THE RELATIVISTIC VLASOV-MAXWELL-BOLTZMANN SYSTEM FOR SHORT RANGE INTERACTION

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Abstract. We are concerned with the Cauchy problem of the relativistic Vlasov-Maxwell-Boltzmann system for short range interaction. For perturbative initial data with suitable regularity and integrability, we prove the large time stability of solutions to the relativistic Vlasov-Maxwell-Boltzmann system, and also obtain as a byproduct the convergence rates of solutions. Our proof is based on a new time-velocity weighted energy method and some optimal temporal decay estimates on the solution itself. The results also extend the case of “hard ball” model considered by Guo and Strain [Comm. Math. Phys. 310: 49–673 (2012)] to the short range interactions.

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1. Introduction.

1.1. The Cauchy problem. We consider the following relativistic Vlasov-Maxwell-Boltzmann system

\[
\begin{align*}
\partial_t F_+ + \vec{p} \cdot \nabla_x F_+ + \left( E + \vec{p} \times \vec{B} \right) \cdot \nabla_p F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\
\partial_t F_- + \vec{p} \cdot \nabla_x F_- - \left( E + \vec{p} \times \vec{B} \right) \cdot \nabla_p F_- &= Q(F_-, F_-) + Q(F_-, F_+).
\end{align*}
\]  

(1.1)

The self-consistent electromagnetic field satisfies the Maxwell system

\[
\begin{align*}
\partial_t E - \nabla_x \times \vec{B} &= -\int_{\mathbb{R}^3} \vec{p} (F_+ - F_-) dp,  \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} (F_+ - F_-) dp, \\
\nabla_x \cdot \vec{B} &= 0.
\end{align*}
\]

(1.2)

Here \( F_\pm = F_\pm(x, p, t) \) are the number density functions for ions (+) and electrons (−) at the phase-space position \( (x, p) = (x_1, x_2, x_3, p_1, p_2, p_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \), at time \( t \in \mathbb{R}_+ \), and \( E(t, x), B(t, x) \) are the electric and magnetic fields, respectively. \( p_0 = \sqrt{1 + |p|^2} \) is the energy of a particle, here and in the sequel, we denote \( \vec{p} = \frac{p}{p_0} \).

The initial data of the coupled system above are given by

\[
F_\pm(x, v, 0) = F_{0, \pm}(x, p), \quad (E, B)(x, 0) = (E_0(x), B_0(x)),
\]

(1.3)

satisfying the compatibility conditions

\[
\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0, +} - F_{0, -}) dp, \quad \nabla_x \cdot B_0 = 0.
\]

The relativistic Boltzmann collision operator \( Q(\cdot, \cdot) \) in (1.1) takes the form of

\[
Q(F, G) = \frac{1}{p_0} \int_{\mathbb{R}^3} dq_0 \int_{\mathbb{R}^3} dq_0' \int_{\mathbb{R}^3} dp_0 \int_{\mathbb{R}^3} dp_0' W[F(p')G(q') - F(p)G(q)],
\]

(1.4)

Here the “transition rate” \( W = W(p, q|p', q') \) is defined as

\[
W = s \sigma(g, \theta) \delta(p_0 + q_0 - p_0' + q_0') \delta^{(3)}(p + q - p' - q').
\]

(1.5)

The delta functions express the conservation of momentum and energy:

\[
p' + q' = p + q, \quad p_0' + q_0' = p_0 + q_0.
\]

(1.6)

The quantity \( s \) in (1.5) is the square of the energy in the “center of momentum”, \( p + q = 0 \), and is given as

\[
s = s(p, q) = -(p_0 + q_0, p + q) \odot (p_0 + q_0, p + q) = 2(p_0q_0 - p \cdot q + 1) \geq 4
\]

where \( \odot \) denotes the Lorentz inner product, which is given by

\[
(p_0, p) \odot (q_0, q) = -p_0 q_0 + p \cdot q.
\]
And the relativistic momentum $g$ in (1.5) is denoted

$$g = g(p,q) = -(p_0 - q_0, p - q) \circ (p_0 - q_0, p - q) = \sqrt{2(p_0q_0 - p \cdot q - 1)} \geq 0.$$ 

The angle $\theta$ in (1.5) is then given by

$$\theta = \theta(p,q) = \frac{(p_0 - q_0, p - q)}{g^2} \circ (p_0 - q_0, p - q).$$

The relativistic differential cross section $\sigma(g, \theta)$ depends only on the relative momentum $g$ and the deviation angle $\theta$. Here, we use the following short range interaction form cf. [13, 31]

$$\sigma(g, \theta) = C_\theta \sigma,$$

where $C_\theta$ is a constant.

In the next sub-sections, we will present the reductions of the collision operator (1.4).

1.2. Center of momentum reduction. One may use Lorentz transformations as described in [37] to reduce the delta functions in (1.4) and obtain

$$Q(F,G) = \frac{1}{p_0} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq' \int_{\mathbb{R}^3} dp' \frac{\sqrt{s}}{p_0q_0} W[F(p')G(q') - F(p)G(q)]$$

$$= \int_{\mathbb{R}^3} dq \int_{S^2} d\omega v(p,q) \sigma(g, \theta)[F(p')G(q') - F(p)G(q)]$$

where $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ and $v = v(p,q)$ is the Møller velocity given by

$$v = v(p,q) = \sqrt{|\bar{p} - \bar{q}|^2 - |\bar{p} \times \bar{q}|^2} = \frac{g\sqrt{s}}{2p_0q_0}.$$ 

The post collision momentum in the expression (1.9) can be written:

$$p' = \frac{p + q}{2} + g \left( \omega + \left( \frac{p_0 + q_0}{\sqrt{s}} - 1 \right) \frac{(p + q) \cdot \omega}{|p + q|^2} \right),$$

$$q' = \frac{p + q}{2} - g \left( \omega + \left( \frac{p_0 + q_0}{\sqrt{s}} - 1 \right) \frac{(p + q) \cdot \omega}{|p + q|^2} \right).$$

The energies are then

$$p'_0 = \frac{p_0 + q_0}{2} + \frac{g}{2\sqrt{s}} (p + q) \cdot \omega,$$

$$q'_0 = \frac{p_0 + q_0}{2} - \frac{g}{2\sqrt{s}} (p + q) \cdot \omega.$$ 

The angle $\theta$ in the reduced expression (1.9) is defined by

$$\cos \theta = \frac{v}{|v|} \cdot \omega,$$

where $v \in \mathbb{R}^3$ has a complicated expression as given in [37] but its precise form will be inessential.

Now we turn to the expression given by Glassey-Strauss in [17].
1.3. **Glassey-Strauss reduction.** Glassey-Strauss have derived an alternative form for relativistic collision operator without using the center-of-momentum. We will skip their argument and write down the result as follows.

\[
Q(F, G) = \frac{1}{p_0} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq' \int_{\mathbb{R}^3} dp' W[F(p')G(q') - F(p)G(q)]
\]

\[
= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s\sigma(g, \theta)}{p_0q_0} \mathbb{B}(p, q, \omega)[F(p'')G(q'') - F(p)G(q)],
\]

where the kernel is

\[
\mathbb{B}(p, q, \omega) = \frac{(p_0 + q_0)^2 |\omega \cdot (p_0q - q_0p)|}{[(p_0 + q_0)^2 - (\omega \cdot (p + q))^2]^2}.
\]

In this expression, the post collisional momentum are given as follows

\[
p'' = p + \alpha(p, q, \omega)\omega, \quad q'' = q - \alpha(p, q, \omega)\omega,
\]

where

\[
\alpha(p, q, \omega) = \frac{2(p_0 + q_0)[\omega \cdot (p_0q - q_0p)]}{(p_0 + q_0)^2 - (\omega \cdot (p + q))^2}
\]

and the energies can be expressed as

\[
p''_0 = p_0 + \alpha(p, q, \omega)\frac{p + q}{p_0 + q_0} \cdot \omega, \quad q''_0 = q_0 - \alpha(p, q, \omega)\frac{p + q}{p_0 + q_0} \cdot \omega,
\]

These formula clearly satisfy the collisional conservations (1.6). The angle (1.7) in (1.12) can then be reduced to

\[
\cos \theta = 1 - \frac{8}{g^2} \frac{[\omega \cdot (p_0q - q_0p)]^2}{(p_0 + q_0)^2 - (\omega \cdot (p + q))^2}.
\]

Moreover, assuming the collisions are elastic as in (1.6), we have the invariance

\[
\omega \cdot (p_0q - q_0p) = \omega \cdot (p''_0q'' - q''_0p'').
\]

Therefore, for fixed \(\omega \in \mathbb{S}^2\), \(\mathbb{B}(p, q, \omega) = \mathbb{B}(p'', q'', \omega)\). The Jacobian for the transformation \((p, q) \rightarrow (p'', q'')\) in these variables [16] is

\[
\frac{\partial(p'', q'')}{\partial(p, q)} = -\frac{p''_0q''_0}{p_0q_0}, \tag{1.13}
\]

Then, it is easy to see that

\[
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s\sigma(g, \theta)}{p_0q_0} \mathbb{B}(p, q, \omega)G(p, q, p'', q'')
\]

\[
= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega v(p, q)\sigma(g, \theta)G(p, q, p', q'), \tag{1.14}
\]

where \((p', q')\) on the right hand side are given by (1.10) and \(G(p, q, p', q') : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}\) is a given function. Moreover the Jacobian (1.13) effectively works for \((p', q')\) in (1.10) as

\[
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega v(p, q)\sigma(g, \theta)G(p, q, p', q')
\]

\[
= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} d\omega v(p, q)\sigma(g, \theta)G(p', q', p, q). \tag{1.15}
\]
1.4. Reformulation, weight functions and norms. We now turn to the presentation of our main result. The global relativistic Maxwellian (the Jüttner solution) is given by

\[ J(p) = \frac{\exp(-p_0)}{4\pi}. \]

We set the perturbation in a standard way

\[ F_{\pm} = J + J^\frac{1}{2}f_{\pm}. \]

Use \([\cdot,\cdot]\) to denote the column vector in \(\mathbb{R}^2\). Set \(F = [F_+, F_-]\) and \(f = [f_+, f_-]\). Then the Cauchy problem (1.1), (1.2), (1.3) can be reformulated as

\[
\begin{aligned}
\partial_t f + \hat{p} \cdot \nabla_x f - E \cdot \hat{p} J^\frac{1}{2} \zeta_1 + Lf &= \frac{\hat{q}}{2} E \cdot \hat{p} f + \zeta \left( E + \hat{p} \times B \right) \cdot \nabla_p f + \Gamma(f, f) \\
\partial_t E - \nabla_x \times B &= \int_{\mathbb{R}^3} \hat{p} J^\frac{1}{2}(f_+ - f_-) dp \\
\partial_t B + \nabla_x \times E &= 0, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} (f_+ - f_-) dp, \\
\nabla_x \cdot B &= 0
\end{aligned}
\]

with initial data

\[ f_{\pm}(0, x, p) = f_{0,\pm}(x, p), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x) \]

satisfying the compatibility condition

\[ \nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (f_{0,+} - f_{0,-}) dp, \quad \nabla_x \cdot B_0 = 0. \]

Here, \(\zeta = \text{diag}(1, -1), \zeta_1 = [1, -1]\), and the linearized collision term \(Lf\) and the nonlinear collision term \(\Gamma(f, f)\) are respectively defined by

\[ Lf = [L_+ f, L_- f], \quad \Gamma(f, g) = [\Gamma_+(f, g), \Gamma_-(f, g)] \]

with

\[
\begin{aligned}
L_{\pm} f &= -2 J^{-\frac{1}{2}} Q(J^\frac{1}{2} f_{\pm}, J) - J^{-\frac{1}{2}} Q(J, J^\frac{1}{2}(f_{\pm} + f_{\mp})), \\
\Gamma_{\pm}(f, g) &= J^{-\frac{1}{2}} Q(J^\frac{1}{2} f_{\pm}, J^\frac{1}{2} g_{\pm}) + J^{-\frac{1}{2}} Q(J^\frac{1}{2} f_{\pm}, J^\frac{1}{2} g_{\mp}).
\end{aligned}
\]

As in [19], the null space of the linearized operator \(L\) is given by

\[ \mathcal{N} = \text{span} \left\{ [1, 0] J^\frac{2}{3}, [0, 1] J^\frac{2}{3}, [p_i, p_i] J^\frac{2}{3} (1 \leq i \leq 3), [p_0, p_0] J^\frac{2}{3} \right\}. \]

Let \(P\) be the orthogonal projection from \(L_{0}^{2} \times L_{0}^{2}\) to \(\mathcal{N}\). Given

\[ f(t, x, p) = [f_+(t, x, p), f_-(t, x, p)], \]

one can decompose \(f\) uniquely as \(f = Pf + \{I - P\} f\) with \(Pf = [P_+ f, P_- f]\) and \(P_{\pm} f = \{a_{\pm}(t, x) + b(t, x) \cdot p + c(t, x) p_0\} J^\frac{1}{2}\).

Here, the coefficient functions \([a_{\pm}, b, c](t, x)\) are determined by \(f\) in the way (3.2). Accordingly, we write \([I-P] f = \{I_+ - P_+\} f, [I-P]_- f = \{I_- - P_-\} f\) with \(I_{\pm} f = f_{\pm}\).

In what follows, we introduce the weight functions and norms used throughout the paper. First of all, we define

\[ w_l = w_l(p) = (p_0)^l, \]
where \( l \) is some nonnegative constant. Denote
\[
|f|_{p}^{2} = \int_{\mathbb{R}^{3}} p|f|^{2} dp, \quad \|f\|_{p}^{2} = \int_{\mathbb{R}^{3}} |f|^{2} dx.
\]
We also use \( \| \cdot \|_{H^{N}} \) to denote the standard Sobolev norm in \( \mathbb{R}^{3} \) with respect to the variables \( x \).

1.5. Main result. To study the global existence by means of the energy method, we define the following temporal energy functional \( \mathcal{E}_{N,\ell}(t) \) as
\[
\mathcal{E}_{N,\ell}(t) \sim \sum_{|\alpha| \leq N} \| \partial^{\alpha} (a_{\pm}, b, c)(t) \|^{2}
+ \sum_{|\alpha| + |\beta| \leq N-1} \| u_{\ell-|\alpha|-|\beta|} \partial_{\beta}^{3} (I - P) f(t) \|^{2}
+ \| (E, B)(t) \|_{H^{N}}^{2},
\]
and the corresponding dissipation rate functional \( \mathcal{D}_{N,\ell}(t) \)
\[
\mathcal{D}_{N,\ell}(t) = \sum_{|\alpha| \leq N-1} \| \partial^{\alpha} \nabla (a_{\pm}, b, c)(t) \|^{2}
+ \sum_{|\alpha| + |\beta| \leq N-1} \| u_{\ell-|\alpha|-|\beta|} \partial_{\beta}^{3} (I - P) f(t) \|^{2}
+ \| (a_{+} - a_{-})(t) \|^{2} + \| E(t) \|_{H^{N-1}}^{2} + \| \nabla B(t) \|_{H^{N-2}}^{2}.
\]
Let us take \( N \geq 9 \) and \( \ell \geq \max \left\{ N - 1 + \frac{1+1}{3}, N_{\ast} - 1 + \frac{2N+3}{4(1-\theta)} \right\} \) with \( 0 < \theta < 1 \), \( l_{1} > N_{\ast} + \frac{3}{2} \) and \( N_{\ast} = \left[ \frac{N-3}{3} \right] \), the exact choice of \( N, \ell, l_{1}, \) and \( \theta \) can be seen in the later proof. Here, we use \( \left[ \frac{N-3}{3} \right] \) to stand for the largest integer which is not larger than \( \frac{N-3}{3} \). In terms of those given constants, the temporal energy norm \( X(t) \) is defined by
\[
X(t) = \sup_{0 \leq \tau \leq t} \left( \mathcal{E}_{N,\ell}(\tau) + \sum_{|\alpha| \leq N-1} (1 + \tau)^{|\alpha|} \| \partial^{\alpha} (f, E, B)(\tau) \|^{2} \right)
+ \sup_{0 \leq \tau \leq t} \sum_{|\alpha| \leq N-1} (1 + \tau)^{|\alpha|+\theta} \left( \| \partial^{\alpha} (I - P) f(\tau) \|^{2} + \| \partial^{\alpha} E(\tau) \|^{2} \right),
\]
where the explicit form of the time decay rates \( r_{|\alpha|}(1 \leq |\alpha| \leq N-1) \) are defined as
\[
r_{|\alpha|} = \begin{cases} |\alpha| + \frac{3}{2}, & |\alpha| \leq N_{\ast}, \\ 1 + \frac{N_{\ast}-1}{N_{\ast} + \frac{3}{2}}, & N_{\ast} + 1 \leq |\alpha| \leq N-1. \end{cases}
\]
The main result of the paper is stated as follows.

