SEPARATION OF VARIABLES FOR SOLITON EQUATIONS VIA THEIR BINARY CONSTRAINED FLOWS

YUNBO ZENG*1 AND WEN-XIU MA†2
*Department of Applied Mathematics, Tsinghua University
Beijing 100084, China
† Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong, China

Abstract. Binary constrained flows of soliton equations admitting $2 \times 2$ Lax matrices have $2N$ degrees of freedom, which is twice as many as degrees of freedom in the case of mono-constrained flows. For their separation of variables only $N$ pairs of canonical separated variables can be introduced via their Lax matrices by using the normal method. A new method to introduce the other $N$ pairs of canonical separated variables and additional separated equations is proposed. The Jacobi inversion problems for binary constrained flows are established. Finally, the factorization of soliton equations by two commuting binary constrained flows and the separability of binary constrained flows enable us to construct the Jacobi inversion problems for some soliton hierarchies.

Keywords: binary constrained flow, separation of variables, Jacobi inversion problem, Lax representation, factorization of soliton equations.

1E-mail: yzeng@tsinghua.edu.cn
2E-mail: mawx@cityu.edu.hk

Typeset by \LaTeX
1. Introduction.

The separation of variables is one of the most universal methods for solving completely integrable (classical and quantum) models. It has been applied successfully to the study of a large number of finite-dimensional integrable Hamiltonian systems (FDIHSs) (see, for example, [1-12]), as well as infinite dimensional integrable Hamiltonian systems in the determination of finite-dimensional quasi-periodic solutions (see, for example, [13-18]). In many cases the separation of variables of integrable classical systems prepares the passage to the corresponding quantum systems. For the classical integrable systems subject to the inverse scattering method, the standard construction of the action-angle variables using the poles of the Baker-Akhiezer function is in fact equivalent to the separation of variables [4].

For a FDIHS, let \( m \) denote the number of degrees of freedom, and \( P_i, i = 1, \ldots, m, \) be functionally independent integrals of motion in involution, the separation of variables means to construct \( m \) pairs of canonical separated variables \( v_k, u_k, k = 1, \ldots, m, [2,3,4] \)

\[
\{u_k, u_l\} = \{v_k, v_l\} = 0, \quad \{v_k, u_l\} = \delta_{kl}, \quad k,l = 1, \ldots, m, \tag{1.1}
\]

and \( m \) functions \( f_k \) such that

\[
f_k(u_k, v_k, P_1, \ldots, P_m) = 0, \quad k = 1, \ldots, m. \tag{1.2}
\]

The equations (1.2) are called separated equations, which give rise to an explicit factorization of the Liouville tori.

For the FDIHSs with the Lax matrices admitting the \( r \)-matrices of the XXX, XXZ and XYZ type, there is a general approach to introduce canonical separated variables [2,3,4,8]. The corresponding separated equations enable us to express the generating function of canonical transformation in completely separated form as an abelian integral on the associated invariant spectral curve. The resulted linearizing map is essentially the Abel map to the Jacobi variety of the spectral curve, thus providing a link, through purely Hamiltonian methods, with the algebro-geometric linearization methods given by [19-22].

An important feature of the separation of variables for a FDIHS is that the number of canonical separated variables \( u_k \) should be equal to the number \( m \) of degrees of freedom. In some cases, the number of \( u_k \) resulted by the normal method may be less than \( m \) and so some additional canonical separated variables should be introduced. So far very few models in these cases have been studied. These cases remain to be a challenging problem [4]. In recent years binary constrained flows of soliton hierarchies have attracted attention (see, for example, [23-29]), whose basic idea was described in [30]. The binary constrained flows are a kind of FDIHSs for which the method presented in [2,3,4,8] is not valid. The degree of freedom for binary constrained flows admitting \( 2 \times 2 \) Lax matrices is an even natural number usually denoted by \( 2N \). The method in [2,3,4,8] allows us to introduce only \( N \) pairs of canonical separated variables \( u_1, \ldots, u_N \) and
v_1, ..., v_N via the Lax matrices. In this paper we propose a new method for determining additional N pairs of canonical separated variables and separated equations for binary constrained flows. The main idea is to construct two functions \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \) defining \( u_{N+1}, ..., u_{2N} \) by the set of zeros of \( \tilde{B}(\lambda) \) and \( v_{N+k} = \tilde{A}(u_{N+k}) \). To keep the canonical conditions (1.1) and the requirement for the separated equations (1.2), it is found that certain commutator relations should be imposed on \( \tilde{B}(\lambda), \tilde{A}(\lambda) \) and \( \tilde{A}(\lambda) \) has some link with the generating function of integrals of motion of binary constrained flows, which provides a way to construct the \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \). In fact, we have to modify the original approach for introducing \( u_1, ..., u_N \) and \( v_1, ..., v_N \) so that \( u_1, ..., u_{2N} \) and \( v_1, ..., v_{2N} \) are canonical conjugated. Having produced the separation of variables, we further construct the Jacobi inversion problems for binary constrained flows. This method is somewhat different from that for introducing canonical variables presented in [31] and can be applied to more binary constrained flows.

Briefly, separation of variables can be characterized as a reduction of a multidimensional problem to a set of one-dimensional ones. The separation of variables of soliton equations in this paper contains two steps of separation of variables. The first step is to factorize \((1 + 1)\)-dimensional soliton equations into two commuting \( x \)- and \( t \)-FDIHSs via binary constrained flows, namely the \( x \)- and \( t \)-dependences of the soliton equations are separated by the \( x \)- and \( t \)-FDIHSs obtained from the \( x \)- and \( t \)-binary constrained flows. The second step is to produce separation of variables for the \( x \)- and \( t \)-FDIHDs by our method to be proposed later on. Finally, combining the factorization of soliton equations with the Jacobi inversion problems for \( x \)- and \( t \)-FDIHSs enables us to establish the Jacobi inversion problems for soliton equations. We will present the separation of variables for the KdV hierarchy, the AKNS hierarchy and the Kaup-Newell hierarchy via their binary constrained flows. In fact, we employ our method in a little different way for those three cases.

In section 2, we recall the binary constrained flows of KdV hierarchy and present factorization of the KdV equations into the \( x \)- and \( t \)-binary constrained flows. By means of the Lax matrix \( M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \) for the binary constrained flows, the method in [2,3,4,8] allows us to define only \( N \) pairs of canonical variables \( u_1, ..., u_N \) by the set of zeros of \( B(\lambda) \) and \( v_k = 2A(u_k) \). We propose a new method to construct two new functions \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \) for introducing \( u_{N+1}, ..., u_{2N} \) by the set of zeros of \( \tilde{B}(\lambda) \) and \( v_{N+k} = \tilde{A}(u_{N+k}) \). The construction of \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \) is based on an observation that the canonical conditions (1.1) need certain commutator relations between \( \tilde{A}(\lambda), \tilde{B}(\lambda), \) and the requirement for the separated equations (1.2) links \( \tilde{A}(\lambda) \) with another generating function of integrals of motion. To guarantee that \( v_1, ..., v_{2N} \) and \( u_1, ..., u_{2N} \) are canonical conjugated, we also have to modify the original way for introducing \( u_1, ..., u_N \) and \( v_1, ..., v_N \). Then we establish the Jacobi inversion problems for the \( x \)- and \( t \)-binary constrained flows. Finally, these Jacobi inversion problems together with the factorization of the KdV equations give rise to the Jacobi inversion problems for the KdV equations. In section 3, the factorization of the AKNS equations is given. Since \( B(\lambda) \)
for the binary constrained AKNS flows, unlike the $B(\lambda)$ for the binary constrained KdV flows, has only $N-1$ zeros, we have to modify the method proposed in section 2 in order to find $2N$ pairs of canonical variables for the binary constrained AKNS flows. In section 4, we present the factorization of the Kaup-Newell equations. Since the commutator relations of $A(\lambda), B(\lambda)$ and $C(\lambda)$ for the binary constrained Kaup-Newell flows are quite different from those for both the binary constrained KdV flows and the binary constrained AKNS flows, we need to further modify the method in sections 2 and 3 in order to find the separation of variables for the Kaup-Newell equations. Finally some remarks are made in section 5.

2. Separation of variables for the KdV equations.

In this section, we use the binary constrained flows of KdV hierarchy to illustrate our method of introducing canonical separated variables. Then we show how to produce the separation of variables for the KdV equations. To make the paper self-contained, we first briefly describe the binary constrained flows of the KdV hierarchy [26].

2.1 Binary constrained flows of the KdV hierarchy.

Let us start from the Schrödinger equation [32]

$$\phi_{xx} + (\lambda + u)\phi = 0$$

which can be rewritten as following spectral problem

$$\phi_x = U(u, \lambda)\phi, \quad U(u, \lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda - u & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (2.1)$$

Its adjoint representation reads

$$V_x = [U, V] \equiv UV - VU. \quad (2.2)$$

Set

$$V = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}. \quad (2.3)$$

Equation (2.2) yields

$$a_0 = b_0 = 0, \quad c_0 = -1, \quad a_1 = 0, \quad b_1 = 1, \quad c_1 = -\frac{1}{2}u,$$

$$a_2 = \frac{1}{4}u_x, \quad b_2 = -\frac{1}{2}u, \quad c_2 = \frac{1}{8}(u_{xx} + u^2), ...,$$

and in general

$$b_{k+1} = Lb_k = -\frac{1}{2}L^{k-1}u, \quad a_k = -\frac{1}{2}b_{k,x}, \quad (2.4a)$$
\[ c_k = -\frac{1}{2} b_{k,xx} - b_{k+1} - b_k u, \quad k = 1, 2, \cdots, \quad (2.4b) \]

where
\[ L = -\frac{1}{4} \partial^2 - u + \frac{1}{2} \partial^{-1} u_x, \quad \partial = \partial_x, \quad \partial^{-1} \partial = \partial \partial^{-1} = 1. \]

Set
\[ V^{(n)}(u, \lambda) = \sum_{i=0}^{n+1} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n+1-i} + \begin{pmatrix} 0 & 0 \\ b_{n+2} & 0 \end{pmatrix}, \quad (2.5) \]

and take the time evolution law of \( \phi \) as
\[ \phi_{t_n} = V^{(n)}(u, \lambda) \phi. \quad (2.6) \]

Then the compatibility condition of the equations (2.1) and (2.6) gives rise to the \( n \)-th KdV equation which can be written as the infinite-dimensional Hamiltonian system
\[ u_{t_n} = -2b_{n+2,x} = \partial L^n u = \partial \frac{\delta H_n}{\delta u}, \quad (2.7) \]

where the Hamiltonian \( H_n \) is given by
\[ H_n = \frac{4b_{n+3}}{2n + 3}, \quad \frac{\delta H_n}{\delta u} = -2b_{n+2}. \]

The matrix \( V \) determined by (2.2) and (2.3) also satisfies the adjoint representation of (2.6)
\[ V_{t_n} = [V^{(n)}, V], \quad (2.8) \]

when \( u \) satisfies (2.7).

For \( n = 1 \) we have
\[ \phi_{t_1} = V^{(1)}(u, \lambda) \phi, \quad V^{(1)} = \begin{pmatrix} \frac{1}{4} u_x \\ -\lambda^2 - \frac{1}{2} u \lambda + \frac{1}{4} u_{xx} + \frac{1}{2} u^2 \end{pmatrix}, \quad (2.9) \]

and the equation (2.7) for \( n = 1 \) is the well-known KdV equation
\[ u_{t_1} = -\frac{1}{4} (u_{xxx} + 6uu_x). \quad (2.10) \]

The adjoint spectral problem reads
\[ \psi_x = -U^T(u, \lambda) \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (2.11) \]

We have [26]
\[ \frac{\delta \lambda}{\delta u} = \beta Tr \left[ \begin{pmatrix} \phi_1 \psi_1 & \phi_1 \psi_2 \\ \phi_2 \psi_1 & \phi_2 \psi_2 \end{pmatrix} \frac{\partial U(u, \lambda)}{\partial u} \right] = -\beta \psi_2 \phi_1, \quad (2.12) \]
where $\beta$ is some constant.

The binary $x$-constrained flows of the KdV hierarchy (2.7) consist of the equations obtained from the spectral problem (2.1) and the adjoint spectral problem (2.11) for $N$ distinct real numbers $\lambda_j$ and the restriction of the variational derivatives for the conserved quantities $H_{k_0}$ (for any fixed $k_0$) and $\lambda_j$:

\[ \Phi_{1,x} = \Phi_2, \quad \Phi_{2,x} = -\Lambda \Phi_1 - u \Phi_1, \]
\[ \Psi_{1,x} = \Lambda \Psi_2 + u \Psi_2, \quad \Psi_{2,x} = -\Psi_1, \]
\[ \frac{\delta H_{k_0}}{\delta u} - \beta^{-1} \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = -2b_{k_0+2} + <\Psi_2, \Phi_1> = 0. \]

Such a constraint (2.13c) has been recognized as a symmetry constraint [25,26,30]. Hereafter we denote the inner product in $\mathbb{R}^N$ by $<.,.>$ and

\[ \Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^T, \quad \Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^T, \quad i = 1, 2, \quad \Lambda = diag(\lambda_1, \cdots, \lambda_N). \]

For $k_0 = 0$, we have

\[ b_2 = -\frac{1}{2} u = \frac{1}{2} <\Psi_2, \Phi_1>, \quad i.e., \quad u = - <\Psi_2, \Phi_1>. \]

By substituting (2.14) into (2.13a) and (2.13b), the first binary $x$-constrained flow becomes a finite-dimensional Hamiltonian system (FDHS) [26]

\[ \Phi_{1x} = \frac{\partial F_1}{\partial \Psi_1}, \quad \Phi_{2x} = \frac{\partial F_1}{\partial \Psi_2}, \quad \Psi_{1x} = -\frac{\partial F_1}{\partial \Phi_1}, \quad \Psi_{2x} = -\frac{\partial F_1}{\partial \Phi_2}, \]

with the Hamiltonian

\[ F_1 = <\Psi_1, \Phi_2> - <\Lambda \Psi_2, \Phi_1> + \frac{1}{2} <\Psi_2, \Phi_1>^2. \]

