Mesoscopic Luttinger Liquid Theory in an Aharonov-Bohm Ring

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A careful study on the mesoscopic persistent current in a Luttinger liquid ring is carried out. It is shown that discreteness plays an important role in calculating the persistent current caused by the magnetic flux. At zero temperature, the current is shown to be independent of the interaction even when \( g = (g_2 - g_1)/2 \) is not zero. The current becomes enhanced at finite temperatures with respect to the non-interacting case, when the parameter \( g \) is positive.

With rapid development of nano-fabrication technology much attention has been paid to the mesoscopic system which exhibits strong quantum phase coherence. Many efforts have been devoted to incorporate the interaction effect between electrons because they are strongly correlated in low dimensions. The Luttinger liquid(LL) theory\textsuperscript{[6]} is a standard method to investigate 1D interacting electron systems. Therefore, it became a starting point to study the mesoscopic property such as the zero mode current operator in the LL. Then \( j(x) \) is represented by

\[
j(x) = i[H, \sum_p \frac{1}{ip} e^{ipx} \rho_p] + \hat{c},
\]

where \( \hat{c} \) is an operator independent of \( x \). The mean current operator, \( I \), can be defined by

\[
I \equiv \frac{1}{pL} \int_0^L dx' j(x').
\]

In the limit when the level spacing \( \Delta p \to 0 \), the mean current \( I \) is given by

\[
I = \frac{1}{pL} \int_0^L dx' \{ i \sum_{p \neq 0} \frac{1}{ip} e^{ipx} [H, \rho_p] + \rho_p c + \rho_p \rho_p c^\dagger \} + \hat{c}
\]

\[
\rho_p c + \rho_p \rho_p c^\dagger = \hat{c}.
\]

since \([H, \rho_p] = 0\). Here, \( I = \hat{c} \) is the uniform zero mode current which can not be obtained through the interaction equation, but only through the definition in terms of total momentum of electrons.

When the total particle number \( N_o \) is odd, there is no current excitation at the ground state. The number of particles at the right(left) branch, \( N_o+ (N_o-) \), becomes \( N_o+ = (N_o - 1)/2 \) and \( N_o- = (N_o - 1 - k_J)/2 \). For the right(left) branch has one more particle at the ground state, \( k_J = 1(-1) \). The case when \( N_o \) is odd, corresponds to \( k_J = 0 \). At finite temperatures, current excitation number \( N_o+ (N_o-) \) is added to \( N_o+ (N_o-) \). For the non-interacting case, the zero mode current, \( I \), can be calculated in terms of total momentum, \(-\frac{e}{L} \sum_{n=0}^{N_o+ + N_o-} \frac{2n}{N_o+ + N_o-} \cdot \hat{c} \), and represented by the excitation numbers,

\[
I = -\frac{e}{N_o} \frac{2\pi}{L} \frac{N_o}{2} (J + k_J) = -\frac{ev_F}{L} (J + k_J).
\]
if internal interactions do not change the total momentum of the system. The zero mode of the Hamiltonian exactly satisfies this condition. The interaction effect is included in obtaining the expectation value, \( \langle \rangle \). However, the current operator should have the interaction independent fermi velocity, \( v_F \), as a prefactor instead of \( v_j \). Here we define the fermi velocity such that \( v_F = N_F \phi_o \), since this corresponds to the product of the particle number at one branch with the momentum discreteness.

In the LL theory, the non-zero and the zero mode in the Hamiltonian are decoupled and the zero mode is given by 
\[
\frac{\pi v_F}{L} (v_F + \frac{2\pi}{\phi_o} k) (J + \frac{2\pi}{\phi_o}^2) + \frac{\pi v_F}{L} (v_F + \frac{2\pi}{\phi_o} k)' N^2. 
\]
The contribution of the flux in the kinetic part comes from changes in the momenta of electrons in accordance to the twisted boundary condition and changes continuously. But, in the interaction part, excitation energy is determined by the current and charge excitation number which are integers. In the continuum field theory, the particle distribution is continuous and, thus, the interaction energy can also be continuous. However, since we are now studying the mesoscopic regime, where the particle discreteness is crucial. We show in the following that a small level shift without charge or current excitation does not cause any change in the interaction energy.

