MAJORITY CHOOSABILITY OF DIGRAPHS

MARcin ANHOLCER, BARTLOMIΕJ BOSEK, AND JAROSŁAw GRYTCZUK

Dedicated to Michał Karoński on the occasion of his 70th birthday.

Abstract. A majority coloring of a digraph is a coloring of its vertices such that for each vertex $v$, at most half of the out-neighbors of $v$ have the same color as $v$. A digraph $D$ is majority $k$-choosable if for any assignment of lists of colors of size $k$ to the vertices there is a majority coloring of $D$ from these lists. We prove that every digraph is majority 4-choosable. This gives a positive answer to a question posed recently by Kreutzer, Oum, Seymour, van der Zypen, and Wood in [3]. We obtain this result as a consequence of a more general theorem, in which majority condition is profitably extended. For instance, the theorem implies also that every digraph has a coloring from arbitrary lists of size three, in which at most $\frac{2}{3}$ of the out-neighbors of any vertex share its color. This solves another problem posed in [3], and supports an intriguing conjecture stating that every digraph is majority 3-colorable.

1. Introduction

Let $D$ be a directed graph. Let $d^+(v)$ denote the number of out-neighbors of vertex $v$. A coloring $c$ of the vertices of $D$ is called majority coloring if for every vertex $v$ the number of its out-neighbors in color $c(v)$ is at most $\frac{1}{2}d^+(v)$. This concept was introduced recently by van der Zypen [7], in connection to neural networks, and studied by Kreutzer, Oum, Seymour, van der Zypen, and Wood in [3]. It is proved there, among other results, that every digraph is majority 4-colorable. The proof is very simple: first, notice that every digraph with no directed cycles is majority 2-colorable (just apply greedy coloring), next, split the edges of a given digraph into two acyclic digraphs, and take the product coloring. It is conjectured in [3] that actually three colors are sufficient for majority coloring of any digraph. This would be best possible since a majority coloring of an odd directed cycle must be a proper coloring of the underlying undirected graph.

Another interesting problem posed in [3] concerns list version of the majority coloring. Suppose that each vertex $v$ of a digraph $D$ is assigned with a list of colors $L(v)$. Then $D$ is majority colorable from these lists if there is a majority coloring $c$ of $D$ with $c(v) \in L(v)$. If $D$ is majority colorable from any lists of size $k$, then we say that $D$ is majority $k$-choosable. The authors of [3] asked if there is a finite number $k$ such that every digraph is majority $k$-choosable. We answer this question in the affirmative by proving a more general theorem which implies that actually every digraph is majority 4-choosable. As another consequence we infer that every digraph is 3-choosable so that at most $\frac{2}{3}d^+(v)$ of the out-neighbors of any vertex $v$ have the same color as $v$. This solves another problem posed in [3], and extends a result of Seymour from [5], asserting that every digraph has 3-coloring in which at least one out-neighbor of each vertex (of positive out-degree) is colored differently.
There are many variants of majority coloring that may be studied in a variety of contexts (see [3]). Perhaps our approach might be useful in some of these situations. We shall discuss briefly these issues in the final section.

2. The results

Our main result reads as follows.

**Theorem 1.** Let $D$ be a directed graph. Suppose that each vertex $v$ is assigned with a list $L(v)$ of four colors. Suppose further that each color $x$ in $L(v)$ is assigned with a real number $r_v(x)$, the rank of color $x$ in $L(v)$. Assume that for every vertex $v$, the color ranks $r_v(x)$ satisfy the following condition:

\[ \sum_{x \in L(v)} r_v(x) \geq 2d^+(v). \]

Then there is a vertex coloring of $D$ from lists $L(v)$ satisfying the following constraint: If $x$ is a color assigned to $v$, then the number of out-neighbors of $v$ in color $x$ is at most $r_v(x)$.

**Proof.** Let us remark first that we do not impose any restrictions on color ranks, except condition (3). These ranks may be positive, negative, or zero. If $r_v(x) = 0$ and $v$ is colored with $x$, then to satisfy the assertion of the theorem, none of the out-neighbors of $v$ may be colored with $x$. If $r_v(x)$ is strictly negative, then actually $v$ cannot be colored with $x$ at all (no set may have negative cardinality).

The proof goes by induction on the number of vertices in $D$. It is not hard to check that the theorem is true for one-vertex digraph. Indeed, by condition (2), at least one color rank in the list must be non-negative, and we may use it to color the only vertex in the digraph. So, let $n \geq 2$, and assume that the assertion of the theorem is true for all digraphs with at most $n - 1$ vertices. Let $D$ be a digraph on $n$ vertices satisfying the assumptions of the theorem, and let $v$ be any vertex of $D$. Consider a new digraph $D'$ obtained by deleting vertex $v$ with color ranks modified as follows. Let $a$ and $b$ be the two colors with highest ranks, $r_v(a)$ and $r_v(b)$, in the list $L(v)$. For each in-coming neighbor $u$ of vertex $v$, decrease the ranks $r_u(a)$ and $r_u(b)$ by one, provided these colors are contained in the list $L(u)$. All the remaining color ranks in these or other lists are left unchanged.

We claim that digraph $D'$ with modified color ranks still satisfies condition (2). Indeed, for each in-coming neighbor $u$ of $v$, the left hand side of (2) decreased by at most two, while the right-hand side of (2) decreased by exactly two (since the out-degree $d^+(u)$ decreased by exactly one). So, by the inductive assumption there is a coloring of $D'$ satisfying the assertion of the theorem.

