REFINEMENT OF SEMINORM AND NUMERICAL RADIUS INEQUALITIES OF SEMI-HILBERTIAN SPACE OPERATORS

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ABSTRACT. We give new inequalities for $A$-operator seminorm and $A$-numerical radius of semi-Hilbertian space operators and show that the inequalities obtained here generalize and improve on the existing ones. Considering a complex Hilbert space $H$ and a non-zero positive bounded linear operator $A$ on $H$, we show with among other seminorm inequalities, if $S, T, X \in B_A(H)$, i.e., if $A$-adjoint of $S, T, X$ exist then

$$2\|S^{T_A}XT\|_A \leq \|SS^{T_A}X + XTT^{T_A}\|_A.$$ 

Further, we prove that if $T \in B_A(H)$ then

$$\frac{1}{4}\|T^{T_A}T + TT^{T_A}\|_A \leq \frac{1}{8}\left(\|T + T^{T_A}\|_A^2 + \|T - T^{T_A}\|_A^2\right), \text{ and}$$

$$\frac{1}{8}\left(\|T + T^{T_A}\|_A^2 + \|T - T^{T_A}\|_A^2\right) + \frac{1}{8}c_A^2(T + T^{T_A}) + \frac{1}{8}c_A^2(T - T^{T_A}) \leq w_A^2(T).$$

Here $w_A(\cdot), c_A(\cdot)$ and $\|\|_A$ denote $A$-numerical radius, $A$-Crawford number and $A$-operator seminorm, respectively.

1. INTRODUCTION

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators acting on a complex Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ is the norm induced from the inner product $\langle \cdot, \cdot \rangle$. For $T \in B(H)$, let $\|T\|$, $w(T)$ and $c(T)$ denote the operator norm, the numerical radius and the Crawford number of $T$, respectively. Note that

$$w(T) = \sup_{x \in H, \|x\|=1} |\langle Tx, x \rangle| \text{ and } c(T) = \inf_{x \in H, \|x\|=1} |\langle Tx, x \rangle|.$$ 

The range space and the null space of $T$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. Let $T^*$ be the adjoint of $T$ and $|T| = (T^*T)^{1/2}$. We reserve the letter $A$ for a non-zero positive bounded linear operator on $H$ and so $\langle Ax, x \rangle \geq 0, \forall x \in H$. Consider the semi-inner product $\langle \cdot, \cdot \rangle_A$ on $H$ induced by $A$, namely,

$$\langle x, y \rangle_A = \langle Ax, y \rangle, \forall x, y \in H.$$ 

The semi-inner product $\langle \cdot, \cdot \rangle_A$ induces a seminorm $\| \cdot \|_A$ on $H$ given by $\|x\|_A = \sqrt{\langle x, x \rangle_A}, \forall x \in H$. This makes $H$ a semi-Hilbertian space. One can verify that

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$\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if $A$ is injective. And $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $A$-bounded if there exists $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$, $\forall x \in \mathcal{H}$. Now we define the $A$-adjoint operator.

**Definition 1.1.** An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be an $A$-adjoint of $T \in \mathcal{B}(\mathcal{H})$ if for every $x, y \in \mathcal{H}$, the identity $(Tx, y)_A = (x, Sy)_A$ holds, i.e., $S$ is a solution of the equation $AX = T^*A$.

The existence of an $A$-adjoint operator is not guaranteed. The set of all operators which admit $A$-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem [12], we have

$$\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : T^* (\mathcal{R}(A)) \subseteq \mathcal{R}(A) \}.$$

If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\sharp A}$. Note that, $T^{\sharp A} = A^*T^*A$ where $A^*$ is the Moore-Penrose inverse of $A$. Again, by applying Douglas theorem, it can be observed that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : T^* (\mathcal{R}(A^{1/2})) \subseteq \mathcal{R}(A^{1/2}) \}.$$

It is easy to check that the collection of all $A$-bounded operators in $\mathcal{B}(\mathcal{H})$ is $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, i.e.,

