Classical capacities of memoryless but not identical quantum channels

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May 26, 2020

Abstract

We study quantum channels that varies on time in a deterministic way, that is they change in an independent but not identical way from one to another use. We derive coding theorems for the classical entanglement assisted and unassisted capacities. We then specialize the theory to lossy bosonic quantum channels and show the existence of contrasting examples where capacities can or cannot be drawn from the limiting behavior of the lossy parameter.

1 Introduction

Any physical process involves a state change and hence can be regarded as a quantum channel, i.e. a stochastic map on the set of density operators [14]. As such it results quite naturally to characterize physical processes in terms of their ability of transmitting information. Hence, much attention has been devoted to quantum channel capacities. They were however mostly confined to the assumption of channels acting in independent and identically distributed way over inputs. Only recently it has been started to go beyond this assumption [3].

Following this line, here we want to address the issue of classical information transmission through memoryless but not identical quantum channels. The study of these channels is motivated as first step to take in venturing into the field of time varying channels which is almost unexplored at quantum level. As matter of fact, in classical setting a deterministic description of time varying channels (especially linear) is preliminarily taken in order to analyze their capacity (see e.g. [10]).

Additionally there are practical situations, of increasing interest for quantum communication, showing deterministically time varying channels. One of these is provided by data transmission from a low-orbit satellite to a geostationary satellite or ground station. In such scenarios, the received signal power increases as the transmitting low-orbit satellite comes into view, and then decreases as it then departs, resulting in a communication link whose time variation is known to sender and receiver (see e.g. [11]).
We shall consider here quantum channels that vary on time in a deterministic way, that is they change in an independent but not identical way from one to another use. To analyze classical information transmission through them, we shall employ smooth quantum relative entropy as main tool, and present it together with other preliminary notions in Section 2. In the core part of the paper we will derive coding theorems for the classical entanglement assisted and unassisted capacities (Sections 3 and 4 respectively). We then specialize the theory to lossy bosonic quantum channels (Section 5) and show the existence of contrasting examples where capacities can or cannot be drawn from the limiting behavior of the lossy parameter (Section 6). Section 7 is for conclusions and Appendices A and ?? contain details on the deviation of smooth relative entropy from standard relative entropy.

2 Preliminaries

Let $H$ be a separable Hilbert space, and let $\mathcal{B}(H)$ be a set of bounded linear operators acting on $H$. A quantum channel $\mathcal{N}_{A\to B}$ is a completely positive trace preserving (CPTP) linear map from $\mathcal{B}(H_A)$ to $\mathcal{B}(H_B)$.

Suppose that we have an infinite sequence $\mathcal{N} = \{\mathcal{N}_k^{A\to B}\}_k$ of quantum channels, known to both the sender and receiver before communication begins. Here we want to address the issue of what are the classical capacities (entangled assisted and unassisted) of such a sequence of channels.

A density operator on $H$ is a positive linear operator with trace equal to one. Let us consider two density operators $\rho$ and $\sigma$ and assume that their spectral decompositions are given by

$$\rho = \sum_{x \in \mathcal{X}} \lambda_x P_x \quad \text{and} \quad \sigma = \sum_{y \in \mathcal{Y}} \mu_y Q_y,$$

where $\mathcal{X}$ and $\mathcal{Y}$ are countable index sets, $\{\lambda_x\}_{x \in \mathcal{X}}$ and $\{\mu_y\}_{y \in \mathcal{Y}}$ are probability distributions with $\sum_{x \in \mathcal{X}} \lambda_x = \sum_{y \in \mathcal{Y}} \mu_y = 1$, and $P_x, Q_y$ are projections such that $\sum_{x \in \mathcal{X}} P_x = \sum_{y \in \mathcal{Y}} Q_y = I$.

Given a Positive Operator Valued Measure (POVM) with two elements, $\Pi$ and $I - \Pi$, aimed at distinguishing $\rho$ from $\sigma$, we consider a smoothed version of quantum relative entropy defined as the negative logarithm of the minimum probability that the ‘test’ $\Pi$ will fail on state $\sigma$, under the constraint that its failure probability on state $\rho$ is not larger than $\epsilon \in (0,1)$ \cite{2, 13}

$$D^\epsilon_H(\rho \parallel \sigma) \equiv \sup_{0 \leq \Pi \leq I, \Tr(\Pi \rho) \geq 1 - \epsilon} \left[- \log \Tr(\Pi \sigma)\right].$$

Throughout this paper log stands for $\log_2$.

