Area Potentials and Deformation Quantization

Thomas L Curtright§, Alexios P Polychronakos♮, and Cosmas K Zachos¶

§ Department of Physics, University of Miami, Box 248046, Coral Gables, Florida 33124, USA
curtright@physics.miami.edu

♮ Physics Department, The Rockefeller University, New York, NY 10021, USA
and
Physics Department, University of Ioannina, 45110 Ioannina, Greece
poly@teorfys.uu.se

¶ High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439-4815, USA
zachos@hep.anl.gov

Abstract

Systems built out of N-body interactions, beyond 2-body interactions, are formulated on the plane, and investigated classically and quantum mechanically (in phase space). Their Wigner Functions—the density matrices in phase-space quantization—are given and analyzed.

In this brief note, we consider systems of N particles on a plane, interacting through N-body potentials, over and above more familiar 2-body potentials. Below, we provide the compact generating function, 

\[ G(a, a^*, b, b^*; z, z^*, p, p^*) = \frac{1}{\pi^{2N}} \exp \left( \frac{1}{\hbar} \left( \frac{|a|^2 + |b|^2}{2} - m(z - a)^* \cdot \mathfrak{M}(z - a) - \frac{(p - b)^* \cdot (p - b)}{m} \right) \right), \]

for their Wigner Functions (WF), the density matrices in phase-space quantization [1, 2, 3, 4]. (For reviews, see [5].)

Consider the hamiltonian of three particles on a plane, interacting in proportion to the (signed) area of the triangle they define:

\[ H_3 = p_1^2 + p_2^2 + p_3^2 + mA, \]

where the signed area (doubled) is

\[ A = (r_1 - r_2) \wedge (r_2 - r_3). \]

1 On leave from the Theoretical Physics Department, Uppsala University, Sweden.
Its (saddle) quadratic form, with the three 2-vectors arrayed in succession, \( \mathbf{R} = (x_1, y_1, x_2, y_2, x_3, y_3) \), is
\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\tag{4}
\]

This may be simplified by elimination of the \( 2 \times 2 \) structure. On the plane, rotational invariance can be exploited by introducing complex variables, \( z \equiv x + iy \), which cuts down the size of such matrices by half on each side, since
\[
\mathbf{r}_1 \wedge \mathbf{r}_2 = x_1 y_2 - x_2 y_1 = \frac{-i}{2} (z_1^*, z_2^*) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\tag{5}
\]

Clearly multiplication of the zs by an arbitrary phase yields a rotated configuration with the same eigenvalue (a double degeneracy of the original problem).

Thus the above quadratic form hermitean matrix (up to 1/2) reduces to the imaginary antisymmetric matrix
\[
\Omega = -i \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.
\tag{6}
\]

Consequently, the above hamiltonian amounts to
\[
H_3 = \frac{1}{2m} \mathbf{p}^* \cdot \mathbf{p} + \frac{m}{2} \mathbf{z}^* \cdot \Omega \mathbf{z},
\tag{7}
\]
for complex 3-vectors \( \mathbf{z} \) and their canonical conjugates \( \mathbf{p}^* \) (with components \( p \equiv p_x + ip_y \)).

The matrix \( \Omega \) consists of Sylvester’s celebrated “nonions” [4], today’s standard clock and shift matrix basis. Specifically, \( \Omega = -i(h - h^2) \), where \( h \) is the cyclic permutation shift matrix, with \( h^3 = \mathbb{1} \), hence \( h^t = h^{-1} = h^2 \). For \( \omega = -1/2 + i\sqrt{3}/2 \), a cube root of unity, \( \Omega \) is diagonalized by the Finite Fourier Transform unitary matrix [7],
\[
U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\
1 & \omega^2 & \omega \\
1 & 1 & 1 \end{pmatrix},
\tag{8}
\]
\[
U^\dagger \Omega U = -i(\omega - \omega^2) \begin{pmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \end{pmatrix},
\tag{9}
\]
i.e., the (real) difference of two clock matrices.

The normal mode frequencies-squared of the hamiltonian then are 0, \( \sqrt{3} \), and \( -\sqrt{3} \). The negative eigenvalues (imaginary frequencies) reflect the cyclic symmetry, 1, 2, 3, anti-mirrored into the antistandard order, 1, 3, 2. For every eigenvalue, the opposite eigenvalue follows for the two-particle permuted (here, antistandard) configuration.

