Photonic Dirac Monopole: Spin-1 Quantization

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We introduce the concept of a photonic Dirac monopole, appropriate for photonic crystals and metamaterials, by utilizing the Dirac-Maxwell correspondence. We show that even in vacuum, the reciprocal momentum space of both Maxwell’s equations and the massless Dirac equation (Weyl equation) possess a magnetic monopole. The critical distinction is the nature of magnetic monopole charges, which are integer valued for photons but half-integer for electrons. We prove that this inherent difference is directly tied to the spin and ultimately connects to the bosonic or fermionic behavior. We also show the presence of photonic Dirac strings, which are line singularities in the underlying Berry gauge potential. Our work sheds light on the recently proposed spin-1 bosonic phase of matter and the fundamental role of photon spin quantization in topological bosonic phases.

Introduction. Dirac’s pioneering paper [1] showed that if magnetic monopoles are found in nature, their charge would be quantized in units of \( Q = n\hbar/e \), with \( n \in \mathbb{Z} \) an integer, \( \hbar \) is Planck’s constant and \( e \) is the electron charge. This is the earliest example of topological quantization - fundamentally different from second quantization. Although there exists no experimental proof of magnetic monopoles till date, there is ample evidence of quantized topological charges in reciprocal (energy-momentum) space. Specifically, the appearance of such monopoles in the band structure of solids indicates the presence of quantized topological invariants, like the Chern number \( Z_2 \) and \( Z_2 \) index [4]. Ultimately, experimental observables such as the quantum Hall conductivity can be traced back to the existence of this quantized topological charge [5].

There have been significant efforts to construct synthetic gauge potentials that mimic the monopole physics in cold atoms [6] and spin ice [7]. One striking example is the realization of non-Abelian gauge theories with Yang-Lee monopoles [8]. The topological field theory of light has been found in knotted solutions of Maxwell’s equations [9, 10], as well as the uncertainty relations for photons [11]. There have also been important recent developments to formulate topological properties for photons utilizing photonic crystals and metamaterials [12–20]. It has been proposed that a degenerate chiral (magneto-electric) medium, if found in nature, will exhibit massless spin-1 quantized edge states and bulk photonic mass [21, 22]. Thus, it is necessary to understand the concept of Dirac monopoles and the influence of integer spin in photonic systems.

In this paper, we elucidate the fundamental difference between the magnetic monopoles appearing in Maxwell’s equations and the Dirac equation. Our work shows that a magnetic monopole appears for both photons and massless fermions in the reciprocal energy-momentum space - even for vacuum. Using a Dirac-Maxwell correspondence, we identify the bosonic and fermionic nature of magnetic monopole charge, which is inherently present in the relativistic theories of both particles. This clearly shows how the integer and half-integer nature of monopoles is ultimately tied to the bosonic and fermionic spin symmetries. Our work sheds light on the fundamental nature of spin-1 photon quantization in bosonic phases of matter.

In the context of geometric phases, the concept of magnetic charges has a rich history starting from the pioneering works of Pancharatam, Berry, Chiao and Wu [23]. Unification of these geometric phases for bosons and fermions was shown for massive particles using a relativistic quantum field theory [24]. In this paper, our focus is on massless particles, as well as the direct demonstration of gauge discontinuities in Maxwell’s and Weyl’s equations. Our derivation does not utilize quantum field theoretic techniques and appeals only to the spin representation of the two massless particles. We note that spin quantization is fundamentally different from topological charges encountered in real space for OAM beams [25], polarization singularities [26] and polarization vortices [27]. This is due to the central concept of gauge discontinuity in magnetic monopole quantization which is related to the topological field theory of bosons and fermions. We function in momentum space of Maxwell’s equations as opposed to real space so our work is specifically suited to develop topological invariants in the band structure of photonic crystals and wave dispersion within metamaterials [28]. One important application of our current technique is for uncovering edge states in bosonic phases of matter displaying the quantum magneto-electric effect [21] and quantum gyrotropic effect [22]. Our unified perspective also sheds light on recent developments of quantized bosonic Hall conductivity and bosonic phases of matter as opposed to fermionic topological phases [29, 30].