**Theorem 1.1.** Assume \( \sigma(g, \theta) \) satisfies (1.8). Take \( N \geq 9 \), \( N_{\ast} = \left[ \frac{N-3}{3} \right] \) and \( \ell \geq \max \{ N - 1 + \frac{l_{1}+1}{2}, N_{\ast} - 1 + \frac{2N+3}{4(1-\theta)} \} \) with \( l_{1} > N_{\ast} + \frac{3}{2} \) and \( 0 < \theta < 1 \). Let \( f_{0} = [f_{0,+}, f_{0,-}] \) satisfy \( F_{\pm}(0, x, p) = J(p) + J_{\pm}^{2}(p) f_{0,\pm}(x, p) \geq 0 \). If
\[
Y_{0} = \sum_{|\alpha| + |\beta| \leq N-1} \| u_{\ell-|\alpha|-|\beta|} \partial_{\beta}^{3} f_{0} \| + \sum_{|\alpha| + |\beta| = N} \| \partial_{\beta}^{3} f_{0} \|
+ \| (E_{0}, B_{0}) \|_{H^{N} \cap L^{1}} + \| w_{l_{1}/2} f_{0} \|_{Z_{l}}
\]
is sufficiently small, then there exists a properly defined energy functional \( \mathcal{E}_{N,\ell}(t) \) in the definition \( (1.19) \) of \( X(t) \)-norm such that the Cauchy problem \( (1.16), (1.17) \) and \( (1.18) \) admits a unique global solution \( (f(t, x, p), E(t, x), B(t, x)) \) satisfying \( F_{\pm}(t, x, p) = J(p) + J_{\pm}^{2}(p) f_{\pm}(t, x, p) \geq 0 \) and
\[
X(t) \lesssim Y_{0}^{2}
\]
for all time $t \geq 0$.

The relativistic Vlasov-Maxwell-Boltzmann system is one of the central equations in relativistic collision kinetic theory. Standard references which discuss relativistic kinetic theory include [1, 2, 9, 12, 14, 15, 16, 17, 18, 22, 23, 24, 26, 35, 37, 32, 38, 39, 45]. We mention some works related to the topic in the paper. When the relativistic effects are not considered, Guo [20] and Strain [30] obtained the global classical solutions of the Vlasov-Maxwell-Boltzmann system with angular cutoff on torus and in the whole space, respectively. Then, the rate of convergence to Maxwellians with any polynomial speed in large time was shown by Guo-Strain [33]. For the Cauchy problem in the whole space, the large-time behavior of classical solutions in the situation of both cutoff and non-cutoff potentials were studied by Duan-Strain [10] and Duan-Liu-Yang-Zhao [7], respectively. And recently, Duan-Lei-Yang-Zhao [5] investigated the the time decay rates of the cutoff Vlasov-Maxwell-Boltzmann system with very soft potential.

However, for the relativistic Vlasov-Maxwell-Boltzmann system, to our best, much less is known. Recently, Guo-Strain [21] studied the global classical existence of the Vlasov-Maxwell-Boltzmann system with “hard ball” condition. Compared with the non-relativistic Vlasov-Maxwell-Boltzmann system, the main difficulty is created by the collision “cross section” and the post collision momentum. Due to the complexity and singularity of both the “cross section” $\sigma(g, \theta)$ and the post collision momentum, it is very hard to study the global classical existence of the system with general “cross section”. More precisely, unlike the non-relativistic Vlasov-Maxwell-Boltzmann system for the non-hard potential [7, 5], one can not directly take momentum derivatives to the collision operators, which must be considered in the investigation of the classical solutions. It should be pointed out that Guo and Strain [21] used two alternative forms of the collision operator to eliminate the singularities caused by the momentum derivatives. The main purpose of the paper is to extend the case of “hard ball” model considered by Guo-Strain [21] to the short range interactions and to obtain the time decay rates of the solutions.

The proof of Theorem 1.1 is based on an interplay of the velocity weighted energy method developed in [19, 34, 4, 6, 7, 8], and also some techniques for obtaining the optimal temporal decay rates used in [41, 42, 43, 44, 40]. Unlike the “hard ball” model studied by Guo-Strain [21], the dissipation of the linearized relativistic Boltzmann collision operator for the short range interaction is weaker in the sense that it is degenerate in the large-velocity domain, which is much similar to the linearized non-relativistic Boltzmann operator with non-hard potentials [34]. To overcome this problem, as it is shown in [19, 34, 4, 6, 7, 8], one solvable way is to introduce the velocity-weight when performing the energy estimates, so as to balance the possible velocity-growth coming from the transport term of the original Boltzmann equation. Nevertheless, compared to the non-relativistic Vlasov-Maxwell-Boltzmann system with non-hard potentials [7, 5], one of the main contribution of the paper is that we do not impose any weight to the highest order derivatives of the solution, since the bad term

$$\sum_{|\beta|=N} \left| \left( \frac{\zeta}{2} E \cdot \tilde{p} \partial_{\beta} f, \partial_{\beta} f \right) \right|$$

can be controlled by $C\|E\|_{L^\infty} \sum_{|\beta|=N} \|\partial_{\beta} f\|^2$ and $\|E\|_{L^\infty}$ enjoys the time decay rates $(1 + t)^{-3/2}$. This phenomenon can be regarded as a kind of the relativistic effects
and do not take place in the case of the non-relativistic Vlasov-Maxwell-Boltzmann system with non-hard potentials where there is always a growth in velocity. Thanks to this observation, we can avoid the trouble coming from the regularity-loss of the solution, cf. [7, 5, 25]. Another important idea of the paper is that we design a new time weighted energy norm $X(t)$ and apply some crucial Sobolev inequalities (see (4.4)) to deduce the time decay rates of the every order derivative of the solution.

The rest of the paper is arranged as follows. In Section 2, we carry out the weighted estimates on $\Gamma$ and $L$. In Section 3, we obtain the time-decay property of the linearized homogeneous system. In Section 4, we shall prove series of lemmas to obtain the closed estimate on $X(t)$-norm so as to conclude the proof of Theorem 1.1 basing on the standard continuity argument.

**Notations.** Throughout this paper, $C$ denotes some generic positive (generally large) constant and $\lambda$ denote some generic positive (generally small) constant, where $C$ and $\lambda$ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $A \gtrsim B$. We use $L^2$ to denote the usual Hilbert spaces $L^2 = L_{x,p}^2$ or $L_{x}^2$ with the norm $\| \cdot \|$, and use $\langle \cdot, \cdot \rangle$ and $\langle \cdot \rangle$ to denote the inner product over $L_{x,p}^2$ and $L_x^2$, respectively. The mixed velocity-space Lebesgue space $Z = L_p^2(L_\lambda^1) = L^2(R^3, L^1(R^3))$ is used. $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial^\beta_\xi = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3} \partial_{p_1}^{\beta_4} \partial_{p_2}^{\beta_5} \partial_{p_3}^{\beta_6}$. The length of $\alpha$ is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and similar for $|\beta|$.  

2. **Weighted estimates on $\Gamma$ and $L$.** This section is concerned with the weighted estimates on $\Gamma$ and $L$ with respect to the weight $w_\epsilon(p)$. Many of these estimates are similar to those in [21].

In light of (1.8), for scalar functions $g_1, g_2$ and $h$, we use the following notations

\[ \mathcal{F}(g_1, g_2) = J^{-\frac{1}{2}} Q(J^{\frac{1}{2}} g_1, J^{\frac{1}{2}} g_2), \]

\[ \mathcal{F}_1(g_1, g_2) = \int_{\mathbb{R}^3} \int_{S^2} v(p, q) \sigma(g, \theta) |g_1(p') g_2(q') - g_1(p) g_2(q)| dq d\omega \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} \frac{Ce_q}{p_0 \sqrt{s}} J^{\frac{1}{2}}(q) g_1(p') g_2(q') dq d\omega \]
\[ - \int_{\mathbb{R}^3} \int_{S^2} \frac{Ce_q}{p_0 \sqrt{s}} J^{\frac{1}{2}}(q) g_1(p) g_2(q) dq d\omega \]
\[ = \mathcal{F}_{1}^{\text{gain}} - \mathcal{F}_{1}^{\text{loss}}, \]

\[ \mathcal{F}_2(g_1, g_2) = \int_{\mathbb{R}^3} \int_{S^2} \frac{s \sigma(g, \theta)}{p_0 \sqrt{s}} B(p, q, w) J^{\frac{1}{2}}(q) |g_1(p'') g_2(q'') - g_1(p) g_2(q)| dq d\omega \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} \frac{C\theta}{p_0 \sqrt{s}} B(p, q, w) J^{\frac{1}{2}}(q) |g_1(p'') g_2(q'') - g_1(p) g_2(q)| dq d\omega \]
\[ = \mathcal{F}_{2}^{\text{gain}} - \mathcal{F}_{2}^{\text{loss}}, \]

and

\[ \mathcal{L} h = - \left\{ \mathcal{F}(h, J^{\frac{1}{2}}) + \mathcal{F}_1(J^{\frac{1}{2}}, h) \right\} \]
\[ = - \int_{\mathbb{R}^3} \int_{S^2} \frac{Ce_q}{p_0 \sqrt{s}} J^{\frac{1}{2}}(q) |h(p') J^{\frac{1}{2}}(q') - h(p) J^{\frac{1}{2}}(q)| dq d\omega \]
we split

From (1.14), one can see that the above operators $\mathcal{T}, \mathcal{L}, \nu,$ and $K$ have the following expressions:

\[
\begin{align*}
\mathcal{T} (g_1, g_2) &= \mathcal{T}_1(g_1, g_2) = \mathcal{T}_2(g_1, g_2), \\
\mathcal{L} h &= - \left\{ \mathcal{T}_2(h, J^\frac{1}{2}) + \mathcal{T}_2(J^\frac{1}{2}, h) \right\}, \\
\nu(p) &= \mathcal{T}_1^{\text{loss}}(1, J^\frac{1}{2}) = \mathcal{T}_2^{\text{loss}}(1, J^\frac{1}{2}), \\
K_2 h &= \left\{ \mathcal{T}_2^{\text{gain}}(h, J^\frac{1}{2}) + \mathcal{T}_2^{\text{gain}}(J^\frac{1}{2}, h) \right\}, \\
K_1 h &= \mathcal{T}_2^{\text{loss}}(J^\frac{1}{2}, h).
\end{align*}
\]

With the above notations, recalling (1.14) and (1.15), it is straightforward to see

\[
\begin{align*}
\Gamma_{\pm}(f, g) &= \Gamma_i(f_{\pm}, g_{\pm}) + \Gamma_1(f, g_{\mp}), \quad i = 1, 2, \quad \text{and} \\
L_{\pm} f &= 2\nu(p) f_{\pm} - 2 \mathcal{T}_1^{\text{gain}}(f_{\pm}, J^\frac{1}{2}) \\
&\quad + \mathcal{T}_1^{\text{loss}}(J^\frac{1}{2}, f_{+} + f_{-}) - \mathcal{T}_1^{\text{gain}}(J^\frac{1}{2}, f_{+} + f_{-}), \quad i = 1, 2.
\end{align*}
\]

2.1. **Weighted estimates on $\Gamma.$** In order to make the weighted estimates on (1.16), particularly on $L$ and $\Gamma,$ we will split the desired estimate into two different cases. Those cases correspond to the following two different integration regions

\[
A = \{|p| \leq 1\} \cup \{|p| \geq 1, |p| \leq 2q_0\}, \quad \overline{A} = \{|p| \geq 1, |p| > 2q_0\}.
\]

To make our presentation more clear, we also introduce the smooth test function $\chi(x) \in C_0^\infty([0, \infty))$ satisfying

\[
\chi(x) = \begin{cases} 
1, & \text{if } x \in [0, 1], \\
0, & \text{if } x \geq 2.
\end{cases}
\]

By the splitting $1 = \chi_A(p, q) + \chi_{\overline{A}}(p, q)$ with

\[
\chi_A(p, q) = \chi(p_0) + (1 - \chi(p_0)) \left( \frac{|p|}{q_0} \right), \quad \chi_{\overline{A}}(p, q) = (1 - \chi(p_0)) \left( 1 - \chi \left( \frac{|p|}{q_0} \right) \right),
\]

we split $\mathcal{T}(g_1, g_2) = \mathcal{T}_{2, A} + \mathcal{T}_{1, \overline{A}}$ using the definitions of $\mathcal{T}_1$ and $\mathcal{T}_2$ as

\[
\begin{align*}
\mathcal{T}_{2, A} &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{C_0}{2p_0q_0 \sqrt{s}} B(q, w) J^\frac{1}{2}(q) [g_1(p'') g_2(q'') - g_1(p) g_2(q)] \chi_A(p, q) dq d\omega, \\
\mathcal{T}_{1, \overline{A}} &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{C_0}{p_0q_0 \sqrt{s}} J^\frac{1}{2}(q) [g_1(p') g_2(q') - g_1(p) g_2(q)] \chi_{\overline{A}}(p, q) dq d\omega.
\end{align*}
\]
By these important decomposition, we will deduce the main estimate:

**Lemma 2.1.** For any $|\beta| \geq 0$, it holds that
\[
\left| w^{|-\beta|} \partial_\beta \Gamma(f_1, f_2, f_3) \right| \\
\lesssim \sum_{\beta_1 + \beta_2 \leq \beta} \left( |w^{1-\beta_1}| \partial_{\beta_1} f_1 \nu |\partial_{\beta_2} f_2 \nu + |\partial_{\beta_1} f_1 \nu |w^{1-\beta_2}| \partial_{\beta_2} f_2 \nu \right) |w^{1-\beta_1}| \partial_{\beta_3} f_3 \nu.
\]

This lemma will follow directly from the later lemmas below.

