A note on the generalized $q$-Euler numbers (2)

By

Kyoung-Ho PARK, Young-Hee KIM, and Taekyun KIM

Abstract. Recently the new $q$-Euler numbers and polynomials related to Frobenius-Euler numbers and polynomials are constructed by Kim (see [3]). In this paper, we study the generalized $q$-Euler numbers and polynomials attached to $\chi$ related to the new $q$-Euler numbers and polynomials which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized $q$-Euler numbers and polynomials attached to $\chi$.

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§1. Introduction

Let $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of real numbers and the complex number field, and let $p$ be a fixed an odd prime number. Assume that $q$ is an indeterminate in $\mathbb{C}$ with $q \in \mathbb{C}$ with $|q| < 1$. As the $q$-symbol $[x]_q$, we denote $[x]_q = \frac{1-q^x}{1-q}$. Recently, $q$-Euler polynomials are defined as

$$\frac{[2]_q}{qe^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} \text{ for } |t + \log q| < \pi, \text{ (see [3]).}$$

In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are call the $n$-th $q$-Euler numbers (see [3]). These $q$-Euler numbers and polynomials are closely relayed to Frobenius-Euler numbers and polynomials and these numbers are studied by Simsek-Cangul-Ozden, Cenkci-Kurt and Can and several authors (see [1-2, 18-26]). In this paper, we study the generalized $q$-Euler numbers and polynomials attached to $\chi$ related to the $q$-Euler numbers and polynomials, $E_{n,q}(x)$, which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized $q$-Euler numbers and polynomials attached to $\chi$.

§2. Congruence for $q$-Euler numbers and polynomials
A note on the generalized $q$-Euler numbers

The ordinary Euler polynomials are defined as

$$e^{xt} \frac{2}{e^t + 1} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1 – 5]),}$$

where we use the technical method notation by replacing $E_n(x)$ by $E_n(x)(n \geq 0)$, symbolically (see [1-2]). Let us consider the generating function of $q$-Euler polynomials $E_{n,q}(x)$ as follows:

$$F_q(x, t) = \left[\frac{2}{q} \right] e^{xt} = \sum_{n=0}^{\infty} \frac{E_{n,q}(x) t^n}{n!}, \quad \text{(1)}$$

and we also note that

$$\sum_{n=0}^{\infty} \frac{E_{n,q}(x) t^n}{n!} = \left[\frac{2}{q} \right] e^{xt} = \frac{1}{e^t - (q^{-1})} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!},$$

where $H_n(-q^{-1}, x)$ are called the $n$-th Frobenius-Euler polynomials (see [3]). From (1), we note that

$$\lim_{q \to 1} F_q(x, t) = 2 e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad \text{(2)}$$

By (1) and (2), we see that

$$\lim_{q \to 1} E_{n,q}(x) = E_n(x).$$

In (1), it is easy to show that

$$\sum_{n=0}^{\infty} \frac{E_{n,q}(x) t^n}{n!} = F_q(x, t) = \left[\frac{2}{q} \right] e^{xt} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} E_{l,q} x^{n-l} \right) \frac{t^n}{n!}.$$ 

By comparing the coefficients on the both sides, we have

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} x^{n-l}, \quad \text{where } E_{l,q} \text{ are the } l\text{-th } q\text{-Euler numbers.} \quad \text{(3)}$$

Let $\chi$ be the Dirichlet’s character with conductor $d \equiv 1 \pmod{2}$. Then we define generating function of the generalized $q$-Euler numbers attached to $\chi$, $E_{n,\chi,q}$ as follows:

$$F_{q,\chi}(t) = \left[\frac{2}{q} \right] \sum_{l=0}^{d-1} \chi(l) q^l (-1)^l e^{lt} = \sum_{n=0}^{\infty} \frac{E_{n,\chi,q} t^n}{n!}. \quad \text{(4)}$$

