Zero-energy states of fermions in the field of Aharonov–Bohm type in 2+1 dimensions

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The quantum-mechanical problem of constructing a self-adjoint Hamiltonian for the Dirac equation in an Aharonov–Bohm field in 2+1 dimensions is solved with taking into account the fermion spin. The one-parameter family of self-adjoint extensions is found for the above Dirac Hamiltonian with particle spin. The correct domain of the self-adjoint Hamiltonian extension selecting by means of acceptable boundary conditions can contain regular and singular (at the point \( r = 0 \)) square-integrable functions on the half-line with measure \( rdr \). We argue that the physical reason of the existence of singular solutions is the additional attractive potential, which appear due to the interaction between the spin magnetic moment of fermion and Aharonov–Bohm magnetic field. For some range of parameters there are bound fermionic states. It is shown that fermion (particle and antiparticle) states with zero energy are intersected what signals on the instability of quantum system and the possibility of a fermion-antifermion pair creation by the static external field.

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I. INTRODUCTION

The quantum Aharonov–Bohm (AB) effect \[1\] has been analyzed in various physical situations in numerous works (see e.g., Ref. \[2\]). When an electron travels in an Aharonov-Bohm field in which the magnetic flux is restricted to a small-radius tube topologically equivalent to a cylinder, the electron wave function acquires a geometric phase. The AB vector potential can produce observable effects because the relative (gauge invariant) phase of the electron wave function, correlated with a nonvanishing gauge vector potential in the domain where the magnetic field vanishes, depends on the total magnetic flux \[3\].

When the external field configuration has the cylindrical symmetry, a natural assumption is that the relevant quantum mechanical system is invariant along the symmetry (z) axis and the system then becomes essentially two-dimensional in the \(xy\) plane. So, such models can be reduced to the (2+1)-dimensional ones. In Refs. \[4\] it was observed that solutions for the Dirac equation in an Aharonov–Bohm field in 2+1 dimensions are the Dirac equation solutions in infinite cosmic strings in 3+1 dimensions. Solutions to the two-component Dirac equation in the AB potential were first obtained and discussed by Alford and Wilczek in Ref. \[5\] in a study of the interaction of cosmic strings with matter. Relativistic quantum AB effect was studied in Ref. \[6\] for the free and bound fermion states by means of exact analytic solutions of the Dirac equation in 2+1 dimensions for a combination of an AB potential and the Lorentz three-vector and scalar Coulomb potentials. The effect of vacuum polarization in the field of infinitesimally thin solenoid is recently investigated in \[7\] and a wonderful phenomenon is revealed: the induced current is finite periodical function of the magnetic flux.

In \[10\] it was observed that the Hamiltonian for the Aharonov–Bohm problem is essentially singular and hence it cannot be immediately defined in the domain \([0, \infty)\) for any differentiable and enough rapidly decreasing (at \(r \to \infty\)) functions in the Hilbert space of square-integrable functions on the half-line with measure \(rdr\). Usually, the Hamiltonians are symmetric operators in its natural domain. The problem of constructing a self-adjoint Hamiltonian is to find all self-adjoint extensions of given symmetric operator and then to select correct self-adjoint extension. The correctness of the known Aharonov–Bohm solutions for scattering problem was analyzed for spinless particles in \[10\], in which a self-adjoint extension of the Hamiltonian is selected by physical condition—"the principle of minimal singularity": the Hilbert-space functions for which the Hamiltonian is defined must not be singular.

A one-parameter self-adjoint extension of the Dirac Hamiltonian in 2+1 dimensions in the pure AB field was constructed by means of acceptable boundary conditions in \[11\]. In \[12\] it was shown that the domain of the self-adjoint Hamiltonian extension can contain, together with regular, square-integrable functions on the half-line with measure \(rdr\) and singular at \(r = 0\) as well as it was constructed a formal solution, which describes a bound fermion state in the field of cosmic string.

Note that the usual four-component Dirac equation in 2+1 dimensions (in the absence of \(z\) coordinate) decouples into two uncoupled two-component Dirac equations for spin projection \(s = +1\) and \(s = -1\). Thus, the two-component Dirac equation describes the planar motion of relativistic electron having only one projection of three-dimensional spin vector. The upper ("large") and lower ("small") components of the two-component wave function are interpreted in terms of positive- and negative-energy solutions of the Dirac equation in 2+1 dimensions. The particle spin in the two-component Dirac equation was artificially introduced in \[13\] as a new parameter. The term including this new parameter appears in the form of an additional delta-function interaction of spin with magnetic field in the Dirac equation squared.

