Holographic relationships in Lovelock type brane gravity

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Abstract. We show that the Lovelock type brane gravity is naturally holographic by providing a correspondence between bulk and surface terms that appear in the Lovelock-type brane gravity action functional. We prove the existence of relationships between the $L_{\text{bulk}}$ and $L_{\text{surf}}$ allowing $L_{\text{surf}}$ to be determined completely by $L_{\text{bulk}}$. In the same spirit, we provide relationships among the various conserved tensors that this theory possesses. We further comment briefly on the correspondence between geometric degrees of freedom in both bulk and surface space.

1. Introduction

Holography is a concept that has received widespread attention in a large number of physical theories. Its application covers a wide extent of scenarios ranging from optics, condensed matter, string theory and gravitation, mainly. In theoretical physics the most common conception we have is that of a correspondence between the dynamics of a system occurring in a $(D-1)$-dimensional space (termed surface, $\partial \mathcal{M}$) and the dynamics in a $D$-dimensional space (termed bulk, $\mathcal{M}$) [1, 2, 3]. In particular, for string theory the concept of holography is used to relate two different theories: one of them describing the surface while the other the bulk. From another point of view, the holography term can be used with a slightly different nuance. It is known in gravitation that the Einstein-Hilbert action, and its geometric generalization viz. the Lanczos-Lovelock gravity action [4], can be splitted into bulk and surface terms closely related each other. This fact leads to a correspondence giving account of the same theory both in the surface and in the bulk which serves to relate the degrees of freedom of the same theory either on the bulk or in the surface. Following the observation of Padmanabhan [4, 6, 7], this latter equivalence can be considered as a holography of the action functional, avoiding confusion with the notion coming from string theory. Certainly, this holographic property of gravitation and its deep connection with the thermodynamics of black holes has been well exploited [8, 9, 10].

In this regard, the Lovelock type brane gravity is a second-order theory that involves a set of geometric invariants defined on a $(p+1)$-dimensional hypersurface, the trajectory
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swept out by a $p$-dimensional brane, embedded within a flat manifold with an extra dimension with the particularity that the associated equations of motion remain of second-order in the derivatives of the fields [11, 12]. This particular aspect makes a given theory free from many of the pathologies that plague higher-order derivative theories [13]. This fact is important because it assures no propagation of extra degrees of freedom. The underlying common structure which gives rise to this theory is that these invariants are polynomials of degree $n \leq p + 1$, but now in the extrinsic curvature of the hypersurface. The interest in this effective field theory is not only for its rich geometric structure but also for its possible implications in the description of cosmological acceleration behaviors in the brane-like universe scenarios [14, 15, 16].

These Lovelock brane invariants are similar in form either to the original Lanczos-Lovelock invariants in pure gravity or to their necessary counterterms in order to have a well posed variational problem [17, 18, 19]. This analogy needs a word of caution. While for even values of $n$ the Lovelock brane invariants look like the Gauss-Bonnet invariants, for odd values of $n$ the corresponding Lovelock brane invariants acquire the form of the Gibbons-Hawking-York-Myers boundary terms [17, 20] which are seen as counterterms if we have the presence of bulk Lovelock invariants. Unlike what happens with the counter-terms either in the Einstein theory or more generally in the Lanczos-Lovelock gravity, for the odd terms in this type of gravity we have the presence of time derivatives of the field variables, which is a sign that we have dynamics on this type of hypersurfaces. This fact leads to a handful of attractive features.

In this spirit, the holographic scheme of our interest considers that from a second-order Lagrangian density $L$ we can establish a splitting of it into two parts; the first being a first-order derivative Lagrangian density, $L_{\text{bulk}}$, and the second a divergence absorbing the second-order derivative content, $L_{\text{surf}}$, with the attribute of being able to establish a specific relationship between these such that it allows to determine $L_{\text{surf}}$ completely in terms of $L_{\text{bulk}}$. This type of correspondences were termed also as holographic relationships [5]. With this understanding, the bulk and the surface terms in Lovelock type brane gravity must encode the same amount of dynamical content. Within the quantum framework of brane-like universes [21, 22, 23, 24, 25, 26] and the Dirac’s extensible model for the electron [27, 28, 29] the usage of this holographic concept has been used in order to extract information from one term of the splitted Lagrangian, based in the other.

This paper is devoted to show that the Lovelock type brane gravity also possesses a holographic nature by providing the existence of holographic relationships. To some extent this fact is reasonable since this type of gravity belongs to the set of theories known as affine in acceleration, i.e., linear in second order derivatives of the fields [30], allowing the identification of a divergence term. In our approach, we have a larger number of holographic relationships since we have at most the double of the geometric invariants in comparison with the original Lanczos-Lovelock gravity. Additionally, we highlight the role played by the different conserved tensors that this theory possesses in our development as well as the relationships among them.
The remainder of the paper is organized as follows. In section 2 we provide an overview of the Lovelock theory for extended objects evolving in a flat Minkowski spacetime. We derive holographic relationships for the Lovelock type brane gravity in section 3. In section 4 we provide some relationships among the different conserved tensors of this theory. In section 5 we provide a discussion about the splitting into two parts of the first Lovelock brane invariant, named $K$-brane action. Conclusions and comments are presented in section 6. Throughout our analysis the convention for the Riemann tensor we follow is $R_{abcd} = -2\partial_{[a} \Gamma_{bc]}^d + 2\Gamma_{[a}^e \Gamma_{b]c}^d$ where $T^{[ab]}$ indicates anti-symmetrization under the convention $T^{[ab]} = (T^{ab} - T^{ba})/2$. Somewhat larger computations were put in an Appendix.

