Quantum control and the challenge of non-Hermitian model-building

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Abstract. In a way inspired by the brief 2002 note “The challenge of nonhermitian structures in physics” by Ramirez and Mielnik (with the text most easily available via arXiv: quant-ph/0211048) the situation in the theory is briefly summarized here as it looks twelve years later. Our text has three parts. In the first one we briefly mention the pre-history (dating back to the Freeman Dyson’s proposal of the non-Hermitian-Hamiltonian method in 1956 and to its subsequent successful “interacting boson model” applications in nuclear physics) and, first of all, the amazing recent progress reached, in the stationary case, using, in essence, an inversion of the Dyson’s approach. The impact on the latter idea upon abstract quantum physics is sampled, first of all, by the reference to papers by Bender et al. (who made the non-Hermitian model-building popular under the nickname of parity-times-time-reflection-symmetric alias PT-symmetric quantum mechanics) and by Mostafazadeh (who reinterpreted PT-symmetry as P-pseudo-Hermiticity). In the second part of our review the emphasis is shifted to the newest, non-stationary upgrade of the formalism which we proposed in the year 2009 and which is characterized by the simultaneous participation of a triplet of Hilbert spaces \( \mathcal{H} \) in the representation of a single quantum system. In the third part of the review we finally emphasize that the majority of applications of our three-Hilbert-space (THS) recipe is still ahead of us because the enhancement of the flexibility is necessarily accompanied by an enhancement of the technical difficulties. An escape out of the technical trap is proposed to be sought in a restriction of attention to quantum models living in finite-dimensional Hilbert spaces \( \mathcal{H} \). As long as the use of such spaces is so typical for the quantum-control considerations, we conclude with conjecture that the THS formalism should start searching for implementations in the field of quantum control.

1. Introduction and summary

1.1. Schrödinger equation and quantum control

In accord with the standard textbooks on quantum theory [1] the evolution of a pure state of a closed quantum system is most comfortably determined by solving Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle^{(P)} = \mathfrak{h}(t)|\psi(t)\rangle^{(P)} \]  

(1)

in which one assumes that the states of the system in question are represented by elements \( |\psi(t)\rangle^{(P)} \) of a properly selected physical Hilbert space \( \mathcal{H}^{(P)} \) and in which the generator of evolution (called Hamiltonian) is self-adjoint, \( \mathfrak{h}(t) = \mathfrak{h}^\dagger(t) \).

Using the notation conventions of Ref. [2] and abbreviating \( |\psi(t)\rangle^{(P)} \equiv |\psi(t)\rangle \) one traditionally assumes that at \( t = 0 \) the system is prepared in an initial state \( |\psi(0)\rangle = |\psi_i\rangle \).
and that it is detected, after some time $T > 0$, in a final state $|\psi(T)\rangle = |\psi_f\rangle$. Thus, in the most conventional approach one knows $h(t)$ and constructs the final state $|\psi_f\rangle \in \mathcal{H}^{(P)}$.

An entirely different task is typical for the so called quantum-control (QC) setup. In its most elementary specification (often called bilinear model – see, e.g., review paper [3] for details) one is given just a desirable final target state $|\psi(T)\rangle = |\psi_f\rangle$. For the purpose, one has to select a suitable “realization Hamiltonian” $h(t) = h^\dagger(t)$ and specify the necessary “realization time” $T > 0$.

For technical reasons people often restrict their attention to finite-dimensional quantum systems living in an $N$-dimensional complex Hilbert spaces $\mathcal{H}^{(P)} = \ell^{(N)}$ in which the admissible self-adjoint QC Hamiltonians have the form of a superposition

$$h(t) = h_0 + \sum_{k=1}^K u_k(t) h_k.$$  

The $(K+1)-plet$ of auxiliary operators $\{h_0, h_1, \ldots, h_K\}$ is assumed time-independent. Moreover, this multiplet of operators is often chosen as a set of generators of a Lie algebra $\mathcal{L}_0$ such that the desired evolution of the system towards a given target state may be proved to exist (one speaks about a “controllability” [4]). In such a setting the target $|\psi_f\rangle$ is to be reached solely via the selection of the real coefficients $u_k(t) \in \mathbb{R}$ called control functions.