The corresponding eigenvectors describe the translation zero mode, \((1,1,1)\), with all three particles moving in coincidence, so without a force; the pulsating cube roots of unity configuration in standard,
1, 2, 3, order, \((\omega, \omega^2, 1)\), or any rotation of it on the plane (with which it can be combined to rotating configurations); and, finally, its mirror image (c.c.), \((\omega^2, \omega, 1)\), in antistandard, 1, 3, 2, order. These last (imaginary frequency) unstable modes lead to the indefinite growing of the area of the antistandard-order triangle, corresponding to a quantum mechanical spectrum unbounded below.

The unstable normal modes of increasing negative area and the concomitant lower unboundedness of the spectrum may be counteracted by considering additional harmonic two-body interactions between each pair of particles, i.e., a positive semi-definite potential component,

\[
V_2 = \frac{m}{2\sqrt{3}} \left( (r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2 \right),
\]

whose corresponding quadratic form matrix,

\[
W = \frac{1}{\sqrt{3}} \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
= \frac{1}{\sqrt{3}} \left( 3 \mathbb{I} - \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \right),
\]

projects out the translation mode and has eigenvalue \(\sqrt{3}\) for the other modes. Thus, the potential \(A + V_2\), corresponds to the modified form \(\mathcal{W} = \Omega + \mathcal{W}\), whose eigenvalues are thus shifted from the above ones to non-negative ones, 0, 2\(\sqrt{3}\), and 0, for the same orthogonal eigenvectors. Consequently, considering the plane-rotational invariance doubling of the modes, the system \(\mathcal{H}_3\) resolves to two harmonic oscillators and four free modes, whose quantization yields a 2-d linear spectrum on the free particles’ continuum.

The system considered so far naturally generalizes to a standard (anticlockwise ordered) \(N\)-gon, with (twice) the area, consisting of a fan of triangles,

\[
A = r_1 \wedge r_2 + r_2 \wedge r_3 + ... + r_N \wedge r_1.
\]

It corresponds to the \(N \times N\) matrix \(\Omega\)

\[
\Omega = -i(h - h^{N-1})... = 2 \sin \left(\ln(-ih)\right),
\]

where \(h\) is the \(N \times N\) cyclic shift matrix, with \(h^N = \mathbb{I}\) and \(h^t = h^{-1} = h^{N-1}\). Extension from \(N\) to \(N + 1\) amounts to the addition of just one triangle to the fan.

Thus, the general eigenvalues for \(\Omega\) are

\[
\lambda_k = -i(\omega^k - \omega^{-k}),
\]

with \(\omega\) the primary \(N\)-th root of unity and \(k = 0, 1, 2, ..., N - 1\). Note the evident mirror-image pairing of the eigenvalues \(\lambda_k = -\lambda_{N-k}\). The corresponding \(N\) complex eigenvectors are

\[
|e_k\rangle = (1, \omega^k, \omega^{2k}, ..., \omega^{k(N-1)}),
\]

which array into the corresponding unitary diagonalizing Finite Fourier Transform matrix \(U\).
The negative eigenvalues may be shifted, as above, by the addition of a 2-body interaction in the potential,
\[ V_2 = \frac{m}{2c} \sum_{i<j} (r_i - r_j)^2 = \frac{m}{2c} \left( N \sum_i r_i^2 - \left( \sum_i r_i \right)^2 \right), \tag{16} \]
where \( c \) is a normalization constant. This interaction corresponds to the matrix
\[ W = \frac{1}{c} \left( N \mathbb{1} - |e_0\rangle \langle e_0| \right). \tag{17} \]
\(|e_0\rangle = (1, 1, 1, \ldots, 1)\) is the translation zero mode projected out, while all other eigenvalues but its own are shifted by \( N/c \). Thus, choosing the normalization \( (c \leq N/2 \) will suffice), all negative eigenvalues of \( W \) may be neutralized to yield a stable system \( H_N \) with a spectrum bounded below.

