The unboundedness of Hausdorff operators on quasi-Banach spaces

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In this note, for a large family of quasi-Banach spaces, including local Hardy space $h^p$ and some inhomogeneous function spaces, we show that the Hausdorff operator $H_\Phi$ is unbounded unless it is a zero operator.

KEYWORDS
Hausdorff operator, local Hardy space, modulation space, unboundedness

MSC CLASSIFICATION
47B38; 42B35

1 INTRODUCTION AND MOTIVATION

Hausdorff operator, in connection with some classical summation methods, has been studied for a long time in the field of real and complex analysis. For the historical background and recent developments of the boundedness on function spaces regarding Hausdorff operators, we refer the reader to survey papers by Chen et al1 and Liflyand.2

For a suitable function $\Phi$, the corresponding Hausdorff operator $H_\Phi$ considered in this paper can be defined by

$$H_\Phi f(x) := \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy.$$ (1.1)

Particularly, when $\Phi$ is taken suitably, Hausdorff operator contains some important operators in the field of harmonic analysis. For instance, the Hardy operator, adjoint Hardy operator,3–5 and the Cesàro operator6,7 in one dimension. The Hardy–Littlewood–Pólya operator and the Riemann–Liouville fractional integral operator can also be derived from the Hausdorff operator.

A more general Hausdorff operator is defined by

$$H_{\Phi,A} f(x) := \int_{\mathbb{R}^n} \Phi(y) f(A(y)x) dy,$$
where $A(y)$ is an $n \times n$ matrix and $\det A(y) \neq 0$ almost everywhere in the support of $\Phi$. The operator $H_{\Phi,A}$ was introduced by Brown and Móricz$^8$ and Lerner and Liflyand.$^9$ It is easy to see that $H_\Phi = H_{\Phi,A}$ with $A(y) = \text{diag}(1/|y|, \ldots, 1/|y|)$. In this sense, $H_\Phi$ is a special case of the general Hausdorff operator $H_{\Phi,A}$.

In recent years, there is an increasing interest on the boundedness of Hausdorff operators on function spaces, one can see Gao et al$^{10,11}$ for the the boundedness of $H_\Phi$ on Lebesgue spaces, Zhao et al$^{12}$ for the boundedness of $H_\Phi$ on modulation and Wiener amalgam spaces, Fan and Lin and Liflyand and Móricz$^{5,13}$ for the boundedness of $H_\Phi$ on $H^1(\mathbb{R})$ and $h^1(\mathbb{R})$, and Ruan and Fan$^{14}$ for the boundedness of $H_{\Phi,A}$ on $H^1(\mathbb{R}^n)$ and $h^1(\mathbb{R}^n)$ with power weights. However, since the argument of using Minkowski’s inequality cannot be applied to quasi-Banach spaces, there are only a few boundedness results considering the boundedness of Hausdorff operators on quasi-Banach spaces.

Among the previous results, the study of Hausdorff operators on $H^p(\mathbb{R}^n)(0 < p < 1)$ has its special status, since $H^p(\mathbb{R}^n)$ is a quasi-Banach space and also an important function space in the field of harmonic analysis. The study of $H_\Phi$ on $H^p(\mathbb{R}^n)$ was first initiated by Kanjin$^{15}$ and continued in Liflyand and Miyachi.$^{16}$ Very recently, in the multidimensional case, Liflyand and Miyachi$^{17}$ establish the multidimensional boundedness results on $H^p(\mathbb{R}^n)$. For the $H^p(\mathbb{R}^n)$ boundedness of another type of Hausdorff operator, we refer the reader to Chen et al. and Ruan and Fan.$^{18,19}$

Liflyand and Miyachi$^{17}$ make a deep research for the boundedness of the general Hausdorff operator $H_{\Phi,A}$ on the homogeneous Hardy space $H^p(\mathbb{R}^n)$. On the one hand, they point out that, unlike the one dimensional case, only smoothness condition $\Phi$ cannot imply the boundedness of $H_{\Phi,A}$, and they also give a class of unbounded $H_\Phi$ on $H^p(\mathbb{R}^n)$ with smooth $\Phi$. On the other hand, with some smoothness assumption on $\Phi$, they introduce an algebraic condition for the matrix $A$ and prove the boundedness of $H_{\Phi,A}$ on $H^p(\mathbb{R}^n)$.

Based on the previous results, an interesting problem is whether we can establish the boundedness result on local Hardy space $h^p(\mathbb{R}^n)$, or even on the general inhomogeneous frequency decomposition spaces such as Triebel–Lizorkin spaces $F^p,q(\mathbb{R}^n)$, Besov spaces $B^p,q(\mathbb{R}^n)$ and modulation spaces $M^p,q(\mathbb{R}^n)$ with $p \in (0, 1)$. In this paper, we will consider the special Hausdorff operator $H_\Phi$ and give a negative answer, showing that the Hausdorff operator $H_\Phi$ is unbounded on a large family of quasi-Banach spaces. In particular, we prove that all the nonzero Hausdorff operators $H_\Phi$ are unbounded on $h^p(\mathbb{R}^n)F^p,q(\mathbb{R}^n), B^p,q(\mathbb{R}^n)$, or $M^p,q(\mathbb{R}^n)$ with $p \in (0, 1)$.

