Grassmann phase-space methods for fermions: uncovering classical probability structure

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The phase-space description of bosonic quantum systems has numerous applications in such fields as quantum optics, trapped ultracold atoms, and transport phenomena. Extension of this description to the case of fermionic systems leads to formal Grassmann phase-space quasiprobability distributions and master equations. The latter are usually considered as not possessing probabilistic interpretation and as not directly computationally accessible. Here, we describe how to construct c-number interpretations of Grassmann phase-space representations and their master equations. As a specific example, the Grassmann B representation is considered. We discuss how to introduce c-number probability distributions on Grassmann algebra and how to integrate them. A measure of size and proximity is defined for Grassmann numbers, and the Grassmann derivatives are introduced which are based on infinitesimal variations of function arguments. An example of c-number interpretation of formal Grassmann equations is presented.

I. INTRODUCTION

Phase-space approach to quantum mechanics has proved to be invaluable tool in such fields as quantum optics and trapped ultracold atoms $^1$-$^4$. This approach allows one to calculate quantum observable properties as averages of classical quantities over certain quasiprobability distributions. At the same time, the full quantum evolution often takes the form of a simple Fokker-Planck equation for these quasiprobability distributions. The latter property turns phase-space techniques into a stochastic simulation tool which was used to conduct Monte Carlo calculations of a number of many-body problems $^5$-$^7$.

When extending phase-space techniques to the case of fermions, a fundamental limitation is faced due to anticommutation of fermionic canonical variables. Because of this, the corresponding canonical operators cannot have c-number eigenvalues except zero. As a consequence it is impossible to construct c number quasiprobability distributions for them.

There are several workarounds to this problem $^3$-$^7$. The most formal and non-classical one is to change the notion of number $^8$. Fermionic canonical operators can have non-zero eigenvalues if we consider these eigenvalues as anticommuting numbers which are conventionally called Grassmann numbers (hereinafter, the term “Grassmann number” will be abbreviated as “g number”). This way it is possible to develop phase-space representations for fermions which bear remarkable analogy to bosonic ones. In particular, there are Grassmann quasiprobability distributions of the same types: $P$, $B$, $Q$ functions, s-ordered representations $^8$-$^{10}$, and also Wigner functions $^{11}$. More over, their master equations also look quite similar to the bosonic case. For example, in the case of real-time quantum dynamics with pairwise interactions, it is possible to derive master equation which looks similar to the Fokker-Planck equation for positive-P distribution $^8$-$^{10}$.

Nevertheless, there is important difference: all fermionic quasiprobability distributions are $g$ numbers. Grassmann numbers are dramatically different from c numbers: the latter are simple, the most basic things. However $g$ number is not simple: it has the structure of many-body correlated state. Every $g$ number defines a hierarchy of $n$-point functions, just as the physical state defines a hierarchy of correlations.

Because of this complexity, Grassmann phase space methods are usually considered as not possessing probabilistic interpretation and as not directly computationally accessible $^5$-$^7$-$^{12}$. At the same time, there are published works in which c number stochastic unravelings are constructed for Grassmann master equations $^8$-$^{10}$. These findings raise a number of questions. Firstly, the possibility of stochastic unraveling means that the Grassmann representations are in fact equivalent to certain c number quasiprobability distributions, with their own correspondence rules for observables, for quantum states, and for evolution equations. These equivalent c number quasiprobabilities were not considered in the literature. Secondly, in the work $^{10}$ there exists controversy with the earlier paper $^8$. This means that the nature of this stochastic unraveling is not completely understood.

The goal of this work is to clarify these questions. In fact, these questions have fundamental dimension: they imply that whatever formal one-time representation of quantum mechanics we invent, with respect to classical computability there are only c-number distributions, and nothing more. Such a unified view opens the way for general theorems to be formulated and proven. For example, generalized phase-space methods solve the sign problem $^{13}$: they allow us to represent quantum evolution as a stochastic process which can be simulated by Monte Carlo methods. Despite this success, in all practical realizations of simulation protocols the problem of quantum complexity reappears in one or another form e.g. numerical instability or exponential spread of Monte Carlo
trajectories. Is that a fundamental limitation or just imperfection of our knowledge?

Another field for which these questions are relevant, is the derivation and \( c \)-number stochastic unraveling of master equations for open quantum systems in fermionic environments [14, 15].

