Relativistic particle interaction with a weak electromagnetic field

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Received

Abstract

Schrodinger equation with two-component wave function which describes a relativistic spin 1/2 particle in a weak electromagnetic field is obtained. In the same approximation Schrodinger equation with traditional norm condition and one-component wave function for a spinless particle is obtained as well. To construct it Foldy-Wouthuysen procedure with the electron charge value as the small parameter is used.

1 Introduction

Attempts to describe the electron which moves with the speed close to the speed of light by two-component wave function appeared after the creation of the relativistic quantum mechanics. It was connected with the fact that in the nonrelativistic case the electron is described by two-component wave function. However, in 1932, Dirac had shown that the relativistic electron is described by four-component wave function which is the solution of the Dirac equation. Trying to find Pauli equation as the limit case of Dirac equation Foldy and Wouthuysen supposed for it the unitary transformation method. This method allowed describing the positive energy states of the free particle by the "large" components of the four-component spinors and negative energy states by the "small" components. It means that the positive energy state is described by the spinor which has third and fourth components equal to zero and the negative energy state is described by spinor with the first and second components equal to zero. In the case of the presence external electromagnetic field such separation had been done approximately with 1/c as the small parameter. In this way Schrodinger equation with two-component wave function was received which is suitable in the weakly relativistic case. Berestetskii and Landau received this equation by other method that differs from Foldy-Wouthuysen procedure.

However the weakly relativistic equation is not good for the particle which moves with the speed very close to the speed of the light. Had used Foldy-Wouthuysen procedure the attempt to describe relativistic particle by two-component wave function in the presence of a weak external electromagnetic
field had been done in 1962 \[3\]. But in the \[3\] higher derivatives with respect to potentials of the electromagnetic field were neglected. In 1995 \[4\] Landau-Berestetskii method was used to separate the "large" and the "small" components in the Dirac equation for the spin 1/2 particle in the weak field approximation. But as for us this solution of the problem is not quite apprehensible.

In this paper Foldy-Wouthuysen procedure is used to construct Schrödinger equation with two-component wave function for relativistic electron in a weak external electromagnetic field. Schrödinger equation for a relativistic spinless particle in a weak external electromagnetic is also obtained. As in \[3\] and \[5\] we take into account the contributions of the first order by small parameter \(e\) but we take also into account higher derivatives with respect to the field potentials. In the special case when we neglect higher orders derivatives we obtain result \[3\], \[5\]. In \[4\] higher orders derivatives is taken into account also, but our method and result differs from method and result \[4\].

2 Relativistic spin-1/2 particle interaction with a weak electromagnetic field

A free relativistic particle with spin 1/2 is described by stationary Dirac equation

\[ E\psi = \hat{H}_D\psi; \] (1)

where \(\hat{H}_D\) - Dirac Hamiltonian

\[ \hat{H}_D = c\vec{\alpha}\vec{p} + \beta mc^2. \] (2)

\(\vec{\alpha}\) and \(\beta\) - 4×4 matrix which could be written in the block form

\[ \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \] (3)

here the sign "1" in the matrix \(\beta\) means 2×2 unit matrix, and \(\vec{\sigma}\) - 2×2 Pauli matrix

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (4)

Foldy and Wouthuysen showed how one could find such representation where the wave function of a free particle with respect to positive energy state is spinor with the third and the fourth components equal to zero. Spinor with the first and the second components equal to zero respond to the negative energy state. In Foldy-Wouthuysen representation the first two components of the wave function are called "large" components and the second two components are called "small" components. Besides that, Dirac Hamiltonian has the diagonal form in Foldy-Wouthuysen representation. It means that Dirac equation can be written as the system of two equations. The first equation of this system contains only the "large" components and the second equation contains only the "small" components. Therefore one can separate "large" and "small"
components. The unitary operator makes transmission to Foldy-Wouthuysen representation \( U_0(\hat{p}) : \Psi = U_0(\hat{p})\psi \)

\[
U_0(\hat{p}) U_0^+(\hat{p}) = U_0^+(\hat{p}) U_0(\hat{p}) = 1; \quad (5)
\]

\[
U_0(\hat{p}) = \frac{1}{\sqrt{2\varepsilon(\hat{p})(\varepsilon(\hat{p}) + mc^2)}} \left( \varepsilon(\hat{p}) + mc^2 + \beta c \alpha \hat{p} \right). \quad (6)
\]

In this way Dirac equation for the free particle will be the next

\[
E \Psi = U_0(\hat{p}) \hat{H}_D U_0^+(\hat{p}) \Psi = \beta \varepsilon(\hat{p}) \Psi; \quad (7)
\]

where \( \varepsilon(\hat{p}) = \sqrt{m^2 c^4 + c^2 \hat{p}^2} \) - the relativistic energy operator of the free particle.

