Towards a Classification of $\text{su}(2) \oplus \cdots \oplus \text{su}(2)$

Modular Invariant Partition Functions

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Abstract. The complete classification of WZNW modular invariant partition functions is known for very few affine algebras and levels, the most significant being all levels of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and level 1 of all simple algebras. Here, we address the classification problem for the nicest high rank semi-simple affine algebras: $(\mathfrak{A}_1^{(1)})^\oplus_r$. Among other things, we explicitly find all automorphism invariants, for all levels $k = (k_1, \ldots, k_r)$, and complete the classification for $\mathfrak{A}_1^{(1)} \oplus \mathfrak{A}_1^{(1)}$, for all levels $k_1, k_2$. We also solve the classification problem for $(\mathfrak{A}_1^{(1)})^\oplus_r$, for any levels $k_i$ with the property that for $i \neq j$ each $\gcd(k_i + 2, k_j + 2) \leq 3$. In addition, we find some physical invariants which seem to be new. Together with some recent work by Stanev, the classification for all $(\mathfrak{A}_1^{(1)})^\oplus_r$ could now be within sight.
1. Introduction

A great deal has been learned in recent years about rational conformal field theories, but in spite of considerable effort relatively little is known about their classification – it is one of the main open problems in the area. A natural first step is to try to classify all Wess-Zumino-Novikov-Witten [29] models. The partition function of a WZNW theory associated with algebra \( \hat{g} \) and level \( k \) can be written as a sesquilinear combination

\[
Z = \sum M_{ab} \chi^k_a \chi^k_b^* \tag{1.1}
\]

of characters \( \chi^k_a \) of the representation of \( \hat{g} \) with (horizontal) highest weight \( a \) and level \( k \). Each \( \chi^k_a \) is a function of a complex vector \( z \) and a complex number \( \tau \). The algebra \( \hat{g} \) is the untwisted affine extension \( g^{(1)} \) of a Lie algebra \( g \) [22].

One curious reason the classifications of these \( Z \) are interesting are the mysterious “coincidences” which appear. For the \( g = A_1 \) classification, there is the ADE pattern [6]; for the \( g = A_2 \) classification the Fermat curves make surprise appearances [25]. These are not fully understood at present, and we can only guess as to their ultimate significance, but they do encourage us to look a little deeper. It is in this spirit that this paper has been written.

There are several conditions the function \( Z \) in (1.1) must satisfy in order to be realized by a physically reasonable conformal field theory. These are discussed in Sect.2. When it does satisfy these conditions, we shall call it a physical invariant.

Many tools (see e.g. [2,4,26,28]) have been developed over the past few years for finding these physical invariants, and it is probably safe to say that almost all physical invariants are now known. But we have been far less successful at actually proving that a given list exhausts all physical invariants belonging to some choice of \( g \) and \( k \). The main completeness proofs at present are: for \( g = A_1 \) and \( g = A_2 \), for any level \( k \) ([6] and [13], respectively); and for \( k = 1 \) for all simple \( g \) [21,11]. In addition, some work in classifying GKO physical invariants [6,5,16] and heterotic physical invariants [12,15] has been done. The result for \( A_1 \) is:

(1) for each \( k \), the diagonal invariant \( A_k \);
(2) for each even \( k \), the complementary invariant \( D_k \);
(3) for \( k = 10, 16, 28 \), the exceptionals \( E_{10}, E_{16}, E_{28} \).

These physical invariants can be found in Ref. [6]. \( A_k \) and \( D_k \) can be thought of as simple current invariants [26]. \( E_{10} \) and \( E_{28} \) are due [4] to conformal embeddings, and \( E_{16} \) is due [24] to an exceptional automorphism of the \( D_{16} \) chiral algebra. The \( A_1 \) result motivates much of the following. For example, in Sects.3 and 4 we find the analogues of the \( A_k, D_k \) invariants, and in Sect.5 we look for the analogues of \( E_{16} \). In principle all conformal embeddings are known [1]; they can occur for \( A_{1,k_1} \oplus \cdots \oplus A_{1,k_r} \) only when a \( k_i = 1, 2, 3, 4, 6, 8, 10 \) or 28 (unfortunately, \textit{a priori} some exceptionals could be due to chiral extensions other than conformal embeddings).

Unfortunately, the complexity increases with the rank, and when the algebra is semi-simple the numbers of physical invariants, including exceptionals, rises significantly. However, new methods developed in Ref. [13], and refined in this paper, make some of the remaining classifications more accessible. In this paper we will focus on the nicest algebra
which is both semi-simple and of arbitrarily high rank: namely, $g^r \overset{\text{def}}{=} A_1 \oplus \cdots \oplus A_1 = (A_1)^{\otimes r}$. There are several known physical invariants for it, which will be discussed in Sect.2. The question this paper addresses is the completeness of this list. This has also been addressed in other papers, e.g. [9,10]. All that has previously been established in this direction is that all level $k = (k_1, k_2)$ physical invariants of $g^2$ have been found for small $k_1, k_2$ by the computer search in Refs. [14] and [15], and for odd $k_1$ and $k_2$ by looking at operator subalgebras of $\hat{A}_{1,k}$ [7]. Important recent progress [27] is the determination of the possible chiral extensions of $g^r$, at least for small $r$.

In Sect.2 we describe most of the physical invariants of $g^r$, summarize our main results, and list the physical invariants of $g^2$. Sect.3 will find all permutation invariants (see equation (3.1)) of $g^r$, for any level $k = (k_1, \ldots, k_r)$. Some of these have not appeared in the literature before. In Sect.4 we explicitly find all simple current invariants (see equation (2.6)), and using this we find in Sect.5 all physical invariants, for most $k$, due to automorphisms of simple current extensions. We also find some new exceptionals (see equations (5.12)). In Sect.6 we illustrate some of this work by finding all physical invariants when the heights $k_i + 2$ are nearly coprime. In Sect.7 we complete the classification for all levels of $g^2$.

There are a number of reasons why a classification of $A_1 \oplus \cdots \oplus A_1$ physical invariants would be interesting. For one thing, there is a duality [16] between the GKO cosets $G_k \oplus G_\ell / G_{k+\ell}$, and a subset of the WZNW $G_k \oplus G_\ell \oplus G_{k+\ell}$. In particular, knowing all $g^3_{(k,\ell,k+\ell)}$ physical invariants will permit one to read off all $A_{1,k} \oplus A_{1,\ell} / A_{1,k+\ell}$ physical invariants. For another thing, it would be nice to know whether the intriguing ADE pattern [6] for $A_1 = g^1$ physical invariants continues in some way in $g^r$ for $r > 1$. So few complete classifications exist at present that a few more should give us more hints of the global pattern applicable to all modular invariant partition functions, which could ultimately lead to a more geometric global understanding of conformal field theories, as well as of the various connections between conformal field theories and other areas of mathematics (e.g. Fermat curves [25]). $A_1 \oplus \cdots \oplus A_1$ is more transparent than any other large rank algebra, so understanding its physical invariants should suggest new techniques or patterns to try on the others. Finally, these $A_1 \oplus \cdots \oplus A_1$ results are also of value in classifying heterotic invariants [15], and have been employed [10] to produce Gepner-type compactifications of the heterotic string.

It should be mentioned that the rank-level duality $C_{n,k} \leftrightarrow C_{k,n}$ [23] implies that our results for $\oplus A_{1,k_i}$ can be carried over to $\oplus i C_{k_i,1}$. This could have applications to the search for exceptionals via conformal embeddings.

Although we obtain a complete classification only for $g^2$, the focus of all but Section 7 is on arbitrary $g^r$. In particular, together with [27], the results of this paper should directly lead to classifications for other $g^r$.

2. The physical invariants of $A_1 \oplus \cdots \oplus A_1$

We begin by introducing notation and terminology. Our attention will be restricted here to the algebra $g^r = (A_1)^{\otimes r}$. Next, we will describe most of the physical invariants of
At the end of this section we summarize the main results of this paper, and review the physical invariants of $g^r$.

Let $\beta_1, \ldots, \beta_r$ denote the fundamental weights of $g^r$. Throughout this paper we will identify the weight $a = a_1 \beta_1 + \cdots + a_r \beta_r$ with its Dynkin labels $(a_1, \ldots, a_r)$. Write $\rho_r = (1, \ldots, 1)$.

An integrable irreducible representation of the affine Lie algebra $\hat{g}^r \overset{\text{def}}{=} (A_1^{(1)})^\oplus_r$ is associated with an $r$-tuple of positive integers $k = (k_1, \ldots, k_r)$ (called the level) and a highest weight $a$ of $g^r$. For most purposes it will be more convenient to use the height $k' = (k'_1, \ldots, k'_r) \overset{\text{def}}{=} (k_1 + 2, \ldots, k_r + 2)$. By $g^r_k$ we mean $g^r$ at level $k$. The set of all possible highest weights corresponding to level $k$ representations is

$$P^r_k \overset{\text{def}}{=} \{(a_1, \ldots, a_r) | a_i \in \mathbb{Z}, \ 0 < a_i < k'_i, \ \forall i\}. \tag{2.1}$$

The character corresponding to the level $k$ representation with weight $a = (a_1, \ldots, a_r) \in P^r_k$ can be written

$$\chi^k_a = \chi^{k_1}_{a_1} \cdots \chi^{k_r}_{a_r},$$

where the $\chi$ denote $A_1$ characters. By including their full variable dependence, for fixed $k$ the $\chi^k_a$ will all be linearly independent [22].

The function $Z$ in (1.1) can and will be identified with its coefficient matrix $M$. Three properties $Z$ must satisfy in order to be the partition function of a physically sensible conformal field theory are:

(P1) **modular invariance.** Letting $S^{(k)}$ and $T^{(k)}$ denote the (unitary) modular matrices of $\hat{g}^r_k$ (see equations (2.3) below), this is equivalent to the two conditions:

$$T^{(k)} \overset{\dagger}{=} MT^{(k)} = M, \quad \text{i.e.} \quad MT^{(k)} = T^{(k)}M; \tag{2.2a}$$

$$S^{(k)} \overset{\dagger}{=} MS^{(k)} = M, \quad \text{i.e.} \quad MS^{(k)} = S^{(k)}M; \tag{2.2b}$$

(P2) **positivity and integrality.** The coefficients $M_{ab}$ must be non-negative integers; and

(P3) **uniqueness of vacuum.** We must have $M_{\rho_r \rho_r} = 1$.

We will call any modular invariant function $Z$ of the form (1.1), an **invariant.** Together they define a complex space, called the **commutant $\Omega^r_k$.** It is closed under matrix multiplication. $Z$ will be called **positive** if in addition each $M_{ab} \geq 0$, and **physical** if it satisfies (P1), (P2), and (P3). Our task is to find all physical invariants corresponding to each algebra $g^r$ and level $k$.

An invariant satisfying (P1), (P2) and (P3) is still not necessarily the partition function of a conformal field theory obeying duality. If it is, we will call it **strongly physical.** These are the invariants of interest to physics. We will discuss the additional properties satisfied by strongly physical invariants (most importantly, that they become automorphism invariants when written in terms of the characters of their maximal chiral algebras) at the beginning of Sect.5.

In any classification it is desirable to use as few assumptions as possible. For one thing this greater generality allows a greater opportunity for the given classification to be
directly relevant to other classifications, some perhaps lying in entirely different domains (e.g. ADE [6], fermat curves [25]). In addition, there is a simple relationship between GKO coset physical invariants and WZNW ones, and additional properties can disturb this relationship. We will try as much as we can to restrict ourselves to (P1)-(P3); when we need to exploit other properties we will clearly say.

The ˆA1 characters ˜χ^k_a behave quite nicely under the modular transformations τ → τ + 1 and τ → −1/τ:

\[
\tilde{\chi}_a^k(z, \tau + 1) = \sum_{b \in P^1_k} (\tilde{T}^{(k)})_{ab} \tilde{\chi}_b^k(z, \tau), \quad \text{where}
\]

\[
(\tilde{T}^{(k)})_{ab} = \exp[\pi i \frac{a^2}{2(k+2)} - \pi i \frac{1}{4}] \delta_{ab};
\]

\[
\tilde{\chi}_a^k(z/\tau, -1/\tau) = \exp[k \pi i z^2/\tau] \sum_{b \in P^1_k} (\tilde{S}^{(k)})_{ab} \tilde{\chi}_b^k(z, \tau), \quad \text{where}
\]

\[
(\tilde{S}^{(k)})_{ab} = \sqrt{\frac{2}{k+2}} \sin[\pi \frac{ab}{k+2}],
\]

For arbitrary rank r, the corresponding modular matrices T^{(k)} and S^{(k)} become ˆT^{(k_1)} \otimes \cdots \otimes ˆT^{(k_r)} and ˆS^{(k_1)} \otimes \cdots \otimes ˆS^{(k_r)} respectively; for all r and k they are real, orthogonal and symmetric.

The easiest source of invariants for g^r_k is through the matrix tensor product M \otimes M'. In particular, let k_s^1 = (k_1, \ldots, k_s) and k_r^s = (k_{s+1}, \ldots, k_r). If M is a physical invariant of g^s level k_s^1 and M' is a physical invariant of g^{r-s} level k_r^s, then M \otimes M' will be a physical invariant of g^r level k. But in general this fails to construct most physical invariants of g^r_k.

Another important source of physical invariants is given by simple currents [26], i.e. outer automorphisms [2] of g^r_k. For g^r_k, a simple current J is just an r-tuple of 0’s and 1’s. They form a group, denoted J, under component-wise addition (mod 2), hence a vector space over Z_2, the integers mod 2. J acts on a weight a \in P^r_k by

\[
(Ja)_i = \begin{cases} 
  a_i & \text{if } J_i = 0 \\
  k'_i - a_i & \text{if } J_i = 1 
\end{cases}
\]

Simple currents play a central role in this paper. Their norms, defined mod 4, and inner products, defined mod 2, are given by

\[
J^2 \overset{\text{def}}{=} \sum_{i=1}^r (J_i)^2 k_i \pmod{4},
\]

\[
J \cdot J' \overset{\text{def}}{=} \sum_{i=1}^r J_i J'_i k_i \pmod{2}.
\]

Another useful quantity is J \cdot (a - \rho_r) = \sum_i J_i(a_i - 1), defined mod 2.
Each simple current $J$ with even norm $J^2$ defines an elementary simple current invariant $M(J)$: for $J^2 \equiv 0$ (called integer-spin), it is given by

$$M(J)_{ab} = \begin{cases} 
\delta_{ab} + \delta_{a,Jb} & \text{if } J \cdot (a - \rho_r) \equiv 0 \pmod{2} \\
0 & \text{otherwise}
\end{cases}; \quad (2.5a)$$

and for $J^2 \equiv 2$ (called half-integer spin), by

$$M(J)_{ab} = \begin{cases} 
\delta_{ab} & \text{if } J \cdot (a - \rho_r) \equiv 0 \pmod{2} \\
\delta_{b,Ja} & \text{otherwise}
\end{cases}. \quad (2.5b)$$

We will usually write the invariant $M(J)$ in (2.5b) as $I^J$, for reasons which will be clearer next section.

More generally, call a physical invariant $M$ a simple current invariant [18] if for all $a, b$ it obeys the selection rule:

$$M_{ab} \neq 0 \Rightarrow b = Ja, \quad (2.6)$$

for some simple current $J$ (depending on $a, b$). Elementary simple current invariants are only a small subset of these, but we will find in Sect.4 that they need only a little help to generate all simple current invariants.

When some of the levels $k_i$ are equal, we may obtain new physical invariants from old ones through conjugation. Call a permutation $\pi$ of $\{1, \ldots, r\}$ a conjugation if $k_i = k_{\pi i}$ for all $i = 1, \ldots, r$. Then for any physical invariant $M$, we define its conjugation $M^\pi$ by the formula

$$(M^\pi)_{a,b} = M_{\pi a,\pi b}. \quad (2.7)$$

For $r = 2$ the notation $M^c$ is common. This conjugation is not to be mistaken for charge conjugation: the charge conjugation matrix $C = S^{(k)^2}$ for $g^k_r$ is just the identity.

Together, simple current invariants and their conjugations constitute what may be called the regular series of physical invariants, and can be expected to exhaust almost all physical invariants of $g^r_r$. For a given $g^r_k$ their number is given by the following remarkable formula [19]

$$\left( \prod_{\ell>1} n_\ell! \right) \prod_{i=0}^{r'} (2^i + 1), \quad (2.8)$$

where $n_\ell$ is the number of $k_i = \ell$, and $r' = r - 1$ or $r - 2$, depending on whether or not all $k_i$ are even (if some $k_i = 2$, there will be some overcounting in (2.8)).

By an exceptional invariant we mean any physical invariant which is not “regular”. For $r > 1$, infinite series of exceptionals exist. They can be constructed for example by taking an exceptional for $g^s_r$, where $s < r$, and tensoring it with any physical invariant from $g^{r-s}_r$. But there may be some exceptionals which cannot be built up from lower rank exceptionals – we will call these sporadic exceptionals. $g^1_1$ has 3 sporatics, and $g^2_2$ has 8 (12, if one treats e.g. $k = (2,10)$ as different from $(10,2)$). The possible existence of sporatics complicates any attempt at classification.

Our main results are (more careful statements of them, along with all relevant definitions, can be found later in the paper):
Theorem 1. Any permutation invariant of any $g_k^r$ is generated, through matrix and tensor products, by conjugations $\pi$, by elementary simple currents $I^J$, and by three different families of “integer-spin” simple current automorphisms.

Theorem 2. Any simple current invariant can be written as the product of a permutation invariant with a number of elementary simple current invariants $M(J)$.

Theorem 4. Provided five anomalous levels are avoided, only simple current invariants and their conjugations obey the following property: the only weights $a$ coupled to $\rho_r$ (i.e. satisfying $M_{a\rho_r} \neq 0$ or $M_{\rho_r a} \neq 0$) are of the form $a \in J\rho_r$.

Theorem 7. Provided each $\gcd(k_i'k_j') \leq 3$ for $i \neq j$, then all physical invariants of $g_k^r$ are known.

Theorem 8. All physical invariants of $g_k^2$, for any $k = (k_1, k_2)$, are known.

