THE SUB-SUPERSOLUTION METHOD FOR THE
FITZHUGH-NAGUMO TYPE REACTION-DIFFUSION SYSTEM
WITH HETEROGENEITY

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Abstract. We construct a heteroclinic solution to the FitzHugh-Nagumo type reaction-diffusion system (FHN RD system) with heterogeneity by the sub-supersolution method due to [5]. \( \sigma(d, \gamma) \) is introduced as the Rayleigh quotient corresponding to a linearized eigenvalue problem of the subsolution, where \( d \) and \( \gamma \) are parameters. The key to construct the solution is the uniform estimate for \( \sigma(\cdot, \cdot) \) from below. In addition, it enables us to analyze an asymptotic behavior of the solution.

1. Introduction and main results. The FitzHugh-Nagumo type reaction-diffusion system (FHN RD system) is introduced in physiology, which essentially describes neural excitability. This is also studied mathematically as a model which generates complex patterns. A typical FHN RD system is the following:

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) &= d \Delta u(x,t) + f(u(x,t)) - v(x,t), \quad x \in \Omega, \ t > 0, \\
\frac{\tau}{\partial t}v(x,t) &= D \Delta v(x,t) + u(x,t) - \gamma v(x,t), \quad x \in \Omega, \ t > 0,
\end{align*}
\]

(1)

where \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\) is domain, \( d, D, \tau, \gamma \) are positive constants and \( f(s) = s - s^3 \).

Throughout this paper, we assume \( D = 1 \) for simplicity. Moreover, in this paper we treat the steady state problem of (1), that is, \( \partial u/\partial t = \partial v/\partial t = 0 \).

In general, the steady state FHN RD system has various solutions and several methods are known to construct a solution to (1). For small \( d > 0 \), there are many works in which the singular perturbation method (SPM) is applied to obtain the solutions. For example, Nishiura [11] applied the SPM for (1) with the assumption that \( \Omega = (0, 1) \) and the boundary condition is the Neumann condition. They proved that for any \( N \in \mathbb{N} \), there exists a stable solution which has \( N \) transition layers by taking \( d > 0 \) small enough. Moreover, Heijster applied the SPM to the three-component model and obtained various localized solutions (e.g. [4]). We remark that they considered the case \( \Omega = \mathbb{R} \) in [4].

For another approach, Oshita [12] constructed a solution as a minimizer of the variational problem corresponding to (1) with the assumption \( \Omega \) is bounded. Oshita
especially showed that the minimizer oscillates rapidly as \( d \to 0 \). Dancer and Yan [6] showed similar results as [12]. We mention that [12] treated the Neumann boundary condition and [6] treated the Dirichlet boundary condition.

In this paper, we apply the sub-supersolution method. We develop the work [5] by Chen, Kung and Morita. In [5], Chen, Kung and Morita considered the case by the sub-supersolution method. Moreover, the solution has just one transition layer from one stable equilibrium point to the other stable equilibrium point.

Motivated by the work [5], we consider the following equation, which involves a heterogeneity \( \mu(x) \):

\[
\begin{cases}
- d u''(x) = \mu(x) f(u(x)) - v(x), & x \in \mathbb{R}_+, x \in (0, \infty), \\
- v''(x) + \gamma v(x) = u(x), & x \in \mathbb{R}_+,
\end{cases}
\]  

(2)

where \( d > 0 \), \( \gamma > 1 \), \( f(s) = s - s^3 \) and \( a_\gamma \) is the largest root of \( f(s) = s/\gamma \). Moreover, \( \mu \) is assumed the following conditions:

(\( \mu_1 \)) \( 1 - \mu \in L^1(\mathbb{R}_+) \cap C^1(\mathbb{R}_+) \).

(\( \mu_2 \)) There exists a constant \( \mu_0 > 0 \) such that \( \mu_0 \leq \mu(x) \) for all \( x \in \mathbb{R}_+ \). In addition, \( \mu \not\equiv 1 \) and \( \mu(x) \leq 1 \) for all \( x \in \mathbb{R}_+ \).

(\( \mu_3 \)) \( \mu'(x) \geq 0 \) for all \( x \in \mathbb{R}_+ \).

(\( \mu_4 \)) There exists a constant \( l_0 > 0 \) such that \( \mu(x) = \mu_0 \) for all \( x \in [0, l_0] \).

The case \( \mu \equiv 1 \) is studied in [5]. For related results, in [8], we constructed a heteroclinic solution to the FHN RD system with heterogeneity defined on \( \mathbb{R} \) instead of \( \mathbb{R}_+ \). Though the equation treated in [8] is slightly different from (2), the argument in [8] can be applied to (2). In fact, if \( \theta = d - 1/\gamma^2 > 0 \) and \( \mu \) satisfies (\( \mu_1 \)) and (\( \mu_2 \)), then one can construct the heteroclinic solution to (2) with the boundary condition \((u, v)(\pm \infty) = (\pm a_\gamma, \pm a_\gamma/\gamma) \). For more detail, see [8]. Moreover, for the Allen-Cahn type equation with heterogeneity, of which (2) is an extended model, Nakashima [10] showed that a solution has a transition layer near the minimum point of \( \mu \) as \( d \to 0 \). We remark that [10] treated the case that \( \Omega \) is bounded and \( \mu \) has non-degenerate local minimum points.

In this paper, we impose the following boundary condition:

\[
(u, v)(0) = (0, 0), \quad (u, v)(\infty) = (a_\gamma, a_\gamma/\gamma).
\]  

(3)

We mention that the solution to (2) with (3) corresponds to a steady state odd solution to (1) with just one transition layer from \((-a_\gamma, -a_\gamma/\gamma)\) to \((a_\gamma, a_\gamma/\gamma)\).

We employ a supersolution \( T \) and a subolution \( W \) defined as follows:

\[
\begin{cases}
- d T''(x) = \mu(x) f(T(x)), & x \in \mathbb{R}_+, x \in \mathbb{R}_+, \\
0 \leq T(x) \leq 1, & x \in \mathbb{R}_+, \\
T'(x) \geq 0, & x \in \mathbb{R}_+, \\
T(0) = 0, T(\infty) = 1.
\end{cases}
\]  

(4)

\[
\begin{cases}
- d W''(x) = \mu_0 f(W(x)) - 1/\gamma, & x \in \mathbb{R}_+, \\
0 \leq W(x) \leq u_\gamma, & x \in \mathbb{R}_+, \\
W'(x) \geq 0, & x \in \mathbb{R}_+, \\
W(0) = 0, W(\infty) = u_\gamma.
\end{cases}
\]  

(5)
where \( u_* \) is the largest root of \( \mu_0 f(s) - 1/\gamma = 0 \). For \( \gamma > 3\sqrt{6}/(2\mu_0) \), we prove the existence of the supersolution \( T \) and the subsolution \( W \) as a minimizer of the following minimizing problems (see Proposition 2.1):

\[
E_1(T) = \inf \{ E_1(u) : u \in X_1 \}, \quad T \in X_1, \tag{6}
\]

\[
E_2(W) = \inf \{ E_2(u) : u \in X_2 \}, \quad W \in X_2. \tag{7}
\]

Here \( E_1(u), E_2(u), X_1 \) and \( X_2 \) are defined as follows:

\[
E_1(u) = \int_{\mathbb{R}^+} \left[ \frac{d}{2} |u'(x)|^2 + \mu(x)F(u(x)) \right] dx,
\]

\[
E_2(u) = \int_{\mathbb{R}^+} \left[ \frac{d}{2} |u'(x)|^2 + G(u(x)) \right] dx,
\]

\[
X_1 = \{ u \in H_{loc}^1(\mathbb{R}^+) : u(0) = 0, \ u(\infty) = 1, \ 0 \leq u(x) \leq 1 \text{ for all } x \in \mathbb{R}^+ \}, \tag{8}
\]

\[
X_2 = \{ u \in H_{loc}^1(\mathbb{R}^+) : u(0) = 0, \ u(\infty) = u_*, \ 0 \leq u(x) \leq u_* \text{ for all } x \in \mathbb{R}^+ \}. \tag{9}
\]

where \( F(u) \) and \( G(u) \) are

\[
F(u) = -\int_1^u f(s) \, ds = \frac{1}{4}(u^2 - 1)^2,
\]

\[
G(u) = -\int_{u_*}^u \left[ \mu_0 f(s) - \frac{1}{\gamma} \right] \, ds.
\]

We note that \( G(u) > 0 \) for all \( u \in [0, u_*) \) under the assumption \( \gamma > 3\sqrt{6}/(2\mu_0) \) (see Proposition 2.1). We write the solution to (5) as \( W_{d,\gamma}(x) \), and we define \( \sigma(d, \gamma) \) as follows:

\[
\sigma(d, \gamma) = \inf \left\{ \frac{J_d(\psi)}{\int_{\mathbb{R}^+} |\xi(x)|^2 \, dx} : \xi \in H_{0}^1(\mathbb{R}^+), \ \xi \neq 0 \right\}, \tag{10}
\]

where \( J_d(\psi) \) is defined as follows:

\[
J_d(\psi) = \int_{\mathbb{R}^+} \left[ \frac{d}{2} |\xi'(x)|^2 - \mu(x)(1 - 3W_{d,\gamma}(x)^2) |\xi(x)|^2 \right] dx. \tag{11}
\]

On the existence of the solution to (2), we obtain the following statement:

**Proposition 1.** Let \( d > 0, \ \gamma > 3\sqrt{6}/(2\mu_0) \) and \( \mu \) be a function satisfying \((\mu1)\) – \((\mu4)\). If the following inequality

\[
\gamma \sigma(d, \gamma) > 1 \tag{12}
\]

holds, then there exists a solution to (2) with (3). Moreover, the solution \((u, v)\) satisfies the following inequalities:

\[
W(x) \leq u(x) \leq T(x) \quad \text{for all } x \in \mathbb{R}^+, \tag{13}
\]

\[
0 \leq v(x) \leq \frac{1}{\gamma} \quad \text{for all } x \in \mathbb{R}^+. \tag{14}
\]

From the above statement, we find that it is necessary to reveal the condition for (12) to be true. We introduce some notations. Define \( \hat{W}_{0,\gamma}(x) = W_{d,\gamma}(\sqrt{d}x) \).

From the definition, \( \hat{W}_{0,\gamma} \) is a solution to
Then fix a small positive constant \( \delta \) and set \( \gamma_0 = 3\sqrt{d}/(2\mu_0) + \delta \). Let \( x_0 = x_0(\mu_0, \gamma_0) \) be a constant such that \( \tilde{W}_{0,\gamma_0}(x_0) = 1/\sqrt{3} \). We note that \( x_0 \) is independent of \( d > 0 \) from (15). Now we state the next statement, which gives a sufficient condition for (12) to be true.

**Proposition 2.** Let \( \delta > 0 \) be a small positive number and \( \mu \) be a function satisfying (\( \mu_1 \)) - (\( \mu_4 \)). Then there exists a positive constant \( \sigma_\delta > 0 \) such that

\[
\sigma(d, \gamma) > \sigma_\delta \quad \text{for all} \quad (d, \gamma) \in (0, (l_0/x_0)^2) \times [\gamma_0, \infty).
\]

From Propositions 1 and 2, we obtain the following main result:

**Theorem 1.1.** Let \( \delta \) be a small positive constant, \( \gamma > \gamma_0 = 3\sqrt{d}/(2\mu_0) + \delta \) be a constant such that \( \gamma > 3\sqrt{d}/(2\mu_0) \) and \( \mu \) be a function satisfying (\( \mu_1 \)) - (\( \mu_4 \)). Then there exists a solution to (2) with (3) for any \( d \in (0, (l_0/x_0)^2) \). Moreover, the solution \( (u_{d,\gamma}, v_{d,\gamma}) \) satisfies (13) and (14).

Our proof is based on [3, 5], but we remark that \( \gamma \) depends on \( d \) in [3, 5]. We emphasize that \( \gamma \) can be chosen independently of \( d \) in Theorem 1.1. Thus Theorem 1.1 gives us some information about the asymptotic behavior of \( u_{d,\gamma} \) as \( d \to 0 \). Namely we obtain the following corollary:

**Corollary 1.** Fix \( \gamma > 3\sqrt{d}/(2\mu_0) \). Let \( \beta \) be a positive constant such that \( \beta < u_\gamma \) and let \( y_1 \) be a point in \( \mathbb{R}_+ \) such that \( u_{d,\gamma}(y_1) = \beta \), where \( u_{d,\gamma} \) is the solution obtained in Theorem 1.1. Then \( y_1 = O(d^{1/2}) \) holds.

