The contact interaction of size-dependent and multimodulus rectangular plate and beam

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Abstract. A mathematical model of contact interaction between a plate and a beam made of a multimodulus material is constructed in the work, taking into account some types of nonlinearities (physical and constructive) and size-dependence. To analyze the stress-strain state of this structure, the variational iteration method is used, which can be used as a basis for methods of consolidation partial differential equations to ordinary differential equations. The physical nonlinearity for structures made from a multimodulus material is taken into account according to the deformation theory of plasticity using the method of variable parameters of I.A. Birger. Numerical examples are given.

1. Introduction
Most modern materials exhibit various elastic reactions under the action of compression and stretching. Materials that have different tensile and compressive modulus are called multimodulus materials [1]. Such materials are used in various fields of industry, for example, in construction - concrete [2] and rocks [3], in mechanical engineering - metal alloy [4] and rubber [5], in biomedicine - biological materials [6], [7], etc. One such material is graphene. It is known that this is the hardest material with a minimum electrical resistivity, it has a higher compression modulus than the tensile modulus [8, 9]. Some composite materials also possess some modularity. These include graphite epoxy resin with graphite as a filler. In this case, the graphite material has a compression modulus four times larger than the tensile modulus [10].

In the second half of the last century, several mathematical models for the study of multimodulus materials were developed. The first is the Bert model [11], which is based on the criterion of positive-negative signs of deformations in longitudinal fibers. This model is widely used in multilayer composites [12-15]. The second is Ambartsumyan's bimodal model [16] for isotropic materials. This model evaluates the various modules in tension and compression on the basis of positive-negative signs of the main stresses, which is especially important for the analysis and design of structures. The modified model proposed by Jones is a model of the weighted correspondence matrix (WCM) for isotropic and orthotropic bimodule materials [17]. A wide review of the models of multimodulus materials is given in [18].
For the first time, the problem of the contact of elastic bodies initially touching at a point was formulated and solved by Hertz [19] in 1882. With the development of technology, contact problems have become relevant and important in the mechanics of a deformable solid. In connection with the complexity of these problems, a large number of approaches and mathematical methods for their solution appeared. Many attempts have been made to solve contact problems, including within the framework of variational inequalities [20]. The current state of research on this issue can be found in [21, 22, 23]. Recently, a new and non-standard contact state has been used, which includes memory effects [24, 25]. Stationary problems of the contact interaction of two full-size plates made of a multimodulus material were considered in [26].

Analysis of the current state of the problem of solving the problems of contact interaction between a plate and a beam, taking into account the diversity of the material, is far from complete. In addition, no mathematical model of such an interaction is constructed with allowance for physical nonlinearity. The present paper makes an attempt for the first time to solve this problem for size-dependent structures.

2. Problem formulation

Let us consider the problem of one-sided mechanical interaction between a rectangular planar size-dependent plate and a size-dependent beam. We consider the plate to be thin, its stress-strain state is described by the classical Kirchhoff theory, and the beam by the Euler-Bernoulli model supplemented by physical nonlinearity in the theory of small elastoplastic deformations (Fig. 1). Suppose that the contact pressure (normal to the stress surface) is much less than the normal stresses in the sections of the plate and the beam, the plate and beam in the contact areas freely slip.

Figure 1. The design scheme of the plate-beam structure.

The choice of the classical theory of the plate of the Kirchhoff model and the Euler-Bernoulli beam is due to the fact that the deformation of the transverse shear influences the stress-strain state and distribution of the contact pressure considerably less than the transverse reduction in the contact zone [27]. The last factor is one of the foundations of the proposed approach. Let us write down the original system of differential equations for contacting plates and beams in the form [27]:

\[
\begin{align*}
A_1w_1(x, y) &= q_1(x, y) - q_2y - \frac{\partial^2 w_1}{\partial t^2} - \frac{\partial^2 w_1}{\partial t}, \\
A_2w_2(x) &= q_2(x) + q_2y - \frac{\partial^2 w_2}{\partial t^2} - \frac{\partial^2 w_2}{\partial t},
\end{align*}
\]

