$W_\infty$ ALGEBRAS IN THE QUANTUM HALL EFFECT

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Abstract

We show that a large class of incompressible quantum Hall states correspond to different representations of the $W_\infty$ algebra by explicit construction of the second quantized generators of the algebra in terms of fermion and vortex operators. These are parametrized by a set of integers which are related to the filling fraction. The class of states we consider includes multilayer Hall states and the states proposed by Jain to explain the hierarchical filling fractions. The corresponding second quantized order parameters are also given.

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1. Introduction

The quantum Hall effect \(^1,^2\) (QHE) appears in two-dimensional systems of electrons in the presence of a strong perpendicular uniform magnetic field \(B\). It is characterized by the existence of a series of plateaux where the Hall conductivity is quantized and the longitudinal conductivity vanishes. The Hall conductivity is proportional to the filling fraction \(\nu\), the ratio between the number of electrons and the degeneracy of the Landau levels. This generic feature which appears in both the integer (IQHE) and the fractional (FQHE) quantum Hall effect is attributed to the existence of a gap, which gives rise to an incompressible ground state. These correspond to incompressible droplet configurations of uniform density \(\rho = \nu B/2\pi\).

Recently the notion of incompressibility has been related to the existence of an infinite dimensional algebraic structure, the \(W_\infty\) algebra, which emerges quite naturally at least in the case of the IQHE \(^3,^4\). In this case \((\nu = n)\) the energy gap is the cyclotron energy separating adjacent Landau levels and the phenomenon can be understood in terms of noninteracting fermions. The underlying \(W_\infty\) algebra emerges as the algebra of unitary transformations which preserve the particle number at each Landau level and it plays the role of a spectrum generating algebra. The corresponding ground state satisfies highest weight conditions\(^4\).

The noninteracting picture is nonapplicable in the case of the FQHE, where the repulsive Coulomb interactions among electrons become important in producing an energy gap. Much of our understanding of the FQHE relies on successful trial wavefunctions. For example, in an attempt to explain the FQHE for \(\nu = 1/m\), where \(m\) is an odd integer, Laughlin proposed\(^5\) a set of wavefunctions which turn out to be quite close to the exact numerical solutions for a large class of repulsive potentials\(^6\). Based on Laughlin wavefunctions, a hierarchy scheme\(^6,^7\) has been developed to explain the other rational values of \(\nu\). A somewhat different approach has been developed by Jain\(^8\). In this approach the FQHE wavefunction is constructed by attaching an even number of magnetic fluxes to
electrons occupying an integer number of Landau levels. This way the incompressibility of the IQHE wavefunctions is carried over to the FQHE wavefunctions.

Following the example of the IQHE, attempts have been made to extend the connection between the incompressibility of the ground state and the $W_\infty$ algebra in the case of the FQHE$^{[4,9,10,11]}$. In ref.$^{[10]}$, using the relation between the $\nu = 1$ and $\nu = 1/m$ ground state, we found that there exists a second quantized expression of a $W_\infty$ algebra which plays the role of a spectrum generating algebra for Laughlin wavefunctions such that the Laughlin ground state satisfies highest weight condition. This provides a specific one-parameter family of $W_\infty$ representations, the parameter being related to the filling fraction.

In this paper we extend these ideas to more general filling fractions $\nu$. We shall derive explicit representations, in terms of second quantized fermion and vortex operators, of the $W_\infty$ algebra generators for the cases of multilayer incompressible states and the states proposed by Jain$^{[8]}$ to explain the hierarchical filling fractions. All these ground states satisfy highest weight conditions. We shall also derive second quantized expressions for the corresponding order parameters.

More generally incompressible states can be thought of as highest weight states of the $W_\infty$ algebra$^{[11]}$. Expressing the $W_\infty$ generators in terms of Fock operators is a useful technique for constructing representations and of course the highest weight state. It is thus possible to go beyond the particular examples we have considered here and obtain other candidate states for describing quantum Hall effect.

2. $W_\infty$ algebras for IQHE

The single-body Hamiltonian for spin polarized (scalar) fermions confined in a two-dimensional plane, in the presence of a uniform magnetic field $B$ perpendicular to the plane is (in units $\hbar = c = e = 1$)

$$H = \frac{1}{2M} (\Pi)^2$$

(2.1)

where $\Pi^i = p_i - A^i(x)$, $i = x, y$ and $\vec{\nabla} \times \vec{A} = -B$. 

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We define the following two sets of independent raising and lowering operators

\[ a = \frac{1}{\sqrt{2B}}(\Pi^x - i\Pi^y) \quad a^\dagger = \frac{1}{\sqrt{2B}}(\Pi^x + i\Pi^y) \quad [a, a^\dagger] = 1 \]

\[ b = \sqrt{\frac{B}{2}}(X + iY) \quad b^\dagger = \sqrt{\frac{B}{2}}(X - iY) \quad [b, b^\dagger] = 1 \]

(2.2)

where \( X, Y \) are the guiding center coordinates

\[ X = x - \frac{1}{B}\Pi^y \quad Y = y + \frac{1}{B}\Pi^x \quad [X, Y] = \frac{i}{B} \]

(2.3)

The two sets of operators \( a, a^\dagger \) and \( b, b^\dagger \) are mutually commuting: \( [a, b] = [a, b^\dagger] = 0 \).

