The structure of Selmer groups of elliptic curves  
and modular symbols

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For an elliptic curve over the rational number field and a prime number $p$, we study the structure of the classical Selmer group of $p$-power torsion points. In our previous paper [12], assuming the main conjecture and the non-degeneracy of the $p$-adic height pairing, we proved that the structure of the Selmer group with respect to $p$-power torsion points is determined by some analytic elements $\tilde{\delta}_m$ defined from modular symbols (see Theorem 1.1.1 below). In this paper, we do not assume the main conjecture nor the non-degeneracy of the $p$-adic height pairing, and study the structure of Selmer groups (see Theorems 1.2.3 and 1.2.5), using these analytic elements and Kolyvagin systems of Gauss sum type.

1 Introduction

1.1 Structure theorem of Selmer groups

Let $E$ be an elliptic curve over $\mathbb{Q}$. Iwasawa theory, especially the main conjecture gives a formula on the order of the Tate Shafarevich group by using the $p$-adic $L$-function (cf. Schneider [24]). In this paper, as a sequel of [10], [11] and [12], we show that we can derive more information than the order, on the structure of the Selmer group and the Tate Shafarevich group from analytic quantities, in the setting of our paper, from modular symbols.

In this paper, we consider a prime number $p$ such that
(i) $p$ is a good ordinary prime $> 2$ for $E$,
(ii) the action of $G_{\mathbb{Q}}$ on the Tate module $T_p(E)$ is surjective where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$,
(iii) the (algebraic) $\mu$-invariant of $(E, \mathbb{Q}_\infty/\mathbb{Q})$ is zero where $\mathbb{Q}_\infty/\mathbb{Q}$ is the cyclotomic $\mathbb{Z}_p$-extension, namely the Selmer group $\text{Sel}(E/\mathbb{Q}_\infty, E[p^\infty])$ (for the definition, see below) is a cofinitely generated $\mathbb{Z}_p$-module,
(iv) $p$ does not divide the Tamagawa factor $\text{Tam}(E) = \Pi_{\ell \text{ bad}}(E(\mathbb{Q}_\ell))$:
We note that the property (iii) is a conjecture of Greenberg since we are assuming (ii).

For a positive integer \(N > 0\), we denote by \(E[p^N]\) the Galois module of \(p^N\)-torsion points, and \(E[p^\infty] = \bigcup_{N>0} E[p^N]\). For an algebraic extension \(F/Q\), \(\text{Sel}(E/F, E[p^N])\) is the classical Selmer group defined by

\[
\text{Sel}(E/F, E[p^N]) = \text{Ker}(H^1(F,E[p^N]) \longrightarrow \prod_v H^1(F_v, E[p^N])/E(F_v)\otimes \Z/p^N),
\]

so \(\text{Sel}(E/F, E[p^N])\) sits in an exact sequence

\[
0 \longrightarrow E(F) \otimes \Z/p^N \longrightarrow \text{Sel}(E/F, E[p^N]) \longrightarrow \text{III}(E/F)[p^N] \longrightarrow 0
\]

where \(\text{III}(E/F)\) is the Tate Shafarevich group over \(F\). We define \(\text{Sel}(E/F, E[p^\infty]) = \lim \text{Sel}(E/F, E[p^N])\).

Let \(P^{(N)}\) be the set of prime numbers \(\ell\) such that \(\ell\) is a good reduction prime for \(E\) and \(\ell \equiv 1 \pmod{p^N}\). For each \(\ell\), we fix a generator \(\eta_\ell\) of \((\Z/\ell\Z)^\times\) and define \(\log_{\eta_\ell}(a) \in \Z/\ell-1)\) by \(\eta^{\log_{\eta_\ell}(a)} = a \pmod{\ell}\).

Let \(f(z) = \sum a_n e^{2\pi i n z}\) be the modular form corresponding to \(E\). For a positive integer \(m\) and the cyclotomic field \(Q(\mu_m)\), we denote by \(\sigma_a \in \text{Gal}(Q(\mu_m)/Q)\) the element such that \(\sigma_a(\zeta) = \zeta^a\) for any \(\zeta \in \mu_m\). We consider the modular element \(\sum_{a=1}^m \frac{[\frac{a}{m}]}{[\frac{a}{m}]}\sigma_a \in \C[\text{Gal}(Q(\mu_m)/Q)]\) of Mazur and Tate (16) where \([\frac{a}{m}] = 2\pi i \int_{a/m} f(z) dz\) is the usual modular symbol. We only consider the real part

\[
\tilde{\theta}_{Q(\mu_m)} = \sum_{a=1}^m \frac{\text{Re}([\frac{a}{m}])}{\Omega_\ell^+} \sigma_a \in Q[\text{Gal}(Q(\mu_m)/Q)]
\]  

(1.1)

where \(\Omega_\ell^+ = \int_{E(\R)} \omega_E\) is the Néron period. Suppose that \(m\) is a squarefree product of primes in \(P^{(N)}\). Since we are assuming the \(G_Q\)-module \(E[p]\) of \(p\)-torsion points is irreducible, we know \(\tilde{\theta}_{Q(\mu_m)} \in \Z_p[\text{Gal}(Q(\mu_m)/Q)]\) (cf. 27).

We consider the coefficient of \(\tilde{\theta}_{Q(\mu_m)}\) of \(\prod_{\ell|m} (\sigma_{\eta_\ell} - 1)\), more explicitly we define

\[
\tilde{\delta}_m = \sum_{a=1}^m \frac{\text{Re}([\frac{a}{m}])}{\Omega_\ell^+} (\prod_{\ell|m} \log_{\eta_\ell}(a)) \in \Z/p^N
\]

(1.2)

where \(\log_{\eta_\ell}(a)\) means the image of \(\log_{\eta_\ell}(a)\) under the canonical homomorphism \(\Z/(\ell-1) \longrightarrow \Z/p^N\). Let \(\text{ord}_p : \Z/p^N \longrightarrow \{0,1,...,N-1,\infty\}\) be the \(p\)-adic valuation normalized as \(\text{ord}_p(p) = 1\) and \(\text{ord}_p(0) = \infty\). We note
that $\text{ord}_p(\tilde{\delta}_m)$ does not depend on the choices of $\eta_\ell$ for $\ell | m$. We define $\tilde{\delta}_1 = \theta_\mathbb{Q} = \text{Re}([0])/\Omega_E^+ = L(E, 1)/\Omega_E^+$.

For a squarefree product $m$ of primes, we define $\epsilon(m)$ to be the number of prime divisors of $m$, namely $\epsilon(m) = r$ if $m = \ell_1 \cdots \ell_r$. Let $\mathcal{N}^{(N)}$ be the set of squarefree products of primes in $\mathcal{P}^{(N)}$. We suppose 1 is in $\mathcal{N}^{(N)}$. For each integer $i \geq 0$, we define the ideal $\Theta_i(Q)^{(N, \delta)}$ of $\mathbb{Z}/p^N$ to be the ideal generated by all $\tilde{\delta}_m$ such that $\epsilon(m) \leq i$ for all $m \in \mathcal{N}^{(N)}$;

$$\Theta_i(Q)^{(N, \delta)} = \{ \{ \tilde{\delta}_m | \epsilon(m) \leq i \text{ and } m \in \mathcal{N}^{(N)} \} \subset \mathbb{Z}/p^N.$$  \hfill (1.3)

We define $n_{i, N} \in \{ 0, 1, \ldots, N - 1, \infty \}$ by $\Theta_i(Q)^{(N, \delta)} = p^{n_{i, N}}(\mathbb{Z}/p^N)$ (we define $n_{i, N} = \infty$ if $\Theta_i(Q)^{(N, \delta)} = 0$).

**Theorem 1.1.1** ([12] Theorem B, Theorem 9.3.1 and (9.14)) We assume that the main conjecture for $(E, Q_\infty/\mathbb{Q})$ (see [12]) and the $p$-adic height pairing is non-degenerate.

1. $n_{i, N}$ does not depend on $N$ when $N$ is sufficiently large (for example, when $N > 2 \text{ord}_p(\eta_0)$ where $\eta_0$ is the leading term of the $p$-adic $L$-function, see §9.4 in [12]). We put $n_i = n_{i, N}$ for $N \gg 0$. In other words, we define $n_i$ by

$$\lim_{\leftarrow} \Theta_i(Q)^{(N, \delta)} = p^{n_i} \mathbb{Z}_p \subset \mathbb{Z}_p.$$

We denote this ideal of $\mathbb{Z}_p$ by $\Theta_i(Q)^{(\delta)}$.

2. Consider the Pontrjagin dual $\text{Sel}(E/Q, E[p^\infty])^\vee$ of the Selmer group. Suppose that

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}(E/Q, E[p^\infty])^\vee = r(\in \mathbb{Z}_{\geq 0}), \text{ and } \dim_{\mathbb{F}_p} \text{Sel}(E/Q, E[p])^\vee = a.$$

Then we have

$$\Theta_0(Q)^{(\delta)} = \ldots = \Theta_{r-1}(Q)^{(\delta)} = 0 \text{ and } \Theta_r(Q)^{(\delta)} \neq 0.$$  \hfill

For any $i \geq r$, $n_i$ is an even number, and

$$p^{nr} = \#(\text{Sel}(E/Q, E[p^\infty])^\vee)_{\text{tors}},$$

$$n_a = 0, \text{ and}$$

$$\text{Sel}(E/Q, E[p^\infty])^\vee \cong \mathbb{Z}_p^{r^2} \oplus (\mathbb{Z}/p^{n_r+2})^{\oplus 2} \oplus (\mathbb{Z}/p^{n_{r+2}-n_r+4})^{\oplus 2} \oplus \ldots \oplus (\mathbb{Z}/p^{n_a-2-n_a})^{\oplus 2}$$

hold.

In particular, knowing $\Theta_i(Q)^{(\delta)}$ for all $i \geq 0$ completely determines the structure of $\text{Sel}(E/Q, E[p^\infty])^\vee$ as a $\mathbb{Z}_p$-module. Namely, the modular symbols determine the structure of the Selmer group under our assumptions.
1.2 Main Results

We define
\[ \mathcal{P}_1^{(N)} = \{ \ell \in \mathcal{P}^{(N)} \mid H^0(F_\ell, E[p^N]) \simeq \mathbf{Z}/p^N \}. \]

This is an infinite set by Chebotarev density theorem since we are assuming (ii) (see [12] §4.3). We define \( \mathcal{N}_1^{(N)} \) to be the set of squarefree products of primes in \( \mathcal{P}_1^{(N)} \). Again, we suppose \( 1 \in \mathcal{N}_1^{(N)} \). We propose the following conjecture.

**Conjecture 1.2.1** There is \( m \in \mathcal{N}_1^{(N)} \) such that \( \tilde{\delta}_m \) is a unit in \( \mathbf{Z}/p^N \), namely
\[ \text{ord}_p(\tilde{\delta}_m) = 0. \]

Numerically, it is easy to compute \( \tilde{\delta}_m \), so it is easy to check this conjecture.

**Theorem 1.2.2** ([12] Theorem 9.3.1) If we assume the main conjecture and the non-degeneracy of the \( p \)-adic height pairing, Conjecture 1.2.1 holds true.

In fact, we obtain Conjecture 1.2.1 considering the case \( i = a \) in Theorem 1.1.1 (cf. \( i = s \) in Theorem 9.3.1 in [12]).

From now on, we do not assume the main conjecture (2.5) nor the non-degeneracy of the \( p \)-adic height pairing.

We define the Selmer group \( \text{Sel}(\mathbf{Z}[1/m], E[p^N]) \) by
\[ \text{Sel}(\mathbf{Z}[1/m], E[p^N]) = \text{Ker}(H^1(Q, E[p^N]) \longrightarrow \prod_{v \mid m} H^1(Q_v, E[p^N])/E(Q_v) \otimes \mathbf{Z}/p^N). \]

If all bad primes and \( p \) divide \( m \), we know \( \text{Sel}(\mathbf{Z}[1/m], E[p^N]) \) is equal to the étale cohomology group \( H^1_{\text{ét}}(\text{Spec} \mathbf{Z}[1/m], E[p^N]) \), which explains the notation “\( \text{Sel}(\mathbf{Z}[1/m], E[p^N]) \)”. (We use \( \text{Sel}(\mathbf{Z}[1/m], E[p^N]) \) for \( m \in \mathcal{N}_1^{(N)} \) in this paper, but \( E[p^N] \) is not an étale sheaf on \( \text{Spec} \mathbf{Z}[1/m] \) for such \( m \).

Let \( \lambda \) be the \( \lambda \)-invariant of \( \text{Sel}(E/Q_\infty, E[p^\infty])^\vee \). We put \( n_\lambda = \min\{n \in \mathbf{Z} \mid p^n - 1 \geq \lambda \} \) and \( d_\lambda = n_\lambda + Nn \) for \( n \in \mathbf{Z}_{\geq 0} \). We define
\[ \mathcal{P}_1^{(N,n)} = \{ \ell \in \mathcal{P}_1^{(N)} \mid \ell \equiv 1 \pmod{p^{d_\lambda}} \} \quad (1.4) \]

(then \( \mathcal{P}_1^{(N,n)} \subset \mathcal{P}_1^{(N)}(Q_{[n]}) \) holds, see the end of §3.1 for this fact, and see §3.1 for the definition of the set \( \mathcal{P}_1^{(N)}(Q_{[n]}) \)). We denote by \( \mathcal{N}_1^{(N,n)} \) the set of squarefree products of primes in \( \mathcal{P}_1^{(N,n)} \).
In this paper, for any finite abelian $p$-extension $K/Q$ in which all bad primes of $E$ are unramified, we prove in [12] the following theorem for $\mathbb{Z}/p^N[\text{Gal}(K/Q)]$-modules $\text{Sel}(E/K, E[p^N])$ and $\text{Sel}(O_K[1/m], E[p^N])$ (see Corollary 4.1.3 and Theorem 4.2.1). We simply state it in the case $K = Q$ below. An essential ingredient in this paper is the Kolyvagin system of Gauss sum type. We construct Kolyvagin systems $\kappa_{m,\ell} \in \text{Sel}(\mathbb{Z}[1/m\ell], E[p^N])$ for $(m,\ell)$ satisfying $\ell \in \mathcal{P}(N, \epsilon(m) + 1)$ and $m\ell \in \mathcal{N}(N, \epsilon(m) + 1)$ (see §3.4 and Propositions 3.4.2) by the method in [12]. (We can construct these elements, using the half of the main conjecture proved by Kato [7].) The essential difference between our Kolyvagin systems $\kappa_{m,\ell}$ of Gauss sum type and Kolyvagin systems in Mazur and Rubin [14] is that our $\kappa_{m,\ell}$ is related to $L$-values. In particular, $\kappa_{m,\ell}$ satisfies a remarkable property $\phi_\ell(\kappa_{m,\ell}) = -\delta_{m\ell}t_{\ell,K}$ (see Propositions 3.4.2 (4)) though we do not explain the notation here.

**Theorem 1.2.3** Assume that $\text{ord}_p(\tilde{\delta}_m) = 0$ for some $m \in \mathcal{N}_1^{(N)}$.

1. The canonical homomorphism

$$s_m : \text{Sel}(E/Q, E[p^N]) \longrightarrow \bigoplus_{\ell|m} E(Q_{\ell}) \otimes \mathbb{Z}/p^N \simeq \bigoplus_{\ell|m} E(Q_{\ell}) \otimes \mathbb{Z}/p^N \simeq (\mathbb{Z}/p^N)^{(m)}$$

is injective.

2. Assume further that $m \in \mathcal{N}_1^{(N,\epsilon(m) + 1)}$ and that $m$ is admissible (for the definition of the notion “admissible”, see the paragraph before Proposition 3.3.2). Then $\text{Sel}(\mathbb{Z}[1/m], E[p^N])$ is a free $\mathbb{Z}/p^N$-module of rank $\epsilon(m)$, and $\{\kappa_{m,\ell}\}_{\ell|m}$ is a basis of $\text{Sel}(\mathbb{Z}[1/m], E[p^N])$.

3. We define a matrix $A$ as in (1.1) in Theorem 4.2.1 using $\kappa_{m,\ell}$. Then $A$ is a relation matrix of the Pontrjagin dual $\text{Sel}(E/Q, E[p^N])^\vee$ of the Selmer group; namely if $f_A : (\mathbb{Z}/p^N)^{(m)} \longrightarrow (\mathbb{Z}/p^N)^{(m)}$ is the homomorphism corresponding to the above matrix $A$, then we have

$$\text{Coker}(f_A) \simeq \text{Sel}(E/Q, E[p^N])^\vee.$$ 

It is worth noting that we get nontrivial (moreover, linearly independent) elements in the Selmer groups.

The ideals $\Theta_i(Q)_{(d)}$ in Theorem 1.1.1 are not suitable for numerical computations because we have to compute infinitely many $\tilde{\delta}_m$. On the other hand, we can easily find $m$ with $\text{ord}_p(\tilde{\delta}_m) = 0$ numerically. Since $s_m$ is injective, we can get information of the Selmer group from the image of $s_m$, which is an advantage of Theorem 1.2.3 and the next Theorem 1.2.5 (see also the comment in the end of Example (5) in §5.3).

We next consider the case $N = 1$, so $\text{Sel}(E/Q, E[p])$. Now we regard $\tilde{\delta}_m$ as an element of $F_p$ for $m \in \mathcal{N}_1^{(1)}$. We say $m$ is $\delta$-minimal if $\tilde{\delta}_m \neq 0$. 

and $\delta_d = 0$ for all divisors $d$ of $m$ with $1 \leq d < m$. Our next conjecture claims that the structure (the dimension) of $\text{Sel}(E/\mathbb{Q}, E[p])$ is determined by a $\delta$-minimal $m$, therefore can be easily computed numerically.

**Conjecture 1.2.4** If $m \in \mathcal{N}_1^{(1)}$ is $\delta$-minimal, the canonical homomorphism

$$s_m : \text{Sel}(E/\mathbb{Q}, E[p]) \longrightarrow \bigoplus_{\ell|m} E(\mathbb{Q}_\ell) \otimes \mathbb{Z}/p \simeq \bigoplus_{\ell|m} E(F_\ell) \otimes \mathbb{Z}/p \simeq (\mathbb{Z}/p^N)^{\epsilon(m)}$$

is bijective. In particular, $\dim_{\mathbb{F}_p} \text{Sel}(E/\mathbb{Q}, E[p]) = \epsilon(m)$.

Therefore, the problem is in showing the other inequality.

We note that the analogue of the above conjecture for ideal class groups does not hold (see §5.4). But we hope that Conjecture 1.2.4 holds for the Selmer groups of elliptic curves. We construct in §5 a modified version $K_{q,q^{'},z_m,\ell}$ of Kolyvagin systems of Gauss sum type for any $(m,\ell)$ with $m\ell \in \mathbb{N}^{(N)}$. (The Kolyvagin system $K_{m,\ell}$ in §3 is defined for $(m,\ell)$ with $m\ell \in \mathbb{N}^{(N,\epsilon(m)+1)}$, but $K_{q,q^{'},z_m,\ell}$ is defined for more general $(m,\ell)$, namely for $(m,\ell)$ with $m\ell \in \mathbb{N}^{(N)}$.) Using the modified Kolyvagin system $K_{q,q^{'},z_m,\ell}$, we prove the following.

