GENUINELY RAMIFIED MAPS AND STABLE VECTOR BUNDLES

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Abstract. Let \( f : X \to Y \) be a separable finite surjective map between irreducible normal projective varieties defined over an algebraically closed field, such that the corresponding homomorphism between étale fundamental groups \( f_* : \pi^\text{et}_1(X) \to \pi^\text{et}_1(Y) \) is surjective. Fix a polarization on \( Y \) and equip \( X \) with the pullback, by \( f \), of this polarization on \( Y \). Given a stable vector bundle \( E \) on \( X \), we prove that there is a vector bundle \( W \) on \( Y \) with \( f_* W \) isomorphic to \( E \) if and only if the direct image \( f_* E \) contains a stable vector bundle \( F \) such that
\[
\frac{\text{degree}(F)}{\text{rank}(F)} = \frac{1}{\text{degree}(f)} \frac{\text{degree}(E)}{\text{rank}(E)}.
\]
We also prove that \( f^* V \) is stable for every stable vector bundle \( V \) on \( Y \).

1. Introduction

Here we continue, from [PS], [BP1], [BP2], the investigations of the direct image \( f_* O_X \), where \( f : X \to Y \) is a separable finite surjective map between irreducible normal projective varieties defined over an algebraically closed field \( k \). First we briefly recall the main results of [BP1], [BP2].

The above map \( f \) is called genuinely ramified if the homomorphism between étale fundamental groups
\[
f_* : \pi^\text{et}_1(X) \to \pi^\text{et}_1(Y)
\]
induced by \( f \) is surjective. Consider the unique maximal pseudostable subbundle \( \mathcal{S} \subset f_* O_X \); the existence of this subbundle is proved in [BP2]. The map \( f \) is genuinely ramified if and only if \( \text{rank}(\mathcal{S}) = 1 \) [BP2]. When \( \dim X = 1 = \dim Y \), this was proved earlier in [BP1]. When \( \dim X = 1 = \dim Y \), if \( E \) is a stable vector bundle on \( Y \), and \( f \) is genuinely ramified, then \( f^* E \) is also stable [BP1].

Here we prove the following (see Theorem 2.4):

**Theorem 1.1.** Let \( f : X \to Y \) be a finite separable surjective map between irreducible normal projective varieties defined over an algebraically closed field \( k \). Then the following four statements are equivalent.

1. The map \( f \) is genuinely ramified.
2. There is no nontrivial étale covering \( g : Y' \to Y \) satisfying the condition that there is a map \( g' : X \to Y' \) for which \( g \circ g' = f \).
3. The fiber product \( X \times_Y X \) is connected.
4. \( \dim H^0(X, (f^* f_* O_X)/(f^* f_* O_X)_{\text{torsion}}) = 1. \)

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This theorem leads to the following generalization, to higher dimensions, of the above mentioned result for curves (see Theorem 2.1):

**Theorem 1.2.** Let \( f : X \to Y \) be a finite separable surjective map between irreducible normal projective varieties such that \( f \) is genuinely ramified. Let \( E \) be a stable vector bundle on \( Y \). Then the pulled back vector bundle \( f^*E \) is also stable.

It should be noted that given a map \( f \) as above, if \( f^*E \) is a stable vector bundle on \( Y \) for a vector bundle \( E \) on \( X \), then \( E \) is stable.

Given a stable vector bundle \( E \) on \( X \) we may ask for a criterion for it to descend to \( Y \) (meaning a criterion for \( E \) to be the pullback of a vector bundle on \( Y \)). The following criterion is deduced using Theorem 1.2 (see Theorem 3.2):

**Theorem 1.3.** Let \( f : X \to Y \) be a genuinely ramified map of irreducible normal projective varieties. Let \( E \) be a stable vector bundle on \( X \). Then there is a vector bundle \( W \) on \( Y \) with \( f^*W \) isomorphic to \( E \) if and only if \( f^*E \) contains a stable vector bundle \( F \) such that

\[
\frac{\text{degree}(F)}{\text{rank}(F)} = \frac{1}{\text{degree}(f)} \cdot \frac{\text{degree}(E)}{\text{rank}(E)}.
\]

Let \( M \) be a smooth projective variety over \( k \). If the étale fundamental group of \( M \) is trivial, then the stratified fundamental group of \( M \) is also trivial [EM]. Let \( f : M \to \mathbb{P}^d_k \) be a finite separable map, where \( d = \dim M \). The above mentioned result of [EM] has the following equivalent reformulation: If \( f \) induces an isomorphism of the étale fundamental groups, then it induces an isomorphism of the stratified fundamental groups. In view of this reformulation, it is natural to ask the following question:

If \( f : X \to Y \) is a finite separable surjective map between irreducible normal projective varieties such that the corresponding homomorphism of the étale fundamental groups is surjective, then does \( f \) induce a surjection of the stratified fundamental groups?