2. Lovelock type brane theory

The system of interest is a $p$-dimensional spacelike extended object, $\Sigma$, probing a $N = (p + 2)$-dimensional Minkowski spacetime $\mathcal{M}$ with metric $\eta_{\mu\nu}$ ($\mu, \nu = 0, 1, \ldots, p + 1$). To specify the $\Sigma$ trajectory, known as worldvolume and denoted by $m$, we set $y^\mu = X^\mu(x^a)$ where $y^\mu$ are the local coordinates of $\mathcal{M}$ and $x^a$ are the local coordinates of $m$, being $X^\mu$ the embedding functions ($a, b = 0, 1, \ldots, p$). Within the geometric framework of extended objects the essential derivatives of $X^\mu$ enter the game through the induced metric $g_{ab} = \eta_{\mu\nu} e^\mu_a e^{\nu}_b$ and the extrinsic curvature $K_{ab} = -\eta_{\mu\nu} n^\mu \nabla_a e^\nu_b = K_{ba}$ where $e^\mu_a = \partial_a X^\mu$ are the tangent vectors to $m$, $n^\mu$ is the spacelike unit normal vector to $m$, and $\nabla_a$ is the worldvolume covariant derivative, $\nabla_a g_{bc} = 0$.

We are interested in the Lovelock type brane gravity action functional \cite{11}

$$S[X] = \int_m d^{p+1}x \sqrt{-g} \sum_{n=0}^{p+1} \alpha_n L_n(g_{ab}, K_{ab}),$$

where

$$L_n(g_{ab}, K_{ab}) = \delta_{b_1b_2b_3\ldots b_n}^{a_1a_2a_3\ldots a_n} K^{b_1}_{a_1} K^{b_2}_{a_2} K^{b_3}_{a_3} \cdots K^{b_n}_{a_n},$$

which leads us to second-order field equations for the field variables $X^\mu$. Here, $\alpha_n$ are constants with appropriate dimensions, $g := \det(g_{ab})$ and $\delta_{b_1b_2b_3\ldots b_n}^{a_1a_2a_3\ldots a_n}$ denotes the alternating tensor known as the generalized Kronecker delta (gKd). Written out in full, the gKd is given by the determinant made of Kronecker delta functions

$$\delta_{b_1b_2\ldots b_{n-1}b_n}^{a_1a_2\ldots a_{n-1}a_n} = \begin{vmatrix} \delta_{b_1}^{a_1} & \delta_{b_2}^{a_1} & \cdots & \delta_{b_{n-1}}^{a_1} & \delta_{b_n}^{a_1} \\ \delta_{b_1}^{a_2} & \delta_{b_2}^{a_2} & \cdots & \delta_{b_{n-1}}^{a_2} & \delta_{b_n}^{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{b_1}^{a_{n-1}} & \delta_{b_2}^{a_{n-1}} & \cdots & \delta_{b_{n-1}}^{a_{n-1}} & \delta_{b_n}^{a_{n-1}} \\ \delta_{b_1}^{a_n} & \delta_{b_2}^{a_n} & \cdots & \delta_{b_{n-1}}^{a_n} & \delta_{b_n}^{a_n} \end{vmatrix}.$$

Regarding the definition \cite{2}, we set $L_0 = 1$. We must to stress that the action functional \cite{1} is invariant under reparametrizations of the worldvolume. In view of the fact that the Lagrangian \cite{2} is a polynomial of degree $n \leq p+1$ in the extrinsic curvature, the action \cite{1} is a second-order derivative theory. The geometrical invariants \cite{2} are
known as Lovelock brane invariants (LBI). By construction, these terms vanish for 

\[ n > p + 1 \]

whereas the term with \( n = p + 1 \) corresponds to a topological invariant not contributing to the field equations. Since the independent variables to describe the worldvolume are the embedding functions instead of the metric, we then have one greater number of Lovelock type brane terms contrary to the pure gravity case. For even \( n \) we recognize the form of the Gauss-Bonnet (GB) terms but expressed now in terms of the worldvolume geometry; for \( n = 0 \) we have the DNG Lagrangian, for \( n = 2 \) we have the Regge-Teitelboim (RT) model \([31, 32, 33, 34, 21, 23, 24, 25]\), for \( n = 4 \) we have the standard GB Lagrangian which for \( p > 3 \) produces non-vanishing equations of motion with ghost-free contribution. On the other side, for odd \( n \) the corresponding LBI are seen as the Gibbons-Hawking-York-Myers boundary terms which may exist if we have the presence of bulk Lovelock invariants (see \([4, 11]\) for further details). In short, for a \( p \)-brane there are at most \( p + 1 \) possible terms leading to second-order equations of motion \([11, 12]\).