The Hamiltonian can be written as

\[ H = \omega(a^\dagger a + 1/2), \quad \omega = B/m \]

(2.4)

and the angular momentum operator is given by \( J = b^\dagger b - a^\dagger a \).

Any state can be now decomposed in terms of the basis \( |n\rangle \otimes |l\rangle \equiv |n,l\rangle \), which is a tensor product of Fock states defined by

\[ a^\dagger a|n\rangle = n|n\rangle \quad \langle n'|n\rangle = \delta_{n',n} \]

\[ b^\dagger b|l\rangle = l|l\rangle \quad \langle l'|l\rangle = \delta_{l',l} \]

(2.5)

Let us also introduce the coherent basis representation for the \( b \) oscillators

\[ b|\zeta\rangle = \zeta|\zeta\rangle \quad \langle \zeta|\zeta'|\rangle = e^{\bar{\zeta}\zeta'} \quad \langle l|\zeta\rangle = \frac{\zeta^l}{\sqrt{l!}} \]

(2.6)

where \( |\zeta\rangle = e^{b^\dagger \zeta}|0\rangle \) and \( \int d^2\zeta e^{-|\zeta|^2} \langle \zeta|\zeta\rangle = 1 \), \( d^2\zeta = \frac{dRe\zeta dIm\zeta}{\pi} \).

The fermionic operator can be similarly decomposed as

\[ \Psi(\vec{x}, t) = \sum_{n,l=0}^{\infty} C^l_n(t)\Psi_{n-l}^l(\vec{x}) \]

(2.7)

where \( \langle l, n| \equiv 0|C^l_n \) and \( \Psi_{n-l}^l(\vec{x}) \equiv \langle \vec{x}|n, l \) is the one-body wavefunction of energy \( \omega(n + 1/2) \) and angular momentum \( l - n \). The subset of states with fixed \( n \) form the \( n \)-th Landau
level which is obviously infinitely degenerate with respect to the angular momentum. In a second quantized language the operators $C^m_l$ satisfy the usual anticommutation relations

\[ \{C^m_l, C^{m'}_{l'}\} = \delta_{m,m'}\delta_{l,l'} \]  

which, as it is obvious from eq. (2.7) corresponds to the standard quantization condition

\[ \{\Psi(\vec{x}, t), \Psi^\dagger(\vec{x}', t)\} = \delta(\vec{x} - \vec{x}') \]

So far we have not made a gauge choice. From now on we choose to work in the symmetric gauge $\vec{A} = \frac{B}{2}(y, -x)$. Using the relation\(^{12}\)

\[ \langle \vec{x}|n, \zeta\rangle = i^n \sqrt{\frac{B}{2\pi}} \frac{(z - \zeta)^n}{\sqrt{n!}} e^{-1/2|z|^2 + \bar{z}\zeta} \]  

where $|n, \zeta\rangle \equiv |n\rangle \otimes |\zeta\rangle$ and $z = \sqrt{\frac{B}{2}}(x + iy)$, we find that the fermion operator is now of the form

\[ \Psi(\vec{x}, t) = \sqrt{\frac{B}{2\pi}} e^{-1/2|z|^2} \sum_{n,l=0}^{\infty} i^n C^m_l(t) \frac{(z - \partial_{\bar{z}})^n}{\sqrt{n!}} \frac{\bar{z}^l}{\sqrt{l!}} \]  

\[ = \sqrt{\frac{B}{2\pi}} e^{-1/2|z|^2} \sum_{n=0}^{\infty} \psi^n(z, \bar{z}, t) \]  

The quantization condition for the fermion operator constrained to be in the $n$-th Landau level is

\[ \{\psi^n(z, \bar{z}, t), \psi^{n'}(z', \bar{z}', t)\} = \frac{1}{n!} (z - \partial_{\bar{z}})^n(z' - \partial_{\bar{z}}')^n e^{\bar{z}\zeta} \]  

The rhs of (2.11) is essentially the projection of the $\delta$-function on the $n$-th Landau level.

Let us now consider the case of noninteracting fermions that occupy the first $n$ Landau levels. In the absence of an external potential the system is symmetric under independent unitary transformations in the space of $C$’s acting at each Landau level, written as:

\[ C^I_l(t) = u_{lk} C^I_k(t) = \langle l|u|k\rangle C^I_k(t) \quad I = 0, 1, \ldots, n-1 \]  

