FOURIENTATIONS AND THE TUTTE POLYNOMIAL

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Abstract. A fourientation of a graph is a choice for each edge of the graph whether to orient that edge in either direction, bidirect it, or leave it unoriented. Fixing a total order on the edges and a reference orientation of the graph, we investigate properties of cuts and cycles in fourientations which give trivariate generating functions that are generalized Tutte polynomial evaluations of the form

$$(k + m)^{n-1}(k + l)^g T \left( \frac{\alpha k + \beta l + m}{k + m}, \frac{\gamma k + l + \delta m}{k + l} \right)$$

for $\alpha, \gamma \in \{0, 1, 2\}$ and $\beta, \delta \in \{0, 1\}$.

We introduce an intersection lattice of 64 cut-cycle fourientation classes enumerated by generalized Tutte polynomial evaluations of this form. We prove these enumerations using a single deletion-contraction argument and classify axiomatically the set of fourientation classes to which our deletion-contraction argument applies. This work unifies and extends earlier results for fourientations due to Gessel and Sagan [20], and results for partial orientations due to the first author [4], and the second author and David Perkinson [27], as well as results for total orientations due to Stanley [46] [48], Las Vergnas [32], Greene and Zaslavsky [24], and Gioan [21], which were previously unified by Gioan [21] and Bernardi [11].

We also investigate a parallel story of edge colorings and produce an intersection lattice of 64 cut-cycle classes for 4-edge-colorings (implicitly based on internal and external activity) which give generalized Tutte polynomial evaluations of the form above. We put forward the problem of finding a unified bijection between fourientations and 4-edge colorings which respects all of these cut-cycle classes. We conclude by describing how several of these classes of fourientations relate to various geometric, combinatorial, and algebraic objects including bigraphical arrangements, cycle-cocycle reversal systems, graphic Lawrence ideals, divisors on graphs, and zonotopal algebras.

1. Introduction

Throughout we use graph to mean finite, undirected graph (although we allow loops and multiple edges). The Tutte polynomial is the most general Tutte-Grothendieck invariant one can associate to a graph; that is, any graph invariant that satisfies a deletion-contraction recurrence is a specialization of the Tutte polynomial. In fact, any graph invariant that satisfies a weighted deletion-contraction recurrence is essentially an evaluation of the Tutte polynomial, as the following theorem, which is sometimes called the recipe theorem, makes precise.

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Key words and phrases. Partial graph orientations, Tutte polynomial, deletion-contraction, hyperplane arrangements, cycle-cocycle reversal system, chip-firing, $G$-parking functions, abelian sandpile model, Riemann-Roch theory for graphs, Lawrence ideals, zonotopal algebras.
Theorem 1.1 (see [52, Theorem 1] and [53, Theorem 2.16]). Let $\mathcal{G}$ be some set of graphs closed under deletion and contraction, let $k$ be a field, and let $f : \mathcal{G} \to k$ be some function that is invariant under graph isomorphism. Suppose that $f$ is normalized so that $f(G) = 1$ if $G$ has no edges. Suppose further that for every graph $G \in \mathcal{G}$ with at least one edge, there is some edge $e \in E(G)$ such that

$$f(G) = \begin{cases} af(G/e) + bf(G \setminus e) & \text{if } e \text{ is neither a bridge nor a loop} \\ x_0f(G \setminus e) & \text{if } e \text{ is a bridge} \\ y_0f(G/e) & \text{if } e \text{ is a loop}, \end{cases}$$

where $G \setminus e$ is graph obtained from $G$ by deleting $e$ and $G/e$ is the graph obtained by contracting $e$. Then for all $G \in \mathcal{G}$ we have

$$f(G) = a^{n-\kappa}b^gT_G(x_0, y_0),$$

where $n := |V(G)|$ is the number of vertices of $G$, $\kappa$ is its number of connected components, $g := |E(G)| - |V(G)| + \kappa$ is its cyclomatic genus, and $T_G(x, y)$ is its Tutte polynomial.

In light of Theorem 1.1, we call an expression of the form $a^{n-\kappa}b^gT_G(x, y)$ a generalized Tutte polynomial evaluation. Note that $n - \kappa$ is the rank of the graphic matroid associated to $G$ and $g$ is its corank. In what follows we assume for simplicity that all graphs are connected. We also write $T(x, y) := T_G(x, y)$ when the graph is implicit.

Our aim in this paper is to systematically exploit Theorem 1.1 in order to enumerate various classes of generalized graph orientations via the Tutte polynomial. A fourientation of a graph is a choice for each edge whether to orient that edge in either direction, bidirect it, or leave it unoriented. (There are $4^{|E(G)|}$ fourientations of a graph $G$ and thus the name.) A $(k, l, m)$-fourientation is obtained from a fourientation by assigning each oriented edge one of $k$ colors, each unoriented edge one of $l$ colors, and each bidirected edge one of $m$ colors. A potential cut (cycle) in a fourientation is the same as a directed cut (cycle) in an ordinary total orientation except that some of the edges may be unoriented (bidirected). In §2 we generate a list of potential cut and cycle properties which mix with one another to give an intersection lattice of 64 cut-cycle properties of $(k, l, m)$-fourientations such that each associated class is enumerated by a generalized Tutte polynomial evaluation. Moreover, we show that our list of properties is exhaustive: we derive the axioms required for our deletion-contraction proof to apply and show that the set of properties satisfying these axioms consists of precisely the potential cut and cycle properties on our list together with two exceptional cases. In §3 we consider specializations of $(k, l, m)$ that recover enumerative results about classes of partial orientations and total orientations obtained by many authors, as detailed in §1.1 below.

In §4 we then investigate an analogous story concerning properties of cuts and cycles in $(k_1, k_2, l, m)$ 4-edge-colorings, which are colorings of the edges of our graph by $k_1$, $k_2$, $l$, and $m$ shades of red, blue, green, and yellow, respectively. We produce a lattice of 64 cut-cycle properties of $(k_1, k_2, l, m)$ 4-edge-colorings (implicitly based on
internal and external activity) whose enumerations agree with those of the \((k, l, m)\)-fourientation cut-cycle classes when \(k_1 = k_2 = k\). In particular, we demonstrate that the specializations of these evaluations to 4-, 3-, and 2-edge-coloring properties agree with the previously obtained tables of fourientations, partial orientations, and total orientations, respectively. The relationship between orientations and 2-edge-colorings, that is, subgraphs, has been previously investigated; for instance, Bernardi [11] devised a unified bijection between orientations and subgraphs that respects cut-cycle classes in the related situation where our graph comes equipped with a combinatorial map. (A combinatorial map is an embedding of the graph in a surface.) At the end of §4 we put forward the problem of finding a unified bijective proof that the various classes of fourientations and 4-edge-colorings are equinumerous.

Our axiomatic approach to orientation properties recovers classes of partial orientations which arose in seemingly unrelated contexts and also suggests interesting new avenues of research. In §5 we outline how several of our cut and cycle properties relate to geometric, combinatorial, and algebraic objects including bigraphical hyperplane arrangements, cycle-cocycle reversal systems, graphic Lawrence ideals, divisors on graphs, and (conjecturally) zonotopal algebras. Recent developments in the study of divisors on graphs, including the commutative algebra of the abelian sandpile model [12], [33], [18], [40], Riemann-Roch theory for graphs [6] [5], and geométrizations of the Matrix-Tree theorem [1], highlight the algebraic significance of the relationship between graph orientations and their indegree sequences. The partial orientation classes we define, which we term \textit{min-edge classes}, appear to arise in many situations where one is interested in indegree sequences. A striking example of this phenomenon, described in detail in §5.6 is that the acyclic-cut free partial orientations point the way towards a monomization of the internal zonotopal algebra [25] associated to a graph \(G\): in particular, their indegree sequences apparently yield a linear basis of this algebra. While there exist constructions of monomizations of the external [44] [16] and central [43] zonotopal algebras associated to \(G\), there is no such construction for the internal algebra that works for all \(G\). We arrived at the definition of the min-edge class cut free only as a result of our abstract machinery, but it conjecturally helps resolve this outstanding problem that we became aware of after we began our research.

1.1. History. Since at least the seminal work of Stanley [46], it has been known that the Tutte polynomial counts classes of graph orientations defined in terms of cuts and cycles. Stanley [46] proved that the number of acyclic orientations of a graph is \(T(2, 0)\), which is also equal to the chromatic polynomial evaluated at \(-1\). Las Vergnas [32] proved that the number of strongly connected orientations, those with no directed cut, is \(T(0, 2)\). Greene and Zaslavsky [24] showed that the number of acyclic orientations of a graph with a unique source \(q\) is \(T(1, 0)\). By fixing a total order on the edges and a reference orientation of the graph, the previous result can be generalized in the following way: the number of acyclic orientations such that the minimum edge in each directed cut is oriented as in the reference orientation is \(T(1, 0)\). These orientations give distinguished representatives for the set of acyclic orientations modulo cut reversals, which can be obtained greedily. Gioan [21] observed that \(T(0, 1)\) counts the number of equivalence classes of strongly connected orientations modulo directed cycle reversals,
or equivalently indegree sequences of strongly connected orientations. Because any two orientations with the same indegree sequence differ by cycle reversals, this theorem is equivalent to the result of Greene and Zaslavsky [24] that the number of strongly connected orientations for which the minimum edge in any directed cycle is oriented as in the reference orientation is $T(0, 1)$ because these orientations give distinguished representatives for the set of strongly connected orientations modulo cycle reversals. Greene and Zaslavsky’s result and its equivalence to Gioan’s result was rediscovered by Chen, Yang, and Zhang [12] who investigated it from a bijective perspective. Stanley [48] observed that the total number of indegree sequences among orientations is counted by $T(2, 1)$, which may be interpreted as a version of the previous result for orientations that are not necessarily strongly connected. Similarly, Gioan [21] proved that the number of (not necessarily acyclic) $q$-connected orientations is $T(1, 2)$, and the number of indegree sequences of these orientations is $T(1, 1)$. Trivially, $T(0, 0) = 0$, the number of strongly connected-acyclic orientations, and $T(2, 2) = 2|E|$, the total number of orientations. Putting all of these enumerations together, Gioan [21] offered a unified framework for interpretations of $T(x, y)$ for all integer values $0 \leq x, y \leq 2$ in terms of of equivalence classes of orientations. He presented separate proofs for the evaluations where either $x$ or $y$ is zero, and then obtained the remaining nonzero evaluations via a convolution formula for the Tutte polynomial due to Kook, Reiner, and Stanton [31]. Gioan later provided a more unified proof [22] making use of matroid duality, while still requiring application of the convolution formula and separate proofs of $T(2, 0)$ and $T(1, 0)$.

Gessel and Sagan [20] used depth-first search to investigate relationships between the Tutte polynomial and partial orientations (which they call “suborientations”) and fourorientations (which they call “subdigraphs”) of a graph. They proved that two generating functions defined by a kind of orientation activity give generalized Tutte polynomial evaluations with the following specializations: the number of acyclic partial orientations of a graph is $2^{g}T(3, 1/2)$, the number of $q$-connected fourorientations is $2^{|E|}T(1, 2)$, and the number of acyclic $q$-connected partial orientations of a graph is $2^{|E|}T(1, 1/2)$. The first result was rediscovered and proven via deletion-contraction by the first author [4], who also showed that the number of strongly connected partial orientations is $2^{n-1}T(1/2, 3)$. It was also shown in [4] that the number of partial orientations modulo cut reversals and modulo cycle reversals are $2^{n-1}T(1, 3)$ and $2^{|E|}T(3, 1)$ respectively. As in the case of total orientations, the partial orientations for which the minimum edge in every directed cut or cycle is oriented in the same direction as the reference orientation give distinguished representatives for these equivalence classes. The second author and Perkinson [27] observed that the number of regions of a generic bigraphical arrangement is $2^{n-1}T(3/2, 1)$ and the number of bounded regions is $2^{n-1}T(1/2, 1)$. They demonstrated that the regions of a bigraphical arrangement are labeled by a certain class of “admissible” acyclic partial orientations. Using exponential parameters to define a generic arrangement, we give an alternate description of these admissible partial orientations as those for which the minimum edge in any potential cycle is neutral. The partial orientations that label bounded region become those which are strongly connected and for which the minimum edge in any potential cycle is neutral.
By specializing \((k, l, m)\) in our main Theorem 2.13 to \((1, 1, 1)\) (fourientations), \((1, 1, 0)\) (type A classes of partial orientations), \((1, 0, 1)\) (type B classes of partial orientations), and \((1, 0, 0)\) (total orientations), we obtain tables (see Figure 4) in which all of the aforementioned results appear as entries. Moreover, for \((k, l, m) = (1, 0, 0)\), our proof of Theorem 2.13 specializes to the first uniform proof of Gioan’s 3 \(\times\) 3 square of total orientation classes, that is, one which treats each of these nine classes on an equal footing.

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2. Fourientations

Let \(G\) be a graph. We use \(V(G)\) to denote the vertex set of \(G\) and \(E(G)\) to denote its edge set. Recall that throughout we will assume that all graphs are connected. (This assumption is justified by the fact that if our graph \(G = G' \sqcup G''\) is the disjoint union of two graphs, then at the level of Tutte polynomials we have \(T_G(x, y) = T_{G'}(x, y) \cdot T_{G''}(x, y)\), so all interesting behavior for objects enumerated by Tutte polynomial evaluations is captured by connected graphs.) We take a moment to describe our notation for the edges of \(G\) and for orientations of these edges. Formally, \(V(G)\) is some finite set and \(E(G)\) is a finite set together with a map \(\varphi(G) : E(G) \rightarrow \binom{V(G)}{2}\), i.e., a map \(\varphi(G)\) from \(E(G)\) to the set of all multisets of \(V(G)\) of size two. By abuse of notation, for an edge \(e \in E(G)\) we write \(e = \{u, v\}\) to mean \(\varphi(G)(e) = \{u, v\}\); however, note that we may have \(e, f \in E\) with \(e = \{u, v\}\) and \(f = \{u, v\}\) but \(e \neq f\) meaning that \(e\) and \(f\) are multiple edges between the same vertices; and it is also possible that \(e = \{u, v\}\) is a loop. In order to talk about orientations of a graph it is helpful to have a reference orientation. A reference orientation \(O_{\text{ref}}\) of \(G\) is a map \(O_{\text{ref}} : E(G) \rightarrow V(G) \times V(G)\) satisfying \(F \circ O_{\text{ref}} = \varphi(G)\) where \(F : V(G) \times V(G) \rightarrow \binom{V}{2}\) is the forgetful map. An orientation of \(e \in E(G)\) is just a formal symbol \(e^+\) or \(e^-\), where we think of \(e^+\) as the orientation of \(e\) that agrees with \(O_{\text{ref}}\) and \(e^-\) as the orientation that disagrees with \(O_{\text{ref}}\). For \(\delta, \varepsilon \in \{+, -\}\) we define \(-\delta\) and \(\delta \cdot \varepsilon\) in the obvious way. When discussing orientations we use the symbols \(\pm\) and \(\mp\) for compactness of notation: any mathematical sentence involving \(\pm\) should be interpreted by replacing all occurrences of \(\pm\) with \(\delta\), all occurrences of \(\mp\) with \(-\delta\), and adding “for \(\delta \in \{-, +\}\)” at the end of the sentence. Let \(E(G) := \{e^\pm : e \in E(G)\}\) be
the set of orientations of edges of $G$. We extend $O_{\text{ref}}$ to a map $E(G) \to V(G) \times V(G)$ by setting $O_{\text{ref}}(e^+) := O_{\text{ref}}(e)$ and $O_{\text{ref}}(e^-) := O_{\text{ref}}(e)^{\text{op}}$, where $(u, v)^{\text{op}} := (v, u)$. Again abusing notation, we write $e^\pm = (u, v)$ to mean $O_{\text{ref}}(e^\pm) = (u, v)$; however, note that if $e = \{u, u\}$ is a loop then $e^+ = (u, u)$ and $e^- = (u, u)$ but we still treat $e^+$ and $e^-$ as different orientations of $e$. We call the pair $(G, O_{\text{ref}})$ an oriented graph.

We also need to review cuts and cycles of graphs as these are fundamental in defining properties of orientations. A cut of $G$ is a partition $Cu = \{U, U^c\}$ for some $U \subseteq V(G)$, where $U^c := V(G) \setminus U$ denotes the complement of $U$, such that both $U$ and $U^c$ are nonempty. We define $E(Cu) := \{e = \{u, v\} : u \in U, v \in U^c, e \in E(G)\}$. The cut $Cu$ is simple if the restriction of $G$ to $U$ and the restriction of $G$ to $U^c$ both remain connected. (What we call simple cuts are often called “bonds.”) We say that an edge $e \in E(G)$ is a bridge if $E(Cu) = \{e\}$ for some (necessarily simple) cut $Cu$. A cycle of $G$ is a list $Cy = v_1, e_1, v_2, e_2, \ldots, v_k, e_k$ with $k \geq 1$, $v_i \in V(G), e_i \in E(G)$, up to rotation and reflection of the indices, such that all edges $e_i$ are distinct and $e_i = \{v_i, v_{i+1 \text{ mod } k}\}$. We define $E(Cy) := \{e_i : 1 \leq i \leq k\}$. The cycle $Cy$ is simple if all the $v_i$ are distinct. We say that an edge $e \in E(G)$ is a loop if $E(Cy) = \{e\}$ for some (necessarily simple) cycle $Cy$.

A directed cut of $(G, O_{\text{ref}})$ is an ordered pair $\vec{Cu} = (U, U^c)$ for some cut $Cu = \{U, U^c\}$ of $G$; let $E(\vec{Cu}) := E(Cu)$ and $E(\vec{C}u) := \{e^\pm = (u, v) : u \in U, v \in U^c, e \in E(G)\}$; i.e., $E(\vec{Cu})$ is the set of edge orientations from $U$ to $U^c$. A directed cycle of $(G, O_{\text{ref}})$ is a list $\vec{Cy} = v_1, e_1^{\delta_1}, \ldots, v_k, e_k^{\delta_k}$, with $\delta_i \in \{+, -\}$, up to rotation but not reflection of indices, for some cycle $Cy = v_1, e_1, \ldots, v_k, e_k$ of $G$ such that $e_i^{\delta_i} = (v_i, v_{i+1 \text{ mod } k})$; let $E(\vec{Cy}) := E(Cy)$ and $E(\vec{Cy}) := \{e_i^{\delta_i} : 1 \leq i \leq k\}$. Note that each cut (cycle) $C$ has two associated directed cuts (cycles) $\vec{C}$ and $-\vec{C}$.

**Definition 2.1.** A fourientation $O$ of an oriented graph $(G, O_{\text{ref}})$ is a subset of $E(G)$.

In other words, a fourientation is a choice for each edge $e$ of a subset of $\{e^+, e^-\}$. If $e^+, e^- \in O$ then we say $e$ is bidirected in $O$, and if $e^+, e^- \notin O$ then we say $e$ is unoriented in $O$. An oriented edge $e$ of $O$ is one for which $e^\pm \in O$ but $e^\mp \notin O$. We emphasize that $O_{\text{ref}}$ is not essential in the definition of a fourientation but $O_{\text{ref}}$ has allowed us to introduce the very useful notation $e^+$ and $e^-$ which we employ throughout this paper. However, when we discuss the properties of cuts and cycles in fourientations which define classes enumerated by generalized Tutte polynomial evaluations the reference orientation will play an indispensable role. Fourientations were introduced and studied in an enumerative context by Gessel and Sagan [20] under the name of “subdigraphs.” They are also superficially similar to the orientations of signed graphs investigated by Zaslavsky [54]; but note that Zaslavsky’s notion of a cycle in a signed graph orientation is quite different from the kinds of cycles of fourientations we investigate. It seems plausible that there is some deeper connection between orientations of signed graphs and fourientations and it would be very interesting to find such a relationship.

**Definition 2.2.** A potential cut (cycle) of a fourientation of an oriented graph is a directed cut (cycle) of the graph such that each edge in that cut (cycle) is either oriented in agreement with the cut (cycle) or is unoriented (bidirected). In symbols, $\vec{Cu}$ is a
potential cut of \( \mathcal{O} \) if \( e^\pm \in E(Cu) \Rightarrow e^\mp \notin \mathcal{O} \) for all \( e \in E \), and \( C^y \) is a potential cycle of \( \mathcal{O} \) if \( e^\pm \in E(Cy) \Rightarrow e^\mp \in \mathcal{O} \) for all \( e \in E \).

**Example 2.3.** Let \((G, \mathcal{O}_{\text{ref}})\) be an oriented graph and \( \mathcal{O} \) be a fourientation of \((G, \mathcal{O}_{\text{ref}})\) as below:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example.png}
\end{array}
\]

Here \( \mathcal{O} = \{ e_2^+, e_2^-, e_3^+, e_4^+, e_5^+, e_6^- \} \). Observe that \( Cu = (\{v_2\}, \{v_1, v_3, v_4, v_5\}) \) is a potential cut of \( \mathcal{O} \) and \( Cy = v_1, e_2^+, v_3, e_4^+, v_4, e_5^+, v_5, e_6^- \) is a potential cycle of \( \mathcal{O} \).