The binary $t_n$-constrained flows of the KdV hierarchy (2.7) are defined by the replicas of (2.6) and its adjoint system for $N$ distinct real number $\lambda_j$

\[ \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)}(u, \lambda_j))^T \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, \ldots, N, \]

as well as the $n$-th KdV equation itself (2.7) in the case of the higher-order constraint for $k_0 \geq 1$. Under the constraint (2.14) and the $x$-FDHS (2.15), the binary $t_1$-constrained flow obtained from (2.16) with $V^{(1)}$ given by (2.9) can also be written as a $t_1$-FDHS

\[ \Phi_{1,t_1} = \frac{\partial F_2}{\partial \Psi_1}, \quad \Phi_{2,t_1} = \frac{\partial F_2}{\partial \Psi_2}, \quad \Psi_{1,t_1} = -\frac{\partial F_2}{\partial \Phi_1}, \quad \Psi_{2,t_1} = -\frac{\partial F_2}{\partial \Phi_2}, \]
with the Hamiltonian

\[ F_2 = -< \Lambda^2 \Psi_2, \Phi_1 > + < \Lambda \Psi_1, \Phi_2 > + \frac{1}{2} < \Psi_2, \Phi_1 > < \Lambda \Psi_2, \Phi_1 > + \frac{1}{2} < \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 > + \frac{1}{8} (< \Psi_2, \Phi_2 > - < \Psi_1, \Phi_1 >)^2. \]

The Lax representation for the $x$-FDHS (2.15) and the $t_1$-FDHS (2.17) can be deduced from the adjoint representation (2.2) and (2.8) by using the method in [33,34]

\[ M_x = [\tilde{U}, M], \quad M_{tn} = [\tilde{V}^{(n)}, M], \quad (2.18) \]

where $\tilde{U}$ and $\tilde{V}^{(n)}$ are obtained from $U$ and $V^{(n)}$ by a substitution of (2.14), and the Lax matrix $M$ is given by

\[ M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \quad (2.19) \]

\[ A(\lambda) = \frac{1}{4} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} - \psi_{2j} \phi_{2j}}{\lambda - \lambda_j}, \quad B(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{1j}}{\lambda - \lambda_j}, \]

\[ C(\lambda) = -\lambda + \frac{1}{2} < \Psi_2, \Phi_1 > + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{2j}}{\lambda - \lambda_j}. \]

The equation (2.18) implies that $\frac{1}{2} Tr M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$ is the generating function of integrals of motion for (2.15) and (2.17). A straightforward calculation yields

\[ A^2(\lambda) + B(\lambda)C(\lambda) \equiv P(\lambda) = -\lambda + \sum_{j=1}^{N} \left[ \frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2} \right], \quad (2.20) \]

where $P_j, j = 1, ..., 2N,$ are $2N$ independent integrals of motion for the FDHSs (2.15) and (2.17)

\[ P_j = \frac{1}{2} \psi_{1j} \phi_{2j} + (-\frac{1}{2} \lambda_j + \frac{1}{4} < \Psi_2, \Phi_1 >) \psi_{2j} \phi_{1j} + \frac{1}{8} \sum_{k \neq j} \frac{1}{\lambda_j - \lambda_k} [ (\psi_{1j} \phi_{1j} - \psi_{2j} \phi_{2j}) (\psi_{1k} \phi_{1k} - \psi_{2k} \phi_{2k}) + 4 \psi_{1j} \phi_{2j} \psi_{2k} \phi_{1k} ], \quad j = 1, ..., N \quad (2.21a) \]

\[ P_{N+j} = \frac{1}{4} (\psi_{1j} \phi_{1j} + \psi_{2j} \phi_{2j}), \quad j = 1, ..., N. \quad (2.21b) \]

It is easy to verify that

\[ F_1 = 2 \sum_{j=1}^{N} P_j, \quad F_2 = 2 \sum_{j=1}^{N} (\lambda_j P_j + P_{N+j}^2). \quad (2.22) \]
With respect to the standard Poisson bracket it is found that

\[ \{ A(\lambda), A(\mu) \} = \{ B(\lambda), B(\mu) \} = \{ C(\lambda), C(\mu) \} = 0, \]  

\[ \{ A(\lambda), B(\mu) \} = \frac{1}{2(\lambda - \mu)} [B(\mu) - B(\lambda)], \]  

\[ \{ A(\lambda), C(\mu) \} = \frac{1}{2(\lambda - \mu)} [C(\lambda) - C(\mu)], \]  

\[ \{ B(\lambda), C(\mu) \} = \frac{1}{\lambda - \mu} [A(\mu) - A(\lambda)]. \]  

(2.23a)

(2.23b)

(2.23c)

(2.23d)

It follows from (2.23) that

\[ \{ A^2(\lambda) + B(\lambda)C(\lambda), A^2(\mu) + B(\mu)C(\mu) \} = 0, \]

which implies that \( P_j, j = 1, \ldots, 2N \), are in involution:

\[ \{ P_k, P_l \} = 0, \quad k, l = 1, \ldots, 2N. \]

Therefore the FDHSs (2.15) and (2.17) are integrable and commute with each other. The construction of (2.15) and (2.17) ensures that if \((\Psi_1, \Psi_2, \Phi_1, \Phi_2)\) satisfies the finite-dimensional integrable Hamiltonian systems (FDIHSs) (2.15) and (2.17) simultaneously, then \( u \) defined by (2.14) solves the KdV equation (2.10).

In general, by substituting (2.14) and using (2.15), the \( t_n \)-constrained flow (2.16) becomes a \( t_n \)-FDIHS and the \( n \)-th KdV equation (2.7) is factorized by the \( x \)-FDIHS (2.15) and the \( t_n \)-FDIHS. Set

\[ A^2(\lambda) + B(\lambda)C(\lambda) = \lambda \sum_{k=0}^{\infty} \tilde{F}_k \lambda^{-k}, \]

(2.24a)

where \( \tilde{F}_k, k = 1, 2, \ldots \), are also integrals of motion for both the \( x \)-FDHSs (2.15) and the \( t_n \)-binary constrained flows (2.16). Comparing (2.24a) with (2.20), one gets

\[ \tilde{F}_0 = -1, \quad \tilde{F}_1 = 0, \quad \tilde{F}_k = \sum_{j=1}^{N} [\lambda_j^{k-2} P_j + (k - 2) \lambda_j^{k-3} P_{N+j}^2], \quad k = 2, 3, \ldots \]

(2.24b)

By employing the method in [34,35], the \( t_n \)-FDIHS obtained from the \( t_n \)-constrained flow (2.16) is found to be of the form

\[ \Phi_{1,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_1}, \quad \Phi_{2,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_2}, \quad \Psi_{1,t_n} = -\frac{\partial F_{n+1}}{\partial \Phi_1}, \quad \Psi_{2,t_n} = -\frac{\partial F_{n+1}}{\partial \Phi_2}, \]

(2.25a)
with the Hamiltonian

\[ F_{n+1} = \sum_{m=0}^{n} \left( \frac{1}{2} \right)^{m-1} \frac{\alpha_m}{m+1} \sum_{l_1+\ldots+l_{m+1}=n+2} \tilde{F}_{l_1} \ldots \tilde{F}_{l_{m+1}}, \quad (2.25b) \]

where \( l_1 \geq 1, \ldots, l_{m+1} \geq 1, \alpha_0 = 1, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{2}, \) and \([34,35]\)

\[ \alpha_m = 2\alpha_{m-1} + \sum_{l=1}^{m-2} \alpha_l \alpha_{m-l-1} - \frac{1}{2} \sum_{l=1}^{m-1} \alpha_l \alpha_{m-l}, \quad m \geq 3. \quad (2.25c) \]

The n-th KdV equation (2.7) is factorized by the \( x \)-FDIHS (2.15) and the \( t_n \)-FDIHS (2.25).

For example, for the second equation in the KdV hierarchy (2.7) with \( n = 2 \)

\[ u_t = \frac{1}{16} (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x, \quad (2.26) \]

the Hamiltonian \( F_3 \) for the \( t_2 \)-FDIHS reads

\[ F_3 = 2\tilde{F}_4 + \frac{1}{2} \tilde{F}_2^2 = 2 \sum_{j=1}^{N} (\lambda_j^2 P_j + 2\lambda_j P_{N+j}^2) + \frac{1}{2} (\sum_{j=1}^{N} P_j)^2. \quad (2.27) \]

Then the second KdV equation (2.26) is factorized by the \( x \)-FDIHS (2.15) and the \( t_2 \)-FDIHS with the Hamiltonian \( F_3 \).

### 2.2 The separation of variables for the KdV equations.

An effective way to introduce the separated variables \( v_k, u_k \) and to obtain the separated equations is to use the Lax matrix \( M \) and the generating function of integrals of motion. For the FDIHSs (2.15) and (2.17), we can define the first \( N \) pairs of the canonical variables \( u_k, v_k, k = 1, \ldots, N \), by the method \([2,3,4,8]\). The commutator relations (2.23) and the generating function of integrals of motion (2.20) enable us to define \( u_1, \ldots, u_N \) by the set of zeros of \( B(\lambda) \)

\[ B(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \psi_{2j} \phi_{1j} = \frac{R(\lambda)}{K(\lambda)}, \quad (2.28a) \]

where

\[ R(\lambda) = \prod_{k=1}^{N} (\lambda - u_k), \quad K(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k), \]
and \( v_1, ..., v_N \) by
\[
v_k = A_1(u_k), \quad k = 1, ..., N,
\]
where
\[
A_1(\lambda) = 2A(\lambda) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} - \psi_{2j} \phi_{2j}}{\lambda - \lambda_j}.
\]

As we will see later, the commutator relations (2.23) guarantee that \( u_1, ..., u_N \) and \( v_1, ..., v_N \) satisfy the canonical conditions (1.1). Then substituting \( u_k \) into (2.20) gives rise to the separated equations
\[
v_k = A_1(u_k) = 2\sqrt{P(u_k)} = 2 \sqrt{-u_k + \sum_{j=1}^{N} \left[ \frac{P_j}{u_k - \lambda_j} + \frac{P^2_{N+j}}{(u_k - \lambda_j)^2} \right]}, \quad k = 1, ..., N.
\]

Now the reason for taking our choice of \( B(\lambda) \) and \( A_1(\lambda) \) becomes apparent.

The FDIHSs (2.15) and (2.17) have \( 2N \) degrees of freedom, therefore we need to introduce the other \( N \) pairs of canonical variables \( v_k, u_k, k = N + 1, ..., 2N \). The main idea is to construct two suitable functions \( \tilde{B}(\lambda), \tilde{A}(\lambda) \) in order to define \( u_{N+1}, ..., u_{2N} \) by the set of zeros of \( \tilde{B}(\lambda) \) and \( v_{N+1}, ..., v_{2N} \) by \( v_{N+k} = \tilde{A}(u_{N+k}) \). The above way for introducing \( u_k, v_k, k = 1, ..., N \) stimulates us to impose two requirements on \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \) in order to construct them. First, the canonical conditions (1.1) require that \( \tilde{B}(\lambda) \) and \( \tilde{A}(\lambda) \) satisfy
\[
\{ \tilde{B}(\lambda), \tilde{B}(\mu) \} = \{ \tilde{A}(\lambda), \tilde{A}(\mu) \} = \{ \tilde{A}(\lambda), B(\mu) \} = \{ \tilde{A}(\lambda), A_1(\mu) \} = 0,
\]
\[
(2.29a)
\]
\[
\{ \tilde{A}(\lambda), \tilde{B}(\mu) \} = \frac{1}{\lambda - \mu} [\tilde{B}(\mu) - \tilde{B}(\lambda)],
\]
\[
(2.29b)
\]
\[
\{ A_1(\lambda), \tilde{B}(\mu) \} = 0.
\]
\[
(2.29c)
\]
The second requirement is that the equation \( v_{N+k} = \tilde{A}(u_{N+k}) \) should give rise to the separated equations. Notice that \( P_{N+j} \) given by (2.21b) are integrals of motion for the FDIHSs (2.15) and (2.17), we can construct another generating function \( \tilde{A}(\lambda) \) of integrals of motion by
\[
\tilde{A}(\lambda) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} + \psi_{2j} \phi_{2j}}{\lambda - \lambda_j} = 2 \sum_{j=1}^{N} \frac{P_{N+j}}{\lambda - \lambda_j}.
\]
\[
(2.30a)
\]
We may use \( \tilde{A}(\lambda) \) to define \( v_{N+1}, ..., v_{2N} \) since substituting \( u_{N+k} \) into the equation (2.30a) immediately leads to the separated equations for \( v_{N+k} \) and \( u_{N+k} \). It is easy to
see that \(\{\tilde{A}(\lambda), B(\mu)\} = \{\tilde{A}(\lambda), \tilde{A}(\mu)\} = \{\tilde{A}(\lambda), A_1(\mu)\} = 0\). We look for \(\tilde{B}(\lambda)\) in the form

\[
\tilde{B}(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} (\delta_1 \phi_{1j}^2 + \delta_2 \phi_{1j} \phi_{2j} + \delta_3 \phi_{2j}^2).
\]

By requiring \(\tilde{B}(\lambda)\) to satisfy (2.29a) and (2.29b), one gets \(\delta_1 = 1, \delta_2 = \delta_3 = 0\), i.e.

\[
\tilde{B}(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j}.
\]  \hspace{1cm} (2.30b)

But \(\tilde{B}(\lambda)\) doesn’t fit (2.29c). In fact, we have

\[
\{A_1(\lambda), \tilde{B}(\mu)\} = \frac{1}{\lambda - \mu} [\tilde{B}(\mu) - \tilde{B}(\lambda)].
\]  \hspace{1cm} (2.31)

However, (2.29a), (2.29b) and (2.31) enable us to replace \(A_1(\lambda)\) by \(\overline{A}(\lambda)\)

\[
\overline{A}(\lambda) \equiv A_1(\lambda) - \tilde{A}(\lambda) = -\sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{\lambda - \lambda_j},
\]  \hspace{1cm} (2.32)

and we will redefine \(v_k\) by \(v_k = \overline{A}(u_k)\).