2.1.1. Estimates in the Glassey-Strauss frame $\mathcal{F}_{2,A}$. To avoid taking derivatives for the singular factor of $|\omega \cdot (\tilde{p} - \tilde{q})|$ inside $\mathbb{B}(p, q, \omega)$ for $\mathcal{F}_{2,A}$, we introduce the following change of variables $q \to u$ (for fixed $p$) as in [21]:
\[
u = p_0 q - pq_0,
\]
from which we have
\[
u = p_0 q + \frac{u}{p_0} = \tilde{p} \left( p \cdot u + \sqrt{(p \cdot u)^2 + p_0^2 + |u|^2} \right) + \frac{u}{p_0},
\]
and
\[q_0 = p \cdot u + \sqrt{(p \cdot u)^2 + p_0^2 + |u|^2}.
\]
According to [21], one also have
\[
\frac{\partial u_i}{\partial q_j} = p_0 \delta_{ij} - \frac{q_i q_j}{q_0}, \quad \left| \frac{\partial u}{\partial q} \right| = \frac{p_0^2}{q_0} (p_0 q_0 - p \cdot q) \geq \frac{p_0^2}{q_0}.
\]
We next express $\mathcal{F}_{2,A}$ as
\[
\mathcal{F}_{2,A} = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{C_0}{p_0 q_0} \left| \omega \cdot u \right| \mathbb{B}(p, q, u) \\
\times J^{\tilde{q}}(q) [g_1(p'' g_2(q'') - g_1(p) g_2(q))] \chi_A(p, q) du d\omega,
\]
where
\[
\mathbb{B}(p, q, u) = \frac{(p_0 + q_0)^2}{[(p_0 + q_0)^2 - (\omega \cdot (p + q))^2]^2}.
\]
Then we have
\[
|\partial_\beta \mathcal{F}_{2,A}| \leq \sum \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{D} u \omega_1 |p| \leq 2q_0 K_{\beta_0}^{A,} \mathbb{B}(p, q, u) \left| \partial_{\beta_1} g_1(p'') \partial_{\beta_2} g_2(q') \mu_{\beta_2} \beta_2 \right| \\
+ \sum \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{D} u \omega_1 |p| \leq 2q_0 K_{\beta_0}^{A,} \mathbb{B}(p, q, u) \left| \partial_{\beta_1} g_1(p) \partial_{\beta_2} g_2(q) \mu_{\beta_2} \right|.
\]
where the sum is over $\beta_0 + \beta_1 + \beta_2 = \beta$, and
\[
K_{\beta_0}^{A} = K_{\beta_0}^{A}(u, p, \omega) = \left| \omega \cdot u \right| \left| \frac{\partial^{p,q}_{\beta_0}}{p_0 q_0} \left( \frac{1}{p_0 q_0} \left| \frac{\partial q}{\partial u} \right| \mathbb{B}(p, q, u) J^{\tilde{q}}(q) \chi_A(p, q) \right) \mu_{\beta_0} \right|.
\]
Also $\partial^{p,q}_{\beta_0}$ denotes the mixed partial derivatives with respect to variables $p$ and $q$, $\mu_{\beta_2}$ and $\mu_{\beta_1}$ are the terms which result from applying the chain ruler to the post-collisional momentum $(p'', q'')$ and momentum $q$ respectively. Here $\mu_{\beta_2}$ and $\mu_{\beta_1}$ contain the sum of products of high order $p$-momentum derivatives of smooth functions $(p'', q'')$ and $q$. The next step is to reverse the change of variables $q \to u$, after the change of variables $u \to q$:
Lemma 2.2. On the set $A$, it holds that
\[
|w^2_{t-|\beta|}K_{\beta_0}^A(p_0q-pq_0,p,\omega) - p_0^{-1}J^\frac{1}{2}(q)| \lesssim q_0^n,
\]
where $n \geq 1$ is a fixed large integer which depends upon $\ell - |\beta| \geq 0$, $\beta, \beta_0, \beta_1$ and $\beta_2$.

Proof. The proof of this lemma can be done in the same way as Lemma 1 in [21]. We omit the details for brevity. □

Lemma 2.3. On the set $A$, it holds that
\[
|w^2_{t-|\beta|}\partial_\beta \mathcal{T}_2,A| \lesssim e^{-\frac{1}{2}p_0} \int_{\mathbb{R}^3} dqd\omega \int_{|p| \leq 2q_0} J^\frac{1}{2}(q) \times \{ |\partial_{\beta_1}g_1(p')| + |\partial_{\beta_1}f_1(p')\partial_{\beta_2}g_2(q')| \}.
\]

Moreover, from the above estimate one can deduce the following uniform upper bound
\[
\langle w^2_{t-|\beta|}\partial_\beta \mathcal{T}_2,A(f_1,f_2,f_3) \rangle \lesssim \sum_{\beta_1+\beta_2 \leq \beta} |\partial_{\beta_1}f_1| |\partial_{\beta_1}f_2| |\partial_{\beta_2}f_3|.
\]

2.1.2. Center of momentum frame. In this subsection, we turn to prove the estimates for $\mathcal{T}_{1,\Pi}$. By a change of variable $p - q \to u$ and using the product rule as well as a reverse change of variables, we have
\[
|\partial_{\beta} \mathcal{T}_{1,\Pi}| \leq \sum_{\beta_1+\beta_2 \leq \beta} \int_{\mathbb{R}^3} dqd\omega \int_{|p| \leq 2q_0} |\partial_{\beta_1}g_1(p')| \partial_{\beta_1}g_2(q' \bar{\beta}_2) | J^\frac{1}{2}(q) |\partial_{\beta_1}g_1(p') \partial_{\beta_2}g_2(q') |.
\]

Here $\bar{\beta}_2$ is the collection of sums of product of momentum derivatives of $p'$ and $q'$, from (1.10), which result from the chain rule of differentiation. Again the sum is over $\beta_0 + \beta_1 + \beta_2 = \beta$. As in [21], one has:

Lemma 2.4. Let $(p,q) \in \mathcal{A}$. For any $|\beta| \geq 0$, it holds that
\[
|\partial_{\beta}g_1| \lesssim \frac{g}{p_0q_0 \sqrt{s}} J^\frac{1}{2}(q),
\]

With the estimates above, we show that

Lemma 2.5. For any $|\beta| \geq 0$, we have the uniform estimate:
\[
|\partial_{\beta} \mathcal{T}_{1,\Pi}| \lesssim \sum_{\beta_1+\beta_2 \leq \beta} \int_{|p| \leq 2q_0} \frac{g}{p_0q_0 \sqrt{s}} J^\frac{1}{2}(q) \times \{ |\partial_{\beta_1}f_1(p') \partial_{\beta_2}f_2(q')| \}.
\]

Moreover, for $\ell - |\beta| \geq 0$, one has
\[
\langle w^2_{t-|\beta|}\partial_\beta \mathcal{T}_{1,\Pi}(f_1,f_2,f_3) \rangle \lesssim \sum_{\beta_1+\beta_2 \leq \beta} \langle w^2_{t-|\beta_1|}\partial_{\beta_1}f_1| \partial_{\beta_2}f_2| + |\partial_{\beta_1}f_1| \partial_{\beta_2}f_2 \rangle |w^2_{t-|\beta_1|}\partial_{\beta_2}f_3|\rangle.
Proof. The first estimate (2.2) follows immediately from Lemma 2.4. Now we show that the second estimate holds. From (2.2), one has the upper bound of

\[ |w_{\ell-|\beta|}^2 \partial_{\beta} \mathcal{F}_{1,\beta}(f_1, f_2), \partial_{\beta} f_3| \]

\[ \lesssim \sum_{\beta_1 + \beta_2 \leq \beta} \int_{\mathbb{R}^6} \int_{S^2} dpdqdw \frac{1}{p_0q_0} J^{\frac{1}{2}}(q) w_{\ell-|\beta|}^2(p) |\partial_{\beta_1} f_1(p') \partial_{\beta_2} f_2(q') \partial_{\beta} f_3(p)| + \sum_{\beta_1 + \beta_2 \leq \beta} \int_{\mathbb{R}^6} \int_{S^2} dpdqdw \frac{1}{p_0q_0} J^{\frac{1}{2}}(q) w_{\ell-|\beta|}^2(p) |\partial_{\beta_1} f_1(p) \partial_{\beta_2} f_2(q) \partial_{\beta} f_3(p)|. \]

The second term clearly has the desired upper bound by using the Cauchy-Schwartz inequality. For the first term, notice that the expressions of $p_0', q_0'$ in (1.11) imply $w_{\ell-|\beta|}^2(p') \lesssim w_{\ell-|\beta|}^2(p') + w_{\ell-|\beta|}^2(q')$. We now use the Cauchy-Schwartz inequality to obtain the upper bound

\[ |w_{\ell-|\beta|}^2 |\partial_{\beta} f_3| \]

\[ \lesssim \sum_{\beta_1 + \beta_2 \leq \beta} \left\{ \left( \int_{\mathbb{R}^6} \int_{S^2} dpdqdw \frac{1}{p_0q_0} w_{\ell-|\beta|}^2(p') |\partial_{\beta_1} f_1(p') \partial_{\beta_2} f_2(q')|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^6} \int_{S^2} dpdqdw \frac{1}{p_0q_0} w_{\ell-|\beta|}^2(q') |\partial_{\beta_1} f_1(p') \partial_{\beta_2} f_2(q')|^2 \right)^{\frac{1}{2}} \right\}. \]

Once again the pre-post change of variables from (1.11) yields the desired estimate. This completes the proof of Lemma 2.5.

With Lemma 2.1 in hand, it is quite easy to show that

**Lemma 2.6.** Let $N \geq 9$, $|\alpha| + |\beta| \leq N$. Then, it holds that

\[ ||(\Gamma(f_1, f_2), \zeta(p))| \lesssim ||f_1||_v ||f_2||_v, \]

for any smooth function $\zeta(p)$ exponentially decay in $p$,

\[ ||\Gamma(f_1, f_2)||_{L_1} \lesssim ||f_1||_v ||f_2||_v, \quad \text{and} \]

\[ \left| (\partial_{\alpha}^{\beta} \Gamma(f, f), w_{\ell-|\alpha|-|\beta|}^{\beta} (I - P) f) \right| \lesssim \mathcal{E}_{N,\ell}^{\frac{1}{2}}(t) D_{N,\ell}(t), \quad |\alpha| + |\beta| \leq N - 1, \]

\[ ||(\partial^{\alpha} \Gamma(f, f), \partial^{\alpha} f)| + ||(\partial_{\beta}^{\alpha} \Gamma(f, f), \partial_{\beta}^{\alpha} (I - P) f) || \]

\[ \lesssim \mathcal{E}_{N,\ell}^{\frac{1}{2}}(t) D_{N,\ell}(t), \quad |\alpha| + |\beta| = N, \quad |\beta| \geq 1. \]

**Proof.** The first estimate follows directly form the definition of $\Gamma$, and Lemmas 2.3, 2.4. And other estimates follow from Lemma 2.6, Sobolev inequalities and definitions of $\mathcal{E}_{N,\ell}$ and $D_{N,\ell}$. \( \square \)

### 2.2. Weighted estimates on $L$

In this subsection, we deduce some weighted estimates on the linearized collision operator $L$ with respect to the weight $w_{\ell}(p)$. we now give the basic estimates for $\nu$ and $K$ whose exact expressions are given at the begging of this section.

**Lemma 2.7.** Under the condition of (1.8), it holds that

\[ \nu(p) \sim p_0^{-1}. \]

Moreover, for $|\beta| \geq 1$, we have

\[ |\partial_{\beta} \nu(p)| \lesssim p_0^{-2}. \]
Lemma 2.3, one has
\[ \nu(p) = \mathcal{F}^{\text{loss}}_1(1, J^\frac{1}{2}) = \mathcal{F}^{\text{loss}}_{1, A}(1, J^\frac{1}{2}) + \mathcal{F}^{\text{loss}}_{2, A}(1, J^\frac{1}{2}). \]

To prove (2.4), it suffices to estimate \( \partial_\beta \mathcal{F}^{\text{loss}}_{2, A}(1, J^\frac{1}{2}) \) and \( \partial_\beta \mathcal{F}^{\text{loss}}_{1, A}(1, J^\frac{1}{2}) \). From Lemma 2.3, one has
\[
\left| \partial_\beta \mathcal{F}^{\text{loss}}_{2, A}(1, J^\frac{1}{2}) \right| \lesssim e^{-\frac{1}{2}p_0} \sum_{\beta_2 \leq \beta} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} dq dw \chi_{|\beta| \leq 2q} J^\frac{1}{2}(q)
\times \left\{ \left| \partial_{\beta_2} J^\frac{1}{2}(q^\prime) \right| + \left| \partial_{\beta_2} J^\frac{1}{2}(q) \right| \right\} \lesssim p_0^{-2}.
\]

On the other hand, we have from (2.1) that
\[
\partial \mathcal{L} \leq \frac{C_\theta}{p_0 q_0 \sqrt{q}} J^\frac{1}{2}(q).
\]

This completes the proof of Lemma 2.7.

With Lemma 2.7 in hand, we can obtain

**Lemma 2.8.** Let \( |\beta| \geq 1 \). For any small \( \eta > 0 \), it holds that
\[
\left< \partial_\beta \{ \nu(p) h \}, w_{l-|\beta|} \partial_\beta h \right> \gtrsim |w_{l-|\beta|} \partial_\beta h|_v^2 - \eta \sum_{|\beta'| \leq |\beta|} |w_{l-|\beta'|} \partial_\beta h|_v^2 - C_\eta |h|_v^2.
\]

By the same arguments used in Proposition 8 of [21], we obtain

**Lemma 2.9.** Let \( |\beta| \geq 0 \). For any \( \eta > 0 \), there exists \( C_\eta > 0 \) such that
\[
\left| \left< w_{l-|\beta|} \partial_\beta K(h_1), \partial_\beta h_2 \right> \right| \lesssim \eta \sum_{|\beta_1| \leq |\beta|} \left| w_{l-|\beta_1|} \partial_\beta h_1 \right|_v + C_\eta \left| h_1 \right|_v \left| w_{l-|\beta|} \partial_\beta h_2 \right|_v.
\]

Basing on Lemmas 2.8 and 2.9, one has

**Lemma 2.10.** For any \( \eta > 0 \), there exists \( C_\eta > 0 \) such that
\[
\left< w_{l-|\beta|} \mathcal{L} h, \partial_\beta h \right> \gtrsim |w_{l-|\beta|} \partial_\beta h|_v^2 - \eta \sum_{|\beta_1| \leq |\beta|} |w_{l-|\beta_1|} \partial_\beta h|_v^2 - C_\eta |h|_v^2.
\]

where \( |\beta| \geq 1 \) and \( l - |\beta| \geq 0 \). For \( |\beta| = 0 \), one also has
\[
\left| \left< w_l^2 \mathcal{L}(h), h \right> \right| \gtrsim |w_l h|_v^2 - C|h|_v^2.
\]

Therefore, from the definitions of the operators \( \mathcal{L} \) and \( L \), we can obtain
\[
\left< w_{l-|\beta|} \partial_\beta L h, \partial_\beta h \right> \gtrsim |w_{l-|\beta|} \partial_\beta h|_v^2 - \eta \sum_{|\beta_1| \leq |\beta|} |w_{l-|\beta_1|} \partial_\beta h|_v^2 - C_\eta |h|_v^2
\]
for \( |\beta| \geq 1 \), and
\[
\left< w_l^2 L h, h \right> \gtrsim |w_l h|_v^2 - C|h|_v^2,
\]
\[
\left< L h, h \right> \gtrsim \{(1 - P) h\}_v^2
\]
for \( |\beta| = 0 \).
3. **Linearized analysis.** Consider the Cauchy problem on the linearized relativistic Vlasov-Maxwell-Boltzmann system with a source:

\[
\begin{aligned}
& \frac{\partial f}{\partial t} + \tilde{p} \cdot \nabla_x f - E \cdot \tilde{p} J_1^2 + L f = S, \\
& \frac{\partial E}{\partial t} - \nabla_x \times B = - \int_{\mathbb{R}^3} \tilde{p} (f_+ - f_-) dp, \\
& \frac{\partial B}{\partial t} + \nabla_x \times E = 0, \\
& \nabla_x \cdot E = \langle J_1^2, f_+ - f_- \rangle, \\
& \nabla_x \cdot B = 0,
\end{aligned}
\]

where \( S = S(t, x, p) = [S_+(t, x, p), S_-(t, x, p)] \), initial data \([f_0, E_0, B_0] \) satisfy the compatibility condition

\[
\nabla_x \cdot E_0 = \langle J_1^2, f_{0,+} - f_{0,-} \rangle, \\
\nabla_x \cdot B_0 = 0.
\]

and the source term \( S \) is assumed to satisfy

\[
\int_{\mathbb{R}^3} (S_+ - S_-) dp = 0.
\]

To consider the solution to the Cauchy problem (3.1), for simplicity, we denote

\[
U = [f, E, B], \quad U_0 = [f_0, E_0, B_0].
\]

Then the solution to the Cauchy problem (3.1) can be formally denoted by

\[
U(t) = U_1(t) + U_2(t),
\]

\[
U_1(t) = [f_1, E_1, B_1](t) = A(t) U_0.
\]

\[
U_2(t) = [f_2, E_2, B_2](t) = \int_0^t A(t - \tau) [S(\tau), 0, 0] d\tau.
\]

where \( A(t) \) is the linear solution operator for the Cauchy problem on the linearized homogeneous system corresponding to (3.1) in the case \( S = 0 \).

