On the existence of infinitely many realization functions of non-global local minima in the training of artificial neural networks with ReLU activation

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Abstract

Gradient descent (GD) type optimization schemes are the standard instruments to train fully connected feedforward artificial neural networks (ANNs) with rectified linear unit (ReLU) activation and can be considered as temporal discretizations of solutions of gradient flow (GF) differential equations. It has recently been proved that the risk of every bounded GF trajectory converges in the training of ANNs with one hidden layer and ReLU activation to the risk of a critical point, by which we mean a zero point of the corresponding gradient function. Taking this into account it is one of the key research issues in the mathematical convergence analysis of GF trajectories and GD type optimization schemes, respectively, to study sufficient and necessary conditions for critical points of the risk function and, thereby, to obtain an understanding about the appearance of critical points in dependence of the problem parameters such as the target function. In the first main result of this work we prove in the training of ANNs with one hidden layer and ReLU activation that for every \(a \in \mathbb{R}, b \in (a, \infty)\) and every arbitrarily large positive \(\delta \in (0, \infty)\) we have that there exists a Lipschitz continuous target function \(f : [a, b] \to \mathbb{R}\) such that for every number \(H \in \mathbb{N} \cap (1, \infty)\) of neurons on the hidden layer we have that the risk function has uncountably many different realization functions of non-global local minimum points whose risks are strictly larger than the sum of the risk of the global minimum points and the arbitrarily large positive real number \(\delta\). In the second main result of this work we show in the training of ANNs with one hidden layer and ReLU activation in the special situation where there is only one neuron on the hidden layer and where the target function is continuous and piecewise polynomial that there exist at most finitely many different realization functions of critical points.
1 Introduction

Gradient descent (GD) type optimization schemes are the standard instruments to train fully connected feedforward artificial neural networks (ANNs) with rectified linear unit (ReLU) activation. Although there are a huge number of numerical simulations which indicate that GD type optimization schemes effectually train ANNs with ReLU activation, until today there is no mathematical theory which rigorously explains the success of GD type optimization schemes in the training of such ANNs (however, cf., e.g., \([4, 10, 13, 16, 26, 38]\) and the references mentioned therein for several promising mathematical analysis approaches for GD type optimization schemes).

GD type optimization schemes can be considered as temporal discretizations of solutions of gradient flow (GF) differential equations and most of the key challenges in the mathematical convergence analysis of GD type optimization schemes seem to already be present in the analysis of GF differential equations. In Eberle et al. \([11, \text{Theorem 1.2}]\) (cf. Bolte & Pauwels \([1, \text{Theorem 4}]\), Davis et al. \([7, \text{Corollary 5.11}]\), Jentzen & Riekert \([19, \text{Item (iv) in Theorem 1.1}]\), and Jentzen & Riekert \([20, \text{Theorem 1.3}]\) it has recently been proved that every non-divergent
GF trajectory converges in the training of ANNs with one hidden layer and ReLU activation to the risk of a critical point, by which we mean a zero point of the corresponding gradient function, and, taking this into account, it is one of the key research issues in the mathematical convergence analysis of GF trajectories and GD type optimization schemes, respectively, to study sufficient and necessary conditions for critical points of the risk function and, thereby, to obtain an understanding about the appearance of critical points in dependence of the problem parameters such as the target function (cf., e.g., Cheridito et al. [3]).

In the training of ANNs with one hidden layer and ReLU activation there appear three types of critical points, that are, saddle points, global minimum points, and non-global local minimum points (cf., e.g., Cheridito et al. [3, Lemma 3.1 and Remark 3.2]). To establish convergence of the risk of a bounded GF trajectory to the risk of a global minimum point, we thus need to exclude the possibilities that the risk of a GF trajectory converges to the risk of a non-global local minimum point or the risk of a saddle point. In the case of saddle points, the articles [13,25,26,29,30] suggest and study a promising approach which might be successful to verify that the risk of an appropriate GF trajectory does not converge to the risk of a saddle point. From this point of view it seems particularly important to analyze the risk function in terms of its non-global local minimum points in order to better understand the success of GD type optimization schemes in the training of ANNs.

The scientific literature has dealt with these non-global local minimum points in a variety of ways. There are several examples for finitely many training data and architectures of ANNs for which the considered risk function has non-global local minimum points (see, e.g., Świrszcz et al. [34]). Moreover, non-global local minimum points could be found in the risk landscape of ANNs with one hidden layer and ReLU activation in special student-teacher setups with the probability distribution of the input data given by the normal distribution (see Safran & Shamir [31]). In other cases, where the target function has a very simple form, the critical points of the risk landscape are fully characterized and thus all local minimum points are known (see Cheridito et al. [2, Corollary 2.15], Cheridito et al. [3], and Jentzen & Riekert [17, Corollary 2.11]). Additionally, in the case of ANNs with linear activation and finitely many training data it was shown that all local minimum points of the risk function corresponding to the squared error loss are global minimum points (cf. Kawaguchi [22] and Laurent & von Brecht [24]). For a connection between the critical points of the risk function and the critical points of the risk function with regard to a larger network width we refer to the articles Zhang et al. [36,37].

Further progress in this regard has been made in the so-called overparameterized regime. In this regime it was demonstrated for different situations that the set of all ANNs with risk equal to 0 forms a high-dimensional submanifold of the parameter space (see Cooper [5]). Analogous results were also shown in the non-overparameterized regime (cf. Dereich & Kassing [8], Fehrmann et al. [12], and Jentzen & Riekert [18]). In addition, for ANNs with one hidden layer and quadratic activation with finitely many training data, using different assumptions, it was shown that all local minimum points in the risk landscape corresponding to the squared error loss are global minimum points (see Du & Lee [9] and Soltanolkotabi et al. [33]). In the case of mild overparameterization in special student-teacher setups for ANNs with one hidden layer and quadratic activation with random training data there are mathematical analyzes for the probability of the occurrence of non-global local minimum points (see Mannelli et al. [28]). In particular, the influence of the number of the training data, the input dimension, and the number of the hidden neurons of the teacher ANN was examined more closely.

We also want to mention approaches to visualize the structure of the risk landscape, which is particularly interesting in a local environment of critical points. The article Li et al. [27] presents different ways to get visual access to the high-dimensional risk landscape, discusses disadvantages for these, and suggests an alternative with the so-called filter normalization.
Without this filter normalization, numerical experiments suggest that arbitrary two-dimensional patterns can be found in the risk landscape of wide and deep ANNs using common training data sets such as FashionMNIST and CIFAR10 (see Skorokhodov & Burtsv [32]). There are different attempts to mathematically explain this phenomenon and to prove that such patterns can be found around approximate global minimum points in special situations, for example, using the universal approximator theorem (see Czarnecki et al. [6]).

In view of these scientific findings, we are in this article particularly interested in the study of non-global local minimum points of the risk functions. In the main results of this work we establish two basic results regarding the appearance of critical points in the training of ANNs with one hidden layer and ReLU activation. Specifically, in the first main result of this work, see Theorem 1.1 below, we prove in the training of ANNs with one hidden layer and ReLU activation that for every \( a \in \mathbb{R} \), \( \delta \in (a, \infty) \) and every arbitrarily large positive \( \delta \in (0, \infty) \) we have that there exists a Lipschitz continuous target function \( f : [a, \delta] \to \mathbb{R} \) such that for every number \( H \in \mathbb{N} \cap (1, \infty) \) of neurons on the hidden layer we have that the risk function has uncountably many different realization functions of non-global local minimum points whose risks are strictly larger than the sum of the risk of the global minimum points and the arbitrarily large positive real number \( \delta \) (see also Figure 1 in Section 4 below for a graphical illustration related to the statement of Theorem 1.1). Theorem 1.1 thus suggests even in the situation where the target function is Lipschitz continuous that the training problem might be very challenging due to the appearance of infinitely many different realization functions of non-global local minimum points. To the best of our knowledge, Theorem 1.1 is the first result in the scientific literature which rigorously proves in the training of fully connected ANNs with ReLU activation that there exists a target function such that the risk function has infinitely many different realization functions of non-global local minimum points. We now present the precise statement of Theorem 1.1.

**Theorem 1.1.** Let \( \delta, a \in \mathbb{R}, \delta \in (a, \infty) \) and let \( N_H^\delta \subset C([a, \delta], \mathbb{R}), \theta \in \mathbb{R}^{3H+1}, H \in \mathbb{N}, \) and \( \mathcal{R}_{f,H} : \mathbb{R}^{3H+1} \to \mathbb{R}, f \in C([a, \delta], \mathbb{R}), H \in \mathbb{N}, \) satisfy for all \( H \in \mathbb{N}, f \in C([a, \delta], \mathbb{R}), \) \( \theta = (\theta_1, \ldots, \theta_{3H+1}) \in \mathbb{R}^{3H+1}, x \in [a, \delta] \) that \( N_H^\delta(x) = \theta_{3H+1} + \sum_{j=1}^{H} \theta_{2H+j} \max\{\theta_{H+j} + \theta_j x, 0\} \) and \( \mathcal{R}_{f,H}(\theta) = \int_a^x (N_H^\delta(y) - f(y))^2 \, dy \). Then there exists a Lipschitz continuous \( f : [a, \delta] \to \mathbb{R} \) such that for all \( H \in \mathbb{N} \cap (1, \infty) \) it holds that

\[
\{ v \in C([a, \delta], \mathbb{R}) : \exists \theta \in \mathbb{R}^{3H+1} : v = N_H^\delta, \varepsilon \in (0, \infty) : \mathcal{R}_{f,H}(\theta) = \inf_{\theta \in [-\varepsilon, \varepsilon]^{3H+1}} \mathcal{R}_{f,H}(\theta + \vartheta) > \delta + \inf_{\vartheta \in \mathbb{R}^{3H+1}} \mathcal{R}_{f,H}(\vartheta) \} \quad (1.1)
\]

is an uncountable set.

Theorem 1.1 is an immediate consequence of Corollary 4.9 in Subsection 4.6. In the second main result of this work, see Theorem 1.2 below, we provide in a special situation sufficient conditions to ensure that there are at most finitely many different realization functions of non-global local minimum points. Specifically, in item (i) in Theorem 1.2 below we show in the training of ANNs with one hidden layer and ReLU activation in the special situation where there is only one neuron on the hidden layer (corresponding to the case \( H = 1 \) in Theorem 1.1 above) and where the target function is continuous and piecewise polynomial that there exist at most finitely many different realization functions of critical points. This enables us to conclude in item (ii) in Theorem 1.2 that (in contrast to the situation of Theorem 1.1 above) there exist at most finitely many different realization functions of (non-global) local minimum points. In addition, item (i) in Theorem 1.2 together with [19, Item (v) in Theorem 1.1] and [11, Theorem 1.2] allows us to conclude in item (iii) in Theorem 1.2 that in training of such ANNs we have that the risk of every non-divergent GF trajectory converges to the risk of a global minimum point provided that the initial risk is sufficiently small. To describe a GF
Let the precise statement of Theorem 1.2.

The risk function is not differentiable in the case of ANNs with ReLU activation (due to the fact that the ReLU activation function \( \mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R} \) fails to be differentiable in the origin). As in [17] (cf., e.g., also Cheridito et al. [2]) we accomplish this by means of an approximation procedure in which the ReLU activation function \( \mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R} \) is approximated through appropriate continuously differentiable functions whose derivatives converge pointwise to the left-derivative of the ReLU activation function; see (1.2) in Theorem 1.2. We now present the precise statement of Theorem 1.2.

**Theorem 1.2.** Let \( n \in \mathbb{N} \), \( r_0, r_1, \ldots, r_n, \alpha \in \mathbb{R}, \theta \in (\alpha, \infty) \), \( f \in C([\alpha, \beta], \mathbb{R}) \) satisfy \( \alpha = r_0 < r_1 < \ldots < r_n = \beta \), assume for all \( j \in \{1, 2, \ldots, n\} \) that \( f|_{[-r_1, r_j]} \) is a polynomial, let \( A_r : \mathbb{R} \rightarrow \mathbb{R}, r \in \mathbb{N} \cup \{\infty\} \), satisfy for all \( x \in \mathbb{R} \) that \( (\cup_{r \in \mathbb{N}} \{A_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R}), A_\infty(x) = \max\{x, 0\}, \sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(A_r)'(y)| < \infty \), and

\[
\lim_{r \to \infty} \left( |A_r(x) - A_\infty(x)| + |(A_r)'(x) - 1_{(0, \infty)}(x)| \right) = 0, \quad (1.2)
\]

let \( \mathcal{N}_r^\alpha \in C([\alpha, \beta], \mathbb{R}), r \in \mathbb{N} \cup \{\infty\}, \theta \in \mathbb{R}^4 \), and \( \mathcal{R}_r : \mathbb{R}^4 \rightarrow \mathbb{R}, r \in \mathbb{N} \cup \{\infty\}, \) satisfy for all \( r \in \mathbb{N} \cup \{\infty\}, \theta = (\theta_1, \ldots, \theta_4) \in \mathbb{R}^4, x \in [\alpha, \beta] \) that \( \mathcal{N}_r^\alpha(x) = \theta_1 + 3_4A_r(\theta_2 + \theta_1 x) \) and \( \mathcal{R}_r(\theta) = \int_\alpha^\beta (\mathcal{N}_r^\alpha(y) - f(y))^2 dy \), and let \( \mathcal{G} : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) satisfy for all \( \theta \in \mathcal{G}^{-1}(\{0\}) : v = \mathcal{N}_\infty^\alpha \) is a finite set, and

(i) it holds that \( \left\{ v \in C([\alpha, \beta], \mathbb{R}) \left( \exists \theta \in \mathcal{G}^{-1}(\{0\}) : v = \mathcal{N}_\infty^\alpha \right) \right\} \) is a finite set,

(ii) it holds that

\[
\left\{ v \in C([\alpha, \beta], \mathbb{R}) \left( \exists \theta \in \mathbb{R}^4 : v = \mathcal{N}_\infty^\alpha \right) \left( \in \mathcal{G}^{-1}(\{0\}) \right) \right\} = \left\{ \mathcal{R}_\infty(\theta) = \inf_{\theta \in [-\varepsilon, \varepsilon]^4} \mathcal{R}_\infty(\theta + \vartheta) \right\} \quad (1.3)
\]

is a finite set, and

(iii) there exists \( \varepsilon \in (0, \infty) \) such that for all \( \Theta = (\Theta_t)_{t \in [0, \infty)} = (\Theta_t^1, \ldots, \Theta_t^n)_{t \in [0, \infty)} \in C([0, \infty), \mathbb{R}^4) \) with \( \lim_{t \to \infty} (\sum_{j=1}^n |\Theta_t^j|) < \infty, \forall t \in [0, \infty) \): \( \Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) ds \), and \( \mathcal{R}_\infty(\Theta_0) \leq \varepsilon + \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta) \) it holds that

\[
\lim_{t \to \infty} \mathcal{R}_\infty(\Theta_t) = \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta). \quad (1.4)
\]

Theorem 1.2 is an immediate consequence of Corollary 5.9 in Subsection 5.6. The remainder of this article is organized in the following way.

In Section 2 we prove in Lemma 2.6 and Lemma 2.7 a few basic differentiability properties for the risk function and we establish in Proposition 2.12 that every local minimum point of the risk function is a critical point (a zero of the generalized gradient function). In Section 3 we recall some basic concepts and elementary results from differential geometry and we collect in Proposition 3.12 some necessary and sufficient conditions for local extremum and saddle points.

In Section 4 we employ Proposition 3.12 from Section 3 to establish in Corollary 4.9 that there exists a Lipschitz continuous target function such that the associated risk function has infinitely many realization functions of non-global local minimum points. Theorem 1.1 above is a direct consequence of Corollary 4.9. We also refer to Figure 1 in Section 4 for a graphical illustration related to the statement of Corollary 4.9. Finally, in Section 5 we prove in Corollary 5.8 in the special situation where the target function is continuous and piecewise polynomial and where both the input layer and hidden layer of the considered ANNs are one-dimensional that there exist only finitely many different realization functions of all critical points of the risk function (of all zeros of the generalized gradient function). Theorem 1.2 above can then be shown by combining Proposition 2.12, Corollary 5.8, [19], Item (v) in Theorem 1.1], and [11, Theorem 1.2]. This is precisely the subject of Corollary 5.9 in Section 5.
2 Values of the generalized gradient function at local minimum points

In this section we establish in Proposition 2.12 in Subsection 2.5 below that every local minimum point \( \theta \in \mathbb{R}^d = \mathbb{R}^{dH+2H+1} \) of the risk function \( \mathcal{R}_\infty : \mathbb{R}^d \to \mathbb{R} \) is a critical point in the sense that it is a zero of the generalized gradient function \( \mathcal{G} : \mathbb{R}^d \to \mathbb{R}^d \). Our proof of Proposition 2.12 uses the essentially well-known representation result for the generalized gradient function \( \mathcal{G} : \mathbb{R}^d \to \mathbb{R}^d \) in Proposition 2.5 and the elementary relationships between the generalized gradient function \( \mathcal{G} : \mathbb{R}^d \to \mathbb{R}^d \) and the first-order partial derivatives of the risk function \( \mathcal{R}_\infty : \mathbb{R}^d \to \mathbb{R} \) in Lemma 2.6 and Lemma 2.7 in Subsection 2.3 below. The proof of Proposition 2.5 can be derived analogously to the proof of [19, Proposition 2.2]. Lemma 2.6 and Lemma 2.7 are slight generalizations of [2, Lemma 2.6] and [2, Lemma 2.7], respectively. Our proof of Proposition 2.5 can be derived analogously to the proof of [19, Proposition 2.2].

In Setting 2.1 in Subsection 2.1 below we describe our mathematical setup to introduce the target function \( f : [\alpha, \beta]^d \to \mathbb{R} \), the unnormalized probability distribution of the input data \( \mu : \mathcal{B}([\alpha, \beta]^d) \to [0, \infty] \), the realization function \( \mathcal{N}_\infty : \mathbb{R}^d \to C(\mathbb{R}^d, \mathbb{R}) \), the risk function \( \mathcal{R}_\infty : \mathbb{R}^d \to \mathbb{R} \), and the generalized gradient function \( \mathcal{G} : \mathbb{R}^d \to \mathbb{R}^d \). For the convenience of the reader we recall the notions of the standard scalar product, of the standard norm, of a local minimum points, of a local maximum point, of a local extremum point, and of a saddle point in Definitions 2.4, 2.8, 2.9, 2.10, and 2.11 in Subsections 2.3 and 2.4 below.

2.1 Artificial neural networks (ANNs) with multidimensional input and hidden layer

Setting 2.1. Let \( d, H, \beta \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty) \) satisfy \( d = DH + 2H + 1 \), let \( f : [\alpha, \beta]^d \to \mathbb{R} \) be measurable, let \( \mathcal{A}_r : \mathbb{R} \to \mathbb{R}, r \in \mathbb{N} \cup \{\infty\} \), satisfy for all \( x \in \mathbb{R} \) that \( (\bigcup_{r \in \mathbb{N}} \{ \mathcal{A}_r \}) \subseteq C^1(\mathbb{R}, \mathbb{R}) \), \( \mathcal{A}_\infty(x) = \max\{x, 0\} \), \( \sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(\mathcal{A}_r)'(y)| < \infty \), and

\[
\lim \sup_{r \to \infty} (|\mathcal{A}_r(x) - \mathcal{A}_\infty(x)| + |(\mathcal{A}_r)'(x) - 1_{[0,\infty)}(x)|) = 0, \tag{2.1}
\]

let \( \mu : \mathcal{B}([\alpha, \beta]^d) \to [0, \infty] \) be a finite measure, for every \( r \in \mathbb{N} \cup \{\infty\} \) let \( \mathcal{N}_r = (\mathcal{N}_r^y)_{y \in \mathbb{R}^d} : \mathbb{R}^d \to C(\mathbb{R}^d, \mathbb{R}) \) and \( \mathcal{R}_r : \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d \), \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) that

\[
\mathcal{N}_r^y(x) = \theta_0 + \sum_{i=1}^{H} \theta_{Hd+i} \mathcal{A}_r(\theta_{Hd+i} + \sum_{j=1}^{d} \theta_{(i-1)d+j} x_j) \tag{2.2}
\]

and \( \mathcal{R}_r(\theta) = \int_{[\alpha, \beta]^d} |\mathcal{N}_r(y) - f(y)|^2 \mu(dy) \), let \( \lambda : \mathcal{B}([\alpha, \beta]^d) \to [0, \infty] \) be the Lebesgue-Borel measure on \([\alpha, \beta]^d\), let \( \mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_d) : \mathbb{R}^d \to \mathbb{R}^d \) satisfy for all \( \theta \in \{ \varphi \in \mathbb{R}^d : ((\nabla \mathcal{R}_r)(\varphi))_{r \in \mathbb{N}} \) is convergent \( \mathcal{G}(\theta) = \lim_{r \to \infty} (\nabla \mathcal{R}_r)(\theta) \), and for every \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d, i \in \{1, 2, \ldots, H\} \) let \( I_i^0 \subseteq \mathbb{R}^d \) satisfy \( I_i^0 = \{ x = (x_1, \ldots, x_d) \in [\alpha, \beta]^d : \theta_{Hd+i} + \sum_{j=1}^{d} \theta_{(i-1)d+j} x_j > 0 \} \).

2.2 Differentiability properties for parameter dependent Lebesgue integrals

Lemma 2.2. Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty) \), let \( \phi : \mathbb{R} \times [\alpha, \beta]^d \to \mathbb{R} \) be measurable, let \( \mu : \mathcal{B}([\alpha, \beta]^d) \to [0, \infty] \) be a measure, assume for all \( x \in \mathbb{R} \) that \( \int_{[\alpha, \beta]^d} |\phi(x, s)| \mu(ds) < \infty \), let \( \Phi : \mathbb{R} \to \mathbb{R} \) satisfy for all \( x \in \mathbb{R} \) that

\[
\Phi(x) = \int_{[\alpha, \beta]^d} \phi(x, s) \mu(ds), \tag{2.3}
\]
let $E \in \mathcal{B}([\alpha, \delta]^d)$ satisfy $\mu([\alpha, \delta]^d \setminus E) = 0$, let $c: E \to \mathbb{R}$ be measurable, let $x \in \mathbb{R}$, $\delta \in (0, \infty)$ satisfy for all $h \in (-\delta, \delta)$, $s \in E$ that $|\phi(x+h, s) - \phi(x, s)| \leq |h||c(s)|$, assume $\int_E |c(s)| \mu(ds) < \infty$, and assume for all $s \in E$ that $\mathbb{R} \ni v \mapsto \phi(v, s) \in \mathbb{R}$ is differentiable at $x$. Then

(i) it holds that $\Phi$ is differentiable at $x$ and

(ii) it holds that

$$\Phi'(x) = \int_E \left( \frac{\partial}{\partial x} \phi \right)(x, s) \mu(ds).$$

**Proof of Lemma 2.2.** Observe that (2.3) and the assumption that $\mu([\alpha, \delta]^d \setminus E) = 0$ demonstrate that for all $h \in \mathbb{R} \setminus \{0\}$ it holds that

$$h^{-1}[\Phi(x + h) - \Phi(x)] = \int_{[\alpha, \delta]^d} h^{-1}[\phi(x + h, s) - \phi(x, s)] \mu(ds)$$

$$= \int_E h^{-1}[\phi(x + h, s) - \phi(x, s)] \mu(ds).$$

In the next step we note that the assumption that for all $s \in E$ it holds that $\mathbb{R} \ni v \mapsto \phi(v, s) \in \mathbb{R}$ is differentiable at $x$ shows that for all $s \in E$ it holds that

$$\limsup_{E \setminus \{0\} \ni h \to 0} h^{-1}[\phi(x + h, s) - \phi(x, s)] - \left( \frac{\partial}{\partial x} \phi \right)(x, s) = 0.$$ 

Furthermore, we observe that the assumption that for all $h \in (-\delta, \delta)$, $s \in E$ it holds that $|\phi(x + h, s) - \phi(x, s)| \leq |h||c(s)|$ proves that for all $h \in (-\delta, \delta) \setminus \{0\}$, $s \in E$ it holds that

$$|h^{-1}[\phi(x + h, s) - \phi(x, s)]| \leq |c(s)|.$$ 

Combining (2.5), (2.6), the assumption that $\int_E |c(s)| \mu(ds) < \infty$, and Lebesgue’s dominated convergence theorem therefore assures that

$$\lim_{E \setminus \{0\} \ni h \to 0} \left[ h^{-1}[\Phi(x + h) - \Phi(x)] - \left( \frac{\partial}{\partial x} \phi \right)(x, s) \right] \mu(ds) = \int_E \left( \frac{\partial}{\partial x} \phi \right)(x, s) \mu(ds).$$

The proof of Lemma 2.2 is thus complete.

**Corollary 2.3.** Let $d, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\delta \in (\alpha, \infty)$, let $\phi: \mathbb{R}^n \times [\alpha, \delta]^d \to \mathbb{R}$ be measurable, let $\mu: \mathcal{B}(\{\alpha, \delta]^d) \to [0, \infty]$ be a measure, assume for all $x \in \mathbb{R}^n$ that $\int_{[\alpha, \delta]^d} |\phi(x, s)| \mu(ds) < \infty$, let $\Phi: \mathbb{R}^n \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^n$ that

$$\Phi(x) = \int_{[\alpha, \delta]^d} \phi(x, s) \mu(ds),$$

let $E \in \mathcal{B}([\alpha, \delta]^d)$ satisfy $\mu([\alpha, \delta]^d \setminus E) = 0$, let $c: E \to \mathbb{R}$ be measurable, let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $j \in \{1, 2, \ldots, n\}$, $\delta \in (0, \infty)$ satisfy for all $h \in (-\delta, \delta)$, $s \in E$ that

$$|\phi(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n, s) - \phi(x_1, \ldots, x_n, s)| \leq |h||c(s)|,$$

assume $\int_E |c(s)| \mu(ds) < \infty$, and assume for all $s \in E$ that $\mathbb{R} \ni v \mapsto \phi(x_1, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n, s) \in \mathbb{R}$ is differentiable at $x_j$. Then

(i) it holds that $\mathbb{R} \ni v \mapsto \Phi(x_1, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n) \in \mathbb{R}$ is differentiable at $x_j$ and

(ii) it holds that

$$\left( \frac{\partial}{\partial x_j} \Phi \right)(x_1, \ldots, x_n) = \int_E \left( \frac{\partial}{\partial x_j} \phi \right)(x_1, \ldots, x_n, s) \mu(ds).$$

**Proof of Corollary 2.3.** Note that Lemma 2.2 shows items (i) and (ii). The proof of Corollary 2.3 is thus complete.
2.3 Differentiability properties for the generalized gradient function

**Definition 2.4.** We denote by \( \langle \cdot, \cdot \rangle : (\cup_{n \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^n)) \to \mathbb{R} \) and \( \| \cdot \| : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \to \mathbb{R} \) the functions which satisfy for all \( \delta \in \mathbb{N} \), \( x = (x_1, \ldots, x_\delta) \), \( y = (y_1, \ldots, y_\delta) \in \mathbb{R}^\delta \) that \((x, y) = \sum_{j = 1}^\delta x_j y_j \) and \( \|x\| = (\sum_{j = 1}^\delta |x_j|^2)^{1/2} \).