**Theorem 1.2.5**

(1) If $\epsilon(m) = 0, 1$, then Conjecture 1.2.4 is true.

(2) If there is $\ell \in \mathcal{P}^{(1)}$ which is $\delta$-minimal (so $\epsilon(\ell) = 1$), then

$$\text{Sel}(E/\mathbb{Q}, E[p^{\infty}]) \simeq \mathbb{Q}_p / \mathbb{Z}_p.$$ 

Moreover, if there is $\ell \in \mathcal{P}^{(1)}$ which is $\delta$-minimal and which satisfies $\ell \equiv 1 \pmod{p^{\lambda'+2}}$ where $\lambda'$ is the analytic $\lambda$-invariant of $(E, \mathbb{Q}_{\infty}/\mathbb{Q})$, then the main conjecture (2.3) for $\text{Sel}(E/\mathbb{Q}_{\infty}, E[p^{\infty}])$ holds true. In this case, $\text{Sel}(E/\mathbb{Q}_{\infty}, E[p^{\infty}])^\vee$ is generated by one element as a $\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})]]$-module.

(3) If $\epsilon(m) = 2$ and $m$ is admissible, then Conjecture 1.2.4 is true.

(4) Suppose that $\epsilon(m) = 3$ and $m = \ell_1\ell_2\ell_3$. Assume that $m$ is admissible and the natural maps $s_{\ell_i} : \text{Sel}(E/\mathbb{Q}, E[p]) \longrightarrow E(F_{\ell_i}) \otimes \mathbb{Z}/p$ are surjective both for $i = 1$ and $i = 2$. Then Conjecture 1.2.4 is true.

In this way, we can determine the Selmer groups by finite numbers of computations in several cases. We give several numerical examples in §5.2.
Remark 1.2.6 Concerning the Fitting ideals and the annihilator ideals of some Selmer groups, we prove the following in this paper. Let $K/Q$ be a finite abelian $p$-extension in which all bad primes of $E$ are unramified. We take a finite set $S$ of good reduction primes, which contains all ramifying primes in $K/Q$ except $p$. Let $m$ be the product of primes in $S$. We prove that the initial Fitting ideal of the $R_K = \mathbb{Z}_p[\text{Gal}(K/Q)]$-module $\text{Sel}(O_K[1/m], E[p^\infty])^\vee$ is principal, and

$$\xi_{K,S} \in \text{Fitt}_0(R_K(\text{Sel}(O_K[1/m], E[p^\infty])^\vee))$$

where $\xi_{K,S}$ is an element of $R_K$ which is explicitly constructed from modular symbols (see (2.13)). If the main conjecture (2.5) for $(E, Q_{\infty}/Q)$ holds, the equality $\text{Fitt}_0(R_K(\text{Sel}(O_K[1/m], E[p^\infty])^\vee)) = \xi_{K,S}R_K$ holds (see Remark 2.3.2). We prove the Iwasawa theoretical version in Theorem 2.2.2.

Let $\vartheta_K$ be the image of the $p$-adic $L$-function, which is also explicitly constructed from modular symbols. We show in Theorem 2.3.1

$$\vartheta_K \in \text{Ann}_{R_K}(\text{Sel}(O_K[1/m], E[p^\infty])^\vee).$$

Concerning the higher Fitting ideals (cf. §2.4), we show

$$\tilde{\delta}_m \in \text{Fitt}_{i,R_K}(\text{Sel}(O_K[1/m], E[p^\infty])^\vee)$$

where $\text{Fitt}_{i,R}(M)$ is the $i$-th Fitting ideal of an $R$-module $M$. We prove a slightly generalized version for $K$ which is in the cyclotomic $\mathbb{Z}_p$-extension $Q_{\infty}$ of $Q$ (see Theorem 2.4.1 and Corollary 2.4.2).

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2 Selmer groups and $p$-adic $L$-functions

2.1 Modular symbols and $p$-adic $L$-functions

Let $E$ be an elliptic curve over $Q$, and $f(z) = \Sigma a_n e^{2\pi inz}$ the modular form corresponding to $E$. In this section, we assume that $p$ is a prime number satisfying (i), (ii), (iii) in §1.1. We define $\mathcal{P}_{\text{good}} = \{ \ell \ | \ \ell \text{ is a good reduction prime for } E \} \setminus \{p\}$. For any finite abelian extension $K/Q$, we denote by $K_{\infty}/K$ the cyclotomic $\mathbb{Z}_p$-extension. For a real abelian field $K$ of conductor
We put $\lim$ any divisor $d$ of primes in $\mathcal{P}$ and put (cf. Mazur and Tate [16] the equation (1) on page 717). The restriction map. Then we know $\mathcal{Q}$ of primes in $\mathcal{P}$ such that the restriction of $\tau$ to $\mathcal{Q}(\mu_d)$ is $\sigma$. Let $m$ be a squarefree product of primes in $\mathcal{P}_{\text{good}}$, and $n$ a positive integer. By our assumption (ii), we know $\hat{\vartheta}^{(m,\mu_n)} \in R_{mp^n}$ (cf. [27]). Let $\alpha \in \mathbb{Z}_p^\times$ be the unit root of $x^2 - a_px + p = 0$ and put

$$\vartheta^{(m,\mu_n)} = \alpha^{-n}(\hat{\vartheta}^{(m,\mu_n)} - \alpha^{-1} \nu_{mp^n, mp^n-1}(\hat{\vartheta}^{(m,\mu_n-1)})) \in R_{mp^n}$$

as usual. Then $\{\vartheta^{(m,\mu_n)}\}_{n \geq 1}$ is a projective system (cf. Mazur and Tate [16] the equation (4) on page 717) and we obtain an element $\vartheta^{(m,\mu_{np^\infty})} \in \Lambda_{\mathcal{Q}(\mu_{mp^\infty})}$, which is the $p$-adic $L$-function of Mazur and Swinnerton-Dyer.

We also use the notation $\Lambda_{np^\infty} = \Lambda_{\mathcal{Q}(\mu_{mp^\infty})}$ for simplicity. Suppose that a prime $\ell$ does not divide $mp$, and $c_{m,\ell,m} : \Lambda_{mp^\infty} \to \Lambda_{mp^\infty}$ is the natural restriction map. Then we know

$$c_{m,\ell,m}(\vartheta^{(m,\mu_{p^\infty})}) = (a_\ell - \sigma_\ell - \sigma_\ell^{-1})\vartheta^{(m,\mu_{p^\infty})} \quad (2.1)$$

(cf. Mazur and Tate [16] the equation (1) on page 717).

We will construct a slightly modified element $\xi^{(m,\mu_{p^\infty})} \in \Lambda_{mp^\infty}$. We put $P_\ell'(x) = x^2 - a_\ell x + \ell$. Let $m$ be a squarefree product of $\mathcal{P}_{\text{good}}$. For any divisor $d$ of $m$ and a prime divisor $\ell$ of $m/d$, $\sigma_\ell \in \text{Gal}(\mathcal{Q}(\mu_{dp^n})/\mathcal{Q}) = \lim \text{Gal}(\mathcal{Q}(\mu_{dp^n})/\mathcal{Q})$ is defined as the projective limit of $\sigma_\ell \in \text{Gal}(\mathcal{Q}(\mu_{dp^n})/\mathcal{Q})$. We consider $P_\ell'(\sigma_\ell) \in \Lambda_{dp^\infty}$. Note that

$$-\sigma_\ell^{-1} = (-\sigma_\ell^{-1}P_\ell'(\sigma_\ell) - (a_\ell - \sigma_\ell - \sigma_\ell^{-1})/(\ell - 1) \in \Lambda_{dp^\infty}. \quad (2.2)$$

We put $\alpha_{d,m} = (\prod_{\ell \mid m}(\sigma_\ell^{-1}))\vartheta^{(m,\mu_{dp^n})} \in \Lambda_{dp^\infty}$ and

$$\xi^{(m,\mu_{p^\infty})} = \sum_{d \mid m} \nu_{m,d}(\alpha_{d,m}) \in \Lambda_{mp^\infty}$$

where $\nu_{m,d} : \Lambda_{dp^\infty} \to \Lambda_{mp^\infty}$ is the norm map defined similarly as above. (This modification $\xi^{(m,\mu_{p^\infty})}$ is done by the same spirit as Greither [5] in which the Deligne-Ribet $p$-adic $L$-functions are treated.) Suppose that $\ell \in$
which are classical Selmer groups. We also use the notation $\text{Sel}(\mathbb{Q})$.

In the case $m = 1$, we denote them by $\text{Sel}(O_F, E[p^\infty])$, $\text{Sel}(O_F, E[p^N])$, which are classical Selmer groups. We also use the notation $\text{Sel}(E/F, E[p^\infty])$, $\text{Sel}(E/F, E[p^N])$ for them, namely

$$\text{Sel}(E/F, E[p^\infty]) = \text{Sel}(O_F, E[p^\infty]), \quad \text{Sel}(E/F, E[p^N]) = \text{Sel}(O_F, E[p^N]).$$
For a finite abelian extension \( K/\mathbb{Q} \), we denote by \( K_\infty/K \) the cyclotomic \( \mathbb{Z}_p \)-extension, and put \( \Lambda_{K_\infty} = \mathbb{Z}_p[[\text{Gal}(K_\infty/\mathbb{Q})]] \). The Pontrjagin dual \( \text{Sel}(O_{K_\infty}, E[p^\infty])^\vee \) is a torsion \( \Lambda_{K_\infty} \)-module (Kato [7] Theorem 17.4).

When the conductor of \( K \) is \( m \), we define \( \vartheta_{K_\infty} \in \Lambda_{K_\infty} \) to be the image of \( \vartheta_{\mathbb{Q}(\mu_{mp^\infty})} \), and also \( \xi_{K_\infty} \in \Lambda_{K_\infty} \) to be the image of \( \xi_{\mathbb{Q}(\mu_{mp^\infty})} \). The Iwasawa main conjecture for \( (E, \mathbb{Q}_\infty/\mathbb{Q}) \) is the equality between the characteristic ideal of the Selmer group and the ideal generated by the \( p \)-adic \( L \)-function;

\[
\text{char}(\text{Sel}(O_{\mathbb{Q}_\infty}, E[p^\infty])^\vee) = \vartheta_{\mathbb{Q}_\infty} \Lambda_{\mathbb{Q}_\infty}.
\] (2.5)

Since we are assuming the Galois action on the Tate module is surjective, we know \( \vartheta_{\mathbb{Q}_\infty} \in \text{char}(\text{Sel}(O_{\mathbb{Q}_\infty}, E[p^\infty])^\vee) \) by Kato [7] Theorem 17.4. Skinner and Urban [26] proved the equality (2.5) under mild conditions. Namely, under our assumptions (i), (ii), they proved the main conjecture (2.5) if there is a bad prime \( \ell \) which is ramified in \( \mathbb{Q}(E[p]) \) (26) Theorem 3.33.

More generally, let \( \psi \) be an even Dirichlet character and \( K \) be the abelian field corresponding to the kernel of \( \psi \), namely \( K \) is the field such that \( \psi \) induces a faithful character of \( \text{Gal}(K/\mathbb{Q}) \). We assume \( K \cap \mathbb{Q}_\infty = \mathbb{Q} \). In this paper, for any finite abelian \( p \)-group \( G \), any \( \mathbb{Z}_p[G] \)-module \( M \) and any character \( \psi : G \to \mathbb{Q}_p^\times \), we define the \( \psi \)-quotient \( M_\psi \) by \( M \otimes_{\mathbb{Z}_p[G]} O_\psi \) where \( O_\psi = \mathbb{Z}_p[[\text{image } \psi]] \) which is regarded as a \( \mathbb{Z}_p[G] \)-module by \( \sigma x = \psi(\sigma)x \) for any \( \sigma \in G \) and \( x \in O_\psi \). We consider \( (\text{Sel}(O_{K_\infty}, E[p^\infty])^\vee)_\psi \), which is a \( \Lambda_\psi \)-module where \( \Lambda_\psi = (\Lambda_{K_\infty})_\psi = O_\psi[[\text{Gal}(K_\infty/K)]] \). We denote the image of \( \vartheta_{K_\infty} \) in \( \Lambda_\psi \) by \( \psi(\vartheta_{K_\infty}) \). Then the main conjecture states

\[
\text{char}( (\text{Sel}(O_{K_\infty}, E[p^\infty])^\vee)_\psi ) = \psi(\vartheta_{K_\infty}) \Lambda_\psi.
\] (2.6)

We also note that \( \psi(\vartheta_{K_\infty}) \Lambda_\psi = \psi(\xi_{K_\infty}) \Lambda_\psi \). By Kato [7], we know \( \psi(\vartheta_{K_\infty}) \Lambda_\psi \in \text{char}( (\text{Sel}(O_{K_\infty}, E[p^\infty])^\vee)_\psi ) \).

Let \( S \subset \mathcal{P}_{\text{good}} \) be a finite set of good primes, and \( K/\mathbb{Q} \) be a finite abelian extension. We denote by \( S_{\text{ram}}(K) \) the subset of \( S \) which consists of all ramifying primes in \( K \) inside \( S \). Recall that \( P_\ell(x) = x^2 - a_\ell x + \ell \). We define

\[
\xi_{K_\infty,S} = \xi_{K_\infty} \prod_{\ell \in S \setminus S_{\text{ram}}(K)} (-\sigma_\ell^{-1} P_\ell(\sigma_\ell)).
\]

So \( \xi_{K_\infty,S} = \xi_{K_\infty} \) if \( S \) contains only ramifying primes in \( K \). Suppose that \( S \) contains all ramifying primes in \( K \) and \( F \) is a subfield of \( K \). We denote by \( c_{K_\infty/F_\infty} : \Lambda_{K_\infty} \to \Lambda_{F_\infty} \) the natural restriction map. Using (2.3) and the above definition of \( \xi_{K_\infty,S} \), we have

\[
c_{K_\infty/F_\infty} (\xi_{K_\infty,S}) = \xi_{F_\infty,S}.
\] (2.7)
For any positive integer \( m \) whose prime divisors are in \( \mathcal{P}_{\text{good}} \), we have an exact sequence
\[
0 \longrightarrow \text{Sel}(O_{K_{\infty}}, E[p^\infty]) \longrightarrow \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty]) \longrightarrow \bigoplus_{v|m} H^1(K_{\infty,v}, E[p^\infty]) \longrightarrow 0
\]
because \( E(K_{\infty,v}) \otimes \mathbb{Q}_p / \mathbb{Z}_p = 0 \) (for the surjectivity of the third map, see Greenberg Lemma 4.6 in [3]). For a prime \( v \) of \( K_{\infty} \), let \( K_{\infty,v,nr} / K_{\infty,v} \) be the maximal unramified extension, and \( \Gamma_v = \text{Gal}(K_{\infty,v,nr}/K_{\infty,v}) \). Suppose \( v \) divides \( m \). Since \( v \) is a good reduction prime, we have \( H^1(K_{\infty,v}, E[p^\infty]) = \text{Hom}_{\text{cont}}(G_{K_{\infty,v,nr}}, E[p^\infty])^\Gamma_v = E[p^\infty](-1)^\Gamma_v \) where \((-1)\) is the Tate twist. By the Weil pairing, the Pontrjagin dual of \( E[p^\infty](-1) \) is the Tate module \( T_p(E) \). Therefore, taking the Pontrjagin dual of the above exact sequence, we have an exact sequence
\[
0 \longrightarrow \bigoplus_{v|m} T_p(E)_{\Gamma_v} \longrightarrow \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee \longrightarrow \text{Sel}(O_{K_{\infty}}, E[p^\infty])^\vee \longrightarrow 0.
\]
(2.8)
Note that \( T_p(E)_{\Gamma_v} \) is free over \( \mathbb{Z}_p \) because \( \Gamma_v \) is profinite of order prime to \( p \).

Let \( K/\mathbb{Q} \) be a finite abelian \( p \)-extension in which all bad primes of \( E \) are unramified. Suppose that \( S \) is a finite subset of \( \mathcal{P}_{\text{good}} \) such that \( S \) contains all ramifying primes in \( K/\mathbb{Q} \) except \( p \). Let \( m \) be a squarefree product of all primes in \( S \).

**Theorem 2.2.1** (Greenberg) \( \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee \) is of projective dimension \( \leq 1 \) as a \( \Lambda_{K_{\infty}} \)-module.

This is proved by Greenberg in [4] Theorem 1 (the condition (iv) in 1.1 in this paper is not needed here, see also Proposition 3.3.1 in [4]). For more general \( p \)-adic representations, this is proved in [12] Proposition 1.6.7. We will give a sketch of the proof because some results in the proof will be used later.

Since we can take some finite abelian extension \( K'/\mathbb{Q} \) such that \( K_{\infty} = K'_\infty \) and \( K' \cap \mathbb{Q}_\infty = \mathbb{Q} \), we may assume that \( K \cap \mathbb{Q}_\infty = \mathbb{Q} \) and \( p \) is unramified in \( K \). Since we are assuming that \( E[p] \) is an irreducible \( G_{\mathbb{Q}} \)-module, we know that \( \text{Sel}(O_{K_{\infty}}, E[p^\infty])^\vee \) has no nontrivial finite \( \mathbb{Z}_p[[\text{Gal}(K_{\infty}/K)]] \)-submodule by Greenberg ([3] Propositions 4.14, 4.15). We also assumed that the \( \mu \)-invariant of \( \text{Sel}(O_{\mathbb{Q}_{\infty}}, E[p^\infty])^\vee \) is zero, which implies the vanishing of the \( \mu \)-invariant of \( \text{Sel}(O_{K_{\infty}}, E[p^\infty])^\vee \) by Hachimori and Matsumo [6]. Therefore, \( \text{Sel}(O_{K_{\infty}}, E[p^\infty])^\vee \) is a free \( \mathbb{Z}_p \)-module of finite rank. By the exact sequence (2.8), \( \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee \) is also a free \( \mathbb{Z}_p \)-module of finite rank.
Put $G = \text{Gal}(K/\mathbb{Q})$. By the definition of the Selmer group and our assumption that all primes dividing $m$ are good reduction primes, we have $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^G = \text{Sel}(O_{Q_{\infty}}[1/m], E[p^\infty])$. Since we assumed that the $\mu$-invariant is zero, $\text{Sel}(O_{Q_{\infty}}[1/m], E[p^\infty])$ is divisible. This shows that the corestriction map $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty]) \to \text{Sel}(O_{Q_{\infty}}[1/m], E[p^\infty])$ is surjective. Therefore, $H^0(G, \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])) = 0$.

