LOGARITHMIC CO-HIGGS BUNDLES

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Abstract. In this article we introduce a notion of logarithmic co-Higgs sheaves associated to a simple normal crossing divisor on a projective manifold, and show their existence with nilpotent co-Higgs fields for fixed ranks and second Chern classes. Then we deal with various moduli problems with logarithmic co-Higgs sheaves involved, such as coherent systems and holomorphic triples, specially over algebraic curves of low genus.

1. Introduction

A co-Higgs sheaf on a complex manifold $X$ is a torsion-free coherent sheaf $E$ on $X$ together with an endomorphism $\Phi$ of $E$, called a co-Higgs field, taking values in the tangent bundle $T_X$ of $X$, i.e. $\Phi \in H^0(\text{End}(E) \otimes T_X)$, such that the integrability condition $\Phi \wedge \Phi = 0$ is satisfied. When $E$ is locally free, it is a generalized vector bundle on $X$, considered as a generalized complex manifold and it is introduced and developed by Hitchin and Gualtieri in [16, 13]. A naturally defined stability condition on co-Higgs sheaves allows one to study their moduli spaces and Rayan and Colmenares investigate their geometry over projective spaces and a smooth quadric surface in [21, 22] and [9]. Indeed it is expected that the existence of stable co-Higgs bundles forces the position of $X$ to be located at the lower end of the Kodaira spectrum, and Corrêa shows in [10] that a Kähler compact surface with a nilpotent stable co-Higgs bundle of rank two is uniruled up to finite étale cover. In [4, 5] the authors suggest a simple way of constructing nilpotent co-Higgs sheaves, based on Hartshorne-Serre correspondence, and obtain some (non-)existence results.

In this article we investigate the existence of nilpotent co-Higgs sheaves with a co-Higgs field vanishing in the normal direction to a given divisor of $X$: for a given arrangement $D$ of smooth irreducible divisors of $X$ with simple normal crossings, the sheaf $T_X(-\log D)$ of logarithmic vector fields along $D$ is locally free and we consider a pair $(E, \Phi)$ of a torsion-free coherent sheaf $E$ and a morphism $\Phi : E \to E \otimes T_X(-\log D)$ with the integrability condition satisfied. The pair is called a $D$-logarithmic co-Higgs sheaf and it is called 2-nilpotent if $\Phi \circ \Phi$ is trivial. Our first result is on the existence of nilpotent $D$-logarithmic co-Higgs sheaves of rank at least two.

Theorem 1.1 (Propositions 3.1, 3.2 and 3.3). Let $X$ be a projective manifold with $\dim(X) \geq 2$ and $D \subset X$ be a simple normal crossing divisor. For fixed $L \in \text{Pic}(X)$
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and an integer \( r \geq 2 \), there exists a \( 2 \)-nilpotent \( D \)-logarithmic co-Higgs sheaf \((E, \Phi)\), where \( \Phi \neq 0 \) and \( E \) is reflexive and indecomposable with \( c_1(E) \cong L \) and \( \text{rank} E = r \).

Indeed, we can strengthen the statement of Theorem 1.1 by requiring \( E \) to be locally free, in cases \( \dim(X) = 2 \) or \( r \geq \dim(X) \), due to the statement of the Hartshorne-Serre correspondence and the dimension of non-locally free locus (see Propositions 3.1 and 3.2). Moreover, in case \( \dim(X) = 2 \), we suggest an explicit number such that a logarithmic co-Higgs bundle exists for each second Chern class at least that number. We notice that the logarithmic co-Higgs sheaves constructed in Theorem 1.1 are highly unstable, which is consistent with the general philosophy on the existence of stable co-Higgs bundles (see [10, Theorem 1.1] for example).

Then we pay our attention to various different types of semistable objects involving logarithmic co-Higgs sheaves. In Section 2 we produce several examples of nilpotent semistable logarithmic co-Higgs sheaves on projective spaces and a smooth quadric surface, using a simple way of constructing in [5]. Since the logarithmic co-Higgs sheaves are co-Higgs sheaves in the usual sense with an additional vanishing condition in the normal direction of divisors, so their moduli space is a closed subvariety of the moduli of the usual co-Higgs sheaves. In Section 3 we describe two moduli spaces of logarithmic co-Higgs bundles of rank two on \( \mathbb{P}^2 \) in two cases.

Then in Section 4 we experiment with extensions of the notion of stability for co-Higgs sheaves and logarithmic co-Higgs sheaves. A key point for the study of moduli spaces was the introduction of parameters for the conditions of stability. We extend two of them, coherent systems and holomorphic triples, to co-Higgs sheaves. Specially in case of holomorphic triples, we show that any holomorphic triple admits the Harder-Narasimhan filtration in Corollary 7.1 and construct the moduli space of \( \nu_\alpha \)-stable \( D \)-logarithmic co-Higgs triples, using Simpson’s idea and quiver interpretation. We always work in cases in which there are non-trivial co-Higgs fields; so in case of dimension one we only consider projective lines and elliptic curves. We call \( \nu_\alpha \)-stability with \( \alpha \in \mathbb{R}_{>0} \), the notion of stability for holomorphic triples. In some cases we prove that the only \( \nu_\alpha \)-stable holomorphic triples are obtained in a standard way from the same holomorphic triple taking the zero co-Higgs field (see Remark 7.23).

It is certain that a logarithmic co-Higgs field is different from a map \( E \to E \otimes T_X(-D) \), unless \( X \) is a curve. We have a glimpse of this map in Section 4 for the cases \( X = \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \). On the contrary, in Section 5 we consider a map \( E \to E \otimes T_X(kD) \) with \( k > 0 \), called a meromorphic co-Higgs field, and describe semistable meromorphic co-Higgs bundles on \( \mathbb{P}^1 \).

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2. Definitions and Examples

Let \( X \) be a smooth complex projective variety of dimension \( n \geq 2 \) with the tangent bundle \( T_X \). For a fixed ample line bundle \( O_X(1) \) and a coherent sheaf \( E \) on \( X \), we denote \( E \otimes O_X(t) \) by \( E(t) \) for \( t \in \mathbb{Z} \). The dimension of cohomology group \( H^i(X, E) \) is denoted by \( h^i(X, E) \) and we will skip \( X \) in the notation, if there is no confusion. For two coherent sheaves \( E \) and \( F \) on \( X \), the dimension of \( \text{Ext}^1_X(E, F) \) is denoted by \( \text{ext}^1_X(E, F) \).
To an arrangement $\mathcal{D} = \{D_1, \ldots, D_m\}$ of smooth irreducible divisors $D_i$'s on $X$ such that $D_i \neq D_j$ for $i \neq j$, we can associate the sheaf $T_X(-\log \mathcal{D})$ of logarithmic vector fields along $\mathcal{D}$, i.e. it is the subsheaf of the tangent bundle $T_X$ whose section consists of vector fields tangent to $\mathcal{D}$. We always assume that $\mathcal{D}$ has simple normal crossings and so $T_X(-\log \mathcal{D})$ is locally free. It also fits into the exact sequence $\{1\}$

$$0 \to T_X(-\log \mathcal{D}) \to T_X \to \oplus_{i=1}^m \varepsilon_i \mathcal{O}_{D_i} \to 0,$$

where $\varepsilon_i : D_i \to X$ is the embedding.

**Definition 2.1.** A $\mathcal{D}$-logarithmic co-Higgs bundle on $X$ is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a holomorphic vector bundle on $X$ and $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X(-\log \mathcal{D})$ with $\Phi \wedge \Phi = 0$. Here $\Phi$ is called the logarithmic co-Higgs field of $(\mathcal{E}, \Phi)$ and the condition $\Phi \wedge \Phi = 0$ is called the integrability.

We say that the co-Higgs field $\Phi$ is 2-nilpotent if $\Phi$ is non-trivial and $\Phi \circ \Phi = 0$. Note that any 2-nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X(-\log \mathcal{D})$ satisfies $\Phi \wedge \Phi = 0$ and so it is a non-zero co-Higgs structure on $\mathcal{E}$, i.e. a nilpotent co-Higgs structure.

Note that if $\mathcal{D}$ is empty, then we get a usual notion of co-Higgs bundle. Indeed for each $\mathcal{D}$-logarithmic co-Higgs bundle we may consider a usual co-Higgs bundle by compositing the injection in $\{1\}$:

$$\mathcal{E} \to \mathcal{E} \otimes T_X(-\log \mathcal{D}) \to \mathcal{E} \otimes T_X.$$

Conversely, for a usual co-Higgs bundle $(\mathcal{E}, \Phi)$ we may composite the surjection in $\{1\}$ to have a map $\mathcal{E} \to \bigoplus_{i=1}^m \mathcal{E} \otimes \mathcal{O}_{D_i}$, whose vanishing would produce a logarithmic co-Higgs structure $\mathcal{E} \to \mathcal{E} \otimes T_X(-\log \mathcal{D})$. Thus our notion of logarithmic co-Higgs bundle capture the notion of a co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ vanishing in the normal direction to the divisors in the support of $\mathcal{D}$; in general it would not be asking for a map $\varphi : \mathcal{E} \to \mathcal{E} \otimes T_X(-\mathcal{D})$ when $\mathcal{D} = \{D\}$. If $\dim(X) = 1$, then we have $T_X(-\log \mathcal{D}) \cong T_X(-D)$. In Section $\{1\}$ we consider a few cases in which we take $T_X(-\mathcal{D})$ with $\mathcal{D}$ smooth, instead of $T_X(-\log \mathcal{D})$.

**Definition 2.2.** For a fixed ample line bundle $\mathcal{H}$ on $X$, a $\mathcal{D}$-logarithmic co-Higgs bundle $(\mathcal{E}, \Phi)$ is $\mathcal{H}$-semistable (resp. $\mathcal{H}$-stable) if

$$\mu(\mathcal{F}) \leq (\text{resp.} <) \mu(\mathcal{E})$$

for every coherent subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ with $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X(-\log \mathcal{D})$. Recall that the slope $\mu(\mathcal{E})$ of a torsion-free sheaf $\mathcal{E}$ on $X$ is defined to be $\mu(\mathcal{E}) := \deg(\mathcal{E}) / \text{rank} \mathcal{E}$, where $\deg(\mathcal{E}) = c_1(\mathcal{E}) \cdot \mathcal{H}^{n-1}$. In case $\mathcal{H} \cong \mathcal{O}_X(1)$ we simply call it semistable (resp. stable) without specifying $\mathcal{H}$.

**Remark 2.3.** Let $(\mathcal{E}, \Phi)$ be a semistable $\mathcal{D}$-logarithmic co-Higgs bundle. For a subsheaf $\mathcal{F} \subset \mathcal{E}$ with $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X$, we have

$$\mathcal{F} \otimes T_X(-\log \mathcal{D}) = (\mathcal{F} \otimes T_X) \cap (\mathcal{E} \otimes T_X(-\log \mathcal{D}))$$

and $\text{Im}(\Phi) \subset \mathcal{E} \otimes T_X(-\log \mathcal{D})$. Thus we get $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X(-\log \mathcal{D})$ and so $(\mathcal{E}, \Phi)$ is semistable as a usual co-Higgs bundle.

