Shift identification in time varying regression quantiles

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Abstract

This article investigates whether time-varying quantile regression curves are the same up to the horizontal shift or not. The errors and the covariates involved in the regression model are allowed to be locally stationary. We formalise this issue in a corresponding non-parametric hypothesis testing problem, and develop a integrated-squared-norm based test (SIT) as well as a simultaneous confidence band (SCB) approach. The asymptotic properties of SIT and SCB under null and local alternatives are derived. We then propose valid wild bootstrap algorithms to implement SIT and SCB. Furthermore, the usefulness of the proposed methodology is illustrated via analysing simulated and real data related to Covid-19 outbreak and climate science.

Keyword: bootstrap, comparison of curves, confidence band, hypothesis testing, locally stationary process, nonparametric quantile regression, Covid-19.
1 Introduction

Consider two regression models with common response variable and the same covariates for two different groups. Formally speaking, suppose that \((y_{i,1}, x_{i,1})_{i=1}^{n_1}\) and \((y_{i,2}, x_{i,2})_{i=1}^{n_2}\) are two sets of data, where the covariates \(x_{i,1} = (x_{i,1,1}, \ldots, x_{i,p_1,1})^\top\) and \(x_{i,2} = (x_{i,1,2}, \ldots, x_{i,p_2,2})^\top\) are \(p_1 \times 1\) and \(p_2 \times 1\) vectors, respectively. Now, for \(\tau \in (0, 1)\) and \(s = 1, 2\), we define the conditional quantiles

\[
Q_{\tau}(y_{i,s}|x_{i,s}) := \inf \{ s : F_{y_{i,s}|x_{i,s}}(s|x_{i,s}) > \tau \} = \theta_{1,\tau,s}(\frac{i}{n_s}) x_{i,1,s} + \cdots + \theta_{p_s,\tau,s}(\frac{i}{n_s}) x_{i,p_s,s}.
\]  

(1.1)

Note that model (1.1) can be written as

\[
y_{i,s} = x_{i,s}^\top \theta_{\tau,s}(\frac{i}{n_s}) + e_{i,\tau,s}, \quad i = 1, \ldots, n_s, \quad s = 1 \text{ and } 2,
\]

(1.2)

where for \(s = 1 \text{ and } 2\), \(\theta_{\tau,s}(t) = (\theta_{1,\tau,s}(t), \ldots, \theta_{p_s,\tau,s}(t))^\top\) are \(p_s \times 1\) vectors with each element being a smooth function on \([0, 1]\), and the errors \(e_{i,\tau,s}\) satisfy \(Q_{\tau}(e_{i,\tau,s}|x_{i,s}) = 0\).

The last condition on the \(\tau\)-th quantile of the conditional distribution of the errors given the covariate ensures the model (1.2) is identifiable, and further technical assumptions on \(x_{i,s}\) and \(e_{i,\tau,s}\) will explicitly be discussed in Section 3. We are now interested in the following hypothesis problem. For a pre-specified vectors \(c_1 \in \mathbb{R}^{p_1 \times 1}\) and \(c_2 \in \mathbb{R}^{p_2 \times 1}\), define \(m_s(t) = c_s^\top \theta_{\tau,s}(t)\) for \(s = 1 \text{ and } 2\), \(c_s = (c_{1,s}, \cdots, c_{p_s,s})^\top\), and we want to test

\[
H_0 : m_1(t) = m_2(t + d) \quad \text{for } 0 < t < 1 - d \text{ and some unknown constant } d \in [0, 1).
\]

(1.3)

Let us now discuss a special case. Note that when \(d = 0\), \(H_0\) will be equivalent to test the equivalence of \(m_1(t)\) and \(m_2(t)\) for \(t \in (0, 1)\). In this case, when \(c_1 = (1, 0, \ldots, 0)^\top\) and \(c_2 = (1, 0, \ldots, 0)^\top\), the problem will coincide with comparing the curves \(\theta_{1,\tau,1}(t)\) and \(\theta_{1,\tau,2}(t)\) for \(t \in (0, 1)\), and such comparison can be carried out by an appropriate functional notion of difference between estimated \(\theta_{1,\tau,1}(t)\) and \(\theta_{1,\tau,2}(t)\). Such types of problems have already been explored in the literature. For example, one can see Munk and Dette (1998). However, our proposed testing of hypothesis problem described in (1.3) is fundamentally...
different from the aforesaid case. Firstly, we are comparing two sets of certain linear combinations of the components of the quantile coefficients of (1.1); it is not a direct comparison between particular quantiles of two different distributions. Secondly, note that in (1.1) and (1.2), the quantiles are time varying, which is entirely different from the usual regression quantiles. Finally, in (1.3), we are checking whether there is any nonnegative shift between two functions \( m_1 \) and \( m_2 \) or not. Note that if \( d = 0 \), as said before, testing equality between \( m_1(t) \) and \( m_2(t) \) can be performed based on the suitable functional difference between estimated \( m_1(t) \) and \( m_2(t) \) over \((0, 1)\). However, in our case, i.e., when \( d > 0 \), the same approach will not work since both functions will not coincide over whole interval \((0, 1)\). Strictly speaking, it will be a comparison between the curves \( m_1(t) \) and \( m_2(t) \) up to a certain shift on the horizontal axis. Such relationship among the quantile curves (at a fixed index value) with respect to the time parameter is often visible in real life also. Exemplary, the Gross Domestic Product (GDP) curves of two nations over a fixed period of time may have this feature. For usual mean based time varying models, such type of problem was studied by Gamboa et al. (2007), Vimond (2010), Collier and Dalalyan (2015) and a few references therein. However, none of them studied such problems in the framework of quantile regression (i.e., (1.1) or (1.3)) for time varying models. This article thoroughly studies this problem, and our major contributions are the following.

The first major contribution is to develop a formal test in checking the hypothesis (1.3) for dependent and non-stationary data. In Section 2, we will establish that the hypothesis (1.3) is equivalent to the equality of \( (m_1^{-1}(.))' \) and \( (m_2^{-1}(.))' \) (for any function \( f, f' \) denotes the derivatives of \( f \), and \( f^{-1} \) denotes its inverse). We here formulate the test statistic based on the \( L^2 \)-norm of a smooth estimator of the difference between \( (m_1^{-1}(.))' \) and \( (m_2^{-1}(.))' \), and the asymptotic distribution of the test statistic is derived under null hypothesis (i.e., the hypothesis described in (1.3)) and local alternatives. The test is denoted as the SIT test as the test is the squared integral test. In this context, we would like to mention that there have been a few research articles on conditional quantiles of independent data, and the readers are referred to Zheng (1998), Horowitz (2002), He and Zhu (2003), Kim (2007) and a few references therein. However, none of the above research articles considered the more widely applicable testing of hypothesis problems that we consider here (see (1.3)) for time varying quantile regression models (see (1.1)).
The second major contribution is to develop the simultaneous confidence band (SCB) for the difference between \((m_1^{-1}(\cdot))'\) and \((m_2^{-1}(\cdot))'\), and an asymptotic property of the SCB is derived, which asserts the form of the confidence band of the difference between \((m_1^{-1}(\cdot))'\) and \((m_2^{-1}(\cdot))'\) for a preassigned level of significance. Recently, Wu and Zhou (2017) studied the limiting properties of simultaneous confidence bands of the corresponding functional considered in their article, which is different than key term of our work. The advantage of proposing such graphical device is that one can expect a certain percentage of time, the value of the difference between \((m_1^{-1}(\cdot))'\) and \((m_2^{-1}(\cdot))'\) will lie inside a band with a specific probability. Our result will ensure that this coverage probability is asymptotically correct.

The third major contribution is to propose a robust Bootstrap procedure to have a good finite sample performance of the SIT and the SCB. In principle, one can carry out the test based on SIT and construct the SCB using the results in Theorems 3.1 and 3.2. However, for small or moderate sample size, directly implementing those results may not produce satisfactory performance due to slow convergence rate, and to overcome this problem, the bootstrap method is proposed, and a better rate of convergence of the Bootstrap method is established as well. In this context, the readers may also look at Zhou and Wu (2010) and a few references therein.

The rest of the article is organized as follows. In Section 2, we characterize the null hypothesis stated in (1.3), which is a key observation in subsequent theoretical studies. Section 2.1 discusses the local linear quantile estimator for time varying regression coefficients, and in Section 2.2 basic ideas related to the estimation of the regression function and its derivative are studied. The two-stage estimator of the shift parameter is also developed in this section along with the formulation of the SIT and the SCB tests. Section 3 investigates the SIT and the SCB tests. Section 4 explores issues related to implementation of the tests and provides the estimation of a complicated expression involved in the implementation of the test described in Section 4.1. The bootstrap version of the tests with the algorithm of implementation is studied in Section 4.2. A real data related to COVID-19 outbreak is analysed in Section 5. Supplementary material contains some bandwidths conditions, the choices of tuning parameters, all technical details and proofs along with simulation studies and an analysis of real data related to climate science.
2 Methodology

Observe that inspecting null hypothesis described in (1.3) is equivalent to checking \((m_1^{-1}(u))' - (m_2^{-1}(u))' = 0\) when \(u\) belongs to a certain interval. Proposition 2.1 states this result explicitly.

**Proposition 2.1** Let \(m_1(t)\) and \(m_2(t)\) be strictly increasing functions on \([0,1]\). Then \(H_0\) holds if and only if \((m_1^{-1}(u))' - (m_2^{-1}(u))' = 0\) for all \(u \in [m_1(0), m_1(1-d)]\) when \(d = m_2^{-1}(m_1(0))\).

In practice, under the null hypothesis (1.3), the estimation of \(d\) and deriving its asymptotic properties is a complicated task. Therefore, asymptotic performance of statistic based on \(\hat{m}_1(t) - \hat{m}_2(t + \hat{d})\), where \(\hat{m}_1, \hat{m}_2\) and \(\hat{d}\) are estimators of \(m_1, m_2\) and \(d\), respectively, could be intractable, and moreover, it is likely to have unsatisfactory results as various issues like different rate of convergences are involved. In contrast, the assertion of Proposition 2.1 suggests that we can test (1.3) based on the estimate of \((m_1^{-1}(u))' - (m_2^{-1}(u))'\). To use Proposition 2.1 in the theoretical results, one needs to know about the various issues such as estimation of the time varying quantile regression functions and their derivatives, formulation of test statistics etc, which are discussed in the following subsections.

2.1 Local linear quantile estimate

We estimate time varying quantile regression coefficients \(\theta_{\tau,s}(t)\) using the concept of local linear quantile estimators. Specifically, for \(s = 1\) and \(2\), the local linear quantile estimate of \((\theta_{\tau,s}(t), \theta_{\tau,s}^\top(t))\) is denoted by \((\hat{\theta}_{\tau,s,b_{n,s}}(t), \hat{\theta}_{\tau,s,b_{n,s}}^\top(t))\), where

\[
(\hat{\theta}_{\tau,s,b_{n,s}}(t), \hat{\theta}'_{\tau,s,b_{n,s}}(t)) = \arg\min_{\beta_0 \in \mathbb{R}^{p_s}, \beta_1 \in \mathbb{R}^{p_s}} \sum_{i=1}^{n_s} \rho_{\tau} \left( y_{i,s} - x_{i,s}^\top \beta_0 - x_{i,s}^\top \beta_1 \left( \frac{i}{n_s} - t \right) \right) K_{b_{n,s}} \left( \frac{i}{n_s} - t \right),
\]

where \(\rho_{\tau}(x) = \tau x 1_{[0,\infty)}(x) - (1 - \tau)x 1_{(-\infty,0)}(x)\), \(K(\cdot)\) is a kernel function with \(K_{b_{n,s}}(\cdot) = K\left( \frac{\cdot}{b_{n,s}} \right)\), and \(b_{n,s}\) is the sequence of bandwidth associated with the \(s\)-th sample \((s = 1\) and \(2\)). Note that the local linear (quantile) estimators have been extensively studied.
in the literature of non-parametric statistics for both independent and dependent data, see for example, Yu and Jones (1998), Chaudhuri (1991), Dette and Volgushev (2008), Qu and Yoon (2015), Wu and Zhou (2017), Wu and Zhou (2018b) among many others. Among them, Wu and Zhou (2017) investigated the estimator (2.1) with locally stationary covariates and errors, and this locally stationary processes have been developed in the literature to model the slowly changing stochastic structure, which can be found in many real world time series data; see for instance, Dahlhaus et al. (1997), Zhou and Wu (2009), Dette and Wu (2020), Dahlhaus et al. (2019). These articles motivated us to work on the hypothesis (1.3) assuming local stationarity. The list of detailed assumptions are deferred to Section 3.

We now estimate \( m_1(t) \) and \( m_2(t) \) through a biased-corrected estimate of \( \tilde{\theta}_{\tau,s} = (\tilde{\theta}_{\tau,s,1}, \ldots, \tilde{\theta}_{\tau,s,p_s})^\top \) for \( s = 1 \) and \( 2 \). That is

\[
\hat{m}_1(t) = c_1^\top \tilde{\theta}_{\tau,1}(t), \quad \hat{m}_2(t) = c_2^\top \tilde{\theta}_{\tau,2}(t),
\]

(2.2)

where for \( s = 1 \) and \( 2 \), and for \( 1 \leq j \leq p_s \),

\[
\tilde{\theta}_{b_{n,s}}^{(b_{n,s})}(t) = 2\tilde{\theta}_{r,s,j}^{(b_{n,s})}(t) - \tilde{\theta}_{r,s,j}(t).
\]

(2.3)

The sup-script inside the parentheses denotes the bandwidth used for the corresponding estimator, and it will be omitted in the rest of the article for the sake of notational simplicity. Notice that \( (\tilde{\theta}_{s,j}^{(b_{n,s})}(t), 1 \leq j \leq p_s, b_{n,s} \leq t \leq 1 - b_{n,s}) \) are equivalent to the local linear quantile estimators using the second-order kernel \( \tilde{K}(x) = 2\sqrt{2}K(\sqrt{2}x) - K(x) \). It can be shown similarly to Section 4.1 of Dette and Wu (2019) that \( \tilde{\theta}_{r,s,j}^{(b_{n,s})}(t) \) has a bias at the order of \( b_{n,s}^3 \), while the unadjusted estimator \( \tilde{\theta}_{r,s,j}(t) \) has a bias of the order \( O(b_{n,s}^2) \), which is non-negligible and hard to evaluate. Therefore, the de-biased estimator has been widely applied in non-parametric inference, see for example Schucany and Sommers (1977) and Wu and Zhao (2007).
2.2 Basic ideas and the tests

Suppose that $H$ is a smooth kernel function, $h_s$ ($s = 1$ and $2$) is a sufficiently small bandwidth, and $N$ is a sufficiently large number. We then estimate $(m_1^{-1})'(t)$ and $(m_2^{-1})'(t)$, which are denoted by $\hat{g}_1(t)$ and $\hat{g}_2(t)$, respectively:

\[
\hat{g}_1(t) = \frac{1}{Nh_1} \sum_{i=1}^{N} H \left( \frac{\hat{m}_1 \left( \frac{t}{N} \right) - t}{h_1} \right), \quad \text{and} \quad \hat{g}_2(t) = \frac{1}{Nh_2} \sum_{i=1}^{N} H \left( \frac{\hat{m}_2 \left( \frac{t}{N} \right) - t}{h_2} \right).
\] (2.4)

Notice that $N$ is not the sample size; it is used for Riemann approximation. Further, observe that

\[
\hat{g}_s(t) \approx \frac{1}{Nh_s} \sum_{i=1}^{N} H \left( \frac{m_s \left( \frac{t}{N} \right) - t}{h_s} \right) \approx \frac{1}{h_s} \int_{(m_s(0)-t)/h_s}^{(m_s(1)-t)/h_s} H \left( \frac{m_s(x) - t}{h_s} \right) dx
\] (2.5)

where $1(A)$ denotes the indicator function of set $A$. Therefore, the estimator defined in (2.4) is a smooth approximation to the step function $((m_s)^{-1})'(t)1(m_s(0) < t < m_s(1))$ and is differentiable with respect to $t$. Such type of estimator was proposed by [Dette et al. (2006)] and studied extensively by [Dette and Wu (2019)] for non-stationary time series models.

