Geometric interpretation of the 2-index potential as twisted de Rham cohomology

TSOU Sheung Tsun*
and
Ioannis P. ZOIS†

Mathematical Institute,
24–29 St. Giles’, Oxford OX1 3LB, UK.

Abstract

It is found that the 2-index potential in nonabelian theories does not behave geometrically as a connection but that, considered as an element of the second de Rham cohomology group twisted by a flat connection, it fits well with all the properties assigned to it in various physical contexts. We also prove some results on the Euler characteristic of the twisted de Rham complex.

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*tsou@maths.ox.ac.uk
†izois@maths.ox.ac.uk, A.S. Onassis Public Benefit Foundation Hellas Scholar
1 The 2-index potential

A skew rank 2 tensor field arises in various contexts: string theory, supergravity, and the loop space formulation of Yang–Mills theory. For notational convenience, we shall consider such a field $B_{\mu\nu}(x)$ interchangeably as a 2-form over spacetime.

In the abelian case, the 2-index field is well studied [1] and fits neatly into the Dirac scheme of fields and potentials for general spin [2]. The field $B_{\mu\nu}(x)$ is usually regarded as a potential transforming under a gauge transformation $\Lambda_{\mu}(x)$ as
\[ \delta B_{\mu\nu}(x) = \partial_{\mu}\Lambda_{\nu}(x) - \partial_{\nu}\Lambda_{\mu}(x), \tag{1} \]

exactly as say the electromagnetic potential but with one more index. One can also readily define the field strength, as a 3-form field
\[ H_{\nu\rho\sigma} = \partial_{\sigma}B_{\nu\rho} + \partial_{\nu}B_{\rho\sigma} + \partial_{\rho}B_{\sigma\nu}. \tag{2} \]

The question immediately arises whether the $B_{\mu\nu}$ field can be interpreted as some sort of connection. This point was investigated by Teitelboim et al [3], and they found that one could regard such a 2-form as a parallel transport of loops (e.g. closed strings), provided the transformation is abelian, as in (1). But for nonabelian $B_{\mu\nu}$ we have to look elsewhere.

Freedman and Townsend [4] proposed a Lagrangian for the nonabelian $B_{\mu\nu}$. Cast in a first-order formulation of the non-linear $\sigma$ model, these fields appear as the dual of the Lagrange multiplier giving the flat connection constraint, thus
\[ \mathcal{L} = \text{Tr} A_{\mu}A_{\mu} + \text{Tr} \ast B_{\mu\nu}F_{\mu\nu}, \tag{3} \]

where $\ast B_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$ is the (Hodge) dual of $B_{\mu\nu}$, and Tr denotes the trace over the nonabelian indices. This action is invariant under the transformation
\[ \delta B_{\mu\nu} = D_{\mu}\Lambda_{\nu} - D_{\nu}\Lambda_{\mu}, \tag{4} \]

with $D_{\mu} = \partial_{\mu} - igA_{\mu}$ the covariant derivative with respect to $A_{\mu}$, but $A_{\mu}$ itself should not transform.

A similar Lagrangian appears in a loop space formulation of Yang–Mills theory as a nonlinear $\sigma$ model [5, 6]:
\[ \mathcal{L} = \text{Tr} A_{\mu}A_{\mu} + \text{Tr} \ast B_{\mu\nu}F_{\mu\nu}, \tag{5} \]
where $A$ is the logarithmic derivative of the holonomy of the gauge potential $A$, and $F$ the covariant curl of $A$. This is in exact analogy with the Freedman–Townsend Lagrangian (3). Although the loop variables $A_\mu$ can be thought of as a connection, under a Yang–Mills gauge transformation (which transforms the potential $A_\mu$ in the usual way), they are invariant. Moreover, the invariance of $L$ under such a gauge transformation demands that the $B$ field transforms as

$$\delta B_{\mu\nu} = D_\mu \Lambda_\nu - D_\nu \Lambda_\mu,$$

where $D_\mu = \delta_\mu - igA_\mu$ is the loop covariant derivative corresponding to $A_\mu$. This is exactly the Freedman–Townsend transformation (4). At the same time, this further confirms the result of [3] that nonabelian $B_{\mu\nu}$ does not behave like a connection, not even in loop space.

