Two-Party Function Computation on the Reconciled Data

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Abstract—In this paper, we initiate a study of a new problem termed function computation on the reconciled data, which generalizes a set reconciliation problem in the literature. Assume a distributed data storage system with two users $A$ and $B$. The users possess a collection of binary vectors $S_A$ and $S_B$, respectively. They are interested in computing a function $\phi$ of the reconciled data $S_A \cup S_B$.

It is shown that any deterministic protocol, which computes a sum and a product of reconciled sets of binary vectors represented as nonnegative integers, has to communicate at least $2^n + n - 1$ and $2^n + n - 2$ bits in the worst-case scenario, respectively, where $n$ is the length of the binary vectors. Connections to other problems in computer science, such as set disjointness and finding the intersection, are established, yielding a variety of additional upper and lower bounds on the communication complexity. A protocol for computation of a sum function, which is based on use of a family of hash functions, is presented, and its characteristics are analyzed.

I. INTRODUCTION

The problem of data synchronization arises in many applications in distributed data storage systems and data networks. For instance, consider a number of users that concurrently access and update a jointly used distributively stored large database. When one of the users makes an update in the data stored locally, the other users are not immediately aware of the change, and thus an efficient method for synchronization of the data is required. This practical problem arises in many systems that store big amounts of data, including those employed by companies such as Dropbox, Google, Amazon, and others.

The problem of data synchronization was studied in the literature over the recent years. A variation of this problem termed two-party set reconciliation considers a scenario, where two users communicate via a direct bi-directional noiseless channel. The users, $A$ and $B$, possess respective sets $S_A$ and $S_B$ of binary vectors. The users execute a communications protocol by sending binary messages to each other. At the end of the protocol, each of the users knows $S_A \cup S_B$. Set reconciliation problem was first studied in [14]. Some of the recent works that investigate this problem are [4], [5], [9], [15], [20]. A number of protocols for set reconciliation were proposed, and their theoretical performance was analyzed.

All aforementioned protocols communicate amount of data, which is asymptotically optimal.

In practical data storage systems, sometimes only a function of the stored data can be requested by some user, and not the data itself. It can be more efficient to compute a function by a group of servers, rather that to provide the full data required for such a computation by the user (see, for instance, Example III.1 below). Therefore, it is an important question how to compute various functions of the data distributed among a number of servers.

The domain of distributed function computation is a mature area, which has been very extensively studied both in computer science and information theory communities. The reader can refer, for example, to [11], [13], [17], [18], [21], and many others. In a standard model, a number of users want to compute jointly a function of the data that they possess. This needs to be achieved by communicating the smallest possible number of bits. This class of problems is very broad, and it covers settings with various types of functions, two versus many users, deterministic and randomized protocols, with or without privacy requirements, etc.

Motivated by the above challenges, in this work, we propose a new problem, which we term function computation on the reconciled data. To the best of our knowledge, this problem was not studied in the literature yet. In this problem, the users compute a function of their reconciled data. It is obvious that this problem can be solved by reconciling the data first, and then by computing the function of this data by the users. However, as we demonstrate in the sequel, this approach is not always optimal in terms of a number of communicated bits.

This paper is structured as follows. In Section II the problem of function computation on the reconciled data is introduced. In Section III known methods for set reconciliation are surveyed. It is shown that using reconciliation as a subroutine does not necessarily yield an optimal solution. A number of bounds on the communication complexity of sum computation on the reconciled data are obtained in Section IV. Connections to some known problems in computer science are established in Section V. A protocol for computation of sum using universal hash functions and its analysis are presented in Section VI. The results are summarized in Section VII.

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1This work is supported in part by the grant EMP133 from the Norwegian-Estonian Research Cooperation Programme and by the grants PUT405 and IUT2-1 from the Estonian Research Council.
II. Problem settings

Let $\mathbb{F} = \{0, 1\}$ be a binary field. Denote by $\mathbb{F}^n$ the vector space of dimension $n$ over $\mathbb{F}$. By slightly abusing the notation, sometimes we treat $\mathbb{F}^n$ as a set of all vectors of length $n$ over $\mathbb{F}$, or, as a set of nonnegative integers in their binary representation. Let the set of all subsets of $\mathbb{F}^n$ be $\mathcal{P}(\mathbb{F}^n)$. We denote $[\ell] \triangleq \{1, 2, \ldots, \ell\}$.

