Linear representations of $\text{Aut}(F_r)$ on the homology of representation varieties

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Abstract
Let $G$ be a compact semisimple linear Lie group. We study the action of $\text{Aut}(F_r)$ on the space $H_*(G^r; \mathbb{Q})$. We compute the image of this representation and prove that it only depends on the rank of $g$. We show that the kernel of this representation is always the Torrelli subgroup $\text{IA}_r$ of $\text{Aut}(F_r)$.

Keywords Representation varieties · Automorphism groups · Homology · Torelli group

Mathematics Subject Classification 57M07

1 Introduction

Let $G$ be a group. The group $G^r$ can be naturally identified with the set of all homomorphisms $\rho : F_r \to G$, where $F_r$ is the free group of rank $r$. As such, the group $\Gamma = \text{Aut}(F_r)$ acts on $G^r$ by precomposition. Explicitly, if $\gamma \in \Gamma$, $x_1, \ldots, x_r$ is a choice of generators for $F_r$ and $\rho \in \text{Hom}(F_r, G)$ then,

$$\gamma : G^r \to G^r$$

$$(\rho(x_1), \ldots, \rho(x_r)) \mapsto (\rho(\gamma x_1), \ldots, \rho(\gamma x_r)).$$

Note that this is a right action. When $G$ is a topological group, this action can be studied using topological tools. One example is to study the induced action of $\Gamma$ on $H_*(G^r; \mathbb{Q})$. This gives rise to a linear representation of $\Gamma$, which we call $\mathcal{H}(G)$. In this paper we calculate the kernel and isomorphism class of this representation when $G$ is a compact semisimple Lie Group.

Theorem 1.1 Let $G$ be a compact semisimple Lie group. Then $\ker(\mathcal{H}(G)) = \text{IA}_r$, where $\text{IA}_r$ is the subgroup of $\Gamma$ that acts trivially on $F_r/[F_r, F_r]$.

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Since \( \text{Inn}(F_r) \subset \text{IA}_r \) we get as a corollary that the representation \( \mathcal{H}(G) \) descends to a representation of \( \text{Out}(F_r) \).

Using the fact that \( \Gamma / \text{IA}_r \cong \text{GL}_r(\mathbb{Z}) \), we are able to get a precise description of these representations as \( \text{GL}_r(\mathbb{Z}) \)-modules. One main feature of this description is that the isomorphism class of \( \mathcal{H}(G) \) depends only on the rank of the Lie algebra of \( G \). We denote by \( \Lambda(V) \) the exterior algebra of the vector space \( V \).

**Theorem 1.2** Let \( G \) be a compact semisimple Lie group, and \( \mathfrak{g} \) be its Lie algebra. Let \( A = \frac{F_r}{[F_r,F_r]} \otimes \mathbb{Q} \), then, as a \( \text{GL}_r(\mathbb{Z}) \) module:

\[
\mathcal{H}(G) \cong \bigotimes_{i=1}^{\text{rank}(\mathfrak{g})} \Lambda(A)
\]

It is possible to generalize the construction of the representations \( \mathcal{H}(G) \) in the following way: given a finite index subgroup \( K < F_r \), we have a finite index subgroup \( \Gamma_K := \{ \gamma \in \Gamma \mid \gamma(K) = K \} < \Gamma \), an inclusion \( \Gamma_K \to \text{Aut}(K) \), and a representation:

\[
\rho_K : \Gamma_K \to \text{GL}(H_*(G^{\text{rank}(K)}; \mathbb{Q}))
\]

We induce this representation to \( \Gamma \) and define:

\[
\mathcal{H}_K(G) = \text{Ind}_{\Gamma_K}^\Gamma (\rho_K)
\]

For these representations, we prove a similar result,

**Theorem 1.3** Let \( G, G' \) be compact semisimple Lie groups, and let \( K < F_r \) be a finite index subgroup. Then:

\[
\ker(\mathcal{H}_K(G)) = \ker(\mathcal{H}_K(G'))
\]

Furthermore, if \( \text{rank}(\mathfrak{g}) = \text{rank}(\mathfrak{g}') \) then

\[
\mathcal{H}_K(G) \cong \mathcal{H}_K(G').
\]

The images of these representations in the case of \( G = SO(1) \) were already studied by Lubotzky and Grunewald [2]. One should note that the generalized representations \( \mathcal{H}_K(G) \) are no longer \( \text{Out}(F_r) \) representations.

