LAMBDA DETERMINANTS

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Abstract.

In this paper we prove a homogenous generalization of the lambda determinant formula of Mills, Robbins and Rumsey. In our formula the parameters depends on two indices. Our result also extends a recent formula of Di Francesco.

1. Introduction

An alternating sign matrix is a square matrix of 0’s 1’s and −1’s such that the sum of each row and column is 1 and the non-zero entries in each row and column alternate in sign. For example:

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Alternating sign matrices arise naturally in Dodgson’s condensation method for calculating \( \lambda \)-determinants \cite{Bre99}.

For each \( k = 0..n \) let us denote by \( x[k] \) the doubly indexed collection of variables \( x[k]_{i,j} \) with indices running from \( i, j = 1..(n - k + 1) \). One should think of these variables as forming a square pyramid with base of dimension \( n + 1 \) by \( n + 1 \). The index \( k \) determines the “height” of the variable in the pyramid.

The variables \( x[0] \) and \( x[1] \) are to be thought of as initial conditions. The remaining \( x[k] \) are defined in terms of the following octahedral recurrence:

\[
x[k+1]_{i,j} = \frac{\mu_{i,n-k-j}x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{i,j}x[k]_{i,j+1}x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}}
\]

The main result of this paper is a closed form expression for \( x[k]_{1,1} \). Our result generalizes the result obtained by Di Francesco \cite{Fra12}, who considered coefficients \( \lambda_{ij} \equiv \lambda_{i-j} \) and \( \mu_{ij} \equiv \mu_{i-j} \).

The outline of this paper is as follows. We begin with some definitions which are necessary in order to write down the closed form expression. In section 3 we introduce left cumulant matrices and a
pair of up / down operators. In section 4 we introduce right cumulant matrices and a second pair of up / down operators which are closely related to the first. Finally in section 5 we prove our main theorem.

2. CLOSED FORM EXPRESSION

For each \( n \) by \( n \) alternating sign matrix \( B \) let \( B \) be the matrix whose \((i, j)\)-th entry is equal to the sum of the entries lying above and to the left of the \((i, j)\)-th entry of \( B \). Similarly, let \( B \) be the matrix whose \((i, j)\)-th entry is equal to the sum of the entries lying above and to the right of the \((i, j)\)-th entry of \( B \). For example:

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
\overline{X} = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 3 \\
1 & 2 & 3 & 4 \\
\end{pmatrix}
\]

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 \\
3 & 2 & 1 & 1 \\
4 & 3 & 2 & 1 \\
\end{pmatrix}
\]

We shall refer to \( B \) as the left cumulant matrix of \( B \) and \( B \) as the right cumulant matrix of \( B \). The original alternating sign matrix may be recovered by the formula:

\[
B_{ij} = B_{ij} + B_{i-1,j} - B_{i,j-1} - B_{i-1,j-1}
\]

\[
= B_{ij} + B_{i-1,j+1} - B_{i,j+1} - B_{i-1,j+1}
\]

If the indices are out of range, then the value of \( B_{ij} \) is taken to be zero.

Lemma 2.1. If \( B' \) is the alternating sign matrix obtained from \( B \) by multiplying on the right by the maximum permutation then \( B' \) is the matrix obtained from \( B \) by multiplying on the right by the maximum permutation.

We shall make use of the notation:

\[
B = (B')'
\]

Lemma 2.2. For all \( i, j \) we have:

\[
B_{i,j} + B_{i,j+1} = i
\]

Proof. The left hand side is equal to the sum of all the entries of the alternating sign matrix \( B \) in the first \( i \) rows. Since the sum of entries in each row of \( B \) is equal to 1, the final result is equal to \( i \) as claimed. \( \square \)

Comparing matrices entrywise, the \( B \) of size \( n \) form a lattice. We remark that this lattice coincides with the completion of the Bruhat order to alternating sign matrices as carried out in Lascoux and Schützenberger [LS96]. One can apply the same operation with \( B \) to form a dual lattice.
Let us define the *lambda weight* of a $k$ by $k$ alternating sign matrix $B$ to be:

$$\lambda^F(B) = \lambda^F - B = \prod_{i,j=1}^k \lambda_{i,j}^{\min(i,j) - B_{i,j}}$$

