A Review of Geometry Investigations of Helicoids

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Abstract. This paper presents research on geometry of helicoids which occurred in different papers. The purpose of the paper is to show a variety in types of helical surfaces which are not well known, and discuss their possible application in architecture and civil engineering. This review is a first stage of a wide investigation on stress-strain analysis methods as applied to helical structures. Author’s attention is paid to the differences in terms, geometry and materials used for these shells, and future possibilities of application of five types of helicoid are also discussed. The work would be interesting for architects, civil engineers, mathematicians and designers, and could lead to a progress in material usage efficiency.

1. Introduction
Helical shells have been known since the time of Archimedes, who proposed to use the so-called ‘Archimedes screw’ to raise water up to 0.5 m in height. Later, these shells were used as auger blades for drilling wells of various purposes: exploration, hydrogeological, seismic, drainage and other technical wells. According to [1] there are five types of helicoids: Right helicoid, Oblique helicoid, Developable helicoid, Convolute helicoid, Pseudo-developable helicoid. These types with different names were found by author in several sources. One of the goals of this paper is to investigate the terms and geometry of different types of helicoids.

After the destruction of the Twin Towers in New York, architects and engineers have set of much thinking which was written in the Y.S. news and world report. Despite the temporary doubts, the demand for skyscrapers has not decreased. One of the reasons of popularity of skyscrapers is the non-availability of the lands for construction and its high cost; especially it concerns the large cities as, the smaller areas are necessary for multistoried buildings, they are very relevant for modern megalopolises in which millions people live. Similar architectural concepts reduce the price of the cost of inhabited office and other rooms.

Besides high-rise buildings in the modern world, there is some kind of polish up for megalopolises which emphasizes their prestige and wealth. William Mitchell, the dean of faculty of architecture and planning of the Massachusetts Institute of Technology, says [2]"Giant buildings are tribute to fashion, prestige and similar."

Due to the continuous scientific and technical progress, it has become possible to embody almost any innovative ideas in architecture. Scientific and technical progress promotes search of new forms of the screw line surfaces applicable for construction combining such qualities as beauty, reliability and technological effectiveness.

Today one of the fashionable architectural concepts is used in construction of screw surfaces which not only have attractive design, but also reduce the area used for construction and efficient material usage.
As the world of surfaces is various and boundless, the innovative ideas in the sphere of architecture are inexhaustible, and geometry of different types of surfaces is worth being investigated from several points (less material usage, optimization of the structure according to stress-strain analysis, etc.).

The helicoid was first ever documented by J. Meusnier [3] around 1775, it was the first unbounded non-periodic minimal surface to be found. Besides the plane, it is the only known ruled minimal surface (zero mean curvature). Important contribution to the theory of helical shells was made by L. Euler (Switzerland) [4], G. Monge (France), E. Catalan [5], G. Darboux [6], D. Hilbert [7] and many others.

The Minimal Surface has a Helix as its boundary. For many years, the helicoid remained the only known example of a complete embedded Minimal Surface of finite topology with infinite Curvature. However, in 1992 a second example, known as Hoffman's Minimal Surface and consisting of a helicoid with a Hole, was discovered [2].

In the 1960s, Scientists began to be interested in calculating the helical shells, and in 1960 the first reinforced concrete helix was built in Poland. In the first half of the 20th century, only single objects were realized, but presently, helical shell structures are becoming popular again [8], as well as helicoids in mathematics [9].

The main idea of this paper is to investigate the differences in helicoids geometry and start discussing the possibilities of their future shape optimization.

2. Geometry investigations

2.1. General information

Geometrically, the helicoid is defined by simultaneously rotating and translating a line at constant speed at an axis to which it is perpendicular [10]. According to [11], the helicoid is the surface generated by the lines parallel to the $xy$ plane which go through each point of the helix and intersect the $z$ axis.

Helicoid is a ruled surface swept out by a line which always intersects a fixed axis at right angles and which rotates uniformly as its point of intersection moves uniformly along the axis and it intersects any cylinder concentric with the axis in a helix. The helicoid has the shape of the thread of a screw.