3.1. **Macro structure.** Before continuing our investigation of the system (3.1), we introduce the notation for some integrals as follows

\[
C_0 = \langle J, p_0 \rangle, \quad C_{00} = \langle J, p_0^2 \rangle, \quad C_{01} = \langle J, p_1^2 \rangle, \quad C_{11} = \langle J, \frac{p_1^2}{p_0} \rangle, \\
C_{21} = \langle J, \frac{p_1^2}{p_0} \rangle, \quad C_{211} = \langle J, \frac{p_1^4}{p_0^2} \rangle, \quad C_{212} = \langle J, \frac{p_1^4}{p_0^2} \rangle.
\]

Recalling the definition of \( Pf \), we see that

\[
\begin{aligned}
o_\pm &= \langle J_1^2, f_\pm \rangle = \langle J_1^2, P_\pm f \rangle - C_0 c, \\
b_i &= \frac{1}{2} (p_i J_1^2, f_+ + f_-) = (p_i J_1^2, P_\pm f), \\
c &= \frac{p_0 \langle J_1^2, f_\pm \rangle - C_0 \langle J_1^2, f_\pm \rangle}{C_{00} - C_0^2}.
\end{aligned}
\]

Taking velocity integrations of (3.1) with respect to the velocity moments

\[
J_1^2, \quad p_i J_1^2 (1 \leq i \leq 3), \quad p_0 J_1^2,
\]
we obtain
\[ \partial_t (a_\pm + C_0 c) + C_{01} \nabla_x \cdot b + \nabla_x \cdot (\hat{p} J^\pm \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) = (J^\pm, S_{\pm}), \]
\[ \partial_t \left[ C_{01} b_i + \langle p_i J^\pm \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f \rangle + \partial_i (C_{11} a_\pm + C_{01} c) + \mathcal{E}_i \right] 
+ \nabla_x \cdot (\hat{p} p_i J^\pm \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) = \langle p_i J^\pm, S_{\pm} - L_{\pm} \rangle f, \]
\[ \partial_t \left[ C_{00} c + C_{01} a_\pm + \langle p_0 J^\pm \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f \rangle + C_{01} \nabla_x \cdot b 
+ \nabla_x \cdot (p J^\pm, \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) \right] 
= \langle p_0 J^\pm, S_{\pm} - L_{\pm} \rangle f. \]

Define the high-order moment functions \( \Theta(f_{\pm}) = (\Theta_{ij}(f_{\pm}))_{3 \times 3} \) and
\[ \Lambda(f_{\pm}) = (\Lambda_1(f_{\pm}), \Lambda_2(f_{\pm}), \Lambda_3(f_{\pm})) \]
by
\[ \Theta_{ij}(f_{\pm}) = \left\langle \left( \frac{p_i p_j}{p_0} - A_1 \right) J^\pm, f_{\pm} \right\rangle, \quad \Lambda_i(f_{\pm}) = \left\langle \left( \frac{1}{p_0} - A_2 \right) J^\pm, f_{\pm} \right\rangle. \]
where \( A_1 \) and \( A_2 \) satisfy
\[ A_1 C_{01} + \frac{C_{01} (C_{11} - A_1)}{C_0} = C_{212}, \quad \text{and} \quad A_2 = \frac{C_{11}}{C_{01}}, \]
respectively.

Further taking velocity integrations of the first equation of (3.1) with respect to the above high-order moments one has
\[ \partial_t \left[ \Theta_{ii}(\{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) - \langle p_0 J^\pm \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f \rangle \right] 
+ \left[ C_{01} - C_{00} A_1 - \frac{C_{00} (C_{11} - A_1)}{C_0} \right] c 
+ (C_{211} - C_{212}) \partial_i b_i 
= \Theta_{ii}(r_{\pm} + S_{\pm}) - \langle p_0 J^\pm, r_{\pm} + S_{\pm} \rangle, \]
\[ \partial_i \Theta_{ij}(\{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) + C_{212} (\partial_i b_j + \partial_j b_i) 
= \Theta_{ij}(r_{\pm} + S_{\pm}) + A_1 \nabla_x \cdot (\hat{p} J^\pm, \{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f), \quad i \neq j, \]
and
\[ \partial_i \Lambda_i(\{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) + \left( C_{21} - \frac{C_{21}^2}{C_{01}} \right) \partial_i a_\pm + \left( C_{21} - \frac{C_{21}^2}{C_{01}} \right) E_i 
= \Lambda_i(r_{\pm} + S_{\pm}), \quad \text{(3.3)} \]
where
\[ r = [r_+, r_-], \quad r_{\pm} = -\hat{p} \cdot \nabla_x (\{ \mathbf{I}_\pm - \mathbf{P}_\pm \} f) - L_{\pm} f. \]
Here we have used (3.3).

In particular, for the nonlinear system (3.1), the non-homogeneous source \( S = [S_+(t, x, p), S_-(t, x, p)] \) takes the form of
\[ S = \frac{\zeta}{2} E \cdot \hat{p} f + \zeta \left( E + \hat{p} \times B \right) \cdot \nabla_p f + \Gamma(f, f). \]
Then, it is straightforward to compute from integration by parts that
\[ \langle J^\pm, S_{\pm} \rangle = 0. \]
3.2. Time decay for the linearized system. For the linearized homogeneous system, one can prove

**Lemma 3.1.** Let \( U = U_1 + U_2 \) be the solution to the Cauchy problem (3.1). Define the velocity weight function \( \mu(p) = \frac{1}{p^2} \). Then, for any multiindex \( \alpha \) with \( |\alpha| = m \) and \( l_1 > m + \frac{3}{2}, l_1 \geq j \), the first part \( U_1 \) of the solution to the linearized homogeneous system satisfies

\[
\| \nabla_x^m U_1(t) \| \lesssim (1 + t)^{-\frac{3}{2} - \frac{j}{2}} \left( \| \mu^1 f_0 \|_{Z_1} + \| (E_0, B_0) \|_{L_x^1} \right) + (1 + t)^{-\frac{3}{2}} \left( \| \mu^j \nabla_x^{m+j+1} f_0 \| + \| \nabla_x^{m+j+1} (E_0, B_0) \| \right). \]

(3.4)

for any \( t \geq 0 \), and the second part of the solution \( U_2 \) corresponding to the solution of the linearized homogeneous system with vanishing initial data satisfies

\[
\| \nabla_x^m U_2(t) \| \lesssim (1 + t)^{-\frac{3}{2} - \frac{j}{2}} \int_0^t \| \mu^1 S(\tau) \|_{Z_1} d\tau + (1 + t)^{-\frac{3}{2}} \int_0^t \| \mu^1 \nabla_x^{m+j+1} S(\tau) \| d\tau + \int_0^t \left( \| \nabla_x^m S(\tau) \| + \| \nabla_x^{m+j+1} S(\tau) \| \right) d\tau.
\]

(3.5)

**Proof.** Basing on the analysis in subsection 3.1, we can apply the same arguments in [4] to have for any \( l \geq 0 \) that

\[
\partial_t E_l(\hat{U}_1(t, k)) + \lambda D_l(\hat{U}_1(t, k)) \lesssim 0,
\]

(3.6)

where \( \hat{U}_1(t, k) \) denotes the Fourier transform of \( U_1 \) with respect to \( x \) and \( k \) is the corresponding variable after Fourier transform. \( E_l(\hat{U}_1(t, k)) \) and \( D_l(\hat{U}_1(t, k)) \) are functionals given by

\[
E_l(\hat{U}_1(t, k)) = \| \hat{f}_1 \|_{L_x^2}^2 + \left( \| \hat{E}_1, \hat{B}_1 \| \right)^2 + \lambda_0 Re\mathcal{E}^{int}(t, k)
\]

\[
+ \lambda_1 \left( \| \mu^1 (I - \mathcal{P}) \hat{f}_1 \|_{L_x^p}^2 \chi_{|k| \leq 1} + \frac{\lambda_2}{1 + |k|^2} \| \mu^l (I - \mathcal{P}) \hat{f}_1 \|_{L_x^2}^2 \chi_{|k| \geq 1} \right)
\]

\[
D_l(\hat{U}_1(t, k)) = \| (I - \mathcal{P}) \hat{f}_1 \|_{L_x^2}^2 + \frac{\lambda_2}{1 + |k|^2} \left( \| \mu^l (I - \mathcal{P}) \hat{f}_1 \|_{L_x^2}^2 \right)
\]

\[
+ \frac{|k|^2}{1 + |k|^2} \left( |\hat{a}_{1, +} + \hat{a}_{1, -}| + |\hat{b}_1|^2 + |\hat{c}_1|^2 \right) + |\hat{a}_{1, +} - \hat{a}_{1, -}|^2
\]

\[
+ \frac{1}{1 + |k|^2} \left( \| \hat{E}_1 \|_{L_x^2}^2 + \| \hat{B}_1 \|_{L_x^2}^2 \right).
\]

Here the constants \( \lambda_0, \lambda_1, \lambda_2 > 0 \) are sufficiently small and \( \mathcal{E}^{int}(t, k) \) is a time-frequency interactive functional satisfying \( |\mathcal{E}^{int}(t, k)| \lesssim |\hat{f}_1|_{L_x^2}^2 + |(\hat{E}_1, \hat{B}_1)|^2 \). From the definitions of \( E_l(\hat{U}_1(t, k)) \) and \( D_l(\hat{U}_1(t, k)) \), we have

\[
D_l(\hat{U}_1(t, k)) \chi_{|k| \leq 1} \lesssim |k|^2 \mathcal{E}^{-1}(\hat{U}_1(t, k)) \chi_{|k| \leq 1}
\]

\[
\lesssim |k|^2 \left( \| \mu^{-1} \hat{f}_1 \|_{L_x^2}^2 + |(\hat{E}_1, \hat{B}_1)|^2 \right) \chi_{|k| \leq 1},
\]
Then for $l_1 > m + \frac{3}{2}$, noticing the fact

$$E_l(\hat{U}_1(t, k)) \lesssim E_{l-1}(\hat{U}_1(t, k)) E_{l_1}^{-\frac{1}{2}}(\hat{U}_1(t, k)),$$

we have from (3.6) that

$$\partial_t E_l(\hat{U}_1(t, k)) \chi_{|k| \leq 1} + \lambda |k|^2 E_{l+1}^{1+1}(\hat{U}_1(t, k)) E_{l_1}^{\frac{1}{2}}(\hat{U}_1(t, k)) \chi_{|k| \leq 1} \lesssim 0,$$

and

$$E_{l_1}^{-\frac{1}{2}}(\hat{U}_1(t, k)) \geq E_{l_1}^{-\frac{1}{2}}(\hat{U}_1(0, k)).$$

The estimates above imply

$$E_l(\hat{U}_1(t, k)) \chi_{|k| \leq 1} \lesssim l E_{l_1}(\hat{U}_1(0, k)) (1 + |k|^2 t)^{-l_1} \chi_{|k| \leq 1}. \quad (3.7)$$

Take $\alpha \geq 0$ with $|\alpha| = m$ to see

$$\int_{|k| \leq 1} |k^\alpha|^2 E_l(\hat{U}_1(t, k)) \, dk \lesssim \int_{|k| \leq 1} |k^\alpha|^2 E_{l_1}(\hat{U}_1(0, k))(1 + |k|^2 t)^{-l_1} \, dk \lesssim \int_{|k| \leq 1} |k^\alpha|^2 (1 + |k|^2 t)^{-l_1} \, dk \sup_{|k| \leq 1} E_{l_1}(\hat{U}_1(0, k)) \lesssim (1 + t)^{-|\alpha| - \frac{3}{2}} \left( \|\mu^{l+1} f_0\|_{L^2}^2 + \|(E_0, B_0)\|_{L^1}^2 \right), \quad (3.8)$$

for $l_1 > m + \frac{3}{2}$.

For the case $|k| \geq 1$, in a similar way, one has

$$E_l(\hat{U}_1(t, k)) \chi_{|k| \geq 1} \lesssim l E_{l_1}(\hat{U}_1(0, k)) \left( 1 + \frac{1}{|k|^2} \right)^{-l_1} \chi_{|k| \geq 1}, \quad (3.9)$$

$$\int_{|k| \geq 1} |k|^2 |k^\alpha|^2 E_l(\hat{U}_1(t, k)) \, dk \lesssim \int_{|k| \geq 1} |k|^2 |k^\alpha|^2 E_{l_1}(\hat{U}_1(0, k)) \left( 1 + \frac{1}{|k|^2} \right)^{-l_1} \, dk \lesssim \int_{|k| \geq 1} |k|^{2(j+1)} |k^\alpha|^2 E_{l_1}(\hat{U}_1(0, k)) \sup_{|k| \geq 1} \left\{ \left( 1 + \frac{1}{|k|^2} \right)^{-l_1} \frac{1}{|k|^2} \right\} \lesssim (1 + t)^{-j} \left( \|\mu^{l+1} \nabla x^{|\alpha|+j+1} f_0\|^2 + \|\nabla x^{|\alpha|+j+1} (E_0, B_0)\|_{L^2}^2 \right), \quad (3.10)$$

for $l_1 \geq j$. Collecting the estimates (3.8) and (3.10), the desired estimate (3.4) follows for $l = 0$ by Parseval's identity.

Now we turn to prove (3.5). For this purpose, we first denote $\hat{U}(t, \tau) = \mathcal{A}(t, \tau) [S(\tau), 0, 0]$ and $\hat{U}(\tau, \tau) = [S(\tau), 0, 0]$. Corresponding to the estimates (3.7) and
For the term $I$, we can obtain

\[ E_\ell(\tilde{U}(t, \tau, k)) \chi_{|k| \leq 1} \lesssim E_{\ell + 1}(\tilde{S}(\tau, k)) \left(1 + |k|^2(t - \tau)\right)^{-\frac{l_1}{2}} \chi_{|k| \leq 1}, \]

\[ E_\ell(\tilde{U}(t, \tau, k)) \chi_{|k| \geq 1} \lesssim E_{\ell + 1}(\tilde{S}(\tau, k)) \left(1 + \frac{1}{|k|^2}(t - \tau)\right)^{-\frac{l_1}{2}} \chi_{|k| \geq 1}. \]

Then we can obtain

\[ \|\nabla_x U_2(t)\|^2 \]

\[ \lesssim \int_0^t \|\nabla_x A(t, \tau)S(\tau)\|^2 \, d\tau \]

\[ \lesssim \int_0^t \left( \int_{|k| \leq 1} |k|^{2m} \left\|\mu^{1/2} \tilde{S}(\tau, k)\right\|^2_{L^2_\mu} \left(1 + |k|^2(t - \tau)\right)^{-\frac{l_1}{2}} \, dk \right. \]

\[ \quad + \left. \int_{|k| \geq 1} |k|^{2m+2} \left\|\mu^{1/2} \tilde{S}(\tau, k)\right\|^2_{L^2_\mu} \left(1 + \frac{1}{|k|^2}(t - \tau)\right)^{-\frac{l_1}{2}} \, dk \right)^{\frac{1}{2}} \, d\tau \]

\[ + \int_0^t \left( \int_{|k| \leq 1} |k|^{2m} \left\|\tilde{S}(\tau, k)\right\|^2_{L^2_\mu} \, dk \right. \]

\[ \quad + \left. \int_{|k| \geq 1} |k|^{2m+2} \left\|\tilde{S}(\tau, k)\right\|^2_{L^2_\mu} \, dk \right)^{\frac{1}{2}} \, d\tau \]

\[ = I_1 + I_2. \]

