Comparison between FEM and Equivalent-Circuit Model Simulations of Superconducting Linear Acceleration System for Pellet Injection

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Received: March 30, 2018; Accepted: July 1, 2018; Published: August 1, 2018

Abstract. Two simulation methods, Equivalent-Circuit Model (ECM) simulation and Finite Element Method (FEM) simulation, are proposed for analyzing the time evolution of the shielding current density in a High-Temperature Superconducting (HTS) film that is part of a pellet container moving in an applied magnetic field. In the ECM simulation, Newton’s equation of motion for the pellet container is solved together with the circuit equations equivalent to the governing equation of the shielding current density in the HTS film. On the other hand, it is solved together with the governing equation in the FEM simulation. Two numerical codes are developed on the basis of the ECM/FEM and the performance of the Superconducting Linear Acceleration (SLA) system is investigated by using the two codes. The results of computations show that, even for the case with a single electromagnet, the SLA system has a possibility to accelerate a pellet container up to over 140 m/s.

Keywords: equivalent-circuit model, finite element method, high-temperature superconducting film, pellet injection, power law, Runge-Kutta method

1. Introduction

Pellet injection is considered as a primary fueling scheme for fusion reactors and a pneumatic pipe-gun type injection system has been investigated experimentally throughout the world [1]. However, this injection system can accelerate pellet velocity only up to 1–1.5 km/s and, therefore, can provide hydrogen particles only to the periphery of the burning plasma by shallow penetration and direct access to the core plasma is not possible, especially in the LHD-type helical fusion reactor.

Recently, a Superconducting Linear Acceleration (SLA) system has been proposed as an alternative to the pneumatic system. Yanagi et al. [2] roughly estimated its acceleration performance using a superconducting bulk. As a result, it was found that a pellet velocity up to 5–10 km/s might be achieved with the SLA system. On the other hand, in order to
numerically simulate the SLA system, an initial-boundary-value coupled problem, which consists of Newton’s equation of motion for a pellet container and the governing equation of the shielding current density in a High-Temperature Superconducting (HTS) film, needs to be solved.

For the purpose of analyzing the shielding current density in High-Temperature Superconductors (HTSs), either the Finite Element Method (FEM) or the Equivalent-Circuit Model (ECM) has been so far used. For example, the FEM is applied to the magnetic shielding analysis [3], the shielding current analysis [4–6], and the magnetic levitation force analysis [7], whereas the ECM is used for analyzing hysteresis/resistive losses [8], HTS current leads [9], and HTS flux flow transistors [10]. However, the acceleration performance of the SLA has never been investigated by either the FEM or the ECM.

The purpose of the present study is to develop the methods for calculating the time evolution of the shielding current density in an HTS film of the SLA system and to numerically investigate the acceleration performance of the SLA system by using the proposed methods.

2. Model of SLA System

A schematic view of the SLA system is shown in Fig. 1. An HTS film is used for accelerating a pellet container by which a hydrogen pellet is carried. While the container moves in the magnetic field generated by an electromagnet, the shielding current density flows in the HTS film. The pellet container is accelerated due to an interaction between the shielding current density and the applied magnetic field. Incidentally, the pellet container is levitated by means of permanent magnets and, hence, it moves only along the direction perpendicular to the HTS film.

Throughout the present paper, the SLA system is assumed to be axially symmetric. By taking the symmetry axis of the electromagnet as $z$-axis and choosing its center of gravity as the origin, we use the cylindrical coordinate system $(O : e_r, e_\theta, e_z)$. In the present study, an electromagnet is assumed to be an infinitesimal-thickness cylindrical sheet of radius $R_c$ and...
height $H_c$, whereas an HTS film is assumed to be a disk shape of thickness $b$ and radius $R_F$ (see Fig. 1). Furthermore, the following coil current $I_{\text{coil}}$ is assumed to flow in the sheet:

$$I_{\text{coil}}(t,Z) = \begin{cases} \alpha t & (0 \leq Z \leq Z_{\text{limit}}) \\ 0 & \text{(otherwise)} \end{cases}$$

(1)

where $\alpha$ is an increasing rate of an electric current and $Z$ denotes $z$-coordinate of the HTS film. As is apparent from (1), the coil current is applied only when $0 \leq Z \leq Z_{\text{limit}}$ is satisfied. In other words, an acceleration region can be expressed as $0 \leq z \leq Z_{\text{limit}}$. Besides, the effect of air resistance is neglected because the SLA experiment is generally executed under the extremely low pressure.