Here is a summary of all the $g_k^2$ physical invariants. Let $A_k$, $D_k$ and $E_k$ denote the physical invariants of $A_{1,k}$. There are either 6 or 2 simple current series for $g_k^2$, depending on whether or not both $k_1, k_2$ are even. These are explicitly given in the Appendix of [15]. When $k_1$ equals 10, 16 or 28, additional physical invariants are the tensor product $E_k \otimes A_k$ for all $k_2$, and the tensor product $E_k \otimes D_k$ and the matrix product $(E_{10} \otimes A_{k_2}) M(J)$ for $J = (1,1)$ if $k_2$ is even. Similarly if instead $k_2 = 10, 16, 28$. If both $k_1, k_2 \in \{10, 16, 28\}$, then in addition there is $E_k \otimes E_k$. If $k_1 = k_2$, the conjugations of all those invariants must also be included, of course. The sporatic exceptions for $A_1 \oplus A_1$ are given in Refs. [14,15]: there is exactly one at each level $(k_1, k_2) = (4,4), (6,6), (8,8), (10,10), (2,10), (3,8), (3,28)$ and $(8,28)$. In particular, see equations (4.3f)-(4.3i) of [14]* and (A.6)-(A.9) of [15].

At the end of Sect.7 we list the numbers of physical invariants of $g^2$, for each $k$.

3. The permutation invariants

By a permutation invariant (or automorphism invariant) we mean a physical invariant of the form

$$Z = \sum_{a \in P_k^r} \chi_a \chi^*_{\sigma a},$$

i.e. $M_{ab} = (I^\sigma)_{ab} \overset{\text{def}}{=} \delta_{b,\sigma a}$

for some permutation $\sigma$ of $P_k^r$. (P3) says $\sigma(\rho_r) = \rho_r$. In this section we will find all $g_k^r$ permutation invariants, for each rank $r$ and level $k$.

Recall that we write $k' = (k_1', \ldots, k_r') = (k_1 + 2, \ldots, k_r + 2)$. For convenience we will often drop the subscript on $\rho_r$.

First let us list some simple examples of $g_k^r$ permutation invariants. The trivial one corresponds to the identity matrix $M = I$. Other examples are its conjugations $M = I^\pi$ (see (2.7)).

* The (4,4) exceptional appeared incorrectly in an early preprint of [14]; the correct invariant reads: $|\chi_{11} + \chi_{15} + \chi_{51} + \chi_{55}|^2 + |\chi_{13} + \chi_{31} + \chi_{35} + \chi_{53}|^2 + 4|\chi_{33}|^2$. 

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Another source of permutation invariants are the \( J^2 \equiv 2 \pmod{4} \) simple currents (see (2.5b)). Matrix products of these \( I^J \) produce many other simple current invariants (2.6) which are permutation invariants. But they do not generate all of them: there are also integer-spin simple current permutation invariants \([18,10]\). For example, choose any \( m \neq n \) with \( k_m \equiv k_n \equiv 0 \pmod{4} \), and define a matrix \( I^{mn} \) by

\[
(I^{mn})_{ab} = \left( \prod_{i \neq m,n} \delta_{a_i,b_i} \right) \cdot \begin{cases}
\delta_{a_m b_m \delta_{a_n b_n}} & \text{if } a_m \equiv a_n \equiv 1 \pmod{2} \\
\delta_{k_m' - a_m b_m \delta_{a_n b_n}} & \text{if } a_m \equiv 1, a_n \equiv 0 \pmod{2} \\
\delta_{a_m b_m \delta_{k_n' - a_n b_n}} & \text{if } a_m \equiv 0, a_n \equiv 1 \pmod{2} \\
\delta_{k_m' - a_m b_m \delta_{k_n' - a_n b_n}} & \text{if } a_m \equiv a_n \equiv 0 \pmod{2}
\end{cases}.
\]

Then \( I^{mn} \) will be a permutation invariant. It was first found in \([18,10]\).

There are two further examples. One, for \( r > 2 \), involves a tensor product of a \( g^3 \) permutation invariant with the identity. Choose any \( \ell, m, n \) with \( k_\ell \equiv 1, k_m \equiv -1 \), and \( k_n \equiv 0 \pmod{4} \), and define a matrix

\[
(I^{\ell mn})_{ab} = \left( \prod_{i \neq \ell,m,n} \delta_{a_i,b_i} \right) \cdot \begin{cases}
\delta_{a_\ell b_\ell \delta_{a_m b_m}} & \text{if } a_n \equiv 1 \pmod{2} \\
\delta_{k_\ell' - a_\ell b_\ell \delta_{k_m' - a_m b_m}} & \text{if } a_n \equiv 0 \pmod{2} \\
\delta_{a_\ell b_\ell \delta_{k_n' - a_n b_n}} & \text{if } a_\ell \equiv a_m \pmod{2} \\
\delta_{k_\ell' - a_\ell b_\ell \delta_{k_n' - a_n b_n}} & \text{if } a_\ell \not\equiv a_m \pmod{2}.
\end{cases}
\]

The other, for \( r > 3 \), involves a tensor product of a \( g^4 \) permutation invariant with the identity. Choose any \( \ell, m, n, p \) with \( k_\ell \equiv k_m \equiv -k_n \equiv -k_p \equiv 1 \pmod{4} \), and define a matrix

\[
(I^{\ell mnp})_{ab} = \left( \prod_{i \neq \ell,m,n,p} \delta_{a_i,b_i} \right) \cdot \begin{cases}
\delta_{a_\ell b_\ell \delta_{a_m b_m \delta_{a_p b_p}}} & \text{if } a_n \equiv 1 \pmod{2} \\
\delta_{k_\ell' - a_\ell b_\ell \delta_{k_m' - a_m b_m \delta_{k_p' - a_p b_p}}} & \text{if } a_n \equiv 0 \pmod{2} \\
\delta_{a_\ell b_\ell \delta_{k_n' - a_n b_n \delta_{k_p' - a_p b_p}}} & \text{if } a_\ell \equiv a_p \pmod{2} \\
\delta_{k_\ell' - a_\ell b_\ell \delta_{k_n' - a_n b_n \delta_{k_p' - a_p b_p}}} & \text{if } a_\ell \not\equiv a_p \pmod{2}.
\end{cases}
\]

Both \( I^{\ell mn} \) and \( I^{\ell mnp} \) are also simple current permutation invariants. They cannot be expressed as (tensor or matrix) products of the permutation invariants \( I^\pi \), \( I^J \) and \( I^{mn} \) listed earlier, or of each other, and seem to have never appeared explicitly in the literature before.

**Theorem 1.** Any permutation invariant \( I^\sigma \) of \( g_k^r \) can be written as the matrix product

\[
I^\sigma = \left( \prod_{J \in \mathcal{A}} I^J \right) \left( \prod_{(m,n) \in \mathcal{B}} I^{mn} \right) \left( \prod_{(\ell,m,n) \in \mathcal{C}} I^{\ell mn} \right) \left( \prod_{(\ell,m,n,p) \in \mathcal{D}} I^{\ell mnp} \right) I^\pi.
\]

Any or all of the sets \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) may be empty, in which case the corresponding product is defined to be the identity matrix \( I \). Of course, \( \mathcal{A} \) is a set of simple currents \( J \) with \( J^2 \equiv 2 \), \( \mathcal{B} \) is a set of pairs \((m,n)\) with \( m \neq n \) and \( k_m \equiv k_n \equiv 0 \pmod{4} \), \( \mathcal{C} \) is a set of triplets \((\ell,m,n)\) with \( k_\ell \equiv 1, k_m \equiv -1 \), and \( k_n \equiv 0 \pmod{4} \), \( \mathcal{D} \) is a 4-tuple \((\ell,m,n,p)\) satisfying \( \ell \neq m, n \neq p, \ell \equiv m \equiv -n \equiv -p \equiv 1 \pmod{4} \), and \( \pi \) is a conjugation.
The set of all permutation invariants of $g_k^r$ forms a (generally nonabelian) group. We will prove below that the $I^1$, $I^{mn}$, $I^{emn}$, $I^{emnp}$ and $I^\pi$ generate this group. From this, it is easy to show that any $I^\sigma$ can be written as in (3.5).

An alternate description of the permutation invariants of $g_k^r$ is given in Lemma 1 proved below. The remainder of this section is devoted to a proof of Thm.1. The proof will be quite explicit, since it will be exploited later in the paper.

Throughout this section we will write $a'$ for $\sigma a$, $b'$ for $\sigma b$, etc.

That the matrix $I^\sigma$ in (3.1b) must commute with $S^{(k)}$ and $T^{(k)}$ (see (P1)) is equivalent to

$$\prod_{i=1}^r \sin(\pi a_i b_i / k_i') = \prod_{i=1}^r \sin(\pi a'_i b'_i / k_i'),$$

$$\sum_{i=1}^r a_i^2 / k_i' \equiv \sum_{i=1}^r a'_i^2 / k_i' \pmod{4},$$

for all $a, b \in P_k^r$.

The fusion coefficients $\tilde{N}_{lmn}^{(k)}$ of $A_1$ level $k$ are well-known [20]:

$$\tilde{N}_{lmn}^{(k)} = \begin{cases} 1 & \text{if } l + m + n \equiv 1 \pmod{2} \text{ and } |l - m| < n < \min\{l + m, 2k' - l - m\} \\ 0 & \text{otherwise} \end{cases}.$$  (3.7a)

Of course, the fusion coefficient $N_{abc}^{(k)}$ of $g_k^r$ is just the product

$$N_{abc}^{(k)} = \prod_{i=1}^r \tilde{N}_{a_i b_i c_i}^{(k_i)}.$$  (3.7b)

Verlinde’s formula implies the useful fact $N_{abc}^{(k)} = N_{a'b'c'}^{(k)}$. Let $f_k(m)$ be the number of $n$ such that $\tilde{N}_{mn}^{(k)} = 1$, then from (3.7a) we find that $f_k(m) = \min\{m, k + 2 - m\}$. But from (3.7b), for any $a \in P_k^r$ the number of $b \in P_k^r$ such that $N_{aab} = 1$ equals $f_k(a_1) \cdots f_k(a_r)$. Thus we get the important equality:

$$\prod_{i=1}^r \min\{a_i, k_i' - a_i\} = \prod_{j=1}^r \min\{a'_j, k'_j - a'_j\}.$$  (3.8)

The first step in the proof of Thm.1 is to “factor off” the conjugation $I^\pi$ in (3.5).

**Claim 1.** Let $e^i = (1, \ldots, 2, \ldots, 1) \in P_k^r$ be the weight whose $j$th component is $(e^i)_j = 1 + \delta_{i,j}$. Let $e^i' = \sigma e^i$. Then $(e^i')_j = J e^\tilde{i}$, for some simple current $J$ (depending on $i$), and some index $\tilde{i}$ satisfying $k_{\tilde{i}} = k_i$.

This is trivial for $k_i = 1$; for future convenience choose $\tilde{i} = i$ when $k_i = 1$.

**Proof.** We may assume $k_i > 1$. Putting $a = e^i$ in (3.8), we see that the LHS equals 2, which forces exactly one of the factors on the RHS (call it $\tilde{i}$) to equal 2, and the remaining factors to equal 1. In other words, for each $j$ either $(e^i')_j = 1 + \delta_{j,\tilde{i}}$ or $k'_j = (1 + \delta_{j,\tilde{i}})$. 

\[8\]
It suffices to show $k_i = k_i$. This follows from (3.6a) with $a = e^i$ and $b = \rho$: we get
\[
\sin(\pi/k'_i) \cdot \sin(2\pi/k'_i) = \pm \sin(\pi/k'_i) \cdot \sin(2\pi/k'_i),
\]
i.e. $\cos(\pi/k'_i) = \pm \cos(\pi/k'_i)$ and hence $k'_i = k'_i$. QED to claim

Claim 1 defines a function $\pi$ of $\{1, \ldots, r\}$, by $\pi i = \bar{i}$. We would like to show $\pi$ is a conjugation of $g'_k$. To do this it suffices now to show that $\pi$ is a permutation – i.e. if $\pi i = \pi j$, then $i = j$.

Suppose for some $i \neq j$, $\pi i = \pi j$. Then by Claim 1, $k_i = k_j = k_{\pi i} = k_{\pi j}$. From (3.6a) with $a = e^i$ and $b = e^j$ we get
\[
\sin(2\pi/k'_i) \cdot \sin(2\pi/k'_j) = \pm \sin(4\pi/k'_i) \cdot \sin(\pi/k'_j),
\]
i.e. $\cos(\pi/k'_i) = \pm \cos(2\pi/k'_i)$, which only has one solution: $k'_i = 3$. But we have defined $\pi i = i$ whenever $k_i = 1$, so $i = \pi i = \pi j = j$.

Therefore $\pi$ is a conjugation. Multiplying $I^\sigma$ on the right by the conjugation $I^\sigma^{-1}$ yields a permutation invariant $I^{\sigma'}$ whose $\pi$ is the identity. Thus we may assume without loss of generality in the following that $\sigma$ is such that $\bar{i} = i$ in Claim 1. This will make our notation somewhat cleaner.

Comparing $b = e^i$ with $b = \rho$ in (3.6a), we see that for any $a, a'_i$ either equals $a_i$ or $k'_i - a_i$. Hence $I^\sigma$ is a simple current invariant (see (2.6)). The simple current permutation invariants have been completely classified in Ref. [18] (subject however to an assumption on the $S$-matrix which does not appear to be satisfied by most $g'_{\ell}$). Our argument, from here to Lemma 1, resembles that of [18], but because we restrict attention here to $g^r$ we are able to explicitly solve the equations of Lemma 1, and obtain in the end equation (3.5).

We have shown that a permutation invariant is completely specified by the simple currents $J^a$ defined for each $a$ by $a' = J^a a$. Define $f^i = J^{\rho} e^i$ and $\tilde{f}_j^i \in \mathcal{J}$ by $\sigma^{-1}(e^i) = \tilde{f}_j^i e^i$. This definition of $J^a$ is ambiguous when $k'_i = 2a_i$ — we are free then to choose any value for $J^a$. It is most convenient however to fix them using equation (3.9) below. When $k_i = 2$, $f^i$ and $\tilde{f}^i$ may be arbitrarily chosen.

For any $a \in P^r_k$ and each $i = 1, \ldots, r$, putting $b = \sigma^{-1}(e^i)$ in (3.6a) gives us
\[
J^a_i \equiv (a - \rho) \cdot \tilde{f}_j^i \underbrace{\sum_{j=1}^{r} (a_j - 1) \tilde{f}_j^i}_{(\text{mod } 2)}. \tag{3.9}
\]
Indeed, we get $\pm \prod_{j=1}^{r} \sin(\pi(e^i) \cdot a_j/k'_j) = \pm \prod_{j=1}^{r} \sin(\pi(e^i) \cdot a_j/k'_j)$, where the LHS sign is $(-1)^{(a - \rho)} \cdot f^i$ and the RHS sign is $(-1)^{J^a_i}$. The product of sin’s vanishes iff $2a_i = k'_i$, in which case as we noted above we may freely choose (3.9) to hold.

Hence $f^i = \tilde{f}^i$, which says that the $f$-matrix for $\sigma^{-1}$ is the transpose of the $f$-matrix for $\sigma$. Also, we have learned that $\sigma$ is completely specified by the $f^i$.

Rewriting equation (3.6b) gives us $\sum_i k'_i (J^a_i)^2 \equiv 2a \cdot J^a \pmod{4}$. Inserting $a = e^i$ gives us
\[
\sum_{j=1}^{r} k'_j f^i_j \equiv 2 \sum_{j=1}^{r} f^i_j + 2f^i_i \pmod{4}. \tag{3.10a}
\]
For $i \neq j$, inserting next $a = e^i + e^j - \rho$ and using (3.9) and (3.10a) gives us
\[
f^i_j + f^j_i \equiv \sum_{\ell} k_{\ell} f^{i}_{\ell} f^{j}_{\ell} \pmod{2}. \tag{3.10b}
\]
Equations (3.9), (3.10) suffice to prove (3.6b) for arbitrary \( a \), so we have extracted all the information we can from it. Similarly, (3.9) and (3.10b) suffice to prove (3.6a). Summarizing our results:

**Lemma 1.** To find all possible permutation invariants \( I^\sigma \) of \( g_k^r \), first find all possible conjugations \( \pi \); then find all possible \( r \times r \) matrices \( F = (f_{ij}) \) of 0’s and 1’s such that:

(i) \( f_{ij} + f_{ji} \equiv \sum_{\ell} k_{\ell} f_{i\ell} f_{j\ell} \) (mod 2), \( \forall i, j \);

(ii) \( \sum_j k_j f_{ji} \equiv 2f_{ii} \) (mod 4), \( \forall i \).

To each such \( F \) define a permutation \( \sigma_F \) by

\[
\sigma_F(a)_i = \begin{cases} 
  a_i & \text{if } \sum_{j=1}^r (a_j - 1)f_{ji} \equiv 0 \pmod{2} \\
  k_i' - a_i & \text{otherwise}
\end{cases}
\] (3.11)

Then to each pair \( (F, \pi) \) there is a permutation invariant given by \( I^{\sigma_F} I^\pi \). This exhausts all permutation invariants of \( g_k^r \). Provided all \( k_i \neq 2 \) and at most one \( k_i = 1 \), then to each different \( (F, \pi) \) will correspond a different permutation invariant.

Actually, we have not yet shown that \( \sigma_F \) will be a permutation of \( P_k^r \). But if \( J^a a = J^b b \), then using (3.9) and (3.10b) to expand out \( J^{c\ell} \cdot (J^a a - \rho) \equiv J^{c\ell} \cdot (J^b b - \rho) \), we get \( J^a_\ell = J^b_\ell \), \( \forall \ell \). Hence \( J^a a = J^b b \) implies \( J^a = J^b \), and thus \( a = b \). So \( \sigma_F \) will indeed be a permutation, and define a permutation invariant.

But whenever \( \sigma \) is a permutation invariant, so will be \( \sigma^{-1} \). We know that the \( F \)-matrix for \( \sigma^{-1} \) is the transpose of that for \( \sigma \). From this we get two other formulas any \( F \) satisfying (i) and (ii) must satisfy:

\[
f_{ij} + f_{ji} \equiv \sum_{\ell=1}^r k_{\ell} f_{i\ell} f_{j\ell} \pmod{2} \quad \forall i, j \quad (3.12a) \\
\sum_{j=1}^r k_j f_{ji} \equiv 2f_{ii} \pmod{4} \quad \forall i \quad (3.12b)
\]

It remains to express each \( \sigma_F \) in the lemma as a product of \( I^J, I^{mn}, I^{\ell mn} \) and \( I^{lmnp} \). We will prove this by induction on \( r \). It is trivial for \( r = 1 \).

