MORE PROPERTIES OF THE FIBONACCI WORD ON AN INFINITE ALPHABET

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ABSTRACT. Recently the Fibonacci word \( W \) on an infinite alphabet was introduced by [Zhang et al., Electronic J. Combinatorics 24–2 (2017) #P2.52] as a fixed point of the morphism \( \phi : (2i) \mapsto (2i)(2i+1), (2i+1) \mapsto (2i+2) \) over all \( i \in \mathbb{N} \). In this paper we investigate the occurrence of squares, palindromes, and Lyndon factors in this infinite word.

1. INTRODUCTION

A word of \( n \) letters is \( x = x[1 \ldots n] \), with \( x[i] \) being the \( i \)-th letter and \( x[i \ldots j] \) the factor consisting of letters from position \( i \) to position \( j \). If \( i = 1 \) then the factor is a prefix and if \( j = n \) it is a suffix. The letters in \( x \) come from some alphabet \( A \). The length of \( x \), written \( |x| \), is the number of occurrences of letters in \( x \) and the number of occurrences of the letter \( a \) in \( x \) is denoted by \( |x|_a \). Two or more adjacent identical factors form a power. A word \( x \) or factor \( x \) is periodic with period \( p \) if \( x[i] = x[i+p] \) for all \( i \) such that \( x[i] \) and \( x[i+p] \) are in \( x \). A periodic word with least period \( p \) and length \( n \) is said to have exponent \( n/p \). The word \( ababa \) has exponent 5/2 and can be written as \( (ab)^{5/2} \). Thus powers have integer exponent at least 2. A factor with exponent 2 is a square. Two words \( x \) and \( y \) are conjugate if there exist words \( u \) and \( v \) such that \( x = uvu \) and \( y = vu \). If \( x = x[1 \ldots n] \) then the reverse of \( x \), written \( R(x) \), is \( x[n \ldots 1] \). A word that equals its own reverse is a palindrome. If \( x = uvu \) we say that \( x \) has border \( u \), and we see that \( x \) has period \( |x| - |u| \). If the alphabet is ordered then words are ordered lexicographically. A word which is lexicographically less than each of its conjugates is a Lyndon word. Lyndon words are necessarily primitive (i.e., not a power of a shorter word).

Recall that the Fibonacci word \( F \) over the binary alphabet \( \{0,1\} \) is the fixed point of the morphism \( \psi(0) = 01, \psi(1) = 0, \) and begins

\[
F = 010010100100101001010\ldots
\]

Recently Zhang, Wen, and Wu [11] introduced an interesting modification of this word. As an alphabet they used the non-negative integers, and as a morphism they used \( \phi(2i) = 2i \cdot 2i + 1 \) and \( \phi(2i + 1) = 2i + 2 \). Here and elsewhere we use ",," to indicate concatenation when it may not be clear from the context. From this we see that \( \phi(0) = 01, \phi^2(0) = 012, \) and so on, with a fixed point of \( \phi \) being the infinite word beginning

\[
W = 012232423244523445456\ldots
\]

We will call this the ZWW word after the authors of [11] and give it the symbol \( W \) as the majority of the names of the authors of [11] begin with \( W \). Some of the properties of the Fibonacci word have parallels with those of the ZWW word. For example, if we reduce the elements of the ZWW word modulo 2 we obtain the Fibonacci word. The well-known finite Fibonacci words are \( F_0, F_1, \ldots \) where \( F_i = \psi^i(0) \). The first few of these words are shown in Table 1 below. Note that these finite words have the property that

\[
F_{i+2} = F_{i+1} \cdot F_i \quad \text{for } i \geq 0.
\]

Analogously, we define the finite ZWW words as \( W_i = \phi^i(0) \). The first few of these are also shown in Table 1 below.

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TABLE 1. Finite Fibonacci and ZWW words

| i   | F_i  | W_i  |
|-----|------|------|
| 0   | 0    | 0    |
| 1   | 01   | 01   |
| 2   | 010  | 012  |
| 3   | 01001| 01223|
| 4   | 01001010 | 01223234 |
| 5   | 0100101001001 | 0122323423445 |

It is easily shown that, for \( i \geq 0 \),

\[
|W_i| = |F_i| = f_{i+2}
\]

where \( f_i \) is the \( i \)-th Fibonacci number defined by:

\[
f_1 = f_2 = 1, \quad f_{i+2} = f_i + f_{i+1} \text{ for } i \geq 1.
\]

For a finite word \( w \), \( n \oplus w \) denotes the word formed from \( w \) by adding \( n \) to each of its members. Analogously to (1), we have the following result.