(We mostly use the notation of ref.\([3]\)) An infinitesimal unitary transformation is generated by a hermitian operator which we write as $\frac{1}{\hbar} \xi(\hat{b}, \hat{b}^\dagger)$ with the antinormal ordering symbol,
where \( \xi \) is a real function when \( \hat{b} \) and \( \hat{b}^\dagger \) are replaced by \( z \) and \( \bar{z} \) respectively. Using eq.(2.12) we obtain the following infinitesimal transformation for the \( I \)-th Landau level fermionic operator:

\[
\delta \Psi^I (\vec{x}, t) = i \langle \vec{x} | \hat{\xi} (b, b^\dagger) | I, k \rangle C_k^I (t) \\
= i \int d^2 \zeta e^{-|\zeta|^2} \langle \vec{x} | \zeta (I, \xi^I (b, b^\dagger)) | I, k \rangle C_k^I (t) \\
= i \hat{\xi} (\partial_{\bar{z}} + \frac{z}{2} \frac{\bar{z}}{2} - \partial_z) \hat{\Psi}^I (\vec{x}, t) \\
= 0 \\
\] (2.13)

where \( \hat{\Psi}^I (\vec{x}, t) = \Psi^I (\vec{x}, t) \Psi^I (\vec{x}, t) \) is the \( I \)-th Landau level fermion density. The generators of the transformations (2.13) are given by

\[
\rho^I [\xi] \equiv i \int d^2 z e^{-|z|^2} \psi^I (z, \bar{z}) \hat{\xi} (\partial_{\bar{z}} \psi^I (z, \bar{z}) - \partial_z) \psi^I (z, \bar{z}) \\
\] (2.15)

where \( d^2 z \equiv \frac{B}{2\pi} dx dy \). These operators satisfy the commutation rules

\[
[\rho^I [\xi_1], \rho^J [\xi_2]] = \delta^{IJ} \frac{i}{B} \rho [\{ \xi_1, \xi_2 \}] \\
\] (2.16)

where

\[
\{ \xi_1, \xi_2 \} = i B \sum_{n=1}^{\infty} \frac{(-)^n}{n!} (\partial^n_{\bar{z}} \xi_1 \partial^n_{\bar{z}} \xi_2 - \partial^n_z \xi_1 \partial^n_z \xi_2) \\
\] (2.17)

\( \{ \} \) is the so-called Moyal bracket. The algebra (2.16)-(2.17) is the direct sum of \( n \) copies of mutually commuting \( W_\infty \) algebras\(^{13-14} \). By choosing \( \xi (z, \bar{z}) = z^l \bar{z}^k \) we obtain

\[
[\rho^I_{rs}, \rho^J_{lk}] = \delta^{IJ} \sum_{n=1}^{\text{min}(s,t)} \frac{(-)^n}{n!} \frac{l! s!}{(l-n)! (s-n)!} \rho_{r+l-n, s+k-n} - (s \leftrightarrow k, r \leftrightarrow l) \\
\] (2.18)

where \( \rho^I_{lk} = \int d^2 z e^{-|z|^2} \psi^I (z, \bar{z}) (\partial_{\bar{z}})^l (\bar{z} - \partial_z)^k \psi^I (z, \bar{z}) \) and it can be expressed in terms of \( C \)'s as

\[
\rho^I_{lk} = \sum_{n=\max(0,l-k)}^{\infty} \frac{(n+k)!}{\sqrt{n! (n+k-l)!}} C_{n+k-l}^I C_n^I \\
\] (2.19)
Let us consider the action of these operators on the ground state $|\Psi_{\nu=n}\rangle_0$, where

$$|\Psi_{\nu=n}\rangle_0 = \prod_{I=0}^{n-1} (C^I_0 \ldots C^I_{N-1}) |0\rangle \quad (2.20)$$

for $N' = nN$ electrons. It is clear that since the operators $\rho^I_{lk}$ decrease the angular momentum for $l > k$ and increase the angular momentum for $l < k$ they satisfy $[4]$:

$$\rho^I_{lk} |\Psi_{\nu=n}\rangle_0 = 0 \quad \text{if} \quad l > k \quad I = 0, 1, \ldots, n - 1$$

$$\rho^I_{lk} |\Psi_{\nu=n}\rangle_0 = |\Psi_{n,I}\rangle \quad \text{if} \quad l \leq k \quad I = 0, 1, \ldots, n - 1 \quad (2.21)$$

where $|\Psi_{n,I}\rangle$ correspond to excitations of higher angular momentum at the $I$-th level. The first line in (2.21) can be considered as the algebraic statement of incompressibility of the $\nu = n$ ground state. As far as excitations are concerned, if $k - l \sim O(1)$, they correspond to edge excitations of the $I$-th level $[15]$. We expect that in the semiclassical limit the $W_\infty$ algebra in eqs. (2.16)-(2.17), reduces to the algebra of area-preserving diffeomorphisms $[3,4,16]$ (see eq.(2.14)). Upon restriction to the low energy edge excitations it gives rise to a $(U(1))^n$ Kac-Moody algebra describing $n$ independent chiral bosons $[15,17-19]$, one for each Landau level.

3. $W_\infty$ algebras in the lowest Landau level

In this section we shall consider cases where the electrons are confined in the lowest Landau level. First we shall briefly review the derivation of $W_\infty$ algebras for $\nu = 1/m$ Laughlin states and their relation to $\nu = 1$ $W_\infty$ algebras, as given in ref.[10], which will be crucial in constructing similar algebraic structures for quantum Hall fluids of general filling fraction.