Note that the matrix \( F^J \) corresponding to \( I^J \) is \( f_{ij} = J_i J_j \). The matrix \( F^{mn} \) corresponding to \( I^{mn} \) equals 0 everywhere, except for \( f_{mn} = f_{nm} = 1 \). The matrix \( F^{\ell mn} \) corresponding to \( I^{\ell mn} \) equals zero everywhere except \( f_{ln}, f_{nl}, f_{mn}, f_{nm} \). The matrix \( F^{lmnp} \) equals zero everywhere except for \( f_{lm}, f_{ln}, f_{ml}, f_{mp}, f_{pm}, f_{nl}, f_{np} \) and \( f_{pn} \). Finally, \( \sigma_F'' = \sigma_F \circ \sigma_F \), where

\[
f''_{ij} = f_{ij} + f'_{ij} + \sum_{\ell=1}^r f_{i\ell} f_{j\ell} k_{\ell} \pmod{2}.
\] (3.13)

Suppose first that some row or column of \( F \) – say the \( i \)th – is identically zero. Then by (i) or (3.12a), respectively, the \( i \)th column or row will also be identically zero. By (3.11) \( \sigma \) fixes the \( i \)th component of each \( a \), and so is the tensor product of \( A^{k_i} \) with some \( g_k^{r-1} \) permutation invariant, where \( \hat{k} = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_r) \). By the induction hypothesis we are done.
Suppose some $f_{ii} = 1$. Let $J_j = f_{ij}$, then $J_j^2 \equiv 2 \pmod 4$, by (ii), so $I^J$ is a permutation invariant. From (3.13) and (ii) we find that $f(\sigma \sigma)_{ij} = 0$ for all $j$ by the previous paragraph we are done.

Thus it suffices to assume all $f_{ii} = 0$. Now suppose some $k_i \equiv 2 \pmod 4$. Then choose as a simple current $J_j = \delta_{ij}$. Then by (3.13), $f(\sigma \sigma)_{ii} = 1$ so by the preceding paragraph we are done. For $a = 0, 1, 2, 3$ let $n_a$ be the number of $k_i \equiv a \pmod 4$. Then we may assume $n_2 = 0$. If $n_1 = n_3 = 0$, i.e. if all $k_i \equiv 0 \pmod 4$, then products of various $I^{mn}$ with $\sigma$ results in an $F$-matrix identically zero, by (i). Also, if exactly one $k_i$ is odd, i.e. $n_1 + n_3 = 1$, then by (ii) the $i$th column of $F$ will be identically zero, and again we are done.

Suppose $k_i \equiv k_j \equiv \pm 1 \pmod 4$, and $f_{ij} = 0$, for $i \neq j$. Then choosing $J_\ell = \delta_{\ell i} + \delta_{\ell j}$ produces $f^{\prime \prime}(\sigma \sigma)_{ii} = 1$, so we are done. Note that if $n_3 \leq 1$ and $n_1 \geq 2$ (or $n_3 \geq 2$ and $n_1 \leq 1$), then by (i) and (ii) this is inevitable. Thus we are reduced to two cases: either $n_1, n_3 \geq 2$; or $n_1 = n_3 = 1$.

Consider now the first case. Choose any $k_i \equiv k_j \equiv +1 \pmod 4$. We may assume $f_{ij} = 1$. Then again by (i) and (ii), there must be a $k_\ell \equiv -1 \pmod 4$ such that $f_{i\ell} = 0$. Choosing any other $k_m \equiv -1 \pmod 4$, we then get from (3.13) that $f(\sigma \sigma \epsilon _m \sigma)_{ii} = 1$.

Finally, consider $k_1 \equiv +1$, $k_2 \equiv -1$ and $k_3 \equiv k_4 \equiv \cdots \equiv k_r \equiv 0 \pmod 4$, where $f_{ij} = 0$ for all $i, j = 1, 2$. By (ii) we get $f_{i1} = f_{i2} \forall i$, so by (i), $f_{ij} = f_{ji} \forall i, j$. If now $f_{ij} = 1$ for any $i, j > 2$, then hit $I^\sigma$ with $I^{ij}$. Thus we have succeeded in reducing the proof to showing that the permutation invariants associated with the matrices

$$F = \begin{pmatrix}
0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 0 & \cdots & 0
\end{pmatrix}$$

(3.14)

can be written as in (3.5). But this is obvious from (3.13): the product $I^{123} \cdots I^{12r}$ works.

This concludes the proof of Thm.1.

4. Simple current invariants

Throughout this section fix $r$ and $k$ and write $S_{ab}$ for $S^{(k)}_{ab}$, and $\rho$ for $\rho_r$. Recall the definition of simple current invariant given in (2.6).

One of the main reason we are interested in simple current invariants is that they seem to exhaust most physical invariants. For example, all but finitely many physical invariants for $A_1$ and $A_2$ are simple current invariants (or their conjugates). This seems to remain true to some extent for $g^r$, though for $r > 1$ we have infinitely many “exceptionals”. It turns out that simple current invariants have the very pleasant property that they are nice enough to classify completely [19], largely because of the important relation

$$S_{Ja, Jb} = -1^{J \cdot (b - \rho) + J \cdot (a - \rho) + J \cdot J'} S_{ab}.$$  

(4.1)
In this section we classify all the simple current invariants of $g_k^r$. In particular we will prove:

**Theorem 2.** Let $M$ be a simple current invariant of $g_k^r$, where at most three $k_i = 2$. Then there exist simple currents $J^1, \ldots, J^m$ and a permutation invariant $I^\sigma$ whose $\pi = \text{id.}$, such that

(a) $J^{i^2} \equiv 0 \pmod{4}$ \forall $i$, and $J^i \cdot J^j \equiv 0 \pmod{2}$ \forall $i, j$;

(b) $M = \left( \prod_{i=1}^{m} M(J^i) \right) I^\sigma$, where $M(J^i)$ is defined in (2.5a).

Another characterization of the simple current invariants is given by equation (4.11) below. The arguments we will use have much in common with the much more general Ref. [19], but because we are focusing here on the special case of $g_k^r$, our conclusion can be made more explicitly, and we are required to assume a little less (in particular, Ref. [19] finds all *strongly physical* simple current invariants, while we find here all *physical* simple current invariants). The strange-looking condition that at most three $k_i = 2$ is unfortunate, and is related to the regularity condition of Ref. [19]. However, it is more than sufficient for the applications we have at present (namely, Sect. 7 below, and [15, 16]). Nevertheless, it is something we would like to return to in the future.

We will begin our proof of Thm. 2 by proving the following general lemma. This lemma will be used throughout this paper, and is clearly of importance in its own right. It holds in a far greater context than just $g_k^r$. To prove it, we will only need to assume (P1)-(P3).

**Lemma 2.** Let $M$ be any physical invariant (not necessarily a simple current invariant). Let $J_L, J_R \subset J$ be defined by $J \in J_L$ iff $M_{J, \rho, \rho} \neq 0$, $J' \in J_R$ iff $M_{J', \rho, \rho} \neq 0$. Define $P_L = \{ a \in P_k^r \mid \exists b$ for which $M_{ab} \neq 0 \}$, $P_R = \{ b \in P_k^r \mid \exists a$ for which $M_{ab} \neq 0 \}$. Then

(i) $J_L$ and $J_R$ are subgroups (hence subspaces over $\mathbb{Z}_2$) of $J$;

(ii) $M_{ab} \neq 0$ implies $J \cdot (a - \rho) \equiv J' \cdot (b - \rho) \equiv 0 \pmod{2}$ \forall $J \in J_L, J' \in J_R$;

(iii) $M_{ab} = M_{J, a, J, b} \forall J \in J_L, J' \in J_R$;

(iv) $J^2 \equiv 0 \pmod{4}, J \cdot J' \equiv 0 \pmod{2} \forall J, J' \in J_L$ (similarly for $J_R$).

(v) Suppose in addition that $M_{a, \rho} \neq 0$ iff $a \in J_L \rho$. Then

\[
a \in P_L \quad \text{iff} \quad J \cdot (a - \rho) \equiv 0 \pmod{2}, \quad \forall J \in J_L;
\]

\[
\| J_L \| = \sum_b M_{pb} S_{pb} / S_{pp}. \quad (4.2b)
\]

To prove this, we begin with the calculation (from $M = S^\dagger MS$ and (4.1))

\[
M_{J, \rho, \rho} = \sum_{a, b} S_{J, \rho, a} M_{ab} S_{b \rho} = \sum_{a, b} S_{pa} M_{ab} S_{b \rho} = -1^{J \cdot (a - \rho)} \sum_{a, b} S_{pa} M_{ab} S_{b \rho} = M_{\rho, \rho} = 1,
\]

(4.3)

for any $J \in J$. Thus $J \in J_L$ iff $J \cdot (a - \rho) \equiv 0 \pmod{2}$, for all $a \in P_L$. This proves (i), (ii), (iii) follows from a calculation similar to (4.3), using (ii). (iv) comes from $T$-invariance, applied to $M_{J, \rho, \rho}, M_{J', \rho, \rho} \neq 0$.
To prove (v), look at the equation $MS = SM$: for any $a$ such that $(a - \rho) \cdot J_L \equiv 0 \pmod{2}$, it implies
\[
\sum_b M_{ab} S_{bp} = \sum_b S_{ab} M_{bp} = \sum_{J \in J_L} S_{ap} \cdot -1^{J \cdot (a - \rho)} = \|J_L\| S_{ap}, \tag{4.4}
\]
the second equality using (iii). But each $S_{bp} > 0$. This gives us (4.2a). (4.2b) follows immediately by restricting (4.4) to $a = \rho$. This completes the proof of the lemma.

Let us begin by deriving some useful expressions. We are interested for the remainder of this section in $M$ being a simple current invariant. Then (4.2b) and (iii) tell us that the cardinalities $\|J_L\| = \|J_R\|$ are equal. For any $a, c \in P_L$, using $SM = MS$ and equations (2.6) and (4.1) we get the important
\[
\sum_{Jc \in Jc} M_{Jc,c} \cdot -1^{J \cdot (a - \rho)} = \sum_{Ja \in J_a} M_{a,Ja} \cdot -1^{J \cdot (c - \rho)}, \tag{4.5}
\]
which holds whenever $S_{ac} \neq 0$. Choosing $a \in P_L$ and $c = \rho$ tells us
\[
\|J_L\| = \sum_{Ja \in J_a} M_{a,Ja} = a\text{th row sum of } M, \tag{4.6}
\]
with a similar expression for the column sum.

Now, choose any $a \in P_L$, $J_R \in J_R$. Then $M_{a,Ja} = M_{a,J_RJa} \neq 0$ for some $Ja \in J$. Then (4.6) tells us both
\[
J_a^2 \equiv 2Ja \cdot (a - \rho) \pmod{4}, \tag{4.7a}
\]
\[
J_R \cdot (a - \rho) \equiv J_a \cdot J_R \pmod{2}. \tag{4.7b}
\]

For any $a$, let $F_a$ ($F^a$) denote the number of $J \in J_L$ ($J \in J_R$) such that $Ja = a$. $a$ is called a fixed point of $J_L$ if $F_a > 1$. A fixed point has the (necessary but not sufficient) property that some of its components $a_i$ must equal $k_i/2$.

Say $a \in P_L$ has property ($*_L$) if there exists a $Ja \in J$ with the property that $M_{ab} \neq 0$ iff $b \in J_RJa$. Similarly, say $c \in P_R$ has property ($*_R$) if $\exists J^c \in J$ such that $M_{bc} \neq 0$ iff $b \in J_LJ^c$. We would like to show that every $a \in P_L$ has property ($*_L$), and every $c \in P_R$ has property ($*_R$). Note that, because of Lemma 2(iii) and (4.6), if $a$ has property ($*_L$) then $M_{ab}$ will either equal $F^a$ or 0.

Consider first $a \in P_L$ which is not a fixed point of $J_R$. Then from (4.6) and Lemma 2(iii), $a$ must have property ($*_L$). Similarly, any $c \in P_R$ which is not a fixed point of $J_L$ must have property ($*_R$).

On the other hand, the treatment of the fixed points of $J_R$, or of $J_L$, is a little more complicated, and in fact because of these complications we will have to assume that only a few $k_i = 2$.

Suppose $c$ is not a fixed point of $J_L$, but $a$ is a fixed point of $J_R$, and suppose $S_{ac} \neq 0$. Then (4.5) reduces to
\[
\|J_L\| \cdot -1^{J^c \cdot (a - \rho)} = \sum_{Ja \in J_a} M_{a,Ja} \cdot -1^{J \cdot (c - \rho)}. \tag{4.8a}
\]
Because the \( \alpha \)th row sum of \( M \) equals \( \|J_L\| \), the only way equality can hold in (4.8a) is if

\[
J^c \cdot (a - \rho) \equiv J \cdot (c - \rho) \pmod{2}
\]

whenever \( M_{a,J_a} \neq 0 \).

Choose any \( J', J'' \in J \) such that \( M_{a,J'_a} \neq 0 \) and \( M_{a,J''_a} \neq 0 \), and let \( J = J'J'' \). Then (4.8b) becomes

\[
J \cdot (c - \rho) \equiv 0 \pmod{2}.
\]

This holds for any \( c \in P_R \) such that \( c \) is not a fixed point of \( J_L \), and \( S_{ac} \neq 0 \). We are free to demand \( J'_i = J''_i = 0 \), and hence \( J_i = 0 \), for any \( i \) for which \( a_i = k'_i/2 \).

Define \( J^a = \{ J \in J \mid J_i = 0 \text{ when } a_i = k'_i/2 \} \), so \( J', J'', J \in J^a \). Let \( J_R^a \) be the projection of \( J_R \) into that space, i.e. \( J_R^a = \{ J \in J^a \mid \exists J_R \in J_R \text{ such that } J_{R_i} = J_i \text{ when } a_i \neq k'_i/2 \} \). Then \( J^a \) is a subspace of \( J \), and \( J_R^a \) is a subspace of \( J^a \). Let \( (J_R^a)\perp \subset J^a \) denote the space

\[
(J_R^a)^\perp = \{ x \in J^a \mid \sum_{i=1}^{r} x_i J_i \equiv 0 \pmod{2}, \ \forall \tilde{J} \in J_R^a \}.
\]

To any \( x \in (J_R^a)^\perp \) assign the weight \( c = c(x) \) by \( c_i = 1 \) if \( x_i \equiv 0 \), and \( c_i = 2 \) if \( x_i \equiv 1 \). Then \( x \in J^a \) implies \( S_{ac} \neq 0 \), and \( x \in (J_R^a)^\perp \) implies \( c \in P_R \). Moreover, provided no \( k_i = 2 \), \( c \) will not be a fixed point of \( J_L \). We will return shortly to the case where some \( k_i = 2 \), but for now assume none do.

Putting this \( c \) in (4.8c) is equivalent to noting that

\[
J \in ((J_R^a)^\perp)^\perp = \{ \tilde{J} \in J^a \mid \sum_{i=1}^{r} \tilde{J}_i x_i \equiv 0 \pmod{2}, \ \forall x \in (J_R^a)^\perp \} = J_R^a,
\]

i.e. \( \exists J^c \in J_R \) such that \( J_i = J^c_i \) when \( a_i \neq k'_i/2 \). Thus \( J''a = J^cJ' \). A similar argument holds for \( (*R) \). But this argument assumed no \( k_i = 2 \).

And what if some \( k_i = 2 \)? Let \( t \) be the number of \( k_i = 2 \). Let \( t(a) \) be the number of \( i \) such that \( a_i = 2 = k_i \). Recall the map \( x \mapsto c(x) \), taking \( (J_R^a)^\perp \) into \( P_R \). Suppose, for a given \( a \in P_L \), we know that every \( x \in (J_R^a)^\perp \) is such that \( c(x) \) obeys property \( (*R) \). Then by the above argument we get that \( a \) obeys \( (*L) \). Now, \( c = c(x) \) will necessarily obey \( (*R) \) if \( t(c) = 0 \) or 1, because then \( c \) could not be a fixed point. Therefore \( a \in P_L \) will necessarily obey \( (*L) \), if either \( t(a) = t \) or \( t - 1 \). Together these two statements tell us that \( (*R) \) is satisfied whenever \( c = c(x) \) has \( t(c) = 0, 1, t - 1 \) or \( t \).

So for any \( t \leq 3 \), i.e. when there are at most \( 3 \) \( k_i = 2 \), we get that all \( a \in P_L \) satisfies \( (*L) \). The same argument applies to \( P_R \).

Equation (4.5) now collapses to the equation

\[
J^c \cdot (a - \rho) \equiv J_a \cdot (c - \rho) \pmod{2}
\]
whenever $a \in \mathcal{P}_L$, $c \in \mathcal{P}_R$, and $S_{ac} \neq 0$. From a similar to argument to that used in equations (4.8c)-(4.9b) above, we get that if $a \equiv a'$ (mod 2), and $a \in \mathcal{P}_L$, then we may choose $J_a = J_a'$. (Put $J = J_a J_a'$; we may demand $J_i = 0$ whenever either $a_i = k_i/2$ or $a_i' = k_i'/2$, etc.) Hence, provided no $k_i = 2$ (we will return shortly to the case where some $k_i = 2$), we may drop the irritating condition $S_{ac} \neq 0$ from (4.10a). (4.10a) also tells us that we may choose $J_a = J_a J_a''$, when $a - \rho \equiv (a' - \rho) + (a'' - \rho)$ (mod 2) and $a', a'' \in \mathcal{P}_L$.

However by definition, $J^c \in \mathcal{J}_R J_b$ for $b = J^c c$. Hence, using (4.7b), we may write (4.10a) in the more convenient form

$$J_b \cdot (a - \rho) + J_a \cdot (b - \rho) \equiv J_a \cdot J_b \pmod{2} \quad \forall a, b \in \mathcal{P}_L. \quad (4.10b)$$

Once again we must return to the case where there are $k_i = 2$. Here however we do not require any restriction on the size of $t$ (= the number of $k_i = 2$), provided we know (4.10a) is satisfied whenever $S_{ac} \neq 0$, and $a \in \mathcal{P}_L$, $c \in \mathcal{P}_R$. The idea is that we are completely free to choose the values of $(J_a)_{i}$ whenever $a_i = 2 = k_i$, so we will try to choose them so that (4.10) holds. The argument is not difficult and we will give only a sketch. Note first that if $t(a) = 0$, then the above argument remains valid and the condition “$S_{ac} \neq 0$” may still be dropped; similarly we can take $J_a = J_a'$ if $a \equiv a'$ (mod 2) and $t(a) = 0$.