Under the above assumptions, the shielding current density $j$ and the electric field $E$ in the HTS film can be given by $j = j_c e_\theta$ and $E = E_c e_\theta$. Moreover, the magnetic flux density $B$ generated by the electromagnet can be expressed as $B = B_r e_r + B_z e_z$.

### 3. Numerical Methods

For the purpose of numerically investigating the acceleration performance of the SLA system, we have to solve an initial-boundary-value coupled problem which are composed of the governing equation for the shielding current density and Newton’s equation of motion for a pellet container. In the present study, the coupled problem is solved by means of two different methods: Equivalent-Circuit Model (ECM) and Finite Element Method (FEM).

In HTS films, the electric field and the shielding current density are closely related to each other through the $J$-$E$ constitutive relation. As the relation, we assume the following power law [4, 5, 11, 12]:

$$E_\theta = E_C \left( \frac{|j_\theta|}{j_c} \right)^N \text{sgn}(j_\theta),$$

(2)

where $j_c$ and $E_C$ denote the critical current density and the critical electric field, respectively, and $N$ is a positive constant. Although (2) is directly incorporated into the FEM simulation, the $I$-$V$ curve derived from (2) is implemented in the ECM simulation.

#### 3.1. ECM simulation

Since the SLA system is assumed to be axially symmetric, the HTS film can be approximated as a set of $n$ current loops, $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, with rectangular cross sections (see Fig. 2). In the following, $I_j$ and $V_j$ are an electric current flowing in $\Lambda_j$ and a voltage induced in $\Lambda_j$, respectively. In addition, the rectangular cross section of $\Lambda_j$ is expressed as $|r - r_j| \leq \Delta r/2$ and $|z - Z| \leq b/2$. Here, $\Delta r$ and $r_j$ denote the width of the cross section and $r$-coordinate of the cross-section center, respectively. Also, the mutual inductance by the electromagnet on $\Lambda_j$ is denoted by $M_j(Z)$.

As expected from (2), there exists the relation between the electric current $I_j$ and the induced voltage $V_j$. The substitution of $V_j = (2\pi r_j)E_\theta$ and $I_j = (b\Delta r)j_\theta$ into (2) yields

$$V_j = V_C j \left( \frac{|I_j|}{I_c} \right)^N \text{sgn}(I_j),$$

(3)
Figure 2: A cross section of the equivalent-circuit model.

where $V_{Cj} = (2\pi r_j)E_C$ and $I_C = (b\Delta r)j_C$. Thus, the $I$-$V$ characteristics for $\Lambda_j$ are given by (3).

Under the above assumptions, Faraday’s law becomes equivalent to the following circuit equations:

$$L \frac{dI}{dt} = -\left\{ M(Z) \frac{dI}{dt} + M'(Z)vI + V \right\},$$

(4)

where $I \equiv (I_1, I_2, \cdots, I_n)^T$, $V \equiv (V_1, V_2, \cdots, V_n)^T$ and $M(Z) \equiv (M_1(Z), M_2(Z), \cdots, M_n(Z))^T$. In addition, $v \equiv dZ/dt$ denotes a velocity of the current loops, i.e., that of the HTS film. Also, $L$ is an inductance matrix, the $(i, j)$ entry of which is an inductance by $\Lambda_j$ on $\Lambda_i$. On the other hand, the dynamic motion of the current loops is governed by Newton’s equation of motion:

$$m \frac{d^2Z}{dt^2} = -2\pi \sum_{j=1}^n r_j B_v(r_j, Z, t) I_j,$$

(5)

where $m$ is a total mass of the pellet container and the HTS film. Note that the right-hand side of (5) denotes the Lorentz force acting on all the current loops. Incidentally, (4) can be also derived by applying the Petrov-Galerkin approach to Faraday’s law. The explicit function spaces containing its test and trial functions are given in Appendix A.