The equation for Dirac particle in the stationary electromagnetic field is obtained from equation (1) by well known substitution \( E \rightarrow E - e \Phi \), \( \hat{p} \rightarrow \vec{\pi} = \hat{p} - e/c \mathbf{A} \)

\[
E \psi = \left[ \alpha \vec{\pi} + \beta mc^2 + e \Phi \right] \psi. \quad (8)
\]

In the presence of the external electromagnetic field it is impossible to separate "large" and the "small" components in the equation (8) by means of Foldy-Wouthuysen procedure exactly. It is possible to do it approximately using a perturbation theory with a small parameter. Foldy and Wouthuysen supposed the inverted value of the light speed \( 1/c \) as the small parameter. It means that one considers the case when the particle energy is not very large. In our work the value of the electron charge \( e \) is the small parameter. This value characterises the electron interaction with the electromagnetic field. Therefore we consider the case of the weak particle interaction with the field and the arbitrary particle energy. Aiming to get the diagonal Hamiltonian let us do the unitary transformation in the equation (8) by means of the operator \( U(\vec{\pi}) : \Psi = U(\vec{\pi})\psi \).

\[
U(\vec{\pi}) = \frac{1}{\sqrt{2\varepsilon(\vec{\alpha}\vec{\pi})(\varepsilon(\vec{\alpha}\vec{\pi}) + mc^2)}} \left( \varepsilon(\vec{\alpha}\vec{\pi}) + mc^2 + \beta c \vec{\alpha} \vec{\pi} \right). \quad (9)
\]

The transformed equation (8) becomes

\[
E \Psi = \left[ \beta \varepsilon(\vec{\Sigma}\vec{\pi}) + e \hat{\Phi}_d + e \hat{\Phi}_{nd} \right] \Psi; \quad (10)
\]

where

\[
\hat{\Phi}_d = \frac{1}{2} \left\{ U_0(\hat{p}) \Phi U_0^+(\hat{p}) + U_0^+(\hat{p}) \Phi U_0(\hat{p}) \right\}; \quad (11)
\]

\[
\hat{\Phi}_{nd} = \frac{1}{2} \left\{ U_0(\hat{p}) \Phi U_0^+(\hat{p}) - U_0^+(\hat{p}) \Phi U_0(\hat{p}) \right\}; \quad (12)
\]

and \( \vec{\Sigma} \) is \( 4 \times 4 \) matrix

\[
\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \quad (13)
\]

The obtained Hamiltonian is diagonal in the zeroth order in the constant of the interaction. The first order contains the diagonal part \( \hat{\Phi}_d \) and the nondiagonal
part $\hat{\Phi}_{nd}$. In order to exclude the nondiagonal part it’s necessary to manage a unitary transformation ones again. After that we would obtain the diagonal Hamiltonian in the zeroth and in the first orders. We also would obtain the terms of the second order in the small parameter. Those terms are small and we do not take them into account. In this way the "large" and the "small" components can be separated in the equation for relativistic spin 1/2 particle in the weak electromagnetic field. But the separation of components can be done by other method. In order to do it let us write operators $\hat{\Phi}_d$ and $\hat{\Phi}_{nd}$ in the next form

$$\hat{\Phi}_d = \begin{pmatrix} \Phi' & 0 \\ 0 & \Phi' \end{pmatrix}; \quad \hat{\Phi}_{nd} = \begin{pmatrix} 0 & \Phi'' \\ -\Phi'' & 0 \end{pmatrix};$$  \hspace{1cm} (14)$$

here

$$\Phi' = \frac{(2\varepsilon(\hat{p}))^{-1/2}}{(\varepsilon(\hat{p}) + mc^2)^{1/2}} \left( (\varepsilon(\hat{p}) + mc^2)\Phi(\varepsilon(\hat{p}) + mc^2) + c^2\vec{\sigma}\hat{p}\Phi\vec{\sigma}\hat{p} \right) \frac{(2\varepsilon(\hat{p}))^{-1/2}}{(\varepsilon(\hat{p}) + mc^2)^{1/2}};$$  \hspace{1cm} (15)$$

$$\Phi'' = \frac{e(2\varepsilon(\hat{p}))^{-1/2}}{(\varepsilon(\hat{p}) + mc^2)^{1/2}} \left( \vec{\sigma}\hat{p}\Phi(\varepsilon(\hat{p}) + mc^2) + (\varepsilon(\hat{p}) + mc^2)\Phi\vec{\sigma}\hat{p} \right) \frac{(2\varepsilon(\hat{p}))^{-1/2}}{(\varepsilon(\hat{p}) + mc^2)^{1/2}}.$$  \hspace{1cm} (16)$$