The quantum relative entropy, its variance and the $T$ quantity are respectively defined as \cite{12, 9, 7}:

$$D(\rho \parallel \sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \Tr(P_x Q_y) \log \left(\frac{\lambda_x}{\mu_y}\right),$$

$$V(\rho \parallel \sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \Tr(P_x Q_y) \left(\log \left(\frac{\lambda_x}{\mu_y}\right) - D(\rho \parallel \sigma)\right)^2,$$

$$T(\rho \parallel \sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \Tr(P_x Q_y) \left|\log \left(\frac{\lambda_x}{\mu_y}\right) - D(\rho \parallel \sigma)\right|^3.$$
For given density operators $\rho$ and $\sigma$ satisfying
\[ D(\rho\|\sigma), V(\rho\|\sigma) \text{ and } T(\rho\|\sigma) < \infty, \] we have the following expansion (for separable Hilbert spaces)
\[ D_H^\varepsilon(n\|\sigma^\otimes n) = nD(\rho\|\sigma) + \sqrt{n}V(\rho\|\sigma)\Phi^{-1}(\varepsilon) + O(\log n), \] where
\[ \Phi(a) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left(-\frac{x^2}{2}\right) dx, \quad \Phi^{-1}(\varepsilon) = \sup\{a \in \mathbb{R} | \Phi(a) < \varepsilon\}. \] Relation (7) was proven for finite dimensional Hilbert spaces in [12, 9], then extended to separable Hilbert spaces in [4, 8]. In Appendix A, we generalize it as follows
\[ D_H^\varepsilon(n\|\sigma^\otimes n) = nD(\rho_i\|\sigma_i) + \sqrt{n}V(\rho_i\|\sigma_i)\Phi^{-1}(\varepsilon) + O(\log n), \] where $\rho_i$’s and $\sigma_i$’s are density operators acting on $\mathcal{H}$ with the additional condition
\[ \lim_{n \to \infty} \frac{1}{6\sum_{i=1}^{n} (T(\rho_i\|\sigma_i))}{\left(\sum_{i=1}^{n} V(\rho_i\|\sigma_i)\right)^3} = 0. \]}

### 3 Entanglement assisted classical capacity

In this Section we present the coding theorem for the entanglement assisted classical capacity of a deterministic sequence of independent channels $\mathfrak{N} = \{\mathcal{N}_k^{A\to B}\}_{k}$. 

**Theorem 1** Given a deterministic sequence of independent channels $\mathfrak{N} = \{\mathcal{N}_k^{A\to B}\}_{k}$, the entanglement assisted classical capacity results
\[ C_E(\mathfrak{N}) = \lim_{n \to \infty} \frac{1}{n} \max_{\rho_{RA}} \left[ \sum_{k=1}^{n} D\left(\mathcal{N}_k^{A\to B}(\rho_{RA})\|\mathcal{N}_k^{A\to B}(\rho_{A})\right) \right], \] where $\rho_{RA}$ is a resource entangled state shared by Alice and Bob.

To prove the Theorem, we will resort to ‘position based encoding’ and ‘sequential decoding strategy’. In other words, Alice and Bob are supposed to share $M$ resource entangled states $\rho_{R_iA_i}$, $i = 1, \cdots, M$ where Bob has $R$ systems and Alice has $A$ systems. If Alice wants to transmit the message $m$ through the channel $\mathcal{N}_k^{A\to B}$, simply selects the $m$’s state in her systems and sends it through the channel so that the marginal state of Bob systems is as follows
\[ \rho_{R_1} \otimes \cdots \otimes \rho_{R_{m-1}} \otimes \mathcal{N}_{A_m\to B}(\rho_{R_mA_m}) \otimes \rho_{R_{m+1}} \otimes \cdots \otimes \rho_{R_M}. \] Bob then to determine which message Alice transmitted, introduces $M$ auxiliary probe systems in the state $|0\rangle\langle 0|$, so that his overall state is
\[ \omega_{BM}^{R_M} \equiv \rho_{R_1} \otimes \cdots \otimes \rho_{R_{m-1}} \otimes \mathcal{N}_{A_m\to B}(\rho_{R_mA_m}) \otimes \rho_{R_{m+1}} \otimes \cdots \otimes \rho_{R_M} \otimes |0\rangle_p \otimes \cdots \otimes |0\rangle_p \otimes |0\rangle_M. \]
He next performs the binary measurements \( \{ \Pi_{R_m B_m P_i}, \hat{\Pi}_{R_m B_m P_i} \} \) sequentially, in the order \( i = 1, \ i = 2, \) etc. With this strategy, the probability that he decodes the \( m \)th message correctly is given by

\[
\text{Tr}\{\Pi_{R_m B_m P_m} \hat{\Pi}_{R_{m-1} B_{m-1} P_{m-1}} \cdots \hat{\Pi}_{R_1 B_1 P_1} \omega_{R M B P M}^m \hat{\Pi}_{R_1 B_1 P_1} \cdots \hat{\Pi}_{R_{m-1} B_{m-1}}\} \quad (14)
\]

Applying the “quantum union bound” \cite{8}, we can bound the complementary probability (error probability)

\[
p_e(m) \equiv 1 - \text{Tr}\{\Pi_{R_m B_m P_m} \hat{\Pi}_{R_{m-1} B_{m-1} P_{m-1}} \cdots \hat{\Pi}_{R_1 B_1 P_1} \omega_{R M B P M}^m \hat{\Pi}_{R_1 B_1 P_1} \cdots \hat{\Pi}_{R_{m-1} B_{m-1}}\} \quad (15)
\]

More specifically by \cite{8} Theorem 5.1, \( p_e(m) \leq \varepsilon \) holds when

\[
\log M = D_H^{\varepsilon, \eta} (N(\rho_R) \| \rho_R \otimes N(\rho_A)) - \log(4\varepsilon/\eta^2),
\]

where \( \eta \in (0, \varepsilon) \) and \( \varepsilon \in (0, 1) \). Now, we are going to use this relation to provide a lower bound on the position based encoding and sequential decoding for the entangled assisted classical capacity of the channel sequence \( \{ N_{A \rightarrow B}^m \}_{k=1}^N \).

