Application of Multifractal Measures to Tehran Price Index

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Abstract

We report an empirical study of Tehran Price Index (TEPIX). To analyze our data we use various methods like as, rescaled range analysis ($R/S$), modified rescaled range analysis (Lo’s method), Detrended Fluctuation Analysis (DFA) and generalized Hurst exponents analysis. Based on numerical results, the scaling range of TEPIX returns is specified, long memory effect or long range correlation property in this market is investigated, characteristic exponent for probability distribution function of TEPIX returns is derived and finally the stage of development in Tehran Stock Exchange is determined.

Keywords: $R/S$ analysis, Hurst exponent, Long memory, Detrended Fluctuation Analysis, Multifractals, Lévy Distributions.

1 Introduction

Financial markets have in recent years been at the focus of physicists’s attempts to apply existing knowledge from statistical mechanics to economic problems. These markets, though largely varying in details of trading rules and traded goods, are characterized by some generic features of their financial time series, called stylized facts. Multifractal processes and the deeply connected mathematics of large deviations and multiplicative cascades have been used in many contexts to account for the time scale dependence of the statistical properties. For example, recent empirical findings [1, 2, 3] suggest that in rough surfaces, this framework is likely to be pertinent. The aim is

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to characterize the statistical properties of the series with the hope that a better understanding of the underlying stochastic dynamics could provide useful information to create new models able to reproduce experimental facts. An important aspect concerns concepts as scaling and the scale invariance of height surface [3, 4]. There is an important volume of data and studies showing self-similarity at short space scales and an apparent breakdown for longer spaces modeled in terms of distributions with truncated tails. Recent studies have shown that the traditional approach based on a Brownian motion picture [5, 6] or other more elaborated descriptions such as Lévy and truncated Lévy distributions [2], all of them relying on the idea of additive process, are not suitable to properly describe the statistical features of these fluctuations. In this sense, there are more and more evidences that a multiplicative process approach is the correct way to proceed, and this line of thought leads in a natural way to multifractality. In fact, this idea was already suggested some years ago when intermittency phenomena in return fluctuations was observed at different length scales which gave rise to some efforts to establish a link with other areas of physics such as turbulence [7, 8]. Nowadays, we know that there are important differences between both systems, as for instance the spectrum of frequencies, but the comparison triggered an intense analysis of the existing data. Multifractal analysis of a set of data can be performed in two different ways, analyzing either the statistics or the geometry. A statistical approach consists of denying an appropriate intensive variable depending on a resolution parameter, then its statistical moments are calculated by averaging over an ensemble of realizations and at random base points. It is said that the variable is multifractal if those moments exhibit a power-law dependence in the resolution parameter [9, 10]. On the other hand, geometrical approaches try to assess a local power-law dependency on the resolution parameter for the same intensive variables at every particular point (which is a stronger statement that just requiring some averages the moments to follow a power law). While the geometrical approach is informative about the spatial localization of self-similar (fractal) structures, it has been much less used because of the greater technical difficulty to retrieve the correct scaling exponents. However, in the latest years an important effort to improve geometrical techniques has been carried out, giving sensible improvement and good performance [11, 12]. We will apply the geometrical approach in this paper as a valuable tool for the understanding of the price return fluctuations.

The main objective of this article is to investigate the characteristics of Tehran Stock Exchange using some multifractal measures. Our purpose is to show how some relatively simple statistics gives us indications on the market situation. The paper is organized as follows. In Section 2 we describe our data. In Section 3, we review the rescaled range ($R/S$) analysis and its modified version, Lo’s $R/S$ analysis. The way of interpretation and empirical results for classical $R/S$ analysis and Lo’s $R/S$ analysis are presented.
In Section 4 a brief description of the Detrended Fluctuation Analysis is given. The results of this analysis for the TEPIX time series are shown in this Section too. In Section 5 we explain the generalized Hurst exponents analysis. Also its results is presented. In Section 6 the concept of characteristic exponent for a probability distribution and its relation to Hurst exponent is reviewed. Moreover, characteristic exponent of TEPIX returns distribution is computed. Then, in Section 7, a simple explanation of long memory process is mentioned and shown how we can use Lo’s R/S analysis to find out presence of such process. The market stage of development and its relation to generalized Hurst exponent is studied in Section 8. Finally, conclusions are given in Section 9.

