Looking beyond the perfect lens

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Abstract. The holy grail of imaging is the ability to see through anything. From the conservation of energy, we can easily see that to see through a lossy material would require lenses with gain. The aim of this paper therefore is to propose a simple scheme by which we can construct a general perfect lens, with gain—one that can restore both the phases and amplitudes of near and far fields.

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1. Introduction

Lenses, since antiquity, have been used for various optical applications—most notably imaging. Conventional lenses only capture the propagating far fields of light waves; hence there is a diffraction limit to the resolution of these lenses—typically the order of a wavelength. The perfect lens, first introduced by Pendry \cite{1}, can surpass this limit by capturing evanescent near fields as well, enabling sub-wavelength resolution.

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Such perfect lenses are made from a class of unconventional materials—‘negative refractive index materials’ (NRIMs) [2, 3]. The principle behind the perfect lens is based on the unusual property of the NRIMs that behave as an ‘optical antimatter’ (or optical complementary) to their conventional counterparts, the ‘positive refractive index materials’ (PRIMs). A particular NRIM lens can reverse the effects of wave propagation in PRIM space such that the net effect for light waves appears as if neither PRIMs nor NRIMs exist in the first place. Perfect lensing can therefore be understood as an optical cancellation, of space so that optical information from one plane is transported to another, forming an exact reconstructed image of the object. This optical cancellation, however, is precise and the requirement for this cancellation can be determined by the ‘perfect lens theorem’ (PLT) [4]. In practice, because of losses the perfect lens is never truly ‘perfect’ and its resolution is limited. Aside from losses within the NRIM, this optical cancellation is also constrained by the conservation of energy; to optically cancel a lossy PRIM space would require an NRIM lens with gain. Therefore, to construct an ‘ideal’ perfect lens—one that can see through anything, even through materials with losses—would require some means to add gain to the NRIM. Various schemes to add gain have been proposed in the past; however, such schemes are not easily implementable [4, 5], since even then the gain added has to be precise.

Recently [6], Pendry made use of the intimate connection between phase conjugation (time reversal) and negative refraction as a novel route to a simple homogeneous NRIM lens. The scheme is simple. By sandwiching a slab of vacuum between two phase conjugating sheets (PCSs), and applying sufficient gain in the PCSs, Pendry showed that this composite slab behaves as a homogeneous NRIM with refractive index \( n = -1 \). This is attractive, in both its simplicity and the fact that gain can be easily incorporated by pumping more power into the PCSs. In addition, it has also been shown how this scheme can be experimentally implemented [7], so this in principle should be realizable (at least for propagating waves).

Although it was hinted [6], it was not clear whether this can also be employed for a general non-homogeneous NRIM. It is therefore the intent of this paper to generalize the results of [6] by introducing a simple strategy for creating the most general inhomogeneous planar NRIM lens using phase conjugation. By adding gain, we can then show that such ‘active’ perfect lenses can surpass the limits of conventional ‘passive’ perfect lenses, allowing us to even look through lossy translucent materials.

2. The strategy

2.1. Statement of the strategy

Here, then, is a simple strategy to construct an NRIM lens that is complementary to a given general PRIM space—even a lossy PRIM:

*Given a PRIM slab, find another copy of this slab that is a mirror reflection to the first. If we sandwich this mirror copy between two PCSs of sufficient gain, then this slab together with the two PCSs becomes an NRIM lens that is complementary to the first PRIM space (see figure 1).*

The scheme above is general and applicable to any spatially inhomogeneous PRIM, even PRIMs that are not time reversal symmetric (lossy materials). The catch in this scheme of course is ‘sufficient gain’, and we shall show more quantitatively later what ‘sufficient’ means.
Figure 1. Illustration of the strategy. The green dashed lines, $\xi_1, \xi_2$, indicate the directions of the eigenvectors/optical axes of the PRIM with material parameters $\epsilon, \mu$. Quantities with overhead ‘bars’ are spatial reflections of ‘un-barred’ ones. A reflected-copy PRIM sandwiched between two PCSs of sufficient gain is equivalent to an effective NRIM lens that behaves as an optical antimatter/complementary to the PRIM space.