We hope that Theorem 1.2 would shed some light on this question.

2. Genuinely ramified maps

Let \( k \) be an algebraically closed field; there is no assumption on the characteristic of \( k \). Following [PS], [BP1], [BHS] we define:

**Definition 2.1.** A separable finite surjective map

\[ f : X \to Y \]

between irreducible normal projective \( k \)-varieties is said to be genuinely ramified if the homomorphism between étale fundamental groups

\[ f_* : \pi_1^{\text{et}}(X) \to \pi_1^{\text{et}}(Y) \]

induced by \( f \) is surjective.
Let
\[ f : X \rightarrow Y \] (2.1)
be a separable finite surjective map between irreducible normal projective varieties. Since \( X \) and \( Y \) are normal, and \( f \) is a finite surjective map, the direct image \( f_* \mathcal{O}_X \) is a reflexive sheaf on \( Y \) whose rank coincides with the degree of the map \( f \).

To define the degree of a torsionfree sheaf on \( Y \), we fix a very ample line bundle \( L_Y \) on \( Y \). Since \( f \) is a finite map, the line bundle \( f^* L_Y \) on \( X \) is ample. Equip \( X \) with the polarization \( f^* L_Y \). In what follows, the degree of the sheaves on \( Y \) and \( X \) are always with respect to the polarizations \( L_Y \) and \( f^* L_Y \) respectively. For a torsionfree sheaf \( F \), define
\[ \mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{Q}. \]

We recall that a torsionfree sheaf \( E \) is called stable (respectively, semistable) if for all subsheaves \( V \subset E \), with \( 0 < \text{rank}(V) < \text{rank}(E) \), the inequality
\[ \mu(V) < \mu(E) \] (respectively, \( \mu(V) \leq \mu(E) \))
holds. If \( E_1 \subset E \) is the first nonzero term of the Harder-Narasimhan filtration of \( E \), then
\[ \mu_{\text{max}}(E) := \mu(E_1) \] \([\text{HL}]\). In particular, if \( E \) is semistable, then \( \mu_{\text{max}}(E) = \mu(E) \).

For any coherent sheaves \( E \) and \( F \) on \( X \) and \( Y \) respectively, the projection formula gives
\[ f_* \text{Hom}(f^* F, E) = f_*((f^* F) \otimes E) = F^* \otimes f_* E = \text{Hom}(F, f_* E). \]
Since \( f \) is a finite map, this gives the following:
\[ H^0(X, \text{Hom}(f^* F, E)) = H^0(Y, f_* \text{Hom}(f^* F, E)) = H^0(Y, \text{Hom}(F, f_* E)). \] (2.2)

Setting \( E = \mathcal{O}_X \) and \( F = \mathcal{O}_Y \) in (2.2) we conclude that the identification \( f^* \mathcal{O}_Y \sim \mathcal{O}_X \) produces an inclusion homomorphism
\[ \mathcal{O}_Y \hookrightarrow f_* \mathcal{O}_X. \] (2.3)

From (2.3) it follows immediately that \( \mu_{\text{max}}(f_* \mathcal{O}_X) \geq 0 \). In fact,
\[ \mu_{\text{max}}(f_* \mathcal{O}_X) = 0 \] (2.4) \([\text{BP1}]\) Lemma 2.2]; although Lemma 2.2 of [BP1] is stated under the assumption that \( \dim Y = 1 \), its proof works for all dimensions.

We recall from [BP2] the definition of a pseudo-stable sheaf.

**Definition 2.2.** A pseudo-stable sheaf on \( Y \) is a semistable sheaf \( F \) on \( Y \) such that \( F \) admits a filtration of subsheaves
\[ 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F \] (2.5)
satisfying the condition that \( F_i/F_{i-1} \) is a stable reflexive sheaf with \( \mu(F_i/F_{i-1}) = \mu(F) \) for every \( 1 \leq i \leq n \).

A pseudo-stable bundle on \( Y \) is a pseudo-stable sheaf \( F \) as above such that

- \( F \) is locally free, and
for each \(1 \leq i \leq n\) the quotient \(F_i/F_{i-1}\) in (2.5) is locally free.

The first one among the above two conditions is actually implied by the second condition.

For the map \(f\) in (2.1) consider the maximal destabilizing subsheaf \(HN_1(f_*\mathcal{O}_X)\) of the reflexive coherent sheaf \(f_*\mathcal{O}_X\) on \(Y\); in other words, \(HN_1(f_*\mathcal{O}_X)\) is the first nonzero term in the Harder-Narasimhan filtration of \(f_*\mathcal{O}_X\). Recall from (2.4) that

\[
\mu(HN_1(f_*\mathcal{O}_X)) = \mu_{\text{max}}(f_*\mathcal{O}_X) = 0.
\]