Rewriting (4) and (5), we get

$$\frac{d}{dt} \begin{pmatrix} I \\ v \\ Z \end{pmatrix} = \begin{pmatrix} -L^{-1} \left\{ M(Z) \frac{dI}{dt} + M'(Z)vI + V \right\} \\ -\frac{2\pi}{m} \sum_{j=1}^n r_j B_v(r_j, Z, t) I_j \\ 0 \end{pmatrix}.$$

(6)

By solving (6) with the following initial condition, $(I^T \ v \ Z) = (0^T \ 0 \ Z_0)$ at $t = 0$, we can determine the time evolution of $I$, $v$ and $Z$. Here, $Z_0$ is an initial position of the HTS film.
3.2. FEM simulation

If the thin-plate approximation is assumed, there exists a scalar function $T(r,t)$ such that

$$j_0 = -\frac{2}{b} \frac{\partial T}{\partial r},$$

and its time evolution is governed by the following integro-differential equation [3–6]:

$$\mu_0 \frac{\partial}{\partial t} (\hat{W}T) = -\frac{1}{r} \frac{\partial}{\partial r} (rE_0) - \frac{\partial}{\partial t} \langle B_z \rangle. \quad (7)$$

Here, $\langle \rangle$ denotes an average operator over the thickness of the HTS film and the operator $\hat{W}$ is defined by

$$\hat{W}T = \frac{2}{b} T(r,t) + \int_0^{R_F} Q(r,r')T(r',t) \, r'dr',$$

where the integral kernel $Q(r,r')$ is given by

$$Q(r,r') = -\frac{2}{\pi b^2 \sqrt{rr'}} \sum_{p=0}^1 (-1)^p k_p K(k_p).$$

Here, $K(k_p)$ is the complete elliptic integral of the 1st kind and its parameter $k_p$ is defined by

$$k_p^2 = \frac{4rr'}{(r+r')^2 + pb^2} \quad (p = 0, 1).$$

The dynamic motion of the pellet container is governed by Newton’s equation of motion:

$$m \frac{d^2Z}{dt^2} = 4\pi \int_0^{R_F} \frac{\partial T}{\partial r} \langle B_z \rangle \, rdr. \quad (8)$$

Note that the right-hand side of (8) is the Lorentz force acting on the film.

The initial and boundary conditions to (7) and (8) is assumed as follows: $(T \ v \ Z) = (0 \ 0 \ Z_0)$ at $t = 0$ and $T = 0$ at $r = R_F$. By solving (7) and (8) together with the initial and boundary conditions, we can simultaneously determine both the time evolution of the shielding current density in the HTS film and the dynamic motion of the pellet container.

After applying the FEM with $n$ elements to the initial-boundary-value coupled problem of (7) and (8), we get the following ordinary differential system:

$$\frac{d}{dt} \begin{pmatrix} T \\ v \\ Z \end{pmatrix} = \begin{pmatrix} -W^{-1}[b_t(Z,t) + v \ b_z(Z) + e(T)] \\ b_z(Z)T \ \\ v \end{pmatrix}. \quad (9)$$

Here, $W$ is an FEM matrix corresponding to $\hat{W}$. In addition, nodal vectors, $T$ and $e(T)$, correspond to $T$ and $E_0$, respectively, whereas $b_z(Z)$, $b_r(Z)$ and $b_t(Z,t)$ are vectors originating from $\langle B_z \rangle$, $\langle B_r \rangle$ and $\langle \frac{\partial B_r}{\partial t} \rangle$, respectively. Incidentally, $e(T)$ can be evaluated by using (2). Throughout the present paper, an interval $I_k \equiv [r_k - \Delta r/2, r_k + \Delta r/2]$ is called the $k$th
element. As the type of finite elements, linear elements are adopted for $I_2, I_3, \cdots, I_n$. In contrast, a different-type element is used for the 1st element $I_1$. Specifically, we assume

$$T(r, t) = T_1(t) \left[ 1 - \left( \frac{r}{\Delta r} \right)^2 \right] + T_2(t) \left( \frac{r}{\Delta r} \right)^2 \quad \text{for } r \in I_1,$$

where $T_k(t) \equiv T(r_k, t)$. Equation (10) is assumed so as to satisfy $j_0 = 0$ at $r = 0$.

If the initial-value problem of (9) is numerically solved, the time evolution of the shielding current density $j$ and the film position $Z$ can be determined.

4. Numerical Results

As mentioned above, an initial-value problem of an ordinary differential system is obtained with either the ECM or the FEM. In the present study, the resulting initial-value problem is solved by means of the 5th order Runge-Kutta method. For the purpose of suppressing the numerical instability, Fehlberg’s adaptive step-size control algorithm [13] is also implemented to the Runge-Kutta method.