For the term $I_1$, we have the upper bound

\[ \int_0^t \left( \int_{|k| \leq 1} |k|^{2m} \left(1 + |k|^2(t - \tau)\right)^{-\frac{l_1}{2}} \, dk \right. \]

\[ \quad + \left. \int_{|k| \geq 1} |k|^{2m+2} \left\|\mu^{1/2} \tilde{S}(\tau, k)\right\|^2_{L^2_\mu} \, dk \right)^{\frac{1}{2}} \, d\tau \]

\[ \times \left( \sup_{|k| \geq 1} \left\{ \left(1 + \frac{1}{|k|^2}(t - \tau)\right)^{-\frac{l_1}{2}} |k|^{-2j}\right\} \right)^{\frac{1}{2}} \, d\tau \]

\[ \lesssim \int_0^t \left( (1 + t - \tau)^{-\frac{2m}{2} - \frac{j}{2}} \left\|\mu^{1/2} S(\tau)\right\|_{L^2_\mu} + (1 + t - \tau)^{-\frac{2}{2}} \left\|\mu^{1/2} \nabla_x^{m+1+j} S(\tau)\right\| \right) \, d\tau. \]

As for the term $I_2$, it is straightforward to see that

\[ I_2 \lesssim \int_0^t \left( \|\nabla_x^m S(\tau)\| + \|\nabla_x^{m+1} S(\tau)\| \right) \, d\tau. \]

Plugging the estimates of $I_1$, $I_2$ into (3.11) gives the desired estimate (3.5). This ends the proof of Lemma 3.1.

\[ \square \]

4. Global a priori estimates. In this section, we are going to prove Theorem 1.1, the main result of the paper. The key point is to deduce the uniform-in-time a priori estimates on solutions to the relativistic Vlasov-Maxwell-Boltzmann system

\[
\begin{aligned}
\partial_t f + p \cdot \nabla_x f - E \cdot \nabla_x J^Z \zeta_1 + Lf &= S, \\
\partial_t E - \nabla_x \times B &= - \int_{\mathbb{R}^3} \hat{p}(f_+ - f_-) \, dp, \\
\partial_t B + \nabla_x \times E &= 0, \\
\nabla_x \cdot E &= \langle J^Z, f_+ - f_- \rangle, \\
\nabla_x \cdot B &= 0,
\end{aligned}
\]

(4.1)
where the nonlinear term \( S = [S_+, S_-] \) is given by
\[
S = \frac{\zeta}{2} E \cdot \hat{p} f + \zeta \left( E + \hat{p} \times B \right) \cdot \nabla_p f + \Gamma(f, f).
\] (4.2)

For this purpose, let \((f, E, B)\) be a smooth solution to (4.1) over the time interval \(0 \leq t \leq T\) with initial data \((f_0, E_0, B_0)\) for some \(0 < T \leq \infty\), and further suppose that \((f, E, B)\) satisfies
\[
X(t) \leq \delta^2,
\] (4.3)

where \(X(t)\) is given in (1.19) and the constant \(\delta > 0\) is sufficiently small. What we want to do in the following is to deduce some a priori estimates on \((f(t, x, p), E(t, x), B(t, x))\) based on the a priori assumption (4.3). To make the presentation easy to follow, we divide this section into several subsections and the first one is concerned with the macro dissipation of the relativistic Vlasov-Maxwell-Boltzmann system.

4.1. Macro dissipation. We first define the macro dissipation \(D_{N,\text{mac}}(t)\) by
\[
D_{N,\text{mac}}(t) = \sum_{|\alpha| \leq N-1} \|\partial^\alpha \nabla_x (a_\pm, b, c)(t)\|^2 + \|a_+ - a_-(t)\|^2 + \|E(t)\|_{H^{N-1}}^2 + \|\nabla_x B(t)\|_{H^{N-2}}^2.
\]

With the above macro structure of the system (4.1) in hand, we have

**Lemma 4.1.** For any integer \(N \geq 9\), there is an interactive energy functional \(E^\text{int}_N(t)\) such that
\[
|E^\text{int}_N(t)| \lesssim \sum_{|\alpha| \leq N} \left( \|\partial^\alpha f(t)\|^2 + \|\partial^\alpha (E, B)(t)\|^2 \right),
\]

and
\[
\frac{d}{dt} E^\text{int}_N(t) + \lambda D_{N,\text{mac}}(t) \lesssim \sum_{|\alpha| \leq N} \|\partial^\alpha (I - P) f(t)\|_{H^\nu}^2 + E_{N,\ell}(t) \mathcal{D}_{N,\ell}(t)
\] (4.4)

hold for \(0 \leq t \leq T\).

**Proof.** Basing on the previous works [10] and [11], it is a quite standard process to obtain the above estimates. We hence omit the details for brevity.

4.2. Uniform spatial energy estimate. In this subsection, we derive the basic energy estimates on the spatial derivatives in \(E_{N,\ell}(t)\). Now we state our main result in this subsection as follows.

**Lemma 4.2.** There exist suitably small positive constants \(\kappa_1, \kappa_2\) and \(\kappa_3\) such that
\[
\frac{d}{dt} \left\{ \frac{1}{2} \sum_{|\alpha| \leq N} \|\partial^\alpha (f, E, B)(t)\|^2 + \kappa_1 E^\text{int}_N(t) \right. \\
+ \left. \frac{1}{2} \sum_{1 \leq |\alpha| \leq N-1} \|w_{\ell-|\alpha|} \partial^\alpha f(t)\|^2 + \kappa_2 \|w_{\ell} (I - P) f(t)\|^2 \right\} \\
+ \lambda \sum_{|\alpha| = N} \|\partial^\alpha (I - P) f(t)\|_{H^\nu}^2 + \mathcal{D}_{N,\text{mac}}(t)
\]
Now we estimate

\[ I \]

Then Lemma 2.10 implies

\[ E \]

where

\[ I \]

It is straightforward to establish the energy identity:

\[ \text{Step 1. Spatial energy estimates without any weight.} \]

From (4.1), it is straightforward to establish the energy identity:

\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq N} \| \partial^\alpha (f, E, B) (t) \|^2 + \sum_{|\alpha| \leq N} (L \partial^\alpha f, \partial^\alpha f) = \sum_{|\alpha| \leq N} (\partial^\alpha S, \partial^\alpha f).
\]

Then Lemma 2.10 implies

\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq N} \| \partial^\alpha (f, E, B) (t) \|^2 + \lambda \sum_{|\alpha| \leq N} \| \partial^\alpha \{ I - P \} f (t) \|_\nu^2 \lesssim I_3,
\]

where \( I_3 = \sum_{|\alpha| \leq N} | (\partial^\alpha S, \partial^\alpha f) | \). We claim that

\[
I_3 \lesssim \mathcal{E}_{N, t}^2 (t) D_{N, t} (t)
+ \sum_{i=1} \| \nabla^i_x (E, B) (t) \|_{L^\infty_x} \left( \| \nabla^N_x f (t) \|_\nu^2 + \| \nabla^N_x \nabla \{ I - P \} f (t) \|_\nu^2 \right).
\]

We first consider the estimate of \( I_3 \) corresponding to \( \Gamma (f, f) \) in \( S \). By using Lemma 2.6, it is straightforward to see that it is bounded up to a generic constant by \( \mathcal{E}_{N, t}^2 (t) D_{N, t} (t) \). For the zero-order term related to the electromagnetic field, one has

\[
\left| \left( \frac{\xi}{2} E \cdot \tilde{\nu} f + \zeta \left( E + \tilde{\nu} \times B \right) \cdot \nabla f \right) \right| \lesssim (\| E \|_{L^\infty_x} \| f (t) \|_\nu^2).
\]

For the term related to \( \partial^\alpha \{ \frac{\xi}{2} E \cdot \tilde{\nu} f \} \) with \( |\alpha| \leq N \), it holds that

\[
\left| \left( \frac{\xi}{2} \partial^\alpha \{ E \cdot f \}, \partial^\alpha f \right) \right|
= \frac{1}{2} \sum_{\alpha_1 \leq \alpha} C_{\alpha_1}^\alpha \left| (\zeta \partial^{\alpha - \alpha_1} E \cdot \tilde{\nu} \partial^{\alpha_1} f, \partial^\alpha f) \right|
\lesssim \sum_{|\alpha - \alpha_1| \leq 2} \| \partial^{\alpha - \alpha_1} E \|_{L^\infty_x} \left( \| \partial^{\alpha_1} f \|_\nu, | \partial^\alpha f | \right)
+ \sum_{|\alpha - \alpha_1| \geq 3} \left( | \partial^{\alpha - \alpha_1} E | \nu^{-\frac{1}{2}} | \partial^{\alpha_1} f \|_\nu, \nu^{\frac{1}{2}} | \partial^\alpha f | \right)
= I_{3, 1} + I_{3, 2}.
\]

Now we estimate \( I_{3, 1} \) and \( I_{3, 2} \) as follows. For the term \( I_{3, 1} \), since

\[
\| \nu^{-\frac{1}{2}} \partial^{\alpha_1} f \|_\nu \lesssim \| w_{t - |\alpha_1|} \partial^{\alpha_1} f \|_\nu
\]
for $|\alpha_1| < |\alpha| \leq N$, $\ell \geq N - 1 + \frac{l_1 + 1}{2}$, one can see that
\[
\sum_{1 \leq |\alpha - \alpha_1| \leq 2} (|\partial^{\alpha_1} f|, |\partial^{\alpha} f|) \\
\lesssim \sum_{1 \leq |\alpha - \alpha_1| \leq 2} \|\nu^{-1} \partial^{\alpha_1} f\|_\nu \|\partial^{\alpha} f(t)\|_\nu \\
\lesssim \sum_{1 \leq |\alpha - \alpha_1| \leq 2} \|w_{\ell - |\alpha_1|} \partial^{\alpha_1} f\|_\nu \|\partial^{\alpha} f(t)\|_\nu \lesssim D_{N, \ell}(t),
\]
from which and the Sobolev inequality in $\mathbb{R}^3$
\[
\|\partial^{\alpha - \alpha_1} E(t)\|_{L_x^\infty} \lesssim \|\nabla_x \partial^{\alpha - \alpha_1} E(t)\|^{\frac{2}{3}} \|\nabla_x^2 \partial^{\alpha - \alpha_1} E(t)\|^{\frac{1}{3}},
\]
it follows that
\[
I_{3.1} \lesssim \|E(t)\|_{L_x^\infty} \|\partial^{\alpha} f(t)\|^2 + \sum_{1 \leq |\alpha - \alpha_1| \leq 2} \|\partial^{\alpha - \alpha_1} \nabla_x E(t)\|^{\frac{2}{3}} \\
\times \|\partial^{\alpha - \alpha_1} \nabla_x^2 E(t)\|^{\frac{2}{3}} \|w_{\ell - |\alpha_1|} \partial^{\alpha_1} f\|_\nu \|\partial^{\alpha} f(t)\|_\nu \\
\lesssim \|E(t)\|_{L_x^\infty} \|\partial^{\alpha} f(t)\|^2 + E_{N, \ell}^1(t)D_{N, \ell}(t).
\]
Similarly, $I_{3.2}$ can be bounded by
\[
I_{3.2} \lesssim \sum_{|\alpha - \alpha_1| \geq 3} \|\partial^{\alpha - \alpha_1} E(t)\| \|\nu^{-1} \partial^{\alpha_1} \nabla_x f\|^{\frac{1}{2}} \|\nu^{-1} \partial^{\alpha_1} \nabla_x^2 f\|^{\frac{1}{2}} \|\partial^{\alpha} f(t)\|_\nu \\
\lesssim E_{N, \ell}^1(t)D_{N, \ell}(t).
\]
Combining the estimates of $I_{3.1}$, $I_{3.2}$ and (4.8), we arrive at
\[
\sum_{|\alpha| \leq N} \left| \left( \frac{c}{2} \partial^\alpha \{E \cdot \hat{p} f\}, \partial^\alpha f \right) \right| \\
\lesssim \|E(t)\|_{L_x^\infty} \sum_{1 \leq |\alpha| \leq N} \|\partial^{\alpha} f(t)\|^2 + E_{N, \ell}^1(t)D_{N, \ell}(t) \\
\lesssim \|E(t)\|_{L_x^\infty} \|\nabla_x^N f(t)\|^2 + E_{N, \ell}^1(t)D_{N, \ell}(t).
\]
In a similar way, one has
\[
\sum_{|\alpha| \leq N} \left| \left( \partial^\alpha \left\{ \left( E + \hat{p} \times B \right) \cdot \nabla_p f \right\}, \partial^\alpha f \right) \right| \\
\lesssim \sum_{|\alpha| \leq N} \left( \|\partial^{\alpha - \alpha_1} (E, B)\|, \|\partial^{\alpha_1} \nabla_p f\|, \|\partial^{\alpha} f\| \right) \\
\lesssim \sum_{1 \leq |\alpha| \leq N} \|\nabla_x (E, B)(t)\|_{L_x^\infty} \|\nabla_x^{\alpha_1 - 1} \nabla_p f(t)\| \|\partial^{\alpha} f(t)\| \\
+ \sum_{1 \leq |\alpha| \leq N} \|\partial^{\alpha - \alpha_1} (E, B)(t)\|_{L_x^\infty} \|\partial^{\alpha_1} \nabla_p f\| \|\partial^{\alpha} f\| \\
+ \sum_{1 \leq |\alpha| \leq N} \|\partial^{\alpha - \alpha_1} (E, B)(t)\| \|\nu^{-\frac{1}{2}} \partial^{\alpha_1} \nabla_p f\|_{L_x^\infty} \|\partial^{\alpha} f\|_\nu \\
\lesssim E_{N, \ell}^1(t)D_{N, \ell}(t) + \|\nabla_x (E, B)(t)\|_{L_x^\infty}
Putting the above estimates into \( I_3 \), we see that the bound \( (4.7) \) is valid.

**Step 2. Higher order spatial energy estimates with weight.** Now we turn to do the weighted energy estimate on \( \partial^\alpha f \) with \( 1 \leq |\alpha| \leq N - 1 \). From \( (4.1) \) and Lemma 2.10, one has

\[
\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N - 1} \left\| w_{t - |\alpha|} \partial^\alpha f(t) \right\|^2 + \lambda \sum_{|\alpha| \leq N - 1} \left\| w_{t - |\alpha|} \partial^\alpha \{1 - P\} f(t) \right\|_\nu^2 \\
\leq \sum_{1 \leq |\alpha| \leq N - 1} \left( \partial^\alpha S, w_{t - |\alpha|}^2 \right) \\
+ \sum_{1 \leq |\alpha| \leq N - 1} \left\| \partial^\alpha \{1 - P\} f(t) \right\|_\nu^2 + \sum_{|\alpha| \leq N - 1} \left( \frac{\zeta}{2} \partial^\alpha E : \tilde{p} J^{\frac{1}{2}}, w_{t - |\alpha|}^2 \partial^\alpha f \right) \tag{4.9}
\]

where \( I_4 = \sum_{1 \leq |\alpha| \leq N - 1} \left( \partial^\alpha S, w_{t - |\alpha|}^2 \partial^\alpha f \right) \). We now prove that

\[
I_4 \lesssim \mathcal{E}_{N,\ell}^{\frac{3}{2}}(t) \mathcal{D}_{N,\ell}(t) + \sum_{\ell \leq 1} \left\| \nabla^\ell (E, B)(t) \right\|_{L^\infty} \times \left( \sum_{|\alpha| \leq N - 1} \left\| w_{t - |\alpha|} \partial^\alpha f(t) \right\|^2 + \sum_{|\alpha| \leq N - 2} \left\| w_{t - |\alpha| - 1} \partial^\alpha \nabla_p f(t) \right\|^2 \right). \tag{4.10}
\]