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \exists \lambda > 0 \text{ such that } \|Tx\|_A \leq \lambda\|x\|_A, \forall x \in \mathcal{H} \}.$$

We note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$. Moreover, the following inclusion holds

$$\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}).$$

Further, the semi-inner product induces the following $A$-operator seminorm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$,

$$\|T\|_A = \sup_{x \in \mathcal{R}(A), x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{x \in \mathcal{H}, \|x\|_A = 1} \|Tx\|_A.$$

For $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, the $A$-numerical radius and the $A$-Crawford number of $T$, denoted as $w_A(T)$ and $c_A(T)$, are defined respectively as

$$w_A(T) = \sup_{x \in \mathcal{H}, \|x\|_A = 1} |(Tx, x)_A|$$

and

$$c_A(T) = \inf_{x \in \mathcal{H}, \|x\|_A = 1} |(Tx, x)_A|.$$

Here we note that if we consider $A = I$ then $\|T\|_A = \|T\|$, $w_A(T) = w(T)$ and $c_A(T) = c(T)$. It is well-known that if $T \in \mathcal{B}_A(\mathcal{H})$ then the following inequality holds

$$\frac{\|T\|_A}{2} \leq w_A(T) \leq \|T\|_A. \quad (1.1)$$

Zamani in [18] improved on the inequality (1.1) to prove that

$$\frac{1}{4} \|T^{\sharp A} + TT^{\sharp A}\|_A \leq w_A^2(T) \leq \frac{1}{2} \|T^{\sharp A} + TT^{\sharp A}\|_A. \quad (1.2)$$
Various other refinements of (1.1) have also been obtained, we refer the interested readers to [5, 6, 7, 8, 10, 14]. It is useful to recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be $A$-selfadjoint if $AT$ selfadjoint, i.e., $AT = T^*A$ and it is called $A$-positive if $AT$ positive. It is clear that if $T \in \mathcal{B}(\mathcal{H})$ is $A$-selfadjoint then $T \in \mathcal{B}_A(\mathcal{H})$. It is well-known that if $T \in \mathcal{B}(\mathcal{H})$ is $A$-selfadjoint then $\|T\|_A = w_A(T)$. Here we also note that if $S, T \in \mathcal{B}_A(\mathcal{H})$ then $(ST)^{2A} = T^{2A}S^{2A}$, $\|ST\|_A \leq \|S\|_A \|T\|_A$ and $\|ST\|_A \leq \|S\|_A w_A(T)$ for all $x \in \mathcal{H}$.

For more information related to $A$-adjoint operators, we refer to [2].

In this paper, we develop many $A$-operator seminorm and $A$-numerical radius inequalities for an operator $T \in \mathcal{B}_A(\mathcal{H})$ improving the existing inequalities. In particular, we show that if $S, T, X \in \mathcal{B}_A(\mathcal{H})$, then

$$2\|S^{2A}XT\|_A \leq \|SS^{2A}X + XTT^{2A}\|_A.$$ 

Further, we obtain that if $T \in \mathcal{B}_A(\mathcal{H})$, then

$$\frac{1}{4}\|T^{2A}T + TT^{2A}\|_A + \frac{1}{8}c_A^2(T + T^{2A}) + \frac{1}{8}c_A^2(T - T^{2A}) \leq w_A^2(T),$$

$$\frac{1}{4}\|T^{2A}T + TT^{2A}\|_A \leq \frac{1}{4\sqrt{2}}\left(\|T + T^{2A}\|_A + \|T - T^{2A}\|_A\right)^\frac{1}{2} \leq w_A^2(T).$$

