Diagonalizing by Fixed–Points

Abstract

A universal schema for diagonalization was popularized by N. S. Yanofsky (2003) in which the existence of a (diagonolized-out and contradictory) object implies the existence of a fixed-point for a certain function. It was shown that many self-referential paradoxes and diagonally proved theorems can fit in that schema. Here, we fit more theorems in the universal schema of diagonalization, such as Euclid’s theorem on the infinitude of the primes and new proofs of G. Boolos (1997) for Cantor’s theorem on the non-equinumerosity of a set with its powerset. Then, in Linear Temporal Logic, we show the non-existence of a fixed-point in this logic whose proof resembles the argument of Yablo’s paradox. Thus, Yablo’s paradox turns for the first time into a genuine mathematico-logical theorem in the framework of Linear Temporal Logic. Again the diagonal schema of the paper is used in this proof; and also it is shown that G. Priest’s inclosure schema (1997) can fit in our universal diagonal/fixed-point schema. We also show the existence of dominating (Ackermann-like) functions (which dominate a given countable set of functions—like primitive recursives) using the schema.

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Diagonal arguments were most likely started from CANTOR, by his (third proof for the) famous theorem on non–denumerability of the reals. What is now well–known as Cantor’s Diagonal Proof is the following argument showing that there can be no surjection from a set $A$ to its powerset $\mathcal{P}(A)$: for any function $F : A \rightarrow \mathcal{P}(A)$ the set $D_F = \{ x \in A \mid x \notin F(x) \}$ is not in the range of $F$ because for any $a \in A$ we have \( a \in D_F \iff a \notin F(a) \), and so $a \in (D_F \setminus F(a)) \cup (F(a) \setminus D_F)$, whence $D_F \neq F(a)$. This argument shows up also in RUSSELL’s Paradox: the collection $R = \{ x \mid x \notin x \}$ of sets is not a set, since for any set $A$ we have $A \in R \iff A \notin A$, so $A \neq R$. One other example is TURING’s Halting Problem in Computability Theory: if $W_0, W_1, W_2, \cdots$ is the family of all RE sets (recursively enumerable subsets of $\mathbb{N}$), then the set $\mathcal{K} = \{ n \in \mathbb{N} \mid n \notin W_n \}$ is not RE, because for any RE set $W_m$ we have $m \in \mathcal{K} \iff m \notin W_m$, and so $m \in (\mathcal{K} \setminus W_m) \cup (W_m \setminus \mathcal{K})$, thus $\mathcal{K} \neq W_m$. It can be seen that the (diagonal) set $\mathcal{K} = \{ n \in \mathbb{N} \mid n \notin W_n \}$ is an RE but undecidable set.

Many other theorems in mathematics (logic, set theory, computability theory, complexity theory, etc.) use diagonal arguments; TARKSI’s theorem on the undefinability of truth, and GÖDEL’s theorem on the incompleteness of sufficiently strong and ($\omega$–)consistent theories are two prominent examples. In 2003, Noson S. Yanofsky published the paper [14] mentioning some earlier descriptions for “many of the classical paradoxes and incompleteness theorems in a categorial fashion”, in the sense that by using “the language of category theory (and of cartesian closed categories in particular)” one can demonstrate some paradoxical phenomena and show the above mentioned theorems of CANTOR, TARKSI AND GÖDEL; the goal of [14] was “to make these amazing results available to a larger audience”. In that paper, a universal schema has been considered in the language of sets and functions (not categories) and the paradoxes of the Liar, the strong liar, RUSSELL, GRELING, RICHARD, Time Travel, and LÖB, and the theorems of CANTOR ($A \not\subseteq \mathcal{P}(A)$), TURING (undecidability of the Halting problem, and existence of a non–re set), BAKER–GILL–SOLOVAY (the existence of an oracle $O$ such that $\mathbf{P}^O \neq \mathbf{NP}^O$), CARNAP (the diagonalization lemma), GÖDEL (first incompleteness theorem), ROSSER (incompleteness of sufficiently strong and consistent theories), TARKSI (undefinability of truth in sufficiently strong languages), PARIKH (existence of sentences with very long proofs), KLEENE (Recursion Theorem), RICE (undecidability of non–trivial properties of recursive functions), and von NEUMANN (existence of self–reproducing machines) are shown as instances.

In this paper, we fit some other theorems and proofs into the above mentioned universal schema of YANOFSKY; these include EUCLID’s Theorem on the infinitude of the primes, BOOLOS’ proof of the existence of some explicitly definable counterexamples to the non–injectivity of functions $F : \mathcal{P}(A) \rightarrow A$ for any set $A$ where $\mathcal{P}(A)$ denotes the powerset of $A$, YABLO’s paradox in a form of a mathematical theorem in the framework of linear temporal logic as a non–existence of some certain fixed–points, and the existence of dominating functions for a given countable set of functions like ACKERMANN’S function which dominates all the primitive recursive functions. In the rest of the introduction we fix our notation and introduce the common framework.

### 1.1 Cantor’s Theorem by Fixed–Points

Let $B$, $C$ and $D$ be arbitrary sets. Any function $f : B \times C \rightarrow D$ corresponds to a function $\hat{f} : C \rightarrow D^B$ where $\hat{f}(c)(b) = f(b,c)$ for any $b \in B$ and $c \in C$ (the set $D^B$ consists of all the functions from $B$ to $D$). Conversely, for any function $F : C \rightarrow D^B$ there exists some $f : B \times C \rightarrow D$ such that $\hat{f} = F$: for any $b \in B$ and $c \in C$ let $f(b,c) = F(c)(b)$. In the other words $\hat{\cdot} : D^{B \times C} \cong (D^B)^C$. Let $f : B \times C \rightarrow D$ be a fixed function. A function $g : B \rightarrow D$ is called representable by $f$ at a fixed $c_0 \in C$, when for any $x \in B$, $g(x) = f(x, c_0)$ holds. In the other words, $g = \hat{f}(c_0)$. So, the function $\hat{f} : C \rightarrow D^B$ is onto if and only if every function $B \rightarrow D$ is representable by $f$ at some $c_0 \in C$. 