Let us denote by $\mathbf{M}_{\mathcal{D},X}(\chi(t))$ the moduli space of semistable $\mathcal{D}$-logarithmic co-Higgs bundles with Hilbert polynomial $\chi(t)$. It exists as a closed subscheme of $\mathbf{M}_X(\chi(t))$ the moduli space of semistable co-Higgs bundles with the same Hilbert polynomial, since the vanishing of co-Higgs fields in the normal direction to $\mathcal{D}$ is a
closed condition. We also denote by $M^p_{D,X}(\chi(t))$ the subscheme consisting of stable ones.

**Example 2.4.** Let $X = \mathbb{P}^1$ and $D = \{p_1, \ldots, p_m\}$ be a set of $m$ distinct points on $X$. Then we have $T_{\mathbb{P}^1}(-\log D) \cong \mathcal{O}_{\mathbb{P}^1}(2 - m)$. Let $E \cong \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(a_i)$ be a vector bundle of rank $r \geq 2$ on $\mathbb{P}^1$ with $a_1 \geq \cdots \geq a_r$ and $(E, \Phi)$ be a semistable $D$-logarithmic co-Higgs bundle, i.e. $\Phi : E \rightarrow E(2 - m)$. If $a_1 = \cdots = a_r$, then the pair $(E, \Phi)$ is semistable for any $\Phi$. If $m \geq 3$, then $\mathcal{O}_{\mathbb{P}^1}(a_1)$ would contradict the semistability of $(E, \Phi)$, unless $a_1 = \cdots = a_r$ and $m \geq 3$, then we have $\Phi = 0$ and so $(E, \Phi)$ is strictly semistable. Assume now that $m \in \{0, 1, 2\}$ and then the corresponding moduli space $M^p_{D, \mathbb{P}^1}(rt + d)$ is projective and $M^p_{D, \mathbb{P}^1}(rt + d)$ is smooth with dimension $(2 - m)r^2 + 1$, where $d = r + \sum_{i=1}^m a_i$ by [17]. The case $m = 0$ is dealt in [21] Theorem 6.1]. Now assume $m = 1$. Adapting the proof of [21] Theorem 6.1], we get Proposition [5.3 which says in the case $\ell = -1$ that the existence of a map $\Phi$ with $(E, \Phi)$ semistable implies that $a_i = a_{i+1} + 1$ for all $i$, while conversely, if $a_i \geq a_{i+1} + 1$ for all $i$, then there is a map $\Phi$ with $(E, \Phi)$ stable and the set of all such $\Phi$ is a non-empty open subset of the vector space $H^0(E_{\text{end}}(E)(1))$.

Now assume that $m = 2$ and so $\Phi \in E_{\text{nd}}(E)$. If $a_1 = \cdots = a_r$, then $\Phi$ is given by an $(r \times r)$-matrix of constants. Since the matrix has an eigenvector, the pair $(E, \Phi)$ is strictly semistable for any $\Phi$. Now assume $a_1 > a_r$ and let $h$ be the maximal integer $i$ with $a_i = 1$. Write $E \cong F \oplus G$ with $F := \oplus_{i=1}^{h_{a_1}} \mathcal{O}_{\mathbb{P}^1}(a_i)$ and $G := \oplus_{i=h_{a_1}+1}^{a_r} \mathcal{O}_{\mathbb{P}^1}(a_i)$. Since any map $F \rightarrow G$ is the zero map, we have $\Phi(F) \subseteq F$ for any $\Phi : E \rightarrow \mathcal{E}$ and so $(E, \Phi)$ is not semistable.

**2.1. Projective spaces.** In [5] we introduce a simple way of constructing nilpotent co-Higgs sheaves $(E, \Phi)$ of rank $r \geq 2$, fitting into the exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus(r-1)}_{\mathbb{X}} \rightarrow E \rightarrow \mathcal{I}_Z \otimes \mathcal{A} \rightarrow 0$$

for a two-codimensional locally complete intersection $Z \subset X$ and $\mathcal{A} \in \text{Pic}(X)$ such that $H^0(T_X \otimes \mathcal{A}^r) \neq 0$. We replace $T_X$ by $T_X(-\log D)$ for a simple normal crossing divisor $D$ in [2] to obtain 2-nilpotent $D$-logarithmic co-Higgs sheaves.

**Example 2.5.** Let $X = \mathbb{P}^n$ with $n \geq 2$ and take $D = \{D_1, \ldots, D_m\}$ with $D_i \in |\mathcal{O}_{\mathbb{P}^n}(1)|$. If $1 \leq m \leq n$, we have $T_{\mathbb{P}^n}(-\log D) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(m-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-m+1)}$ by [12], and in particular we have $h^0(T_{\mathbb{P}^n}(-\log D)(-1)) > 0$. Thus we may apply the proof of [5] Theorem 1.1] to get the following: here the invariant $x_E$ is defined to be the maximal integer $x$ such that $h^0(E(-x)) \neq 0$.

**Proposition 2.6.** The set of nilpotent maps $\Phi : E \rightarrow E \otimes T_{\mathbb{P}^n}(-\log D)$ on a fixed stable reflexive sheaf $E$ of rank two on $\mathbb{P}^n$ is an $(n - m - 1)$-dimensional vector space only if $c_1(E) + 2x_E = -3$. In the other cases the set is trivial.

**Remark 2.7.** Consider the case $m = n+1$ in Example [2.5] with $\bigcup_{i=1}^{n+1} D_i = \emptyset$. Then we have $T_{\mathbb{P}^n}(-\log D) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$. Let $E$ be a reflexive sheaf of rank $r \geq 2$ on $\mathbb{P}^n$ with a semistable (resp. stable) logarithmic co-Higgs structure $(E, \Phi)$. Note that if $\Phi$ is trivial, the semistability (resp. stability) of $(E, \Phi)$ is equivalent to the semistability (resp. stability) of $E$. Note that if $\Phi \neq 0$, then $T_{\mathbb{P}^n}(-\log D) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$. This implies that if $\Phi$ is not trivial, then $E$ is not stable. We claim that $E$ is semistable. If not, call $G$ the first step of the Harder-Narasimhan filtration of $E$. By a property of the Harder-Narasimhan filtration there is no non-zero map $G \rightarrow E/G$ and so no non-zero map $G \rightarrow (E/G) \otimes T_{\mathbb{P}^n}(-\log D)$. Thus we get $\Phi(G) \subseteq G \otimes T_{\mathbb{P}^n}(-\log D)$,
Example 2.8. Let $X = \mathbb{P}^2$ and take $\mathcal{D} = \{D\}$ with $D$ a smooth conic. Since $h^0(T_{\mathbb{P}^2}) = 8$ and $h^0(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D(2)) = 5$, we have $h^0(T_{\mathbb{P}^2}(-\log D)) > 0$ from (1). By taking $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^2}$ in [5] Equation (1) of Condition 2.2, we get a strictly semistable logarithmic co-Higgs bundle $(\mathcal{E}, \Phi)$ with a non-zero co-Higgs field $\Phi$, where $\mathcal{E}$ is strictly semistable of any arbitrary rank $r \geq 2$ and any non-negative integer $c_2(\mathcal{E}) = \deg(Z)$. Moreover, for any integer $c_2(\mathcal{E}) \geq r - 1$ we may find an indecomposable one.

Example 2.9. Let $X \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. Let $D \subset X$ be a smooth hyperplane section of $X$ with $H \subset \mathbb{P}^{n+1}$ the hyperplane such that $D = X \cap H$ and take $\mathcal{D} = \{D\}$. If $p \in \mathbb{P}^{n+1}$ is the point associated to $H$ by the isomorphism between $\mathbb{P}^{n+1}$ and its dual induced by an equation of $X$, then we have $p \notin X$ since $X$ is smooth. Letting $\pi_p : X \rightarrow \mathbb{P}^n$ denote the linear projection from $p$, we have $T_X(-\log D) \cong \pi_p^*(\Omega^{\log}_D(2))$ by [3] Corollary 4.6. Since $\Omega^{\log}_D(2)$ is globally generated, so is $T_X(-\log D)$ and in particular $H^0(T_X(-\log D)) \neq 0$. By taking $\mathcal{A} \cong \mathcal{O}_X$ in [5] Equation (1) of Condition 2.2, we get a strictly semistable logarithmic co-Higgs bundle $(\mathcal{E}, \Phi)$ with a non-zero co-Higgs field $\Phi$, where $\mathcal{E}$ is strictly semistable of any arbitrary rank $r \geq 2$.

2.2. Smooth quadric surfaces. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface and we may assume for a vector bundle $\mathcal{E}$ of rank two that

$$\det(\mathcal{E}) \in \{\mathcal{O}_X, \mathcal{O}_X(-1,0), \mathcal{O}_X(0,-1), \mathcal{O}_X(-1,-1)\}.$$ 

The case of the usual co-Higgs bundle with $\mathcal{D} = \emptyset$ is done in [9] Theorem 4.3. We assume either

(i) $\mathcal{D} \subset \{\mathcal{O}_X(1,0), \mathcal{O}_X(2,0), \mathcal{O}_X(0,1), \mathcal{O}_X(0,2)\}$, or

(ii) $\mathcal{D} = L \cup R$ with $L \in \{\mathcal{O}_X(1,0)\}$ and $R \in \{\mathcal{O}_X(0,1)\}$.

In the latter case $T_X(-\log D)$ fits into the exact sequence

$$(3) \quad 0 \rightarrow T_X(-\log D) \rightarrow \mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,2) \rightarrow \mathcal{O}_L \oplus \mathcal{O}_R \rightarrow 0,$$

because $\mathcal{O}_L(L) \cong \mathcal{O}_L$, $\mathcal{O}_R(R) \cong \mathcal{O}_R$ and $T_X \cong \mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,2)$. In particular, we have $h^0(T_X(-\log D)(i,j)) > 0$ for all $(i,j) \in \{(0,0), (0,-1), (1,0), (0,-1)\}$. We may also consider the following cases:

(iii) $\mathcal{D} = L \cup L' \cup R$ with $L, R$ as above and $L \neq L' \in \{\mathcal{O}_X(1,0)\}$; we still have $h^0(T_X(-\log D)(i,j)) > 0$ for $(i,j) \in \{(0,0), (0,-1)\}$.

(iv) $\mathcal{D} = L \cup L' \cup R \cup R'$ with $L, L', R$ as above and $R \neq R' \in \{\mathcal{O}_X(0,1)\}$.

Indeed, if $\mathcal{D}$ consists of $a$ lines in $\{\mathcal{O}_X(1,0)\}$ and $b$ lines in $\{\mathcal{O}_X(0,1)\}$, then we have $T_X(-\log D) \cong \mathcal{O}_X(2-a,0) \oplus \mathcal{O}_X(0,2-b)$ by [3] Proposition 6.2.