**Lemma 2** A message \( m \) can be sent through the channels \( \otimes_{k=1}^n N_k \) with \( p_e(m) \leq \varepsilon \) by choosing

\[
\log M = \sum_{k=1}^n D \left( N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right)
+
\sqrt{\sum_{k=1}^n V \left( N_{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right) \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n)}. \quad (17)
\]

**Proof.** By replacing \( N^{A \rightarrow B} \) with \( \otimes_{k=1}^n N_{k}^{A \rightarrow B} \) and letting \( \Lambda_{R^n B^n} \) be a measurement operator such that

\[
\Lambda_{R^n B^n} = \arg \max_{\Lambda} \left( D_H^{\varepsilon, \eta} \left( \otimes_{k=1}^n N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \otimes_{k=1}^n \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right) \right), \quad (18)
\]

we can get from \cite{16}

\[
\log M = D_H^{\varepsilon, \eta} \left( \otimes_{k=1}^n N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \otimes_{k=1}^n \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right) - \log \left( \frac{4\varepsilon}{\eta^2} \right). \quad (19)
\]

Next, setting \( \eta = 1/\sqrt{n} \), with the second order asymptotic relation \cite{9} we arrive at

\[
\log M = \sum_{k=1}^n D \left( N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right)
+
\sqrt{\sum_{k=1}^n V \left( N_{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right) \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n)}, \quad (20)
\]

with the condition

\[
\lim_{n \to \infty} \frac{6}{\sqrt{n}} \sum_{k=1}^n \frac{T \left( N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right)}{\left[ \sum_{k=1}^n V \left( N_{k}^{A \rightarrow B}(\rho_{RA}) \left\| \rho_R \otimes N_{k}^{A \rightarrow B}(\rho_A) \right\| \right) \right]^3} = 0. \quad (21)
\]
Proof. Theorem 1. The direct part is based on the result of Lemma 2 which provides a lower bound for the capacity, namely

\[ n C_E(\mathcal{R}) \geq \max_{\rho_{RA}} \sum_{k=1}^{n} D \left( \mathcal{N}_k^{A\rightarrow B}(\rho_{RA}) \parallel \rho_R \otimes \mathcal{N}_k^{A\rightarrow B}(\rho_A) \right) \]

\[ + \sqrt{\sum_{k=1}^{n} V \left( \mathcal{N}_k^{A\rightarrow B}(\rho_{RA}) \parallel \rho_R \otimes \mathcal{N}_k^{A\rightarrow B}(\rho_A) \right)} \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n). \]  

(22)

For the converse part, since the channels are independent, though not identical, it is like to have them used in parallel, hence

\[ n C_E(\mathcal{R}) \leq \max_{\rho_{RA}} \sum_{k=1}^{n} D \left( \mathcal{N}_k^{A\rightarrow B}(\rho_{RA}) \parallel \rho_R \otimes \mathcal{N}_k^{A\rightarrow B}(\rho_A) \right). \]  

(23)

4 Unassisted classical capacity

This Section is devoted to the coding theorem for the unassisted classical capacity of a deterministic sequence of independent channel \( \mathcal{N} = \{\mathcal{N}_k^{A\rightarrow B}\}_k \).

Theorem 3 Given a deterministic sequence of independent channel \( \mathcal{N} = \{\mathcal{N}_k^{A\rightarrow B}\}_k \), the unassisted classical capacity with separable inputs results

\[ C(\mathcal{R}) = \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \max_{\rho_X^n, A^n} \sum_{k=1}^{n} D \left( \mathcal{N}_k^{A\rightarrow B}(\rho_{X^n A^n}) \parallel \rho_X \otimes \mathcal{N}_k^{A\rightarrow B}(\rho_{A^n}) \right) \right], \]

(24)

where \( \rho_{X^n A^n} = \sum_{x^n \in X^n} p(x^n) |x^n\rangle\langle x^n| \otimes \rho_{A^n}^{x^n} \)

(25)

is a classical-quantum state, with \( |x^n\rangle \in \mathcal{H}_X^{\otimes n} \) orthonormal states (being \( \mathcal{H}_X \) a separable Hilbert space) and \( \rho_{A^n}^{x^n} = \left( \rho_{A_1}^{x_1} \otimes \cdots \otimes \rho_{A_n}^{x_n} \right) \).