Now, using $\hat{g}_s$ ($s = 1$ and $2$), one can estimate $m_s^{-1}(t)$ ($s = 1$ and $2$) by

\[
\hat{G}_s(t) = \int_{m_s(0)}^{t} \hat{g}_s(u) du.
\] (2.6)

This fact motivates us to estimate the horizontal shift $d$ under null hypothesis as follows. Note that for $0 \leq t \leq 1 - d$, we have $m_1(t) = m_2(t + d) = u$ when $m_1(0) \leq u \leq m_1(1 - d)$, and therefore,

\[
d = m_2^{-1}(u) - m_1^{-1}(u), \quad m_1(0) \leq u \leq m_1(1 - d).
\] (2.7)
This fact drives us to estimate $d$ by

$$
\hat{d} = \frac{\hat{m}_1(1-\tilde{d})}{\hat{m}_1(1-\tilde{d}) - \hat{m}_1(0)} \int_{\hat{m}_1(0)}^{\hat{m}_1(1-\tilde{d})} \left[ \hat{G}_2(u) - \hat{G}_1(u) \right] du,
$$

where $\tilde{d} = \hat{m}_2^{-1}(\hat{m}_1(0))$ is a preliminary estimator of $d$ by letting $u = m_1(0)$ in (2.7). With this $\hat{d}$, one can therefore estimate the endpoints of intervals in (2.7), i.e., $a := m_1(0)$ and $b := m_1(1 - \hat{d})$. Let $\hat{a}$ and $\hat{b}$ be the estimators of $a$ and $b$, respectively, where $\hat{a} := \hat{m}_1(0)$ and $\hat{b} := \hat{m}_1(1 - \hat{d})$ under null hypothesis, and their properties under null and alternative are discussed in Proposition E.2 of the supplementary material.

Next, to formulate the test statistic, we use the fact in Proposition 2.1 and propose the SIT and the SCB tests to check the hypothesis described in (1.3) based on $\hat{g}_1(t) - \hat{g}_2(t)$. For the SIT test, the test statistics is defined as

$$
T_{n_1, n_2} = \int (\hat{g}_1(t) - \hat{g}_2(t))^2 \hat{w}(t) dt, \quad \text{where} \quad \hat{w}(t) = 1(\hat{a} + \eta \leq t \leq \hat{b} - \eta),
$$

and $\eta = \eta_{n_1, n_2}$ is a positive sequence diminishes sufficiently slow as $n_1, n_2 \to \infty$. For instance, one may consider $\eta_{n_1, n_2}$ vanishes at the rate of $\frac{1}{\log(n_1 + n_2)}$. The purpose of introducing $\eta$ here is to avoid the issues related to the boundary points; for details, see remark E.1 of the supplementary material. Observe that $T_{n_1, n_2}$ is an estimate of distance between $(m_1^{-1})'(t)$ and $(m_2^{-1})'(t)$ in $L_2$ sense, and we shall reject the null hypothesis when $T_{n_1, n_2}$ is a large enough. The second test is the simultaneous confidence band centered around $\hat{g}_1(t) - \hat{g}_2(t)$, whose detailed expression is provided in the statement of Theorem 3.2. Using the relation between the testing of hypothesis and the confidence band, it is easy to see that the SCB test is rejected at significance level $\alpha$ if the $[\hat{a} + \eta, \hat{b} - \eta]$ is not entirely contained by the 100$(1 - \alpha)$% SCB.

### 3 Asymptotic Results

In this section, we investigate the asymptotic properties of $T_{n_1, n_2}$ and the asymptotic form of the SCB at a presumed significance level $\alpha$. We start from a few concepts and
assumptions for the model described in (1.2). Let \((\zeta_i^{(1)})_{i \in \mathbb{Z}}, (\zeta_i^{(2)})_{i \in \mathbb{Z}}, (\eta_i^{(1)})_{i \in \mathbb{Z}}\) and \((\eta_i^{(2)})_{i \in \mathbb{Z}}\) be i.i.d. random vectors, and the filtrations for \(s = 1\) and 2 are the following: \(\mathcal{F}_{i,s} = (\zeta_{-\infty}^{(s)}, \ldots, \zeta_{i-1}^{(s)}, \zeta_i^{(s)})\) and \(\mathcal{G}_{i,s} = (\eta_{-\infty}^{(s)}, \ldots, \eta_{i-1}^{(s)}, \eta_i^{(s)})\). We assume that the covariates and errors are both locally stationary process in the sense of [Zhou and Wu (2009)], i.e.,

\[
x_{i,s} = H_s \left( \frac{i}{n}, \mathcal{G}_{i,s} \right), \quad e_{i,s} = L_s \left( \frac{i}{n}, \mathcal{F}_{i,s}, \mathcal{G}_{i,s} \right)
\]

for \(s = 1\) and 2, where \(H_s\) and \(L_s\) are the marginal filters. We list some basic assumptions of processes \(x_{i,s}\) and \(e_{i,s}\) in conditions (A3) and (A4). Now, write \(\mathcal{F}^*_{i,s} = (\mathcal{F}_{-1,s}, \zeta_0^{(s)}, \ldots, \zeta_i^{(s)}, \zeta_{i+1}^{(s)})\), where \((\zeta_i^{(s)})_{i \in \mathbb{Z}}\) is an i.i.d. copy of \((\zeta_i^{(s)})_{i \in \mathbb{Z}}\) and define \(\mathcal{G}^*_{i,s}\) also in a similar way. For a \(p\) dimensional (random) vector \(v := (v_1, \ldots, v_p)\top\), let \(|v| = \sqrt{\sum_{i=1}^{p} v_i^2}\), and for any random vector \(v\), write \(\|v\|_q = (\mathbb{E}(\|v\|^q))^{1/q}\), which is its \(L_q\)-norm \(q \geq 1\). Let \(\chi \in (0, 1)\) be a fixed constant, and suppose that \(M\) and \(\eta\) are sufficiently large and sufficiently small positive constants, respectively; though it may vary from line to line. For any positive semi-definite matrix \(\Sigma\), write \(\lambda_{\min}(\Sigma)\) as its smallest eigenvalue.

We first give out the following set of conditions, which enable us to study the deviation of the nonparametric quantile estimator, \(\hat{\theta}_\tau - \theta_\tau\).

(A1) Define \(\theta''_{\tau,s}(t) = (\theta''_{1,\tau,s}(t), \ldots, \theta''_{p_s,\tau,s}(t))\top\) for \(s = 1\) and 2. Assume that \((\theta''_{i,\tau,s}, 1 \leq i \leq p_s, s = 1, 2)\) is Lipschitz continuous on \([0, 1]\).

(A2) Define \(Q_\tau(L_s(t, \mathcal{F}_{i,s}, \mathcal{G}_{i,s})|\mathcal{H}_{i,s}) := \inf_x \{\mathbb{P}(L_s(t, \mathcal{F}_{i,s}, \mathcal{G}_{i,s}) \geq x|\mathcal{G}_{i,s}) \geq \tau\}\). Assume that \(Q_\tau(L_s(t, \mathcal{F}_{i,s}, \mathcal{G}_{i,s})|\mathcal{G}_{i,s}) = 0\).

(A3) For errors process, we assume that for \(s = 1\) and 2,

\[
i) \quad \delta(L, i) := \sup_{t \in [0,1]} \|L_s(t, \mathcal{F}^*_{i,s}, \mathcal{G}^*_{i,s}) - L_s(t, \mathcal{F}_{i,s}, \mathcal{G}_{i,s})\|_1 = O(\chi^i) \quad \text{for} \quad i \geq 0. \quad (3.1)
\]

\[
ii) \quad \sup_{0 \leq t_1, t_2 \leq 1} \|L_s(t_1, \mathcal{F}_{0,s}, \mathcal{G}_{0,s}) - L_s(t_2, \mathcal{F}_{0,s}, \mathcal{G}_{0,s})\|_v \leq M|t_1 - t_2| \quad (3.2)
\]

for a constant \(v \geq 1\).

(A4) For covariate processes, we assume that for \(s = 1\) and 2, there exists a constant

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\( t_x > 0 \) such that,

\[ i) \quad \max_{1 \leq i \leq n_s} \mathbb{E}(\exp(t_x|x_{i,s})) \leq M < \infty, \]

\[ ii) \quad \delta(H, i) := \sup_{t \in [0, 1]} \| H_s(t, t_i^*) - H_s(t, G_i) \|_1 = O(\chi_i) \quad \text{for} \quad i \geq 0, \]

\[ iii) \quad \sup_{0 \leq t_1, t_2 \leq 1} \| H_s(t_1, F_{0,s}, G_{0,s}) - H(t_2, F_{0,s}, G_{0,s}) \|_2 \leq M |t_1 - t_2|. \]

(A5) For conditional densities, we define for \( s = 1 \) and 2, and for \( 0 \leq q \leq 2p_s + 1 \),

\[ F_s^{(q)}(t, x|F_{i-1,s}, G_{i,s}) = \frac{\partial^q}{\partial x^q} \mathbb{P}(L_s(t, F_{i,s}, G_{i,s}) \leq x|F_{i-1,s}, G_{i,s}). \]

In particular, we write \( f_s(t, x|F_{i-1,s}, G_{i,s}) = F_s^{(1)}(t, x|F_{i-1,s}, G_{i,s}) \) for brevity. Assume that \( \sup_{t \in [0, 1], x \in \mathbb{R}} |F_s^{(q)}(t, x|F_{i-1,s}, G_{i,s})| \leq M. \)

Further, define

\[ \delta_{l,s}^{(q)}(i - 1) = \sup_{t \in [0, 1], x \in \mathbb{R}} \| F_s^{(q)}(t, x|F_{i-1,s}, G_{i,s}) - F_s^{(q)}(t, x|F_{i-1,s}, G_{i,s}) \|_t, \]

and assume that \( \delta_{l,s}^{(q)}(i - 1) = O(\chi_i) \) for \( i \geq 0. \)

(A6) Define for \( s = 1 \) and 2, conditional on \( G_{i,s} \), the conditional quantity and the quantile design matrix as

\[ f_s(t, x|G_{i,s}) = \frac{\partial}{\partial x} \mathbb{P}(e_{i,s}(t) \leq x|G_i), \quad \Sigma_s(t) = \mathbb{E}(f_s(t, 0|G_{i,s})H_s(t, G_{i,s})H_s^T(t, G_{i,s})). \]

Assume that

\[ \sup_{t \in [0, 1]} \| f_s(t, 0|G_i) - f_s(t, 0|G_i^*) \|_2 = O(\chi_i) \quad \text{for} \quad i \geq 0, \]

\[ \inf_{t \in [0, 1]} \lambda_{\min}((\Sigma_s(t)) \geq \eta > 0 \quad \text{and} \quad \sup_{t \in [0, 1]} |f_s^{(1)}(t, 0|G_0)| \leq M. \]

(A7) Let \( \psi(x) = \tau - 1(x \leq 0) \) be the left derivative of \( \rho(x) \). For \( s = 1 \) and 2, define the
gradient vector process

\[ U_s(t, F_{i,s}, G_{i,s}) = \psi_\tau(L_s(t, F_{i,s}, G_{i,s}))H_s(t, G_{i,s}). \]  

(3.11)

Notice that by definition, \( U(i/n_s, F_{i,s}, G_{i,s}) = \psi_\tau(e_{i,s})x_{i,s} \), which is the gradient vector.

Now define the long run covariance matrices for \( U \), which is

\[ V^2_s(t) = \sum_{j=-\infty}^{\infty} \text{Cov}(U_s(t, F_{0,s}, G_{0,s}), U_s(t, F_{j,s}, G_{j,s})). \]  

(3.12)

Assume that for \( s = 1 \) and 2,

\[ \inf_{t \in (0,1)} \lambda_{\text{min}}(V^2_s(t)) \geq \tilde{\eta} > 0. \]  

(3.13)

(A8) The kernel functions \( K(\cdot) \) and \( H(\cdot) \) are symmetric and twice differentiable functions with support \([-1,1]\). Also, \( \int_{-1}^{1} K(x)dx = 1 \), \( \int_{-1}^{1} H(x)dx = 1 \), and \( K'' \), \( H'' \) are Lipschitz continuous on \([-1,1]\).

Conditions (A1)-(A8) are associated with the smoothness for the quantile regression coefficients, conditional quantiles, errors and covariates. The quantities \( \delta(L, i) \) and \( \delta(G, i) \) are called ‘physical dependence measure’ in the literature (see Zhou and Wu (2009)), and ii) of conditions (A3) and (A4) postulate stochastic Lipschitz continuity \( L_s \) and \( H_s \), respectively. In fact, conditions (A3) and (A4) ensure that the error and covariates are both short range locally stationary processes with geometrically decaying dependence measure. The verification of these conditions is uncomplicated for a general class of locally stationary processes; refer to Zhou and Wu (2009) for more details. Condition (A5) is a standard assumption on the dependence measures of the derivatives of the errors’ conditional densities for non-stationary time series quantile regression, see Wu and Zhou (2017), Wu and Zhou (2018a) among many others for details and Zhou and Wu (2009) for the verification of this condition on representative examples. Assumption (A6) ensures that the process \( \sum_{i=1}^{n_s} f_s(i/n_s, 0|G_i)x_{i,s}x_{i,s}^T K_{b_{n,s}}(i/n_s - t)/(n_s b_{n,s}) \) converges to the non-degenerate quantile design matrix. Similar conditions are also assumed in Kim (2007), Qu (2008) and a few references therein. Condition (A7) means that the long-run covariance matrices of the
gradient vectors $\mathbf{U}_s(t, \mathcal{F}_{i,s}, \mathcal{G}_{i,s})$ are non-degenerate. Condition (A8) is a mild condition for kernels, and the well known Epanechnikov and many more kernel functions satisfy the assumptions stated in (A8). Notice that conditions (A1)-(A7) generalize the conditions (A1)-(A5) of Wu and Zhou (2017) for multiple curves.