**Lemma 1.** For \( i \geq 2 \),

\[
W_{i+1} = W_i \cdot 2 \oplus W_{i-1}.
\]

**Proof.** We note that for any number \( i \)

\[
\phi(2 \oplus i) = 2 \oplus \phi(i).
\]

We use induction on \( i \). It is easily checked that the statement holds for \( i = 2 \). Suppose it holds for all \( i \) with \( 2 \leq i \leq j \). Then

\[
W_{j+1} = \phi(W_j)
= \phi(W_{j-1} \cdot 2 \oplus W_{j-2}) \text{ by the induction hypothesis}
= W_j \cdot \phi(2 \oplus W_{j-2})
= W_j \cdot 2 \oplus \phi(W_{j-2})
= W_j \cdot 2 \oplus W_{j-1}
\]

as required. □

**Lemma 2.** For all \( i \geq 0 \), the first letter of \( W_i \) is 0 and the last letter of \( W_i \) is \( i \). Moreover, \( i \) occurs only once in \( W_i \) and all the other letters in \( W_i \) are less than \( i \).

**Proof.** This follows easily by induction using the previous lemma. □

We now present two ways of factorising \( W_k \). These will be used later.

**Lemma 3.** For \( k \geq 2 \),

\[
W_k = 01 \cdot \prod_{i=0}^{k-2} 2 \oplus W_i = 01 \cdot 2 \oplus \prod_{i=0}^{k-2} W_i
\]

where \( \prod \) indicates concatenation.

**Proof.** The second and third parts of the display are clearly equal. We will use induction on \( k \) to show the first and second are. The theorem clearly holds when \( k = 2 \). We assume it hold up to \( k - 1 \) so that

\[
W_{k-1} = 01 \cdot \prod_{i=0}^{k-3} 2 \oplus W_i.
\]
Now using Lemma 1, we have
\[
W_k = W_{k-1} \cdot 2 \oplus W_{k-2}
\]
\[
= 01 \cdot \Pi_{i=0}^{k-3} 2 \oplus W_i \cdot 2 \oplus W_{k-2}
\]
\[
= 01 \cdot \Pi_{i=0}^{k-2} 2 \oplus W_i
\]
and the result follows. \qed

As an example of Lemma 3 consider,
\[
W_5 = 01 \cdot 2 \oplus W_0 \cdot 2 \oplus W_1 \cdot 2 \oplus W_2 \cdot 2 \oplus W_3
\]
\[
= 01 \cdot 2 \oplus 0 \cdot 2 \oplus 01 \cdot 2 \oplus 012 \cdot 2 \oplus 01223
\]
\[
= 01 \cdot 23 \cdot 234 \cdot 2345.
\]
The second factorisation also uses iteration of Lemma 1.

**Lemma 4.** For \( k \geq 1 \),
\[
(4) \quad W_{2k-1} = \left[ \prod_{j=k}^{1} 2(k - j) \oplus W_{2j-2} \right] \cdot (2k - 1),
\]
\[
(5) \quad W_{2k} = \left[ \prod_{j=k}^{1} 2(k - j) \oplus W_{2j-1} \right] \cdot (2k),
\]
where \( \prod_{j=k}^{1} \) indicates concatenation in the order \( j = k, k-1, \ldots, 1 \).

**Proof.** We give the proof for \( 2k - 1 \); the proof for \( 2k \) is similar. Since \( W_1 = 0 \oplus W_0 \cdot 1 = 01 \), the result holds for \( k = 1 \) \((2k - 1 = 1)\). Suppose it is true for some \( 2k - 1, k \geq 1 \). Then
\[
W_{2k+1} = W_{2k} \cdot 2 \oplus W_{2k-1}, \text{ by Lemma 1}
\]
\[
= W_{2k} \cdot 2 \oplus \left[ \prod_{j=k}^{1} 2(k - j) \oplus W_{2j-2} \cdot (2k - 1) \right] \text{ by the induction hypothesis}
\]
\[
= W_{2k} \cdot \left[ \prod_{j=k}^{1} 2(k+1 - j) \oplus W_{2j-2} \right] \cdot (2k + 1)
\]
\[
= \left[ \prod_{j=k+1}^{1} 2(k+1 - j) \oplus W_{2j-2} \right] \cdot (2k + 1),
\]
and so the result holds for all odd indices \( 2k+1 \). \qed

As an example of Lemma 4, consider
\[
W_5 = W_4 \cdot 2 \oplus W_2 \cdot 4 \oplus W_0 \cdot 5
\]
\[
= 0122324 \cdot 2 \oplus 012 \cdot 4 \oplus 0 \cdot 5
\]
\[
= 0122324 \cdot 234 \cdot 4 \cdot 5.
\]