**Remark 1.4** The spaces \( \mathcal{G}' \) are called representation varieties, as they classify representations into a given group \( G \). Representation varieties are related to a similar object called a character varieties. The group \( G \) acts on \( \mathcal{G}' \) by coordinate wise conjugation. The invariant theoretic quotient \( \mathcal{G}' // G \) is called a character variety. It classifies representations from \( F_r \) to \( G \) up to conjugacy. When \( F_r \) is viewed as the fundamental group of a surface \( S \), the mapping class group \( \text{Mod}(S) \) acts on \( \mathcal{G}' // G \). There are many analogies between mapping class groups and automorphism groups of free groups. Capell et al. [1] carry out a calculation with a scope that is reminiscent of our own. They take a surface group instead of \( F_r \), a character variety instead of a representations variety, a mapping class group instead of \( \Gamma \), and \( G = SU(2) \) instead of a general \( G \). In contrast to our result, they find that the kernel of the action is an infinite index subgroup of the Torelli group.

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2 The rational homology of Lie groups

2.1 Pontrjagin and intersection structures

Given a compact Lie group $G$, consider the continuous map $\mu : G \times G \to G$ given by $\mu(a, b) = ab$. This induces a linear map: $\mu_* : H(G; \mathbb{Q}) \otimes H(G; \mathbb{Q}) \to H(G; \mathbb{Q})$. The map $\mu_*$ gives $H_*(G; \mathbb{Q})$ the structure of a ring, called the Pontrjagin ring. We denote Pontrjagin ring multiplication using the $\cdot$ symbol.

The structure of $(H_*(G; \mathbb{Q}), \cdot)$ is classical and has been calculated by Pontrjagin [4] for simple non-exceptional Lie groups. For the sake of simplicity, we first state it in these cases. The root system of such Lie groups belong to the one of the infinite families: $A_n$, $B_n$, $C_n$, $D_n$, where $n$ is the rank of the Lie algebra of $G$. This root system entirely determines the above structure. To each such root system, assign a finite sequence $\{d_k\}_{k=1}^n$ where $n = \text{rank}(g)$ in the following way:

- $A_n : d_k = 2k + 1$
- $B_n, C_n : d_k = 4k - 1$
- $D_n : d_k = \begin{cases} 4(k + 1) - 5 & k = 1, \ldots, n - 1 \\ 2n - 1 & k = n \end{cases}$

Let $S := S^{d_1} \times \ldots \times S^{d_n}$ where $S^d$ is the $d$-dimensional sphere. Then there is an isomorphism

$$\chi : H_*(S, \mathbb{Q}) \to H_*(G; \mathbb{Q})$$  \hspace{1cm} (3)

This isomorphism can be further described as follows. Choose basepoints $p_k \in S^{d_k}$ for each $1 \leq k \leq n$, then for an index $1 \leq l \leq n$, there is an inclusion given by

$$I^l : S^{d_l} \hookrightarrow S$$

$$x \mapsto (p_1, \ldots, p_{l-1}, x, p_{l+1}, \ldots, p_n)$$  \hspace{1cm} (4)

Denote by $R^k = I^k(S^{d_k}) \subset S$. For each $k$, there exists an $d_k$-dimensional submanifold of $G$, which we call $T^k$, such that $\chi[R^k] = [T^k]$ in $H_*(G, \mathbb{Q})$. We denote this homology class by $I^k$.