Next we will show that $H^1(G, \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])) = 0$. Let $N_E$ be the conductor of $E$ and put $m' = \text{mp}_E$. We know $\text{Sel}(O_{K_{\infty}}[1/m'], E[p^\infty])$ is equal to the étale cohomology group $H^1_{\text{et}}(\text{Spec } O_{K_{\infty}}[1/m'], E[p^\infty])$. We have an exact sequence

$$0 \to \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty]) \to \text{Sel}(O_{K_{\infty}}[1/m'], E[p^\infty]) \to \bigoplus_{v|m'} H^2_v(K_{\infty,v}) \to 0 \quad (2.9)$$

where $H^2_v(K_{\infty,v}) = H^1(K_{\infty,v}, E[p^\infty])/(E(K_{\infty,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, and the surjectivity of the third map follows from Greenberg Lemma 4.6 in [3]. Let $E[p^\infty]^0$ be the kernel of $E[p^\infty] = E(\mathbb{T})[p^\infty] \to E(\mathbb{F}_p)[p^\infty]$ and $E[p^\infty]_{et} = E[p^\infty]/E[p^\infty]^0$. For a prime $v$ of $K_{\infty}$ above $p$, we denote by $K_{\infty,v,nr}$ the maximal unramified extension of $K_{\infty,v}$, and put $\Gamma_v = \text{Gal}(K_{\infty,v,nr}/K_{\infty,v})$. We know the isomorphism $H^2_v(K_{\infty,v}) \cong H^1(K_{\infty,v,nr}, E[p^\infty]_{et})^{\Gamma_v}$ by Greenberg [2] §2. If $v$ is a prime of $K_{\infty}$ not above $p$, we know $H^2_v(K_{\infty,v}) = H^1(K_{\infty,v}, E[p^\infty])$. Therefore, we get an isomorphism

$$\left( \bigoplus_{v|m'} H^2_v(K_{\infty,v}) \right)^G = \bigoplus_{u|m'} H^2_u(\mathbb{Q}_u)$$

where $v$ (resp. $u$) runs over all primes of $K_{\infty}$ (resp. $\mathbb{Q}_\infty$) above $m'/m = pN_E$. Thus, $\text{Sel}(O_{K_{\infty}}[1/m'], E[p^\infty])^G \to \bigoplus_{v|m'} H^2_v(K_{\infty,v})^G$ is surjective. On the other hand, we have $H^2_{\text{et}}(\text{Spec } O_{K_{\infty}}[1/m'], E[p^\infty]) = 0$ (see [2] Propositions 3, 4). This implies that

$$H^1(G, H^1_{\text{et}}(\text{Spec } O_{K_{\infty}}[1/m'], E[p^\infty])) = H^1(G, \text{Sel}(O_{K_{\infty}}[1/m'], E[p^\infty])) = 0.$$  

Taking the cohomology of the exact sequence $2.9$, we get

$$H^1(G, \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])) = 0 \quad (2.10)$$

Therefore, $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])$ is cohomologically trivial as a $G$-module by Serre [25] Chap. IX Théorème 8. This implies that $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee$ is also cohomologically trivial. Since $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee$ has no nontrivial finite submodule, the projective dimension of $\text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^\vee$ as a $\Lambda_{K_{\infty}}$-module is $\leq 1$ by Popescu [20] Proposition 2.3.
Theorem 2.2.2 Let $K/Q$ be a finite abelian $p$-extension in which all bad primes of $E$ are unramified. We take a finite set $S$ of good reduction primes which contains all ramifying primes in $K/Q$ except $p$. Let $m$ be the product of primes in $S$. Then

(1) $\xi_{K_{\infty},S}$ is in the initial Fitting ideal $\text{Fitt}_{0,\Lambda_{K_{\infty}}}(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee})$.

(2) We have

$$\text{Fitt}_{0,\Lambda_{K_{\infty}}}(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee}) = \xi_{K_{\infty},S}\Lambda_{K_{\infty}}$$

if and only if the main conjecture (2.3) for $(E, Q_{\infty}/Q)$ holds.

Proof. As we explained in the proof of Theorem 2.2.1, we may assume that $K \cap Q_{\infty} = Q$. We recall that $\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee}$ is a free $Z_p$-module of finite rank under our assumptions.

(1) Let $\psi : \text{Gal}(K/Q) \to \overline{Q}^\times_p$ be a character of $\text{Gal}(K/Q)$, not necessarily faithful. We study the Fitting ideal of the $\psi$-quotient $(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi} = \text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee} \otimes Z_p[\text{Gal}(K/Q)] O_{\psi}$. We denote by $F$ the subfield of $K$ corresponding to the kernel of $\psi$. We regard $\psi$ as a faithful character of $\text{Gal}(F/Q)$. Since $\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\text{Gal}(K/F)} = \text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])$, we have

$$(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi} = (\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee} \otimes Z_p[\text{Gal}(F/Q)]) O_{\psi}.$$ 

where the right hand side is defined to be $\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee} \otimes Z_p[\text{Gal}(F/Q)]$.

We put $\Lambda_{\psi} = (\Lambda_{F_{\infty}})_{\psi}$. The group homomorphism $\psi$ induces the ring homomorphism $\Lambda_{K_{\infty}} \to \Lambda_{\psi}$ which we also denote by $\psi$. The composition with $\psi_{K_{\infty}/F_{\infty}} : \Lambda_{K_{\infty}} \to \Lambda_{F_{\infty}}$ and the above ring homomorphism $\psi$ is also denoted by $\psi : \Lambda_{K_{\infty}} \to \Lambda_{\psi}$. Note that $F/Q$ is a cyclic extension of degree a power of $p$. We denote by $F'$ the subfield of $F$ such that $[F : F'] = p$. We put $N_0 = N_{\text{Gal}(F/F')} = \Sigma_{\sigma \in \text{Gal}(F/F')} \sigma$. If we put $[F : Q] = p^e$ and take a generator $\gamma$ of $\text{Gal}(F/Q)$, $N_0 = \Sigma_{i=0}^{p-1} \gamma^{p^{-i}}$ is a cyclotomic polynomial and $O_{\psi} = Z_p[N_{\psi}] \simeq Z_p[\text{Gal}(F/Q)]/N_0$. For any $Z_p[\text{Gal}(F/Q)]$-module $M$, we define $M^\psi = \ker(N_0 : M \to M)$. Then the Pontrjagin dual of $M^\psi$ is $(M^\psi)^\vee = (M^\vee)/N_0 = (M^\vee)^{\psi}$. By the same method as the proof of (2.10), we have $H^1(\text{Gal}(F/F'), \text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])) = 0$. Therefore, $\sigma - 1 : \text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}]) \to \text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\psi}$ is surjective where $\sigma = \gamma^{p^{-1}}$ is a generator of $\text{Gal}(F/F')$. Therefore, taking the dual, we know that there is an injective homomorphism from $(\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi}$ to $\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee}$ which is a free $Z_p$-module. Therefore, $(\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi}$ contains no nontrivial finite $\Lambda_{\psi}$-submodule. This shows that

$$\text{Fitt}_{0,\Lambda_{\psi}}((\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi}) = \text{char}((\text{Sel}(O_{F_{\infty}}[1/m], E[p^{\infty}])^{\vee})_{\psi}).$$
Consider the $\psi$-quotient of the exact sequence (2.8):

$$(\bigoplus_{v|m} T_p(E)_{\Gamma_v})_{\psi} \rightarrow (\text{Sel}(O_{F_{\infty}[1/m]}, E[p^{\infty}])^\vee)_{\psi} \rightarrow (\text{Sel}(O_{F_{\infty}}, E[p^{\infty}])^\vee)_{\psi} \rightarrow 0$$

where $v$ runs over all primes of $F_{\infty}$ above $m$. Since $\text{Ext}^1_{\mathbb{Z}_p}(\text{Gal}(F/Q), (\text{Sel}(O_{F_{\infty}}, E[p^{\infty}])) = \hat{H}^0(\text{Gal}(F/Q), (\text{Sel}(O_{F_{\infty}}, E[p^{\infty}]))$ is finite, the first map of the above exact sequence has finite kernel.

Suppose that $\ell$ is a prime divisor of $m$. If $\ell$ is unramified in $F$, we have

$$\text{Fitt}_{0,\Lambda_{\psi}}((\bigoplus_{v|\ell} T_p(E)_{\Gamma_v})_{\psi}) = P'_\ell(\sigma_{\ell})\Lambda_{\psi}$$

where $P'_\ell(x) = x^2 - a_{\ell}x + \ell$. If $\ell$ is ramified in $F$, $\psi(\ell) = 0$ and $((\bigoplus_{v|\ell} T_p(E)_{\Gamma_v})_{\psi}$ is finite. Therefore, we have

$$\text{char}((\bigoplus_{v|m} T_p(E)_{\Gamma_v})_{\psi}) = (\prod_{\ell \in S \setminus \text{ram}(F)} P'_\ell(\sigma_{\ell}))\Lambda_{\psi}.$$ 

Using the above exact sequence and Kato’s theorem $\psi(\xi_{F_{\infty}}) \in \text{char}((\text{Sel}(O_{F_{\infty}}, E[p^{\infty}])^\vee)_{\psi})$, we have

$$\text{char}((\text{Sel}(O_{F_{\infty}[1/m]}, E[p^{\infty}])^\vee)_{\psi}) \supset \psi(\xi_{F_{\infty}})(\prod_{\ell \in S \setminus \text{ram}(F)} P'_\ell(\sigma_{\ell}))\Lambda_{\psi}.$$ 

Since $\xi_{F_{\infty}}(\prod_{\ell \in S \setminus \text{ram}(F)} P'_\ell(\sigma_{\ell})) = \xi_{F_{\infty}}, S$ modulo unit and $c_{K_{\infty}/F_{\infty}}(\xi_{K_{\infty}, S}) = \xi_{F_{\infty}}, S$ by (2.7), we obtain

$$\psi(\xi_{K_{\infty}, S}) \in \text{Fitt}_{0,\Lambda_{\infty}}((\text{Sel}(O_{K_{\infty}[1/m]}, E[p^{\infty}])^\vee)_{\psi}) \tag{2.11}$$

for any character $\psi$ of $\text{Gal}(K/Q)$. Since the $\mu$-invariant of $\text{Sel}(O_{K_{\infty}[1/m]}, E[p^{\infty}])^\vee$ is zero as we explained above, (2.11) implies

$$\xi_{K_{\infty}, S} \in \text{Fitt}_{0,\Lambda_{\infty}}((\text{Sel}(O_{K_{\infty}[1/m]}, E[p^{\infty}])^\vee)$$

(see Lemma 4.1 in [9], for example).

(2) We use the same notation $\psi$, $F$, etc. as above. At first, we assume (2.5). Then the algebraic $\lambda$-invariant of $\text{Sel}(E/F_{\infty}, E[p^{\infty}])^\vee$ equals the analytic $\lambda$-invariant by Hachimori and Matsumo [6, 13], so the main conjecture $\text{char}((\text{Sel}(O_{F_{\infty}}, E[p^{\infty}])^\vee)_{\psi}) = \psi(\xi_{F_{\infty}})\Lambda_{\psi}$ also holds. Therefore, we have

$$\text{char}((\text{Sel}(O_{F_{\infty}[1/m]}, E[p^{\infty}])^\vee)_{\psi}) = \psi(\xi_{F_{\infty}})(\prod_{\ell \in S \setminus \text{ram}(F)} P'_\ell(\sigma_{\ell}))\Lambda_{\psi}$$

$$= \psi(\xi_{F_{\infty}, S})\Lambda_{\psi} = \psi(\xi_{K_{\infty}, S})\Lambda_{\psi}.$$
and

$$\text{Fitt}_{0}(\Lambda_{K_{\infty}}) \psi ((\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}]))^\vee) = \psi(\xi_{K_{\infty}}, s\Lambda_{K_{\infty}})\Lambda_{\psi}.$$ 

It follows from [9] Corollary 4.2 that

$$\text{Fitt}_{0}\Lambda_{K_{\infty}}(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}]))^\vee = \xi_{K_{\infty}}, s\Lambda_{K_{\infty}}.$$ 

On the other hand, if we assume the above equality, taking the $\text{Gal}(K/Q)$-invariant part of $\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}])$, we get

$$\text{Fitt}_{0}\Lambda_{Q_{\infty}}(\text{Sel}(O_{Q_{\infty}}[1/m], E[p^{\infty}]))^\vee = \xi_{Q_{\infty}}, s\Lambda_{Q_{\infty}}, s,$$

which implies (2.5).

### 2.3 An analogue of Stickelberger’s theorem

Let $K/Q$ be a finite abelian $p$-extension. When the conductor of $K$ is $m$, we define $\vartheta_{K} \in R_{K} = \mathbb{Z}[t][\text{Gal}(K/Q)]$ to be the image of $\vartheta_q(\mu_{mp^{\infty}}) \in \Lambda_{Q(\mu_{mp^{\infty}})}$. Therefore, if $m$ is prime to $p$, $\vartheta_{K}$ is the image of $\vartheta_q(\mu_{m}) = (1 - 2\sigma^{-1})(1 - \frac{1}{m})\vartheta_{Q(\mu_{m})}$ by (2.4). If $m = m'p^{n}$ for some $m'$ which is prime to $p$ and for some $n \geq 2$, $\vartheta_{K}$ is the image of $\vartheta_{Q(\mu_{m'p^{n}})} = \alpha^{-n}(\vartheta_{Q(\mu_{m'p^{n-1}}) - \alpha^{-1}\nu_{m'p^{n}, m'p^{n-1}}(\vartheta_{Q(\mu_{m'p^{n-1}})}}))$.

For any positive integer $n$, we denote by $Q(n)$ the maximal $p$-subextension of $Q$ in $Q(\mu_{n})$.

**Theorem 2.3.1** For any finite abelian $p$-extension $K$ in which all bad primes of $E$ are unramified, $\vartheta_{K}$ annihilates $\text{Sel}(O_{K}, E[p^{\infty}])^\vee$, namely we have

$$\vartheta_{K}\text{Sel}(O_{K}, E[p^{\infty}])^\vee = 0.$$ 

Proof. We may assume $K = Q(mp^{n})$ for some squarefree product of primes in $\mathcal{P}_{\text{good}}$ and for some $n \in \mathbb{Z}_{\geq 0}$. By Theorem 2.2.2 (1), taking $S$ to be the set of all prime divisors of $m$, we have $\xi_{K_{\infty}} \in \text{Fitt}_{0}\Lambda_{K_{\infty}}(\text{Sel}(O_{K_{\infty}}[1/m], E[p^{\infty}]))^\vee$, which implies $\xi_{K_{\infty}}\text{Sel}(O_{K_{\infty}}, E[p^{\infty}])^\vee = 0$. Let $\xi_{K} \in K_{\mathbb{Q}} = \mathbb{Z}[\text{Gal}(K/Q)]$ be the image of $\xi_{K_{\infty}}$. Since the natural map $\text{Sel}(O_{K}, E[p^{\infty}]) \rightarrow \text{Sel}(O_{K_{\infty}}, E[p^{\infty}])$ is injective, we have $\xi_{K}\text{Sel}(O_{K}, E[p^{\infty}])^\vee = 0$.

By the definitions of $\xi_{Q(\mu_{mp^{n}})}$, $\xi_{Q(mp^{n})}$, $\vartheta_{Q(\mu_{mp^{n}})}$, we can write

$$\xi_{K} = \xi_{Q(mp^{n})} = \vartheta_{Q(mp^{n})} + \sum_{d|m, d \neq m} \lambda_{d}\nu_{m,d}(\vartheta_{Q(dp^{n})})$$

for some $\lambda_{d} \in R_{Q(mp^{n})}$ where $\nu_{m,d} : R_{Q(dp^{n})} \rightarrow R_{Q(mp^{n})}$ is the norm map defined similarly as in (2.4). We will prove this theorem by induction on
m. Since \( d < m \), we have \( \vartheta_{Q(dp^n)} \in \text{Ann}_{R_{Q(dp^n)}}(\text{Sel}(Q_{dp^n}, E[p^\infty])^\vee) \) by the hypothesis of the induction. This implies that \( \nu_{m,d}(\vartheta_{Q(dp^n)}) \) annihilates \( \text{Sel}(Q_{dp^n}, E[p^\infty])^\vee \). Since \( \xi_K \) is in \( \text{Ann}_{R_K}(\text{Sel}(K, E[p^\infty])^\vee) \), the above equation implies that \( \vartheta_K \) is in \( \text{Ann}_{R_K}(\text{Sel}(K, E[p^\infty])^\vee) \).

**Remark 2.3.2** Let \( K, S, m \) be as in Theorem 2.2.2 Under our assumptions, the control theorem works completely:

\[
\text{Sel}(O_K[1/m], E[p^\infty]) \xrightarrow{\sim} \text{Sel}(O_{K_{\infty}}[1/m], E[p^\infty])^{\text{Gal}(K_{\infty}/K)}.
\]

Therefore, Theorem 2.2.2 (1) implies that \( \text{Fitt}_{0,R_K}(\text{Sel}(O_K[1/m], E[p^\infty])^\vee) \) is principal and

\[
\xi_{K,S} \in \text{Fitt}_{0,R_K}(\text{Sel}(O_K[1/m], E[p^\infty])^\vee) \tag{2.13}
\]

where \( \xi_{K,S} \) is the image of \( \xi_{K_{\infty},S} \) in \( R_K \).

Theorem 2.2.2 (2) implies that if we assume the main conjecture 2.5, we have

\[
\text{Fitt}_{0,R_K}(\text{Sel}(O_K[1/m], E[p^\infty])^\vee) = \xi_{K,S}R_K. \tag{2.14}
\]

### 2.4 Higher Fitting ideals

For a commutative ring \( R \) and a finitely presented \( R \)-module \( M \) with \( n \) generators, let \( A \) be an \( n \times m \) relation matrix of \( M \). For an integer \( i \geq 0 \), \( \text{Fitt}_{i,R}(M) \) is defined to be the ideal of \( R \) generated by all \((n-i) \times (n-i)\) minors of \( A \) (cf. [19]; this ideal \( \text{Fitt}_{i,R}(M) \) does not depend on the choice of a relation matrix \( A \)).

Suppose that \( K/Q \) is a finite extension such that \( K \) is in the cyclotomic \( Z_p \)-extension \( Q_\infty \) of \( Q \), and that \( m \) is a squarefree product of primes in \( \mathcal{P}^{(N)} \). We define \( K(m) \) by \( K(m) = Q(m)K \).

We put \( G_\ell = \text{Gal}(Q(\ell)/Q) \) and \( G_m = \text{Gal}(Q(m)/Q) = \prod_{\ell \mid m} G_\ell \). We have \( \text{Gal}(K(m)/K) = G_m \). We put \( n_\ell = \text{ord}_p(\ell-1) \). Suppose that \( m = \ell_1 \cdots \ell_r \). We take a generator \( \tau_{\ell_i} \) of \( G_{\ell_i} \) and put \( S_i = \tau_{\ell_i} - 1 \in R_{K(m)} \). We write \( n_i \) for \( n_{\ell_i} \). We identify \( R_{K(m)} \) with

\[
R_K[G_m] = R_K[S_1, \ldots, S_r]/((1 + S_1)^{p^{n_1}} - 1, \ldots, (1 + S_r)^{p^{n_r}} - 1).
\]

We consider \( \vartheta_{K(m)} \in R_{K(m)} \) and write

\[
\vartheta_{K(m)} = \sum_{i_1, \ldots, i_r \geq 0} a_{i_1, \ldots, i_r}^{(m)} S_1^{i_1} \cdots S_r^{i_r}
\]

where \( a_{i_1, \ldots, i_r}^{(m)} \in R_K \). Put \( n_0 = \min\{n_1, \ldots, n_r\} \). For \( s \in \mathbb{Z}_{\geq 0} \), we define \( c_s \) to be the maximal positive integer \( c \) such that

\[
T^{-1}((1 + T)^{p^{n_0}} - 1) \in p^{c_s} \mathbb{Z}_p[T] + T^{s+1} \mathbb{Z}_p[T].
\]
Lemma 3.1.1 in [12] we have λ for some α. Write ξ.