Consider two users, $A$ and $B$, possessing sets $S_A, S_B \subseteq \mathbb{F}^n$, respectively. We denote the intersection of these two sets as $S_0 = S_A \cap S_B$. The sizes of these sets are given as $m_0 = |S_0|$, $m_A = |S_A|$ and $m_B = |S_B|$. Additionally, it is assumed that $\max\{m_A, m_B\} \leq \kappa$. Denote the sizes of the set differences as $d_A = |S_A \setminus S_0|$, $d_B = |S_B \setminus S_0|$ and $d = d_A + d_B$. We assume hereby that $A$ knows the values of $d_A$ and $m_0$, and that $B$ knows the values of $d_B$ and $m_0$.

The users $A$ and $B$ want to compute cooperatively a function $f : \mathcal{P}(\mathbb{F}^n) \times \mathcal{P}(\mathbb{F}^n) \to V$, where $V$ is the range of $f$. The functions that we consider in this work are all defined over the reconciled data, namely they have the form $f(S_A, S_B) = \phi(S_A \cup S_B)$, where $S_A \cup S_B$ is a standard set-theoretic union of the two sets, and $\phi : \mathcal{P}(\mathbb{F}^n) \to V$.

In order to do so, $A$ and $B$ jointly execute a communications protocol, according to which they send binary messages to each other. Specifically, the protocol $F$ consists of the messages

$$M_1 = (w_{1,1}, w_{1,2}, \ldots, w_{1,p_1}) \in \mathbb{F}^{p_1},$$
$$M_2 = (w_{2,1}, w_{2,2}, \ldots, w_{2,p_2}) \in \mathbb{F}^{p_2},$$
$$\vdots$$
$$M_r = (w_{r,1}, w_{r,2}, \ldots, w_{r,p_r}) \in \mathbb{F}^{p_r},$$

which are sent alternately between $A$ and $B$. After the message $M_r$ is sent, both users can compute the value of $f(S_A, S_B)$. The number of messages $r$ is called the number of rounds of the protocol.

Communication complexity $\text{COMM}(F)$ of the protocol $F$ is defined as the minimum total number of bits $\sum_{i=1}^r p_i$ that are sent between the users in the worst-case scenario for all $S_A, S_B \in \mathcal{P}(\mathbb{F}^n)$.

There are different models of how the protocols use randomness. In deterministic protocol, we assume that all computations and messages sent by the users are deterministic, and they are uniquely determined by the sets $S_A$ and $S_B$. By following the discussion in [6], we consider several randomized protocol models. In a protocol with shared randomness, both users $A$ and $B$ have access to an infinite sequence of independent unbiased random bits. The users are expected to compute the function correctly with probability close to 1. By contrast, in a protocol with private randomness, each user possesses its own string of random bits. Finally, in the “Las-Vegas”-type protocol, at the end of the protocol the users always compute the function correctly, but the number of communicated bits is a random variable, and the complexity is measured as the expected number of the communicated bits.

III. Connection to set reconciliation

The set reconciliation problem can be viewed as a function computation problem on the reconciled data, where the function $\phi$ is an identity, namely, $f(S_A, S_B) = S_A \cup S_B$. A number of protocols were proposed in the literature for efficient distributed set reconciliation with two users. In [14], interpolation of characteristic polynomials over a Galois field is used. The proposed deterministic protocol assumes the knowledge of approximate values of $d_A$ and $d_B$, and it achieves $\text{COMM}(F) = O(dn)$, which is asymptotically communication-optimal. In particular, when $d$ is small compared to $n$, that protocol clearly outperforms a naive reconciliation scheme, where the users simply exchange their data.

Another randomized protocol, which employs invertible Bloom filters, was presented in [4], [5]. Alternatively, it was proposed to use so-called biff codes for randomized set reconciliation in [15]. Finally, a randomized protocol that uses techniques akin to linear network coding were employed in [20] leading to yet another reconciliation protocol. The latter method assumes existence of certain family of pseudo-random hash functions. All mentioned randomized protocols have asymptotically optimal communication complexity $\text{COMM}(F) = O(dn)$.