We also prove that if $T \in \mathcal{B}_A(\mathcal{H})$ and $A(T + T^{2A})^2(T - T^{2A}) = 0$, then

$$w_A^2(T) = \frac{1}{2}\|T^{2A}T + TT^{2A}\|_A.$$ 

The technique that we use in developing the inequalities is little different from the one used in earlier works like [5, 17, 18], we briefly discuss it here. The semi-inner product $\langle \cdot, \cdot \rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ defined as

$$[\overline{x}, \overline{y}] = \langle Ax, y \rangle,$$

for all $\overline{x} = x + \mathcal{N}(A), \overline{y} = y + \mathcal{N}(A) \in \mathcal{H}/\mathcal{N}(A)$. Note that $(\mathcal{H}/\mathcal{N}(A), [\cdot, \cdot])$ is not complete unless $\mathcal{R}(A)$ is closed in $\mathcal{H}$. L. de Branges and J. Rovnyak [11] showed that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ with the inner product

$$\langle A^{1/2}x, A^{1/2}y \rangle = \langle P_{\mathcal{R}(A)}x, P_{\mathcal{R}(A)}y \rangle, \quad \forall \ x, y \in \mathcal{H}.$$ 

Here $P_{\mathcal{R}(A)}$ denotes the projection onto $\overline{\mathcal{R}(A)}$. The Hilbert space $(\mathcal{R}(A^{1/2}), \langle \cdot, \cdot \rangle)$ is denoted by $\mathcal{R}(A^{1/2})$ and we use the symbol $\| \cdot \|_{\mathcal{R}(A^{1/2})}$ to represent the norm induced by the inner product $\langle \cdot, \cdot \rangle$. For more information related to the Hilbert space $\mathcal{R}(A^{1/2})$, we refer the interested readers to [1]. Note that the fact $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$ implies that $(Ax, Ay) = \langle x, y \rangle_A$. This implies the useful relation

$$\|Ax\|_{\mathcal{R}(A^{1/2})} = \|x\|_A, \quad \forall \ x \in \mathcal{H}.$$ 

To proceed further we need the following lemma which gives a nice connection between $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and $\overline{T} \in \mathcal{B}(\mathcal{R}(A^{1/2}))$.

**Lemma 1.2.** ([1, Prop. 3.6]) Let $T \in \mathcal{B}(\mathcal{H})$ and let $Z_A : \mathcal{H} \to \mathcal{R}(A^{1/2})$ be defined by $Z_Ax = Ax, \quad \forall \ x \in \mathcal{H}$. Then $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists unique $\overline{T} \in \mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A T = \overline{T} Z_A$. 
There are many important well-known relations between $T$ and $\tilde{T}$, we mention a few of them in the form of the following lemma.

**Lemma 1.3.** ([16, Prop. 2.9]) Let $T \in \mathcal{B}_A(\mathcal{H})$. Then

(i) $\tilde{T}^{zA} = (\tilde{T})^*$ and $(\tilde{T}^{zA})^{zA} = \tilde{T}$.

(ii) $\|T\|_A = \|\tilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}$, $w_A(T) = w(\tilde{T})$ and $c_A(T) = c(\tilde{T})$.

The following lemma is also obvious from the definition of $\tilde{T}$.

**Lemma 1.4.** Let $S, T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and let $\lambda \in \mathbb{C}$ be any scalar. Then

$$S + \lambda T = \tilde{S} + \lambda \tilde{T} \quad \text{and} \quad \tilde{S}T = \tilde{S}\tilde{T}.$$ 

We end this section by elaborating the steps that will be used to develop the $A$-operator seminorm and the $A$-numerical radius inequalities of semi-Hilbertian space operators.

**Step 1.** Begin with $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

**Step 2.** Consider the corresponding $\tilde{T} \in \mathcal{B}(\mathcal{R}(A^{1/2}))$.

**Step 3.** Look at the classical operator norm and numerical radius inequalities for the operator $\tilde{T}$ acting on the Hilbert space $\mathcal{R}(A^{1/2})$.

**Step 4.** Go back to $A$-operator seminorm and $A$-numerical radius inequalities for the operator $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

### 2. Main results

We begin this section with the following theorem that gives an $A$-operator seminorm inequality of the product of semi-Hilbertian space operators.