The helicoid has a wide variety of shapes, depending on the pitch, the proximity of the lines and the points, and whether the pairs of points connected by the rulings are in fact not quite diametrically opposite.

The helicoid and the catenoid are parts of a family of helicoid-catenoid minimal surfaces [12].

Helicoids play a very important role as minimal surfaces. In the technical area generalized helicoids are used for staircases, screws and helical pipes [13, 14].

A generalized helicoid is a surface in Euclidean space generated by rotating and simultaneously displacing a curve, the profile curve, along a line, its axis. Any point of the given curve is the starting point of a circular helix. If the profile curve is contained in a plane through the axis, it is called the meridian of the generalized helicoid. Simple examples of generalized helicoids are the helicoids. The meridian of a helicoid is a line which intersects the axis orthogonally.

Very important types of generalized helicoids are:

✓ The ruled generalized helicoids. Their profile curves are lines and the surfaces are ruled surfaces.

✓ The circular generalized helicoids. Their profile curves are circles.

The helicoid is shaped like Archimedes screw, but extends infinitely in all directions. According to [10], it can be described by the following parametric equations in Cartesian coordinates:

\[ x = \rho \cos(\alpha \theta) \]  
\[ y = \rho \sin(\alpha \theta) \]  
\[ z = \theta \]
where \( \rho \) and \( \theta \) are range from negative infinity to positive infinity, while \( \alpha \) is a constant. If \( \alpha \) is positive, then the helicoid is called right-handed; if negative then it is called left-handed. The helicoid has principal curvatures

\[
\pm \frac{\alpha}{(1 + \alpha^2 \rho^2)^{3/2}}.
\]

The sum of these quantities gives the mean curvature (zero since the helicoid is a minimal surface) and the product gives the Gaussian curvature [15]. The helicoid is homeomorphic to the plane \( \mathbb{R}^2 \). To see this, let \( \alpha \) decrease continuously from its given value down to zero. Each intermediate value of \( \alpha \) will describe a different helicoid, until \( \alpha = 0 \) is reached and the helicoid becomes a vertical plane.

Conversely, a plane can be turned into a helicoid by choosing a line, or axis, on the plane, then twisting the plane around that axis:

\[
\pi \left[ R \sqrt{R^2 + h^2} + h^2 \ln \left( \frac{R + \sqrt{R^2 + h^2}}{h} \right) \right]
\]

2.2. Five types of helicoid

The helicoid is a special case of an ordinary screw surface. The ordinary screw surface is formed by the ordinary screw movement and characterized by points trajectory which represent the cylindrical screw lines with a constant step lying on coaxial round cylinders. As stated earlier, this paper is focusing on five types of helical surfaces which are shown on figures 1-5.

2.2.1. Right helicoid. Right helicoid is called the screw line surface described by a straight line which crosses a helicoid axis at right angle rotates with a constant angular speed around this axis and at the same time moves progressively along the same axis [10]. Sometimes it is also called ‘straight’, as in [16].

Parametric form of right helicoid (figure 1) and its coefficients of quadratic forms [10]:

\[
\begin{align*}
x &= (r, v) = r \cos v \\
y &= (r, v) = r \sin v \\
z &= c \\
A &= 1 \\
F &= 0 \\
B^2 &= r^2 + c^2 \\
L &= N = 0 \\
M &= -C/B
\end{align*}
\]

2.2.2. Inclined helicoid. Inclined helicoid is called ‘oblique’ helicoid in [1, 10]. Formation of inclined helicoid is similar to formation of a screw conoid: formation moves on the screw line and on his axis, remaining at the same time, all the time parallel consecutive forming, a direct circular cone (figure 2).
If the height of the directing cone equal to zero, then the inclined helicoid will turn into a screw conoid. Thus, the screw conoid is a special case of an inclined helicoid; forming a screw conoid are perpendicular to a surface axis, with reference to this surface, is called differently a direct or right helicoid.