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Theorem 1.1 (Cantor) Assume the function $\alpha : D \to D$, for a set $D$, does not have any fixed point (i.e., $\alpha(d) \neq d$ for all $d \in D$). Then for any set $B$ and any function $f : B \times B \to D$ there exists a function $g : B \to D$ that is not representable by $f$ (i.e., for all $b \in B$, $g(b) \neq f(\langle a, b \rangle)$).

Proof. The desired function $g : x \mapsto \alpha(f(x, x))$ can be constructed as follows:

$\begin{array}{c}
B \times B \xrightarrow{\triangle_B} B \xrightarrow{\alpha} D \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
2 Euclid’s Theorem on the Infinitude of the Primes

Our first instance of Cantor’s Diagonal Proof starts with a surprise: the ancient theorem of Euclid stating that there are infinitely many prime numbers. We use (almost) the classical proof of Euclid which seems far from being a diagonal argument. Indeed there are many different proofs of this theorem in the literature, and ours is not a new one; we just fit a proof in a diagonal diagram as above.

Theorem 2.1 (Euclid) There are infinitely many prime numbers in \( \mathbb{N} \).

Proof. Define the function \( f : \mathbb{N} \times \mathbb{N} \to 2 \) as follows:

\[
    f(n, m) = \begin{cases} 
    1 & \text{if all the prime factors of } (n! + 1) \text{ are less than } m \\
    0 & \text{if some prime factor of } (n! + 1) \text{ is greater than or equal to } m
    \end{cases}
\]

For example, \( f(4, 9) = 1 \) because \( 4! + 1 = 25 \) and it has no other prime factor but 5 and 5 < 9; it can be seen that \( f(4, m) = 0 \) for all \( m \leq 5 \) and \( f(4, m) = 1 \) for all \( m > 5 \). Indeed, for any \( n \in \mathbb{N} \) we have \( f(n, n) = 0 \) because no prime factor of \( n! + 1 \) can be less than \( n \): for any \( d < n \) if \( d \mid (n! + 1) \) then from \( d \mid n! \) it follows that \( d \mid 1 \) so \( d \) cannot be a prime. Now, consider the function \( g : \mathbb{N} \to 2 \) constructed as

\[
\begin{array}{ccc}
\mathbb{N} \times \mathbb{N} & \xrightarrow{f} & 2 \\
\downarrow & & \downarrow \neg \\
\Delta_{\mathbb{N}} & \xleftarrow{g} & 2
\end{array}
\]

If all the prime numbers are less than \( p \in \mathbb{N} \) then the function \( g \) is representable by \( f \) at \( p \): for any \( n \in \mathbb{N} \), \( f(n, p) = 1 \) and \( g(n) = \neg(f(n, n)) = 1 \); whence \( g(n) = f(n, p) \) for all \( n \in \mathbb{N} \). A contradiction follows as before: if such a number \( p \) exists, then \( f(p, p) \) becomes a fixed–point of \( \neg \). So, there exists no \( p \in \mathbb{N} \) such that all the primes are non–greater than \( p \); whence there must be infinitely many prime numbers. \( \square \)

This surprising argument, we believe, deserves another closer look: define the function \( F : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \) by

\[
    F(n) = \{ x \in \mathbb{N} \mid n \text{ is greater than or equal to all the prime factors of } (x! + 1) \}.
\]

Cantor’s Theorem says that \( F \) cannot be surjective, or, more explicitly, \( D_F = \{ n \mid n \notin F(n) \} \) the (anti–diagonal) set is not equal to any \( F(m) \). A number–theoretic argument shows that \( D_F = \mathbb{N} \) because for any \( n \) all the prime factors of \( (n! + 1) \) are greater than \( n \) (see the proof of the above Theorem 2.1). On the other hand if \( p \in \mathbb{N} \) is the greatest prime, then \( F(p) = \mathbb{N} = D_F \), a contradiction!

3 Some Other Proofs for Cantor’s Theorem

In 1997 late George Boolos published another proof \([1]\) for Cantor’s Theorem, by showing that there cannot be any injection from the powerset of a set to the set. This proof has been (implicitly or explicitly) mentioned also in \([7, 11]\) (but without referring to the earlier \([1]\)). The first proof is essentially Cantor’s Diagonal Argument.

