Dynamics of a degenerately damped stochastic Lorenz-Stenflo system

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Dynamics of a degenerately damped stochastic Lorenz-Stenflo system

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Abstract Little seems to be known about the sensitivity of steady states for stochastic systems. This paper discusses such dynamics of a degenerately damped stochastic Lorenz-Stenflo model. Precisely, the solution is proved to be a nice diffusion via the Lie bracket technique and non-trivial Lyapunov functions. The finiteness of the expected positive recurrence time entails the existence problem. On the other hand, a cut-off function is constructed to show the non-existence result via proof by contradiction. For other interesting cases, the expected recurrence time is shown to be infinite.

Keywords Lorenz-Stenflo system · Invariant measure · Lyapunov function · Noise-induced stabilization

1 Introduction

To describe the low-frequency, short-wavelength acoustic-gravity perturbations in the atmosphere, Stenflo [1] has derived a four-dimensional continuous-time dynamical system, given by

\[
\begin{align*}
\frac{dx}{dt} &= a(y-x) + rw, \\
\frac{dy}{dt} &= cx - y - xz, \\
\frac{dz}{dt} &= xy - bz, \\
\frac{dw}{dt} &= -x - aw,
\end{align*}
\]

(1)

where \(x, y, z, w\) are state variables of the so-called Lorenz-Stenflo equation (1), and positive parameters \(a, c, r\) are the Prandtl, generalized Rayleigh, rotation numbers, respectively and \(b\) is the geometric parameter.

Obviously, one can reduce it to the usual Lorenz system in [2] with interesting mathematical properties if the rotation of the earth is not considered. For the past decades, many scholars have studied its complex dynamical behaviors such as boundedness [3,4], periodicity [5,6], bifurcation [7–9], synchronization [10], chaotic and hyperchaotic dynamics [11–14] and influences by Lévy noise [15].

Notice that the geometric parameter \(b\) is strictly positive as in the derivation of (1), but it will tend to zero under sufficiently large generalized Rayleigh number. On the other hand, the so-called Homogeneous Rayleigh-Bénard (HRB) system was established with \(b \leq 0\) appearing in the temperature equation [16,17]. Indeed, the similar degeneracy effect was observed in a certain zero Prandtl limit to model mantle convection [18,19]. Therefore, it is natural to investigate the corresponding dynamics of (1) when \(b \leq 0\). However, it is straightforward that the corresponding solution to (1) on the \(z\)-direction explodes in finite time under the initial conditions \((x_0 = y_0 = w_0 = 0, z_0 \neq 0)\) provided that \(b < 0\). While for the case \(b = 0\), any point on the \(z\)-axis becomes equilibrium and thus one can prove the existence of singularly degenerate heteroclinic cycles but there is no compact global attractor in this situation. Therefore, both embarrassing cases motivate us to investigate the possibility for stabilizing the dynamics by adding external noise perturbations.

It is well-known that an arbitrary small additive noise can stabilize an explosive ordinary differential equation (ODE) [20,21]. If, in addition, the corresponding Markov process admits an invariant probability measure, it corresponds to the so-called noise-induced stabilization problem. In this respect, there is already con-
siderable interest in studying on stationary state, stable oscillations, and related work [22–32].

Motivated by the aforementioned discussion, we are interested in the stochastic Lorenz–Stenflo system

\[
\begin{align*}
    dx &= (a(y-x)+rw)dt + \sqrt{2\kappa_1}dB_1, \\
    dy &= (cx-y-xz)dt + \sqrt{2\kappa_2}dB_2, \\
    dz &= (xy-bz)dt + \sqrt{2\kappa_3}dB_3, \\
    dw &= (-x+aw)dt + \sqrt{2\kappa_4}dB_4,
\end{align*}
\]

where \(B_i, i = 1, 2, 3, 4\) are independent, standard Brownian motions, \(\kappa_i \geq 0, i = 1, 2, 3, 4\) represent the intensity of random noise and other parameters are in conformity with the ones in system (1). To ensure system (2) is genuinely stochastic, we require that at least one \(\kappa_i\) is positive.

In the absence of noise, we know that the solutions are explosive or have no compact global attractor when \(b \leq 0\). The interesting question here is whether the presence of noise induces the existence and number of invariant probability measure for the generated Markov transition semigroup.

The goal of this paper is to apply the approaches in [31, 32] to solve the noise-induced stabilization problem of (1) with additive Brownian noise. More precisely, we first state the philosophy in proving the existence a unique invariant probability measure to Markov transition semigroup generated by (2). In fact, the first step is to prove the non-explosion of the solution to (2) under a suitable Lyapunov function and Young’s inequality. Then Lie bracket over the vector fields allows us to show that such solution is further a nice diffusion. The final step is to acquire the globally finite expected returns to some compact set by constructing another Lyapunov function. As for the non-existence under highly degenerate noise, we construct a cut-off function and set forth proof by contradiction. While for the case \(b < 0\), we shall directly prove the expected recurrence time is infinite.

The rest of this paper is organized as follows. Section 2 collects necessary definitions, notations and criteria. Section 3 contains our main results together with detailed proofs.

2 Preliminaries

Let \(M_{nk}\) be an \(n \times k\) real matrix and consider the following Itô stochastic differential equation

\[
dX_t = F(X_t)dt + G(X_t)dB_t,
\]

where \(F = (F_1, \ldots, F_n) \in C^2(\mathbb{R}^n; \mathbb{R}^n), \ G = (G_1, \ldots, G_k) \in C^2(\mathbb{R}^n; M_{nk}),\) and \(B_t = (B^{1}_t, \ldots, B^{k}_t)^T\) represents a standard \(k\)-dimensional Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Given a function \(V \in C^2(\mathbb{R}^n; \mathbb{R}),\) the infinitesimal generator of the process along \(V\) for system (3) is defined by

\[
\mathcal{L}V(X) = F(X)\nabla V(X) + \frac{1}{2}G^T(X)\nabla^2 V(X)
\]

\[
= \sum_{j=1}^{n} F_j(X) \partial_x^j V(X)
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{k} \sum_{l=1}^{k} G_{il} G_{jl}(X) \partial_x^2 X_{ij} V(X).
\]

In general, the smoothness of \(F\) and \(G\) can not guarantee the existence of global solution to (3), but one can define a local unique pathwise solution, which is denoted by \(X_t = X(0,x,t)\) under the initial condition \(X_0 = x\). We next introduce a stopping time \(\tau\)

\[
\tau = \lim_{n \to \infty} \tau_n,
\]

where \(\tau_n = \inf\{t \geq 0 : |X_t| \geq n\}\) for \(n \in \mathbb{N}^+\). Thus there is a unique solution \(X_t\), for all times \(t < \tau, \mathbb{P}\)-almost surely. Herein, \(\tau\) stands for the explosion time of the process \(X_t\) and by which \(X_t\) is said to be non-explosive if

\[
\mathbb{P}_x\{\tau < \infty\} = 0 \quad \text{for all initial conditions } x \in \mathbb{R}^n.
\]