Dirac-Maxwell correspondence. The correspondence between Dirac’s and Maxwell’s equations is best expressed in the Reimann-Silberstein (R-S) basis [31, 32], which utilizes a vector wavefunction for light. Using this representation, we develop a topological field theory of the vacuum photon. In the R-S basis, we combine the electric \( \vec{E} \) and magnetic \( \vec{H} \) fields into a complex superposition,

\[
\Psi = (\vec{E} + i\vec{H}) / \sqrt{2},
\]

where \( i = \sqrt{-1} \) is the imaginary unit and the electromagnetic fields are associated with plane waves. We strongly emphasize that relativity requires vectorial representations for spin-1 bosonic fields and spinor-½ representations for fermionic fields. Spin-0 particles constitute scalar fields while spin-2 particles, such as gravitons, are described by tensor fields. Therefore, to unravel the topological bosonic properties of
light, we cannot work in a restricted sub-space ignoring components of the electromagnetic field. Simultaneously, we do not describe polarizations separately. In the R-S basis, Maxwell’s equations in vacuum can be combined into a first-order wave problem as follows,

$$i\vec{k} \times \vec{\Psi} = H_1 \vec{\Psi} = \omega \vec{\Psi},$$

(2)

which we label as spin $s = 1$. Here, $\omega$ is the frequency of light and we consider dynamical fields over all frequencies and wavevectors, not simple static fields. We can thus unambiguously identify a Hamiltonian for light,

$$H_1(\vec{k}) = \vec{k} \cdot \vec{S} = k_x S_x + k_y S_y + k_z S_z.$$  

(3)

$\vec{k} = (k_x, k_y, k_z)$ is the momentum of the plane wave in vacuum and $\vec{S} = (S_x, S_y, S_z)$ are the set of SO(3) antisymmetric matrices,

$$S_x = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, S_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}, S_z = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

(4)

These operators obey the familiar Lie algebra $[S_i, S_j] = i\epsilon_{ijk} S_k$ which encode information about integer spin. Notice our photonic Hamiltonian $H_1 = \vec{k} \cdot \vec{S}$ represents optical helicity, i.e., the projection of spin $\vec{S}$ along the direction of momentum $\vec{k}$. This is further clarified on direct comparison with massless Dirac fermions (Weyl fermions), which is the supersymmetric partner of the massless photon [33]. The Weyl equation is expressed as,

$$H_\frac{1}{2} \psi = E \psi,$$

(5)

where the massless Dirac Hamiltonian $H_\frac{1}{2}$, corresponding to spin $s = \frac{1}{2}$, is identified with electronic helicity,

$$H_\frac{1}{2}(\vec{k}) = \vec{k} \cdot \vec{\sigma} = k_x \sigma_x + k_y \sigma_y + k_z \sigma_z.$$  

(6)

$\vec{\sigma}$ are the Pauli matrices of SU(2) and obey the identical Lie algebra $[\sigma_i/2, \sigma_j/2] = i\epsilon_{ijk} \sigma_k/2$,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

(7)

Both particles are massless and satisfy an analogous helicity equation. However, the critical difference is revealed in the group operations of the particular particle; encapsulated by the SO(3) antisymmetric matrices for the spin-1 photon (Eq. (4) and the SU(2) Pauli matrices for the spin-1/2 Weyl fermion (Eq. (7)).