Since \(\mathcal{O}_Y \subset HN_1(f_*\mathcal{O}_X)\) (see (2.3)) is a stable locally free subsheaf of degree 0, we conclude that [BP2, Theorem 4.3] applies to \(f_*\mathcal{O}_X\). Let

\[
\mathcal{S} \subset f_*\mathcal{O}_X
\]

be the unique maximal pseudo-stable bundle of degree zero such that \((f_*\mathcal{O}_X)/\mathcal{S}\) is torsionfree [BP2, Theorem 4.3]. Note that we have

\[
\mathcal{O}_Y \subset \mathcal{S}. \tag{2.7}
\]

**Notation 2.3.** For any \(k\)-variety \(Z\), and a coherent sheaf \(F\) on \(Z\), we write

\[F_{\text{tor}} \subset F\]

for the torsion subsheaf of \(F\). Also, define

\[H^0_\varphi(W, F) := H^0(W, F/F_{\text{tor}}). \tag{2.8}\]

In particular,

\[H^0_\varphi(X, f^*f_*\mathcal{O}_X) = H^0(X, (f^*f_*\mathcal{O}_X)/(f^*f_*\mathcal{O}_X)_{\text{tor}}). \tag{2.9}\]

Consider the Cartesian diagram

\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{p_2} & X \\
\downarrow p_1 & \square & \downarrow f \\
X & \xrightarrow{\tilde{f} := f} & Y
\end{array}
\]

(2.10)

where \(p_1\) and \(p_2\) are the natural projections to the first and second factors respectively. Since \(f\) is a finite morphism, by the proper base change theorem of quasi-coherent sheaves we have

\[
\tilde{f}^*f_*\mathcal{O}_X = f^*f_*\mathcal{O}_X = (p_1)_*\mathcal{O}_{X \times_Y X} \tag{2.11}
\]

(see [St, Part 3, Ch. 59, § 59.55, Lemma 59.55.3]). Pulling back the inclusion map in (2.3) to \(X\) we obtain

\[
\mathcal{O}_X \subset f^*f_*\mathcal{O}_X \cong (p_1)_*\mathcal{O}_{X \times_Y X}.
\]

This implies that

\[
\dim H^0_\varphi(X, f^*f_*\mathcal{O}_X) \geq 1 \tag{2.12}
\]

(see (2.9)).
We can also describe $H^0_F(X, f^*f_*O_X)$ in terms of the cohomology of a sheaf on the reduced scheme $(X \times_Y X)_{\text{red}}$. Let
\[ Z := (X \times_Y X)_{\text{red}} \subset X \times_Y X \]
be the reduced subscheme. So $O_Z$ is a quotient of $O_{X \times_Y X}$. Let
\[ j : Z \longrightarrow X \times_Y X \]
be the closed immersion. Let
\[ p'_1, p'_2 : Z \longrightarrow X \]
be the restrictions to $Z$ of the projections $p_1, p_2$ respectively in (2.10). For $i = 1, 2$, we have $p_i \circ j = p'_i$, where $j$ is the map in (2.14). Then
\[ (p'_1)_*O_Z = (p_1)_*j_*O_Z = (p_1)_*(O_{X \times_Y X} / (O_{X \times_Y X})_{\text{tor}}) = ((p_1)_*O_{X \times_Y X}) / ((p_1)_*O_{X \times_Y X})_{\text{tor}}. \]
(2.16)

To explain the last equality in (2.16), note the following:

- If $\mathcal{V}$ is a torsionfree sheaf on $X \times_Y X$, then $(p_1)_*\mathcal{V}$ is a torsionfree sheaf on $X$.
- If $\mathcal{V}$ is a torsion sheaf on $X \times_Y X$, then $(p_1)_*\mathcal{V}$ is a torsion sheaf on $X$.

Hence (2.16) holds.

In view of (2.16), invoking the isomorphism in (2.11) it is concluded that
\[ (p'_1)_*O_Z = (f^*f_*O_X) / (f^*f_*O_X)_{\text{tor}}. \]

This implies that
\[ H^0(X, (p'_1)_*O_Z) = H^0_F(X, f^*f_*O_X) = H^0(X, (p'_2)_*O_Z) \]
(2.17)

(The notation in (2.8) is used).

**Theorem 2.4.** As in (2.1), let $f : X \longrightarrow Y$ be a finite separable surjective map between irreducible normal projective varieties. Then the following five statements are equivalent.

1. The map $f$ is genuinely ramified.
2. There is no nontrivial étale covering $g : Y' \longrightarrow Y$ satisfying the condition that there is a map $g' : X \longrightarrow Y'$ for which $g \circ g' = f$.
3. The fiber product $X \times_Y X$ is connected.
4. $\dim H^0_F(X, f^*f_*O_X) = 1$ (see (2.9)).
5. The subsheaf $\mathcal{S} \subset f_*O_X$ in (2.6) coincides with the subsheaf $\mathcal{O}_Y$ in (2.3); in other words, the inclusion in (2.7) is an equality. This is equivalent to the condition that $\text{rank}(\mathcal{S}) = 1$.