For the purpose of this work, we choose a particular \( g \)-number phase-space method, the Grassmann \( B \) representation [10]. This representation is analog of Drummond-Gardiner positive-\( P \) representation [16]. This specific choice does not reduce the generality of results: had we chosen another \( g \)-number representation, we would apply to it the same techniques as described in this work, and would come to analogous conclusions.

In section II we begin with a brief exposition of Grassmann \( B \) representation as it is known in literature and simultaneously recalling basic facts about Grassmann calculus. Next, we are going to construct probability and stochastic calculus on Grassmann algebra. In order to accomplish this, in section III we discuss such notions as: function of arbitrary \( g \)-number; proximity and size of \( g \) number; Grassmann derivatives based on infinitesimal variation of function argument; probability distribution on Grassmann algebra and its integral. In section IV we introduce \( g \)-number probability distribution into the Grassmann \( B \) representation. Actually, this way we obtain a novel \( g \) number representation which we call “stochastic Grassmann \( B \) representation”. It is shown that evolution of the emerging quasiprobability distributions is governed by Fokker-Planck equation (for systems with pairwise interactions). The corresponding stochastic equations are found to coincide with that derived in [10]. We conclude in section V.

II. GRASSMANN \( B \) REPRESENTATION

The \( g \)-number analog of Drummond-Gardiner positive-\( P \) representation [16] is Grassmann \( B \) representation. In this section we briefly review main results about \( B \) representation, simultaneously recalling basic notions of Grassmann calculus.

A. Definition of representation

Suppose we have a fermionic system with \( M \) modes (single-particle states). For each mode \( j \), there are associated creation \( \hat{a}_j^\dagger \) and annihilation \( \hat{a}_j \) operators. A Bargmann coherent state is defined as

\[
|e\rangle = \exp \left( - \sum_j e_j \hat{a}_j^\dagger \right) |0\rangle, \tag{1}
\]

where \( e_j \) is a Grassmann number. We consider \( g \) numbers \( e_j \) as linearly independent basis elements, with anticommuting multiplication law

\[
e_i e_j = -e_j e_i, \tag{2}
\]

and which generate the algebra of arbitrary \( g \) numbers

\[
g = G(0) + \sum_i G(i) e_i + \sum_{i_1 < i_2} G(i_1 i_2) e_i_1 e_i_2 + \ldots + \sum_{i_1 < \ldots < i_M} G(i_1 \ldots i_M) e_i_1 \ldots e_i_M. \tag{3}
\]

Every Grassmann number \( g \) can be unambiguously decomposed into even and odd parts,

\[
g = g^+ + g^-, \tag{4}
\]

where the even part \( g^+ \) consists of even powers of \( e_i \) in representation Eq. (3), and the odd part \( g^- \) consists of odd powers of \( e_i \) respectively. We also suppose that there is Grassmann complex conjugation operation

\[
()^*: e_j \rightarrow e^*_j, \tag{5}
\]

which is analog of complex conjugation for \( c \) numbers. The elements \( e_j^* \) generate arbitrary conjugated \( g \) numbers \( g^* \), a conjugated variant of \( g \). Although the most general \( g \) number contains both elements \( e_i \) and their conjugates \( e^*_i \), we do not encounter such \( g \) numbers in our problem, thus we assume that all \( g \) numbers contain either \( e_i \) or \( e^*_i \). To put it in other way, we are dealing only with “analytic” \( g \) numbers. We assume that complex conjugation respects anticommutativity: for any \( g \) numbers \( \alpha, \beta, \gamma \) we have:

\[
(\alpha \beta \gamma)^* = \gamma^* \beta^* \alpha^*. \tag{6}
\]

Due to this rule, Grassmann complex conjugation can be interpreted in a way which is consistent with Hermitian conjugation. For example:

\[
(\alpha \hat{a}_j \gamma)^\dagger = \gamma^* \hat{a}_j^\dagger \alpha^*. \tag{7}
\]