So if \(X_t\) is non-explosive, it can generate a Markov process and its transition probability measure is defined as \(\mathcal{P}_t(x, \cdot ) = \mathbb{P}_x\{X_t \in \cdot \}\). Denote by \(\mathcal{B}\) the Borel \(\sigma\)-field of subsets of \(\mathbb{R}^n\), the Markov transition semigroup satisfies

\[
\mathcal{P}_t V(X) = \mathbb{E}_x V(X_t) = \int_{\mathbb{R}^n} V(Y) \mathcal{P}_t (X, dY), \quad X \in \mathbb{R}^n,
\]

and

\[
\pi \mathcal{P}_t(A) = \int_{\mathbb{R}^n} \pi(dX) \mathcal{P}_t (X, A) \quad A \in \mathcal{B}.
\]

for the bounded, \(\mathcal{B}\)-measurable functions \(V : \mathbb{R}^n \to \mathbb{R}\), where \(\mathbb{E}\) denotes the corresponding expectation. Indeed, a positive measure \(\pi\) is invariant for \(\mathcal{P}_t\) if \(\pi \mathcal{P}_t = \pi\) for all \(t \geq 0\). An invariant measure \(\pi\) for \(\mathcal{P}_t\) is invariant probability measure for \(\mathcal{P}_t\) provided that \(\pi(\mathbb{R}^n) = 1\).

To our purpose, we next introduce two concepts to tell either the existence or the non-existence of an invariant probability measure, and several notations related to Lie bracket.

Definition 1 On an open set \(U \subseteq \mathbb{R}^n\), the differential operator \(\mathcal{L}\) is called hypo-elliptic if for any distribution \(u \in V \subseteq U\) such that \(\mathcal{L}u \in C^0(V),\) it yields \(u \in C^0(V)\).

Definition 2 Assume that the solution \(X_t\) to (3) is non-explosive and further satisfies

(i) \(F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)\) and \(G \in C^\infty(\mathbb{R}^n; M_{nk});\)
(iii) the support $\text{supp}(\mathscr{P}(X,t)) = \mathbb{R}^n$, $\forall t > 0$, $X \in \mathbb{R}^n$.

Then $X_t$ is called a nice diffusion.

On the other hand, for two (smooth) vector fields

$$
\begin{align*}
U(X) &= \sum_{j=1}^{n} U_j(X) \frac{\partial}{\partial x_j}, \\
W(X) &= \sum_{j=1}^{n} W_j(X) \frac{\partial}{\partial x_j},
\end{align*}
$$

the Lie bracket of them is defined by

$$
[ U, W ](X) = \sum_{j,k=1}^{n} \left( U^k(X) \frac{\partial W_j(X)}{\partial x_k} - W^k(X) \frac{\partial U_j(X)}{\partial x_k} \right) \frac{\partial}{\partial x_j}.
$$

It allows us to introduce the following notations:

$$
\text{ad}^O U(W) = W, \\
\text{ad} U(W) = [U, W], \\
\text{ad}^m U(W) = \text{ad} U(\text{ad}^{m-1} U(W)), \text{ for } m \geq 2.
$$

In particular, when $W$ depends polynomially on the components of $X$ for any $X \in \mathbb{R}^n$, one further denotes

$$
n(X, W) := \max_{j=1, \ldots, n} \deg(p_j) \quad \text{where } p_j(\lambda) := W_j(\lambda X).
$$

Thus, for any collection of vector fields $\mathscr{G}$ on $\mathbb{R}^n$, let

$$
\text{cone}_{\geq 0} \mathscr{G} = \left\{ \sum_{j=1}^{N} \lambda_j U_j : \{ \lambda_1, \ldots, \lambda_N \} \subset [0, \infty), \\
\text{and } \{ U_1, \ldots, U_N \} \subset \mathscr{G} \right\}.
$$

Since we are only interested in the situation that $G$ is independent of $X$ and $F$ is a polynomial, and further let

$$
\mathscr{G}_0 := \text{span} \{ G_0, \ldots, G_k \}, \\
\mathscr{G}_O^O := \mathscr{G}_0 \cup \left\{ \text{ad}^{n(G,F)}(G) : G \in \mathscr{G}_0, n(G,F) \text{ is odd} \right\}, \\
\mathscr{G}_O^F := \left\{ G \in \mathscr{G}_O^O : G \text{ is a constant vector field} \right\}, \\
\mathscr{G}_E := \left\{ \text{ad}^{n(G,F)}(G) : G \in \mathscr{G}_0, n(G,F) \text{ is even} \right\}, \\
\mathscr{G}_F := \text{span} \left( \mathscr{G}_O^O + \text{cone}_{\geq 0} \mathscr{G}_E^F \right).
$$

For $j > 1$ we pause to denote that

$$
\mathscr{G}_j^O := \left\{ \text{ad}^{n(G,F)}(G) : G \in \mathscr{G}_j^O, H \in \mathscr{G}_j, n(G,H) \text{ is odd} \right\}, \\
\mathscr{G}_j^E := \left\{ \text{ad}^{n(G,F)}(G) : G \in \mathscr{G}_j^O, H \in \mathscr{G}_j, n(G,H) \text{ is even} \right\}.
$$

by which, one lets

$$
\mathscr{G}_j^O := \mathscr{G}_j^O \cup \mathscr{G}_j^O, \\
\mathscr{G}_j^O := \{ G \in \mathscr{G}_j^O : G \text{ is a constant vector field} \}, \\
\mathscr{G}_j^E := \text{span} \left( \mathscr{G}_j^O + \text{cone}_{\geq 0} \mathscr{G}_j^E \right).
$$

We next extract some criteria from [32], which are useful to conclude the existence or non-existence of an invariant probability measure.

Lemma 1 [32, Proposition 2.1] Given $F, G \in C^2$ and let $X_t$ is the solution to (3) with the corresponding infinitesimal generator $\mathcal{L}$ defined in (4).

(i) Suppose that there is a function $V \in C^2(\mathbb{R}^n; [0, +\infty))$ such that $V(X) \to \infty$ as $|X| \to \infty$ and

$$
\mathcal{L}V(X) \leq pV(X) + q, \forall X \in \mathbb{R}^n,
$$

for $p, q > 0$. Then $X_t$ is non-explosive.

(ii) Assume that $X_t$ is non-explosive and there is a function $V \in C^2(\mathbb{R}^n; [0, +\infty))$, a compact set $\mathscr{K} \subseteq \mathbb{R}^n$ and constants $p, q > 0$ such that

$$
\mathcal{L}V(X) \leq -p + q \mathbb{1}_{\mathscr{K}}(X), \forall X \in \mathbb{R}^n,
$$

then

$$
\mathbb{E} \xi_{\mathscr{K}} \leq \frac{V(X)}{p}, \forall X \in \mathbb{R}^n,
$$

where $\xi_{\mathscr{K}} := \inf \{ t \geq 0 : X_t \in \mathscr{K} \}$ represents the first hitting time of $\mathscr{K}$ by $X_t$.