Parametric form of inclined helicoid (figure 2):

\[
\begin{align*}
  x &= x(r, \nu) = r \cos \nu \\
  y &= y(r, \nu) = r \sin \nu \\
  z &= cv + kr
\end{align*}
\]

(14) \hspace{1cm} (15) \hspace{1cm} (16)

Coefficients of the basic quadratic forms of the surface:

\[
\begin{align*}
  A^2 &= 1 + k^2 \\
  F &= ck \\
  B^2 &= r^2 + c^2 \\
  L &= 0 \\
  M &= \frac{-c}{\sqrt{B^2-F^2}} \\
  N &= \frac{kr^2}{\sqrt{B^2-F^2}} \\
  k_r &= 0 \\
  k_\nu &= \frac{N}{B^2} \\
  \cos \chi &= \frac{ck}{\sqrt{r^2+c^2}} \hspace{1cm} (25)
\end{align*}
\]

where \( k \) = angular coefficient, \( k = ctg \alpha \), \( \chi \) = angle between coordinate lines \( r \) and \( \nu \). The end section of the helicoid at \( z = 0 \) yields \( \rho = cvt g \alpha \), that is, the Archimedean spiral.

2.2.3. Deployable helicoid. The torso surface formed by tangents to the screw line of a constant step on the circular cylinder is called the developable helicoid \([1, 10, 17]\) and torso helicoids in \([10]\), but may be also called deployable helicoid (figure 3). There were found several forms of parametric equations of this helicoid.

a. Parametric form of deployable helicoid.

\[
\begin{align*}
  x &= x(u, \nu) = \frac{acosv-au \sin v}{m} \\
  y &= y(u, \nu) = \frac{asinu+au \cos v}{m} \\
  z &= z(u, \nu) = \frac{bv+bu}{m} \\
  \text{where } m &= \sqrt{a^2+b^2} \hspace{1cm} (29)
\end{align*}
\]

\( b \) = pitch of helix \( u = 0 \) (return edges), \( \nu \) = angle, counted from the axis \( Ox \).

The coefficients of the basic quadratic forms of the surface and its principal curvatures:

\[
\begin{align*}
  A &= 1 \\
  F &= m \\
  B^2 &= \frac{m^2+u^2a^2}{m^2} \hspace{1cm} (32)
\end{align*}
\]
The coordinate lines coincide with the rectilinear generators of the helicoid, and the lines \( v \) represent coaxial helical lines. A conjugate but non-orthogonal system of curvilinear coordinates.

b. Parametric form of deployable helicoid (figure 3):

\[
\begin{align*}
    x &= x(u, v) = a \cos v - u \cos \varphi \sin v \\
    y &= y(u, v) = a \sin v + u \cos \varphi \cos v \\
    z &= z(u, v) = a \cot \varphi + u \sin \varphi
\end{align*}
\]

(35) (36) (37)

where \( \varphi \) = the angle of inclination of the rectilinear generators of the helicoid to the \( xOy \) plane, then

\[
tg \varphi = \frac{b}{a}
\]

(38)

The coefficients of the basic quadratic forms of the surface:

\[
\begin{align*}
    A &= 1 \\
    F &= \frac{a}{\cos \varphi} \\
    B^2 &= F^2 + u^2 \cos^2 \varphi \\
    B^2 - F^2 &= u^2 \cos^2 \varphi \\
    L &= M = 0
\end{align*}
\]

(39) (40) (41) (42) (43) (44)

2.2.4. Convoluted helicoid. The convoluted helicoid is formed by means of the straight line of \( AB \) moving in space, and which all the time is crossed with the screw line (figure 4), concerning at the same time the side surface of the direct circular cylinder with radius \( a \). Axes of the screw line and the cylinder coincide and form straight line and axis are not crossed at the right angle.