From this corollary, we can see that \( u_{d,\gamma} \) “transits” from 0 to \( \beta \) on the interval whose width is \( O(d^{1/2}) \).

Our method can be applied to the following problem:

\[
\begin{aligned}
- \frac{\partial^2 u}{\partial x^2} &= \mu(x)(f(u(x)) - v(x)), & x &\in \mathbb{R}_+ = (0, \infty), \\
- \frac{\partial^2 v}{\partial x^2} + \gamma v(x) &= u(x), & x &\in \mathbb{R}_+, \\
u(x), v(x) &> 0, & x &\in \mathbb{R}_+, \\
u(0) &= (0, 0), & u, v(\infty) &= (\alpha, \alpha_\gamma/\gamma),
\end{aligned}
\]

(16)

For the equation, we employ \( T \) defined in (4) as a supersolution and \( \tilde{W} \) as a subsolution defined as follows:

\[
\begin{aligned}
- \frac{\partial^2 \tilde{W}}{\partial x^2} &= \mu(x)(f(\tilde{W}(x)) - 1/\gamma), & x &\in \mathbb{R}_+, \\
0 &\leq \tilde{W}(x) \leq u_\gamma, & x &\in \mathbb{R}_+, \\
\tilde{W}(x) &> 0, & x &\in \mathbb{R}_+, \\
\tilde{W}(0) &= 0, & \tilde{W}(\infty) &= \tilde{u}_\gamma,
\end{aligned}
\]

(17)

where \( \tilde{u}_\gamma \) is the largest root of \( f(s) = 1/\gamma = 0 \). By the variational approach as (5), for \( \gamma > 3\sqrt{d}/2 \) we can prove the existence of \( \tilde{W} \). We write the solution to (17) as \( \tilde{W}_{d,\gamma} \), and we define \( \tilde{\sigma}(d, \gamma) \) as follows:
Theorem 1.2. Let $\delta$ be a small positive constant, $\gamma > \gamma_1 = 3\sqrt{6}/2 + \delta$ be a constant such that $\gamma \sigma_\delta > 1$ and $\mu$ be a function satisfying $(\mu 1) - (\mu 4)$. Then there exists a solution to (16) for any $d \in (0, (l_0/x_1)^2)$. Moreover, the solution $(\bar{u}_{d,\gamma}, \bar{v}_{d,\gamma})$ satisfying the following inequalities:

$$
\bar{W}(x) \leq \bar{u}_{d,\gamma}(x) \leq T(x) \quad \text{for all } x \in \mathbb{R}_+,
$$

$$
0 \leq \bar{v}_{d,\gamma}(x) \leq \frac{1}{\gamma} \quad \text{for all } x \in \mathbb{R}_+
$$

If we employ $\bar{W}_{d,\gamma}$ as a subsolution, then we can obtain almost the same statement by repeating almost the same arguments as Theorem 1.1. However, we can see a slightly better estimate of $\bar{u}_{d,\gamma}$ by employing $\bar{W}_{d,\gamma}$ as a subsolution since $\bar{W}_{d,\gamma} \leq \bar{W}_{d,\gamma}$ holds.

This paper is arranged as follows. In Section 2, we collect basic lemmas. The proof of Theorem 1.1 is presented in Section 3. We first prove Proposition 1 in Section 3.1 and next prove Proposition 2 and Theorem 1.1 in Section 3.2. The proof of Theorem 1.2 is presented in Section 4. We mainly state the differences between Theorems 1.1 and 1.2. In Appendix, we give the proofs of some auxiliary lemmas.

2. Preliminaries and basic estimates. In this section, we prove Theorem 1.1. We prepare some basic lemmas in this section. This section consists of four parts.

2.1. Existence of $T$ and $W$, comparison lemma of $T$ and $W$. We first show the existence of the supersolution $T(x)$ and the subsolution $W(x)$. We mention that the proof is based on [1].

Lemma 2.1. If $\gamma > 3\sqrt{6}/(2\mu_0)$, then there exist minimizers $T$ and $W$ to the minimizing problems (6) and (7). $T$ and $W$ satisfy (4) and (5) respectively.
Proof. We only prove the existence of \( W(x) \) since one can prove the existence of \( T(x) \) similarly. First, we show that \( G(0) > G(u_{\gamma}) = 0 \) if \( \gamma > 3\sqrt{6}/(2\mu_0) \). By a direct calculation, we have

\[
G(0) = \int_{u_{\gamma}}^{0} \left[ \mu_0 (s^3 - s) + \frac{1}{\gamma} \right] \, ds = \frac{\mu_0}{4} u_{\gamma}^4 + \frac{\mu_0}{2} u_{\gamma}^2 - \frac{u_{\gamma}}{\gamma} = \frac{1}{4} u_{\gamma} \left( -\mu_0 u_{\gamma}^3 + \mu_0 u_{\gamma} + u_{\gamma} - \frac{4}{\gamma} \right).
\]

With the equation \( \mu_0 (u_{\gamma} - u_{\gamma}^3) = 1/\gamma \), it follows that

\[
G(0) = \frac{1}{4} u_{\gamma} (u_{\gamma} - 3/\gamma).
\]

We note that

\[
u_{\gamma} > \frac{3}{\gamma} \iff \frac{3}{\gamma} - \left( \frac{3}{\gamma} \right)^3 > \frac{1}{\mu_0 \gamma} \iff \gamma > \frac{3\sqrt{6}}{2\mu_0},
\]

and thus we arrive at \( G(0) > G(u_{\gamma}) = 0 \). Next, we show the existence of the minimizer. It suffices to show that \( E \) has a minimizer in \( X_2' \) defined as follows;

\[X_2' = \{ u \in H^1_{loc}(\mathbb{R}_+) : u(0) = 0, u(\infty) = u_{\gamma} \}.
\]

To see this, we assume that \( E_2 \) has a minimizer \( W \in X_2' \) and \( W \) is not a nondecreasing function. Then there exists points \( x_1 \) and \( x_2 \) such that \( 0 < x_1 < x_2 < \infty \) and \( W(x_1) = W(x_2) \) hold. We set \( \omega \) as follows:

\[
\omega(x) = \begin{cases} 
W(x), & 0 \leq x < x_1, \\
W(x + x_2 - x_1), & x_1 \leq x < \infty.
\end{cases}
\]

From the positivity of \( G \) on \([0, u_{\gamma}]\), we can see that \( E_2(\omega) < E_2(W) \). Hence the minimizer \( W \) should be a nondecreasing function. Moreover, it implies \( W \in X_2 \), where \( X_2 \) is defined as (9).

Now we shall prove the existence of a minimizer of \( E_2 \) in \( X_2' \). Let \( \{ u_n \}_{n \in \mathbb{N}} \) be a minimizing sequence in \( X_2 \). As mentioned above, we may assume that \( u_n \in X_2 \) and \( u_n \) is a nondecreasing function. It is easy to see that \( ||u_n'||_{L^2(\mathbb{R}_+)} = ||(u_n - u_{\gamma})'||_{L^2(\mathbb{R}_+)} \) is uniformly bounded. In addition, we readily see that

\[
E_2(u_n) = \int_{\mathbb{R}_+} \left[ \frac{d}{2} |u_n'|^2 + G(u_n) \right] \, dx \geq C_1 \int_{\mathbb{R}_+} (u_n - u_{\gamma})^2 \, dx
\]

since there exists a constant \( C_1 > 0 \) such that \( G(u) \geq C_1 (u - u_{\gamma}^2) \). Hence we can see that \( ||u_n - u_{\gamma}||_{L^2(\mathbb{R}_+)} \) is also uniformly bounded. It follows that there exists \( \phi \in H^1(\mathbb{R}_+) \subset C(\mathbb{R}_+) \) such that

\[
(u_n - u_{\gamma}) \rightharpoonup \phi \text{ weakly in } H^1(\mathbb{R}_+)
\]

and

\[
(u_n - u_{\gamma}) \to \phi \text{ strongly in } C_{loc}(\mathbb{R}_+).
\]

We set \( W = u_{\gamma} + \phi \). Then we obtain

\[
E_2(W) \leq \liminf_{n \to \infty} \int_{\mathbb{R}_+} \frac{d}{2} |u_n'|^2 \, dx + \liminf_{n \to \infty} \int_{\mathbb{R}_+} G(u_n) \, dx = \inf_{u \in X_2} E_2(u)
\]

from the lower semicontinuity of \( L^2 \) space and Fatou’s lemma.
Now we check that $W$ satisfies $W(\infty) = u_\gamma$. Let $\rho$ be a small constant and $s_n$ be a point such that $u_n(s_n) = u_\gamma - \rho$. Then we have

$$E_2(u_n) \geq \min\{G(t) : 0 \leq t \leq u_\gamma - \rho\} s_n,$$

It follows that $\{s_n\}$ is uniformly bounded in $\mathbb{R}_+$. This implies that there exist a constant $s < \infty$ and a subsequence $\{s_{n_j}\}_{j \in \mathbb{N}}$ such that $s_{n_j} \to s$ as $j \to \infty$. Hence we can see $u(s) = u_\gamma - \rho$, which means $u(\infty) = u_\gamma$. Moreover, it is clear that $u(0) = 0$, and thus we have proved that $W$ is the minimizer of (7). With a standard argument, we readily see that $W(x)$ is a solution to (5).

With attention to the monotonicity of $\mu$, we can prove the existence of $T$ by repeating the same argument.

Next, we prove a comparison lemma, which is necessary for the sub-supersolution method.

**Lemma 2.2.** For any $d > 0$ and $\gamma > 3\sqrt{6}/(2\mu_0)$, $W(x) \leq T(x)$ holds, where $T$ and $W$ are the solutions to (4) and (5), respectively.

**Proof.** $W$ satisfies $-dW'' \leq \mu(x)f(W)$, and then we have

$$-dW''T \leq \mu(x)f(W)T. \tag{20}$$

Similarly we have

$$-dT''W = \mu(x)f(T)W. \tag{21}$$

We now derive a contradiction under the assumption that there exists an interval $[a,b] \subset \mathbb{R}_+$ such that

$$W(a) = T(a), \ W(b) = T(b), \ W'(a) \geq T'(a), \ W'(b) \leq T'(b), \ W(x) > T(x) \text{ for all } x \in (a,b).$$

From (20) and (21), we can see that

$$\int_a^b [-dW''T + dT''W] \, dx \leq \int_a^b \mu(x)TW \left( \frac{f(W)}{W} - \frac{f(T)}{T} \right) \, dx.$$

We note that $f(t)/t = 1 - t^2$ is a decreasing function on $[0,1]$. Thus the right hand side is negative from the assumption. We calculate the left hand side:

$$\int_a^b [-W''T + T''W] \, dx = [-W''T + T''W]_a^b = T(b)(T'(b) - W'(b)) + W(a)(W'(a) - T'(a)).$$

It is easy to check that the left hand side is nonnegative, but it clearly contradicts that the right hand side is negative. Therefore we conclude $W \leq T$ on $\mathbb{R}_+$. \hfill $\square$

### 2.2. Monotonicity of $\sigma(d, \gamma)$ on $\gamma$

We next show the monotonicity of $\sigma(d, \gamma)$.