As an aside, we note that fourientations are canonically in bijection with total orientations of certain labeled graph minors by contracting the bidirected edges and deleting the undirected edges. This perspective seems quite natural as potential cycles and cuts are mapped to directed cycles and cuts, respectively. The awkward part of this construction is that we must remember how the oriented graph minor was obtained, i.e., which edges were contracted and which were deleted, even if various choices produce the same oriented graph minor.

In this section, we define various classes of fourientations of \((G, \mathcal{O}_{\text{ref}})\) in terms of potential cuts and cycles. We will need more input data to define these classes. Specifically, we will also need \(<\), a total order on the edges of \( G \). Such an edge order often appears in investigations of the Tutte polynomial because it allows one to define the (internal and external) activity of spanning trees of a graph. It may be possible to extend our work to allow for other notions of activity; for instance, see the recent paper \cite{14} which develops a unified perspective for various kinds of activity. However, we will stick to the most classical case of activity defined in terms of a total edge order here. We call the triple \( G = (G, \mathcal{O}_{\text{ref}}, <) \) an ordered, oriented graph. A fourientation of \( G \) is of course a fourientation of the underlying oriented graph \((G, \mathcal{O}_{\text{ref}})\).

The classes of fourientations we will define are enumerated by the Tutte polynomial, so we now review deletion and contraction. For \( e = \{u, v\} \in E(G) \), the graph obtained by **deleting** \( e \) is denoted \( G \setminus e \); this graph has \( V(G \setminus e) := V(G) \) and \( E(G \setminus e) := E \setminus \{e\} \). The graph obtained by **contracting** \( e \) is denoted \( G/e \); now we set \( V(G/e) := V/\sim \) where \( \sim \) is the identification \( u \sim v \), and again \( E(G/e) := E \setminus \{e\} \). Of course, we can also form the deletion \( G \setminus e \) and contraction \( G/e \) of an ordered, oriented graph \( G \) by keeping track of the extra data in the obvious way. We similarly define the deletion \( \mathcal{O} \setminus e \) and contraction \( \mathcal{O}/e \) of a fourientation \( \mathcal{O} \) (which in fact are both just equal to \( \mathcal{O} \setminus \{e^+, e^-\} \)). In particular note that if \( e^+, e^- \notin \mathcal{O} \) then we will often treat \( \mathcal{O} \) as a fourientation of \( G \setminus e \) and of \( G/e \). For a subset of edges \( H \subseteq E(G) \) we also define \( G \setminus H \) to be the graph obtained by deleting (contracting) all the edges in \( H \) in any order. Of course we similarly define \( \mathcal{O} \setminus H \) and \( \mathcal{O}/H \). For a simple cut \( Cu \) of \( G \), we define the **contraction to \( Cu \)**, denoted \( G_{Cu} \), to be \( G/(E(G) \setminus E(Cu)) \). The contraction to a
simple cut always yields a \textit{banana graph} $B_n$ for some $n \geq 1$, where the family of banana graphs is

\begin{center}
\begin{tikzpicture}
\draw (0,0) circle (.1cm);
\draw (1,.5) circle (.1cm);
\draw (2,1) circle (.1cm);
\draw (3,1.5) circle (.1cm);
\end{tikzpicture}
\end{center}

$B_1$ $B_2$ $B_3$ 

Similarly, for a simple cycle $C_y$ of $G$, we define the \textit{restriction to $C_y$}, which we denote $G_{C_y}$, to be the graph obtained from $G \setminus (E(G) \setminus E(C_y))$ by removing all isolated vertices. The restriction to a simple cycle always yields a \textit{cycle graph} $C_n$ for some $n \geq 1$, where the family of cycle graphs is

\begin{center}
\begin{tikzpicture}
\draw (0,0) circle (.1cm);
\draw (1,.5) circle (.1cm);
\draw (2,1) circle (.1cm);
\draw (3,1.5) circle (.1cm);
\end{tikzpicture}
\end{center}

$C_1$ $C_2$ $C_3$ 

We define the restriction to a simple cycle $G_{C_y}$ and the contraction to a simple cut $G_{Cu}$ of an ordered, oriented graph $G$ in the obvious way by keeping track of the extra data.

A fundamental notion for graph orientations is that of reachability by directed paths. In an ordinary total orientation $O$ we say that the vertex $v$ is reachable from the vertex $u$ if we can walk from $u$ to $v$ along a series of edges that are oriented in $O$ consistently with our walk. Because we are viewing a bidirected edge as the union of both orientations of an edge we will allow a bidirected edge to be traversed in either direction. On the other hand, unoriented edges cannot be traversed in either direction (because they are not present). Thus we define a \textit{potential path} $P$ of a fourientation $O$ of $(G, O_{\text{ref}})$ to be a list $v_1, e^\delta_1, \ldots, e^\delta_{k-1}, v_k$ for some $k \geq 1$ such that the $e_i$ are distinct, $e^\delta_i = (v_i, v_{i+1})$, and $e^\delta_i \in O$ for all $1 \leq i \leq k$. We say that $P$ is a potential path \textit{from} $v_1$ \textit{to} $v_k$ and we set $E(P) := \{e_i: 1 \leq i \leq k-1\}$. If there is a potential path $P$ of $O$ from $u$ to $v$ then we say $v$ is \textit{reachable} from $u$ in $O$. It is a classical fact, which can be seen by considering reachability classes, that every edge in a total orientation belongs to a directed cycle or a directed cut but not both. It remains the case in fourientations that every oriented edge belongs to either a potential cut or potential cycle but not both, as we show in Proposition 2.5 below. First we present a more basic lemma that says that potential cuts and cycles of a fourientation are disjoint.

**Lemma 2.4.** Let $O$ be a fourientation of $(G, O_{\text{ref}})$. Let $\overrightarrow{Cu}$ be a potential cut of $O$ and $\overrightarrow{Cy}$ a potential cycle of $O$. Then $E(\overrightarrow{Cu}) \cap E(\overrightarrow{Cy}) = \emptyset$.

**Proof.** Certainly if $e$ is bidirected in $O$ then it does not belong to a potential cut and if $e$ is unoriented in $O$ then it does not belong to a potential cycle. So suppose that $e^\pm = (u, v) \in O$ but $e^\mp \notin O$ and let $\overrightarrow{Cu} = (U, U^c)$ be a potential cut of $O$ with $e \in E(\overrightarrow{Cu})$. Note that $u$ is not reachable from $v$ because any potential path from $v$ to $u$ would have to go through an edge in $E(U, U^c)$ and these are all either
unoriented or oriented from $U$ into $U^c$. Thus there is no potential cycle $\overrightarrow{Cy}$ of $O$ with $e \in E(\overrightarrow{Cy})$.

**Proposition 2.5.** Let $O$ be a fourientation of $(G, O_{ref})$ and $e$ an oriented edge of $O$. Then $e \in E(\overrightarrow{Cu})$ for some potential cut $\overrightarrow{Cu}$ of $O$ or $e \in E(\overrightarrow{Cy})$ for some potential cycle $\overrightarrow{Cy}$ of $O$ but not both.

**Proof.** Let $e^\pm = (u, v) \in O$ with $e^\mp \notin O$. Let $U$ be set of vertices reachable from $v$ in $O$. If $u \in U$ then $e$ belongs to a potential cycle. Otherwise $(U, U^c)$ is a potential cut containing $e$. By Lemma 2.4 we know that $e^\pm$ cannot belong to both a potential cut and a potential cycle. □

In general we cannot partition all of the edges of a fourientation into potential cuts and cycles. However, the following two propositions offer two dual ways to extend the partition of the oriented edges in Proposition 2.5 to a decomposition of the entire edge set of our graph.

**Proposition 2.6.** Let $O$ be a fourientation of $(G, O_{ref})$. Then there is a unique decomposition $E(G) = E_{cu}(O) \cup E_{cy}(O)$ such that

(a) for any $e \in E_{cu}(O)$ we have $e \in E(\overrightarrow{Cu})$ for some potential cut $\overrightarrow{Cu}$ of $O / E_{cu}(O)$;

(b) there is no $e \in E_{cy}(O)$ with $e \in E(\overrightarrow{Cy})$ for any potential cycle $\overrightarrow{Cy}$ of $O / E_{cy}(O)$.

**Proposition 2.7.** Let $O$ be a fourientation of $(G, O_{ref})$. Then there is a unique decomposition $E(G) = E_{cy}(O) \cup E_{cu}(O)$ such that

(a) for any $e \in E_{cy}(O)$ we have $e \in E(\overrightarrow{Cy})$ for some potential cycle $\overrightarrow{Cy}$ of $O \setminus E_{cy}(O)$;

(b) there is no $e \in E_{cu}(O)$ with $e \in E(\overrightarrow{Cy})$ for any potential cycle $\overrightarrow{Cy}$ of $O / E_{cy}(O)$.

**Proof of Proposition 2.6.** We first show existence. Let $E_{cu}(O)$ be the set of edges which belong to a potential cut of $O$ and $E_{cy}(O)$ the complement of this set. Clearly condition (a) is satisfied. To show (b) let $e \in E_{cu}(O)$. If $e$ is bidirected in $O$ then certainly it does not belong to a potential cut of $O / E_{cu}(O)$. Next suppose $e$ is oriented in $O$. Then since $e$ does not belong to a potential cut of $O$, by Proposition 2.6 it belongs to a potential cycle $\overrightarrow{Cy}$ of $O$. Note that $E(\overrightarrow{Cy}) \cap E_{cu}(O) = \emptyset$ by Lemma 2.4. Thus $\overrightarrow{Cy}$ persists as a potential cycle in $O / E_{cy}(O)$, so again by Lemma 2.4 we have that $e$ belongs to no potential cycle. Finally, suppose that $e = \{u, v\}$ is unoriented in $O$. Note that because $e$ does not belong to a potential cut, there is a potential path $P$ from $u$ to $v$ and another potential path $P'$ from $v$ to $u$. All of the edges in $E(P) \cup E(P')$ either belong to potential cycles or are bidirected; at any rate, none of them belong to potential cuts. Thus $P$ and $P'$ persist as potential paths in $O / E_{cy}(O)$. Finally, note that the paths $P$ and $P'$ prevent $e$ from belonging to any potential cut of $O / E_{cy}(O)$. So indeed regardless of how $e$ is fouriented it does not belong to any potential cut of $O / E_{cy}(O)$.

For proving uniqueness of this decomposition, suppose $E(G) = A \cup B$ and every edge of $G / B$ belongs to a potential cut of $O / B$ while no edges of $G \setminus A$ belong to a potential cut of $O \setminus A$. First suppose that there exits some $e \in A \setminus E_{cu}(O)$. We know that $e$ does not belong to a potential cut in $O$, hence it certainly does not belong to a potential
cut in \( \mathcal{O}/B \), which is a contradiction. Therefore we may assume that \( A \subseteq E_{cu}(\mathcal{O}) \) and \( E_{cu}(\mathcal{O}) \subseteq B \). Next suppose there is some edge in \( e \in B \setminus E_{cu}(\mathcal{O}) \). We know that \( e \) belongs to a potential cut in \( \mathcal{O} \), and therefore \( e \) belongs to a potential cut in \( \mathcal{O} \setminus A \), which is a contradiction. Thus \( A = E_{cu}(\mathcal{O}) \) and \( B = E_{cu}(\mathcal{O}) \). \( \square \)

**Proof of Proposition 2.6.** The proof is analogous to the proof of Proposition 2.6. We define \( E_{cy}(\mathcal{O}) \) to be the set of edges which belong to a potential cycle. \( \square \)

We now define our main objects of study in this paper, namely *Tutte fourientation properties*. These properties are defined axiomatically so as to satisfy exactly those conditions necessary for us to carry out a deletion-contraction argument that invokes Theorem 1.1 and proves that they are enumerated by generalized Tutte polynomial evaluations. The key observation that motivates this definition is that if some objects associated to our graph \( G \) are enumerated by a generalized Tutte polynomial evaluation, then there is some way of recursively deleting and contracting edges of \( G \) so that at each step our enumeration respects the relevant weighted deletion-contraction recurrence. We may as well assume the order that we delete and contract is dictated by \( < \). Thus we force the weighted deletion-contraction recurrence to hold with respect to the maximum edge of our graph. As we will see later, we can also give explicit descriptions of these properties that focuses instead on the status of the minimum edge.

**Definition 2.8.** A *fourientation property* is a map

\[ \{ (G, \mathcal{O}) : \mathcal{O} \text{ is a fourientation of the ordered, oriented graph } G \} \rightarrow \{ \text{good, bad} \} \]

that is invariant under isomorphism of the input.\(^1\) When \((G, \mathcal{O})\) is good we say that \( \mathcal{O} \) is a good fourientation of \( G \) with respect to the property, and similarly when \((G, \mathcal{O})\) is bad. In general we identify a fourientation property with its set of good fourientations. We call a fourientation property a *cut (cycle) property* if \( \mathcal{O} \) is a good fourientation of \( G \) if and only if \( \mathcal{O}_C \) is a good fourientation of \( G_C \) for each simple cut (cycle) \( C \) of \( G \). We call a cut (cycle) property a *Tutte cut (cycle) property* if for all ordered, oriented graphs \( G \) we have

T1 \( \mathcal{O} \) is a bad fourientation of \( G \) only if \( \mathcal{O} \) has a potential cut (cycle);

T2 if the maximum edge \( e \) of \( G \) is neither a bridge nor a loop, then

(a) if \( \mathcal{O} \) is a good fourientation of \( G \setminus e \) (\( G/e \)) then \( \mathcal{O} \cup \{ e^+ \} \) and \( \mathcal{O} \cup \{ e^- \} \) are both good fourientations of \( G \);

(b) if \( \mathcal{O} \) is a bad fourientation of \( G \setminus e \) (\( G/e \)) but \( \mathcal{O} \cup \{ e^+, e^- \} \) \((\mathcal{O})\) is a good fourientation of \( G \), then exactly one of \( \mathcal{O} \cup \{ e^+ \} \) or \( \mathcal{O} \cup \{ e^- \} \) is a good fourientation of \( G \);

\( \)\(^1\)Formally, we say that \((G^1, \mathcal{O}^1_{ref}, <^1), \mathcal{O}^1\) is isomorphic to \((G^2, \mathcal{O}^2_{ref}, <^2), \mathcal{O}^2\) if there exist bijections \( \nu : V(G^1) \rightarrow V(G^2) \) and \( \eta : E(G^1) \rightarrow E(G^2) \) such that for all \( e, e' \in E(G^1) \):

- \( \varphi(G^2)(\eta(e)) = \nu(\varphi(G^1)(e)) \);
- \( \mathcal{O}^1_{ref}(\eta(e)) = \nu(\mathcal{O}^1_{ref}(e)) \);
- \( e <^1 e' \) if and only if \( \eta(e) <^2 \eta(e') \);
- \( e^\pm \in \mathcal{O}^1 \) if and only if \( \eta(e)^\pm \in \mathcal{O}^2 \).
(c) \( O \) is a good fourientation of \( G \setminus e \) if and only if \( O \cup \{ e^+, e^- \} \) is a good fourientation of \( G \).

A Tutte cut-cycle property is the intersection of a Tutte cut property with a Tutte cycle property.

The following lemma says that condition \( T2(c) \) applies to Tutte cut-cycle properties as well as individual Tutte cut or cycle properties.

**Lemma 2.9.** Let \( e \) be the maximum edge of \( G \), and assume \( e \) is not a bridge or loop. Then \( O \) is a good fourientation of \( G \setminus e \) with respect to some Tutte cut-cycle property if and only if \( O \) is good for \( G \). Similarly, \( O/e \) is good for \( G/e \) if and only if \( O \cup \{ e^+, e^- \} \) is good for \( G \).

**Proof.** Suppose \( O \) is bad for \( G \setminus e \) with respect to the cycle property: then there is a simple cycle \( C_y \) of \( G \setminus e \) so that \( O \cup C_y \) is bad for the cycle property; this cycle \( C_y \) persists in \( G \) and in fact \( G \cup C_y \) is isomorphic to \( (G \setminus e) \cup C_y \); therefore \( O \) is also bad for \( G \). Similarly, if \( O \) is bad for \( G \) with respect to the cycle property and \( e^+, e^- \notin O \), then there is a simple cycle \( C_y \) of \( G \) so that \( O \cup C_y \) is bad for the cycle property. If we had \( e \in E(C_y) \), then \( O \cup C_y \) could not have a potential cycle and so could not be bad by condition \( T1 \). So indeed \( e \notin E(C_y) \), and thus \( C_y \) remains a simple cycle of \( G \setminus e \), and thus \( O \) remains bad for \( G \setminus e \). On the other hand, that \( O \) is bad for \( G \setminus e \) with respect to the cut property if and only if \( O \) is bad for \( G \) is exactly condition \( T2(c) \). The proof for \( G/e \) is analogous. \( \square \)

**Lemma 2.10.** Let \( e \) be the maximum edge of \( G \) and let \( O \) be a good fourientation of \( G \) with respect to some Tutte cut-cycle property. Then either \( O \setminus e \) is good for \( G \setminus e \) or \( O/e \) is good for \( G/e \).

**Proof.** If \( e \) is a bridge or loop then \( O \setminus e \) and \( O/e \) are both clearly good. So assume that \( e \) is neither a bridge nor a loop. By Lemma 2.9 if \( e \) is unoriented in \( O \) then \( O \setminus e \) is good and if \( e \) is bidirected in \( O \) then \( O/e \) is good. So assume further that \( e^\delta \in O \) and \( e^{-\delta} \notin O \) for some \( \delta \in \{+, -\} \). Now suppose to the contrary that both \( O \setminus e \) and \( O/e \) are bad. Note as in the proof of Lemma 2.9 that \( O \setminus e \) is certainly good for \( G \setminus e \) with respect to the cycle property; and \( O/e \) is certainly good for \( G/e \) with respect to the cut property. So \( O \setminus e \) is bad with respect to the cut property and \( O/e \) is bad with respect to the cycle property. Then by condition \( T2(b) \) we can extend \( O \setminus e \) to a bad fourientation of \( G \) with respect to the cut property by orienting \( e \) in a certain direction, and it must be that \( O' := (O \setminus e) \cup \{ e^{-\delta} \} \) is this extension as we know \( O \) is good. Analogously we know that \( O' = (O/e) \cup \{ e^{-\delta} \} \) is bad for \( G \) with respect to the cycle property. But that means that there must be some cut \( C_u \) of \( G \) so that \( O'_{Cu} \) is bad with respect to the cut property. It must be that \( e \in E(C_u) \) or else \( O'_{Cu} \) would also be bad. And it must be that there is a way of directing \( C_u \) to become a potential cut \( C_u \) of \( O' \) or else the contraction \( O'_{Cu} \) could not be bad by condition \( T1 \). Analogously we can find a potential cycle \( C_y \) of \( O' \) with \( e \in E(C_y) \). However, this contradicts Lemma 2.4 which says the edge sets of potential cuts and potential cycles are disjoint. \( \square \)

The deletion-contraction argument we use to count the good fourientations with respect to some Tutte cut-cycle property will actually work for fourientations that are...
weighted by the number of oriented, bidirected, and unoriented edges they contain. Thus we define the following chromatic extension of fourientations.

**Definition 2.11.** For \( k, l, m \in \mathbb{Z}_{\geq 0} \), a \((k, l, m)\)-fourientation is obtained from a fourientation by assigning each of the oriented edges one of \( k \) colors, each of the unoriented edges one of \( l \) colors, and each of the bidirected edges one of \( m \) colors. If a variable is set equal to zero, we naturally require that the associated fourientations have no edges with the corresponding fourientation type. We say a \((k, l, m)\)-fourientation is good with respect to a fourientation property if the underlying fourientation is good.

**Definition 2.12.** We define the bad bridge set \( X \subseteq \{\emptyset, \{-\}, \{+\}\} \) of a Tutte cut property in the following way: let \((B_1, O_{\text{ref}}, <)\) be the unique (up to isomorphism) ordered, oriented graph whose underlying graph is the banana graph \( B_1 \) and let \( e \) denote the unique edge of \( B_1 \); define \( X \) to be the set of all \( S \) such that \( \{e^\varepsilon : \varepsilon \in S\} \) is a bad fourientation for \((B_1, O_{\text{ref}}, <)\) with respect to the cut property. (Observe that by \([T]\) it is impossible for \( \{e^+, e^-\} \) to be bad with respect to a Tutte cut property.) For a Tutte cycle property we define its bad loop set \( Y \subseteq \{-\}, \{+\}, \{-+, +\}\) analogously in terms of the fourientations that are bad for \( C_1 \). The bad bridge (loop) set of a Tutte cut-cycle property is the bad bridge (loop) set of its underlying cut (cycle) property.

**Theorem 2.13.** For a fourientation \( O \), let \( o(O) \) denote the number of oriented edges, \( u(O) \) the number of unoriented edges, and \( b(O) \) the number of bidirected edges. Fix a Tutte cut-cycle property with bad bridge set \( X \) and bad loop set \( Y \). Then for \( G \), an ordered oriented graph whose underlying graph \( G \) has \( n \) vertices and cyclomatic genus \( g \), we have

\[
\sum_O k^{o(O)} l^{u(O)} m^{b(O)} = (k + m)^{n-1} (k + l)^g T_G \left( \frac{x_0}{k + m}, \frac{y_0}{k + l} \right)
\]

where the sum is over all good fourientations \( O \) of \( G \), and where

\[
x_0 := \delta(\{+\} \notin X)k + \delta(\{-\} \notin X)l + \delta(\emptyset \notin X)l + m
\]
\[
y_0 := \delta(\{+\} \notin Y)k + \delta(\{-\} \notin Y)k + \delta(\emptyset \notin Y)l + \delta(\{-+, +\} \notin Y)m
\]

and \( \delta(P) \) is \( 1 \) if \( P \) is true and \( 0 \) if \( P \) is false. In other words, the number of good \((k, l, m)\)-fourientations of \( G \) is given by \([T]\).

