We discuss some aspects of string cosmology with an emphasis on the role played by the dilaton. A cosmological scenario based on the assumption that all spatial dimensions are periodic so that winding modes play an important role is reviewed. A possibility of a transition from a ‘string phase’ to the ‘standard’ cosmology via a radiation dominated era and inflation is analysed. We also summarise some recent results about time dependent solutions of tree level string equations.

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1. Introduction

One of the few universal predictions of string theory is the existence of a scalar field (dilaton) which is coupled to matter. The presence of the dilaton along with the graviton (and the antisymmetric tensor) in the string theory effective action was first pointed out in ref.1. The analogy with \( \omega = -1 \) Jordan-Brans-Dicke theory

\[
S = \frac{1}{2} \int d^D x \sqrt{-G} \left[ \eta R - \omega \eta^{-1} (\partial \eta)^2 \right] + S_m[G, \psi]
\]

and an apparent conflict with solar system observations (imposing the constraint \( |\omega| > 500 \), i.e. ruling out the presence of a scalar component of gravity) was noted.

The full non-polynomial structure of the dilaton couplings in the tree level string effective action (which was difficult to determine using the methods of ref.1) was first inferred indirectly after the construction of \( D = 10 \) supergravity (interpreted as a zero slope limit of a superstring theory) and later understood from string theory. In contrast to the original JBD action, the string action contains the dilaton couplings to other “matter” fields already in the ‘Jordan’ or ‘string’ frame

\[
S = \frac{1}{2} \int d^D x \sqrt{-G} \ e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} H^2_{\lambda\mu\nu} - \frac{1}{4} F^2_{\mu\nu} 
- 2V(\phi) - (\partial \psi)^2 - m^2 \psi^2 + ... \right].
\] (1.2)

For generality we have included a dilaton potential term and also a scalar field with a tree level mass \( m \). The corresponding action in the Einstein frame is

\[
S_E = \frac{1}{2} \int d^D x \sqrt{-g} \left[ R - 2p(\partial \phi)^2 - \frac{1}{12} e^{-4p\phi} H^2_{\lambda\mu\nu} - \frac{1}{4} e^{-2p\phi} F^2_{\mu\nu} 
- 2\tilde{V}(\phi) - (\partial \psi)^2 - e^{2p\phi} m^2 \psi^2 + ... \right],
\] (1.3)

\[
g_{\mu\nu} = e^{-2p\phi} G_{\mu\nu}, \quad \tilde{V}(\phi) = e^{2p\phi} V(\phi), \quad p \equiv 2/(D - 2) \]. (1.4)
The Einstein frame form of the JBD action (1.1) is

\[
S_E' = \frac{1}{2} \int d^D x \sqrt{-g} \left[ R - q_0 (\partial \phi)^2 - \frac{1}{4} e^{q_1 \phi} F^2_{\mu \nu} - e^{q_2 \phi} (\partial \psi)^2 - e^{q_3 \phi} m^2 \psi^2 + ... \right], \quad (1.5)
\]

\[
q_0 = 4 \omega + 2p(D - 1), \quad q_1 = (D - 4)p, \quad q_2 = 2, \quad q_3 = Dp. \quad (1.6)
\]

The actions (1.3) and (1.5) belong to a general class of actions describing interactions of a massless scalar “universally” coupled to matter. While in (1.3) the dilaton coupling constant is of order one (there is no “hierarchy” of scales of the dilaton and gravitational couplings at the string tree level) it is the value of \( \omega^{-1/2} \) that sets the scale of dilaton couplings in (1.5).

There are two basic types of observational restrictions on a scale (denoted by \( \omega^{-1/2} \)) of interactions of a massless scalar with matter (see Sec.2.3 and references there). The first comes from solar system post-newtonian experiments like radar time delay measurements. The second is related to a cosmological evolution of the scalar field. If the scalar changes with time, this produces a time variation of effective particle masses, or, equivalently, in the Jordan frame, of the gravitational constant. The latter variation can be constrained from consideration of primordial nucleosynthesis. To satisfy both post-newtonian and nucleosynthesis bounds \( |\omega| \) in (1.1) should be greater than a few hundreds. Much stronger constraint applies in the case when the scalar (like the string theory dilaton or one combination of it with moduli) is coupled to the gauge field kinetic terms in the action. The time variation of the electromagnetic coupling should be extremely small to satisfy bounds on stability of nuclear isotopes. In the simplest model this gives the restriction \( |\omega| > 10^7 \).

The crucial point that may help to avoid conflict with observations is that non-perturbative string corrections\(^6,7\) should modify the structure of low energy interactions of the dilaton (and of other massless scalars which are present in a \( D = 4 \) string spectrum). In particular, a non-trivial dilaton potential (and hence a mass for a fluctuation near a minimum) should be generated. Also, the matter mass terms should have a non-perturbative
origin so that their coupling to the dilaton ($\sim e^{\exp(-ae^{-2\phi})}$) should be different from that in (1.3).

While the dilaton is most probably “frozen” at the minimum of its potential at the present time, it could have played an important role in early Universe. It seems reasonable to assume that the potential term becomes essential (and supersymmetry is broken) only at rather late stage of evolution so that at earlier times the dilaton can be treated as massless (and is described approximately by the tree level effective lagrangian). Then one can try to analyse possible models of string cosmology using a “phenomenological” approach, i.e. accounting for the effects of a gas of string modes by adding some “matter” terms in the tree level metric - dilaton action. We shall discuss a cosmological scenario based on such an approach and the issue of its correspondence with the ‘standard’ cosmology in Sec.2.

One may also hope to gain some useful information about string cosmology by studying time dependent solutions of the vacuum metric - dilaton equations and their exact conformal field theory generalizations. While looking for “cosmological” conformal field theories may be of limited importance this approach may be “complementary” to the “phenomenological” one (based on a low energy effective action containing only leading terms in the $\alpha'$ expansion but including non-perturbative corrections and “matter” terms).

We shall review some recent results about time dependent solutions of the string tree level equations in Sec.3.

2. Cosmological Scenario

The “standard” cosmological scenario based on an inflationary phase followed by a hot Universe phase does not give answers some basic “initial condition” questions

\[a\] A conformal theory corresponds only to a perturbative (classical) solution of a superstring (Bose string) theory. It is not known whether non-perturbative solutions (e.g. extremals of an effective action which contains non-perturbative corrections like a dilaton potential) can be described in terms of 2d conformal theories.
like why the Universe was expanding and why this expansion was taking place in three spatial dimensions. One may expect that the string theory being a fundamental theory should provide answers to such questions. If the basic objects are closed strings it is natural to define them in a compact space where they can wind around possible nontrivial cycles and thus have additional solitonic “winding” states in their spectrum. Flat space is then considered as a limiting case of a torus. Our starting point for a discussion of string cosmology will be to assume that all spatial dimensions are compact and are of characteristic string (i.e. Planck) scale $\sqrt{\alpha'} \sim M_p^{-1}$. The aim is then to understand why only three dimensions have expanded while others remained “internal”, i.e. of Planck size. A particular mechanism\(^8\) which may provide an explanation of why only three (or less) dimensions are likely to expand is based on the fact that the winding modes oppose the expansion.\(^8,9\)

If the expansion did happen the Universe should eventually reach a radiation dominated phase in which all massive string modes have decoupled. Later on, when nonperturbative corrections to the string effective action will become important and supersymmetry will be broken the Universe we may enter an inflationary phase (with one of the scalars in the low energy spectrum playing the role of an “inflaton”).