Repeated application of the Gauss-Codazzi integrability condition for extended objects, \( R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc} \), where \( R_{abcd} \) denotes the worldvolume Riemann tensor, gives rise to express the LBI for the even and odd cases in terms of \( R_{\alpha \beta \gamma \delta} \). Indeed, for \( n = 0, 1, 2, 3, \ldots \) we have

\[
L_{(2n)} = \frac{1}{2n} \partial^{a_1 a_2 \cdots a_{2n-1} a_{2n}} R_{b_1 b_2 \cdots b_{2n-1} b_{2n}} R_{a_1 a_2} \cdots R_{a_{2n-1} a_{2n}},
\]

\[
L_{(2n+1)} = \frac{1}{2n} \partial^{a_1 a_2 \cdots a_{2n+1}} R_{b_1 b_2 a_1 a_2} \cdots R_{a_{2n-1} a_{2n} a_{2n+1}},
\]

These relations will prove useful to carry out the computations leading to holographic relationships in this type of gravity. Indeed, when the LBI are expressed in this fashion, suggest to consider alternative sets of independent variables instead of \( X^\mu \).

While varying the action \([11]\) in order to obtain the field equations we encounter the worldvolume tensor \([11]\)

\[
J_{(n)}^{a} := \partial_{b_1 b_2 b_3 \cdots b_n} K_{b_1 a_1} K_{b_2 a_2} K_{b_3 a_3} \cdots K_{b_n a_n},
\]

for the \( n \)th order Lovelock type brane invariant. These are symmetric because inherits the symmetries from the extrinsic curvature, and conserved because \( \nabla_a J_{(n)}^{ab} = 0 \), courtesy of the Codazzi-Mainardi integrability condition for extended objects, \( \nabla_a K_{bc} = \nabla_b K_{ac} \). These were named Lovelock brane tensors. For a \((p + 1)\)-dimensional worldvolume there are at most an equal number of conserved tensors \( J_{(n)}^{ab} \). In fact, we have that \( J_{(n)}^{ab} = (1/n) (\partial L_n / \partial K_{ab}) \), for \( n \neq 0 \).

Similarly, we have a close relationship between the \( J_{(n)}^{ab} \) and the worldvolume Riemann tensor. A repeated application of the Gauss-Codazzi integrability condition in the definition \([6]\) yields the handy identities

\[
J_{(2n)}^{a} = \frac{1}{2n} \partial^{a_1 a_2 \cdots a_{2n-1} a_{2n}} R_{b_1 b_2 \cdots b_{2n-1} b_{2n}} R_{a_1 a_2} \cdots R_{a_{2n-1} a_{2n}},
\]

\[
J_{(2n+1)}^{a} = \frac{1}{2n} \partial^{a_1 a_2 \cdots a_{2n} a_{2n+1}} R_{b_1 b_2 a_1 a_2} \cdots R_{a_{2n-1} a_{2n} a_{2n+1}},
\]

for \( n = 0, 1, 2, 3, \ldots \).
We argue that, except for the $K$ term, all the LBI satisfy the structure
\[ \sqrt{-g}L = \sqrt{-g} Q_a^{bcd} R^a_{bcd}, \] (9)
where $Q_a^{bcd}$ is a tensor that has all the symmetries of the worldvolume Riemann tensor, made from the induced metric, the Riemann tensor and the extrinsic curvature itself, besides a zero divergence, $\nabla_c Q_a^{bcd} = 0$. This will be proved shortly.

It need be pointed out that due to from the fact that the worldvolume is an oriented timelike manifold embedded in $\mathcal{M}$, the odd Lovelock type brane invariants, unlike what happens with the counter-terms in the pure Lovelock gravity, they have the presence of time derivatives, which indicates that we have dynamical content on this type of hypersurfaces.

3. Holographic relationships

Rather than being just a notational trick the geometrical information provided by (4) and (5) becomes important because, except for the $K$ brane term, any $n$th Lovelock type brane density acquires the form
\[ L_n = \sqrt{-g} (n) Q_a^{bcd} R^a_{bcd}, \] (10)
where $(n) Q_a^{bcd}$ is a tensor with specific properties. We turn now to prove that any $n$th order Lovelock type brane model allows a decomposition in terms of both a bulk and surface components of the action functional in which both terms are directly related. To do this we will proceed in two parts.