In this paper we shall present a geometric framework in which the $B_{\mu\nu}(x)$ field is not regarded as a gauge potential but as a cohomological freedom intimately related to the existence of a flat connection $A_\mu$. We go on to explore further mathematical consequences of this construction which may have useful physical applications.

## 2 Flat connections and the twisted de Rham complex

For ease of presentation, in this section we shall use almost exclusively the index-free notation of differential forms.

By a flat connection we mean one with zero curvature. This means that we shall include the more general case where the base space $X$ (e.g. spacetime) need not be simply connected, in which case a flat connection may have non-trivial holonomy. In fact, it is well known that gauge equivalent classes of flat connections are in 1–1 correspondence with conjugacy classes of irreducible representations of $\pi_1(X)$ into the gauge group $G$ [7].

If we denote the exterior covariant derivative and curvature of the connection $A$ by $d_A$ and $F_A$ respectively, then on any form $\omega$ we have

$$\begin{align*}
d_A^2 \omega &= d_A(d_A \omega) \\
&= d_A(d\omega + A \wedge \omega)
\end{align*}$$
\[ d(d\omega + A \wedge \omega) + A \wedge (d\omega + A \wedge \omega) = (dA + A \wedge A) \wedge \omega = F_A \wedge \omega. \tag{7} \]

Hence if \( A \) is flat, \( d^2_A = 0 \). This means that the exterior covariant derivative can actually be used as the differential in a differential complex, in direct contrast to the general Yang–Mills case.

Recall that associated to the principal \( G \) bundle over \( X \), with flat connection \( A \), we have a flat vector bundle \( E \) (with fibre the Lie algebra of \( G \)).

We can consider the space \( \Omega^p(X, E) \) of \( p \)-forms with values in \( E \), which is by definition the space of global sections of the vector bundle \( \Lambda^p T^* X \otimes E \), the tensor product of the \( p \)-th exterior power of the cotangent bundle \( T^* X \) and the vector bundle \( E \). Locally over an open set \( U \subset X \) such a \( p \)-form is given by

\[ \omega = \sum \omega_i \otimes e^i, \tag{8} \]

where \( \omega_i \) are \( p \)-forms on \( U \) and \( e^i \) are sections of \( E \) over \( U \), and the tensor product is over the algebra of \( C^\infty \) functions on \( U \). In our case of a flat vector bundle \( E \), we can extend the usual de Rham complex \( \Omega^*(X, d) \) over \( X \) to a complex \( \Omega^*(X, E, d_A) \) using the flat connection \( A \). The flatness guarantees the existence of locally constant sections \( e_1^U, \ldots, e_n^U \) (\( n=\text{rank of } E \)) with \( d_A e_i^U = 0 \). We can then define the exterior derivative \( d_A \omega \) of the form \( \omega \) by

\[ d_A(\sum \omega_e \otimes e_e^U) = \sum (d\omega_e) \otimes e_e^U \tag{9} \]

over the open set \( U \). Since the sections \( e_e^U \) are locally constant, it can readily be seen that \( d_A \omega \) agrees on overlaps and hence is globally defined \[.\]

Moreover, \( d_A^2 = 0 \). It therefore makes sense to define the cohomology groups \( H^*_A(X, E) \) as \( d_A \)-closed forms modulo \( d_A \)-exact forms in the usual way. It is easy to see that if \( E \) is a trivial bundle of rank \( n \) with the trivial flat connection, then \( H^*_\text{trivial}(X, E) \) is just \( n \) copies of the usual de Rham groups \( H^*(X) \).

It is generally recognized that cohomology group elements correspond to physically interesting quantities \[.\] If we now think of the \( B \) field not as a 2-from but as an element of \( H^2 \), then its transformation is nothing but the cohomological freedom of an exact 2-form:

\[ \delta B = d_A \Lambda \tag{10} \]
with $\Lambda$ a 1-form, in other words, the transformation (4). It is therefore *not* a
gauge freedom of the usual Yang–Mills type. Moreover, (10) reduces to (11) in
the abelian case, which need not therefore be interpreted as a gauge (in
the electromagnetism sense) transformation. In addition, the 3-form $d_A B$ is
as that discussed [10] for the ‘curvature’ of $B$.