Similar to the estimate of \( I_3 \), by using Lemma 2.6, the corresponding weighted estimate of \( \Gamma(f, f) \) can be bounded up to a generic constant by \( \mathcal{E}_{N,\ell}^{\frac{3}{2}}(t) \mathcal{D}_{N,\ell}(t) \). For the term \( \frac{\zeta}{2} E : \tilde{p} f \) in \( S \), one has

\[
\sum_{1 \leq |\alpha| \leq N - 1} \left| \left( \frac{\zeta}{2} \partial^\alpha \{E : \tilde{p} f\}, w_{t - |\alpha|}^2 \partial^\alpha f \right) \right| \\
\lesssim \| E(t) \|_{\mathcal{L}^\infty} \sum_{1 \leq |\alpha| \leq N - 1} \left\| w_{t - |\alpha|} \partial^\alpha f(t) \right\|^2 \\
+ \sum_{1 \leq |\alpha| \leq N - 1} \left\| \partial^{-\alpha} E(t) \right\|_{L^\infty} \left\| \nu^{-\frac{1}{2}} w_{t - |\alpha|} \partial^{\alpha_1} f(t) \right\|_{L^\infty} \left\| \partial^{\alpha} f(t) \right\|_{L^\infty} \\
+ \sum_{1 \leq |\alpha| \leq N - 1} \left\| \partial^{-\alpha - 1} E(t) \right\| \left\| \nu^{-\frac{1}{2}} w_{t - |\alpha|} \partial^{\alpha_1} f(t) \right\|_{L^\infty} \left\| \nu^{\frac{1}{2}} \partial^\alpha f(t) \right\| \\
\lesssim \| E(t) \|_{\mathcal{L}^\infty} \sum_{|\alpha| \leq N - 1} \left\| w_{t - |\alpha|} \partial^\alpha f(t) \right\|^2 + \mathcal{E}_{N,\ell}^{\frac{3}{2}}(t) \mathcal{D}_{N,\ell}(t).
\]

Next, we treat the estimate related to \( \zeta (E + \tilde{p} \times B) \cdot \nabla_p f \). Noticing that

\[
\left\| \nabla_p \left( w_{t - |\alpha|}^2 \right) \right\| \lesssim \nu^{\frac{1}{2}} w_{t - |\alpha|}^2,
\]

and

\[
\left\| \nu^{-\frac{1}{2}} w_{t - |\alpha|} \partial^{\alpha_1} \nabla_p f \right\| \lesssim \left\| w_{t - |\alpha| - 1} \partial^{\alpha_1} \nabla_p f \right\|_{\nu},
\]
for $|\alpha - \alpha_1| \geq 2$. We get from Hölder’s inequality and Sobolev’s inequality that

$$
\sum_{1 \leq |\alpha| \leq N - 1} \left| \left( \zeta \partial^{\alpha} \left\{ \left( E + \tilde{p} \times B \right) \cdot \nabla_p f \right\}, w_{\ell - |\alpha|} \partial^{\alpha} f \right) \right|
\lesssim \sum_{1 \leq |\alpha| \leq N - 1} \left| \left( \zeta (E + \tilde{p} \times B) \cdot \nabla_p \partial^{\alpha} f, w_{\ell - |\alpha|} \partial^{\alpha} f \right) \right|
+ \sum_{1 \leq |\alpha| \leq N - 1} \sum_{|\alpha_1| < |\alpha|} \left( |\partial^{\alpha - \alpha_1} (E, B)| \left| w_{\ell - |\alpha|} \partial^{\alpha_1} \nabla_p f \right|, \left| w_{\ell - |\alpha|} \partial^{\alpha} f \right| \right)
\lesssim \sum_{1 \leq |\alpha| \leq N - 1} \| (E, B) (t) \|_{L_\infty} \left( \left| \nabla_p \left( w_{\ell - |\alpha|}^2 \right) \right|, \left| \partial^{\alpha} f (t) \right| \right)
+ \sum_{1 \leq |\alpha| \leq N - 1} \sum_{2 \leq |\alpha - \alpha_1| \leq 3} \left\| \partial^{\alpha - \alpha_1} (E, B) (t) \right\|_{L_\infty} \left\| \nu^{-\frac{1}{2}} w_{\ell - |\alpha|} \partial^{\alpha_1} \nabla_p f \right\| \left\| w_{\ell - |\alpha|} \partial^{\alpha} f \right\|_\nu
+ \sum_{1 \leq |\alpha| \leq N - 1} \sum_{|\alpha_1| \geq 4} \left\| \nu^{-\frac{1}{2}} w_{\ell - |\alpha|} \partial^{\alpha_1} \nabla_p f (t) \right\|_{L_2} \left\| w_{\ell - |\alpha|} \partial^{\alpha} f \right\|_\nu
\lesssim \mathcal{E}_{N, \ell}^\frac{3}{2} (t) \mathcal{D}_{N, \ell} (t) + \sum_{|\alpha| \leq N - 1} \sum_{i \leq 1} \left\| \nabla_x^i (E, B) (t) \right\|_{L_\infty}
+ \left[ \left\| w_{\ell - |\alpha|} \nabla_x^{|\alpha|} f (t) \right\|^2 + \left\| w_{\ell - |\alpha|} \nabla_x^{|\alpha| - 1} \nabla_p f (t) \right\|^2 \right].
$$

Then (4.10) follows from the above estimates.

**Step 3. Zero energy estimates on the micro component with weight.**
To complete the proof of (4.5), it remains now to deduce the energy estimates on $w_\ell (I - P) f$, for this, let us first apply the micro projection $\{ I - P \}$ to the first equation of (4.1) to obtain

$$
\partial_t (I - P) f + \tilde{p} \cdot \nabla_x (I - P) f - E \cdot \tilde{p} \partial^{\frac{1}{2}} \zeta_1 + L f = (I - P) S + P (\tilde{p} \cdot \nabla_x \partial^{\alpha} f) - \tilde{p} \cdot \nabla_x Pf,
$$

(4.11)

then perform the standard energy estimate on (4.11) to derive

$$
\frac{1}{2} \frac{d}{dt} \| w_\ell (I - P) f (t) \|^2 + \lambda \| w_\ell (I - P) f (t) \|^2_\nu
\lesssim \mathcal{E}_{N, \ell}^\frac{3}{2} (t) \mathcal{D}_{N, \ell} (t) + \left\| (I - P) f (t) \right\|_{L_\infty}^2 + \| E (t) \|^2
+ \sum_{|\alpha| = 1} \left\| w_{\ell - |\alpha|} \partial^{\alpha} f (t) \right\|^2_\nu + \left\| [E, B] (t) \right\|_{L_\infty} \| w_\ell f (t) \|^2.
$$

(4.12)

Finally, choosing suitably small positive constants $\kappa_1, \kappa_2$ and $\kappa_3$ satisfying $\kappa_1 \gg \kappa_2 \gg \kappa_3$, and taking the summation of (4.4) $\times \kappa_1$, (4.9) $\times \kappa_2$, (4.12) $\times \kappa_3$ and (4.6), we obtain (4.5). This completes the proof of Lemma 4.2. \( \square \)
4.3. Mixed space-velocity derivatives energy estimate. In this subsection, we are devoted to deriving the mixed space-velocity derivatives energy estimate. With this done, we can combine the estimates in this subsection and the previous one to obtain a uniform estimate of \( E_{N,t}(t) \). For result in this direction, we have

**Lemma 4.3.** For some large constants \( C_{1|\beta|} > 0 \) and \( C'_{1|\beta|} > 0 \), it holds that

\[
\frac{d}{dt} \left[ \sum_{|\alpha|+|\beta|=N} C_{1|\beta|} \| \partial^\alpha \{ I - P \} f(t) \|_2^2 \right. \\
\left. + \sum_{|\alpha|+|\beta|=N-1} C'_{1|\beta|} \| w_{t-|\alpha|-|\beta|} \partial^\alpha \{ I - P \} f(t) \|_2^2 \right] \\
+ \lambda \left( \sum_{|\alpha|+|\beta|=N} \| \partial^\alpha \{ I - P \} f(t) \|_\nu \right. \\
\left. + \sum_{|\alpha|+|\beta|\leq N-1} \| w_{t-|\alpha|-|\beta|} \partial^\alpha \{ I - P \} f(t) \|_\nu \right) \\
\leq E_{N,t}(t) D_{N,t}(t) + \sum_{|\alpha|\leq N} \| \partial^\alpha \{ I - P \} f(t) \|_\nu^2 \\
\left. + \sum_{|\alpha|\leq N-2} \| w_{t-|\alpha|-|\beta|} \partial^\alpha f(t) \|_\nu^2 \right) \\
\left. \times \left( \sum_{|\alpha|+|\beta|=N} \| \partial^\alpha f(t) \|_2^2 + \sum_{|\alpha|+|\beta|\leq N-1} \| w_{t-|\alpha|-|\beta|} \partial^\alpha \{ I - P \} f(t) \|_2^2 \right) \right). \tag{4.13}
\]

**Proof.** Acting \( \partial^\alpha \) with \(|\alpha| + |\beta| = N \) and \(|\beta| \geq 1 \) to (4.11), we have

\[
\partial_t \partial^\alpha \{ I - P \} f + \tilde{\rho} \cdot \nabla_x \partial^\alpha \{ I - P \} f + \sum_{1 \leq |\beta| \leq |\beta|} C^\beta_{1|\beta|} \partial_{\beta} \tilde{\rho} \cdot \nabla_x \partial^\alpha_{\beta-\beta} \{ I - P \} f \\
- \partial^\alpha E \cdot \partial_{\beta} \tilde{J}^{\frac{1}{2}} \zeta_1 + \partial^\alpha L f \\
= \partial^\alpha \{ I - P \} S + \partial_{\beta} \left[ \tilde{\rho} \cdot \nabla_x \partial^\alpha f - \tilde{\rho} \cdot \nabla_x P \partial^\alpha f \right]. \tag{4.14}
\]

The direct energy estimate of (4.14) gives

\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \{ I - P \} f(t) \|_2^2 + \lambda \| \partial^\alpha \{ I - P \} f(t) \|_\nu^2 \\
\leq C \sum_{1 \leq |\beta| \leq |\beta|} \| \partial^\alpha_{\beta} \nabla_x \{ I - P \} f(t) \|_\nu^2 \\
\]

from which and Lemma 2.10 and Cauchy-Schwartz’s inequality, it follows that

\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \{ I - P \} f(t) \|_2^2 + \lambda \| \partial^\alpha \{ I - P \} f(t) \|_\nu^2 \\
\leq C \sum_{1 \leq |\beta| \leq |\beta|} \| \partial^\alpha_{\beta} \nabla_x \{ I - P \} f(t) \|_\nu^2 \\
\]

The final result follows from this inequality.
Performing the similar calculations as estimating $I_3$, one has

$$\sum_{|\alpha|+|\beta|=N} \left( \left\| \partial_\beta \{I - P\} f(t) \right\|^2 + \lambda \left\| \partial_\beta \{I - P\} f(t) \right\|^2 \right)$$

$$+ \sum_{|\alpha|+|\beta|=N} \left( \left\| \partial_\beta \{I - P\} f(t) \right\|^2 + \left\| \partial_\beta \{I - P\} f(t) \right\|^2 \right)$$

Taking a suitably linear combination of the resulting inequalities over $1 \leq \beta \leq N$, we see that there exist $C_\beta$ ($1 \leq \beta \leq N$) such that

$$\sum_{|\alpha|+|\beta|=N} \left( \left\| \partial_\beta \{I - P\} f(t) \right\|^2 + \lambda \left\| \partial_\beta \{I - P\} f(t) \right\|^2 \right)$$

Now we turn to obtain the weighted energy estimate concerning $\partial_\beta \{I - P\} f$ with $|\alpha|+|\beta| \leq N-1$ and $|\beta| \geq 1$. Taking the inner product of (4.14) with $w_{t-|\alpha|-|\beta|} \partial_{\beta_1} \{I - P\} f$ over $\mathbb{R}^3 \times \mathbb{R}^3$, applying Lemma 2.10 and Cauchy-Schwartz’s inequality with $\eta > 0$, we deduce

$$\frac{1}{\nu} \left\| \frac{d}{dt} \right\| \left\| w_{t-|\alpha|-|\beta|} \partial_{\beta_1} \{I - P\} f(t) \right\| \right\|_\nu^2 + \lambda \left\| \frac{d}{dt} \right\| \left\| \partial_{\beta_1} \{I - P\} f(t) \right\| \right\|_\nu^2$$

$$\leq \eta \sum_{|\beta_1| \leq |\beta|} \left\| \partial_{\beta_1} \{I - P\} f(t) \right\| \right\|_\nu^2 + C_\eta \left\| \partial_{\beta} \{I - P\} f(t) \right\| \right\|_\nu^2$$

$$+ \left\| \partial_{\beta} \left\{ \left( \frac{\zeta}{2} E \cdot \tilde{\nu} f + P (\tilde{\nu} \cdot \nabla x f) - \tilde{\nu} \cdot \nabla x P f \right) \right\} \partial_{\beta} \{I - P\} f(t) \right\| \right\|_\nu^2$$

$$\leq \eta \sum_{|\beta_1| \leq |\beta|} \left\| \partial_{\beta_1} \{I - P\} f(t) \right\| \right\|_\nu^2 + C_\eta \left\| \partial_{\beta} \{I - P\} f(t) \right\| \right\|_\nu^2$$

$$+ \sum_{|\alpha| \leq N-2} \left( \left\| \partial_\beta \{I - P\} S \left\| \partial_\beta \{I - P\} f(t) \right\| \right\|_\nu^2 + \lambda \left\| \partial_\beta \{I - P\} f(t) \right\| \right\|_\nu^2$$

$$+ \left\| \partial_\beta \{I - P\} S \left\| \partial_\beta \{I - P\} f(t) \right\| \right\|_\nu^2 $$
On the other hand, similar to the estimate of $I_4$, we have
\[
\left| \left( \partial_\beta^2 \{ I - P \} S, w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f \right) \right| \lesssim E_{N,\ell}(t) D_{N,\ell}(t) + \sum_{i \leq 1} \| \nabla_x^i (E, B)(t) \|_{L^\infty}^2 \times \sum_{|\alpha| + |\beta| \leq N - 1, |\beta| \geq 1} \| w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f(t) \|_2^2.
\]
Substituting this inequality into (4.18) and making a proper linear combination of $\| w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f(t) \|_2^2$ with $|\alpha| + |\beta| \leq N - 1$ and $|\beta| \geq 1$, we have for some constants $C'_{\beta} > 0$ that
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| + |\beta| \leq N - 1, |\beta| \geq 1} C'_{\beta} \| w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f(t) \|_2^2 + \lambda \sum_{|\alpha| + |\beta| \leq N - 1, |\beta| \geq 1} \| w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f(t) \|_2^2 \lesssim \sum_{|\alpha| \leq N - 2} (\| \partial^\alpha E(t) \|_2^2 + \| \| w_{\ell - |\alpha| - 1} \nabla_x \partial^\alpha f(t) \|_{L^2} + \| \partial^\alpha \{ I - P \} f(t) \|_{L^2}^2) + E_{N,\ell}(t) D_{N,\ell}(t) + \sum_{i \leq 1} \| \nabla_x^i (E, B)(t) \|_{L^\infty}^2 \times \sum_{|\alpha| + |\beta| \leq N - 1, |\beta| \geq 1} \| w_{\ell - |\alpha| - |\beta|} \partial_\beta^2 \{ I - P \} f(t) \|_2^2.
\]
Then (4.17) and (4.19) yield (4.13). This finishes the proof Lemma 4.3. \hspace{1cm} \Box

With Lemmas 4.2 and 4.3 in hand, we make a proper linear combination of (4.5) and (4.13) to have
\[
\frac{d}{dt} E_{N,\ell}(t) + \lambda D_{N,\ell}(t) \lesssim \sum_{i \leq 1} \| \nabla_x^i (E, B)(t) \|_{L^\infty} E_{N,\ell}(t) + E_{N,\ell}(t) D_{N,\ell}(t).
\]
Recalling the definitions of $E_{N,\ell}(t)$ and $X(t)$ and using the smallness assumption of $X(t)$ in (4.3), we can obtain
\[
\frac{d}{dt} E_{N,\ell}(t) + \lambda D_{N,\ell}(t) \lesssim (1 + t)^{-\frac{3}{2}} X^\frac{1}{2}(t) E_{N,\ell}(t).
\]
From the Gronwall inequality, (4.20) further implies
\[
E_{N,\ell}(t) + \lambda \int_0^t D_{N,\ell}(\tau) d\tau \lesssim E_{N,\ell}(0).
\]

4.4. Decay of pure spatial derivatives. In this part, basing on the definition of $X(t)$ and the estimates in the previous subsections, we will derive the time-decay of pure spatial derivatives $\partial^\alpha (f, E, B)$ with $|\alpha| \leq N - 1$. Before this, we cite the following calculus inequalities in the Sobolev spaces, which will be used frequently.