3.1. Even case

Clearly, (4) furnishes us with the structure given in (10) for $n = 1, 2, 3, \ldots$, 
\[ L_{(2n)} = (2n) Q_{b_{2n-1}}^{a_{2n-1} a_{2n}} R_{b_{2n-1}}^{a_{2n-1} a_{2n}}, \] (11)
with 
\[ (2n) Q_{b_{2n-1}}^{a_{2n-1} a_{2n}} := \frac{1}{2^n} \delta^{a_1 a_2 \ldots a_{2n-1} a_{2n}} g^{a_{2n}} R_{b_1 b_2}^{b_{2n-3} b_{2n-2}} \ldots R_{a_{2n-3} a_{2n-2}}^{a_1 a_2} g_{b_1 b_2}^{a_{2n-1} a_{2n}}. \] (12)
With the indices conveniently placed, it is evident that this tensor inherits the symmetries from the Riemann tensor
\[ (2n) Q^{abcd} = -(2n) Q^{dcba} = -(2n) Q^{bacd} = (2n) Q^{cdab}, \] (13)
and it is constructed from $g^{ab}$ and $R^a_{bcd}$ or, from a brane point of view, from $g^{ab}$ and $K_{ab}$ when the worldvolume Riemann tensor is expressed in terms of the extrinsic curvature via the Gauss-Codazzi equations for the worldvolume. The first values of this $Q$ tensor are
\[ (2) Q^{abcd} = g^{a[c} g^{d]b} = J_{(0)}^{a[c} g^{d]b}, \] (14)
\[ (4) Q^{abcd} = R^{abcd} - 2G^{a[c} g^{d]b} - 2g^{a[c} R^{d]b}, \]
\[ = J_{(2)}^{a[c} g^{d]b} - 2g^{a[c} J_{(1)}^{d]e} R^b_e + R^{abcd}, \] (15)
with $G^{ab} = R^{ab} - (R/2)g^{ab}$ being the worldvolume Einstein tensor, and $J_{(0)}^a, J_{(1)}^a$ and $J_{(2)}^a$ are the first Lovelock brane tensors given by (6), (11).

Consequently, relationship (11) allows us to express the Lagrangian densities for the even case as

$$\mathcal{L}_{(2n)} = \sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} R_{b c d}.$$  \hspace{1cm} (16)

With the aid of the Riemann tensor, $R_{abc}^d = -2\partial_a\Gamma^d_{b c} + 2\Gamma^e_{c[a} \Gamma^d_{b c]}$, the former equation can be written in the form

$$\mathcal{L}_{(2n)} = \partial_c \left(2\sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{c}_{d e} \right) - 2 \left(\partial_c \sqrt{-g}\right) \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{c}_{d e}$$

$$- 2\sqrt{-g} \left(\partial_c \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \right) \Gamma^{c}_{d e} - 2\sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{c}_{d e} \Gamma^{d}_{e c}.$$  \hspace{1cm} (17)

Now, by virtue of the identity $\partial_a \sqrt{-g} = \sqrt{-g} \Gamma^b_{a b}$, we have

$$\mathcal{L}_{(2n)} = \partial_c \left(2\sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \right) + 2\sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{c}_{d e} \Gamma^{d}_{e c}$$

$$- 2\sqrt{-g} \left(\nabla_c \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \right) \Gamma^{c}_{d e}.$$  \hspace{1cm} (18)

It follows therefore that, as long as the condition $\nabla_c \langle 2n \rangle Q_a \Gamma^{a}_{b c d} = 0$ is satisfied, the previous equation takes the form

$$\mathcal{L}_{(2n)} = \mathcal{L}_{(2n)}^{\text{bulk}} + \mathcal{L}_{(2n)}^{\text{sur}},$$  \hspace{1cm} (19)

where

$$\mathcal{L}_{(2n)}^{\text{bulk}} = 2\sqrt{-g} \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{c}_{d e} \Gamma^{d}_{e c}.$$  \hspace{1cm} (20)

Some aspects of this framework are in order. It should be notice that for the RT model (18) is quadratic in $\Gamma^a_{b c}$, this evident by using (14). In this approach the form of $R_{abc}^d$, expressed entirely in terms of the Christoffel symbols and its derivatives without requiring $g^{ab}$ and $K_{ab}$, has been an important ingredient in arriving to (17). This fact, allows the decomposition of the Lagrangian densities in terms of a surface and bulk terms. Relation (18) can be considered as a variant of the so-called Dirac-Schrödinger Lagrangian, or $\Gamma\Gamma$ Lagrangian for short, for Einstein theory [35]. According to the symmetries (13) the divergence of the tensor $Q$ on any of the indices vanishes. This is proved in Appendix A.1.

It is necessary to stress that $Q_a \Gamma^{a}_{b c d} = Q_a \Gamma^{a}_{b c d} (g^{ab}, R_{abcd})$ so the differentiation of (18) with respect to the Christoffel symbols infers that

$$\delta^r_s \frac{\partial \mathcal{L}_{(2n)}^{\text{bulk}}}{\partial \Gamma^r_{st}} = 2\sqrt{-g} \left[ \langle 2n \rangle Q_r \Gamma^{b c s} \Gamma_{b c} + \langle 2n \rangle Q_a \Gamma^{a}_{b c d} \Gamma^{d}_{e c} \right],$$  \hspace{1cm} (20)

where $\partial \Gamma^a_{b c}/\partial \Gamma^r_{st} = \delta^a_r \delta^b_s \delta^c_t$ has been considered. Due to the symmetry properties of the tensor (12), it is necessary that the second term of this expression should vanish. Clearly, comparison of the former expression with relation (19) furnishes us with a holographic relation

$$\mathcal{L}_{(2n)}^{\text{sur}} = -\partial_a \left(\delta^c_b \frac{\partial \mathcal{L}_{(2n)}^{\text{bulk}}}{\partial \Gamma^c_{ab}}\right).$$  \hspace{1cm} (21)
We find then that (17), with the aid of the identity (21), can be expressed as
\[
\mathcal{L}_{(2n)} = \mathcal{L}_{(2n)\text{ bulk}} - \partial_a \left( \delta_b^c \frac{\partial \mathcal{L}_{(2n)\text{ bulk}}}{\partial \Gamma_{ab}^c} \right). \tag{22}
\]
This way of approaching the problem is similar in the spirit to the one carried out in [5] for the case of pure Lovelock gravity. It remains to verify that \( \nabla_c \left[ (2n)Q_{abcd} \right] = 0 \). In fact, on geometrical grounds, the the invariance under reparametrizations of the worldvolumes in this type of gravity framework requires this condition.