Now, for $s = 1$ and 2, we define $\mathcal{T}_{n,s} = (b_{n,s}, 1-b_{n,s})$ and $M_{e_s}(t) = ((c_s^\top \Sigma_s^{-1}(t))V_s^2(c_s^\top \Sigma_s^{-1}(t))^\top)^{1/2}$. We then have the following result related to uniform approximation of $\hat{m}_s(t) - m_s(t)$ through a Gaussian process. The proof of this proposition is based on a Bahadur Representation of $\hat{m}_s(t) - m_s(t)$. We further consider a few more conditions (B) on the bandwidth and the regression functions and state it in Section A of the supplementary material.

Before stating the main results on $T_{n_1,n_2}$ and SCB, we introduce a few more notations. Define for $s = 1$ and 2,

$$\tilde{g}_s(t) = M_{e_s}(m_s^{-1}(t))(m_s^{-1}(t))^{2}\int_\mathbb{R} H'(y)dy, \quad (3.14)$$

$$\tilde{g}_s(m_s(u)) = \tilde{g}_s^2(m_s(u))w(m_s(u))m_s'(u), \tilde{V}_s = \int_\mathbb{R} \int_\mathbb{R} \tilde{g}_s^2(m_s(u))\tilde{K}'(x)\tilde{K}'(y)^2du dy. \quad (3.15)$$

Write $m_{2,1}(u) = m_2^{-1}(m_1(u))$, $\tilde{g}_{1,2}(m_1(u)) = \tilde{g}_1(m_1(u))\tilde{g}_2(m_1(u))w(m_1(u))m_1'(u)$ and $\tilde{V}_{12}(r) = \int_\mathbb{R} \int_\mathbb{R} \tilde{g}_{1,2}^2(m_1(u))\int_\mathbb{R} \tilde{K}'(x)\tilde{K}'(rm_{21}(u)x+y)dx dudy$. For $s = 1$ and 2, let $\tilde{B}_s = \left(\int_\mathbb{R} H'(y)dy\right)^2 \int_\mathbb{R} \tilde{K}^2(x)dx \int_\mathbb{R} \tilde{K}'(x)\tilde{K}'(y)^2du dy$. Notice that under the null hypothesis (1.3), $m_{2,1}'(u) \equiv 1$.

**Theorem 3.1** Assume the conditions stated in (A1)-(A8) and (B1), (B2), (B3) in the supplementary material. Now, suppose that $n_1/n_2 \to \gamma_0$, $b_{n,1}/b_{n,2} \to \gamma_1$, $\eta^{-1} = O(\log(n_1 + n_2))$, $\eta = o(1)$. Further, let $(m_1^{-1})'(t) - (m_2^{-1})'(t) = \rho_n \kappa(t)$ for some bounded non-zero function $\kappa(t)$, and $\rho_n := \rho_{n_1,n_2} = (n_1 b_{n,1}^5/2)^{-1/2}$. We then have

$$n_1 b_{n,1}^5/2T_{n_1,n_2} - b_{n,1}^{-1/2}(\tilde{B}_1 + \gamma_0 \gamma_1^3 \tilde{B}_2) - \int_\mathbb{R} \kappa(t)w(t)dt \Rightarrow N(0, V_T), \quad (3.16)$$

where $V_T = \tilde{V}_1 + \gamma_0^2 \gamma_1^5 \tilde{V}_2 + 4\gamma_0 \gamma_1^3 \tilde{V}_{12}(\gamma_1)$.

Under null hypothesis, $\kappa \equiv 0$. Therefore, Theorem 3.1 suggests to reject null hypothesis.
of (1.3) whenever

\[
T_{n_1,n_2} > \frac{b_{n,1}^{-1/2} (\hat{B}_1 + \gamma_0 \gamma_1^3 \hat{B}_2) + z_{1-\alpha} \sqrt{V}}{n_1 b_{n,1}^{5/2}},
\]

(3.17)

where \(\alpha\) is the significance level, \(z_{1-\alpha}\) is the \((1-\alpha)\)-th quantile of a standard normal distribution, \(\hat{B}_1, \hat{B}_2\) and \(\hat{V}\) are appropriate estimates of asymptotic bias parameter \(B_1, B_2\) and the asymptotic variance \(V\), respectively. Moreover, Theorem 3.1 shows that the SIT test is able to detect the alternative which converges to null at a rate of \(\sqrt{(n_1 b_{n,1}^{5/2})}\), with asymptotic power

\[
1 - \Phi\left(z_{1-\alpha} - \frac{\int \kappa^2(t) w(t) dt}{V^{1/2}}\right),
\]

(3.18)

where \(\Phi(\cdot)\) denotes the CDF of a standard normal random variable.

**Theorem 3.2** Assume the conditions stated in (A1)-(A8) and (B1), (B2), (B3) in the supplementary material. Further, assume that for \(s = 1\) and 2, \(b_{n,s} \to c_{b,s} \in (0,1)\), \(\hat{n}_1 \to n_1 \in (0,1)\), \(\hat{n}_2 \to n_2 \in (0,1)\), \(\eta = o(1)\), \(\eta^{-1} = O(\log(n_1+n_2))\), and let \(c_{b,1} = c_{n,1} = c_{h,1} = 1\). Define

\[
K_1(t) = \sum_{s=1}^{2} \frac{M_{c_s}(m_s(t)) (m_s^{-1}(t))^2}{c_{h,s} c_{b,s}^3 (m_s^{-1}(t))^2} \left( \int_{\mathbb{R}} K^2(y) dy \right) \left( \int_{\mathbb{R}} H'(x) dx \right)^2,
\]

(3.19)

\[
K_2(t) = \sum_{s=1}^{2} \frac{M_{c_s}(m_s(t))}{c_{h,s} c_{b,s}^4} \left( \int_{\mathbb{R}} K^2(y) dy \right) \left( \int_{\mathbb{R}} H''(x) dx \right)^2.
\]

(3.20)

Then, if \((m_1^{-1})'(t) - (m_2^{-1})'(t) = \rho_{n_1,n_2} \kappa(t)\) for some non-zero bounded function \(\kappa(t)\) and \(\rho_{n_1,n_2} = o(\eta)\), as \(\min(n_1,n_2) \to \infty\), we have

\[
\mathbb{P}\left( \sup_{t \in I_{a,b}} \frac{\sqrt{(n_1 b_{n,1}^3)}(\hat{m}_1^{-1}(t) - (\hat{m}_2^{-1})'(t) - ((m_1^{-1})'(t) - (m_2^{-1})'(t)))}{K_1(t)} \geq \sqrt{-2 \log(\pi \alpha / \kappa_0)} \right) \to \alpha,
\]

(3.21)
where \( I_{\hat{a}, \hat{b}} = (\hat{a} + \eta, \hat{b} - \eta) \) and

\[
\kappa_0 = \frac{b_{n,1}}{h_1^2} \int_{m_1(0)}^{m_1(1-m_2^{-1}(m_1(0)))} \left( \frac{K_2(t)}{K_1(t)} \right)^{1/2} dt.
\] (3.22)

Theorem 3.2 gives us the following simultaneous confidence band of \((m_1^{-1})'(t) - (m_2^{-1})'(t)\):

\[
(m_1^{-1})'(t) - (m_2^{-1})'(t) \pm \frac{\hat{K}^{1/2}(t)}{\sqrt{(n_1 b_{n,1}^b)}} \sqrt{(-2 \log(\pi \alpha / \hat{\kappa}_0))}, \quad t \in I_{\hat{a}, \hat{b}},
\] (3.23)

where \( \hat{\kappa}_0 = \frac{b_{n,1}}{h_1^2} \int_{\hat{a} + \eta}^{\hat{b} - \eta} \sqrt{\frac{\hat{K}_2(t)}{\hat{K}_1(t)}} dt \), and \( \hat{K}_1(t) \) and \( \hat{K}_2(t) \) are appropriate estimators of \( K_1(t) \) and \( K_2(t) \), respectively. Therefore we can reject the null hypothesis (1.3) at significance level \( \alpha \). Furthermore, it follows from condition (B) and (3.22) that the width of (3.23) is \( \sqrt{\frac{\log n - \log \alpha}{\sqrt{n_1 b_{n,1}^b}}} \), which indicates the SIT test is asymptotically more powerful than the SCB test when bandwidths are of the same order. However, for moderately large sample size, the SCB test performs well when \((m_1^{-1})'(t) - (m_2^{-1})'(t)\) is 'bumpy', or equivalently the majority part of two curves \( m_1(t) \) and \( m_2(t) \) are same up to the horizontal shift while minor parts of \( m_1(t) \) and \( m_2(t) \) have notably different shapes so that their differences cannot be eliminated by a horizontal shift.

## 4 Implementation of the tests

### 4.1 Estimation of \( M_{c_s}(t) \)

The implementation of the SIT test and the SCB test require the estimation of \( M_{c_s}(t) \). For \( s = 1 \) and \( 2 \), let \( \hat{e}_{i,s}(t) = y_{i,s} - x_{i,s}^{\top} \hat{\theta}_{r,s}(t) \), and

\[
\Sigma_s(t) = \frac{1}{n_s b_{n,s} w_{n,s}} \sum_{i=1}^{n_s} \phi \left( \frac{\hat{e}_{i,s}(t)}{w_{n,s}} \right) x_{i,s} x_{i,s}^{\top} K_{b_{n,s}}(i/n_s - t),
\] (4.24)
where bandwidth $w_{n,s}$ is such that $w_{n,s} = o(1)$, $n_s w_{n,s} \to \infty$, and $\phi(\cdot)$ is the probability density function of the standard normal distribution. It has been shown in Theorem 6 in Wu and Zhou (2017) that with appropriate choices of $w_{n,s}$, $\hat{\Sigma}(t)$ is a consistent estimator of $\Sigma(t)$ uniformly on $[b_{n,s}, 1 - b_{n,s}]$. To estimate $V_s(t)$, we define

$$
\hat{\Xi}_{i,s} = \sum_{j=-M_s}^{M_s} \left( \tau - 1 \left( \hat{e}_{i+j,s} \left( \frac{i + j}{n_s} \right) \leq 0 \right) \right) x_{i+j,s},
$$

where $M_s \to \infty$, and $M_s = o(n_s)$ is the window size.

Furthermore, let $\hat{\Delta}_i = \frac{\hat{\Xi}_{i,s} \hat{\Xi}_{i,s}^\top}{2M_s + 1}$, and

$$
\hat{V}_s(t) = \frac{1}{n_s b_{n,s}} \sum_{i=1}^{n_s} K_{b_{n,s}}(i/n_s - t) \hat{\Delta}_i.
$$

With appropriate choices of bandwidth, Theorem 5 in Wu and Zhou (2017) shows that $\hat{V}_s(t)$ converges to $V(t)$ uniformly on $[b_{n,s}, 1 - b_{n,s} - M_s + 1]$. We then estimate $M_{c_s}(t)$ by

$$
\hat{M}_{c_s}(t) := \left( \left( c_s^\top \hat{\Sigma}_{c_s}^{-1}(t) \right) \hat{V}_{s}^2 \left( c_s^\top \hat{\Sigma}_{c_s}^{-1}(t) \right) \right)^{1/2}
$$

for $t \in [b_{n,s}, 1 - b_{n,s} - M_s + 1]$, and $\hat{M}_{c_s}(t) = \hat{M}_{c_s}(b_{n,s} + M_s + 1) / n_s$ for $0 \leq t < b_{n,s} + M_s + 1 / n_s$, $\hat{M}_{c_s}(t) = \hat{M}_{c_s}(1 - b_{n,s} - M_s + 1)$ for $t \in (1 - b_{n,s} - M_s + 1, 1]$. Consequently, the estimator $\hat{M}_{c_s}(t)$ is a consistent estimator of $M_{c_s}(t)$ under appropriate choices of $M_s$ and $w_{n,s}$. We move the choice of $M_s$ and $w_{n,s}$ to Section B of the supplementary material.

### 4.2 Bootstrap-Based Test

Let $\{V_{j,1}, j \in \mathbb{Z}\}$ and $\{V_{j,2}, j \in \mathbb{Z}\}$ be i.i.d. standard normal random variables. Theorems 3.1 and 3.2 are built on the fact that the distribution of $\hat{g}_1(t) - \hat{g}_2(t)$ can be well approximated by

$$(m_1^{-1}(t))' - (m_2^{-1}(t))' + Z(t, \{V_{j,s}\}_{j \in \mathbb{Z}, s=1,2}),$$

15
where \( Z(t, \{ V_{j,s} \}_{j \in \mathbb{Z}, s=1,2}) \) is a Gaussian process defined by \( Z(t, \{ V_{j,s} \}_{j \in \mathbb{Z}, s=1,2}) := Z_1(t, \{ V_{j,1} \}_{j \in \mathbb{Z}}) - Z_2(t, \{ V_{j,2} \}_{j \in \mathbb{Z}}) \), and for \( s = 1 \) and \( 2 \),

\[
Z_s(t, \{ V_{j,s} \}_{j \in \mathbb{Z}}) = \sum_{j=1}^{n_s} W_s(m_s, j, t) V_{j,s}
\]

and

\[
W_s(m_s, j, t) = \frac{1}{n_s b_{n,s} N h_s^2} \sum_{i=1}^{N} M_{\mathbb{C}_s}(i/N) H' \left( \frac{m_s(i/N) - t}{h_s} \right) \tilde{K}_{b_{n,s}}(j/n_s - i/N).
\]

The limiting distribution is established by the asymptotic limit of quadratic form of the Gaussian process \( Z(t, \{ V_{j,s} \}_{j \in \mathbb{Z}, s=1,2}) \) for Theorem 3.1 and the convergence of extreme values of \( Z(t, \{ V_{j,s} \}_{j \in \mathbb{Z}, s=1,2}) \) for Theorem 3.2. However, the direct implementation of Theorem 3.1 and Theorem 3.2 is difficult. The former involves a complicated bias term of the order \( (b_n^{-1/2}) \) to be estimated, and the latter has a slow convergence rate \( O \left( \frac{1}{\sqrt{\log n_s}} \right) \), which follows from the proof of Theorem 3.2. To circumvent this difficulty, we propose the following Bootstrap-assisted algorithm based on \( Z(t, \{ V_{j,s} \}_{j \in \mathbb{Z}, s=1,2}) \).

**Algorithm 4.1 (Bootstrap-SIT)**

(a) Estimate \( m_1 \) and \( m_2 \) by (2.2) and estimate \( M_{\mathbb{C}_s}(\cdot) \).