The main point in this derivation is the simple observation that the $\nu = 1/m$ Laughlin ground state wavefunction is related to the $\nu = 1$ wavefunction by attaching $2p$ (where $m = 2p + 1$) flux quanta to each electron

$$\Psi^0_{\nu=1/m} = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2p} \Psi^0_{\nu=1} \quad (3.1)$$
In a second quantized language we have that (in this section we have dropped the superscript “0” used to denote the lowest Landau level fermion operators)

\[ |\Psi_{\nu=1/m}\rangle_0 = U_{2p} |\Psi_1\rangle_0, \quad |\Psi_1\rangle_0 = \tilde{U}_{2p} |\Psi_{\nu=1/m}\rangle_0 \]

(3.2)

where

\[ |\Psi_1\rangle_0 = \int d^2 z_1 \ldots d^2 z_N e^{-\sum_i |z_i|^2 - \sum_{i<j} (\bar{z}_i - \bar{z}_j)^2 |z_1 \ldots z_N\rangle} \]

(3.3)

and

\[ U_{2p} = \sum_{N=2}^{\infty} \int d^2 z_1 \ldots d^2 z_N e^{-\sum_i |z_i|^2 \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2p} |z_1 \ldots z_N\rangle \langle z_1 \ldots z_N|} \]

\[ \tilde{U}_{2p} = \sum_{N=2}^{\infty} \int d^2 z_1 \ldots d^2 z_N e^{-\sum_i |z_i|^2 \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{-2p} |z_1 \ldots z_N\rangle \langle z_1 \ldots z_N|} \]

(3.4)

where \( |z_1 \ldots z_N\rangle = \frac{1}{\sqrt{N!}} \psi^\dagger(z_1) \ldots \psi^\dagger(z_N)|0\rangle \). The operator \( \tilde{U}_{2p} \) is in general singular but its action on the space of the \( \nu = 1/m \) Laughlin states is well defined and \( \tilde{U}_{2p} U_{2p} = U_{2p} \tilde{U}_{2p} = 1 \). It is clear now that

\[ W_{2p}[\xi] \equiv U_{2p} \rho[\xi] \tilde{U}_{2p} \]

(3.5)

are the corresponding \( W_\infty \) algebra generators for \( \nu = 1/m \). They are mainly the original \( \nu = 1 \) generators transformed by a similarity transformation.* Corresponding to (3.5) there is a second quantized expression in terms of fermion and quasihole operators

\[ W_{2p}[\xi] = \int d^2 z e^{-|z|^2} \psi^\dagger(z) e^{2p\alpha(z)} \frac{\partial}{\partial \bar{z}} \psi^\dagger(\bar{z}) e^{-2p\alpha(\bar{z})} \psi(\bar{z}) \]

(3.6)

where

\[ \alpha(\bar{z}) = \int d^2 z' e^{-|z'|^2} \ln(\bar{z} - \bar{z}') \psi^\dagger(\bar{z}') \psi(\bar{z}') \]

(3.7)

and \( e^{\alpha(\bar{z})} \) is the quasihole operator since

\[ e^{\alpha(\bar{z})} |\Psi\rangle = \int d^2 z_1 \ldots d^2 z_N e^{-\sum_i |z_i|^2 \prod_{i<j} (\bar{z}_i - \bar{z}_j)^2 |z_1 \ldots z_N\rangle} \]

(3.8)

* Such transformations have been recently used in the context of quantum gravity in ref.[27].
where 

$$|\Psi\rangle = \int d^2z_1...d^2z_N e^{-\sum_i |z_i|^2} F(\bar{z}_1,...,\bar{z}_N)|z_1\cdots z_N\rangle \quad (3.9)$$

We see that the appropriate insertion of the vortex (quasihole) operators in (3.6) reflects the underlying similarity transformation (3.5). Using the fact that

$$\psi(\bar{z}') e^{n\alpha(\bar{z})} = (\bar{z} - \bar{z}')^n e^{n\alpha(\bar{z})} \psi(\bar{z}')$$

$$e^{n\alpha(\bar{z})} \psi^\dagger(\bar{z}') = (\bar{z} - \partial_\bar{z}')^n \psi^\dagger(\bar{z}') e^{n\alpha(\bar{z})} \quad (3.10)$$

where \(n\) is an integer and \([\alpha(\bar{z}), \alpha(\bar{z}')] = 0\) we can show that the operators \(W_{2p}\) satisfy a strong \(W_\infty\) algebra

$$[W_{2p}[\xi_1], W_{2p}[\xi_2]] = W_{2p}[\{\xi_1, \xi_2\}] \quad (3.11)$$

where \(\{\}\) is the Moyal bracket defined earlier in (2.17). They further play the role of a spectrum generating algebra* in the space of Laughlin states, since

$$(W_{2p})_{lk}|\Psi_{\nu=1/m}\rangle_0 = 0 \quad \text{if} \quad l > k$$

$$(W_{2p})_{lk}|\Psi_{\nu=1/m}\rangle_0 = |\Psi_{\nu=1/m}\rangle \quad \text{if} \quad l \leq k \quad (3.12)$$

where \(|\Psi_{\nu=1/m}\rangle\) is a higher angular momentum state of the form

$$|\Psi_{\nu=1/m}\rangle = \int d^2z_1...d^2z_N e^{-\sum_i |z_i|^2} \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m P(\bar{z}_1,...,\bar{z}_N)|z_1\cdots z_N\rangle \quad (3.13)$$

and \(P(\bar{z}_1,...,\bar{z}_N)\) is a homogeneous symmetric polynomial.

