Semiparametric Contextual Bandits

Akshay Krishnamurthy *, Zhiwei Steven Wu †, and Vasilis Syrgkanis ‡

2Microsoft Research NYC, New York, NY
3Microsoft Research New England, Cambridge, MA

July 17, 2018

Abstract

This paper studies semiparametric contextual bandits, a generalization of the linear stochastic bandit problem where the reward for an action is modeled as a linear function of known action features confounded by a non-linear action-independent term. We design new algorithms that achieve \( \tilde{O}(d\sqrt{T}) \) regret over \( T \) rounds, when the linear function is \( d \)-dimensional, which matches the best known bounds for the simpler unconfounded case and improves on a recent result of Greenewald et al. [19]. Via an empirical evaluation, we show that our algorithms outperform prior approaches when there are non-linear confounding effects on the rewards. Technically, our algorithms use a new reward estimator inspired by doubly-robust approaches and our proofs require new concentration inequalities for self-normalized martingales.

1 Introduction

A number of applications including online personalization, mobile health, and adaptive clinical trials require that an agent repeatedly makes decisions based on user or patient information with the goal of optimizing some metric, typically referred to as a reward. For example, in online personalization problems, we might serve content based on user history and demographic information with the goal of maximizing user engagement with our service. Since counterfactual information is typically not available, these problems require algorithms to carefully balance exploration—making potentially suboptimal decisions to acquire new information—with exploitation—using collected information to make better decisions. Such problems are often best modeled with the framework of contextual bandits, which captures the exploration-exploitation tradeoff and enables rich decision making policies but ignores the long-term temporal effects that make general reinforcement learning challenging. Contextual bandit algorithms have seen recent success in applications, including news recommendation [22] and mobile health [34].

Contextual bandit algorithms can be categorized as either parametric or agnostic, depending on whether they model the relationship between the reward and the decision or not. Parametric approaches typically assume that the reward is a (generalized) linear function of a known decision-specific feature vector [17, 11, 1, 4]. Once this function is known to high accuracy, it can be used to make near-optimal decisions. Exploiting
this fact, algorithms for this setting focus on learning the parametric model. Unfortunately, fully parametric assumptions are often unrealistic and challenging to verify in practice, and these algorithms may perform poorly when the assumptions do not hold.

In contrast, agnostic approaches make no modeling assumptions about the reward and instead compete with a large class of decision-making policies [21, 3]. While these policies are typically parametrized in some way, these algorithms provably succeed under weaker conditions and are generally more robust than parametric ones. On the other hand, they typically have worse statistical guarantees, are conceptually much more complex, and have high computational overhead, technically requiring solving optimization problems that are NP-hard in the worst case. This leads us to a natural question:

Is there an algorithm that inherits the simplicity and statistical guarantees of the parametric methods and the robustness of the agnostic ones?

Working towards an affirmative answer to this question, we consider a semiparametric contextual bandit setup where the reward is modeled as a linear function of the decision confounded by an additive non-linear perturbation that is independent of the decision. This setup significantly generalizes the standard parametric one, allowing for complex, non-stationary, and non-linear rewards (See Section 2 for a precise formulation). On the other hand, since this perturbation is just a baseline reward for all decisions, it has no influence on the optimal one, which depends only on the unknown linear function. In the language of econometrics and causal modeling, the treatment effect is linear.

In this paper, we design new algorithms for the semiparametric contextual bandits problem. When the linear part of the reward is $d$-dimensional, our algorithms achieve $\tilde{O}(d\sqrt{T})$ regret over $T$ rounds, even when the features and the confounder are chosen by an adaptive adversary. This guarantee matches the best results for the simpler linear stochastic bandit problem up to logarithmic terms, showing that there is essentially no statistical price to pay for robustness to confounding effects. On the other hand, our algorithm and analysis is quite different, and it is not hard to see that existing algorithms for stochastic bandits fail in our more general setting. Our regret bound also improves on a recent result of Greenewald et al. [19], who consider the same setup but study a weaker notion of regret. Our algorithm, main theorem, and comparisons are presented in Section 3.

We also compare our algorithm to approaches from both parametric and agnostic families in an empirical study (we use a linear policy class for agnostic approaches). In Section 5, we evaluate several algorithms on synthetic problems where the reward is (a) linear, and (b) linear with confounding. In the linear case, our approach learns, but is slightly worse than the baselines. On the other hand, when there is confounding, our algorithm significantly outperforms both parametric and agnostic approaches. As such, these experiments demonstrate that our algorithm represents a favorable trade off between statistical efficiency and robustness.

On a technical level, our algorithm and analysis require several new ideas. First, we derive a new estimator for linear models in the presence of confounders, based on recent and classical work in semiparametric statistics and econometrics [28, 10]. Second, since standard algorithms using optimism principles fail to guarantee consistency of this new estimator, we design a new randomized algorithm, which can be viewed as an adaptation of the action-elimination method of Even-Dar et al. [16] to the contextual bandits setting. Finally, analyzing the semiparametric estimator requires an intricate deviation argument, for which we derive a new self-normalized inequality for vector-valued martingales using tools from de la Peña et al. [14, 15].

2 Preliminaries

We study a generalization of the linear stochastic bandit problem with action-dependent features and action-independent confounder. The learning process proceeds for $T$ rounds, and in round $t$, the learner receives a
context $x_t \triangleq \{z_{t,a}\}_{a \in \mathcal{A}}$ where $z_{t,a} \in \mathbb{R}^d$ and $\mathcal{A}$ is the action set, which we assume to be large but finite. The learner then chooses an action $a_t \in \mathcal{A}$ and receives reward

$$r_t(a_t) \triangleq \langle \theta, z_{t,a_t} \rangle + f_t(x_t) + \xi_t,$$

where $\theta \in \mathbb{R}^d$ is an unknown parameter vector, $f_t(x_t)$ is a confounding term that depends on the context $x_t$ but, crucially, does not depend on the chosen action $a_t$, and $\xi_t$ is a noise term that is centered and independent of $a_t$.

For each round $t$, let $a_t^* \triangleq \arg\max_{a \in \mathcal{A}} \langle \theta, z_{t,a} \rangle$ denote the optimal action for that round. The goal of our algorithm is to minimize the regret, defined as

$$\text{Reg}(T) \triangleq \sum_{t=1}^T r_t(a_t^*) - r_t(a_t) = \sum_{t=1}^T \langle \theta, z_{t,a_t^*} - z_{t,a_t} \rangle.$$

Observe that the noise term $\xi_t$, and, more importantly, the confounding term $f_t(x_t)$ are absent in the final expression, since they are independent of the action choice.

We consider the challenging setting where the context $x_t$ and the confounding term $f_t(\cdot)$ are chosen by an adaptive adversary, so they may depend on all information from previous rounds. This is formalized in the following assumption.

**Assumption 1 (Environment).** We assume that $x_t = \{z_{t,a}\}_{a \in \mathcal{A}}, f_t, \xi_t$ are generated at the beginning of round $t$ before $a_t$ is chosen. We assume that $x_t$ and $f_t$ are chosen by an adaptive adversary, and that $\xi_t$ satisfies $\mathbb{E}[\xi_t | x_t, f_t] = 0$ and $|\xi_t| \leq 1$.

We also impose mild regularity assumptions on the parameter, the feature vectors, and the confounding functions.

**Assumption 2 (Boundedness).** Assume that $\|\theta\|_2 \leq 1$ and that $\|z_{t,a}\|_2 \leq 1$ for all $a \in \mathcal{A}, t \in [T]$. Further assume that $f_t(\cdot) \in [-1, 1]$ for all $t \in [T]$.

For simplicity, we assume an upper bound of 1 in these conditions, but our algorithm and analysis can be adapted to more generic regularity conditions.

**Related work.** Our setting is related to linear stochastic bandits and several variations that have been studied in recent years. Among these, the closest is the work of Greenewald et al. [19] who consider the same setup and provide a Thompson Sampling algorithm using a new reward estimator that eliminates the confounding term. Motivated by applications in medical intervention, they consider a different notion of regret from our more-standard notion and, as such, the results are somewhat incomparable. For our notion of regret, their analysis can produce a $T^{2/3}$-style regret bound, which is worse than our optimal $\sqrt{T}$ bound. See Section 3.3 for a more detailed comparison.

Other results for linear stochastic bandits include upper-confidence bound algorithms [29, 11, 1], Thompson sampling algorithms [4, 30], and extensions to generalized linear models [17, 23]. However, none of these models accommodate arbitrary and non-linear confounding effects. Moreover, apart from Thompson sampling, all of these algorithms use deterministic action-selection policies (conditioning on the history), which provably incurs $\Omega(T)$ regret in our setting, as we will see.

One can accommodate confounded rewards via an agnostic-learning approach to contextual bandits [5, 21, 3]. In this framework, we make no assumptions about the reward, but rather compete with a class of parameterized policies (or experts). Since a $d$-dimensional linear policy is optimal in our setting, an
agnostic algorithm with a linear policy class addresses precisely our notion of regret. However there are two
disadvantages. First, agnostic algorithms are all computationally intractable, either because they enumerate
the (infinitely large) policy class, or because they assume access to optimization oracles that can solve NP-hard
problems in the worst case. Second, most agnostic approaches have regret bounds that grow with $\sqrt{K}$, the
number of actions, while our bound is completely independent of $K$.

We are aware of one approach that is independent of $K$, but it requires enumeration of an infinitely
large policy class. This method is based on ideas from the adversarial linear and combinatorial bandits
literature [13, 2, 8, 9]. Writing $\theta_t \triangleq (\theta, f_t(x_t)) \in \mathbb{R}^{d+1}$ and $z'_{t,a} \triangleq (z_{t,a}, 1) \in \mathbb{R}^{d+1}$, our setting can
be re-formulated in the adversarial linear bandits framework. However, standard linear bandit algorithms
compete with the best fixed action vector in hindsight, rather than the best policy with time-varying action
sets. To resolve this, one can use the linear bandits reward estimator [32] in a contextual bandit algorithm
like EXP4 [5], but this approach is not computationally tractable with the linear policy class. For our setting,
we are not aware of any computationally efficient approaches, even oracle-based approaches, that achieve
$\text{poly}(d) \sqrt{T}$ regret with no dependence on the number of actions.

We resolve the challenge of confounded rewards with an estimator from the semiparametric statistics
literature [35], which focuses on estimating functionals of a nonparametric model. Most estimators are based
on Neyman Orthogonalization [24], which uses moment equations that are insensitive to nuisance parameters
in a method-of-moments approach [10]. These orthogonal moments typically involve a linear correction
to an initial nonparametric estimate using so-called influence functions [7, 27]. Robinson [28] used this
approach for the offline version of our setting (known as the partially linear regression (PLR) model) where
he demonstrated a form of double-robustness [26] to poor estimation of the nuisance term (in our case $f_t(x_t)$).
We generalize Robinson’s work to the online setting, showing how orthogonalized estimators can be used for
adaptive exploration. This requires several new techniques, including a novel action selection policy and a
self-normalized inequality for vector-valued martingales.

3 Algorithm and Results

In this section, we describe our algorithm and present our main theoretical result, an $\tilde{O}(d\sqrt{T})$ regret bound
for the semiparametric contextual bandits problem.

