Two-point mean value formulas for some elliptic equations spaces with constant curvature

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Abstract. An analogue of the consequence of Bitsadze & Nakhushiev’s mean value formula was obtained in the Lobachevskiy space and on a sphere.

1. Introduction

The terms ”mean value theorem” and ”mean value formula” often do not mean similar facts to some extent in various sections and applied problems. Nevertheless varied results for different types of equations have in common mean of function at some set of points, as a finite set of points of the phase plane in mean formula for the equation of a vibrating string, a sphere, as in mean formula for a harmonic function \cite{1}, pair of spheres as in the Asgeirsson theorem \cite{2} or the set which is structured more complicatedly. Mean value theorems for elliptic equations are known most widely. The most famous of those is the mean value theorem (by Gauss) for the Laplace equation. This fact applies to general elliptic equations. Mean value theorem is carried over to ordinary differential equations. Mean value formulae are used while studying a proper spectral decomposition of elliptic operators. In this context see works by Ilyin V. A. and Moiseev E. I. (the papers \cite{3}–\cite{4}). Mean value theorem for Laplace’s equation in circular sector is proved in the \cite{5}. As well above mean value theorem generalizes to Riemannian manifolds (see the book \cite{6}). We should point out the papers \cite{7}, \cite{8}, where the condition of converse mean value theorem for harmonic functions is weakened.

Among mean value theorems for hyperbolic equations the mentioned above Asgeirsson theorem is one of the best known. The A.V. Bitsadze and A.M. Nakhushiev mean value theorem for the wave equation \cite{9} is also well known. In this we derived a two-point mean value formula for elliptic equation in the Lobachevskiy space and on the sphere from the mean value formula for the wave equation in those spaces.

2. Materials and methods

The paper uses classical methods of mathematical analysis, the theory of equations of mathematical physics and differential geometry
3. The mean value theorem for the wave equation in Euclidean space

Consider in space $R^{n+1}$ a pair of points $(\chi^{(j)}, \tau^{(j)})$, $j = 1, 2$, where $\chi^{(j)} = (\chi_1^{(j)}, \chi_2^{(j)}, \ldots, \chi_n^{(j)})$, $j = 1, 2$, that satisfy the condition

$$|\chi^{(1)} - \chi^{(2)}| < |\tau^{(1)} - \tau^{(2)}|. \quad (1)$$

We construct a matrix $A$ by the following rule. We fix an index $i \in \{1, \ldots, n\}$ and set

$$a_{ij} = \frac{\chi^{(1)}_j - \chi^{(2)}_j}{|\chi^{(1)} - \chi^{(2)}|}, \quad j = 1, \ldots, n.$$ 

The remaining elements of this matrix can be constructed from the conditions $AA^T = I$, $\det A = 1$, where $A^T$ is the matrix, transposed to the matrix $A$, $I$ is a unit matrix. Let $A_i$ denote the matrix obtained from the matrix by replacing the $i$-th column by zeros. Following the paper [9], we introduce an averaging operator $S_\tau$ by the formula

$$S^n v = S^n_\tau v = \gamma(n) \int_{|\xi| = \tau} v(\eta, \sigma) \, d\omega_\xi, \quad |\chi^{(1)} - \chi^{(2)}| > 0, \quad (2)$$

where $\gamma(n) = \sqrt{\pi^{1-n}}$,

$$\eta = \frac{|\tau^{(1)} - \tau^{(2)}| \xi_i (\chi^{(1)} - \chi^{(2)})}{|\chi^{(1)} - \chi^{(2)}| \sqrt{(\tau^{(1)} - \tau^{(2)})^2 - |\chi^{(1)} - \chi^{(2)}|^2}} + \frac{\chi^{(1)} + \chi^{(2)}}{2} A_i \xi, \quad \sigma = \frac{\tau^{(1)} + \tau^{(2)}}{2} - \frac{|\chi^{(1)} - \chi^{(2)}| \xi_i}{\sqrt{(\tau^{(1)} - \tau^{(2)})^2 - |\chi^{(1)} - \chi^{(2)}|^2}},$$

d$\omega_\xi$ is a sphere $|\xi| = t$ surface area element in $R^n$. When $\chi^{(1)} = \chi^{(2)}$ we define the operator $S_\tau$ by the formula

$$S^n = S^n_\tau v = \gamma(n) \int_{|\xi| = \tau} v\left(\chi^{(2)} + \xi, \frac{\tau^{(1)} + \tau^{(2)}}{2}\right) \, d\omega_\xi. \quad (3)$$