* We have used the infinite plane geometry where the electrons are confined by the existence of an external confining potential\([4,10]\), for example a central harmonic oscillator potential. Such a potential can be thought of controlling the maximum single-particle angular momentum \(L\) available within a Landau level, which also determines the size of the droplet corresponding to the ground state configuration. In the case of the \(\nu = 1/m\) ground state, eqs.(3.3)-(3.13), \(N\) and \(L\) are related by \(L = m(N-1)\).
The operators $W_{2p}$, $p = 0, 1, \ldots$, form a one-parameter family of representations for $W_\infty$ algebra and the corresponding Laughlin ground state is the highest weight state. This provides an algebraic statement of incompressibility for the Laughlin ground states.

We would like now to extend these ideas to include other incompressible states corresponding to filling fractions $\nu \neq 1/m$. We still consider the case of lowest Landau level fermions. First we identify the $W_\infty$ algebra structure corresponding to the state with filling fraction $\nu = 1 - 1/m$. Using the idea of particle-hole conjugation, we can write the $\nu = 1 - 1/m$ ground state, in the thermodynamic limit and up to normalization factors, as

$$|\Psi_{\nu=1-1/m}\rangle_0 \sim \int d^2 z_1 \ldots d^2 z_M e^{-\sum |z_i|^2} \prod_{i<j} (z_i - z_j)^m \psi(z_1) \ldots \psi(z_M)|\Psi_{\nu=1}\rangle_0$$  \hspace{1cm} (3.14)

Let us now introduce the operator $\beta(z)$

$$\beta(z) = \int d^2 z' e^{-|z'|^2} \ln(z - z') \psi(z') \psi^\dagger(z')$$  \hspace{1cm} (3.15)

The operator $e^{\beta(z)}$ satisfies equations similar to (3.10):

$$e^{n\beta(z)} \psi(z') = (z - \partial z')^n \psi(z') e^{n\beta(z)}$$

$$\psi^\dagger(z') e^{n\beta(z)} = (z - z')^n e^{n\beta(z)} \psi^\dagger(z')$$  \hspace{1cm} (3.16)

Based on these relations and the fact that $\psi^\dagger(z_i)|\Psi_{\nu=1}\rangle_0 = 0$, we find that the operators

$$\tilde{W}_{2p}[\xi] = \int d^2 z e^{-|z|^2} \psi(z) e^{2p\beta(z)} \frac{1}{4} \xi(z, \partial z) \frac{1}{4} e^{-2p\beta(z)} \psi^\dagger(z)$$  \hspace{1cm} (3.17)

satisfy a $W_\infty$ algebra and the state (3.14) satisfies a highest weight condition

$$(\tilde{W}_{2p})_{lk} |\Psi_{\nu=1-1/m}\rangle_0 = 0, \quad l > k$$  \hspace{1cm} (3.18)

The operator $\tilde{W}_{2p}$ is essentially the charge-conjugated version of $W_{2p}$.

In order to obtain more general filling fractions, one may consider systems where two distinct species of electrons are involved (cases where the electrons have different spin or they are in separate layers). Trial ground state wavefunctions of the form

$$\Psi^{m_1, m_2, n}(\vec{x}_i, \vec{w}_i) = \prod_{i<j} (\vec{x}_i - \vec{x}_j)^{m_1} \prod_{i<j} (\vec{w}_i - \vec{w}_j)^{m_2} \prod_{i,j} (\vec{x}_i - \vec{w}_j)^n e^{-1/2 \sum_i (|z_i|^2 + |w_i|^2)}$$  \hspace{1cm} (3.19)
have been suggested\textsuperscript{[21]} as candidates for describing incompressible Hall states for a twolayer system at \( \nu = \frac{m_1 + m_2 - 2n}{m_1 m_2 - n^2} \). The integers \( m_1 \) and \( m_2 \) are odd so that the wavefunctions are antisymmetric under exchange of identical fermions.

