i \leq C_0 e^{-\gamma i/\alpha}$ for all $i \in I$. Then all $0 < \lambda \leq e^{-1}$ we have:

$$
df(\lambda) \leq Q \log(\lambda^{-1})^\alpha$$

where $Q = \gamma^{-\alpha} \left[ 1 + C_0 \int_1^\infty \frac{(\log(u) + 1)^{\alpha-1}}{C_0 u + u^2} du \right]$

Proof. By definition of $df(\lambda)$ we have

$$df(\lambda) \leq \sum_{m \geq 1} \frac{C_0}{C_0 + \lambda e^{\gamma m\frac{1}{\alpha}}}.$$ 

Moreover as the $x \rightarrow \frac{C_0}{C_0 + \lambda e^{\gamma x\frac{1}{\alpha}}}$ is positive and decreasing, therefore we have

$$df(\lambda) \leq \int_0^{+\infty} \frac{C_0}{C_0 + \lambda e^{\gamma u\frac{1}{\alpha}}} du.$$ 

Let us consider the following substitution

$$u = e^{\gamma \lambda^{\frac{1}{\alpha}}}.$$ 

Therefore we have

$$df(\lambda) \leq \int_{\lambda}^{\infty} C_0 \gamma^{-\alpha} \frac{(\log(u\lambda^{-1}))^{\alpha-1}}{u} du \leq \int_{\lambda}^{1} C_0 \gamma^{-\alpha} \frac{(\log(u\lambda^{-1}))^{\alpha-1}}{C_0 u + u^2} du + \int_{1}^{\infty} C_0 \gamma^{-\alpha} \frac{(\log(u\lambda^{-1}))^{\alpha-1}}{C_0 u + u^2} du$$

Therefore for all $\lambda \leq e^{-1}$, we obtain that

$$df(\lambda) \leq \gamma^{-\alpha} \left[ \log(\lambda^{-1})^\alpha \right] + \gamma^{-\alpha} \left[ \log(\lambda^{-1})^\alpha + \int_1^{\infty} \frac{(\log(u) + 1)^{\alpha-1}}{C_0 u + u^2} du \right]$$

Finally we obtain that

$$df(\lambda) \leq \gamma^{-\alpha} \left[ \log(\lambda^{-1})^\alpha \right] + \gamma^{-\alpha} \left[ \log(\lambda^{-1})^\alpha + \int_1^{\infty} \frac{(\log(u) + 1)^{\alpha-1}}{C_0 u + u^2} du \right]$$

Therefore we have

$$df(\lambda) \leq Q \log(\lambda^{-1})^\alpha$$

where $Q = \gamma^{-\alpha} \left[ 1 + C_0 \int_1^{\infty} \frac{(\log(u) + 1)^{\alpha-1}}{C_0 u + u^2} du \right]$

□

Let us now split the error $\|f_{H,z,\lambda} - f_\rho\|_\rho^2$ into two parts.

$$\|f_{H,z,\lambda} - f_\rho\|_\rho^2 \leq 2\|f_{H,z,\lambda} - f_{H,\rho,\lambda}\|_\rho^2 + 2\|f_{H,\rho,\lambda} - f_\rho\|_\rho^2$$

The following Lemma provides a control of the approximation error.

Lemma 2. Let $0 < \beta \leq 2$, $\rho$ a probability measure on $X \times Y$ and $H$ a separable RKHS on $X$ with respect to a bounded and measurable kernel $k$. If there exist $g \in L^2_{d\rho}(X)$ such that $f_\rho = T^{\beta/2}_\rho g$, then for all $\lambda > 0$ it holds

$$\|f_{H,\rho,\lambda} - f_\rho\|_\rho^2 \leq \lambda^{\beta} \|g\|_\rho^2$$
Proof. By denoting \( a_i := (f, e_i)_{L^2_{d\rho}} \) we have
\[
\| f_{H, \rho, \lambda} - f_{\rho} \|_\rho^2 = \left\| \sum_{i \in I} \frac{\mu_i}{\mu_i + \lambda} a_i e_i - \sum_{i \in I} a_i e_i \right\|_\rho^2 \\
= \lambda^2 \sum_{i \in I} \left( \frac{\mu_i}{\mu_i + \lambda} \right)^2 \mu_i^{-\beta} a_i^2
\]
Let us consider the following function for \( \lambda > 0 \) and \( 0 \leq \gamma \leq 1 \): \( f_{\lambda, \gamma} : t \in \mathbb{R}_+ \rightarrow \frac{t^\gamma}{t^\gamma + \lambda} \). By considering the derivative of \( f_{\lambda, \gamma} \), we obtain that \( \sup_{t \in [0,1]} f_{\lambda, \gamma} \leq \lambda^{-1} \), therefore we have
\[
\| f_{H, \rho, \lambda} - f_{\rho} \|_\rho \leq \lambda^\beta \sum_{i \in I} \frac{\mu_i}{\mu_i + \lambda} \mu_i^{-\beta} a_i^2 \\
\leq \lambda^\beta \| g \|_\rho^2
\]
□

Let us now focus on the estimation error. Let \( \tau \geq 1, \lambda > 0 \). Thanks to theorem F.1 we have with \( \rho^\lambda \)-probability \( \geq 1 - 4e^{-\tau} \),
\[
\| f_{z, \lambda} - f_{\rho, \lambda} \|_\rho \leq 128 \frac{\tau^2}{L} \left( \frac{5df(\lambda)\sigma^2_\lambda + K \frac{L_\lambda}{L}}{L} \right)
\] (9)
as soon as \( \ell \geq N_{\lambda, \tau} \) where \( K = \sup_{x \in X} k(x, x) \) and
\[
N_{\lambda, \tau} = \max \left( \frac{256\tau^2 Kdf(\lambda)}{\lambda}, \frac{16\tau K}{\lambda}, \tau \right) \\
\sigma_\lambda = \max(\sigma, \| f_{\rho} - f_{\rho, \lambda} \|_{L^2_{d\rho}}) \\
L_\lambda = \max(L, \| f_{\rho} - f_{\rho, \lambda} \|_{L^2_{d\rho}})
\]
In the next Lemma, we exhibit a control \( N_{\lambda, \tau}, L^2_\lambda \) and \( \sigma^2_\lambda \).

**Lemma 3.** Let \( \rho \in \mathcal{F}_{H, \alpha, \beta} \) be a probability measure. Then there are constants \( N, V > 0 \) depending only on \( \mathcal{F}_{H, \alpha, \beta} \) such that \( N_{\lambda, \tau} \leq N \tau^2 \log(\lambda^{-1})_\alpha \) and \( L_\lambda^2, \sigma^2_\lambda \leq \frac{V}{\lambda} \) for all \( 0 < \lambda \leq e^{-1} \) and \( \tau \geq 1 \).

**Proof.** Thanks to Lemma 1 we have that
\[
df(\lambda) \leq Q \log(\lambda^{-1})^\alpha
\]
where \( Q = \gamma^\alpha \left[ 1 + C_0 \int_1^\infty \frac{\log(u) + 1)}{C_0 u + u^2} du \right] \)
And by definition we have
\[
N_{\lambda, \tau} \leq \max \left( \frac{256\tau^2 Kdf(\lambda)}{\lambda}, \frac{16\tau K}{\lambda}, \tau \right) \\
\leq N \tau^2 \log(\lambda^{-1})_\alpha
\]
where \( N = \max(256 KQ, 16 K, 1) \)
Moreover we have
\[
\| f_{\rho} - f_{H, \rho, \lambda} \|_{L^2_{d\rho}} \leq 2(\| f_{\rho} \|_{L^2_{d\rho}} + \| f_{\rho, \lambda} \|_{L^2_{d\rho}}) \\
\leq 2B_{\infty} + 2K \| f_{H, \rho, \lambda} \|_{L^2_{d\rho}}
\]

But by denoting \( a_i := \langle f, e_i \rangle_{L^2_{d\xi}(X)} \) we remark that

\[
\|f_{H,\rho,\lambda}\|_H^2 = \left\| \sum_{i \in I} \frac{\mu_i}{\mu_i + \lambda} a_i e_i \right\|_H^2 = \sum_{i \in I} \left( \frac{\mu_i}{\mu_i + \lambda} \right)^2 a_i^2 \mu_i^{-1} = \sum_{i \in I} \left( \frac{\frac{\mu_i^{\beta+1}}{\mu_i + \lambda}}{\mu_i^{\beta} + 1} \right)^2 a_i^2 \mu_i^{-\beta}
\]

Indeed the first equality is due to the fact that we have assumed the existence of \( \beta > 0 \) and \( g \in L^2_{d\xi}(X) \) such that \( f_\rho = T_\rho^{\beta/2} g \). Let us now consider the following function for \( \lambda > 0 \) and \( 0 \leq \gamma \leq 1 \): \( f_{\lambda,\gamma} : t \in \mathbb{R}_+ \rightarrow \frac{t^\gamma}{\ell^{\gamma+1}} \).

By considering the derivative of \( f_{\lambda,\gamma} \), we obtain that \( \sup_{t \in \mathbb{R}_+} f_{\lambda,\gamma} \leq \lambda^{\gamma-1} \). Therefore if \( 0 < \beta \leq 1 \) we have that

\[
\|f_{H,\rho,\lambda}\|_H^2 \leq \lambda^{\beta-1} \|g\|_\rho^2 \leq \lambda^{\beta-1} B
\]

Finally if \( \beta > 1 \) then \( f_\rho \in H \) and we have

\[
\|f_\rho - f_{H,\rho,\lambda}\|_{L^2_{d\xi}} \leq K \|f_\rho - f_{H,\rho,\lambda}\|_H^2
\]

But we have

\[
\|f_\rho - f_{H,\rho,\lambda}\|_H^2 = \left\| \sum_{i \in I} \frac{\lambda}{\mu_i + \lambda} a_i e_i \right\|_H^2 = \sum_{i \in I} \left( \frac{\lambda}{\mu_i + \lambda} \right)^2 a_i^2 \mu_i^{-1} = \lambda^2 \sum_{i \in I} \left( \frac{\mu_i^{\beta+1}}{\mu_i^{\beta} + 1} \right)^2 a_i^2 \mu_i^{-\beta}
\]

And as \( \beta > 1 \), we obtain that

\[
\|f_\rho - f_{H,\rho,\lambda}\|_H^2 \leq \lambda^{\beta-1} \|g\|_\rho \leq B
\]

Finally by choosing \( V = \max(L^2, \sigma^2, 2BK \) + \( 2B_\infty \)) we obtain that \( L^2_{\lambda, 2}, \sigma^2_{\lambda} \leq \frac{V}{\lambda \max(1, \beta, \sigma)} \).

We can now prove the Theorem. If \( \ell \geq N \frac{\tau^2 \log(\lambda^{-1})^\alpha}{\ell \lambda} \) with \( N \) a constant from Lemma 3 we obtain that

\[
\|f_{H,\rho,\lambda} - f_{H,\rho,\lambda}\|_\rho^2 \leq C_1 \left[ \frac{\tau^2}{\ell \lambda \max(0, 1-\beta)} \left( \log(\lambda^{-1})^\alpha + 1 \right) \right]
\]

where \( C_1 = 128 \cdot V \max(5 \ast Q, K) \)

with a \( \rho^\ell \)-probability \( \geq 1 - e^{-4\tau} \). We finally have

\[
\|f_{H,\rho,\lambda} - f_\rho\|_\rho^2 \leq C \left[ \lambda^\beta + \frac{\tau^2}{\ell \lambda \max(0, 1-\beta)} \left( \log(\lambda^{-1})^\alpha + 1 \right) \right]
\]

where \( C = 2 \cdot \max(B, 128 \cdot V \max(5 \ast Q, K)) \)

with a \( \rho^\ell \)-probability \( \geq 1 - e^{-4\tau} \). Now if we assume that \( \beta > 1 \) and \( \lambda_\ell = \frac{1}{\ell^{\gamma+1}} \), we obtain that

\[
\|f_{H,\rho,\lambda} - f_\rho\|_\rho^2 \leq C \left[ \frac{1}{\ell} + \frac{\tau^2}{\ell} \left( \frac{1}{\beta^\alpha} \log(\ell)^\alpha + \frac{1}{\ell^1 - \gamma} \right) \right]
\]

\[
\leq 3C \frac{\tau^2 \log(\ell)^\alpha}{\ell}
\]

with a \( \rho^\ell \)-probability \( \geq 1 - e^{-4\tau} \) provided that

\[
\ell \geq \max \left( e^\beta, N \frac{\tau^2 \log(\lambda^{-1})^\alpha}{\lambda_\ell} \right)
\]