Then a straightforward calculation shows that \(\overline{B}(\lambda) = B(\lambda), \overline{A}(\lambda), \tilde{B}(\lambda), \tilde{A}(\lambda)\) satisfy the following required commutator relations

\[
\{\overline{B}(\lambda), \overline{B}(\mu)\} = \{\tilde{B}(\lambda), \tilde{B}(\mu)\} = \{\overline{A}(\lambda), \overline{A}(\mu)\} = \{\tilde{A}(\lambda), \tilde{A}(\mu)\} = 0, \hspace{1cm} (2.33a)
\]

\[
\{\overline{B}(\lambda), \tilde{B}(\mu)\} = \{\overline{B}(\lambda), \tilde{A}(\mu)\} = \{\tilde{B}(\lambda), \overline{A}(\mu)\} = \{\tilde{B}(\lambda), \tilde{A}(\mu)\} = 0, \hspace{1cm} (2.33b)
\]

\[
\{\overline{A}(\lambda), \overline{B}(\mu)\} = \frac{1}{\lambda - \mu} [\overline{B}(\mu) - \overline{B}(\lambda)], \hspace{1cm} \{\tilde{A}(\lambda), \tilde{B}(\mu)\} = \frac{1}{\lambda - \mu} [\tilde{B}(\mu) - \tilde{B}(\lambda)]. \hspace{1cm} (2.33c)
\]

We have the following proposition.

**Proposition 1.** Assume that \(\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N\). Introduce the separated variables \(u_1, \ldots, u_{2N}\) by the set of zeros of \(\overline{B}(\lambda)\) and \(\tilde{B}(\lambda)\):

\[
\overline{B}(\lambda) = B(\lambda) + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{1j}}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)}, \hspace{1cm} (2.34a)
\]

\[
\tilde{B}(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j} = \frac{\overline{T}(\lambda)}{K(\lambda)}, \hspace{1cm} (2.34b)
\]
with
\[ R(\lambda) = \prod_{k=1}^{N} (\lambda - u_k), \quad \overline{R}(\lambda) = \prod_{k=1}^{N} (\lambda - u_{N+k}), \]
and \( v_1, \ldots, v_{2N} \) by
\[ v_k = A(u_k) = A_1(u_k) - \tilde{A}(u_k) = -\sum_{j=1}^{N} \frac{\psi_{2j}\phi_{2j}}{u_k - \lambda_j}, \quad k = 1, \ldots, N, \quad (2.34c) \]
\[ v_{N+k} = \tilde{A}(u_{N+k}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}}{u_{N+k} - \lambda_j}, \quad k = 1, \ldots, N. \quad (2.34d) \]

If \( u_1, \ldots, u_N \), are single zeros of \( \overline{B}(\lambda) \), then \( v_1, \ldots, v_{2N} \) and \( u_1, \ldots, u_{2N} \) are canonically conjugated, i.e., they satisfy (1.1).

Proof. Notice that
\[ \lim_{\lambda \to \lambda_j - 0} \tilde{B}(\lambda) = -\infty, \quad \lim_{\lambda \to \lambda_j + 0} \tilde{B}(\lambda) = \infty, \]
it is easy to see that
\[ u_{N+1} < \lambda_1 < u_{N+2} < \lambda_2 < \ldots < u_{2N} < \lambda_N. \quad (2.35) \]
We have \( \overline{B}'(u_k) \neq 0, \tilde{B}'(u_{N+k}) \neq 0 \). Hereafter the prime denotes the differentiation with respect to \( \lambda \). In what follows, we take \( k, l = 1, \ldots, N \). It follows from (2.33b) that
\[ 0 = \{u_k, \tilde{B}(u_{N+l})\} = \tilde{B}'(u_{N+l})\{u_k, u_{N+l}\} + \{u_k, \tilde{B}(\mu)\}_{\mu = u_{N+l}}, \]
\[ 0 = \{\overline{B}(u_k), u_{N+l}\} = \overline{B}'(u_k)\{u_k, u_{N+l}\} + \{\overline{B}(\lambda), u_{N+l}\}_{\lambda = u_k}, \]
\[ 0 = \{\overline{B}(u_k), \tilde{B}(u_{N+l})\} = \overline{B}'(u_k)\tilde{B}'(u_{N+l})\{u_k, u_{N+l}\} + \overline{B}'(u_k)\{u_k, \tilde{B}(\mu)\}_{\mu = u_{N+l}} \]
\[ + \tilde{B}'(u_{N+l})\overline{B}(\lambda), u_{N+l}\}_{\lambda = u_k} + \{\overline{B}(\lambda), \tilde{B}(\mu)\}_{\lambda = u_k, \mu = u_{N+l}} \]
\[ = \overline{B}'(u_{N+l})\{\overline{B}(\lambda), u_{N+l}\}_{\lambda = u_k} \]
which together lead to \( \{u_k, u_{N+l}\} = 0 \). Similarly, \( \{u_1, u_l\} = 0, \{u_{N+k}, u_{N+l}\} = 0 \).

Using (2.33b), (2.33c) and above results, one gets
\[ \{v_k, \overline{B}(\mu)\}_{\mu = u_l} = \{\overline{A}(u_k), \overline{B}(\mu)\}_{\mu = u_l} \]
\[ = \overline{A}'(u_k)\{u_k, \overline{B}(\mu)\}_{\mu = u_l} + \{\overline{A}(\lambda), \overline{B}(\mu)\}_{\lambda = u_k} \}_{\mu = u_l}, \]
\[ A'(u_k)[\{u_k, B(u_l)\}] - B'(u_l)[\{u_k, u_l\}] + [(\overline{A}(\lambda), \overline{B}(\mu))]_{\lambda=u_k} \] 
\[ = \frac{B(\mu) - B(u_k)}{u_k - \mu} \] 
\[ \text{and} \]
\[ 0 = \{v_k, B(u_l)\} = B'(u_l)[\{v_k, u_l\}] + \{v_k, B(\mu)\}_{\mu=u_l}, \]

then
\[ \{v_k, u_l\} = -\frac{1}{B'(u_l)} \{v_k, B(\mu)\}_{\mu=u_l} = \delta_{kl}. \]

In the same way, one gets \( \{v_{N+k}, u_{N+l}\} = \delta_{kl} \). The following equalities
\[ \{v_k, u_{N+l}\} = \overline{A}(u_k), u_{N+l} = \{\overline{A}(\lambda), u_{N+l}\}_{\lambda=u_k}, \]
\[ 0 = \{\overline{A}(\lambda), \overline{B}(u_{N+l})\} = \overline{B}'(u_{N+l})\{\overline{A}(\lambda), u_{N+l}\}. \]

yield \( \{v_k, u_{N+l}\} = 0 \) and similarly \( \{v_{N+k}, u_l\} = 0. \)

Finally,
\[ \{v_k, v_{N+l}\} = \{\overline{A}(u_k), \overline{A}(u_{N+l})\} \]
\[ = \overline{A}'(u_k)[\{u_k, \overline{A}(\mu)\}_{\mu=u_{N+l}} + \overline{A}'(u_{N+l})\{\overline{A}(\lambda), u_{N+l}\}_{\lambda=u_k} \]
\[ = \overline{A}'(u_k)[\{u_k, v_{N+l}\} - \overline{A}'(u_{N+l})\{u_k, u_{N+l}\}] \]
\[ + \overline{A}'(u_{N+l})[\{v_k, u_{N+l}\} - \overline{A}'(u_k)\{u_k, u_{N+l}\}] = 0, \]

similarly
\[ \{v_k, v_l\} = \overline{A}(u_k)[\{u_k, v_l\} + \overline{A}'(u_l)[v_k, u_l] = -\overline{A}'(u_k)\delta_{kl} + \overline{A}'(u_l)\delta_{kl} = 0, \]

and \( \{v_{N+k}, v_{N+l}\} = 0. \) This completes the proof.

It follows from (2.34a) and (2.34b) that
\[ \psi_{2j} \phi_{1j} = 2 \frac{R(\lambda_j)}{K'(\lambda_j)}, \quad \phi_{1j}^2 = 2 \frac{\overline{R}(\lambda_j)}{K'(\lambda_j)}, \quad j = 1, \ldots, N, \]
or
\[ \phi_{1j} = \sqrt{\frac{2R(\lambda_j)}{K'(\lambda_j)}}, \quad \psi_{2j} = \frac{\sqrt{2R(\lambda_j)}}{\sqrt[3]{R(\lambda_j)K'(\lambda_j)}}, \quad j = 1, \ldots, N. \] (2.36)

Also (2.34a) results
\[ u = - \langle \Psi_2, \Phi_1 \rangle = 2 \sum_{j=1}^{N} (u_j - \lambda_j). \] (2.37)
We now present the separated equations. By substituting $u_k$ into (2.20), $u_{N+k}$ into (2.30a) and using (2.34), one gets the separated equations

$$v_k = A_1(u_k) - \tilde{A}(u_k) = 2\sqrt{P(u_k)} - \tilde{A}(u_k)$$

$$= 2\sqrt{-u_k + \sum_{j=1}^{N} \left[ \frac{P_j}{u_k - \lambda_j} + \frac{P_{N+j}^2}{(u_k - \lambda_j)^2} \right] - 2\sum_{j=1}^{N} \frac{P_{N+j}}{u_k - \lambda_j}}, \quad k = 1, ..., N, \quad (2.38a)$$

$$v_{N+k} = \tilde{A}(u_{N+k}) = 2\sum_{j=1}^{N} \frac{P_{N+j}}{u_{N+k} - \lambda_j}, \quad k = 1, ..., N. \quad (2.38b)$$

Replacing $v_k$ by the partial derivative $\frac{\partial S}{\partial u_k}$ of the generating function $S$ of the canonical transformation and interpreting the $P_i$ as integration constants, the equations (2.38) give rise to the Hamilton-Jacobi equations which are completely separable and may be integrated to give the completely separated solution

$$S(u_1, ..., u_{2N}) = \sum_{k=1}^{N} \left[ \int_{u_k}^{u_k} (2\sqrt{P(\lambda)} - \tilde{A}(\lambda)) d\lambda + \int_{u_{N+k}}^{u_{N+k}} \tilde{A}(\lambda) d\lambda \right]$$

$$= 2\sum_{k=1}^{N} \left[ \int_{u_k}^{u_k} \sqrt{P(\lambda)} d\lambda - \sum_{i=1}^{N} P_{N+i} \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| \right]. \quad (2.39)$$

The linearizing coordinates are then

$$Q_i = \frac{\partial S}{\partial P_i} = \sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}} d\lambda, \quad i = 1, ..., N, \quad (4.00a)$$

$$Q_{N+i} = \frac{\partial S}{\partial P_{N+i}} = 2\sum_{k=1}^{N} \left[ \int_{u_k}^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| \right], \quad i = 1, ..., N. \quad (4.00b)$$

By using (2.22), the linear flow induced by (2.15) is then given by

$$Q_i = \gamma_i + x \frac{\partial F_1}{\partial P_i} = \gamma_i + 2x, \quad Q_{N+i} = 2\gamma_{N+i} + x \frac{\partial F_1}{\partial P_{N+i}} = 2\gamma_{N+i}, \quad i = 1, ..., N. \quad (4.01)$$

Hereafter $\gamma_i, i = 1, ..., 2N$, are arbitrary constants. Combining the equation (2.40) with the equation (2.41) leads to the Jacobi inversion problem for the FDIHS (2.15)

$$\sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}} d\lambda = \gamma_i + 2x, \quad i = 1, ..., N. \quad (4.20a)$$
The $\phi_{1j}, \psi_{2j}$ and $<\Psi_2, \Phi_1>$ defined by (2.36) and (2.37) are the symmetric functions of $u_k, k = 1, \ldots, 2N$. If, by using the Jacobi inversion technique [19], $\phi_{1j}, \psi_{2j}$ and $<\Psi_2, \Phi_1>$ can be obtained from (2.42), then $\phi_{2j}, \psi_{1j}$ can be found from the first and the last equation in (2.15) by an algebraic calculation, respectively. The $(\phi_{1j}, \phi_{2j}, \psi_{1j}, \psi_{2j})$ provides the solution to the FDIHS (2.15).

By using (2.22), the linear flow induced by (2.17) is then given by

$$Q_i = \gamma_i + \frac{\partial F_2}{\partial P_i} t_1 = \gamma_i + 2\lambda_i t_1,$$

$$Q_{N+i} = 2\gamma_{N+i} + \frac{\partial F_2}{\partial P_{N+i}} t_1 = 2\gamma_{N+i} + 4P_{N+i} t_1, \quad i = 1, \ldots, N,$$  \hspace{1cm} (2.43)

where $\gamma_i$ are arbitrary constants. Combining the equation (2.40) with the equation (2.44) leads to the Jacobi inversion problem for the FDIHS (2.17)

$$\sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{1}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda = \gamma_i + 2\lambda_i t_1, \quad i = 1, \ldots, N,$$  \hspace{1cm} (2.44a)

$$\sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| = \gamma_{N+i} + 2P_{N+i} t_1, \quad i = 1, \ldots, N.$$  \hspace{1cm} (2.44b)

Finally, since the KdV equation (2.10) is factorized by the FDIHSs (2.15) and (2.17), combining the equation (2.42) with the equation (2.44) and using (2.37) give rise to the Jacobi inversion problem for the KdV equation (2.10)

$$\sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{1}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda = \gamma_i + 2\lambda_i t_1, \quad i = 1, \ldots, N,$$  \hspace{1cm} (2.45a)

$$\sum_{k=1}^{N} \int_{u_k}^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| = \gamma_{N+i} + 2P_{N+i} t_1, \quad i = 1, \ldots, N.$$  \hspace{1cm} (2.45b)

Notice that $u$ defined by (2.37) is the symmetric function of $u_k, k = 1, \ldots, N$. If, by using the Jacobi inversion technique [19], $u$ can be found in terms of Riemann theta functions by solving (2.45), then $u$ provides the solution of the KdV equation (2.10).