Lemma 2 [32, Theorem 2.2] Let $V_1, V_2 \in C^2(\mathbb{R}^n; \mathbb{R})$. If

(i) $\limsup_{|X| \to \infty} V_1(X) = \infty; \\
(ii) V_2$ is strictly positive outside of a compact set;

(iii) $\limsup_{s \to \infty} \frac{\max_{X | s} V_1(X)}{s V_2(X)} = 0$;

(iv) there exists an $R > 0$ such that

$$
\mathcal{L}V_1(X) \geq 0 \quad \text{and} \quad \mathcal{L}V_2(X) \leq 1, \quad \forall |X| > R,
$$

where $\xi_R := \inf \{ t \geq 0 : |X_t| \leq R \}$. Then there is $M \geq 0$ such that $\mathbb{E} \xi_R \xi_R = \infty$, whenever $|X_0| \geq R$ and $V_1(X_t) \geq M$.

Lemma 3 [32, Theorem 2.6] Let $F$ is a polynomial and $G$ is $X$-independent. If the solution $X_t$ to (3) is non-explosive and

$$
\text{span} \left\{ H \in \bigcup_{ j \geq 0} \mathscr{G}_j^O : H \text{ is a constant vector} \right\} = \mathbb{R}^n.
$$

Then $X_t$ is a nice diffusion.
Lemma 4  [32, Proposition 2.5] Suppose that $X_t$ is a nice diffusion, the following statements hold for $\mathcal{P}_t$.

(i) There is at most one invariant probability measure if and only if there exists $R > 0$ such that $E_X\xi_R < \infty, \forall \ X \in \mathbb{R}^n$ and the mapping $X \mapsto E_X\xi_R$ is bounded on compact subsets of $\mathbb{R}^n$.

(ii) $\mathcal{P}_t$ has an invariant probability measure if and only if $\mathcal{P}_t$ is a nice diffusion under specific parameters via Lemma 3. Thanks to Lemma 1(ii), we are able to prove the existence of a unique invariant probability measure based on Lemma 4. Namely, we prove the solution is non-explosive by Lemma 1(i). Furthermore, we focus on the hypo-ellipticity and irreducibility. Namely, we prove the solution is non-explosive by Lemma 1(i) and further a nice diffusion under specific parameters via Lemma 3. Thanks to Lemma 1(ii), we construct a suitable Lyapunov function and show that it satisfies the inequality of Lemma 1(ii), by which we are able to prove the existence of a unique invariant probability measure based on Lemma 4. Nevertheless, we demonstrate the non-existence result through Lemma 2.

In the sequel, we will denote by $X_t = (x_t, y_t, z_t, w_t)$ the solution to (2). We always assume $\sum_{i=1}^{2}\kappa_i^2 \neq 0$ to guarantee that system (2) is genuinely stochastic.

3.1 Nice diffusion

Theorem 1  For $a, b, c, r \in \mathbb{R}$ and $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$, solution $X_t$ to (2) is non-explosive. Moreover, it is a nice diffusion provided that $\kappa_1, \kappa_3 > 0$ and $\kappa_2^2 + \kappa_4^2 \neq 0$.

Proof  To prove the first assertion, we take

$$H(x, y, z, w) = \frac{1}{2} \left[ \frac{1}{r} x^2 + y^2 + \left( z - \frac{a}{r} \right)^2 + w^2 \right],$$

and the corresponding infinitesimal generator reads

$$L = (a(y - x) + rw)\partial_t + (cx - y - xz)\partial_x + (xy - bz)\partial_y + (-x - aw)\partial_w + \kappa_1\partial_x^2 + \kappa_2\partial_y^2 + \kappa_3\partial_z^2 + \kappa_4\partial_w^2.$$  

A direct computation leads to

$$LH = -\frac{a}{r} x^2 - y^2 - bz^2 + b \left( c + \frac{a}{r} \right) z - aw^2 + \kappa_1 \partial_x + \kappa_2 \partial_y + \kappa_3 \partial_z + \kappa_4 \partial_w.$$  

Thus the required condition in Lemma 1(i) is reached when taking $V = H$ with Young’s inequality.

So it remains to show that system (2) satisfies the spanning condition in Lemma 3, then it turns out that $X_t$ is a nice diffusion. To this aim, using the Lie bracket, we have

$$F = [a(y - x) + rw]\partial_t + (cx - y - xz)\partial_x + (xy - bz)\partial_y + (-x - aw)\partial_w,$$

$$G_1 = \sqrt{2\kappa_1}\partial_x,$$

$$G_2 = \sqrt{2\kappa_2}\partial_y,$$

$$G_3 = \sqrt{2\kappa_3}\partial_z,$$

$$G_4 = \sqrt{2\kappa_4}\partial_w.$$  

By considering $F, G_1, G_2, G_3, G_4$ as vectors, we can get

$$F(\lambda G_1) = \left(-a\lambda \sqrt{2\kappa_1}, c\lambda \sqrt{2\kappa_1}, 0, -\lambda \sqrt{2\kappa_1}\right)^T.$$  

Therefore we can verify that $n(G_1, F) = 1$, then we have

$$G_1 := \text{ad}^1 G_1(F) = [G_1, F] = -a\sqrt{2\kappa_1}\partial_t + (c - z)\sqrt{2\kappa_1}\partial_y + y\sqrt{2\kappa_1}\partial_z - \sqrt{2\kappa_1}\partial_w \in \mathfrak{g}_0^\mathbb{R}.$$  

Hence, from $n(G_2, G_1) = 1$ it follows

$$G_2 := \text{ad}^1 G_2(G_1) = [G_2, G_1] = \sqrt{4\kappa_1 \kappa_2}\partial_z \in \mathfrak{g}_2^\mathbb{R}.$$  

And on the other hand, from $n(G_3, G_2) = 1$ it follows

$$G_3 := \text{ad}^1 G_3(G_2) = [G_3, G_2] = -\sqrt{4\kappa_1 \kappa_2}\partial_z \in \mathfrak{g}_2^\mathbb{R}.$$  

Along the same line, there are only three cases

(a) $G_1, G_2, G_3, G_4 \in \bigcup_{j \geq 1} \mathfrak{g}_j^\mathbb{R}$ if $\kappa_1, \kappa_2, \kappa_4 > 0$,

(b) $G_1, G_2, G_3, G_4 \in \bigcup_{j \geq 1} \mathfrak{g}_j^\mathbb{R}$ if $\kappa_1, \kappa_2, \kappa_4 > 0$,

(c) $G_1, G_2, G_3, G_4 \in \bigcup_{j \geq 1} \mathfrak{g}_j^\mathbb{R}$ if $\kappa_1, \kappa_2, \kappa_4 > 0$.

which satisfy the required spanning condition and therefore the proof of Theorem 1 is complete.

