Black holes as effective geometries

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Abstract
Gravitational entropy arises in string theory via coarse graining over an underlying space of microstates. In cases with enough supersymmetry, it has been possible to explicitly construct such microstates in spacetime and understand how coarse graining of non-singular, horizon-free objects can lead to an effective description as an extremal black hole. We discuss how these results arise in type II string theory on AdS\(_5\) \times S\(_5\) and on AdS\(_3\) \times S\(^3\) \times T\(^4\) that preserve 16 and eight supercharges, respectively. For such a picture of black holes as effective geometries to extend to cases with a finite horizon area, the scale of quantum effects in gravity would have to extend well beyond the vicinity of the singularities in the effective theory. By studying examples in M-theory on AdS\(_3\) \times S\(^2\) \times \text{CY} that preserve four supersymmetries, we show how this can happen.

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1. Introduction

A spacetime geometry can carry an entropy in string theory via coarse graining over an underlying set of microstates. Since the initial success of string theory in accounting for the entropy of supersymmetric black holes by counting states in a field theory [1], there has been an ongoing effort to understand exactly what the structure of these microstates is and how they manifest themselves in gravity. It has been shown that in examples with enough supersymmetry, including some extremal black holes, one can construct a basis of 'coherent' microstates whose spacetime descriptions in the $\hbar \to 0$ limit approach non-singular, horizon-free geometries which resemble a topologically complicated 'foam'. Conversely, in these cases the quantum Hilbert space of states can be constructed by directly quantizing a moduli space of smooth classical solutions. Nevertheless, the typical states in these Hilbert spaces respond to semiclassical probes as if the underlying geometry was singular, or an extremal black hole. In this sense, these black holes are effective, coarse-grained descriptions of underlying non-singular, horizon-free states. We discuss how these results arise for states in type II string theory on $\text{AdS}_5 \times S^5$ and on $\text{AdS}_3 \times S^3 \times T^4$ that preserve 16 and eight supercharges, respectively. We also discuss the connection between ensembles of microstates and coarse-grained effective geometries. Such results suggest the idea that all black-hole geometries in string theory, even those with a finite horizon area, can be seen as the effective coarse-grained descriptions of complex underlying horizon-free states which have an extended spacetime

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3 It is important to note that the notion of state here is quite general and need not be restricted to semiclassical states or states accessible in supergravity alone. Thus the notion of horizon-free must also be generalized to mean entropy-less.
structure [2, 3]. This idea seems initially unlikely because one might expect that the quantum effects that correct the classical black-hole spacetime would be largely confined to regions of high curvature near the singularity, and would thus not modify the horizon structure. To study this we examine states of M-theory on $\text{AdS}_3 \times S^2 \times \text{CY}$ with four supercharges, where a finite horizon area can arise. We work with a large class of these states whose spacetime descriptions are accessible using split attractor flows and give rise to ‘long throats’ of the kind needed to give effective black-hole behaviour. These states are related to distributions of D-branes in spacetime. Surprisingly, it turns out that the quantized solution space has large fluctuations even at macroscopic proper distances, suggesting that the scale of quantum effects in gravity could extend beyond the vicinity of singularities in the effective theory. Thus, the idea that all black holes might simply be effective descriptions of underlying horizon-free objects tentatively survives this test.

1.1. Background

In string theory, black holes can often be constructed by wrapping $D$-branes on cycles in a compact manifold $X$ so they appear as point-like objects in the spatial part of the non-compact spacetime, $\mathbb{R}^{1,d-1}$. As the string coupling is increased, these objects backreact on spacetime and can form supersymmetric spacetimes with macroscopic horizons. The entropy associated with these objects can be determined ‘microscopically’ by counting BPS states in a field theory living on the branes, and this has been shown in many cases to match the count expected from the horizon area (see [1, 4] for the prototypical calculations). Although the field-theory description is only valid for very small values of the string coupling $g_s$, the fact that the entropy counting in the two regimes coincides can be attributed to the protected nature of BPS states that persist in the spectrum at any value of the coupling unless a phase transition occurs or a wall of stability is crossed. The fact that the (leading) contribution to the entropy of the black hole could be reproduced from counting states in a sector of the field theory suggests that the black-hole microstates dominate the entropy in this sector.

While it is very helpful that these states can be counted at weak coupling, understanding the nature of these states in gravity at finite coupling remains an open problem. As $g_s$ is increased the branes couple to gravity and we expect them to start backreacting on the geometry. The main tools we have to understand the spacetime or closed-string picture of the system are the AdS/CFT correspondence and the field-theory description of D-branes.

Within the framework of the AdS/CFT correspondence black holes with near-horizon geometries of the form $\text{AdS}_m \times S^n$ must correspond to objects in a dual conformal field theory that have an associated entropy. A natural candidate is a thermal ensemble or density matrix, in the CFT, composed of individual pure states (see, e.g., [5]). AdS/CFT then suggests that there must be corresponding pure states in the closed string picture and that these would comprise the microstates of the black hole. It is not clear, however, that such states are accessible in the supergravity description. First, the dual objects should be closed string states and may not admit a classical description. Even if they do admit a classical description they may involve regions of high curvature and hence be inherently stringy. For BPS black holes, however, we may restrict to the BPS sector in the Hilbert space where the protected nature of the states suggests that they should persist as we tune continuous parameters (barring phase transitions 

\[4 \text{ More generally objects with horizons, microscopic or macroscopic, are expected to have an associated entropy which should manifest itself in the CFT.}

\[5 \text{ Here ‘BPS’ can mean either 1/2, 1/4 or 1/8 BPS states or black holes in the full string theory. The degree to which states are protected depends on the amount of supersymmetry that they preserve and our general remarks should always be taken with this caveat.} \]
transitions or wall crossings). We may then hope to see a supergravity manifestation of these states (and indeed this turns out to be the case for systems with sufficient supersymmetry). However, the large \( N \) limit, which must be taken for supergravity to be a valid description, bears many similarities with the \( \hbar \rightarrow 0 \) limit in quantum mechanics where we know that most states do not have a proper classical limit. As we will see, if a supergravity description can be obtained at all, it will only be for appropriately ‘semiclassical’ or ‘coherent’ states.

Even if this is the case and some CFT states are dual to smooth geometries these may not be distinguishable by their profile in the non-compact direction alone, and much of the interesting geometry may reside in the backreaction of the compactification manifold (which is generally quite complicated). Despite these potential problems, recently, a very fruitful programme has been undertaken to explore and classify the smooth supergravity duals of the CFT states making up the black-hole ensemble. Smoothness here is important because if these geometries exhibit singularities we expect these to either be resolved by string-scale effects, making them inaccessible in supergravity, or enclosed by a horizon implying that the geometry corresponds, not to a pure state, but rather an ensemble with some associated entropy.

Large classes of such smooth supergravity solutions, asymptotically indistinguishable from black-hole solutions, have indeed been found [6–13] (and related to previously known black-hole composites [14, 15]). These are complete families of solutions preserving a certain amount of supersymmetry with fixed asymptotic charges and with no (or very mild) singularities.

In constructing such solutions it has often been possible to start with a suitable probe brane solution with the correct asymptotic charges in a flat background and to generate a supergravity solution by backreacting the probe [6, 9, 16]. In a near-horizon limit these backreacted probe solutions are asymptotically AdS, and by identifying the operator corresponding to the probe and the state it makes in the dual CFT, the backreacted solution can often be understood as the spacetime realization of an appropriately coherent microstate in the CFT. Lin, Lunin and Maldacena [6] showed that the backreaction of such branes (as well their transition to flux) was identified with a complete set of asymptotically AdS\(_5\) supergravity solutions (as described above) suggesting that the latter should be related to 1/2 BPS states of the original D3 probes generating the geometry. Indeed, in [17, 18], it was shown that quantizing the space of such supergravity solutions as a classical phase space reproduces the spectrum of BPS operators in the dual \( \mathcal{N} = 4 \) superconformal Yang–Mills (at \( N \rightarrow \infty \)).

In a different setting Lunin and Mathur [8] were able to construct supergravity solutions related to configurations of a D1–D5 brane in six dimensions (i.e., compactified on a \( T^4 \)) by utilizing dualities that relate this system to an F1–P system (see also [7]). The latter system is nothing more than a BPS excitation of a fundamental string quantized in a flat background. The back reaction of this system can be parametrized by a profile \( F_i(z) \) in \( \mathbb{R}^4 \) (the transverse directions). T-duality relates configurations of this system to that of the D1–D5 system.

Recall that the naive backreaction of a bound state of D1–D5 branes is a singular or ‘small’ black hole in five dimensions. The geometries arising from the F1–P system, on the

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6 \( N \) measures the size of the system. For black holes it is usually related to mass in the bulk and conformal weight in the CFT.

7 The question of which asymptotic charges of the microstates should match those of the black hole is somewhat subtle and depends on which ensemble the black hole is in. In principle some of the asymptotic charges might be traded for their conjugate potentials. Moreover, the solutions will, in general, only have the same isometries asymptotically.

8 Throughout this paper we will be discussing ‘microstates’ of various objects in string theory but the objects will not necessarily be holes (i.e., spherical horizon topology) nor will they always have a macroscopic horizon. In fact, there is no 1/2 BPS solution in AdS\(_5 \times S^5\) with any kind of a horizon. We will, nonetheless, somewhat carelessly continue to refer to these as ‘microstates’ of a black hole for the sake of brevity.
other hand, are smooth after dualizing back to the D1–D5 frame, though they have the same asymptotics as the naive solution [7]. Each \( F_{1–P} \) curve thus defines a unique supergravity solution with the same asymptotics as the naive \( D1–D5 \) black hole but with a different subleading structure. Smoothness of these geometries led Lunin and Mathur to propose that these solutions should be mapped onto individual states of the \( D1–D5 \) CFT. The logic of this idea was that individual microstates do not carry any entropy, and hence should be represented in spacetime by configurations without horizons. Lunin and Mathur also conjectured that the naive black-hole geometry is somehow a coarse graining over all these smooth solutions, i.e., that the black hole itself is simply an effective, coarse-grained description. This idea is sometimes termed as the fuzz-ball proposal, though to make this precise requires a more careful statement as we discuss in section 1.2.

The focus of these proceedings will be \([19–28]\), which use well-controlled supersymmetric examples to explore the idea that black holes might be simply ‘effective geometries’, i.e., that they are effective coarse-grained description of underlying horizon-free objects. This will also involve understanding the nature of typical black-hole microstates and how they may be resolved by probes \([19]\) as this is an integral part of the information loss paradox. The discussion will involve several different systems ranging from \(1/2\) BPS states in \(AdS_5 \times S^5\) to the \(1/4\) BPS states of the \(D1–D5\) system to the least controlled \(1/8\) BPS case where we will study bound multicentre configurations in four and five dimensions. Only in the final case will genuine macroscopic horizons emerge but the \(1/2\) and \(1/4\) BPS cases are under more technical control and hence important to study. In all cases we will try to understand how the BPS spectrum emerges in supergravity, how it is related to the BPS spectrum of the CFT (or more generally the brane theory) and how such states might contribute to the ensembles characterizing black holes in string theory. Much of what will be discussed is a review of work by other authors \([2, 6, 9, 11, 13, 14, 17]\) (see also the other references cited earlier and throughout the remainder of the text).

As the related literature is voluminous and complicated we attempt to provide, in section 7, a brief survey of the various works and which branches of the field they fit into. This survey is by no means exhaustive and no doubt neglects many important works, but we feel it can, nonetheless, serve as a useful reference point for readers attempting to orient themselves within the field.

1.2. Some answers to potential objections

The idea that black holes are simply effective descriptions of underlying horizon-free objects is confusing because it runs counter to well-established intuitions in effective field theory, most importantly the idea that near the horizon of a large black hole the curvatures are small and hence so are the effects of quantum gravity. Indeed, it is not easy to formulate a precisely stated conjecture for black holes with a finite horizon area, although for extremal black holes with enough supersymmetry a substantial amount of evidence has accumulated for the correctness of the picture, as reviewed in this paper. To clarify some potential objections, we transcribe below a dialogue between the authors, addressing some potential objections and representing our current point of view. Also see \([3, 29, 30]\).

(1) How can a smooth geometry possibly correspond to a ‘microstate’ of a black hole?

Smooth geometries do not exactly correspond to states. Rather, as classical solutions they define points in the phase space of a theory (since a coordinate and a momenta define a history and hence a solution; see section 2.1 for more details) which is isomorphic to the solution space. In combination with a symplectic form the phase space defines the Hilbert space of the theory upon quantization. As always in quantum mechanics, it is not
possible to write down a state that corresponds to a point in phase space. The best we can
do is to construct a state which is localized in one unit of phase-space volume near a point.
We will refer to such states as coherent states. Very often (but not always, as we will
see later in these notes) the limit in which supergravity becomes a good approximation
corresponds exactly to the classical limit of this quantum-mechanical system, and in this
limit coherent states localize at a point on phase space. It is in this sense, and only in this
sense, that smooth geometries correspond to microstates. Clearly, coherent states are very
special states, and a generic state will not be describable in terms of a smooth geometry.

2) How can a finite-dimensional solution space provide an exponential number of states?
The number of states obtained by quantizing a given phase space is roughly given by
the volume of the phase space as measured by the symplectic form \( \omega, N \sim \int \omega^k / k! \)
for a \( 2k \)-dimensional phase space. Thus, all we need is an exponentially growing volume
which is relatively easy to achieve.

3) Why do we expect to be able to account for the entropy of the black hole simply by
studying smooth supergravity solutions?
Well, actually, we do not really expect this to be true. In cases with enough supersymmetry,
one does recover all BPS states of the field theory by quantizing the space of smooth
solutions, but there is no guarantee that the same will remain true for large black holes,
and the available evidence does not support this point of view. We do however expect that
by including stringy degrees of freedom we should be able to accomplish this, in view of
open/closed string duality.

4) If black-hole ‘microstates’ are stringy in nature then what is the content of the ‘fuzzball
proposal’?
The content of the fuzzball proposal, as promulgated by the authors, is that the closed
string description of a generic microstate of a black hole, while possibly highly stringy
and quantum in nature, has an interesting structure that extends all the way to the horizon
of the naive black-hole solution, and is well approximated by the black-hole geometry
outside of the horizon. Moreover, while the naive black-hole geometry corresponds to
a thermal state (e.g., in the dual field theory when it exists) the actual “black hole” will
really be in a given (generic) pure state.

More precisely the naive black-hole solution is argued to correspond to a thermodynamic
ensemble of pure states. The generic constituent state will not have a good geometrical
description in classical supergravity; it may be plagued by regions with a string-scale
curvature and may suffer large quantum fluctuations. These, however, are not restricted
to the region near the singularity but extend all the way to the horizon of the naive
geometry. This is important as it might shed light on information loss via the Hawking
radiation from the horizon as near-horizon processes would now encode information about
this state that, in principle, distinguish it from the ensemble average.

5) Why would we expect a string-scale curvature or large quantum fluctuations away from
the singularity of the naive black-hole solution? Why would the classical equations of
motion break down in this regime?
As mentioned in the answer to question 1.2, it is not always true that a solution to the
classical equations of motion is well described by a coherent state, even in the supergravity
limit. In particular there may be some regions of phase space where the density of states
is too low to localize a coherent state at a particular point. Such a point, which can
be mapped onto a particular solution of the equations of motion, is not a good classical
solution because the variance of any quantum state whose expectation values match the
solution will necessarily be large.
Another way to understand this is to recall that the symplectic form effectively discretizes the phase space into $\hbar$-sized cells. In general all the points in a given cell correspond to classical solutions that are essentially indistinguishable from each other at large scales. It is possible, however, for a cell to contain solutions to the equation of motion that do differ from each other at very large scales. Since a quantum state can be localized at most to one such cell it is not possible to localize any state to a particular point within the cell. Only in the strict $\hbar \to 0$ limit will the cell size shrink to a single point suggesting there might be states corresponding to a given solution but this is just an artefact of the limit. A specific explicit example of such a scenario is discussed later in this paper. Thus, even though the black-hole solution satisfies the classical equation of motion all the way to the singularity this does not necessarily imply that when quantum effects are taken into account that this solution will correspond to a good semiclassical state with very small quantum fluctuations.

(6) So is a black hole a pure state or a thermal ensemble?

In a fundamental theory we expect to be able to describe a quantum system in terms of pure states. This applies to a black hole as well. At first glance, since the black hole carries an entropy, it should be associated with a thermal ensemble of microstates. But, as we know from statistical physics, the thermal ensemble can be regarded as a technique for approximating the physics of the generic microstate in the microcanonical ensemble with the same macroscopic charges. Thus, one should be able to speak of the black hole as a coarse grained effective description of a generic underlying microstate. Recall that a typical or generic state in an ensemble is very hard to distinguish from the ensemble average without doing impossibly precise microscopic measurements. The entropy of the black hole is then, as usual in thermodynamics, a measure of the ignorance of macroscopic observers about the nature of the microstate underlying the black hole.

(7) What does an observer falling into a black hole see?

This is a difficult question which cannot be answered at present. The above picture of a black hole does suggest that the observer will gradually thermalize once the horizon has been passed, but the rate of thermalization remains to be computed. It would be interesting to do this and to compare it to recent suggestions that black holes are the most efficient scramblers in nature [31, 32].

(8) Does the fuzzball proposal follow from AdS/CFT?

As we have defined it the fuzzball proposal does not follow from AdS/CFT. The latter is obviously useful for many purposes. For example, given a state or density matrix, we can try to find a bulk description by first computing all one-point functions in the state, and by subsequently integrating the equations subject to the boundary conditions imposed by the one-point function. If this bulk solution is unique and has a low variance (so that it represents a good saddle-point of the bulk path integral) then it is the right geometric dual description. In particular, this allows us to attempt to find geometries dual to superpositions of smooth geometries. What it does not do is provide a useful criterion for which states have good geometric dual descriptions; it is not clear that there is a basis of coherent states that all have decent dual geometric descriptions, and it is difficult to determine the way in which bulk descriptions of generic states differ from each other. In particular, it is difficult to show that generic microstates have a non-trivial structure all the way up to the location of the horizon of the corresponding black hole.