**Proof.** Fix a Tutte cut-cycle property. For an ordered, oriented graph \( G \), define

\[
T(G) := \{O : O \text{ is a good fourientation of } G \text{ with respect to our property}\}.
\]

For a set \( O \) of fourientations, define \( \hat{O} := \sum_{O \in \hat{O}} k^{o(O)} l^{u(O)} m^{b(O)} \). Let \( f(G) := \hat{T}(G) \) where \( G \) has \( G \) as its underlying graph. The proof will also establish inductively that \( f(G) \) is well-defined, that is, that this generating function does not depend on what reference orientation and edge order we give \( G \).

Let \( G \) be a graph and set \( \hat{G} := (G, O_{\text{ref}}, <) \) for arbitrary \( O_{\text{ref}} \) and \(<\). If \( G \) has no edges then certainly \( f(G) = 1 \). So assume \( G \) has an edge and let \( e \) be the maximum edge of \( G \). If \( e \) is a bridge, then the simple cycles of \( G \) are the same as those of \( G \setminus e \) and there is one additional simple cut: namely, the cut that has \( e \) as its only edge. So any good fourientation \( O \) of \( G \setminus e \) extends to a good fourientation \( O \cup \{e^\varepsilon : \varepsilon \in S\} \) of \( G \).
as long as $S \notin X$, and we get all good fourientations of $G$ this way. Therefore if $e$ is a bridge then $f(G) = x_0f(G \setminus e)$. Similarly if $e$ is a loop then $f(G) = y_0f(G/e)$. So from now on assume that $e$ is neither a bridge nor loop.

We want to show $f(G) = (k+l)f(G\setminus e) + (k+m)f(G/e)$ from which the result follows via Theorem 1.1. For $O \in T(G \setminus e)$, set $De(O) := \{O' \in T(G) : O' \setminus e = O\}$. Similarly, for $O \in T(G/e)$, set $Co(O) := \{O' \in T(G) : O'/e = O\}$. Set $De := \bigcup_{O \in T(G \setminus e)} De(O)$ and $Co := \bigcup_{O \in T(G/e)} Co(O)$. We claim that for $O \in T(G \setminus e)$,

(i) either $De(O) \subseteq De \setminus Co$ and $De(O) = \{O, O \cup \{e^\varepsilon\}\}$ for some $\varepsilon \in \{-, +\}$;

(ii) or $De(O) \subseteq De \cap Co$ and $De(O) = \{O, O \cup \{e^+, e^-\}, O \cup \{e^+, e'\}, O \cup \{e^+, e^-\}\}$.

From this claim it follows that $f(G \setminus e) = \frac{De \setminus Co}{k+l} + \frac{De \cap Co}{2k+l+m}$. So let us prove the claim. First of all, by Lemma 2.9 we know that $O \in T(G)$ which agrees with our claim. Assume first $O \cup \{e^+, e^-\} \in De(O)$; we need to show $O \in T(G/e)$ and $O \cup \{e^\varepsilon\} \in T(G)$ for any $\varepsilon \in \{-, +\}$. For any simple cut $Cu$ of $G/e$ we have a corresponding simple cut $Cu'$ of $G \setminus e$ so that $(G/e)Cu$ and $(G \setminus e)Cu'$ are isomorphic. Thus $O$ is good for $G/e$ with respect just to the cut property. Because $O \cup \{e^+, e^-\}$ is good for $G$ with respect to the cycle property, by condition $T2(c)$ we get that $(O \cup \{e^+, e^-\})/e = O$ is also good for $G/e$ with respect to the cycle property and therefore $O \in T(G/e)$. Using condition $T2(a)$ with respect to the cut and cycle properties gives $O \cup \{e^+\} \in T(G)$ and $O \cup \{e^-\} \in T(G)$ so we are done. Next assume $O \cup \{e^+, e^-\} \notin De(O)$; we need to show $O \notin T(G/e)$ and $O \cup \{e^\varepsilon\} \in T(G)$ for a unique $\varepsilon \in \{-, +\}$. Lemma 2.9 gives $O \notin T(G/e)$. We know that $O \cup \{e^+\}$ and $O \cup \{e^-\}$ are good for $G$ with respect to just the cut property by condition $T2(a)$. As before, we know $O$ is good for $G/e$ with respect to the cut property. So it must be that $O$ is bad for $G/e$ with respect to the cycle property. Condition $T2(c)$ says that since $O$ is not good for $G/e$ with respect to the cycle property, exactly one of $O \cup \{e^+\}$ or $O \cup \{e^-\}$ is good for $G$ with respect to the cycle property. So the claim is proved.

We can similarly show that $f(G/e) = \frac{Co \setminus De}{k+l} + \frac{De \cap Co}{2k+l+m}$. Then by Lemma 2.10 we have

$$f(G) = De \setminus Co + Co \setminus De + De \cap Co$$

$$= (k+l)f(G \setminus e) + (k+m)f(G/e),$$

thus completing the proof. \hfill \square

**Remark 2.14.** Theorem 2.13 says that there are the same number of good fourientations of $(G, O^1_{ref}, <1)$ and $(G, O^2_{ref}, <2)$ with respect to some fixed Tutte cut-cycle property. It would be interesting to find a simple bijection between these sets of fourientations; that is, it would be interesting to understand how the set of good fourientations changes as we modify the reference orientation and total order.

A priori it is not clear that there are any nontrivial Tutte cut properties. We will now show that there exist Tutte cut properties for all bad bridge sets $X \subseteq \{\emptyset, \{+\}, \{-\}\}$. Moreover, we will show that the Tutte cut properties are almost classified by $X$ (specifically, for each choice of $X$ there are one, two, or three Tutte cut properties with bad bridge set $X$). Of course the situation is analogous for Tutte cycle properties.
Definition 2.15. A min-edge cut (cycle) property is defined by \( X \subseteq \{\emptyset, \{-\}, \{+\}\} \) \((\{\{-\}, \{+\}, \{-+,\}\})\) and \( \delta \in \{+, -, \}\). A potential cut (cycle) \( \mathcal{C} \) of a fourientation \( \mathcal{O} \) of an ordered, oriented graph \( G \) is bad with respect to the min-edge cut (cycle) property defined by \( (X, \delta) \) if it satisfies both of the following conditions:

(i) \( \{e : e_{\min} \in \mathcal{O}\} = S \) for some \( S \in X \), where \( e_{\min} \) is the minimum edge in \( E(\mathcal{C}) \);
(ii) if \( e_{\min} \) is unoriented (bidirected) in \( \mathcal{O} \) then \( e_{\min}^\delta \in E(\mathcal{C}) \).

If the potential cut (cycle) is not bad, then it is good. A fourientation \( \mathcal{O} \) of \( G \) is good with respect to the min-edge cut (cycle) property defined by \( (X, \delta) \) if and only if all of its potential cuts (cycles) are good.

A heuristic explanation for the emphasis on the status of minimum edges in potential cuts is that in checking whether a cut is good or bad with respect to a Tutte cut property we repeatedly peel off maximal edges until we reduce to the base case where the cut’s minimum edge becomes a bridge. The point of \( \delta \) is that in order to satisfy \( T2(b) \) we want one of the ways of extending a bad cut by an oriented edge to be bad and the other way to be good: if the bad cut consists only of unoriented edges then both ways of extending it by an oriented edge still yield potential cuts and so could potentially be bad by \( T1 \); in this case \( \delta \) tells us which of these in fact is bad.

There are 12 essentially different min-edge cut properties because the choice of \( \delta \) is relevant only if \( \emptyset \in X \). Let \( -X \) denote the set we get by swapping \( + \leftrightarrow - \) and define \(-\mathcal{O}\) similarly. Clearly \( \mathcal{O} \) is a good fourientation of \( G \) with respect to the min-edge cut property defined by \( (X, \delta) \) if and only if \(-\mathcal{O}\) is good with respect to \((-X, -\delta)\).

So we may as well fix \( \delta = - \) and thus reduce the list to the following eight properties, which we call the min-edge cut classes of fourientations:

1. **Cut general** \((X = \emptyset)\): There are no restrictions on potential cuts.
2. **Cut directed** \((X = \{\emptyset\})\): For each potential cut, if the minimum edge of the cut is unoriented then cut contains some oriented edge directed in agreement with the reference orientation of this minimum edge.
3. **Cut negative** \((X = \{\{-\}\})\): The minimum edge in each potential cut is unoriented or is oriented in agreement with its reference orientation.
4. **Cut positive** \((X = \{\{+\}\})\): The minimum edge in each potential cut is unoriented or is oriented in disagreement with its reference orientation.
5. **Cut connected** \((X = \emptyset, \{-\})\): Each potential cut contains an oriented edge directed in agreement with the reference orientation of the minimum edge in the cut.
6. **Cut co-connected** \((X = \emptyset, \{+\})\): For each potential cut, either the minimum edge of the cut is unoriented and the cut contains an oriented edge directed in agreement with the reference orientation of this minimum edge, or the minimum edge is oriented in disagreement its the reference orientation.
7. **Cut neutral** \((X = \{\{-\}, \{+\}\})\): The minimum edge in each potential cut is unoriented.
8. **Cut free** \((X = \emptyset, \{-\}, \{+\})\): The minimum edge in each potential cut is unoriented and the cut contains an oriented edge directed in agreement with the reference orientation of this minimum edge.
We omit the description of the min-edge cycle classes which are exactly analogous. Observe that the poset of the above eight classes ordered by containment is of course isomorphic to the Boolean lattice on three elements. A min-edge class of fourientations is the intersection of a min-edge cut class and min-edge cycle class. Note that an arbitrary intersection of min-edge classes remains a min-edge class.

**Theorem 2.16.** Any min-edge cut (cycle) property is a Tutte cut (cycle) property.

*Proof.* Let us work with cut properties; the cycle properties are exactly analogous. First let us prove that a min-edge cut property is a Tutte cut property. Fix some min-edge cut property. Clearly the property is defined in an isomorphism invariant way and so it is indeed a fourientation property. A potential simple cut being good with respect to our min-edge cut property is the same as the corresponding contraction to a banana graph being good (and in light of \[T1\] we only care about potential cuts). So certainly if a fourientation is good, its contractions to its simple cuts are good. Conversely, assume the fourientation \(\mathcal{O}\) is bad. Then there is a bad potential cut \(\overrightarrow{Cu}\) for \(\mathcal{O}\). In fact we have \(E(\overrightarrow{Cu}) = E(\overrightarrow{Cu_1}) \cup \cdots \cup E(\overrightarrow{Cu_k})\) for potential cuts \(\overrightarrow{Cu_i}\) whose underlying undirected cuts \(Cu_i\) are simple. So if \(e_{\min}^\delta \in E(\overrightarrow{Cu})\) with \(e_{\min}\) being the minimal edge of \(E(\overrightarrow{Cu})\) then \(e_{\min}^\delta \in E(\overrightarrow{Cu_i})\) for some \(i\), which means \(\mathcal{O}_{Cu_i}\) is bad. Thus our min-edge cut property is indeed a cut property. What remains to check are the conditions \[T1\] and \[T2\]. Condition \[T1\] holds trivially. Now let us deal with condition \[T2\] so let \(e\) be the maximal edge of \(G\) with \(e\) neither a bridge nor a loop, and let \(\mathcal{O}\) be a fourientation of \(G\ \setminus\ e\). Condition \[T2(a)\] holds because in this case \(e\) cannot be the minimum edge in any potential cut, so any good potential cuts of \(\mathcal{O}\) which it becomes a part of remain good in \(\mathcal{O} \cup \{e^\pm\}\). Condition \[T2(c)\] holds for much the same reason: since \(e\) is not the minimum edge in any potential cut, any good potential cuts for \(\mathcal{O}\) which it becomes a part of remain good potential cuts in \(\mathcal{O}\) when considered as a fourientation of \(G\) and any bad potential cuts remain bad potential cuts.

The least obvious condition is \[T2(b)\]. First of all, if \(\mathcal{O}\) is bad for \(G\ \setminus\ e\) then one of \(\mathcal{O} \cup \{e^+\}\) or \(\mathcal{O} \cup \{e^-\}\) is bad because \(G\ \setminus\ e\) has at least one bad potential cut and so if we orient \(e\) as \(e^\pm\) in a way consistent with this cut it will remain a bad potential cut in \(\mathcal{O} \cup \{e^\pm\}\). In order to complete the proof that \[T2(b)\] holds, we claim that if \(\mathcal{O} \cup \{e^+, e^-\}\) is good for \(G\) then at least one of \(\mathcal{O} \cup \{e^+\}\) or \(\mathcal{O} \cup \{e^-\}\) is good. Suppose that to the contrary \(\mathcal{O} \cup \{e^+, e^-\}\) is good but \(\mathcal{O} \cup \{e^+\}\) and \(\mathcal{O} \cup \{e^-\}\) are both bad. Since \(\mathcal{O} \cup \{e^+, e^-\}\) is good, it cannot be that there is a bad potential cut \(\overrightarrow{Cu}\) of \(\mathcal{O} \cup \{e^\pm\}\) with \(e \notin E(\overrightarrow{Cu})\). So it must be that there is a bad potential...
cut \( \overrightarrow{Cu}_+ = (U_+, U_+^c) \) of \( O \cup \{e^+\} \) and a bad potential cut \( \overrightarrow{Cu}_- = (U_-, U_-^c) \) of \( O \cup \{e^-\} \) with \( e \in E(\overrightarrow{Cu}_+) \cap E(\overrightarrow{Cu}_-) \). The idea, as depicted in Figure 1, is to glue the cuts \( \overrightarrow{Cu}_+ \) and \( \overrightarrow{Cu}_- \) together to find a bad potential cut which does not involve \( e \). Let \( e_{\min} \) be the minimum edge of \( E(\overrightarrow{Cu}_+) \cup E(\overrightarrow{Cu}_-) \). By supposition that \( e \) is not a bridge, we have \( e \neq e_{\min} \). Suppose by symmetry that \( e_{\min}^\delta \in \overrightarrow{E}(\overrightarrow{Cu}_+) \) for some \( \delta \in \{+, -\} \).

We claim that \( e_{\min}^\delta \neq (u, v) \) for any \( u \in U_+ \cap U_-^c \) and \( v \in U_+^c \cap U_- \). Suppose to the contrary. Then \( e_{\min} \in E(\overrightarrow{Cu}_+) \cap E(\overrightarrow{Cu}_-) \), which means that in order for \( \overrightarrow{Cu}_+ \) to be a potential cut of \( O \cup \{e^+\} \) and \( \overrightarrow{Cu}_- \) a potential cut of \( O \cup \{e^-\} \) we need \( e_{\min} \) to be unoriented in \( O \). But then we have \( e_{\min}^\delta \in \overrightarrow{E}(\overrightarrow{Cu}_+) \) and \( e_{\min}^{\delta-} \in \overrightarrow{E}(\overrightarrow{Cu}_-) \), so by part (ii) of the min-edge cut definition we cannot have that \( \overrightarrow{Cu}_+ \) and \( \overrightarrow{Cu}_- \) are both bad potential cuts, a contradiction. So indeed \( e_{\min}^{\delta-} \neq (u, v) \) for any \( u \in U_+ \cap U_-^c \) and \( v \in U_+^c \cap U_- \). One consequence of this is that \( U_+ \neq U_-^c \). So at least one of \( U_+ \cap U_-^c \) or \( U_-^c \cap U_- \) is nonempty. And on the other hand if \( e = \{w, x\} \) then \( \{w, x\} \subseteq (U_+ \cap U_-^c) \cup (U_-^c \cap U_-^c)^c \), so \( U_+ \cap U_-^c \) and \( (U_-^c \cap U_-^c)^c \) are both nonempty. Thus if we define \( \overrightarrow{Cu}' := (U_+ \cap U_-^c, (U_+ \cap U_-^c)^c) \) and \( \overrightarrow{Cu}'' := (U_-^c \cap U_-^c, (U_-^c \cap U_-^c)^c) \) at least one of these must genuinely be a directed cut. Our discussion of \( e_{\min} \) also establishes that \( e_{\min}^{\delta-} \in \overrightarrow{E}(\overrightarrow{Cu}') \) or \( e_{\min}^{\delta-} \in \overrightarrow{E}(\overrightarrow{Cu}'') \). Whichever of \( \overrightarrow{Cu}' \) or \( \overrightarrow{Cu}'' \) it belongs to is a potential cut of \( O \cup \{e^+, e^-\} \) because we have \( e \in E(\overrightarrow{Cu}') \cap E(\overrightarrow{Cu}'') \). But then one of \( \overrightarrow{Cu}' \) or \( \overrightarrow{Cu}'' \) is a bad potential cut of \( O \cup \{e^+, e^-\} \), contradicting our assumption that \( O \cup \{e^+, e^-\} \) is good. \( \square \)

In order to give a near converse to Theorem 2.16 and to completely classify Tutte cut properties we need to introduce two anomalous properties: cut weird and cut co-weird. The cut weird fourientations of an ordered, oriented graph are those such that every potential cut contains at least one oriented edge and the minimum oriented edge in the cut is oriented in agreement with its reference orientation. The cut co-weird fourientations are those such that every potential cut contains at least one oriented edge and the minimum oriented edge in the cut is oriented in disagreement with its reference orientation.

**Theorem 2.17.** Any Tutte cut property is either a min-edge cut property or is cut weird or cut co-weird. There is a completely analogous classification for Tutte cycle properties.

**Proof.** Again we work with the cut case. Fix some Tutte cut property. Because it is a cut property, this property is determined by the values it takes on ordered, oriented banana graphs. It is not hard to see that if our Tutte property agrees with some min-edge cut property \((X, \delta)\) on all ordered, oriented banana graphs then it is agrees with \((X, \delta)\) on all graphs (here we again use the fact that a cut decomposes into simple cuts). Our goal is to find \((X, \delta)\). Clearly we should define \( X \) to be bad bridge set of our Tutte cut property. In order to define \( \delta \) we need to consider some small banana graphs. Let us view the banana graph \( B_n \) as having vertex set \( V(B_n) := \{u, v\} \) and edge set \( E(B_n) := \{e_1, \ldots, e_n\} \) where \( e_1 := \cdots := e_n := \{u, v\} \). Define the edge order \( < \) by \( e_1 < \cdots < e_n \). If \( \emptyset \notin X \), then we define \( \delta \) arbitrarily. If \( \emptyset \in X \), then we
define δ as follows: define a reference orientation \( O_{2,\text{ref}}^2 \) by \( O_{2,\text{ref}}^2 (e_1^+) := O_{2,\text{ref}}^2 (e_2^-) := (u, v) \); then let δ ∈ \{+, −\} be so that \( O^2 := \{e_δ^3\} \) is a bad fourientation of \((B_2, O_{2,\text{ref}}^2, <)\).

We need to check that our property agrees with the min-edge cut property \((X, δ)\) on all banana graphs. So let \((B_n, O_{\text{ref}}^1, <)\) be an ordered, oriented banana graph and assume \( n > 1 \) since we know our Tutte property agrees with \((X, δ)\) for \( n = 1 \). Let \( O \) be a fourientation of \((B_n, O_{\text{ref}}^1, <)\). If \( O \) has any bidirected edges, we know it is good by condition \( T_1 \) because it has no potential cuts and this agrees with \((X, δ)\). So now assume \( O \) has no bidirected edges. If \( O \setminus e_n \) is good for \((B_n, O_{\text{ref}}^1, <)\) then we know by conditions \( T_2(a) \) and \( T_2(c) \) that \( O \) is good no matter how \( e_n \) is fouriented, which again agrees with \((X, δ)\). If \( O \setminus e_n \) is bad but contains at least one oriented edge, then we know by conditions \( T_1 \) and \( T_2(c) \) that \( O \) is good if and only if \( e \) is oriented to disagree with that oriented edge and make it so that \( O \) has no potential cuts. This agrees with \((X, δ)\). So finally assume that \( O \setminus e_n \) is bad and contains no oriented edges. Note by repeated application of \( T_2(c) \) that this is possible only if \( \emptyset \in X \). Certainly by \( T_2(c) \) if \( e_n^- \notin O \) then \( O \) is bad; so the status of \( O \) is only at issue if \( e_n^+ \in O \) for some \( ε \in \{+, −\} \). We claim that in this case the status of \( O \) is still consistent with \((X, δ)\) unless we are in an exceptional case that we will address at the end.

From now on assume \( \emptyset \in X \) (or else we cannot have that \( O \setminus e_n = \emptyset \) is bad). Using the + ↔ − symmetry assume additionally from now on that \( δ = − \). The proof that follows is technical and requires constructing several auxiliary graphs; Figure 2 offers a pictorial aid for the various subclaims made below. We must now consider how our Tutte property behaves with respect to the other reference orientation for \( B_2 \).