Next we introduce the operator $B_\tau$ by formulas

$$B^n v = B^n_\tau v = \tau \left(\frac{\partial}{2 \tau \partial \tau}\right)^{\frac{n+1}{2}} \frac{1}{\tau} S_\tau v, \quad n = 1(\text{mod}2), \quad (4)$$

$$B^n v = B^n_\tau v = \frac{1}{\sqrt{\pi}} \left(\frac{\partial}{2 \tau \partial \tau}\right)^{\frac{n}{2}} \int_0^\tau S_\tau v \, d\theta \sqrt{\tau^2 - \theta^2}, \quad n = 0(\text{mod}2), \quad (5)$$

**Theorem 1.** (the mean value theorem for the wave equation [9]). If a function $v(\chi_1, \ldots, \chi_n, \tau)$ is a regular solution of the wave equation

$$\frac{\partial^2 v}{\partial \tau^2} = \sum_{k=1}^n \frac{\partial^2 v}{\partial \chi_k^2}, \quad (6)$$


then for any pair of points \((\chi^{(j)}, \tau^{(j)})\), \(j = 1, 2\), satisfying condition (1), we have the equality

\[ v(\chi^{(1)}, \tau^{(1)}) + v(\chi^{(2)}, \tau^{(2)}) = B_\varrho \varrho, \tag{7} \]

\( \varrho = \sqrt{\Delta \tau^2 - |\Delta \chi|^2/2}, \ \Delta \tau = \tau^{(1)} - \tau^{(2)}, \ \Delta \chi = \chi^{(1)} - \chi^{(2)}. \)

There is an assertion for sufficiently smooth functions that is inverse to theorem 3. Let

\[ l = \max \left\{ n - 1, \frac{n + 3}{2} \right\}, \ n = 1(\text{mod} \ 2), \tag{8} \]

\[ l = \max \left\{ n, \left[ \frac{n + 3}{2} + 1 \right] \right\}, \ n = 0(\text{mod} \ 2). \tag{9} \]

**Theorem 2.** (the inverse mean value theorem for the wave equation \([10]\)). Let the function \(v(\chi, \tau) \in C^l(\mathbb{R}^{n+1})\) satisfies relation (7) for any pair of points \((\chi^{(j)}, \tau^{(j)})\), \(j = 1, 2\), which satisfy condition (1). Then \(v(\chi, \tau)\) is a regular solution of the wave equation (6).

### 4. The wave equation in the Lobachevskiy space and transformations of P. Lax and R. Phillips

Let’s consider the realization of the Lobachevski geometry on half-space

\[ R^n_{+} = \{ x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) : x_n > 0 \}, n \geq 2, \]

with a metric

\[ ds^2 = dx^2 / x_n^2, \]

where \(dx = (dx_1, \ldots, dx_{n-1}, dx_n) = (dx', dx_n), dx^2 = dx \cdot dx = dx_1^2 + \cdots + dx_n^2. \) This model of Lobachevski geometry will be designated by \( \Pi. \) In other words, components of the metric (contra-variant) tensor and the inverse metric (covariant) tensor in our case, respectively, have the form

\[ g_{ij} = \delta_{ij} x_n^2, \quad g^{ij} = \delta_{ij} x_n^2, \quad i, j = 1, \ldots, n, \tag{10} \]

where \( \delta_{ij} \) is the Kronecker symbol. As is well known (see for ex. the book \([11]\)) the Laplace-Beltrami operator in the metric \( \|g_{ij}\| \) is defined by the formula

\[ \Delta_\omega = \frac{1}{|g|} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} g^{ik} \sqrt{|g|} \frac{\partial}{\partial x_j}, \]

where \( g = \det \|g_{ij}\|. \) The Laplace-Beltrami operator in metric (10) has the form

\[ \Delta_\omega = x_n \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + x_n \frac{\partial}{\partial x_n} \left( x_n^{2-n} \frac{\partial}{\partial x_n} \right). \tag{11} \]

We consider the wave equation in the Lobachevskiy space, which has the form

\[ \frac{\partial^2 u}{\partial t^2} = \Delta_\omega u + \left( \frac{n - 1}{2} \right)^2 u. \tag{12} \]

This equation was considered by P. Lax and R. Phillips in the scattering theory \([12]\). It was also studied by I.A. Kipriyanov and L.A.Ivanov (see the papers \([13]\), \([14]\), \([16]\)). The concept of the
wave equation in a space with constant curvature is clarified by these authors. Following these authors we call equation (12) the wave equation in the Lobachevskiy space.