In this paper we would like to study how the fermion spin affects the properties of a bound Dirac fermion in an Aharonov-Bohm field in 2+1 dimensions. We find the wave function of bound states, derive an equation implicitly determining the fermion energy and study the behavior of relativistic energy levels of spin-one-half fermion in the AB field. It is shown that the lowest energy levels of particles and antiparticles intersect upon adiabatic variation of the magnetic flux \(\Phi\) between two integers \(n\), i.e. when \(\Phi = \Phi_0(n + 1/2)\), where \(\Phi_0 \equiv 2\pi/|e|\) is the elementary magnetic flux and \(e\) is the fermion (electron) charge.

The spectrum of Dirac’s equation under consideration is symmetric with respect to the change of the sign of energy and the states with precisely zero energy exist. Jackiw and Rebbi \[12\] were observed that, in a charge conjugation symmetric theory of one-dimensional Dirac fermions interacting with a solitonic background field (the kink), the vacuum acquires a fractional fermion charge \(\pm 1/2\) due to the existence of fermion states with zero energy (zero modes). The states with zero energy exactly in the middle of spectrum have been known to exist when the mass-term forms a vortex in the configurational space \[13\]. The problem of zero-energy states of the two-dimensional Dirac Hamiltonian with a unit vortex in the mass-term is considered in the presence of pseudo magnetic field in the context of fractionalization by Jackiw and Pi in \[14\] and in the presence of pseudo as well as true magnetic field in \[15\].
In the presence of a vector potential, the Dirac Hamiltonian does not exhibit a charge conjugation symmetry since a charge coupling treats particles and antiparticles differently. So the existence of fermion states with zero energy does not necessarily imply a fractional fermion number [16] but the intersection of energy levels signals on the instability of quantum system.

It is well to note that the possibility of existence of weakly bound electron states was shown in [17, 18] due to the interaction between the three-dimensional spin magnetic moment of electron and magnetic field of infinitely thin solenoid with applying solutions of the Pauli equation in 3+1 dimensions. We shall adopt the units where \( c = \hbar = 1 \).

II. SOLUTIONS TO THE DIRAC EQUATION IN 2+1 DIMENSIONS IN AN AHARONOV-BOHM FIELD FOR THE SCATTERING PROBLEM

The Dirac equation for a fermion of mass \( m \) and charge \( e > 0 \) in 2+1 dimensions in the potential \( A_\mu \) is

\[
(\gamma^\mu P_\mu - m)\Psi = 0,
\]

(1)

Here the Dirac \( \gamma^\mu \) matrices are conveniently defined in terms of the Pauli spin matrices as (see, [11])

\[
\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2,
\]

(2)

and \( s \) is a new parameter characterizing twice the spin value \( s = \pm 1 \) for spin “up” and “down”, respectively, (see, [11]), \( \hat{P}_\mu = -i\partial_\mu - eA_\mu \) is the generalized electron momentum operator.

We seek solutions of Eq. (1) in an Aharonov–Bohm field

\[
A^0 = 0, \quad A_r = 0, \quad A_\varphi = \frac{B}{r}, \quad r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x)
\]

(3)
in the form

\[
\Psi(t, x) = \frac{1}{\sqrt{2\pi}} \exp(-iEt + il\varphi)\psi(r, \varphi),
\]

(4)

where \( E \) is the electron energy, \( l \) is an integer, and \( \psi(r, \varphi) \) is a two-component function (i.e. a 2-spinor)

\[
\psi(r, \varphi) = \left( f_1(r) f_2(r)e^{is\varphi} \right).
\]

(5)

The wave function \( \Psi \) is an eigenfunction of the conserved total angular momentum \( J_z \equiv L_z + \frac{s}{2} \sigma_3 \), where \( L_z \equiv -i\partial/\partial \varphi \) with eigenvalue \( j = l + s/2 \).