**Proof.** We will show that (1) $\iff$ (2), (1) $\iff$ (5), (3) $\iff$ (4), (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4).

Let
\[ f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y) \]
be the homomorphism between the étale fundamental groups induced by the map $f$. Since the map $f$ is dominant, $f_*(\pi^\mathrm{\text{et}}_1(X))$ is a subgroup of $\pi^\mathrm{\text{et}}_1(Y)$ of finite index. In fact, the index of this subgroup is at most $\deg(f)$. Let

$$g : Y' \longrightarrow Y$$

be the étale covering for this subgroup $f_*(\pi^\mathrm{\text{et}}_1(X)) \subset \pi^\mathrm{\text{et}}_1(Y)$. Then there is a morphism $g' : X \longrightarrow Y'$ such that $g \circ g' = f$. To explain this map $g'$, fix a point $y_0 \in Y$ and also fix points $x_0 \in X$ and $y_1 \in Y'$ over $y_0$. Let $X'$ be the connected component of the fiber product $X \times_Y Y'$ containing the point $(x_0, y_1)$. The given condition that $f_*(\pi^\mathrm{\text{et}}_1(X, x_0)) = g_*(\pi^\mathrm{\text{et}}_1(Y', y_1))$ implies that the étale covering $f' : X' \longrightarrow X$, where $f'$ is the natural projection, induces an isomorphism of étale fundamental groups. Hence $f'$ is actually an isomorphism. If $g'$ is the composition of maps

$$X \sim \longrightarrow X' \longrightarrow Y',$$

where $X' \longrightarrow Y'$ is the natural projection, then clearly, $g \circ g' = f$.

Therefore, the statement (2) implies the statement (1). Conversely, if there is a nontrivial étale covering $g : Y' \longrightarrow Y$ and a morphism $g' : X \longrightarrow Y'$ such that $g \circ g' = f$, then the homomorphism $f_*$ is not surjective, because the homomorphism $g_*$ is not surjective. So the statements (1) and (2) are equivalent.

It is known that the statements (1) and (5) are equivalent; see [BP2, Proposition 1.3]. We will now prove that the statements (3) and (4) are equivalent.

The fiber product $X \times_Y X$ is connected if and only if $\dim H^0(Z, \mathcal{O}_Z) = 1$, where $Z$ is constructed in (2.13). Therefore, $X \times_Y X$ is connected if and only if

$$\dim H^0(X, (p'_1)_*\mathcal{O}_Z) = 1,$$

where $p'_1$ is the map in (2.15). Consequently, from (2.17) it follows immediately that the statements (3) and (4) are equivalent.

Next we will show that the statement (3) implies the statement (2).

Assume that (2) does not hold. So there is a nontrivial étale covering

$$g : Y' \longrightarrow Y$$

and a map $g' : X \longrightarrow Y'$ such that $g \circ g' = f$. Since $g$ is étale, the fiber product $Y' \times_Y Y'$ is a reduced normal projective scheme. However, $Y' \times_Y Y'$ is not connected. In fact, the image of the diagonal embedding $Y' \hookrightarrow Y' \times_Y Y'$ is a connected component of $Y' \times_Y Y'$. Note that this diagonal embedding is not surjective because $g$ is nontrivial. Since the map

$$(g', g') : X \times_Y X \longrightarrow Y' \times_Y Y'$$

is surjective, and $Y' \times_Y Y'$ is not connected, we conclude that $X \times_Y X$ is not connected. Hence the statement (3) implies the statement (2).

Finally, we will prove that the statement (5) implies the statement (4).

Assume that the statement (4) does not hold. So from (2.12) it follows that

$$n := \dim H^0_b(X, f^*f_*\mathcal{O}_X) \geq 2.$$  

(2.18)
Choose a finite Galois field extension \( k(Y) \subset k(X) \subset L \). Let
\[
\Gamma = \text{Gal}(L/k(Y))
\]
be the corresponding Galois group. Let \( M \) be the normalization of \( X \) in \( L \). So \( M \) is an irreducible normal projective variety with \( k(M) = L \), and
\[
f : M \rightarrow Y
\]
is a Galois (possibly branched) cover dominating the map \( f; \) the Galois group for \( \hat{f} \) is \( \text{Gal}(L/k(Y)) = \Gamma \). Let
\[
g : M \rightarrow X
\]
be the morphism induced by the inclusion map \( k(X) \hookrightarrow L \), so \( f \circ g = \hat{f} \).