2.1. Basic assumptions

Our basic assumptions will be the following:

(i) *weak coupling*: string interactions are small; the dilaton, i.e the effective string coupling should not increase with time.

(ii) *adiabaticity*: the metric and dilaton are evolving slowly with time so that higher derivative terms in the effective action can be ignored.

(iii) *space is a torus*: spatial dimensions are periodic so that winding modes are present in the string spectrum.
We shall assume also that the metric and the dilaton depend only on time

\[ ds^2 = -dt^2 + \sum_{i=1}^{N} a_i^2(t) dx_i^2 , \] (2.1)

\[ a_i = e^{\lambda_i(t)} , \quad \phi = \phi(t) , \quad N = D - 1 . \]

It is useful to introduce the “shifted” dilaton field \( \varphi \) (which is invariant under the duality transformations\(^{12,13,14}\))

\[ \varphi \equiv 2\phi - \sum_{i=1}^{N} \lambda_i , \quad \sqrt{-G} e^{-2\phi} = e^{-\varphi} . \] (2.2)

Then the action for the gravitational degrees of freedom interacting with string “matter”

\[ S = \frac{1}{2} \int d^D x \sqrt{-G} e^{-2\phi} \left[ c + R + 4(\partial \phi)^2 \right] + S_m[G, \phi] , \] (2.3)

takes the form

\[ S = \frac{1}{2} \int dt e^{-\varphi} \left[ c + \sum_{i=1}^{N} \dot{\lambda}_i^2 - \dot{\varphi}^2 - 2U(\lambda_i, \varphi) \right] . \]

For generality we have included the central charge deficit term which is set equal to zero in the most of the present section. The resulting equations for the scale factors and the dilaton are\(^{13,14,9,15}\)

\[ c - \sum_{i=1}^{N} \ddot{\lambda}_i^2 + \dot{\varphi}^2 = 2U , \] (2.4)

\[ \ddot{\lambda}_i - \dot{\varphi} \dot{\lambda}_i = -\frac{\partial U}{\partial \lambda_i} , \] (2.5)

\[ \ddot{\varphi} - \sum_{i=1}^{N} \dot{\lambda}_i^2 = \frac{\partial U}{\partial \varphi} . \] (2.6)

Eq. (2.4) is a constraint which is conserved as a consequence of (2.5),(2.6). This is the generic form of cosmological equations after all other possible variables (like matter fields, temperature, etc) are eliminated. Consider, for example, the case when the matter part of (2.3) is represented by a (one-loop) free energy of a gas of string modes in thermal equilibrium at temperature \( \beta^{-1} \),

\[ S_m = -\int dt \ F(\lambda, \beta) . \]
Then
\[ c - \sum_{i=1}^{N} \dot{\lambda}_i^2 + \dot{\varphi}^2 = 2e^2E, \]  
(2.7)
\[ \ddot{\lambda}_i - \dot{\varphi} \dot{\lambda}_i = e^2 P_i, \]  
(2.8)
\[ \ddot{\varphi} - \sum_{i=1}^{N} \dot{\lambda}_i^2 = e^2 E, \]  
(2.9)
where
\[ E = F + \beta \frac{\partial F}{\partial \beta}, \quad P_i = -\frac{\partial F}{\partial \lambda_i}, \quad \dot{E} + \sum_{i=1}^{N} \dot{\lambda}_i P_i = 0. \]  
(2.10)

Since \( F = F(\lambda(t), \beta(t)) \) eq.(2.10) is equivalent to the conservation of the entropy \( s = \beta^2 \frac{\partial F}{\partial \beta}, \) \( \dot{E} + \sum_{i=1}^{N} \dot{\lambda}_i P_i = \dot{s}/\beta. \) Solving the adiabaticity condition one can in principle express the temperature in terms of \( \lambda_i. \) Then eqs. (2.7)–(2.9) take the form of (2.4)–(2.6) with
\[ U = e^2 E(\lambda, \beta(\lambda)). \]

The system (2.4)–(2.6) has an obvious mechanical interpretation: it describes a particle moving in the potential \( U. \) Since the dilaton should not grow during the expansion of the spatial dimensions \( \dot{\varphi} \) should be negative (\( \dot{\varphi} \) does not change sign if the r.h.s. of (2.6) is positive). Then the dilaton term in (2.5) can be interpreted as a friction force.

2.2. String phase

Let us now assume that a toroidal universe is filled with classical strings in momentum and winding states which are out of thermodynamical equilibrium. Since the masses of momentum modes \( (\sim a_i^{-1}) \) grow with a decrease of \( \lambda_i \) while the masses of winding modes \( (\sim a_i) \) grow with an increase of \( \lambda_i \) the “energy” \( U \) in (2.4) will grow at both positive and negative \( \lambda_i \) (and will not depend on \( \varphi \) since we are discussing classical contributions). This is easy to see, for example, representing momentum and winding states as “tachyonic” scalar string modes
\[ S_m = \frac{1}{2} \int dt \, e^\varphi \left[ |\dot{\psi}|^2 - \left( \sum_{i=1}^{N} m_i^2 e^{-2\lambda_i} - 4 \right) |\psi|^2 + |\dot{\psi}|^2 - \left( \sum_{i=1}^{N} \tilde{m}_i^2 e^{2\lambda_i} - 4 \right) |\tilde{\psi}|^2 + ... \right]. \]  
(2.11)
Solving the system (2.4)–(2.6) with such $U(\lambda_i)$ one concludes\(^9\) that $\dot{\varphi}$ remains negative if it was negative at the initial moment and the trajectory of the system on the energy – $\lambda_i$ plot is going down (reflecting from the walls of the potential $U$) towards the minimum of $U$ at $\lambda_i = 0$. As a result, $a_i(t)$ are oscillating near the Planck scale with the amplitude of oscillations decreasing because of the dilaton damping.

Since the presence of classical winding modes prevents penetration to large radius region they must first “annihilate” (winding and anti-winding state hitting each other and producing a momentum state) to make the expansion possible. The interaction of classical strings occurs only when their world surfaces intersect but such process is most probable when the number of space-time dimensions is less or equal to 2+2=4. This is the idea of the mechanism\(^8\) suggesting an explanation of why only three (or less) spatial dimensions can expand to “macroscopic” sizes. If, by a fluctuation, some three dimensions started expanding, winding modes will start annihilating and that will make further expansion possible.