**Helical eigenstates.** We now solve for the eigenstates of the above Hamiltonians. As expected, Maxwell and Weyl’s equations possess two helical degrees of freedom. For the photon in Eq. (3), these are conventional right and left-handed circular polarization $H_1 \vec{e}_\pm = \pm \omega \vec{e}_\pm$.

$$\vec{e}_\pm(\vec{k}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\theta} \pm i \hat{\phi} \end{bmatrix},$$

(8)

where $\theta$ and $\phi$ are the spherical polar coordinates of $\vec{k}$ and $k = \sqrt{\vec{k} \cdot \vec{k}}$ is the magnitude of the wavevector. The photon is massless and therefore linearly dispersing in vacuum $\omega_\pm = \pm k$. Similarly, the eigenstates of the Weyl equation in Eq. (5) are comprised of two massless helical spinors $H_\frac{1}{2} \psi_\pm = \pm \vec{k} \psi_\pm$, which are represented as,

$$\psi_+(\vec{k}) = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{bmatrix}, \quad \psi_-(\vec{k}) = \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\phi} \end{bmatrix}.$$  

(9)

Indeed, these states are also linearly dispersing $E_\pm = \pm k$. An important observation can be made in $\vec{e}_\pm$ and $\psi_\pm$. The eigenstates are ill-defined at the origin of the momentum space $k = 0$, since they are arbitrarily dependent on $\theta$ and $\phi$ at this point. In fact, by parameterizing $\theta$ as the inclination from the $k_z$-axis, the eigenstates are not well-behaved at the north $\theta = 0$ or south $\theta = \pi$ poles either - they are multi-valued at both points. Such discontinuous behavior is impossible to remove and results from choosing a particular gauge for the eigenstates. This is the underlying source for Dirac monopoles and strings. The linear dispersion (light cone) of the massless helical states is displayed in Fig. 1.

**Spin quantization in photonic Dirac monopoles and strings.** In vacuum $\vec{k}$-space, we discover a magnetic Dirac monopole for both Maxwell’s and Weyl’s equations but with intrinsic differences. This is demonstrated by first defining the magnetic flux in momentum space - i.e. the Berry curvature. For the photon, the Berry curvature of either right or left-handed helicity can be found from the circular eigenstates derived in Eq. (5).

$$\vec{F}_1^\pm = -i \vec{\nabla}_k \times [\vec{e}_\pm \cdot (\vec{\nabla}_k \vec{e}_\pm)].$$  

(10)

For the massless electron, the analogous Berry curvature is found by evaluating the spinor eigenstates in Eq. (9).

$$\vec{F}_\frac{1}{2}^\pm = -i \vec{\nabla}_k \times [\psi_\pm \cdot (\vec{\nabla}_k \psi_\pm)].$$  

(11)

Here, $\vec{\nabla}_k = \sum_j \hat{j} \partial_k$ is the gradient operator in momentum space. On evaluating the Berry curvature for both particles...
with positive and negative helicities ($\pm$), we find that $\vec{F}_s$ possesses a Dirac monopole,
\begin{equation}
\vec{F}_s(\vec{k}) = Q_s \hat{F}(\vec{k}).
\end{equation}
$\vec{F}$ being the magnetic field of a Dirac monopole in $\vec{k}$-space,
\begin{equation}
\vec{F}(\vec{k}) = \frac{\vec{k}}{k^3}.
\end{equation}
Note that $Q_s$ in Eq. (12) is the topological magnetic charge and is fundamentally different for the two particles,
\begin{equation}
Q_s = s.
\end{equation}
$s$ is precisely the spin of the particle, which takes integer or half-integer values for bosons or fermions respectively. We emphasize that the magnetic monopole charge is naturally quantized,
\begin{equation}
Q_s = \frac{1}{4\pi} \iiint \vec{F}_s \cdot d^2\vec{k}.
\end{equation}
The charge is located at the origin $k = 0$ of the momentum space, exactly where the eigenstates are ill-defined, and acts as a source for the magnetic field $\nabla_k \cdot \vec{F}_s = 4\pi Q_s \delta^3(\vec{k})$. Notice that the magnetic monopole charge of the photon,
\begin{equation}
Q_1 = 2Q_2 = 1,
\end{equation}
is twice the electron due to integer spin. Magnetic monopole charges for the two helicities have opposite signs $Q_s^\pm = \pm Q_s$. This ensures the net Berry curvature vanishes $\vec{F}_s^+ + \vec{F}_s^- = 0$, as expected due to time-reversal symmetry in vacuum. A visualization of the magnetic flux is shown in Fig. 2.