Define the projection $p(a) \in \mathcal{J}$ by $p(a)_i \equiv a_i \pmod{2}$ if $k_i = 2$, and $p(a)_i \equiv 0$ if $k_i \neq 2$. Begin by finding a set of $a^\alpha \in \mathcal{P}_L$ for which $p(a^\alpha - \rho)$ forms a basis of $p(\mathcal{P}_L - \rho)$; for convenience make it “upper triangular”. Then, by choosing the free values of $(J_a^\alpha)_i$ appropriately, it is possible to force (4.10b) to be satisfied for any $a = a^\alpha$, $b = b^\beta$. Now, any $a \in \mathcal{P}_L$ can be written as $a - \rho \equiv \sum m^\alpha (a^\alpha - \rho) + a' - \rho \pmod{2}$, for some $m^\alpha \in \mathbb{Z}$ and $a' \in \mathcal{P}_L$ with $t(a') = 0$ and $a'_i \neq k'_i/2$ for all $i$. Define $J'_a = \sum m^\alpha J_{a^\alpha} + J_{a'}$. Then the usual argument shows that $J = J'_a J_a$ satisfies (4.9b), so we are free to choose $J_a = J'_a$. In this way we get that (4.10b) is satisfied for all $a, b \in \mathcal{P}_L$.

Let us summarize our results so far. What specifies our simple current invariant $M$? Choose two subgroups $\mathcal{J}_L, \mathcal{J}_R \subset \mathcal{J}$, with equal orders $\|\mathcal{J}_L\| = \|\mathcal{J}_R\|$. For each $a \in \mathcal{P}_L$ find a $J_a \in \mathcal{J}$. Define $M$ by

$$M_{ac} = \begin{cases} F^a & \text{if } a \in \mathcal{P}_L \text{ and } c \in \mathcal{J}_R J_a \setminus \mathcal{J}_a \\ 0 & \text{otherwise} \end{cases} \quad (4.11)$$

Both $\mathcal{J}_L$ and $\mathcal{J}_R$ must satisfy Lemma 2(iv). Also, in order for $M$ to be a modular invariant, equations (4.10b) and (4.7) must be satisfied. There are various consistency conditions which must also be satisfied (e.g. $\mathcal{J}_L a = \mathcal{J}_L b$ iff $\mathcal{J}_R J_a a = \mathcal{J}_R J_b b$), but these will be automatically satisfied if these other conditions are.

Our remaining task is to solve these conditions, i.e. to show the matrix $M$ in (4.11) satisfying these conditions must be as in Thm.2. To this effect, note that if we had $\mathcal{J}_L = \mathcal{J}_R$, and in addition each $J_a = 0$, then we may write $M$ as

$$M = \prod_{J \in \beta} M(J), \quad (4.12)$$

where $\beta$ is any basis of $\mathcal{J}_L$ and $M(J)$ denotes the simple current invariant defined in equation (2.5a). This can be proven by explicitly expanding (4.12), using the fact that $J_L \cdot J_L' = 0 \forall J_L, J_L' \in \mathcal{J}_L$. 

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So our task is to find a permutation invariant $I^\sigma$ (3.11), whose conjugation $\pi = id$, such that $I^\sigma$ takes $J_R\rho$ to $J_L\rho$ and each $J_a a$ into $J_L a$. Inspecting (3.9), we may expect to take as $F$ the transpose of the matrix which sends $a - \rho$ to $J_a$. This is indeed the right idea, though this matrix is only defined for $a \in P_L$.

Let $n = ||J_L|| = ||J_R||$. Find $a^i \in P_L$, $i = 1, \ldots, r - n$, so that the $a^i - \rho$ form a basis for the vector space spanned by $P_L - \rho$ (mod 2). It is possible to choose bases $\{J^i_L\}$ of $J_L$ and $\{J^i_R\}$ of $J_R$, and a set of vectors $\{b^j\}$, for $j = 1, \ldots, n$, such that $j^i \cdot J^j_R \equiv b^i \cdot J^j_R \equiv \delta_{ij}$ (mod 2). (As before, dot products between $b^i$’s and $J^i$’s will look like $\sum b_i J_i$, and those between $J^i$’s will be like $\sum J_i J^j_i k_i$.) Then the $\{a^i - \rho\}$ and $\{b^j\}$ together form a basis for all of $J$. Define $M_{ij} = J^i_a \cdot b^j$.

Finally, define $J'_{c}$ linearly for all $c \in J$, by putting

\[
J'_{a^i - \rho} = J_{a^i} + \sum_{j=1}^{n} M_{ij} J^j_R, \quad i = 1, \ldots, r - n, 
\]

\[
J'_{b^j} = 0, \quad j = 1, \ldots, n.
\]

Hence for all $a \in P_L - \rho$ and all $b \in \text{span of } b^j$, we have $J'_{a^i} \cdot b \equiv 0$ (mod 2) and $J'_{b^j} = 0$. Also, $J_a$ is only defined mod $J_R$, so we may replace $J_a$ with $J'_{a - \rho}$ without affecting the $M$ in (4.11).

An easy calculation now shows

\[
J'_{a} \cdot c + J'_{c} \cdot d \equiv J'_{a} \cdot J'_{d} \quad (\text{mod } 2),
\]

\[
J'_{c}^2 \equiv 2 J'_{c} \cdot c \quad (\text{mod } 4),
\]

$\forall c, d \in J$. Define a matrix $F = (f_{ij})$ of 0’s and 1’s by $(J_{c})_i = \sum_{j=1}^{r} f_{ij} c_j$. Then (4.14) becomes equations (3.12), so $\sigma_F$ is a permutation invariant. Also, $\sigma_F(J'_{a^i - \rho} a) = a$ for all $a \in P_L$:

\[
(J'_{a^i - \rho} a) = \sum_{j} f_{j i} ((J'_{a^i - \rho} a) - \rho) = \sum_{j} f_{j i} k_j + a_j - 1
\]

so $\sigma_F(J'_{a^i a} a) = J'_{a - \rho} (J'_{a - \rho}) a = a$. This means the set $P_F^R$ for the simple current invariant $M^F = M M^\sigma_F$ equals the set $P_L = P_F^L$, which implies $J_R^F = J_L = J^F_L$. Also, the $J^F_a$ for this invariant can all be taken to be 0. This then concludes the proof of Thm.2.

We know of no examples of simple current invariants of $g'_k$, where there are more than 3 $k_i = 2$, which does not satisfy (i) and (ii) of Thm.2. To remove the condition on the number $t$ of $k_i = 2$ in Thm.2, it suffices to prove that for all $c \in P_R$ with $c_i = 1, 2$, there exists a $J^c \in J$ satisfying: $M_{bc} \neq 0$ iff $b \in J_L J^c$. In this section we have only proven it for $c$ with $t(c) = 0, 1, t - 1$ or $t$, where $t(c)$ is the number of $i$ with $c_i = k_i = 2$. That is the source of our restriction $t \leq 3$. 

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5. Exceptional invariants

(P1)-(P3) are not the only properties a physically acceptable partition function must satisfy. We learn in [24] that it will look like

\[ Z = \sum_{i=1}^{\alpha} c_{i} c'_{i} \tau \]

\[ \text{where } c_{i} = \sum_{a \in P_{r}} m_{ia} \chi_{a}^{k}, \quad c'_{j} = \sum_{b \in P_{r}} m'_{jb} \chi_{b}^{k}, \]

where \( m_{ia}, m'_{jb} \) are non-negative integers and \( \tau \) is some bijection. The \( c_{i}, c'_{j} \) are characters for the LHS, RHS maximally extended chiral algebras \( C_{L}, C_{R} \). We may label them so that \( m_{i\rho} = \delta_{i1}, m'_{j\rho} = \delta_{j1} \). The characters \( c_{i}, c'_{j} \) transform modularly with unitary symmetric matrices \( S_{e}^{ij}, T_{e}^{ij}, S'_{e}^{ij}, T'_{e}^{ij} \), and satisfy

\[ S_{1i}^{e} \geq S_{11}^{e} > 0, \quad S_{ij}^{e} \geq S_{11}^{e} > 0, \quad \forall i, j. \]  

In this section, by a strongly physical invariant \( M \) we mean a physical invariant satisfying equations (5.1). There are other conditions \( Z \) and its chiral algebras must obey in order to be physically acceptable – e.g. the fusion rules of its chiral algebras must be non-negative integers. In Sect.7 another condition will be introduced.

Each simple current invariant is strongly physical, and defines chiral extensions \( C_{L} = C(J_{L}), C_{R} = C(J_{R}) \) of the affine algebra. The simple current invariants and their conjugations can be thought of as the regular series of physical invariants for \( g_{k}^{r} \), generalizing the \( A_{k}, D_{k} \) series of \( A_{1} \). Any other physical invariants can be called exceptional. An exceptional invariant comes in two basic kinds: either it corresponds to an exceptional automorphism \( \tau \) of simple current extensions (e.g. \( E_{16} \) for \( A_{1} \)); or one of its (maximally extended) chiral algebras \( C_{L}, C_{R} \) are exceptional, i.e. at least one is not a simple current extension (e.g. \( E_{10} \) and \( E_{28} \) of \( A_{1} \)). We will call exceptionals of the first kind \( E_{16} \)-like, and those of the second kind \( E_{10} \)-like.

In this section we investigate the nature and existence of both these kinds of exceptional invariants for \( g_{k}^{r} \), aided by the powerful machinery of Ref. [24]. We can expect in our WZNW classifications that “almost every” level \( k \) will be nicely behaved; our goal in this section is to find these generic results and to isolate those levels where sporatics may occur. We will be able to say much more here about the \( E_{16} \)-like exceptionals, than about the \( E_{10} \)-like exceptionals. The latter are most easily handled using additional machinery [27].

The main examples of chiral extensions are due to simple currents. By a simple current chiral extension \( C(J_{L}) \) we simply mean one whose characters, when projected down to the affine variables, look like

\[ ch_{[a]_{i}} = \frac{G([a]_{i})}{F_{a}} \sum_{J \in J_{L}} \chi_{Ja}, \quad i = 1, \ldots, f(a), \]
where as usual $F_a$ is the number of $J \in \mathcal{J}_L$ with $Ja = a$, where $a \in \mathcal{P}_L$, and where $[a]$ denotes the orbit $\mathcal{J}_L a$. The coefficients $G([a]_i)$ are positive integers satisfying

$$\sum_{i=1}^{f(a)} (G([a]_i))^2 = F_a. \quad (5.2b)$$

When $f(a) = 1$ we will often write $ch_{[a]_1}$ as $ch_{[a]}$. We are most familiar with the case where all $G([a]_i) = 1$, in which case $f(a) = F_a$, but we will see that it is important to allow $G > 1$ as well. We will have more to say about this shortly. Equations (5.2) tell us that the invariant $\sum |ch_{[a]_i}|^2$ will equal the simple current invariant in (4.12); indeed they are the only sets of characters obeying (5.1) with that property. We will usually use primes to denote the corresponding quantities for $C_R = C(\mathcal{J}_R)$, e.g. $[b]' = \mathcal{J}_R b$.

Equations (5.2) give us

$$\sum_{i=1}^{f(a)} G([a]_i) S_{[a],i,[b],j}^e = \frac{\|\mathcal{J}_L\| G([b]_j)}{F_b} S_{ab}, \quad (5.3a)$$

where $S_{ab} = S_{ab}^{(k)}$ is the $S$-matrix for $g_k^e$. Equation (5.3a) fixes $S_{[a],i,[b],j}^e$ if either $f(a) = 1$ or $f(b) = 1$. Formally defining the fusion rules $N_{[a],h,[b],i,[c],j}^e$ by Verlinde's formula, we get from (5.3a) that if $f(a) = f(b) = 1$, then

$$N_{[a],i,[b],j,[c],j}^e = \frac{G([c]_j)}{G([a]) G([b])} \sum_{d \in \mathcal{J}_L c} N_{abd}. \quad (5.3b)$$

The remaining values of $N^e$ and $S^e$ for simple current extensions are harder to compute in general, though (5.3a) does tell us

$$\sum_{h=1}^{f(a)} \sum_{i=1}^{f(b)} G([a]_h) G([b]_i) N_{[a],h,[b],i,[c],j}^e = G([c]_j) \sum_{d \in \mathcal{J}_L c} N_{a,b,d}. \quad (5.3c)$$

We can see from these equations that some $G([a]_i) \neq 1$ will often lead to non-integer fusion rules. Indeed, it is easy to show that for $r = 2$, integer fusion rules require that all $G([a]_i) = 1$. However, this is not true for all $r - \{5,7\}$ for a counter-example when $r = 8$.

Apparently, the constraints of unitarity, symmetry, the relations $S^e 2 = (S^e T^e)^3 = C^e$, and integrality of the fusion rules do not suffice to essentially fix $S^e$, and hence $\mathcal{C}(\mathcal{J}_L)$, for a given $\mathcal{J}_L$. Again, see (5.12b) for a counter-example. This seems like an important observation, since it is not hard to show that for any simple current extension $\mathcal{C}(\mathcal{J}_L)$, any physical invariant $M^e$ of $\mathcal{C}(\mathcal{J}_L)$ will produce a physical invariant $M$ of $g_k^e$, through the formula

$$M_{ab} = \sum_{i=1}^{f(a)} \sum_{j=1}^{f(b)} G([a]_i) G([b]_j) M_{[a],i,[b],j}^e. \quad (5.3d)$$
All that is needed to prove that (5.3d) is a physical invariant is the unitarity and symmetry of $S^e$, along with (5.3a). For a given $\mathcal{J}_L$ there will be several such $S^e$. Indeed, the $g_{i,4}^2$ exceptional is a realization of this. It corresponds to a symmetry of a simple current chiral extension with all $G([a]_i) = 1$ except $G([33]) = 2$; the matrix $S^e$ is real and obeys $(S^e T^e)^3 = I$, as it should. However, $N^e_{[33],[33],[33]} = 1/2$. The $g_{2,4}^2$ exceptional can also be interpreted as the diagonal invariant for an exceptional chiral extension, but again the fusion rules will not be integers [10].

We will begin by giving a characterization of $\mathcal{E}_{10}$-like exceptional.

**Theorem 3.** Let $M$ be any strongly physical invariant of $g_k^e$, where at most three $k_i = 2$ and at most one $k_i = 4$. Suppose $M_{\alpha \rho} \neq 0$ iff $a \in J_L \rho$. Then its LHS chiral algebra $C_L$ will be a simple current chiral extension $C(\mathcal{J}_L)$.

**Proof.** Define $Z'$ by

$$Z' = \sum_{i=1}^{\alpha} |ch_i|^2 = \sum_{i=1}^{\alpha} \sum_{a,b \in P^r} m_{ia} m_{ib} \chi_a \chi_b^*, \text{ i.e. } M'_{ab} = \sum_{i=1}^{\alpha} m_{ia} m_{ib}. \quad (5.4a)$$

Then $M$ strongly physical implies $M'$ will be a physical invariant. $M'$ will have left and right chiral algebras equal to $C_L$. It is more convenient to work with $M'$ than with $M$. Our main tools will be Lemma 2 and Perron-Frobenius theory [17].

Write $M'$ as a direct sum of indecomposable submatrices $B_{\ell}, \ell = 1, \ldots, \beta$. Let $W_\ell \subset P_L$ be the weights contained in the block $B_{\ell}$. Call $W_1$ the unique block containing $\rho$. The Perron-Frobenius eigenvalue of $B_1$ is $r(B_1) = \|J_L\|$, so by Lemma 3 of [13] $r(B_{\ell}) \leq \|J_L\|$ for each $\ell$. Consider now any block $B_{\ell}$ with some $a \in W_\ell$ having $F_a = 1$. Defining

$$(B_{\ell}^a)_{bc} = \begin{cases} 1 & \text{if } b, c \in J_L a \\ 0 & \text{otherwise} \end{cases}, \quad (5.4b)$$

we see that $B_{\ell} \geq B_{\ell}^a$, by Lemma 2 and $M'_{\alpha \alpha} \geq 1$, and hence that $r(B_{\ell}) \geq r(B_{\ell}^a)$, with equality iff $B_{\ell} = B_{\ell}^a$ [17]. But $r(B_{\ell}^a) = \|J_L\|$. Therefore whenever $F_a = 1$, there exists a unique $\ell(a)$ such that $m_{i,a} = \delta_{i,\ell(a)}$, and $W_{\ell(a)} = J_L a$.

Note from Lemma 2 that $M'_{aa} = M_{J_a,a} = M'_{J_a,a}$ for any $J \in J_L$. Rewriting this in terms of the $m_{ib}$, the triangle inequality tells us $m_{ia} = m_{i,a}$, for all $1 \leq i \leq \alpha$, $a \in P_L$, and $J \in J_L$.

From $MS = SM$, we get that

$$\sum_{b \in W_\ell} (B_{\ell})_{ab} S_{bp} = \sum_{b \in W_1} S_{ab} M'_{bp} = \|J_L\| S_{a \rho}, \quad (5.5a)$$

for all $a \in W_\ell$. In other words, the vector $x_\ell$ defined by $(x_\ell)_a = S_{a \rho}$, for $a \in W_\ell$, is the Perron-Frobenius eigenvector of $B_{\ell}$, and $r(B_{\ell}) = \|J_L\|$. Now consider $B_{\ell}^\infty = \lim_{n \to \infty} (M'/\|J_L\|)^n$. This limit will exist, and in fact will be the direct sum of the matrices $B_{\ell}^\infty$ defined by

$$(B_{\ell}^\infty)_{ab} = \frac{(x_\ell)_a (x_\ell)_b}{\sum_{c \in W_\ell} (x_\ell)_c} = C_{\ell} S_{a \rho} S_{b \rho}, \text{ where } C_{\ell} = \frac{1}{\sum_{c \in W_\ell} S_{c \rho}^2}. \quad (5.5b)$$
and \(a, b \in W_e\). \(M^\infty\) will be a modular invariant. This means

\[
C_i \sum_{b \in W_i} S_{ap} S_{bp} S_{bc} = C_j \sum_{b \in W_j} S_{ab} S_{bp} S_{cp}, \quad \forall a \in W_i, c \in W_j. \tag{5.5c}
\]

Consider the special case where \(F_a = 1\). Then (5.5c) simplifies to the statement: the ratio \(S_{ab}/S_{bp}\) will be independent of whichever \(b \in W_j\) is chosen, for any fixed \(j\).

