We propose a new method to generate the internal isospin degree of freedom by non-local bound states. This can be seen as motivated by Bargmann–Wigner like considerations, which originated from local spin coupling. However, our approach is not of purely group theoretical origin, but emerges from a geometrical model. The rotational part of the Lorentz group can be seen to mutate into the internal iso-group under some additional assumptions. The bound states can thereafter be characterized by either a triple of spinors ($\xi_1, \xi_2, \eta$) or a pair of an average spinor and a “gauge” transformation ($\phi, R$). Therefore, this triple can be considered to be an isospinor. Inducing the whole dynamics from the covariant gauge coupling we arrive at an isospin gauge theory and its Lagrangian formulation. Clifford algebraic methods, especially the Hestenes approach to the geometric meaning of spinors, are the most useful concepts for such a development. The method is not restricted to isospin, which served as an example only.

1 Introduction

Composite systems are of extraordinary importance in modelling physical systems. Using composites as building blocks one can build up more involved systems. However, an exact treatment of the bound systems show up to be not only difficult but impossible in general. While two body problems are usually solvable in classical mechanics, it has been proved that this cannot be done in general already for a three body configuration. Moreover, such a three body, or three degrees of freedom, situation turns out to be in general chaotic. This might be called the bound state problem. On the other hand, it is possible only to design exact solvable models in very idealized situations which have but a few contacts to real systems.

Already Luis de Broglie suggested the photon to be composite. Bargmann and Wigner derived locally coupled systems of spins. This story entered quantum field theory via Heisenbergs unified non-linear spinor field theory, which took the radical point to start with only one fundamental spinor field considering all (other) observable fields as composed quantities. A quite elaborate

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and extended version of a non-linear spinor theory of principally unobservable preon fields was developed by Stumpf, a Heisenberg and Bopp pupil, which succeeded in deriving, by exact calculations, the electro-weak interaction and the corresponding bosons on a quantum level as composite fields including their dynamics. However, due to the complicated and quite involved computations the strong and gravitational interactions resisted up to now an exact treatment. Nevertheless, the Stumpf theory of weak mapping can fairly well be applied to approximations, when the \( n \)-particle wave functions are not known exactly. This led to very fruitful calculations deriving QCD and a linearized version of gravity and a successful application to superconductivity including the important fact of symmetry breaking. Our criticism of this method, especially of the tedious composite wave function calculations and their algebraic ill-defined behaviour, was formulated in.

The method proposed in this paper relays, as the Stumpf theory, on an extension of the Bargmann–Wigner idea of local spin coupling. Since Stumpf tried to calculate non-local bound state wave functions as exact solutions of Bethe–Salpeter like equations, we surmount this difficulty by a simple geometric postulate. Hence, no dynamic origin of the bound mechanism is given.

Beside the fact, that also in the Bargmann–Wigner local coupling no such mechanism was proposed, we will see later, that this step puts the bound state problem in a nutshell and opens thereby the study of bound-state dynamics even if the details are not known exactly. This is much more important in non-abelian gauge theories like \( \text{SU}(2) \) electro-weak and \( \text{SU}(3) \) colour of QCD which are chaotic in the classical formulation.

2 Geometry of Hestenes spinors

The main idea, proposed by Hestenes since 1966, is to formulate Dirac theory representation independent. In usual, old fashioned notation, e.g. Bjorken and Drell, Diracs equations for a free field are written as

\[
i\gamma^\mu \partial_\mu \psi - m \psi = 0.
\]