Similarly, let us define the *mu weight* of an $k$ by $k$ alternating sign matrix $B$ to be:

$$\mu^G(B) = \mu^G - B = \prod_{i,j=1}^k \mu_{i,j}^{\max(i-j+1,0) - B_{i,j+1-k}}$$

The *standard weight* of an alternating sign matrix $B$ is simply:

$$x^B = \prod_{i,j=1}^k x_{i,j}^{B_{i,j}}$$

Robbins and Rumsey [RR86] defined two multiplicity free operators acting on the vector space spanned by alternating sign matrices, which we shall discuss in section 3:

$$\Omega : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n+1)]$$

$$\Delta : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n-1)]$$

Our closed form expression for $x[k]_{1,1}$ now takes the form:

(4) $$x[k]_{1,1} = \sum_{\substack{(A,B) \mid |B|=k,|A|=k-1 \\ A \in \Delta(B)}} \lambda^F(B) s(\lambda)^{-F(A)} \mu^G(B) t(\mu)^{-G(A)} x[1]^B s(x[0])^{-A}$$

where:

(5) $$s(z)_{i,j} = z_{i+1,j+1}$$

(6) $$t(z)_{i,j} = z_{i+1,j-1}$$

Note that this formula shows that $x[k]_{1,1}$ is a Laurent polynomial, and not just a rational function as would be expected from its recursive definition. This is an example of the Laurent phenomenon. See, for example [FZ02].
3. UP AND DOWN OPERATORS

We shall now define the multiplicity free operators acting on the vector space spanned by alternating sign matrices mentioned in the previous section:

\[ \mathcal{U} : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n+1)] \]
\[ \mathcal{D} : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n-1)] \]

These operators have the property that if \( B \in \text{ASM}(n) \) contains \( r \) ones and \( s \) negative ones then number of terms occuring in \( \mathcal{U}(B) \) is \( 2^r \) while the number of terms occuring in \( \mathcal{D}(B) \) is \( 2^s \).

If we fix an order on the \(-1\)'s of \( B \) then each element \( A \) of \( \mathcal{D}(B) \) is naturally indexed by a binary string. Similarly if we fix an order of the \( 1 \)'s in \( B \) then each element \( C \) of \( \mathcal{U}(B) \) is indexed by a binary string.

To define these operators we shall need the notion of left interlacing matrices:

\[
\begin{pmatrix}
B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} \\
A_{1,1} & A_{1,2} & A_{1,3} & \ \\
B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} \\
A_{2,1} & A_{2,2} & A_{2,3} & \ \\
B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} \\
A_{3,1} & A_{3,2} & A_{3,3} & \ \\
B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} \\
\end{pmatrix}
\]

The conditions on the matrix \( \overrightarrow{A} \) are as follows:

\[
\begin{pmatrix}
x & y \\
a & \ \\
z & \ \\
\end{pmatrix}
\begin{array}{cc}
x, w - 1 \leq a \leq y, z
\end{array}
\]

An example:

\[
\begin{pmatrix}
\{0,1\} & 1 & 1 & 1 \\
1 & \{1,2\} & 2 & 2 \\
1 & 2 & \{1,2\} & 3 \\
1 & 2 & 3 & 4
\end{pmatrix}
\]

Above and to the left of a \(-1\) in the alternating sign matrix \( B \) there are two possible choices for the corresponding value of the left cumulant matrix \( \overrightarrow{A} \). At all other positions there is a single choice \([RR86]\).
Here are the corresponding alternating sign matrices:

\[
\begin{align*}
A_{11} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
A_{01} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
A_{10} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
A_{00} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\end{align*}
\]

One may check that adding one at position \((i, j)\) in the left cumulant matrix \(A\) is equivalent to adding the matrix \(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\) with upper left hand corner at position \((i, j)\) to the corresponding alternating sign matrix \(A\) [RR86].