In order to identify the quantum \( W_\infty \) algebra structure associated with (3.19) we introduce two independent lowest Landau level fermion operators such that

\[
\{ \psi^I(\bar{z}, t), \psi^J(\bar{z}', t) \} = \delta^{IJ} e^{\bar{z} z'} \quad I = 1, 2
\]

and the corresponding quasi-hole operators \( e^{\alpha^I(\bar{z})} \) where

\[
\alpha^I(\bar{z}) = \int d^2 z' e^{-|z'|^2} \ln(\bar{z} - z') \psi^I(z') \psi^J(z')
\]

A relation similar to (3.1) applies between (3.19) and the \( \nu = 1 \) state. The main difference now is that vortices from different species are involved. This suggests that the corresponding \( W_\infty \) generators have the form

\[
W^1[\xi] = \int d^2 z e^{-|z|^2} \psi^{1\dagger}(z) e^{[(m_1-1)\alpha^1(\bar{z}) + n\alpha^2(\bar{z})]} \frac{\partial}{\partial \bar{z}} e^{-[(m_1-1)\alpha^1(\bar{z}) + n\alpha^2(\bar{z})]} \psi^1(\bar{z})
\]

\[
W^2[\xi] = \int d^2 z e^{-|z|^2} \psi^{2\dagger}(z) e^{[n\alpha^1(\bar{z}) + (m_2-1)\alpha^2(\bar{z})]} \frac{\partial}{\partial \bar{z}} e^{-[n\alpha^1(\bar{z}) + (m_2-1)\alpha^2(\bar{z})]} \psi^2(\bar{z})
\]

It is straightforward to show that the operators in (3.22) give rise to two commuting \( W_\infty \) algebras and that the corresponding ground state (3.19) satisfies highest weight condition. The expressions (3.22) provide a family of representations of \( W_\infty \) parametrized by three integers.

Using the expressions (3.21) for the quasi-hole operators we can also construct second quantized operators for the order parameters\textsuperscript{[22,23]} of the system

\[
q_1^\dagger = \int d^2 z e^{-|z|^2} \psi^{1\dagger}(z) e^{m_1 \alpha^1(\bar{z})} e^{n \alpha^2(\bar{z})}
\]

\[
q_2^\dagger = \int d^2 z e^{-|z|^2} \psi^{2\dagger}(z) e^{m_2 \alpha^2(\bar{z})} e^{n \alpha^1(\bar{z})}
\]

This is a straightforward generalization of the order parameter operator for the \( \nu = 1/m \) Laughlin state\textsuperscript{[10]}.
Using eqs.(3.10) and the commutativity of $\alpha$’s we can show that

\[ [q_1^\dagger, q_1^\dagger] = [q_2^\dagger, q_2^\dagger] = 0 \]

\[ q_1^\dagger q_2^\dagger = -(-)^n q_2^\dagger q_1^\dagger \]  (3.24)

Thus for $n$ odd the operators $q_I^\dagger$ are bosonic, as there should be in order to describe order parameters. For $n$ even they are fermionic and so we shall consider the redefined operators

\[ Q_I^\dagger = q_I^\dagger e^{i \pi S}, \quad S = \int d^2 z e^{-|z|^2} (\psi_I^\dagger(z)\psi^\dagger(\bar{z}) - \psi^2_2(z)\psi^2(\bar{z})) \]  (3.25)

where $S$ is a “cocycle” factor needed to impose commutativity conditions$^{[23]}$ for $Q_1^\dagger$, $Q_2^\dagger$

\[ [Q_I^\dagger, Q_J^\dagger] = 0, \quad I, J = 1, 2 \]  (3.26)

The many-body ground state (3.19) is now created out of the vacuum by the action of the bosonic operators

\[ |\Psi_{m_1,m_2,n}\rangle = \frac{1}{\sqrt{N!M!}} (Q_2^\dagger)^M (Q_1^\dagger)^N |0\rangle \]  (3.27)

The above construction can now be generalized to an $r$-layer system. The generalized ground state wavefunction is$^{[24]}$

\[ \Psi^K(\vec{z}_i) = \prod_{I=1}^r \prod_{i<j}^r (\bar{z}_i^I - \bar{z}_j^I)^{K_{IJ}} \prod_{I<J} \prod_{i,j}^r (\bar{z}_i^I - \bar{z}_j^J)^{K_{IJ}} e^{-1/2 \sum_i |z_i|^2} \]  (3.28)

where $K_{IJ}$ are odd integers. The corresponding filling fraction is given by $\nu = \sum_{I,J} (K^{-1})_{IJ}$ where $K$ is an $r \times r$ symmetric matrix. The $W_\infty$ generators are

\[ W_I^K[\xi] = \int d^2 z e^{-|z|^2} \psi_I^\dagger(z) \xi^{\alpha^I}(z) \bar{\xi}(\partial_z, \bar{z}) \bar{\xi}^{\alpha^I}(\bar{z}) e^{-\sum_j K_{IJ} \bar{z}_j^I \partial_j \xi^{\alpha^I}(z)} \]  (3.29)

where $I = 1, ..., r$ and $\bar{K} = K - \Pi$ ($\Pi$ is the identity matrix). We can show, as before, that the operators $W_I^K$ give rise to $r$ commuting copies of $W_\infty$ algebras and the ground states corresponding to (3.28) satisfy highest weight conditions. Expressions (3.29) provide a family of representations of $W_\infty$ algebra parametrized by the symmetric matrix $K$.  