We introduce mapping $\Pi \times R \rightarrow R^{n+1}$ by means of formulas

$$s = e^t,$$

$$\tau = \tau(x,t) = \frac{s}{2} \left( \frac{1}{x_n} + x_n + \frac{|x'|^2}{x_n} \right),$$

$$\chi_n = \chi_n(x,t) = \frac{s}{2} \left( \frac{1}{x_n} - x_n - \frac{|x'|^2}{x_n} \right),$$

$$\chi' = \chi'(x,t) = s \frac{x'}{x_n},$$

where $|x'| = \sqrt{x_1^2 + \cdots + x_{n-1}^2}, \chi' = (\chi_1, \ldots, \chi_{n-1}), \chi = (\chi', \chi_n)$.

Let note that the inequality

$$|\chi|^2 < \tau^2$$

in terms of $(x, t)$ variables is equivalent to the inequality $x_n^2 > 0$, that holds for all points $(x, t) \in \Pi \times R$. Therefore, the mapping defined by means of equalities (13)–(16), realizes a one-to-one correspondence between the space $\Pi \times R$ and a cone in the space $R^{n+1}$ where the cone is defined by condition (17).

We add (14) and (15). We get

$$\tau + \chi_n = \frac{s}{x_n}.$$  

By subtraction (15) from (14), we get

$$\tau - \chi_n = \frac{s|x|^2}{x_n}.$$  

From (15) we find

$$x' = \frac{x_n}{s} \chi',$$

from there, taking into account (18), we get

$$x' = \frac{\chi'}{\tau + \chi_n}.$$  

We substitute (20) in (19). Then

$$\tau - \chi_n = \frac{s}{x_n} \left( \frac{x_n^2}{s\tau} |\chi'|^2 + x_n^2 \right),$$

whence

$$x_n = \frac{\sqrt{\tau^2 - |\chi|^2}}{\tau - \chi_n}.$$  

Further, taking into account (18) we have

$$s = x_n (\tau + \chi_n).$$
and considering (21), we get
\[ s = \sqrt{\tau^2 - |\chi|^2}, \]
and taking into account (13), we obtain
\[ t = \ln \sqrt{\tau^2 - |\chi|^2}. \]  (22)

Thus, the transformation defined by formulas (20), (21), (22), is the inverse of the transformation defined by formulas (13)–(16).

In addition to transformations of the variables (13)–(16) and their inverse (20), (21), (22), we also introduce a change of function
\[ v(\chi, \tau) = (\tau^2 - |\chi|^2)^{(1-n)/4} u(x(\chi, \tau), t(\chi, \tau)). \]  (23)

Transformations (13)–(16), their inverse (20), (21), (22), and also (23) were introduced for \( n = 2 \) in the book [12]. Therefore, it is appropriate to call them the Lax-Phillips transforms.

**Theorem 3.** A function \( u(x, t) \in C^2(\Pi \times R) \) is a regular solution of the wave equation (12) in the Lobachevskiy space if and only if the function (23), obtained from \( u \) by means of Lax-Phillips transforms is a regular solution of the wave equation (6) in the cone \( |\chi| < |\tau| \) of space \( R^{n+1} \) of variables \( \chi = (\chi_1, \ldots, \chi_n), \tau \).

**Proof.** We can make a direct substitution into the equation, but we will carry out the proof according to the scheme of P. Lax and R. Phillips. In accordance with this scheme, we consider the wave operator
\[ \Lambda = \frac{\partial^2}{\partial \tau^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \]
as the Laplace-Beltrami operator corresponding to an indefinite Lorentz metric
\[ d\sigma^2 = d\tau^2 - |d\chi|^2. \]  (24)

In metric (24) we pass to the coordinates \((s, x)\). For the derivatives of \( \tau \) we have:
\[ \frac{\partial \tau}{\partial s} = \frac{1}{2} \left( \frac{1}{x_n} + x_n + \frac{|x'|^2}{x_n} \right) = \frac{1 + |x|^2}{2x_n}, \]
\[ \frac{\partial \tau}{\partial x_j} = \frac{s x_j}{x_n}, \quad j = 1, 2, \ldots, n - 1, \]
\[ \frac{\partial \tau}{\partial x_n} = \frac{s (x_n^2 - |x'|^2 - 1)}{2x_n^2}. \]