For each \(\ell = 1, \ldots, r\) with \(k_\ell \neq 1, 4\), define \((f^\ell)_j = 1 + 2\delta_j^\ell\). Then \(f^\ell \in \mathcal{P}_L\) and \(F_{f^\ell} = 1\). For any \(b, c \in W_j\), for any \(j\), we then have

\[
\frac{S_{f^\ell b}}{S_{bp}} = \frac{S_{f^\ell c}}{S_{cp}}, \quad \text{i.e.} \quad \frac{\sin(3\pi b_\ell/k_\ell')}{\sin(\pi b_\ell/k_\ell')} = \frac{\sin(3\pi c_\ell/k_\ell')}{\sin(\pi c_\ell/k_\ell')}. \tag{5.5d}
\]

But \(\sin(3x)/\sin(x) = 3 - 4\sin^2(x)\), so \(\sin(\pi b_\ell/k_\ell') = \pm \sin(\pi c_\ell/k_\ell')\), i.e. \(c_\ell = b_\ell\) or \(k_\ell' - b_\ell\).

This conclusion is automatic when \(k_\ell = 1\). Finally, \(T\)-invariance forces it for \(k_\ell = 4\) (provided there is only one such \(k_\ell\)).

Thus, \(M'\) is a simple current invariant. From the analysis of Sect.4, we get that \(M'\) can be written in the form of (4.11), for \(J_L = J_R\) and all \(J_a = 0\). Therefore \(\sum_i m_{ia}^2 = F_a\). Also, for each \(i = 1, \ldots, \beta\) there is a \(a^i \in \mathcal{P}_L\) such that \(W_i = J_L a^i\). QED

What this tells us is that in order for \(M\) to be \(\mathcal{E}_{10}\)-like, there must be some weight \(a \notin J_\rho\), such that either \(M_{pa} \neq 0\) or \(M_{ap} \neq 0\). That something strange can happen when two \(k_i = 4\), is shown by the \(g^2_{2,4}\) exceptional.

Next, we will investigate the existence of \(\mathcal{E}_{16}\)-like exceptions. The following theorem says that they only exist for certain small \(k_i\). This is one of the main results of this paper.

**Theorem 4.** Let \(M\) be a strongly physical invariant of \(g^r_k\), where no \(k_i = 2\) and at most one \(k_i = 4\). Assume \(M_{ap} \neq 0\) iff \(a \in J_{1p}\), and \(M_{pb} \neq 0\) iff \(b \in J_{2p}\). Suppose for any \(k_i = 4, 8, 12\) or \(16\) that there does not exist a \(J' \in J_R\) with \(J_j' = \delta_{ij}\), and for any \(k_i = 4\) or \(8\) that there does not exist a \(J \in J_L\) with \(J_j = \delta_{ij}\). Then there exists a permutation invariant \(I^\sigma\) such that

\[
M = M(J^1) \cdots M(J^m) \cdot I^\sigma, \tag{5.6}
\]

for some basis \(\{J^1, \ldots, J^m\}\) of \(J_L\).

**Proof.** Thm.3 tells us \(M\) can be thought of as a bijection \(\tau\) between simple current extensions \(C(J_L)\) and \(C(J_R)\). Our proof will follow as closely as possible the proof of Thm.1.

For each \(i = 1, \ldots, r\) with \(k_i \neq 1\), define \((f^i)_j = 1 + 2\delta^i_j\). Then \(f^i \in \mathcal{P}_L\), and \(F_{f^i} = 1\). Our first task will be to show \([b^i]_j' = \tau([f^i])\) is not a fixed point of \(J_R\).

Suppose for contradiction that \(b^i\) is a fixed point of \(J_R\), and call \(F'_{b^i} = F''_{b^i}\). Then

\[
S_{\rho f^i} = \frac{1}{F'_i} S_{\rho b^i}. \tag{5.7a}
\]
$F'_i > 1$ implies $b^i_j = k'_j/2$ for $j$ belonging to some set, call it $I^i$. Then (5.7a) becomes either

$$\sin(3\pi/k'_i) \prod_{\ell \in I^i} \sin(\pi/k'_\ell) \geq \frac{1}{F'_i} \sin(\pi/k'_i) \quad \text{if} \ i \notin I^i, \quad (5.7b)$$

$$\sin(3\pi/k'_i) \prod_{\ell \in I^i} \sin(\pi/k'_\ell) \geq \frac{1}{F'_i} \quad \text{if} \ i \in I^i. \quad (5.7c)$$

The hypotheses of this theorem were designed to ensure equations (5.7b), (5.7c) can never be satisfied. Thus $b^i$ will not be a fixed point of $\mathcal{J}_R$, so $G'([b^i][b]) = F'_i = 1$.

Assume for now that $k_i \neq 3$. Then by (5.3b) there will be exactly three $[a]$ such that the fusion rule $N_{[f^i],[f^i],[a]}^L = 1$. Thus there must be exactly three $[c]^i_j$ such that $N_{[b]^i,[b]^i],[c]^i_j}^R = 1$. We claim this forces $b^i = J^i f^i$ for some simple current $J^i$, and some index $i$. Indeed, if for example there was a $\ell$ for which $3 < (b^i)\ell < k'_i - 3$, then there would be at least four such $c$. The other possibilities are eliminated similarly.

In fact we have $k_i = k_i$. This follows from equation (5.7a):

$$\sin(3\pi/k'_i) \cdot \sin(\pi/k'_i) = \sin(\pi/k'_i) \cdot \sin(3\pi/k'_i), \quad \text{i.e.} \ \sin(\pi/k'_i) = \pm \sin(\pi/k'_i). \quad (5.8a)$$

A similar conclusion holds if instead $k_i = 3$. Then there must be exactly two $[c]^i_j$ such that $N_{[b^i],[b^i],[c]^i_j}^R = 1$.

This defines a function $\pi$ on $\{1, \ldots, r\}$, sending $i$ to $\bar{i}$ (define $\pi i = i$ whenever $k_i = 1$). We want to show $\pi$ is a conjugation, i.e. that it is also a bijection. If for some $i \neq j$ we have $\pi i = \pi j$, then

$$\sin(3\pi/k'_i) \cdot \sin(\pi/k'_i) = \sin(9\pi/k'_i) \cdot \sin(\pi/k'_i), \quad (5.8b)$$

i.e. $\sin(\pi/k'_i) = \pm \sin(3\pi/k'_i)$, which only has $k'_i = 4$ as a solution. So $\pi$ defines a conjugation, and can be factored off. For now on take $\pi = id$.

What we have shown is that for all $i$ with $k_i \neq 1$, $\tau([f^i]) = [J^i f^i]'$ for some simple current $J^i$. Now take any $[a]_j$, and write $[b]^i_\ell = \tau([a]_j)$. Dividing $S_{[a]_j,[f^i]}^L = S_{[b]^i_\ell,[b]}^R$ by $S_{[a]_j,[\rho]}^L = S_{[b]^i_\ell,[\rho]}^R$, gives us $b_i = a_i$ or $k'_i - a_i$, in the usual way (see (5.5d)). A similar conclusion occurs automatically whenever $k_i = 1$. Thus $M$ (or more precisely $M \cdot \bar{I}^{\pi - 1}$) is a simple current invariant, so by Thm.2 can be written in the form (5.6).

QED

The known $g_k^- \mathcal{E}_{16}$-like exceptionals occur at levels where at least 1 $k_i$ equals 16, or at least 2 equal 8, or at least two equal 4, or (see equations (5.12)) at least 8 equal 2. Certainly there is no question that with a little more work the hypotheses in Thm.4 can be weakened – two results along these lines are given below.

**Theorem 5.** Let $M$ be a physical invariant. Suppose $M_{a\rho} \neq 0$ iff $a \in \mathcal{J}_{L\rho}$, and $M_{b\rho} \neq 0$ iff $b \in \mathcal{J}_{R\rho}$. Assume in addition that $\mathcal{J}_L$ and $\mathcal{J}_R$ have no fixed points. Then there exists a conjugation $\pi$ such that $M\pi$ is a simple current invariant.
Proof. Look at the matrix $M' = MM^T$; as usual write it in block form as $M' = \oplus B_i$, where each $B_i$ is indecomposable, and where $B_1$ contains the $(\rho, \rho)$-entry. Then by Lemma 2(iii) $B_1$ is the $||J_L|| \times ||J_L||$ matrix

$$B_1 = ||J_L|| \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad (5.9a)$$

so we know the Perron-Frobenius eigenvalues all satisfy $r(B_i) \leq ||J_L||^2$.

Now choose any $a \in \mathcal{P}_L$, then there exists a $b \in \mathcal{P}_R$ such that $M_{ab} \neq 0$. Suppose the $(a, a)$-entry of $M'$ is in $B_i$. Then $\forall J, J' \in J_L$,

$$(B_i)_{Ja,J'a} = (B_i)_{aa} = \sum_{c \in \mathcal{P}_R} M_{ac}^2 \geq \sum_{c \in J_R b} M_{ac}^2 = ||J_L||M_{ab}^2. \quad (5.9b)$$

Let $B^a_i$ denote the $||J_L|| \times ||J_L||$ matrix

$$(B^a_i)_{cd} = \begin{cases} ||J_L||M_{ab}^2 & \text{if } c, d \in J_L a \\ 0 & \text{otherwise} \end{cases}. \quad (5.9c)$$

Then $(B_i)_{cd} \geq (B^a_i)_{cd} \geq 0 \forall c, d$, so $r(B_i) \geq r(B^a_i)$, with equality iff $B_i = B^a_i$. But $r(B^a_i) = ||J_L||^2 M_{ab}^2$. Therefore $B^a_i = B_i$, and $M_{ab} = 1$.

What this means is that there is a bijection $\tau$ from $\mathcal{P}_L/J_L$ onto $\mathcal{P}_R/J_R$, defined by $\tau[a] = [b]'$ iff $M_{ab} \neq 0$. The equation $SM = MS$ says that, for any $a, a' \in \mathcal{P}_L$,

$$S_{aa'} = S_{\tau a, \tau a'}. \quad (5.10a)$$

In order to understand this bijection $\tau$, we will mimic as closely as possible the proof of Thm.1 in Sect.3. One complication is that here the weights are constrained to come from $\mathcal{P}_L$ and $\mathcal{P}_R$.

Use (5.3b) to formally define the fusion rules $N^L_{[a][b][c]}$. Now, the statement that $J_L$ has no fixed points means that any $J \in J_L$ has $J_i = 1$ for some $i$ with $k_i$ odd. This means that at most one $d \in [c]$ will have $N_{a bd} \neq 0$, so $N^L_{[a][b][c]} = 0$ or 1, as in Sect.3. Similarly for $N^R_{[a]'[b]'[c]}'$.

Choose $a = c$. We are interested in $[b]$ for some $b$ with all coefficients odd, otherwise $N^L_{[a][a][b]}$ will vanish. Such $[b]$ will automatically lie in $\mathcal{P}_L$. The number of such $[b]$ is precisely given by the LHS of equation (3.8). Thus (3.8) remains valid.

Consider first the odd $k_i$. Define $g^i = (1, \ldots, 1, k'_i - 2, 1, \ldots, 1)$. Then $g^i \in \mathcal{P}_L$, and the argument of Thm.1 applies, and we find that there is a conjugation $\pi_o$ acting only on the odd $k_i$ such that

$$(\pi_o \circ \tau(a))_i = a_i \text{ or } k'_i - a_i, \quad \text{for all } i \text{ with } k_i \text{ odd.} \quad (5.10b)$$

It remains to find a conjugation $\pi_e$ acting only on the even $k_i$ for which the analogue of (5.10b) holds for even $k_i$; then $\pi = \pi_e \circ \pi_o$ would be the desired conjugation, and $MI^\pi$ would be a simple current invariant.
Consider any even $k_i$, then there exists a vector $e^i \in \mathcal{P}_L$ such that $(e^i)_j = 1 + \delta_{ij}$ when $k_i$ is even (this again follows because there are no fixed points). (3.8) applied to these, and using (5.10b), defines $\pi_e$, and the remainder of the argument follows as before. QED

This theorem tells us e.g. when all $k_i$ are odd, the only exceptional physical invariants are $\mathcal{E}_{10}$-like. We will use it in Sect.7. It should be possible to strengthen this result somewhat.

Note that if $M$ is strongly physical, with chiral algebras $C_L = \mathcal{C}(\mathcal{J}_L), \mathcal{C}_R = \mathcal{C}(\mathcal{J}_R)$, then $\mathcal{J}_L$ will have fixed points iff $\mathcal{J}_R$ will. The reason is that if $\mathcal{C}(\mathcal{J}_L)$ has no fixed points then it will have exactly $\alpha = \prod (k_i + 1)/\|J_L\|^2$ extended characters $\chi_i$, while if it does have fixed points, then it will have more than this number (even if some $G([a]_i) > 1$). Because $\|\mathcal{J}_L\| = \|\mathcal{J}_R\|$ (Lemma 2), and because there must be a bijection between the extended characters, the desired result follows.

Finally, we will now look at the simplest case with fixed points. The motivation is to see if $\mathcal{E}_{16}$ belongs to an infinite series of $\mathcal{E}_{16}$-like exceptionals (it will also be used in Sect.7).

**Theorem 6.** Let $M$ be a strongly physical invariant, with no $k_i = 2$ and at most one $k_i = 4$. Suppose $M_{a\rho} \neq 0$ iff $a \in \{\rho, J\}$, and $M_{\rho b} \neq 0$ iff $b \in \{\rho, J'\}$. Then there exists a permutation invariant $I^\sigma$ such that either $M = M(J) I^\sigma$, or (rearranging the levels if necessary) $M = (\mathcal{E}_{16} \otimes A_k) I^\sigma$, where $A_k$ denotes the diagonal invariant of $\hat{k} = (k_2, k_3, \ldots, k_r)$.

**Proof.** The case where both $J$ and $J'$ have no fixed points is considered in Thm.5, so we may suppose both $J$ and $J'$ have fixed points. Without loss of generality put $J_i = 1$ iff $i \leq n$ for some $n$. Then all $k_i$, for $i \leq n$, will be even, and $\sum_{i \leq n} k_i \equiv 0 \pmod{4}$. For any $i > n$ consider $(e^i)_j = 1 + \delta_{ij}$. The usual argument shows that it cannot get mapped by $\tau$ to a fixed point, and in fact will get mapped to $[b^i]' = [J' e^i]'$ for some $i$ with $J_i' = 0$ and $k_i = k_i$ ($J_i^i = 0$ follows because $J' e^i$ must be in $\mathcal{P}_R$). This defines a conjugation, as usual, and factoring it off allows one to consider $J = J'$. Then for any $a \in \mathcal{P}_L$, there exists a simple current $J^a$ such that $\tau([a]_j)_i = [J^a a]_i$, for all $i > n$, and $1 \leq j \leq f(a)$. In fact a similar calculation to that of equation (3.9) gives that there exist numbers $f_{ij} \in \{0, 1\}$ such that

$$J^a_i \equiv \sum_{j=1}^r (a_j - 1) f_{ji} \pmod{2}, \quad \forall i > n. \quad (5.11)$$

Consider first the case $n > 1$. Define $(f^i)_j = 1 + 2\delta_{ij}$, for each $i \leq n$. It suffices to show $\tau([f^i])$ can never be a fixed point. But this follows from (5.7): because $\|I^i\| \geq n > 1$, and since there is no $k_j = 2$ and at most one $k_j = 4$, we find that (5.7c) cannot have a solution. Therefore $M$ will be a simple current invariant, and we are done.

Now consider the case when $n = 1$. By Thm.4 it suffices to consider $k_1 = 4, 8, 12$ or 16. Again define $f^1 = (3, 1, 1, \ldots, 1)$. Then $\tau(f^1)_i = 1$ for all $i > 1$, by (5.11). Then by $T$-invariance, for $k_1 = 4$ both $[f^1]$ and $\tau[f^1]$ will be fixed points, and for $k_1 = 8$ or 12 neither will be. So for $k_1 \neq 16$, $M$ will be a simple current invariant, hence of the desired form.

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The remaining \( n = 1 \) case, with \( k_1 = 16 \), can be handled directly using (5.11), or by using a somewhat simplified argument along the lines of the \( n = 1 \) part of the proof of Thm.7, given below. QED

In other words, when \( \|J_L\| = 2 \) there really is only one \( E_{16} \)-like exceptional, namely \( E_{16} \) itself. Since there is another \( E_{16} \)-like exceptional at level (8,8), it is natural to guess that they will also be found at \( g_4 \cdot 4,4,4 \) and \( g_8 \cdot 2,2,2,2,2,2,2 \) (Thm.2 tells us there will not be an exceptional at \( g_8^{16,16} \)). In fact, this is the case: let \( J_4 \) be the set of all \( r = 4 \) simple currents, and let \( J^a_s \) denote those \( r = 8 \) ones with an even number of nonzero components, and let \( \langle \chi_a \rangle_4 \) denote the sum \( \sum_{b \in J_4} \chi^a_b \) (similarly for \( J^a_s \)), then

\[
Z(4^1) = |\langle \chi_{1111} \rangle_4|^2 + 2|\langle \chi_{3333} \rangle_4 + \langle \chi_{1313} \rangle_4 + \langle \chi_{1131} \rangle_4 + \langle \chi_{1113} \rangle_4\chi^a_3333 + cc| \\
2|\langle \chi_{1133} \rangle_4 + \langle \chi_{3311} \rangle_4|^2 + 2|\langle \chi_{1313} \rangle_4 + \langle \chi_{3131} \rangle_4|^2 + 2|\langle \chi_{3311} \rangle_4 + \langle \chi_{1313} \rangle_4|^2 \\
+ 8(|\langle \chi_{3333} \rangle_4|^2 + |\langle \chi_{1333} \rangle_4|^2 + |\langle \chi_{3133} \rangle_4|^2 + |\langle \chi_{3313} \rangle_4|^2 + |\langle \chi_{3331} \rangle_4|^2); (5.12a)
\]

\[
Z(2^s) = |\langle \chi_{11111111} \rangle_8|^2 + 8|\langle \chi_{33111111} \rangle_8 \chi^a_{22222222} + cc| + 64|\chi_{22222222}^a|^2. \quad (5.12b)
\]

There is an analogue of \( Z(2^s) \) for each \( r = 8s \) replace ‘8’ in (5.12b) with \( 2^{4s-1} \) and 64 with \( 2^{8s-2} \). For \( s > 1 \) these are not fundamentally new, however, since they can be obtained from \( Z(2^8) \) by tensor products and multiplication by the elementary simple current invariant associated with \( J^a_{8s} \).

