High integer spins beyond the Fierz-Pauli Framework

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The description of higher integer spins takes its origin from the 1939 paper by Fierz and Pauli ¹ who suggested to consider spin s as the highest spin in the symmetric traceless tensor $\varphi_{\mu_1\mu_2...\mu_s}(x)$ belonging to the $(\frac{s}{2}, \frac{s}{2})$ representation of the Lorentz Group. This representation is reducible and contains all values of spin from spin 0 to s. All the lower spin values should be eliminated. A result which can be accomplished by imposing the divergenless condition in addition to the Klein-Gordon condition, i.e. the field $\varphi_{\mu_1\mu_2...\mu_s}(x)$ should satisfy

$$\varphi_{\mu_1\mu_2...\mu_j...\mu_s}(x) = \varphi_{\mu_1\mu_2...\mu_j...\mu_s}(x),$$

$$(p^2 - m^2)\varphi_{\mu_1\mu_2...\mu_s}(x) = 0,$$

$$p^{\mu_1}\varphi_{\mu_1\mu_2...\mu_s}(x) = 0$$

It was noticed in ¹ that minimally coupling electromagnetism in Eqs.¹ lead to immediate algebraic inconsistencies which can be avoided by requiring that all equations involving derivatives must be obtained from a Lagrangian. Many attempts have been made in the past to construct such Lagrangians ² a procedure that becomes involved because of the need of the use auxiliary fields in the formalism in order to obtain Eqs.¹ from the Lagrangian (see Ref.¹).

A similar formalism was also developed in ¹ for the fermion case. However, an alternative framework using symmetric spinor-vector quantities $\psi_{\mu_1\mu_2...\mu_s}$ to describe spin $s = k + \frac{1}{2}$ was formulated in ³ where a linear equation of motion is obtained from a variational principle without the need for auxiliary fields. The corresponding subsidiary conditions which eliminate the redundant components are obtained from this equation. Soon after its formulation it was discovered that the Rarita-Schwinger formalism suffer from serious inconsistencies ⁴, ⁵, ⁶, ⁷, ⁸ a problem which has been unresolved since.

In a previous work ⁹ two of the authors proposed a formalism to avoid these problems in the fermionic case. Indeed, based on the facts shown in ⁹ that i) Dirac equation is just the projector over parity eigensubspaces contained in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ ii) Proca equation is just the projector over the negative parity eigensubspace contained in $(\frac{1}{2}, \frac{1}{2})$; and the fact shown in ⁹ that Proca equation can also be seen as the projector over the $-2m^2$ eigensubspace of $W^2$ contained in $(\frac{1}{2}, \frac{1}{2})$, it was concluded that the correct equation of motion for a field should be obtained by projecting over the corresponding eigensubspace of $W^2$ and parity in a given representation of the Lorentz Group. Guided by this principle, a new equation of motion for a spin 3/2 particle was formulated in ⁹ which yields the appropriate subsidiary conditions and can be derived from a variational principle without the need for auxiliary fields. Furthermore, based on a previous study on the propagation of higher spin waves in the framework of projectors over eigensubspaces of $W^2$ alone ¹⁰, it was argued that this equation is free of the Velo-Zwanziger pathologies.

In this work we apply the same principle to the bosonic case. We construct the equation of motion corresponding to the covariant projectors onto invariant subspaces of the squared Pauli-Lubanski vector and parity. Specifically, we explicitly construct covariant projectors in the representation $(\frac{1}{2}, \frac{1}{2})^*$ and derive the condition that fixes the invariant subspace of eigenvalue $(-m^2s(s + 1))$. For the sake of transparency and without any loss of generality in the following
we carry out all considerations in momentum space and we focus in the case \( s = 2 \). The generalization to arbitrary integer \( s \) is straightforward.