As a matter of notation, the material parameters of PRIM space are denoted by $\epsilon(x), \mu(x)$. The material parameters for the mirror-copy PRIM are given by $\bar{\epsilon}(\bar{x}), \bar{\mu}(\bar{x})$ where the ‘barred’ quantities are spatial reflections of the ‘un-barred’ ones (i.e. if $x = [x, y, z]$, then $\bar{x} = [-x, y, z]$ when reflected about the $x = 0$ plane; also $\bar{\epsilon}$ has eigenvectors/optical axes that are spatially reflected with respect to $\epsilon$) (see figure 1).

2.2. Relationship between eigensolutions

To prove the above strategy (section 2.1), we first need to establish a simple relationship of field solutions between the two PRIM slabs separated by a PCS.

Consider two planar slabs of materials described by a cartesian coordinate system $\{x, y, z\} = \{x, x_\perp\}$, where the $x$-axis is normal to the slabs and $x_\perp$ is the perpendicular direction. Let $x < 0$ be filled with the first medium with material parameters $\epsilon(x), \mu(x)$ (generally inhomogeneous and may be dissipative), let $x > 0$ be filled with the second medium $\bar{\epsilon}(\bar{x}), \bar{\mu}(\bar{x})$ (where the ‘barred’ quantities are the spatial reflection about $x = 0$, of the ‘un-barred’ ones) and let $x = 0$ be filled with a PCS. Now (time harmonic, spatial) Maxwell’s equations for $x < 0$ are given by

\begin{align*}
\nabla \times E &= -i\omega \mu \cdot H, \\
\nabla \times H &= i\omega \epsilon \cdot E, \quad x < 0.
\end{align*}

Suppose also that the spatial solutions (of the electric field) to Maxwell’s equations for $x < 0$ are given by $E = f_n^\pm(x, x_\perp)$,\(^2\) where the $\pm$ superscript represents the left going ($-$) and the right going ($+$) waves and the $n$ subscript indicates it is the $n$th eigensolution to the equations. Now because we are only interested in propagation in the $x$-direction, we can ignore $x_\perp$, simplifying the notation to $E = f_n^\pm(x)$.

\(^2\) Where $f_n^\pm(x, x_\perp)$ in general is a vector valued function. In general, we can always find an orthogonal complete set of basis functions, and in this paper we will assume that $\{f_n^\pm(x, x_\perp)\}$ is such a set.
In general, the presence of the PCS at $x = 0$ has two effects on the eigenwaves. Firstly, the sheet would perform phase conjugation to the spatial parts of the eigenwave. We note that this spatial phase conjugation is also equivalent to frequency inversion $^3 \omega \rightarrow -\omega$ \cite{6, 8}. Secondly, the PCS can amplify/attenuate and mix the eigenwaves. This second effect accounts for gain due to phase conjugation, as well as the fact that phase conjugation in general is dependent on the linear momentum of the incoming wave. For the moment, we will delay the second consideration to the next section, while we focus on the first point that passing through PCS would perform frequency reversal.

The presence of the PCS then implies that Maxwell’s equations for $x > 0$ are given by
\[
\nabla \times E = -i(\omega)\vec{\mu} \cdot H,
\]
\[
\nabla \times H = i(\omega)\vec{\epsilon} \cdot E,
\]
\[0 < x. \tag{2}\]

We note that $\vec{\epsilon} = -\vec{\epsilon}; \vec{\mu} = -\vec{\mu}$ is the optical complementary (according to PLT \cite{9} (see appendix \ref{sec:appendix:a} of $\epsilon, \mu$); this implies that if $g^\pm_n(x)$ is the (spatial) solution to Maxwell’s equations for $x > 0$, then according to PLT, $g^\pm_n(x)$ will be a mirror image/spatial reflection of $f^\pm_n(x)$. Now for the field solutions to obey causality or the radiation condition, we need $\lim_{x \rightarrow -\infty} |g^\pm_n(x)| \rightarrow 0$, that is fields going away from the interface must decay to zero. Since we note that $\lim_{x \rightarrow -\infty} |f^\pm_n(x)| \rightarrow 0$, this means that the right-going eigenwave for $x > 0$ is the spatial reflection of the left-going eigenwave for $x < 0$, that is $g^\pm_n(x) = f^\mp_n(x)$. In turn, this also means that $g^\pm_n(x) = \tilde{f}^\mp_n(x)$.