In general, since the $n$-th KdV equation (2.7) is factorized by the $x$-FDIHS (2.15) and the $t_n$-FDIHS (2.25), the above procedure can be applied to find the Jacobi inversion problem for the $n$-th KdV equation (2.7). We have the following proposition.
Proposition 2. The Jacobi inversion problem for the n-th KdV equation (2.7) is given by

\[ \sum_{k=1}^{N} \int_{u_k}^{u} \frac{1}{(\lambda - \lambda_i) \sqrt{P(\lambda)}} d\lambda = \gamma_i + 2x \]

\[ + t_n \sum_{m=0}^{n} \left( \frac{1}{2} \right)^{m-1} \alpha_m \sum_{l_1+\ldots+l_{m+1}=n+2} \lambda_i^{l_{m+1}-2} \tilde{F}_{l_1} \ldots \tilde{F}_{l_m}, \quad i = 1, \ldots, N, \]  

(2.46a)

\[ \sum_{k=1}^{N} \int_{u_k}^{u} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| = \gamma_{N+i} \]

\[ + t_n \sum_{m=0}^{n} \left( \frac{1}{2} \right)^{m-2} \alpha_m \sum_{l_1+\ldots+l_{m+1}=n+2} (l_{m+1} - 2) \lambda_i^{l_{m+1}-3} P_{N+i} \tilde{F}_{l_1} \ldots \tilde{F}_{l_m}, \]

\[ i = 1, \ldots, N, \]  

(2.46b)

where \( l_1 \geq 1, \ldots, l_{m+1} \geq 1 \) and \( \tilde{F}_{l_1} \ldots \tilde{F}_{l_m} \), are given by (2.24b).

For example, by using (2.27), the Jacobi inversion problem for the second KdV equation (2.26) is given by

\[ \sum_{k=1}^{N} \int_{u_k}^{u} \frac{1}{(\lambda - \lambda_i) \sqrt{P(\lambda)}} d\lambda = \gamma_i + 2x + (2\lambda_i^2 + \sum_{j=1}^{N} P_j) t_2, \quad i = 1, \ldots, N, \]  

(2.47a)

\[ \sum_{k=1}^{N} \int_{u_k}^{u} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| = \gamma_{N+i} + 4\lambda_i P_{N+i} t_2, \quad i = 1, \ldots, N. \]  

(2.47b)

The \( u \) solved from the Jacobi inversion problem (2.47) provides the solution for the second KdV equation (2.26).

The Jacobi inversion problem for the KdV hierarchy in our case is somewhat different from that derived by means of the stationary equations of the KdV hierarchy [36], since there is an additional term \( -\ln|u_k - \lambda_i| + \ln|u_{N+k} - \lambda_i| \) in (2.46b).

3. The separation of variables for the AKNS equations.

3.1 Binary constrained flows of the AKNS equations.

For the AKNS spectral problem [37]

\[ \phi_x = U(u, \lambda) \phi, \quad U(u, \lambda) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (3.1) \]
its adjoint representation (2.2) and (2.3) yield
\[ a_0 = -1, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = q, \quad c_1 = r, \quad a_2 = \frac{1}{2} qr, \ldots, \]
and in general
\[
\begin{pmatrix} c_{k+1} \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} c_k \\ b_k \end{pmatrix}, \quad a_k = \partial^{-1}(qc_k - rb_k), \quad k = 1, 2, \ldots \tag{3.2}
\]
\[
L = \frac{1}{2} \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix}.
\]
Take
\[
\phi_{t_n} = V^{(n)}(u, \lambda)\phi, \quad V^{(n)}(u, \lambda) = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i}. \tag{3.3}
\]
Then the compatibility condition of equations (3.1) and (3.3) gives rise to the AKNS hierarchy
\[
u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = JL^n \begin{pmatrix} r \\ q \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \quad n = 1, 2, \ldots, \tag{3.4}
\]
where the Hamiltonian \( H_n \) and the Hamiltonian operator \( J \) are given by
\[
J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad H_n = 2a_{n+1} + \frac{c_n}{n+1}, \quad \begin{pmatrix} c_n \\ b_n \end{pmatrix} = \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \ldots.
\]
For \( n = 2 \) we have
\[
\phi_{t_2} = V^{(2)}(u, \lambda)\phi, \quad V^{(2)} = \begin{pmatrix} -\lambda^2 + \frac{1}{2} qr & q\lambda - \frac{1}{2} qx \\ r\lambda + \frac{1}{2} rx & \lambda^2 - \frac{1}{2} qr \end{pmatrix}, \tag{3.5}
\]
and the AKNS equation (3.4) for \( n = 2 \) reads
\[
q_{t_2} = -\frac{1}{2} q_{xx} + q^2 r, \quad r_{t_2} = \frac{1}{2} r_{xx} - r^2 q. \tag{3.6}
\]
The adjoint AKNS spectral problem is the equation (2.11) with \( U \) given by (3.1). We have [25]
\[
\frac{\delta \lambda}{\delta u} = \begin{pmatrix} \frac{\delta \lambda}{\delta q} \\ \frac{\delta \lambda}{\delta r} \end{pmatrix} = \beta Tr\left[\begin{pmatrix} \phi_1 \psi_1 & \phi_1 \psi_2 \\ \phi_2 \psi_1 & \phi_2 \psi_2 \end{pmatrix} \frac{\partial U(u, \lambda)}{\partial u}\right] = \beta \begin{pmatrix} \psi_1 \phi_2 \\ \psi_2 \phi_1 \end{pmatrix}. \tag{3.7}
\]
The binary \( x \)-constrained flows of the AKNS hierarchy (3.4) are defined by [25,29]
\[
\Phi_{1,x} = -\Lambda \Phi_1 + q \Phi_2, \quad \Phi_{2,x} = r \Phi_1 + \Lambda \Phi_2, \tag{3.8a}
\]
\[ \Psi_{1,x} = \Lambda \Psi_1 - r \Psi_2, \quad \Psi_{2,x} = -q \Psi_1 - \Lambda \Psi_2, \] 

\[ \frac{\delta H_{k_0}}{\delta u} - \beta^{-1} \sum_{j=1}^{N} \delta \lambda_j = (c_{k_0}) - \left( < \Psi_1, \Phi_2 > \right) = 0. \] 

For \( k_0 = 1 \), we have

\[ \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix} = \left( < \Psi_1, \Phi_2 > \right) = 0. \] 

By substituting (3.9) into (3.8a) and (3.8b), the first binary \( x \)-constrained flow becomes a \( x \)-FDHS [25]

\[ \begin{align*} 
\Phi_{1x} &= \frac{\partial F_1}{\partial \Psi_1}, \\
\Phi_{2x} &= \frac{\partial F_1}{\partial \Psi_2}, \\
\Psi_{1x} &= -\frac{\partial F_1}{\partial \Phi_1}, \\
\Psi_{2x} &= -\frac{\partial F_1}{\partial \Phi_2}, 
\end{align*} \]

with the Hamiltonian

\[ F_1 = < \Lambda \Psi_2, \Phi_2 > - < \Lambda \Psi_1, \Phi_1 > + < \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 >. \]

Under the constraint (3.9) and the FDHS (3.10), the binary \( t_2 \)-constrained flow obtained from (3.3) with \( V^{(2)} \) given by (3.5) and its adjoint equation for \( N \) distinct real number \( \lambda_j \) can also be written as a \( t_2 \)-FDHS

\[ \begin{align*} 
\Phi_{1,t_2} &= \frac{\partial F_2}{\partial \Psi_1}, \\
\Phi_{2,t_2} &= \frac{\partial F_2}{\partial \Psi_2}, \\
\Psi_{1,t_2} &= -\frac{\partial F_2}{\partial \Phi_1}, \\
\Psi_{2,t_2} &= -\frac{\partial F_2}{\partial \Phi_2}, 
\end{align*} \]

with the Hamiltonian

\[ F_2 = < \Lambda^2 \Psi_2, \Phi_2 > - < \Lambda^2 \Psi_1, \Phi_1 > + < \Psi_2, \Phi_1 > < \Lambda \Psi_1, \Phi_2 > + < \Lambda \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 > - \frac{1}{2} ( < \Psi_2, \Phi_2 > - < \Psi_1, \Phi_1 > ) < \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 >. \]

The Lax representation for the FDHSs (3.10) and (3.11) which can also be deduced from the adjoint representation (2.2) and (2.8) are presented by (2.18) with the entries of the Lax matrix \( M \) given by [29]

\[ A(\lambda) = -1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} - \psi_{2j} \phi_{2j}}{\lambda - \lambda_j}, \]

\[ B(\lambda) = \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{1j}}{\lambda - \lambda_j}, \quad C(\lambda) = \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{2j}}{\lambda - \lambda_j}. \]
A straightforward calculation yields

\[
A^2(\lambda) + B(\lambda)C(\lambda) \equiv P(\lambda) = 1 + \sum_{j=1}^{N} \left[ \frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2} \right],
\]

(3.13)

where \( P_j, j = 1, \ldots, 2N \), are \( 2N \) independent integrals of motion for the FDHSs (3.10) and (3.11)

\[
P_j = \psi_{2j}\phi_{2j} - \psi_{1j}\phi_{1j}, \quad j = 1, \ldots, N
\]

(3.14a)

\[
P_{N+j} = \frac{1}{2} (\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}), \quad j = 1, \ldots, N.
\]

(3.14b)

It is easy to verify that

\[
F_1 = \sum_{j=1}^{N} (\lambda_j P_j + P_{N+j}^2) - \frac{1}{4} (\sum_{j=1}^{N} P_j)^2,
\]

(3.15a)

\[
F_2 = \sum_{j=1}^{N} (\lambda_j^2 P_j + 2\lambda_j P_{N+j}^2) - \frac{1}{2} \left( \sum_{j=1}^{N} P_j \right) \left( \sum_{j=1}^{N} (\lambda_j P_j + P_{N+j}^2) \right) + \frac{1}{8} (\sum_{j=1}^{N} P_j)^3.
\]

(3.15b)

With respect to the standard Poisson bracket it is found that

\[
\{ A(\lambda), A(\mu) \} = \{ B(\lambda), B(\mu) \} = \{ C(\lambda), C(\mu) \} = 0,
\]

(3.16a)

\[
\{ A(\lambda), B(\mu) \} = \frac{1}{\lambda - \mu} [B(\mu) - B(\lambda)],
\]

(3.16b)

\[
\{ A(\lambda), C(\mu) \} = \frac{1}{\lambda - \mu} [C(\lambda) - C(\mu)],
\]

(3.16c)

\[
\{ B(\lambda), C(\mu) \} = \frac{2}{\lambda - \mu} [A(\mu) - A(\lambda)].
\]

(3.16d)

Then \( \{ A^2(\lambda) + B(\lambda)C(\lambda), A^2(\mu) + B(\mu)C(\mu) \} = 0 \) implies that \( P_j, j = 1, \ldots, 2N \), are in involution. The AKNS equation (3.6) is factorized by the \( x \)-FDIHS (3.10) and the \( t_2 \)-FDIHS (3.11), namely, if \( (\Psi_1, \Psi_2, \Phi_1, \Phi_2) \) satisfies the FDIHSs (3.10) and (3.11) simultaneously, then \( (q, r) \) given by (3.9) solves the AKNS equation (3.6). In general, the factorization of the \( n \)-th AKNS equations (3.4) will be presented in the end of section 3.2.
3.2 The separation of variables for the AKNS equations.

In contrast with the $B(\lambda)$ in the Lax matrix $M$ for the constrained KdV flows, the $B(\lambda)$ given by (3.12b) has only $N - 1$ zeros, one has to define the canonical variables $u_k, v_k, k = 1, \ldots, 2N$, in a little different way. The commutator relations (3.16) and the generating function of integrals of motion (3.13) enable us to introduce $u_1, \ldots, u_N$ by means of $B(\lambda)$ in the following way:

$$B(\lambda) = \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{1j}}{\lambda - \lambda_j} = e^{u_N} \frac{R(\lambda)}{K(\lambda)},$$

where

$$R(\lambda) = \prod_{k=1}^{N-1} (\lambda - u_k), \quad K(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k),$$

and $v_1, \ldots, v_N$ by

$$v_k = A(u_k), \quad k = 1, \ldots, N - 1, \quad v_N = \frac{1}{2}(<\Psi_2, \Phi_1> - <\Psi_1, \Phi_2>).$$

The equation (3.17a) yields

$$u_N = \ln <\Psi_2, \Phi_1>.$$  

Then it is easy to verify that

$$\{u_N, B(\mu)\} = \{v_N, A(\mu)\} = 0, \quad \{v_N, u_N\} = 1,$$

$$\{u_N, A(\mu)\} = -\frac{B(\mu)}{<\Psi_2, \Phi_1>}, \quad \{v_N, B(\mu)\} = B(\mu).$$

As we will show later, the commutator relations (3.16) and (3.18) guarantee that $u_1, \ldots, u_N, v_1, \ldots, v_N$ satisfy the canonical conditions (1.1).