We note that a problem of computing any function $f$ can be solved by $A$ and $B$ by reconciling their data first, and then by computing $f$ by each user separately (or by one of the users). By using this method, the communication complexity is determined by the complexity of the underlying set reconciliation protocol. For example, for each of the aforementioned protocols, $\text{COMM}(F) = O(dn)$. Sometimes, an improvement in communication complexity can be obtained by using one-directional reconciliation, namely, when the data is reconciled by only one user, and then the function value is sent back to the other user. However, if $d_A = d_B$, this approach does not lead to asymptotic improvement.

As the following example illustrates, some functions can be computed by a deterministic protocol with much smaller communication complexity.

**Example III.1.** Assume that $A$ and $B$ are interested in computing $f(S_A, S_B) = \max\{S_A \cup S_B\}$, where all entries in $S_A \cup S_B$ are viewed as non-negative integer numbers in their binary representation. The following protocol requires only $2n$-bit communication.

1) The users $A$ and $B$ compute $x_A = \max\{S_A\}$ and $x_B = \max\{S_B\}$, respectively.
2) The users $A$ and $B$ exchange the values of $x_A$ and $x_B$.
3) Each user computes $\max\{x_A, x_B\}$.

Analogous protocol can be used to compute a number of other idempotent functions $\phi$, such as minimum, bit-wise logical or and bit-wise logical and. It is an interesting question, however, what is the worst-case number of communicated bits for computing different functions on the reconciled data. We partly answer this question for some of the functions in the sequel.
IV. LOWER BOUNDS USING f-MONOCROMATIC RECTANGLES

A. Sum over integers

In this section, we consider the function \( f \) with the integer range, defined as follows:

\[
f(S_A, S_B) = \sum_{x \in S_A \cup S_B} x,
\]

where every string \( x \in S_A \cup S_B \) can be viewed as an integer in its binary representation.

We introduce the following definition, which is taken from [10] Definition 1.

**Definition IV.1.** Let \( \eta \in \mathbb{N} \) and \( f: \mathbb{F}^n \times \mathbb{F}^n \rightarrow V \) be a function with range \( V \). A rectangle is a subset of \( \mathbb{F}^n \times \mathbb{F}^n \) of the form \( X_1 \times X_2 \), where \( X_1, X_2 \subseteq \mathbb{F}^n \). A rectangle \( X_1 \times X_2 \) is called \( f \)-monochromatic if for every \( x \in X_1 \) and \( y \in X_2 \), the value of \( f(x, y) \) is the same.

**Lemma IV.2.** [11] Proposition 1.13 Let \( R \subseteq \mathbb{F}^n \times \mathbb{F}^n \). Then \( R \) is a rectangle if and only if

\[
(x_1, y_1) \in R \quad \text{and} \quad (x_2, y_2) \in R \quad \implies \quad (x_1, y_2) \in R.
\]

**Definition IV.3.** [10] Let \( f: \mathbb{F}^n \times \mathbb{F}^n \rightarrow V \) be a function. Denote by \( \mathcal{R}(f) \) the minimum number of \( f \)-monochromatic rectangles that partition the space of \( \mathbb{F}^n \times \mathbb{F}^n \).

We use the following lemma, which is stated in [10] Lemma 2]. It allows to reformulate the problem of lower-bounding communication complexity as a problem in combinatorics.

**Lemma IV.4.** Let \( f: \mathbb{F}^n \times \mathbb{F}^n \rightarrow V \) be a function, which is computed using protocol \( F \). Then,

\[
\text{COMM}(F) \geq \log_2(\mathcal{R}(f)).
\]

The proof of the lemma is given in [11].

In order to be able to use Lemma IV.4, we need to represent the inputs \( S_A \) and \( S_B \) as binary vectors. A natural way to do that is by using binary characteristic vectors \( a \) and \( b \) of length \( \eta = 2^n \).

**Theorem IV.5.** The number of bits communicated between \( A \) and \( B \) in any deterministic protocol \( F \) that computes the function \( f \) defined in (1) is at least

\[
\text{COMM}(F) \geq 2^n + n - 1.
\]

**Proof.** The proof is done by estimating the number of \( f \)-monochromatic rectangles, where \( f \) is given by (1).