Therefore
\[ d\tau = \frac{1 + |x|^2}{2x_n} ds + \frac{s}{x_n} x' \cdot dx' + \frac{s (x_n^2 - |x'|^2 - 1)}{2x_n^2} dx_n, \]  (25)

where \( dx' = (dx_1, \ldots, dx_{n-1}) \), \( x' \cdot dx' \) denotes the scalar product of the vectors \( x' \) and \( dx' \) in the space \( R^{n-1} \). Squaring equality (25), we get
\[ d\tau^2 = \frac{(1 + |x|^2)^2}{4x_n^2} ds^2 + \frac{s^2}{x_n^2} (x' \cdot dx')^2 + \frac{s^2 (x_n^2 - |x'|^2 - 1)^2}{4x_n^4} dx_n^2 + \]
\[ + \frac{s(1 + |x|^2)}{x_n} ds x' \cdot dx' + \frac{s^2 (x_n^2 - |x'|^2 - 1)}{x_n^2} dx_n x' \cdot dx'. \]
Further, for the derivatives of $\chi$,

$$
\frac{\partial \chi_\kappa}{\partial s} = \frac{x_\kappa}{x_n}, \quad \kappa = 1, \ldots, n - 1,
$$

$$
\frac{\partial \chi_\kappa}{\partial x_j} = \frac{s \delta_\kappa^j}{x_n}, \quad \kappa = 1, \ldots, n - 1, \quad j = 1, \ldots, n - 1,
$$

$$
\frac{\partial \chi_\kappa}{\partial x_n} = -\frac{s x_\kappa}{x_n^2}, \quad \kappa = 1, \ldots, n - 1,
$$

we get

$$
d\chi_\kappa = \frac{s}{x_n} dx_\kappa + \frac{x_\kappa}{x_n} ds - \frac{s x_\kappa}{x_n^2} dx_n, \quad \kappa = 1, \ldots, n - 1. \tag{27}
$$

Squaring equality (27), and then summing over the $\kappa$, we get

$$
|d\chi'|^2 = \frac{s^2}{x_n^2} |dx'|^2 + \frac{|x'|^2}{x_n^2} ds^2 + \frac{s |x'|^2}{x_n^4} dx_n^2 + \frac{2s}{x_n^2} ds x' \cdot dx' -
$$

$$
- \frac{2s^2}{x_n^6} dx_n x' \cdot dx' - \frac{2s |x'|^2}{x_n^3} ds dx_n. \tag{28}
$$

For the last coordinate we have

$$
d\chi_n = -\frac{s}{x_n} x' \cdot dx' + \frac{1 - |x|^2}{2 x_n} ds + \frac{s}{2 x_n^2} (|x'|^2 - 1 - x_n^2) dx_n,
$$

whence, after squaring, we get

$$
d\chi_n^2 = \frac{s^2}{x_n^2} (x' \cdot dx')^2 + \frac{(1 - |x|^2)^2}{4 x_n^2} ds^2 +
$$

$$
+ \frac{s^2 (|x'|^2 - 1 - x_n^2)^2}{4 x_n^4} dx_n^2 - \frac{s (1 - |x|^2)}{x_n^2} ds (x' \cdot dx') -
$$

$$
- \frac{s^2 (|x'|^2 - 1 - x_n^2)^2}{4 x_n^6} dx_n (x' \cdot dx') + \frac{s (1 - |x|^2) (|x'|^2 - 1 - x_n^2)}{2 x_n^3} ds dx_n. \tag{29}
$$

Replacing (26), (28) and (29) in (24), after identical transformations we obtain

$$
d\sigma^2 = ds^2 - \frac{s^2 |dx|^2}{x_n^2}. \tag{30}
$$

The transition from the coordinate $s$ to the coordinate $t$ by the formula $s = e^t$ leads expression (30) to the form

$$
d\sigma^2 = e^{2t} \left( dt^2 - \frac{|dx|^2}{x_n^2} \right). \tag{31}
$$
It follows that the Laplace-Beltrami operator corresponding to the indefinite metric (31), has the form
\[
\Lambda = e^{-(n+1)t} \frac{\partial}{\partial t} \left( e^{(n-1)t} \frac{\partial}{\partial t} \right) - x_n^2 e^{-2t} \Delta x_n - x_n^n e^{-2t} \frac{\partial}{\partial x_n} \left( x_n^{2-n} \frac{\partial}{\partial x_n} \right).
\] (32)

Thus, the fact that the function \(v(\chi, \tau)\) satisfies the Euclidean wave equation (6) in cone (17), is equivalent to the fact that the function \(u(x, t) = v(\chi(x, t), \tau(x, t))\) satisfies the equation
\[
\frac{e^{(1-n)t}}{2} \frac{\partial}{\partial t} \left( e^{(n-1)t} \frac{\partial v}{\partial t} \right) = \Delta u,
\] (33)
where the operator \(\Delta u\) is defined by formula (11).