The Pontrjagin structure of $H_*(G, \mathbb{Q})$ is also known: it is a graded exterior algebra with a unit, generated by $I^1, \ldots, I^n$. Under the isomorphism $\chi$ in Eq. (3), the Pontrjagin structure corresponds to the direct product structure.

**Example 2.1** Let $G = SO_3(\mathbb{R})$. Then $H_*(G) \cong H_*(S^3 \times S^5)$. Recall that the Poincaré polynomial of a finite dimensional manifold $Y$ is the polynomial $\sum_i \dim H_i(Y, \mathbb{Q})t^i$. The Poincaré polynomial of $H_*(G)$ is $1 + x^3 + x^5 + x^8$. As generators for the Pontrjagin structure, we have $t^i \in H_i(G)$ for $i = 3, 5$, with dimensions 3, 5, and unit, which we call 1. The space $H_6(G)$ is generated by $t^1 \cdot t^2 = -t^2 \cdot t^1$.

The situation for general semisimple groups $G$ (including the simple exceptional Lie groups) is similar and was worked out in [5]. As in the simple case, $H_*(G)$ is isomorphic to the homology of a product of rank($g$) odd-dimensional spheres. Furthermore, the Pontrjagin structure is once again an exterior algebra of rank($g$) generators which we denote $t^1, \ldots, t^n$ that correspond to the odd dimensional sub-manifolds $T^1, \ldots, T^n$.

**Remark 2.2** In this paper we will deal with the product of a number of copies of the same object such as $S^r$ or $\bigoplus_{i=1}^r H_*(G, \mathbb{Q})$. In such a situation, say $X^r$, each copy of $X$ often
Consider $G'$, for $1 \leq i \leq r$ we have natural inclusions $\Delta_i : G \to G'$ which induce maps $\Delta_i : H_*(G) \to H_*(G')$ (throughout the paper we shall abuse notation denoting both the map on the level of the groups and the map on the level of the homology by the same notation). Given $t^k \in H_n(G)$ we denote $\Delta_i(t^k) = t^k_i$. The elements $t^k_i$ are the same generators of $H_*(G')$ discussed above, but are indexed differently. A product of the $t^k_i$'s is called a monomial. Notice that the set of monomials is finite.

**Example 2.3** The Poincaré polynomial of $G = SU(2)$ is $1 + x^3$. Now for $G' = SU(2)'$, the Poincaré polynomial is $\sum_{j=1}^r \binom{r}{k} x^{3k}$. As a ring, it is generated by the elements $t^1_1, \ldots, t^1_r$, all of which have dimension 3.

### 2.2 Diagonal inclusions

In addition to the maps $\Delta_i : G \to G'$ we define maps $\Delta_{i,j} : G \to G'$ for $1 \leq i \neq j \leq n$ by sending $g \in G$ to the element with $g$ in the $i$ and $j$ components, and $1 \in G$ in all other components. We call the maps of the form $\Delta_{i,j}$ diagonal inclusions.

The map assigning a group to its Pontrjagin ring is functorial, and the homomorphisms $\Delta_{i,j}$ induce ring homomorphisms $\Delta_{i,j} : (H_*(G), \cdot) \to (H_*(G'), \cdot)$. Thus, to give an explicit description of these maps, it is enough to calculate them for the generators $t = t^k \in H_*(G)$. We recall that $\Delta_i(t^k) = t^k_i \in H_*(G')$.

**Proposition 2.4** For $1 \leq i \neq j \leq r$, $1 \leq k \leq \text{rank}(g)$, $\Delta_{i,j}(t^k) = t^k_i + t^k_j$

**Proof** We find it convenient to work in the setting of a product of spheres. Let $P_i : S^r \to S$ denote the projection to the $i$-th component. We define $I^k_i : R^k \to S^r$ to be the map so that $P_i \circ I^k_i$ is the map $I^k$ of Eq. (4), and $P_m \circ I^k_i$ is the constant basepoint map for $m \neq i$. Similarly, let $I^k_{ij} : R^k \to S^r$ be the map so that $P_m \circ I^k_{ij}$ is equal to the map $I^k$ for $m = i, j$ and otherwise it is the constant basepoint map. We denote by $s$ the fundamental class of $R^k$, and by $s^k_i = I^k_i(s) \in H_*(S^r)$. The proposition reduces to showing that $I^k_{ij}(s) = s^k_i + s^k_j$.