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As far as the order parameters are concerned we can appropriately generalize the expressions (3.25)-(3.27). We start by noticing that

\[ q_I^\dagger q_J^\dagger = -( - )^{K_{IJ}} q_J^\dagger q_I^\dagger \]  

(3.30)

where

\[ q_I^\dagger = \int d^2z e^{-|z|^2} \psi_I^\dagger(z) e^{\sum_L K_{IL} \alpha^L(z)} \]  

(3.31)

We then find that the appropriate bosonic order parameter operators are

\[ Q_I^\dagger = q_I^\dagger e^{i \pi S}, \quad S = \sum_I \Lambda_I \int d^2z e^{-|z|^2} \psi_I^\dagger(z) \psi_I^\dagger(z) \]  

(3.32)

where \( \Lambda_I \) are integers such that \( K_{IJ} + \frac{\Lambda_I - \Lambda_J}{2} \) = odd.

As we noticed in ref.[10] the \( W_{\infty} \) generators, eq.(3.29), can be further written as

\[ W_K^I[\xi] = \int d^2z e^{-|z|^2} \psi_I^\dagger(z) \xi(\partial \bar{z} - \sum_J \tilde{K}_{IJ} \int d^2z' \rho^J(\bar{z}', \bar{z}', \bar{z} - \bar{z}') \psi_I^\dagger(\bar{z})] \]  

(3.33)

The term \( \sum_J \tilde{K}_{IJ} \int d^2z' \rho^J(\bar{z}', \bar{z}', \bar{z} - \bar{z}') \) plays the role of a gauge potential and it is similar to the one induced by the Chern-Simons interaction[25]. Analogous first quantized expressions were also derived in ref.[9].

Further by applying particle-hole conjugation as in (3.14)-(3.18) we can derive second quantized expressions for the \( W_{\infty} \) generators for \( \nu = 1 - \nu \) by charge-conjugating \( W_K^I \).

The wavefunctions (3.28) are also associated to the standard hierarchy scheme[6,7,24]. They are written in terms of electron coordinates and the coordinates of the quasiparticles formed at each level of the hierarchy. According to the hierarchy scenario, the quasiparticles at each level are the condensates of the quasiparticles of the previous level. Although it is clear from eq.(3.33) that at a first quantized level there is a \( W_{\infty} \) structure[9], the precise second quantized representation of \( W_{\infty} \) generators is not clear. The main difficulty in this case is the assignment of creation and annihilation operators to the quasiparticles at each level of the hierarchy. Given this difficulty we are going to use in the next section, the
alternative description of the FQHE, proposed by Jain, where everything is formulated in terms of electrons. Such a framework will be very convenient in displaying the $W_\infty$ algebra structure corresponding to the FQHE states, as described by Jain wavefunctions.

4. $W_\infty$ algebras for Jain states

One of the theories proposed to explain the observed fractions for the FQHE is the one suggested by Jain\cite{8}, in which there is a strong connection between IQHE and FQHE. As far as the hierarchical filling fractions $\nu = \frac{n}{2pn+1}$ are concerned, the essential idea of this scenario is that the corresponding incompressible FQHE ground state wavefunction $\Psi_0^{\nu}$, is related to the IQHE wavefunction $\Psi_0^{n}$ as

$$\Psi_0^{\nu} = \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{2p}\Psi_0^{n}$$  \hspace{1cm} (4.1)

This relation is a straightforward generalization of (3.1) and we can therefore use previous ideas in order to write down the second quantized $W_\infty$ generators which would express the incompressibility of (4.1). The main difference now is that fermions in higher Landau levels are involved and some of the previously used operator relations might change.

The state corresponding to (4.1) can be written as $(N' = nN)$ \footnote{N is related to the maximal single-particle angular momentum available within a Landau level as it was mentioned in the footnote in pg.8. For example by counting powers of $\bar{z}_i^0$ we find that $2p(N'-1) + (N-1) = L$, where $L$ is the maximal single-particle angular momentum in the lowest Landau level.}

$$|\Psi_\nu\rangle \sim \int \prod_{I=0}^{n-1} \left[ (\prod_{i=1}^{N} d^2 z_i^I e^{-|z_i^I|^2}) \prod_{i<j}^{I<J}(z_i^I - z_j^J)^{2p} \prod_{i,j}^{I<J}(\bar{z}_i^I - \bar{z}_j^J)^{2p} \right] \prod_{I=0}^{n-1} [\Delta^I[\bar{x}_1^I, ..., \bar{x}_N^I]\psi^I(z_1^I, \bar{z}_1^I) ... \psi^I(z_N^I, \bar{z}_N^I)] |0\rangle$$  \hspace{1cm} (4.2)

where $\Delta^I[\bar{x}_1, ..., \bar{x}_N]$ \footnote{In defining $\Delta^I[\bar{x}_1, ..., \bar{x}_N]$ we have factored out all the exponential factors $e^{-\frac{d}{2}|z_i|^2}$. For example for the lowest Landau level $\Delta^0[\bar{x}_1, ..., \bar{x}_N] = \prod_{i<j}(\bar{z}_i - \bar{z}_j)$.} is the properly antisymmetrized $N$-body wavefunction and $\psi^I(z, \bar{z})$
is the fermion operator at the $I$-th Landau level. Antisymmetrization is done independently at each Landau level. We first notice that the incompressibility condition for $|\Psi_{\nu}\rangle_0$ in the first line of (2.21) can be expressed as a differential equation on $\Delta$, namely

$$\sum_{i=1}^{N}(\partial \bar{z}_i)^l(\bar{z}_i - \partial z_i)^k \Delta^I[\bar{x}_1, ..., \bar{x}_N] = 0 \quad l > k, \quad \forall I \quad (4.3)$$