**Theorem 2.1.** Let $S, T, X \in \mathcal{B}_A(\mathcal{H})$. Then

$$\|S^{z_A}XT\|_A \leq \frac{1}{2}\|SS^{z_A}X + XTT^{z_A}\|_A.$$ 

**Proof.** Since $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$, so $S, T, X \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Therefore, it follows from Lemma 1.2 that there exists unique $\tilde{S}$ in $\mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A S = \tilde{S} Z_A$. Similarly, there exist $\tilde{T}$ and $\tilde{X}$ in $\mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A T = \tilde{T} Z_A$ and $Z_A X = \tilde{X} Z_A$. Following [4], we have if $S, T, X \in \mathcal{B}(\mathcal{H})$ then $2\|S^* XT\| \leq \|SS^* X + XTT^*\|$. It follows that

$$\|\tilde{(S)}^* \tilde{X}\tilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \leq \frac{1}{2}\|\tilde{S}\tilde{(S)}^* \tilde{X} + \tilde{X}\tilde{T}\tilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.3(i)}$$

$$\Rightarrow \|\tilde{S}^{z_A} \tilde{X}\tilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \leq \frac{1}{2}\|\tilde{S} S^{z_A} \tilde{X} + \tilde{X}\tilde{T} T^{z_A}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.3(ii)}$$

$$\Rightarrow \|S^{z_A}XT\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \leq \frac{1}{2}\|SS^{z_A}X + XTT^{z_A}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.4}$$

This completes the proof. 

Considering $X = I$ in the above theorem we get the following corollary.

**Corollary 2.2.** Let $S, T \in \mathcal{B}_A(\mathcal{H})$. Then

$$\|S^{z_A}T\|_A \leq \frac{1}{2}\|SS^{z_A} + TT^{z_A}\|_A.$$
Then we have that

\[ \|S + T\|_A \leq \left( \|S\|_A^2 + \|T\|_A^2 + \|S^{2\lambda} T + T^{2\lambda} S\|_A \right)^{\frac{1}{2}} \leq \|S\|_A + \|T\|_A, \]

\[ (ii) \|S + T\|_A \leq \left( \|S\|_A^2 + \|T\|_A^2 + \|S\|_A \|T\|_A + \min \{ w_A(S^{2\lambda} T), w_A(ST^{2\lambda}) \} \right)^{\frac{1}{2}} \]

\[ (iii) \|S + T\|_A^2 \leq \|S\|_A^2 + \|T\|_A^2 + \frac{1}{2}\|S^{2\lambda} S + T^{2\lambda} T\|_A + w_A(S^{2\lambda} T), \]

\[ (iv) \|S + T\|_A^2 \leq \|S\|_A^2 + \|T\|_A^2 + \frac{1}{2}\|S S^{2\lambda} + TT^{2\lambda}\|_A + w_A(ST^{2\lambda}), \]

and

\[ (v) \|ST^{2\lambda}\|_A \leq \frac{1}{\sqrt{3}} \left( \frac{S^{2\lambda} S + T^{2\lambda} T}{2} \right)^2 + \|ST^{2\lambda}\|_A^2 \| + \|ST^{2\lambda}\|_A \left( \frac{S^{2\lambda} S + T^{2\lambda} T}{2} \right) \right|_A \]

\[ \leq \frac{1}{2} \|S^{2\lambda} S + T^{2\lambda} T\|_A. \]

**Remark 2.4.** It follows from Theorem 2.3(i) that if \( \|S + T\|_A = \|S\|_A + \|T\|_A \) then \( \|S^{2\lambda} T + T^{2\lambda} S\|_A = 2\|S\|_A\|T\|_A \). Also from Theorem 2.3(ii) it follows that if \( \|S + T\|_A = \|S\|_A + \|T\|_A \) then \( w_A(S^{2\lambda} T) = \|S\|_A\|T\|_A \) and \( w_A(ST^{2\lambda}) = \|S\|_A\|T\|_A \).

We next obtain a lower bound for the A-numerical radius that improves on [15, Th. 1].