1.2. The plan and summary of the paper

In our present paper we intend to expose the standard, above-outlined formulation of the quantum control problem to a modification. It will be inspired by the recent developments in quantum theory in which one complements the description of dynamics (i.e., of the Hamiltonian) by the possibility of an independent alteration of the Hilbert space itself.

The essence of the latter developments will be explained in sections 2, 3 and 4. Firstly, in section 2 we shall follow, for pedagogical reasons, older reviews [5, 6] and introduce the amendment $\mathfrak{h} \rightarrow G \neq G^\dagger$ of the generator of evolution in its simplified, time-independent version with $\mathfrak{h} \neq h(t)$ replaced by $G \neq G(t)$ in Schrödinger Eq. (1). Marginally, let us add that from the present perspective, sections 2 and 3 should be read as a mere contextual introduction. They offer a review of recent updates of quantum theory which became widely known as $\mathcal{PT}$-symmetric Quantum Mechanics (PTSQM, cf. the Bender’s review paper [5]) or, in a slightly more general form, as Pseudo-Hermitian representation of Quantum Mechanics (PHRM, cf. the Mostafazadeh’s review paper [6]). As long as in both of these approaches the operators of observables must remain stationary (or, at best, quasi-stationary [7]) none of these formalism is directly applicable in the QC context.

Subsequently, section 4 will outline the upgraded and generalized (a.k.a. “three-Hilbert-space”, THS) formalism of Refs. [2,8]. Our discussion will cover the case in which the manifest time dependence of $\mathfrak{h} = h(t)$ and of $G = G(t)$ is permitted. Although some of the preceding ideas remain unchanged, it will be necessary to shift the emphasis. Indeed, only a change of perspective will enable us to address the QC-related conceptual questions. In this sense, the next section 5 should be read as a more technical addendum reviewing a few aspects of necessary mathematics. The key message is that the most general time-dependent Hamiltonian-like operators may still be required to generate the standard unitary evolution of a given quantum system in time. We shall also explain why, in contrast to the PTSQM or PHRM scenarios, the spectra of our present Hamiltonian-like generators $G(t) \neq G^\dagger(t)$ are, in general, complex.

In the key part and climax of our message in section 6 we shall return to the problems of quantum control, outlining briefly the possible use of the whole THS machinery for an enhancement of the flexibility and efficiency of the specific QC tasks. Preliminarily, our proposal may be summarized as opening a new approach to quantum control in which one extends the
model-building freedom via a transfer of Schrödinger equation from its representation (1) in the "primary" Hilbert space $\mathcal{H}^{(P)}$ (which is assumed to appear, for any reason, unfriendly) to some of its alternative though, by assumption, equivalent and technically friendlier forms.

A few complementary comments on such a possibility will be finally formulated in our last section 7. We shall emphasize that the THS-representation-mediated introduction of the manifestly time-dependent non-Hermitian generators of evolution $G(t)$ is in fact necessary in the QC context. We believe that our considerations will offer a sufficiently strong encouragement for a more concrete model-building activity in the nearest future.

2. Time-independent non-Hermitian Hamiltonians in Quantum Mechanics

2.1. Modified Schrödinger equation

It is well known [1] that in principle, the constructive solution of Schrödinger Eq. (1) is particularly straightforward for Hamiltonians which are Hermitian, diagonalized and not time-dependent. Even in these cases, there exist quantum systems (like, for example, heavy atomic nuclei) for which even the brute-force numerical diagonalization of a given Hermitian $\not h = h(t)$ yields, typically, very poorly convergent results.