Our paper is organized as follows. In Section 2, we collect some notations and basic properties of function spaces. The unboundedness results, and their proofs will be presented in Section 3.

Throughout this paper, we will adopt the following notations. We use $X \lesssim Y$ to denote the statement that $X \lesssim CY$, with a positive constant $C$ that may depend on $n, p$, but it might be different from line to line. The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$. We use $X \lesssim Y$ to denote $X \lesssim C_\lambda Y$, meaning that the implied constant $C_\lambda$ depends on the parameter $\lambda$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we denote $|x|_\infty := \max_{i=1,2,\ldots,n} |x_i|$ and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

## 2 | PRELIMINARY

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx, \quad \mathcal{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} \, d\xi.
$$

The local Hardy space was introduced by Goldberg.$^{20}$ Let $0 < p < \infty$ and let $\psi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \psi(x) \, dx \neq 0$. Define $\psi_t(x) := t^{-n} \psi(x/t)$. The local Hardy space is defined by

$$
h^p := \{ f \in \mathcal{S}' : \|f\|_{h^p} = \| \sup_{0<t<1} |\psi_t \ast f|\|_{L^p} < \infty \}.
$$

We note that the definition of the local Hardy spaces is independent of the choice of $\psi \in \mathcal{S}$.

To define Besov space and Triebel–Lizorkin space, we introduce the dyadic decomposition of $\mathbb{R}^n$. Let $\phi$ be a smooth bump function supported in the ball $\{ \xi : |\xi| < 3/2 \}$ and be equal to 1 on the ball $\{ \xi : |\xi| \leq 4/3 \}$. Denote

$$
\phi(\xi) = \phi(\xi) - \phi(2\xi).
$$

(2.1)
and a function sequence

\[
\begin{align*}
\phi_j(\xi) &= \phi(2^{-j}\xi), \quad j \in \mathbb{Z}^+,
\phi_0(\xi) &= 1 - \sum_{j \in \mathbb{Z}^+} \phi_j(\xi) = \varphi(\xi).
\end{align*}
\]

(2.2)

For integers \(j \geq 0\), we define the Littlewood–Paley operators

\[
\Delta_j = \mathcal{F}^{-1} \phi_j \mathcal{F}.
\]

(2.3)

Let \(0 < p, q \leq \infty, s \in \mathbb{R}\). For a tempered distribution \(f\), we set the norm

\[
\|f\|_{B_{p,q}^s} = \left( \sum_{j=0}^{\infty} 2^{jq\|\Delta_j f\|_{L^p}^q} \right)^{1/q},
\]

(2.4)

with the usual modifications when \(q = \infty\). The (inhomogeneous) Besov space \(B_{p,q}^s\) is the space of all tempered distributions \(f\) for which the quantity \(\|f\|_{B_{p,q}^s}\) is finite. Let \(0 < p < \infty, 0 < q \leq \infty, s \in \mathbb{R}\). For a tempered distribution \(f\), we set the norm

\[
\|f\|_{F_{p,q}^s} = \left\| \left( \sum_{j=0}^{\infty} 2^{jq|\Delta_j f|^q} \right)^{1/q} \right\|_{L^p},
\]

(2.5)

with the usual modifications when \(q = \infty\). The Triebel–Lizorkin space \(F_{p,q}^s\) is the space of all tempered distributions \(f\) for which the quantity \(\|f\|_{F_{p,q}^s}\) is finite. We recall that the local Hardy space \(h^p\) is equivalent with the inhomogeneous Triebel–Lizorkin space \(F_{0,2}^p\) for \(p \in (0, \infty)\).

Next, we introduce the modulation space. Denote by \(Q_k\) the unit cube with the center at \(k\). Then, the family \(\{Q_k\}_{k \in \mathbb{Z}^n}\) constitutes a decomposition of \(\mathbb{R}^n\). Let \(\eta : \mathbb{R}^n \rightarrow [0, 1]\) be a smooth function satisfying that \(\eta(\xi) = 1\) for \(|\xi|_\infty \leq 1/2\) and \(\eta(\xi) = 0\) for \(|\xi|_\infty \geq 3/4\). Let

\[
\eta_k(\xi) = \eta(\xi - k), \quad k \in \mathbb{Z}^n
\]

be a translation of \(\eta\). Since \(\eta_k(\xi) = 1\) in \(Q_k\), we have that \(\sum_{k \in \mathbb{Z}^n} \eta_k(\xi) \geq 1\) for all \(\xi \in \mathbb{R}^n\). Denote

\[
\sigma_k(\xi) = \eta_k(\xi) \left( \sum_{l \in \mathbb{Z}^n} \eta_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.
\]

It is easy to know that \(\{\sigma_k\}_{k \in \mathbb{Z}^n}\) constitutes a smooth partition of the unity, and \(\sigma_k(\xi) = \sigma(\xi - k)\). The frequency-uniform decomposition operators can be defined by

\[
\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}
\]

for \(k \in \mathbb{Z}^n\). Now, we give the (discrete) definition of modulation space \(M_{p,q}^s\).