**Proposition 2.5.** Assume Setting 2.1 and let \( \theta = (\theta_1, \ldots, \theta_\delta) \in \mathbb{R}^\delta \), \( e_1, e_2, \ldots, e_\delta \in \mathbb{R}^d \) satisfy \( e_1 = (1, 0, 0, \ldots, 0) \), \( e_2 = (0, 1, 0, \ldots, 0) \), \ldots, \( e_\delta = (0, 0, 0, \ldots, 0, 1) \in \mathbb{R}^d \). Then it holds for all \( i \in \{1, 2, \ldots, H\} \), \( j \in \{1, 2, \ldots, d\} \) that

\[
G_{(i-1)d+j}(\theta) = 2\theta_{Hd+i} \int_{\mathbb{R}^d} (\phi(x) - \mu(x)) \mu(dx),
\]

\[
G_{Hd+i}(\theta) = 2\theta_{Hd+i} \int_{\mathbb{R}^d} (\phi(x) - \mu(x)) \mu(dx),
\]

\[
G_{Hd+i}(\theta) = 2 \int_{[\alpha, \beta]^d} \left( \mathcal{A}_\infty([\theta_{Hd+i} + \sum_{k = 1}^d \theta_{(i-1)d+k} (e_k, x)]) (\mathcal{N}_{\infty}^g(x) - f(x)) \right) \mu(dx),
\]

and

\[
G_\delta(\theta) = 2 \int_{[\alpha, \beta]^d} (\mathcal{N}_{\infty}^g(x) - f(x)) \mu(dx)
\]

(cf. Definition 2.4).

**Proof of Proposition 2.5.** Observe that (2.1) and (2.2) establish (2.12) (cf., e.g., [19, Proposition 2.2] and [15, Items (v) and (vi) in Proposition 2.5]). The proof of Proposition 2.5 is thus complete.

**Lemma 2.6.** Assume Setting 2.1 and let \( \theta = (\theta_1, \ldots, \theta_\delta) \in \mathbb{R}^\delta \). Then

(i) it holds for all \( i \in \mathbb{N} \cap (Hd + H, \delta) \) that \( \mathbb{R} \ni v \mapsto \mathcal{R}_\infty(\theta_1, \ldots, \theta_i-1, v, \theta_{i+1}, \ldots, \theta_\delta) \in \mathbb{R} \) is differentiable at \( \theta_i \) and

(ii) it holds for all \( i \in \mathbb{N} \cap (Hd + H, \delta) \) that \( \left( \frac{\partial}{\partial \theta_i} \mathcal{R}_\infty(\theta) \right)(\theta) = G_i(\theta) \).

**Proof of Lemma 2.6.** Throughout this proof let \( \phi: \mathbb{R}^\delta \times [\alpha, \beta]^d \to \mathbb{R} \) satisfy for all \( \vartheta \in \mathbb{R}^\delta \), \( x \in [\alpha, \beta]^d \) that \( \phi(\vartheta, x) = \mathcal{N}_{\infty}^g(x) - f(x) \) and let \( e_1, e_2, \ldots, e_\delta \in \mathbb{R}^d \) satisfy \( e_1 = (1, 0, 0, \ldots, 0) \), \( e_2 = (0, 1, 0, \ldots, 0) \), \ldots, \( e_\delta = (0, 0, \ldots, 0, 1) \in \mathbb{R}^d \). Note that the fact that \( \mathcal{A}_\infty \) is Lipschitz continuous establishes that

\[
\mathbb{R}^\delta \times [\alpha, \beta]^d \ni (\vartheta, x) \mapsto \mathcal{N}_{\infty}^g(x) \in \mathbb{R}
\]

is locally Lipschitz continuous. The fact that for all \( v, w \in \mathbb{R} \) it holds that \( v^2 - w^2 = (v - w)(v + w) \) hence ensures that for all \( v, w \in \{\vartheta \in \mathbb{R}^\delta : \|\vartheta - \vartheta\| \leq 1\} \), \( x \in [\alpha, \beta]^d \) it holds that

\[
|\phi(v, x) - \phi(w, x)| = \left| \left[ \mathcal{N}_{\infty}^g(x) - f(x) \right]^2 - \left[ \mathcal{N}_{\infty}^w(x) - f(x) \right]^2 \right| = \left[ \left( \mathcal{N}_{\infty}^g(x) - \mathcal{N}_{\infty}^w(x) \right) \left( \mathcal{N}_{\infty}^g(x) + \mathcal{N}_{\infty}^w(x) - 2f(x) \right) \right] \leq 2 |\mathcal{N}_{\infty}^g(x) - \mathcal{N}_{\infty}^w(x)| \left| f(x) \right| + \sup_{\vartheta \in [\alpha, \beta]^d} \| \vartheta - \vartheta \| \leq 1 \sup_{y \in [\alpha, \beta]^d} |\mathcal{N}_{\infty}^g(y)| < \infty
\]

(cf. Definition 2.4). Furthermore, observe that Hölder’s inequality and the fact that for all \( \vartheta \in \mathbb{R}^\delta \) it holds that

\[
\mathcal{R}_\infty(\vartheta) = \int_{[\alpha, \beta]^d} |\mathcal{N}_{\infty}^g(x) - f(x)|^2 \mu(dx) = \int_{[\alpha, \beta]^d} \phi(\vartheta, x) \mu(dx) < \infty
\]

 assure that for all \( \vartheta \in \mathbb{R}^\delta \) it holds that

\[
\int_{[\alpha, \beta]^d} |f(x)| \mu(dx) \leq \left[ \mu([\alpha, \beta]^d) \right]^{1/2} \left[ \int_{[\alpha, \beta]^d} |f(x)|^2 \mu(dx) \right]^{1/2} \leq \left[ \mu([\alpha, \beta]^d) \right]^{1/2} \left[ \int_{[\alpha, \beta]^d} |\mathcal{N}_{\infty}^g(x)|^2 \mu(dx) \right]^{1/2} \leq \left[ \mu([\alpha, \beta]^d) \right]^{1/2} \left[ \mathcal{R}_\infty(\vartheta) \right]^{1/2} + \left[ \mu([\alpha, \beta]^d) \right] \left[ \sup_{x \in [\alpha, \beta]^d} |\mathcal{N}_{\infty}^g(x)| \right] < \infty.
\]
In addition, note that the chain rule and the fact that for all \(i \in \mathbb{N} \cap (Hd + H, \mathfrak{d})\), \(x \in [\alpha, \beta]^{d}\) it holds that \(\mathbb{R} \ni v \mapsto \mathcal{N}_{\infty}^{d(\theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_{2})}(x) \in \mathbb{R}\) is differentiable at \(\theta_{i}\) imply that for all \(i \in \mathbb{N} \cap (Hd + H, \mathfrak{d})\), \(x \in [\alpha, \beta]^{d}\) it holds that \(\mathbb{R} \ni v \mapsto \phi(\theta_{i}, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_{2}, x) \in \mathbb{R}\) is differentiable at \(\theta_{i}\). Combining this, \((2.13), (2.14), (2.15), (2.16)\), and Corollary 2.3 demonstrates that for all \(i \in \mathbb{N} \cap (Hd + H, \mathfrak{d})\) it holds that \(\mathbb{R} \ni v \mapsto \mathcal{R}_{\infty}(\theta_{1}, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_{2}) \in \mathbb{R}\) is differentiable at \(\theta_{i}\) and

\[
(\frac{\partial}{\partial \theta_{i}} \mathcal{R}_{\infty})(\theta) = \int_{[\alpha, \beta]^{d}} (\frac{\partial}{\partial \theta_{i}} \phi)(\theta_{1}, \ldots, \theta_{b}) \mu(dx). \tag{2.17}
\]

Next observe that the chain rule establishes that for all \(j \in \{1, 2, \ldots, H\}\), \(x = (x_{1}, \ldots, x_{d}) \in [\alpha, \beta]^{d}\) it holds that

\[
\left(\frac{\partial}{\partial \theta_{j}} \mathcal{R}_{\infty}\right)(\theta_{1}, \ldots, \theta_{b}, x) = 2(\mathcal{N}_{\infty}^{d}(x) - f(x))[\mathcal{A}_{\infty}(\theta_{Hd + j} + \sum_{k=1}^{d} \theta_{(j-1)d+k} x_{k})] \tag{2.18}
\]

and

\[
(\frac{\partial}{\partial \theta_{j}} \phi)(\theta_{1}, \ldots, \theta_{b}, x) = 2(\mathcal{N}_{\infty}^{d}(x) - f(x)). \tag{2.19}
\]

This and \((2.17)\) prove that for all \(j \in \{1, 2, \ldots, H\}\) it holds that

\[
(\frac{\partial}{\partial \theta_{j}} \mathcal{R}_{\infty})(\theta) = 2 \int_{[\alpha, \beta]^{d}} [\mathcal{A}_{\infty}(\theta_{Hd + j} + \sum_{k=1}^{d} \theta_{(j-1)d+k} x_{k})] \mathcal{N}_{\infty}^{d}(x) - f(x) \mu(dx). \tag{2.20}
\]

and

\[
(\frac{\partial}{\partial \theta_{j}} \mathcal{R}_{\infty})(\theta) = 2 \int_{[\alpha, \beta]^{d}} (\mathcal{N}_{\infty}^{d}(x) - f(x)) \mu(dx). \tag{2.21}
\]

Combining this with Proposition 2.5 establishes for all \(i \in \mathbb{N} \cap (Hd + H, \mathfrak{d})\) that \((\frac{\partial}{\partial \theta_{i}} \mathcal{R}_{\infty})(\theta) = \mathcal{G}_{i}(\theta)\). The proof of Lemma 2.6 is thus complete. \(\square\)

**Lemma 2.7.** Assume Setting 2.1, assume \(\mu \ll \lambda\), and let \(\theta = (\theta_{1}, \ldots, \theta_{b}) \in \mathbb{R}^{b}\). Then

(i) it holds for all \(j \in \{1, 2, \ldots, H\}\), \(i \in \mathbb{N} \cap ((j - 1)d, jd] \cup \{Hd + j\}\) with \(|\theta_{Hd + j}| + \sum_{k=1}^{d} |\theta_{(j-1)d+k}| > 0\) that \(\mathbb{R} \ni v \mapsto \mathcal{R}_{\infty}(\theta_{1}, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_{b}) \in \mathbb{R}\) is differentiable at \(\theta_{i}\) and

(ii) it holds for all \(j \in \{1, 2, \ldots, H\}\), \(i \in \mathbb{N} \cap ((j - 1)d, jd] \cup \{Hd + j\}\) with \(|\theta_{Hd + j}| + \sum_{k=1}^{d} |\theta_{(j-1)d+k}| > 0\) that \((\frac{\partial}{\partial \theta_{i}} \mathcal{R}_{\infty})(\theta) = \mathcal{G}_{i}(\theta)\).

**Proof of Lemma 2.7.** Throughout this proof let \(\phi: \mathbb{R}^{b} \times [\alpha, \beta]^{d} \to \mathbb{R}\) satisfy for all \(\vartheta \in \mathbb{R}^{b}\), \(x \in [\alpha, \beta]^{d}\) that \(\phi(\vartheta, x) = |\mathcal{N}_{\infty}^{d}(x) - f(x)|^{2}\), let \(e_{1}, e_{2}, \ldots, e_{d} \in \mathbb{R}^{d}\) satisfy \(e_{1} = (1, 0, 0, \ldots, 0), e_{2} = (0, 1, 0, \ldots, 0), \ldots, e_{d} = (0, 0, 0, \ldots, 1) \in \mathbb{R}^{d}\), and let \(E_{j} \subseteq \mathbb{R}^{d}, j \in \{1, 2, \ldots, H\}\), satisfy for all \(j \in \{1, 2, \ldots, H\}\) that

\[
E_{j} = \{x = (x_{1}, \ldots, x_{d}) \in [\alpha, \beta]^{d}: \theta_{Hd + j} + \sum_{k=1}^{d} \theta_{(j-1)d+k} x_{k} \neq 0\}. \tag{2.22}
\]

Note that the integral transformation theorem ensures that for all \(\vartheta \in \mathbb{R}^{d}\setminus\{0\}\), \(c \in \mathbb{R}\) it holds that

\[
\int_{\mathbb{R}^{d}} \mathbbm{1}_{\{x \in \mathbb{R}^{d}: c + \langle \vartheta, x \rangle = 0\}}(y) dy = \int_{\mathbb{R}^{d}} \mathbbm{1}_{\{0\}}(c + \langle \vartheta, y \rangle) dy = \int_{\mathbb{R}^{d}} \mathbbm{1}_{\{0\}}(\langle \vartheta, y + c \|\vartheta\|^{-2} \vartheta \rangle) dy = \int_{\mathbb{R}^{d}} \mathbbm{1}_{\{0\}}(\langle \vartheta, y \rangle) dy = \int_{\mathbb{R}^{d}} \mathbbm{1}_{\{x \in \mathbb{R}^{d}: \langle \vartheta, x \rangle = 0\}}(y) dy \tag{2.23}
\]

(cf. Definition 2.4). Moreover, observe that the rank-nullity theorem demonstrates that for all \(\vartheta \in \mathbb{R}^{d}\setminus\{0\}\) it holds that

\[
\dim_{\mathbb{R}}(\{x \in \mathbb{R}^{d}: \langle \vartheta, x \rangle = 0\}) = d - \dim_{\mathbb{R}}(\{y \in \mathbb{R}: \exists x \in \mathbb{R}^{d}: y = \langle \vartheta, x \rangle\}) = d - 1. \tag{2.24}
\]
Therefore, we obtain for all \( \vartheta \in \mathbb{R}^{d}\{0\} \) that \( \int_{\mathbb{R}^{d}} 1_{\{x \in \mathbb{R}^{d}: \vartheta(x) = 0\}}(y) \, dy = 0 \). Combining this with (2.23) and the fact that for all \( \vartheta \in \mathbb{R}^{d} \), \( c \in \mathbb{R} \) it holds that \( \{ x \in [a, \vartheta]^d : c + \vartheta(x) = 0 \} \subseteq \{ x \in \mathbb{R}^{d} : c + \vartheta(x) = 0 \} \) shows that for all \( \vartheta \in \mathbb{R}^{d}\{0\} \), \( c \in \mathbb{R} \) it holds that \( \lambda(\{ x \in [a, \vartheta]^d : c + \vartheta(x) = 0 \}) \leq \int_{\mathbb{R}^{d}} 1_{\{x \in \mathbb{R}^{d}: c + \vartheta(x) = 0\}}(y) \, dy = 0 \). This and (2.22) prove that for all \( j \in \{1, 2, \ldots, H\} \) with \( |\theta_{Hd+j}| + \sum_{k=1}^{d} |\theta_{(j-1)d+k}| > 0 \) it holds that

\[
\lambda([a, \vartheta]^d \setminus E_j) = 0. 
\] (2.25)

Next note that the fact that \( \mathcal{A}_\infty \) is Lipschitz continuous implies that

\[
\mathbb{R}^{d} \times [a, \vartheta]^d \ni (\vartheta, x) \mapsto \mathcal{N}_\infty^0(x) \in \mathbb{R} 
\] (2.26)
is locally Lipschitz continuous. The fact that for all \( v, w \in \mathbb{R} \) it holds that \( v^2 - w^2 = (v - w)(v + w) \) hence shows that for all \( v, w \in \{ \vartheta \in \mathbb{R}^{d} : \| \vartheta - \vartheta \| \leq 1 \} \), \( x \in [a, \vartheta]^d \) it holds that

\[
|\phi(v, x) - \phi(w, x)| = \left| \left[ \mathcal{N}_\infty^v(x) - f(x) \right]^2 - \left[ \mathcal{N}_\infty^w(x) - f(x) \right]^2 \right| = \left[ \left( \mathcal{N}_\infty^v(x) - \mathcal{N}_\infty^w(x) \right) \left( \mathcal{N}_\infty^v(x) + \mathcal{N}_\infty^w(x) - 2f(x) \right) \right] \leq 2|\mathcal{N}_\infty^v(x) - \mathcal{N}_\infty^w(x)| \left[ f(x) \right] + \sup_{\vartheta \in \mathbb{R}^{d}, \| \vartheta - \vartheta \| \leq 1} \sup_{y \in [a, \vartheta]^d} |\mathcal{N}_\infty^y(y)| < \infty. 
\] (2.27)

Moreover, observe that Hölder’s inequality and the fact that for all \( \vartheta \in \mathbb{R}^{d} \) it holds that

\[
\mathcal{R}_\infty(\vartheta) = \int_{[a, \vartheta]^d} \left| \mathcal{N}_\infty^v(x) - f(x) \right|^2 \mu(dx) = \int_{[a, \vartheta]^d} \phi(\vartheta, x) \mu(dx) < \infty 
\] (2.28)
prove that for all \( \vartheta \in \mathbb{R}^{d} \) it holds that

\[
\int_{[a, \vartheta]^d} \left| \mathcal{N}_\infty^v(x) - f(x) \right|^2 \mu(dx) = \left[ \mu([a, \vartheta]^d) \right]^{1/2} \left[ \int_{[a, \vartheta]^d} \left| f(x) \right|^2 \mu(dx) \right]^{1/2} 
\] ≤ \left[ \mu([a, \vartheta]^d) \right]^{1/2} \left[ \int_{[a, \vartheta]^d} \left| \mathcal{N}_\infty^v(x) - \mathcal{N}_\infty^w(x) \right|^2 \mu(dx) \right]^{1/2} 
\] + \left[ \mu([a, \vartheta]^d) \right]^{1/2} \left[ \int_{[a, \vartheta]^d} \left| \mathcal{N}_\infty^w(x) - f(x) \right|^2 \mu(dx) \right]^{1/2} 
\] ≤ \left[ \mu([a, \vartheta]^d) \right]^{1/2} \left[ \mathcal{R}_\infty(\vartheta) \right]^{1/2} + \left[ \mu([a, \vartheta]^d) \right] \left[ \sup_{x \in [a, \vartheta]^d} |\mathcal{N}_\infty^0(x)| \right] < \infty.
\] (2.29)

In addition, note that the chain rule and the fact that for all \( x \in \mathbb{R} \{0\} \) it holds that \( \mathcal{A}_\infty \) is differentiable at \( x \) and \( (\mathcal{A}_\infty)’(x) = 1_{(0,\infty)}(x) \) demonstrate that for all \( j \in \{1, 2, \ldots, H\} \), \( i \in \mathbb{N} \cap (j-1)d, jd] \cup \{ Hd + j \} \), \( x \in E_j \) it holds that \( \mathbb{R} \ni v \mapsto \mathcal{N}_\infty^{(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_d, x)}(x) \in \mathbb{R} \) is differentiable at \( x \). This ensures that for all \( j \in \{1, 2, \ldots, H\} \), \( i \in \mathbb{N} \cap (j-1)d, jd] \cup \{ Hd + j \} \), \( x \in E_j \) it holds that

\[
\mathbb{R} \ni v \mapsto \phi(\theta_1, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_d, x) \in \mathbb{R} 
\] (2.30)
is differentiable at \( x \). Combining this, \( (2.25), (2.26), (2.27), (2.28), (2.29) \), the assumption that \( \mu \ll \lambda \) and Corollary 2.3 establishes that for all \( j \in \{1, 2, \ldots, H\} \), \( i \in \mathbb{N} \cap (j-1)d, jd] \cup \{ Hd + j \} \) with \( |\theta_{Hd+j}| + \sum_{\ell=1}^{d} |\theta_{(j-1)d+\ell}| > 0 \) it holds that \( \mathbb{R} \ni v \mapsto \mathcal{R}_\infty(\theta_1, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_H) \in \mathbb{R} \) is differentiable at \( \theta_i \) and

\[
\left( \frac{\partial}{\partial \theta_i} \mathcal{R}_\infty \right)(\theta) = \int_{E_j} \left( \frac{\partial}{\partial \theta_i} \phi \right)(\theta_1, \ldots, \theta_d, x) \mu(dx). 
\] (2.31)

Next observe that the chain rule demonstrates that for all \( j \in \{1, 2, \ldots, H\} \), \( k \in \{1, 2, \ldots, d\} \), \( x = (x_1, \ldots, x_d) \in E_j \) it holds that

\[
\left( \frac{\partial}{\partial x_k} \mathcal{R}_\infty \right)(\theta_1, \ldots, \theta_d, x) = 2(\mathcal{N}_\infty^\theta(x) - f(x))\theta_{Hd+j}x_k1_{(0,\infty)}(\theta_{Hd+j} + \sum_{\ell=1}^{d} \theta_{(j-1)d+\ell}x_\ell) 
\] (2.32)
= \left(\mathcal{N}_\infty^\theta(x) - f(x)\right)\theta_{Hd+j}x_k1^\theta(x)
Proposition 2.12. 2.5 Values of the generalized gradient function at local minimum (cf. Definition 2.10). Let \( R \) and Definition 2.11. Let \( f \) assume that \( \theta \) is a local minimum point of \( f \) \( x \) \( H \) and \( j \). Then we say that \( x \) is a local minimum point of \( f \). This and (2.31) prove for all \( j \in \{1, 2, \ldots, H\} \), \( k \in \{1, 2, \ldots, d\} \) with \( |\theta_{H+d+j}| + \sum_{\ell=1}^{d} |\theta_{(j-1)d+\ell}| > 0 \) that

\[
\left( \frac{\partial}{\partial \theta_{H+d+j}} R_{\infty} \right)(\theta) = 2\theta_{H+d+j} \int_{E_j} (e_k, x) (N^0_{\infty}(x) - f(x)) 1_{I^\theta_j}(x) \mu(dx) \\
= 2\theta_{H+d+j} \int_{I^\theta_j} (e_k, x) (N^0_{\infty}(x) - f(x)) \mu(dx)
\]

and

\[
\left( \frac{\partial}{\partial \theta_{H+d+j}} R_{\infty} \right)(\theta) = 2\theta_{H+d+j} \int_{E_j} (N^0_{\infty}(x) - f(x)) 1_{I^\theta_j}(x) \mu(dx) \\
= 2\theta_{H+d+j} \int_{I^\theta_j} (N^0_{\infty}(x) - f(x)) \mu(dx)
\]

Combining this with Proposition 2.5 establishes that for all \( j \in \{1, 2, \ldots, H\} \), \( i \in \mathbb{N} \cap ((j-1)d, j+1) \cup \{Hd+j\} \) with \( |\theta_{H+d+j}| + \sum_{\ell=1}^{d} |\theta_{(j-1)d+\ell}| > 0 \) it holds that \( \left( \frac{\partial}{\partial \theta_{H}} R_{\infty} \right)(\theta) = G_i(\theta) \). The proof of Lemma 2.7 is thus complete. \( \square \)

2.4 Local extrema and saddle points

Definition 2.8. Let \( \mathfrak{d} \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^3 \) be a set, let \( f : U \to \mathbb{R} \) be a function, and let \( x \in U \). Then we say that \( x \) is a local minimum point of \( f \) if and only if there exists \( \varepsilon \in (0, \infty) \) such that \( f(x) = \inf_{y \in \{u \in U : ||u-x|| \leq \varepsilon\}} f(y) \) (cf. Definition 2.4).

Definition 2.9. Let \( \mathfrak{d} \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^3 \) be a set, let \( f : U \to \mathbb{R} \) be a function, and let \( x \in U \). Then we say that \( x \) is a local maximum point of \( f \) if and only if there exists \( \varepsilon \in (0, \infty) \) such that \( f(x) = \sup_{y \in \{u \in U : ||u-x|| \leq \varepsilon\}} f(y) \) (cf. Definition 2.4).

Definition 2.10. Let \( \mathfrak{d} \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^3 \) be a set, let \( f : U \to \mathbb{R} \) be a function, and let \( x \in U \). Then we say that \( x \) is a local extremum point of \( f \) if and only if \( x \in \{y \in \mathbb{R}^3 : (y \text{ is a local minimum point of } f) \} \cup \{y \in \mathbb{R}^3 : (y \text{ is a local maximum point of } f) \} \) (cf. Definitions 2.8 and 2.9).

Definition 2.11. Let \( \mathfrak{d} \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^3 \) be open, let \( f : U \to \mathbb{R} \) be a function, let \( x \in U \), and assume that \( f \) is differentiable at \( x \). Then we say that \( x \) is a saddle point of \( f \) if and only if we have that

(i) it holds that \( x \) is not a local extremum point of \( f \) and

(ii) it holds that \( (\nabla f)(x) = 0 \)

(cf. Definition 2.10).

2.5 Values of the generalized gradient function at local minimum points

Proposition 2.12. Assume Setting 2.1, assume \( \mu \ll \lambda \), and let \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^3 \) be a local minimum point of \( R_{\infty} \) (cf. Definition 2.8). Then \( G(\theta) = 0 \).
**Proof of Proposition 2.12.** Note that Lemma 2.6 ensures that for all $i \in \mathbb{N} \cap (Hd + H, \mathfrak{d}]$ it holds that $\mathbb{R} \ni v \mapsto \mathcal{R}_\infty(\theta_1, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_d) \in \mathbb{R}$ is differentiable at $\theta_i$ and $(\frac{\partial}{\partial \theta_i} \mathcal{R}_\infty)(\theta) = \mathcal{G}_i(\theta)$. This and the assumption that $\theta$ is a local minimum point of $\mathcal{R}_\infty$ implies that for all $i \in \mathbb{N} \cap (Hd + H, \mathfrak{d}]$ it holds that

$$\mathcal{G}_i(\theta) = (\frac{\partial}{\partial \theta_i} \mathcal{R}_\infty)(\theta) = 0. \quad (2.36)$$

Moreover, observe that Lemma 2.7 ensures that for all $j \in \{1, 2, \ldots, H\}, i \in \mathbb{N} \cap ((j-1)d, jd] \cup \{Hd + j\}$ with $|\theta_{Hd + j}| + \sum_{k=1}^{d}|\theta_{(j-1)d + k}| > 0$ it holds that $\mathbb{R} \ni v \mapsto \mathcal{R}_\infty(\theta_1, \ldots, \theta_{i-1}, v, \theta_{i+1}, \ldots, \theta_d) \in \mathbb{R}$ is differentiable at $\theta_i$ and $(\frac{\partial}{\partial \theta_i} \mathcal{R}_\infty)(\theta) = \mathcal{G}_i(\theta)$. This and the assumption that $\theta$ is a local minimum point of $\mathcal{R}_\infty$ implies that for all $j \in \{1, 2, \ldots, H\}, i \in \mathbb{N} \cap ((j-1)d, jd] \cup \{Hd + j\}$ with $|\theta_{Hd + j}| + \sum_{k=1}^{d}|\theta_{(j-1)d + k}| > 0$ it holds that

$$\mathcal{G}_i(\theta) = (\frac{\partial}{\partial \theta_i} \mathcal{R}_\infty)(\theta) = 0. \quad (2.37)$$

In addition, note that Proposition 2.5 and the fact that for all $j \in \{1, 2, \ldots, H\}$ with $|\theta_{Hd + j}| + \sum_{k=1}^{d}|\theta_{(j-1)d + k}| = 0$ it holds that $\Lambda_j = \emptyset$ demonstrate that for all $j \in \{1, 2, \ldots, H\}, i \in \mathbb{N} \cap ((j-1)d, jd] \cup \{Hd + j\}$ with $|\theta_{Hd + j}| + \sum_{k=1}^{d}|\theta_{(j-1)d + k}| = 0$ it holds that $\mathcal{G}_i(\theta) = 0$. This and (2.37) assure that for all $j \in \{1, 2, \ldots, H\}, i \in \mathbb{N} \cap ((j-1)d, jd] \cup \{Hd + j\}$ it holds that $\mathcal{G}_i(\theta) = 0$. Therefore, we obtain that for all $i \in \mathbb{N} \cap (0, Hd + H]$ it holds that $\mathcal{G}_i(\theta) = 0$. Combining this with (2.36) establishes that for all $i \in \{1, 2, \ldots, \mathfrak{d}\}$ it holds that $\mathcal{G}_i(\theta) = 0$. The proof of Proposition 2.12 is thus complete. \qed

# 3 Differential geometric preliminaries

This section is devoted to establish some essentially well-known necessary and sufficient conditions for local extremum and saddle points in Proposition 3.12 in Subsection 3.5 below. Our proof of Proposition 3.12 uses the well-known rank bound for the Hessian matrix in Lemma 3.7 in Subsection 3.4 below, whose proof can be found, e.g., in [14, Chapter 2], the essentially well-known sufficient condition for a local minimum point in Lemma 3.10, and the well-known necessary condition for a local minimum point in Lemma 3.11 in Subsection 3.5 below. In the proof of Lemma 3.10 we employ the well-known Taylor-type estimate from Lemma 3.8 in Subsection 3.5.