In the year 1956, one of the most unexpected ways out of similar difficulties has been proposed by Dyson [9]. He proposed a reparametrization $|\psi(t)\rangle^{(P)} = \Omega |\psi(t)\rangle^{(F)}$ of wave functions in which a "friendlier" ket $|\psi(t)\rangle^{(F)}$ was assumed to belong to a "friendlier" Hilbert space $\mathcal{H}^{(F)}$. Moreover, the time-independent mapping $\Omega$ was chosen, in contrast to common practice, non-unitary, yielding a nontrivial operator product $\Omega \Omega^\dagger \neq I$. In this way, the insertion in Eq. (1) led to a potentially friendlier Schrödinger equation defined in the new Hilbert space,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle^{(F)} = H |\psi(t)\rangle^{(F)}, \quad H = \Omega^{-1} \not h \Omega \neq \not h(t).$$

(3)

Naturally, in the numerical setting the Dyson’s trick and Hilbert-space invertible mapping $\Omega : \mathcal{H}^{(F)} \rightarrow \mathcal{H}^{(P)}$ only made sense if it led to an accelerated convergence but its enormous success may be found confirmed, e.g., in the recent nuclear-physics-devoted review paper [10].

A not entirely pleasant consequence of the non-Hermiticity of the Dyson’s mapping may be seen in the emergence of a manifest non-Hermiticity of $H$. Indeed, in the new language the old Hermiticity rule reads

$$\not h^\dagger = [\Omega^{-1}]^\dagger H^\dagger \Omega^\dagger = \Omega H \Omega^{-1}$$

(4)

and may be re-written in a more compact form

$$H^\dagger \Theta = \Theta H, \quad \Theta = \Omega^\dagger \Omega \neq I.$$  

(5)

Thus, the new Hamiltonian may only be declared “quasi-Hermitian” [11].

2.2. The coexistence and mutual relations of the triplet of simultaneous representation Hilbert spaces

A new life of the same old trick has been conceived in 1998 when Bender with Boettcher [12] proposed the use of certain non-Hermitian $H$ with real spectrum in the role of a standard quantum energy observable. Subsequently, the consistent PTSQM formalism (with its physics-inspired emphasis on the additional feature of parity-times-time-reversal symmetry) has been born, in its final form, in the year 2004 [5,13]. In parallel, also the more general, less restrictive PHRQM version of the formalism as already known to nuclear physicists before 1992 [10] was given new life and popularity by Mostafazadeh (see his numerous publications and/or their summary in his comprehensive review paper [6]).

For our present purposes the PHRQM relations between the above-mentioned P- and F-superscripted Hilbert spaces (and between these two spaces and the third, S-superscripted space
which only differs from the F-space by the use of the metric-mediated, i.e., $\Theta$-mediated inner product) may be summarized using the following diagram,

3. A note on the history and applications

3.1. The birth and the resolution of the puzzle.

Although the very compact review-like 2002 note “The challenge of non-Hermitian structures in physics” by A. Ramirez and B. Mielnik [14] is merely twelve years old, the subject and its applications in the various branches of physics developed, in between, so quickly that one should (and, in what follows, we are going to) update some of their conclusions. Pars pro toto, today, the Ramirez’s and Mielnik’s citation of the 2001 note [15] offering a vague indication of the non-Hermitian Hamiltonian’s having “link with pseudo-euclidean structures” [14] would have to be complemented by the reference to the subsequent 2004 paper [16]. In the latter text the authors considered the same illustrative non-Hermitian square-well Hamiltonian but they already were able to explain its full compatibility with the first principles of conventional quantum theory. In this manner the latter authors provided a virtually exhaustive resolution of all of the related apparent paradoxes. Thus, in brief, one can only repeat that after the year 2004, the “consistent interpretation” of non-Hermitian quantum Hamiltonians $H \neq H^\dagger$ of Bender with Boettcher [12] could not have been declared “missing” by the authors of Ref. [14] anymore.

