Coagulation Fragmentation Laws Induced By General Coagulations of Two-Parameter Poisson-Dirichlet Processes

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Pitman (1999) describes a duality relationship between fragmentation and coagulation operators. An explicit relationship is described for the two-parameter Poisson-Dirichlet laws, say \( PD(\alpha, \theta) \), with parameters \((\alpha, \theta)\) and \((\beta, \theta/\alpha)\), wherein \( PD(\alpha, \theta) \) is coagulated by \( PD(\beta, \theta/\alpha) \) for \( 0 < \alpha < 1 \), \( 0 \leq \beta < 1 \) and \(-\beta < \theta/\alpha\). This remarkable explicit agreement was obtained by combinatorial methods via exchangeable partition probability functions (EPPF). It has been noted that such a method is not easy to employ for more general processes. This work discusses an alternative analysis which can feasibly extend the characterizations above to more general models of \( PD(\alpha, \theta) \) coagulated with some law \( Q \). The analysis exploits distributional relationships between compositions of species sampling random probability measures and coagulation operators. It is shown, based on results of Vershik, Yor and Tsilevich (2004) and James (2002), how the calculation of generalized Cauchy-Stieltjes transforms of random probability measures provides a blueprint to obtain explicit characterizations of \( PD(\alpha, \theta) \) coagulated with some law \( Q \). We use this to obtain explicit descriptions in the case where \( Q \) corresponds to a large class of power tempered Poisson Kingman models described in James (2002). That is, explicit results are obtained for models outside of the \( PD(\beta, \theta/\alpha) \) family. We obtain a new proof of Pitman’s result as a by-product. Furthermore, noting an obvious distinction from the class of \( PD(\alpha, \theta) \) derived from a stable subordinator, we discuss briefly the case of Dirichlet processes coagulated by various \( Q \).

1 Introduction

Let \( P_1^+ = \{ p = (p_i) : p_1 \geq p_2 \geq p_3 \ldots \geq 0; \sum_{i=1}^{\infty} p_i = 1 \} \). Furthermore, for a sequence \((x_1, x_2, \ldots)\) of non-negative real numbers with \( \sum_{i=1}^{\infty} x_i = 1 \), let \( RANK(x_1, x_2, \ldots) \in P_1^+ \) be the decreasing rearrangement of terms of the sequence. Pitman (1999, 2005), in particular section 5.4 of Pitman (2005), gives the following definition of coagulation and fragmentation kernels on \( P_1^+ \). For each probability measure \( Q \) on \( P_1^+ \), two Markov-transition kernels \( Q-COAG \) and \( Q-FRAG \) can be defined for \( P_1^+ \) as follows. For \( p \in P_1^+ \); \( (Q-COAG)(p, \cdot) \) is the distribution on \( P_1^+ \) of \( RANK(\sum_i p_i (U_i \in I_i^Q)) \), \( j \geq 1 \) where \( (I_i^Q) \) is a \( Q \)-partition of \([0,1]\) and the \( U_i \) are i.i.d. uniform on \([0,1]\) independent of \( (I_i^Q) \). Note that for brevity we refer the reader to Pitman (2005, ch.5) for further explanations of the above quantities. \( (Q-FRAG)(p, \cdot) \) is the distribution of \( RANK(p_i Q_{ij}, i, j \geq 1) \) where \( (Q_{ij})_{j \geq 1} \) has distribution \( Q \) for each \( i \), and these sequences are independent as \( i \) varies.

For a probability measure \( P \) on \( P_1^+ \), let \( R := P(Q - COAG) \). That is the random probability measure on \( P_1^+ \) defined by

\[
R(\cdot) = \int_{P_1^+} P(dr)Q - COAG(r, \cdot)
\]

which may be called \( P \ coagulated \) by \( Q \). In principle, there are many ways to characterize the laws \( R, P, Q-COAG, Q-FRAG \). For example, one may do this via their corresponding exchangeable partition probability functions (EPPF) on the space of partitions of the integers. However this is, in

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general, a non-trivial matter. Ideally one wants to identify such laws which may be described nicely via the graph of Pitman (2005),

\[
\begin{align*}
X \xrightarrow{\tilde{Q}} Y \\
\end{align*}
\]

where one reads (1) as \(P(Y \in \cdot | X) = \tilde{Q}(X, \cdot)\) and \(P(X \in \cdot | Y) = \dot{Q}(Y, \cdot)\). With respect to the present context, \(X\) has distribution \(P\), \(\tilde{Q}(X, \cdot) = Q - COAG(X, \cdot)\), \(Y\) has distribution \(R\) and \(\dot{Q}(Y, \cdot) = Q - FRAG(Y, \cdot)\). The general task, that we shall consider, is given \(P\) and \(Q - COAG\), find \(R\) and \(Q - FRAG\).