Lemma 4.4. \cite{29} For any $m \in \mathbb{Z}$, $u, v \in L^\infty \cap H^m(\mathbb{R}^N)$, it holds that
\[
\| \nabla_x^m (uv) \| \lesssim \| u \|_{L^\infty} \| \nabla_x^m v \| + \| v \|_{L^\infty} \| \nabla_x^m u \|,
\]
\[
\| \nabla_x^m (uv) - u \nabla_x^m v \| \lesssim \| \nabla_x u \|_{L^\infty} \| \nabla_x^{m-1} v \| + \| v \|_{L^\infty} \| \nabla_x^m u \|.
\]
The main result of this subsection can be stated as follows:
Lemma 4.5. Suppose $\ell \geq N - 1 + \frac{l_1 + 1}{2}$ with $l_1 > N_* + \frac{3}{2}$. For $0 \leq t \leq T$, it holds that

$$
\|\nabla^m_x (f, E, B)(t)\|^2 \lesssim (1 + t)^{-r_m} \left[ Y_0^2 + X^2(t) \right], \quad m \leq N - 1. \tag{4.22}
$$

Proof. Recalling the definition of $r_m$, we first prove that

$$
\|\nabla^m_x (f, E, B)(t)\|^2 \lesssim (1 + t)^{-m-3/2} \left[ Y_0^2 + X^2(t) \right], \quad m \leq N_*. \tag{4.23}
$$

To verify (4.23), we will deduce the decay rates of the $N_*$-th and zeroth order derivatives of solution explicitly then obtain the corresponding decays rates of the other ones by the method of interpolation. For this, we start from the following mild form

$$
U(t) = A(t) U_0 + \int_0^t A(t - \tau) [S(\tau), 0, 0] d\tau = U_1(t) + U_2(t),
$$

which denotes the solution $U = [f, E, B]$ to the Cauchy problem on system (3.1) with the nonlinear term $S$ given in (4.2) and initial data $U_0 = [f_0, E_0, B_0]$. Then for $m = N_*$, it follows from Lemma 3.1 that

$$
\begin{align*}
\|\nabla^N_x U(t)\| &\leq \|\nabla^N_x U_1(t)\| + \|\nabla^N_x U_2(t)\| \\
& \lesssim (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \left( \|\mu^l f_0\|_{L^1} + \|(E_0, B_0)\|_{L^1} \right) \\
& \quad + (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \left( \|\mu^l \nabla^{N_* + 1} f_0\| + \|\nabla^{N_* + 1} (E_0, B_0)\| \right) \\
& \quad + (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \int_0^t \|\mu^l S(\tau)\|_{L^1} d\tau \\
& \quad + \int_0^t \left( \|\nabla_x^{N_*} S(\tau)\| + \|\nabla_x^{N_* + 1} S(\tau)\| \right) d\tau \\
& \lesssim (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} Y_0 + I_{5,1} + I_{5,2} + I_{5,3}.
\end{align*}
$$

(4.24)

Here we have taken

$$
j = \frac{3}{2} + N_*, \quad l_1 > N_* + \frac{3}{2}, \quad N_* \geq 2,
$$

and denoted

$$
\begin{align*}
I_{5,1} &= (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \int_0^t \|\mu^l S(\tau)\|_{L^1} d\tau, \\
I_{5,2} &= (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \int_0^t \|\mu^l \nabla^{N_* + 1} S(\tau)\| d\tau, \\
I_{5,3} &= \int_0^t \left( \|\nabla_x^{N_*} S(\tau)\| + \|\nabla_x^{N_* + 1} S(\tau)\| \right) d\tau.
\end{align*}
$$

In view of Lemmas 2.6, (4.2) and (4.21), we use the definition of $X(t)$ to obtain

$$
\begin{align*}
I_{5,1} &\lesssim (1 + t)^{-\frac{3}{2} - \frac{N_*}{2}} \int_0^t \left[ \|S(\tau)\| (\|\mu^l f(\tau)\| + \|\mu^l (I - P) f(\tau)\|) \right] d\tau \\
& \quad + \|\mu^l \nabla_x P f(\tau)\| d\tau
\end{align*}
$$

(4.25)
Here we used $p_0 = \mu^2(p)$ and $\nu^{-\frac{1}{2}} w_{l_1} \lesssim w_{l-1}$ for $\ell \geq l_1 + 3/2$.
We further take
$$\ell \geq N - 1 + \frac{l_1 + 1}{2}, \quad 2N_\ast + \frac{7}{2} \leq N - 1.$$ with this, one can show that
$$\int_0^t \|\mu^1 \nabla_x^{2N_\ast + \frac{7}{2}} \nabla_p (I - P) f(\tau)\| \|(E, B)\|_{L^\infty_x(L^2_p)} d\tau$$
$$\lesssim \int_0^t \|w_{l-(2N_\ast + \frac{7}{2})} \nabla_x^{2N_\ast + \frac{7}{2}} \nabla_p (I - P) f(\tau)\|_{\nu} \sum_{1 \leq i \leq 2} \|\nabla_x^{i}(E, B)\| d\tau$$
$$\lesssim \int_0^t D_{N, \ell}(\tau) d\tau \lesssim \mathcal{E}_{N, \ell}(0),$$
from which and Lemma 4.4 and (4.21), we then bound $I_{5,2}$ by
$$I_{5,2} \lesssim (1 + t)^{-\frac{3}{2} - \frac{N_\ast}{2}} \int_0^t D_{N, \ell}(\tau) d\tau \lesssim (1 + t)^{-\frac{3}{2} - \frac{N_\ast}{2}} \mathcal{E}_{N, \ell}(0).$$
For the term $I_{5,3}$, on the one hand, applying Lemma 4.4 again, one has
$$I_{5,3} \lesssim \int_0^t \left[ (\|\nabla_x^{N_\ast} E(\tau)\| + \|\nabla_x^{N_\ast + 1} E(\tau)\|) \|f(\tau)\|_{L^\infty_x(L^2_p)} \right] d\tau$$
$$+ \left[ (\|\nabla_x^{N_\ast} f(\tau)\| + \|\nabla_x^{N_\ast + 1} f(\tau)\|) \|E(\tau)\|_{L^\infty_x(L^2_p)} \right] d\tau$$
$$+ \int_0^t \left[ (\|\nabla_x^{N_\ast} (E, B)(\tau)\| + \|\nabla_x^{N_\ast + 1} (E, B)(\tau)\|) \|\nabla_p f(\tau)\|_{L^\infty_x(L^2_p)} \right] d\tau$$
$$+ \int_0^t \left[ (\|\nabla_x^{N_\ast} f(\tau)\|_{\nu} + \|\nabla_x^{N_\ast + 1} f(\tau)\|_{\nu}) \|\nu \frac{3}{2} f(\tau)\|_{L^\infty_x(L^2_p)} \right] d\tau.$$ On the other hand, from the Sobolev inequalities
$$\|g(\tau)\|_{L^\infty_x(L^2_p)} \lesssim \|\nabla_x g(\tau)\|_{L^2_p} \|\nabla_x^2 g(\tau)\|_{L^\infty_x(L^2_p)},$$
$$\|\nabla_x^{i} \nabla_p f(\tau)\| \lesssim \|\nabla_x^{i} \nabla_x^{N_\ast - 1} f(\tau)\|^{\frac{i}{N_\ast - 1}} \|\nabla_x^{i} f(\tau)\|^{\frac{N_\ast - 1 - i}{N_\ast - 1}}, \quad i < N - 1,$$ and the definition of $X(t)$, it follows that
$$\|f(\tau)\|_{L^\infty_x(L^2_p)} + \|(E, B)(\tau)\|_{L^\infty_x(L^2_p)} \lesssim (1 + \tau)^{-\frac{3}{2}} X^\frac{3}{2}(t),$$
$$\|\nabla_x^{N_\ast} (f, E, B)(\tau)\| \lesssim (1 + \tau)^{-\frac{3}{2} - \frac{N_\ast}{2}} X^\frac{3}{2}(t).$$
\[ \| \nabla_x^{N+1} (f, E, B)(\tau) \| \lesssim (1 + \tau)^{-\frac{1}{2} - \frac{N(N-2)}{2(N-1)(N-2)}} (N+\frac{1}{2}) \mathcal{X}^\frac{1}{2} (t), \]

\[ \| \nabla_x^{N+1} \nabla_p f(\tau) \| \lesssim \| \nabla_x^N \nabla_p^{N-1} f(\tau) \| \lesssim \mathcal{E}_{N,\ell}^{-\frac{N(N-4)}{2}} (0) (1 + \tau)^{-\frac{N-4}{2} + \frac{N-2}{N-1}} \mathcal{X}^\frac{N-2}{N-1} (t), \]

Continuing, we set
\[ N_\# = \left\lceil \frac{N - 3}{3} \right\rceil, \quad N \geq 9, \]

which yields
\[ \frac{r_{N,1} + 1}{2} \geq \frac{N_\# + 3}{4} + \left[ \frac{1}{2} - \frac{3}{2(N-1)} - \frac{N_\# + 1/2}{2(N-1 - N_\#)} \right] \]

and
\[ \frac{r_{N,1} + 1}{2} \geq \frac{N_\# + 3}{4}. \]

Therefore, we can further bound \( I_{5,3} \) by
\[ I_{5,3} \lesssim \int_{\frac{1}{2}}^t \left[ (1 + \tau)^{-\frac{r_{N,1} + 3}{2} + \frac{12N-17}{8(N-1)(N-2)}} + (1 + \tau)^{-\frac{r_{N,1} + 3}{2} + \frac{12N-17}{8(N-1)(N-2)}} \right] d\tau 
\times \left[ X(t) + \mathcal{E}_{N,\ell}(0) \right]
\lesssim (1 + t)^{-\frac{N_\#}{4}} \left[ X(t) + \mathcal{E}_{N,\ell}(0) \right], \]

for \( N \geq 9 \) and \( N_\# = \left\lceil \frac{N - 3}{4} \right\rceil \). Plugging the estimates of \( I_{5,1}, I_{5,2} \) and \( I_{5,3} \) into (4.24) gives
\[ \| \nabla_x^N U(t) \| \lesssim (1 + t)^{-\frac{1}{2} - \frac{N_\#}{4}} [X(t) + Y_0]. \quad (4.25) \]

Then (4.22) follows for the case \( m = N_\# \).

For the case \( m = 0 \), in a similar way, it is more easier to show that
\[ \| U(t) \| \lesssim (1 + t)^{-\frac{1}{2}} [X(t) + Y_0]. \quad (4.26) \]

Applying (4.25) and (4.26), we get from the interpolation inequality that for 1 \( \leq m \leq N_\# \),
\[ \| \nabla_x^{N-1} U(t) \| \lesssim \| \nabla_x^N U(t) \| \lesssim \| U(t) \|^{\frac{N-m}{N}} \| U(t) \|^{\frac{N-m}{N}} \lesssim (1 + t)^{-\frac{1}{2} - \frac{m}{2N}} [X(t) + Y_0]. \quad (4.27) \]

It remains now to prove (4.22) in the case of \( N_\# + 1 \leq m \leq N - 1 \). To do this, we first establish the following estimate:
\[ \| \nabla_x^{N-1} U(t) \| \lesssim (1 + t)^{-1} [X(t) + \mathcal{E}_{N,\ell}(0)]. \quad (4.28) \]
Moreover, one has

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=N-1} \|\partial^\alpha (f, E, B)(t)\|^2 + \lambda \sum_{|\alpha|=N-1} \|\partial^\alpha \{I-P\} f(t)\|^2 \lesssim \sum_{|\alpha|=N-1} |(\partial^\alpha S, \partial^\alpha f)|$$

$$\lesssim \sum_{|\alpha|=N-1} \left| \left( \partial^\alpha \left\{ \frac{\zeta}{2} E \cdot \bar{p} f \right\}, \partial^\alpha f \right) \right| + \sum_{|\alpha|=N-1} \left| \left( \partial^\alpha \{\Gamma(f, f)\}, \partial^\alpha f \right) \right|$$

$$+ \sum_{|\alpha|=N-1} \left| \left( \partial^\alpha \left\{ \zeta (E + \bar{p} \times B) \cdot \nabla f \right\}, \partial^\alpha f \right) \right|.$$

By Lemmas 2.3, 2.5 and 4.4 and the definition of $D_{N,\ell}(t)$, we have

$$\sum_{|\alpha|=N-1} |(\partial^\alpha \Gamma(f, f), \partial^\alpha f)| \lesssim \|\nabla_x^{N-1} f(t)\|_{L^2_x(L^\infty_t)}^2 \lesssim \sum_{1 \leq i \leq 2} \|\nabla_x^i f(t)\| D_{N,\ell}(t),$$

and

$$\sum_{|\alpha|=N-1} \left| \left( \partial^\alpha \left\{ \frac{\zeta}{2} E \cdot \bar{p} f \right\}, \partial^\alpha f \right) \right|$$

$$\lesssim \left[ \|\nabla_x^{N-1} E(t)\| \|f(t)\|_{L^2_x(L^\infty_t)} + \|\nabla_x^{N-1} f(t)\| \|E(t)\|_{L^\infty_x} \right] \|\nabla_x^{N-1} f(t)\|$$

$$\lesssim \left[ \|\nabla_x^{N-1} E(t)\| \sum_{1 \leq i \leq 2} \|\nabla_x^i f(t)\| + \|\nu^{-\frac{1}{2}} \nabla_x^{N-1} f(t)\| \nu \sum_{1 \leq i \leq 2} \|\nabla_x^i E(t)\| \right]$$

$$\times \|\nu^{-\frac{1}{2}} \nabla_x^{N-1} f(t)\| \nu$$

$$\lesssim \sum_{1 \leq i \leq 2} \|\nabla_x^i (f, E)(t)\| D_{N,\ell}(t).$$

Moreover, one has

$$\sum_{|\alpha|=N-1} \left| \left( \partial^\alpha \left\{ \zeta (E + \bar{p} \times B) \cdot \nabla f \right\}, \partial^\alpha f \right) \right|$$

$$\lesssim \left[ \|\nabla_x^{N-1} (E, B)(t)\| \|\nabla_p f(t)\|_{L^2_x(L^\infty_t)} + \|\nabla_x^{N-2} \nabla_p f(t)\| \|\nabla_x (E, B)(t)\|_{L^\infty_x} \right]$$

$$\times \|\nabla_x^{N-1} f(t)\|$$

$$\lesssim \sum_{1 \leq i \leq 2} (\|\nabla_x^i \nabla_p f(t)\| + \|\nabla_x^i (E, B)(t)\|) D_{N,\ell}(t).$$

Combining the above estimates, we have

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=N-1} \|\partial^\alpha (f, E, B)(t)\|^2 + \lambda \sum_{|\alpha|=N-1} \|\partial^\alpha \{I-P\} f(t)\|^2 \lesssim \sum_{1 \leq i \leq 2} \left( \|\nabla_x^i \nabla_p f(t)\| + \|\nabla_x^i (f, E, B)(t)\| \right) D_{N,\ell}(t)$$
Proof. For $0 < \rho < 1$ and $0 \leq m \leq N_* - 1$.

\begin{align*}
\left\| \nabla_x^m f(t) \right\|_2^2 &\lesssim \left\| \nabla_x^m [(I - P) f(t)] \right\|_2^2 \\
&\lesssim \left\| \nabla_x^m [(I - P) f(t)] \right\|_2^2 + \left\| \nabla_x^m E(t) \right\|_2^2 + \lambda \left\| \nabla_x^m [(I - P) f(t)] \right\|_2^2 \\
&\lesssim \sum_{|\alpha| = m} |\langle \alpha, \nabla_x^m [(I - P) f(t)] \rangle| \\
&\quad + \sum_{|\alpha| = m} |\langle \nabla_x \cdot \nabla_x^m [(I - P) f(t)] \rangle| \\
&\quad + \sum_{|\alpha| = m} |\langle \nabla_x \times \nabla_x^m [(I - P) f(t)] \rangle| \\
&\quad + \sum_{|\alpha| = m} |\langle \nabla_x \cdot \nabla_x^m [(I - P) f(t)] \rangle| \\
&\quad + \sum_{|\alpha| = m} |\langle \nabla_x \times \nabla_x^m [(I - P) f(t)] \rangle|,
\end{align*}