We now need to construct two functions $\tilde{B}(\lambda), \tilde{A}(\lambda)$ to define $u_{N+1}, \ldots, u_{2N}$ by means of $B(\lambda)$ and $v_{N+1}, \ldots, v_{2N}$ by $v_{N+k} = \tilde{A}(u_{N+k})$. By the exactly same argument as in the previous section, we construct $\tilde{A}(\lambda)$ by

$$\tilde{A}(\lambda) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} + \psi_{2j} \phi_{2j}}{\lambda - \lambda_j} = \sum_{j=1}^{N} \frac{P_{N+j}}{\lambda - \lambda_j},$$

since the equation (3.19a) enable us to obtain immediately the separated equations (1.2) for $v_{N+k}$ and $u_{N+k}$, and $\tilde{B}(\lambda)$ by

$$\tilde{B}(\lambda) = \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j}.$$
Then it is easy to verify that $A(\lambda), B(\lambda), \tilde{A}(\lambda), \tilde{B}(\mu)$ satisfy the commutator relations
\begin{align}
\{\tilde{B}(\lambda), B(\mu)\} &= \{\tilde{B}(\lambda), A(\mu)\} = \{\tilde{A}(\lambda), B(\mu)\} = \{\tilde{A}(\lambda), A(\mu)\} = 0, \\
\{\tilde{A}(\lambda), B(\mu)\} &= \frac{1}{\lambda - \mu} [\tilde{B}(\mu) - \tilde{B}(\lambda)], \\
\{A(\lambda), \tilde{B}(\mu)\} &= \frac{1}{\lambda - \mu} [\tilde{B}(\mu) - \tilde{B}(\lambda)].
\end{align}

The relation (3.20c) doesn’t fit the requirement for the canonical conditions (1.1). According to (3.20) and (3.16) we can replace $A(\lambda)$ by $\overline{A}(\lambda)$
\begin{equation}
\overline{A}(\lambda) \equiv A(\lambda) - \tilde{A}(\lambda) = -1 - \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{\overline{\lambda} - \lambda_j},
\end{equation}

namely, we redefine $v_1, ..., v_N$ by
\begin{equation}
v_k = \overline{A}(u_k) = A(u_k) - \tilde{A}(u_k) = -1 - \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{u_k - \lambda_j}, \quad k = 1, ..., N - 1, \tag{3.22a}
\end{equation}
\begin{equation}
v_N = - < \Psi_2, \Phi_2 >. \tag{3.22b}
\end{equation}

We now define $u_{N+1}, ..., u_{2N}$ by $\tilde{B}(\lambda)$ as follows:
\begin{equation}
\tilde{B}(\lambda) = \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\overline{\lambda} - \lambda_j} = e^{v_{2N}} \frac{\overline{R}(\lambda)}{K(\lambda)}, \quad \overline{R}(\lambda) = \prod_{k=1}^{N-1} (\lambda - u_{N+k}), \tag{3.23a}
\end{equation}
and $v_{N+1}, ..., v_{2N}$ by
\begin{equation}
v_{N+k} = \tilde{A}(u_{N+k}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} + \psi_{2j} \phi_{2j}}{u_{N+k} - \lambda_j}, \quad k = 1, ..., N - 1, \tag{3.23b}
\end{equation}
\begin{equation}
v_{2N} = \frac{1}{2} (< \Psi_1, \Phi_1 > + < \Psi_2, \Phi_2 >). \tag{3.23c}
\end{equation}

The equation (3.23a) leads to
\begin{equation}
u_{2N} = ln < \Phi_1, \Phi_1 >. \tag{3.23d}
\end{equation}

Then a straightforward calculation shows that $\overline{B}(\lambda) = B(\lambda), \overline{A}(\lambda), \tilde{B}(\lambda), \tilde{A}(\lambda)$ satisfy the commutator relations (2.33) and
\begin{equation}
\{u_N, \overline{B}(\mu)\} = \{u_N, \tilde{B}(\mu)\} = \{u_N, \tilde{A}(\mu)\} = 0, \quad \{u_N, \overline{A}(\mu)\} = - \frac{\overline{B}(\mu)}{< \Psi_2, \Phi_1 >}, \tag{3.24a}
\end{equation}
\[
\{v_N, \overline{A}(\mu)\} = \{v_N, \overline{B}(\mu)\} = \{v_N, \overline{A}(\mu)\} = 0, \quad \{v_N, \overline{B}(\mu)\} = \overline{B}(\mu),
\]
\[
\{u_{2N}, \overline{B}(\mu)\} = \{u_{2N}, \overline{A}(\mu)\} = \{u_{2N}, \overline{B}(\mu)\} = 0, \quad \{u_{2N}, \overline{A}(\mu)\} = -\frac{\overline{B}(\mu)}{<\psi_1, \Phi_1>},
\]
\[
\{v_{2N}, \overline{B}(\mu)\} = \{v_{2N}, \overline{A}(\mu)\} = \{v_{2N}, \overline{A}(\mu)\} = 0, \quad \{v_{2N}, \overline{B}(\mu)\} = \overline{B}(\mu),
\]
\[
\{v_{2N}, u_{2N}\} = 1, \quad \{v_{2N}, u_{2N}\} = 1, \quad \{v_{2N}, u_{2N}\} = \{v_{2N}, u_{2N}\} = \{v_{2N}, v_{2N}\} = 0.
\]
conditions (1.1). By using the similar method, for example, it is found from (3.24) that for \( k = 1, \ldots, N - 1 \), we have

\[
0 = \{u_N, B(u_k)\} = B'(u_k)\{u_N, u_k\} + \{u_N, B(\mu)\}_{\mu = u_k} = B'(u_k)\{u_N, u_k\},
\]

\[
\{u_N, v_k\} = \{u_N, A(u_k)\} = A'(u_k)\{u_N, u_k\} - \frac{B'(\mu)}{<\Psi_2, \Phi_1>_{\mu = u_k}} = 0,
\]

which gives rise to \( \{u_N, u_k\} = \{u_N, v_k\} = 0 \) and so on. In this way we complete the proof.

It follows from (3.25a) and (3.25b) that

\[
\psi_2 j \phi_1 j = e^{u_N N} \frac{R(\lambda_j)}{K'(\lambda_j)}, \quad \phi_1^2 j = e^{u_{2N} N} \frac{R(\lambda_j)}{K'(\lambda_j)}, \quad j = 1, \ldots, N,
\]

or

\[
\phi_1 j = \sqrt{\frac{e^{u_{2N} N} R(\lambda_j)}{K'(\lambda_j)}}, \quad \psi_2 j = \frac{e^{u_N N} R(\lambda_j)}{\sqrt{e^{u_{2N} N} R(\lambda_j) K'(\lambda_j)}}, \quad j = 1, \ldots, N. \tag{3.26}
\]

The equation (3.9) and (3.17c) results

\[
q = e^{u_N N}. \tag{3.27}
\]

We now present the separated equations. By substituting \( u_k \) into (3.13), \( u_{N+k} \) into (3.19a) and using (3.25c) and (3.25e), one gets the separated equations

\[
v_k = A(u_k) - \tilde{A}(u_k) = \sqrt{P(u_k)} - \tilde{A}(u_k)
\]

\[
= \sqrt{1 + \sum_{j=1}^{N} \frac{P_j}{u_k - \lambda_j} + \frac{P_{N+j}}{(u_k - \lambda_j)^2}} - \sum_{j=1}^{N} \frac{P_{N+j}}{u_k - \lambda_j}, \quad k = 1, \ldots, N - 1, \tag{3.28a}
\]

\[
v_{N+k} = \tilde{A}(u_{N+k}) = \sum_{j=1}^{N} \frac{P_{N+j}}{u_{N+k} - \lambda_j}, \quad k = 1, \ldots, N - 1. \tag{3.28b}
\]

It is easy to see from (3.14) that

\[
<\Psi_2, \Phi_2> - <\Psi_1, \Phi_1> = \sum_{i=1}^{N} P_i, \quad <\Psi_1, \Phi_1> + <\Psi_2, \Phi_2> = 2 \sum_{i=1}^{N} P_{N+i},
\]

which together with (3.25d) and (3.25f) leads to

\[
v_N = -\frac{1}{2} \sum_{i=1}^{N} (P_i + 2P_{N+i}), \quad v_{2N} = \sum_{i=1}^{N} P_{N+i}. \tag{3.28c}
\]
Replacing $v_k$ by the partial derivative $\frac{\partial S}{\partial u_k}$ of the generating function $S$ of the canonical transformation and interpreting the $P_i$ as integration constants, the equations (3.28) may be integrated to give the generating function of the canonical transformation

$$S(u_1, \ldots, u_{2N}) = \sum_{k=1}^{N-1} \int_{u_k}^{u_{N+k}} \left( \sqrt{P(\lambda)} - \tilde{A}(\lambda) \right) d\lambda + \int_{u_{N+k}}^{u_{2N}} \tilde{A}(\lambda) d\lambda$$

$$- \frac{1}{2} \sum_{i=1}^{N} (P_i + 2P_{N+i}) u_N + \sum_{i=1}^{N} P_{N+i} u_{2N}$$

$$= \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{N+k}} \sqrt{P(\lambda)} d\lambda - \sum_{i=1}^{N} P_{N+i} ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| \right]$$

$$- \frac{1}{2} \sum_{i=1}^{N} (P_i + 2P_{N+i}) u_N + \sum_{i=1}^{N} P_{N+i} u_{2N}. \quad (3.29)$$

The linearizing coordinates are then

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{k=1}^{N-1} \int_{u_k}^{u_{N+k}} \frac{1}{(\lambda - \lambda_i) \sqrt{P(\lambda)}} d\lambda - \frac{1}{2} u_N, \quad i = 1, \ldots, N, \quad (3.30a)$$

$$Q_{N+i} = \frac{\partial S}{\partial P_{N+i}} = \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{N+k}} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| \right] - u_N + u_{2N}, \quad i = 1, \ldots, N. \quad (3.30b)$$

By using (3.15a), the linear flow induced by the FDIHS (3.10) together with the equations (3.30) leads to the Jacobi inversion problem for the FDIHS (3.10)

$$\sum_{k=1}^{N-1} \int_{u_k}^{u_{N+k}} \frac{1}{(\lambda - \lambda_i) \sqrt{P(\lambda)}} d\lambda - u_N = \gamma_i + (2\lambda_i - \sum_{k=1}^{N} P_k)x, \quad i = 1, \ldots, N, \quad (3.31a)$$

$$\sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{N+k}} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| \right] - u_N + u_{2N} = \gamma_{N+i} + 2P_{N+i}x, \quad i = 1, \ldots, N. \quad (3.31b)$$

By using (3.15b), the linear flow induced by the FDIHS (3.11) and the equations (3.30) result in the Jacobi inversion problem for the FDIHS (3.11)

$$\sum_{k=1}^{N-1} \int_{u_k}^{u_{N+k}} \frac{1}{(\lambda - \lambda_i) \sqrt{P(\lambda)}} d\lambda - u_N$$
\[= \bar{\gamma}_i + [2\lambda_i^2 - \sum_{k=1}^{N}(\lambda_k P_k + \lambda_i P_k + P_{N+k}^2)] + \frac{3}{4} \left(\sum_{k=1}^{N} P_k^2\right)t_2, \quad i = 1, \ldots, N, \]  
\label{eq:3.32a}

\[= \bar{\gamma}_{N+i} + P_{N+i}(4\lambda_i - \sum_{k=1}^{N} P_k)t_2, \quad i = 1, \ldots, N. \]  
\label{eq:3.32b}

Then, since the AKNS equations (3.6) are factorized by the FDIHSs (3.10) and (3.11), combining the equations (3.31) with the equations (3.32) gives rise to the Jacobi inversion problem for the AKNS equations (3.6)

\[\sum_{k=1}^{N-1} \int_{u_k}^{u_{N+i}} \frac{P_{N+i}}{(\lambda - \lambda_i)^2 \sqrt{P(\lambda)}} d\lambda - \ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| - u_N + u_{2N} \]

\[= \bar{\gamma}_{N+i} + P_{N+i}(4\lambda_i - \sum_{k=1}^{N} P_k)t_2, \quad i = 1, \ldots, N. \]  
\label{eq:3.33b}

If \(\phi_{1j}, \psi_{2j}, q\) defined by (3.26) and (3.27) can be solved from (3.33) by using the Jacobi inversion technique, then \(\phi_{2j}, \psi_{1j}\) can be obtained from the first equation and the last equation in (3.10) by an algebraic calculation, respectively. Finally \(q\) and \(r = <\Psi_1, \Phi_2>\) provides the solution to the AKNS equations (3.6).

In general, in order to obtain the Jacobi inversion problem for the n-th AKNS equations (3.4), we set

\[A^2(\lambda) + B(\lambda)C(\lambda) = \sum_{k=0}^{\infty} \tilde{F}_k \lambda^{-k}, \]  
\label{eq:3.34}

where \(\tilde{F}_k, k = 1, 2, \ldots,\) are also integrals of motion for both the FDHS (3.10) and the \(t_n\)-binary constrained flow. Comparing (3.34) with (3.13), one gets

\[\tilde{F}_0 = 1, \quad \tilde{F}_k = \sum_{j=1}^{N} [\lambda_j^{k-1} P_j + (k-1)\lambda_j^{k-2} P_{N+j}^2], \quad k = 1, 2, \ldots \]  
\label{eq:3.35}
The n-th AKNS equations (3.4) are factorized by the \(x\)-FDIHS (3.10) and the \(t_n\)-FDIHS with the Hamiltonian \(F_n\) given by [25]

\[
F_n = 2 \sum_{m=0}^{n} \left(-\frac{1}{2}\right)^{m} \frac{\alpha_m}{m+1} \sum_{l_1+\ldots+l_{m+1}=n+1} \tilde{F}_1 \ldots \tilde{F}_{m+1},
\]

(3.36)

where \(l_1 \geq 1, \ldots, l_{m+1} \geq 1\), and \(\alpha_m\) are given by (2.25c). In the same way, we have the following proposition.

**Proposition 4.** The Jacobi inversion problem for the n-th AKNS equations (3.4) is of the form

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{1}{(\lambda - \lambda_i)^{1/2}} \sqrt{P(\lambda)} d\lambda - u_N = \gamma_i + (2\lambda_i - \sum_{k=1}^{N} P_k)x
\]

\[+ 2t_n \sum_{m=0}^{n} \left(-\frac{1}{2}\right)^{m} \alpha_m \sum_{l_1+\ldots+l_{m+1}=n+1} \lambda_i^{l_{m+1}-1} \tilde{F}_1 \ldots \tilde{F}_m, \quad i = 1, \ldots, N,
\]

(3.37a)

\[
\sum_{k=1}^{N} \int_{u_k}^{u_{k+1}} \frac{P_{N+i}}{(\lambda - \lambda_i)^{1/2}} \sqrt{P(\lambda)} d\lambda - ln \left| \frac{u_k - \lambda_i}{u_{N+k} - \lambda_i} \right| - u_N + u_{2N} = \gamma_{N+i} + 2P_{N+i}x
\]

\[+ 4t_n \sum_{m=0}^{n} \left(-\frac{1}{2}\right)^{m} \alpha_m \sum_{l_1+\ldots+l_{m+1}=n+1} (l_{m+1} - 1)\lambda_i^{l_{m+1}-2} P_{N+i} \tilde{F}_1 \ldots \tilde{F}_m, \quad i = 1, \ldots, N,
\]

(3.37b)

where \(l_1 \geq 1, \ldots, l_{m+1} \geq 1\), and \(\tilde{F}_1, \ldots, \tilde{F}_m\), are given by (3.35).