(9) To what degree does it make sense to consider quantizing a (sub)space of supergravity solutions?

In some instances a subspace of the solution space corresponds to a well-defined symplectic manifold and is hence a phase space in its own right. Quantizing such a
space defines a Hilbert space which sits in the larger Hilbert space of the full theory. Under some favourable circumstances this Hilbert space may be decoupled in the sense that the total Hilbert space can be approximated as a product of the subspace and another space. For instance, in determining BPS states we can imagine imposing BPS constraints on the Hilbert space of the full theory, generated by quantizing the full solution space, and expect that the resulting states will be supported primarily on the locus of points that corresponds to the BPS phase space; that is, the subset of the solution space corresponding to classical BPS solutions. It is therefore possible to first restrict the phase space to this subspace and then quantize it in order to determine the BPS sector of the Hilbert space.

2. States, geometries, and quantization

Throughout these proceedings we will be exploring the relationship between families of smooth supergravity solutions and appropriately coherent microstates of supersymmetric black holes in string theory. The logic of this relationship is illustrated in figure 1.

The first important component is a (complete) family of supergravity solutions preserving the same supersymmetry and with the same asymptotic conserved charges (see footnote 7) as a BPS black hole. As solutions they can be related to points in a phase space and, as a family, they define a submanifold of the full phase space (this notion is elaborated upon in the next section and references therein). Because they are BPS they can be argued to generate a proper decoupled phase space of their own9 [17, 18], at least for the purpose of enumerating states. Indeed, in the cases we consider, one can check that the restriction of the symplectic form to this space is non-degenerate implying that the space is actually a symplectic submanifold10.

For a thorough and detailed discussion of the subtleties involved in this ‘on-shell quantization’ the reader is referred to [18, sections 2.7–2.10]. Quantizing the space of such solutions as a phase space yields a Hilbert space populated by putative BPS microstates of the black hole. In [17], this was done for 1/2 BPS geometries asymptotic to AdS5 × S5 and found to reproduce the 1/2 BPS spectrum of the dual CFT.

Viewed another way, these geometries are simply well-defined classical solutions to supergravity that are asymptotically indistinguishable from a black-hole solution. If they can be included in a decoupled AdS throat, then they are amenable to study via the AdS/CFT duality where the CFT provides access to the quantum structure of the theory. The smooth solutions should be seen as approximating semiclassical states in the supergravity regime of the bulk (closed string) theory. Such solutions have CFT duals that can be determined by studying the subleading asymptotics of the geometry. Because these geometries are smooth, we expect the dual states to be pure states rather than generic density matrices.

A natural question that emerges is: what is the relationship between the BPS sector of the CFT and the states arising from quantizing this space of smooth BPS supergravity solutions? In some cases, with a large amount of supersymmetry, where the full BPS sector of the CFT is known, it was found that the latter matched the states arising from the direct quantization of the supergravity BPS phase space [17, 18]. However, as mentioned in section 1.2 (question 3), we do not necessarily expect that, in cases with less supersymmetry, the entire BPS Hilbert space can be recovered without including stringy corrections. However, if stringy degrees of freedom are included, it is natural to assume that the resultant states can be matched to supersymmetric states in the CFT.

9 Though these arguments are for the 1/2 and 1/4 BPS cases they should extend to 1/8 BPS bearing in mind the possible discontinuities in the spectrum at walls of marginal stability.

10 As noted in [18], this is related to the fact that the solutions are stationary but not static so the momenta conjugate to the spatial components of the metric are non-vanishing.
The states arising from quantizing only smooth supergravity solutions, without any stringy degrees of freedom, should thus correspond to some restricted subspace of the full BPS Hilbert space of the theory (which corresponds to the BPS sector of the CFT). We would like to understand how these pure states relate to black holes. The latter have entropy, so we expect them to be dual to density matrices in the CFT. The ensemble of states making up such a density matrix should be a suitable thermodynamic average over pure states in the same sector of the Hilbert space as the black hole. As our ‘microstates’ arise from geometries that preserve the same supersymmetries and carry some of the same asymptotic charges as the black hole, they provide suitable candidates for the states that form the ensemble. It is extremely likely (and has been shown to be the case for the 1/4 BPS states) that the generic state in this ensemble will not be semiclassical and is thus only accessible in the CFT or by directly quantizing the BPS phase space. It is less clear whether a large portion of these states or indeed any typical states at all arise from quantizing only supergravity modes. As mentioned above, this has proven to be the case when sufficient supersymmetry is present, but it is not clear if it will continue to do so for less supersymmetric black holes. As the states arising from supergravity are relatively
accessible (compared to more stringy states), it is important to explore their structure and also
to determine how they are related to the typical states making up the black-hole ensemble.

In some cases, there is reason to believe that the Hilbert space enjoys a more refined
decomposition in subsectors than just by macroscopic quantum numbers alone. In [33–35],
for instance, a decomposition based on split attractor trees is conjectured (this will be discussed
further in section 5.3). In these cases, it is possible that of all the states with the correct
asymptotic charges the black-hole ensemble will only include microstates from a single
subsector. In [22] another (related) constraint on the constituents of the black-hole ensemble
was found. There it was argued that any geometries that do not survive a near-horizon
decoupling limit should not contribute states to the black-hole ensemble because they do not
correspond to bona fide bound configurations of the original $D$-brane system generating the
black hole.

In order to study the spacetime structure of the microstates of a black hole, it is desirable
to have an inverse map between the states in the CFT and classical geometries. Of course
this map is not injective as many states in the CFT are not semiclassical, so we would also
like to define a criterion for determining which such states yield good classical geometries
and which yield geometries with large quantum fluctuations. Possible criteria were discussed
in [24, 36] and will play a role in some of the arguments that follow. The point of view
that we would like to assume is based on the need for a classical observer to measure the
system [19]. Thus we would like to identify a set of operators, $O_\alpha$, in the CFT that are dual to
‘macroscopic observables’. The requirement that a state yield a good classical geometry can
be translated into a constraint on the variance of the expectation values for these observables
in the semiclassical limit.

2.1. Phase-space quantization

The space of classical solutions of a theory is generally isomorphic to its classical phase space.
Heuristically, this is because a given point in the phase space, composed of a configuration
and associated momenta, can be translated into an entire history by integrating the equations
of motion against these initial data; likewise, by fixing a spatial foliation, any solution can be
translated into a unique point in the phase space by extracting a configuration and momentum
from the solution evaluated on a fixed spacial slice. This observation can be used to quantize
the theory using a symplectic form, derived from the Lagrangian, on the space of solutions
rather than on the phase space. This is an old idea [37] (see also [38] for an extensive list of
references and [39–41] for more recent work) which was used in [17, 18, 42, 43] to quantize
the LLM [6] and Lunin–Mathur [8] geometries. An important subtlety in these examples is
that it is not the entire solution space which is being quantized but rather a subspace of the
solutions with a certain amount of supersymmetry.

In general, quantizing a subspace of the phase space will not yield the correct physics as
it is not clear that the resultant states do not couple to states coming from other sectors. It
is not even clear that a given subspace will be a symplectic manifold with a non-degenerate
symplectic pairing. As discussed in [18], we expect the latter to be the case only if the
subspace contains dynamics; for gravitational solutions we thus expect stationary solutions,
for which the canonical momenta are not trivial, to possibly yield a non-degenerate phase
space. This still does not address the issue of consistency as states in the Hilbert space derived
by quantizing fluctuations along a constrained submanifold of the phase space might mix
with modes transverse to the submanifold. When the submanifold corresponds to the space
of BPS solutions, one can argue, however, that this should not matter. The number of BPS
states is invariant under continuous deformations that do not cross a wall of marginal stability
or induce a phase transition. Thus if we can quantize the solutions in a regime where the interaction with transverse fluctuations is very weak, then the energy eigenstates will be given by perturbations around the states on the BPS phase space, and although these will change character as parameters are varied, the resultant space should be isomorphic to the Hilbert space obtained by quantizing the BPS sector alone. If a wall of marginal stability is crossed, states will disappear from the spectrum, but there are tools that allow us to analyse this as it occurs (see section 5.3).

Let us emphasize that the validity of this decoupling argument depends on what questions one is asking. If we were interested in studying dynamics, then we would have to worry about how modes on the BPS phase space interact with transverse modes. For the purpose of enumerating or determining general properties of states, however, as we have argued, it should be safe to ignore these modes. For an example of the relation between states obtained by considering the BPS sector of the fully quantized Hilbert space and the states obtained by quantizing just the BPS sector the phase space see [23, 44].

As mentioned, the LLM and Lunin–Mathur geometries have already been quantized and the resultant states were matched with states in the dual CFTs. We will have occasion to mention this briefly in the following, but we will ultimately focus on the quantization of \( N = 2 \) solutions in four (or \( N = 1 \) in five) dimensions. For such solutions, although a decoupling limit has been defined [23], the dual \( N = (0, 4) \) CFT is rather poorly understood. Thus quantization of the supergravity solutions may yield important insights into the structure of the CFT and will be important in studying the microstates of the corresponding extremal black objects.

2.2. Black holes, AdS throats and dual CFT

One of the most powerful tools to study properties of black holes in string theory is the AdS/CFT correspondence [45]. This conjecture relates string theory on backgrounds of the form \( \text{AdS}_{p+1} \times M \) to a CFT \( p \) that lives on the boundary of the \( \text{AdS}_{p+1} \) space. Such backgrounds arise from taking a particular decoupling limit of geometries describing black objects such as black holes, black strings, black tubes, etc. This limit amounts to decoupling the physics in the near-horizon region\(^{11} \) of the black object from that of the asymptotically flat region by scaling the appropriate Planck length, \( l_p \), to decouple the asymptotic gravitons from the bulk. At the same time one needs to scale appropriate spatial coordinates with powers of \( l_p \) to keep the energies of some excitations finite. This procedure should be equivalent to the field-theory limit of the brane-bound states generating the geometry under consideration.

We are interested in black objects which describe normalizable deformations in the \( \text{AdS}_{p+1} \) background. These correspond to a state/density matrix on the dual CFT according to the following dictionary:

| Bulk | Boundary |
|------|----------|
| \( \exp(-S_{\text{bulk}}) \) | \( \text{Tr}(\rho \mathcal{O}_1 \ldots \mathcal{O}_n) = \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\rho} \) |
| Classical geometries | Semiclassical states? |
| Black hole | \( \rho \sim \exp(-\sum \beta_i \mathcal{O}_i) \) |
| Entropy \( S \) | \( S = -\text{Tr}(\rho \log \rho) \) |
| Bulk isometry \( D \) | \( [\rho, \hat{D}] = 0 \) |
| ADM quantum numbers of \( D \) | \( \text{Tr}(\rho \hat{D}) = \langle \hat{D} \rangle = D_{\text{ADM}} \) |

\(^{11}\) In some of the cases treated in these proceedings the region will not be an actual near-horizon region as the original solutions may be horizon-free, but the decoupling limits are motivated by analogy with genuine black holes where the relevant region is the near-horizon one.
In the first line $O_i$ are operators dual to sources turned on in the boundary. They are included in the bulk calculation of $\langle s_{\text{bulk}}^{\text{on shell}} \rangle$. The second line can be seen as the definition of the dual semiclassical state. More specifically, a semiclassical state is the one that has an unambiguous dual bulk geometry (i.e., that in the classical limit ($N \to \infty$ and $\hbar \to 0$) macroscopic observables take on a fixed expectation value with vanishing variance). In some ideal situations such semiclassical states turn out to be the analogue of coherent states in the harmonic oscillator. In the third line, we describe a typical form of a density matrix that we expect to describe black holes. This form is motivated by the first law of thermodynamics: the entropy as defined in the fourth line obeys $dS = \sum \beta_i d\langle O_i \rangle$, and by matching this to the first law as derived from the bulk description of the black hole, we can identify the relevant set of operators $O_i$ and potentials $\mu_i$ and guess the corresponding density matrix. The fourth line simply states that we expect a relation between the bulk and boundary entropies. In the fifth and the last line, $\hat{D}$ is the current/operator dual to the bulk isometry $D$.

One question that we want to shed some light on in these proceedings is ‘Given a density matrix $\rho$ on the CFT side, is there a dual geometry in the bulk?’. On general grounds one could have expected that a general density matrix $\rho$ should be dual to a suitably weighted sum over geometries, each of which could be singular, have regions with high curvature and perhaps not have good classical limits. As a result, the dual gravitational description of a general density matrix will not generally be trustworthy. However, under suitable circumstances, it can happen that there is a dual ‘effective’ geometry that describes the density matrix $\rho$ very well. This procedure of finding the effective geometry is what we will call ‘coarse graining’. In the gravity description, this amounts to neglecting the details that a classical observer cannot access anyway due to limitations associated with the resolution of their apparatus. So, one can phrase our question in the opposite direction, ‘What are the characteristics of a density matrix on the CFT side, so that there is a good dual effective geometry that describes the physics accurately?’.

One can try to construct the dual effective geometry following the usual AdS/CFT prescription. To do so, one should first calculate all the non-vanishing expectation values of all operators dual to supergravity modes (assuming one knows the detailed map between the two). On the CFT side, these vevs are simply given by

$$\langle O_i \rangle = \text{Tr}(\rho O_i),$$

and they determine the boundary conditions for all the supergravity fields. The next step is to integrate the gravity equations of motion subject to these boundary conditions to get the dual geometry. This is in principle what has to be done according to AdS/CFT prescription. The problem with this straightforward approach is that it is not terribly practical, and we will therefore revert to a different approach\(^\text{12}\). Before describing various examples in more detail, we first describe the main idea in general terms. We will first start by describing the connection between quantum physics and the classical phase space. After that, we are going to briefly describe the philosophy behind constructing effective geometries.

### 2.3. Phase-space distributions

To have an idea about what it means to average over ensembles of geometries, or ‘coarse grain’ as we will refer to it, we need to understand some general features of the bulk theory. In general, we will assume that we are dealing with a supergravity theory in the bulk\(^\text{13}\). Recall

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\(^\text{12}\) Though it would be interesting to study in some detail the connection between the two.

\(^\text{13}\) Although in principle we would like to extend this discussion to include stringy degrees of freedom (when relevant) we do not, at present, have any control over the latter.
that solutions to the supergravity equation of motion can be associated with points in a phase space (see, for example, [41]). The boundary theory, on the other hand, is generally studied as a quantum conformal field theory. As a result we are looking for a map between quantum states (CFT) and classical objects living in a phase space (bulk). A well-known example of such a map is the map between quantum states and their corresponding classical phase-space densities (see the review [46] and references there in to the original literature). A good guess then is that the map that we are looking for should be a ‘dressed’ version of the former. Let us pause for a moment to discuss the phase-space distribution idea [46]. A particle (or statistical system) in a quantum theory is described by giving its density matrix $\rho$. The result of any measurement can be seen as an expectation value of an appropriate operator which is given explicitly by

$$\langle O \rangle_\rho = \text{Tr}(\rho O). \quad (2.1)$$

This is reminiscent of classical statistical mechanics where the measurements are averages of appropriate quantities using some statistical distribution

$$\langle O \rangle_w = \int dp \, dq w(p, q) O(p, q), \quad (2.2)$$

where the integration is over the full phase space. One can wonder at this point if it is possible to construct a density $w(p, q)$ so that one can rewrite equation (2.1) as equation (2.2). The answer is affirmative: for every density matrix $\rho$ there is an associated phase-space distribution $w_\rho$ such that for all operators $A$ the following equality holds:

$$\int dp \, dq w_\rho(p, q) A(p, q) = \text{Tr}(\rho A(\hat{p}, \hat{q})). \quad (2.3)$$

What about the uniqueness of $w_\rho$? Recall that in a quantum theory, we have to face the question of operator ordering. This comes about because the operators $\hat{q}$ and their dual momenta $\hat{p}$ do not commute with each other. This means that the distribution $w_\rho$ should somehow include information about the chosen order of $\hat{p}$ and $\hat{q}$. As a result, there does not exist a unique phase-space distribution. For example, the distribution corresponding to the Weyl ordering is the Wigner distribution, which is given by

$$w(p, q) \sim \int dy \langle q - y | \rho | q + y \rangle e^{2ipy}. \quad (2.4)$$

This distribution suffers from the fact that it is not positive definite in general. It is also quite sensitive to the physics at a quantum scale [19, 46] as it usually has fluctuations of order $\hbar$. Another drawback of this distribution is that it is difficult to work with from a computational standpoint. There is another commonly used distribution which is positive definite: the Husimi distribution. It is roughly the convolution of the Wigner distribution with a Gaussian. This eliminates most of the fluctuations of order $\hbar$. The price that one pays for this is that the resulting operators must be anti-normal ordered. However, for semiclassical states, which by definition are states for which the classical limit is unambiguous, $w_\rho(p, q)$ should be independent of the choice of ordering prescription in the classical limit as well, so this is not actually much of a problem.

2.4. Typical states versus coarse-grained geometry

Let us recapitulate what we have discussed so far and what the missing steps are to achieve our goal. On the gravity side we have geometries with certain asymptotics that in principle yield the one-point functions of the dual operators in the CFT. On top of that we have in principle a way to quantize the reduced phase space of solutions by using the induced symplectic form.
On the CFT side, we can consider density matrices $\rho$ and find the corresponding expectation values of operators. The only missing ingredient is to construct the dual effective geometry. At this stage we have two options: (i) we first select a typical representative from the CFT ensemble of states and then map this typical state directly onto the geometry, or (ii) we somehow average over all the geometries dual to the states in the ensemble.