3.2 Existence of invariant probability measure

We will be led in the sequel to consider the degenerately damped situation, where $b = 0$. Thus (2) reduces to

$$\begin{cases}
  dx = (a(y - x) + rw)dt + \sqrt{2\kappa_1}dB_1, \\
  dy = (cx - y - xz)dt + \sqrt{2\kappa_2}dB_2, \\
  dz = xydt + \sqrt{2\kappa_3}dB_3, \\
  dw = (-x - aw)dt + \sqrt{2\kappa_4}dB_4.
\end{cases}$$

Our task here is to construct a suitable Lyapunov function resulting in a globally finite expected return subject to some compact set. To this purpose, we take

$$V = \frac{1}{2} \left[ \frac{1}{r} x^2 + y^2 + z^2 - 2 \left( c + \frac{a}{r} \right) z + w^2 + \kappa_0 \right] + \frac{n_1 y \theta_1(x, y, z, w)}{x^2} + \frac{n_2 \theta_2(x, y, z, w)}{2\kappa_1} \left( R_1^2 - x^2 \right),$$

whose detailed construction is put in the Appendix (for the sake of reader’s convenience).
Theorem 2  Assume that $b = 0$. If $\kappa_1 > 0$ and $\kappa_2, \kappa_3, \kappa_4 \geq 0$, there exists an $R > 0$ such that for any $S > 0$

$$\sup_{|x| \leq S} \mathbb{E}X \xi_R < \infty,$$

where $\xi_R$ is return time to the ball of radius $R$. Furthermore, system (5) possesses a unique invariant probability measure provided that $\kappa_1, \kappa_4 > 0$ and $\kappa_2^2 + \kappa_3^2 \neq 0$.

Proof  Before proceeding any further, we emphasize that $C > 0$ is independent of $R_0, R_1, R_2, R_3$ and $n_0, n_1, n_2$ unless explicitly stated otherwise. Also, we let $X' := x^2 + y^2 + w^2$ and $X'' := |x||z|^{1/3}$ for the sake of simplicity.

Regarding Lemma 1(ii), we begin by observing that

$$\mathcal{M}(\mathcal{V}) = \mathcal{M}(\mathcal{H}) + \sum_{i=1}^{2} (\theta_i \mathcal{M}(\psi_i) + \psi_i \mathcal{M}(\theta_i)) + 2\mathcal{V}_\kappa \theta_i \cdot \mathcal{V}_\kappa \psi_i,$$

where $\mathcal{V}_\kappa = (\kappa_1 \partial_z, \kappa_2 \partial_z, \kappa_3 \partial_z, \kappa_4 \partial_z)$ and $\mathcal{M}(\psi_1$ and $\psi_2$ are as in (22), (27) and (29) in the Appendix, respectively. To estimate each term in (6), we proceed to their derivatives fall on the cut-off functions $\theta_1$ and $\theta_2$. In fact, it follows that for $\theta_1$

$$\begin{align*}
\partial_t \theta_1 &\leq \frac{C}{R_0^3} |R_0 \leq X' | R_0 \leq R_1, \\
\partial_t \theta_1 &\leq \frac{C}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t \theta_1 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}}, \\
\partial_t \theta_1 &\leq \frac{C}{R_0^2} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t^2 \theta_1 &\leq \frac{C(R_0 + 1)}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t^2 \theta_1 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}}, \\
\partial_t^2 \theta_1 &\leq \frac{C(R_0 + 1)}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t^2 \theta_1 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}},
\end{align*}$$

and for $\theta_2$

$$\begin{align*}
\partial_t \theta_2 &\leq \frac{C}{R_0^3} |R_0 \leq X' | R_0 \leq R_1, \\
\partial_t \theta_2 &\leq \frac{C}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t \theta_2 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}}, \\
\partial_t \theta_2 &\leq \frac{C}{R_0^2} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t \theta_2 &\leq \frac{C(R_0 + 1)}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t \theta_2 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}}, \\
\partial_t \theta_2 &\leq \frac{C(R_0 + 1)}{R_0} |R_0 \leq X' | R_0 \leq R_0, \\
\partial_t \theta_2 &\leq \frac{C|z| |R_0 \leq X' | R_1 | z|^{1/3} + C}{R_3 |z| |R_0 \leq X' | R_1 | z|^{1/3}},
\end{align*}$$

where again $C > 0$ is independent of $R_0, R_1, R_2$ and $R_3$.

We now are ready to expand $\mathcal{M}(\psi_i) \mathcal{M}(\theta_i)$ as

$$\psi_i \mathcal{M}(\theta_i) = \frac{m_i y}{xz} [(ay - ax + mw) \partial_z \theta_1 + (-x - xz) \partial_z \theta_1 + xz \partial_z \theta_1 + (-y - xz) \partial_z \theta_2 + xz \partial_z \theta_2 + \kappa_2 \partial_z \theta_1 + \kappa_3 \partial_z \theta_1 + \kappa_4 \partial_z \theta_2].$$

Since on $\mathcal{F}_1$ one has

$$|\psi_i| = n_1 \frac{y}{xz} \leq \frac{n_1 R_0^2}{R_1} \frac{1}{|z|^{1/3}}.$$

Using the estimates of derivatives on $\theta_i$, we have

$$\begin{align*}
|\psi_i \partial_z \theta_1| &\leq \frac{n_1 R_0^2}{R_1} \frac{C}{R_1} \leq Cn_1 K_4, \\
|\psi_i \partial_z \theta_1| &\leq \frac{n_1 R_0^2}{R_1} \frac{C}{R_1} \leq Cn_1 K_4, \\
|\psi_i \partial_z \theta_1| &\leq |n_1 \partial_z \theta_1| \leq Cn_1 R_0, \\
|\kappa_1 \psi_i \partial_z \theta_1| &\leq Cn_1 \left( \frac{K_4 + R_0^2}{R_1} \right), \\
|\psi_i (\kappa_2 \partial_z \theta_1 + \kappa_3 \partial_z \theta_1 + \kappa_4 \partial_z \theta_1)| &\leq Cn_1 (K_4 + \|\partial_z\|).
\end{align*}$$
where \( K_{R_3} \), a constant that might depend on \( R_0, R_1 \) and \( R_2 \) such that
\[
\lim_{R_3 \to \infty} K_{R_3} = 0.
\]

Hence, we obtain
\[
|\psi_1, \mathcal{M}(\theta)| \leq C_n \left( \underline{\mathcal{M}}_0 + K_{R_3} + \frac{R_0^2}{R_1^2} \right). \tag{9}
\]

We next turn to estimate
\[
|\nabla \kappa_{\theta_1} \cdot \nabla \kappa_{\psi_1}| \leq n_1 \left( \frac{y}{x^2} \left| \partial_\theta \theta_1 \right| + \frac{1}{x^2} \left| \partial_\theta \theta_1 \right| + \frac{y}{x^2} \left| \partial_\theta \theta_1 \right| \right) \leq C_n \left( \frac{R_1^2}{R_0^2} \right) \left( \frac{1}{R_1} \frac{|z|^2}{R_1^2} \right) + \frac{1}{R_0^2} \left( \frac{|x|^2}{R_1^2} + \frac{1}{R_1} \right) \leq C_n \left( \frac{R_1^2 + R_0}{R_1^2} + K_{R_3} \right). \tag{10}
\]