Now we replace the function in equation (33) by formula (23). Taking into account (22), in the coordinates \((x, t)\), we get
\[
v = e^{(1-n)/2} u.
\] (34)
The substitution (34) gives equation (33) to equation (12), which completes the proof of the theorem.

5. The mean value theorem for the wave equation in the Lobachevskiy space

Now we prove an analog of theorem 3 for the wave equation in the Lobachevskiy space, basing on the connection of the wave equations in the Lobachevskiy space and the Euclidean space, which is established by theorem 4.

**Theorem 4.** Let a function \(u(x, t)\) be a regular solution of the wave equation (12) in the Lobachevskiy space. Let \((x^{(j)}, t^{(j)}) = (x_1^{(j)}, \ldots, x_{n-1}^{(j)}, x_n^{(j)}, t^{(j)}) = (x'_{(j)}, x'^{(j)}_{(j)}, t^{(j)}), j = 1, 2\), be a pair of points in the space \(\Pi \times R\) that is satisfying the condition
\[
(x_2^{(2)} e^{t'_{(1)}} - x_1^{(1)} e^{t'_{(2)}}) (x_2^{(1)} e^{t_{(1)}} - x_1^{(2)} e^{t_{(2)}}) - e^{(1+n)t}|\Delta x'|^2 > 0.
\] (35)

Then the following equality holds (the mean value formula)
\[
e^{(1+n)/2} u(x^{(1)}, t^{(1)}) + e^{(1+n)/2} u(x^{(2)}, t^{(2)}) = B^\pi_t (fu),
\] (36)
where \(f = (\tau^2 - |\chi|^2)^{(1-n)/4}, \chi^{(j)} = \chi(x^{(j)}, t^{(j)}), \tau^{(j)} = \tau(x^{(j)}, t^{(j)}), j = 1, 2, u = u(x, t) = u(x(\chi, \tau), t(\chi, \tau)), \) the operator \(B^\pi_t\) is defined by formulas (4) and (5).

Conversely, if the function \(u(x, t)\) belongs to \(C^l(\Pi \times R)\), where \(l\) is defined by formulas (8) and (9) satisfies the mean value formula (36), then it is a regular solution of the wave equation (12) in the Lobachevskiy space.

**Proof.** Let \(\Delta \tau = \tau^{(1)} - \tau^{(2)}\). Then we have
\[
\Delta \tau = \tau^{(1)} - \tau^{(2)} = \frac{s^{(1)}}{2x_n^{(1)}} (1 + |x^{(1)}|^2) - \frac{s^{(2)}}{2x_n^{(2)}} (1 + |x^{(2)}|^2) = \\
= \frac{s^{(1)}x_n^{(2)} + s^{(1)}|x^{(1)}|^2x_n^{(2)} - s^{(2)}x_n^{(1)} - s^{(2)}|x^{(2)}|^2x_n^{(1)}}{2x_n^{(1)}x_n^{(2)}}.
\]
The increment \(\Delta \chi_n = \chi_n^{(1)} - \chi_n^{(2)}\) is expressed by the formula
\[
\Delta \chi_n = \frac{s^{(1)}x_n^{(2)} - s^{(1)}|x^{(1)}|^2x_n^{(2)} - s^{(2)}x_n^{(1)} + s^{(2)}|x^{(2)}|^2x_n^{(1)}}{2x_n^{(1)}x_n^{(2)}}.
\]
Hence we obtain
\[ \Delta \tau^2 - \Delta \chi_n^2 = (\Delta \tau - \Delta \chi_n)(\Delta \tau + \Delta \chi_n) = \]
\[ = \left( s(1)x_n(2) - s(2)x_n(1) \right) \left( s(1)x_n(1)^2 - s(2)x_n(2)^2 \right) \]
\[ \left( x_n(1)^2 x_n(2)^2 \right) \].

Further, since
\[ \Delta \chi' = \chi'(1) - \chi'(2) = \frac{s(1)x_n(1)x_n(2) - s(2)x_n(1)x_n(2)}{x_n(1)^2 x_n(2)^2} \],
scalar squaring, we obtain
\[ |\Delta \chi'|^2 = \frac{(s(1))^2 |x_n(1)|^2 \left( x_n(2)^2 \right)^2 - (s(2))^2 |x_n(2)|^2 \left( x_n(1)^2 \right)^2}{\left( x_n(1)^2 x_n(2)^2 \right)^2} \].