**Lemma 2.3.** If $\gamma_1 < \gamma_2$, then $\sigma(d, \gamma_1) \leq \sigma(d, \gamma_2)$ for all $d > 0$.

**Proof.** It suffices to show that $W_{d,\gamma_1} \leq W_{d,\gamma_2}$ on $\mathbb{R}_+$ if $\gamma_1 < \gamma_2$. For simplicity, we write $W_i = W_{d,\gamma_i}$ ($i = 1, 2$). We note that $W_1(x) < W_2(x)$ for sufficiently large $x > 0$ since $W_1(\infty) < W_2(\infty)$. Let $x_m$ be

$$x_m = \inf\{y \geq 0 : W_1(x) < W_2(x) \text{ for all } x \in (y, \infty)\}.$$
From the definition, it is clear that \( x_m \geq 0 \). We now derive a contradiction under the assumption \( x_m > 0 \). It is easy to see that
\[
W_1(x_m) = W_2(x_m), \quad \text{and} \quad W'_1(x_m) \leq W'_2(x_m).
\]
On the other hand, \( W_i \) (\( i = 1, 2 \)) satisfies that
\[
-dW''_i(x) = \mu_0 f(W_i(x)) - \frac{1}{\gamma_i},
\]
Thus we obtain that
\[
-d(W''_2(x_m) - W''_1(x_m)) = \mu_0 \{f(W_2(x_m)) - f(W_1(x_m))\} + \frac{1}{\gamma_1} - \frac{1}{\gamma_2} = \frac{1}{\gamma_1} - \frac{1}{\gamma_2} > 0.
\]
This implies that
\[
(W'_2(x_m) - W'_1(x_m)) > 0.
\]
Hence there exists \( x_s < x_m \) defined as follows:
\[
x_s = \inf\{y \geq 0 : W_1(x) > W_2(x) \text{ for all } x \in (y, x_m)\}.
\]
We readily see that
\[
W_1(x_s) = W_2(x_s), \quad W'_1(x_s) \geq W'_2(x_s).
\]
From (22), for \( i = 1, 2 \) we have
\[
\left[ -\frac{d}{2}(W'_i)^2 \right]_{x_s}^{\infty} = \int_{x_s}^{\infty} \left( \mu_0 f(W_i(x)) - \frac{1}{\gamma_i} \right) W'_i(x) \, dx.
\]
By straightforward calculation, we have
\[
\frac{d}{2}(W'_i(x_s))^2 = G_i(W_i(x_s)),
\]
where \( G_i(u) = \int_{u_{\gamma_i}}^{u} [-\mu_0 f(s) + 1/\gamma_i] \, ds \). With attention to \( G_1(s) < G_2(s) \) for all \( 0 < s < u_{\gamma_1} \), we have
\[
W'_1(x_s) < W'_2(x_s).
\]
However, this contradicts \( W'_1(x_s) \geq W'_2(x_s) \). This leads to the conclusion that \( W_1 < W_2 \) on \( \mathbb{R}_+ \). \( \square \)

2.3. Continuity of \( W_{d, \gamma} \) and \( \sigma(d, \gamma) \) on \( d \). In this subsection, we prove the uniform continuity of \( \sigma(d, \gamma) \) respect to \( d \). To see this, we present Lemmas 2.4 and 2.5. We next show the continuity of \( W_{d, \gamma} \) respect to \( d \). The proofs of Lemmas 2.4 and 2.5 are presented in Appendix.

**Lemma 2.4.** Let \( \gamma > 3\sqrt{6}/(2\mu_0) \) and \( d_s \in (0, l_0/x_0) \). For any \( \epsilon > 0 \), there exists a positive constant \( \delta \) such that
\[
\|W_{d_1, \gamma} - W_{d_2, \gamma}\|_{L^\infty(\mathbb{R}_+)} < \epsilon
\]
for any \( d_1, d_2 \in [d_s, l_0/x_0] \) such that \( |d_1 - d_2| < \delta \).

**Lemma 2.5.** There exists a positive constant \( M \) independent of \( d \) and \( \gamma \) such that
\[
\sigma(d, \gamma) < M \quad \text{for all } (d, \gamma) \in (0, \infty) \times \left( \frac{3\sqrt{6}}{2\mu_0}, \infty \right).
\]
From Lemmas 2.4 and 2.5, the continuity of \( \sigma(\cdot, \gamma) \) can be proved. This lemma is applied for the uniform estimate of \( \sigma(d, \gamma) \) in Section 3.2.

**Lemma 2.6.** Let \( d_0 \) and \( \gamma \) be constants such that \( d_0 \in (0, l_0/x_0) \) and \( \gamma > 3\sqrt{6}/(2\mu_0) \). Then \( \sigma(\cdot, \gamma) \) is uniformly continuous on \([d_0, l_0/x_0] \).

**Proof.** We fix a small constant \( \epsilon > 0 \). Let \( d_1 \) and \( d_2 \) be constants such that \( d_0 < d_1 < d_2 \leq l_0/x_0 \). Moreover we assumed that \( |d_2 - d_1| \) is sufficiently small. Then there exists \( \xi \in H_0^1(\mathbb{R}_+) \) such that \( \|\xi\|_{L^2(\mathbb{R}_+)} = 1 \) and

\[
\sigma(d_1, \gamma) + \epsilon > J_{d_1}(\xi) = \int_{\mathbb{R}_+} \left[ \frac{d_1}{2} |\xi'|^2 + \mu(x)(3W_1^2 - 1)|\xi|^2 \right] dx, \tag{23}
\]

where \( W_i = W_{d_i, \gamma} \) \((i = 1, 2)\). We note that we may assume \( \|W_1 - W_2\|_{L^\infty(\mathbb{R}_+)} < \epsilon \) from Lemma 2.4. Hence we obtain that

\[
J_{d_1}(\xi) - J_{d_2}(\xi) \geq \int_{\mathbb{R}_+} \left[ -\frac{d_2 - d_1}{2} |\xi'|^2 - 3\epsilon^2 |\xi|^2 \right] dx. \tag{24}
\]

Now we evaluate \( \|\xi'\|_{L^2(\mathbb{R}_+)}^2 \): From the definition of \( \xi \), we easily see

\[
\frac{d_1}{2} \|\xi'\|_{L^2(\mathbb{R}_+)}^2 < \sigma(d_1, \gamma) + 1 - \int_{\mathbb{R}_+} (3W_1^2 - 1)|\xi|^2 dx.
\]

Moreover, we calculate as follows from Lemma 2.5:

\[
\|\xi'\|_{L^2(\mathbb{R}_+)}^2 < \frac{2}{d_1}(M + 2) < \frac{2}{d_0}(M + 2), \tag{25}
\]

where \( M \) is defined in Lemma 2.5. Combining (23) – (25), we obtain

\[
\sigma(d_1, \gamma) - \sigma(d_2, \gamma) + \epsilon > -\frac{d_2 - d_1}{d_0}(M + 2) - 3\epsilon^2,
\]

that is,

\[
\sigma(d_2, \gamma) - \sigma(d_1, \gamma) < \frac{d_2 - d_1}{d_0}(M + 2) + 3\epsilon^2 + \epsilon.
\]

By the same argument, we also have

\[
\sigma(d_2, \gamma) - \sigma(d_1, \gamma) > -\frac{d_2 - d_1}{d_0}(M + 2) - 3\epsilon^2 - \epsilon.
\]

As a consequence, we conclude that \( \sigma(\cdot, \gamma) \) is uniformly continuous. \( \square \)

2.4. **Existence lemma for minimizing Problem (10).** The next lemma is applied to show the existence of a minimizer of (10). We mention that the proof is based on [2].

**Lemma 2.7.** Let \( a(x) \) be a continuous function satisfying \( a(x) \to 0 \) as \( x \to \infty \). \( I(u) \) is defined by

\[
I(u) = \int_{\mathbb{R}_+} \left[ \frac{1}{2} |u'(x)|^2 + a(x)|u(x)|^2 \right] dx. \tag{26}
\]

If there exists a function \( u_0 \in H_0^1(\mathbb{R}_+) \) such that \( I(u_0) < 0 \), then there exists a minimizer of the minimizing problem

\[
\sigma = \inf \{ I(u) : u \in H_0^1(\mathbb{R}_+), \|u\|_{L^2(\mathbb{R}_+)} = 1 \}. \tag{27}
\]

Moreover, the minimizer is nonnegative on \( \mathbb{R}_+ \).
Proof. For simplicity, we write \( Y = \{ u \in H^1_0(\mathbb{R}_+) : \| u \|_{L^2(\mathbb{R}_+)} = 1 \} \). Let \( \{ u_n \}_{n \in \mathbb{N}} \) be a minimizing sequence in \( Y \). From the definition of \( I(u) \), we may assume that \( u_n \geq 0 \) on \( \mathbb{R}_+ \). Since

\[
\int_{\mathbb{R}_+} \frac{1}{2} |u'(x)|^2 \, dx \leq I(u_n) + \| a \|_{L^\infty(\mathbb{R}_+)} ,
\]

we can see that \( \| u' \|_{L^2(\mathbb{R}_+)} \) is uniformly bounded. Thus there exists a function \( u \in H^1_0(\mathbb{R}_+) \) such that \( u_n \rightharpoonup u \) weakly in \( H^1_0(\mathbb{R}_+) \) and \( u_n \rightarrow u \) strongly in \( C_{\text{loc}}(\mathbb{R}_+) \).

From the lower semi-continuity of \( L^2 \) space, \( u \) satisfies

\[
\int_{\mathbb{R}_+} |u'(x)|^2 \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}_+} |u'_n(x)|^2 \, dx .
\]

We then prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}_+} a(x) |u_n(x)|^2 \, dx = \int_{\mathbb{R}_+} a(x) |u(x)|^2 \, dx.
\] (28)

By direct calculation,

\[
\left| \int_{\mathbb{R}_+} a(x) |u_n(x)|^2 \, dx - \int_{\mathbb{R}_+} a(x) |u(x)|^2 \, dx \right|
\leq \int_{\{ x \leq r \}} a(x) \left( |u_n| + |u| \right) \left( |u_n| - |u| \right) \, dx + \int_{\{ x > r \}} a(x) \left( |u_n|^2 + |u|^2 \right) \, dx
\leq 2 \| a \|_{L^\infty(\mathbb{R}_+)} \| u_n - u \|_{L^2(0, r)} + 2 \sup_{x > r} |a(x)| .
\]

Fix \( r > 0 \) large enough that \( \sup_{x > r} |a(x)| \) is sufficiently small. Since the embedding \( H^1_0(\mathbb{R}) \subset L^2(0, r) \) is compact, \( \{ u_n \}_n \) has a subsequence which converges to \( u \) in \( L^2(0, r) \). Thus we obtain (28). As a consequence, we can see that

\[
I(u) \leq \liminf_{n \to \infty} \int_{\mathbb{R}} |u'_n(x)|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}_+} a(x) |u_n(x)|^2 \, dx \leq \sigma .
\]

It is easy to see that \( u \neq 0 \) from the assumption \( \sigma < 0 \). Moreover, if \( \| u \|_{L^2(\mathbb{R}_+)} < 1 \), then \( \bar{u} = u/\| u \|_{L^2(\mathbb{R}_+)} \in Y \) satisfies that

\[
I(\bar{u}) \leq \frac{\sigma}{\| u \|_{L^2(\mathbb{R}_+)}^2} < \sigma .
\]

It clearly contradicts the definition of \( \sigma \). Therefore, \( \| u \|_{L^2(\mathbb{R}_+)} \) should be equal to 1 and we complete the proof. \( \square \)

3. Proof of Theorem 1.1. In this section, we present the proof of Theorem 1.1. First, we prove Proposition 1 in Section 3.1. Next, we show Proposition 2 in Section 3.2.

3.1. Construction of the solution to (2). We give a method to construct a solution with the sub-supersolution method in this subsection. Namely our goal in this subsection is to show Proposition 1. We mention that the framework to construct a solution is based on [3, 5].

We first introduce functions \( \hat{u}, \hat{v} \in C^\infty(\mathbb{R}_+) \) as follows:

\[
\hat{v}(x) = \begin{cases} 
0, & 0 \leq x < 1/2, \\
\gamma, & x > 1.
\end{cases}
\]
Lemma 3.1. Let \( \tilde{v} \) be a function satisfying that \( \tilde{v}(0) = 0, \ \tilde{v}(\infty) = a_\gamma / \gamma \) and \( \tilde{v} - \tilde{v} \in H^1_0(\mathbb{R}_+) \cap H^3(\mathbb{R}_+) \). Then there exists a solution \( u \in C^2(\mathbb{R}_+) \) to the equation

\[
\begin{aligned}
-du''(x) &= \mu(x)f(u(x)) - \tilde{v}(x), \quad x \in \mathbb{R}_+, \\
W(x) &\leq u(x) \leq T(x), \quad x \in \mathbb{R}_+, \\
u(0) &= 0, \quad u(\infty) = a_\gamma.
\end{aligned}
\] (29)

Moreover, \( u - \hat{u} \in H^2(\mathbb{R}_+) \cap H^0(\mathbb{R}_+) \) holds.