Alternatively, there is another form of express the holographic relation (21). By using the well known relation for the Christoffel symbols in terms of the geometry of the worldvolume, \( \Gamma_{ab}^c = g^{cd} e_{\mu a} \partial_b e^{\mu c} \), when a fixed metric exists in the bulk; this is the case, for example, when the Minkowski metric is expressed in spherical coordinates. As before, the differentiation of (18) with respect to the gradients of the tangent vectors leads to
\[
e^\mu_a \frac{\partial \mathcal{L}_{(2n)\text{ bulk}}}{\partial (\partial_a e^\mu_c)} = -2\sqrt{-g} Q_{a \text{ bcd}} \Gamma_{bd}. \tag{23}
\]
In arriving to the last equality we have used (13) and the symmetries of the Christoffel symbols. In this sense, comparison with the relation (19) furnishes us with another holographic relationship
\[
\mathcal{L}_{(2n)\text{ surf}} = -\partial_a \left( e^\mu_b \frac{\partial \mathcal{L}_{(2n)\text{ bulk}}}{\partial (\partial_a e^\mu_b)} \right). \tag{24}
\]
Such transformation leads to express (17) as
\[
\mathcal{L}_{(2n)} = \mathcal{L}_{(2n)\text{ bulk}} - \partial_a \left( e^\mu_b (2n) \mathcal{P}_{\mu}^{ab} \right). \tag{25}
\]
This particular expression serves to make contact with the Hamiltonian framework behind this effective field theory. With support with the Ostrogradsky Hamiltonian formalism [36, 38, 39], the canonical momentum conjugate to \( e^\mu_a = \partial_a X^\mu \) is
\[
(2n) \mathcal{P}_{\mu}^{ab} := \frac{\partial \mathcal{L}_{(2n)\text{ bulk}}}{\partial (\partial_a e^\mu_b)}, \tag{26}
\]
so that the relation (25) becomes
\[
\mathcal{L}_{(2n)} = \mathcal{L}_{(2n)\text{ bulk}} - \partial_a \left( e^\mu_b (2n) \mathcal{P}_{\mu}^{ab} \right). \tag{27}
\]
This is precisely the form of the so-called “d(qp)” structure introduced by Padmanabhan in [5, 6, 10], which assumes that it is valid for many gravitational theories.

Regarding the last alternative, for the RT model when the definition of the Christoffel symbols, \( \Gamma_{bc}^a = e^{\mu a} \partial_b e^\mu_c \), is introduced into (18) and after a arrangement of the various terms floating around, the corresponding bulk Lagrangian density (18) becomes
\[
\mathcal{L}_\text{RT bulk} = \sqrt{-g} M_{\mu \nu}^{abcd} \partial_a e^\mu_b \partial_b e^\nu_d, \tag{28}
\]
where
\[
M_{\mu \nu}^{abcd} := -2g^{[b \text{ c}]} e_\mu \text{ e}^\nu [d]. \tag{29}
\]
Notice that (28) is now quadratic in \( e^\mu_a \).
3.2. Odd case

In analogy with the analysis performed for the even case, from the expression (5) it follows that the structure (10) holds in the odd case. For \( n = 1, 2, 3, \ldots \)

\[
L_{(2n+1)} = \frac{1}{2n} \delta_{b_1b_2b_3b_4 \cdots b_{2n+1}}^{a_1a_2a_3a_4 \cdots a_{2n+1}} g^{bb_{2n}} g^{cb_{2n+1}} R_{a_1a_2} b_1 b_2 \cdots \\
\cdots R_{a_{2n-3}a_{2n-2}} b_{2n-3} b_{2n-2} R_{b_{2n-1}} b_{a_{2n-1}a_{2n}} K_{c a_{2n+1}},
\]

where

\[
(2n+1) Q_{b_{2n-1}}^{a_2a_{n-1}a_{2n}} = \frac{1}{2n} \delta_{b_1b_2b_3b_4 \cdots b_{2n+1}}^{a_1a_2a_3a_4 \cdots a_{2n+1}} g^{a_{2n} b_{2n}} R_{a_1 a_2} b_1 b_2 \cdots \\
\cdots R_{a_{2n-3}a_{2n-2}} b_{2n-3} b_{2n-2} K_{b_{2n+1}} {a_{2n+1}}.
\]

This tensor also inherits the symmetries from the Riemann tensor

\[
(2n+1) Q^{abcd} = - (2n+1) Q^{dbac} = - (2n+1) Q^{bacd} = (2n+1) Q^{cdab}.
\]