On the basis of the ECM/FEM, two numerical codes have been developed for analyzing the time evolution of the shielding current density and the dynamic motion of the pellet container. In this section, we investigate the acceleration performance of the SLA system by using the two codes. Throughout the present study, the geometrical and physical parameters are fixed as follows: $R_c = 5 \text{ cm}$, $H_c = 10 \text{ cm}$, $b = 1 \text{ mm}$, $R_F = 4 \text{ cm}$, $m = 10 \text{ g}$, $Z_0 = 1 \text{ mm}$, $E_C = 1 \text{ mV/m}$, $j_C = 1 \text{ MA/cm}^2$ and $N = 20$. Also, the number of current loops and the number of finite elements are assumed as $n = 500$. The reason why we assume $n = 500$ is given in Appendix B.
Figure 4: Time dependences of the coil current $I_{\text{coil}}$ and the film position $Z$ for the case with $\alpha = 20 \, \text{ka/ms}$. In this figure, the green and blue curves show the time dependence of $I_{\text{coil}}$ and that of $Z$, respectively.

4.1. ECM simulation

In order to determine the acceleration region, $0 \leq z \leq Z_{\text{limit}}$, let us first investigate the $Z$-dependence of the mutual inductance vector $M$. To this end, the maximum component\(^1\) $M_n$ of $M$ is evaluated as a function of $Z$ and is depicted in Fig. 3. Apparently, $M_n$ decreases monotonously with $Z$ until it almost vanishes for $Z \gtrsim 30 \, \text{cm}$. Hence, for $Z \gtrsim 30 \, \text{cm}$, the shielding currents, $I_1, I_2, \cdots, I_n$, are hardly influenced by the coil current $I_{\text{coil}}$. For this reason, the value of $Z_{\text{limit}}$ is assumed as $Z_{\text{limit}} = 30 \, \text{cm}$. Typical time dependences of the coil current $I_{\text{coil}}$ and the film position $Z$ are depicted in Fig. 4.

Next, both the total shielding current $I$ and the velocity $v$ are evaluated as functions of time. The results of computations are shown in Figs. 5(a) and 5(b). The magnitude of $I$ develops with time for $t \lesssim 5 \, \text{ms}$. After that, $|I|$ rapidly decreases with time until it almost vanishes for $t \gtrsim 7 \, \text{ms}$. On the other hand, the HTS film is accelerated only while the total shielding current does not vanish. Furthermore, its velocity reaches up to around $117 \, \text{m/s}$ for $t \gtrsim 7 \, \text{ms}$. Hereafter, a velocity at the time when the inequality $Z \gtrsim Z_{\text{limit}}$ is first satisfied is called a final velocity and is denoted by $v_f$.

Finally, we investigate how the increasing rate $\alpha$ of the coil current affects the acceleration performance of the SLA system. For this purpose, the final velocity $v_f$ is calculated for various values of $\alpha$ and is depicted in Fig. 6. We see from this figure that $v_f$ increases monotonously with $\alpha$. Especially for $\alpha = 30 \, \text{ka/ms}$, it amounts to $143 \, \text{m/s}$.

From the above results, we conclude that, even if a single electromagnet is used, the pellet container can be accelerated efficiently. Therefore, if multiple electromagnets were implemented to the SLA system, a hydrogen pellet might be provided directly to core plasma in the LHD.

\(^1\)Because of $r_1 < r_2 < \cdots < r_n$, the inequality $M_1 < M_2 < \cdots < M_n$ can be easily proved. Hence, $M_n$ is the maximum component of $M$.\[215\]
Figure 5: Time dependences of (a) the total shielding current $I$ and (b) the velocity $v$ for the case with $\alpha = 20$ kA/ms. In these figures, the blue and red curves are obtained by the ECM code and by the FEM code, respectively.

4.2. FEM simulation

Let us compare the results obtained by the FEM code with those by the ECM code. The time variation of the total shielding current $I$ and that of the velocity $v$ are first determined by using the FEM code. The results of computations are also depicted in Figs. 5(a) and 5(b). These figures indicate that almost the same but slightly different results are obtained by both codes. Such slightly different results in Fig. 5 can be attributed to the following two factors:

- The FEM is among the Galerkin approach, whereas the ECM is a kind of the Petrov-Galerkin approach (see Appendix A).
- The shape functions for the ECM are totally different from those for the FEM (see Section 3.2 and Appendix A).
Next, the dependence of the final velocity $v_f$ on $\alpha$ is determined by means of the FEM code, and it is also depicted in Fig. 6. This figure also indicates that the values of $v_f$ obtained by the FEM code are in fair agreement with those by the ECM code. Specifically, the relative error between the values of $v_f$ obtained by both codes is always less than 2.3%. This result suggests the validity of the simulated results by both codes.