Proof. Let \( \{ R_i \}_{i \in \mathbb{N}} \) be a monotone increasing sequence such that \( R_i \to \infty \) as \( i \to \infty \). In addition, for any fixed \( i \in \mathbb{N} \), we define \( \{ r_{ij} \}_{j \in \mathbb{N}} \) that \( r_{ij} > R_i \) and \( r_{ij} \to \infty \) as \( j \to \infty \). Then we can construct a solution \( \{ u_{ij} \}_{(i,j) \in \mathbb{N}^2} \) to the equation

\[
\begin{aligned}
-du_{ij}''(x) &= \mu(x)f(u_{ij}(x)) - \tilde{v}(x), \quad x \in (0, r_{ij}), \\
W(x) &\leq u_{ij}(x) \leq T(x), \quad x \in [0, r_{ij}], \\
u_{ij}(0) &= 0, \quad u_{ij}(r_{ij}) = a_\gamma
\end{aligned}
\] (30)

with a standard argument in the sub-supersolution method (e.g. [7]).

Now we prove that for fixed \( i \in \mathbb{N} \), there exists a subsequence of \( \{ u_{ij} \}_{j \in \mathbb{N}} \) which converges strongly in \( C^2(\tilde{I}_i) \), where \( \tilde{I}_i = (0, R_i) \). It suffices to show that \( \{ \| u_{ij} \|_{C^2(\tilde{I}_i)} \}_{j \in \mathbb{N}} \) is uniformly bounded and that \( u_{ij}'' \) is equicontinuous respect to \( j \in \mathbb{N} \). First, we readily see that \( \| u_{ij} \|_{L^\infty(\tilde{I}_i)} \) is uniformly bounded since \( W(x) \leq u_{ij}(x) \leq T(x) \) for all \( x \in \tilde{I}_i \). Then it follows that \( \| u_{ij}'' \|_{L^\infty(\tilde{I}_i)} \) is also uniformly bounded from (30). We can also derive the boundedness of \( \{ \| u_{ij}'' \|_{L^\infty(\tilde{I}_i)} \}_{j \in \mathbb{N}} \) with the boundedness of \( \{ \| u_{ij} \|_{L^\infty(\tilde{I}_i)} \}_{j \in \mathbb{N}} \) and \( \{ \| u_{ij}'' \|_{L^\infty(\tilde{I}_i)} \}_{j \in \mathbb{N}} \). Thus we show that \( \{ \| u_{ij} \|_{C^2(\tilde{I}_i)} \}_{j \in \mathbb{N}} \) is uniformly bounded. It is easy to see that \( \{ u_{ij}'' \}_{j \in \mathbb{N}} \) is equicontinuous since \( u_{ij} \) is a solution to (30) and

\[
g(x, u) = \mu(x)f(u) - \tilde{v}(x)
\]

is uniformly continuous over \( \mathbb{R}_+ \times [0, 1] \). As a consequence, there exists a function \( U_i \in C^2(\tilde{I}_i) \) and subsequence \( \{ u_{ij(k)} \}_{k \in \mathbb{N}} \subset \{ u_{ij} \}_{j \in \mathbb{N}} \) such that

\[
u_{ij(k)} \to U_i \quad \text{in} \quad C^2(\tilde{I}_i) \quad \text{as} \quad k \to \infty.
\]

We rewrite the subsequence by \( \{ u_{ij} \}_{j \in \mathbb{N}} \). We can also construct a subsequence of \( \{ u_{i+1,j} \}_{j \in \mathbb{N}} \) that converges to a function \( U_{i+1} \in C^2(\tilde{I}_{i+1}) \) strongly in \( C^2(\tilde{I}_{i+1}) \). We note that \( U_{i+1}|_{\tilde{I}_i} = U_i \). Thus the diagonal sequence \( \{ u_{ii} \}_{i \in \mathbb{N}} \) converges to a function \( u \in C^2(\mathbb{R}_+) \) strongly in \( C^0_{loc}(\mathbb{R}_+) \). It is clear that \( u \) satisfies

\[
\begin{aligned}
-du''(x) &= \mu(x)f(u(x)) - \tilde{v}(x), \quad x \in \mathbb{R}_+, \\
W(x) &\leq u(x) \leq T(x), \quad x \in \mathbb{R}_+, \\
u(0) &= 0.
\end{aligned}
\]

It suffices to show that \( u(\infty) = a_\gamma \) to see that \( u \) is the solution to (29). We prove by contradiction. Namely we assume that there exists a positive number \( \delta \) and a sequence \( \{ \tilde{x}_i \}_{i \in \mathbb{N}} \) such that \( \tilde{x}_i \to \infty \) \( (i \to \infty) \) and

\[
|u(\tilde{x}_i) - a_\gamma| > \delta.
\]

However, there does not exist a positive numbers \( \delta_* \) and \( \tilde{x}_* \) such that

\[
|u(x) - a_\gamma| > \delta_* \quad \text{for all} \quad x > \tilde{x}_*.
\]
In fact, if there exist $\delta_*$ and $\bar{x}_*$ satisfying the above inequality, then there exists $\delta_* > 0$ such that
\[ |u''(x)| = |\mu(x)f(u(x)) - \bar{v}(x)| > \delta_* \quad \text{for all } x \in (\bar{x}_*, \infty), \]
but it contradicts $W(x) \leq u(x) \leq T(x)$. Thus there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ such that $x_i \to \infty \ (i \to \infty)$ and
\[ |u(x_i) - a_\gamma| = \delta \quad \text{for all } i \in \mathbb{N}. \tag{31} \]

We note that for large $x > 0$, the following inequalities hold:
\[ u(x) > u_\gamma, \]
\[ \left| \bar{v}(x) - \frac{a_\gamma}{\gamma} \right| < \delta \mu_0 \cdot \frac{3u_\gamma^2 - 1}{2}. \]

We now assume that $u(x_i) = a_\gamma + \delta$ holds for a fixed $i \in \mathbb{N}$. Then we can see that
\[ \mu(x)f(u(x_i)) - \bar{v}(x_i) = \mu(x) \left( f(a_\gamma) + f(u(x_i)) - f(a_\gamma) \right) - \left( \frac{a_\gamma}{\gamma} + \bar{v}(x_i) - \frac{a_\gamma}{\gamma} \right) \]
\[ = \mu(x) \left( 1 - u(x_i)^2 - u(x_i)a_\gamma - a_\gamma^2 \right) \delta - \bar{v}(x_i) + \frac{a_\gamma}{\gamma}, \]
\[ < -\mu_0(3u_\gamma^2 - 1)\delta + \delta \mu_0 \cdot \frac{3u_\gamma^2 - 1}{2} < 0. \]

Remark that the above inequality is true for any large $x > 0$ such that $u(x) \geq a_\gamma + \delta$. Hence we can see that there exists $y_{i-1} \in [x_{i-1}, x_i)$ such that
\[ \begin{cases} 
\mu(x)f(u(x)) - \bar{v}(x) < 0, & x \in (y_{i-1}, x_i), \\
u(y_{i-1}) \leq a_\gamma + \delta 
\end{cases} \]
with attention to (31). This implies that
\[ \begin{cases} 
\mu(x) > 0, & x \in (y_{i-1}, x_i), \\
u(y_{i-1}) \leq a_\gamma + \delta, \\
u(x_i) = a_\gamma + \delta 
\end{cases} \]
and hence $u$ does not attain a maximum on $(y_{i-1}, x_i)$ from the maximum principle, that is, $u < a_\gamma + \delta$ on $(y_{i-1}, x_i)$. Thus $u(x_i) > 0$ from Hopf’s lemma. However, if $u(x)$ is monotone increasing on $[x_i, \infty)$ since $u''(x) > 0$ as long as $u(x) \geq a_\gamma + \delta$. It contradicts the existence of the sequence $\{x_i\}_{i \in \mathbb{N}}$. Similarly we derive contradiction in the case $u(x_i) = a_\gamma - \delta$. As a consequence, we conclude $u(x) \to a_\gamma$ as $x \to \infty$.

Finally, we prove $u - \bar{u} \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R})$. It suffices to show $\phi = u - a_\gamma \in H^2(\mathbb{R}_+)$ since $u(0) = 0$. We fix a constant $l > 0$ large enough that
\[ 1 - u(x)^2 - u(x)a_\gamma - a_\gamma^2 \leq -\frac{1}{2}(3a_\gamma^2 - 1) \quad \text{for all } x \in [l, \infty) \]
and set $L > l$. Moreover, define $\psi = \bar{v} - a_\gamma/\gamma$. Then we have
\[ \int_l^L -d\phi'' \phi \, dx = \int_l^L \mu(x) \left( f(u) - \frac{a_\gamma}{\gamma} - \psi \right) \phi \, dx - \int_l^L \left( 1 - \mu(x) \right) \frac{a_\gamma}{\gamma} \phi \, dx. \]
The left hand side is calculated as
\[ \int_I -d\phi'' \phi \, dx = [-d\phi' \phi]_I^L + d \int_I |\phi'|^2 \, dx. \]

The first term of the right hand side is computed as
\[ \int_I \mu(x) \left( f(u) - \frac{a_\gamma}{\gamma} - \psi \right) \phi \, dx = \int_I \left[ \mu(x)(1 - u^2 - u\gamma - a_\gamma^2) |\phi|^2 - \psi\phi \right] \, dx \]
\[ \leq -\mu_0 \int_I \frac{1}{2} (3a_\gamma^2 - 1) |\phi|^2 \, dx + \int_I |\psi| \phi \, dx \]
\[ \leq -\frac{\mu_0}{2} (3a_\gamma^2 - 1 - \delta) \int_I |\phi|^2 \, dx + \frac{1}{2\delta} \int_{\mathbb{R}_+} |\psi|^2 \, dx, \]

where \( \delta \) is a small positive number. Therefore, we obtain
\[ d \int_I |\phi'| \, dx + \frac{\mu_0}{2} (3a_\gamma^2 - 1 - \delta) \int_I |\phi|^2 \, dx \]
\[ \leq [d\phi' \phi]_I^L + \frac{1}{2\delta} \|\psi\|^2_{L^2(\mathbb{R}_+)} + \frac{a_\gamma}{\gamma} \|1 - \mu\|_{L^1(\mathbb{R}_+)}. \]

Noting that \( \phi \) is smooth and uniformly bounded in \( \mathbb{R} \), we can see that there exists a constant \( C > 0 \) such that \( \|\phi\|_{H^1(\mathbb{R}_+)} < C \). Moreover, it is easy to see \( \phi'' \in L^2(\mathbb{R}_+) \) since
\[ \mu(x)f(u) - v = \mu(x)\phi(1 - u^2 - u\gamma - a_\gamma^2) - \frac{a_\gamma}{\gamma} (1 - \mu(x)) + \psi \quad \text{on} \quad (l, \infty). \]

Therefore we complete the proof.

Lemma 3.1 is on the method of construction of \( u \). The next lemma is on the method of construction of \( v \).

**Lemma 3.2.** Let \( u \) be a function such that \( u - \tilde{u} \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \) and \( W(x) \leq u(x) \leq T(x) \) for all \( x \in \mathbb{R}_+ \). Then there exists a unique solution \( v \) to the equation

\[
\begin{cases}
- v''(x) + \gamma v(x) = u(x), & x \in \mathbb{R}_+, \\
0 \leq v(x) \leq \frac{1}{\gamma}, & x \in \mathbb{R}_+, \\
v(0) = 0, & v(\infty) = \frac{a_\gamma}{\gamma}, \\
v - \tilde{u} \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+).
\end{cases}
\]

**Proof.** Let \( \psi \) be a solution to the following equation:
\[
\begin{cases}
- \psi''(x) + \gamma \psi(x) = u(x) - \tilde{u}(x), & x \in \mathbb{R}_+, \\
\psi \in H^1_0(\mathbb{R}_+) \cap H^2(\mathbb{R}_+).
\end{cases}
\]

With a standard argument, we can see that the solution is unique. Moreover, we can also see that
\[ 0 \leq \psi(x) \leq \frac{1}{\gamma} (u(x) - \tilde{u}(x)) \quad \text{for all} \quad x \in \mathbb{R}_+ \]
from the maximum principle. Thus \( v = \hat{v} + \psi \) is the desired solution.