Consider the following commutative diagram, where where $D(x) = (x, x)$ is the diagonal map and $R = R^k$,

$$
\begin{array}{ccc}
R & \xrightarrow{I^k_{ij}} & S^r \\
\downarrow{D} & & \downarrow{I^k_i \times I^k_j} \\
R \times R & &
\end{array}
$$

Let $d = \dim(R)$, then $H_d(R \times R, \mathbb{Q}) \cong H_d(R, \mathbb{Q}) \oplus H_d(R, \mathbb{Q})$ and denote by $s_1$ and $s_2$ the images of $s \times 1, 1 \times s$ under this isomorphism. Note that $D(R)$ is a $d$-dimensional manifold so $D(R) = as_1 + bs_2$, where the coefficients are given by the intersection pairing. Notice that $D(R)$ intersects $R \times p$ in exactly one point and similarly for $p \times R$. Thus $a, b = \pm 1$. Moreover, $D(R)$ is invariant under switching the components, thus, $a = b$. Finally notice that composing $D$ with the projection to the first factor equals the identity. Thus $a = b = 1$, hence $D(R) = s_1 + s_2$ and therefore, $I^k_{ij}(s) = s^k_i + s^k_j$. \(\square\)
3 Calculating the action of generators of $\Gamma$

Choose a basis $\{a_1, \ldots, a_r\}$ for $F_r$. A (right or left) Nielsen transformation with respect to this basis is an automorphism of the form:

$$N^L_{i,j}(a_k) = \begin{cases} a_j a_i & k = i \\ a_k & k \neq i \end{cases}$$

or

$$N^R_{i,j}(a_k) = \begin{cases} a_i a_j & k = i \\ a_k & k \neq i \end{cases}$$

Moreover, let $\sigma \in \Gamma$ be the automorphism fixing $a_k$ for $k \neq 1, 2$ and switching $a_1$ and $a_2$. Nielsen proved [3] that the group $\Gamma$ is generated by $\sigma$ and by Nielsen transformations. As explained in the introduction, every automorphism $\gamma \in \text{Aut}(F_r)$, induces an automorphism of the group $G^r$ (see Eq. 1) and hence an automorphism of the Pontrjagin ring $H_*(G^r, \mathbb{Q})$. The action of $\sigma$ on $H_*(G^r)$ is given by:

$$\sigma(t^k_i) = t^k_{\sigma(i)}$$

We now compute the action of the Nielsen transformations.

**Proposition 3.1** The maps $N^R_{ij}, N^L_{ij}$ induce automorphisms of the Pontrjagin ring $H_*(G^r, \mathbb{Q})$. For $i \neq j, l \in \{1, \ldots, r\}$ and $1 \leq k \leq \text{rank}(g)$, we have

$$N^R_{i,j}(t^k_l) = N^L_{i,j}(t^k_l) = \begin{cases} t^k_l & l \neq j \\ t^k_i + t^k_j & l = j \end{cases}$$

Therefore the actions of $N^R_{ij}$ and $N^L_{ij}$ are equal.

**Proof** We work out the case of $N^R_{ij}$, the case of $N^L_{ij}$ is similar and left to the reader. For $1 \leq i \leq r$ define $p_i : G^r \to G$ to be the $i$-th coordinate projection. Let $\text{trunc}_i : G^r \to G^r$ by the rule:

$$\text{trunc}_i(g_1, \ldots, g_r) = (g_1, \ldots, 1, \ldots g_r)$$

where the 1 is in the $i$-th coordinate.

