Primordial perturbations and inflation in holographic cosmology

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We consider an inflationary scenario in the holographic braneworld with a cosmological fluid occupying the 3+1 dimensional brane located at the holographic boundary of an asymptotic $\text{AdS}_5$ bulk. The contribution of the boundary conformal field can be represented as a modification of Einstein’s equations on the boundary. Using these effective Einstein equations we calculate the cosmological perturbations and derive the corresponding power spectra assuming a general $k$-essence type of inflaton. We find that the braneworld scenario affects the scalar power spectrum only in the speed of sound dependence on the slow-roll parameters whereas there is no change in the tensor power spectrum. This implies that the changes in the spectral indices appear at the second order in the slow-roll parameter expansion.

I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] goes beyond pure string theory and links many important theoretical and phenomenological issues. In particular, a simple physically relevant model related to AdS/CFT is the Randall-Sundrum (RS) braneworld model [4, 5] and its cosmological applications. In the braneworld scheme, the RS brane provides a cutoff regularization for the infrared divergences of the on shell bulk action [6–13]. A related scheme is based on the holographic braneworld scenario [14–17] in which the cosmology is derived from the effective four-dimensional Einstein equations on the holographic boundary of $\text{AdS}_5$.

In a braneworld scenario, matter is confined on a brane moving in the higher dimensional bulk, with only gravity allowed to propagate in the bulk [18–20]. If the brane is located at the boundary of a 5-dimensional asymptotically AdS space-time, we refer to this type of braneworld as the holographic braneworld [14, 15]. In the study of the holographic braneworld, a crucial property of an asymptotically AdS bulk is that AdS space is dual to a conformal field theory at its boundary. A connection between AdS/CFT correspondence and cosmology has been studied in a different approach based on a holographic renormalization group flows in quantum field theory [21, 22].

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The holographic cosmology has a property that the universe evolution starts from a point at which the energy density and cosmological scale are both finite \[23, 24\], rather than from the usual Big Bang singularity of the standard cosmology. Then the inflation phase proceeds naturally immediately after \( t = 0 \). This type of inflation has been recently studied in a holographic braneworld scenario with an effective tachyon field on the brane \[17\]. In that paper the first order cosmological perturbations have been calculated in an approximate scheme in which the modification of Einstein equations were treated as an approximate modification of the effective energy momentum tensor at the boundary. Here we study the complete first-order perturbation theory on the holographic brane with field equations on the brane modified due to the dual conformal theory on the AdS boundary.

Our main motivation is to show how the modification of Einstein’s equation on the holographic braneworld affects the primordial power spectra. As we shall shortly demonstrate, the modification of Einstein equations on the holographic brane is substantial and yields a modified background cosmology with a quartic term \( \propto H^4 \) in Friedmann equations, as shown previously in a different way \[14, 15\]. In view of that, one would expect the first order perturbation to be substantially modified yielding a modified power spectra. We assume that inflation is driven by a general \( k \)-essence \[25, 26\]. The term \( k \)-essence denotes a fluid described by a field theory with Lagrangian that is a general function of the field and its kinetic term. This form of field theory, which includes a canonical scalar field as a special case, was first explored as a generalized scalar field model of inflation \[27, 28\], with speed of sound being a nontrivial function of the field. Study of a general \( k \)-essence is important as it covers many models of inflation (including canonical scalar field inflation) with predictions in agreement with observations (for recent works see \[29–33\]).

The remainder of the paper is organized as follows. In Sec. II we introduce the effective Einstein equations on the holographic boundary from which we derive the corresponding Friedmann equation. In Sec. III we study the primordial scalar and tensor perturbations and calculate the power spectra and spectral indices. In Appendix A we demonstrate equivalence of the energy-momentum conservation and the equation of motion for a general \( k \)-essence field theory. In Appendix B we directly verify the \( k \)-essence perturbations without using the fluid notation. In Appendix C we briefly discuss the background-field solutions for a few examples of \( k \)-essence.

II. HOLOGRAPHIC EINSTEIN EQUATIONS AND COSMOLOGY

A general asymptotically AdS\(_5\) metric in Fefferman-Graham coordinates is of the form \[34\]

\[
 ds^2 = G_{ab} dx^a dx^b = \ell^2 \left( g_{\mu\nu} dx^\mu dx^\nu \right) ,
\]

(1)
where the constant $\ell$ of dimension of length is the AdS curvature radius. We use the Latin alphabet for 4+1 and the Greek alphabet for 3+1 spacetime indices and the metric sign convention is (+ − − − −). Near $z = 0$ the metric can be expanded as

$$g_{\mu\nu}(z, x) = g_{\mu\nu}^{(0)}(x) + z^2 g_{\mu\nu}^{(2)}(x) + z^4 g_{\mu\nu}^{(4)}(x) + \cdots.$$  \hfill (2)

Then, the four-dimensional Einstein equations on the holographic boundary are \cite{9, 15, 24}

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}^{(0)} = 8\pi G_N (\langle T_{\mu\nu}^{\text{CFT}} \rangle + T_{\mu\nu}),$$  \hfill (3)

where $R$ and $R_{\mu\nu}$ are respectively the Ricci scalar and the Ricci tensor associated with the metric $g_{\mu\nu}^{(0)}$. The Newton’s constant is related to the AdS curvature radius $\ell$ and five-dimensional gravitational constant $G_5$ as

$$G_N = \frac{2G_5}{\ell}.$$  \hfill (4)

In the following analysis we shall consider $G_N$ and $\ell$ as fixed fundamental physical parameters and $G_5$ as a derived quantity.