Proof of Theorem 2.4.1. We may assume K/Q, where K

Let (see [12] §6.3). Taking s = 1 and i = r in Theorem 2.4.1 we get

Corollary 2.4.2 Let K/Q be a finite extension such that K ⊂ Q∞. We have

δm ∈ Fittr,K/pN(Sel(E/K, E[pN]))∨

where m = ℓ1 · · · ℓr.

Proof of Theorem 2.4.1 We may assume K = Q(pn) for some n ≥ 0, so K(m) = Q(mp2n). First of all, we consider the image ξK(m) ∈ RK(m) of ξK(m)∞. Since Sel(E/K(m), E[p∞]) → Sel(E/K(m)∞, E[p∞]) is injective, ξK(m) is in Fitt0,RK(m)(Sel(E/K(m), E[p∞]))∨ by Theorem 2.2.2 (1). We write

\[ ξK(m) = \sum_{i_1, \ldots, i_r ≥ 0} α_{i_1, \ldots, i_r}^{(m)} S_{i_1}^{i_1} \cdot \ldots \cdot S_{i_r}^{i_r} \]

where α_{i_1, \ldots, i_r}^{(m)} ∈ RK. Assume that i1, ..., ir ≤ s and i1 + ... + ir ≤ i. Then by Lemma 3.1.1 in [12] we have

\[ α_{i_1, \ldots, i_r}^{(m)} ∈ Fitt_{i_1, \ldots, i_r} R_K/p^{cs} (Sel(E/K, E[p^{cs}]))^∨. \]

On the other hand, since K(m) = Q(mp2n) for some n ≥ 0, we have

\[ ξK(m) = \bar{θ}K(m) + \sum_{d|m, d≠m} λ_d v_m,d(\bar{θ}Q(dp^n)) \]

for some λd ∈ RK(m) by (2.12). This implies that the images of ξK(m) and \( \bar{θ}K(m) \) under the canonical homomorphism

\[ R_K(m) = R_K[S_1, \ldots, S_r]/I \rightarrow R_K[[S_1, \ldots, S_r]]/J \]
written as
\[ Z \text{ is also isomorphic to } P_0 - \rightarrow Z \]
\[ \ell \text{ is in } P \]
We define \( E \) acts on the group for a positive integer prime for \( E \)
Recall that in §3.1 Some definitions
§3
systems of Gauss sum type in the case of elliptic curves. From this section
We recall the results in [12] on Euler system s and Kolyvagin
This completes the proof of Theorem 2.4.1.

3 Review of Kolyvagin systems of Gauss sum type for elliptic curves

In this section, we recall the results in [12] on Euler systems and Kolyvagin systems of Gauss sum type in the case of elliptic curves. From this section we assume all the assumptions (i), (ii), (iii), (iv) in [11]

3.1 Some definitions
Recall that in §2 we defined \( \mathcal{P}_{good} \) by \( \mathcal{P}_{good} = \{ \ell \mid \ell \text{ is a good reduction prime for } E \} \setminus \{ p \} \), and \( \mathcal{P}^{(N)} \) by
\[ \mathcal{P}^{(N)} = \{ \ell \in \mathcal{P}_{good} \mid \ell \equiv 1 \text{ (mod } p^N) \} \]
for a positive integer \( N > 0 \). If \( \ell \) is in \( \mathcal{P}_{good} \), the absolute Galois group \( G_{\mathbb{F}_\ell} \) acts on the group \( E[p^N] \) of \( p^N \)-torsion points, so we consider \( H^1(\mathbb{F}_\ell, E[p^N]) \).
We define
\[ \mathcal{P}^{(N)}_{0} = \{ \ell \in \mathcal{P}^{(N)} \mid H^0(\mathbb{F}_\ell, E[p^N]) \text{ contains an element of order } p^N \} \],
\[ (\mathcal{P}^{(N)}_{0})' = \{ \ell \in \mathcal{P}^{(N)} \mid H^0(\mathbb{F}_\ell, E[p^N]) = E[p^N] \}, \]
\[ \mathcal{P}^{(N)}_{1} = \{ \ell \in \mathcal{P}^{(N)} \mid H^0(\mathbb{F}_\ell, E[p^N]) \simeq \mathbb{Z}/p^N \}. \]
So \( \mathcal{P}^{(N)}_{0} \supset (\mathcal{P}^{(N)}_{0})' \supset \mathcal{P}^{(N)}_{1} \), and \( (\mathcal{P}^{(N)}_{0})' \cap \mathcal{P}^{(N)}_{1} = \emptyset \). Suppose that \( \ell \) is in \( \mathcal{P}^{(N)}_{1} \). Then, since \( \ell \equiv 1 \text{ (mod } p^N) \), we have an exact sequence
\[ 0 \rightarrow \mathbb{Z}/p^N \rightarrow E[p^N] \rightarrow \mathbb{Z}/p^N \rightarrow 0 \]
of \( G_{\mathbb{F}_\ell} \)-modules where \( G_{\mathbb{F}_\ell} \) acts on \( \mathbb{Z}/p^N \) trivially. So the action of the Frobenius \( \text{Frob}_\ell \) at \( \ell \) on \( E[p^N] \) can be written as \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) for a suitable basis of \( E[p^N] \). Therefore, \( H^1(\mathbb{F}_\ell, E[p^N]) \) is also isomorphic to \( \mathbb{Z}/p^N \) for \( \ell \in \mathcal{P}^{(N)}_{1} \).
Let \( t \in E[p^N] \) be an element of order \( p^N \). We define
\[ \mathcal{P}_{0,t}^{(N)} = \{ \ell \in \mathcal{P}^{(N)} \mid t \in H^0(\mathbb{F}_\ell, E[p^N]) \}, \]
\[ \mathcal{P}_{1,t}^{(N)} = \{ \ell \in \mathcal{P}^{(N)} \mid H^0(\mathbb{F}_\ell, E[p^N]) = (\mathbb{Z}/p^N)_{t} \}. \]
So, \( \mathcal{P}_{0}^{(N)} = \bigcup_{t} \mathcal{P}_{0,t}^{(N)} \) and \( \mathcal{P}_{1}^{(N)} = \bigcup_{t} \mathcal{P}_{1,t}^{(N)} \) where \( t \) runs over all elements of order \( p^N \). Since we assumed that the Galois action on the Tate module is
surjective, both \((P_0')^{(N)}\) and \(P_1^{(N)}\) are infinite by Chebotarev density theorem ([12] §4.3).

We define \(K_p\) to be the set of number fields \(K\) such that \(K/Q\) is a finite abelian \(p\)-extension in which all bad primes of \(E\) are unramified. Suppose that \(K\) is in \(K_p\). We define

\[
(P_0')^{(N)}(K) = \{ \ell \in (P_0')^{(N)} \mid \ell \text{ splits completely in } K \},
\]

\[
P_1^{(N)}(K) = \{ \ell \in P_1^{(N)} \mid \ell \text{ splits completely in } K \}.
\]

Again by Chebotarev density theorem, both \((P_0')^{(N)}(K)\) and \(P_1^{(N)}(K)\) are infinite (see [12] §4.3).

Suppose \(\ell \in P_{good}\). For a prime \(v\) above \(\ell\), we know \(H^1(K_v, E[p^N])/(E(K_v) \otimes \mathbb{Z}/p^N) = H^0(\kappa(v), E[p^N](-1))\) where \(\kappa(v)\) is the residue field of \(v\). We put

\[
H^2_\ell(K) = \bigoplus_{v|\ell} H^0(\kappa(v), E[p^N](-1)). \tag{3.1}
\]

If \(\ell\) is in \((P_0')^{(N)}(K)\) (resp. \(P_1^{(N)}(K)\)), \(H^2_\ell(K)\) is a free \(R_K/p^N\)-module of rank 2 (resp. rank 1) where \(R_K = \mathbb{Z}_p[\text{Gal}(K/Q)]\) as before.

From now on, for a prime \(\ell \in P_0^{(N)}\), we fix a prime \(\ell_{\overline{Q}}\) of an algebraic closure \(\overline{Q}\) above \(\ell\). For any algebraic number field \(F\), we denote the prime of \(F\) below \(\ell_{\overline{Q}}\) by \(\ell_F\), so when we consider finite extensions \(F_1/k\), \(F_2/k\) such that \(F_1 \subset F_2\), the primes \(\ell_{F_2}, \ell_{F_1}\) satisfy \(\ell_{F_2} | \ell_{F_1}\).

We take a primitive \(p^n\)-th root of unity \(\zeta_{p^n}\) such that \((\zeta_{p^n})_{n \geq 1} \in \mathbb{Z}_p(1) = \lim_{\rightarrow} \mu_{p^n}\), and fix it.

In the following, for each \(\ell\) in \((P_0')^{(N)}(K)\), we take \(t_\ell \in H^0(F_\ell, E[p^N])\) and fix it. We define

\[
t_{\ell,K} = (t_\ell \otimes \zeta_{p^N}^\otimes(-1), 0, ..., 0) \in H^2_\ell(K) \tag{3.2}
\]

where the right hand side is the element whose \(\ell_K\)-component is \(t_\ell \otimes \zeta_{p^N}^\otimes(-1)\) and other components are zero.

Suppose that \(K\) is in \(K_p\). Let \(K_{\infty}/K\) be the cyclotomic \(\mathbb{Z}_p\)-extension, and \(K_n\) be the \(n\)-th layer. Since \(\text{Sel}(O_{K_{\infty}}, E[p^\infty])^\vee\) is a finitely generated \(\mathbb{Z}_p\)-module, the corestriction map \(\text{Sel}(O_{K_{n+1}}, E[p^N]) \to \text{Sel}(O_K, E[p^N])\) is the zero map if \(m\) is sufficiently large. We take the minimal \(m > 0\) satisfying this property, and put \(K_{[1]} = K_m\). We define inductively \(K_{[n]}\) by \(K_{[n]} = (K_{[n-1]})_{[1]}\) where we applied the above definition to \(K_{[n-1]}\) instead of \(K\).
We can compute how large $K_{[n]}$ is. Let $\lambda$ be the $\lambda$-invariant of $\Sel(O_{K_{\infty}},E[p^\infty])^\vee$. We take $a \in \mathbb{Z}_{\geq 0}$ such that $p^{a+1} - p^a \geq \lambda$. Suppose that $K = K'_m$ ($m$-th layer of $K'_\infty/K'$) for some $K'$ such that $p$ is unramified in $K'$. The corestriction map $\Sel(O_{K'_m+1},E[p]) \to \Sel(O_{K'_m},E[p])$ is the zero map. Therefore, $\Sel(O_{K_{a'+N}},E[p^N]) \to \Sel(O_{K'_m},E[p^N])$ is the zero map. Put $a' = \max(a - m,0)$. Then $\Sel(O_{K_{a'+N}},E[p^N]) \to \Sel(O_{K_{a'}},E[p^N])$ is the zero map. Therefore, we have $K_{[1]} \subset K_{a'+N}$. Also we know $K_{[n]} \subset K_{a'+nN}$.

Let $n_\lambda$, $d_n$ be the numbers defined just before (1.4) in [14]. Then we can show that if $\ell \in \mathcal{P}_1^{(N)}$ satisfies $\ell \equiv 1 \pmod{p^{d_n}}$, $\ell$ is in $\mathcal{P}_1^{(N)}(\mathbb{Q}_{[n]})$ by the same method as above.

### 3.2 Euler systems of Gauss sum type for elliptic curves

We use the following lemma which is the global duality theorem (see Theorem 2.3.4 in Mazur and Rubin [14]).

**Lemma 3.2.1** Suppose that $m$ is a product of primes in $\mathcal{P}_{\text{good}}$. We have an exact sequence

$$0 \to \Sel(O_K,E[p^N]) \to \Sel(O_K[1/m],E[p^N]) \to \bigoplus_{\ell|m} \mathcal{H}_{\ell}^2(K) \to \Sel(O_K,E[p^N])^\vee.$$ 

We remark that we can take $m$ such that the last map is surjective in our case (see Lemma 3.4.1 below).

Let $K$ be a number field in $\mathcal{K}(p)$ and $\ell \in \mathcal{P}_0^{(N)}(K_{[1]})$. We apply the above lemma to $K_{[1]}$ and obtain an exact sequence

$$\Sel(O_{K_{[1]}}[1/\ell],E[p^N]) \xrightarrow{\partial_{K_{[1]}}} \mathcal{H}_{\ell}^2(K_{[1]}) \xrightarrow{w_{\ell}} \Sel(O_{K_{[1]}},E[p^N])^\vee.$$ 

Consider $\partial_{K_{[1]}},t_\ell,K_{[1]} \in \mathcal{H}_{\ell}^2(K_{[1]})$. By Theorem 2.3.1 we know $w_{\ell}(\partial_{K_{[1]}},t_\ell,K_{[1]}) = \partial_{K_{[1]}},t_\ell,K_{[1]} = 0$. Therefore, there is an element $g \in \Sel(O_{K_{[1]}},E[p^N])$ such that $\partial_{\ell}(g) = \partial_{K_{[1]}},t_\ell,K_{[1]}$. We define

$$g_{\ell,t_\ell}^{(K)} = \Cor_{K_{[1]}},t_\ell,K_{[1]}(g) \in \Sel(O_{K_{[1]}},E[p^N]).$$ (3.3)

This element $g_{\ell,t_\ell}^{(K)}$ does not depend on the choice of $g \in \Sel(O_{K_{[1]}},E[p^N])$ ([12] §5.4). We write $g_{\ell}$ instead of $g_{\ell,t_\ell}^{(K)}$ when no confusion arises.

**Remark 3.2.2** To define $g_{\ell}$, we used in [12] the $p$-adic $L$-function $\theta_{K_{\infty}}$ whose Euler factor at $\ell$ is $1 - \frac{4}{\ell} \sigma_{\ell}^{-1} + \frac{1}{\ell^2} \sigma_{\ell}^{-2}$. The element $\theta_{K_{\infty}}$ can be constructed from $\theta_{K_{\infty}}$ by the same method as when we constructed $\xi_{K_{\infty}}$ in [24]. In the above definition (3.3), we used $\theta_{K}$ (namely $\theta_{K_{\infty}}$) instead of $\theta_{K_{\infty}}$. 

20
3.3 Kolyvagin derivatives of Gauss sum type

Let $\ell$ be a prime in $\mathcal{P}_{\text{good}}$. We define $\partial_\ell$ as a natural homomorphism

$$\partial_\ell : H^1(K, E[p^N]) \to \mathcal{H}_K^2(K) = \bigoplus_{v|\ell} H^0(\kappa(v), E[p^N](-1))$$

where we used $H^1(K_v, E[p^N])/(E(K_v) \otimes \mathbb{Z}/p^N) = H^0(\kappa(v), E[p^N](-1))$.

Next, we assume $\ell \in \mathcal{P}_1^{(N)}(K)$. We denote by $Q_{\ell}(\ell)$ the maximal $p$-subextension of $Q_{\ell}$ inside $Q_{\ell}(\mu_\ell)$. Put $G_\ell = \text{Gal}(Q_{\ell}(\ell)/Q_{\ell})$. By Kummer theory, $G_\ell$ is isomorphic to $\mu_p^{n_\ell}$ where $n_\ell = \text{ord}_p(\ell - 1)$. We denote by $\tau_\ell$ the corresponding element of $G_\ell$ to $\zeta_p^{n_\ell}$ that is the primitive $p^{n_\ell}$-th root of unity we fixed.

We consider the natural homomorphism $H^1(Q_{\ell}, E[p^N]) \to H^1(Q_{\ell}(\ell), E[p^N])$ and denote the kernel by $H^1_{\text{tr}}(Q_{\ell}, E[p^N])$. Let $Q_{\ell, \text{nr}}$ be the maximal unramified extension of $Q_{\ell}$. We identify $H^1(F_{\ell}, E[p^N])$ with $H^1(\text{Gal}(Q_{\ell, \text{nr}}/Q_{\ell}), E[p^N])$, and regard it as a subgroup of $H^1(Q_{\ell}, E[p^N])$. Then both $H^1(F_{\ell}, E[p^N])$ and $H^1_{\text{tr}}(Q_{\ell}, E[p^N])$ are isomorphic to $\mathbb{Z}/p^N$, and we have decomposition

$$H^1(Q_{\ell}, E[p^N]) = H^1(F_{\ell}, E[p^N]) \oplus H^1_{\text{tr}}(Q_{\ell}, E[p^N])$$

as an abelian group. We also note that $H^1(F_{\ell}, E[p^N])$ coincides with the image of the Kummer map and is isomorphic to $E(Q_{\ell}) \otimes \mathbb{Z}/p^N$. We consider the homomorphism

$$\phi' : H^1(Q_{\ell}, E[p^N]) \to H^1(F_{\ell}, E[p^N]) \quad (3.4)$$

which is obtained from the above decomposition.

Note that $H^1(F_{\ell}, E[p^N]) = E[p^N]/\langle \text{Frob}_{\ell} - 1 \rangle$ where Frob$_{\ell}$ is the Frobenius at $\ell$. Since $\ell$ is in $\mathcal{P}_1^{(N)}$, Frob$_{\ell}^{-1} - 1 : E[p^N]/\langle \text{Frob}_{\ell} - 1 \rangle \to E[p^N]$ is an isomorphism. We define $\phi'' : H^1(Q_{\ell}, E[p^N]) \to H^0(F_{\ell}, E[p^N])$ as the composition of $\phi'$ and $H^1(F_{\ell}, E[p^N]) \xrightarrow{\text{Frob}_{\ell}^{-1} - 1} H^0(F_{\ell}, E[p^N])$. We define

$$\phi_\ell : H^1(K, E[p^N]) \to \mathcal{H}_K^2(K)(1)$$

as the composition of the natural homomorphism $H^1(K, E[p^N]) \to \bigoplus_{v|\ell} H^1(K_v, E[p^N])$ and $\phi''$ for $K_v$. Using the primitive $p^N$-th root of unity $\zeta_{p^N}$ we fixed, we regard $\phi_\ell$ as a homomorphism

$$\phi_\ell : H^1(K, E[p^N]) \to \mathcal{H}_K^2(K).$$

For a prime $\ell \in \mathcal{P}_1^{(N)}(K)$, we put $G_\ell = \text{Gal}(Q(\ell)/Q)$. We identify $G_\ell$ with $\text{Gal}(Q(\ell)/Q_{\ell})$. Recall that we defined $n_\ell$ by $p^{n_\ell} = [Q(\ell) : Q]$, and we
took a generator $\tau_\ell$ of $\mathcal{G}_\ell$ above. We define
\[ N_\ell = \sum_{i=0}^{p^\alpha-1} \tau_\ell^i \in \mathbb{Z}[\mathcal{G}_\ell], \quad D_\ell = \sum_{i=0}^{p^\alpha-1} i\tau_\ell^i \in \mathbb{Z}[\mathcal{G}_\ell] \]
as usual.