It turns out that (31) is constructed from \( g^{ab}, R^a_{bcd} \) and \( K_{ab} \) unlike what happens with the even case. A few values of the \( Q \) tensor are

\[
(3) Q^{abcd} = J^{[a}_{(1)} g^{d]b} - g^{a[c} K^{d]b}, \\
(5) Q^{abcd} = J^{[a[c} d]b} - 3g^{a[c} J^{d]e}_{(2)} K^{e} b + 6K^{a e} J^{[c]e} K^{d]b} - 3 R^{e e c d} K^{b} e,
\]

where \( J^{ab}_{(1)}, J^{ab}_{(2)} \) and \( J^{ab}_{(3)} \) are conserved Lovelock brane tensors. It is therefore possible to write a Lagrangian density for the odd case, analogous to (16), by writing

\[
L_{(2n+1)} = \sqrt{-g} (2n+1) Q^{abcd} R^a_{bcd}.
\]

In building the splitting of the Lagrangian density for this case into a bulk term and a surface term, is clear that the treatment performed for the even case, is applied directly. We thus find

\[
L_{(2n+1)} = L_{(2n+1) \text{ surf}} + L_{(2n+1) \text{ bulk}}
\]

where

\[
L_{(2n+1) \text{ bulk}} = 2\sqrt{-g} (2n+1) Q^{abcd} g^{e c} \Gamma^{a}_{b c} \Gamma^{c}_{e d}, \\
L_{(2n+1) \text{ surf}} = \partial_{c} \left( 2\sqrt{-g} (2n+1) Q^{abcd} g^{e c} \Gamma^{a}_{b c} \Gamma^{c}_{e d} \right),
\]

as long as the condition \( \nabla_{e} \left[ (2n+1) Q^{abcd} \right] = 0 \) is satisfied.

It must be pointed out that from the functional dependence of the variables of the tensor (31), \( Q = Q(g^{ab}, R^a_{bcd}, K_{ab}) \), it immediately follows that the structure of (20), (21) and (22), holds for the odd case. Certainly, the structures given in (21) and (22) are still maintained for this case

\[
L_{(2n+1) \text{ surf}} = -\partial_{a} \left( \delta^{c}_{b} \frac{\partial L_{(2n+1) \text{ bulk}}}{\partial \Gamma^{a}_{b c}} \right).
\]

We find then that (34), with the aid of the identity (37), can be expressed as

\[
L_{(2n+1)} = L_{(2n+1) \text{ bulk}} - \partial_{a} \left( \delta^{c}_{b} \frac{\partial L_{(2n+1) \text{ bulk}}}{\partial \Gamma^{a}_{b c}} \right),
\]
thus establishing holographic relationship for the odd case.

In analogy with the even case, the expression (23) remains valid in this case as long as we take into account that the tensor (31) does not explicitly depend on the gradient of the tangent vectors. This serves to establish another holographic relationship for the odd case. Additionally, the relationship (27) is still valid for this case so we can also ensure a “d(qp)” structure inherent in this context. The proof that the tensor (31) is conserved is provided in Appendix A.2.

4. Relationships among conserved Lovelock type brane tensors

We do not attempt to attribute any sophisticated interpretation, either physical or geometrical, to the \( Q \) tensors. In turn, we believe that tensors \( J \) have a clearer geometric interpretation since they come from the invariance under reparametrizations of the worldvolume. In fact, it turns out that these tensors are anchored in some manner.

It follows from the contraction of definition (6) with the extrinsic curvature, an expression for the \( n \)th order Lovelock type brane invariant

\[
L_{(p+1)} = J^{ab}_{(p)}K_{ab},
\]

with \( p = 0, 1, 2, \ldots \). With this identity we shall relate tensors \( Q \) in favor of tensors \( J \). Certainly, when the Gauss-Codazzi integrability condition, \( R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc} \), is inserted into the expression (11) and by invoking the symmetries of the \( Q \) tensor, we get

\[
L_{(2n)} = 2(2n)Q^{abcd}K_{ac}K_{bd}.
\]

Now, when the expression (39) enters the game, this allows to identify a relationship among the conserved tensors of the Lovelock brane gravity theory

\[
J^{ab}_{(2n-1)} = 2(2n)Q^{acbd}K_{cd}.
\]

It is important to remark that the conservation of the tensors (12) and (31) is due to the invariance under reparametrizations of the worldvolume. Knowing that \( \nabla_b J^{ab}_{(n)} = 0 \) must be fulfilled, it is necessary that the \( Q \) tensors must be conserved, as it happens.

Clearly, a similar procedure with the relations (30) and (39), helps to identify another relationship among the conserved tensors of the theory

\[
J^{ab}_{(2n)} = 2(2n+1)Q^{acbd}K_{cd}.
\]

It is worthwhile to remark that (41) and (42) are valid for \( n = 1, 2, 3, \ldots \).

In [11, 12] was proved that for a \( p \)th order Lovelock type brane model, the associated field equation is

\[
L_{(p+1)} = J^{ab}_{(p)}K_{ab} = 0.
\]

In this sense, from the relations (41) and (42), it is fairly easy to express the equations of motion in terms of the conserved \( Q \) tensors.