In general, on the left-hand side of (3) there is a cosmological term $\Lambda g_{\mu\nu}^{(0)}$ related to the brane tension and the AdS bulk cosmological constant \cite{14, 15}. This term can be made small or entirely removed by imposing the RS fine tuning condition \cite{15} and if it were present would be important for the late Universe cosmology and irrelevant for our early universe considerations.

The AdS curvature radius $\ell$ is constrained by tests of Newton’s law at small distances. For the Randall-Sundrum braneworld it has been shown \cite{35} that for $r \gg \ell$ the extra-dimension effects strengthen Newton’s gravitational field by a factor $1 + \mathcal{O}(\ell^2/r^2)$. Table-top tests of Newton’s laws currently find no deviations of Newton’s potential at distances greater than about 0.1 mm \cite{36} and about 0.05 mm \cite{37} yielding an upper bound on the AdS$_5$ curvature $\ell \lesssim 0.1$ mm or $\ell^{-1} \gtrsim 10^{-12}$ GeV.

The energy-momentum tensor $T_{\mu\nu}$ describes matter on the brane. The vacuum expectation value $\langle T_{\mu\nu}^{\text{CFT}} \rangle$ can be obtained in terms of the quantities $g_{\mu\nu}^{(2n)}$ related to the bulk metric \cite{9}

$$\langle T_{\mu\nu}^{\text{CFT}} \rangle = - \frac{\ell^2}{2\pi G_N} \left\{ g_{\mu\nu}^{(4)} - \frac{1}{8} \left[ (\text{Tr} g^{(2)})^2 - \text{Tr}(g^{(2)})^2 \right] g_{\mu\nu}^{(0)} - \frac{1}{2} (g^{(2)})^2 \nu\nu + \frac{1}{4} \text{Tr} g^{(2)} g^{(2)} \right\}.  \hfill (5)$$

Here

$$g_{\mu\nu}^{(2)} = \frac{1}{2} \left( R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}^{(0)} \right)$$  \hfill (6)

and

$$g_{\mu\nu}^{(4)} = g_{\mu\nu}^{c(4)} + \tilde{g}_{\mu\nu}^{(4)},$$  \hfill (7)

where

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where
\[ g^{c(4)}_{\mu\nu} = \frac{1}{4} (g^{(2)}_{\alpha\beta})^2 \equiv \frac{1}{4} g^{(0)}_{\mu\rho} g^{(2)}_{\rho\sigma} g^{(2)}_{\sigma\nu}. \] (7a)
The tensor \( \tilde{g}^{(4)} \) depends on the boundary metric \( g^{(0)}_{\mu\nu} \) and vanishes if the metric is conformally flat. Then, we can write
\[ \langle T_{\mu\nu}^{\text{CFT}} \rangle = -\frac{\ell^2}{32\pi G_N} \left[ \frac{2}{3} RR_{\mu\nu} - R_{\mu\rho} R^\rho_{\nu} + \frac{1}{4} \left( 2 R_{\alpha\beta} R^{\alpha\beta} - R^2 \right) g^{(0)}_{\mu\nu} \right] + t_{\mu\nu} \] (8)
where
\[ t_{\mu\nu} = -\frac{\ell^2}{2\pi G_N} \tilde{g}^{(4)}_{\mu\nu}. \] (8a)

From (3) and (8) we obtain the holographic Einstein equations
\[ R_{\mu\nu} - \frac{1}{2} R_{\mu\nu}^{(0)} + \frac{\ell^2}{4} \left[ \frac{2}{3} RR_{\mu\nu} - R_{\mu\rho} R^\rho_{\nu} + \frac{1}{4} \left( 2 R_{\alpha\beta} R^{\alpha\beta} - R^2 \right) g^{(0)}_{\mu\nu} \right] = 8\pi G_N (T_{\mu\nu} + t_{\mu\nu}), \] (9)
where the term \( t_{\mu\nu} \) on the right-hand side gives no contribution if the boundary spacetime represented by the metric \( g^{(0)}_{\mu\nu} \) is conformally flat.

Now we consider the consequences of this scenario for cosmology. To this end, we specify the boundary metric to have a Friedmann-Robertson-Walker (FRW) form
\[ ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) d\Omega_k^2, \] (10)
where
\[ d\Omega_k^2 = d\chi^2 + \frac{\sin^2(\sqrt{k} \chi)}{k} (d\theta^2 + \sin^2 \theta d\varphi^2) \] (11)
is the spatial line element for a closed \((k = 1)\), open hyperbolic \((k = -1)\), or open flat \((k = 0)\) space. Next, we assume
\[ T^\mu_\nu = \text{diag}(\rho, -p, -p, -p) \] (12)
and using the effective Einstein equations (9) we obtain the holographic Friedmann equations [14, 15]
\[ H^2 + \frac{k}{a^2} - \frac{\ell^2}{4} \left( H^2 + \frac{k}{a^2} \right)^2 = \frac{8\pi G_N}{3} \rho, \] (13)
\[ \left( \dot{H} - \frac{k}{a^2} \right) \left[ 1 - \frac{\ell^2}{2} \left( H^2 + \frac{k}{a^2} \right) \right] = -4\pi G_N (p + \rho). \] (14)
Equations (13) and (14) imply
\[ \dot{\rho} + 3H(p + \rho) = 0, \] (15)
which also follows from energy-momentum conservation \( T^\mu_{\nu,\nu} = 0 \).
III. PERTURBATIONS IN THE HOLOGRAPHIC COSMOLOGY

Here we derive the spectra of the cosmological perturbations for the holographic cosmology with matter in the form of general $k$-essence. We shall closely follow J. Garriga and V. F. Mukhanov [28] and adjust their formalism to account for the Holographic cosmological perturbations.