Now we consider, instead, a finite slab $-d < x < 0$ of a material with material parameters given by $\epsilon(x), \mu(x)$ and another slab $0 < x < d$ with material $\tilde{\epsilon}(\tilde{x}), \tilde{\mu}(\tilde{x})$, together with two PCSs at $x = 0$ and $x = d$. The same results would hold with eigenwaves for $-d < x < 0$ being a spatial reflection about $x = 0$ of eigenwaves in $0 < x < d$ (see figure \ref{fig:2}(a)). In a later section we will see that the presence of the second PCS plays a crucial role in transforming the medium in $0 < x < d$ into a true optical complementary medium to the medium in $-d < x < 0$.

### 2.3. Proof of the principle

We shall calculate the overall transmission and reflection coefficients of the two slabs of materials we have previously considered, and show that, under certain conditions, the second slab of material (with the two PCSs) would behave as an optical complementary to the first, so that the overall transmission matrix would be $1$.

Here we use the bra and ket notation to write the eigenwaves as $|f^\pm_n(x)\rangle$ and $|g^\pm_n(x)\rangle$.

Note that using this notation, $\langle f^\pm_n(x) |$ is defined as $\langle f^\pm_n | f^\mp_m \rangle = \delta_{nm}$ (the same for $\langle g^\pm_n(x) |$).

Unless otherwise stated, we shall assume Einstein’s summation convention to hold throughout the paper.

Now we can define a propagation matrix/operator for $-d < x < 0$, as $P^\pm(x', x) = |f^\pm_n(x')\rangle \langle f^\pm_n(x)|$, where the propagation matrix allows us to propagate arbitrary wave function $|\Psi\rangle$ from $x$ to $x'$ by $P^\pm(x', x)|\Psi\rangle$ (if $x < x'$, that is right going) and by $P^\mp(x', x)|\Psi\rangle$ (if $x > x'$, that is left going). Similarly, for $0 < x < d$, the propagation matrix is given by $Q^\pm(x', x) = |g^\pm_n(x')\rangle \langle g^\pm_n(x)|$. Now using the results of the previous section, this propagation matrix can be written more precisely as $Q^\pm(x', x) = |\tilde{f}^\mp_n(x')\rangle \langle \tilde{f}^\mp_n(x)|$.

\[\]
Figure 2. (a) The relationship between eigensolutions on both sides of the PCS, where eigensolutions on the right of the PCS are the spatial reflection of those on the left. The directions of the arrow show the flow of energy. In particular, the right propagating wave is changed from $f^+_n(x)$ on the left of the PCS to $\bar{f}^-_{-n}(x)$ on the right. (b) The effect of channel mixing in general, so that an incoming wave $f^+_n(x)$ is scattered to $T_{nm}\bar{f}^-_{m}(x)$, where $T_{nm}$ encodes the gain/attenuation due to the PCS.

As we only consider propagation from the $x = -d$ to $x = d$ plane, we only need the following:

\begin{align*}
P^+(0, -d) &= |f^+_n(0)\rangle \langle f^+_n(-d)| = P^+, \\
P^-(d, 0) &= |f^-_n(-d)\rangle \langle f^-_n(0)| = P^-, \\
Q^+(d, 0) &= |\bar{f}^-_{-n}(d)\rangle \langle \bar{f}^-_{-n}(0)| = Q^+, \\
Q^-(0, d) &= |\bar{f}^+_{n}(0)\rangle \langle \bar{f}^+_{n}(d)| = Q^-.
\end{align*}

Now we can take into account the second consideration of the PCS—that there is an amplifying/attenuation effect and that phase conjugation in general is a function of momentum—implying that it can also mix eigenwaves. To model the transmission and reflection of waves at the PCS, we need to define transmission and reflection matrices at the PCS. Without loss of generality, we can assume that the transmission (reflection) of a PCS is symmetric: in the sense that transmitting (reflecting) a wave travelling from left to right of the PCS is the same in reverse. With this we can define the following transmission matrices:

\begin{align*}
T &= |\bar{f}^-_{-n}(0)\rangle T_{nm} \langle f^+_m(0)|, \\
&= |f^-_{n}(0)\rangle T_{nm} \langle \bar{f}^+_{m}(0)|, \quad x = 0, \tag{4}
\\
T' &= |f^+_n(d)\rangle T'_{nm} \langle \bar{f}^-_{m}(d)|, \quad x = d. \tag{5}
\end{align*}

where the matrix elements $T_{nm}$ and $T'_{nm}$ describes how waves from channel $m$ are scattered to waves in channel $n$, for the PCS at $x = 0$ and $x = d$, respectively (see figure 2(b)); in general, these elements are non-diagonal (for the explicit calculation of $T_{nm}$ and $T'_{nm}$, see appendix A.1).
Figure 3. Illustration of the various matrices in section 2.3. \( P^\pm, Q^\pm \) are the propagation matrices, which propagate fields between planes \( x = -d \) to \( x = 0 \) and \( x = 0 \) to \( x = d \), respectively. \( T, T' \) (and \( R, R' \)) are the transmission (and reflection) matrices of the PCS at \( x = 0 \) and \( x = d \), respectively.

