On squares of spaces and $F_\sigma$-sets

Arnold W. Miller

Abstract: We show that the continuum hypothesis implies there exists a Lindelöf space $X$ such that $X^2$ is the union of two metrizable subspaces but $X$ is not metrizable. This gives a consistent solution to a problem of Balogh, Gruenhage, and Tkachuk. The main lemma is that assuming the continuum hypothesis there exist disjoint sets of reals $X$ and $Y$ such that $X$ is Borel concentrated on $Y$, i.e., for any Borel set $B$ if $Y \subseteq B$ then $X \setminus B$ is countable, but $X^2 \setminus \Delta$ is relatively $F_\sigma$ in $X^2 \cup Y^2$.

In Balogh, Gruenhage, and Tkachuk the following question is asked:

Question 4.1. Let $X$ be a regular paracompact space $X$ such that $X \times X$ is the union of two metrizable subspaces. Must $X$ be metrizable? What if $X$ is Lindelöf?

**Theorem 1** Assume the continuum hypothesis. Then there exists a non-metrizable regular Lindelöf space $X$ such that $X^2$ is the union of two metrizable subspaces.

We first prove the following Lemma.

**Lemma 2** (CH) There are uncountable disjoint sets $X, Y \subseteq 2^\omega$ such that

1. $X$ is Borel concentrated on $Y$, i.e., every Borel set in $2^\omega$ containing $Y$ contains all but countably many elements of $X$,

2. $Y^2 \setminus \Delta$ is $F_\sigma$ in $X^2 \cup Y^2$, and

3. $X^2 \setminus \Delta$ is $F_\sigma$ in $X^2 \cup Y^2$.

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Here \( \Delta = \{(x, x) : x \in 2^\omega \} \).

Proof
We identify the Cantor space \( 2^\omega \) with the power set of \( \omega, P(\omega) \). We use \([\omega]^\omega\) to stand for the infinite subsets of \( \omega \). Define for \( y \in [\omega]^\omega \)
\[ [y]^{*\omega} = \{ x \in [\omega]^\omega : x \subseteq^* y \} \]
where \( \subseteq^* \) stands for inclusion mod finite. Let \( \langle B_\alpha : \alpha < \omega_1 \rangle \) be all Borel subsets of \([\omega]^\omega\). We construct \( y_\alpha \) for \( \alpha < \omega_1 \) so that
1. \( \alpha < \beta \) implies \( y_\beta \subseteq^* y_\alpha \) and \( y_\beta \neq^* y_\alpha \) and
2. either \( y_\alpha \notin B_\alpha \) or \([y_\alpha]^{*\omega} \subseteq B_\alpha \).

These conditions are easy to get. Given \( y_\beta \) for \( \beta < \alpha \) be arbitrary with \( y \subseteq^* y_\beta \) but \( y_\beta \neq^* y \) for each \( \beta < \alpha \). If \([y]^{*\omega} \) is a subset of \( B_\alpha \), then simply take \( y_\alpha \in [y]^{*\omega} \setminus B_\alpha \), otherwise take \( y_\alpha = y \).

Let
\[ X = \{ y_\alpha \setminus y_{\alpha+1} : \alpha < \omega_1 \} \] and \( Y = \{ y_\alpha : \alpha < \omega_1 \} \)

Iff \( B \) is any Borel set containing \( Y \), then choose \( \alpha \) so that \( B = B_\alpha \). At stage \( \alpha \) of the construction it must have been that \([y_\alpha]^{*\omega} \subseteq B_\alpha \). But this means that \( x_\beta \in B_\alpha \) for all \( \beta \geq \alpha \). So \( X \) is Borel concentrated on \( Y \).

If we define
\[ F = \{(u, v) \in P(\omega) \times P(\omega) : (u \subseteq^* v \text{ or } v \subseteq^* u) \text{ and } u \neq v \} \]
Then \( F \) is an \( F_\sigma \) set and
\[ F \cap (X^2 \cup Y^2) = (Y^2 \setminus \Delta) \]
Also if we define
\[ H = \{(u, v) \in P(\omega) \times P(\omega) : u \cap v =^* \emptyset \} \]
Then \( H \) is an \( F_\sigma \) set and
\[ H \cap (X^2 \cup Y^2) = (X^2 \setminus \Delta) \]
This finishes the proof of the Lemma.
QED
Now define the following Michael-line like topology. Suppose that \( M \) is a topological space and \( X \subseteq M \). Then \( M(X) \) is the topological space on the same set but with the following topology. For \( x \in X \) we make \( x \) an isolated point, i.e., add \( \{x\} \) to the topology of \( M(X) \). For any point \( y \in M \setminus X \) neighborhoods in \( M \) form a neighborhood basis for \( y \) in \( M(X) \). It is easy to see that \( M(X) \) is regular for any regular space \( M \) and subset \( X \subseteq M \).