Define \( O_{3,\text{ref}}^2 \) by \( O_{3,\text{ref}}^2 (e_1^+) := O_{3,\text{ref}}^2 (e_2^-) := (u, v) \) and define \( O^{3'} := \{e_3^+\} \). There are two cases to address: either \( O^{3'} \) is bad for \((B_2, O_{3,\text{ref}}^2, <)\) or it is good.

**Case I:** \( O^{3'} \) is a bad fourientation of \((B_2, O_{3,\text{ref}}^2, <)\).

Note that this case is consistent with the min-edge cut property defined by \((X, δ)\).

We will show that indeed our Tutte property is this min-edge cut property.

**Subclaim 1.** Set \( O_{3,\text{ref}}^3 (e_1^+) := O_{3,\text{ref}}^3 (e_2^-) := O_{3,\text{ref}}^3 (e_3^+) := (u, v) \). Then \( O^3 := \{e_3^+\} \) is a bad fourientation of \((B_3, O_{3,\text{ref}}^3, <)\).

**Proof.** Suppose to the contrary. Define the auxiliary graph \( G^* \) by \( V(G^*) := \{u, v, w\} \) and \( E(G^*) := \{e_1, e_2, e_3, e_4, e_5\} \) where \( e_1 := e_3 := \{u, w\} \) and \( e_2 := e_4 := e_5 := \{u, v\} \).

Set \( O_{\text{ref}}^*(e_1^-) := O_{\text{ref}}^* (e_2^-) := (u, w) \) and \( O_{\text{ref}}^*(e_4^+) := O_{\text{ref}}^*(e_5^+) := (u, v) \).

Then \( O^* := \{e_3^+, e_4^+, e_5^+\} \) must be good for \( G_* := (G^*, O^*_{\text{ref}}, <) \): the graph \( G_* \) has two simple cuts \( C_{u_1} := \{\{u, w\}, \{v\}\} \) and \( C_{u_2} := \{\{u, w\}, \{v\}\} \); and the contraction to these cuts are \((G_*^*_{u_1}, O^*_{\text{ref}}) \simeq ((B_2, O_{2,\text{ref}}^2, <), O^2) \) and \((G_*^*_{u_2}, O^*_{\text{ref}}) \simeq ((B_3, O_{3,\text{ref}}^3, <), O^3) \), both of which are good by supposition. Let \( G^* \) be the graph obtained from \( G_* \) by adding an edge \( e_6 := \{v, w\} \) and let \( O_{\text{ref}}^* \) be any extension of \( O^*_{\text{ref}} \). By the Tutte condition (2c), we have that \( O^* \) remains good for \( G^* := (G^*, O^*_{\text{ref}}, <) \). Set \( C_{u_3} := \{\{u\}, \{v, w\}\} \), a cut of \( G^* \). Since we are working with a cut property, we know the contraction \((G_{C_{u_3}}^*, O^*_{\text{ref}_{C_{u_3}}}) \) is good; by removing \( e_5 \) and \( e_4 \) from this contraction using conditions \( T_2(a) \) and \( T_2(c) \) we get that something isomorphic to \((B_3, O_{3,\text{ref}}^3, <), −O^3 \) is good. But \( O^3 \) and −\( O^3 \)
Assume throughout by symmetry $\delta = -$:

\[\text{bad}\]

**Case I:**

\[\text{bad}\]

**Subclaim 1:**

\[\text{good} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad}\]

**Subclaim 2:**

\[\text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad}\]

**Subclaim 3:**

\[\text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good}\]

**Subclaim 4:**

\[\text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad} \quad \Rightarrow \quad \text{bad}\]

**Subclaim 5:**

\[\text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good}\]

**Induct on $n$:**

\[\text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{good}\]

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**Figure 2.** A visual aid for the proof of Theorem 2.17. The smaller arrows in the middle of an edge indicate the reference orientation (if there are no arrows in the middle of an edge then the reference orientation is arbitrary). The larger arrows at the end of an edge are edge orientations that belong to the fourientation. In general edges are ordered from left-to-right (with leftmost being minimal) but edge labels are included when this is not the case and the order is important.

both being good for $(B_3, \mathcal{O}_\text{ref}^3, <)$ contradicts the Tutte condition $T2(b)$. So indeed it must have been that $\mathcal{O}^3$ was bad.

\[\square\]

**Subclaim 2.** Let $n > 1$. Fix some $(B_n, \mathcal{O}_\text{ref}, <)$. Suppose $\mathcal{O} = \{e_n^\varepsilon\}$ for $\varepsilon \in \{-, +\}$ with $\mathcal{O}_\text{ref}(e_n^+) = \mathcal{O}_\text{ref}(e_n^-)$. Then $\mathcal{O}$ is bad for $(B_n, \mathcal{O}_\text{ref}, <)$. 


Proof. Assume without loss of generality that $O_{\text{ref}}(e_i^+) = (u, v)$. Define a reference orientation $O_{\text{ref}}^{n+3}$ of $B_{n+3}$ by $O_{\text{ref}}^{n+3}(e_1^+) := O_{\text{ref}}^{n+3}(e_2^-) := O_{\text{ref}}^{n+3}(e_3^-) := (u, v)$ and $O_{\text{ref}}^{n+3}(e_i^+) := O_{\text{ref}}^{n+3}(e_i^-)$ for all $4 \leq i \leq n + 3$. Since $O^3$ is bad for $(B_3, O_{\text{ref}}^3, <)$, repeated use of condition $T2(c)$ and one application of $T2(b)$ says that $O^* := \{ e_3^+, e_{n+3}^- \}$ is bad for $(B_{n+3}, O_{\text{ref}}^{n+3}, <)$. Define the auxiliary graph $G^*$ by $V(G^*) := \{ u, v, w \}$ and $E(G^*) := \{ e_1, \ldots, e_{n+4} \}$, where $e_1 := e_4 := \ldots := e_{n+3} := \{ u, v \}$, $e_{n+4} := \{ v, w \}$ and $e_2 := e_3 := \{ u, w \}$. Let $O_{\text{ref}}^*$ be any extension of $O_{\text{ref}}^{n+3}$. Because the contraction of $((G^*, O_{\text{ref}}^*, <), O^*)$ to the cut $\{ u, \{ v, w \} \}$ is isomorphic to $((B_{n+3}, O_{\text{ref}}^{n+3}, <), O^*)$, we get that $O^*$ is bad for $G^* := (G^*, O_{\text{ref}}^*, <)$. So by condition $T2(c)$, $O^* \setminus e_{n+4}$ is bad for $G^* \setminus e_{n+4}$. Note that $G^* \setminus e_{n+4}$ has two simple cuts: $C_{u_1} := \{ \{ u, v \}, \{ w \} \}$ and $C_{u_2} := \{ \{ u, w \}, \{ v \} \}$. The contraction of $G^* \setminus e_{n+4} : (G^*, O_{\text{ref}}^*, <)$ to $C_{u_1}$ is isomorphic to $((B_2, O_{\text{ref}}^2, <), O^2)$, which is good. So it must be that the contraction of $G^* \setminus e_{n+4} : (G^*, O^*, <)$ to $C_{u_2}$, which is isomorphic to $((B_n, O_{\text{ref}}^3, <), O)$, is bad. □

Under the assumptions of the previous subclaim we have by condition $T2(b)$ that $-O$ is good. Recall by considerations at the beginning of the proof that the status of any fourientation $O$ was only at issue if $O = \{ e^\epsilon \}$ for some $\epsilon \in \{ +, - \}$ and $O \setminus e_3$ was bad. But we just showed that in this case the status of $O$ still agrees with the min-edge cut property defined by $(X, \delta)$. So we are done with Case I.

Case II: $O^{2'}$ is a good fourientation of $(B_2, O_{\text{ref}}^{2'}, <)$.

Note that this case is in contradiction with the min-edge cut property defined by $(X, \delta)$. We claim that our Tutte property must be cut weird.

Subclaim 3. We have $\{-\} \in X$.

Proof. Suppose to the contrary. Define the auxiliary graph $G^*$ by $V(G^*) := \{ u, v, w \}$ and $E(G^*) := \{ e_1, e_2, e_3 \}$ where $e_1 := e_3 := \{ u, v \}$ and $e_2 := \{ u, w \}$. Define $O_{\text{ref}}^*$ by $O_{\text{ref}}^*(e_1^+) := O_{\text{ref}}^*(e_2^-) := (u, v)$ and $O_{\text{ref}}^*(e_3^-) := (w, u)$. Then $O^* := \{ e_2^-, e_3^+ \}$ is good for $G^* := (G^*, O_{\text{ref}}^*, <)$: the graph $G^*$ has two simple cuts $C_{u_1} := \{ \{ u, v \}, \{ w \} \}$ and $C_{u_2} := \{ \{ u, w \}, \{ v \} \}$. The contraction of $G^* \setminus e_{n+4} : (G^*, O_{\text{ref}}^*, <)$ to $C_{u_1}$ is isomorphic to $((B_2, O_{\text{ref}}^2, <), O^2)$, both of which are good by supposition. Let $G^{*'}$ be the graph obtained from $G^*$ by adding an edge $e_4 := \{ v, w \}$ and let $O_{\text{ref}}^{*'}$ be any extension of $O_{\text{ref}}^*$. Then $O^*$ is good for $G^{*'} := (G^{*'}, O_{\text{ref}}^{*'}, <)$ by condition $T2(a)$. Set $C_{u_3} := \{ \{ u \}, \{ v, w \} \}$, a cut of $G^{*'}$. The contraction of $(G_{C_{u_3}}^{*'}, O_{C_{u_3}}^{*'})$ is good; by removing $e_3$ from this contraction using $T2(b)$ we get that something isomorphic to $((B_2, O_{\text{ref}}^{2'}, <), -O^{2'})$ is good. But $O^{2'}$ and $-O^{2'}$ both being good for $(B_2, O_{ref}^{2'}, <)$ contradicts $T2(b)$. So $\{-\} \in X$. □

Subclaim 4. We have $\{+\} \notin X$.

Proof. Define the auxiliary graph $G^*$ by $V(G^*) := \{ u, v, w \}$ and $E(G^*) := \{ e_1, e_2 \}$ where $e_1 := \{ u, v \}$ and $e_2 := \{ u, w \}$. Define $O_{\text{ref}}^*(e_1^+) := (u, v)$ and $O_{\text{ref}}^*(e_2^+) := (u, w)$. Set $O^* := \{ e_2^+ \}$. Then $O^*$ is bad for $G^* := (G^*, O_{\text{ref}}^*, <)$ because its contraction to $C_{u_1} := \{ \{ u, v \}, \{ v \} \}$ is bad. Let $G^{*'}$ be the graph obtained from $G^*$ by adding an edge $e_3 := \{ v, w \}$, let $O_{\text{ref}}^{*'}$ be the extension of $O_{\text{ref}}^*$ with $O_{\text{ref}}^{*'}(e_3^+) := (v, w)$, and
let $O' := O \cup \{e_3^+, e_3^-\}$. Note that $O'$ is good for $G' := (G', O', <)$: the contractions to $Cu_1$ and $Cu_2 := \{(u, v), \{w\}\}$ no longer give potential cuts, and the contraction to $Cu_3 := \{(u), \{v, w\}\}$ is isomorphic to $((B_2, O_2^\ref, <), O_2^\ref)$.

So by condition (2b), one of $O^* \cup \{e_3^+\}$ or $O^* \cup \{e_3^-\}$ must be good for $G^*$. Note that $O^* \cup \{e_3^+\}$ is not good because the contraction of $(G', O^* \cup \{e_3^+\})$ to $Cu_1$ is isomorphic to $((B_2, O_2^\ref, <), -O_2^\ref)$, which is bad by condition $T_2(b)$ since $((B_2, O_2^\ref, <), O_2^\ref)$ is good. So $O^* \cup \{e_3^-\}$ must be good; but then $((B_2, O_2^\ref, <), \{e_3^-, e_3^+\})$, which is isomorphic to the contraction of $(G^*, O^* \cup \{e_3^-\})$ to $Cu_2$, is good. Then by $T_2(b)$ and $T_1$ we get $\{+\} \notin X$. □

Therefore we must have $X = \{\emptyset, \{-\}\}$. This indeed is possible. In this case, the good fourientations are the cut weird fourientations. To see that these are exactly the good fourientations, again we can just check agreement on banana graphs. The only case not addressed by above considerations is when $O$ is a fourientation of $(B_n, O, <)$ for some $n > 1$ where $e_n^\ref \in O$ for $\in \{\-\,+\}$ and $O \setminus e_n = \emptyset$ is bad.

**Subclaim 5.** Let $n > 1$. Set $O := \{e_1^\ref\}$. Then $O$ is good for any $(B_n, O, <)$.

**Proof.** We prove this by induction on $n$. The case $n = 2$ is true by our suppositions. So assume $n > 2$ and the result holds for smaller $n$. Assume without loss of generality that $O_\ref(e_1^\ref) = (u, v)$. Suppose $O_\ref(e_1^\ref) = (u, v)$ for $\in \{\-\,+\}$. Define the auxiliary graph $G^*$ by $V(G^*) := \{u, v, w\}$ and $E(G^*) := \{e_1, \ldots, e_{n+1}\}$ where $e_1 := \ldots := e_{n-2} := e_n := \{u, v\}$ and $e_{n-1} := e_{n+1} := \{u, w\}$. Define $O_\ref(e_i) := O_\ref(e_i)$ for all $1 \leq i \leq n - 2$ and $O_\ref(e_{n-1}^\ref) := O_\ref(e_{n+1}^\ref) := (u, w)$ and $O_\ref(e_{n}^\ref) := (u, v)$. Let $O^* := \{e_1^+, e_{n+1}^\ref\}$. Then $O^*$ is good for $G^* := (G^*, O^*_\ref, <)$. The graph $G^*$ has two simple cuts $Cu_1 := \{(u, v), \{w\}\}$ and $Cu_2 := \{(u, w), \{v\}\}$; the contraction to $Cu_1$ is isomorphic to $((B_2, O_2^\ref, <), O_2^\ref)$ or to $((B_2, O_2^\ref, <), O_2^\ref)$, which are good, and the contraction to $Cu_2$ is isomorphic to $((B_{n-1}, O_{n-1}^\ref, <), \{e_{n-1}^+, e_{n+1}^\ref\})$, which is good by our inductive hypothesis. Let $G'$ be the graph obtained from $G^*$ by adding an edge $e_{n+2} := \{v, w\}$ and let $O_\ref(e_i)$ be any extension of $O^*_\ref$. By condition $T_2(c)$ we have that $O^*$ remains good for $G' := (G^*, O_\ref(e_i), <)$. Set $Cu_3 := \{(u), \{v, w\}\}$, a cut of $G'$. The contraction $(G'_{Cu_3}, O_{Cu_3}^\ref)$ is good; by removing $e_{n+1}^\ref$ from this contraction using condition $T_2(a)$ we get that something isomorphic to $((B_n, O_\ref, <), O)$ is good. □

Under the assumptions of the previous subclaim we have by condition $T_2(b)$ that $-O$ is bad. So indeed any property that lands in Case II would have to be cut weird. By mimicking the proof of Theorem 2.16 one can show that cut weird actually defines a consistent Tutte cut property. Finally note that if $\delta = +$ then by a completely symmetric argument either our Tutte property is still a min-edge cut property or we arrive at the other exceptional case where our property is cut co-weird. □

**Remark 2.18.** Define a signed, ordered, oriented graph to be $(G, O, <, \sigma)$, where the triple $(G, O, <)$ is an ordered, oriented graph, and $\sigma : E(G) \rightarrow \{+, -\}$ is any map from the edges of $G$ to $\{+, -\}$. We could extend our notion of fourientation property to take as input fourientations of signed, ordered, oriented graphs and only require invariance under isomorphism of these more decorated structures. Then we could
extend the min-edge cut (cycle) property defined by \((X, \delta)\) to signed, ordered, oriented graphs by saying that a potential cut \((\vec{C}, \delta)\) to a fourientation \(O\) of \((G, O_{\text{ref}}, <, \sigma)\) is bad if it satisfies both of the following conditions:

\[(i') \{ \varepsilon: e_{\varepsilon}^{\min} \in O \} = S \text{ for some } S \in X, \text{ where } e_{\min} \text{ is the minimum edge in } E(\vec{C}) ; \]

\[(i'') \text{ if } e_{\min} \text{ is unoriented (bidirected) in } O \text{ then } e_{\min}^{\delta-\sigma(e_{\min})} \in E(\vec{C}). \]

The arguments already given in this section establish that the number of good \((k, l, m)\)-fourientations of \((G, O_{\text{ref}}, <, \sigma)\) with respect to the intersection of the min-edge cut property defined by \((X, \delta_1)\) and the min-edge cycle property defined by \((Y, \delta_2)\) is still given by formula (1) in the statement of Theorem 2.13. However, a classification of Tutte properties where we allow the extra decoration \(\sigma\) appears to be significantly more involved than Theorem 2.17, and it is unclear what is gained by this extra level of generality. It would certainly be interesting to find a simple bijection from the good fourientations of \((G, O_{\text{ref}}, <, \sigma_1)\) to the good fourientations of \((G, O_{\text{ref}}, <, \sigma_2)\) with respect to some fixed min-edge cut property \((X, \delta)\).

3. Specializations

In this section we consider \((k, l, m)\)-fourientations for special values of \((k, l, m)\). Let us call a fourientation with no bidirected edges a Type A fourientation, and a fourientation with no unoriented edges a Type B fourientation. In other words, a Type A fourientation is a \((1, 1, 0)\)-fourientation and a Type B fourientation is a \((1, 0, 1)\)-fourientation. The fourientations that are both Type A and Type B, the \((1, 0, 0)\)-fourientations, are precisely the total orientations. The impetus for this research was actually to unify the study of classes of partial orientations that arose in various contexts. We now explain how Tutte fourientation properties give rise to many interesting classes of partial orientations.

3.1. Partial orientations.

**Definition 3.1.** A partial orientation of \((G, O_{\text{ref}})\) is a subset \(O\) of \(E(G)\) such that for each \(e \in E(G)\) at least one of \(e^+\) or \(e^-\) is not in \(O\). If \(e^+ \notin O\) and \(e^- \notin O\) then we say \(e\) is neutral in \(O\) and we write \(e \notin O\). If \(e^+ \in O\) then we say \(e\) is oriented in \(O\).

So a partial orientation is just a Type A fourientation where we call the unoriented edges neutral. However, when studying partial orientations we actually want to consider Type A and Type B fourientations “simultaneously.” Let us call the images of the min-edge classes of fourientations under the identity map from Type A fourientations to partial orientations the Type A classes of partial orientations. There is also an obvious bijection from the set of Type B fourientations of \(G\) to the set of partial orientations of \(G\) where we treat bidirected edges as neutral. Let us call the images of the min-edge classes of fourientations under this second bijection the Type B classes of partial orientations. Many (but not all) of the min-edge classes of partial orientations have been studied before, as we detail in §5. In order to explicitly describe the Type A and B classes of partial orientations, let us give some preliminary definitions.
Definition 3.2. By abuse of language, a directed cut (cycle) of a partial orientation is a directed cut (cycle) of the underlying oriented graph for which all edges are oriented in agreement with the cut (cycle). A potential cut (cycle) of a partial orientation is a directed cut (cycle) of the underlying oriented graph for which all oriented edges are oriented consistently with the cut (cycle), but neutral edges are allowed. In symbols, $\overrightarrow{C}$ is a directed cut (cycle) of $\mathcal{O}$ if $e^\pm \in E(\overrightarrow{C}) \Rightarrow e^\pm \in \mathcal{O}$ for all $e \in E$, whereas $\overrightarrow{C}$ is a potential cut (cycle) of $\mathcal{O}$ if $e^\pm \in E(\overrightarrow{C}) \Rightarrow e^\mp \not\in \mathcal{O}$ for all $e \in E$.

Using these notions of potential and directed cuts and cycles, we give the following names to the Type A classes of partial orientations of an ordered, oriented graph:

1. **Cut/cycle general**: There are no restrictions on cuts/cycles.
2. **Cycle minimal**: The minimum edge in each directed cycle is oriented in agreement with its reference orientation.
3. **Cycle maximal**: The minimum edge in each directed cycle is oriented in disagreement with its reference orientation.
4. **Acyclic**: There are no directed cycles.
5. **Cut directed**: For each potential cut, if the minimum edge of the cut is neutral then the cut contains an oriented edge directed in agreement with the reference orientation of this minimum edge.
6. **Cut negative**: The minimum edge in each potential cut is neutral or is oriented in agreement with its reference orientation.
7. **Cut positive**: The minimum edge in each potential cut is neutral or is oriented in disagreement with its reference orientation.
8. **Cut connected**: Each potential cut contains an oriented edge directed in agreement with the reference orientation of the minimum edge in the cut.
9. **Cut co-connected**: For each potential cut, either the minimum edge of the cut is neutral and the cut contains an oriented edge directed in agreement with the reference orientation of this minimum edge, or the minimum edge is oriented in disagreement with its reference orientation.
10. **Cut neutral**: The minimum edge in each potential cut is neutral.
11. **Cut free**: The minimum edge in each potential cut is neutral and the cut contains an oriented edge directed in agreement with the reference orientation of this minimum edge.