We note that the photonic Dirac monopole is accompanied by a string of singularities in the underlying gauge potential. This Dirac string is unobservable as it is a gauge dependent phenomenon but sheds light on the fundamental differences between electrons and photons. The Berry gauge potential for the massless photon can be expressed using the eigenstates in Eq. (5),
\begin{equation}
\vec{A}_1^\pm = -ie^\pm_\pm (\nabla_k e^\pm_\pm).
\end{equation}
Likewise, the Berry potential of the massless electron is found from Eq. (5),
\begin{equation}
\vec{A}_s^\pm = -i\psi^\pm_\pm \nabla_k \psi^\pm_\pm.
\end{equation}
Upon solving for $\vec{A}_s^\pm = \pm \vec{A}_s$, we again find a clear dependence on the magnetic monopole charge $Q_s$ which is different for bosons and fermions,
\begin{equation}
\vec{A}_s(\vec{k}) = Q_s \frac{1 - \cos \theta}{k \sin \theta} \hat{\phi},
\end{equation}
and $\vec{F}_s = \nabla_k \times \vec{A}_s$ reproduces the Berry curvature in Eq. (12). The gauge potential is singular along the $k_z$-axis, at $\theta = 0$ and $\pi$, where the eigenstates are multi-valued. This line singularity that originates at the monopole and extends to infinity is known as a Dirac string. Fig. 2 displays a visualization of the Dirac monopole and strings for both the massless particles with differing spin. We note that the above equations are traditionally found in the theory of magnetic charges in real space - not momentum space. Following this, quantization of magnetic charge naturally emerges from the requirement of a single-valued wavefunction in the presence of singular (multi-valued) gauge potentials. Our rigorous derivation is unique as it unifies the momentum space [34] of Maxwell’s equations and the Weyl equation. This makes it ideally suited for extension to topological theories of band structure in photonic crystals and wave dispersion in metamaterials.

**Berry phase.** We now provide a detailed comparison of $\vec{k}$-space Pancharatnam-Berry phase (hereon called geometric phase) for photons and electrons, that arises from their corresponding spin properties. The geometric phase calculated for any closed path on the $\vec{k}$-sphere is gauge invariant,
\begin{equation}
\gamma_s = \oint A_s \cdot d\vec{k} = \iint \vec{F}_s \cdot d^2\vec{k}.
\end{equation}$\gamma_s$ is the geometric phase and is equivalent to the flux of Berry curvature $\vec{F}_s$ through a surface bounded by the path. In this case, we see that $\iint \vec{F}_s \cdot d^2\vec{k} = Q_s \oint d\Omega$ is exactly the solid angle $\Omega(C)$ traced along the $\vec{k}$-sphere,
\begin{equation}
\gamma_s = Q_s \Omega(C),
\end{equation}
where $C$ designates the bounded path. We now consider a closed path around a great circle of the $\vec{k}$-sphere (eg: the equatorial path). For massless particles, this is equivalent to rotating the particle back into itself. The accumulated phase must be quantized,
\begin{equation}
\gamma_s = 2\pi Q_s.
\end{equation}
Ultimately, the geometric phase of antisymmetric particle species. This is due to the fact that fermions are fermions vs. bosonic. This photonic Dirac monopole has a spin dependent Berry curvature and photon: $Q_1 = 1$) which arises from the differing spin symmetries (fermionic vs. bosonic). It is accompanied by a string of singularities in the underlying gauge potential $\hat{A}_s$. Any closed path around the equator of the string produces a quantized Berry phase $\gamma_s = \hat{\vec{A}}_s \cdot d\vec{k} = 2\pi Q_s$. The accumulated phase in $\hat{k}$-space is fundamentally tied to the spin of the particle $\mathcal{R}_s(2\pi) = e^{i\gamma_s} = (-1)^{2s}$.