Annihilation and creation operators act upon coherent state as:

\[
\hat{a}_j |e\rangle = e_j |e\rangle, \quad \hat{a}_j^\dagger |e\rangle = -\hat{\gamma}^* j |e\rangle = |e\rangle \hat{\gamma}^* j, \tag{8}
\]

where \( \hat{\gamma}^* j \) and \( \hat{\gamma} j \) are the usual left and right Grassmann derivative operators with respect to element \( e_j \). Left and right derivatives with respect to complex conjugate elements \( e^*_j \) are denoted as \( \hat{\gamma}^* j \) and \( \hat{\gamma} j \). In order to maintain consistency with the properties [10] and [11], the complex conjugation of derivatives is defined as

\[
[\hat{\gamma}^* j]^* = \hat{\gamma} j, \quad [\hat{\gamma} j]^* = \hat{\gamma}^* j. \tag{9}
\]

For example:

\[
(\alpha \hat{\gamma} j^* \beta)^* = \gamma^* \beta^* \hat{\gamma}^* j \alpha^*. \tag{10}
\]
and
\[ (\alpha \tilde{a}_j^* \gamma \tilde{a}_j^\dagger) = \tilde{\gamma} \tilde{a}_j^\dagger \alpha^* \] \tag{11}

Analogously to bosonic Drummond-Gardiner positive-
\P_{\text{representation}} \cite{16}, we can double the dimension of
our Grassmann algebra by introducing additional basis elements \(e'_j, j = 1 \ldots M\). Then, one introduces the
non-diagonal coherent state projections \(|e\rangle \langle e'|\), where
\[ \langle e'| = (\langle e'\rangle^\dagger = \langle 0| \exp \left( - \sum_j \tilde{a}_j e'_j \right) \] 
Every number-conserving density operator can be expanded over these
projections as
\[ \hat{\rho} = \int de_1^* \ldots de_M^* de_1 \ldots de_1 B(e, e^*) |\langle e'| \rangle \] \tag{12}

where \(\int de_j\) is a standard Grassmann integration;
\(B(e, e^*)\) is even \(q\) number which is called \(B\) representa-
tion of density operator. We will denote the relation \[(12)\]
symbolically as
\[ B(e, e^*) = \{\tilde{\rho}\}_B \] \tag{13}

In work \cite{16} it is shown that \(B(e, e^*)\) always exists and
is unique.

\subsection*{B. Equation of motion}

Let us consider a quantum system with Hamiltonian
\[ \hat{H} = \tilde{a}_p^\dagger T_{pq} \tilde{a}_q - \frac{1}{4} \tilde{a}_p^\dagger \tilde{a}_q^\dagger V_{pqrs} \tilde{a}_r \tilde{a}_s \] \tag{14}

From now on, summation over repeated indices is implied. Real
time evolution of density operator is governed by von Neumann equation
\[ i \partial_t \hat{\rho} = \left[ \hat{H}, \hat{\rho} \right] \] \tag{15}

We can use the properties of coherent states \cite{13} in order
to find master equation for the corresponding \(B\) representa-
tion. In particular, by Grassmann integration by parts it can be shown that \cite{16}
\[ \{\tilde{a}_j \hat{\rho}\}_B = e_j \{\tilde{\rho}\}_B, \quad \{\tilde{a}_j^\dagger \hat{\rho}\}_B = \tilde{\gamma} \tilde{a}_j \{\tilde{\rho}\}_B, \] \tag{16}
\[ \{\tilde{\rho} \tilde{a}_j\}_B = \{\tilde{\rho}\}_B \tilde{\gamma}^\dagger, \quad \{\tilde{\rho} \tilde{a}_j^\dagger\}_B = \{\tilde{\rho}\}_B e_j^* \] \tag{17}

Here, \(\tilde{\gamma}^\dagger\) is the right Grassmann derivative with respect
to element \(e'_j\). We can apply these rules for von
Neumann equation \[(15)\], and find:
\[ \partial_t \{\tilde{\rho}\}_B = - \partial_q \left( i T_{pq} e'_q \right) B - \{\tilde{\rho}\}_B \left[ \tilde{\gamma} \tilde{a}_j \frac{i}{2} V_{pqrs} e_r e_s \right]^* \] 
\[ + \frac{1}{2} \partial_q \tilde{\gamma} \left( \frac{i}{2} V_{pqrs} e_r e_s \right) \{\tilde{\rho}\}_B \] 
\[ + \frac{1}{2} \{\tilde{\rho}\}_B \left[ \partial_q \tilde{\gamma} e_j^* \left( \frac{i}{2} V_{pqrs} e_r e_s \right) \right]^* \] \tag{18}