For an even Lovelock type brane model, \( L_{(2n)} \), the equation of motion results \( L_{(2n+1)} = J^{ab}_{(2n)}K_{ab} = 0 \). That is, \( 2(2n+1)Q^{acbd}K_{cd}K_{ab} = 0 \), where the expression (42)
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has been invoked. Thus, by using back the Gauss-Codazzi condition, we have that the associated equation of motion is

\[(2n+1)Q_{a}^{bcd}R_{b}^{a}_{bcd} = 0.\]  \hspace{1cm} (44)

Similarly, for an odd Lovelock type brane model, \(L_{(2n+2)}\), the equation of motion is

\[L_{(2n+2)} = J_{(2n+1)}^{ab}K_{ab} = 0.\]  \hspace{1cm} (41)

In such a case, by considering (41), this equation of motion also takes a compact form

\[(2n+2)Q_{a}^{bcd}R_{b}^{a}_{bcd} = 0.\]  \hspace{1cm} (45)

Doubtless, (44) and (45) make up a rendering of the expression (43) in terms of the Riemann tensor.

Furthermore, in passing we would like to mention the relation among the conserved stress tensor \(f_{a}^{\mu}(n)\) for the \(n\)th order LBI with the \(Q\) tensors in dependence of the nature of \(n\), [11]. This is quite straightforward by considering identities (11) and (12). For the even case we have

\[f_{(2n)}^{a\mu} = 2(2n+1)Q^{abcd}K_{cd}e_{b}^{\mu},\]  \hspace{1cm} (46)

whereas for the odd case, we get

\[f_{(2n-1)}^{a\mu} = 2(2n-1)Q^{abcd}K_{cd}e_{b}^{\mu}.\]  \hspace{1cm} (47)

What remains to be done is an analysis of both the physical utility as well as a deep geometric interpretation of this type of holographic relationships.

5. On the \(K\) brane action

Within the framework behind (17) and (34), the dependence on the curvature tensor has been essential in our development when we perform the covariant separation of the Lovelock type brane invariants. A glance at (5) shows that for \(n = 1\), the scheme adopted here rules out the case of the \(K\) brane model because this cannot be written in terms of the Riemann tensor. This is quite evident since this model is linear in the mean extrinsic curvature. Thence, this particular case needs a careful handling, or different, since the simple expression, \(\sqrt{-g}K\), does not help to reach the purpose. Indeed, the Lagrangian density for this model in presence of a fixed background spacetime is

\[L_{K} = \sqrt{-g}g^{ab}K_{ab},\]

\[= -\sqrt{-g}G_{\mu\nu}n^{\mu}g^{ab}\partial_{a}e_{b}^{\nu} - \sqrt{-g}G_{\mu\nu}n^{\mu}\Gamma_{\alpha\beta}^{\nu}\mathcal{H}^{\alpha\beta},\]  \hspace{1cm} (48)

where \(G_{\mu\nu}\) is the bulk metric, \(\Gamma_{\alpha\beta}^{\nu}\) are the background Christoffel symbols and \(\mathcal{H}^{\mu\nu} := g^{ab}\epsilon_{\mu}^{\alpha}\epsilon_{\nu}^{\alpha}\) is the projection operator in \(\mathcal{M}\) onto \(m\). Although the strategy of identifying a surface term seems feasible, here is not directly of much use since by considering an integration by parts, and making use of the definition of the extrinsic curvature this fails due to one faces the orthonormality of the worldvolume basis, \(e_{a} \cdot n = 0\).
An alternative way of handling this issue is by breaking the worldvolume covariance and thus perform a Hamiltonian analysis supported by a geometric ADM framework. Then, inspired by the geometric approach made in [37, 38] we can express (48) as
\[
\mathcal{L}_K = -\sqrt{-g} g^{AB} G_{\mu \nu} n^\mu \partial_A e^\nu_B - 2\sqrt{-g} \frac{N^A}{N^2} G_{\mu \nu} n^\mu \partial_A \dot{X}^\nu + \frac{\sqrt{h}}{N} G_{\mu \nu} n^\mu \dot{X}^\nu
\]
\[\quad - \sqrt{-\tilde{g}} G_{\mu \nu} n^\mu \Gamma^\nu_{\alpha \beta} \mathcal{H}^{\alpha \beta},\]
where \(\dot{X}^\mu = \partial_t X^\mu\), \(t\) is a coordinate that labels the leaves of the foliation of the worldvolume by \(\Sigma_t\) and \(h := \det(h_{AB})\) with \(h_{AB}\) being the spacelike metric defined on \(\Sigma_t\) such that \(\sqrt{-\tilde{g}} = N \sqrt{h}\). Further, \(N\) and \(N^A\) are the lapse function and the shift vector, respectively. Here is more explicit the linear dependence of this Lagrangian density on the accelerations of the brane so that this model also is affine in acceleration [30]. A possibility to continue with the computation is to take advantage from the fact that for a codimension one worldvolume, the normal vector can be written in terms of the velocity of the brane, \(\dot{X}^\mu\), the Levi-Civita symbols and the geometry of the brane \(\Sigma_t\) in order to reduce the expression. However, a further develop gives rise to a splitting rather difficult at this time so that we are unable to immediately assess the accuracy of this alternative. Therefore, this issue needs to be further analyzed before to provide a total conclusion. We will report this development elsewhere.