3.1 A Lower Bound

Before turning to the algorithm, we first present a lower bound against deterministic algorithms. Since the
functions $f_t$ may be chosen by an adaptive adversary, it is not hard to show that this setup immediately
precludes the use of deterministic algorithms.

**Proposition 3.** Consider an algorithm that, at round $t$, chooses an action $a_t$ as a deterministic function of
the observable history $H_t \triangleq \{x_{1:t}, a_{1:t-1}, r_{1:t-1}\}$. There exists a semiparametric contextual bandit instance
with $d = 2$ and $K = 2$ where the regret of the algorithm is at least $T/2$.

See Appendix B for the proof, which resembles the standard argument against deterministic online
learning algorithms [12]. The main difference is that the adversary uses the confounding term to corrupt
the information that the learner receives, whereas, in the standard proof, the adversary chooses the optimal
action in response to the learner. In fact, deterministic algorithms can succeed in the full information version
of our setting, since taking differences between rewards eliminates the confounder. Thus, bandit feedback
plays a crucial role in our construction and the bandit setting is considerably more challenging than the full
information analogue.
was chosen at round \( t \) which also applies to randomized algorithms, holds in our more general setting as well.

### 3.2 The Algorithm

**Parameter estimation.** For simplicity, we use \( z_t \triangleq z_{t,a_t} \) to denote the feature vector for the action that was chosen at round \( t \), and similarly we use \( r_t \triangleq r_t(a_t) \). Using all previously collected data, specifically \( \{z_\tau, r_\tau\}_{\tau=1}^T \) at the end of round \( t \), we would like to estimate the parameter \( \theta \). First, if \( f_\tau(x_\tau) \) were identically

**Algorithm 1: BOSE (Bandit orthogonalized semiparametric estimation)**

| Input: \( T, \delta \in (0, 1) \). |
|---|
| 1. Set \( \lambda \leftarrow 4d \log(9T) + 8 \log(4T/\delta) \) and \( \gamma(T) \leftarrow \sqrt{\lambda} + \sqrt{27d \log(1 + 2T/d)} + 54 \log(4T/\delta) \). |
| 2. Initialize \( \hat{\theta} \leftarrow 0 \in \mathbb{R}^d, \Gamma \leftarrow \lambda I_{d \times d} \). |
| 3. for \( t = 1, \ldots, T \) do |
| 4. Observe \( x_t = \{z_{t,a}\}_{a \in A} \). |
| 5. Filter \( \mathcal{A}_t \leftarrow \\{a \in A \mid \forall b \in A, \langle \hat{\theta}, z_{t,b} - z_{t,a} \rangle \leq \gamma(T)\|z_{t,a} - z_{t,b}\|_{\Gamma^{-1}} \} \). (2) |
| 6. Find distribution \( \pi_t \in \Delta(A_t) \) such that \( \forall a \in A_t \) (We use \( \text{Cov}_{b \sim \pi_t}(z_{t,b}) \equiv \mathbb{E}[z_{t,b}z_{t,b}^\top] - (\mathbb{E}z_{t,b})(\mathbb{E}z_{t,b})^\top) \)
\[ \|z_{t,a} - \mathbb{E}_{b \sim \pi_t} z_{t,b}\|_{\Gamma^{-1}} \leq \text{tr}(\Gamma^{-1} \text{Cov}(z_{t,b})). \] (3) |
| 7. Sample \( a_t \sim \pi_t \) and play \( a_t \). Observe \( r_t(a_t) \). \( r_t(a_t) = \langle \theta, z_{t,a_t} \rangle + f_\tau(x_t) + \xi_t \). |
| 8. Let \( \mu_t = \mathbb{E}_{a \sim \pi_t}[z_{t,a} \mid x_t] \) and update parameters \[
\Gamma \leftarrow \Gamma + (z_{t,a_t} - \mu_t)(z_{t,a_t} - \mu_t)^\top, \quad \hat{\theta} \leftarrow \Gamma^{-1} \sum_{\tau=1}^t (z_{\tau,a_\tau} - \mu_\tau)r_\tau(a_\tau).
\] \] (4) |

We emphasize that, except for the Thompson Sampling approach [4], essentially all algorithms for the linear stochastic bandit problem use deterministic strategies, so they provably fail in the semiparametric setting. As we mentioned, Thompson Sampling was analyzed in our setting by Greenewald et al. [19], but they do not obtain the optimal \( \sqrt{T} \)-type regret bound (See Section 3.3 for a more quantitative and detailed comparison). In contrast, our algorithm is quite different from all of these approaches; it ensures enough randomization to circumvent the lower bound and also achieves the optimal \( \sqrt{T} \) regret.

To conclude this discussion, we remark that the \( \Omega(d\sqrt{T}) \) lower bound for linear stochastic bandits [13], which also applies to randomized algorithms, holds in our more general setting as well.

### 3.2 The Algorithm

Pseudocode for the algorithm, which we call BOSE, for “Bandit Orthogonalized Semiparametric Estimation,” is displayed in Algorithm 1. The algorithm maintains an estimate \( \hat{\theta} \) for the true parameter \( \theta \), which it uses in each round to select an action via two steps: (1) an action elimination step that removes suboptimal actions, and (2) an optimization step that finds a good distribution over the surviving actions. The algorithm then samples and plays an action from this distribution and uses the observed reward to update the parameter estimate \( \hat{\theta} \). This parameter estimation step is the third main element of the algorithm. We now describe each of these three components in detail.
zero, by exploiting the linear parametrization we could use ridge regression, which with some \( \lambda > 0 \) gives

\[
\hat{\theta}_{\text{Ridge}} \triangleq \left( \lambda I + \sum_{\tau=1}^{t} z_{\tau} z_{\tau}^\top \right)^{-1} \sum_{\tau=1}^{t} z_{\tau} r_{\tau}.
\]

This estimator appears in most prior approaches for linear stochastic bandits [29, 11, 1]. Unfortunately, since \( f_\tau(x_\tau) \) is non-zero, \( \hat{\theta}_{\text{Ridge}} \) has non-trivial and non-vanishing bias, so even in benign settings it is not a consistent estimator for \( \theta \).

Our approach to eliminating the bias from the confounding term \( f_\tau(x_\tau) \) is to center the feature vectors \( z_\tau \). Intuitively, in the ridge estimator, if \( z_\tau \) is centered, then \( z_\tau (r_\tau - \langle \theta^*, z_\tau \rangle) \) is mean zero, even when there is non-negligible bias in the second term. As such, the error of the corresponding estimator can be expected to concentrate around zero. In the semiparametric statistics literature, this is known as Neyman Orthogonalization [24], which was analyzed in the context of linear regression by Robinson [28] and in a more general setting by Chernozhukov et al. [10].

To center the feature vector, we will, at round \( t \), choose action \( a_t \) by sampling from some distribution \( \pi_t \in \Delta(A) \). Let \( \mu_t \triangleq \mathbb{E}_{a_t \sim \pi_t, x_t \sim \mathcal{P}} [z_{t,a_t} x_t] \) denote the mean feature vector, taking expectation only over our random action choice. With this notation, the orthogonalized estimator is

\[
\Gamma = \lambda I + \sum_{\tau=1}^{t} (z_\tau - \mu_\tau)(z_\tau - \mu_\tau)^\top, \quad \hat{\theta} = \Gamma^{-1} \sum_{\tau=1}^{t} (z_\tau - \mu_\tau) r_\tau.
\]

\( \hat{\theta} \) is a Ridge regression version of Robinson’s classical semiparametric regression estimator [28]. The estimator was originally derived for observational studies where one might not know the propensities \( \mu_\tau \) exactly, and the standard description involves estimates \( \hat{f}_\tau \) and \( \hat{\mu}_\tau \) for the confounding term \( f_\tau \) and the propensities \( \mu_\tau \), respectively. Informally, the estimator achieves a form of double-robustness, in the sense that it is accurate if either of these auxiliary estimators are. In our case, since we know the propensities \( \mu_\tau \) exactly, we can use an inconsistent estimator for the confounding term, so we simply set \( f_\tau(x_\tau) \equiv 0 \).

In Lemma 5, we prove a precise finite sample concentration inequality for this orthogonalized estimator, showing that the confounding term \( f_t(x_t) \) does not introduce any bias. While the estimator has been studied in prior works [28], to our knowledge, our error guarantee is novel.

The convergence rate of the orthogonalized estimator depends on the eigenvalues of the matrix \( \Gamma \), and we must carefully select actions to ensure these eigenvalues are sufficiently large. To see why, notice that any deterministic action-selection approach with the orthogonalized estimator (including confidence based approaches), will fail, since \( z_\tau = \mu_\tau \), so the eigenvalues of \( \Gamma \) do not grow rapidly and in fact the estimator is identically 0. This argument motivates our new action selection scheme which ensure substantial conditional covariance.

**Action selection.** Our action selection procedure has two main elements. First using our estimate \( \hat{\theta} \), we eliminate any action that is provably suboptimal. Based on our analysis for the estimator \( \hat{\theta} \), at round \( t \), we can certify action \( a \) is suboptimal, if we can find another action \( b \) such that

\[
\langle \hat{\theta}, z_{t,b} - z_{t,a} \rangle > \gamma(T) \| z_{t,b} - z_{t,a} \| \Gamma^{-1}.
\]
Here $\gamma(T)$ is the constant specified in the algorithm, and $\|x\|_M \triangleq \sqrt{x^\top M x}$ denotes the Mahalanobis norm. Using our confidence bound for $\hat{\theta}$ in Lemma 5 below, this inequality certifies that action $b$ has higher expected reward than action $a$, so we can safely eliminate $a$ from consideration.

The next component is to find a distribution over the surviving actions, denoted $A_t'$ at round $t$, with sufficient covariance. The distribution $\pi_t \in \Delta(A_t')$ that we use is the solution to the following feasibility problem

$$\forall a \in A_t', \quad \|z_{t,a} - \mathbb{E}_{b \sim \pi_t} z_{t,b}\|_\Gamma^2 \leq \text{tr}(\Gamma^{-1} \text{Cov}_{b \sim \pi_t}(z_{t,b})).$$

For intuition, the left hand side of the constraint for action $a$ is an upper bound on the expected regret if $a$ is the optimal action on this round. Thus, the constraints ensure that the regret is related to the covariance of the distribution, which means that if we incur high regret, the covariance term $\text{Cov}_{b \sim \pi_t}(z_{t,b})$ will be large. Since we use a sample from $\pi_t$ to update our parameter estimate, this means that whenever the instantaneous regret is large, we must learn substantially about the parameter. In this way, the distribution $\pi_t$ balances exploration and exploitation. We will see in Lemma 8 that this program is convex and always has a feasible solution.

Our action selection scheme bears some resemblance to action-elimination approaches that have been studied in various bandit settings [16]. The main differences are that we adapt these ideas to the contextual setting and carefully choose a distribution over the surviving actions to balance exploration and exploitation.

### 3.3 The Main Result

We now turn to the main result, a regret guarantee for BOSE.

**Theorem 4.** Consider the semiparametric contextual bandit problem under Assumption 1 and Assumption 2. For any parameter $\delta \in (0, 1)$, with probability at least $1 - \delta$, Algorithm 1 has regret at most $O(d \sqrt{T \log(\frac{TK}{\delta})})$.