In [11] the authors investigated the growth of \( W \); in particular, they showed that the \( n \)-th letter in \( W \) is less than \( c \log n \) for some constant \( c \) and that the sum of the letters of \( W_k \) is
\[
\frac{k(\gamma^{k+1} + \psi^{k+1})(\psi^2 + \gamma)}{(\gamma - \psi)^2} + \frac{\psi^k - \gamma^k}{(\gamma - \psi)^3}
\]
where \( \gamma = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} \). Much of their paper was devoted to presenting various factorisations of \( W \) using singular words. They also showed that the only palindromes in \( W \) are those of the form \( 2i \cdot 22, \)
$2i \oplus 232$, and $2i \oplus 323$ where $i$ is any non-negative integer. In the present paper we are mainly concerned with the finite words $W_k$. In Section 2 we count the numbers of occurrences of letters in $W_k$, in Section 3 we count palindromes, and in Sections 4 and 5 we count squares. In the final two sections we count the Lyndon factors in $W_k$ and describe the Lyndon array of $W_k$.

2. Number of occurrences of a letter in $W_k$

We set $N(i, n)$ to be the number of occurrences of the letter $n$ in $W_i$. It is clear that

(6) $N(i, 0) = 1$ for all $i \geq 0$,

(7) $N(i, 1) = 1$ for all $i \geq 1$,

(8) $N(0, n) = 0$ for all $n \geq 1$,

(9) $N(1, n) = 0$ for all $n \geq 2$.

Theorem 5.

$$N(i, n) = \left( i - n + \left\lfloor \frac{n}{2} \right\rfloor \right),$$

where $(\frac{x}{y}) = 0$ if $y > x$, $x < 0$ or $y < 0$.

Proof. Note that each of (6), (7), (8) and (9) is satisfied by (10). By Lemma 1, we have, for $i \geq 1$,

$$N(i + 1, n) = |W_i \cdot 2 \oplus W_{i-1}|_n$$

$$= |W_i|_n + |2 \oplus W_{i-1}|_n$$

$$= |W_i|_n + |W_{i-1}|_{n-2}$$

$$= N(i, n) + N(i - 1, n - 2).$$

Also, by Lemma 2, $N(i, n) = 0$ whenever $n > i$. Now we prove that (10) holds when $i \geq n$. We use induction on $i$. This is true when $i = 0$ and $i = 1$ by (8) and (9). Suppose it holds for $i = 1, \ldots, k$. Then

$$N(k + 1, n) = \left( k - n + \left\lfloor \frac{n}{2} \right\rfloor \right) + \left( k - n + \left\lfloor \frac{n}{2} \right\rfloor - 1 \right)$$

$$\left( k - n + \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$$

as required.

3. Palindromes

Zhang et al. [11, Prop. 27] showed that there are no non-trivial palindromes in the ZWW word other than $22 \oplus 2i$, $232 \oplus 2i$, and $323 \oplus 2i$ for $i \geq 0$. There are also trivial palindromes consisting of a single letter. In the next two theorems we count the total number of palindromes in $W_i$ and the number of distinct palindromes in $W_i$. By Lemma 1, we have

$$W_i = W_{i-1} \cdot 2 \oplus W_{i-2} \quad \text{for } i \geq 0.$$

Palindromes in $W_i$ therefore come in three types: those contained in $W_{i-1}$, those contained in $2 \oplus W_{i-2}$ and those which straddle the boundary between $W_{i-1}$ and $2 \oplus W_{i-2}$. If a palindrome is of the last type for some $i$ we say it is straddling.

Lemma 6. The only straddling palindromes in the ZWW word are $22$, $232$, and $323$ which occur only in $W_3$ and $W_4$. 

Proof. A straddling palindrome must include the last letter of $W_{i-1}$ and the first letter of $2 \oplus W_{i-2}$ for $i \geq 2$. By Lemma 2 these are $i - 1$ and 2 respectively. Therefore both $2 \cdot (i - 1)$ and $(i - 1) \cdot 2$ must appear in the palindrome. Since 2 is always followed by 2 or 3 in the ZW W word, we must have $i = 3$ or 4. When $i = 3$ we have $W_i = 01223$ and when $i = 4$ we have $W_i = 01223234$. These contain the straddling palindromes 22, 232 and 323. There can be no other straddling palindromes in the ZWW word. □

Theorem 7. The total number of palindromes in $W_i$, including single letter palindromes, is 1, 2, 3, 6 for $i = 0, 1, 2, 3$ respectively, and $f_{i+3} - 2f_{i-1}$ for $i > 3$.

