PARABOLIC LIMITS OF RENORMALIZATION

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ABSTRACT. In this paper we give a combinatorial description of the renormalization limits of infinitely renormalizable unimodal maps with essentially bounded combinatorics admitting quadratic-like complex extensions. As an application we construct a natural analogue of the period-doubling fixed point. Dynamical hairiness is also proven for maps in this class. These results are proven by analyzing parabolic towers: sequences of maps related either by renormalization or by parabolic renormalization.

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1. Introduction

In this paper we extend the well-known combinatorial description of renormalization limits of unimodal maps with bounded combinatorics to renormalization limits of maps with essentially bounded combinatorics. This class of maps was introduced by Lyubich [L2] and their complex geometry was studied in [LY]. Roughly speaking the high renormalization periods of such maps are due to their renormalizations being small perturbations of parabolic maps. Although this leads to the creation of unbounded combinatorics, the essential geometry of these maps remains bounded away from zero.

Let us state our results (see §2 for background). In §3.1 we construct a countable collection of maximal tuned Mandelbrot copies \( \{ M_{n_1}^{(3)} \}_{n=1}^\infty \) that accumulate at \( c = -1.75 \), the root point of \( M^{(3)} \). These copies have “essentially period tripling” combinatorics. Our first result produces the analog of the renormalization fixed point in the essentially period tripling situation:

**Theorem 1.1.** There is a unique quadratic-like germ \( F \) such that

\[ \mathcal{R}^n(f) \to F \]

for any quadratic-like map \( f \) in the hybrid class of an infinitely renormalizable real quadratic with a tuning invariant

\[ \tau(f) = (M_{n_1}^{(3)}, M_{n_2}^{(3)}, \ldots, M_{n_k}^{(3)}, \ldots) \]

satisfying \( n_k \to \infty \) as \( k \to \infty \). Any quadratic-like representative of \( F \) is hybrid equivalent to \( z^2 - 1.75 \) and hence has a period three parabolic orbit.

In order to state our second theorem we need to fix some notation. Let \( \Omega \) denote the space of unimodal non-renormalizable permutations, or shuffles, and let \( pe(\sigma) \) be the essential period of \( \sigma \in \Omega \). Let

\[ \Omega_p = \{ \sigma \in \Omega : pe(\sigma) \leq p \} \]

and let \( \Omega^\text{cpt}_p \) be the compactification of \( \Omega_p \) defined in §3.3. Let

\[ \Sigma_p = \Pi_{-\infty}^\infty \Omega^\text{cpt}_p \]

with coordinate projections \( \pi_n : \Sigma_p \to \Omega^\text{cpt}_p \) and let \( \omega : \Sigma_p \to \Sigma_p \) be the left shift operator. Let \( \Omega^\text{cpt}_{p,*} \) denote the space \( \Omega^\text{cpt}_p \) with the symbol * adjoined and let \( \Sigma^*_p = \Pi_{-\infty}^\infty \Omega^\text{cpt}_{p,*} \). We will denote the left shift on \( \Sigma^*_p \) by \( \omega \) as well. For any quadratic-like map \( f \) hybrid equivalent to an infinitely renormalizable real polynomial let

\[ \tilde{\sigma}(f) = (\ldots, *, *, \sigma_0, \sigma_1, \sigma_2, \ldots) \]

where \( \sigma_n \) is the shuffle corresponding to the \( n \)-th Mandelbrot copy in \( \tau(f) \). Let \( \tilde{p}_e(f) = \sup_{n \geq 0} p_e(\pi_n(\tilde{\sigma}(f))) \). Let \( G\text{Quad}(m) \) be the space of quadratic-like germs with modulus at least \( m \). We can now state the combinatorial classification of all limits of renormalization of an infinitely renormalizable real quadratic with essentially bounded combinatorics.

**Theorem 1.2.** There is an \( m > 0 \) so that for any \( p > 1 \) there exists a continuous map

\[ h : \Sigma_p \to G\text{Quad}(m) \]

with the following property. Let \( f \) be a quadratic-like map in the hybrid class of an \( \infty \)-renormalizable real quadratic with \( \tilde{p}_e(f) \leq p \) and let \( \tilde{\sigma} \in \Sigma_p \) be a limit point of \( \tilde{\sigma}_n = \omega^n(\tilde{\sigma}(f)) \).
If $\bar{\sigma}_n \to \bar{\sigma}$ then
\[
\mathcal{R}^n(f) \to h(\bar{\sigma}).
\] (1.1)

Furthermore, if $\sigma_0 = \pi_0(\bar{\sigma}) \in \Omega_p$ then $h(\bar{\sigma})$ is renormalizable by the shuffle type $\sigma_0$ and $h$ is a conjugacy between $\omega$ and $\mathcal{R}_{\sigma_0}$. If $\sigma_0 \notin \Omega_p$ then the inner class of $h(\bar{\sigma})$ is the root of a maximal tuned Mandelbrot copy $M(\sigma_0)$.

Let us comment on the ideas involved in the paper. Recall that the central objects of McMullen’s argument \cite{McM2} are towers: sequences of quadratic-like maps related by renormalization. A forward tower is a one-sided sequence and a bi-infinite tower is a two-sided infinite sequence. The question of convergence of renormalization is equivalent to the question of combinatorial rigidity of the corresponding limiting bi-infinite towers. However, for maps with essentially bounded combinatorics the limiting towers may contain parabolic maps and we lose the renormalization relation between levels. In this case a new relation appears: parabolic renormalization. That is, the maps in the limiting towers are related by either classical or parabolic renormalization. A tower which contains a parabolic renormalization is called a parabolic tower. Our proof of the rigidity of bi-infinite parabolic towers with definite modulus and essentially bounded combinatorics consists of first analyzing forward towers and then analyzing bi-infinite towers.

Our analysis of forward parabolic towers was motivated by the work of A. Epstein \cite{E}, which considered general holomorphic dynamical systems (with maximal domains of definition) and their geometric limits. The phenomenon studied there was the renormalization (different from the sense used in this paper) of a parabolic orbit at the ends of its Écalle-Voronin cylinders. The phenomenon we study occurs away from the ends and as a result the forward infinite towers in this paper look in many ways like infinitely renormalizable real quadratic maps.

The combinatorial rigidity of forward parabolic towers with polynomial base map follows from the theory of quadratic-like families and from the combinatorial rigidity of quadratic polynomials with complex bounds and real combinatorics (see Proposition \ref{prop:quadratic-like}). After analyzing the Julia set of a forward tower we prove any quasiconformal conjugacy of a forward infinite parabolic tower with essentially bounded combinatorics and complex bounds is a hybrid conjugacy (see \S6.4).

Then following the arguments of McMullen we prove in \S6.5 the rigidity of bi-infinite towers. That is, we first prove

**Theorem 1.3** (Dynamical Hairiness). The union of the Julia sets of the forward infinite sub-towers of a bi-infinite tower with essentially bounded combinatorics and complex bounds is dense in the plane.

Then we prove

**Theorem 1.4.** Any quasiconformal equivalence of a bi-infinite tower with essentially bounded combinatorics and complex bounds is affine.

Let us mention a parallel with critical circle maps. The theory of renormalization of unimodal maps is closely related to renormalization theory of critical circle maps. The rotation number $\rho$, more specifically its continued fraction expansion, determines the combinatorics of a circle map. If the factors in its expansion are bounded then the map has bounded combinatorics and has unbounded combinatorics otherwise. If a circle map has unbounded
combinatorics then the rotation numbers of the renormalizations contain rational limit points and the corresponding limit of renormalization contain parabolic periodic points. That is, the only kind of unbounded combinatorics in the theory of critical circle maps is the essentially bounded combinatorics. DeFaria [deF] analyzed the renormalization limits of critical circle maps with bounded combinatorics and Yampolsky [Y] proves complex bounds for arbitrary combinatorics. We expect the techniques in this paper can be adapted to analyze renormalization limits of critical circle maps with arbitrary combinatorics.

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1.1. \textbf{Notation.}

- $\mathbb{H} \subset \mathbb{C}$ denotes the complex upper half-plane, $\hat{\mathbb{C}}$ the Riemann sphere, $\mathbb{N} = \mathbb{N}_0$ the non-negative integers and $\mathbb{N}_+$ the positive integers.
- $[a,b]$ will also denote the interval $[b,a]$ if $b < a$.
- $\text{diam}(U)$ denotes the euclidean diameter of $U \subset \mathbb{C}$ and $|I|$ the diameter of $I \subset \mathbb{R}$.
- $\text{cl}(X)$, $\text{int}(X)$ and $\partial X$ denote the closure, interior and boundary of $X$ in $\mathbb{R}$ if $X \subset \mathbb{R}$ and in $\mathbb{C}$ otherwise.
- $U \Subset V$ means $U$ is compactly contained in $V$. Namely $\text{cl}(U)$ is compact and $\text{cl}(U) \subset V$.
- in a dynamical context $f^n$ denotes $f$ composed with itself $n$ times.
- if $V$ is a simply connected domain and $U \subset V$ then $\text{mod}(U,V) = \sup_A \text{mod}(A)$ where $A$ is an annulus separating $U$ from $\partial V$.
- $\text{Dom}(f)$ and $\text{Range}(f)$ denote the domain and range of $f$.
- $\text{Comp}(X)$ denotes the collection of connected components of $X$ and $\text{Comp}(X,Y)$ denotes the components of $X$ intersecting $Y$.
- $P_c(z) = z^2 + c$.

2. \textbf{Background}

2.1. \textbf{Quadratic-like maps.} We will assume the reader is familiar with the theory of quasiconformal maps and the Measurable Riemann Mapping Theorem (see [LV]).

A holomorphic map $f : U \to V$ is \textit{quadratic-like} if $U$ and $V$ are topological disks in $\mathbb{C}$ with $U \Subset V$ and $f$ is a branched double cover of $U$ onto $V$. By topological disk we mean a simply connected domain in $\mathbb{C}$. A topological disk whose boundary is a Jordan curve will be called a \textit{Jordan disk}. Unless otherwise indicated we will assume the critical point of a quadratic-like map is at the origin. A point $z \in U$ is \textit{non-escaping} if $f^n(z)$ is defined for all $n \geq 0$. For a quadratic-like $f : U \to V$ define

- The \textit{filled Julia set} $K(f) = \text{cl}\{z \in U : z$ is non-escaping\}$
- The \textit{Julia set} $J(f) = \partial K(f)$
- The \textit{post-critical set} $P(f) = \text{cl}\{\bigcup_{n \geq 1} f^n(0)\}$

An actual quadratic polynomial can be considered quadratic-like by taking $V = \{z : |z| < R\}$ for some large $R$. Following [McM2], define $\text{Quad}$ to be the union of all quadratic-like maps $f : U \to V$ and all quadratic polynomials $f : \mathbb{C} \to \mathbb{C}$ with a non-escaping critical point at the origin.

Impose on $\text{Quad}$ the \textit{Carathéodory topology}. That is, a sequence $f_n : U_n \to V_n$ converges to $f : U \to V$ iff $(U_n,0)$ and $(V_n,f_n(0))$ converge to $(U,0)$ and $(V,f(0))$, respectively, in
Define the subspaces

\[ \text{Quad}(m) = \{ f \in \text{Quad} : f \text{ is a polynomial or } \text{mod}(U, V) \geq m \} \]

and

\[ R\text{Quad} = \{ f \in \text{Quad} : f(z) = \overline{f(z)} \}. \]

The following compactness lemma is a basic tool in renormalization theory:

**Lemma 2.1 ([McM1], Theorem 5.8).** For any \( C_0 > 0, C_1 < \infty, m > 0 \), the set

\[ \{ f \in \text{Quad}(m) : C_0 \leq \text{diam} K(f) \leq C_1 \} \]

is compact.

Define \( G\text{Quad}(m) \) to be the quotient space of \( \text{Quad}(m) \) by the relation \( f \sim g \) iff \( f = g \) on a neighborhood of zero. Define the set of quadratic-like germs to be \( G\text{Quad} = \cup_m G\text{Quad}(m) \). Convergence of germs will always take place in some \( G\text{Quad}(m) \). The germ of \( f \) will be denoted by \( [f] \). From [McM2, Lemma 7.1] the (filled) Julia set of a quadratic-like germ is well defined and consequently if \( f \) and \( g \) are two quadratic-like representatives of a germ in \( G\text{Quad}(m) \) then \( f = g \) on an \( \epsilon(m/2) \)-scaled neighborhood of \( K(f) = K(g) \). Since \( K(f) \) is an upper semi-continuous function on \( \text{Quad} \), if \( f_k \in G\text{Quad}(m) \) converges to \( f \) then for any sequence of representatives \( g_k \in \text{Quad}(m) \) it follows \( g_k \) converges to \( g \) on a definite neighborhood of \( K(f) \). Given \( f \in G\text{Quad} \) let

\[ \text{mod}(f) = \sup \text{mod}(U, V) \]

where the supremum is taken over all quadratic-like representations of \( f \).

Let \( f \in \text{Quad} \). For a given \( x \neq 0 \) let \( x' = f^{-1}(f(x)) \setminus \{x\} \). If \( x = 0 \) let \( x' = 0 \). There are two fixed points \( \alpha \) and \( \beta \) of \( f \) counted with multiplicity and labeled so that \( J(f) \setminus \{\beta\} \) is connected. The only case when \( \alpha = \beta \) is when \( I(f) = 1/4 \). We say \( f \in \text{Quad} \) is normalized if \( \beta(f) = 1 \). We normalize a germ by normalizing any quadratic-like representative.

A quasi-conformal equivalence \( \phi \) between quadratic-like maps \( f \) and \( g \) is a quasiconformal map from a neighborhood of \( K(f) \) to a neighborhood of \( K(g) \) such that \( \phi \circ f = g \circ \phi \). A quasi-conformal equivalence is a hybrid equivalence if \( \partial \phi|_{K(f)} = 0 \) as a distribution.

**Proposition 2.2 (Straightening, [DH2]).** Any quadratic-like map \( f \) is hybrid equivalent to a quadratic polynomial. If \( K(f) \) is connected the polynomial is unique up to affine conjugacy.
Moreover, if \( f \in \text{Quad}(m) \) then the equivalence can be chosen to be a conjugacy on an \( \epsilon(m) \)-scaled neighborhood of \( K(f) \) and with dilatation bounded above by \( K(m) < \infty \).

The inner class of a map \( f \in \text{Quad} \), denoted \( I(f) \), is the unique \( c \) value such that \( f \) is hybrid equivalent to \( P_c \). The inner class of a germ \( I([f]) \) is the inner class of any quadratic-like representative. The Mandelbrot set, \( M \), is the set of \( c \in \mathbb{C} \) such that the Julia set of \( z^2 + c \) is connected. Let \( \mathcal{H}(c) = I^{-1}(c) \) for \( c \in M \). Note \( \mathcal{H}(c) \subseteq \text{Quad} \).

**Proposition 2.3** ([DH2], [McM2, Proposition 4.7]). \( I : \text{Quad} \to M \) is continuous.

### 2.2. Renormalization

A parameter value \( c \in \mathbb{C} \) is called super-stable if 0 is periodic under \( P_c \). To each super-stable \( c \neq 0 \) there is associated a homeomorphic copy of \( M \) containing \( c \) called the Mandelbrot set tuned by \( c \), or, briefly, an \( M \)-copy, and denoted by \( c \star M \). The root of \( c \star M \) is the point corresponding to \( 1/4 \) and the center is the point \( c \). For every copy \( c \star M \) there is a \( p > 1 \) such that for any \( c' \in c \star M \), except possibly the root, and any \( f \in \mathcal{H}(c') \) there is a domain \( U \ni 0 \) such that \( f^p|_U \in \text{Quad} \). The map \( f^p|_U \) is called a (complex) pre-renormalization of \( f \) and \( f \) is said to be renormalizable of period \( p \). This pre-renormalization is always simple, meaning the iterates of \( J(f^p|_U) \) under \( f \) are either disjoint or intersect only along the orbit of \( \beta(f^p|_U) \). The period of the copy, \( p(c \star M) \), is the maximal such \( p \) and we say \( c \star M \) is maximal if there is only one such \( p \). We say \( c \star M \) is real if \( c \) is real. The only real maximal \( M \)-copy for which the root point is not renormalizable is the period two copy \( M(2) \). We will denote the real period three copy by \( M(3) \). Define \( \mathcal{H}(c \star M) \) to be the set of renormalizable \( f \in I^{-1}(c \star M) \). In Fig. 1 we have drawn the Mandelbrot set highlighting \( M(2) \) and \( M(3) \). The root points are \( c = -0.75 \) and \( c = -1.75 \), respectively.

![Figure 1. The Mandelbrot set.](image-url)
$\mathcal{R}(f)$ to be the normalized quadratic-like germ of any pre-renormalization of period $p$. We define the renormalization of a germ $\mathcal{R}([f])$ to be the renormalization of a quadratic-like representative. A map $f \in \text{Quad}$ is infinitely renormalizable if $\mathcal{R}^n(f)$ is defined for all $n \geq 0$, or, equivalently, if $I(f)$ is contained in infinitely many $M$-copies. The tuning invariant of an infinitely renormalizable map $f \in \text{Quad}$ is

$$\tau(f) = (M_0, M_1, M_2, \ldots)$$

where $M_n$ is the maximal $M$-copy containing $I(\mathcal{R}^n(f))$. We say $f$, even if it is only finitely renormalizable, has real combinatorics if all $M$-copies in $\tau(f)$ are real. See [2] for a more complete description of tuning.

Let us turn to renormalization in real dynamics. Let $I \subset \mathbb{R}$ be a closed interval. A continuous map $f : I \to I$ is unimodal if $f(\partial I) \subset \partial I$ and there is a unique extremum $c$ of $f|_I$. For $f \in \text{RQuad}$ let $B(f) = [\beta, \beta']$, and $A(f) = [\alpha, \alpha'] \subset B(f)$. Note that $K(f) \cap \mathbb{R} = B(f)$. The next lemma follows from Lemma [2] and the continuity of $\beta$ and $\beta'$.

**Lemma 2.4.** For $m > 0$, $|\beta(f)|$ and $|B(f)|$ are $C(m)$-commensurable to $\text{diam} K(f)$ for any $f \in \text{RQuad}(m)$.

Any $f \in \text{RQuad}$ is unimodal on $B(f)$ (and this interval is maximal). We say a unimodal map $f|_{[a,b]}$ with $a < b$ is positively oriented if $f(b) = b$. The quadratic family $P_c$ is positively oriented. A unimodal map $f : I \to I$ is real-renormalizable if there is an interval $I' \ni c$ and an $n \geq 1$ such that $f^n|_{I'}$ is unimodal. Unlike complex renormalization, we can canonically define real-renormalization as acting on unimodal maps as follows. Define the real pre-renormalization $f_1$ of a unimodal map $f$ as $f^n|_{I'}$ where $n$ is minimal and $I$ is maximal and define the real-renormalization $\mathcal{R}(f)$ as $f_1$ conjugated by $x \mapsto x/\beta(f_1)$ where $\beta(f)$ is the boundary fixed point of $f$.

Suppose $f \in \text{RQuad}$ is real-renormalizable and positively oriented. Let $f_1$ be a pre-renormalization and let $\sigma(f)$ be the permutation induced on the orbit of $B(f_1)$ labeled from left to right. Any permutation that can be so realized is called a unimodal non-renormalizable permutation, or a shuffle. The permutation on two symbols we will denote by $\sigma^{(2)}$. If $\sigma(f) = \sigma^{(2)}$ we say $f$ is immediately renormalizable. The map $c \ast M \mapsto \sigma(P_c)$ from the set of real maximal $M$-copies to the set of shuffles is a bijection. We will denote the shuffle corresponding to $c \ast M$ by $\sigma(c \ast M)$ and the real maximal $M$-copy corresponding to $\sigma$ by $M(\sigma)$. We will occasionally use the notation $\mathcal{R}_\sigma$ to denote the complex renormalization operator acting on $\mathcal{H}(M(\sigma))$ and on its germs. If $g \in \mathcal{H}(M(\sigma))$ then define $\sigma(g) = \sigma$. For an infinitely renormalizable $f \in \text{Quad}$ with real combinatorics define

$$\bar{\sigma}(f) = (\sigma(M_0), \sigma(M_1), \sigma(M_2), \ldots)$$

where $\tau(f) = (M_0, M_1, M_2, \ldots)$.