Similarly, we can define the reflection matrices \( R \) and \( R' \) for the PCS at \( x = 0 \) and \( x = d \), respectively.

With all these in place (see figure 3), we can now calculate the overall transmission/transfer matrix, \( \hat{T}_0 \), which is given by

\[
\hat{T}_0 = T' \cdot Q^+ \left( 1 + \left[ R \cdot Q^- \cdot R' \cdot Q^+ \right] + \left[ R \cdot Q^- \cdot R' \cdot Q^+ \right]^2 + \cdots \right) \cdot T \cdot P^+
\]

\[
= T' \cdot Q^+ \left( \sum_{n=0}^{\infty} \left[ R \cdot Q^- \cdot R' \cdot Q^+ \right]^n \right) \cdot T \cdot P^+ \tag{6}
\]

\[
= T' \cdot Q^+ \left( 1 - \left[ R \cdot Q^- \cdot R' \cdot Q^+ \right] \right)^{-1} \cdot T \cdot P^+.
\]

where we used the formulae for geometric progression for the second line, and \( 1 \) is the identity matrix.

Now if we allow the gain of the reflection matrices to be sufficiently large such that the diagonal elements of the matrices in square brackets \( M = [R \cdot Q^- \cdot R' \cdot Q^+] \) are much larger than \( 1 \)—that is \( M_{nn} = \sum_m R_{nm}R'_{mn} \gg 1 \) (no sum over \( n \))—we obtain from (6) the following \( \hat{T}_0 \):

\[
\hat{T}_0 \approx -T' \cdot Q^+ \cdot \left( [R \cdot Q^- \cdot R' \cdot Q^+] \right)^{-1} \cdot T \cdot P^+
\]

\[
= -T' \cdot Q^+ \cdot [Q^+]^{-1} \cdot R'^{-1} \cdot [Q^-]^{-1} \cdot R^{-1} \cdot T \cdot P^+ \tag{7}
\]

\[
= - \left( T' R'^{-1} \right) \cdot [Q^-]^{-1} \cdot (R^{-1} T) \cdot P^+.
\]

Again if we design the PCS such that \( (T' R'^{-1}) = (R^{-1} T) = 1 \) (generally this is true for a thin PCS where \( T \approx R \)), we obtain

\[
\hat{T}_0 \approx -[Q^-]^{-1} P^+ \tag{8}
\]
Since from (3) \( Q^- = |\tilde{f}^+(0)\rangle \langle \tilde{f}^+(d)| \), this implies that \([Q^-]^{-1} = |\tilde{f}^+(d)\rangle \langle \tilde{f}^+(0)|\). Now using again the fact that \( \tilde{f}^+(x) \) is the spatial reflection of \( f^+(x) \), this implies that \([Q^-]^{-1} = |f^+(-d)\rangle \langle f^+(0)| = [P^+]^{-1} \); thus
\[
\hat{T}_0 \approx -1.
\tag{9}
\]

Equation (9) implies that fields at the \( x = -d \) plane is ideally replicated at \( x = d \)—the combined effect of these two slabs therefore cancels each other optically. Finally, it should be noted that for the above analysis to be valid, the simplification from (6) to (7) requires the diagonal elements of \( M = [R \cdot Q^- \cdot R' \cdot Q^+] \) to be much larger than 1, that is written explicitly:
\[
M_{nn} = \sum_m R_{nm}R'_{mn} \gg 1 \quad (\text{no summation over } n),
\tag{10}
\]
where the matrix elements \( R_{nm} \) and \( R'_{mn} \) can be calculated in the same way as in equation (A.1) and (A.2). This condition given by (10) then tells us quantitatively how much gain the two PCS slabs require in order for the composite slab to behave effectively as an NRIM.