Assume that $\mathcal{E}$ fits into the following exact sequence as in [9] Equation (3.1)

$$(4) \quad 0 \rightarrow \mathcal{O}_X(r,d) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(r',d') \oplus \mathcal{I}_Z \rightarrow 0,$$

where $Z \subset X$ is a zero-dimensional scheme, $\det(\mathcal{E}) \cong \mathcal{O}_X(r + r', d + d')$ and $c_2(\mathcal{E}) = \deg(Z) + rd' + r'd$. Note that that any logarithmic co-Higgs bundle is co-Higgs in the usual sense and so the set of all $(c_1, c_2)$ allowed for $\mathcal{D}$ is contained in the one allowed for $\mathcal{D} = \emptyset$. In particular, if we are concerned only in $\mathcal{O}_X(1,1)$-semistability, the possible pairs $(c_1, c_2)$ are contained in the one described in [9].
Theorem 4.3]. Moreover, any existence for the case $D = L \cup R$ implies the existence for $D \in \{O_X(1,0), O_X(0,1)\}$.

(a) First assume det$(E) \cong O_X$ and we prove the existence for $c_2 \geq 0$. In this case we take $r = d = r' = d' = 0$ and the 2-nilpotent co-Higgs structure induced by $I_Z \to T_X(-\log D)$, i.e. by a non-zero section of $T_X(-\log D)$. This construction gives $(E, \Phi)$ with $E$ strictly semistable for any polarization.

(b) Assume det$(E) \cong O_X(-1,0)$ by symmetry and see the existence for $c_2 \geq 0$. In case $h^0(T_X(-\log D)(-1,0)) > 0$, we take $(r, r', d, d') = (-1,0,0,0)$ and $\Phi$ induced by a non-zero map $I_Z \to T_X(-\log D)(-1,0)$. Then $E$ is stable for every polarization, unless $Z = \emptyset$ and $E$ splits, because $Z \neq \emptyset$ would imply $h^0(E) = 0$; even when $Z = \emptyset$ and so $E \cong O_X \oplus O_X(-1,0)$, the pair $(E, \Phi)$ is stable for every polarization.

(c) Assume det$(E) \cong O_X(-1,-1)$ and take $(r, d) = (-1,0)$ and $(r', d') = (0, -1)$ with $D \in |O_X(1,0)|$. Then we have $h^0(T_X(-\log D)(-1,1)) > 0$ and $c_2(E) = \text{deg}(Z) + 1$. We get that $E$ is semistable with respect to $O_X(1,1)$.

3. Existence

Proposition 3.1. Assume dim$(X) = 2$ and let $D \subset X$ be a simple normal crossing divisor. For fixed $L \in \text{Pic}(X)$ and an integer $r \geq 2$, there exists an integer $n = n_{X,D}(L,r)$ such that for all integers $c_2 \geq n$ there is a 2-nilpotent $D$-logarithmic co-Higgs bundle $(E, \Phi)$ with $\Phi \neq 0$, where $E$ is an indecomposable vector bundle of rank $r$ with Chern classes $c_1(E) \cong L$ and $c_2(E) = c_2$. 

Proof. Fix a very ample $R \in \text{Pic}(X)$ such that

- $h^0(\omega_X \otimes (L^{\otimes (r-1)} \otimes R^\otimes r)^\vee) = 0$;
- $h^0(T_X(-\log D) \otimes L^{\otimes (r-1)} \otimes R^\otimes r) > 0$;
- $L \otimes R$ is spanned.

Set $n = n_{X,D}(r, L) := r - (r-1)(r-2)L^2 - (r-1)^2R^2 - (2r-3)(r-1)L \cdot R$.

For each $c_2 \geq n$, let $S \subset X$ be a union of general $(c_2 + r - n)$ points and consider a general extension

$$0 \to (L \otimes R)^{\oplus (r-1)} \to E \to I_S \otimes (L^{\otimes (r-2)} \otimes R^{\otimes (r-1)})^\vee \to 0.$$ 

From the choice of $R$ the Cayley-Bacharach condition is satisfied and so $E$ is locally free with $c_1(E) \cong L$ and $c_2(E) = c_2$. Now from a non-zero section in $H^0(T_X(-\log D) \otimes L^{\otimes (r-1)} \otimes R^\otimes r)$ we have a non-zero map $\varphi : I_S \otimes (L^{\otimes (r-2)} \otimes R^{\otimes (r-1)})^\vee \to L \otimes R \otimes T_X(-\log D)$, inducing a non-zero map $\Phi : E \to E \otimes T_X(-\log D)$ that is 2-nilpotent and so integrable.

Thus to complete the proof it is sufficient to prove that $E$ is indecomposable for a suitable $R$. Assume $E \cong E_1 \oplus \cdots \oplus E_k$ with $k \geq 2$ and each $E_i$ indecomposable and locally free of positive rank. Since $R$ is very ample and $L \otimes R$ is spanned, the image of the evaluation map $H^0(E) \otimes O_X \to E$ is isomorphic to $(L \otimes R)^{\oplus (r-1)}$ and its cokernel is isomorphic to $I_S \otimes (L^{\otimes (r-2)} \otimes R^{\otimes (r-1)})^\vee$. Thus, up to a permutation of the factors, we have $(L \otimes R)^{\oplus (r-1)} \cong E_1 \oplus \cdots \oplus E_{k-1} \oplus F$ with $F$ a vector bundle and $E_i/F \cong I_S \otimes (L^{\otimes (r-2)} \otimes R^{\otimes (r-1)})^\vee$. Since $E_1$ is indecomposable, we get that $E_1 \cong L \otimes R$. But since $\sharp(S) \geq r$, we have $\text{ext}^1_I(I_S \otimes (L^{\otimes (r-2)} \otimes R^{\otimes (r-1)})^\vee, O_X) \geq r$ and so we may choose $E$ so that $L \otimes R$ is not a factor of $E$. \hfill $\Box$
**Proposition 3.2.** Assume \( n = \dim(X) \geq 3 \) and let \( D \subset X \) be a simple normal crossing divisor. For a fixed \( L \in \text{Pic}(X) \) and an integer \( r \geq n \), there exists a 2-nilpotent \( D \)-logarithmic co-Higgs bundle \((E, \Phi)\), where \( E \) is an indecomposable vector bundle of rank \( r \) on \( X \) with \( \det(E) \cong L \).

**Proof.** We first assume that \( \mathcal{L}^\vee \) is very ample with

- \( h^1(\mathcal{L}^\vee) = h^2(\mathcal{L}^\vee) = 0 \), where we use the assumption \( n \geq 3 \);
- \( h^0(\mathcal{L}^\vee) \geq r - 1 \) and \( h^0(\mathcal{L}^\vee \otimes T_X(-\log D)) > 0 \).

Fix a very ample line bundle \( H \) on \( X \) such that \( h^0(H^0 \otimes \mathcal{L}^\vee) = h^1((H^0)^{\otimes 2} \otimes \mathcal{L}^\vee) = 0 \), e.g., by taking \( H \cong (\mathcal{L}^\vee)^{\otimes 2} \) and applying Kodaira’s vanishing. Let \( Y \subset X \) be a general complete intersection of two elements of \( |H| \) and then \( Y \) is a non-empty connected manifold of codimension 2 with normal bundle \( N_Y \), isomorphic to \( H^{\otimes 2}_{|Y|} \).

The line bundle \( \mathcal{R} := \wedge^2 N_Y \otimes \mathcal{L}^\vee|_Y \cong (H^{\otimes 2} \otimes \mathcal{L}^\vee)|_Y \) is a very ample line bundle on \( Y \) and we have \( h^0(Y, \mathcal{R}) \geq h^0(Y, (\mathcal{L}^\vee)|_Y) \). From the exact sequence

\[
0 \to (H^0)^{\otimes 2} \to (H^0)^{\otimes 2} \to I_Y \to 0
\]

we get \( h^0(I_Y \otimes \mathcal{L}^\vee) = 0 \) and so \( h^0(Y, \mathcal{R}) \geq h^0(Y, (\mathcal{L}^\vee)|_Y) \geq r - 1 \). Since \( \mathcal{R} \) is spanned and \( \dim(Y) = n - 2 \), a general \((n - 1)\)-dimensional linear subspace \( V \subset H^0(Y, \mathcal{R}) \) spans \( \mathcal{R} \). Hence there are linearly independent sections \( s_1, \ldots, s_{r-1} \) of \( H^0(Y, \mathcal{R}) \) spanning \( \mathcal{R} \). Since \( H^2(\mathcal{L}^\vee) = 0 \), by the Hartshorne-Serre correspondence the sections \( s_1, \ldots, s_{r-1} \) give a vector bundle \( E \) of rank \( r \) fitting into an exact sequence (see [2, Theorem 1.1])

\[
0 \to \mathcal{O}_X^{\oplus (r-1)} \to E \to I_Y \otimes \mathcal{L} \to 0.
\]

In particular we have \( \det(E) \cong \mathcal{L} \). Any non-zero section of \( H^0(\mathcal{L}^\vee \otimes T_X(-\log D)) \) gives a 2-nilpotent logarithmic co-Higgs structures on \( E \) with \( \Phi \neq 0 \). Now it remains to show that \( E \) is indecomposable. Assume \( E \cong G_1 \oplus G_2 \) with \( G_i \) non-zero. Let \( G_i' \) be the image of the evaluation map \( H^0(G_i) \otimes \mathcal{O}_X \to G_i \) for \( i = 1, 2 \). Since \( \mathcal{L}^\vee \) is very ample, we have \( h^0(E) = r - 1 \) and the image of the evaluation map \( H^0(E) \otimes \mathcal{O}_X \to E \) is isomorphic to \( \mathcal{O}_X^{\oplus (r-1)} \) and so \( G_i' \cong G_i' \oplus G_i'' \cong \mathcal{O}_X^{\oplus (r-1)} \). In particular, we have \( G_i \cong G_i' \) for some \( i \) and so at least one of the factors of \( E \) is trivial. Set \( \mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{F} \) with \( \text{rank} (\mathcal{F}) = r - 1 \). By [2, Theorem 1.1] the bundle \( \mathcal{F} \) comes from a subbundle of \( H^0(Y, \mathcal{R}) \) and so \( \mathcal{E} \) is induced by the sections \( u_1, \ldots, u_{r-2}, 0 \). Since \( H^1(\mathcal{L}^\vee) = 0 \), the uniqueness part of [2, Theorem 1.1] gives that \( s_1, \ldots, s_{r-1} \) generate the linear subspace of \( H^0(Y, \mathcal{R}) \) spanned by \( u_1, \ldots, u_{r-2} \) and so they are not linearly independent, a contradiction.

Now we drop any assumption on \( \mathcal{L} \). Take an integer \( m \gg 0 \) and set \( \mathcal{L}' := \mathcal{L} \otimes (\mathcal{H}^\vee)^{\otimes mr} \). Then we get that \( (\mathcal{L}')^\vee \) is very ample and \( H^2((\mathcal{L}')^\vee) = 0 \). By the first part there is \( (\mathcal{E}', \Phi') \) with \( \det(\mathcal{E}') \cong \mathcal{L}' \). We may take \( \mathcal{E} := \mathcal{E}' \otimes (\mathcal{H}^\vee)^{\otimes m} \) and let \( \Phi : \mathcal{E} \to \mathcal{E} \otimes T_X(-\log D) \) be the non-zero map induced by \( \Phi' \).