It is seen that the radial Hamiltonian \( h_r \) is singular at point \( r = 0 \) in the Aharonov–Bohm field and it cannot be immediately defined for the class of functions to be self-adjoint operator. So, we need to solve the eigenvalue problem for this operator, which is

\[
h_r \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = \begin{bmatrix} m & sdf/dr + (l + \mu + s)/r \\ -sdf/dr + (l + \mu)/r & -m \end{bmatrix} \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = E \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix},
\]

(6)

where \( \mu \equiv eB \).

Because of the existence of finite magnetic flux inside solenoid \( \Phi = 2\pi B \) the term including the spin parameter appears in the form of an additional delta-function interaction of spin with magnetic field of solenoid

\[
\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} = B\pi \delta(\mathbf{r})
\]

(7)
in the Dirac equation (6) squared. Note that if \( \mu \) is an integer \( n \), then the magnetic field flux is quantized as \( \Phi = \Phi_0 n \). For the cosmic strings considered in [7], \( \Phi = e/Q \), where \( Q \) is the Higgs charge. The additional (spin) potential

\[
-seB\delta(\mathbf{r})/r
\]

(8)
in the Dirac equation (6) squared will be taken into account by boundary conditions.

If \( \mu \) is nonintegral then the regular solutions at \( r = 0 \) of Eq. (6) for \( E^2 > m^2 \) are

\[
\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = \frac{1}{N} \begin{pmatrix} \sqrt{E + mJ_\nu(pr)} \\ \sqrt{E - mJ_{\nu+s}(pr)} \end{pmatrix},
\]

(9)
Here $N$ is a normalization factor, $\nu = |l + \mu|$, $p = \sqrt{E^2 - m^2}$, $\nu + s > 0$ and $J_\nu(pr)$ is the regular Bessel function. Singular Bessel functions at $r = 0$ but square-integrable on the half-line with measure $rdr$ also are admissible quantum-mechanical solutions of Eq. (9). It is seen from Eq. (9) that singular solutions (localized, obviously, at the origin) can appear if only additional potential (8) is attractive. Besides, rejecting singular solutions leads to a loss of completeness in the angular basis [5, 19].

Since the spin term is invariant with respect to transformations $e \rightarrow -e$, $s \rightarrow -s$, we can consider the case $e > 0$ only. Let $\mu > 0$, then potential (8) is attractive for $s = 1$ and repulsive for $s = -1$. So, singular solutions (localized at $r = 0$) can appear if $\mu > 0$, $B > 0$, $s = 1$ (particle state) or $\mu < 0$, $B > 0$, $s = -1$ (antiparticle state). Written

$$\mu = [\mu] + \gamma \equiv n + \gamma,$$

where $[\mu] \equiv n$ denotes the largest integer $\leq \mu$, and

$$1 > \gamma > 0,$$

we can easily find, that singular, square-integrable functions are the Bessel functions of the order $\gamma - 1$ (upper spinor component) and $-\gamma$ (lower spinor component) with $l = -n - 1$, $n \geq 0$.

III. SELF-ADJOINT EXTENSIONS FOR THE RADIAL DIRAC HAMILTONIAN

We must construct the self-adjoint extensions of the radial Dirac Hamiltonian $h_r$ and, then, select a needed extension by means of some physical condition. The problem is solved for symmetric operators by the method of deficiency indices developed by von Neumann (see, for example, [20, 22]). In our problem $h_r$ is symmetric operator if, for arbitrary spinors $f(r)$ and $g(r)$,

$$\int_0^\infty g^\dagger(r)h_rf(r)rdr = \int_0^\infty [h_rg(r)]^\dagger f(r)rdr,$$

what leads to the following boundary condition [5]

$$\lim_{r \to 0} rg^\dagger(r)\sigma_2 f(r) = 0.$$  

A symmetric operator $h$ is self-adjoint, if its domain $D(h)$ coincides with the domain of its adjoint operator. Since the defect subspace contains functions, which are singular at $r = 0$, the adjoint operator has a larger domain. So, it is natural to posit the boundary condition (13) in the defect subspace.