**Definition 2.1.** Let \(s \in \mathbb{R}, 0 < p, q \leq \infty\). The modulation space \(M_{p,q}^s\) consists of all \(f \in \mathcal{S}'\) such that the (quasi-)norm

\[
\|f\|_{M_{p,q}^s} := \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_{L^p}^q \right)^{1/q}
\]

is finite. Note that this definition is independent of the choice of \(\{\sigma_k\}_{k \in \mathbb{Z}^n}\). We also recall a basic fact that \(C^\infty_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset M_{p,q}^s\) for any \(s \in \mathbb{R}, 0 < p, q \leq \infty\).

We say \(X \in L^p_0(\mathbb{R}^n)\), if for every \(\varphi \in C^\infty_c(\mathbb{R}^n)\), we have

\[
\|\varphi * f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_X
\]
for all \( f \in X \), where the constant \( C \) is only dependent on \( \varphi \). We would like to point that \( L^p_0 \) above is not a space, and we only adopt the notation \( X \rightarrow L^p_0(\mathbb{R}^n) \) to denote the connections between \( X \) and \( L^p \).

Following, we collect some basic embedding results of function spaces. We shall abbreviate \( M^{p,q}_0, B^{p,q}_0, \) and \( F^{p,q}_0 \) as \( M^{p,q}, B^{p,q}, \) and \( F^{p,q} \), respectively.

**Lemma 2.2.** Let \( 0 < p \leq 1, 0 < q \leq \infty, s \in \mathbb{R} \).

1. For \( X = h^p, M^{p,q}_0, B^{p,q}_0, F^{p,q}_0 \), we have \( \|g\|_{L^p} \leq \|g\|_X \) for all measurable functions \( g \in X \).
2. For \( Y = M^{p,q}, \mathcal{F} M^{p,q}, B^{p,q}, F^{p,q} \), we have \( Y \rightarrow L^p_0(\mathbb{R}^n) \).

**Proof.** We first verify that \( \|g\|_{L^p} \leq \|g\|_{h^p} \) for measurable functions \( g \in h^p \). Take a \( C_c^\infty \) function \( \psi \) with \( \int_{\mathbb{R}^n} \psi(x)dx = 1 \). We have

\[
\lim_{t \to 0} \psi_t \ast g = g \text{ a.e. on } \mathbb{R}^n.
\]

Thus,

\[
\|g\|_{L^p} = \|\lim_{t \to 0} \psi_t \ast g\|_{L^p} \leq \|\sup_{0 < t < 1} |\psi_t \ast g|\|_{L^p} = \|g\|_{h^p},
\]

where we use the definition of \( h^p \) in the last equality.

For \( X = M^{p,q}, B^{p,q}, F^{p,q} \), the inequality \( \|g\|_{L^p} \leq \|g\|_X \) follows directly by the definition of function space and the triangle inequality, we only show the details for \( B^{p,q} \):

\[
\|f\|_{L^p} = \left( \sum_{j=0}^{\infty} \|\Delta_j f\|_{L^p}^p \right)^{1/p} \leq \left( \sum_{j=0}^{\infty} \|\Delta_j f\|_{L^p}^p \right)^{1/p} = \|f\|_{B^{p,q}} = \|f\|_{F^{p,q}}.
\]

Next, we turn to the proof of statement (2). First, we deal with the case \( Y = M^{p,q}_0 \). Using the triangle inequality, we have

\[
\|\varphi \ast f\|_{L^p} = \left\| \sum_{k \in \mathbb{Z}^n} \square_k(\varphi \ast f) \right\|_{L^p} \leq \left( \sum_{k \in \mathbb{Z}^n} \|\square_k(\varphi \ast f)\|_{L^p}^p \right)^{1/p}.
\]

Moreover, there exists a constant \( c_n \) such that \( \square_k = \sum_{|l| \leq c_n} \square_{k+l} \square_l \), then

\[
\|\square_k(\varphi \ast f)\|_{L^p} = \left\| \sum_{|l| \leq c_n} \square_{k+l} \varphi \ast f \right\|_{L^p} \leq \sum_{|l| \leq c_n} \|\square_k \varphi\|_{L^p} \|\square_l f\|_{L^p}.
\]

The above two estimates then yield that

\[
\|\varphi \ast f\|_{L^p} \leq \left( \sum_{k \in \mathbb{Z}^n} \left( \sum_{|l| \leq c_n} \|\square_{k+l} \varphi\|_{L^p} \|\square_l f\|_{L^p} \right) \right)^{1/p} \leq \sup_{k \in \mathbb{Z}^n} \langle k \rangle^d \|\square_k f\|_{L^p} \left( \sum_{k \in \mathbb{Z}^n} \left( \sum_{|l| \leq c_n} \|\square_{k+l} \varphi\|_{L^p} \right)^p \right)^{1/p} \leq \|f\|_{M^{p,q}} \|\varphi\|_{M^{p,q}} \leq \|f\|_{M^{p,q}},
\]

where we use the fact \( \varphi \in M^{p,q}_0 \) and \( M^{p,q}_0 \subset M^{p,q}_0 \).