The constants, and indeed a bound depending on $\lambda$ and $\gamma(T)$ can be extracted from the proof, provided in the appendix. To interpret the regret bound, it is worth comparing with several related results:

**Comparison with linear stochastic bandits.** While most algorithms for linear stochastic bandits provably fail in our setting (via Proposition 3), the best regret bounds here are $O(\sqrt{dT \log(TK/\delta)})$ [11] and $O(d \sqrt{T \log(T) + \sqrt{T \log(\log(1/\delta))}})$ [1] depending on whether we assume that the number of actions $K$ is small or not. This latter result is optimal when the number of actions is large [13], which is the setting we are considering here. Since our bound matches this optimal regret up to logarithmic factors, and since linear stochastic bandits are a special case of our semiparametric setting, our result is therefore also optimal up to logarithmic factors. An interesting open question is whether an $\tilde{O}(\sqrt{dKT} \log(K/\delta))$ regret bound is achievable in the semiparametric setting.

**Comparison with agnostic contextual bandits.** The best oracle-based agnostic approaches achieve $\tilde{O}(\sqrt{dKT})$ regret [3], incurring a polynomial dependence on the number of actions $K$, although there is one inefficient method that can achieve $\tilde{O}(d \sqrt{T})$ as we discussed previously. To date, all efficient methods in the agnostic setting require some form of i.i.d. [3] or transductive assumption [33, 25] on the contexts, which we do not assume here.

---

2This follows easily by combining ideas from Auer et al. [5] and Cesa-Bianchi and Lugosi [9].
Comparison with Greenewald et al. [19]. Greenewald et al. [19] consider a very similar setting to ours, where rewards are linear with confounding, but where one default action \( a_0 \) always has \( z_{t,a_0} = 0 \in \mathbb{R}^d \). Applications in mobile health motivate a restriction that the algorithm choose the \( a_0 \) action with probability \( \in [p, 1 - p] \) for some small \( p \in (0, 1) \). Their work also introduces a new notion of regret where they compete with the policy that also satisfies this constraint but otherwise chooses the optimal action \( a^*_t \). In this setup, they obtain an \( \tilde{O}(d^2\sqrt{T}) \) regret bound, which has a worse dimension dependence than Theorem 4.

While the setup is somewhat different, we can still translate our result into a regret bound in their setting, since BOSE can support the probability constraint, and by coupling the randomness between BOSE and the optimal policy, the regret is unaffected.\(^3\) On the other hand, since the constant in their regret bound scales with \( 1/p \), their results as stated are vacuous when \( p = 0 \) which is precisely our setting. For our more challenging regret definition, their analysis can produce a suboptimal \( T^{2/3} \)-style regret bound, and in this sense, Theorem 4 provides a quantitative improvement.

Summary. BOSE achieves essentially the same regret bound as the best linear stochastic bandit methods, but in a much more general setting. On the other hand, the agnostic methods succeed under even weaker assumptions, but have worse regret guarantees and/or are computationally intractable. Thus, BOSE broadens the scope for computationally efficient contextual bandit learning.

4 Proof Sketch

We sketch the proof of Theorem 4 in the two-action case (\(|A| = 2\)), which has a much simpler proof that preserves the main ideas. The technical machinery needed for the general case is much more sophisticated, and we briefly describe some of these steps at the end of this section, with a complete proof in the Appendix.

In the two arm case, one should set \( \gamma(T) \triangleq \sqrt{\lambda} + \sqrt{9d \log(1 + T/(d \lambda))} + 18 \log(T/\delta) \) and \( \lambda = O(1) \), which differs slightly from the algorithm pseudocode for the more general case. Additionally, note that with two actions, the uniform distribution over \( A_t \) is always feasible for Problem (3). Specifically, if the filtered set has cardinality 1, we simply play that action deterministically, otherwise we play one of the two actions uniformly at random.

The proof has three main steps. First we analyze the orthogonalized regression estimator defined in (4). Second, we study the action selection mechanism and relate the regret incurred to the error bound for the orthogonalized estimator. Finally, using a somewhat standard potential argument, we show how this leads to a \( \sqrt{T} \)-type regret bound. For the proof, let \( \theta_t, \Gamma_t \) be the estimator and covariance matrix used on round \( t \), both based on \( t - 1 \) samples.

For the estimator, we prove the following lemma for the two action case. The main technical ingredient is a self-normalized inequality for vector-valued martingales, which can be obtained using ideas from de la Peña et al. [15].

**Lemma 5.** Under Assumption 1 and Assumption 2, let \( K = 2 \) and \( \gamma(T) \triangleq \sqrt{\lambda} + \sqrt{9d \log(1 + T/(d \lambda))} + 18 \log(T/\delta) \). Then, with probability at least \( 1 - \delta \), the following holds simultaneously for all \( t \in [T] \):

\[
\|\hat{\theta}_t - \theta\|_{\Gamma_t} \leq \gamma(T).
\]

\(^3\)Technically it is actually smaller by a factor of \((1 - p)\).
Proof. Using the definitions and Assumption 1, it is not hard to re-write
\[ \hat{\theta}_t = \Gamma^{-1}_t (\Gamma_t - \lambda I) \theta + \Gamma^{-1}_t \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau}, \]
where \( Z_{\tau} \triangleq z_{\tau, a} - \mu_t \) and \( \zeta_{\tau} \triangleq \langle \theta, \mu_t \rangle + f_t(x_{\tau}) + \xi_{\tau} \). Further define \( S_t \triangleq \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau} \). Then, applying the triangle inequality the error is at most
\[ \| \hat{\theta}_t - \theta \|_{\Gamma_t^{-1}} \leq \| \lambda \theta \|_{\Gamma_t^{-1}} + \| S_t \|_{\Gamma_t^{-1}}. \]
The first term here is at most \( \sqrt{\lambda} \) since \( \Gamma_t \succeq \lambda I \). To control the second term, we need to use a self-normalized concentration inequality, since \( Z_{\tau} \) is a random variable, and the normalizing term \( \Gamma_t = \lambda I + \sum_{\tau=1}^{t-1} Z_{\tau} Z_{\tau}^\top \) depends on the random realizations. In Lemma 10 in the appendix, we prove that with probability at least \( 1 - \delta \), for all \( t \in [T] \)
\[ \| S_t \|_{\Gamma_t^{-1}}^2 \leq 9d \log(1 + T/(d \lambda)) + 18 \log(T/\delta). \tag{5} \]
The lemma follows from straightforward calculations. \( \square \)

Before proceeding, it is worth commenting on the difference between our self-normalized inequality (5) and a slightly different one used by Abbasi-Yadkori et al. [1] for the linear case. In their setup, they have that \( \zeta_{\tau} \) is conditionally centered and sub-Gaussian, which simplifies the argument since after fixing the \( Z_{\tau} \)'s (and hence \( \Gamma_t \)), the randomness in \( \zeta_{\tau} \)'s suffices to provide concentration. In our case, we must use the randomness in \( Z_{\tau} \) itself, which is more delicate, since \( Z_{\tau} \) affects the numerator \( S_t \) but also the normalizer \( \Gamma_t \). In spite of this additional technical challenge, the two self-normalized processes admit similar bounds.

Next, we turn to the action selection step, where recall that either a single action is played deterministically, or the actions are played uniformly at random.

Lemma 6. Let \( \mu_t \triangleq \mathbb{E}_{a \sim \pi_t} z_{t, a} \) where \( \pi_t \) is the solution to (3), and assume that the conclusion of Lemma 5 holds. Then with probability at least \( 1 - \delta \)
\[ \text{Reg}(T) \leq \sqrt{2T \log(1/\delta)} + 2\gamma(T) \sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} \text{Cov}(z_{t,b}))}. \]

Proof. We first study the instantaneous regret, taking expectation over the random action. For this, we must consider two cases. First, with Lemma 5, if \( |A_t| = 1 \), we argue that the regret is actually zero. This follows from the Cauchy-Schwarz inequality since assuming \( A_t = \{a\} \) we get
\[ \langle \theta, z_{t,a} - z_{t,b} \rangle \geq (\hat{\theta}_t, z_{t,a} - z_{t,b}) - \gamma(T) \| z_{t,a} - z_{t,b} \|_{\Gamma_t^{-1}} \]
which is non-negative using the fact that \( b \) was eliminated. Therefore \( a \) is the optimal action and we incur no regret. Since \( \pi_t \) has no covariance, the upper bound holds.

On the other rounds, we set \( \pi_t \) = Unif(\( \{a, b\} \)) and hence \( \mu_t = (z_{t,a} + z_{t,b})/2 \). Assuming again that \( a \) is the optimal action, the expected regret is
\[ \langle \theta, z_{t,a} - a \rangle = \frac{1}{2} \langle \theta, z_{t,a} - z_{t,b} \rangle \leq \frac{1}{2} \left( (\hat{\theta}_t, z_{t,a} - z_{t,b}) + \gamma(T) \| z_{t,a} - z_{t,b} \|_{\Gamma_t^{-1}} \right) \]
\[ \leq \gamma(T) \| z_{t,a} - z_{t,b} \|_{\Gamma_t^{-1}} \leq 2\gamma(T) \sqrt{\text{tr}(\Gamma_t^{-1} \text{Cov}(z_{t,b}))}. \]
Here the first inequality uses Cauchy-Schwarz, the second uses (2), since neither action was eliminated, and the third uses (3). This bounds the expected regret, and the lemma follows by Azuma’s inequality.

The last step of the proof is to control the sequence
\[
\sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} \text{Cov}(z_{t,b}))}.
\]
First, recall that
\[
\text{Cov}(z_{t,b}) \triangleq \mathbb{E}_{b \sim \pi_t}[ (z_{t,b} - \mu_t)(z_{t,b} - \mu_t)^\top]
\]
with \(\mu_t \triangleq \mathbb{E}_{b \sim \pi_t}[z_{t,b}]\). Since in the two-arm case \(\pi_t\) either chooses an arm deterministically or uniformly randomizes between the two arms, the following always holds:
\[
\text{Cov}(z_{t,b}) = (z_{t,a_t} - \mu_t)(z_{t,a_t} - \mu_t)^\top.
\]
It follows that \(\Gamma_{t+1} \triangleq \Gamma_t + \text{Cov}_{b \sim \pi_t}(z_{t,b})\), and with \(\Gamma_1 \triangleq \lambda I\), the standard potential argument for online ridge regression applies. We state the conclusion here, and provide a complete proof in the appendix.

**Lemma 7.** Let \(\Gamma_t, \pi_t\) be defined as above and define \(M_t \triangleq (z_{t,a_t} - \mu_t)(z_{t,a_t} - \mu_t)^\top\). Then
\[
\sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} M_t)} \leq \sqrt{dT(1 + 1/\lambda) \log(1 + T/(d\lambda))}.
\]

Combining the three lemmas establishes a regret bound of
\[
\text{Reg}(T) \leq O\left(\sqrt{dT \log(T/\delta) \log(T/d)} + d\sqrt{T} \log(T/d)\right)
\]
with probability at least \(1 - \delta\) in the two-action case.

