Minimization and Synthesis of the Tail in Sequential Compositions of Mealy machines

Alberto Larrauri
Graz University of Technology

Roderick Bloem
Graz University of Technology

Abstract

We consider a system consisting of a sequential composition of Mealy machines, called head and tail. We study two problems related to these systems. In the first problem, models of both head and tail components are available, and the aim is to obtain a replacement for the tail with the minimum number of states. We introduce a minimization method for this context which yields an exponential improvement over the state of the art. In the second problem, only the head is known, and a desired model for the whole system is given. The objective is to construct a tail that causes the system to behave according to the given model. We show that, while it is possible to decide in polynomial time whether such a tail exists, there are instances where its size is exponential in the sizes of the head and the desired system. This shows that the complexity of the synthesis procedure is at least exponential, matching the upper bound in complexity provided by the existing methods for solving unknown component equations.

2012 ACM Subject Classification Hardware → Circuit optimization

Keywords and phrases

Acknowledgements
Minimization and Synthesis of the Tail in Sequential Compositions of Mealy machines

1 Introduction

The optimization of completely specified and incompletely specified Finite State Machines are classical problems [12, 7, 15]. In practice, systems are usually decomposed into multiple FSMs that can be considered more or less separately. These decompositions lead to problems that have been studied intensively [9, 10, 24]. First, to optimize a system, we can optimize its components separately. However, there may be more room for optimization when considering them together. This way, it is possible to find $T$ and $T'$ of different sizes that are not equivalent in isolation, but that can be used interchangeably in the context of another component $H$ [23]. Second, for a context $H$ and an overall desired behaviour $M$, we may look for a component $T$ such that $T \circ H \equiv M$ [2, 27, 24]. This problem may occur, for instance, in rectification, where a designer wants to repair or change the behaviour of a system by only modifying a part of it.

In this paper, we consider machines $H$ (head) and $T$ (tail), such that $T$ receives inputs from $H$ but not vice-versa. We study both the problem of minimizing $T$ without changing the language of the composition $T \circ H$ (Tail Minimization Problem) and the problem of building $T$ when we are given $H$ and a desired model for the system $M$ (Tail Synthesis Problem). We note that despite the simplicity of this setting, in various cases it is possible to tackle problems in more complex two-component networks using one-way compositions via simple reductions [25]. We show that, through polynomial reductions, our findings can be applied to any kind of composition between $H$ and $T$ where all of $T$'s outputs can be externally observed.

The first exact solution to the Tail Minimization Problem was given by Kim and Newborn [14]. They compute the smallest $T'$ by minimizing an incompletely specified (IS) Mealy machine $N$ with size proportional to $2^{\left|H\right|\left|T\right|}$. Minimization of IS machines is a known computationally hard problem [20]. Known algorithms for this task take exponential time in the size of $N$ [21, 1, 19], which cannot be improved under standard complexity-theoretic assumptions. This yields a doubly exponential complexity for the whole Kim-Newborn procedure. This is unsatisfactory as deciding whether a replacement $T'$ for the tail $T$ of a given size exists is an NP problem: given a candidate $T'$, we can compute the composition $T' \circ H$ and check language equivalence with the original system $T \circ H$ in polynomial time. The reason for the blow-up in the complexity of the Kim-Newborn method lies in a determinization step that results in $N$ being exponentially larger than $H$. Various non-exact methods have been studied to avoid this determinization step. For instance, Rho and Somenzi [15] present an heuristic in which a “summarized” incompletely specified machine is obtained, and in [13] Wang and Brayton avoid performing exact state minimization and in turn perform optimizations at the net-list logic level.

The observation that the Tail Minimization Problem can be solved efficiently with access to an NP oracle yields a straightforward solution via iterative encodings into SAT. As an improvement, we propose a modification of the Kim-Newborn procedure and that runs in singly exponential time. To develop our method, we introduce the notion of observation machines (OMs), which can be regarded as IS machines with universal branching. Our algorithm yields a polynomial reduction of the Tail Minimization Problem to the problem of minimizing an OM. To carry out this last task, we generalize the procedure for minimizing IS machines shown in [1] to OMs. Preliminary experimental results show that the proposed approach is much more efficient than the naive encoding into SAT on random benchmarks.

We then turn to the Tail Synthesis Problem. Here, we are given $H$ and $M$ and we look for a $T$ such that $T \circ H \equiv M$. Apart from finding such $T$, we may simply be interested
in deciding whether it exists. In this case we say that the instance of the problem is feasible. This problem is a particular case of so-called missing component equations \cite{24}, where arbitrary connections between \( H \) and \( T \) are allowed. The general approach given in \cite{27}, is based on the construction of a deterministic automaton representing the “flexibility” of \( T \), called the E-machine. Both the task of deciding whether an equation is feasible and the one of computing a solution in the affirmative case take linear time in the size of the E-machine. However, this automaton is given by a subset construction, and has size \( O(2^{|M|}|M|) \). This yields an exponential upper bound for the complexity of solving general unknown component equations as well as the complexity of only deciding their feasibility.

We show the surprising result that deciding the feasibility of the Tail Synthesis Problem takes polynomial time, but computing an actual solution \( T \) has exponential complexity, matching the bound given by the E-machine approach \cite{27}. This follows from the existence of instances of the synthesis problem where all solutions \( T \) have exponential size. We give a family of such instances constructively. Additionally, we show that it is possible to represent all solutions of the Tail Synthesis Problem via an OM. This representation avoids any kind of subset construction and hence is exponentially more succinct than the E-machine.

\section{Preliminaries}

\subsection*{General Notation}

We write \([k]\) for \([0, \ldots, k-1]\), \(2^X\) for the power set of \( X \), and \( X^*\) for the set of finite words of arbitrary length over \( X \). We use overlined variables \( \overline{x} = x_0 \ldots x_{n-1} \) for words, and write \( e \) for the empty word. Given two words of the same length we define \( (\overline{x}, \overline{y}) := (x_0, y_0) \ldots (x_n, y_n) \in (X \times Y)^* \).

\subsection*{Mealy Machines}

Let \( X \) and \( Y \) be finite alphabets. A Mealy machine \( M \) from \( X \) to \( Y \) is a tuple \((X,Y,S_M,D_M,\delta_M,\lambda_M,r_M)\), where \( S_M \) is a finite set of states, \( D_M \subseteq S_M \times X \) is a specification domain, \( \delta_M : D_M \to S_M \) is the next state function, \( \lambda_M : D_M \to Y \) is the output function and \( r_M \in S_M \) is the initial state. We say that an input string \( \overline{x} := x_0x_1 \ldots x_n \in X^* \) is \textit{defined} at a state \( s_0 \) if there are states \( s_1, \ldots, s_{n+1} \) satisfying both \( (s_i, x_i) \in D_M \) and \( s_{i+1} = \delta_M(s_i, x_i) \) for all \( 0 \leq i \leq n \). We call the sequence \( s_0, x_0, y_0, s_1, \ldots, s_n, x_n, y_n, s_{n+1} \), where \( y_i = \lambda_M(s_i, x_i) \), the \textit{run} of \( M \) on \( \overline{x} \) from \( s_0 \). When \( s_0 = r_M \), we simply call this sequence the run of \( M \) on \( \overline{x} \). We write \( \Omega_M(s) \) for the set of input sequences defined at state \( s \), and \( \Omega_M \) for \( \Omega_M(r) \). We lift \( \delta_M \) and \( \lambda_M \) to defined input sequences in the natural way. We define \( \delta_M(s, \epsilon) = s \) and \( \lambda_M(s, \epsilon) = \epsilon \) for all \( s \). Given \( \overline{x} \in \Omega_M(s) \), if \( s' = \delta_M(s, \overline{x}) \) and \( (s', x') \in D_M \) then \( \delta_M(s, \overline{x}x') = \delta_M(s', x') \) and \( \lambda_M(s, \overline{x}x') = \lambda_M(s, \overline{x}) \lambda_M(s', x') \). We write \( \delta_M(\overline{x}) \) and \( \lambda_M(\overline{x}) \) for \( \delta_M(r_M, \overline{x}) \) and \( \lambda_M(r_M, \overline{x}) \) respectively. We define \( \text{Out}(M) \) as the set of words \( \lambda_M(\overline{x}) \), for all \( \overline{x} \in \Omega_M \).