**Theorem 2.5.** Let \( T \in B_A(\mathcal{H}) \). Then

\[ \frac{1}{8} \|T^{2\lambda} T + TT^{2\lambda}\|_A \leq \left( \|T + T^{2\lambda}\|_A^2 + \|T - T^{2\lambda}\|_A^2 \right), \]

and

\[ \frac{1}{8} \left( \|T + T^{2\lambda}\|_A^2 + \|T - T^{2\lambda}\|_A^2 \right) + \frac{1}{8} c_A^2 (T + T^{2\lambda}) + \frac{1}{8} c_A^2 (T - T^{2\lambda}) \leq w_A(T). \]

**Proof.** Let \( T = \text{Re}_A(T) + i \text{Im}_A(T) \), where \( \text{Re}_A(T) = \frac{T + T^{2\lambda}}{2} \) and \( \text{Im}_A(T) = \frac{T - T^{2\lambda}}{2i} \). Then we have that

\[ \frac{1}{4} \|T^{2\lambda} T + TT^{2\lambda}\|_A = \frac{1}{2} \|\text{Re}_A^2(T) + \text{Im}_A^2(T)\|_A \leq \frac{1}{2} \left( \|\text{Re}_A(T)\|_A^2 + \|\text{Im}_A(T)\|_A^2 \right). \]

This implies the first inequality of the theorem. Now let \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \). Then from the decomposition \( T = \text{Re}_A(T) + i \text{Im}_A(T) \), we have that

\[ \|\langle Tx, x \rangle_A \|^2 = \|\langle \text{Re}_A(T)x, x \rangle_A \|^2 + \|\langle \text{Im}_A(T)x, x \rangle_A \|^2. \]

This implies that

\[ \|\langle Tx, x \rangle_A \|^2 \geq \|\langle \text{Re}_A(T)x, x \rangle_A \|^2 + c_A^2 (\text{Im}_A(T)). \]

Since \( \text{Re}_A(T) \) is A-selfadjoint, taking supremum over \( \|x\|_A = 1 \), we get

\[ w_A^2(T) \geq \|\text{Re}_A(T)\|_A^2 + c_A^2 (\text{Im}_A(T)). \quad (2.1) \]
Similarly, $\text{Im}_A(T)$ is $A$-selfadjoint and we get
\[ w_A^2(T) \geq \| \text{Im}_A(T) \|^2_A + c^2_A(\text{Re}_A(T)). \] (2.2)
Combining (2.1) and (2.2), we have
\[ c^2_A(\text{Im}_A(T)) + c^2_A(\text{Re}_A(T)) + \| \text{Im}_A(T) \|^2_A + \| \text{Re}_A(T) \|^2_A \leq 2w_A^2(T). \]
This implies the second inequality of the theorem, and hence completes the proof.

\[ \square \]

**Remark 2.6.** (i) It follows from Theorem 2.5 that if $T \in \mathcal{B}_A(\mathcal{H})$, then
\[ \frac{1}{4} \| T^{z_A}T + TT^{z_A} \|_A + \frac{1}{8} c^2_A(T + T^{z_A}) + \frac{1}{8} c^2_A(T - T^{z_A}) \leq w_A^2(T). \] (2.3)
(ii) Feki in [15, Th. 1] proved that if $T \in \mathcal{B}_A(\mathcal{H})$, then
\[ \frac{1}{4} \| T \|_A^2 + \frac{1}{4} \max \{ m_A^2(T), m_A^2(T^{z_A}) \} \leq w_A(T), \]
where $m_A(T) = \inf \{ \| Tx \|_A : x \in \mathcal{H}, \| x \|_A = 1 \}$. In [10, Th. 2.9], we proved that
\[ \frac{1}{4} \| T \|_A^2 + \frac{1}{4} \max \{ m_A^2(T), m_A^2(T^{z_A}) \} \leq \frac{1}{4} \| T^{z_A}T + TT^{z_A} \|_A. \]
Therefore, the first inequality in (1.2) is sharper than [15, Th. 1]. Clearly, the inequality (2.3) is sharper than the first inequality in (1.2) and so it is sharper than [15, Th. 1].

We next prove the following theorem.