Allowing non-locally free sheaves, we may extend Proposition 3.2 to all ranks at least two in the following way.

**Proposition 3.3.** Under the same assumption as in Proposition 3.2 with \( 2 \leq r \leq n - 1 \), there exists a 2-nilpotent \( D \)-logarithmic co-Higgs reflexive sheaf \((E, \Phi)\), where \( E \) is indecomposable of rank \( r \) with \( \det(E) \cong \mathcal{L} \) and non-locally free locus of dimension at most \( (n - r - 1) \).
Remark 4.2. (1) It is likely that we may not apply our method of construction of 2-nilpotent co-Higgs structure to the case when $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, -1)$, because it requires a non-zero section in $H^0(T_X(-D)(-1, 0))$, which is trivial.

(2) Take $D = L \cup R$ with $L, R \in |\mathcal{O}_X(1, 0)|$ and $L \neq R$; the case with $L, R \in |\mathcal{O}_X(0, 1)|$ is similar. Then the existence for the case $c_1(\mathcal{E}) = \mathcal{O}_X(0, 0)$ can be done for any $c_2 \geq 0$ as above.

4. VANISHING ALONG DIVISORS

As observed, the notion of logarithmic co-Higgs bundle is not asking for a map $\varphi : \mathcal{E} \to \mathcal{E} \otimes T_X(-D)$ if $\dim(X) \geq 2$. In this section we study vector bundles of rank two on a projective plane and a smooth quadric surface with sections in $H^0(\mathcal{E}^{\text{ad}}(\mathcal{E}) \otimes T_X(-D))$.

4.1. Projective plane. Let $X = \mathbb{P}^2$ and take $D \in |\mathcal{O}(1)|$ a projective line. Then we have $T_{\mathbb{P}^2}(-D) = T_{\mathbb{P}^2}(-1)$ and so $h^0(T_{\mathbb{P}^2}(-D)) = 3$. We may give a 2-nilpotent co-Higgs structure on a vector bundle $\mathcal{E}$ of rank 2 fitting into the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_Z \to 0$$

from a non-zero section in $H^0(T_{\mathbb{P}^2}(-D))$. Thus there exists a strictly semistable co-Higgs bundle of rank two for all $c_2 \geq 0$, which is indecomposable for $c_2 > 0$. Indeed for any such bundles with positive $c_2$ we have a three-dimensional vector space of 2-nilpotent co-Higgs structures. On the contrary we have some results on non-existence of co-Higgs bundles on projective spaces in [5, Section 3]. Applying the same argument to $T_{\mathbb{P}^n}(-1)$, we get the following, as in Proposition 2.4.

Proposition 4.1. If $\mathcal{E}$ is a stable reflexive sheaf of rank two on $\mathbb{P}^n$ with $n \geq 2$, then any nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^n}(-1)$ is trivial.

4.2. Quadric surface. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and take $D \in |\mathcal{O}_X(1, 0)|$; by symmetry the case $D \in |\mathcal{O}_X(0, 1)|$ is similar. We have $T_X(-D) \cong \mathcal{O}_X(1, 0) \oplus \mathcal{O}_X(-1, 2)$.

(a) In case $\det(\mathcal{E}) \cong \mathcal{O}_X$ we prove the existence for $c_2 \geq 0$. By taking $r = d = r' = d' = 0$, we obtain a 2-nilpotent co-Higgs structure induced by $\mathcal{I}_Z \to T_X(-D)$, i.e. by a non-zero section of $T_X(-D)$. This construction gives $(\mathcal{E}, \Phi)$ with $\mathcal{E}$ strictly semistable for any polarization.

(b) In case $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, 0)$ we also see the existence for $c_2 \geq 0$. Since $h^0(T_X(-D)(-1, 0)) > 0$, we take $(r, r', d, d') = (-1, 0, 0, 0)$ and $\Phi$ induced by a non-zero map $\mathcal{I}_Z \to T_X(-D)(-1, 0)$. Then $\mathcal{E}$ is stable for every polarization, unless $Z = \emptyset$ and $\mathcal{E}$ splits, because $Z \neq \emptyset$ would imply $h^0(\mathcal{E}) = 0$; even when $Z = \emptyset$ and so $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-1, 0)$, the pair $(\mathcal{E}, \Phi)$ is stable for every polarization.

(c) Assume $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, -1)$ and take $(r, d) = (-1, 0)$ and $(r', d') = (0, -1)$ with $D \in |\mathcal{O}_X(1, 0)|$. Note that $h^0(T_X(-D)(-1, 1)) > 0$ and $c_2(\mathcal{E}) = \deg(Z) + 1$. Then we get that $\mathcal{E}$ is semistable with respect to $\mathcal{O}_X(1, 1)$.

Remark 4.2. (1) It is likely that we may not apply our method of construction of 2-nilpotent co-Higgs structure to the case when $\det(\mathcal{E}) \cong \mathcal{O}_X(0, -1)$, because it requires a non-zero section in $H^0(T_X(-D)(-1, 0))$, which is trivial.

(2) Take $D = L \cup R$ with $L, R \in |\mathcal{O}_X(1, 0)|$ and $L \neq R$; the case with $L, R \in |\mathcal{O}_X(0, 1)|$ is similar. Then the existence for the case $c_1(\mathcal{E}) = \mathcal{O}_X(0, 0)$ can be done for any $c_2 \geq 0$ as above.
5. Extension of co-Higgs bundles

Fix an ample line bundle $\mathcal{H}$ on $X$ and a vector bundle $\mathcal{G}$. Then we may define $\mathcal{H}$-(semi)stability for a pair $(\mathcal{E}, \Phi)$ with $\mathcal{E}$ a torsion-free sheaf and $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{G}$, similarly as in Definition 2.2 with $\mathcal{G}$ instead of $T_X(-\log D)$. Then the definition of (logarithmic) co-Higgs bundle is obtained by taking $\mathcal{G} \in \{ T_X, T_X(-\log D), T_X(-D) \}$ with the integrability condition $\Phi \wedge \Phi = 0$. Note that it is enough to check the integrability condition on a non-empty open subset $U$ of $X$.

**Definition 5.1.** Fix an effective divisor $D \subset X$ and a positive integer $k$, for which we take $\mathcal{G} := T_X(kD)$. A pair $(\mathcal{E}, \Phi)$ is called a **meromorphic co-Higgs sheaf** with poles of order at most $k$ contained in $D$, if it satisfies the integrability condition on $U := X \setminus D$.

Via the inclusion $T_X \hookrightarrow T_X(kD)$ induced by a section of $O_X(kD)$ with $kD$ as its zeros, we see that any co-Higgs sheaf is also a meromorphic co-Higgs for any $k$ and $D$. A meromorphic co-Higgs sheaf with poles contained in $D$ induces an ordinary co-Higgs sheaf $(\mathcal{F}, \varphi)$ on the non-compact manifold $U$ and our definition of meromorphic co-Higgs sheaves captures the extension of $(\mathcal{F}, \varphi)$ to $X$ with at most poles on $D$ of order at most $k$.

**Remark 5.2.** We may generalize the definition of a meromorphic co-Higgs sheaf as follows: take $D = \cup_{i=1}^{s} D_i$ with each $D_i$ irreducible and consider $\sum_{i=1}^{s} k_i D_i$, $k_i$ a positive integer, instead of $kD$. Then we get the co-Higgs sheaves $(\mathcal{F}, \varphi)$ on $X \setminus D$, which extends meromorphically to $X$ with poles of order at most $k$ on each $D_i$.

Our method used in constructing 2-nilpotent co-Higgs sheaves (see [5, Condition 2.2]) can be applied to construct 2-nilpotent meromorphic co-Higgs sheaves, if $h^0(T_X(kD)) > 0$; we may easily check when the construction gives locally free ones. In the set-up of Sections 2.2 and 4 we immediately see how to construct examples filling in several Chern classes.

Assume that $\dim(X) = 1$ and let $D = p_1 + \cdots + p_s$ be $s$ distinct points on $X$. Set $\ell := \deg(\sum_{i=1}^{s} k_i p_i)$ and $r := \text{rank}(\mathcal{E})$. We adapt the proof of [21, Theorem 6.1] with only very minor modifications to prove the following result. To cover the case needed in Example 2.3, we allow as $\ell$ an integer at least $-1$.

**Proposition 5.3.** Let $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ be a vector bundle of rank $r \geq 2$ on $\mathbb{P}^1$ with $a_1 \geq \cdots \geq a_r$.

(i) If $(\mathcal{E}, \Phi)$ is semistable with a map $\Phi : \mathcal{E} \to \mathcal{E}(2 + \ell)$, then we have $a_{i+1} \geq a_i - \ell - 2$ for each $i \leq r - 1$.

(ii) Conversely, if $a_{i+1} \geq a_i - \ell - 2$ for each $i \leq r - 1$, then there is a map $\Phi : \mathcal{E} \to \mathcal{E}(2 + \ell)$ such that no proper subbundle $\mathcal{F} \subset \mathcal{E}$ satisfies $\Phi(\mathcal{F}) \subseteq \mathcal{F}(2 + \ell)$, and in particular $(\mathcal{E}, \Phi)$ is stable. The set of all such $\Phi$ is non-empty open subset of the vector space $H^0(\text{End}(\mathcal{E})(2 + \ell))$.

**Proof.** Assume the existence of an integer $i$ such that $a_{i+1} \leq a_i - \ell - 3$ and take $\Phi : \mathcal{E} \to \mathcal{E}(2 + \ell)$. Set $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ with $\mathcal{F} := \oplus_{j=1}^{i} \mathcal{O}_{\mathbb{P}^1}(a_j)$ and $\mathcal{G} := \oplus_{j=i+1}^{s} \mathcal{O}_{\mathbb{P}^1}(a_j)$. Since any map $\mathcal{F} \to \mathcal{G}(2 + \ell)$ is the zero map, we have $\Phi(\mathcal{F}) \subseteq \mathcal{F}(2 + \ell)$ and so $(\mathcal{E}, \Phi)$ is not semistable.