Taking into account an equality
\[ |\Delta \chi|^2 = |\Delta \chi'|^2 + \Delta \chi_n^2 \],
from (37) and (38) we get
\[ \Delta \tau^2 - |\Delta \chi|^2 = \frac{\left( s(1)x_n(2) - s(2)x_n(1) \right) \left( s(1)x_n(1) - s(2)x_n(2) \right)}{x_n(1)^2 x_n(2)^2} - s(1)s(2) |\Delta x'|^2 \].

Taking into account that \( s(j) = e^{t(j)} \) and the fact that \( x_n(j) > 0, j = 1, 2 \), we obtain that condition (35) is equivalent to condition \( \Delta \tau^2 - |\Delta \chi|^2 > 0 \). Therefore, the points \((\chi(j), \tau(j))\), \( j = 1, 2 \), satisfy the condition of theorem 3. Let
\[ v(\chi, \tau) = f(\chi, \tau)u(x(\chi, \tau), t(\chi, \tau)) = (\tau^2 - |\chi|^2)^{(1-n)/4}u(x(\chi, \tau), t(\chi, \tau)). \]

In accordance to Theorem 4 function (39) satisfies the wave equation (6) in Euclidean space, and hence it satisfies the mean value formula. (7). If we express the function \( v(\chi, \tau) \) in the formula (7) in terms of the function \( u(x(\chi, \tau), t(\chi, \tau)) \) by formula (39), we obtain the statement of the theorem. The proof is complete.

6. The Hadamard descent method and the two-point mean value theorem for the Laplace equation in Euclidean space

A two-point formula of the mean for the Laplace equation was obtained in [17] by means of the Hadamard descent method.

Suppose now that the function \( v(\chi) \) satisfies the Laplace equation
\[ \frac{\partial^2 v}{\partial \chi_1^2} + \ldots + \frac{\partial^2 v}{\partial \chi_n^2} = 0 \]
in a neighborhood of the ball \(|\chi| < r\).

We carry out the fictitious variable \( \tau \) in accordance with the Hadamard descent method assuming
\[ v(\chi, \tau) = v(\chi, 0) = v(\chi), \quad |\chi| < \tau, \quad \tau \in R \]
Without loss of generality, we set
\[ \tau_1 = 0, \quad \tau_2 = r > 0. \]

The function \( v(\chi, \tau) \) will satisfy equation (6), which means that the mean value formula (7) is valid for it. Taking into account our assumptions the operator \( S_\tau \) will act as follows:
\[
S_\tau v = S_\tau^n v = \gamma(n) \int_{|\xi| = \tau} v(\eta) \, d\omega_\xi, \; |\chi^{(1)} - \chi^{(2)}| > 0,
\]
and formula (7) is transformed to the form
\[
v(\chi^{(1)}) + v(\chi^{(2)}) = B\varphi v.
\]

Let us clarify the geometric meaning of the formula (43). Without loss of generality, we set
\[ \chi^{(1)} = \alpha = (\alpha_1, 0, \ldots, 0), \quad \chi^{(2)} = 0 = (0, 0, \ldots, 0). \]

In (41) integration is carried on the sphere \( \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = \tau^2. \) We replace the first variable
\[
\nu = \frac{\alpha_1}{2} + \frac{r\xi_1}{\sqrt{r^2 - \alpha_1^2}}.
\]
Expressing the \( \xi_1 \) in terms of \( \nu \) from formula (44), we obtain
\[
\xi_1 = \frac{\sqrt{r^2 - \alpha_1^2}}{r}(\nu - \frac{\alpha_1}{2}).
\]
Substituting expression (45) into the sphere equation \( |\xi| = r \), we obtain in the new variables \( \nu, \xi_2, \ldots, \xi_n \) the ellipsoid \( \Phi \) equation:
\[
\frac{4(\nu - \frac{\alpha_1}{2})}{r^2} + \frac{4(\xi_2^2 + \cdots + \xi_n^2)}{r^2 - \alpha_1^2} = 1.
\]
Integration in formula (43) is carried out on the ellipsoid \( \Phi \), which is given by equation (46). Thus, the following assertion is proved.