Denote \( \Phi \triangleq \mathbb{F}^n \\setminus \{0\} \), where the elements of \( \Phi \) can be viewed as integers in \([2^n - 1]\). We use the following set of pairs of subsets

\[
\mathcal{F}_0 = \{(Y, \Phi \setminus Y) : Y \subseteq \Phi \} \triangleq \{(Y_i, Y'_i) : i \in [2^{2^n-1}]\}.
\]

Then, for every \((Y_i, Y'_i) \in \mathcal{F}_0\), we have

\[
f(Y_i, Y'_i) = \sum_{i=1}^{2^{2^n-1}} i = 2^{2^n-1}(2^n - 1).
\]

On the other hand, take \( i, j \in [2^{2^n-2}] \), such that \( i \neq j \). We have two cases:

- If \( Y_i \cup Y'_i \neq \Phi \), then there exists \( x \in \Phi \), such that \( x \notin Y_i \cup Y'_i \).

Thus, \( x \notin Y'_i \cup Y_j \), and therefore

\[
f(Y_j, Y'_j) < 2^{2^n-1}(2^n - 1).
\]

Therefore, due to Lemma IV.2, there are at least \( 2^{2^n-1} \) different \( f \)-monochromatic rectangles consisting of the elements of \( \mathcal{F}_0 \).

Additionally, for any \( \ell \in [2^n - 1] \), denote \( \Phi_\ell \triangleq \mathbb{F}^n \\setminus \{0, \ell\} \). We use the following pairs

\[
\mathcal{F}_\ell = \{(Z, \Phi_\ell \setminus Z) : Z \subseteq \Phi_\ell \} \triangleq \{(Z_i, Z'_i) : i \in [2^{2^n-2}]\}.
\]

Then, for every \((Z_i, Z'_i) \in \mathcal{F}_\ell\), we have

\[
f(Z_i, Z'_i) = \sum_{i=1}^{2^{2^n-1}} i - \ell = 2^{2^n-1}(2^n - 1) - \ell.
\]

On the other hand, take \( i, j \in [2^{2^n-1}] \), such that \( i \neq j \). Similarly to the previous case, it can be shown that either

\[
f(Z_j, Z'_j) < 2^{2^n-1}(2^n - 1) - \ell \quad \text{or} \quad f(Z_i, Z'_i) < 2^{2^n-1}(2^n - 1) - \ell.
\]

Therefore, due to Lemma IV.2, there are at least \( 2^{2^n-2} \) different \( f \)-monochromatic rectangles consisting of the elements of \( \mathcal{F}_\ell \). Since \( \ell \) can be chosen in \( 2^{2^n} - 1 \) ways, we conclude that the number of different \( f \)-monochromatic rectangles is at least

\[
\mathcal{R}(f) \geq 2^{2^n-1} + (2^n - 1) \cdot (2^{2^n-2}) = (2^{2^n-2}) \cdot (2^n + 1) > 2^{2^n+n-2}.
\]

Finally, by applying Lemma IV.4 and by rounding the result up to the next bit, we obtain that \( \text{COMM}(F) \geq 2^n + n - 1 \).
sets of \(f\)-monochromatic rectangles, \(F_0, F_1, F_2\) and \(F_3\), are shown in four different colors. Each set contains a number of a single-entry \(f\)-monochromatic rectangles.

We see that the total number of monochromatic rectangles is at least

\[
R(f) \geq |F_0| + |F_1| + |F_2| + |F_3| = 8 + 4 + 4 + 4 = 20.
\]

By using Lemma IV.3, the communication complexity is at least \(\log_2(R(f)) = \log_2(20)\) bits. By rounding up to the next integer, we obtain that \(\text{COMM}(f) \geq 5\).

We remark that the result can be slightly improved by using the fact that there are additional rectangles corresponding to the values 0, 1 and 2. However, that improvement is relatively small, and thus we omit it for the sake of simplicity.

We also note that there is a trivial deterministic protocol that computes \(f\) by using \(2^n + 2n - 2\) bits: first, \(A\) sends the characteristic vector \(a\) of \(S_A\) of length \(2^n - 1\) (note that zero does not affect the sum) to \(B\), then \(B\) computes \(f\) and sends the result back to \(A\). Since the sum requires \(2n - 1\) bits to represent, the claimed result follows.

