A NOTE ON \( q \)-EULER NUMBERS AND POLYNOMIALS

TAEKYUN KIM

ABSTRACT. The purpose of this paper is to construct \( q \)-Euler numbers and polynomials by using \( p \)-adic \( q \)-integral equations on \( \mathbb{Z}_p \). Finally, we will give some interesting formula related to these \( q \)-Euler numbers and polynomials.

§1. Introduction

The usual Bernoulli numbers are defined by

\[
\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1},
\]

which can be written symbolically as \( e^{Bt} = \frac{t}{e^t - 1} \), interpreted to mean \( B^k \) must be replaced by \( B_k \) when we expand on the left. This relation can also be written \( e^{(B+1)t} - e^{Bt} = t \), or, if we equate power of \( t \),

\[
B_0 = 1, \quad (B + 1)^k - B_k = \begin{cases} 
1 & \text{if } k = 1 \\
0 & \text{if } k > 1,
\end{cases}
\]

where again we must first expand and then replace \( B^i \) by \( B_i \), cf. [6,8,9,10,11].

Carlitz’s \( q \)-Bernoulli numbers \( \beta_k \) can be determined inductively by

\[
\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 
1 & \text{if } k = 1 \\
0 & \text{if } k > 1,
\end{cases}
\]

with the usual convention about replacing \( \beta^i \) by \( \beta_{i,q} \) (see [1,2,3,4,5,12]).

Carlitz also defined \( q \)-Euler numbers and polynomials as

\[
H_0(u; q) = 1, \quad (qH + 1)^k - uH_k(u; q) = 0 \quad \text{for } k \geq 1,
\]

where \( u \) is a complex number with \( |u| > 1 \); and for \( k \geq 0, \)

\[
H_k(u, x; q) = (q^x H + [x]_q)^k, \quad \text{cf. } [2, 3, 4, 9, 10, 11, 12],
\]

Typeset by \textsc{AMS-\TeX}
where $[x]_q$ is defined by $[x]_q = \frac{1-q^x}{1-q}$.

It was known that the ordinary Euler polynomials are defined by
\[ 2 \frac{e^t + 1}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi), \tag{3} \]
we note that $E_n = E_n(0)$ are called $n$-th Euler numbers, cf. [6,7,8].

Let $p$ be a fixed odd prime, and let $\mathbb{C}_p$ denote the $p$-adic completion of the algebraic closure of $\mathbb{C}_p$. For $d$ a fixed positive integer $(p,d) = 1$, let
\[ X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z}, \]
\[ X_1 = \mathbb{Z}_p, \]
\[ X^* = \bigcup_{0 < a < dp \atop (a,p) = 1} (a + dp\mathbb{Z}_p), \]
\[ a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \}, \]
where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let $q$ be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we always assume $|q - 1|_p < p^{-1/p-1}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Throughout this paper we use the notation:
\[ [x]_q = \frac{q^x - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{x-1}. \]

We say that $f$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients
\[ F_f(x, y) = \frac{f(x) - f(y)}{x - y} \]
have a limit $l = f'(a)$ as $(x, y) \to (a, a)$.

For $f \in UD(\mathbb{Z}_p)$, let us start with expression
\[ \frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \tag{4} \]
representing $q$-analogue of Riemann sums for $f$, cf. [5]. The integral of $f$ on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of those sums, when it exists. The $q$-Volkenborn integral of function $f \in UD(\mathbb{Z}_p)$ is defined by
\[ I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} f(x)q^x, \quad (\text{see [5]}). \tag{5} \]
From the definition of $[x]_q$, we derive

$$[x]_q = \frac{1 - (-q)^x}{1 + q} = 1 - q + q^2 - q^3 + \cdots + (-1)^{x-1}q^{x-1}.$$  

In [4,5], it was known that

$$\int_{\mathbb{Z}_p} [x + y]_q^m d\mu_q(y) = \beta_{m,q}(x), \quad (6)$$

where $\beta_{m,q}(x)$ are called Carlitz’s $q$-Bernoulli polynomials. By (5), we easily see that

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \frac{[2]_q}{2} \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x q^x. \quad (7)$$

The purpose of this note is to construct $q$-Euler numbers which can be uniquely determined by

$$q(qE_q + 1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases} \quad (8)$$

with the usual convention about replacing $E_i^q$ by $E_{i,q}$. From these numbers, we will derive some interesting formulae.

§2. A note on $q$-Euler numbers and polynomials

From (7), we derive formulae as follows:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (9)$$

where $f_1(x)$ is translation with $f_1(x) = f(x + 1)$.