2 Data Description

We analyze the values of the TEPIX for the period of almost 9 years: from 20th may 1995 to 18th march 2004. Before 1995 the Tehran Price Index was rather fixed because of government controls. The data have been recorded at each trading day. So that our database consists of 2342 values and 2341 daily returns. The sources of this data is the center of research and development of Iran capital market and the paper utilizes only the closing prices. In Fig.1 we present a time series corresponding to daily values of the TEPIX index. It must be mentioned that TEPIX tracks the performance of more than 350 listed local firms.

![Figure 1: Daily closure TEPIX index history (1995-2004).](image)
Also, Table 1 provides summary statistics of logarithmic returns. According to data in Table 1, it is seen that, the probability distribution of TEPIX returns does not look like a Gaussian distribution and belongs to stable Lévy distributions.

| Mean  | Std.Dev. | Skewness | Kurtosis |
|-------|----------|----------|----------|
| 0.0011| 0.0046   | 1.0619   | 20.827   |

### 3 Rescaled Range (R/S) analysis

#### 3.1 Classical R/S Analysis

The Hurst rescaled range (R/S) analysis is a technique proposed by Henry Hurst in 1951 [13] to test presence of correlations in empirical time series. The main idea behind the R/S analysis is that one looks at the scaling behavior of the rescaled cumulative deviations from the mean, or the distance the system travels as a function of time. This is compared to the null-hypothesis of a random walk. For an independent system, the distance covered increases, on average, by the square root of time. If the system covers a larger distance than this, it cannot be independent by definition, and the changes must be influencing each other; they have to be correlated. Although there may be autoregressive process present that can cause short-term correlations, we will see that when adjusting for such short-term correlations, there may be other forms of memory effects present which need to be examined.

Consider a time series in prices of length $P$. This time series is then transformed into a time series of logarithmic returns of length $N = P - 1$ such that

$$N_i = \log\left(\frac{P_{i+1}}{P_i}\right), \quad i = 1, 2, ..., P - 1. \quad (1)$$

Time period is divided into $m$ contiguous sub-periods of length $n$, such that $m \times n = N$. Each sub-period is labelled by $I_a$, with $a = 1, 2, ..., m$. Then, each element in $I_a$ is labelled by $N_k$, a such that $k = 1, 2, ..., n$. For each sub-period $I_a$ of length $n$ the average is calculated as

$$M_a = \frac{1}{n} \sum_{k=1}^{\tau} N_{k, a} \quad (2)$$
Thus, $M_a$ is the mean value of the $N'_i$'s contained in the sub-period $I_a$ of length $n$. Then, we calculate the time series of accumulated departures from the mean $(X_{k,a})$ for each sub-period $I_a$, defined as

$$X_{k,a} = \sum_{i=1}^{k} (N_{i,a} - M_a), \quad k = 1, 2, \ldots n. \quad (3)$$

As can be seen from Eq. (3), the series of accumulated departures from the mean always will end up with zero. Now, the range that the time series covers relative to the mean within each sub-period is defined as

$$R_{I_a} = \max(X_{k,a}) - \min(X_{k,a}), \quad 1 < k < n. \quad (4)$$

The next step is to calculate the standard deviation for each sub-period $I_a$,

$$S_{I_a} = \sqrt{\frac{1}{n} \sum_{k=1}^{n} (N_{k,a} - M_a^2)}. \quad (5)$$

Then, the range for each sub-period ($R_{I_a}$) is rescaled by the corresponding standard deviation ($S_{I_a}$). Recall that we had $m$ contiguous sub-periods of length $n$. Thus, the average $R/S$ value for length or box $n$ is

$$(R/S)_n = \frac{1}{m} \sum_{a=1}^{m} \frac{R_{I_a}}{S_{I_a}}. \quad (6)$$

Now, the calculations from Eqs. (1)-(6) must be repeated for different time horizons. This is achieved by successively increasing $n$ and repeating the calculations until we have covered all integer $ns$. One can say that $R/S$ analysis is a special form of box-counting for time series. However, the method was developed long before the concepts of fractals. After having calculated $R/S$ values for a large range of different time horizons $n$, we plot $\log(R/S)_n$ against $\log(n)$. By performing a least-squares regression with $\log(R/S)_n$ as the dependent variable and $\log(n)$ as the independent one, we find the slope of the regression which is the estimate of the Hurst exponent $H$. The Hurst exponent ($H$) and the fractal dimension $D_f$ are related as

