Abstract

Let $P$ and $T$ be disjoint sets of prime numbers with $T$ finite. A simple formula is given for the natural density of the set of square-free numbers which are divisible by all of the primes in $T$ and by none of the primes in $P$. If $P$ is the set of primes congruent to $r$ modulo $m$ (where $m$ and $r$ are relatively prime numbers), then this natural density is shown to be 0 and if $P$ is the set of Mersenne primes (and $T = \emptyset$), then it is approximately $0.3834$.

1. Main results

Gegenbauer proved in 1885 that the natural density of the set of square-free integers, i.e., the proportion of natural numbers which are square-free, is $6/\pi^2$ [3, Theorem 333; reference on page 272]. In 2008 J. A. Scott conjectured [8] and in 2010 G. J. O. Jameson proved [5] that the natural density of the set of odd square-free numbers is $4/\pi^2$ (so the proportion of natural numbers which are square-free and even is $2/\pi^2$). Jameson’s argument was adapted from one used to compute the natural density of the set of all square-free numbers. In this note we use the classical result for all square-free numbers to reprove Jameson’s result and indeed to generalize it.

**Theorem 1.** Let $P$ and $T$ be disjoint sets of prime numbers with $T$ finite. Then the proportion of all numbers which are square-free and divisible by all of the primes in $T$ and by none of the primes in $P$ is

$$
\frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1 + p} \prod_{p \in P} \frac{1}{1 + p}.
$$

As in the above theorem, throughout this paper $P$ and $T$ will be disjoint sets of prime numbers with $T$ finite. The letter $p$ will always denote a prime number. The term *numbers* will always refer to positive integers. Empty products, such as occurs in the first product above when $T$ is empty, are understood to equal 1. If $P$ is infinite, we will argue in Section 4 that the second product above is well-defined.
Examples. 1. Setting $P = \{2\}$ and $T$ equal to the empty set $\emptyset$ in the theorem we see that the natural density of the set of odd square-free numbers is $\frac{6}{\pi^2} \frac{2}{3+1} = \frac{4}{7\pi^2}$; taking $T = \{2\}$ and $P = \emptyset$ we see that the natural density of the set of even square-free numbers is $\frac{6}{\pi^2} \frac{1}{2+1} = \frac{2}{7\pi^2}$. Thus one third of the square-free numbers are even and two thirds are odd. (These are Jameson’s results of course.)

2. Taking $P = \{101\}$ and $T = \emptyset$ in the theorem we see that the set of square-free numbers not divisible by 101 has natural density $\frac{6}{\pi^2} \frac{101}{101+1}$. Thus slightly over 99% of square-free numbers are not divisible by 101.

3. Set $T = \{2, 5\}$ and $P = \{3, 7\}$ in the theorem. Then the theorem says that the natural density of the set of square-free numbers divisible by 10 but not by 3 or 7 is $\frac{6}{\pi^2} \frac{1}{3} \frac{3}{5} \frac{7}{101} = \frac{7}{32\pi^2}$, so the proportion of square-free numbers which are divisible by 10 but not by 3 or 7 is $7/192$.

Our interest in the case that $P$ is infinite arose in part from a question posed by Ed Bertram: what is the natural density of the set of square-free numbers none of which is divisible by a prime congruent to 1 modulo 4? The answer is zero; more generally we have the following.

Theorem 2. Let $r$ and $m$ be relatively prime numbers. Then the natural density of the set of square-free numbers divisible by no prime congruent to $r$ modulo $m$ is zero.

This theorem is a corollary of the previous theorem since, as we shall see in Section 5, for any $r$ and $m$ as above, $\prod_{p \equiv r \pmod{m}} p/(1 + p) = 0$.

By way of contrast with Theorem 2 we will prove in Section 6 a lemma giving a simple condition on an infinite set of primes $P$ which will guarantee that the set of square-free numbers not divisible by any element of $P$ has positive natural density. An immediate corollary is that the set of square-free numbers not divisible by any Mersenne prime has positive natural density; we will also show how to closely approximate the natural density of this set.

2. A Basic Lemma

Notation. For any finite set $S$ of primes we set $d_S = \prod_{p \in S} p$.

For any real number $x$ and set $B$ of numbers, we let $B[x]$ denote the number of elements $t$ of $B$ with $t \leq x$. Recall that if $\lim_{x \to \infty} B[x]/x$ exists, then it is by definition the natural density of $B$ [6, Definition 11.1]. In case it exists we will denote the natural density of $B$ by $B^*$. We will also let $|B|$ denote the cardinality of $B$.