**Theorem 5.** (two-point mean value theorem for a harmonic function). Let a domain \( \Omega \subset \mathbb{R}^n \), in which the function \( v(\chi) \) is harmonic, contains the points \( \chi^{(1)} \) and \( \chi^{(2)}, \; |\chi^{(1)} - \chi^{(2)}| < r \). Let another domain bounded by the ellipsoid \( \Phi \) defined (in suitable coordinate system) by equation (46), lies in the domain \( \Omega \) together with its closure. Then the two-point mean value formula (43) is valid.

For \( n = 3 \) this theorem takes on a very transparent form.

**Theorem 5'.** (two-point mean value theorem for a harmonic function of three variables). For \( n = 3 \) in the conditions of the theorem 6 we have the formula
\[
v(\alpha) + v(0) = \frac{1}{2\pi} \iiint_{n_1^2 + n_2^2 + n_3^2 = 1} v\left(\frac{\alpha_1}{2}, \sqrt{r^2 - \alpha_1^2} \frac{n_1}{2}, \sqrt{r^2 - \alpha_1^2} \frac{n_2}{2}, \sqrt{r^2 - \alpha_1^2} \frac{n_3}{2}\right) \, d\omega_n +
\]
\[ + \frac{\sqrt{r^2 - \alpha_1^2}}{4\pi} \oint\oint u(\frac{\alpha_1}{2}, \frac{r\eta_1}{2}, \frac{\sqrt{r^2 - \alpha_1^2}\eta_2}{2}, \frac{\sqrt{r^2 - \alpha_1^2}\eta_3}{2}) \cdot \mu \, d\omega, \]

where \( \mu = (\frac{r\eta_1}{\sqrt{r^2 - \alpha_1^2}}, \eta_2, \eta_3) \), the sign \( \cdot \) denotes the scalar product in \( \mathbb{R}^3 \).

**Theorem 6.** For \( \chi^{(1)} = \chi^{(2)} = \chi^{(0)} \) in the case \( n = 1(\text{mod}2) \) the mean value formula (43) coincides with the well-known Gauss formula for the harmonic function for a sphere of radius \( r/2 \):

\[ v(\chi^{(0)}) = \frac{1}{|S_n|\delta^{n-1}} \oint_{|\xi - \chi^{(0)}| = \delta} v(\xi) \, d\omega, \delta = r/2. \]

This theorem is proved by the same scheme as in [10].

7. The Hadamard descent method and the two-point formula of the mean value for an elliptic equation in the Lobachevskiy space

We now apply the Hadamard descent method to the mean value formula (36). Let the function \( u = u(x) \) be a regular solution of an equation

\[ \Delta_\omega u + \left( \frac{n-1}{2} \right)^2 u = 0. \quad (47) \]

In accordance to the Hadamard descent method we carry out a fictitious variable \( t \), assuming

\[ u(x, t) = u(x, 0) = u(x), t \in R. \]

Then formally the function \( u(x, t) \) satisfies the wave equation (12) in the Lobachevskiy space. Therefore, it also satisfies the mean value formula (36). In this formula \( t^{(1)} \) and \( t^{(2)} \) should already be perceived as two parameter values satisfying condition (35). In fact, we have proved the two-point formula of the mean for equation (47): for any two points \( x^{(1)}, x^{(2)} \in \Pi \) and any two values \( t^{(1)} \) and \( t^{(2)} \in R \), satisfying condition

\[ (x_n^{(2)} e^{(1)} - x_n^{(1)} e^{(2)})(x_n^{(1)} e^{(1)} - x_n^{(2)} e^{(2)}) - e^{(1)+t^{(2)}}|\Delta x'|^2 > 0, \quad (48) \]

the regular solution \( u(x) \) of equation (47) satisfies the mean value formula

\[ e^{-1}(1-n)/2 u(x^{(1)}, t^{(1)}) + e^{-2}(1-n)/2 u(x^{(2)}, t^{(2)}) = B^\varphi_u(fu). \quad (49) \]

8. The Hadamard descent method and the two-point formula of the mean value for an elliptic equation on a sphere.

We consider the wave equation on a sphere, which has the following form

\[ \partial^2 u/\partial t^2 = \Delta_\omega u - ((n-1)/2)^2 u. \quad (50) \]

This equation was considered by P. Lax and R. Phillips [15], by I.A. Kipriyanov and L.A. Ivanov (see [13], [14], [16]). These authors refined the concept of the wave equation in a space with constant curvature. Following these authors, we call equation (50) the wave equation on the sphere.