Proof. We write $P(W_i)$ and $S(W_i)$ for the total number of palindromes in $W_i$ and the number of straddling palindromes in $W_i$ respectively. It is easily checked that, for $i = 0, 1, 2$ and 3, $P(W_i)$ equals 1, 2, 3 and 6 respectively. Since the total number of palindromes in $W_i$ equals the total number in $2 \oplus W_i$ we get, from Lemma 1, that

$$P(W_{i+2}) = P(W_{i+1}) + S(W_{i+2}) + P(W_i).$$

From Lemma 6, $S(W_i) = 0$ when $i > 4$, so for $i > 2$, we have

$$P(W_{i+2}) = P(W_{i+1}) + P(W_i).$$

This recurrence gives $P(W_i) = f_{i+3} - 2f_{i-1}$. □

We write $D(W_i)$ for the number of distinct palindromes in $W_i$. It is easily checked that $D(W_i) = 0$ for $i = 0, 1$ and 2 and $D(W_3) = 1$.

Theorem 8. For $i \geq 2$, we have

\begin{equation}
D(W_i) = \lfloor 5i/2 \rfloor - 2.
\end{equation}

Proof. Since $W_{i+2} = W_{i+1} \cdot 2 \oplus W_i$ and there are no straddling palindromes in $W_i$ when $i > 4$, any palindrome appearing in $W_{i+2}$ for the first time must occur in $2 \oplus W_i$. It must equal $2 \oplus p$ where $p$ is a palindrome that occurred for the first time in $W_i$. From this we see that 22 and 3 first occur in $W_3$, 44 and 5 first in $W_5$, 66 and 7 first in $W_7$, and so on. Similarly 4, 232 and 323 occur for the first time in $W_4$, 6, 454 and 545 for the first time in $W_6$, and so on. From this we see that $D(W_{i+2}) = 5 + D(W_i)$ for $i \geq 3$. With the initial conditions $D(W_3) = 3$ and $D(W_4) = 8$ we obtain (11). □

4. Distinct Squares in $W_k$

Squares in the finite Fibonacci words have been characterised in [3] and [7]. In this section we characterise the distinct squares in $W_k$. Our analysis here depends on the factorisation of $W_k$ obtained in Lemma 1. We will also need the following lemma.

Lemma 9. Let $i$ and $k$ be integers such that $0 \leq i \leq \lfloor k/2 \rfloor$. Then the length $|W_{k-2i}|$ suffix of $W_k$ is $2i \oplus W_{k-2i}$.

Proof. We use induction on $k$. The lemma holds vacuously when $k = 0$ and $k = 1$. Suppose it holds up to $k - 1$. By Lemma 1,

$$W_k = W_{k-1} \cdot 2 \oplus W_{k-2}.$$ 

We see immediately that the lemma holds when $i = 1$. For $1 < i \leq \lfloor k/2 \rfloor$ the length $|W_{k-2i}|$ suffix of $W_k$ is the length $|W_{k-2-2(i-1)}|$ suffix of $2 \oplus W_{k-2}$ which is $2 \oplus$ the length $|W_{k-2i}|$ suffix of $W_{k-2}$. By the induction hypothesis this is

$$2 \oplus (2(i - 1) \oplus W_{k-2-2(i-1)}) = 2i \oplus W_{k-2i}$$

and the lemma is proved. □

With Lemma 4 this gives the following result.
Lemma 10. For $k \geq 0$,

$$W_{k+1} = S_{k,k} \cdot S_{k,k-2} \cdots S_{k,k-2[k/2]} \cdot (k+1)$$

where $S_{k,i}$ is the length $|W_i|$ suffix of $W_k$.

As an example of the factorisation presented in the above lemma, consider the following:

$$W_4 = 01223234$$
$$W_5 = 01223234 \cdot 23 \cdot 4 \cdot 5$$

We now prove the main result of this section. Since $W_k$ is a prefix of $W_{k+1}$ the set of squares in $W_{k+1}$ contains all those squares that appeared in $W_k$. Other squares in $W_{k+1}$ we call new squares.

Theorem 11. For every $k \geq 0$:

(a) $W_{k+1}$ introduces $\lfloor k/2 \rfloor$ new squares of periods $f_k, f_{k-2}, \ldots, f_{k-2[k/2]+2}$;

(b) $W_{k+1}$ contains exactly $\lfloor k/2 \rfloor \lfloor k/2 \rfloor$ distinct squares.