After the expansion had happened, the universe enters the second stage of evolution in which the “matter” is represented by a gas of string modes (defined on an $N$-torus) which are in thermal equilibrium. $N$ is now the number of expanding spatial dimensions (i.e. three) and for simplicity we shall assume that all $\lambda_i$ are equal ($\lambda_i = \lambda$). The properties of the corresponding “energy” function $E(\lambda) = E(\lambda, \beta(\lambda))$ were studied using the microcanonical ensemble\(^8,16\) and can be summarised as follows. Containing the contributions of both momentum and winding modes, this function (as well as the partition function and the temperature) is duality symmetric $E(\lambda) = E(-\lambda)$. It reaches its maximum in the ‘Hagedorn region’ near $\lambda = 0$ where it is almost constant. For large enough $\lambda$ the temperature drops below the Hagedorn temperature $T_H$ and the massive string modes go out of thermodynamical equilibrium ($T_H$ is of the same Planck order as the masses of the string modes). The behaviour of $E(\lambda)$ at large $\lambda$ is thus the same as in the ‘radiation dominated region’ (where only the massless string modes contribute to the partition function), i.e. $E(\lambda)$ exponentially goes to zero, $E \sim T^{N+1}a^N \sim e^{-\lambda}$.  

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Though the transition from the Hagedorn to the radiation dominated region is not fully understood, it is possible to solve the resulting system (2.4)–(2.6) or (2.7)–(2.9)

\[ c - N\dot{\lambda}^2 + \dot{\phi}^2 = 2e^{\phi}E , \tag{2.12} \]

\[ \ddot{\lambda} - \dot{\phi}\dot{\lambda} = -N^{-1}e^{\phi} \frac{\partial E}{\partial \lambda} , \tag{2.13} \]

\[ \ddot{\phi} - N\dot{\lambda}^2 = e^{\phi}E , \tag{2.14} \]

separately in the two regions and to match the resulting solutions\(^9\). Since in the Hagedorn era \( E = const \), i.e. the “potential” is flat and nothing prevents the expansion to continue. One finds that the dilaton is decreasing while the scale factor is growing with a slow-down, asymptotically approaching a constant

\[ \varphi = \varphi_0 - \ln|t^2 - b^2| , \quad \lambda = \lambda_0 + \frac{1}{\sqrt{N}} \ln|\frac{t - b}{t + b}| . \tag{2.15} \]

Expanding, \( \lambda \) naturally reaches the intermediate region where \( E \) drops and the system starts ‘rolling down’ over the potential in the direction of large \( \lambda \). In the radiation dominated era

\[ P = N^{-1}E , \quad E = E_0e^{-\lambda} , \quad \rho = E_0e^{-(N+1)\lambda} \]

there exists a special ‘power law’ solution with a constant dilaton \( \phi \)

\[ \lambda = \lambda_0 + q \ln t , \quad \varphi = \varphi_0 + s \ln t , \tag{2.16} \]

\[ q = 2/(N + 1) , \quad s = -2N/(N + 1) , \quad \phi = \frac{1}{2}(\varphi + N\lambda) = \text{const} . \]

The important point is that this solution is an attractor\(^9\), i.e. all solutions (with the initial conditions \( \dot{\phi} < 0 \) and \( \dot{\lambda} > 0 \)) approach it asymptotically. The conclusion is that without any unnatural fine tuning we reach the standard ‘radiation dominated’ cosmological era with the dilaton remaining constant at late times.
2.3. Transition to ‘standard’ cosmology: dilaton potential and inflation

In the radiation dominated era the massive string modes have already decoupled and the dynamics is governed essentially by the low energy effective field theory for the “light” fields. Then non-perturbative corrections should become important and, in particular, a supersymmetry breaking phase transition (generating a potential for the dilaton) should happen at some time.

The effect of a non-perturbative dilaton potential on the cosmological evolution was studied e.g. in refs.17,18,19. In addition to the dilaton, the effective action may include other scalar modes as well (which may play an important role in a possible inflationary phase, see below). In general, a $D = 4, N = 1$ supersymmetric effective action can be parametrised by the scalar fields $(S, T)$ of chiral supermultiplet(s) and by a Kahler potential and superpotential$^{21,6}$. In the case of the model corresponding to the heterotic string compactified on a Ricci flat 6-dimensional space (e.g. torus) with a time-dependent scale $b = e^{\sigma(t)}$

\[ ReS = e^{-2\Phi}, \quad ReT = e^{\sigma/2}, \quad \Phi \equiv \phi - 3\sigma, \quad \varphi \equiv 2\phi - 3\lambda - 6\sigma = 2\Phi - 3\lambda. \quad (2.17) \]

The effective action resulting from compactification on a 6-torus has the following form in the string frame

\[ S = \int d^4x\sqrt{-G} \; e^{-2\Phi} \left\{ \frac{1}{2} \left[ R + 4(\partial\Phi)^2 - 6(\partial\sigma)^2 - 2V(\Phi, \sigma) \right] + L_m \right\}, \quad (2.18) \]

where $L_m$ contains the contributions of the “axions” (imaginary parts of the scalars $S$ and $T$) as well as other “light” matter fields. It is $\Phi$ that we shall play the role of the dilaton in the $D = 4$ theory. For the isotropic spatially flat 4-metric $ds^2 = -dt^2 + e^{2\lambda(t)}dx_idx^i$ we get ($N = 3$)

\[ S = \frac{1}{2} \int dt \; e^{-\varphi} \left[ N\dot{\lambda}^2 - \dot{\varphi}^2 + 6\dot{\sigma}^2 - 2V(\Phi, \sigma) + \ldots \right] \quad (2.19) \]

\[ b \] Cosmological solutions in the presence of an explicit mass term for the dilaton were first discussed in ref.20.
(\(\sigma\) plays the role of 6 of \(\lambda_i\) in (2.3)–(2.6)). The resulting system of equations can be found from (2.4)–(2.6). One should also include the contributions of the energy and pressure of the radiation (see (2.12)–(2.14)). The evolution of \(\sigma\) is not important and can be ignored in the first approximation.

The potential corresponding to the supersymmetry breaking due to gaugino condensation and a non-trivial antisymmetric tensor (axion) background has the following dependence on \(\Phi\) (see refs.7,22 and references therein)

\[
V = \sum_i e^{a_i e^{-2\Phi}}(A_i + B_i e^{-2\Phi} + C_i e^{-4\Phi}) .
\]

In the case of the two gaugino condensates

\[
V = d_1^2 e^{-a_1 Y} [(a_1 Y + 1)^2 - 3] + d_2^2 e^{-a_2 Y} [(a_2 Y + 1)^2 - 3] - 2d_1d_2 e^{\frac{1}{2}(a_1+a_2)Y} [a_1a_2 Y^2 + (a_1 + a_2)Y - 2] , \quad Y \equiv e^{-2\Phi} . \tag{2.20}
\]

The constants \(d_i\) and \(a_i\) depend on a gauge group of the hidden sector (for example, \(d_i \sim 10^{-2}, a_i \sim 10\)). This potential starts from zero in the weak coupling region of large negative \(\Phi\), grows and reaches a local maximum, then decreases to a local minimum (with negative \(V\)), then has the second local maximum and finally goes to \(-\infty\) at large positive \(\Phi\). Since the potential has a local minimum it may fix the value of the dilaton. This would suppress time variations of effective masses and couplings and give a mass to the fluctuating part of the dilaton.