Now, if we compare this equation with the classical proba-
bility \(c\)-number Fokker-Planck equation, expressed in
terms of complex variables \cite{17},
\[ \partial_t P = - \partial_p \partial_q \partial_{q'} A_{pq} + \frac{1}{2} \partial_p \partial_q B_{pq} P \] 
\[ + \partial_q \partial_{q'} B_{pq} P + \frac{1}{2} \partial_p \partial_{q'} B_{pq} P \] \tag{19}

we observe that \(B\) representation master equation \[(18)\]
looks like anticommuting analog of Fokker-Planck equation \[(19)\]. This analogy encourages us to find Grassmann
stochastic process which has \(\{\tilde{\rho}\}_B\) as its “probability”
density. It fact, it has been done in \cite{16}, but without
considering the emerging probability distributions. In
the following sections we will do it by explicitly intro-
ducing \(c\)-number probability distribution into the \(B\) rep-
resentation \[(12)\].

\section*{III. Grassmann Calculus Revisited}

We want to construct a classical stochastic interpreta-
tion of the \(B\) master equation \[(18)\]. Before we do it, we
need to carry out some preparatory work. The classical
stochastic process is defined through infinitesimal incre-
ments of the process variables. The appearance of term
“infinitesimal” means we need to discuss how to intro-
duce the norm of arbitrary \(g\) number \(g\). Moreover, the
behaviour under infinitesimal variations is described in
terms of derivatives. However, the conventional Grass-
mann derivative operators \(\partial_j\) and \(\partial_{j'}\) are defined as
formal algebraic manipulations on the basis elements \(e_j\).
Therefore, we need to find Grassmann derivatives which
are connected with infinitesimal variations. Next, in order
to introduce probability distributions on Grassmann
algebra, we need to discuss the notion of function of
arbitrary \(g\) number and how to integrate it.

\subsection*{A. Norm of Grassmann number}

In \cite{16} p. 49 it is argued that \(g\) numbers do not have
notions of size and magnitude, thus there is no notion of
proximity for them. Nevertheless, we believe that this
is not correct.

Since “analytic” Grassmann numbers are defined ac-
\textit{cording to} \textit{Eq.} \[(15)\], we see that each \(g\) number is equivalent to a hierarchy of \(n\)-point functions \(G(t_1 \ldots i_n)\).
Physically, we can interpret \(g\) number as a quantum
many-body state, and the functions \(G(t_1 \ldots i_n)\) can be interpreted as its \(n\)-particle amplitudes. Due to anticom-
mutation between the basis elements, \(G(t_1 \ldots i_n)\) are not
unique: we can represent \(n\)-point function as a sum
\[ G(t_1 \ldots i_n) = G_A (t_1 \ldots i_n) + Z (t_1 \ldots i_n), \] \tag{20}

where \(G_A (t_1 \ldots i_n)\) is completely antisymmetric, and
\(Z (t_1 \ldots i_n)\) is arbitrary but which has the symmetry of
any Young tableau except complete antisymmetry. We can introduce the norm of \( g \) number as the sum of norms of its \( n \)-particle amplitudes
\[
\|g\|^2 \equiv |G_A(0)|^2 + |G_A(1)|^2 + \ldots + |G_A(M)|^2 .
\] (21)

Then, the distance between two \( g \)-numbers \( g \) and \( h \) is defined as \( \|g - h\|^2 \). Such a definition is appealing from physical point of view, since the two quantum states should be regarded as similar if all their \( n \)-point functions (correlations) are similar. If we choose \( n \)-point-function norm as the Hilbert-Schmidt norm,
\[
\|G_A(n)\|^2 = \sum_{i_1 < \ldots < i_n} |G_A(i_1 \ldots i_n)|^2 ,
\] (22)
then our \( g \) number norm satisfies all the expected and reasonable inequalities,
\[
\|g + h\| \leq \|g\| + \|h\| , \quad \|gh\| \leq \|g\| \|h\| ,
\] (23)
and if
\[
\|g - h\| = 0 \quad \text{then} \quad g = h .
\] (24)

From a physical point of view, the norm \(\|g\|^2\) has the meaning of (unnormalized) probability of observing any \( n \)-point configuration, and \(\|g\|^2\) is its normalization factor.