We introduce a map \( S_n \times R \rightarrow R^{n+1} \) by means of formulas

\[ T = \tau_\pm(x, z, t) = \pm \sin t/(z \mp \cos t), \quad \chi = \chi_\pm(x, z, t) = \pm x/(z \mp \cos t). \quad (51) \]
where $\chi = (\chi_1, \ldots, \chi_n)$. Along with the coordinate transformations (51) we introduce a substitution of function

$$u(x, z, t) = (z - \cos t)^{-(n-1)/2}v(\chi(x, z, t), \tau(x, z, t)).$$

(52)

Transforms (51)–(52) are equivalent to the transformations introduced in the paper [15] in spherical coordinates on the sphere $S_n$. Therefore, it is appropriate to call them the P. Lax and R. Phillips transforms.

**Theorem 7.** The function $v(\chi, \tau)$ is a regular solution of the wave equation (6) in the cone $|\chi| < |\tau|$ of space $R^{n+1}$ of variables $\chi = (\chi_1, \ldots, \chi_n)$, $\tau$ if and only if the function (52), obtained from $v$ by means of Lax-Phillips transformations (51)–(52), is a regular solution of the wave equation (50) in the part of the space $S_n \times R$, determined by the condition $|z| > |\cos t|$, $t \in (0, 2\pi]$.

It is possible to prove the analogue of Theorem 1 for the wave equation on the sphere basing on the connection between the wave equations on the sphere and in Euclidean space, which is established by theorem 7.

**Theorem 8.** Let the function $u(x, t)$ be a regular solution of the spherical wave equation (50) in a part of the space $S_n \times R$, defined by the condition $|z| > |\cos t|$, $t \in (0, 2\pi]$. Let $(x^{(j)}, z^{(j)}, t^{(j)})$, $j = 1, 2$ be a pair of points in this part of the space $S_n \times R$, satisfying the condition

$$(z^{(1)}z^{(2)} + x^{(1)}x^{(2)} - \cos(t^{(1)} - t^{(2)}))/((z^{(1)} - \cos t^{(1)})(z^{(2)} - \cos t^{(2)})) > 0.$$  

(53)

Then following equality (the mean value formula)

$$B_\circ(f(x, \tau), z(\chi, \tau), t(\chi, \tau)) = B_\circ(f(\chi, \tau), u(x(\chi, \tau), z(\chi, \tau), t(\chi, \tau))),$$

(54)

occurs, where $f = \frac{((\tau^2 + |\chi|^2 + 2)^2 - 4\tau^2|\chi|^2/2)^{(1-n)/2}}{\sqrt{\Delta \tau^2 - |\Delta \chi|^2/2}}$, $\Delta \chi = \chi^{(1)} - \chi^{(2)}$, $\Delta \tau = \tau^{(1)} - \tau^{(2)}$, $\chi^{(j)} = \chi(x^{(j)}, t^{(j)})$, $\tau^{(j)} = \tau(x^{(j)}, t^{(j)})$, $j = 1, 2$, $u = u(x, t) = u(x(\chi, \tau), t(\chi, \tau))$, the operator $B_\circ$ is defined by formulas (4) and (5). Conversely, if the function $u(x, t) \in C^l(S_n \times R)$, where $l$ is defined by (8), satisfies the mean value formula (54), then it is a regular solution of the spherical wave equation (50).

We now apply the Hadamard descent method to the mean value formula (54). Let a function $u = u(x)$ be a regular solution of the equation

$$\Delta_\omega u + \left(\frac{n-1}{2}\right)^2 u = 0.$$  

(55)

In accordance to the Hadamard descent method, we introduce the dummy variable $t$ assuming $t \in R$.

Then formally the function $u(x, t)$ satisfies the wave equation (50) on the sphere. Therefore, it also satisfies the mean value formula (54). In this formula $t^{(1)}$ and $t^{(2)}$ should already be perceived as two parameter values that satisfy condition (53). Thus we proved the two-point mean value formula for equation (55): for any two points $(x^{(1)}, z^{(1)}), (x^{(2)}, z^{(2)}) \in S_n$ and any two values of $t^{(1)}, t^{(2)} \in (0, 2\pi)$ that satisfy condition (53) the regular solution $u(x)$ of equation (55) satisfies the mean value formula

$$(z^{(1)} + \cos t^{(1)})^{(n-1)/2}u(x^{(1)}, z^{(1)}, t^{(1)}) + (z^{(2)} + \cos t^{(2)})^{(n-1)/2}u(x^{(2)}, z^{(2)}, t^{(2)}) =$$

$$= B_\circ(f(x, \tau), u(x(\chi, \tau), z(\chi, \tau), t(\chi, \tau))).$$


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