Theorem 3.1 No function \( h : \mathcal{P}(A) \to A \) can be injective.
**Proof.** Let \( h : \mathcal{P}(A) \to A \) be a function. Define \( f : \mathcal{P}(A) \times \mathcal{P}(A) \to 2 \) by

\[
f(X,Y) = \begin{cases} 
  1 & \text{if } h(X) \not\in Y \\
  0 & \text{if } h(X) \in Y 
\end{cases}
\]

and let \( g : \mathcal{P}(A) \to 2 \) be the following function

\[
\begin{array}{ccc}
\mathcal{P}(A) \times \mathcal{P}(A) & \xrightarrow{f} & 2 \\
\Delta_{\mathcal{P}(A)} & \xrightarrow{\text{neg}} & \\
\mathcal{P}(A) & \xrightarrow{g} & 2 \\
\end{array}
\]

Let \( \mathcal{D}_h = \{ a \in A \mid \exists Y \subseteq A ( h(Y) = a \land a \not\in Y) \} \). Note that for any \( X \subseteq A \) we have \( h(X) \not\in X \to h(X) \in \mathcal{D}_h \). We show that if \( h \) is one-to-one then \( g \) is representable by \( f \) at \( \mathcal{D}_h \). For, if \( h \) is injective then for any \( X \subseteq A \),

\[
\begin{align*}
  h(X) \in \mathcal{D}_h & \quad \implies \quad \exists Y \subseteq A ( h(Y) = h(X) \land h(X) \not\in Y) \\
  & \quad \implies \quad \exists Y ( Y = X \land h(X) \not\in Y) \\
  & \quad \implies \quad h(X) \not\in X
\end{align*}
\]

Whence, \( h(X) \not\in X \iff h(X) \in \mathcal{D}_h \) for all \( X \subseteq A \). So, for any \( X \subseteq A \),

\[
\begin{align*}
f(X, \mathcal{D}_h) = 0 & \quad \iff \quad h(X) \in \mathcal{D}_h \\
& \quad \iff \quad h(X) \not\in X \\
& \quad \iff \quad f(X, X) = 1 \\
& \quad \iff \quad g(X) = \text{neg}(f(X, X)) = 0
\end{align*}
\]

Thus, \( g(X) = f(X, \mathcal{D}_h) \). The contradiction (that \( \text{neg} \) possesses a fixed-point) follows as before, implying that the function \( h \) cannot be injective.

In fact the proof of the above theorem gives some more information than mere non-injectivity of any function \( h : \mathcal{P}(A) \to A \), i.e., the existence of some \( C, D \subseteq A \) such that \( h(C) = h(D) \) and \( C \neq D \).

**Corollary 3.2** For any function \( h : \mathcal{P}(A) \to A \) there are some \( C, D \subseteq A \) such that \( h(C) = h(D) \in D \setminus C \) (and so \( C \neq D \)).

**Proof.** For any \( X \subseteq A \) we had \( h(X) \not\in X \to h(X) \in \mathcal{D}_h \), whence \( h(\mathcal{D}_h) \not\in \mathcal{D}_h \to h(\mathcal{D}_h) \in \mathcal{D}_h \), and so \( h(\mathcal{D}_h) \in \mathcal{D}_h \). Thus, there exists some \( \mathcal{C}_h \) such that \( h(\mathcal{C}_h) = h(\mathcal{D}_h) \) and \( h(\mathcal{D}_h) \not\in \mathcal{C}_h \). So, for these \( \mathcal{C}_h, \mathcal{D}_h \subseteq A \) we have \( h(\mathcal{C}_h) = h(\mathcal{D}_h) \in \mathcal{D}_h \setminus \mathcal{C}_h \).

Boolos notes in [1] that, in the above proof, though the set \( \mathcal{D}_h \) had an explicit definition:

\[
\mathcal{D}_h = \{ a \in A \mid \exists Y \subseteq A ( h(Y) = a \land a \not\in Y) \},
\]

the set \( \mathcal{C}_h \) was not defined explicitly, and its mere existence was shown. So, this proof of non-injectivity was not constructive (did not explicitly construct two sets \( C \) and \( D \) such that \( h(C) = h(D) \) and \( C \neq D \)). For a constructive proof, Boolos [1] proceeds as follows (cf. [7] [11]).

Fix a function \( h : \mathcal{P}(A) \to A \). Call a subset \( B \subseteq A \) an \( h \)-woset (\( h \) well ordered set) when there exists a well ordering \( \prec \) on \( B \) such that \( b = h(\{ x \in B \mid x \prec b \}) \) for any \( b \in B \). For example, \( \{ h(\emptyset) \} \) is an \( h \)-woset, and indeed any non-empty \( h \)-woset must contain \( h(\emptyset) \). Some other examples of \( h \)-wosets are the following:

\[
\{ h(\emptyset), h(\{ h(\emptyset) \}) \} \text{ and } \{ h(\emptyset), h(\{ h(\emptyset) \}), h(\{ h(\emptyset), h(\{ h(\emptyset) \}) \}) \}.
\]

We need the following two facts about the \( h \)-wosets:
(1) If $B$ and $C$ are two $h$–wosets with the well ordering relations $\prec_B$ and $\prec_C$ then exactly one (and only one) of the following holds:
   (i) $(B, \prec_B)$ is an initial segment of $(C, \prec_C)$, or
   (ii) $(C, \prec_C)$ is an initial segment of $(B, \prec_B)$, or
   (iii) $(B, \prec_B) = (C, \prec_C)$.

(2) For any $h$–woset $B$, if $h(B) \notin B$ then the set $\Phi(B) = B \cup \{h(B)\}$ is an $h$–woset, and $B$ is an initial segment of $\Phi(B)$.

The statement (1) corresponds to Zermelo’s theorem that any two well ordered sets are comparable to each other: either they are isomorphic or one of them is isomorphic to an initial segment of the other one. It follows from (1) that the union of all $h$–wosets is an $h$–woset, denoted by $\mathcal{W}_h$; thus $\mathcal{W}_h$ is the greatest $h$–woset. For (2) let $B$ be an $h$–woset with the well ordering $\prec_B$ such that $h(B) \notin B$. Then $\Phi(B)$ is an $h$–woset with the well ordering $\prec_{\Phi(B)} = \prec_B \cup (B \times \{h(B)\})$.