Let us again consider the operators $\alpha^I(\bar{z}), I = 0, 1, ..., n - 1$.

$$\alpha^I(\bar{z}) = \int d^2z e^{-|z|^2} \ln(\bar{z} - \partial z)\psi^{\dagger I}(z', \bar{z})\psi^I(z', \bar{z}) \quad (4.4)$$

Using the commutation relations (2.11) we can show that

$$[\alpha^I(\bar{z}), \psi^{\dagger I}(z', \bar{z}')] = \ln(\bar{z} - \partial z')\psi^{\dagger I}(z', \bar{z}')$$

$$[\alpha^I(\bar{z}), \psi^I(z', \bar{z}')] = -\ln(\bar{z} - \partial z')\psi^I(z', \bar{z}') \quad (4.5)$$

Using these relations along with the fact that wavefunctions in different Landau levels are orthogonal we find that the operators

$$W^{I}_{2p} = \int d^2z e^{-|z|^2} \psi^{\dagger I}(z, \bar{z})e^{2p} \sum_J \frac{\alpha^J(\bar{z})}{4}(\partial \bar{z}_J, \bar{z} - \partial z_J)e^{-2p} \sum_J \alpha^J(\bar{z})\psi^I(z, \bar{z}) \quad (4.6)$$

where $I = 0, ..., n - 1$, give rise to $n$ commuting copies of $W_\infty$ algebras and the corresponding Jain state for $\nu = \frac{n}{2pm+1}$ satisfies the highest weight condition

$$(W^{I}_{2p})_{lk}|\Psi_{\nu}\rangle_0 = 0, \quad l > k \quad I = 0, ..., n - 1 \quad (4.7)$$

As before we can use particle-hole conjugation to derive the charge-conjugated $\tilde{W}^{I}_{2p}$ operators which express the incompressibility of the states with $\tilde{\nu} = 1 - \nu$.

5. Discussion

In this paper we have given an explicit second quantized representation of the $W_\infty$ algebra structure characterizing the incompressibility of a large class of fractional Hall states. The corresponding generators are expressed in terms of fermion and vortex operators. In general incompressible states can be thought of as highest weight states of $W_\infty$ algebra\cite{11}.
Expressing the $W_\infty$ generators in terms of Fock operators is useful in constructing representations and of course the highest weight state. It is thus possible to obtain new candidate states for describing quantum Hall effect.

For the specific examples we considered, the multilayer incompressible states and the Jain states, we find that there are $r$ commuting copies of $W_\infty$ algebras, where $r$ is the number of layers in a multilayer system or the number of Landau levels in the Jain framework[8]. This $W_\infty$ algebra, besides characterizing the incompressibility of the ground state, provides a spectrum generating algebra for excitations. We expect that in the semiclassical limit the $W_\infty$ algebra reduces to the algebra of area-preserving diffeomorphisms, which upon restriction to the low energy edge excitations gives rise to a $((U(1))^r$ Kac-Moody algebra describing $r$ independent chiral bosons, one for each layer. So far there is the underlying assumption that the number of electrons at each layer is conserved. If this is relaxed by considering tunneling between different layers the spectrum generating algebra is enlarged by an embedding in a nonabelian structure.*

Further in constructing the $W_\infty$ generators for multilayer systems and the hierarchical states based on Jain’s construction we made use of the special mapping (4.1) between integer and fractional quantum Hall states. In Jain’s scenario there are more generalized incompressible states[8], for example states which can be written as products of integer quantum Hall states. A generic feature of the corresponding incompressible fluids is the existence of nonabelian edge excitations[26], which again suggests a nonabelian $W_\infty$ algebra structure. The precise fermionic representation of this is under investigation.

An approach somewhat similar in spirit, in linking the incompressibility of fractional quantum Hall states to the existence of a $W_\infty$ algebra structure, has been advocated in ref.[11]. Using the representation theory of a centrally extended $W_\infty$ algebra (corresponding to the one-dimensional edge degrees of freedom) the authors of [11] were able to classify the hierarchical (abelian) quantum Hall states. Although the relation between the

* This was pointed to me by Bunji Sakita.
two-dimensional centerless $W_\infty$ algebra and the one-dimensional centrally extended one is straightforward in the $\nu = 1$ case, it is not so for other filling fractions. In the $\nu = 1$ case, the normal ordering of the full two-dimensional $W_\infty$ algebra with respect to the $\nu = 1$ ground state gives the centrally extended one-dimensional $W_\infty$ algebra\cite{19}. It is interesting to see whether the two $W_\infty$ structures can be similarly related in the case of other filling fractions.

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