$$D_f = 2 - H. \quad (7)$$

In theory, $H = 0.5$ means that the time series is independent, but as mentioned above the process need not be Gaussian. If $H = 0.5$, the process may in fact be a non-Gaussian process as e.g. the Student-t or gamma. If $H \in (0.5, 1.0]$ it implies that the time series is persistent which is characterized by long memory effects on all time scales. For example, all daily price changes are correlated with future daily price changes; all weekly price
changes are correlated with all future weekly price changes and so on. This is one of the key characteristics of fractal time series. It is also a main characteristic of non-linear dynamical systems that there is a sensitivity to initial conditions which implies that such a system in theory would have an infinite memory. The persistence implies that if the series has been up or down in the last period then the chances are that it will continue to be up or down, respectively, in the next period. This behavior is also independent of the time scale we are looking at. The strength of the trend-reinforcing behavior, or persistence, increases as $H$ approaches 1.0. This impact of the present on the future can be expressed as a correlation function $(C)$, 

$$C = 2^{(2H-1)} - 1.$$  

(8)

In the case of $H = 0.5$ the correlation $C$ equals zero, and the time series is uncorrelated. However, if $H = 1.0$ we see that $C = 1$, indicating perfect positive correlation. On the other hand, when $H \in [0, 0.5)$ we have anti-persistence. This means that whenever the time series have been up in the last period, it is more likely that it will be down in the next period. Thus, an anti-persistent time series will be more choppier than a pure random walk with $H = 0.5$. The $R/S$ analysis can also uncover average non-periodic cycles in the system under study. If there is a long memory process at work, for a natural system this memory is often finite, even though long memory processes theoretically are supposed to last forever, as was the case for mathematical fractals and the logistic map. When the long term memory is lost, or the memory of the initial conditions has vanished, the system begins to follow a random walk; this is also called the crossover point. Thus, a crucial point in the estimation of the Hurst exponent is to use the proper range for which there is non-normal scaling behavior. This is the range for which the scaling behavior is linear in the $\log(R/S)_n$ versus $\log(n)$ plot. If there is a crossover-point, this can be seen as a break in the plot where the slope changes for a certain value, $\log(n_{\text{max}})$. If this is the case, it is an indication of a non-periodic cycle with average cycle length equal to $n_{\text{max}}$.

A plot of the rescaled range $R/S$ as a function of $\tau$ for the TEPIX returns over the mentioned period is shown in the curve of Fig.2. The data in this case show a scaling regime that goes from $\tau = 2$ up to 7 (in log scale) approximately. It is equal to 128 trading days or 180 days. A linear regression in this region yields the value $H = 0.79 \pm 0.03$. It must be mentioned that the Hurst method tends to overestimate the Hurst exponent for time series of small sizes [15].

### 3.2 Lo’s Modified R/S Analysis

The classical $R/S$ test has been proven to be too weak to indicate a true long memory process, in fact it tends to indicate a time series has long memory
when it does not. In 1991 Lo [16] introduced a stronger test based on a modified R/S statistics, which is known to be too strong to indicate a true long memory process. Lo’s modified R/S test is described in brief below.

For a given series \( N_i, i = 1, 2, ..., n \), Lo defines the modified R/S statistic as,

\[
Q_n = \frac{\max \sum_{j=1}^{k} (N_j - \bar{N}_n) - \min \sum_{j=1}^{k} (N_j - \bar{N}_n)}{\sigma_n(q)}, \quad k = 1, n \tag{9}
\]

where the denominator is expressed as,

\[
\sigma_n^2(q) = \frac{1}{n} \sum_{j=1}^{k} (N_j - \bar{N}_n)^2 + \frac{2}{n} \sum_{j=1}^{q} w_j(q) \left[ \sum_{i=j+1}^{n} (N_i - \bar{N}_n)(N_{i-j} - \bar{N}_n) \right] \tag{10}
\]

in which

\[
w_j(q) = 1 - \frac{j}{q+1}, \quad q < n \tag{11}
\]

Lo finally standardizes the statistic \( Q_n \) by dividing by \( \sqrt{n} \) and is denoted as \( V_n(q) \). The numerator of \( V_n(q) \) is the range of deviation from the approximate linear trend line in a given interval and the denominator is the sample variance augmented with weighted autocovariances up to a lag determined \( q \). For \( q = 0 \), this is same as the classical R/S statistic. This autocovariance part of the denominator is non zero for series exhibiting short term memory.
and this makes the statistic robust to heteroscedasticity.