B. Functions of Grassmann numbers

1. Algebraic functions

The major objects of our theory, \( B \) function \( B(e, e^\dagger) \), coherent state dyadic \( \langle e | e^\dagger \rangle \), and master equation \( [8] \), are formulated as depending on basis elements \( e_j \) and \( e^*_j \). This means that in a stochastic interpretation, \( e_j \) and \( e^*_j \) should be replaced with stochastic process variables \( g_j \) and \( g^*_j \), which should be considered as arbitrary \( g \) numbers. Therefore, we need to consider functions of arbitrary grassmann numbers, e.g. \( |g\rangle \langle g^*| \). General analytic function \( f \) of arbitrary \( g \) numbers \( g_j \) is a sum of monomials
\[
(g^{+}_{i_1})^{p_1} \ldots (g^{+}_{i_p})^{p_n} g^{-}_{j_1} \ldots g^{-}_{j_m} ,
\] (25)
where \( p_k \) are nonnegative integer powers since in general \( (g^{+}_{k})^{2} \neq 0 \); however the indices \( j_1 \ldots j_m \) should all be different since \( (g^{-}_{k})^{2} = 0 \). We call such functions algebraic since they can be expressed in terms of algebraic operations: multiplication, addition, and taking even/odd parts.

2. Non-algebraic functions

In order to define \( c \)-number stochastic process, we also need to introduce classical probabilities on Grassmann numbers. Apparently they cannot be expressed in terms of algebraic operations. However, such non-algebraic functions of grassmann numbers naturally depend on \( n \)-point functions. For a given \( g \) number \( g \), we will denote its \( n \)-point function by the corresponding capital letter, \( G(i_n) \), where \( i_n = (i_1 \ldots i_n) \); the set of all \( G(i_n) \) of a given order \( n \), for all values of \( i_n \), will be denoted by \( G(n) \); and the full hierarchy \( (G(0), \ldots , G(M)) \) will be designated by \( G \). Therefore, a classical probability \( P \) depending on \( g \) will be denoted as \( P(G) \). Observe that we take antisymmetric part of \( G \).

C. Metric Grassmann derivatives

Now we have the notion of proximity and magnitude. We can introduce the Grassmann derivatives which are based on infinitesimal variations of arguments. In order to distinguish them from the ordinary formal Grassmann derivatives, we call them “metric Grassmann derivatives”. According to standard calculus, the derivative of function \( f \) is defined through its local behaviour
\[
f(g + \delta) - f(g) = \sum_{i_n} \Delta_A(i_n) \partial_{G_A(i_n)} f(g) + O(\|\delta\|^2) ,
\] (26)
where \( \Delta_A(i_n) \) is antisymmetric part of \( n \)-point function \( \Delta(i_n) \) of \( \delta \). This definition is precise. However, it is insufficient since it ignores the algebraic structure and the commutation properties of \( \delta \). This is because we can write
\[
\delta = \delta^+ + \delta^- ,
\] (27)
and substitute it into \( f(g + \delta) \). Since \( f(g + \delta^+ + \delta^-) \) is a polynomial, we expand it, and move all \( \delta^+ \) and \( \delta^-\) to the left (or to the right) respecting their commutation properties. Keeping only the first order terms in \( \delta^+ \) and \( \delta^- \), we arrive at the following representations of local behaviour:
\[
f(g + \delta) - f(g) = \delta^+ \partial^+_g f(g) + \delta^- \partial^-_g f(g) + O\left(\|\delta\|^2\right) ,
\] (28)
and
\[
f(g + \delta) - f(g) = f(g) \partial^+_g \delta^+ + f(g) \partial^-_g \delta^- + O\left(\|\delta\|^2\right) ,
\] (29)
where we introduce the odd left \( \partial^-_g \), the odd right \( \partial^-_g \), the even left \( \partial^+_g \), and the even right \( \partial^+_g \) metric Grassmann derivatives. The left even Grassmann derivative \( \partial^+_g \) has the following properties:
\[
\partial^+_g c = 0 , \quad \partial^+_g g^*_j = \delta_{ij} , \quad \partial^+_g g_j = 0 ,
\] (30)
where \( c \) is a \( g \) number constant. More complex objects are differentiated according to linearity,
\[
\partial^+_g \{c_1 f_1(g) + c_2 f_2(g)\} = c_1 \partial^+_g f_1(g) + c_2 \partial^+_g f_2(g) ,
\] (31)
and by employing the following commutation relation:

\[ \overrightarrow{\partial}_g f = \left( \overrightarrow{\partial}_g f \right) + f \overrightarrow{\partial}_g. \] (32)