Given a classical-quantum channel \( x \rightarrow \rho_B^x \), we know from [15] that there exists an encoding and position based decoding for choosing

\[ \log M = D_H^{\varepsilon - \eta} (\rho_{X^n B} \parallel (\rho_X \otimes \rho_B)) - \log(4\varepsilon/\eta^2), \]

(26)

where \( \eta \in (0, \varepsilon) \), \( \varepsilon \in (0, 1) \) and

\[ \rho_{X^n B} = \sum_x p(x) |x\rangle\langle x| \otimes \rho_B^x, \]

(27)

with \( \rho_B^x \equiv \mathcal{N}(\rho_A^x) \). In other words, there exist an encoding \( m \rightarrow \rho_{A^m}^x \) and position based POVM \( \{\Omega_B^m\}_{m=1}^{M} \) as decoder such that

\[ p_\varepsilon(m) \equiv \frac{1}{M} \sum_{m=1}^{M} \text{Tr} \left\{ (I_B - \Omega_B^m) \mathcal{N}(\rho_A^m) \right\} \leq \varepsilon. \]

(28)
Lemma 4 A message $m$ can be sent through the channels $\otimes_{k=1}^n N_k$ with $p_e(m) \leq \varepsilon$ by choosing

$$\log M = \sum_{k=1}^n D \left( N_k^{A \rightarrow B}(\rho_{RA}) || \rho_R \otimes N_k^{A \rightarrow B}(\rho_A) \right) + \sqrt{\sum_{k=1}^n V \left( N_k^{A \rightarrow B}(\rho_{RA}) || \rho_R \otimes N_k^{A \rightarrow B}(\rho_A) \right)} \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n).$$  \hspace{1cm} (29)

Proof. If we replace $\rho_{XB}$ with $\otimes_{i=1}^n \rho_{XB_i}$ in Eq. (26), then we have

$$\log M = D^{\varepsilon-\eta} \left( \bigotimes_{k=1}^n (\rho_{XB_k} || \rho_X \otimes \rho_{B_k}) \right) - \log(4 \varepsilon / \eta^2).$$  \hspace{1cm} (30)

As a consequence, following the arguments of Lemma 2 we can get

$$\log M(\varepsilon) = \sum_{k=1}^n D(\rho_{XB_k} || \rho_X \otimes \rho_{B_k}) + \sqrt{\sum_{k=1}^n V(\rho_{XB_k} || \rho_R \otimes \rho_{B_k})} \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n),$$  \hspace{1cm} (31)

with the condition

$$\lim_{n \to \infty} \frac{6 \sum_{k=1}^n (T(\rho_{XB_k} || \rho_X \otimes \rho_{B_k}))}{\sqrt{\sum_{k=1}^n V(\rho_{XB_k} || \rho_X \otimes \rho_{B_k})}^3} = 0.$$  \hspace{1cm} (32)

Proof. Theorem 3 The direct part is based on the result of Lemma 4 which provides a lower bound for the capacity, namely

$$n C(\mathcal{M}) \geq \max_{\rho_{XA^n}} \left[ \sum_{k=1}^n D(\rho_{XB_k} || \rho_X \otimes \rho_{B_k}) + \sqrt{\sum_{k=1}^n V(\rho_{XB_k} || \rho_R \otimes \rho_{B_k})} \Phi^{-1} \left( \varepsilon - \frac{1}{\sqrt{n}} \right) + O(\log n) \right].$$  \hspace{1cm} (33)

For the converse part, since the channels are independent, though not identical, it is like to have them used in parallel, hence from the Holevo bound we have

$$n C(\mathcal{M}) \leq \max_{\rho_{XA^n}} \sum_{k=1}^n D(\rho_{XB_k} || \rho_X \otimes \rho_{B_k}).$$  \hspace{1cm} (34)

5 Memoryless but not identical Gaussian lossy channels

Since the coding Theorems 1 and 3 were derived without any restriction on the dimensionality of Hilbert spaces, they can be straightforwardly applied to continuous variable (bosonic) quantum channels.
We shall focus on a sequence of Gaussian lossy channels (each acting on a single bosonic mode) \( \{ \mathcal{N}_{\eta_k} \}_{k=1}^{\infty} \), where \( \eta_k \in (0, 1) \) is the transmissivity characterizing the \( k \)th channel. As customary we shall also consider an average energy \( N \) per channel use, so to have the constraint

\[
\sum_{k=1}^{n} N_k = nN, \tag{35}
\]
on the effective energy \( N_k \) employed at \( k \)th use.

### 5.1 Entangled assisted classical capacity

We consider Alice and Bob sharing \( M \) two-mode squeezed state each with photon mean number \( N_k \). We want to see how the capacity resulting from Theorem 1 is approached over channel uses.

In Ref.\[1\] it has been shown that

\[
D \left( \mathcal{N}_{\eta_k}^{k} (\rho_{RA}) \parallel \rho_R \otimes \mathcal{N}_{\eta_k}^{k} (\rho_A) \right) = g(N_k) + g(\eta_k N_k) - g((1 - \eta_k)N_k), \tag{36}
\]

where \( g(x) \equiv (x + 1) \log(x + 1) - x \log x \). In addition the quantum relative entropy variance is computed in \[7\] as

\[
V \left( \mathcal{N}_{\eta_k}^{k} (\rho_{RA}) \parallel \rho_R \otimes \mathcal{N}_{\eta_k}^{k} (\rho_A) \right) = (1 - \eta_k)N_k ((1 - \eta_k)N_k + 1) \left[ \log \left( 1 + \frac{1}{(1 - \eta_k)N_k} \right) \right]^2
- 2(1 - \eta_k)N_k(N_k + 1) \log \left( 1 + \frac{1}{(1 - \eta_k)N_k} \right) \log \left( 1 + \frac{1}{N_k} \right)
+ N_k(N_k + 1) \left[ \log \left( 1 + \frac{1}{N_k} \right) \right]^2. \tag{37}
\]