For the convenience of the reader we also recall in this section the notion of the spectrum of a matrix as well as some basic differential geometric concepts such as the notions of an immersion, of a submanifold, of the unique projection on a nonempty set, and of the tangent space; see Definitions 3.1, 3.2, 3.3, 3.4, 3.5, and 3.9.

## 3.1 Immersions

**Definition 3.1 (Immersion).** Let $\mathfrak{d}, k \in \mathbb{N}, n \in \mathbb{N} \cup \{\infty\}$ and let $U \subseteq \mathbb{R}^k$ be open. Then we say that $\varphi$ is a $C^n$-immersion from $U$ to $\mathbb{R}^n$ if and only if we have that

(i) it holds that $\varphi \in C^n(U, \mathbb{R}^n)$ and

(ii) it holds for all $x \in U$ that $\text{rank}(\varphi'(x)) = k$.

## 3.2 Submanifolds of Euclidean spaces

**Definition 3.2 (Submanifold).** Let $\mathfrak{d}, k \in \mathbb{N}, n \in \mathbb{N} \cup \{\infty\}$. Then we say that $\mathcal{M}$ is a $k$-dimensional $C^n$-submanifold of $\mathbb{R}^n$ if and only if it holds for all $x \in \mathcal{M}$ that there exist $U \in \{V \subseteq \mathbb{R}^k : V$ is open$\}, \varepsilon \in (0, \infty), \varphi \in C(U, \mathbb{R}^n)$ such that
(i) it holds that \( M \subseteq \mathbb{R}^3 \),
(ii) it holds that \( \varphi \) is a \( C^1 \)-immersion from \( U \) to \( \mathbb{R}^3 \),
(iii) it holds that \( \varphi(U) = M \cap \{ y \in \mathbb{R}^3 : \| x - y \| < \varepsilon \} \), and 
(iv) it holds that \( U \ni y \mapsto \varphi(y) \in \varphi(U) \) is a homeomorphism (cf. Definitions 2.4 and 3.1).

3.3 Nonlinear projections

**Definition 3.3.** Let \( d \in \mathbb{N} \) and let \( M \subseteq \mathbb{R}^3 \) satisfy \( M \neq \emptyset \). Then we denote by \( \mathcal{P}_M \subseteq \mathbb{R}^d \) the set given by
\[
\mathcal{P}_M = \{ x \in \mathbb{R}^d : (\exists z \in M : \| x - y \| = \inf_{z \in M} \| x - z \|) \} \tag{3.1}
\]
and we denote by \( \mathcal{P}_M : \mathcal{P}_M \rightarrow \mathbb{R} \) the function which satisfies for all \( x \in \mathcal{P}_M \) that \( \mathcal{P}_M(x) \in M \) and
\[
\| x - \mathcal{P}_M(x) \| = \inf_{y \in M} \| x - y \| \tag{3.2}
\]
(cf. Definition 2.4).

**Definition 3.4.** Let \( d, k \in \mathbb{N} \), let \( M \subseteq \mathbb{R}^3 \) be a \( k \)-dimensional \( C^2 \)-submanifold of \( \mathbb{R}^3 \), and assume \( M \neq \emptyset \) (cf. Definition 3.2). Then we denote by \( \mathcal{P}_M \subseteq \mathbb{R}^d \) the set given by
\[
\mathcal{P}_M = \cup_{U \subseteq \mathbb{R}^d \text{ is open}, \cup_{\mathcal{P}_M, U} U \text{ and } \mathcal{P}_M(U) \in C^1(U, \mathbb{R}^3)} \tag{3.3}
\]
(cf. Definition 3.3).

3.4 Tangent spaces associated to submanifolds of Euclidean spaces

**Definition 3.5.** Let \( d \in \mathbb{N} \), let \( M \subseteq \mathbb{R}^3 \) be a set, and let \( x \in M \). Then we denote by \( \mathcal{T}^d_M \) the set given by
\[
\mathcal{T}^d_M = \{ v \in \mathbb{R}^d : (\exists \gamma \in C^1(\mathbb{R}, \mathbb{R}^d) : (\gamma(\mathbb{R}) \subseteq \mathcal{M}) \land \gamma(0) = x \land \gamma'(0) = v) \} \}. \tag{3.4}
\]

**Lemma 3.6.** Let \( d, k \in \mathbb{N} \), let \( M \) be a \( k \)-dimensional \( C^1 \)-submanifold of \( \mathbb{R}^3 \), and let \( x \in M \) (cf. Definition 3.2). Then it holds that \( \mathcal{T}^d_M \) is a \( k \)-dimensional vector subspace of \( \mathbb{R}^d \).

**Proof of Lemma 3.6.** Observe that, e.g., [14, Chapter 2] ensures that \( \mathcal{T}^d_M \) is a \( k \)-dimensional vector subspace of \( \mathbb{R}^d \). The proof of Lemma 3.6 is thus complete. \( \square \)

**Lemma 3.7.** Let \( d, k \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^3 \) be open, let \( f \in C^2(U, \mathbb{R}) \), let \( M \subseteq U \) satisfy \( M = \{ x \in U : (\nabla f)(x) = 0 \} \), assume that \( M \) is a \( k \)-dimensional \( C^1 \)-submanifold of \( \mathbb{R}^3 \), and let \( x \in M \) (cf. Definition 3.2). Then
(i) it holds for all \( v \in \mathcal{T}^d_M \) that \( ((\text{Hess } f)(x))v = 0 \) and
(ii) it holds that \( \text{rank}((\text{Hess } f)(x)) \leq d - k \)
(cf. Definition 3.5).

**Proof of Lemma 3.7.** Throughout this proof let \( v \in \mathcal{T}^d_M \) and let \( \gamma \in C^1(\mathbb{R}, \mathcal{M}) \) satisfy \( \gamma(0) = x \) and \( \gamma'(0) = v \) (cf. Definition 3.5). Note that the fact that for all \( y \in M \) it holds that \( (\nabla f)(y) = 0 \) proves that
\[
0 = \frac{d}{dt}(\nabla f)(\gamma(t))|_{t=0} = ((\text{Hess } f)(x))\gamma'(0) = ((\text{Hess } f)(x))v. \tag{3.5}
\]
Hence, we obtain for all \( v \in \mathcal{T}^d_M \) that \( ((\text{Hess } f)(x))v = 0 \). Lemma 3.6 therefore demonstrates that \( \text{rank}((\text{Hess } f)(x)) \leq d - k \). The proof of Lemma 3.7 is thus complete. \( \square \)
3.5 Necessary and sufficient conditions for local extremum and saddle points

Lemma 3.8. Let \( d \in \mathbb{N} \), let \( U \subseteq \mathbb{R}^d \) be open, let \( f \in C^2(U, \mathbb{R}) \) have locally Lipschitz continuous derivatives, and let \( K \subseteq U \) be compact. Then there exists \( c \in \mathbb{R} \) such that for all \( x, y \in K \) with \( U_{t \in [0,1]}((1-t)x+ty) \subseteq K \) it holds that

\[
|f(y) - f(x) - \langle (\nabla f)(x), y-x \rangle - \frac{1}{2} \langle y-x, (\text{Hess } f)(x)(y-x) \rangle| \leq c \|x-y\|^3 \tag{3.6}
\]

(cf. Definition 2.4).

Proof of Lemma 3.8. Throughout this proof let \( \varphi_{x,y} : \mathbb{R} \to \mathbb{R} \), \( x, y \in K \), satisfy for all \( x, y \in K \), \( t \in [0,1] \) with \( (1-t)x + ty \in K \) that \( \varphi_{x,y}(t) = f((1-t)x + ty) \). Observe that Lebesgue’s number lemma and the assumption that \( f \in C^2(U, \mathbb{R}) \) has locally Lipschitz continuous derivatives ensure that there exists \( L \in \mathbb{R} \) which satisfies for all \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in K \), \( i, j \in \{1, 2, \ldots, d\} \) that

\[
|\langle \frac{\partial^2}{\partial x_i \partial x_j} f \rangle(x) - \langle \frac{\partial^2}{\partial y_i \partial y_j} f \rangle(y)| \leq L \|x-y\| \tag{3.7}
\]

(cf. Definition 2.4). Moreover, note that the chain rule and the assumption that \( f \in C^2(U, \mathbb{R}) \) ensure that for all \( x, y \in K \), \( t \in [0,1] \) with \( (1-t)x + ty \in K \) it holds that \( \varphi_{x,y} \) is twice continuously differentiable at \( t \). Taylor’s theorem hence proves that for all \( x, y \in K \) with \( U_{t \in [0,1]}((1-t)x+ty) \subseteq K \) it holds that

\[
\varphi_{x,y}(1) = \varphi_{x,y}(0) + (\varphi_{x,y})'(0) + \int_0^1 (1-t) ((\varphi_{x,y})''(t)) \, dt. \tag{3.8}
\]

In addition, observe that the chain rule shows that for all \( x, y \in K \), \( t \in [0,1] \) with \( (1-t)x + ty \in K \) it holds that

\[
(\varphi_{x,y})'(t) = \langle (\nabla f)((1-t)x + ty), y-x \rangle \tag{3.9}
\]

and

\[
(\varphi_{x,y})''(t) = \langle y-x, (\text{Hess } f)((1-t)x + ty)(y-x) \rangle. \tag{3.10}
\]

This, (3.7), the Cauchy-Schwarz inequality, and the fact that for all \( A = (A_{i,j})_{i,j \in \{1,2,\ldots,d\}} \in \mathbb{R}^{d \times d} \), \( x \in \mathbb{R}^d \) it holds that \( \|Ax\| \leq (\sum_{i=1}^{d} \sum_{j=1}^{d} |A_{i,j}|^2)^{1/2} \|x\| \) establish that for all \( x, y \in K \), \( t \in [0,1] \) with \( (1-t)x + ty \in K \) it holds that

\[
|(\varphi_{x,y})''(t) - (\varphi_{x,y})''(0)| = ||y-x, [(\text{Hess } f)((1-t)x + ty) - (\text{Hess } f)(x)](y-x)|| \leq \|x-y\| \|[(\text{Hess } f)((1-t)x + ty) - (\text{Hess } f)(x)](y-x)|| \leq \|x-y\|^3 \tag{3.11}
\]

Combining this, (3.8), (3.9), and (3.10) demonstrates that for all \( x, y \in K \) with \( U_{t \in [0,1]}((1-t)x+ty) \subseteq K \) it holds that

\[
|f(y) - f(x) - \langle (\nabla f)(x), y-x \rangle - \frac{1}{2} \langle y-x, (\text{Hess } f)(x)(y-x) \rangle| \\
= |\varphi_{x,y}(1) - \varphi_{x,y}(0) - (\varphi_{x,y})'(0) - \frac{1}{2} (\varphi_{x,y})''(0)| = \left| \int_0^1 (\varphi_{x,y})''(t)(1-t) \, dt - \frac{1}{2} (\varphi_{x,y})''(0) \right| \tag{3.12} \\
= \left| \int_0^1 [(\varphi_{x,y})''(t) - (\varphi_{x,y})''(0)](1-t) \, dt \right| \leq \|x-y\|^3 \int_0^1 (1-t) \, dt = \frac{1}{2} \|x-y\|^3.
\]

The proof of Lemma 3.8 is thus complete. \( \square \)

Definition 3.9. Let \( d \in \mathbb{N}, A \in \mathbb{R}^{d \times d} \). Then we denote by \( \sigma(A) \subseteq \mathbb{R} \) the set given by

\[
\sigma(A) = \{ \lambda \in \mathbb{R} : (\exists v \in \mathbb{R}^d \setminus \{0\} : Av = \lambda v) \}. \tag{3.13}
\]
Lemma 3.10. Let $\mathfrak{d}, k \in \mathbb{N}$, let $U \subseteq \mathbb{R}^d$ be open, let $f \in C^2(U, \mathbb{R})$ have locally Lipschitz continuous derivatives, let $M \subseteq U$ satisfy $M = \{x \in U: (\nabla f)(x) = 0\}$, assume that $M$ is a $k$-dimensional $C^2$-submanifold of $\mathbb{R}^d$, and let $x \in M$ satisfy $\text{rank}((\text{Hess } f)(x)) = \mathfrak{d} - k$ and $\sigma((\text{Hess } f)(x)) \subseteq [0, \infty)$ (cf. Definitions 3.2 and 3.9). Then it holds that $x$ is a local minimum point of $f$ (cf. Definition 2.8).

Proof of Lemma 3.10. Throughout this proof let $\mathbb{R}^{R, \mathfrak{d}} \subseteq \mathbb{R}^d$, $R, \mathfrak{d} \in (0, \infty)$, satisfy for all $R, \mathfrak{d} \in (0, \infty)$ that

$$\mathbb{R}^{R, \mathfrak{d}} = \{z \in \mathbb{R}^d : \exists y \in M \cap \{w \in \mathbb{R}^d : \|x - w\| \leq R\} : (\exists v \in (T_M y)^{\perp} \cap \{w \in \mathbb{R}^d : \|w\| < \mathfrak{d} : z = y + v\})\}$$

(3.14)

(cf. Definitions 2.4 and 3.5). In the following we distinguish between the case $k = \mathfrak{d}$ and the case $k < \mathfrak{d}$. We first prove in the case

$$k = \mathfrak{d}$$

(3.15)
that $x$ is a local minimum point of $f$. Observe that the assumption that $\text{rank}((\text{Hess } f)(x)) = \mathfrak{d} - k$ and (3.16) demonstrate that $\text{rank}((\text{Hess } f)(x)) > 0$. Moreover, note that the assumption that $f \in C^2(U, \mathbb{R})$ ensures that for all $y \in U$ it holds that $(\text{Hess } f)(y)$ is symmetric. Lemma 3.6 and item (i) in Lemma 3.7 therefore establish that there exist $\lambda_i : U \rightarrow \mathbb{R}, i \in \{1, 2, \ldots, \mathfrak{d}\}$, and $v_i : U \rightarrow \mathbb{R}^d$, $i \in \{1, 2, \ldots, \mathfrak{d}\}$, which satisfy that

(i) it holds for all $y \in U$ that \(\{v_{\mathfrak{d} - k + 1}(y), v_{\mathfrak{d} - k + 2}(y), \ldots, v_{\mathfrak{d}}(y)\}\) is a Hamel basis of $T_M y$,

(ii) it holds for all $y \in U$, $i \in \{1, 2, \ldots, \mathfrak{d}\}$ that $(\text{Hess } f)(y)v_i(y) = \lambda_i(y)v_i(y)$, and

(iii) it holds for all $y \in U$, $i, j \in \{1, 2, \ldots, \mathfrak{d}\}$ that

$$\langle v_i(y), v_j(y) \rangle = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$$

(3.17)

Observe that items (i), (ii), and (iii), the assumption that $\text{rank}((\text{Hess } f)(x)) = \mathfrak{d} - k$, and item (i) in Lemma 3.7 show that $\min_{i \in \{1, 2, \ldots, \mathfrak{d} - k\}} \lambda_i(x) > 0$. Moreover, note that, e.g., [18, Proposition 4.5] ensures that $M \subseteq P_M$. This, the fact that $P_M \subseteq \mathbb{R}^d$ is open, the fact that $\min_{i \in \{1, 2, \ldots, \mathfrak{d} - k\}} \lambda_i(x) > 0$, and the fact that the eigenvalues depend continuously on a matrix (cf., e.g., Kato [21, Theorem 5.2]) demonstrate that there exist $\varepsilon \in (0, \infty)$ and an open set $W \subseteq P_M$ which satisfy that

(a) it holds that $x \in W$,

(b) it holds for all $y \in M \cap W$ that $\min_{i \in \{1, 2, \ldots, \mathfrak{d} - k\}} \lambda_i(y) > \varepsilon$,

(c) it holds that $M \cap W$ is connected,

(d) it holds that $\overline{W}$ is compact, and
(e) it holds that $\overline{W} \subseteq U$

(cf. Definition 3.4). Observe that items (i) and (iii) ensure that for all $y \in M \cap W$, $v \in (T^I_M)\perp$ there exist $u_1, \ldots, u_{d-k} \in \mathbb{R}$ such that $v = \sum_{i=1}^{d-k} u_i v_i(y)$. This implies that for all $y \in M \cap W$, $v \in (T^I_M)\perp$ there exist $u_1, \ldots, u_{d-k} \in \mathbb{R}$ such that

$$\langle v, \text{Hess } f(y) v \rangle = \sum_{i=1}^{d-k} \sum_{j=1}^{d-k} u_i u_j \lambda_j(y) \langle v_i(y), v_j(y) \rangle = \sum_{i=1}^{d-k} \lambda_i(y) |u_i|^2 \geq \varepsilon \sum_{i=1}^{d-k} |u_i|^2 = \varepsilon \sum_{i=1}^{d-k} \sum_{j=1}^{d-k} u_i u_j \langle v_i(y), v_j(y) \rangle = \varepsilon \|v\|^2. \quad (3.18)$$

Next note that Lemma 3.8 (applied with $K \cap \overline{W}$ in the notation of Lemma 3.8) proves that there exists $c \in (0, \infty)$ which satisfies for all $y, z \in \overline{W}$ with $\cup_{t \in [0,1]} \{(1-t)y + tz \} \subseteq \overline{W}$ that

$$|f(y) - f(z) - \langle (\nabla f)(z), y - z \rangle - \frac{1}{2} \|y-z\|^2, (\text{Hess } f)(z)(y-z)\rangle| \leq c\|y-z\|^3. \quad (3.19)$$

Furthermore, observe that the assumption that $M \cap W$ is connected and the fact that for all $y \in M$ it holds that $(\nabla f)(y) = 0$ show that for all $y \in M \cap W$ it holds that $f(y) = f(x)$. Next note that [18, Proposition 4.10] ensures that there exist $R, \delta \in (0, \infty)$ which satisfy for all $y \in M \cap \{w \in \mathbb{R}^3 \mid \|x-w\| \leq R\}$, $v \in (T^I_M)\perp \cap \{w \in \mathbb{R}^3 \mid \|w\| < \delta\}$ that $p_M(y+v) = y$, $\delta < \frac{c}{2\varepsilon}$, $B^{R,\delta} \subseteq W$, and

$$B^{R,\delta} = \{a \in \mathbb{R}^3 \mid \inf_{b \in M} \|a-b\| = \inf_{b \in M \cap \{w \in \mathbb{R}^3 \mid \|x-w\| \leq R\}} \|a-b\| < \delta\} \quad (3.20)$$

(cf. Definition 3.3). Observe that (3.20) implies that for all $y \in B^{R,\delta}$, $t \in [0,1]$ it holds that $y - p_M(y) \in (T^I_M)p_M(y)\perp$ and $p_M(y) + t(y - p_M(y)) \in B^{R,\delta}$. Combining this with (3.18), (3.19), the fact that for all $y \in B^{R,\delta}$ it holds that $(\nabla f)(p_M(y)) = 0$, and the fact that for all $y \in B^{R,\delta}$ it holds that $f(p_M(y)) = f(x)$ establishes that for all $y \in B^{R,\delta}$ it holds that

$$f(y) \geq f(p_M(y)) + \langle (\nabla f)(p_M(y)), y - p_M(y) \rangle + \frac{1}{2} \|y - p_M(y), (\text{Hess } f)(p_M(y))(y - p_M(y))\rangle - c\|y - p_M(y)\|^3 \geq f(x) + \frac{c}{2}\|y - p_M(y)\|^2 - c\|y - p_M(y)\|^3 \geq f(x) + \|y - p_M(y)\|^2 \left(\frac{c}{2} - c\delta\right) \geq f(x). \quad (3.21)$$

This proves in the case $k < d$ that $x$ is a local minimum point of $f$. The proof of Lemma 3.10 is thus complete.

**Lemma 3.11.** Let $d \in \mathbb{N}$, let $U \subseteq \mathbb{R}^3$ be open, let $f \in C^2(U, \mathbb{R})$, $x \in U$, and assume that $x$ is a local minimum point of $f$ (cf. Definition 2.8). Then

$$\sigma(\text{Hess } f(x)) \subseteq [0, \infty) \quad (3.22)$$

(cf. Definition 3.9).

**Proof of Lemma 3.11.** We prove (3.22) by contradiction. In the following we thus assume that there exist $\lambda \in (-\infty, 0)$, $v \in \mathbb{R}^3 \setminus \{0\}$ which satisfy

$$(\text{Hess } f(x))v = \lambda v. \quad (3.23)$$

Note that the assumption that $U \subseteq \mathbb{R}^3$ is open and the assumption that $f \in C^2(U, \mathbb{R})$ ensure that there exist $\varepsilon \in (0, \infty)$, $g \in C^2((-\varepsilon, \varepsilon), \mathbb{R})$ which satisfy for all $t \in (-\varepsilon, \varepsilon)$ that

$$x + tv \in U \quad \text{and} \quad g(t) = f(x + tv). \quad (3.24)$$
Observe that (3.24), the assumption that $f \in C^2(U, \mathbb{R})$, and the chain rule show that for all $t \in (-\varepsilon, \varepsilon)$ it holds that
\[ g'(t) = \langle v, (\nabla f)(x + tv) \rangle \quad \text{and} \quad g''(t) = \langle v, (\Hess f)(x + tv)v \rangle \]  \quad (3.25)
(cf. Definition 2.4). Moreover, note that the assumption that $x$ is a local minimum point of $f$ demonstrates that $(\nabla f)(x) = 0$. Combining this, (3.23), and (3.25) establishes that
\[ g'(0) = \langle v, (\nabla f)(x) \rangle = 0 \quad \text{and} \quad g''(0) = \langle v, (\Hess f)(x)v \rangle = \langle v, \lambda v \rangle = \lambda \|v\|^2 < 0. \]  \quad (3.26)
Hence, we obtain that there exists $\delta \in (0, \varepsilon)$ which satisfies for all $t \in (-\delta, \delta)$ that
\[ g''(t) < g''(0)/2. \]  \quad (3.27)
Observe that (3.26), (3.27), and the fundamental theorem of calculus establish that for all $t \in (-\delta, \delta) \{0\}$ it holds that
\[ g(t) = g(0) + \int_0^t g'(s) \, ds = g(0) + \int_0^t (g'(0) + \int_0^s g''(r) \, dr) \, ds \]
\[ = g(0) + \int_0^t \int_0^s g''(r) \, dr \, ds < g(0). \]  \quad (3.28)
Therefore, we obtain that for all $t \in (-\delta, \delta) \{0\}$ it holds that $f(x + tv) = g(t) < g(0) = f(x)$. This is a contradiction to the assumption that $x$ is a local minimum point of $f$. The proof of Lemma 3.11 is thus complete.

\[ \square \]

**Proposition 3.12.** Let $\mathfrak{d}, k \in \mathbb{N}$, let $U \subseteq \mathbb{R}^p$ be open, let $f \in C^2(U, \mathbb{R})$ have locally Lipschitz continuous derivatives, let $\mathcal{M} \subseteq U$ satisfy $\mathcal{M} = \{x \in U : (\nabla f)(x) = 0\}$, assume that $\mathcal{M}$ is a $k$-dimensional $C^2$-submanifold of $\mathbb{R}^p$, and let $x \in \mathcal{M}$ satisfy $\text{rank}((\Hess f)(x)) \geq \mathfrak{d} - k$ (cf. Definition 3.2). Then

(i) it holds that $\text{rank}((\Hess f)(x)) = \mathfrak{d} - k$,

(ii) it holds that $x$ is a local minimum point of $f$ if and only if $\sigma((\Hess f)(x)) \subseteq [0, \infty)$,

(iii) it holds that $x$ is a local maximum point of $f$ if and only if $\sigma((\Hess f)(x)) \subseteq (-\infty, 0)$, and

(iv) it holds that $x$ is a saddle point of $f$ if and only if
\[ \min\{\#(\sigma((\Hess f)(x)) \cap (0, \infty)) \#(\sigma((\Hess f)(x)) \cap (-\infty, 0))\} > 0 \]  \quad (3.29)
(cf. Definitions 2.8, 2.9, 2.11, and 3.9).

**Proof of Proposition 3.12.** Note that Lemma 3.7 and the assumption that $\text{rank}((\Hess f)(x)) \geq \mathfrak{d} - k$ demonstrate that $\text{rank}((\Hess f)(x)) = \mathfrak{d} - k$. This establishes item (i). Observe that item (i) and Lemma 3.10 prove that
\[ (\sigma((\Hess f)(x)) \subseteq [0, \infty]) \rightarrow [x \text{ is a local minimum point of } f]. \]  \quad (3.30)
Moreover, note that Lemma 3.11 establishes that
\[ ([x \text{ is a local minimum point of } f] \rightarrow [\sigma((\Hess f)(x)) \subseteq [0, \infty])]. \]  \quad (3.31)
Combining this and (3.30) establishes item (ii). Observe that item (i) and Lemma 3.10 (applied with $f \rightsquigarrow -f$ in the notation of Lemma 3.10) ensure that
\[ (\sigma((\Hess f)(x)) \subseteq (-\infty, 0]) \rightarrow [x \text{ is a local maximum point of } f]. \]  \quad (3.32)
In addition, note that Lemma 3.11 (applied with $f \rightsquigarrow -f$ in the notation of Lemma 3.11) demonstrates that
\[ ([x \text{ is a local maximum point of } f] \rightarrow [\sigma((\Hess f)(x)) \subseteq (-\infty, 0])]. \]  \quad (3.33)
This and (3.32) establish item (iii). Observe that items (ii) and (iii) prove item (iv). The proof of Proposition 3.12 is thus complete.

\[ \square \]
4 On infinitely many realization functions of non-global local minimum points

In this section we employ Proposition 3.12 from Section 3 above to establish in Corollary 4.9 in Subsection 4.6 below that there exists a Lipschitz continuous target function such that the associated risk function has infinitely many realization functions of non-global local minimum points. Corollary 4.9 is a simple consequence of Corollary 4.8 in Subsection 4.6. Our proof of Corollary 4.8, in turn, makes use of Lemma 4.2 in Subsection 4.2 below, Lemma 4.6 in Subsection 4.4 below, and Lemma 4.7 in Subsection 4.5 below.