To study the approach to the constant regime and the correspondence with the ‘standard’ cosmology it is natural to use the Einstein frame. A cosmological background in the string frame theory (1.2)

\[
ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -dt^2 + e^{2\lambda(t)} d\Omega^2 , \quad \Phi = \Phi(t) , \tag{2.21}
\]

corresponds to the following background in the Einstein frame theory (1.3)

\[
ds^2_\text{E} = g_{\mu\nu}dx^\mu dx^\nu = e^{-2\phi(t)}(-dt^2 + e^{2\lambda(t)} d\Omega^2 ) , \tag{2.22}
\]
\[ ds^2_E = -d\tau^2 + e^{2\Lambda(\tau)} \, d\Omega^2 \quad , \]  
\[ d\tau = dt \, e^{-p\Phi(t)} , \quad \Lambda \equiv \lambda - p\Phi , \quad p = 2/(N - 1) . \]

\( d\Omega^2 \) is the interval of a maximally symmetric 3-space with curvature \( k \). The cosmological equations in the Einstein frame have the form \((N = 3; \text{cf.}(2.4)-(2.6),(2.12)-(2.14))\)

\[ N(N - 1)\dot{\Lambda}^2 - 4(N - 1)^{-1}\dot{\Phi}^2 = 2\dot{U} \quad , \]  
\[ (N - 1)\ddot{\Lambda} + 4(N - 1)^{-1}\dot{\Phi}^2 = N^{-1} \frac{\partial \dot{U}}{\partial \Lambda} \quad , \]  
\[ \ddot{\Phi} + N\dot{\Lambda}\dot{\Phi} = -\frac{1}{4}(N - 1) \frac{\partial \dot{U}}{\partial \Phi} \quad , \]  
\[ \dot{U}(\Lambda, \Phi) \equiv e^{4\Phi/(N-1)} \left[ U(\lambda, \Phi) - c \right] . \]

Here the dots denote derivatives over the Einstein frame time \( \tau \). Note that the structure of eqs.(2.26),(2.27) is different from that of the string frame equations (2.5),(2.6). The dilaton \( \Phi \) is not damping the evolution of the scale factor \( e^{\Lambda} \); it is expanding \( \Lambda \) that provides a friction term in the equation for the dilaton (2.27).

In general, \( U \) in (2.4) may contain the contributions of the spatial curvature, antisymmetric tensor background \( (H_{ijk} = h\epsilon_{ijk} \text{ in the string frame}) \), radiation \( (\rho = E_0e^{-(N+1)\lambda}) \) and the dilaton potential,

\[ U = -\frac{1}{2}kN(N - 1) \, e^{-2\lambda} + \frac{1}{4}h^2 \, e^{-2N\lambda} + E_0 \, e^{2\Phi-(N+1)\lambda} + V(\Phi) \quad (2.29) \]

(one can also include contributions due to scalar non-relativistic matter\(^{23,18}\)). Then \( \dot{U} \) in (2.28) is given by

\[ \dot{U} = -\frac{1}{2}kN(N - 1) \, e^{-2\lambda} + \frac{1}{4}h^2 \, e^{-4\Phi-2N\lambda} + E_0 \, e^{-(N+1)\lambda} \]

\[ + e^{4\Phi/(N-1)} \, V(\Phi) - c \, e^{4\Phi/(N-1)} . \]  
\[ (2.30) \]
In the absence of a dilaton potential and radiation the asymptotic solution of (2.25)–(2.28) is the same as in the $\hat{U} = 0$ case, i.e. a ‘power law’ expansion and dilaton changing logarithmically in time,

$$\Lambda = \Lambda_0 + N^{-1} \ln \tau, \quad \Phi = \Phi_0 + \frac{1}{2} N^{-1/2}(N - 1) \ln \tau.$$ 

Such a dilaton behaviour would produce unacceptable variations of particle masses and couplings. As we have discussed above, if the universe is dominated by the radiation the dilaton approaches a constant value$^9$ (see also refs.23,18). However, the radiation dominated era cannot last forever. As the universe will enter the matter dominated era the dilaton will eventually restart changing with time if it is not suppressed by a potential$^{18}$. The conclusion is that to avoid conflict with observations (in particular, with the nucleosynthesis bound) the dilaton should be already “fixed” by a potential at the time the universe enters the matter dominated phase.

When the non-perturbative dilaton potential is “turned on” during the radiation dominated era $\Phi(\tau)$ goes through a transitional period and starts approaching a (different) constant corresponding to the minimum of the potential. As it is clear from (2.25), if $V$ and hence $\hat{U}$ (i.e. the effective cosmological constant) is positive at the minimum, the solution with $\Phi$ sitting at the minimum and the scale factor exponentially expanding is a stable one (the expansion of the universe rapidly damps the dilaton oscillations near the minimum).

The negative value of the potential (2.20) at the minimum suggests that the minimum may be unstable$^{19}$ (with the universe eventually starting contracting and the dilaton starting moving away from the minimum). However, the contributions of other scalars (matter fields) may shift the value of the effective cosmological constant making it zero or positive. In the first case the dilaton relaxes to the minimum while the universe expands according to the radiation era law ($e^\Lambda \sim \tau^{2/(N+1)}$). The exponential inflation one finds in the second case can be also achieved of course by artificially fine tuning the value of $c$ in (2.30) to make $\hat{U}_{\text{min}} > 0$.$^{24,18}$
A possibility to have an inflationary period strongly depends on details of supersymmetry breaking and structure of non-perturbative terms in the string low energy effective action (see refs.25,26 and references therein). Among the conditions necessary in order to have a period of inflation is the existence of a local minimum in the potential of some scalar matter field (“inflaton”). Let us assume that $\sigma$ in (2.18) corresponds to a flat direction of $V$ and that the potential $V(\Phi)$ has a minimum, i.e. it generates a mass term for the dilaton, $V \sim (\Phi - \Phi^*)^2 + \ldots$ (we shall assume that $V_{\text{min}} = 0$). Considering the model of ref.25 in which $L_m$ contains a minimally coupled scalar field $\psi$ with a Higgs-type potential we get the following action in the Einstein frame ($N = 3$, $p = 1$; the gravitational constant is set equal to one)

$$S_E = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - p(\partial \Phi)^2 - 3(\partial \sigma)^2 - e^{2p\Phi} \left[ \mu^2 (\Phi - \Phi^*)^2 + \frac{1}{2} (\partial \psi)^2 + (M^4 - \frac{1}{2} m^2 \psi^2 + \frac{1}{4} h \psi^4) + \ldots \right] \right\}. \quad (2.31)$$

Here $\mu$ is related to a supersymmetry breaking scale and $M$ is of order of a GUT scale (a possible choice of parameters is $\mu \sim 10^{-6} M_P$, $M \sim 10^{-3} M_P$, $m \sim 10^{-4} M_P$). If initially $\Phi \neq \Phi^*$ and the inflaton $\psi$ is in the metastable minimum of its potential ($\psi = 0$) the effective cosmological constant contains two relevant contributions: $O(\mu^2)$ and $O(M^4)$. The first term leads to a period of chaotic inflation which is followed (after the dilaton relaxes to its minimum) by a period of “old” inflation driven by the second term (for a similar two - scalar inflationary model see ref.29). The dynamics of $\sigma$ and $\psi$ is mostly irrelevant. Solving the equation for $\sigma$ we get an extra $O(e^{-2N\Lambda})$ term in $\hat{U}$ in (2.30). Ignoring all $O(e^{-n\Lambda})$ terms ($n > 0$) in $\hat{U}$ we find

$$\hat{U} = e^{2p\Phi} \left[ \mu^2 (\Phi - \Phi^*)^2 + M^4 \right] + \ldots \quad . \quad (2.32)$$

The above conclusion about the inflationary periods then follows from the analysis of solutions of the system (2.25)–(2.27).

