A Comparison of Performance Measures for Online Algorithms

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Abstract

This paper provides a systematic study of several recently suggested measures for online algorithms in the context of a specific problem, namely, the two server problem on three colinear points. We examine how these measures evaluate the Greedy Algorithm and Lazy Double Coverage, commonly studied algorithms in the context of server problems. We examine Competitive Analysis, the Max/Max Ratio, the Random Order Ratio, Bijective Analysis and Relative Worst Order Analysis and determine how they compare the two algorithms. We find that by the Max/Max Ratio and Bijective Analysis, Greedy is the better algorithm. Under the other measures Lazy Double Coverage is better, though Relative Worst Order Analysis indicates that Greedy is sometimes better. Our results also provide the first example of an algorithm that is optimal under Relative Worst Order Analysis.

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1 Introduction

Since its introduction by Sleator and Tarjan in 1985 [14], Competitive Analysis has been the most widely used method for evaluating online algorithms. A problem is said to be online if the input to the problem is given a piece at a time, and the algorithm must commit to parts of the solution over time before the entire input is revealed to the algorithm. Competitive Analysis evaluates an online algorithm in comparison to the optimal offline algorithm which receives the input in its entirety in advance and has unlimited computational power in determining a solution. Informally speaking, we look at the worst-case input which maximizes the ratio of the cost of the online algorithm for that input to the cost of the optimal offline algorithm on that same input. The maximum ratio achieved is called the Competitive Ratio. Thus, we factor out the inherent difficulty of a particular input (for which the offline algorithm is penalized along with the online algorithm) and measure what is lost in making decisions with partial information.

Despite the popularity of Competitive Analysis, researchers have been well aware of its deficiencies and have been seeking better alternatives almost since the time that it came into wide use. Many of the problems with Competitive Analysis stem from the fact that it is a worst case measure and fails to examine the performance of algorithms on instances that would be expected in a particular application. Other problems come from the fact that the ratios tend to be artificially high since online algorithms are compared against such a powerful alternative. It has also been observed that Competitive Analysis sometimes fails to distinguish between algorithms which have very different performance in practice and intuitively differ in quality.

Over the years, researchers have devised alternatives to Competitive Analysis, each designed to address one or all of its shortcomings. This paper is a study of several alternative measures for evaluating online algorithms that have been recently suggested in the literature. We perform this comparison in the context of a particular problem: the 2-server problem on the line with three possible request points, nick-named here the baby server problem. We look at two algorithms (Greedy and Lazy Double Coverage [6]) and four different measures: Bijective Analysis, the Max/Max Ratio, Random Order Ratio and Relative Worst Order Analysis. Our results include the first example of an algorithm which is optimal for a particular problem under Relative Worst Order Analysis.
In the baby server problem, three distinct points on the line are determined in advance. There are two mobile servers, each of which occupies a single point on the line at any time. A sequence of requests arrives. Each request is a name of one of the three designated points on the line. After each request, one of the two servers must move to that location. The algorithm decides which server to send and the goal is to minimize the total distance traveled by the two servers. The Greedy algorithm always sends the server that is closest to the requested point. Lazy Double Coverage (to be defined more formally later) may move the server that is farther away if the closer server has traveled a certain amount since the farther server has last moved.

In investigating this problem, we find that according to some quality measures for online algorithms, Greedy is better than Ldc, whereas for others, Ldc is better than Greedy.

The ones that conclude that Ldc is best are focused on a worst-case sequence for the ratio of an algorithm’s cost compared to Opt. In the case of Greedy and Ldc, this conclusion makes use of the fact that there exists a family of sequences for which Greedy’s cost is unboundedly larger than the cost of Opt, whereas Ldc’s cost is always at most a factor two larger than the cost of Opt.

On the other hand, the measures that conclude that Greedy is best compare two algorithms based on the multiset of costs stemming from the set of all sequences of a fixed length. In the case of Greedy and Ldc, this makes use of the fact that for any fixed $n$, both the maximum as well as the average cost of Ldc over all sequences of length $n$ are greater than the corresponding values for Greedy.

Using Relative Worst Order Analysis a more nuanced result can be obtained. We show that Ldc can be a factor at most two worse than Greedy, while Greedy can be unboundedly worse than Ldc.

The simplicity of this problem enables us to give the first proof of optimality in Relative Worst Order Analysis: Ldc is an optimal algorithm for this case.

2 Preliminaries

In this section, we define the server problem used throughout this paper as the basis for our comparison. We also define the the server algorithms used, and the quality measures which are the subject of this study.
2.1 The Server Problem

Server problems [3] have been the objects of many studies. In its full generality, one assumes that some number \( k \) of servers are available in some metric space. Then a sequence of requests must be treated. A request is simply a point in the metric space, and a \( k \)-server algorithm must move servers in response to the request to ensure that at least one server is placed on the request point. A cost is associated with any move of a server (this is usually the distance moved in the given metric space), and the objective is to minimize total cost. The initial configuration (location of servers) may or may not be a part of the problem formulation.

In investigating the strengths and weaknesses of the various measures for the quality of online algorithms, we define the simplest possible nontrivial server problem:

**Definition 1** The *baby server problem* is a 2-server problem on the line with three possible request points \( A \), \( B \), and \( C \), in that order from left to right, with distance one between \( A \) and \( B \) and distance \( d > 1 \) between \( B \) and \( C \). The cost of moving a server is defined to be the distance it is moved. We assume that initially the two servers are placed on \( A \) and \( C \).

All results in the paper pertain to this problem.

2.2 Server Algorithms

First, we define some relevant properties of server algorithms:

**Definition 2** A server algorithm is called

- *noncrossing* if servers never change their relative position on the line.
- *lazy* [13] if it never moves more than one server in response to a request and it does not move any servers if the requested point is already occupied by a server.

A server algorithm fulfilling both these properties is called *compliant*.

Given an algorithm, \( \mathcal{A} \), we define the algorithm *lazy \( \mathcal{A} \), \( \mathcal{L}A \), as follows: \( \mathcal{L}A \) will maintain a virtual set of servers and their locations as well as the
real set of servers in the metric space. There is a one-to-one correspondence of real servers and virtual servers. The virtual set will simulate the behavior of \( A \). The initial server positions of the virtual and real servers are the same. Whenever a virtual server reaches a request point, the corresponding real server is also moved to that point (unless both virtual servers reach the point simultaneously, in which case only the physically closest is moved there). Also, if a request is made and there is a virtual server but no real server already on the requested point then the real server that corresponds to the virtual server already on the request point is moved to serve the request. Otherwise the virtual servers do not move.

**Proposition 1** For any 2-server algorithm, there exists a noncrossing algorithm with the same cost on all sequences.

**Proof** Observation made in [6].

The following was also observed in [13]:

**Proposition 2** For an algorithm \( A \), let the lazy version of it be \( \mathcal{L}A \). For any sequence \( I \) of requests, \( A(I) \geq \mathcal{L}A(I) \).