We say \( M \) is \textbf{completely specified}, if \( D_M = S_M \times X \). Otherwise we say that \( M \) is \textbf{incompletely specified}. From now on, we refer to completely specified Mealy machines simply as Mealy machines, and to incompletely specified ones as IS Mealy machines. For considerations of computational complexity, we consider alphabets to be fixed and the size \(|M|\) of a Mealy machine \( M \) to be proportional to its number of states.

We say that a \textbf{(completely specified) Mealy machine} \( N \) \textbf{implements} an IS Mealy machine \( M \) with the same input/output alphabets as \( N \) if \( \lambda_N(\overline{x}) = \lambda_M(\overline{x}) \) for all \( \overline{x} \in \Omega_M \). The problem of minimizing an IS Mealy machine \( M \) consists of finding a minimal implementation...
We define a cascade composition of Mealy machines. Let $T$ and $H$ be Mealy machines, where $T$ is the head and $H$ is the tail. The cascade composition $T \circ H$ is defined as follows:

- The initial state of $T \circ H$ is the initial state of $H$.
- The transition function of $T \circ H$ is defined as follows:
  
  $$(s, x) \rightarrow (s', y)$$

where $s, s' \in S_T$, $x \in X$, and $y \in Y$. We define $s_H := (s_T, x) := (s_H, x)$ and $s_T' := (s_T, y) := (s_T', y)$.

- The output function of $T \circ H$ is defined as follows:
  
  $$(s, x) \rightarrow (s', y)$$

where $s, s' \in S_T$, $x \in X$, and $y \in Y$. We define $y_H := (s_T, x) := (y_H, x)$ and $y_T' := (s_T, y) := (y_T', y)$.

Given a cascade composition $T \circ H$, we say that $T \circ H$ is a replacement for $T$ if $T' \circ H \equiv T \circ H$. We say that $T'$ is $n$-replaceable in $T \circ H$ if there is a replacement $T'$ for $T$ with at most $n$ states.

We study two problems related to the tail component $T$ of a cascade composition. In the first, both $H$ and $T$ are given and the goal is to find a minimal replacement $T'$ for $T$ that leaves the behaviour of the system unaltered. In the second, $H$ and $M$ are given instead and one is asked to find $T$ such that $T \circ H \equiv M$. We also consider two related decision problems.

**Problem statements**

We define a cascade composition of Mealy machines as a system consisting of two Mealy machines $H$, the head, and $T$, the tail, that work in sequential composition as shown in Figure 1a. We write $T \circ H$ to refer to this sequential composition. The behaviour such cascade composition can be described via another Mealy machine $M$ resulting from a standard product construction: Set $S_M := S_H \times S_T$ and $r_M := (r_H, r_T)$. Let $s_H \in S_H, s_T \in S_T, x \in X$ and $y := \lambda_H(s_H, x)$. We define $s_H := (s_T, x) := (s_H', s_T')$ where $s_H' := \delta_H(s_H, x)$ and $s_T' := \delta_T(s_T, y)$, and $\lambda_M((s_H, s_T), x) := \lambda_T(s_T, y)$. Given a cascade composition $T \circ H$, we say that a Mealy machine $T'$ is a replacement for $T$ if $T' \circ H \equiv T \circ H$. We say that $T'$ is $n$-replaceable in $T \circ H$ if there is a replacement $T'$ for $T$ with at most $n$ states.

We study two problems related to the tail component $T$ of a cascade composition. In the first, both $H$ and $T$ are given and the goal is to find a minimal replacement $T'$ for $T$ that leaves the behaviour of the system unaltered. In the second, $H$ and $M$ are given instead and one is asked to find $T$ such that $T \circ H \equiv M$. We also consider two related decision problems.
1. **Problem 1** (Tail Minimization Problem). Given a cascade composition $T \circ H$, find a replacement $T'$ for $T$ with the minimum amount of states.

2. **Problem 2** (Tail Synthesis Problem). Given Mealy machines $H$ and $M$ sharing the same input alphabet, construct a Mealy machine $T$ so that $T \circ H \equiv M$.

3. **Problem 3** ($n$-Replaceability of the Tail). Given a cascade composition $T \circ H$ and a number $n \in \mathbb{N}$, decide whether there is a replacement $T'$ for $T$ with at most $n$ states.

4. **Problem 4** (Feasibility of the Tail Synthesis Problem). Given Mealy machines $H$ and $M$ with the same input alphabet, decide whether there exists some Mealy machine $T$ such that $T \circ H \equiv M$.

4. **Observation Machines**

We will now define observation machines, which can be regarded as IS Mealy machines [10] with universal branching. This construction allows us to express the solutions of the Tail Minimization Problem and the Tail Synthesis Problem while avoiding the determinization steps in [14] and [27]. An observation machine (OM) from $X$ to $Y$ is a tuple $M = (X, Y, S_M, D_M, \Delta_M, \lambda_M, r_M)$ defined the same way as a Mealy machine except for the next-state function $\Delta_M$, which now maps elements of the specification domain to sets of states $\Delta_M : D_M \rightarrow 2^{S_M \setminus \{0\}}$. A run of $M$ over $\pi \in X^*$ starting from $s_0 \in S_M$, is a sequence $s_0, x_0, y_0, s_1, \ldots, s_n, x_n, y_n, s_{n+1}$, where $(s_i, x_i) \in D_M$, $s_{i+1} \in \Delta_M(s_i, x_i)$ and $y_i = \lambda(s_i, x_i)$ for all $0 \leq i \leq n$. We call such run simply a run of $M$ on $\pi$ when $s_0 = r_M$. We say that a sequence $\pi$ is defined at a state $s$ if there is a run of $M$ on $\pi$ starting from $s$. As with Mealy machines, we put $\Omega_M(s)$ and $\Omega_M$ for the sets of defined sequences at $s$ and at $r_M$ respectively. We say that $M$ is consistent if all runs of $M$ on any given defined input sequence $\pi \in \Omega_M$ have the same output. Unless otherwise specified, we assume all OMs to be consistent. In this case we can lift $\lambda$ to defined input sequences, writing $\lambda_M(\pi)$ for the unique output sequence $\overline{\pi}$ corresponding to all runs of $M$ over $\pi$. We also lift the transition function $\Delta_M$ to defined sequences and sets of states as done previously for NFAs. Note that when $M$ has no branching, i.e. $|\Delta_M(s, x)| = 1$ for all $(s, x) \in D_M$, we end up with a construction equivalent to an IS Mealy machine. Again, for considerations of computational complexity, we consider alphabets to be fixed, and the size $|M|$ to be proportional to $|S_M| + \sum_{(s,a) \in D_M} |\Delta_M(s, a)|$.

We can use consistent OMs to represent specifications over Mealy machines. We say that a machine $N$ implements a OM $M$ with the same input/output alphabets as $N$ if $\lambda_M(\pi) = \lambda_N(\pi)$ for all $\pi \in \Omega_M$. Note that it is straightforward to test this property in $O(|N||M|)$ time via a product construction. We say that $M$ is $n$-implementable if it has an implementation with at most $n$ states. The problem of minimizing an OM $M$ is the one of finding an implementation of it with the minimum number of states.