With these lemmas, we now prove Proposition 1.
Proof of Proposition 1. Define \( v_0 = \hat{v} \) and \((u_j, v_j) \ (j = 1, 2, \ldots)\) as follows:

\[
\begin{align*}
- du_j''(x) &= \mu(x)f(u_j) - v_j(x), \quad x \in \mathbb{R}_+,
- v_j''(x) + \gamma v_j(x) &= u_j(x), \quad x \in \mathbb{R}_+,
W(x) &\leq u_j(x) \leq T(x), \quad x \in \mathbb{R}_+,
0 \leq v_j(x) \leq 1/\gamma, \quad x \in \mathbb{R}_+,
(u_j, v_j)(0) &= (0, 0), \quad (u_j, v_j)(\infty) = (a_\gamma, a_\gamma/\gamma),
(u_j - \hat{u}, v_j - \hat{v}) &\in (H^2(\mathbb{R}_+) \times H^1_0(\mathbb{R}_+))^2.
\end{align*}
\]

The existence of \((u_j, v_j)\) is guaranteed by Lemmas 3.1 and 3.2. We define \((\phi_j, \psi_j) = (u_j - \hat{u}, v_j - \hat{v})\) and we now show that \(\{(\phi_j, \psi_j)\}_{j \in \mathbb{N}}\) is a Cauchy sequence in \(H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+))^2\).

From the definition,

\[
\begin{align*}
- du_{j+1}'(x) &= \mu(x)f(u_{j+1}(x)) - v_j(x), \quad x \in \mathbb{R}_+,
- du_j''(x) &= \mu(x)f(u_j(x)) - v_{j-1}(x), \quad x \in \mathbb{R}_+.
\end{align*}
\]

Then we have

\[
-d\xi_j''(x) = \mu(x)|\xi_j(x)|^2(1 - u_{j+1}^2(x) - u_{j+1}u_j(x) - u_j^2(x)) - \eta_{j-1}(x), \quad (32)
\]

where \((\xi_j, \eta_j) = (\phi_{j+1} - \phi_j, \psi_{j+1} - \psi_j)\). Integrate (32) over \(\mathbb{R}_+\):

\[
\int_{\mathbb{R}_+} \left[ d|\xi_j|^2 - \mu(x)|\xi_j|^2 (1 - u_{j+1}^2 - u_{j+1}u_j - u_j^2) \right] dx = -\int_{\mathbb{R}_+} \xi_j\eta_{j-1} dx.
\]

It is easy to check that the left hand side and the right hand side are respectively evaluated as follows:

\[
\text{l.h.s.} \geq \int_{\mathbb{R}_+} \left[ d|\xi_j|^2 - \mu(x)|\xi_j|^2 (1 - 3W_{d, \gamma}^2) \right] dx \geq \sigma(d, \gamma) \|\xi_j\|_{L^2(\mathbb{R}_+)}^2,
\]

\[
\text{r.h.s.} \leq \frac{1}{\gamma} \|\xi_j\|_{L^2(\mathbb{R}_+)} \|\xi_j\|_{L^2(\mathbb{R}_+)}.
\]

Thus we obtain

\[
\|\xi_j\|_{L^2(\mathbb{R}_+)} \leq \frac{1}{\sigma(d, \gamma) \gamma} \|\xi_{j-1}\|_{L^2(\mathbb{R}_+)} = \kappa^{j-1} \|\xi_1\|_{L^2(\mathbb{R}_+)},
\]

where \(\kappa = 1/\sigma(d, \gamma) > 1\). We readily see

\[
\|\eta_j\|_{L^2(\mathbb{R}_+)} \leq \frac{1}{\gamma} \|\xi_j\|_{L^2(\mathbb{R}_+)} \leq \frac{\kappa^{j-1}}{\gamma} \|\xi_1\|_{L^2(\mathbb{R}_+)}.
\]

Moreover, from (32) we have

\[
d|\xi_j''|_{L^2} \leq 2 \|\xi_j\|_{L^2(\mathbb{R}_+)} + \|\eta_{j-1}\|_{L^2(\mathbb{R}_+)} \leq C_1 \kappa^{j-2} \|\xi_1\|_{L^2(\mathbb{R}_+)}.\]

Similarly we can see

\[
\|\eta_j''\|_{L^2(\mathbb{R}_+)} \leq C_2 \kappa^{j-1} \|\xi_1\|_{L^2(\mathbb{R}_+)},
\]

from the equation

\[
-\eta_j'' = \xi_j - \gamma \eta_j.
\]

It is easy to check

\[
\|\xi_j''\|_{L^2(\mathbb{R}_+)} \leq \|\xi_j''\|_{L^2(\mathbb{R}_+)} \|\xi_j\|_{L^2(\mathbb{R}_+)} \leq C_3 \kappa^{j-1} \|\xi_1\|_{L^2(\mathbb{R}_+)}^2,
\]

\[
\|\eta_j''\|_{L^2(\mathbb{R}_+)} \leq C_4 \kappa^{2(j-1)} \|\xi_1\|_{L^2(\mathbb{R}_+)}^2.
\]
As a consequence, we can see that \((\phi_j, \psi_j)\) is a Cauchy sequence in \((H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+))^2\). Therefore we set
\[
(U, V) = \lim_{j \to \infty} (\hat{u} + \phi_j, \hat{v} + \psi_j),
\]
and \((U, V)\) is a solution to (2). \(\square\)

3.2. Evaluation of \(\sigma(d, \gamma)\) from below. In this subsection, we give a proof of Proposition 2 and Theorem 1.1. To prove Proposition 2, it is important to show the positivity of \(\sigma(d, \gamma)\) and the uniform estimate of \(\sigma(d, \gamma)\) respect to small \(d > 0\) from below. For reader’s convenience, we post (10) again:
\[
\sigma(d, \gamma) = \inf \left\{ \frac{J_d(\psi)}{\int_{\mathbb{R}_+} |\xi(x)|^2 \, dx} : \xi \in H^1_0(\mathbb{R}_+), \xi \not\equiv 0 \right\},
\]
where \(J_d(\psi)\) is defined as (11). Throughout this section, \(W_{d, \gamma}\) denotes the solution to (5). Moreover, we remark that from Lemma 2.3, it suffices for Proposition 2 to show the following proposition:

**Proposition 3.** Let \(\gamma\) be a constant such that \(\gamma > 3\sqrt{6}/(2\mu_0)\). Then there exists a positive constant \(\sigma_\gamma\) such that
\[
\sigma(d, \gamma) > \sigma_\gamma \quad \text{for all } d \in (0, l_0/x_0).
\]

The next lemma is the key lemma to show Proposition 2. Before we state the lemma, recall \(x_0\) is a constant such that \(\hat{W}_{0, \gamma}(x_0) = 1/\sqrt{3}\), where \(\gamma = 3\sqrt{6}/2 + \delta\), \(\delta\) is a small positive constant and \(W_{0, \gamma_0}(x) = W_{d, \gamma_0}(\sqrt{d} x)\). We remark that \(W_{d, \gamma}(x) > 1/\sqrt{3}\) holds for all \(x > \sqrt{d} x_0\) and \(\gamma \geq \gamma_0\) since \(W_{d, \gamma}(x)\) is a nondecreasing function respect to \(x\) and also an increasing function respect to \(\gamma\) (see Lemma 2.3).

**Proposition 4.** For \(\gamma > 3\sqrt{6}/2\), \(x_\gamma\) is defined as the point in \(\mathbb{R}_+\) such that \(\hat{W}_{0, \gamma}(x_\gamma) = 1/\sqrt{3}\), where \(\hat{W}_{d, \gamma}(x) = W_{d, \gamma}(\sqrt{d} x)\). If \(d < (l_0/x_0)^2\), then \(\sigma(d, \gamma)\) is positive.

**Proof.** We prove by contradiction. Namely we assume \(\sigma(d, \gamma) \leq 0\). Define
\[
\tilde{V}(x) = \mu(x)(3W_{d, \gamma}(x)^2 - 1) - (3\gamma^2 - 1),
\]
\[
\tilde{J}(\xi) = \int_{\mathbb{R}_+} \left[ \frac{d}{2} |\xi'(x)|^2 + \tilde{V}(x) |\xi(x)|^2 \right] \, dx.
\]
We note that \(\tilde{V}(x) \to 0\) as \(x \to \infty\). Thus from Lemma 2.7, we find that the following minimizing problem
\[
\tilde{\sigma}(d, \gamma) = \inf \{ \tilde{J}(\xi) : \xi \in H^1_0(\mathbb{R}_+), ||\xi||_{L^2(\mathbb{R}_+)} = 1 \}
\]
has a minimizer \(\xi_0 \in H^1_0(\mathbb{R}_+\) since \(\tilde{\sigma}(d, \gamma) = \sigma(d, \gamma) - (3\gamma^2 - 1) < 0\). Moreover, \(\xi_0\) is also a minimizer of (10) since
\[
J(\xi) = \tilde{J}(\xi) + \int_{\mathbb{R}_+} (3\gamma^2 - 1)|\xi|^2 \, dx,
\]
where \(J(\xi)\) is defined as (11). It is clear that the minimizer \(\xi_0\) is a solution to
\[-d\xi''_{0}(x) + \mu(x)(3W_{d, \gamma}(x)^2 - 1)\xi_0(x) = \sigma\xi_0(x).
\]
Multiplying the above equation by \(W_{d, \gamma}'\) and integrating over \(\mathbb{R}_+\), we have
\[- \int_{\mathbb{R}_+} d\xi''_{0}W_{d, \gamma} \, dx = \int_{\mathbb{R}_+} \left[ -\mu(x)(3W_{d, \gamma}^2 - 1)W_{d, \gamma}'\xi_0 + \sigma W_{d, \gamma}'\xi_0 \right] \, dx.
\]
We note that
\[
(1.\text{h.s.}) = -d[\xi' W_{d,\gamma}]_0^\infty + d \int_{\mathbb{R}_+} \xi'' W_{d,\gamma} dx \\
= d\xi'(0) W_{d,\gamma}(0) + d \int_{\mathbb{R}_+} \xi'' W_{d,\gamma} dx,
\]
and thus we obtain
\[
d\xi'(0) W_{d,\gamma}(0) + d \int_{\mathbb{R}_+} \xi'' W_{d,\gamma} dx = \int_{\mathbb{R}_+} [\mu(x) (3 W_{d,\gamma}^2 - 1) W_{d,\gamma} \xi_0 + \sigma W'_{d,\gamma} \xi_0] dx.
\]

On the other hand, by differentiating (5), we obtain
\[
-d W''_{d,\gamma} = \mu_0 (1 - 3 W_{d,\gamma}^2) W'_{d,\gamma}.
\]
Moreover, multiplying the above equation by $\xi_0$ and integrating over $\mathbb{R}_+$, we can see that
\[
d \int_{\mathbb{R}_+} W''_{d,\gamma} \xi_0 dx = \int_0^\infty \mu_0 (1 - 3 W_{d,\gamma}^2) W'_{d,\gamma} \xi_0 dx.
\]
Subtracting (34) from (33), we obtain
\[
d\xi'(0) W_{d,\gamma}(0) = \sigma \int_{\mathbb{R}_+} \xi_0 W'_{d,\gamma} dx - \int_{\mathbb{R}_+} (\mu(x) - \mu_0) (3 W_{d,\gamma}^2 - 1) W'_{d,\gamma} \xi_0 dx,
\]
that is,
\[
\sigma \int_{\mathbb{R}_+} \xi_0 W'_{d,\gamma} dx = d\xi'(0) W_{d,\gamma}(0) + \int_{\mathbb{R}_+} (\mu(x) - \mu_0) (3 W_{d,\gamma}^2 - 1) W'_{d,\gamma} \xi_0 dx.
\]

From the assumption, the left hand side is non-positive. However, since $3 W_{d,\gamma}(x) - 1 > 0$ holds for all $x > l_0$ from the assumption $d < (l_0/x_\gamma)^2$, the right hand side is positive. Therefore we conclude that $\sigma(d, \gamma) > 0$. \hfill \Box

**Remark 1.** With attention to that there exists $x_\infty \in (x_0, \infty)$ such that $x_\gamma \to x_\infty$ as $\gamma \to \infty$, we can see that for any $d < (l_0/x_\gamma)^2$ there exists $\gamma_2 = \gamma_2(d)$ such that $d < (l_0/x_\gamma)^2$. Hence combining Lemma 2.3 and Proposition 4, we find that for given $d \in (l_0/x_\gamma)^2$, there exists $\gamma_3 > \gamma_2$ such that $\gamma_3 > \gamma_2$ for all $\gamma > \gamma_3$. However, $\gamma_3$ obviously depends on $d$.