Table 2 gives the results from the modified $R/S$ statistic. $R/S$ analysis is extremely sensitive to the order of truncation $q$ and there is no statistical criteria for choosing $q$ in the framework of this statistic. Since there is no data driven guidance for the choice of this parameter, we consider different values for $q = 0, 2, 4, 6, 8, 10$ and 15. More explanations related to long memory effect and interpretation of data in Table 2 will be mentioned in Section 7. In brief, the starred values in Table 2 reject null hypothesis of short memory.

Table 2: Modified rescaled range ($R/S$) statistic for the returns, absolute and squared returns.

| Lag order | $R/S$ statistic | absolute returns | squared returns |
|-----------|----------------|------------------|-----------------|
| $q$       | returns        |                  |                 |
| 0         | 1.0513         | 0.7857*          | 0.9573          |
| 2         | 0.8219         | 0.5393*          | 0.7528*         |
| 4         | 0.7475*        | 0.4475*          | 0.6654*         |
| 6         | 0.7156*        | 0.3952*          | 0.6111*         |
| 8         | 0.6964*        | 0.3608*          | 0.5734*         |
| 10        | 0.6807*        | 0.3362*          | 0.5451*         |
| 15        | 0.6724*        | 0.2972*          | 0.4980*         |

4 Detrended Fluctuation Analysis

Detrended fluctuation analysis (DFA) is a scaling analysis technique providing a simple quantitative parameter—the scaling exponent $\alpha$—to represent the correlation properties of a time series [17]. The advantage of DFA over many techniques are that it permits the detection of long-range correlations embedded in seemingly non-stationary time series, and also avoids the spurious detection of apparent long-range correlations that are an artifact of non-stationarity. Additionally, the advantages of DFA in computation of $H$ over other techniques (for example, the Fourier transform) are:

- inherent trends are avoided at all time scales;
- local correlations can be easily probed.
To implement the DFA, let us suppose we have a time series, \( N(i) (i = 1, ..., N_{\text{max}}) \). We integrate the time series \( N(i) \):

\[
y(j) = \sum_{i=1}^{j} [N(i) - \langle N \rangle]
\]

where:

\[
\langle N \rangle = \frac{1}{N_{\text{max}}} \sum_{j=1}^{N_{\text{max}}} N(i).
\]

Next we break up \( N(i) \) into \( K \) non-overlapping time intervals, \( I_n \), of equal size \( \tau \) where \( n = 0, 1, ...K - 1 \) and \( K \) corresponds to the integer part of \( N_{\text{max}}/\tau \). In each box, we fit the integrated time series by using a polynomial function, \( y_{pol}(i) \), which is called the local trend. For order-\( l \) DFA (DFA-1 if \( l=1 \), DFA-2 if \( l=2 \), etc.), the \( l \)-order polynomial function should be applied for the fitting. We detrend the integrated time series \( y(i) \) in each box, and calculate the detrended fluctuation function:

\[
Y(i) = y(i) - y_{pol}(i).
\]

For a given box size \( s \), we calculate the root mean square fluctuation:

\[
F(s) = \sqrt{\frac{1}{N_{\text{max}}} \sum_{i=1}^{N_{\text{max}}} [Y(i)]^2}
\]

The above computation is repeated for box sizes \( s \) (different scales) to provide a relationship between \( F(s) \) and \( s \). A power law relation between \( F(s) \) and \( s \) indicates the presence of scaling: \( F(s) \sim s^{\alpha} \). The parameter \( \alpha \), called the scaling exponent or correlation exponent, represents the correlation properties of the signal: if \( \alpha = 0.5 \), there is no correlation and the signal is an uncorrelated signal [17]; if \( \alpha < 0.5 \), the signal is anticorrelated; if \( \alpha > 0.5 \), there are positive correlations in the signal. In the two latest cases, the signal can be well approximated by the fractional Brownian motion law [15].

