AN ELEMENTARY PROOF OF THE 3 DIMENSIONAL SIMPLEX MEAN WIDTH CONJECTURE

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Abstract. After a Hessian computation, we quickly prove the 3D simplex mean width conjecture using classical methods in section 2. Then, we generalize some components to $d$ dimensions in section 4.

1. Introduction

Let $B^d_2 \subset \mathbb{R}^d$ be the standard Euclidean ball, with basis $(e_1, \ldots, e_d)$. Denote $\mathbb{S}^{d-1} := \partial B^d_2$. Let $\delta(X, Y)$ be the arclength on $\mathbb{S}^{d-1}$ and $\mu$ be the uniform probability measure on $\mathbb{S}^{d-1}$.

The support function of a convex body $K \subset B^d_2$ is

$$h_K(u) := \max_{x \in K} u \cdot x$$

and the mean width

$$w(K) := 2 \int_{\mathbb{S}^{d-1}} h_K(u) d\mu(u)$$

Conjecture 1.1. (Simplex Mean Width Conjecture) Of all simplices contained in $B^d_2$, the inscribed regular simplex has the maximum mean width, and is unique up to isometry.

Conjecture 1.1 was mentioned in the survey by Gritzmann and Klee [4] and by Klee several times in his talks. Litvak surveyed the problem more recently in arXiv:math/0606350 [math.DG]. It’s related to the problem of recovering transmissions from a noisy signal and has been assumed to be true by information theorists [2, 3]. The $d + 1$ dimensional Gaussian Random Vector Maximum Conjecture is equivalent and was proved in 4 dimensions arXiv:2008.04827v2 [math.PR]. In section 2 we give a simpler and more classical proof. The following claim will be useful throughout.

Claim 1.2. In order that $\Delta := \text{conv}(v_0, \ldots, v_d)$ maximizes $w(\cdot)$ over all simplices contained in $B^d_2$, it must be that

(a) $B^d_2$ is the smallest ball containing $\Delta$.
(b) $v_i \in \mathbb{S}^{d-1}$ for $0 \leq i \leq n$, i.e. $\mathbb{S}^{d-1}$ is the circumsphere of $\Delta$.

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(c) The closed hemispheres centered at \( v_i \) cover \( S^{d-1} \).

Proof. (a) If \( \triangle \subset Q + rB_2^d \) with \( r < 1 \), then \((\triangle - Q)/r\) is a simplex contained in \( B_2^d \) and
\[
w((\triangle - Q)/r) = w(\triangle/r) = w(\triangle)/r > w(\triangle)
\]
(b) Suppose that \(|v_0| < 1\). Extend the edge \( v_1v_0 \) past \( v_0 \) until it meets \( S^{d-1} \) at a point \( v_{n+1} \). The simplex \( \tilde{\triangle} := \text{conv}\{v_1, \ldots, v_{n+1}\} \subset B_2^d \) contains \( v_0 \) as a convex combination of \( v_{n+1} \) and \( v_1 \), so it strictly contains \( \triangle \). Since \( w \) is strictly increasing under set inclusion, \( w(\triangle) < w(\tilde{\triangle}) \). Therefore, if \(|v_i| < 1 \) for any \( i \), then \( w(\triangle) \) is not maximum.

(c) Suppose \( v \in S^{d-1} \) is such that \( \max v \cdot v_i = m < 0 \) for all \( i \). It follows that each \( v_i \) is contained in the spherical cap \( \{v \cdot x \leq m\} \), which has diameter less than 1. Part (a) implies there is no such \( v \).