Now assume $a_{i+1} \geq a_i - \ell - 2$ for all $i$. Write $\Phi$ as an $(r \times r)$-matrix $B$ with entries $b_{i,j} \in \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_i), \mathcal{O}_{\mathbb{P}^1}(a_j + 2 + \ell))$. For fixed homogeneous coordinates $z_0, z_1$ on $\mathbb{P}^1$ with $\infty = [1 : 0]$ and $0 = [0 : 1]$, see a homogeneous polynomial of
degree $d$ in the variables $z_0, z_1$ as a polynomial of degree at most $d$ in the variable $z := z_0/z_1$. Take

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & z \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

so that $b_{i,j} = 0$ unless either $(i,j) = (1,r)$ or $j = i+1$; we take $b_{i,i+1} = 1$ for all $i$, i.e. the elements of $\mathbb{C}[z]$ associated to $z_1^{a_i+1-a_i+2+\ell}$, and $b_{1,r} = z$, the element of $\mathbb{C}[z]$ associated to $z_0 z_1^{a_0-a_0+1+\ell}$. Then there is no proper subbundle $F \subseteq E$ with $\Phi(F) \subseteq F(2+\ell)$, because the characteristic polynomial of $B$ is $\det(tI - B) = (-1)^r z + t^r$, which is irreducible in $\mathbb{C}[z,t]$.

\[\square\]

**Remark 5.4.** Assume the genus $g$ of $X$ is at least $2$ and that $2 - 2g + \ell < 0$. Then there exists no semistable meromorphic co-Higgs bundle $(E, \Phi)$ with $\Phi \neq 0$. Indeed, for any pair $(E, \Phi)$, the map $\Phi$ would be a non-zero map between two semistable vector bundles with the target having lower slope.

## 6. Moduli over projective plane

Let $X = \mathbb{P}^n$ and fix $D = \{D\}$ with $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$. Then we have $T_{\mathbb{P}^n}(-\log D) \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$ and

$$\Phi = (\varphi_1, \ldots, \varphi_n) : E \rightarrow E \otimes T_{\mathbb{P}^n}(-\log D)$$

with $\varphi_i : E \rightarrow \mathcal{O}(1)$ for $i = 1, \ldots, n$. Assume that $(E, \Phi)$ is a semistable co-Higgs bundle of rank $r$ along $D$. If $E \cong \oplus_{i=1}^n \mathcal{O}(a_i)$ is a direct sum of line bundles on $\mathbb{P}^n$ with $a_1 \geq a_i+1$ for all $i$, then we get $a_i \leq a_{i+1} + 1$ for all $i$ by adapting the proof of [21] Theorem 6.1. Thus in case rank$(E) = r = 2$, by a twist we fall into two cases: $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

We denote by $\text{End}_D(E)$ the kernel of the trace map $\text{End}(E) \rightarrow \mathcal{O}_X$, the trace-free part, and then we have

$$\text{End}(E) \otimes T_X(-\log D) \cong (\text{End}_D(E) \otimes T_X(-\log D)) \oplus T_X(-\log D).$$

Thus any co-Higgs field $\Phi$ can be decomposed into $\Phi_1 + \Phi_2$ with $\Phi_1 \in H^0(\text{End}_D(E) \otimes T_X(-\log D))$ and $\Phi_2 \in H^0(T_X(-\log D))$. Note that $(E, \Phi)$ is (semi)stable if and only if $(E, \Phi_1)$ is (semi)stable. Thus we may pay attention only to trace-free logarithmic co-Higgs bundles. Let us denote by $M_D(c_1, c_2)$ the moduli of semistable trace-free $D$-logarithmic co-Higgs bundles of rank two on $\mathbb{P}^2$ with Chern classes $(c_1, c_2)$. In case $D = \emptyset$ we simply denote the moduli space by $M(c_1, c_2)$.

**Proposition 6.1.** $M_D(-1,0)$ is isomorphic to the total space of $\mathcal{O}_D(-2)^{\oplus 6}$.

**Proof.** By [14] Lemma 3.2 $E$ is not semistable for $(E, \Phi) \in M_D(-1,0)$ and so we get an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(t) \rightarrow E \rightarrow I_Z(-t-1) \rightarrow 0$ with $t \geq 0$. Here $\Phi(\mathcal{O}_{\mathbb{P}^2}(t)) \subset I_Z(-t)$ is a non-trivial subsheaf and so we get $t = 0$ and $Z = \emptyset$. Thus we get $E \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Then following the proof of [22] Theorem 5.2 verbatim, we see that

$$M_D(-1,0) \cong H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \times (H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \setminus \{0\}) / C^*,$$

where $C^*$ acts on $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ with weight $-2$ and on $H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \setminus \{0\}$ with weight $1$. Thus we get that $M_D(-1,0)$ is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}$. Indeed,
from the sequence \((1)\) twisted by \(-1\), we can identify \(PH^0(\mathcal{O}_D^{\oplus 2})\) with \(D\) and so \(M_\alpha(-1,0)\) can be obtained by restricting \(\mathcal{O}_{D_2}(-2)^{\oplus 6}\) to \(D\) as a closed subscheme of \(M(-1,0)\), which is isomorphic to the total space of \(\mathcal{O}_{D_2}(-2)^{\oplus 6}\) (see \cite[Theorem 5.2]{22}).

Recall in \cite[Page 1447]{22} that \(M(0,0)\) is 8-dimensional and non-isomorphic to \(M(-1,0)\), with an explicitly described open dense subset. On the contrary to Proposition \ref{prop:7.1}, we obtain two-codimensional subspace \(M_D(0,0)\) of \(M(0,0)\).

**Proposition 6.2.** \(M_D(0,0)\) contains the total space of \(\mathcal{O}_{D_2}(-2)\) with the zero section contracted to a point, as an open dense subset.

**Proof.** Take \((\mathcal{E}, \Phi) \in M_D(0,0)\). From \(c_2 = c_2^2\), we get that \(\mathcal{E}\) is not stable and so it fits into the following exact sequence

\[0 \rightarrow \mathcal{O}_{D_2}(t) \rightarrow \mathcal{E} \rightarrow I_Z(-t) \rightarrow 0\]

with \(t \geq 0\) and \(\deg(Z) = t^2\). First assume \(t > 0\). Since every map \(\mathcal{O}_{D_2}(t) \rightarrow I_Z(-t) \otimes \mathcal{O}_{D_2}(-\log D)\) is the zero-map, we get \(\Phi(\mathcal{O}_{D_2}(t)) \subset \mathcal{O}_{D_2}(t) \otimes \mathcal{O}_{D_2}(-\log D)\), contradicting the semistability of \((\mathcal{E}, \Phi)\). Now assume \(t = 0\) and so we get \(\mathcal{E} \cong \mathcal{O}_{D_2}^{\oplus 2}\).

Then we follow the argument in \cite[Theorem 5.3]{22} to get the assertion. \(\square\)

**7. Coherent system and Holomorphic triple**

If \(\mathcal{F} \subset \mathcal{E}\) is a non-trivial subsheaf, then its saturation \(\mathcal{F}^\rho\) is defined to be the maximal subsheaf of \(\mathcal{E}\) containing \(\mathcal{F}\) with rank \(\mathcal{F} = \text{rank} \mathcal{F}\); \(\mathcal{F}\) is the only subsheaf of \(\mathcal{E}\) containing \(\mathcal{F}\) with \(\mathcal{E}/\mathcal{F}\) torsion-free.

**7.1. Coherent system.** Inspired by the theory of coherent systems on smooth algebraic curves in \cite{S}, we consider the following definition. Let \(\mathcal{E}\) be a torsion-free sheaf of rank \(r \geq 2\) on \(X\) and \((\mathcal{E}, \Phi)\) be a \(\mathcal{D}\)-logarithmic co-Higgs structure. Then we define a set

\[S = S(\mathcal{E}, \Phi) := \{(\mathcal{F}, \mathcal{G}) \mid 0 \subseteq \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{E} \text{ with } \Phi(\mathcal{F}) \subseteq \mathcal{G} \otimes T_X(-\log D)\}.

For a fixed real number \(\alpha \geq 0\) and \((\mathcal{F}, \mathcal{G}) \in S\), set

\[\mu_\alpha(\mathcal{F}, \mathcal{G}) = \mu(\mathcal{F}) + \alpha \left(\frac{\text{rank} \mathcal{F}}{\text{rank} \mathcal{G}}\right),\]

\[\mu'_\alpha(\mathcal{F}, \mathcal{G}) = \mu(\mathcal{F}) + \alpha \left(\frac{\text{rank} \mathcal{F}}{\text{rank} \mathcal{F} + \text{rank} \mathcal{G}}\right)\]

Note that \(\mu_\alpha(\mathcal{E}, \mathcal{E}) = \mu(\mathcal{E}) + \alpha\) and \(\mu'_\alpha(\mathcal{E}, \mathcal{E}) = \mu(\mathcal{E}) + \alpha/2\). From now on we use \(\mu_\alpha\), but \(\mu'_\alpha\) does the same job. In general, we have \(\mu_\alpha(\mathcal{F}, \mathcal{G}) \leq \mu(\mathcal{F}) + \alpha\) for \((\mathcal{F}, \mathcal{G}) \in S\) and equality holds if and only if rank \(\mathcal{F} = \text{rank} \mathcal{G}\), i.e. \(\mathcal{G}\) is contained in the saturation \(\mathcal{F}^\rho\) of \(\mathcal{F}\) in \(\mathcal{E}\).

**Definition 7.1.** The pair \((\mathcal{E}, \Phi)\) is said to be \(\mu_\alpha\)-stable (resp. \(\mu_\alpha\)-semistable) if \(\mu_\alpha(\mathcal{F}, \mathcal{G}) < \mu_\alpha(\mathcal{E}, \mathcal{E})\) (resp. \(\mu_\alpha(\mathcal{F}, \mathcal{G}) \leq \mu_\alpha(\mathcal{E}, \mathcal{E})\)) for all \((\mathcal{F}, \mathcal{G}) \in S \setminus \{(\mathcal{E}, \mathcal{E})\}\).

A similar definition is given with \(\mu'_\alpha\).

Note that if \(\mathcal{E}\) is semistable (resp. stable), then a pair \((\mathcal{E}, \Phi)\) is \(\mu_\alpha\)-semistable (resp. \(\mu_\alpha\)-stable) for any \(\alpha\) and \(\Phi\). The converse also holds for \(\Phi = 0\).
Remark 7.2. We have $\Phi(\mathcal{F}) \subseteq \tilde{G} \otimes T_X(-\log D)$ for $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$ and so to test the $\mu_\alpha$-(semi)stability of $(\mathcal{E}, \Phi)$, it is sufficient to test the pairs $(\mathcal{F}, \mathcal{G}) \in \mathcal{S} \setminus \{(\mathcal{E}, \mathcal{E})\}$ with $\tilde{G}$ saturated in $\mathcal{E}$. Moreover, if $\mathcal{G}$ is saturated in $\mathcal{E}$, then $\mathcal{G} \otimes T_X(-\log D)$ is saturated in $\mathcal{E} \otimes T_X(-\log D)$. Since $\Phi(\mathcal{F})$ is a subsheaf of $\Phi(\mathcal{F})$ with the same rank we have $\Phi(\mathcal{F}) \subseteq \mathcal{G} \otimes T_X(-\log D)$. So to test the $\mu_\alpha$-(semi)stability of $(\mathcal{E}, \Phi)$ it is sufficient to test the pairs $(\mathcal{F}, \mathcal{G}) \in \mathcal{S} \setminus \{(\mathcal{E}, \mathcal{E})\}$ with both $\mathcal{F}$ and $\mathcal{G}$ saturated in $\mathcal{E}$.