B. Multiplication over integers

As before, let \(S_A, S_B \subseteq \mathbb{F}^n\). Consider the function \(f\) with the integer range, defined as follows:

\[
f(S_A, S_B) = \prod_{x \in S_A \cup S_B} x.
\] (3)

The following theorem presents a lower bound on the communication complexity of a two-party deterministic protocol for computation of this \(f\).

**Theorem IV.6.** The number of bits communicated between \(A\) and \(B\) in any deterministic protocol \(F\) that computes the function \(f\) defined in (5) is at least

\[
\text{COMM}(F) \geq 2^n + n - 2.
\]

**Proof.** The proof is analogous to the proof of Theorem IV.5.

We estimate the number of different \(f\)-monochromatic rectangles, and then apply Lemma IV.3 to obtain a lower bound on the communication complexity.

Denote \(\Phi \triangleq \mathbb{F}^n \setminus \{0, 1\}\). At first, we count the number of rectangles on the main diagonal. We define:

\[
F_0 = \{(Y, \Phi \setminus Y) : Y \subseteq \Phi\} \triangleq \{(Y, Y') : i \in [2^{n-2}]\}.
\]

Then, for every \((Y_i, Y'_i) \in F_0:\)

\[
f(Y_i, Y'_i) = \prod_{i=2}^{2^n-1} i = (2^n - 1)!. \tag{4}
\]

Take \(i, j \in [2^{n-2}]\) such that \(i \neq j\). We consider two cases:

- If \(Y_i \cup Y'_i \neq \Phi\), since \(Y_i \neq Y_j\), there exists \(x \in Y_i \cap Y'_j\), thus \(x \notin Y_i' \cup Y_j\). Then,

\[
f(Y_i, Y'_j) < (2^n - 1)!. \tag{5}
\]

- If \(Y_i \cup Y'_i = \Phi\), since \(Y_i \neq Y_j\), there exists \(x \in Y_i \cap Y'_j\), thus \(x \notin Y_i' \cup Y_j\). Then,

\[
f(Y_i, Y'_j) < (2^n - 1)!
\]

Due to Lemma IV.3 there exist at least \(2^{2^n-2}\) different \(f\)-monochromatic rectangles in \(F_0\).

Additional \(f\)-monochromatic rectangles can be constructed as follows. For every \(\ell \in \{2, \ldots, 2^n - 1\}\), denote \(\Phi_\ell \triangleq \mathbb{F}^n \setminus \{0, 1, \ell\}\). We define the pairs

\[
F_\ell = \{(Z, \Phi_\ell \setminus Z) : Z \subseteq \Phi_\ell\} \triangleq \{(Z_i, Z'_i) : i \in [2^{n-3}]\}.
\]

Then, for every pair \((Z_i, Z'_i) \in F_\ell\) we have that

\[
f(Z_i, Z'_i) = \prod_{i=2}^{2^n-1} i = \frac{(2^n - 1)!}{\ell}.
\]

Take \(i, j \in [2^{n-3}]\) such that \(i \neq j\). Then, similarly to the proof of Theorem IV.5 either

\[
f(Z_j, Z'_i) = \frac{(2^n - 1)!}{\ell} \quad \text{or} \quad f(Z_i, Z'_j) = \frac{(2^n - 1)!}{\ell}. \tag{6}
\]

From Lemma IV.3 the set \(F_\ell\) contains \(2^{2^n-3}\) \(f\)-monochromatic rectangles. We can choose \(\ell\) in \(2^n - 2\) ways, and thus the number of \(f\)-monochromatic rectangles in \(F_\ell, \ell \neq 0\), is

\[
(2^n - 2) \cdot (2^{2^n-3}). \tag{7}
\]

There is at least one additional \(f\)-monochromatic rectangle corresponding to the value 0 of the function \(f\). By summing things up, we obtain that the total number of \(f\)-monochromatic rectangles is at least

\[
\text{COMM}(F) \geq 2^{n-2} + (2^n - 2) \cdot (2^{2^n-3}) + 1 = 2^{n+n-3} + 1.
\]

Due to Lemma IV.3 by rounding up to the next integer, the communication complexity of a protocol \(F\) computing \(f\) as defined in Equation IV.3 is at least \(\text{COMM}(F) \geq 2^n + n - 2\).