The support function of the simplex \( \triangle = \text{conv}\{v_0, \ldots, v_d\} \) reduces to a maximum over the vertex set:
\[
\max_{x \in \triangle} u \cdot x = \max_{0 \leq i \leq d} u \cdot v_i
\]
As such, for each \( i \) define the \( i \)th Voronoi cell on the sphere to be
\[
S_i := \{u \in S^{d-1} : u \cdot v_i = \max_{x \in \triangle} x \cdot u\}
\]
The mean width of a simplex is then
\[
w(\triangle) = 2 \int_{S^{d-1}} \max_{0 \leq i \leq d} X \cdot v_i \, d\mu(X)
\]
\[
= 2 \sum_{i=0}^{d} \int_{S_i} X \cdot v_i \, d\mu(X) 
\]
(1.1)
Now define
\[
M_A(R) := \int_R X \cdot A \, d\mu(X)
\]
(1.2)
2. SMWC DIMENSION 3
**Theorem 2.2** (Napier). Suppose \( \triangle ABC \) is a right spherical triangle. Then,

\[
(2.1) \quad \cos c = \cos a \cos b \\
(2.2) \quad \cos A = \cos a \sin B \\
(2.3) \quad \cos A = \tan b \cot c \\
(2.4) \quad \tan a = \tan A \sin b
\]

and, of course, the same if \((A, a)\) and \((B, b)\) are interchanged.

Claim 2.3 gives a parametrization of an arc as measured from the opposite vertex and Theorem 2.4 a simple formula reminiscent of the area of a Euclidean triangle.

**Claim 2.3.** Let \( \triangle ABC \subset S^2 \) be a right spherical triangle with \( A = (1, 0, 0) \) and the right angled vertex \( C = (\cos b, \sin b, 0) \). Side \( a \) is parametrized by \((\cos \Phi(\theta), \cos \theta \sin \Phi(\theta), \sin \theta \sin \Phi(\theta))\) as \( \theta \in [0, A] \) where

\[
\sin^2 \Phi(\theta) = \frac{\tan^2 b}{\tan^2 b + \cos^2 \theta}
\]

**Proof.** Napier’s rule 2.3 says \( \cos \theta = \tan b \cot \Phi(\theta) \), so

\[
\sin^2 \Phi(\theta) = \frac{1}{1 + \cot^2 \Phi(\theta)} = \frac{\tan^2 b}{\tan^2 b + \cos^2 \theta}
\]

\[\Box\]

**Theorem 2.4.** Let \( \triangle ABC \) be a right spherical triangle with \( C = \pi/2 \). Then,

\[
4\pi \sigma(\triangle ABC) M_A(\triangle ABC) = \frac{1}{2} a \sin b
\]

**Proof.** Set \( A = (1, 0, 0) \). Uniform measure on \( S^2 \) is \( d\mu = (4\pi)^{-1} \sin \varphi d\varphi d\theta \), giving
\[8\pi \int_{\triangle ABC} X \cdot A d\mu(X) = \int_0^A \int_0^{\Phi(\theta)} \cos \varphi \sin \varphi d\varphi d\theta\]
\[= \int_0^A \sin^2 \Phi(\theta) d\theta\]
\[= \int_0^A \tan^2 b/(\tan^2 b + \cos^2 \theta) d\theta\]
\[= \tan^2 b \int_0^A \sec^2 \theta / (1 + \tan^2 b + \tan^2 b \tan^2 \theta) d\theta\]
\[= \tan^2 b \int_0^{\tan A} 1/(\sec^2 b + u^2 \tan^2 b) du\]
\[= \sin b \int_0^{\tan A \sin b} 1/(1 + u^2) du\]
\[= \sin b (\tan^{-1}(\sin b \tan A))\]
\[= a \sin b\]

Where Claim 2.3 was used in the third line and the last line follows from Napier’s rule 2.4.

\[\square\]

**Remark 2.5.** Denote the area of \(\triangle ABC\) as \([ABC]\). The centroid \(G\) satisfies

\([ABC]G = \int_{\triangle ABC} X d\mu(X)\)

Let \(d\) be the length of the altitude from angle \(A\) to side \(a\). Either \(\triangle ABC\) is the union of two right triangles with common side \(d\), or the difference. Either way, use the previous claim for the second line and Theorem 2.1 in the third of the following,

\[2[ABC](G \cdot A) = 2M_A(\triangle ABC)\]
\[= a \sin d\]
\[= a \sin b \sin C\]