According to Theorem 1 and (36) we now need to maximize the quantity

\[
\sum_{k=1}^{n} \left[ g(N_k) + g(\eta_k N_k) - g((1 - \eta_k)N_k) \right], \tag{38}
\]

with respect to \( N_k \). This amounts to set

\[
\delta \left\{ \text{Eq. (38)} \right\} = \sum_{k=1}^{n} \left[ g' (N_k) + \eta_k g' (\eta_k N_k) - (1 - \eta_k) g' ((1 - \eta_k)N_k) \right] \delta N_k = 0, \tag{39}
\]

where \( g' \) stands for the derivative of \( g \) with respect to its argument. From the energy constraint (35) we further have

\[
\delta \left\{ \sum_{k=1}^{n} N_k \right\} = \sum_{k=1}^{n} \delta N_k = 0. \tag{40}
\]

Using a Lagrange multiplier \( \beta \) we get

\[
\sum_{k=1}^{n} \left[ g' (N_k) + \eta_k g' (\eta_k N_k) - (1 - \eta_k) g' ((1 - \eta_k)N_k) - \beta \right] \delta N_k = 0. \tag{41}
\]
Solving the set of \( n + 1 \) equations

\[
\begin{aligned}
g'(N_k) + \eta_k g'(\eta_k N_k) - (1 - \eta_k)g'((1-\eta_k)N_k) - \beta &= 0 \\
\sum_{k=1}^{n} N_k &= nN
\end{aligned}
\]  \hspace{1cm} (42)

allows us to find the \( N_k \) and \( \beta \) giving

\[
C_{E,n} \equiv \max_{N_k} \sum_{k=1}^{n} [g(N_k) + g(\eta_k N_k) - g((1-\eta_k)N_k)]. \hspace{1cm} (43)
\]

Clearly \( \lim_{n \to \infty} C_{E,n}/n = C_E(\{N_{\eta_k}\}) \).

The variance of the quantum relative entropy \( C_{E,n}/n \) can be obtained by means of (37) as

\[
\frac{1}{n^2} \sum_{k=1}^{n} V(\mathcal{N}_{A\to B}(\rho_{RA})\|\rho_R \otimes \mathcal{N}_{A\to B}^k(\rho_A)). \hspace{1cm} (44)
\]

5.2 Unassisted classical capacity

We want to see how the capacity resulting from Theorem 3 is approached over channel uses.

From Ref. [16], we know that

\[
D(\rho_{X_{B_k}}\|\rho_X \otimes \rho_{B_k}) = g(\eta_k N_k), \hspace{1cm} (45)
\]

and

\[
V(\rho_{X_{B_k}}\|\rho_X \otimes \rho_{B_k}) = \eta_k N_k(\eta_k N_k + 1) [\log (\eta_k N_k + 1) - \log (\eta_k N_k)]^2. \hspace{1cm} (46)
\]

Then, according to Theorem 1 and (36) we now need to maximize the quantity

\[
\sum_{k=1}^{n} g(\eta_k N_k) \hspace{1cm} (47)
\]

with respect to \( N_k \).

Proceeding like in the previous section, using a Lagrange multiplier \( \beta \) and imposing (35), we get

\[
\begin{aligned}
g'(N_k) - \beta &= 0 \\
\sum_{k=1}^{n} N_k &= nN
\end{aligned}
\]  \hspace{1cm} (48)

Solving this set of \( n + 1 \) equations allows us to find the \( N_k \) and \( \beta \) giving

\[
C_n \equiv \max_{N_k} \sum_{k=1}^{n} [g(N_k)]. \hspace{1cm} (49)
\]

Clearly \( \lim_{n \to \infty} C_n/n = C(\{N_{\eta_k}\}) \).

The variance of the quantum relative entropy \( C_n/n \) can be obtained by means of (46) as

\[
\frac{1}{n^2} \sum_{k=1}^{n} V(\rho_{X_{B_k}}\|\rho_X \otimes \rho_{B_k}). \hspace{1cm} (50)
\]
6 Examples

Here we want to apply the results of Sec.5 to some specific cases study, i.e. specific sequences of lossy channels.

6.1 Example 1

Consider

\[
\eta_k = \eta + \eta e^{-(k-1)^2/\Delta}, \quad 0 < \eta, \bar{\eta} < \frac{1}{2}.
\]

After a transient (whose extension is determined by \(\Delta\)) the channel reaches a transmissivity \(\eta\) (see Fig.1). The distribution of input energy shows a similar behavior to transmissivity (see Fig.1). Note however that the sequence \(\{N_k^{(n)}\}_k\) depends on total number \(n\) of channel uses.

![Figure 1: Quantities \(\eta_k\) (dashed line) and \(N_k^{(n)}\) (solid line) vs \(k\) for \(n = 100\). On the left (resp. right) is the case for entanglement assisted (resp. unassisted) classical communication. It is \(\Delta = 5\). The values of other parameters are \(\eta = 0.4, \bar{\eta} = 0.1, N = 1\).](image)

For this example the capacity \(\lim_{n \to \infty} (C_{E,n}/n)\) can be guessed by simply computing \(C_E\) for \(\lim_{k \to \infty} \eta_k\) (see Fig.2 left). Analogous argument holds true for \(\lim_{n \to \infty} (C_n/n)\), i.e. it can be guessed by simply computing \(C\) for \(\lim_{k \to \infty} \eta_k\) (see Fig.2 right).