We investigate the relative strength of a number of algorithms for the baby server problem:

We define a number of algorithms by defining their behavior on the next request point, \( p \). For all algorithms, no moves are made if a server already occupies the request point (though internal state changes are sometimes made in such a situation).

**Greedy** moves the closest server to \( p \). Note that due to the problem formulation, ties cannot occur (and the server on \( C \) is never moved).

If \( p \) is in between the two servers, **Double Coverage** (Dc), moves both servers at the same speed in the direction of \( p \) until at least one server reaches the point. If \( p \) is on the same side of both servers, the nearest server moves to \( p \).

We define \( a \)-Dc to work in the same way as Dc, except that the right-most server moves at a speed \( a \leq d \) times faster than the left-most server.

We refer to the lazy version of Dc as \( \text{Ldc} \) and the lazy version of \( a \)-Dc as \( a \)-Ldc.
The balance algorithm [13], BAL, makes its decisions based on the total distance travelled by each server. For each server, \( s \), let \( d_s \) denote the total distance travelled by \( s \) from the initiation of the algorithm up to the current point in time. On a request, BAL moves a server, aiming to obtain the smallest possible \( \max_s d_s \) value after the move. In case of a tie, BAL moves the server which must move the furthest.

If \( p \) is in between the two servers, DUMMY moves the server that is furthest away to the request point. If \( p \) is on the same side of both servers, the nearest server moves to \( p \). Again, due to the problem formulation, ties cannot occur (and the server on \( A \) is never moved).

2.3 Quality Measures

In analyzing algorithms for the baby server problem, we consider input sequences \( I \) of request points. An algorithm, \( A \), which treats such a sequence has some cost, which is the total distance moved by the two server. This cost is denoted by \( A(I) \). Since \( I \) is of finite length, it is clear that there exists an offline algorithm with minimal cost. By OPT, we refer to such an algorithm and \( \text{OPT}(I) \) denotes the unique minimal cost of processing \( I \).

All of the measures described below can lead to a conclusion as to which algorithm of two is better. In contrast to the others, Bijective Analysis does not indicate how much better the one algorithm might be; it does not produce a ratio, as the others do.

2.3.1 Competitive Analysis

In Competitive Analysis [9 14 10], we define an algorithm \( A \) to be \( c \)-competitive if there exists a constant \( \alpha \) such that for all input sequences \( I \), \( A(I) \leq c \text{OPT}(I) + \alpha \).

2.3.2 The Max/Max Ratio

The Max/Max Ratio [2] compares an algorithm’s worst cost for any sequence of length \( n \) to OPT’s worst cost for any sequence of length \( n \). The Max/Max Ratio of an algorithm \( A \), \( w_M(A) \), is \( M(A)/M(\text{OPT}) \), where

\[
M(A) = \limsup_{t \to \infty} \max_{|I|=t} A(I)/t.
\]
2.3.3 The Random Order Ratio

Kenyon [11] defines the Random Order Ratio to be the worst ratio obtained over all sequences, comparing the expected value of an algorithm, $A$, over all permutations of a given sequence to the expected value of $\text{Opt}$ over all permutations of the given sequence:

$$\limsup_{\text{OPT}(I) \to \infty} \frac{E_{\sigma}[A(\sigma(I))]}{\text{OPT}(I)}$$

The original context for this definition is Bin Packing for which the optimal packing is the same, regardless of the order in which the items are presented. Therefore, it does not make sense to take an average over all permutations for $\text{Opt}$. For server problems, however, the order of requests in the sequence may very well change the cost of $\text{Opt}$. We choose to generalize the Random Order Ratio as follows:

$$\limsup_{\text{OPT}(I) \to \infty} \frac{E_{\sigma}[A(\sigma(I))]}{E_{\sigma}[\text{OPT}(I)]}$$

If, on the other hand, one chooses the following generalization, the results presented here are the same:

$$\limsup_{\text{OPT}(I) \to \infty} E_{\sigma}\left[\frac{A(\sigma(I))}{\text{OPT}(\sigma(I))}\right]$$

2.3.4 Bijective Analysis

In [1], Bijective and Average Analysis are defined, as methods of comparing two online algorithms directly. As with the Max/Max Ratio and Relative Worst Order Analysis, the two algorithms are not necessarily compared on the same sequence. Here, we adapt those definition to the notation used here.

In Bijective Analysis, the sequences of a given length are mapped, using a bijection onto the same set of sequences. The performance of the first algorithm on a sequence, $I$, is compared to the performance of the second algorithm on the sequence $I$ is mapped to. If $I_n$ denotes the set of all input sequences of length $n$, then an online algorithm $A$ is no worse than an online algorithm $B$ according to Bijective Analysis if there exists an integer $n_0 \geq 1$ such that for each $n \geq n_0$, there is a bijection $f : I_n \to I_n$ satisfying $A(I) \leq B(f(I))$ for each $I \in I_n$. 
Average Analysis can be viewed as a relaxation of Bijective Analysis. An online algorithm $A$ is no worse than an online algorithm $B$ according to Average Analysis if there exists an integer $n_0 \geq 1$ such that for each $n \geq n_0$, $\sum_{I \in I_n} A(I) \leq \sum_{I \in I_n} B(I)$.

2.3.5 Relative Worst Order Analysis

Relative Worst Order Analysis was introduced in [4] and extended in [5]. It compares two online algorithms directly. As with the Max/Max Ratio, it compares two algorithms on their worst sequence in the same part of a partition. The partition is based on the Random Order Ratio, so that the algorithms are compared on sequences having the same content, but possibly in different orders.

**Definition 3** Let $I$ be any input sequence, and let $n$ be the length of $I$. If $\sigma$ is a permutation on $n$ elements, then $\sigma(I)$ denotes $I$ permuted by $\sigma$. Let $A$ be any algorithm. Then, $A(I)$ is the cost of running $A$ on $I$, and

$$A_W(I) = \max_\sigma A(\sigma(I)).$$

**Definition 4** For any pair of algorithms $A$ and $B$, we define

$$c_l(A, B) = \sup \{ c \mid \exists b: \forall I: A_W(I) \geq c B_W(I) - b \} \quad \text{and}$$

$$c_u(A, B) = \inf \{ c \mid \exists b: \forall I: A_W(I) \leq c B_W(I) + b \}.$$  

If $c_l(A, B) \geq 1$ or $c_u(A, B) \leq 1$, the algorithms are said to be comparable and the Relative Worst-Order Ratio $WR_{A, B}$ of algorithm $A$ to algorithm $B$ is defined. Otherwise, $WR_{A, B}$ is undefined.