### 2.4.1. Typical states/geometries

A typical state in an ensemble is one for which the expectation values of macroscopic observables agree to within the observable accuracy with the average of the observable in the entire ensemble. Obviously, this notion depends on the appropriate notions of macroscopic observables and observable accuracy, but in the examples we describe we will usually have a reasonably educated guess regarding what the typical states are. Given a typical state, we can try to map it directly onto a solution of supergravity (this may still be a formidable task), after which one still needs to verify that the resulting geometry has no pathologies. This approach was followed for example in [3, 19].

### 2.4.2. Average/coarse graining

Alternatively, we can try to average over states and geometries directly. On the CFT side, this is trivial since it essentially involves constructing a density matrix and following the usual rules of quantum mechanics. But the coarse-graining procedure is difficult to implement on the bulk side because gravity is a non-linear theory. However, in all examples that we will study, the equations of motion of supergravity in the BPS sector will effectively be linearized, which allows us to solve the equations in terms of harmonic functions with sources. In addition, the space of solutions will be in one-to-one correspondence with distributions of the sources. This immediately suggests a suitable coarse-graining procedure: we simply smear the harmonic functions against the phase-space density which describes the density matrix in question. This will be the basic idea in the three cases we discuss in sections 3–5, but the details will be quite different in each case. It would be interesting to explore in more detail whether this method gives rise to the appropriate averaging of the one-point functions, and to what extent it agrees with the approach based on typical states that we described in the previous paragraph.

### 2.5. Entropic suppression of variances

One might object any picture where the microstates of a black hole are individually realized in spacetime as extended horizon-free bound states would inevitably lead to radically different results for probe measurements. If this were so, then the usual black hole could not be a good effective description, and there would be a massive violation of our usual expectations from effective field theory. Fortunately, one can show that in any scenario where the entropy of a black hole has a statistical interpretation in terms of states in a microscopic Hilbert space, the variance of finitely local observables over the Hilbert will be suppressed by a power of $e^{-S}$.

To see this, consider a quantum mechanical Hilbert space of states with energy eigenvalues lying between $E$ and $E + \Delta E$ with a basis

$$\mathcal{M}_{\text{hor}} = \{|s\rangle : H|s\rangle = e_s|s\rangle; E \leq e_s \leq E + \Delta E\}. \quad (2.4)$$

Thus the Hilbert consists of states

$$\mathcal{M}_{\text{sup}} = \left\{|\psi\rangle = \sum_s c_s^\psi |s\rangle\right\}. \quad (2.5)$$
with $|s\rangle$ as in (2.4) and $\sum_s |c_s|^2 = 1$. The expectation value of the Hamiltonian $H$ in any state in $\mathcal{M}_{\text{sup}}$ also lies between $E$ and $E + \Delta E$. If entropy of the system is $S(E)$, then the basis in (2.4) has dimension $e^{S(E)}$:

$$1 + \text{dim } \mathcal{M}_{\text{sup}} = |\mathcal{M}_{\text{bas}}| = e^{S(E)}. \quad (2.6)$$

Now take $\mathcal{O}$ to be any local operator and consider finitely local observables of the form

$$c_0^2 = \langle \psi | 1 | \psi \rangle, \quad c_1^2 = \langle \psi | \mathcal{O} | \psi \rangle, \quad c_2^2 = \langle \psi | \mathcal{O}^2 | \psi \rangle, \quad c_3^2 = \langle \psi | \mathcal{O}^3 | \psi \rangle, \ldots \quad (2.7)$$

We would like to measure how these moments vary over the ensemble $\mathcal{M}_{\text{sup}}$. The ensemble averages of the moments (2.7) and their variances over the ensemble are given by

$$\langle c^k \rangle_{\mathcal{M}_{\text{sup}}} = \int D\psi c_\psi^k \quad (2.8)$$

$$\text{var}[c^k]_{\mathcal{M}_{\text{sup}}} = \int D\psi (c_\psi^k)^2 - \langle c^k \rangle^2_{\mathcal{M}_{\text{sup}}}. \quad (2.9)$$

The differences between states in the ensemble of microstates in their responses to local probes are quantified by the standard deviation to mean ratios

$$\frac{\sigma[c^k]_{\mathcal{M}_{\text{sup}}}}{\langle c^k \rangle_{\mathcal{M}_{\text{sup}}}} = \sqrt{\frac{\text{var}[c^k]_{\mathcal{M}_{\text{sup}}}}{\langle c^k \rangle^2_{\mathcal{M}_{\text{sup}}}}}. \quad (2.10)$$

It was shown in [27] that

$$\text{var}[c^k]_{\mathcal{M}_{\text{sup}}} < \frac{1}{e^S + 1} \text{var}[c^k]_{\mathcal{M}_{\text{bas}}}. \quad (2.11)$$

where the variance on the right-hand side is computed just over a set of basis elements, while the variance on the left-hand side is over the entire Hilbert space. This result follows because the generic state in the Hilbert space is a random superposition of the form (2.5), and in the computation of correlation functions the phases in the coefficients $c_\psi^k$ lead to cancellations. Thus the only avenue to having a variance large enough to distinguish microstates by defeating the $e^S$ suppression in (2.11) is to find probe operators that have exponentially large correlation functions. Finitely local correlation functions in real time typically do not grow in this way and hence, for black holes, with their enormous entropy, a semiclassical observer will have no hope of telling microstates apart from each other. This is especially so because, as we will discuss in subsequent section, even the elements of the basis of microstates (2.4) for a black hole will usually possess the property of typicality, namely that they will be largely indistinguishable using coarse probes.

Thus, even if the microstates of a black hole are realized in spacetime as some sort of horizon-free bound states, finitely local observables with finite precision, of the kind that are accessible to semiclassical observers, would fail to distinguish between these states. Indeed, the semiclassical observer, having finite precision, might as well take an ensemble average of the observables over the microstates, as this would give the same answer. The ensemble of microstates gives a density matrix with entropy $S$, and will be described in spacetime as a black-hole geometry. In this sense, the black-hole geometry will give the effective description of measurements made by semiclassical observers.
3. AdS$_5 \times S^5$

We start with the best understood case AdS$_5 \times S^5$ whose AdS/CFT dictionary is well developed. The dual CFT is $\mathcal{N} = 4$ U(N) super Yang–Mills, where $N$ is the number of D3 branes that generate the geometry. Many supergravity solutions that asymptote to AdS$_5 \times S^5$ are known, including ones with 1/2, 1/4, 1/8 and 1/16 of the original supersymmetries preserved. Black holes with a macroscopic horizon only exist either in the 1/16 BPS case [47] or without any supersymmetry. The latter include AdS–Schwarzschild black holes and some of their qualitative properties can be reproduced from the dual CFT [19]. However, we are going to restrict ourselves to the 1/2-BPS case, where completely explicit descriptions of both the supergravity solutions and the CFT states are known. Therefore, this provides an excellent testing ground to test the general philosophy that we have been advocating. Our exposition will be necessarily brief, we refer the reader to [19] for further details.

3.1. The 1/2-BPS sector in field theory

The Hilbert space of 1/2-BPS states in $\mathcal{N} = 4$ U(N) super Yang–Mills is isomorphic to the Hilbert space of $N$ fermions in a harmonic oscillator potential as shown in [48, 49]. The latter can conveniently be enumerated in terms of Young diagrams with $N$ rows as follows. The ground state is composed of fermions (labelled by $i = 1, \ldots, N$) with energies $E_i^g = [(i-1)+1/2]h$; this is the Fermi sea of the system. When we excite these fermions, the energies become $E_i = (e_i + 1/2)h$ for some positive disjoint integers $e_i \geq i - 1$. Because we are dealing with fermions, wavefunctions are completely antisymmetrized, and we can always order $\{e_i\}$ in a ascending order $e_1 < e_2 < \cdots < e_N$. As a result, the numbers $r_i$ defined by

$$r_i = e_i - i + 1$$

form a non-decreasing set of integers which can be encoded in a Young diagram where $r_i$ describes the length of the $i$th row. It is convenient to also introduce variables $c_j$ which count the number of columns of length $j$. They are related to the $r_i$ via

$$c_N = r_1, \quad c_{N-i} = r_{i+1} - r_i, \quad i = 1, 2, \ldots, (N-1),$$

and clearly

$$r_{i+1} = e_{i+1} - i = c_{N-i} + \cdots + c_N.$$
It turns out that in the thermodynamical limit $N \gg 1$ where one rescales the Young diagram by a factor of $\sqrt{N}$, the Young diagram approaches a ‘limiting shape’ with probability 1 [50]. In other words, in the large $N$ limit almost all states/operators belonging to the canonical ensemble under study will have associated Young diagrams that have vanishingly small fluctuations around this limit curve. One can check this claim by calculating the variance, see [19] for further details. The limiting shape Young diagram describes a ‘typical’ Young diagram in this ensemble (see section 2.4).

To describe this limiting shape in some more detail, let us introduce two coordinates $x$ and $y$ along the rows and columns of the Young diagram. We adopt the convention where the origin $(0,0)$ is the bottom left corner of the diagram, and $x$ increases going up while $y$ increases to the right. In fermion language, $x$ labels the particle number and $y$ its excitation above the vacuum. One then has

$$y(x) = \langle y(x) \rangle = \sum_{i=N-x}^{N} (c_i).$$

In the large $N$ limit, $x$ and $y$ can be treated as continuous variables and the summation above becomes an integral. Since the $c_i$ are independent random variables in the canonical ensemble, it is straightforward to evaluate $y(x)$ explicitly, and one obtains an equation for the limit shape of the form

$$(1 - q^N)q^y + q^{N-x} = 1,$$ (3.1)

where $q$ is related to the temperature of the system.

3.2. The 1/2-BPS sector in supergravity

All 1/2-BPS solutions in supergravity are given by the LLM geometries [6]

$$dx^2 = -h^{-2} (dt + V_i \, dx^i)^2 + h^2 (dy_1^2 + dy_2^2) = \eta \, e^G \, d\Omega_1^2 = \eta \, e^{-G} \, d\Omega_1^2$$

$$h^{-2} = 2\eta \cosh G, \quad \eta \partial_i V_j = \epsilon_{ij} \partial_j z, \quad \eta (\partial_i V_j - \partial_j V_i) = \epsilon_{ij} \partial_i z$$

$$z = \frac{1}{2} \tanh G, \quad \bar{z}(\eta, x_1, x_2) = \frac{\eta^2}{\pi} \int d\eta_1 d\eta_2 \frac{1 - u(0; \eta_1, \eta_1)}{[(x - \bar{y})^2 - \eta^2]},$$ (3.4)

where $i = 1, 2$. In addition there is a self-dual 5-form field strength that depends on the function $\bar{z}$. It is clear from the above equations that the full geometry is specified by choosing a boundary function $u(0; y_1, y_2)$. The requirement of smoothness of the geometry forces $u \in [0,1]$. So one can see the function $u$ as defining a droplet in the $(x_1, x_2)$-plane whose boundary separates the region where $u = 1$ from the region where $u = 0$. This means smooth 1/2 BPS geometries are in one-to-one correspondence with droplets on the $(x_1, x_2)$-plane.

Following [6], in order to match these solutions with states in the field theory, consider geometries for which the regions in the $(x_1, x_2)$-plane where $u = 1$ are compact. Quantization of the flux in the geometry leads to the following identifications:

$$h \leftrightarrow 2\pi \ell_p^4, \quad N = \int \frac{d^3x}{2\pi \hbar} u(0; x_1, x_2).$$ (3.5)

The conformal dimension$^{15}$ $\Delta$ of a given configuration is

$$\Delta = \frac{1}{2} \int \frac{d^3x}{2\pi \hbar} \left( \frac{x_1^2 + x_2^2}{h} \right) u(0; x_1, x_2) = \frac{1}{2} \left( \int \frac{d^3x}{2\pi \hbar} u(0; x_1, x_2) \right)^2.$$ (3.6)

$^{15}$ The bulk interpretation of the conformal dimension is energy.
The formulae above suggest interpreting \( u(0; x_1, x_2) \) as a density. They indeed have a remarkably simple interpretation \([6]\) in terms of the hydrodynamic limit of the phase space of the dual fermionic system, once we identify the \((x_1, x_2)\)-plane with the single particle phase space of the fermions. This has been confirmed by directly quantizing the phase space of smooth gravitational solutions \([17, 51, 52]\). According to the general strategy, we should therefore try to identify \( u(0; x_1, x_2) \) directly with the one-particle phase-space density for any density matrix in the quantum-mechanical fermion system \([19]\).

### 3.3. Geometry versus field-theory states

Let us explore the map between states and geometries in some detail. In particular, we are interested in states and ensembles with a well-defined classical limit. We claim that a sufficient condition for having a well-defined semiclassical limit is that the Young diagrams approach a fixed limiting shape with probability one in the large \(N\) limit. For the canonical ensemble, this limiting shape was given in \((3.1)\), but for other states and ensembles different limit curves may arise in the large \(N\) limit. We will continue to denote those curves by \(y(x)\). They will describe the effective, coarse-grained geometry corresponding to the states/ensembles. To extract the geometry, we use the fact that it should be rotationally invariant, and by matching energy \(\leftrightarrow\) conformal dimension, flux \(\leftrightarrow\) rank of the gauge group (=number of fermions), we get

\[
N = \int dx = \int \frac{u(0; r^2)}{2\hbar} dr^2, \quad E = \int (x + y(x)) dx = \int \frac{r^2 u(0; r^2)}{4\hbar^2} dr^2 = \Delta. \quad (3.7)
\]

The above equations should not just hold at the level of integrals but also at the level of integrands, so that

\[
\frac{u(0; r^2)}{2\hbar} dr^2 = dx, \quad \frac{r^2 u(0; r^2)}{4\hbar^2} dr^2 = (y(x) + x) dx.
\]

Combining these yields

\[
y(x) + x = \frac{r^2}{2\hbar},
\]

and taking derivatives with respect to \(x\), we obtain the identification \([19]\)

\[
u(0; r^2) = \frac{1}{1 + y'}\quad (3.8)
\]

So given a Young diagram with a limit shape \(y(x)\), one can associate to it a geometry generated by \(u(0; r^2)\) according to \((3.8)\). For the limit curve of the canonical ensemble \((3.1)\), the resulting phase-space distribution is just that of a finite temperature Fermi gas system which was considered in \([53]\). Unfortunately, no one has succeeded in writing the corresponding metric explicitly in a closed form. Since it would describe our best guess for a ‘1/2-BPS black hole,’ it would be interesting to know what it looks like.

Note that in our conventions \(y' \geq 0\) and therefore the associated geometry generically has null singularities with \(0 < u(0; r^2) < 1\); the unphysical cases with \(u < 0\) or \(u > 1\), which give rise to naked timelike singularities \([54]\), do not appear.

We conclude this section with some remarks:

- Most states yield ambiguous ‘quantum foam’ geometries with string scale signature.
- Semiclassical states yield well-defined but still mildly singular spacetimes.
- Geometries could be coarse grained over thanks to the linear description of the solutions of the field equations.
- Non-rotationally invariant configurations can also be studied but do not correspond to a single limiting Young diagram.
• Given a limiting shape, we can associate an entropy to it by counting all the states that approach the limiting shape in the large $N$ limit (in other words, that are macroscopically indistinguishable from it). It would be interesting to provide an explicit expression for this entropy and to match it to the bulk geometry\textsuperscript{16}.

4. AdS$_3 \times S^3$

In this section, we are going to discuss the bound states of D1 and D5 branes in type II-B string theory compactified on\textsuperscript{17} $T^4 \times S^1$. These are 1/2-BPS states (preserving eight supercharges) that describe a black hole without a classical horizon in five dimensions. However, we are going to work in six dimensions, keeping explicitly track of the $S^1$. One of the reasons behind this decision is that in this way one gets solutions that are asymptotically AdS$_3 \times S^3$ after taking a suitable decoupling limit. Thus we can employ the AdS/CFT machinery and benefit from the known properties of the dual two-dimensional conformal field theory.

As is well known, the 1/2-BPS states in the CFT dual to the D1–D5 system can be identified with the states at level $L_0 = N_1 N_5$ in a system with $b_1 + b_3$ chiral fermions and $b_0 + b_2 + b_4$ chiral bosons, where $b_i = \dim H^i(M_4)$. Here $N_1$ and $N_5$ are the quantized number of D1 and D5 branes. Note that this identification of 1/2 BPS states with a system of free bosons and fermions is only valid at the level of the Hilbert space, not at the level of correlation functions. Thus, we would ideally like to be able to find a detailed map between states/ensembles in this auxiliary theory of free bosons and fermions and half-BPS solutions of six-dimensional supergravity. In what follows, we will describe such a map. We will first review the known supergravity solutions and their quantization, and then propose a map which is again based on the notion of phase-space densities. We conclude this section by discussing various relevant examples.

4.1. The supergravity solution and its quantization

Starting with a fundamental string with transversal profile $F(s) \subset \mathbb{R}^4$ then dualizing, one gets the following solutions [8, 7, 43], written in the string frame:\textsuperscript{18}

\begin{align}
\frac{1}{\sqrt{f_1 f_5}} & \left[ - (dt + A)^2 + (dy + B)^2 \right] + \frac{1}{\sqrt{f_1 f_5}} dx^2 + \frac{1}{\sqrt{f_1 f_5}} dz^2 \\
e^{2\phi} &= \frac{f_1}{f_5}, \quad C = \frac{1}{f_1} (dt + A) \wedge (dy + B) + C, \quad (4.1)
\end{align}

where $y$ parametrizes a circle with coordinate radius $R$, $z^i$ are coordinates on $T^4$ with coordinate volume $V_4$, the Hodge star $*_4$ is defined with respect to the four-dimensional noncompact space spanned by $x^i$ and

\begin{align}
&dB = *_4 dA, \quad dC = -*_4 df_5, \quad A = \frac{Q_5}{L} \int_0^L \frac{F'(s) \, ds}{|x - F(s)|^2}, \\
f_5 &= 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|x - F(s)|^2}, \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|F'(s)|^2 \, ds}{|x - F(s)|^2}. \quad (4.2)
\end{align}

The solutions are asymptotically $\mathbb{R}^{1,4} \times S^1 \times T^4$. We can take a decoupling limit which simply amounts to erasing the 1 from the harmonic functions. The resulting metric will then be asymptotically AdS$_3 \times S^3 \times T^4$.