Since these quantities are independent of coordinates, the two other cases \((A \to B \to C)\) follow similarly. If \(A, B, C\) have no common great circle, \(G\) is the unique point satisfying these equations. The formula

\[2[ABC]G \det(A, B, C) = \sum_{\text{cyc}} (B \times C) a \sin b \sin C\]

is verified by taking dot products with each of \(A, B, C\) separately. Recall the triple product \(\det(A, B, C) = A \times B \cdot C\). Now, arbitrarily putting \(A = (1, 0, 0)\) and \(C_2 = 0\) and \(\sigma(\triangle ABC) = 1\), we see that \(C \times B \cdot A = C_1 B_2 = \)
\[ \sin b(\sin c \sin A), \text{ the volume of the zonotope generated by } A, B, C, \text{ is the ubiquitous quantity } n \text{ associated to a spherical triangle in } [6]. \text{ Dividing the equation } \text{ thru by this quantity becomes Brock’s formula } [7]. \]

\[ 2[ABC]G = \sum_{\text{cyc}} (B \times C) \frac{a}{\sin a} \]

Consider a right spherical triangle \( \triangle \) with \( C = \pi/2 \). We use equation 2.2 to convert the formula from Theorem 2.4 in terms of \( A \) and \( B \).

**Theorem 2.6.** The function defined by

\[ f(A, B) := a \sin b = \left( \cos^{-1} \frac{\cos B}{\sin A} \right) \left( 1 - \frac{\cos^2 A}{\sin^2 B} \right)^{1/2} \]

is negative definite (i.e. \( -f \) is convex) in the region

\[ R := \{-\pi/2 < A, B < \pi/2\} \cap \{\cos^2 A + \cos^2 B < 1\} \]

*Proof.* See section 5 □

**Theorem 2.7.** The Simplex Mean Width Conjecture is true for \( d = 3 \)

*Proof.* Let the orientation of a spherical triangle be \( \sigma(\triangle ABC) = +1 \) for a counterclockwise ordering of \( A, B, C \) and \(-1 \) for clockwise. If \( \sigma(\triangle ABC) = -1 \), measure both the angles and edges as negative.

Suppose \( \triangle = \text{conv}(v_0, \ldots, v_3) \) is any simplex with \( |v_i| = 1 \) for each \( i \) (by Claim 1.2). The Voronoi cells

\[ S_i(\triangle) := \{ u \in \mathbb{S}^2 : i \in \text{argmax}_i u \cdot v_i \} \]

are each the intersection of three hemispheres and so are spherical triangles tiling \( \mathbb{S}^2 \). For each \( i \), label the vertices of \( S_i \) as \( A, B, C \) (suppressing subscript \( i \)) in the counterclockwise orientation. Draw arcs from \( v_i \) to \( A, B, C \) and drop altitudes from \( v_i \) to the edges \( a, b, c \) bounding \( S_i \) (they may leave the interior of \( S_i \)) with feet \( D, E, F \) on \( a, b, c \) (resp.). Form the collection \( T \) of right triangles \( \triangle v_i XY \) such that \( X, Y \) come from the list \( A, F, B, D, C, E \) and \( Y \) is immediately to the right of \( X \). Since

\[ \sigma(\triangle XYZ)1_{\triangle XYZ} + \sigma(\triangle XZW)1_{\triangle XZW} = \sigma(\triangle XYW)1_{\triangle XYW} \]

we can write

\[ M_{v_i}(S_i) = \sum_{T \in T} \sigma(T)M_{v_i}(T) \]