It should also be emphasized that although the derivation for (6) holds even for \( \|M\| \gg 1 \) (refer to appendix A.2 for proof and further discussion), the minus sign in the denominator of (6), that is \((1 - M)^{-1}\), can be a potential problem: especially for some eigen modes where \( M = 1 \) would mean that \( \hat{T} \) is singular. This problem can be circumvented by designating \( R = -R' \) for the two PCS sheets. Then the denominator of (6) would be replaced by \((1 + M)^{-1}\) and the result would be regular for all values of gain.

3. Applications

Now the above principle can be simply applied to a slab of vacuum, where \( |f_q^\pm(x)\rangle = |q\rangle e^{\pm ik_1 x} \) and \( k_1 = \sqrt{k_0^2 - q^2} = \alpha + i\beta, \; k_0^2 = \omega^2/c^2 \) and \( q \) is the transverse momentum state. We can easily show that the transmission and reflection matrices are given by \( T = | -q\rangle t(q)\langle q| \), \( T' = | -q\rangle t(q)\exp(-2i \cdot ad)\langle q| \) and \( R = | -q\rangle r(q)\exp(-2i \cdot ad)\langle q| \), \( R = | -q\rangle r(q)\langle q| \), and these together with the propagation matrices give
\[
\hat{T}_0 = T' \cdot Q^+ \cdot (1 - [R \cdot Q^- \cdot R' \cdot Q^+])^{-1} \cdot T \cdot P^+
\tag{11}
\]
where \( b = |\Im(k_1)| \), and (11) is essentially the same result as Pendry’s [6].

Note the term in the denominator; when \( M_{qq} = r^2 \exp(-2b \cdot d) \gg 1 \) the exponential terms in the numerator and denominator of \( \hat{T}_0 \) cancel, and when \( t(q) = r(q) \), then \( \hat{T}_0 \approx -1 \), as we have discussed above.

Now if we place a lossy medium between two PCSs, it would be ‘inverted’ and become like an effective medium with gain instead. So in place of a vacuum, if we place a homogeneous lossy slab we would basically get the same result as (11) except that \( k_0^2 = \alpha + i\beta \), where \( \beta \) is some measure of losses in the slabs. Once again with sufficient gain, \( \hat{T}_0 \approx -1 \) as well. This is a remarkable result, as we have a simple scheme to create an ‘active perfect lens’ that can see through even a lossy slab of material, something that cannot be done by a conventional passive perfect lens (see figure 4).
Figure 4. Illustration of the active perfect lens. A suitably reflected PRIM sandwiched between a pair of PCSs with sufficient gain is effectively an ‘ideal’ perfect lens that can restore the fields of the object even when these fields have propagated through a lossy medium.

4. Conclusion

We have proposed and proved a simple scheme by which one can make an ‘active’ perfect lens. The proof relies on general eigensolutions of any PRIM, implying that this single generic scheme can be used to construct any inhomogeneous NRIM lens we want.

In addition, because there is no distinction between the evanescent near fields and the propagating far fields in the proof, this implies that theoretically, with sufficient gain, such perfect lenses can also exhibit sub-wavelength imaging. In practice however, restoration of the near fields requires inaccessibly large gain in the PCS, and it remains to be seen if this is possible. Nevertheless, restoration of the decaying far fields—that exist in the lossy PRIM—can be shown to be easily achievable with current technology. Thus, we should still be able to use these lenses to see through lossy materials.

Finally, the generality of the NRIM obtained through this scheme implies that we can use transformation optics to deform this planar NRIM to other geometries, thus obtaining a more general perfect lens.

Appendix A

A.1 Explicit calculation of elements of transmission matrices

In general, because phase conjugation depends only on transverse momentum, the transmission matrix is diagonal only in the momentum basis; hence these elements are given explicitly by

\[ T_{nm} = \sum_k \langle \tilde{f}_n^{-} (0) | -k \rangle t(k) \hat{\Phi}(k | f_m^{+} (0) \rangle, \quad \text{(no sum over m)}, \quad (A.1) \]

and

\[ T_{nm}' = \sum_k \langle f_n^{+} (d) | -k \rangle t(k) \hat{\Phi}(k | \tilde{f}_m^{-} (d) \rangle, \quad \text{(no sum over m)}, \quad (A.2) \]

where \( \langle k | f_m^{\pm} (x) \rangle = \int dx_\perp f_m^{\pm} (x , x_\perp) e^{i k \cdot x_\perp} \) is the projection of the eigenwave from the \( | f_m^{\pm} (x) \rangle \) basis to the \( | k \rangle \) basis,\(^5\) and \( \langle f_m^{\pm} (x) | k \rangle \) is the reverse. \( t(k) \) is a real function that determines the