Taking into the account that

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix};$$  \hspace{1cm} (17)$$

we can write the equation (10) in the next form

$$E\phi = \varepsilon(\vec{\sigma}\vec{\pi})\phi + e\Phi'\phi + e\Phi''\chi;$$  \hspace{1cm} (18)$$

$$E\chi = -\varepsilon(\vec{\sigma}\vec{\pi})\chi + e\Phi'\chi - e\Phi''\phi.$$  \hspace{1cm} (19)$$

Let us express the function $\chi$ from the equation (19) in the case of the positive energy, and the function $\phi$ from the equation (18) in the case of the negative energy

$$\chi = -(E + \varepsilon(\vec{\sigma}\vec{\pi}) - e\Phi')^{-1}e\Phi''\phi,$$  \hspace{1cm} (20)$$

$$\phi = (E - \varepsilon(\vec{\sigma}\vec{\pi}) - e\Phi')^{-1}e\Phi''\chi;$$  \hspace{1cm} (21)$$

i. e. $\chi \sim e\phi$ when $E > 0$, and $\phi \sim e\chi$ when $E < 0$. After putting (20) into the equation (18) and (21) into the equation (14) and neglecting the terms of the second orders we get the approximate Dirac equation in Foldy-Wouthuysen representation

$$E \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \left[ \beta\varepsilon(\vec{\Sigma}\vec{\pi}) + e\hat{\Phi}_d \right] \begin{pmatrix} \phi \\ \chi \end{pmatrix}. $$ \hspace{1cm} (22)$$

Here the function $\phi$ responds to the positive energy state and the function $\chi$ responds to the negative energy state.
If we neglect the contribution of the second order then the norm condition of the wave function $\Psi$

$$\int \Psi^+ \Psi \, dv = \int (\phi^+ \phi + \chi^+ \chi) \, dv = 1; \quad (23)$$

can be written using only the function $\phi$ in the case $(E > 0)$

$$\int \phi^+ \phi \, dv = 1. \quad (24)$$

Therefore we can write two-component Schrodinger equation for the relativistic electron in the weak external electromagnetic field

$$E\phi = \hat{H}\phi; \quad (25)$$

where

$$\hat{H} = \varepsilon(\vec{\pi}) + e\Phi'. \quad (26)$$

From (26) it is easy to get Pauli spin and weakly relativistic Hamiltonians [2]. For that one should expand (26) in series $1/c$ and take the terms of the zeroth and the first orders. Therefore result (25) and (26) is a generalisation of Pauli spin equation and weakly relativistic equation.

It is also possible to manage Weil transformation in the (22). Weil transformation for the operator $\hat{A}$ is function $a(q, p), (a \leftrightarrow \hat{A})$ which is determined by the expression [3]

$$a(q, p) = \int du e^{\frac{iq\cdot u}{\hbar}} \langle p + \frac{1}{2} u | \hat{A} | p - \frac{1}{2} u \rangle;$$

here $| p - \frac{1}{2} u \rangle$ - is eigenvector of the impulse operator which responds to eigenvalue $p - \frac{1}{2} u$. If the operator $\hat{A}$ is the function which depends only on $\hat{p}$ or only on $\hat{r}$ then Weil transformation is the same function which depends on $p$ or $q$ correspondly. Weil transformation of the multiplication of two operators responds expression which is determined by Weil transformation of every operator. In order to find Weil transformation for operator $\Phi_d$ which is determined by (11) one should know Weil transformation which responds to multiplication of three operators. According to [3] the formula

$$\hat{A}\hat{B}\hat{C} \leftrightarrow \exp\left\{ \frac{i\hbar}{2} \left( \frac{\partial (b) \partial (c)}{\partial q \partial p} - \frac{\partial (a) \partial (b)}{\partial q \partial p} \right) \right\} \sum_{\mu, \nu, k, \mu} a_{k, \mu} b_{\mu, \nu} c_{\nu, L}$$

is true for the multiplication of three operators $\hat{A}(p), \hat{B}(r), \hat{C}(p)$ which act on spinors and have Weil transformation $a_{k, \mu}(p), b_{\mu, \nu}(r), c_{\nu, L}(p)$. Let us write Weil transformation for the Hamiltonian (22). The transmission formulas to this representation contain higher derivatives with respect to the potentials of the electromagnetic field. If they are neglected then Blount result is obtained [3, 3]

$$\hat{H} \leftrightarrow \beta\varepsilon(\vec{\pi}) + e\Phi - \mu_B \frac{mc^2}{\varepsilon(p)} \beta \Sigma B - \mu_B \frac{mc^3}{\varepsilon(p)(\varepsilon(p) + mc^2)} \left[ p \times \Sigma \right] E \quad (27)$$
where $\mu_B$ - Bor magneton.