In Lemma 4.2 we calculate the risks for suitable ANN realizations, in Lemma 4.6 we establish some properties of the Hessian matrix of the risk function at points in \( M \subseteq \mathbb{R}^p \), and in Lemma 4.7 we prove that every \( \theta \in \mathcal{M} \) has the same risk value and is a critical point of \( R^\theta_r : \mathbb{R}^p \to \mathbb{R}^p \). In our proof of Lemma 4.7 we use Lemma 4.4, whose proof is partially inspired by [18, Item (ii) in Lemma 2.15]. Our proof of Lemma 4.6 uses well-known rank properties presented in Lemma 4.5 in Subsection 4.4, whose proof can be found, e.g., in [23, Chapter 2]. Some of the computations in Lemma 4.6 were aided by WOLFRAM MATHEMATICA (see [35]).

In Setting 4.1 in Subsection 4.1 below we introduce the mathematical objects considered in this section such as the realization functions \( N_r^\theta \in C([a, \theta], \mathbb{R}) \), \( r \in \mathbb{N} \cup \{ \infty \} \), \( \theta \in \mathbb{R}^p \), the risk functions \( R^\theta_r : \mathbb{R}^p \to \mathbb{R}^p \), \( r \in \mathbb{N} \cup \{ \infty \} \), \( f \in C([a, \theta], \mathbb{R}) \), the specific target function \( f \in C([a, \theta], \mathbb{R}) \), the \( (d-2) \)-dimensional \( C^\infty \)-submanifold \( \mathcal{M} \) of \( \mathbb{R}^p \) (see Lemma 4.3), and the realization functions \( M \subseteq C([a, \theta], \mathbb{R}) \) associated to \( \mathcal{M} \subseteq \mathbb{R}^p \).

In Figure 1 in Subsection 4.6 we present numerical simulations associated to Corollary 4.8 in the case where \( H = 4, d = 13, a = 0, b = 1, \alpha = 1/3 \), and \( \beta = 2/3 \). In these simulations we randomly initialize 50 ANNs with the Xavier initialization, then we approximately train these ANNs with the GD optimization method using a learning rate of \( 1/20 \) until the maximum norm of the generalized gradient function evaluated at the current position of the GD process is strictly less than \( 10^{-4} \), and, thereafter, we gradually plot the realization functions of the resulting ANNs whereby a realization function is not drawn if a realization function with an \( L^2 \)-distance strictly less than \( 10^{-4} \) has already been drawn. We also refer to Listing 1 for the PYTHON source code used to create Figure 1.

4.1 ANNs with one-dimensional input and multidimensional hidden layer

Setting 4.1. Let \( H, d \in \mathbb{N}, \alpha, \beta \in \mathbb{R}, \theta \in (\alpha, \infty) \) satisfy \( d = 3H + 1 \) and \( 0 < \alpha < \beta < 1 \), let \( A_r : \mathbb{R} \to \mathbb{R} \), \( r \in \mathbb{N} \cup \{ \infty \} \), satisfy for all \( x \in \mathbb{R} \) that \( \bigcup_{r \in \mathbb{N}} \{ A_r \} \subseteq C^1(\mathbb{R}, \mathbb{R}) \), \( A_\infty(x) = \max\{x, 0\} \), \( \sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(A_r)'(y)| < \infty \), and

\[
\limsup_{r \to \infty} \left( |A_r(x) - A_\infty(x)| + |(A_r)'(x) - 1_{(0,\infty)}(x)| \right) = 0, \tag{4.1}
\]

let \( N_r^\theta \in C([\alpha, \theta], \mathbb{R}) \), \( r \in \mathbb{N} \cup \{ \infty \} \), \( \theta \in \mathbb{R}^p \), and \( R^\theta_r : \mathbb{R}^p \to \mathbb{R} \), \( r \in \mathbb{N} \cup \{ \infty \} \), \( f \in C([a, \theta], \mathbb{R}) \), \( \theta = (\theta_1, \ldots, \theta_6) \in \mathbb{R}^6 \), \( x \in [a, \theta] \) that \( N_r^\theta(x) = \theta_6 + \sum_{j=1}^H \theta_{2H + j} A_r(\theta_{H+j} + \theta_j x) \) and \( R^\theta_r(\theta) = \int_a^\theta (N_r^\theta(y) - f(y))^2 \, dy \), let \( f \in C([0, 1], \mathbb{R}) \) satisfy for all \( x \in [0, 1] \) that

\[
f(x) = \begin{cases} 
\frac{4(x-\alpha) + 3\alpha^2 - 1}{4(1-\alpha)^1/2(1+3\alpha)^{1/2}} & : x \in [0, \alpha] \\
\frac{3x^2 - 1}{4(1-x)^1/2(1+3x)^{1/2}} & : x \in (\alpha, \beta] \\
\frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^1/2(1+3\beta)^{1/2}} & : x \in (\beta, 1].
\end{cases}
\tag{4.2}
\]
let $f \in C([\alpha, \beta], \mathbb{R})$ satisfy for all $x \in [\alpha, \beta]$ that $f(x) = f\left(\frac{\beta - x}{\beta - \alpha}\right)$, let $M \subseteq \mathbb{R}^d$ satisfy

$$
M = \left\{ \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d : \exists (x, y) \in (\alpha, \beta) \times (0, \infty) : \\
\left( \forall j \in \{2, 3, \ldots, H\} : \max\{\theta_j \alpha + \theta_{H+j}, \theta_j \beta + \theta_{H+j}\} < 0 \right) \wedge (\theta_1 = \frac{y}{\beta - \alpha}) \right\}. 
$$

(4.3)

let $M \subseteq C([\alpha, \beta], \mathbb{R})$ satisfy $M = \{v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in M : \psi = N^\theta\}$, and let $(\theta_0)_{\theta \in M} \subseteq \mathbb{R}$, $(\theta_2)_{\theta \in M} \subseteq \mathbb{R}$ so that for all $\theta = (\theta_1, \ldots, \theta_2) \in M$ that $\theta_1 = \theta_2/\theta_1$ and $\theta_2 = (\theta_2 - \alpha)/\theta_2 - \alpha$.

4.2. Risks for suitable ANN realization functions

**Lemma 4.2.** Assume Setting 4.1, let $q \in (\alpha, \beta)$, and let $N \in C([0, 1], \mathbb{R})$ satisfy for all $x \in [0, 1]$ that

$$
N(x) = -\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} + \frac{\max\{x-q,0\}}{2(1-q)^{1/2}(1+3q)^{1/2}} = \frac{2\max\{x-q,0\} - (1-q)^2}{4(1-q)^{1/2}(1+3q)^{1/2}}.
$$

(4.4)

Then

(i) it holds that

$$
\int_0^q (N(x) - f(x)) \, dx = \int_q^1 (N(x) - f(x)) \, dx = \int_q^1 x(N(x) - f(x)) \, dx = 0
$$

(4.5)

and

(ii) it holds that

$$
\int_0^q (N(x) - f(x))^2 \, dx = \int_0^1 [f(x)]^2 \, dx - \frac{1}{4x^2}
$$

(4.6)

**Proof of Lemma 4.2.** Note that the chain rule ensures that for all $x \in (0, 1)$ it holds that

$$
\left[ -\frac{x(1-x)^{1/2}}{4(1+3x)^{1/2}} \right]' = -\frac{(1-x)^{1/2} - (x/2)(1-x)^{-1/2}[(1+3x)^{1/2} - (x-1)^{1/2}]}{4(1+3x)}
$$

$$
= -\frac{(2(1-x) - x)(1+3x - 3x(1-x))}{8(1-x)^{1/2}(1+3x)^{1/2}} = \frac{3x^2 - 1}{4(1-x)^{1/2}(1+3x)^{1/2}}
$$

(4.7)

and

$$
\left[ -\frac{(3x^2 + 2x + 1)(1-x)^{1/2}}{24(1+3x)^{1/2}} \right]' = -\frac{(6x + 2)(1-x)^{1/2} - (3x^2 + 2x + 1)(1/2)(1-x)^{-1/2}[(1+3x)^{1/2} - (3x^2 + 2x + 1)(1-x)^{1/2}]}{24(1+3x)}
$$

$$
= -\frac{(6x + 2)(2-2x - 3(x^2 + 2x + 1)(1+3x - (3x^2 + 2x + 1)(1-3x))}{48(1-x)^{1/2}(1+3x)^{1/2}} = \frac{(3x + 1)(2(1-x) - (3x^2 + 2x + 1))}{12(1-x)^{1/2}(1+3x)^{3/2}}
$$

(4.8)

Hence, we obtain that

$$
\int_0^q (N(x) - f(x)) \, dx = \int_0^q (N(x) - f(x)) \, dx + \int_0^q (N(x) - f(x)) \, dx
$$

$$
= \int_0^x \left( -\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} - \frac{4x^2 - (4x^2 - 3x^2 - 4x - 1)}{4(1-x)^{1/2}(1+3x)^{1/2}} \right) \, dx + \int_q^1 \left( -\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} - \frac{3x^2 - 1}{4(1-x)^{1/2}(1+3x)^{1/2}} \right) \, dx
$$

$$
= \left[ -\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} - \frac{4x^2 - (4x^2 - 3x^2 - 4x - 1)}{4(1-x)^{1/2}(1+3x)^{1/2}} \right] x = a
$$

$$
= \int_q^1 \left( -\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} - \frac{3x^2 - 2x^2 - 2x}{4(1-x)^{1/2}(1+3x)^{1/2}} \right) \, dx = 0.
$$

(4.9)
\[
f_q^1(\mathcal{N}(x) - f(x)) \, dx
= f_q^\beta(\mathcal{N}(x) - f(x)) \, dx + \int_q^1 f_q^\beta(\mathcal{N}(x) - f(x)) \, dx
= f_q^\beta \left( \frac{2x^2 - 1}{(1-q)^{3/2}(1+3q)^{1/2}} \right) - \left( \frac{3x^2 - 1}{(1-x)^{1/2}} \right) \, dx
+ \int_q^1 f_q^\beta \left( \frac{2x^2 - 1}{(1-q)^{3/2}(1+3q)^{1/2}} \right) - \left( \frac{12x^2 - (18x^2 + 8x - 2)x + 3x^4 + 10x^2 - 1}{(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \, dx
= \left[ \frac{x^2 - (1+q^2)x}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=1 \quad x=\beta
+ \left[ \frac{x^2 - (1+q^2)x}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=q
+ \int_q^1 \left[ \frac{x^2 - (1+q^2)x}{4(1-q)^{3/2}(1+3q)^{1/2}} \right] - \left( \frac{12x^2 - (18x^2 + 8x - 2)x + 3x^4 + 10x^2 - 1}{(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \, dx
= \left[ \frac{2\beta^2 - (1+q^2)x}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=1 \quad x=\beta
+ \left[ \frac{2\beta^2 - (1+q^2)x}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=q
- \left( \frac{3\beta^2 - 3\beta^2}{4(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \beta
- \left( \frac{3\beta^2 - 3\beta^2}{4(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \beta
- \left( \frac{q(1-q^2)}{4(1-q)^{1/2}(1+3q)^{1/2}} \right) \beta
+ \left( \frac{q(1-q^2)}{4(1-q)^{1/2}(1+3q)^{1/2}} \right) \beta
= \left( \frac{2(1-\beta)^3(1+3\beta)^3 + (\beta+3\beta^3)(\beta-3\beta^2+3\beta-1)}{4(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) = 0,
\]
and
\[
f_q^1 x(\mathcal{N}(x) - f(x)) \, dx
= \int_q^1 x(\mathcal{N}(x) - f(x)) \, dx - \int_q^1 f_q^\beta x(\mathcal{N}(x) - f(x)) \, dx
= \int_q^1 \left[ \frac{x(2x^2 - 1)}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=1 \quad x=\beta
+ \left[ \frac{x(2x^2 - 1)}{4(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=q
- \int_q^1 \left( \frac{12x^2 - (18x^2 + 8x - 2)x + 3x^4 + 10x^2 - 1}{(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \, dx
= \left[ \frac{x^2 - (1+q^2)x}{24(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=1 \quad x=\beta
+ \left[ \frac{x^2 - (1+q^2)x}{24(1-q)^{3/2}(1+3q)^{1/2}} \right]_x=q
- \left( \frac{3\beta^2 - 3\beta^2}{24(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \beta
- \left( \frac{3\beta^2 - 3\beta^2}{24(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) \beta
- \left( \frac{q(1-q^2)}{24(1-q)^{1/2}(1+3q)^{1/2}} \right) \beta
+ \left( \frac{q(1-q^2)}{24(1-q)^{1/2}(1+3q)^{1/2}} \right) \beta
= \left( \frac{(1-\beta)(1+5\beta + 9\beta^2 + 9\beta^3) + (9\beta^3 - 3\beta^2 - 3\beta - 2)(\beta^2)}{24(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right) = 0.
\]
This establishes item (i). Moreover, observe item (i) shows that
\[
\int_q^1 [\mathcal{N}(x) f(x)] \, dx = \int_q^1 [\mathcal{N}(x)]^2 \, dx - \int_q^1 \mathcal{N}(x)[\mathcal{N}(x) - f(x)] \, dx
= \int_q^1 [\mathcal{N}(x)]^2 \, dx + \frac{(1-q)^{1/2}}{4(1+3q)^{1/2}} \int_q^1 (\mathcal{N}(x) - f(x)) \, dx
- \frac{1}{(1-q)^{3/2}(1+3q)^{1/2}} \int_q^1 f_q^2(2x - 1 - q^2)(\mathcal{N}(x) - f(x)) \, dx
= \int_q^1 [\mathcal{N}(x)]^2 \, dx
\]
and
\[
\int_q^1 [\mathcal{N}(x)]^2 \, dx = \int_q^q [-\frac{(1-q)^{1/2}}{4(1+3q)^{1/2}}]^2 \, dx + \int_q^1 \left[ \frac{2x^2 - 1}{4(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right] \, dx
= \frac{q(1-q)}{16(1+3q)} + \left[ \frac{(2x^2 - 1)^3}{96(1-\beta)^{1/2}(1+3\beta)^{1/2}} \right]_x=1 \quad x=q
= \frac{1}{48}.
\]
Therefore, we obtain that
\[
\int_0^1 (\mathcal{N}(x) - f(x))^2 \, dx = \int_0^1 [f(x)]^2 \, dx - 2 \int_0^1 [\mathcal{N}(x)f(x)] \, dx + \int_0^1 [\mathcal{N}(x)]^2 \, dx = \int_0^1 [f(x)]^2 \, dx - \frac{1}{4},
\]
(4.14)
This establishes item (ii). The proof of Lemma 4.2 is thus complete. □

4.3 On a submanifold of the ANN parameter space

Lemma 4.3. Assume Setting 4.1. Then it holds that \( \mathcal{M} \) is a \((d - 2)\)-dimensional \( C^\infty \)-submanifold of \( \mathbb{R}^d \) (cf. Definition 3.2).

Proof of Lemma 4.3. Throughout this proof let \( z = (z_1, \ldots, z_d) \in \mathcal{M}, \varepsilon \in (0, \infty) \), let \( \mathcal{U} \subseteq \mathbb{R}^{d-2} \) satisfy
\[
\mathcal{U} = \left\{ x = (x_1, \ldots, x_{d-2}) \in \mathbb{R}^{d-2} : \left[ (\alpha < x_1 < \beta) \land (x_2 > 0) \right] \land (\forall j \in \{2, 3, \ldots, H\}: \max\{x_{j+1}\alpha + x_{H+j}, x_{j+1}\beta + x_{H+j}\} < 0) \right\},
\]
(4.15)
let \( U \subseteq \mathbb{R}^{d-2} \) satisfy
\[
U = \left\{ x = (x_1, \ldots, x_{d-2}) \in \mathcal{U} : \left( \exists y = (y_1, \ldots, y_d) \in \mathbb{R}^d : \|y - z\| < \varepsilon \right) \land \left[ y_1 = \frac{x_2}{\beta - \alpha} \right] \land \left[ y_{H+1} = -x_1 x_2 - \frac{x_2 \alpha}{\beta - \alpha} \right] \land \left[ y_{2H+1} = \frac{x_2}{2x_2(1-x_1)^{1/2}(1+3x_1)^{1/2}} \right] \land \left[ y_d = \frac{(1-x_1)^{1/2}}{4(1+3x_1)^{1/2}} \right] \land \left[ \forall j \in \{2, 3, \ldots, H\} : |y_j - x_{j+1}| < \varepsilon \right] \right\},
\]
(4.16)
and let \( \varphi = (\varphi_1, \ldots, \varphi_d) : U \to \mathbb{R}^d \) satisfy for all \( x = (x_1, \ldots, x_{d-2}) \in U \) that
\[
\begin{align*}
\varphi_1(x) &= \frac{x_2}{\beta - \alpha}, & \varphi_{H+1}(x) &= -x_1 x_2 - \frac{x_2 \alpha}{\beta - \alpha}, \\
\varphi_{2H+1}(x) &= \frac{x_2}{2x_2(1-x_1)^{1/2}(1+3x_1)^{1/2}}, & \varphi_d(x) &= \frac{(1-x_1)^{1/2}}{4(1+3x_1)^{1/2}},
\end{align*}
\]
(4.17)
and \( (\forall j \in \{2, 3, \ldots, H\} : (\varphi_j(x) = x_{j+1}) \land (\varphi_{H+j}(x) = x_{H+j}) \land (\varphi_{2H+j}(x) = x_{2H+j}) ) \)
(4.18)
(cf. Definition 2.4). Note that (4.16) assures that \( U \) is open. Next observe that (4.17) and (4.18) ensure that for all \( x = (x_1, \ldots, x_{d-2}) \in U \) it holds that
\[
\frac{d}{dx} (\varphi_{H+1}(x), \varphi_1(x), \ldots, \varphi_H(x), \varphi_{H+2}(x), \ldots, \varphi_{2H}(x), \varphi_{2H+2}(x), \ldots, \varphi_{3H}(x))
\]
\[
= \begin{pmatrix}
-2 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \in \mathbb{R}^{(d-2) \times (d-2)}.
\]
(4.19)
This shows that for all \( x \in U \) it holds that \( \operatorname{rank}(\varphi'(x)) = d - 2 \). Combining this with the fact that \( \varphi \in C^\infty(U, \mathbb{R}^d) \) implies that \( U \ni x \mapsto \varphi(x) \in \mathbb{R}^d \) is a \( C^\infty \)-immersion from \( U \) to \( \mathbb{R}^d \) (cf. Definition 3.1). Next note that (4.16), (4.17), and (4.18) ensure that for all \( x \in U \) it holds that \( \|\varphi(x) - z\| < \varepsilon \). Combining this with (4.15), (4.16), (4.17), and (4.18) assures that
\[
\varphi(U) = \mathcal{M} \cap \{ x \in \mathbb{R}^d : \|x - z\| < \varepsilon \}.
\]
(4.20)
Next observe that (4.17) and (4.18) show that for all \( x = (x_1, \ldots, x_{d-2}), y = (y_1, \ldots, y_{d-2}) \in U \) with \( \varphi(x) = \varphi(y) \) it holds that \( x_1 = y_1, x_2 = y_2, \) and \( \forall j \in \{2, 3, \ldots, H\} : (|x_{j+1} - y_{j+1}| < \varepsilon \)
Proof of Lemma 4.4. Throughout this proof let
\[ \theta = (x, \ldots, \theta) \in \mathbb{R}^n : \left( \prod_{i=1}^{k} \prod_{x \in \{0, \theta \}} \theta(x + \theta) \neq 0 \right) \].

(4.22)

Note that (4.22) shows that \( V \) is open. Moreover, observe that (4.3) ensures that for all \( \theta = (\theta_1, \ldots, \theta_n) \in \mathcal{M} \) there exists \( y \in (0, \infty) \) such that \( \theta_1 \sigma + \theta_2 \sigma + \ldots + \theta_n \sigma = y(1 - q_0) \neq 0 \), and (4.22) demonstrates that \( \mathcal{M} \subseteq V \). Combining this with the fact that \( V \) is open and [18, Item (ii) in Lemma 2.15] proves that \( \mathcal{M} \subseteq \mathcal{V} \). The proof of Lemma 4.4 is thus complete. \( \Box \)

4.4 On the rank of the Hessian of the risk function

Lemma 4.4. Assume Setting 4.1. Then there exists an open \( V \subseteq \mathbb{R}^n \) such that \( \mathcal{M} \subseteq V \) and \( (\mathcal{R}_n^f)^{|V} \subseteq C^2(V, \mathbb{R}) \).

Proof of Lemma 4.4. Throughout this proof let \( V \subseteq \mathbb{R}^n \) satisfy
\[ V = \{ \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n : \left( \prod_{i=1}^{k} \prod_{x \in \{0, \theta \}} \theta(x + \theta) \neq 0 \right) \}. \]

(4.22)

Then \( \text{rank}(\mathfrak{A}) = \text{rank}(A) \).

Proof of Lemma 4.5. Note that, e.g., [23, Chapter 2] ensures that \( \text{rank}(\mathfrak{A}) = \text{rank}(A) \). The proof of Lemma 4.5 is thus complete. \( \Box \)

Lemma 4.6. Assume Setting 4.1, let \( (\mathcal{H})_{\theta \in \mathcal{M}} \subseteq \mathbb{R}^4 \) satisfy for all \( \theta = (\theta_1, \ldots, \theta_n) \in \mathcal{M} \) that
\[
\mathcal{H}_\theta = \begin{pmatrix}
\left( \frac{\partial^2}{\partial \theta_1^2} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_4} \mathcal{R}_\infty^f \right)(\theta) \\
\left( \frac{\partial^2}{\partial \theta_2 \partial \theta_1} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_2^2} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_2 \partial \theta_3} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_2 \partial \theta_4} \mathcal{R}_\infty^f \right)(\theta) \\
\left( \frac{\partial^2}{\partial \theta_3 \partial \theta_1} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_3 \partial \theta_2} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_3 \partial \theta_3} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_3 \partial \theta_4} \mathcal{R}_\infty^f \right)(\theta) \\
\left( \frac{\partial^2}{\partial \theta_4 \partial \theta_1} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_4 \partial \theta_2} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_4 \partial \theta_3} \mathcal{R}_\infty^f \right)(\theta) & \left( \frac{\partial^2}{\partial \theta_4 \partial \theta_4} \mathcal{R}_\infty^f \right)(\theta)
\end{pmatrix},
\]

(4.24)

and let \( \theta = (\theta_1, \ldots, \theta_n) \in \mathcal{M} \) (cf. Lemma 4.4). Then

(i) it holds for all \( x \in [a, b] \) that
\[
\mathcal{L}^\theta_n(x) = -\frac{(1-q_0)^{1/2}}{2(1+3q_0)^{1/2}} + \frac{1}{2(1+3q_0)^{1/2}} \max \left\{ \frac{\epsilon - \delta}{\delta - \epsilon} - q_0, 0 \right\},
\]

(4.25)
(ii) it holds that
\[ \int_{\alpha}^{\beta} (\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = \int_{\alpha}^{\beta} (\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = \int_{\alpha}^{\beta} x(\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = 0, \] (4.26)

(iii) it holds that \( \text{rank}(\mathcal{H}_\theta) = 2 \),

(iv) it holds that \( \sigma(\mathcal{H}_\theta) \subseteq [0, \infty) \),

(v) it holds that \( \text{rank}((\text{Hess} \mathcal{R}_\infty^\beta)(\theta)) = 2 \), and

(vi) it holds that \( \sigma((\text{Hess} \mathcal{R}_\infty^\beta)(\theta)) \subseteq [0, \infty) \)

(cf. Definition 3.9).