As emphasized in [8] and obvious from the definition (9), the cohomological
groups depend in general on the particular trivialization chosen for $E$.
This means that, if we think of $A$ as a connection in a principal bundle (as
in the gauge case), then gauge equivalent $A$’s may give rise to different $B$’s.
This makes perfect sense for the theory in hand, because the term $\text{Tr } A^2$ in (3)
makes it immediately obvious that the Lagrangian $\mathcal{L}$ is *not* ‘gauge invariant’.
This is why whereas $B$ transforms as in (4), $A$ must remain invariant.

The same observations apply to the loop space formulation of Yang–Mills
theory. Since the phase factor is Yang–Mills gauge invariant, the loop space
connection $\mathcal{A}$ is also gauge invariant. So is of course the Lagrangian in (3).
On the other hand, there is no freedom in transforming the loop connection
$\mathcal{A}$, because that would mean changing the phase factor which is a physically
measurable quantity.

The twisted de Rham cohomology groups $H^*_A(X, E)$ are topological in-
variants which are defined whenever there is a flat connection on a vector
bundle $E$. Now a flat connection appears in many contexts which may be
physically interesting, notably in integrable systems. This is not surpris-
ing: a flat connection ensures integrability of lifts. The following results are
easy consequences of the definitions and may prove useful in studying the
invariances of such systems.

In analogy with the usual Euler characteristic of a manifold, we make the
following definition.[4]

**Definition** The *Euler characteristic* of the twisted de Rham complex
$\Omega^*(X, E, d_A)$ is defined to be

$$
\chi(X, E) = \sum_i (-1)^i \dim(H^i_A(X, E)).
$$

[4]Applying noncommutative geometry methods to the flat foliation induced by the flat
connection $A$, one can study the $\eta$-invariant (related to global anomalies) which is more
sensitive than the Euler characteristic defined here but less sensitive than the de Rham
cohomology groups: it depends on the gauge equivalence class of $A$. This will be reported
elsewhere [12].

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The notation makes sense because of the following result.

**Proposition** Let $E$ be the adjoint vector bundle $\text{ad}P$, associated to the principal bundle $P$ over a manifold $X$ with structure group $G$ assumed compact and connected, equipped with a flat connection $A$. With respect to the induced connection $E$ is a flat vector bundle. Then the Euler characteristic $\chi(X, E)$ is independent of the flat connection used in calculating the cohomology groups $H^i_A(X, E)$.

**Proof** We shall prove this by calculating the Euler characteristic $\chi(X, E)$ using the symbol of an elliptic operator associated to the differential $d_A$.

For simplicity we shall use the same symbol $d_A$ for all the differentials in the complex:

$$(d_A)_p: (\Lambda^p T^*X) \otimes E \rightarrow (\Lambda^{p+1} T^*X) \otimes E.$$  \hfill (12)

We can ‘assemble’ the bundle (for details see [[]]) by defining the single operator

$$D_{d_A}: \Omega^{\text{even}} \rightarrow \Omega^{\text{odd}},$$  \hfill (13)

where

$$\begin{align*}
\Omega^{\text{even}} &= \Gamma(\bigoplus_p (\Lambda^{2p} T^*X) \otimes E), \\
\Omega^{\text{odd}} &= \Gamma(\bigoplus_p (\Lambda^{2p+1} T^*X) \otimes E),
\end{align*}$$

defined by

$$D_{d_A} = d_A \oplus d_A^*,$$  \hfill (14)

that is,

$$D_{d_A}(\omega_0, \omega_2, \ldots) = (d_A \omega_0 + d_A^* \omega_2, d_A \omega_2 + d_A^* \omega_4, \ldots),$$  \hfill (15)

where $d_A^*$ is the formal adjoint of $d_A$ with respect to some Riemannian metric on $X$.

Recall the symbol $\sigma(D)$ of a differential operator from sections of a vector bundle $E$ to sections of a vector bundle $F$

$$D: \Gamma(E) \rightarrow \Gamma(F)$$  \hfill (16)

is a vector bundle homomorphism

$$\sigma(D): \pi^*(E) \rightarrow \pi^*(F),$$  \hfill (17)
where, for $SX$ the unit sphere bundle in the tangent bundle, $\pi$ is the canonical projection $SX \to X$. In local coordinates, since $D$ is first order in this case, $\sigma(D)$ is obtained by replacing $\partial/\partial \mu$ with $i\xi_\mu$, where $\xi_\mu$ is the $\mu$th coordinate in the cotangent bundle $T^*X$. Furthermore, $D$ is elliptic if its symbol is invertible. This can be extended to a differential complex $E$ which is elliptic if the corresponding sequence of symbols $\sigma(E)$ is exact outside the zero section of $TX$.