Finally, we compare the computational cost of the ECM with that of the FEM. To this end, the CPU times are measured while the simulation of the SLA system is performed from $t = 0$ to $t = 20 \text{ ms}$. For the case with $\alpha = 20 \text{ kA/ms}$, the CPU time of the FEM is about 1.7 times as great as that of the ECM. This speed difference is caused by the adaptive step-size control in the Runge-Kutta method (see Appendix C).

5. Conclusion

We have proposed two simulation methods, ECM simulation and FEM simulation, for solving an initial-boundary-value coupled problem that is composed of a governing equation of the shielding current density in an HTS film and Newton’s equation of motion for a pellet container. Even if the coupled problem is approximated by either of the proposed methods, it reduces to an initial-value problem of an ordinary differential system that can be numerically solved by the Runge-Kutta method.

On the basis of the ECM/FEM, we have developed two numerical codes whose numerical results agree well with each other. By using the two codes, we have investigated the acceleration performance of the SLA system. Conclusions obtained in the present study are summarized as follows:

- Although both the ECM and the FEM are useful tools to simulate the SLA system,
the ECM is faster than the FEM. Moreover, the ECM is easy to understand physically because an inductance matrix and a mutual inductance vector appear in the governing equations. Consequently, we can conclude that the ECM is recommended for analyzing the SLA system.

- The mutual inductance by the coil current on the current loops rapidly diminishes as the pellet container separates form the electromagnet. Therefore, the coil current has only to be applied only when the mutual inductance does not vanish.
- Even for the case where a single electromagnet is used in the SLA system, the pellet container can be accelerated efficiently.

Acknowledgement

The authors would like to express their gratitude to anonymous reviewers whose comments improved the readability of this paper considerably. This work was supported in part by Japan Society for the Promotion of Science under a Grant-in-Aid for Scientific Research (C) No. 15K05926. A part of this work was also carried out with the support and under the auspices of the NIFS Collaboration Research program (NIFS16KNXN341, NIFS16KECA047, NIFS17KNTS051).

Appendix A. ECM as Petrov-Galerkin Approach

As mentioned in Section 3, the ECM is derived from the integral form of Faraday’s law. In this appendix, we show that (4) can be also obtained by applying the Petrov-Galerkin approach to Faraday’s law. Throughout this appendix, \(B_S\) is a magnetic flux density generated by the shielding current density \(j = j_0 e_\theta\).

First of all, we define the inner product of two functions, \(f(r)\) and \(g(r)\), as

\[ \langle f, g \rangle \equiv 2\pi \int_0^{R_T} f(r)g(r) r dr. \]

We further define the functions, \(\phi_1(r), \phi_2(r), \ldots, \phi_n(r)\), by

\[ \phi_i(r) \equiv \begin{cases} 1 & (0 \leq r \leq r_i) \\ 0 & (\text{otherwise}) \end{cases} \quad (i = 1, 2, \ldots, n). \]

Then, the inner product \(\langle \phi_i, g \rangle\) can be rewritten as the following surface integral:

\[ \langle \phi_i, g \rangle = \iint_{S_i} g(r) dS, \]  

(11)

where \(S_i \equiv \{(r, z) : 0 \leq r \leq r_i, z = Z(t)\}\) is a circular domain on the plane \(z = Z(t)\). Hence, \(\Phi_i \equiv \langle \phi_i, B \cdot e_z \rangle\) and \(\Phi_{Si} \equiv \langle \phi_i, B_S \cdot e_z \rangle\) denote the magnetic flux, through \(S_i\), generated
by the electromagnet and that by the shielding current density, respectively. Moreover, by
using Stokes’ theorem, we get
\[
\langle \phi_i, (\nabla \times E) \cdot e_z \rangle = \int_{S_i} (\nabla \times E) \cdot e_z \, dS = \oint_{\partial S_i} E \cdot e_\theta \, ds = V_i,
\]
(12)
where \(s\) is an arclength along the boundary \(\partial S_i\) of \(S_i\).