We define $N_1^{(N)}(K)$ to be the set of squarefree products of primes in $\mathcal{P}_1^{(N)}(K)$. We suppose $1 \in N_1^{(N)}(K)$. For $m \in N_1^{(N)}(K)$, we put $\mathcal{G}_m = \text{Gal}(\mathbb{Q}(m)/\mathbb{Q})$, $N_m = \prod_{\ell|m} N_\ell \in \mathbb{Z}[\mathcal{G}_m]$, and $D_m = \prod_{\ell|m} D_\ell \in \mathbb{Z}[\mathcal{G}_m]$. Assume that $\ell$ is in $(\mathcal{P}_0^{(N)}(K(m)[1]))$ and consider $g_{\ell,t,t}^{(K(m))} \in \text{Sel}(O_{K(m)[1/\ell]}[E[p^N]])$.

We can check that $D_m g_{\ell,t,t}^{(K(m))}$ is in $\text{Sel}(O_{K(m)[1/m\ell]}[E[p^N]]^{\mathcal{G}_m})$. Using the fact that $\text{Sel}(O_{K(m)[1/m\ell]}[E[p^N]]) \xrightarrow{\sim} \text{Sel}(O_{K(m)[1/m\ell]}[E[p^N]])^{\mathcal{G}_m}$ is bijective by Lemma 3.3.1 below (cf. also [12] Lemma 6.3.1), we define
\[ \kappa_{m,\ell} = \kappa_{m,\ell,t,t} \in \text{Sel}(O_{K[1/m\ell]}[E[p^N]]) \]
to be the unique element whose image in $\text{Sel}(O_{K(m)[1/m\ell]}[E[p^N]])$ is $D_m g_{\ell,t,t}^{(K(m))}$.

The following lemma will be also used in the next section.

**Lemma 3.3.1** Suppose that $K$, $L \in K_{(p)}$ and $K \subset L$. For any $m \in \mathbb{Z}_{>0}$, the restriction map $\text{Sel}(O_{K[1/m]}[E[p^N]]) \xrightarrow{\sim} \text{Sel}(O_{L[1/m]}[E[p^N]]^{\text{Gal}(L/K)})$ is bijective.

Proof. Let $N_E$ be the conductor of $E$, $m' = mpN_E$, and $m''$ the product of primes which divide $pN_E$ and which do not divide $m$. Put $G = \text{Gal}(L/K)$.

We have a commutative diagram of exact sequences
\[
\begin{array}{cccc}
0 & \rightarrow & \text{Sel}(O_{K[1/m]}[E[p^N]]) & \rightarrow & \text{Sel}(O_{K[1/m']}[E[p^N]]) & \rightarrow & \bigoplus_{v|m''} H_{K,v}^2 \bigg|_{\alpha_1} \\
0 & \rightarrow & \text{Sel}(O_{L[1/m]}[E[p^N]])^{G} & \rightarrow & \text{Sel}(O_{L[1/m']}[E[p^N]])^{G} & \rightarrow & \bigoplus_{w|m''} H_{L,w}^2 \bigg|_{\alpha_3}
\end{array}
\]
where $H_{K,v}^2 = H^1(K_v,E[p^N])/E(K_v) \otimes \mathbb{Z}/p^N)$ and $H_{L,w}^2 = H^1(L_w,E[p^N])/E(L_w) \otimes \mathbb{Z}/p^N)$. Since $\text{Sel}(O_{L[1/m']}[E[p^N]]) = H^1_{G}(\text{Spec}O_{L[1/m']}[E[p^N]])$ and $H^0(L,E[p^N]) = 0$, $\alpha_2$ is bijective. Suppose that $v$ divides $m''$ and $w$ is above $v$. When $v$ divides $N_E$, since $v$ is unramified in $L$ and $p$ is prime to $\text{Tam}(E)$, $H_{K,v}^2 \rightarrow H_{L,w}^2$ is injective (Greenberg [42] §3). When $v$ is above $p$, $H_{K,v}^2 \rightarrow H_{L,w}^2$ is injective because $a_p \not\equiv 1 \pmod{p}$ (Greenberg [42] §3). Hence $\alpha_3$ is injective. Therefore, $\alpha_1$ is bijective.

In [11], if $m$ has a factorization $m = \ell_1 \cdots \ell_r$ such that $\ell_{i+1} \in \mathcal{P}_1^{(N)}(K(\ell_1 \cdots \ell_i))$ for all $i = 1, \ldots, r-1$, we called $m$ well-ordered. But the word “well-ordered” might cause confusion, so we call $m$ admissible in this paper if $m$
satisfies the above condition. Note that we do not impose the condition \( \ell_1 < \ldots < \ell_r \) in the above definition, and that \( m \) is admissible if there is one factorization as above. We sometimes call the set of prime divisors of \( m \) admissible if \( m \) is admissible.

Suppose that \( m = \ell_1 \cdot \ldots \cdot \ell_r \). We define \( \delta_m \in R_K/p^N \) by

\[
\vartheta_K(m) \equiv \delta_m \prod_{i=1}^{r} (1 - \tau_{\ell_i}) \pmod{p^N, (\tau_{\ell_1} - 1)^2, \ldots, (\tau_{\ell_r} - 1)^2} \tag{3.6}
\]

(see [12] §6.3).

We simply write \( \kappa_{m, \ell} \) for \( \kappa_{m, \ell, t_\ell} \). We have the following Proposition ([12] Propositions 6.3.2, 6.4.5 and Lemma 6.3.4).

**Proposition 3.3.2** Suppose that \( m \) is in \( N_1^{(N)}(K) \), and \( \ell \in (\mathcal{P}_0^{(N)}(K(m))_{[1]} \) satisfies some properties. In this subsection we construct \( \kappa_{m, \ell} \) for \( \ell \in \mathcal{P}_{\infty}^{(N)}(K) \) satisfying some properties (see Proposition 3.4.2). The property (4) in Proposition 3.4.2 is a beautiful property of our Kolyvagin systems of Gauss sum type, which is unique for Kolyvagin systems of Gauss sum type.

For a squarefree product \( m \) of primes, we define \( \epsilon(m) \) to be the number of prime divisors of \( m \), namely \( \epsilon(m) = r \) if \( m = \ell_1 \cdot \ldots \cdot \ell_r \).

For any prime number \( \ell \), we write

\[
\mathcal{H}_2^{\ell}(K) = \bigoplus_{v|\ell} H^1(K_v, E[p^N])/(E(K_v) \otimes \mathbb{Z}/p^N),
\]

and consider the natural map

\[
w_K : \bigoplus_{\ell} \mathcal{H}_2^{\ell}(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee
\]

which is obtained by taking the dual of \( \text{Sel}(O_K, E[p^N]) \rightarrow \bigoplus_v E(K_v) \otimes \mathbb{Z}/p^N \). We also consider the natural map

\[
\partial_K : H^1(K, E[p^N]) \longrightarrow \bigoplus_{\ell} \mathcal{H}_2^{\ell}(K).
\]
We use the following lemma which was proved in \cite{12} Proposition 4.4.3 and Lemma 6.2.1 (2).

**Lemma 3.4.1** Suppose that $K \in \mathcal{K}(p)$ and $r_1, \ldots, r_s$ are $s$ distinct primes in $\mathcal{P}_1^{(N)}(K)$. Assume that for each $i = 1, \ldots, s$, $\sigma_i \in H^2_r(\mathbb{F}_p)$ is given, and also $x \in \text{Sel}(O_K, E[p^N])$ is given. Let $K'/K$ be an extension such that $K' \in \mathcal{K}(p)$. Then there are infinitely many $\ell \in \mathcal{P}_0^{(N)}(K)$ such that $w_K(t_{\ell,K}) = x$. We take such a prime $\ell$ and fix it. Then there are infinitely many $\ell' \in (\mathcal{P}_0')^{(N)}(K')$ which satisfy the following properties:

(i) $w_K(t_{\ell,K}) = w_K(t_{\ell,K}) = x$.

(ii) There is an element $z \in \text{Sel}(O_K[1/\ell\ell'], E[p^N])$ such that $\partial_K(z) = t_{\ell,K} - t_{\ell,K}$ and $\phi_{t_{1}}(z) = \sigma_i$ for each $i = 1, \ldots, s$.

Assume that $m\ell$ is in $\mathcal{N}_1^{(N)}(K_{[\ell=m\ell]})$. By Lemma 3.4.1 we can take $\ell' \in (\mathcal{P}_0')^{(N)}$ satisfying the following properties:

(i) $\ell' \in (\mathcal{P}_0')^{(N)}(K_{[\ell=m\ell]/[m]=[1]} K_{[\ell=1/\ell\ell']}$ where $n_0$ is as in Proposition 3.5.2.

(ii) $w_{K_{[\ell=m\ell]}}(t_{\ell',K_{[\ell=m\ell]}}) = w_{K_{[\ell=m\ell]}}(t_{\ell,K_{[\ell=m\ell]}})$.

(iii) Let $\phi_{r(K_{[\ell=m\ell]}]} : H^1(K_{[\ell=m\ell]}, E[p^N]) \to H^2_r(K_{[\ell=m\ell]})$ be the map $\phi_r$ for $K_{[\ell=m\ell]}$. There is an element $b'$ in $\text{Sel}(O_{K_{[\ell=m\ell]}}[1/\ell\ell'], E[p^N])$ such that

$$\partial_{K_{[\ell=m\ell]}}(b') = t_{\ell',K_{[\ell=m\ell]}} - t_{\ell,K_{[\ell=m\ell]}}$$

and $\phi_r^{K_{[\ell=m\ell]}}(b') = 0$ for all $r$ dividing $m$.

We have already defined $\kappa_{m,\ell'}$ in the previous subsection. We put $b = \text{Cor}_{K_{[\ell=m\ell]}/K}(b')$ and define

$$\kappa_{m,\ell} = \kappa_{m,\ell'} - \delta_m b.$$

(3.7)

Then this element does not depend on the choice of $\ell'$ and $b'$ (see \cite{12} §6.4). In \cite{12}, we took $b'$ which does not necessarily satisfy $\phi_r^{K_{[\ell=m\ell]}}(b') = 0$ in the definition of $\kappa_{m,\ell}$. But we adopted the above definition here because it is simpler and there is no loss of generality.

The next proposition was proved in \cite{12} Propositions 6.4.3, 6.4.5, 6.4.6.

**Proposition 3.4.2** Suppose that $m\ell$ is in $\mathcal{N}_1^{(N)}(K_{[\ell=m\ell]})$. Then

1. $\kappa_{m,\ell} \in \text{Sel}(O_{K_{[\ell=m\ell]}}[1/m, E[p^N]])$.
2. $\partial_{r}(\kappa_{m,\ell}) = \phi_r(\kappa_{m,\ell})$ for any prime divisor $r$ of $m$.
3. Assume further that $m$ is admissible. Then $\phi_r(\kappa_{m,\ell}) = 0$ for any prime divisor $r$ of $m$.
4. Assume further that $m\ell$ is admissible, and $m\ell$ is in $\mathcal{N}_1^{(N)}(K_{[\ell=m\ell]+1})$. Then we have

$$\phi_{\ell}(\kappa_{m,\ell}) = -\delta_{m\ell} t_{\ell,K}.$$
4 Relations of Selmer groups

In this section, we prove a generalized version of Theorem 1.2.3.

4.1 Injectivity theorem

Suppose that \( K \in \mathcal{K}(p) \) and that \( m \in \mathcal{N}_1(N)(K) \). For a prime divisor \( r \) of \( m \), we denote by

\[
\lambda_r : \mathcal{H}_r^2(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee
\]

the homomorphism which is the dual of \( \text{Sel}(O_K, E[p^N]) \to \bigoplus_{v|r} E(K_v) \otimes \mathbb{Z}/p^N \). Recall that \( \mathcal{H}_r^2(K) \) is a free \( R_K/p^N \)-module of rank 1, generated by \( t_{r,K} \).

Proposition 4.1.1 We assume that \( \delta_m \) is a unit of \( R_K/p^N \) for some \( m \in \mathcal{N}_1(N)(K) \). Then the natural homomorphism

\[
\bigoplus_{r|m} \lambda_r : \bigoplus_{r|m} \mathcal{H}_r^2(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee
\]

is surjective.

Remark 4.1.2 We note that \( \delta_m \) is numerically computable, in principle.

Proof of Proposition 4.1.1 Let \( x \) be an arbitrary element in \( \text{Sel}(O_K, E[p^N])^\vee \). Let

\[
\lambda_r : \mathcal{H}_r^2(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee
\]

be the natural homomorphism for each \( r \mid m \). We will prove that \( x \) is in the submodule generated by all \( \lambda_r(t_{r,K}) \)’s. Using Lemma 3.4.1, we can take a prime \( \ell \in \mathcal{P}_1^N(K(m)[1]K_0+N) \) such that \( \ell(t_{\ell,K}) = x \) and \( \ell \) is prime to \( m \). We consider the Kolyvagin derivative \( \kappa_m,\ell \) which was defined in (3.5). Consider the exact sequence

\[
\text{Sel}(O_K[1/\ell], E[p^N]) \to \bigoplus_{\ell'|\ell^m} \mathcal{H}_{\ell'}^2(K) \xrightarrow{w_{K_0}} \text{Sel}(O_K, E[p^N])^\vee
\]

(see Lemma 3.2.1) where \( \partial = (\oplus \partial v)_{v|\ell^m} + \sum_{\ell'|\ell^m} w_{E(K_v)} \). For each \( r \mid m \) we define \( \lambda_r \in R_K/p^N \) by \( \partial_{v}(\kappa_m,\ell) = \lambda_r t_{r,K} \in \mathcal{H}_{\ell}^2(K) \). The above exact sequence and Proposition 3.3.2 (2) imply that

\[
\delta_m x + \sum_{r|m} \lambda_r w_{r}(t_{r,K}) = 0
\]

in \( \text{Sel}(O_K, E[p^N])^\vee \). Since we assumed that \( \delta_m \) is a unit, \( x \) is in the submodule generated by all \( \lambda_r(t_{r,K}) \)’s. This completes the proof of Proposition 4.1.1.

For a prime \( \ell \in \mathcal{P}_1^N(K) \), we define

\[
\mathcal{H}_{\ell,f}^1(K) = \bigoplus_{v|\ell} E(\kappa(v)) \otimes \mathbb{Z}/p^N.
\]
Since \( \kappa(v) = F_\ell, E(\kappa(v)) \otimes \mathbb{Z}/p^N \) is isomorphic to \( \mathbb{Z}/p^N \) and \( H^1_{\ell,f}(K) \) is a free \( R_K/p^N \)-module of rank 1.

**Corollary 4.1.3** Suppose that \( m = \ell_1 \cdot \ldots \ell_a \) is in \( \mathcal{N}^{(N)}_1(K) \). We assume that \( \delta_m \) is a unit of \( R_K/p^N \). Then the natural homomorphism

\[
 s_m : \text{Sel}(O_K, E[p^N]) \longrightarrow \bigoplus_{i=1}^a H^1_{\ell_i,f}(K)
\]

is injective.

**Proof.** This is obtained by taking the dual of the statement in Proposition 4.1.1.

### 4.2 Relation matrices

**Theorem 4.2.1** Suppose that \( m = \ell_1 \cdot \ldots \ell_a \) is in \( \mathcal{N}^{(N)}_1(K_{[a+1]}) \). We assume that \( m \) is admissible and that \( \delta_m \) is a unit of \( R_K/p^N \). Then

1. \( \text{Sel}(O_K[1/m], E[p^N]) \) is a free \( R_K/p^N \)-module of rank \( a \).
2. \( \{\kappa_{m/\ell_1}, \ell_1\}_{1 \leq i \leq a} \) is a basis of \( \text{Sel}(O_K[1/m], E[p^N]) \).
3. The matrix

\[
 A = \begin{pmatrix}
 \delta_{\ell_1} & \phi_{\ell_1}(\kappa_{m/\ell_1}, \ell_2) & \ldots & \phi_{\ell_1}(\kappa_{m/\ell_1}, \ell_a) \\
 \phi_{\ell_2}(\kappa_{m/\ell_2}, \ell_1) & \delta_{\ell_2} & \ldots & \phi_{\ell_2}(\kappa_{m/\ell_2}, \ell_a) \\
 \vdots & \vdots & \ddots & \vdots \\
 \phi_{\ell_a}(\kappa_{m/\ell_a}, \ell_1) & \phi_{\ell_a}(\kappa_{m/\ell_a}, \ell_2) & \ldots & \delta_{\ell_a}
\end{pmatrix}
\]  

(4.1)

is a relation matrix of \( \text{Sel}(E/K, E[p^N])^\vee \).

In particular, if \( a = 2 \), the above matrix is

\[
 A = \begin{pmatrix}
 \delta_{\ell_2} & \phi_{\ell_1}(g_{\ell_2}) \\
 \phi_{\ell_2}(g_{\ell_1}) & \delta_{\ell_1}
\end{pmatrix}.
\]

This is described in Remark 10.6 in [11] in the case of ideal class groups.