The proof of Boolos [1] continues as follows (see also [7]): since $\Phi(\mathcal{W}_h) = \mathcal{W}_h$ then $h(\mathcal{W}_h) \in \mathcal{W}_h$. Also for $\mathcal{V}_h = \{x \in \mathcal{W}_h \mid x \prec_{\mathcal{W}_h} h(\mathcal{W}_h)\}$ we have $h(\mathcal{V}_h) = h(\mathcal{V}_h)$ and $\mathcal{W}_h \neq \mathcal{V}_h$ because $h(\mathcal{W}_h) \notin \mathcal{V}_h$. Indeed, the result is stronger than this (and Corollary 3.2) since the sets $\mathcal{W}_h$ and $\mathcal{V}_h$ were explicitly defined in a way that $\mathcal{V}_h \subsetneq \mathcal{W}_h$ holds and $h(\mathcal{V}_h) = h(\mathcal{V}_h) \in \mathcal{W}_h \setminus \mathcal{V}_h$. As another partial surprise we show that this proof is also diagonal and fits in our universal framework.

**Theorem 3.3 (Boolos)** For any set $A$ and function $h : \mathcal{P}(A) \rightarrow A$ there exist explicitly definable subsets $V, W \subseteq A$ such that $V \subsetneq W$ and $h(V) = h(W) \in W \setminus V$.

**Proof.** Let $\mathcal{W}_h$ be the class of all $h$–wosets; i.e., all subsets $B \subseteq A$ on which there exists a (unique) well ordering $\prec_B$ such that $b = h(\{x \in B \mid x \prec_B b\})$ for all $b \in B$. Define $\Phi : \mathcal{W}_h \rightarrow \mathcal{W}_h$ by

\[
\Phi(X) = \begin{cases} 
X \cup \{h(X)\} & \text{if } h(X) \notin X \\
X & \text{if } h(X) \in X 
\end{cases}
\]

with $\prec_{\Phi(X)} = \begin{cases} 
\prec_X \cup (X \times \{h(X)\}) & \text{if } h(X) \notin X \\
\prec_X & \text{if } h(X) \in X 
\end{cases}$

Define the function $f : \mathcal{W}_h \times \mathcal{W}_h \rightarrow 2$ by

\[
f(X, Y) = \begin{cases} 
1 & \text{if } \Phi(X) \text{ is isomorphic to } Y \text{ or an initial segment of it } \left(\Phi(X) \sqsubseteq Y\right) \\
0 & \text{if } Y \text{ is isomorphic to an initial segment of } \Phi(X) \left(\Phi(X) \sqsubset Y\right) 
\end{cases}
\]

Let $\mathcal{W}_h$ be the greatest element of $\mathcal{W}_h$ (as above). Then $f(X, \mathcal{W}_h) = 1$ for all $X \in \mathcal{W}_h$. We claim that

(∗) there exists some $Z \in \mathcal{W}_h$ such that $h(Z) \in Z$ or equivalently $\Phi(Z) = Z$.

Assume (for a moment) that the claim is false. Then for all $X \in \mathcal{W}_h$, $X$ is (isomorphic to) an initial segment of $\Phi(X)$; whence $f(X, X) = 0$. Let $g : \mathcal{W}_h \rightarrow 2$ be as

\[
\begin{array}{c}
\mathcal{W}_h \times \mathcal{W}_h \\
\trianglearrow
\end{array}
\begin{array}{c}
\downarrow g \\
\downarrow g
\end{array}
\begin{array}{c}
\uparrow f
\\
\uparrow f
\\
\downarrow f
\\
\downarrow f
\end{array}
\begin{array}{c}
2 \\
2
\end{array}
\]

It follows from assuming the falsity of the claim (∗) that

\[g(X) = \text{neg}(f(X, X)) = 1 = f(X, \mathcal{W}_h).\]
Thus $g$ is representable by $f$ (at $\mathcal{W}_h$) and the usual contradiction (the existence of a fixed-point for $\text{neg}$) follows. So, the claim (*) is true, and there exists some $Z \in \mathcal{W}_h$ such that $h(Z) \in Z$ or equivalently $\Phi(Z) = Z$. It can be seen that then $\mathcal{W}_h = Z$, so $\Phi(\mathcal{W}_h) = \mathcal{W}_h$ and $h(\mathcal{W}_h) = h(\mathcal{W}_h) \subseteq \mathcal{W}_h$. Whence, as above, for the subset $\mathcal{V}_h = \{x \in \mathcal{W}_h \mid x <_{\mathcal{W}_h} h(\mathcal{W}_h)\}$ ($\subseteq A$) we will have $\mathcal{V}_h \not\subseteq \mathcal{W}_h$ and $h(\mathcal{V}_h) = h(\mathcal{W}_h) \in \mathcal{W}_h \setminus \mathcal{V}_h$. Note that both $\mathcal{W}_h$ and $\mathcal{V}_h$ were defined explicitly.

Let us iterate what was proved:

(Corollary 3.2) For any function $h : \mathcal{P}(A) \to A$ a subset $\mathcal{D}_h \subseteq A$ was explicitly defined in such a way that there exists some $\mathcal{C}_h \subseteq A$ (without an explicit definition) such that $\mathcal{C}_h \neq \mathcal{D}_h$ and $h(\mathcal{C}_h) = h(\mathcal{D}_h) \in \mathcal{D}_h \setminus \mathcal{C}_h$.