V. CONNECTIONS TO KNOWN PROBLEMS

A. Lower Bounds using Results for Set Disjointness

Given two sets \(S_A, S_B \subseteq \mathbb{F}^n\), the binary set disjointness function \(\text{DISJ}(S_A, S_B)\) is defined as follows:

\[
\text{DISJ}(S_A, S_B) = \begin{cases} 1 & \text{if } S_A \cap S_B = \emptyset, \\ 0 & \text{otherwise.} \end{cases}
\]

**Set disjointness problem:** there are two users \(A\) and \(B\) that possess the sets \(S_A, S_B \subseteq \mathbb{F}^n\), respectively. The users want to compute jointly the function \(\text{DISJ}(S_A, S_B)\).

We show a simple reduction from the set disjointness problem to the sum computation problem.

**Reduction:** assume that \(F\) is a protocol for computing \(f\) in (1) by \(A\) and \(B\). Then, given \(S_A\) and \(S_B\), the set disjointness problem can be solved by \(A\) and \(B\) as follows.
1) The user $A$ sends to $B$ a special bit, indicating if $0 \in A$. If $0 \in A \cap B$, then $B$ announces that $\text{Disj} (S_A, S_B) = 0$. Halt.
2) The users $A$ and $B$ compute $x_A = \sum x \in S_A x$ and $x_B = \sum x \in S_B x$, respectively.
3) The users $A$ and $B$ run the protocol $F$ to find $y = f(S_A, S_B)$.
4) User $B$ sends $x_B$ to $A$.
5) If $x_A + x_B = y$, then $A$ concludes that $\text{Disj} (S_A, S_B) = 1$. Otherwise, if $x_A + x_B \neq y$, then $\text{Disj} (S_A, S_B) = 0$.

The correctness of the protocol is straightforward, given that $S_A \cap S_B = \emptyset$ and if only if $x_A + x_B = y$ and $0 \notin A \cap B$.

A single bit is sent in Step 1 and $2n - 1$ bits are required to represent the integer value of $x_B$ in Step 4. Thus, the communication complexity of the proposed protocol for the set disjointness problem is $\text{COMM}(F) + 2n$. Then, the upper bound for set disjointness problem is $\text{COMM}(F) + 2n \geq \Theta(2^n)$.

There is a variety of known bounds on communication complexity of the two-party protocols for the set disjointness problem. For example, for deterministic protocols, there is a lower bound of $2^n + 1$ bits [11] using fooling sets, and for randomized protocols the asymptotically tight bound is $\Theta(2^n)$ [1], [6], [8], [19]. From these bounds, we obtain the lower bounds $\text{COMM}(F) \geq 2^n - 2n + 1$ for deterministic and $\text{COMM}(F) = \Omega(2^n)$ for randomized case of function computation problem.

Recall that for the deterministic case, there is an upper bound of $O(2^n)$ for sum computation problem (see discussion at the end of Section IV-A), which is also an upper bound on complexity of any randomized protocol, thus yielding an asymptotically tight bound of $\Theta(2^n)$ for randomized settings.

B. Upper Bound using Finding the Intersection Problem

Another related problem is finding the intersection [2], in which the users $A$ and $B$ are interested in finding the intersection of the sets that they possess.

Finding the intersection problem: there are two users $A$ and $B$ that possess the sets $S_A, S_B \subseteq \mathbb{F}^n$, respectively. The users want to compute jointly the function $S_A \cap S_B$.

A protocol for this problem can be used to compute a sum (or, for example, a product) of the reconciled sets.

The following result is proved in [2] for the sets of size at most $\kappa$.

Theorem V.1. [2, Theorem 3.1] There exists an $O(\sqrt{\kappa})$-round constructive randomized protocol for finding the intersection problem with success probability $1 - 1/\text{POLY}(\kappa)$. In the model of shared randomness the total communication complexity is $O(\kappa)$ and in the model of private randomness it is $O(\kappa + \log \kappa)$.

Assume that there is a protocol for computing the intersection $S_A \cap S_B$. Then, the users can run the following protocol for computing the sum on the reconciled data.

1) $A$ and $B$ compute $S_A \cap S_B$.
2) $A$ and $B$ compute $x_A = \sum x \in S_A x$ and $x_B = \sum x \in S_B x$, respectively.
3) $A$ and $B$ exchange the values of $x_A$ and $x_B$.
4) Each user computes the result by computing $x_A + x_B - \sum x \in S_A \cap S_B x$.