Let \( Y = \mathcal{F} M^{p,q} \). By the conclusion for \( Y = M^{p,q}_0 \), we have \( \|\varphi \ast f\|_{L^p} \leq \|f\|_{M^{p,q}} \). Note that \( \mathcal{F} M^{p,q} \) is equal to the Wiener amalgam space \( W^{p,q} \) (see Guo et al., 21, p. 10), and recall the embedding relations (see lemma 2.5 of Guo et al.):

\[
\mathcal{F} M^{p,q} \subset \mathcal{F} M^{p,q}_0 = W^{p,q}_0 \subset M^{p,q}_0.
\]

The conclusion follows by

\[
\|\varphi \ast f\|_{L^p} \leq \|f\|_{M^{p,q}} \leq \|f\|_{\mathcal{F} M^{p,q}}.
\]
For $Y = B^{p,q}_s$, there exists a constant $\delta(p,q) > 0$ such that $B^{p,q}_s \subset M^{p,q}_{s-\delta(p,q)}$ (see theorem 1.2 of Guo and Zhao\textsuperscript{23}). This and the conclusion for $Y = M^{p,q}_{s-\delta(p,q)}$ imply that

$$\|\varphi * f\|_{L^p} \lesssim \|f\|_{M^{p,q}_{s-\delta(p,q)}} \lesssim \|f\|_{B^{p,q}_s}.$$  

For $Y = F^{p,q}_s$, take a constant $\delta > 0$, then $F^{p,q}_s \subset B^{p,q}_s$ (see Triebel,\textsuperscript{24} p. 47). This and the conclusion for $Y = B^{p,q}_s$ imply that

$$\|\varphi * f\|_{L^p} \lesssim \|f\|_{B^{p,q}_s} \lesssim \|f\|_{F^{p,q}_s}.$$  

\[\square\]

3 | MAIN THEOREMS

In this section, we give our main theorems and their proofs. Suppose $X$ is a (quasi-)Banach space with translation invariant property: $\|T_y f\|_X = \|f\|_X$, where $T_y f(x) := f(x - y)$ is the translation of $f$.

**Theorem 3.1.** Let $0 < p < 1$, $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Suppose $C_0^\infty(\mathbb{R}^n) \subset X$ and $\|g\|_{L^p} \lesssim \|g\|_X$ for all measurable function $g \in X$. We have

$$H_\Phi \text{ is bounded on } X \iff H_\Phi = 0.$$  

**Proof.** The “if” part is trivial. We only present the proof for the “only if” part.

We first point out that $H_\Phi f$ is pointwise well defined for any smooth function $f \in C_0^\infty(\mathbb{R}^n)$ supported away from the origin. In fact, in this case, we denote $E_x := \{y : f\left(\frac{x}{|y|}\right) \neq 0\}$. Observe that for any fixed $x \in \mathbb{R}^n$, $E_x$ is a bounded measurable set away from the origin. Recalling that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, then the following integral is convergent:

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y)f\left(\frac{x}{|y|}\right) \, dy = \int_{E_x} \Phi(y)f\left(\frac{x}{|y|}\right) \, dy.$$  

Moreover, $H_\Phi f$ is a measurable function on $\mathbb{R}^n$. Using the polar coordinates, we write

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y)f\left(\frac{x}{|y|}\right) \, dy = \int_0^\infty \int_{S^{n-1}} \Phi(ry') f(x/r) r^{n-1} \, d\sigma(y') \, dr =: \int_0^\infty \phi(r) f(x/r) \, dr,$$

where

$$\phi(r) := \int_{S^{n-1}} \Phi(ry') r^{n-1} \, d\sigma(y').$$

It follows by $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n)$. Since $\phi$ is a measurable function on $(0, \infty)$, almost every point in $(0, \infty)$ is a Lebesgue point. Hence, it suffices to verify that if there exists a Lebesgue point $r_0 > 0$ such that $\phi(r_0) \neq 0$, then $H_\Phi$ is unbounded.

Without loss of generality, we assume that $r_0 = 1$ is a Lebesgue point of $\phi$, satisfying $\phi(1) = 1$. The proof for other cases is similar.

Taking $\theta = \frac{1-p}{2}$, we set

$$A_j = [1 - 2^{-\theta j}, 1 + 2^{-\theta j}].$$

Take $g$ to be a nonnegative smooth function supported on $B(0, 2)$, satisfying $g = 1$ on $B(0, 1)$. Denote by $g_e(x) := g(x - 2e_0)$ the translation of $g$, where $e_0 = (1, 0, 0, \ldots, 0)$ be the unit vector on $\mathbb{R}^n$. For sufficiently large $j$, we have

$$(2^j + 2)(1 - 2^{-\theta j}) + 1 < (2^j - 2)(1 + 2^{-\theta j}) - 1.$$  

Denote

$$E_j := \bigcup_{a \in \mathbb{R} : (2^j + 2)(1 - 2^{-\theta j}) + 1 \leq a \leq (2^j - 2)(1 + 2^{-\theta j}) - 1} B(ae_0, 1/2).$$
We have $|E_j| \sim 2^{(1 - \theta_0)}$. For $x \in E_j$, we have

$$(2^j + 2)(1 - 2^{-\theta_0}) \leq |x| \leq (2^j - 2)(1 + 2^{-\theta_0}), \ |x| \sim 2^j.$$ 