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\(^{c}\) For some other papers on inflation in string cosmology see refs.27,28.
Note that $M^4$ plays the role of a negative central charge deficit $c$ in (2.28). If the dilaton potential term in (2.32) (which eventually fixes the dilaton) was not included, we would have found that asymptotically the solutions approach the well-known "linear dilaton, flat string frame metric" solution\textsuperscript{30} which in the Einstein frame takes the form\textsuperscript{31}

$$\Lambda = \Lambda_0 + \ln \tau, \quad \Phi = \Phi_0 - \frac{1}{2} (N - 1) \ln \tau,$$

(2.33)

$$(N - 1)^2 = -c e^{4\Phi_0/(N-1)} = M^4 e^{2p\Phi_0}.$$ 

With $O(e^{-n\Lambda})$ terms in $\hat{U}$ included, the solution (2.33) is an attractor only at infinity, i.e. the scale factor will be growing rather slowly ($\dot{a} < 1$) so that the horizon problem will not be solved\textsuperscript{32}. The dilaton potential is thus necessary also in order to get a sufficient inflation in this model.

A different mechanism for a realisation of a period of (extended\textsuperscript{33}) inflation was considered in ref.26 (cf. ref. 28). Here the main role is played by a scalar $\sigma$ corresponding to a flat direction of the dilaton - moduli potential. The existence of a flat direction is a necessary condition for extended inflation. Though in general the scalar corresponding to a flat direction may be a non-trivial combination of $\Phi$ and the modulus ($\sigma$ in (2.17)) we shall use the same notation ($\sigma$ for a flat direction and $\Phi$ for the "orthogonal" one) as in (2.31). The new element as compared to the model (2.31) is that we shall include possible couplings of $\sigma$ to matter. Let us assume for simplicity that $\sigma$ couples exponentially to the kinetic terms and masses of the matter fields\textsuperscript{6,22,26}. Then the relevant part of the effective action is given by (we shall ignore the dependence on $\Phi$ since it is fixed by the potential)

$$S_E = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} R - 3 (\partial \sigma)^2 - \frac{1}{2} \sum_n \left[ e^{\gamma_n \sigma} (\partial \psi_n)^2 + e^{\beta_n \sigma} m_n^2 \psi_n^2 \right] + \ldots \right\}. \quad (2.34)$$

In a matter dominated phase the kinetic terms are not important while the mass terms produce a potential for $\sigma$ given by a sum of exponentials. If we further assume that the potential is dominated by one of the terms $e^{\beta \sigma}$ ($\beta = \beta_i$), the resulting system is similar to (2.30),(2.31),(2.32) with $\sigma$ playing the role of $\Phi$ and the matter density – the role of $-c$
(with this identification $\beta = \sqrt{6}$ in (2.30),(2.31)).$ The cosmological equations have again the ‘power law’ solution

$$\Lambda = \Lambda_0 + 12\beta^{-2} \ln \tau, \quad \sigma = \sigma_0 - 2\beta^{-1} \ln \tau.$$  (2.35)

Since $\sigma$ is a massless scalar field the values of the coefficients $\gamma_n$, $\beta_n$ of its couplings to matter are in principle strongly constrained by the post - newtonian experiments of radar echo delay$^{34}$. An additional constraint comes from the condition that there should be no significant time variation of $\sigma$, i.e. of masses, for a consistency of the primordial nucleosynthesis scenario$^{26,35}$. The post - newtonian bound need not apply to the coefficients of those fields $\psi_n$ in (3.34) which may correspond to a dark matter$^{36,35,37}$. This suggest to identify $\beta$ with the constant of the dark matter coupling to $\sigma$. If it is the dark matter that governs the cosmological evolution in the matter dominated era $\beta$ is still constrained by the primordial nucleosynthesis bound$^{36,37,35}$. The value of $\beta$ is also subject to the condition of getting sufficient inflation$^{37,26}$. It is not clear whether these constraints can be naturally satisfied in string models.

3. Time Dependent Solutions of String Tree Level Equations

In this section we shall discuss time–dependent ("cosmological") solutions of the string tree level equations corresponding to the action (1.2) (with $V = -c/2$) or (2.3),

$$R_{\mu\nu} + 2D_\mu D_\nu \phi + \frac{1}{2} \alpha' R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} + \ldots = 0,$$  (3.1)

$^d$ In general, one can trade one of the coefficients in the exponentials for the JBD constant in (1.1) by making a Weyl transformation, i.e. by going into the Jordan frame as in ref.26.

$^e$ We are assuming that the scalar field corresponding to the flat direction is not coupled to the gauge field kinetic terms in the action (it is only one ‘dilaton’ combination of the original dilaton and the moduli that couples to the gauge terms in an essential way). If this scalar was coupled to the gauge terms, its time dependence would be severely constrained$^{38,26,35}$ by the bound on a variation of the electromagnetic coupling$^{39}$. 

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\[ c + 2D^2 \phi - 4(\partial \phi)^2 - \frac{1}{4} \alpha' R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} + \ldots = 0. \] 

(\( c = -\frac{2}{3\alpha'} (D - 26) \) in the bosonic string). One of the motivations for studying time dependent solutions of (3.1), (3.2) is the following. In the previous section we were assuming the “adiabaticity” of evolution, i.e. that the fields change slowly enough so that higher derivative terms in the effective action can be ignored. It does not look sensible to include only a finite number of terms in the \( \alpha' \) - expansion (once one of them gives significant contribution others should be important as well). To go beyond the “adiabaticity” assumption (what may be necessary in order to clarify the issue of cosmological singularity, etc) one should really look for exact (all orders in \( \alpha' \)) solutions of the string equations. At the level of (super)string perturbation theory this is equivalent to finding the corresponding conformal theories which admit a “cosmological” interpretation.