Both \( Z(4^1) \) and \( Z(2^8) \) seem to be new. \( Z(2^8) \) seems particularly interesting, since it corresponds to a simple current chiral algebra \( C(J^a_8) \), with \( f(22222222) = 2 \) (not 128), and \( G([22222222])_1 = G([22222222])_2 = 8 \). The \( S \)-matrix for this extension is

\[
S^e = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

It is unitary and symmetric, and obeys the relations \( S^{e2} = (S^e T^e)^3 = I \), as it should. The corresponding fusion rules can be computed from Verlinde’s formula, and are all found to be non-negative integers.

6. An example: all \( gcd(k', k'_j) \leq 3 \)

Let us review what we have learned. For a given modular invariant \( M \) of \( g_k \), let \( K^R_M(a) \) be the set of all \( b \in P^r_k \) such that \( M_{ab} \neq 0 \); define \( K^L_M(b) \) to be those \( a \in P^r_k \) for which \( M_{ab} \neq 0 \). We know (Thm.3 of [11] together with equation (4.2b) above) that for a given physical invariant \( M \), \( K^L_M(\rho_r) = \{\rho_r\} \) if \( K^R_M(\rho_r) = \{\rho_r\} \) if \( M \) is a permutation invariant, in which case it is listed in (3.5). If for all \( a \), both \( K^L_M(a), K^R_M(a) \subset J a \), then \( M \) is a simple current invariant, and will appear in Thm.2.

We are interested in finding all physical invariants for a given \( k \). Let \( K^L_k(\rho_r) \) denote the \( a \)-couplings of \( g_k^r \), i.e. the union over all \( g_k^r \) physical invariants \( M \) of the set \( K^L_M(a) \), or equivalently of the set \( K^R_M(a) \). For almost every \( k \), experience tells us \( K^L_k(\rho_r) \subset J \rho_r \),
Lemma 3. (a) No assumptions beyond (P1)-(P3) were needed to derive them. For any RCFT [8], and plays an important role in the search in [14], and the heterotic classification in [15]. We call it the parity rule.

There are three main constraints. Let \( b \in \mathcal{K}_k^r(a) \). Then by \( T \)-invariance it must satisfy

\[
\sum_{i=1}^{r} \frac{a_i^2}{k_i} \equiv \sum_{i=1}^{r} \frac{b_i^2}{k_i} \pmod{4}.
\]

(6.1a)

Also, the relation \( MS = SM \) and the fact that \( S_{\rho c}^{(k)} > 0 \) for all \( c \) implies for any positive invariant \( M \) that

\[
s(b) \overset{\text{def}}{=} \sum_{c \in P_k^r} M_{\rho c} S_{cb}^{(k)} \geq 0, \quad \forall b \in P_k^r,
\]

(6.1b)

with \( s(b) = 0 \) iff \( M_{cb} = 0 \forall c \in P_k^r \).

The third tool is the parity rule [11,25,8]. To formulate it, we must follow the following definitions.

Choose any \( a_i \in \mathbb{Z} \), \( i = 1, \ldots, r \). For any \( x, y \), by \( \{x\}_y \) we will mean the unique number \( \equiv x \pmod{y} \) lying between 0 and \( y \). For each \( i \) define \( a^+_i \) and \( \epsilon_i \) by:

\[
a^+_i = \{a_i\}_{2k_i}, \quad \epsilon_i = +1 \text{ if } 0 < \{a_i\}_{2k_i} < k_i; \quad a^+_i = 2k_i - \{a_i\}_{2k_i} \text{ and } \epsilon_i = -1 \text{ if } k_i' = \{a_i\}_{2k_i} < 2k_i';
\]

\[
a^+_i = 0 \text{ and } \epsilon_i = 0 \text{ if } \{a_i\}_{2k_i} = k_i'. \quad \text{We call } \epsilon(a) = \epsilon_1 \cdots \epsilon_r \text{ the parity of } a.
\]

Note that \( \epsilon(a) \in \{\pm 1, 0\} \), and when it is nonzero, \( a^+ \in P_k^r \).

Now choose any \( a, b \in P_k^r \). Let \( L_k \) be the set of all integers \( \ell \) coprime to \( 2k_1' \cdots k_r' \). Then \( \epsilon(\lambda a) \epsilon(\lambda b) \neq 0 \). More importantly [11] for any level \( k \) modular invariant \( M \), and any \( \lambda \in L_k \),

\[
M_{ab} = \epsilon(\lambda a) \epsilon(\lambda b) M_{(\lambda a)^+, (\lambda b)^+}.
\]

(6.2a)

Thus \( \mathcal{K}_k^r((\lambda a)^+) = (\lambda \mathcal{K}_k^r(a))^+ \), for all \( \lambda \in L_k \), \( a \in P_k^r \). A similar statement to (6.2a) holds for any RCFT [8], and plays an important role in the \( \hat{A}_2 \) classification [13], the computer search in [14], and the heterotic classification in [15]. We call it the parity rule.

Its main value for our purpose lies in its immediate consequence:

\[
b \in \mathcal{K}_k^r(a) \Rightarrow \epsilon(a) = \epsilon(b) \quad \forall \lambda \in L_k.
\]

(6.2b)

Together, (6.1) and (6.2b) constitute extremely strong constraints on the sets \( \mathcal{K}_k^r(\lambda) \). No assumptions beyond (P1)-(P3) were needed to derive them.

We will first look at (6.2b) for the case \( r = 1 \).

Lemma 3. (a) Let \( n \) be odd, and choose any integer \( 0 < a < n \). Suppose \( 0 < b < n \) satisfies, for all \( \ell \) coprime to \( 2n \), the relation

\[
\{\ell a\}_{2n} < n \quad \text{iff} \quad \{\ell b\}_{2n} < n.
\]

(6.3)

Then either \( b = a \) or \( b = n - a \).

(b) Let \( n \) be even. Then for \( a = 1 \), the solutions \( b \) to (6.3) are:

for \( n \neq 6, 10, 12, 30 \): \( b = 1 \) and \( n - 1 \);
for $n = 6$: $b = 1, 3, 5$;
for $n = 10$: $b = 1, 3, 7, 9$;
for $n = 12$: $b = 1, 5, 7, 11$; and
for $n = 30$: $b = 1, 11, 19, 29$.

Proof. (a) For $n$ odd, each $\ell_i = 2^i + n$ is coprime to $2n$. Write the binary expansions $x = a/n = \sum_{i=1}^{\infty} x_i 2^{-i}$, $y = b/n = \sum_{i=1}^{\infty} y_i 2^{-i}$, where each $x_i, y_i \in \{0, 1\}$. Then for $i = 1, 2, \ldots$, taking $\ell = \ell_i$ in (6.3) gives us $a + x_i \equiv b + y_i \pmod{2}$. If $a \equiv b \pmod{2}$, this forces $a = b$; if $a \not\equiv b \pmod{2}$ this forces $a = n - b$.

(b) For $n$ even and $a = 1$, all the work was done in Claim 1 of Sect.4 of [13]. There we found all odd $b$ solutions to (6.3). That any solution $b$ to (6.3) must be odd, when $n$ is even and $a = 1$, follows by taking $\ell = n - 1$ there. QED

Remember in using Lemma 3 that $n$ there plays the role of height $k + 2$, not level $k$. If we define an anomalous coupling here to be solutions $a, b$ to (6.3), where both $a \not\equiv b$ and $a + b \not\equiv n$, then Lemma 3(a) says that for $n$ odd there are no anomalous couplings. However, for $n$ even they are common. The list of all anomalous couplings for $n \leq 30$ are:

$$n = 6: \{1, 3, 5\};$$

$$n = 10: \{1, 3, 7, 9\};$$

$$n = 12: \{1, 5, 7, 11\}, \{2, 6, 10\};$$

$$n = 18: \{3, 9, 15\};$$

$$n = 20: \{2, 6, 14, 18\};$$

$$n = 24: \{2, 10, 14, 22\}, \{4, 12, 20\};$$

$$n = 30: \{1, 11, 19, 29\}, \{3, 9, 21, 27\}, \{5, 15, 25\}, \{7, 13, 17, 23\}. \quad (6.4)$$

Equation (6.4) should be read as follows. If $a \not\equiv b$ is a solution to (6.3), and $a + b \not\equiv n$ then $a$ and $b$ will both belong to one of the sets listed in (6.4). Conversely, two elements $a \not\equiv b$ of a common set in (6.4), which do not satisfy $a + b = n$, will be an anomalous coupling for that height $n$.

When $r = 1$ it is convenient to identify a weight $a$ with its Dynkin label $a_1$. Also, when the level $k$ is understood we will write $\bar{a} = k' - a$.

One useful consequence of Lemma 3 is that it will permit us to find all the positive invariants of $A_{1,k}$. Recall that a modular invariant is called positive if all its coefficients $M_{\lambda \mu} \geq 0$ (but they need not be integers). Equations (6.1),(6.2) hold for any $b \in K^R_M(a)$, for any positive invariant $M$. The main reason the positive invariants of $A_{1,k}$ will be useful is through projecting [3] out $r - 1$ of the $A_1$ factors. See [7] for an example.

Lemma 4. Any positive invariant of $A_{1,k}$ can be written as a linear combination (with positive coefficients) of the 1, 2, or 3 physical invariants at that level.

Proof. The three exceptional levels can be done on a computer, or by hand using (6.4). For $k$ odd, the result follows from Lemma 3(a) and $T$-invariance, and the fact that the only modular invariants which are diagonal matrices will be, for any algebra and level, a scalar multiple of the identity. Therefore we need to consider only even $k \neq 10, 16, 28$. 

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First consider \( k \equiv 2 \mod 4 \). By Lemma 3(b) and \( T \)-invariance, \( \mathcal{K}_M^R(1) = \mathcal{K}_M^L(1) = \{1\} \) for any positive invariant \( M \).

Write \( \mathcal{K}_M(a) = \mathcal{K}_M^R(a) \cup \mathcal{K}_M^L(a) \). Suppose we know \( \mathcal{K}_M(a) \subset \{a, \bar{a}\} \) for all \( a \). Then the \((1, a), (a, 1), (1, \bar{a})\) and \((\bar{a}, 1)\) entries of \( SM = MS \) tell us \( M_{11} = M_{aa} \) for \( a \) odd, and \( M_{11} = M_{aa} + M_{a\bar{a}}, M_{aa} = M_{\bar{a}a}, M_{a\bar{a}} = M_{\bar{a}a} \) for \( a \) even. Also \( (SM)_{2a} = (MS)_{2a} \) tells us \( M_{22} - M_{2\bar{a}} = M_{aa} - M_{\bar{a}a}, \) for \( a \) even. Then \( M = M_{22}A_k + M_{2\bar{a}}D_k \), and we would be done. So we want to show \( \mathcal{K}_N(a) \subset \{a, \bar{a}\} \).

By an argument similar to that used in deriving (5.5c), by looking at \( N^\infty = \lim_{n \to \infty} \{(M + M^T)/(2M_{11})\}^n \) we get the following expression:

\[
C_a \sum_{c \in W(a)} S_{1a}S_{1c}S_{cb} = C_b \sum_{d \in W(b)} S_{ad}S_{1d}S_{1b} \quad \text{where} \quad C_a = \sum_{c \in W(a)} S_{1c}^2
\]

(6.5a)

for all \( a, b \), where \( W(a) \supset \mathcal{K}_M(a) \) is some set of weights (namely, the weights in the indecomposable block of \( N^\infty \) containing \( a \)). Suppose we know \( W(a) = \{a\} \) for some \( a \), then for any \( b \)

\[
\frac{S_{ab}}{S_{1b}} = \frac{S_{ad}}{S_{1d}} \quad \forall d \in W(b).
\]

(6.5b)

If instead we know \( W(a) = \{a, \bar{a}\} \) for some \( a \), then (6.5b) will hold for any odd \( b \).

If in equation (6.5b) we have \( a = 3 \), then \( d = b \) or \( \bar{b} \) and we are done, so it suffices here to show \( W(3) = \{3\} \). \( W(a) = \{a\} \) does hold for any \( a \) coprime to \( 2k' \), by (6.2a) and the fact that \( W(1) = \{1\} \). If \( 3 \) does not divide \( k' \), we are done. Otherwise (6.5b) gives us \( S_{3a}/S_{13} = S_{ad}/S_{1d} \) for any \( a \) coprime to \( 2k' \) and \( d \in W(3) \). But the LHS will always be \( \geq 1 \) in absolute value, which forces either \( d \) or \( \bar{d} \) to divide \( k' \). \( T \)-invariance now forces \( d = 3 \) or \( \bar{3} \), and we are done.

The proof for \( k \equiv 0 \mod 4 \) is similar. Here we get \( \mathcal{K}_M(3) \subset W(3) \subset \{3, \bar{3}\} \), provided we avoid the exceptional case \( k = 16 \). By (6.5b) this forces \( \mathcal{K}_M(a) \subset \{a, \bar{a}\} \) for all odd \( a \). From the usual \( MS = SM \) arguments, we get \( M_{aa} = M_{11} \geq M_{a\bar{a}} = M_{\bar{a}a} \) for all odd \( a \). Replace \( M \) with \( M' = M - M_{11}D_k \). \( M' \) will be positive, it will have only 1 as a \( \rho \)-coupling, and \( a = 3 \) in (6.5b) together with \( MS = SM \) forces \( M' = M_{22}A_k \).

QED

Consider now the case where each \( k'_i \) is coprime to each \( k'_j, \ i \neq j \). Given any \( \ell \in L_k \), choose any \( \ell_i \equiv \ell \mod 2k'_i \). Then each \( \ell_i \) is coprime to \( 2k'_i \). The converse is also true: given any set \( \{\ell_1, \ldots, \ell_r\} \), where each \( \ell_i \) is coprime to \( 2k'_i \), there is an \( \ell \in L_k \) such that \( \ell_i \equiv \ell \mod 2k'_i \) (this follows from the Chinese Remainder Theorem).

Fix \( i \). For each \( \ell_i \) coprime to \( 2k'_i \), \( \ell \in L_k \) correspond to the \( r \)-tuple \((1, \ldots, \ell_i, \ldots, 1)\). Then (6.2b) collapses to the one-dimensional

\[
\epsilon(\ell_i a_i) = \epsilon(\ell_i b_i), \quad \forall \ell_i \text{ coprime to } 2k'_i,
\]

(6.6)

which was considered in Lemma 3. A little more care shows that (6.6) still holds, if all we have here is each \( gcd(k'_i, k'_j) \leq 3 \). Indeed, for a given \( \ell_i \) coprime to \( 2k'_i \), choose \( \ell_j = \pm 1, \ j \neq i \), according to the following rule: if \( g_{ij} = gcd(k'_i, k'_j) = 1 \), choose \( \ell_j = +1 \); otherwise choose \( \ell_j \equiv \ell_i \mod 2g_{ij} \). Then, again by the Chinese Remainder Thm., \( \exists \ell \in L_k \) such that \( \ell \equiv \ell_j \mod 2k'_j, \ \forall j \). Since \( \epsilon_{k_j}(\pm a_j) = \pm \epsilon_{k_j}(a_j) \), for any \( a_j \), (6.2b) reduces to (6.6)
for this $\ell$. We now know enough for a quick proof of the following result; the only hard
part as usual is handling the 3 exceptional levels.

**Theorem 7.** Suppose each $\gcd(k'_i, k'_j) \leq 3$, for $i \neq j$. Then the only strongly physical
invariants belonging to these levels $k$ are the simple current invariants, classified in Thm.2,
together with the products (reordering the levels if necessary) $(E_{10} \otimes A_k)D_k$, $E_{16} \otimes D_k$, and
$E_{28} \otimes D_k$, where $A_k$ denotes the diagonal invariant of level $k$, $D_k$ denotes any simple current
invariant of level $k$, and $\hat{k} = (k_2, \ldots, k_r)$.

**Proof.** We would like to use Lemma 3 and (6.6) to get that $K^r_k(\rho) \subset J\rho$, and then Thm.4
to get the desired result. But the first step requires that we avoid $k_i$ equal to 4, 8, 10 or 28,
and the second step that we avoid 2, 4, 8, 12 and 16.

Choose any $a$, and any $b \in K^R_k(a)$. Consider first that $k_i = 2$ or 12. Because it
(i.e. $n = 4, 14$) is not on the list of (6.4), (6.6) implies $b_i = a_i$ or $\bar{a}_i$. If $k_i = 8$, then (6.4)
and $T$-invariance (in particular (6.1a) multiplied by $\prod_j k'_j/5$, taken mod 1) also implies
$b_i = a_i$ or $\bar{a}_i$.

Next suppose $k_i = 4$. Then the gcd condition says no $k_j = 10, 16, 28$, so for all $j \neq i$,
$a_j = 1$ implies $b_j = 1$ or $\bar{1}$. Then by $T$-invariance $K^r_k(\rho) \subset J\rho$. The familiar argument
shows that $f^i = (1, \ldots, 3, \ldots, 1)$, can only couple to $Jf^i$ for some simple current $J$. As
usual this forces $b_i = a_i$ or $\bar{a}_i$.