\textsuperscript{16}By dividing the phase space into Planck-size cells, the obvious guess for the information-theory inspired bulk entropy would be $S \sim - \int d^4 v \log n$.

\textsuperscript{17}The same story carries over to the case of K3 $\times S^1$.

\textsuperscript{18}We are going to follow the conventions of [43].
As mentioned above, the solutions are parametrized in terms of a closed curve
\[ x_i = F_i(s), \quad 0 < s < L, \quad i = 1, \ldots, 4. \] (4.3)

In the following we are going to ignore oscillations in the \( T^4 \) direction as well as fermionic excitations, for a further discussion of these degrees of freedom see [55, 56]. The D1 (D5) charge \( Q_1 \) (\( Q_5 \)) satisfy
\[ L = \frac{2\pi Q_5}{R}, \quad Q_1 = \frac{Q_5}{L} \int_0^L |F'(s)|^2 \, ds. \] (4.4)

It turns out that the space of classical solutions has finite volume and therefore will yield a finite number of quantum states. To see this, on first starts by expanding \( F \) in oscillators:
\[ F(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left( c_k e^{i2\pi s} + c_k^\dagger e^{-i2\pi s} \right), \] (4.5)

where \( \mu = \frac{g_s}{\sqrt{V_4}} \). Then one computes the restriction of the Poisson bracket to the space of solutions (4.1) which turns out to be [42, 43]
\[ [c_k, c_{k'}^\dagger] = \delta_{kk'} \delta_{kk'}. \] (4.6)

After quantization, the relation between \( Q_1 \) and \( Q_5 \) reads
\[ \left\langle \int_0^L :|F'(s)|^2 : \right\rangle = \left( \frac{2\pi}{L} \right)^2 \mu^2 N, \] (4.7)

where
\[ N_1 = \frac{g_s}{V_4} Q_1, \quad Q_5 = g_s N_5, \quad N = N_1 N_5 = \sum_{k=1}^{\infty} k \langle c_k^\dagger c_k \rangle. \] (4.8)

\( N_1, N_5 \) count the number of D1 and D5 branes, respectively. The modes \( c_k \) become the creation and annihilation modes of four of the total of \( b_0 + b_2 + b_4 \) bosons; one can check that the four that appear are precisely the ones associated with the \( H^{0,0}(M), H^{2,0}(M), H^{0,2}(M), H^{2,2}(M) \) cohomology. Finally, note that the number of states and hence the entropy can easily be extracted from the known partition functions of chiral bosons and fermions.

4.2. Geometries from states

The Hilbert space is spanned by
\[ |\psi\rangle = \prod_{i=1}^{4} \prod_{k=1}^{\infty} \langle c_k^\dagger \rangle^{N_k}|0\rangle, \quad \sum k N_k = N. \] (4.9)

Given a state, or more generically a density matrix in the CFT
\[ \rho = \sum_{ij} c_{ij} |\psi_i\rangle \langle \psi_j |, \] (4.10)

we wish to associate with it a density on phase space. The phase space is given by classical curves which we will parametrize as (note that \( d \) and \( \bar{d} \) are now complex numbers, not operators)
\[ F(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} (d_k e^{i2\pi s} + \bar{d}_k e^{-i2\pi s}) \] (4.11)

and which obey the classical constraint (4.4).
We now propose to associate to a density matrix of the form (4.10) a phase-space density (compare to the general discussion in section 2.1) of the form [21]

$$ f(d, \tilde{d}) = \sum_i \frac{\langle 0 | e^{d_{ki}c_{k}^†} | \psi_i \rangle e^{d_{ki}c_{k}^i} | 0 \rangle}{\langle 0 | e^{d_{ki}c_{k}^i} | 0 \rangle}. $$

(4.12)

The distribution corresponding to a generic state $|\psi\rangle = \prod_{k=1}^{\infty} (c_{k}^i)^{N_k} |0\rangle$ can be easily computed:

$$ f(d, \tilde{d}) = \prod_{k,i} (d_{k,i}^{\tilde{d}_{k,i}})^{N_k} \, e^{-\tilde{d}_i}. $$

(4.13)

Note that our phase-space density (4.12), as written, is a function on a somewhat larger phase space as $d, \tilde{d}$ do not have to obey (4.4). To cure this discrepancy we are going to include an extra factor $\exp(-\beta \tilde{N})$ in the calculations, where we choose $\beta$ such that the expectation value of $\tilde{N}$ is precisely $N$. This is just like passing from a microcanonical ensemble to a canonical one, and for many purposes this is probably a very good approximation. For a thorough discussion of this point see [21].

To further motivate (4.12), we note that it associates to a coherent state a density which is not exactly a Gaussian centred around the classical curve, but there are some corrections due to the finite $N$ projections. Obviously, these corrections will vanish in the $N \to \infty$ limit.

The density (4.12) has the property that for any function $g(d, \tilde{d})$

$$ \int \int_{d, \tilde{d}} f(d, \tilde{d}) g(d, \tilde{d}) = \sum_i \langle \psi_i | : g(c, c^i) :_A | \psi_i \rangle, $$

(4.15)

where $: g(c, c^i) :_A$ is the anti-normal ordered operator associated with $g(c, c^i)$, and $\int_{d, \tilde{d}}$ is an integral over all variables $d_i$. Since the theory behaves like a $(1+1)$-dimensional field theory the natural thing to do is to calculate expectation values of normal ordered operators in order to avoid infinite normal ordering contributions. Besides, everything we do is limited by the fact that our analysis is in classical gravity and therefore can at best be valid up to quantum corrections. As a result a further modification to our proposal will be to redefine $g(d, \tilde{d})$ by subtracting the anti-normal ordering effects.

Since the harmonic functions appearing in (4.2) can be arbitrarily superposed, we finally propose to associate to (4.10) the geometry

$$ f_5 = 1 + \frac{Q_5}{L \tilde{N}} \int_0^{L} \int_{d, \tilde{d}} f(d, \tilde{d}) \frac{ds}{|x - F(s)|^2} $$

$$ f_1 = 1 + \frac{Q_5}{L \tilde{N}} \int_0^{L} \int_{d, \tilde{d}} \frac{f(d, \tilde{d}) |F'(s)|^2 ds}{|x - F(s)|^2} $$

$$ A' = \frac{Q_5}{L \tilde{N}} \int_0^{L} \int_{d, \tilde{d}} f(d, \tilde{d}) F'(s) ds \frac{1}{|x - F(s)|^2} $$

(4.16)

with the normalization constant

$$ N^{-1} = \int_{d, \tilde{d}} f(d, \tilde{d}). $$

(4.17)
In [7] it was shown that the geometries corresponding to a classical curve are regular provided \(|F(s)|\) is different from 0 and the curve is not self-intersecting. In our setup we sum over continuous families of curves with some weighing factor which can introduce singularities. We expect these singularities to be rather mild, certainly for semiclassical density matrices, and in addition in various examples the averages will turn out to be completely smooth anyway (see section 4.3). Another point worth mentioning is that the average will no longer solve the vacuum type IIB equations of motion, instead a small source will appear on the right-hand side of the equations. Since these sources are subleading in the \(1/N\) expansion and vanish in the classical limit, they are in a regime where classical gravity is not valid and they may well be cancelled by higher order contributions to the equations of motion. To have an idea about these sources let us study the circular profile.

We consider the following profile:

\[
F^1(s) = a \cos \frac{2\pi k}{L} s, \quad F^2(s) = a \sin \frac{2\pi k}{L} s, \quad F^3(s) = F^4(s) = 0, \tag{4.18}
\]

which describes a circular curve winding \(k\) times around the origin in the 12-plane. In order to simplify our discussion, we focus on the simplest harmonic function \(f_5\). In order to evaluate the various integrals it will be convenient to Fourier transform the \(x\)-dependence using

\[
\frac{1}{|x|^2} = \frac{1}{4\pi^2} \int d^4 \nu \frac{e^{i x \nu}}{|\nu|^2}. \tag{4.19}
\]

Classically \(\Box f_5\) is a delta function with a source at the location of the classical curve, to be precise

\[
\Box f_5 = -4\pi Q s \delta(x_1^2 + x_2^2 - a^2) \delta(x_3) \delta(x_4). \tag{4.20}
\]

Now in the quantum theory, we associate to the classical circular curve \((4.18)\) the density matrix \((4.14)\) and subsequently the phase-space density \((4.12)\). Working this out we find out that

\[
f(d, \bar{d}) = \left(\left(\bar{d}_k^1 + id_k^2\right)\left(\bar{d}_k^1 - id_k^2\right)\right)^N e^{-\sum_n \nu_n \bar{d}_n d_n}. \tag{4.21}
\]

We have ignored the delta function coming from the projection here and expect \((4.21)\) to be valid for large values of \(N/k\). It is therefore better thought of as a semiclassical profile rather than the full quantum profile.

According to \((4.16)\) the harmonic function \(f_5\) is now given by

\[
f_5 = 1 + \frac{Qs}{4\pi^2} N \int_0^L ds \int_{d, \bar{d}} f(d, \bar{d}) \int d^4 \nu \frac{1}{|\nu|^2} e^{i x \nu - \frac{F(s) + \sum_n \nu_n^2 \bar{d}_n d_n}{2}}. \tag{4.22}
\]

where we have used \((4.19)\) and the constant \(\sum_n \nu_n^2 \bar{d}_n d_n\) appears due to the fact that we want to compute a normal ordered quantity instead of an anti-normal ordered one. The function \(F(s)\) depends on an infinite set of complex oscillators \(d_n\). It can be easily seen that the contribution for the oscillators different from \(d_1^1\) and \(d_2^1\) cancels exactly against the normal ordering constant \(u^5 \bar{\mu}^2 / 2t\) mentioned above.

So \(\Box f_5\) for this case reads

\[
\Box f_5 = -4\pi Q s \delta(x_3) \delta(x_4) A(x_1, x_2) \tag{4.23}
\]

\[
A(x_1, x_2) = \int_0^\infty d\rho \rho J_0\left(\sqrt{\frac{x_1^2 + x_2^2}{\rho}}\right) L_{N/2} \left(\frac{a^2 \rho^2}{4N/k}\right). \tag{4.24}
\]

Until here we have not used any approximation. Using the identity

\[
L_N(x) = \frac{e^x}{N^1} \int_0^\infty e^{-t} t^N J_0(2\sqrt{tx}) \, dt
\]
and approximating \( \exp \left( \frac{d^2 \varphi^2}{2N/k} \right) \approx 1 \), one obtains
\[
A(x_1, x_2) = \frac{e^{-N/k} r^2}{(N/k - 1)!a^2}
\]
with \( r^2 = x_1^2 + x_2^2 \). In the limit \( N/k \to \infty \), \( A(x_1, x_2) \) approaches \( \frac{\delta(r^2/a^2)}{a^2} \) and the classical and quantum results agree. For large \( N/k \), \( A(x_1, x_2) \) is approximately a Gaussian around \( r^2 \approx a^2 \) and width \( 1/\sqrt{N/k} \), indeed, using Stirling’s formula
\[
A(x_1, x_2) \approx \frac{\sqrt{N/k}}{\sqrt{2\pi}} e^{-N/k(r^2/a^2 - 1)/2} (r^2/a^2)^{N/k}. \tag{4.26}
\]
So the geometry corresponds to a solution of the equations of motion in presence of smeared sources. The width of the smeared source goes to zero in the limit \( N/k \to \infty \), as expected.

4.3. Thermal ensembles

In the following, we consider the geometry of some thermal ensembles of interest.

4.3.1. \( M = 0 \) BTZ. The corresponding thermal ensemble is characterized by the following density matrix:
\[
\rho = \sum_{N_k, \tilde{N}_k} \frac{|N_k \rangle \langle N_k| e^{-\beta \hat{\mathcal{H}}_{N_k}} \langle \tilde{N}_k| \langle \tilde{N}_k|}{\text{Tr} e^{-\beta \hat{\mathcal{H}}_{N_k}}},
\]
where \( |N_k \rangle \) is a generic state labelled by collective indices \( N_k \)
\[
|N_k \rangle = \prod_k \frac{1}{\sqrt{N_k!}} (c^\dagger_k)^{N_k} |0\rangle
\]
and we have chosen a normalization so that \( \langle N_k | \tilde{N}_k \rangle = \delta_{N_k, \tilde{N}_k} \). The value of the potential \( \beta \) has to be adjusted such that \( \langle \tilde{N} \rangle = N \). It is clear that
\[
\rho = \prod_k \rho_k, \quad \rho_k = (1 - e^{-\beta}) \sum_{n=0}^{\infty} e^{-nk} |k, n \rangle \langle k, n| \tag{4.28}
\]
with \( |k, n \rangle = \frac{1}{\sqrt{n!}} (c^\dagger_k)^{n} |0\rangle \). Then the full distribution will simply be the product \( f(d, \bar{d}) = \prod_k f_{d_k}\bar{d}_k \) with
\[
f_{d_k}\bar{d}_k^{(k)} = (1 - e^{-\beta}) e^{-d_k\bar{d}_k} \sum_{n=0}^{\infty} \frac{e^{-nk}}{n!} (d_k\bar{d}_k)^n = (1 - e^{-\beta}) \exp(-(1 - e^{-\beta})d_k\bar{d}_k). \tag{4.29}
\]
The needed harmonic functions 4.2 are deduced from the following generating function
\[
 f_s = \frac{Q_5}{4\pi^2 L^2} \int d^4 u \int_0^L dv \int d\varphi d\bar{\varphi} f(d, \bar{d}) e^{\sum \frac{2d^2 \varphi^2 - 2\pi^2 \alpha^2 \varphi^2}{L^2} e^{i\varphi x}} |u|^{-2}, \tag{4.30}
\]
which gives
\[
f_5 = Q_5 \frac{1 - e^{-\beta}}{x^2}, \quad f_1 = Q_1 \frac{1 - e^{-\beta}}{x^2} \tag{4.31}
\]
\[
A_i = 0, \quad \beta \approx \pi \sqrt{\frac{2}{3N}}.
\]
\(^{19}\) We are going to ignore the \( i \)-index in some equations where it does not play any role. We hope that this will not create any confusion.
A final comment is in order. The geometry obtained differs from the classical $M = 0$ BTZ black hole by an exponential piece. Following [3, 57] we could put a stretched horizon at the point where this exponential factor becomes of order one, so that the metric deviates significantly from the classical $M = 0$ BTZ solution. Thus, using this criterion we find for the radius of the stretched horizon\(^{20}\)

$$r_0 \approx \frac{\mu}{\beta^{1/2}}$$

(4.32)

with corresponding entropy proportional to $N^{3/4}$. This exceeds the entropy of the mixed state from which the geometry was obtained, the latter grows as $N^{1/2}$. This does not contradict any known laws of physics, and in addition we should remember that the notion of stretched horizon depends on the choice of observer. It is quite likely that for a suitable choice of observer, the entropy of the stretched horizon agrees with the entropy obtained from the dual CTFT. For a further discussion of this point see [21, 58].

4.3.2. The small black ring. In this section we consider a slightly more complicated example, namely an ensemble consisting of a condensate of $J$ oscillators of level $q$ plus a thermal ensemble of effective level $N - qJ$. As argued in [20, 25, 59, 60] such an ensemble should describe (in a certain region of parameter space) a small black ring of angular momentum $J$ and dipole (or the Kaluza–Klein) charge $q$.

Using the techniques developed in the previous sections we can compute the generating harmonic function for this case as well and we find

$$f_v = Q_5 L_1 \left( \frac{\mu^2}{4q} \left( \left( \frac{2\pi q}{L} v_2 + i\theta_1 \right)^2 + \left( \frac{2\pi q}{L} v_1 - i\theta_2 \right)^2 \right) \right) e^{-\frac{\sqrt{2\pi q^2 N - qJ}}{L}} \frac{1 - e^{-\frac{2\pi q^2}{\mu^2}}}{|x|^2},$$

(4.33)

where $D = \pi \sqrt{2/3} (N - qJ)^{1/2}$ so that the geometry is purely expressed in terms of the macroscopic quantities $N, J$ and $q$.

We would like to make contact between this geometry and the geometry corresponding to small black rings studied in [20]. As we will see, in the limit of large quantum numbers both geometries reproduce the same one-point functions.

In order to see this, first note that the exponential factor $e^{-\frac{2\pi q^2}{\mu^2}}$ will not contribute (as it vanishes faster than any power at asymptotic infinity). Second one has the formal expansion

$$L_1 \left( \frac{\mu^2}{4q} \right) = J_0 \left( \mu \sqrt{\frac{J}{q}} \right)^{1/2} + \cdots.$$  

(4.34)

In order to estimate the validity of this approximation we can think of $O$ as being proportional to $1/|x|^2$. On the other hand, $\mu \sqrt{J/q}$ can be roughly interpreted as the radius of the black ring (see [20, 61], where this parameter is called $R$). Hence the approximation is valid for large values of $J$ at a fixed distance compared to the radius of the ring.