In our example we have:

\[\mathcal{D}(X) = A_{0,0} + A_{1,0} + A_{0,1} + A_{1,1}\]

We shall be especially interested in the “smallest” matrix \(A\) which is left interlacing with the matrix \(B\) and which we denote by \(A^{\text{min}} = A_{00} \ldots 0\). We have, by construction:

\[
\overline{A}^{\text{min}}_{ij} = \max(B_{ij}, B_{i+1,j+1} - 1)
\]

The \(\Phi\) operator is defined similarly.
The rule for constructing all possible matrices $\overline{C}$ for a given matrix $\overline{B}$ is the last row and last column must be strictly increasing from 1 to $n$ as well as that:

$$\begin{pmatrix} x & y \\ c & \; \\ z & w \end{pmatrix} \quad \text{with} \quad y, z \leq c \leq w, x + 1$$

Here is an example:

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \overline{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Interlacing matrices:

$$\begin{pmatrix} 0 & \{0, 1\} & 1 \\ \{0, 1\} & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Above and to the left of a 1 in the alternating sign matrix $\overline{B}$ there are two possible choices for the corresponding value of $\overline{C}$. At all other positions there is a single choice [RRSC].

$$\overline{C}_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$
$$\overline{C}_{01} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$
$$\overline{C}_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$
$$\overline{C}_{00} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Here are the corresponding alternating sign matrices:
One may check that, as expected, subtracting one at position \((i, j)\) in the left cumulant matrix \(C\) is equivalent to subtracting the matrix
\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
with upper left hand corner at position \((i, j)\) from the corresponding alternating sign matrix \(C\) \[RR86\].

In our example we have:

\[
\Upsilon(Y) = C_{0,0} + C_{0,1} + C_{1,0} + C_{1,1}
\]

This time we shall be especially interested in the “largest” matrix \(C\) which is left interlacing with \(B\) and which we denote by \(C^{\text{max}} = C_{11\ldots1}\).

We have, by construction:

\[
C_{ij}^{\text{max}} = \min(B_{ij}, B_{i-1,j-1} + 1)
\]

4. More up-down operators

We will need a second set of up and down operators which are closely related to the first.

\[
\Upsilon^* : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n + 1)]
\]
\[
\Upsilon^* : \text{ASM}(n) \to \mathbb{Z}[\text{ASM}(n - 1)]
\]

To define these operators we make use of right interlacing matrices:
In the right interlacing case, the conditions on the matrix $A^*$ are:

\[
\begin{pmatrix}
  x & y \\
  a & z
\end{pmatrix}
\]

\[y, z - 1 \leq a \leq x, w\]

Continuing with our example matrix $X$:

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  1 & 1 & 0 & 0 \\
  1 & 1 & 2 & 2 \\
  2 & 1 & 1 & 0 \\
  3 & 2 & 1 & 1 \\
  4 & 3 & 2 & 1
\end{pmatrix}
\]

Above and to the right of a $-1$ in the alternating sign matrix $B$ there are two possible choices for the corresponding value of the right cumulant matrix $A^*$. Again, if we fix an order on the $-1$’s of $B$ then each element $A^*$ of $D(B)$ is naturally indexed by a binary string determining the position in the right cumulant matrix $A^*$ where the larger of the two possible values was chosen.

\[
\begin{align*}
A^*_{11} &= \begin{pmatrix}
  1 & 1 & 0 \\
  2 & 1 & 1 \\
  3 & 2 & 1
\end{pmatrix} \\
A^*_{01} &= \begin{pmatrix}
  1 & 0 & 0 \\
  2 & 1 & 1 \\
  3 & 2 & 1
\end{pmatrix} \\
A^*_{10} &= \begin{pmatrix}
  1 & 1 & 0 \\
  2 & 1 & 0 \\
  3 & 2 & 1
\end{pmatrix} \\
A^*_{00} &= \begin{pmatrix}
  1 & 0 & 0 \\
  2 & 1 & 1 \\
  3 & 2 & 1
\end{pmatrix}
\end{align*}
\]

Here are the corresponding alternating sign matrices:
Adding one at position \((i, j)\) in the right cumulant matrix \(A^*\) is equivalent to adding the matrix \(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\) with upper right hand corner at position \((i, j)\) to the alternating sign matrix \(A^*\).

If \(B\) is an \(n\) by \(n\) alternating sign matrix then \(D^*(B)\) is the sum of all \((n-1)\) by \((n-1)\) alternating sign matrices \(A^*\) such that \(A^*\) is right interlacing with \(B^*\).