An $\infty$-renormalizable map $f \in \text{Quad}$ has complex bounds if there is some $m > 0$ such that the domain $U_k$ and range $V_k$ of the $k$-th complex pre-renormalization $f_k$ can be chosen to satisfy $\text{mod}(U_k, V_k) \geq m$ for all $k \geq 1$. The following theorem establishes combinatorial rigidity of infinitely renormalizable maps with real combinatorics and complex bounds.

**Theorem 2.5** ([3]). If $P_c$ and $P_{c'}$ are two $\infty$-renormalizable quadratics with complex bounds and the same real combinatorics then $c = c'$.

Complex bounds are proven to exist for real quadratics:
Theorem 2.6 ([LY, LZ, S, LS]). Real infinitely renormalizable quadratics have complex bounds. That is, if \( f_k : U_k \rightarrow V_k \) is a complex renormalization of an \( \infty \)-renormalizable real quadratic \( f \) then \( [f_k] \in \text{GQuad}(m) \) for some \( m > 0 \) independent of \( k \). Moreover, \( U_k \) and \( V_k \) can be chosen to be \( K \)-quasidisks,

\[
diam(V_k) \leq C \cdot |B(f_k)|,
\]

and, if \( \sigma(\mathcal{R}^{k-1}(f)) \neq \sigma^{(2)} \) then the unbranched condition holds:

\[
P(f) \cap V_k = P(f_k).
\]

The values \( m, C \) and \( K \) are independent of \( f \).

When we make an additional assumption on the combinatorics we obtain the unbranched condition on all levels.

Lemma 2.7. Let \( \epsilon > 0 \). Suppose \( f \) is an infinitely renormalizable real quadratic with \( I(\mathcal{R}^k(f)) \geq -2 + \epsilon \) for all \( k \geq 0 \). Then there is an \( m > 0 \) such that the domain \( U_k \) and range \( V_k \) of the \( k \)-th pre-renormalization can be chosen to satisfy

- \( \text{mod}(U_k, V_k) \geq m \)
- \( U_k \) and \( V_k \) are \( K \)-quasidisks
- \( \text{diam}(V_k) \leq C \cdot |B(f_k)| \)
- \( P(f) \cap V_k = P(f_k) \)

for all \( k \geq 1 \). The constants \( m \) and \( K \) then depend on \( \epsilon \).

Proof. If \( \sigma(\mathcal{R}^{k-1}(f)) \neq \sigma^{(2)} \) then let \( U_k \) and \( V_k \) be from Theorem 2.4. So assume \( \sigma(\mathcal{R}^{k-1}(f)) = \sigma^{(2)} \). Let \( h : U_k \rightarrow V_k \) and \( h_1 : U_k \rightarrow V_k \) be the \( (k-1) \)-st and \( k \)-th pre-normalization from Theorem 2.3 rescaled so that \( \text{diam}(K(h_1)) = 1 \). Let \( E = P(h) \setminus P(h_1) \). From the following lemma, Proposition 2.3 and the assumption \( I(\mathcal{R}^k(f)) \geq -2 + \epsilon \) we obtain

\[
\text{dist}(E, B(h_1)) = |h^3(0) - \alpha(h)| \geq C(\epsilon, m) > 0.
\]

From a construction of Sands, \( V_k \) can be chosen to be the union of a euclidean disk centered at \( 0 \) of radius \( |\beta(h_1)| \) and two small euclidean disks centered at \( \pm \beta(h_1) \) of radius \( \epsilon' > 0 \). The modulus \( \text{mod}(U_k', V_k') \) is bounded below by a function \( m'(\epsilon') > 0 \). Choose \( \epsilon' < C(\epsilon, m) \).  

2.3. Generalized quadratic-like maps. A holomorphic map \( f \) is generalized quadratic-like if \( \text{Range}(f) = V \) is a topological disk, each \( U \in \text{Comp}(\text{Dom}(f)) \) is a topological disk compactly contained in \( V \) and \( f|_U \) is a conformal isomorphism except for a distinguished component \( U_0 \), the central component, where \( f|_{U_0} \) is a branched double cover onto \( V \). We will consider only generalized quadratic-like maps whose domain has of finitely many components. Define the filled Julia set, \( K(f) \), the Julia set, \( J(f) \), and the post-critical set, \( P(f) \), as for quadratic-like maps.

Let \( \text{Gen} \) be the union of \( \text{Quad} \) and the space of generalized quadratic-like maps with a non-escaping critical point at the origin. Let \( \text{RGen} \) be the space of real-symmetric maps in \( \text{Gen} \) with real-symmetric domains. Define

\[
\text{Gen}(m) = \{ f \in \text{Gen} : \text{mod}(\text{Dom}(f), \text{Range}(f)) \geq m \}.
\]

Impose on \( \text{Gen} \) the Carathéodory topology as follows. For a given \( f \in \text{Gen} \) let \( f(0) \) be the basepoint of \( \text{Range}(f) \) and let \( u_f = f^{-1}(f(0)) \) be the basepoints of \( \text{Comp}(\text{Dom}(f)) \). A sequence \( f_n \in \text{Gen} \) converges to \( f \) iff

- \( u_{f_n} \) converges in the Hausdorff topology to \( u_f \)


• if $X$ is any Hausdorff limit point of $\widehat{\mathbb{C}} \setminus \text{Dom}(f_n)$ then
  \[ \text{Dom}(f) = \text{Comp}(\widehat{\mathbb{C}} \setminus X, u_f) \]
• $f_n \to f$ on compact subsets of $\text{Dom}(f)$.

The space of generalized quadratic-like germs is the quotient space of $\text{Gen}$ by the relation $f \sim g$ iff $f = g$ on a neighborhood of $u_f = u_g$.

Define the geometry of $f \in \text{Gen}$ as
  \[ \text{geo}(f) = \inf_{U \in \text{Comp}(\text{Dom}(f))} \frac{\text{diam}(K(f) \cap U)}{\text{diam} K(f)}. \]

The following lemma is a direct generalization of Lemma 2.1.

**Lemma 2.8.** For a given $m > 0$, $C_0 > 0$ and $C_1$ the set
  \[ \{ f \in \text{Gen}(m) : \text{geo}(f) \geq C_0 \text{ and } C_0 \leq \text{diam} K(f) \leq C_1 \} \]

is compact.

Suppose $f : \cup U_j \to V$ is a generalized quadratic-like map with critical point at the origin and suppose $U \subset \mathbb{C}$ is open. Define the open sets $D_0$ and $D_+$ by
  \[ D_{0,+} = \{ z : f^n(z) \in U \text{ for some } n \in \mathbb{N}_{0,+} \} \]
and the maps $L_{0,+}(f,U) = L_{0,+} : D_{0,+} \to U$ by $L_{0,+}(z) = f^n(z)$ for the minimal landing time $\text{ltime}(z) = n \in \mathbb{N}_{0,+}$ such that $f^n(z) \in U$. We call $L_0$ the first landing map and $L_+$ the strict first landing map.

Define the first return map to $U$, $R(f,U) = R$ by
  \[ R = L_+ |_{D_+ \cap U} \]

Let $R = R(f,U_0)$ and suppose $0 \in \text{Dom}(R)$. Define the generalized renormalization of $f$ as $R$ restricted to $\text{Comp}(\text{Dom}(R), P(f) \cup \{0\})$.

**Lemma 2.9.** Let $\lambda > 0$, $m > 0$ and $r \in \mathbb{N}$. Suppose $f \in \text{Gen}(m)$ satisfies $\text{geo}(f) \geq \lambda$ and suppose $g \in \text{Gen}$ is a restriction of $R(f,U_0)$ such that
  \[ \text{Dom}(g) = \text{Comp}(\text{Dom}(R), u_g) \text{ and } \sup_{z \in \text{Dom}(g)} \text{ltime}(z) \leq r. \]

Then there exists $C(\lambda,m,r) > 0$ such that $\text{geo}(g) \geq C$.

**Proof.** Assume $\text{diam} K(f) = 1$. Let $\cup U_j = \text{Dom}(f)$ and let $K_j = U_j \cap K(f)$. Since $\text{diam} K_j \geq \lambda$ and $\text{mod}(K_j, U_j) \geq m$ it follows that $U_j$ contains an $\epsilon(\lambda,m)$ neighborhood of $K_j$. For each $z \in u_g$ let $U_{z,0}, U_{z,1}, \ldots, U_{z,k} = U_0$ be the pull back of $U_0$ along the orbit $z, f(z), \ldots, f^k(z) = g(z)$. Assume $f(z) \in \cup U_j$. From the Koebe Distortion Theorem and the fact that $k \leq r$ it follows $U_{z,1}$ contains a definite neighborhood of $f(z)$ and that $f^{k-1} : U_{z,1} \to U_0$ has bounded distortion. Hence each $U_{z,0}$ contains a definite neighborhood of $z$ by Lemma 2.8. The lemma follows by pulling $\cup U_{z,0}$ back to each $U_{z,1}$ by a map with bounded distortion and bounded derivative and then to $U_{z,0}$. \[\Box\]

Let $L_0 = L_0(f,U)$ and suppose $0 \in \text{Dom}(L_0)$. Define the first through map, $T = T(f,U)$, of $f$ by
  \[ T = f \circ L_0. \]

We shall analyze the geometry of certain first through maps in §2.5.
2.4. Families of generalized quadratic-like maps. In this section we summarize the theory of holomorphic families of generalized quadratic-like maps. For further details see [4]. Let \( D \subset \mathbb{C} \) be a Jordan disk and fix \(* \in D\). Let \( \pi_1 \) and \( \pi_2 \) be the coordinate projections of \( \mathbb{C}^2 \) to the first and second coordinates. Given a set \( X \subset \mathbb{C}^2 \) let \( X_\lambda = \pi_2(X \cap \pi_1^{-1}(\lambda)) \). An open set \( X \subset \mathbb{C}^2 \) is a Jordan bidisk over \( D \) if \( \pi_1(X) = D \) and \( X_\lambda \) is a Jordan disk for all \( \lambda \in D \). We say \( X \) admits an extension to the boundary if \( \text{cl}(X) \) is homeomorphic over \( \text{cl}(D) \) to \( \text{cl}(D) \times \text{cl}(\mathbb{D}) \). A section \( \Psi : \text{cl}(D) \to \text{cl}(X) \) is a trivial section if there is a fiber-preserving homeomorphism \( h : \text{cl}(X) \to \text{cl}(D) \times \text{cl}(\mathbb{D}) \) such that \((h \circ \Psi)(\lambda) = (\lambda, 0)\). Given a Jordan bidisk \( X \) which admits an extension to the boundary we define the frame \( \delta X \) as the torus \( \bigcup_{\lambda \in \partial D} \cup_{z \in \partial X_\lambda} (\lambda, z) \). A section \( \Phi : D \to X \) is proper if it admits a continuous extension to \( \partial D \) and \( \Phi(\partial D) \subset \delta X \). Let \( \Phi \) be a proper section and let \( \Psi \) be a trivial section. Let \( \phi = \pi_2 \circ \Phi \) and \( \psi = \pi_2 \circ \Psi \). Define the winding number of \( \Phi \) to be the winding number of the curve \((\phi - \psi)|_{\partial D}\) around the origin.

Lemma 2.10 (Argument Principle). Let \( X \) be a Jordan bidisk over \( D \) that admits an extension to the boundary. Let \( \Phi : D \to X \) be a proper section and let \( \Psi : \text{cl}(D) \to \text{cl}(X) \) be a continuous section, holomorphic on \( D \). Let \( \phi = \pi_2 \circ \Phi \) and \( \psi = \pi_2 \circ \Psi \). Suppose there are no solutions to \( \phi = \psi \) on \( \partial D \). Then the number of solutions to \( \phi = \psi \) counted with multiplicity is equal to the winding number of \( \Phi \).

Let \( \bigcup_j U_j \) be a pairwise disjoint collection of Jordan bidisks over \( D \) with \( 0 \in U_\lambda = U_{0, \lambda} \). Let \( \mathbb{V} \) be a Jordan bidisk over \( D \) such that each \( U_{j, \lambda} \) is compactly contained in \( V_\lambda \). Let

\[
f : \bigcup_j U_j \to \mathbb{V}
\]

be a fiber-preserving holomorphic map such that each fiber map \( f_\lambda : \bigcup_j U_{j, \lambda} \to V_\lambda \) is a generalized quadratic-like map with critical point at the origin and which on each branch \( f_\lambda|_{U_{j, \lambda}} \) admits a holomorphic extension to a neighborhood of \( U_{j, \lambda} \). Let \( h \) be a holomorphic motion

\[
h_\lambda : (\partial V_\lambda, \partial U_{j, \lambda}) \to (\partial V_\lambda, \partial U_{j, \lambda})
\]

over \( D \) with basepoint \(* \in D \) which respects the dynamics. We say \((f, h)\) is a holomorphic family of generalized quadratic-like maps over \( D \). When \( \bigcup U_j \) consists of only one bidisk then the family is a DH quadratic-like family. A family is proper if

1. \( \mathbb{V} \) admits an extension to the boundary
2. for each \( z \in \bigcup_j \partial U_{j, \star} \) the section \( \lambda \mapsto (\lambda, h_\lambda(z)) \) extends continuously to \( \partial D \) and is a trivial section
3. the critical-value section \( \Phi(\lambda) = (\lambda, f_\lambda(0)) \) is proper.

The winding number of a proper family is the winding number of the critical value section.

Theorem 2.11 ([DH2]). If \((f, h)\) is a proper DH quadratic-like family over \( D \) with winding number 1 then

\[
M(f, h) = \{ \lambda \in D : J(f_\lambda) \text{ is connected} \}
\]

is homeomorphic to the standard Mandelbrot set \( M \). The homeomorphism is given by the inner class map \( \lambda \mapsto I(f_\lambda) \).

We finish this section with the renormalization of a family. Let \((f : \bigcup U_j \to \mathbb{V}, h)\) be a proper holomorphic family of generalized quadratic-like maps over \( D \) with winding number 1. If \( 0 \in R(f_\lambda, U_{0, \lambda}) \) let \( \tilde{i}_\lambda \) be the return itinerary of \( f_\lambda \): the (possibly empty) sequence of indices of off-critical pieces \( \{U_{j, \lambda}\} \) through which the critical point passes before returning to
$U_{0,\lambda}$. For such an $f_\lambda$ we can define a holomorphic motion $h'$ of the boundaries of the domain and range of the return map to $U_{0,\lambda}$ by pulling back the holomorphic motion $h$ by $f_\lambda$. The motion $h'$ has basepoint $\lambda$ and is defined over the neighborhood of $\lambda$ having the itinerary $\tilde{\iota}_\lambda$.

**Lemma 2.12** ([L4, Lemma 3.6]). Let $(f: \cup_j U_j \to \cup_j, h)$ be a proper generalized quadratic-like family over $D$ with winding number 1. Let $* \in D$ be the basepoint and let $g_* = R(f_*, U_{0,*})$. Suppose $0 \in \text{Dom}(g_*)$. Then the set

$$D' = \{ \lambda \in D : \tilde{\iota}_\lambda = \tilde{\iota}_* \}$$

is a Jordan disk and the family of first return maps $(g, h')$ over $D'$ is proper and has winding number 1.

### 2.5. Parabolic periodic points.

The limits of maps with unbounded but essentially bounded combinatorics are maps with parabolic periodic points. This section reviews the local theory near parabolic orbits and their perturbations. The main results are the existence and continuity of Fatou coordinates. These results were proven in [DH1] and [La] for perturbations lying in an analytic family and later generalized in [Sh]. Our presentation is based on [Sh].

Throughout this section we give the space of holomorphic maps the “compact-open topology with domains”. A basis for this topology is given by the sets

$$\mathcal{N}(f, K, \epsilon) = \{ g : |g(z) - f(z)| < \epsilon \text{ for } z \in K \}$$

where $K \subset \text{Dom}(f)$ is compact and $\epsilon > 0$. If a sequence of quadratic-like maps converges to $f : U \to \mathbb{C}$ in the Carathéodory topology then it also converges to $f : U \to \mathbb{C}$ in this topology.

Let $\mathcal{P}_0$ be the space of holomorphic maps $f_0$ with a fixed point $\xi_0$ that is parabolic and non-degenerate: $f_0'(\xi_0) = 1$ and $f_0''(\xi_0) \neq 0$. For example, choose any quadratic-like map $f_0$ hybrid equivalent to $z^2 + 1/4$. Choose a neighborhood $N \ni \xi_0$ so that $f_0|_N$ is a diffeomorphism and maps $N$ onto a neighborhood $N' \ni \xi_0$.

**Proposition 2.13** (Fatou coordinates). Let $f_0 \in \mathcal{P}_0$ and choose $N$ and $N'$ as above. Then there exist topological disks $D_\pm \in N \cap N'$, whose union forms a punctured neighborhood of $\xi_0$ and which satisfy

$$f_0^{+1}(\text{cl}(D_\pm)) \subset D_\pm \cup \{ \xi_0 \} \text{ and } \cap_{n \geq 0} f_0^{+n}(\text{cl}(D_\pm)) = \{ \xi_0 \}.$$

Moreover, there exist univalent maps $\Phi_\pm : D_\pm \to \mathbb{C}$ such that

1. $\Phi_\pm$ are unique up to post-composing with a translation
2. $\text{Range}(\Phi_+)$ and $\text{Range}(\Phi_-)$ contain a right and left half-plane, respectively
3. $\Phi_\pm(f_0(z)) = \Phi_\pm(z) + 1$

The disks $D_\pm$ are called **incoming** and **outgoing petals** and the maps $\Phi_\pm$ are called the Fatou coordinates. The Fatou coordinates induce conformal isomorphisms between the Écalle-Voronin cylinders $\mathcal{C}_\pm = D_\pm/f_0$ and $\mathbb{C}/\mathbb{Z}$. Let $\pi_\pm$ denote the projection of $D_\pm$ to $\mathcal{C}_\pm$ and extend $\pi_+$ to the attracting basin of $\xi_0$ by $\pi_+(z) = \pi_+(f_0^n(z))$ for a large enough $n$.

A **transit map** $g : \mathcal{C}_+ \to \mathcal{C}_-$ is a conformal isomorphism which respects the ends $\pm \infty$. A holomorphic map $\tilde{g} : U \to \mathbb{C}$ is a local lift of a transit map $g$ if $\text{cl}(U) \subset D_+$, $\text{Range}(\tilde{g}) \subset D_-$, and

$$g \circ \pi_+ = \pi_- \circ \tilde{g}.$$ 

When written in Fatou coordinates, $\tilde{g}$ is a translation $T_a$ by a complex number $a$. The quantity $\tilde{a} = a \mod \mathbb{Z}$, called the **phase**, depends only on $g$ (and the normalization of Fatou
coordinates) and uniquely specifies $g$. We will use the notation $g_\bar{a}$ to denote the transit map with phase $\bar{a}$.

To simplify future notation, let $\Phi = \Phi_+$ and $\phi = \Phi_1^{-1}$. Also, we shall freely use the notation $\Phi_n, \Phi_f, \mathcal{C}_{n,\pm}$, etc to indicate a dependence on an index $n$ or map $f$.