Informally, the reason we call the branching “universal” is that the i/o language of an OM $M$ is given by an universal automata of the same size, but not by a non-deterministic one. The intuitive argument is that once an input word leaves $\Omega_M$, all behaviours are allowed. In a way, this means that $M$ accepts the complement of $\Omega_M$, which is given by an NFA.

5. **Optimization of the Tail**

In this section we introduce a novel solution for the Tail Minimization Problem. This solution improves over the state of the art, represented by the Kim-Newborn (K-N) method [14], by...
Minimization and Synthesis of the Tail in Sequential Compositions of Mealy machines

Avoiding an expensive determinization step during the process. Two important observations are the following.

▶ **Observation 1.** The \( n \)-replaceability decision problem Problem 3 is in NP: Given a candidate \( T' \) with \(|S_{T'}| \leq n \) it takes polynomial time to build Mealy models for \( T \circ H \) and \( T \circ H' \), and to decide whether they are equivalent.

▶ **Observation 2.** A Mealy machine \( T' \) is a replacement for \( T \) if and only if \( \lambda_{T}(\overline{y}) = \lambda_{T'}(\overline{y}) \) for all \( y \in \text{Out}(H) \).

Because of Observation 1, there are exponential-time algorithms for Problem 1 and Problem 3, there is a straightforward ("naive") polynomial reduction of the \( n \)-replaceability problem into a satisfiability problem along the lines of bounded synthesis. This encoding can be used to optimize the tail of a cascade composition by finding the minimum \( n \) for which the resulting CNF formula is satisfiable. The exponential complexity of this procedure contrasts with the double-exponential complexity of the K-N algorithm. This reasoning also applies to more general networks of Mealy machines, implying that the approach for component minimization based on optimizing the "E-machine" is not optimal in theory, as it takes doubly exponential time.

In the following, we study the problem in more detail and present an novel approach that we will compare against the naive encoding in the experimental section.

### 5.1 Proposed Minimization Algorithm

We give an overview of our minimization method here. The algorithm is divided in three steps: (1) We compute an NFA \( A \) which accepts the language \( \text{Out}(H) \), (2) using \( A \) and \( T \) we build an OM \( M \) whose set of implementations is precisely the set of replacements for \( T \). By Observation 2, this is ensured by \( \Omega_{M} = \text{Out}(H) \) and \( \lambda_{M}(\overline{y}) = \lambda_{T}(\overline{y}) \) for all \( \overline{y} \in \Omega_{M} \). Lastly, (3) we find an implementation \( T' \) of \( M \) with the minimum number of states. This machine \( T' \) is a minimal replacement for \( T \).

This algorithm follows the K-N procedure but skips an expensive determinization step. Indeed, the difference is that in the K-N method the NFA \( A \) obtained in (1) is determined before performing (2). This removes the branching from \( M \), making it an IS machine, but it yields \( |M| = O(2^{|H||T|}) \) rather than \( |M| = O(|H||T|) \) in our method. Step (3), the last one, is responsible of the overall complexity of our algorithm as well as the K-N one, and takes \( 2^{|M|^{O(1)}} \) time. This makes the total time costs of our procedure and the K-N one \( 2^{(|H||T|)^{O(1)}} \).
The Image Automaton

Let $H$ be a Mealy machine from $X$ to $Y$. In this section we describe how to obtain an NFA whose language is $\text{Out}(H)$. The image automaton of $H$ (sometimes called the inverse automaton), written $\text{Im}(H)$, is the NFA over $Y$ defined as follows: Let $S_{\text{Im}(H)} := S_H$ and $r_{\text{Im}(H)} := r_H$. We set $\Delta_{\text{Im}(H)}(s_1, y) := \{s_2 \in S_H \mid \exists x \in X \text{ s.t. } \lambda(s_1, x) = y, \delta(s_1, x) = s_2 \}$. It holds that $\mathcal{L}(\text{Im}(H)) = \text{Out}(H)$. Essentially, to obtain $\text{Im}(H)$ one deletes the input labels from $H$’s transitions, as shown in Figure 3a. The time and space complexity of this construction is $O(|H|)$.

The Restriction Machine

Let $A$ be an NFA over $Y$, and $T$ a machine from $Y$ to $Z$. In this section our goal is to build an OM $M$ whose set of defined sequences $\Omega_M$ is precisely $\mathcal{L}(A)$ and that satisfies $\lambda_M(\overline{y}) = \lambda_T(\overline{y})$ for all $\overline{y} \in \Omega_M$. The restriction of $T$ to $A$, denoted by $T|_A$, is the OM $M$ from $Y$ to $Z$ defined as follows. Let $S_M := S_T \times S_A$ and $r_M := (r_T, r_A)$. Given a state $s_M := (s_T, s_A) \in S_M$ and an input $y \in Y$ there are two possibilities: (1) $\Delta_A(s_A, y) \neq \emptyset$. In this case we mark the transition as defined $(s_M, y) \in D_M$, and set $\Delta_M(s_M, y) = \{(s_T', s_A') \mid s_T' = \delta_T(s_T, y), s_A' \in \Delta_A(s_A, y)\}$. Alternatively, (2) $\Delta_A(s_A, y) = \emptyset$. Here we just mark the transition as undefined $(s_M, y) \notin D_M$. It is direct to see that both $\Omega_M = \mathcal{L}(A)$ and $\lambda_M(\overline{y}) = \lambda_T(\overline{y})$ for all $\overline{y} \in \Omega_M$.

An example is given in Figure 3b. This product construction generalizes the one in the K-N algorithm: when $A$ is deterministic, so is $M$, yielding an IS Mealy machine. The construction of $M := T|_A$ can be performed in $O(|T||A|)$ time. In the case where $A := \text{Im}(H)$ results from the head $H$ of a cascade, we can substitute $|A| = O(|H|)$. 

\[ \begin{array}{c}
\text{(a) The image automaton } \text{Im}(H) \text{ of the machine } H \text{ in Figure 2a.} \\
\text{(b) The restriction } M := T|_{\text{Im}(H)} \text{ of the Machine } \\
T \text{ in Figure 2a to the automata } A \text{ in Figure 2a.}
\end{array} \]
Minimization and Synthesis of the Tail in Sequential Compositions of Mealy Machines

Reduction to a Covering Problem

Let $M$ be an OM from $Y$ to $Z$. Our objective is to find a minimal implementation of $M$. In order to do this, we generalize the theory of [8] for minimization of IS machines. The basic idea is that we can define a compatibility relation $\sim$ over $S_M$ and use it to reduce the task to a covering problem over $S_M$. Two states $s_1, s_2 \in S_M$ are compatible, written $s_1 \sim s_2$, if $\lambda_M(s_1, \gamma) = \lambda_M(s_2, \gamma)$ for all $\gamma \in \Omega_M(s_1) \cap \Omega_M(s_2)$. We note that this relation between states is symmetric and reflexive, but not necessarily transitive. A set $Q \subseteq S_M$ is called a compatible if all its states are pairwise compatible. We say that $Q$ is incompatible at depth $k$ if $k$ is the length of the shortest word $\gamma \in Y^*$ such that $\gamma$ is defined for two states $s_1, s_2 \in Q$, and $\lambda_M(s_1, \gamma) \neq \lambda_M(s_2, \gamma)$. A convenient characterization is as follows:

**Lemma 3.** Let $Q \subseteq S_M$. The following statements hold: (1) $Q$ is incompatible at depth 1 if and only if for some $s_1, s_2 \in Q$ there is a defined input $y \in Y$ with $\lambda_M(s_1, y) \neq \lambda_M(s_2, y)$. (2) If $Q$ is incompatible at depth $i > 1$ then $\Delta_M(Q, y)$ is incompatible at depth $i - 1$ for some $y \in Y$.