(10)
Moreover as $\beta > 1$, \[ \frac{\ell \lambda_\ell}{\log(\lambda_\ell^{-1})^\alpha} = \frac{\ell^{1/\beta}}{\frac{\log(\ell)^\nu}{\log(\ell)^\beta}} \] goes to infinity as $\ell$ goes to infinity, we conclude that there exist $\ell_\tau$ such that for all $\ell \geq \ell_\tau$, the condition (10) is satisfied and we finally have
\[
\lim_{\tau \to +\infty} \lim_{\ell \to \infty} \sup_{\rho \in \mathcal{F}_{H,\alpha,\beta}} \rho^\ell \left( z : \| f_{H,z} - f_\rho \|_\rho^2 > \frac{\tau \log(\ell)^\alpha}{\ell^\beta} \right) = 0
\]

Now if we consider the case where $0 < \beta < 1$, by considering $\lambda_\ell = \frac{\log(\ell)^{\frac{\nu}{\beta}}}{\log(\ell)^\beta - \log(\ell)^{\frac{\nu}{\beta}}}$ we obtain
\[
\| f_{H,z} - f_\rho \|_\rho^2 \leq C \left[ \log(\ell)^\alpha \frac{\tau^2}{\ell^\beta} \left( \log(\ell) - \log(\log(\ell)) \right) + \frac{1}{\log(\ell)^{\frac{\nu}{\beta}}} \right]
\]
with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ provided that
\[
\ell \geq \max \left( N \frac{\tau^2 \log(\lambda_\ell^{-1})^\alpha}{\lambda_\ell}, e^{\frac{1}{\ell^\beta}} \right)
\]

Moreover we have that
\[
\frac{\ell \lambda_\ell}{\log(\lambda_\ell^{-1})^\alpha} = \frac{\log(\ell)^{\frac{\nu}{\beta}}}{\log(\ell)^\beta - \log(\ell)^{\frac{\nu}{\beta}}} \]

And as $0 < \beta < 1$, we finally have that
\[
\lim_{\tau \to +\infty} \lim_{\ell \to \infty} \sup_{\rho \in \mathcal{F}_{H,\alpha,\beta}} \rho^\ell \left( z : \| f_{H,z} - f_\rho \|_\rho^2 > \frac{\tau \log(\ell)^\alpha}{\ell^\beta} \right) = 0
\]

Finally let consider the case where $\beta = 1$. By considering $\lambda_\ell = \frac{\log(\ell)^\nu}{\ell}$ with $\mu > \alpha > 0$ we obtain
\[
\| f_{H,z} - f_\rho \|_\rho^2 \leq 3C \frac{\tau^2 \log(\ell)^\mu}{\ell^\beta}
\]
with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ provided that
\[
\ell \geq \max \left( N \frac{\tau^2 \log(\lambda_\ell^{-1})^\alpha}{\lambda_\ell}, e^{\frac{1}{\ell^\beta}} \right)
\]

Finally as $\mu > \alpha > 0$ we have that
\[
\lim_{\tau \to +\infty} \lim_{\ell \to \infty} \sup_{\rho \in \mathcal{F}_{H,\alpha,\beta}} \rho^\ell \left( z : \| f_{H,z} - f_\rho \|_\rho^2 > \frac{\tau \log(\ell)^\alpha}{\ell^\beta} \right) = 0
\]

As a reminder, we have obtained the result for $C = 2 \max(B,128 \star V \max(5 \star Q, K))$ and $N = \max(256KQ, 16K, 1)$ where $Q = \gamma^{-\alpha} \left[ 1 + C_0 \int_1^\infty \frac{(\log(u)+1)^{n-1}}{e^{u+u^2}} du \right]$, $V = \max(L^2, \sigma^2, 2BK + 2B_\infty)$, and $K = \sup_{x \in \chi} k(x,x)$.

\[\Box\]

A.2 Proof of Theorem 3.2

Proof. We follow the suggestion of presented in section 3 of [13] in order to construct a family of probability measures $\rho_f \in \mathcal{F}_{H,\alpha,q,\beta}$ parametrized by suitable vectors $f \in H$. Let $\nu \in \mathcal{P}_{H,\alpha,q}$ and let us denote $\tilde{\sigma} = \min(\sigma, L)$. Then we define for a measurable function $f : \chi \to \mathcal{Y}$ and $x \in \chi$ the distribution $\rho_f(\cdot|x) := \mathcal{N}(f(x), \tilde{\sigma}^2)$ as the normal distribution on $\mathcal{Y} = \mathbb{R}$ with mean $f(x)$ and variance $\tilde{\sigma}^2$. Hence
ρ_f(A) = \int_X \int_Y 1_A(x,y) d\rho_f(y|x) d\nu(x) defines a probability measure on \( X \times Y \) with marginal distribution \( \nu \) on \( X \), i.e. \( (\rho_f)_X = \nu \). For \( f \in L_2^\nu(X) \) we have \( \int_X \int_Y y^2 d\rho_f(y|x) = \sigma^2 + \|f\|_{L_2^\nu(X)}^2 < \infty \) and \( f_{\rho_f} = f \).

Moreover, the properties of the normal distribution implies \( \int_X |y - f(x)|^m d\rho_f(y|x) \leq \frac{1}{2} m! \sigma^m \) for all \( x \in X \).

Hence if \( \|f\|_{L_2^{\rho_f}(X)}^2 < B_\infty \) and there exist \( g \in L_2^{\rho_f}(X) \) such that \( f_{\rho_f} = T_\nu^{\beta/2} g \) and \( \|g\|_{L_2^{\rho_f}(X)}^2 \leq B \), then \( \rho_f \in \mathcal{F}_{\mathcal{H},\alpha,q,\beta} \). So we reduced the construction of probability measures to the construction of appropriate functions \( f \). To this end we use binary strings \( \omega = (\omega_1, \ldots, \omega_m) \in \{-1,1\}^m \) and define for \( 0 < \epsilon < 1 \)

\[
\begin{align*}
g_\omega &:= \left(\frac{\epsilon}{m}\right)^{1/2} \frac{1}{m} \sum_{i=1}^{m} \omega_i \mu_i^{-\beta/2} e_i \\
\text{and} \quad f_\omega &:= T_\nu^{\beta/2} g_\omega = \left(\frac{\epsilon}{m}\right)^{1/2} \sum_{i=1}^{m} \omega_i e_i
\end{align*}
\]

Because \( f_\omega \) is a finite linear combination of the eigenvectors \( e_i \) of \( T_\nu \) it holds \( \|f_\omega\|_{L_2^{\rho_f}(X)}^2 < \infty \) and \( \|g_\omega\|_{L_2^{\rho_f}(X)}^2 < +\infty \). First we want to establish sufficient conditions on \( \epsilon \) and \( m \) such that \( \|f_\omega\|_{L_2^{\rho_f}(X)}^2 < B_\infty \) and \( \|g_\omega\|_{L_2^{\rho_f}(X)}^2 \leq B \). In the following we denote \( \|\cdot\|_\nu := \|\cdot\|_{L_2^{\rho_f}(X)} \).

**Lemma 4.** Let \( H \) be a separable RKHS on \( X \) with respect to a bounded and measurable kernel \( k \), \( q \geq \gamma > 0 \), \( \alpha > 0 \), \( 0 < \beta \leq 2 \) such that \( \mathcal{P}_{H,\alpha,q} \) is not empty and let \( \nu \in \mathcal{P}_{H,\alpha,q} \). Then there is an constants \( U, v > 0 \) and \( 0 < \epsilon_1 \leq 1 \) depending only on \( \mathcal{P}_{H,\alpha,q,\beta} \), such that \( \|f_\omega\|_{L_2^{\rho_f}(X)}^2 < B_\infty \) and \( \|g_\omega\|_{L_2^{\rho_f}(X)}^2 \leq B \) holds for all \( 0 < \epsilon \leq \epsilon_1 \) and all \( m \leq U \log(\nu^{-1})^\alpha \).

**Proof.** Let \( m \in \mathbb{N} \) and \( 0 < \epsilon < 1 \). Therefore we have

\[
\|g_\omega\|_{L_2^{\rho_f}(X)}^2 = \frac{\epsilon}{m} \sum_{i=1}^{m} \omega_i \mu_i^{-\beta} \leq \epsilon \mu^{-\beta} \leq c^{-\beta} \epsilon q \beta m^{1/\alpha}
\]

Moreover we have

\[
\|f_\omega\|_{L_2^{\rho_f}(X)}^2 \leq K \|f_\omega\|_H^2 \leq \frac{K}{c} \epsilon q \beta m^{1/\alpha}
\]

Therefore by considering \( U := \min\left(\frac{1}{q^{\beta}}, \frac{1}{q^{\beta/2}}\right) \) and \( v := \min\left(\frac{BC}{K}, B_\infty c^{\beta}\right) \), we obtain that for all \( \epsilon \leq \epsilon_1 := \min(1, v) \) and all \( m \leq U \log(\nu^{-1})^\alpha \), \( \|f_\omega\|_{L_2^{\rho_f}(X)}^2 < B_\infty \) and \( \|g_\omega\|_{L_2^{\rho_f}(X)}^2 \leq B \).

If \( \omega' = (\omega_1', \ldots, \omega_m') \in \{-1,1\}^m \) is another binary string, we investigate the norm of the difference \( f_\omega - f_{\omega'} \).

We obtain that

\[
\|f_\omega - f_{\omega'}\|_{L_2^{\rho_f}(X)}^2 = \frac{\epsilon}{m} \sum_{i=1}^{m} (\omega_i - \omega_i')^2
\]

Therefore as \( (\omega_i - \omega_i')^2 \leq 4 \) we obtain that:

\[
\|f_\omega - f_{\omega'}\|_{L_2^{\rho_f}(X)}^2 \leq 4 \epsilon
\]

In order to obtain a lower bound, we assume that \( \sum_{i=1}^{m} (\omega_i - \omega_i')^2 \geq m \), i.e. the distance between \( \omega \) and \( \omega' \) is large, and finally we obtain

\[
\|f_\omega - f_{\omega'}\|_{L_2^{\rho_f}(X)}^2 \geq \epsilon
\]

The following theorem is a restatement of Theorem 3.1 of [10] in our setting.
**Theorem A.1.** Let $H$ be a separable RKHS on $X$ with respect to a bounded and measurable kernel $k$, $q \geq \gamma > 0$, $\alpha > 0$, $0 < \beta \leq 2$ such that $\mathcal{P}_{H,\alpha,q}$ is not empty and let $\nu \in \mathcal{P}_{H,\alpha,q}$. Let also $z \rightarrow f_z$ an arbitrary measurable learning method for $\ell \in \mathbb{N}$ and $z \in Z^\ell$. Then there exist $0 < \epsilon_0 \leq 1$ such that for all $\epsilon \leq \epsilon_0$ and for all $\ell \in \mathbb{N}$ there is a $\rho \in \mathcal{F}_{H,\alpha,q,\beta}$ such that

$$\rho^\ell \left( z : \|f_z - f_\rho \|_\rho^2 > \frac{\epsilon}{4} \right) \geq \min \left( \frac{N^*_\rho}{N^*_\rho + 1}, \hat{\eta} \sqrt{N^*_\rho e^{-\frac{4\epsilon}{2\sqrt{\epsilon}}} \log(\nu^{-1})} \right)$$

where $N^*_\rho = e^{\frac{\ell}{2\alpha} \log(\nu^{-1})}$ and $\hat{\eta} = e^{-3/\epsilon}$.