This implies that

$$\frac{|x|}{2^j - 2} \leq 1 + 2^{-\theta_0}, \ \frac{|x|}{2^j + 2} \geq 1 - 2^{-\theta_0}. \tag{3.1}$$

Recall $\text{supp} \subseteq B(0, 2)$. For every $x \in E_j$, set

$$E_{j,x}^0 := \{ r : g_j \left( \frac{x}{r} \right) \neq 0 \}, \ E_{j,x}^1 := \{ r : g_j \left( \frac{x}{r} \right) = 1 \}.$$ 

By a direct calculation and (3.1), we deduce that for $x \in E_j$,

$$E_{j,x}^1 \subset E_{j,x}^0 \subset \left( \frac{|x|}{2^j + 2}, \frac{|x|}{2^j - 2} \right) \subset A_j. \tag{3.2}$$

Since $|x| \sim 2^j$ for $x \in E_j$, we have the upper estimate of $|E_{j,x}^0|$:

$$|E_{j,x}^0| \leq \left| \frac{|x|}{2^j - 2} - \frac{|x|}{2^j + 2} \right| \sim 2^{-j}.$$ 

Next, we turn to the lower estimate of $|E_{j,x}^1|$. For $x \in E_j$, write $x = ae_0 + y$ for $y \in B(0, 1/2)$, we have

$$r \in E_{j,x}^1 \iff \left| \frac{x}{r} - 2^j e_0 \right| \leq 1 \iff \left| \frac{ae_0 + y}{r} - 2^j e_0 \right| \leq 1 \iff \left| \frac{ae_0}{r} - 2^j e_0 \right| + \left| \frac{y}{r} \right| \leq 1 \iff \left| \frac{a}{r} - 2^j \right| \leq \frac{1}{4}, \text{ (if } r \geq \frac{2}{3}) \iff r \in \left[ \frac{a}{2^j + 1/4}, \frac{a}{2^j - 1/4} \right].$$

Observe that

$$\lim_{j \to \infty} \frac{a}{2^j + 1/4} = \lim_{j \to \infty} \frac{a}{2^j - 1/4} = 1$$

for $a \in [(2^j + 2)(1 - 2^{-\theta_0}) + 1, (2^j - 2)(1 + 2^{-\theta_0}) - 1]$. For sufficiently large $j$, we deduce that $\left[ \frac{a}{2^j + 1/4}, \frac{a}{2^j - 1/4} \right] \subset [2/3, \infty)$. Hence,

$$\left[ \frac{a}{2^j + 1/4}, \frac{a}{2^j - 1/4} \right] \subset E_{j,x}^1.$$ 

The lower estimate of $|E_{j,x}^1|$ follows by

$$|E_{j,x}^1| \geq \left| \frac{a}{2^j - 1/4} - \frac{a}{2^j + 1/4} \right| \sim 2^{-j}.$$ 

The combination of lower estimate of $|E_{j,x}^1|$ and upper estimate of $|E_{j,x}^0|$ yields that

$$2^{-j} \leq |E_{j,x}^1| \leq |E_{j,x}^0| \leq 2^{-j}, \text{ or equivalently, } |E_{j,x}^1| \sim |E_{j,x}^0| \sim 2^{-j}.$$
Next, we divide the Hausdorff operator into the main term and the error term by

\[ H_\Phi g_j(x) = \int_0^\infty \phi(r)g_j(x/r)dr = \int_0^\infty \phi(1)g_j(x/r)dr + \int_0^\infty (\phi(r) - \phi(1))g_j(x/r)dr \]

\[ =: H^M_\Phi g_j(x) + H^E_\Phi g_j(x). \]

Let us first turn to the estimate of main term. For \( x \in E_j \), we have

\[ H^M_\Phi g_j(x) = \int_0^\infty g_j(x/r)dr \geq \int_{E_{j,x}} g_j(x/r)dr = |E_{j,x}| \sim 2^{-j}. \]

Recalling \(|E_j| \sim 2^{(1-\theta)j}\) and \( \theta = \frac{1-p}{2} \), we have following estimate of the main term:

\[ \|H^M_\Phi g_j\|_{L^p(E_j)} \geq 2^{-jp}|E_j| \sim 2^{-j}2^{(1-\theta)j} = 2^{\frac{(1-p)j}{2}}. \]

(3.3)

On the other hand,

\[ \|H^E_\Phi g_j\|_{L^p(E_j)} \leq \|H^E_\Phi g_j\|_{L^p(E_j)} |E_j|^{-1} \]

\[ \leq \left( \int_{E_{j,x}} \phi(r) - \phi(1)|g_j(x/r)drdx \right)^p |E_j|^{1-p} \]

(3.4)

\[ = \left( \int_{E_{j,x}} \phi(r) - \phi(1)|g_j(x/r)drdx \right)^p |E_j|^{-p}. \]