Parametric form of convoluted helicoid (fig d.).

\[
\begin{align*}
    x &= x(t, v) = a \cos v - t \sin \gamma \sin v \\
    y &= y(t, v) = a \sin v + t \sin \gamma \cos v \\
    z &= z(t, v) = pv + t \cos \gamma
\end{align*}
\]

(45) (46) (47)

where \( a \) = the shortest distance between the line \( AB \) and the axis \( Oz \); \( \gamma \) = the angle between the generator of the straight line \( AB \) and the screw axis \( Oz \); The parameter \( t \) determines the position of the point \( M \) on the rectilinear generator. The equations give both cavities of the helicoid with

\[
t > 0 \text{ and } t < 0
\]

(48)

The coefficients of the basic quadratic forms of the surface:

\[
\begin{align*}
    A &= 1 \\
    F &= a \sin \gamma + p \cos \gamma \\
    B^2 &= a^2 + p^2 + t^2 \sin^2 \gamma \\
    L &= 0
\end{align*}
\]

(49) (50) (51) (52)

\[
M = \frac{\sin \gamma (act \gamma - p)}{\sqrt{(act \gamma - p)^2 + t^2}}
\]

(53)

\[
N = \frac{[a(act \gamma - p) + t^2 \sin \gamma \cos \gamma]}{\sqrt{(act \gamma - p)^2 + t^2}}
\]

(54)

\[
K = -\frac{(act \gamma - p)^2}{[(act \gamma - p) + t^2]^2} < 0
\]

(55)

The angle \( \chi \) between the coordinate lines \( t \) and \( v \) is calculated by equation

\[
\cos \chi = \frac{(a \sin \gamma + p \cos \gamma)}{\sqrt{a^2 + p^2 + t^2 \sin^2 \gamma}}
\]

(56)
Straight-line generators will be orthogonal to the helix lines $v$, if the constant angle $\gamma$ is taken by equation

$$tg\gamma = -\frac{b}{a}$$  \hspace{1cm} (57)

A direct convolutional helicoid can be obtained by taking $\gamma = \frac{\pi}{2}$.  \hspace{1cm} (58)

2.2.5. Elliptic helicoid. The elliptic helicoid – is a linear surface of negatively Gaussian curvature. This type of a helicoid can be carried also to a class of spiral surfaces (figure 5), and to a class of linear surfaces to group of line surfaces of negative Gaussian curvature.

Parametric form of elliptic helicoid (figure 5) \cite{12}:

$$x = x(u, v) = av\cos u$$  \hspace{1cm} (59)

$$y = y(u, v) = bv\sin u$$  \hspace{1cm} (60)

$$z = z(u) = cu$$  \hspace{1cm} (61)

where $a$ and $b = \text{constants}$.

Coefficients of the basic quadratic forms of the surface and its curvature:

$$A^2 = a^2v^2\sin^2u + b^2v^2\cos^2u + c^2$$  \hspace{1cm} (62)

$$F = v(b^2 - a^2)\sin u \cos u$$  \hspace{1cm} (63)

$$B^2 = a^2\cos^2u + b^2\sin^2u$$  \hspace{1cm} (64)

$$A^2B^2 - F^2 = a^2b^2v^2 + c^2(a^2\cos^2u + b^2\sin^2u)$$  \hspace{1cm} (65)

$$L = 0$$  \hspace{1cm} (66)

$$M = \frac{abc}{\sqrt{a^2b^2v^2 + c^2(a^2\cos^2u + b^2\sin^2u)}}$$  \hspace{1cm} (67)

$$N = 0$$  \hspace{1cm} (68)

$$k_u = k_v = 0$$  \hspace{1cm} (69)

$$K = \frac{-a^2b^2c^2}{(A^2B^2 - F^2)^2} < 0$$  \hspace{1cm} (70)

$$H = -\frac{abcF}{(A^2B^2 - F^2)^2} = -\frac{abc(v(b^2 - a^2)\sin u \cos u)}{(A^2B^2 - F^2)^2}$$  \hspace{1cm} (71)