The common interpretation is to look at the \( \gamma_{\alpha\beta}^\mu \) matrices as matrix valued components of a four-vector \( \vec{\gamma} \). With this preconception one runs immediately into difficulties. The spin indices \( \alpha, \beta \) cannot be considered to characterize components of the same space \( \mathbb{M}_{1,3} \) (Minkowski space) as the \( \mu \) index does. A so-called “internal” spin space, no comment what “internal” does mean here or is related to, is introduced. Denoting spin space \( \mathcal{S} \), one has \( \vec{\gamma} \) to be an element of a mixed tensor space

\[
\vec{\gamma} = \gamma_{\alpha\beta}^\mu e_\mu \otimes \xi_\alpha \otimes \xi_\beta \in \mathbb{M}_{1,3} \otimes \mathcal{S} \otimes \mathcal{S}^* \cong \gamma_{\alpha\beta}^\mu.
\]
Using Ockham’s razor, Hestenes simply identified the $\gamma$-matrices with the spinor representation matrices of the Minkowski basis vectors. Denoting the half integral spinor representations as $D_{1/2}$ and $D_{1/2}$, we can give a representation $\pi$ of an arbitrary but fixed basis $\{e^\mu\} \in M_{1,3}$ as spin-tensors

$$\pi(e^\mu) = \gamma^\mu \in D_{1/2} \otimes D_{1/2} = \gamma^\mu_\alpha \beta \in S^\alpha \otimes S^\ast_\beta.$$ (3)

Here $\mu$ is a label, not an index! Observe that these representation spaces include the Minkowski space. In this way, no additional internal space is needed.

Since the $\gamma$-matrices constitute a Clifford algebra over the reals $\mathbb{R}$, which can be denoted $\mathbb{R}_{1,3}$, one has the following decomposition

$$\mathbb{R}_{1,3} = 1 \oplus M_{1,3} \oplus B \oplus T \oplus V = S \otimes S^\ast,$$ (4)

where $B = M_{1,3} \wedge M_{1,3}$ is the space of bi-vectors, $T$ are tri-vectors and $V$ is the four-vector volume element. A complexification can be achieved as $\mathbb{C} \otimes \mathbb{R}_{1,3} \cong \mathbb{C}_{1,3}$ if desired. From this decomposition it is clear that the multivector spaces are invariant subspaces of the Clifford-Lipschitz group and especially of the various pin and spin groups, e.g.: $\text{spin}_{1,3} \cdot M_r \subset M_r$. Turning Dirac’s equations into a representation free scheme, yields the Dirac-Hestenes equation in terms of an arbitrary basis $\{e_\mu\}$ of the Minkowski space

$$\partial \Psi e_{12} - m \Psi e_0 = 0.$$ (5)

Notations are: $e_{12} = e_1 e_2$, $\partial = e^\mu \partial_\mu$ and $\Psi \in \mathbb{R}_{1,3}^+$ an even operator spinor. The condition to be even reduces the number of degrees of freedom to the correct value and splits the spin-tensor into two parts $S \otimes S = (S \otimes S)^+ \oplus (S \otimes S)^-$, where only the even part is again a subalgebra. However since one is interested in these spaces as representation spaces, only the linear structure is in use. The equation respects this decomposition if and only if the mass term vanishes, see. (8) Introducing $\Psi' = \Psi e_{12}$, we arrive at the equation $\partial \Psi' = 0$.

The important thing is, that $\Psi$ as an algebra element obtains an operational interpretation. For $1, i_4$ or $\gamma^0$ one gets

$$\Psi_1 \gamma^{i_4} = \Omega_1, \quad \Psi i_4 \Psi^{i_4} = \Omega_2 i_4, \quad J^\mu = <\Psi \gamma^0 \Psi^{\gamma^0} >_1 = <\Psi \gamma^0 \Psi^{\gamma^0} >_1 \gamma^\mu, \quad (6)$$

where $\gamma$ is the reversion antiautomorphism and $< \ldots >_1$ the projection to the one-vector part. We can give a polar decomposition of $\Psi$:

$$\Psi = (\Psi \Psi')^{1/2} \frac{\Psi}{(\Psi \Psi')^{1/2}} = \rho^{1/2} \mathbb{V},$$ (7)
where $V$ is in the Clifford-Lipschitz group. Since spinors are defined only up to a phase, we extract the duality rotation with Yvon-Takabayasi angle $\beta$ from $V$, which Hestenes interprets as statistical. The polar form is then

$$\Psi = \rho^{\frac{1}{2}} V = \rho^{\frac{1}{2}} e^{i \beta} R.$$  \hspace{1cm} (8)