We will work with the \( (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) \) representation of the Lorentz Group. This representation is reducible and can be decomposed into irreducible representations as

\[
\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) = [(1, 0) \oplus (0, 1)] \oplus (1, 1) \oplus (0, 0),
\]
corresponding to the direct sum of an antisymmetric second rank tensor, a symmetric traceless second rank tensor and a scalar. It is the \(-6m^2\) eigensubspace of \( W^2 \) (spin 2 in the rest frame) contained in the \( (1, 1) \) representation space (traceless symmetric tensor) which we are interested in here. It is the goal of this paper to show that the highest spin in \( T^{(s)}_{\mu_1 \ldots \mu_s}(x) \), can be pinned down by one sole covariant equation quadratic in the momenta.

We begin with recalling that the Pauli–Lubanski (PL) vector for a given Homogeneous Lorentz Group (HLG) representation is defined as

\[
W_\mu = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} M^{\nu \alpha} P^\beta,
\]
where \( \epsilon_{0123} = 1 \), while \( M^{\nu \alpha} \) are the corresponding generators. This operator can be shown to satisfy the commutators

\[
[W_\alpha, M_{\mu \nu}] = i(g_{\alpha \mu} W_\nu - g_{\alpha \nu} W_\mu), \quad [W_\alpha, P_\mu] = 0,
\]
i.e. it transforms as a four-vector under Lorentz transformations. Here \( g_{\alpha \mu} \) is a flat metric.

We first construct the generators of the HLG for the \( (\frac{1}{2}, \frac{1}{2}) \) representation as

\[
(M^{\mu \nu})_\alpha^\beta = i(g_{\alpha}^\mu g^{\nu \beta} - g^{\mu \beta} g_{\alpha}^\nu).
\]
The Pauli-Lubanski operator in \( (\frac{1}{2}, \frac{1}{2}) \) space denoted by \( w_\mu \). In substituting Eq. (8) into the defining expression (4) one finds

\[
(w_\mu)_\alpha^\beta = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} (M^{\nu \rho})_\alpha^\beta p^\sigma = i \epsilon_{\mu \alpha \beta \sigma} p^\sigma.
\]
From the latter equation one easily deduces \( w^2 \) as

\[
(w^2)_\alpha^\beta = -2 (g_{\alpha}^\beta p^\sigma - p_\alpha p^\beta).
\]
This operator possesses the two different eigenvalues, 0, and \(-2p^2\), respectively. States belonging to the latter eigensubspace satisfy

\[
w^2 A = -2m^2 A.
\]
The matrix form of \( w^2 \) implies

\[
(w^2)A = (w^2)A.
\]
Substitution of Eq. (4) into Eq. (4) under usage of Eq. (4) amounts to

\[
(g_{\alpha}^\beta p^\sigma - p_\alpha p^\beta) A_\beta = m^2 A_\alpha,
\]
or in more conventional form

\[
p^\beta F_{\beta \alpha} - m^2 A_\alpha = 0,
\]
where \( F_{\beta \alpha} = p_{\beta} A_\alpha - p_\alpha A_\beta \). The conclusion is that Proca equation originates directly from the frame independent projector onto the \( W^2 \) eigensubspace that gives spin 1 at rest. As shown in [9], Proca operator is just the covariant parity projector for this representation, hence in this space, projection over parity eigensubspaces is equivalent to the projection over \( w^2 \) eigensubspaces.

Next we construct the Pauli-Lubanski vector for the \( (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) \) representation. The HLG generators for the \( (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) \) representation is obtained as

\[
M^{\mu \nu}_{RS} = M^{\mu \nu}_{(\frac{1}{2}, \frac{1}{2})} \otimes 1_4 + 1_4 \otimes M^{\mu \nu}_{(\frac{1}{2}, \frac{1}{2})},
\]
where $M^{\alpha\beta}_{\mu\nu}$ denote the generators in the four vector space, while $1_4$ is the four dimensional unit matrix. The Pauli-Lubanski vector for this representation is then calculated as

$$ (W_\mu)_{\alpha\beta ab} = (w_\mu)_{\alpha\beta} g_{ab} + g_{\alpha\beta} (w_\mu)_{ab}, $$

where we consider that both Greek and Roman indices are Lorentz ones and we make a distinction only to keep track of quantities coming from the different $(\frac{1}{2}, \frac{1}{2})$ subspaces. The squared operator is calculated as

$$ (W^2)_{\alpha\beta ab} = (w^2)_{\alpha\beta} g_{ab} + g_{\alpha\beta} (w^2)_{ab} + 2[(w_\mu)_{\alpha\gamma} g_{ac}](w^\mu)_{cb} g_{\gamma\beta}. $$