If $c_l(A, B) \geq 1$, then $WR_{A, B} = c_u(A, B)$, and

if $c_u(A, B) \leq 1$, then $WR_{A, B} = c_l(A, B)$.  

If $WR_{A, B} < 1$, algorithms $A$ and $B$ are said to be comparable in $A$’s favor. Similarly, if $WR_{A, B} > 1$, the algorithms are said to be comparable in $B$’s favor.  

**Definition 5** Let $c_u$ be defined as in Definition 4. If at least one of the ratios $c_u(A, B)$ and $c_u(B, A)$ is finite, the algorithms $A$ and $B$ are $(c_u(A, B), c_u(B, A))$-related.  

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| Measure                        | Value                                                                 |
|-------------------------------|----------------------------------------------------------------------|
| Competitive Ratio             | $CR_A = \max_I \frac{A(I)}{\text{OPT}(I)}$                           |
| Max/Max Ratio                 | $MR_A = \frac{\max_{|I|=n} A(I)}{\max_{|I'|=n} \text{OPT}(I')}$        |
| Random Order Ratio            | $RR_A = \max_I \frac{E_\sigma[A(\sigma(I))]}{\text{OPT}(I)}$          |
| Relative Worst-Order Ratio    | $WR_{A,B} = \max_I \frac{\max_\sigma A(\sigma(I))}{\max_{\sigma'} B(\sigma'(I))}$ |

Table 1: Comparison of measures

**Definition 6** Let $c_u(A, B)$ be defined as in Definition 5. Algorithms $A$ and $B$ are *weakly comparable in A’s favor*,

- if $A$ and $B$ are comparable in A’s favor,
- if $c_u(A, B)$ is finite and $c_u(B, A)$ is infinite, or
- if $c_u(A, B) \in o(c_u(B, A))$.

\[\square\]

### 3 Competitive Analysis

The $k$-server problem has been studied using Competitive Analysis starting in [12]. In [6], it is shown that the competitive ratios of DC and LDC are $k$, which is optimal, and that Greedy is not competitive.

### 4 The Max/Max Ratio

In [2], a concrete example is given with two servers and three non-collinear points. It is observed that the Max/Max Ratio favors the greedy algorithm over the balance algorithm, BAL.
BAL behaves similarly to LDC and identically on LDC’s worst case sequences. The following theorem shows that the same conclusion is reached when the three points are on the line.

**Theorem 1** Greedy is better than LDC on the baby server problem with regards to the Max/max Ratio.

**Proof** Given a sequence of length $n$, Greedy’s maximum cost is $n$, so $M(\text{Greedy}) = 1$.

Opt’s cost is at most $n$ (since it is at least as good as Greedy). Opt’s maximal cost over all sequences of length $n = k(2([d] + 1) + 1)$, for some $k$, is at least $\frac{2([d]+1)}{2([d]+1)+1} n$. since its cost on $(AB)^{[d]+1}C)^k$ is this much. Thus, we can bound $M(\text{Opt})$ by $\frac{2d}{2d+1} \leq \frac{2([d]+1)}{2([d]+1)+1} \leq M(\text{Opt}) \leq 1$.

We now prove a lower bound on $M(\text{LDC})$.

Consider the sequence $s = (BABA...BC)^pX$, where the length of the alternating $A/B$-sequence before the $C$ is $2[d]+1$, $X$ is a possibly empty alternating sequence of $A$s and $B$s starting with a $B$, $|X| = n \mod (2[d] + 2)$, and $p = \frac{n-|X|}{2[d]+2}$.

If $|X| \neq 2[d]+1$, then $\text{LDC}(s) = p(2[d]+2d) + |X| = n + \frac{(d-1)(n-|X|)}{[d]+1}$.

Otherwise, $\text{LDC}(s) = p(2[d]+2d) + |X| = n + \frac{(d-1)(n-|X|)}{[d]+1} + d - 1$.

Since we are taking the supremum, we restrict our attention to sequences where $|X| = 0$. Thus, $M(\text{LDC}) \geq \frac{n+(d-1)n}{n} = 1 + \frac{d-1}{[d]+1} \geq 1 + \frac{d-1}{d+1} = \frac{2d}{d+1}$.

Finally, $w_M(\text{Greedy}) = \frac{M(\text{Greedy})}{M(\text{Opt})} \leq \frac{1}{M(\text{Opt})}$,

while $w_M(\text{LDC}) = \frac{M(\text{LDC})}{M(\text{Opt})} \geq \frac{2d}{(2d+1)M(\text{Opt})}$.

Since $M(\text{Opt})$ is bounded, $\frac{w_M(\text{LDC})}{w_M(\text{Greedy})} \geq \frac{2d}{d+1}$, which is greater than 1 for $d > 1$.

Observe from the proof that Greedy is close to optimal with respect to the Max/max Ratio, since the cost of Greedy divided by the cost of Opt tends toward one for large $d$. 

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Since Ldc and Dc perform identically on their worst sequences of any given length, they also have the same Max/Max Ratio.

5 The Random Order Ratio

**Theorem 2** Ldc is better than Greedy on the baby server problem with regards to the Random Order Ratio.

**Proof** The Random Order Ratio is the worst ratio obtained over all sequences, comparing the expected value of an algorithm over all permutations of a given sequence to the expected value of Opt over all permutations of the given sequence.

Since the competitive ratio of Ldc is two, on any given sequence, Ldc’s cost is bounded by two times the cost of Opt on that sequence, plus an additive constant. Thus, the Random Order Ratio is also at most two.

Consider all permutations of the sequence \((BA)^n\). We consider positions from 1 through \(n\) in these sequences. Refer to a maximal subsequence consisting entirely of either As or Bs as a run.

Given a sequence containing \(h\) As and \(t\) Bs, the expected number of runs is \(1 + \frac{2ht}{h+t}\). (A problem in [8] gives that the expected number of runs of As is \(\frac{h(t+1)}{h+t}\), so the expected number of runs of Bs is \(\frac{t(h+1)}{h+t}\). Adding these gives the result.) Thus, with \(h = t = \frac{n}{2}\), we get \(\frac{n}{2} + 1\) expected number of runs.

The cost of Greedy is equal to the number of runs if the first run is a run of Bs. Otherwise, the cost is one smaller. Thus, Greedy’s expected cost on a permutation of \(s\) is \(\frac{n}{2} + \frac{1}{2}\).

The cost of Opt for any permutation of \(s\) is \(d\), since it simply moves the server from \(C\) to \(B\) on the first request to \(B\) and has no other cost after that.

Thus, the Random Order Ratio is \(\frac{n+1}{2d}\), which, as \(n\) tends to infinity, is unbounded. \(\square\)

6 Bijective Analysis

Bijective analysis correctly distinguishes between Dc and Ldc, indicating that the latter is the better algorithm. This follows from the following
general theorem about lazy algorithms, and the fact that there are some sequences where one of Dc’s servers repeatedly moves from C towards B, but moves back to C before ever reaching B, while Ldc’s server stays on C.