Let the radial Dirac Hamiltonian $h_r$ (8) has the domain $D(f)$, where $f(r)$ is absolutely continuous functions, square integrable on the half-line $[0, \infty)$ with measure $rdr$ and regular at $r = 0$. We must construct the defect subspaces $D^\pm$ of adjoint operator $h^\dagger$ with eigenvalue $\pm im$ ($m$ is the fermion mass)

$$h^\dagger f(r) \equiv \begin{bmatrix} m & sdf/dr + (l + \mu + s)/r \\ -m & -df/dr + (l + \mu - s)/r \end{bmatrix} \begin{pmatrix} f^\pm_1(r) \\ f^\pm_2(r) \end{pmatrix} = \pm im \begin{pmatrix} f^\pm_1(r) \\ f^\pm_2(r) \end{pmatrix}.$$  

Then, any self-adjoint extension of $h_r$ can be constructed by the isometries $D^+ \rightarrow D^-$. This extension can be fixed by a parameter $\theta$:

$$f^+(r) \rightarrow e^{i\theta}f^-(r).$$  

The correct domain for the self-adjoint extension $h^\theta$ of $h_r$ is given by

$$D(h^\theta) = D(h_r) + C[f^+(r) + e^{i\theta}f^-(r)],$$  

where $C$ is arbitrary complex constant, and $\theta$ is arbitrary parameter but fixed for given extension ($2\pi > \theta > 0$).

In our case, it is simpler to use solutions of Eq. (12), with $m = 0$ in the body of operator $h^\dagger$:

$$\begin{pmatrix} f^\pm_1(r) \\ f^\pm_2(r) \end{pmatrix} = N \begin{pmatrix} K_{\nu}(mr) \\ se^{\pm i\nu/2}K^\nu_{\nu+1}(mr) \end{pmatrix},$$  

where $N$ is a normalization factor, and $K_\nu$ is the MacDonald function. Singular, square integrable functions are the MacDonald functions of the order $\gamma - 1$ (upper spinor component) and $\gamma$ (lower spinor
component) with $l = -n - 1, \ n \geq 0$. Parameterized spinors of the defect subspaces in the correct domain of self-adjoint extension \([10]\) can be easily constructed in the simple form

$$2C e^{i\theta/2} \left( \frac{K_{\gamma-1}(mr) \cos(\theta/2)}{sK_{\gamma}(kr)} \right). \tag{18}$$

Taking into account formula $K_{\nu}(z) = K_{-\nu}(z)$ and asymptotic behavior $K_{\nu}(x) \approx 2^{\nu-1}\Gamma(\nu)/x^{\nu}$ at $x \to 0$, where $\Gamma(\nu)$ is the gamma function, we can easily find that boundary condition \([13]\) will be satisfied for arbitrary spinor with its asymptotic behavior

$$\lim_{mr \to 0} f(mr) \sim \left( \frac{\gamma \sin(\theta^*/2)}{-s(mr)^{-\gamma} \cos(\theta^*/2)} \right). \tag{19}$$

Here

$$\tan(\theta^*/2) = \frac{\Gamma(1-\gamma)}{2\Gamma(\gamma)} \tan(\theta/2) \tag{20}$$

and $2\pi > \theta^* > 0$.

\section*{IV. Bound Fermion States in the Field of a Cosmic String}

For nonintegral $\mu$ there exists a formal solution of Eq. \([6] \) with $m^2 > E^2$ that describes a bound fermion state with the spin $s$

$$\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = N \begin{pmatrix} \sqrt{m+E}K_{\gamma-1}(kr) \\ s\sqrt{m-E}K_{\gamma}(kr) \end{pmatrix}, \tag{21}$$

where $N$ is a normalization factor, $k = \sqrt{m^2-E^2}$. This bound fermion state exists for the range $2\pi > \theta^* > \pi$. The appearance of bound state “suggests that this range of parameters in the effective Hamiltonian parameterizes nontrivial effects in the core” of cosmic string \([3]\). We see that the physical reason for the existence of bound state can be the appearance of additional attractive (for instance, $D\delta(r)$ type) potential in the core of cosmic string.