Proof. Consider the factorisation in the last lemma. Since $S_{k,i+2}$ is a suffix of $S_{k,i}$, $W_{k+1}$ contains squares $S_{k,k-2i}$ for $i = 1, \ldots, \lfloor k/2 \rfloor$. Each of these squares has period $|W_{k-2i}|$ which equals $f_{k-2i+2}$ for $i = 1, \ldots, \lfloor k/2 \rfloor$. We will show that these are all new squares and that they are the only new squares in $W_{k+1}$. Note that no new square can occur as a factor of $S_{k,k-2i}$ since such a square would be a factor of $W_k$ and therefore not new. Neither can a new square containing the letter $k+1$ since this occurs only once in $W_{k+1}$. So any new square must contain the last letter of $S_{k,k-2i}$ for some $i$. This letter is always $k$ and no square in $S_k$ contains $k$ since $k$ occurs only once in $S_k$. It follows that the squares described above are indeed new.

Now we show there are no other new squares in $W_{k+1}$. If there were others they would contain at least two copies of $k$. Such a square cannot contain more than two copies since the distances between consecutive occurrences of $k$ are different. Thus a new square must contain the $k$ at the end of $S_{k,k-2i}$ for some $i$, and the $k$ at the start of $S_{k,k-2i+1}$. If the square is not $S_{k,k-2i}$ then its first half contains the $k$ at the end of $S_{k,k-2i}$ and the letter following that $k$ which is $2i$. For it to be a square $2i$ should equal the letter at the end of $S_{k,k-2i+2}$ which is $i+1$ or $k$ and does not equal $2i$. Therefore we cannot have such a square and the only new squares are those noted above. This completes the proof of part (a).

The number of distinct squares in $W_{k+1}$ is then $\sum_{i=0}^{\lfloor k/2 \rfloor} |i/2|$ which equals $\lfloor k/2 \rfloor \lfloor k/2 \rfloor$. This is part (b). \qed

Recall that a run is a periodic factor whose length is at least twice its period. The computation of runs is important algorithmically [8, 9]. The maximum number of runs in any word of length $n$ is denoted $\rho(n)$: recently Bannai et al. [2] proved the long-standing conjecture that $\rho(n) < n$ for all $n$. If $X[1..2p+l]$ is a run with period $p$ and $l > 0$ then $X[1..2p]$ and $X[2..2p+1]$ are both squares. We saw in the proof of Theorem 11 that such pairs of squares do not exist in $W_k$. We therefore have the following result.

Corollary 12. Every run in the ZWW word is a square.
Claim. For $i < 3$, we have $S(W_i) = 0$, and for $i \geq 3$, we have $S(W_i) = 1$.

Proof of Claim. Recall that $|W_i| = f_{i+2}$. The cases for $i < 3$ are easily checked so we assume $i \geq 3$. To simplify notation we temporarily write $w$ for $W_i$ so that a straddling square must begin in $w[1..f_{i+1}]$ and end in $w[f_{i+1} + 1..f_{i+2}]$. We first show that, for such $i$, $S(w) \geq 1$. Note that, for $i \geq 2$,

\[
\begin{align*}
\text{for } i &< 3, \\
\text{for } i \geq 3, \\
\end{align*}
\]

Using (2) we have
\[
\begin{align*}
[1..f_i] &= W_{i-2} \\
w[f_i + 1..f_i + f_{i-1}] &= w[f_i + 1..f_{i+1}] = 2 \oplus W_{i-3} \\
w[f_{i+1} + 1..f_{i+1} + f_{i-1}] &= 2 \oplus W_{i-3} \\
w[f_{i+1} + f_{i+1} + f_{i-1} + f_{i-1}] &= w[f_{i+1} + f_{i-1} + 1..f_{i+2}] = 4 \oplus W_{i-4}
\end{align*}
\]

From (2) and Lemma 2, we see that
\[
[w[f_{i+1}] = w[f_{i+1} + f_{i-1}] = i - 1
\]
and there are no occurrences of $i - 1$ in $w$ before position $f_{i+1}$. So each half of the Square $(2 \oplus W_{i-3})^2$ ends in $i - 1$. Thus any straddling square must contain both these occurrences of $i - 1$. If it contained more than two occurrences of $i - 1$ it would have to contain both of these and its first half would contain $w[f_{i+1}..f_{i+1} + f_{i-1}]$ and so its period would be at least $f_{i-1} + 1$. But its second half would be contained in $w[f_{i+1} + f_{i-1} + 1..f_{i+2}]$ which has length

\[
f_{i+2} - f_{i+1} - f_{i-1} = f_{i-2}
\]