Next we choose the appropriate $W^2$ and parity eigensubspace by restricting the state field $h_{\alpha a}$ to satisfy

$$ (w^2)_{\alpha\beta} g_{ab} h^{\beta b} = -2m^2 h_{\alpha a}, \quad g_{\alpha\beta} (w^2)_{ab} h^{\beta b} = -2m^2 h_{\alpha a}, $$

i.e to be composed as the direct product of fields in $(\frac{1}{2}, \frac{1}{2})$ belonging to the $-2m^2$ eigensubspace of $w^2$. Finally, we impose that $h^{\beta b}$ belongs to the $-6m^2$ eigensubspace in the full $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$ space, thus we get

$$ -6m^2 h_{\alpha a} = -2m^2 h_{\alpha a} - 2m^2 h_{\alpha a} + 2[(w_\mu)_{\alpha\gamma} g_{ac}](w^\mu)_{cb} g_{\gamma\beta} h^{\beta b}. $$

Using the explicit form in Eq. (16) we obtain the equation of motion for a massive spin 2 particle with well defined parity as

$$ [\epsilon_{\mu a\beta\gamma} \epsilon^{\rho\epsilon\beta\gamma} - m^2 g_{a\beta} g_{\rho\epsilon}] h^{\beta b} = 0. $$

Equation (17) is our prime result. It defines uniquely the $-6m^2$ eigensubspace of $W^2$ with well defined parity in the $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$ representation space. It follows directly from the symmetries of space-time and is unambiguous. Furthermore, in general second order equations containing factors of $p^\mu p^\rho$ are ambiguous with respect to the order of these factors under gauging (see e.g. [13]). This is not the case for our equation since we keep track of the $p$ factors come from.

In principle $h^{\beta b}$ contains more components than necessary to describe a spin 2 particle. However, from Eq. (17) we obtain subsidiary conditions which eliminate the redundant components. Indeed, the first property of this field which is obvious from this equation is its symmetry under the exchange of Lorentz indices

$$ h_{\alpha \beta} = h_{\beta \alpha}. $$

The second set of conditions are obtained contracting our Eq. (17) with $p^\alpha$, due to the anti-symmetry of the Levi-Civita tensor one easily finds

$$ p_\beta h^{\beta \alpha} = 0. $$

Finally going to the rest frame we can easily convince ourselves that in this frame

$$ h^{\alpha} = 0, $$

and since this is a scalar quantity this condition is fulfilled in every frame. Eqs. (17) [13][18][19] eliminate 11 of the 16 components contained in $h_{\alpha \beta}$ and we are left with only 5 degrees of freedom which are the same number of d.o.f. for a massive spin 2 particle.

It is worth remarking that in the massless case this equation coincides with the one satisfied by the graviton field in sourceless linearized gravity which describes a free graviton in a flat Minkowski space [11] (see also [12]). The propagation of spin 2 waves under minimal coupling in our formalism is presently under investigation. In this concern, our work may be also useful to clarify some aspects about the relation between the mass of the graviton and the cosmological constant $\Lambda$ which recently has been subject of some interest [13][14] in connection with causality for the propagation of a graviton in an electromagnetic background. In this direction, it is worth mentioning that it has been shown [15] that linearized gravity with cosmological constant admits an $S$ duality prescription $\Lambda \rightarrow \frac{1}{\Lambda}$ or a strong coupling limit $[16]$ via the duality $l_p \rightarrow \frac{1}{l_p}$. It may be interesting for further research to see if our present study is useful in this direction.

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