where: \(w_1(x, y)\), \(w_2(x)\) - stand for the deflection of the plate and the beam, respectively; \(q_1\) - damping coefficient, \(q_2\) - damping coefficient, \(\Omega_1 = \{a_1(x, y) \leq z \leq h_1(x, y)\}\), \(\Omega_2 = \{a_2(x) \leq z \leq b_2(x)\}\) - areas occupied by a plate and a beam, respectively; \(\partial\Omega_1\) - borders of the areas, \(z = a_2(x), \quad z = b_2(x), \quad (x) \in \Omega_2\) - the equation of the external surface of the beam, which allows us to take into account the variability of the thickness in the
calculations. Accounting multimodulus material beams carried out on the basis of work Zhao [28], He [29]. The beam is divided into two under the "zero-surface" $z = c_i(x)$: 
$\Omega^+ = \{a_i(x) \leq z \leq c_i(x)\} \cup \Omega^- = \{c_i(x) \leq z \leq b_i(x)\}$. Material in the areas $\Omega^+$ is determined $E_2^i(x, z, e^{i+}_2)$ - Young’s modulus, appropriate stretching (+), in the areas $\Omega^-$ is determined $E_1^i(x, z, e^{i-}_1)$ - Young’s modulus, appropriate compression (-), $e^{i+}_2$ - intensity of deformation. The stress-strain state according to [29], is determined from the relations - variable modulus of elasticity of the beam: $\sigma^{i+}_x = E_2^i \varepsilon^{i+}_x$.

According to the modified couple stress theory proposed by Yang [30], the operator for a size-dependent plate has the form [31-32]:
\[
A_i w_i(x, y) = \left\{ \frac{p_i}{2} \frac{y^2}{\gamma_i} + \frac{1}{2} \frac{\partial^2 w_i}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w_i}{\partial y^2} + 2 \frac{\partial^2 w_i}{\partial x \partial y} \right\}, \quad \gamma_i = \frac{a}{h}, \quad \gamma_2 = \frac{l}{h}.
\]

The operator for a physically nonlinear problem describing the stress-strain state of a size-dependent beam of variable thickness made of a multimodulus material has the form:
\[
A_i w_i(x) = \left\{ \frac{p_i}{2} \frac{y^2}{\gamma_i} + \frac{1}{2} \frac{\partial^2 w_i}{\partial x^2} \right\} \frac{\partial^2}{\partial x^2} \left( B_{11,2} \frac{\partial^2 w_i}{\partial x^2} \right),
\]
where $B_{11,2} = \frac{1}{2} \left[ \frac{E_{11,2}}{E_{01,2}} - \frac{E_{21,2}}{E_{02,2}} - E_{21,2} - E_{22,2} \right]$, $E_{0m,2} = \int_0^l E_2^m z^m \, dz + \int_0^l E_2^m \, dz$. In the method of variable elasticity parameters of Birger [33], used to solve a physically nonlinear problem, $E_2^{i+}(x, z, e_2^{i+})$ - is not a constant, but a function that depends on the deformed state at a point, and is determined by means of formula [33]:
\[
E_2^{i+} = \frac{9K_{12}}{3K_{12} + G_2^{i+}},
\]
here we assume that the volume deformation $K_{12} = const$. In the deformation theory of plasticity, the shear modulus is determined:
\[
G_2^{i+} = \frac{1}{3} \frac{\sigma^{(i+)}_2}{e^{(i+)}_2},
\]
where $\sigma^{(i+)}_2$, $e^{(i+)}_2$ - intensity of stresses and deformations of the beam, where
\[
e^{(i+)}_2 = \frac{2}{3} \left[ \varepsilon^{(i+)}_x \right] \varepsilon^{(i+)}_x = -z \frac{\partial^2 w_i}{\partial x^2}.
\]

Boundary conditions of two types:
\[\text{a) articulated support:} \quad w_i \bigg|_{\Omega} = \frac{\partial^2 w_i}{\partial n^2} \bigg|_{\Omega} = 0, \quad i = 1, 2. \]
\[\text{b) rigid pinch:} \quad w_i \bigg|_{\Omega} = \frac{\partial w_i}{\partial n} \bigg|_{\Omega} = 0, \quad i = 1, 2. \]

In the system (1), the contact pressure proportional to the transverse compression $w_i - h_i - w_2$ in the contact zone of the plate and the beam has the form:
\[
q_i(x, y) = K \frac{E_2^{i+}}{h} (w_i - h_i - w_2),
\]
where we write the function $\psi$ in the form:

$$\psi = [1 + \text{sign}(w_i - h_k - w_j)]/2,$$

here $h_k$ - clearance between the elements, $K$ - coefficient of rigidity of transverse compression.

Due to the presence of a multiplier $\psi$ in equation (1), the system of equations becomes nonlinear, and the problem as a whole is constructively nonlinear. Constructive nonlinearity is associated with the possibility of switching on and off one-way links. Adding to this task a physical nonlinearity leads to the impossibility of solving it by direct methods. Therefore, the solution is carried out using the method of successive approximations together with the process of refining the boundaries of contact zones.