3.2. The current state of art and the continuing emergence of new puzzles

A brief recollection of the developments in the field during the last twelve years reveals that the related research activities did not stop after 2004. Naturally, an understanding of the basic idea was already available but multiple open questions survived. Many of them were already asked around the end of the millennium, i.e., immediately after the publication of the inspiring letter [12]. During a few years, many non-Hermitian Hamiltonians $H^{(\text{NH})}$ with real spectra were then analyzed by many authors. Still, using the words of loc. cit., “in all of these designs” the proper “statistical interpretation [was] still missing” [14].

Fortunately, as we already mentioned, the progress was quick. Around the year 2004, virtually all of the essential connections between the exotic-looking $H^{(\text{NH})}$ and the conventional quantum theory seem to have been already established. Still, new ideas kept emerging even after the year 2004. Typically, the ambiguity problems concerning the assignment of a metric $\Theta$ to a given Hamiltonian $H$ were never completely abandoned. Between the years 2007 - 2010 people also re-opened [17] and solved [18] the puzzling unitarity/non-unitarity conflict in the scattering arrangement. Similarly, due to the apparent failure of the semi-classical approximations, a new crisis emerged very recently [19]. Last though not least, even the lasting conflict between the intuitive and rigorous quantum-theoretical perception of the concept of locality did also hit the PTSQM theory in the past [20] as well as very recently [21]. Nevertheless, in a way
paralleling these fluctuations, various versions of the general THS theory may be now declared to have acquired, under several sophisticated technical assumptions, a more or less closed and final-looking form.

4. The challenge of time-dependent non-Hermitian Hamiltonians in Quantum Mechanics

4.1. A remark on terminology

In the next-to-perfect Mostafazadeh’s review paper [6] the physicists read, with satisfaction, that the “time-dependent quasi-Hermitian Hamiltonians arise naturally in the application of pseudo-Hermitian quantum mechanics in quantum cosmology”. At the same time the mathematicians could feel puzzled when reading there that “in pseudo-Hermitian quantum mechanics we are bound to use quasi-stationary Hamiltonians” defined as “admitting a time-independent metric”. Puzzling as a comparison of these two statements may sound (cf. also the unpublished discussion of this topic in arXiv [22]), it in fact merely reflects the Mostafazadeh’s unexplained decision of working, exclusively, with the observable generators of the quantum time evolution. In other words, the scope of Mostafazadeh’s PHRQM formulation remains restricted to the above-mentioned Schrödinger Eq. (3) in which the generators of evolution \( H \) remain compatible with the Dieudonné’s quasi-Hermiticity constraint (5). Then, together with the standard requirements of the unitarity of the theory this would really imply that we must have \( \Theta \neq \Theta(t) \), indeed.

In this sense, the THS time-dependent-metric representation of a quantum system as proposed in Refs. [2,8] may be perceived as a further nontrivial generalization of the Bender’s time-independent-metric PTSQM frame as well as of the Mostafazadeh’s time-independent-metric formalism of PHRQM.

4.2. The challenge of time-dependent metrics

Once we admit that the crypto-Hermitian time-evolution time-independent-metric law (3) may be further generalized, our constructive considerations become straightforward (cf. [2,8] for details). First of all, admitting the explicit time-variability of the Dyson’s map \( \Omega = \Omega(t) \) of the ket-vector spaces \( \mathcal{H}^{(F)} \rightarrow \mathcal{H}^{(P)} \), i.e., postulating the relation

\[
|\psi(t)\rangle^{(P)} = \Omega(t) |\psi(t)\rangle^{(F)}
\]

an elementary insertion of this ansatz in the original Schrödinger Eq. (1) immediately yields the properly modified form of its equivalent representation in the friendlier Hilbert space \( \mathcal{H}^{(F)} \),

\[
i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle^{(F)} = G(t) |\psi(t)\rangle^{(F)}, \quad G(t) = H(t) - \Sigma(t).
\]