Lemma 3 gives a straightforward way to compute the compatibility relation $\sim$ over $S_M$. We begin by finding all pairs $s_1, s_2$ incompatible at depth 1. Afterwards we propagate the incompatible pairs backwards: if $s_1 \sim s_2$ and $s_1 \in \Delta_M(s_1', y)$, $s_2 \in \Delta_M(s_2', y)$, then $s_1' \sim s_2'$. This process can be carried out in $O(|M|^2)$ time (see Appendix A for a reduction to Horn SAT). When $M = T_{\text{im}(H)}$, it holds $|M| = O(|H||T|)$, and the required time to compute the $\sim$ relation over $B$ is $O(|H|^2|T|^2)$.

Let $F \subseteq 2^S_M$ be a family where all $C \in F$ are compatibles. We call $F$ a closed cover of compatibles over $M$ if the following are satisfied: (1) $r_M \in C$ for some $C \in F$, and (2) for any $C \in F$ and $y \in Y$ there is some $C' \in F$ such that $\Delta(C, y) \subseteq C'$. The following theorem implies that the problem of finding a minimal implementation for $M$ is polynomially equivalent to the problem of finding a closed cover of compatibles over it with the minimum size. This equivalence is illustrated in Figure 4a.

**Theorem 4.** Let $M$ be an OM from $Y$ to $Z$. Let $|M| := \sum_{s,y} |\Delta_M(s, y)|$ be the number of transitions in $M$. The following two statements hold: (1) Let $N$ be an implementation of $M$. Then it is possible to build a closed cover of compatibles $F$ over $M$ with $|F| \leq |S_N|$ in $O(|M||N|)$ time. (2) Let $F$ be a closed cover of compatibles over $M$. Then it is possible to build an implementation $N$ of $M$ with $|S_M| = |F|$ in $O(|M||F|)$ time.

**Proof.** We prove (1) and (2) separately. (1): Let $N$ be an implementation of $M$. For each $s_N \in S_N$ we define the set $Q(s_N) \subseteq S_M$ as follows: $Q(s_N) = \{ s_M \in S_M \mid \exists \gamma \in \Omega_M(s_N) s.t. \delta_M(\gamma) = s_N, s_M \in \Delta_M(\gamma) \}$. It holds that each $Q(s_N)$ is a compatible, as the output of each state $s_M \in Q(s_N)$ over a defined input sequence has to coincide with the output of $s_N$ in $N$. Moreover, $r_M \in Q(r_M)$, and for any $s_N \in S_N, y \in Y$ it holds that $\Delta_M(Q(s_N), y) \subseteq \Delta_M(s_N, y)$. Thus, the family $F := \{ Q(s_N) \mid s_N \in S_N \}$ is a closed cover of compatibles over $N$, and $|F| \leq |S_N|$. Note that $|F|$ may be strictly less than $|S_N|$, as for some $s_N^1, s_N^2 \in S_N$ it may happen that $Q(s_N^1) = Q(s_N^2)$. The sets $Q(s_N)$, and with them the family $F$, can be derived from a product construction between $M$ and $N$ which takes $O(|M||N|)$ time. (2): Let $F$ be a closed cover of compatibles over $M$. We build an implementation $N$ of $M$: Set $S_N := F$. Thus, each state in $N$ corresponds to compatible $C \in F$. We define $r_N$ as an arbitrary $C \in F$ satisfying $r_M \in C$. Let $C \in F$ and $y \in Y$. We define $\delta_N(C, y)$ as an arbitrary $C' \in F$ such that $\Delta_M(C, y) \subseteq C'$. To define $\lambda_N(C, y)$ we take into account two possibilities: (1) $(s_M, y) \in D_M$ for some $s_M \in C$. Then we set $\lambda_N(C, y) = \lambda_M(s_M, y)$. Note that $\lambda_N(C, y)$ is independent of the particular choice of $s_M$,
as \( M \) being consistent implies that all choices yield the same output. (2) \( (s_M, y) \notin D_M \) for all \( s_M \in C \). In this case put an arbitrary output for \( \lambda_N(C, y) \). It can be checked that \( N \) is indeed an implementation of \( M \) (Appendix \( B \)). The Mealy machine \( N \) given by this construction satisfies \(|S_N| = |F|\), and can be built in \( O(|M||N|) \) time and space.

\[\begin{array}{c}
\text{(a) The relation of the OM } M \text{ in Figure 3b,}\\
together with a closed cover of compatibles } P \text{ and } Q
\end{array}\]

\[\begin{array}{c}
\text{(b) A Mealy machine } T' \text{ corresponding to the cover in Figure 4a.}
\end{array}\]

\[\begin{array}{c}
\text{Figure 4 Final step of the minimization of } T \text{ in Figure 2. The machine } T' \text{ in Figure 4b is a}\\
\text{minimal replacement for } T.
\end{array}\]

**Obtaining a Minimal Replacement**

Let \( M \) be an OM from \( Y \) to \( Z \). Further suppose that \( M = T|_{\text{Im}(H)} \) for some machines \( T, H \). Replacements of \( T \) in \( T \circ H \) are precisely the implementations of \( M \). Thus, solving the Tail Minimization Problem amounts to finding a minimal implementation of \( M \). As shown in the previous section, this is equivalent to finding a minimal closed cover of compatibles over \( M \). This reduction has been widely employed in the study of the analogous minimization problem for IS Mealy machines [21, 1, 16]. We propose an adaptation of the method in [1] as a tentative approach for finding a minimal implementation of \( M \).

Given a bound \( n \), we reduce the problem of finding a closed cover \( F \) of \( n \) compatibles over \( M \) to a SAT instance. The CNF encoding follows closely the one in [1], and the details can be found at Appendix \( C \). As them, we also compute in advance a so-called partial solution. This is clique of pairwise incompatible states \( Cl \subseteq S_M \) obtained via a greedy algorithm. Each state in \( Cl \) must belong to a different compatible in \( F \), so the clique can be used for adding symmetry breaking predicates to the encoding and thus to reduce solving times.

In order to obtain the minimal replacement for \( T \) in \( T \circ H \) we look for the minimum \( n \) that yields a satisfiable CNF encoding. It is clear that such \( n \) must lie in the interval \([|Cl|, |S_T|]\). In particular, when \(|Cl| = |S_T|\), the machine \( T \) is already optimal and no encoding is needed. Otherwise, any searching strategy for \( n \) in \([|Cl|, |S_T|]\) may be employed. In our case, we simply employ linear search from \(|Cl|\) upwards, which is expected to perform well when \(|Cl|\) is a good estimate for the optimal \( n \), as is the case in [1].
5.2 Complexity of the Tail Minimization Problem

As evidenced in Observation 1, deciding whether the tail in a cascade composition is n-replaceable is an NP problem. For completeness sake we show that this result is tight, meaning that the problem is NP-hard as well. To the best of our knowledge, this complexity result has not been shown elsewhere.