Now for the \(\Lambda^*\) operator.

\[
\begin{pmatrix}
C^*_{1,1} & C^*_{1,2} & C^*_{1,3} \\
B_{1,1} & B_{1,2} & B_{1,3} \\
C^*_{2,1} & C^*_{2,2} & C^*_{2,3} \\
B_{2,1} & B_{2,2} & B_{2,3} \\
C^*_{3,1} & C^*_{3,2} & C^*_{3,3}
\end{pmatrix}
\]

The rule for constructing all possible matrices \(C^*\) for a given matrix \(B\) is the last column must be strictly increasing from 1 to \(n+1\), the last row must be strictly decreasing from \(n\) to 1, and:

\[
\begin{pmatrix}
x & y \\
c & \end{pmatrix}
\]

\[
\begin{pmatrix}
w \\
x \leq c \leq y + 1,
\end{pmatrix}
\]

Here is an example:

\[
Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y^* = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}
\]
Interlacing matrices:

\[
\begin{pmatrix}
1 & 1 & \{0,1\} \\
1 & 1 \\
2 & \{1,2\} & 1 \\
2 & 1 \\
3 & 2 & 1
\end{pmatrix}
\]

Above and to the right of a 1 in the alternating sign matrix \(B\) there are two possible choices for the corresponding value of \(C^*\). At all other positions there is a single choice.

Fixing an order on the 1’s of \(B\), for each element \(C^*\) of \(\mathcal{U}^*(B)\) is naturally indexed by a binary string determining the position in the right cumulant matrix \(C^*_r\) where the larger of the two possible values was chosen.

\[
C_{11}^* = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}
\]

\[
C_{01}^* = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad C_{10}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}
\]

\[
C_{00}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}
\]

Here are the corresponding alternating sign matrices:

\[
C_{11}^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

\[
C_{01}^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_{10}^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]

\[
C_{00}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]
Subtracting one at position \((i, j)\) in the right cumulant matrix \(C^*\) is equivalent to subtracting the matrix \(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\) with upper right hand corner at position \((i, j)\) from the alternating sign matrix \(C^*\).

**Proposition 4.1.**

\[ A_{\min}^* = A_{\max} \]

**Proof.** Consider the following segments of interlacing matrices:

\[
\begin{pmatrix}
  a & b & c & d \\
  x & y & z & \\
  e & f & g & h \\
  w & u & v & \\
  i & j & k & l \\
  m & n & o & p
\end{pmatrix}
\begin{pmatrix}
  a^* & b^* & c^* & d^* \\
  x^* & y^* & z^* & \\
  e^* & f^* & g^* & h^* \\
  w^* & u^* & v^* & \\
  i^* & j^* & k^* & l^* \\
  m^* & n^* & o^* & p^*
\end{pmatrix}
\]

The elements \(a, b, c, \text{ etc...}\) belong to the left cumulant matrix \(\overline{B}\) while the elements \(x, y, z \text{ etc...}\) belong to the left-interlacing matrix \(\overline{A}_{\max}\).

Similarly the elements \(a^*, b^*, c^*, \text{ etc...}\) belong to the right cumulant matrix \(\overline{B}^*\) while the elements \(x^*, y^*, z^* \text{ etc...}\) belong to the right-interlacing matrix \(\overline{A}_{\min}^*\).

We wish to show that the value of the entry of \(A_{\max}\) at position \(u\) is equal to the value of \(A_{\min}^*\) at position \(u^*\). That is, by equations \((2)\) and \((3)\) we want to show that:

\[ x + u - w - y = u^* + z^* - y^* - v^* \]

As a consequence of lemma \([2.2]\) there is some \(\gamma\) such that:

\[
\begin{align*}
  a + b^* &= b + c^* = c + d^* = \gamma \\
  e + f^* &= f + g^* = g + h^* = \gamma + 1 \\
  i + j^* &= j + k^* = k + \ell^* = \gamma + 2 \\
  m + n^* &= n + o^* = o + p^* = \gamma + 3
\end{align*}
\]
Now, by construction, we have:

\[ x + u - w - y = \min(a, f - 1) + \min(f, k - 1) - \min(e, j - 1) - \min(b, g - 1) \]
\[ = \min(\gamma - b^*, \gamma - g^*) + \min(\gamma + 1 - g^*, \gamma + 1 - \ell^*) \]
\[ - \min(\gamma + 1 - f^*, \gamma + 1 - k^*) - \min(\gamma - c^*, \gamma - h^*) \]
\[ = -\max(b^*, g^*) - \max(g^*, \ell^*) + \max(f^*, k^*) + \max(c^*, h^*) \]
\[ = -y^* - v^* + u^* + z^* \]