Lemma 7.3. If $(\mathcal{E}, \Phi)$ is not semistable (resp. stable), then it is not $\mu_\alpha$-semistable (resp. not $\mu_\alpha$-stable) for any $\alpha$.

Proof. Take $\mathcal{F} \subset \mathcal{E}$ such that $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X(-\log D)$ and $\mu(\mathcal{F}) > \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$). We have $(\mathcal{F}, \mathcal{F}) \in \mathcal{S}$ and so $\mu_\alpha(\mathcal{F}, \mathcal{F}) = \mu(\mathcal{F}) + \alpha > (\text{resp. } \geq) \mu(\mathcal{E}) + \alpha = \mu_\alpha(\mathcal{E}, \mathcal{E})$, proving the assertion. □

Remark 7.4. Lemma 7.3 shows that $\mu_\alpha$-stability is stronger than the stability of the pairs $(\mathcal{E}, \Phi)$ in the sense of [20, 21, 22] and so they form a bounded family if we fix the Chern classes of $\mathcal{E}$. However, if $(\mathcal{E}, \Phi)$ is not $\mu_\alpha$-semistable, a pair $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$ with $\mu_\alpha(\mathcal{F}, \mathcal{G}) > \mu(\mathcal{E}) + \alpha$ and maximal $\mu_\alpha$-slope may have $\text{rank}(\mathcal{G}) > \text{rank}(\mathcal{F})$, i.e. $\Phi(\mathcal{F}) \not\subseteq \mathcal{F} \otimes T_X(-\log D)$ and so we do not define the Harder-Narasimhan filtration of $\mu_\alpha$-unstable pairs $(\mathcal{E}, \Phi)$.

Proposition 7.5. Let $(\mathcal{E}, \Phi)$ be a $\mathcal{D}$-logarithmic co-Higgs bundle on $X$ with $\mathcal{E}$ not semistable. Then there exist two positive real numbers $\beta$ and $\gamma$ such that

(i) $(\mathcal{E}, \Phi)$ is not $\mu_\alpha$-semistable for all $\alpha < \beta$, and

(ii) if $(\mathcal{E}, \Phi)$ is semistable in the sense of Definition 2.2, it is $\mu_\alpha$-semistable for all $\alpha > \gamma$.

Proof. Assume that $\mathcal{E}$ is not semistable and take a subsheaf $\mathcal{G}$ with $\mu(\mathcal{G}) > \mu(\mathcal{E})$. Note that $(\mathcal{G}, \mathcal{E}) \in \mathcal{S}$. Then there exists a real number $\beta > 0$ such that $\mu_\alpha(\mathcal{G}, \mathcal{E}) > \mu(\mathcal{E}) + \alpha = \mu_\alpha(\mathcal{E}, \mathcal{E})$ for all $\alpha$ with $0 < \alpha < \beta$. Thus $(\mathcal{E}, \Phi)$ is not $\mu_\alpha$-semistable if $\alpha < \beta$.

Now assume that $\mathcal{E}$ is not semistable, but that $(\mathcal{E}, \Phi)$ is semistable. Define

$$\Delta = \{\text{the saturated subsheaves } \mathcal{A} \subset \mathcal{E} \mid \mu(\mathcal{A}) > \mu(\mathcal{E})\}.$$

Let $\mu_{\text{max}}(\mathcal{E})$ be the maximum of the slopes of subsheaves of $\mathcal{E}$, which exists as a finite real number by the existence of the Harder-Narasimhan filtration of $\mathcal{E}$. Since $\mathcal{E}$ is not semistable, we have $\mu_{\text{max}}(\mathcal{E}) > \mu(\mathcal{E})$ and set $\gamma := r(\mu_{\text{max}}(\mathcal{E}) - \mu(\mathcal{E})) > 0$. Fix any real number $\alpha \geq \gamma$. Now take $\mathcal{A} \in \Delta$ and set $s := \text{rank} \mathcal{A}$. Since $(\mathcal{E}, \Phi)$ is semistable, we get $\text{rank} \mathcal{B} > s$. Thus we have

$$\mu_\alpha(\mathcal{A}, \mathcal{B}) \leq \mu(\mathcal{A}) + \alpha s/(s + 1) \leq \mu(\mathcal{A}) + \alpha(r - 1)/r \leq \mu_\alpha(\mathcal{E}, \mathcal{E}),$$

and so $(\mathcal{E}, \Phi)$ is $\mu_\alpha$-semistable for all $\alpha \geq \gamma$. □

Remark 7.6. For $s = 1, \ldots, r - 1$, let $\Delta_s$ be the set of all $\mathcal{G} \in \Delta$ with rank $s$. If $\mu(\mathcal{G}) < \mu_{\text{max}}(\mathcal{E})$ for all $\mathcal{G} \in \Delta_{r-1}$, we may use a lower real number instead of $\gamma$ in the proof of Proposition 7.3.

Example 7.7. Let $X = \mathbb{P}^1$ and take $\mathcal{D} = \{p\}$ with $p$ a point. Then we have $T_{\mathbb{P}^1}(-\log D) \cong T_{\mathbb{P}^1}(-p) \cong \mathcal{O}_p(1)$. Let $(\mathcal{E}, \Phi)$ be a semistable $\mathcal{D}$-logarithmic co-Higgs bundle of rank $r \geq 2$ on $\mathbb{P}^1$ with $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_p(a_i)$ with $a_1 \geq \cdots \geq a_r$ and $a_i - a_{i+1} \leq 1$ for all $i = 1, \ldots, r + 1$ as in Example 2.4. We assume that $\mathcal{E}$ is
not semistable, i.e. $a_r < a_1$. The value $\gamma$ in Proposition 7.3 could depend on $\Phi$, although it is the same for all general $\Phi$. Up to a twist we may assume $a_1 = 0$. We have $\mu(\mathcal{E}) = c_1/r$ with $c_1 = a_1 + \cdots + a_r$. For each $s = 1, \ldots, r - 1$, set $b_s = (a_1 + \cdots + a_s)/s$ and define

$$\gamma_0 := \max_{1 \leq s \leq r - 1} (s + 1)(b_s - c_1/r).$$

We have $\mu(\mathcal{F}) \leq b_s$ for all $\mathcal{F} \in \Delta_s$ and so $\mu_\alpha(\mathcal{F}, \mathcal{G}) \leq \mu_\alpha(\mathcal{E}, \mathcal{E})$ for all $(\mathcal{F}, \mathcal{G})$ with rank $\mathcal{F} = s$ and $\Phi(\mathcal{F}) \not\subseteq \mathcal{F} \otimes T_{\mathbb{P}^1}(-\log \mathcal{D})$. Hence $(\mathcal{E}, \Phi)$ is $\mu_\alpha$-semistable for all $\alpha \geq \gamma_0$.

**Example 7.8.** Similarly as in Example 7.7 we take $X = \mathbb{P}^1$ and $\mathcal{D} = \emptyset$. Then we have $T_{\mathbb{P}^1}(-\log \mathcal{D}) \cong T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. We argue as in Example 7.7, except that now we only require that $a_i - a_{i+1} \leq 2$ for all $i = 1, \ldots, r - 1$.

**Example 7.9.** Take $X = \mathbb{P}^n$ with $n \geq 2$ and assume that $(\mathcal{E}, \Phi)$ is a semistable logarithmic co-Higgs reflexive sheaf of rank two with $\mathcal{E}$ not semistable. Up to a twist we may assume $c_1(\mathcal{E}) \in \{-1, 0\}$. Set $c_1 := c_1(\mathcal{E})$. Since $\mathcal{E}$ is not semistable, we have an exact sequence

$$(6) \quad 0 \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{E} \to \mathcal{I}_Z(c_1 - t) \to 0$$

with either $Z = \emptyset$ or $\dim(Z) = n - 2$, and $t \geq 0$ and $t > 0$ if $c_1(\mathcal{E}) = 0$. Since $(\mathcal{E}, \Phi)$ is semistable, there is no saturated subsheaf $A \subset \mathcal{E}$ of rank one with $(A, A) \in \mathcal{S}$ and $\mu(A) > -1$. Note that $\mu_\alpha(\mathcal{O}_{\mathbb{P}^n}(t), \mathcal{E}) = t + \alpha/2$ and so $(\mathcal{E}, \Phi)$ is $\mu_\alpha$-stable (resp. $\mu_\alpha$-semistable) if and only if $\alpha > 2t - c_1$ (resp. $\alpha \geq 2t - c_1$).

Now we discuss the existence of such a pair $(\mathcal{E}, \Phi)$. Since $(\mathcal{E}, \Phi)$ is semistable, we should have $\Phi(\mathcal{O}_{\mathbb{P}^n}(t)) \not\subseteq \mathcal{O}_{\mathbb{P}^n}(t) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$ and so there is a non-zero map $\mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{I}_Z(c_1 - t) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$. Since $t > c_1 - t$ and $h^0(T_{\mathbb{P}^n}(-2)) = 0$, we get $t = 0$ and $c_1 = -1$. Then we also get $H^0(\mathcal{I}_Z(-1) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})) \neq 0$, which gives restrictions on the choice of $\mathcal{D}$ and $Z$. Assume that $\mathcal{D} = \{D\}$ with $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ a hyperplane, so that $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$. In this case we get $Z = \emptyset$ and so $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$. See Proposition 6.1 for the associated moduli space in case $n = 2$.

### 7.2. Holomorphic triple

We may also consider a holomorphic triple of logarithmic co-Higgs bundles and define its semistability as in [7].

**Definition 7.10.** A holomorphic triple of $\mathcal{D}$-logarithmic co-Higgs bundles is a triple $((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$, where each $(\mathcal{E}_i, \Phi_i)$ is a $\mathcal{D}$-logarithmic co-Higgs sheaf with each $\mathcal{E}_i$ torsion-free on $X$ and $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a map of sheaves such that $\Phi_2 \circ f = f \circ \Phi_1$, where $f : \mathcal{E}_1 \otimes T_X(-\log \mathcal{D}) \to \mathcal{E}_2 \otimes T_X(-\log \mathcal{D})$ is the map induced by $f$.

For any real number $\alpha \geq 0$, define the $\nu_\alpha$-slope of a triple $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ to be the $\nu_\alpha$-slope of the triple $((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ in the sense of [7], i.e.

$$\nu_\alpha((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f) = \deg_\alpha(A) \frac{\deg_{\alpha}(\mathcal{A})}{\deg_{\alpha}(\mathcal{A})}$$

where $\deg_{\alpha}(\mathcal{A}) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_2) + \alpha \deg \mathcal{E}_1$. A holomorphic subtriple $\mathcal{B} = ((\mathcal{F}_1, \Psi_1), (\mathcal{F}_2, \Psi_2), g)$ of $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ is a holomorphic triple with $\mathcal{F}_i \in \mathcal{E}_i, \Psi_i = \Phi_i|\mathcal{F}_i$ and $g = f|\mathcal{F}_1$. Since $\Phi_i$ is integrable, so is $\Psi_i$. 