\(^5\) The Fourier transform of the eigenwave at the sheet.
gain for plane waves with transverse momentum \( k \). \( \Phi \) is a phase conjugation operator that acts only on \( \mathbb{C} \) numbers, performing complex conjugation on the right of the operator. (If \( a \) is a complex number, then \( \Phi a = a^* \).) This phase conjugation operator is necessary to impose the right boundary conditions at the PCSs. We note that we can move \( \mathbb{C} \) numbers to the right of the operator by performing complex conjugation on that number so that \( \Phi a b = a^* \Phi b \). This means that equations (A.1) and (A.2) can be rewritten as

\[
T_{nm} = \sum_k \Phi \left[ \langle \vec{f}_{l}^- (0) | - k \rangle \right]^* t(k) \langle k | \vec{f}_{m}^+ (0) \rangle, \quad \text{(no sum over } m) \tag{A.3}
\]

\[
T'_{nm} = \sum_k \Phi \left[ \langle f_{n}^+ (d) | - k \rangle \right]^* t(k) \langle k | \vec{f}_{m}^- (d) \rangle. \quad \text{(no sum over } m) \tag{A.4}
\]

Now the expressions for the reflection matrices are the same so that the explicit elements of \( M = [R \cdot Q^- \cdot R' \cdot Q^+] \) given by \( M_{in} = R_{lm} R'_{mn} \) are given by

\[
M_{in} = \sum_m R_{lm} R'_{mn}
\]

\[
= \sum_{m} \left[ \sum_{k'} \langle \vec{f}_{l}^- (0) | - k \rangle r(k) \hat{\Phi} \langle k | \vec{f}_{m}^+ (0) \rangle \sum_{k} \langle \vec{f}_{m}^+ (d) | - k \rangle r(k) \hat{\Phi} \langle k | \vec{f}_{n}^- (d) \rangle \right]
\]

\[
= \sum_{m} \left[ \sum_{k'} \langle \vec{f}_{l}^- (0) | - k \rangle r(k) \left[ \langle k | \vec{f}_{m}^+ (0) \rangle \right]^* \sum_{k} \left[ \langle \vec{f}_{m}^+ (d) | - k \rangle \right]^* r(k) \langle k | \vec{f}_{n}^- (d) \rangle \right], \tag{A.5}
\]

where we used the property illustrated in equations (A.3) and (A.4) and the fact that \( \hat{\Phi} \hat{\Phi} = 1 \) to remove the complex conjugation operator. The diagonal elements of \( M \) obtained from (A.5) are important in determining how much gain the PCSs require in order to observe the effect of negative refraction—that is, \( M_{nn} = \sum_m R_{nm} R'_{mn} \gg 1 \).
A common objection might be raised about the derivation of (6). The key to deriving (6) is the summation \( \sum_{n=0}^{\infty} M^n = (1 - M)^{-1} \) for all \( M \) except for \( M = 1 \). Some might object that the summation identity only converges for \( M \) such that \( |M_{nn}| < 1 \). Strictly speaking, one can overcome this by appealing to analytic continuation, as long as we include all terms up to \( n = \infty \), and this summation identity should still hold.

In anticipation of this objection, this subsection gives an alternative derivation of (6). Using the method of undetermined coefficients. Define the following undetermined matrices \( A, B, C \) and \( D \) such that (if \( |\psi\rangle \) is the field at \( x = -d \) \( |A\rangle = A|\psi\rangle \), is the field at \( x = 0^- \), similarly \( |B\rangle = B|\psi\rangle \) at \( x = 0^+ \), \( |C\rangle = C|\psi\rangle \) at \( x = d^- \) and \( |D\rangle = D|\psi\rangle \) at \( x = d^+ \). Taking into account the boundary conditions, the undetermined matrices then satisfies the following coupled simultaneous equations:

\[
\begin{align*}
A &= R \cdot P^+ + T \cdot Q^+ C, \\
B &= T \cdot P^+ + R \cdot Q^+ C, \\
C &= R' Q^+ B, \\
D &= T' Q^+ B.
\end{align*}
\] (A.6)

Solving for these we obtain

\[
\begin{align*}
A &= R \cdot P^+ + T \cdot Q^- R' Q^+ (1 - R Q^- R' Q^+)^{-1} T P^+, \\
B &= (1 - R Q^- R' Q^+)^{-1} T P^+, \\
C &= R' Q^+ (1 - R Q^- R' Q^+)^{-1} T P^+, \\
D &= T' Q^+ (1 - R Q^- R' Q^+)^{-1} T P^+.
\end{align*}
\] (A.7)

where we note that \( D = \hat{T}_0 \) in equation (6).