Let $f(x) = e^{[x]_q t}$. Then we see that

$$q \int_{\mathbb{Z}_p} e^{[x+1]_q t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q}(x) = [2]_q. \quad (10)$$

First, we consider the following integral :
\[
\int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q}(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l \int_{\mathbb{Z}_p} q^{xl} d\mu_{-q}(x) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l \sum_{m=0}^{\infty} (-1)^m (l+1)^m \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{(l+1)m} \frac{t^n}{n!}
\]

By (11), we easily see that

\[
\int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) = E_{n,q}.
\]

Note that

\[
\lim_{q \to 1} E_{n,q}(x) = E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x), \quad \text{see[7].}
\]

By the same method, we note that

\[
\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \left( \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l q^{lx} \int_{\mathbb{Z}_p} q^{yl} d\mu_{-q}(y) \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \left( \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l q^{lx} \frac{t^n}{1+q^{l+1}} \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{t^n}{n!}
\]

Thus, we have

\[
\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = E_{n,q}(x), \quad n \geq 0.
\]

4
Note that
\[
\lim_{q \to 1} E_{n,q}(x) = E_n(x) = \int_{Z_p} (x + y)^n d\mu_{-1}(y).
\]

Since
\[
[x + 1]_q = \frac{1 - q^{x+1}}{1 - q} = \frac{1 - q}{1 - q} + \frac{1 - q^x}{1 - q} q = 1 + q[x]_q.
\] (14)

From (10), (12) and (14), we derive
\[
[2]_q = \sum_{n=0}^{\infty} \left( q \int_{Z_p} [x + 1]_q^n d\mu_{-q}(x) + \int_{Z_p} [x]_q^n d\mu_{-q}(x) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( q \sum_{l=0}^{n} \binom{n}{l} q^l \int_{Z_p} [x]_q^l d\mu_{-q}(x) + \int_{Z_p} [x]_q^n d\mu_{-q}(x) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( q \sum_{l=0}^{n} \binom{n}{l} q^l E_{l,q} + E_{n,q} \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} (q(qE_q + 1)^n + E_{n,q}) \frac{t^n}{n!},
\] (15)

with the usual convention about replacing $E_{i,q}^q$ by $E_{i,q}$.

By (15), we easily see that
\[
q(qE_q + 1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}
\] (16)

with the usual convention about replacing $E_{i,q}^q$ by $E_{i,q}$. When we compare Eq.(16) and Eq.(1), the Eq.(16) seems to be interesting formula. In particular, these numbers seem to be new, which are different than Carlitz’s $q$-Euler numbers. From (7), we derive
\[
q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),
\] (16 - 1)

where $n \in \mathbb{N}$, $f_n(x) = f(x + n)$. When $n$ is an odd positive integer, we note that
\[
q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l).
\] (17)

From (17), we derive
\[
q^n \int_{Z_p} [x + n]_q^m d\mu_{-q}(x) + \int_{Z_p} [x]_q^m d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m,
\] (18)
where \( n \) is an odd positive integer. By (12) and (13), we easily see that

\[
[2]_q \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m = q^n E_{m,q}(n) + E_{m,q},
\]

(19)

where \( n \) is an odd positive integer. If \( n \) is an even integer, then we have in Eq.(16-1) that

\[
q^n \int_{\mathbb{Z}_p} [x + n]_q^m d\mu_{-q}(x) - \int_{\mathbb{Z}_p} [x]_q^m d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l [l]_q^m.
\]

(20)

From (12), (13) and (20), we derive

\[
q^n E_{m,q}(n) - E_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l [l]_q^m,
\]

(21)

where \( n \) is a positive even integer. It seems to be interesting to compare (19) and (21).

**References**

[1] L. C. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.

[2] L. C. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc. **76** (1954), 332–350.

[3] M. Cenkci, M. Can and V. Kurt, *p-adic interpolation functions and Kummer-type congruences for q-twisted Euler numbers*, Advan. Stud. Contemp. Math. **9** (2004), 203–216.

[4] M. Cenkci, M. Can, *Some results on q-analogue of the Lerch zeta function*, Adv. Stud. Contemp. Math. **12** (2006), 213–223.

[5] T. Kim, *q–Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288–299.

[6] T. Kim, *A note on p-adic invariant integral in the rings of p-adic integers*, Advan. Stud. Contemp. Math. **13** (2006), 95–99.

[7] T. Kim, *q-generalized Euler numbers and polynomials*, Russian J. Math. Phys. **13** (2006), 293–298.

[8] T. Kim, *A note on q-Euler and Genocchi numbers*, Proc. Japan Acad. Ser. A **77** (2001), 139–141.

[9] N. Koblitz, *q-Bernoulli numbers*, J. Number Theory **14** (1982), 332–339.

[10] Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. **11** (2005), 205–218.

[11] Y. Simsek, *Q-Dedekind type sums related to q-zeta function and basic L-series*, J. Math. Anal. Appl. **318** (2006), 333–351.

[12] H. M. Srivastava, T. Kim and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12** (2005), 241–268.

Taekyun Kim
EECS, Kyungpook National University, Taegu 702-701, S. Korea
e-mail: tkim@knu.ac.kr; tkim64@hanmail.net