Also, the signed sum of the six angles meeting at \( v_i \) is \( 2\pi \) while the signed sum of the other six non-right angles equals the sum of the angles of \( S_i \), which exceeds the area of \( S_i \) by \( \pi \) (Girard’s Theorem). It follows that the
whole of $\mathbb{S}^2$ may be divided into 24 right triangles where the signed sum of the 24 angles measured at $\{v_0, \ldots, v_3\}$ is $4*2\pi$ and the sum of the other 24 non-right angles is $4\pi + 4\pi$. To see that none of these 24 right triangles contains an angle exceeding $\pi/2$, note that the spherical Pythagorean (2.1) implies there would be two such edges from the same triangle, in particular one meeting at a vertex $v_i$. But, from (Claim. 1.2(c)) every point in $\mathbb{S}^2$ is within $\pi/2$ from a vertex $v_i$. As no edge exceeds $\pi/2$, neither does any angle. Theorems 2.4 and 2.6 imply that the maximum mean width occurs when all these angles are equal, i.e. a regular tetrahedron. □

3. Integration on Spheres

To generalize Theorem 2.4 we need to generalize a right triangle to a right angled simplex, called a path simplex, on higher dimensional spheres. Then, we find the marginal mean of an RV uniform over the simplex. Let us start with the measure of spherical caps. Denote a spherical cap of geodesic radius $r$, centered at $e_1$, by $B^{d-1}_S(r) \subset \mathbb{S}^{d-1}$. Define the incomplete Wallis integral by

$$W^d(x) = \int_0^x \sin^d t dt$$

and denote $W^d := W^d(\pi)$.

**Theorem 3.1.** The uniform measure of a spherical cap is

$$\mu(B^{d-1}_S(r)) = \frac{W^{d-2}(r)}{W^{d-2}}$$

where

$$W^d(r) = \frac{d-1}{d}W^{d-2}(r) - \frac{1}{d}\cos r \sin^{d-2} r$$

**Proof.** Cut the sphere into thin slices perpendicular to the radius at the center of the cap. The cross section at geodesic radius $\varphi$ is $\sin \varphi \cdot \mathbb{S}^{d-2}$, so the measure of a thin slice is proportional to $\sin^{d-2} \varphi d\varphi$, and equation 3.1 follows.

For the recursion formula, consider $(\sin^d \varphi)^n$ for $d \geq 2$:

$$(d \cos \varphi \sin^{d-1} \varphi)' = d(d-1) \cos^2 \varphi \sin^{d-2} \varphi - n \sin^d \varphi$$

$$= d(d-1) \sin^{d-2} \varphi - d^2 \sin^d \varphi$$

Integrating gives 3.2. □

**Remark 3.2.** Equation 3.2 may be used to find the trigonometric series for $W^d(r)$. Note that $W^0(r) = r$ and $W^1(r) = 1 - \cos r$. The denominator of 3.1 is the well known Wallis Integral. When $r = \pi$, we can multiply equation 3.2 through by $W^{d-1}$ to get
\[ d \cdot W^d W^{d-1} = (d - 1) W^{d-1} W^{d-2} \]

which shows
\[ d \cdot W^d W^{d-1} = W^1 W^0 = 2\pi \]

for all \( d \). Since \( W^d \) is decreasing with \( d \), we come upon the bound (see also [9]),
\[
\sqrt{\frac{2\pi}{d+1}} < W^d < \sqrt{\frac{2\pi}{d}} \tag{3.3}
\]

**Corollary 3.3.** Averaging the marginal of an RV uniform over a cap,
\[
M_{e_1}(B_{S^d-1}(r)) = \int_0^r \cos t \, d\mu(B_{S^{d-1}}(t))
\]
\[
= \int_0^r \cos t \sin^{d-2} t dt
\]
\[
= \frac{\sin^{d-1} r}{W^{d-2}}
\]

Now, fix an axis \( A \in S^{d-1} \) and parametrize \( S^{d-1} \) by \( X = (r, \theta) \in [0, \pi] \times S^{d-2} \) where \( \cos r = X \cdot A \) and
\[
\theta = \frac{X - A \cos r}{|X - A \cos r|}
\]

