Pure $R^2$ gravity can gravitate about a flat background

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Abstract. Pure $R^2$ gravity ($R^2$ gravity by itself with no Einstein-Hilbert term) has attracted attention because it is different from other quadratic gravity theories. In a curved de Sitter (dS) or anti-de Sitter (AdS) background, it is equivalent to Einstein gravity with an additional massless scalar and with a cosmological constant. In contrast to other higher-derivative theories, it is therefore unitary. The equivalence with Einstein gravity is not valid for a flat background. In fact, it has been shown that linearizations of pure $R^2$ gravity about flat spacetime does not produce a graviton. In other words, it does not gravitate about flat space. Pure $R^2$ gravity is invariant under restricted Weyl transformations where the metric is scaled by a conformal factor that obeys a harmonic condition. In this work we consider an action composed of pure $R^2$ gravity, a massless scalar field $\phi$ non-minimally coupled to gravity plus other terms. The entire action is invariant under restricted Weyl transformations. We show that when the scalar field $\phi$ acquires a non-zero vacuum expectation value (VEV), flat spacetime now becomes a viable gravitating background solution. The restricted Weyl symmetry becomes broken, not explicitly but spontaneously. In other words, when $\phi$ acquires a non-zero VEV, the equivalent Einstein action has now the possibility of having a zero cosmological constant and therefore solutions in a Minkowski background. The action can also have, as before, a non-zero cosmological constant, so that solutions in a dS and AdS background are still possible.

1. Introduction

Pure $R^2$ gravity is a gravity theory where the action contains an $R^2$ term but no Einstein-Hilbert term (no Ricci scalar) or cosmological constant. This theory has recently attracted considerable interest. This is largely due to the fact that when the background is curved, it is equivalent to Einstein gravity plus a massless scalar field and a non-zero cosmological constant [1–4]. This fact was recently used to show that critical gravity could be obtained from four-dimensional scale-invariant gravity [5] which does not require the inclusion of an Einstein-Hilbert term explicitly in contrast to the original work on critical gravity [6]. It follows then that pure $R^2$ gravity is necessarily unitary in contrast to other higher-derivative quadratic gravity theories [4]. The theory is scale-invariant i.e. it does not change under scale transformations $g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu}$ where $\lambda$ is a constant. Moreover, it has recently been discovered
that it is invariant under the restricted Weyl transformations \[7,8\] \( g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu} \) where the conformal factor \( \Omega(x) \) obeys the harmonic condition \( \Box \Omega(x) \equiv g^{\mu\nu} \partial_\mu \partial_\nu \Omega(x) = 0 \). Therefore \( \Omega(x) \) is not limited to being a constant so that scale symmetry is smaller than restricted Weyl symmetry.

Pure \( R^2 \) gravity is equivalent to Einstein gravity when the restricted Weyl symmetry is broken spontaneously. This occurs when the vacuum (background) spacetime has \( R \neq 0 \) which excludes a flat background but allows for a de Sitter (dS) or anti-de Sitter (AdS) background. The unbroken sector corresponding to an \( R = 0 \) vacuum (background) is separate and has no relation to Einstein gravity. In fact, it has been demonstrated that linearization about Minkowski spacetime yields no gravitons but only a propagating scalar. Pure \( R^2 \) gravity does not gravitate about a flat spacetime \[4\].

One can of course simply add an Einstein-Hilbert term to \( R^2 \) to make the theory gravitate about flat spacetime. However, that action would not maintain the restricted Weyl symmetry. In this work, we construct an action that includes \( R^2 \) gravity, a massless scalar field \( \phi \) non-minimally coupled to gravity together with a \( \lambda \phi^4 \) term. The action is invariant under restricted Weyl transformations. For the restricted Weyl symmetry to hold, the field \( \phi \) must be originally massless. It is then demonstrated that this action is equivalent to an action that includes an Einstein-Hilbert term, a massive field \( \phi \) non-minimally coupled to gravity, a cosmological constant, plus an extra massless scalar field \( \varphi \). The field \( \varphi \) is expressed later in terms of a field \( \psi \) that has a canonical kinetic term. In going over to the Einstein action, the \( \phi \) field acquires a mass. The restricted Weyl symmetry is subsequently broken spontaneously.