Note first that the above derivation does not require any summation identity. In addition, since all the undetermined matrices \( A, B, C \) and \( D \) are finite even for \( M \) such that \( M_{nn} > 1 \), therefore (A.7) is a valid solution to the scattering problem. By the uniqueness theorem then, since this is a valid solution it is also the only unique solution.

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A.3 PLT

This section contains a short proof of the PLT. Firstly, Maxwell’s equations in the absence of free charges and currents are given by

\[
\nabla \times E = -i\omega B, \quad \nabla \cdot D = 0, \\
\nabla \times H = i\omega D, \quad \nabla \cdot B = 0.
\]

(A.8)

We can clearly see that under reflection transformation, \(\{E, H\}\) transforms as vectors while \(\{D, B\}\) transforms as pseudo-vectors—that is under reflection \(R:\{E_\perp, E_z\} \rightarrow \{E_\perp, -E_z\}\), while \(\mathcal{R} : \{D_\perp, D_z\} \rightarrow \{-D_\perp, D_z\}\). From the constitutive equations (assuming linearity)

\[
D = \epsilon \cdot E, \\
B = \mu \cdot H.
\]

(A.9)

We can therefore see that in order to map the vectors on the right side of the equation to pseudo-vectors on the left, the permittivity (permeability) tensor must transform under reflection as \(\mathcal{R}\):

\[
\tilde{\epsilon} = \mathcal{R} \epsilon \mathcal{R}^{-1},
\]

(A.10)

in order for the Maxwell’s equation to be covariant. This can also be simply written as \(\tilde{\epsilon} = -\epsilon\), where \(\tilde{\epsilon} = \mathcal{R} \epsilon \mathcal{R}^{-1}\) (where \(\mathcal{R} = \text{Diag}[1, 1, -1]\) is the reflection transformation operator).

Geometrically, the eigenvectors of \(\tilde{\epsilon}\) are reflected copies of \(\epsilon\) while the eigenvalues of \(\tilde{\epsilon}\) are the same as \(\epsilon\). Physically this means that a material with \(\tilde{\epsilon}\) can be interpreted as a spatially reflected copy of another material with permittivity \(\epsilon\).

Now if we were to reverse the argument and ask the question—given a half space \((z < 0)\) filled with a material \(\epsilon(x), \mu(x)\), what material must we fill the other half space \((z > 0)\) with so that the field solution in this other half space is the reflected copy of the first, then the answer is simply \(\tilde{\epsilon}(\tilde{x}), \tilde{\mu}(\tilde{x})\). So then the PLT tells us that given a slab of material \(\epsilon(x), \mu(x)\), we can always find it’s optical complementary \(\tilde{\epsilon}(\tilde{x}), \tilde{\mu}(\tilde{x})\), such that the net effect of propagation through these two materials is cancelled out.

References

[1] Pendry J B 2000 Negative refraction makes a perfect lens Phys. Rev. Lett. 85 3966–9
[2] Smith D R, Padilla W J, Vieer D C, Nemat-Nasser S C and Schultz S 2000 Composite medium with simultaneously negative permeability and permittivity Phys. Rev. Lett. 84 4184–7
[3] Pendry J B 2004 A new route to negative refraction Science 306 1353–5
[4] Ramakrishna S A and Pendry J B 2003 Removal of absorption and increase in resolution in a near-field lens via optical gain Phys. Rev. B 67 201101
[5] Popov A K and Shalaev V M 2006 Compensating losses in negative-index metamaterials by optical parametric amplification Opt. Lett. 31 2169–71
[6] Pendry J B 2008 Time reversal and negative refraction Science 322 71–3
[7] Aubry A and Pendry J B 2010 Mimicking a negative refractive slab by combining two phase conjugators J. Opt. Soc. Am. B 27 72–84
[8] Maslovski S and Tretyakov S 2003 Phase conjugation and perfect lensing J. Appl. Phys. 94 4241–3
[9] Pendry J B and Ramakrishna S A 2003 Focussing light using negative refraction J. Phys.: Condens. Matter 15 6345–64

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