**Proof.** Thanks to Lemma 4 there exists $1 \geq \epsilon_1 > 0$ and $U, v > 0$ such that for all $\omega$, $0 < \epsilon \leq \epsilon_1$ and $m \leq U \log(\nu^{-1})^\alpha$ we have $\rho_{f_\omega} \in \mathcal{F}_{H,\alpha,q,\beta}$. Let us now fix $0 < \epsilon < \epsilon_1$ and consider the case where $m = \lfloor U \log(\nu^{-1}) \rfloor$. Moreover Theorem F.2 suggests that there are many binary strings with large distances. Indeed as soon as $m \geq 16$, there exists $f_\omega^1, ..., f_\omega^{N_\epsilon}$ where $N_\epsilon \geq \epsilon^m/24 \geq N^*_\epsilon$ such that $\rho_{f_\omega^1}, ..., \rho_{f_\omega^{N_\epsilon}} \in \mathcal{F}_{H,\alpha,q,\beta}$, and for all $i \neq j$ in $[1,N_\epsilon]$ we have:

$$4\epsilon \geq \|f_{\omega^i} - f_{\omega^j}\|_\nu^2 \geq \epsilon$$

In fact if we assume that $\epsilon \leq \epsilon_0 := \min(\epsilon_1, \nu e^{-\left(\frac{q}{2}\right)\log(\nu)})$, then $m \geq 16$ is satisfied and the above results hold. Now, given $\ell \in \mathbb{N}$, let

$$A_i := \left\{ z : \|f_z - f_{\omega^i}\|_\nu^2 < \frac{\epsilon}{4} \right\}$$

for all $i = 1, ..., N_\epsilon$. Thanks to the lower bound eq.11 we obtain that $A_i \cap A_j = \emptyset$ if $i \neq j$, therefore thanks to Lemma 8 we have that there exist $i \in [1,N_\epsilon]$, such that $\rho = \rho_{f_{\omega^i}}$ and that either

$$p := \rho^\ell (A_i) > \frac{N_\epsilon}{N_\epsilon + 1} \geq \frac{N^*_\epsilon}{N^*_\epsilon + 1}$$

or

$$\frac{4\epsilon \ell}{2\sigma^2} \geq - \log(p) + \log(\sqrt{N^*_\epsilon}) - 3/\epsilon$$

The left-hand side of the latter inequality comes from the fact that we can describe the Kullback-Leibler divergence for these measures. Indeed thanks to Lemma 8 for $f, f' \in L_{2\sigma^2}(X)$ and $\ell \geq 1$ it holds $\rho_f^\ell \ll \rho_{f'}^\ell$, and $\rho_f^\ell \gg \rho_{f'}^\ell$. Furthermore, the KL divergence fulfills

$$KL(\rho_f^\ell, \rho_{f'}^\ell) = \frac{\ell}{2\sigma^2} \|f - f'\|_{L_{2\sigma^2}(X)}^2$$

And as $f_\rho = f_{\rho_{f_{\omega^i}}} = f_{\omega^i}$ we obtain that

$$\rho^\ell \left( z : \|f_z - f_\rho\|_\rho^2 > \frac{\epsilon}{4} \right) \geq \min \left( \frac{N^*_\rho}{N^*_\rho + 1}, \hat{\eta} \sqrt{N^*_\rho e^{-\frac{4\epsilon}{2\sqrt{\epsilon}}} \log(\nu^{-1})} \right)$$

We can now give a proof of the theorem. Given $\tau > 0$, for all $\ell \in \mathbb{N}$, let $\epsilon_\ell = 4\log(\ell)^\alpha/\ell$. Since $\epsilon_\ell$ goes to 0 when $\ell$ goes to $\infty$, for $\ell$ large enough, $\epsilon_\ell \leq \epsilon_0$, and we can apply Theorem A.1 and we obtain

$$\inf_{f_\rho \in \mathcal{F}_{H,\alpha,q,\beta}} \sup_{\rho \in \mathcal{F}_{H,\alpha,q,\beta}} \rho^\ell \left( z : \|f_z - f_\rho\|_\rho^2 > \tau \log(\ell)^\alpha/\ell \right) \geq \min \left( \frac{N^*_\epsilon}{N^*_\epsilon + 1}, \hat{\eta} \sqrt{N^*_\epsilon e^{-\frac{4\epsilon_\ell}{2\sqrt{\epsilon}}} \log(\nu^{-1})} \right)$$
Moreover we have
\[
\sqrt{N_{\ell}^* e^{-\frac{4\ell t}{2\pi^2}}} = \varepsilon e^{\frac{L}{\pi^2} \left[ \log(t) - \frac{\log(\varepsilon)}{\pi^2} \right] t} \times e^{-\frac{8\pi \log(t) n}{2\varepsilon}}
\]
Therefore if $\tau$ is small enough such that $\tau < \frac{L^2}{268}$, the quantity $\sqrt{N_{\ell}^* e^{-\frac{4\ell t}{2\pi^2}}}$ goes to $0$ as $\ell$ goes to $\infty$. Also if $\ell$ goes to $\infty$, $\frac{n}{N_{\ell}^* e^{-\frac{4\ell t}{2\pi^2}}} - 1$ goes to $1$. Finally we obtain that:
\[
\lim_{\tau \to 0^+} \lim_{\ell \to \infty} \inf_{T} \inf_{t} \sup_{x, \rho} \rho \left( \| f_{z} - f_{\rho} \|_{p} > \tau b_{t} \right) = 1
\]

\[\square\]

B Multi-Layer Perceptron

B.1 Proof of Lemma 4.1

Proof. Let $N \geq 0$ be the number of layers, $(m_1, ..., m_N) \in \mathbb{N}^N$ and let $(\sigma_i)_{i=1}^N$ be a sequence of $N$ functions which admits a Taylor decomposition in $0$ on $\mathbb{R}$ such that for every $i \in [1, N]$ and $x \in \mathbb{R}$

\[
\sigma_i(x) = \sum_{t \geq 0} a_{i,t} x^t
\]

We can now define the sequence $(f_i)_{i=1}^N$ such that for every $i \in [1, N]$ and $x \in \mathbb{R}$

\[
f_i(x) := \sum_{t \geq 0} |a_{i,t}| x^t
\]

Let us now introduce two sequence of functions $(\phi_i)_{i=1}^N$ and $(\psi_i)_{i=1}^N$ such that for all $i \in [1, N]$ and $x \in \ell_2$

\[
\phi_i(x) := \left( \sqrt{|a_{i,t}| x_{k_1} \ldots x_{k_t}} \right)_{t \in \mathbb{N}, k_1, \ldots, k_t \in \mathbb{N}}
\]
\[
\psi_i(x) := \left( \frac{a_{i,t}}{\sqrt{|a_{i,t}|}} x_{k_1} \ldots x_{k_t} \right)_{t \in \mathbb{N}, k_1, \ldots, k_t \in \mathbb{N}}
\]

with the convention that $\frac{0}{0} = 0$. Moreover as a countable union of countable sets is countable and $(\sigma_i)_{i=1}^N$ are defined on $\mathbb{R}$, we have that for all $x \in \ell_2$ and $i \in [1, N]$, $\phi_i(x)$, $\psi_i(x) \in \ell_2$. Indeed there exists a bijection $\mu : \mathbb{N} \to \bigcup_{i \geq 0} \mathbb{N}^2$, therefore we can denote for all $i \in [1, N]$ and $x \in \ell_2$, $\phi_i(x) = (\phi_i(x)_{\mu(j)})_{j \in \mathbb{N}}$ and $\psi_i(x) = (\psi_i(x)_{\mu(j)})_{j \in \mathbb{N}}$. We have then

\[
\langle \phi_i(x), \phi_i(x') \rangle_{\ell_2} = \sum_{j \in \mathbb{N}} \phi_i(x)_{\mu(j)} \phi_i(x')_{\mu(j)}
\]
\[
= \sum_{t \geq 0} |a_{i,t}| \sum_{k_1, \ldots, k_t} x_{k_1} \ldots x_{k_t} x'_{k_1} \ldots x'_{k_t}
\]
\[
= \sum_{t \geq 0} |a_{i,t}| \langle x, x' \rangle_{\ell_2}^t
\]
\[
= f_i(\langle x, x' \rangle_{\ell_2})
\]

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Moreover the same calculations lead also to the fact that
\[ \langle \psi_i(x), \psi_i(x') \rangle_{\ell_2} = f_i(\langle x, x' \rangle_{\ell_2}) \]
Therefore \( \phi_i \) and \( \psi_i \) are a feature maps of the positive semi-definite kernel \( k_i : x, x' \in \ell_2 \times \ell_2 \to f_i(\langle x, x' \rangle_{\ell_2}) \) and we have
\[ \langle \phi_i(x), \phi_i(x) \rangle_{\ell_2} = \langle \psi_i(x), \psi_i(x) \rangle_{\ell_2} = f_i(\|x\|_{\ell_2}^2) < \infty \]
Finally let us define the sequence of kernels \( (K_i)_{i=1}^N \) defined on \( \mathcal{X} \times \mathcal{X} \) such that, for all \( x, x' \in \mathcal{X} \)
\[ K_i(x, x') := f_1 \circ ... \circ f_i(\langle x, x' \rangle_{\mathbb{R}}^\times) \]
and let us denote \( (H_i)_{i=1}^N \) the sequence of RKHS associated. Moreover in the following, we consider \( \mathbb{R}^d \) as a subset of \( \ell_2 \). One can easily show by induction on \( i \in [1, N] \) that for all \( x, x' \in \mathcal{X} \)
\[ K_i(x, x') = \langle \phi_1 \circ ... \circ \phi_i(x), \phi_1 \circ ... \circ \phi_i(x') \rangle_{\ell_2} \]
Let us denote \( F_{\mathcal{X},(\sigma_i)^N_{i=1}} \) the function space defined by a neural network where the activations are the \( (\sigma_i)^N_{i=1} \). Moreover let \( (W^k)_{k=1}^{N+1} \) be any sequence such that for all \( k \in [1, N+1] \), \( W^k := (u^k_j)_{j=1}^{m_k} \in \mathcal{M}_{m_{k-1} \times m_k} (\mathbb{R}) \) and \( \mathcal{N} \) the function in \( F_{\mathcal{X},(\sigma_i)^N_{i=1}} \) associated. Let us now show by induction that at each layer \( i \in [1, N] \) of the neural network, for \( k \in [1, m_i] \), the coordinate \( N^i_k \) is a function which lives in \( H_i \) such that for all \( x \in \mathcal{X} \)
\[ N^i_k(x) = \left\langle \psi_i \left( \sum_{j_{i-1}=1}^{m_{i-1}} W^i_{j_{i-1}, k} \psi_{i-1} \left( \sum_{j_1=1}^{m_1} W^{2}_{j_1, j_2} \psi_1(w^1_{j_1}) \right) \right), \phi_1 \circ ... \circ \phi_1(x) \right\rangle_{\ell_2} \]
For \( i = 1 \), we have for all \( k \in [1, m_1] \)
\[ N^i_k(x) = \sigma_1(\langle x, w^1_k \rangle) \]
where \( w^1_k \in \mathbb{R}^d \subset \ell_2 \). In fact we can show that for every \( w \in \mathbb{R}^d, x \in \mathcal{X} \to \sigma_1(\langle x, w \rangle) \) lives in \( H_1 \). Indeed we have
\[ \sigma_1(\langle x, w \rangle) = \sum_{t \geq 0} a_{1,t} \langle x, w \rangle^t = \sum_{t \geq 0} a_{1,t} \sum_{k_1, ..., k_t} x_{k_1} ... x_{k_t} w_{k_1} ... w_{k_t} = \sum_{t \geq 0} \sum_{k_1, ..., k_t} \sqrt{a_{1,t} x_{k_1} ... x_{k_t}} \frac{a_{1,t}}{\sqrt{a_{1,t}}} w_{k_1} ... w_{k_t} = \langle \phi_1(x), \psi_1(w) \rangle_{\ell_2} \]
Thanks to Theorem \ref{thm:kernel}, the following application \( x \in \mathcal{X} \to \sigma_1(\langle x, w \rangle) \) lives in \( H_1 \) and finally we have for all \( k \in [1, m_1] \):
\[ N^i_k(x) = \langle \phi_1(x), \psi_1(w^1_k) \rangle_{\ell_2} \in H_1 \]

Let us assume that the result hold for \( i \in [1, N-1] \) and let \( k \in [1, m_{i+1}] \), then we have by definition of the neural network that for all \( x \in \mathcal{X} \)
\[ N^{i+1}_k(x) = \sigma_{i+1} \left( \langle N^i(x), w^{i+1}_k \rangle_{\mathbb{R}^m_i} \right) = \sigma_{i+1} \left( \sum_{j_i=1}^{m_i} W^{i+1}_{j_i,k} N^i_j(x) \right) \]
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Then by induction we have that

\[ N_{i+1}^{i+1}(x) = \sigma_{i+1} \left( \sum_{j=1}^{m_i} W_{i,k}^{i+1} V_j^i, \phi_i \circ \ldots \circ \phi_1(x) \right) \]

where \( V_j^i := \psi_i \left( \sum_{j=1}^{m_i} W_{j,j-1,j}^i \psi_{j-1} \left( \sum_{j=1}^{m_j} W_{j,j}^2 \psi_1(w_{j,1}^j) \right) \ldots \right) \) \( \in \ell_2 \)

Therefore we obtain

\[ N_{i+1}^{i+1}(x) = \left\langle \psi_{i+1} \left( \sum_{j=1}^{m_i} W_{i,k}^{i+1} V_j^i \right), \phi_{i+1} \circ \ldots \circ \phi_1(x) \right\rangle_{\ell_2} \]