In pure \( R^2 \) gravity without the \( \phi \) field, the symmetric vacuum corresponds to \( R = 0 \); this is the unbroken sector where Minkowski space is not a viable background. However, the restricted Weyl symmetry is broken spontaneously when the \( \phi \) field, which is non-minimally coupled, acquires a non-zero vacuum expectation value (VEV). We then can obtain an \( R = 0 \) background in the equivalent Einstein action. In Einstein gravity, it is well known that linearizations about flat spacetime lead to gravitational waves. In other words, Einstein gravity obviously gravitates about a flat spacetime. Therefore linearizations of pure \( R^2 \) gravity about Minkowski spacetime now yield gravitons when \( \phi \) acquires a non-zero VEV.

2. The \( R^2 \) action with non-minimally coupled scalar

The action that we start with has pure \( R^2 \) gravity together with a non-minimally coupled massless field \( \phi \):

\[
S_0 = \int d^4x \sqrt{-g} \left( \alpha R^2 - \xi \phi^2 - \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4 \right)
\]

(1)

where \( \alpha, \xi \) and \( \lambda \) are free dimensionless parameters. The above action is restricted Weyl invariant \[3,7\] i.e. it is invariant under the transformation

\[
g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \quad \phi \rightarrow \phi/\Omega \quad \text{with} \quad \Box \Omega = 0
\]

(2)

where the conformal factor \( \Omega(x) \) is a real smooth function. This symmetry does not allow in (1) the following: an Einstein-Hilbert term, a mass term for the scalar field \( \phi \) and a cosmological constant. The above action is equivalent to

\[
S_1 = \int d^4x \sqrt{-\tilde{g}} \left( -\alpha (c_1 \varphi + R + \frac{c_2}{\alpha} \phi^2)^2 + \alpha R^2 - \xi R \phi^2 - \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4 \right)
\]

(3)
where \( \varphi \) is an auxiliary field and \( c_1 \) and \( c_2 \) are arbitrary constants\(^1\). Expanding the above action we obtain

\[
S_2 = \int d^4x \sqrt{-g} \left( -c_1^2 \alpha \varphi^2 - 2\alpha c_1 \varphi R - (\xi + 2c_2) R\phi^2 - \partial_\mu \phi \partial^\mu \phi - 2c_1 c_2 \varphi \phi^2 - (\alpha^{-1} c_2^2 + \lambda) \phi^4 \right).
\]

(4)

Action (4) is equivalent to the original action (1) and is restricted Weyl invariant as long as \( \varphi \) transforms accordingly; it is invariant under the transformations \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \varphi \to \varphi/\Omega^2, \phi \to \phi/\Omega \) with \( \Box \Omega = 0 \).

After performing the conformal (Weyl) transformation

\[
g_{\mu\nu} \to \varphi^{-1} g_{\mu\nu}, \quad \phi \to \varphi^{1/2} \phi
\]

action (4) turns into

\[
S_3 = \int d^4x \sqrt{-g} \left( -\alpha c_1^2 - 2\alpha c_1 R - \partial_\mu \phi \partial^\mu \phi - 2c_1 c_2 \phi^2 - (\alpha^{-1} c_2^2 + \lambda) \phi^4 - (\xi + 2c_2) R \phi^2 \\
+ 3 \alpha c_1 \varphi \partial_\mu \partial^\mu \phi + (6(\xi + 2c_2) - 1) \varphi^{1/2} \Box \varphi^{-1/2} \phi^2 \right).
\]

(6)

We now have an Einstein-Hilbert action that contains a non-minimally coupled massive \( \phi \) field plus other terms. The constants \(-2\alpha c_1\) and \(-\alpha c_1^2\) determine Newton’s constant and the cosmological constant respectively and can be chosen freely by adjusting the parameters \( \alpha \) and \( c_1 \) (with \( \alpha c_1 < 0 \) to ensure the correct sign for Newton’s constant)\(^2\). We can then choose \( \xi, c_2 \) and \( \lambda \) to select the mass of \( \phi \), the coefficient \(-\alpha^{-1} c_2^2 + \lambda\) and the coefficient of \( R \phi^2 \) respectively. Define \( \psi = \sqrt{-3\alpha c_1} \ln \varphi \). The kinetic term for \( \varphi \) is now equal to \(-\partial_\mu \psi \partial^\mu \psi \) which has the appropriate canonical form.