**Theorem 3** The lazy version of any algorithm for the baby server problem is at least as good as the original algorithm according to Bijective Analysis.

**Proof** By Proposition[2], the identity function, \(id\), is a bijection such that \(LA(I) \leq A(id(I))\) for all sequences \(I\).

**Theorem 4** Greedy is better than any lazy algorithm Lazy (including LDC) for the baby server problem according to Bijective Analysis.

**Proof** Since Greedy has cost zero for the sequences consisting of only the point A or only the point C and cost one for the point B, it is easy to define a bijection \(f\) for sequences of length one, such that \(\text{Greedy}(I) \leq \text{Lazy}(f(I))\). Suppose that for all sequences of length \(k\) that we have a bijection, \(f\), from Greedy’s sequences to Lazy’s sequences, such that for each sequence \(I\) of length \(k\), \(\text{Greedy}(I) \leq \text{Lazy}(f(I))\). To extend this to length \(k+1\), consider the three sequences formed from a sequence \(I\) of length \(k\) by adding one of the three requests A, B, or C to the end of \(I\), and the three sequences formed from \(f(I)\) by adding each of these points to the end of \(f(I)\). At the end of sequence \(I\), Greedy has its two servers on different points, so two of these new sequences have the same cost for Greedy as on \(I\) and one has cost exactly 1 more. Similarly, Lazy has its two servers on different points at the end of \(f(I)\), so two of these new sequences have the same cost for Lazy as on \(f(I)\) and one has cost either 1 or \(d\) more. This immediately defines a bijection \(f'\) for sequences of length \(k+1\) where \(\text{Greedy}(I) \leq \text{Lazy}(f'(I))\) for all \(I\) of length \(k+1\).

**Corollary 1** Greedy is the unique optimal algorithm.

**Proof** Note that the proof of Theorem 4 shows that Greedy is strictly better than any lazy algorithm which ever moves the server away from C, so it is better than any other lazy algorithm. By Theorem 3, it is better than any algorithm.

**Observation 1** If an algorithm is better than another algorithm with regards to Bijective Analysis, then it is also better with regards to Average Analysis[1].

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Corollary 2 GREEDY is the unique optimal algorithm with respect to Average Analysis.

Proof Follows directly from Corollary 1 and Observation 1.

Clearly, by definition, the algorithm Dummy permanently leaves a server on A.

Theorem 5 Dummy is the unique worst algorithm among compliant server algorithms for the baby server problem according to Bijective Analysis.

Proof Similar to the proof of Theorem 4, but now with cost $d$ for every actual move.

Lemma 1 If $a \leq b$, then there exists a bijection $\sigma_n : \{A, B, C\}^n \rightarrow \{A, B, C\}^n$ such that $a$-LDC($I$) $\leq$ $b$-LDC($\sigma_n(I)$) for all sequences $I \in \{A, B, C\}^n$.

Proof We use the bijection from the proof of Theorem 4 showing that Greedy is the unique best algorithm, but specify the bijection completely, as opposed to allowing some freedom in deciding the mapping in the cases where we are extending by a request where the algorithms already have a server. Suppose that $\sigma_n$ is already defined. Consider a sequence $I_n$ of length $n$ and the three possible ways, $I_nA$, $I_nB$, and $I_nC$, of extending it to length $n + 1$. Suppose that $a$-LDC has servers on points $X_a, Y_a \in \{A, B, C\}$ after handling the sequence $I_n$, and $b$-LDC has servers on points $X_b, Y_b \in \{A, B, C\}$ after handling $\sigma_n(I_n)$. Let $Z_a$ be the point where $a$-LDC does not have a server and $Z_b$ the point where $b$-LDC does not. Then $\sigma_{n+1}(I_nZ_a)$ is defined to be $\sigma_n(I_n)Z_b$. In addition, since the algorithms are lazy, both algorithms have their servers on two different points of the three possible, so there must be at least one point $P$ where both algorithms have a server. Thus, $P \in \{X_a, Y_a\} \cap \{X_b, Y_b\}$. Let $U_a$ be the point in $\{X_a, Y_a\} \setminus \{P\}$ and $U_b$ be the point in $\{X_b, Y_b\} \setminus \{P\}$. Then, $\sigma_{n+1}(I_nP)$ is defined to be $\sigma_n(I_n)P$ and $\sigma_{n+1}(I_nU_a)$ to be $\sigma_n(I_n)U_b$.

Consider running $a$-LDC on a sequence $I_n$ and $b$-LDC on $\sigma_n(I_n)$ simultaneously. The sequences are clearly constructed so that, at any point during this simultaneous execution, both algorithms have servers moving or neither does.

Clearly, the result follows if we can show that $b$-LDC moves away from and back to $C$ at least as often as $a$-LDC does. By construction, the two
sequences, $I_n$ and $\sigma_n(I_n)$, will be identical up to the point where $b$-LDC (and possibly $a$-LDC) moves away from $C$ for the first time. In the remaining part of the proof, we argue that if $a$-LDC moves away from and back to $C$, then $b$-LDC will also do so before $a$-LDC can do it again. Thus, the total cost of $b$-LDC will be at least that of $a$-LDC.

Consider a request causing the slower algorithm, $a$-LDC, to move a server away from $C$.

If $b$-LDC also moves a server away from $C$ at this point, both algorithms have their servers on $A$ and $B$, and the two sequences continue identically until the faster algorithm again moves a server away from $C$ (before or at the same time as the slower algorithm does).

If $b$-LDC does not move a server away from $C$ at this point, since, by construction, it does make a move, it moves a server from $A$ to $B$. Thus, the next time both algorithms move a server, $a$-LDC moves from $B$ to $C$ and $b$-LDC moves from $B$ to $A$. Then both algorithms have servers on $A$ and $C$. Since $a$-LDC has just moved a server to $C$, whereas $b$-LDC must have made at least one move from $A$ to $B$ since it placed a server at $C$, $b$-LDC must, as the faster algorithm, make its next move away from $C$ strictly before $a$-LDC does so. In conclusion, the sequences will be identical until the faster algorithm, $b$-LDC, moves a server away from $C$.

\[\Box\]

**Theorem 6** According to Bijective Analysis and Average Analysis, slower variants of LDC are better than faster variants for the baby server problem.

**Proof** Follows immediately from Lemma 1 and the definition of the measures.

\[\Box\]

Thus, the closer a variant of LDC is to GREEDY, the better Bijective and Average Analysis predict that it is.

## 7 Relative Worst Order Analysis

Let $I_\mathcal{A}$ denote a worst ordering of the sequence $I$ for the algorithm $\mathcal{A}$.