And finally, thanks to Theorem F.3, we have that \( x \in X \rightarrow N_{i+1}^{i+1}(x) \) lives in \( H_{i+1} \) and we have for all \( k \in [1, m_{i+1}] \)

\[ N_{i+1}^{i+1}(x) = \left\langle \psi_{i+1} \left( \sum_{j=1}^{m_i} W_{i,k}^{i+1} V_j^i \right), \phi_{i+1} \circ \ldots \circ \phi_1(x) \right\rangle_{\ell_2} \]

Now let us show that \( N \) lives in \( H_N \) and that we can bound its RKHS norm. Indeed by definition of \( N \) we have that for all \( x \in X \)

\[ N(x) = \langle N^N(x), W^{N+1} \rangle_{\mathbb{R}^{m_N}} \]

\[ = \sum_{j=1}^{m_N} W_j^{N+1} N_{jN}^N(x) \]

And thanks to what precedes, we have that for all \( j_N \in [1, m_N] \), \( N_{jN}^N \in H_N \), then as a linear combination of the \( (N_{jN}^N)_{jN=1}^{m_N} \), we finally have that \( N \in H_N \). Moreover thanks to what precedes we have

\[ N(x) = \left\langle \sum_{j_N=1}^{m_N} W_j^{N+1} \psi_N \left( \sum_{j_{N-1}=1}^{m_{N-1}} W_{j_N,j_{N-1}}^N \psi_{N-1} \left( \sum_{j_{N-2}=1}^{m_{N-2}} W_{j_{N-1},j_{N-2}}^N \psi_{N-2} \right) \ldots \right) \right\rangle_{\ell_2} \]

Therefore thanks to the Theorem F.3 we have that

\[ \|N\|^2_{H_N} \leq \left\| \sum_{j_N=1}^{m_N} W_j^{N+1} \psi_N \left( \sum_{j_{N-1}=1}^{m_{N-1}} W_{j_N,j_{N-1}}^N \psi_{N-1} \left( \sum_{j_{N-2}=1}^{m_{N-2}} W_{j_{N-1},j_{N-2}}^N \psi_{N-2} \right) \ldots \right) \right\|^2_{\ell_2} \]

Moreover, as the \( (\phi_i)_{i=1}^{N} \) are respectively feature maps of the kernels \( (k_i)_{i=1}^{N} \), one can show by induction that

\[ \left\| \sum_{j_N=1}^{m_N} W_j^{N+1} \psi_N \left( \sum_{j_{N-1}=1}^{m_{N-1}} W_{j_N,j_{N-1}}^N \psi_{N-1} \left( \sum_{j_{N-2}=1}^{m_{N-2}} W_{j_{N-1},j_{N-2}}^N \psi_{N-2} \right) \ldots \right) \right\|^2_{\ell_2} \]

\[ = (W^{N+1})^T f_N \left( (W^{N})^T f_{N-1} \left( (W^{N-1})^T f_{N-2} \left( (W^{2})^T f_1 \left( (W^1)^T W^1 \right) \ldots W^{N-1} \right) W^{N} \right) \right) \]

where the \( (f_i)_{i=1}^{N} \) are functions acting coordinate-wise. Thanks to the Schur Inequality \[ \text{[5]} \], we obtain that for all \( k \in \mathbb{N} \) and any \( A \in \mathcal{M}_n(\mathbb{R}) \)

\[ \|A^k\| \leq \|A\|^k \]

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And as for all $i \in [1,N]$, $(|a_i,t|)_{t \geq 0}$ are non negative sequences, we obtain that for all $i \in [1,N]$ and $A \in M_n(\mathbb{R})$
\[
\|f_i(A)\| \leq f_i(||A||)
\]

Finally by a simple induction we obtain that
\[
(W^{N+1})^T f_N ((W^N)^T f_{N-1} ((W^{N-1})^T f_2 ((W^2)^T f_1 ((W^1)^T W^1) W^2) ... W^{N-1}) W^N) W^{N+1}
\leq ||W^{N+1}||^2 f_N(||W^N||^2 f_{N-1}(... f_1(||W^1||^2)...)\]

And we obtain our upper bound of the RKHS norm of $N$.

Finally let us assume that for every $i \in [1,N]$ and $n \in \mathbb{N}$ we have $\sigma_i^{(n)}(0) \neq 0$. Then by denoting by $(b_m)_{m \geq 0}$ the coefficients of the Taylor decomposition of $f_N \circ ... \circ f_1$, the following Lemma ensures that for every $m \geq 0$, $b_m > 0$ (see proof section G.1).

**Lemma 5.** Let $(f_i)_{i=1}^N$ a family of functions that can be expand in their Taylor series in 0 on $\mathbb{R}$ such that for all $k \in [1, N]$, $(f_k^{(n)}(0))_{n \geq 0}$ are positives. Let us define also $\phi_1,...,\phi_{N-1} : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ such that for every $k \in [1, N-1]$ and $l,m \geq 0$:
\[
\phi_k(l,m) := \frac{d^m}{dt^m}|_{t=0} f_k^{(l)}(t) m!
\]

Then $g := f_N \circ ... \circ f_1$ can be expanded in its Taylor series in 0 on $\mathbb{R}$ such that for all $t \in \mathbb{R}$:
\[
g(t) = \sum_{l_1,..,l_N \geq 0} \sum_{l_1,..,l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N,l_{N-1})\times \phi_1(l_2,l_1) t^{l_1}
\]

Moreover $(g^{(n)}(0))_{n \geq 0}$ is a positive sequence.

Therefore thanks to the positivity of the coefficient $(b_m)_{m \geq 0}$ the cc-universality of $K_N$ follows directly from Theorem E.4.

### C Regularized Least-squares for Multi-Layer Perceptrons

#### C.1 Notations

Let $P_m(d)$ be the space of homogeneous polynomials of degree $m$ in $d$ variables with real coefficients, $H_m(d)$ be the space of homogenious polynomials defined by
\[
H_m(d) := \{ P \in P_m(d) | \Delta P = 0 \}
\]

$H_m(S^{d-1})$ the space of (real) spherical harmonics of degree $m$, be the set of restrictions of harmonic polynomials in $H_m(d)$ to $S^{d-1}$, $L_2^{d_{d-1}}(S^{d-1})$ be the space of (real) square-integrable functions on the sphere $S^{d-1}$ endowed with its induced Lebesgue measure $d\sigma_{d-1}$ and $|S^{d-1}|$ the surface area of $S^{d-1}$. Moreover, $L_2^{d_{d-1}}(S^{d-1})$ endowed with its natural inner product is a separable Hilbert space. The family of spaces $(H_m(S^{d-1}))_{m \geq 0}$, yields a Hilbert space direct sum decomposition
\[
L_2^{d_{d-1}}(S^{d-1}) = \bigoplus_{m \geq 0} H_m(S^{d-1})
\]

which means that the summands are closed, pairwise orthogonal. Moreover, each $H_m(S^{d-1})$ has a finite dimension which is
\[
\text{if } m \geq 2 \alpha_{m,d} = \left( \begin{array}{c} d-1 + m \cr m \end{array} \right) - \left( \begin{array}{c} d-1 + m - 2 \cr m - 2 \end{array} \right)
\]
\[\text{with } \alpha_{0,d} = 1, \text{ and } \alpha_{1,d} = d.\]
Therefore for all $m \geq 0$, given any orthonormal basis of $H_m(S^{d-1})$, $(Y_{m}^1, ..., Y_{m}^{\alpha_{m,d}})$, we can build an Hilbertian basis of $L^2_{d\sigma_{d-1}}(S^{d-1})$ by concatenating these orthonormal basis. Let us denote in the following $(Y_{m}^l)_{m,l}$ such an Hilbertian basis of $L^2_{d\sigma_{d-1}}(S^{d-1})$. Let us now define the integral operator on $L^2_{d\sigma_{d-1}}(S^{d-1})$ associated with a positive semi-definite kernel $k$ on $S^{d-1}$

$$T_k : L^2_{d\sigma_{d-1}}(S^{d-1}) \to L^2_{d\sigma_{d-1}}(S^{d-1})$$

$$f \to \int_{S^{d-1}} k(x,\cdot)f(x) d\sigma_{d-1}(x)$$

As soon as $\int_{S^{d-1}} k(x,x) d\sigma_{d-1}(x)$ is finite, $T_k$ is well defined, self-adjoint, positive semi-definite and trace-class.

C.2 Proof of Thorem 5.1 and 5.2

Proof. Here the main goal is to control the rate of decay of the eigenvalues associated with the integral operator $T_p$. Indeed to show the upper rate, we show that there exists $\alpha, C_0, \gamma > 0$ such that for any $\rho \in \mathcal{G}_{\omega,\beta}$, the eigenvalues, $(\mu_i)_{i \in I}$ where $I$ is at most countable, of the integral operator $T_p$ associated with $K_N$ fulfill the following upper bound for all $i \in I$:

$$\mu_i \leq C_0 e^{-\gamma i^{1/\alpha}}$$

therefore $\mathcal{G}_{\omega,\beta} \subset \mathcal{F}_{H_N,\alpha,\beta}$ and the result will follow from Theorem 3.1. Moreover to show the minimax-rate, we show that $I = \mathbb{N}$ and that there exists $c > 0$ and $q \geq \gamma > 0$ such that for any $\rho \in \mathcal{G}_{\omega,h,\beta}$, the eigenvalues, $(\mu_i)_{i \in I}$ of the integral operator $T_p$ associated with $K_N$ fulfill the following lower bound for all $i \in I$

$$\mu_i \geq ce^{-q i^{1/\alpha}}$$

and the result will follow from Theorem 3.2 By definition of $(b_m)_{m \geq 0}$, we have

$$K_N(x,x') = \sum_{m \geq 0} b_m((x,x')_{\mathbb{R}^d})^m$$

Moreover thanks to theorem F.3 we have an explicit formula of the eigenvalues of $T_{K_N}$, the integral operator associated with the kernel $K_N$ defined on $L^2_{d\sigma_{d-1}}(S^{d-1})$. Indeed each spherical harmonics of degree $m$, $Y_m \in H_m(S^{d-1})$, is an eigenfunction of $T_{K_N}$ with associated eigenvalue given by the formula

$$\lambda_m = \frac{|S^{d-2}| \Gamma((d-1)/2)}{2^{m+1}} \sum_{s \geq 0} b_{2s+m} \frac{(2s+m)!}{(2s)!} \frac{\Gamma(s+1/2)}{\Gamma(s+m+d/2)}$$

The following Proposition gives a control the rate of convergence of $(\lambda_m)_{m \geq 0}$

**Proposition C.1.** If there exist $1 > r > 0$ and $0 < c_2 \leq c_1$ constants such that for all $m \geq 0$

$$c_2 r^m \leq b_m \leq c_1 r^m$$

then by denoting $C_1 = \frac{|S^{d-2}| \Gamma((d-1)/2)}{2^{m+1}} \frac{s^{d-1}}{1-r^d}$ and $C_2 = |S^{d-2}| \Gamma((d-1)/2) \Gamma(1/2) L \frac{\pi}{2}$ where $L$ is a constant only depending on $d$, we have that all $m \geq 0$

$$C_2 \left( \frac{r}{4} \right)^m \leq \lambda_m \leq C_1 r^m$$

**Proof.** Let us denote $\theta_{s,m} = b_{2s+m} \frac{(2s+m)!}{(2s)!} \frac{\Gamma(s+1/2)}{\Gamma(s+m+d/2)}$. Therefore we have:

$$\theta_{s,m} = b_{2s+m} \frac{(2s+m)...(2s+1)}{(s+m+\frac{d-2}{2})...(s+\frac{d}{2})} \times 2^{m+\frac{d+1}{2}}$$

$$\leq 2^{\frac{d+1}{2}} c_1 (2r)^m r^{2s}$$

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The last inequality comes from the fact that \( \frac{(2s+m)(2s+1)}{2s+2m+d-2}(2s+1) \leq 1 \) and the upper bound given in eq. (13). Therefore we have

\[
\sum_{s \geq 0} \theta_{s,m} \leq (2r)^m \frac{2^{2-s}c_1}{1-r^2}
\]

We also have

\[
\lambda_m = \frac{|S|^{d-2} |\Gamma((d-1)/2)|}{2^{m+1}} \sum_{s \geq 0} \theta_{s,m} \leq r^m \frac{|S|^{d-2} |\Gamma((d-1)/2)|}{2} \frac{2^{2-s}c_1}{1-r^2}
\]

Moreover we have

\[
\lambda_m = \frac{|S|^{d-2} |\Gamma((d-1)/2)|}{2^{m+1}} \sum_{s \geq 0} \theta_{s,m} \geq \frac{|S|^{d-2} |\Gamma((d-1)/2)|}{2^{m+1}} \theta_{0,m}
\]

\[
\geq b_m \frac{|S|^{d-2} |\Gamma((d-1)/2)|}{2^{m+1}} \frac{m! \Gamma(1/2)}{\Gamma(m+d/2)}
\]

\[
\geq |S|^{d-2} |\Gamma((d-1)/2)| \Gamma(1/2) c_2 \left( \frac{r}{2} \right)^m \frac{m!}{\Gamma(m+d/2)}
\]

The last inequality comes from the lower bound given in eq. (13). Moreover thanks to the Stirlings approximation formula we have

\[
\Gamma(x) \sim \sqrt{2\pi x} x^{-1/2} e^{-x}
\]

which leads to

\[
\frac{m!}{\Gamma(m+d/2)} \sim e^{d/2} \left( 1 - \frac{d/2}{m+d/2} \right)^{m} \frac{m^{1/2}}{(m+d/2)^{d-1/2}}
\]

Finally we obtain

\[
\frac{m!}{\Gamma(m+d/2)} \sim \frac{1}{m^{d-1/2}}
\]

Therefore there exists a constant \( L > 0 \) only depending on \( d \) such that for all \( m \geq 0 \) we have

\[
\frac{m!}{\Gamma(m+d/2)} \geq L \frac{1}{2^m}
\]

Finally we have

\[
\lambda_m \geq |S|^{d-2} |\Gamma((d-1)/2)| \Gamma(1/2) L \frac{c_2}{2} \left( \frac{r}{4} \right)^m
\]

\[\square\]

We can now derive a tight control of the eigenvalues of \( T_{K_N} \) ranked in the non-decreasing order with their multiplicities.