Unfortunately, very few examples of such theories are known at present. Apart from the trivial (flat metric, linear dilaton) solution\(^{30}\) and the \( R \times S^3 \) solution\(^{31}\) (based on \( SU(2) \) WZW theory)\(^f\) other known exact solutions which are based on gauged WZW theories\(^{45}\) do not look very appealing as cosmological backgrounds: they have very few (abelian) symmetries and are often singular\(^{46-53,9}\)\(^g\). It appears as if they correspond to a rather small and special subclass of \( D > 2 \) solutions of eqs. (3.1), (3.2). For example, only a subset of the simplest “toroidal” cosmological solutions of (3.1), (3.2) with \( N = D - 1 \) commuting isometries\(^{54}\) has an identified coset conformal field theory counterpart\(^{50}\).

\(^{f}\) An exact “time–dependent” solution is represented of course by any WZW theory\(^{40}\) with a group \( G \) which has one non-compact generator. An example is provided by \( SU(1,1) \) WZW model\(^{41}\) which can be interpreted as the \( D=3 \) anti de Sitter space-time with a vector field background\(^{42}\). It is likely, however, that one should necessarily gauge a subgroup of \( G \) (i.e. to consider a non-compact coset model) to get rid of the negative norm states\(^{43,44}\).

\(^{g}\) One can give a “cosmological” interpretation to a static “black hole” - type solution by rotating a space-like direction into a time-like\(^{13,9,50,52,53}\).
Standard cosmological backgrounds have their spatial sections represented by maximally symmetric $N$ - dimensional manifolds (e.g. a sphere, a flat space or a pseudosphere). If there are such regular solutions of the leading order string effective equations (3.1),(3.2) then there should exist the corresponding “maximally symmetric” conformal field theories.$^h$ The first step towards understanding of some features of these hypothetic conformal theories is to study general solutions of the leading order equations (3.1),(3.2) which have a high degree of symmetry. In particular, it seems important to generalise the solution of ref.54 (its isotropic limit) to the case when the spatial sections have a non-zero curvature$^{10}$. In what follows we shall describe some known “cosmological” solutions of (3.1), (3.2) starting with the most symmetric ones and proceeding in the direction of decreasing symmetry.

3.1. Solutions with maximally symmetric space

Let us first note that the only perturbative solution of eqs. (3.1),(3.2) with a maximal space – time symmetry of the metric is the flat solution of ref.30

$$G_{\mu\nu} = \delta_{\mu\nu}, \quad \phi = \phi_0 - bt, \quad 4b^2 = -c. \quad (3.3)$$

In fact, if the space-time curvature is constant the only solution of the leading order form of eq.(3.1) is (3.3). Assuming $\phi = const$ and $R \neq 0$ one may hope to solve eq.(3.1) by trying to compensate the leading order term $R$ by the $\alpha' R^2$ - correction$^{55}$. The two terms

$h$ The only example of a solution with a maximally symmetric space which has known conformal field theory interpretation is the “static” $N=3$ solution of ref.31. $N=3$ (pseudo)sphere is special being equivalent to a group space. Higher dimensional spheres and de Sitter spaces do not directly correspond to conformal theories. For example, the “(anti) de Sitter string” of ref.44 based on gauged SO(D,1)/SO(D-1,1) ( SO(D-1,2)/SO(D-1,1) ) WZW model with D larger (smaller) than 26 has a space-time interpretation not in terms of the (anti) de Sitter space-time but in terms of a background which does not have a maximally symmetric subspace (see refs.48,49,53).

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cancel each other if the curvature is negative, $\alpha'R = -N(N - 1)$. However, the resulting $\alpha'R$ is not small so that all higher loop corrections to the $\beta$-function (3.1) are equally important and we fail to find a consistent solution.$^i$

Next, let us consider the cosmological backgrounds with maximally symmetric space

$$ds^2 = -dt^2 + a^2(t)\, d\Omega^2 , \quad d\Omega^2 = g_{bc}dx^bdx^c ,$$

$$a = e^{\lambda(t)} , \quad \phi = \phi(t) , \quad b, c = 1, \ldots, N , \quad D = 1 + N .$$

g_{bc} is a metric of a maximally symmetric $N$ - dimensional space with the radius of curvature $k^{-1}$ ($k = -1, 0, 1$), i.e. $R_{bc} = k(N - 1)g_{bc}$. The flat solution (3.3) corresponds to

$$k = 0 , \quad \lambda = \lambda_0 = \text{const} , \quad \phi = \phi_0 - bt , \quad \varphi \equiv 2\phi - N\lambda = \varphi_0 - 2bt .$$

The resulting system of equations is (2.4)–(2.6), i.e.

$$c - N\dot{\lambda}^2 + \dot{\varphi}^2 = 2U , \quad (3.6)$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = -N^{-1}\frac{\partial U}{\partial \lambda} , \quad (3.7)$$

$$\ddot{\varphi} - N\dot{\lambda}^2 = \frac{\partial U}{\partial \varphi} , \quad (3.8)$$

where

$$U = -kN(N - 1)\, e^{-2\lambda} . \quad (3.9)$$

We shall first ignore other possible contributions to $U$ which may come from the antisymmetric tensor and gauge field backgrounds (cf.(2.29)). The simplest solution corresponds

$^i$ In general, one should not expect a solution to exist since it is believed that sigma models with maximally symmetric target spaces have mass generation, i.e. are not conformal theories. Still, in the case of a negative curvature there is a formal possibility that the beta function may have a non-perturbative zero.
to the flat space sections \((k = 0, \text{i.e. } U = 0)\) and is the isotropic case of the solution of ref.54

\[
\varphi = \varphi_0 - \ln \sinh 2bt , \quad 4b^2 = -c ,
\]

\[
\lambda = \lambda_0 + \frac{1}{\sqrt{N}} \ln \tanh bt ,
\]

i.e.

\[
\phi = \frac{1}{2}(\varphi + N\lambda) = \varphi_0 - \frac{1}{2} \ln [(\sinh bt)^{\sqrt{N}+1}(\cosh bt)^{-\sqrt{N}+1}].
\]

Asymptotically at large \(t\) it approaches the flat solution (3.5).

One may ask how “close” can string solutions with \(k \neq 0\) resemble the maximally symmetric \(D\) - dimensional de Sitter space. Naively, one could expect that the role of \(c\) in (3.6) is similar to that of the cosmological constant in the corresponding Einstein equation.

In fact, rewriting (3.6) in terms of the original dilaton \(\phi\) we get

\[
N(N-1)\dot{\lambda}^2 + 4\dot{\phi}^2 - 4N\dot{\phi}\dot{\lambda} = -c - N(N-1)k e^{-2\lambda} .
\]

If \(\phi = \text{const} \) (3.13) has the usual de Sitter \((c < 0)\)

\[
\lambda = \lambda_0 + \ln \cosh Ht \quad (k = +1) ; \quad \lambda = \lambda_0 + \ln \sinh Ht \quad (k = -1) ;
\]

\[
\lambda = \lambda'_0 + Ht \quad (k = 0) ,
\]

\[
c = -N(N-1)H^2 , \quad \lambda_0 = -\ln H
\]

or anti de Sitter \((c > 0)\)

\[
\lambda = \lambda_0 + \ln \sin Ht \quad (k = -1) , \quad c = N(N-1)H^2
\]

solutions. The point, however, is that while \(\phi = \text{const} \) and the (anti) de Sitter metric solve (3.6) and (3.7) they do not satisfy the remaining dilaton equation (3.8). That is why the dilaton should necessarily change with time producing a “deformation” of the de Sitter metric\(^9,10\). In fact, it turns out that it is the time variation of the dilaton and not that of
the scale factor that “compensates” for the presence of the “cosmological constant” \(-c\) in (3.13) in the asymptotic region of large \(t\).