In this evolution equation the new, time-dependent isospectral image

\[
H(t) = \Omega^{-1}(t) H(t) \Omega(t) = H^{\dagger}(t) = \Theta^{-1}(t) H^{\dagger}(t) \Theta(t)
\]

of the original Hamiltonian enters the time-dependent generator of quantum evolution in combination with the so called [23] quantum Coriolis force

\[
\Sigma(t) = i\Omega^{-1}(t) \dot{\Omega}(t), \quad \dot{\Omega}(t) = \partial_t \Omega(t).
\]

We should add that the emergence of the Coriolis term \( \Sigma(t) \) simply reflects the emergence of the manifest time-dependence of the inner products in the alternative physical and nontrivial-metric-endowed third Hilbert space \( \mathcal{H}^{(S)} \). Secondly, we should emphasize that the latter space
still coincides with $\mathcal{H}^{(F)}$ up to the metric, i.e., as a topological vector space [6]. Thirdly, we may return now to the first paragraph of this section and see that separately, both the “virtual-force” Coriolis operator $\Sigma(t)$ and the related time-dependent generator $G(t) = H(t) - \Sigma(t)$ become, in general, unobservable. In other words, the requirement of the manifest time-dependence of the generator in our most general but still unitarity-guaranteeing Schrödinger Eq. (7) implies that the spectrum of such an operator $G(t)$ may cease to be real.

In this sense, the standard textbook Quantum Theory admits its phenomenologically most general but still mathematically fully consistent THS representation as introduced in Ref. [8], reviewed in Ref. [2] and described by the following amended diagram

5. The properties of the sophisticated physical Hilbert space $\mathcal{H}^{(S)}$

5.1. A mixed blessing of the use of the time-dependent Dyson maps

The two-step realization $P \to F \to S$ of the unitary equivalence between the two alternative physical Hilbert spaces $\mathcal{H}^{(P)}$ and $\mathcal{H}^{(S)}$ has its merits (e.g., a simplification $h(t) \to H(t)$ of the observable of the instantaneous but still, in principle, measurable $P -$space-based energy of the system) and shortcomings (e.g., the use of a frequently rather misleading terminology). Still, the above-cited emergence of the non-Hermitian plus time-dependent forms of the hiddenly unitary quantum evolution law (7) “in ... quantum cosmology” [6] offers a sufficiently persuasive motivation for the mathematical study as well as for new proposals of phenomenological applications of the THS representations of quantum systems which are made more flexible by the permission of time-dependence in the underlying Dyson’s maps $\Omega(t)$.

Naturally, the price to be paid for the maximally enhanced flexibility of Eq. (7) is not too low. In particular, the original motivation of the formalism (which proved fairly persuasive in theory, plus strong in applications) gets perceptively weakened in the time-dependent THS case [23]. Moreover, the technical difficulties further increase if we decide to invert the original “Dyson’s” direction $P \to F \to S$ of the construction as incorporated in the PHRQM formalism of Refs. [6,9,10] in which one started all considerations from a given pair of operators $h \neq h(t)$ and $\Omega \neq \Omega(t)$. Indeed, even in the perceivably more restrictive but still time-independent-metric-unsing PTSQM formalism as summarized in Ref. [5] the situation appeared complicated since the pair of operators $h \neq h(t)$ and $\Omega \neq \Omega(t)$ only had to be reconstructed at the very end of all of the constructive manipulations (cf., e.g., an exactly solvable model [24] for illustration).

All this explains why the current progress in the cosmological applications of the THS formalism (cf., e.g., their first preliminary samples in [25]) still remains so deplorably slow. At the same time, the current tradition of the use of just finite-dimensional Hilbert spaces in the context of quantum control seems to open new perspectives for applications of the time-dependent-THS Schrödinger Eq. (7). Let us, therefore, complement our preceding introductory THS outline by a few most relevant further technicalities.
5.2. Ad hoc notation conventions