**Proposition C.2.** Let us denote \( (\eta_m)_{m=0}^M \) the positive eigenvalues of the integral operator \( T_{K_N} \) associated to the kernel \( K_N \) ranked in a non-increasing order with their multiplicities, where \( M \in \mathbb{N} \cup \{ +\infty \} \). If there exist \( 1 > r > 0 \) and \( 0 < c_2 \leq c_1 \) constants such that for all \( m \geq 0 \)

\[
c_2 r^m \leq b_m \leq c_1 r^m,
\]

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then \( M = +\infty \) and there exits \( 0 < Q_1 < Q_2 \) constants only depending on \( d \) such that for all \( m \geq 0 \)

\[
C_2 e^{-Q_2 \log((4/r)m)^{\frac{1}{d-1}}} \leq \eta_m \leq C_1 e^{-Q_1 \log((1/r)m)^{\frac{1}{d-1}}},
\]

where \( C_1 = \frac{|S^{d-2}|\Gamma((d-1)/2)^2}{2(d-2)\Gamma(1/2)^2} \) and \( C_2 = |S^{d-2}|\Gamma((d-1)/2)\Gamma(1/2)L_{d-2}^{\frac{2}{d-2}} \) where \( L \) is a constant only depending on \( d \).

**Proof.** From Theorem [F.5] the \((Y_k^{l})_{k,l_k}\) is an orthonormal basis of eigenfunctions of \(T_{K_N}\) associated with the eigenvalues \((\lambda_{k,l_k})_{k,l_k}\) such that for all \( k \geq 0 \) and \( 1 \leq l_k \leq \alpha_{k,d}, \lambda_{k,l_k} := \lambda_k \geq 0 \) where \( \lambda_k \) is given by the formula \((12)\). Moreover the assumption \((13)\) guarantees that \( b_m > 0 \) for all \( m \geq 0 \), and thanks to the formula of \((12)\), we deduce that \((\lambda_{k,l_k})_{k,l_k}\) are exactly the positive eigenvalues of \(T_{K_N}\) with their multiplicities and that \( M = +\infty \). Let us now define an indexation of the sequence of eigenvalues of \(T_{K_N}\). For that purpose we define this following application.

\[
\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \\
(k,q) \to \sum_{i=0}^{k-1} \alpha_{i,d} + q - 1
\]

with the convention that \( \sum_{i=0}^{-1} \alpha_{i,d} = 0 \). Therefore by defining \( E := \{(k,l_k) : k \in \mathbb{N} \text{ and } l_k \in [1,\alpha_{k,d}]\} \) we have first that \( \phi(E) = \mathbb{N} \) and \( \phi\mid E \) is injective. Therefore we can define \((\delta_m)_{m \geq 0}\) the sequence of eigenvalues of \(T_{K_N}\) with their multiplicities such that for every \( m \geq 0 \) there exists a unique \((k,l_k) \in E\) such that

\[
m := \phi(k,l_k) = \sum_{i=0}^{k-1} \alpha_{i,d} + l_k - 1
\]

\[
\delta_m := \lambda_{k,l_k}
\]

Thanks to Proposition [C.1] we have that for all \( k \geq 0 \)

\[
C_2 \left(\frac{r^{d-1}}{d-1} \right)^k \leq \lambda_k \leq C_1 r^k
\]

Let \( m \geq 0 \) and \( k, l_k \) defined as in eq. \((14)\). First we have that for all \( k \geq 2 \)

\[
\alpha_{k,d} = \binom{d-1+k}{k} - \binom{d-1+k-2}{k-2}
\]

Therefore we have that

\[
\sum_{i=0}^{k} \alpha_{i,d} \sim \frac{2k^{d-1}}{(d-1)!}
\]

Then there exist \( Q_2 > Q_1 > 0 \) constants only depending on \( d \) such that

\[
Q_1 m^{\frac{1}{d-1}} \leq k \leq Q_2 m^{\frac{1}{d-1}}
\]

Then we have

\[
C_2 \left(\frac{r^{d-1}}{d-1} \right)^Q_2 m^{\frac{1}{d-1}} \leq \delta_m \leq C_1 r^Q m^{\frac{1}{d-1}}
\]

And finally by definition of the \((\eta_m)_{m \geq 0}\) and as the upper and lower bounds of \( \delta_m \) are decreasing functions, we obtain that for all \( m \geq 0 \)

\[
C_2 e^{-Q_2 \log((4/r)m)^{\frac{1}{d-1}}} \leq \eta_m \leq C_1 e^{-Q_1 \log((1/r)m)^{\frac{1}{d-1}}}.
\]

\( \square \)
We can now derive a tight control of the eigenvalues of $T_{\rho}$, denoted $(\mu_{m})_{m \in \mathbb{N}}$, which will conclude the proof. Let us first show that $I = \mathbb{N}$. Indeed as $S^{d-1}$ is compact and $K_{N}$ continuous, the Mercer theorem guarantees that $H_{N}$ and $L^{d \sigma_{d-1}}(S^{d-1})$ are isomorphic. Moreover let first recall the two key assumptions to obtain a control on the eigenvalues of $T_{\rho}$. Indeed we have assumed that

$$\frac{d\nu}{d\sigma_{d-1}} < \omega \quad \text{and} \quad \frac{d\nu}{d\sigma_{d-1}} > \omega$$

(16)

Let us now define

$$T_{\omega} : L^{d \sigma_{d-1}}(S^{d-1}) \quad \rightarrow \quad f \quad \rightarrow \quad \int_{S^{d-1}} K_{N}(x,.) f(x) d\sigma_{d-1}(x) - \int_{S^{d-1}} K_{N}(x,.) f(x) d\nu(x)$$

and let us denote $E^{k}$, the span of the greatest $k$ eigenvalues strictly positive of $T_{\rho}$ with their multiplicities. Thanks to the min-max Courant-Fischer theorem we have that

$$\mu_{k} = \max_{V \subset G_{k}, \|x\|=1} \min \langle T_{\rho} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})}$$

where $G_{k}$ is the set of all s.e.v of dimension $k$ in $L^{d \sigma_{d-1}}(S^{d-1})$. Therefore we have

$$\eta_{k} \geq \frac{1}{\omega} \min_{x \in E^{k} \setminus \{0\}, \|x\|=1} \langle \omega \times T_{\rho} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})}$$

$$= \frac{1}{\omega} \min_{x \in E^{k} \setminus \{0\}, \|x\|=1} \left\{ \langle T_{\rho} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})} + \langle T_{\omega} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})} \right\}$$

$$\geq \frac{1}{\omega} \min_{x \in E^{k} \setminus \{0\}, \|x\|=1} \langle T_{\rho} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})} + \frac{1}{\omega} \min_{x \in E^{k} \setminus \{0\}, \|x\|=1} \langle T_{\omega} x, x \rangle_{L^{d \sigma_{d-1}}(S^{d-1})}$$

Then if $T_{\omega}$ is positive we obtain that

$$\eta_{k} \geq \frac{1}{\omega} \mu_{k}$$

Let us now show the positivity of $T_{\omega}$. Thanks to the assumption [16], we have that for all $f \in L^{d \sigma_{d-1}}(S^{d-1})$

$$T_{\omega}(f) = \int_{S^{d-1}} \left[ \omega - \frac{d\nu}{d\sigma_{d-1}} \right] K_{N}(x,.) f(x) d\sigma_{d-1}(x)$$

Therefore $v := \omega - \frac{d\nu}{d\sigma_{d-1}(x)}$ is positive and by denoting $M = \int_{S^{d-1}} v(x) d\sigma_{d-1}(x)$ and by re-scaling the above equality by $\frac{1}{M}$, we have that $V : x \rightarrow \frac{v(x)}{M}$ is a density function and by denoting $d\Gamma = V d\sigma_{d-1}$ we have

$$\frac{1}{M} \times T_{\omega}(f) = \int_{S^{d-1}} K_{N}(x,.) f(x) d\Gamma(x)$$

Therefore $T_{\omega}$ is positive and thanks to Proposition C.2, we have

$$\mu_{m} \leq \omega \eta_{m} \leq \omega C_{1} e^{-Q_{1} \log(1/r) m^{1/r}}$$

Moreover if we assume in addition that the assumption [16], we obtain by an analogue reasoning that for all $k \geq 0$

$$\eta_{k} \leq \frac{1}{h} \mu_{k}$$

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And we have that for all $m \geq 0$

$$hC_2 e^{-Q_2 \log(4/r)m^{1/\alpha}} \leq h\eta_m \leq \mu_m \leq \omega\eta_m \leq \omega C_1 e^{-Q_1 \log(1/r)m^{1/\alpha}}.$$  \hspace{1cm} (17)

**Upper Rate.** Let us now prove Theorem 5.1. Let $w \geq 1$ and $0 < \beta \leq 2$ and let us denote $\alpha = d - 1$ and $\gamma = Q_1 \log(1/w)$. Thanks to the RHS of eq. (17) we have that for any $\rho \in \mathcal{G}_{\omega, \beta}$, the eigenvalues, $(\mu_i)_{i \geq 0}$, of the integral operator $T_\rho$ associated with $K_N$ fulfill the following upper bound for all $i$:

$$\mu_i \leq \omega C_1 e^{-\gamma i^{1/\alpha}}$$

where $C_1 = \frac{|S^{d-2}| \Gamma((d-1)/2) 2^{d-1} \Gamma(1-\frac{d}{2})}{2}$. Therefore $\mathcal{G}_{\omega, \beta} \subset \mathcal{F}_{H, \alpha, \beta}$ and by applying theorem 3.1, we obtain the results for $C = 2 \max(B, 128 \cdot V \max(5 \cdot Q, K))$ and $A = \max(256KQ, 16K, 1)$ where $Q = \gamma^{-\alpha} \left[ 1 + \omega C_1 \int_1^\infty \frac{\log(u+1)^{\alpha-1}}{u^{d/2+1/\alpha}} du \right]$, $V = \max(L^2, \sigma^2, 2BK + 2B_\infty)$, $K = \sup_{x \in X} k(x, x)$.

**Lower Rate.** Let us now prove Theorem 5.2. Let $0 < \delta < 1 \leq \omega$ and let us denote $q = Q_2 \log(1/w)$. Thanks to the LHS of eq. (17) we have that for any $\rho \in \mathcal{G}_{\omega, \beta}$, the eigenvalues, $(\mu_i)_{i \geq 0}$, of the integral operator $T_\rho$ associated with $K_N$ fulfill the following lower bound for all $i$:

$$\mu_i \geq hC_2 e^{-q i^{1/\alpha}}$$

where $C_2 = |S^{d-2}| \Gamma((d-1)/2) \Gamma(1/2)L^{\omega \gamma}$. Therefore $\mathcal{G}_{\omega, \beta} \subset \mathcal{F}_{H, \alpha, q, \beta}$ and we can apply Theorem 3.2.