**Proof of Theorem 4.2.1 (1).** By Proposition 4.1.1 \( \bigoplus_{i=1}^a H^2_{\ell_i}(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee \) is surjective. Therefore, by Lemma 3.2.1 we have an exact sequence

\[
 0 \longrightarrow \text{Sel}(O_K, E[p^N]) \longrightarrow \text{Sel}(O_K[1/m], E[p^N]) \overset{\partial}{\longrightarrow} \bigoplus_{i=1}^a H^2_{\ell_i}(K) \longrightarrow \text{Sel}(O_K, E[p^N])^\vee \longrightarrow 0.
\]  

(4.2)

It follows that \( \# \text{Sel}(O_K[1/m], E[p^N]) = \# \bigoplus_{i=1}^a H^2_{\ell_i}(K) = \#(R_K/p^N)^a \).
Let \( m_{R_{K}} \) be the maximal ideal of \( R_{K} \). By Lemma 3.3.1, \( \text{Sel}(\mathbf{Z}[1/m], E[p^{N}]) \xrightarrow{\sim} \text{Sel}(O_{K}[1/m], E[p^{N}])^{\text{Gal}(K/Q)} \) is bijective. Since \( H^{0}(Q, E[p^{\infty}]) = 0 \), the kernel of the multiplication by \( p \) on \( \text{Sel}(\mathbf{Z}[1/m], E[p^{N}]) \) is \( \text{Sel}(\mathbf{Z}[1/m], E[p]) \). Therefore, we have an isomorphism \( \text{Sel}(O_{K}[1/m], E[p^{N}])^{\vee} \otimes_{R_{K}} R_{K}/m_{R_{K}} \simeq \text{Sel}(\mathbf{Z}[1/m], E[p])^{\vee} \). From the exact sequence

\[
0 \to \text{Sel}(\mathbf{Z}, E[p]) \to \text{Sel}(\mathbf{Z}[1/m], E[p]) \to \bigoplus_{i=1}^{a} \mathcal{H}_{\ell_{i}}^{2}(Q) \to \text{Sel}(\mathbf{Z}, E[p])^{\vee} \to 0,
\]

and \( \mathcal{H}_{\ell_{i}}^{2}(Q) = H^{0}(F_{\ell_{i}}, E[p]) \simeq F_{p} \), we know that \( \text{Sel}(\mathbf{Z}[1/m], E[p]) \) is generated by \( a \) elements. Therefore, by Nakayama’s lemma, \( \text{Sel}(O_{K}[1/m], E[p^{N}])^{\vee} \) is generated by \( a \) elements. Since \( \# \text{Sel}(O_{K}[1/m], E[p^{N}])^{\vee} = \#(R_{K}/p^{N})^{a} \), \( \text{Sel}(O_{K}[1/m], E[p^{N}])^{\vee} \) is a free \( R_{K}/p^{N} \)-module of rank \( a \). This shows that \( \text{Sel}(O_{K}[1/m], E[p^{N}]) \) is also a free \( R_{K}/p^{N} \)-module of rank \( a \) because \( R_{K}/p^{N} \) is a Gorenstein ring.

(2) We identify \( \bigoplus_{i=1}^{a} \mathcal{H}_{\ell_{i}}^{2}(K) \) with \( (R_{K}/p^{N})^{a} \), using a basis \( \{t_{\ell_{i}, K}\}_{1 \leq i \leq a} \). Consider \( \phi_{\ell_{i}} : \text{Sel}(O_{K}[1/m], E[p^{N}]) \to \mathcal{H}_{\ell_{i}}^{2}(K) \) and the direct sum of \( \phi_{\ell_{i}} \), which we denote by \( \Phi \);

\[
\Phi = \oplus_{i=1}^{a} \phi_{\ell_{i}} : \text{Sel}(O_{K}[1/m], E[p^{N}]) \to \bigoplus_{i=1}^{a} \mathcal{H}_{\ell_{i}}^{2}(K) \simeq (R_{K}/p^{N})^{a}.
\]

Recall that \( \kappa_{\mathcal{M}, \ell_{i}} \) is an element of \( \text{Sel}(O_{K}[1/m], E[p^{N}]) \) (Proposition 3.4.2 (0)). By Proposition 3.4.2 (3), (4), we have

\[
\Phi(\kappa_{\mathcal{M}, \ell_{i}}) = -\delta_{m}e_{i}
\]

for each \( i \) where \( \{e_{i}\}_{1 \leq i \leq a} \) is the standard basis of the free module \( (R_{K}/p^{N})^{a} \). Since we are assuming that \( \delta_{m} \) is a unit, \( \Phi \) is surjective. Since both the target and the source are free modules of the same rank, \( \Phi \) is bijective. This implies Theorem 1.2.1 (2).

(3) Using the exact sequence 4.2 and the isomorphism \( \Phi \), we have an exact sequence

\[
(R_{n}/p^{N})^{a} \xrightarrow{\partial \Phi^{-1}} \bigoplus_{1 \leq i \leq a} \mathcal{H}_{\ell_{i}}^{2}(K_{n}) \xrightarrow{r} \text{Sel}(O_{K}, E[p^{N}])^{\vee} \to 0.
\]

We take a basis \( \{-\delta_{m}e_{i}\}_{1 \leq i \leq a} \) of \( (R_{n}/p^{N})^{a} \) and a basis \( \{t_{\ell_{i}, K}\}_{1 \leq i \leq a} \) of \( \bigoplus_{1 \leq i \leq a} \mathcal{H}_{\ell_{i}}^{2}(K_{n}) \). Then the \((i,j)\)-component of the matrix corresponding to \( \partial \circ \Phi^{-1} \) is \( \delta_{t_{\ell_{j}, K}}(\kappa_{\mathcal{M}, \ell_{i}}) \). If \( i = j \), this is \( \delta_{m}^{2} \) by Proposition 3.4.2 (2). If \( i \neq j \), we have \( \partial_{t_{\ell_{i}, K}}(\kappa_{\mathcal{M}, \ell_{j}}) = \phi_{\ell_{i}}(\kappa_{\mathcal{M}, \ell_{j}}) \) by Proposition 3.4.2 (1). This completes the proof of Theorem 1.2.1.
Remark 4.2.2 Suppose that $\ell$ is in $P_1^{(N)}(K)$. We define

$$\Phi'_\ell : H^1(K, E[p^N]) \rightarrow \mathcal{H}^1_{\ell,f}(K)$$

as the composition of the natural map $H^1(K, E[p^N]) \rightarrow \bigoplus_{v \mid \ell} H^1(K_v, E[p^N])$ and $\phi' : H^1(K_v, E[p^N]) \rightarrow H^1(\kappa(v), E[p^N]) = E(\kappa(v)) \otimes \mathbb{Z}/p^N$ in [3.4]. For $m \in \mathcal{N}_1^{(N)}(K)$, we define

$$\Phi'_m : H^1(K, E[p^N]) \rightarrow \bigoplus_{\ell \mid m} \mathcal{H}^1_{\ell,f}(K)$$

as the direct sum of $\Phi'_\ell$ for $\ell \mid m$. By definition, the restriction of $\Phi'_m$ to $S = \text{Sel}(E/K, E[p^N])$ coincides with the canonical map $s_m$;

$$(\Phi'_m)_S = s_m : \text{Sel}(E/K, E[p^N]) \rightarrow \bigoplus_{\ell \mid m} \mathcal{H}^1_{\ell,f}(K).$$

Since $\mathcal{H}^1_{\ell,f}(K)$ and $\mathcal{H}^2_{\ell}(K)$ are Pontrjagin dual each other, we can take the dual basis $t^*_{\ell,K}$ of $\mathcal{H}^1_{\ell,f}(K)$ as an $R_K/p^N$-module from the basis $t_{\ell,K}$ of $\mathcal{H}^2_{\ell}(K)$. Under the assumptions of Theorem [4.2.1], using the basis $\{t^*_{\ell,K}\}_{1 \leq i \leq a}$ of $\bigoplus_{i=1}^a \mathcal{H}^1_{\ell,f}(K)$, $\{t_{\ell,K}\}_{1 \leq i \leq a}$ of $\bigoplus_{i=1}^a \mathcal{H}^2_{\ell}(K)$ and the isomorphism $\Phi'_m$, we have an exact sequence $\bigoplus_{\ell \mid m} \mathcal{H}^1_{\ell,f}(K) \xrightarrow{f} \bigoplus_{i=1}^a \mathcal{H}^2_{\ell}(K) \rightarrow \text{Sel}(E/K, E[p^N])^\vee \rightarrow 0$. Then the matrix corresponding to $f$ is an organizing matrix in the sense of Mazur and Rubin [15] (cf. [12] §9).

5 Modified Kolyvagin systems and numerical examples

5.1 Modified Kolyvagin systems of Gauss sum type

In [3.4] we constructed Kolyvagin systems $\kappa_{m, \ell}$ for $(m, \ell)$ such that $m\ell \in \mathcal{N}_1^{(N)}(K_{(m\ell)+1})$. But the condition $\ell \in P_1^{(N)}(K_{(m\ell)+1})$ is too strict, and it is not suitable for numerical computation. In this subsection, we define a modified version of Kolyvagin systems of Gauss sum type for $(m, \ell)$ such that $m\ell \in \mathcal{N}_1^{(N)}(K)$.

Suppose that $K$ is in $K(p)$. For each $\ell \in P_1^{(N)}(K)$, we fix $t_\ell \in H^0(F_\ell, E[p^N])$ of order $p^N$, and consider $t_{\ell,K} \in \mathcal{H}^2_{\ell}(K)$, whose $\ell_K$-component is $t_\ell \otimes \zeta^{(N-1)}$ and other components are zero. Using $t_{\ell,K}$, we regard $\partial_\ell$ and $\phi_\ell$ as homomorphisms $\partial_\ell : H^1(K, E[p^N]) \rightarrow R_K/p^N$ and $\phi_\ell : H^1(K, E[p^N]) \rightarrow R_K/p^N$.

We will define an element $\kappa_{m, \ell}^{q, q', z}$ in $\text{Sel}(O_K[1/m\ell], E[p^N])$ for $(m, \ell)$ such that $m\ell \in \mathcal{N}_1(K)$ (and for some primes $q, q'$ and some $z$ in $\text{Sel}(O_K[1/qq'], E[p^N])$).
Consider \((m, \ell)\) such that \(\ell \) is a prime and \(m\ell \in \mathcal{N}_1(K)\). We take \(n_0\) sufficiently large such that every prime of \(K_{n_0}\) dividing \(m\ell\) is inert in \(K_{\infty}/K_{n_0}\). Then by Proposition \([3.3.2](1)\), for any \(q \in (\mathcal{P}_0'(N))(K(m\ell)[1]K_{n_0+N})\), \(\kappa_{m\ell,q} \in \text{Sel}(O_K[1/m\ell q], E[p^N])\) satisfies
\[
\partial_r(\kappa_{m\ell,q}) = \phi_r(\kappa_{m\ell,q})
\]
for all \(r\) dividing \(m\ell\). By Lemma \([3.4.1]\) we can take \(q, q' \in (\mathcal{P}_0'(N))(K(m\ell)[1]K_{n_0+N})\) satisfying
- \(w_K(t_q, K) = w_K(t_{q'}, K)\), and
- there is \(z \in H^1(O_K[1/qq'], E[p^N])\) such that \(\partial_K(z) = t_q K - t_{q'}, K\), \(\phi(z) = 1\) and \(\phi_r(z) = 0\) for any \(r\) dividing \(m\).

For any \(m \in \mathcal{N}_1(K)\), let \(\delta_m\) be the element defined in \((3.6)\). We define
\[
\kappa_{m\ell,q}^{q,q'} = \kappa_{m\ell,q} - \kappa_{m\ell,q'} - \delta_{mz}^r. \quad (5.1)
\]
By Proposition \([3.3.2]\) \((2)\), we have \(\kappa_{m\ell,q}^{q,q'} \in \text{Sel}(O_K[1/m\ell], E[p^N])\).

**Proposition 5.1.1** \((0)\) \(\kappa_{m\ell,q}^{q,q'}\) is in \(\text{Sel}(O_K[1/m\ell], E[p^N])\).

1. The element \(\kappa_{m\ell,q}^{q,q'}\) satisfies \(\partial_r(\kappa_{m\ell,q}^{q,q'}) = \phi_r(\kappa_{m\ell,q}^{q,q'})\) for any prime divisor \(r\) of \(m\).
2. We further assume that \(m\ell\) is admissible in the sense of the paragraph before Proposition \([3.3.2]\). Then we have \(\phi_r(\kappa_{m\ell,q}^{q,q'}) = 0\) for any prime divisor \(r\) of \(m\).
3. Under the same assumptions as \((2)\), \(\phi_r(\kappa_{m\ell,q}^{q,q'}) = -\delta_{mz}\) holds.

Proof. \((1)\) Using the definition of \(\kappa_{m\ell,q}^{q,q'}\) and Proposition \([3.3.2]\) \((1)\), we have
\[
\partial_r(\kappa_{m\ell,q}^{q,q'}) = \partial_r(\kappa_{m\ell,q} - \kappa_{m\ell,q'}) = \phi_r(\kappa_{m\ell,q}^{q,q'}) = \phi_r(\kappa_{m\ell,q}^{q,q'}) = \phi_r(\kappa_{m\ell,q}^{q,q'}) + \delta_{mz}^r = \phi_r(\kappa_{m\ell,q}^{q,q'}).
\]
These computations imply \((1)\).

\((2)\) We have \(\phi_r(\kappa_{m\ell,q}) = 0\) by Proposition \([3.3.2]\) \((3)\). This together with \(\phi_r(z) = 0\) implies \(\phi_r(\kappa_{m\ell,q}^{q,q'}) = 0\).

\((3)\) We again use Proposition \([3.3.2]\) \((3)\) to get \(\phi_r(\kappa_{m\ell,q}^{q,q'}) = 0\). Since \(\phi(z) = 1\), we have \(\phi_r(\kappa_{m\ell,q}^{q,q'}) = \phi_r(\kappa_{m\ell,q}^{q,q'}) = -\delta_{mz}\) holds. This completes the proof of Proposition 5.1.1

**5.2 Proof of Theorem 1.2.5**

In this subsection we take \(K = \mathbb{Q}\). For \(m \in \mathcal{N}(\mathbb{Q}) = \mathcal{N}(\mathbb{Q})\), we consider \(\delta_m \in \mathbb{Z}/p^N\), which is defined from \(\partial_{\mathbb{Q}(m)}\) by \((3.6)\). We define \(\delta_m \in \mathbb{Z}/p^N\) by
\[
\delta_m \equiv \delta_m \prod_{i=1}^r (\tau_{\ell_i} - 1) \mod p^N, (\tau_{\ell_i} - 1)^2, \ldots, (\tau_{\ell_r} - 1)^2 \quad (5.2)
\]
where \( m = \ell_1 \cdots \ell_r \). By (2.4), \( \bar{\theta}_{Q(m)} = u \theta_{Q(m)} \) for some unit \( u \in R_{Q(m)}^\times \).

This together with (3.6) and (5.2) implies that

\[
\text{ord}_p(\tilde{\delta}_m) = \text{ord}_p(\delta_m). \tag{5.3}
\]

We take a generator \( \eta_{\ell} \in (\mathbb{Z}/\ell\mathbb{Z})^\times \) such that the image of \( \sigma_{\eta_{\ell}} \in \text{Gal}(Q(\mu_\ell)/Q) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times \) in \( \text{Gal}(Q(\ell)/Q) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times \otimes \mathbb{Z}_p \) is \( \tau_\ell \) which is the generator we took.

Then, using (5.2) and (1.1), we can easily check that the equation (1.2) in §1.1 holds.

In the rest of this subsection, we take \( N = 1 \). We simply write \( P_1 \) for \( P_{1(1)} \), so

\[
P_1 = \{ \ell \in P_{\text{good}} \mid \ell \equiv 1 \pmod{p} \text{ and } E(F_{\ell}) \simeq \mathbb{Z}/p \}.
\]

The set of squarefree products of primes in \( P_1 \) is denoted by \( N_1 \).

We first prove the following lemma which is related to the functional equation of an elliptic curve.

**Lemma 5.2.1** Let \( \epsilon \) be the root number of \( E \). Suppose that \( m \in N_1 \) is \( \delta \)-minimal (for the definition of \( \delta \)-minimalness, see the paragraph before Conjecture 1.2.4). Then we have \( \epsilon = (-1)^{\epsilon(m)} \).

**Proof.** By the functional equation (1.6.2) in Mazur and Tate [16] and the above definition of \( \tilde{\delta}_m \), we have \( \epsilon(-1)^{\epsilon(m)} \tilde{\delta}_m \equiv \tilde{\delta}_m \pmod{p} \). Since \( \tilde{\delta}_m \not\equiv 0 \pmod{p} \) by (5.3), we get the conclusion.

For each \( \ell \in P_1 \), we fix a generator \( t_{\ell} \in H^2(c)(Q) = H^0(F_{\ell}, E[p])(-1) \simeq \mathbb{Z}/p = \mathbb{F}_p \), and regard \( \phi_{\ell} \) as a map \( \phi_{\ell} : H^1(Q, E[p]) \rightarrow \mathbb{F}_p \). Note that the restriction of \( \phi_{\ell} \) to \( \text{Sel}(E/Q, E[p]) \) is the zero map if and only if the natural map \( s_{\ell} : \text{Sel}(E/Q, E[p]) \rightarrow E(F_{\ell}) \otimes \mathbb{Z}/p \simeq \mathbb{F}_p \) is the zero map.

I) Proof of Theorem 1.2.5 (1), (2).

Suppose that \( \epsilon(m) = 0 \), namely \( m = 1 \). Then \( \delta_1 = \theta_\ell \mod{p} = L(E, 1)/\Omega_E \mod{p} \). If \( \delta_1 \neq 0 \), \( \text{Sel}(E/Q, E[p]) = 0 \) and \( s_1 \) is trivially bijective. Suppose next \( \epsilon(m) = 1 \), so \( m = \ell \in P_1 \). It is sufficient to prove the next two propositions.

**Proposition 5.2.2** Assume that \( \ell \in P_1 \) is \( \delta \)-minimal. Then \( \text{Sel}(E/Q, E[p]) \) is 1-dimensional over \( \mathbb{F}_p \), and \( s_{\ell} : \text{Sel}(E/Z, E[p]) \rightarrow \mathbb{F}_p \) is bijective. Moreover, the Selmer group \( \text{Sel}(E/Q, E[p^\infty])^\vee \) with respect to the p-power torsion points \( E[p^\infty] \) is a free \( \mathbb{Z}_p \)-module of rank 1, namely \( \text{Sel}(E/Q, E[p^\infty])^\vee \simeq \mathbb{Z}_p \).
Proof. We first assume \( \text{Sel}(E/\mathbb{Q}, E[p]) = 0 \) and will obtain the contradiction. We consider \( \kappa_{1,\ell}^{q,q',z} = \kappa_{\ell,q} - \kappa_{\ell,q'} - \delta_{\ell}z \), which was defined in (5.1). By Proposition 5.3.2 (1), we know \( \partial_{\ell}(\kappa_{1,\ell}^{q,q',z}) = \phi_{\ell}(g_q - g_{q'}) \). Consider the exact sequence (see Lemma 5.2.1)

\[
0 \longrightarrow \text{Sel}(E/\mathbb{Q}, E[p]) \longrightarrow \text{Sel}(\mathbb{Z}[1/r], E[p]) \longrightarrow \mathcal{H}_r^2(\mathbb{Q})
\]

for any \( r \in \mathcal{P}_1 \) where \( \text{Sel}(\mathbb{Z}[1/r], E[p]) \longrightarrow \mathcal{H}_r^2(\mathbb{Q}) \cong \mathbb{F}_p \) is nothing but \( \partial_r \). Since we assumed \( \text{Sel}(E/\mathbb{Q}, E[p]) = 0 \), \( \text{Sel}(\mathbb{Z}[1/r], E[p]) \longrightarrow \mathcal{H}_r^2(\mathbb{Q}) \cong \mathbb{F}_p \) is injective for any \( r \in \mathcal{P}_1 \). So \( \partial_q(g_q) = \delta_1 = 0 \) implies that \( g_q = 0 \). By the same method, we have \( g_{q'} = 0 \). Therefore, \( \partial_{\ell}(\kappa_{1,\ell}^{q,q',z}) = \phi_{\ell}(g_q - g_{q'}) = 0 \), which implies that \( \kappa_{1,\ell}^{q,q',z} \in \text{Sel}(E/\mathbb{Q}, E[p]) \).