Using the above approximations it is straightforward to compute the harmonic functions

$$f_5 = \frac{Q_5}{r^2 + \mu^2 L^2 \cos \theta}, \quad f_1 = \frac{Q_1}{r^2 + \mu^2 L^2 \cos \theta},$$

(4.35)

where we have used the following coordinate system:

$$x_1 = (r^2 + a^2)^{1/2} \sin \theta \cos \phi, \quad x_2 = (r^2 + a^2)^{1/2} \sin \theta \sin \phi, \quad x_3 = r \cos \theta \cos \psi, \quad x_4 = r \cos \theta \sin \psi.$$  

(4.36)

\(^{20}\) The same value is obtained if we compute the average size of the curve in $\mathbb{R}^4$, $r_0^2 = \langle |\mathbf{r}|^2 \rangle$. 

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Hence in this approximation the geometry reduces exactly to that of the small black ring studied in [20].

4.3.3. Generic thermal ensemble and the no-hair theorem. In the following we consider a generic thermal ensemble, where each oscillator $c_k$ is occupied thermally with a temperature $\beta_k$. We further will assume that $\beta_{k+}$ for the directions 1, 2 is equal to $\beta_{k-}$ for the directions 3, 4. Restricting to, say, directions 1, 2 we are led to consider the following distribution:

$$f(d, \bar{d}) = \exp\left( -\left(1 - e^{-\beta_{k+}}\right)d_k^* \bar{d}_k^* - \left(1 - e^{-\beta_{k-}}\right)d_k \bar{d}_k \right).$$

(4.37)

Following the same steps as for the case of the small black ring, we obtain

$$f_5 = Q_5 \frac{1 - e^{-\frac{2\pi^2}{\nu^2}}}{|x|^2},$$

(4.38)

$$f_1 = Q_1 \left( \frac{1 - e^{-\frac{2\pi^2}{\nu^2}}}{|x|^2} - \frac{\mu^2}{4N\mu^2D^2} e^{-\frac{2\pi^2}{\nu^2}} \right),$$

(4.39)

$$A = \frac{\mu^2 J}{2} \left( 2 e^{-\frac{2\pi^2}{\nu^2}} - \frac{1 - e^{-\frac{2\pi^2}{\nu^2}}}{|x|^2} \right) (\cos^2 \theta \ d\phi + \sin^2 \theta \ d\psi),$$

(4.40)

where $(|x|, \theta, \phi, \psi)$ are standard spherical coordinates on $\mathbb{R}^4$.

We see that, rather surprisingly, the geometry depends only on a few quantum numbers $N, J, D$ which are given in terms of the temperatures by

$$N = 2 \sum_k k \left( \frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right),$$

(4.41)

$$J = 2 \sum_k \left( \frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} - \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right),$$

(4.42)

$$D = 2 \sum_k \frac{1}{k} \left( \frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right).$$

(4.43)

As a result, the information carried by the geometry is much less than that carried by the ensemble of microstates. In fact, only $N$ and $J$ are visible at infinity, while $D$ sets the size of the ‘core’ of the geometry. We also find that $D$ is precisely the expectation value of the dipole operator introduced in [20]. Its presence in the density matrix is supported by an analysis of the first law of thermodynamics [62]. It is a non-conserved charge which makes its extension to interacting theories an interesting open problem.

We interpret the above remark as a manifestation of the no-hair theorem for black holes. The derivation in this section assumes that the temperatures are all sufficiently large. By tuning the temperatures, it is possible to condense one (like in the small black ring case) or more oscillators. If this happens, we should perform a more elaborate analysis, and we expect that the dual geometrical description\textsuperscript{21} corresponds to concentric small black rings. In this case the configuration will depend on more quantum numbers than just $N, J, D$, in particular we

\textsuperscript{21} It is not difficult to see that the harmonic functions will now take the form of multiple Laguerre polynomials with differential operator arguments acting on the generating harmonic function of the $M = 0$ BTZ solution.
will find solutions where the small black rings carry arbitrary dipole charge. Thus, once we try to put hair on the small black hole by tuning chemical potentials appropriately, we instead find a phase transition to a configuration of concentric small black rings, each of which still is characterized by just a few quantum numbers.

5. AdS$_3 \times$ S$^2$

Although the extremal $D1$–$D5$ system has proved a fertile example in which to test the idea that black holes are simply effective geometries, the extremal solution has a horizon coincident with the singularity with a finite area possibly being generated by higher derivative corrections. Thus this example has some special features that do not generalize. It would be desirable to be able to study a similar scenario for a system where the total charge corresponds to a black hole with a macroscopic horizon (i.e., a three charge black hole). Such black holes are $1/8$ BPS solutions in the full string theory or can emerge as $1/2$ BPS solutions of $\mathcal{N} = 2$ four-dimensional or $\mathcal{N} = 1$ five-dimensional supergravity (i.e., string or $M$-theory reduced on a Calabi–Yau).

5.1. Solution spaces

In four dimensions Bates and Denef [14] have constructed general multicentred BPS solutions of generic $\mathcal{N} = 2$ supergravity theories, and in [10] Bena and Warner classify the full set of BPS solutions for the special case of the five-dimensional $\mathcal{N} = 2$ supergravity theory which is the truncation of the $\mathcal{N} = 8$ theory (i.e., the theory is invariant under eight instead of 32 supersymmetries). The latter require specifying a four-dimensional base metric that is restricted to be hyper-Kähler. A particularly appealing class of hyper-Kähler manifolds are Gibbons–Hawking or multi-Taub–NUT geometries which are asymptotically $\mathbb{R}^3 \times S^1$ and for which we have explicit metrics. Moreover, it has been shown that the five-dimensional solutions constructed using a Gibbons–Hawking base manifold [9] correspond to the four-dimensional ones via the 4D/5D connection [15, 63, 64] making them an interesting class of solutions to study [12].

The five-dimensional solutions, although relatively complicated, are determined entirely in terms of $2b_2 + 2$ harmonic functions where $b_2$ is the second Betti number of the compactification Calabi–Yau, $X$,

$$\begin{align*}
H^0 &= \sum_a \frac{p^0_a}{|x_a|} + h^0, \\
H^A &= \sum_a \frac{p^A_a}{|x_a|} + h^A, \\
H_A &= \sum_a q^A_a + h_A, \\
H_0 &= \sum_a q^0_a + h_0.
\end{align*}
$$

(5.1)

Here the coordinate vector $x_a$ gives the position in the spatial $\mathbb{R}^3$ of the $a$th centre with charge $\Gamma_a = (p^0_a, p^A_a, q^A_a, q^0_a)$ (note here $A$ runs from 1, \ldots, $b_2$). The IIA interpretation of these charges is $(p_0^A, p_2, q^A_2, q^0_2)$ wrapping cycles of $X$ while in M-theory the charge vector is $(KK, M5, M2, P)$. Note that the harmonics have $2b_2 + 2$ constants $h = (h^0, h^A, h_A, h_0)$ that together determine the asymptotic behaviour of the harmonics and hence the solutions. We will also have frequent occasion to use the notation $\Gamma = (p^0, p^A, q_A, q_0)$ to refer to the total charge $\Gamma = \sum_a \Gamma_a$.

The position vectors have to satisfy the integrability constraints

$$\sum_b \frac{(\Gamma_a, \Gamma_b)}{|x_a - x_b|} = \langle h, \Gamma_a \rangle.$$  

(5.2)
where we define the symplectic intersection product
\[
\langle \Gamma_1, \Gamma_2 \rangle := -p_0^0 q_0^2 + p_1^1 q_2^3 - q_0^1 p_2^0 + q_0^1 q_1^0.
\] (5.3)

By summing (5.2) over \(a\) we find that the constants \(h\) have to obey
\[
\langle h, \Gamma \rangle = 0.
\]

Note that even once the charges of each centre have been fixed, there is a large space of solutions that may even have several disconnected components. In particular, the constraint (5.2) implies that the positions of the centres are generally restricted, defining a complicated moduli space of (generically) bound solutions.

The metric, gauge field and Kähler scalars of the solution are now given in terms of the harmonics by
\[
ds^2 = 2^{-2/3} Q^{-2} \left[ -(H^0)^2 (dt + \omega)^2 - 2L(dt + \omega)(d\psi + \omega_0) + \Sigma^2 (d\psi + \omega_0)^2 \right]
+ 2^{-2/3} Q \, dx^i \, dx^i,
\]
\[
A^A = -\frac{H^0}{Q^{1/2}} (dt + \omega) + \frac{1}{H^0} \left( H^A - \frac{L y^A}{Q^{1/2}} \right) (d\psi + \omega_0) + A^A_d,
\] (5.4)
\[
Y^A = \frac{2^{1/3} y^A}{\sqrt{Q}},
\]
where \(x^i \in \mathbb{R}^3\) and \(\psi\) is an angular coordinate with period \(4\pi\), and the functions appearing satisfy the relations
\[
d\omega_0 = \star dH^0,
\]
\[
dA^A_d = \star dH^A,
\]
\[
\star d\omega = (dH, H)
\]
\[
\Sigma = \sqrt{Q^{-3} - L^2 (H^0)^2},
\] (5.5)
\[
L = H_0 (H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0,
\]
\[
Q = \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right)^{2/3},
\]
\[
D_{ABC} y^A y^B = -2H_C H^0 + D_{ABC} H^A H^B.
\]

Here the Hodge star is with respect to the flat \(\mathbb{R}^3\) spanned by the coordinates \(x^i\) and \(D_{ABC}\) are the triple intersection numbers of the chosen basis of \(H^2(X)\). Note that the only equation in (5.5) for which there is no general solution in a closed form is the last one. In some cases, e.g., when \(b_2 = 1\), it is even possible to obtain a solution, in a closed form, to this equation.

The function \(\Sigma\) appearing in the metric in (5.4) is known as the entropy function. When evaluated at \(\vec{x}_a\) it is proportional to the entropy of the black hole whose horizon lies at \(\vec{x}_a\). This follows from the Bekenstein–Hawking relation and the fact that \(\Sigma\) determines the area of the horizon at \(\vec{x}_a\). If this area is zero then the centre at \(\vec{x}_a\) does not have any macroscopic entropy and, if the associated geometry does not suffer from a large curvature in this region, then there is no reason to believe stringy corrections will change this. In this case the centre is taken to correspond to a genuine ‘zero-entropy bit’ and if all centres in a given solution enjoy this property then the associated geometry is smooth and, according to the philosophy promulgated throughout these notes, we would like to associate it to a particular semiclassical state (potentially in a dual CFT). In such cases there are often good microscopic arguments for the absence of entropy. A centre with charge \(\Gamma = (1, p/2, p^2/8, p^3/48)\) corresponds to a single D6-brane wrapping the Calabi–Yau with all lower dimensional charges induced by...
the Abelian flux. A configuration with a single such centre can be spectral flowed (see, e.g., [22, 65]) to a single D6-brane with no flux and hence no additional degrees of freedom in the Calabi–Yau; thus ‘integrating out’ the Calabi–Yau degrees of freedom does not generate an entropy and the associated five-dimensional solution is smooth.

Multicentre configurations with every centre constrained to be of the above form have been studied in [11, 12, 66] and numerous other works by the same authors. Note that the associated four-dimensional solution is not smooth, but this is simply a result of the Kaluza–Klein reduction on a non-trivial $S^1$ fibration. This highlights an important distinction: while the entropy, determined by replacing $H^A$ in the definition of $\Sigma^1(H)$ with $\Gamma^A$, is a duality invariant notion, the smoothness of the resulting supergravity solution is not (see, e.g., [67]). Thus we will assume solutions with all centres satisfying $\Sigma^1(\Gamma^A) = 0$ are good candidate ‘microstate geometries’ (in the sense described in section 2) even though they may have naked singularities in some duality frames.

From (5.4) and (5.5) it may seem that the solutions are singular if $H^0$ vanishes, but this is not the case as various terms in $Q$ and $L$ cancel any possible divergences due to negative powers of $H^0$ (in fact, the BTZ black hole can, in the decoupling limit introduced in the next section, be mapped onto such a solution with $H^0$ vanishing everywhere).

An additional complication is the fact that even solutions satisfying the constraint equation (5.2) may still suffer from various pathologies, most notably closed timelike curves (CTCs). For instance, the prefactor to the $d\psi^2$ term in the metric may become negative if $\Sigma$ becomes imaginary. Unfortunately there is no simple criterion which can be used to determine if a given solution is pathology free. To fill in this gap [33, 35] devised the attractor flow conjecture, a putative criterion for the existence of (well-behaved) solutions which we will describe in section 5.3.

An essential feature of these solutions is that they are stationary but not static. In particular they carry quantized intrinsic angular momentum associated with the crossed electric and magnetic fields of the dyonic centres [33]

$$\tilde{j} = \frac{1}{2} \sum_{a < b} \langle \Gamma_a, \Gamma_b \rangle \tilde{x}_{ab}.$$  (5.6)

This will be important when quantizing the solution space as it is a necessary (but not sufficient) condition for the latter to be a proper phase space with a non-degenerate symplectic form. A solution space with vanishing angular momentum does not enjoy this property and must be completed to a phase space by the addition of velocities (see, e.g., [68]).

5.2. Decoupling

Some specific five-dimensional solutions obtained from an $\mathcal{N} = 2$ truncation of $\mathcal{N} = 8$ (i.e., compactification on $T^6$) have been studied using AdS/CFT [59, 61] by taking a decoupling limit of a dualized form of the solutions. This cannot be generalized to an arbitrary compact CY because the duality group of the latter is not known. To study the full class of solutions using AdS/CFT it is desirable to have a more general decoupling limit that can embed a larger subset of these solutions in an AdS$_3$ throat. The limit we will describe can be taken in a proper $\mathcal{N} = 2$ theory and will yield an AdS$_3 \times S^2$ near-horizon geometry [23]. This decoupling limit only works for solutions whose total charge does not contain any overall D6/KK-monopole charge so the relevant CFT is essentially the MSW CFT. Although the latter is not under

22 As described in [15] this would also imply that the four-dimensional metric associated with this 5D solution (via the 4D/5D connection of [63]) becomes imaginary as $\Sigma$ appears directly in the former.

23 The ‘MSW’ CFT is the theory that arises as the low-energy description of $M5$-branes wrapping an ample divisor in the Calabi–Yau. It is an $\mathcal{N} = (0, 4)$ superconformal field theory and it owes its name to the three authors of [4].
very good control, it is nonetheless possible to determine, from the asymptotics of a given geometry, the CFT quantum numbers of the corresponding state. It is also possible to use general CFT properties to determine various quantities such as the number of states in a given charge sector.

In [23] the decoupling limit of the solutions described above is defined by taking $\ell_5 \to 0$ ($\ell_5$ is the 5D plank length) while keeping fixed the mass of $M_2$ branes stretched between the various centres and wrapping the $M$-theory circle. In doing so we also fix the volume of the Calabi–Yau in 5D plank units and the length of the $M$-theory circle, $R$. Since the mass of such membranes is given by $m_{M2} \sim R \Delta x/\ell_5^3$, the coordinate distances between the centres, $\Delta x$, must be rescaled as $\ell_5^3$.

Alternatively, we can see this limit as a rescaling of the 5D metric under which the Einstein part of the action is invariant.

We now define new rescaled coordinates, $x'$, and harmonic functions, $H$, as

$$x^i = \ell_5^{-3} x^i, \quad H = \ell_5^{3/2} H,$$

(5.7)

By restricting to the region of finite $x'$ we are essentially keeping the mass of transverse, open membranes fixed while rescaling the original coordinates, $x^i$. One can see that, in these new variables, the structure of the solution (in terms of the harmonics) does not change in the decoupling limit except for an overall scaling of the metric by a factor of $\ell_5^{-2}$.

The rescaled harmonics do take a new form, however,

$$H_0^a = \sum_a p_0^a |x - x_a|, \quad H_A^a = \sum_a p_A^a |x - x_a|, \quad H_0 = \sum_a q_0^a |x - x_a| - R^{3/2}/4.$$

(5.8)

In particular note that all the constants have disappeared except the $D_0$-brane constant which now takes a fixed value. Related to this is the fact that the asymptotic value of the moduli are forced to the attractor point, $Y^A \sim p^A$ (this corresponds to sending the 4D Kähler moduli to $J^A = \infty p^A$).

Recall that the coordinate locations of the centres must satisfy the integrability constraint (5.2) and that this constraint depends on the value of the constants in the harmonic functions. As a consequence it is possible that, in taking the decoupling limit, some solutions cease to exist. For instance if we consider two $D4$–$D2$–$D0$ centres, $\Gamma_a$ and $\Gamma_b$, then the constraint equation in the decoupling limit is

$$\langle \Gamma_a, \Gamma_b \rangle_{x_{ab}} = h_0 p^0 = 0,$$

(5.9)

implying that $x_{ab}$, the inter-centre distance, must be infinite (unless the charges are parallel, $\langle \Gamma_a, \Gamma_b \rangle = 0$), so one of the centres is forced out of the finite region of the rescaled coordinates, $x'$, as we take the limit. We interpret this as implying that such centres cannot sit in the same decoupled AdS3 throat.

Because only the $D0$ constant survives, centres without $D6$ charge are no longer bound together. A related fact, which will emerge presently, is that the solution space of such centres does not have a non-degenerate symplectic form (because, in the decoupling limit, intrinsic angular momentum is proportional to $D6$ dipole moment) on it and hence cannot be quantized without the addition of additional degrees of freedom.

Even if a set of charges admits a solution that satisfies the constraint equations in the decoupling limit, the solution may develop other pathologies such as CTCs. To study these we may resort once more to the attractor flow conjecture of [33] which is argued, in [22], to extend to the decoupled solutions. We will not discuss this in any depth here except to
note that the fact that the asymptotic moduli, \( Y^A \), are forced to the attractor point for the total charge implies that only attractor flow trees which can be extended to trees starting from this value of the asymptotic moduli will survive the decoupling limit. Further discussion can be found in section 5.3 and [22].