**C.3 Proof of Theorem 5.3**

*Proof.* Our first goal is to obtain a control of $\text{df}(\lambda)$ associated with $T_\rho$. For that purpose we first control the rate of convergence of the eigenvalues of $T_{K_N}$, then we deduce a control on the eigenvalues of $T_\rho$ to finally obtain a control of $\text{df}(\lambda)$. Once we obtain the control of $\text{df}(\lambda)$, similar arguments of Theorem 3.1 can be applied in this case to obtain high-probability bounds. First we derive a control in the super-exponential regime for any $\delta > 0$ of the $(\lambda_m)_{m \geq 0}$ defined as

$$\lambda_m = \frac{|S^{d-2}| \Gamma((d-1)/2)}{2m+1} \sum_{s \geq 0} b_{2s+m} \frac{(2s+m)! \Gamma(s+1/2)}{(2s)! \Gamma(s+m+d/2)}$$

**Proposition C.3.** If there exist $\delta > 0$ such that

$$\left| \frac{b_m}{b_{m-1}} \right| \in O(m^{-\delta})$$

then if $\delta \geq 1/2$ we have

$$\lambda_m \in O \left( \frac{b_m}{2m+1 m(d-2)/2} \right) .$$

Otherwise, if $0 < \delta < 1/2$ we have

$$\lambda_m \in O \left( \frac{m^{2\alpha} b_m}{2m+1 m(d-2)/2} \right)$$

where $\alpha := \frac{1}{1-2\delta}$.  

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Proof. We consider only the case when $0 < \delta < 1/2$ as [1] show the result for $\delta \geq 1/2$. Let us denote $\theta_{s,m} = b_{2s+m} \cdot \frac{(2s+m)!}{\Gamma(s+1/2)^{2s+m+d/2+1}}$. Therefore we have

$$\left| \frac{\theta_{s+1,m}}{\theta_{s,m}} \right| = \frac{b_{2s+2+m} \cdot (2s+m+2)(2s+m+1)}{b_{2s+m} \cdot (2s+2)(2s+2m+d)}$$

Let $0 < \delta < 1/2$ such that [18] hold. There exists a $\gamma$ such that for all $m \geq 1$

$$\frac{|b_m|}{|b_{m-1}|} \leq \frac{\gamma}{m^{\delta}}.$$

Therefore we obtain

$$\left| \frac{\theta_{s+1,m}}{\theta_{s,m}} \right| \leq \gamma \frac{(2s+m+2)^{1-\delta} (2s+m+1)^{1-\delta}}{(2s+2)(2s+2m+d)}.$$

Yet since $0 < \delta < 1/2$ we have

$$\left| \frac{\theta_{s+1,m}}{\theta_{s,m}} \right| \leq \gamma^2 \frac{(2s+m+2)^{2-2\delta}}{(2s+2)(2s+2m+d)} \leq \gamma^2 \frac{(2s+m+2)^{2-2\delta}}{(2s+2)(2s+m+2)} \leq \gamma^2 \frac{(2s+m+2)^{1-2\delta}}{(2s+2)}.$$

Let $\alpha := \frac{1}{1-2\delta}$, therefore we have

$$\left| \frac{\theta_{s+1,m}}{\theta_{s,m}} \right|^\alpha \leq \gamma^{2\alpha} \frac{(2s+m+2)}{(2s+2)^\alpha} \leq \frac{\gamma^{2\alpha}}{(2s+2)^{\alpha-1}} + \frac{\gamma^{2\alpha} m}{(2s+2)^\alpha}.$$

Therefore we have for all $m, s \geq 0$

$$\left| \theta_{s+1,m} \right|^\alpha \leq \left| \theta_{0,m} \right| \prod_{i=0}^{s} \left( 1 + \frac{m}{2i+2} \right) \frac{\gamma^{2\alpha(s+1)}}{2^{(\alpha-1)(s+1)}((s+1)!)^{\alpha-1}}.$$

Moreover, as soon as $s + 1 \geq m\delta/2$, we have

$$1 + \frac{m}{2s+2} \leq 1 + \frac{1}{\delta}.$$

Therefore for all $m, s \geq 0$ we obtain that

$$\prod_{i=0}^{s} \left( 1 + \frac{m}{2i+2} \right) \leq \left( 1 + \frac{1}{\delta} \right)^{s+1} \times \prod_{i=0}^{s} \left( 1 + \frac{m}{2i+2} \right)$$

For all $m \geq 2$, and $s \geq 0$, we obtain finally

$$\left| \theta_{s+1,m} \right|^\alpha \leq \left| \theta_{0,m} \right| \left( \frac{2^{\alpha-1}}{2^{\alpha-1}} \right)^{s+1} \times \frac{1}{((s+1)!)^{\alpha-1}}.$$
Let \( C := \frac{(1+\delta)+\gamma^2}{2^\alpha} \). Therefore we obtain that, for all \( m \geq 2 \)
\[
|\theta_{s,m}| \leq |\theta_{0,m}| m^{\frac{1}{2} + \frac{1}{2}} \times \frac{(C \delta)^s}{(s!)^{\frac{1}{2}}}.
\]
And finally for all \( m \geq 2 \)
\[
\sum_{s \geq 0} |\theta_{s,m}| \leq |\theta_{0,m}| m^{\frac{1}{2} + \frac{1}{2}} \times \sum_{s \geq 0} \frac{(C \delta)^s}{(s!)^{\frac{1}{2}}}.
\]
Therefore thanks to Theorem 12 we have
\[
|\lambda_m| \leq \sigma_{d-2} \Gamma((d-1)/2) \frac{b_m m!}{\Gamma(m+d/2)} |\theta_{0,m}| m^{\frac{1}{2} + \frac{1}{2}} \times \sum_{s \geq 0} \frac{(C \delta)^s}{(s!)^{\frac{1}{2}}}
\]
But as \(|\theta_{0,m}| = \Gamma(1/2) \frac{b_m m!}{\Gamma(m+d/2)}\), then we obtain
\[
|\lambda_m| \leq \sigma_{d-2} \Gamma((d-1)/2) \frac{b_m m!}{\Gamma(m+d/2)} \frac{m!}{\Gamma(m+d/2)} \times \sum_{s \geq 0} \frac{(C \delta)^s}{(s!)^{\frac{1}{2}}}
\]
And as we have
\[
\frac{m!}{\Gamma(m+d/2)} \in O \left( \frac{1}{m^{(d-2)/2}} \right)
\]
Finally we have
\[
\lambda_m \in O \left( \frac{m^{\frac{1}{2} + \frac{1}{2} - \frac{(d-2)}{2}} b_m}{2m+1 \Gamma(m+d/2)} \right).
\]

From the control given by Proposition C.3 we are now able to control the eigenvalues of \( T_{KN} \) ranked a non-increasing order with their multiplicities.

**Proposition C.4.** Let us denote \((\eta_m)_{m=0}^M\) the positive eigenvalues of the integral operator \( T_{KN} \) associated to the kernel \( K_N \) ranked in a non-increasing order with their multiplicities, where \( M \in \mathbb{N} \cup \{+\infty\} \). If there exist \( \delta > 0 \) such that
\[
|b_m| b_m^{-1} \in O(m^{-\delta}) \quad (19)
\]
then \( M = +\infty \) and there exist constant \( 1 \geq G > 0 \) independent of \( \delta \) such that
\[
\eta_m \in O \left( m^{-\frac{4(d-1)}{G}} \right),
\]
where \( s := \frac{4(d-1)}{G} \).

**Proof.** The assumption \((19)\) guarantees that \( b_m > 0 \) for all \( m \geq 0 \), therefore \( M = +\infty \) and as in the proof of Proposition C.2 we can define \((\delta_m)_{m \geq 0}\) the sequence of eigenvalues of \( T_{KN} \) with their multiplicities such that for every \( m \geq 0 \) there exists a unique \((k, l_k) \in E := \{(k, l_k) : k \in \mathbb{N} \text{ and } l_k \in [1, \alpha_{k,d}]\}\) such that
\[
m := \phi(k, l_k) = \sum_{i=0}^{k-1} \alpha_{i,d} + l_k - 1
\]
\[
\delta_m := \lambda_{k,l_k}
\]
Let us define $q(\delta, k) = \begin{cases} \frac{k^2}{2\alpha} + \frac{1}{\alpha} & \text{if } \frac{1}{2} > \delta > 0 \\ 0 & \text{otherwise.} \end{cases}$

Thanks to Proposition C.3, there exists a constant $C$ independent of $k$ such that:

$$\lambda_k \leq C \frac{b_k k^q(\delta, k)}{2^{k+1}k^{(d-2)/2}}$$

Therefore we have:

$$\delta_m \leq C \frac{b_k k^q(\delta, k)}{2^{k+1}k^{(d-2)/2}}$$

But there exist a constant $Q \geq 2$ such that, for all $k \geq 1$, we have

$$b_k \leq \frac{Q}{k^\alpha} b_{k-1} \leq \frac{Q}{\alpha} b_0$$

Therefore we have

$$\delta_m \leq C b_0 \frac{Q^k k^q(\delta, k)}{2^{k+1}k^{(d-2)/2} (k!)^\alpha}$$

Then by applying the Stirling formula, there exist a constant $L > 0$ such that

$$\delta_m \leq CLb_0 \frac{e^{\delta k} Q^k}{2^{k+1}k^{(d-2)/2+\delta/2}} \frac{k^q(\delta, k)}{k^\delta k}$$

Moreover we remarks that if $\delta > \frac{1}{2}$,

$$\delta k - q(\delta, k) = \delta k$$

Otherwise we have that

$$\delta k - q(\delta, k) = \delta k \left(1 - \frac{1}{2\alpha}\right) - \frac{1}{\alpha}$$

But as soon as $0 < \delta < 1/2$, $\alpha > 1$, therefore we have that for all $\delta > 0$

$$\delta_m \leq CLb_0 \frac{e^{\delta k} Q^k}{2^{k+1}k^{(d-2)/2+\delta/2}} \frac{1}{k^{\frac{4k}{2}-1}}$$

Moreover there exist $Q_2 \geq 1 \geq Q_1 > 0$ constants such that

$$Q_1 m^{\frac{1}{d-1}} \leq k \leq Q_2 m^{\frac{1}{d-1}}$$

Therefore we obtain that

$$\delta_m \leq \frac{CLb_0}{2Q_1^{(d-2)/2}} \frac{e^{\delta Q}}{Q_1^m m^{\frac{1}{d-1}}} Q_{2m}^{\frac{1}{d-1}}$$

Finally by denoting $s := \frac{4(d-1)}{Q_1}$ which is independent of $\delta$ we obtain that

$$\delta_m \times m^{\frac{1}{d} \frac{1}{d-1}} \leq \frac{CLb_0}{2Q_1^{(d-2)/2}} \frac{e^{\delta Q}}{Q_1^m m^{\frac{1}{d-1}}} Q_{2m}^{\frac{1}{d-1}}$$
And as the Right-Hand side of the inequality is bounded we obtain that:

$\delta_m \in O \left( m^{-\frac{d}{2}} \right)$

Finally the assumption guarantees that eventually the sequence $(\delta_m)_{m \geq 0}$ is exactly the sequence of eigenvalues ordered in the non-increasing order. Indeed we have the following Proposition.

**Lemma 6.** If $(b_m)_{m \geq 0}$ is a non-increasing sequence eventually, then $(\delta_m)_{m \geq 0}$ is a non-increasing sequence eventually.

**Proof.** For a fixed $k$ we have that

$$\frac{2s + k + 1}{2s + 2k + d} < 1 \text{ for all } s \geq 0$$

and as $(b_m)_{m \geq 0}$ is a non increasing sequence eventually, then there exists $k_0 \geq 0$ such that for all $k \geq k_0$ and $s \geq 0$

$$b_{2s+k+1} \leq b_{2s+k}$$

Therefore for all $k \geq k_0$, we have $\lambda_k \geq \lambda_{k+1} > 0$ and the result follows. \(\square\)

Therefore the sequence $(\delta_m)_{m \geq 0}$ coincides eventually with the sequence of eigenvalues of the integral operator $T_{K_N}$ ordered in the non-increasing order and the result follows. \(\square\)

In the next Proposition, we obtain a tight control of the degrees-of-freedom.