(b) Generate \( Q \) copies of i.i.d. standard normal random variables \( \{ V_{j,1}^{(Q)} \}_{j=1}^{n_1} \) and \( \{ V_{j,2}^{(Q)} \}_{j=1}^{n_2} \) to obtain the statistic

\[
M_Q = \int \left( Z_1^{(Q)}(t, \{ V_{j,1} \}_{j \in \mathbb{Z}}) - Z_2^{(Q)}(t, \{ V_{j,2} \}_{j \in \mathbb{Z}}) \right)^2 \tilde{w}(t) dt.
\]

(c) Let \( M_{(1)} \leq M_{(2)} \leq \ldots \leq M_{(B)} \) be the ordered statistics of \( \{ M_s, 1 \leq s \leq Q \} \). We reject the null hypothesis \( (1.3) \) at level \( \alpha \), whenever

\[
T_{n_1,n_2} > M_{(B(1-\alpha))}.
\]

The \( p \)-value of this test is given by \( 1 - Q^*/Q \), where \( Q^* = \max \{ r : M_{(r)} \leq T_{n_1,n_2} \} \).

**Algorithm 4.2 (Bootstrap-SCB)**
(a) Estimate $m_1$ and $m_2$ by (2.2) and estimate $M_{c_s} (\cdot)$.

(b) Generate $Q$ copies of i.i.d. standard normal random variables $\{V_{j,1}\}_{j=1}^{n_1}$ and $\{V_{j,2}\}_{j=1}^{n_2}$ to obtain the statistic

$$
\hat{M}_Q = \sup_{t \in \bar{T}_{a,b}} \left| (Z_1^{(Q)}(t, \{V_{j,1}\}_{j \in \mathbb{Z}}) - Z_2^{(Q)}(t, \{V_{j,2}\}_{j \in \mathbb{Z}}))/K_1^{1/2}(t) \right|
$$

(c) Let $\hat{M}_1(1) \leq \hat{M}_2(1) \leq \ldots \leq \hat{M}_B(1)$ be the ordered statistics of $\{\hat{M}_s, 1 \leq s \leq Q\}$. Then, the $(1 - \alpha)$-SCB of $(m_1^{-1})'(t) - (m_2^{-1})'(t)$ is

$$
(m_1^{-1})'(t) - (m_2^{-1})'(t) \pm \hat{M}_{(1-B(1-\alpha))}K_1^{1/2}(t).
$$

To apply Algorithm 4.1, there is no need to estimate the bias term $b_{n,1}^{-1/2}(\hat{B}_1 + \gamma_0 \gamma_3 \hat{B}_2)$ as well as the asymptotic variance $V_\gamma$. The validity of these algorithms are based on the approximation of $Z(t, \{V_{j,s}\}_{j \in \mathbb{Z}, s=1,2})$ to $\hat{g}_1(t) - \hat{g}_2(t)$ (see (2.4) for the expressions of $\hat{g}_1(t)$ and $\hat{g}_2(t)$), which is discussed in detail in the proof of Theorem 3.1. Notice that the cutoff values $M_{(1-B(1-\alpha))}$ and $\hat{M}_{(1-B(1-\alpha))}$ are obtained for fixed $n_1$ and $n_2$, while the critical values in Theorem 3.1 and Theorem 3.2 are based on the limiting distribution. Therefore, similar to Zhao and Wu (2008), we expect that Algorithms 4.1 and 4.2 will outperform the test using the critical values of Theorems 3.1 and 3.2. Finally, to implement Algorithm 4.2, we need to estimate $K_1$, which consists of the estimate of $m_{s}^{-1}$, $(m_{s}^{-1})'$ and $m_{s}'$ for $s = 1$ and 2. We suggest to estimate these quantities by $\int_{m_s(0)}^{t} \hat{g}_s(u)du$, $\hat{g}_s(u)$, and $c_s^{(r,s)}\hat{\theta}_{r,s}(t)$ for $s = 1$ and 2, respectively, where $\hat{\theta}_{r,s}^{(r,s)}(t) = (\hat{\theta}_{r,s,1}^{(r,s)}(t), \ldots, \hat{\theta}_{r,s,p}^{(r,s)}(t))^\top$ with $2\hat{\theta}_{r,s,j}^{(r,s)}(t) - \hat{\theta}_{r,s,j}^{(r,s)}(t)$, and $\hat{\theta}_{r,s}^{(r,b_{n,s})}(t) = (\hat{\theta}_{r,s,1}^{(r,b_{n,s})}(t), \ldots, \hat{\theta}_{r,s,p}^{(r,b_{n,s})}(t))^\top$ is defined in (2.1) having bandwidth $b_{n,s}$.

5 Real Data Analysis

Cumulative infected cases and deaths due to COVID-19: This data set consists of two variables, namely, the cumulative number of infected cases and the cumulative number of deaths due to Covid-19 outbreak in a particular country for the period from December
31, 2019 to October 7, 2020, i.e., \( n = 282 \) days. We here consider two countries, namely, France and Germany as they are from the same continent. Our analysis is based on the log transformed data since the data is varying from a small values to a quite large values. The data set is available at [https://ourworldindata.org/coronavirus-source-data](https://ourworldindata.org/coronavirus-source-data). The analysis has three parts, namely, (A) Analysis of cumulative infected cases and deaths in France due to Covid-19 outbreak, (B) Analysis of cumulative infected cases in France and Germany due to Covid-19 outbreak and (C) Analysis of Cumulative deaths in France and Germany due to Covid-19 outbreak. All three analyses are done for \( \tau = 0.5 \), i.e., based on the median curve of the cumulative infected cases and deaths. Besides, in order to implement our proposed methodology, we consider \( n = 282 \) equally spaced points on \([0, 1]\), and the plots are prepared on the time interval \([0, 1]\).

### 5.1 Cumulative infected cases and deaths in France: Covid-19 outbreak

Let us first discuss a few observations. The left diagram in Figure 1 indicates that both cumulative infected cases and deaths are increasing over time in France, which is also expected as the new cases are added to the data on every day. In fact, it is observed in the right diagram in Figure 1 that the median curves (i.e., \( \tau = 0.5 \)) of the cumulative infected cases and the deaths have an increasing trend over time. First, we now implement the SIT and the SCB tests on the full data, and the tests are carried out using the procedure explained in Section 4.2. In this context, we would like to mention that there is no co-variate here, and hence, the choice of \( c \) does not have any role. For \( B = 1000 \), the \( p \)-values of the SIT and the SCB tests are computed when \( \tau = 0.5 \), and the \( p \)-values are 0.067 and 0.072 for the SIT and the SCB tests, respectively. These \( p \)-values of both tests indicate the rejection of the null hypothesis at 8% level of significance, i.e., in other words, the cumulative infected cases and deaths in France due to Covid-19 do not follow the model described in (1.3).

However, we obtain the large \( p \)-values as 0.473 and 0.446 for the SIT and the SCB tests, respectively when the tests are implements on the data corresponds to time on \([0, 0.4]\), i.e., the data for the period from December 31, 2019 to April 13, 2020, i.e., altogether for the
Figure 1: The left diagram plots the cumulative infected cases (solid line) and deaths (dashed line) in France. The right diagram plots the median curves of cumulative infected cases (solid line) and deaths (dashed line) in France.

period of 105 days. These $p$-values indicate that the cumulative infected cases and deaths in France have pattern like the model described in (1.3). In this study, we obtain $\hat{d} = 0.0564$, i.e., in other words, till April 13, 2020, in France, the curves of cumulative infected cases and deaths followed the same pattern but the cumulative infected cases was ahead about six days (as $0.0564 \times 105 = 5.922$) compared to the cumulative deaths. Afterwards, as the death rate went down, the same shift difference was not observed till October 7, 2020.

5.2 Cumulative infected cases in France and Germany: Covid-19 outbreak

In this case, we implement the SIT and the SCB tests on the cumulative infected cases of France and Germany. For $B = 1000$ and $\tau = 0.5$, we obtain the $p$-values as 0.523 and 0.458 of the SIT test and the SCB test, respectively, which indicates that the cumulative infected cases in France and Germany follows the model described in (1.3). Moreover, we obtain $\hat{d} = 0.003$, i.e., in other words, one can conclude that the cumulative infected cases in France have the same pattern as that of Germany, but they are approximately ahead of a day (as $0.003 \times 282 = 0.846$) compared to Germany’s number for the period from December 31, 2019 to October 7, 2020, i.e., altogether the period of 282 days.
Figure 2: The left diagram plots the cumulative infected cases of France (solid line) and that of Germany (dashed line). The right diagram plots the median curves of cumulative infected cases of France (solid line) and that of Germany (dashed line).

5.3 Cumulative deaths in France and Germany: Covid-19 outbreak

First observe that the left diagram in Figure 3 indicates that the cumulative death cases in both France and Germany are increasing over time, and it is also expected as the new cases are added to the data on every day. In addition, it is observed in the right diagram in Figure 3 that the median curves (i.e., $\tau = 0.5$) of the cumulative deaths in France and Germany have an increasing trend over time. We now implement the SIT and the SCB tests on the full data, and the tests are carried out as the earlier cases. For $B = 1000$ and $\tau = 0.5$, the $p$-values are obtained as 0.084 and 0.081 of the SIT and the SCB tests, respectively. These $p$-values of both tests indicate the rejection of the null hypothesis at 9% level of significance, i.e., in other words, the cumulative deaths due to Covid-19 in France and Germany do not follow the model described in (1.3).

However, we see the opposite scenario when the SIT and the SCB tests are implemented on the data corresponds to time on $[0, 0.4]$, i.e., the data for the period from December 31, 2019 to April 13, 2020, i.e., altogether the period of 105 days. For this period of time, the $p$-values are 0.633 and 0.596 for the SIT test and the SCB test, respectively. These large $p$-values indicate that the cumulative deaths in France and Germany have pattern like the
model described in (1.3). In this study, we obtain $\hat{d} = 0.075$, i.e., in other words, till April 13, 2020, the cumulative deaths in France have the same pattern as that of Germany, but they were approximately ahead of eight days ($0.075 \times 105 = 7.875$) compared to Germany’s number. Afterwards, as the death rate went down, the same shift difference was not observed till October 7, 2020.

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**6 SUPPLEMENTARY MATERIAL**

The supplementary material contains the conditions on bandwidth, the procedure of bandwidth selection, all results of simulation study, an analysis of real data related to climate science which considers various covariates, and all technical details.
Supplementary material to
“Shift identification in time varying regression quantiles”

Abstract

In this supplementary material, we discuss some bandwidth conditions in Section A and the choice of bandwidths in Section B, presenting simulation studies in Section C and a real temperature data study in Section D, and display all proofs with some propositions in Section E.

A Additional assumptions (B) on bandwidths and regression functions

We consider a few more conditions (B) on the bandwidth and the regression function.

(B1) For \( s = 1 \) and 2, \( \frac{\log^{3/2} n_s}{\sqrt{b_n^2 n_s}} = o(1) \), \( h_s/b_n = o(1) \), \( \pi_{n,s} = o(\log n_s) \), \( \frac{\log^{4/3} n_s}{\sqrt{(n_s b_n)^3}} = o(1) \), and \( h_s/b_n^2 \to \infty \). Let \( \Omega_n = 2 \sum_{s=1}^{2} (\Theta_{n,s} + h_s + \frac{1}{Nh_s} + \frac{\log^4 n_s}{(n b_n)^{3/2} h_s^2}) \) and assume that

\[ \Omega_n = o \left( \sum_{s=1}^{2} (n_s b_n^{5/2})^{-1/2} \right). \]

(B2) \( (m'_l)^{-1}(m_1(0)) \in (0, 1) \).

(B3) \( b_{n,s} \to 0, \frac{n b_{n,s}^4}{\log^4 n} \to \infty, n^t b_{n,s} = o(1) \) for some \( t > 0 \).

The condition (B1) implies that in practice we should choose \( h_s \) small, which was remarked by Dette et al. (2006) also. Further, (B1) ensures that our proposed estimator \( \hat{b} \) is well defined under alternative hypothesis, and under null, (B2) clearly means that \( d \in (0, 1) \). Condition (B3) guarantees that the nonparameteric estimate \( \hat{m}_s \) approximate well \( m_s \), \( s = 1, 2 \).

B Bandwidth Selection

In this section, we first discuss the choices of the smoothing parameters, namely, \( b_{n,s} \) and \( h_s \) (\( s = 1 \) and 2) for calculating \( T_{n_1,n_2} \). According to Dette et al. (2006), when \( h_s \)
is sufficiently small, it has a negligible impact to the test, and therefore, by considering
bandwidth conditions \((B)\), we recommend choosing \(h_s = n_s^{-1/3}\) as a rule of thumb. For \(b_{n,s}\),
we propose to choose this tuning parameter by a corrected-Generalized Cross Validation (C-GCV)
method (see Craven and Wahba (1978)). Notice that for the local linear regression
with bandwidth \(b\), the estimator can be written as \(\hat{Y}_s = D_s(b)Y_s\) for some \(n_s \times n_s\) matrix
\(D_s(b)\), and then the GCV selects

\[
\hat{b}_{n,s,\text{mean}} = \arg \min_b GCV(b), \text{ where } GCV(b) = \frac{n_s^{-1}(\hat{Y}_s(b) - Y_s)^\top(\hat{Y}_s(b) - Y_s)}{(1 - \text{trace}(D_s(b)/n_s))^2}. \tag{B.1}
\]

Following the arguments of Yu and Jones (1998), it is appropriate to select \(\hat{b}_{n,s}\) by correcting
\(\hat{b}_{n,s,\text{mean}}\). First, we define \(\hat{b}_{n,s} = 2C_s\hat{b}_{n,s,\text{mean}}\) with

\[
C_s = \left(\frac{\int_0^1 \text{trace}(M_1_s(t)) dt}{\int_0^1 \text{trace}(\tilde{M}_s(t)) dt}\right)^{1/5}, \tag{B.2}
\]

where \(M_1_s(t) = ((1_s^\top \Sigma_s^{-1}(t))V_s(t)^2(1_s^\top \Sigma_s^{-1}(t))^\top)^{1/2}, 1_s\) is a \(p_s\)-dimensional identity vector,
\(V_s(t)\) is the same as defined in \([3.12]\) of the main article, \(\tilde{M}_s(t) = \tilde{\Sigma}_s^{-1}(t)\tilde{A}_s(t)\tilde{\Sigma}_s^{-1}(t)\) with

\[
\tilde{A}_s(t) = \sum_{j=\infty}^{\infty} \text{Cov}(H_s(t,G_{0,s})\tilde{L}(t,F_{0,s},G_{0,s}),H_s(t,G_{j,s})\tilde{L}(t,F_{j,s},G_{j,s})), \tag{B.3}
\]

(\(\tilde{L}(t,F_{j,s},G_{j,s}), 1 \leq j \leq n_s\)) is the errors process of local linear regression for \(s\)-th sample
\((s = 1\ and\ 2)\), and \(\tilde{\Sigma}_s(t) = E(H_s(t,G_0)H_s(t,G_0)^\top)\). We refer to Zhou and Wu (2010)
for the estimation of \(\tilde{M}_s(t)\). Then as recommended by Zhou (2010), for the SIT test, we use
\(\hat{b}_{n,s} = \hat{b}_{n,s}^o \times n_s^{-1/45}\) while for the SCB test, we use \(\hat{b}_{n,s} = \hat{b}_{n,s}^o\).