We now consider perturbations of $f_0 \in \mathcal{P}_0$. Since $\xi_0$ is a non-degenerate parabolic fixed point the generic perturbation will cause it to bifurcate into two nearby fixed points $\xi_f$ and $\xi'_f$ with multipliers $\lambda_f$ and $\lambda'_f$, respectively. Let $\mathcal{N}$ be the neighborhood of $\xi_0$ chosen for Proposition 2.14 and let $\mathcal{P}$ be the space of holomorphic maps which are diffeomorphisms of $\mathcal{N}$. Let $\mathcal{P}_1$ be the set of $f \in \mathcal{P}$ with exactly two fixed points $\xi_f$ and $\xi'_f$ in $\mathcal{N}$ satisfying

$$\arg(1-\lambda_f), \arg(1-\lambda'_f) \in [\pi/4, 3\pi/4] \cup [-3\pi/4, -\pi/4]. \quad (2.1)$$

**Theorem 2.14** (Douady coordinates). Let $f_0 \in \mathcal{P}_0$. There is a neighborhood $\mathcal{N}$ of $f_0$ such that if $f \in (\mathcal{N} \cap \mathcal{P}_1)$ then there exist univalent maps $\Phi_f = \Phi_{f,+}$ and $\phi_f = (\Phi_{f,-})^{-1}$, unique up to translation, and a constant $a_f \in \mathbb{C}$ satisfying

1. $\Phi_f(f(z)) = \Phi_f(z) + 1$ and $\phi_f(w + 1) = f(\phi_f(w))$ where defined
2. $\mathcal{C}_{f,+} = \text{Dom}(\Phi_f)/f$ and $\mathcal{C}_{f,-} = \text{Range}(\phi_f)/f$ are conformally cylinders and one can choose fundamental domains $S_{f,\pm}$ to depend on $f \in \mathcal{P}_1$ continuously in the Hausdorff topology.
3. (see Fig. 2) for $z \in S_{f,+}$ there is an $n > 0$ such that $f^n(z) \in S_{f,-}$ and for $n$ minimal

$$f^n(z) = (\phi_f \circ T_{a_f+n} \circ \Phi_f)(z). \quad (2.2)$$

If we fix points $z_\pm \in D_\pm$ and normalize $\Phi_{f,\pm}$ by $\Phi_{f,\pm}(z_\pm) = 0$ then $\Phi_{f,\pm}$ depend continuously on $f \in \mathcal{N} \cap (\mathcal{P}_0 \cup \mathcal{P}_1)$.

Suppose $f_0 \in \mathcal{P}_0$ and $f \in \mathcal{P}_1 \cap \mathcal{N}$ where $\mathcal{N}$ is from Theorem 2.14. The discontinuous map from $S_{f,+}$ to $S_{f,-}$ defined by equation 2.2 projects to a transit map $g_f : \mathcal{C}_{f,+} \to \mathcal{C}_{f,-}$ with phase $\bar{a}_f = a_f \mod \mathbb{Z}$. This map describes how a long orbit of $f$ “passes through the gate” between $\xi_f$ and $\xi'_f$. The following lemma relates the convergence of $\bar{a}_f$ to the convergence of local lifts.

**Lemma 2.15.** Let $f_k \in \mathcal{P}_1$ converge to $f_0 \in \mathcal{P}_0$ and suppose $\bar{a}_{f_k} \to \bar{a}$. Then for any local lift $\bar{g}$ of $g_\bar{a}$ there exists a sequence $n_k$ such that

$$f_k^{n_k} \to \bar{g}$$

uniformly as $k \to \infty$.

**Proof.** Let $K = \text{cl}(\text{Dom}(\bar{g}))$ and define $a \in \mathbb{C}$ by $\bar{g} = \phi \circ T_a \circ \Phi$. Let $K_1$ be a compact set in $\mathbb{C}$ containing $\Phi(K)$ in its interior and let $K_2$ be a compact set in $\text{Dom}(\phi)$ containing $T_a(K_1)$ in its interior. Let $a_k$ be the constant $a_{f_k}$ in Proposition 2.14. Since $\bar{a}_{f_k} \to \bar{a}$ there exists a sequence $n_k$ so that $a_k + n_k \to a$.

For $k$ large enough $K \subset \text{Dom}(\Phi_{f_k}), \Phi_{f_k}(K) \subset K_1, T_{a_k+n_k}(K_1) \subset K_2$ and $K_2 \subset \text{Dom}(\phi_{f_k})$. The lemma follows from equation (2.2) and since $\Phi_{f_k}, T_{a_k+n_k}, \phi_{f_k}$ converge to $\Phi, T_a, \phi$ uniformly on $K, K_1, K_2$, respectively. \qed

The following lemma gives a simple condition under which perturbed Fatou coordinates exist.
Lemma 2.16. Suppose $f_n$ is a sequence of quadratic-like maps converging in the Carathéodory topology to a quadratic-like map $f \in \mathcal{P}_0$. Suppose the fixed points of $f_n$ are repelling. Then $f_n \in \mathcal{P}_1$ for $n$ large enough.

Proof. Using the holomorphic index (see [M1]) one can prove that

$$\frac{1}{1 - \lambda_{f_n}'} + \frac{1}{1 - \lambda_{f_n}''}$$

converges as $f_n \to f$. Since $\lambda, \lambda' \in \mathbb{C} \setminus \mathbb{D}$ it follows

$$|\arg(1 - \lambda_{f_n})| \to \pi/2 \text{ and } |\arg(1 - \lambda_{f_n}')| \to \pi/2$$
as \( n \to \infty \). In particular, \( f_n \in \mathcal{P}_1 \) for \( n \) large.

For any \( z \in D_+ \cap D_- \) define the \( \hat{\text{Ecalle-Voronin}} \) transformation \( \mathcal{E} \) by

\[
\mathcal{E}(\pi_-(z)) = \pi_+(z).
\]

One can show that \( \mathcal{E} \) extends holomorphically to the two ends of \( \mathcal{C}_- \) by using the Fatou coordinates and the standard isomorphism \( \pi(z) = \exp(2\pi iz) \) of \( \mathbb{C}/\mathbb{Z} \) to \( \mathbb{C} \setminus 0 \). The following lemma is useful for controlling the dynamics near the ends of the \( \hat{\text{Ecalle-Voronin}} \) cylinders.

**Lemma 2.17.** Suppose \( f_0 \in \mathcal{H}(1/4) \) and \( g : \mathcal{C}_+ \to \mathcal{C}_- \) is a transit map such that the critical point of \( f_0 \) escapes \( K(f_0) \) under iterates of \( f_0 \) and local lifts of \( g \). Then

\[
|(g \circ \mathcal{E})'(\pm \infty)| > 1.
\]

**Proof.** We will prove the lemma with the critical point escaping after just one iterate of a local lift of \( g \). Assume the critical point of \( f \) is at the origin. Let \( R = g \circ \mathcal{E} \) and \( J_- = \pi_-(J(f_0)) \).

Let \( V_{\pm \infty} \) denote the connected components of \( (\mathcal{C}_- \setminus J_-) \cup \{ \pm \infty \} \) containing \( \pm \infty \) and let \( U_{\pm \infty} = g^{-1}(V_{\pm \infty}) \) (see Fig. 3).

Note that \( \mathcal{E} \) can be extended to \( V_{\pm \infty} \) as a branched cover. The set of critical points is the backward orbit of 0 and the only critical value is \( \pi_+(0) \).

Since \( \pi_+(0) \notin U_{\pm \infty} \) and each \( U_{\pm \infty} \) is simply connected there is a branch of \( \mathcal{E}^{-1} \) defined on \( U_{\pm \infty} \) preserving \( \pm \infty \). Composing \( \mathcal{E}^{-1} \circ g^{-1} \) we have constructed a branch of \( R^{-1} \) which maps each \( V_{\pm \infty} \) strictly inside itself and fixes \( \pm \infty \). The lemma follows from the Schwarz lemma.

We close this section with a lemma on the geometry of some particular first through maps. Let \( m > 0 \) and let

\[
X = \{ f \in \text{Quad}(m) : f \in \mathcal{H}(1/4) \text{ and } \text{diam } K(f) = 1 \}.
\]

From Lemma 2.14, \( X \) is compact (in the Carathéodory topology). For each \( f \in X \) choose a neighborhood \( N \ni \beta(f) \) on which \( f \) is a diffeomorphism and let \( \mathcal{N}_1, \ldots, \mathcal{N}_k \) be a finite cover of \( X \) by the neighborhoods from Theorem 2.14. In order to preserve certain compactness properties, we will need \( \mathcal{N}_i \) to be closed neighborhoods. By rescaling we can extend the neighborhoods \( \mathcal{N}_i \) to be a finite cover of \( \{ f \in \text{Quad}(m) : f \in \mathcal{H}(1/4) \} \). Note the coordinates do not necessarily agree on the overlaps \( \mathcal{N}_i \cap \mathcal{N}_j \).

Now suppose \( f \in \text{Gen}(m) \) is a generalized quadratic-like map such that the critical point escapes the central component \( U_0 \). Suppose \( f \in (\mathcal{N}_i \cap \mathcal{P}_1) \) for some \( 1 \leq i \leq k \) and suppose the critical point of \( f \) passes once through the gate before landing in the off-critical pieces. In this case we say \( f \) has a saddle-node cascade. Let \( T = T(f, \cup_{j \neq 0} U_j) \) be the first through map of \( f \) and define the modified landing time \( l(z) \) of \( z \in \text{Dom}(T) \) as follows. For each \( \mathcal{N}_i \ni f \), there is a choice of fundamental domains \( S_{f, \pm} \). Write \( T(z) \) as a composition of \( f, (f|_{\mathcal{N}_i})^{-1} \) and of the discontinuous map \( \tilde{g}_f : S_{f, +} \to S_{f, -} \) defined in equation (2.2). Define \( l_i(z) \) as the minimal number of maps in this composition and define \( l(z) = \max_i l_i(z) \).

**Lemma 2.18.** Let \( f_k \in \text{Gen}(m) \) be a sequence of maps with saddle-node cascades such that the modified landing times \( l(0) \) are bounded. Suppose \( f_k \to f \) and \( K(f|_{U_0}) \) is connected. Then \( f|_{U_0} \in \mathcal{H}(1/4) \).

**Proof.** Fix some neighborhood \( \mathcal{N}_i \) containing \( f_k \) and \( f \). Let \( n_k \) be the transit time defined by equation (2.2) for the orbit of the origin. If \( f|_{U_0} \notin \mathcal{H}(1/4) \) then \( n_k \) is bounded from
above. But then the origin escapes $U_{k,0}$ under a bounded number of iterates of $f_k$, which is a contradiction. \hfill \qed

**Lemma 2.19.** Let $\lambda > 0$, $m > 0$ and $r \in \mathbb{N}$. Suppose $f \in \text{Gen}(m)$ has a saddle-node cascade and $\text{geo}(f) \geq \lambda$. Then the phase $\tilde{a}_f$ of the induced transit map lies in a pre-compact subset of $\mathbb{C}/\mathbb{Z}$. Suppose $g \in \text{Gen}$ is a restriction of the first through map $T$ such that $\text{Dom}(g) = \text{Comp}(\text{Dom}(T), u_g)$ and for $z \in u_g$,

$$l(z) \leq r.$$

Then there exists $C(\lambda, m, r) > 0$ such that $\text{geo}(g) \geq C$.

**Proof.** Let us prove the first statement. Suppose $f \in \mathcal{N}_i$. Let $c_1 = f^{r_1}(0)$ be the first moment when the orbit of 0 lands in $S_{f, +}$. We can assume $r_1$ is uniform over the neighborhood $\mathcal{N}_i$. Then $c_1$ lies in a pre-compact subset of $\mathcal{C}_{f, +}$. Let $c_2 = \tilde{g}_f(c_1)$. Since $f^n(c_2) \in \cup_{j \neq 0} U_j$ for
some $n \leq r$, it follows $c_2$ lies in a pre-compact subset of $C_{f_n}$. Hence the phase $a_f$, measured in the coordinates from $N_i$, lies in a pre-compact subset of $\mathbb{C}/\mathbb{Z}$.

The bound on the geometry is clear since the perturbed Fatou coordinates converge and the transit maps $g_f$ lie in a pre-compact subset and the number of iterates of $g_f$, $f$ and $(f|_{N_i})^{-1}$ is bounded. □

3. Combinatorics

3.1. Essentially period tripling. An $\infty$-renormalizable map $f$ has bounded combinatorics if $\tau(f)$ contains a finite number of distinct maximal tuned Mandelbrot sets, or, equivalently, if $\bar{\sigma}(f)$ contains a finite number of distinct shuffles. In this section we construct an infinite set of maximal tuned Mandelbrot sets with bounded essential period. Hence any map whose tuning invariant is chosen from these Mandelbrot sets will have essentially bounded combinatorics. On the other hand if the tuning invariant contains an infinite number of distinct Mandelbrot sets then the map will not have bounded combinatorics. The simplest way to construct such a collection of maximal tuned Mandelbrot sets is by perturbing in a particular way the map $z^2 - 1.75$, the root point of the period three tuned copy. For this reason we say these copies are essentially period tripling.

Let $f(x) = x^2 - 1.75$ and let $\xi$ be the parabolic periodic orbit of period three. Recall $A = A(f) = [\alpha, \alpha']$. Let $g$ be the first return map of $f$ on $A$ (see Fig. 3). Let $I^1$ and $I^1_1$ be the two indicated intervals satisfying $g|_{I^1} = f^3$ and $g|_{I^1_1} = f^2$.

![Figure 4. The first return map for $x^2 - 1.75$.](image)

Fix a small $\epsilon > 0$ and consider $c \in (-1.75, -1.75 + \epsilon)$. The periodic point $\xi$ bifurcates and the orbit of the critical point under $f^3$ now escapes the interval $I^1$. Let $c_n$ be the parameter value (see Fig. 3) so that for $f = f_{c_n}$,

- $f^{3i}(0) \in I^1$ for $i = 1, \ldots, n - 1$,
- $f^{3n}(0) \in I^1_1$. 

\bullet \ f^{3n+2}(0) = 0.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The first return map for $x^2 + c_5$ and the orbit of the origin.}
\end{figure}

In the next section we will justify the claim that $c_n$ exists and is the center of a maximal tuned Mandelbrot set, denoted $M_n^{(3)}$. Equivalently, if we let $\sigma_n^{(3)}$ be the permutation induced on the orbit by $f_{c_n}$ of the origin labeled from left to right, then $\sigma_n^{(3)}$ is a shuffle. In Fig. 6 we have drawn the period three tuned Mandelbrot set and a few of the $M_n^{(3)}$ accumulating at its root point. Any map $f_c$ for $c \in M_n^{(3)}$ will be renormalizable with essential period $p_c(f_c) = 5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The Mandelbrot set near the real period three tuned copy.}
\end{figure}

In Fig. 7 we have drawn the filled Julia sets for $z^2 - 1.75$ and for $z^2 - c_n$ for some $c_n$ with $n$ large. Fig. 8 shows two blow-ups of the Julia set of $f = z^2 - c_n$. The “ghost” boundary of the basin of $\xi$ is visible in the left picture and the pre-images of this ghost boundary nest down to $J(\mathcal{R}(f))$ in the right picture.
3.2. Essentially bounded combinatorics. In this section we describe the return type sequence of a given shuffle $\sigma$ and define the essential period $p_e(\sigma)$.

Suppose $f \in Quad$ is renormalizable, has real combinatorics, and $\sigma(f) \neq \sigma^{(2)}$. In real contexts we will assume $f \in RQuad$. Define the complex principal nest $V^0 \supset V^1 \supset V^2 \supset \ldots$ of $f$ as follows. Choose a straightening of $f$ to a polynomial $f_c$ and pull the equipotential and external ray foliations of $f_c$ back to $f$. Cut the domain $D$ bounded by a fixed equipotential level by the closure of the rays that land at $\alpha$ and at $\alpha'$. The resulting set of connected components is called the initial Yoccoz puzzle. Let $V^0$ be the component containing 0 and let

$$V^m = \text{Comp}(\text{Dom}(R(f, V^{m-1})), 0).$$

For $m \geq 0$ let $I^m = V^m \cap \mathbb{R}$. 

---

**Figure 7.** The filled Julia set of the map $z^2 - 1.75$ (above) and of $f = z^2 - c_n$ for some large $n$.

**Figure 8.** Blow-ups of $J(f)$ near the origin.
For $m \geq 1$, let $g_m : \cup_i V_i^m \to V^{m-1}$ be the generalized renormalization of $f$ on $V^{m-1}$. We will also denote the restriction to the real line $g_m : \cup_i I_i^m \to I^{m-1}$ by $g_m$. Number the intervals $\cup_i I_i^m$ (and domains $V_i^m$) from left to right and so that $0 \in I_0^m = I^{m+1}$. See Fig. 5 for an example of the first two levels of the real principal nest and the graph of $g_1$.

**Lemma 3.1 (L3).** Let $m > 0$, $n > 0$ and let $f \in Quad(m)$. Suppose $f$ has real combinatorics, is not immediately renormalizable, and the return time of any $z \in Dom(g_1)$ through the initial Yoccoz puzzle until the first return to $V^0$ is bounded above by $n$. Then $\text{mod}(\text{Dom}(g_1), \text{Range}(g_1)) \geq m_0(m, n) > 0$, geo$(g_1) \geq C(m, n) > 0$ and diam $K(g_1)/ \text{diam } K(f) \geq C'(m, n) > 0$.

The return type of $g_m$ is defined as follows (see L4 for details). Let $g \in \text{RGen}$ have finite type and let $\cup_i I_i = \text{Dom}(g) \cap \mathbb{R}$ numbered from left to right with $0 \in I_0$. Let $(\Gamma, \epsilon)$ be the free ordered signed semigroup generated by $\{I_i\}$ where $\epsilon : \{I_i\} \to \{\pm 1\}$ is the sign function defined for $i \neq 0$ by $\epsilon(I_i) = +1$ iff $g|_{I_i}$ is orientation preserving and for $i = 0$ by $\epsilon(I_0) = +1$ iff $0$ is a local minimum of $g$. Let $h \in \text{RGen}$ be a restriction of $R(g, I_0)$ to finitely many components of its domain and let $\cup_j J_j = \text{Dom}(h) \cap \mathbb{R}$. Let $(\Gamma', \epsilon')$ be the corresponding signed semigroup for $h$. Let $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ be the homomorphism generated by assigning to each $J_j$ the word $I_{i_1} I_{i_2} \cdots I_{i_n}$ where $I_{i_k}$ is the interval containing $g^{k}(J_j)$ and $n$ is the return time of $J_j$ to $I_0$. The homomorphism $\chi : \Gamma' \to \Gamma$ is the return type of $h$.

A homomorphism $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ between free ordered signed semigroups is called unimodal if the image of every generator is a word ending with the central interval and if the map is strictly monotone on the intervals to the right and left of center and has an extremum at the center. We say a unimodal $\chi$ is admissible if

$$\epsilon'(I_j^i) = \text{sgn}(j)\epsilon(\chi(I_j^i)) \text{ for } j \neq 0 \text{ and } \epsilon'(I_j^0) = \epsilon(\chi(I_j^0)) \text{ for } j = 0.$$ 

Let us describe the initial combinatorics of $f$. Let $(\Gamma_0, \epsilon_0)$ be the signed semigroup generated by $+I^0$ and two intervals $-I^0_1$ and $+I^0_2$. We say a unimodal homomorphism $\chi : (\Gamma, \epsilon) \to (\Gamma_0, \epsilon_0)$ is zero-admissible if it is admissible and additionally for each $I_i$ there is a $p_i \geq 0$ with $p_0 \geq 1$ and such that

$$\chi(I_i) = I_{-1}^0 I_{p_i}^0 I_0.$$ 

The initial combinatorics of $f$ is described by the homomorphism assigning to each $I_i^1$ its itinerary by $f$ through the intervals $I^0$ and the connected components of $B(f) \setminus I^0$. In general if $h_1$ is any restriction of the first return map to $V^0$ then the return type of $h_1$ is the homomorphism mapping to any interval in its domain its itinerary through the above intervals. Note that if $f$ has negative orientation then repeat the construction with all signs reversed.

The combinatorics of $f$ up to level $m$ is described by the sequence $S_m$ of admissible unimodal homomorphisms

$$\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0$$

where $\chi_m$ is the return type of $g_m$ and $\chi_1$ is zero-admissible. Each $S_m$ is irreducible, meaning the orbit of the critical point enters every interval $I_j^m$. Since $f$ is renormalizable there exists an $m'$ such that $\Gamma_m$ is the semigroup with one generator for all $m \geq m'$. Let $S(\sigma) = S_{m'}$ for the smallest such value of $m'$. Then the shuffle $\sigma(f)$ is uniquely specified by $S_{m'}$. Moreover, we have the following
**Theorem 3.2 ([L6]).** Let $S$ be an irreducible finite sequence of admissible unimodal homomorphisms:

$$
\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0.
$$

Suppose $\Gamma_m$ is the only semigroup with one generator, $\Gamma_0$ is as above and $\chi_1$ is zero-admissible. Then there is a unique shuffle $\sigma$ such that $S(\sigma) = S$.