A. Background equations

In the following we consider a spatially flat background with Friedmann equations (13) and (14). The beginning of inflation is characterized by the so called slow-roll regime with slow-roll parameters satisfying $\varepsilon_i \ll 1$. We use the following recursive definition of the slow roll parameters [38, 39]

$$\varepsilon_{i+1} = \frac{\dot{\varepsilon}_i}{H\varepsilon_i},$$

starting with

$$\varepsilon_1 = -\frac{\dot{H}}{H^2}.$$  \hfill (17)

Next we assume that apart from the conformal field there is matter on the holographic brane described by a general $k$-essence action

$$S_{\text{matt}} = \int d^4x \sqrt{-g} \mathcal{L}(X, \theta)$$  \hfill (18)

where $\mathcal{L} = \mathcal{L}(X, \theta)$ is an arbitrary function of the field $\theta$ and the kinetic term $X$,

$$X \equiv g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta.$$  \hfill (19)

The energy-momentum tensor associated with $S_{\text{matt}}$ is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}} = 2\mathcal{L}_X \partial_\mu \theta \partial_\nu \theta - g_{\mu\nu} \mathcal{L}$$  \hfill (20)

where the subscript , $X$ denotes a partial derivative with respect to $X$.

Formally, one may proceed by solving the $\theta$ field equation in conjunction with Einstein's equations. However, it proves advantageous to pursue the hydrodynamic picture. Using standard notation based on the similarity with perfect fluids [28] (see also [40–42] for further details on the $k$-essence fluid correspondence), the pressure $p$ and energy energy density $\rho$ are given by

$$p = \mathcal{L}, \quad \rho = 2X \mathcal{L}_X - \mathcal{L}.$$  \hfill (21)
Then, the energy-momentum tensor can be expressed as

\[ T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \tag{22} \]

where

\[ u_\mu \equiv \frac{\partial \mu}{\sqrt{X}}. \tag{23} \]

The definition above implies that \( u^\mu u_\mu = 1 \).

### B. Scalar perturbations and power spectrum

Assuming a spatially flat background with line element (10) with \( k = 0 \), we introduce the perturbed line element in the Newtonian gauge

\[ ds^2 = (1 + 2\Psi)dt^2 - (1 - 2\Phi)a^2(t)(dr^2 + r^2d\Omega^2). \tag{24} \]

This perturbed spacetime is no longer conformally flat so the perturbed Einstein equations will have a nonvanishing contribution of the tensor \( \delta t_{\mu\nu} \). However, for the moment we assume that the contribution of \( \delta t_{\mu\nu} \) is of higher order in the slow roll parameter expansion and we justify this assumption a posteriori. Inserting the above metric components in the field equations (9) and ignoring \( \delta t_{\mu\nu} \) we obtain

\[
\frac{2}{a^2} \left( 1 - \frac{\ell^2}{2} H^2 \right) \left( \nabla^2 \Phi - 3a^2 H (\dot{\Phi} + H \Psi) \right) = 8\pi G_N \delta T^0_0 \tag{25}
\]

\[
2 \left( 1 - \frac{\ell^2}{2} H^2 \right) \partial_i (\dot{\Phi} + H \Psi) = 8\pi G_N \delta T^i_0 \tag{26}
\]

\[
-2 \left[ \ddot{\Phi} + H(\dot{\Psi} + 3\dot{\Phi}) + \Psi(3H^2 + 2H) + \frac{1}{2a^2} \nabla^2 (\Psi - \Phi) \right] \delta^i_j + \ell^2 \left[ H^2 \ddot{\Phi} + (3H^3 + 2H \dot{H}) \dot{\Phi} + H^3 \dot{\Psi} + (3H^4 + 4H^2 \dot{H}) \Psi - \frac{1}{6a^2} (3H^2 + 5\dot{H}) \nabla^2 \Phi + \frac{1}{6a^2} (3H^2 + \dot{H}) \nabla^2 \Psi \right] \delta^i_j
\]

\[
- \frac{1}{a^2} \partial^i \partial_j (\Phi - \Psi) + \frac{\ell^2}{2a^2} (H^2 + \dot{H}) \partial^i \partial_j (\Phi - \Psi) = 8\pi G_N \delta T^i_j \tag{27}
\]

The perturbed energy-momentum tensor components are found from Eq. (22) and coincide with those of a perfect fluid. Besides, following standard Newtonian gauge conventions, the coordinates are chosen such that \( u^i \) is a first order perturbative quantity (\( u^i = \delta u^i \)). Hence, up to the first perturbative order,

\[ \delta T^0_0 = \delta \rho \tag{28} \]
\[ \delta T^0_i = (\rho + p) \delta u_i, \]  

(29)

\[ \delta T^i_j = -\delta^i_j \delta p. \]  

(30)

Then, from the off-diagonal part of (27)

\[ \left(1 - \frac{\ell^2}{2} (H^2 + \dot{H})\right) \partial^i \partial_j (\Phi - \Psi) = 0 \]  

(31)

we read off the slip parameter

\[ \eta \equiv \frac{\Phi}{\Psi} = 1, \]  

(32)

as in general relativity (GR). Hence, in the following we can work in longitudinal gauge with \( \Psi = \Phi \).

The slip parameter is defined in Fourier \( k \)-momentum space but, following the common practice \[43\], we omit the dependence on \( k \) in Eq. (32).