Let
\[
\varphi : \mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X) \rightarrow (f^*f_*\mathcal{O}_X)/(f^*f_*\mathcal{O}_X)_{\text{tor}}
\]
be the evaluation homomorphism. To prove this, first note that for a semistable sheaf \( V \) on \( Y \), the pullback \( (f^*V)/(f^*V)_{\text{tor}} \) is semistable (see [Bun-Par 1, Remark 2.1]; the proof in [Bun-Par 1, Remark 2.1] works for all dimensions). Therefore, from (2.4) we conclude that
\[
\mu_{\text{max}}((f^*f_*\mathcal{O}_X)/(f^*f_*\mathcal{O}_X)_{\text{tor}}) = 0.
\]
Any coherent subsheaf \( W \subset \mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X) \) such that
\begin{enumerate}
\item \( \text{degree}(W) = 0 \), and
\item the quotient \( (\mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X))/W \) is torsionfree
\end{enumerate}
must be of the form
\[
\mathcal{O}_X \otimes W \subset \mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X)
\]
for some subspace
\[
W \subset H^0_F(X, f^*f_*\mathcal{O}_X).
\]
Indeed, this follows from the fact that \( \text{degree}(W) \) is a nonzero multiple of the degree of the pullback of the tautological bundle by the rational map from \( X \) to the Grassmannian corresponding to \( W \); if \( \text{rank}(W) = a \), then \( W \) is given by a rational map from \( X \) to the Grassmannian parametrizing \( a \) dimensional subspace of \( k^a \) (this map is defined on the open subset over which \( W \) is a subbundle of \( \mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X) \)). Therefore, from the given two conditions on \( W \) it follows that it is of the form \( \mathcal{O}_X \otimes W \) for some subspace \( W \subset H^0_F(X, f^*f_*\mathcal{O}_X) \). For the homomorphism \( \varphi \) in (2.20) if \( \text{kernel}(\varphi) \neq 0 \), then \( \text{degree}(\text{kernel}(\varphi)) = 0 \); this follows from (2.21) and the fact that \( \text{degree}(\mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X)) = 0 \) (these two together imply that the image of \( \varphi \) lies in the maximal semistable subsheaf of \( (f^*f_*\mathcal{O}_X)/(f^*f_*\mathcal{O}_X)_{\text{tor}} \)). Also, the quotient \( (\mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X))/\text{kernel}(\varphi) \) is torsionfree, because it is contained in \((f^*f_*\mathcal{O}_X)/(f^*f_*\mathcal{O}_X)_{\text{tor}}\). Therefore, from the earlier observation we conclude that \( \text{kernel}(\varphi) \) is of the form \( \mathcal{O}_X \otimes W \subset \mathcal{O}_X \otimes H^0_F(X, f^*f_*\mathcal{O}_X) \) for some subspace \( W \subset H^0_F(X, f^*f_*\mathcal{O}_X) \). From this it follows that the evaluation map on any \( w \in W \subset H^0_F(X, f^*f_*\mathcal{O}_X) \) is identically zero. But this implies that \( w = 0 \). Therefore, we conclude that \( W = 0 \), and the homomorphism \( \varphi \) in (2.20) is injective.
Let
\[ \mathcal{S} := \text{image}(\varphi) \subset (f^* \mathcal{O}_X)/(f^* \mathcal{O}_X)_{tor} \] (2.22)
be the image of \( \varphi \). Since \( \varphi \) is injective, we conclude that \( \mathcal{S} \) is isomorphic to \( \mathcal{O}_X^{\oplus n} \) (see (2.13)). The pullback \( g^* \mathcal{S} \) by the map \( g \) in (2.19) is free (equivalently, it is a trivial bundle) because \( \mathcal{S} \) is so. The pulled back homomorphism
\[ g^* \varphi : g^* \mathcal{S} \longrightarrow (g^* f^* \mathcal{O}_X)/(g^* f^* \mathcal{O}_X)_{tor} = (\hat{f}^* f^* \mathcal{O}_X)/(\hat{f}^* f^* \mathcal{O}_X)_{tor} \]
is injective because \( \varphi \) is injective and \( \mathcal{S} \) is free (or the bundle is trivial). Using \( g^* \varphi \) we would consider \( g^* \mathcal{S} \) as a subsheaf of \( (\hat{f}^* f^* \mathcal{O}_X)/(\hat{f}^* f^* \mathcal{O}_X)_{tor} \).

The Galois group \( \Gamma = \text{Gal}(\hat{f}) \) has a natural action on \( \hat{f}^* f^* \mathcal{O}_X \) because it is pulled back from \( Y \). This action of \( \Gamma \) on \( \hat{f}^* f^* \mathcal{O}_X \) produces an action of \( \Gamma \) on the torsionfree quotient \( (\hat{f}^* f^* \mathcal{O}_X)/(\hat{f}^* f^* \mathcal{O}_X)_{tor} \).