Thus as long as no $k_i = 10, 16, 28$, the proof of Thm.4 will carry through, and the
strongly physical invariant will be a simple current invariant, listed in Thm.2. So it suffices
to consider $k_1$, say, equal to one of 10, 16, 28. The gcd condition then says that no other
$k_i = 4, 10, 16, 28$, so

$$b_i = a_i \text{ or } \bar{a}_i, \quad \forall i > 1. \tag{6.7a}$$

Consider first $k_1 = 16$. Then $K^r_k(\rho) \subset J\rho$, so by Thm.3 we have an automorphism
$\tau$ between simple current chiral algebras $C(J_L), C(J_R)$. Let $M$ be any strongly physical
invariant, and write $f^1 = (3, 1, \ldots, 1)$. If $\tau[f^1]$ is not a fixed point, then $M$ is a simple
current invariant by the usual argument. So suppose $\tau[f^1]$ is a fixed point, i.e. the first
component ($\tau[f^1])_1 = 9$, and $J^1 = (1, 0, \ldots, 0) \in J_L \cap J_R$. Write $\hat{a} = (a_2, \ldots, a_r)$,
and $J\hat{R}$ be that subset of $J_R$ which fixes the first component. So $J_R$ is spanned by $J^1$ and
$J\hat{R}$. Let $P_L$ be the set of all $\hat{a} \in P^\ast_{\hat{R}}$ such that there exists an $x < 18$, $b \in P_R$, such that
$M_{x\bar{a}, b} \neq 0$. First note that for any $a$, (6.4) tells us $(\tau[a]_1 = a_1$ or $\bar{a}_1$, if $a_1 = 1, 5$ or
7. Choose any $\hat{a} \in P_L$, and suppose $\tau[(1\hat{a})_1] = [1\hat{J}a]_\hat{e}$. Choose any $[x\hat{a}]$ and any $\hat{c} \in \hat{P}_L$
with components $\hat{c}_\ell = 1$ or 2 (so $F\hat{c} = 1)$, and write $\tau([x\hat{a}]_j) = [y\hat{J}a]_\ell$ and $\tau[1\hat{c}] = [1\hat{c}']$. Comparing

$$\frac{G([1\hat{a}]_1)}{F_{\hat{a}}}S_{1\hat{a}, \hat{c}} = \frac{G'([1\hat{J}a]_i)}{F_i J\hat{a}}S_{1\hat{J}a, \hat{c}_i} \quad \text{i.e. } \hat{S}_{\hat{a}\hat{c}} = \hat{S}_{\hat{a}\hat{c}'} \cdot \frac{g_{\hat{a}\hat{c}}}{\hat{J}_{\hat{a}}} 1^\prime \cdot \hat{\rho} \alpha \tag{6.7b}$$

$$\frac{G([x\hat{a}]_j)}{F_{x\hat{a}}}S_{x\hat{a}, \hat{c}} = \frac{G'([y\hat{J}a]_\ell)}{F_{\hat{y}\hat{a}}}S_{y\hat{J}a, \hat{c}_\ell} \quad \text{i.e. } \hat{S}_{\hat{a}\hat{c}} = \hat{S}_{\hat{a}\hat{c}'} \cdot \frac{g_{\hat{a}\hat{c}}}{\hat{J}_{\hat{a}}} 1^\prime \cdot \hat{\rho} \beta. \tag{6.7c}$$

where $\hat{S}$ is the $S$-matrix for $g_{\hat{k}}^{-1}$, and $\alpha, \beta$ are the obvious combinations of $G, G', F, F'$. By
the usual argument (see (4.9)) we see that we can choose $\hat{J}$ and $\hat{J}'$ so that $(\hat{J}, \hat{J}') \cdot (\hat{\rho} - \hat{\rho}) \equiv 0$
Let any $\tilde{a} \in \hat{\mathcal{P}}_L$ be congruent (mod 2) to some $\tilde{c} \in \hat{\mathcal{P}}_L$ whose components are 1's and 2's. So we get from the (4.3) argument that $\hat{J} \hat{J}' \in \hat{\mathcal{J}}_R$. In other words,

$$\tau([1\tilde{a}]_1) = [1\tilde{b}]_i' \Rightarrow \forall x, j, \exists y, \ell \text{ such that } \tau([x\tilde{a}]_j) = [y\tilde{b}]_\ell'. \quad (6.7d)$$

Now define the projection $M'$ of $M$ by $A_{16}$:

$$M'_{\tilde{a}\tilde{b}} = \sum_{x=1}^{17} M_{x\tilde{a}, x\tilde{b}}. \quad (6.8a)$$

Then $M'$ is a positive invariant. $(6.7a)$ tells us that $M'$ is a (non-physical) simple current invariant. Note that $M'_{\tilde{b}\tilde{b}} = 7$. In fact $(6.7d)$ implies $M'_{J\tilde{b}, \tilde{b}} = 0$ unless $\hat{J} \in \hat{\mathcal{J}}_L$, in which case it equals 7; similarly, $M'_{J'\tilde{b}, \tilde{b}} = 0$ unless $\hat{J}' \in \hat{\mathcal{J}}_R$, in which case it also equals 7. Then (4.3) says $\hat{J}_L \cdot (\hat{\mathcal{P}}_L - \tilde{b}) \equiv 0 \pmod{2}$, so by the (4.4) calculation we get that

$$\sum_{\tilde{\mu}} M'_{\tilde{\mu} \tilde{\mu}} = 7\|\hat{J}_L\|. \quad (6.8b)$$

From $(6.8b)$ we see $(\tau[3\tilde{a}])_1 = 9$. Gathering all the results we have, we get that $M = \mathcal{E}_{16} \otimes D_k$ for the (physical) simple current invariant $D_k = \frac{1}{7} M'$.

Next consider $k_1 = 28$. Again, $(6.7a)$ is satisfied. Without loss of generality suppose $K_M^R(\rho) \not\subset J_\rho$.

Claim 2. Suppose $M_{\rho, 11\tilde{b}} \neq 0$ or $M_{\rho, 19\tilde{b}} \neq 0$. Then

$$K_M^L(\rho) = \hat{J}_L\{(1\tilde{\rho}), (11\tilde{\rho}), (19\tilde{\rho}), (29\tilde{\rho})\}, \quad K_M^R(\rho) = \hat{J}_R\{(1\tilde{\rho}), (11\tilde{\rho}), (19\tilde{\rho}), (29\tilde{\rho})\},$$

for some groups $\hat{J}_L, \hat{J}_R$ of simple currents fixing the first component.

Proof of claim. The proof is based on a somewhat tedious application of $(6.1b)$. We will sketch some of the details. Define $\hat{J}_R$ to be those $\hat{J}$ such that $M_{\rho, 1J\rho} \neq 0$. It will be a group, by Lemma 2. Let $\|\hat{J}_R\| n_x = \sum_{\tilde{b}} M_{\rho, x\tilde{b}}$ for $x = 11, 19, 29$. These $n_x$ will be integers by Lemma 2. Putting $a = (3\tilde{\rho})$ in $(6.1b)$ gives us the relation $1 - n_{11} - n_{19} + n_{29} \geq 0$. Continuing in this way we force $n_{11} = n_{19} = n_{29} = 1$. Then there exists simple currents $\hat{J}_x$ such that $M_{\rho, x\tilde{b}} \neq 0$ iff $\tilde{b} \in \hat{J}_R\hat{J}_x$, in which case $M_{\rho, x\tilde{b}} = 1$. To show $\hat{J}_x \in \hat{J}_R$ for each $x$, it suffices to show that for each $\tilde{a}$ satisfying $\hat{J}_R \cdot (\tilde{a} - \tilde{\rho}) \equiv 0 \pmod{2}$, then $\hat{J}_x \cdot (\tilde{a} - \tilde{\rho}) \equiv 0$. This follows from $(6.1b) - e.g.$ if for some such $\tilde{a}$ we had $\hat{J}_x \cdot (\tilde{a} - \tilde{\rho}) \equiv 1 \pmod{2}$ for all $x = 11, 19, 29$, then take $a = (1, \tilde{a})$ in $(6.1b)$.

That $K_M^L(\rho)$ must also contain anomalous $\rho$-couplings follows from $(SM)_{\rho \rho} = (MS)_{\rho \rho}$. QED to claim

$(6.1b)$ and $T$-invariance (using $(4.7a)$) now tell us that $M_{x\tilde{a}, y\tilde{b}} \neq 0$ implies either $x, y \in \{1, 11, 19, 29\}$ or $x, y \in \{7, 13, 17, 23\}$.

Define the projection

$$M'_{\tilde{a}\tilde{b}} = \frac{1}{8} \sum_{x=1}^{29} M_{x\tilde{a}, x\tilde{b}} = M_{1\tilde{a}, 1\tilde{b}}; \quad (6.9a)$$

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the second equality follows from (6.2a). Obviously $M'$ is a (physical) simple current invariant. Project again, giving

$$M''_{xy} = \frac{1}{\|J_R\|^2} \sum_{\tilde{a}, \tilde{b}} M_{x\tilde{a},y\tilde{b}} M'_{\tilde{a},\tilde{b}}.$$  \hspace{1cm} (6.9b)

Lemma 4 and the fact that $K_R = \{\} \implies M'' = \mathcal{E}_{28}$. This implies that $M = \mathcal{E}_{28} \otimes M'$.

Finally, consider $k_1 = 10$. Again, (6.7a) is satisfied. A similar argument to the one used in the proof of Claim 2 gives us:

**Claim 3.** Suppose $K^R_M(\rho) \not\subset J_\rho$. Then either

- **case 1:** $K^R_M(\rho) = \mathcal{J}_R((1\hat{\rho}), (7\hat{\rho}))$ and $K^L_M(\rho) = \mathcal{J}_L((1\hat{\rho}), (7\hat{\rho}))$;
- **case 2:** $K^R_M(\rho) = \mathcal{J}_R((1\hat{\rho}), (7\hat{\rho}), (5\hat{\rho}^R), (11\hat{\rho}^R))$

and $K^L_M(\rho) = \mathcal{J}_L((1\hat{\rho}), (7\hat{\rho}), (5\hat{\rho}^L), (11\hat{\rho}^L))$;

where $\mathcal{J}_L, \mathcal{J}_R$ are groups of simple currents fixing the first component and $\hat{J}^L, \hat{J}^R$ are simple currents.

First look at case 1. (6.2a) tells us that e.g. $M_{1\hat{a},1\hat{b}} = M_{x\hat{a},x\hat{b}}$ for $x = 5, 7, 11$. Also, the equation $M = SM \otimes M$ produces expressions like

$$M_{1\hat{a},c} = \sqrt{\frac{1}{6}} \left\{ \sin(\pi/12) (A^4_{\hat{a}c} + A^1_{\hat{a}c}) + \sin(5\pi/12) (A^5_{\hat{a}c} + A^7_{\hat{a}c}) + \sin(\pi/3) (A^4_{\hat{a}c} + A^8_{\hat{a}c}) \right\},$$

where $A^x_{\hat{a}c} = \sum_{\hat{b}, \hat{d}} \hat{S}_{\hat{a}, \hat{b}} M_{x\hat{b}, \hat{d}} S_{\hat{d}c}$.

(6.10)

Comparing the expressions for $M_{x\hat{a},c}$ for $x = 1, 5, 7, 11$, and noting from $T$-invariance that at least two of these vanish, we get that $M_{1\hat{a},c} = M_{7\hat{a},c}$ and $M_{5\hat{a},c} = M_{11\hat{a},c}$. Familiar arguments tell us that all row and column sums must be equal to $2\|J_L\|$. It can be shown, by evaluating $MS = SM$ at $(1\hat{\rho}, 4\hat{\rho})$, that there is no triple $\hat{a}, \hat{b}, \hat{c}$ such that both $M_{1\hat{a},1\hat{b}} \equiv 0$ and $M_{1\hat{a},5\hat{c}} \equiv 0$. Suppose there is a $\hat{a}, \hat{c}$ such that $M_{1\hat{a},5\hat{c}} \equiv 0$. The same calculation tells us there exists a simple current $\mathcal{J}$ such that $M_{4\hat{j},\hat{\rho},4\hat{\rho}} = 1$, and it satisfies $\mathcal{J} \cdot (\hat{a} - \hat{\rho}) \equiv 1 \pmod{2}$. By $T$-invariance, the simple current $J = (1, \mathcal{J})$ has norm $J^2 \equiv 2 \pmod{4}$; write $\mathcal{M} = M I^J$. Then $\mathcal{M}_{1\hat{a},5\hat{d}} = 0$ for all $\hat{b}, \hat{d}$. It is now easy to show $\mathcal{M} = \mathcal{E}_{10} \otimes \mathcal{M}'$, where

$$\mathcal{M}'_{\hat{a}\hat{b}} = \frac{1}{6} \sum_{x=1}^{11} \mathcal{M}_{x\hat{a},x\hat{b}} = \frac{2}{3} \mathcal{M}_{1\hat{a},1\hat{b}} + \frac{1}{3} \mathcal{M}_{4\hat{a},4\hat{b}}.$$ \hspace{1cm} (6.11)

All that remains is case 2. Then $J = (1, \mathcal{J}^R) \in J_R$, so by Lemma 2 $M_{1\hat{a},1\hat{b}} = M_{i\hat{a},i\hat{b},j\hat{\rho}}$, etc. The proof that $M = (\mathcal{E}_{10} \otimes A_{\hat{k}}^1 M(J) (A_{10} \otimes D_{\hat{k}})$ for some simple current invariant $D_k$ now proceeds as in (6.10) above. QED

Throughout the proof of Thm.7, we used the easily verified fact that if $M = M' \otimes M''$ and $M$ and $M'$ are both modular invariants, then so will be $M''$.  

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What is so special about the numbers \( \ell \leq 3 \) is that they are the only ones with the following property: any \( n \) coprime to \( 2\ell \) satisfies \( n \equiv \pm 1 \pmod{2\ell} \). Nevertheless, Thm.7 could be strengthened with more effort, but as it stands it provides an easy example of the power of equations (6.1),(6.2). Those equations will also be used in the following section. Note that Thm.5 tells us that when all \( k_i \) are odd, Thm.7 will apply to any (not necessarily strongly) physical invariant. For \( r = 2 \), it is also possible to show that Thm.7 holds for any physical invariant.

7. The \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \) classification

In this section we apply all that we have learned in the previous sections, to find all strongly physical invariants for \( A_1 \oplus A_1 \) at all levels \( k = (k_1,k_2) \). It suffices to find all \( \mathcal{E}_{10} \)-like and \( \mathcal{E}_{16} \)-like exceptionals; the regular series (i.e. the simple current invariants and their conjugations) are listed in the Appendix of [15]. First we will find all \( \mathcal{E}_{16} \)-like exceptionals, as well as find a characterization of all of the obvious (i.e. non-sporadic) \( \mathcal{E}_{10} \)-like exceptionals. The previous sections suggest to look at the \( \rho \)-couplings \( \mathcal{K}_M = \mathcal{K}^L_M(\rho) \cup \mathcal{K}^R_M(\rho) \).

Throughout this section, \( M \) will denote a strongly physical invariant of \( g^2 \). Let \( \mathcal{A}_k, \mathcal{D}_k, \mathcal{E}_k \) denote the \( A_{1,k} \) physical invariants. Write \( J^1 = (1,0), \ J^2 = (0,1) \), and \( J^{12} = (1,1) \). A valuable result is that for all \( 21 \geq k_1 \geq k_2 \), along with \( k_1 = 28 \) and \( 21 \geq k_2 \), all physical invariants have been found using explicit calculations using the lattice method [14,15]. Small levels often appear below as special cases, and this permits us to dismiss them.

**Proposition.** (a) Suppose \( \mathcal{K}_M \subset \mathcal{J}\rho = \{1, k_1 - 1\} \times \{1, k_2 - 1\} \). Then \( M \) will lie on one of the regular series, or will be one of the exceptionals \( \mathcal{E}_{16} \otimes \mathcal{A}_{k_2}, \ \mathcal{E}_{16} \otimes \mathcal{D}_{k_2}, \ \mathcal{E}_{16} \otimes \mathcal{E}_{16}, \ \mathcal{A}_{k_1} \otimes \mathcal{E}_{16}, \ \mathcal{D}_{k_1} \otimes \mathcal{E}_{16} \) (or their conjugations, if \( k_1 = k_2 = 16 \), or the \( k = (4,4) \) or \( (8,8) \) exceptionals.

(b) Suppose for \( k = (28,k_2) \), we have \( \mathcal{K}_M \subset \{1,11,19,29\} \times \{1,k_2 - 1\} \). Then either \( M \) will be listed in (a), or \( M = \mathcal{E}_{28} \otimes \mathcal{A}_{k_2} \) or (if \( k_2 \) is even) \( \mathcal{E}_{28} \otimes \mathcal{D}_{k_2} \) or (if \( k_2 = 16 \) \( \mathcal{E}_{28} \otimes \mathcal{E}_{16} \), or (if \( k_2 = 3 \) the \( k = (28,3) \) exceptional.

(c) Suppose for \( k = (10,k_2) \), we have \( \mathcal{K}_M \subset \{1,5,7,11\} \times \{1,k_2 - 1\} \). Then either \( M \) will be listed in (a), or \( M = \mathcal{E}_{10} \otimes \mathcal{A}_{k_2} \), or (if \( k_2 \) is even) \( \mathcal{E}_{10} \otimes \mathcal{D}_{k_2} \) or (\( \mathcal{E}_{10} \otimes \mathcal{A}_{k_2} \)) \( M(J^{12}) \), or (if \( k_2 = 16 \) \( \mathcal{E}_{10} \otimes \mathcal{E}_{16} \), or (if \( k_2 = 2 \) the \( k = (10,2) \) exceptional.

**Proof.** (a) Thms.4 and 6 of Sect.5 prove this for almost all levels. The remaining cases are:

(i) \( k = (2,k_2), k_2 \equiv 2 \pmod{4}, \mathcal{J}_L = \mathcal{J}_R = \{0,J^{12}\} \);

(ii) \( k = (2,k_2), k_2 \equiv 0 \pmod{4}, \mathcal{J}_L = \mathcal{J}_R = \{0,J^2\} \);

(iii) \( k = (k_1,k_2), k_1 \equiv k_2 \equiv 0 \pmod{4}, \mathcal{J}_L = \mathcal{J}_R = \{0,J^1,J^2,J^{12}\} \), and \( k_1 \leq 16 \);

(iv) \( k = (4,4), \mathcal{J}_L, \mathcal{J}_R \neq \{0\} \).

(iv) has already been worked out explicitly [14]. In all other cases, Thm.3 applies, and we have a bijection \( \tau \) between simple current extensions.

(i): First note that \( \tau \) will not send \( f^2 = (1,3) \) to a fixed point, except possibly for \( k = (2,10) \) (the usual (5.7a) argument works here). The familiar fusion rule argument then says \( \tau[f^2] = [Jf^2] \) for some simple current \( J \), and \( T \)-invariance then forces \( J \in \mathcal{J}_L \).
Then for any \( \lambda \in \mathcal{P}_L \), the \((5.5d)\) argument says \((\tau[a])_2 = a_2 \) or \( \bar{a}_2 \), and \( T \)-invariance then forces \((\tau[a])_1 = a_1 \) or \( \bar{a}_1 \). Therefore \( M \) is a simple current invariant, and we are done.