The names of the Type B classes of partial orientations are similar (with “strongly connected” being dual to acyclic). The point of considering Type A and Type B classes simultaneously is that there are interesting containment relations between classes across types: Figure 3 depicts these relations. Theorem 2.13 tells us that all Type A and Type B classes of partial orientations are enumerated by generalized Tutte polynomial evaluations (but note that it is not true in general that an intersection of a Type A and a Type B class is enumerated by a generalized Tutte polynomial evaluation). Figure 4 below displays these specific evaluations. The containment relations depicted in Figure 3 imply inequalities among the generalized Tutte polynomial evaluations in Figure 4; for instance, for any graph $G$ on $n$ vertices with cyclomatic genus $g$ we have $2^g T_G(2, \frac{3}{2}) \leq 2^{n-1} T_G(1, 3)$. 
3.2. **Total orientations.** Of course, we can also set \((k, l, m) = (1, 0, 0)\). The \((1, 0, 0)\)-fourientations of \(G\) are precisely the total orientations. The min-edge cut classes for total orientations are:

1. **Cut general:** There are no restrictions on cuts.
2. **Cut minimal:** The minimum edge in each directed cut is oriented in agreement with its reference orientation.
3. **Cut maximal:** The minimum edge in each directed cut is oriented in disagreement with its reference orientation.
4. **Strongly connected:** There are no directed cuts.

The poset of these four classes ordered by containment is isomorphic to the Boolean lattice on two elements. By intersecting min-edge cut and cycle classes of total orientations we realize all values of \(T(x, y)\) for integral \(0 \leq x, y \leq 2\) as explained in the unifying work of Gioan [21] and Bernardi [11]. Bernardi connects this 3 \(\times\) 3 table with a corresponding table for subgraphs, just as we will do in §4. Note, however, that the input data of [11] is different than what we are working with here: Bernardi uses an embedding of the graph into a surface rather than \(O_{ref}\) and \(<\) to define his classes and in particular to define a notion of internal and external activity. The middle row and column of the 3 \(\times\) 3 table have various equivalent descriptions (see §1.1 above):

- Middle row:
  - cycle minimal total orientations;
  - cycle maximal total orientations;
  - equivalence classes of total orientations modulo cycle reversals;
– indegree sequences of total orientations.

• Middle column:
  – cut minimal total orientations;
  – cut maximal total orientations;
  – equivalence classes of total orientations modulo cocycle reversals;
  – $q$-connected total orientations.

Informally, the indegree sequence of an orientation is the list of the numbers of incoming edges at each vertex, and a $q$-connected orientation is one with a directed path from $q$ to every other vertex. (Co)cycle reversals are explained in §4.2 below. At first sight, it might appear pointless to include both cut minimal and cut maximal orientations in this list since one can be obtained from the other by reversing the orientation of all edges. Our primary motivation for considering them simultaneously is the potential bijective correspondence with edge coloring classes suggested in §4.2.

3.3. Subgraphs. If we set $(k, l, m) = (0, 1, 1)$ we get something rather trivial, namely fourientations with no oriented edges. We may identify such fourientations with subgraphs by thinking of the bidirected edges as belonging to our subgraph and the unoriented edges as being absent. Here a subgraph just means a subset $H \subseteq E(G)$ of the edges of $G$. The min-edge cut classes for subgraphs become:

1. **Cut general**: There are no restrictions on cuts.
2. **Spanning**: The subgraph intersects every cut of the graph nontrivially.

Of course the poset of these two classes ordered by containment is isomorphic to the Boolean lattice on one element. Dually, the min-edge cycle classes become:

1. **Cycle general**: There are no restrictions on cuts.
2. **Forest**: The subgraph has no cycles.

These same subgraph classes will also appear in §4 where we consider edge colorings.

4. Edge Colorings

In this section we show how the enumeration of classes of 4-edge-colorings of $G$ parallels the story of fourientations developed above. To define these classes, we still require input data of $<$, a total order on the edges of $G$, but we no longer require a reference orientation $O_{ref}$. Let us call the pair $(G, <)$ an ordered graph.

4.1. **rygb-edge-colorings**.

**Definition 4.1.** A **rygb-edge-coloring** of an ordered graph is an assignment of the colors red, yellow, green, or blue to the edges of the graph. A **potential cut (cycle)** of an rygb-edge-coloring of an ordered graph is a cut (cycle) of the underlying graph such that all non-minimal edges in the cut (cycle) are blue or green (red or yellow), and the minimum edge is blue, green, or red (red, yellow, or blue).

Note that our definitions of potential cut and cycle for edge colorings depend on the total edge order, whereas our definitions of potential cut and cycle for fourientations did not. Conceptually, subgraphs are simpler than orientations and to compensate we have “shifted” some of the complexity of orientations onto the edge order.
Definition 4.2. A color min-edge cut (cycle) property is defined by a choice of colors \( X \subseteq \{r, g, b\}\) \(\{r, y, b\}\)\(^2\). A potential cut (cycle) of a rygb-edge-coloring is bad with respect to the color min-edge cut (cycle) property associated to \( X \) if the color of its minimum edge belongs to \( X \), and is good otherwise.

Definition 4.3. We obtain a \((k_1, k_2, l, m)\)-rygb-edge-coloring from a rygb-edge-coloring by assigning each of the red edges one of \(k_1\) “shades” of red, each of the blue edges one of \(k_2\) shades of blue, each of the green edges one of \(l\) shades of green, and each of the yellow edges one of \(m\) shades of yellow. A \((k_1, k_2, l, m)\)-rygb-edge-coloring has a color min-edge cut (cycle) property if its underling rygb-edge-coloring has it.

Theorem 4.4. For an rygb-edge-coloring \( C \), let \( r(C) \) denote the number of red edges, \( b(C) \) denote the number of blue edges, \( g(C) \) denote the number of green edges, and \( y(C) \) denote the number of yellow edges. Fix a color min-edge cut property \( X \subseteq \{r, g, b\}\) and color min-edge cycle property \( Y \subseteq \{r, y, b\}\). Then for \((G, <)\), an ordered graph with \( n \) vertices whose cyclomatic genus is \( g \) we have
\[
\sum_C k_1^{r(C)} k_2^{b(C)} l^{g(C)} m^{y(C)} = (k_1 + m)^{n-1}(k_2 + l)^g T_G\left(\frac{x_0}{k_1 + m}, \frac{y_0}{k_2 + l}\right)
\]
where the sum is over all good rygb-edge-colorings \( C \) of \((G, <)\) with respect to \( X \) and \( Y \), and where
\[
x_0 := \delta(r \notin X) k_1 + \delta(b \notin X) k_2 + \delta(g \notin X) l + m
\]
\[
y_0 := \delta(r \notin Y) k_1 + \delta(b \notin Y) k_2 + l + \delta(y \notin Y) m
\]
and \( \delta(P) \) is 1 if if \( P \) is true and 0 if \( P \) is false. In other words, the number of good \((k_1, k_2, l, m)\)-rygb-edge-colorings of \((G, <)\) is given by \(2\).

Proof. The proof is very similar to, but much simpler than, that of Theorem 2.13. Again a key observation is that potential cuts and potential cycles of rygb-edge-colorings do not intersect. After making this observation the same reasoning as in \(2\) will allow us to apply Theorem 1.1. \(\square\)

If we set \(k_1 := k_2 := k\), we see that the generalized Tutte polynomial evaluations arising in Theorem 4.4 are exactly the same as those arising in Theorem 2.13. We give the following names to the eight classes of color min-edge cut properties (where “green-blue cut” means a cut all of whose edges are either green or blue and so on):

1. **Cut general** \((X = \emptyset)\): There are no restrictions on cuts.
2. **g-quasi ry spanning** \((X = \{g\})\): The coloring has no green-blue cut whose minimum edge is green.
3. **b-quasi ry spanning** \((X = \{b\})\): The coloring has no green-blue cut whose minimum edge is blue.
4. **r external** \((X = \{r\})\): The coloring has no red-green-blue cut whose minimum edge is the unique red edge of the cut.
5. **ry spanning** \((X = \{b, g\})\): The coloring has no green-blue cut.

\(\)\(^2\)We hope the reader can distinguish our use of \( g \) for the color green and our use of \( g \) for genus.
(6) **g-quasi ry spanning + r external** \((X = \{r, g\})\): The coloring has no green-blue cut whose minimum edge is green, and no red-green-blue cut whose minimum edge is the unique red edge of the cut.

(7) **b-quasi ry spanning + r external** \((X = \{r, b\})\): The coloring has no green-blue cut whose minimum edge is blue, and no red-green-blue cut whose minimum edge is the unique red edge of the cut.

(8) **ry spanning + r external** \((X = \{r, g, b\})\): The coloring has no green-blue cut, and no red-green-blue cut whose minimum edge is the unique red edge of the cut.

The cycle properties are similar. The poset of the above eight properties ordered by containment is of course isomorphic to the Boolean lattice on three elements. These color min-edge properties correspond exactly to the min-edge cut and cycle properties of fourientations. The top table in Figure 4 shows the generalized Tutte polynomial evaluations that enumerate all the classes of fourientations and rygb-edge-colorings when \((k, l, m) = (1, 1, 1)\). We may plug in \((k, l, m) = (1, 1, 0)\) or \((k, l, m) = (1, 0, 1)\) to arrive at rgb- or ryb-edge-colorings which correspond to Type A or Type B classes of partial orientations. The middle two tables in Figure 4 show the relevant enumerations for these 3-edge-coloring classes (which we will not define explicitly). If we set \((k, l, m) = (1, 0, 0)\), we get rb-edge-colorings, which we can identify with subgraphs by thinking of the red edges as belonging to our subgraph and the blue edges as being absent. For a subgraph \(H \subseteq E(G)\) of an ordered graph \((G, <)\), let us say \(e \in H\) is internally active if there is some cut \(Cu\) of \(G\) with \(H \cap E(Cu) = \{e\}\) and where \(e\) is the minimum element of \(E(Cu)\). Similarly, let us say \(e \in H\) is externally active if there is some cycle \(Cy\) of \(G\) with \(H \setminus E(Cy) = \{e\}\) and where \(e\) is the minimum element of \(E(Cy)\). These notions of internal and external activity go back to Tutte [51]. The subgraph min-edge cut properties we arrive at are:

1. **Cut general**: No restriction on cuts.
2. **Spanning**: The subgraph intersects every cut of the graph nontrivially.
3. **External**: The subgraph contains no internally active edges.
4. **Spanning + External**: The subgraph intersects every cut of the graph nontrivially and the subgraph contains no internally active edges.

The poset of these subgraph properties ordered by containment is isomorphic to the Boolean lattice on two elements. The cycle properties are analogous. The bottom 3 × 3 in Figure 4 table then shows the classical orientation/subgraph enumerations. If \(G\) is a planar graph and \(G^*\) is its dual graph then there is a natural correspondence between orientations of \(G\) and \(G^*\) (via the “right-hand rule”). Moreover, we can extend this duality to fourientations by mapping a bidirected edge in \(G\) to the unoriented dual edge in \(G^*\) and vice versa. The important observation about this duality is that it interchanges potential cuts and cycles and it respects the minimum edge in each cut and cycle. It is well-known that the Tutte polynomial of \(G^*\) is obtained from the Tutte polynomial of \(G\) by interchanging the variables \(x\) and \(y\); this follows from the fact that deletion and contraction are planar dual. As an immediate consequence, we see that this duality maps min-edge classes of fourientations of \(G\) to min-edge classes of fourientations of \(G^*\) by transposing each table in Figure 4 and then swapping the
Type A and Type B tables. This suggests that perhaps the correct way of presenting these tables is to place them in one large $7 \times 7$ table such that planar duality can be seen as a single transposition. For rygb-edge-colorings the planar duality is even simpler: we map red edges to blue dual edges and vice-versa, and we map green edges to yellow dual edges and vice-versa.

4.2. **A master bijection.** Our work suggests that there should be a master bijection between fourientations and rygb-edge-colorings of $G$ that respects the min-edge classes.

**Conjecture 4.5.** There is a bijection $\zeta$ between fourientations of $G = (G, \mathcal{O}_{\text{ref}}, <)$ and rygb-edge-colorings of $(G, <)$ that maps bidirected edges to yellow edges, unoriented edges to green edges, and such that

\[
\begin{align*}
\mathcal{O} \text{ is} & \quad \text{if and only if } \zeta(\mathcal{O}) \text{ is} \\
\text{cut directed} & \quad g\text{-quasi ry spanning} \\
\text{cut negative} & \quad b\text{-quasi ry spanning} \\
\text{cut positive} & \quad r \text{ external} \\
\text{cut connected} & \quad r \text{y spanning} \\
\text{cut co-connected} & \quad g\text{-quasi ry spanning} + r \text{ external} \\
\text{cut neutral} & \quad b\text{-quasi ry spanning} + r \text{ external} \\
\text{cut free} & \quad r \text{y spanning} + r \text{ external}
\end{align*}
\]

and similarly for min-edge cycle properties.

In principle, the deletion-contraction arguments above may give a complicated recursive definition of such a bijection. However, what we really desire is a simple non-recursive procedure, perhaps building on some basic tree algorithm like depth-first search. Equivalently, we could ask for a four parameter (i.e., $(k_1, k_2, l, m)$) extension of our enumerative results for fourientations. Bernardi [11] defines a bijection between total orientations and subgraphs, but again, his input data is different from ours. For more partial results towards this bijection at the level of total orientations see [30], [12], [10], and [1].

**Example 4.6.** Let $G$ be the triangle graph as below:

Take $\mathcal{O}_{\text{ref}}$ as above and let $<$ be given by $e_1 < e_2 < e_3$. Figure 5 has a table with the fourientations of $G$ divided into some of their min-edge classes. Each cell of the table is filled with the fourientations that belong to the cut class of the corresponding column and cycle class of the corresponding row, but not to any further refined classes: for instance, the cycle free-cut directed cell contains all those fourientations that are cycle free and cut directed but not cycle free and cut connected. This figure also has a table with the rygb-edge-colorings of $(G, <)$ similarly divided into their corresponding min-edge classes. Figures 6 and 7 have tables for the remaining classes of fourientations and
rygb-edge-colorings. Note that one bijection $\zeta$ satisfying the conditions of Conjecture 4.5 is given by taking a fourientation to the rygb-edge-coloring in the same spot below.

Let us conclude this section by explaining how the desired bijection $\zeta$ of Conjecture 4.5, even when restricted to total orientations, must be somewhat subtle. We now recall the cycle/cocycle reversal systems of Gioan [21]. Given a total orientation $O$ of $G$, a (co)cycle reversal is the operation of replacing $O$ by $(O \setminus E(\overrightarrow{C})) \cup E(-\overrightarrow{C})$ for some directed cycle (cut) $\overrightarrow{C}$ of $O$. We write $O \overset{Cy}{\sim} O'$ ($O \overset{Cu}{\sim} O'$) if $O$ is related to $O'$ by a series of (co)cycle reversals, and write $O \sim O'$ if $O$ is related to $O'$ by a series of cycle and cocycle reversals. The equivalence relations $\overset{Cy}{\sim}, \overset{Cu}{\sim}$, and $\sim$ define the cycle, cocycle, and cycle-cocycle reversal systems respectively. Each equivalence class in the (co)cycle reversal system contains a unique cycle (cut) minimal orientation, and each equivalence class in the cycle-cocycle reversal system contains a unique cut minimal-cycle minimal orientation.

We can define some similar operations and equivalence relations for subgraphs. Specifically, given a subgraph $H \subseteq E(G)$, an internal (external) toggle is the operation of replacing $H$ by $H \Delta \{e\}$ for some edge $e \in E(G)$ that is internally (externally) active in $H$ or in $H \Delta \{e\}$. We write $H \overset{in}{\overset{ex}{\sim}} H'$ ($H \overset{ex}{\overset{in}{\sim}} H'$) if $H$ is related to $H'$ by a series of internal (external) toggles, and write $H \sim H'$ if $H$ is related to $H'$ by a series of internal and external toggles. Each equivalence class with respect to $\overset{in}{\sim}$ contains a unique spanning subgraph and each equivalence class with respect to $\overset{ex}{\sim}$ contains a unique subgraph that is a forest. The partition of subgraphs of $G$ into equivalence classes with respect to $\sim$ gives the tree-interval decomposition of $G$ (see Bernardi [11, §4] and [15, 7, 23]): this decomposition is so-called because each equivalence class with respect to $\sim$ contains a unique spanning tree of $G$.

Thus we see that the desired bijection $\zeta$ maps representatives (i.e. cut minimal, cycle minimal, and cut minimal-cycle minimal orientations) of these three orientation equivalence classes to corresponding representatives (i.e. spanning subgraphs, forests, and spanning trees) of the corresponding three subgraph equivalence classes. However, note that $\zeta$ cannot simply transform (co)cycle reversals into external (internal) toggles because in particular the cycle-cocycle reversal system and the tree-interval decomposition of $G$ can have different partition structures, as the next example demonstrates.

**Example 4.7.** Let $G := (G, O_{ref}, <)$ be as in Example 4.6. The cycle-cocycle reversal system yields the following partition of the total orientations of $G$ (with cut minimal-cycle minimal orientations in red):

\[
\left\{ \begin{array}{c}
\begin{array}{c}
\Delta \\
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
\Delta \\
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
\Delta \\
\end{array}
\end{array} \right\}
\]

while the tree-interval decomposition yields the following partition of the subgraphs of $G$ (with spanning trees in red):

\[
\left\{ \begin{array}{c}
\begin{array}{c}
\triangle \\
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
- \\
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
\triangle \\
\end{array}
\end{array} \right\}.
\]


### Fourorientations – *rygb-edge-colorings*

| General | Cut pos./neg. | Cut neutral | Cut free |
|---------|--------------|-------------|---------|
|         | Cut dir. | (co)-con. |         |         |
| General | $2^E|T(2, 2)$ | $2^E|T(\frac{3}{2}, 2)$ | $2^E|T(1, 2)$ | $2^E|T(\frac{1}{2}, 2)$ |
| Cycle pos./neg. | $2^E|T(2, \frac{3}{2})$ | $2^E|T(\frac{3}{2}, \frac{3}{2})$ | $2^E|T(1, \frac{3}{2})$ | $2^E|T(\frac{1}{2}, \frac{3}{2})$ |
| Cycle directed | $2^E|T(2, 1)$ | $2^E|T(\frac{3}{2}, 1)$ | $2^E|T(1, 1)$ | $2^E|T(\frac{1}{2}, 1)$ |
| Cycle neutral | $2^E|T(2, \frac{1}{2})$ | $2^E|T(\frac{3}{2}, \frac{1}{2})$ | $2^E|T(1, \frac{1}{2})$ | $2^E|T(\frac{1}{2}, \frac{1}{2})$ |
| Cycle (co)-con. |       |        |        |         |

| General | b/g-quasi | ry span | + r ext |
|---------|-----------|---------|--------|
|         | ry spanning |         |        |
|         | r ext. |         |        |
|         | + r external |         |        |

### Type A classes of partial orientations

**rygb-edge-colorings**

| General | Cut pos./neg. | Cut neutral | Cut free |
|---------|--------------|-------------|---------|
|         | Cut dir. | (co)-con. |         |         |
| Gen.    | $2^g|T(3, \frac{3}{2})$ | $2^g|T(2, \frac{3}{2})$ | $2^g|T(1, \frac{3}{2})$ | $2^g|T(0, \frac{3}{2})$ |
| Cycle min. | $2^g|T(3, 1)$ | $2^g|T(2, 1)$ | $2^g|T(1, 1)$ | $2^g|T(0, 1)$ |
| Cycle max. | $2^a|T(3, \frac{1}{2})$ | $2^a|T(2, \frac{1}{2})$ | $2^a|T(1, \frac{1}{2})$ | $2^a|T(0, \frac{1}{2})$ |
| Acyc. | b/g-quasi | ry span. | + r ext. | ry span. |
|        | ry span. | + r external |         |        |

### Type B classes of partial orientations

**rygb-edge-colorings**

| General | Cut pos./neg. | Cut neutral | Cut free |
|---------|--------------|-------------|---------|
|         | Cut dir. | (co)-con. |         |         |
| Gen.    | $2^{n-1}|T(\frac{3}{2}, 3)$ | $2^{n-1}|T(1, 3)$ | $2^{n-1}|T(\frac{1}{2}, 3)$ |
| Cycle pos./neg. | $2^{n-1}|T(\frac{3}{2}, 2)$ | $2^{n-1}|T(1, 2)$ | $2^{n-1}|T(\frac{1}{2}, 2)$ |
| Cycle directed | $2^{n-1}|T(\frac{3}{2}, 1)$ | $2^{n-1}|T(1, 1)$ | $2^{n-1}|T(\frac{1}{2}, 1)$ |
| Cycle neutral | $2^{n-1}|T(\frac{1}{2}, 0)$ | $2^{n-1}|T(1, 0)$ | $2^{n-1}|T(\frac{1}{2}, 0)$ |
| Cycle (co)-con. |       |        |        |         |
| Cycle free | b/g-quasi | ry span. | + r ext. | ry span. |
|        | ry span. | + r external |         |        |

### Total orientations – *Subgraphs (rb-edge-colorings)*

| General | Cut pos./neg. | Cut neutral | Cut free |
|---------|--------------|-------------|---------|
|         | Cut dir. | (co)-con. |         |         |
| General | $T(2, 2)$ | $T(1, 2)$ | $T(0, 2)$ |
| Cycle min. | $T(2, 1)$ | $T(1, 1)$ | $T(0, 1)$ |
| Acyclic | $T(2, 0)$ | $T(1, 0)$ | $T(0, 0)$ |

| General | Spanning | + External |
|---------|----------|------------|
| External | Spanning | + External |

**Figure 4.** Four tables showing how the min-edge classes of generalized orientations, and their associated classes of edge colorings, are enumerated by generalized Tutte polynomial evaluations.
|                  | General | Cut directed | Cut connected | Cut free |
|------------------|---------|--------------|---------------|---------|
| General          |         |              |               |         |
| Cycle directed   |         |              |               |         |
| Cycle connected  |         |              |               |         |
| Cycle free       |         |              |               |         |

|                  | General  | g-quasi ry span. | ry span. | ry span. + r ext. |
|------------------|-----------|------------------|---------|-------------------|
| General          |           |                  |         |                   |
| y-quasi ry forest|           |                  |         |                   |
| ry forest         |           |                  |         |                   |
| ry forest + b int.|           |                  |         |                   |

Figure 5. Certain classes of fourientations and the corresponding classes of rygb-edge-colorings for Example 4.6.
Figure 6. Certain classes of fourientations and the corresponding classes of rygb-edge-colorings for Example 4.6.
Figure 7. Certain classes of fourientations and the corresponding classes of rygb-edge-colorings for Example 4.6.
5. Connections to previous work and future directions

In this section we explore how the min-edge classes of partial orientations are connected to various geometric, combinatorial, and algebraic structures. Throughout we fix an ordered, oriented graph \( G = (G, \mathcal{O}_{ref}, <) \) with \( n := |V(G)| \) vertices and cyclomatic genus \( g := |E(G)| - |V(G)| + 1 \). We will suppress mention of the reference orientation and edge order where it is not necessary (and thus for example speak of fourientations or partial orientations of \( G \)).