![Figure 2: (Left) \(C_{E,n}/n\) vs \(n\) for \(\Delta = 5\) (solid line). The bottom (resp. top) dashed line represents the capacity \(g(N) + g(\eta N) - g((1-\eta)N)\) (resp. \(g(N) + g((\eta + \bar{\eta}) N) - g((1-(\eta + \bar{\eta}))N)\)). (Right) \(C_n/n\) vs \(n\) for \(\Delta = 5\) (solid line). The bottom (resp. top) dashed line represents the capacity \(g(N)\) (resp. \(g((\eta + \bar{\eta})N)\)). The values of other parameters are \(\eta = 0.4, \bar{\eta} = 0.1, N = 1\).](image)
In Fig. 3 we report the variances (44) and (50) as functions of \( n \). As one can see the variance of \( C_{E,n}/n \) converges faster than that of \( C_n/n \).

![Figure 3: Variance of \( C_{E,n}/n \) (solid line) and of \( C_n/n \) (dashed line) vs \( n \) for \( \Delta = 5, \eta = 0.4, \overline{\eta} = 0.1, N = 1 \).](image)

6.2 Example 2

Consider

\[
\eta_k = \eta + \overline{\eta} \left| \sin \left( \frac{k - \frac{1}{2}}{\Delta} + \frac{\pi}{2} \right) \right|, \quad 0 < \eta, \overline{\eta} < \frac{1}{2}
\]

In this case we have an oscillatory behavior of \( \eta_k \) (whose frequency is determined by \( \Delta \)), as can be seen in Fig. 4. The distribution of input energy still follows the behavior of transmissivity.

![Figure 4: Quantities \( \eta_k \) (dashed line) and \( N^{(n)}_k \) (solid line) vs \( k \) for \( n = 100 \). On the left (resp. right) is the case for entanglement assisted (resp. unassisted) classical communication. It is \( \Delta = 5 \). The values of other parameters are \( \eta = 0.4, \overline{\eta} = 0.1, N = 1 \).](image)

In this case the capacity cannot be guessed by the \( \lim_{k \to \infty} \eta_k \), because this latter does not exist, however the capacity \( C_{E,n}/n \), after transient oscillations depending on \( \Delta \), converges to a well definite value for \( n \to \infty \) (see Fig. 5 left). The same happens for the capacity \( C_n/n \) (see Fig. 5 right).
Figure 5: (Left) $C_{E,n}/n$ vs $n$ for $\Delta = 5$ (solid line). The bottom (resp. top) dashed line represents the capacity $g(N) + g(\eta N) - g((1 - \eta)N)$ (resp. $g(N) + g((\eta + \bar{\eta})N) - g((1 - (\eta + \bar{\eta}))N)$). (Right) $C_n/n$ vs $n$ for $\Delta = 5$ (solid line). The bottom (resp. top) dashed line represents the capacity $g(\eta N)$ (resp. $g((\eta + \eta)N)$). The values of other parameters are $\eta = 0.4, \bar{\eta} = 0.1, N = 1$.

As for what concerns the variance of $C_{E,n}/n$ and of $C_n/n$, the behavior is quite similar to that of Fig.3

7 Conclusion

In conclusion, we studied quantum channels that vary from one to another use in a deterministic way. To analyze their ability in transmitting classical information we resorted to a smoothed version of quantum relative entropy. As first result we derived a generalization of the relation between smooth and standard quantum relative entropies to the case of tensor product of non identical density operators, which is valid also for separable Hilbert spaces (Eq.(9)). We then proved coding theorems for the classical entanglement assisted and unassisted capacities (Theorems 1 and 3 respectively). For that we used position based coding and quantum union bound applied to sequential decoding. The results were then adapted, with the help of input energy constraints, to continuous variable quantum channels, specifically lossy bosonic. Finally enlightening examples were put forward in this context. They show that only when the sequence of channels parameter has a well defined limit, the capacities can be easily evaluated. The approach taken allowed us to evaluate the maximum transmission rate for any number of channel uses and estimate the error.

The natural extension of this work would be the study of time varying channels in a non-deterministic (still memoryless) way. We are confident that the mathematical tools developed here will be useful to this end. On another direction one can pursue the quantum capacity of the introduced sequences of channels, which however needs slightly different tools.

Acknowledgments

The authors are grateful to Mark M. Wilde for useful discussions.