2.3. Helicoids transformations

Helicoid and Catenoid are members of the same associate family of surfaces therefore, one can bend a catenoid into a portion of a helicoid without stretching \cite{12,18}. In other words, one can make a (mostly) continuous and isometric deformation of a catenoid to a portion of the helicoid such that every member of the deformation family is minimal (having a mean curvature of zero). A parametrization of such a deformation is given by the system below:

$$x(u, v) = \cos \theta \sinh v \sin u + \sin \theta \cosh v \cos u$$

$$y(u, v) = -\cos \theta \sinh v \cos u + \sin \theta \cosh v \sin u$$

$$z(u, v) = u \cos \theta + v \sin \theta$$

For $(u, v) \in (-\pi, \pi) \times (-\infty, \infty)$, with deformation parameter $-\pi < \theta \leq \pi$, where $\theta = \pi$ corresponds to a right-handed helicoid $\theta = \pm \frac{\pi}{2}$ corresponds to a catenoid, and $\theta = 0$ corresponds to a left-handed helicoid.

3. Discussion and future recommendations

Despite developed geometrical characteristics of five types of helicoids mentioned above, there are still some gaps that could lead to misunderstandings from other researchers. For example, during the process of review some differences in determinations of helicoids named by separate authors were
found out. In this paper, the most mentioned and logical (in author’s opinion) terms are used, however more research is needed to prove it.

Thus, the helicoid on figure 1 is called ‘right’ helicoid in [1,10,19,20], while the same type of helicoid is mentioned as ‘straight’ helicoid in [12]. The helicoid on figure 2 is generally called ‘oblique’ helicoid [1,10], but it is described as ‘inclined’ helicoid in [21]. As the helicoid on figure 3 is concerned, despite the term suggested in this paper (‘deployable’ helicoid) it is usually called ‘developable’ helicoid [1,10], because it logically coincides with the term ‘developable surface’ that is widely used in practice and research papers [22], and it is often mentioned as ‘torso’ helicoid [23]. Some researchers call helicoids as ‘pre-twisted beams’ [24]. It may be caused by difficulties in translation or differences in understanding, but it is obviously required to put all terms in accordance with some etalon on the base of mathematical or engineering logic. However, the terms of two other types of helicoids which are not widely used are even less clear for researchers, and there were found much more terms for different types of helicoids in [10,12]. That is why it is needed to create a clear classification of helicoid types according to their geometry before starting investigation on searching the optimal shape for specific parameters.

In recent times, the most complete overviews on the geometry, calculation and application of helical shells of a general type were presented by Krivoshapko [23] and Rynkovskaya [8], and a review for linear helical shells is given in [25], where, in particular, it is shown that the geometry of all five types of helicoids has been studied quite well. At the same time, not all types of ruled helical shells are used in practice, which is associated, in a number of cases, with insufficient information. Investigators are aware of the state of affairs in the area of determining the stress-strain state of these shells. That is why it would also be necessary to investigate these helicoids stress-strain state for future choice of the type of helicoid and widen their application in architecture and civil engineering.

In mechanical engineering, right, oblique and deployable helicoids are often used, however, their strength is determined in most cases on the basis of experiments, in particular, due to the relatively small size of the screws, augers and threads used, but some research is also done [26-30]. Obviously, to use these surfaces in architecture and civil engineering, it is required to conduct more research [31] on methods of stress-strain calculations of helicoids.

4. Conclusions

The geometry of five types of helicoids are presented in the article. Some of them are well developed and some of them were not investigated too much, and most of differences are unknown by architects, civil engineers and designers. However, these differences could lead to impact on the stress-strain state of a real structure, as well as help to use material more efficiently. Besides, knowledge of the differences in geometry could provide a proper information for parametric investigations according to maintenance of a real structure.

The investigation on stress-strain state of these five types of helicoids is also required to supply other researchers, architects and engineers with data for choosing an optimal type of helicoid for determined parameters in practice.

As it is shown in the article, some differences in terms were found by author which could lead to misunderstanding between other researchers, and it is suggested to put all terms into an order according to a determined logic.

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