If \( \varphi = 0 \), the Weyl transformation (5) is not valid. This implies that the equivalence of the theories assumes a vacuum with \( \varphi \neq 0 \). Under the transformation \( \varphi \to \varphi/\Omega^2 \), this vacuum is not invariant which implies that the restricted Weyl symmetry is broken spontaneously. This is clear from the fact that the final action (6) has an Einstein-Hilbert term and a massive scalar \( \phi \). The massless scalar \( \psi \) is the Goldstone boson associated with the broken symmetry. In spontaneously broken theories the original symmetry is realized as a shift symmetry of the Goldstone bosons [9]. This is realized in this case. The action (6) is invariant under \( g_{\mu\nu} \to g_{\mu\nu}, \varphi \to \varphi/\Omega^2 \) and \( \phi \to \phi \) with condition\(^3\) \( \Box \Omega - \partial_\mu (\ln \varphi) \partial^\mu \Omega = 0 \). The field \( \psi \) then transforms in the following fashion: \( \psi \to \psi - 2\sqrt{-3\alpha c_1} \ln \Omega \). This shift symmetry places constraints on the nature of the interaction term between \( \varphi \) (or \( \psi \)) and \( \phi \) in (6). The shift symmetry also does not allow a mass term for \( \psi \).

3. Background or vacuum solutions

We now determine the vacuum solutions to the Einstein action (6) where \( \varphi \) is a non-zero constant. The vacuum solutions to (6) can be divided into two types: one where the scalar

\(^1\) The squared term yields a Gaussian integral over \( \varphi \) in the path integral and does not affect anything i.e. \( \int D\varphi e^{-i \alpha c_1^2 \int d^4x \sqrt{-g} (\varphi - (\langle \varphi \rangle))^2} = \text{const} \).

\(^2\) The constant \( c_1 \) is dimensionful and has units of \( \text{length}^{-2} \). This stems from the fact that \( c_1 \varphi \) in (3) has units of \( \text{length}^{-2} \) and \( \varphi \) is assumed dimensionless in (5). In contrast, the constant \( c_2 \) is dimensionless.

\(^3\) The original restricted Weyl symmetry (with metric tensor denoted with a hat) required the condition \( \Box \Omega = 0 \). This condition, after the replacement \( g_{\mu\nu} = \varphi^{-1} g_{\mu\nu} \) becomes \( \Box \Omega - \partial_\mu (\ln \varphi) \partial^\mu \Omega = 0 \). Note that \( \varphi^{1/2} \Box \varphi^{-1/2} \) is invariant under \( \varphi \to \varphi/\Omega^2 \) when the condition \( \Box \Omega - \partial_\mu (\ln \varphi) \partial^\mu \Omega = 0 \) holds.
field $\phi$ acquires a non-zero VEV and one where the vacuum is $\phi = 0$. The former can have either an $R = 0$ background (Minkowski space) or an $R \neq 0$ background (dS or AdS space) whereas the latter can only have an $R \neq 0$ background (dS or AdS space but no Minkowski space).

We will actually first begin by describing a solution that is possible in the original action (1) which is not a solution to the Einstein action (6). This corresponds to a solution where the restricted Weyl symmetry is unbroken.

3.1. Unbroken sector: no equivalence to Einstein gravity

As already mentioned, the equivalence between the final Einstein action and the original $R^2$ action requires that $\phi \neq 0$. We now consider separately the case $\phi = 0$. We will now see that in many cases the situation actually reduces to the case with $\phi \neq 0$ where the restricted Weyl invariance can be broken spontaneously either by non-zero $R$ or $\phi$.

The equation of motion for $\phi$ in action (4) yields $R = -\frac{c^2}{16} \phi^2$ when $\phi = 0$. Therefore, non-zero $R$ and $\phi$ are still permitted here so that the restricted Weyl invariance can be broken. However, in such cases it is possible to make $\phi$ non-zero. Since $\phi$ is arbitrary, one can introduces a different $c_2$ as $\tilde{c}_2$. It then follows that $\phi = \frac{1}{\tilde{c}_1} (-R - \frac{\tilde{c}_2}{\tilde{c}_1} \phi^2)$ becomes non-zero. In other words, the equivalence to the Einstein action still holds in this case but with different $c_2$.

The special case is when $\phi = 0$ and $R = 0$. When $\phi = 0$ the original action (1) has a flat background solution and this corresponds to a vacuum with $\phi = 0$ in (4). This cannot be turned into $\phi \neq 0$ by changing $c_2$. In this case, the restricted Weyl symmetry is preserved and there is no equivalent Einstein action. As an example, the Schwarzschild black hole is a solution to the original action (1) with an $R = 0$ and $\phi = 0$ background. However, in this unbroken sector, we already saw that Minkowski spacetime does not gravitate [4] and we therefore do not pursue this direction any further.