References

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A Second-order asymptotic

In this Appendix we derive Eq. (9), which represents a generalization of the relation between smooth and standard quantum relative entropies to the case of tensor product of non identical density operators, which is valid also for separable Hilbert spaces.
Let us assume that $\rho_i, \sigma_i$, for $i = 1, 2, \ldots$ be full rank density operators on Hilbert spaces $\mathcal{H}$. Let us consider their spectral decompositions as follows

$$\rho_i = \sum_{x_i} \lambda_{x_i} P_{x_i}, \quad (53)$$

and

$$\sigma_i = \sum_{y_i} \mu_{y_i} Q_{y_i}. \quad (54)$$

We also define two distributions for any two density operators $\rho = \sum_x \lambda_x P_x$ and $\sigma = \sum_x \mu_x Q_x$ as follows

$$P_{\rho,\sigma} = \lambda_x \text{Tr}(P_x Q_y), \quad (55)$$

and

$$Q_{\rho,\sigma} = \mu_y \text{Tr}(P_x Q_y). \quad (56)$$

We then introduce

$$P_{\rho,\sigma}(x_1, \ldots, x_n) = \prod_i P_{\rho,\sigma}(x_i), \quad (57)$$

and

$$Q_{\rho,\sigma}(x_1, \ldots, x_n) = \prod_i Q_{\rho,\sigma}(y_i). \quad (58)$$

**Lemma 5** Let $\rho$ and $\sigma$ be two density operators acting on a separable Hilbert space $\mathcal{H}$. For a given number $L$, there exists a measurement operator $T_L$ ($0 \leq T_L \leq I$) such that

$$\text{Tr}(T_L \rho) \geq \Pr(Z \geq \log L), \quad \text{Tr}(T_L \sigma) \leq \frac{1}{L}, \quad (59)$$

where $Z$ is a random variable defined by $Z \equiv \log P_{\rho,\sigma}(X) - \log Q_{\rho,\sigma}(X)$.

Let $\bar{Z} = \frac{1}{n} \sum_i Z_i$ be the average over $n$ independent but not identical random variables $Z_i = \log P_{\rho,\sigma}(X_i) - \log Q_{\rho,\sigma}(X_i)$. For each $i$, set

$$\mu_i \equiv D(P_{\rho_i,\sigma_i} \parallel Q_{\rho_i,\sigma_i}) = D(\rho_i \parallel \sigma_i), \quad s_i^2 \equiv V(P_{\rho_i,\sigma_i} \parallel Q_{\rho_i,\sigma_i}) = V(\rho_i \parallel \sigma_i). \quad (60)$$

and,

$$t_n \equiv \mathbb{E}\left( (Z_n - \mu_i)^3 \right) = \mathbb{E}\left( [\log P_i(X_i) - \log Q_i(X_i) - D(\rho \parallel \sigma)]^3 \right). \quad (61)$$

Now, we use the following Theorem which is a generalization of the central limit Theorem for independent but not identical random variables.

**Theorem 6** [3]. Let the $\{X_n\}$ be random variables such that

$$\mathbb{E}(X_n) = 0, \quad \mathbb{E}(X_n^2) = s_n^2, \quad \mathbb{E}(|X_n|^3) = t_n, \quad (62)$$

Put

$$\frac{s_n^2}{s_1^2 + \cdots + s_n^2}, \quad \frac{t_n}{t_1 + \cdots + t_n}, \quad (63)$$
and denoted by $P^n$ the distribution of the normalized sum $(X_1 + \cdots + X_n)/\bar{s}_n$, then under the following condition

$$\lim_{n \to \infty} \frac{6\tilde{r}_n}{\bar{s}_n} = 0,$$  \quad (64)

for all $x$ and $n$, we have

$$\left| P^n \left( \frac{X_1 + \cdots + X_n}{\bar{s}_n} \leq x \right) - \Phi(x) \right| \leq \frac{6\tilde{r}_n}{\bar{s}_n},$$

where

$$\Phi(x) \equiv \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$  \quad (66)

**Proposition 7** Let $\rho_1, \rho_2, \ldots$ and $\sigma_1, \sigma_2, \ldots$ denote states acting on a separable Hilbert space $\mathcal{H}$. Suppose that $D(\rho_i || \sigma_i), V(\rho_i || \sigma_i), T(\rho_i || \sigma_i) < \infty$ and $V(\rho_i || \sigma_i) > 0$, for each $i = 1, 2, 3, \ldots$. Suppose $n$ is sufficiently large such that $\varepsilon - \frac{6\sum_{i=0}^n [T(\rho_i || \sigma_i)]}{\sqrt{\sum_{i=0}^n V(\rho_i || \sigma_i)}} > 0$. Then

$$D^\varepsilon_H \left( \bigotimes_{i=1}^n \rho_i \bigotimes_{i=1}^n \sigma_i \right) = \sum_{i=1}^n D(\rho_i || \sigma_i) + \sum_{i=1}^n V(\rho_i || \sigma_i) \Phi^{-1} \left( \varepsilon - \frac{6\sum_{i=0}^n [T(\rho_i || \sigma_i)]}{\sqrt{\sum_{i=0}^n V(\rho_i || \sigma_i)}} \right),$$

$$= \sum_{i=1}^n D(\rho_i || \sigma_i) + \sum_{i=1}^n V(\rho_i || \sigma_i) \Phi^{-1}(\varepsilon) + O(\log n).$$  \quad (68)