(Theorem 3.3) For any function $h : \mathcal{P}(A) \to A$ two subset $\mathcal{V}_h \subseteq A$ and $\mathcal{W}_h \subseteq A$ were explicitly defined in such a way that $\mathcal{V}_h \not\subseteq \mathcal{W}_h$ and $h(\mathcal{V}_h) = h(\mathcal{W}_h) \in \mathcal{W}_h \setminus \mathcal{V}_h$.

4 Yablo’s Paradox

To counter a general belief that all the paradoxes stem from a kind of circularity (or involve some self-reference, or use a diagonal argument) Stephen Yablo designed a paradox in 1985 that seemingly avoided self-reference (12, 13). Let us fix our reading of Yablo’s Paradox. Consider the sequence of sentences $\{Y_n\}_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$:

$Y_n$ is true $\iff \forall k > n (Y_k$ is untrue).

The paradox follows from the following deductions. For each $n \in \mathbb{N}$,

- $Y_n$ is true $\implies \forall k > n (Y_k$ is untrue)
- $\implies (Y_{n+1}$ is untrue) and $\forall k > n + 1 (Y_k$ is untrue)
- $\implies (Y_{n+1}$ is untrue) and $(Y_{n+1}$ is true),
- $\implies$ Contradiction!

Thus $Y_n$ is not true. So, $\forall k (Y_k$ is untrue),

and in particular $\forall k > 0 (Y_k$ is untrue),

and so $Y_0$ must be true (and untrue at the same time); contradiction!

4.1 Propositional Linear Temporal Logic

The propositional linear temporal logic (LTL) is a logical formalism that can refer to time; in LTL one can encode formulae about the future, e.g., a condition will eventually be true, a condition will be true until another fact becomes true, etc. LTL was first proposed for the formal verification of computer programs in 1977 by Amir Pnueli [9]. For a modern introduction to LTL and its syntax and semantics see e.g. [8].

Two modality operators in LTL that we will use are the “next” modality $\odot$ and the “always” modality $\Box$. The formula $\odot \varphi$ holds (in the current moment) when $\varphi$ is true in the “next step”, and the formula $\Box \varphi$ is true (in the current moment) when $\varphi$ is true “now and forever” (“always in the future”). In the other words, $\Box$ is the reflexive and transitive closure of $\odot$. It can be seen that the formula $\odot \neg \varphi \iff \neg \odot \varphi$ is always true (is a law of LTL, see T1 on page 27 of [8]), since $\varphi$ is untrue in the next step if and only if it is not the case that “$\varphi$ is true in the next step”. Also the formula $\Box \Box \psi$ is true when $\psi$ is true from the next step onward, that is $\psi$ holds in the next step, and the step after that, and the step after that, etc. The same holds for $\Box \odot \psi$; indeed the formula $\odot \Box \psi \iff \Box \odot \psi$ is a law of LTL (T12 on page 28 of [8]). Whence, we have the equivalences $\odot \Box \neg \varphi \iff \odot \neg \odot \varphi \iff \Box \neg \odot \varphi$ in LTL.

The intended model (semantics) of LTL is a system $\langle \mathbb{N}, \models \rangle$ where $\models \subseteq \mathbb{N} \times \text{Atoms}$ is an arbitrary relation which can be extended to all formulas as follows:
Here we can readily check the validity of the formula \( \circ \neg \varphi \iff \neg \circ \varphi \) as follows:

\[
\begin{align*}
n \models \circ \neg \varphi & \iff (n + 1) \models \neg \varphi \iff (n + 1) \not\models \varphi \iff n \not\models \circ \varphi \iff n \models \neg \circ \varphi.
\end{align*}
\]

Also the validity of \( \circ \varphi \iff \diamond \circ \varphi \) can be readily checked:

\[
\begin{align*}
n \models \circ \varphi & \iff (n + 1) \models \circ \varphi \\
& \iff \forall k \geq n + 1 \,(k \models \varphi) \\
& \iff \forall k \geq n \,(k + 1) \models \varphi) \\
& \iff \forall k \geq n \,(k \models \circ \varphi) \\
& \iff n \models \circ \varphi.
\end{align*}
\]

Now we show the non–existence of a formula \( Y \) that satisfies the equivalence

\[
Y \iff \circ \neg Y \iff \diamond \neg \circ Y;
\]

in other words \( Y \) is a fixed–point of the operator \( x \mapsto \circ \neg x \equiv \diamond \neg \circ x \). Following [14] we can demonstrate this by the following diagram

\[
\begin{array}{c}
\text{LTL} \times \text{LTL} \xrightarrow{f} 2 \\
\text{LTL} \xrightarrow{g} 2
\end{array}
\]

where \( \text{LTL} \) is the set of sentences in the language of LTL and \( f \) is defined by

\[
f(X, Y) = \begin{cases} 
1 & \text{if } X \not\equiv \circ \neg Y \\
0 & \text{if } X \equiv \circ \neg Y
\end{cases}
\]

Here, \( g \) is the characteristic function of all the Yablo–like sentences, the sentences which claim that all they say in the future (from the next step onward) is untrue.

**Theorem 4.1** For any \( \varphi \), the formula \( (\varphi \iff \circ \circ \neg \varphi) \) is not provable in LTL.

**Proof.** If LTL proves \( \psi \iff \circ \circ \neg \psi \) for some (propositional) formula \( \psi \), then for a model \( \langle \mathbb{N}, \models \rangle \):

(i) If \( m \models \psi \) for some \( m \), then \( m \models \circ \circ \neg \psi \) so \( (m + 1) \models \circ \neg \psi \), hence \( (m + i) \models \neg \psi \) for all \( i \geq 1 \).