But Proposition 5.1.1 (3) tells us that \( \phi_{\ell}(\kappa_{1,\ell}^{q,q',z}) = -\delta_\ell \neq 0 \). Therefore, \( \kappa_{1,\ell}^{q,q',z} \neq 0 \), which contradicts our assumption \( \text{Sel}(E/\mathbb{Q}, E[p]) = 0 \). Thus we get \( \text{Sel}(E/\mathbb{Q}, E[p]) \neq 0 \).

On the other hand, by Corollary 4.1.3 we know that \( s_{\ell} : \text{Sel}(E/\mathbb{Q}, E[p]) \longrightarrow \mathbb{F}_p \) is injective, therefore bijective.

By Lemma 5.2.1 the root number \( \epsilon \) is \( -1 \). This shows that \( \text{Sel}(E/\mathbb{Q}, E[p^\infty])^\vee \) has positive \( \mathbb{Z}_p \)-rank by the parity conjecture proved by Nekovář ([15]). Therefore, we finally have \( \text{Sel}(E/\mathbb{Q}, E[p^\infty])^\vee \cong \mathbb{Z}_p \), which completes the proof of Proposition 5.2.2.

If we assume a slightly stronger condition on \( \ell \), we also obtain the main conjecture. Let \( \lambda' = \lambda_{an} \) be the analytic \( \lambda \)-invariant of the \( p \)-adic \( L \)-function \( \vartheta_{\mathbb{Q}_\infty} \). We put \( n_{\lambda'} = \min\{n \in \mathbb{Z} \mid p^n - 1 \geq \lambda' \} \).

**Proposition 5.2.3** Suppose that there is \( \ell \in \mathcal{P}_1 \) such that

\[ \ell \equiv 1 \pmod{p^{n_{\lambda'}+2}} \text{ and } \tilde{\delta}_\ell \neq 0. \]

Then the main conjecture for \( (E, \mathbb{Q}_\infty/\mathbb{Q}) \) is true and \( \text{Sel}(E/\mathbb{Q}_\infty, E[p^\infty])^\vee \) is generated by one element as a \( \Lambda_{\mathbb{Q}_\infty} \)-module.

Proof. We use our Euler system \( g_{1}(K) \) in 3.2 instead of \( \kappa_{1,\ell}^{q,q',z} \) which was used in the proof of Proposition 5.2.2. Let \( \lambda \) be the algebraic \( \lambda \)-invariant, namely the rank of \( \text{Sel}(E/\mathbb{Q}_\infty, E[p^\infty])^\vee \). Then \( \lambda \leq \lambda' \) and \( \vartheta_{\mathbb{Q}_\infty} \in \text{char}(\text{Sel}(\mathcal{O}_{\mathbb{Q}_\infty}, E[p^\infty])^\vee) \) by Kato’s theorem.

Put \( K = \mathbb{Q}_{n_{\lambda'}} \) and \( f = p^{n_{\lambda'}} \). Consider the group ring \( R_K/p = \mathbb{F}_p[\text{Gal}(K/\mathbb{Q})] \). We identify a generator \( \gamma \) of \( \text{Gal}(K/\mathbb{Q}) \) with \( 1 + t \), and identify \( R_K/p \) with \( \mathbb{F}_p[[t]]/(t^f) \). The norm \( N_{\text{Gal}(K/\mathbb{Q})} = \Sigma_{i=0}^{f-1} \gamma^i \) is \( t^{f-1} \) by this identification, so our assumption \( \lambda' \leq f - 1 \) implies that the corestriction map
Sel(E/K, E[p]) \rightarrow Sel(E/Q, E[p]) is the zero map because \( \lambda \leq \lambda' \). Therefore, we have \( Q_{[1]} \subset K \). Since \( p^{\lambda' + 1} - p^{\lambda'} > p^{\lambda'} - 1 \geq \lambda' \geq \lambda \), the corestriction map \( Sel(E/Q_{\lambda'+1}, E[p]) \rightarrow Sel(E/Q_{\lambda'}, E[p]) = Sel(E/K, E[p]) \) is also the zero map. This shows that \( Q_{[2]} \subset Q_{\lambda'+1} \).

Our assumption \( \ell \equiv 1 \pmod{p^{\lambda' + 2}} \) implies that \( \ell \) splits completely in \( Q_{\lambda'+1} \), so we have \( \ell \in \mathcal{P}_1(Q_{[2]})) = \mathcal{P}_1(K_{[1]}) \). Therefore, we can define

\[
g_\ell^{(K)} \in Sel(O_K[1/\ell], E[p])
\]

in \( \mathbf{3.2} \). Since \( \ell \in \mathcal{P}_1(Q_{[2]}) \), we also have

\[
\phi_\ell(g_\ell^{(Q)}) = -\delta_\ell^{(Q)} = -\delta_\ell
\]

by Proposition \( \mathbf{3.4.2} \) (4). It follows from our assumption \( \delta_\ell \neq 0 \) that \( g_\ell^{(Q)} \neq 0 \). Since Cor_{K/Q}(g_\ell^{(K)}) = g_\ell^{(Q)} and the natural map \( i : Sel(Z[1/\ell], E[p]) \rightarrow Sel(O_K[1/\ell], E[p]) \) is injective, we get

\[
i(\ell^{(Q)}) = N_{Gal(K/Q)}g_\ell^{(K)} = t^{\ell^{-1}}g_\ell^{(K)} \neq 0.
\]

Consider \( \partial_\ell : Sel(O_K[1/\ell], E[p]) \rightarrow R_K/p \). By definition, we have \( \partial_\ell(g_\ell^{(K)}) = ut^\lambda \) for some unit \( u \) of \( R_K/p \). This shows that \( \partial_\ell(t^{\ell^{-1}}g_\ell^{(K)}) = 0 \), which implies that \( t^{\ell^{-1}}g_\ell^{(K)} \in Sel(E/K, E[p]) \). The fact \( t^{\ell^{-1}}g_\ell^{(K)} \neq 0 \) implies the submodule generated by \( t^{\ell^{-1}}g_\ell^{(K)} \) is isomorphic to \( R_K/(p, t^\lambda) \) as an \( R_K \)-module. Namely, we have

\[
Sel(E/K, E[p]) \supset \langle t^{\ell^{-1}}g_\ell^{(K)} \rangle \simeq R_K/(p, t^\lambda).
\]

This implies that \( \lambda = \lambda' \), and \( Sel(E/K, E[p]) \simeq R_K/(p, t^\lambda) \). Therefore, we have \( Sel(E/Q_\infty, E[p])^\vee \simeq \Lambda_{Q_\infty}/p, \vartheta_{Q_\infty} \). This together with Kato’s theorem we mentioned implies that \( Sel(E/Q_\infty, E[p^\infty])^\vee \simeq \Lambda_{Q_\infty}/(\vartheta_{Q_\infty}) \).

II) Proof of Theorem \( \mathbf{1.2.5} \) (3).

Suppose that \( m = \ell_1\ell_2 \in \mathcal{N}_1 \) and \( m \) is \( \delta \)-minimal. As in the proof of Proposition \( \mathbf{5.2.2} \) we assume \( Sel(E/Q, E[p]) = 0 \) and will get the contradiction. We consider \( \kappa^{q,q',z} \_\ell_1\ell_2 \) defined in \( \mathbf{5.1} \). Consider the exact sequence (see Lemma \( \mathbf{3.2.3} \))

\[
0 \rightarrow Sel(E/Q, E[p]) \rightarrow Sel(Z[1/\ell_1\ell_2qq'], E[p]) \xrightarrow{\partial} \bigoplus_{v \in \{\ell_1, \ell_2, q, q'\}} H^2_v(Q).
\]

By the same method as the proof of Proposition \( \mathbf{5.2.2} \), \( g_q = g_{q'} = 0 \). Therefore, \( \partial_{\ell_1}(\kappa_{\ell_1, q} - \kappa_{\ell_1, q'}) = \phi_{\ell_1}(g_q - g_{q'}) = 0 \) by Proposition \( \mathbf{3.3.2} \) (1). We have \( \partial_{q}(\kappa_{\ell_1, q}) = \delta_{\ell_1} = 0, \partial_{q'}(\kappa_{\ell_1, q'}) = 0, \partial_{q}(\kappa_{\ell_1, q'}) = 0, \partial_{q'}(\kappa_{\ell_1, q'}) = \delta_{\ell_1} = 0. \)

32
Therefore, \( \partial(\kappa_{\ell_1,q} - \kappa_{\ell_1,q'}) = 0 \). This together with \( \text{Sel}(E/Q, E[p]) = 0 \) shows that \( \kappa_{\ell_1,q} - \kappa_{\ell_1,q'} = 0 \). Therefore, using Proposition 5.2.2 (1), we have

\[
\partial_{\ell_2}(\kappa_{\ell_1,\ell_2}^{q,q';z}) = \partial_{\ell_2}(\kappa_{m,q} - \kappa_{m,q'}) = \phi_{\ell_2}(\kappa_{\ell_1,q} - \kappa_{\ell_1,q'}) = 0.
\]

By the same method as the above proof of \( \kappa_{\ell_1,q} - \kappa_{\ell_1,q'} = 0 \), we get \( \kappa_{1,\ell_2}^{q,q';z} = 0 \). This implies that \( \partial_{\ell_1}(\kappa_{1,\ell_2}^{q,q';z}) = \phi_{\ell_1}(\kappa_{1,\ell_2}^{q,q';z}) = 0 \) by Proposition 5.1.1 (1). It follows that \( \partial(\kappa_{1,\ell_2}^{q,q';z}) = 0 \), which implies \( \kappa_{1,\ell_2}^{q,q';z} \in \text{Sel}(E/Q, E[p]) \). But this is a contradiction because we assumed \( \text{Sel}(E/Q, E[p]) = 0 \) and

\[
\phi_{\ell_2}(\kappa_{1,\ell_2}^{q,q';z}) = -\delta_m \neq 0
\]

by Proposition 5.1.1 (3). Thus, we get \( \text{Sel}(E/Q, E[p]) \neq 0 \).

Now the root number is 1 by Lemma 5.2.1, therefore, by the parity conjecture proved by Nekovář ([18]), we obtain \( \dim_{\mathbf{F}_p} \text{Sel}(E/Q, E[p]) \geq 2 \). On the other hand, by Corollary 5.1.3 we know that \( s_m : \text{Sel}(E/Q, E[p]) \to (\mathbf{F}_p)^{\oplus 2} \) is injective. Therefore, the injectivity of \( s_m \) implies the bijectivity of \( s_m \). This completes the proof of Theorem 1.2.5 (3).

We give a simple corollary.

**Corollary 5.2.4** Suppose that there is \( m \in N_1 \) such that \( m \) is \( \delta \)-minimal and \( e(m) = 2 \). We further assume that the analytic \( \lambda \)-invariant \( \lambda' \) is 2. Then the main conjecture for \( (E, Q_{\infty}/Q) \) holds.

**Proof.** Put \( t = \gamma - 1 \) and identify \( \Lambda_{Q_{\infty}/Q} \) with \( \mathbf{F}_p[[t]] \). Let \( \mathcal{A} \) be the relation matrix of \( S = \text{Sel}(E/Q_{\infty}, E[p]) \). Since \( S/p, t) = \text{Sel}(E/Q, E[p])' \simeq \mathbf{F}_p \oplus \mathbf{F}_p \), \( t^2 \) divides \( \det \mathcal{A} \mod p \). Therefore, the algebraic \( \lambda \)-invariant is also 2. This implies the main conjecture because \( \det \mathcal{A} \) divides \( \delta_{Q_{\infty}} \) in \( \Lambda_{Q_{\infty}} \) (Kato [27]).

III) Proof of Theorem 1.2.5 (4).

**Lemma 5.2.5** Suppose that \( \ell, \ell_1, \ell_2 \) are distinct primes in \( P_1 \) satisfying \( \delta_\ell = \delta_{\ell_1} = \delta_{\ell_2} = 0 \). Assume also that \( s_\ell : \text{Sel}(E/Q, E[p]) \to \mathbf{F}_p \) is bijective, and that \( \ell \ell_1, \ell \ell_2 \) are both admissible. We take \( q, q' \) such that they satisfy the conditions when we defined \( \kappa_{\ell_1,\ell_2,\ell}^{q,q';z} \). Then we have

1. \( \text{Sel}(E/Q, E[p]) = \text{Sel}(Z[1/\ell], E[p]) \),
2. \( \kappa_{1,\ell_1,\ell}^{q,q';z} = 0 \), \( \kappa_{1,\ell_2,\ell}^{q,q';z} = 0 \), and
3. \( \kappa_{1,\ell_1,\ell_2,\ell}^{q,q';z} \in \text{Sel}(E/Q, E[p]) \).
Proof. (1) Since $s_{\ell}$ is bijective, taking the dual, we get the bijectivity of $\mathcal{H}_{\ell}(Q) \longrightarrow \text{Sel}(E/Q, E[p])^\vee = \text{Sel}(Z, E[p])^\vee$. By the exact sequence

$$0 \longrightarrow \text{Sel}(Z, E[p]) \longrightarrow \text{Sel}(Z[1/\ell], E[p]) \xrightarrow{\partial_1} \mathcal{H}_{\ell}(Q) \longrightarrow \text{Sel}(Z, E[p])^\vee \longrightarrow 0$$

in Lemma 3.2.1, we get $\text{Sel}(E/Q, E[p]) = \text{Sel}(Z, E[p]) = \text{Sel}(Z[1/\ell], E[p])$.

(2) We first note that the bijectivity of $s_{\ell} : \text{Sel}(E/Q, E[p]) \longrightarrow F_p$ implies the bijectivity of $\phi_{\ell} : \text{Sel}(E/Q, E[p]) \longrightarrow F_p$. Since $\partial_1(\kappa_{\ell,q}) = \delta_\ell = 0$, $\kappa_{\ell,q} \in \text{Sel}(Z[1/\ell], E[p]) = \text{Sel}(E/Q, E[p])$ where we used the property (1) which we have just proved. Proposition 5.3.2(3) implies $\phi_{\ell}(\kappa_{\ell,q}) = 0$, which implies $\kappa_{\ell,q} = 0$ by the bijectivity of $\phi_{\ell}$. By the same method, we have $\kappa_{\ell,q} = 0$. Therefore, we have

$$\kappa_{1,\ell,q}^{q,q',z} = \kappa_{\ell,q} - \kappa_{\ell,q'} - \delta_\ell z = 0.$$

Therefore, Proposition 5.1.1 (1) implies $\partial_1(\kappa_{1,\ell,q}^{q,q',z}) = \phi_1(\kappa_{1,\ell,q}^{q,q',z}) = 0$. This implies $\kappa_{1,\ell,q}^{q,q',z} \in \text{Sel}(Z[1/\ell], E[p]) = \text{Sel}(E/Q, E[p])$. Using Proposition 5.1.1 (3), we have

$$\phi_1(\kappa_{1,\ell,q}^{q,q',z}) = -\delta_\ell \ell_1 = 0,$$

which implies $\kappa_{1,\ell,q}^{q,q',z} = 0$ by the bijectivity of $\phi_1$. The same proof works for $\kappa_{2,\ell,q}^{q,q',z}$.

(3) It follows from Proposition 5.1.1(1) and Lemma 5.2.5(2) that $\partial_1(\kappa_{2,\ell,q}^{q,q',z}) = \phi_1(\kappa_{2,\ell,q}^{q,q',z}) = 0$ for each $i = 1, 2$. This implies $\kappa_{2,\ell,q}^{q,q',z} \in \text{Sel}(Z[1/\ell], E[p])$.

Using $\text{Sel}(Z[1/\ell], E[p]) = \text{Sel}(E/Q, E[p])$ which we proved in (1), we get the conclusion. This completes the proof of Lemma 5.2.5.

We next prove Theorem 1.2.5 (4). Assume that $m = \ell_1\ell_2\ell_3 \in N_1$, $m$ is $\delta$-minimal, $m$ is admissible, and $s_{\ell_i} : \text{Sel}(E/Q, E[p]) \longrightarrow F_p$ is surjective for each $i = 1, 2$.

We assume $\dim_{F_p} \text{Sel}(E/Q, E[p]) = 1$ and will get the contradiction. By this assumption, $s_{\ell_i} : \text{Sel}(E/Q, E[p]) \longrightarrow F_p$ for each $i = 1, 2$ is bijective. This implies that $\phi_{\ell_i} : \text{Sel}(E/Q, E[p]) \longrightarrow F_p$ for each $i = 1, 2$ is also bijective. By Lemma 5.2.5 (3) we get $\kappa_{2,\ell_3,\ell_1}^{q,q',z} \in \text{Sel}(E/Q, E[p])$, taking $q$, $q'$ satisfying the conditions when we defined this element. By Proposition 5.1.1 (3), we have $\phi_{\ell_1}(\kappa_{2,\ell_3,\ell_1}^{q,q',z}) = -\delta_m \neq 0$, which implies $\kappa_{2,\ell_3,\ell_1}^{q,q',z} \neq 0$. But by Proposition 5.1.1 (2), we have $\phi_{\ell_3}(\kappa_{2,\ell_3,\ell_1}^{q,q',z}) = 0$. This contradicts the bijectivity of $\phi_{\ell_3}$. Therefore, we obtain $\dim_{F_p} \text{Sel}(E/Q, E[p]) > 1$.

By Lemma 5.2.1 and our assumption that $m$ is $\delta$-minimal, we know that the root number $\epsilon$ is $-1$. This shows that $\dim_{F_p} \text{Sel}(E/Q, E[p]) \geq 3$ by the parity conjecture proved by Nekovář ([15]). On the other hand, Corollary
generated by one element. The main conjecture holds by Proposition 5.2.3. In this case, we can check that $F_{\ell}^1 P_{\text{root}}$ is injective. Therefore, the above map $s_m$ is bijective. This completes the proof of Theorem 1.2.5 (4).

5.3 Numerical examples

In this section, we give several numerical examples.

Let $E = X_0(11)^{(d)}$ be the quadratic twist of $X_0(11)$ by $d$, namely $dy^2 = x^3 - 4x^2 - 160x - 1264$. We take $p = 3$. Then if $d \equiv 1 \pmod{p}$, $p$ is a good ordinary prime which is not anomalous (namely $a_p(=a_3)$ for $E$ satisfies $a_p \neq 1 \pmod{p}$), and $p = 3$ does not divide $\text{Tam}(E)$, and the Galois representation on $T_3(E)$ is surjective. In the following examples, we checked $\mu' = 0$ where $\mu'$ is the analytic $\mu$-invariant. Then this implies that the algebraic $\mu$-invariant is also zero (Kato [7] Theorem 17.4 (3)) under our assumptions. In the computations of $\delta_m$ below, we have to fix a generator of $\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$ for a prime $\ell$. We always take the least primitive root $\eta_\ell$ of $(\mathbb{Z}/\ell\mathbb{Z})^\times$. We compute $\delta_m$ using the formula in (12).