Note that

\[ \{ (x, r) : x \in E_j, r \in E_{j,x}^0 \} \subset \{ (x, r) : r \in A_j, x \in \mathbb{R}^n \}. \]

We deduce that

\[ \int_{E_{j,x}} \phi(r) - \phi(1)|g_j(x/r)drdx \leq \int_{A_j} \phi(r) - \phi(1)|g_j(x/r)dxdr \]

\[ = \|g\|_{L^1} \int_{A_j} \phi(r) - \phi(1)|^rdr \leq \int_{A_j} |\phi(r) - \phi(1)|dr = \epsilon_j |A_j| \leq \epsilon_j^2 2^{-\theta j}, \]

where \( \epsilon_j \to 0^+ \) as \( j \to \infty \). Combining this with (3.4), we have

\[ \|H^E_\Phi g_j\|_{L^p(E_j)} \leq \epsilon_j^p 2^{-\theta p j} |E_j|^{1-p} \leq \epsilon_j^p 2^{-\theta p j} 2^{(1-\theta)(1-p)j} = \epsilon_j^p 2^{\frac{(1-p)j}{2}}. \]

(3.5)

By (3.3) and (3.5), there exist two constants \( C_1 \) and \( C_2 \) such that

\[ \|H^M_\Phi g_j\|_{L^p(E_j)} \geq C_1 2^{\frac{(1-p)j}{2}}, \quad \|H^E_\Phi g_j\|_{L^p(E_j)} \leq C_2 \epsilon_j^p 2^{\frac{(1-p)j}{2}}. \]

For sufficiently large \( j \) such that \( C_2 \epsilon_j^p \leq \frac{1}{C_1} / 2 \), we have

\[ \|H_\Phi g_j\|_{L^p} \geq \|H^M_\Phi g_j\|_{L^p(E_j)} \]

\[ \geq \|H^M_\Phi g_j\|_{L^p(E_j)} - \|H^E_\Phi g_j\|_{L^p(E_j)} \]

\[ \geq C_1 2^{\frac{(1-p)j}{2}} - C_2 \epsilon_j^p 2^{\frac{(1-p)j}{2}} \geq (\frac{1}{C_1} / 2)^{\frac{(1-p)j}{2}}. \]
However, the boundedness of $H_\Phi$ yields that

$$\|H_\Phi g_j\|_{L^p} \leq \|H_\Phi g\|_{L^p} \leq \|g\|_{L^p} = \|g\|_{L^p},$$

which leads to a contradiction.

Recall that all the spaces $L^p, h^p, M^p, B^p, F^p$ are translation invariant. The following corollary is a direct conclusion of Lemma 2.2 and Theorem 3.1.

**Corollary 3.2.** Let $0 < p < 1, \Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. We have

$$H_\Phi \text{ is bounded on } X \iff H_\Phi = 0,$$

where $X = L^p(\mathbb{R}^n), h^p(\mathbb{R}^n), M^p(\mathbb{R}^n), B^p(\mathbb{R}^n), F^p(\mathbb{R}^n)$.

For more general frequency decomposition spaces such as $X = B^p, F^p, M^p$, the embedding condition $\|g\|_{L^p} \leq \|g\|_X$ is no longer valid. We establish the following theorem with the help of the modified embedding (see Lemma 2.2).

**Theorem 3.3.** Let $0 < p < 1, \Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Suppose $\Psi_\Phi(\mathbb{R}^n) \subset X$ and $X \hookrightarrow L^p(\mathbb{R}^n)$. We have

$$H_\Phi \text{ is bounded on } X \iff H_\Phi = 0.$$

**Proof.** The “if” part is trivial. We focus on the proof for the “only if” part. As in the proof of Theorem 3.1, we write $H_\Phi f(x) = \int_0^\infty \phi(r) f(x/r) dr$, and assume $r_0 = 1$ is the Lebesgue point of $\Phi$ with $\Phi(1) = 1$. Let $g_j, a_j, E_j, F_j$, $\forall x \in \mathbb{R}^n$, $H_\Phi g_j, H_\Phi^M g_j, H_\Phi^L g_j, \Phi$ be as in the proof of Theorem 3.1. Take a $C^\infty(\mathbb{R}^n)$ nonnegative function $\phi$ satisfying $\phi(x) = 1$ on $B(0, 1)$ and supp$\phi \subset B(0, 2)$. By the assumption, we have

$$\|\phi * H_\Phi g_j\|_{L^p(\mathbb{R}^n)} \leq \|H_\Phi g_j\|_{L^p} \leq \|g_j\|_{L^p} = \|g_j\|_{L^p}.$$

Set

$$\Xi_j = \{x : B(x, 2) \subset E_j\}.$$

We have

$$|\Xi_j| \sim |E_j| \sim 2^{(1-\theta)j} \quad (j \to \infty).$$

Recall that $H_\Phi^M g_j(x) \geq 0$ for $x \in \mathbb{R}^n$ and $H_\Phi^M g_j(x) \geq 2^{-j}$ for $x \in E_j$. For $x \in \Xi_j$, we have

$$\phi * H_\Phi^M g_j(x) = \int_{\mathbb{R}^n} \phi(x-z) H_\Phi^M g_j(z) dz \geq \int_{B(x, 1)} H_\Phi^M g_j(z) dz \geq \int_{B(x, 1)} 2^{-j} dz \sim 2^{-j}.$$