By using Theorem V.1 the total number of communicated bits is $O(\kappa) + 4n$ in the shared randomness model and $O(\kappa) + 4n + O(\log n)$ in the private randomness model.

VI. USING HASH FUNCTIONS

A. Setting

In this section, we construct a “Las Vegas” type randomized protocol for computing the function $f$ as defined in (1).

The proposed protocol is based on the use of universal hash functions [3], as follows. Let $H \triangleq \mathbb{F}^k$ and $\mathcal{H} = \{h\}$ be a family of all hash functions $h : \mathbb{F}^n \rightarrow H$, such that

$$\forall K \in H, \forall h \in \mathcal{H} : \{x : h(x) = K\} \leq 2^{n-k}.\quad (6)$$

Assume that functions $h \in \mathcal{H}$ are chosen randomly uniformly from $\mathcal{H}$, and independently from the previous choices. Hereafter, we can assume that before the protocol is executed, $A$ and $B$ agree on some random order of $h_0, h_1, h_2, \cdots \in \mathcal{H}$, which are used in the protocol.

B. Protocol

The pseudocode of the proposed protocol is presented as Algorithm [1]

\begin{algorithm}
\caption{Protocol pseudocode}
1: \textbf{procedure} PROTOCOL
2: \hspace{0.5cm} for $i = 0$; true; $i = i + 1$ do
3: \hspace{1.0cm} $B$ sends the set $K_i = \{h_i(x) : x \in S_B\}$ to $A$
4: \hspace{1.0cm} $A$ creates empty set $L_i$
5: \hspace{1.0cm} for $x \in S_A$ do
6: \hspace{1.5cm} if $h_i(x) \notin K_i$ then
7: \hspace{2.0cm} $A$ adds $x$ to $L_i$
8: \hspace{1.5cm} end if
9: \hspace{1.0cm} end for
10: \hspace{1.0cm} if $|L_i| = d_A$ then
11: \hspace{1.5cm} break
12: \hspace{1.0cm} end if
13: \hspace{1.0cm} end for
14: \hspace{1.0cm} $A$ sends $s = \sum x \in L_i x$ to $B$
15: \hspace{1.0cm} $B$ computes $s' = s + \sum x \in S_B x$
16: \hspace{1.0cm} $B$ sends $s'$ to $A$
17: \textbf{end procedure}
\end{algorithm}

C. Communication complexity

Below, we estimate communication complexity of the proposed protocol. While the main idea of the protocol is relatively straightforward, the detailed analysis requires some nontrivial elaboration.

There are three statements, where the data is sent between the users: in lines [3] [14] and [16]. We denote the corresponding
number of bits sent during each statement as \( t_0, t_1 \) and \( t_2 \).
We have:
\[
\begin{align*}
t_0 &= km_B, \\
t_1 &= 2n - 1, \\
t_2 &= 2n - 1. 
\end{align*}
\]

### D. Success Probability

Below, we estimate the probability of the loop in lines 2–13 to end with a break statement in line 11. The number of loops determines the total number of communicated bits.

In this analysis, we assume that the hash functions satisfy 6. Then, the collision probability for a randomly chosen \( h \in \mathcal{H} \) is
\[
\Pr[\text{collision}] = \Pr[h(x) = h(y) | x \in \mathbb{F}^n, y \in \mathbb{F}^n, x \neq y] = \frac{2^{n-k} - 1}{2^n - 1}.
\]

The break statement in line 11 is activated when \( |L_i| = d_A \) for some \( i \).

If \( x \in S_0 \), then \( h(x) \in K_i \). Otherwise, if \( x \in S_A \setminus S_0 \), then \( h(x) \notin K_i \) only if there is no collision with an element in \( K_i \):
\[
\Pr[|L_i| = d_A] = \Pr[\text{no collision for every } x \in S_A \setminus S_0] = \left(1 - \frac{2^{n-k} - 1}{2^n - 1}\right)^{d_A}.
\]

### E. Number of communicated bits

Next, we compute the number of communicated bits \( T_r \) during \( r \in \mathbb{N} \) rounds. For brevity, we denote
\[
\begin{align*}
p_a &= \Pr[\text{accept}] = \Pr[|L_i| = d_A] \\
p_n &= \Pr[\text{not accept}] = 1 - p_a.
\end{align*}
\]

Here, \( p_a \) is a probability that the protocol succeeds in computing the sum of all elements.