**Proposition C.5.** Let $\omega > 0$ and $\nu \in W_\omega$. Let also $(\mu_i)_{i \in I}$, where $I$ is at most countable, be the sequence of eigenvalues with their multiplicities associated with $T_\nu$ ranked in the non decreasing order. If there exist $1 > \delta > 0$ such that

$$\left| \frac{b_m}{b_{m-1}} \right| \in O(m^{-\delta})$$

Then we have:

$$\text{df}(\lambda) \in O \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right)$$

**Proof.** From Proposition there exists $0 < G \leq 1$ such that

$$\eta_m \in O \left( m^{-\frac{d}{2}} \right)$$

where $s := \frac{4(d-1)}{d}$. Moreover as $S^{d-1}$ is compact and $K_N$ continuous, the Mercer theorem guarantees that $H_N$ and $L^2(S^{d-1})$ are isomorphic, and $I = \mathbb{N}$. Therefore as in the proof of Proposition the min-max Courant-Fisher theorem allows us to obtain that for all $m \geq 0$:

$$\mu_m \leq \omega \eta_m \in O \left( m^{-\frac{d}{2}} \right)$$

Let $\lambda > 0$ and let us now compute $\text{df}(\lambda)$. There exist $\gamma > 0$ such that

$$\text{df}(\lambda) = \sum_{i \in I} \frac{\mu_i}{\mu_i + \lambda} \leq \sum_{m \geq 1} \frac{\gamma}{\gamma + \lambda e^{\frac{d}{2} \log(m)m^{\frac{1}{d-1}}}}$$

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Moreover as the application \( x \in [1, +\infty[ \rightarrow \frac{\gamma}{\gamma + \lambda e^{\frac{d}{\log(x)}}} \) is positive and decreasing, therefore we have

\[
df(\lambda) \leq \int_{1}^{+\infty} \frac{\gamma}{\gamma + \lambda e^{\frac{d}{\log(x)}}} \, dx
\]

Let us denote \( \alpha = \left( \frac{\lambda}{\gamma} \right)^{d-1} \), \( g : x \in ]1, +\infty[ \rightarrow \log(x)^{d-1} x \) and let us consider the following substitution

\[
v = \lambda e^{\frac{d}{\log(x)}} \frac{1}{x}
\]

Therefore we have

\[
\int_{1}^{+\infty} \frac{\gamma}{\gamma + \lambda e^{\frac{d}{\log(x)}}} \, dx = \int_{\lambda}^{+\infty} \frac{\alpha^{d-1}}{\gamma + \frac{\alpha^{d-1} \log(v \lambda^{-1})}{(\alpha \log(v \lambda^{-1})^{d-1})}} \frac{(g^{-1})(\alpha \log(v \lambda^{-1})^{d-1})}{\gamma + v} \, dv
\]

Moreover we have for any \( \lambda \leq 1 \) and \( v \geq e^{\alpha^{\frac{1}{d-1}}} \)

\[
g\left(\alpha^{\frac{d}{\log(v \lambda^{-1})}}\right) = \log(\alpha^{\frac{d}{\log(v \lambda^{-1})}}) = \frac{\alpha^{\frac{d}{\log(v \lambda^{-1})}}}{d-1} \alpha^{\frac{d}{\log(v \lambda^{-1})}} \geq \alpha \log(v \lambda^{-1})
\]

Indeed the latter inequality comes from the fact that \( 0 \leq \log(x) \leq x \) for \( x \geq 1 \). Therefore by monotonicity of \( g \) we obtain that

\[
g^{-1}(\alpha \log(v \lambda^{-1})^{d-1}) \geq \alpha^{\frac{d}{\log(v \lambda^{-1})}}
\]

Moreover as \( g'(x) = \log(x)^{d-1} + (d - 1) \log(x)^{d-2} \) and it is an increasing function, we obtain that

\[
g' \circ g^{-1}(\alpha \log(v \lambda^{-1})^{d-1}) \geq g'\left(\alpha^{\frac{d}{\log(v \lambda^{-1})}}\right) \geq \log(\alpha^{\frac{d}{\log(v \lambda^{-1})}})^{d-1}
\]

where the last inequality comes from

\[
\alpha^{\frac{d}{\log(v \lambda^{-1})}}^{d-1} \geq 1.
\]

Finally we obtain that

\[
(g^{-1})'\left(\alpha \log(v \lambda^{-1})^{d-1}\right) \leq \frac{1}{\log(\alpha^{\frac{d}{\log(v \lambda^{-1})}})^{d-1}}.
\]

Therefore we have

\[
df(\lambda) \leq \int_{1}^{+\infty} \frac{\gamma \alpha^{d-1}}{\gamma + \frac{\alpha^{d-1} \log(v \lambda^{-1})}{(\alpha \log(v \lambda^{-1})^{d-1})}} \, dv
\]

\[
\leq \int_{\lambda}^{e^{\alpha^{\frac{1}{d-1}}}} \frac{\alpha^{d-1}}{\gamma + \frac{\alpha^{d-1} \log(v \lambda^{-1})}{(\alpha \log(v \lambda^{-1})^{d-1})}} \, dv + \int_{e^{\alpha^{\frac{1}{d-1}}}}^{+\infty} \frac{\gamma \alpha^{d-1}}{(\gamma + v) \left[\log(\alpha^{\frac{d}{\log(v \lambda^{-1})}})^{d-1}\right]} \, dv
\]

\[
\leq g^{-1}\left(\alpha \log(e^{\frac{1}{d-1}}) \lambda^{-1}\right)^{d-1} + \int_{\lambda}^{e^{\alpha^{\frac{1}{d-1}}}} \frac{\gamma \alpha^{d-1}}{(\gamma + v) \left[\log(\alpha^{\frac{d}{\log(v \lambda^{-1})}})^{d-1}\right]} \, dv
\]

But by considering \( \lambda \leq e^{-1} \), we obtain that

\[
\log(v \lambda^{-1})^{d-1} \leq (\log(v) + 1)^{d-1} \log(\lambda^{-1})^{d-1}
\]
Moreover we have for all $v \geq e^{\alpha \frac{d-1}{d}} \geq 1$

$$\log(\alpha^\frac{1}{d} \log(v \lambda^{-1}) \frac{d-1}{d-1})^d = \left[ \frac{1}{d} \log(\alpha) + \frac{d-1}{d} \log(\log(v \lambda^{-1})) \right]^{d-1} \geq \left[ \frac{1}{d} \log(\alpha) + \frac{d-1}{d} \log(\log(\lambda^{-1})) \right]^{d-1}$$

And as soon as

$$\lambda \leq \min \left( e^{-1}, \exp(\frac{d}{d-1} \log(\alpha^\frac{1}{d})) \right)$$

We have that

$$\log(\alpha^\frac{1}{d} \log(v \lambda^{-1}) \frac{d-1}{d-1})^d \geq \left( \frac{d-1}{d} \right)^{d-1} (\log(\log(\lambda^{-1})))^{d-1}$$

Finally we obtain that

$$\int_{e^{\alpha \frac{d-1}{d}}}^{+\infty} \frac{\gamma \left[ \frac{d-1}{d} \log(v \lambda^{-1}) \right]}{(\log(\log(\lambda^{-1})))^{d-1} (\gamma + v)^d} dv \leq \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \gamma \alpha (d-1)^{d-1} \int_{e^{\alpha \frac{d-1}{d}}}^{+\infty} \frac{(\log(v) + 1)^{d-1}}{(\gamma v + v^2)} dv$$

Moreover we have:

$$g \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right) \geq \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \log \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right)^{d-1}$$

And we have:

$$\log \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right)^{d-1} = [(d-1) \log(\log(\lambda^{-1})) - (d-1) \log(\log(\log(\lambda^{-1})))]^{d-1}$$

But as $y \to \frac{d}{d-1} y - \log(y)$ is positive on $\mathbb{R}_+$, we obtain that:

$$\log \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right)^{d-1} \geq \left( \log(\log(\lambda^{-1})) \right)^{d-1}$$

And finally we have that:

$$\frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \geq g^{-1}(\log(\lambda^{-1})^{d-1})$$

Therefore we obtain that:

$$df(\lambda) \leq g^{-1} \left( \alpha \log(e^{\frac{1}{d-1}} \lambda^{-1})^{d-1} \right) + \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \gamma \alpha (d-1)^{d-1} \int_{e^{\alpha \frac{d-1}{d}}}^{+\infty} \frac{(\log(v) + 1)^{d-1}}{(\gamma v + v^2)} dv$$

And thanks to the eq. (20), we obtain that:

$$df(\lambda) \in O \left( \frac{\log(\lambda^{-1})^{d-1}}{(\log(\log(\lambda^{-1})))^{d-1}} \right)$$

□
Let us now prove the theorem. Thanks to Proposition C.5, we have an explicit control of $df(\lambda)$, which allows us to derive a analogue proof of Theorem 3.1. Indeed there exists a constant $Q > 0$ such that for any $\lambda > 0$ we have:

$$df(\lambda) \leq Q \frac{\log(\lambda^{-1})^{d-1}}{\lambda (\log(\log(\lambda^{-1})))^{d-1}}$$

The only changes that we have to take care about is the changes due to the new formulation of the $df(\lambda)$, that is to say, $N_{\lambda, \tau}$ and the RHS term of the inequality in eq.9. Indeed in that case by the exact same arguments of the proof of Lemma 3 we have for $\tau \geq 1$ and $\lambda > 0$:

$$N_{\lambda, \tau} \leq N \tau^2 \frac{\log(\lambda^{-1})^{d-1}}{\lambda (\log(\log(\lambda^{-1})))^{d-1}}$$

where $N = \max(256KQ, 16K, 1)$

Moreover if $\ell \geq N_{\lambda, \tau}$, a analogue proof as in Theorem 3.1 gives

$$\|f_{H_{\lambda, \sigma}} - f_{\rho, \lambda}\|_{\rho}^2 \leq C_1 \left[ \frac{\tau^2}{\ell \max(0, 1 - \beta)} \left( \frac{\log(\lambda^{-1})^{d-1}}{\lambda (\log(\log(\lambda^{-1})))^{d-1}} + \frac{1}{\ell \lambda} \right) \right]$$

where $C_1 = 128 \cdot V \max(5 \cdot Q, K)$

with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$. Moreover from Lemma 2 we finally have:

$$\|f_{H_{\lambda, \sigma}} - f_{\rho, \lambda}\|_{\rho}^2 \leq C \left[ \frac{\tau^2}{\ell \max(0, 1 - \beta)} \left( \frac{\log(\lambda^{-1})^{d-1}}{\lambda (\log(\log(\lambda^{-1})))^{d-1}} + \frac{1}{\ell \lambda} \right) \right]$$

where $C = 2 \cdot \max(B, 128 \cdot V \max(5 \cdot Q, K))$

with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$. Now if we assume that $\beta > 1$ and $\lambda_\ell = \frac{1}{\tau^{1/3}}$, we obtain that:

$$\|f_{H_{\lambda, \sigma}} - f_{\rho, \lambda}\|_{\rho}^2 \leq C \left[ \frac{1}{\ell^2} + \frac{\tau^2}{\ell} \left( \frac{1}{\beta \ell^{d-1}} \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}} + \frac{1}{\ell^{1-\beta}} \right) \right]$$

$$\leq 3C \tau^2 \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$$

with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ provided that

$$\ell \geq \left( \frac{N}{\beta^{d-1}} \right) \tau^{\frac{d}{d-1}} \left( \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}} \right) \tau^{\frac{d}{d-1}} := \ell_\tau$$

A similar reasoning leads to the results in the cases where $\beta = 1$ and $0 < \beta < 1$. \hfill \Box

**Corollary C.1.** Under the exact same assumptions of theorem 5.3, it holds:

$$\lim_{\tau \to +\infty} \lim_{\ell \to \infty} \sup_{\rho \in \mathcal{F}_{H_{\lambda, \sigma}}} \sup_{\rho \in \mathcal{F}_{H_{\lambda, \sigma}}} \rho^\ell (z : \|f_{\lambda, \sigma} - f_{\rho, \lambda}\|_{\rho}^2 > \tau a_\ell) = 0$$

if one of the following conditions hold:

- $\beta > 1$, $\lambda_\ell = \frac{1}{\ell^{1/3}}$ and $a_\ell = \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$

- $\beta = 1$, $\lambda_\ell = \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$ and $a_\ell = \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$ for $\mu > d - 1 > 0$

- $\beta < 1$, $\lambda_\ell = \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$ and $a_\ell = \frac{\log(\lambda^{-1})^{d-1}}{\log(\log(\lambda^{-1}))^{d-1}}$
D  Example

Lemma 7. Let $N \geq 2$. If $f_N$ is the exponential function we have the following relation for all $m \geq 0$

$$b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} b_k b_{m+1-k} (m + 1 - k)$$

Moreover if $f_1(t) = ... = f_N(t) = \exp(t)$ the result above hold for all $n \in [2, N]$.