The map $N^R_{ij}$ factors through the maps shown in the following commutative diagram.

$$
\begin{array}{cccccc}
G^r & \xrightarrow{\Delta_{1,2}} & G^r \times G^r & \xrightarrow{(\text{trunc}_j, Id)} & G^r \times G^r & \xrightarrow{(Id, \Delta_{i,j} \circ p_j)} & G^r \times G^r \xrightarrow{\mu} G^r \\
& & N^R_{i,j} & & & \\
\end{array}
$$

Where $\mu$ is the Pontrjagin product. The first three maps in the diagram are homomorphisms, and as such the induced maps on homology are Pontrjagin ring homomorphisms. Thus, to calculate the action of these homomorphisms, it suffices to calculate their action on generators. Notice that the inclusion of $G^r$ into $G^r \times G^r$ necessitates the addition of an index. We will use the notation $(t)_1$ or $(t)_2$ to denote the image of $t$ in $H_*(G^r \times G^r)$ under the maps $\Delta_1, \Delta_2$. By Proposition 2.4, we have that:

$$\Delta_{1,2}(t^k_i) = (t^k_i)_1 + (t^k_i)_2$$
Let $\mu$ be a $\Delta_1,2$-module. The reason for this is that $\mu((x)_1(y)_2) = x \cdot y = \mu((x \cdot y)_1)$.

4 Describing the image and the kernel of $\mathcal{H}(\Gamma)$

The group $\Gamma$ acts on the abelianization of $F_r$. Therefore, $\Gamma$ acts on the vector space $A = [F_r/F_r] \otimes \mathbb{Q} \cong \mathbb{Q}^r$, and on its dual space $A^*$. We have the corresponding representations by $\alpha : \Gamma \to GL_r(\mathbb{Q}), \alpha^* : \Gamma \to GL_r(\mathbb{Q})$ (note that $\alpha^*$ is an opposite representation).

In this section we will show that for every $G$, the representation $\mathcal{H}(G)$ factors through $\alpha^*$ and describe $\mathcal{H}(G)$ as a $GL_r(\mathbb{Q})$-module. Surprisingly, the isomorphism class of $\mathcal{H}(G)$ only depends on $\operatorname{rank}(g)$ and its kernel does not depend on $G$ at all.

Fix an index $1 \leq k \leq \operatorname{rank}(g)$, and consider the linear map $\mathcal{E}_k : \Lambda(A) \to H_s(G^r)$ given on a basis by:

$$\mathcal{E}_k(e_{j_1} \wedge \cdots \wedge e_{j_m}) = t_{j_1}^k \cdots t_{j_m}^k$$

$$\mathcal{E}_k(1_{\Lambda(A)}) = 1_{H_s(G^r)}$$

Given a subset $S = \{k_1, \ldots, k_l\} \subset \{1, \ldots, \operatorname{rank}(g)\}$, define the map

$$\mathcal{E}_S : \bigotimes_{a=1}^l \Lambda(A) \to H_s(G^r)$$

by setting:

$$\mathcal{E}_S(v_1 \otimes \cdots \otimes v_l) = \prod_{a=1}^l \mathcal{E}_{k_a}(v_a).$$

Notice that for $S = \{1, \ldots, \operatorname{rank}(g)\}$ the linear map $\mathcal{E}_S : \bigotimes_{k=1}^{\operatorname{rank}(g)} \Lambda(A) \to H_s(G^r)$ is an isomorphism since it is surjective and the dimensions over $\mathbb{Q}$ are equal. Thus $\mathcal{E}_S$ induces an isomorphism $GL(\bigotimes_{k=1}^{\operatorname{rank}(g)} \Lambda(A)) \to GL(H_s(G^r))$. We will denote this isomorphism too by $\mathcal{E}_S$.