**Extending to many actions.** Several more technical steps are required for the general setting. First, the martingale inequality used in Lemma 5 requires that the random vectors are symmetric about the origin. This is only true for the two-action case, and in fact a similar inequality does not hold in general for the non-symmetric situation that arises with more actions. In the non-symmetric case, both the empirical and the population covariance must be used in the normalization, so the analogue of (5) is instead
\[
\|S_t\|_{(\Gamma_t + \mathbb{E}\Gamma_t)^{-1}}^2 \leq 27d \log(1 + 2T/d) + 54 \log(4T/\delta).
\]
On the other hand, the error term for our estimator depends only on the empirical covariance \(\Gamma_t\). To correct for the discrepancy, we use a covering argument\(^4\) to establish
\[
\lambda I + \Gamma_t \succeq (\lambda - 6d \log(T/\delta)) I + (\Gamma_t + \mathbb{E}\Gamma_t)/3.
\]
With this semidefinite inequality, we can translate from the Mahalanobis norm in the weaker self-normalized bound to one with just \(\Gamma_t\), which controls the error for the estimator.

We also argue that problem (3) is always feasible, which is the contents of the following lemma.

\(^4\)For technical reasons, the Matrix Bernstein inequality does not suffice here since it introduces a dependence on the maximal variance. See Appendix for details.
Lemma 8. Problem (3) is convex and always has a feasible solution. Specifically, for any vectors \( z_1, \ldots, z_n \in \mathbb{R}^d \) and any positive definite matrix \( M \), there exists a distribution \( w \in \Delta([n]) \) with mean \( \mu_w \) such that

\[
\forall i \in [n], \| z_i - \mu_w \|_M^2 \leq \text{tr}(M \text{Cov}(z_i)).
\]

The proof uses convex duality. Integrating these new arguments into the proof for the two-action case leads to Theorem 4.

5 Experiments

We conduct a simple experiment to compare BOSE with several other approaches. We simulate three different environments that follow the semiparametric contextual bandits model with \( d = 10, K = 2 \). In the first setting the reward is linear and the action features are drawn uniformly from the unit sphere. In the latter two settings, we set \( f_t(x_t) = -\max_a \langle \theta, z_t, a \rangle \), which is related to the construction in the proof of Proposition 3. One of these semiparametric settings has action features sampled from the unit sphere, while for the other, we sample from the intersection of the unit sphere and the positive orthant.

In Figure 1, we plot the performance of Algorithm 1 against four baseline algorithms: (1) OFUL: the optimistic algorithm for linear stochastic bandits [1], (2) THOMPSON sampling for linear contextual bandits [4], (3) EPSGREEDY: the \( \epsilon \)-greedy approach [21] with a linear policy class, (4) ILTCB: a more sophisticated agnostic algorithm [3] with linear policy class. The first algorithm is deterministic, so can have linear regret in our setting, but is the natural baseline and one we hope to improve. Thompson Sampling is another natural baseline, and a variant was used by Greenewald et al. [19] in essentially the same setting as ours. The latter two have \( (Kd)^{1/3}T^{2/3} \) and \( \sqrt{KdT} \) regret bounds respectively under our assumptions, but require solving cost-sensitive classification problems, which are NP-hard in general. Following prior empirical evaluations [20], we use a surrogate loss formulation based on square loss minimization in the implementation.

The results of the experiment are displayed in Figure 1, where we plot the cumulative regret against the number of rounds \( T \). All algorithms have a single parameter that governs the degree of exploration. In BOSE

\[\text{http://github.com/akshaykr/oracle_cb/}.\]
and OFUL, this is the constant $\gamma(T)$ in the confidence bound, in THOMPSON it is the variance of the prior, and in ILTCB and EPSGREEDY it is the amount of uniform exploration performed by the algorithm. For each algorithm we perform 10 replicates for each of 20 values of the corresponding parameter, and we plot the best average performance, with error bars corresponding to $\pm$2 standard deviations.

In the linear experiment (Figure 1, left panel), BOSE performs the worst, but is competitive with the agnostic approaches, demonstrating a price to pay for robustness. The experimental setup in the center panel is identical except with confounding, and BOSE is robust to this confounding, with essentially the same performance, while the three baselines degrade dramatically. Finally, when the features lie in the positive orthant (right panel), OFUL degrades further, while BOSE remains highly effective.

Regarding the baselines, we make two remarks:

1. Intuitively, the positive orthant setting is more challenging for OFUL since there is less inherent randomness in the environment to overcome the confounding effect.
2. The agnostic approaches, despite strong regret guarantees, perform somewhat poorly in our experiments, and we believe this for three reasons. First, our surrogate-loss implementation is based on an implicit realizability assumption, which is not satisfied here. Second, we expect that the constant factors in their regret bounds are significantly larger than those of BOSE or OFUL. For computational reasons, we only solve the optimization problem in ILTCB every 50 rounds, which causes a further constant factor increase in the regret.

Overall, while BOSE is worse than other approaches in the linear environment, the experiment demonstrates that when the environment is not perfectly linear, approaches based on realizability assumptions (either explicitly like in OFUL, or implicitly like in implementations of ILTCB and EPSGreedy), can fail. We emphasize that linear environments are rare in practice, and such assumptions are typically impossible to verify. We therefore believe that trading off a small loss in performance in the specialized linear case for significantly more robustness, as BOSE demonstrates, is desirable.

6 Discussion

This paper studies a generalization of the linear stochastic bandits setting, where rewards are confounded by an adaptive adversary. Our new algorithm, BOSE, achieves the optimal regret, and also matches (up to logarithmic factors) the best algorithms for the linear case. Our empirical evaluation shows that BOSE offers significantly more robustness than prior approaches, and performs well in several environments.
A Using the OLS Estimator

Here we construct an example problem to demonstrate how using the standard OLS estimator can fail in the semiparametric setting. While not a comprehensive proof against all asymptotically biased approaches, similar examples can be constructed for related estimators.

Consider a two-dimensional problem with two actions and no stochastic noise, where \( \theta = e_2 \), the second standard basis vector. On the even rounds, the actions are \( z_1 = (1, 1) \), \( z_2 = (1, 1/3) \) and the confounding term is \( f = -1 \). On the odd rounds, the actions are \( z_1 = z_2 = (1, 0) \) and the confounding term is \( f = 1 \). For any policy for selecting actions, the OLS estimator before round \( t \) (for even \( t \)) is the solution to the following optimization problem:

\[
\min_{w \in \mathbb{R}^2} \alpha (w_1 + w_2)^2 + (1 - \alpha) (w_1 + w_2/3 + 2/3)^2 + (w_1 - 1)^2 = L(w)
\]

where \( \alpha \in [0, 1] \) corresponds to the fraction of the even rounds (up to round \( t \)) where the policy chose \( z_1 \). We will argue that, for any \( \alpha \), the solution to this problem \( \hat{w} \) has \( \hat{w}_2 < 0 \). Since there is no stochastic noise, there is no need for confidence bounds once the covariance is full rank, which happens after the second round. Together, this implies that any sensible policy based on \( \hat{w} \) will prefer \( z_2 \) to \( z_1 \) on the even rounds, but \( z_1 \) yields higher reward by a fixed constant. Thus using OLS in a confidence-based approach leads to linear regret.

We now show that \( \hat{w}_2 \) is strictly negative. We have

\[
\frac{\partial L(w)}{\partial w_1} = 2 \alpha (w_1 + w_2) + 2(1 - \alpha)(w_1 + w_2/3 + 2/3) + 2(w_1 - 1),
\]

\[
\frac{\partial L(w)}{\partial w_2} = 2 \alpha (w_1 + w_2) + \frac{2}{3}(1 - \alpha)(w_1 + w_2/3 + 2/3).
\]

Setting both equations equal to zero yields the following system:

\[
4w_1 + (2/3 + 4\alpha/3)w_2 = 2/3 + 4\alpha/3, \quad (2/3 + 4\alpha/3)w_1 + (2/9 + 16\alpha/9)w_2 = 4\alpha/9 - 4/9.
\]

The solution to this system is

\[
w_1 = \frac{(2\alpha + 1)^2}{-4\alpha^2 + 12\alpha + 1}, \quad w_2 = \frac{4\alpha^2 + 5}{4\alpha^2 - 12\alpha - 1},
\]

provided that \( 4\alpha^2 \neq 12\alpha + 1 \), which is not possible with \( \alpha \in [0, 1] \). In the interval \( [0, 1] \) we have that
\( 4\alpha^2 - 12\alpha - 1 < 0 \), and hence \( w_2 < 0 \). Thus, the OLS estimator incorrectly predicts that \( z_2 \) receives higher reward than \( z_1 \) on the even rounds. Since confidence intervals are not needed, the algorithm suffers linear regret.

B Proof of Proposition 3

We consider two possible values for the true parameter: \( \theta_1 = e_1 \in \mathbb{R}^2, \theta_2 = e_2 \in \mathbb{R}^2 \). At all rounds, the context \( x_t = \{e_1, e_2\} \) contains just two actions, and we further assume that the noise term \( \xi_t = 0 \) almost surely. Since the action \( a_t \) is a deterministic function of the history, it can also be computed by the adaptive adversary at the beginning of the round, and the adversary chooses

\[
f_t(x_t) = -1 \{a_t = \arg\max_a (\theta, z_t, a)\}.
\]
We first focus on the simpler two action case. Before turning to the main analysis, we prove two supporting lemmas. We show that the algorithm must choose the suboptimal action at least \( d \) times, leading to the lower bound.

**C Proof for the Two-Action Case**

We first focus on the simpler two action case. Before turning to the main analysis, we prove two supporting lemmas. The first is an algebraic inequality relating matrix determinants to traces. This inequality also appears in Abbasi-Yadkori et al. [1].

**Lemma 9.** Let \( X_1, \ldots, X_n \) denote vectors in \( \mathbb{R}^d \) with \( \| X_i \|_2 \leq L \) for all \( i \in [n] \). Define \( \Gamma = \lambda I + \sum_{i=1}^n X_i X_i^\top \). Then

\[
\det(\Gamma) \leq (\lambda + nL^2/d)^d.
\]

**Proof.** We will apply the following standard argument:

\[
\det(\Gamma)^{1/d} \leq \frac{1}{d} \text{tr}(\Gamma) = \frac{1}{d} \text{tr}(\lambda I) + \frac{1}{d} \sum_{i=1}^n \text{tr}(X_i X_i^\top) = \lambda + \frac{1}{d} \sum_{i=1}^n \| X_i \|_2^2 \leq \lambda + nL^2/d.
\]

The first step is a spectral version of the AM-GM inequality and the remaining steps use linearity of the trace operator and the boundedness conditions.

The second lemma is a new self-normalized concentration inequality for vector valued martingales.