Proof of Lemma 4.6. Observe that the fact that for all \( x \in \mathbb{R} \) it holds that \( \mathcal{A}_\infty(x) = \max\{x, 0\} \) ensures that for all \( x \in [\alpha, \beta] \) it holds that
\[ \mathcal{N}_\infty^\beta(x) = \theta_3 + \theta_{2H+1} \max\{\theta_{H+1} + \theta_1, 0\}. \] (4.27)

Next note that (4.3) demonstrates that there exists \( y \in (0, \infty) \) such that
\[ \alpha < q_\theta < \beta, \quad \alpha < q_\theta < \beta, \quad \theta_{H+1} = -q_\theta \theta_1, \quad q_\theta = (\beta - \alpha)q_\theta + \alpha, \quad \theta_1 = \frac{y}{\theta - \alpha}, \] (4.28)
\[ \theta_{H+1} = -\theta_1((\beta - \alpha)q_\theta + \alpha) \quad \text{and} \quad \theta_{2H+1} = \frac{1}{2\theta_1(\theta - \alpha)^{3/2}(1+3q_\theta)^{1/2}}, \] (4.29)

Hence, we obtain that
\[ \theta_{H+1} = -\theta_1((\beta - \alpha)q_\theta + \alpha) \quad \text{and} \quad \theta_{2H+1} = \frac{1}{2\theta_1(\theta - \alpha)^{3/2}(1+3q_\theta)^{1/2}}. \] (4.30)

Combining this with (4.27), (4.28), and (4.29) ensures that for all \( x \in [0, 1] \) it holds that
\[ \mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) = -\frac{(1-q_\theta)^{1/2}}{4(1+3q_\theta)^{1/2}} + \frac{\max\{x-q_\theta, 0\}}{2(1-q_\theta)^{3/2}(1+3q_\theta)^{1/2}}. \] (4.31)

This establishes item (i). Observe that (4.31) assures that for all \( x \in [0, 1] \) it holds that
\[ \mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) = -\frac{(1-q_\theta)^{1/2}}{4(1+3q_\theta)^{1/2}} + \frac{\max\{x-q_\theta, 0\}}{2(1-q_\theta)^{3/2}(1+3q_\theta)^{1/2}}. \] (4.32)

The integral transformation theorem, item (i) in Lemma 4.2, and the fact that \( \alpha < q_\theta < \beta \) hence ensure that
\[ \int_{\alpha}^{\beta} (\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = (\beta - \alpha) \int_{\frac{\alpha}{\beta - \alpha}}^{\frac{\beta - \alpha}{\beta - \alpha}} (\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = (\beta - \alpha) \int_{0}^{\frac{1}{\beta - \alpha}} ((\beta - \alpha)x + \alpha) (\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = 0, \] (4.33)
\[ \int_{\alpha}^{\beta} x(\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = (\beta - \alpha) \int_{0}^{\frac{1}{\beta - \alpha}} ((\beta - \alpha)x + \alpha) x(\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = (\beta - \alpha) \int_{0}^{\frac{1}{\beta - \alpha}} ((\beta - \alpha)x + \alpha) x(\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = 0, \] (4.34)
and
\[ \int_{\alpha}^{\beta} x(\mathcal{N}_\infty^\beta(x) - f(x)) \, dx = (\beta - \alpha)^2 \int_{0}^{\frac{1}{\beta - \alpha}} ((\beta - \alpha)x + \alpha)^2 (\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = (\beta - \alpha)^2 \int_{0}^{\frac{1}{\beta - \alpha}} ((\beta - \alpha)x + \alpha)^2 (\mathcal{N}_\infty^\beta((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha)) \, dx = 0. \] (4.35)
This establishes item (ii). Note that (4.28), (4.29), and item (i) show that
\[
\mathcal{N}_{\infty}^\theta(q_\theta) = -\frac{(1-q_\theta)^{1/2}}{4(1+3q_\theta)^{1/2}} \quad \text{and} \quad f(q_\theta) = f((\ell - \epsilon)q_\theta + \epsilon) = f(q_\theta) = \frac{3[q_{\theta}]^2 - 1}{4(1-q_{\theta})^{1/2}(1+3q_{\theta})^{1/2}}.
\] (4.36)

This implies that
\[
\mathcal{N}_{\infty}^\theta(q_\theta) - f(q_\theta) = -\frac{(1-q_\theta)^{1/2}}{4(1+3q_\theta)^{1/2}} - \frac{3[q_{\theta}]^2 - 1}{4(1-q_{\theta})^{1/2}(1+3q_{\theta})^{1/2}} = \frac{q_\theta(1-q_\theta)^{1/2}}{4(1-q_{\theta})^{1/2}(1+3q_{\theta})^{1/2}} = \frac{q_\theta}{2(1-q_{\theta})^{1/2}(1+3q_{\theta})^{1/2}}. \quad (4.37)
\]

Item (ii), [18, Lemma 2.15], (4.28), (4.29), and (4.30) therefore assure that
\[
\left(\frac{\partial^2}{\partial \theta^2} R_{\infty}^\theta(\theta)\right) = 2[\theta_{2H+1}]^2 \int q_\theta^2 x^2 \, dx - \frac{2[\theta_{2H+1}]^2}{\theta_1^2} [q_{\theta}(\mathcal{N}_{\infty}^\theta(q_\theta) - f(q_\theta))
\[
= \frac{\theta-q_{\theta}}{2[\theta_1]^2(\ell - \epsilon)^2(1-q_{\theta})^3(1+3q_{\theta})^3} - \frac{q_{\theta}}{2[\theta_1]^2(\ell - \epsilon)(1-q_{\theta})^2(1+3q_{\theta})^2} - \frac{1}{2[\theta_1]^2(\ell - \epsilon)(1-q_{\theta})^2(1+3q_{\theta})^2}.$
\] (4.39)

\[
\left(\frac{\partial^2}{\partial \theta_1 \partial \theta^2} R_{\infty}^\theta(\theta)\right) = 2 \int q_\theta^2 (x + \theta_{H+1})^2 \, dx = 2[\theta_1]^2 \int q_\theta (x - q_\theta)^2 \, dx = \frac{2}{3}[\theta_1]^2 (\ell - \epsilon)^3 (1-q_{\theta})^3,
\] (4.40)

\[
\left(\frac{\partial^2}{\partial \theta_1 \partial \theta^2} R_{\infty}^\theta(\theta)\right) = 2 \left[\frac{\partial^2}{\partial \theta^2} q_\theta(x_{\theta_{H+1}})^2 \, dx + \frac{2[\theta_{2H+1}]}{\theta_1} q_\theta(\mathcal{N}_{\infty}^\theta(q_\theta) - f(q_\theta))
\[
= \frac{\theta-q_{\theta}}{4[\theta_1]^2(\ell - \epsilon)^2(1-q_{\theta})^3(1+3q_{\theta})^3} - \frac{[\ell - \epsilon)(1-q_{\theta})]^2}{2[\theta_1]^2(\ell - \epsilon)(1-q_{\theta})^2(1+3q_{\theta})^2} - \frac{1}{2[\theta_1]^2(\ell - \epsilon)(1-q_{\theta})^2(1+3q_{\theta})^2}.$
\] (4.42)
\[
\left(\frac{\partial^2}{\partial \theta^2} \mathcal{R}_\infty^\theta\right)(\theta) = 2 \theta_{2H+1} \int_{q_0}^\theta x(x + \theta_{H+1}) \, dx + 2 \int_{q_0}^\theta x(N_\infty^\theta (x) - f(x)) \, dx \\
= 2 \theta_{2H+1} \theta_1 \int_{q_0}^\theta x(x - q_0) \, dx = 2 \theta_{2H+1} \theta_1 \left(\frac{\beta - [q_0]}{\theta_0} - \frac{\beta^2 - [q_0]^2}{2}\right) \\
= \frac{1}{(\theta - \theta_0)^1/2(1 + \theta_{2H+1}/2)^1/2} \left[2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2\right] (4.43) \\
= \frac{(\theta - \theta_0)(\beta - \theta_0)(\theta_0 - \theta_0^2)(2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2)}{6(\theta - \theta_0)(1 - \theta_{2H+1}/2)^1/2(1 + \theta_{2H+1}/2)^1/2} \\
= \frac{(\theta - \theta_0)(\beta - \theta_0)(\theta_0 - \theta_0^2)(2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2)}{6(\theta - \theta_0)(1 - \theta_{2H+1}/2)^1/2(1 + \theta_{2H+1}/2)^1/2},
\]

\[
\left(\frac{\partial^2}{\partial \theta^2} \mathcal{R}_\infty^\theta\right)(\theta) = 2 \theta_{2H+1} \theta_1 \int_{q_0}^\theta (x - q_0) \, dx = \theta_{2H+1}(\beta - q_0)^2 \\
= \frac{(\theta - \theta_0)(\beta - \theta_0)(\theta_0 - \theta_0^2)(2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2)}{2(1 - \theta_{2H+1}/2)^1/2(1 + \theta_{2H+1}/2)^1/2},
\]

\[
\left(\frac{\partial^2}{\partial \theta^2} \mathcal{R}_\infty^\theta\right)(\theta) = 2 \theta_{2H+1} \int_{q_0}^\theta (x - q_0) \, dx = \theta_{2H+1}(\beta - q_0)^2 \\
= \frac{(\theta - \theta_0)(\beta - \theta_0)(\theta_0 - \theta_0^2)(2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2)}{2(1 - \theta_{2H+1}/2)^1/2(1 + \theta_{2H+1}/2)^1/2},
\]

and

\[
\left(\frac{\partial^2}{\partial \theta^2} \mathcal{R}_\infty^\theta\right)(\theta) = 2 \int_{q_0}^\theta (x - q_0) \, dx = \theta_{1}(\beta - q_0)^2 \\
= \frac{(\theta - \theta_0)(\beta - \theta_0)(\theta_0 - \theta_0^2)(2(\beta^2 + \theta q_0 + [q_0]^2) - 3\theta q_0 - 3[q_0]^2)}{2(1 - \theta_{2H+1}/2)^1/2(1 + \theta_{2H+1}/2)^1/2}.
\]

Therefore, we obtain that

\[
\mathcal{H}_\theta = \begin{pmatrix}
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2}
\end{pmatrix}.
\]

This and Lemma 4.5 (applied with \(m \lor 4, n \lor 4, r_1 \lor [\theta_1]^2, r_2 \lor [\theta_1]^2, r_3 \lor 1, r_4 \lor \theta_1, c_1 \lor 1, c_2 \lor 1, c_3 \lor [\theta_1]^{-2}, c_4 \lor [\theta_1]^{-1}, \epsilon \lor 0, \phi \lor 0\) in the notation of Lemma 4.5) ensure that \(\mathcal{H}_\theta\) and

\[
\mathcal{H}_\theta = \begin{pmatrix}
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} \\
\frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2} & \frac{\beta - \theta_0}{\theta_0 - \theta_0^2}
\end{pmatrix}.
\]
have the same rank. Combining this with Lemma 4.5 (applied with \(m \sim 4, n \sim 4, r_1 \sim (1 - q_\theta)^1/2(1 + 3q_\theta)^1/2, r_2 \sim (1 - q_\theta)^1/2(1 + 3q_\theta)^1/2, r_3 \sim 1, r_4 \sim 1, c_1 \sim (1 - q_\theta)^1/2(1 + 3q_\theta)^1/2, c_2 \sim (1 - q_\theta)^1/2(1 + 3q_\theta)^1/2, c_3 \sim 1, c_4 \sim 1, \epsilon \sim 0, f \sim 0\) \(0 \in \) in the notation of Lemma 4.5) shows that \(\mathcal{H}_\theta\) and

\[
\left(\begin{array}{c}
\frac{\epsilon^2(1-q_\theta)^2+\epsilon^2(1+2q_\theta)^2+\delta(1+4q_\theta-5q_\theta)^2}{6(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta)^2}{4(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta+q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} \\
\frac{(\epsilon-\delta)(1-q_\theta)^3}{2(\epsilon-\delta)(1-q_\theta)^3} \\
\frac{2(\delta-\epsilon)(1-q_\theta)^2}{2(\delta-\epsilon)(1-q_\theta)^2}
\end{array}\right)
\]  

(4.50)

have the same rank. This and Lemma 4.5 (applied with \(m \sim 4, n \sim 4, r_1 \sim (1 - q_\theta)^{-1}, r_2 \sim (1 - q_\theta)^{-1}, r_3 \sim (1 - q_\theta)^{-1}, r_4 \sim 1, c_1 \sim 1, c_2 \sim 1, c_3 \sim (1 - q_\theta)^{-1}, c_4 \sim 1, \epsilon \sim 0, f \sim 0\) \(0 \in \) in the notation of Lemma 4.5) imply that \(\mathcal{H}_\theta\) and

\[
\left(\begin{array}{c}
\frac{\epsilon^2(1-q_\theta)^2+\epsilon^2(1+2q_\theta)^2+\delta(1+4q_\theta-5q_\theta)^2}{6(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta)^2}{4(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta+q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} \\
\frac{(\epsilon-\delta)(1-q_\theta)^3}{2(\epsilon-\delta)(1-q_\theta)^3} \\
\frac{2(\delta-\epsilon)(1-q_\theta)^2}{2(\delta-\epsilon)(1-q_\theta)^2}
\end{array}\right)
\]  

(4.51)

have the same rank. Combining this with Lemma 4.5 (applied with \(m \sim 4, n \sim 4, r_1 \sim (1 - q_\theta)^{-1}, r_2 \sim (1 - q_\theta)^{-1}, r_3 \sim (1 - q_\theta)^{-1}, r_4 \sim 1, c_1 \sim -1, c_2 \sim 1, c_3 \sim 1, c_4 \sim 1, \mathcal{H}_1 \sim 1, \mathcal{H}_2 \sim 2, \epsilon \sim 0, f \sim 0\) \(\in \) in the notation of Lemma 4.5) proves that \(\mathcal{H}_\theta\) and

\[
\left(\begin{array}{c}
\frac{\epsilon^2(1-q_\theta)^2+\epsilon^2(1+2q_\theta)^2+\delta(1+4q_\theta-5q_\theta)^2}{6(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta)^2}{4(1-q_\theta)(1+3q_\theta)} \\
\frac{\epsilon(1-q_\theta)^2+\delta(1+4q_\theta+q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} \\
\frac{(\epsilon-\delta)(1-q_\theta)^3}{2(\epsilon-\delta)(1-q_\theta)^3} \\
\frac{2(\delta-\epsilon)(1-q_\theta)^2}{2(\delta-\epsilon)(1-q_\theta)^2}
\end{array}\right)
\]  

(4.52)

have the same rank. This, Lemma 4.5 (applied with \(m \sim 4, n \sim 4, r_1 \sim (1 - q_\theta)^{-1}, r_2 \sim (1 - q_\theta)^{-1}, r_3 \sim (1 - q_\theta)^{-1}, r_4 \sim 1, c_1 \sim -1, c_2 \sim 1, c_3 \sim 1, c_4 \sim 1, \mathcal{H}_1 \sim 1, \mathcal{H}_2 \sim 2, \epsilon \sim 0, f \sim 0\) \(\in \) \(\) the fact that

\[
- \left[ \frac{\epsilon^2(1-q_\theta)^2+\delta^2(1+2q_\theta)^2+\epsilon\delta(1+3q_\theta-5q_\theta)^2}{6(1-q_\theta)(1+3q_\theta)} \right] - \frac{\epsilon\delta(1-q_\theta)^2+\delta^2(1+3q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} = -2\epsilon^2(1-q_\theta)^2-2\epsilon\delta(1+2q_\theta)^2-2\epsilon(1+4q_\theta-5q_\theta)^2+3\epsilon\delta(1-q_\theta)^2+3\delta^2(1+4q_\theta)^2
\]

(4.53)

the fact that

\[
- \left[ \frac{\epsilon(1-q_\theta)^2+\delta(1+3q_\theta)^2}{4(1-q_\theta)(1+3q_\theta)} \right] - \frac{\delta(1+2q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} = -2\epsilon(1-q_\theta)^2+\delta^2(1+2q_\theta)^2-2\epsilon\delta(1+4q_\theta-5q_\theta)^2+3\epsilon\delta(1-q_\theta)^2+3\delta^2(1+4q_\theta)^2
\]

(4.54)

the fact that

\[
- \left[ \frac{\epsilon(1-q_\theta)^2+\delta(1+3q_\theta)^2}{4(1-q_\theta)(1+3q_\theta)} \right] - \frac{\delta(1+2q_\theta)^2}{2(1-q_\theta)(1+3q_\theta)} = -2\epsilon(1-q_\theta)^2+\delta^2(1+2q_\theta)^2-2\epsilon\delta(1+4q_\theta-5q_\theta)^2+3\epsilon\delta(1-q_\theta)^2+3\delta^2(1+4q_\theta)^2
\]

(4.55)
and the fact that
\[- \left[ \frac{\epsilon (1 - q_0) + \delta (1 + q_0)}{2} \right] = \frac{(\epsilon - \delta) (1 - q_0)}{2} \] (4.56)
show that $H_\theta$ and
\[
\begin{pmatrix}
\frac{(\epsilon - \delta) (2e(1 - q_0) + 6(1 + 5q_0))}{12(1 + 3q_0)} & \frac{e(1 - q_0^2) + 6(1 + 4q_0 + |q_0|^2)}{4(1 - q_0)(1 + 3q_0)} & \frac{e(1 - q_0^2) + 6(2 + q_0^2)}{6} & \frac{e(1 - q_0) + \delta (1 + q_0)}{2} \\
\frac{\delta (1 - q_0)}{6} & \frac{\delta (1 + q_0) + \epsilon (1 + 5q_0)}{4(1 + 3q_0)} & \frac{\delta (1 - q_0)}{2} & 1 \\
\frac{\delta (1 - q_0)}{2} & \frac{\delta (1 - q_0)}{2} & 1 & (1 - q_0)
\end{pmatrix}
\] (4.57)
have the same rank. Combining this with (4.54), (4.55), (4.56), the fact that
\[- \left[ \frac{(\epsilon - \delta) (2e(1 - q_0) + 6(1 + 5q_0))}{12(1 + 3q_0)} \right] = \frac{\delta (\epsilon - \delta) (1 + q_0)}{6(1 + 3q_0)} \]
and Lemma 4.5 (applied with $m \sim n \sim 4$, $r_1 \sim -1$, $r_2 \sim 1$, $r_3 \sim 1$, $r_4 \sim 1$, $c_1 \sim 1$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $c_1 \sim 1$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $c_1 \sim 1$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $c_1 \sim 1$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $\epsilon \sim 0$, $f \sim 0$ in the notation of Lemma 4.5) ensures that $H_\theta$ and
\[
\begin{pmatrix}
\frac{(\epsilon - \delta) (1 - q_0)}{6(1 + 3q_0)} & \frac{\delta (\epsilon - \delta) (1 + q_0)}{4(1 + 3q_0)} & \frac{(\epsilon - \delta) (1 - q_0)}{6} & \frac{(\epsilon - \delta) (1 - q_0)}{2} \\
\frac{\delta (1 - q_0)}{6} & \frac{\delta (1 + q_0) + \epsilon (1 + 5q_0)}{4(1 + 3q_0)} & \frac{\delta (1 - q_0)}{2} & 1 \\
\frac{\delta (1 - q_0)}{2} & \frac{\delta (1 - q_0)}{2} & 1 & (1 - q_0)
\end{pmatrix}
\] (4.59)
have the same rank. This and Lemma 4.5 (applied with $m \sim n \sim 4$, $r_1 \sim (\delta - \epsilon)^{-1}$, $r_2 \sim 1$, $r_3 \sim 1$, $r_4 \sim 1$, $c_1 \sim (\beta - \delta)^{-1}$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $c_1 \sim 1$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $\epsilon \sim 0$, $f \sim 0$ in the notation of Lemma 4.5) assure that $H_\theta$ and
\[
\begin{pmatrix}
\frac{1 - q_0}{6(1 + 3q_0)} & \frac{1 + q_0}{4(1 + 3q_0)} & \frac{1 - q_0}{6} & \frac{1 - q_0}{2} \\
\frac{1 + q_0}{6(1 + 3q_0)} & \frac{1 + 2q_0}{4(1 + 3q_0)} & \frac{1}{2} & 1 \\
\frac{1 - q_0}{6} & \frac{1}{2} & \frac{2}{3} (1 - q_0) (1 - q_0) & 1 \\
\frac{1 - q_0}{2} & 1 & (1 - q_0) & 2
\end{pmatrix}
\] (4.60)
have the same rank. Combining this and Lemma 4.5 (applied with $m \sim 4$, $n \sim 4$, $r_1 \sim (1 - q_\theta)^{-1}$, $r_2 \sim 1$, $r_3 \sim 1$, $r_4 \sim 1$, $c_1 \sim (1 - q_\theta)^{-1}$, $c_2 \sim 1$, $c_3 \sim 1$, $c_4 \sim 1$, $\epsilon \sim 0$, $f \sim 0$ in the notation of Lemma 4.5) shows that $H_\theta$ and
\[
\begin{pmatrix}
\frac{1}{6(1 - q_0)(1 + 3q_\theta)} & \frac{1 + q_0}{4(1 - q_0)(1 + 3q_\theta)} & \frac{1}{2} & \frac{1}{2} \\
\frac{1 + q_0}{6(1 - q_0)(1 + 3q_\theta)} & \frac{1 + 2q_0}{4(1 - q_0)(1 + 3q_\theta)} & \frac{1}{2} & 1 \\
\frac{1}{6} & \frac{1}{2} & \frac{2}{3} (1 - q_\theta) (1 - q_\theta) & 1 \\
\frac{1}{2} & 1 & (1 - q_\theta) & 2
\end{pmatrix}
\] (4.61)
have the same rank. This, the fact that
\[
\frac{1 + q_\theta}{4(1 - q_0)(1 + 3q_\theta)} - \frac{1}{6(1 - q_0)(1 + 3q_\theta)} = \frac{3(1 + q_\theta) - 2}{12(1 - q_0)(1 + 3q_\theta)} = \frac{1}{12(1 - q_\theta)}
\] (4.62)
the fact that
\[
\frac{1 + 2q_\theta}{2(1 - q_0)(1 + 3q_\theta)} - \frac{1 + q_\theta}{4(1 - q_0)(1 + 3q_\theta)} = \frac{2(1 + 2q_\theta) - 1 - q_\theta}{4(1 - q_0)(1 + 3q_\theta)} = \frac{1}{4(1 - q_\theta)}
\] (4.63)
and Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \ell_1 \bowtie 2, \ell_2 \bowtie 1, \epsilon \bowtie -1, f \bowtie 0\) in the notation of Lemma 4.5) prove that \(H_\theta\) and

\[
\begin{pmatrix}
\frac{1}{6(1-q_\theta)(1+3q_\theta)} & \frac{1}{12(1-q_\theta)} & \frac{1}{12(1-q_\theta)} & \frac{1}{12(1-q_\theta)} \\
\frac{1}{12(1-q_\theta)} & \frac{1}{4(1-q_\theta)} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{2} & \frac{2}{3}(1-q_\theta)(1-q_\theta) \\
\frac{1}{2} & 1 & (1-q_\theta) & 2
\end{pmatrix}
\] (4.64)

have the same rank. Combining this, (4.62), the fact that \(\frac{1}{4(1-q_\theta)} - \frac{1}{12(1-q_\theta)} = \frac{1}{6(1-q_\theta)}\), and Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \ell_1 \bowtie 2, \ell_2 \bowtie 1, \epsilon \bowtie 0, f \bowtie -1\) in the notation of Lemma 4.5) demonstrates that \(H_\theta\) and

\[
\begin{pmatrix}
\frac{1}{6(1-q_\theta)(1+3q_\theta)} & \frac{1}{12(1-q_\theta)} & \frac{1}{12(1-q_\theta)} & \frac{1}{12(1-q_\theta)} \\
\frac{1}{12(1-q_\theta)} & \frac{1}{4(1-q_\theta)} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{2} & \frac{2}{3}(1-q_\theta)(1-q_\theta) \\
\frac{1}{2} & 1 & (1-q_\theta) & 2
\end{pmatrix}
\] (4.65)

have the same rank. Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \ell_1 \bowtie 3, \ell_2 \bowtie 2, \epsilon \bowtie -1, f \bowtie -1\) in the notation of Lemma 4.5) therefore ensures that \(H_\theta\) and

\[
\begin{pmatrix}
\frac{1}{6(1-q_\theta)(1+3q_\theta)} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\
\frac{1}{6} & \frac{2}{3}(1-q_\theta) & 0 & (1-q_\theta) \\
\frac{1}{6} & 0 & 0 & 0 \\
\frac{1}{2} & (1-q_\theta) & 0 & 2
\end{pmatrix}
\] (4.66)

have the same rank. Combining this with Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \ell_1 \bowtie 0, \ell_2 \bowtie 0, \epsilon \bowtie 0, f \bowtie 0\) in the notation of Lemma 4.5) assures that \(H_\theta\) and

\[
\begin{pmatrix}
\frac{1}{6(1-q_\theta)(1+3q_\theta)} & \frac{1}{6} & 0 & \frac{1}{2} \\
\frac{1}{6} & \frac{2}{3}(1-q_\theta) & 0 & (1-q_\theta) \\
0 & 0 & 0 & 0 \\
\frac{1}{2} & (1-q_\theta) & 0 & 2
\end{pmatrix}
\] (4.67)

have the same rank. Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 6(1-q_\theta), r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \ell \bowtie 0, f \bowtie 0\) in the notation of Lemma 4.5) hence implies that \(H_\theta\) and

\[
\begin{pmatrix}
\frac{1}{1+3q_\theta} & (1-q_\theta) & 0 & 3(1-q_\theta) \\
\frac{1}{6} & \frac{2}{3}(1-q_\theta) & 0 & (1-q_\theta) \\
0 & 0 & 0 & 0 \\
\frac{1}{2} & (1-q_\theta) & 0 & 2
\end{pmatrix}
\] (4.68)

have the same rank. Combining this and Lemma 4.5 (applied with \(m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 6, r_3 \bowtie 1, r_4 \bowtie 2, c_1 \bowtie (1+3q_\theta), c_2 \bowtie (1-q_\theta)^{-1}, c_3 \bowtie 1, c_4 \bowtie 1, \epsilon \bowtie 0, f \bowtie 0\) in the notation of Lemma 4.5) shows that \(H_\theta\) and

\[
\begin{pmatrix}
1 & 1 & 0 & 3(1-q_\theta) \\
1+3q_\theta & 4 & 0 & 6(1-q_\theta) \\
0 & 0 & 0 & 0 \\
1+3q_\theta & 2 & 0 & 4
\end{pmatrix}
\] (4.69)
have the same rank. This and Lemma 4.5 (applied with \( m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \dot{\varepsilon}_1 \bowtie 2, \dot{\varepsilon}_2 \bowtie 1, \epsilon \bowtie -4, \bar{f} \bowtie 0 \) in the notation of Lemma 4.5) ensure that \( \mathcal{H}_\theta \))

\[
\begin{pmatrix}
1 & 1 & 0 & 3(1-q_\theta) \\
3q_\theta -3 & 0 & 0 & -6(1-q_\theta) \\
0 & 0 & 0 & 0 \\
1+3q_\theta & 2 & 0 & 4
\end{pmatrix}
\] (4.70)

have the same rank. Combining this with Lemma 4.5 (applied with \( m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \dot{\varepsilon}_1 \bowtie 4, \dot{\varepsilon}_2 \bowtie 2, \epsilon \bowtie -(3q_\theta - 3), \bar{f} \bowtie 0 \) in the notation of Lemma 4.5) assures that \( \mathcal{H}_\theta \))

\[
\begin{pmatrix}
1 & 1 & 0 & 3(1-q_\theta) \\
3q_\theta -3 & 0 & 0 & -6(1-q_\theta) \\
0 & 0 & 0 & 0 \\
3q_\theta -1 & 0 & 0 & -2+6q_\theta
\end{pmatrix}
\] (4.71)

have the same rank. Combining this with Lemma 4.5 (applied with \( m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \dot{\varepsilon}_1 \bowtie 4, \dot{\varepsilon}_2 \bowtie 2, \epsilon \bowtie -(3q_\theta - 3), \bar{f} \bowtie 0 \) in the notation of Lemma 4.5) prove that \( \mathcal{H}_\theta \))

\[
\begin{pmatrix}
1 & 1 & 0 & 3(1-q_\theta) \\
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 4
\end{pmatrix}
\] (4.72)

have the same rank. Combining this with Lemma 4.5 (applied with \( m \bowtie 4, n \bowtie 4, r_1 \bowtie 1, r_2 \bowtie 1, r_3 \bowtie 1, r_4 \bowtie 1, c_1 \bowtie 1, c_2 \bowtie 1, c_3 \bowtie 1, c_4 \bowtie 1, \dot{\varepsilon}_1 \bowtie 4, \dot{\varepsilon}_2 \bowtie 2, \epsilon \bowtie -(3q_\theta - 3), \bar{f} \bowtie 0 \) in the notation of Lemma 4.5) demonstrates that \( \mathcal{H}_\theta \))

\[
\begin{pmatrix}
1 & 1 & 0 & 3(1-q_\theta) \\
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (4.73)

have the same rank. Hence, we obtain that \( \text{rank}(\mathcal{H}_\theta) = 2 \). This establishes item (iii). Next observe that the fact that \( \alpha < q_\theta < \beta \) ensures that

\[
\begin{align*}
(1 + 4q_\theta - 5|q_\theta|^2)^2 - 4(1 - q_\theta)^2(1 + 2q_\theta)^2 \\
= (1 + 4q_\theta - 5|q_\theta|^2 - 2(1 + q_\theta - 2|q_\theta|^2))(1 + 4q_\theta - 5|q_\theta|^2 + 2(1 + q_\theta - 2|q_\theta|^2)) \\
= (-1 + 2q_\theta - |q_\theta|^2)3(1 + 2q_\theta - 3|q_\theta|^2) = -3(1 - q_\theta)^3(1 + 3q_\theta) < 0.
\end{align*}
\] (4.74)

This assures that

\[
\alpha^2(1 - q_\theta)^2 + \beta^2(1 + 2q_\theta)^2 + \alpha \beta(1 + 4q_\theta - 5|q_\theta|^2) \geq 0.
\] (4.75)