We now discuss the selection of \(w_{n,s}\) and \(M_s\) for the estimation of the quantity \(\hat{M}_{c_s}(t)\)
in Section 4 of the main article. As a rule of thumb, we propose to choose \(M_s = \lfloor n_s^{1/3}\rfloor\)
and select \(w_{n,s}\) by minimum volatility method. Specifically, consider a grid of possibly
\(w_{n,s}: \{w_{s,1}, ..., w_{s,k}\}\). Together with \(M_s\) and \(b_{n,s}\), one can calculate \(\hat{M}_{c_{s,1}}, ..., \hat{M}_{c_{s,k}}\) using
\{w_{s,1}, \ldots, w_{s,k}\}, respectively. Then, for a positive integer \( u \) (\( u = 5 \) say), define

\[
ise(\hat{M}_{cs,u}(l)) = \frac{1}{u-1} \sum_{v=1}^{l+u-1} \left( \hat{M}_{cs,v}(t) - \frac{1}{u-1} \sum_{v=l}^{l+u-1} \hat{M}_{cs,v}(t) \right)^2.
\] (B.4)

Now, let \( l' \) be the minimiser of \( ise(\hat{M}_{cs,u}(l)) \), and we select \( w_{s,l'+\lfloor u/2 \rfloor} \) as \( w_{n,s} \). The validity of these methods for choosing \( w_{n,s} \) and \( M_s \) are given in Wu and Zhou (2017), which also proposed methods of tuning parameters for refinement. For simplicity, we omit the detailed description of the tuning procedure for refinement in our paper. Our empirical study finds that our choices of tuning parameters \( b_{n,s}, h_s, M_s, w_{n,s} \) and the estimate of \( M_{cs}(t) \) work reasonably well.

C Finite Sample Simulation Studies

This section studies the finite sample performance of the test based on \( T_{n_1,n_2} \), i.e., the SIT test and the SCB test. The performance of the tests is carried out for \( n_1 = n_2 = n = 50, 100, 200 \) and \( 500 \), the number of repetitions = 1000 and the number of Bootstrap replication \( (i.e., B) = 500 \). In this study, we consider the Epanechnikov kernel (e.g., see Silverman (1998)) unless mentioned otherwise, and the upper limit of the Riemann sum is the same as the sample size, i.e., \( N = n \). Apart from these choices, \( h_s = n_s^{-1/3} \) is chosen, and we choose \( b_{n,1} \) and \( b_{n,2} \) as described in Section B.

The covariate random variable and the error random variable are generated by the following way. For \( s = 1 \) and \( 2 \),

\[
x_{i,s} = H_s \left( \frac{i}{n}, G_{i,s} \right), e_{i,s} = L_s \left( \frac{i}{n}, F_{i,s}, G_{i,s} \right),
\]

where the notations are defined at the beginning of Section 3 of the main article. Here \( \xi_i \)'s and \( \eta_i \)'s (involved in the expression of \( H_s \) and \( L_s \)) are i.i.d. standard normal random variables and \( t_{5}/\sqrt{(5/3)} \) random variables, respectively, and \( t_5 \) denotes the \( t \)-distribution with 5 degrees of freedom.
**Example 1:** Let
\[
y_{1,i} = \theta_{1,\tau,1} \left( \frac{i}{n} \right) x_{1,i,1} + \theta_{2,\tau,1} \left( \frac{i}{n} \right) x_{2,i,1} + (e_{i,1} - e_{i,\tau,1})
\]
and
\[
y_{2,i} = \theta_{1,\tau,2} \left( \frac{i}{n} \right) x_{1,i,2} + \theta_{2,\tau,2} \left( \frac{i}{n} \right) x_{2,i,2} + (e_{i,2} - e_{i,\tau,2}).
\]
Here $e_{i,\tau,1}$ and $e_{i,\tau,2}$ are the same as defined in (1.2) in the main manuscript. Suppose that $\theta_{1,\tau,1}(t) = t$, $\theta_{2,\tau,1}(t) = \log t$, and $\theta_{1,\tau,2}(t) = t - 0.1$, $\theta_{2,\tau,2}(t) = \log(t - 0.1)$. Further, consider $c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = 1$. In the numerical studies, we consider $\tau = 0.5, 0.7$ and $0.8$.

**Example 2:** Let
\[
y_{1,i} = \theta_{1,\tau,1} \left( \frac{i}{n} \right) x_{1,i,1} + \theta_{2,\tau,1} \left( \frac{i}{n} \right) x_{2,i,1} + \theta_{3,\tau,1} \left( \frac{i}{n} \right) x_{3,i,1} + (e_{i,1} - e_{i,\tau,1})
\]
and
\[
y_{2,i} = \theta_{1,\tau,2} \left( \frac{i}{n} \right) x_{1,i,2} + \theta_{2,\tau,2} \left( \frac{i}{n} \right) x_{2,i,2} + \theta_{3,\tau,2} \left( \frac{i}{n} \right) x_{3,i,2} + (e_{i,2} - e_{i,\tau,2}).
\]
Here $e_{i,\tau,1}$ and $e_{i,\tau,2}$ are the same as defined in (1.2) in the main manuscript. Suppose that $\theta_{1,\tau,1}(t) = t^2$, $\theta_{2,\tau,1} = \sin \frac{\pi t}{2}$, $\theta_{3,\tau,1} = e^t$, and $\theta_{1,\tau,2} = (t - 0.1)^2$, $\theta_{2,\tau,2} = \sin \frac{\pi(t - 0.1)}{2}$ and $\theta_{3,\tau,2} = e^{t - 0.1}$. Further, consider $c_{1,1} = c_{2,1} = c_{3,1} = c_{1,2} = c_{2,2} = c_{3,2} = 1$, and here also, $\tau = 0.5, 0.7$ and $0.8$ are considered in the numerical study.

Note that for both Examples 1 and 2, the choices of time varying coefficients (i.e, $\theta(t)$’s) satisfy the null hypothesis described in (1.3). Tables 1 and 2 show the rejection probabilities of the SIT test and the SCB test for Examples 1 and 2, respectively when the level of significance is 5% and 10%.

For power study, we consider the the same error and covariate processes and used the following examples :

**Example 3:** Let
\[
y_{1,i} = \theta_{1,\tau,1} \left( \frac{i}{n} \right) x_{1,i,1} + \theta_{2,\tau,1} \left( \frac{i}{n} \right) x_{2,i,1} + (e_{i,1} - e_{i,\tau,1})
\]
and

\[ y_{2,i} = \theta_{1,\tau,2} \left( \frac{i}{n} \right) x_{1,i,2} + \theta_{2,\tau,2} \left( \frac{i}{n} \right) x_{2,i,2} + (e_i - e_{i,\tau,2}). \]

Here \( e_{i,\tau,1} \) and \( e_{i,\tau,2} \) are the same as defined in (1.2) in the main manuscript. Suppose that \( \theta_{1,\tau,1}(t) = t, \theta_{2,\tau,1}(t) = \log t, \) and \( \theta_{1,\tau,2}(t) = t^2, \theta_{2,\tau,2}(t) = (\log t)^2. \) Further, consider

| model | n = 50 | n = 100 | n = 200 | n = 500 |
|-------|--------|--------|--------|--------|
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.063 | 0.061 | 0.058 | 0.051 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.119 | 0.116 | 0.110 | 0.103 |
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.062 | 0.061 | 0.057 | 0.052 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.118 | 0.117 | 0.113 | 0.104 |
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.063 | 0.060 | 0.057 | 0.052 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.115 | 0.114 | 0.109 | 0.106 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.067 | 0.060 | 0.059 | 0.052 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.122 | 0.120 | 0.114 | 0.107 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.065 | 0.059 | 0.056 | 0.051 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.120 | 0.119 | 0.111 | 0.104 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.061 | 0.059 | 0.056 | 0.053 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.117 | 0.115 | 0.109 | 0.105 |

Table 1: The estimated size of the SIT test for different sample sizes \( n_1 = n_2 = n \). The levels of significance (denoted as \( \alpha \)) are 5% and 10%.

| model | n = 50 | n = 100 | n = 200 | n = 500 |
|-------|--------|--------|--------|--------|
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.067 | 0.063 | 0.057 | 0.053 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.122 | 0.111 | 0.107 | 0.102 |
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.064 | 0.060 | 0.056 | 0.051 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.125 | 0.119 | 0.111 | 0.105 |
| Example 1 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.062 | 0.061 | 0.055 | 0.050 |
| Example 1 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.124 | 0.117 | 0.106 | 0.101 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.065 | 0.062 | 0.057 | 0.049 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.129 | 0.118 | 0.111 | 0.104 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.067 | 0.063 | 0.054 | 0.050 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.126 | 0.118 | 0.110 | 0.102 |
| Example 2 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.065 | 0.057 | 0.053 | 0.050 |
| Example 2 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.123 | 0.116 | 0.107 | 0.101 |

Table 2: The estimated size of the SCB test for different sample sizes \( n_1 = n_2 = n \). The levels of significance (denoted as \( \alpha \)) are 5% and 10%.
In the numerical studies, we consider \( \tau = 0.5, 0.7 \) and \( 0.8 \).

**Example 4:** Let

\[
y_{1,i} = \theta_{1,\tau,1} \left( \frac{i}{n} \right) x_{1,i,1} + \theta_{2,\tau,1} \left( \frac{i}{n} \right) x_{2,i,1} + \theta_{3,\tau,1} \left( \frac{i}{n} \right) x_{3,i,1} + (e_{i,1} - e_{i,\tau,1})
\]

and

\[
y_{2,i} = \theta_{1,\tau,2} \left( \frac{i}{n} \right) x_{1,i,2} + \theta_{2,\tau,2} \left( \frac{i}{n} \right) x_{2,i,2} + \theta_{3,\tau,2} \left( \frac{i}{n} \right) x_{3,i,2} + (e_{i,2} - e_{i,\tau,2}).
\]

Here \( e_{i,\tau,1} \) and \( e_{i,\tau,2} \) are the same as defined in (1.2) in the main manuscript. Suppose that \( \theta_{1,\tau,1} = t^2, \theta_{2,\tau,1} = \sin \frac{\pi t}{2}, \theta_{3,\tau,1} = e^t, \) and \( \theta_{1,\tau,2} = t^3, \theta_{2,\tau,2} = \cos \frac{\pi t}{2} \) and \( \theta_{3,\tau,2} = \log t \). Further, consider \( c_{1,1} = c_{2,1} = c_{3,1} = c_{1,2} = c_{2,2} = c_{3,2} = 1 \), and here also, \( \tau = 0.5, 0.7 \) and \( 0.8 \) are considered in the numerical study.

Note that in Examples 3 and 4, the choices of the time varying regression coefficients do not satisfy the assertion of null hypothesis described in (1.3). Table 4 shows the rejection probabilities of the test based on \( T_{n_1,n_2} \) when data follow the models described in Examples 3 and 4.

| model | \( n = 50 \) | \( n = 100 \) | \( n = 200 \) | \( n = 500 \) |
|-------|--------------|--------------|--------------|--------------|
| Example 3 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.367 | 0.401 | 0.546 | 0.777 |
| Example 3 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.445 | 0.517 | 0.699 | 0.901 |
| Example 3 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.422 | 0.499 | 0.627 | 0.818 |
| Example 3 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.497 | 0.563 | 0.776 | 0.923 |
| Example 3 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.378 | 0.444 | 0.622 | 0.816 |
| Example 3 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.412 | 0.535 | 0.701 | 0.888 |
| Example 4 (\( \alpha = 5\% \), \( \tau = 0.5 \)) | 0.422 | 0.477 | 0.661 | 0.825 |
| Example 4 (\( \alpha = 10\% \), \( \tau = 0.5 \)) | 0.447 | 0.500 | 0.708 | 0.917 |
| Example 4 (\( \alpha = 5\% \), \( \tau = 0.7 \)) | 0.475 | 0.503 | 0.688 | 0.848 |
| Example 4 (\( \alpha = 10\% \), \( \tau = 0.7 \)) | 0.510 | 0.582 | 0.727 | 0.949 |
| Example 4 (\( \alpha = 5\% \), \( \tau = 0.8 \)) | 0.398 | 0.419 | 0.589 | 0.801 |
| Example 4 (\( \alpha = 10\% \), \( \tau = 0.8 \)) | 0.419 | 0.475 | 0.623 | 0.878 |

Table 3: The estimated power of the test of the test based on \( T_{n_1,n_2} \), i.e., the SIT test for different sample sizes \( n_1 = n_2 = n \). The levels of significance (denoted as \( \alpha \)) are 5% and 10%.

It follows from the results of Examples 1 and 2 that the test based on \( T_{n_1,n_2} \), i.e., the
Example 3 ($\alpha = 5\%, \tau = 0.5$)

| n   | 0.343 | 0.376 | 0.519 | 0.743 |
|-----|-------|-------|-------|-------|

Example 3 ($\alpha = 10\%, \tau = 0.5$)

| n   | 0.421 | 0.487 | 0.665 | 0.874 |
|-----|-------|-------|-------|-------|

Example 3 ($\alpha = 5\%, \tau = 0.7$)

| n   | 0.395 | 0.470 | 0.592 | 0.789 |
|-----|-------|-------|-------|-------|

Example 3 ($\alpha = 10\%, \tau = 0.7$)

| n   | 0.466 | 0.525 | 0.739 | 0.888 |
|-----|-------|-------|-------|-------|

Example 3 ($\alpha = 5\%, \tau = 0.8$)

| n   | 0.344 | 0.417 | 0.589 | 0.786 |
|-----|-------|-------|-------|-------|

Example 3 ($\alpha = 10\%, \tau = 0.8$)

| n   | 0.387 | 0.509 | 0.668 | 0.849 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 5\%, \tau = 0.5$)

| n   | 0.434 | 0.496 | 0.675 | 0.842 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 10\%, \tau = 0.5$)

| n   | 0.472 | 0.521 | 0.735 | 0.946 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 5\%, \tau = 0.7$)

| n   | 0.499 | 0.530 | 0.723 | 0.880 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 10\%, \tau = 0.7$)

| n   | 0.538 | 0.611 | 0.752 | 0.981 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 5\%, \tau = 0.8$)

| n   | 0.421 | 0.443 | 0.614 | 0.824 |
|-----|-------|-------|-------|-------|

Example 4 ($\alpha = 10\%, \tau = 0.8$)

| n   | 0.445 | 0.498 | 0.651 | 0.901 |
|-----|-------|-------|-------|-------|

Table 4: The estimated power of the SCB test for different sample sizes $n_1 = n_2 = n$. The levels of significance (denoted as $\alpha$) are 5% and 10%.