Now, the procedure described in Ref. \[28\] (see also the appendix of \[17\] for more details) can be applied, keeping in mind that the background evolution is governed by Eqs. (13) and (14) with \( k = 0 \). The relevant Einstein equations at linear order are given by

\[ a^{-2} \nabla^2 \Phi - 3H \dot{\Phi} + 3H^2 \Phi = 4\pi G_N \delta T^0_0 (1 - h^2/2)^{-1}, \]  

(33)

\[ (\dot{\Phi} + H \Phi)_i = 4\pi G_N \delta T^0_i (1 - h^2/2)^{-1}, \]  

(34)

where we have abbreviated

\[ h \equiv \ell H. \]  

(35)

Equations (33) and (34) are sufficient for deriving the scalar power spectra. However, to check the consistency of our assumptions, we will also consider the \( ij \) components of the Einstein equations at linear order,

\[ \ell^2 \left( \Phi \ddot{H} + \frac{\dot{H}}{H} \dot{\Phi} - \frac{1}{3a^2} \frac{\dot{H}}{H^2} \Delta \Phi \right) \delta^i_j - \left( \dot{\Phi} + 4H \dot{\Phi} + 2\Phi \ddot{H} + 3H^2 \Phi \right) \left(1 - \frac{h^2}{2}\right) \delta^i_j = 4\pi G_N \delta T^j_j. \]  

(35a)

The perturbations of the stress tensor components \( \delta T^\mu_\nu \) are induced by the perturbations of the scalar field \( \theta(t, x) = \theta(t) + \delta \theta(t, x) \) and by the metric perturbation \[28\] (see also \[44, 45\]). Keeping up with the fluid analogy we define the speed of sound

\[ c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_\theta = \frac{p}{\rho} = \frac{p}{\rho + 2X\rho,XX} = \frac{p + \rho}{2X\rho,XX}. \]  

(36)
With this, the component $\delta T^0_0$ at first perturbative order can be written as
\[
\delta T^0_0 = \rho_cX \delta X + \rho_c \delta \theta = \frac{\rho + p}{2X c_s^2} \left( \delta g^{\mu \nu} \partial_\mu \theta \partial_\nu \theta + 2g^{\mu \nu} \partial_\mu \theta \partial_\nu \delta \theta - \frac{\dot{X}}{\theta} \delta \theta \right) + \frac{\dot{\theta}}{\theta} \delta \theta .
\] (37)

Since $k$-essence is described by an action, diffeomorphism invariance implies that the energy-momentum tensor associated with the action must be conserved. At background level the conservation equation in fluid notation takes the usual form given by Eq. (15). Using this and the fact that $X = \dot{\theta}^2$ at background level, from (37) we obtain
\[
\delta T^0_0 = \frac{p + \rho}{c_s^2} \left[ \left( \frac{\delta \theta}{\theta} \right)^2 - \Phi \right] - 3H(p + \rho) \frac{\delta \theta}{\theta} .
\] (38)

Similarly, from (29) we obtain
\[
\delta T^0_i = (p + \rho) \left( \frac{\delta \theta}{\theta} \right) ,
\] (39)

Equations (36-39) are no novelty [28], and are reviewed here only for convenience. For the $ij$ components, Eq. (30) combined with (38) yields
\[
\delta T^i_j = -c_s^2 \delta T^0_0 \delta^i_j + \left( c_s^2 \rho_c - p_c \right) \delta \theta \delta^i_j .
\] (39a)

In particular, for a pure $k$-essence the second term on the right hand side of (39a) vanishes in which case we have
\[
\delta T^i_j = -c_s^2 \delta T^0_0 \delta^i_j .
\] (39b)

In the above derivation it was not necessary to explicitly employ the field equation obtained from the variation with respect to $\theta$ because, as we have demonstrated in Appendix A, the equation of motion of the $\theta$ field is equivalent to the energy momentum conservation law. Therefore, the fluid-based notation, as used above, is fully equivalent to using the field equations derived from the fundamental fields $\theta$ and $g_{\mu \nu}$. Alternatively, we could have derived Eq. (38) and (39) starting directly from the definition of the energy-momentum tensor (20) and employing the equation of motion of the $\theta$ field, without resorting to the hydrodynamic description. We show this in Appendix B.

In the slow roll regime the sound speed deviates slightly from unity and may be expressed in terms of slow-roll parameters. First, by making use of the definition (17) and modified Friedmann equations (13) and (14) with (21), we can express the variable $X$ in the slow roll regime as
\[
X = -\frac{2p(2 - h^2)}{3p_c(4 - h^2)} \varepsilon_1 + O(\varepsilon^2_1) .
\] (40)
Then from (36) we find

\[ c_s^2 = 1 + \frac{4(2 - h^2)}{3(4 - h^2)} \frac{pp,XX}{p_r^2} \varepsilon_1 + \mathcal{O}(\varepsilon_1^2). \]  

(41)

For example, in the tachyon model with Lagrangian \( L = -V \sqrt{1 - X} \) one finds [17]

\[ c_s^2 = 1 - \frac{4(2 - h^2)}{3(4 - h^2)} \varepsilon_1 + \mathcal{O}(\varepsilon_1^2). \]  

(42)

Using (38) and (39) equations (33) and (34) take the form

\[ \left( \frac{\delta \theta}{\theta} \right)^\cdot = \Phi + \frac{c_s^2}{4 \pi G_N a^2 (\bar{p} + \bar{\rho})} \nabla^2 \Phi, \]  

(43)

\[ (a \Phi)^\cdot = 4 \pi G_N a (\bar{p} + \bar{\rho}) \frac{\delta \theta}{\theta}, \]  