In the sequel, we focus on the cut-off terms involving \( \psi_2 \). In this respect, we expand \( \psi_2, \mathcal{M}(\theta_2) \) as
\[
\psi_2, \mathcal{M}(\theta_2) = n_2 \left( \frac{4R_1^2}{z^2} - x^2 \right) \left( \frac{ay - ax + \kappa_0}{z^2} \right) \left( \frac{1}{R_1^2} \right) \left( \frac{|z|^2}{R_1^2} \right) + \frac{1}{R_0^2} \frac{1}{R_1^2} \left( \frac{|x|^2}{R_1^2} + \frac{1}{R_1} \right) \leq C_n \left( \frac{R_1^2 + R_0}{R_1^2} + K_{R_3} \right). \tag{11}
\]

(12)

Notice that each term in (12) is supported on the set \( \{ x' \leq 2R_1 \} \), and therefore the estimate (30) applies. Then it leads to
\[
\left\{ \begin{array}{l}
|\psi_2, \partial_\theta \theta_2| \leq C_n R_0^2 \left( \frac{1}{|z|^2} \right) + C_n R_0 \left( \frac{1}{|z|^2} \right) \leq C_n K_{R_3} , \\
|\psi_2, \partial_\theta \theta_2| \leq C_n R_0 \left( \frac{1}{|z|^2} \right) \leq C_n K_{R_3} , \\
|\psi_2, \partial_\theta \theta_2| \leq C_n R_0 \left( \frac{1}{|z|^2} \right) \leq C_n K_{R_3} , \\
|\psi_2, \partial_\theta \theta_2| \leq C_n R_0 \left( \frac{1}{|z|^2} \right) \leq C_n K_{R_3} , \end{array} \right.
\]

and
\[
|\kappa_1 \psi_2 \partial_\theta \theta_2 | \leq C_n \left( K_{R_3} + \underline{\mathcal{M}}_0 \right), \leq C_n \left( K_{R_3} + \underline{\mathcal{M}}_0 \right). \tag{13}
\]

Hence, we have
\[
|\psi_2, \mathcal{M}(\theta_2)| \leq C_n \left( \frac{R_3}{R_2} + K_{R_3} + \theta_2 + \underline{\mathcal{M}}_0 \right), \tag{14}
\]

where \( K_{R_3} \) is as in (9). Recall that \( R_2 \geq R_1 \) and \( \mathcal{M}_0 = \{ x^2 + y^2 + w^2 \geq R_0 \} \) we have
\[
|\nabla \kappa_{\theta_2} \cdot \nabla \kappa_{\psi_2}| \leq n_2 \left( \frac{x}{R_1^2} \right) \left( \frac{1}{R_1^2} \right) \leq C_n \left( \frac{R_2^2}{R_1} \right) \left( \frac{1}{R_1^2} \right) \leq C_n \left( \frac{1}{\mathcal{M}_0} + K_{R_3} \right). \tag{15}
\]

Let us gather the estimates (10), (11), (13), (14), (25), (28), (31), to obtain for \( R_2 \geq R_0 \),

\[
\mathcal{M}(V) \leq -\frac{a}{r} x^2 - y^2 - aw^2 + \kappa - n_1 \theta_1 \left( 1 - C_n \frac{R_0^2}{R_1} \right) \leq n_2 \theta_2 \left( 1 - C_n \frac{R_0^2}{R_1} \right) + C(n_1 + n_2) \underline{\mathcal{M}}_0 \]

(16)

where \( \kappa = 2 \left( \kappa_0 + \kappa_2 + \kappa_3 + \kappa_4 \right) \). Let us fix \( n_2 = 16\kappa, n_1 \geq \max\{8\kappa, 2Cn_2\} \) and \( R_0 > 1 \) such that in \( \mathcal{M}_0 = \{ x^2 + y^2 + w^2 \geq R_0 \} \), it follows that
\[
\frac{a}{r} x^2 + y^2 + aw^2 \geq 4\kappa + 2C(n_1 + n_2). \tag{17}
\]

Then, choose \( R_1 > 1 \) such that
\[
C_n \frac{R_0^3}{R_1^3} \leq \frac{\kappa}{3} \quad \text{and} \quad \frac{R_0^4}{R_1^4} \leq \frac{1}{2}, \tag{18}
\]

(19)

and \( R_2 \geq R_0 \) such that
\[
C_n \frac{R_0^3}{R_1^3} \leq \frac{\kappa}{3}, \quad \frac{R_0^4}{R_1^4} \leq \frac{1}{2}, \tag{20}
\]

(21)

Finally, choose \( R_3 \) such that
\[
K_{R_3}(n_1 + n_2) \leq \frac{\kappa}{3} \quad \text{and} \quad C \frac{R_0^4 R_1^2}{R_2^3} \leq \frac{3}{4}, \tag{22}
\]

(23)

With these parameter selections and referring back to (32), (33) we therefore have
\[
\mathcal{M}(V) \leq -4\kappa \underline{\mathcal{M}}_0 + 2\kappa - \frac{1}{n_1} \underline{\mathcal{M}}_0 - \frac{1}{4} n_2 \underline{\mathcal{M}}_0 \leq -2\kappa + 4\kappa (1 - \underline{\mathcal{M}}_0) \leq -2\kappa + 4\kappa \underline{\mathcal{M}}_0 \tag{24}
\]
Since $R_2 \geq R_0$, one has $\{x^2 + y^2 + w^2 \leq R_0, \ |z| \geq R_3 \} \subset \mathcal{H}_1 \cup \mathcal{H}_2$, and therefore $1 - \mathbb{P}_{\mathcal{H}_1 \cup \mathcal{H}_2} = 0$, where $\mathcal{H}_i = \{x^2 + y^2 + w^2 \leq R_0, \ |z| \leq R_3 \}$. Consequently, (23) follows with $p = 2\mathcal{K}$ and $q = 4\mathcal{K}$.

Finally, it remains to check the non-negativity of $V$. Notice that our selection of the parameters $R_0, R_1, R_2, R_3$ and of $n_1, n_2$ was made independent of the value $\kappa_0$ (see (34)). Moreover, by (8) and (30), we have

\[ |\theta_1 \psi_1| \leq Cn_1 \frac{R_0^3}{R_1 R_3^\gamma} \]

and

\[ |\theta_2 \psi_2| \leq Cn_2 \frac{R_2^2}{R_3^\gamma} \]

respectively. Thus having fixed $R_0, R_1, R_2, R_3, n_1, n_2$ and referring back to (34) we have

\[
V \geq \frac{1}{2} \left\{ \frac{x^2 + y^2 + z^2}{r} - 2 \left( c + \frac{a}{r} \right) z + w^2 + \kappa_0 \right\} - Cn_1 \frac{R_0^3}{R_1 R_3^\gamma} + Cn_2 \frac{R_2^2}{R_3^\gamma},
\]

which can be always positive for every $(x, y, z, w) \in \mathbb{R}^4$ by choosing large enough $\kappa_0$. Therefore, the proof of Theorem 2 is now complete. \[\square\]

3.3 Non-existence of invariant probability measure

In this subsection, we devote ourselves to the non-existence issue. Our results are slightly different for $b = 0$ and $b < 0$, so we state them separately.