Consider \( Z \) defined in (2.13) together with the projections \( p'_1 \) and \( p'_2 \) from it in (2.15). Both \( (p'_1)^* \mathcal{S} \) and \( (p'_2)^* \mathcal{S} \) are free because \( \mathcal{S} \) is so. The natural isomorphism
\[ (p'_1)^* f^* f_* \mathcal{O}_X \longrightarrow (p'_2)^* f^* f_* \mathcal{O}_X, \]
given by the fact that they are the pull backs of a single sheaf on \( Y \), produces an isomorphism
\[ (((p'_1)^* f^* f_* \mathcal{O}_X))/((p'_1)^* f^* f_* \mathcal{O}_X)_{tor} \longrightarrow (((p'_2)^* f^* f_* \mathcal{O}_X))/((p'_2)^* f^* f_* \mathcal{O}_X)_{tor}. \] (2.23)
It can be shown that the isomorphism in (2.23) takes the subsheaf \( (p'_1)^* \mathcal{S} \subset (((p'_1)^* f^* f_* \mathcal{O}_X))/((p'_1)^* f^* f_* \mathcal{O}_X)_{tor} \), where \( \mathcal{S} \) is defined in (2.22), to the subsheaf
\[ (p'_2)^* \mathcal{S} \subset (((p'_2)^* f^* f_* \mathcal{O}_X))/((p'_2)^* f^* f_* \mathcal{O}_X)_{tor}. \]
Indeed, we have
\[ H^0(X, (p'_1)_* \mathcal{O}_Z) = H^0(Z, \mathcal{O}_Z) = H^0(X, (p'_2)_* \mathcal{O}_Z) \]
(because \( p'_1 \) and \( p'_2 \) are finite maps), and therefore from (2.17) it follows that both \( (p'_1)^* \mathcal{S} \) and \( (p'_2)^* \mathcal{S} \) coincide with \( \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \). The isomorphism in (2.23) takes the subsheaf \( (p'_1)^* \mathcal{S} \) to \( (p'_2)^* \mathcal{S} \), and the homomorphism \( (p'_1)^* \mathcal{S} \longrightarrow (p'_2)^* \mathcal{S} \) given by the isomorphism in (2.23) coincides with the following automorphism of \( \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \) (after we identify \( (p'_1)^* \mathcal{S} \) and \( (p'_2)^* \mathcal{S} \) with \( \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \)). The involution \( (x, y) \longmapsto (y, x) \) of \( X \times_Y X \) produces an involution of \( Z = (X \times_Y X)_{\text{red}} \). This involution of \( Z \) in turn produces an involution
\[ \delta : H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Z, \mathcal{O}_Z). \]
Define the automorphism
\[ \text{Id} \otimes \delta : \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \longrightarrow \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z). \]
The homomorphism \( (p'_1)^* \mathcal{S} \longrightarrow (p'_2)^* \mathcal{S} \) given by the isomorphism in (2.23) coincides with the automorphism \( \text{Id} \otimes \delta \) of \( \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \), after we identify \( (p'_1)^* \mathcal{S} \) and \( (p'_2)^* \mathcal{S} \) with \( \mathcal{O}_Z \otimes H^0(Z, \mathcal{O}_Z) \).

Using the above observation that the isomorphism in (2.23) takes the subsheaf \( (p'_1)^* \mathcal{S} \) to \( (p'_2)^* \mathcal{S} \) we will now show that the natural action of the Galois group \( \Gamma = \text{Gal}(\hat{f}) \) on
(\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor} \text{ preserves the subsheaf } g^* \mathcal{S}, \text{ where } g \text{ is the map in (2.19); it was noted earlier that } \Gamma \text{ acts on } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor}.

To prove that the action of } \Gamma \text{ on } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor} \text{ preserves } g^* \mathcal{S}, \text{ let } Y_0 \subset Y \text{ be the largest open subset such that}

- the map } f \text{ is flat over } Y_0, \text{ and}
- the map } g \text{ is flat over } f^{-1}(Y_0).

Define } X_0 := f^{-1}(Y_0) \text{ and } M_0 := \hat{f}^{-1}(X_0). \text{ The conditions on } Y_0 \text{ imply that the restriction of } \hat{f} \text{ to } M_0 \text{ is flat [Ha, Ch III, § 9]. In view of the descent criterion for sheaves under flat morphisms, [Ga, Vi], the above observation, that the isomorphism in (2.23) takes the subsheaf } (p_1^* \mathcal{S}) \text{ to the subsheaf } (p_2^* \mathcal{S}), \text{ implies that the restriction } \mathcal{S}|_{X_0} \text{ descends to a subsheaf of } (f_* \mathcal{O}_X)|_{Y_0}. \text{ Consequently, the action of } \Gamma \text{ on } (\hat{f}^* f_* \mathcal{O}_X)|_{M_0} \text{ preserves } (g^* \mathcal{S})|_{M_0} \text{ (as it is the pullback of a sheaf on } Y_0); \text{ note that } \hat{f}^* f_* \mathcal{O}_X \text{ is torsionfree over } M_0 \text{ because the map } \hat{f} \text{ is flat over } M_0. \text{ The codimension of the complement } M \setminus M_0 \text{ is at least two. Therefore, given two vector bundle } A \text{ and } B \text{ on } M \text{ together with an isomorphism}