We can now justify our construction of the essentially period tripling shuffles $\sigma^{(3)}_m$ from §3.1. Consider the following signed semigroups generated by the specified intervals

$$
\Gamma = \langle +I_{-1}, -I_0 \rangle \\
\Gamma' = \langle -I_0 \rangle
$$

and consider the following homomorphisms

\begin{align*}
\chi_0 : \Gamma &\to \Gamma_0 \quad \text{generated by} \quad I_{-1} \mapsto I_{-1}I_0 \text{ and } I_0 \mapsto I_{-1}I_0 \\
\chi : \Gamma &\to \Gamma \quad \text{generated by} \quad I_{-1} \mapsto I_{-1}I_0 \text{ and } I_0 \mapsto I_0 \\
\chi' : \Gamma' &\to \Gamma \quad \text{generated by} \quad I_0 \mapsto I_{-1}I_0.
\end{align*}

Then the sequence corresponding to the essentially period tripling combinatorics $\sigma^{(3)}_m$ is

$$
\Gamma' \xleftarrow{\chi'} \Gamma \xleftarrow{\chi} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0
$$

where $\chi$ is repeated $n - 1$ times.

A level $m > 0$ is called *non-central* iff

$$
g_m(0) \in V^{m-1} \setminus V^m.
$$

Let $m(0) = 0$ and let $0 < m(1) < m(2) < \cdots < m(\kappa)$ enumerate the non-central levels, if any exist, and let $h_k \equiv g_{m(k)+1}, k = 0, \ldots, \kappa$.

The nest of intervals (or the corresponding nest of pieces $V^m$)

$$
I^{m(k) + 1} \supset I^{m(k) + 2} \supset \cdots \supset I^{m(k+1)}
$$

is called a *central cascade*. The length $l_k$ of the cascade is defined as $m(k+1) - m(k)$. Note that a cascade of length 1 corresponds to a non-central return to level $m(k)$.

A cascade in $\kappa$ is called a *saddle-node* if $0 \not\in h_k I^{m(k)+1}$. Otherwise it is called *Ulam-Neumann*. For a long saddle-node cascade the map $h_k$ is combinatorially close to $z \mapsto z^2 + 1/4$. For a long Ulam-Neumann cascade it is close to $z \mapsto z^2 - 2$.

The next lemma shows that for a long saddle-node cascade, the map $h_k : I^{m(k)+1} \to I^{m(k)}$ is a small perturbation of a map with a parabolic fixed point.

**Lemma 3.3 ([L2]).** Let $h_k : U_k \to V_k$ be a sequence of real-symmetric quadratic-like maps with $\text{mod}(h_k) \geq \epsilon > 0$ having saddle-node cascades of length $l_k \to \infty$. Then any limit point of this sequence in the Carathéodory topology $f : U \to V$ is hybrid equivalent to $z \mapsto z^2 + 1/4$, and thus has a parabolic fixed point.

**Proof.** It takes $l_k$ iterates for the critical point to escape $U_k$ under iterates of $h_k$. Hence the critical point does not escape $U$ under iterates of $f$. By the kneading theory ([MT]) $f$ has on the real line topological type of $z^2 + c$ with $-2 \leq c \leq 1/4$. Since small perturbations of $f$ have escaping critical point, the choice for $c$ boils down to only two boundary parameter values, $1/4$ and $-2$. Since the cascades of $h_k$ are of saddle-node type, $c = 1/4$. \qed
Since both fixed points of such a sequence $h_k$ are repelling, it follows from Lemma 2.16 that for $k$ large enough $h_k$ has perturbed Fatou coordinates and so $h_k$ has a saddle-node cascade in the sense described in §2.3.

Let $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$ and let $h_kx \in I^j \setminus I^{j+1}$. Set
\[
d(x) = \min\{j - m(k), m(k + 1) - j\}.
\]
This parameter shows how deep the orbit of $x$ lands inside the cascade. Let us now define $d_k$ as the maximum of $d(x)$ over all $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$. Given a saddle-node cascade (§3.3), let us call all levels $m(k) + d_k < l < m(k + 1) - d_k$ neglectable.

Let $f$ be renormalizable and $f_1$ a pre-renormalization of $f$. Define the essential period $p_e = p_e(f)$ as follows. Let $p$ be the period of the periodic interval $J = B(f_1)$, and set $J_k = f^kJ$, for $0 \leq k \leq p - 1$. Let us remove from the orbit $\{J_k\}_{k=0}^{p-1}$ all intervals whose first landing to some $I^{m(k)}$ belongs to a neglectable level, to obtain a sequence of intervals $\{J_k\}_{k=1}^{m}$. The essential period is the number of intervals which are left, $p_e(f) = m$. Note the essential period of a shuffle is well-defined and in this way we can define $p_e(f)$ for any $f \in Quad$ with real combinatorics.

Let us give some examples of combinatorial types involving long saddle-node cascades with neglectable levels. Let $\Gamma, \Gamma', \chi$ and $\chi_0$ be from §3.1 and §3.2.

**Example 3.1 (Goes Through Twice).** Let $\chi_2 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_0$ and $I_{-1} \mapsto I^2_{-1}I_0$. Then any sequence of the form
\[
\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi^2} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi^2} \Gamma \xrightarrow{\chi} \Gamma_0
\]
will correspond to a shuffle where the critical orbit moves up through the cascade until the top, returns to the level of $\chi_2$, moves up through the cascade again and then returns to the renormalization interval. If the total number of levels in the sequence is $m$ then the number of neglectable levels will be roughly $m - 2 \min(d, m - d)$ where $d$ is the level of $\chi_2$.

**Example 3.2 (Two Cascades).** As a second example imagine perturbing the right-hand picture in Fig. 8 so that the renormalization becomes hybrid equivalent to $z^2 + \frac{1}{4}$. Now any further perturbation will cause the parabolic orbit to bifurcate and we can create another long cascade. More specifically, let $\chi_3 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_{-1}I_0$ and $I_{-1} \mapsto I^2_{-1}I_0$ and consider a sequence of the form
\[
\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi^3} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi^4} \Gamma \xrightarrow{\chi} \Gamma_0.
\]
Since $\chi_3$ has a non-central return the two long sequences of $\chi$ form two separate saddle-node cascades, each with a long sequence of neglectable levels.

### 3.3. Parabolic shuffles.** Let $\Omega_p$ be the space of shuffles $\sigma$ satisfying $p_e(\sigma) \leq p$. In this section we construct a compactification $\Omega_p^{cpt}$ of $\Omega_p$ which will form the elements of our combinatorial description of renormalization limits.

Suppose $f \in RQuad$ is renormalizable and let
\[
\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_m^{-1}} \cdots \xrightarrow{\chi_1} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0
\]
be its sequence of return types. Let $l$ be a neglectable level and let $\chi_l : (\Gamma_l, \epsilon_l) \to (\Gamma_{l-1}, \epsilon_{l-1})$ be the return type of $g_l$. It is clear that if both level $l - 1$ and $l + 1$ are neglectable then
$(\Gamma_l, \epsilon_l)$ and $(\Gamma_{l-1}, \epsilon_{l-1})$ are generated by configurations of the form
\[ \pm I_{-p}, \pm I_{-p+1}, \ldots, \pm I_{-1}, -I_0 \]
or by
\[ +I_0, \pm I_1, \ldots, \pm I_{p-1}, \pm I_p \]
for some $p \geq 1$. We claim that $(\Gamma_l, \epsilon_l) \cong (\Gamma_{l-1}, \epsilon_{l-1})$ and that $\chi_l$ is defined by $I_i \mapsto I_i0$ for $i \neq 0$ and $I_0 \mapsto I_0$. First it is clear $I_0 \mapsto I_0$. Now if $\chi_l(I_i)$ contained more than one off-critical interval then $l$ would not be a neglectable level. Since $\chi_l$ is unimodal it follows $\Gamma_{l-1}$ contains at least as many intervals as $\Gamma_l$. Since the return type sequence is irreducible $\Gamma_{l-1}$ contains exactly the same number of intervals as $\Gamma_l$. Hence $I_i \mapsto I_i0$. The claim that the signs agree follows from the condition that $\chi_l$ be admissible.

Hence we can “insert” another neglectable level into $S$ before $l$ to obtain another irreducible sequence $S'$ of return types:
\[ \Gamma_m \xrightarrow{\chi_m} \cdots \xrightarrow{\chi_l} \Gamma_{l-1} \cong \Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1} \cdots \xrightarrow{\chi_1} \Gamma_0. \]

From Theorem 3.4 there is a unique shuffle $\sigma'$ such that $S(\sigma') = S'$.

We say two shuffles $\sigma$ and $\sigma'$ in $\Omega_p$ are essentially equivalent if one can insert a finite number of neglectable levels into $\sigma$ and $\sigma'$ and obtain equal shuffles. Let $\Xi$ be the partition of $\Omega_p$ into essentially-equivalent equivalence classes. Let $U \in \Xi$ be a non-trivial equivalence class. Then there is an $n = n_U > 0$ such that for any $\sigma \in U$ the return type sequence $S(\sigma)$ has exactly $n$ different cascades $S_1, S_2, \ldots, S_n$, canonically ordered, containing neglectable levels. Let $l_k$, $k = 1, \ldots, n$, denote the number of neglectable levels in the cascade $S_k$. The map $\theta_U : U \to \mathbb{N}_+^n$ given by $\sigma \mapsto (l_1, l_2, \ldots, l_n)$ is a homeomorphism. Let
\[ \overline{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{+\infty\} \]

be the one-point compactification of $\mathbb{N}$. Define $U^{cpt} \supset U$ as the unique space such that $\theta_U$ extends to a homeomorphism $\theta_U : U^{cpt} \to \overline{\mathbb{N}}_+^n$. Define $\Omega_p^{cpt} \supset \Omega_p$ as the union of the trivial classes of $\Xi$ and of the spaces $U^{cpt}$ for non-trivial $U \in \Xi$. An element of $\Omega_p^{cpt} \setminus \Omega_p$ is called an end and can be represented by a “sequence” of return types where infinitely long sequences of neglectable levels are allowed:
\[ \Gamma_m \xrightarrow{\chi_m} \cdots \xrightarrow{\chi_{l+1}} \Gamma_{l+1} \xrightarrow{\chi_{l+1}} (\Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1})^\infty \xrightarrow{\chi_{l-1}} \Gamma_{l-2} \xrightarrow{\chi_{l-2}} \cdots \xrightarrow{\chi_1} \Gamma_0. \]

The following lemma is evident from the definition of essential period and $\Omega_p^{cpt}$.

**Lemma 3.4.** For any $p > 1$ the space $\Omega_p^{cpt}$ is metrizable and compact.

Let $\mathcal{M}_p = \{ M(\sigma) \}_{\sigma \in \Omega_p}$ be the collection of $M$-copies corresponding to $\Omega_p$ and let $\mathcal{C}_p = \{ c(\sigma) \}_{\sigma \in \Omega_p}$ be the corresponding collection of centers. We now describe the topology of $\mathcal{C}_p$ and how $\operatorname{cl}(\mathcal{C}_p)$ compares to $\Omega_p^{cpt}$. For any $U \in \Xi$ with $n = n_U \geq 1$ let $\mathcal{C}_U \subset \mathcal{C}_p$ denote the collection of centers of $\{ M(\sigma) \}_{\sigma \in U}$. Since $\Xi$ is a finite partition it sufﬁces to describe the topology of the sets $\mathcal{C}_U$. We claim for each non-trivial $U \in \Xi$ there is a homeomorphism of $\mathbb{R}$ which maps $\mathcal{C}_U$ to the image of the function $F : \mathbb{N}_+^n \to \mathbb{R}$ given by
\[ F(x_1, x_2, \ldots, x_n) = 2^{-x_1} + 2^{-x_1x_2 - 1} + \cdots + 2^{-x_1x_2 \cdots x_n - n + 1} \]
where $n = n_U$ (see Fig. 9).
irreducible then simply remove all intervals $I$.

One can check that point and shorten the sequence if necessary. We obtain a unique shuffle

$σ$ truncated at level $l$.

Let $(Γ_T, ε_T)$ be the semigroup generated by $I_0$ with $ε_T(I_0) = ε(I_0)$. Let $χ_T$ be the homomorphism defined by $I_0 ↦ χ(I_0)$. Let $S'$ be the sequence

$Γ_T χ_T ↦ Γ_{l-1} χ_T ↦ ... ↦ Γ_1 χ_T ↦ Γ_0$.

One can check that $S'$ is a sequence of admissible unimodal return types. If $S'$ is not irreducible then simply remove all intervals $I_l^m$ not in the combinatorial orbit of the critical point and shorten the sequence if necessary. We obtain a unique shuffle $σ' = |σ|_l$, the shuffle $σ$ truncated at level $l$.

Let $U ∈ Ξ$ satisfy $n = n_U ≥ 1$. Any shuffle $σ ∈ U$ has $n$ cascades with neglectable levels of lengths $x_1, ..., x_n$ respectively. As $x_1 → ∞$, the corresponding centers accumulate at the root of the tuned $M$-copy corresponding to any $σ ∈ U$ truncated at the first neglectable level. If we fix $x_1$ and let $x_2 → ∞$ the corresponding centers accumulate at the root of the $M$-copy corresponding to truncating at the second cascade of neglectable levels. In general if we fix the lengths of the first $k$ sequences of neglectable levels and let the length of the $k + 1$-st sequence grow the centers converge to the root of the $M$-copy corresponding to truncating at the $(k + 1)$-st neglectable sequence.

Given an end $τ ∈ Ω_{p}^{εpt}$ let

$c(τ) = root(|σ|_l)$

where $σ ∈ Ω_p$ is in a sufficiently small neighborhood of $τ$, $l$ is a neglectable level of $σ$ which belongs to the first infinitely long cascade of $τ$, and $root(σ)$ is the root of the $M$-copy $M(σ)$. The map $c : Ω_{p}^{εpt} → ℝ$ is continuous and its image is $cl(Γ_p)$.

We return to our examples. Choose a large $p$ so that the shuffles from Example 3.1 and Example 3.2 are contained in $Ω_p$.

First consider the essentially period tripling shuffles $σ_n^{(3)}$. Then $c(σ_n^{(3)}) → root(σ^{(3)})$ where $σ^{(3)}$ is the period tripling shuffle. Moreover, $σ_n^{(3)}$ converges to an end $τ_1 ∈ Ω_{p}^{εpt}$.

Now consider the shuffles $σ_{m,d}$ from Example 3.1 (Goes Through Twice). First fix $d > 1$ and let $m → ∞$. Then $c(σ_{m,d}) → root(|σ_{m,d}|_l) ≠ root(σ^{(3)})$ where $l$ is any neglectable level and, in much the same spirit as essential period tripling, $σ_{m,d}$ converges in $Ω_{p}^{εpt}$ to an end. Now fix $m − d > 1$ and let $m → ∞$. Then $c(σ_{m,d}) → root(σ^{(3)})$ and $σ_{m,d}$ converges to an end $τ_2 ∈ Ω_{p}^{εpt}$.

Finally consider the shuffles $σ_{l_1,l_2}$ from Example 3.2 (Two Cascades). Fix $l_1 > 1$ and let $l_2 → ∞$. Then $c(σ_{l_1,l_2}) → r_{l_1} = root(|σ_{l_1,l_2}|_l)$ where $l$ is any neglectable level in the second cascade. The sequence $r_{l_1} → root(σ^{(3)})$ as $l_1 → ∞$. Moreover, for any sequence of $l_2$ if we
Let \( l_1 \to \infty \) then \( c(\sigma_{l_1,l_2}) \to \text{root}(\sigma^{(3)}) \). Now consider the limits of \( \sigma_{l_1,l_2} \) in \( \Omega_{p}^{\text{cpt}} \). If we fix \( l_2 \) and let \( l_1 \to \infty \) the shuffles will converge to an end \( \tau_{\infty,l_2} \).

This completes our description of the topology of \( \mathcal{M}_{p} \) and how \( \text{cl}(\mathcal{C}_{p}) \) compares to \( \Omega_{p}^{\text{cpt}} \).

4. Parabolic Renormalization

Let \( c \star M \) be a maximal tuned Mandelbrot set with root \( c' \) and suppose \( f \in \mathcal{H}(c') \) is renormalizable and let \( f_{0} \) be a pre-renormalization. Let \( \xi = \beta(f_{0}) \). Choose incoming and outgoing petals \( D_{\pm} \) around the parabolic point \( \xi \) and let \( \mathcal{C}_{\pm} \) denote the respective Écalle-Voronin cylinders and \( \pi_{\pm} \) the projections with \( \pi_{+} \) extended to \( B = \text{int}(K(f_{0})) \). Fix a transit map \( g : \mathcal{C}_{+} \to \mathcal{C}_{-} \) satisfying

\[
g(\pi_{+}(0)) \notin \pi_{-}(K(f_{0})).
\]

Given a collection \( \{f_{\alpha}\} \) of homomorphic maps let \( (f_{\alpha}) \) denote the set of restrictions of all finite compositions of \( \{f_{\alpha}\} \). Let

\[
\mathcal{F}(f,g) = \langle f \cup \{\text{all local lifts of } g \to D_{\pm}\} \rangle.
\]

Note that \( \mathcal{F}(f,g) \) is independent of the choice of petals \( D_{\pm} \). A collection \( \mathcal{F} \) of holomorphic maps closed under composition and restriction is called a conformal dynamical system. Define the orbit of a point \( z \in \mathbb{C} \) as

\[
\text{orb}(z) = \text{orb}(\mathcal{F},z) = \bigcup_{h \in \mathcal{F}} h(z).
\]

We say \( \mathcal{F} \) is contained in any geometric limit of a sequence \( \mathcal{F}_{n} \) if for any \( f \in \mathcal{F} \) there are \( f_{n} \in \mathcal{F}_{n} \) such that \( f_{n} \to f \) on compact sets.

We say the pair \( (f,g) \) is parabolic renormalizable if there is a neighborhood \( U \supseteq 0 \) and a map \( h \in (\mathcal{F}(f,g) \setminus \langle f \rangle) \) such that

\[
h|_{U} \in \text{Quad}.
\]

We call such an \( h|_{U} \) a parabolic pre-renormalization of \( (f,g) \) and we call the germ of a normalized pre-renormalization a parabolic renormalization of \( (f,g) \). In the next section we will show that the domain \( U \) of the pre-renormalization can be canonically chosen.

4.1. Essentially period tripling. In this section we describe a construction from \[DD\] for finding a canonical representation of the parabolic renormalization in the essentially period tripling case. For simplicity we will state the construction for the quadratic map \( P_{-1.75} \). However, it is clear how to generalize this construction to any map \( f \in \mathcal{H}(-1.75) \).

Recall from \[\S 3.1\] the sequence of maximal tuned Mandelbrot sets \( c_{n} \star M \) with essentially period tripling combinatorics accumulate at the root of the period three tuned copy, \( c = -1.75 \). Let \( f = P_{-1.75} \) and choose \( f_{0} \) and \( D_{\pm} \) as above. Let \( B = \text{int}(K(f_{0})) \) and let \( f_{n} = P_{c_{n}} \).

Choose \( n_{0} \) sufficiently large and choose

\[
U_{-} \in \bigcap_{n \geq n_{0}} \text{Comp}(\text{int}(K(f)) \setminus B, P(f_{n}))
\]

such that \( U_{-} \subseteq D_{-} \). Let \( t \) be the landing time of \( U_{-} \) to \( B \) under \( f \).

Let

\[
\mathcal{D}_{f} = \{ g : \mathcal{C}_{+} \to \mathcal{C}_{-} | g \text{ is a transit map and } g(\pi_{+}(0)) \in \pi_{-}(U_{-}) \}.
\]

The phase map gives a conformal isomorphism of \( \mathcal{D}_{f} \) to a disk \( D_{f} \subseteq \mathbb{C}/\mathbb{Z} \). Note that \( D_{f} \) is a Jordan domain. Choose a branch of \( \pi^{-1} \) so that \( \text{Range}(\pi^{-1}) \supseteq U_{-} \). For \( g \in \mathcal{D}_{f} \) let \( W_{g} \) be
the connected component of \((\pi_{+}^{-1} \circ g^{-1} \circ \pi_{-})(U_{-})\) containing 0. Since \(\pi_{-}(U_{-})\) is a topological disk, it follows the map \(R_{\tilde{a}} : W_{g} \to B\) given by

\[
R_{\tilde{a}} = f^{t} \circ \pi_{-}^{-1} \circ g_{a} \circ \pi_{+}
\]

is quadratic-like with possibly disconnected Julia set. If \(J(R_{\tilde{a}})\) is connected then we have constructed a parabolic pre-renormalization of \((f, g)\).