The classical equations of motion are
\[ \ddot{z} = -\Omega z. \tag{18} \]
Note the vanishing of the torque (and hence conservation of angular momentum) for the system \( H_N \) (manifestly rotational invariant!), even though the interaction is not two-body. This follows from summing the torque on each particle, since \( \Omega \) is hermitean,
\[ \tau = \sum_i r_i \wedge m \dot{r}_i = \frac{im}{2} (z^* \cdot \Omega z - (\Omega z)^* \cdot z) = 0. \tag{19} \]

Further note the \( N \) independent time-invariant complex vector combinations,
\[ \exp(-i\sqrt{\Omega} t) \left( p + i\sqrt{\Omega} \cdot z \right). \tag{20} \]
In the less compact notation of \( 2n \)-dimensional phase space \( (n = 2N) \) there are \( 2n \) real independent time-invariant combinations. Eliminating \( t \) among them yields \( 2n-1 \) independent real time-invariant quantities\( Q_i \), which characterize maximally superintegrable systems in phase space \( [9] \). For these, time evolution is completely specified by flow perpendicular to all the phase-space gradients \( \nabla Q_i \). Thus, the time evolution of the phase-space variables, and so the time derivative of any phase-space function, is proportional to the corresponding Jacobian determinant for the full \( 2n \)-dimensional phase space,
\[ \frac{df}{dt} \propto \frac{\partial(f, Q_1, \ldots, Q_{2n-1})}{\partial(x_1, p_{x_1}, y_1, p_{y_1}, \ldots, x_N, p_{x_N}, y_N, p_{y_N})}, \tag{21} \]
i.e., the phase-space Nambu Bracket \([9]\).

There is an alternate way to stabilize the system by shifting the negative eigenvalues of \( \Omega \) without the above introduction of 2-body interactions, but, instead, through the introduction of a magnetic field \( B \) pointing into the plane in question, and consideration of the particles as charged. Such a magnetic field breaks parity and differentiates between left- and right-spinning cyclotron orbits. The magnetic field modifies the complex equations of motion \([8]\),
\[ \ddot{z} = iBz - \Omega z, \tag{22} \]
\[ \frac{p_{x_2} + i2\sqrt{3} x_2}{p_{y_2} + i2\sqrt{3} y_2}; \quad (p_{x_2} + i2\sqrt{3} x_2) \exp(-2\sqrt{3} x_1/m p_{x_1}); \quad p_{x_1}, p_{y_1}, p_{x_3}, p_{y_3}; \quad \frac{p_{y_1} x_1}{p_{x_1} y_1}, \frac{p_{x_3} x_1}{p_{x_3} y_1}, \frac{p_{y_3} x_1}{p_{x_3} y_3}. \]
and thus shifts the eigenfrequencies of the modes from $\sqrt{\lambda_k}$ to $w_k$, solutions of
\[ w_k^2 - Bw_k - \lambda_k = 0. \tag{23} \]
The modified mode frequencies
\[ w_k = \frac{B}{2} \pm \sqrt{\frac{B^2}{4} + \lambda_k} \tag{24} \]
are real for $B^2 \geq -4 \min(\lambda_k)$, which equals 8 for even $N$, and is smaller for odd $N$. Thus, the system is stable for a field at least as strong as $|B| > 2\sqrt{2}$. (In the small mass limit, where the strength of the area potential and the magnetic field are kept unchanged, the left-hand side of (22) drops out and hence $w_k = \lambda_k/B$, describing pure Landau levels.)

The zero mode $w_0 = 0$ coincides with $\sqrt{\lambda_0} = 0$ and represents Landau level degeneracy. The mode frequencies $w_k$ can be positive or negative, corresponding to the chirality of the cyclotron orbits. At the critical field, $|B_c| = 2\sqrt{2}$, (taking $-i\omega_k = \exp(2\pi ik/N - i\pi/2)$), the above square root acts on a perfect square, so that
\[ w_k/2 = \text{sgn}(B_c) \cos(\pi/4) \pm \cos \left( \frac{\pi k}{N} - \frac{\pi}{4} \right), \tag{25} \]
where $\text{sgn}(B_c) = \pm 1$. So, essentially (for large $N$ where the vanishing and nonvanishing $w$ corresponding to $\lambda_0 = 0$ are neglected), 25% of the modes are of one chirality and 75% are of the opposite. For the above $N = 3$ case, the critical field is $B_c = 2(3)^{1/4}$; beyond the zero mode, there is one mode with $w = B_c$, two with $w = (3)^{1/4}(1 \pm \sqrt{2})$, and the two ones stabilized by the field to $w = B_c/2$.