\[ \mathcal{I} : A|_{M_0} \sim \sim B|_{M_0} \]

over } M_0, \text{ there is a unique isomorphism}

\[ \tilde{\mathcal{I}} : A \sim B \]

that restricts to } \mathcal{I} \text{ on } M_0; \text{ recall that } M \text{ is normal. For any } \gamma \in \Gamma, \text{ set } g^* \mathcal{S} = A = B \text{ and } \mathcal{I} \text{ to be the action of } \gamma \text{ on } (g^* \mathcal{S})|_{M_0}; \text{ recall that } g^* \mathcal{S} \text{ is locally free and the action of } \Gamma \text{ on } (\hat{f}^* f_* \mathcal{O}_X)|_{M_0} \text{ preserves } (g^* \mathcal{S})|_{M_0}. \text{ Now from the above observation that } \mathcal{I} \text{ extends uniquely to } M \text{ we conclude that the action of } \gamma \text{ on } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor} \text{ preserves the subsheaf } g^* \mathcal{S}.

From the above observation that } g^* \mathcal{S} \text{ is preserved by the natural action of } \Gamma \text{ on } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor} \text{ we will now deduce that there is a locally free subsheaf}

\[ V \subset f_* \mathcal{O}_X \]

such that the image of } \hat{f}^* V \text{ in } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor} \text{ coincides with the image of } g^* \mathcal{S} \text{ in } (\hat{f}^* f_* \mathcal{O}_X)/(\hat{f}^* f_* \mathcal{O}_X)_\text{tor}.

To prove this, consider the free sheaf

\[ \mathcal{O}_M \otimes H^0(M, g^* \mathcal{S}) \longrightarrow M; \]

it corresponds to the trivial vector bundle on } M \text{ with fiber } H^0(M, g^* \mathcal{S}). \text{ Let}

\[ \Phi : \mathcal{O}_M \otimes H^0(M, g^* \mathcal{S}) \longrightarrow g^* \mathcal{S} \quad (2.24) \]

be the evaluation map. This } \Phi \text{ is an isomorphism, because } g^* \mathcal{S} \text{ is free, or equivalently, it is a trivial vector bundle (recall that } \mathcal{S} \text{ is free). The action of } \Gamma \text{ on } g^* \mathcal{S} \text{ produces an action of } \Gamma \text{ on } H^0(M, g^* \mathcal{S}) \text{ which, coupled with the action of } \Gamma \text{ on } M, \text{ produces an action of } \Gamma \text{ on } \mathcal{O}_M \otimes H^0(M, g^* \mathcal{S}). \text{ The isomorphism } \Phi \text{ in (2.24) is evidently } \Gamma\text{–equivariant for the actions of } \Gamma.
Let

$$\Gamma_0 \subset \Gamma$$

be the normal subgroup that acts trivially on the vector space $H^0(M, g^*S)$. Now using $\Phi$ we conclude that $g^*S$ descends to a vector bundle on the quotient $M/\Gamma_0$. Let

$$S_1 \longrightarrow M/\Gamma_0$$

be the descent of $g^*S$. In other words, the pullback of $S_1$ to $M$ has a $\Gamma_0$–equivariant isomorphism with $g^*S$. Note that $\mathcal{O}_M \otimes H^0(M, g^*S)$ descends to $\mathcal{O}_{M/\Gamma_0} \otimes H^0(M, g^*S)$ on $M/\Gamma_0$, because $\Gamma_0$ acts trivially on $H^0(M, g^*S)$, and hence $S_1$ is a trivial vector bundle.

For the action of $\Gamma$ on $M$, the isotropy subgroup $\Gamma_m \subset \Gamma$ any point $m \in M$ is actually contained in the subgroup $\Gamma_0$ in (2.25). Indeed, this follows immediately from the fact that $\Gamma_m$ acts trivially on the fiber of $(\hat{f}^*f_*\mathcal{O}_X)/(\hat{f}^*f_*\mathcal{O}_X)_{tor}$ over $m$; recall that $\mathcal{S}$ is the image of the injective map $\varphi$ from a trivial vector bundle (see (2.20) and (2.22)). Consequently, the natural map

$$M/\Gamma_0 \longrightarrow Y$$

given by $\hat{f}$ is actually étale Galois with Galois group $\Gamma/\Gamma_0$. The action of $\Gamma$ on $g^*S$ produces an action of $\Gamma/\Gamma_0$ on $S_1$ in (2.26); it is a lift of the action of $\Gamma/\Gamma_0$ on $M/\Gamma_0$. Hence $S_1$ descends to a vector bundle $S_2$ on $Y$.

We have $S_2 \subset \mathcal{S}$, where $\mathcal{S}$ is constructed in (2.6), and also

$$\text{rank}(S_2) = \text{rank}(\mathcal{S}) = n \geq 2$$

(see (2.18)). Hence the statement (5) in the theorem fails. Therefore, the statement (5) implies the statement (4). This completes the proof. 