Let $e_1, \ldots, e_r \in A$ be the images of the free generating basis $x_1, \ldots, x_r$ of $F_r$. Let $de_1, \ldots, de_n \in A^*$ be a dual basis. Let $t : GL(A^*) \to GL(\bigotimes_{i=1}^{\operatorname{rank}(g)} \Lambda(A))$ be the composition of the natural map $GL(A^*) \to GL(\bigotimes_{i=1}^{\operatorname{rank}(g)} \Lambda(A^*))$ with the dualizing map $GL(\bigotimes_{i=1}^{\operatorname{rank}(g)} \Lambda(A^*)) \to GL(\bigotimes_{i=1}^{\operatorname{rank}(g)} \Lambda(A))$. 

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Proposition 4.1 When $S = \{1, \ldots, \text{rank}(g)\}$, the following diagram commutes.

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\mathcal{H}(G)} & GL(H_*(G')) \\
\alpha^* & \uparrow & \mathcal{E}_S \\
GL(A^*) & \xrightarrow{\iota} & GL(\bigotimes_{i=1}^{\text{rank}(g)} \Lambda(A))
\end{array}
\]

Proof It is enough to show this for a set of generators of $\Gamma$. The action of a Nielsen transformation $N_{i,j}$ on $A^*$ is given in the standard basis by

\[
N_{i,j}^R(de_l) = \begin{cases}
de_i + de_j & l = j \\
de_l & l \neq j
\end{cases}
\]

The action of $N_{i,j}^L$ is the same. The map $\sigma$ permutes the first and second coordinates. The action of $GL_r(\mathbb{Q})$ commutes with the wedge and tensor product, and hence $\mathcal{E}_S \circ \alpha^*$ is exactly the same as the action calculated in Proposition 3.1.

Corollary 4.2 The kernel of the representation $\mathcal{H}(G)$ is precisely $IA_r$. The image depends only on $\text{rank}(g)$.

Proof Consider the diagram in Lemma 4.1. The map $\iota$ is injective, because $A$ injects into $\bigotimes_{i=1}^{\text{rank}(g)} \Lambda(A)$. Therefore the kernel of $\mathcal{H}(G)$ equals the kernel of $\alpha^*$ and this is the Torelli subgroup $IA_r$.

5 Describing the image and kernel of $\mathcal{H}_K(G)$

Let $K < F_r$, define $\Gamma_K = \{\gamma \in \Gamma \mid \gamma K = K\}$. We can view the group $\Gamma_K$ as a subgroup of $\text{Aut}(K)$. As such Proposition 4.1 gives the following corollary.

Corollary 5.1 The following diagram commutes:

\[
\begin{array}{ccc}
\Gamma_K & \xrightarrow{\rho_K} & GL(H_*(G^{\text{rank}(K)})) \\
\downarrow & & \uparrow \\
GL(\text{rank}(K))(\mathbb{Q}) & \xrightarrow{\iota} & GL\left(\bigotimes_{i=1}^{\text{rank}(g)} \Lambda\left(\frac{K}{[K,K]} \otimes \mathbb{Q}\right)\right)
\end{array}
\]

If $K$ has finite index in $F_r$ then $\Gamma_K$ has finite index in $\Gamma$. Define the representation $\mathcal{H}_K(G) = \text{Ind}_{\Gamma_K}^{\Gamma}(\rho_K)$ to be the induced representation of $\rho_K$. Let $IA_{\Gamma}(K)$ be the subgroup of $\Gamma$ that acts trivially on the abelianization of $K$.

Corollary 5.2 We have that:

\[
\ker(\mathcal{H}_K(G)) = \Gamma_K \cap IA_{\Gamma}(K)
\]

Furthermore, if $G'$ is another compact semisimple Lie group whose Lie algebra has the same rank as $g$ then $\mathcal{H}_K(G') \cong \mathcal{H}_K(G)$
**Proof** For the first claim, we have that since $H_K(G)$ is a representation on $\Gamma$ induced from $K$, it is clear that $\ker(H_K(G)) \subset \Gamma_K$. By Corollary 5.1, we have that $\ker(H_K(G)) \cap \Gamma_K = IA_{\Gamma}(K) \cap \Gamma_K$.

For the second claim, it follows directly from the diagram, since we are inducing isomorphic representations from the same groups.

\[ \square \]

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