which is less than $f_{i-1} + 1$ so no such square can exist. We conclude that any straddling square in $w$ contains exactly these two instances of $i - 1$. Its period must therefore be $f_{i-1}$. We are therefore asking whether there is a square of length $2f_{i-1}$ in $(2 \oplus W_{i-3})^2 \cdot 4 \oplus W_{i-4}$ which is not a prefix. If there were such a square there would also be one in $W_{i-3}^2 \cdot 2 \oplus W_{i-4}$. This would necessarily contain the 0 at the start of the second $W_{i-3}$ but this is the only 0 in the word other than the initial 0. We conclude that no such square exists and the only straddling square is $(2 \oplus W_{i-3})^2$. So for $i \geq 3, S(W_i) = 1$.

Using this lemma, and noting that $T(2 \oplus W_i) = T(W_i)$, equation (12) becomes

\[
(13) \quad T(W_{i+2}) = T(W_{i+1}) + T(W_i) + 1
\]

with $T(W_1) = T(W_2) = 0$. The solution to this is $T(W_i) = f_i - 1$. $\square$

6. Lyndon factors

Theorem 14. For all $n \geq 0$, $W_n$ is a Lyndon word.

Proof. For each $n \geq 0$, $W_n$ begins with 0 and contains no other occurrences of 0 by Lemmas 1 and 2; hence $W_n$ is a Lyndon word. $\square$

Theorem 15. Let $L_k(W_n)$ denote the number of Lyndon factors beginning with the letter $k$ in $W_n$.

(i) For all $n \geq 0$, $L_0(W_n) = f_{n+2}$.

(ii) For odd $k \geq 1$ and $n \geq k$, we have $L_k(W_n) = f_{n+3-k} - 1$.

(iii) For all $n \geq 2$, $L_2(W_n) = \sum_{j=0}^{n-2} \left\{ \sum_{i=j}^{n-2} f_{i+2} - (n - 2 - j)f_{j+2} - f_{j+1} \right\} + 1$. 
(iv) For even \( k \geq 4 \) and \( n \geq k \), we have \( \mathcal{L}_k(W_n) = \mathcal{L}_2(W_{n+2-k}) \).

**Proof.**

(i) For each \( n \geq 0 \), \( W_n \) begins with 0 and contains no other occurrences of 0 (by Lemmas 1 and 2), so each of the \( f_{n+2} = |W_n| \) prefixes of \( W_n \) is a Lyndon word. Hence \( \mathcal{L}_0(W_n) = f_{n+2} \).

(ii) For each \( n \geq 1 \), \( W_n \) begins with 01 and contains no other occurrences of 0 and 1 (by Lemmas 1 and 2), so each of the \( f_{n+2} - 1 \) prefixes of \( 0^{-1}W_n \) is a Lyndon word. Hence \( \mathcal{L}_1(W_n) = f_{n+2} - 1 \). For odd \( k \geq 3 \), it follows from Lemma 2 that \( \mathcal{L}_k(W_n) = 0 \) for \( n < k \), and for \( n \geq k \), \( \mathcal{L}_k(W_n) \) is equal to the number of Lyndon factors beginning with 1 in \( W_{n-(k-1)} \), i.e., \( \mathcal{L}_k(W_n) = f_{n+2-(k-1)} - 1 \).

(iii) This part is trickier than the others and is proved separately in §6.1 below.

(iv) For even \( k \geq 4 \), \( \mathcal{L}_k(W_n) = 0 \) for \( n < k \) (by Lemma 2), and for \( n \geq k \), it follows from Lemma 1 that \( \mathcal{L}_k(W_n) \) is equal to the number of Lyndon factors beginning with 2 in \( W_{n-(k-2)} \), i.e., \( \mathcal{L}_k(W_n) = \mathcal{L}_2(W_{n-(k-2)}) \).

\[ \square \]

6.1. **Proof of part (iii) of Theorem 15.** For \( i = 2, 3, 4, 5, 6, 7, 8 \), we observe that \( \mathcal{L}_2(W_i) \) is equal to \( 1, 3, 7, 18, 42, 93, 195 \), respectively. These numbers look mysterious, without any obvious connection to the Fibonacci numbers, as one might expect. But, indeed, the Fibonacci numbers are involved in the formula for \( \mathcal{L}_2(W_i) \) that we determine below.