Remark 7.11. As before, we may use the slope $\nu_\alpha$ to define the $\nu_\alpha$-(semi)stability for $D$-logarithmic co-Higgs triples. If $h : A \to B$ is a non-zero map of $\nu_\alpha$-semistable holomorphic triples, then we have $\nu_\alpha(B) \geq \mu_\alpha(A)$. Moreover, if $A$ is $\nu_\alpha$-stable, then either $\nu_\alpha(B) > \nu_\alpha(A)$ or $h$ is injective; in addition, if $B$ is also $\nu_\alpha$-stable, then $h$ is an automorphism.

Remark 7.12. The degenerate holomorphic triple $((E_1, \Phi_1), (E_2, \Phi_2), 0)$ with $f = 0$ is $\nu_\alpha$-semistable if and only if $\alpha = \mu(E_2) - \mu(E_1)$ and both $(E_i, \Phi_i)$’s are semistable as in [6] Lemma 3.5. Moreover such triples are not $\nu_\alpha$-stable (see [6] Corollary 3.6). Note that if $\Phi_1 = \Phi_2 = 0$, then we fall into the usual holomorphic triples. We also have an analogous statement for the case $r_2 = \text{rank} E_2 = 1$ as in [6] Lemma 3.7.

Remark 7.13. For subtriples $B$ and $B'$ of $A$, we may define their sum and intersection $B + B'$ and $B \cap B'$; let $B = ((F_1, \Psi_1), (F_2, \Psi_2), g)$ and $B' = ((F'_1, \Psi'_1), (F'_2, \Psi'_2), g)$. Then we may use $F_1 + F'_1$ and $F_1 \cap F'_1$ with the restrictions of $\Phi_i$ and $f$ to them. Now call $\bar{F}_i$ the saturation of $F_i$ in $E_i$. Since $\bar{F}_1 \cap T_X(-\log D)$ is saturated in $E_i \cap T_X(-\log D)$, we have $\Phi_i(\bar{F}_i) \subseteq \bar{F}_i \cap T_X(-\log D)$. Since $f(\bar{F}_1) \subseteq \bar{F}_2$, we have $f(\bar{F}_1) \subseteq \bar{F}_2$ and so we may also define the saturation $\bar{B}$ of $B$ with $\nu_\alpha(\bar{B}) = \nu_\alpha(B)$.

Fix $\alpha \in \mathbb{R}_{>0}$ and let $A = ((E_1, \Phi_1), (E_2, \Phi_2), f)$ be a holomorphic triple. We define $\beta(A)$ to be the maximum of the set of the $\nu_\alpha$-slopes of all subtriples of $A$ and let

$$B := \{ B \subseteq A \mid \nu_\alpha(B) = \beta(A) \}.$$  

Lemma 7.14. The set of the $\nu_\alpha$-slopes of all subtriples of $A$ is upper bounded and so $\beta(A)$ exists. Moreover, the set $B$ has a unique maximal element.

Proof. The ranks of any non-zero subsheaf of $E_i$ is upper bounded by $r_i := \text{rank} E_i$ and lower bounded by 1. The existence of the Harder-Narasimhan filtration of $E_i$ gives the existence of positive rational numbers $\gamma_i$ with denominators between 1 and $r_i$ such that $\mu(F_1) \leq \gamma_i$ for all non-zero subsheaves $F_1$ of $E_i$. We may use the definition of $\nu_\alpha$-slope to get an upper-bound for the $\nu_\alpha$-slopes of the subtriples of $A$. There are only finitely many possible $\nu_\alpha$-slopes greater than $\nu_\alpha(A)$, because the ranks are upper and lower bounded and each $\deg(G)$ for a subsheaf $G$ of $E_i$ is an integer, upper bounded by $\text{max}(r_1\mu(E_1), r_2\mu(E_2))$. Thus the set of the $\nu_\alpha$-slopes of all subtriples of $A$ has a maximum $\beta(A)$.

If $\nu_\alpha(A) = \beta(A)$, then $A$ itself is the maximum element of $B$. Now assume $\nu_\alpha(A) > \beta$ and that there are $B_1, B_2 \in B$ with each $B_i$ maximal and $B_1 \neq B_2$. Since $B_i$ is maximal, it is saturated and so $A_i := A/B_i$ is a holomorphic triple for each $i$. Since $B_2 \neq B_1$, the inclusion $B_2 \subset A$ induces a non-zero map $u : B_2 \to A/B_1$. Since $\nu_\alpha(\ker(u)) \leq \beta(A)$ if $u$ is not injective, we have $\nu_\alpha(u(A/B_1)) \geq \beta(A)$. Thus we get $\nu_\alpha(B_1 + B_2) \geq \beta(A)$, contradicting the maximality of $B_1$ and the assumption $B_2 \neq B_1$.

Assume that $A$ is not $\nu_\alpha$-semistable. By Lemma 7.14 there is a subtriple $D(A) = ((F_1, \Psi_1), (F_2, \Psi_2), g) \in B$ such that every $G \in B$ is a subtriple of $D(A)$ and each $F_i$ is saturated in $E_i$. Note that $D(A)$ is $\nu_\alpha$-semistable. Since $F_i$ is saturated in $E_i$ and $\Psi_i = \Phi_i|_{F_i}$, for each $i$, $\Phi_i$ induces a co-Higgs field $\tau_i : E_i/F_i \to (E_i/F_i) \otimes T_X(-\log D)$. Since $\Phi_i$ is integrable, so is $\tau_i$. Since $g = f|_{F_1}$, $f$ induces a map $f' : E_1/F_1 \to E_2/F_2$ such that $A/D(A) := ((E_1/F_1, \tau_1), (E_2/F_2, \tau_2), f')$ is a holomorphic triple. Now we
may check that each subtriple of $\mathcal{A}/D(\mathcal{A})$ has $\nu_\alpha$-slope less than $\beta(\mathcal{A})$ and so $D(\mathcal{A})$ defines the first step of the Harder-Narasimhan filtration of $\mathcal{A}$. The iteration of this process allows us to have the Harder-Narasimhan filtration of $\mathcal{A}$ with respect to $\nu_\alpha$.

**Corollary 7.15.** Any holomorphic triple admits the Harder-Narasimhan filtration with respect to $\nu_\alpha$-slope.

**Remark 7.16.** Let $Z$ denote a projective completion of $T_X(-\log D)$, e.g. $Z = \mathbb{P}(\mathcal{O}_X \oplus T_X(-\log D))$, and call $D_\infty := Z \setminus T_X(-\log D)$ the divisor at infinity. By [24] Lemma 6.8 a co-Higgs sheaf $(\mathcal{E}, \Phi)$ on $X$ is the same thing as a coherent sheaf $\mathcal{E}_Z$ with $\text{Supp}(\mathcal{E}_Z) \cap D_\infty = \emptyset$. Due to [24] Corollary 6.9 we may interpret a $\nu_\alpha$-semistable holomorphic triple of logarithmic co-Higgs bundles on $X$ as a $\nu_\alpha$-semistable holomorphic triple of vector bundles on $Z$ with support not intersecting $D_\infty$ as in [7].

Based on Remark 7.16 we may consider a $\nu_\alpha$-semistable triple of $D$-logarithmic co-Higgs sheaves as a $\nu_\alpha$-semistable quiver sheaf for the quiver $\overset{1}{\Phi} \overset{2}{\delta}$ on $Z$ with empty intersection with $D_\infty$. This interpretation ensures the existence of moduli space of $\nu_\alpha$-stable triples of $D$-logarithmic co-Higgs sheaves on $X$, say $\mathcal{M}_{D, \alpha}(r_1, r_2, d_1, d_2)$ with $(r_1, d_1)$ a pair of rank and degree of the $i$th-factor of the triples; indeed we may consider Gieseker-type semistability of quiver sheaves to $\nu$-semistable holomorphic triple of vector bundles on $Z$ with support not intersecting $D_\infty$ as in [23]. As noticed in [23] Remark in page 17, the $\nu_\alpha$-stability implies the Gieseker-type stability and so $\mathcal{M}_{D, \alpha}(r_1, r_2, d_1, d_2)$ can be considered as a quasi-projective subvariety of the one in [23]. Now let us define

$$\alpha_m := \mu(\mathcal{E}_2) - \mu(\mathcal{E}_1), \quad \alpha_M := \left(1 + \frac{r_1 + r_2}{r_1 - r_2}\right) \left(\mu(\mathcal{E}_2) - \mu(\mathcal{E}_1)\right)$$

for $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ as in [19]. Then we have

**Proposition 7.17.** [7] Proposition 2.2] If $\alpha > \alpha_M$ with rank $\mathcal{E}_1 \neq \text{rank} \mathcal{E}_2$ or $\alpha < \alpha_m$, then there exists no $\nu_\alpha$-semistable triple of $D$-logarithmic co-Higgs sheaves.

**Proof.** Due to [24] Corollary 6.9, it is sufficient to check the assertion for $\nu_\alpha$-semistability for a triple of coherent sheaves on $Z$. While the proof of [7] Proposition 2.2 is for curves, the proof is numerical involving rank and degree with respect to a fixed ample line bundle so that it works also for $Z$. \hfill \Box

From now on we assume that $X$ is a smooth projective curve of genus $g$ and let $D = \{p_1, \ldots, p_m\}$ be a set of $m$ distinct points on $X$. Take $g \in \{0, 1\}$ and assume that $T_X(-\log D) \cong \mathcal{O}_X$, i.e. $(g, m) \in \{(0, 2), (1, 0)\}$. For any triple $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ and $c \in \mathbb{C}$, set

$$\mathcal{A}_c := ((\mathcal{E}_1, \Phi_1 - c \text{Id}_{\mathcal{E}_1}), (\mathcal{E}_2, \Phi_2 - c \text{Id}_{\mathcal{E}_2}), f)$$

and then $\mathcal{A}_c$ is also a triple. In particular, if $\mathcal{E}_1 \cong \mathcal{E}_2$ and $f \cong c \text{Id}_{\mathcal{E}_1}$, then the study of the $\nu_\alpha$-(semi)stability of $\mathcal{A}$ is reduced to the known case $f = 0$.

**Remark 7.18.** Assume that $f$ is not injective. Since $\hat{f} \circ \Phi_1 = \Phi_2 \circ f$, we have $\Phi_1(\text{ker}(f)) \subseteq \text{ker}(\hat{f})$ and $\mathcal{B} := ((\text{ker}(f), \Phi_1|_{\text{ker}(f)}), (0, 0), 0)$ is a subtriple of $\mathcal{A}$. Set $\rho := \text{rank}(\text{ker}(f))$ and $\delta := \text{deg}(\text{ker}(f))$. If we have

$$\nu_\alpha(\mathcal{B}) = \delta/\rho + \alpha > \frac{r_1\alpha + d_1 + d_2}{r_1 + r_2},$$

then $\mathcal{A}$ would not be $\nu_\alpha$-semistable.
Remark 7.19. For any triple $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$, we get a dual triple $\mathcal{A}^\vee = ((\mathcal{E}_1^\vee, \Phi_1^\vee), (\mathcal{E}_2^\vee, \Phi_2^\vee, f^\vee))$, where $\Phi_i^\vee$ and $f^\vee$ are the transpose of $\Phi_i$ and $f$, respectively. Then $\mathcal{A}$ is $\nu_\alpha$-(semi)stable if and only if $\mathcal{A}^\vee$ is $\nu_\alpha$-(semi)stable (see [6 Proposition 3.16]).