4. The separation of variables for the Kaup-Newell equations.

4.1 Binary constrained flows of the Kaup-Newell hierarchy.

For the Kaup-Newell spectral problem [38]

\[
\phi_x = U(u, \lambda)\phi, \quad U(u, \lambda) = \begin{pmatrix} -\lambda^2 & q \lambda \\ r \lambda & \lambda^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix},
\]

(4.1)

its adjoint representation (2.2) and (2.3) yields

\[
a_0 = 1, \quad a_2 = -\frac{1}{2} qr, \quad b_1 = -q, \quad c_1 = -r, \quad b_3 = \frac{1}{2} (q^2 r + q), \quad c_3 = \frac{1}{2} (qr^2 - r),
\]

and in general \(a_{2k+1} = b_{2k} = c_{2k} = 0\)

\[
\begin{pmatrix} c_{2k+1} \\ b_{2k+1} \end{pmatrix} = L \begin{pmatrix} c_{2k-1} \\ b_{2k-1} \end{pmatrix}, \quad a_{2k} = \frac{1}{2} \partial^{-1} (qc_{2k-1,x} + rb_{2k-1,x}), \quad k = 1, 2, \ldots,
\]

(4.2)
\[ L = \frac{1}{2} \begin{pmatrix} \partial - r \partial^{-1} q \partial & -r \partial^{-1} r \partial \\ -q \partial^{-1} q \partial & -\partial - q \partial^{-1} r \partial \end{pmatrix}. \]

Take
\[
\phi_{t_n} = V^{(n)}(u, \lambda) \phi, \quad V^{(n)}(u, \lambda) = \sum_{i=0}^{n-1} \begin{pmatrix} a_{2i+1} \lambda^{2n-2i} & b_{2i+1} \lambda^{2n-2i-1} \\ c_{2i+1} \lambda^{2n-2i} & -a_{2i} \lambda^{2n-2i} \end{pmatrix}.
\] (4.3)

Then the compatibility condition of equations (4.1) and (4.3) gives rise to the Kaup-Newell hierarchy
\[
\phi_t = \begin{pmatrix} q \\ r \end{pmatrix}, \quad J \left( \begin{array}{c} c_{2n-1} \\ b_{2n-1} \end{array} \right) = \frac{\delta H_{2n}}{\delta u}, \quad n = 1, 2, \ldots, \] (4.4)

where the Hamiltonian \( H_n \) and the Hamiltonian operator \( J \) are given by
\[
J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad H_{2n} = \frac{4a_{2n+2} - rc_{2n+1} - qb_{2n+1}}{2n}, \quad \left( \begin{array}{c} c_{2n+1} \\ b_{2n+1} \end{array} \right) = \frac{\delta H_{2n}}{\delta u}.
\]

For \( n = 2 \) we have
\[
\phi_{t_2} = V^{(2)}(u, \lambda) \phi, \quad V^{(2)} = \begin{pmatrix} \lambda^4 - \frac{1}{2} qr \lambda^2 & -q \lambda^3 + \frac{1}{2} (q^2 r + q_x) \lambda \\ -r \lambda^3 + \frac{1}{2} (qr^2 - q_x) \lambda & -\lambda^4 + \frac{1}{2} qr \lambda^2 \end{pmatrix},
\] (4.5)

and the coupled derivative nonlinear Schrödinger (CDNS) equations obtained from the equation (4.4) for \( n = 2 \) read
\[
q_{t_2} = \frac{1}{2} q_{xx} + \frac{1}{2} (q^2 r)_x, \quad r_{t_2} = -\frac{1}{2} r_{xx} + \frac{1}{2} (r^2 q)_x.
\] (4.6)

The adjoint Kaup-Newell spectral problem is the equation (2.11) with \( U \) given by (4.1). We have [25]
\[
\frac{\delta \lambda}{\delta u} = \left( \begin{array}{c} \frac{\delta \lambda}{\delta q} \\ \frac{\delta \lambda}{\delta r} \end{array} \right) = \beta Tr \left[ \begin{pmatrix} \phi_1 \psi_1 & \phi_1 \psi_2 \\ \phi_2 \psi_1 & \phi_2 \psi_2 \end{pmatrix} \frac{\partial U(u, \lambda)}{\partial u} \right] = \beta \left( \begin{array}{c} \lambda \psi_1 \phi_2 \\ \lambda \psi_2 \phi_1 \end{array} \right).
\] (4.7)

The binary \( x \)-constrained flows of the Kaup-Newell hierarchy (4.4) are defined by
\[
\Phi_{1,x} = -\Lambda^2 \Phi_1 + q \Lambda \Phi_2, \quad \Phi_{2,x} = r \Lambda \Phi_1 + \Lambda^2 \Phi_2, \] (4.8a)
\[
\Psi_{1,x} = \Lambda^2 \Psi_1 - r \Lambda \Psi_2, \quad \Psi_{2,x} = -q \Lambda \Psi_1 - \Lambda^2 \Psi_2, \] (4.8b)
\[
\frac{\delta H_{k_0}}{\delta u} - \beta^{-1} \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \left( \begin{array}{c} c_{2k_0+1} \\ b_{2k_0+1} \end{array} \right) - \frac{1}{2} \left( \begin{array}{c} \langle \Lambda \Psi_1, \Phi_2 \rangle \\ \langle \Lambda \Psi_2, \Phi_1 \rangle \end{array} \right) = 0.
\] (4.8c)
For $k_0 = 1$, we have

\[
\begin{pmatrix}
  c_1 \\
  b_1
\end{pmatrix} = -\begin{pmatrix}
  r \\
  q
\end{pmatrix} = \frac{1}{2} \left( <ΛΨ_1, Φ_2 > - <ΛΨ_2, Φ_1 > \right) = 0.
\] (4.9)

By substituting (4.9) into (4.8a) and (4.8b), the first binary $x$-constrained flow becomes a FDHS

\[
Φ_{1x} = \frac{∂F_1}{∂Ψ_1}, \quad Φ_{2x} = \frac{∂F_1}{∂Ψ_2}, \quad Ψ_{1x} = -\frac{∂F_1}{∂Φ_1}, \quad Ψ_{2x} = -\frac{∂F_1}{∂Φ_2},
\] (4.10)

with the Hamiltonian

\[
F_1 = <Λ^2Ψ_2, Φ_2 > - <Λ^2Ψ_1, Φ_1 > - \frac{1}{2} <ΛΨ_2, Φ_1 > <ΛΨ_1, Φ_2 >.
\]

Under the constraint (4.9) and the FDHS (4.10), the binary $t_2$-constrained flow obtained from (4.3) with $V^{(2)}$ given by (4.5) and its adjoint equation for $N$ distinct real numbers $λ_j$ can also be written as a FDHS

\[
Φ_{1,t_2} = \frac{∂F_2}{∂Ψ_1}, \quad Φ_{2,t_2} = \frac{∂F_2}{∂Ψ_2}, \quad Ψ_{1,t_2} = -\frac{∂F_2}{∂Φ_1}, \quad Ψ_{2,t_2} = -\frac{∂F_2}{∂Φ_2},
\] (4.11)

with the Hamiltonian

\[
F_2 = - <Λ^4Ψ_2, Φ_2 > + <Λ^4Ψ_1, Φ_1 > + \frac{1}{2} <ΛΨ_2, Φ_1 > <ΛΨ_1, Φ_2 >
\]

\[
+ \frac{1}{2} <Λ^3Ψ_2, Φ_1 > <ΛΨ_1, Φ_2 > - \frac{1}{32} <ΛΨ_2, Φ_1 >^2 <ΛΨ_1, Φ_2 >^2
\]

\[
+ \frac{1}{8} (<Λ^2Ψ_2, Φ_2 > - <Λ^2Ψ_1, Φ_1 >) <ΛΨ_2, Φ_1 > <ΛΨ_1, Φ_2 >.
\]

The Lax representation for the FDHSs (4.10) and (4.11) are presented by (2.18) with the entries of the Lax matrix $M$ given by

\[
A(λ) = 1 + \frac{1}{4} \sum_{j=1}^{N} \frac{λ^2 j (ψ_{1j} φ_{1j} - ψ_{2j} φ_{2j})}{λ^2 - λ_j^2},
\] (4.12a)

\[
B(λ) = \frac{1}{2} λ \sum_{j=1}^{N} \frac{λ_j ψ_{2j} φ_{1j}}{λ^2 - λ_j^2}, \quad C(λ) = \frac{1}{2} λ \sum_{j=1}^{N} \frac{λ_j ψ_{1j} φ_{2j}}{λ^2 - λ_j^2}.
\] (4.12b)

A straightforward calculation yields

\[
A^2(λ) + B(λ)C(λ) \equiv P(λ) = 1 + \sum_{j=1}^{N} \left[ \frac{P_j}{λ^2 - λ_j^2} + \frac{λ^4 j^2 P_j^2}{(λ^2 - λ_j^2)^2} \right],
\] (4.13)
where \( P_j, j = 1, \ldots, 2N \), are \( 2N \) independent integrals of motion for the FDIHSs (4.10) and (4.11)

\[
P_j = -\frac{1}{2}\lambda_j^2(\psi_{2j}\phi_{2j} - \psi_{1j}\phi_{1j}) + \frac{1}{8} < \Lambda \Psi_2, \Phi_1 > \lambda_j \psi_{1j}\phi_{2j} + \frac{1}{8} < \Lambda \Psi_1, \Phi_2 > \lambda_j \psi_{2j}\phi_{1j}
\]

\[
+ \frac{1}{8} \sum_{k \neq j} \frac{1}{\lambda_j^2 - \lambda_k^2} \left[ \lambda_j^2 \lambda_k^2(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) + 2\lambda_j \lambda_k (\lambda_j^2 + \lambda_k^2) \psi_{1j}\phi_{2j} \psi_{2k}\phi_{1k} \right],
\]

\[j = 1, \ldots, N\]  \hspace{1cm} (4.14a)

\[
P_{N+j} = \frac{1}{4}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}), \quad j = 1, \ldots, N. \hspace{1cm} (4.14b)
\]

It is easy to verify that

\[
F_1 = -2 \sum_{j=1}^{N} P_j, \quad F_2 = 2 \sum_{j=1}^{N} (\lambda_j^2 P_j + \lambda_j^4 P_{N+j}^2) - \frac{1}{2} (\sum_{j=1}^{N} P_j)^2, \hspace{1cm} (4.15a)
\]

\[
< \Psi_2, \Phi_2 >= + < \Psi_1, \Phi_1 >= 4 \sum_{j=1}^{N} P_{N+j}. \hspace{1cm} (4.15b)
\]

By inserting \( \lambda = 0 \), (4.13) leads to

\[
1 + \frac{1}{4} (< \Psi_2, \Phi_2 > - < \Psi_1, \Phi_1 >) = \sqrt{P(0)} = \sqrt{1 + \sum_{j=1}^{N} [-P_j \lambda_j^{-2} + P_{N+j}^2].} \hspace{1cm} (4.16)
\]

With respect to the standard Poisson bracket it is found that

\[
\{ A(\lambda), A(\mu) \} = \{ B(\lambda), B(\mu) \} = \{ C(\lambda), C(\mu) \} = 0, \hspace{1cm} (4.17a)
\]

\[
\{ A(\lambda), B(\mu) \} = \frac{\mu}{2(\lambda^2 - \mu^2)} [\mu B(\mu) - \lambda B(\lambda)], \hspace{1cm} (4.17b)
\]

\[
\{ A(\lambda), C(\mu) \} = \frac{\mu}{2(\lambda^2 - \mu^2)} [\lambda C(\lambda) - \mu C(\mu)], \hspace{1cm} (4.17c)
\]

\[
\{ B(\lambda), C(\mu) \} = \frac{\lambda \mu}{\lambda^2 - \mu^2} [A(\mu) - A(\lambda)]. \hspace{1cm} (4.17d)
\]

Then \( A^2(\lambda) + B(\lambda)C(\lambda), A^2(\mu) + B(\mu)C(\mu) \) = 0 implies that \( P_j, j = 1, \ldots, 2N \), are in involution. The CDNS equations (4.6) are factorized by the \( x \)-FDIHS (4.10) and the \( t_2 \)-FDIHS (4.11), namely, if \( (\Psi_1, \Psi_2, \Phi_1, \Phi_2) \) satisfies the FDIHSs (4.10) and (4.11) simultaneously, then \( (q, r) \) given by (4.9) solves the CDNS equations (4.6). The factorization of the \( n \)-th Kaup-Newell evuations (4.4) will be presented in the end of section 4.2.
4.2 The separation of variables for the Kaup-Newell equations.

Since the commutator relations (4.17) are quite different from (2.23) and (3.16), we have to modify a little bit of the method presented in sections 2 and 3. Let us denote \( \tilde{\lambda} = \lambda^2, \tilde{\lambda}_j = \lambda_j^2 \). The entries of the Lax matrix \( M \) given by (4.12) can be rewritten as

\[
\begin{align*}
A(\tilde{\lambda}) &= 1 + \frac{1}{4} (\langle \Psi_2, \Phi_2 \rangle - \langle \Psi_1, \Phi_1 \rangle) + \frac{1}{2} \tilde{\lambda} A_1(\tilde{\lambda}), \\
B(\tilde{\lambda}) &= \frac{1}{2} \sqrt{\tilde{\lambda}} B(\tilde{\lambda}),
\end{align*}
\] (4.18a)

where

\[
\begin{align*}
A_1(\tilde{\lambda}) &= \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} - \psi_{2j} \phi_{2j}}{\tilde{\lambda} - \tilde{\lambda}_j}, \\
B(\tilde{\lambda}) &= \sum_{j=1}^{N} \sqrt{\tilde{\lambda}_j} \psi_{2j} \phi_{1j}.
\end{align*}
\] (4.18b)

It is easy to see that

\[
\begin{align*}
\{ A_1(\tilde{\lambda}), A_1(\tilde{\mu}) \} &= \{ B(\tilde{\lambda}), B(\tilde{\mu}) \} = 0, \quad (4.19a) \\
\{ A_1(\tilde{\lambda}), B(\tilde{\mu}) \} &= \frac{1}{\tilde{\lambda} - \tilde{\mu}} [B(\tilde{\mu}) - B(\tilde{\lambda})]. \quad (4.19b)
\end{align*}
\]

It follows from (4.16) and (4.18a) that

\[
A(\tilde{\lambda}) = \sqrt{1 + \sum_{j=1}^{N} [-P_j \tilde{\lambda}_j^{-1} + P_{N+j}^2] + \frac{1}{2} \tilde{\lambda} A_1(\tilde{\lambda})}. \quad (4.19c)
\]

The commutator relations (4.19) and the generating function of integrals of motion (4.13) enable us to introduce \( u_1, \ldots, u_N \) in the following way:

\[
B(\tilde{\lambda}) = \sum_{j=1}^{N} \sqrt{\tilde{\lambda}_j} \psi_{2j} \phi_{1j} = e^{u_N} \frac{R(\tilde{\lambda})}{K(\tilde{\lambda})}, \quad (4.20)
\]

where

\[
R(\tilde{\lambda}) = \prod_{k=1}^{N-1} (\tilde{\lambda} - u_k), \quad K(\tilde{\lambda}) = \prod_{k=1}^{N} (\tilde{\lambda} - \tilde{\lambda}_k),
\]

and \( v_1, \ldots, v_N \) by \( A_1(\tilde{\lambda}) \).