In the large $N$ limit, the above system reduces to a closed noncovariant string undulating on the plane. The particle index becomes a continuous periodic variable, $\sigma = 2\pi n/N$, and the complex $N$-vector of the particles’ position goes into a scalar closed string field of that variable, $z(t) \mapsto \phi(\sigma, t)$. The potential is the (signed) area enclosed by the string. The resulting noncovariant lagrangian density is
\[ \mathcal{L} = \frac{m}{2} \dot{\bar{\phi}} \dot{\phi} + i \bar{\phi} \partial_\sigma \phi. \tag{26} \]
This amounts to Schrödinger’s Lagrangian, with the roles of space and time reversed. The dispersion relation of Schrödinger’s action, $E = p^2/2$, here reverses to
\[ w^2 + 2k = 0, \tag{27} \]
with $k$ the integer Fourier modes of the compact variable $\sigma$. This specifies an infinite tower of stable and unstable modes. As in the finite case, introducing an additional harmonic two-body potential for the particles amounts to a term,
\[ \int_{\sigma < \sigma'} d\sigma d\sigma' |\phi(\sigma) - \phi(\sigma')|^2 = \int d\sigma |\phi(\sigma)|^2 - \int d\sigma |\phi(\sigma)|^2, \tag{28} \]
especially an external harmonic potential minus a center-of-mass term. This would correspond to an external constant potential in the Schrödinger equation. The modes are shifted by a constant, but still an infinity of them remain unstable.

We finally quantize the original system in phase-space [5]. Groenewold’s associative $\star$-product [10], in our condensed notation of complex vectors,
\[ \star \equiv e^{i\hbar(\hat{\partial}_z \cdot \hat{\partial}_p + \hat{\partial}_p \cdot \hat{\partial}_z - \hat{\partial}_p \cdot \hat{\partial}_z - \hat{\partial}_p \cdot \hat{\partial}_p)}, \tag{29} \]
is the cornerstone of phase-space quantization. (It allows c-number functions in phase space to multiply with each other and with WFs associatively and noncommutatively, in perfect parallel to operator manipulations in the standard, Hilbert space, formulation of quantum mechanics [5].) The Poisson Brackets which are exponentiated in the \( \ast \)-product are

\[
\{ z_i^\ast, p_j \} = 2 \delta_{ij},
\]

and their complex conjugate, for \( i, j = 1, \ldots, N \).

The linear contact transformation \( U \) discussed above, which leads to the normal mode variables \( U^\dagger z \) and \( U^\dagger p \), is a linear canonical transformation, and therefore preserves Poisson Brackets and \( \ast \)-products [14]. (To be contrasted to nonlinear canonical transformations which actually transform \( \ast \)-products in a covariant fashion [12]). That is to say, for these transformations the \( \ast \) above is a scalar, i.e. equals to the same expression for the normal mode variables \( U^\dagger z \). All formal manipulations below, then, may be equivalently conducted in the space of the original or the normal mode variables, and no distinction between either variables in \( \ast \) is necessary.

Thus, e.g., in terms of the normal modes, the WF for the \( N = 3 \) case \( \mathcal{G}_3 \) is a product of the six WFs of each normal mode: four unnormalizable free ones and two oscillators of frequency-squared 2

\[
f = \delta(p_1 - \hbar k_1)\delta(p_2 - \hbar k_2) \frac{(-)^{n+s}}{\pi^2} \exp \left( -\frac{1}{\hbar} \frac{p_1^2}{mw} + mw k_1^2 \right) L_n \left( 2 \frac{\hbar^2}{m} \frac{p_2^2}{mw} + mw k_2^2 \right) L_s \left( 2 \frac{\hbar^2}{m} \frac{p_3^2}{mw} + mw k_3^2 \right),
\]

where the \( L_n \)'s are Laguerre polynomials with \( n, s = 0, 1, 2, \ldots \), and the two \( k_s \)s comprise four arbitrary real constants. These WFs obey the phase-space stargenvalue equation for the complete spectrum of the quantum system [14],

\[
\mathcal{G}_3 \ast f = \left( \frac{\hbar^2 k_1^2 + k_3^2}{2m} + \left( \frac{3}{4} \right)^{1/4} \hbar (1 + n + s) \right) f.
\]