Solutions of (3.6)–(3.8) with positive \(k\) or positive \(c\) appear to be singular\(^\text{10}\) (though in some cases the singularity may be a coordinate one as in the anti de Sitter case, the dilaton always starts growing in a finite period of time making such solutions unphysical). If \(c \leq 0\) and the space has a negative curvature one finds a regular solution with the dilaton always decreasing with time (we assume that \(\dot{\varphi} < 0\) at \(t = 0\)). If \(\lambda\) is contracting at the initial moment it eventually reflects from the potential wall and expands to infinity. The expansion is with slow-down due to the damping effect of the dilaton. The large \(t\) asymptotics of the solution\(^\text{10}\) is different from (3.5)

\[
\lambda \simeq \lambda_1 + \frac{1}{2} \ln t ,
\]

\[
\varphi \simeq \varphi_1 - 2bt - \frac{1}{4} N \ln t , \quad \phi \simeq \phi_1 - bt + \frac{1}{4} N \ln t ,
\]

i.e. the scale factor is slowly growing while the dilaton is linearly decreasing as in (3.5) in order to compensate for the non-vanishing \(c\).

Let us now consider the case of non-vanishing antisymmetric tensor background. The equation for the antisymmetric tensor of rank \(n - 1\)

\[
D_{\lambda_1} (e^{-2\phi} H^{\lambda_1 \ldots \lambda_n}) = 0 .
\]

has two classes of non-trivial solutions consistent with symmetries of the ansatz (3.4). The first is found if the number of space dimensions \(N\) is equal to the rank \(n\) of the antisymmetric tensor field strength\(^{56}\)

\[
H_{0a_1 \ldots a_{N-1}} = 0 , \quad H_{a_1 \ldots a_N} = h \varepsilon_{a_1 \ldots a_N} , \quad h = \text{const} .
\]

Then \(U\) in (3.6) takes the form

\[
U = -\frac{1}{2} k N (N - 1) e^{-2\lambda} + \frac{1}{4} h^2 e^{-2N\lambda} .
\]
The two particular cases relevant for string theory correspond to \( N = 2 \) (vector field background\textsuperscript{57}) and \( N = 3 \) (rank 2 antisymmetric tensor background\textsuperscript{56}).

If the spatial curvature is positive \( k > 0 \) the potential \( U \) has the minimum at \( \lambda = \lambda_0 \)

\[
h^2 = 2k(N - 1)e^{2(N - 1)\lambda_0}, \quad U(\lambda_0) = -\frac{1}{4}(N - 1)h^2e^{-2N\lambda_0} < 0 \ . \tag{3.21}
\]

Then if \( c < 0 \) the system (3.6)–(3.8) has the following “static” solution

\[
\lambda = \lambda_0, \quad \varphi = \varphi_0 - 2bt, \quad \phi = \phi_0 - bt, \quad 4b^2 = |c| - \frac{1}{2}(N - 1)h^2e^{-2N\lambda_0} \ . \tag{3.22}
\]

For \( N = 3 \) it has the well known conformal field theory generalisation represented by the direct product of the \( D = 1 \) ‘time’ theory with linear dilaton and the \( SU(2) \) WZW theory (i.e. \( S^3 \) parallelised by the antisymmetric tensor background)\textsuperscript{31}. The \( N = 2 \) case (with \( b = 0 \)) was considered, e.g., in ref.\textsuperscript{57}. In this case the constant \( F_{ab} \)-flux “compensates” for the curvature of \( S^2 \). It is possible to interpret the \( SU(1,1) \) WZW model as an exact conformal field theory which generalises this solution to all orders in \( \alpha’ \) expansion\textsuperscript{42}.

The general solution is regular if \( c_{eff} \equiv c - 2U(\lambda_0) \leq 0 \). Then \( \dot{\varphi} \) remains negative if it was negative at \( t = 0 \), i.e. the dilaton term in (3.7) plays the role of a damping force. As a result, the solution (3.21) is an attractor, i.e. it is the asymptotic form of solutions with \( \dot{\varphi} < 0 \) (the space-time is asymptotically \( R \times S^N \))\textsuperscript{57,31}.

If the spatial curvature is negative or zero \( k \leq 0 \), \( U \) in (3.19) is positive and has no local minima. The qualitative behaviour of solutions is then the same as in the absence of the antisymmetric tensor background (3.9), i.e. for \( c \leq 0 \) the dilaton is decreasing while the scale factor expands with slow-down (or first contracts to a minimal value and then expands to infinity)\textsuperscript{10}.

The second class of solutions of (3.18) exists if \( N = n - 1 \) (i.e. \( N = 1, 2 \))\textsuperscript{56}

\[
H_{\lambda_1...\lambda_{N+1}} = h \ e^{2\phi} \epsilon_{\lambda_1...\lambda_{N+1}}, \quad h = \text{const} \ . \tag{3.23}
\]

Then

\[
U = -\frac{1}{2}kN(N - 1)e^{-2\lambda} + \frac{1}{4}h^2e^{4\phi} = -\frac{1}{2}kN(N - 1)e^{-2\lambda} + \frac{1}{4}h^2e^{2\phi + 2N\lambda} \ . \tag{3.24}
\]
The antisymmetric tensor contribution to $U$ in this case is equivalent to the “two-loop” term in the dilaton potential. If the space is flat ($k = 0$) the system (3.6)–(3.8) with $U$ given by (3.24) has a simple analytic solution\textsuperscript{10}. The expression for $\varphi$ is the same as in (3.10) while $\lambda$ and the original dilaton $\phi$ are given by

$$\lambda = \lambda_0 - \frac{1}{N} \ln[A^{-1}(\tanh bt)^{-\sqrt{N}} + A (\tanh bt)^{\sqrt{N}}] \, ,$$

$$\phi = \phi_0 - \frac{1}{2} \ln(\sinh bt \cosh bt \ [A^{-1}(\tanh bt)^{-\sqrt{N}} + A (\tanh bt)^{\sqrt{N}}] ) \, ,$$

$A^2 = \frac{h^2}{32b^2} e^{2\phi_0 + N\lambda_0} = \frac{h^2}{8|c|} e^{4\phi_0}$

(eq.(3.25) reduces to (3.11) in the limit of $h = 0$). The large $t$ behaviour of this solution is the same as in (3.5), i.e. the scale factor approaches its maximal value ($\lambda_0 - N^{-1} \ln(A + A^{-1})$) while the dilaton is linear. This is not surprising since the effect of the $O(h^2)$ term in $U$ (3.24) becomes negligible because of the decrease of the dilaton. In the special case of $c = 0$ one finds

$$\varphi = \varphi_0 - \ln t \, , \quad \lambda = \lambda_0 - \frac{1}{N} \ln(A^{-1}t^{-\sqrt{N}} + A t^{\sqrt{N}}) \, ,$$

so that the scale factor grows at small $t$ until it reaches its maximum at $t_\ast = A^{-1/\sqrt{N}}$ and then asymptotically contracts to zero. The dilaton $\phi$ first grows and then starts decreasing.