By the exactly same argument as in sections 2 and 3, we construct \( \tilde{A}(\tilde{\lambda}) \) and \( \tilde{B}(\tilde{\lambda}) \) by

\[
\begin{align*}
\tilde{A}(\tilde{\lambda}) &= \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \phi_{1j} + \psi_{2j} \phi_{2j}}{\tilde{\lambda} - \tilde{\lambda}_j} = 2 \sum_{j=1}^{N} \frac{P_{N+j}}{\lambda - \lambda_j}, \quad (4.21) \\
\tilde{B}(\tilde{\lambda}) &= \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j}, \quad (4.22)
\end{align*}
\]
and, for the same reason, we have to replace \( A_1(\tilde{\lambda}) \) by \( \overline{A}(\tilde{\lambda}) \)

\[
\overline{A}(\tilde{\lambda}) \equiv A_1(\tilde{\lambda}) - \tilde{A}(\tilde{\lambda}) = - \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{\lambda - \tilde{\lambda}_j}.
\] (4.23)

Then we have the following proposition.

**Proposition 5.** Assume that \( \lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, ..., N \). Introduce the separated variables \( u_1, ..., u_{2N} \) by the \( \overline{B}(\tilde{\lambda}) \) and \( \overline{B}(\tilde{\lambda}) \):

\[
\overline{B}(\tilde{\lambda}) = \sum_{j=1}^{N} \frac{\sqrt{\lambda_j \psi_{2j} \phi_{1j}}}{\lambda - \tilde{\lambda}_j} = e^{u_N} \frac{R(\tilde{\lambda})}{K(\lambda)},
\] (4.24a)

\[
\overline{B}(\tilde{\lambda}) = \sum_{j=1}^{N} \frac{\phi_{ij}^2}{\lambda - \tilde{\lambda}_j} = e^{u_{2N}} \frac{\overline{R}(\tilde{\lambda})}{K(\lambda)},
\] (4.24b)

with

\[
R(\tilde{\lambda}) = \prod_{k=1}^{N-1} (\tilde{\lambda} - u_k), \quad \overline{R}(\tilde{\lambda}) = \prod_{k=1}^{N-1} (\tilde{\lambda} - u_{N+k}),
\]

and \( v_1, ..., v_{2N} \) by

\[
v_k = \overline{A}(u_k) = A_1(u_k) - \tilde{A}(u_k) = - \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{u_k - \tilde{\lambda}_j}, \quad k = 1, ..., N - 1,
\] (4.24c)

\[
v_N = - \langle \Psi_2, \Phi_2 \rangle,
\] (4.24d)

\[
v_{N+k} = \tilde{A}(u_{N+k}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j} \phi_{2j}}{u_{N+k} - \tilde{\lambda}_j}, \quad k = 1, ..., N - 1,
\] (4.24e)

\[
v_{2N} = \frac{1}{2} (\langle \Psi_1, \Phi_1 \rangle + \langle \Psi_2, \Phi_2 \rangle).
\] (4.24f)

If \( u_1, ..., u_N \) are single zeros of \( \overline{F}(\lambda) \), then \( v_1, ..., v_{2N} \) and \( u_1, ..., u_{2N} \) are canonically conjugated, i.e., they satisfy (1.1).

**Proof.** It follows from (4.24a) and (4.24b) that

\[
u_N = \ln < \Lambda \Psi_2, \Phi_1 >,
\] (4.25)

\[
u_{2N} = \ln < \Psi_1, \Phi_1 >.
\] (4.26)
By a straightforward calculation, it is found that \( \overline{B}(\lambda), \overline{A}(\lambda), \tilde{B}(\lambda), \tilde{A}(\lambda) \) satisfy the commutator relations (2.33) with \( \lambda, \mu \) replaced by \( \tilde{\lambda}, \tilde{\mu} \), as well as the following commutator relations:

\[
\begin{align*}
\{u_N, \overline{B}(\mu)\} &= \{u_N, \tilde{B}(\mu)\} = \{u_N, \tilde{A}(\mu)\} = 0, \quad \{u_N, \overline{A}(\mu)\} = -\frac{\overline{B}(\mu)}{<\Lambda \Psi_2, \Phi_1>}, \\
\{v_N, \overline{A}(\mu)\} &= \{v_N, \tilde{B}(\mu)\} = \{v_N, \tilde{A}(\mu)\} = 0, \quad \{v_N, \overline{B}(\mu)\} = \overline{B}(\mu), \\
\{u_{2N}, \overline{B}(\mu)\} &= \{u_{2N}, \overline{A}(\mu)\} = \{u_{2N}, \tilde{B}(\mu)\} = 0, \quad \{u_{2N}, \tilde{A}(\mu)\} = -\frac{\tilde{B}(\mu)}{<\Psi_1, \Phi_1>}, \\
\{v_{2N}, \overline{B}(\mu)\} &= \{v_{2N}, \overline{A}(\mu)\} = \{v_{2N}, \tilde{A}(\mu)\} = 0, \quad \{v_{2N}, \tilde{B}(\mu)\} = \tilde{B}(\mu), \\
\{v_N, u_N\} &= 1, \quad \{v_{2N}, u_{2N}\} = 1, \\
\{u_{2N}, u_N\} &= \{u_{2N}, v_N\} = \{v_{2N}, u_N\} = \{v_{2N}, v_N\} = 0.
\end{align*}
\] (4.27a–f)

Then in the exactly same way as for the proposition 1 and 3, we can complete the proof.

It follows from (4.24a) and (4.24b) that

\[
\lambda_j \psi_{2j} \phi_{1j} = e^{u_N} \frac{R(\lambda_j^2)}{K'(\lambda_j^2)}, \quad \phi_{1j}^2 = e^{u_{2N}} \frac{\overline{R}(\lambda_j^2)}{K'(\lambda_j^2)}, \quad j = 1, \ldots, N,
\] (4.28)

or

\[
\phi_{1j} = \sqrt{\frac{e^{u_{2N}} \overline{R}(\lambda_j^2)}{K'(\lambda_j^2)}}, \quad \psi_{2j} = \frac{e^{u_N} R(\lambda_j^2)}{\lambda_j \sqrt{e^{u_{2N}} \overline{R}(\lambda_j^2) K'(\lambda_j^2)}}, \quad j = 1, \ldots, N.
\] (4.29)

The equations (4.9) and (4.25) result

\[
q = -\frac{1}{2} e^{u_N}.
\] (4.30)

We now present the separated equations. By substituting \( u_k \) into (4.13), \( u_{N+k} \) into (4.21) and using (4.19c), (4.24c) and (4.24e), one gets the separated equations

\[
v_k = A_1(u_k) - \tilde{A}(u_k) = \frac{2}{u_k} \sqrt{\tilde{P}(u_k)} - \sqrt{P(0)} - 2 \sum_{j=1}^{N} \frac{P_{N+j}}{u_k - \lambda_j^2}, \quad k = 1, \ldots, N - 1,
\] (4.31a)

\[
v_{N+k} = \tilde{A}(u_{N+k}) = 2 \sum_{j=1}^{N} \frac{P_{N+j}}{u_{N+k} - \lambda_j^2}, \quad k = 1, \ldots, N - 1,
\] (4.31b)
where $P(0)$ are given by (4.16) and

$$
\tilde{P}(\tilde{\lambda}) = 1 + \sum_{j=1}^{N} \left[ \frac{P_j}{\lambda - \lambda_j^2} + \frac{\lambda_j^4 P_{N+j}^2}{(\lambda - \lambda_j^2)^2} \right].
$$

It follows from (4.15b), (4.16), (4.24d) and (4.24f) that

$$
v_N = 2 - 2\sqrt{P(0)} - 2 \sum_{i=1}^{N} P_{N+i}, \quad v_{2N} = 2 \sum_{i=1}^{N} P_{N+i}. \quad (4.31c)
$$

Replacing $v_k$ by the partial derivative $\frac{\partial S}{\partial u_k}$ of the generating function $S$ of the canonical transformation and interpreting the $P_i$ as integration constants, the equations (4.31) may be integrated to give the generating function of the canonical transformation

$$
S(u_1, \ldots, u_{2N}) = \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{k+1}} \frac{2}{\lambda} \sqrt{\tilde{P}(\tilde{\lambda})} - \frac{2}{\lambda} \sqrt{P(0)} - \tilde{A}(\tilde{\lambda}) d\tilde{\lambda} + \int_{u_N}^{\infty} \tilde{A}(\tilde{\lambda}) d\tilde{\lambda} \right]
$$

$$
+ (2 - 2\sqrt{P(0)} - 2 \sum_{i=1}^{N} P_{N+i}) u_N + 2 \sum_{i=1}^{N} P_{N+i} u_{2N}
$$

$$
= \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{k+1}} \frac{2}{\lambda} \sqrt{\tilde{P}(\tilde{\lambda})} d\tilde{\lambda} - 2\sqrt{P(0)} ln|u_k| - 2 \sum_{i=1}^{N} P_{N+i} ln \left| \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right| \right]
$$

$$
+ (2 - 2\sqrt{P(0)} - 2 \sum_{i=1}^{N} P_{N+i}) u_N + 2 \sum_{i=1}^{N} P_{N+i} u_{2N}. \quad (4.32)
$$

The linearizing coordinates are then

$$
Q_i = \frac{\partial S}{\partial u_i} = \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{k+1}} \frac{1}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)} \sqrt{\tilde{P}(\tilde{\lambda})} d\tilde{\lambda} + \frac{1}{\tilde{\lambda}_i^2 \sqrt{P(0)}} ln|u_k| \right] + \frac{1}{\tilde{\lambda}_i^2 \sqrt{P(0)}} u_N,
$$

$$
i = 1, \ldots, N, \quad (4.33a)
$$

$$
Q_{N+i} = \frac{\partial S}{\partial u_{N+i}} = \sum_{k=1}^{N-1} \left[ \int_{u_k}^{u_{k+1}} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2} \sqrt{\tilde{P}(\tilde{\lambda})} d\tilde{\lambda} - \frac{2P_{N+i}}{\sqrt{P(0)}} ln|u_k| - 2ln \left| \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right| \right]
$$

$$
- 2(\frac{P_{N+i}}{\sqrt{P(0)}} + 1) u_N + 2u_{2N}, \quad i = 1, \ldots, N. \quad (4.33b)
$$
Finally, since the CDNS equations (4.6) are factorized by the FDIHS (4.10) and (4.11), combining the equation (4.34) with the equation (4.35) gives rise to the Jacobi inversion problem for the CDNS equations (4.6)

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{1}{\lambda(\tilde{\lambda} - \lambda_i^2)\sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} + \frac{1}{\lambda_i^2 \sqrt{P(0)}} ln|u_k| + \frac{1}{\lambda_i^2 \sqrt{P(0)}} u_N = \gamma_i - 2x, \\
i = 1, \ldots, N,
\]

\[\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2 \sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} - \frac{2P_{N+i}}{\sqrt{P(0)}} ln|u_k| - 2ln \left[ \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right] \\
- 2\left( \frac{P_{N+i}}{\sqrt{P(0)}} + 1 \right)u_N + 2u_{2N} = \gamma_{N+i}, \quad i = 1, \ldots, N.
\]

By using (4.15a), the linear flow induced by (4.11) and the equation (4.34) yield the Jacobi inversion problem for the FDIHS (4.11)

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{1}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)\sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} + \frac{1}{\lambda_i^2 \sqrt{P(0)}} ln|u_k| + \frac{1}{\lambda_i^2 \sqrt{P(0)}} u_N = \bar{\gamma}_i + (2\lambda_i^2 - \sum_{k=1}^{N} P_k)t_2, \\
i = 1, \ldots, N,
\]

\[\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2 \sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} - \frac{2P_{N+i}}{\sqrt{P(0)}} ln|u_k| - 2ln \left[ \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right] \\
- 2\left( \frac{P_{N+i}}{\sqrt{P(0)}} + 1 \right)u_N + 2u_{2N} = \bar{\gamma}_{N+i} + 4\lambda_i^4 P_{N+i}t_2, \quad i = 1, \ldots, N.
\]

Finally, combining the equation (4.34) with the equation (4.35) gives rise to the Jacobi inversion problem for the CDNS equations (4.6)

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{1}{\lambda(\tilde{\lambda} - \lambda_i^2)\sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} + \frac{1}{\lambda_i^2 \sqrt{P(0)}} ln|u_k| + \frac{1}{\lambda_i^2 \sqrt{P(0)}} u_N \\
= \gamma_i - 2x + (2\lambda_i^2 - \sum_{k=1}^{N} P_k)t_2, \quad i = 1, \ldots, N,
\]

\[\sum_{k=1}^{N-1} \int_{u_k}^{u_{k+1}} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2 \sqrt{P(\tilde{\lambda})}} d\tilde{\lambda} - \frac{2P_{N+i}}{\sqrt{P(0)}} ln|u_k| - 2ln \left[ \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right] \\
- 2\left( \frac{P_{N+i}}{\sqrt{P(0)}} + 1 \right)u_N + 2u_{2N} = \gamma_{N+i} + 4\lambda_i^4 P_{N+i}t_2, \quad i = 1, \ldots, N.
\]
If $\phi_{1j}, \psi_{2j}, q$ defined by (4.29) and (4.30) can be solved from (4.36) by using the Jacobi inversion technique, then $\phi_{2j}, \psi_{1j}$ can be obtained from the first equation and the last equation in (4.10), respectively. Finally $q$ and $r = -<\Lambda \Psi_1, \Phi_2>$ provides the solution to the CDNS equations (4.6).