Multiplying this inequality by $(1 + t)$ yields

\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| = N - 1} (1 + t) \left\| \partial^\alpha (f, E, B)(t) \right\|_2^2 \\
&\quad + \lambda \sum_{|\alpha| = N - 1} \left\| \partial^\alpha \{ I - P \} f(t) \right\|_2^2 \\
&\lesssim \sum_{|\alpha| = N - 1} \left\| \partial^\alpha (f, E, B)(t) \right\|_2^2 + (1 + t)^{-\frac{N}{2}} X^{\frac{1}{2}}(t) D_{N,\ell}(t) \lesssim D_{N,\ell}(t).
\end{align*}

We integrate this inequality over $[0, t]$ for $0 \leq t \leq T$ and use (4.21) to obtain (4.28). Once (4.28) is obtained, we then get from (4.25) that

\begin{align*}
\left\| \nabla_x U(t) \right\|_2^2 &\lesssim \left\| \nabla_x U(t) \right\|_2^2 \lesssim \left\| \nabla_x U(t) \right\|_2^2 + \left\| \nabla_x U(t) \right\|_2^2 \\
&\lesssim (1 + t)^{-\frac{1}{2}} \left\| \nabla_x U(t) \right\|_2^2 + \left\| \nabla_x U(t) \right\|_2^2,
\end{align*}

(4.29) together with (4.27) gives (4.22), this completes the proof of Lemma 4.5. \(\square\)

4.5. Sharper decay of the electric field and the microscopic part. In this subsection, we are concerned with the sharper time-decay of the electric field and the microscopic part of the solution $f$. The main result of this subsection can be stated as follows:

Lemma 4.6. Suppose $\ell \geq \max \left\{ N - 1 + \frac{l_1 + 1}{2}, N_* - 1 + \frac{2N + 3}{4(1 - \rho)} \right\}$ with $l_1 > N_* + \frac{3}{2} = \left[ \frac{N - 3}{4} \right] + \frac{3}{2}$ and $N \geq 9$. For $0 \leq t \leq T$, it holds that

\begin{align*}
\left\| \nabla_x^m [(I - P) f(t)] \right\|_2^2 \lesssim X^2(t) + Y_0^2 \left( 1 + t \right)^{-(m + \frac{1}{2} + \rho)},
\end{align*}

(4.30)

where $0 < \rho < 1$ and $0 \leq m \leq N_* - 1$.

Proof. For $|\alpha| = m$ with $0 \leq m \leq N_* - 1$, apply $\partial^\alpha$ of (4.11) and (4.1) and take the direct energy estimate to have

\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \left\| \nabla_x^m (I - P) f(t) \right\|_2^2 + \left\| \nabla_x^m E(t) \right\|_2^2 \right] + \lambda \left\| \nabla_x^m (I - P) f(t) \right\|_2^2 \\
\lesssim \sum_{|\alpha| = m} \left| \langle \alpha, \nabla_x^m (I - P) f(t) \rangle \right| \\
+ \sum_{|\alpha| = m} \left| \langle \nabla_x \cdot \nabla_x^m (I - P) f(t) \rangle \right| \\
+ \sum_{|\alpha| = m} \left| \langle \nabla_x \times \nabla_x^m (I - P) f(t) \rangle \right|.
\end{align*}

(4.31)
where we used Lemma 2.6. For the second term and third term on the R.H.S. of (4.31), we use Lemma 4.5 to get
\[
\sum_{|\alpha|=m} |(P (\tilde{\partial} \cdot \nabla_x \tilde{\partial}^\alpha f) - \tilde{\partial} \cdot \nabla_x \tilde{\partial}^\alpha P f, \tilde{\partial}^\alpha \{ I - P \} f)|
\leq C_\eta \| \nabla_x^{m+1} f(t) \|_2^2 + \eta \| \nabla_x^m \{ I - P \} f(t) \|_2^2
\leq \eta \| \nabla_x^m \{ I - P \} f(t) \|_2^2 + C_\eta (1 + t)^{-m - \frac{3}{2}} \left[ X^2(t) + Y_0^2 \right],
\]
for any \( \eta > 0 \). And for the first term on the R.H.S. of (4.31), we obtain from Lemmas 2.6, 4.4 and 4.5 that
\[
\sum_{|\alpha|=m} |(\tilde{\partial}^\alpha \{ I - P \} S, \tilde{\partial}^\alpha \{ I - P \} f)|
\leq \| f(t) \|_{L^\infty_x(L^2_t)} \| \nabla_x^m f(t) \|_2 \| \nabla_x^m \{ I - P \} f(t) \|_2
+ \left[ \| f(t) \|_{L^\infty_x(L^2_t)} \| \nabla_x^m E(t) \|_2 + \| E(t) \|_{L^\infty_x} \| \nabla_x^m f(t) \|_2 \right] \| \nabla_x^m \{ I - P \} f(t) \|_2
+ \| \nabla_x^m \{ I - P \} f(t) \|_2 \left[ \| \nabla_x^m (E, B)(t) \|_2 \| \nu^{-\frac{3}{2}} \nabla_x^m f(t) \|_{L^\infty_x(L^2_t)} \right]
+ \chi_{m\geq 1} \| \nu^{-\frac{3}{2}} \nabla_x^m \nabla_x f(t) \|_2 \| \nabla_x (E, B)(t) \|_{L^\infty_x}
+ \chi_{m\geq 1} \| \nu^{-\frac{3}{2}} \nabla_x^{m-1} \nabla_x P f(t) \|_2 \| \nabla_x (E, B)(t) \|_{L^\infty_x}
\leq (1 + t)^{-r_m - \frac{3}{2}} \left[ X^2(t) + Y_0^2 \right] + \eta \| \nabla_x^m \{ I - P \} f(t) \|_2^2
+ C_\eta (1 + t)^{-m - \frac{3}{2}} \left[ X^2(t) + Y_0^2 \right]
+ C_\eta \chi_{m\geq 1} (1 + t)^{-r_m - \frac{3}{2}} \left[ X^2(t) + Y_0^2 \right].
\]
Here we used the following estimates:
\[
\| \nu^{-\frac{3}{2}} \nabla_x \nabla_x f(t) \|_2 \| \nu^{-\frac{3}{2}} \nabla_x^2 f(t) \|
\leq \| \nu^{-N} \nabla_x^{2N} \nabla_x f(t) \|_2 \| \nabla_x f(t) \|_2^{\frac{2N - 1}{2N - 2}} \| \nabla_x f(t) \|_2^{\frac{2N - 1}{2N - 2}}
\leq \| \nu^{-N} \nabla_x^{2N} \nabla_x f(t) \|_2 \| \nabla_x f(t) \|_2^{\frac{2N - 1}{2N - 2}} \| \nabla_x f(t) \|_2^{\frac{2N - 1}{2N - 2}}
\leq (1 + t)^{-\frac{3}{2}} X(t),
\]
\[
\chi_{m\geq 1} \| \nu^{-\frac{3}{2}} \nabla_x^{m-1} \nabla_x (I - P) f(t) \|_2 \| \nabla_x (E, B)(t) \|_{L^\infty_x}
\leq |\nu^{-N} \nabla_x^{2N} \nabla_x^{m-1} (I - P) f(t) \|_2 \| \nabla_x^{m-1} (I - P) f(t) \|_2^{\frac{2N - 1}{2N - 2}} \| (1 + t)^{-3 - \frac{3}{2}} X(t)
\leq (1 + t)^{-\frac{2N - 1}{2N - 2} (r_m - 1) + \psi - \frac{3}{4}} X^2(t) \leq (1 + t)^{-r_m - \psi - \frac{3}{4}} X^2(t),
\]
and
\[
\chi_{m\geq 1} \| \nu^{-\frac{3}{2}} \nabla_x^{m-1} \nabla_x P f(t) \|_2 \| \nabla_x (E, B)(t) \|_{L^\infty_x}
\leq (1 + t)^{-r_m - \frac{3}{2} - \frac{3}{4}} X^2(t) \leq (1 + t)^{-r_m - \frac{3}{2} - \frac{3}{4}} X^2(t),
\]
for \( 0 < \psi < 1 \) and \( N_* \geq 2 \).
Therefore, we can take \( \eta \) small enough to further estimate (4.31) by

\[
\frac{d}{dt} \left[ \|\nabla_x^m \{I - P\} f(t)\|^2 + \|\nabla_x^m E(t)\|^2 \right] + \lambda \|\nabla_x^m \{I - P\} f\|^2 \leq \eta \|\nabla_x^m E(t)\|^2 + C \eta (1 + \tau)^{-m-\frac{3}{2}} \left[ Y_0^2 + X^2(t) \right],
\]

(4.32)

On the other hand, it follows from (3.3) that for \( 0 \leq m \leq N_* - 1 \),

\[
2 \left| C_{21} - \frac{C_{11}}{C_{01}} \right|^2 \|\nabla_x^m E(t)\|^2 + \left| C_{21} - \frac{C_{11}}{C_{01}} \right|^2 \|\nabla_x^m (a_+ - a_-) (t)\|^2
\]

\[
= - \frac{d}{dt} \sum_{1 \leq i \leq 3} \left( \Lambda_i(\{I - P\} \nabla_x^m f) - \Lambda_i(\{I - P\} \nabla_x^m f) \right) + C \sum_{1 \leq i \leq 3} \left( \Lambda_i(\{I - P\} \nabla_x^m f) - \Lambda_i(\{I - P\} \nabla_x^m f) \right)
\]

\[
- \sum_{1 \leq i \leq 3} \left( \nabla_x^m \{\Lambda_i(r_+ + S_+)\} - \nabla_x^m \{\Lambda_i(r_- + S_-)\} \right).
\]

(4.33)

Now we use the terms in (4.33). For \( 0 \leq m \leq N_* - 1 \), we use (4.1) and Lemma 4.5 to bound the second term in the R.H.S. of (4.33) by

\[
\frac{1}{2} \left| C_{21} - \frac{C_{11}}{C_{01}} \right|^2 \|\nabla_x^m E(t)\|^2 + C \|\nabla_x^m \{I - P\} f\|^2 + \|\nabla_x^{m+1} f\|^2
\]

(4.34)

Noticing the definitions of \( S \) and \( r \), we use Lemma 2.6 and Lemma 4.5 to bound the third term in the R.H.S. of (4.33) by

\[
\frac{1}{2} \left| C_{21} - \frac{C_{11}}{C_{01}} \right|^2 \|\nabla_x^m E(t)\|^2 + C \|\nabla_x^m \{I - P\} f\|^2
\]

\[
+ C \|\nabla_x^{m+1} f\|^2 + \|E, B\| \leq L^2 \left( \frac{1}{2} \right)
\]

(4.35)

Plugging (4.34) and (4.35) into (4.33) gives

\[
\frac{d}{dt} \sum_{1 \leq i \leq 3} \left( \Lambda_i(\{I_+ - P_+\} \nabla_x^m f) - \Lambda_i(\{I_- - P_-\} \nabla_x^m f) \right) + C \sum_{1 \leq i \leq 3} \left( \Lambda_i(\{I_+ - P_+\} \nabla_x^m f) - \Lambda_i(\{I_- - P_-\} \nabla_x^m f) \right)
\]

\[
- \sum_{1 \leq i \leq 3} \left( \nabla_x^m \{\Lambda_i(r_+ + S_+)\} - \nabla_x^m \{\Lambda_i(r_- + S_-)\} \right).
\]

(4.36)

Making a proper linear combination of (4.32) and (4.36), and letting \( \eta \) sufficiently small, we have

\[
\frac{d}{dt} E_m(\{I - P\} f, E)(t) + \lambda \left( \|\nabla_x^m \{I - P\} f\|^2 + \|\nabla_x^m E(t)\|^2 \right)
\]

\[
\leq \eta \|\nabla_x^m E(t)\|^2 + C \eta (1 + \tau)^{-m-\frac{3}{2}} \left[ Y_0^2 + X^2(t) \right].
\]
where
\[ \mathcal{E}_m((I - P)f, E)(t) \sim \|\nabla_x^m (I - P)f(t)\|^2 + \|\nabla_x^m E(t)\|^2. \]

Let \(0 < \rho < 1\), define the low-velocity domain and the corresponding high-velocity domain:
\[ \Sigma(t) = \{p_0 \leq t^{1-\rho}\}, \quad \Sigma^c(t) = \{p_0 > t^{1-\rho}\}. \]

Then it follows that
\[ \frac{d}{dt}\mathcal{E}_m((I - P)f, E)(t) + \lambda t^{\rho-1}\mathcal{E}_m((I - P)f, E)(t) \]
\[ \lesssim (1 + t)^{-m-\frac{2}{5}} \left[ Y_0^2 + X^2(t) \right] + \lambda t^{\rho-1}\|\chi_{\Sigma(t)}\nabla_x^m (I - P)f(t)\|^2, \]
which implies
\[ \|\nabla_x^m (I - P)f(t)\|^2 + \|\nabla_x^m E(t)\|^2 \]
\[ \lesssim e^{-t^\rho} \left[ \|\nabla_x^m (I - P)f(t)\|^2 + \|\nabla_x^m E(t)\|^2 \right] \]
\[ + \int_0^t e^{t^\rho - t^\rho} (1 + \tau)^{-m-\frac{2}{5}} d\tau \left[ Y_0^2 + X^2(t) \right] \]
\[ + \int_0^t e^{t^\rho - t^\rho} \tau^{\rho-1}\|\chi_{\Sigma(t)}\nabla_x^m (I - P)f(\tau)\|^2 d\tau. \]

We now set
\[ \ell \geq \max \left\{ N - 1 + \frac{l_1 + 1}{2}, N_\ast - 1 + \frac{2N_\ast + 3}{4(1 - \rho)} \right\}, \]
then
\[ \|\chi_{\Sigma(t)}\nabla_x^{2N_\ast+\frac{1}{5}} \nabla_x^{N_\ast-1} (I - P)f(t)\|^2 \leq \|w_{l-(N_\ast-1)} \nabla_x^{N_\ast-1} (I - P)f(t)\|^2. \]

We get from the above inequality, (4.21) and Lemma 4.5 that
\[ \|\chi_{\Sigma(t)}\nabla_x^m (I - P)f(t)\|^2 \lesssim \tau^{-m-\frac{2}{5}} \|\chi_{\Sigma(t)}\nabla_x^{2m+\frac{3}{5}} \nabla_x^m (I - P)f(t)\|^2 \]
\[ \lesssim \tau^{-m-\frac{2}{5}} \left[ \|\chi_{\Sigma(t)}\nabla_x^{2m+\frac{3}{5}} \nabla_x^m (I - P)f(t)\|^2 \right] \]
\[ \lesssim \tau^{-m-\frac{2}{5}} \mathcal{E}_{N, \ell}(0), \]
for \(m \leq N_\ast - 1\). Plugging (4.38) into (4.37), we arrive at
\[ \|\nabla_x^m (I - P)f(t)\|^2 + \|\nabla_x^m E(t)\|^2 \lesssim (1 + t)^{-m-\frac{2}{5}} \left[ Y_0^2 + X^2(t) \right], \]
which gives the desired estimate (4.30). This completes the proof of Lemma 4.6. \(\square\)

4.6. **Global existence.** We are now in a position to complete the proof of Theorem 1.1. Recall \(X(t)\)-norm (1.19). From (4.21), (4.22) and (4.30), it follows that
\[ X(t) \lesssim Y_0^2 + X^2(t). \]
Since \(Y_0\) is sufficiently small, (1.20) holds true. The global existence follows further from the local existence (cf. [32, 21]) and the continuity argument in the usual way. This completes the proof of Theorem 1.1.
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