$R$ has six degrees of freedom (eight of $\Psi$ minus $\rho, \beta$) and is an element of the double cover of the Lorentz group $\text{spin}_{1,3}$, inducing Lorentz transformations on vectors. A Dirac-Hestenes spinor is thus an operational object, which transforms a reference object into locally given ones. As an example

$$J^\mu(x) = \Psi(x) e^\mu \Psi^\dagger(x) = \rho(x) e^{i \beta(x)} R(x) e^{\mu} R^\dagger(x) = \rho(x) e^{i \beta(x)} L^\mu_\nu(x) e^\nu.$$  \hspace{1cm} (9)

The current density is seen to be $\rho(x)$ times a duality rotation times a local Lorentz transformation $L^\mu_\nu(x)$ of the fixed reference system $\{e^\mu\}$. This interpretation will be the starting point for our geometric composite model.

### 3 Geometric model for bound states

#### 3.1 Bargmann-Wigner bound states

Recalling the idea of Bargmann and Wigner, we start with free spinors. How are dynamical equations derived for spin tensors? Restricting ourself to two particle systems, every rank two spin tensor can be decomposed into irreducible parts using the symmetric group. We obtain

$$\xi_1 \otimes \xi_2 = \xi_{(1} \otimes \xi_{2)} \oplus \xi_{[1} \otimes \xi_{2]}$$  \hspace{1cm} (10)

where [12] and (12) means (anti) symmetrization of the tensors. Since the $\xi_i$ are supposed to carry a $D_\frac{1}{2}$ spin one-half representation, we end up with a spin one triplet (symmetric case) and a spin zero singlet (antisymmetric case).

Concentrating on the symmetric case, we can give the equations of motion as

$$(\gamma^\mu \partial_\mu - m)_{\alpha \alpha'} \Psi_{(\alpha' \beta)} = 0.$$  \hspace{1cm} (11)

Expanding this in a symmetric basis of $\gamma$-matrices $\gamma^\mu C$, $\Sigma^{\mu \nu} C = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) C$, where $C \gamma^\mu C^{-1} = -\gamma^\mu T$, leads to a decomposition of $\Psi$ into

$$\Psi_{(\alpha \beta)} = A_\mu (\gamma^\mu C)_{\alpha \beta} + F_{\mu \nu} (\Sigma^{\mu \nu} C)_{\alpha \beta}.$$  \hspace{1cm} (12)

Since $A_\mu$ has four and $F_{\mu \nu}$ has six components, they cannot be independent. One finds the abelian $U(1)$ gauge theory

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \cong dA$$  \hspace{1cm} (13)

$$\Box A_\mu - \partial_\mu (\partial_\nu A_\nu) = m^2 A_\mu.$$
In a suitable gauge and the limit of vanishing mass one arrives at the theory of vacuum electrodynamics. Spin zero particles or spin $3/2$ Proca fields emerging from third rank spin-tensors can be treated analogously. However, it can be shown, that e.g. gravitational forces cannot be modelled in this way. Also several other difficulties remain unsolved. Finally we remark, that there is no direct contact between the dynamics of the constituents and the composite. Bargmann and Wigner assume simply the same differential operator to act on every tensor of any rank.

3.2 Non-local geometric bound states

A geometric composite model is proposed, which is based on the Hestenes interpretation of Dirac spinors. This model is also \textit{ad hoc} in some sense, but cures some of the above mentioned problems of local Bargmann-Wigner states.

Given two Dirac particles $\xi_1$ and $\xi_2$ forming a non-local bound system, we do not concentrate on informations about the internal dynamics, but try to give a dynamics for the entire bound system. We propose: i) The bound object shall be characterizable by the usual quantum numbers. ii) The dynamics of composites shall follow from the dynamics of the free constituents.