Nevertheless, the generic \( N \) case may be approached in a compact way through the WF generating functions of [13], which generalize the WFs for coherent states [11] (the multiparticle generalization for quadratic systems may also be found in [14]).

\[
G(a, a^\ast, b, b^\ast; z, z^\ast, p, p^\ast) = \frac{1}{\pi^{2N}} \exp \left( \frac{1}{\hbar} \frac{|a|^2 + |b|^2}{2} - m(z - a)^\ast \cdot \mathfrak{M}(z - a) - \frac{(p - b)^\ast \cdot (p - b)}{m} \right).
\]

For

\[
\mathfrak{M} = \frac{1}{2m} p^\ast \cdot p + \frac{m}{2} z^\ast \cdot \mathfrak{M} z,
\]

this function satisfies

\[
\mathfrak{M} \ast G \simeq \frac{\hbar}{2} (2N + (a - b) \cdot (\partial_a - \partial_b) + (a^\ast - b^\ast) \cdot (\partial_{a^\ast} - \partial_{b^\ast})) G,
\]

\[
G \ast \mathfrak{M} \simeq \frac{\hbar}{2} (2N + (a + b) \cdot (\partial_a + \partial_b) + (a^\ast + b^\ast) \cdot (\partial_{a^\ast} + \partial_{b^\ast})) G.
\]

The \( \mathfrak{M} \) may be diagonalized by unitarily transforming all variables and shifts \( z, p, a, b, \partial_z, \partial_p \), through \( U^\dagger z \), and their c.c.s through \( U^\dagger z^\ast \), \( U^\dagger \partial_z \), etc. All quantities \( \mathfrak{M}, G, \ast \), are scalar dot products. Applications of such generating functions are detailed in [13].
Several constructions based on the potentials introduced here will be presented in future publications.

Acknowledgments
This work was supported in part by the US Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38, and the NSF Award 0073390.

References

[1] J Moyal, Proc Camb Phil Soc 45 (1949) 99-124
[2] E Wigner, Phys Rev 40 (1932) 749-759
[3] H Weyl, Z Phys 46 (1927) 1; also reviewed in H Weyl (1931) The Theory of Groups and Quantum Mechanics, Dover, New York
[4] F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer, Ann Phys 111 (1978) 61; ibid. 111
[5] C Zachos, hep-th/0110114;
   M Hillery, R O’Connell, M Scully, and E Wigner, Phys Repts 106 (1984) 121;
   H-W Lee, Phys Repts 259 (1995) 147;
   N Balazs and B Jennings, Phys Repts 104 (1984) 347;
   R Littlejohn, Phys Repts 138 (1986) 193;
   M Berry, Philos Trans R Soc London A287 (1977) 237;
   M Gadella, Fortschr Phys 43 (1995) 3, 229-264;
   L Cohen, Time-Frequency Analysis (Prentice Hall PTR, Englewood Cliffs, 1995);
   F Berezin, Sov Phys Usp 23 (1980) 763-787
[6] J Sylvester, Johns Hopkins University Circulars I (1882) 241-242; ibid. II (1883) 46; ibid. III (1884) 7-9. Summarized in The Collected Mathematics Papers of James Joseph Sylvester (Cambridge University Press, 1909) v. III
[7] T Santhanam and A Tekumalla, Found Phys 6 (1976) 583-587;
   E Floratos and G Leontaris, Phys Lett B412 (1997) 35-41
[8] D Fairlie and C Zachos, Phys Lett B224 (1989) 101;
   E Floratos, Phys Lett B228 (1989) 335-340
[9] R Chatterjee, Lett Math Phys 36 (1996) 117-126;
   Y Nambu, Phys Rev D7 (1973) 2405-2441;
   N Mukunda and E Sudarshan, Phys Rev D13 (1976) 2846-2850;
   C Gonera and Y Nutku, Phys Lett A285 (2001) 301-306
[10] H Groenewold, Physica 12 (1946) 405-460
[11] D Han, Y Kim, and M Noz, Phys Rev A40 (1989) 902-912
[12] T Curtright, D Fairlie, and C Zachos, Phys Rev D58 (1998) 025002
[13] T Curtright, T Uematsu, and C Zachos, J Math Phys 42 (2001) 2396-2415 hep-th/0011137
[14] V Dodonov and V Man’ko, Physica 137A (1986) 306-316