The result follows. \(\square\)

A similar argument to the above may be used to show that \(A_{\max}^* = A_{\min}\) as well as \(C_{\min}^* = C_{\max}\) and \(C_{\max}^* = C_{\min}\). More precisely:

**Proposition 4.2.** If \(s\) is a binary string, and \(\overline{s}\) is its complement, then \(A_s = A_{\overline{s}}^*\) and \(C_s = C_{\overline{s}}^*\).

In other words, for any alternating sign matrix \(B\) the partial order of matrices occurring in \(D^*(B)\) is precisely the dual of the partial order of matrices occurring in \(D(B)\). Similarly for \(U(B)\) and \(U^*(B)\).

5. **Proof of main theorem**

Our proof is almost identical to that given in [RR86]. Let us recall the recurrence:

\[ x[k + 1]_{i,j} = \frac{\mu_{i,n-k-j}x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{i,j}x[k]_{i,j+1}x[k]_{i+1,j}}{x[k - 1]_{i+1,j+1}} \] (7)

To simplify things, let us introduce the notation:

\[ D(x[k])_{i,j} = \mu_{i,n-k-j}x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{ij}x[k]_{i,j+1}x[k]_{i+1,j} \]

so that we may rewrite equation (7) as:

\[ x[k + 1]_{i,j} = \frac{D(x[k])_{i,j}}{s(x[k - 1])_{i,j}} \]

**Theorem 5.1.** For \(2 \leq k \leq n\) we have:

\[ x[k]_{1,1} = \sum_{(A,B)}^{(A,B)} \lambda^{F(B)}(\lambda)^{-F(A)} \mu^{G(B)}(\mu)^{-G(A)}x[1]^B s(x[0])^{-A} \]

The sum is over all pairs of matrix \((A,B)\) such that \(A\) occurs in the expansion of \(D(B)\).
Proof. The result is trivially true when $k = 2$. Making use of the invariance in $k$, followed by the recurrence, we may obtain $x[k + 1]_{1,1}$ from $x[k]_{i,j}$ as follows:

\[
x[k + 1]_{1,1} = \sum_{(A, B)} \lambda^F(B) s(\lambda)^{-F(A)} \mu^G(B) t(\mu)^{-G(A)} x[2]^B s(x[1])^{-A}
\]

\[
= \sum_{(A, B)} \lambda^F(B) s(\lambda)^{-F(A)} \mu^G(B) t(\mu)^{-G(A)} (D(x[1]))^B s(x[1])^{-A}
\]

We must show that this is equal to:

\[
\sum_{(B, C)} \lambda^F(C) s(\lambda)^{-F(B)} \mu^G(C) t(\mu)^{-G(B)} x[1]^C s(x[0])^{-B}
\]

To do this, we fix some alternating sign matrix $B$ with $|B| = k$ and take the coefficient of $s(x[0])^{-B}$ on both sides. We must now prove that:

\[
(9) \quad \sum_{|A| = k-1} \lambda^F(B) s(\lambda)^{-F(A)} \mu^G(B) t(\mu)^{-G(A)} D(x[1])^B s(x[1])^{-A}
\]

\[
= \sum_{|C| = k+1} \lambda^F(C) s(\lambda)^{-F(B)} \mu^G(C) t(\mu)^{-G(B)} x[1]^C
\]

Here the sum is over all $A$ (resp. $C$) which may be found in the expansion of $D(B)$ (resp $\mathcal{U}(B)$).