In order to account the level discreteness correctly, the twisted boundary condition should be implemented from the beginning of the bosonization process. Also, a careful analysis on the discreteness of the particle distribution is required, since the parity effect comes from the discreteness of the particle numbers. When the fermion field is expanded, \( \psi_r(x) \equiv \left( \frac{2\pi}{L} \right)^{\frac{1}{2}} \sum_{k=\infty} e^{-ikx} c_{kr} \), the twisted boundary condition gives a condition that 
\[
k = \frac{2\pi}{L} (n + \phi/\phi_o). 
\]
Here, \( n \) is an integer. With this modification on \( k \), one can proceed the bosonization process exactly same as when \( \phi = 0 \). Considering the flux range, 
\[
-\frac{\pi}{\phi_o} < \frac{\phi}{2\phi_o} < \frac{\pi}{\phi_o},
\]
we note that positive magnetic flux causes an upward(downward) shift of the right(left) branch electrons. Now, the zero mode contribution to the kinetic energy excitation can be obtained from the prescription 
\[
H_k^p = \sum_r H_k^p \rightarrow H_k^p = \sum_k c_{kr}^\dagger c_{kr},
\]
where \( \dagger \) signifies the normal ordering.

To obtain the zero mode bosonic form in \( H_k^p \), we consider the ladder operator which increases the number of electrons above the Fermi level, such that \( U_r |N_r, N_{-r} \rangle = |N_r + 1, N_{-r} \rangle \). The ladder operator can be constructed as 
\[
\sum_k c_{kp}^\dagger \delta \left[ r k - \left( k_F + \frac{2\pi}{L} (N_r - \frac{1}{2} + r \phi/\phi_o) \right) \right].
\]

The zero mode part of the kinetic energy excitation at each branch is given by
\[
\frac{2\pi v_F}{L} \sum_{n=1}^{N_r} (n - \frac{1}{2} + r \phi/\phi_o) = \frac{\pi v_F}{L} N_r^2 + r \frac{2\pi v_F}{L} \phi_o N_r. 
\]

For the even parity, application of the magnetic flux removes the degeneracy of the uppermost particle occupation. Since positive \( \phi \) shifts the level to the right side, the uppermost particle takes a level at the left branch. Thus, the energy cost for the excitation in the left branch become increased by the level width, \( v_F \frac{2\pi}{\phi_o} \), for each excited particle. Considering this effect and Eq. (5), we obtain for the kinetic part
\[
\frac{2\pi v_F}{2L} N^2 + \frac{2\pi v_F}{2L} \left( (J + k_j)^2 + 2(J + k_j) \frac{2\phi}{\phi_o} \right),
\]
where \( k_j = -1 (+1) \) for positive(negative) \( \phi \) for the even parity and 0 for the odd parity.

The zero mode in the interacting part, \( H_{int}^p \), is directly deducible from the Hamiltonian. The wave vector shift due to the twisted boundary condition makes change in the bosonic density operator as 
\[
\rho(p) = \sum_k c_k^\dagger \frac{2\pi}{2\phi_o} \frac{2\pi}{\phi_o} + \frac{2\pi}{\phi_o} \phi^2 + \frac{2\pi}{\phi_o} \phi^2 \] \[
\frac{2\pi v_F}{2L} \sum_{r,k,k'} c_{r,k}^\dagger c_{r,k'} \cdot \frac{2\pi}{\phi_o} \phi^2 + \frac{2\pi}{\phi_o} \phi^2 + \frac{2\pi}{\phi_o} \phi^2 \] \[
+ \frac{2\pi v_F}{2L} \sum_{r,k,k'} c_{r,k}^\dagger c_{r,k'} \cdot \frac{2\pi}{\phi_o} \phi^2 + \frac{2\pi}{\phi_o} \phi^2 + \frac{2\pi}{\phi_o} \phi^2 \]

where the normal ordering subtracts the infinite ground state density of type-r fermions. We observe that \( N_r = \sum_{r,k,k'} c_{r,k}^\dagger c_{r,k'} \) is an integer number of the excited fermions for any value of \( \phi \). Also, \( N_r \) is a good quantum number because \( [H, N_r] = 0 \) and does not change continuously as implied in the continuum approximation. Here, we note that the interaction parameter \( g_2(q_j) \) is introduced to describe the forward scattering between different(same) branch particles excited from the fermi level. In order to perform the thermal average of the PC, we consider the energy levels of the excited states. Because the excitation energy in the interaction part is completely determined by the charge excitation number, \( N \), and the current excitation number, \( J \), the parity effect does not appear. Therefore, we obtain for the zero mode of the interaction part
\[
H_{int}^p = \frac{1}{2L} \left( \frac{g_1 + g_2}{2} \right) N^2 + \frac{1}{2L} \left( \frac{g_1 - g_2}{2} \right) J^2.
\]
exclusion degree is decoupled from the current part, it is sufficient to consider the total current excitation contribution only. 

\[ H_j = \frac{v_F \pi}{2L} (J + k_J)^2 + 2(J + k_J) \frac{2\phi}{\phi_o} + g \frac{2}{2L} J^2, \tag{9} \]

where \( g = \frac{2e_2}{2L} \).