(44)

where we have defined

\[ \bar{p} + \bar{\rho} = (p + \rho)(1 - h^2 / 2)^{-1}. \]  

(45)

To check the consistency of our assumptions, we have examined the \( ii \) components of Einstein’s equations. Using Eqs. (35a) and (39b) for \( i = j \) we have derived an equation similar to (43). Combining with (33), (34), and (38), we have obtained

\[ \left( \frac{\delta \theta}{\theta} \right)^\cdot = \Phi + \frac{c_s^2}{4 \pi G_N a^2 (\bar{p} + \bar{\rho})} \left( 1 + \frac{h^2}{3(1 - h^2 / 2)c_s^2 \bar{H}^2} \right) \nabla^2 \Phi, \]  

(45a)

which departs from (43) by an addition of the order of \( \varepsilon_1 \). Clearly, this discrepancy is a consequence of neglecting the contribution of \( \delta t_{\mu\nu} \) terms in the perturbed Einstein equations (25)-(27). As we have anticipated, this analysis justifies our assumption that the contribution of \( \delta t_{\mu\nu} \) will be of higher order in the slow roll parameter expansion. In the rest of the paper we will use Eq. (43) to calculate the power spectra at the lowest orders in epsilon parameters.

So far we have not used the modified Friedmann cosmology so equations (43) and (44) will coincide with those derived in [28] if \( p + \rho \) in [28] is replaced by \( \bar{p} + \bar{\rho} \). We now use the equation

\[ \dot{H} = -4 \pi G_N (\bar{p} + \bar{\rho}). \]  

(46)

which follows from (14). As in Ref. [28], we introduce new functions

\[ \xi = \frac{a \Phi}{4 \pi G_N H}, \quad \zeta = \Phi + H \frac{\delta \theta}{\theta}. \]  

(47)
The quantity $\zeta$ is gauge invariant and measures the spatial curvature of comoving (or constant-$\theta$) hyper-surfaces. Substituting the definitions (47) into (43) and (44) and using (46) we find

$$a\dot{\zeta} = z^2c_s^2\zeta,$$  \hspace{1cm} (48)

$$a\dot{\zeta} = z^{-2}\nabla^2\zeta,$$  \hspace{1cm} (49)

where

$$z = \frac{a(\bar{p} + \bar{\rho})^{1/2}}{c_sH} = \frac{a}{c_s}\sqrt{\frac{\varepsilon_1}{4\pi G_N}}.$$  \hspace{1cm} (50)

Here we have used the definition (17) of the slow roll parameter $\varepsilon_1$. Equations (48) and (49) with (50) are identical to those obtained in the standard $k$-inflation [28].

Note that the only change in Eqs. (48) and (49) compared with those in Ref. [28] comes from the factor $1 - h^2/2$ that multiplies $\rho + p$ inside $z$ per Eq. (50). However, this factor is absorbed into $\varepsilon_1$ due to Eq. (46). As a result, the second equality in Eq. (50) is exactly the same as in Ref. [28]. This feature is due to the fact that the factor $1 - h^2/2$ appears on the right-hand side of the perturbation equations (48) and (49) in the same way as in the background evolution equation (46). In contrast, in Ref. [17], where an approximate scheme was used, the expression for $z$ as a function of $\varepsilon_1$, explicitly contains the factor $1 - h^2/2$, thus leading to different equations than those of Ref. [28].

By introducing the conformal time $\tau = \int dt/a$ and a new variable $v = z\zeta$, it is straightforward to show from equations (48) and (49) that $v$ satisfies a second order differential equation

$$v'' - c_s^2\nabla^2 v - \frac{z''}{z}v = 0,$$  \hspace{1cm} (51)

where the primes denote derivatives with respect to $\tau$. By making use of the Fourier transformation, we also obtain the mode-function equation

$$v_q'' + \left(c_s^2q^2 - \frac{z''}{z}\right)v_q = 0.$$  \hspace{1cm} (52)

As we are looking for a solution to this equation in the slow-roll regime, it is useful to express the quantity $z''/z$ in terms of slow-roll parameters $\varepsilon_i$. In the slow-roll regime one can use the relation [46]

$$\tau = -\frac{1 + \varepsilon_1}{aH} + \mathcal{O}(\varepsilon_1^2),$$  \hspace{1cm} (53)
which follows from the definition (17) expressed in terms of the conformal time. At linear order in $\varepsilon_i$ we find

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\tau^2},$$

where

$$\nu^2 = \frac{9}{4} + 3\varepsilon_1 + \frac{3}{2}\varepsilon_2.$$  \hfill (55)

We look for a solution to (52) which satisfies the positive frequency asymptotic limit

$$\lim_{\tau \to -\infty} v_q = \frac{e^{-i\omega_q\tau}}{\sqrt{2\omega_q}}.$$  \hfill (56)

Then the solution which up to a phase agrees with (56) is

$$v_q = \sqrt{\pi} \left(-\tau\right)^{1/2} H^{(1)}_{\nu}(-\omega_q \tau),$$

where $H^{(1)}_{\nu}$ is the Hankel function of the first kind of rank $\nu$.  \hfill (57)