Theorem 3 When $b = \kappa_1 = \kappa_0 = 0$, and one of $\kappa_2, \kappa_3$ is positive, there is no invariant measure for system (5).

Proof Assume that there is an invariant probability measure $\mu$ of (5) and let $(x, y, z, w)$ have law $\mu$. Thus there exists an increasing sequence of integers $(N_j)_{j=1}^{\infty}$ with $N_{j+1} - N_j \geq 2$ such that

\[
\lim_{j \to \infty} \mathbb{P}(\{2az - x^2 - rw^2\} \subset [N_j, N_j + 2]) = 0. \tag{15}
\]

Based on the construction and properties of $F_N$ as defined by (35) in Appendix, we apply Itô's formula to $F_N(2az - x^2 - rw^2)$

\[
\mathbb{E}_\mu F_N(2az_t - x_t^2 - rw^2_t) = -2aR_0 \mathbb{E}_\mu F_N(2az_t - x_t^2 - rw^2_t) + \mathbb{E}_\mu \int_0^t \left( 2a x^2 + 2aw \right) F_N'(2az - x^2 - rw^2) ds,
\]

which further implies

\[
\mathbb{E}_\mu (x^2 + rw^2) F_N'(2az_t - x_t^2 - rw^2_t) = -2aR_0 \mathbb{E}_\mu F_N'(2az_t - x_t^2 - rw^2_t).
\]

The monotone convergence theorem further indicates

\[
\mathbb{E}[x^2 + rw^2] = \lim_{j \to \infty} \mathbb{E}[x^2 + rw^2] F_N'(2az - x^2 - rw^2) = -2aR_0 \lim_{j \to \infty} \mathbb{E} F_N'(2az - x^2 - rw^2). \tag{16}
\]

Finally, it follows from $|F_N''| \leq c^*, F_N'' = 0$ on the complement of $[N, N + 2]$, and (15) that

\[
|\mathbb{E}[F_N''(2az - x^2 - rw^2)]| \leq c^* \mathbb{P}(\{2az - x^2 - rw^2\} \in [N_j, N_j + 2]) = 0. \tag{17}
\]

Combining (16) and (17) yields $\mathbb{E}[x^2 + rw^2] = 0$, then $x, w = 0$ almost surely. Moreover, if $\kappa_3 > 0$, we have $z(t) = z(0) + \sqrt{2\kappa_3}B_3(t)$. This contradicts invariance. While, if $\kappa_2 > 0$, we have

\[
dx = ay dt, \quad dy = -yd t + \sqrt{2\kappa_2}dB_3,
\]

Using that $x = 0$ almost surely, we obtain

\[
ay = \frac{dx}{dt} = 0,
\]

then $y = 0$ which contradicts to $\kappa_2 > 0$. Therefore, the proof of Theorem 3 is complete. 

Theorem 4 When $b < 0$, for any $\mathcal{H} \subset \mathbb{R}^4$ compact, there exists $(x, y, z, w) \notin \mathcal{H}$ such that

\[
\mathbb{E}_{(x, y, z, w)} \mathbb{1}_{\mathcal{H}^c} = \infty
\]

where

\[
\xi_{\mathcal{H}} = \inf\{t \geq 0 : (x_t, y_t, z_t, w_t) \notin \mathcal{H} \}
\]

If we further let $\kappa_1, \kappa_2 > 0$ and $\kappa_2 + \kappa_3 \neq 0$, then (2) does not possess an invariant probability measure.

Proof We proceed it in four steps to construct $V_1$ and $V_2$ satisfying the conditions in Lemma 2.

Step 1. Fix $R > 1$ such that $\tilde{H}(x, y, z, w) > 1$ ($\tilde{H}(x, y, z, w)$ is as in (24)) for any $|(x, y, z, w)| > R$. Thus we take $W_2 \in C^2(\mathbb{R}^4)$ via

\[
W_2(x, y, z, w) = \ln \tilde{H}(x, y, z, w),
\]

for $|(x, y, z, w)| > R$. Then, $W_2 > 0$ outside of a compact set. Moreover, standard calculations give that

\[
\mathcal{L} W_2(x, y, z, w) = \frac{1}{\tilde{H}(x, y, z, w)} \left[ |b|^2 - y^2 - \frac{a^2 x^2}{r} - 2|b| \left( c + \frac{a}{r} \right) z + \kappa_3 + \kappa_2 + \kappa_1 \right]
\]

\[
+ \frac{1}{\tilde{H}^2(x, y, z, w)} \left( \kappa_2 x^2 + \kappa_3 y^2 + \kappa_2 \left( z - \frac{a}{r} \right)^2 + \kappa_3 w^2 \right).
\]
Consequently, there exists a constant $K > 0$ such that
\[
\mathcal{L}W_2(x, y, z, w) \leq K \text{ for all } (x, y, z, w) \in \mathbb{R}^4,
\]
which motivates us to define $V_2 = W_2/K$.

Step 2. Denote by
\[
A = \frac{2k_1 + 2r_2 + 2}{|\mathbf{b}|}, \quad m = \max \left\{ \frac{2k_1}{a}, 2a^2k_3, \frac{2r_2}{a} \right\},
\]
and let
\[
f(\zeta) := (1 - \cos \zeta)^2.
\]
One can check that $f(0) = f'(0) = f''(0) = 0$ and $f$ is (strictly) increasing on $(0, \pi)$. In particular, $f'(\frac{2\pi}{3}) > 0 = f''(\frac{2\pi}{3})$. Let $B > \frac{2\pi}{3}$ close to $\frac{2\pi}{3}$ such that $f' \geq -mf''$ on $(\frac{2\pi}{3}, B)$. Next, we define
\[
\Psi(\zeta) = \begin{cases} 
0 & \zeta < 0, \\
(1 - \cos \zeta)^2 = f(\zeta) & \zeta \in [0, B], \\
c_0 \ln(\zeta + c_1) + c_2 & \zeta > B,
\end{cases}
\]
where constants $c_0, c_1, c_2$ are determined in a moment.

We now claim that $\Psi$ is a $C^2$ function. Clearly, $\Psi$ is a $C^2$ function at $0$, it remains to show that
\[
c_0 \ln(B + c_1) + c_2 = f(B) > 0,
\]
\[
\frac{c_0}{(B + c_1) \ln(B + c_1)} = f'(B) > 0,
\]
\[
-\frac{c_0 (1 + \ln(B + c_1))}{[(B + c_1) \ln(B + c_1)]^2} = f''(B) < 0.
\]
Substituting the second equation into the third one, we obtain
\[
\frac{1 + \ln(B + c_1)}{(B + c_1) \ln(B + c_1)} = -\frac{f''(B)}{f'(B)} > 0.
\]

However, the function
\[
z \mapsto \frac{1 + \ln z}{z \ln z}
\]
is positive and decreasing on $(1, \infty)$ with a vertical asymptote at $z = 1$ and decaying at infinity. Thus, there exists (unique) $c_1$ such that $B + c_1 > 1$ and (18) holds true.