Now we shall prove Proposition 3. We divide Proposition 3 into the following two statements:

**Proposition 5.** Let $\gamma$ be a constant such that $\gamma > 3 \sqrt{6}/(2 \mu_0)$. Then there exists a constant $d_0 = d_0(\gamma) \in (0, l_0/x_\gamma)$ and $\sigma_{\gamma}^{(1)} > 0$ such that
\[
\sigma(d, \gamma) > \sigma_{\gamma}^{(1)} \quad \text{for all } d \in (0, d_0].
\]

**Proposition 6.** Let $\gamma$ be a constant such that $\gamma > 3 \sqrt{6}/(2 \mu_0)$. Then there exists a constant $\sigma_{\gamma}^{(2)} > 0$ such that
\[
\sigma(d, \gamma) > \sigma_{\gamma}^{(2)} \quad \text{for all } d \in [d_0, l_0/x_\gamma],
\]
where $d_0$ is defined in Proposition 5.
Proof of Proposition 5. We prove by contradiction. Namely we assume that there exists a sequence \( \{d_n\}_{n \in \mathbb{N}} \) such that \( d_n \to 0 \) and \( \sigma_n \to 0 \) as \( n \to \infty \), where \( \sigma_n = \sigma(d_n, \gamma) \). With a similar argument in Proposition 4, we can check that the minimizing problem (10) with \( d = d_n \) has a minimizer \( \xi_n \in H^1_0(\mathbb{R}_+) \). With a standard argument, we find that

\[-d_n\xi_n''(x) + V_n(x)\xi_n(x) = \sigma_n\xi_n(x),\]

where \( V_n(x) = \mu(x)(3W_{d_n, \gamma}(x)^2 - 1) \). Set \( \tilde{\xi}_n \) be \( \tilde{\xi}_n(x) = d_n^{1/4}\xi_n(\sqrt{d_n}x) \). We note that \( \|\xi_n\|_{L^2(\mathbb{R}_+)} = 1 \). Then it is easy to see

\[-\tilde{\xi}_n''(x) + V_n(\sqrt{d_n}x)\tilde{\xi}_n(x) = \sigma_n\tilde{\xi}_n(x).\]

Multiplying the equation by \( \tilde{\xi}_n \) and integrating over \( \mathbb{R}_+ \), we obtain the following equation:

\[\int_{\mathbb{R}_+} \left| \tilde{\xi}_n' \right|^2 \, dx = \int_{\mathbb{R}_+} (\sigma_n - V_n(\sqrt{d_n}x)) \left| \tilde{\xi}_n \right|^2 \, dx.\]

We readily find that \( \{\sup_n |\sigma_n - V_n(\sqrt{d_n}x)|\} \) is uniformly bounded from the assumption on \( \sigma_n \). It follows that \( \{\|\xi_n\|_{H^1_0(\mathbb{R}_+)}\} \) as also uniformly bounded. Thus there exists \( \tilde{\xi}_0 \in H^1_0(\mathbb{R}_+) \) such that \( \tilde{\xi}_n \to \tilde{\xi}_0 \) weakly in \( H^1_0(\mathbb{R}_+) \) as \( n \to \infty \). Moreover, with attention to \( W_{d_n, \gamma}(\sqrt{d_n}x) = W_{0, \gamma}(x) \), we compute

\[V_n(\sqrt{d_n}x) - \sigma_n = \mu(\sqrt{d_n}x)(3W_{0, \gamma}(x)^2 - 1) - \sigma_n.\]

Hence we set \( r > 0 \) large enough, and then we have

\[V_n(\sqrt{d_n}x) - \sigma_n \geq \frac{\mu_0}{2} (3u_0^2 - 1)\]

for any large \( n \in \mathbb{N} \) and any \( x > r \). Now we set \( K = \mu_0(3u_0^2 - 1)/2 \) and define \( \phi \) as follows:

\[
\begin{cases}
    - \phi''(x) + K\phi(x) = 0, & x \in (r, \infty), \\
    \phi(r) > \sup_{n \in \mathbb{N}} \|\xi_n\|_{L^\infty(\mathbb{R}_+)}. 
\end{cases}
\]

We find that

\[\tilde{\xi}_n(x) \leq \phi(x) \leq Ce^{-\sqrt{K}(x-r)} \quad \text{for all } x > r.\]

Hence taking sufficiently large \( R > r \) for any small \( \delta > 0 \), we obtain

\[\left\|\tilde{\xi}_n\right\|_{L^2(0, R)} > 1 - \delta \quad \text{for all } n \in \mathbb{N}.\]

The compactness of the embedding \( H^1_0(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \) leads that

\[\left\|\tilde{\xi}_0\right\|_{L^2(\mathbb{R}_+)} \geq \left\|\tilde{\xi}_0\right\|_{L^2(0, R)} > 1 - \delta.\]

This implies that \( \left\|\tilde{\xi}_0\right\|_{L^2(\mathbb{R}_+)} = 1 \). In addition, \( V_n(\sqrt{d_n}x) \) tends to \( \mu_0(3W_{0, \gamma}(x)^2 - 1) \) as \( n \to \infty \) for any \( x \in \mathbb{R}_+ \). As a consequence, we can derive that

\[-\tilde{\xi}_0''(x) + \mu_0 (3W_{0, \gamma}(x)^2 - 1) \tilde{\xi}_0(x) = 0.\]

We deduce from the equation that \( \tilde{\xi}_0'(0)\tilde{W}_{0, \gamma}(0) = 0 \) similarly as (35). It follows \( \tilde{\xi}_0'(0) = 0 \), but this means \( \tilde{\xi}_0 = 0 \). Clearly it is a contradiction. Therefore we conclude that there exists a constant \( \sigma_0 > 0 \) and \( d_0 = d_0(\gamma) \in (0, (\mu_0/\mu)^2) \) such that \( \sigma(d, \gamma) > \sigma_0 \) for all \( d \in (0, d_0) \). \( \square \)
Proof of Proposition 6. The statement follows from the uniform continuity of \(\sigma(\cdot, \gamma)\) on \([d_0, l_0/x_0]\) which has been already proved in Lemma 2.6. Indeed, there exists a point \(d_0 \in [d_0, l_0/x_0]\) such that

\[
\sigma(d_0, \gamma) = \min_d \sigma(d, \gamma) = \sigma^{(2)}_{\gamma}
\]

from the continuity of \(\sigma(\cdot, \gamma)\). The positivity of \(\sigma^{(2)}_{\gamma}\) is followed from Proposition 4. \(\square\)

Proposition 3 follows from Propositions 5 and 6. Moreover, Proposition 1 follows from Proposition 3 as mentioned in the beginning in this subsection. Combining Propositions 1 and 2, we easily prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that \(\gamma \sigma(d, \gamma) > 1\) holds for any \(d \in (0, (l_0/x_0)^2)\) from the definition of \(\sigma_0\). Thus it follows that there exists a solution to (2) for any \(d \in (0, (l_0/x_0)^2)\) from Proposition 1. Moreover, it is easy to see that the solution \((u, \gamma, v, \gamma)\) satisfies (13) and (14) from Proposition 1. \(\square\)

4. Proof of Theorem 1.2. In this section, we give a proof of Theorem 1.2. Basically we can prove it as in the proof of Theorem 1.1, and thus we state only differences between Theorems 1.1 and 1.2. First, we state the key statements to see Theorem 1.2.

Proposition 7. Let \(d > 0\), \(\gamma > 3\sqrt{6}/2\) and \(\mu\) be a function satisfying \((\mu 1) - (\mu 4)\). If the following inequality

\[
\gamma \sigma(d, \gamma) > 1
\]

holds, then there exists a solution to (16). Moreover, the solution \((u, v)\) satisfies the following inequalities:

\[
W(x) \leq u(x) \leq T(x) \quad \text{for all } x \in \mathbb{R}_+,
\]

\[
0 \leq v(x) \leq \frac{1}{\gamma} \quad \text{for all } x \in \mathbb{R}_+.
\]

Proposition 8. Let \(\delta\) be a small positive constant and \(\mu\) be a function satisfying \((\mu 1) - (\mu 4)\). Then there exists a positive constant \(\bar{\sigma}_\delta\) such that

\[
\sigma(d, \gamma) > \bar{\sigma}_\delta \quad \text{for all } (d, \gamma) \in (0, d_1) \times \left(\frac{3\sqrt{6}}{2} + \delta, \infty\right).
\]

Proposition 7 and Proposition 8 corresponds to Proposition 1 and Proposition 2 respectively. We shall prove these propositions with the similar argument in Section 2. Now we collect lemmas to show the propositions.

Lemma 4.1. If \(\gamma > 3\sqrt{6}/2\), then there exists minimizers \(\bar{W}\) and \(\hat{W}\) to the minimizing problems (36) and (37). \(\bar{W}\) and \(\hat{W}\) satisfy (17) and (19) respectively.

\[
\bar{E}_2(\bar{W}) = \inf \left\{ \bar{E}_2(u) : u \in X_3 \right\}, \quad \bar{W} \in X_3, \quad (36)
\]

\[
\hat{E}_2(\hat{W}) = \inf \left\{ \hat{E}_2(u) : u \in X_3 \right\}, \quad \hat{W} \in X_3. \quad (37)
\]

Here \(\bar{E}_2, \hat{E}_2\) and \(X_3\) are defined as follows:

\[
\bar{E}_2(u) = \int_{\mathbb{R}_+} \left[ |u'(x)|^2 + \mu(x)G(u(x)) \right] dx,
\]
Moreover, for any \( x \) holds for all \( x \in \mathbb{R}_+ \), where \( \tilde{G}(u) = -\int_{\mathbb{R}_+} [f(s) - 1/\gamma] ds \).

Proof. By repeating the same argument as in the proof of Lemma 2.1, we can prove the statement. We only note that \( \tilde{G}(0) > \tilde{G}(\bar{u}_\gamma) = 0 \) if \( \gamma > 3\sqrt{6}/2 \).

The next lemma corresponds to Lemma 2.2. We omit the proof.

**Lemma 4.2.** For any \( d > 0 \) and \( \gamma > 3\sqrt{6}/(2\mu_0) \), \( \bar{W}(x) \leq T(x) \) holds, where \( T \) and \( \bar{W} \) are the solutions to (4) and (17) respectively.

As in Lemma 2.3, we can see the monotonicity of \( \tilde{\sigma}(d, \gamma) \) respect to \( \gamma \) as follows. We present the proof in Appendix. We remark that we employ a different strategy to show Lemma 4.3. The strategy is based on [9].

**Lemma 4.3.** If \( \gamma_1 < \gamma_2 \), then \( \tilde{\sigma}(d, \gamma_1) \leq \tilde{\sigma}(d, \gamma_2) \).

We shall show the behavior of \( \bar{W}_{d, \gamma} \) in Lemmas 4.4 – 4.7. The next lemma corresponds to Lemma 2.4. We omit the proof.

**Lemma 4.4.** Let \( \gamma > 3\sqrt{6}/2 \) and \( d_* \in (0, (l_0/x_1)^2) \) be constants. For any \( \epsilon > 0 \), there exists a positive constant \( \delta \) such that

\[
\| \bar{W}_{d_1, \gamma} - \bar{W}_{d_2, \gamma} \|_{L^\infty(\mathbb{R}_+)} < \epsilon
\]

for any \( d_1, d_2 \in [d_*, l_0/x_1] \) such that \( |d_1 - d_2| < \delta \).

We next prove \( \bar{W}_{d, \gamma}(x) \leq \bar{W}_{d, \gamma}(x) \).