In general, the above procedure can be applied to the whole Kaup-Newell hierarchy (4.4). Set

$$A^2(\lambda) + B(\lambda)C(\lambda) = \sum_{k=0}^{\infty} \tilde{F}_k \lambda^{-2k},$$

(4.37a)

where $\tilde{F}_k, k = 1, 2, \ldots$, are also integrals of motion for both the $x$-FDHSs (4.10) and the $t_n$-binary constrained flows (2.16). Comparing (4.37a) with (4.13), one gets

$$\tilde{F}_0 = 1, \quad \tilde{F}_k = \sum_{j=1}^{N} [\lambda_j^{2k-2}P_j + (k - 1)\lambda_j^{2k}P_{N+j}], \quad k = 1, 2, \ldots.$$  

(4.37b)

By employing the method in [34,35], the $t_n$-FDIHS obtained from the $t_n$-constrained flow is of the form

$$\Phi_{1,t_n} = \frac{\partial F_n}{\partial \Psi_1}, \quad \Phi_{2,t_n} = \frac{\partial F_n}{\partial \Psi_2}, \quad \Psi_{1,t_n} = -\frac{\partial F_n}{\partial \Phi_1}, \quad \Psi_{2,t_n} = -\frac{\partial F_n}{\partial \Phi_2},$$

(4.38a)

with the Hamiltonian

$$F_n = 2 \sum_{m=0}^{n-1} \left(-\frac{1}{2}\right)^m \frac{\alpha_m}{m+1} \sum_{l_1 + \ldots + l_{m+1} = n} \tilde{F}_{l_1} \ldots \tilde{F}_{l_{m+1}},$$

(4.38b)

where $l_1 \geq 1, \ldots, l_{m+1} \geq 1$, and $\alpha_m$ are given by (2.25c). Since the $n$-th Kaup-Newell equations (4.4) is factorized by the $x$-FDIHS (4.10) and the $t_n$-FDIHS (4.38). We have the following proposition.

**Proposition 6.** The Jacobi inversion problem for the $n$-th Kaup-Newell equations (4.4) is given by

$$\sum_{k=1}^{N-1} \left[ \int^{u_k} \frac{1}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)} \sqrt{P(\tilde{\lambda})} d\tilde{\lambda} + \frac{1}{\lambda_i^2 \sqrt{P(0)}} ln|u_k| \right] + \frac{1}{\lambda_i^2 \sqrt{P(0)}} u_N = \gamma_i - 2x$$

$$+ 2t_n \sum_{m=0}^{n-1} \left(-\frac{1}{2}\right)^m \alpha_m \sum_{l_1 + \ldots + l_{m+1} = n} \lambda_i^{2l_{m+1}-2} \tilde{F}_{l_1} \ldots \tilde{F}_{l_m}, \quad i = 1, \ldots, N.$$  

(4.39a)
The Jacobi inversion problem for the equations (4.40) is given by

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_k+1} d\tilde{\lambda} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2 \sqrt{P(\tilde{\lambda})}} \left[ \frac{2P_{N+i}}{\sqrt{P(0)}} \ln|u_k| \right] - 2\ln \left| \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right| \]

\[-2\left( \frac{P_{N+i}}{\sqrt{P(0)}} + 1 \right)u_N + 2u_{2N} = \gamma_{N+i}
\]

where \( l_1 \geq 1, \ldots, l_{m+1} \geq 1, \) and \( \tilde{F}_{l_1}, \ldots, \tilde{F}_{l_{m+1}}, \) are given by (4.37b).

For example, the third equations in the Kaup-Newell hierarchy with \( n = 3 \) are of the form

\[
q_{t_3} = -\frac{1}{4}q_{xxx} - \frac{3}{8}(q^3 r^2 + 2qrq_x)_x, \quad r_{t_3} = -\frac{1}{4}r_{xxx} - \frac{3}{8}(r^3 q^2 - 2qr)^2 r_x. \quad (4.40)
\]

The Kaup-Newell equations (4.40) can be factorized by the \( x \)-FDIHS (4.10) and \( t_3 \)-FDIHS with the Hamiltonian \( F_3 \) defined by

\[
F_3 = \sum_{j=1}^{N} \left( 2\lambda_j^4 P_j + 4\lambda_j^6 P_{N+j}^2 \right) - \left( \sum_{j=1}^{N} (\lambda_j^2 P_j + \lambda_j^4 P_{N+j}^2) \right) \sum_{j=1}^{N} P_j + \frac{1}{4}\left( \sum_{j=1}^{N} P_j \right)^3. \quad (4.41)
\]

The Jacobi inversion problem for the equations (4.40) is given by

\[
\sum_{k=1}^{N-1} \int_{u_k}^{u_k+1} d\tilde{\lambda} \frac{2\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda} - \lambda_i^2)^2 \sqrt{P(\tilde{\lambda})}} \left[ \frac{2P_{N+i}}{\sqrt{P(0)}} \ln|u_k| \right] - 2\ln \left| \frac{u_k - \lambda_i^2}{u_{N+k} - \lambda_i^2} \right| \]

\[-2\left( \frac{P_{N+i}}{\sqrt{P(0)}} + 1 \right)u_N + 2u_{2N} = \gamma_{N+i} + \left[ 8\lambda_i^6 P_{N+i} - 2\lambda_i^4 P_{N+i} \right] \sum_{j=1}^{N} P_j t_3, \quad i = 1, \ldots, N.
\]
5. Concluding remarks.

The method in [2,3,4,8] is not valid for the separation of variables for binary constrained flows of soliton hierarchies. This paper proposed a new method to solve this problem. For a certain kind of integrable models, a general approach to the separation of variables was proposed in [2,3,4] by taking the poles of the properly normalized Baker-Akhiezer function and the corresponding eigenvalues of the Lax operator as separated variables. As pointed out in [4], there is no guarantee that the separated variables so constructed satisfy the canonical conditions (1.1). The method proposed in this paper is to start directly from the canonical conditions (1.1) and the requirement for the separated equations (1.2). We introduced $2N$ pairs of separated variables by means of four functions $B(\lambda), A(\lambda), \tilde{B}(\lambda)$ and $\tilde{A}(\lambda)$, which are constructed in such way that they satisfy certain commutator relations required by the canonical conditions (1.1) and $A(\lambda)$ and $\tilde{A}(\lambda)$ are linked to the generating functions of the integrals of motion for the models. This method ensures that the separated variables are canonically conjugated. We produced two sets of separated equations directly from two generating functions of integrals of motion. It seems that the separated equations are intimately connected with the generating functions of integrals of motion.

The finite gap solutions or finite-dimensional quasiperiodic solutions for the KdV equation was studied in [36] by means of the stationary equation of the KdV hierarchy called the Lax-Novikov equation. By the standard Jacobi inversion technique [19], the finite-dimensional quasiperiodic solution can be given in an explicit form in terms of the Riemann theta functions associated with the invariant spectral curve. The Jacobi inversion problem (2.45) for the KdV equation is somewhat different from that in [36] due to some additional terms. The Jacobi inversion problems for some binary constrained flows require the development of the standard Jacobi inversion technique in order to solve them explicitly.

Acknowledgments.

This work was supported by the City University of Hong Kong and the Research Grants Council of Hong Kong and the Chinese Basic Research Project “Nonlinear Science”. One of the authors (Y.B.Zeng) wishes to express his gratitude to Department of Mathematics of the City University of Hong Kong for warm hospitality.

References.
1. Arnol’d, V.I.: Mathematical methods of classical mechanics, 2nd edition, New-York: Springer, 1994.
2. Sklyanin, E.K.: Separation of variables in the Gaudin model, J. Soviet. Math. 47, 2473-2488 (1989).
3. Kuznetsov, V.B.: Quadrics on real Riemannian spaces of constant curvature: separation of variables and connection with Gaudin magnet, J. Math. Phys. 33, 3240-3254
4. Sklyanin, E.K.: Separation of variables, Prog. Theor. Phys. Suppl. 118, 35-60 (1995).
5. Babelon, O. and Talon, M.: Separation of variables for the classical and quantum Neumann model, Nucl. Phys. B 379, 321-339 (1992).
6. Eilbeck, J.C., Enol’skii, V.Z., Kuznetsov, V.B. and Tsiganov, A.V.: Linear $r$-matrix algebra for classical separable systems, J. Phys. A: Math. Gen. 27, 567-578 (1994).
7. Kalnins, E.G., Kuznetsov, V.B. and Willard Miller, Jr: Quadrics on complex Riemannian spaces of constant curvature, separation on variables, and the Gaudin magnet, J. Math. Phys. 35, 1710-1731 (1994).
8. Harnad, J. and Winternitz, P.: Classical and quantum integrable systems in $\tilde{gl}(2)^+$ and separation of variables, Commun. Math. Phys. 172, 263-285 (1995).
9. Kulish, P.P., Rauch-Wojciechowski, S. and Tsiganov, A.V.: Stationary problems for equation of the KdV type and dynamical $r$-matrices, J. Math. Phys. 37, 3463-3482 (1996).
10. Zeng, Yunbo: The separability and dynamical $r$-matrix for the constrained flows of the Jaulent-Miodek hierarchy, Phys. Lett. A 216, 26-32 (1996).
11. Zeng, Yunbo: A family of separable Hamiltonian systems and their classical dynamical $r$-matrix, Inverse Problems 12, 1-13 (1996).
12. Zeng, Yunbo: Separation of variables for the constrained flows, J. Math. Phys. 38, 321-329 (1997).
13. Adams, M.R., Harnad, J. and Hurtubise, J.: Darboux coordinates and Liouville-Arnold integration in loop algebras, Commun. Math. Phys. 155, 385-413 (1993).
14. Adams, M.R., Harnad, J. and Hurtubise, J.: Liouville generating function for isospectral Hamiltonian flow in Loop algebras, in: Integrable and superintegrable systems, ed. B.A. Kuperschmidt, Singapore: World Scientific, 1990.
15. Harnad, J. and Wisse, M.A.: Isospectral flow in Loop algebras and quasiperiodic solution to the sine Gordon equation, J. Math. Phys. 34, 3518-3526 (1993).
16. Wisse, M.A.: Darboux coordinates and isospectral Hamiltonian flow for the massivevethirring model, Lett. Math. Phys. 28, 287-294 (1993).
17. Zeng, Yunbo: Using factorization to solve soliton equation, J. Phys. A: Math. Gen. 30, 3719-3724 (1997).
18. Zeng, Yunbo: The Jacobi inversion problem for soliton equations, J. Phys. Soc. Jpn. 66, 2277-2282 (1997).
19. Dubrovin, B.A.: Theta functions and nonlinear equations, Russian Math. Survey 36, 11-92 (1981).
20. Krichever, I.M. and Novikov, S.P.: Holomorphic bundles over algebraic curves and nonlinear equations, Russian Math. Surveys 32, 53-79 (1980).
21. Adler, M. and van Moerbeke, P.: Completely integrable systems, Euclidean Lie algebras and curve, Adv. Math. 38, 267-317 (1980).
22. Adler, M. and van Moerbeke, P.: Linearization of Hamiltonian systems, Jacobi variables and representation theory, ibid. 38, 318-379 (1980).
23. Ragnisco, O. and Rauch-Wojciechowski, S.: Restricted flows of the AKNS hierarchy, Inverse Problems 8, 245-262 (1992).
24. Zeng, Yunbo: The higher-order constraint and integrable systems related to Boussinesq equation, Chinese Science Bulletin 37, 1937-1942 (1992).
25. Ma, W.X. and Strampp, W.: An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, Phys. Lett. A 185, 277-286 (1994).
26. Ma, W.X.: New finite-dimensional integrable systems by symmetry constraint of the KdV equations, J. Phys. Soc. Jpn. 64, 1085-1091 (1995).
27. Ma, W.X., Fuchssteiner, B. and Oevel, W.: A 3 × 3 matrix spectral problem for AKNS hierarchy and its binary nonlinearization, Physica A 233, 331-354 (1996).
28. Ma, W.X., Ding, Q., Zhang, W.G. and Lu, B.Q.: Binary nonlinearization of Lax pairs of Kaup-Newell soliton hierarchy, IL Nuovo Cimento B 111, 1135-1149 (1996).
29. Li, Yishen and Ma, W.X.: Binary nonlinearization of AKNS spectral problem under higher-order symmetry constraints, to appear in Chaos, Solitons and Fractals.
30. Ma, W.X. and Fuchssteiner, B.: Binary nonlinearization of Lax pairs, in: Nonlinear Physics, ed. E. Alfinito, M. Boiti, L. Martina and F. Pempielli, Singapore: World Scientific, 1996, pp217-224.
31. Zeng, Yunbo and Ma, W.X.: The construction of canonical separated variables for binary constrained AKNS flow, preprint.
32. Newell, A.C.: Solitons in mathematics and physics, Philadelphia: SIAM, 1985.
33. Zeng, Yunbo and Li, Yishen: The deduction of the Lax representation for constrained flows from the adjoint representation, J. Phys. A: Math. Gen. 26, L273-L278 (1993).
34. Zeng, Yunbo: New factorization of the Kaup-Newell hierarchy, Physica D 73, 171-188 (1994).
35. Zeng, Yunbo: An approach to the deduction of the finite-dimensional integrability from the infinite-dimensional integrability, Phys. Lett. A 160, 541-547 (1991).
36. Novikov, S.P.: A method for solving the periodic problem for the KdV equation and its generalizations, in: Solitons, Topics in current physics, ed. R. Bullough and P. Caudrey, New York: Springer-Verlag, 1980, pp. 325-338.
37. Ablowitz, M. and Segur, H.: Solitons and the inverse scattering transform, Philadelphia: SIAM, 1981.
38. Kaup, D.J. and Newell. A.C.: An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19, 798-801 (1978).