Formula (8) is written for the case of contact between a plate and a beam of the same thickness $h$ and with the same values of $E$, $K$. For contact problems of the theory of Kirchhoff plates and Euler-Bernoulli beams, Winkler's connection between compression and contact pressure is used. If there is a thin gasket between the plate and the beam, it is taken into account by changing the value $K$.

If the initial position of the plate and the beam (gap function $h_k$) and the load are such that they do not enter into contact during deformation, then $\psi = 0$ and the system (1) disintegration into two independent systems. Otherwise (1) is connected. We substitute (8) into (1):

$$\begin{align*}
    A_1w_1(x, y) &= q_1(x, y) - K \frac{E_1^+}{h} (w_1 - h_k - w_j)\psi - \frac{\partial^2 w_1}{\partial t^2} - \epsilon \frac{\partial w_1}{\partial t}, \\
    A_2w_2(x, y) &= q_2(x, y) + K \frac{E_2^+}{h} (w_1 - h_k - w_j)\psi - \frac{\partial^2 w_2}{\partial t^2} - \epsilon \frac{\partial w_2}{\partial t}.
\end{align*}$$

The system (10) should be considered in complex with the boundary conditions (6, 7) and the condition of non-penetration of the layers, this system constitutes a constructively and physically nonlinear problem.

To solve a stationary constructively nonlinear problem obtained from the dynamical system (10), it is possible to construct an iterative process that allows one to solve only one of the equations of the system (10) successively at each step of the loading. This allows you to reduce the order of the system of equations exactly 2 times for a two-layer package, if the $n$ layers, then in $n$ times. We write the iterative procedure in the form:

$$\begin{align*}
    A_1(w_1^{(n)}) + K \frac{E_1^+}{h} \psi_{(n-1)} w_1^{(n)} &= q_1(x, y) + K \frac{E_1^+}{h} (h_k + w_2^{(n-1)})\psi_{(n-1)}, \\
    A_2(w_2^{(n)}) + K \frac{E_2^+}{h} \psi_{(n-1)} w_2^{(n)} &= q_2(x, y) + K \frac{E_2^+}{h} (w_1^{(n-1)} - h_k)\psi_{(n-1)}.
\end{align*}$$

The corresponding boundary conditions (6) or (7) and the non-penetration condition of the layers should be added to the system (11). For the dynamic task, the iterative procedure is performed at each step in time.

If the values $E$, $h$ are different, then for the upper plate values are denoted by $E_1$ and $h_1$, and for the lower beam - $E_2$ and $h_2$:

$$q_1(x, y) = 2K \frac{E_1^+}{h_1} (1 + \frac{E_1^+h_1}{E_2^+h_2})(w_1 - h_k - w_2)$$

For a fixed contact zone, it is necessary to solve system (11) by one of the methods of reducing partial differential equations to ordinary differential equations.
3. Numerical results

As a numerical experiment, let us consider the contact interaction of two physically nonlinear Euler-Bernoulli beams made of a multimodulus material under the action of an external transverse load \( q = q_0 \sin(\omega_p t) \) under the following parameters: \( \omega_p = 5.1, \alpha^- = 0.8, \alpha^+ = 1 \). The material of beams is pure aluminum, the nonlinear diagram for which is described by the dependence:

\[
\sigma_s^{(i)\pm} = \alpha^\pm \sigma_s^{(i)} \left[ 1 - \exp \left( - \frac{e_i^{(i)\pm}}{\varepsilon_i^{(i)\pm}} \right) \right].
\]

Initially, the beams enter into a contact interaction at the amplitude of the load \( q_0 = 0.458 \), the character of the oscillations of both beams is harmonic at the frequency of external excitation \( \omega_p = 5.1 \). Further, at intensity \( q_0 = 0.7 \), the Hopf bifurcation occurs. As the load amplitude increases \( q_0 = 1.1 \), the oscillations go into a chaotic state through intermittency (Fig. 2).

![Figure 2](image)

Figure 2. The graph of the general oscillations of the beams (a), the phase portrait of the upper beam (b), the Fourier power spectrum (c) the upper beams at \( q_0 = 1.1 \).

A study was made of the effect of taking account of the different modularity on the contact interaction of beams. The analysis showed that the contact interaction of the beams from the multimodulus material occurs at lower loading amplitudes than in the case of a single-modulus material.

![Figure 3](image)

Figure 3. Dependence on the load of the deflection of the upper beam of a multimodulus material (red line), from a single-modulus material (blue dotted line).

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