In particular, \((m + 1) \models \circ \neg \psi \) and \((m + j) \models \neg \psi \) for all \( j \geq 2 \) which implies \((m + 2) \models \circ \neg \psi \) or \((m + 1) \models \circ \neg \psi \) so \((m + 1) \models \neg \psi \), a contradiction!

(ii) So for all \( k \) we have \( k \models \neg \psi \) or equivalently \( k \models \neg \circ \circ \neg \psi \) or \( k \models \circ \circ \neg \psi \), thus \( (k + 1) \models \neg \circ \circ \neg \psi \); hence \((k + n) \models \neg \varphi \) for some \( n \geq 1 \), contradicting (i)!

So, LTL \( \not\models (\varphi \iff \circ \circ \neg \varphi) \) for all formulas \( \varphi \). \( \square \)

The above proof is very similar to Yablo’s argument (in his paradox) presented at the beginning of this section, and this goes to say that Yablo’s paradox has turned into a genuine mathematico–logical theorem (in LTL) for the first time in Theorem 4.1.
4.2 Priest’s Inclosure Schema

In 1997 Priest [10] has shown the existence of a formula $Y(x)$ which satisfies $Y(n) \iff \forall k > n \ T(⌜Y(k)⌝)$ for every $n \in \mathbb{N}$, where $T(x)$ is a (supposedly truth) predicate; here $\psi$ is the (Gödel) code of the formula $\varphi$ and for a $k \in \mathbb{N}$, $k$ is a term representing $k$ (e.g. $1 + \cdots + 1 [k \text{ times}]$). Rigorous proofs for the existence of such a formula $Y(x)$ (and its construction) can be found in [3, 4]. Here we construct a formula $Y(x)$ which, for every $n \in \mathbb{N}$, satisfies the formula $Y(n) \iff \forall k > n \ \bar{\Psi}(⌜Y(k)⌝)$ for some $\Pi_1$ formula $\bar{\Psi}$, by using the Recursion Theorem (of Kleene); for recursion–theoretic definitions and theorems see e.g. [5]. Let $T$ denote Kleene’s $T$ Predicate, and for a fixed $\Pi_1$ formula $\Psi(x)$ let $r$ be the recursive function defined by $r(x, y) = \mu z (z > x \land \neg \Psi(⌜\exists u T(y, z, u)⌝))$; note that $\neg \Psi$ is a $\Sigma_1$ formula. By the S–m–n theorem there exists a primitive recursive function $s$ such that $\varphi_{s(e)}(x) = r(x, y)$; here $\varphi_n$ denotes the unary recursive function with (Gödel) code $n$, so $\varphi_0, \varphi_1, \varphi_2, \cdots$ lists all the unary recursive functions. By Kleene’s Recursion Theorem, there exists some (Gödel code) $e$ such that $\varphi_e = \varphi_{s(e)}$. Whence,

$$\varphi_e(x) = \varphi_{s(e)}(x) = r(x, e) = \mu z (z > x \land \neg \Psi(⌜\exists u T(e, z, u)⌝)).$$

So, for any $x \in \mathbb{N}$ we have $\exists u T(e, z, u) \iff \varphi_e(x) \downarrow \iff \exists z(z > x \land \neg \Psi(⌜\exists u T(e, z, u)⌝))$, or in the other words we have the equivalence

$$\neg \exists u T(e, z, u) \iff \forall z > x \ \Psi(⌜\exists u T(e, z, u)⌝).$$

Thus if we let $\mathcal{Y}(n) = \neg \exists z T(e, z, z)$, then for any $n \in \mathbb{N}$ we have

$$\mathcal{Y}(n) \iff \forall k > n \ \bar{\Psi}(⌜\mathcal{Y}(k)⌝).$$

Let us note that Yablo’s paradox occurs when $\Psi$ is taken to be an untruth (or non-satisfaction) predicate; in fact one might be tempted to take $\neg \text{Sat}_{\Pi_1}(x, \emptyset)$ (see Theorem 1.75 of [6]) as $\Psi(x)$; but by construction $\text{Sat}_{\Pi_1}(x, \emptyset)$ is $\Pi_1$ and so $\neg \text{Sat}_{\Pi_1}(x, \emptyset)$ is $\Sigma_1$, and our proof works for $\Psi \in \Pi_1$ only (otherwise the function $r$ could not be recursive). Actually, the above construction shows that the predicate $\text{Sat}_{\Pi_1}(x, \emptyset)$ (in [6]) cannot be $\Sigma_1$, which is equivalent to saying that the set of (arithmetical) true $\Pi_1$ sentences cannot be recursively enumerable, and this is a consequence of Gödel’s first incompleteness theorem (cf. [3, 4]).

In [10] Priest also introduced his Inclosure Schema and showed that Yablo’s paradox is amenable in it (see also [2]). In the following, we show that Priest’s Inclosure Schema can fit in Yanofsky’s framework [13]. With some inessential modification for better reading, Priest’s inclosure schema is defined to be a triple $(\Omega, \Theta, \delta)$ where

- $\Omega$ is a set of objects;
- $\Theta \subseteq \mathcal{P}(\Omega)$ is a property of subsets of $\Omega$ such that $\Omega \in \Theta$;
- $\delta : \Theta \to \Omega$ is a function such that for each $X \in \Theta$, $\delta(X) \notin X$.

That any inclosure schema is contradictory can be seen from the fact that by the second item $\delta(\Omega)$ must be defined and belong to $\Omega$, but at the same time by the third item $\delta(\Omega) \notin \Omega$. We show how this can be proved by the non–existence of a fixed–point for the negation function.