Let $A$ denote the set of square-free numbers. Then we let $A(T, P)$ denote the set of elements of $A$ which are divisible by all elements of $T$ and by no element of
The set of square-free numbers analyzed in Theorem 1 is $A(T, P)$.

The above notation is used in the next lemma, which shows how the calculation of the natural density of the sets $A(T, P)$ reduces to the calculation of the natural density of sets of the form $A(\emptyset, S)$ and, when $P$ is finite, also reduces to the calculation of the natural density of sets of the form $A(S, \emptyset)$.

**Lemma 1.** For any finite set of primes $S$ disjoint from $T$ and from $P$ and for any real number $x$, we have $A(T, S \cup P)[x] = A(T \cup S, P)[x d_S]$. Moreover, the set $A(T \cup S, P)$ has a natural density if and only if $A(T, P \cup S)$ has a natural density, and if these natural densities exist, then $A(T, P \cup S)^* = d_S A(T \cup S, P)^*$.

This lemma generalizes Lemmas 1 and 2 of [2].

**Proof.** The first assertion is immediate from the fact that multiplication by $d_S$ gives a bijection from the set of elements of $A(T, S \cup P)$ less than or equal to $x$ to the set of elements of $A(T \cup S, P)$ less than or equal to $x d_S$. This implies that

$$
\frac{A(T, P \cup S)[x]}{x} = d_S \frac{A(T \cup S, P)[x d_S]}{x d_S}.
$$

The lemma follows by taking the limit as $x$ (and hence $x d_S$) goes to infinity.  

**Remark 1.** We might note that if we assume that for all $T$ the sets $A(T, \emptyset)$ have natural densities, then it is easy to compute these natural densities. After all, for any $T$ the set $A$ is the disjoint union over all subsets $S$ of $T$ of the sets $A(T \setminus S, S)$, so by Lemma 1

$$
\frac{6}{\pi^2} = A^* = \sum_{S \subseteq T} A(T \setminus S, S)^* = \sum_{S \subseteq T} d_S A(T, \emptyset)^* = A(T, \emptyset)^* \sum_{S \subseteq T} d_S.
$$

But

$$
\sum_{S \subseteq T} d_S = \sum_{d|d_T} d = \prod_{p \in T} (1 + p)
$$

[6, Theorem 4.5], so indeed

$$
A(T, \emptyset)^* = \frac{6}{\pi^2} \prod_{p \in \mathbb{P}} \frac{1}{1 + p}.
$$

We could now use Lemma 1 to obtain the formula of Theorem 1 in the case that $P$ is finite.
3. Proof of Theorem 1 when $P$ is Finite

The theorem in this case was proved in [2]. For the convenience of the reader we sketch the proof using the notation of this paper.

**Lemma 2.** Let $p$ be a prime number not in $T$. If the set $A(T, \emptyset)$ has natural density $D$, then the set $A(\{p\} \cup T, \emptyset)$ has natural density $\frac{1}{p+1}D$.

**Proof.** For any real number $x$ we set $E(x) = A(\{p\} \cup T, \emptyset)[x]$. Let $\epsilon > 0$.

Note that $A(T, \emptyset)$ is the disjoint union of $A(\{p\} \cup T, \emptyset)$ and $A(T, \{p\})$. Hence by Lemma 1 (applied to $A(T, \{p\})$) for any real number $x$,

$$A(T, \emptyset)[x/p] = E(x/p) + E(x)$$

and so by the choice of $D$ there exists a number $M$ such that if $x > M$ then

$$\left| \frac{E(x)}{x/p} + \frac{E(x/p)}{x/p} - D \right| < \epsilon/3.$$

We next pick an even number $k$ such that $\frac{1}{p^k} < \frac{\epsilon}{3}$. Then

$$|E(x/p^k)| \leq x/p^k < \frac{\epsilon x}{3}$$

and also (using the usual formula for summing a geometric series)

$$-Dx \sum_{i=1}^{k} \left( -\frac{1}{p} \right)^i - Dx \frac{1}{p+1} = Dx \left| \frac{(-\frac{1}{p}) - (-\frac{1}{p})^{k+1}}{1 - (-\frac{1}{p})} + \frac{1}{p+1} \right|$$

$$= Dx \left| -1 + \frac{p^x}{p+1} + \frac{1}{p+1} \right| < \frac{1}{p^k}Dx < \frac{\epsilon x}{3}.$$