**Lemma 10 (Symmetric self-normalized inequality).** Let \( \{ F_t \}_{t=1}^T \) be a filtration and let \( \{ (Z_t, \zeta_t) \}_{t=1}^T \) be a stochastic process with \( Z_t \in \mathbb{R}^d \) and \( \zeta_t \in \mathbb{R} \) such that (1) \((Z_t, \zeta_t)\) is \( F_t \) measurable, (2) \(|\zeta_t| \leq M \) for all \( t \in [T] \), (3) \( Z_t \perp \zeta_t | F_t \), (4) \( \mathbb{E}[|Z_t| | F_t] = 0 \), and (5) for all \( x \in \mathbb{R}^d \), \( \mathcal{L}(\langle x, Z_t \rangle | F_t) = \mathcal{L}(\langle -x, Z_t \rangle | F_t) \) where \( \mathcal{L} \) denotes the probability law, so that \( Z_t \) is conditionally symmetric. Let \( \Sigma \triangleq \sum_{t=1}^T Z_t Z_t^\top \). Then for any positive definite matrix \( Q \) we have

\[
\mathbb{P} \left[ \left\| \sum_{t=1}^T Z_t \zeta_t \right\|_{(Q + M^2 \Sigma)^{-1}}^2 \geq 2 \log \left( \frac{\det(Q + M^2 \Sigma)}{\det(Q)} \right) \right] \leq \delta.
\]

**Proof.** The proof follows the recipe in de la Peña et al. [15] (See also de la Peña et al. [14] for a more comprehensive treatment including the univariate case). We start by applying the Chernoff method. Let \( \Sigma \triangleq Q + M^2 \Sigma \). We can write

\[
\mathbb{P} \left[ \left\| \sum_{t=1}^T Z_t \zeta_t \right\|_{\Sigma^{-1}}^2 \geq 2 \log \left( \frac{\det(\Sigma)}{\det(Q)} \right) \right] \leq \mathbb{P} \left[ \exp \left( \frac{1}{2} \left\| \sum_{t=1}^T Z_t \zeta_t \right\|_{\Sigma^{-1}}^2 \right) \geq \frac{1}{\delta} \sqrt{\frac{\det(\Sigma)}{\det(Q)}} \right] \leq \delta \mathbb{E} \left[ \sqrt{\frac{\det(Q)}{\det(\Sigma)}} \exp \left( \frac{1}{2} \left\| \sum_{t=1}^T Z_t \zeta_t \right\|_{\Sigma^{-1}}^2 \right) \right].
\]
Therefore, if we prove that this latter expectation is at most one, we will arrive at the conclusion. A similar statement appears in Theorem 1 of de la Peña et al. [15], but our process is slightly different due to the presence of \( \zeta_t \). To bound this latter expectation, fix some \( \lambda \in \mathbb{R}^d \) and consider an exponentiated process with the increments

\[
D^\lambda_t \triangleq \exp \left( \langle \lambda, Z_t \zeta_t \rangle - \frac{M^2 \langle \lambda, Z_t \rangle^2}{2} \right).
\]

Observe that \( \mathbb{E}[D^\lambda_t | F_t] \leq 1 \) since by the conditional symmetry of \( Z_t \), we have

\[
\mathbb{E}[D^\lambda_t | F_t] = \mathbb{E} \left[ \mathbb{E} \left[ D^\lambda_t \mid F_t, \zeta_t \right] \mid F_t \right]
\]

\[
= \mathbb{E} \left[ \exp \left( -\frac{M^2 \langle \lambda, Z_t \rangle^2}{2} \right) \right] \leq 1.
\]

This argument first uses the conditional symmetry of \( Z_t \) and the conditional independence of \( Z_t, \zeta_t \), then the identity \((e^x + e^{-x})/2 = \cosh(x)\) and finally the analytical inequality \( \cosh(x) \leq e^{x^2/2} \). Finally in the last step we use the bound \( |\zeta_t| \leq M \). This implies that the martingale \( U^\lambda_t \triangleq \prod_{t=1}^T D^\lambda_t \) is a super-martingale with \( \mathbb{E}[U^\lambda_t] \leq 1 \) for all \( t \), since by induction

\[
\mathbb{E}[U^\lambda_t] = \mathbb{E}[U^\lambda_{t-1} | \mathbb{E}[D^\lambda_t | F_t]] \leq \mathbb{E}[U^\lambda_{t-1}] \leq \ldots \leq 1.
\]

Now we apply the method of mixtures. In a standard application of the Chernoff method, we would choose \( \lambda \) to maximize \( \mathbb{E}[U^\lambda_t] \), but since we still have an expectation, we cannot swap expectation and maximum. Instead, we integrate the inequality \( \mathbb{E}[U^\lambda_t] \leq 1 \), which holds for any \( \lambda \), against a Gaussian distribution with covariance \( Q^{-1} \). By Fubini’s theorem, we can swap the expectations to obtain

\[
1 \geq \mathbb{E}_{\lambda \sim \mathcal{N}(0, Q^{-1})} \mathbb{E}[U^\lambda_t] = \mathbb{E} \int U^\lambda_t (2\pi)^{-d/2} \sqrt{\det(Q)} \exp(-\lambda^T Q \lambda/2) d\lambda
\]

\[
= \mathbb{E} \int (2\pi)^{-d/2} \sqrt{\det(Q)} \exp \left( \sum_{t=1}^T \langle \lambda, Z_t \zeta_t \rangle - \frac{M^2 \lambda^T \sum_{t=1}^T Z_t Z_t^T \lambda + \lambda^T Q \lambda}{2} \right) d\lambda
\]

\[
= \mathbb{E} \int (2\pi)^{-d/2} \sqrt{\det(Q)} \exp \left( \langle \lambda, S \rangle - \frac{M^2 \lambda^T \Sigma \lambda + \lambda^T Q \lambda}{2} \right) d\lambda,
\]

where \( S \triangleq \sum_{t=1}^T Z_t \zeta_t \) and recall that \( \Sigma \triangleq \sum_{t=1}^T Z_t Z_t^T \). By completing the square, the term in the exponent can be rewritten as

\[
\langle \lambda, S \rangle - \frac{M^2 \lambda^T \Sigma \lambda + \lambda^T Q \lambda}{2} = \frac{1}{2} (-\langle \lambda - \Sigma^{-1} S \rangle^T \Sigma (\lambda - \Sigma^{-1} S) + S^T \Sigma^{-1} S),
\]

where recall that \( \Sigma \triangleq M^2 \Sigma + Q \). As such we obtain

\[
1 \geq \mathbb{E} \left[ \exp \left( S^T \Sigma^{-1} S/2 \right) \right] \times \int (2\pi)^{-d/2} \sqrt{\det(Q)} \exp \left( -\langle \lambda - \Sigma^{-1} S \rangle^T \Sigma (\lambda - \Sigma^{-1} S) \right) d\lambda
\]

\[
= \mathbb{E} \left[ \frac{\sqrt{\det(Q)}}{\det(\Sigma)} \exp (S^T \Sigma^{-1} S) \right].
\]
This proves the lemma.

Equipped with the two lemmas, we can now turn to the analysis of the influence-adjusted estimator.

**Lemma 11 (Restatement of Lemma 5).** Under Assumption 1 and Assumption 2, with probability at least \(1 - \delta\), the following holds simultaneously for all \(t \in [T]\):

\[
\| \hat{\theta}_t - \theta \|_{\Gamma_t} \leq \sqrt{\lambda} + \sqrt{9d \log(1 + T/(d \lambda))} + 18 \log(T/\delta).
\]

**Proof.** Recall that we define \(\hat{\theta}_t, \Gamma_t\) to be the estimator and matrix used in round \(t\), based on \(t - 1\) examples. Fixing a round \(t\), we start by expanding the definition of \(\hat{\theta}_t\). We use the shorthand \(z_\tau \triangleq z_{\tau,a_\tau}, \mu_\tau \triangleq \mathbb{E}_{a \sim \pi_\tau}[z_{\tau,a}], \) and \(r_\tau \triangleq r_\tau(a_\tau).

\[
\hat{\theta}_t = \Gamma_t^{-1} \sum_{\tau=1}^{t-1} (z_\tau - \mu_\tau) r_\tau = \Gamma_t^{-1} \sum_{\tau=1}^{t-1} (z_\tau - \mu_\tau)(\langle \theta, z_\tau \rangle + f_\tau(x_\tau) + \xi_\tau)
\]

\[
= \Gamma_t^{-1} \sum_{\tau=1}^{t-1} (z_\tau - \mu_\tau)(\langle \theta, z_\tau - \mu_\tau \rangle + \langle \theta, \mu_\tau \rangle + f_\tau(x_\tau) + \xi_\tau)
\]

\[
= (\Gamma_t)^{-1}(\Gamma_t - \lambda I)\theta + \Gamma_t^{-1} \sum_{\tau=1}^{t-1} (z_\tau - \mu_\tau)(\langle \theta, \mu_\tau \rangle + f_\tau(x_\tau) + \xi_\tau).
\]

Let \(Z_\tau \triangleq z_\tau - \mu_\tau\) and \(\xi_\tau \triangleq \langle \theta, \mu_\tau \rangle + f_\tau(x_\tau) + \xi_\tau\). With this expansion, we can write

\[
\| \hat{\theta}_t - \theta \|_{\Gamma_t} = \| - \lambda \Gamma_t^{-1}\theta + \Gamma_t^{-1} \sum_{\tau=1}^{t-1} Z_\tau \xi_\tau \|_{\Gamma_t} \leq \| \lambda \theta \|_{\Gamma_t^{-1}} + \| \sum_{\tau=1}^{t-1} Z_\tau \xi_\tau \|_{\Gamma_t^{-1}} \leq \sqrt{\lambda} + \| \sum_{\tau=1}^{t-1} Z_\tau \xi_\tau \|_{\Gamma_t^{-1}}.
\]

To finish the proof, we apply Lemma 10 to this last term. To verify the preconditions of the lemma, let \(\mathcal{F}_\tau \triangleq \sigma(x_1, \ldots, x_\tau, a_1, \ldots, a_{\tau-1}, \xi_1, \ldots, \xi_{\tau-1})\) denote the \(\sigma\)-algebra corresponding to the \(\tau\)th round, after observing the context \(x_\tau\). Then the policy \(\pi_\tau\) and hence the action \(a_\tau\) are \(\mathcal{F}_\tau\) measurable and so is the noise term \(\xi_\tau\). Therefore, \(Z_\tau = z_{\tau,a_\tau} - \mathbb{E}_{a \sim \pi_\tau}[z_{\tau,a}]\) is measurable, which verifies the first precondition. Using the boundedness properties in Assumption 2, we know that \(|\xi_\tau| \leq 3 \triangleq M\), and by construction of the random variables, we have \(Z_\tau \perp \xi_\tau|\mathcal{F}_\tau\) and \(\mathbb{E}[Z_\tau|\mathcal{F}_\tau] = 0\). Finally, for the symmetry property, either \(Z_\tau|\mathcal{F}_\tau \equiv 0\) if one action is eliminated, or otherwise we have \(\mu_\tau = \frac{1}{2}(z_{\tau,1} + z_{\tau,2})\) since there are only two actions. In this case the random variable \(Z_\tau|\mathcal{F}_\tau = \epsilon_\tau (z_{\tau,1} - z_{\tau,2})/2\) where \(\epsilon_\tau\) is a Rademacher random variable. By inspection this is clearly conditionally symmetric. As such, we may apply Lemma 10, which reveals that with probability at least \(1 - \delta\),

\[
\| \sum_{\tau=1}^{t-1} Z_\tau \xi_\tau \|_{\Gamma_t^{-1}} \leq 2M^2 \log \left( \frac{1}{\delta} \sqrt{\frac{\det(M^2 \Gamma_t)}{\det(M^2 \lambda I)}} \right)
\]

\[
= 18 \log \left( \sqrt{\lambda^{-d} \det(\Gamma_t)/\delta} \right).
\]

The inequality here is Lemma 10 with \(Q = M^2 \lambda I\), and for the last equality we use that \(\det(cQ) = c^d \det(Q)\) for a \(d \times d\) positive semidefinite matrix \(Q\). As two final steps, we apply Lemma 9 and take a union bound.
over all rounds $T$. Combining these, we get that for all $T$,

$$||\hat{\theta}_t - \theta||_{\Gamma_t} \leq \sqrt{\lambda} + \left\| \sum_{\tau=1}^{t-1} Z_\tau \zeta_\tau \right\|_{\Gamma_t^{-1}} \leq \sqrt{\lambda} + \sqrt{18 \left( \log(\lambda^{-d} \det(\Gamma_t)) + \log(T/\delta) \right) + 9d \log(1 + T/(d\lambda)) + 18 \log(T/\delta)}. \quad \square$$

Therefore, with $\gamma(T) \triangleq \sqrt{\lambda} + \sqrt{9d \log(1 + T/(d\lambda)) + 18 \log(T/\delta)}$ we can apply Lemma 6 to bound the regret by

$$\text{Reg}(T) \leq \sqrt{2T \log(1/\delta)} + 2\gamma(T) \sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} \text{Cov}(z_{t,b})).}$$

Via a union bound, this inequality holds with probability at least $1 - 2\delta$. To finish the proof we need to analyze this latter term. This is the contents of the following lemma. A related statement, with a similar proof, appears in Abbasi-Yadkori et al. [1].