It is of interest to show explicitly that the perturbation of the scalar field $\theta$ and metric perturbation $\Phi$ are smooth functions of time. The perturbations $\delta \theta$ and $\Phi$ are encoded in the functions $\zeta$ and $\xi$ that satisfy the first order differential equations (48) and (49). These equations are equivalent to a single second order differential equation (51) for the function $v$. Hence, the functions $\zeta$ and $\xi$ can be expressed in terms of $v$ and its derivative $\dot{v}$. This in turn implies that the perturbations $\delta \theta$ and $\Phi$ can be expressed in terms of $v$ and $\dot{v}$. More explicitly, from Eqs. (47)-(49) it follows in momentum space

$$\delta \theta = \left[ \left( \frac{1}{zH} - \frac{4\pi G_N \dot{\zeta} \dot{v}_q}{q^2} \right) v_q + \frac{4\pi G_N z}{q^2} \dot{v}_q \right] \dot{\theta},$$

$$\Phi = \frac{4\pi G_N}{q^2} \left( \dot{v}_q - z \dot{\zeta}_q \right),$$

where $v_q$ is a solution to (52) in momentum space. In the slow-roll regime the solution is expressed in terms of the Hankel functions of the first kind (57). This type of solution appears in all scalar models of inflation (see, e.g., [47]). From our solutions depicted in Fig. 1, the slope $\dot{\theta}$ is everywhere finite and asymptotically approaches a finite constant or zero. Hence, there are reasonable $k$-essence models that yield both $\Phi$ and $\theta$ as smooth functions in the domain of inflation.

Applying the standard canonical quantization [48] the field $v_q$ is promoted to an operator and the power spectrum of the field $\zeta_q = v_q/z$ is obtained from the two-point correlation function

$$\langle \zeta_q \zeta_{q'} \rangle = \langle \dot{v}_q \dot{v}_{q'} \rangle / z^2 = (2\pi)^3 \delta(q + q')|\zeta_q|^2.$$  \hfill (60)
The spectral density
\[ P_S(q) = \frac{q^3}{2\pi^2} |\zeta_q|^2 = \frac{q^3}{2\pi^2 \varepsilon_2^2} |v_q|^2, \] (61)
with \( v_q \) given by (57), characterizes the primordial scalar fluctuations.

Next, we evaluate the scalar spectral density at the horizon crossing, i.e., for a wave-number satisfying \( q = aH \). Following Refs. [39, 49] we make use of the expansion of the Hankel function for \( c_s q \tau \ll 1 \),
\[ H^{(1)}_\nu(-c_s q \tau) \simeq -\frac{i}{\pi} \Gamma(\nu) \left( -\frac{c_s q \tau}{2} \right)^{-\nu}, \] (62)
where the conformal time \( \tau < 0 \) and \( q \) is the comoving wave number. Using this we find at the lowest order in \( \varepsilon_1 \) and \( \varepsilon_2 \)
\[ P_S \simeq \frac{G_N H^2}{\pi c_s \varepsilon_1} \left[ 1 - 2 (1 + C) \varepsilon_1 - C \varepsilon_2 \right], \] (63)
where \( C = \gamma - 2 + \ln 2 \simeq -0.73 \) and \( \gamma \) is the Euler constant, so we recover the standard expression [39, 49]. However, it should be stressed that the relation between the sound speed and \( \varepsilon_1 \) is not the usual one, as shown in Eq. (41). Hence, in our case the contribution of the sound speed will be altered in comparison with the standard result.

C. Tensor perturbations

The tensor perturbations are related to the production of gravitational waves during inflation. The metric perturbation is, in this case, written as
\[ ds^2 = dt^2 - a^2(t) (\delta_{ij} + h_{ij}) dx^i dx^j, \] (64)
where \( h_{ij} \) is traceless and transverse. Inserting the metric components in the field equations (9) we obtain
\[ \left( 1 - \frac{\ell^2}{2} (H^2 + \dot{H}) \right) \left( \ddot{h}_{ij} + 3H \dot{h}_{ij} - \frac{1}{a^2} \nabla^2 h_{ij} \right) = 8\pi G_N \delta T_{ij}. \] (65)
In the absence of anisotropic stress, the gravitational waves are decoupled from matter and the right-hand side of the above equation is zero. Hence, our braneworld scenario does not introduce changes to the gravitational waves dynamics. The spectral density which characterizes the primordial tensor fluctuations, in the approximation \( q \tau \ll 1 \), is given by
\[ P_T \simeq \frac{16G_N H^2}{\pi} \left[ 1 - 2 (1 + C) \varepsilon_1 \right]. \] (66)
Hence, the tensor perturbation spectrum is given by the usual expression [46] in the same way as our scalar spectrum \( P_S \).
D. Scalar spectral index and tensor to scalar ratio

The scalar spectral index $n_S$ and tensor to scalar ratio $r$ are given by

$$n_S - 1 = \frac{d \ln P_S}{d \ln q} \approx \frac{1}{H(1 - \varepsilon_1)} \frac{d \ln P_S}{dt}, \quad (67)$$

$$r = \frac{P_T}{P_S}, \quad (68)$$

where $P_S$ and $P_T$ are evaluated at the horizon crossing with $q = aH$. From (63) and (66) keeping the terms up to the quadratic order in $\varepsilon_i$ we obtain

$$r = 16 \varepsilon_1 \left[1 + C \varepsilon_2 + \frac{2(2 - h^2)}{3(4 - h^2)} \frac{p_{p,X}}{p_{p,X}^2} \varepsilon_1\right] \quad (69)$$

and

$$n_s = 1 - 2\varepsilon_1 - \varepsilon_2 - \left(2 + \frac{8h^2}{3(4 - h^2)} \frac{p_{p,X}}{p_{p,X}^2}\right) \varepsilon_1^2$$

$$- \left(3 + 2C + \frac{2(2 - h^2)}{3(4 - h^2)} \frac{p_{p,X}}{p_{p,X}^2}\right) \varepsilon_1 \varepsilon_2 - C \varepsilon_2 \varepsilon_3. \quad (70)$$

Note that our results at linear order in $\varepsilon_i$ agree with the standard scalar field inflation as well as with the general $k$-essence inflation [49]. At quadratic order the standard $k$-inflation results are recovered in the limit $\ell \to 0$ ($h \to 0$). For example, if we specify the $k$-essence Lagrangian to the tachyon, in the limit $\ell \to 0$ we recover the standard tachyon inflation (e.g., compare with Ref. [39]).