In Fig.3 we plot in double-logarithmic scale the corresponding fluctuation function \( F(s) \) against the box size \( s \). Using the above procedure, we obtain the following estimate for the Hurst exponent: \( H = 0.72 \pm 0.01 \). Since \( H > 0.5 \) it is concluded that the TEPIX returns show persistence; i.e., strong correlations between consecutive increments. It is seen that for \( s \sim 115 \) the empirical data deviate from the initial scaling behavior. This indicates that the TEPIX tends to loose its memory after a period of about 162 days. Based on overestimating the Hurst exponent in the \( R/S \) analysis it may be explained why the exponent \( H \) obtained via the Hurst method is usually larger than that of the DFA method [18].
Figure 3: Fluctuation Function $F(s)$ as a function of box size for the returns of the TEPIX index in the period of 1994-2004.

5 Generalized Hurst Exponents Approach

A generalization of the approach proposed by Hurst should be associated with the scaling behavior of statistically significant variables constructed from the time series [19]. Therefore we analyze the $q$-order moments of the distribution of the increments which is a good characterization of the statistical evolution of a stochastic variable $P(t)$. The generalized Hurst exponents, $H_q \equiv H(q)$, for a time series $P(t)(t = 1, 2, ...)$ are defined by the scaling properties of its structure functions $S_q(\tau)$

\[ S_q(\tau) = \langle |P(t+\tau) - P(t)|^q \rangle_T^{\frac{1}{q}} \sim \tau^{H(q)} \]  

where $q > 0$, $\tau$ is the time lag and averaging is over the time box (window) $T \gg \tau$, usually the largest time scale of the system. The function $H(q)$ contains information about averaged generalized volatilities at scale $\tau$ (only $q = 1, 2$ are used to define the volatility). In particular, the $H(1)$ exponent indicates persistent ($H(1) > 0.5$) or anti-persistent ($H(1) < 0.5$) behavior of the trend. For the Brownian random walk one gets $H(1) = 0.5$. For the popular Lévy stable and truncated Lévy processes with parameter $\alpha$, it has been found that $H(q) = q/\alpha$ for $q < \alpha$ and $H(q) = 1$ for $q \geq \alpha$. In this framework, we can distinguish between two kinds of processes:

- a process where $H(q) = H$, constant independent of $q$;
- a process with $H(q)$ not constant.
The first case is characteristic of unifractal processes where $qH(q)$ is linear and completely determined by its index $H$. In the second case, when $H(q)$ depends on $q$, the process is commonly called multi-fractal and different exponents characterize the scaling of different $q$-moments of the distribution.

Eq. (13) is studied numerically in order to analyze the generalized $q$th-order Hurst exponents in the structure function $S_q(\tau)$. Table 3 includes the values of the generalized Hurst exponents $H(q)$ in the structure function for the TEPIX. The values $H(q)$ versus $q$ for $q = 1, ..., 10$ are plotted in Fig. 3 for the TEPIX.

![Figure 4: Generalized Hurst exponents versus the $q$ for the returns of the TEPIX in the period of 1995-2004.](image)

Table 3: Values of the generalized $q$th-order Hurst exponents $H(q)$ for the TEPIX.

|   | $H(1)$ | $H(2)$ | $H(3)$ | $H(4)$ | $H(5)$ |
|---|--------|--------|--------|--------|--------|
|   | 0.8622 | 0.7935 | 0.7373 | 0.6929 | 0.6592 |
| $H(6)$ | 0.6340 | 0.6149 | 0.6001 | 0.5885 | 0.5792 |

It should be noted that the methods listed above, that is, $R/S$ analysis, Lo’s $R/S$ analysis and DFA can only extract a single scaling exponent from a time series. However, it is possible that the given time series may be governed by more than one scaling exponents, in which case a single scaling
exponent would be unable to capture the complex dynamics inherent in the data. Analysis using generalized Hurst exponents method, elucidates the dependence of $H(q)$ on $q$, which is a hallmark of multifractal processes. Such processes are far more than one exponent to characterize their scaling properties [20].