At first, we look at the cases where we limit the number of rounds to 1, 2 and 3. To express the expected number of communicated bits in an instance of the protocol, which succeeds after at most \( r \) rounds, we use the random variable \( T_r, r \in \mathbb{N} \). We have:
\[
\begin{align*}
E[T_1] &= p_a(t_0 + t_1) + t_2, \\
E[T_2] &= p_a(t_0 + t_1) + p_n p_a(t_0 + t_0 + t_1) + t_2, \\
E[T_3] &= p_a(t_0 + t_1) + p_n p_a(t_0 + t_0 + t_1) + p_n p_a(t_0 + t_0 + t_0 + t_1) + t_2.
\end{align*}
\]

In general, when bounding the number of rounds by \( r \), the number of the communicated bits is
\[
E[T_r] = \sum_{i=0}^{r-1} p_n^i p_a ((i + 1) t_0 + t_1) + t_2. \tag{14}
\]

By allowing an unbounded number of rounds, we obtain
\[
E[T_{\infty}] = \sum_{i=0}^{\infty} p_n^i p_a ((i + 1) t_0 + t_1) = p_a t_0 \sum_{i=0}^{\infty} p_n^i (i + 1) + p_a t_1 \sum_{i=0}^{\infty} p_n^i = p_a t_0 \sum_{i=0}^{\infty} \frac{p_n^i}{p_n^i} + p_a t_1 \frac{1}{1 - p_n} = p_a t_0 \frac{p_n}{p_n} + p_a t_1 \frac{1}{1 - p_n} = t_0 p_a^{-1} + t_0 + t_1 = t_0 \left(1 - \frac{2^{n-k} - 1}{2^n - 1}\right)^{-d_A} + t_1. \tag{15}
\]

By using equations 7–9, we obtain
\[
E[T_{\infty}] = k m_B \left(1 - \frac{2^{n-k} - 1}{2^n - 1}\right)^{-d_A} + 4n - 2. \tag{16}
\]

Given \( m_B, d_A \) and \( n \), we next find
\[
\arg \min_k k m_B \left(1 - \frac{2^{n-k} - 1}{2^n - 1}\right)^{-d_A} + 4n - 2,
\]
in order to determine the optimal value of \( \text{COMM}(F) \), which minimizes the total number of communicated bits.

For simplicity, we assume that \( k \ll n \) (otherwise, the hashing approach is not efficient). Under that assumption,
\[
\text{COMM}(F) = \arg \min_k \left\{ k m_B (1 - 2^{-k})^{-d_A} + 4n - 2 \right\}.
\]
By substituting $k = \log_2\left(\frac{d_A}{c}\right)$, where $c$ is a constant, we obtain:

$$km_B\left(1 - 2^{-k}\right)^{-d_A} + 4n - 2
≈ km_B \left(1 - \frac{c}{d_A}\right)^{-d_A} + 4n - 2
= O(m_B \cdot \log d_A + n).$$

VII. SUMMARY AND FUTURE WORK

In this work, we initiated a study of a new problem called function computation on the reconciled data. The problem considers a scenario where two users possess sets of vectors $S_A$ and $S_B$, respectively, and they aim at computing the value of $\phi(S_A \cup S_B)$ for some function $\phi$. We considered simple cases of $\phi$, such as identity, maximum, minimum, sum, product. Specifically, for sum, we derived a number of lower and upper bounds on communication complexity (for different models of randomness). We showed connections to some known problems in communication complexity. Finally, we proposed a “Las Vegas” type randomized algorithm and analyzed its communication complexity.

Many intriguing questions are still left open. Specifically, it would be interesting to obtain tight bounds, and to design efficient protocols, for computation of various functions. Different models of randomness can be considered. Finally, protocols for a number of users larger than two can also be investigated.

VIII. ACKNOWLEDGEMENTS

The authors wish to thank Dirk Oliver Theis for helpful discussions and for pointing out the connection to the set disjointness problem.

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