Fix any \(* \in D_{f}\). Define the holomorphic motion

\[
h_{\tilde{a}} : (\partial B, \partial W_{g_{a}}) \to (\partial B, \partial W_{g_{a}})
\]

on \(\partial B\) by the identity and locally on \(\partial W_{g_{a}}\) by pulling back under \(R_{\tilde{a}}\). Let \(V = \{(\bar{a}, z) : \bar{a} \in D_{f}, z \in B\}\) and \(U = \{(\bar{a}, z) : \bar{a} \in D_{f}, z \in W_{g_{a}}\}\). Let \(f : U \to V\) be defined by

\[
R(\bar{a}, z) = (\bar{a}, R_{\tilde{a}}(z)).
\]

**Lemma 4.1.** The family \((R, h)\) is a proper DH quadratic-like family with winding number 1.

**Proof.** The map \(f^{t}\) is a conformal isomorphism of a neighborhood of \(U_{-}\) onto a neighborhood of \(B\). There is a branch of \(\pi_{-}^{-1}\) such that the map \((\pi_{-}^{-1} \circ g_{a} \circ \pi_{+})(0)\) is a conformal isomorphism of a neighborhood of \(D_{f}\) onto a neighborhood of \(U_{-}\). The lemma follows.

The following lemma states that the renormalization operators \(R_{\sigma_{n}^{(4)}}\) converge to essentially period tripling parabolic renormalization.

**Lemma 4.2.** Let \(f \in \mathcal{H}(-1.75)\). Suppose \(f_{k} \in \text{Quad}\) is a sequence of renormalizable maps with \(f_{k} \to f\) and \(\sigma(f_{k}) \to \tau\). Let \(g_{k} : \mathcal{C}_{f_{k},+} \to \mathcal{C}_{f_{k},-}\) be the induced transit maps with phases \(\tilde{a}_{k}\). Then

1. \(\{\tilde{a}_{k}\}\) is pre-compact
2. If \(\tilde{a}_{k_{j}} \to \bar{a}\) is a convergent subsequence then \(J(R_{\tilde{a}})\) is connected and
   \[
   [h_{k_{j}}] \to [R_{\tilde{a}}]
   \]
   where \(h_{k}\) is a pre-renormalization of \(f_{k}\)
3. \(\mathcal{F}(f, g_{a})\) is contained in any geometric limit of \(\langle f_{k}\rangle\)

**Proof.** Let \(h_{k}\) be a pre-renormalization of \(f_{k}\). Since \(\sigma(f_{k}) \to \tau\) and \(f_{k} \to f\) we can write

\[
h_{k} = f_{k}^{N_{1}} \circ \tilde{g}_{k} \circ f_{k}^{N_{2}}
\]

on some neighborhood of the origin for some fixed \(N_{1}, N_{2}\) and some choice of local lift \(\tilde{g}_{k}\) of the induced transit maps \(g_{k} : \mathcal{C}_{f_{k},+} \to \mathcal{C}_{f_{k},-}\). The first claim is that \(h_{k}\) can be chosen in \(\text{Quad}(m')\) for some \(m' > 0\). Let \(V'\) be an \(\epsilon\)-neighborhood of the central basin \(B\) of \(f\) for some small \(\epsilon > 0\). Choose \(\epsilon\) small enough and \(N_{1}\) and \(N_{2}\) large enough so that for large \(k\) the right-hand side of \((\text{I})\) can be used to define a pre-renormalization \(h_{k}\) with range \(V'\). Let \(U'_{k} = h_{k}^{-1}(V')\). By taking \(k\) larger still we can assume \(U'_{k}\) is contained in an \(\epsilon/2\) neighborhood of \(B\). It follows there is an \(m' > 0\) so that \(\text{mod}(U'_{k}, V') \geq m'\). Moreover, \(\text{diam}(U'_{k}) \geq C > 0\) for some \(C\) independent of \(k\). Hence \((\text{I})\) holds on a definite neighborhood of the origin.

From the convergence of Fatou coordinates and the convergence of \(f_{k}\) it follows that \(\{\tilde{a}_{k}\}\) is pre-compact. Let \(\tilde{a}_{k_{j}} \to \bar{a}\) be a convergent subsequence. Then \(h_{k_{j}}\) converges on a definite neighborhood of the origin to the map \(f^{N_{1}} \circ \tilde{g}_{a} \circ f^{N_{2}}\) for an appropriate local lift \(\tilde{g}_{a}\) of \(g_{a}\). Since the origin is non-escaping under all \(h_{k}\) it follows \(J(R_{\tilde{a}})\) is connected. The last statement follows from the fact that \(f_{k} \to f\) and Lemma 2.13.
Moreover, the proof of the previous lemma can be modified to prove the following

**Lemma 4.3.** Suppose \( f \in \mathcal{H}(-1.75) \) and \( f_k \in \mathcal{H}(-1.75) \) satisfy \( f_k \to f \). Suppose \( g_k : C_{f_k,+} \to C_{f_k,-} \) is a transit map with phase \( \bar{a} \) such that \( R_k = R_{\bar{a}_k} \) is defined. Then

1. \( \{ \bar{a}_k \} \) is pre-compact
2. if \( \bar{a}_{k_j} \to \bar{a} \) is a convergent subsequence then
   \[ R_{k_j} \to R_\bar{a}. \]
3. \( \mathcal{F}(f,g) \) is contained in any geometric limit of \( \mathcal{F}(f_{k_j},g_{k_j}) \)

We finish this section with two useful properties of parabolic renormalization. The first property is that open sets intersecting the Julia set of the parabolic pre-renormalization iterate under \( \mathcal{F}(f,g) \) to open sets intersecting \( J(f) \).

**Lemma 4.4.** Let \( f \in \mathcal{H}(-1.75) \) and \( g : C_+ \to C_- \) be a transit map with phase \( \bar{a} \) such that \( J(R_\bar{a}) \) is connected. Suppose \( U \) is an open set satisfying
\[ U \cap J(R_\bar{a}) \neq \emptyset. \]
Then there is an \( h \in \mathcal{F}(f,g) \) such that \( U \cap \text{Dom}(h) \neq \emptyset \) and
\[ h(U) \supset J(f). \]

*Proof.* From the construction of \( R_\bar{a} \) it is clear that there is an \( h \in \mathcal{F}(f,g) \) such that \( h \) is a quadratic-like extension of \( R_\bar{a} \) to a small neighborhood of \( B = \text{Range}(R_\bar{a}) \). It follows that there an \( m \geq 0 \) such that \( h^m(U) \cap \partial B \neq \emptyset \). But \( \partial B \subset J(f) \). Iterating \( f \) further covers all of \( J(f) \).

The second property is that no quadratic-like representative of \( [R_\bar{a}] \) can have too large a domain.

**Lemma 4.5.** Let \( f \in \mathcal{H}(-1.75) \) and \( g : C_+ \to C_- \) be a transit map with phase \( \bar{a} \) such that \( R_\bar{a} \) is defined. If \( (\tilde{f} : U \to V) \in \text{Quad} \) satisfies \( [\tilde{f}] = [R_\bar{a}] \) then
\[ U \subset \text{Range}(R_\bar{a}). \]

*Proof.* Let \( f_1 \) be a pre-renormalization of \( f \) and let \( B = \text{Range}(R_\bar{a}) \). Suppose \( U \cap \partial B \neq \emptyset \) and let \( U' \) be the connected component of \( U \cap B \) containing \( 0 \). Since \( f_1 \)-preimages of \( 0 \) accumulate on \( J(f_1) = \partial B \) there exists an \( n > 0 \) and \( z_0 \in U' \) such that \( f_1^n(z_0) = 0 \). Since \( [\tilde{f}] = [R_\bar{a}] \) it follows \( \tilde{f} \) has a critical point at \( z_0 \), which is a contradiction.

4.2. **Generalized parabolic renormalization.** In this section we modify the construction of parabolic renormalization to act on generalized quadratic-like maps.

Let \( f : \bigcup_j U_j \to V \) be a generalized quadratic-like map with \( f_0 = f|_{U_0} \in \mathcal{H}(1/4) \). Let \( \xi = \beta(f_0) \). Choose incoming and outgoing petals \( D_\pm \) around the parabolic point \( \xi \) and let \( C_\pm \) denote the respective Écalle-Voronin cylinders and \( \pi_\pm \) the projections with \( \pi_+ \) extended to \( B = \text{int}(K(f_0)) \).

For a given \( g : C_- \to C_+ \) let \( L_0 \) be the first landing map under \( \mathcal{F}(f,g) \) to \( \bigcup_{j \neq 0} U_j \). Note that if \( C \) is a connected component of \( \text{Dom}(L_0) \) then there is an \( h \in \mathcal{F}(f,g) \) such that \( C \in \text{Dom}(h) \) and \( h(z) = L_0(z) \) for all \( z \in C \). Let \( T \) be the first through map
\[ T = f \circ L_0. \]
Note that $T$ is not a generalized quadratic-like map. However, $T$ has at most one critical value, and, with a slight abuse of notation we will treat $T$ as a generalized quadratic-like map.

Let $X \subset \mathbb{C}/\mathbb{Z}$ be the set of phases $\bar{a}$ such that for $g = g_{\bar{a}}$,

$$0 \in \text{Dom}(L_0).$$

It is clear that $X$ is a countable pairwise disjoint collection of Jordan disks. Let $D$ be a connected component of $X$. Let $T$ denote the family over $D$ of first through maps $T$. The construction of the holomorphic motion $h$ described before Lemma 4.1 carries over unchanged to this situation. Moreover, one can modify the proof of Lemma 4.1 to prove

**Lemma 4.6.** For any connected component $D$ of $X$, the family $(T, h)$ over $D$ is a proper generalized quadratic-like family with winding number 1.

The following two lemmas are generalizations of Lemma 4.2 and Lemma 4.3, respectively. We omit the proofs.

**Lemma 4.7.** Suppose $f \in \text{Gen}$ satisfies $f|_{U_0} \in \mathcal{H}(1/4)$ and $f_k \in \text{Gen}$ is a sequence converging to $f$. Let $T_k$ be the first through map for $f_k$ and suppose that $T_k \in \text{Gen}$. Let $g_k$ be the induced transit maps of $f_k$ with phase $\bar{a}_k$. Suppose $\bar{a}_{k_j} \to \bar{a}$ is a convergent subsequence and suppose $0 \in \text{Dom}(T_{\bar{a}})$ where $T_{\bar{a}}$ is the first through map for $\mathcal{F}(f, g_{\bar{a}})$, Then

$$T_{k_j} \to T_{\bar{a}}.$$

Moreover, $\mathcal{F}(f, g_{\bar{a}})$ is contained in any geometric limit of $(f_{k_j})$. 
\textbf{Lemma 4.8.} Suppose \( f \in \text{Gen} \) and \( f_k \in \text{Gen} \) satisfy \( f|_{\mathcal{U}_0} \in \mathcal{H}(1/4) \), \( f_k|_{\mathcal{U}_k,0} \in \mathcal{H}(1/4) \), and \( f_k \to f \). Let \( g_k \) be a transit map of \( f_k \) with phases \( \bar{a}_k \). Let \( T_k \) be the first through map for \( \mathcal{F}(f_k, g_k) \) and suppose \( T_k \in \text{Gen} \). Suppose \( \bar{a}_k \to \bar{a} \) is a convergent subsequence and suppose \( 0 \in \text{Dom}(T_{\bar{a}}) \) where \( T_{\bar{a}} \) is the first through map for \( \mathcal{F}(f, g_{\bar{a}}) \). Then
\[
T_{k_j} \to T_{\bar{a}}
\]
Moreover, \( \mathcal{F}(f, g_{\bar{a}}) \) is contained in any geometric limit of \( \mathcal{F}(f_{k_j}, g_{k_j}) \).

5. Towers

Let \( S \subset \mathbb{Z} \) be a set of consecutive integers containing \( \mathbb{N}_0 \) and let \( f_n \) be a sequence of maps in \( \text{Gen} \) indexed by \( n \in S \). Let \( U_{n,0} \) be the central component of \( f_n \). Let
\[
S_C = \{ n \in S : f_n|_{\mathcal{U}_{n,0}} \in \mathcal{H}(1/4) \}
\]
and let
\[
S_Q = \{ n \in S : f_n \in \text{Quad} \}.
\]
For \( n \in S_C \) let \( g_n : C_{n,+} \to C_{n,-} \) be a transit map between the Écalle-Voronin cylinders \( C_{n,\pm} \) of \( f_n \). The collection of maps
\[
\mathcal{T} = \{ f_n : n \in S \} \cup \{ g_n : n \in S_C \}
\]
is called a tower iff for each pair \( n, n+1 \in S \) one of the following conditions hold:

T1: \( n \in S_Q \), \( f_n \) is immediately renormalizable and \([f_{n+1}] = [h]\) where \( h \) is a pre-renormalization of \( f_n \) of minimal period

T2: \( n \in S_Q \), \( f_n \) is not immediately renormalizable and \([f_{n+1}] = [h]\) where \( h \) is a restriction of the first return map to the initial central puzzle piece of \( f_n \)

T3: \( n \notin (S_Q \cup S_C) \) and \([f_{n+1}] = [h]\) where \( h \) is a restriction of the first return map \( R(f_n, U_{n,0}) \) or first through map \( T(f_n, \cup_{j \neq 0} U_{n,j}) \)

T4: \( n \in S_C \) and \([f_{n+1}] = [h]\) where \( h \) is a restriction of the first through map of the pair \((f_n, g_n)\).

We shall often identify \( g_n \) with the set of local lifts of \( g_n \) for some choice of incoming and outgoing petals \( D_{n,\pm} \). If \( S_C \neq \emptyset \) then \( \mathcal{T} \) is a parabolic tower.

Let \( \text{Tow} \) be the space of towers with the following topology: a sequence \( \mathcal{T}_m = \{ f_{m,n}, g_{m,n} \} \) converges to \( \mathcal{T} = \{ f_n, g_n \} \) iff

- \( S_m \to S \) and \( S_{m,C} \to S'_{C} \subset S_{C} \)
- if \( n \in S \setminus S_C \) then \( f_{m,n} \to f_n \)
- if \( n \in S'_{C} \) then \( f_{m,n} \to f_n \) and \( g_{m,n} \to g_n \)
- if \( n \in S_C \setminus S'_{C} \) then \( f_{m,n}|_{U_{m,n,0}} \) has both fixed points repelling, \( f_{m,n} \to f_n \) and \( h_{m,n} \to g_n \)

where \( h_{m,n} \) is the induced transit map on the perturbed Écalle-Voronin cylinders.

If \( S = \mathbb{Z} \) then \( \mathcal{T} \) is a bi-infinite tower and otherwise \( \mathcal{T} \) is a forward tower. The map \( f_{\min(S)} \) in a forward tower is called the base map. Define \( \text{Dom}(\mathcal{T}) \) and \( \text{Range}(\mathcal{T}) \) to be the domain and range of the base map.

Let \( \mathcal{T} \) be a forward tower and let \( f_m \) be the base map of \( \mathcal{T} \). Let
\[
\mathcal{F}(\mathcal{T}) = \langle \mathcal{T} \setminus \{ f_n : n > m \} \rangle
\]
where recall \( \langle f_n \rangle \) denotes the set of restrictions of all finite compositions of \( \{ f_n \} \). Define the orbit of \( z \in \text{Dom}(\mathcal{T}) \) by
\[
\text{orb}(z) = \text{orb}(\mathcal{F}(\mathcal{T}), z).
\]
Note that if $\text{Range}(f_n) \subset \text{Range}(f_m)$ and $z \in \text{Dom}(f_n)$ then $f_n(z) \in \text{orb}(z)$. We say $\text{orb}(z)$ escapes if $\text{orb}(z) \cap (\text{Range}(\mathcal{T}) \setminus \text{Dom}(\mathcal{T})) \neq \emptyset$. Define the filled Julia set, $K(\mathcal{T})$, the Julia set, $J(\mathcal{T})$, and the post-critical set, $P(\mathcal{T})$, as for quadratic-like maps. For a bi-infinite tower $\mathcal{T}$ define the post-critical set

$$P(\mathcal{T}) = \text{cl} \bigcup_{S' \subset S} P(\mathcal{T}|_{S'})$$

where $\mathcal{T}|_{S'}$ ranges over forward subtowers of $\mathcal{T}$ and where the closure is taken as a subset of $\hat{\mathbb{C}}$.

Two towers $\mathcal{T}$ and $\mathcal{T}'$ with $S(\mathcal{T}) = S(\mathcal{T}')$ are quasi-conformally equivalent if there is a quasi-conformal map $\phi$ such that

1. $\phi$ is a quasi-conformal conjugacy of $f_n$ and $f'_n$ on a neighborhood of $K(f_n)$ to a neighborhood of $K(f'_n)$ for all $n \in S$,

2. $\phi$ induces a quasi-conformal conjugacy of the transit maps $g_n$ and $g'_n$ for $n \in S_C$.

A quasi-conformal equivalence $\phi$ between two forward towers is a hybrid equivalence if $\overline{\partial \phi}|_{K(\mathcal{T})} \equiv 0$ and is a holomorphic equivalence if $\phi$ is holomorphic. The following proposition is the analogue of Proposition 2.2 for towers.

**Proposition 5.1 (Straightening).** Let $\mathcal{T}$ be a forward tower such that its base map is quadratic-like. Then $\mathcal{T}$ is hybrid equivalent to a tower with a quadratic base map.

**Proof.** Let $f_m$ be the base map of $\mathcal{T}$. From Proposition 2.2 there is a hybrid equivalence $\phi$ between $f_m$ and a unique polynomial of the form $z^2 + c$. Let $u(z)$ be the complex dilatation of $\phi$ and let $\mu = u(z)dz/dz$ be the corresponding Beltrami differential. Since $\phi$ is quasi-conformal there is a $k < 1$ such that $\|u(z)\|_\infty \leq k$. Let $U \supset K(f_m)$ be the domain on which $\phi$ is a conjugacy.

Define the Beltrami differential $\mu'$ by

$$\mu'|_{K(\mathcal{T})} \equiv 0$$

and if $z \in (U \setminus K(\mathcal{T}))$ by

$$\mu'|_{U'} = h^*(\mu)$$

where $h \in \mathcal{F}(\mathcal{T})$ and $U' \ni z$ satisfy $h(U') \subset (U \setminus K(f_m))$. There are restrictions $f'_n$ of $f_n$ such that $[f'_n] = [f_n]$ and $\mu'$ is invariant under the forward tower $\mathcal{T}' = \{f'_n, g_n\}$.

Write $\mu'(z) = u'(z)dz/dz$. Since all maps in $\mathcal{T}'$ are holomorphic $\|u'(z)\|_\infty \leq k < 1$. Let $\phi_1$ be the solution to the Beltrami equation

$$\overline{\partial \phi_1} = u' \cdot \overline{\partial \phi_1}$$

and let

$$\mathcal{T}'' = \{\phi_1 \circ h \circ \phi_1^{-1} : h \in \mathcal{T}'\}.$$  

We claim $\mathcal{T}''$ is again a forward tower and that $\phi_1$ is a hybrid equivalence between $\mathcal{T}$ and $\mathcal{T}''$. Let $n \in S_C$ and $g_n \in \mathcal{T}'$. Let $g''_n = \phi_1 \circ g_n \circ \phi_1^{-1}$ and $f''_n = \phi_1 \circ f'_n \circ \phi_1^{-1}$. Since $\phi_1$ conjugates forward and backward orbits of $f_n$ to orbits of $f''_n$, it follows that $g''_n$ is a map on the Écalle-Voronin cylinders of $f''_n$. Since $\phi_1$ is a homeomorphism, it is evident that $g''_n$ is a homeomorphism. Moreover, $\mu'$ is invariant under $g_n$, and so $g''_n$ is conformal. That is, the conjugate of a transit map in $\mathcal{T}$ is a transit map in $\mathcal{T}''$. The other properties of a tower are clear.

The base map of $\mathcal{T}''$ is holomorphically equivalent to a polynomial. Hence $\mathcal{T}''$ is holomorphically equivalent to a tower with a polynomial base map. 