Similarly to bijective analysis, relative worst order analysis correctly distinguishes between DC and LDC, indicating that the latter is the better algorithm. This follows from the following general theorem about lazy algorithms, and the fact that there are some sequences where one of DC’s
servers repeatedly moves from $C$ towards $B$, but moves back to $C$ before ever reaching $B$, while Ldc’s server stays on $C$. If $d$ is just marginally larger than some integer, even on Ldc’s worst ordering of this sequence, it does better than Dc.

**Theorem 7** The lazy version of any algorithm for the baby server problem is at least as good as the original algorithm according to Relative Worst Order Analysis.

**Proof** By Proposition 2 for any request sequence $I$, $\mathcal{L}A(I_{L\mathcal{A}}) \leq \mathcal{A}(I_{L\mathcal{A}}) \leq \mathcal{A}(I_{L\mathcal{A}})$. □

**Theorem 8** Greedy and Ldc are $(\infty, 2)$-related and are thus weakly comparable in Ldc’s favor for the baby server problem according to Relative Worst Order Analysis.

**Proof** First we show that $c_u(\text{Greedy}, \text{Ldc})$ is unbounded. Consider the sequence $(BA)^n$. As $n$ tends to infinity, Greedy’s cost is unbounded, whereas Ldc’s cost is at most $3d$ for any permutation.

Next we turn to $c_u(\text{Ldc, Greedy})$. Since the competitive ratio of Ldc is 2, for any sequence $I$, there exists a constant $b$ such that $\text{Ldc}(I_{\text{Ldc}}) \leq 2\text{Greedy}(I_{\text{Ldc}}) + b \leq 2\text{Greedy}(I_{\text{Greedy}}) + b$. Thus, $c_u(\text{Ldc, Greedy}) \leq 2$.

For the lower bound of 2, consider the family of sequences $I_p = (BABA...BC)^p$, where the length of the alternating $A/B$-sequence before the $C$ is $2 \lfloor d \rfloor + 1$.

$Ldc(I_p) = p(2 \lfloor d \rfloor + 2d)$. An worst ordering for Greedy alternates $A$s and $B$s. Since there is no cost for the $C$s and the $A/B$ sequences start and end with $B$s, $\text{Greedy}(\sigma(I_p)) \geq p(2 \lfloor d \rfloor) + 1$ for any permutation $\sigma$.

Then, $c_u(\text{Ldc, Greedy}) \geq \frac{p(2 \lfloor d \rfloor) + 2d}{p(2 \lfloor d \rfloor) + 1} \geq \frac{p(4d)}{p(2d) + 1}$. As $p$ goes to infinity, this approaches 2.

Thus, Greedy and Ldc are weakly comparable in Ldc’s favor. □

Recalling the definition of $\alpha$-Ldc, a request for $B$ is served by the right-most server if it is within a virtual distance of no more than $\alpha$ from $B$. Thus, when the left-most server moves and its virtual move is over a distance of $l$, then the right-most server virtually moves a distance $\alpha l$. When the right-most

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server moves and its virtual move is over a distance of \( al \), then the left-most server virtually moves a distance of \( l \).

In the results that follow, we frequently look at the worst ordering of an arbitrary sequence.

**Definition 7** The canonical worst ordering of a sequence, \( I \), for algorithm \( a \) is the sequence produced by allowing the cruel adversary (the one which always lets the next request be the unique point where \( A \) does not currently have a server) to choose requests from the multiset defined from \( I \). This process continues until there are no requests remaining in the multiset for the point where \( A \) does not have a server. The remaining points from the multiset are concatenated to the end of this new request sequence in any order.

It is easy to see what the canonical worst ordering of a sequence for \( a \)-LDC is.

**Proposition 3** Consider an arbitrary sequence \( I \) containing \( n_A \) As, \( n_B \) Bs, and \( n_C \) Cs. A canonical worst ordering of \( I \) for \( a \)-LDC is \( I_a = (BABA...BC)^{p_a} X \), where the length of the alternating \( A/B \)-sequence before the \( C \) is \( 2 \left\lfloor \frac{a}{d_a} \right\rfloor + 1 \). Here, \( X \) is a possibly empty sequence. The first part of \( X \) is an alternating sequence of As and Bs, starting with a B, until there are not both As and Bs left. Then we continue with all remaining As or Bs, followed by all remaining Cs. Finally,

\[
p_a = \min \left\{ \left\lfloor \frac{n_A}{\left\lfloor \frac{a}{d_a} \right\rfloor} \right\rfloor, \left\lfloor \frac{n_B}{\left\lfloor \frac{a}{d_a} \right\rfloor} + 1 \right\rfloor, n_C \right\}.
\]

**Lemma 2** Let \( I_a \) be the canonical worst ordering of \( I \) for \( a \)-LDC. \( I_a \) is a worst permutation of \( I \) for \( a \)-LDC, and the cost for \( a \)-LDC on \( I_a \) is \( c_a \), where

\[
p_a(2 \left\lfloor \frac{a}{d_a} \right\rfloor + 2d) \leq c_a \leq p_a(2 \left\lfloor \frac{a}{d_a} \right\rfloor + 2d) + 2 \left\lfloor \frac{a}{d_a} \right\rfloor + d.
\]

**Proof** Consider a request sequence, \( I \). Between any two moves from \( B \) to \( C \), there must have been a move from \( C \) to \( B \). Consider one such move. Between the last request to \( C \) and this move, the other server must move from \( A \) to \( B \) \( \left\lfloor \frac{a}{d_a} \right\rfloor + 1 \) times, which requires some first request to \( B \) in this subsequence, followed by at least \( \left\lfloor \frac{a}{d_a} \right\rfloor \) occurrences of requests to \( A \), each followed by a request to \( B \). (Clearly, extra requests to \( A \) or \( B \) could also occur, either causing moves or not.) Thus, for every move from \( B \) to \( C \),
there must be at least \( \left\lfloor \frac{d}{a} \right\rfloor + 1 \) Bs, \( \left\lfloor \frac{d}{a} \right\rfloor \) As and one C. Thus, the number of moves from B to C is bounded from above by \( p_a \). There can be at most one more move from C to B than from B to C. If such a move occurs, there are no more Cs after that in the sequence. Therefore, the sequences defined above give the maximal number of moves of distance \( d \) possible. More As or Bs in any alternating A/B-sequence would not cause additional moves (of either distance 1 or \( d \)), since each extra point requested would already have a server. Fewer As or Bs between two Cs would eliminate the move away from C before it was requested again. Thus, the canonical worst ordering is a worst ordering of I.