Proof. We remarks that:

$$f_N(t) = (f_N \circ (f_{N-1})') (t) = f_{N-1}'(t) \times f_N(t)$$

Therefore by applying the Leibniz formula we obtained that:

$$f_{m+1}^N (t) = \sum_{k=0}^{m} \binom{m}{k} f_k^N (t) f_{m+1-k}^N (t)$$

Therefore we obtain:

$$\frac{f_{m+1}^N (0)}{(m+1)!} = \sum_{k=0}^{m} \binom{m}{k} \frac{k!}{(m+1)!} f_k^N (0) f_{m+1-k}^N (0)$$

$$= \frac{1}{m+1} \sum_{k=0}^{m} f_k^N (0) f_{m+1-k}^N (0) (m+1-k)$$

The results follows. □

E  Background

Definition E.1. (Positive semi-definite kernel) A positive semi-definite kernel on $X$ is an application $k : X \times X \to \mathbb{R}$ symmetric, such that for all finite families of points in $X$, the matrix of pairwise kernel evaluations is positive semi-definite.

Definition E.2. (Positive definite kernel [20]) A positive definite kernel on $X$ is an application $k : X \times X \to \mathbb{R}$ symmetric, such that for all finite families of distinct points in $X$, the matrix of pairwise kernel evaluations is positive definite.

Definition E.3. (c-universal [27]) A continuous positive semi-definite kernel $k$ on a compact Hausdorff space $X$ is called c-universal if the RKHS, $H$ induced by $k$ is dense in $C(X)$ w.r.t. the uniform norm, i.e., for every function $g \in C(X)$ and all $\epsilon > 0$, there exists an $f \in H$ such that $\|f - g\|_u \leq \epsilon$.

Definition E.4. (cc-universal [27]) A continuous positive semi-definite kernel $k$ on a Hausdorff space $X$ is said to be cc-universal if the RKHS, $H$ induced by $k$ is dense in $C(X)$ endowed with the topology of compact convergence, i.e., for any compact set $Z \subset X$, for any $g \in C(Z)$ and all $\epsilon > 0$, there exists an $f \in H|_Z$ such that $\|f - g\|_u \leq \epsilon$.

Remark 3. It is important to notice that a c- or cc- universal kernel is necessarily a positive definite kernel.

Definition E.5. (Dot product kernel on the sphere [25]) Let consider a function $f : [-1, 1] \to \mathbb{R}$ that can be expended into its Taylor series in 0, i.e. for all $t \in [-1, 1]$:

$$f(t) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} t^n$$

Let us now define the general dot product kernel $k$ associated with $f$ on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ by:

$$k(x, y) := f((x, y)_{\mathbb{R}^d}).$$
Remark 4. If \( f^{(n)}(0) \geq 0 \) for every \( n \geq 0 \) then \( k \) is a continuous positive semi-definite kernel on \( S^{d-1} \).

F Useful Theorems

Theorem F.1. \([13]\) Let \( 0 < \beta \leq 2 \) and \( H \) a separable RKHS on \( X \) with respect to a bounded and measurable kernel \( k \) and \( \rho \) a probability measure on \( X \times Y \) with \( \int_X \mu (x) \gamma^2 \, d\rho (x,y) < \infty \). We assume that \( \| f_\rho \|_L^2 \gamma < \infty \) and that there exist \( g \in L^2(\rho) \) such that \( f_\rho = T_\rho^{1/2} g \). Furthermore, we assume that there exist \( \sigma > 0 \) and \( L > 0 \) such that
\[
\int_Y |y - f_\rho (x)|^m \, d\rho (y|x) \leq \frac{1}{2} m! L^{m-2}
\]
for \( \rho_X \)-almost all \( x \in X \) and all \( m \geq 2 \). Then for \( \tau \geq 1, \lambda > 0 \) and \( \ell \geq N_{\lambda, \tau} \) we have with \( \rho^\ell \)-probability \( \geq 1 - 4e^{-\tau} \),
\[
\| f_{n, \lambda} - f_{\rho, \lambda} \|_\rho^2 \leq \frac{128}{\ell} \left( 5d(\lambda)\sigma_\lambda^2 + K \frac{L_\lambda}{\lambda} \right)
\]
where \( K = \sup_{x \in X} k(x,x) \)
\[
N_{\lambda, \tau} = \max \left( \frac{256\tau^2 K d(\lambda)}{\lambda}, \frac{16\tau K}{\lambda}, \tau \right)
\]
\[
\sigma_\lambda = \max(\sigma, \| f_\rho - f_{\rho, \lambda} \|_{L^2(\rho)})
\]
\[
L_\lambda = \max(L, \| f_\rho - f_{\rho, \lambda} \|_{L_2(\rho)})
\]

Theorem F.2. \([8]\) For every \( m > 16 \) there exist \( N \in \mathbb{N} \) and \( \omega^1, \ldots, \omega^N \in \{-1, +1\}^m \) such that
\[
\sum_{i=1}^m (\omega_i^k - \omega_i^j)^2 \geq m \quad k \neq j = 1, \ldots, N
\]
\[
N \geq e^{m/24}
\]

Theorem F.3. \([22]\) Let \( \phi : X \to H \) be a feature map to a Hilbert space \( H \), and let \( K(z, z') := \langle \phi(z), \phi(z') \rangle_H \) a positive semi-definite kernel on \( X \). Then \( \mathcal{H} := \{ f_\alpha : z \in X \to \langle \alpha, \phi(z) \rangle_H, \; \alpha \in H \} \) endowed with the following norm:
\[
\| f_\alpha \|^2 := \inf_{\alpha' \in \mathcal{H}} \{ \| \alpha' \|^2_H \; s.t \; f_{\alpha'} = f_\alpha \}
\]
is the RKHS associated to \( K \).

Theorem F.4. \([23]\) Let \( 0 < r \leq +\infty \) and \( f : (-r, r) \to \mathbb{R} \) be a \( C^\infty \)-function that can be expanded into its Taylor series in \( 0 \), i.e.
\[
f(x) = \sum_{m=0}^\infty a_m x^m.
\]
Let \( X := \{ x \in \mathbb{R}^d : \| x \|_2 < \sqrt{r} \} \). If we have \( a_n > 0 \) for all \( n \geq 0 \) then \( k(x,y) := f(\langle x, y \rangle) \) defines a \( c \)-universal kernel on every compact subset of \( X \).

Theorem F.5. \([7]\) Each spherical harmonics of degree \( m \), \( Y_m \in H_m(S^{d-1}) \), is an eigenfunction of \( T_{K_m} \) with associated eigenvalue given by the formula:
\[
\lambda_m = \frac{S^{d-2}}{2^{m+1}} \frac{\Gamma((d-1)/2)}{\Gamma(s+1/2)} \sum_{s \geq 0} b_{2s+m} \frac{(2s+m)!}{(2s)!} \frac{\Gamma(s+m+d/2)}{\Gamma(s+m+d/2)}
\]


G  Technical Lemmas

Lemma 8. \[13\] For \(f, f' \in L^2_2(X)\) and \(\ell \geq 1\) it holds \(\rho_\ell^f \ll \rho_\ell^{f'}\) and \(\rho_\ell^f \gg \rho_\ell^{f'}\). Furthermore, the Kullback-Leibler divergence fulfills
\[
KL(\rho_\ell^f, \rho_\ell^{f'}) = \frac{\ell}{2\sigma^2} \|f - f'\|^2_{L^2_2(X)}
\]

Lemma 9. \[10\] Let \(\mathcal{A}\) be a sigma algebra on the space \(\Omega\). Let \(A_i \in \mathcal{A}, i \in \{0, 1, ..., n\}\) such that \(\forall i \neq j, A_i \cap A_j = \emptyset\). Let \(P_i, i \in \{0, 1, ..., n\}\) be \(n+1\) probability measures on \((\Omega, \mathcal{A})\). If
\[
p := \sup_{i=1, ..., n} P_i(\Omega \setminus A_i)
\]
then either \(p > \frac{n}{n+1}\) or
\[
\min_{j=1, ..., n} \frac{1}{n} \sum_{i \neq j} KL(P_i, P_j) \geq \psi_n(p)
\]
where
\[
\psi_n(p) := \log(n) + (1 - p) \log \left(\frac{1 - p}{p}\right) - p \log \left(\frac{n - p}{p}\right)
\]

G.1 Proof of Lemma \[5\]

Proof. Let us show the result by induction on \(N\). For \(N = 1\) the result is clear as \(f_1\) can be expand in its Taylor series in 0 on \(\mathbb{R}\) with positives coefficients. Let \(N \geq 2\), therefore we have:
\[
g(t) = (f_N \circ ... \circ f_2) \circ (f_1(t))
\]
By induction, we have that for all \(t \in \mathbb{R}\):
\[
f_N \circ ... \circ f_2(t) = \sum_{l_2, ..., l_N \geq 0} \frac{f^{(l_N)}_N(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1}) ... \times \phi_2(l_3, l_2) l^{l_2}
\]
Therefore we have that:
\[
g(t) = \sum_{l_2, ..., l_N \geq 0} \frac{f^{(l_N)}_N(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1}) ... \times \phi_2(l_3, l_2) (f_1(t))^{l_2}
\]
Moreover, for all \(l_2 \geq 0\), \(f_1^{l_2}\) can be expand in its Taylor series in 0 on \(\mathbb{R}\) with non negative coefficients, and we have that for all \(l_2 \geq 0\) and \(t \in \mathbb{R}\):
\[
(f_1(t))^{l_2} = \sum_{l_1 \geq 0} \phi_1(l_2, l_1) t^{l_1}
\]
And we obtain that:
\[
g(t) = \sum_{l_1, ..., l_N \geq 0} \frac{f^{(l_N)}_N(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1}) ... \times \phi_1(l_2, l_1) t^{l_1}
\]
Finally we have by unicity of the decomposition that for all \( l_1 \geq 0 \):

\[
\frac{g^{(l_1)}(0)}{l_1!} = \sum_{l_2, \ldots, l_N \geq 0} \frac{f^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1}) \times \phi_1(l_2, l_1)
\]

Moreover let \( k \in [1, N - 1] \), \( l \geq 1 \) and let us denote \((a_k^i)_{i \geq 0}\) the coefficients in the Taylor decomposition of \( f_k \). Then we have:

\[
f_k^l(t) = \sum_{n_1, \ldots, n_l \geq 0} \prod_{i=1}^{l} (a_k^{n_i}) x^{n_1 + \ldots + n_l}
\]

But as \( a_k^i > 0 \) for all \( i \geq 0 \), we obtain that by unicity of the decomposition that for all \( m \geq 0 \):

\[
\phi_k(l, m) = \frac{d^m}{dt^m} |_{t=0} f_k^l(t) > 0
\]

and the result follows.

\[\square\]

**H Experiments**

The MNIST dataset is a dataset of handwritten digits from 0 to 9 which has a training set of 60,000 examples, and a test set of 10,000 examples. In order to train networks on this classification task, we consider the Cross-Entropy loss. Figures 2, 3 show the eigenvalue decay of the kernel associated with MLPs where the number of weight at each layer is fixed to be 128 and 256 respectively. We observe that, as the number of layers increases, the eigenvalue decay is slower. The eigenvalue decay gives a concrete notion of the complexity of the function space considered. Indeed, given an eigensystem \((\mu_m)_{m \geq 0}\) and \((e_m)_{m \geq 0}\) of positive eigenvalues and eigenfunctions respectively of the integral operator \( T_\rho \), associated with the Kernel \( K_N \), defined on \( L^2_{d\rho_{d-1}}(S^{d-1}) \), the RKHS \( H_N \) associated is defined as:

\[
H_N = \left\{ f \in L^2_{d\rho_{d-1}}(S^{d-1}) : f = \sum_{m \geq 0} a_m e_m \quad \text{with} \quad \left( \frac{a_m}{\sqrt{\mu_m}} \right) \in \ell^2 \right\}
\]

endowed with the following inner product:

\[
\langle f, g \rangle = \sum_{m \geq 0} \frac{a_m b_m}{\mu_m}
\]

From this definition, we see immediately that as the eigenvalues of the integral operator decreases slower, the RKHS becomes larger. Therefore composing layers allows the function space generated by the network to grow. Moreover the source condition on the target function \( f_\rho \) assumes that that there exists \( g \in L^2_{d\rho_{d-1}}(S^{d-1}) \) such that \( f_\rho = T_\rho^{\beta/2} g \) and we have an explicit formulation of the space where the target function is allowed to live:

\[
H_\beta := \left\{ f \in L^2_{d\rho_{d-1}}(S^{d-1}) : f = \sum_{m \geq 0} a_m e_m \quad \text{with} \quad \left( \frac{a_m}{\beta/2} \right) \in \ell^2 \right\}
\]
Figure 2: Rate of eigenvalue decay for different depths in semi-log scale. The number of weights is fixed to 256 at each layer.

Figure 3: Rate of eigenvalue decay for different depths in semi-log scale. The number of weights is fixed to 128 at each layer.

Therefore as soon as the eigenvalues of the integral operator decrease slower, the space where the target function $f_\rho$ is allowed to live grows and the network gets better at approximating the target function $f_\rho$. 