From \([19]\) and \([21]\) we derive an equation, which implicitly determines the bound state energies of particle ($m \leq E \leq 0$) and antiparticle ($0 \leq E \leq -m$) in the form

$$\frac{(m+E)^\gamma}{(m-E)^{1-\gamma}} = -(2m)^{2\gamma-1} \frac{\Gamma(\gamma)}{\Gamma(1-\gamma)} \tan \frac{\theta^*}{2}. \tag{22}$$

Indeed, from the continuity consideration, we can conclude that particle states are the states that tend to the boundary of the continuous spectrum $E = m$ upon infinitely slow switching off the external field; under such switching off antiparticle states tend to the boundary $E = -m$ (see, e.g., \([22]\)). So Eq. \([22]\) as a function of $\gamma$ describes two energy (particle and antiparticle) curves.

For $\theta^* = 3\pi/2$ these curves are symmetric with respect to the horizontal line $E = 0$ and when parameter $\gamma$ changes from $0$ to $1/2$, Eq. \([22]\) determines the energy of bound particle state in the region $m \leq E \leq 0$.

Denoting $x = E/m$ Eq. \([22]\) can be written for each curve in the region $0 < \gamma < 1/2$ as

$$\pm x = b(1-x^2)^{1-\gamma} - 1, \quad b = 2^{3\gamma-1}\pi^{-1}\Gamma^2(\gamma) \sin\pi\gamma > 0. \tag{23}$$

Two curves intersect the horizontal line $E = 0$ at $\gamma = 1/2$.

We see that in the AB field, indeed the Dirac Hamiltonian in 2+1 dimensions does not exhibit a charge conjugation but fermion states with zero energy exist. For $\gamma = 1/2, \ s = 1$ the particle wave function (zero mode) in the lowest energy state is

$$\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = N \begin{pmatrix} K_{1/2}(mr) \\ K_{1/2}(mr) \end{pmatrix}. \tag{24}$$

The antiparticle wave function with $E = 0$ for $\gamma = 1/2, \ s = -1$ can be obtained by means the charge conjugation operator, which, in our case, is $C = i\sigma_2$. Hence, the antiparticle wave function is

$$\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = N \begin{pmatrix} K_{1/2}(mr) \\ -K_{1/2}(mr) \end{pmatrix}. \tag{25}$$
Thus, when the parameter $\gamma$ changes adiabatically from 0 to $1/2$, (the magnetic flux $\Phi$ changes from $\Phi = 2\pi n/|e|$ to $\Phi = 2\pi(n + 1/2)/|e|$) the energy split between bound states of particle and antiparticle vanishes and their lowest energy levels intersect. The intersection of energy levels signals on the instability of quantum system, i.e. the vacuum, and the possibility of creation of a fermion-antifermion pair from the vacuum by the static external field. The latter is not valid for the electron-positron pair production by the real Aharonov–Bohm field of thin solenoid since the electron energy levels in the AB field has to be defined from the corresponding Dirac equation in 3+1 dimensions. But solutions of the Dirac equation in an Aharonov–Bohm field in 2+1 dimensions describe relativistic fermions in the field of cosmic string in 3+1 dimensions, so, one can hope that obtained results will be helpful in studying the behavior of fermions in the field of cosmic string.

It is helpful to compare the fermion’s creation in the AB field with the positron creation from the vacuum by a strong Coulomb potential field $U(r) = -a/r, a > 0$ in the quantum electrodynamics (QED). When $a$ changes adiabatically to “the critical charge” $a_{cr}$ the lowest level of electron (with charge $e < 0$) tends to the boundary of the lower continuum of energies. For $a > a_{cr}$ it intersects the above boundary and the vacuum becomes unstable, which leads to the appearance of a bound (vacuum) state in the lower continuum and results in the positron production in a free state (see, for example, [24]). Due to the existence of bound state the vacuum simultaneously can acquire negative electric charge [24, 25].

In the AB field in 2+1 dimensions, when $\gamma$ changes adiabatically to $1/2$, the zero-energy (fermion and antifermion) states simultaneously exist and the vacuum, evidently, can acquire a magnetic moment equal to the two spin magnetic moment of the fermion. It should be noted that the existence of induced charge density, which can change the critical charge, in a strong Coulomb field due to vacuum polarization does not significantly change results concerning to the positron production from the vacuum by a Coulomb field (see, [24]); the induced charge density in the field of infinitesimally thin solenoid due to vacuum polarization is exactly equal to zero [0]. Of course, all the discussed questions require more subtle study since the effects at $\gamma \geq 1/2$ are of a many-particle nature and to describe them, we need a quantum field theory formalism.

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