Combining this with (4.38) and the fact that \( \alpha < \beta \) demonstrates that

\[
\det\left(\frac{\partial^2}{\partial \theta^4} \mathcal{R}_\theta(\theta)\right) = \det\left(\frac{\text{e}^2(1-q_\theta)^2+\text{e}^2(1+2q_\theta)^2+\text{e}^4(1+4q_\theta-5|q_\theta|^2)}{6|\theta|^2(\text{e}^2-\alpha)(1-q_\theta)^2(1+3q_\theta)^2}\right)
\] (4.76)

\[
= \text{e}^2(1-q_\theta)^2+\text{e}^2(1+2q_\theta)^2+\text{e}^4(1+4q_\theta-5|q_\theta|^2)\cdot \frac{1}{6|\theta|^2(\text{e}^2-\alpha)(1-q_\theta)^2(1+3q_\theta)^2} > 0.
\]

Furthermore, note that (4.39), (4.40), (4.41), the fact that \( \alpha < \beta \), and the fact that \( \alpha < q_\theta < \beta \) show that

\[
\det\left(\frac{\partial^2}{\partial \theta^4} \mathcal{R}_\theta(\theta)\right) = \frac{1+2q_\theta}{2|\theta|^2(\text{e}^2-\alpha)(1-q_\theta)^2(1+3q_\theta)^2} > 0,
\] (4.77)
\[
\det \left( \left( \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right) \right) = \frac{2}{3} [\theta_1]^2 (\theta - \alpha)^3 (1 - q_\theta)^3 > 0, \tag{4.78}
\]
and
\[
\det \left( \left( \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right) \right) = 2(\theta - \alpha) > 0. \tag{4.79}
\]
Moreover, observe that the fact that
\[
4(1 - q_\theta)^2 (1 + 2q_\theta) - 3(1 - [q_\theta]^2)^2
= (1 - q_\theta)^2 (4(1 + 2q_\theta) - 3(1 + q_\theta)^2) = (1 - q_\theta)^2 (4 + 8q_\theta - 3 - 6q_\theta - 3[q_\theta]^2)
= (1 - q_\theta)^2 (1 + 2q_\theta - 3[q_\theta]^2) = (1 - q_\theta)^3 (1 + 3q_\theta), \tag{4.80}
\]
the fact that
\[
4(1 + 2q_\theta)^3 - 3(1 + 4q_\theta + [q_\theta]^2)^2 = 4(1 + 6q_\theta) + 40q_\theta^2 - 8[q_\theta]^3 - 3(1 + 16[q_\theta]^2 + [q_\theta]^4 + 4q_\theta + 2[q_\theta]^2 + 8[q_\theta]^3)
= 1 - 6[q_\theta]^2 + 8[q_\theta]^3 - 3[q_\theta]^4 = (1 - q_\theta)(1 + q_\theta) - 5[q_\theta]^2 (1 - q_\theta) + 3[q_\theta]^3 (1 - q_\theta)
= (1 - q_\theta)(1 + q_\theta - 5[q_\theta]^2 + 3[q_\theta]^3) = (1 - q_\theta)^2 (1 + 2q_\theta - 3[q_\theta]^2) = (1 - q_\theta)^3 (1 + 3q_\theta), \tag{4.81}
\]
and the fact that
\[
3(1 - [q_\theta]^2)(1 + 4q_\theta + [q_\theta]^2) - 2(1 + 4q_\theta - 5[q_\theta]^2)(1 + 2q_\theta)
= (1 - q_\theta)(3(1 + 4q_\theta + [q_\theta]^2) + q_\theta + 4[q_\theta]^2 + [q_\theta]^3) - 2(1 + 7q_\theta + 10[q_\theta]^2))
= (1 - q_\theta)(1 + q_\theta - 5[q_\theta]^2 + 3[q_\theta]^3) = (1 - q_\theta)^3 (1 + 3q_\theta), \tag{4.82}
\]
equaint that
\[
4[\alpha^2 (1 - q_\theta)^2 + \beta^2 (1 + 2q_\theta)^2 + \alpha \beta (1 + 4q_\theta - 5[q_\theta]^2)][1 + 2q_\theta]
- 3[\alpha^2 (1 - [q_\theta]^2)^2 + \beta^2 (1 + 4q_\theta + [q_\theta]^2)^2 + 2 \alpha \beta (1 - [q_\theta]^2)(1 + 4q_\theta + [q_\theta]^2)]
= \alpha^2 (4(1 - q_\theta)^2 (1 + 2q_\theta) - 3(1 - [q_\theta]^2)^2 + \beta^2 (4(1 + 2q_\theta)^3 - 3(1 + 4q_\theta + [q_\theta]^2)^2)
- 2 \alpha \beta (3(1 - [q_\theta]^2)(1 + 4q_\theta + [q_\theta]^2) - 2(1 + 4q_\theta - 5[q_\theta]^2)(1 + 2q_\theta)]
= \alpha^2 (1 - q_\theta)^3 (1 + 3q_\theta) + \beta^2 (1 - q_\theta)^3 (1 + 3q_\theta) - 2 \alpha \beta (1 - q_\theta)^3 (1 + 3q_\theta)
= (\theta - \alpha)^2 (\theta - q_\theta)^2 (1 + 3q_\theta). \tag{4.83}
\]
The fact that \(\alpha < q_\theta < \beta\), (4.38), (4.39), and (4.42) therefore show that
\[
\det \left( \left( \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right) \left( \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right) \right) = \left( \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right) \left[ \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right] - \left[ \frac{\partial^2}{\partial \theta^2} R^f_{\infty}(\theta) \right]^2
= \frac{[\alpha^2 (1 - q_\theta)^2 + \beta^2 (1 + 2q_\theta)^2 + \alpha \beta (1 + 4q_\theta - 5[q_\theta]^2)][1 + 2q_\theta] - [\alpha (1 - [q_\theta]^2) + \beta (1 + 4q_\theta + [q_\theta]^2)]^2}{12[\theta_1]^4 (\theta - \alpha)^2 (1 - q_\theta)^2 (1 + 3q_\theta)^2}
+ \frac{\alpha^2 (4(1 - q_\theta)^2 (1 + 2q_\theta) - 3(1 - [q_\theta]^2)^2 + \beta^2 (4(1 + 2q_\theta)^3 - 3(1 + 4q_\theta + [q_\theta]^2)^2 - 2 \alpha \beta (3(1 - [q_\theta]^2)(1 + 4q_\theta + [q_\theta]^2) - 2(1 + 4q_\theta - 5[q_\theta]^2)(1 + 2q_\theta)]}{16[\theta_1]^4 (\theta - \alpha)^2 (1 - q_\theta)^2 (1 + 3q_\theta)^2}
= \frac{4\alpha^2 (1 - q_\theta)^2 + \beta^2 (1 + 2q_\theta)^2 + \alpha \beta (1 + 4q_\theta - 5[q_\theta]^2)[1 + 2q_\theta] - 3[\alpha^2 (1 - [q_\theta]^2)^2 + \beta^2 (1 + 4q_\theta + [q_\theta]^2)^2 + 2 \alpha \beta (1 - [q_\theta]^2)(1 + 4q_\theta + [q_\theta]^2)]}{48[\theta_1]^4 (\theta - \alpha)^2 (1 - q_\theta)^2 (1 + 3q_\theta)^2}
= \frac{1}{48[\theta_1]^4 (1 - q_\theta)(1 + 3q_\theta)^2} > 0. \tag{4.84}
\]
Next note that the fact that \(\alpha < q_\theta < \beta\), (4.38), (4.40), (4.43), the fact that
\[
4(1 + 2q_\theta)^2 - (1 + 3q_\theta)(2 + q_\theta)^2 = 4(1 + 4q_\theta + 4[q_\theta]^2) - (1 + 3q_\theta)(4 + 4q_\theta + [q_\theta]^2)
= 4 + 16q_\theta + 16[q_\theta]^2 - 4 - 4q_\theta - [q_\theta]^2 - 12q_\theta - 12[q_\theta]^2 - 3[q_\theta]^3 = 3[q_\theta]^2 (1 - q_\theta), \tag{4.85}
\]
and the fact that
\[
4(1 + 4q_\theta - 5[q_\theta]^2) - 2(1 + 3q_\theta)(1 - q_\theta)(2 + q_\theta) = 4(1 - q_\theta)(1 + 5q_\theta) - 2(1 + 3q_\theta)(1 - q_\theta)(2 + q_\theta) \\
= 2(1 - q_\theta)[2(1 + 5q_\theta) - (1 + 3q_\theta)(2 + q_\theta)] \\
= 2(1 - q_\theta)[2 + 10q_\theta - 2 - q_\theta - 6q_\theta - 3[q_\theta]^2] = 6q_\theta(1 - q_\theta)^2
\]
prove that
\[
\det \left( \begin{array}{c} \frac{\partial^2 R^\prime}{\partial \theta^2}(\theta) \\ \frac{\partial^2}{\partial \theta^2}(\theta) \\ \frac{\partial^2}{\partial \theta^2}(\theta) \\ \frac{\partial^2}{\partial \theta^2}(\theta) \end{array} \right) = \left( \frac{\partial^2}{\partial \theta^2}(\theta) \right) \left( \frac{\partial^2}{\partial \theta^2}(\theta) \right) - \left( \frac{\partial^2}{\partial \theta^2}(\theta) \right)^2
\]
\[
= \left[ a^2(1-q_\theta)^2 + (1+2q_\theta)^2 + a(1+4q_\theta - 5[q_\theta]^2) \right] \left[ \frac{2}{3}[\theta] + (1-q_\theta)^3 \right] - \left[ \frac{1-\theta^2}{6(1-q_\theta)^2} \right] \left[ (\theta + 2q_\theta)^2 + (1+4q_\theta - 5[q_\theta]^2) \right]
\]
\[
= \frac{(\delta - e)^2(1-q_\theta)^2}{36(1+3q_\theta)^2} \left[ a^2(1-q_\theta)^2 + (1+2q_\theta)^2 + a(1+4q_\theta - 5[q_\theta]^2) \right]
\]
\[
= \frac{(\delta - e)^2(1-q_\theta)^2}{36(1+3q_\theta)^2} \left[ a^2(1-q_\theta)^2 + (1+2q_\theta)^2 + a(1+4q_\theta - 5[q_\theta]^2) \right]
\]
Furthermore, observe that (4.38), (4.41), (4.44), the fact that
\[
4(1 - q_\theta)^2 - 3(1 - q_\theta)^3(1 + 3q_\theta) = (1 - q_\theta)^2(4 - 3(1 - q_\theta)(1 + 3q_\theta))
\]
\[
= (1 - q_\theta)^2(4 - 3(1 + 2q_\theta - 3[q_\theta]^2)) = (1 - q_\theta)^2(1 - 6q_\theta + 9[q_\theta]^2)
\]
\[
= (1 - q_\theta)^2(1 - 3q_\theta)^2
\]
the fact that
\[
4(1 + 4q_\theta - 5[q_\theta]^2) - 6(1 - q_\theta)^2(1 + 3q_\theta)(1 + q_\theta) = 4(1 - q_\theta)(1 + 5q_\theta)
\]
\[
= 2(1 - q_\theta)(2 + 10q_\theta - 3(1 + 4q_\theta + 3[q_\theta]^2 - q_\theta - 4[q_\theta]^2 - 3[q_\theta]^3))
\]
and the fact that
\[
4(1 + 4q_\theta) - 3(1 - q_\theta)(1 + 3q_\theta)(1 + q_\theta)^2 = 4 + 16q_\theta + 16[q_\theta]^2 - 3(1 - [q_\theta]^2)(1 + 4q_\theta + 3[q_\theta]^2)
\]
\[
= 4 + 16q_\theta + 16[q_\theta]^2 - 3(1 + 4q_\theta + 3[q_\theta]^2 - [q_\theta]^2 - 4[q_\theta]^3 - 3[q_\theta]^4)
\]
\[
= 1 + 4q_\theta + 10[q_\theta]^2 + 12[q_\theta]^3 + 9[q_\theta]^4 = (1 + 2q_\theta + 3[q_\theta]^2)^2
\]
ensure that

\[
\det \left( \begin{array}{ccc}
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty^\prime(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty^\prime(\theta) \\
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty(\theta)
\end{array} \right) = \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty^\prime(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty^\prime(\theta) \right) \right] - \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty(\theta) \right) \right]^2
\]

\[
= \left[ \frac{\alpha^2 (1 - q_0)^2 + \beta^2 (1 + 2 q_0)^2 + \alpha \delta (1 + 4 q_0 - 5 q_0^2)}{6 [\theta_1]^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} \right] \left[ 2 (\beta - \alpha) \right] - \left[ \frac{\alpha (1 - q_0) + \beta (1 + q_0)}{2 [\theta_1]^2 (1 - q_0)^2 (1 + 3 q_0)^2} \right]^2
\]

Next note that (4.39), (4.40), (4.41), (4.45), (4.46), (4.47), the fact that \( \alpha < \beta \), and the fact that \( \alpha < q_0 < \beta \) show that

\[
\det \left( \begin{array}{ccc}
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty^\prime(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty^\prime(\theta) \\
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty(\theta)
\end{array} \right) = \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty^\prime(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty^\prime(\theta) \right) \right] - \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty(\theta) \right) \right]^2
\]

\[
= \left[ \frac{2 [\theta_1]^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} {2 \theta_1^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} \right] \left[ 2 (\beta - \alpha) \right] - \left[ \frac{1 \theta_1 (1 - q_0)^2 (1 + 3 q_0)^2} {2 \theta_1^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} \right]^2
\]

\[
= \frac{1 + 2 q_0}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} - \frac{1}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} = \frac{3 q_0^2}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} > 0,
\]

and

\[
\det \left( \begin{array}{ccc}
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty^\prime(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty^\prime(\theta) \\
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty(\theta)
\end{array} \right) = \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty^\prime(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty^\prime(\theta) \right) \right] - \left[ \left( \frac{\partial^2}{\partial \theta H} R_\infty(\theta) \right) \left( \frac{\partial^2}{\partial \theta H^2} R_\infty(\theta) \right) \right]^2
\]

\[
= \left[ \frac{2 [\theta_1]^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} {2 \theta_1^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} \right] \left[ 2 (\beta - \alpha) \right] - \left[ \frac{1 \theta_1 (1 - q_0)^2 (1 + 3 q_0)^2} {2 \theta_1^2 (\theta_1 - 1 q_0)^2 (1 + 3 q_0)^2} \right]^2
\]

\[
= \frac{1 + 2 q_0}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} - \frac{1}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} = \frac{3 q_0^2}{\theta_1^2 (1 - q_0)^2 (1 + 3 q_0)^2} > 0,
\]

In addition, observe that (4.24) and the fact that \( \text{rank}(H_\theta) = 2 \) assure that for all \( i, j, k \in \{1, 2, H + 1, \beta \} \) it holds that

\[
\det \left( \begin{array}{ccc}
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty^\prime(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty^\prime(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty^\prime(\theta) \\
\left( \frac{\partial^2}{\partial \theta H} \right) R_\infty(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty(\theta) & \left( \frac{\partial^2}{\partial \theta H^2} \right) R_\infty(\theta)
\end{array} \right) = 0 = \det(H_\theta).
\]
Combining this, (4.76), (4.77), (4.78), (4.79), (4.84), (4.87), (4.91), (4.92), (4.93), and (4.94) with the Sylvester’s criterion demonstrates that $\mathcal{H}_\theta$ is positive-semidefinite. Therefore, we obtain that $\sigma(\mathcal{H}_\theta) \subseteq [0, \infty)$. This establishes item (iv). Note that (4.27) shows that for all $i, j \in \{2, 3, \ldots, H\}$, $x, y \in \{\theta_i, \theta_j, \theta_{H+i}, \theta_{H+j}, \theta_{2H+i}, \theta_{2H+j}\}$ it holds that
\[
(\frac{\partial^2}{\partial x \partial y} \mathcal{R}_\infty^f)(\theta) = (\frac{\partial^2}{\partial x \partial y} \mathcal{R}_\infty^f)(\theta) = 0.
\] (4.96)
Combining this with items (iii) and (iv) shows that $\sigma((\text{Hess} \mathcal{R}_\infty^f)(\theta)) = \sigma(\mathcal{H}_\theta) \subseteq [0, \infty)$ and rank $((\text{Hess} \mathcal{R}_\infty^f)(\theta)) = \text{rank}(\mathcal{H}_\theta) = 2$. This establishes items (v) and (vi). The proof of Lemma 4.6 is thus complete.

4.5 On a submanifold of local minimum points of the ANN parameter space

Lemma 4.7. Assume Setting 4.1. Then

(i) it holds that $\mathcal{M}$ is an uncountable set,

(ii) it holds for all $\theta \in \mathcal{M}$ that $\mathcal{R}_\infty^f$ is differentiable at $\theta$,

(iii) it holds for all $\theta \in \mathcal{M}$ that $(\nabla \mathcal{R}_\infty^f)(\theta) = 0$,

(iv) it holds for all $\theta, \vartheta \in \mathcal{M}$ that $\mathcal{R}_\infty^f(\theta) = \mathcal{R}_\infty^f(\vartheta)$, and

(v) it holds that

\[
\mathcal{M} = \{v \in C([\alpha, \beta], \mathbb{R}) : 
\exists \theta \in \Theta : v = \mathcal{N}_\infty^g, \varepsilon \in (0, \infty), \mathcal{R}_\infty^g(\theta) = \inf_{\vartheta \in [-\varepsilon, \varepsilon]} \mathcal{R}_\infty^f(\theta + \vartheta)\}.
\] (4.97)

Proof of Lemma 4.7. Observe that item (i) in Lemma 4.6 and the fact that
\[
\{q \in \mathbb{R} : \exists \theta_0 \in \mathcal{M} : q = \mathcal{N}_\infty^g\} = (\alpha, \beta)
\] (4.98)
ensure that $\mathcal{M}$ is an uncountable set. This establishes item (i). Note that Lemma 4.4 shows that for all $\theta \in \mathcal{M}$ it holds that $\mathcal{R}_\infty^f$ is differentiable at $\theta$. This establishes item (ii). Observe that [17, Item (v) in Proposition 2.3] and item (ii) in Lemma 4.6 assure that for all $\theta = (\theta_1, \ldots, \theta_3) \in \mathcal{M}$ it holds that
\[
(\frac{\partial}{\partial \theta_1} \mathcal{R}_\infty^f)(\theta) = 2\theta_{2H+1} \int_{q_0}^6 x \mathcal{N}_\infty^g(x) - f(x) \, dx = 0,
\] (4.99)
\[
(\frac{\partial}{\partial \theta_{H+1}} \mathcal{R}_\infty^f)(\theta) = 2\theta_{2H+1} \int_{q_0}^6 (\mathcal{N}_\infty^g(x) - f(x)) \, dx = 0,
\] (4.100)
\[
(\frac{\partial}{\partial \theta_{2H+1}} \mathcal{R}_\infty^f)(\theta) = 2\theta_1 \int_{q_0}^6 (x - q)(\mathcal{N}_\infty^g(x) - f(x)) \, dx = 0,
\] (4.101)
and
\[
(\frac{\partial}{\partial \theta_j} \mathcal{R}_\infty^f)(\theta) = 2 \int_{q_0}^6 (\mathcal{N}_\infty^g(x) - f(x)) \, dx = 0.
\] (4.102)
Lemma 4.4 and the fact that for all $\theta = (\theta_1, \ldots, \theta_3) \in \mathcal{M}$, $j \in \{2, 3, \ldots, H\}$ it holds that
$(\frac{\partial}{\partial \theta_j} \mathcal{R}_\infty^f)(\theta) = (\frac{\partial}{\partial \theta_{H+j}} \mathcal{R}_\infty^f)(\theta) = (\frac{\partial}{\partial \theta_{2H+j}} \mathcal{R}_\infty^f)(\theta) = 0$ therefore show that for all $\theta \in \mathcal{M}$ it holds that $(\nabla \mathcal{R}_\infty^f)(\theta) = 0$. This establishes item (iii). Note that the integral transformation theorem and item (i) in Lemma 4.6 prove that for all $\theta \in \mathcal{M}$ it holds that
\[
\mathcal{R}_\infty^f(\theta) = \int_{q_0}^6 (\mathcal{N}_\infty^g(x) - f(x))^2 \, dx
\] (4.103)
\[
= (\theta - \alpha) \int_{q_0}^1 (\mathcal{N}_\infty^g((\theta - \alpha)x + \alpha) - f((\theta - \alpha)x + \alpha))^2 \, dx
\] (4.103)
\[
= (\theta - \alpha) \int_{q_0}^1 \left(\frac{(1-q_0)^{1/2}}{4(1+3q_0)^{1/2}} + \max(x-q_0,0) \right) \left(1 - \frac{1-q_0}{2(1+3q_0)^{1/2}} - f(x)\right)^2 \, dx.
\]
Combining this and item (ii) in Lemma 4.2 with the fact that for all \( \theta \in \mathcal{M} \) it holds that \( \alpha < q_\theta < \beta \) demonstrates that for all \( \theta \in \mathcal{M} \) it holds that
\[
R_\infty^f(\theta) = (\beta - \alpha)(\int_0^1 [f(x)]^2 \, dx - \frac{1}{48} ).
\] (4.104)

Hence, we obtain that for all \( \theta, \vartheta \in \mathcal{M} \) it holds that
\[
R_\infty^f(\theta) = R_\infty^f(\vartheta).
\]
This establishes item (iv).

Observe that Proposition 3.12, Lemma 4.3, and items (v) and (vi) in Lemma 4.6 demonstrate that for all \( \theta \in \mathcal{M} \) it holds that \( \theta \) is a local minimum point of \( \mathbb{R}^d \ni \vartheta \mapsto R_\infty^f(\vartheta) \in \mathbb{R} \) (cf. Definition 2.8). This establishes item (v). The proof of Lemma 4.7 is thus complete. \( \square \)

4.6 On infinitely many realization functions of non-global local minimum points

Corollary 4.8. Assume Setting 4.1 and assume \( H > 1 \). Then there exists \( \delta \in (0, \infty) \) such that
\[
\begin{align*}
\{v & \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \mathbb{R}^d : v = \mathcal{N}_\infty^\theta, \varepsilon \in (0, \infty) : \\
& \quad R_\infty^f(\theta) = \inf_{\vartheta \in [-\varepsilon, \varepsilon]} R_\infty^f(\theta + \vartheta) > \delta + \inf_{\vartheta \in \mathbb{R}^d} R_\infty^f(\vartheta) \} 
\end{align*}
\] (4.105)
is an uncountable set.

![Numerical simulations associated to Corollary 4.8](image)

Figure 1: Numerical simulations associated to Corollary 4.8 in the case where \( H = 4, \beta = 13, \alpha = 0, \beta = 1, \alpha = 1/3, \) and \( \beta = 2/3 \) in Corollary 4.8: On the left picture we approximately plot the target function \( f : [0, 1] \to \mathbb{R} \) (cf. (4.2)) and 10 different realization functions of non-global local minimum points (cf. (4.3)) of \( R_\infty^f : \mathbb{R}^{13} \to \mathbb{R} \). On the right picture we plot the result from a simulation with the following setting. We randomly initialize 50 ANNs with the Xavier initialization (we initialize the weights normal distributed with mean 0 and variance \( 2/5 \) and we initialize the biases with 0), then we approximately train these ANNs with the GD optimization method using a learning rate of \( 1/20 \) until the maximum norm of the generalized gradient function (cf. (2.12)) evaluated at the current position of the GD process is strictly less than \( 10^{-4} \), and, thereafter, we gradually plot the realization functions of the resulting ANNs whereby a realization function is not drawn if a realization function with a \( L^2 \)-distance strictly less than \( 10^{-4} \) has already been drawn. This simulation resulted in four different realization functions (red, orange, green, and purple) on the right picture whereby one realization function (orange) can also approximately be found in the left picture. We also refer to Listing 1 for the PYTHON source code used to create Figure 1.