**Theorem 2.5.** Let $f : X \longrightarrow Y$ be a finite separable surjective map between irreducible normal projective varieties such that $f$ is genuinely ramified. Let $E$ be a stable vector bundle on $Y$. Then the pulled back vector bundle $f^*E$ is also stable.

**Proof.** In view of Theorem 2.4, the proof is exactly identical to the proof of Theorem 1.1 of [BP1]. The only point to note that in Proposition 3.5 of [BP1], the sheaves $\mathcal{L}_i$ are no longer locally free. Now they are sheaves properly contained in $\mathcal{O}_X$. But this does not affect the arguments. We omit the details. 

3. A Characterization of Stable Pullbacks

**Lemma 3.1.** Let $f : X \longrightarrow Y$ be a separable finite surjective map of irreducible normal projective varieties. Let $E$ be a semistable vector bundle on $X$. Then

$$\mu_{\text{max}}(f_*E) \leq \frac{1}{\text{degree}(f)} \cdot \mu(E).$$

**Proof.** Let $F \subset f_*E$ be the first nonzero term of the Harder-Narasimhan filtration of $f_*E$, so

$$\mu_1 := \mu_{\text{max}}(f_*E) = \mu(F).$$
Therefore, \((f^*F)/(f^*F)_{\text{tor}}\) is semistable \([BP1, \text{Remark 2.1}]\), and
\[\mu((f^*F)/(f^*F)_{\text{tor}}) = \text{degree}(f) \cdot \mu_1.\]

The inclusion map \(F \hookrightarrow f_*E\) gives a homomorphism \(f^*F \hookrightarrow E\) (see (2.2)), which in turn produces a homomorphism
\[\beta : (f^*F)/(f^*F)_{\text{tor}} \hookrightarrow E.\]

Since \(\beta \neq 0\), and both \(E\) and \((f^*F)/(f^*F)_{\text{tor}}\) are semistable, we have
\[\mu(E) \geq \mu(\text{image}(\beta)) \geq \mu((f^*F)/(f^*F)_{\text{tor}}) = \text{degree}(f) \cdot \mu_1.\]

This proves the lemma. \(\square\)

**Theorem 3.2.** Let \(f : X \rightarrow Y\) be a genuinely ramified map of irreducible normal projective varieties. Let \(E\) be a stable vector bundle on \(X\). Then there is a vector bundle \(W\) on \(Y\) with \(f^*W\) isomorphic to \(E\) if and only if \(f_*E\) contains a stable vector bundle \(F\) such that
\[\mu(F) = \frac{1}{\text{degree}(f)} \cdot \mu(E).\]

**Proof.** First assume that there is a vector bundle \(W\) on \(Y\) such that \(f^*W\) is isomorphic to \(E\). It can be shown that \(W\) is stable. Indeed, if
\[S \subset W\]
is a nonzero coherent subsheaf such that \(\text{rank}(W/S) > 0\) and \(\mu(S) \geq \mu(W)\), then we have \(\text{rank}((f^*W)/(f^*S)) > 0\) and
\[\mu(f^*S) = \mu(S) \cdot \text{degree}(f) \geq \mu(W) \cdot \text{degree}(f) = \mu(f^*W),\]
which contradicts the stability condition for \(f^*W = E\).

Using the given condition \(f^*W = E\), and the projection formula, we have
\[f_*E = f_*f^*W = W \otimes f_*O_X.\]

Since \(O_Y \subset f_*O_X\) (see (2.3)), we have
\[W \subset W \otimes f_*O_X = f_*E.\]

Note that
\[\mu(W) = \frac{1}{\text{degree}(f)} \cdot \mu(E).\]

To prove the converse, assume that \(f_*E\) contains a stable vector bundle \(F\) such that
\[\mu(F) = \frac{1}{\text{degree}(f)} \cdot \mu(E).\]

Then \(f^*F\) is stable by Theorem 2.5. Let
\[h : f^*F \rightarrow E\]
be the natural homomorphism; this \(h\) is given by the inclusion map \(F \hookrightarrow f_*E\) using the isomorphism in (2.2). Since
\[(1) \text{ both } f^*F \text{ and } E \text{ are stable},\]
\[(2) \mu(E) = \text{degree}(f) \cdot \mu(F) = \mu(f^*F), \text{ and}\]
we conclude that $h$ is an isomorphism.

\begin{remark}

Note that the proof of Theorem 3.2 shows the following. Let $f : X \to Y$ be a finite surjective map of irreducible normal projective varieties. Let $E$ be a stable vector bundle on $X$. If there is a vector bundle bundle $W$ on $Y$ with $f^*W$ isomorphic to $E$, then $f_*E$ contains a stable vector bundle $F$ such that $\mu(F) = \frac{1}{\text{degree}(f)} \cdot \mu(E)$.

\end{remark}

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