From this and the fact $|\Xi_j| \sim 2^{(1-\theta)j}$, we deduce that

$$\|\phi * H_\Phi^M g_j\|_{L^p(\Xi_j)} \geq 2^{-j} |\Xi_j| \geq 2^{-j} 2^{(1-\theta)j} = 2^{(1-\theta)j}.$$
On the other hand, observing that $E_{j,z}^0 \subset A_j$ for $z \in B(x, 2)$ with $x \in \Xi_j$, we have

$$
\| \varphi \ast H_{\Phi g_j}^{E_j} \|_{L(\Xi_j)}^p \leq \| \varphi \ast H_{\Phi g_j}^{E_j} \|_{L(\Xi_j)}^p \|_1^{1-p}
$$

$$
\leq \left( \int_{\Xi_j} \int_{B(x,2)} \varphi(x-z) \int_0^\infty |\phi(r) - \phi(1)| g_j(z/r) dr dz \right)^p |\Xi_j|^{1-p}
$$

$$
= \left( \int_{\Xi_j} \int_{B(x,2)} \varphi(x-z) \int_0^\infty |\phi(r) - \phi(1)| g_j(z/r) dr dz \right)^p |\Xi_j|^{1-p}
$$

$$
\leq \left( \int_{A_j} |\phi(r) - \phi(1)| \int_{\mathbb{R}^n} g_j(z/r) \int_{\mathbb{R}^n} \varphi(x-z) dx dz dr \right)^p |\Xi_j|^{1-p}
$$

$$
\leq \left( \int_{A_j} |\phi(r) - \phi(1)| dr \right)^p |\Xi_j|^{1-p} \sim \epsilon_j^p |A_j|^p |\Xi_j|^{1-p} \sim \epsilon_j^p \frac{(\frac{1}{2})^{\frac{1}{p}}}{2}.
$$

where $\epsilon_j \to 0^+$ as $j \to \infty$. Now, we have finished the estimates of main term $\| \varphi \ast H_{\Phi g_j}^{E_j} \|_{L(\Xi_j)}^p$ and error term $\| \varphi \ast H_{\Phi g_j}^{E_j} \|_{L(\Xi_j)}^p$, the remainder of this proof is the same as that of Theorem 3.1.

Using Theorem 3.3 and Lemma 2.2, we have following corollary.

**Corollary 3.4.** Let $0 < q \leq \infty$, $s \in \mathbb{R}$, $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. If $0 < p < 1$, we have $H_\Phi$ is bounded on $X \iff H_\Phi = 0$.

where $X = F^p_q(\mathbb{R}^n)$, $B^p_q(\mathbb{R}^n)$ or $M^p_q(\mathbb{R}^n)$.

In particular, due to the time-frequency symmetry of modulation space, we have following corollary.

**Corollary 3.5.** Let $0 < p, q \leq \infty$. Let $\Phi$ be a measurable function satisfying

$$
\int_{\mathbb{R}(0,1)} |y|^n \Phi(y) dy < \infty, \quad \int_{\mathbb{R}(0,1)^f} \Phi(y) dy < \infty.
$$

If $0 < p < 1$ or $0 < q < 1$, we have $H_\Phi$ is bounded on $M^p_q(\mathbb{R}^n) \iff H_\Phi = 0$.

**Proof.** The “if” part is trivial. We focus on the proof for the “only if” part.

If $0 < p < 1$, we have $M^p_q(\mathbb{R}^n) \hookrightarrow L^p_0(\mathbb{R}^n)$, then the conclusion follows by Theorem 3.3.

If $p \geq 1, 0 < q < 1$, we will use the Fourier transform to exchange the time and frequency space. It follows by remark 1.4 in Zhao et al.\(^{12}\) that

$$
\widehat{H_\Phi f} = \widehat{H_\Phi} \hat{f},
$$

where

$$
\widehat{H_\Phi} f(x) = \int_{\mathbb{R}^n} \Phi(y) |y|^n f(|y|x) dy.
$$

Thus,

$$
\|H_\Phi f\|_{M^p_q} = \|\widehat{H_\Phi} \hat{f}\|_{M^p_q} = \|\widehat{H_\Phi} \hat{f}\|_{M^p_q}.
$$

If $H_\Phi$ is bounded on $M_{p,q}$, we have

$$
\|\widehat{H_\Phi} \hat{f}\|_{M^p_q} = \|H_\Phi \hat{f}\|_{M^p_q} \leq \|\hat{f}\|_{M^p_q} = \|f\|_{M^p_q}.
$$
Write
\[ \overline{H}_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y) |y|^n f(|y|x) dy \]
\[ = \int_0^\infty \int_{S^{n-1}} \Phi(ry') r^n f(rx') d\sigma(y') r^{n-1} dr \]
\[ = \int_0^\infty \int_{S^{n-1}} \Phi(\frac{x'}{r}) r^{-1-2n} f(\frac{x}{r}) d\sigma(y') dr =: \int_0^\infty \tilde{\Phi}(r) f(x/r) dr, \]
where
\[ \tilde{\Phi}(r) = \int_{S^{n-1}} \Phi(\frac{y'}{r}) r^{-1-2n} d\sigma(y'). \]