SIT test and the SCB test can achieve the nominal level of significance when $\tau = 0.5$, 0.7 and 0.8. In terms of estimated power, the results of Examples 3 and 4 indicate that the SIT and the SCB tests can achieve the maximum power as the sample size increases. Precisely speaking, for Example 3, the SIT test is marginally more powerful than the SCB test whereas for Example 4, the SCB test is faintly more powerful than the SIT test. We also observe the same phenomena for unequal $n_1$ and $n_2$ but for the sake of concise presentation, we have not here reported the values of the estimated size and power.

D Average Temperature Anomaly

This data set consists of four variables, namely, average temperature anomaly, the carbon emission in the form of gas, solid and liquid. We here consider two regions, namely, the northern hemisphere and the southern hemisphere since the feature of average temperature anomaly and the carbon emission in the form of gas, solid and liquid are different in two hemispheres, and they are monotonically increasing over time which causes interest of study in climate science (see, e.g., [Raupach et al. (2014)]). The data set for these two regions of the aforementioned four variables are available in [https://ourworldindata.org/](https://ourworldindata.org/)
These yearly data sets reported the values of the variables for the period from 1850 to 2018, i.e., \( n = 169 \). In this study, the average temperature anomaly is considered as the response variable (denoted as \( y \)), and the carbon emission in the form of gas (denoted as \( x_1 \)), solid (denoted as \( x_2 \)) and liquid (denoted as \( x_3 \)) are the covariates.

Here also, we would like to discuss a few more observations on this data: The diagrams in Figure 4 indicate that for both northern and southern hemispheres, \( y, x_1, x_2 \) and \( x_3 \) increase over time, which is a well-known feature in climate science. Moreover, we observe from Figure 5 that the fitted quantile coefficient curves associated with \( x_1, x_2 \) and \( x_3 \) are monotonically increasing over time for a given quantile (in Figure 5, \( \tau = 0.5 \)). We now investigate the performance of the test based on \( T_{n_1,n_2} \) (here \( n_1 = n_2 = n = 169 \)) to check whether this data favors \( H_0 \) (see (1.3)) or not when \( c = (c_{1,1}, c_{2,1}, c_{3,1}, c_{1,2}, c_{2,2}, c_{3,2}) = (1, 0, 0, 1, 0, 0) \) and \((0, 0, 1, 0, 0, 1)\), and the test is carried out following the procedure described in Section 4.2. For \( B = 1000 \), we here also computed the \( p \)-values for \( \tau \) at 5% level of significance. For \( \tau = 0.5 \) and \( c = (1, 0, 0, 1, 0, 0) \), the \( p \)-values of the SIT test and the SCB test are 0.347 and 0.299, respectively. Next, when \( c = (0, 0, 1, 0, 0, 1) \) and \( \tau = 0.5 \), the \( p \)-values of the SIT test and the SCB test are 0.574 and 0.513, respectively. These \( p \)-values of both the tests SIT and SCB indicate that this data set favors the null hypothesis for \( c = (1, 0, 0, 1, 0, 0) \) and \( (0, 0, 1, 0, 0, 1) \) when \( \tau = 0.5 \), which is consistent with the nature of the curves drawn on the left and the right diagrams of Figure 5.

However, for \( \tau = 0.5 \) and \( c = (c_{1,1}, c_{2,1}, c_{3,1}, c_{1,2}, c_{2,2}, c_{3,2}) = (0, 1, 0, 0, 1, 0) \), the \( p \)-values of the SIT test and the SCB tests are 0.087 and 0.081, respectively, which indicate a rejection of the null hypothesis at 9% level of significance. These small \( p \)-values obtained in the last case is consistent with the feature of the curves illustrated in the middle diagram in Figure 5. Specifically for \( \tau = 0.5 \), the case \( c = (0, 1, 0, 0, 1, 0) \) corresponds to the quantile coefficient curve presented in the middle diagram of Figure 5, which clearly indicates that there is no any constant shift between the quantile coefficient curves for the northern and the southern hemispheres. This fact leads to relatively small \( p \)-values.
Figure 4: Plots of temperature anomaly, carbon (gas, liquid and solid) emission in northern and southern hemispheres. In each diagram, the line curve represents for the northern hemisphere, and the dotted curve represents for the southern hemisphere.
Figure 5: The plots of fitted quantile coefficients curves associated with $x_1$ (left diagram), $x_2$ (middle diagram) and $x_3$ (right diagram). Here $\tau$ denotes the index of the quantile. In each diagram, the line curve represents for the northern hemisphere, and the dotted curve represents for the southern hemisphere.

E Proofs

Sketch of the proofs. The properties of the estimators $\hat{a}$ and $\hat{b}$ follow from the stochastic expansion of the deviation of the local linear quantile estimator $\hat{m}_s(t) - m_s(t)$, (cf. Proposition 1 of Wu and Zhou (2017)) as well as the monotonicity of functions $\mu_s$, $s = 1, 2$. The proof of Theorem 3.1 have two steps. The first step is to expand the deviation of the test statistics under null and local alternatives, and approximate it by some Gaussian process using the extended argument of proof of Theorem 4.1 of Dette and Wu (2019). Notice that in our case, we consider two samples as well as the local linear quantile regression. The second step is to using Theorem 2.1 of de Jong (1987) to figure out the asymptotic behavior of a quadratic Gaussian integrals via tedious calculations. Based on step 1 of the proof of Theorem 3.1 we further show Theorem 3.2 using the approximation formula in Proposition 1 of Sun and Loader (1994) to obtain the simultaneous confidence band. Intensive calculations are provided in our proof to determine the parameters in the approximation formula of Sun and Loader (1994).
E.1 Some propositions

Proposition E.1 Assume conditions (A1)-(A8) and (B3). Then on a possibly richer probability space, there exists i.i.d. sequence of standard normal random variables \((V_{i,1})_{i \in \mathbb{Z}}\) and \((V_{i,2})_{i \in \mathbb{Z}}\), such that for \(s = 1\) and 2,

\[
\sup_{t \in T_{n,s}} \left| \hat{m}_s(t) - m_s(t) - \frac{M_{c_s}(t) \sum_{i=1}^{n_s} V_{i,s} \bar{K}_{b_{n,s}}(i/n_s - t)}{nb_{n,s}} \right| = O_p(\Theta_{n,s}),
\]

(E.2)

where \(\Theta_{n,s} = n_{1/4} b_{n,s}^{1/4} + \frac{\pi_{n,s} n_{1/4}}{\sqrt(n_{s} b_{n,s})} \), and \(\pi_{n,s} = b_{n,s} \log^6 n_{s} + (n_{s} b_{n,s})^{-1/4} \log^3 n_{s} + b_{n,s}^{3/2} \sqrt(n_{s} b_{n,s}) \log^3 n_{s}.\)

Proof. The proposition follows immediately from Proposition 1 of Wu and Zhou (2017).

The following proposition provides the convergence rate of \(\hat{a}\) and \(\hat{b}\) under the null and the local alternatives.

Proposition E.2 Under the conditions of Proposition E.1, assuming (B1), (B2) and \(\eta = o(1), \eta^{-1} = O(\log(n_1 + n_2))\). Then if \((m_1^{-1})'(t) - (m_2^{-1})'(t) = \rho_{n_1,n_2} \kappa(t)\) for some non-zero bounded function \(\kappa(t)\) and \(\rho_{n_1,n_2} = o(\eta)\), we have that

\[
i) \max(|\hat{a} - m_1(0)|, |\hat{b} - m_1(1 - m_2^{-1}(m_1(0)))|) = O_p \left( \sum_{s=1}^{2} \frac{\sqrt(n_{s})}{\sqrt(n_{s} b_{n,s})} b_{n,s} + \rho_{n_1,n_2} \right).
\]

(E.3)

Notice that under null, the LHS of (E.3) will be reduced to \(\max(|\hat{a} - a|, |\hat{b} - b|)\). Moreover,

\[
ii) \lim_{n_1 \to \infty, n_2 \to \infty} P(m_{s}^{-1}(t) \in (b_n, 1 - b_n), \ s = 1, 2, \ \text{for all} \ t \in (\hat{a} + c_1 \eta, \hat{b} - c_2 \eta)) = 1. \quad (E.4)
\]

for any given positive constant \(c_1, c_2 > 0\).

Remark E.1 Note that i) in Proposition E.2 shows that \(\hat{w}(t)\) in (2.9) is consistent under null hypothesis and local alternatives. Further, observe that ii) in Proposition E.2 shows that by introducing \(\eta\), we avoid bandwidth conditions since ii) excludes regions where \(m_s(t)\) \((s = 1\ and 2)\ are close to 0 and 1.\)
Proof of Proposition 2.1: We extend the proof of Lemma 2.1 in Dette et al. (2019). We shall show (1.3) is equivalent to

\[
(m_2^{-1}(u) - m_1^{-1}(u))' = 0 \quad \text{(E.5)}
\]

for \(m_1(0) < u < m_1(1 - d)\).

If \(m_1(t) = m_2(t + d)\) for \(0 < t < 1 - d\) for some unknown \(d\), and \(m_1(t)\) and \(m_2(t)\) are monotonically increasing for \(0 < t < 1 - d\), one then can write \(u = m_1(t) = m_2(t + d)\) for \(m_1(0) < u < m_1(1 - d)\), which implies that \(t = m^{-1}(u)\) and \(t + d = m_2^{-1}(u)\). Therefore,

\[
(m_2^{-1}(u) - m_1^{-1}(u)) = d
\]

for \(m_1(0) < u < m_1(1 - d)\). Since \(d\) is a constant, we have proven that (1.3) implies (E.5).

On the other hand, by (E.5), one can see that for any \(t \in (m_1(0), m_1(1 - d))\),

\[
\int_{m_1(0)}^{t} (m_1^{-1}(u))' du = \int_{m_1(0)}^{t} (m_2^{-1}(u))' du. \quad \text{(E.6)}
\]

As a result, we have \(m_1^{-1}(t) = m_2^{-1}(t) - m_2^{-1}(m_1(0))\), and \(d = m_2^{-1}(m_1(0))\). Therefore, by rearranging the equation and taking \(m_2'\) on both sides of it, one can conclude that

\[
t = m_2(m_1^{-1}(t) + m_2^{-1}(m_1(0))) = m_1(m_1^{-1}(t)) \quad \text{(E.7)}
\]

for \(0 < m_1^{-1}(t) < 1 - d\), which finishes the proof by setting \(u = m_1^{-1}(t)\).

Proposition E.3 Along with the conditions of Proposition E.1, assume that \(\pi_{n,s} = o(\sqrt{\log n_s})\), \(n_s b_{n,s}^2 \log^2 n_s \to \infty\) for \(s = 1\) and 2. Then, we have that for \(s = 1\) and 2,

\[
\sup_{t \in T_{n,s}} |\hat{m}_s(t) - m_s(t)| = O_p \left( \frac{\sqrt{\log n_s}}{\sqrt{n_s b_{n,s}}} \right). \quad \text{(E.8)}
\]
Proof. It follows from Lemma 1 of Zhou and Wu (2010) that
\[
\sup_{t \in T_{n,s}} \left| \sum_{i=1}^{n_s} V_{i,s} \bar{K}_{b_{n,s}}(i/n_s - t) \right| = O_p \left( \frac{\sqrt{\log n}}{\sqrt{n_s b_{n,s}}} \right).
\] (E.9)
Then the assertion of this proposition follows from Proposition E.1. □

Proof of Proposition E.2: Notice that \((m^2_2)^{-1}(t) - (m'_1)^{-1}(t) = \rho_{n_1,n_2} \kappa(t)\) implies that
\[
m_2^{-1}(u) - m_1^{-1}(u) = m_2^{-1}(m_1(0)) + \int_{m_1(0)}^{u} \rho_{n_1,n_2} \kappa(t) dt.
\] (E.10)

Proof of assertion i): By proposition E.3 and (E.10), it is sufficient to show that for \(s = 1\) and 2,
\[
\sup_{u \in [a,b]} |\hat{g}_s(u) - g_s(u)| = O_p \left( \frac{\sqrt{\log n_s}}{\sqrt{(n_s b_{n,s})b_{n,s}}} \right),
\] (E.11)
where \(g_s(t) = \frac{1}{Nh_s} \sum_{i=1}^{N} H \left( \frac{\hat{m}_s(i/n_s) - t}{h_s} \right)\). After carefully inspecting the proof of Theorem 3.1, one can find that \(\sup_{t \in [a,b]} |\hat{g}_s(t) - g_s(t)| = O_p \left( \sup_{t \in [a,b]} Z_s(t) \right)\), where \(Z_s(t)\) is defined in (E.28). Then the proposition follows from (E.48) in the proof of Theorem 3.1. The assertion in (ii) follows from condition (B2), (E.10) (with \(u = \hat{m}_1 \left( 1 - \hat{\alpha} \right)\) especially), strict monotonicity of \(m_s\) \(s = 1\) and 2, the mean value theorem and the fact that \(b_{n_1} = o(\eta), b_{n_2} = o(\eta)\) and \(\rho_{n_1,n_2} = o(\eta)\), which can be explained as follows. Note that using (i), with probability tending to 1, we have
\[
m_1(0) < \hat{\alpha}_1 + c_1 \eta.
\] (E.12)
Now, since \(b_{n_1} = o(\eta)\), and \(m_1^{-1}\) is differentiable, using mean value theorem with probability tending to 1, we have
\[
m_1(b_{n_1}) < \hat{\alpha} + c_1 \eta,
\] (E.13)
and hence, \(\lim_{n \to \infty} \mathbb{P}(b_{n_1} < m_1^{-1}(\hat{\alpha} + c_1 \eta)) = 1\). Arguing in a similar way, one can establish
that \( \lim_{n \to \infty} \mathbb{P}(1 - b_{n1} > m_1^{-1}(\hat{b} - c_2 \eta)) = 1 \). Then (ii) holds since the function \( m^{-1}(\cdot) \) is monotone.