From our consideration we conclude the following model: Let $(\xi_1, \xi_2, \eta)$ be a triple of spinors. The $\xi_i$ are the spinors of the constituents and $\eta$ is a reference spinor connected to the $\xi_i$ by $\xi_i = \rho^{1/2}_i \exp(i\beta/2) R_i \eta$. Assume that there is a very small vector $\vec{a}$ satisfying $\vec{x}_2 = 2 \star \vec{a} + \vec{x}_1$. The time average over an internal period of $\vec{a}$ shall be constant. Since the $\xi_i$ spins are space-like separated, they can have \textit{any alignment}. However we will discard here reflections, boosts and assume that rotations can map the spinors onto one another. We introduce hence an average spinor $\phi$ and a Lorentz transformation $R$ which describes all internal degrees of freedom of the composite beside internal radial oscillations, boosts and mutual internal reflections. In fact this removes an internal phase. One observes the correspondence

$$(\xi_1, \xi_2, \eta) \cong (\phi, R) \quad (14)$$

$$# 7 + 7 + 0 = 7 + 3 + (3 \text{ boosts} + 1 \text{ radial}),$$

where the second line gives the degrees of freedom. The reference spinor $\eta$ is not subjected to change, and has no freedom.

4 Induced gauge theory

We derive the covariant coupling of $\phi$ and $R$ as a direct consequence of our geometric composite model. Thereafter the theory is completed by inducing
the full action from the covariant coupling. The curvature two-form or kinetic field energy term is added to turn the local Lorentz field $R$ into a dynamical one, which has consequently to be chosen to be a boson.

From our geometrical model we derive that there exists a Lorentz transformation $S$ which connects the two spinors $\xi_i$, see Figure 1:

$$S\xi_2 = \xi_1, \quad S^{-1}\xi_1 = \xi_2, \quad S^* = S^{-1}. \quad (15)$$

The inverse exists because of the group structure. But observe, that we can also build the square root if we restrict ourselves to the compact part of the Lorentz group (rotations). More generally this can be done if the spin directions are not on the light-cone, which is impossible here. Hence we define

$$R^2 = S, \quad R^* = R^{-1}. \quad (16)$$

Introduce a centre of mass or average spinor $\phi$ which satisfies the following two equations (which actually defines $\phi$)

$$\xi_1 = R\phi, \quad \xi_2 = R^{-1}\phi. \quad (17)$$

More precisely we should include the translations $T_{\vec{a}}$ also, but in a dilute gas at moderate temperatures the particles will usually be far away from another and lock point like, so internal vibrations are not important.

In deriving the dynamics we have to remember that $R(x)$ is a field. Furthermore, we use the free field dynamics for the $\xi_i$ to derive the composite dynamics. Starting from the Dirac-Hestenes equation

$$\nabla \xi_1 e_{12} + m \xi_1 e_0 = 0 \quad (18)$$

inserting $\xi_1 = R\phi$, we get

$$\nabla (R\phi)e_{12} + m(R\phi)e_0 = 0. \quad (19)$$
Using $RR^* = 1$ and the Leibnitz rule yields $(R'(\nabla R) + \nabla)\phi e_{12} + m\phi e_0 = 0$ which can be summarized with help of the covariant derivative $D := \nabla + R'(\nabla R)$ to

$$D\phi e_{12} + m\phi e_0 = 0. \quad (20)$$

Indeed, $R'(\nabla R)$ can be considered to be a vector field. Following Hestenes we might introduce $\Omega = 2R'(\nabla R)$ to obtain the derivative of $R$ by $\nabla R = R/2(2R'(\nabla R)) = 1/2 R\Omega$. The covariant derivative reads now

$$D = \nabla + \frac{1}{2} \Omega, \quad (21)$$

where the coupling constant is included in $\Omega = g\Omega'$. The field strength can easily be calculated as usual by computing the commutator of covariant derivatives, which yields a non-abelian $SU(2)$ from the covering group of the rotational part of the Lorentz group,

$$F = [D, D] = [\nabla + \frac{1}{2} \Omega, \nabla + \frac{1}{2} \Omega]. \quad (22)$$