Then, for fixed $c_1$ we set
\[
c_0 = f'(B)(B + c_1) \ln(B + c_1) > 0
\]
and define $V_1$ by
\[
V_1(x, y, z, w) = \Psi \left( \frac{2a_2 - x^2 - y^2}{\lambda} \right),
\]
then $V_1$ is a $C^2(\mathbb{R}^4)$ function and
\[
\mathcal{L}V_1 = 2a|b|z + ax^2 + arw^2 - k_1 - r_2k_1 \Psi' + 4(k_1x^2 + a^2k_3 + kr^2w^2) \lambda^2 \Psi''.
\]

Step 3. We claim that
\[
\mathcal{L}V_1 \geq 0.
\]

To this goal, we let
\[
\zeta = \frac{2az - x^2 - y^2 - A}{\lambda}.
\]
First, if $\zeta \leq 0$, then $\Psi' = \Psi'' = 0$ and (20) follows. While if $\zeta \geq 0$, it follows that
\[
2az \geq 2ax - y^2 - 2A = \frac{2k_1 + 2r_2}{|\mathbf{b}|},
\]
and thus $a|b|z - k_1 - r_2k_1 \geq 1$. Hence,
\[
(4a_2x^2 + 4a_2k_3 + kr^2w^2) \leq 2m(ax^2 + arw^2 + 1).
\]

Thus, from (19) and (21) follows
\[
(1) \text{ If } \zeta \in [0, \frac{4\pi}{3}], \text{ then } \Psi' = \Psi'' = 0, \text{ and the non-negativity of coefficients of } \Psi', \Psi'' \text{ implies (20).}
\]

(2) If $\zeta \in (\frac{4\pi}{3}, B)$, then $\Psi' > 0$ and $\Psi'' < 0$. Thus, (19) and (21) follows
\[
\frac{1}{\lambda} \mathcal{L}V_1 \geq 2a|b|z + ax^2 + arw^2 - k_1 - 2r_2k_1 \Psi' + \lambda (4a_2x^2 + 4a_2k_3 + kr^2w^2) \Psi''
\]
\[
\geq 2(ax^2 + arw^2 + 1) \Psi' + 2\lambda m(ax^2 + arw^2 + 1) \Psi''
\]
\[
\geq 0.
\]

(3) If $\zeta \in [B, \infty)$, then $\Psi(\zeta) = c_0 \ln(\zeta + c_1) + c_2$. Since $c_0 > 0$, one has $\Psi'(\zeta) > 0, \Psi''(\zeta) < 0$. Then, (19) and (21) follows
\[
\frac{1}{\lambda} \mathcal{L}V_1 \geq 2(ax^2 + arw^2 + 1) \Psi' + 2\lambda m(ax^2 + arw^2 + 1) \Psi''
\]
\[
\geq 2c_0 (ax^2 + arw^2 + 1) \left( \frac{1}{(\frac{4\pi}{3} + c_1) \ln(\zeta + c_1)} \right) \left( 1 - \frac{m(1 + \ln(\zeta + c_1))}{(\frac{4\pi}{3} + c_1) \ln(\zeta + c_1)} \right)
\]
\[
\geq 2c_0 (ax^2 + arw^2 + 1) \left( \frac{1}{(\frac{4\pi}{3} + c_1) \ln(\zeta + c_1)} \right) \left( 1 - \frac{m(1 + \ln(B + c_1))}{(B + c_1) \ln(B + c_1)} \right)
\]
\[
\geq 0.
\]
Step 4. Let us verify that the assumptions of Lemma 2 are satisfied with $V_1$ and $V_2$. Clearly, (iv) follows from the construction of $V_1$ and $V_2$ and (ii) is sure due to the fact that $\lim_{(x,y,z,w)\to\infty} H(x,y,z,w) = \infty$. As for (i),

$$\limsup_{(x,y,z,w)\to\infty} V_1(x,y,z,w) \geq \lim_{z\to\infty} V_1(0,0,z,0) = \lim_{z\to\infty} \Psi(\lambda(2az - A)) = \lim_{z\to\infty} c_0 \ln \left( \frac{\lambda(2az - A) + c_1 + c_2}{\lambda} \right) = \infty,$$

while for (iii)

$$\limsup_{R \to \infty} \sup_{(x,y,z,w) \in R} V_1(x,y,z,w) \leq \limsup_{R \to \infty} \ln \left( \frac{1}{2} (R - c - \frac{a}{T})^2 \right) \leq c_0 \ln \left( \frac{\lambda(2\sigma R - A) + c_1 + c_2}{\lambda} \right) \leq 0,$$

using the fact that $z \mapsto V_1(x,y,z,w)$ is increasing for large $z$ and $(x,y,w) \mapsto V_1(x,y,z,w)$ is non-increasing. This finishes the proof based on Lemma 2. \qed

A Appendix

A.1 Derivation of the Lyapunov function

It is notoriously difficult to check that the infinitesimal generator of (5) leads to

$$\mathcal{M} = (a(y - x) + rw) \partial_x + (-y - xz) \partial_y + \lambda \partial_z + (r - aw) \partial_w + \kappa_1 \partial_x + \kappa_2 \partial_z + \kappa_3 \partial_w,$$

by which our immediate goal is to acquire the inequality

$$\mathcal{M} \leq -p + q \partial_x,$$

for some constants $p, q > 0$ and some compact set $\mathcal{X} \subseteq \mathbb{R}^4$.

We first choose the following Lyapunov function

$$\tilde{H}(x,y,z,w) = \frac{1}{2} \left[ \frac{1}{r} x^2 + y^2 + z^2 - 2 \left( e + \frac{a}{T} \right) z + w^2 + k_0 \right],$$

where $k_0 > 0$ is large enough so that $\tilde{H} \geq 0$. Since

$$\mathcal{M}(\tilde{H}) = -a x^2 - y^2 - aw^2 + 2 \left( \frac{w}{r} + k_2 + k_3 + k_4 \right),$$

the required inequality (23) is sure on the set where $|(x,y,w)| := \sqrt{x^2 + y^2 + w^2}$ is large. More specifically, let the region

$$\mathcal{R}_0 := \{ x^2 + y^2 + w^2 \geq R_0 \}$$

with a sufficiently large $R_0 \geq 0$, that is

$$R_0 \geq \frac{2 \tilde{k}}{\min \{ \tilde{r}, 1, a \}} = \frac{2 (\tilde{k} + k_2 + k_3 + k_4)}{\min \{ \tilde{r}, 1, a \}},$$

we have

$$\mathcal{M}(\tilde{H}) \leq -\tilde{k} \text{ in } \mathcal{R}_0.$$
Consequently, for any fixed $R_0 \geq 1$, we can choose suitably large $n_1 \geq 1 + 4\sqrt[4]{4K}$ and $R_1 \geq 1$ so that

$$\mathcal{M}(\hat{H} + \psi_1) \leq -\frac{1}{2} n_1$$
on the region $\mathcal{R}_1$.