Now suppose that $x > p^kM$. Then for all $0 \leq i \leq k$ we have $x/p^i > M$ and hence (applying the choice of $M$ above),

$$\left| (-1)^iE \left( \frac{x}{p^i} \right) + (-1)^iE \left( \frac{x}{p^{i+1}} \right) - (-1)^iD \frac{x}{p^{i+1}} \right| < \frac{\epsilon x}{3p^{i+1}}.$$  (3)

Using the triangle inequality to combine the inequalities (1) and (2) and all the inequalities (3) for $0 \leq i < k$ and dividing through by $x$, we can conclude that

$$\left| \frac{E(x)}{x} - \frac{1}{p+1}D \right| < \frac{\epsilon}{3} \left( \sum_{i=1}^{k} \frac{1}{p^i} \right) + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.$$

Hence the natural density of $A(\{p\} \cup T, \emptyset)$ is $\frac{1}{p+1}D$. \qed
Theorem 1 now follows in the case that \( P \) is empty from the above lemma by induction on the number of elements of \( T \). That it is true when \( P \) is finite but not necessarily empty follows from Lemma 1: in the statement of that lemma replace \( P \) by \( \emptyset \) and \( S \) by \( P \); then we see that the natural density of \( \mathcal{A}(T, P) \) is indeed

\[
d_{P, \mathcal{A}(T \cup P, \emptyset)^*} = \frac{6}{\pi^2} \prod_{p \in P} p \prod_{p \in T \cup P} \frac{1}{1 + p} = \frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1 + p} \prod_{p \in P} \frac{1}{1 + p}.
\]

4. Proof of Theorem 1 when \( P \) is Infinite

In this and the next section we will use the simple fact that for all \( x > 1 \) we have

\[
\frac{1}{x} > \log(x + 1) - \log x = \log \frac{x + 1}{x} > \frac{1}{1 + x} > \frac{1}{2x}. \tag{4}
\]

We first show that the expression \( \prod_{p \in P} \frac{p}{1 + p} \) is well-defined when \( P \) is infinite. Let \( p_1, p_2, p_3, \ldots \) be the strictly increasing sequence of elements of \( P \). Since all the quotients \( p_i/(1 + p_i) \) are less than 1, the partial products of the infinite product \( \prod_i p_i/(1 + p_i) \) form a strictly decreasing sequence bounded below by 0; thus the infinite product \( \prod_i p_i/(1 + p_i) \) converges, say to \( \alpha \). Its limit is also unchanged by any rearrangement of its factors; this is easy to check if \( \alpha \) is zero. Otherwise the infinite sum

\[
\sum_i \log \frac{p_i}{1 + p_i} = \sum_i \log(p_i) - \log(1 + p_i)
\]

converges absolutely (to \( \log(\alpha) \)) and therefore its value is unchanged under rearrangements; hence the corresponding fact is also true of the infinite product \( \prod_i p_i/(1 + p_i) \). Thus in all cases the expression \( \prod_{p \in P} \frac{p}{1 + p} \) is well-defined.

We now prove the theorem in the case that \( T \) is empty.

First suppose that \( \alpha \neq 0 \). Then \( \sum_{p \in P} 1/p < \infty \). After all, we have

\[
-\log \alpha = \sum_{p \in P} \log(1 + p) - \log p > \frac{1}{2} \sum_{p \in P} \frac{1}{p}.
\]

Now observe that \( \mathcal{A} \setminus \mathcal{A}(\emptyset, P) \) is the disjoint union

\[
\mathcal{A} \setminus \mathcal{A}(\emptyset, P) = \cup_{k \geq 1} \mathcal{A}(\{p_k\}, \{p_1, \cdots, p_{k-1}\})
\]

since for all \( b \in \mathcal{A} \setminus \mathcal{A}(\emptyset, P) \) there exists a least \( k \geq 1 \) with \( p_k \) dividing \( b \), so that \( b \in \mathcal{A}(\{p_k\}, \{p_1, \cdots, p_{k-1}\}) \).