**Theorem 4.2** If an inclosure schema exists, then negation has a fixed-point.

**Proof.** Assume $(\Omega, \Theta, \delta)$ is a (hypothetical) inclosure schema. Put $f : \Theta \times \Theta \to 2$ as

$$f(X, Y) = \begin{cases} 1 & \text{if } \delta(X) \in Y \\ 0 & \text{if } \delta(X) \notin Y \end{cases}$$

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and let \( g : \Theta \to 2 \) be defined as

\[
\begin{array}{c}
\Theta \times \Theta \xrightarrow{f} 2 \\
\text{△}_\Theta \xrightarrow{\neg} 2 \\
\Theta \xrightarrow{g} 2
\end{array}
\]

We show that \( g \) is representable by \( f \) at \( \Omega \). For every \( X \in \Theta \) we have \( f(X, \Omega) = 1 \). On the other hand by the property of \( \delta \), for any \( X \in \Theta \), \( \delta(X) \not\in X \), and so \( f(X, X) = 0 \), thus \( g(X) = \neg(f(X, X)) = 1 \). Whence \( g(X) = f(X, \Omega) \) for all \( X \in \Theta \).

## 5 Dominating Functions

Ackermann’s function is a recursive (computable) function which is not primitive recursive (see e.g. [5]). The class of primitive recursive functions is the smallest class which contains the initial functions, i.e.,

- the constant zero function \( z(x) = 0 \),
- the successor function \( s(x) = x + 1 \) and
- the projection functions \( p^i_n(x_1, \ldots, x_n) = x_i \) for any \( 1 \leq i \leq n \in \mathbb{N} \),

and is closed under

- composition and
- primitive recursion,

i.e., for primitive recursive functions \( f, f_1, \ldots, f_n \) the function \( \text{comp}(f; f_1, \ldots, f_n) \) defined by \( (x_1, \ldots, x_m) \mapsto f(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) \) is also primitive recursive, and also for primitive recursive functions \( g \) and \( h \) the function \( \text{prim-rec}(g, h) \) defined by \( (x_1, \ldots, x_n, 0) \mapsto g(x_1, \ldots, x_n) \) and

\[
(x_1, \ldots, x_n, x + 1) \mapsto h(\text{prim-rec}(g, h)(x_1, \ldots, x_n, x), x_1, \ldots, x_n, x)
\]

is also primitive recursive.

The class of recursive functions contains the same initial functions and is closed under composition, primitive recursion, and also

- minimization,

i.e., for recursive function \( f \) the function \( \text{min}(f) \) defined by \( (x_1, \ldots, x_n) \mapsto y \) where \( y \) is the least natural number that satisfies \( f(x_1, \ldots, x_n, y) = 0 \) is also recursive; note that then for all \( z < y \) we have \( f(x_1, \ldots, x_n, z) \neq 0 \), and if there is no such \( y \) then \( \text{min}(f) \) is undefined on \( x_1, \ldots, x_n \).

In fact, Ackermann’s function is not only a non–primitive recursive (and a recursive) function, but it also dominates all the primitive recursive functions (see e.g. [5]). A function \( g \) is said to dominate a function \( f \) (or \( f \) is dominated by \( g \)) when for all but finitely many \( x \)’s the inequality \( g(x) > f(x) \) holds. Here we show a way of dominating a given enumerable list of functions by diagonalization. Before that let us note that the set of all primitive recursive functions can be (recursively) enumerated: let \( \#(f) \) denote the (Gödel) code of the function \( f \) and define the Gödel code of a primitive recursive function inductively:

- \( \#(z) = 1 \),
- \( \#(s) = 2 \),
- \( \#(\text{comp}(f; f_1, \ldots, f_n)) = 5 \cdot \#(f) \cdot 7 \cdot \#(f_1) \cdot \cdots \cdot \#(f_n) \) and
- \( \#(\text{prim-rec}(g, h)) = 3 \cdot \#(g) \cdot 5 \cdot \#(h) \),

where \( p_i \) is the \( i \)–th prime number (thus, \( p_0 = 2, p_1 = 3, p_2 = 5, p_3 = 7, \ldots \)). Let \( \nu_n \) be the primitive recursive function with code \( n \), if \( n \) is a code of such a function; if \( n \) is not a code for a primitive recursive
function (such as \( n = 3 \) or \( n = 10 \)) then let \( \nu_n \) be the constant zero function \( z \). So, \( \nu_0, \nu_1, \nu_2, \cdots \) lists all the primitive recursive functions. We show the existence of a unary function that dominates all the functions \( \nu_i \)'s in the above list.

**Theorem 5.1** For a list of functions \( f_1, f_2, f_3, \cdots : \mathbb{N} \rightarrow \mathbb{N} \), there exists a unary function \( \mathbb{N} \rightarrow \mathbb{N} \) that dominates them all.