**Proof.** (Part $\geq$) Applying the Berry–Esseen theorem [5] to the random sequence $Z_1 - D(\rho_1 || \sigma_1), \ldots, Z_n - D(\rho_n || \sigma_n)$, we find that

$$\left| \Pr \left\{ \frac{Z^n}{\sqrt{\sum_{i=0}^n V(\rho_i || \sigma_i)}} \leq x \right\} - \Phi(x) \right| \leq \alpha(n),$$

where $Z^n \equiv \frac{1}{n} \sum_{i=1}^n \left[ Z_i - D(\rho_i || \sigma_i) \right]$ and $\alpha(n) \equiv \frac{6\sum_{i=0}^n [T(\rho_i || \sigma_i)]}{\sqrt{\sum_{i=0}^n V(\rho_i || \sigma_i)}}$, which implies that

$$\Pr \left\{ \sum_{i=1}^n Z_i \leq \sum_{i=1}^n D(\rho_i || \sigma_i) + nx \sqrt{\sum_{i=0}^n V(\rho_i || \sigma_i)} \right\} \leq \Phi(x) + \alpha(n).$$  \quad (70)

Picking $x = \Phi^{-1} \left( \varepsilon - \frac{6\tilde{r}_n}{\bar{s}_n} \right)$, this becomes

$$\Pr \left\{ \sum_{i=1}^n Z_i \leq \sum_{i=1}^n D(\rho_i || \sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i || \sigma_i)} \Phi^{-1}(\varepsilon - \alpha(n)) \right\} \leq \varepsilon.$$  \quad (71)

Choosing $L$ such that

$$\log L = \sum_{i=1}^n D(\rho_i || \sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i || \sigma_i)} \Phi^{-1}(\varepsilon - \alpha(n))$$  \quad (72)
and applying Lemma 5, we find that
\[
\text{Tr}\{T^n \bigotimes_{i=1}^n \rho_i\} \geq \Pr\left\{\sum_{i=1}^n Z_i \geq \log L\right\} = 1 - \Pr\left\{\sum_{i=1}^n Z_i \leq \log L\right\} \geq 1 - \varepsilon, \tag{73}
\]
while
\[
\text{Tr}\{T^n \bigotimes_{i=1}^n \sigma_i\} \leq \frac{1}{L} = \exp\left\{-\left[\sum_{i=1}^n D(\rho_i||\sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i||\sigma_i)\Phi^{-1}(\varepsilon - \alpha(n))}\right]\right\}. \tag{74}
\]
This implies that
\[
-\log \text{Tr}\left\{T^n \bigotimes_{i=1}^n \sigma_i\right\} \geq \sum_{i=1}^n D(\rho_i||\sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i||\sigma_i)\Phi^{-1}(\varepsilon - \alpha(n))}. \tag{75}
\]
Since \(D_\varepsilon^n(\bigotimes_{i=1}^n \rho_i||\bigotimes_{i=1}^n \sigma_i)\) involves an optimization over all possible measurement operators \(T^n\) satisfying \(\text{Tr}\{T^n \bigotimes_{i=1}^n \rho_i\} \geq 1 - \varepsilon\), we conclude that the bound \(\geq\) in [67] holds true. The equality \(\approx\) follows from expanding \(\Phi^{-1}\) at the point \(\varepsilon\) using Lagrange’s mean value theorem.

(Part \(\leq\)) We use the following theorem from Ref. [6].

**Theorem 8** [6] Let \(\rho\) and \(\sigma\) be faithful states acting on a separable Hilbert space \(\mathcal{H}\), let \(T\) be a measurement operator acting on \(\mathcal{H}\) and such that \(0 \leq T \leq I\), and let \(\nu, \theta \in \mathbb{R}\). Then
\[
e^{-\theta} \text{Tr}\{(I - T)\rho + \text{Tr}\{T\sigma\} \geq \frac{e^{-\eta}}{1 + e^{\nu - \theta}} \Pr\{X \leq \nu\}, \tag{76}
\]
where \(X\) is a random variable taking values \(\log(\lambda_x/\mu_y)\) with probability \(\lambda_x \text{Tr}(P_xQ_y)\).

Then, the proof closely follows the proof of Proposition 2 in [7], which is based on [4]. Choosing
\[
\nu_n = \sum_{i=1}^n D(\rho_i||\sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i||\sigma_i)\Phi^{-1}\left(\varepsilon + \frac{2}{\sqrt{n}} + \alpha(n)\right)}, \tag{77}
\]
and \(\theta_n = \nu_n + \frac{1}{2}\log n\), we get
\[
\text{Tr}\{T^n \bigotimes_{i=1}^n \sigma_i\} \geq \left\{e^{-\sum_{i=1}^n D(\rho_i||\sigma_i) - \sqrt{\sum_{i=1}^n V(\rho_i||\sigma_i)\Phi^{-1}(\varepsilon + 2n^{-1/2} + \alpha(n))} - \frac{1}{2}\log n}\right\}\left(\frac{1}{1 + n^{-1/2}}\right), \tag{78}
\]
where \(\text{Tr}\{(I^n - T^n) \bigotimes_{i=1}^n \rho_i\} \leq \varepsilon\). Then, we find
\[
-\log \text{Tr}\{T^n \bigotimes_{i=1}^n \sigma_i\} \leq \sum_{i=1}^n D(\rho_i||\sigma_i) + \sqrt{\sum_{i=1}^n V(\rho_i||\sigma_i)\Phi^{-1}(\varepsilon + 2n^{-1/2} + \alpha(n))} + \frac{1}{2}\log n - \log \frac{1}{1 + n^{-1/2}}. \tag{79}
\]