**Lemma 4.5.** Let \( \bar{W}_{0, \gamma} \) be a solution to (19), that is, \( \bar{W}_{0, \gamma} \) satisfies

\[
\begin{aligned}
-\bar{W}_{0, \gamma}''(x) &= \mu_0(f(\bar{W}_{0, \gamma}(x)) - 1/\gamma), & x \in \mathbb{R}_+, \\
0 &\leq \bar{W}_{0, \gamma}(x) \leq u_\gamma, & x \in \mathbb{R}_+, \\
\bar{W}_{0, \gamma}'(x) &\geq 0, & x \in \mathbb{R}_+, \\
\bar{W}_{0, \gamma}(0) &= 0, \quad \bar{W}_{0, \gamma}(\infty) = u_\gamma.
\end{aligned}
\tag{19}
\]

Moreover, \( \bar{W}_{d, \gamma} \) is defined as \( \bar{W}_{d, \gamma}(x) = \bar{W}_{0, \gamma}(x/\sqrt{d}) \). Then \( \bar{W}_{d, \gamma}(x) \leq \bar{W}_{d, \gamma}(x) \) holds for all \( x \in \mathbb{R}_+ \).

Proof. From (17), we have

\[
\left[ -d^2 (\bar{W}_{d, \gamma}'(x))^2 \right]_0^\infty = \int_0^\infty \mu(x) \left( f(\bar{W}_{d, \gamma}(x)) - \frac{1}{\gamma} \right) \bar{W}_{d, \gamma}'(x) dx.
\]

Thus we obtain

\[
\frac{d}{2} (\bar{W}_{d, \gamma}'(0))^2 = \mu_0 \bar{G}(0) + \int_0^\infty \mu'(x) \bar{G}(\bar{W}_{d, \gamma}(x)) dx. \tag{38}
\]

On the other hand, \( \bar{W}_{d, \gamma} \) satisfies that

\[
\begin{aligned}
- d\bar{W}_{d, \gamma}(x) &= \mu_0 \left( f(\bar{W}_{d, \gamma}(x)) - \frac{1}{\gamma} \right), & x \in \mathbb{R}_+, \\
0 &\leq \bar{W}_{d, \gamma}(x) \leq u_\gamma, & x \in \mathbb{R}_+, \\
\bar{W}_{d, \gamma}(0) &= 0, \quad \bar{W}_{d, \gamma}(\infty) = u_\gamma
\end{aligned}
\]
from the definition. We can derive
\[
\frac{d}{2}(\hat{W}_{d,\gamma}(0))^2 = \mu_0 \hat{G}(0)
\]
with the same argument as (38). Thus it follows that \(\hat{W}_{d,\gamma}(0) < W'_{d,\gamma}(0)\) and \(\hat{W}_{d,\gamma} < W_{d,\gamma}\) on \((0, \delta)\) for small \(\delta > 0\).

Here we assume that there exists \(x_2 > 0\) such that \(\hat{W}_{d,\gamma}(x_2) = \hat{W}_{d,\gamma}(x_2)\) and \(\hat{W}_{d,\gamma}(x) < \hat{W}_{d,\gamma}(x)\) for all \(x \in (0, x_2)\). Then we have \(\hat{W}'_{d,\gamma}(x_2) \leq \hat{W}'_{d,\gamma}(x_2)\). On the other hand, it is easy to see that
\[
\frac{d}{2}(\hat{W}'_{d,\gamma}(x_2))^2 = \mu(x_2) \hat{G}(\hat{W}_{d,\gamma}(x_2)) + \int_{x_2}^{\infty} \mu'(x) \hat{G}(\hat{W}_{d,\gamma}(x)) \, dx,
\]
\[
\frac{d}{2}(\hat{W}'_{d,\gamma}(x_2))^2 = \mu \hat{G}(\hat{W}_{d,\gamma}(x_2)) < \frac{d}{2}(\hat{W}'_{d,\gamma}(x_2))^2.
\]
However, it contradicts \(\hat{W}'_{d,\gamma}(x_2) \leq \hat{W}'_{d,\gamma}(x_2)\). Hence we have \(\hat{W}_{d,\gamma} \leq \hat{W}_{d,\gamma}\) on \(\mathbb{R}_+\).

From Lemma 4.5, we immediately obtain the following statement. We omit the proof.

**Lemma 4.6.** Let \(\delta\) be a small positive constant, \(\gamma_1 = 3\sqrt{6}/2 + \delta\) and \(x_1\) be a point in \(\mathbb{R}_+\) such that \(\hat{W}_{0,\gamma_1}(x_1) = \beta_1\), where \(\beta_1\) is the positive smaller root of \(f(s) = 2/(3\sqrt{6})\). If \(d < (l_0/x_1)^2\) holds, then \(\hat{W}_{d,\gamma_1}(x) > \beta_1\) for all \(x > l_0\).

The next lemma shows the asymptotic behavior of \(\hat{W}_{d,\gamma}\).

**Lemma 4.7.** Let \(\gamma > 3\sqrt{6}/2\) and \(\hat{W}_{d,\gamma}(x) = \hat{W}_{d,\gamma}(\sqrt{d}x)\). Then \(\hat{W}_{d,\gamma}(x) \to \hat{W}_{0,\gamma}(x)\) holds for all \(x \in \mathbb{R}_+\) as \(d \to 0\), where \(\hat{W}_{0,\gamma}\) is defined as (19).

**Proof.** \(\hat{W}_{d,\gamma}\) is obtained as a minimizer of \(\hat{E}_2(u)\). Moreover, \(\hat{W}_{d,\gamma}\) is characterized as
\[
\int_{\mathbb{R}_+} \left[ \frac{1}{2} |\hat{W}_{d,\gamma}'|^2 + \mu(\sqrt{d}x) \hat{G}(\hat{W}_{d,\gamma}) \right] \, dx = \hat{E}_2(\hat{W}_{d,\gamma}) = \inf_{u \in X_3} \hat{E}_2(u).
\]

Remarking that there exists a constant \(\tilde{C} > 0\) such that \(\hat{G}(u) > \tilde{C}|u - u_\gamma|^2\) for all \(u > 0\), we find that \(\{||\hat{W}_{d,\gamma} - u_\gamma||_{H^1_0(\mathbb{R}_+)}\}\) is uniformly bounded from the above equation. Hence there exists \(\psi \in H^1_0(\mathbb{R}_+)\) such that \(\hat{W}_{d,\gamma} - u_\gamma \to \psi\) weakly in \(H^1_0(\mathbb{R}_+)\) and \(\hat{W}_{d,\gamma} - u_\gamma \to \psi\) strongly in \(C_{loc}(\mathbb{R}_+)\). Let \(\hat{w}_{0,\gamma} = u_\gamma + \psi\) and then we can see that
\[
\begin{align*}
-\hat{w}_{0,\gamma}'(x) &= \mu_0 \left(f(\hat{w}_{0,\gamma}(x)) - \frac{1}{\gamma}\right), & x \in \mathbb{R}_+, \\
\hat{w}_{0,\gamma}(0) &= 0, \\
\hat{w}_{0,\gamma}(\infty) &= u_\gamma.
\end{align*}
\]

As a consequence, we conclude that \(\hat{w}_{0,\gamma} = \hat{W}_{0,\gamma}\) from the uniqueness of the solution to (19).

The next lemma corresponds to Lemma 2.5. We omit the proof.

**Lemma 4.8.** There exists a positive constant \(\bar{M}\) independent of \(d\) and \(\gamma\) such that
\[
\bar{d}(d, \gamma) < \bar{M} \quad \text{for all } (d, \gamma) \in (0, \infty) \times \left(\frac{3\sqrt{6}}{2}, \infty\right).
\]
The next lemma corresponds to Lemma 3.1. By the same argument as in the proof of Lemma 3.1, we can prove the lemma.

**Lemma 4.9.** Let \( \bar{v} \) be a function satisfying that \( \bar{v}(0) = 0, \bar{v}(\infty) = \alpha \gamma / \gamma \) and \( \bar{v} - \hat{v} \in H^1_0(\mathbb{R}_+) \cap H^1(\mathbb{R}_+) \). Then there exists a solution \( u \in C^2(\mathbb{R}_+) \) to the equation
\[
\begin{aligned}
- \frac{d^2u}{dx^2}(x) &= \mu(x)(f(u(x)) - \bar{v}(x)), \quad x \in \mathbb{R}_+, \\
\bar{W}(x) &\leq u(x) \leq T(x), \quad x \in \mathbb{R}_+, \\
u(0) &= 0, \ u(\infty) = \alpha \gamma
\end{aligned}
\]
Moreover, \( u - \hat{u} \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \) holds.

With these lemmas, we can prove Proposition 7.

**Proof of Proposition 7.** As in the proof of Proposition 1, we can easily prove the proposition. \( \square \)

We next show the positivity of \( \bar{\sigma}(d, \gamma) \). Basically we can prove same as Lemma 4, but we need some modification.

**Proposition 9.** Let \( d < (l_0/x_1)^2 \) and \( \gamma > 3\sqrt{6}/2 \). Then \( \bar{\sigma}(d, \gamma) \) is positive.

**Proof.** We prove by contradiction. Namely we assume that \( \bar{\sigma}(d, \gamma) \leq 0 \). Under this assumption, we can show that the minimizing problem (18) has a minimizer \( \xi_1 \in H^1_0(\mathbb{R}_+) \) as in Lemma 4. Moreover, with attention to that \( \bar{W}_{d, \gamma} \) satisfies
\[
d \int_{\mathbb{R}_+} \bar{W}''_{d, \gamma} \xi_1' \, dx
= \int_{\mathbb{R}_+} \mu(x) \left( 1 - 3 \bar{W}^2_{d, \gamma} \right) \bar{W}'_{d, \gamma} \xi_1 \, dx
+ \int_{\mathbb{R}_+} \mu'(x) \left( \bar{W}_{d, \gamma} - \bar{W}^3_{d, \gamma} - \frac{1}{\gamma} \right) \xi_1 \, dx,
\]
which corresponds to (34), we obtain
\[
\sigma \int_{\mathbb{R}_+} \xi_1 \bar{W}_{d, \gamma} \, dx = d \xi_1'(0) \bar{W}''_{d, \gamma}(0) + \int_{\mathbb{R}_+} \mu'(x) \left( \bar{W}_{d, \gamma} - \bar{W}^3_{d, \gamma} - \frac{1}{\gamma} \right) \xi_1 dx
\]
by repeating the same argument as in Lemma 4. Since \( \bar{W}_{d, \gamma}(x) > \beta_1 \) holds for all \( x > l_0 \) from Lemma 4.6, the right hand side of (39) is positive. However, the left hand side is non-positive. Clearly it is a contradiction. Thus we conclude \( \bar{\sigma}(d, \gamma) > 0 \). \( \square \)

We now prove Proposition 8. From Lemma 4.3, Proposition 8 can be proved from the following propositions:

**Proposition 10.** Let \( \gamma \) be a constant such that \( \gamma > 3\sqrt{6}/2 \). Then there exist constants \( d_1 = d_1(\gamma) \in (0, l_0/x_1) \) and \( \bar{\sigma}^{(1)}(\gamma) > 0 \) such that
\[
\bar{\sigma}(d, \gamma) > \bar{\sigma}^{(1)}(\gamma) \quad \text{for all} \quad d \in (0, d_0).
\]

**Proposition 11.** Let \( \gamma \) be a constant such that \( \gamma > 3\sqrt{6}/2 \). Then there exists \( \bar{\sigma}^{(2)}(\gamma) > 0 \) such that
\[
\bar{\sigma}(d, \gamma) > \bar{\sigma}^{(2)}(\gamma) \quad \text{for all} \quad d \in [d_0, l_0/x_1].
\]
Proof of Proposition 10. We prove by contradiction. Namely we assume that there exists a sequence \( \{d_n\} \) such that \( d_n \to 0 \) and \( \sigma_n \to 0 \) as \( n \to \infty \), where \( \sigma_n = \sigma(d_n, \gamma) \). We can see that there exists a minimizer \( \xi_n \in H^1_0(\mathbb{R}_+) \) of (18) with \( d = d_n \). Let \( \bar{\xi}_n(x) = d_n^{1/4} \xi_n(\sqrt{d_n} x) \). Then \( \bar{\xi}_n \) satisfies
\[
-\bar{\xi}_n''(x) + V_n(\sqrt{d_n} x) \bar{\xi}_n(x) = \sigma_n \bar{\xi}_n(x),
\]
where \( V_n(x) = \mu(x)(3\tilde{W}_{d_n, \gamma}(x)^2 - 1) \). As in the proof of Proposition 5, we can prove the existence of a function \( \bar{\xi}_0 \in H^1_0(\mathbb{R}_+) \). It is easy to check \( ||\xi_0||_{L^2(\mathbb{R}_+)} \). With attention to Lemma 4.7, we can see that \( \bar{\xi}_0 \) satisfies
\[
-\bar{\xi}_0''(x) + \mu_0(3\tilde{W}_{0, \gamma}(x)^2 - 1)\bar{\xi}_0(x) = 0.
\]
Thus we deduce that \( \bar{\xi}_0 \tilde{W}_{0, \gamma}(0) = 0 \) similarly as (35). However, it is a contradiction. Therefore we conclude the statement. \( \square \)

Proof of Proposition 11. We can prove the statement similarly as Proposition 6. \( \square \)

Theorem 1.2 is followed from Propositions 7 and 8 and we omit the proof.