The time evolution of the shielding current density \(j_0\) is governed by Faraday’s law:
\[
\frac{\partial (B \cdot e_z)}{\partial t} + \frac{\partial (B \cdot e_z)}{\partial t} + (\nabla \times E) \cdot e_z = 0.
\]
(13)
After taking the inner product of both sides in (13) with \(\phi_i(r)\) and, subsequently, substituting
(12) into the resulting equation, we get
\[
\frac{d\Phi_{S_i}}{dt} = -\frac{d\Phi_i}{dt} - V_i.
\]
(14)

In the ECM, the shielding current density is assumed to be homogeneous in the cross
section of \(\Lambda_i\) with a plane \(\theta = \text{const}\). In other words, \(j_0\) is assumed as
\[
j_0 = \sum_{i=1}^{n} \frac{I_i}{b \Delta r} \psi_i(r),
\]
(15)
where \(\psi_i(r)\) is given by
\[
\psi_i(r) = \begin{cases} 
1 & (|r - r_i| \leq \Delta r) \\
0 & \text{(otherwise)} 
\end{cases} \quad (i = 1, 2, \cdots, n).
\]

By substituting (15) into (14), we obtain the equivalent-circuit equation (4). In the above
derivation of (4), (15) can be regarded as a trial function, whereas \(\phi_i(r)\)\(\big|_{i=1}^{n}\) are treated
as test functions. Note that, although the test and trial functions are contained in \(W_\phi \equiv \text{span}(\phi_1, \phi_2, \cdots, \phi_n)\) and \(W_\psi \equiv \text{span}(\psi_1, \psi_2, \cdots, \psi_n)\), respectively, \(W_\phi \neq W_\psi\) is satisfied. In
other words, the test and trial functions do not belong to the same function space. Hence, the
above derivation of the equivalent-circuit equation is based on the Petrov-Galerkin approach.

**Appendix B. Convergence of Velocity**

As mentioned in Section 4, we adopt \(n = 500\) as the number of current loops or as the
number of finite elements. In this appendix, we show the reason why \(n\) is fixed as \(n = 500\)
especially for the ECM.

Once the value of \(n\) is specified, a \(v-t\) curve can be uniquely determined by using the
ECM. We first investigate the convergence of \(v-t\) curves with respect to \(n\). To this end, time-
dependences of \(v\) are numerically determined for various values of \(n\) and are depicted in Fig.
7. This figure indicates that \(v-t\) curves tend to converge with an increase in \(n\).
Figure 7: Time dependences of the velocity $v$ for the case with $a = 20 \text{ kA/ms}$. The inset of this figure shows the dependence of the relative error $\epsilon$ on the number $n$ of current loops for the same case.

For the purpose of quantitatively investigating convergence property of $v$–$t$ curves, we define a relative error by

$$\epsilon \equiv \frac{\|v_n - v_{10}\|}{\|v_{10}\|},$$

where $v_n(t)$ is a velocity for the case of $n$ current loops. Moreover, for the definition of $\|\|$,

the following 2-norm is adopted:

$$\|v\|^2 = \int_0^{t_{\text{max}}} [v(t)]^2 \, dt,$$

where $t_{\text{max}} = 20 \text{ ms}$.

Next, the dependence of the relative error on $n$ is investigated and it is depicted in the inset of Fig. 7. For the case with $10^2 \leq n \leq 10^4$, the relative error vibrates with an increase in $n$ before it tends to converge. In addition, the relative error is always less than 2%. Especially for the case with $n = 500$, it is less than 0.7%. Hence, the sufficient accuracy is achieved for this case. From the above reason, $n$ is assumed as $n = 500$ in the present study.

Appendix C. Computational Cost for Runge-Kutta Method with Adaptive Step-Size Control

As mentioned in Section 4, the ECM is about 1.7 times faster than the FEM. The difference in speed of both methods is caused by Fehlberg’s adaptive step-size control in the Runge-Kutta method. At each time step of the Runge-Kutta method with the adaptive step-size control, the subdivision of the time step size is repeated until the predetermined accuracy is achieved. In this appendix, the number of the subdivision is called a subdivision count.
Subdivision counts both for the ECM and for the FEM are numerically determined as functions of time step and they are depicted in Fig. 8. In addition, the ratio of the subdivision count $N_{\text{FEM}}$ for the FEM to that for the ECM is calculated as a function of time step. The results of computations are shown in the inset of Fig. 8. We see from this figure that $N_{\text{FEM}}$ is even larger than $N_{\text{ECM}}$ at almost every time step. Hence, the difference between the CPU time for the ECM and that for the FEM must be attributed to the total subdivision counts, which is a summation of subdivision counts over all time steps.

Figure 8: Dependence of the subdivision counts on time step for the case with $\alpha=20 \text{ kA/ms}$. The inset shows the dependence of the ratio $N_{\text{FEM}}/N_{\text{ECM}}$ on time step for the same case.

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