For the flat bundle $E$, if we denote by $\Delta_p$ the Laplacian on $\Omega^p(X, E)$, thus

$$\Delta_p = (d_A)_{p-1}(d_A^*)_{p-1} + (d_A^*)_p(d_A)_p,$$  

then (since $d_A^2 = d_A^*2 = 0$)

$$D_A D_A^* = \bigoplus \Delta_{2p}$$

$$D_A^* D_A = \bigoplus \Delta_{2p+1}.$$  

The exactness of the symbol complex $\sigma(\Omega(X, E))$ off the zero section then implies that $\sigma(\Delta_p)$ is an isomorphism (off the zero section). Therefore, $\Delta_p$ and hence $D_A$ are elliptic. Then it follows from the usual Hodge theory that

$$\text{Ker } D_A = \bigoplus h^{2i}$$

$$\text{Coker } D_A = \bigoplus h^{2i+1}$$

where $h^i$ are the harmonic sections of the bundle $\bigoplus (\Lambda^p T^* X) \otimes E$, namely elements of $\text{Ker } \Delta^i$. This means we have found an elliptic operator $D_A$ whose index gives the required Euler characteristic:

$$\text{ind}(D_A) = \chi(X, E).$$  

(19)

By the Atiyah–Singer index formula $[\square]$, we know that this depends only on the symbol of $D_A$ and not on $D_A$ itself.

It is obvious from the above that the symbol of $d_A$ is independent of the flat connection $A$ used, since the term of highest degree is $\partial/\partial \mu$. The symbol of $D_A$ is given by $i\xi - i\xi^*$ (where $\xi^*$ is contraction with $\xi$), also independent of $A$. The index of $D_A$ then gives the Euler characteristic as above, which is therefore independent of the flat connection $A$ used. □

**Corollary 1** When $X = \mathbb{R}^4$, $\chi(X, E) = \dim G$.  

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Proof Obvious. □

Corollary 2 Under a usual gauge transformation, the Freedman–Townsend theory remains in the same stable isomorphism class (of complex vector bundles over $X$).

Proof The homotopy classes of symbols of elliptic pseudo-differential operators are in one-one correspondence [11] with elements of $K^0(TX)$, where $TX$ is the tangent bundle of $X$, which in turn can be identified with the analytic $K$-homology group $K_0(X)$ [13]. The former is by definition stable isomorphism classes of vector bundles over $X$. □

3 Remarks

In the case when $B_{\mu\nu}$ is abelian, it can be shown easily [4] that the theory is equivalent to a massless scalar field. This is an example of the general duality between scalar fields and $(d-2)$-form fields, where $d$ is the dimension of spacetime [12]. Here $d = 4$. This duality also interchanges Bianchi identities (topology) and equations of motion (dynamics), reminiscent of the Wu–Yang treatment of electric and magnetic charges [15, 6]. Similar considerations apply in the nonabelian case, giving the equivalence between the first-order and second-order formulations of the non-linear $\sigma$ model [4]. Here the scalar field is obtained from the flatness condition of $A_\mu$, which is locally of the form $g^{-1}\partial_\mu g$, with $g$ an element of the group $G$.

One may ask where the extra degrees of freedom of a spin 2 field have gone to, if it is equivalent to a scalar field. This is exactly accounted for by its cohomological freedom [14]. Suppressing the Lie algebra indices, the 6 degrees of freedom of a skew rank 2 tensor are taken up by the 4 degrees of the vector $\Lambda_\mu$, plus its cohomological freedom of an additive scalar, leaving just the one degree of freedom of a scalar field.

In the case of Yang–Mills theory in loop space, this extra freedom gives rise to a dual gauge symmetry which is magnetic in nature if the original symmetry is considered to be electric. This leads to a fascinating electric–magnetic dual symmetry for Yang–Mills theory which is somewhat unexpected [10].

In conclusion, the interpretation of the 2-index field as a twisted de Rham cohomology group element, together with its inherent cohomological freedom,
gives a geometric explanation of many of its properties. This is particularly interesting for the nonabelian case and may serve as a guide for studying its possible interactions. Furthermore, this geometric interpretation gives a satisfying picture for the loop space formulation of Yang–Mills theory, with particular regard to its symmetry properties.

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