Observe \( \tilde{\Phi} \in L^1_{loc}(\mathbb{R}^+) \) and recall \( \mathcal{F} M^{p,q}(\mathbb{R}^n) \hookrightarrow L^q_0(\mathbb{R}^n) \) with \( q \in (0, 1) \). By the same argument as in the proofs of Theorems 3.1 and 3.3, we conclude that \( \tilde{\Phi} = 0 \) and complete this proof. \( \square \)

**Final comments:**

By Corollary 3.4 and the fact \( h^p(\mathbb{R}^n) = F^p_0(\mathbb{R}^n) \), we have the unboundedness characterization of \( H_\Phi \) on \( h^p(\mathbb{R}^n) \):

\[ H_\Phi \text{ is bounded on } h^p(\mathbb{R}^n) \iff H_\Phi = 0. \]

By a boundedness result on \( H^p(\mathbb{R}) \) (see theorem C in Liflyand and Miyachi\(^\text{17}\)), we find that if \( \Phi \) is a smooth function with compact support on \( \mathbb{R} \setminus \{0\} \), then \( H_\Phi \) is bounded on \( H^p(\mathbb{R}) \). However, our unboundedness characterization tells us that there is no chance for a nonzero Hausdorff operator \( H_\Phi \) to be bounded on \( h^p(\mathbb{R}) \). From this, one can find that the boundedness of \( H_\Phi \) on \( h^p(\mathbb{R}) \) is even worse than that on \( H^p(\mathbb{R}) \).

In the multidimensional case, by example 5.2 in Liflyand and Miyachi\(^\text{17}\) if \( \Phi \) is a nonnegative smooth function with compact support on \( (0, \infty) \), the following Hausdorff operator is bounded on \( H^p(\mathbb{R}^n) \):

\[ Hf(x) = \int_{(0,\infty) \times SO(n,\mathbb{R})} \Phi(r) f(xP/r) dr d\mu(P). \]

However, we find that the above operator is unbounded on \( h^p(\mathbb{R}^n) \). Let us give a short proof as follows.

Without loss of generality, we assume that \( \Phi(r) \geq 1 \) for \( r \in [1-\delta, 1+\delta] \), where \( \delta \) is a small positive constant. Take \( g \) to be a nonnegative smooth function supported on \( B(0, 2) \), satisfying \( g = 1 \) on \( B(0, 1) \). Denote by \( g_j(x) = g(x - 2^j e_0) \) the translation of \( g \), where \( e_0 = (1, 0, 0, \ldots, 0) \). Note that \( g_j \in h^p(\mathbb{R}^n) \) with \( \|g_j\|_{h^p(\mathbb{R}^n)} = \|g\|_{h^p(\mathbb{R}^n)} \). Denote \( E^1_{j,x} := \{(r, P) : g_j(|x|e_0 P/r) = 1\} \). For \( x \in E_j = \{x \in \mathbb{R}^n : (2^j + 1/2)(1-\delta) \leq |x| \leq (2^j - 1/2)(1+\delta)\} \), one can verify that

\[ \left\{ (r, P) : \frac{|x|}{2^j + 1/2} \leq r \leq \frac{|x|}{2^j - 1/2}, |e_0(P - I)| \leq \frac{1}{2^{j+2}} \right\} \subset E^1_{j,x}. \]

Hence, we have
\[ Hg_j(x) = \int_{(0,\infty) \times SO(n,\mathbb{R})} \Phi(r) g_j(xP/r) dr d\mu(P) \]
\[ = \int_{(0,\infty) \times SO(n,\mathbb{R})} \Phi(r) g_j(|x|e_0 P/r) dr d\mu(P) \]
\[ \geq \int_{\|x\|/2^{j+2}}^{\|x\|/2^{j-2}} \int_{|e_0(P-I)| \leq \frac{1}{2^{j+1}}} \Phi(r) g_j((|x|/r)e_0 P) d\mu(P) dr \]
\[ \geq \int_{\|x\|/2^{j+2}}^{\|x\|/2^{j-2}} \int_{|e_0(P-I)| \leq \frac{1}{2^{j+1}}} d\mu(P) dr \geq 2^{-jn}. \]

Then, for \( j \to \infty \),
\[ \|Hg_j\|^p_{L^p(E_j)} \geq 2^{-jn} |E_j| \sim 2^{-jn} 2^j = 2^{jn(1-p)} \to \infty, \]
which implies the unboundedness of \( H \). From this, in some sense, we find that in the multidimensional case, the boundedness of \( H_{\Phi,A} \) on \( h^p(\mathbb{R}^n) \) is worse than that on \( H^p(\mathbb{R}^n) \).

By the above arguments, one may ask a natural question: for \( n \geq 2 \), is there a nontrivial Hausdorff operator \( H_{\Phi,A} \) bounded in \( h^p(\mathbb{R}^n) \)?
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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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