For all \( n \) and \( k \) we have

\[
\frac{\mathcal{A}(\{p_k\}, \{p_1, \cdots, p_{k-1}\})[n]}{n} \leq \frac{|\{j : 1 \leq j \leq n \text{ and } p_k | j\}|}{n} \leq \frac{1}{p_k}.
\]
Hence by Tannery’s theorem (see [9, p. 292] or [4, p. 199]) the natural density of \( A \setminus A(\emptyset, P) \) is therefore

\[
\lim_{n \to \infty} \frac{(A \setminus A(\emptyset, P))[n]}{n} = \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{A(\{p_k\}, \{p_1, \ldots, p_{k-1}\})[n]}{n}
\]

\[
= \lim_{k \to \infty} \sum_{n=1}^{\infty} \frac{6}{\pi^2} \frac{1}{1 + p_k} \prod_{i<k} \frac{p_i}{1 + p_i}
\]

by the proof in the previous section of the theorem in the case that \( P \) is finite. Writing \( 1/(1 + p_k) = 1 - p_k/(1 + p_k) \) we can see that the natural density of \( A \setminus A(\emptyset, P) \) is therefore a limit of telescoping sums

\[
\frac{6}{\pi^2} \lim_{L \to \infty} \sum_{k=1}^{L} \left( \prod_{i<k} \frac{p_i}{1 + p_i} - \prod_{i<k+1} \frac{p_i}{1 + p_i} \right)
\]

\[
= \frac{6}{\pi^2} \lim_{L \to \infty} \left( 1 - \prod_{i \leq L} \frac{p_i}{1 + p_i} \right) = \frac{6}{\pi^2} (1 - \alpha)
\]

and thus

\[
A(\emptyset, P)^* = \frac{6}{\pi^2} - \frac{6}{\pi^2} (1 - \alpha) = \frac{6}{\pi^2} \prod_{p \in P} \frac{p}{1 + p},
\]

which proves the theorem in the case that \( \alpha \neq 0 \) and \( T = \emptyset \).

We next consider the case that \( \alpha = \prod_{p \in P} \frac{p}{1 + p} = 0 \). Suppose that \( \epsilon > 0 \). By hypothesis there exists a number \( M \) with \( \frac{6}{\pi^2} \prod_{i \leq M} \frac{p_i}{1 + p_i} < \epsilon/2 \). Then by our proof of the theorem in the case that \( P \) is finite there exists a number \( L \) such that if \( n > L \) then

\[
A(\emptyset, \{p_1, p_2, \ldots, p_M\})[n] < \frac{\epsilon}{2} + \frac{6}{\pi^2} \prod_{i \leq M} \frac{p_i}{1 + p_i}.
\]

Thus if \( n > L \) we have

\[
0 \leq \frac{A(\emptyset, P)[n]}{n} \leq \frac{A(\emptyset, \{p_1, p_2, \ldots, p_M\})[n]}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{A(\emptyset, P)[n]}{n} = 0 = \frac{6}{\pi^2} \prod_{p \in P} \frac{p}{1 + p},
\]

which proves the theorem if \( \alpha = 0 \) and \( T = \emptyset \).

This completes the proof of the theorem in the case that \( T = \emptyset \). The general case where \( T \) is arbitrary then follows from Lemma 1, applied with \( T \) and \( S \) replaced respectively by \( \emptyset \) and \( T \): \( A(T, P)^* \) equals
$A(\emptyset, P \cup T)^* / d_T = \frac{6}{\pi^2} \prod_{p \in T} \frac{1}{p} \prod_{p \in P} \frac{p}{1 + p} \prod_{p \in P} \frac{p}{1 + p} = \frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1 + p} \prod_{p \in P} \frac{p}{1 + p},$

which completes the proof of Theorem 1.

5. Proof of Theorem 2

Let us set $P = \{p : p \equiv r \ (mod \ m)\}$. It then suffices by Theorem 1 to prove that $\prod_{p \in P} p / (1 + p) = 0$. For any real number $x > 3$ we have

$$\sum_{x > p \in P} \frac{1}{p} - \frac{\log \log x}{\phi(m)} = O(1)$$

(see [1, Exercise 6, page 156]). Hence $\sum_{p \in P} 1/p = \infty$. Theorem 2 now follows from the next lemma, which applies to any infinite set $P$ of primes.

**Lemma 3.** Let $P$ be an infinite set of primes. Then $\sum_{p \in P} 1/p = \infty$ if and only if $\prod_{p \in P} p / (1 + p) = 0$. Moreover, if $\sum_{p \in P} 1/p \leq S < \infty$ for a real number $S$, then $\prod_{p \in P} p / (1 + p) \geq e^{-S}$.