The decoupled solutions are asymptotically AdS3 \( \times S^2 \) and their asymptotic form is given by

\[
ds_{5D}^2 = -\frac{\rho^2}{4U^3} \, dv \, d\psi + \frac{U^{-4}}{4} \left[ -R^2 (d^0)^2 \, dv^2 + D \, d\psi^2 \right] + 4U^2 \frac{d\rho^2}{\rho^2} + U^2 (d\theta^2 + \sin^2 \theta \, d\alpha^2) + O\left(\frac{1}{\rho^2}\right),
\]

\[
A^A_{5D} = -p^A \cos \theta \, d\alpha - D^{AB} q_B \, d\psi + O\left(\frac{1}{\rho^2}\right),
\]

\[
Y^A = \frac{p^A}{U} + O\left(\frac{1}{r^2}\right),
\]

where we have introduced some coordinate redefinitions. We first introduce standard spherical coordinates, \((r, \theta, \phi)\), on the base spatial \( R^3 \) of the solution with the axis of the sphere (the \( z \)-axis) aligned with the \( D_6 \) dipole moment of the solution

\[
\vec{d}_0 := \sum_a p_a \vec{x}_a.
\]

For brevity we introduce the notation \( d^0 = |\vec{d}_0| \) and \( \vec{e} = \vec{r} / r \), so \( d^0 \) is the norm of the dipole moment and \( \vec{e} \) is a unit vector in the radial direction. We then make a further coordinate redefinition

\[
v = t = \frac{R}{4} \psi, \quad \alpha = \phi + R d^0 \left(\frac{p^3}{3}\right)^{-1} v,
\]

\[
U^3 = \frac{p^3}{6} \quad \text{and} \quad D = \frac{p^3}{3} (D^{AB} q_A q_B - 2q_0),
\]

and define a new radial coordinate \( \rho \) via

\[
\rho^2 = \frac{U^{-4}}{4} \left[ R \left( \frac{\varepsilon \cdot d^A \varepsilon_{ABC} p^B p^C}{3} - \frac{p^A q_A d^0 \cos \theta}{3} \right) + \frac{R}{U} r, \right]
\]

where \( \varepsilon^A \) is the \( D_4 \) dipole moment, defined analogously to \( \vec{d}_0 \). To make a connection to the dual CFT the solution needs to be reduced along the \( S^2 \) to give a theory defined purely on AdS3 with KK modes in representations of \( SU(2) \). This procedure is reviewed in some detail in [69] with particular attention to the subtleties involved in reducing the 5D Chern–Simons terms in the supergravity action.

The resultant metric is asymptotically AdS3 and from this, and the asymptotic form of the gauge field, we can determine the charges in dual field theory (see [23] for details). In particular we find that

\[
L_0 = \left(\frac{p^A q_A}{2p^3}\right)^2 - q_0 + \frac{p^3}{24},
\]

\[
\tilde{L}_0 = \left(\frac{p^A q_A}{2p^3}\right)^2 + \frac{p^3}{24},
\]

\( \Box \)
and that the $SU(2)$ R-symmetry charge associated with a solution is determined entirely in terms of its $D6$ dipole moment

$$J_0^3 = -\frac{R^2 d^0}{8}.$$  \hfill (5.17)

While (5.16) gives the charges expected from general considerations of the MSW CFT associated with the total charge $\Gamma$, the $SU(2)$ charge, $J_0^3$, depends on a dipole moment and, as such, is absent in the single-centre solution. In fact, it is nothing more than the intrinsic angular momentum, $J$, defined in (5.6) which, in the decoupling limit, can be shown to be proportional to the $D6$ dipole moment.

In the next section we will see that this charge plays a distinguished role in the quantization of the system as its presence is necessary in order to have a non-trivial symplectic form on the phase space.

5.3. Split attractors and state counting

In [33] a conjecture is proposed whereby pathology-free solutions are those with a corresponding attractor flow tree in the moduli space. This conjecture was first posed for multicentred four-dimensional solutions, so we will introduce some four-dimensional terminology here. The four-dimensional moduli, $t^A(\vec{x}) = B^A(\vec{x}) + iJ^A(\vec{x})$, are the complexified Kähler moduli of the Calabi–Yau. The relation between these moduli and their five-dimensional counterparts can be found in [22, 63]. To each charge vector, $\Gamma_i$, we can associate a complex number, the central charge, as

$$Z(\Gamma_i; t) := \langle \Gamma_i, \Omega(t) \rangle \quad \Omega(t) := -\frac{e^t}{\sqrt{4J^3}},$$  \hfill (5.18)

Note that, since $t^A$ is a two-form, $\Omega$ is a sum of even degree forms. The phase of the central charge, $\alpha(\Gamma_i) := \arg[Z(\Gamma_i; t)]$, encodes the supersymmetry preserved by that charge at the given value of the moduli. The even form $\Omega$ is related, asymptotically, to the constants in the harmonics (5.1) (which define both the 4D and 5D solutions) as

$$h = -2 \text{Im}(e^{-\alpha(\Gamma)}\Omega)|_{\infty}.$$  \hfill (5.19)

For more details of the 4D solution the reader should consult [14].

An attractor flow tree is a graph in the Calabi–Yau moduli space beginning at the moduli at infinity, $t^A|_{\infty}$, and ending at the attractor points for each centre. The edges correspond to single-centre flows towards the attractor point for the sum of charges further down the tree. Vertices can occur where single-centre flows (for a charge $\Gamma = \Gamma_1 + \Gamma_2$) cross walls of marginal stability where the central charges are all aligned ($|Z(\Gamma)| = |Z(\Gamma_1)| + |Z(\Gamma_2)|$). The actual flow of the moduli $t^A(\vec{x})$ for a multicentred solution will then be a thickening of this graph (see [33, 35] for more details). According to the conjecture a given attractor flow tree will correspond to a single connected set of solutions to the equations (5.2), all of which will be well behaved. An example of such a flow is given in figure 2.

The intuition behind this proposal is based on studying the two-centre solution for charges $\Gamma_1$ and $\Gamma_2$. The constraint equations (5.2) imply that when the moduli at infinity are moved near a wall of marginal stability (where $Z_1$ and $Z_2$ are parallel), the centres are forced infinitely far apart

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{\langle h, \Gamma_1 \rangle} = \frac{\langle \Gamma_1, \Gamma_2 \rangle |Z_1 + Z_2|}{2 \text{Im}(Z_2 Z_1)}|_{\infty}.$$  \hfill (5.20)
Figure 2. Sketch of a three-centre attractor flow tree from [22, 35]. Lines with arrows indicate single-centre attractor flows while straight lines without arrows are walls of marginal stability.

The tree starts at the circle on top (the moduli at infinity) and flows towards the attractor points indicated by the boxes. Note here that $\Gamma_4 = \Gamma_1 + \Gamma_2$ and $\Gamma = \Gamma_4 + \Gamma_5$. On the walls of marginal stability the moduli are such that $|Z(\Gamma; t)| = |Z(\Gamma_1; t)| + |Z(\Gamma_2; t)|$ (horizontal wall on top) and $|Z(\Gamma_4; t)| = |Z(\Gamma_1; t)| + |Z(\Gamma_2; t)|$ (diagonal wall on bottom left).

(This figure is in colour only in the electronic version)

In this regime the actual flows in moduli space are well approximated by the split attractor trees, since the centres are so far apart that the moduli will assume single-centre behaviour in a large region of spacetime around each centre. Thus in this regime the conjecture is well motivated. Varying the moduli at infinity continuously should not alter the BPS state count, which corresponds to the quantization of the two-centre moduli space, so unless the moduli cross a wall of marginal stability we expect solutions smoothly connected to these to also be well defined. Extending this logic to the general $N$-centre case requires an assumption that it is always possible to tune the moduli such that the $N$ centres can be forced to decay into two clusters that effectively mimic the two-centre case. There is no general argument that this should be the case, but one can run the logic in reverse, building certain large classes of solutions by bringing in charges pairwise from infinity, and this can be understood in terms of attractor flow trees. What is not clear is that all solutions can be constructed in this way. For more discussion the reader should consult [34].

Although this conjecture was initially proposed for asymptotically flat solutions, in [23] it was argued that the essential features of the attractor flow conjecture continue to hold in the decoupling limit.

For generic charges the attractor flow conjecture also provides a way to determine the entropy of a given solution space. The idea is that the entropy of a given total charge is the sum of the entropy of each possible attractor flow tree associated with it. Thus the partition function receives contributions from all possible trees associated with a given total charge and specific moduli at infinity. An immediate corollary of this is that, as emphasized in [35], the partition function depends on the asymptotic moduli. As the latter are varied, certain attractor trees will cease to exist; specifically, a tree ceases to contribute when the moduli at infinity cross a wall of marginal stability (MS) for its first vertex, $\Gamma \rightarrow \Gamma_1 + \Gamma_2$, as evident from (5.20).

For two-centre solutions one can determine the entropy most easily near marginal stability where the centres are infinitely far apart. In this regime locality suggests that the Hilbert state contains a product of three factors [35]

$$\mathcal{H}(\Gamma_1 + \Gamma_2; t_{\text{ms}}) \supset \mathcal{H}_{\text{int}}(\Gamma_1, \Gamma_2; t_{\text{ms}}) \otimes \mathcal{H}(\Gamma_1; t_{\text{ms}}) \otimes \mathcal{H}(\Gamma_2; t_{\text{ms}}).$$

(5.21)

24 Since attractor flow trees do not have to split at walls of marginal stability, there will in general be other contributions to $\mathcal{H}(\Gamma_1 + \Gamma_2; t_{\text{ms}})$ as well.
Since the centres move infinitely far apart as $t_{\text{inf}}$ is approached, we do not expect them to interact in general. There is, however, conserved angular momentum carried in the electromagnetic fields sourced by the centres and this also yields a non-trivial multiplet of quantum states. Thus the claim is that $\mathcal{H}_{\text{int}}$ is the Hilbert space of a single spin $J$ multiplet where $J = \frac{1}{2}(|\langle \Gamma_1, \Gamma_2 \rangle| - 1).$\textsuperscript{25} $\mathcal{H}(\Gamma_1)$ and $\mathcal{H}(\Gamma_2)$ are the Hilbert spaces associated with BPS brane excitations in the Calabi–Yau and their dimensions are given in terms of a suitable entropy formula for the charges $\Gamma_1$ and $\Gamma_2$ valid at $t_{\text{inf}}$.

Thus, if the moduli at infinity were to cross a wall of marginal stability for the two-centre system above the associated Hilbert space would cease to contribute to the entropy (or the index). A similar analysis can be applied to a more general multicentred configuration like that in figure 2 by working iteratively down the tree and treating subtrees as though they correspond to single centres with the combined total charge of all their nodes. The idea is, once more, that we can cluster charges into two clusters by tuning the moduli and then treat the clusters effectively like individual charges. We can then iterate these arguments within each cluster. This counting argument mimics the constructive argument for building the solutions by bringing in charges from infinity and is hence subject to the same caveats, discussed above.

Altogether the above ideas allow us to determine the entropy associated with a particular attractor tree, which, by the split attractor flow conjecture corresponds to a single connected component of the solutions space. The entropy of a tree is the product of the angular momentum contribution from each vertex (i.e., $|\langle \Gamma_1, \Gamma_2 \rangle|$) times the dimension of $\mathcal{H}(\Gamma_1)$ times the entropy associated with each node. When we want to compare against the number of states derived from quantizing the classical phase space, however, the latter factor (from the nodes) will not be included as it is not visible in the supergravity solutions.

In the following sections, we will show that it is also possible, in the two- and three-centre cases, to quantize the solution space directly and to match the entropy so derived with the entropy calculated using the split attractor tree. This provides a non-trivial check of both calculations.

Before proceeding to count the number of states associated with an attractor flow tree we should mention an important subtlety in using the attractor flow conjecture to classify and validate solutions. Certain classes of charges will admit so called scaling solutions\textsuperscript{[35, 70]} which are not amenable to study via attractor flows. These solutions are characterized by the fact that the constraint equations (5.2) have solutions that continue to exist at any value of the asymptotic moduli. We will discuss these solutions in greater detail in the next subsection, but it is important to note here that the general arguments given in this section (such as counting of states via attractor flow trees) do not apply to scaling solutions.

5.4. Scaling solutions

As noted in [35, 70], for certain choices of charges it is possible to have points in the solution space where the coordinate distances between the centres go to zero. Moreover, this occurs for any choice of moduli, so it is, in fact, a property of the charges alone.

Such solutions occur as follows. We take the inter-centre distances to be given by $r_{ab} = \lambda \Gamma_{ab} + \mathcal{O}(\lambda^2)$ (fixing the order of the $ab$ indices by requiring the leading term to always be positive). As $\lambda \to 0$ we can always solve (5.2) by tweaking the $\lambda^2$ and higher terms. The leading behaviour will be $r_{ab} \sim \lambda \Gamma_{ab}$, but clearly this is only possible if the $\Gamma_{ab}$ satisfy the triangle inequality. Thus any three centres with intersection products $\Gamma_{ab}$ satisfying the triangle inequalities define a scaling solution.

\textsuperscript{25} The unusual $-1$ in the definition of $J$ comes from quantizing additional fermionic degrees of freedom [23, 44].
We will in general refer to such solutions as scaling solutions meaning, in particular, supergravity solutions corresponding to $\lambda \sim 0$. The space of supergravity solutions continuously connected (by varying the $\vec{x}_p$ continuously) to such solutions will be referred to as scaling solution spaces. We will, however, occasionally lapse and use the term scaling solution to refer to the entire solution space connected to a scaling solution. We hope the reader will be able to determine, from the context, whether a specific supergravity solution or an entire solution space is intended.

These scaling solutions are interesting because (a) they exist for all values of the moduli; (b) the coordinate distances between the centres go to zero; and (c) an infinite throat forms as $\lambda^{-2}$. Combining (b) and (c) we see that, although the centres naively collapse on top of each other, the actual metric distance between them remains finite in the $\lambda \rightarrow 0$ limit. In this limit an infinite throat develops looking much like the throat of a single-centre black hole with the same charge as the total charge of all the centres. Moreover, as this configuration exists at any value of the moduli, it looks a lot more like a single-centre black hole (when the latter exists) than generic non-scaling solutions. As a consequence of the moduli independence of these solutions, it is not clear how to understand them in the context of attractor flows; the techniques developed in [23] provide an alternative method to quantize these solutions that applies even though the attractor tree does not.

Unlike the throat of a normal single-centre black hole, the bottom of the scaling throat has a non-trivial structure. If the charges, $\Gamma_a$, are zero entropy bits (e.g., D6s with an Abelian flux), then the five-dimensional uplifts of these solutions will yield smooth solutions in some duality frame and the throat will not end in a horizon but will be everywhere smooth, even at the bottom of the throat. Outside the throat, however, such solutions are essentially indistinguishable from single-centre black holes. Thus such solutions have been argued to be ideal candidate spacetime realizations of microstates corresponding to single-centre black holes. In [35], it was noticed that some of these configurations, when studied in the Higgs branch of the associated quiver gauge theory, enjoy an exponential growth in the number of states unlike their non-scaling cousins which have only polynomial growth in the charges.

5.5. Quantization

As anticipated in [18], since the dual (0,4) CFT of $N = 2$ black holes (lifted to five dimensions) is less well understood than its (4,4) cousin, in this case one can hope to make progress by quantizing the phase space of the supergravity solutions directly. The quantization we will perform will be quite general in that it will cover the original 4D multicentre black-hole configurations [14], their 5D uplift discussed in section 5.1 and the decoupled version of the latter (which can be related to the (0,4) CFT). For $N = 2$ black holes coming from Calabi–Yau compactifications, it is likely that a large portion of the entropy will come from degrees of freedom in the Calabi–Yau, so it is likely not be possible to account for a finite fraction of the entropy using supergravity alone. Whether this is the case or not is an important question, though the answer may be difficult to determine as the solutions are rather complicated and even the classical phase space is quite non-trivial.

A picture where black-hole microstates are realized in spacetime as extended bound states would associate to a given total charge, $\Gamma$, all possible decompositions of $\Gamma$ into $N$ charges, $\Gamma_a$, positioned at different centres, such that the solutions are smooth. The associated phase space for each decomposition would then be the (possibly disconnected) solution space for the charges. Quantization of this phase space should, in principle, yield certain microstates.
of the black hole and the set of (supergravity) microstates should come from summing over all possible decompositions. The notion of smoothness is not necessarily (duality frame) invariant, so a more precise criterion might be that the constituent charges, \( \Gamma_a \), should have no entropy (even microscopic) associated with them.

For a given decomposition into \( N \) centres, the phase space will be the \((2N - 2)\)-dimensional submanifold of \( \mathbb{R}^{3N-3} \) given by solving the constraint equations (5.2) for the positions \( \vec{x}_a \). Note that in arriving at this counting we have subtracted the three-centre of mass degrees of freedom; while these are present they generally decouple. As mentioned, this manifold may be disconnected and may possess a rather complicated topology. Moreover, for a given total charge, \( \Gamma \), there will be many possible decompositions into different numbers of centres implying that the total phase space will be a disconnected sum of many manifolds of different dimension. Determining the symplectic structure of even the lowest dimensional solution spaces is already challenging [23].

5.6. Symplectic form

In order to quantize the phase spaces described in section 5.5 we will need to determine the symplectic structure on these spaces. This can, in principle, be derived from the supergravity action as done, for instance, in [17]. In this case, however, it is far more tractable to take a different approach [23]. As discussed in [44], the four-dimensional multicentred solutions can also be analysed in the probe approximation by studying the quiver quantum mechanics of D-branes in a multicentred supergravity background. Moreover, a non-renormalization theorem [44] implies that the terms in the quiver quantum mechanics Lagrangian linear in the velocities do not receive corrections, either perturbatively or non-perturbatively. We can use this fact to calculate the symplectic form in the probe regime and extend it to the fully backreacted solution; this is because, for time-independent solutions, the symplectic form depends only on the terms in the action linear in the velocity.