In the LL Hamiltonian, the zero modes are decoupled from the non-zero mode. Here, we write down the zero mode of the partition function

\[ Z_0 = C \sum_m e^{-\beta \frac{v_F}{4}(2m+k_J)^2 + 2(2m+k_J) \frac{2\phi}{\phi_o} + \frac{2\pi}{(2m)^2}} , \tag{10} \]

where terms irrelevant in the current calculation are absorbed in constant \( C \). When there is no charge excitation, the eigenvalue of operator, \( J \), becomes an even integer, \( 2m \). It is necessary to carry out the exact summation process instead of the integration process of the continuum approximation to incorporate the AB flux effect and the parity effect correctly. The result is given by

\[ Z_0 = C \theta_3 \left( \frac{i\pi v_F \beta}{L} (k_J + \frac{2\phi}{\phi_o}) e^{-\frac{2\pi}{v_F} k_J \frac{\phi}{\phi_o}} \right) e^{-\frac{2\pi v_F \beta k_J \frac{\phi}{\phi_o}}}, \tag{11} \]

where \( C \) is a constant which does not depend on the flux \( \phi \) and \( \theta_3(v, q) = \sum_{n=-\infty}^{\infty} q^n e^{2\pi inv} \) is the Jacobi theta function.

The current can be obtained as before calculating the total momentum, \( -\frac{e}{2} \sum_{n=1}^{N_a} \frac{2\pi}{L} \left( n + \frac{\phi}{\phi_o} \right) \), and represented by the excitation numbers,

\[ I = -\frac{v_F}{L} \left( J + k_J + \frac{2\phi}{\phi_o} \right) , \tag{12} \]

where \( v_F = \frac{N_a \phi_o}{2\pi} \). Here we note that \( I_o \equiv \frac{v_F}{L} = \frac{2\pi v_F}{\phi_o} \) is the flux in unit \( h = 1 \). With Eq. (11) and Eq. (12), we readily obtain the expectation value of PC given by

\[ \langle I \rangle = -\frac{\partial}{\partial \phi} \left[ -\frac{1}{\beta} \ln Z_0 \right] - I_o \frac{2\phi}{\phi_o} , \tag{13} \]

which has a final form

\[ \langle I \rangle = -I_o \left[ k_J + \frac{2\phi}{\phi_o} - 2 \sum_{n=1}^{\infty} (-1)^n \sinh(k_J + \frac{2\phi}{\phi_o}) \frac{2\pi}{\sinh \frac{2\pi}{v_F}} \right] , \tag{14} \]

where \( T^* = \frac{2\pi v_F}{k_B L} \) is the characteristic temperature. Here, we used

\[ \frac{\partial_3}{\theta_3} (2m) = 4\pi i \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1 - q^{2n}} \sin 2\pi n, \tag{15} \]

which is valid when \( |n| \leq -\frac{1}{2\pi} \ln q \).
with the Legget’s theorem\cite{14}.

Several groups have carried out numerical calculations for PCs in interacting systems with disorder\cite{13,14,17}. In those cases interactions preserve the left-right symmetry and, thus, they correspond to the case, $g = 0$ in LL scheme. Lanczos method\cite{13} showed that, for a clean ring, the amplitude of PC is independent of the interaction strength away from half-filling. In the presence of disorder, however, the amplitude become dependent on the interaction strength. Lanczos method for a dirty ring\cite{13} and Monte Carlo simulation for an electron interacting with a diffuse environment\cite{16} demonstrated that the amplitude is suppressed, while it is enhanced in DMRG calculation for a disordered ring\cite{17}.

In summary, we have developed a LL theory in a mesoscopic ring. In mesoscopic systems like AB ring, the level discreteness plays an important role. We obtained the zero mode in the LL Hamiltonian by exact summation instead of continuum integration. At zero temperature, the PC generally does not depend on the interaction, which is consistent with microscopic derivations. At finite temperatures, amplitude of the PC is enhanced with respect to the non-interacting case and its behavior confirms the Legget’s theorem.

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