In what follows, we set

\[ X_i = 2 \oplus W_i \quad \text{for} \quad i \geq 0 \]

so that

\[
\begin{align*}
X_0 & = 2 \\
X_1 & = 23 \\
X_2 & = 234 \\
X_3 & = 23445
\end{align*}
\]

**Lemma 16.** For \( k \geq 2 \), we have

\[ W_k = 01X_0X_1\cdots X_{k-2}. \]

**Proof.** This is immediate from Lemma 3. \( \square \)

**Example:** \( W_5 = 01 \cdot 2 \cdot 23 \cdot 234 \cdot 23445 = 01 \cdot X_0 \cdot X_1 \cdot X_2 \cdot X_3. \)

**Lemma 17.** A factor of the ZWW word which begins with 2 is Lyndon unless it is bordered.

**Proof.** For the sake of contradiction suppose that \( u \) is an unbordered factor of the ZWW word with \( u[1] = 2 \) which is not Lyndon. Since it is not Lyndon, it has a suffix \( s \) which is lexicographically less than \( u \). Let \( p \) be the prefix of \( u \) of length \( |s| \). Since \( u \) is unbordered \( p \neq s \) and we must have \( s \) less than \( p \). Since \( p \) begins with 2 so must \( s \) and each therefore begins at the start of \( X_i \) for some \( i \). Say that \( p \) is a prefix of \( X_iX_{i+1}\cdots \) and \( s \) is a prefix of \( X_{i+j}X_{i+j+1}\cdots \). By Lemma 16, \( X_i \) is a prefix of \( X_{i+j} \). Let \( X' \) be the length \( |X_{i+j}| - |X_i| \) suffix of \( X_{i+j} \). Then we have a prefix of \( X'X_{i+j+1}\cdots \) which is lexicographically less than a prefix of \( X_{i+1}\cdots \). But that is impossible since \( X_{i+1}\cdots \) begins with 2 and \( X'X_{i+j+1}\cdots \) begins with something larger than 2. This contradiction completes the proof. \( \square \)

**Theorem 18.** The number of Lyndon words beginning with 2 in \( W_i \) is

\[
\sum_{j=0}^{i-2} \left\{ \sum_{k=j}^{i-2} f_{k+2} - (i - 2 - j)f_{j+2} - f_{j+1} \right\} + 1.
\]
Proof. Each Lyndon word beginning with 2 in $W_i$ must begin at the the beginning of $X_j$ for some $j$ with $0 \leq j \leq i - 2$. We first calculate the number of such words for a particular value of $j$ and then sum over $j$.

Let $u$ be a word beginning at the start of $X_j$ and ending in $W_i$. By (14) and Lemma 16, we see that each pair of consecutive 2s in $W_i$ is separated by a different distance. It follows that a border of $u$ cannot contain more than one 2. It therefore contains exactly one 2 and has length at most $|X_j|$. By Lemma 1, $X_j$ is a prefix of $X_k$ for all $k$ greater than $j$ so $u$ has a border if and only if $u$ ends inside a length $|X_j|$ prefix of some $X_k$ for $j + 1 \leq k \leq i - 2$. There are thus $(i - 2 - j)|X_j|$ words $u$ which are not Lyndon. For $j > 0$ the total number of words in $W_i$ that begin at the start of $X_j$ and finish after the length $|X_{j-1}|$ prefix of $X_j$ is

$$\sum_{k=j}^{i-2} f_{k+2} - |X_{j-1}| = \sum_{k=j}^{i-2} f_{k+2} - |X_{j-1}|.$$ 

The length $|X_{j-1}|$ prefix is excluded because any Lyndon word in it will duplicate one in $X_{j-1}$. Subtracting the number of non-Lyndon words from this gives the total number of Lyndon words in $W_i$ that begin at the start of $X_j$:

$$\sum_{k=j}^{i-2} f_{k+2} - (i - 2 - j)|X_j| - |X_{j-1}| = \sum_{k=j}^{i-2} f_{k+2} - (i - 2 - j)f_{j+2} - |X_{j-1}|.$$ 

To get the total number of Lyndon words in $W_i$ we sum this over $j$. In doing so we replace $|X_{j-1}|$ with $f_{j+1}$ when $j > 0$. When $j = 0$ there is nothing to subtract, however to make the formula nicer we subtract $f_{j+1}$ in this case as well and compensate by adding 1 to the final formula. The total number of Lyndon words beginning with 2 in $W_i$ is therefore

$$\sum_{j=0}^{i-2} \left\{ \sum_{k=j}^{i-2} f_{k+2} - (i - 2 - j)f_{j+2} - f_{j+1} \right\} + 1.$$

\[\square\]

7. LYNDON ARRAY OF $W_k$

The Lyndon array $\lambda = \lambda_X[1..n]$ (equivalently, $\mathcal{L} = \mathcal{L}_X[1..n]$) of a given non-empty word $X = X[1..n]$ gives at each position $i$ the length (equivalently, the end position) of the longest Lyndon word starting at $X[i]$. For example:

$$X = a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b$$

$$\lambda = 2\ 1\ 5\ 2\ 1\ 2\ 1\ 3\ 2\ 1$$

$$\mathcal{L} = 2\ 2\ 7\ 5\ 7\ 7\ 10\ 10\ 10$$

Clearly $\lambda[i] = \mathcal{L}[i] - i + 1$.