5.1. Bi(co)graphical arrangements and cycle (cut) neutral partial orientations. Cycle neutral partial orientations are related to the bigraphical arrangements originally defined by the second author and Perkinson [27]. We explain this relationship precisely here, and also define for the first time the object dual to the bigraphical arrangement, namely the bicographical arrangement. The bi(co)graphical arrangement depends on \( G \) as well as a parameter list \( A = (a_e \pm) \in \mathbb{R}_{>0}^{E(G)} \), which is a list of positive real parameters \( a_e^+, a_e^- \in \mathbb{R}_{>0} \) for each \( e \in E(G) \). For an appropriate choice of parameters, the regions of the bi(co)graphical arrangement are in bijection with cycle (cut) neutral partial orientations; moreover the regions that avoid a certain generic hyperplane are in bijection with cut minimal-cycle neutral (cycle minimal-cut neutral) partial orientations, and the bounded regions are in bijection with strongly connected-cycle neutral (acyclic-cut neutral) partial orientations. The result concerning bounded regions of the bigraphical arrangement was essentially already proved in [27], albeit in slightly different language. In general, for any hyperplane arrangement \( \mathcal{A} \) these three region counts (total number of regions, number of regions avoiding a generic hyperplane, number of bounded regions) are given (up to sign) by evaluating the characteristic polynomial \( \chi_{\mathcal{A}}(t) \) at \( t = -1, 0, 1 \). These three characteristic polynomial evaluations allow us to explain an entire row (resp., column) in one of the tables in Figure 4 in terms of the bi(co)graphical arrangement.

The degenerate case of the bi(co)graphical arrangement where we set all the parameters \( a_e \pm \) to 0 recovers the (co)graphical arrangement. Many of the results here are extensions from total orientations to partial orientations of results obtained by Greene and Zaslavsky in [24], especially §8 of that paper which explores the cographical arrangement. In the proofs in this subsection we assume some familiarity with the theory of hyperplane arrangements, especially the notions of the intersection poset and characteristic polynomial of a hyperplane arrangement; see [19] for all the relevant definitions and background information.

**Definition 5.1.** Let \( A = (a_e \pm) \in \mathbb{R}_{>0}^{E(G)} \) be a parameter list. Let \( W \simeq \mathbb{R}^{V(G)} \) be a real vector space with basis \( x_v \) for \( v \in V \). Let \( U \subseteq W \) be the subspace of \( W \) where \( \sum_{v \in V(G)} x_v = 0 \). The *bigraphical arrangement* \( \Sigma_{(G, \mathcal{O}_{ref})} \subseteq U \) is

\[
\Sigma_{(G, \mathcal{O}_{ref})}(A) := \{ H_{e^+} \cap U, H_{e^-} \cap U : e \in E(G) \text{ with } e \text{ not a loop} \}
\]

where for a non-loop \( e \in E(G) \) with \( e^\pm = (u, v) \) we define \( H_{e^\pm} := x_v - x_u = a_e \pm \). Note that the bigraphical arrangement is an essential arrangement of \( 2|E(G)| \) hyperplanes in \( (n - 1) \)-dimensional space.
Definition 5.2. Let \( A \in \mathbb{R}^{|E(G)|} \) be a parameter list. Let \( W \simeq \mathbb{R}^{|E(G)|} \) be a real vector space with basis \( x_e \) for \( e \in E(G) \), with the convention that \( x_e = -x_{e'} \). Let \( U \subseteq W \) be the subspace of \( W \) where for every \( v \in V(G) \) we have \( \sum_{e \in E(G) \setminus \{v\}} x_e = 0 \). The bicographical arrangement \( \Sigma_{\text{bic}}(G, O_{\text{ref}})(A) \subseteq U \) is

\[
\Sigma_{\text{bic}}(G, O_{\text{ref}})(A) := \{ H_e^+ \cap U, H_e^- \cap U : e \in E(G) \}
\]

where for \( e \in E(G) \) we define \( H_e := x_e = a_e \). Note that the bicographical arrangement is an essential arrangement of \( 2|E(G)| \) hyperplanes in \( g \)-dimensional space. (This is because \( U \) is determined by \( n \) linear equations, any \( n - 1 \) of which are linearly independent, so its dimension is \(|E(G)| - (n - 1) = g\).)

A region of a hyperplane a hyperplane arrangement \( A \) in \( \mathbb{R}^k \) is a connected component of \( \mathbb{R}^k \setminus A \). In both the bigraphical and bicographical arrangements, the hyperplanes \( H_e^+ \) and \( H_e^- \) cut out a “sandwich” in space for each \( e \in E(G) \), so that for any region \( R \) of the arrangement exactly one of the following holds:

(a) \( R \) is in the half-space of \( U \setminus H_e^+ \) opposite from \( H_e^- \);
(b) \( R \) is in the half-space of \( U \setminus H_e^- \) opposite from \( H_e^+ \);
(c) \( R \) is between \( H_e^+ \) and \( H_e^- \).

Thus there is a natural map \( R \mapsto O_R \) that associates to any region \( R \) of either the bigraphical or bicographical arrangement a partial orientation \( O_R \) of \( (G, O_{\text{ref}}) \) whereby \( e \in E(G) \) is oriented as \( e^+ \) in case [a] it is oriented as \( e^- \) in case [b] and it is left neutral in case [c]. The second author and Perkinson [27] show that for a generic parameter list \( A \) the number of regions of \( \Sigma_{\text{bic}}(G, O_{\text{ref}})(A) \) is given by a generalized Tutte polynomial evaluation. In order to make their input data compatible with the edge order \( < \) used to define classes of partial orientation above we will fix a particular choice of generic parameters, namely, exponential parameters. We define the exponential parameter list associated to \( < \) to be \( A^\leq := (a_{e}^\leq) \) where for each \( e \in E(G) \) we set \( a_{e}^\leq := (1/2)^i \) if \( e \) is the \( i \)th smallest edge according to \( < \). That is, if \( e_1 < e_2 < \cdots < e_m \) are the elements of \( E(G) \), then \( a_{e_i}^\leq = a_{e_i}^\leq = (1/2)^i \).

Proposition 5.3. The map \( R \mapsto O_R \) is a bijection between the regions of \( \Sigma_{(G, O_{\text{ref}})}(A^\leq) \) and the cycle neutral partial orientations of \( G \).

Proof. Cleary \( R \mapsto O_R \) is injective. It is shown in [27] that for a bigraphical arrangement \( \Sigma_{(G, O_{\text{ref}})}(A) \) with arbitrary parameter list \( A \), the image of this map \( R \mapsto O_R \) is the set of so-called \( A \)-admissible partial orientations. A partial orientation \( O \) is \( A \)-admissible if every potential cycle of \( O \) has a positive score with respect to \( A \), where

\[ \text{In the case of the bigraphical arrangement, if } e \text{ is a loop we did not include hyperplanes } H_e \text{ for } e \text{ because they would lead to contradictory equations, but we can in fact consider these as hyperplanes “at infinity” and thus treat any region as “between” } H_e^+ \text{ and } H_e^- \text{. Thus for a region } R \text{ of } \Sigma_{(G, O_{\text{ref}})}(A), \text{ a loop will always be neutral in } O_R. \text{ It can similarly be seen that for a region } R \text{ of } \Sigma_{(G, O_{\text{ref}})}(A), \text{ a bridge will always be neutral in } O_R. \]
the score $\nu_A(C, \mathcal{O})$ of a potential cycle $\overrightarrow{Cy}$ is given by

$$\nu_A(\overrightarrow{Cy}, \mathcal{O}) := \sum_{e^\pm \in E(\overrightarrow{Cy}), e \notin \mathcal{O}} a_{e^\pm} - \sum_{e^\pm \in E(\overrightarrow{Cy}), e \in \mathcal{O}} a_{e^\pm}.$$ 

We are interested in the case of the exponential parameter list $A^<$. There is a simpler description of admissibility in this special case: a partial orientation is $A^<$-admissible precisely when the minimal edge in every potential cut is neutral because the contribution of this minimal edge in a potential cycle dominates the score of that cycle. In other words, a partial orientation is $A^<$-admissible precisely when it is cycle neutral. So indeed the image of $R \mapsto \mathcal{O}_R$ is the set of cycle neutral orientations.

**Proposition 5.4.** The map $R \mapsto \mathcal{O}_R$ is a bijection between the regions of $\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)$ and the cut neutral partial orientations of $G$.  

**Proof.** This proposition is formally dual to the previous one. Using the same techniques as in [27] we can describe when a partial orientation is in the image of $R \mapsto \mathcal{O}_R$ in terms of scores associated to potential cuts, and we will see that with $A^<$ the $A$-coadmissible partial orientations will be precisely the cut neutral ones. Alternatively, one could also prove, as in [27], that because $A^<$ is generic the characteristic polynomial of $\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)$ is $\chi_{\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)}(t) = (-2)^g T_G(1, 1 - t/2)$, which would show via Zaslavsky’s theorem [54, 49, Theorem 2.5] that there are at least the same number of cut neutral partial orientations as regions of $\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)$. Then it is easy to see that $\mathcal{O}_R$ for $R$ a region of $\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)$ cannot have a potential cut whose minimum edge is directed, proving that the map is indeed a bijection. \qed

Compare the following propositions to [24, Corollary 8.2].

**Proposition 5.5.** Let $M > 0$ be some large positive constant and define the hyperplane $H_0 \subseteq \mathbb{R}^{n-1}$ by $H_0 := \sum_{e \in E(G), e^+ = (u, v)} q^<(x_v - x_u) = -M$. Then the map $R \mapsto \mathcal{O}_R$ is a bijection between the regions $R$ of $\Sigma_{(G, \mathcal{O}_{rel})}^*(A^<)$ with $R \cap H_0 = \emptyset$ and the cut minimal-cycle neutral partial orientations of $G$. 

**Proof.** Let $\mathcal{A}$ be an essential arrangement of hyperplanes in $\mathbb{R}^k$. Let us say the hyperplane $H$ is generic with respect to $\mathcal{A}$ if for any $H_1, \ldots, H_m \in \mathcal{A}$, we have that $H$ has nonempty intersection with $H_1 \cap \ldots \cap H_m$ if and only if $\dim(H_1 \cap \ldots \cap H_m) \geq 1$. Suppose $H$ is generic with respect to $\mathcal{A}$. Then the number of regions $R$ of $\mathcal{A}$ such that $H \cap R = \emptyset$ is given by $(-1)^k \chi_\mathcal{A}(0)$, where $\chi_\mathcal{A}$ is the characteristic polynomial of $\mathcal{A}$. This assertion is [24, Theorem 3.1].

We say that a parameter list $A$ is generic if the arrangement $\Sigma_{(G, \mathcal{O}_{rel})}(A)$ is generic in the sense of [49, §2]. It is shown in [27] that for generic $A$, the characteristic polynomial of $\Sigma_{(G, \mathcal{O}_{rel})}(A)$ is $\chi_{\Sigma_{(G, \mathcal{O}_{rel})}(A)}(t) = (-2)^{n-1} T_G(1 - t/2, 1)$. Thus the previous claim tells us that the number of regions of $\Sigma_{(G, \mathcal{O}_{rel})}(A)$ for generic $A$ that avoid a generic hyperplane is $2^{n-1} T_G(1, 1)$, which is precisely the number of cut minimal-cycle positive partial orientations of $G$ by Theorem 2.13. It is easy to see that $A^<$ is a generic parameter list and $H_0$ is generic with respect to $\Sigma_{(G, \mathcal{O}_{rel})}(A^<)$. 


To finish the proof, we show that if $\mathcal{O}_R$ is not cut minimal, then $R$ must have nonempty intersection with $H_0$. Suppose $\mathcal{O}_R$ is not cut minimal. Then there is a directed cut $\overrightarrow{Cu} = (W, W^c)$ of $\mathcal{O}_R$ such that the orientation of the minimal edge in $E(\overrightarrow{Cu})$ disagrees with $\mathcal{O}_{\text{ref}}$. Let $1_{W^c} := \sum_{e \in W^e} x_e$ and let $p$ be a point in $R$. Let $L_0$ denote the linear form $\sum_{e \in E, e^+(u,v)} a_{e^+}^<(x_v - x_u)$. Observe that $L_0(p + t1_{W^c}) = L_0(p) + Nt$ where

$$N := \sum_{e^+ \in E(\overrightarrow{Cu})} a_{e^+}^c - \sum_{e^- \in E(\overrightarrow{Cu})} a_{e^-}^c.$$  

Note that $N$ is negative because the orientation of the minimal edge in $E(\overrightarrow{Cu})$ disagrees with $\mathcal{O}_{\text{ref}}$. And note also that $p + t1_{W^c} \in R$ for all $t \in [0, \infty)$. Thus we can find a point $q \in R$ with $L_0(q)$ arbitrarily small. This means $R$ intersects $H_0$ nontrivially as long as $M$ is taken to be sufficiently large. \hfill \Box

**Proposition 5.6.** Let $M \gg 0$ be some large positive constant and define the hyperplane $H_0 \subseteq \mathbb{R}^9$ by $H_0 := \sum_{e \in E(G)} a_{e^+}^c x_e^+ = -M$. Then the map $R \mapsto \mathcal{O}_R$ is a bijection between the regions $R$ of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ with $R \cap H_0 = \emptyset$ and the cycle minimal-cut neutral partial orientations of $\mathcal{G}$.

**Proof.** Again, this proposition is formally dual to the previous one and the key is to compute the characteristic polynomial of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$. \hfill \Box

The following propositions should be seen as analogous to the fact that there are no strongly connected-acyclic total orientations, which agrees with there being no bounded regions of the (co)graphical arrangement.

**Proposition 5.7.** The map $R \mapsto \mathcal{O}_R$ is a bijection between the regions of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ that are bounded and the strongly connected-cycle neutral partial orientations of $\mathcal{G}$.

**Proof.** It is shown in [27] that the bounded regions $R$ of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A)$ for any parameter list $A$ are those for which $\mathcal{O}_R$ is $A$-admissible and such that every oriented edge in $\mathcal{O}_R$ belongs to a potential cycle. Although they did not describe it in these terms, that is equivalent to saying that the bounded regions $R$ are those for which $\mathcal{O}_R$ is $A$-admissible and strongly connected because, in light of Proposition 2.5, each oriented edge in a partial orientation either belongs to a potential cycle or to a directed cut, but not both. Thus indeed the bounded regions $R$ of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ are those for which $\mathcal{O}_R$ is strongly connected and cycle neutral. \hfill \Box

**Proposition 5.8.** The map $R \mapsto \mathcal{O}_R$ is a bijection between the regions of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ that are bounded and the acyclic-cut neutral partial orientations of $\mathcal{G}$.

**Proof.** This proposition is again dual to the previous one. \hfill \Box

**Example 5.9.** Let $\mathcal{G} := (G, \mathcal{O}_{\text{ref}}, <)$ be as in Example 4.6. The Tutte polynomial of $G$ is $T_G(x, y) = x^2 + x + y$. Figure 8 shows the bigraphical arrangement of $\mathcal{G}$ together with a labeling of its regions by partial orientations. Note that there are $2^{n-1}T_G(\frac{3}{2}, 1) = 19$ regions of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ and their labels are the cycle neutral partial orientations. There are $2^{n-1}T_G(1, 1) = 12$ regions of $\Sigma^*_{(G, \mathcal{O}_{\text{ref}})}(A^<)$ that avoid $H_0$, shaded in light and dark.
The hyperplane $H_0 = \frac{1}{2}(x_{v_2} - x_{v_1}) + \frac{1}{4}(x_{v_3} - x_{v_1}) + \frac{1}{8}(x_{v_2} - x_{v_3}) = -M$ from Proposition 5.5 is depicted in red.

Figure 8. The bigraphical arrangement $\Sigma_{(G,\mathcal{O}_{\text{ref}})}(A^<)$ in Example 5.9.

The hyperplane $H_0 = \frac{1}{2}(x_{v_2} - x_{v_1}) + \frac{1}{4}(x_{v_3} - x_{v_1}) + \frac{1}{8}(x_{v_2} - x_{v_3}) = -M$ from Proposition 5.5 is depicted in red.

Figure 9. The bicographical arrangement $\Sigma^*_{(G,\mathcal{O}_{\text{ref}})}(A^<)$ in Example 5.9.

The hyperplane $H_0 = \frac{1}{2}(x_{e_1}^-) + \frac{1}{4}(x_{e_2}^+) + \frac{1}{8}(x_{e_3}^+) = -M$ from Proposition 5.6 is depicted in red.

Remark 5.10. Let $W \simeq \mathbb{R}^{E(G)}$ be a real vector space with basis $x_{e^+}$ for $e \in E(G)$ with the convention that $x_{e^-} = -x_{e^+}$. Let $\mathcal{E}_{(G,\mathcal{O}_{\text{ref}})}(A) := \{H_{e^\pm}\}$ be the “perturbed coordinate hyperplane arrangement” in $W$ with $H_{e^\pm} := x_{e^\pm} = a_{e^\pm}$. For a directed cut or cycle $\overrightarrow{C}$ of $G$ define the vector $x_{\overrightarrow{C}} := \sum_{e^\pm \in E(C)} x_{e^\pm} \in W$. The bicographical arrangement is the projection of $\mathcal{E}_{(G,\mathcal{O}_{\text{ref}})}(A)$ to the subspace $U \subseteq W$ where $x_{\overrightarrow{C}} = 0$ for gray, and their labels are the cut minimal-cycle neutral partial orientations. There are $2^{n-1}T_G(\frac{1}{2}, 1) = 7$ bounded regions of $\Sigma_{(G,\mathcal{O}_{\text{ref}})}(A^<)$, shaded in dark gray, and their labels are the strongly connected-cycle neutral partial orientations. Figure 9 depicts the bicographical arrangement of $G$ similarly labeled.
all directed cuts $\overrightarrow{Cu}$ of $G$. In order to make the bigraphical arrangement look more like the bicographical arrangement, one can also view it as the projection of $E(G,O_{\text{ref}})(A)$ to the subspace $U' \subseteq W$ where $x_{\overrightarrow{Cy}} = 0$ for all directed cycles $\overrightarrow{Cy}$ of $G$. Consequently one might wonder which other min-edge classes of partial orientations can be described by projecting $E(G,O_{\text{ref}})(A^<)$ to various subspaces: for instance, the set of all partial orientations is naturally in bijection with the regions of $E(G,O_{\text{ref}})(A^<)$.

**Remark 5.11.** There is another notion of acyclicity for partial orientations which is not to be confused with our acyclic partial orientations. In a recent paper of Iriarte [28] this other kind of acyclic partial orientation is called a “partial acyclic orientation.” A partial acyclic orientation is one for which the contraction of all neutral edges yields an acyclic total orientation. There is a bijection between the partial acyclic orientations of a graph and the faces (i.e., the regions and the faces of lower dimension) of its ordinary graphical arrangement (see Greene-Zaslavsky [24, Lemma 7.2] or Zaslavsky [55, Corollary 4.6], who proves a stronger version of this result that holds at the level of signed graphs). Recast in our terminology, these partial acyclic orientations are the partial orientations whose only potential cycles consist of all neutral edges. Apparently the partial acyclic orientations are not enumerated in a simple way by the Tutte polynomial. However, we remark that these partial acyclic orientations are precisely the partial orientations that are cycle neutral for all choices of edge order $<$. They are also the partial orientations that are $A$-admissible for all choices of parameter list $A \in \mathbb{R}_{>0}^{E(G)}$.