Remark 7.20. Assume $(g, m) = (1, 0)$ and take a triple $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ with each $\mathcal{E}_i$ simple. By Atiyah’s classification of vector bundles on elliptic curves, the semistability of $\mathcal{E}_i$ is equivalent to its stability and also equivalent to its indecomposability and with degree and rank coprime. Then each $\Phi_i$ is the multiplication by a constant, say $c_i$. We get that the two triples $\mathcal{A}$ and $((\mathcal{E}_1, 0), (\mathcal{E}_2, 0), f)$ share the same subtriples and so these two triples are $\nu_\alpha$-(semi)stable for the same $\alpha$ simultaneously. There is a good description of this case in [18 Section 7].

Now we suggest some general description on $\nu_\alpha$-(semi)stable triples on $X$ in case of $r_1 = r_2 = 2$ from $(a) \sim (c)$ below; we exclude the case described in Remark [9.20] and silently use Remark [7.19] to get a shorter list. In some case we stop after reducing to a case with $f$ not injective, i.e. to a case in which $\mathcal{A}$ is not $\nu_\alpha$-semistable for $\alpha \gg 0$ (see Remark [7.18]).

(a) Assume $r_1 = r_2 = 2$ and that at least one of $\mathcal{E}_i$ is not semistable, say $\mathcal{E}_1$. Then, due to Segre-Grothendieck theorem and Atiyah’s classification of vector bundles on elliptic curves, we have $\mathcal{E}_1 \cong L_1 \oplus R_1$ with $\deg(L_1) > \deg(R_1)$ and $E_2 \cong L_2 \oplus R_2$ with $\deg(L_2) \geq \deg(R_2)$, or $g = 1$ and $\mathcal{E}_2$ is a non-zero extension of the line bundle $L_2$ by itself; in the latter case we put $R_2 := L_2$. If $\mathcal{E}_2$ is indecomposable, then it has a unique line bundle isomorphic to $L_2$ and so $F_2(L_2) \subseteq L_2$. We have

$$\nu_\alpha(A) = \alpha/2 + (\deg(L_1) + \deg(L_2) + \deg(R_1) + \deg(R_2))/4.$$ 

The map $\Phi_i : \mathcal{E}_i \to \mathcal{E}_i$ induces a map $\Phi_i|_{L_i} : L_i \to L_i$, which is induced by the multiplication by a constant, say $c_i$. Then we get two triples $A_i$, for $i = 1, 2$. Since $A = (A_1, A_2)$ is a triple, we get $f(L_1) \subseteq L_2$ and so we may define a subtriple $A_1 := ((L_1, \Phi_1|_{L_1}), (L_2, \Phi_2|_{L_2}, f|_{L_1})$ with

$$\nu_\alpha(A_1) = \alpha + (\deg(L_1) + \deg(L_2))/2$$
$$> \alpha/2 + (\deg(L_1) + \deg(L_2) + \deg(R_1) + \deg(R_2))/4 = \nu_\alpha(A),$$

which implies that $A$ is not $\nu_\alpha$-semistable.

(b) Form now we assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable. We also assume that $f$ is non-zero so that $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}_2)$. We are in a case with $r_1 = r_2 = 2$ and we look at a proper subtriple $B = ((F_1, \Phi_1|_{F_1}), (F_2, \Phi_2|_{F_2}, f|_{F_1})$ with maximal $\nu_\alpha(B)$. In particular, each $F_i$ is saturated in $\mathcal{E}_i$, i.e. either $F_i = \mathcal{E}_i$ or $F_i = 0$ or $\mathcal{E}_i/F_i$ is a line bundle. Set $s_i := \text{rank}(F_i)$ and then we have $1 \leq s_1 + s_2 \leq 3$. If $s_2 = 2$, i.e. $F_2 = \mathcal{E}_2$, then we have $\nu_\alpha(B) < \nu_\alpha(A)$ for all $\alpha > 0$, because $\mathcal{E}_1$ is semistable and $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}_2)$. If $s_2 = 0$, then $f$ is not injective. If $s_1 = 0$ we just exclude the case $\alpha \leq \alpha_m$ with subtriple $((0, 0), (\mathcal{E}_2, \Phi_2), 0)$. In the case $s_1 = s_2 = 1$ we know that $\nu_\alpha(B) \leq \nu_\alpha(A)$ and that equality holds if and only if both $\mathcal{E}_1$ and $\mathcal{E}_2$ are strictly semistable and each $F_i$ is a line subbundle of $\mathcal{E}_i$ with maximal degree. Note that the injectivity of $f$ implies $s_1 \leq s_2$. Thus when $f$ is injective, it is sufficient to test the case $s_1 = s_2 = 1$. Then we have the following, when $f$ is injective.

- If $\alpha > \alpha_m$ and at least one of $\mathcal{E}_i$’s is stable, then $A$ is $\nu_\alpha$-stable.
- If $\alpha \geq \alpha_m$ and $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable, then $A$ is $\nu_\alpha$-semistable.
Lemma 7.21. For a general map \( f : \mathcal{E}_1 \to \mathcal{E}_2 \) with \( \mathcal{E}_i := \mathcal{O}_{\mathbb{P}^1}(a_i)^{\otimes 2} \) and \( a_2 \geq a_1 + 2 \), there exists no subsheaf \( \mathcal{O}_{\mathbb{P}^1}(a_1) \subset \mathcal{E}_1 \) such that the saturation of its image in \( \mathcal{E}_2 \) is a line bundle isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(a_2) \).

Proof. Up to a twist we may assume that \( a_1 = 0 \). If we fix homogeneous coordinates \( x_0, x_1 \) on \( \mathbb{P}^1 \), then the map \( f \) is induced by two forms \( u(x_0, x_1) \) and \( v(x_0, x_1) \) of degree \( a_2 \). Then it is sufficient to prove that there is no point \( (a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\} \) with which \( au(x_0, x_1) + bv(x_0, x_1) \) is either identically zero or with a zero of multiplicity \( a_2 \). This is true for general \( u(x_0, x_1) \) and \( v(x_0, x_1) \), e.g. we may take \( u(x_0, x_1) = x_0^{a_2} + x_0x_1^{a_2-1} \) and \( v(x_0, x_1) = x_0x_1^{a_2-1} + x_1^{a_2} \).

The next is an analogue of Lemma 7.21 for elliptic curves.

Lemma 7.22. Let \( X \) be an elliptic curve with two line bundles \( L_i \) for \( i = 1, 2 \) such that \( \deg(L_2) \geq \deg(L_1) + 4 \). For a general map \( f : L_1^{\otimes 2} \to L_2^{\otimes 2} \), there is no subsheaf \( L_1 \subset L_1^{\otimes 2} \) such that the saturation of its image in \( L_2^{\otimes 2} \) is isomorphic to \( L_2 \).

Proof. It is sufficient to find an injective map \( h : L_1^{\otimes 2} \to L_2^{\otimes 2} \) for which no subsheaf \( L_1 \subset L_1^{\otimes 2} \) has its image under \( h \) whose saturation in \( L_2^{\otimes 2} \) is isomorphic to \( L_2 \). Up to a twist we may assume \( L_1 \cong \mathcal{O}_X \) and so \( l := \deg(L_2) \geq 4 \). First assume \( l = 4 \) and write \( L_2 \cong M^{\otimes 2} \) with \( \deg(M) = 2 \). If \( \varphi : X \to \mathbb{P}^1 \) be a morphism of degree two, induced by \( |M| \), then we may set \( h := \varphi^*(h_1) \) for a general \( h_1 : \mathcal{O}_{\mathbb{P}^1}^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^1}(2)^{\otimes 2} \) with Lemma 7.21 applied to \( h_1 \).

Now assume \( l \geq 5 \) and fix an effective divisor \( D \subset X \) of degree \( l - 4 \). Then we may take as \( h \) the composition of a general map \( \mathcal{O}_X^{\otimes 2} \to L_2(-D)^{\otimes 2} \) with the map \( L_2(-D)^{\otimes 2} \to L_2^{\otimes 2} \) obtained by twisting with \( \mathcal{O}_X(D) \).

Remark 7.23. Let \( D \) be an arrangement with \( T_X(-\log D) \cong \mathcal{O}_X \) on \( X \) with arbitrary dimension. For two line bundles \( L_1 \) and \( L_2 \) with \( L_2 \otimes L_1^* \) globally generated, set a triple \( \mathcal{B} = ((\mathcal{E}_1, 0), (\mathcal{E}_2, 0), f) \) with \( \mathcal{E}_i \cong L_i^{\otimes r} \) and \( f \) injective. As in \( \mathcal{B} \) we may generate other triples \( \mathcal{B}_c \) for each \( c \in \mathbb{C} \), but often there are no other \( D \)-logarithmic co-Higgs triples with \( \mathcal{B} \) as the associated triple of vector bundles. For example, assume \( X \) is a smooth projective curve of genus \( g \in \{0, 1\} \). For a fixed co-Higgs field \( \Phi_1 : \mathcal{E}_1 \to \mathcal{E}_1 \) with the associated \((r \times r)\)-matrix \( A_1 \) of constants, we are looking for \( f \) and \( \Phi_2 : \mathcal{E}_2 \to \mathcal{E}_2 \) with the associated matrix \( A_2 \) such that \( A = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f) \) is a \( D \)-logarithmic co-Higgs triple. Let \( M \) be the \((r \times r)\)-matrix with coefficient in \( H^0(L_2 \otimes L_1^*) \) associated to \( f \). Then we need \( A_2 \) and \( M \) such that \( A_2 M = M A_1 \). Assume that \( A_1 \) has a unique Jordan block. If \( L_1 \cong L_2 \) and \( M \) is general, then we get a \( D \)-logarithmic co-Higgs triple if and only if \( A_2 \) is a polynomial in \( A_1 \). If \( L_1 \not\cong L_2 \) and \( f \) is general, then there is no such \( A_2 \). We check this for the case \( r = 2 \) and the general case can be shown similarly. With no loss of generality we may assume that the unique eigenvalue of \( A_1 \) is zero. Assume the existence of \( f \) and \( \Phi_2 \) with associated \( M \) and \( A_2 \). We have \( \ker(\Phi_1) \cong L_1 \) and \( f(\ker(\Phi_1)) \subseteq \ker(\Phi_2) \). Thus we get that \( f(L_1) \) has ker(\( \Phi_2 \)) \cong L_2 \) as its saturation, contradicting Lemmas 7.21 and 7.22 for a general \( f \).

Remark 7.24. In the same way as in \( \mathcal{B} \) one can define \( D \)-logarithmic co-Higgs holomorphic chains with parameters, but if the maps are general, then very few logarithmic co-Higgs fields \( \Phi_1 \) are allowed.
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