We clearly see that geometric phases in $\hat{k}$-space are dependent on the spin of the particle,

$$e^{i\gamma_s} = (-1)^{2s}.$$  \hspace{1cm} (23)

Notice that $e^{i\gamma_{\frac{1}{2}}} = -1$ and $e^{i\gamma_1} = +1$ are antisymmetric or symmetric under a $2\pi$ rotation depending on the spin $Q_s = s$. Ultimately, the geometric phase of $\gamma_{\frac{1}{2}} = \pi$ or $\gamma_1 = 2\pi$ is tied to the fermionic or bosonic statistics of the particle. We emphasize that this geometric phase $\gamma_1 = 2\pi$ is routinely encountered for massless Dirac fermions in graphene [35,36]. However, the direct correspondence with spin-1 massless photons $\gamma_1 = 2\pi$ has not been pointed out till date.

**Rotational symmetry.** The nuance behind integer and half-integer geometric phases (Eq. (22)) is explained more rigorously by considering the operations of the rotational (spin) groups. Maxwell’s equations (Eq. (2)) transform under the $SO(3)$ group $\mathcal{R}_1(\alpha) = \exp[i\alpha n \cdot \hat{S}]$, where $\alpha$ is the angle subtended about an axis $\vec{n}$. This is true for all vector fields. Conversely, the Weyl equation (Eq. (3)) transforms under the $SU(2)$ group $\mathcal{R}_2(\alpha) = \exp[i\alpha \vec{n} \cdot \vec{\sigma}/2]$, characteristic of spinors. Although $SO(3)$ and $SU(2)$ obey the same Lie algebra, the group representations are inequivalent. The distinction is evident under a cyclic rotation,

$$\mathcal{R}_s(2\pi) = (-1)^{2s}.$$ \hspace{1cm} (24)

Notice that the accumulated phase is different depending on the particle species. This is due to the fact that fermions are antisymmetric $\mathcal{R}_{\frac{1}{2}}(2\pi) = -1$ under rotations, while bosons are symmetric $\mathcal{R}_1(2\pi) = +1$ and this behavior is guaranteed by the spin-statistics theorem. The difference fundamentally changes the interpretation of fermionic and bosonic topologies [37].

Our results suggest that massless electrons in a twisted Dirac fiber would yield Chiao-Tomita phases [38] exactly half the value of photons. We also note that spin-momentum locking is a universal property in photonics which arises entirely from the transversality $\vec{k} \cdot \vec{\Psi} = 0$ of electromagnetic waves. This phenomenon can be explained with causal boundary conditions on evanescent fields and does not necessarily require topological considerations [39–41]. For example, conventional surface plasmon polaritons on metallic layers and waveguide modes show spin-momentum locking but these are not related to any topologically protected edge states or non-trivial phases.

**Conclusions.** In conclusion, we have introduced the concept of a photonic Dirac monopole appropriate for the field of spin photonics, topological photonic crystals and metamaterials. It shows magnetic monopole charge quantization in momentum space arising solely from spin-1 properties of the photon. We elucidated this phenomenon using a Dirac-Maxwell correspondence in the Reimann-Silberstein basis. Our work sheds light on the role of photon spin in the recently proposed topological bosonic phase for light [21,22]. The edge states of such a topological phase exhibit spin-1 quantization as opposed to spin-$\frac{1}{2}$ quantization in fermionic phases of matter. This is ultimately connected to the presence of quantized monopole charges (bosonic or fermionic type [42,43]) in the dispersion of bulk matter. Experimentally probing monopole charge in momentum space can shed light on fundamental symmetries in spin electrodynamics of bosons and fermions.

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