**Proof.** The inequalities of display (4) imply that

$$\frac{1}{2} \sum_{p \in P} \frac{1}{p} \leq - \log \prod_{p \in P} \frac{p}{1 + p} = \sum_{p \in P} \log \frac{p + 1}{p} \leq \sum_{p \in P} \frac{1}{p}.$$

Our conclusions follow easily. \qed

6. Mersenne Primes

**Lemma 4.** Let $d$ be a number. Let $P = \{p_1, p_2, \cdots\}$ be an infinite set of primes such that for all $i \geq 1$ we have $p_i \geq 2^i - d$. Then the set of square-free numbers not divisible by any element of $P$ has positive natural density.

**Proof.** The infinite sum $\sum_{p \in P} 1/p$ converges since it is less than or equal to the infinite sum $\sum_{i<\infty} 1/(2^i - d)$, which itself converges by the limit comparison test. Thus by Lemma 3, $\prod_{p \in P} p / (1 + p) > 0$; the lemma now follows from Theorem 1. \qed

An immediate corollary of this lemma (or just of Theorem 1 if the relevant set of primes turns out to be finite) is that the natural density of the set of square-free numbers not divisible by any Fermat prime is positive [3, Section 2.5]. Similarly,
the natural density of the set of square-free numbers not divisible by any Mersenne prime is positive. The next theorem allows us to approximate this natural density closely.

Let \( q_i \) denote the \( i \)-th prime number and let \( P \) denote the set of Mersenne primes. For any number \( M \) let

\[
P_M = \{ p \in P : p = 2^i - 1 \text{ for some } i \leq M \}
\]

(so, for example, \( P_5 = \{3, 7, 31, 127\} \)). Let \( A_M = \prod_{p \in P_M} p/(1+p) \) (so, for example, \( A_5 = \frac{3 \cdot 7 \cdot 31 \cdot 127}{4 \cdot 8 \cdot 32 \cdot 128} \approx .63078 \)). If the number of Mersenne primes is finite, then there exists some \( M \) with \( P = P_M \) and so \( A_M = \prod_{p \in P} \frac{p}{1+p} \). In general, we have the following.

**Theorem 3.** Whether \( P \) is finite or infinite, we have

\[
A_M \geq \prod_{p \in P} \frac{p}{1+p} \geq A_M \exp \left(-\frac{1}{2M}\right).
\]

**Proof.** For any number \( i \) we have \( q_i \geq i + 1 \) so

\[
\sum_{p \in P \setminus P_M} \frac{1}{p} \leq \sum_{i > M} \frac{1}{2^{q_i} - 1} \leq \sum_{i > M} \frac{1}{2^i} = \frac{1}{2^M},
\]

so by Lemma 3

\[
A_M \geq \prod_{p \in P} \frac{p}{1+p} = \prod_{p \in P_M} \frac{p}{1+p} \prod_{p \in P \setminus P_M} \frac{p}{1+p} \geq A_M \exp \left(-\frac{1}{2M}\right).
\]

Thus, for example, taking \( M = 5 \) we have

\[
.631 \geq A_5 \geq \prod_{p \in P} \frac{p}{1+p} \geq \exp \left(-\frac{1}{2^5}\right) A_5 \geq .611
\]

so the natural density \( D \) of the set of square-free numbers divisible by no Mersenne prime satisfies

\[
.384 \geq .631 \frac{6}{\pi^2} \geq D \geq .611 \frac{6}{\pi^2} \geq .371.
\]

If we repeat this calculation with \( M = 17 \) (so that now

\[
P_M = \{3, 7, 31, 127, 2^{13} - 1, 2^{17} - 1, 2^{19} - 1, 2^{31} - 1\}\)

[3, Section 2.5], then we find that

\[
.38342 > D > .38341.
\]
Remark 2. An easy induction shows that for $i > 1$, $q_i - 2i$ is an increasing function of $i$; hence if $i > M > 1$ then $q_i \geq 2i + q_M - 2M$. Using this bound on $q_i$ the argument in the proof of the above theorem can be modified to show that

$$\sum_{p \in P \setminus P_M} \frac{1}{p} \leq 3(2^{q_M} - 1)$$

so that by Lemma 3

$$D \geq \frac{6}{\pi^2} A_M \exp\left(-\frac{1}{3(2^{q_M} - 1)}\right) \geq \frac{6}{\pi^2} A_M \exp\left(-\frac{1}{3(M^{M \log 2} - 1)}\right)$$

where we have used Rosser’s Theorem [7] to deduce the last inequality. Thus, for example, if $M = 5$ (so $q_M = 11$) we can compute that

$$D \geq \frac{6}{\pi^2} A_5 \exp\left(-\frac{1}{3(2^{11} - 1)}\right) > .383403.$$

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