3.2. Broken sector with vacuum $\phi = 0$ and $R \neq 0$

In action (6), when the vacuum has $\phi = 0$, the cosmological constant is given by $\Lambda = -c_1/4$ and the Ricci scalar is equal to

$$R = 4\Lambda = -c_1.$$  

If $c_1 < 0$ then $R$ is positive (dS space) and if $c_1 > 0$ then $R$ is negative (AdS space). In particular, the constant $c_1$ must be either positive or negative; it cannot be identically zero. This excludes flat space as a possible background\(^4\).

3.3. Broken sector where $\phi$ has non-zero VEV: viable Minkowski background

The case where $\phi$ has a non-zero VEV yields the vacuum solution

$$\phi^2 = \frac{-\alpha \xi}{\xi c_2 - 2\alpha \lambda}; \quad R = \frac{2\alpha c_1 \lambda}{\xi c_2 - 2\alpha \lambda}. \quad (8)$$

\(^4\) The constant $c_1$ can be made arbitrarily small (but not identically zero) while keeping $\alpha c_1$ fixed. In this limiting procedure, one has either dS or AdS spacetime depending on whether $c_1$ is negative or positive respectively. Minkowski spacetime corresponds to $c_1$ being identically zero. Another way to view this distinction is that the Einstein vacuum equations with zero cosmological constant are scale-invariant whereas the Einstein vacuum equations with arbitrarily small cosmological constant are not scale-invariant. As pointed out in [10, 11] there is a discontinuity between the limit as $\Lambda$ tends to zero and $\Lambda = 0$. Only the latter corresponds to asymptotically Minkowski spacetime.
To obtain the correct sign for Newton’s constant one requires that $\alpha c_1 < 0$. Since $\phi^2$ must be positive this implies that when $\xi > 0$, $\xi c_2 - 2\alpha \lambda \xi$ is positive and is negative when $\xi < 0$. It follows that we have AdS space when $\xi > 0$ and dS space when $\xi < 0$.

Note that $R = 0$ and $\phi^2 = \frac{-\alpha c_1}{c_2^2}$ when $\lambda = 0$ (positivity of $\phi^2$ implies here that $c_2 > 0$). Even though $R = 0$, the restricted Weyl symmetry is spontaneously broken because $\phi$ has a non-zero VEV. This Minkowski solution is now a perfectly viable background since it represents the Minkowski space of Einstein gravity. The crucial point is that when the scalar field $\phi$ acquires a non-zero VEV, this provides $R^2$ gravity with a gravitating flat background, making the theory richer and more appealing.

4. Conclusion

Pure $R^2$ gravity by itself (i.e. not supplemented by any other terms) suffers from not having a viable Minkowski background. In this work we showed that when one adds a non-minimally coupled scalar field $\phi$ to the action and it acquires a non-zero VEV, then the theory has a perfectly good Minkowski background (linearizations about it now yield a graviton). The original $R^2$ action possesses restricted Weyl symmetry: it is invariant under the transformation $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $\phi \rightarrow \phi/\Omega$ with $\Box \Omega = 0$. The equivalent Einstein action clearly does not possess this symmetry and hence the equivalence can only be valid if the symmetry is spontaneously broken. Under a restricted Weyl transformation, an $R = 0$ vacuum remains at $R = 0$, and the symmetry is not spontaneously broken. This is why it was originally thought that the equivalence can only hold if the vacuum had $R \neq 0$ which excluded a Minkowski background in the Einstein action. However, with the addition of a non-minimally coupled scalar $\phi$, the symmetry is spontaneously broken if $\phi$ acquires a non-zero VEV. Therefore the vacuum with $R = 0$ and $\phi \neq 0$ breaks spontaneously the symmetry and this vacuum is a solution of the Einstein action. It follows that when $\phi \neq 0$, the original $R^2$ action now has a viable gravitating Minkowski background.

It is worth noting that one can extend this work to include magnetic monopoles in Einstein gravity by including non-abelian gauge fields (Yang-Mills fields) as was done originally in the early 90’s in a Minkowski background [12–14] and then later in curved backgrounds [15–19]. In that case, when the scalar field acquires a non-zero VEV it leads to the spontaneous breaking of both the gauge symmetry and the restricted Weyl symmetry.

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