Within each of the \( p_a \) repetitions of (BABABABA...BC), each of the requests for A and all but the last request for B cause a move of distance 1, and the last two requests each cause a move of distance \( d \), giving the lower bound on \( c_a \). Within X, each of the first \( 2 \left\lfloor \frac{d}{a} \right\rfloor \) requests could possibly cause a move of distance 1, and this could be followed by a move of distance \( d \). After that, no more moves occur. Thus, adding costs to the lower bound gives the upper bound on \( c_a \).

**Theorem 9** If \( a \leq b \), then \( a\text{-LDC} \) and \( b\text{-LDC} \) are \( (2 \left\lfloor \frac{d}{a} \right\rfloor + 2d) n_c \)-related for the baby server problem according to Relative Worst Order Analysis.

**Proof** By Lemma 2 in considering \( a\text{-LDC} \)’s performance in comparison with \( b\text{-LDC} \)’s, the asymptotic ratio depends only on the values \( p_a \) and \( p_b \) defined for the canonical worst orderings \( I_a \) and \( I_b \) for \( a\text{-LDC} \) and \( b\text{-LDC} \), respectively. Since \( a > b \), the largest value of \( \frac{p_a}{p_b} \) occurs when \( p_a = n_c \), since more Cs would allow more moves of distance \( d \) by \( b\text{-LDC} \). Since the contribution of X to \( a\text{-LDC} \)’s cost can be considered to be a constant, we may assume that \( n_A = n_C \left\lfloor \frac{d}{a} \right\rfloor \) and \( n_B = n_C \left( \left\lfloor \frac{d}{a} \right\rfloor + 1 \right) \).

When considering \( b\text{-LDC} \)’s canonical worst ordering of this sequence, all the Cs will be used in the initial part. By Lemma 2 we obtain the following ratio, for some constant \( c \):

\[
\frac{(2 \left\lfloor \frac{d}{a} \right\rfloor + 2d) n_c}{(2 \left\lfloor \frac{d}{b} \right\rfloor + 2d) n_c + c}
\]

Similarly, a sequence giving the largest value of \( \frac{p_a}{p_b} \) will have \( p_b = \frac{n_A}{\left\lfloor \frac{d}{a} \right\rfloor} \), since more As would allow \( a\text{-LDC} \) to have a larger \( p_a \). Since the contribution
of $X$ to $b$-LDC can be considered to be a constant, we may assume that $n_A = n_C \left\lfloor \frac{d}{b} \right\rfloor$, $n_B = n_C(\left\lfloor \frac{d}{b} \right\rfloor + 1)$, and $p_b = n_C$.

Now, when considering $a$-LDC’s worst permutation of this sequence, the number of periods, $p_a$, is restricted by the number of $A$s. Since each period has $\left\lfloor \frac{d}{a} \right\rfloor A$s, $p_a = \left\lfloor n_C \frac{d}{a} \right\rfloor$. After this, there are a constant number of $A$s remaining, giving rise to a constant additional cost $c'$.

Thus, the ratio is the following:

$$\frac{(2 \left\lfloor \frac{d}{b} \right\rfloor + 2d)n_C}{(2 \left\lfloor \frac{d}{a} \right\rfloor + 2d)n_C} + c'$$

Considering the above ratios asymptotically as $n_C$ goes to infinity, we obtain that $a$-LDC and $b$-LDC are $(\left\lfloor \frac{d}{a} \right\rfloor + d, \left\lfloor \frac{d}{b} \right\rfloor + d, 2)-related.$

We provide strong indication that LDC is better than $b$-LDC for $b \neq 1$. If $b > 1$, this is always the case, whereas if $b < 1$, it holds in many cases, including all integer values of $d$.

**Theorem 10** Consider the baby server problem evaluated according to Relative Worst Order Analysis. For $b > 1$, if LDC and $b$-LDC behave differently, then they are $(r, r_b)$-related, where $1 < r < r_b$. If $a < 1$, $a$-LDC and LDC behave differently, and $d$ is a positive integer, then they are $(r_a, r)$-related, where $1 < r_a < r$.

**Proof** By Theorem 11 $a$-LDC and $b$-LDC are $(\left\lfloor \frac{d}{a} \right\rfloor + d, \left\lfloor \frac{d}{b} \right\rfloor + d, 2)-related.$

Subtracting these two values and comparing to zero, we get

$$\left(\left\lfloor \frac{d}{a} \right\rfloor + d - \left(\left\lfloor \frac{d}{b} \right\rfloor + d\right)\right) < 0$$

$$\Updownarrow$$

$$E = \left(\left\lfloor \frac{d}{a} \right\rfloor + d\right)\left(\left\lfloor \frac{d}{a} \right\rfloor \left\lfloor \frac{d}{b} \right\rfloor - (\left\lfloor \frac{d}{a} \right\rfloor + d)(\left\lfloor \frac{d}{b} \right\rfloor)\right) < 0$$

We can rewrite $E = \left(\left\lfloor \frac{d}{a} \right\rfloor - \left\lfloor \frac{d}{b} \right\rfloor\right)\left(\left\lfloor \frac{d}{a} \right\rfloor \left\lfloor \frac{d}{b} \right\rfloor - d^2\right)$.

Observe that the first term is positive when $b > a$ and the algorithms behave differently.
We now analyze LDC compared with variants of other speeds. First, assume that \( a = 1 \) and \( b > 1 \).

Since the algorithms are different, \( \left\lfloor \frac{d}{b} \right\rfloor \leq \left\lfloor \frac{d}{a} \right\rfloor - 1 \). Thus, the second factor, \( \left\lfloor \frac{d}{a} \right\rfloor \left\lfloor \frac{d}{b} \right\rfloor - d^2 \), is bounded from above by \( d(d - 1) - d^2 < 0 \). Hence, if LDC and \( b\text{-LDC} \) are \((c_1, c_2)\)-related, then \( c_1 < c_2 \).

Now, assume that \( a < 1 \) and \( b = 1 \).

Again, the algorithms are different, so \( \left\lfloor \frac{d}{b} \right\rfloor \leq \left\lfloor \frac{d}{a} \right\rfloor - 1 \).

Note that if \( \left\lfloor \frac{d}{a} \right\rfloor \left\lfloor \frac{d}{b} \right\rfloor > d^2 \), then \( a\text{-LDC} \) and LDC are \((c_1, c_2)\)-related, where \( c_2 > c_1 \). This holds when \( d \) is a positive integer.

There is an indication that \( a\text{-LDC} \) and \( \frac{1}{a}\text{-LDC} \) are in some sense of equal quality:

**Corollary 3** When \( \frac{d}{a} \) and \( \frac{d}{b} \) are integers, then \( a\text{-LDC} \) and \( b\text{-LDC} \) are \((b, b)\)-related when \( b = \frac{1}{a} \).

We now set out to prove that LDC is an optimal algorithm in the following sense: there is no other algorithm \( A \) such that LDC and \( A \) are comparable and \( A \) is strictly better or such that LDC and \( A \) are weakly comparable in \( A \)'s favor.