Let us consider the operator $A$ of Dirac theory in Foldy-Wouthuysen representation for the positive energy states. In Dirac theory operator $A$ act on four component wave function. Let us respond it operators which act only on "large" components. If operator $UAU^+$ is diagonal the matrix element $\langle \Psi'|UAU^+|\Psi'' \rangle$ can be written as

$$\langle \phi'|UAU^+|\phi'' \rangle + \langle \chi'|UAU^+|\chi'' \rangle$$  \hspace{1cm} (28)

here

$$|\phi\rangle = \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix}; \quad |\chi\rangle = \begin{pmatrix} 0 \\ \chi \end{pmatrix}.$$  

The second term in the (28) is of the second order in small parameter $e$ and can be dropped down. Therefore the operator of the Dirac theory $A$ should be replaced by the projection of the operator $UAU^+$ on the subspace of "large" components. If the operator $UAU^+$ is nondiagonal then the matrix element $\langle \Psi'|UAU^+|\Psi'' \rangle$ can be written as

$$\langle \phi'|UAU^+|\chi'' \rangle + \langle \chi'|UAU^+|\phi'' \rangle.$$  \hspace{1cm} (29)

Here we can neglect nothing. But from expression (20) we can find

$$|\chi\rangle = \rho_1 \frac{e}{E + \varepsilon(\hat{p})}\Phi''|\phi\rangle;$$  \hspace{1cm} (30)

where $\rho_1$ is $4 \times 4$ matrix

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (31)

Therefore, from expressions (30) (29) the operator of Dirac theory $A$ should be replaced by the projection of the operator

$$\left(\rho_1 \frac{e}{E + \varepsilon(\hat{p})}\Phi''\right)^+ UAU^+ + UAU^+ \rho_1 \frac{e}{E + \varepsilon(\hat{p})}\Phi'';$$  \hspace{1cm} (32)

on the subspace of "large" components. In this place the remark must be done. The operators $(E + \varepsilon(\hat{p}))^{-1}$ and $\Phi''$ in the expressions (29) and (30) should be permuted. It can be done when we use Fourier transformation to the potentials of the field. After that the value $E$ should be replaced by the operator $\varepsilon(\hat{p})$.

### 3 Interaction of the relativistic spinless particle with a weak electromagnetic field

The problem of description of spinless particle in a weak electromagnetic field by one-component wave function with traditional norm condition is quite analogous to the description of Dirac particle. That is why it is not necessary to explain it in detail and we will write only the main results.

Klein-Gordon equation for a spinless particle in an electromagnetic field

$$(E - e\Phi)^2 \psi = (c^2\pi^2 + m^2 c^4)\psi;$$  \hspace{1cm} (33)
can be written in the next form

$$E\Psi = \left[ (\eta + \rho) \frac{\vec{p}^2}{2m} + \eta mc^2 + e\Phi \right] \Psi; \quad (34)$$

where

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + (E - U)/mc^2 \\ 1 - (E - U)/mc^2 \end{pmatrix} \psi; \quad (35)$$

and $\eta, \rho$ are $4 \times 4$ matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (36)$$

If external electromagnetic field is absent we can separate the "large" and the "small" components in the equation (34) using the transformation $\Psi' = U_0(\vec{p})\Psi$, where

$$U_0(\vec{p}) = \frac{1}{2\sqrt{mc^2\varepsilon(\vec{p})}} \left( \varepsilon(\vec{p}) + mc^2 + \eta \rho (\varepsilon(\vec{p}) - mc^2) \right). \quad (37)$$

In the presence of an external electromagnetic field it is not possible to separate "large" and "small" components in the equation (34) exactly. In order to make it approximately let us use the operator $U(\vec{p}) = U_0(\vec{p})$ for transmission to Foldy-Wouthuysen representation. Taking into account only values of the zeroth and the first order in small parameter $e$ we get Shrodinger equation which describe a spinless particle in a electromagnetic field

$$E\psi = \hat{H}\psi; \quad (38)$$

where

$$\hat{H} = \varepsilon(\vec{p}) + \frac{e}{2} \left( \frac{1}{\sqrt{\varepsilon(\vec{p})}} \Phi \sqrt{\varepsilon(\vec{p})} + \sqrt{\varepsilon(\vec{p})} \Phi \frac{1}{\sqrt{\varepsilon(\vec{p})}} \right). \quad (39)$$

The wave function $\psi$ is one component function and satisfy a traditional norm condition

$$\int \psi^+ \psi dv = 1. \quad (40)$$

The Hamiltonian (39) can be expend in series in small parameter $1/c$. If we took into account several first contributions of this series we get the relativistic corrections to Hamiltonian of the nonrelativistic spinless particle [6].

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