First of all, let us remind the readers of our review paper [2] that in parallel to the above-mentioned formal coincidence of kets \( |\psi(t)\rangle^{(F)} = |\psi(t)\rangle^{(S)} \) (i.e., to the formal coincidence of the two ket-vector spaces \( \mathcal{H}^{(F)} \) and \( \mathcal{H}^{(S)} \)) one has to keep in mind that the respective conjugate dual-space elements alias primed-vector-space elements alias linear functionals (i.e., in the Dirac’s terminology, the bra-vectors \( \langle \psi| \in [\mathcal{H}^{(F)}]’ \) and \( (\psi)| \in [\mathcal{H}^{(S)}]’ \) which are assigned to the corresponding ket vectors via the respective Hermitian-conjugation antilinear operations \( T \) ) remain different,

\[
(\psi)| \equiv \langle \psi| T \neq \langle \psi| .
\]

In order to suppress confusion we shall also accept another convention that all of the eligible Hermitian conjugations \( T \) : \( |\psi⟩ \rightarrow |\varphi⟩ \) will occur without superscripts, i.e., they will always be understood as performed solely in the trivial-metric spaces, i.e., just in our \( P \) - or \( F \) -superscripted Hilbert spaces. This means that in our present paper we shall never employ the abbreviated and metric-dependent Hermitian-conjugation operation \( T^{(S)} \). Thus, for example, the inverse conjugation \( T^{-1} \) : \( ⟨\varphi| → |\varphi⟩ \) will be always understood as performed just in the friendly, \( F \) -superscripted Hilbert space, etc.

The use of such notation conventions enables us to characterize the unitary equivalence between our \( P \) - and \( S \) -superscripted Hilbert space in an extremely compact manner, viz., via the following coincidence of the respective inner products,

\[
\langle \psi| \varphi⟩ \ (\equiv ⟨(P)| \psi| \varphi⟩^{(P)}) = \langle \psi| \varphi⟩ \ (\equiv ⟨(S)| \psi| \varphi⟩^{(S)}) .
\]

Moreover, the conventional textbook use of an orthonormalized basis \( \{ |n⟩ \} \) in \( \mathcal{H}^{(P)} \) may be immediately paralleled by its \( S \) -superscripted-Hilbert-space (bi)orthonormal-basis descendant with the respective kets \( |n⟩ \) and bras \( ⟨n| \), etc. On these grounds one characterizes a physical state of a given quantum system either by the kets \( |ψ(t)⟩ \) and bras \( ⟨ψ(t)| \) in the \( P \) -superscripted representation or, alternatively, by the “simpler” kets \( |ψ(t)⟩ \) and bras \( ⟨ψ(t)| \) in their preferable but formally strictly unitarily equivalent \( S \) -superscripted representation.

5.3. Quantum time-evolution and its control by the two Schrödinger equations

In the constructive mathematical perspective a decisive THS-representation advantage is that one never has to leave the auxiliary friendly space, treating the structure-reflecting concepts and symbols like, e.g., \( ⟨ψ(t)| \) or \( H^† \) as the mere metric-containing abbreviations. Moreover, as we already mentioned, a key benefit of our conventions is that after an ultimate return to the friendly Hilbert space we have got rid of all of the superscripts. In particular, the fully general non-Hermitian THS quantum evolution process as described in Ref. [2] may be now perceived as initiated, at time \( t = 0 \) and in its friendly \( \mathcal{H}^{(F)} \) representation, by the choice of two initial ket-vectors \( |ψ(0)⟩ \) and \( |ψ(0)⟩ \), with the latter one being formally expressible, in the cases when we know the metric, as the metric-multiple \( Θ(0) |ψ(0)⟩ \). Next, in the Dyson-inspired direct \( P \rightarrow F \rightarrow S \) recipe one has to know the generator \( G(t) \) and, as long as \( G(t) \neq G^†(t) \), one must solve the two time-evolution Schrödinger equations,

\[
iℏ\frac{∂}{∂t} |ψ(t)⟩ = G(t) |ψ(t)⟩
\]

\[
iℏ\frac{∂}{∂t} ⟨ψ(t)| = G^†(t) ⟨ψ(t)|
\]
(incidentally, notice an unfortunate misprint in [2]). One can also find another benefit of our notation in the subsequent elementary re-derivation of formula
\[ \partial_t \langle \psi(t) | \psi(t) \rangle = 0, \]
i.e., in a reconfirmation of conservation law for the norm of state \( \psi(t) \) when considered in its amended physical \((S)\)–superscripted representation.