**Proof of Theorem 3.1** Write

\[
\tilde{T}_{n1,n2} = \int_{\mathbb{R}} (\hat{g}_1(t) - \hat{g}_2(t))^2 w(t) dt, \quad \text{where} \quad w(t) = 1(a + \eta \leq t \leq b - \eta). \tag{E.14}
\]

In the following, we shall prove that, under conditions of Theorem 3.1,

\[
n_1 b_{n1}^{5/2} \tilde{T}_{n1,n2} - b_{n1}^{-1/2} (\hat{B}_1 + \gamma_0 \gamma_1^2 \hat{B}_2) - \int_{\mathbb{R}} \kappa^2(t) w(t) dt \Rightarrow N(0, V_T), \tag{E.15}
\]

\[
n_1 b_{n1}^{5/2} (\tilde{T}_{n1,n2} - T_{n1,n2}) = o_p(1). \tag{E.16}
\]

**Proof of (E.15):**

Define \( g_s(t) = \frac{1}{Nh_s} \sum_{i=1}^{N} H\left(\frac{m_s(\frac{t+}{h_s}) - t}{h_s}\right) \). Then we have the following decomposition:

\[
\tilde{T}_{n1,n2} = \int_{\mathbb{R}} (J_1(t) - J_2(t) + J_3(t))^2 w(t) dt, \tag{E.17}
\]

\[
J_s(t) = \hat{g}_s(t) - g_s(t), s = 1 \text{ and } 2, J_3(t) = g_1(t) - g_2(t). \tag{E.18}
\]

Using the similar argument of page 471 of Dette et al. (2006), we have

\[
J_3(t) = (m_1^{-1})'(t) - (m_2^{-1})'(t) + O\left( h_1 + h_2 + \frac{1}{Nh_1} + \frac{1}{Nh_2} \right) \\
= \rho_n \kappa(t) + O\left( h_1 + h_2 + \frac{1}{Nh_1} + \frac{1}{Nh_2} \right). \tag{E.19}
\]

Next, by Taylor series expansion, we have that for \( s = 1 \) and 2, the following decomposition holds:

\[
J_s(t) = J_{s,1}(t) + J_{s,2}(t), \tag{E.20}
\]
where
\[
J_{s, 1}(t) = \frac{1}{Nh_s^2} \sum_{i=1}^{N} H'(\frac{m_s(i/N) - t}{h_s})(\hat{m}_s(i/N) - m_s(i/N)),
\]
\[
J_{s, 2}(t) = \frac{1}{2Nh_s^3} \sum_{i=1}^{N} H''(\frac{m_s(i/N) - t + \nu_s^*(\hat{m}_s(i/N) - m_s(i/N))}{h_s})(\hat{m}_s(i/N) - m_s(i/N))^2
\]
for some \(\nu_s^* \in [-1, 1]\) (s = 1 and 2). Notice that \(\frac{\log^{4/3} n_s}{\sqrt{(n_s b_{n,s})h_s}} = o(1)\), and thus the number of non-zero summands in \(J_{s, 2}(t)\) is of order \(O\left(\frac{\log^{4/3} n_s}{(n_s b_{n,s})^{3/2}}h_s^2\right)\) with probability 1. Using Propositions [E.1] and [E.3] with the same arguments in the proof of Theorem 4.1 in the online supplement of [Dette and Wu (2019)] for obtaining bound for \(\Delta_{2,N}\) in their paper, we obtain that for \(s = 1\) and 2, and uniformly for \(t \in [a + \eta, b - \eta]\),
\[
J_{s, 2}(t) = O_P\left(\frac{\log^4 n_s}{(n_s b_{n,s})^{3/2}}h_s^2\right). \tag{E.21}
\]
Therefore, by applying the assertion of Proposition [E.1] to \(J_{s, 1}(t)\), equations (E.19)–(E.21), on a possibly richer probability space, there exists a sequence of i.i.d. standard normal random variables \((V_{s,i})_{i \in \mathbb{Z}}\), \(s = 1\) and 2 such that \(\tilde{T}_{n_1,n_2}\) can be written as
\[
\tilde{T}_{n_1,n_2} = \int_{\mathbb{R}} [(m_1^{-1})'(t) - (m_2^{-1})'(t) + Z(t) + R(t)]^2 w(t)dt
\]
\[
= \int_{\mathbb{R}} [(p_n \kappa(t) + Z(t) + R(t)]^2 w(t)dt, \tag{E.22}
\]
where
\[
Z(t) = \sum_{s=1}^{2} (-1)^{s-1} \frac{n_s}{n_s b_{n,s} N h_s^2} \sum_{i=1}^{N} \sum_{j=1}^{n_s} M_{c_s}(i/N) H'(\frac{m_s(i/N) - t}{h_s}) \bar{K}_{b_{n,s}}(j/n_s - i/N) V_{j,s}, \tag{E.23}
\]
and \(\sup_{t \in [a + \eta, b - \eta]} |R(t)| = O_P(\Omega_n). \tag{E.24}\)
As
\[ n_1 b_{n,1}^{5/2} \rho_n^2 \int_{\mathbb{R}} \kappa^2(t)w(t)dt = \int_{\mathbb{R}} \kappa^2(t)w(t)dt, \tag{E.25} \]
to prove (E.15), it is sufficient to show that
\[ n_1 b_{n,1}^{5/2} \int_{\mathbb{R}} Z(t)w(t)dt - b_{n,1}^{-1/2}(\bar{B}_1 + \gamma_0 \gamma_1^2 \bar{B}_2) - \int_{\mathbb{R}} \kappa^2(t)w(t)dt \Rightarrow N(0, V_T), \tag{E.26} \]
\[ n_1 b_{n,1}^{5/2} \left( \int_{\mathbb{R}} R(t)(R(t) + \rho_n \kappa(t) + Z(t))w(t)dt + \int_{\mathbb{R}} \rho_n \kappa(t)Z(t)w(t)dt \right) = o_p(1). \tag{E.27} \]

Proof of (E.26):

We decompose \( Z(t) \) by \( Z(t) := Z_1(t) - Z_2(t) \). Here for \( s \) = 1 and 2, we have
\[ Z_s(t) = \sum_{j=1}^{n_s} W_s(m_s, j, t)V_{j,s}, \text{ where} \tag{E.28} \]
\[ W_s(m_s, j, t) = \frac{1}{n_s b_{n,s}Nh_s^2} \sum_{i=1}^{N} M_{e_s}(i/N)H'\left( \frac{m_s(i/N) - t}{h_s} \right) K_{b_{n,s}}(j/n_s - i/N). \tag{E.29} \]

As a result, we have
\[ \int_{\mathbb{R}} (Z_1(t) - Z_2(t))^2w(t)dt = A_1 + A_2 - 2A_{12}, \text{ where} \tag{E.30} \]
\[ A_s = \int_{\mathbb{R}} \left( \sum_{j=1}^{n_s} W_s(m_s, j, t)V_{j,s} \right)^2 w(t)dt, \text{ } s = 1 \text{ and } 2, \tag{E.31} \]
\[ A_{12} = \int_{\mathbb{R}} \left( \sum_{j=1}^{n_1} W_1(m_1, j, t)V_{j,1} \right) \left( \sum_{j=1}^{n_2} W_2(m_2, j, t)V_{j,2} \right) w(t)dt. \tag{E.32} \]

We first prove the results for \( A_1 \), and the result for \( A_2 \) can be evaluated in a similar
way. Notice that \( A_1 = A_{1,a} + A_{1,b}, \) where

\[
A_{1,a} = \sum_{j=1}^{n_1} \left[ \int_{\mathbb{R}} W_1^2(m_1, j, t) w(t) dt \right] V_{j,1}^2, \tag{E.33}
\]

\[
A_{1,b} = 2 \sum_{1 \leq j_1 < j_2 \leq n_1} \left[ \int_{\mathbb{R}} W_1(m_1, j_1, t) W_1(m_1, j_2, t) w(t) dt \right] V_{j_1,1} V_{j_2,2}. \tag{E.34}
\]

Now, using the bandwidth condition \( h_s = o(b_{n,s}) \), calculating the Riemann sum with widths of summands \( 1/N \) and change of variable \( y = (m_s(u) - t)/h_s \), a few steps algebraic calculations show that (note that we use the similar calculation in Proposition \[E.2\])

\[
W_s(m_s, j, t) = \frac{1}{n_s b_{n,s} h_s} \int_{\mathbb{R}} M_{c_s}(m_s^{-1}(t + h_s y)) H'(y) \times \\
\bar{K}_{b_{n,s}}(j/n_s - m_s^{-1}(t + h_s y))(m_s^{-1})(t + h_s y) dy + O(R(j, s, n, t)), \tag{E.35}
\]

where

\[
R(j, s, n, t) = \frac{1}{n_s b_{n,s} h_s^2 N} 1 \left( \left| \frac{j/n_s - m_s^{-1}(t)}{b_{n,s} + Mh_s} \right| \leq 1 \right) \tag{E.36}
\]

for a sufficiently large constant \( M \). Since \( H \) is chosen to be symmetric, we have \( \int_{\mathbb{R}} H'(x) dx = 0 \). Therefore, by Taylor series expansion, for \( t \) with \( w(t) \neq 0 \), it follows that for \( s = 1 \) and \( 2 \), the leading term of \( W_s(m_s, j, t) \) can be written as

\[
\tilde{W}_s(m_s, j, t) = \frac{-1}{n_s b_{n,s}^2} M_{c_s}(m_s^{-1}(t)) K' \left( \frac{j/n_s - m_s^{-1}(t)}{b_{n,s}} \right) \left( (m_s^{-1})'(t) \right)^2 \int_{\mathbb{R}} H'(y) y dy
\]

\[
= \frac{-1}{n_s b_{n,s}^2} \tilde{g}_s(t) K' \left( \frac{j/n_s - m_s^{-1}(t)}{b_{n,s}} \right). \tag{E.37}
\]

Next, by using (E.37), we have (note that we use the similar calculation in Proposition
\( \mathbb{E}(A_{1,a}) = \sum_{j=1}^{n_1} \int_{\mathbb{R}} \tilde{W}_1^2(m_1, j, t)w(t)dt(1 + o(1)) \)

\[ = \frac{1}{n_1 b_{n,1}^3} \left( \int_{\mathbb{R}} H'(y)dy \right)^2 \int_{\mathbb{R}} K^2(x)dx \int_{\mathbb{R}} M_{c_1}^2(m^{-1}_s(t))(m^{-1}_s(t))^4w(t)dt(1 + o(1)) \]

\[ = \frac{1}{n_1 b_{n,1}^3} \left( \int_{\mathbb{R}} H'(y)dy \right)^2 \int_{\mathbb{R}} K^2(x)dx \int_{\mathbb{R}} M_{c_1}^2(u)u^4m'_1(u)w(m_1(u))du(1 + o(1)) \]

\[ = \frac{1}{n_1 b_{n,1}^3} \tilde{B}_1(1 + o(1)). \]  \hspace{1cm} (E.38)

On the other hand, the similar calculations show that

\[ \text{Var}(A_{1,a}) = \sum_{j=1}^{n_1} \left[ \int_{\mathbb{R}} W_1^2(m_1, j, t)w(t)dt \right]^2 = O \left( \frac{n_1 b_{n,1}^2}{n_1 b_{n,1}^8} \right) = O \left( \frac{1}{n_1 b_{n,1}^8} \right). \]  \hspace{1cm} (E.39)

Then for \( A_{1,b} \), we have that

\[ \text{Var}(A_{1,b}) = \sum_{j_1=1}^{n_s} \sum_{j_2=1, j_2 \neq j_1}^{n_s} \left[ \int_{\mathbb{R}} W_1(m_1, j_1, t)W_1(m_1, j_2, t)w(t)dt \right]^2 \]

\[ = \sum_{j_1=1}^{n_s} \sum_{j_2=1, j_2 \neq j_1}^{n_s} \left[ \int_{\mathbb{R}} \tilde{W}_1(m_1, j_1, t)\tilde{W}_1(m_1, j_2, t)w(t)dt \right]^2 (1 + o(1)) \]

\[ = \frac{1}{n_s^2 b_{n,s}^8} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \tilde{g}_1^2(t)K' \left( \frac{u - m_1^{-1}(t)}{b_{n,1}} \right) K' \left( \frac{v - m_1^{-1}(t)}{b_{n,1}} \right) w(t)dt \right]^2 du dv \]

\[ \times (1 + o(1)). \]  \hspace{1cm} (E.40)

Now, by changing variable, i.e., letting \( \frac{u - m_1^{-1}(t)}{b_{n,1}} = x \), and using the fact that \( \tilde{g}_1(m_1(u)) = \)

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\[ g_1^2(m_1(u))w(m_1(u))m_1'(u), \text{ we have} \]

\[
\begin{align*}
\text{Var}(A_{1,b}) & = \frac{1}{n_1^2 b_{n,1}^6} \int \int \left[ \int \hat{g}_1^2(m_1(u)) \hat{K}'(x) \hat{K}' \left( x + \frac{v - u}{b_{n,1}} \right) w(m(u)) m'(u) dx \right]^2 dudv(1 + o(1)) \\
& = \frac{1}{n_1^2 b_{n,1}^6} \int \int \hat{g}_1^2(m_1(u)) \left[ \int \hat{K}'(x) \hat{K}' \left( x + \frac{v - u}{b_{n,1}} \right) dx \right]^2 dudv(1 + o(1)) \\
& = \frac{1}{n_1^2 b_{n,1}^6} \int \int \hat{g}_1^2(m_1(u)) (\hat{K}' \ast \hat{K}'(y))^2 dydudv(1 + o(1)) = \frac{1}{n_1^2 b_{n,1}^6} \hat{V}_1(1 + o(1)). \quad (E.41)
\end{align*}
\]

Combining (E.38) (E.39) and (E.41), it follows that

\[
\begin{align*}
\mathbb{E}(A_{1,a}) & = \frac{1}{n_1^2 b_{n,1}^3} \hat{B}_1(1 + o(1)), \quad (E.42) \\
\text{Var}(A_{1,a}) & = O \left( \frac{1}{n_1^2 b_{n,1}^6} \right), \quad (E.43) \\
\text{Var}(A_{1,b}) & = \frac{1}{n_1^2 b_{n,1}^5} \hat{V}_1(1 + o(1)).
\end{align*}
\]

Similarly,

\[
\begin{align*}
\mathbb{E}(A_{2,a}) & = \frac{1}{n_2 b_{n,2}^3} \hat{B}_2(1 + o(1)), \quad \text{Var}(A_{2,a}) = O \left( \frac{1}{n_2 b_{n,2}^6} \right), \quad \text{Var}(A_{2,b}) = \frac{1}{n_2 b_{n,2}^5} \hat{V}_2(1 + o(1)). \quad (E.44)
\end{align*}
\]

On the other hand, for \( A_{12} \), we have

\[
\begin{align*}
\text{Var}(A_{12}) & = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \left[ \int W_1(m_1, j_1, t) W_2(m_2, j_2, t) w(t) dt \right]^2 \\
& = \frac{1}{n_1 n_2 b_{n,1}^4 b_{n,2}^4} \int \int \left[ \int \hat{g}_1(t) \hat{g}_2(t) \hat{K}' \left( \frac{u - m_{1}^{-1}(t)}{b_{n,1}} \right) \hat{K}' \left( \frac{v - m_{2}^{-1}(t)}{b_{n,2}} \right) w(t) dt \right]^2 dudv(1 + o(1)). \quad (E.45)
\end{align*}
\]
Now, by change of variable using \(x = (u - m_1^{-1}(t))/b_{n,1}\), we have that

\[
\int \int \int \tilde{g}_1(t) \tilde{g}_2(t) \tilde{K}' \left( \frac{u - m_1^{-1}(t)}{b_{n,1}} \right) \tilde{K}' \left( \frac{v - m_2^{-1}(t)}{b_{n,2}} \right) w(t) dt \right]^2 dudv
\]

\[
= b_{n,1}^2 \int \int \tilde{g}_{1,2}^2(m_1(u)) \left[ \int \tilde{K}'(x) \tilde{K}' \left( \frac{m_1'(u) b_{n,1} x + v - m_2(u)}{b_{n,2}} \right) dx \right]^2 dudv(1 + o(1))
\]

\[
= b_{n,1}^2 b_{n,2} \int \int \tilde{g}_{1,2}^2(m_1(u)) \left[ \int \tilde{K}'(x) \tilde{K}' \left( \frac{m_2'(u) b_{n,1} x + y}{b_{n,2}} \right) dx \right]^2 dudy(1 + o(1)). \tag{E.46}
\]

Therefore, we have that

\[
\text{Var}(A_{12}) = \frac{\tilde{V}_{12}(\gamma_1)}{n_1 n_2 b_{n,1}^3 b_{n,2}^3} (1 + o(1)) \tag{E.47}
\]

Notice that \(A_{1,b}, A_{2,b}, A_{12}\) are mutually uncorrelated. Using this fact, bandwidth conditions and \((E.30), (E.42), (E.44)\) and \((E.47)\), equation \((E.26)\) follows from Theorem 2.1 of de Jong (1987) and tedious calculations.