Proof of Corollary 4.8. Throughout this proof let \( p \in (\alpha, \beta) \) and let \( \mathcal{N}^\theta \in C([0, 1], \mathbb{R}), \theta \in \mathbb{R}^d \), satisfy for all \( \theta \in \mathbb{R}^d, x \in [0, 1] \) that \( \mathcal{N}^\theta(x) = \mathcal{N}_\infty^\theta((\beta - \alpha)x + \alpha) \). Note that Lemma 4.7 ensures that

\[34\]
(i) it holds that $\mathcal{M}$ is uncountable set

(ii) it holds that

$$
\mathcal{M} = \{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \vartheta \in \mathcal{M} : v = \mathbb{N}_d (x) , \varepsilon \in (0, \infty) : \mathcal{R}_{d}^{\varepsilon} (\vartheta) = \inf_{\vartheta \in [-\varepsilon, \varepsilon]} \mathcal{R}_{d}^{\varepsilon} (\vartheta + \vartheta) \} \}. \quad (4.106)
$$

Next observe that item (i) in Lemma 4.6 shows that for all $\theta = (\theta_1, \ldots, \theta_b) \in \mathcal{M}$, $x \in [0, 1]$ it holds that

$$
\mathcal{N}^{\theta}_d (x) = \mathbb{N}^{\theta}_d ((\beta - \alpha)x + \alpha) = -\frac{(1-\varrho q_0)^{1/2}}{4(1+3\varrho q_0)^{1/2}} + \frac{\max \{ x-\varrho q_0, 0 \}}{2(1-\varrho q_0)^{3/2}(1+3\varrho q_0)^{1/2}}. \quad (4.107)
$$

Combining this with the integral transformation theorem assures that for all $\theta = (\theta_1, \ldots, \theta_b) \in \mathcal{M}$ it holds that

$$
\mathcal{R}_{d}^{\theta} (x) = \int_a^{\theta} \mathbb{N}_d^{\theta} (x - f(x)) \, d\varrho \hspace{1cm} = (\beta - \alpha) \int_a^{\theta} (\mathbb{N}_d^{\theta} ((\beta - \alpha)x + \alpha) - f((\beta - \alpha)x + \alpha))^2 \, d\varrho 
$$

$$
= (\beta - \alpha) \int_a^{\theta} (\mathbb{N}_d^{\theta} (x) - f(x))^2 \, d\varrho 
$$

$$
= (\beta - \alpha) \int_a^{\theta} \left( -\frac{(1-\varrho q_0)^{1/2}}{4(1+3\varrho q_0)^{1/2}} + \frac{\max \{ x-\varrho q_0, 0 \}}{2(1-\varrho q_0)^{3/2}(1+3\varrho q_0)^{1/2}} - f(x) \right)^2 \, d\varrho. \quad (4.108)
$$

The fact that for all $\theta \in \mathcal{M}$ it holds that $\alpha < q_\theta < \beta$, item (iv) in Lemma 4.7, and item (ii) in Lemma 4.2 hence demonstrate that for all $\theta \in \mathcal{M}$ it holds that

$$
\mathcal{R}_{d}^{\theta} (x) = (\beta - \alpha) \int_a^{\theta} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, d\varrho. \quad (4.109)
$$

Next note that (4.2) and the assumption that $\alpha < p < \beta$ ensure that

$$
f(p) = \frac{3\varrho^2 - 1}{4(1-p)^{1/2}(1+3p)^{1/2}} > -\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}}. \quad (4.110)
$$

The fact that $f \in C([0, 1], \mathbb{R})$ hence implies that there exists $\varepsilon \in (0, \infty)$ which satisfies for all $x \in (p - \varepsilon, p + \varepsilon)$ that $(p - \varepsilon, p + \varepsilon) \subseteq (\alpha, \beta)$ and

$$
f(x) > -\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{x-p+\varepsilon}{4(1-p)^{1/2}(1+3p)^{1/2}} > -\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}}. \quad (4.111)
$$

In the following let $\vartheta = (\vartheta_1, \ldots, \vartheta_b) \in \mathbb{R}^b$ satisfy for all $j \in \{ 1, 2, \ldots, d \} \setminus \{ 1, 2, H + 1, H + 2, H + 1, 2H + 1, 2H + 2 \}$ that

$$
\vartheta_1 = \vartheta_2 = \frac{1}{\beta - \alpha} \hspace{1cm} \vartheta_{H+1} = -\frac{\alpha}{\beta - \alpha} - p + \varepsilon 
$$

$$
\vartheta_{2H+1} = \vartheta_{2H+2} = \frac{1}{4(1-p)^{1/2}(1+3p)^{1/2}} \hspace{1cm} \vartheta_b = -\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} \hspace{1cm} \text{and} \hspace{1cm} \vartheta_j < 0. \quad (4.112)
$$

Observe that (4.112) and (4.113) ensure that for all $x \in [0, 1]$ it holds that

$$
\mathcal{N}_d^{\theta} (x) = \mathbb{N}_d^{\theta} ((\beta - \alpha)x + \alpha) = -\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p+\varepsilon, 0 \}}{4(1-p)^{3/2}(1+3p)^{1/2}} + \frac{\max \{ x-p-\varepsilon, 0 \}}{4(1-p)^{3/2}(1+3p)^{1/2}}. \quad (4.114)
$$

Combining this with (4.109) and (4.111) proves that for all $\theta \in \mathcal{M}$ it holds that

$$
\mathcal{R}_{d}^{\theta} (x) = (\beta - \alpha) \int_0^1 (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, dx 
$$

$$
= (\beta - \alpha) \left[ \int_0^{p+p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} - f(x))^2 \, dx + \int_{p-p}^{p-p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, dx + \int_{p+p}^{p+p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, dx \right] \quad (4.115)
$$

$$
> (\beta - \alpha) \left[ \int_0^{p+p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} - f(x))^2 \, dx + \int_{p-p}^{p-p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, dx + \int_{p+p}^{p+p} (-\frac{(1-p)^{1/2}}{4(1+3p)^{1/2}} + \frac{\max \{ x-p, 0 \}}{2(1-p)^{3/2}(1+3p)^{1/2}} - f(x))^2 \, dx \right] 
$$

$$
= (\beta - \alpha) \int_0^1 (\mathcal{N}_d^{\theta}(x) - f(x))^2 \, dx = \mathcal{R}_{d}^{\theta} (x) \geq \inf_{\theta \in \mathbb{R}^b} \mathcal{R}_{d}^{\theta} (x). \quad (4.116)
$$

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This, (4.106), and item (iv) in Lemma 4.7 show that there exists \( \delta \in (0, \infty) \) which satisfies
\[
M = \left\{ v \in C([\alpha, \beta], \mathbb{R}) : \begin{array}{l}
\exists \theta \in \{ \Theta \in \mathcal{M} : v = N^{\alpha}_H \}, \epsilon \in (0, \infty) : \\
R_{\alpha}^\ell (\theta) = \inf_{\Theta \in [-\epsilon, \epsilon]^R} R_{\alpha}^\ell (\theta + \Theta) > \delta + \inf_{\Theta \in \mathbb{R}^3} R_{\alpha}^\ell (\Theta) \}
\right\}
\]
(4.116)
\[
\subseteq \{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \Theta \in \mathbb{R}^3 : v = N^{\alpha}_H \}, \epsilon \in (0, \infty) : \\
R_{\alpha}^\ell (\theta) = \inf_{\Theta \in [-\epsilon, \epsilon]^R} R_{\alpha}^\ell (\theta + \Theta) > \delta + \inf_{\Theta \in \mathbb{R}^3} R_{\alpha}^\ell (\Theta) \}
\}
\]
(4.117)
Note that (4.116) and the fact that \( M \) is an uncountable set demonstrate that
\[
\{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \Theta \in \mathbb{R}^3 : v = N^{\alpha}_H \}, \epsilon \in (0, \infty) : \\
R_{\alpha}^\ell (\theta) = \inf_{\Theta \in [-\epsilon, \epsilon]^R} R_{\alpha}^\ell (\theta + \Theta) > \delta + \inf_{\Theta \in \mathbb{R}^3} R_{\alpha}^\ell (\Theta) \}
\}
(4.118)
is an uncountable set. The proof of Corollary 4.8 is thus complete.

**Corollary 4.9.** Let \( \delta, \alpha, \beta \in \mathbb{R} \), \( \delta \in (\alpha, \infty) \) and for every \( H \in \mathbb{N} \) let \( N^\alpha_H \in C([\alpha, \beta], \mathbb{R}) \), \( \theta \in \mathbb{R}^{3H+1} \), and \( R_{f,H} : \mathbb{R}^{3H+1} \to \mathbb{R} \), \( f \in C([\alpha, \beta], \mathbb{R}) \), satisfy for all \( f \in C([\alpha, \beta], \mathbb{R}) \), \( \theta = (\theta_1, \ldots, \theta_{3H+1}) \in \mathbb{R}^{3H+1} \), \( x \in [\alpha, \beta] \) that \( N^\alpha_H (x) = \theta_{3H+1} + \sum_{j=1}^{H} \theta_{2H+j} \max \{ \theta_{H+j} + \theta_{j}x, 0 \} \) and \( R_{f,H} (\theta) = \int_{[\alpha, \beta]} |f(y) - N^\alpha_H (y)|^2 \, dy \). Then there exists a Lipschitz continuous \( f : [\alpha, \beta] \to \mathbb{R} \) such that for all \( H \in \mathbb{N} \cap (1, \infty) \) it holds that
\[
\{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \Theta \in \mathbb{R}^{3H+1} : v = N^\alpha_H \}, \epsilon \in (0, \infty) : \\
R_{f,H} (\theta) = \inf_{\Theta \in [-\epsilon, \epsilon]^R} R_{f,H} (\theta + \Theta) > \delta + \inf_{\Theta \in \mathbb{R}^{3H+1}} R_{f,H} (\Theta) \}
\}
(4.119)
is an uncountable set.

**Proof of Corollary 4.9.** Throughout this proof let \( \alpha, \beta \in (0, 1) \) satisfy \( \alpha < \beta \), let \( f \in C([0, 1], \mathbb{R}) \) satisfy for all \( x \in [0, 1] \) that
\[
f(x) = \begin{cases}
\frac{4(\beta - \alpha) + 3\alpha^2 - 1}{4(1-\alpha)^{3/2}(1+3\alpha)^{3/2}} & : x \in [0, \alpha] \\
\frac{3\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}} & : x \in (\alpha, \beta] \\
\frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}} & : x \in (\beta, 1],
\end{cases}
(4.120)
\]
(4.119)
let \( f \in C([\alpha, \beta], \mathbb{R}) \) satisfy for all \( x \in [\alpha, \beta] \) that \( f(x) = f\left(\frac{x-a}{b-a}\right) \), let \( \delta \in \mathbb{N}, H \in \mathbb{N} \cap (1, \infty) \) satisfy \( \delta = 3H + 1 \), and \( \epsilon \in (0, \infty) \) satisfy that
\[
\{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \Theta \in \mathbb{R}^3 : v = N^\alpha_H \}, \epsilon \in (0, \infty) : \\
R_{f,H} (\theta) = \inf_{\Theta \in [-\epsilon, \epsilon]^R} R_{f,H} (\theta + \Theta) > \delta + \inf_{\Theta \in \mathbb{R}^3} R_{f,H} (\Theta) \}
\}
(4.121)
is an uncountable set (cf. Corollary 4.8), and let \( \bar{f} \in C([\alpha, \beta], \mathbb{R}) \) satisfy for all \( x \in [\alpha, \beta] \) that
\[
f(x) = \begin{cases}
\left[\frac{\delta}{2}\right]^{1/2}f(x) & : \delta \geq 0, \\
f(x) & : \delta < 0.
\end{cases}
(4.122)
\]
Observe that the choice of \( \delta \) ensures that for all \( x \in (0, 1) \) it holds that
\[
\left[\frac{4(\beta - \alpha) + 3\alpha^2 - 1}{4(1-\alpha)^{3/2}(1+3\alpha)^{3/2}}\right]' = \frac{1}{(1-\alpha)^{3/2}(1+3\alpha)^{3/2}},
(4.123)
\]
\[
\frac{3\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}}' = \frac{3\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}},
(4.124)
\]
\[
\frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}}' = \frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}},
(4.125)
\]
\[
\frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}}' = \frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}},
(4.126)
\]
\[
\frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}}' = \frac{12\beta^2 - (18\beta^2 + 8\beta - 2)x + 3\beta^4 + 10\beta^2 - 1}{4(1-\beta)^{3/2}(1+3\beta)^{3/2}},
(4.127)
\]
and

$$\left[\frac{12\beta x^2-(18\beta^2+8\beta-2)x+3\beta^4+10\beta^3-1}{4(1-\beta)^{5/2}(1+3\beta)^{3/2}}\right]' = \frac{24\beta x-18\beta^2-8\beta+2}{(1-\beta)^{5/2}(1+3\beta)^{3/2}} = \frac{12\beta x-9\beta^2-4\beta+1}{2(1-\beta)^{5/2}(1+3\beta)^{3/2}}.$$

(4.124)

The fact that \(\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} f(x) = f(\alpha)\), the fact that \(\lim_{x\to\beta} f(x) = \lim_{x\to\beta} f(x) = f(\beta)\), the fact that for all \(x \in (\alpha, \beta)\) it holds that

$$0 < \frac{1}{(1-x)^{5/2}(1+3\alpha)^{3/2}} < \frac{1}{(1-x)^{5/2}(1+3\alpha)^{3/2}},$$

(4.125)

and the fact that for all \(x \in (\beta, 1)\) it holds that

$$\frac{3\beta^2-4\beta+1}{2(1-\beta)^{5/2}(1+3\beta)^{3/2}} < \frac{12\beta x-9\beta^2-4\beta+1}{2(1-\beta)^{5/2}(1+3\beta)^{3/2}} < \frac{-9\beta^2+8\beta+1}{2(1-\beta)^{5/2}(1+3\beta)^{3/2}}$$

(4.126)

therefore show that \([0, 1] \ni x \mapsto f(x) \in \mathbb{R}\) is Lipschitz continuous. This implies that \([\alpha, \beta] \ni x \mapsto f(x) \in \mathbb{R}\) is Lipschitz continuous. Next note that the fact that for all \(N \in \mathbb{N}\), \(f \in C([\alpha, \beta], \mathbb{R})\), \(\theta \in \mathbb{R}^{3H+1}\) it holds that \(\mathcal{R}_{f, \delta}(\theta) = \int_{0}^{\infty} (f(x) - N_{H}^{\theta}(x))^2 dx\) ensures that for all \(f \in C([\alpha, \beta], \mathbb{R})\), \(c \in \mathbb{R}\), \(\theta = (\theta_1, \ldots, \theta_{b})\), \(\vartheta = (\vartheta_1, \ldots, \vartheta_{b}) \in \mathbb{R}^b\) with \(\forall j \in \{1, 2, \ldots, b\}\): \([\vartheta_{2H+j}] \wedge [\theta_j] = [\theta_{2H+j}] \wedge [\theta_{2H+j}]\) it holds that \(\mathcal{R}_{f, H}(\vartheta) = |c|^2 \mathcal{R}_{f, H}(\theta)\). Combining this with (4.120) demonstrates that

$$\left\{ v \in C([\alpha, \beta], \mathbb{R}) : \exists \theta \in \{ \theta \in \mathbb{R}^{3H+1} : v = N_{H}^{\theta}, \vartheta \in (0, \infty) : \mathcal{R}_{f, H}(\theta) = \inf_{\vartheta \in \mathbb{R}^{3H+1}} \mathcal{R}_{f, H}(\theta, \vartheta) > \delta \right\}$$

is an uncountable set. The proof of Corollary 4.9 is thus complete.

\[\square\]
(theta[j]*x+theta[H+j]), 0, 1, limit=100)[0]
g[2*H+j] = 2*integrate.quad(lambda x:
np.maximum(theta[j]*x+theta[H+j],0) * (realization(theta,x)-f(x)), 0, 1, limit=100)[0]
g[3*H] = 2*integrate.quad(lambda x:
realization(theta,x)-f(x), 0, 1, limit=100)[0]
return g

# vectorized versions of f, g, and the realization function
vec_f = np.vectorize(f)
vec_g = np.vectorize(g)
vec_realization = np.vectorize(realization, excluded=['theta'])

# initialize each ANN with the Xaviar initialization
parameters = np.random.normal(scale=math.sqrt(2/(H+1)), size=(3*H+1)*M)
for i in range(M):
    parameters[i*(3*H+1)+H:i*(3*H+1)+2*H] = parameters[i*(3*H+1)+3*H] = 0
theta = np.reshape(parameters, (M,3*H+1))

# train each ANN with the GD optimization method
for i in range(M):
    while (True):
        grad = gen_grad(theta[i])
        if np.linalg.norm(grad,np.inf) < 1e-4:
            break
        theta[i] = theta[i] - eta * grad

# divide the trained ANNs into groups with a similar realization function
groups = []
for i in range(M):
    for group in groups:
        if(integrate.quad(lambda x: (realization(theta[i],x) - realization(group[1],x))**2, 0, 1)[0] < 1e-4):
            group.append(theta[i])
            break
    else:
        groups.append([integrate.quad(lambda x: (realization(theta[i],x)-f(x))**2, 0, 1)[0], theta[i]])
groups = sorted(groups, key=lambda g: g[0])

# plot the simulation with one representative realization function per group
plt.rcParams["axes.prop_cycle"] = plt.cycler('color', ['tab:blue','tab:red','tab:orange','tab:green','tab:purple'])
plt.figure()
plt.margins(x=0)
plt.xticks(np.arange(0,1.1,0.1))
plt.ylim((-0.25,0.55))
x = np.linspace(0,1,10000)
plt.plot(x, vec_f(x), label="risk"+r'\$=0$ (target function)", zorder=M+1)
for group in groups:
    plt.plot(x, vec_realization(theta=group[1],x=x),
             label="risk"+r'\$\approx$'+format(group[0], 'f'))
plt.legend()
plt.savefig("simulation.pdf", bbox_inches="tight")

# plot ten realization funct. of the considered non-global loc. min. points
plt.figure()
plt.margins(x=0)
plt.xticks(np.arange(0,1.1,0.1))
plt.ylim((-0.25,0.55))
5 On finitely many realization functions of critical points

In this section we prove in Corollary 5.8 in Subsection 5.6 below in the special situation where the target function \( f : [a, b] \to \mathbb{R} \) is continuous and piecewise polynomial and where both the input layer and the hidden layer of the considered ANNs are one-dimensional that there exist only finitely many different realization functions \( \mathcal{N}_\infty^\theta \subset C([a, b], \mathbb{R}), \theta \in \mathbb{R}^4 \), of all critical points of the risk function \( R_\infty : \mathbb{R}^4 \to \mathbb{R}^4 \) in the sense that there exist only finitely many different realization functions \( \mathcal{N}_\infty^\theta \subset C([a, b], \mathbb{R}), \theta \in \mathbb{R}^4 \), of zeros of the generalized gradient function \( G : \mathbb{R}^4 \to \mathbb{R}^4 \). In Corollary 5.9 in Subsection 5.6 we extend Corollary 5.8 by using [19, Item (v) in Theorem 1.1] and [11, Theorem 1.2] to establish that in the training of such ANNs we have that the risk of every non-divergent GF trajectory converges to the risk of a global minimum point provided that the initial risk is sufficiently small.

The remainder of this section is organized in the following way. In Setting 5.1 in Subsection 5.1 below we present our mathematical setup of ANNs with one-dimensional input and hidden layer, in the elementary result in Lemma 5.2 in Subsection 5.2 below we analyze critical points with constant realization functions, in the elementary result in Lemma 5.3 in Subsection 5.3 below we analyze critical points with affine linear realization functions, in Lemma 5.6 in Subsection 5.4 below we analyze critical points with non-decreasing non-affine linear realization functions, in Subsection 5.5 below we analyze critical points with non-increasing non-affine linear realization functions. In Subsection 5.6 we combine Lemma 5.2, Lemma 5.3, Lemma 5.6, and Lemma 5.7 to establish Corollary 5.8.

5.1 ANNs with one-dimensional input and hidden layer

Setting 5.1. Let \( n \in \mathbb{N}, f_0, f_1, \ldots, f_n \in \mathbb{R}, a \in \mathbb{R}, \theta \in (a, \infty) \) satisfy \( a = f_0 < f_1 < \ldots < f_n = \theta \), let \( f \in C((a, \theta], \mathbb{R}) \) satisfy for all \( j \in \{1, 2, \ldots, n\} \) that \( f|_{[f_{j-1}, f_j]} \) is a polynomial, for every \( \theta = (\theta_1, \ldots, \theta_4) \in \mathbb{R}^4 \) let \( I^\theta \subseteq \mathbb{R} \) satisfy \( I^\theta = \{ x \in [a, \theta] : \theta_2 + \theta_3 x \geq 0 \} \), let \( A_r : \mathbb{R} \to \mathbb{R}, r \in \mathbb{N} \cup \{ \infty \}, \) satisfy for all \( x \in \mathbb{R} ) \) that \( (\cup_{r\in \mathbb{N}} \{A_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R}), A_\infty(x) = \max\{x, 0\}, \sup_{r\in \mathbb{N}} \sup_{y \in [-\|x\|, \|x\|]} \|\mathcal{A}_r\|'(y)\} < \infty, \) and

\[
\limsup_{r \to \infty} (|A_r(x) - A_\infty(x)| + |(A_r)'(x) - 1_{(0, \infty)}(x)|) = 0, \quad (5.1)
\]

for every \( r \in \mathbb{N} \cup \{ \infty \} \) let \( \mathcal{N}_r^\theta \subset C((a, \theta], \mathbb{R}), \theta \in \mathbb{R}^4 \), and \( \mathcal{N}_r^\theta : \mathbb{R}^4 \to \mathbb{R} \) satisfy for all \( \theta = (\theta_1, \ldots, \theta_4) \in \mathbb{R}^4, x \in [a, \theta] \) that \( \mathcal{N}_r^\theta(x) = \theta_1 + \theta_3 A_r(\theta_2 + \theta_3 x) \) and \( \mathcal{N}_r^\theta(\theta) = \int_0^\theta (\mathcal{N}_r^\theta(y) - f(y))^2 dy, \) and let \( G : \mathbb{R}^4 \to \mathbb{R}^4 \) satisfy for all \( \theta \in \{ \theta \in \mathbb{R}^4 : ((\nabla \mathcal{R}_r)(\theta))_{r\in \mathbb{N}} \) is convergent \} that \( G(\theta) = \lim_{r \to \infty} (\nabla \mathcal{R}_r)(\theta). \)

5.2 Critical points with constant realization functions

Lemma 5.2. Assume Setting 2.1. Then

\[
\#\{v \in C([a, \theta]^d, \mathbb{R}) : (\exists \theta \in \mathcal{G}^{-1}(\{0\}) : \forall x, y \in [a, \theta]^d : v(x) = \mathcal{N}_r^\theta(y))\} = 1. \quad (5.2)
\]
Proof of Lemma 5.2. Throughout this proof let $V \subseteq \mathbb{R}^3$ satisfy

$$V = \{ \theta \in \mathcal{G}^{-1}(\{0\}) : \forall x, y \in [a, b]^d : \mathcal{N}_\infty^0(x) = \mathcal{N}_\infty^0(y) \}. \tag{5.3}$$

Observe that Lemma 2.6 and Lemma 2.7 ensure that

(i) it holds for all $\theta \in V$ that $[a, b]^d \ni x \mapsto \mathcal{N}_\infty^0(x) \in \mathbb{R}$ is constant,

(ii) it holds for all $\theta = (\theta_1, \ldots, \theta_3) \in V$ that $\mathbb{R} \ni t \mapsto \mathcal{R}_\infty(\theta_1, \theta_2, \ldots, \theta_3, t) \in \mathbb{R}$ is differentiable, and

(iii) it holds for all $\theta \in V$ that

$$\left(\frac{\partial}{\partial \theta_4^i}\mathcal{R}_\infty\right)(\theta) = 2 \int_{[a, b]^d} (\mathcal{N}_\infty^0(x) - f(x)) \mu(dx) = 0. \tag{5.4}$$

Therefore, we obtain that for all $\theta \in V$, $x \in [a, b]^d$ it holds that

$$\mathcal{N}_\infty^0(x) = \frac{1}{\mu([a, b]^d)} \int_{[a, b]^d} (\mathcal{N}_\infty^0(y) - f(y)) \mu(dy) + \int_{[a, b]^d} f(y) \mu(dy). \tag{5.5}$$

This shows that for all $\theta, \vartheta \in V$ it holds that $\mathcal{N}_\infty^0 = \mathcal{N}_\infty^0$. This establishes (5.2). The proof of Lemma 5.2 is thus complete. \hfill \Box

5.3 Critical points with affine linear realization functions

Lemma 5.3. Assume Setting 5.1. Then

$$\#(\{v \in C([a, b], \mathbb{R}) : (\exists \theta = (\theta_1, \ldots, \theta_4) \in \mathcal{G}^{-1}(\{0\}) : [(\theta_1 \theta_3 \neq 0) \land (I^\theta = [a, b]) \land (v = \mathcal{N}_\infty^0)]\}) = 1. \tag{5.6}$$

Proof of Lemma 5.3. Throughout this proof let $V \subseteq \mathbb{R}^4$ satisfy

$$V = \{ \theta = (\theta_1, \ldots, \theta_4) \in \mathcal{G}^{-1}(\{0\}) : [(\theta_1 \theta_3 \neq 0) \land (I^\theta = [a, b])]\}. \tag{5.7}$$

Note that Lemma 2.6 and Lemma 2.7 show that

(i) it holds for all $\theta \in V$ that $\mathcal{R}_\infty$ is differentiable at $\theta$ and

(ii) it holds for all $\theta \in V$ that $(\nabla \mathcal{R}_\infty)(\theta) = \mathcal{G}(\theta) = 0$.

Hence, we obtain that for all $\theta = (\theta_1, \ldots, \theta_4) \in V$ it holds that

$$\left(\frac{\partial}{\partial \theta_4^i}\mathcal{R}_\infty\right)(\theta) = 2 \int_{[a, b]^d} (\mathcal{N}_\infty^0(x) - f(x)) \mu(dx) = 0$$

$$= 2 \int_{[a, b]^d} (\theta_1 x + \theta_2)(\mathcal{N}_\infty^0(x) - f(x)) \mu(dx) = \left(\frac{\partial}{\partial \theta_4^i}\mathcal{R}_\infty\right)(\theta). \tag{5.8}$$

This implies that for all $\theta \in V$ it holds that

$$\int_{[a, b]^d} (\mathcal{N}_\infty^0(x) - f(x)) \mu(dx) = 0 = \int_{[a, b]^d} x(\mathcal{N}_\infty^0(x) - f(x)) \mu(dx). \tag{5.9}$$

Combining this with the fact that for all $\theta = (\theta_1, \ldots, \theta_4)$, $x \in [a, b]$ it holds that $\mathcal{N}_\infty^0(x) = \theta_1 \theta_3 x + (\theta_4 + \theta_3 \theta_2)$ ensures that for all $\theta = (\theta_1, \ldots, \theta_4) \in V$ it holds that

$$\int_{[a, b]^d} (\theta_1 \theta_3 x + (\theta_4 + \theta_3 \theta_2) - f(x)) \mu(dx) = \theta_1 \theta_3 \left[\frac{\theta_4^2 - \theta_3^2}{2}\right] + (\theta_4 + \theta_3 \theta_2)(\vartheta - a) - \int_{[a, b]^d} f(x) \mu(dx) = 0 \tag{5.10}$$
 throughout this proof let
\[ \theta_1 \theta_3 x + (\theta_4 + \theta_2) = f(x) = \theta_1 \theta_3 \left[ \frac{\theta^3 - \theta^2}{3} + (\theta_4 + \theta_3 \theta_2) \frac{\theta^2 - \theta^2}{2} \right] - \int_{-\theta}^x f(x) \, dx = 0. \]  
(5.11)

The fact that
\[ \left( \frac{\theta^3 - \theta^2}{4} \right)^2 - [\theta - \alpha \left( \frac{\theta^3 - \theta^2}{3} \right)] = -\frac{\theta - \alpha}{12} \left[ 4(\theta^2 + \theta \alpha + \alpha^2) - 4(\theta + \alpha) \right] = \frac{(\theta - \alpha)^4}{12} \neq 0 \]  
(5.12)

therefore shows that there exist \( c_1, c_2 \in \mathbb{R} \) which satisfy for all \( \theta = (\theta_1, \ldots, \theta_4) \in V \) that
\[ \theta_1 \theta_3 = c_1 \quad \text{and} \quad \theta_4 + \theta_3 \theta_2 = c_2. \]  
(5.13)

Observe that (5.13) implies that for all \( \theta, \alpha \) it holds that \( \mathcal{N}_{\infty}(x) = c_1 x + c_2 \). The proof of Lemma 5.3 is thus complete.