**Lemma 12.** Let $X_1, \ldots, X_T$ be a sequence of vectors in $\mathbb{R}^d$ with $\|X_t\|_2 \leq 1$ and define $\Gamma_1 \triangleq \lambda I$, $\Gamma_t \triangleq \Gamma_{t-1} + X_{t-1}X_{t-1}^\top$. Then

$$\sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} X_tX_t^\top)} \leq \sqrt{T d(1 + 1/\lambda) \log(1 + T/(d\lambda))}. \quad \text{Proof.} \quad \text{First, apply the Cauchy-Schwarz inequality to the left hand side to obtain}$$

$$\sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} X_tX_t^\top)} \leq \sqrt{T} \sqrt{\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} X_tX_t^\top)}. \quad \text{For the remainder of the proof we work only with the second term. Let us start by analyzing a slightly different quantity, tr}(\Gamma_{t+1}^{-1}X_tX_t^\top). \text{ By concavity of log det}(M), \text{ we have}$$

$$\log\det(\Gamma_t) \leq \log\det(\Gamma_{t+1}) + \text{tr}(\Gamma_{t+1}^{-1}(\Gamma_t - \Gamma_{t+1})), \quad \text{which implies}$$

$$\text{tr}(\Gamma_{t+1}^{-1}X_tX_t^\top) = \text{tr}(\Gamma_{t+1}^{-1}(\Gamma_{t+1} - \Gamma_t)) \leq \log\det(\Gamma_{t+1}) - \log\det(\Gamma_t) \quad \text{As such, we obtain a telescoping sum}$$

$$\sum_{t=1}^{T} \text{tr}(\Gamma_{t+1}^{-1}X_tX_t^\top) \leq \log\det(\Gamma_{T+1}) - \log\det(\Gamma_1) \leq d \log(\lambda + T/d) - d \log \lambda = d \log(1 + T/(d\lambda)) \quad \text{The first inequality here uses the concavity argument and the second uses Lemma 9. To finish the proof, we must translate back to } \Gamma_t^{-1}. \text{ For this, we use the Sherman-Morrison-Woodbury identity, which reveals that}$$

$$X_t^\top \Gamma_{t+1}^{-1} X_t = X_t^\top (\Gamma_t + X_tX_t^\top)^{-1} X_t = X_t^\top \left( \Gamma_t^{-1} - \frac{\Gamma_t^{-1}X_tX_t^\top \Gamma_t^{-1}}{1 + \|X_t\|_{\Gamma_t^{-1}}^2} \right) X_t$$

$$= \frac{\|X_t\|_{\Gamma_t^{-1}}^2}{1 + \|X_t\|_{\Gamma_t^{-1}}^2} \geq (1 + 1/\lambda)^{-1}\|X_t\|_{\Gamma_t^{-1}}^2.$$
Here in the last step we use that \( \|X_t\|_{\Gamma_t}^2 \leq \|X_t\|_{(\lambda I)^{-1}}^2 \leq 1/\lambda \). Overall, we obtain
\[
\sum_{t=1}^T \text{tr}(\Gamma_t^{-1}X_tX_t^\top) \leq (1 + 1/\lambda)d \log(1 + T/(d\lambda)),
\]
and combined with the first application of Cauchy-Schwarz, this proves the lemma.

Combining the lemmas, we have that with probability at least \( 1 - 2\delta \), the regret is at most
\[
\text{Reg}(T) \leq \sqrt{2T \log(1/\delta)} + 2\gamma(T)\sqrt{Td(1 + 1/\lambda) \log(1 + T/(d\lambda))}.
\]

With \( \lambda = 1 \), this bound is \( O(\sqrt{Td \log(T/d)} \log(T/d)) \).

### D Proof for the General Case

We now turn to the more general case. We need several additional lemmas.

**Lemma 13 (Restatement of Lemma 8).** Problem (3) is convex and always has a feasible solution. Specifically, for any vectors \( z_1, \ldots, z_n \in \mathbb{R}^d \) and any positive definite matrix \( M \), there exists a distribution \( w \in \Delta([n]) \) with mean \( \mu_w = \mathbb{E}_{b \sim w}[z_b] \) such that
\[
\forall i \in [n], \quad \|z_i - \mu_w\|_M^2 \leq \text{tr}(M \text{Cov}_w(z_b)).
\]

**Proof.** We analyze the minimax program
\[
\min_{w \in \Delta([n])} \max_{i \in [n]} \|z_i - \mu_w\|_M^2 - \text{tr}(M \text{Cov}_w(z)).
\]

The goal is to show that the value of this program is non-negative, which will prove the result. Expanding the definitions, we have
\[
\min_{w \in \Delta([n])} \max_{i \in [n]} \|z_i - \mu_w\|_M^2 - \text{tr}(M \text{Cov}_w(z)) = \min_{w \in \Delta([n])} \max_{i \in [n]} \sum_i v_i \left( \|z_i - \mu_w\|_M^2 + \mu_w^\top M \mu_w - \sum_j w_j z_j^\top M z_j \right).
\]
The last equivalence here is Sion’s Minimax Theorem [31], which is justified since both domains are compact convex subsets of \( \mathbb{R}^n \) and since the objective is linear in the maximizing variable \( v \), and convex in the minimizing variable \( w \). This convexity is clear since \( \mu_w \) is a linear in \( w \), and hence the first two terms are convex quadratics (since \( M \) is positive definite), while the third term is linear in \( w \). Thus Sion’s theorem lets us swap the order of the minimization and maximization.
Now we upper bound the solution by setting $w = v$. This gives
\[
\leq \max_{v \in \Delta(n)} \sum_i v_i \left( \| z_i - \mu_v \|_M^2 + \mu_v^\top M \mu_v - \sum_j v_j z_j^\top M z_j \right)
\]
\[
= \max_{v \in \Delta(n)} \sum_i v_i \left( (z_i - \mu_v)^\top M (z_i - \mu_v) + \mu_v^\top M \mu_v - \sum_j v_j z_j^\top M z_j \right) = 0. \quad \square
\]

To prove the analog of Lemma 10, we need several additional tools. First, we use Freedman’s inequality to derive a positive-semidefinite inequality relating the sample covariance matrix to the population matrix.

**Lemma 14.** Let $X_1, \ldots, X_n$ be conditionally centered random vectors in $\mathbb{R}^d$ adapted to a filtration $\{\mathcal{F}_t\}_{t=1}^n$ with $\|X_i\|_2 \leq 1$ almost surely. Define $\hat{\Sigma} \triangleq \sum_{i=1}^n X_i X_i^\top$ and $\Sigma \triangleq \sum_{i=1}^n \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i]$. Then, with probability at least $1 - \delta$, the following holds simultaneously for all unit vectors $v \in \mathbb{R}^d$:
\[
v^\top \Sigma v \leq 2v^\top \hat{\Sigma} v + 9d \log(9n) + 8 \log(2/\delta).
\]

This lemma is related to the Matrix Bernstein inequality, which can be used to control $\|\Sigma - \hat{\Sigma}\|_2$, a quantity that is quite similar to what we are controlling here. The Matrix Bernstein inequality can be used to derive a high probability bound of the form
\[
\forall v \in \mathbb{R}^d, \|v\|_2 = 1, \quad v^\top (\Sigma - \hat{\Sigma}) v \leq \frac{1}{2} \|\Sigma\|_2 + c \log(dn/\delta),
\]
for a constant $c > 0$. On one hand, this bound is stronger than ours since the deviation term depends only logarithmically on the dimension. However, the variance term involves the spectral norm rather than a quantity that depends on $v$ as in our bound. Thus, Matrix Bernstein is worse when $\Sigma$ is highly ill-conditioned, and since we have essentially no guarantees on the spectrum of $\Sigma$, our specialized inequality, which is more adaptive to the specific direction $v$, is crucial. Moreover, the worse dependence on $d$ is inconsequential, since the error will only appear in a lower order term.

**Proof.** First consider a single unit vector $v \in \mathbb{R}^d$, we will apply a covering argument at the end of the proof. By assumption, the sequence of sums $\{\sum_{i=1}^n v^\top (X_i X_i^\top - \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i])v \mid \mathcal{F}_i\}^n_{i=1}$ is a martingale, so we may apply Freedman’s inequality [18, 6], which states that with probability at least $1 - \delta$
\[
|v^\top (\hat{\Sigma} - \Sigma)v| \leq 2 \sqrt{\sum_{i=1}^n \text{Var}(v^\top (X_i X_i^\top - \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i])v \mid \mathcal{F}_i)} \log(2/\delta) + 2 \log(2/\delta).
\]

Let us now upper bound the variance term: for each $i$,
\[
\text{Var}(v^\top (X_i X_i^\top - \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i])v \mid \mathcal{F}_i) \leq \mathbb{E}[(v^\top (X_i X_i^\top - \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i])v \mid \mathcal{F}_i)^2 \mid \mathcal{F}_i] \leq \mathbb{E}[(v^\top X_i)^4 \mid \mathcal{F}_i] \leq v^\top \mathbb{E}[X_i X_i^\top \mid \mathcal{F}_i] v,
\]
where the last inequality follows from the fact that $\|X_i\|_2 \leq 1$ and $\|v\|_2 \leq 1$. Therefore, the cumulative conditional variance is at most $v^\top \Sigma v$. Plugging this into Freedman’s inequality gives us
\[
|v^\top (\hat{\Sigma} - \Sigma)v| \leq 2 \sqrt{v^\top \Sigma v \log(2/\delta) + 2 \log(2/\delta)}.
\]
Now, using the fact that \(2\sqrt{ab} \leq \alpha a + b/\alpha\) for any \(\alpha > 0\), with the choice \(\alpha = 1/2\), we get
\[
|v^\top (\Sigma - \Sigma)v| \leq v^\top \Sigma v/2 + 4\log(2/\delta).
\]
Re-arranging, this implies
\[
v^\top \Sigma v \leq 2v^\top \hat{\Sigma} v + 8\log(2/\delta), \tag{7}
\]
which is what we would like to prove, but we need it to hold simultaneously for all unit vectors \(v\).

To do so, we now apply a covering argument. Let \(N\) be an \(\epsilon\)-covering of the unit sphere in the projection pseudo-metric \(d(u, v) = \|uu^\top - vv^\top\|_2\), with covering number \(N(\epsilon)\). Then via a union bound, a version of (7) holds simultaneously for all \(v \in N\), where we rescale \(\delta \to \delta/N(\epsilon)\).