$\square$
6. Limiting Towers

In this paper we study the parabolic towers that are limits of certain McMullen towers. To be precise we make the following definition. For a given $\kappa > 0$ let $Tow(\kappa)$ be the closure of the set $Tow_0(\kappa)$ of $\mathcal{T} \in Tow$ satisfying

1. $S_C = \emptyset$
2. $f_0 \in Quad$ is normalized
3. $f_n \in Gen(1/\kappa)$ for all $n \in S$
4. if $n \in S_Q$ then $f_n$ has real combinatorics and $p_e(f_n) \leq \kappa$
5. if $n \notin S_Q$ then $[f_n] = [h_n]$ where $h_n$ is either the generalized renormalization of $f_{n-1}$ or the first through map $T$ of $f_{n-1}$ restricted to $\text{Comp}(\text{Dom}(T), P(f_{n-1}))$
6. if $f_n$ is a first through map then $f_{n-1}$ has a saddle-node cascade in the sense described in §2.5
7. quadratic-like levels are at most $\kappa$ apart: if $n, m \in S_Q$ are adjacent quadratic-like levels then $|m - n| \leq \kappa$.
8. $V_n = \text{Range}(f_n)$ is a $\kappa$-quasidisk
9. $\text{diam} V_n \leq \kappa \text{diam} K(f_n)$
10. Unbranched Property: $V_n \cap P(f_m) = P(f_n)$ for $n \geq m$

We will refer to towers in $Tow(\kappa)$ as towers with essentially bounded combinatorics and complex bounds. Over the next several sections we will analyze the basic properties of towers in $Tow(\kappa)$.

The combinatorics of a tower $\mathcal{T} \in Tow_0(\kappa)$ is the sequence $\tilde{\sigma}(\mathcal{T})$ indexed by $n \in S_Q$ of shuffles $\sigma_n = \sigma(f_n)$. Recall $\Omega^{\text{opt}}(\kappa)$ is the compactification of the space $\Omega(\kappa)$ of shuffles $\sigma$ with $p_e(\sigma) \leq \kappa$. Suppose $\mathcal{T}_m \in Tow_0(\kappa)$ is a sequence of towers converging to the parabolic tower $\mathcal{T} \in Tow(\kappa)$. The combinatorics of $\mathcal{T}$ is the sequence $\tilde{\sigma}(\mathcal{T})$ indexed by $n \in S_Q$ of shuffles and ends given by $\lim_{m \to \infty} \sigma_{m,n}$. Clearly the combinatorics of a tower is invariant under hybrid equivalence.

Two towers $\mathcal{T} = \{f_n, g_n\}$ and $\mathcal{T'} = \{f'_n, g'_n\}$ are combinatorially equivalent if $S(\mathcal{T}) = S(\mathcal{T'})$ and $\tilde{\sigma}(\mathcal{T}) = \tilde{\sigma}(\mathcal{T'})$.

Proposition 6.1 (Forward Combinatorial Rigidity). Let $\mathcal{T}$ and $\mathcal{T}'$ be forward towers hybrid equivalent to towers in $Tow(\kappa)$. Let $f_n$ and $f'_n$ be the respective base maps. Suppose $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ and $[f_n] = [f'_n]$. Then $[f_n] = [f'_n]$ for all $n \in S$ and $g_n = g'_n$ for $n \in S_C$.

Proof. First, it is clear that $[f_n] = [f'_n]$ for $m \leq n \leq \min\{S_C\}$ where $\min\{\emptyset\} = \infty$. Now suppose by induction that $n \in S_C$ and $[f_n] = [f'_n]$. We claim $g_n = g'_n$. Let $L_a$ be the first landing map of $\mathcal{F}(f_n, g_a)$ to the off-critical pieces of $f_n$ and let

$$X = \{a : 0 \in \text{Dom}(L_a)\}.$$ 

Let $(\mathcal{T}, h)$ be the holomorphic family over the component $D \subset X$ containing $g_n$ of generalized quadratic-like maps constructed in §2.4. Construct the sequence of families of first return maps as described in §2.4 until the next level where $f_n|_{V_{n,0}} \in Quad$. Similarly construct the families containing $g'_n$ using $f'_n$. Since $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ it follows from Theorem 2.3, uniqueness of root points and Theorem 2.11 that $g_n = g'_n$. \qed

Combining this result with straightening we have the following
Corollary 6.2. Any two combinatorially equivalent forward towers $\mathcal{T}, \mathcal{T}' \in \text{Tow}(\kappa)$ are hybrid equivalent.

Proof. Straighten $\mathcal{T}$ and $\mathcal{T}'$ to the towers $\mathcal{T}_1$ and $\mathcal{T}_2$ with quadratic base maps. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are combinatorially equivalent it follows from Theorem 2.4 and the uniqueness of root points that the base maps are equal. Hence by Proposition 6.1 $\mathcal{T}_1$ and $\mathcal{T}_2$ are hybrid equivalent.

We now prove compactness:

Proposition 6.3. For any $\kappa > 0$ the space $\text{Tow}(\kappa)$ is compact.

Proof. Let $\mathcal{T}_m = \{f_{m,n}, g_{m,n}\}$ be a sequence in $\text{Tow}(\kappa)$. By selecting a subsequence we may assume there are an infinite number of maps $f_{m,n}$ that for all $n \in S$ to some generalized quadratic-like maps $f_n$. Let $S_C \subset S$ be the levels with $f_n|_{U_{n,0}} \in \mathcal{H}(1/4)$. From Lemma 2.19 we can choose a subsequence so that the transit maps on each level $n \in S_C$ converge. From Lemma 4.7 and Lemma 3.1, there exists a function $C(\kappa)$ such that for all $f_{m,n}$,

$$\text{geo}(f_{m,n}) \geq C(\kappa) > 0.$$  

From Lemma 2.8 we can select a subsequence $\mathcal{T}_{m_k}$ so that $f_{m_k,n}$ converges on all levels $n \in S$ to some generalized quadratic-like maps $f_n$. Let $S_C \subset S$ be the levels with $f_n|_{U_{n,0}} \in \mathcal{H}(1/4)$. From Lemma 3.1, we can choose a subsequence so that the transit maps on each level $n \in S_C$ converge. From Lemma 4.7 and Lemma 6.4, the limiting collection of maps will form a tower. The other properties of a tower are clear.

Lemma 6.4. Let $\mathcal{T} \in \text{Tow}(\kappa)$. Then $\text{diam } K(f_n) \rightarrow 0$ as $n \rightarrow \infty$. If $\mathcal{T}$ is a bi-infinite tower then $\text{diam } K(f_n) \rightarrow \infty$ as $n \rightarrow -\infty$.

Proof. Let us prove the first statement. We can assume there are an infinite number of levels $n \rightarrow \infty$ where $f_n$ is not immediately renormalizable, for otherwise $\mathcal{T}$ is eventually a McMullen tower with period-doubling combinatorics and the result follows. Choose a subsequence $f_{n_k}$, $n_k \rightarrow \infty$, of generalized quadratic-like with at least one off-critical piece.

Suppose by contradiction that $\text{diam } K(f_{n_k}) \geq \epsilon > 0$. Let $\cup_i U_{k,i} = \text{Dom}(f_{n_k})$ and $K_{k,j} = K(f_{n_k}) \cap U_{k,j}$. We may assume $K_{k+1,j} \subset K_{k,0}$ by selecting levels of first return.

Then since $\text{geo}(f_{n_k}) \geq C(\kappa) > 0$ and $\text{mod}(K_{k,j}, U_{k,j}) \geq 1/\kappa$ it follows that $U_{k,j}$ contains a definite neighborhood of $K_{k,j}$. Hence there is eventually some $j_1, j_2 \neq 0$ and $k_2 > k_1$ with $K_{k_2,j_2} \cap U_{k_1,j_1} \neq \emptyset$. But this is a contradiction since $K_{k_2,j_2} \subset K_{k_1,0}$ and $K_{k_1,0} \cap U_{k_1,j_1} = \emptyset$.

The second statement is analogous.

Proposition 6.5 (McM2, Corollary 5.12). The postcritical set $P(\mathcal{T})$ varies continuously with $\mathcal{T} \in \text{Tow}(\kappa)$.

Proof. Let $\mathcal{T}_m$ be a sequence of towers in $\text{Tow}(\kappa)$ converging to a tower $\mathcal{T}$. Assume $\mathcal{T}$ is a forward tower. If $z \in \text{orb}(\mathcal{T}, 0)$ then $d(z, P(\mathcal{T}_m)) \rightarrow 0$ as $m \rightarrow \infty$ since $\mathcal{F}(\mathcal{T})$ is contained in any geometric limit of $\mathcal{T}_m$. Hence $P(\mathcal{T}) \subset \liminf_m P(\mathcal{T}_m)$. We must show $\limsup_m P(\mathcal{T}_m) \subset P(\mathcal{T})$.

For $n \in S_Q$ let $K_n(0) = K(f_n)$ and let $K_n(i)$ enumerate the orbit of $K(f_n)$ by $\mathcal{T}$. That is,

$$\cup_i K_n(i) = \{h(z) : z \in K(f_n), h \in \mathcal{F}(\mathcal{T})\}.$$
Let \( \delta_n = \sup_i \text{diam} \, K_n(i) \). The arguments proving \( \text{diam} \, K_n(0) \to 0 \) can be adapted to prove \( \delta_n \to 0 \). Let \( \epsilon > 0 \) and let \( N \) be large enough so that \( \delta_N < \epsilon \). Let
\[
\bigcup_i K_{m,n}(i) = \{ h(z) : z \in K(f_{m,n}), h \in \mathcal{F}(T_m) \}.
\]
Since \( T_m \to T \) it follows that for \( m > N \) large enough \( \bigcup_i K_{m,n}(i) \) is contained in an \( \epsilon \)-neighborhood of \( \bigcup_i K_n(i) \). Hence \( P(T_m) \) is contained in a \( 2\epsilon \)-neighborhood of \( P(T) \).

Now suppose \( T \) is a bi-infinite tower. From the continuity of \( P(T) \) for forward towers and the unbranched property for \( \text{Tow}_0(\kappa) \) it follows that \( P(T|_{S_n}) = V_n \cap P(T) \) where \( S_n \subset S \) is any index set of a forward tower. Since \( V_n \) contains an \( \epsilon(\kappa) \)-scaled neighborhood of \( K(f_n) \) and \( \text{diam} \, K(f_n) \to \infty \) as \( n \to -\infty \), it follows that \( P(T) = \{ \infty \} \cup_{S' \subset S} P(T|_{S'}) \). \hfill \Box

### 6.1. Expansion of the hyperbolic metric.

One of the central ideas in McMullen’s arguments is that maps in a tower expand the hyperbolic metric on the complement of the post-critical set. In this section we prove similar propositions.

**Lemma 6.6.** There are continuous increasing functions \( C_1(s) \) and \( C_2(s) \) such that if \( f : X \to Y \) is an inclusion between two hyperbolic Riemann surfaces and \( x \in X \) then, letting \( s = d(x, Y \setminus X) \),
\[
0 < C_1(s) \leq \| Df(x) \| \leq C_2(s) < 1.
\]
Moreover, \( C_2(s) \to 0 \) as \( s \to 0 \).

**Proof.** The inequality \( \| Df(x) \| \leq C_2(s) < 1 \) and the properties of \( C_2(s) \) are found in \cite{McM2}. Lift \( f \) to the universal cover \( \pi : \mathbb{D} \to Y \) and normalize so that \( x = f(x) = 0 \). The inclusion \( B_s = \{ z : d_{\mathbb{D}}(0, z) < s \} \to \mathbb{D} \) factors through \( f \) and so \( \| Df(0) \| \geq 1/r(s) \) where \( r(s) \) is the radius of \( B_s \) measured in the euclidean metric. \hfill \Box

The following Proposition states when maps in a forward tower \( T \in \text{Tow}(\kappa) \) expand the hyperbolic metric on \( \text{Range}(T) \setminus P(T) \) and gives an estimate on the amount of expansion and the variation of expansion.

Recall if the base map of \( T \) is \( f_m : U_m \to V_m \) then \( \text{Range}(T) = V_m \) and \( \text{Dom}(T) = U_m \). We will use the notation \( \rho_m, \| \cdot \|_m, d_m(\cdot, \cdot) \) and \( \ell_m(\cdot) \) to denote the hyperbolic metric, norm, distance and length on \( \text{Range}(T) \setminus P(T) \).

**Proposition 6.7.** Let \( T \in \text{Tow}(\kappa) \) be a forward tower with base map \( f_m : U_m \to V_m \). Suppose that \( h \in \mathcal{F}(T) \) and let \( Q_h = h^{-1}(P(T)) \). Then
\[
\| Dh(z) \|_m > 1
\]
for any \( z \in (\text{Dom}(h) \setminus Q_h) \). Moreover, if \( Q_h \cap P(T) \neq \emptyset \) then
\[
C^{-1}_{2}(s_2) \leq \| Dh(z) \|_m \leq C^{-1}_{1}(s_1)
\]
where \( s_1 = d_m(z, Q_h \cup \partial\text{Dom}(h)) \) and \( s_2 = d_m(z, Q_h) \). Finally, if \( \gamma \) is a path in \( \text{Dom}(h) \setminus Q_h \) with endpoints \( z_1 \) and \( z_2 \), then
\[
\| Dh(z_2) \|_m^{1/\alpha} \leq \| Dh(z_1) \|_m \leq \| Dh(z_2) \|_m^\alpha
\]
where \( \alpha = \exp(M\ell_m(h(\gamma))) \) for a universal \( M > 0 \).

**Proof.** We apply McMullen’s argument to the approximations of \( h \). Let \( T_j \in \text{Tow}_0(\kappa) \) converge to \( T \). We can assume \( S_j = S \) and \( m = 0 \). Let \( \rho_0 \) be the hyperbolic metric on \( V_0 \setminus P(T) \) and let \( \rho_{j,0} \) be the hyperbolic metric on \( V_{j,0} \setminus P(f_{j,0}) \).
Since
\[ f_0 : (U_0 \setminus f_0^{-1}(P(T))) \rightarrow (V_0 \setminus P(T)) \]
is a covering map and the inclusion
\[ \iota : (U_0 \setminus f_0^{-1}(P(T))) \hookrightarrow (V_0 \setminus P(T)) \]
is a contraction by the Schwarz Lemma, we see \( f_0 \) expands \( \rho_0 \). That is, \( \|Df_0(z)\|_0 > 1 \) for \( z \in (U_0 \setminus f_0^{-1}(P(T))) \). Similarly \( f_{j,0} \) expands \( \rho_{j,0} \).

Suppose \( h \in F(T) \). Let \( z \in (\text{Dom}(h) \setminus h^{-1}(P(T))) \). Choose compact sets \( K_1 \subset (\text{Dom}(h) \setminus P(T)) \) and \( K_2 \subset (\text{Range}(h) \setminus P(T)) \) which contain neighborhoods of \( z \) and \( h(z) \), respectively. Since the domains \( \text{cl}(V_{j,0}) \) converge in the Hausdorff topology to \( \text{cl}(V_0) \) and the post-critical sets \( P(f_{j,0}) \) converge to \( P(T) \), the hyperbolic metrics \( \rho_{j,0} \) converge uniformly on \( K_1 \) and \( K_2 \) to \( \rho_0 \). For large enough \( j \) we have \( \text{Range}(h) \subset V_{j,0} \) since \( \text{Range}(h) \subset V_0 \). Hence there are iterates \( t_j \) such that
\[ f_{j,0}^{t_j} \rightarrow h \]
uniformly on \( K_1 \) in the \( C^1 \) topology as \( j \rightarrow \infty \). Thus maps arbitrarily close to \( h \) expand metrics arbitrarily close to \( \rho_0 \). Hence \( h \) is non-contracting: \( \| Dh(z) \|_0 \geq 1 \). To prove \( h \) is expanding, it suffices to assume \( h \) is a local lift of a transit map. For this we use induction on \( n \in S_C \). First the base case. Let \( n = \min S_C \) and let \( h' \) be another local lift of \( g_n \) such that
\[ h = f_0 \circ h'. \]
Since \( f_0 \) is expanding and \( \tilde{h} \) is non-contracting it follows \( h \) is expanding and the base case holds. Now suppose by induction that local lifts of \( g_{n_1}, \ldots, g_{n_k} \) expand \( \rho_0 \) for the first \( k \) levels in \( S_C \). Let \( h \) be a local lift of \( g_{n_{k+1}} \) where \( n_{k+1} \) is the next level in \( S_C \) after \( n_k \). There is a restriction \( f \) of \( f_{n_{k+1}} \) so that \( f \in F(T) \) and we can assume the attracting and repelling petals \( D_s \) were chosen to lie in \( \text{Dom}(f) \). But then like before there is another local lift \( h' \) so that \( h = f \circ h' \) and we again see \( h \) must be expanding.

Now we estimate how much \( h \) expands \( \rho_0 \). Choose \( z \in (\text{Dom}(h) \setminus Q_h) \) and let \( K_1 \) and \( K_2 \) be closed neighborhoods of \( z \) and \( h(z) \) as above. Then just as above for large \( j \), we can find iterates \( t_j \) such that \( f_{j,0}^{t_j} \rightarrow h \) uniformly on \( K_1 \) as \( j \rightarrow \infty \). Let \( V_j^{-n} = f_{j,0}^{-n}(V_{j,0}) \) and \( P_j^{-n} = f_{j,0}^{-n}(P(f_{j,0})) \). Since
\[ f_{j,0}^{t_j} : V_j^{-t_j} \setminus P_j^{-t_j} \rightarrow V_{j,0} \setminus P(f_{j,0}) \]
is a local isometry we can apply Lemma 6.6 to the inclusion
\[ \iota : V_j^{-t_j} \setminus P_j^{-t_j} \hookrightarrow V_{j,0} \setminus P(f_{j,0}) \]
to get the inequalities
\[ C_2^{-1}(s) \leq \| Df_{j,0}^{t_j}(z) \|_{\rho_{j,0}} \leq C_1^{-1}(s) \]
where
\[ s = d_{\rho_{j,0}}(z, P_j^{-t_j} \cup \partial V_j^{-t_j}). \]
Since \( C_1 \) and \( C_2 \) are increasing,
\[ C_2^{-1}(s') \leq \| Df_{j,0}^{t_j}(z) \|_{\rho_{j,0}} \leq C_1^{-1}(s') \]
where
\[ s_1' = d_{\rho_{j,0}}(z, P_j^{-t_j} \cup K_1) \] and \( s_2' = d_{\rho_{j,0}}(z, P_j^{-t_j} \cap K_1). \]
But \( f_{j,0}^t \to h \) uniformly on \( K_1, (P_j^{t_j} \cap K_1) \to (Q_h \cap K_1) \) and \( \rho_{j,0} \to \rho_0 \) uniformly on \( K_1 \) and \( K_2 \) as \( j \to \infty \). Thus the second statement of the Proposition follows if we let \( K_1 \) range over larger and larger compact sets in \( \text{Dom}(h) \setminus P(\mathcal{T}) \).

To conclude let us prove the last statement about the variation of expansion. From [McM1, Cor 2.27] the variation in \( \|Df_{j,0}^t(z)\|_0 \) is controlled by the distance between \( z_1 \) and \( z_2 \) measured in the hyperbolic metric on \( V_j^{t_j} \setminus P_j^{t_j} \). Since \( f_{j,0}^t \) is a covering map, this distance is bounded above by the length of \( f_{j,0}^t(\gamma) \) measure on \( V_{j,0} \setminus P(f_{j,0}) \). As \( j \to \infty \) this length converges to \( \ell_0(h(\gamma)) \). The statement follows.

The following corollary can be used to control the expansion of the hyperbolic metric on one level with bounds from a deeper level.

**Corollary 6.8.** Let \( \mathcal{T} \in \text{Tow}(\kappa) \) be a forward tower with base map \( f_m : U_m \to V_m \). Suppose \( n \in S_Q \) is a level such that \( V_n \subset V_m \) and let \( \mathcal{T}' \) be the tower \( \mathcal{T} \) restricted to the levels \( n' \geq n \). Let \( h \in \mathcal{F}(\mathcal{T}') \) and let \( Q_h = h^{-1}(P(\mathcal{T}')) \). Then if \( (Q_h \setminus P(\mathcal{T}')) \neq \emptyset \) and \( z \in \text{Dom}(h) \setminus Q_h \),

\[
C_2^{-1}(s_2) \leq \|Dh(z)\|_m
\]

where \( s_2 = d_n(z, Q_h) \).