5.2. The cycle/cocycle reversal systems. Gioan [21] investigated the set of total orientations modulo directed cycle and/or directed cut (cocycle) reversals, and he used this setup to give a unified framework for understanding the evaluations $T(x, y)$ for $0 \leq x, y \leq 2$ integral. Each equivalence class in the cycle/cocycle reversal systems contains a unique cycle/cut minimal orientation and so these objects give distinguished representatives. Thus the $3 \times 3$ table at the bottom of Figure 4 is equivalent to Gioan’s $3 \times 3$ square. Furthermore, Gioan also showed that the two orientations are in the same equivalence class of the cycle-cocycle reversal system if and only if their associated indegree sequences are equivalent by chip-firing moves, which we now describe: given a chip configuration, which is simply a function from the vertices to the integers, a vertex fires by sending a chip to each of its neighbors and losing its degree number of chips in the process; we say that two chip configurations $D$ and $D'$ are chip-firing equivalent if we can get from one to the other by a sequence of chip-firings moves. Equivalently, if we view $D$ and $D'$ as vectors, then they are chip-firing equivalent when their difference is in the integer span of the columns of the Laplacian matrix of $G$.

In [4] and [5] the first author investigated two different extensions of Gioan’s cycle-cocycle reversal systems for partial orientations. One extension, which we call the cycle/cocycle reversal systems for partial orientations describes the set of partial orientations modulo cycle and/or cocycle reversals. The definition of (co)cycle reversals for partial orientations are exactly the same as for total orientations: given a partial orientation $\mathcal{O}$ of $G$, a (co)cycle reversal is the operation of replacing $\mathcal{O}$ by $(\mathcal{O} \setminus E(\overrightarrow{C})) \cup E(-\overrightarrow{C})$ for some directed cycle (cut) $\overrightarrow{C}$ of $\mathcal{O}$. These cycle/cocycle reversal systems are related to the graphic and cographic Lawrence ideals from combinatorial commutative algebra.
and in [4] it was demonstrated that they define equivalence classes of partial orientations counted by generalized Tutte polynomial evaluations. The other extension, which we call the generalized cycle/cocycle reversal systems for partial orientations was introduced in [3] for the study of chip-firing in the context of Baker and Norine’s combinatorial Riemann-Roch theorem [6]. Here we briefly recall these two different frameworks and describe how the generalized cycle/cocycle reversal systems extend naturally to the setting of fourientations. In particular, we explain how this extension allows for a direct and aesthetically pleasing interpretation of the graphical Riemann-Roch duality. At the time of writing, the precise connection between the Tutte polynomial and the generalized cycle/cocycle reversal systems remains a mystery.

5.3. The cycle/cocycle reversal systems for partial orientations, cycle/cut minimal partial orientations, and graphic/cographic Lawrence ideals. As in Remark 5.10, let $W \simeq \mathbb{R}^{E(G)}$ be the vector space with basis $x_e^+ = -x_e^-$ for $e \in E(G)$. Consider the lattice $\mathbb{Z}^{E(G)}$ in $W$. Given any element $u = \sum_{e \in E(G)} c_e(x_e^+) \in \mathbb{Z}^{E(G)}$, let $u^+ := \sum_{c_e \geq 0} c_e(x_e^+)$ and $u^- := -\sum_{c_e \leq 0} c_e(x_e^+)$ be the positive and negative parts of $u$. Fix a field $k$ and let $S = k[y_e^+, y_e^- : e \in E(G)]$ be a polynomial ring in $2|E(G)|$ variables. To $u \in \mathbb{Z}^{E(G)}$ we associate a binomial $b(u) := y_u^+ y_u^- - y_u^+ y_u^- \in S$, where we use the notation $y_u^+ := \prod_{e \in E(G)} (y_e^+)^{c_e}$ for $c = \sum_{e \in E(G)} c_e(x_e) \in \mathbb{N}^{E(G)}$. Let $L$ be a sublattice of $\mathbb{Z}^{E(G)}$. To $L$ we associate the binomial Lawrence ideal $I_L := (b(u) : u \in L)$.

Recall that for a directed cut or cycle $\tilde{C}$ of $G$ we defined $x_{\tilde{C}} := \sum_{e} x_e \in W$. We define the cut lattice of $G$ to be $\langle x_{\tilde{C}u} : \tilde{C}u \text{ is a directed cut} \rangle_\mathbb{Z}$ and the cycle lattice to be $\langle x_{\tilde{C}y} : \tilde{C}y \text{ is a directed cycle} \rangle_\mathbb{Z}$. See [3] for a more organic homological description of the cut and cycle lattices. The graphic and cographic Lawrence ideals, which we will denote $I_{\tilde{C}u}$ and $I_{\tilde{C}y}$, are the Lawrence ideals associated to the cut and cycle lattices respectively. The observation which relates these ideals to cycle/cocycle reversal systems is the following: we can encode a partial orientation $O$ of $G$ as a square free monomial $y^O := \prod_{e} y_e^+ \in S$; then division of $y^O$ by some $b(x_{\tilde{C}u})$ or $b(x_{\tilde{C}y})$ corresponds to a (co)cycle reversal of $\tilde{C}u$ ($\tilde{C}y$) in $O$. These ideals have been previously studied in the context of algebraic combinatorics and algebraic statistics [8], [19], [29], [39]. One can show that $\{b(x_{\tilde{C}u}) : \tilde{C}u \text{ is a directed cut}\}$ and $\{b(x_{\tilde{C}y}) : \tilde{C}y \text{ is a directed cycle}\}$ are minimal binomial generating sets for the Lawrence ideals they generate and a theorem of Sturmfels [50] Theorem 7.1 then gives that they are universal Gröbner bases for these ideals. We find that the square free standard monomials with respect to the reverse lexicographic order correspond directly to the cut and cycle minimal partial orientations. It follows that these objects can be computed greedily since the division with respect to a Gröbner basis always yields a unique remainder. For a self contained combinatorial proof of this corollary, see [3].

If we define the $S$-modules
\[
M_{\tilde{C}u} := S/I_{\tilde{C}u} \otimes S/\langle (y_e^+)^2, (y_e^-)^2, y_e^+ y_e^- : e \in E(G) \rangle,
\]
\[
M_{\tilde{C}y} := S/I_{\tilde{C}y} \otimes S/\langle (y_e^+)^2, (y_e^-)^2, y_e^+ y_e^- : e \in E(G) \rangle,
\]
we see that $\mathcal{M}_{C_y}^{\pm}(\mathcal{M}_{C_y}^{\pm})$ is spanned by $y^O$ over all partial orientations $O$ of $G$ up to (co)cycle reversals. In [4] it was shown that the cycle (cut) minimal partial orientations serve as representatives for their classes in the (co)cycle reversal system. Thus a direct application of Theorem 2.13 yields the following expressions for the Hilbert series of these modules:

$$\text{Hilb}(\mathcal{M}_{C_y}^{\pm}; y) = (y + 1)^{n-1}y^O T(1, 2y + 1, \frac{1}{y});$$

$$\text{Hilb}(\mathcal{M}_{C_y}^{\pm}; y) = y^{n-1}(y + 1)^O T(2y + 1, 1).$$

5.4. The generalized cycle/cocycle reversal systems and Riemann-Roch theory for fourientations. In [5], an edge pivot for partial orientations was defined as follows: given an edge $e$ oriented towards a vertex $v$ and $e'$ a neutral edge incident to $v$, we may unorient $e$ and orient $e'$ towards $v$. This name is motivated by the image of an oriented edge nailed down at its head which can pivot to other unoriented edges. The generalized cycle, cocycle and cycle-cocycle reversal systems for partial orientations are defined to be these systems extended to partial orientations by the inclusion of edge pivots.

We now introduce generalized edge pivots for fourientations, which we will refer to as simply edge pivots. Let $e$ and $e'$ be a pair of edges incident to $v$. Suppose that $e$ is bidirected or is oriented towards $v$ and $e'$ is either unoriented or oriented away from $v$. Then we can remove the orientation of $e$ towards $v$ and add an orientation of $e'$ towards $v$. That is, if $O$ is a fourientation with $e_1 = (v, u) \in O$ but $e_2 = (w, u) \notin O$, then an edge pivot is the operation of replacing $O$ by $O' = (O \setminus \{e_1\}) \cup \{e_2\}$. See Figure 10 for the different combinatorial types of edge pivots. The generalized cycle, cocycle and cycle-cocycle reversal systems for fourientations are defined to be these systems extended to partial orientations by the inclusion of edge pivots.

We write $O \sim O'$ if the fourientations $O$ and $O'$ are equivalent in the generalized cocycle reversal system.

**Remark 5.12.** A cycle reversal in a fourientation can be performed by a sequence of generalized edge pivots as depicted in Figure 11. Thus the generalized cycle-cocycle reversal system and the generalized cocycle reversal system for fourientations agree.

For a fourientation $O$ of $G$ and a vertex $v \in V(G)$ we define the indegree of $O$ at $v$ to be $\text{indeg}_O(v) := |\{e_\pm = (u, v) \in O\}|$. In keeping with the terminology of algebraic geometry, we define the divisor associated to the fourientation $O$ to be $D_O := \sum_{v \in V(G)}(\text{indeg}_O(v) - 1)(v)$ viewed as a formal sum of the vertices with integer coefficients. Similarly, given two divisors $D$ and $D'$ we write $D \sim D'$ if they are equivalent by chip-firing moves and say they are linearly equivalent. See [42] for
background on linear equivalence of divisors. We note that our terminology is justified by the rich connection between combinatorial divisor theory for graphs and chip-firing \[6\], \[13\], \[38\], \[3\]. Lemma 3.1 of \[5\] says that two partial orientations are equivalent in the generalized cycle reversal system if and only if they have the same associated divisors, which extends Gioan’s \[21\] Proposition 4.10] from total to partial orientations. We now further extend this result to the setting of fourientations.

**Lemma 5.13.** Two fourientations \(O\) and \(O'\) are equivalent by edge pivots if and only if \(D_O = D_{O'}\).

**Proof.** It is clear that if \(O\) and \(O'\) are equivalent in the generalized cycle reversal system then \(D_O = D_{O'}\). We now demonstrate the converse. First suppose that there exists some edge \(e = (u, v)\) such that \(e\) is oriented towards \(v\) in \(O\), but \(e\) is bidirected in \(O'\). Because \(D_O = D_{O'}\) we know that there exists some \(e'\) incident to \(u\) such that \(e'\) is not oriented towards \(u\) in \(O\). We can perform a pivot from \(e\) to \(e'\) in \(O\). By induction on the symmetric difference of \(O\) and \(O'\) we may assume that no such edge exists. Therefore their symmetric difference is a Type A fourientation and we reduce to Lemma 3.1 of \[5\]. \(\square\)

In \[5\] the first author introduced a “nonlocal” extension of an edge pivot called a Jacob’s ladder cascade and employed this operation repeatedly. We now extend this operation to fourientations. Let \(P\) be a directed path from \(u\) to \(v\) in the fourientation \(O\) (i.e., \(P\) is a path from \(u\) to \(v\) that walks along oriented edges). Let \(e_1\) and \(e_2\) be edges not in \(P\) such that \(e_1^O = (x, u)\), \(e_2^O = (y, v)\) with \(e_1^O \in O\) and \(e_2^O \notin O\). Then we can perform successive edge pivots along \(P\) to so that \(e_1^O \notin O\) and \(e_2^O \in O\) and we call this operation a Jacob’s ladder cascade; see Figure 12. We note that our definition allows for \(e_1 = e_2 = \{u, v\}\) and hence a cycle reversal may be viewed as a special case of a Jacob’s ladder cascade.

Given a fourientation \(O\), we define \(O\) to be the fourientation obtained by reversing the orientation of each directed edge, replacing each unoriented edge with a bidirected edge, and replacing each bidirected edge with an unoriented edge. We recall that the canonical divisor of \(G\) is \(K = \sum_{v \in V(G)}(\deg(v) - 2)(v)\) where the degree of \(v \in V(G)\) is \(\deg(v) := |\{e = \{u, v\}: e \in E(G), u \in V(G)\}|\). Baker and Norine’s Riemann-Roch formula for graphs \[6\] investigates the rank of a divisor \(D\), written \(r(D)\), in comparison to \(r(K - D)\). We do not review the definition of rank here, nor the Riemann-Roch formula, but we note the following important observation.

**Remark 5.14.** If \(O\) is a fourientation, then \(K - D_O = D_{\overline{O}}\). Thus the divisors associated to complementary fourientations are Riemann-Roch dual.
Figure 12. A Jacob’s ladder cascade.

Lemma 5.15. If $O$ and $O'$ are fourientations, then $O \sim O'$ if and only if $\overline{O} \sim \overline{O'}$.

Proof. This is trivial. \hfill \Box

Lemma 5.16. Let $O$ be a fourientation, then

(i) $O \sim O'$ with $O'$ a Type A fourientation if and only if $\deg(D_O) \leq g - 1$;

(ii) $O \sim O'$ with $O'$ a Type B fourientation if and only if $\deg(D_O) \geq g - 1$.

Proof. We have that $\deg(D_O) \leq g - 1$ if and only if $\deg(D_O) \geq g - 1$, and $O$ is Type A if and only if $\overline{O}$ is Type B, thus Lemma 5.15 shows that (ii) is equivalent to (i).

We now verify (i). It is clear that if $O \sim O'$ with $O'$ a type A partial orientation, then $\deg(D_O) \leq g - 1$. Conversely, suppose $\deg(D_O) \leq g - 1$ and $O$ is not a Type A fourientation. Let $S$ be the set of vertices incident to a bidirected edge and $T$ be the set of edges incident to an unoriented edge. By assumption, both $S$ and $T$ are non-empty. Furthermore, we take $\overline{S}$ to be the set of vertices which are reachable from $S$ by a (possibly empty) directed path. If $\overline{S} \cap T \neq \emptyset$ then we may perform a Jacob’s ladder cascade to decrease the number of bidirected edges. By induction on the number of bidirected edges in $O$ we can assume that eventually $\overline{S} \cap T = \emptyset$. Therefore $(\overline{S'}, S)$ is fully oriented towards $\overline{S}$, and we can reverse this directed cut enlarging $\overline{S}$. By induction on $|\overline{S'}|$ this process must terminate. \hfill \Box

Theorem 3.4 of [5] states that two partial orientations are equivalent in the generalized cycle-cocycle reversal system if and only if their associated divisors are chip-firing equivalent. This extends Gioan’s [21, Proposition 4.13] from total orientations to partial orientations. We now extend this theorem further to the setting of fourientations.

Theorem 5.17. If $O$ and $O'$ are fourientations, then $O \sim O'$ if and only if $D_O \sim D_O'$.

Proof. It is clear that if $O$ and $O'$ are equivalent in the generalized cocycle reversal system, then $D_O \sim D_O'$. We now demonstrate the converse. Lemma 5.15 in conjunction with the fact that $D_O \sim D_O'$ if and only if $D_{\overline{O}} \sim D_{\overline{O'}}$ allows us to assume that $\deg(D_O) \leq g - 1$. By Lemma 5.16 $O \sim O_1$ and $O' \sim O_2$ such that both $O_1$ and $O_2$ are Type A fourientations. We know that $D_{\overline{O}} \sim D_{\overline{O}_1}$ and $D_{\overline{O'}} \sim D_{\overline{O}_2}$, thus by transitivity $D_{O_1} \sim D_{O_2}$. Now by Theorem 3.4 in [5] we have $O_1 \sim O_2$ and again by transitivity $O \sim O'$. \hfill \Box
5.5. Indegree sequences of partial orientations. In this subsection and the next we assume all graphs are loopless. With this assumption, any acyclic partial orientation of a graph can be completed to an acyclic total orientation. For a fourientation \( \mathcal{O} \) of \( G \), define \( \overline{D}_\mathcal{O} := \sum_{v \in \tilde{V}(G)} \text{indeg}_\mathcal{O}(v)v \in \mathbb{Z}V(G) \). We call \( \overline{D}_\mathcal{O} \) the inddegree sequence of \( \mathcal{O} \). Recall the divisor associated to \( \mathcal{O} \) is \( D_\mathcal{O} := \sum_{v \in V(G)} (\text{indeg}_\mathcal{O}(v) - 1)(v) \in \mathbb{Z}V(G) \); the distinction between the divisor associated to a fourientation and its inddegree sequence is just one of normalization. We might hope that the number of inddegree sequences among partial orientations in a min-edge class of partial orientations is also given by a generalized Tutte polynomial evaluation. But as observed in [4], the number of inddegree sequences among all partial orientations of \( G \) cannot be a generalized Tutte polynomial evaluation for \( G \) itself: for example, the path on three edges has 21 inddegree sequences while the star on three edges has 20, but the Tutte polynomials of all trees on \( n \) vertices are the same. This also shows that the number of inddegree sequences of all acyclic partial orientations of \( G \) is not a generalized Tutte polynomial evaluation, as of course all partial orientations of a tree are acyclic. One way to get around this obstruction is by considering the Tutte polynomial of graphs related to \( G \). Let us denote by \( G^* \) the cone over \( G \), which is the graph obtained from \( G \) by adding an extra vertex \( v_0 \) and connecting it by an edge to every other vertex in \( G \). Note that the cone over the path on three edges and the cone over the star on three edges have different Tutte polynomials. It turns out that the set \( \{ \overline{D}_\mathcal{O} : \mathcal{O} \text{ an acyclic partial orientation of } G \} \) is the set of \( (G^*, v_0) \)-parking functions and thus the cardinality of this set is \( T_{G^*}(1, 1) \).

We first need some terminology to explain why this is.

**Definition 5.18.** Let \( G \) be a graph and designate a special sink vertex \( q \in V(G) \). Set \( \tilde{V}(G) := V(G) \setminus \{ q \} \). A \((G, q)\)-parking function is an element \( c = \sum_{v \in \tilde{V}(G)} c_v(v) \) of \( \mathbb{Z}\tilde{V}(G) \) so that for every non-empty \( U \subseteq \tilde{V}(G) \), there is \( u \in U \) with \( 0 \leq d_U(u) < c_u \), where \( d_U(u) := |\{ e = \{ u, v \} \in E(U, V(G) \setminus U) \}| \). The set of \((G, q)\)-parking functions inherits a partial order from \( \mathbb{Z}\tilde{V}(G) \). A maximal \((G, q)\)-parking function is one that is maximal among \((G, q)\)-parking functions with respect to this order.

A source of a partial orientation is a vertex with no incoming directed edges. There is a well-known bijection between acyclic total orientations of \( G \) with unique source \( q \) and maximal \((G, q)\)-parking functions given by \( \mathcal{O} \mapsto (D_\mathcal{O})_{\overline{Z}\tilde{V}(G)} \) where \( (\cdot)_{\overline{Z}\tilde{V}(G)} \) means to ignore the \(-1\) coefficient of \( q \) and treat the expression as an element of \( \mathbb{Z}\overline{\tilde{V}}(G) \). (The inverse map of this bijection is essentially given by Dhar’s burning algorithm [17]; see for example [11].) Observe that the set \( \{ \overline{D}_\mathcal{O} : \mathcal{O} \text{ an acyclic total orientation of } G \} \) is also equal to \( \{(D_\mathcal{O})_{V(G)} : \mathcal{O} \text{ an acyclic total orientation of } G^* \text{ with unique source } v_0 \} \), so the inddegree sequences of acyclic total orientations of \( G \) are the maximal \((G^*, v_0)\)-parking functions. But then observe that \( \{ \overline{D}_\mathcal{O} : \mathcal{O} \text{ an acyclic partial orientation of } G \} \) is the same as \( \{ c \in \mathbb{Z}V(G) : 0 \leq c \leq \overline{D}_\mathcal{O} \text{ for some acyclic total orientation } \mathcal{O} \text{ of } G \} \) because any acyclic partial orientation can be completed to an acyclic total orientation. It is a simple fact that \( c \in \mathbb{Z}V(G) \) is a \((G, q)\)-parking function if and only if \( 0 \leq c \leq c' \) for some maximal \((G, q)\)-parking function \( c' \). Thus indeed the set of \((G^*, v_0)\)-parking functions is \( \{ \overline{D}_\mathcal{O} : \mathcal{O} \text{ an acyclic partial orientation of } G \} \), as claimed. It is a classical
ated by Gioan \cite{21} and Kuperberg, and Shokrieh \cite{1}, who proved that these are exactly the break divisors of Mikhalkin and Zharkov \cite{38} offset by a chip at $q$. These break divisors were discovered originally in the context of divisor theory for tropical curves, where they can be seen to provide canonical representatives for the set of divisors of degree $g$ modulo linear

fact (again, see \cite{10}) that the number of $(G, q)$-parking functions is $T_G(1, 1)$, the number of spanning trees of $G$. So the number of indegree sequences of acyclic partial orientations of $G$ is $T_G^\bullet(1, 1)$.

The main result of \cite{27} is that the set of indegree sequences of acyclic partial orientations of $G$ is equal to $\{D_{O_R} : R \text{ a region of } \Sigma_{(G, O_{ref})}(A)\}$ for any parameter list $A \in \mathbb{R}_{>0}^{E(G)}$ (see also Mazin \cite{35}, who extends this result, which was originally proven only for simple graphs, to multigraphs; and for more on the connection between parking functions and partial orientations when $G = K_n$ is the complete graph, see \cite{9}). Therefore the number of indegree sequences of cycle neutral partial orientations of $G$ is also given by $T_G^\bullet(1, 1)$. It would be interesting to see if we can count indegree sequences for other classes of partial orientations by evaluating the Tutte polynomial of graphs related to $G$, or by using more complicated expressions involving the Tutte polynomial of $G$ itself.