For this approach to be consistent, it is necessary that the BPS solution space, which we interpret as a phase space, of the four- and five-dimensional supergravity theories, as well as the probe theory, all match. This follows from the fact that they are all governed by the same equation, (5.2) [44]. For instance, one can see that a probe brane of charge \( \Gamma_a \) in the background generated by a charge \( \Gamma_b \) is forced off to infinity as a wall of marginal stability is approached [44], analogous to what was described below equation (5.9) for the corresponding supergravity solutions.

In [23], the symplectic form on the solution space is determined. We will not review the derivation in detail, but simply note that it arises from the term coupling the probe brane to the background gauge field, \( \dot{x}^i A_i \), giving

\[
\dot{\Omega} = \frac{1}{2} \sum_p \delta x_p^i \wedge \langle \Gamma_p, \delta A_d(x_p) \rangle, \tag{5.22}
\]

where \( A_d \) is the ‘spatial’ part of the gauge field defined in (5.4) (this descends naturally to the spatial part of the 4D gauge field). Using the definition of \( A_d \), we can further manipulate this expression [23] and put it in the form

\[
\dot{\Omega} = \frac{1}{4} \sum_{p \neq q} \langle \Gamma_p, \Gamma_q \rangle \epsilon_{ijk} \frac{\delta(x_p - x_q)^i \wedge \delta(x_p - x_q)^j (x_p - x_q)^k}{|x_p - x_q|^3}. \tag{5.23}
\]

Actually, the full set of supergravity microstates probably requires considering more general solutions than those with Gibbons–Hawking base, but one can at least hope that the latter set contributes a finite fraction of the entropy.
This is a two-form on the \((2N - 2)\)-dimensional solution space which is a submanifold of \(\mathbb{R}^{3N-3}\) defined by (5.2). Moreover, one can show that, on this submanifold, this form is closed and, in the cases we will investigate below, non-degenerate. Thus it endows the solution space with the structure of a phase space. Note that, as anticipated, the centre of mass degrees of freedom do not appear in the symplectic form above and hence decouple in the quantization of the system. They will, in principle, yield an overall pre-factor in the partition function which we will not take into account.

Although the constraint equations (5.2) are invariant under global SO(3) rotations, these are nonetheless (generically) degrees of freedom of the system and this is reflected in the symplectic form. If we contract (5.23) with the vector field that generates rotations around the 3-vector \(n_i\) (i.e., we take \(\delta x^i_{pq} = \epsilon^{ijk} n_j x^k_{pq}\)), then the symplectic form reduces to

\[
\tilde{\Omega} \rightarrow n^i \delta J^i, \tag{5.24}
\]

where \(J^i\) are the components of the angular momentum vector defined in (5.6).

This is nothing more than the statement that the components \(J^i\) are the conjugate momenta associated with global SO(3) rotations. In general the symplectic form on any of our phase spaces\(^{27}\) will have terms like the above coming from the global SO(3) rotations, in addition to terms depending on other degrees of freedom. As advertised, (5.24) implies that solution spaces with any \(J^i = 0\) will have a degenerate symplectic form and will therefore not constitute a proper phase space.\(^{28}\)

### 5.7. Quantizing the two-centre phase space

The inter-centre position of a two-centre configuration is fixed in terms of the charges and the moduli at infinity, but the axis of the centres can still be rotated, so, neglecting the centre of mass degree of freedom, we are left with a solution space that is topologically a two-sphere with diameter

\[
x_{12} = \frac{\langle h, \Gamma_1 \rangle}{\langle \Gamma_1, \Gamma_2 \rangle}. \tag{5.25}
\]

The symplectic form (5.23) is proportional to the standard volume form on the two-sphere and is entirely of the form (5.24) (note here that, as mentioned in footnote 27, collinearity of the solution implies that one \(U(1) \subset SO(3)\) decouples). In terms of standard spherical coordinates, it is given by

\[
\tilde{\Omega} = \frac{1}{2} (\Gamma_1, \Gamma_2) \sin \theta \, d\theta \wedge d\phi = |J| \sin \theta \, d\theta \wedge d\phi. \tag{5.26}
\]

We can now quantize the moduli space using the standard rules of geometric quantization. We introduce a complex variable \(z\) by

\[
z^2 = \frac{1 + \cos \theta}{1 - \cos \theta} e^{2i\phi} \tag{5.27}
\]

and find that the Kähler potential corresponding to \(\tilde{\Omega}\) is given by

\[
K = -2|J| \log(\sin \theta) = -|J| \log \left( \frac{z \overline{z}}{(1 + z \overline{z})^2} \right). \tag{5.28}
\]

The holomorphic coordinate \(z\) represents a section of the line bundle \(L\) whose first Chern class equals \(\tilde{\Omega}/(2\pi)\). The Hilbert space consists of global holomorphic sections of this line bundle

\(^{27}\)This does not hold for solution spaces with unbroken rotational symmetries, such as solution spaces containing only collinear centres or only a single centre. In these cases some SO(3) rotations act trivially, do not correspond to genuine degrees of freedom and do not appear in the symplectic form.

\(^{28}\)As mentioned in footnote 27, this does not hold in the two-centre case where some SO(3) directions decouple. There are also potential subtleties with solution spaces where \(J = 0\) at a single point, but we neglect these for now.
and a basis of these is given by \( \psi_m(z) = z^m \). However, not all of these functions are globally well behaved. By examining the norm of \( \psi_m \) given by

\[
|\psi_m|^2 \sim \int \mathrm{dvol} e^{-K} |\psi_m(z)|^2 \sim \int \mathrm{d} \cos \theta \mathrm{d}\phi (1 + \cos \theta)^{|J|+m} (1 - \cos \theta)^{|J|-m}
\]

(5.29)

one finds that \( \psi_m \) only has a finite norm if \( m \geq -|J| \) and \( m \leq |J| \). The total number of states equals \( 2|J| + 1 \). This is in agreement with the wall-crossing formula up to a shift by 1. It can be shown that the inclusion of fermionic degrees of freedom will get rid of this unwanted shift [23].

The integrand in (5.29) is a useful quantity as it is also the phase-space density associated with the state \( \psi_m \). According to the logic in [19, 21, 24] (reviewed in section 2), the right bulk description of one of the microstates \( \psi_m \) should be given by smearing the gravitational solution against the appropriate phase-space density, which here is naturally given by the integrand in (5.29). Since there are only \( 2|J| + 1 \) microstates, we cannot localize the angular momentum arbitrarily sharply on the \( S^2 \); rather, it will be spread out over an area of approximately \( \pi/|J| \) on the unit two-sphere. It is therefore only in the limit of large angular momentum that we can trust the description of the two-centred solution (with two centres at fixed positions) in supergravity.

5.8. Quantizing the three-centre phase space

The phase space of the three-centre case is four dimensional. Placing one centre at the origin (fixing the translational degrees of freedom) leaves six coordinate degrees of freedom, but these are constrained by two equations. This leaves four degrees of freedom, of which three correspond to rotations in \( \text{SO}(3) \) and one of which is related to the separation of the centres.

This space is most easily visualized in the decoupling limit29 for the case when one of the centres has no \( D_6 \)-brane charge. In that case, the solution has an angular momentum vector \( J^i \) directed between the two centres with \( D_6 \) charge and the orientation of the direction of this vector defines an \( S^2 \) in the phase space. The third centre is free to rotate around the axis defined by this vector providing an additional \( U(1) \), which we will coordinatize by an angle \( \sigma \), fibred non-trivially over the \( S^2 \). Finally the angular momentum vector has a magnitude which may be bounded from both below and above, and this provides the final coordinate in the phase space. This construction is perhaps not the most obvious one from a coordinate space perspective, but in these coordinates the symplectic form takes a simple and convenient form. This can also be used when all the centres have \( D_6 \) charge, but then the relation between these coordinates and the locations of the centres are less straightforward.

The symplectic form in these coordinates is (see [23] for a derivation)

\[
\tilde{\Omega} = j \sin \theta \, d\theta \wedge d\phi - d j \wedge D\sigma
\]

(5.30)

with \( D\sigma = d\sigma - A, j = |J| \), and \( dA = \sin \theta \, d\theta \wedge d\phi \), so that \( A \) is a standard monopole one-form on \( S^2 \). The gauge field \( A \) implements the non-trivial fibration of \( \sigma \) over the \( S^2 \). A convenient choice for \( A \) is \( A = -\cos \theta \, d\phi \), so that finally the symplectic form can be written as a manifestly closed two-form

\[
\tilde{\Omega} = -d(j \cos \theta) \wedge d\phi - d j \wedge d\sigma.
\]

(5.31)

Let us consider the shape of the solution space spanned by coordinates \([\theta, \phi, j, \sigma]\). \( \theta \) and \( \phi \) are standard spherical coordinates with the latter defining a \( U(1) \) that degenerates

29 We will work in the decoupling limit simply because in this limit \( \hat{J} \) is directly proportional to the \( D_6 \) dipole moment, so it is easier to visualize it. The general characteristics of the solution space described hold for all incarnations of the solutions we have considered—the 4D solutions, their 5D uplift and the decoupling limit of the latter. We will use the language of 4D charges simply for brevity.
at $\theta = 0, \pi$. As has already been mentioned, the angular momentum usually spans some range $j \in [j_-, j_+]$ though there are cases where, for fixed charges, the angular momentum spans two separate ranges resulting in a solution space with two disconnected components (which must be quantized separately). On the boundaries of these regions, at $j = j$ or $j = j_+$, the centres are collinear, so the coordinate $\sigma$ degenerates. Thus the phase space is a symplectic manifold with two $U(1)$ actions (corresponding to rotations in $\sigma$ and $\phi$) which degenerate at special points. In fact the manifold is a toric Kähler manifold and can thus be quantized using the technology of [71, 72].

This is done in detail in [23], and here we will report only the results. Using the technology of geometric quantization, we can determine the basis of states spanning the Hilbert space (defined as the space of normalizable holomorphic sections of an appropriate line bundle over the phase space). It turns out that the number of such states is given by

$$\mathcal{N} = (j_+ - j_- + 1)(j_+ + j_- + 1).$$

(5.32)

In fact this is not quite correct as we have neglected fermionic degrees of freedom in defining our phase space. It is possible to include these degrees of freedom and quantize the resulting system [23], and this slightly changes the set of states. The final result becomes

$$\mathcal{N} = (j_+ - j_-)(j_+ + j_-).$$

(5.33)

This is the number we now wish to compare to the entropy determined by the wall-crossing formula.

Let us consider the attractor flow tree depicted in figure 2. For the given charges, $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$, there are in fact many different possible trees, but, in terms of determining the relevant number of states, the only thing that matters is the branching order. In figure 2, the first branching is into charges $\Gamma_1$ and $\Gamma_2 = \Gamma_1 + \Gamma_2$, so the degeneracy associated with this split is $|\langle \Gamma_4, \Gamma_3 \rangle|$ and the degeneracy of the second split is $|\langle \Gamma_1, \Gamma_2 \rangle|$ giving a total number of states

$$\mathcal{N}_{\text{tree}} = |\langle \Gamma_1, \Gamma_2 \rangle||\langle \Gamma_3 + \Gamma_{23} \rangle|,$$

(5.34)

where we have adopted an abbreviated notation, $\Gamma_{ij} = \langle \Gamma_i, \Gamma_j \rangle$.

To compare this with the number of states arising from geometric quantization of the solution space, equation (5.33), we need to determine $j_+$ and $j_-$. Recall that the latter correspond to two different collinear arrangements of the centres and, in a connected solution space, there can be only two such configurations [23]. To relate this to a given attractor flow tree, we will assume part of the attractor flow conjecture, namely that we can tune the moduli to force the centres into two clusters as dictated by the tree. For the configuration in figure 2, for instance, this implies we can move the moduli at infinity close to the first wall of marginal stability (the horizon dark blue line) which will force $\Gamma_3$ very far apart from $\Gamma_1$ and $\Gamma_2$. In this regime, it is clear that the only collinear configurations are $\Gamma_1 = \Gamma_2 = \Gamma_3$ and $\Gamma_2 = \Gamma_1 = \Gamma_3$; it is not possible to have $\Gamma_3$ in between the other two charges. Since $j_+$ and $j_-$ always correspond to collinear configurations they must, up to signs, each be one of

$$j_1 = \frac{1}{2}(\Gamma_{12} + \Gamma_{13} + \Gamma_{23})$$

(5.35)

$$j_2 = \frac{1}{2}(-\Gamma_{12} + \Gamma_{13} + \Gamma_{23}).$$

(5.36)

$j_+$ will correspond to the larger of $j_1$ and $j_2$ and $j_-$ to the smaller, but, from the form of (5.33), we see that this will only effect $\mathcal{N}$ by an overall sign (which we are not tracking carefully in any case). Thus

$$\mathcal{N} = \pm(j_1 - j_2)(j_1 + j_2) = \pm\Gamma_{12}(\Gamma_{13} + \Gamma_{23}),$$

(5.37)

which nicely matches (5.34).
Of course to obtain this matching we have had to assume the attractor flow conjecture itself (in part), so it does not serve as an entirely independent verification. Demonstrating this matching more carefully would help validate both methods of state counting [23].

5.9. More than three centres?

The symplectic form (5.23), when non-degenerate, defines a phase-space structure on the solution space for an arbitrary number of centres. Analysing the solution space for a generic set of charges however is quite difficult as the constraint equations (5.2) imbue this space with a complicated geometric structure. In the two- and three-centre case we were able to do this because the space had a toric structure. Fortunately there is a much larger class of charges that also enjoy this property; namely any configuration with two generic charges, $\Gamma_1, \Gamma_2$, interacting with any number, $N$, of mutually BPS particles $\Gamma_i$.

From the constraint equation (5.2),

$$\frac{\langle \Gamma_i, \Gamma_1 \rangle}{x_{1i}} + \frac{\langle \Gamma_i, \Gamma_2 \rangle}{x_{2i}} = \alpha$$

and

$$\frac{\langle \Gamma_1, \Gamma_2 \rangle}{x_{12}} + \sum_i \frac{\langle \Gamma_1, \Gamma_i \rangle}{x_{1i}} = \beta,$$

it is clear that the $U(1)$ around the $\vec{x}_{12}$ axis do not appear in the constraint equations (i.e., the separation, $x_{ij}$, between mutually BPS centres decouples) so each new coordinate $x_i$ comes with an additional $U(1)$ isometry.

Two interesting examples in this class were studied in [23]. The first case is obtained by setting all the $\Gamma_i = \Gamma_1 \neq \Gamma_2$. This corresponds to having a single-centre $\Gamma_1$ surrounded by a gas of particles of charge $\Gamma_2$ which do not interact with each other and sit at a fixed distance $\langle \Gamma_2, \Gamma_1 \rangle/\alpha$ from the $\Gamma_1$. From (5.2) the solution space can be determined to be a product of $S^2$'s corresponding to the position of each of the $\Gamma_i$'s on a sphere centred at $\vec{x}_1$. This ‘halo’ configuration is interesting and has occurred before in the literature [35, 44] because it corresponds to a system with a non-primitive charge $(N + 1)\Gamma_2$. This is, in fact, a two charge system rather than an $N + 2$ charge system, since $N + 1$ centres have parallel charges. In this case [23] was able to use the technology of geometric quantization of toric manifolds to compute the degeneracy in this setting and match it to that computed using attractor trees [33] once more providing a nice consistency check between the two techniques.

A further generalization of this corresponds to setting $\Gamma_1, \Gamma_2 = (\pm 1, p/2, \pm p^3/8, p^3/48)$ with the + and − corresponding to centre 1 and 2, respectively and setting $\Gamma_i = (0, 0, 0, -q_i)$ (with $p, q_i > 0$ and $N = \sum_i q_i$). This system is of physical interest because the total charge corresponds to a D4–D0 black hole if we take $N > p^3/24$. Moreover, note that the charges we have selected satisfy $\Sigma (\Gamma_a) = 0$ so these geometries have no entropy associated with them and are candidate microstate geometries for the D4–D0 black hole.

Letting $I = p^3/6$ the regime $N < I/4$ is referred to as the polar regime where single-centred D4–D0 black holes do not exist. The regime $N > I/2$, on the other hand, correspond to the regime in which the total charges, $\Gamma_1, \Gamma_2, \Gamma_0 = \sum_i \Gamma_i$, are scaling in the sense defined in section 5.4. Recall that in the scaling regime the charges can collapse to a single point in the solution space and form an infinitely deep throat that strongly resembles a black hole to an outside observer. Hence, for $N > I/2$, it is the scaling solutions that are the most likely candidate microstate geometries.

This system was studied in [73, 74] where it was argued that D0 should yield the leading contribution to the black hole entropy after they expand into elliptic D2 via the Myers effect. It
is thus interesting to see how many states are captured by these smooth supergravity solutions. This computation was done in [23] and the entropy of these configurations was shown to have a leading behaviour of $N^{2/3}$ in the regime $N < I/2$. The entropy of a black hole in this regime, $I/4 < N < I/2$, is of the order $\sqrt{NI} \sim N$, so clearly these configurations cannot account for all the states.

It would be interesting to compute the entropy associated with these configurations in the regime $N > I/2$ and compare this with the entropy of a black hole in this regime.

5.10. Large scale quantum effects: scaling solutions

Although it has long been understood how to account for the number of black-hole microstates in string theory [1], this has generally been done in a dual field theory making it difficult to address some fundamental questions in black-hole quantum mechanics such as information loss via the Hawking radiation. For some microscopic black holes (such as those discussed in section 4), the ability to dualize to an F1–P system has allowed for a more detailed analysis of the structure of the microstates. For these black holes, it has been argued [57] that the average microstate is a highly quantum superposition of states with the corresponding spacetime a wildly fluctuating ‘fuzzball’. The very interesting part of this claim is that these fluctuations extend over a region of spacetime circumscribed by the putative black-hole horizon. The ‘metrics’ corresponding to the states in the superposition are all very different within the region which would be enclosed by a horizon in the naive black-hole solution, but they settle down very quickly to the same metric outside the horizon. Thus the remarkable claim of [57] is that the generic state in the black-hole ensemble has quantum fluctuations over a large region of spacetime reaching all the way to the black-hole horizon.