\section{Critical points with non-decreasing non-affine linear realization functions}

\begin{lemma}
Let \( n \in \mathbb{N} \), \( x_0, x_1, \ldots, x_n \in \mathbb{R} \) satisfy \( 0 = x_0 < x_1 < \ldots < x_n = 1 \), let \( \ell \in C([0,1], \mathbb{R}) \) satisfy for all \( j \in \{1, 2, \ldots, n\} \) that \( \ell_{\left| [x_{j-1}, x_j] \right.} \) is a polynomial, and let \( j \in \{1, 2, \ldots, n\} \). Then
\[ \{ q \in [x_{j-1}, x_j] \setminus \{0,1\} : \right. \]
\[ \left. \begin{array}{c}
[1 - q_1 \int_0^q f(x) \, dx] = \int_0^1 (\frac{x+1}{3} - x) f(x) \, dx \right. \\
\left. \left( \int_0^1 f(x) \, dx \neq \frac{1}{q} \int_0^q f(x) \, dx \right) \right. \\
\left. \left( \int_0^1 f(x) \, dx \neq \frac{1}{q} \int_0^q f(x) \, dx \right) \right. \\
\right. \]
\[ \left. \right. \]
\[ \left. \right. \]  
(5.14)

is a finite set.
\end{lemma}

\begin{proof}
Throughout this proof let \( d \in \mathbb{N}_0, q \in [x_{j-1}, x_j] \setminus \{0,1\} \) satisfy
\[ \frac{1 - q^2}{6q} \int_0^q f(x) \, dx = \int_0^1 (\frac{q+1}{3} - x) f(x) \, dx \quad \text{and} \quad \int_0^1 f(x) \, dx \neq \frac{1}{q} \int_0^q f(x) \, dx, \]  
(5.15)

and \( d = \deg(\ell_{\left| [x_{j-1}, x_j] \right.}) \). In the following we distinguish between the case \( d = 0 \) and the case \( d > 0 \). First we prove (5.14) in the case \( d = 0 \). Note that the assumption that \( d = 0 \) implies that there exists \( c \in \mathbb{R} \) which satisfies for all \( x \in [x_{j-1}, x_j] \) that \( f(x) = c \). Therefore, we obtain that
\[ (1 - q)^2 \int_0^q f(x) \, dx = \int_0^1 (\frac{q+1}{3} - x) f(x) \, dx + \int_0^1 c \, dx \]  
and
\[ 2q \int_0^1 (q + 2 - 3x) c \, dx = 2q \int_0^1 c \, dx + \int_0^1 (q + 2 - 3x) f(x) \, dx. \]  
(5.16)

The fact that
\[ (1 - q)^2 \left[ \int_0^1 f(x) \, dx + \int_0^1 c \, dx \right] \]
\[ = (q^2 - 2q + 1) \left[ \int_0^1 f(x) \, dx + cq - cx_{j-1} \right] = cq^3 + q^2 \left[ \int_0^1 f(x) \, dx - cx_{j-1} - 2c \right] \]  
(5.17)

and the fact that
\[ 2q \left[ \int_0^q (q + 2 - 3x) c \, dx + \int_0^1 (q + 2 - 3x) f(x) \, dx \right] \]
\[ = 2q \left[ 2cx_{j-1} - 2cq + 2cx_{j} - 2cq - \frac{2}{3} c \right] + \frac{2}{3} c q^2 + \int_0^1 f(x) \, dx + 2 \int_0^1 f(x) \, dx \]
\[ - 3 \int_0^1 x f(x) \, dx \]
\[ = cq^3 + c \left[ 2cx_{j} - 4c + 2 \int_0^1 f(x) \, dx \right] \]
\[ + q(4cx_{j-1} - 3c \left[ x_{j-1} \right]^2 + 4 \int_0^1 f(x) \, dx - 6 \int_0^1 x f(x) \, dx \]  
(5.18)
hence show that
\[
q^2 \left[ \int_0^{x_{j-1}} f(x) \, dx - c \, x_{j-1} + 2c - 2c \, x_j - 2 \int_{x_j} f(x) \, dx \right] + q \left[ c + 2c \, x_{j-1} - 4c \, x_j + 3c[x_j]^2 - 2 \int_0^{x_{j-1}} f(x) \, dx - 4 \int_{x_j} f(x) \, dx + 6 \int_{x_j} x \, f(x) \, dx \right] + \left[ \int_0^{x_{j-1}} f(x) \, dx - c \, x_{j-1} \right] = 0. \tag{5.19}
\]

Next observe that (5.15) ensures that
\[
\int_0^{x_{j-1}} f(x) \, dx - \frac{1}{q} \int_0^{x_j} f(x) \, dx = \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} f(x) \, dx - \frac{1}{q} \left[ \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} f(x) \, dx \right] \tag{5.20}
\]
This shows that
\[
\left[ \int_0^{x_{j-1}} f(x) \, dx - c \, x_{j-1} + 2(c - c \, x_j - \int_{x_j} f(x) \, dx) \right]^2 + \left[ \int_0^{x_{j-1}} f(x) \, dx - c \, x_{j-1} \right] \neq 0. \tag{5.21}
\]
Combining this with (5.19) establishes (5.14) in the case \( d = 0 \). In the next step we prove (5.14) in the case \( d > 0 \). Note that the assumption that \( d > 0 \) assures that there exist \( a \in \mathbb{R} \setminus \{0\} \) and \( Q: \mathbb{R} \to \mathbb{R} \) which satisfy for all \( x \in [x_{j-1}, x_j] \) that \( Q \) is a polynomial with \( \deg(Q) \leq d - 1 \) and \( f(x) = ax^d + Q(x) \). Hence, we obtain that
\[
(1 - q)^2 \left[ \int_0^{x_j} f(x) \, dx \right] = (1 - q)^2 \left[ \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} (ax^d + Q(x)) \, dx \right] \tag{5.22}
\]
and
\[
2q \int_q^{x_j} (q + 2 - 3x) f(x) \, dx = 2q \left[ \int_q^{x_j} (q + 2 - 3x)(ax^d + Q(x)) \, dx + \int_{x_j} (q + 2 - 3x)f(x) \, dx \right]. \tag{5.23}
\]
Combining this with (5.15) shows that
\[
(1 - q)^2 \left[ \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} (ax^d + Q(x)) \, dx \right] = 2q \left[ \int_q^{x_j} (q + 2 - 3x)(ax^d + Q(x)) \, dx + \int_{x_j} (q + 2 - 3x)f(x) \, dx \right]. \tag{5.24}
\]
Next observe that the fact that \( \deg(Q) \leq d - 1 \) demonstrates that there exist polynomials \( \mathcal{P}: \mathbb{R} \to \mathbb{R} \) and \( \mathcal{Y}: \mathbb{R} \to \mathbb{R} \) which satisfy \( \max\{\deg(\mathcal{P}), \deg(\mathcal{Y})\} \leq d + 2 \),
\[
(1 - q)^2 \left[ \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} (ax^d + Q(x)) \, dx \right] = (q^2 - 2q + 1) \left[ \frac{a}{d+1} \left(q^{d+1} - [x_{j-1}]^{d+1} + \int_0^{x_{j-1}} f(x) \, dx + \int_{x_j} Q(x) \, dx \right] \tag{5.25}
\]
and
\[
2q \left[ \int_q^{x_j} (q + 2 - 3x)(ax^d + Q(x)) \, dx + \int_{x_j} (q + 2 - 3x)f(x) \, dx \right] = 2q \left[ \int_q^{x_j} (q - 3x)ax^d \, dx + \int_{x_j} (q - 3x)Q(x) \, dx + 2 \int_q^{x_j} (ax^d + Q(x)) \, dx \right] + \int_{x_j} (q + 2 - 3x)f(x) \, dx \right] \tag{5.26}
\]
and
\[
= q^{d+3} \left[ \frac{2a}{d+1} + \mathcal{Y}(q) \right] + \mathcal{P}(q) = q^{d+3} \left[ \frac{2a(d+1)}{(d+1)(d+2)} \right] + \mathcal{P}(q). \tag{5.27}
\]
Combining this with (5.24) ensures that
\[
q^{d+3} \left[ \frac{2a(d+1)}{(d+1)(d+2)} \right] + \mathcal{P}(q) = q^{d+3} \left[ \frac{3a}{(d+1)(d+2)} \right] + \mathcal{P}(q) = 0. \tag{5.27}
\]
The fact that \( a \neq 0 \), the fact that \( d > 0 \), and the fact that \( \max\{\deg(\mathcal{P}), \deg(\mathcal{Y})\} \leq d + 2 \) hence establish (5.14) in the case \( d > 0 \). The proof of Lemma 5.4 is thus complete. \( \square \)
Corollary 5.5. Let $n \in \mathbb{N}$, $x_0, x_1, \ldots, x_n \in \mathbb{R}$ satisfy $0 = x_0 < x_1 < \ldots < x_n = 1$ and let $\varphi \in C([0,1], \mathbb{R})$ satisfy for all $j \in \{1,2,\ldots,n\}$ that $\varphi_{|x_{j-1},x_j}$ is a polynomial. Then

$$q \in (0,1): \left\{ \left( \int_0^q \varphi(x) \, dx = f_1^1 q \frac{q^2}{3} - x \right) \varphi(x) \, dx \right\} \bigcap \left\{ \int_0^q \varphi(x) \, dx \neq \frac{1}{q} \int_0^q \varphi(x) \, dx \right\} \right \} \quad (5.28)$$

is a finite set.

Proof of Corollary 5.5. Throughout this proof for every $j \in \{1,2,\ldots,n\}$ let $S_j \subseteq (0,1)$ satisfy

$$S_j = \left\{ q \in [x_{j-1}, x_j] \setminus \{0,1\} : \left\{ \left( \int_0^q \varphi(x) \, dx = f_1^1 q \frac{q^2}{3} - x \right) \varphi(x) \, dx \right\} \bigcap \left\{ \int_0^q \varphi(x) \, dx \neq \frac{1}{q} \int_0^q \varphi(x) \, dx \right\} \right \} \quad (5.29)$$

Note that Lemma 5.4 and (5.29) ensure that for all $j \in \{1,2,\ldots,n\}$ it holds that $S_j$ is a finite set. The fact that

$$\bigcup_{j \in \{1,2,\ldots,n\}} S_j = \left\{ q \in (0,1) : \left\{ \left( \int_0^q \varphi(x) \, dx = f_1^1 q \frac{q^2}{3} - x \right) \varphi(x) \, dx \right\} \bigcap \left\{ \int_0^q \varphi(x) \, dx \neq \frac{1}{q} \int_0^q \varphi(x) \, dx \right\} \right \} \quad (5.30)$$

hence establishes (5.28). The proof of Corollary 5.5 is thus complete. \hfill \Box

Lemma 5.6. Assume Setting 5.1. Then

$$\left\{ v \in C([a,b], \mathbb{R}) : (\exists \theta = (\theta_1, \ldots, \theta_4) \in G^{-1}(\{0\}) \cdot \left\{ (\theta_1 > 0 \neq \theta_3) \wedge (a < -(\theta_2/\theta_1) < b) \wedge (v = N_\infty^q) \right\} \right\} \quad (5.31)$$

is a finite set.

Proof of Lemma 5.6. Throughout this proof let $\varphi \in C([0,1], \mathbb{R})$ satisfy for all $x \in [0,1]$ that

$$\varphi(x) = f(x/(\theta - a) + a), \text{ let } \Theta = (\Theta_1, \ldots, \Theta_4) \in \{ (\theta = (\theta_1, \ldots, \theta_4) \in G^{-1}(\{0\}) : \left\{ (\theta_1 > 0 \neq \theta_3) \wedge (a < -(\theta_2/\theta_1) < b) \right\} \right\} \right\}$$

let $\theta = (w, b, w, c) \in \mathbb{R}^4$ satisfy $w = \Theta_1(\theta - a)$, $b = \Theta_1 a + \Theta_2$, $v = \Theta_3$, and $c = \Theta_4$, and let $q \in \mathbb{R}$ satisfy $q = -b/w$. Observe that the fact that $|\Theta_1| + |\Theta_2| > 0$, item (i) in Lemma 2.6, and item (i) in Lemma 2.7 show that $R_\infty$ is differentiable at $\Theta$. The fact that $G(\Theta) = 0$, Lemma 2.6, Lemma 2.7, and the integral transformation theorem therefore ensure that

$$\left( \frac{\partial}{\partial \Theta_1} R_\infty \right)(\Theta) = 2 \Theta_3 \int_{\theta} \varphi(x)(N_\infty^q(x) - f(x)) \, dx = 2 \Theta_3 \int_{\theta} \varphi(x)(\Theta_1 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx$$

$$= 2(\theta - a)\Theta_3 \int_{\theta} \left( (\theta - a)x + a \right)(\Theta_4 + \Theta_3[\Theta_2 + \Theta_1((\theta - a)x + a)]) - f((\theta - a)x + a)) \, dx$$

$$= 2(\theta - a)v \int_{\theta} \left( (\theta - a)x + a \right)c + vw(x - q) - f(x) \, dx = 0,$$  

(5.32)

$$\left( \frac{\partial}{\partial \Theta_2} R_\infty \right)(\Theta) = 2 \Theta_3 \int_{\theta} \varphi(x)(N_\infty^q(x) - f(x)) \, dx = 2 \Theta_3 \int_{\theta} \varphi(x)(\Theta_2 + \Theta_1 x, 0) - f(x)) \, dx$$

$$= 2(\theta - a)\Theta_3 \int_{\theta} \left( \Theta_1 + \Theta_3[\Theta_2 + \Theta_1((\theta - a)x + a)]) - f((\theta - a)x + a)) \, dx$$

$$= 2(\theta - a)v \int_{\theta} \left( c + vw(x - q) - f(x) \right) \, dx = 0,$$  

(5.33)

$$\left( \frac{\partial}{\partial \Theta_4} R_\infty \right)(\Theta) = 2 \int_{\theta} \varphi(x)(N_\infty^q(x) - f(x)) \, dx = 2 \int_{\theta} \varphi(x)(\Theta_1 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx$$

$$= 2(\theta - a)\Theta_3 \int_{\theta} \left( (\theta - a)x + a \right)(\Theta_4 + \Theta_3[\Theta_2 + \Theta_1((\theta - a)x + a)]) - f((\theta - a)x + a)) \, dx$$

$$= 2(\theta - a)v \int_{\theta} \left( (\theta - a)x + a \right)c + vw(x - q) - f(x) \, dx = 0,$$  

(5.34)
and
\[
(\frac{\partial}{\partial \Theta_1} R_\infty)(\Theta) = 2 \int_{\Theta_1^0} (N(\nabla_\infty^0(x) - f(x)) \, dx = 2 \int_{\Theta_1^0} (\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx
\]
\[
= 2(\Theta - \Theta_3)\int_{\Theta_1^0} (\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 ((\Theta - \Theta_3)x + \Theta_3), 0\} - f((\Theta - \Theta_3)x + \Theta_3)) \, dx
\]
\[
= 2(\Theta - \Theta_3)\int_{\Theta_1^0} (c + vw \max\{x - q, 0\} - f(x)) \, dx = 0.
\]

Hence, we obtain that
\[
\int_{\Theta_1^0} (c - f(x)) \, dx = \int_{\Theta_1^0} (c + vw(x - q) - f(x)) \, dx = \int_{\Theta_1^0} x(c + vw(x - q) - f(x)) \, dx = 0.
\]

This implies that
\[
cq - \int_{\Theta_1^0} f(x) \, dx = 0, \quad c(1 - q) + vw\left[\frac{1 - q^2}{2} - q(1 - q)\right] - \int_{\Theta_1^0} f(x) \, dx = 0,
\]
\[
\text{and} \quad c\left[\frac{1 - q^2}{2}\right] + vw\left[\frac{1 - q^2}{3} - \frac{q(1 - q^2)}{2}\right] - \int_{\Theta_1^0} x f(x) \, dx = 0.
\]

Therefore, we obtain that
\[

c = \frac{1}{q} \left[\int_{\Theta_1^0} f(x) \, dx\right], \quad \text{and} \quad \frac{1 - q^2}{2q} \left[\int_{\Theta_1^0} f(x) \, dx - \frac{1}{q} \int_{\Theta_1^0} f(x) \, dx\right],
\]
\[
\text{and} \quad \frac{(1 - q)^2}{3q^2} \int_{\Theta_1^0} f(x) \, dx = \int_{\Theta_1^0} \left[\frac{2q^2}{3} - x\right] f(x) \, dx.
\]

Moreover, note that the fact that \(\Theta_1 \Theta_3 \neq 0\) assures that \(vw \neq 0\). Combining this with (5.39) shows that \(\int_{\Theta_1^0} f(x) \, dx \neq \frac{1}{q} \int_{\Theta_1^0} f(x) \, dx\). Corollary 5.5, (5.39), and (5.40) hence establish (5.31). The proof of Lemma 5.6 is thus complete.

\[\square\]

### 5.5 Critical points with non-increasing non-affine linear realization functions

**Lemma 5.7.** Assume Setting 5.1. Then

\[
\{v \in C([\Theta, 1], \mathbb{R}) : (\exists \theta = (\theta_1, \ldots, \theta_4) \in G^{-1}\{\{0\}\}) : \quad [(\theta_1 < 0 \neq \theta_3) \wedge (\theta < -(\theta_2/\theta_1) < \theta_3) \wedge (v = N_\infty^0)]\}
\]

is a finite set.

**Proof of Lemma 5.7.** Throughout this proof let \(f, f \in C([0, 1], \mathbb{R})\) satisfy for all \(x \in [0, 1]\) that \(f(x) = f(x(\Theta - \Theta_3) + a)\) and \(f(x) = f(1 - x)\), let \(\Theta = (\Theta_1, \ldots, \Theta_4) \in \{\theta = (\theta_1, \ldots, \theta_4) \in G^{-1}\{\{0\}\}) : [(\theta_1 < 0 \neq \theta_3) \wedge (\theta < -(\theta_2/\theta_1) < \theta_3)]\}, \theta = (w, b, v, c) \in \mathbb{R}^4\) satisfy \(w = \Theta_1(\Theta - \Theta_3)\), \(b = \Theta_1(\Theta - \Theta_3)\), \(v = \Theta_3\), and \(c = \Theta_4\), and let \(q, q \in \mathbb{R}\) satisfy \(q = -\frac{b}{w}\) and \(q = 1 - q\). Observe that the fact that \(|\Theta_1| + |\Theta_2| > 0\), item (i) in Lemma 2.6, and item (i) in Lemma 2.7 show that \(R_\infty\) is differentiable at \(\Theta\). The fact that \(G(\Theta) = 0\), Lemma 2.6, Lemma 2.7, and the integral transformation theorem therefore ensure that

\[
(\frac{\partial}{\partial \Theta_1} R_\infty)(\Theta) = 2\Theta_3 \int_{\Theta_1^0} x(N_\infty^0(x) - f(x)) \, dx = 2\Theta_3 \int_{\Theta_1^0} x(\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx
\]
\[
= 2(\Theta - \Theta_3)\Theta_3 \int_{\Theta_1^0} (\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 ((\Theta - \Theta_3)x + \Theta_3), 0\} - f((\Theta - \Theta_3)x + \Theta_3)) \, dx
\]
\[
= 2(\Theta - \Theta_3)v \int_{\Theta_1^0} ((\Theta - \Theta_3)x + \Theta_3)(c + vw(x - q) - f(x)) \, dx = 0,
\]

(5.42)
Assume Setting 5.1. Then Corollary 5.8.

5.6 On finitely many realization functions of critical points of Lemma 5.7 is thus complete.

The fact that $\Theta$ is a finite set.

\[ \frac{\partial}{\partial \Theta_0} \mathcal{R}_\infty(\Theta) \]
\[ = 2\Theta_3 \int_\Theta (\mathcal{N}_\infty^\Theta(x) - f(x)) \, dx = 2\Theta_3 \int_\Theta (\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx \]
\[ = 2(\beta - \alpha) \Theta_3 \int_0^\Theta (\Theta_4 + \Theta_3 \{\Theta_2 + \Theta_1 ((\beta - \alpha)x + \alpha)\} - f((\beta - \alpha)x + \alpha)) \, dx \]
\[ = 2(\beta - \alpha) v \int_0^\Theta (c + vw(x - q) - f(x)) \, dx = 0, \]
\[(\frac{\partial}{\partial \Theta_1} \mathcal{R}_\infty(\Theta) \)
\[ = 2 \int_\Theta (\Theta_2 + \Theta_3 \max\{\Theta_2 + \Theta_1 x, 0\} - f(x)) \, dx = 2 \int_\Theta (\Theta_4 + \Theta_3 \max\{\Theta_2 + \Theta_1 ((\beta - \alpha)x + \alpha)\} - f((\beta - \alpha)x + \alpha)) \, dx \]
\[ = 2(\beta - \alpha) \int_0^\Theta (c + vw \min\{x, 0\} - f(x)) \, dx = 0. \]

Hence, we obtain that
\[ \int_q^1 (c - f(x)) \, dx = \int_0^\Theta (c + vw(x - q) - f(x)) \, dx = \int_0^\Theta x(c + vw(x - q) - f(x)) \, dx = 0. \] (5.46)

This implies that
\[ c(1 - q) - \int_q^1 f(x) \, dx = 0, \quad cq + vw[\frac{q^2}{2} - q^2] - \int_0^\Theta f(x) \, dx = 0, \] (5.47)
and
\[ c\left[\frac{q^2}{2}\right] + vw\left[\frac{q^2}{2} - \frac{q^3}{2}\right] - \int_0^\Theta x f(x) \, dx = 0. \] (5.48)

Therefore, we obtain that
\[ c = \frac{1}{1-q} \left[ \int_q^1 f(x) \, dx \right], \quad vw = \frac{2}{q} \left[ \frac{1}{1-q} \int_q^1 f(x) \, dx - \int_0^\Theta f(x) \, dx \right], \] (5.49)
and
\[ \frac{q^2}{6(1-q)} \left[ \int_q^1 f(x) \, dx \right] = \int_0^\Theta (x - \frac{q}{2}) f(x) \, dx. \] (5.50)

Combining this with the integral transformation theorem ensures that
\[ vw = \frac{2}{(1-q)^2} \left[ \frac{1}{q} \int_0^\Theta \hat{f}(y) \, dy - \int_0^1 \hat{f}(y) \, dy \right] \quad \text{and} \quad \frac{(1-q)^2}{q^2} \int_0^\Theta \hat{f}(y) \, dy = \int_0^1 (\frac{q+2}{3} - y) \hat{f}(y) \, dy. \] (5.51)

The fact that $\Theta_1 \Theta_3 \neq 0$, Corollary 5.5, (5.49), and (5.50) therefore establish (5.41). The proof of Lemma 5.7 is thus complete. \( \square \)

5.6 On finitely many realization functions of critical points

Corollary 5.8. Assume Setting 5.1. Then
\[ \{v \in C([\alpha, \beta], \mathbb{R}) : (\exists \theta \in G^{-1}(\{0\}) : v = \mathcal{N}_\infty^\Theta) \} \]
(5.52)
is a finite set.
Proof of Corollary 5.8. Throughout this proof let $\theta = (\theta_1, \ldots, \theta_3) \in \mathcal{G}^{-1}(\{0\})$. In the following we distinguish between the case $\forall x, y \in [\alpha, \beta]: N^0_{qc}(x) = N^0_{qc}(y)$, the case $\forall x, y \in (\alpha, \beta): (N^0_{qc})'(x) = (N^0_{qc})'(y) \neq 0$, and the case $\exists x, y \in (\alpha, \beta): (N^0_{qc})'(x) \neq (N^0_{qc})'(y)$. We first prove (5.52) in the case
\[ \forall x, y \in [\alpha, \beta]: N^0_{qc}(x) = N^0_{qc}(y). \tag{5.53} \]
Note that (5.53) and Lemma 5.2 establish (5.52) in the case $\forall x, y \in [\alpha, \beta]: N^0_{qc}(x) = N^0_{qc}(y)$. Next we prove (5.52) in the case
\[ \forall x, y \in (\alpha, \beta): (N^0_{qc})'(x) = (N^0_{qc})'(y) \neq 0. \tag{5.54} \]
Observe that (5.54) ensures that $\theta_1 \theta_3 \neq 0$ and $I^\theta = [\alpha, \beta]$. Lemma 5.3 hence establishes (5.52) in the case $\forall x, y \in (\alpha, \beta): (N^0_{qc})'(x) = (N^0_{qc})'(y) \neq 0$. In the next step we prove (5.52) in the case
\[ \exists x, y \in (\alpha, \beta): (N^0_{qc})'(x) \neq (N^0_{qc})'(y). \tag{5.55} \]
Note that (5.55) assures that $\theta_1 \theta_3 \neq 0$ and $\alpha < -\theta_2/\theta_1 < \beta$. Lemma 5.6 and Lemma 5.7 therefore establish (5.52) in the case $\exists x, y \in (\alpha, \beta): (N^0_{qc})'(x) \neq (N^0_{qc})'(y)$. The proof of Corollary 5.8 is thus complete. \hfill \square

Corollary 5.9. Assume Setting 5.1. Then

(i) it holds that $\{v \in C([\alpha, \beta], \mathbb{R}): \exists \theta \in \mathcal{G}^{-1}(\{0\}): v = N^0_{qc}\}$ is a finite set,

(ii) it holds that
\[ \{v \in C([\alpha, \beta], \mathbb{R}): \exists \theta \in \mathbb{R}^4: v = N^0_{qc}, \varepsilon \in (0, \infty): \mathcal{R}_\infty(\theta) = \inf_{\vartheta \in [-\varepsilon, \varepsilon]} \mathcal{R}_\infty(\theta + \vartheta)\} \tag{5.56} \]

is a finite set, and

(iii) there exists $\varepsilon \in (0, \infty)$ such that for all $\Theta = (\Theta_t)_{t \in [0, \infty)} \in C([0, \infty), \mathbb{R}^4)$ with $\liminf_{t \to \infty} \|\Theta_t\| < \infty$, $\forall t \in [0, \infty)$: $\Theta_t = \Theta_0 - \int_0^t G(\Theta_s) \, ds$, and $\mathcal{R}_\infty(\Theta_0) \leq \varepsilon + \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta)$ it holds that
\[ \limsup_{t \to \infty} \mathcal{R}_\infty(\Theta_t) = \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta). \tag{5.57} \]

(cf. Definition 2.4).

Proof of Corollary 5.9. Observe that Corollary 5.8 establishes item (i). Moreover, note that Proposition 2.12 implies that
\[ \{v \in C([\alpha, \beta], \mathbb{R}): \exists \theta \in \mathbb{R}^4: v = N^0_{qc}, \varepsilon \in (0, \infty): \mathcal{R}_\infty(\theta) = \inf_{\vartheta \in [-\varepsilon, \varepsilon]} \mathcal{R}_\infty(\theta + \vartheta)\} \tag{5.58} \]

is a finite set. Hence, we obtain that there exists a finite set $S \subseteq \mathbb{R}^4$ which satisfies
\[ \{v \in \mathbb{R}: (\exists \theta \in \mathcal{G}^{-1}(\{0\}) \cap (\mathcal{R}_\infty)^{-1}(\{\inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta), \infty\}): v = \mathcal{R}_\infty(\theta)\} = \cup_{\theta \in S} \{\mathcal{R}_\infty(\theta)\}. \tag{5.59} \]

In the following let $\varepsilon \in (0, \infty)$ satisfy $\varepsilon < [\min_{\theta \in S} \mathcal{R}_\infty(\theta)] - \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta)$ and let $\Theta = (\Theta_t)_{t \in [0, \infty)} \in C([0, \infty), \mathbb{R}^4)$ satisfy
\[ \liminf_{t \to \infty} \|\Theta_t\| < \infty, \quad \forall t \in [0, \infty): \Theta_t = \Theta_0 - \int_0^t G(\Theta_s) \, ds, \tag{5.60} \]

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and \( \mathcal{R}_\infty(\Theta_0) \leq \varepsilon + \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta) \) (5.62) (cf. Definition 2.4). Note that \([11, \text{Theorem 1.2}]\) ensures that \(\sup_{t \in [0, \infty)} \|\Theta_t\| < \infty\). The fact that for all \(\theta \in \mathcal{G}^{-1}(\{0\}) \cap (\mathcal{R}_\infty)^{-1}(\inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta), \infty))\) it holds that

\[ \mathcal{R}_\infty(\Theta_0) \leq \varepsilon + \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta) < \mathcal{R}_\infty(\theta) \] (5.63)

and \([19, \text{Item (v)} \text{in Theorem 1.1}]\) therefore show that

\[ \limsup_{t \to \infty} \mathcal{R}_\infty(\Theta_t) = \inf_{\vartheta \in \mathbb{R}^4} \mathcal{R}_\infty(\vartheta). \] (5.64)

This establishes item (iii). The proof of Corollary 5.9 is thus complete. \(\square\)

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