We now extend this coloring to the deleted vertex $v$ in the following way. First notice that

\[ r_v(a) + r_v(b) \geq d^+(v). \]

Indeed, by the maximality of ranks of colors $a$ and $b$ in the list $L(v)$, the inequality $r_v(a) + r_v(b) < d^+(v)$ would imply $\sum_{x \in L(v)} r_v(x) < 2d^+(v)$, contrary to the assumption. Let $n_a$ and $n_b$ denote the number of out-neighbors of $v$ colored with colors $a$ and $b$, respectively. Obviously, $n_a + n_b \leq d^+(v)$. Hence, by (1), at least one of the following inequalities must be satisfied:

\[ r_v(a) \geq n_a \quad \text{or} \quad r_v(b) \geq n_b. \]

We chose a color whose rank satisfies one of these inequalities, and assign that color to $v$. 

We claim that the extended coloring satisfies the assertion of the theorem. First, let \( u \) be arbitrary in-coming neighbor of \( v \). Let \( x \) denote the color assigned to \( u \) in coloring of \( D' \). If \( x \) is one of the colors \( a \) or \( b \), then the number of out-neighbors of \( u \) in \( D' \) colored with \( x \) is at most \( r_u(x) - 1 \), by inductive assumption. Thus, their number in \( D \) after coloring the vertex \( v \) is still bounded by \( r_u(x) \). If \( x \) is neither equal to \( a \) nor to \( b \), then the constraint is fulfilled even more. If \( u \) is an arbitrary out-neighbor of \( v \), or any other vertex of \( D' \), then the corresponding constraint holds by induction, since out-neighborhoods and color ranks for such vertices remained unchanged in \( D' \). Finally, for the vertex \( v \) we have chosen color \( a \) or \( b \) so that the corresponding inequality of (2) is satisfied. This completes the proof.

We obtain now easily the aforementioned consequences for majority choosability of digraphs.

**Corollary 1.** Every digraph is majority 4-choosable.

**Proof.** Put \( r_v(x) = \frac{1}{2}d^+(v) \) for each vertex \( v \) and for every color \( x \) from its list \( L(v) \), and apply the theorem. \( \square \)

**Corollary 2.** Let \( D \) be a digraph with color lists of size three assigned to the vertices. Then there is a coloring from these lists such that for each vertex \( v \), at most \( \frac{2}{3} \) of its out-neighbors have the color of \( v \).

**Proof.** Let \( 0 < \varepsilon < \frac{1}{3} \) be a real number. Let \( v \) be a vertex in \( D \), and let \( L(v) \) denote its list with three colors. For each color \( x \) in \( L(v) \) assign the rank \( r_v(x) = \frac{2}{3}d^+(v) + \varepsilon \). Now, add a new fictitious color \( f \) with the rank \( r_v(f) = -3\varepsilon \) to each list \( L(v) \). The assertion of the corollary follows now directly from Theorem 1. \( \square \)

### 3. Discussion

There are many variants of majority coloring that may be studied for various combinatorial structures (see [3]). For instance, in a multi-color version considered in [3], the majority constraint is strengthened to \( \frac{1}{k}d^+(v) \), where \( k \geq 2 \) is a fixed integer. It is easy to see that \( k \) colors are sufficient for acyclic digraphs, and thus \( k^2 \) colors suffice for arbitrary digraph (by product coloring). It is conjectured in [3] that \( k + 1 \) colors are actually enough. As noted by David Wood (personal communication), the proof of Theorem 1 can be easily extended to the multi-color setting, however, it only gives the same quadratic bound in the list version of the problem.

The situation looks much simpler for undirected graphs. An old result of Lovász [1] asserts that every graph is majority 2-colorable, and more generally, it is \( k \)-colorable so that at most \( 1/k \) neighbors of each vertex share its color, for every \( k \geq 2 \). The proof is very simple: just take a coloring that minimizes the total number of monochromatic edges. The same argument works in the list version, and after slight modification it gives a result similar to Theorem 1 (with color ranks in each list summing up to at least the degree of the corresponding vertex).

Majority coloring may be studied for infinite graphs as well (see [1]). For undirected graphs it is known as the problem of *unfriendly partitions* (see [2]). As proved by Shelah and Milner [6], every infinite graph is majority 3-colorable, but there are graphs on uncountably many vertices that are not majority 2-colorable. Whether every countably infinite graph has a majority 2-coloring remains a mystery. Perhaps it would be interesting to consider similar questions for infinite directed graphs.
We conclude the paper with the following strengthening of the majority coloring conjecture from [3].

**Conjecture 1.** Every digraph is majority 3-choosable.

**References**

[1] R. Aharoni, E. C. Milner, K. Prikry, Unfriendly partitions of a graph. J. Combin. Theory Ser. B 50 (1990), no. 1, 1-10.
[2] R. Cowan, W. Emerson, Unfriendly Partitions. [http://www.openproblemgarden.org/](http://www.openproblemgarden.org/)
[3] S. Kreutzer, S. Oum, P. Seymour, D. van der Zypen, D. R. Wood, Majority Closuring of Digraphs, arXiv:1608.03040.
[4] L. Lovász, On decomposition of graphs, Studia Sci. Math. Hungar, 1966, 237-238.
[5] P. Seymour, On the two-colouring of hypergraphs, Quarterly J. Math, 25 1974, 303-311.
[6] S. Shelah, E.C. Milner, Graphs with no unfriendly partitions. A tribute to Paul Erdős, 373–384, Cambridge Univ. Press, Cambridge, 1990.
[7] D. van der Zypen, Majority coloring for directed graphs, 2016. MathOverflow [http://mathoverflow.net/](http://mathoverflow.net/).