6 Characteristic Exponent

Paul Lévy, the French mathematician, proposed a general approach with the Gaussian as only a special case, to identify probability distributions which their sum has the same probability distribution. A stable Lévy distribution is represented by [21]

$$L_\alpha(N, \triangle t) \equiv \frac{1}{\pi} \int_0^\infty \exp(-\gamma \triangle t q^\alpha) \cos(qN) dq$$

(17)

where $\alpha$ is the characteristic exponent $0 < \alpha \leq 2$, $N$ the return, $\gamma$ the scale factor, and $\triangle t$ the time interval. This distribution obeys below scaling relations:

$$N_{\triangle t} = N_s(\triangle t)^{1/\alpha}$$

(18)

and

$$L_\alpha(N_s, \triangle t) \equiv L_\alpha(N_s, 1)(\triangle t)^{-1/\alpha}.$$  

(19)

If $\alpha = 2$ the distribution is Gaussian, and there is a finite second moment. If $\alpha = 1$ we have the Cauchy distribution with both infinite first and second moments. In the region for which $1 < \alpha < 2$, the second moment becomes infinite, but with a stable mean. Lévy stable distributions are self-similar and this means that the probabilities of return are the same for all time intervals once we adjust for the time scale. Roughly speaking, an agent with 1 min time interval faces the same risk as a 100 min agent in his time interval when adjusted for scale. The $\alpha$ exponent takes this scaling relationship into account.

The fractal dimension of the probability space, $\alpha$, used in above Equations is related to the Hurst exponent of the time series as:

$$\alpha = \frac{1}{H}.$$  

(20)

In this way, characteristic exponent for return distribution of TEPIX can be calculated. $\alpha$ exponents derived by using all of above methods has been shown in Table 4.

These estimates of $\alpha$ are relatively close to each other. Based on $\alpha$ values, we observe a non-normal scaling behavior and all estimates indicate that the process is different from a pure random walk. In fact, these results
Table 4: Values of the $\alpha$ exponents resulted from Hurst exponents.

|        | $\alpha_{R/S}$ | $\alpha_{Lo}$ | $\alpha_{DFA}$ | $\alpha_{GHE}$ |
|--------|----------------|---------------|----------------|----------------|
|        | 1.266          | 1.389         | 1.163          | 1.160          |

are evidence of a non-linear chaotic system. The distribution of TEPIX returns can be fitted by a stable Lévy distribution. For a better comparison of the return distribution with a Gaussian PDF and evaluating derived Hurst exponents, we have performed a maximum likelihood estimation of stable parameters. The parameters of this fitted Lévy distribution is presented in Table 5.

Alpha is the same characteristic exponent. Beta is the skewness in the range [-1,1] and gamma and delta are straightforward scale and shift parameters respectively. A probability distribution function of returns against Gaussian distribution with the mean and standard deviation of fitted Lévy distribution is depicted in Fig.5. Fitted Lévy distribution itself, is plotted in the Fig.6. It can be seen from this figures that the the real distribution (or the Lévy fitted ones) of returns is different from a Gaussian PDF (random walk).

![Figure 5: Probability distribution function of returns against a Gaussian distribution.](image)

Based on maximum likelihood parameter estimation of Lévy distribution
Table 5: The parameters of the fitted Lévy distribution.

| α   | β    | γ       | δ       |
|-----|------|---------|---------|
| 1.316 | 0.2049 | 0.0018  | 0.00076 |

(direct estimation), the best characteristic exponent has been resulted by classical $R/S$ analysis and Lo’s $R/S$ analysis respectively.

7 Long Memory Process

A random process is called a long memory process if it has an autocorrelation function that is not integrable. This happens, for example, when the autocorrelation function decays asymptotically as a power law of the form $\tau^{-\alpha}$ with $\alpha < 1$. This is important because it implies that values from the distant past can have a significant effect on the present, and implies anomalous diffusion. A process is defined as long memory if in the limit $k \to \infty$

$$
\gamma(k) = \sim k^{-\alpha} L(k) \quad (21)
$$

where $0 < \alpha < 1$ and $L(x)$ is a slowly varying function at infinity. The degree of long memory dependence is given by the exponent $\alpha$; the smaller $\alpha$, the longer the memory.

The Hurst exponent simply is related to $\alpha$. For a long memory process
\[ H = 1 - \frac{\alpha}{2} \text{ or } \alpha = 2 - 2H. \] Short memory processes have \( H = \frac{1}{2} \), and the autocorrelation function decays faster than \( k^{-1} \). A positively correlated long memory process is characterized by a Hurst exponent in the interval \((0.5, 1)\).

As an application of results provided by Lo’s modified \( R/S \) statistic, it can be said [22], at the 5% significance level, the null hypothesis of a short memory process is rejected if the modified \( R/S \) statistic does not fall within the confidence interval \([0.809, 1.862]\). For returns, the null hypothesis of short memory is rejected at any lags, except for 0 and 2. For absolute and squared returns, the null hypothesis of short memory is rejected for all lag orders.