Appendix. Proofs of Lemma 2.4, 2.5 and 4.3. In this part, we present the proofs of Lemmas 2.4, 2.5 and 4.3.

Lemma 2.4. Let \( \gamma > 3\sqrt{6}/(2\mu_0) \) and \( d_* \in (0, l_0/x_0) \) be a constant. For any \( \varepsilon > 0 \), there exists a positive constant \( \delta \) such that
\[
||W_{d_1, \gamma} - W_{d_2, \gamma}||_{L^\infty(\mathbb{R}_+)} < \varepsilon
\]
for any \( d_1, d_2 \in [d_*, l_0/x_0] \) such that \( |d_1 - d_2| < \delta \).

Proof. It suffices to show that for given \( \gamma \in (3\sqrt{6}/(2\mu_0), \infty) \),
\[
||W_{d_1, \gamma} - W_{d_2, \gamma}||_{H^1(\mathbb{R}_+)} = o(1) \quad \text{as} \quad |d_2 - d_1| \to 0.
\]
For simplicity \( W_1 \) and \( W_2 \) denotes \( W_{d_1, \gamma} \) and \( W_{d_2, \gamma} \), respectively. \( W_1 \) and \( W_2 \) satisfy
\[
-d_1 W_1'' = \mu_0 f(W_1) - 1/\gamma,
-d_2 W_2'' = \mu_0 f(W_2) - 1/\gamma.
\]
Thus we obtain
\[
-d_1 (W_1'' - W_2'') + (d_2 - d_1) W_2'' = \mu_0 (f(W_1) - f(W_2))
= \mu_0 (W_1 - W_2)(1 - W_1^2 - W_1 W_2 - W_2^2).
\]
Multiplying by \( (W_1 - W_2) \) and integrating over \( \mathbb{R}_+ \), we can see that
\[
\int_{\mathbb{R}_+} [-d_1 (W_1'' - W_2'') (W_1 - W_2) + (d_2 - d_1) W_2'' (W_1 - W_2)] dx
= \mu_0 \int_{\mathbb{R}_+} (W_1 - W_2)^2 (1 - W_1^2 - W_1 W_2 - W_2^2) dx.
\]
Thus we have
\[
d_1 \int_{\mathbb{R}_+} (W_1^2 - W_2^2)^2 dx \leq C_1 (d_2 - d_1) \int_{\mathbb{R}_+} W_2''^2 dx + C_2 \int_{\mathbb{R}_+} (W_1 - W_2)^2 dx.
\]
We shall show
\[
\int_{\mathbb{R}_+} (W_1(x) - W_2(x))^2 dx = o(1) \quad \text{as} \quad |d_2 - d_1| \to 0.
\]
We note that
\[ \int_{\mathbb{R}^+} (W_1(x) - W_2(x))^2 \, dx = \int_{\mathbb{R}^+} \left\{ \tilde{W}_{0, \gamma} \left( \frac{x}{\sqrt{d_2}} \right) - \tilde{W}_{0, \gamma} \left( \frac{x}{\sqrt{d_1}} \right) \right\}^2 \, dx. \]

From the above equation, it is clear that \( W_2(x) \to W_1(x) \) for all \( x \in \mathbb{R}^+ \) as \( d_2 \to d_1 \).

If \( z_i(x) = \tilde{W}_{0, \gamma}(x/\sqrt{d_i}) - u_\gamma \in L^2(\mathbb{R}^+) \) \( (i = 1, 2) \), then it follows (41) from the Lebesgue dominated convergence theorem. For simplicity, we write \( \tilde{W} = \tilde{W}_{0, \gamma} \).

We set \( z = W_0 - u_\gamma \). Then we have
\[ -z'' = \mu_0 f(W_0) - \frac{1}{\gamma} \mu_0(f(W_0) - f(u_\gamma)) = \mu_0(1 - W_0^2 - W_0u_\gamma - u_\gamma^2)z. \]

We remark that \( 1 - W_0(x)^2 - W_0(x)u_\gamma - u_\gamma^2 < -(3u_\gamma^2 - 1)/2 < 0 \) holds for large \( x > 0 \). Hence we can deduce that there exists constants \( C_3 \) and \( C_4 \) such that
\[ z(x) = W_0(x) - u_\gamma < C_4e^{-C_4x} \text{ for all } x > 0. \]

It is easy to check that \( z_i \in L^2 \). As a consequence, we deduce (41). The statement is followed from (40) and (41).

**Lemma 2.5.** There exists a positive constant \( M \) independent of \( d \) and \( \gamma \) such that
\[ \sigma(d, \gamma) < M \quad \text{for all } (d, \gamma) \in (0, \infty) \times \left( \frac{3\sqrt{6}}{2\mu_0}, \infty \right). \]

**Proof.** Let \( J_d(\psi) \) be a functional defined as (11). By changing variables \( x = \sqrt{d}X \) and setting \( \xi(X) = \xi(\sqrt{d}X) \), we calculate \( J_d(\psi) \) as follows:
\[ J_d(\xi) = \sqrt{d} \int_{\mathbb{R}^+} \left[ \frac{1}{2} |\xi'|^2 + \mu(\sqrt{d}X)(3W_{d, \gamma}(x)^2 - 1) |\xi|^2 \right] \, dx \leq \sqrt{d} \int_{\mathbb{R}^+} \left[ \frac{1}{2} |\xi'|^2 + 2 |\xi|^2 \right] \, dx. \]

Moreover, it is clear that \( \|\xi\|_{L^2(\mathbb{R}^+)} = \sqrt{d} \|\xi\|_{L^2(\mathbb{R}^+)}. \) Fix \( \xi_0 \in H^1_0(\mathbb{R}^+) \) and set \( \xi_0(x) = \xi_0(x/\sqrt{d}) \). Then we obtain
\[ \sigma(d, \gamma) \leq \frac{J_d(\xi_0)}{\|\xi_0\|_{L^2(\mathbb{R}^+)}} \leq \int_{\mathbb{R}^+} \left[ \frac{1}{2} |\xi_0'|^2 + 2 |\xi_0|^2 \right] \, dx. \]

The right hand side is clearly independent of \( d \) and \( \gamma \).

**Lemma 4.3.** If \( \gamma_1 < \gamma_2 \), then \( \bar{\sigma}(d, \gamma_1) \leq \bar{\sigma}(d, \gamma_2) \).

**Proof.** It suffices to show that \( \bar{W}_{d, \gamma_1} \leq \bar{W}_{d, \gamma_2} \) on \( \mathbb{R}^+ \) if \( \gamma_1 < \gamma_2 \). For simplicity, we write \( \bar{W}_i = \bar{W}_{d, \gamma_i} \) \( (i = 1, 2) \). Since \( W_1(x) < W_2(x) \) for sufficiently large \( x > 0 \), there exists a positive constant \( L > 0 \) such that
\[ W_1(x) < W_2(x) \quad \text{for all } x > L. \]

We may assume that \( W_1(L) = W_2(L) \equiv \alpha \). Thus it suffices to show
\[ W_1(x) \leq W_2(x) \quad \text{for all } x \in [0, L]. \]
We prove by contradiction. We set \( \Gamma = \{ x > L : W_1(x) > W_2(x) \} \) and assume \( \Gamma \neq \emptyset \). Let \( g_i \) and \( \bar{G}_i \) \( (i = 1, 2) \) be as follows:

\[
g_i(t) = -\left( f(t) - \frac{1}{\gamma_i} \right),
\]

\[
\bar{G}_i(s) = \int_0^s g_i(t) \, dt.
\]

From the definition, it follows that \( g_1(s) \geq g_2(s) \) and \( \bar{G}_1(s) \geq \bar{G}_2(s) \) hold for all \( s > 0 \). Moreover, we note that

\[
\bar{G}_i(s) = \bar{G}_i(s) + \int_0^{u_{\gamma_i}} g_i(t) \, dt,
\]

where \( \bar{G}_i(s) = -\int_{u_{\gamma_i}} [f(t) - 1/\gamma_i] \, dt \). We define \( \tilde{E}_{i,L}(u) \) \( (i = 1, 2) \) and \( X_L \) as follows:

\[
\tilde{E}_{i,L}(u) = \int_0^L \left[ \frac{d}{2} |u'(x)|^2 + \mu(x) \bar{G}_i(u) \right] \, dx,
\]

\[
X_L = \{ u \in H^1(0,L) : u(0) = 0, \ u(L) = \alpha \}.
\]

We recall that \( W_i \) is a minimizer of \( \tilde{E}_{i}(u) \) in \( X_L \), where \( \tilde{E}_i \) denotes \( E_2 \) defined as (36) with \( \gamma = \gamma_i \) \( (i = 1, 2) \). This implies that \( W_i \) is a minimizer of \( \tilde{E}_{i,L}(u) \) in \( X_L \). Moreover, we define

\[
\tilde{\tilde{E}}_{i,L}(u) = \int_0^L \left[ \frac{d}{2} |u'(x)|^2 + \mu(x) \bar{G}_i(u) \right] \, dx,
\]

and then \( W_i \) is also a minimizer of \( \tilde{\tilde{E}}_{i,L}(u) \) in \( X_L \) because of (42).

Set \( \phi(x) = (W_1(x) - W_2(x))_+ \). Then \( \phi \in H^1_0(0,L) \) and \( \phi \neq 0 \). In addition, the following inequalities hold:

\[
0 \leq \tilde{E}_{1,L}(W_1 - \phi) - \tilde{E}_{1,L}(W_1), \quad (43)
\]

\[
0 \leq \tilde{E}_{2,L}(W_2 + \phi) - \tilde{E}_{2,L}(W_2). \quad (44)
\]

Now we evaluate \( \tilde{E}_{2,L}(W_2 + \phi) - \tilde{E}_{2,L}(W_2) \) by a direct calculation. From the definition, we calculate as

\[
\tilde{E}_{2,L}(W_2 + \phi) - \tilde{E}_{2,L}(W_2) = \int_0^L \frac{d}{2} \left( |W'_2 + \phi'|^2 - |W'_2|^2 \right) \, dx + \int_0^L \mu(x) \int_{W_2(x)}^{W_2(x) + \phi(x)} g_2(t) \, dt \, dx.
\]

We note that \( \phi(x) = 0 \) holds for any \( x \in \Gamma^c = \{ x < L : W_1(x) \leq W_2(x) \} \). In addition, we remark that \( W_2 \) can be represented as \( W_2(x) = W_1(x) - \phi(x) \) for any \( x \in \Gamma \). Hence we evaluate \( \tilde{E}_{2,L}(W_2 + \phi) - \tilde{E}_{2,L}(W_2) \) as follows:

\[
\tilde{E}_{2,L}(W_2 + \phi) - \tilde{E}_{2,L}(W_2) = \int_\Gamma \frac{d}{2} \left( |W'_1|^2 - |W'_1 - \phi'|^2 \right) \, dx + \int_\Gamma \mu(x) \int_{W_1(x) - \phi(x)}^{W_1(x)} g_2(t) \, dt \, dx
\]

\[
< \int_\Gamma \frac{d}{2} \left( |W'_1|^2 - |W'_1 - \phi'|^2 \right) \, dx + \int_\Gamma \mu(x) \int_{W_1(x) - \phi(x)}^{W_1(x)} g_1(t) \, dt \, dx
\]
\[
\int_0^L \frac{d}{2} \left( |W_1'|^2 - |W_1' - \phi'|^2 \right) \, dx + \int_0^L \left[ \mu(x) \int_{W_1(x) - \phi(x)}^{W_1(x)} g_1(t) \, dt \right] \, dx
\]
\[
= \tilde{E}_{1,L}(W_1) - \tilde{E}_{1,L}(W_1 - \phi) \leq 0.
\]

However, it clearly contradicts (43). \qed

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