The left odd derivative \( \overrightarrow{\partial}_{g_i} \) has the following properties:

\[ \overrightarrow{\partial}_{g_i} c = 0, \quad \overrightarrow{\partial}_{g_i} g_j = \delta_{ij}, \quad \overrightarrow{\partial}_{g_i} g_j^+ = 0. \] (33)

Compound objects are differentiated according to the anti-\[ \text{linearity} \]

\[ \overrightarrow{\partial}_g \{c1 f1 (g) + c2 f2 (g)\} = \overleftarrow{\partial}_g f1 (g) + \overleftarrow{\partial}_g f2 (g), \] (34)

and the anticommuta relation

\[ \overrightarrow{\partial}_g f = \left( \overrightarrow{\partial}_g f \right) + \overleftarrow{\partial}_g f. \] (35)

Here, for each Grassmann number \( g \) we have introduced its involution \( \overline{g} \) as negation of its odd part:

\[ \overline{g} = g^+ - g^- . \] (36)

The properties of the right derivatives are obtained through complex conjugation, according to the following relations:

\[ \left[ \overrightarrow{\partial}^\pm \right]^* = \overleftarrow{\partial}^\pm, \quad \left[ \overleftarrow{\partial}^\pm \right]^* = \overrightarrow{\partial}^\pm . \] (37)

Different derivatives have the following commutation relations:

\[ \overrightarrow{\partial}_{g_i} \overleftarrow{\partial}_{g_j} = \overleftarrow{\partial}_{g_j} \overrightarrow{\partial}_{g_j} = (\pm 1) \overrightarrow{\partial}_{g_i} \overleftarrow{\partial}_{g_j} . \] (38)

Left and right derivatives are related as:

\[ \overrightarrow{\partial}_g f (g) = f (g) \overleftarrow{\partial}_g , \quad \overleftarrow{\partial}_g f (g) = - \overrightarrow{\partial}_g f (g) . \] (39)

There is relation between the ordinary calculus derivatives and the metric Grassmann derivatives:

\[ \sum_{i_n} \Delta_A (i_n) \partial_{G_A(i_n)} f (g) = \left\{ \delta^+ \overrightarrow{\partial}_g^+ + \delta^- \overleftarrow{\partial}_g^- \right\} f (g) \]

\[ = f (g) \left\{ \overrightarrow{\partial}_g^+ \delta^+ + \overleftarrow{\partial}_g^- \delta^- \right\} . \] (40)

D. Integration over Grassmann algebra

In order to work with classical probability we need to integrate it over Grassmann numbers. Therefore, we introduce integration in the space of \( n \)-point functions

\[ \int dG_A P (G_A) := \prod_{i=1}^M \prod_{i_n \subset C} dG_A (i_n) dG^*_A (i_n) P (G_A) , \] (41)

where \( \prod_{i_n} \) means the product over all the ordered sequences \( i_1 < \ldots < i_n \). Using our definitions, it can be shown that there is the following integration by parts formula

\[ \int dG_A f (g) \left\{ \sum_{i_n} \partial_{G_A (i_n)} H (i_n) \right\} P (G_A) \]

\[ = - \int dG_A P (G_A) \left\{ h^+ \overrightarrow{\partial}_g^+ + h^- \overleftarrow{\partial}_g^- \right\} f (g) \]

\[ - \int dG_A f (g) \left\{ \overrightarrow{\partial}_g^+ h^+ + \overleftarrow{\partial}_g^- h^- \right\} P (G_A) . \] (42)

From now on we assume that \( n \)-point functions are always antisymmetric, and the subscript \( A \) will be omitted.