**Theorem 2.7.** Let $S, T \in \mathcal{B}_A(\mathcal{H})$. Then
\[ w_A(T^{z_A}S) \leq \frac{1}{2} \| S^{z_A}S + T^{z_A}T \|_A. \]

**Proof.** It was proved in [13] that if $S, T \in \mathcal{B}(\mathcal{H})$, then $2w(T^*S) \leq \| S^*S + T^*T \|$. Since $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$, so $S, T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Therefore, it follows from Lemma 1.2 that there exists unique $\tilde{S}$ in $\mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A S = \tilde{S} Z_A$. Similarly, there exists unique $\tilde{T}$ in $\mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A T = \tilde{T} Z_A$. Therefore, from the above inequality we have that
\[ w\left( (\tilde{T})^* \tilde{S} \right) \leq \frac{1}{2} \| (\tilde{S})^* \tilde{S} + (\tilde{T})^* \tilde{T} \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \]
\[ \Rightarrow w\left( T^{z_A} \tilde{S} \right) \leq \frac{1}{2} \| \tilde{S}^{z_A} \tilde{S} + \tilde{T}^{z_A} \tilde{T} \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.3(i)} \]
\[ \Rightarrow w\left( S^{z_A} \tilde{S} \right) \leq \frac{1}{2} \| S^{z_A} \tilde{S} + T^{z_A} \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.4} \]
\[ \Rightarrow w\left( \tilde{T}^{z_A} \tilde{S} \right) \leq \frac{1}{2} \| S^{z_A} \tilde{S} + T^{z_A} \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \text{ by Lemma 1.4} \]
\[ \Rightarrow w_A(T^{z_A}S) \leq \frac{1}{2} \| S^{z_A}S + T^{z_A}T \|_A, \text{ by Lemma 1.3(ii)} \]
This completes the proof. \[ \square \]

Proceeding in the same way we can prove the following theorem by using the corresponding results from [9, Th. 2.10, 2.13, 2.18] and [9, Cor. 2.21, 3.5], respectively.
Theorem 2.8. Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following inequalities hold:

(i) $w_A^2(T) \geq \frac{1}{8} \left( \max \left\{ \|T + T^{\sharp A}\|_A^2, \|T - T^{\sharp A}\|_A^2 \right\} + \|T + T^{\sharp A}\|_A \|T - T^{\sharp A}\|_A \right)$.

(ii) $w_A^2(T) \geq \frac{1}{4\sqrt{2}} \left( \|T + T^{\sharp A}\|_A^4 + \|T - T^{\sharp A}\|_A^4 \right)^{1/4} \geq \frac{1}{4}\|T^{\sharp A}T + TT^{\sharp A}\|_A$.

(iii) $w_A^2(T) \geq \frac{1}{8} \left[ \left( \|T + T^{\sharp A}\|_A^2 + \|T - T^{\sharp A}\|_A^2 \right)^2 + \frac{1}{2} \left( \|T + T^{\sharp A}\|_A^2 - \|T - T^{\sharp A}\|_A^2 \right)^2 \right]^{1/2}$.

(iv) $\frac{1}{2}\|T^{\sharp A}T + TT^{\sharp A}\|_A - \frac{1}{4}\left( T + T^{\sharp A} \right)^2(T - T^{\sharp A})^2 \leq w_A^2(T) \leq \frac{1}{2}\|T^{\sharp A}T + TT^{\sharp A}\|_A$.

(v) $w_A^2(T) \leq \frac{1}{\sqrt{3}} \left( \frac{T^{\sharp A}T + TT^{\sharp A}}{2} \right)^2 w_A^2(T) \leq \frac{1}{2}\|T^{\sharp A}T + TT^{\sharp A}\|_A$.

Remark 2.9. (a) It is clear that

$\|T + T^{\sharp A}\|_A^2 + \|T - T^{\sharp A}\|_A^2 \leq \max \left\{ \|T + T^{\sharp A}\|_A^2, \|T - T^{\sharp A}\|_A^2 \right\} + \|T + T^{\sharp A}\|_A \|T - T^{\sharp A}\|_A$.