Consider another unit vector \(u\) and let \(v\) be the covering element. We have
\[
u^\top \Sigma u = \text{tr}(\Sigma(uu^\top - vv^\top)) + v^\top \Sigma v \leq \text{tr}(\Sigma(uu^\top - vv^\top)) + 2v^\top \hat{\Sigma} v + 8\log(2N(\epsilon)/\delta)
\leq \|\Sigma - 2\Sigma\|_\ast \epsilon + 2u^\top \hat{\Sigma} u + 8\log(2N(\epsilon)/\delta).
\]
Here \(\|\cdot\|_\ast\) denotes the nuclear norm, which is dual to the spectral norm \(\|\cdot\|_2\). Since all vectors are bounded by 1, we obtain
\[
\|\Sigma - 2\Sigma\|_\ast \leq d\lambda_{\text{max}}(\Sigma - 2\Sigma) \leq 3dn.
\]
Overall, the following bound holds simultaneously for all unit vectors \(v \in \mathbb{R}^d\), except with probability at most \(\delta\):
\[
v^\top \Sigma v \leq 3dn\epsilon + 2v^\top \hat{\Sigma} v + 8\log(2N(\epsilon)/\delta).
\]

The last step of the proof is to bound the covering number \(N(\epsilon)\). For this, we argue that a covering of the unit sphere in the Euclidean norm suffices, and by standard volumetric arguments, this set has covering number at most \((3/\epsilon)^d\). To see why this suffices, let \(u\) be a unit vector and let \(v\) be the covering element in the Euclidean norm, which implies that \(\|u - v\|_2 \leq \epsilon\). Further assume that \(\langle u, v \rangle > 0\), which imposes no restriction since the projection pseudo-metric is invariant to multiplying by \(-1\). By definition we also have \(\langle u, v \rangle \leq 1\). Note that the projection norm is equivalent to the sine of the principal angle between the two subspaces, which once we restrict to vectors with non-negative inner product means that \(\|uu^\top - vv^\top\|_2 = \sin \angle(u, v)\). Now
\[
\sin \angle(u, v) = \sqrt{1 - \langle u, v \rangle^2} = \sqrt{(1 + \langle u, v \rangle)(1 - \langle u, v \rangle)}
\leq \sqrt{2(1 - \langle u, v \rangle)} = \sqrt{\|u\|^2_2 + \|v\|^2_2 - 2\langle u, v \rangle} = \|u - v\|_2 \leq \epsilon.
\]
Using the standard covering number bound, we now have
\[
v^\top \Sigma v \leq 3dn\epsilon + 2v^\top \hat{\Sigma} v + 8d\log(3/\epsilon) + 8\log(2/\delta).
\]
Setting \(\epsilon = 1/(3n)\) gives
\[
v^\top \Sigma v \leq d + 2v^\top \hat{\Sigma} v + 8d\log(9n) + 8\log(2/\delta) \leq 2v^\top \hat{\Sigma} v + 9d\log(9n) + 8\log(2/\delta).
\]
\[\square\]
With the positive semidefinite inequality, we can work towards a self-normalized martingale concentration bound. The following is a restatement of Lemma 7 from de la Peña et al. [15].

**Lemma 15** (Lemma 7 of de la Peña et al. [15]). Let \( \{X_i\}_{i=1}^n \) be a sequence of conditionally centered vector-valued random variables adapted to the filtration \( \{\mathcal{F}_t\}_{t=1}^n \) and such that \( \|X_i\|_2 \leq B \) for some constant \( B \). Then

\[
U_n(\lambda) = \exp \left( \lambda^\top \sum_{i=1}^n X_i - \lambda^\top \left( \sum_{i=1}^n X_i X_i^\top + \mathbb{E}[X_i X_i^\top | \mathcal{F}_i] \right) \frac{\lambda}{2} \right)
\]

is a supermartingale with \( \mathbb{E}[U_n(\lambda)] \leq 1 \) for all \( \lambda \in \mathbb{R}^d \).

The lemma is related to (6), but does not require that conditional probability law for \( X_i \) is symmetric, which we used previously. To remove the symmetry requirement, it is crucial that the quadratic self-normalization has both empirical and population terms. With this lemma, the same argument as in the proof of Lemma 10, yields a self-normalized tail bound.

**Lemma 16.** Let \( \{\mathcal{F}_t\}_{t=1}^T \) be a filtration and let \( \{(Z_t, \zeta_t)\}_{t=1}^T \) be a stochastic process with \( Z_t \in \mathbb{R}^d \) and \( \zeta_t \in \mathbb{R} \) such that (1) \( (Z_t, \zeta_t) \) is \( \mathcal{F}_t \) measurable, (2) \( |\zeta_t| \leq M \) for all \( t \in [T] \), (3) \( Z_t \perp \perp \zeta_t | \mathcal{F}_t \), and (4) \( \mathbb{E}[\zeta_t | \mathcal{F}_t] = 0 \). Let \( \hat{\Sigma} \equiv \sum_{t=1}^T Z_t Z_t^\top \) and \( \Sigma \equiv \sum_{t=1}^T \mathbb{E}[Z_t Z_t^\top | \mathcal{F}_t] \). Then for any positive definite matrix \( Q \) we have

\[
\Pr \left[ \left\| \sum_{t=1}^T Z_t \zeta_t \right\|_{(Q+M^2(\hat{\Sigma} + \Sigma))^{-1}} \geq 2 \log \left( \frac{1}{\delta} \sqrt{\det(Q + M^2(\hat{\Sigma} + \Sigma))} \right) \right] \leq \delta.
\]

**Proof.** The proof is identical to Lemma 10, but uses Lemma 15 in lieu of (6). \( \square \)

We can now analyze the influence-adjusted estimator.

**Lemma 17.** Under Assumption 1 and Assumption 2 and assuming that \( \lambda \geq 4d \log(9T) + 8 \log(4T/\delta) \), with probability at least \( 1 - \delta \), the following holds simultaneously for all \( t \in [T] \):

\[
\|\hat{\theta}_t - \theta\|_{\Gamma_t} \leq \sqrt{\lambda} + \sqrt{27d \log(1 + 2T/d) + 54 \log(4T/\delta)}.
\]

**Proof.** Using the same argument as in the proof of Lemma 5, we get

\[
\|\hat{\theta}_t - \theta\|_{\Gamma_t} \leq \sqrt{\lambda} + \left\| \sum_{\tau=1}^{t-1} Z_\tau \zeta_\tau \right\|_{\Gamma_t^{-1}},
\]

where \( Z_\tau \equiv z_\tau - \mu_\tau \) and \( \zeta_\tau \equiv \langle \theta, \mu_\tau \rangle + f_\tau(x_\tau) + \xi_\tau \), just as before. Now we must control this error term, for which we need both Lemma 14 and Lemma 16. Apply Lemma 14 to the vectors \( Z_\tau \), setting \( \hat{\Sigma}_t \equiv \sum_{\tau=1}^{t-1} Z_\tau Z_\tau^\top \) and \( \Sigma_t \equiv \sum_{\tau=1}^{t-1} \mathbb{E}[Z_\tau Z_\tau^\top | \mathcal{F}_\tau] \). With probability at least \( 1 - \delta/(2T) \), we have that for all unit vectors \( v \in \mathbb{R}^d \)

\[
v^\top \Sigma_t v \leq 2v^\top \hat{\Sigma}_t v + 9d \log(9T) + 8 \log(4T/\delta) \leq 2v^\top \Sigma_t v + 9d \log(9T) + 8 \log(4T/\delta).
\]

This implies a lower bound on all quadratic forms involving \( \hat{\Sigma}_t \), which leads to positive semidefinite inequality

\[
\lambda I + \hat{\Sigma}_t \geq (\lambda - 3d \log(9T) - 8/3 \log(4T/\delta))I + (\hat{\Sigma}_t + \Sigma_t)/3.
\]
This means that for any vector \( v \), we have
\[
\| v \|_{(\lambda I + \Sigma_t)}^2 \leq \| v \|_{((\lambda - 3d \log(9T) - 8/3 \log(4T/\delta)) I + (\hat{\Sigma}_t + \Sigma_t)/3)^{-1}}^2
\leq 3\| v \|_{((3\lambda - 9d \log(9T) - 8 \log(4T/\delta)) I + \hat{\Sigma}_t + \Sigma_t)^{-1}}^2.
\]

Before we apply Lemma 16, we must introduce the range parameter \( M \). Fix a round \( t \) and let \( A \triangleq ((3\lambda - 9d \log(9T) - 8 \log(4T/\delta)) I + \hat{\Sigma}_t + \Sigma_t) \) denote the matrix in the Mahalanobis norm. Then,
\[
\left\| \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau} \right\|_{(\lambda I + \Sigma_t)^{-1}}^2 = M^2 \left\| \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau} \right\|_{((M^2 A)^{-1})}^2.
\]

Now apply Lemma 16 with \( Q \triangleq M^2((3\lambda - 9d \log(9T) - 8 \log(4T/\delta)) I). \) Since we require \( Q > 0 \), this requires \( \lambda > 3d \log(9T) - 8/3 \log(4T/\delta) \), which is satisfied under the preconditions for the lemma. Under this assumption, we get
\[
\left\| \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau} \right\|_{(\lambda I + \Sigma_t)}^2 \leq 3M^2 \left\| \sum_{\tau=1}^{t-1} Z_{\tau} \zeta_{\tau} \right\|_{(Q + M^2(\Sigma_t + \Sigma_t))^{-1}}^2
\leq 6M^2 \log \left( \frac{4T}{\delta} \frac{\det(Q + M^2(\Sigma_t + \Sigma_t))}{\det(Q)} \right),
\]
with probability at least \( 1 - \delta/(2T) \). With a union bound, the inequality holds simultaneously for all \( T \), with probability at least \( 1 - \delta \).

The last step is to analyze the determinant. Using the same argument as in the proof of Lemma 9, it is not hard to show that
\[
\left( \frac{\det(Q + M^2(\Sigma_t + \Sigma_t))}{\det(Q)} \right)^{1/d} \leq 1 + \frac{2(t - 1)}{d(3\lambda - 9d \log(9T) - 8 \log(4T/\delta))}.
\]

If we impose the slightly stronger condition that \( \lambda \geq 4d \log(9T) + 8 \log(4T/\delta) \), then the term in the denominator is at least 1, and then we have that
\[
\| \hat{\theta}_t - \theta \|_{\Gamma_t} \leq \sqrt{\lambda} + \sqrt{6M^2 \log(4T/\delta)} + 3dM^2 \log(1 + 2T/d).
\]

Finally, as in the two-action case, we use the fact that \( |\zeta_t| \leq 3 \triangleq M \).

Recall the setting of \( \gamma(T) \triangleq \sqrt{\lambda} + \sqrt{27d \log(1 + 2T/d)} + 54 \log(4T/\delta) \) and the definition of \( \lambda \triangleq 4d \log(9T) + 8 \log(4T/\delta) \). For the remainder of the proof, condition on the probability \( 1 - \delta \) event that Lemma 17 holds. We now turn to analyzing the regret.