The following is exercise 5.5.2 from Engelking [2]:

**Proposition 3** Suppose \( M \) is a metric space and \( X \subseteq M \). Then \( M(X) \) is metrizable iff \( X \) is an \( F_\sigma \) set in \( M \).

Our example is \( M(X) \) where \( X \) and \( Y \) are from the Lemma and \( M = X \cup Y \) has its usual (separable metric) topology as a subspace of \( 2^\omega \). It follows from the Proposition that \( M(X) \) is not metrizable.

\( M(X) \) is a Lindelöf space. Take any open cover \( \mathcal{U} \) of \( M(X) \). Open sets in \( M(X) \) have the form \( U \cup Z \) where \( U \) is open in \( M \) and \( Z \subseteq X \) is arbitrary. Then since \( Y \) has its standard topology, countably many elements of \( \mathcal{U} \) will cover \( Y \), say

\[
\{ (U_n \cup X_n : n < \omega) \subseteq \mathcal{U} \}
\]

where each \( U_n \) open in \( M \) and \( X_n \subseteq X \). But since \( X \) is Borel concentrated on \( Y \) we have that \( X \setminus \cup\{U_n : n < \omega\} \) is countable, so we need only add countably many more elements of \( \mathcal{U} \) to cover all of \( M(X) \).

\( M(X) \) is the union of two metrizable subspaces. Let

\[
M_1 = (X^2 \setminus \Delta) \cup Y^2
\]
\[M_2 = (X \times Y) \cup (Y \times X) \cup (X^2 \cap \Delta).
\]

Note that \( M_1 \) is \( N(X^2 \setminus \Delta) \) where \( N = (X^2 \setminus \Delta) \cup Y^2 \) in its separable metric topology as a subspace of \( 2^\omega \times 2^\omega \). By the Lemma we have that \( X^2 \setminus \Delta \) is relatively \( F_\sigma \) in \( N \) and so by Proposition \( M_1 \) is metrizable.

To see that \( M_2 \) is metrizable use the Bing Metrization Theorem:

A topological space is metrizable iff it is regular and has a \( \sigma \)-discrete basis.

A family \( \mathcal{B} \) of subsets of \( X \) is discrete iff every point of \( X \) has a neighborhood meeting at most one element of \( \mathcal{B} \). \( \sigma \)-discrete means the countable union of discrete families.
Note that for each \( x \in X \) the sets \( \{x\} \times Y \) and \( Y \times \{x\} \) are open in \( M_2 \). Let \( B \) be a countable open basis for \( Y \). Then

\[
\mathcal{C} = \{ U \times \{x\}, \{x\} \times U, \{(x, x)\} : x \in X, U \in B\}
\]

is an open basis for \( M_2 \). It is \( \sigma \)-discrete. The family \( \{(x, x)\} : x \in X \) is discrete in \( M_2 \) since \( X^2 \cap \Delta \) is closed in \( M_2 \). And for each fixed \( U \in B \) the family \( \{U \times \{x\} : x \in X\} \) is discrete in \( M_2 \). (For \((x, x) \in X \) use the neighborhood \( \{x\} \times \{x\} \). For \((y, x) \in Y \times \{x\}\) with \( y \in Y \) and \( x \in X \) use the neighborhood \( Y \times \{x\} \) and for \((x, y) \in \{x\} \times Y \) use the neighborhood \( \{x\} \times Y \).) Similarly, for each \( U \in B \) the family \( \{\{x\} \times U : x \in X\} \) is discrete in \( M_2 \). Since \( B \) is countable, \( M_2 \) has a \( \sigma \)-discrete basis and is therefore metrizable.

This proves Theorem 1.

QED

The next Theorem is an easy generalization of Theorem 1 using the tower cardinal \( t \) which is defined as follows. \( t \) is the minimum cardinality of a set \( T \subseteq [\omega]^{\omega} \) which is linearly ordered by \( \subseteq^* \) but there does not exist \( z \in [\omega]^{\omega} \) with \( z \subseteq^* y \) for every \( y \in T \). Martin’s axiom implies that \( t = c \).

**Theorem 4** Suppose \( t = c \). Then there exists a nonmetrizable regular paracompact space \( X \) such that \( X^2 \) is the union of two metrizable subspaces.