**Theorem 5.** Deciding whether the tail in a cascade composition is n-replaceable is an NP-hard problem.

**Proof.** Let $N$ be an IS machine and let $n \in \mathbb{N}$. We build machines $H$ and $T$ in polynomial time satisfying that $N$ is n-implementable if and only if $T$ is n-replaceable in $T \circ H$. As deciding whether $N$ is n-implementable is an NP-complete problem \[20\], this reduction proves the theorem. Informally, the aim is to build $H$ whose output language coincides with $\Omega_N$, and use an arbitrary implementation of $N$ as $T$. This idea does not quite work because it requires $\Omega_N$ to have no maximal words, which not may be the case, but the problem can be fixed adding some extra output symbols and transitions. Let $\bar{Y}$ and $\bar{Z}$ be the input and output alphabets of $N$ respectively. Let $\bar{Y} := Y \cup \{\perp\}$, $\bar{Z} := Z \cup \{\perp\}$, where $\perp$ is a fresh symbol. We begin by building a machine $H$ from $Y$ to $\bar{Y}$. We set $S_H := S_N \cup \{\ast\}$, where $\ast$ is a fresh state, and $r_H := r_N$. Given $(s, y) \in S_N \times Y$, we define $\delta_H(s, y) := \delta_N(s, y)$ and $\lambda_H(s, y) := \lambda_N(s, y)$ if $(s, y) \in D_N$, and $\delta_H(s, y) := \ast$ and $\lambda_H(s, y) := \perp$ otherwise. We also define $\delta_H(\ast, y) := \ast$ and $\lambda_H(\ast, y) := \perp$ for all $y$. It is direct to see that $\text{Out}(H) = \Omega_N \{\perp\}^\ast$. Now we build another machine $T$ from $\bar{Y}$ to $\bar{Z}$. Let $N'$ be an arbitrary implementation of $N$, built in polynomial time by simply adding the missing transitions. To construct $T$ we add self loops $\delta_T(s, \perp) = s$, $\lambda_T(s, \perp) = \perp$ to $N'$ at each state $s \in S_N$.

Now we show that $T$ is $n$-replaceable in $T \circ H$ if and only if $N$ is n-implmentable. As exposed in Observation 2, a machine $T'$ is a valid replacement for $T$ if and only if for all $\overline{y} \in \text{Out}(H)$ $\lambda_{T'}(\overline{y}) = \lambda_T(\overline{y})$. Moreover, any word $\overline{y} \in \text{Out}(H)$ is of the form $\overline{\pi}(\perp)^k$, where $\overline{\pi} \in \Omega_N$. By construction $\lambda_{T'}(\overline{\pi}(\perp)^k) = \lambda_N(\overline{\pi})(\perp)^k$. This implies the following: (1) Let $T'$ be a replacement for $T$. Then removing all transitions on input $\perp$ from $T'$ yields an implementation of $N$ of the same size. (2) Let $T'$ be an implementation of $N$. Then adding self-loops $\delta_T(s, \perp) = s$, $\lambda_T(s, \perp) = \perp$ to all states of $T'$ yields a replacement for $T$ in $T \circ H$ with the same number of states. This proves the result.

6 Synthesis of the Tail

In this second part of the paper we study the Tail Synthesis Problem, and its associated Feasibility Problem. Our main result is the fact that while the Feasibility Problem has polynomial time complexity, there are instances of the Synthesis Problem where minimal solutions have exponential size, and hence the problem itself has exponential complexity. The proof of this result relies on a construction that, given a feasible instance of the Synthesis Problem, produces an OM that represents all its solutions.

The Tail Synthesis problem is a particular case of an “unknown component equation”, \[24\]. In the general problem, a component $H$ and a system model $M$ are known, and the goal is to find $T$ that connected to $H$ in a given way yields $M$. The general approach for solving these equations is given in \[27\]. This method is based on the computation of the “E-machine”, which is a DFA $E$ over input/output satisfying the following condition. A Mealy machine $T$ is a solution for the component equation if and only if all its traces $(\overline{y}, \lambda_T(\overline{y}))$ are accepted by $E$. Deciding whether such $T$ exists amounts to determining the winner of a safety game on $A$, and synthesising $T$ is equivalent to giving a winning-strategy for the Protagonist
player in this game. Both these tasks can be carried out in $O(|A|)$ time \[3,4,17\]. However, building the E-machine involves the determinization of a product construction, resulting in $|A| = O(2^{|H|} |M|)$. This procedure gives an $O(2^{|H|} |M|)$ upper complexity bound both for checking whether an equation is feasible and for synthesising the missing component. The case where the unknown component is the head of a cascade composition deserves special attention, as in that instance the E-machine can be obtained with no determinization \[24\], making the feasibility check and the synthesis task possible within polynomial time in this scenario.

6.1 Feasibility of the Synthesis Problem

In this section we characterize the feasible instances of the Tail Synthesis Problem, and show that the feasibility check can be carried out in polynomial time.

Let $H$ and $M$ be Mealy machines from $X$ to $Y$ and from $X$ to $Z$ respectively. A solution $T$ to the synthesis problem (that is, $T \circ H \equiv M$) must satisfy $\lambda_T(\lambda_H(\tau)) = \lambda_M(\tau)$ for all $\tau \in X^*$. In particular, if such solution $T$ exists, then there cannot be two words $\tau, \tau' \in X^*$ with $\lambda_H(\tau) = \lambda_H(\tau')$ but $\lambda_M(\tau) \neq \lambda_M(\tau')$. We argue that the converse holds as well.

\[\textbf{Proposition 6.} \text{ There exists a Mealy machine } T \text{ with } T \circ H \equiv M \text{ if and only if for any two words } \tau, \tau' \in X^* \text{ with } \lambda_H(\tau) = \lambda_H(\tau') \text{ it holds } \lambda_M(\tau) = \lambda_M(\tau') \text{ as well.} \]

\[\text{Proof.} \text{ See Appendix D.} \]

As a consequence of this result, deciding the feasibility of the Tail Synthesis Problem given by $H$ and $M$ is equivalent to checking whether $\lambda_H(\tau) = \lambda_H(\tau')$, but $\lambda_T(\tau) \neq \lambda_T(\tau')$ for some $\tau, \tau' \in X^*$. The existence of such words $\tau, \tau'$ can be easily computed in $O(|H|^2 |M|^2)$ time via a fixed point procedure on the synchronous product of $H$ and $M$.

6.2 Representing all Solutions

We give the construction of an OM with size $O(|H||M|)$ encoding all solutions for a feasible instance of the Tail Synthesis Problem. We note that this is an exponentially more succinct representation of the solutions than the E-machine from \[27\]. In fact, this construction is equivalent to the NDE-machine introduced in \[10,26\] but with universal acceptance conditions, rather than non-deterministic ones.

We define an OM $N$ from $Y$ to $Z$ as follows. Let $S_N \subseteq S_H \times S_M$ be the set of pairs $(s_H, s_M)$ that are reachable in the synchronous product $H \times M$. That is, those satisfying $\delta_H(\tau) = s_H$ and $\delta_M(\tau) = s_M$ for some $\tau \in X^*$. Let $r_N := (r_H, r_M)$. We define the transition and output functions $\Delta_N, \lambda_N$ for $N$. Fix a transition $(s_H, s_M) \in S_N, y \in Y$. Let $V$ be the set of $x \in X$ satisfying $\lambda_H(s_H, x) = y$. We take two cases into consideration. If the set $V$ is empty, then we set the transition as undefined $((s_H, s_M), y) \notin D_N$. Otherwise, if $V \neq \emptyset$, we mark the transition as defined $((s_H, s_M), y) \in D_N$. In this case $\lambda_M(s_M, x)$ takes a unique value $z$ for all $x \in V$. Indeed, the opposite would yield two sequences $\tau, \tau' \in X^*$ with $\lambda_H(\tau) = \lambda_H(\tau')$ and $\lambda_H(\tau) \neq \lambda_M(\tau')$, making the Tail Synthesis Problem given by $H$ and $M$ infeasible. Hence, we can define $\lambda_N((s_H, s_M), y)$ as $z$, and $\Delta_N((s_H, s_M), y)$ as $\{ (s_H', s_M') \mid \exists x \in V \text{ s.t. } s_H' = \delta_H(s_H, x), s_M' = \delta_M(s_M, x) \}$.

\[\textbf{Proposition 7.} \text{ Let } N \text{ be the OM defined above. A machine } T \text{ satisfies } T \circ H \equiv M \text{ if and only if } T \text{ implements } N. \]
Minimization and Synthesis of the Tail in Sequential Compositions of Mealy machines

Proof. The machine $T$ satisfies $T \circ H \equiv M$ if and only if for all $\bar{y} \in \text{Out}(H)$ and $\bar{x} \in X^*$ with $\lambda_H(\bar{x}) = \bar{y}$, it holds $\lambda_T(\bar{y}) = \lambda_M(\bar{x})$. Note that by construction $\Omega_N = \text{Out}(H)$. Moreover, there is a run of $N$ over $\bar{y}$ whose output is $\bar{x}$ if and only if there is some $\bar{x} \in X^*$ satisfying $\lambda_H(\bar{x}) = \bar{y}$ and $\lambda_M(\bar{x}) = \bar{x}$. This shows the result.