Besides, the Hurst exponent is linked to the modified \( R/S \) statistic by

\[
\lim_{T \to +\infty} E[\frac{R_T}{S_T}(q)]/(aT^H) = 1 \tag{22}
\]

with \( a > 0 \). With this link it is possible to obtain the following approximate relationship:

\[
\log E[\frac{R_T}{S_T}(q)] \approx \log(a) + H \log(t) \tag{23}
\]

In order to estimate the value of the Hurst exponent, \( H \), we first determine a series of estimates of the Hurst exponent by fitting an ordinary least squares regression between \( \log[R_T;l/S_T;l(q)], l = 1, \ldots, j \) and \( \log(l), l = 1, \ldots, j \) for every \( j = 2, \ldots, T^* \), where \( R_{T;l} \) and \( S_{T;l}(q) \) are quantities related to \( R_T \) and \( S_T(q) \), respectively. Then we choose the optimal estimate in this series. As a result of such procedure, Hurst exponent has obtained equal to 0.721 ± 0.001.

Therefore, the Lo’s method verifies long memory process in returns of the TEPIX, based on above discussions.

8 Market Stage of Development

Based on recent research for characterizing the stage of development of markets [23] it is shown that the exponent \( H(2) \) has sensitivity to the degree of development of the market. As far as Stock markets are concerned, the generalized Hurst exponents \( H(1), H(2) \) show remarkable differences between developed and emerging markets. At one end of the spectrum there are stocks like as the Nasdaq 100 (US), the S&P 500 (US), the Nikkei 225 (Japan) and so on. Whereas, at the other end, there are Russian AK&M, the Indonesian JSXC, the Peruvian LSEG, etc. This sensitivity of the scaling exponents to the market conditions provides a new and simple way of empirically characterizing the development of financial markets.

Roughly speaking, emerging markets are associated with high value of \( H(1) \) and developed markets are associated with low values of \( H(1) \). Besides, it is found that all the emerging markets have \( H(2) \geq 0.5 \) whereas all the developed have \( H(2) \leq 0.5 \).
Considering all of above discussions and results, we notice that Tehran Stock Exchange belongs to emerging markets category and it is far from an efficient and developed market. Hurst exponent calculated by $R/S$ and DFA methods in one hand and generalized Hurst exponents ($H(1)$ and $H(2)$) in the other hand, both present this fact.

For the sake of comparison between various stock markets, the two first generalized Hurst exponents are shown in Table 5. It must be noticed that all of data in Table 6, except those corresponds to TEPIX, have been calculated during 1997 to 2001 period [23], while those corresponds to TEPIX have been calculated in the time period from 1995 to 2004.

Table 6: Hurst exponents $H(1)$ and $H(2)$ for stock market indices.

| Stock Market indices | $H(1)$ | $H(2)$ |
|----------------------|--------|--------|
| Nasdaq 100           | 0.47   | 0.45   |
| S&P 500              | 0.47   | 0.44   |
| Nikkei 225           | 0.46   | 0.43   |
| AK&M                 | 0.65   | 0.51   |
| JSXC                 | 0.57   | 0.53   |
| LSEG                 | 0.61   | 0.58   |
| TEPIX                | 0.86   | 0.79   |

These results indicate that, financial market characteristics in Iran do not show developed situations. In fact, Tehran Stock Exchange belongs to the category of emerging financial markets.

9 Conclusions

In this paper the concept of multifractality has been applied to Tehran Stock Exchange data. This market show a fractal scaling behavior significantly different from what a random walk would produce. For TEPIX time series we have obtained a Hurst exponent greater than 0.5, indicating that the TEPIX has long term dependence (persistence). This memory effect seems to last for up to about 6 months (115-128 trading days). Analysis using generalized Hurst exponents method, indicates the dependence of $H(q)$ on $q$, which is an evidence of multifractal processes. Also, we show that based on generalized Hurst exponents, financial market characteristics in Iran do not indicate a developed market. In other words, we are dealing with an emerging capital market. These findings imply that there are patterns, or trends in returns that persist over time. This provides a theoretical platform supporting the use of technical analysis to produce above average returns. The findings may be used to improve the current models or to make new
ones which use the concept of fractal scaling.

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