(1) $d = 13$. We take $N = 1$. Since $\delta_7 = 20 \neq 0 \pmod{3}$, we know that $\text{Fitt}_{1,F_3}(\text{Sel}(E/\mathbb{Q}, E[3])^\vee) = F_3$ by Theorem 2.4.1, so $\text{Sel}(E/\mathbb{Q}, E[3])$ is generated by one element.

The root number is $\epsilon = (\frac{13}{11}) = -1$, so $L(E, 1) = 0$. We compute $P_1 = \{7, 31, 73, \ldots\}$. Therefore, $\delta_7 \equiv 0 \pmod{3}$ implies $\text{Sel}(E/\mathbb{Q}, E[3]) \simeq F_3$ and $\text{Sel}(E/\mathbb{Q}, E[3\infty])^\vee \simeq \mathbb{Z}_3$ by Proposition 5.2.2. Also, it is easily computed that $\lambda' = 1$ in this case. This implies that $\text{Sel}(E/\mathbb{Q}_\infty, E[3\infty])^\vee \simeq \mathbb{Z}_3$, so the main conjecture also holds.

We can find a point $P = (7045/36, -574201/216)$ of infinite order on the minimal Weierstrass model $y^2 + y = x^3 - x^2 - 1746x - 50295$ of $E = X_0(11)^{(13)}$. Therefore, we know $\text{III}(E/\mathbb{Q})[3\infty] = 0$. We can easily check that $E(F_7)$ is cyclic of order 6, and that the image of the point $P$ in $E(F_7)/3E(F_7)$ is non-zero. So we also checked numerically that $s_7 : \text{Sel}(E/\mathbb{Q}, E[3]) \longrightarrow E(F_7)/3E(F_7)$ is bijective as Proposition 5.2.2 claims.

(2) $d = 40$. We know $\epsilon = (\frac{40}{11}) = -1$. We take $N = 1$. We can compute $P_1 = \{7, 67, 73, \ldots\}$, and $\delta_7 = -40 \neq 0 \pmod{3}$. This implies that $\text{Sel}(E/\mathbb{Q}, E[3]) \simeq F_3$ and $\text{Sel}(E/\mathbb{Q}, E[3\infty])^\vee \simeq \mathbb{Z}_3$ by Proposition 5.2.2.

In this case, we know $\lambda' = 7$. Therefore, $n_{\lambda'} = 2$. We can check $5347 \in P_1$ (where $5347 \equiv 1 \pmod{3^5}$) and $\delta_{5347} = -412820 \neq 0 \pmod{3}$. Therefore, the main conjecture holds by Proposition 5.2.3. In this case, we can check
that the $p$-adic $L$-function $\vartheta_{\mathbb{Q}_{\infty}}$ is divisible by $(1 + t)^3 - 1$, so we have

$$\text{rank}_{\mathbb{Z}} \text{Sel}(E/\mathbb{Q}_{1}, E[3^\infty])^\vee = 3$$

where $\mathbb{Q}_{1}$ is the first layer of $\mathbb{Q}_{\infty}/\mathbb{Q}$.

In the following, for a prime $\ell \in \mathcal{P}$, we take a generator $\tau_{\ell}$ of $\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$ and put $S = \tau_{\ell} - 1$. We write $\vartheta_{\mathbb{Q}(\ell)} = \Sigma a_i^{(\ell)} S^i$ where $a_i^{(\ell)} \in \mathbb{Z}_p$.

Note that $\tilde{c} = a_1^{(\ell)}$.

(3) $d = 157$. We know $\epsilon = \frac{155}{11} = 1$ and $L(E, 1)/\Omega^+_E = 45$. We take $N = 1$. We compute $a_2^{(37)} = -14065/2 \not\equiv 0 \pmod{3}$. Since $37 \equiv 1 \pmod{3^2}$, $c_2 = 2 - 1 = 1$ and $a_2^{(37)}$ is in $\text{Fitt}_2\mathbb{F}_3(\text{Sel}(E/\mathbb{Q}, E[3]))$ by Theorem 2.4.1 which implies that $\text{Fitt}_2\mathbb{F}_3(\text{Sel}(E/\mathbb{Q}, E[3])) = \mathbb{F}_3$. Therefore, $\text{Sel}(E/\mathbb{Q}, E[3])$ is generated by at most two elements.

We compute $\mathcal{P}_1 = \{7, 67, 73, 127, \ldots\}$. Since $127 \equiv 1 \pmod{7}$, $7 \times 127$ is admissible. We compute $\tilde{\delta}_{7 \times 127} = 83165 \not\equiv 0 \pmod{3}$. Therefore, $7 \times 127$ is $\delta$-minimal. It follows from Theorem 1.2.3 (3) that $\text{Sel}(E/\mathbb{Q}, E[3]) \simeq \mathbb{F}_3 \oplus \mathbb{F}_3$. In this example, we can check $\lambda' = 2$, so Corollary 5.2.3 together with the above computation implies the main conjecture. Since $L(E, 1)/\Omega^+_E = 45 \not\equiv 0$, $\text{rank } E(\mathbb{Q}) = 0$ by Kato, which implies $\text{Sel}(E/\mathbb{Q}, E[3^\infty]) = \text{III}(E/\mathbb{Q})[3^\infty]$. Since $45 \in \text{Fitt}_0\mathbb{Z}_3(\text{Sel}(E/\mathbb{Q}, E[3^\infty])^\vee)$, we have $\# \text{III}(E/\mathbb{Q})[3^\infty] \leq 9$, and

$$\text{III}(E/\mathbb{Q})[3^\infty] \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

(4) $d = 265$. In this case, $\epsilon = \frac{265}{11} = 1$ and $L(E, 1) = 0$. We take $N = 1$. As in Example (3), we compute $a_2^{(37)} = 16985 \not\equiv 0 \pmod{3}$, which implies that $\text{Sel}(E/\mathbb{Q}, E[3])$ is generated by at most two elements as above. We compute $\mathcal{P}_1 = \{7, 13, 31, 67, 103, 109, 127, \ldots\}$. For an admissible pair $\{7, 127\}$, we have $\tilde{\delta}_{7 \times 127} = -138880 \not\equiv 0 \pmod{3}$. Therefore, $7 \times 127$ is $\delta$-minimal and $\text{Sel}(E/\mathbb{Q}, E[3]) \simeq \mathbb{F}_3 \oplus \mathbb{F}_3$ by Theorem 1.2.3 (3). Since $\lambda' = 2$ in this case, by Corollary 5.2.3 we know that the main conjecture holds.

Since $L(E, 1) = 0$, we know $\text{rank } \text{Sel}(E/\mathbb{Q}, E[3^\infty])^\vee > 0$ by the main conjecture. This implies that

$$\text{Sel}(E/\mathbb{Q}, E[3^\infty])^\vee \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

Now $E$ has a minimal Weierstrass model $y^2 + y = x^3 - x^2 - 725658x - 430708782$. We can find rational points $P = (2403, 108146)$ and $Q = (5901, -448036)$ on this curve. We can also easily check that $E(\mathbb{F}_{7})$ is cyclic group of order 6 and $E(\mathbb{F}_{31})$ is cyclic of order 39. The image of $P$ in $E(\mathbb{F}_7)/3E(\mathbb{F}_7) \simeq \mathbb{Z}/3\mathbb{Z}$ is 0 (the identity element), and the image of
for even Dirichlet characters of order 6, and $E(F_{13})$ is cyclic of order 18. Both of the images of $P$ in $E(F_{13})$ are of order 3. Therefore, $s_{\ell} : \text{Sel}(E/Q, E[3]) \to E(F_{\ell})/3E(F_{\ell})$ is surjective for each $\ell = 7, 13$. Since $13 = -1 \equiv (F_{13}^*)^3$, $463 = 1 \equiv (F_{13}^*)^3$ and $463 = 8 \equiv (F_{13}^*)^3$, the image of $P$ in $E(F_{13})$ is cyclic and coincides.

*Proof (5)*. Take $N = 1$ at first. For $\ell = 271$, we have $a_3^{(271)} = 900852395/2 \equiv 0$ (mod 3), which implies that $\dim_{\mathbb{F}_3} \text{Sel}(E/Q, E[3]) \leq 3$. We can find a rational point $P = (1194979057/51984, 40988136480065/11852352)$ on the minimal Weierstrass equation $y^2 + y = x^3 - x^2 - 7518626x - 14370149754$ of $E = E_0(11)^{(853)}$. We know that $E(F_{271})$ is cyclic of order 6, and $E(F_{13})$ is cyclic of order 18. Both of the images of $P$ in $E(F_{13})$ are of order 3. Therefore, $s_{\ell} : \text{Sel}(E/Q, E[3]) \to E(F_{\ell})/3E(F_{\ell})$ is surjective for each $\ell = 7, 13$. Since $13 = 1 \equiv (F_{13}^*)^3$, $463 = 1 \equiv (F_{13}^*)^3$ and $463 = 8 \equiv (F_{13}^*)^3$, the image of $P$ in $E(F_{13})$ is cyclic and coincides. We can compute $\delta_{7 \times 13 \times 463} = -8676400 \not\equiv 0$ (mod 3), and can check that $m = 7 \times 13 \times 463$ is $\delta$-minimal.

By Theorem 1.2.5 (4), we have

$$\text{Sel}(E/Q, E[3]) \simeq F_3 \oplus F_3 \oplus F_3.$$  \hfill (5.4)

We have a rational point $P$ of infinite order, so the rank of $E(Q)$ is $\geq 1$. Take $N = 3$ and consider $\ell = 271$. Since $\delta_{271} = a_1^{(271)} = 35325 \equiv 9$ (mod 27), $9$ is in $\text{Fitt}_{1, Z[\rho^2 Z]}(\text{Sel}(E/Q, E[3]))$ by Corollary 2.1.2. This implies that $\text{rank } E(Q) = 1$ and $\#\text{III}(E/Q)[3^\infty] \leq 9$. This together with (5.4) implies that

$$\text{III}(E/Q)[3^\infty] \simeq Z/3Z \oplus Z/3Z.$$  \hfill (5.5)

Note that if we used only Theorem 1.1.1 and these computations, we could not get (5.4) or (5.5) because we could not determine $\Theta_1(Q)^{(D)}$ by finite numbers of computations. We need Theorem 1.2.5 to obtain (5.4) and (5.5).

*Proof (6)*. For positive integers $d$ which are conductors of even Dirichlet characters (so $d = 4m$ or $d = 4m + 1$ for some $m$) satisfying $1 \leq d \leq 1000$, $d \equiv 1$ (mod 3), and $d \not\equiv 0$ (mod 11), we computed $\text{Sel}(E/Q, E[3])$. Then $\dim \text{Sel}(E/Q, E[3]) = 0, 1, 2, 3$, and the case of dimension $3$ occurs only for $d = 853$ in Example (5).
(7) We also considered negative twists. Take \( d = -2963 \). In this case, we know \( L(E, 1) \neq 0 \) and \( L(E, 1)/\Omega_E^+ = 81 \). We know from the main conjecture that the order of the 3-component of \( \Sha(E/\Q) \) is 81, but the main conjecture does not tell the structure of this group. Take \( N = 1 \) and \( \ell = 19 \). Then we compute \( a_2^{(19)} = 2753/2 \neq 0 \) (mod 3) (we have \( \vartheta_Q(19) \equiv -432S + (2753/2)S^2 \) mod (9, \( S^2 \)). Since \( c_2 = 1 \), this shows that \( a_2^{(19)} \) is in \( \text{Fitt}_2 \{ \text{Sel}(E/\Q, E[3]) \} \) by Theorem 2.4.1. Therefore, we have \( \text{Fitt}_2 \{ \text{Sel}(E/\Q, E[3]) \} = F_3 \), which implies that \( \text{Sel}(E/\Q, E[3]) \simeq (F_3)^{\oplus 2} \). This denies the possibility of \( \Sha(E/\Q)[3^\infty] \simeq (\Z/3\Z)^{\oplus 4} \), and we have

\[ \Sha(E/\Q)[3^\infty] \simeq \Z/9\Z \oplus \Z/9\Z. \]

(8) Let \( E \) be the curve \( y^2 + xy + y = x^3 + x^2 - 15x + 16 \) which is 563A1 in Cremona’s book [1]. We take \( p = 3 \). Since \( a_3 = -1 \), Tam(E) = 1, \( \mu = 0 \) and the Galois representation on \( T_3(E) \) is surjective, all the conditions we assumed are satisfied. We know \( \epsilon = 1 \) and \( L(E, 1) = 0 \). Take \( N = 1 \). We compute \( \mathcal{P}_1 = \{ 13, 61, 103, 109, 127, 139, \ldots \} \). For admissible pairs \( \{ 13, 103 \}, \{ 13, 109 \} \), we compute \( \delta_{13 \times 103} = -6819 \equiv 0 \) (mod 3) and \( \delta_{13 \times 109} = -242 \neq 0 \) (mod 3). From the latter, we know that

\[ \text{Sel}(E/\Q, E[3]) \simeq (F_3)^{\oplus 2}. \]

is bijective by Theorem 1.2.5 (3). Since \( \lambda = 2 \), the main conjecture also holds by Corollary 5.2.4. We know \( L(E, 1) = 0 \), so \( \text{Sel}(E/\Q, E[3^\infty]) \simeq (\Z_3)^{\oplus 2} \).

Numerically, we can find rational points \( P = (2, -2) \) and \( Q = (-4, 7) \) on this elliptic curve. We can check that \( E(F_{13}) \) is cyclic of order 12, \( E(F_{103}) \) is cyclic of order 84, and \( E(F_{109}) \) is cyclic of order 102. The points \( P \) and \( Q \) have the same image and do not vanish in \( E(F_{13})/3E(F_{13}) \), but the image of \( P \) in \( E(F_{109})/3E(F_{109}) \) is zero, and the image of \( Q \) in \( E(F_{109})/3E(F_{109}) \) is non-zero. This shows that \( P \) and \( Q \) are linearly independent over \( \Z_3 \), and \( s_{13 \times 109} \) is certainly bijective. Since all the elements in \( \text{Sel}(E/\Q, E[3^\infty]) \) come from the points, we have \( \Sha(E/\Q)[3^\infty] = 0 \). On the other hand, the image of \( P \) in \( E(F_{103})/3E(F_{103}) \) coincides with the image of \( Q \), so \( s_{13 \times 103} \) is not bijective. This is an example for which \( \delta_{13 \times 103} \equiv 0 \) (mod 3) and \( s_{13 \times 103} \) is not bijective.

(9) Let \( E \) be the elliptic curve \( y^2 + xy + y = x^3 + x^2 - 10x + 6 \) which has conductor 18097. We take \( p = 3 \). We know \( a_3 = -1 \), Tam(E) = 1, \( \mu = 0 \) and the Galois representation on \( T_3(E) \) is surjective, so all the conditions we assumed are satisfied. In this case, \( \epsilon = -1 \) and \( L(E, 1) = 0 \). Take \( N = 1 \). We compute \( \mathcal{P}_1 = \{ 7, 19, 31, 43, 79, \ldots, 601, \ldots \} \). We know \( \{ 7, 43, 601 \} \) is
admissible. We have \( \delta_{7 \times 43 \times 601} = -2424748 \not\equiv 0 \pmod{3} \), and \( 7 \times 43 \times 601 \) is \( \delta \)-minimal. We thank K. Matsuno heartily for his computing this value for us. The group \( E(\mathbb{F}_7) \) is cyclic of order 9 and \( E(\mathbb{F}_{43}) \) is cyclic of order 42. The point \((0, 2)\) is on this elliptic curve, and has non-zero image both in \( E(\mathbb{F}_7)/3E(\mathbb{F}_7) \) and \( E(\mathbb{F}_{43})/3E(\mathbb{F}_{43}) \). So both \( s_7 \) and \( s_{43} \) are surjective, and we can apply Theorem 1.2.5 (4) to get

\[
\delta_{7 \times 43 \times 601} : \text{Sel}(E/\mathbb{Q}, E[3]) \xrightarrow{\cong} (\mathbb{F}_3)^{\oplus 3}
\]

is bijective.

Numerically, we can find 3 rational points \( P = (0, 2) \), \( Q = (2, -1) \), \( R = (3, 2) \) on this elliptic curve, and easily check that the restriction of \( \delta_{7 \times 43 \times 601} \) to the subgroup generated by \( P, Q, R \) in \( \text{Sel}(E/\mathbb{Q}, E[3]) \) is surjective. Therefore, we have checked numerically that \( \delta_{7 \times 43 \times 601} \) is bijective. This also implies that \( \text{rank } E(\mathbb{Q}) = 3 \) since \( E(\mathbb{Q})_{\text{tors}} = 0 \). Therefore, all the elements of \( \text{Sel}(E/\mathbb{Q}, E[3]) \) come from the rational points, and we have \( \Omega(E/\mathbb{Q})[3^\infty] = 0 \).

### 5.4 A Remark on ideal class groups

We consider the classical Stickelberger element

\[
\tilde{\delta}_{\mathbb{Q}(\mu_m)}^{St} = \sum_{\sigma} \frac{1}{\sigma(a)} - \frac{a}{m} \sigma_a^{-1} \in \mathbb{Q}[\text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]
\]

(cf. (1.1)). Let \( K = \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field with conductor \( d \), and \( \chi \) be the corresponding quadratic character. Let \( m \) be a squarefree product whose prime divisors \( \ell \) split in \( K \) and satisfy \( \ell \equiv 1 \pmod{p} \). Using the above classical Stickelberger element, we define \( \tilde{\delta}_{m,K}^{St} \) by

\[
\tilde{\delta}_{m,K}^{St} = - \sum_{\sigma} \frac{a}{m \sigma_a} \chi(a) (\prod_{\ell | m} \log_{\mathbb{F}_p}(a))
\]

(cf. (1.2)). We denote by \( Cl_K \) the class group of \( K \), and define the notion "\( \tilde{\delta}_{m,K}^{St} \)-minimality" analogously. We consider the analogue of Conjecture 1.2.4 for \( \tilde{\delta}_{m,K}^{St} \) and \( \text{dim}_{\mathbb{F}_p}(Cl_K/p) \). Namely, we ask whether \( \text{dim}_{\mathbb{F}_p}(Cl_K/p) = \epsilon(m) \) for a \( \delta_{m,K}^{St} \)-minimal \( m \). Then the analogue does not hold. For example, take \( K = \mathbb{Q}(\sqrt{-23}) \) and \( p = 3 \). We know \( Cl_K \cong \mathbb{Z}/3\mathbb{Z} \). Put \( \ell_1 = 151 \) and \( \ell_2 = 211 \). We compute \( \tilde{\delta}_{\ell_1,K}^{St} = -270 \equiv 0 \pmod{3} \), \( \tilde{\delta}_{\ell_2,K}^{St} = -1272 \equiv 0 \pmod{3} \), and \( \tilde{\delta}_{\ell_1 \cdot \ell_2,K}^{St} = -415012 \equiv 2 \pmod{3} \). This means that \( \ell_1 \cdot \ell_2 \) is \( \delta_{K}^{St} \)-minimal. But, of course, we know \( \text{dim}_{\mathbb{F}_p}(Cl_K/p) = 1 < 2 = \epsilon(\ell_1 \cdot \ell_2) \).
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