6.3 Lower Bounds for Synthesising the Tail

In this section we show that there are instances of the synthesis problem (Problem 2) where all solutions have at least exponential size.

**Theorem 8.** There exist finite alphabets $X,Y,Z$ and families of increasingly large Mealy machines $\{M_n\}_n$, from $X$ to $Y$, and $\{H_n\}_n$, from $X$ to $Z$, for which the size of any $T_n$ satisfying $T_n \circ H_n \equiv M_n$ is bounded from below by an exponential function of $|M_n||H_n|$.

The proof of this result has two parts. First, we show an infinite family of OMs for which all implementations have exponential size (Lemma 9). Afterwards we prove that any OM $N$ can be “split” into Mealy machines $M$ and $H$ with the same number of states (plus one) as $N$ for which any $T$ with $T \circ H \equiv M$ provides an implementation of $N$ (Lemma 10). These two results together prove Theorem 8. Some care has to be employed in order to obtain fixed alphabets in Theorem 8. The alphabets for $H$ and $T$ in the splitting construction of Lemma 10 depend on the alphabets of $N$ and its degree, defined as $d(N) := \max_{s,y \in D_N}|\Delta_N(s,y)|$. Hence, it is necessary to ensure in Lemma 9 that the resulting OMs have bounded degree.

**Lemma 9.** There are finite alphabets $Y,Z$ and increasingly large OMs $\{N_n\}_n$ from $Y$ to $Z$ for which the size of a machine $T_n$, implementing $N_n$, is bounded from below by an exponential function of $|N_n|$. Moreover, it is possible to build $N_n$ in such a way that $\max_{n} d(N_n) = 2$.

Proof. See Appendix E.

**Lemma 10.** Let $N$ be a consistent OM from $Y$ to $Z$, and let $k := d(N)$. Let $X := Y \times [k]$, $\hat{Y} := Y \cup \{\bot\}$, and $\hat{Z} := Z \cup \{\bot\}$, where $\bot$ is a fresh symbol. Then there exist Mealy machines $H$, from $X$ to $\hat{Y}$, and $M$, from $X$ to $\hat{Z}$, such that (1) $|S_H|, |S_M| = |S_N| + 1$, (2) there are machines $T$ with $T \circ H \equiv M$, and (2) any of those machines $T$ satisfies $\lambda_T(\bar{y}) = \lambda_N(\bar{y})$ for all $\bar{y} \in \Omega_N$.

Proof. We give a explicit construction for $H$ and $M$. We define the edge set of $N$, $G_N \subseteq S_N \times Y \times S_N$, as the set consisting of the triples $(s,y,s')$ where $(s,y) \in D_N$ and $s' \in \Delta_N(s,y)$. By the definition of $k$, we can build a map $L : G_N \rightarrow [k]$ satisfying $L(s,y,s') \neq L(s,y,t_2)$, for any $(s,y) \in D_T$ and any two different states $t_1,t_2 \in \Delta_N(s,y)$. The map $L$ assigns a number $0 \leq i < k$ to each edge $(s,y,s') \in G_N$, giving different labels to each edge corresponding to a pair $(s,y) \in D_N$. We define $H$ and $M$ at the same time. Set $S_H, S_M := S_N \cup \{\ast\}$, where $\ast$ is a fresh state, and $r_H, r_M := r_N$. Let $s \in S_N \cup \{\ast\}$ and let $x := (y,i) \in X = Y \times [k]$. We take into account three cases: (1) If $s = \ast$, then $\delta_H(\ast, x), \delta_M(\ast, x) = \ast$, and $\lambda_H(\ast, x), \lambda_M(\ast, x) = \bot$. (2) If $s \in S_N$ and there exists some $t \in S_N$ with $L(s,y,t) = i$, then $\delta_H(s,x), \delta_M(s,x) = t$, $\lambda_H(s,x) = y$, and $\lambda_M(s,x) = \lambda_N(s,y)$. Finally, (3) if $s \in S_N$ and there is no $t$ with $L(s,y,t) = i$, then $\delta_H(s,x), \delta_M(s,x) = \ast$, $\lambda_H(s,x), \lambda_M(s,x) = \bot$. We claim that $H$ and $M$ built this way satisfy the theorem’s statement (see Appendix E).
in a more general setting through polynomial reductions. Consider a system consisting of two interconnected components $H$ and $T$, where all of $T$’s output signals can be externally observed. Without loss of generality we can assume that (1) $T$’s input signals coincide with $H$’s output signals, (2) the system’s output signals coincide with $T$’s output signals, and (3) $H$’s input signals are the system input signals plus $T$’s output ones. This situation is depicted in Figure 1b. We claim that the minimization and synthesis of $T$ in this context can be polynomially reduced to those of the tail in a cascade composition. The reductions are similar to the ones in [25], and can be found in Appendix G.

8 Experimental Evaluation

Since the Tail Minimization Problem is in NP, it allows for a straightforward reduction to SAT, which already improves over the doubly-exponential complexity of the K-N algorithm. We show in Figure 5a preliminary experimental results comparing this solution with our proposed one. The baseline method uses a “naive” CNF encoding to decide whether the tail component is $n$-replaceable. Like in our proposed approach, this is done for increasing values of $n$ until a satisfiable CNF is obtained. The encoding can be seen as analogous to ours but without the information about the incompatibility graph and the partial solution. The experiments were run on cascade compositions consisting of independently randomly generated machines $M$ and $T$ with $n = |S_M| = |S_T|$ and input/output alphabets of size 4. Mean CPU times of all runs for each $n$ are plotted for both the baseline algorithm and ours. The baseline implementation is not able to complete any instance with $n \geq 12$ with a timeout of 10 minutes, while our algorithm solved all instances in under a minute. We conclude that our approach has a clear benefit over a straightforward approach for this class of random instances. Both the implementation of our algorithm and the baseline algorithm use CryptoMiniSat [22]. All experiments were run on a Intel Core i5-6200U (2.30GHz) machine.

One reason for the good performance is that our algorithm skips the CNF encoding entirely in about half the instances by using the size of the partial solution, as discussed in ???. To illustrate this we ran 200 additional experiments where $T$ and $H$ were generated independently with random sizes between 12 and 60 states. The running times are shown in Figure 5b where orange points correspond to instances where it was possible to avoid any CNF encoding, and the blue ones correspond to the rest. It can be observed that orange and blue points form two separate clouds of points. It is apparent that the algorithm is significantly more efficient when it does not call the SAT solver, but the approach uses under a minute regardless of whether the SAT solver is employed.
References

1. Andreas Abel and Jan Reineke. MEMIN: SAT-based exact minimization of incompletely specified Mealy machines. In 2015 IEEE/ACM International Conference on Computer-Aided Design (ICCAD), pages 94–101. IEEE, 2015. doi:10.1109/ICCAD.2015.7372555

2. M. D. Di Benedetto, A. Sangiovanni-Vincentelli, and T. Villa. Model matching for finite-state machines. *IEEE Transactions on Automatic Control*, 46(11):1726–1743, 2001. doi:10.1109/9.106483

3. Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and reactive synthesis. In *Handbook of Model Checking*, pages 921–962. Springer, 2018.

4. Roderick Bloem, Robert Könighofer, and Martina Seidl. Sat-based synthesis methods for safety specs. In *International Conference on Verification, Model Checking, and Abstract Interpretation*, pages 1–20. Springer, 2014.