Making use of proposition 4.2 we may rewrite the right hand side of equation (9) as:

\[
\sum_{|C| = k+1} \lambda^F(C) s(\lambda)^{-F(B)} \mu^G(C) t(\mu)^{-G(B)} x[1]^C
\]

\[
= s(\lambda)^{-F(B)} t(\mu)^{-G(B)} x[1]^C \lambda^{F(C)_{\text{max}}} \mu^{G(C)_{\text{min}}} \prod_{B_{ij}=1} (\mu_{i,n-k-j} + \lambda_{ij} \frac{x[1]_{i,j} x[1]_{i+1,j+1}}{x[1]_{i,j} x[1]_{i+1,j+1}})
\]

\[
(10) \quad = s(\lambda)^{-F(B)} t(\mu)^{-G(B)} x[1]^C \lambda^{F(C)_{\text{max}}} \mu^{G(C)_{\text{min}}} \prod_{B_{ij}=1} \frac{D(x[1]_{i,j})}{x[1]_{i,j} s(x[1]_{i,j})}
\]
while the left hand side of equation (9) may be written as:

\[
\sum_{|\lambda|=k-1} \lambda^{F(B)} s(\lambda) - F(A) \mu^{G(B)} t(\mu) - G(A) D(x[1]) B s(x[1])^{-A} \\
= \lambda^{F(B)} \mu^{G(B)} D(x[1]) B s(\lambda) - F(A_{min}) t(\mu) - G(A_{max}) s(x[1])^{-A_{min}} \prod_{B_{i,j}=-1} (\mu_{i,n-k-j} + \lambda_{ij} \frac{x[1]_{i+1,j}}{x[1]_{i,j}} x[1]_{i+1,j+1}) \\
= \lambda^{F(B)} \mu^{G(B)} D(x[1]) B s(\lambda) - F(A_{min}) t(\mu) - G(A_{max}) s(x[1])^{-A_{min}} \prod_{B_{i,j}=-1} \frac{D(x[1]_{i,j})}{x[1]_{i,j}s(x[1]_{i,j})} \\
= \lambda^{F(B)} \mu^{G(B)} s(\lambda) - F(A_{min}) t(\mu) - G(A_{max}) s(x[1])^{-A_{min}} \prod_{B_{i,j}=-1} D(x[1]_{i,j}) \prod_{B_{i,j}=-1} \frac{1}{x[1]_{i,j}s(x[1]_{i,j})} \\
\tag{11}
\]

Comparing equation (10) with equation (11), we must show that:

\[
s(\lambda) F(A_{min}) \lambda^{F(C_{max})} t(\mu) G(A_{max}) \mu^{G(C_{max})} s(x[1])^{C_{max}} s(x[1])^{A_{min}} \\
= s(\lambda) F(B) \lambda^{F(B)} t(\mu) G(B) \mu^{G(B)} (x[1] s(x[1]))^{B}
\]

To complete the proof one need only observe that:

\[
\min(x + 1, y) + \max(x, y - 1) = x + y
\]

More precisely, we have, by construction, that:

\[
A_{i,j}^{min} = \max(\overline{B}_{i,j}, \overline{B}_{i+1,j+1} - 1) \\
C_{i,j}^{max} = \min(\overline{B}_{i-1,j-1} + 1, \overline{B}_{i,j})
\]

and so:

\[
C_{i,j}^{max} + A_{i-1,j-1}^{min} = \min(\overline{B}_{i-1,j-1} + 1, \overline{B}_{i,j}) + \max(\overline{B}_{i-1,j-1}, \overline{B}_{i,j} - 1) \\
= \overline{B}_{i,j} + \overline{B}_{i-1,j-1}
\]

This gives us the same power of \( \lambda_{i,j} \) on both sides. By equations 3 and 4 we also have:

\[
C_{i,j}^{max} + A_{i-1,j-1}^{min} = B_{i,j} + B_{i-1,j-1}
\]

This gives us the same power of \( x_{i,j} \) on both sides.

The power of \( \mu_{i,j} \) on the left hand side is given by:

\[
C_{i,k-j+1}^{min} + A_{i-1,k-j}^{max} = \max(\overline{B}_{i,k-j+1}, \overline{B}_{i+1,k-j+2}) + \min(\overline{B}_{i,k-j+1}, \overline{B}_{i+1,k-j+2}) \\
= \overline{B}_{i,k-j+1} + \overline{B}_{i+1,k-j+2}
\]

which is the power of \( \mu_{i,j} \) on the right hand side. The result follows. \( \square \)
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