Proof

The main Lemma changes to:

**Lemma 5** \((t = c)\) There are disjoint sets \( X, Y \subseteq 2^\omega \) of cardinality \( c \) such that

1. \( X \) is Borel \( c \)-concentrated on \( Y \), i.e., for every Borel set \( B \) in \( 2^\omega \), if \( Y \subseteq B \) then \( |X \setminus B| < c \),
2. \( Y^2 \setminus \Delta \) is \( F_\sigma \) in \( X^2 \cup Y^2 \), and
3. \( X^2 \setminus \Delta \) is \( F_\sigma \) in \( X^2 \cup Y^2 \).

The proof is similar. The space \( M = X \cup Y \) is the same. Since \( X \) is not relatively Borel in \( M \) we have by Proposition 3 that \( M(X) \) is not metrizable. But \( M(X) \) is regular and paracompact for any \( X \subseteq M \) and metric \( M \), see example 5.1.22 Engelking [2].
Remark. The Michael line is the topological space $M(X)$ where $M$ is the unit interval, $[0, 1]$, and $X$ the irrationals in $[0, 1]$. Michael Granado in unpublished work has shown that the square of the Michael line is not the union of two metrizable subspaces.

**Question 6** (Using just ZFC) Do there exist disjoint sets of reals $X$ and $Y$ such that $X$ is not $F_\sigma$ in $X \cup Y$ but $X^2 \setminus \Delta$ is $F_\sigma$ in $X^2 \cup Y^2$?

**References**

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[2] Engelking, Ryszard; General topology. Translated from the Polish by the author. Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60] PWN—Polish Scientific Publishers, Warsaw, 1977. 626 pp.

Arnold W. Miller
miller@math.wisc.edu
http://www.math.wisc.edu/~miller
University of Wisconsin-Madison
Department of Mathematics, Van Vleck Hall
480 Lincoln Drive
Madison, Wisconsin 53706-1388
The appendix is not intended for final publication but for the electronic version only.

Appendix

Suppose $M$ is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff $X$ is an $F_\sigma$ in $M$. (Engelking 5.5.2)

Proof
Suppose $X$ is not $F_\sigma$ in $M$, then $Y = M \setminus X$ is closed in $M(X)$ (since the points of $X$ are isolated, $X$ is open). But $Y$ is not $G_\delta$ in $M(X)$. To see, this suppose that $Y = \cap_{n \in \omega} U_n$ were each $U_n$ is open in $M(X)$. Then there would exists $V_n$ open in $M$ and $X_n \subseteq X$ with $U_n = V_n \cup X_n$. But then $Y = \cap_{n \in \omega} V_n$ which contradicts $Y$ is not $G_\delta$ in $M$.

For the converse, suppose $X$ is $F_\sigma$ in $M$ and write it as the union of closed sets $X = \cup_{n < \omega} C_n$. $M(X)$ is regular so it is enough by the Bing Metrization Theorem to check that it has a $\sigma$-discrete base. Let $B$ be a $\sigma$-discrete base for $M$. We claim that

$$B \cup \{\{x\} : x \in X\}$$

which is a basis for $M(X)$ is $\sigma$-discrete in $M(X)$. $B$ is $\sigma$-discrete in $M$ so it is also $\sigma$-discrete in $M(X)$.

$$\{\{x\} : x \in X\} = \cup_{n < \omega} C_n$$

where $C_n = \{\{x\} : x \in C_n\}$ shows that it is $\sigma$-discrete, since for any $n$ if $x \notin C_n$ then $M \setminus C_n$ is a neighborhood of $x$ missing all elements of $C_n$.

$M(X)$ is regular paracompact, whenever $M$ is metric. (Engelking 5.1.22)

Proof
Regular: Given $p \in M$ if $p \in X$ then it is has the clopen neighborhood $\{p\}$, if $p \notin X$, then the neighborhoods of $p$ in $M$ are also a neighborhood basis in $M(X)$.

Paracompact: Let $U$ be an open cover of basic open sets in $M(X)$. We may assume it has the form:

$$U = \mathcal{V} \cup \{\{x\} : x \in Z\}$$

where $\mathcal{V}$ is a family of basic open sets in $M$ and $Z = X \setminus \mathcal{V}$. Since metric spaces are hereditarily paracompact, there exists a locally finite refinement $\mathcal{W}$ of $\mathcal{V}$ with $\cup \mathcal{V} = \cup \mathcal{W}$. But then $\mathcal{W} \cup \{\{x\} : x \in Z\}$ is a locally finite refinement of $U$. 

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