Unfortunately the black hole discussed in [57] is microscopic and has no horizon in supergravity (without higher derivative corrections); it would thus be very desirable to be able to demonstrate this type of behaviour in a system with a macroscopic black hole. In [23] an attempt was made to do exactly this. Scaling multicentre solutions can classically form arbitrarily deep throats that become infinitely deep in the strict $\lambda \to 0$ limit where the coordinate separation of the centres vanishes. We expect, however, that quantum effects will prohibit us from localizing the centres arbitrarily close together and will thus cap off the throat. What is remarkable about this is that the symplectic form, and hence the quantum exclusion principle, is not renormalized, as we increase $g_s$ (to interpolate between quiver quantum mechanics and gravity), so, even though gravitational effects increase the distance between the centres as the throat forms, the phase-space volume stays very small. Thus gravitational backreaction essentially blows up these quantum effects to a macroscopic scale. This is important not only because it is reminiscent of the large scale quantum fluctuations of the D1–D5 black hole, but also because a smooth geometry with an infinite throat would be hard to understand in the context of AdS/CFT. Many solution spaces with a scaling point persist and continue to exhibit scaling behaviour even after we take a decoupling limit making all the solutions asymptotically $\text{AdS}_3 \times S^7$. This is problematic as general arguments suggest that an infinitely deep throat in a smooth geometry that is asymptotically AdS would imply a continuous spectrum in the CFT. Thus it is comforting that the analysis of [23] reveals the infinite throat to be an artefact of the classical limit.

Before discussing this phenomena in more detail let us note some caveats. The states defined by quantizing the scaling solution spaces are not necessarily generic black-hole

30 Most of the relevant states are stringy states, so the term metric is not really appropriate. A more precise statement would be expectation values of a profile of the string in the F1–P system. See, e.g., [3, 57] for more details.
microstates (in fact, it is probable that such states require including additional stringy degrees of freedom in the phase space), so they may not reflect the behaviour of the actual black-hole ensemble. Also, the symplectic form was computed in the gauge theory and extended to gravity via a supersymmetric non-renormalization theory; it would be more insightful to have a direct supergravity computation of the symplectic form. These caveats notwithstanding, it is remarkable that these solutions exhibit a quantum structure on a large scale even though they are smooth with a small curvature everywhere. We will return to this point shortly.

What is actually determined in [23] is the maximal depth of an effective throat generated by trying to localize a state as much as possible in the small $\lambda \to 0$ region of the phase space (recall $\lambda \to 0$ is the scaling point where the centres coincide in coordinate space and an infinitely deep throat forms in the geometry). Specifically, a three-centre solution similar to the one described in the previous section with a pure fluxed D6–D6 pair and a single D0 with charge $-N$ is considered in its lowest angular momentum eigenstate and the expectation value of the harmonic $H_0$ and the D6–D0 separation is computed. The latter is shown to be of order $\epsilon \sim N/I \geq 1/2$ implying that the centres cannot be localized arbitrarily close to each other, so an infinite throat never forms. Rather a cap is expected to form at a scale set by the D0–D6 distance. A wave equation analysis for a scalar field in a simplified asymptotically AdS background with a capped of throat of order $\epsilon$ reveals that the corresponding mass gap in the CFT goes as $\epsilon/c$ where $c = 6I$ is the central charge of the CFT. The expected result, from comparison with the D1–D5 system (see, e.g., [3]), is a mass gap of order $1/c$ which is indeed what is found in this case as $\epsilon$ is bounded from below by $1/2$ (since $N \geq I/2$ for the solution to be scaling).

While the computation above is heuristic in many ways, it yields two very important qualitative lessons. The first is that quantization of these solution spaces as phase spaces resolves several classical pathologies such as infinitely deep throats and also clarifies the issue of bound states (see [23] for a discussion of this). More importantly, however, it demonstrates that classical solutions may be invalid even though they do not suffer from large curvature scales or singularities. This is an important point, so let us explore it further.

In this particular system, the phase-space structure of the supergravity theory can be related to that of quiver quantum mechanics by a non-renormalization theorem. In the latter the scaling solutions (at weak coupling) are analogous to electron–monopole bound states. The Heisenberg uncertainty implies the minimum inter-centre distance is of order $x_{ij} \sim \hbar$. Moreover because the solution space is a phase space rather than a configuration space, the coordinates are conjugate to other coordinates rather than velocities, so it is not possible to localize all coordinate directions with arbitrary precision by constructing delta-function states\(^{31}\). Thus this quantity will have a large variance, so $\delta x_{ij}/x_{ij} \sim 1$ for very small $x_{ij}$. At weak coupling, this is nothing more than the standard uncertainty principle and is not particularly surprising.

What is surprising is that this behaviour persists even once gravity becomes strong and the centres backreact stretching the infinitesimal coordinate distance between them to a macroscopic metric distance. Moreover, in this regime the depth of the throat is extremely sensitive to the precise value of $x_{ij}$ (see [75] for a numerical example), thus the large relative value of $\delta x_{ij}$ translates into wildly varying depths for the associated throat. The associated expectation values for any component of the metric have an extremely large variance $\delta g/g$ and so cannot possibly correspond to good semiclassical states. It is somewhat unusual to have classical configurations that cannot be well approximated by semiclassical states (i.e.,

\(^{31}\) Even if this were not the case, delta function localized states have a large spread in momentum and would thus destroy any bound state.
those with low variance), but here this can be seen to follow from the very small phase-space volume this class of classical solutions occupy.

6. Conclusions

One obvious limitation of our discussion has been the restriction to extremal supersymmetric black holes. Although these are the most tractable, eventually we would like to deal with non-supersymmetric, realistic black holes that Hawking radiate. In an interesting recent work, a connection between non-supersymmetric black holes and interacting fluid dynamics was found [76, 77], which suggests that the type of dual descriptions in terms of free gases of particles that we used in the supersymmetric cases may not be sufficient. This does not mean that such black holes cannot be studied using the approaches discussed in this review, but it does indicate that the nature of microstates as well as the extrapolation from weak to strong coupling is much more complicated. For some recent progress in obtaining the Hawking radiation from a microstate point of view, see, e.g., [78]. See also [19] for a naive and qualitative description of the microstates of an AdS–Schwarzschild in the weakly coupled gauge theory.

A key feature of the idea that black holes are effective descriptions of underlying extended bound states is that these bound states should roughly have an extent that agrees with that of the black hole. In particular, they should grow as the string coupling is increased in the same way as the black-hole horizon grows, a property emphasized and explicitly shown for the three-charge supertube in [79]. In [12] it was shown that a large class of multicentred bound solutions with the same asymptotic charges as black holes shows precisely this sort of growth with the string coupling. Another useful hint comes from the fluctuations at large proper distances in solutions of the kind needed to given effective black hole behaviour (section 5 and [23]). It would be interesting to show whether the spacetime realization of the generic black-hole microstate can grow in this way too, especially since such a growth would have important consequences for resolving the information paradox [29].

For large black holes, it is unlikely that the underlying microstates can be described in supergravity alone. To see this, recall that in AdS/CFT the states that are dual- to single-particle supergravity modes in the bulk are BPS states and their descendants. Therefore, one might expect that supergravity solutions only contain information about products of BPS operators and their descendants which are the duals of general multiparticle supergraviton states. Now in general, the phase in which AdS contains a thermal gas of supergravitons is separated from the phase in which the AdS space contains a black hole via a phase transition. This seems to indicate that supergravitons alone do not have enough degrees of freedom to account for the black hole entropy. For example, for AdS$^3 \times S^3$ one can show explicitly that this is the case [80]. Thus if it does turn out that all black holes are understood as the effective descriptions of extended bound states, the latter will likely involve many stringy degrees of freedom.

Along similar lines, it would be worthwhile exploring in more detail to what extent the multicentred solutions we considered in section 5 can account for a finite fraction of the entropy of the corresponding macroscopic black hole. They are not the most general 5D supersymmetric solutions, as we took special hyper-Kähler manifolds as our base, and in addition we ignored Calabi–Yau excitations and stringy excitations. Even if they do not contribute a finite fraction of the entropy, one may wonder whether one may at least find some typical black-hole microstates in this class. This question is very difficult to answer, because it is not clear what set of macroscopic observables we should precisely include in our definition of typicality. Finally, we should be careful to not view all smooth multicentred solutions as
associated with a single black hole. Many are more properly thought of as being associated with multicentred black holes. One may expect that only the solutions which are described by a single-centred attractor flow (this will include in particular many scaling solutions) are honest microstates of a single black hole [23]. Obviously, much more work is required in order to apply the philosophy outlined in these proceedings in greater detail to macroscopic supersymmetric black holes.

Another open problem is to provide a more detailed map between black objects in the bulk and ensembles in the boundary CFT. What distinguishes semiclassical ensembles from non-classical ones? What possible first laws of thermodynamics can these black object have? How many chemical potentials do we typically need to include in their dual description? Is there a natural way to describe multicentred bound states in the dual field theory? We have only begun to see some glimpses of answers to these questions in the examples we have described.

Among the many possible generalizations, we would like to mention the following results. For the $\text{AdS}_5 \times S^5$ case, a significant progress has been made in identifying the space of $1/4$ BPS geometries [81], and perhaps an extension of the state-geometry map onto the $1/4$ BPS case is now feasible. There was also recent progress in generalizing the notion of typical states from $N = 4$ SYM theory to more general four-dimensional SCFT [26].

7. Literature survey

Since the literature on this subject is voluminous, we will not attempt a comprehensive review, but will rather merely point the reader in the direction of many relevant works and attempt to give a sense of the status of various branches of the field. In so doing, we will no doubt miss some important developments but hope that references to these are contained in the works cited below.

7.1. The D1–D5(–P) and related systems

(i) The D1–D5 system

This configuration arises in type-IIB string compactified on $\mathcal{M} \times S^1$, where $\mathcal{M}$ is a four-dimensional Ricci-flat manifold. The supergravity solution describes $N_5$ D5-branes wrapping the full compact space and $N_1$ D1-branes wrapping the $S^1$; the most general metric corresponding to this configuration was constructed in [8] (also see [82–84]). Originally there were neither internal nor fermionic excitations. In [7], the internal excitations were included, and [55] included the fermionic excitations. The near-horizon geometry of this system is $\text{AdS}_3 \times S^3 \times \mathcal{M}$, so the dual theory is a CFT$_2$. It was in the context of this system that the fuzzball proposal was first proposed [2].

A nice, but slightly out-dated, review of the D1–D5 system and the fuzzball proposal is contained in [3]. A significant progress has occurred since this review: the supergravity phase space has been directly quantized [42, 43], and the idea that black holes are simply effective geometries has been studied in detail in the context of this system [21, 25, 56, 58, 85–87]. Supporting evidence for the stretched horizon idea was given in [58, 85]. In [85], it was argued that quantum gravity effects become important at scales larger than the Planck/string scales. In [58], a sub-ensemble of the original thermal ensemble was considered. It was shown that, even in this case, the area of stretched horizon matches the microscopic entropy of the subsystem to leading order. This continues to hold even after the inclusion of 1-loop string corrections. In [21, 25, 56, 87] various quantities in the CFT and supergravity were compared.
(ii) **Beyond the D1–D5 system**

Further generalizations of the D1–D5 system followed. They can be collected under two general themes: adding more charges and breaking supersymmetry. In the first case the aim was to add some charges (momentum most of the time) to reduce the amount of preserved supersymmetry without breaking all the supersymmetry. An incomplete list of references in this direction includes \[13, 88–94\].

In \[88\] a first attempt to construct dual geometries for D1–D5–P microstates was undertaken; here the momentum was added as a small perturbation of the D1–D5 system. The first success was achieved in \[13\], which was followed by other works \[89, 90, 94\]. In \[93\] some arguments were given to support the claim that higher order correction to these three-charge geometries would not generate a horizon or a singularity if they were not present at tree level.

There are also studies of microstates using a 1/4-BPS probe approach, see, e.g., \[38, 95\].

A class of five-dimensional smooth solutions, closely related to the D1–D5–P system, have also been the focus of an extended research programme \[9–12, 22, 23, 59, 66, 67, 70, 73, 75, 96–98\]. These solutions are discussed in section 5 of these proceedings.

On the other hand, less work has been done on non-BPS solutions \[78, 99–102\]. In \[99\] smooth non-supersymmetric geometries were constructed. They were then proven to be classically unstable in \[100\]. This classical instability was shown to give rise to the Hawking radiation in \[78\]. More details appeared in \[103\]. Other properties of these solutions were studied in \[101\]. Another set of non-supersymmetric solutions in four dimensions appeared in \[102\]. Another interesting study was the tunnelling of a collapse of a shell to fuzz-ball geometries \[104\] which is needed for a possible fuzz-ball like proposal for more realistic four-dimensional black holes.

(iii) **Bound versus unbound systems**

To avoid overcounting the states responsible for a black hole’s entropy, one needs to know that the solution one is dealing with describes a bound system. For the D1–D5 system, this can be achieved by adding D3 charge. This system has been studied in \[105\] in the context of the fuzzball proposal.

Other works in this direction are \[67, 86, 91\]. In \[91\] known D1–D5 (D1–D5–P) geometries were rewritten in a fibre × base form in order to gain some insight into their structure. In \[86\] another route was followed. Using the D1–D5 system as a prototype, this paper studied the dynamical behaviour of such systems and put forth a conjecture to distinguish bound systems. In \[22\] three charge systems were studied in an AdS$_3$ × S$^2$ decoupling limit and a criterion for bound configurations was proposed in terms of split attractor trees \[33\]. In \[23\] some of these solutions were quantized allowing a direct analysis of whether the relevant state is bound or not. In \[67\] another criterion was given using spectral flow in AdS$_3$ × S$^3$.

7.2. **Asymptotically AdS$_5$ × S$^5$ and related solutions**

In AdS$_5$ × S$^5$ one can, in principle, consider 1/2, 1/4, 1/8 and 1/16 excitations of the full string theory. The only known example of a supersymmetric black hole in AdS$_5$ with a finite area is the Gutowski–Reall black hole \[47, 106\], and this solution is only 1/16 BPS. There are 1/2 BPS black solutions \[107–110\], known as superstars, but these do not have a horizon in two-derivative supergravity. Nonetheless, a significant effort has gone into studying the possible smooth, asymptotically AdS$_5$ geometries with varying amounts of supersymmetry. For the 1/2 BPS case the solutions were completely classified in \[6\].

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(i) The $1/2$ BPS case

Various generalizations of the LLM [6] geometries have been considered. The notion of
typical states in an ensemble was explored in [19], while in [24] a ‘metric’ operator was
defined in the CFT and used to establish a criterion to determine which states will have
a well-defined classical dual geometry. Another direction of research, closely related
to topics discussed in these proceedings, involved quantization of the original LLM
geometries using phase space techniques [17, 18].

Thermal properties and Wilson loops in the CFT dual to these geometries were studied
in [111, 112], while in [113–115] $1/2$ BPS geometries corresponding to defects in the
CFT were considered. In [116] the relation between phase-space densities in the fermion
formulation of the theory and generalized Young tableaux is studied. In [117] the on-shell
action for the LLM geometries was derived.

In [118] different methods were used to calculate the entropy of a ‘black hole’ resulting
from coarse graining over LLM geometries. There was a perfect agreement between
entropies calculated by CFT and gravity coarse grainings.

There is, in fact, a large volume of literature on the $1/2$ BPS case, and our short survey
is by no means intended to be comprehensive. For more references the reader should
consult the works cited above.

(ii) Less SUSY

Backgrounds which preserve only $1/4$, $1/8$ and $1/16$ supersymmetry have also been
considered and explicit supergravity solutions have been constructed. For instance
smooth $1/4$ BPS solutions were constructed in [119–122] and an LLM-like prescription
to derive them from droplets on the plane is related to constraints on brane webs in
[81, 123].

Probe solutions (giant gravitons) preserving $1/8$ of the supersymmetry were studied
in [124–128] and the backreaction of such probes was worked out in [129, 130]. The $1/8$
BPS sector of the dual CFT was explored in [131].

As mentioned above, the $1/16$ BPS sector is distinguished by having black holes with
macroscopic horizons [47, 106, 132–134]. The $1/16$ sector of the CFT has also been
studied; operators potentially related to the black hole have been identified [135], and the
entropy has been (qualitatively) reproduced [136]. Giant gravitons preserving $1/16$ of
the supersymmetry were found in [137].

Attempts to treat the full set of $1/2$, $1/4$ and $1/8$ solutions in a common framework
can be found in [138, 139]. Other related work includes [140–142]. Once more, this list
is only intended to serve as an introduction to the literature and no doubt has failed to
include many important works.

(iii) No supersymmetry

Another direction of investigation involves breaking all supersymmetries. Some research
in this direction includes [143–145].

The considerations described above for the D1–D5 system and the $1/2$-BPS states of
AdS5s were applied to other backgrounds (an M-theory solution) in [146]. In [27], general
considerations are applied to study the differences between correlators of an operator in a
typical state and a thermal state. In this paper, it is also shown that for a system with an
entropy $S$, the variance in finitely local correlation functions over the entire Hilbert space will
be suppressed by $e^{-S}$. Because of this, regardless of the detailed origin of black hole entropy,
if there is any statistical interpretation of the underlying degeneracy, semiclassical observers
will have difficulty telling the microstates apart.
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