IV. SUMMARY AND CONCLUSIONS

We have studied the early cosmology on the holographic brane where the effective Einstein equations are modified due to the dual conformal theory on the AdS boundary. We have developed the complete perturbation theory at linear order for gravity on the holographic brane together with a general $k$-essence field.

We have derived the scalar and tensor power spectra up to the second order in the slow-roll parameter expansion and calculated the scalar spectral index $n_s$ and scalar to tensor ratio $r$. We have found that the expressions for both the scalar and tensor power spectra have the same form as those in the standard $k$-inflation. This is a remarkable result in view of quite nontrivial modification of Einstein’s equations in the holographic braneworld. However, the functional dependence of the sound speed on the slow-roll parameters is altered and, as a result, the second order terms in
the slow-roll parameter expansion of $n_s$ and $r$ are modified compared to those in the standard $k$-inflation, as shown in Eqs. (69) and (70).

It would be of considerable interest to investigate the holographic inflation for a particular $k$-essence model or any other interesting model with a nontrivial sound speed. However, this would go beyond the scope of the present paper as our purpose here is to show how the holographic braneworld scenario affects inflation driven by a general $k$-essence.

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**Appendix A: Energy conservation and equation of motion**

Consider the $k$-essence action

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}(X, \theta)$$

(A1)

where

$$X = g^{\mu \nu} \partial_\mu \theta \partial_\nu \theta.$$  

(A2)

In the cosmological context it is appropriate to assume $X > 0$. Variation of (A1) yields the equation of motion

$$(2 \mathcal{L}_X g^{\mu \nu} \theta_{,\nu})_{,\mu} - \mathcal{L}_{,\theta} = 0$$

(A3)

where we denote

$$\mathcal{L}_X \equiv \frac{\partial \mathcal{L}}{\partial X}, \quad \mathcal{L}_{,\theta} \equiv \frac{\partial \mathcal{L}}{\partial \theta}.$$  

(A4)

The energy-momentum tensor associated with $S_{\text{matter}}$ can be expressed in the perfect fluid form

$$T_{\mu \nu} = (p + \rho) u_\mu u_\nu - pg_{\mu \nu}.$$  

(A5)
where we use the hydrodynamic variables
\[ p = \mathcal{L}, \quad \rho = 2X\mathcal{L}_X - \mathcal{L}_\theta, \]  
(A6)

and
\[ u_\mu = \frac{\partial_\mu \theta}{\sqrt{X}} \]  
(A7)

The energy-momentum conservation equation
\[ T^{\mu\nu} = 0 \]  
(A8)

contracted with \( u^\mu \) and applied to (A5) yields another expression of the energy conservation
\[ \dot{\rho} + 3H(p + \rho) = 0, \]  
(A9)

where \( \dot{\rho} \equiv u^\mu \rho_{,\mu} \) and the expansion rate \( H = u^{\mu,\mu}/3 \).

For a general k-essence theory one can proof the following theorem: *Equation of motion for a k-essence field is satisfied if and only if \( T^{\mu\nu},_\nu = 0 \). For our purpose it is sufficient to prove that the equation of motion (A3) is equivalent to the energy conservation equation (A9) provided \( X > 0 \).

**Proof**

Using (A7) Eq. (A3) can be written as
\[ (2\sqrt{X} \mathcal{L}_X u^\mu),_\mu - \mathcal{L}_\theta = 0. \]  
(A10)

Multiplying by \( \sqrt{X} \), equation (A10) can be recast in the form
\[ 2X\mathcal{L}_X u^\mu,;_\mu + \mathcal{L}_X u^\mu X,_{\mu} + 2X u^\mu (\mathcal{L}_X),_{\mu} - \sqrt{X} \mathcal{L}_\theta = 0. \]  
(A11)

Then by making use of the Leibniz rule and (A7) we obtain
\[ 2X\mathcal{L}_X u^\mu,;_\mu + u^\mu (2X \mathcal{L}_X),_{\mu} - \mathcal{L}_X u^\mu X,_{\mu} - u^\mu \theta_{,\mu} \mathcal{L}_\theta = 0. \]  
(A12)

Assembling the last two terms we obtain
\[ 2X\mathcal{L}_X u^\mu,;_\mu + u^\mu (2X \mathcal{L}_X - \mathcal{L}),_{\mu} = 0. \]  
(A13)

Using (A6) this can be written as
\[ u^\mu,;_\mu (p + \rho) + u^\mu \rho_{,\mu} = 0 \]  
(A14)

Identifying \( u^{\mu,\mu} = 3H \) and \( u^{\mu} \rho_{,\mu} = \dot{\rho} \) we obtain the energy conservation equation in the usual form (A9). Hence, we have shown that the equation of motion (A10) implies the energy conservation
equation (A14). The reverse statement is also true since every step in our derivation from (A10) to (A14) is reversible and hence, equations (A10) and (A14) are equivalent.