IV. STOCHASTIC GRASSMANN \( B \) REPRESENTATION

A. Definition

Now we are ready to introduce the stochastic interpretation of formal Grassmann \( B \) representation master equation [18]. The idea is that we introduce random \( g \) number vectors \( g \) and \( g' \). The coherent state dyadic is considered to be a function of these vectors, \( \langle g | g' \rangle \). These coherent states have the following properties:

\[ \hat{a}_i^\dagger |g\rangle = - \overrightarrow{\partial}_{g_i} |g\rangle = (\mp 1) |g\rangle \overrightarrow{\partial}_{g_i} , \] (43)

\[ \hat{a}_i |g\rangle = \hat{a}_i \left( 1 - g_p \hat{a}_p^\dagger \right) |0\rangle = - \overleftarrow{g}_\hat{a}_i \hat{a}_p^\dagger |0\rangle = - \overleftarrow{g}_i |0\rangle . \] (44)

The last equation is problematic: its form is not suitable for construction of a phase-space representation. However, if \( g \) is odd, so that \( g_p = g_p^- \), then we obtain

\[ \hat{a}_i |g^-\rangle = g_i^- |0\rangle = g^+ \left( 1 - g_p^- \hat{a}_p^\dagger \right) |0\rangle = g_i^- |0\rangle . \] (45)

We see that suitable differential correspondences are realized only when \( g \) belongs to the odd sector. Therefore, from now on we impose this restriction on \( g \) and \( g' \).

The conjugated relations are:

\[ \langle (g^-)\rangle \hat{a}_i = - \langle (g^-)\rangle \overrightarrow{\partial}_{g_i} = \overleftarrow{\partial}_{g_i} \langle (g^-)\rangle \] (46)

\[ \langle (g^-)\rangle \hat{a}_i^\dagger = \langle (g^-)\rangle (g_i^-)^* \] (47)

At the time moment \( t = 0 \), the random vectors \( g \) and \( g' \) should coincide with the vectors of basis elements, \( g = e \) and \( g' = e' \). However, at later time they begin to diffuse. We express this fact by inserting integration over probability distribution into Grassmann \( B \) representation [12]:

\[ \hat{\rho} (t) = \int dG dG^* P (G, G^* : t) \]

\[ \times \int de_1^* \ldots de_M^* de_M \ldots de_1 B (e, e^*) |g\rangle \langle g'| , \] (48)
with the initial condition

$$P(G, G^*; 0) = \delta(G - E) \delta(G^* - E^*)$$ \hspace{1cm} (49)$$

Here, bold capital letters designate vectors $G = (G_0 \ldots G_M), E = (E_0 \ldots E_M)$ etc.; symbol $G_j$ means hierarchy of $n$-point functions for $g_j$. In fact, our Grassmann representation is equivalent to ordinary $c$ number phase space representation

$$\hat{\rho}(t) = \int dGdG^* P(G, G^*; t) \hat{\Lambda}(G, G^*)$$ \hspace{1cm} (50)$$

with the overcomplete operator basis

$$\hat{\Lambda}(G, G^*) = \int de_1^* \ldots de_M^* de_e^{*} B(e, e^*) |g \rangle \langle g^*|.$$ \hspace{1cm} (51)$$

We call this representation “stochastic Grassmann B representation”.

**B. Equation of motion**

We denote symbolically the relation (50) as

$$P(G, G^*; t) = \{\hat{\rho}(t)\}_P(G, G^*)$$ \hspace{1cm} (52)$$

In order to construct master equation for stochastic grassmann $B$ representation, we proceed analogously to section 113; we find expressions for \{\hat{a}_i^\dagger \hat{a}_j \hat{\rho}(t)\}_P$ etc. Note that since integration by parts formula (42) contains only the combinations $h^{-1} \partial_g^{-1} g$ and $\partial_h^{-1} h^{-1}$, there is no rules for non-conserving terms like \{\hat{a}_j \hat{\rho}(t)\}_P. Using the coherent state properties (43), (45), (46), and (47), and integrating by parts according to (42), we find:

$$\{\hat{a}_i^\dagger \hat{a}_j \hat{\rho}(t)\}_P = - \sum_{i_n} \partial G_j(i_n) G_j(i_n) \{\hat{\rho}(t)\}_P,$$ \hspace{1cm} (54)$$

$$\{\hat{\rho}(t) \hat{a}_i^\dagger \hat{a}_j\}_P = - \sum_{i_n} \partial G^*_j(i_n) G^*_j(i_n) \{\hat{\rho}(t)\}_P.$$ \hspace{1cm} (55)$$

Representation for quartic terms like \{\hat{a}_l^\dagger \hat{a}_k \hat{a}_i \hat{a}_j \hat{\rho}(t)\}_P can be found by repeated application of Eqs. (54)-(55) and by using the anticommutation relation

$$\partial_{g_s} g_s = \delta_{ps} - g_s \partial_{g_p}.$$ \hspace{1cm} (56)$$

In the stochastic Grassmann B representation, von Neumann equation (17) assumes form:

Stratonovitch sense

$$dG_p(i_n) = -i \sum_q T_{pq} G_q(i_n) dt + \frac{i}{4} \sum_{lq} V_{plq} G_q(i_n) dt$$
$$+ \sqrt{\frac{\omega_p}{2l}} \sum_{\gamma q} O^{(\gamma)}_{pq} G_q(i_n) dX_\gamma, \hspace{1cm} (60)$$

$$dG'_p(i_n) = -i \sum_q T_{pq} G'_q(i_n) dt + \frac{i}{4} \sum_{lq} V_{plq} G'_q(i_n) dt$$
$$+ \sqrt{\frac{\omega_p}{2l}} \sum_{\gamma q} O^{(\gamma)}_{pq} G'_q(i_n) dY_\gamma. \hspace{1cm} (61)$$
Here we have decomposed the pair potential as \[ V_{pqrs} = \sum_{\gamma} \omega_{\gamma} O^{(\gamma)}_{pr} O^{(\gamma)}_{qs}. \] (62)

The real Wiener increments \( dX_{\gamma} \) and \( dY_{\gamma} \) obey to the standard statistics

\[
E[dX_{\gamma}] = E[dY_{\gamma}] = E[dX_{\gamma}dY_{\mu}] = 0, \tag{63}
\]

\[
E[dX_{\gamma}dX_{\mu}] = E[dY_{\gamma}dY_{\mu}] = \delta_{\gamma\mu}. \tag{64}
\]

We note that Eqs. (60) and (61) actually form a set of equations for each of the \( n \)-point functions \( G_p(i_1 \ldots i_n) \) and \( G'_p(i_1 \ldots i_n) \), which are uncoupled for different \( n \) and even for different values of \( i_1 \ldots i_n \). We can multiply each equation for \( G_p(i_1 \ldots i_n) \) and \( G'_p(i_1 \ldots i_n) \) by \( \varepsilon_i \ldots \varepsilon_n \) and \( \varepsilon'_i \ldots \varepsilon'_n \), then sum them up over \( i_1 \ldots i_n \) and over \( n \), and obtain a system of coupled stochastic equations for odd grassmann numbers \( g_1 \ldots g_M \) and \( g'_1 \ldots g'_M \):

\[
dg_p = -i \sum_q T_{pq} q dt + \frac{i}{4} \sum_{lq} V_{lpql} q dt + \sqrt{\frac{\omega_q}{2i}} \sum_{\gamma q} O^{(\gamma)}_{pq} q dX_{\gamma},
\]

\[
dg'_p = -i \sum_q T_{pq} q' dt + \frac{i}{4} \sum_{lq} V_{lpql} q' dt + \sqrt{\frac{\omega_q}{2i}} \sum_{\gamma q} O^{(\gamma)}_{pq} q' dY_{\gamma}. \tag{65}
\]

In fact, the equations are the same as those obtained in \[10\], except the notational difference for Hamiltonian terms \[14\] and that our equations are in Stratonovich form, whereas equations in \[21\] are in Itô form. However, for numerical calculations we always have to interpret these equations in the \( n \)-point picture [Eqs. (60) and (61)].

V. CONCLUSIONS

A few conclusions can be drawn from the results of this study.

Grassmann numbers are objects of high computational complexity but they are not as abstract as they are usually considered. We can interpret \( g \) number as a physical many-body state, with a hierarchy of correlations. This leads to natural notions of size and proximity between them. With the help of these notions, we were able to develop a \( c \)-number stochastic calculus on Grassmann algebra.

Each \( g \)-number phase-space representation can be converted into \( c \)-number phase-space representation by introducing probability distributions on Grassmann algebra. This way many of \( g \)-number methods can be made accessible to computations.

We put forward a conjecture that whatever abstract algebraic one-time representation of quantum mechanics is invented, with respect to classical computability we can always reformulate it as a \( c \)-number phase-space representation. This opens up the road for general results to be established.

We believe that the methods introduced in this work will be useful when considering such problems as interpretation and \( c \)-number stochastic unraveling of (markovian or non-markovian) master equations for open systems in fermionic environment.

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