**Proof.** Define the function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) as 
\[
f(n, m) = \max_{i \leq n} f_i(m)
\]
and let \( g \) be defined by the following diagram where \( s \) is the successor function:

\[
\begin{array}{ccc}
\mathbb{N} \times \mathbb{N} & \xrightarrow{f} & \mathbb{N} \\
\Delta_{\mathbb{N}} & & \\
\mathbb{N} & \xrightarrow{s} & \mathbb{N} \\
\end{array}
\]

In fact, the function \( g : \mathbb{N} \rightarrow \mathbb{N} \) is defined as 
\[
g(x) = \max_{i \leq x} f_i(x) + 1.
\]
Since the successor function does not have any fixed-point, the function \( g \) is not equal to any of \( f_i \)'s. Moreover, \( g \) dominates all the \( f_i \)'s, since for any \( m \in \mathbb{N} \) and any \( x \geq m \) by the definition of \( g \) we have 
\[
g(x) > \max_{i \leq x} f_i(x) \geq f_m(x).
\]

For dominating the primitive recursive functions (some of which are not unary) we can consider their unarized version: let \( \rho_0, \rho_1, \rho_2, \cdots \) be the list of unary functions \( \mathbb{N} \rightarrow \mathbb{N} \) defined as 
\[
\rho_i(x) = \nu_i(x, \cdots, x).
\]
Whence \( \rho_0, \rho_1, \rho_2, \cdots \) lists all the unary primitive recursive functions, and the construction of Theorem 5.1 produces a unary function which dominates all the unary primitive recursive functions. Let us note that the function \( g \) obtained in the proof of Theorem 5.1 is computable (intuitively) and so recursive (by Church’s Thesis); one can show directly that the above function \( g \) is recursive (without appealing to Church’s Thesis) by some detailed work through Recursion Theory (cf. e.g. [5]).

6 Conclusions

There are many interesting questions and suggestions for further research at the end of [14] which motivated the research presented in this paper; most of the questions remain unanswered as of today. The proposed schema, i.e., the diagram of the proof of Theorem 1.1

\[
\begin{array}{ccc}
B \times B & \xrightarrow{f} & D \\
\Delta_B & & \\
B & \xrightarrow{g} & D \\
\end{array}
\]

can be used as a criterion for testing whether an argument is diagonal or not. What makes this argument (of the non-existence of a fixed-point for \( \alpha : D \rightarrow D \)) diagonal is the diagonal function \( \Delta_B : B \rightarrow B \times B \). In most of our arguments we had \( D = 2 = \{0, 1\} \) and \( \alpha = \text{neg} \) by which the proof was constructed by diagonalizing out of the function \( f : B \times B \rightarrow D \). Only in Theorem 5.1 we had \( D = \mathbb{N} \) and \( \alpha = s \) (the successor function) which was used for generating a dominating function. We could have used the
diagonalizing out argument by setting $D = 2 = \{0, 1\}$ and $\alpha = \text{neg}$ for the function $\tilde{f} : \mathbb{N} \times \mathbb{N} \to 2$, defined by

$$\tilde{f}(n, m) = \begin{cases} 0 & \text{if } f_n(m) = 0 \\ 1 & \text{if } f_n(m) \neq 0 \end{cases}$$

Then the constructed function $\check{g} : \mathbb{N} \to 2$ by $\check{g}(n) = \text{neg}(\tilde{f}(n, n))$ differs from all the functions $f_i$’s (because $\check{g}(i) \neq f_i(i)$ for all $i$). So, this way one could construct a non-primitive recursive (but recursive) function, though this function does not dominate all the primitive recursive functions.

For other exciting questions and examples of theorems or paradoxes which seem to be self-referential we refer the reader to the last section of [14]. It will be nice to see some of those proposals or other more phenomena fit in the above universal diagonal schema.

References

[1] George Boolos, Constructing Cantorian Counterexamples, *Journal of Philosophical Logic* 26:3 (1997) 237–239.

[2] Otávio Bueno & Mark Colyvan, Paradox without Satisfaction, *Analysis* 63:2 (2003) 152–156.

[3] Cezary Cieśliński, “Yablo Sequences in Truth Theories”, in: Kamal Lodaya (ed.) *Logic and Its Applications*, Proceedings of the 5th Indian Conference, ICLA 2013, Chennai, India, January 10–12, 2013, Lecture Notes in Computer Science, Volume 7750, Springer (2013) pp. 127–138.

[4] Cezary Cieśliński & Rafał Urbaniak, Gödelizing the Yablo Sequence, *Journal of Philosophical Logic* 42:5 (2013) 679–695.

[5] Richard L. Epstein & Walter A. Carnielli, *Computability: computable functions, logic, and the foundations of mathematics*, Advanced Reasoning Forum (3rd ed. 2008).

[6] Petr Hájek & and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*, Springer (2nd. print. 1998).

[7] Akicho Kanamori & David Pincus, “Does GCH Imply AC Locally?”, in: Gabor Halasz & Laszlo Lovasz & Miklos Simonovits & Vera T. Sós (eds.) *Paul Erdős and His Mathematics II*, Bolyai Society for Mathematical Studies, Vol. 11, János Bolyai Mathematical Society & Springer (2002) pp. 413–426.

[8] Fred Kröger & Stephan Merz, *Temporal Logic and State Systems*, Springer (2008).

[9] Amir Pnueli, “The Temporal Logic of Programs”, in: Proceedings of the 18th Annual Symposium on Foundations of Computer Science (SFCS’77) IEEE Computer Society, USA (1977) pp. 46–57.

[10] Graham Priest, Yablo’s Paradox, *Analysis* 57:4 (1997) 236–242.

[11] Natarajan Raja, “Yet Another Proof of Cantor’s Theorem”, in: Jean-Yves Béziau & Alexandre Costa-Leite (eds.) *Dimensions of Logical Concepts*, Coleção CLE: Volume 54 (2009) pp. 209–217.

[12] Stephen Yablo, Truth and Reflection, *Journal of Philosophical Logic* 14:3 (1985) 297–349.

[13] Stephen Yablo, Paradox without Self-Reference, *Analysis* 53:4 (1993) 251–252.

[14] Noson S. Yanofsky, A Universal Approach to Self-Referential Paradoxes, Incompleteness and Fixed Points, *Bulletin of Symbolic Logic* 9:3 (2003) 362–386.