**Proof of \((E.27)\):** By \((E.28), (E.29), (E.35)-(E.37)\), we obtain for \(s = 1, 2\)

\[
\sup_{t \in [0,1]} |Z_s(t)| = \frac{\sqrt{\log n_s}}{\sqrt{(n_s b_{n,s}) b_{n,s}}}, \tag{E.48}
\]

which together with bandwidth conditions yields \((E.27)\). Hence \((E.15)\) holds.

**Proof of Theorem 3.1 (conclusion part):** By the proof of \((E.15)\) and the bandwidth conditions, it suffices to show that

\[
n_1 b_{n,1}^{5/2} \int \limits_{\mathbb{R}} (Z(t))^2 (\dot{w}(t) - w(t)) dt = o_p(1). \tag{E.49}
\]

Meanwhile, assertion in \((E.49)\) follows from \((E.48), Proposition \(\text{E.2}\)\) and bandwidth conditions. Now, by \((E.15)\) and \((E.16)\), Theorem 3.1 follows. \(\square\)

**Proof of Theorem 3.2:**
We prove the theorem in two steps.

Step 1: Recall $Z(t)$ defined in (E.23). We first evaluate $\mathbb{E}(Z^2(t))$, $\mathbb{E}(Z'^2(t))$ and $\mathbb{E}(Z(t)Z'(t))$. Since $h_s = o(b_{n,s})$, uniformly for $t \in [a + \eta, b - \eta]$, we have

$$\mathbb{E}(Z^2(t))$$

$$\begin{align*}
= & \sum_{s = 1}^{2} \frac{1}{(n_s b_{n,s} N h_s^2)^2} \sum_{j = 1}^{n_s} \left( \sum_{i = 1}^{N} M_{c_s}(i/N) H' \left( \frac{m_s(i/N) - t}{h_s} \right) \bar{K}_{b_{n,s}}(j/n_s - i/N) \right)^2 \\
= & \sum_{s = 1}^{2} \frac{1}{(n_s b_{n,s} h_s^2)^2} \sum_{j = 1}^{n_s} \left( \int M_{c_s}(u) H' \left( \frac{m_s(u) - t}{h_s} \right) \bar{K}_{b_{n,s}}(j/n_s - u) du + O \left( \frac{1}{n_s^2 b_{n,s} h_s^2} \right) \right)^2 \\
= & \sum_{s = 1}^{2} \frac{1}{n_s (b_{n,s} h_s^2)^2} \int \left( \int M_{c_s}(u) H' \left( \frac{m_s(u) - t}{h_s} \right) \bar{K}_{b_{n,s}}(v - u) du \right)^2 dv + O \left( \frac{1}{n_s^2 b_{n,s} h_s^2} \right) \\
= & \sum_{s = 1}^{2} \frac{M_{c_s}^2(m_s^{-1}(t))}{n_s (b_{n,s} h_s^2)^2 (m_s'(m_s^{-1}(t)))^2} \int \int H'(x) \bar{K}_{b_{n,s}}(v - m_s^{-1}(t + x h_s)) dx dv + O \left( \frac{1}{n_s b_{n,s} h_s} \right) \\
= & \sum_{s = 1}^{2} \frac{M_{c_s}^2(m_s^{-1}(t))(m_s^{-1})^2(t)}{n_s b_{n,s}^3 (m_s'(m_s^{-1}(t)))^2} \int \bar{K}'^2(y) dy \left( \int H'(x) dx \right)^2 \left( 1 + O \left( \frac{h_s}{b_{n,s}} \right) \right) + O \left( \frac{1}{n_s b_{n,s} h_s} \right). 
\end{align*}$$

\[42\]
Next, we have

\[ \mathbb{E}(Z^2(t)) \]

\[ = \sum_{s=1}^{2} \frac{1}{(n_s b_{n,s} N h_s^3)^2} \sum_{j=1}^{n_s} \left( \sum_{i=1}^{N} M_{c_s}(i/N) H'' \left( \frac{m_s(i/N) - t}{h_s} \right) \tilde{K}_{b_{n,s}}(j/n_s - i/N) \right)^2 \]

\[ = \sum_{s=1}^{2} \frac{1}{(n_s b_{n,s} h_s^3)^2} \sum_{j=1}^{n_s} \left( \int M_{c_s}(u) H'' \left( \frac{m_s(u) - t}{h_s} \right) \tilde{K}_{b_{n,s}}(j/n_s - u) du \left( 1 + O \left( \frac{1}{Nh_s} \right) \right) \right)^2 \]

\[ = \sum_{s=1}^{2} \frac{1}{(n_s b_{n,s} h_s^3)^2} \int \left( \int M_{c_s}(u) H'' \left( \frac{m_s(u) - t}{h_s} \right) \tilde{K}_{b_{n,s}}(v - u) du \right)^2 dv \left( 1 + O \left( \frac{1}{Nh_s} \right) \right) \]

\[ + O \left( \frac{1}{n_s b_{n,s} h_s^4} \right) \]

\[ = \sum_{s=1}^{2} \frac{M_{c_s}^2(m_s^{-1}(t))}{n_s(b_{n,s} h_s^3)^2(m_s'(m_s^{-1}(t)))^2} \int \left( \int H''(x) \tilde{K}_{b_{n,s}}(v - m_s^{-1}(t + x h_s)) dx \right)^2 dv \left( 1 + O \left( \frac{1}{Nh_s} \right) \right) \]

\[ + O \left( \frac{1}{n_s b_{n,s} h_s^3} \right) \]

\[ = \sum_{s=1}^{2} \frac{M_{c_s}^2(m_s^{-1}(t))}{n_s b_{n,s} h_s^4(m_s'(m_s^{-1}(t)))^2} \int \tilde{K}^2(y) dy \left( \int H''(x) dx \right)^2 \left( 1 + O \left( \frac{1}{Nh_s} + \frac{h_s}{b_{n,s}} \right) \right) + O \left( \frac{1}{n_s b_{n,s} h_s^4} \right). \]
Finally, by the symmetry of $K$ and $H$, we have

$$\mathbb{E}(Z(t)Z'(t))$$

$$= \sum_{s=1}^{2} \frac{1}{(n_b n_s h_s^2)^2} \sum_{j=1}^{n_s} \left( \sum_{i=1}^{N} M_{c_{i}}(i/N) H'\left(\frac{m_s(i/N) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(j/n_s - i/N) \right)$$

$$\times \left( \sum_{i=1}^{N} M_{c_{i}}(i/N) H''\left(\frac{m_s(i/N) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(j/n_s - i/N) \right)$$

$$= \sum_{s=1}^{2} \frac{1}{(n_b n_s h_s^2)^2} \sum_{j=1}^{n_s} \left( \int_{\mathbb{R}} M_{c_{s}}(u) H'\left(\frac{m_s(u) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(j/n_s - u)du + O\left(\frac{1(|j/n_s - t| \leq \bar{M}b_{n,s})}{N}\right) \right)$$

$$\left( \int_{\mathbb{R}} M_{c_{s}}(u) H''\left(\frac{m_s(u) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(v - u)du + O\left(\frac{1(|j/n_s - t| \leq \bar{M}b_{n,s})}{N}\right) \right)$$

$$= \sum_{s=1}^{2} \frac{1}{n_s(b_n h_s) h_s^2} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} M_{c_{s}}(u) H'\left(\frac{m_s(u) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(v - u)du \right)$$

$$\left( \int_{\mathbb{R}} M_{c_{s}}(u) H''\left(\frac{m_s(u) - t}{h_s}\right) \tilde{K}_{b_{n,s}}(v - u)du \right)$$

$$= \sum_{s=1}^{2} \frac{1}{n_s(b_n h_s) h_s^2} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{M_{c_{s}}(m_s^{-1}(t))}{h_s m_s'(m_s^{-1}(t))} \left( \int_{\mathbb{R}} H'(x) \tilde{K}_{b_{n,s}}(v - m_s^{-1}(t + xh_s))dx \right) \right)$$

$$\left( \int_{\mathbb{R}} H''(x) \tilde{K}_{b_{n,s}}(v - m_s^{-1}(t + xh_s))dx \right)$$

$$= \frac{1}{n_s(b_n h_s^2 h_s^3)} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} H'(x) \tilde{K}_{b_{n,s}}(v - m_s^{-1}(t + xh_s))dx \right)$$

$$\left( \int_{\mathbb{R}} H''(x) \tilde{K}_{b_{n,s}}(v - m_s^{-1}(t + xh_s))dx \right)$$

$$= O\left(\frac{h_s + (h_s/b_{n,s})^2}{n_s(b_n h_s^2 h_s^3)}\right).$$

**Step 2:** We use Proposition 1 of Sun and Loader (1994) to evaluate the maximum deviation of $Z(t)$. For any two $p$-dimensional vectors $u = (u_1, \ldots, u_p)^\top$ and $v = (v_1, \ldots, v_p)^\top$, write $< u, v > = \sum_{i=1}^{p} u_i v_i$, and $\|u\|_E = < u, u >$. Define

$$T_{j,s}(t) = w_s(m_s, j, t), s = 1 \text{ and } 2, 1 \leq j \leq n_s, \quad (E.50)$$

$$T(x) = \left( T_{1,1}(x), \ldots, T_{n_1,1}(x), T_{1,2}(x), \ldots, T_{n_2,2}(x) \right)^\top, \quad (E.51)$$
and $V = (V_1, ..., V_{n_1}, V_{n_1+1}, ..., V_{n_1+n_2})^\top$, where $\{V_i\}_{i=1}^{n_1+n_2}$ is a i.i.d. sequence of standard normal random variables. Then, for any $0 < a < b < 1$ and $\eta > 0$, $\sup_{t \in [a, b]} |Z(t)|$ has the same distribution as $\sup_{t \in [a+\eta, b-\eta]} | < T(t), V > |$. Therefore, by Proposition 1 of Sun and Loader (1994), we have that

$$
\lim_{c \to \infty} P\left( \sup_{t \in [a, b]} | < T(t), V > | \geq c \right) = \kappa_0(a + \eta, b - \eta) \exp \left( -\frac{c^2}{2} \right) + 2(1 - \phi(c)) + o(\exp(-c^2/2)), \tag{E.52}
$$

where $\kappa_0(a + \eta, b - \eta) = \int_{a+\eta}^{b-\eta} \left\| \frac{\partial}{\partial x} \left( \frac{T(x)}{\|T(x)\|_E} \right) \right\|_E dx$. Notice that

$$
\|T(x)\|_E = \|Z(x)\|, \quad \left\| \frac{\partial T(x)}{\partial x} \right\|_E = \left\| \frac{\partial}{\partial x} Z(x) \right\|, \tag{E.53}
$$

$$
\frac{\partial}{\partial x} \|T(x)\|_E = \frac{< T(x), \frac{\partial}{\partial x} T(x) >}{\|T(x)\|_E} = \frac{\mathbb{E}(Z(t)Z'(t))}{\|Z(t)\|}, \quad \text{and} \tag{E.54}
$$

$$
\left\| \frac{\partial}{\partial x} \left( \frac{T(x)}{\|T(x)\|_E} \right) \right\|_E = \left\| \frac{T(x)}{\|T(x)\|_E} - \frac{\partial}{\partial x} \frac{T(x)}{\|T(x)\|_E} \right\|_E. \tag{E.55}
$$

By using (E.53)–(E.55) and the results of $\mathbb{E}(Z^2(t))$, $\mathbb{E}(Z'^2(t))$ and $\mathbb{E}(Z(t)Z'(t))$ from Step 1, we have that

$$
\left\| \frac{\partial}{\partial x} \left( \frac{T(x)}{\|T(x)\|_E} \right) \right\|_E = \frac{b_{n,1}}{h_1^2} \sqrt{\frac{K_2(t)}{K_1(t)}} (1 + o(1)). \tag{E.56}
$$

Furthermore, by (E.3), we have

$$
\lim_{n_1 \to \infty} P\left( \max(|\hat{a} - m_1(0)|, |\hat{b} - m_1(1 - m_2^{-1}(m_1(0)))|) \leq \eta/2 \right) = 1. \tag{E.57}
$$

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Therefore,

\begin{align*}
\mathbb{P}\left( \sup_{t \in [\hat{m}_1(0)+3\eta/2,\hat{m}_1(1-m_2^{-1}(m_1(0)))-3\eta/2]} | < T(t), V > | \geq c \right) \\
\leq \mathbb{P}\left( \sup_{t \in [\hat{a}+\eta,\hat{b}-\eta]} | < T(t), V > | \geq c \right) \\
\leq \mathbb{P}\left( \sup_{t \in [\hat{m}_1(0)+\eta/2,\hat{m}_1(1-m_2^{-1}(m_1(0)))-\eta/2]} | < T(t), V > | \geq c \right). \quad (E.58)
\end{align*}

As a consequence, by (E.56), (E.52) and (E.58), and the fact that \( \eta = o(1) \), we have that

\begin{align*}
\lim_{n_1 \to \infty} \left( \frac{b_{n,1}}{\hat{h}_1^2} \right)^{-1} \lim_{c \to \infty} \mathbb{P}\left( \sup_{t \in [\hat{a}+\eta,\hat{b}-\eta]} | < T(t), V > | \geq c \right) \\
= \left( \frac{b_{n,1}}{\hat{h}_1^2} \right)^{-1} \kappa_0(m_1(0),m_1(1-m_2^{-1}(m_1(0)))) \exp \left( -\frac{c^2}{2} \right), \quad (E.59)
\end{align*}

which completes the proof by solving \( \frac{\kappa_0(m_1(0),m_1(1-m_2^{-1}(m_1(0))))}{\pi} \exp \left( -\frac{c^2}{2} \right) = \alpha. \quad \square \)

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