6. Concluding remarks

We have shown that Lovelock type brane gravity has holographic relationships similar in spirit to those of pure Lanczos-Lovelock gravity. These relationships allow us to extract information about a part of the splitted Lagrangian density, in terms of the other one. A distinctive feature of this framework is the strong dependence on the conservation condition on the \(Q\) tensors, (12) and (31). As shown, this is a consequence of the invariance under reparametrizations of the action functional in this type of gravity. In each surface governed by this action, there is only one degree of freedom, corresponding to the breathing mode of the worldvolume so its nature is only geometric [12]. In this sense, it is expected that the Lagrangians, either \(\mathcal{L}_{\text{bulk}}\) or \(\mathcal{L}_{\text{surf}}\) reflect this fact, thereby encoding the same amount of dynamical content by describing the same degree of freedom. The next step in this development is to explore the physical consequences coming from the holographic relationships. We expect to show eventually how these relationships get reflected at the boundary surface of the worldvolumes governed by this gravity. Furthermore, the holographic relationships that were found here also preserve the so-called "\(d(qp)\)" structure [5, 6], which can be thought of as a part of the transition from a coordinate representation to a momentum representation in dependence of the chosen variables. Additionally, in order to continue the quantum analysis for these brane models, the Hamiltonian formalism can benefit from this approach by considering only the Lagrangian \(\mathcal{L}_{\text{bulk}}\) since the second-order nature of the full Lagrangian density is encoded in the Lagrangian \(\mathcal{L}_{\text{surf}}\) thus avoiding a cumbersome Ostrogradski Hamiltonian approach.
In view of the antisymmetry of the gKd it follows the relation

\[ \nabla_c \left( \partial (2n) Q_{a \, b c d} \right) = 0. \]

\[ \nabla_c \left( \partial (2n) Q_{a \, b c d} \right) = \frac{1}{2n} \delta_{a b c d}^{a_1 a_2 \ldots a_{2n-3} a_{2n-2} c d} g^{bb_{2n}} \left[ \nabla_c \mathcal{R}_{a_1 a_2} b_{1 b_2} c_{a_3 a_4} b_{b_3} \ldots \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}} + \ldots + \mathcal{R}_{a_1 a_2} b_{1 b_2} c_{a_3 a_4} b_{b_3} \ldots \nabla_c \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}} \right], \]

\[ \nabla_c \left( \partial (2n) Q_{a \, b c d} \right) = \left( n - 1 \right) \frac{1}{2n} \delta_{a b c d}^{a_1 a_2 \ldots a_{2n-3} a_{2n-2} c d} g^{bb_{2n}} \mathcal{R}_{a_1 a_2} b_{1 b_2} c_{a_3 a_4} b_{b_3} \ldots \nabla_c \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}}. \]

In view of the antisymmetry of the gKd it follows the relation

\[ \nabla_c \left( \partial (2n) Q_{a \, b c d} \right) = \frac{1}{2n} \delta_{a b c d}^{a_1 a_2 \ldots a_{2n-3} a_{2n-2} c d} g^{bb_{2n}} \mathcal{R}_{a_1 a_2} b_{1 b_2} c_{a_3 a_4} b_{b_3} \ldots \nabla_c \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}}. \]

From the usual Bianchi identity, \( \nabla_{[a} \mathcal{R}_{b c] d e} = 0 \), we infer that

\[ \nabla_c \left( \partial (2n) Q_{a \, b c d} \right) = 0, \quad \text{(A.1)} \]

so that, the assertion is proved.

\[ \nabla_c \left( \partial (2n+1) Q_{a \, b c d} \right) = 0. \]

\[ \nabla_c \left( \partial (2n+1) Q_{a \, b c d} \right) = \frac{1}{2n} \delta_{a b c d}^{a_1 a_2 \ldots a_{2n-3} a_{2n-2} c d} g^{bb_{2n+1}} \left( \nabla_c \mathcal{R}_{a_1 a_2} b_{1 b_2} \ldots \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}} K_{a_{2n+1}} + \ldots + \mathcal{R}_{a_1 a_2} b_{1 b_2} \ldots \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}} \nabla_c K_{a_{2n+1}} \right), \]

\[ \nabla_c \left( \partial (2n+1) Q_{a \, b c d} \right) = \left( n - 1 \right) \frac{1}{2n} \delta_{a b c d}^{a_1 a_2 \ldots a_{2n-3} a_{2n-2} c d} g^{bb_{2n+1}} \mathcal{R}_{a_1 a_2} b_{1 b_2} \ldots \nabla_c \mathcal{R}_{a_{2n-3} a_{2n-2}} b_{2n-3 b_{2n-2}} K_{a_{2n+1}}. \]
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and the symmetry of the extrinsic curvature, it follows that this proves our assertion.

By invoking the Bianchi identity, \( \nabla_c [ (2n+1) Q_{a}^{bcd} ] = 0 \), as well the antisymmetry of the gKd and the symmetry of the extrinsic curvature, it follows that

\[
\nabla_c \left[ (2n+1) Q_{a}^{bcd} \right] = 0. \tag{A.2}
\]

This proves our assertion.

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