**Proof.** We may assume \( m = 0 \). Since \( V_n \subset V_0 \) and \( P(\mathcal{T}') = P(\mathcal{T}) \cap V_n \) we see

\[
(V_n \setminus P(\mathcal{T}')) \subset (V_0 \setminus P(\mathcal{T}))
\]

and so

\[
d_0(z, Q_h) \leq d_n(z, Q_h).
\]

Since the function \( C_2 \) in Proposition 6.7 is increasing,

\[
C_2^{-1}(d_n(z, Q_h)) \leq C_2^{-1}(d_0(z, Q_h)).
\]

Finally, since \( \text{Range}(h) \subset V_n \) and \( V_n \cap P(\mathcal{T}) = P(\mathcal{T}') \),

\[
h^{-1}(P(\mathcal{T}')) = h^{-1}(P(\mathcal{T})).
\]

Since \( h \in \mathcal{F}(\mathcal{T}) \) it follows from Proposition 6.7 that

\[
C_2^{-1}(d_0(z, Q_h)) \leq \|Dh(z)\|_m.
\]

\[\square\]

In order to apply this corollary we need to get a bound on \( s_2 = d_n(z, Q_h) \). This is done by compactness:

**Lemma 6.9.** Let \( \mathcal{T} \in \text{Tow}(\kappa) \), \( n \in S_Q \) and \( z \in f_n^{-1}(V_n \setminus U_n) \). Then \( d_n(z, Q_{f_n}) \leq C(\kappa) \).

**Proof.** By shifting we may assume \( n = 0 \). Since \( U_0 \) and \( V_0 \) are \( \kappa \)-quasidisks, the set \( V_0' = \text{cl}(f_0^{-1}(V_0 \setminus U_0)) \) varies continuously with \( \mathcal{T} \in \text{Tow}(\kappa) \). Since \( P(\mathcal{T}) \) varies continuously the hyperbolic metric \( \rho_0 \) and the set \( Q_{f_0} \) vary continuously. Therefore the function \( F \) on \( \text{Tow}(\kappa) \) given by

\[
F(\mathcal{T}) = \sup_{z \in V_0'} d_0(z, Q_{f_0})
\]

is continuous. Since \( \text{Tow}(\kappa) \) is compact by Lemma 6.3, there is a \( C(\kappa) \) such that \( F(\mathcal{T}) \leq C(\kappa) \).

\[\square\]
6.2. Equivalent definitions of the Julia set. There are several equivalent definitions of the Julia set of a rational map. In this section we present the analogous result for forward towers.

Fix a forward tower $T \in \text{Tow}(\kappa)$. The full orbit of a point under $T$, much like the full orbit of a point very near the origin in the Feigenbaum map, can be dissected to reveal much more structure. For forward towers this can be done by iterating deeper maps when possible.

By shifting we may assume $S = \mathbb{N}_0$. By restricting each $f_n \in T$ construct a tower $T' = \{f'_n, g'_n\}$ such that

1. $[f_n] = [f'_n]$ and $g_n = g'_n$
2. $V'_{n+1} \subset U'_{n,0}$ for each $n \in S$ except $V'_{n+1} = V'_n$ for each $n \in S$ such that $f_{n+1}$ is a first through map
3. $U_0 = U'_0$.

Note that $T'$ may no longer be a tower in $\text{Tow}(\kappa)$ but that $\mathcal{F}(T') = \mathcal{F}(T)$. For any non-zero $z \in U_0$ define the depth of $z$ to be

$$\text{depth}(z) = \max\{n \in S : z \in U'_{n,0}\}.$$ 

For a point $z \in U_0$ we say a (possibly finite) sequence $(z_0, z_1, z_2, \ldots)$ is a sub-orbit of $z$ (in $T'$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i \in V_0 \setminus U_0$ then $z_{i+1}$ is not defined
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in \text{Dom}(\tilde{g}_n)$ then $z_{i+1} = \tilde{g}_n(z_i)$ for some local lift $\tilde{g}_n \in T$
- otherwise $z_{i+1} = f'_\text{depth}(z_i)(z_i)$

Note any sub-orbit of $z$ is a subset of $\text{orb}(z)$ and $\text{orb}(z)$ escapes iff there exists a sub-orbit $z_0, \ldots, z_N$ such that $z_N \in V_0 \setminus U_0$.

A point $z \in U_0$ is called periodic (in $T$) if there exists $h \in \mathcal{F}(T)$ such that $h(z) = z$. Equivalently, $z \neq 0$ is periodic iff there is an $x \in \text{orb}(z)$ such that $z \in \text{orb}(x)$ and a sub-orbit $x_0, x_1 = h_1(x), \ldots, x_N = h_N(x)$ of $x$ such that $x_0 = x_N$ and $x_0 \neq x_i$ for $0 < i < N$. The multiplier, $\lambda$, of the periodic orbit through $z$ is defined to be $Dh_N(x)$. The multiplier does not depend on the sub-orbit. A periodic orbit is called superattracting, attracting, repelling, neutral if $\lambda$ satisfies $\lambda = 0$, $|\lambda| < 1$, $|\lambda| > 1$, $|\lambda| = 1$, respectively.

**Lemma 6.10.** Let $T \in \text{Tow}(\kappa)$. The only non-repelling periodic orbits in $T$ are the orbits through the parabolic points of $f_n$ for $n \in S_C$.

**Proof.** Let $z_0, \ldots, z_N$ be the periodic orbit. Since the only non-repelling periodic orbits in $P(T)$ are the orbits through the parabolic points, we can assume the orbit is disjoint from $P(T)$. By Proposition 5.7,

$$\|Dh_N(z)\|_0 > 1$$

But then

$$|\lambda| = |Dh_N(z)| > 1$$

in the euclidean metric as well. 

For a given level $n \in S_C$ let $B_n = K(f_n|_{U_{n,0}})$ be the central basin of level $n$. A connected compact set $K \subset U_0$ is iterable if $K \cap \partial B_n = \emptyset$ for all central basins $B_n$. Mimicking the definition of sub-orbits of points, we say a (possibly finite) sequence of compact sets $(K_0, K_1, K_2, \ldots)$ is a sub-orbit of $K$ (in $T'$) if the following conditions are satisfied:
Proof. By shifting we may assume $K_0 = K$. Let $i$ be iterable except possibly the last one, if it exists. If $i_i \subset \text{Dom}(g_n)$ then $K_{i+1} = \tilde{g}_n(K_i)$ for some local lift $\tilde{g}_n \in \mathcal{T}$. Otherwise $K_{i+1} = f^{\prime}_i(K_i)$ where $d = \text{min}_{z \in K_i} \text{depth}(z)$.

Now that we have said what it means to iterate an iterable compact set, we can prove the following

**Proposition 6.11.** Suppose $\mathcal{T} \in \text{Tow}(\kappa)$ and let $y \in J(\mathcal{T})$. The following are two equivalent definitions of the Julia set:

1. $J(\mathcal{T}) = \text{cl}\{z \in \text{Dom}(\mathcal{T}) : z \text{ is a repelling periodic point}\}$
2. $J(\mathcal{T}) = \text{cl}\{z \in \text{Dom}(\mathcal{T}) : z \text{ is a pre-image of } y\}$

**Proof.** By shifting we may assume $S = \mathbb{N}_0$. We may also assume $S_C \neq \emptyset$. Let $z \in \partial K(\mathcal{T})$ and let $W$ be a connected neighborhood of $z$. We can assume $W \subset \text{int}(K(f_0))$. Let $K = \text{cl}(W)$. We can form the suborbit $K_i = h_i(K)$ from $K$ until the first moment when $K_i$ is not iterable. Such a moment must exist since the orbit of $z \in K$ never escapes but the orbit of some other point in $K$ does escape.

Case 1: Suppose $\text{int}(K_i) \cap \partial B_n \neq \emptyset$ for some $n \in S$. Then by arguing as in Lemma 4 there is an open set $W' \subset K_i$ and composition $h \in \mathcal{F}(\mathcal{T})$ defined on $W'$ such that $h(W') \cap J(f_0) \neq \emptyset$. There is then an open set $W'' \subset h(W')$ and an $N \geq 0$ such that $K(f_0) \subset f^N(W'')$. Since $W \subset K(f_0)$ there exists a point $z_0 \in W$ such that

$$f^N_0 \circ h \circ h_i(z_0) = z_0.$$  

By Lemma 6.10, if we chose $W$ to be small enough, $z_0$ must be repelling.

Case 2: If $K_i$ is not iterable because $K_i \cap (V_0 \setminus U_0) \neq \emptyset$, then by perhaps choosing a smaller neighborhood $W$ and iterating $f_0$ more, we can assume that the moment when $K_i$ is not iterable is because $\text{int}(K_i) \cap \partial B_n \neq \emptyset$ for $n = \text{min} S_C$ and we can argue as in case 1.

Case 3: Suppose $\text{int}(K_i) \cap \partial B_n = \emptyset$ for some $n \in S_C$ but that $\partial K_i \cap \partial B_n \neq \emptyset$. Then by choosing a slightly smaller neighborhood $W$ we can assume $K_i$ is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time $K_i$ is not iterable $K_i$ falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood $W'$ of $z$ we see that the orbit of $W'$ is defined for all iterates. But this is impossible since then $W'$ never escapes, contradicting the fact that $z \in \partial K(\mathcal{T})$. Thus after a finite number of restrictions, the non-iterable set $K_i$ must fall into the cases considered above. Thus

$$J(\mathcal{T}) = \text{cl}\{z \in U_0 : z \text{ is a repelling periodic point}\}.$$  

Let $z \in K(\mathcal{T})$ and let $W$ be a connected neighborhood of $z$. Suppose $W$ contains a repelling periodic point $z_0$. Again let $K = \text{cl}(W)$ and start forming the sub-orbit $K_i = h_i(K)$ through $K$. Claim there is a moment when $K_i$ is not iterable. For otherwise the maps $h_i$ form a normal family on $W$ and that contradicts the fact that $W$ contains a repelling periodic point. Thus there is a non-iterable iterate $K_i$.

Case 1: Just as case 1 above, there is a open set $W' \subset K_i$ and composition $h \in \mathcal{F}(\mathcal{T})$ defined on $W'$ such that $h(W') \cap J(f_0) \neq \emptyset$. But then there is a point in $h(W')$ that escapes and thus there is a point in $W$ that escapes as well.

Case 2: If $K_i$ is not iterable because $K_i \cap (V_0 \setminus U_0) \neq \emptyset$, then we have found a point in $W$ that escapes.
Case 3: Suppose \( \text{int}(K_i) \cap \partial B_n = \emptyset \) but that \( \partial K_i \cap \partial B_n \neq \emptyset \). Then by choosing a slightly smaller neighborhood \( W \) that still contains the repelling periodic point \( z_0 \), we can assume \( K_i \) is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time \( K_i \) is not iterable \( K_i \) falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood \( W' \) containing \( z_0 \) we see that the orbit of \( W' \) is defined for all iterates. But this is impossible since the iterates of \( W' \) cannot form a normal family. Thus after a finite number of restrictions, the non-iterable set \( K_i \) must fall into the cases considered above. Thus

\[
J(T) \supset \text{cl} \{ z \in U_0 : z \text{ is a repelling periodic point} \}.
\]

To prove the second statement, notice that the argument proving the first also proves that if \( y \in J(T) \) then any point in \( U_0 \) has a pre-image arbitrarily close to \( y \). That is,

\[
J(T) \subset \text{cl} \{ z \in U_0 : \text{there is an } h \text{ such that } h(z) = y \}.
\]

The reverse inclusion follows from the fact that \( J(T) \) is closed and backward invariant and that \( y \in J(T) \).

6.3. The interior of the filled Julia set. An infinitely renormalizable quadratic-like map \( f \in RQuad \) has a filled Julia set with empty interior. The same statement holds for forward towers:

**Proposition 6.12.** For any \( T \in \text{Tow}(\kappa) \),

\[
\text{int}(K(T)) = \emptyset.
\]

As a corollary we have

**Proposition 6.13.** The Julia set \( J(T) \) varies continuously with \( T \in \text{Tow}(\kappa) \).

**Proof.**

The proof of Proposition 6.12 is broken into propositions Proposition 6.16 and Proposition 6.17 and will occupy the rest of this section.

By shifting we may assume \( S = \mathbb{N}_0 \). Suppose by contradiction that \( \mathcal{O} = \text{Comp}(\text{int}(K(T))) \) is non-empty. Let \( U \in \mathcal{O} \) and \( z \in U \). Let \( K \subset U \) be a compact and connected neighborhood of \( z \). Recall \( B_n \) are the central basins of \( T \). Since \( \partial B_n \subset J(T) \) for all \( n \in S_C \) it follows that \( K \) is iterable. Since \( J(T) \) is backward invariant we see that all the iterates of \( K \) are iterable as well. Thus the orbit of \( K \) is well defined and contains the orbit of \( z \) and so, letting \( K \) range over larger and larger compact subset of \( U \), we can define the orbit of \( U \), \( \text{orb}(U) \), to be components containing the orbit of \( K \).

A component \( U \in \mathcal{O} \) is called **periodic** if \( U' \subset \text{orb}(U) \) implies \( U \subset \text{orb}(U') \). A component \( U \in \mathcal{O} \) is called **pre-periodic** if \( U \) is not itself periodic but there is a periodic component in \( \text{orb}(U) \).

The classification of periodic components is based on the following

**Proposition 6.14.** Let \( h : U \to U \) be an analytic transform of a hyperbolic Riemann surface \( U \). The we have one of the following possibilities:

1. \( h \) has an attracting or superattracting fixed point in \( U \) to which all orbits converge
Proposition 6.14. Let \( U \) be a hyperbolic domain on the sphere, and \( h : U \to U \) an analytic transform continuous up to the boundary. Suppose that the set of fixed points of \( h \) on \( \partial U \) is totally disconnected. Then in case 2) of Proposition 6.14 there is a fixed point \( \alpha \in \partial U \) such that \( h_m(z) \to \alpha \) for every \( z \in U \).

We shall use these two propositions to prove

Proposition 6.15. Let \( U \) be a hyperbolic domain on the sphere, and \( h : U \to U \) a conformal homeomorphism of finite order. Suppose all iterates tend to infinity. Then \( \partial U \) is repelling. Thus there cannot be a superattracting or attracting orbits. Suppose all iterates tend to \( \partial U \). Now the set of points on \( \partial U \) fixed by \( h \) is isolated, since otherwise \( h \) would be the identity on an open set and that would contradict Proposition 6.7. Applying Proposition 6.15 again contradicts Lemma 6.10.

Proof. Suppose \( U \in \mathcal{O} \) is periodic or pre-periodic.

Since \( \text{cl}(U) \) is disjoint from \( P(T) \), Lemma 6.10 implies any periodic point in \( \text{cl}(U) \) must be repelling. Thus there cannot be a superattracting or attracting orbits. Suppose all iterates tend to \( \partial U \). Now the set of points on \( \partial U \) fixed by \( h \) is isolated, since otherwise \( h \) would be the identity on an open set and that would contradict Proposition 6.7. Applying Proposition 6.15 again contradicts Lemma 6.10.

The other possibilities in Proposition 6.14 are ruled out because \( h \) expands the hyperbolic metric on \( U_0 \setminus P(T) \) and any map conjugate to a rotation will have high iterates arbitrarily close to the identity.

Now suppose there is a component \( U' \) from \( \text{orb}(U) \) such that \( \text{cl}(U') \) is not iterable. To simplify the exposition we will assume \( T \) is a real-symmetric tower. However, this is not essential. Since \( U \) is periodic we may assume \( U = U' \). Since \( U \subset K(f_0) \) there must be an \( n \in S_C \) such that \( \text{cl}(U) \cap \partial(B_n) \neq \emptyset \). Since \( U \cap J(T) = \emptyset \) it follows that \( U \subset B_n \) and if \( n' \in S_C \) is the next parabolic level above \( n \) then \( \text{cl}(U) \cap B_{n'} = \emptyset \).

Let \( K = \text{cl}(U), f = f_n|_{U_0} \) and \( \xi = \beta(f) \). Since \( B_n \) and \( \partial B_n \) are invariant by \( f \), it follows \( K_k = f^k(K) \subset \text{cl}(B_n) \setminus B_{n'} \) and \( \partial K_k \cap \partial B_n \neq \emptyset \) for all \( k \geq 0 \). Let

\[
\mathcal{B} = \text{Comp}(B_n \setminus \left( \bigcup_{k \geq 0} f^{-k}(\mathbb{R}) \right))
\]

be the collection of components of the partition pictured in Fig. 11.

First we claim that \( U \cap \mathbb{R} = \emptyset \). Let \( n' \in S_C \) be the largest quadratic-like level before \( n \). Let \( B_{n''} \) be the central basin of the first level \( n'' \in S_C \) after \( n' \). Then the \( f_{n''} \) pre-images of \( B_{n''} \) cover a dense subset of \( \mathbb{R} \cap K(T_{n'}) \) where \( T_{n'} \subset T \) is the forward tower with levels \( m \geq n' \). It follows that the pre-images by \( \mathcal{F}(T) \) cover a dense subset of \( \mathbb{R} \cap B_n \) and accumulate at \( \xi \). Since \( \partial B_{n''} \subset J(T) \) the claim is established. Since \( U \) is periodic under \( T \), we can assume \( U \subset A \) where \( A \in \mathcal{B} \) satisfies \( \xi \in \partial A \). Without loss of generality assume \( A \subset \mathbb{H} \).

Let \( \gamma = \partial A \). Let

\[
\gamma_1 = \bigcup_{g_n} \widetilde{g}_{n}^{-1}(\gamma).
\]
Since $g_n$ is a real translation, $\gamma_1 \subset \mathbb{H}$. Let

$$\gamma_2 = \bigcup_{k \geq 0} (f^{-1})^k(\gamma_1)$$

where the branch of $f^{-1}$ is chosen so that $f^{-1}(\mathbb{H} \cap B_n) \subset \mathbb{H} \cap B_n$ (see Fig. 12). It follows from Lemma 4.4 that $U$ is contained in the domain $A_1$ bounded by $\gamma_2$. Continue this process. That is, the pre-image of $\gamma_2$ by $\tilde{g}_n$ is contained in $A_1$ and pulling back by $f^{-1}$ we see that $U$ is contained in a domain $A_2 \subset A_1$. By Lemma 2.17,

$$\bigcap_{m \geq 1} A_m = \emptyset$$

and so a non-iterable periodic component $U$ cannot exist.

A component $U \in \mathcal{O}$ that is neither periodic nor pre-periodic is called wandering.

**Proposition 6.17.** No $U \in \mathcal{O}$ is wandering.

**Proof.** Suppose $U \in \mathcal{O}$ is wandering. Let $K \subset U$ be compact and connected. Then $K$ is iterable and all iterates of $K$ are iterable. Fix an $z \in \text{int}(K)$. Since each map $h$ from the
orbit of $K$ is defined on a neighborhood of $K$ and since \( Q_h = h^{-1}(P(T)) \subset J(T) \), it follows from Proposition 6.7 that
\[
\sup_h \| Dh(z) \|_0 < \infty. \tag{6.1}
\]

Suppose there is an \( \epsilon > 0 \) such that there are an infinite number of iterates \( h_n \) satisfying
\[
d(h_n(z), P(T)) > \epsilon
\]
where the distance is just the euclidean distance. Order the \( h_n \) to match the ordering on the orbit. That is, if \( n < m \) then \( h_m(z) \in \text{orb}(h_n(z)) \). Since each \( h_n(z) \) lies in a compact subset of the hyperbolic surface \( V_0 \setminus P(T) \),
\[
d_0(h_n(z), Q_{f_0}) \leq C \epsilon,
\]
and so from Proposition 6.7
\[
\| Df_0(h_n(z)) \|_0 \geq C > 1. \tag{6.2}
\]

But then
\[
\| Dh_{n+1}(z) \|_0 \geq \| D(f_0 \circ h_n)(z) \|_0 = \| Df_0(h_n(z)) \|_0 \cdot \| Dh_n(z) \|_0 \geq C \| Dh_n(z) \|_0
\]
which as \( n \to \infty \) contradicts equation (6.1).