**Lemma 18.** Let \( \mu_t \triangleq \mathbb{E}_{a \sim \pi_t} z_{t,a} \) where \( \pi_t \) is the solution to (3) and assume the conclusion of Lemma 17 holds. Then with probability at least \( 1 - \delta \)
\[
\text{Reg}(T) \leq (1 + 6\gamma(T)) \sqrt{2T \log(2T/\delta)} + 3\gamma(T) \sqrt{T \sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1}(z_{t,a} - \mu_t)(z_{t,a} - \mu_t)^\top)}.
\]
This lemma is slightly more complicated than Lemma 6.

Proof. First, using the same application of Azuma’s inequality as in the proof of Lemma 6, with probability $1 - \delta/2$, we have

$$\text{Reg}(T) \leq \sqrt{2T \log(2/\delta)} + \sum_{t=1}^{T} \mathbb{E}_{a \sim \pi_t}[(\theta, z_{t,a_t}^* - z_{t,a}) \mid F_t].$$

Now we work with this latter expected regret

$$\sum_{t=1}^{T} \mathbb{E}_{a \sim \pi_t}[(\theta, z_{t,a_t}^* - z_{t,a}) \mid F_t] = \sum_{t=1}^{T} \langle \hat{\theta}, z_{t,a_t}^* - z_{t,a} \rangle \leq \sum_{t=1}^{T} \langle \hat{\theta}, z_{t,a_t}^* - z_{t,a} \rangle + \gamma(T) \|z_{t,a_t}^* - \mu_t\|_{\Gamma_t^{-1}}.$$  

For the first term, we use the filtration condition (2)

$$\langle \hat{\theta}, z_{t,a_t}^* - \mu_t \rangle = \sum_{a \in A_t} \pi_t(a) \langle \hat{\theta}, z_{t,a_t}^* - z_{t,a} \rangle \leq \gamma(T) \sum_{a \in A_t} \pi_t(a) \|z_{t,a_t}^* - z_{t,a}\|_{\Gamma_t^{-1}}$$

$$\leq \gamma(T) \|z_{t,a_t}^* - \mu_t\|_{\Gamma_t^{-1}} + \gamma(T) \sum_{a \in A_t} \pi_t(a) \|z_{t,a} - \mu_t\|_{\Gamma_t^{-1}}.$$

Applying the feasibility condition in (3), we can bound the expected regret by

$$\sum_{t=1}^{T} \mathbb{E}_{a \sim \pi_t}[(\theta, z_{t,a_t}^* - z_{t,a}) \mid F_t] \leq 3\gamma(T) \sum_{t=1}^{T} \sqrt{\text{tr}(\Gamma_t^{-1} \text{Cov}_{a \sim \pi_t}(z_{t,a}))} \leq 3\gamma(T) \sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \text{Cov}_{a \sim \pi_t}(z_{t,a})).$$

To complete the proof, we need to relate the covariance, which takes expectation over the random action, with the particular realization in the algorithm, since this realization affects the term $\Gamma_{t+1}$. Let $Z_t \triangleq z_{t,a_t} - \mu_t$ denote the centered realization, then the covariance term is

$$\text{Cov}_{a \sim \pi_t}(z_{t,a}) = \mathbb{E}[Z_t Z_t^\top | F_t]$$

In order to derive a bound on $\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \text{Cov}_{a \sim \pi_t}(z_{t,a}))$, we first consider the following

$$\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t]) - \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top).$$

Observe that sequence of sums $\{\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t]) - \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top)\}_{t=1}^{T}$ is a martingale. Also, each term $\text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t]) - \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top)$ is bounded by 1 because $\Gamma_1 = \lambda I$ and $\lambda > 1$. Applying the Freedman’s inequality reveals that with probability at least $1 - \delta/2$

$$\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t]) - \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top) \leq 2 \sqrt{\sum_{t=1}^{T} \mathbb{E}[(Z_t^\top \Gamma_t^{-1} Z_t)^2 | F_t] \log(2/\delta) + 2 \log(2/\delta)}$$

$$\leq 2 \sqrt{\sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t]) \log(2/\delta) + 2 \log(2/\delta)}$$

$$\leq 1 \sqrt{2 \sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} \mathbb{E}[Z_t Z_t^\top | F_t])} + 4 \log(2/\delta).$$
Then rearranging and plugging back into our regret bound, we have

\[
\text{Reg}(T) \leq \sqrt{2T \log(2/\delta)} + 3\gamma(T) \sqrt{2T \left( \sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top) + 4 \log(2/\delta) \right)}
\]

\[
\leq (1 + 6\gamma(T)) \sqrt{2T \log(2/\delta)} + 3\gamma(T) \sqrt{2T \sum_{t=1}^{T} \text{tr}(\Gamma_t^{-1} Z_t Z_t^\top)}.
\]

To conclude the proof of the theorem, apply Lemma 7, which applies on the last term on the RHS of Lemma 18. Overall, with probability at least \(1 - 2\delta\), we get

\[
\text{Reg}(T) \leq (1 + 6\gamma(T)) \sqrt{2T \log(2/\delta)} + 3\gamma(T) \sqrt{2T d \log(1 + T/(d\lambda))}.
\]

Since \(\lambda = \Theta(d \log(T/\delta))\) and \(\gamma(T) = O(\sqrt{d \log(T)} + \sqrt{\log(T/\delta)})\), we get with probability \(1 - \delta\),

\[
\text{Reg}(T) \leq O \left( d \sqrt{T \log(T)} + \sqrt{d T \log(T/\delta)} + \sqrt{T \log(T/\delta)} \log(1/\delta) \right).
\]

References

[1] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In *Advances in Neural Information Processing Systems*, 2011.

[2] Jacob D Abernethy, Elad Hazan, and Alexander Rakhlin. Competing in the dark: An efficient algorithm for bandit linear optimization. In *Conference on Learning Theory*, 2008.

[3] Alekh Agarwal, Daniel Hsu, Satyen Kale, John Langford, Lihong Li, and Robert E Schapire. Taming the monster: A fast and simple algorithm for contextual bandits. In *International Conference on Machine Learning*, 2014.

[4] Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. In *International Conference on Machine Learning*, 2013.

[5] Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire. The nonstochastic multiarmed bandit problem. *SIAM Journal on Computing*, 2002.

[6] Alina Beygelzimer, John Langford, Lihong Li, Lev Reyzin, and Robert E Schapire. Contextual bandit algorithms with supervised learning guarantees. In *International Conference Artificial Intelligence and Statistics*, 2011.

[7] Peter J Bickel, Chris AJ Klaassen, Ya’acov Ritov, and Jon A Wellner. *Efficient and adaptive estimation for semiparametric models*. Springer New York, 1998.

[8] Sébastien Bubeck, Nicolo Cesa-Bianchi, and Sham M Kakade. Towards minimax policies for online linear optimization with bandit feedback. In *Conference on Learning Theory*, 2012.

[9] Nicolo Cesa-Bianchi and Gábor Lugosi. Combinatorial bandits. *Journal of Computer and System Sciences*, 2012.
[10] Victor Chernozhukov, Denis Chetverikov, Mert Demirer, Esther Duflo, Christian Hansen, Whitney Newey, and James M. Robins. Double machine learning for treatment and causal parameters. \textit{arXiv:1608.00060}, 2016.

[11] Wei Chu, Lihong Li, Lev Reyzin, and Robert E Schapire. Contextual bandits with linear payoff functions. In \textit{International Conference on Artificial Intelligence and Statistics}, 2011.

[12] Thomas M Cover. Behavior of sequential predictors of binary sequences. In \textit{Conference on Information Theory, Statistical Decision Functions and Random Processes}, 1965.

[13] Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under bandit feedback. In \textit{Conference on Learning Theory}, 2008.

[14] Victor H de la Peña, Tze Leung Lai, and Qi-Man Shao. \textit{Self-normalized processes: Limit theory and statistical applications}. Springer Science & Business Media, 2008.

[15] Victor H de la Peña, Michael J Klass, and Tze Leung Lai. Theory and applications of multivariate self-normalized processes. \textit{Stochastic Processes and their Applications}, 2009.

[16] Eyal Even-Dar, Shie Mannor, and Yishay Mansour. Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. \textit{Journal of Machine Learning Research}, 2006.

[17] Sarah Filippi, Olivier Cappe, Aurélien Garivier, and Csaba Szepesvári. Parametric bandits: The generalized linear case. In \textit{Advances in Neural Information Processing Systems}, 2010.

[18] David A Freedman. On tail probabilities for martingales. \textit{The Annals of Probability}, 1975.

[19] Kristjan Greenewald, Ambuj Tewari, Susan Murphy, and Predag Klasnja. Action centered contextual bandits. In \textit{Advances in Neural Information Processing Systems}, 2017.

[20] Akshay Krishnamurthy, Alekh Agarwal, and Miroslav Dudík. Contextual semibandits via supervised learning oracles. In \textit{Advances in Neural Information Processing Systems}, 2016.

[21] John Langford and Tong Zhang. The epoch-greedy algorithm for multi-armed bandits with side information. In \textit{Advances in Neural Information Processing Systems}, 2008.

[22] Lihong Li, Wei Chu, John Langford, and Robert E Schapire. A contextual-bandit approach to personalized news article recommendation. In \textit{International Conference on World Wide Web}, 2010.

[23] Lihong Li, Yu Lu, and Dengyong Zhou. Provable optimal algorithms for generalized linear contextual bandits. In \textit{International Conference on Machine Learning}, 2017.

[24] Jerzy Neyman. \textit{C(α) tests and their use}. \textit{Sankhyā: The Indian Journal of Statistics, Series A}, 1979.

[25] Alexander Rakhlin and Karthik Sridharan. Bistro: An efficient relaxation-based method for contextual bandits. In \textit{International Conference on Machine Learning}, 2016.

[26] James M Robins and Andrea Rotnitzky. Recovery of information and adjustment for dependent censoring using surrogate markers. In \textit{AIDS Epidemiology}. Springer, 1992.

[27] James M Robins, Lingling Li, Eric Tchetgen Tchetgen, and Aad van der Vaart. Higher order influence functions and minimax estimation of nonlinear functionals. In \textit{Probability and Statistics: Essays in Honor of David A. Freedman}. Institute of Mathematical Statistics, 2008.
[28] Peter M Robinson. Root-n-consistent semiparametric regression. *Econometrica: Journal of the Econometric Society*, 1988.

[29] Paat Rusmevichientong and John N Tsitsiklis. Linearly parameterized bandits. *Mathematics of Operations Research*, 2010.

[30] Daniel Russo and Benjamin Van Roy. Learning to optimize via posterior sampling. *Mathematics of Operations Research*, 2014.

[31] Maurice Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 1958.

[32] Adith Swaminathan, Akshay Krishnamurthy, Alekh Agarwal, Miroslav Dudík, John Langford, Damien Jose, and Imed Zitouni. Off-policy evaluation for slate recommendation. In *Advances in Neural Information Processing Systems*, 2017.

[33] Vasilis Syrgkanis, Akshay Krishnamurthy, and Robert E Schapire. Efficient algorithms for adversarial contextual learning. In *International Conference on Machine Learning*, 2016.

[34] Ambuj Tewari and Susan A Murphy. From ads to interventions: Contextual bandits in mobile health. In *Mobile Health*. Springer, 2017.

[35] Anastasios Tsiatis. *Semiparametric theory and missing data*. Springer Science & Business Media, 2007.