The second situation is that $\alpha \in (\frac{1}{4}, 1)$, where the dominant term in (26) becomes $\kappa_1 \lambda^{2\alpha} \frac{d_1^2}{z^\alpha} + \kappa_2 \lambda^{2\alpha} \frac{d_2^2}{z^\alpha}$. This allows us to consider

$$dX = \sqrt{2K} \frac{d_b}{b_1}, \quad \dot{y} = 0, \quad \dot{Z} = 0, \quad dW = \sqrt{2K} \frac{d_b}{b_1},$$

To our goal, we define the region

$$\mathcal{R}_2 := \{x^2 + y^2 + w^2 \leq R_2, |x| |z| \leq R_1, |z| \leq R_3\}.$$

Similarly, we are interested in finding a function $\psi_2$ that solves

$$(\kappa_1 \frac{d_1^2}{z^\alpha} + \kappa_2 \frac{d_2^2}{z^\alpha}) \psi_2 = -n_2.$$

Clearly, a particular solution to the above equation is

$$\psi_2 = \frac{n_2}{2K} \left( \frac{4R_1^2}{z^\alpha} - x^2 \right),$$

implying the estimate

$$|\psi_2| \leq \frac{n_2 R_1^2}{2K |z|^\alpha}, \quad \text{whenever} \quad |x| |z| \leq 2R_1,$$

Thus we obtain

$$\frac{1}{n_2} \mathcal{M}(\psi_2)$$

$$= (ay - ax + rw) \left( -\frac{x}{K} \right) + xy \left( -\frac{4R_1^2}{3K} |z|^\alpha \right) - 1 + \frac{20K \psi_1}{9K |z|^\alpha},$$

$$\leq \frac{CR^2 R_0}{R_1} - 1,$$

since

$$|y - x| \leq \frac{2R_2 R_1^2}{|z|^\alpha} \leq \frac{2R_2 R_1^2}{R_1^2},$$

$$|w| \leq \frac{R_1 |w|}{|z|^\alpha} \leq \frac{R_1 R_1^2}{R_1^2},$$

where $C = C(a, c, r, \alpha_1, \alpha_2)$ is independent of $R_1, R_2, R_3$ and $n_2$. Hence, we get

$$\mathcal{M}(\hat{H} + \psi_2) \leq -\frac{1}{2} n_2$$
on the region $\mathcal{R}_2$.

by choosing large $R_1 \geq 1$ and $n_2 \geq 1 + 4\sqrt[4]{4K}$. For the critical situation $\alpha = \frac{1}{2}$, the dominant dynamics is

$$\lambda \frac{d_1^2}{z} - \lambda \frac{d_2^2}{z} x \dot{d}_1 + \lambda \frac{1}{2} \dot{d}_2,$$

whereas $\psi_2$ is also valid cause it is independent of $y$.

Besides, it is easy to check

$$\limsup_{|X| \to \infty} \psi_1(X) = 0$$

for $i = 1, 2$, so that the inequality (23) is sure where

$$\mathcal{A} := \{x^2 + y^2 + w^2 \leq R_0, |z| \leq R_1\}.$$

Collecting the above discussion, we arrive at the preliminary candidate

$$V := \hat{H} + \psi_1 + \psi_2,$$

where 1 stands for the indicator function. It therefore remains only to smooth this Lyapunov function. To do this, we introduce non-negative $C^\infty(\mathbb{R})$ functions $\chi$ and $\tilde{\chi}$:

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

and

$$\tilde{\chi}(x) = \begin{cases} 1 & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| \leq 1/2, \end{cases}$$

by which, we define

$$\theta_1 := \chi \left( \frac{x^2 + y^2 + w^2}{R_0} \right) \tilde{\chi} \left( \frac{|x| |z|}{R_1} \right) \tilde{\chi} \left( \frac{|z|}{R_1} \right),$$

and

$$\theta_2 := \chi \left( \frac{x^2 + y^2 + w^2}{R_2} \right) \tilde{\chi} \left( \frac{|x| |z|}{R_2} \right) \tilde{\chi} \left( \frac{|z|}{R_1} \right).$$

Therefore, we obtain the gluing version

$$V := \hat{H} + \theta_1 \psi_1 + \theta_2 \psi_2$$

$$= \left( \frac{1}{2} + \frac{\alpha}{z} \right) \frac{c + \alpha}{r} \left( x^2 + y^2 + w^2 + R_0 \right) + n_1 \theta_1(x, y, z, w) + n_2 \theta_2(x, y, z, w) \left( R_1^2 - x^2 \right).$$

with specific parameters $R_0, R_1, R_2, R_3 \geq 1$ and $\kappa_0, \kappa_1, \kappa_2 > 0$.

A.2 Construction of cut-off function $F_N$

For each $N \geq 1$, we define a $C^2$ function $F_N : \mathbb{R} \to \mathbb{R}$ as

$$F_N(x) :=$$

$$= \left\{ \begin{array}{ll}
    x, & x \in [0, N), \\
    h(x - N) + N, & x \in [N, N + 1), \\
    N + 1, & x \geq N + 2,
  \end{array} \right.$$ (35)

where $h : [0, 2] \to \mathbb{R}$ be a non-decreasing $C^2$ function such that

$$h(0) = h''(0) = h'(2) = h''(2) = 0, h(0) = 1, h(2) = 1, \max_{[0, 2]} |h'| \leq 1.$$

Denote $c^* = \max_{[0, 2]} |h'|$. Clearly,

$$F_N''' \geq 0, \max_{[0, 2]} |F_N'''| \leq 1, \quad \text{and} \quad \max_{[0, 2]} |F_N''| = c^*.$$

We next claim that $F_{N_j + 1} \geq F_{N_j}$ for any $j$. In fact, for $|\xi| \leq N_j$ one has $1 = F_{N_j}'(\xi) = F_{N_j}'(\xi)$ and for $|\xi| \geq N_j + 2$ one has $F_{N_j}'(\xi) = 0 \leq F_{N_j + 1}'(\xi)$. Finally, since $N_j + 1 \geq N_j + 2$, for any $|\xi| \in [N_j, N_j + 2]$. We have $F_{N_j}'(\xi) = 1 = F_{N_j + 1}'(\xi)$. Thus, $(F_N')$ is a non-decreasing sequence of non-negative functions that converges pointwise to 1 on $\mathbb{R}$. In order to facilitate the readers, we carry out the following simulation. To this aim, we take

$$h(x) := \frac{1}{16} x^4 - \frac{1}{4} x^3 + x,$$

for any $x \in [0, 2]$. Obviously, the required conditions for $h(x)$ are satisfied, and the phase diagram is shown in Fig. 1.
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This page contains references and acknowledgments. The text appears to be a continuation of the previous page, discussing various references and their details. The page includes a phase diagram of $F_N$ and its derivatives, along with references to various scientific works on the topic of the Lorenz-Stenflo system. The page also mentions the conflict of interest, data availability statement, and references section. The document appears to be a scientific paper discussing the dynamics of a degenerately damped stochastic Lorenz-Stenflo system.