So we can assume
\[
\lim sup_h d(h(z), P(T)) = 0.
\]

Let \( T_n = T|_{s_n} \) be the restriction of \( T \) to levels \( m \geq n \). Let \( \mathcal{K}_n \) be the collection of little filled Julia sets \( \mathcal{K}_n = \text{orb}(T, K(T_n)) \).

From Lemma 2.17 we see \( \text{orb}(z) \) must accumulate on some \( z' \not\in \xi_0 \) where \( \xi_0 \) is the parabolic orbit of \( f_0 \). But then \( z' \) is contained in a little filled Julia set in \( \mathcal{K}_1 \). By iterating forward we can assume \( z' \in K(T_1) \). It follows that there is a \( y_1 \in \text{orb}(z) \) such that \( y_1 \in K(T_1) \). Now again there is an accumulation point \( \text{orb}(y_1) \) disjoint from \( \xi_1 \), the parabolic orbit of \( f_1 \), and, repeating the whole argument inductively, there is a sequence of iterates \( y_n \in K(T_n) \).

Each \( y_n \) has a moment \( x_n \in \text{orb}(z) \) when \( \text{orb}(z) \) enters the collection of little filled Julia sets \( \mathcal{K}_n \). Once \( \text{orb}(z) \) enters \( \mathcal{K}_n \) it never leaves. It can happen that different \( y_n \) have the same moment \( x_n \). However, since
\[
\bigcap_{n \geq 0} K(T_n) = \{0\}
\]
there must be an infinite number of distinct entry moments \( x_n \).

Let \( z_n \in \text{orb}(z) \) satisfy \( f_0(z_n) = x_n \). Thus the relation between the points \( z, x_n, y_n \) and \( z_n \) is given by: \( z_n \in \text{orb}(z), x_n = f_0(z_n) \) is the time \( \text{orb}(z) \) enters \( \mathcal{K}_n \) and \( y_n \in \text{orb}(x_n) \) is the first time \( x_n \) enters \( K(T_n) \). Claim
\[
d_0(z_n, Q_{f_0}) \leq C'.
\]

Let \( K'_n \) be the component of \( f_0^{-1}(\mathcal{K}_n) \setminus \mathcal{K}_n \) containing \( z_n \). The set \( K'_n \) is called a companion filled Julia set of level \( n \). Since \( Q_{f_0} \cap K'_n \neq \emptyset \), it is enough to show
\[
\text{diam}_0(K'_n) \leq C'.
\]

Consider the sets \( U'_n \) and \( V'_n \) containing \( K'_n \) which are pull-backs of \( \text{Dom}(f_n) \) and \( V_n \) by the map sending \( z_n \) to \( y_n \). By the unbranched property this pull-back is univalent. Since
mod(Dom(f_n), V_n) ≥ 1/κ, we have mod(U'_n, V'_n) ≥ 1/κ and so, from [McM1, Theorem 2.4], the diameter D_n of U'_n in the hyperbolic metric on V_n is bounded. But V'_n ⊂ (V_0 \ P(T)). Thus
\[ \text{diam}_0(K'_n) \leq \text{diam}_0(U'_n) \leq D_n \leq C(\kappa) \]
and the claim is established.

But then equations 6.2 and 6.3 hold along the sequence z_n = h_n(z), and we again get a contradiction to 6.1. \[ \square \]

6.4. Line fields and forward towers. A line field is a measurable Beltrami differential with |u(z)| = 1 on a set of positive measure and |u(z)| = 0 otherwise. A line field is invariant under \( T \) iff for every \( h \in T \), \( Dh \) maps the line at \( x \) to the line at \( h(x) \) for almost every \( x \in \text{Dom}(h) \). Using Proposition 6.11 and Proposition 6.12 we can rephrase Proposition 6.1 in terms of invariant line fields.

Before doing so, we need the following

**Lemma 6.18.** [71] Let \( T \in \text{Tow}(\kappa) \). The group \( G \) of homeomorphisms of \( J(T) \) that commute with all maps \( h \in T \) is totally disconnected.

**Proof.** Let \( \phi \in G \) be a map in the connected component of the identity. Suppose \( z_0 \) is a repelling periodic point with \( h(z_0) = z_0 \) for some \( h \in \mathcal{F}(T) \). Since the solutions to \( h(z) = z \) are isolated \( \phi \) must fix \( z_0 \). The lemma follows from density of repelling cycles: Proposition 6.11. \[ \square \]

We now prove the following version of forward tower rigidity:

**Proposition 6.19 (No Line Fields for Forward Towers).** No forward tower \( T \) hybrid equivalent to a tower in \( \text{Tow}(\kappa) \) supports an invariant line field on its filled Julia set.

**Proof.** By Proposition 6.11 it suffices to consider a forward tower \( T \) having a base map of the form \( z^2 + c_0 \) and by shifting we may assume \( S = \mathbb{N}_0 \). Since \( T \) is hybrid equivalent to a tower in \( \text{Tow}(\kappa) \) it follows from Proposition 6.12 that \( K(T) = J(T) \). Suppose by contradiction that \( T \) did admit an invariant line field
\[ \mu = u(z)dz/dz \]
supported on \( J(T) \). For any \( w \in \mathbb{D} \) consider the invariant Beltrami differential
\[ \mu_w = w \cdot u(z)dz/dz \]
on \( \hat{\mathbb{C}} \). Let \( \phi_w \) be a solution to the corresponding Beltrami equation normalized so that the map
\[ f_{w,0} = \phi_w \circ f_0 \circ \phi_w^{-1} \]
is again a rational map of the form \( z^2 + c_w \) for some \( c_w \in \mathbb{C} \). Let \( T_w \) be the tower
\[ T_w = \{ \phi_w \circ h \circ \phi_w^{-1} : h \in T \}. \]

From Proposition 2.3 and the uniqueness of root points, \( c_w = c_0 \) for all \( w \in \mathbb{D} \). Proposition 6.1 implies \( f_{w,n} = f_n \) for all \( n \in S \) and \( g_{w,n} = g_n \) for all \( n \in S_C \) and \( w \in \mathbb{D} \). So \( \phi_w \) is a holomorphic family of quasi-conformal maps with \( \phi_0 = \text{id} \) and \( \phi_w \) mapping \( J(T) \) homeomorphically to itself commuting with the dynamics of \( T \). From Lemma 6.18 \( \phi_w|_{J(T)} = \text{id} \). But then the complex dilatation of \( \phi_w \) is zero at all points of Lebesgue density of \( J(T) \) and so \( \mu \) is not supported on \( J(T) \), a contradiction. \[ \square \]
6.5. **Line fields and bi-infinite towers.** In this section we move from studying properties of forward towers to studying bi-infinite towers. The plan of attack again follows [McM2].

Let $Tow^\infty(\kappa)$ denote the set of bi-infinite towers in $Tow(\kappa)$. Given $T \in Tow^\infty(\kappa)$ define $S_N \subset S_Q$ as follows. Let $S_{N,0} = \{0\}$. Then inductively let $S_{N,n+1} = S_{N,n} \cup \{m_{n+1}\}$ where $m_{n+1} = \max\{m \in S_Q| m < m', U_m \supset V_{m'}\}$ and $m' = \min S_{N,n}$. Define $S_N = \cup_{n=\infty} S_{N,n}$. That is, $S_N$ is the minimal set of nested levels approaching $-\infty$. From Lemma 6.4 we see $S_N$ is unbounded below.

Define the depth of a non-zero point $z \in \mathbb{C}$ by

$$\text{depth}(z) = \max\{m \in S_N : z \in U_m\}.$$ 

For a point $z \in \mathbb{C}$ we say a (possibly finite) sequence $(z_0, z_1, z_2, \ldots)$ is a sub-orbit of $z$ (in $T$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in \text{Dom}(\tilde{g}_n)$ then $z_{i+1} = \tilde{g}_n(z_i)$ for some $\tilde{g}_n \in T$
- otherwise $z_{i+1} = f_{\text{depth}(z_i)}(z_i)$

Let $\rho^{-\infty}$ be the hyperbolic metric on $\mathbb{C} \setminus P(T)$ and as in §6.4 let $\rho_n$ be the hyperbolic metric on $V_n \setminus P(T|S_n)$. From Lemma 6.4 and the unbranched property the metrics $\rho_n$ converge uniformly on compact sets to $\rho^{-\infty}$. Using the expansion from §6.1, we now prove

**Theorem** 6.3. For any $T \in Tow^{-\infty}(\kappa)$

$$\lim_{n \to -\infty} J(T|S_n) = \mathring{\mathbb{C}}$$

in the Hausdorff topology.

**Proof.** Let $T_n = T|S_n$. Let $z \notin \cup_{s \leq 0} J(T_s)$. Without loss of generality we may assume $z \in U_0$. Then $\text{orb}(T_s, z)$ escapes $U_s$ for any $s \in S_N$. Let $z_s = h_s(z)$ be the orbit point just before the first moment of escape on level $s$. That is, $f_s(z_s) \in V_s \setminus U_s$ and if $z' \in \text{orb}(z)$ also satisfies $f_s(z') \in V_s \setminus U_s$ then $z' \in \text{orb}(z_s)$. For a given $s \in S_N$ let $\gamma_s$ be a hyperbolic geodesic in $V_s \setminus P(T_s)$ connecting $z_s$ with $J(T_s)$. From Lemma 6.9, there is a $C$ independent of $s$ such that $\ell_s(\gamma_s) \leq C$. Fix a small $\epsilon > 0$ and let $A$ be an $\epsilon$-scaled neighborhood of $P(T_s)$. Then $h_s$ has an extension $h \in F(T)$ that is a covering map onto $V_s \setminus A$. Let $\gamma_s$ be the connected component of $h^{-1}(\gamma_s')$ containing $z$.

We now argue $\ell_s(\gamma_s)$ shrinks as $s \to -\infty$. The proposition would follow since $\rho_s$ converges to $\rho^{-\infty}$ near $z$ and since Julia sets are backward invariant. Fix an $s \in S_N$ and let $N_s = |\{s, \ldots, 0\} \cap S_N|$ be the minimal number of moments when the orbit of $z$ escapes a nested level. It follows from Lemma 6.9 and Corollary 6.8 that there is a $C > 1$ such that

$$C \leq \|Df_t(z_t)\|_s$$

for any $t \in \{s, \ldots, 0\} \cap S_N$. Hence

$$C^{N_s} \leq \|Dh_s(z)\|_s. \quad (6.4)$$

Hence the derivative at the endpoint $z$ grows exponentially in $N_s$. From Proposition 6.7, there exists a $C > 1$ such that equation (6.4) holds along $\gamma_s$ and hence the length of $\gamma_s$ shrinks as $s \to -\infty$. \qed
A measurable line field $\mu$ on an open set $U$ is called univalent if there is a univalent map $h: U \to \mathbb{C}$ such that $\mu = h^*(d\bar{z}/dz)$. The main statement in this section is the following extension of Proposition 6.19.

**Theorem 6.20** (No Line Fields for Bi-infinite Towers). Let $\mathcal{T} \in \text{Tow}_\infty(\kappa)$. There does not exist a measurable line field $\mu$ in the plane such that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n)$, $n \in S_N$.

**Proof.** Suppose to the contrary that $\mu = u(z)d\bar{z}/dz$ is a measurable invariant line field which is non-zero on a set, $B$, of positive measure. Let $z \in B$ be a point of almost continuity of $u$ and satisfying $|u(z)| = 1$. That is, for each $\epsilon > 0$, the chance of randomly choosing a point $y$ a distance $r$ from $z$ that satisfies $|u(y) - u(z)| > \epsilon$ tends to 0 as $r$ tends to 0:

$$\lim_{r \to 0} \frac{\text{area}\{y \in B(z, r) : |u(y) - u(z)| > \epsilon\}}{\text{area } B(z, r)} = 0$$

where $B(z, r)$ is the euclidean ball of radius $r$ centered at $z$. By Proposition 6.19, we can assume $z \notin K(\mathcal{T}_n)$ for any $n$. Let $z_n$ be an infinite sub-orbit from $z$ and for each $s \in S_N$ let $z_{ns} = h_{ns}(z)$ denote the moments in the sub-orbit when $z_{ns+1}$ first satisfies $z_{ns+1} \in V_s \setminus U_s$.

For a given $s \in S_N$ let $\mathcal{T}_s$ denote $\mathcal{T}$ shifted so that level $s$ is moved to level 0 and let $w_s$ and $u_s$ denote $z_{ns}$ and $u$ shifted by $s$. That is, if $|B(f_s)| = \alpha_s$, then $w_s = \alpha_s^{-1}z_{ns}$ and $u_s(z) = u(\alpha_sz)$. Then since $\text{Tow}^{\infty}(\kappa)$ is compact the sequence $\mathcal{T}_s$ has a subsequence which as $s \to -\infty$ converges to some $\mathcal{T}' \in \text{Tow}^{\infty}(\kappa)$. By choosing a further subsequence we may assume $w_s$ converges to a $w \in \text{cl}(\{(f_0^s)^{-1}(V_0^s \setminus U_0^s)\})$ and, from [McM2], $\mu_s$ converges weak*, and hence pointwise almost everywhere, to a measurable line field $\mu'$ invariant by $\mathcal{T}'$ in the sense that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}'_n)$, $n \in S_N(\mathcal{T}')$.

Let $D$ be a small disk around $w$ in $V_0' \setminus P(\mathcal{T}_0')$. The hyperbolic diameter of $D$ in $V_0' \setminus P(\mathcal{T}_0')$ is close to that of $D_s = \alpha_s^{-1}(D)$ in the metric on $V_s \setminus P(\mathcal{T}_s)$ for $s$ near $-\infty$. Since $D_s$ is disjoint from $P(\mathcal{T}_s)$, there is, by the argument given in Proposition 6.3, a univalent pullback $D_s'$ of $D_s$ by the map $h_{ns}$. By equation 6.4 and the variation of expansion in Proposition 6.7, we see $D_s'$ is a sequence of open sets containing $z$ such that in the euclidean metric diam($D_s'$) → 0 and $B(z,C \text{diam}(D_s')) \subset D_s'$ as $s \to -\infty$ for some constant $C$. Therefore from [McM1], Theorem 5.16 we can choose $\mu'$ to be univalent on $D$.

By Proposition 6.3 there is an $s \in S_N(\mathcal{T}')$ such that $J(\mathcal{T}'_s) \cap D \neq \emptyset$. By invariance, if $Dh(z) \neq 0$ and $\mu'$ is locally univalent around $z$ then $\mu'$ agrees almost everywhere with a locally univalent line field around $h(z)$ for any composition $h \in \mathcal{F}(\mathcal{T}_s')$. From Proposition 6.11 the orbit of $D$ by $\mathcal{T}_s'$ covers all of $V_s'$. So $\mu'$ agrees almost everywhere with a line field that is locally univalent on the set $V_s' \setminus P(\mathcal{T}_s')$. Since $f_s'$ is injective on $P(\mathcal{T}_s')$ every point in $P(\mathcal{T}_s')$ except $f_s'(0)$ has an $f_s'$ pre-image around which $\mu'$ agrees (a.e.) with a locally univalent line field. Hence $\mu'$ agrees (a.e.) with a locally univalent line field around $(f_s')^2(0)$ and 0, which is a contradiction, since then we obtain contradictory behavior of $\mu'$ around $f_s'(0)$. □

As a corollary we obtain

**Theorem 4.4**. If $\mathcal{T}, \mathcal{T}' \in \text{Tow}^{\infty}(\kappa)$ are normalized combinatorially equivalent towers then $[f_n] = [f'_n]$ for all $n \in S$ and $g_n = g'_n$ for $n \in S_S$.

**Proof.** Let $S_N$ be the set of nested levels of $\mathcal{T}$ as constructed above. For each $n \in S_N$, let $\phi_n$ be a hybrid equivalence between $\mathcal{T}_n$ and $\mathcal{T}'_n$ coming from straightening (see Corollary 6.2). The dilatation of $\phi_n$ is bounded above by a constant depending only on $\kappa$ and $\phi_n$ fixes 0 and $\infty$ and maps $\beta(f_0)$ to $\beta(f'_0)$. Thus we can pass to a convergent subsequence $\phi_{n_k} \to \phi$ as
\( n \to -\infty. \) Since \( \phi_n \) restricts to a quasi-conformal equivalence of \( f_s \) and \( f'_s \) for \( s > n, s \in S_N \), on a definite neighborhood of \( K(f_s) \), it follows that \( \phi \) is a quasi-conformal equivalence. Let \( \mu \) be the line field defined by \( \phi \) and \( \mu_n \) the line field defined by \( \phi_n \). Since \( h_n(\mu_n) = \mu_n \) for all \( h \in \mathcal{F}(T_n) \) it follows that \( h_n(\mu) = \mu \) for all \( h \in \mathcal{F}(T_n), n \in S_N \).

From Theorem 5.20, \( \mu = 0 \) and so \( \phi \) is conformal. Since \( \lim_{n \to -\infty} U_n = \hat{\mathbb{C}} \), \( \phi \) is linear and since \( T \) and \( T' \) are normalized \( \phi \) is the identity. \( \square \)

7. **Proof of Theorem 1.1 and Theorem 1.2**

Let \( p > 1 \). Let \( f \) be an \( \infty \)-renormalizable real quadratic polynomial with \( \tilde{p}_r(f) \leq p \). The first step in the proof of Theorem 1.2 is to construct a tower \( T \in \text{Toow}(\kappa) \) from \( f \).

It follows from Lemma 2.7 that there is a \( \kappa > 0 \) depending only on \( p \), and a forward tower \( \mathcal{T} = \{ f_n \} \in \text{Toow}_0(\kappa) \) with the following property. For \( n \in S_0 \) let \( [f'_n] \) be \( \{ f_n \} \) normalized and let \( k(n) = |S_Q \cap \{ 1, \ldots, n \}| \). Then

\[
[f'_n] = \mathcal{R}^{k(n)}(f).
\]

Hence renormalization acts on towers by shifting. Let \( T_n \) denote the tower \( T \) shifted by \( n \) so that \( f_n \) is normalized and has index 0. By compactness there exists a limiting tower \( T' \) and by Theorem 1.4 the germ \([f_0]\) is uniquely specified by the combinatorics of \( T' \): a bi-infinite sequence of \( \sigma \in \Omega_p^{\text{cpl}} \). Hence if \( f \) has essentially period tripling combinatorics the germs \( \mathcal{R}^{k}(f) \) converge to a unique germ \( F \), which proves Theorem 1.1.

To prove Theorem 1.2 suppose \( \bar{\sigma} \in \Sigma_p \) is a bi-infinite sequence of shuffles and ends in \( \Omega_p^{\text{cpl}} \). Let \( \sigma_n = \pi_n(\bar{\sigma}) \) denote the \( n \)-th element of \( \bar{\sigma} \). For each \( \sigma_n \) let \( \sigma_{m,n} \) be a sequence in \( \Omega_\pi \) converging to \( \sigma_n \). Define the sequence \( (\tau) \in \Pi^\infty \Sigma_p \) by

\[ \tau = (\sigma_{0,0}, \sigma_{1,-1}, \sigma_{1,0}, \sigma_{1,1}, \sigma_{2,-2}, \sigma_{2,-1}, \sigma_{2,0}, \sigma_{2,1}, \sigma_{2,2}, \ldots, \sigma_{n,-n}, \ldots, \sigma_{n,n}, \ldots, \ldots) \]

and let \( \tilde{\tau}_n = \omega^{j(n)}(t\tilde{a}u) \) where \( \omega \) is the left-shift operator and \( j(n) = 1 + 3 + 5 + \cdots + (2n - 1) + n \). Then by construction \( \tilde{\tau}_n \to \bar{\sigma} \). Let \( f \) be a real quadratic polynomial with shuffle sequence \( \tilde{\tau} \) and let \( T \) be a tower in \( \text{Toow}(\kappa) \) constructed from \( f \). By compactness of towers let \( T' = \{ f'_n, g'_n \} \) be a limiting tower of \( T_{j(n)} \). Define the function \( h : \Sigma_p \to GQuad(m) \) by

\[ h(\bar{\sigma}) = [f'_0]. \]

From Theorem 1.3 \( h \) is well-defined and is continuous. The other properties of \( h \) are clear.

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