**Theorem 11** LDC is optimal for the baby server problem according to Relative Worst Order Analysis.

**Proof** Since the competitive ratio of LDC is 2, for any algorithm \( A \) and any sequence \( I \), there is a constant \( b \) such that \( \text{LDC}(I_{\text{LDC}}) \leq 2A(I_{\text{LDC}}) + b \leq 2\text{A}(I_{A}) + b \). Thus, \( c_u(LDC, A) \leq 2 \), so \( A \) is not weakly comparable to LDC in \( A \)'s favor.

For \( A \) to be a better algorithm than LDC according to Relative Worst Order Analysis, it has to be comparable to LDC and perform more than an additive constant better on some sequences (which then necessarily must be an infinite set).

Assume that there exists a family of sequences \( S_1, S_2, \ldots \) such that for any constant \( c \) there exists an \( i \) such that \( \text{LDC}_W(S_i) \geq \text{A}_W(S_i) + c \).

Then we prove that there exists another family of sequences \( S'_1, S'_2, \ldots \) such that for any constant \( c \) there exists an \( i \) such that \( \text{A}_W(S'_i) \geq \text{LDC}_W(S'_i) + c \).
This establishes that if \( A \) performs more than a constant better on its worst permutations of some family of sequences than LDC does on its worst permutations, then there exists a family where LDC has a similar advantage over \( A \) which implies that the algorithms are not comparable.

Now assume that we are given a constant \( c \). Since we must find a value greater than any \( c \) to establish the result, we may assume without loss of generality that \( c \) is large enough that
\[
3dc \geq \left\lfloor \frac{3d+1}{d} \right\rfloor (\lfloor d \rfloor + 2d) + 3d.
\]

Consider a sequence \( S \) from the family \( S_1, S_2, \ldots \) such that \( \text{LDC}_{W}(S) \geq A_{W}(S) + 3dc \). From \( S \) we create a member \( S' \) of the family \( S'_1, S'_2, \ldots \) such that \( A_{W}(S') \geq \text{LDC}_{W}(S') + c \).

Assume that \( S \) contains \( n_A \) As, \( n_B \) Bs, and \( n_C \) Cs.

The idea behind the construction is to allow the cruel adversary to choose points from the multiset defined from \( S \) as in the definition of canonical worst orderings. This process continues until the cruel adversary has used all of either the As, Bs, or Cs in the multiset, resulting in a sequence \( S' \). If the remaining points from the multiset are concatenated to \( S' \) in any order, this creates a permutation of \( S \). The performance of \( A \) on this permutation must be at least as good as its performance on its worst ordering of \( S \).

We now consider the performance of LDC and \( A \) on \( S' \) and show that LDC is strictly better.

Let \( n'_A, n'_B, \) and \( n'_C \) denote the number of As, Bs, and Cs in \( S' \), respectively.

Let \( p = \min \left\{ \left\lfloor \frac{n'_A}{d} \right\rfloor, \left\lfloor \frac{n'_B}{d+1} \right\rfloor, n'_C \right\} \).

By Lemma 2 the cost of LDC on its canonical worst ordering of \( S' \) is at most \( p(2\lfloor d \rfloor + 2d) + 3d \).

The cost of \( A \) is \( 2dn'_C + n'_A + n'_B - n'_C \), since every time there is a request for \( C \), this is because a server in the step before moved away from \( C \). These two moves combined have a cost of \( 2d \). Every request to an \( A \) or a \( B \) has cost one, except for the request to \( B \) immediately followed by a request to \( C \), which has already been counted in the \( 2dn'_C \) term. A similar argument shows that LDC’s cost is bounded from above by the same term.

Assume first that \( \frac{n'_A}{d} = \frac{n'_B}{d+1} = n'_C \). Then \( S' \) can be permuted so that it is a prefix of LDC’s canonical worst ordering on \( S \) (see Lemma 2 with \( a = 1 \)). Since, by construction, we have run out of either As, Bs, or Cs (that is, one type is missing from \( S \) minus \( S' \) as multisets), LDC’s cost on its worst ordering of \( S \) is at most its cost on its worst ordering on \( S' \) plus \( 3d \). Thus, \( \text{LDC}_{W}(S_i) \geq A_{W}(S_i) + c \) does not hold in this case, so we may assume that
these values are not all equal.

Case $n_A' > \lfloor d \rfloor p$:
LDC’s cost on $S'$ is at most the cost of $A$ minus $(n_A' - \lfloor d \rfloor p)$ plus $3d$.

Case $n_B' > (\lfloor d \rfloor + 1)p$:
LDC’s cost on $S'$ is at most the cost of $A$ minus $(n_B' - (\lfloor d \rfloor + 1)p)$ plus $3d$.

Case $n_C' > p$:
LDC’s cost on $S'$ is at most the cost of $A$ minus $(2d - 1)(n_C' - p)$ plus $1$.

We compare LDC’s canonical worst orderings of $S$ and $S'$. In both cases, the form is as in Lemma 2, with $a = 1$. Thus, for $S'$ the form is $(BA)^{\lfloor d \rfloor BC}X$, and for $S$, it is $((BA)^{\lfloor d \rfloor BC})^l Y$ for some positive integer $l$. The string $X$ must contain all of the $As$, all of the $Bs$, and/or all of the $Cs$ contained in $((BA)^{\lfloor d \rfloor BC})^l$, since after this the cruel adversary has run out of something. Thus, it must contain at least $l \lfloor d \rfloor$ $As$, $l(\lfloor d \rfloor + 1)$ $Bs$ or $l$ $Cs$. The extra cost that LDC has over $A$ on $S$ is at most its cost on $((BA)^{\lfloor d \rfloor BC})^l Y$ minus cost $l \lfloor d \rfloor$ for the $As$, $Bs$ or $Cs$ contained in $X$, so at most $l(2 \lfloor d \rfloor + 2d) - l \lfloor d \rfloor + 3d = l(\lfloor d \rfloor + 2d) + 3d$.

Thus, $LDC_W(S) - A_W(S) \leq l(\lfloor d \rfloor + 2d) + 3d$ while $A_W(S') - LDC_W(S') \geq l \lfloor d \rfloor - 3d$.