In a symbolic notation this can be written as

$$F = d\Omega + \frac{1}{4} \Omega \times \Omega \quad (23)$$

where the first term is structural equivalent to the abelian field strength and the second term is the non-abelian left regular action of $\text{spin}_3 \equiv SU(2)$ on the Lie group generators. This sort of action can now be seen to constitute an internal isospin of the geometric composite proposed above. The full action of this theory can be written as

$$\mathcal{L} = \langle \phi(D + \frac{1}{2} \Omega)\phi^* + \frac{1}{4} F^2 \rangle_0. \quad (24)$$

It occurs now, that from a geometrical composite model, using operational spinors, we were able to derive a $SU(2)$ gauge covariant coupling and a $SU(2)$ gauge theory by inducing the gauge field dynamics. The reversed statement would be, that every gauge field might be seen as an internal motion. Since only compact groups are used, this internal degrees of freedom are due to bound structures. Our approach should be compared with the purely gauge theoretic treatment of Daviau which shows clearly the isospin acting on different representations.

Remark that for a single spinor it is not necessary to introduce a reference spinor $\eta$, while this is un-evitable, due to relative adjustments, for two or n-particle systems. It is possible to account for internal radial oscillatory motions in our approach, which then would “gauge” $T_d$. 
Since we omitted boosts, which are however not compact, our bound state concept is not relativistic invariant, in spite of the resulting theory which is. This has to be considered elsewhere.

The geometric bound model is in accord with some very general considerations on linear forms of multiparticle systems which were investigated in [14]. It furthermore explains a factor 2 not understood in the Stumpf weak mapping formalism which there forced the temporal gauge to be used [33].

Finally a strong support is given, that gauge theories might arise always from dynamical effects in theories of compound objects.

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References

1. L. de Broglie, Théorie général des particules à spin Gautier–Villars/Paris, 2nd. ed. (1954)
2. V. Bargmann, E. Wigner, Proc. Nat. Acad. Sci. (USA) 34, 211–223 (1948)
3. W. Heisenberg, Einführung in die einheitliche Feldtheorie der Elementarteilchen Hirzel/Stuttgart (1967)
4. H. Stumpf, Th. Borne,Composite particle theory in quantum field theory Vieweg/Braunschweig (1994)
5. B. Fauser, H. Stumpf, Adv. in Appl. Cliff. Alg. 7(sup.), 399–418 (1994)
6. D. Hestenes, Space time algebra Gordon and Beach (1966)
7. J.D. Bjorken, S.D. Drell, Relativistische Quantenmechanik Mc Graw-Hill inc. 1964, BL-Wissenschaftsverlag/Mannheim (1966)
8. B. Fauser, Proc. of the 4th Int. Conf. on Clifford alg. and their appl. in Math. Phys., Aachen, Kluwer/Dordrecht, 89–107 (1996)
9. D. Hestenes, R. Gurtler, J. Math. Phys. 16(3), 556 (1975)
10. R. Gurtler, D. Hestenes, J. Math. Phys. 16(3), 573 (1975)
11. D. Lurie, Particle and Fields Interscience Publ. (1968)
12. D. Hestenes, New foundation for classical mechanics Kluwer/Dorchrecht (1986)
13. C. Daviau, Equation de Dirac non lineaire, Thesis, Univ. Nantes (1993)
14. B. Fauser, Clifford geometric parameterization of inequivalent vacua, J. Phys. A: Math. Gen., submitted (1999) hep-th/9710047
15. W. Pfister, Yang-Mills-Dynamik als effektive Theorie von vektoriellen Spinor-Isospinor-Bindungszuständen in einem Preonfeldmodell, Thesis, Univ. Tübingen (1990)