Therefore, the inequality in Theorem 2.8(i) improves on the first inequality in (1.2).

(b) By using the first inequality in Theorem 2.5, we conclude that the inequality in Theorem 2.8(iii) is stronger than the left hand inequality in (1.2).

(c) It follows from Theorem 2.8(iv) that if $\|T + T^{\sharp A}\|_A^2(T - T^{\sharp A})^2 = 0$, i.e., if $A(T + T^{\sharp A})^2(T - T^{\sharp A})^2 = 0$ then $w_A^2(T) = \frac{1}{2}\|T^{\sharp A}T + TT^{\sharp A}\|_A$.

(d) Clearly, the inequality in Theorem 2.8(v) is stronger than the right hand inequality in (1.2).

Our next theorem is an improvement of [14, Th. 7].

Theorem 2.10. Let $T \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_A(T) \leq \frac{1}{\sqrt{2}} \left( \frac{1}{2} \|T\|_A^2 + \frac{1}{2} \|T^2\|_A^2 \right) + \frac{1}{2}\|TT^{\sharp A} + T^{\sharp A}T\|_A \right)^{1/2}$$

$$\leq \frac{1}{2}\|T\|_A + \frac{1}{2}\|T^2\|_A^1.$$ 

Proof. Since $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$, so $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Therefore, it follows from Lemma 1.2 that there exists unique $\tilde{T}$ in $\mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_AT = \tilde{T}Z_A$. Therefore, following [3, Cor. 1] we have

$$w_A^2(\tilde{T}) \leq \frac{1}{2}\|\tilde{T}^2\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}^2 \left( \frac{1}{2}\|\tilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}^2 + \frac{1}{2}\|\tilde{T}^2\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}^2 \right) + \frac{1}{4}\|\tilde{T}^*\tilde{T} + \tilde{T}\tilde{T}^*\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}.$$ 

Now by using Lemma 1.4 and Lemma 1.3 we get the first inequality of the theorem. The second inequality follows from [3, Remark 5] by using similar arguments as above. □
Remark 2.11. Feki in [14, Th. 7] proved that
\[ w_A(T) \leq \frac{1}{2} \|T\|_A + \frac{1}{2} \|T^2\|_A^2. \]
Clearly, the first inequality in Theorem 2.10 is sharper than that in [14, Th. 7].

The final theorem of this paper is a well-known result [5, 17], which is given here to highlight the advantage of the technique used in this paper.

Theorem 2.12. Let \( T \in \mathcal{B}_A(\mathcal{H}) \). Then
\[
\frac{1}{16} \|TT^*A + T^*T\|_A^2 + \frac{1}{4} c_A \left( \left( \text{Re}_A(T^2) \right)^2 \right) \leq w_A^4(T)
\]
\[
\leq \frac{1}{8} \|TT^*A + T^*T\|_A^2 + \frac{1}{2} w_A^2(T^2).
\]

Proof. Since \( \mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), so \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). Therefore, it follows from Lemma 1.2 that there exists unique \( \tilde{T} \) in \( \mathcal{B}(\mathcal{R}(A^{1/2})) \) such that \( Z_A T = \tilde{T} Z_A \). It follows from [3, Th. 8] that
\[
\frac{1}{16} \|T^* \tilde{T} + \tilde{T} (T^*)^2 \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} + \frac{1}{4} c \left( \left( \text{Re}_{A^{1/2}}(T^2) \right)^2 \right) \leq w_A^4(\tilde{T})
\]
and
\[
w_A^4(\tilde{T}) \leq \frac{1}{8} \|T^* \tilde{T} + \tilde{T} (T^*)^2 \|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} + \frac{1}{2} w_A^2(T^2).
\]
Thus, the required inequalities follow from Lemma 1.4 and Lemma 1.3. \( \square \)

Remark 2.13. The proof of the inequalities given here is simple and short in comparison to the proofs of the same in [5, 17]. We conclude that the same holds true for many existing proofs on \( A \)-operator seminorm and \( A \)-numerical radius inequalities for semi-Hilbertian space operators.

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