(ii): The argument is the same as (i); the exceptional levels here is \( k = (2, 16) \) and \((2, 4)\).

(iii): Suppose \( k_2 > 16 \) (the remaining ten levels \( k \in \{4, 8, 12, 16\} \times \{4, 8, 12, 16\} \) have already been done explicitly). Then as usual looking at \( f^2 = (1, 3) \) implies \((\tau[a])_2 = a_2 \) or \( \bar{a}_2 \), \( \forall a \in \mathcal{P}_L \). Looking at \( f^1 = (3, 1) \) then implies \((\tau[a])_1 = a_1 \) or \( \bar{a}_1 \), for any \( a \), except possibly if \( k_1 = 16 \). Therefore \( M \) will be a simple current invariant, except when \( k_1 = 16 \) and \((\tau[f^1])_1 = 9 \). The remainder of the argument, namely that in the latter case \( M \) must equal \( \mathcal{E}_{16} \otimes \mathcal{D}_{k_2} \), is as in Thm.6 or Thm.7.

(b) The usual arguments from (6.1b) give us that, assuming \( K_M \not\subset J \rho \), \( \mathcal{K}^L_M = \mathcal{K}^R_M = J_L \{(1, 1), (11, 1)\} \), where either \( J_L = \{0, J^1\} \) or \( J_L = \{0, J^1, J^2, J^{12}\} \). Also, each \( M_{\rho b} = M_{b \rho} \) is 0 or 1.

Consider the block diagonal form \( M' = \sum |ch_i|^2 \). Then Lemma 2 and Perron-Frobenius tells us that its blocks \( B'_i \) will either be a \((2||J_L||) \times (2||J_L||) \) block of 1’s, a \( ||J_L|| \times ||J_L|| \) block of 2’s, or perhaps a \((||J_L||/2) \times (||J_L||/2) \) block of 4’s. By an argument as in (6.10), we get that \( M'_{1x,b} = M'_{11x,b} \) etc. From this we find that \( M' = \mathcal{E}_{28} \otimes A_{k_2} \) (if \( ||J_L|| = 2 \)) or \( M' = \mathcal{E}_{28} \otimes \mathcal{D}_{k_2} \) (if \( ||J_L|| = 4 \)).

Now return to the given \( M \). It will correspond to some automorphism \( \tau \) of the chiral algebra defined by \( M' \). The \( S \)-matrix of that chiral algebra can be easily computed, it turns out to be \( S' \otimes S'' \), where \( S' \) is the \( 2 \times 2 \) \( S \)-matrix defined by \( \mathcal{E}_{28} \), and \( S'' \) is either the \( S \)-matrix for \( A_{k_2} \) or \( \mathcal{D}_{k_2} \). From these explicit values one quickly finds that \( \tau \) also must factorize in this way (apart from \( k_2 = 3 \)), which means \( M \) must be in the desired form.

(c) The proof here is similar to that of (b). QED

If both \( k_1, k_2 \) are odd, then Thm.5 says that the conclusion of Proposition (a) still holds if we consider there, instead of strongly physical invariants \( M \), any physical invariant \( \mathcal{K}^L_M \).

This proposition covers most of the physical invariants of \( g_k^2 \). It says that we are done the \( g_k^2 \) classification if we can show, apart from some small \( k = (k_1, k_2) \) which can be handled individually, that the following holds:

\[
(a, b) \in \mathcal{K}_M \Rightarrow a \in \{1, k'_1 - 1\} \ \forall k_1, \text{ or} \\
a \in \{1, 3, 5\} \text{ for } k_1 = 4, \text{ or} \\
a \in \{1, 3, 7, 9\} \text{ for } k_1 = 8, \text{ or} \\
a \in \{1, 5, 7, 11\} \text{ for } k_1 = 10, \text{ or} \\
a \in \{1, 11, 19, 29\} \text{ for } k_1 = 28. \tag{7.1}
\]

The reason is that Lemma 3 and (6.2a) then constrain \( b \) similarly, and with \( T \)-invariance \( \mathcal{K}_M \) is reduced to the possibilities considered in the proposition, with three exceptions. They are \( k = (10, 10), (10, 28), \) and \((28, 28)\). The first two are explicitly worked out by computer in [14] and [15], respectively.

The remaining special case \( k = (28, 28) \) can be handled by the now-familiar arguments. If \( \mathcal{K}_M \not\subset \{1, 11, 19, 29\} \times \{1, 29\} \) and \( \mathcal{K}_M \not\subset \{1, 29\} \times \{1, 11, 19, 29\} \), then from (6.1b) we
get $K_M^L = K_M^R = \{1, 11, 19, 29\} \times \{1, 11, 19, 29\}$, and all $M_{\rho,b}$ and $M_{a,p}$ equal 0 or 1. So $J_L = J_R = \{0, J^1, J^2, J^{12}\}$. The (6.10) argument tells us $M_{1x,1y} = M_{11x,1y}$ etc. Equation (6.1b) tells us precisely which rows and columns will be non-zero. Each non-zero row and column of $M$ must sum to 16. Putting all this together, we find that $M$ is completely fixed once we know whether $M_{17,17} = 1$ or 0 – then $M = \mathcal{E}_{28} \otimes \mathcal{E}_{28}$ or $(\mathcal{E}_{28} \otimes \mathcal{E}_{28})^c$, resp.

So it suffices to find all levels $k$ where $K_M$ does not satisfy (7.1). We can expect there to only be finitely many such $k$ – they will be where to find the remaining $\mathcal{E}_{10}$-like exceptionals. If we were to follow the approach of [13], we would have three main tools to constrain $K_M$: equations (6.1),(6.2). The analysis is in fact somewhat simpler here, with one important qualification: there are two independent levels, $k_1$ and $k_2$, here while in [13] we had only one. This makes the number of different cases to be considered quite large.

There is an alternative, though it requires that we impose the full machinery of [24]. By analysing the polynomial solutions of the corresponding Knizhnik-Zamolodchikov equations, Stanev [27] has found the list of all possible chiral extensions of $A_{1,k_1} \oplus A_{1,k_2}$. The only sporadic levels on this list turn out to be $(2,2)$, $(3,1)$, $(6,6)$, $(8,3)$, $(10,2)$, $(10,10)$, and $(28,8)$, corresponding to conformal embeddings of $A_{1,k_1} \oplus A_{1,k_2}$ into $A_3$, $G_2$, $B_4$, $C_3$, $D_4$, $D_5$, and $F_4$, respectively.

These levels have all been explicitly checked in [14] and [15]. The remaining levels will then have $\rho$-couplings given in the proposition (apart from the 3 exceptionals noted previously). The conclusion is:

**Theorem 8.** The complete list of strongly physical invariants of $A_{1,k_1} \oplus A_{1,k_2}$ is given in Sect.2.

In particular, for each $k = (k_1, k_2)$ there is $N(k)$ physical invariants, where

(i) $N(k) = 1$ for $k_1 = 2$ and $k_2$ odd (or vice versa);

(ii) $N(k) = 2$ for $k_1$ odd and $k_2 \notin \{k_1, 2, 10, 16, 28\}$, (or vice versa), and $k \neq (3, 8)$;

(iii) $N(k) = 3$ for $k_1$ odd and $k_2 \in \{10, 16, 28\}$ (or vice versa), and $k \neq (3, 28)$;

(iv) $N(k) = 4$ for $k_1 = k_2$ both odd, provided $k \neq (1, 1)$;

(v) $N(k) = 6$ for $k_1 \neq k_2$ both even, unless either $k_1$ or $k_2$ lies in $\{2, 10, 16, 28\}$;

(vi) $N(k) = 8$ for $k_1 \in \{16, 28\}$ and $k_2 \notin \{2, 10, 16, 28\}$ is also even (or vice versa), unless $k = (28, 8)$;

(vii) $N(k) = 9$ for $k_1 = 10$ and $k_2 \notin \{2, 10, 16, 28\}$;

(viii) $N(k) = 11$ for $k = (16, 28)$;

(ix) $N(k) = 12$ for $k_1 = k_2$ even, unless $k_1 = k_2 \in \{2, 4, 6, 8, 10, 16, 28\}$;

(x) $N(k) = 13$ for $k = (4, 4), (6, 6)$ or $(8, 8)$;

(xi) $N(k) = 22$ for $k = (16, 16)$ or $(28, 28)$;

(xii) $N(k) = 27$ for $k = (10, 10)$.
The term “vice versa” in (i), (ii), (iii) and (vi) refers to the obvious fact that $N(k_1, k_2) = N(k_2, k_1)$. All the physical invariants for $k_1 = k_2$ are explicitly given in [14]; all those for $k_1 \neq k_2$ are in [15].

8. Conclusion

In this paper we classify all partition functions for $A_{1,k_1} \oplus A_{1,k_2}$ WZNW theories. The result, given in Sect.2, can be summarized as follows. There are the analogues of the $A$ and $D$ series, 2 or 6 of them for each $k = (k_1, k_2)$, depending on whether or not one of the $k_i$ is odd. If $k_1$ or $k_2$ is 10, 16 or 28, there are also the invariants that can be built up from the $A_1$ exceptionals $E_{10}, E_{16}$ or $E_{28}$, respectively. When $k_1 = k_2$, the conjugations of all these invariants must be included. Finally, there is one additional exceptional at each level $k = (4,4), (6,6), (8,8), (10,10), (2,10), (3,8), (3,28)$ and $(8,28)$. The number of physical invariants for a given $k$ will usually be 2 or 6, but gets as high as 27 (for $k = (10,10)$).

Our main focus however has not been on $A_1 \oplus A_1$, but rather on the arbitrary rank case $g^r = A_1 \oplus \cdots \oplus A_1$, where we have obtained a number of results (see Thms.1-7). The main remaining obstacle to the classification for arbitrary rank $r$ is to find all possible exceptional chiral extensions. To this end, the recent work of Stanev [27] is most interesting, and together with the work in this paper should permit a classification at least for $g^3$ and $g^4$, along with “almost every” level $k$ for each $r > 4$. Rank $r = 3$ is of particular interest, since it should lead to a classification of all $su(2)_k \oplus su(2)_\ell / su(2)_{k+\ell}$ GKO cosets [16]. In this paper we have found that the most difficult levels for the $g^r_k$ classification seem to be those for which some $k_i = 2, 4, 8, 10, 12, 16$ or 28. But we should be able to handle these anomalous cases explicitly, as was done in Sect.3 for $k_i = 2$, or in the proposition in Sect.7.

Some of the results of this paper have already found direct application (see [15] and [16]). More important, much of the techniques developed in this paper can be carried over to other RCFT classifications – e.g. Lemma 2 here permits a significant simplification of Sect.5 of [13]. In fact, thanks to this, it is now possible to rewrite that section of [13] in such a way that that paper finds all physical invariants of $su(3)_k$ (previously, for half the levels $k$ some results from [24] were needed, so for those levels the argument applied to only strongly physical invariants). In other words, the classification for $su(3)_k$, $\forall k$, now requires only (P1)-(P3).

This paper is designed to probe the largely unknown realm of large rank semi-simple physical invariants. We find enormous numbers of physical invariants, but most of these are easily tractible, using for example simple currents. The great hope, albeit one with little hard justification at present, is that for any affine algebras and levels, irregularities really only appear when simultaneously the rank and levels are small; all other invariants can be obtained from these and from generic physical invariants, using the standard constructions. In other words, the hope is that a finite amount of information captures all physical invariants. For example for $su(N)_k$, the “generic” physical invariants would be the $A$ and $D$ series and their conjugations, together with the conformal embeddings at $k = N - 2, N$, and $N + 2$; irregularities have only appeared so far at $(N,k) = (2,10), (2,16), (2,28), (3,9), (3,21), (4,8), (5,5), (6,6), (8,4), (8,10), (9,3)$ and $(16,10)$. This paper is consistent with this vision, but much, much work remains.

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The \( su(2) \) classification yielded the A-D-E problem \[6\]. The \( su(3) \) classification produced the Fermat curve coincidences \[25\]. Both of these hint of a deep, rich structure underlying these problems. It would be very interesting to discover if any further “coincidences” arise, connected to the \( su(2) \oplus su(2) \) classification given here.

Acknowledgments This work is supported in part by the Natural Sciences and Engineering Research Council of Canada. During the course of writing this paper, I have benefitted from conversations with Antoine Coste, Quang Ho-Kim, Philippe Ruelle, Yassen Stanev and Mark Walton; in particular without Prof. Ho-Kim’s programming skills it would have been very difficult to complete this paper. I also appreciate the hospitality shown by the IHES and the Carleton mathematics department.

References

1. Bais, F. A., Bouwknegt, P. G.: A classification of subgroup truncations of the bosonic string. Nucl. Phys. \textbf{B279} 561-570 (1987);
   Schellekens, A. N., Warner, N. P.: Conformal subalgebras of Kac-Moody algebras. Phys. Rev. \textbf{D34} 3092-3096 (1986)
2. Bernard, D.: String characters from Kac-Moody automorphisms. Nucl. Phys. \textbf{B288} 628-648 (1987);
   Ahn, C., Walton, M. A.: Spectra on nonsimply-connected group manifolds. Phys. Lett. \textbf{B223} 343-348 (1989)
3. Bouwknegt, P.: On the construction of modular invariant partition functions. Nucl. Phys. \textbf{B290 [FS20]} 507-526 (1987)
4. Bouwknegt, P., Nahm, W.: Realizations of the exceptional modular invariant \( A_1^{(1)} \) partition functions. Phys. Lett. \textbf{B184} 359-362 (1987)
5. Cappelli, A.: Modular invariant partition functions of superconformal theories. Phys. Lett. \textbf{B185} 82-88 (1987)
6. Cappelli, A., Itzykson, C., Zuber, J.-B.: Modular invariant partition functions in two dimensions. Nucl. Phys. \textbf{B280 [FS18]} 445-465 (1987); The A-D-E classification of \( A_1^{(1)} \) and minimal conformal field theories. Commun. Math. Phys. \textbf{113} 1 (1987);
   Kato, A.: Classification of modular invariant partition functions in two dimensions. Mod. Phys. Lett. \textbf{A2} 585 (1987);
   Gepner, D., Qui, Z.: Modular invariant partition functions for parafermionic theories. Nucl. Phys. \textbf{B285} 423-453 (1987)
7. Cleaver, G. B., Lewellen, D. C.: On Modular invariant partition functions for tensor products of conformal field theories. Phys. Lett. \textbf{B300} 354-360 (1993)
8. Coste, A., Gannon, T.: Galois symmetry in RCFT. Phys. Lett. \textbf{B} (to appear)
9. Degiovanni, P.: Modular invariance with a non simple symmetry algebra. Nucl. Phys. \textbf{B} (Proc. Suppl.) \textbf{5B} 71-86 (1988)
10. Fuchs, J., Klemm, A., Schmidt, M. G., Versteegen, D.: New exceptional \( (A_1^{(1)})^r \) invariants and the associated Gepner models. Int. J. Mod. Phys. \textbf{A7} 2245-2264 (1992)
11. Gannon, T.: WZW commutants, lattices, and level-one partition functions. Nucl. Phys. B396 708-736 (1993)
12. Gannon, T.: Partition functions for heterotic WZW conformal field theories. Nucl. Phys. B402 729-753 (1993)
13. Gannon, T.: The classification of affine SU(3) modular invariant partition functions. Commun. Math. Phys. (to appear)
14. Gannon, T., Ho-Kim, Q.: The low level modular invariant partition functions of rank-two algebras. Int. J. Mod. Phys. A (to appear)
15. Gannon, T., Ho-Kim, Q.: The rank-four heterotic modular invariant partition functions. Preprint, IHES
16. Gannon, T., Walton, M. A.: The classification of GKO modular invariants (work in progress)
17. Gantmacher, F.R.: The theory of matrices Vol. II. New York: Chelsea Publishing Co. 1964
18. Gato-Rivera, B., Schellekens, A. N.: Complete classification of simple current automorphisms. Nucl. Phys. B353 519-537 (1991);
Schellekens, A. N.: Fusion rule automorphisms from integer spin simple currents. Phys. Lett. B244 255-260 (1990)
19. Gato-Rivera, B., Schellekens, A. N.: Complete classification of simple current modular invariants for RCFT’s with a center \((\mathbb{Z}_p)^k\). Commun. Math. Phys. 145 85-121 (1992);
Kreuzer, M., Schellekens, A. N.: Simple currents versus orbifolds with discrete torsion – a complete classification. Nucl. Phys. 411 97-121 (1994)
20. Gepner, D., Witten, E.: String theory on group manifolds. Nucl. Phys. B278 493-549 (1986)
21. Itzykson, C.: Level one Kac-Moody characters and modular invariance. Nucl. Phys. (Proc. Suppl.) 5B 150-165 (1988);
Degiovanni, P.: Z/NZ conformal field theories. Commun. Math. Phys. 127 71-99 (1990)
22. Kač, V. G.: Infinite Dimensional Lie Algebras, 3rd ed. Cambridge: Cambridge University Press 1990
23. Kač, V. G., Wakimoto, M.: Modular and conformal invariance constraints in repre-
sentation theory of affine algebras. Adv. Math. 70 156-236 (1988)
Verstegen, D.: Conformal embeddings, rank-level duality, and exceptional modular invariants. Commun. Math. Phys. 137 567-586 (1991)
Walton, M. A.: Conformal branching rules and modular invariants. Nucl. Phys. B322 775-790 (1989)
24. Moore, G., Seiberg, N.: Naturality in conformal field theory. Nucl. Phys. B313 16-40 (1989)
25. Ruelle, Ph., Thiran, E., Weyers, J.: Implications of an arithmetical symmetry of the commutant for modular invariants. Nucl. Phys. B402 693-708 (1993)
26. Schellekens, A. N., Yankielowicz, S.: Extended chiral algebras and modular invariant partition functions. Nucl. Phys. B327 673-703 (1989)
27. Stanev, Y.: Local extensions of the \(SU(2) \times SU(2)\) conformal current algebra. (in preparation); (private communication)
28. Verstegen, D.: New exceptional modular invariant partition functions for simple Kac-Moody algebras. Nucl. Phys. B346 349-386 (1990)
29. Witten, E.: Non-abelian bosonization in two dimensions. Commun. Math. Phys. 92 455-472 (1984);
Novikov, S. P.: Usp. Mat. Nauk 37 3 (1982)