From (4), we note that

$$\lim_{q \to 1} F_{q,\chi}(t) = \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a e^{at}}{e^{at} + 1} = \sum_{n=0}^{\infty} \frac{E_{n,\chi} t^n}{n!}. \quad \text{(5)}$$
where $E_{n,\chi}$ are the $n$-th ordinary Euler numbers attached to $\chi$. By (4) and (5), we see that
\[
\lim_{q \to 1} E_{n,\chi,q} = E_{n,\chi}.
\]
From (5), we can also derive
\[
\sum_{n=0}^\infty E_{n,\chi,q} \frac{t^n}{n!} = F_{q,\chi}(t) = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^ke^{kt}
\[
= \sum_{n=0}^\infty \left([2]_q \sum_{k=0}^\infty \chi(k)(-q)^k\right) \frac{t^n}{n!}
\[
= \sum_{n=0}^\infty \left(d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,qd}(\frac{a}{d})\right) \frac{t^n}{n!}.
\]
By comparing the coefficients on the both sides of (6), we have
\[
E_{n,\chi,q} = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^k = d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,qd}(\frac{a}{d}).
\]
Finally, we define the generating function of the generalized $q$-Euler polynomials attached to $\chi$, $E_{n,\chi,q}(x)$ as follows:
\[
F_{q,\chi}(x,t) = \sum_{n=0}^\infty E_{n,\chi,q}(x) \frac{t^n}{n!} = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^{k}e^{(x+k)t}.
\]
By (8), we easily see that
\[
\sum_{n=0}^\infty E_{n,\chi,q}(x) \frac{t^n}{n!} = F_{q,\chi}(x,t) = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^k (x+k)^n
\[
= \sum_{n=0}^\infty \left([2]_q \sum_{k=0}^\infty \chi(k)(-q)^k(x+k)^n\right) \frac{t^n}{n!}
\[
= \sum_{n=0}^\infty \left(d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,qd}(\frac{a+x}{d})\right) \frac{t^n}{n!}.
\]
Thus, we have
\[
E_{n,\chi,q}(x) = d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,qd}(\frac{a+x}{d}) = \sum_{\ell=0}^n \left(\begin{array}{c} n \\ \ell \end{array}\right) x^{n-\ell} E_{\ell,\chi,q} = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^k (x+k)^n.
\]
Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then, we see that
\[
q^d F_{q,\chi}(d,t) + F_{q,\chi}(t) = [2]_q \sum_{k=0}^\infty \chi(k)(-q)^{k}e^{(d+k)t} + [2]_q \sum_{k=0}^\infty \chi(k)(-q)^{k}e^{kt}
\[
= [2]_q \sum_{k=0}^{d-1} \chi(k)(-q)^{k}e^{kt}.
\]
From (11), we have
\[
\sum_{n=0}^{\infty} \left( q^d E_{n,\chi,q}(d) + E_{n,\chi,q} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ [2]_q \sum_{k=0}^{d-1} \chi(k)(-q)^k k^n \right\} \frac{t^n}{n!}.
\]
Therefore, we obtain the following theorem.

**Theorem 1.** For \(q \in \mathbb{C}\) with \(|q| < 1\), \(n \in \mathbb{Z}_+\) and \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\), we have
\[
q^d E_{n,\chi,q}(d) + E_{n,\chi,q} = [2]_q \sum_{k=0}^{d-1} \chi(k)(-q)^k k^n.
\]

Let \(p\) be a positive odd integer and let \(N \in \mathbb{N}\). Then we have
\[
[2]_q \sum_{a=0}^{dp^N-1} \chi(a)(-q)^a a^n = q^{dp^N} E_{n,\chi,q}(dp^N) + E_{n,\chi,q}
\]
\[
= q^{dp^N} \sum_{j=0}^{n} \binom{n}{j} (dp^N)^j E_{n-j,\chi,q} + E_{n,\chi,q}
\]
\[
= q^{dp^N} \sum_{j=1}^{n} \binom{n}{j} (dp^N)^j E_{n-j,\chi,q} + (q^{dp^N} + 1) E_{n,\chi,q}
\]
\[
\equiv 2E_{n,\chi,q} \pmod{dp^N},
\]
because \(q^{ndp^N} \equiv 1 \pmod{dp^N}\). Therefore, we obtain the following theorem.

**Theorem 2.** Let \(p\) be a positive odd integer and \(q \in \mathbb{C}\) with \(|q| < 1\) and \((q - 1, dp) = 1\). For \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\), we have
\[
[2]_q \sum_{a=0}^{dp^N-1} \chi(a)(-q)^a a^n \equiv 2E_{n,\chi,q} \pmod{dp^N}.
\]

**Remark.** Define
\[
L_{E,q}(s, \chi|x) = [2]_q \sum_{n=0}^{\infty} \frac{(-q)^n \chi(n)}{(n+x)^s},
\]
where \(s \in \mathbb{C}\), and \(x \neq 0, -1, -2, \ldots\). For \(k \in \mathbb{Z}_+\), we have \(L_{E,q}(-k, \chi|x) = E_{k,\chi,q}(x)\).

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Kyoung-Ho PAK
Department of Mathematics,
Sogang University, Seoul 121-741, S. Korea
E-mail: sagamath@yahoo.co.kr

Young-Hee KIM
Division of General Education-Mathematics,
Kwangwoon University, Seoul 139-701, S. Korea
E-mail: yhkim@kw.ac.kr

Taekyun KIM
Division of General Education-Mathematics,
Kwangwoon University, Seoul 139-701, S. Korea
E-mail: tkkim@kw.ac.kr