**Corollary 3.4.** Let a region \( R \subset S^{d-1} \) be defined by \( 0 \leq r \leq \varphi(\theta) \) under the above parameterization for some measurable \( \varphi \). Then
\[
M_A(R) = \int_{S^{d-2}} \frac{\sin^{d-1} \varphi(\theta)}{(d-1) W^{d-2}} \, d\mu(\theta)
\]

When this region is a spherical simplex, say \( T \subset S^{d-1} \), it is the intersection of \( d \) hemispheres. Arrange the vertices as columns in the matrix \( V \). We describe \( T \) with matrix \( H \in \mathbb{R}^{d \times d} \), whose rows are the centers of the \( d \) hemispheres, ordered so that the \( i \)th vertex is in the interior of the \( i \)th hemisphere. Then \( x/|x| \in T \) iff
\[
Hx \geq 0
\]

Since each vertex \( v_i \) is on the boundary of the other \( d - 1 \) hemispheres, \( H \) is a diagonal multiple of \( V^{-1} \). Further, if \( e_1 \) is a vertex, we may assume the first column of \( H \) is \((1, 0, 0, \ldots, 0)^t\). Also, \( T \) is a union of arcs originating at \( e_1 \), and \( T \) has a parameterization \( 0 \leq r \leq \varphi(\theta) \) as in Corollary 3. To find \( \varphi(\theta) \), note that the first row of \( H \) describes the facet opposite \( e_1 \).
\[
\cos \varphi(\theta) + \theta \cdot (H_{12}, \ldots, H_{1d}) \sin \varphi(\theta) = 0
\]
giving
\[
\tan \varphi(\theta) = -\frac{1}{\theta \cdot (H_{12}, \ldots, H_{1d})}
\]
or
\[
\sin^2 \varphi(\theta) = \frac{1}{1 + (\theta \cdot (H_{12}, \ldots, H_{1d}))^2}
\]

The support \( \tilde{T} \) of \( \varphi \) is the simplex described by omitting the first row and column of \( H \). Applying Corollary 3, we get
\[
M_{e_1}(T) = \frac{1}{(d-2)W_{d-2}} \int_{\tilde{T}} (1 + (\theta \cdot (H_{12}, \ldots, H_{1d}))^2)^{(1-d)/2} d\mu(\theta)
\]

4. SMWC For \( d > 3 \)

In this section, we assume the set of points \( V = \{v_0, \ldots, v_d\} \subset S^{d-1} \) is in general position so that the feet of the altitudes to a face never land in the boundary of that face.

**Definition 4.1.** A path simplex in \( S^{d-1} \) is a spherical simplex with \( d \) edges forming a path such that any pair of them determine great circles meeting at right angles. The endpoints of the path are called end vertices.

**Definition 4.2.** Let \( T \) be a spherical simplex with vertices \( \{p_1, \ldots, p_d\} \) and inward facing normals \( \{g_1, \ldots, g_d\} \) corresponding to the opposite facets. Let \( G_T = [g_1 \ldots g_d] \) and the (angle) Gram matrix be \( G_T^t G_T \).

Note, many authors use \( 2I_{d \times d} - G_T \) as the Gram matrix. It is known that \( G \) determines \( T \) up to isometry, e.g. [8].

**Claim 4.3.** If \( T \) is a path simplex in \( S^{d-1} \), it has a tridiagonal Gram matrix.

**Proof.** Say the vertices along the path are \( p_1, \ldots, p_d \), in order. The \( i \)th row of \( P^{-1} = [p_1, \ldots, p_d]^{-1} \) is normal to \( F_i \), so there is a diagonal matrix \( D \) with
\[
G = (DP^{-1})(DP^{-1})^t = D(P^t P)^{-1} D
\]
The path simplex condition may be restated as the arc \( p_i p_{i+1} \) must be orthogonal to span\( \{p_{i+1}, \ldots, p_d\} \) for \( 1 \leq i \leq d - 1 \). That is, the vector \( p_i - (p_i \cdot p_{i+1})p_{i+1} \) tangent to the arc at \( p_{i+1} \) is orthogonal to each of \( p_{i+1}, \ldots, p_d \). It follows that
is upper triangular. Substitute this into equation 4.1. Since $G$ is symmetric, $G$ must be tridiagonal. □