Another way to obtain Tutte polynomial enumerations of indegree sequences for min-edge classes of partial orientations is by restricting to special input data. Choose a sink $q \in V(G)$. Let $\tilde{V}(G) := V(G) \setminus \{q\}$ denote the non-sink vertices. Also choose a $q$-rooted, ordered spanning tree $T$ of $G$. By this we mean that $T$ is a directed spanning tree of $G$ rooted at $q$, with edges oriented away from $q$ and totally ordered in some way consistent with the partial order of ancestry so that edges closer to $q$ in $T$ are less than those further away. Let us say that $O_{ref}$ and $\prec$ are compatible with the data of $(q, T)$ if reference orientation $O_{ref}$ is obtained by extending the orientation of the edges of $T$ to all the edges in $E(G)$ arbitrarily, and the edge order $\prec$ is obtained by extending the order on the edges of $T$ to an order of all the edges in $E(G)$ in some way so that edges not in $T$ are all greater than edges in $T$. If $O_{ref}$ and $\prec$ are compatible with $(q, T)$, then we call the ordered, oriented graph $G = (G, O_{ref}, \prec)$ a $(q, T)$-connected graph. Now assume $G$ is a $(q, T)$-connected graph. In this case, the cut connected partial orientations of $G$ are the same as the $q$-connected partial orientations, i.e., those partial orientations for which there exists a directed path from $q$ to every vertex $v \in \tilde{V}(G)$. And the acyclic $q$-connected total orientations of $G$ are the same as the acyclic total orientations of $G$ with unique source $q$. So, arguing as before, the set of $(G, q)$-parking functions is also equal to $\{(D_O)_{\Sigma_V(G)} : O \text{ an acyclic cut connected partial orientation of } G\}$. Thus, the number of indegree sequences of acyclic cut connected partial orientations of a $(q, T)$-connected graph is $T_G(1, 1)$. Note that the number of indegree sequences of acyclic cut connected partial orientations of an arbitrary ordered, oriented graph $G = (G, O_{ref}, \prec)$ is not necessarily given by $T_G(1, 1)$.

For $G$ a $(q, T)$-connected graph, the cut-cycle minimal total orientations enumerated by $T(1, 1)$ become the cycle minimal $q$-connected orientations. These objects are in bijection with their associated divisors which were introduced by Gioan \cite{21} and further investigated by Bernardi \cite{11}. These divisors were rediscovered by An, Baker, Kuperberg, and Shokrieh \cite{1}, who proved that these are exactly the break divisors of Mikhalkin and Zharkov \cite{38} off set by a chip at $q$. These break divisors were discovered originally in the context of divisor theory for tropical curves, where they can be seen to provide canonical representatives for the set of divisors of degree $g$ modulo linear
equivalence. In particular, this implies that by adding a chip at \( q \) to the divisors associated to \( q \)-connected orientations, we lose all dependence on \( q \). Interestingly, there exist tropical proofs of the existence and uniqueness of break divisors, which are not combinatorial in nature.

**Remark 5.19.** In contrast to the complicated situation with partial orientations described above, the number of indegree sequences among a min-edge class of total orientations is certainly given by a Tutte polynomial evaluation as outlined in [11]. In the other direction, it also might be interesting to investigate the number of indegree sequences among fourientations in a min-edge class. Again, this value is not necessarily a generalized Tutte polynomial evaluation. However, nevertheless we can sometimes get a simple expression for this value: for instance, it is easily seen that the number of indegree sequences among all fourientations of a graph \( G \) is \( \prod_{v \in V(G)} \deg(v) \).

### 5.6. Monomizations of zonotopal algebras and cut free partial orientations

One can extend the enumeration of \( (G,q) \)-parking functions via the Tutte polynomial to an expression for the generating function of \( (G,q) \)-parking functions by degree. For \( c = \sum_{v \in \tilde{V}(G)} c_v(v) \in \mathbb{Z}\tilde{V}(G) \), define the degree of \( c \) to be \( \deg(c) := \sum_{v \in \tilde{V}(G)} c_v \). A famous result of Merino [37] is that \( T_G(1,y) = \sum_c g^\deg(c) \) where the sum is over all \( (G,q) \)-parking functions \( c \). (Merino [36] used this generating function result to resolve a special case of a 1977 conjecture of Stanley [47] on the \( h \)-vector of a matroid complex.) Merino’s theorem can also be expressed succinctly using commutative algebra. Fix some field \( \mathbf{k} \) and let \( R := \mathbf{k}[x_v : v \in \tilde{V}(G)] \) be the polynomial ring with generators indexed by non-sink vertices. For \( U \subseteq \tilde{V}(G) \), define \( x^U := \prod_{v \in U} x_v^{d_v(u)} \) where as before we have \( d_v(u) := |\{e = \{u,v\} \in E(U,U^c)\}| \). Then define the monomial ideal \( I_{(G,q)} := \langle x^U : U \subseteq \tilde{V}(G) \text{ with } U \neq \emptyset \rangle \). We use the notation \( x^c := \prod_{v \in \tilde{V}(G)} x_v^{c_v} \) for \( c = \sum_{v \in \tilde{V}(G)} c_v(v) \in \mathbb{N}\tilde{V}(G) \). It is not difficult to see that a linear basis of \( R/I_{(G,q)} \) is \( \{x^c : c \text{ a } (G,q) \text{-parking function}\} \). A restatement of Merino’s theorem is then that the Hilbert series of \( R/I_{(G,q)} \) is \( \text{Hilb}(R/I_{(G,q)};y) = y^\deg \cdot T_G(1,1/y) \).

We conjecture analogous results using acyclic cut free partial orientations in place of acyclic cut connected ones. As before, we must work with a \( (q,T) \)-connected graph \( G \). In fact, we will need to restrict our input data further by assuming that \( T \) is a \( q \)-rooted, DFS ordered spanning tree. This means that \( T \) is a directed spanning tree of \( G \) built up by performing a depth-first search traversal of \( G \) starting at \( q \), with edges oriented away from \( q \) and ordered according the order in which they are first visited in this traversal. Let us call \( \{(d_G)_{\tilde{e}} : \mathcal{O} \text{ an acyclic cut free partial orientation of } G\} \) the set of \( (G,q,T) \)-subparking functions. This set depends on \( T \), whereas the set of \( (G,q) \)-parking functions does not. For a subset \( U \subseteq \tilde{V}(G) \), define \( x^{U,-1} := \prod_{u \in U} x_u^{d_u^{-1}(u)} \) with \( d_u^{-1}(u) := |\{e = \{u,v\} \in E(U,U^c) \text{ and } e \text{ is not the minimum edge in } E(U,U^c)\}| \). So in particular the degree of \( x^{U,-1} \) is one less than the degree of \( x^U \). Then define the monomial ideal \( I^{-1}_{(G,q,T)} := \langle x^{U,-1} : U \subseteq \tilde{V}(G) \text{ with } U \neq \emptyset \rangle \).
Conjecture 5.20. We have $\text{Hilb}(R/I_{G,q,T}^{-1}; y) = y^g \cdot T_G(0, 1/y)$ and a linear basis of $R/I_{G,q,T}^{-1}$ is $\{x^c : c \text{ a } (G, q, T)\text{-subparking function}\}$.

Example 5.21. Let $G$ be the graph below:

Let $G$ be a $(q, T)$-connected graph where $T$ is as above with $e_1 < e_2$. The Tutte polynomial of our graph is $T_G(x, y) = y^3 + x^2 + 2xy + 2y^2 + x + y$. The 9 acyclic cut free partial orientations of $G$ are the following:

Therefore the set of $(G, q, T)$-subparking functions is $\{0, (v_1), (v_2), 2(v_1)\}$. We can compute that $I_{G,q,T}^{-1} = \langle x_{v_1}x_{v_2}, x_{v_2}x_{v_1} \rangle$. A linear basis of $R/I_{G,q,T}^{-1}$ is $\{1, x_{v_1}, x_{v_2}, x_{v_2}^2\}$. And indeed, $\text{Hilb}(R/I_{G,q,T}^{-1}; y) = 1 + 2y + y^2 = y^3 \cdot T_G(0, 1/y)$.

Motivated by questions in Schubert calculus \[45\], Postnikov and Shapiro \[43\] studied the monomial ideal $I_{G,q}$ as well as a deformation of this ideal generated by powers of homogenous linear forms. Specifically, setting $d_U := \sum_{u \in U} d_U(u)$, they defined the power form ideal $J_{G,q} := \langle (\sum_{u \in U} x_u)^{d_U} : U \subseteq \overline{V}(G) \text{ with } U \neq \emptyset \rangle$. One of the main results of \[43\] is that $R/I_{G,q}$ and $R/J_{G,q}$ have the same Hilbert series. The quotient $R/J_{G,q}$ is a so-called zonotopal algebra \[25\]; in fact it is a central zonotopal algebra. One can define $J'_{G,q} := \langle (\sum_{u \in U} x_u)^{d_U+r} : U \subseteq \overline{V}(G) \text{ with } U \neq \emptyset \rangle$ for $r \geq -1$. The quotients $R/J'_{G,q}$ in the special cases $r = +1, 0, -1$ yield the external, central and internal algebras associated to $G$. From Ardila and Postnikov \[2\] it follows that

$$\text{Hilb}(R/J'_{G,q}^{+1}; y) = y^g T_G(1+y, 1/y);$$
$$\text{Hilb}(R/J'_{G,q}^{-1}; y) = y^g T_G(0, 1/y).$$

One might wonder whether there are some analogous monomial ideals $I'_{G,q}$ for $r = \pm 1$ with $\text{Hilb}(R/J'_{G,q}^{r}; y) = \text{Hilb}(R/I'_{G,q}^{r}; y)$; in this case we say the monomial ideal is
A restatement of Conjecture 5.20 is the assertion that the monomial ideal \( I(G,q,T) \) is a monomization of the internal power form ideal \( I(G,q) \).

For the complete graph \( G = K_{n+1} \), Postnikov-Shapiro-Shapiro \cite{postnikov2010} found such an external monomial ideal \( I(G,q) \) (and indeed this external case was the one they were originally interested in). The idea is that if the vertices \( V(K_{n+1}) \) come with a total order \( \prec \) then we can simply define \( x^{U,-1,\prec} := x^U \cdot x_u \) where \( u \) is minimal in \( U \) with respect to \( \prec \) and then \( I_{(K_{n+1},\prec)} := \langle x^{U,-1,\prec}; U \subseteq V(K_{n+1}) \text{ with } U \neq \emptyset \rangle \) is a monomization of \( J_{(K_{n+1},\prec)} \). Desjardins \cite{desjardins2010} succeeded in extending their construction to obtain an external monomial ideal \( I_{(G,q)} \) for any \( G \), but showed that certain assumptions on \( G \) were necessary to mimic this construction and obtain an appropriate internal monomial ideal \( I_{(G,q)} \). Specifically, suppose \( V(G) = \{v_0, v_1, \ldots, v_n\} \) with sink \( q := v_0 \) and we have a total order \( v_0 \prec v_1 \prec \cdots \prec v_n \). Then define \( x^{U,-1,\prec} := x^U \cdot x_u^{-1} \) where \( u \) is minimal in \( U \) with respect to \( \prec \). If \( q \) is connected by an edge to each other vertex then \( x^{U,-1,\prec} \) is always a monomial. In this case, set \( I_{(G,\prec)} := \langle x^{U,-1,\prec}; U \subseteq \tilde{V}(G) \text{ with } U \neq \emptyset \rangle \). With the crucial further assumption that there is at least one edge between every pair of vertices in \( G \), Desjardins \cite{desjardins2010} \S 4 showed that indeed \( \text{Hilb}(R/J_{(G,q)}; y) = \text{Hilb}(R/I_{(G,\prec)}; y) \). But observe that if every pair of vertices is connected by an edge in \( G \), then in fact \( I_{(G,\prec)} = I_{(G,q,T)} \) where \( T \) is the DFS tree built up by visiting \( v_0 \), then \( v_1 \), then \( v_2 \), and so on up to \( v_n \). So Conjecture 5.20 includes Desjardin’s result about \( I_{(G,\prec)} \) as a special case.

Recently there has been a great deal of interest in understanding minimal free resolutions of \( I_{(G,q)} \) and minimal free resolutions of a certain binomial ideal for which \( I_{(G,q)} \) is a distinguished initial ideal \cite{federico2010}. It would be interesting to find a combinatorial description of a minimal free resolution of \( I_{(G,q,T)} \) or to compute its Betti numbers combinatorially. Even more interesting would be to find some combinatorially-meaningful binomial ideal which has \( I_{(G,q,T)} \) as an initial ideal for an appropriate choice of weight.

References

1. Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem. *Forum Math. Sigma*, 2:e24, 25, 2014.
2. Federico Ardila and Alexander Postnikov. Combinatorics and geometry of power ideals. *Trans. Amer. Math. Soc.*, 362(8):4357–4384, 2010.

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4For \( \mathcal{I} \) to qualify as a monomization of \( \mathcal{J} \), the monomial ideal \( \mathcal{I} \) should serve as a model for the power form ideal \( \mathcal{J} \) in more ways than simply having the same Hilbert series. For instance, we would want the standard basis of \( R/\mathcal{I} \) to be a basis of \( R/\mathcal{J} \). We would also want the Betti tables of the two ideals to agree. Conjecturally \cite[Conjecture 6.10]{federico2010}, if \( \mathcal{I} \) is a monotone monomial ideal, as is the case for \( I_{(G,q)} \) and is likely the case for our \( I_{(G,q,T)} \), then \( \text{dim}_k(R/\mathcal{I}) = \text{dim}_k(R/\mathcal{J}) \) implies that the Betti tables of \( \mathcal{I} \) and \( \mathcal{J} \) are the same. However, we will only consider Hilbert series here.
3. Roland Bacher, Pierre de la Harpe, and Tatiana Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. Bull. Soc. Math. France, 125(2):167–198, 1997.
4. Spencer Backman. Partial graph orientations and the Tutte polynomial. arXiv:1408.3962 August 2014.
5. Spencer Backman. Riemann-Roch theory for graph orientations. arXiv:1401.3309 January 2014.
6. Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. Adv. Math., 215(2):766–788, 2007.
7. Ruth A. Bari. Chromatic polynomials and the internal and external activities of Tutte. In Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977), pages 41–52. Academic Press, New York-London, 1979.
8. Dave Bayer, Sorin Popescu, and Bernd Sturmfels. Syzygies of unimodular Lawrence ideals. J. Reine Angew. Math., 534:169–186, 2001.
9. Matthias Beck, Ana Berrizbeitia, Michael Dairyko, Claudia Rodriguez, Amanda Ruiz, and Schuyler Veeneman. Parking functions, Shi arrangements, and mixed graphs. arXiv:1405.5587 2014.
10. Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. Discrete Math., 310(8):1340–1353, 2010.
11. Olivier Bernardi. Tutte polynomial, subgraphs, orientations and sandpile model: new connections via embeddings. Electron. J. Combin., 15(1):Research Paper 109, 53, 2008.
12. Beifang Chen, Arthur L. B. Yang, and Terence Y. J. Zhang. A bijection for Eulerian-equivalence classes of totally cyclic orientations. Graphs Combin., 24(6):519–530, 2008.
13. Filip Cools, Jan Draisma, Sam Payne, and Elina Robeva. A tropical proof of the Brill-Noether theorem. Adv. Math., 230(2):759–776, 2012.
14. Julien Courtiel. A general notion of activity for the Tutte polynomial. arXiv:1412.2081 December 2014.
15. Henry H. Crapo. The Tutte polynomial. Aequationes Math., 3:211–229, 1969.
16. Craig J. Desjardins. Monomization of Power Ideals and Parking Functions. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Massachusetts Institute of Technology.
17. Deepak Dhar. Self-organized critical state of sandpile automaton models. Phys. Rev. Lett., 64(14):1613–1616, 1990.
18. Anton Dochtermann and Raman Sanyal. Laplacian ideals, arrangements, and resolutions. J. Algebraic Combin., 40(3):805–822, 2014.
19. Mathias Drton, Bernd Sturmfels, and Seth Sullivant. Lectures on algebraic statistics, volume 39 of Oberwolfach Seminars. Birkhäuser Verlag, Basel, 2009.
20. Ira M. Gessel and Bruce E. Sagan. The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions. Electron. J. Combin., 3(2):Research Paper 9, approx. 36 pp. 1996.
21. Emeric Gioan. Enumerating degree sequences in digraphs and a cycle-cocycle reversing system. European J. Combin., 28(4):1351–1366, 2007.
22. Emeric Gioan. Circuit-cocircuit reversing systems in regular matroids. Ann. Comb., 12(2):171–182, 2008.
23. Gary Gordon and Lorenzo Traldi. Generalized activities and the Tutte polynomial. Discrete Math., 85(2):167–176, 1990.
24. Curtis Greene and Thomas Zaslavsky. On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. Trans. Amer. Math. Soc., 280(1):97–126, 1983.
25. Olga Holtz and Amos Ron. Zonotopal algebra. Adv. Math., 227(2):847–894, 2011.
26. Sam Hopkins. Another proof of Wilmes’ conjecture. Discrete Math., 323:43–48, 2014.
27. Sam Hopkins and David Perkinson. Bigraphical arrangements. arXiv:1212.4398 Forthcoming Trans. Amer. Math. Soc., December 2012.
28. Benjamin Iriarte. Acyclic orientations and spanning trees. arXiv:1412.8114 December 2014.
29. Maria Kateri, Fatemeh Mohammadi, and Bernd Sturmfels. A family of quasisymmetry models. arXiv:1403.0547 Forthcoming Journal of Algebraic Statistics, March 2014.
[30] Daniel J Kleitman and Kenneth J Winston. Forests and score vectors. *Combinatorica*, 1(1):49–54, 1981.
[31] W. Kook, V. Reiner, and D. Stanton. A convolution formula for the Tutte polynomial. *J. Combin. Theory Ser. B*, 76(2):297–300, 1999.
[32] Michel Las Vergnas. Convexity in oriented matroids. *J. Combin. Theory Ser. B*, 29(2):231–243, 1980.
[33] Madhusudan Manjunath, Frank-Olaf Schreyer, and John Wilmes. Minimal free resolutions of the $G$-parking function ideal and the toppling ideal. *Trans. Amer. Math. Soc.*, 367(4):2853–2874, 2015.
[34] Madhusudan Manjunath and Bernd Sturmfels. Monomials, binomials and Riemann–Roch. *J. Algebraic Combin.*, 37(4):737–756, 2013.
[35] Mikhail Mazin. Multigraph hyperplane arrangements and parking functions. arXiv:1501.01225, January 2015.
[36] Criel Merino. The chip firing game and matroid complexes. In *Discrete models: combinatorics, computation, and geometry (Paris, 2001)*, Discrete Math. Theor. Comput. Sci. Proc., AA, pages 245–255 (electronic). Maison Inform. Math. Discrèt. (MIMD), Paris, 2001.
[37] Criel Merino López. Chip firing and the Tutte polynomial. *Ann. Comb.*, 1(3):253–259, 1997.
[38] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008.
[39] Fatemeh Mohammadi. Divisors on graphs, orientations, syzygies, and system reliability. arXiv:1405.7972, May 2014.
[40] Fatemeh Mohammadi and Farbod Shokrieh. Divisors on graphs, binomial and monomial ideals, and cellular resolutions. arXiv:1306.5351, June 2013.
[41] Fatemeh Mohammadi and Farbod Shokrieh. Divisors on graphs, connected flags, and syzygies. In 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), Discrete Math. Theor. Comput. Sci. Proc., AS, pages 885–896. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013.
[42] David Perkinson, Jacob Perlman, and John Wilmes. Primer for the algebraic geometry of sandpiles. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 211–256. Amer. Math. Soc., Providence, RI, 2013.
[43] Alexander Postnikov and Boris Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142 (electronic), 2004.
[44] Alexander Postnikov, Boris Shapiro, and Mikhail Shapiro. Chern forms on flag manifolds and forests. In *Proceedings of the 10-th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 1998)*, Fields Institute, Toronto, 1998.
[45] Alexander Postnikov, Boris Shapiro, and Mikhail Shapiro. Algebras of curvature forms on homogeneous manifolds. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 227–235. Amer. Math. Soc., Providence, RI, 1999.
[46] Richard P. Stanley. Acyclic orientations of graphs. *Discrete Math.*, 5:171–178, 1973.
[47] Richard P. Stanley. Cohen-Macaulay complexes. In *Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, pages 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31. Reidel, Dordrecht, 1977.
[48] Richard P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980.
[49] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.
[50] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
[51] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.
[52] D. J. A. Welsh and C. Merino. The Potts model and the Tutte polynomial. J. Math. Phys., 41(3):1127–1152, 2000.

[53] Dominic Welsh. The Tutte polynomial. Random Structures Algorithms, 15(3-4):210–228, 1999.

[54] Thomas Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. Mem. Amer. Math. Soc., 1(issue 1, 154):vii+102, 1975.

[55] Thomas Zaslavsky. Orientation of signed graphs. European J. Combin., 12(4):361–375, 1991.

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