6. Time-dependent Dyson maps in quantum control setup

6.1. A sample of the realization of the project of generalized non-Hermitian quantum control

In the THS generalization the traditional QC superposition ansatz (2) may be made less restrictive in several directions involving, first of all, several alternative real-control-function assumptions. Thus, the traditional Hermitian QC-related postulate (2) may be replaced, say, by its analogues describing the “evolution Hamiltonian” with a complex spectrum
\[ G(t) = G_0 + \sum_{k=1}^{K_G} u_k(t) G_k \]
and/or the “observable Hamiltonian” with the (time-varying but, in principle, measurable) instantaneous-energy real spectrum,
\[ H(t) = H_0 + \sum_{m=1}^{K_H} z_m(t) H_m \]
etc. Naturally, the most fundamental innovation may be expected to result from the highly nontrivial nature of the non-unitary, manifestly time-dependent Dyson’s maps, say, of the same multinomial form
\[ \Omega(t) = \Omega_0 + \sum_{n=1}^{K_\Omega} v_n(t) \Omega_n. \]
Obviously, as long as the knowledge of \( \Omega(t) \) implies the knowledge of the Coriolis term \( \Sigma(t) \), the role of assumption (17) seems fundamental. Only when we choose \( K_\Omega = 1 \) and set \( \Omega_0 = 0 \) we still obtain a transparent multinomial-operator toy model with metric \( \Theta(t) = v^2(t) \Theta_1 \) and with a diagonal-matrix Coriolis operator \( \Sigma(t) = iv(t)/v(t) I = -iw(t) I. \)

7. The ultimate reconstruction challenge

In the context of our preceding illustrative example we may prolong our methodical analysis and choose, say, \( K_H = 1. \) This will enable us to insert all ansatzs in the Dieudonné’s observability requirement (5). With the real control function \( z_1(t) = z(t) \), this requirement becomes time-variation-independent and it may be separated and solved elementwise, yielding two conditions
\[ H_0^\dagger \Theta_1 = \Theta_1 H_0, \quad H_1^\dagger \Theta_1 = \Theta_1 H_1. \]
If the solution \( \Theta_1 \) exists we shall be already able to derive the closed form of the generator
\[ G(t) = H_0 + u(t) H_1 + w(t) H_2 \]
with \( K_G = 2 \) and \( H_2 = i I. \) Thus, in a way, we shall return to a more or less standard QC scenario, with the main difference and innovation resulting from our new freedom of having the generator \( G(t) \) which is non-Hermitian and which is even non-quasi-Hermitian (i.e., which does have complex eigenvalues).
In the latter context let us finally recall an unpublished preprint [26] in which Hynek B´ıla tried to study a few more concrete non-Hermitian toy models living in the two-dimensional, i.e., in the first nontrivial friendly complex Hilbert space $H^{(F)} = \ell^2$. This study revealed that one could also avoid the reference to the THS Schrödinger equations completely. In such an approach the necessary time-dependent metric $\Theta(t)$ has to be reconstructed via direct solution of the corresponding operator evolution differential equation of the Heisenberg-representation-resembling form which follows immediately from the definition of $\Sigma(t)$,

$$i\partial_t \Theta(t) = G^\dagger(t)\Theta(t) - \Theta(t) G(t).$$  \hspace{1cm} (20)

Unfortunately, the B´ıla’s preliminary results were never completed (cf. also [27,28]). Perhaps, the project itself could still acquire a new life in the non-Hermitian QC context.