Claim 4.4. Let $\triangle$ be spherical simplex in $\mathbb{S}^{d-1}$. For almost every $O \in \mathbb{S}^{d-1}$, there is a collection $\mathcal{T}$ of $d!$ path simplices with end vertex $O$ and a sign $\sigma : \mathcal{T} \to \{-1, 1\}$ satisfying

$$1_\triangle = \sum_{T \in \mathcal{T}} \sigma(T) 1_T \quad (a.e.)$$

Proof. Proceed by induction; $d = 2$ is trivial. Suppose the lemma is true at level $d$, and we will show it is also true at the level $d + 1$.

Let $\triangle \subset \mathbb{S}^d$ be a simplex. Drop an altitude from $O \in \mathbb{S}^{d-1}$ to any facet $F$ of $\triangle$ with foot $O_F \notin \partial F$ (excluding a measure 0 case). Both $F$ and $O_F$ lie in a common great sphere $S_F$ of dimension $d - 1$. By induction, we have a collection $\mathcal{T}_F$ of $d!$ path simplices with paths beginning at $O_F$ and satisfying

$$1_F = \sum_{T_F \in \mathcal{T}_F} \sigma(T_F) 1_{T_F}$$

Since $OO_F$ is perpendicular to $S_F$, each of these paths may append $OO_F$ to the beginning to represent the path simplices $T^F := \text{conv}(O, T_F)$. Form the collection $\mathcal{T}_F := \{T^F : T_F \in \mathcal{T}_F\}$, and the simplex $\triangle^F := \text{conv}(O, F)$. Now, identify points $X \in S_F$ with segments $OX$, so that

$$1_{\triangle^F} = \sum_{T^F \in \mathcal{T}_F} \sigma(T^F) 1_{T^F}$$

(4.3)

$$1_\triangle = \sum_{F} \begin{cases} -1 & \text{if } \mu(\triangle^F \cap \triangle) = 0 \\ +1 & \text{o.w.} \end{cases} 1_{\triangle^F}$$

Since there are $d+1$ faces $F$ (in the second line above), the total sum involves $(d + 1)!$ path simplices. □

Claim 4.5. The Voronoi cells generated by $V$ are simplices. The set $\mathcal{C}$ of faces across all $d + 1$ cells corresponds to the nonempty subsets of $V$ in that each face is the set of points equidistant from the points in its corresponding subset. Also, the intersection of any two of these faces is again a face ($\mathcal{C}$ is a simplicial complex).
Proof. The Voronoi cells are bounded by perpendicular bisectors between each pair of vertices $v_i, v_j$. That’s $d$ facets for each cell, i.e. a simplex. Each face of a simplex corresponds to an intersection of facets, or in this case, perpendicular bisectors. Thus, each facet is the set of points equidistant from some subset of $V$. It follows that the intersection of two faces corresponds to the union of the two corresponding subsets. □

For a set $V = \{v_0, \ldots, v_d\} \subset S^{d-1}$ and a maximal chain $\tau$ in the power set of $V$, denote by $S_\tau$ the path simplex starting at the vertex in the singleton set of $\tau$ and landing successively in the faces determined by the subsets in $\tau$, increasing by inclusion.

Claim 4.6. Let $C$ be as in the previous claim. To each face $F \in C$ (and hence every subset $Q \subset V$) call $p(F) \in \text{span}(F)$ the point that minimizes the distance from the points in $Q$. Then, $p(F)$ is a vertex of $S_\tau$ iff $Q \in \tau$.

Proof. $\text{span}(F)$ is the set of points equidistant from each point in $Q$. Since the composition of projections onto subspaces is the projection onto the intersection of the subspaces, each point in $Q$ projects onto $\text{span}(F)$ with foot $p(F)$. Finally, $p(F)$ is a vertex of $S_\tau$ if $p(F)$ is equidistant from points in $Q$, but no other points in $V$. □

The following is the main result.