From the choice of $c$ and the bound just derived, we find that

$$\frac{3d + 1}{\lfloor d \rfloor - 1} (\lfloor d \rfloor + 2d) + 3d \leq 3dc \leq LDC_W(S) - A_W(S) \leq l(\lfloor d \rfloor + 2d) + 3d$$

Thus, $l \geq \frac{3d + 1}{\lfloor d \rfloor - 1}$, which implies the following:

$$l \geq \frac{3d + 1}{\lfloor d \rfloor - 1}$$

$\uparrow$

$$l(\lfloor d \rfloor - 1) \geq 3d + 1$$

$\uparrow$

$$l \lfloor d \rfloor - 3d \geq l + 1$$

$\uparrow$

$$l \lfloor d \rfloor - 3d \geq \frac{l(\lfloor d \rfloor + 2d) + 3d}{3d}$$

Now,

$$A_W(S') - LDC_W(S') \geq l \lfloor d \rfloor - 3d \geq \frac{l(\lfloor d \rfloor + 2d) + 3d}{3d} \geq \frac{3dc}{3d} = c.$$
The proof of optimality only depends on the form of the canonical worst orderings for the optimal algorithm and the costs associated with these. More specifically, the canonical worst orderings produced using the cruel adversary are worst orders, and these sequences have the form \( R \cdot Q_1 \cdot Q_2 \ldots Q_k \cdot X \), where \( R \) has constant cost, \( Q_i \) has the form \((BA)^{r_i}BC\) with \( l \leq r_i \leq l + 1 \) for some integer \( l \), and \( X \) cannot include enough \( A \)'s, \( B \)'s and \( C \)'s so that the sequence could instead have been written \( R \cdot Q_1 \cdot Q_2 \ldots Q_k \cdot Q_{k+1} \cdot X' \) for some \( Q_{k+1} \) having the same form as the others and for some \( X' \). Since the cruel adversary is used, the patterns \( Q_i \) are uniquely defined by the algorithm and every request in each of them has a cost. In the above proof, all the \( Q_i \) were identical. As an example of where they are not all identical, consider \( d = \frac{2}{3} \), where the canonical worst ordering for \( BAL \) has the following prefix:

\[
BABC(BABABABCBABABC)^*.
\]

Note that this implies that \( a-Ldc \) and \( BAL \) are also optimal algorithms.

In the definitions of \( Ldc \) and \( BAL \) given in Section \( a \), different decisions are made as to which server to use in cases of ties. In \( Ldc \) the server which is physically closer is moved in the case of a tie (equal virtual distances from the point requested). The rationale behind this is that the server which would have the least cost is moved. In \( BAL \) the server which is further away is moved to the point. The rationale behind this is that, since \( d > 1 \), when there is a tie, the total cost for the closer server is already significantly higher than the total cost for the other, so moving the server which is further away evens out how much total cost they have, at least temporarily. With these tie-breaking decisions, the two algorithms behave very similarly when \( d \) is an integer.

**Theorem 12** \( Ldc \) and \( BAL \) are not comparable on the baby server problem with respect to Relative Worst Order Analysis, except when \( d \) is an integer, in which case they are equivalent.

**Proof** One can see that a canonical worst ordering of a sequence \( I \) for \( BAL \) has the following prefix: \( R \cdot Q_1 \cdot Q_2 \ldots Q_k \), where for all \( i, Q_i = (BA)^{r_i}BC \), where \( \lfloor d \rfloor \leq r_i \leq \lceil d \rceil \) and \( R \) contains at most two \( C \)'s.

Suppose that \( d \) is an integer. Consider any request sequence \( I \). \( Ldc \)'s canonical worst ordering has the prefix \((BA)^dBC\)^{\( k \)}, while \( BAL \)'s canonical worst ordering has the prefix \( BABC(BA)^{\lfloor \frac{d}{2} \rfloor + 1}BC((BA)^dBC)^{k'} \). These prefixes of \( Ldc \)'s and \( BAL \)'s canonical worst orderings of \( I \) are identical,
except that $R$ is empty for Ldc, while Bal has $R = BABC(BA)^{⌊\frac{d}{2}⌋−1}BC$. This leads to some small difference at the end, where Bal might have at most two fewer of the $(BA)^dBC$ repetitions than Ldc. Thus, their performance on their respective worst orderings will be identical up to an additive constant due to the differences at the start and end of the prefixes.

Now consider the case where $d$ is not an integer. Following the proof of Theorem 11, consider a sequence, $I$, where the canonical worst ordering for Bal consists entirely of the prefix form described above. Since $d$ is not an integer, not all of the $Q_i$s will have the same length. The difference from Ldc’s canonical worst ordering is that there are some $Q_i$ of the form $(BA)^{⌈d⌉}BC$, so in their respective prefixes, Bal has more As and Bs, compared to the number of Cs, than Ldc does. This means that Ldc will have a lower cost on its worst ordering. However, the sequence, $I'$, produced by letting the cruel adversary for Ldc choose points from the multiset defined by this sequence, $I$, has all $Q_i = (BA)^{⌊d⌋}BC$. Bal’s canonical worst ordering of $I'$ will have fewer $Q_i$s because there will not be enough As to make the longer repetitions required. Therefore, the two algorithms are incomparable when $d$ is not an integer.

8 Open Questions

It appears that both Bijective Analysis and Relative Worst Order Analysis have the advantage over Competitive Analysis that if one algorithm does at least as well as another on every sequence, then they decide in favor of the better algorithm. This works for the lazy vs. non-lazy version of the baby server problem, and also for some paging problems—LRU vs. FWF and look-ahead vs. no look-ahead. See [7] and [5] with regards to these results on Bijective Analysis and Relative Worst Order Analysis, respectively.

However, Bijective Analysis does not reflect the fact that when comparing on the same sequence, even though Greedy can be a factor two better than Lazy Double Coverage, LDC is sometimes much better than Greedy for the baby server problem. Similarly, according to Bijective Analysis, LIFO and LRU are equivalent for paging, but LRU is often significantly better than LIFO, which keeps the first $k - 1$ pages it sees in cache forever.

In both of these cases, Relative Worst Order Analysis says that the algorithms are weakly comparable in favor of the “better” algorithm. It would be interesting to see if there are problems where Bijective Analysis gives better results than Relative Worst Order Analysis. For instance, can local-
ity of reference be added to Relative Worst Order Analysis and succeed in separating LRU from FIFO, as one can by adding locality of reference to Bijective Analysis \[1\]?

Is it possible to develop a set of general guidelines to determine which of the measures studied here is best used in which context?

So far there seems to be basically one technique for proving results in Bijective Analysis, and it works when you can compare two algorithms and say that among all the next possible requests, the one algorithm has at least as many low cost requests as the other. This works for demand paging \[3\], since all such algorithms have the same number of pages in the cache and the same number currently not in the cache. This works for Greedy vs. other algorithms for our problem. And this works for list accessing \[7\], since each algorithm has exactly one item with each possible cost in the list. It seems hard to apply this to bin packing or even to more complicated \(k\)-server problems. It would be interesting to come up with a new technique.

Finally, we have investigated the smallest possible server problem in order to evaluate the differences between various measures. We believe, we would obtain the same classifications on more general server problems, but generalizations of some of the results presented here would require significant technical effort. It would also be interesting to prove optimality of some algorithm for a more general problem.

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