In a cosmological background described by the metric (10) we have \( X = \dot{\theta}^2 \). Then, the equation of motion (A3) takes the form

\[
2(2X\mathcal{L}_{,XX} + \mathcal{L}_{,X})\ddot{\theta} + 6H\mathcal{L}_{,X}\dot{\theta} + 2X\mathcal{L}_{,X}\theta - \mathcal{L}_{,\theta} = 0. \tag{A15}
\]

It may be easily verified that the same equation is obtained also from the energy conservation equation (A9).

**Appendix B: Alternative derivation of the energy-momentum perturbations**

Here we derive the perturbations of the energy momentum tensor directly from it’s definition

\[
T^\mu_\nu = 2\mathcal{L}_{,X} g^{\mu\alpha} \partial_\alpha \theta \partial_\nu \theta - \delta^\mu_\nu \mathcal{L}. \tag{B1}
\]

At linear order we find

\[
\delta T^\mu_\nu = 2(\mathcal{L}_{,XX} \delta X + \mathcal{L}_{,X\theta} \delta \theta) g^{\mu\alpha} \partial_\alpha \theta \partial_\nu \theta + 2\mathcal{L}_{,X} (\delta g^{\mu\alpha} \partial_\alpha \theta \partial_\nu \theta + 2g^{\mu\alpha} \partial_\alpha \theta \partial_\nu \delta \theta) - \delta^\mu_\nu (\mathcal{L}_{,X} \delta X + \mathcal{L}_{,\theta} \delta \theta) \tag{B2}
\]

and owing to

\[
\delta X = \delta g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta + 2g^{\mu\nu} \partial_\mu \theta \partial_\nu \delta \theta = -2X\Phi + 2\dot{\theta}\ddot{\theta}, \tag{B3}
\]

we obtain

\[
\delta T^0_0 = (2X\mathcal{L}_{,XX} + \mathcal{L}_{,X})(2\dot{\theta}\ddot{\theta} - 2X\Phi) + (2X\mathcal{L}_{,X\theta} - \mathcal{L}_{,\theta}) \delta \theta \tag{B4}
\]

and

\[
\delta T^i_0 = (2X\mathcal{L}_{,X})(\frac{\delta \theta}{\theta}). \tag{B5}
\]

Then, combining Eq. (B4) with \( \theta \) field equation (A15) derived in Appendix A, we obtain

\[
\delta T^0_0 = 2(2X\mathcal{L}_{,XX} + \mathcal{L}_{,X})(\dot{\theta}\ddot{\theta} - \ddot{\theta}\dot{\theta} - X\Phi) - 6H\mathcal{L}_{,X}\dot{\theta}\ddot{\theta}. \tag{B6}
\]

It may be easily verified that Eqs. (B6) and (B5) will be identical to Eqs. (38) and (39), respectively, if the fluid variables \( p, \rho \), and \( c_s^2 \) in (38) and (39) are replaced by their original field theory expressions.
Appendix C: Background evolution of the field in particular $k$-essence models

In order to show the behavior of the $\theta$ field and for illustration purposes, we study here a few examples of $k$-essence in the holographic braneworld scenario. We consider two classes of $k$-essence: the canonical scalar field and tachyon condensate models. The Lagrangian density for the canonical case is given by

$$\mathcal{L} = \frac{1}{2} X - V(\theta),$$  \hfill (C1)

and for the tachyon condensate by [50]

$$\mathcal{L} = -V(\theta)\sqrt{1-X}. \hfill (C2)$$

In order to proceed with numerical calculations, we need to specify $V(\theta)$. For the canonical model we make use of

$$V(\theta) = \frac{1}{2} m^2 \theta^2 . \tag{C3}$$

as the simplest potential being used for the large field or chaotic inflation [47, 51] For the tachyon condensate we take the exponential potential which has been extensively exploited in the tachyon literature [17, 39, 52, 53]

$$V(\theta) = \ell^{-4} e^{-\omega \theta} \hfill (C4)$$

and the inverse power-law potential [54–56]

$$V(\theta) = m^{4-n} \theta^{-n}, \quad n > 0. \hfill (C5)$$

We start from the $\theta$-field equation (A15) together with the expansion rate $h \equiv \ell H$ expressed in terms of $\theta$. From Eq. (13) with $k = 0$ we obtain

$$h^2 = 2 \left( 1 - \sqrt{1 - \frac{8\pi G_N}{3} \ell^2 \rho} \right), \tag{C6}$$

as a solution to a quadratic equation that meets the requirement that $H$ should be zero for $\rho = 0$. Initial conditions are taken at $t = 0$ where we fix $h_i = \sqrt{2}$ and set $\dot{\theta}(0) = 0$ in all examples. The initial $\theta_0 \equiv \theta(0)$ is then fixed by the first Friedmann equation.

In Fig. 1 we depict our numerical results as four plots of $\theta$ as a function of time. The canonical field, as expected, vanishes asymptotically with damped oscillations whereas the tachyon field grows linearly for large $t$. In the lower right panel we compare the tachyon model with exponential
FIG. 1. Plots of the field $\theta$ versus time (in units of $\ell$) for different choices of $k$-essence. Upper left: canonical scalar field with quadratic potential (C3); upper right: tachyon $k$-essence field with exponential potential (C4); lower left: tachyon $k$-essence with inverse power-law potential (C5); lower right: comparison between the holographic braneworld and GR scenarios. The field $\theta$ in the canonical model, the mass $m$, and the parameter $\omega$ are in units of $\ell^{-1}$, whereas $\theta$ in the tachyon model is in units of $\ell$.

potential in the holographic braneworld and in general relativity. The initial condition for the GR model is adjusted so that two models agree in the low density limit, i.e., at large $t$. These two scenarios disagree substantially only at small $t$, as expected.

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