In the conclusion let us add that the metric-determining “parallel Cauchy problem” (20) could be addressed by various techniques and under a multitude of approximations but in the QC context, the use of a finite-dimensional Hilbert-space approximation seems most promising, especially because it parallels the common practice used in the standard Hermitian models [3]. Thus, we believe that in the nearest future, the traditional Hermiticity condition $H(t) = H^\dagger(t)$ need not remain obligatory and uncircumventable, anymore.

References

[1] Landau L D and Lifshitz E M 1999 Quantum Mechanics (Non-relativistic Theory) (Oxford: Butterworth-Heinemann)

[2] Znojil M 2009 Symm. Integ. Geom. Meth. Appl. SIGMA05(2009)001 (Preprint arXiv:0901.0700)

[3] Dong D-Y and Petersen I R 2010 IET Control Theory & Applications 4 2651

[4] D’Alessandro D 2007 Introduction to Quantum Control and Dynamics (Boca Raton, FL: Chapman & Hall/CRC)

[5] Bender C M 2007 Rep. Prog. Phys. 70 947

[6] Mostafazadeh A 2010 Int. J. Geom. Meth. Mod. Phys. 7 1191

[7] Mostafazadeh A 2007 Phys. Lett. B 650 288

[8] Znojil M 2008 Phys. Rev. D 78 085003

[9] Dyson F J 1956 Phys. Rev. 102 1217

[10] Scholtz F G, Geyer H B and Hahne F J W 1992 Ann. Phys. (NY) 213 74

[11] Dieudonné J 1961 Proc. Int. Symp. Lin. Spaces (Oxford: Pergamon) pp 115-122.

[12] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243

[13] Bender C M, Brody D C and Jones H F 2004 Phys. Rev. Lett. 92 119902

[14] Ramirez A and Mielnik B 2003 Rev. Mex. Fis. S. 2 49 130

[15] Znojil M 2001 Phys. Lett. A 285 7

[16] Mostafazadeh A and Batal A 2004 J. Phys. A: Math. Theor. 37 11645

[17] Jones H F 2007 Phys. Rev. D 76 125003; Cannata F, Dedonder J-P and Ventura A 2007 Ann. Phys. (NY) 322 397

[18] Znojil M 2009 Phys. Rev. D. 80 045009

[19] Davies B 2000 Bull. London Math. Soc. 32 432; Trefethen L N and Embree M 2005 Spectra and pseudospectra. The behavior of nonnormal matrices and operators (Princeton: University Press); Siegl P and Krejcirik D 2012 Phys. Rev. D 86 121702(R)

[20] Günther U and Samsonov B 2008 Phys. Rev. Lett. 101 230404

[21] Lee Y-C, Hsieh M-H, Flammia S T and Lee R-K 2014 Phys. Rev. Lett. 112 130404

[22] Mostafazadeh A 2007 Comment on “Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution” (Preprint arXiv:0711.0137v1); Znojil M 2007 Reply to Comment on “Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution” (Preprint arXiv:0711.0514); Znojil M 2007 Which operator generates time evolution in Quantum Mechanics? (Preprint arXiv:0711.0535)

[23] Znojil M 2013 Int. J. Theor. Phys. 52 2038

[24] Znojil M 2009 J. Math. Phys. 50 122105

[25] Andrianov A A, Cannata F, Kamenshchik A Y 2006 J. Phys. A: Math. Gen. 39 9975

Znojil M 2012 J. Phys.: Conf. Series 343 012136

[26] Bila H 2009 Adiabatic time-dependent metrics in PT-symmetric quantum theories (Preprint arXiv:0902.0474)
[27] Bila H 2008 *Non-Hermitian Operators in Quantum Physics* (Prague: Charles University; PhD thesis)
[28] Gong J-B and Wang Q-H 2010 *Phys. Rev. A* 82 012103; Gong J-B and Wang Q-H 2012 Time-Dependent PT-Symmetric Quantum Mechanics (*Preprint* arXiv:1210.5344)