Theorem 4.7. Let $T \subset S^{d-1}$ be the path simplex with $\theta_{ij}$ the angle between facets $F_i$ and $F_j$. If $A$ is an end vertex of $T$ and the Hessian of $M_A(T)$ from equation 3.4 as a function of $\theta_{12}, \ldots, \theta_{d,d+1}$ is negative definite, then the simplex mean width conjecture is true.

Proof. Let $\triangle$ be a simplex in $\mathbb{R}^d$ with vertices $V = \{v_0, \ldots, v_d\} \subset S^{d-1}$ (Claim [12b]). The altitudes from $v_i$ and $v_j$ to the perpendicular bisector between them meet at the same point. The rest of the triangulations of each cell, as in Claim 4.4, coincide along facets, forming a $d+1$ times larger simplicial complex $C^\dagger$ where every face is the intersection of cells.

By Claim 4.6 the vertices (or facets or acute dihedral angles) of any path simplex $S_\tau$ may be ordered as $\tau$ is ordered. Given three consecutive subsets $Q, Q \cup \{x\}, Q \cup \{x, y\}$ in a maximal chain $\tau$ of the subsets of $V$, the only replacement subset for $Q \cup \{x\}$ to keep $\tau$ a chain is $Q \cup \{y\}$. It follows that if a dihedral angle at a $d-3$ dimensional face in a path simplex $S_\tau$ is between two nonadjacent facets, there are $2 \times 2 = 4$ path simplices meeting at that face, at right angles as we know from Claim 4.3. Similarly, a dihedral angle between two adjacent facets has $3 \times 2 = 6$ path simplices meeting at that face.

$C^\dagger$ has a total of $(d + 1)!$ path simplices, each with $d$ acute dihedral angles meeting together at the $d-3$ dimensional faces, according to their position
in the $\tau$ ordering. So, for each $i$, the sum of the angles $\theta_{i,i+1}$ over all path simplices is $2\pi(d + 1)/6$. If $M_A(T)$ is negative definite as a function of these adjacent dihedral angles, the mean width from equation 1.1 will be maximum when all the adjacent dihedral angles are equal to $\pi/3$, that is when $\triangle$ is regular.

5. PROOF OF THEOREM 2.3

Use Napier’s formula [2.2] to phrase quantities in terms of ‘intermediate variables’ $a, b$. Differentiating this formula with respect to $A$ and using the spherical law of sines 2.1,

$$\sin A = \sin a \sin B \frac{da}{dA} = \sin A \sin b \frac{da}{dA}$$

Now with respect to $B$, and using again [2.1] and [2.2] (cycled $A \rightarrow B$),

$$0 = \sin a \sin B \frac{da}{dB} - \cos a \cos B$$

$$= \sin A \sin b \frac{da}{dB} - \cos a \cos b \sin A$$

The same holds if $(A, a)$ and $(B, b)$ are interchanged, yielding

$$\frac{\partial a}{\partial A} = \frac{1}{\sin b} \sin a \cos b$$

$$\frac{\partial b}{\partial A} = \frac{1}{\sin a} \cos a \cos b$$

$$\frac{\partial a}{\partial B} = \frac{\cos a \cos b}{\sin b}$$

$$\frac{\partial b}{\partial B} = \frac{1}{\sin a}$$

The first order partials are then

$$f_A = \frac{da}{dA} \sin b + a \cos b \frac{db}{dA} = 1 + a \cos a \cos^2 b$$

$$f_B = \frac{da}{dB} \sin b + a \cos b \frac{db}{dB} = \cos b \left( \cos a + \frac{a}{\sin a} \right)$$