5. William F Dowling and Jean H Gallier. Linear-time algorithms for testing the satisfiability of propositional horn formulae. *The Journal of Logic Programming*, 1(3):267–284, 1984.

6. Bernd Finkbeiner and Sven Schewe. Bounded synthesis. *Int. J. Softw. Tools Technol. Transf.*, 15(5-6):519–539, 2013. doi:10.1007/s10009-012-0228-z

7. Seymour Ginsburg. On the reduction of superfluous states in a sequential machine. *J. ACM*, 6(2):259–282, 1959. doi:10.1145/320964.320983

8. A. Grasselli and F. Luccio. A Method for Minimizing the Number of Internal States in Incompletely Specified Sequential Networks. *IEEE Transactions on Electronic Computers*, EC-14(3):350–359, June 1965. doi:10.1109/PGEC.1965.264140

9. Gary D. Hachtel and Fabio Somenzi. *Logic synthesis and verification algorithms*. Kluwer, 1996.

10. M.S. Harris. Synthesis of finite state machines: Functional optimization. *Microelectronics Journal*, 29(6):364–365, 1998. doi:10.1016/S0026-2692(97)00075-X

11. Frederick C Hennie. *Finite-state models for logical machines*. Wiley, 1968.

12. John E. Hopcroft and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley, 1979.

13. Huey-Yih Wang and R. K. Brayton. Multi-level logic optimization of FSM networks. In Proceedings of IEEE International Conference on Computer Aided Design (ICCAD), pages 728–735, November 1995. doi:10.1109/ICCAD.1995.480254

14. Joonki Kim and M.M. Newborn. The Simplification of Sequential Machines with Input Restrictions. *IEEE Transactions on Computers*, C-21(12):1440–1443, 1972. doi:10.1109/T-C.1972.2238521

15. June-Kyung Rho and F. Somenzi. Don’t care sequences and the optimization of interacting finite state machines. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 13(7):865–874, 1994. doi:10.1109/43.293943

16. T. Kam, T. Villa, R. Brayton, and A. Sangiovanni-Vincentelli. A fully implicit algorithm for exact state minimization. In 31st Design Automation Conference, page 684–690, Jun 1994. doi:10.1145/196244.196615

17. Andreas Morgenstern, Manuel Gesell, and Klaus Schneider. Solving games using incremental induction. In *International Conference on Integrated Formal Methods*, pages 177–191. Springer, 2013.

18. Marvin C. Paull and Stephen H. Unger. Minimizing the number of states in incompletely specified sequential switching functions. *IRE Trans. Electron. Comput.*, 8(3):356–367, 1959. doi:10.1109/TEC.1959.5222697

19. J.M. Pena and A.L. Oliveira. A new algorithm for exact reduction of incompletely specified finite state machines. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 18(11):1619–1632, Nov./1999. doi:10.1109/43.806807

20. C. P. Pfleeger. State reduction in incompletely specified finite-state machines. *IEEE Transactions on Computers*, C-22(12):1099–1102, Dec 1973. doi:10.1109/T-C.1973.223655
21 June-Kyung Rho, Gary D Hachtel, Fabio Somenzi, and Reily M Jacoby. Exact and heuristic algorithms for the minimization of incompletely specified state machines. *IEEE transactions on computer-aided design of integrated circuits and systems*, 13(2):167–177, 1994.

22 Mate Soos, Karsten Nohl, and Claude Castelluccia. Extending SAT solvers to cryptographic problems. In Oliver Kullmann, editor, *Theory and Applications of Satisfiability Testing - SAT 2009, 12th International Conference, SAT 2009, Swansea, UK, June 30 - July 3, 2009. Proceedings*, volume 5584 of *Lecture Notes in Computer Science*, pages 244–257. Springer, 2009. doi:10.1007/978-3-642-02777-2\_24

23 Stephen H Unger. Flow table simplification-some useful aids. *IEEE Transactions on Electronic Computers*, (3):472–475, 1965.

24 Tiziano Villa, Nina Yevtushenko, Robert K Brayton, Alan Mishchenko, Alexandre Petrenko, and Alberto Sangiovanni-Vincentelli. *The unknown component problem: theory and applications*. Springer Science & Business Media, 2011.

25 H. Wang and R. K. Brayton. Input don’t care sequences in fsm networks. In *Proceedings of 1993 International Conference on Computer Aided Design (ICCAD)*, page 321–328, Nov 1993. doi:10.1109/ICCAD.1993.580076

26 Yosinori Watanabe. *Logic Optimization of Interacting Components in Synchronous Digital Systems*. PhD thesis, EECS Department, University of California, Berkeley, Apr 1994. URL: [http://www2.eecs.berkeley.edu/Pubs/TechRpts/1994/2554.html](http://www2.eecs.berkeley.edu/Pubs/TechRpts/1994/2554.html)

27 Yosinori Watanabe and Robert K. Brayton. The maximum set of permissible behaviors for FSM networks. In *Proceedings of the 1993 IEEE/ACM International Conference on Computer-Aided Design*, ICCAD ’93, pages 316–320. IEEE Computer Society Press, 1993.
A Horn CNF Encoding for the Incompatibility Relation

Let \( M \) be an OM from \( Y \) to \( Z \). In this section we describe a way of computing the compatibility relation over \( M \) defined in Section 5.1 by finding the minimal satisfying assignment of a Horn formula. Let \( m = |S_M| \). We identify states in \( S_M \) with numbers \( j \in [m] \). We describe our Horn encoding as follows. For each \( i,j \in [m] \) with \( i \leq j \) we introduce a variable \( X_{i,j} \) meaning \( i \sim j \). We give now the clauses of the formula. For each pair \( i \leq j \) we consider two scenarios: (1) If \( i \) and \( j \) are incompatible at depth one, we add the clause \( X_{i,j} \). (2) Otherwise, for all \( y \in Y, i', j' \in \Delta_M(i,y), j' \in \Delta_M(j,y) \) we add clauses \( X_{i,j'} \implies X_{i,j} \), where the values of \( i' \) and \( j' \) are swapped if necessary to ensure \( i' \leq j' \).

It follows from Lemma 3 that two states \( i \leq j \) are compatible if and only if \( X_{i,j} \) is true in the minimal satisfying assignment for this encoding. It is well-known that assignment can be obtained in time linear in the size of the formula [5]. Moreover, the encoding has size \( O(|M|^2) \). Hence, the compatibility relation over \( M \) can be computed in \( O(|M|^2) \) time.

B Correctness of the Covering Reduction

Proof that the \( N \) constructed in Theorem 4 implements \( M \): Let \( \overline{y} \in \Omega_M \). Consider a run \( s_0:=r_M, y_0, s_1, \ldots, s_n, y_n = s_{n+1} \) of \( M \) on \( \overline{y} \), and let \( C_0:=r_N, y_0, z_0, C_1, \ldots, y_n, z_n, C_{n+1} \) be the run of \( N \) on \( \overline{y} \). By construction, \( s_i \in C_i \) for all \( 0 \leq i \leq n + 1 \). Moreover, as \( \lambda_M(s_i, y_i) = z_i \) it also holds \( \lambda_N(C_i, y_i) = z_i \). Thus, \( \lambda_M(\overline{y}) = \lambda_N(\overline{y}) \) and \( N \) implements \( M \). □