When $k < 0$ the asymptotic behaviour of the solution is determined by the first term in the potential (3.24), i.e. it coincides with (3.16),(3.17). This conclusion seems to be valid in the general case of $c \leq 0 \, , \, k < 0$ and a dilaton potential $V(\phi)$ given by a sum of exponentials $e^{r\phi}$, $r > 0$ with positive coefficients. In fact, a slow growth of $\lambda$ and a rapid decrease of the dilaton $\phi$ with time implies that the dilaton potential term in $U$ will be negligible at late times.

We conclude that there are three basic asymptotic regimes of regular solutions with maximally symmetric space (dilaton is always linear at large $t$): (i) $k = 0$: flat metric (eq.(3.5)); (ii) $k > 0$: a sphere of fixed radius “parallelised” by a background antisymmetric
tensor field strength (eq.(3.22)); (iii) $k < 0$: the non-trivial expanding ($a \sim t^{1/2}$) spacetime (eqs.(3.16),(3.17)). It is an interesting question which conformal theory corresponds to the third asymptotics.

Let us note that we have described the solutions in the string frame which is most appropriate for a discussion of correspondence with conformal theories. The form of the solutions in the Einstein frame can be found using (2.22)–(2.24). It is the rapid (linear) decrease of the dilaton that determines the asymptotic behaviour of the scale factor in the Einstein frame. As a result, all asymptotic solutions (3.5), (3.22) and (3.16) look the same being transformed into the Einstein frame. Namely, if in the string frame

$$
\lambda = \lambda_0 + q \ln t \, , \quad \phi = \phi_0 - bt \, , \quad \varphi \simeq \varphi_0 - 2bt \, , \quad b > 0 \, ,
$$

(3.27)

then in the Einstein frame we get

$$
\phi = \phi_1 - \frac{1}{2} (N - 1) \ln(\tau - \tau_0) \, ,
$$

(3.28)

$$
\Lambda = \Lambda_1 + \ln(\tau - \tau_0) + q \ln \ln(\tau - \tau_0) \, .
$$

(3.29)

While $\phi(\tau)$ is decreasing much slower than $\phi(t)$, $\Lambda(\tau)$ is still growing logarithmically with the coefficient of the leading logarithm being *universal*, i.e. independent of $b$ in the dilaton $\varphi(t)$ or $q$ in $\lambda(t)$. This implies that looking at asymptotics of solutions in the Einstein frame is not sufficient in order to identify different exact solutions corresponding to different conformal theories.

### 3.2. Anisotropic solutions

The simplest class of anisotropic solutions is given by the spatially flat metric (2.1) (we are assuming $c < 0$)

$$
\lambda_i = \lambda_{i0} + q_i \ln \tanh bt \, , \quad \sum_{i=1}^{N} q_i^2 = 1 \, ,
$$

(3.30)
with the dilaton $\varphi$ being the same as in (3.10). At large times the metric is flat (scale factors approach constants) while the dilaton is linear (the corresponding asymptotic solution in the Einstein frame is again the isotropic ‘Milne’ universe $e^A \sim \tau$). This solution can be generalised (e.g. by using duality transformations) to the case of non-vanishing antisymmetric tensor backgrounds$^{58,50}$.

A subclass of the backgrounds (3.30) can be identified as representing the leading form of the exact solutions corresponding to some gauged WZW theories (e.g. $SL(2, R) \times SO(1, 1)^{D-2}/SO(1, 1)$ coset models$^{50,52}$). Given that the $D = 2$ “black hole” metric$^{59,45}$ is related to the $D = 2$ cosmological solution$^{54}$ by a complex rotation$^{13,9}$ one can construct, for example, a $D = 4$ cosmological solution by taking the direct product$^{52}$ of rotated black hole theory ($SL(2, R)/SO(1, 1)$ with negative level number) with $R^2$. The corresponding leading order metric is the following particular case of (2.1),(3.30)

$$ds^2 = -dt^2 + \tanh^2 bt \ dx_1^2 + dx_2^2 + dx_3^2 . \quad (3.31)$$

One can obtain other anisotropic solutions by taking various direct products of simple WZW models, e.g. of the “rotated” $SL(2, R)/SO(1, 1)$ theory with the $SU(2)/U(1)$ euclidean black hole theory$^{51,52}$.

Less trivial but less symmetric (inhomogeneous) and singular anisotropic solutions correspond to $SO(D - 1, 2)/SO(D - 1, 1)$ or $SO(D - 1, 2)/SO(D - 1, 1)$ WZW models$^{44,47,48,49,53}$. The metric and the dilaton of the simplest $D = 3$ model which solve (3.1),(3.2) in the leading order approximation can be represented, for example, in the form$^{49}$

$$ds^2 = -dt^2 + b^{-2}(x_1^2 + x_2^2 - 1)^{-1}\left[\tanh^2 bt \ dx_1^2 + \coth^2 bt \ dx_2^2 \right] , \quad (3.32)$$

$$\phi = \phi_0 - \ln \sinh 2bt - \frac{1}{2} \ln(x_1^2 + x_2^2 - 1) , \quad b^2 = k^{-1} , \quad 4b^2 = -c . \quad (3.33)$$

One can take a direct product of the $D = 3$ model with a gaussian model to shift the value of the central charge. The large $t$ asymptotics of (3.32),(3.33) is the product of the time line and the $D = 2$ euclidean “black hole” background.
4. Concluding remarks

Though the dilaton should be “frozen” at the minimum of a non-perturbative potential at late times (or large distances) it may play an essential role in early string cosmology. It is important to study the transitional period during which the dilaton is switching from its perturbation theory regime to a non-perturbative one. To be able to analyse in detail this transition and late time string cosmology one needs to have better understanding of the non-perturbative structure of the low energy string effective action.

One of the most characteristic properties of the dilaton coupling is its damping or stabilisation effect on cosmological solutions. In general (assuming that the time symmetry is broken by the initial condition $\dot{\phi} < 0$) the dilaton coupling introduces a kind of dissipation into the system. As a consequence, the second order string effective equations reduce to the first order renormalisation group equations in the case of large dilaton damping (see e.g. refs.60,10). The coupling of the dilaton (dark) matter may lead to an additional source of matter entropy production\textsuperscript{35}.

As for the tree level exact solutions, it would be interesting to identify new conformal field theories which may correspond to the maximally symmetric solutions of the leading order string equations. The analysis of exact time dependent string solutions and a test string propagation on the corresponding backgrounds may shed some light on a number of conceptional problems (like which metric one should use to measure singularities\textsuperscript{61,53}, expansion\textsuperscript{31,27,28}, etc) related to the fact that the string gravity is described by the metric and a scalar dilaton field.

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