For second order partials,

$$f_{AA} = \frac{\sin a}{\cos a \cos^2 b} \frac{1}{\sin b} - \left( 2 \sin b \cos a + \frac{1}{\sin b \cos a} \right) \frac{a}{\sin a}$$

$$f_{BB} = - \sin b + \frac{\cos a \cos^2 b}{\sin b} - \left( \frac{\sin b}{\cos a} + \frac{\cos^2 b}{\sin b \cos a} \right) \frac{a}{\sin a}$$

$$f_{AB} = \frac{\cos^2 b}{\sin b} \cos a - \left( 1 + \frac{\sin^2 b}{\sin b} \right) \frac{a}{\sin a}$$

Uniquely represent
\[
\frac{\sin^2 a}{\cos^2 a \cos^2 b} \det \text{Hess}(f) = \frac{\sin^2 a}{\cos^2 a \cos^2 b} (f_{AA} f_{BB} - f_{AB}^2)
\]
\[
= P(\sin a, \sin b) a^2 \frac{\cos a}{\sin a} + R(\sin a, \sin b) \frac{a^2}{\sin^2 a}
\]
for some rational functions \(P, Q, R\). Now we will compute \(P, Q, R\).

\[
P(x, y) = \left( \frac{1}{y} \right) \left( -y + \frac{(1 - x^2)(1 - y^2)}{y} \right) - \left( \frac{(1 - y^2)^2}{y^2} - (1 - x^2) \right)
\]
\[
= -1 + \frac{(1 - x^2)(1 - y^2)(1 - y^2)}{y^2}
\]
\[
=(1 - x^2)(1 - y^2) - 1
\]
\[
Q(x, y) = - \frac{1}{y} \left( \frac{y}{1 - x^2} + \frac{1 - y^2}{y} \right)
\]
\[
- \left( 2y + \frac{1}{y(1 - x^2)} \right) \left( -y + \frac{(1 - x^2)(1 - y^2)}{y} \right)
\]
\[
+ 2 \left( \frac{1 - y^2}{y} \right) \left( \frac{1 + y^2}{y} \right)
\]
\[
= 2 \left( y^2 - \frac{1 - y^2}{y^2} - (1 - x^2)(1 - y^2) + \frac{1 - y^4}{y^2} \right)
\]
\[
= 2(1 - (1 - x^2)(1 - y^2))
\]
\[
R(x, y) = \left( 2y + \frac{1}{y(1 - x^2)} \right) \left( y + \frac{(1 - y^2)(1 - x^2)}{y} \right) - \left( y + \frac{1}{y} \right)^2
\]
\[
= 2y^2 + \frac{1}{1 - x^2} + 2(1 - x^2)(1 - y^2) + \frac{1}{y^2} - 1 - \left( y + \frac{1}{y} \right)^2
\]
\[
= -y^2 - 1 - 2x^2 + 2x^2 y^2 + \frac{1}{1 - x^2}
\]
\[
= -2 + (1 - 2x^2)(1 - y^2) + \frac{1}{1 - x^2}
\]
\[
= (2x^2 - 1) \left( y^2 - 1 + \frac{1}{1 - x^2} \right)
\]
\[
= \left( \frac{x^2}{1 - x^2} - \frac{1 - x^2}{x^2} \right) x^2 (1 - (1 - x^2)(1 - y^2))
\]

From the spherical law of sines \([2.1]\) and spherical Pythagorean \([2.1]\)

\[
\frac{\sin^2 a}{\sin^2 A} = \sin^2 c = 1 - \cos^2 c = 1 - \cos^2 a \cos^2 b
\]

and so we reduce the Hessian to
This can be seen to be positive by following the chain of inequalities backward:

\[ \frac{\sin^2 A}{\cos^2 a \cos^2 b} \text{detHess} f = (\tan^2 a - \cot^2 a)a^2 + (2 \cot a)a - 1 = a^2 \tan^2 a - (1 - a \cot a)^2 \]

for \( |a| < \pi/2 \), which follows since \( x/\sin x \) (resp. \( x/\tan x \)) is tangent to \( y = 1 \) and increasing (resp. decreasing) away from 0 for \( -\pi < x < \pi \).

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