C CNF Encoding for OM Minimization

We describe how to reduce the problem of finding a closed cover of \( n \) over \( M \) to SAT, following the CNF encoding given in [1]. We identify the set \( S_M \) with the set of integers \([m]\), where \( m := |S_M| \). Finding a closed cover of \( n \) compatibles is equivalent to finding two maps \( C : [n] \to 2^{|m|} \) and \( \text{Succ} : [n] \times Y \to 2^{|m|} \) that satisfy: (1) for each \( i \in [n] \) the set \( C(i) \subseteq [m] \) is a compatible, (2) \( r_M \in C(i) \) for some \( i \), (3) \( \Delta_M(C(i), y) \subseteq C(j) \) for all \( j \in \text{Succ}(i, y) \), and (4) \( \text{Succ}(i, y) \neq \emptyset \) for each \( i \in [n] \) and \( y \in Y \). The propositional variables of the CNF encoding are the following: A variable \( L_{s,i} \) for each \( s \in [m], i \in [n] \), encoding that \( s \in C(i) \). A variable \( N_{i,j,y} \) for each pair \( i,j \in [n], y \in Y \), encoding that \( j \in \text{Succ}(i, y) \). The clauses of our encoding are as follows: A clause \( \neg L_{s_1,i} \lor \neg L_{s_2,i} \) for each \( i \in [n], s_1, s_2 \in [m] \) with \( s_1 \leq s_2 \) and \( s_1 \sim s_2 \). This encodes condition (1). A clause \( \lor_{i \in [n]} L_{r,i} \), which encodes condition (2). We have a clause \( \lor_{j \in [n]} N_{i,j,y} \) for each \( i \in [n], y \in Y \), encoding condition (3). Finally, there is a clause \( \lor_{s \in [m]} (N_{i,j,y} \land L_{s,i}) \implies \lor_{s' \in [m]} L_{s',j} \) for each \( s \in [m], y \in Y \) with \( (s,y) \in D_M, s' \in \Delta_M(s,y), i,j \in [n] \). These clauses encode condition (4). The CNF obtained so far already encodes the desired problem. However, we also use a partial solution \( \mathcal{C} \subseteq S_M \) for adding symmetry breaking predicates. If \( \mathcal{C} = \{s_1, \ldots, s_l\} \) is a set of \( l \leq n \) pair-wise incompatible states, we can add the clauses \( L_{s,i} \) for each \( 0 \leq i \leq l \).

D Characterization of Feasible Synthesis Instances

Proof of Proposition 5: The fact that \( T \)’s existence implies the second part of the statement is straightforward, as discussed above. We prove the other implication. Suppose that any two words \( \overline{x}, \overline{x} \) with \( \lambda_M(\overline{x}) = \lambda_M(\overline{x}) \) also satisfy \( \lambda_M(\overline{x}) = \lambda_M(\overline{x}) \). This defines a map \( F : \text{Out}(H) \to Z^* \) by setting \( F(\overline{y}) = \lambda_M(\overline{x}) \) if \( \overline{y} = \lambda_M(\overline{x}) \). Let us define the language
\[ L_F \subseteq (Y \times Z)^* \] as the one consisting of the words \( \langle y, F(y) \rangle \) for all \( y \in \text{Out}(H) \). Clearly this language is regular and prefix-closed. Hence, there is some Mealy machine \( T \) form \( Y \) to \( Z \) satisfying \( F(y) = \lambda_T(y) \) for all \( y \in \text{Out}(H) \). By definition of \( F \), we have \( \lambda_T(\lambda_H(\pi)) = \lambda_M(\pi) \) for all \( \pi \in X^* \), and \( T \circ H \equiv M \). This proves the result.

**E** Observation Machines with no Small Implementations

Fix \( n > 0 \). Figure 6 shows the construction of an OM \( M \) (bottom right) from \( Y := \{a, b\} \) to \( Z := \{a, b, \top\} \) for which all implementations have at least \( 2^n \) states. Moreover, \( d(M) = 2 \).

Let \( y_0y_1 \ldots y_{2n-1} \) be a word in \( Y^* \). \( M \) behaves the following way: (1) It outputs \( \top \) in response to the first \( n \) inputs \( y_0, \ldots, y_{n-1} \). (2) Starting from \( y_n, M \) responds with \( \top \) until \( b \) is received as an input. (3) If \( y_{n+1} \) is the first input equal to \( b \) since \( y_n \), then \( M \) outputs \( y_i \) in response. Intuitively, an implementation of \( M \) has to have at least \( 2^n \) states because it has to store the first \( n \) inputs in order to carry out (3).

![Figure 6](image)

- Figure 6 The construction of an OM with no small implementations (bottom right). The symbol * stands for both \( a, b \).

**F** Correctness of the splitting construction

Correctness of the constructions in Lemma 10 Here we show that the given constructions for \( H \) and \( M \) indeed fulfill the result. We have to show that (1) some \( T \) satisfies \( T \circ H \equiv M \) and (2) any such \( T \) also satisfies \( \lambda_X(y) = \lambda_T(y) \) for all \( y \in \Omega_N \). Remember that characters \( x \in X \) are (input,label) pairs \( (y, l) \in Y \times [k] \). We say that a word \( \pi := (y_0, l_0) \ldots (y_n, l_n) \in X^* \) labels a run of \( N \) if \( N \) has a run \( r_N = s_0, y_0, z_0, s_1, \ldots, s_n, y_n, z_n, s_{n+1} \) where \( L(s_i, y_i, s_{i+1}) = l_i \) for all \( 0 \leq i \leq n \). Given such \( \pi \) by construction \( s_0, (y_0, l_0), y_0, s_1, \ldots, s_n, (y_n, l_n), y_n, s_{n+1} \) and
Minimization and Synthesis of the Tail in Sequential Compositions of Mealy machines

$s_0, (y_0, l_0), s_1, \ldots, s_n, (y_n, l_n), z_n, s_{n+1}$ are the runs of $H$ and $M$ on $\pi$, respectively. Let $\pi \in X^*$ be arbitrary. We can write $\pi = \pi \overline{\nu}$, where $\pi = \langle y, l \rangle$ is the largest prefix of $\pi$ which labels a run of $N$. The previous observation yields $\lambda_H(\overline{\nu}) = \overline{y}$, $\lambda_M(\overline{\nu}) = \lambda_N(\overline{y})$. Moreover, by construction $\lambda_H(s, \overline{\nu}), \lambda(s, \overline{\nu})$ are both sequences of only $\bot$ symbols, where $s = \delta_H(\overline{\nu}) = \delta_M(\overline{\nu})$. This way we have shown that the language $E(H, M) = \{ \langle \lambda_H(x), \lambda_M(x) \rangle \mid x \in X^* \}$ equals $\{ \langle \overline{y}, \lambda_N(\overline{y}) \rangle \mid \overline{y} \in \Omega_N \}{\bot}^*$. This identity proves both (1) and (2). Indeed, if $\lambda_H(x) = \lambda_H(x')$, then necessarily $\lambda_M(x) = \lambda_M(x')$. By Proposition 6 this implies (1). Also, if $\lambda_H(x) = \overline{y}$ and $\overline{y} \in \Omega_N$, then $\lambda_M(x) = \lambda_N(\overline{y})$, which shows (2).

G Reductions to Cascade Compositions

We put $T \odot H$ for the composition between two Mealy machines $H$ and $T$ shown in Figure 1b. This composition is not well-defined in general [10, Section 6.2], but a sufficient requirement is that $H$ is a Moore machine, for example. Analogously to Problem 1, we consider the problem of finding a minimal replacement for $T$ in $T \odot H$ when both $T$ and $H$ are given. In Figure 7a it is shown how to build a machine $H'$ such that finding a minimal replacement for $T$ in $T \odot H$ is equivalent to finding a minimal replacement for $T$ in the cascade composition $T \circ H'$. Similarly, when $M$ and $H$ are given, we can study the problem of finding $T$ with $T \odot H \equiv M$. In Figure 7b it is shown how to build $H'$ in a way that $T$ satisfying $T \odot H \equiv M$ is equivalent to $T \circ H' \equiv M$.

Figure 7 Polynomial transformations to cascade compositions.