The Lyndon array has only recently been clearly defined [4], but turns out to have an intimate connection with the suffix array [10] that so far is not well understood: the suffix array can be computed from $\lambda$ in linear time using the Next Smaller Value algorithm [6, 4]; on the other hand, a sorted version of $\lambda$ is computed by Phase I of the first non-recursive linear-time suffix array computation algorithm due to Baier [1]. To date nine algorithms, including Baier’s, have been discovered to compute the Lyndon array of $X$, of which the fastest in practice is apparently a brute force approach that requires $\Omega(n^2)$ time in the worst case [4].

It is thus perhaps of interest to investigate the Lyndon arrays $\lambda_k/\mathcal{L}_k$ of $W_k$ for given finite $k$. Trivially, since for all positive $k$, $W_k[1..2] = 01$, we have $\mathcal{L}_k[1] = \mathcal{L}_k[2] = w_k$ where $w_k = |W_k| = f_{k+2}$. If for $k \geq 2$, \[\phantom{\text{MORE PROPERTIES OF THE FIBONACCI WORD ON AN INFINITE ALPHABET\;}}\]
we define $W_{k-1}^{k-2,2} = 2 \oplus W_{k-2}$ to express the suffix of $W_k$ in terms of a prefix of length $w_{k-2}$ of $W_{k-1}$, Lemma 3 yields the following.

**Remark 19.** For $k \geq 2$, $W_k = W_{k-1} \cdot W_{k-1}^{k-2,2}$.

Thus, using $w_{k-2}$, each $W_k$ can be computed by a direct calculation from $W_{k-1}$. The first few values, for $k = 2, 3, 4$, are

\[
\begin{align*}
W_2 &= W_1 \cdot W_1^{0,2} = 01 \cdot 2 \\
W_3 &= W_2 \cdot W_2^{1,2} = 012 \cdot 23 \\
W_4 &= W_3 \cdot W_3^{2,2} = 01223 \cdot 234
\end{align*}
\]

The following results are then immediate.

**Remark 20.**

(a) For $k \geq 2$, the digit 2 occurs only as the first digit of $W_{k-1}^{k-2,2}$.

(b) Since $W_{k-1}^{k-2,2}$ is a prefix of $W_{k-1}^{k-1,2}$, we have $W_{k-1}^{k-2,2} < W_{k-1}^{k-1,2}$.

Thus for $i \geq 2$, $W_k[i] = 2 \implies L_i = w_k$. Otherwise, $W_k[i] > 2$, and so for all such $i$, $\lambda_k[i]$ must take the same values assumed by the corresponding positions in $W_{k-1}$. More precisely:

**Lemma 21.** For $k \geq 1$,

(a) $1 \leq i \leq w_{k-1} \implies \lambda_k[i] = \lambda_{k-1}[i]$;

(b) $w_{k-1} + 1 \leq i \leq w_k \implies \lambda_k[i] = \lambda_{k-1}[i - w_{k-1}]$.

Based on this result, the algorithm shown in Figure 1 computes $L_{W_k}$ in time $\Theta(w_k)$; implementing the statements in square brackets also yields the computation of $W_k$. Apart from storage for $W_k$ and $L$, Algorithm LA requires only constant space.

**procedure LA** $(k)$

* Suppose that $w_k = f_{k+2}$ has been precomputed.

$[W[1..2] \leftarrow 01]; \ L[1..2] \leftarrow w_k w_k; \ w_- \leftarrow 2; \ w_{-2} \leftarrow 1$

for $j \leftarrow 2$ to $k$ do

$w \leftarrow w_- + w_{-2}$

for $i \leftarrow w_- + 1$ to $w$ do

$[W[i] \leftarrow W[i - w_-] + 2]$

if $i = w_{-1} + 1$ then

$L[i] \leftarrow w_k$

elsif $L[i - w_-] = w_k$ then

$L[i] \leftarrow w$

else

$L[i] \leftarrow L[i - w_-] + w_- n$

$w_{-2} \leftarrow w_-; \ w_- \leftarrow w$

**Figure 1.** Compute $W_k$ and its Lyndon array $L_{W_k}, k \geq 0$.

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