Pitman (1999) establishes the most general known coagulation/fragmentation duality of this type using the two-parameter Poisson Dirichlet distribution on \(P_1\). The two-parameter Poisson Dirichlet distribution, denoted as \(PD(\alpha, \theta)\), for the separate ranges \(0 \leq \alpha < 1\), \(\theta > -\alpha\) and \(\alpha = -\kappa\), \(\theta = m\kappa\) for \(\kappa > 0\) and some integer \(m = 1, 2, \ldots\) is discussed in for instance Pitman and Yor (1997) and Pitman (2005) and has numerous applications and interpretations. We shall provide more details shortly. First Theorem 12 of Pitman (1999) may be described in terms of the following diagram as given in Pitman (2005); for \(0 < \alpha < 1\), \(0 \leq \beta < 1\), \(-\beta < \theta/\alpha\),

\[
\begin{align*}
PD(\alpha, \theta) \xrightarrow{PD(\beta/\alpha)} \xleftarrow{PD(\alpha\beta, \theta)} PD(\alpha\beta, \theta) \xrightarrow{PD(\alpha, -\alpha\beta)} FRAG
\end{align*}
\]

where the notation \(PD(\alpha, \theta)\) and \(PD(\alpha\beta, \theta)\) in (2) is to be understood as some \(X\) and \(Y\) having these respective laws. In other words, this gives the explicit description of the coagulation/fragmentation duality in relation to \(PD(\alpha, \theta)\) coagulated by \(PD(\beta, \theta/\alpha)\). That is one can set, \(P = PD(\alpha, \theta)\), \(Q = PD(\beta, \theta/\alpha)\) and \(R = PD(\alpha\beta, \theta)\). Pitman (1999) proves this result via a combinatorial argument involving the respective EPPF’s of the various two-parameter Poisson-Dirichlet models. The argument used exploited the Gibbs structure of these EPPF’s and as noted by Pitman (1999) is not obviously extendable to obtain explicit expressions for other laws.

In this paper we show how one may replace the combinatorial argument by an argument involving generalized Cauchy-Stieltjes transform and moreover extend Pitman’s result to more general models of \(PD(\alpha, \theta)\) coagulated by some \(Q\). That is to say given \(Q\) we want to complete the description of the following diagram

\[
\begin{align*}
PD(\alpha, \theta) \xrightarrow{Q - COAG} Y \xleftarrow{Q - FRAG} PD(\alpha\beta, \theta)
\end{align*}
\]

A key to our exposition is the following characterization via exchangeable random probability measures. First every random sequence \((P_i) \in P_1\) has a law \(P\) which determines and is determined by the law of the random probability measure

\[
\tau_P(\cdot) = \sum_{i=1}^{\infty} P_i \delta_{U_i}(\cdot)
\]

where \(U_i\) are iid uniform \([0, 1]\). This representation is equivalent to saying that \(\tau_P\) is a species sampling random probability measure [see Pitman (1996)] based on a Uniform distribution. Now associating the definition of random probability measure \(\tau_Q\) with \(Q\) in an obvious way, Lemma 5.18 of Pitman (2005) states that the law \(R = P(Q - COAG)\) is the unique probability distribution on \(P_1\) such that

\[
(\tau_R(u), 0 \leq u \leq 1) \overset{d}{=} (\tau_P(\tau_Q(u)), 0 \leq u \leq 1)
\]
where it is assumed that $\tau_P(u), 0 \leq u \leq 1$ and $(\tau_Q(u), 0 \leq u \leq 1)$ are independent. We shall also use the notation $\tau_R = \tau_P \circ \tau_Q$ to denote composition. See also Bertoin and Pitman (2000), Bertoin and Le Gall (2003, 2005) for a related discussion.

Our approach is to try to ascertain directly the distribution of $\tau_R = \tau_P \circ \tau_Q$, when $\tau_P$ is determined by a $PD(\alpha, \theta)$ model. The main tool will be the explicit evaluation of the generalized Cauchy-Stieltjes transform for $\tau_R$. As we shall show, this approach is particularly well suited for the $PD(\alpha, \theta)$ models due to results of Vershik, Yor and Tsilevich (2004) model in conjunction with the results of James (2002). We apply the Cauchy-Stieltjes transforms in James (2002) to extend (2) to a family of $Q$ belonging to a class of power tempered Poisson Kingman laws. This constitutes a large class of models which are derived from rather arbitrary continuous infinitely divisible random variables. As an important example, we show that when $Q$ is a Dirichlet process, Pitman’s result in (2) for $Q = PD(0, \theta/\alpha)$ follows from the identity of Cifarelli and Regazzini (1990) and a new more general result for $Q = PD(0, \nu), \nu > \theta/\alpha$, follows by a characterization of the Dirichlet process given in James (2005).

Some other notable, but not exhaustive, list of references for various types of coagulation/fragmentation models include Aldous and Pitman (1998), Bolthausen and Sznitman (1998), Bertoin (2002), Bertoin and Pitman (2000), Bertoin and Goldschmidt (2004), Dong, Goldschmidt and Martin (2005) and Schweinsberg (2000).

2 Construction of Poisson Kingman Type Random Probability Measures

We first describe the class of models $Q$ we shall explicitly consider. Let $T$ denote a strictly positive random variable with density denoted as $f_T$ and Laplace transform

$$\mathbb{E}[e^{-\lambda T}] = e^{-\psi(\lambda)} = \int_0^\infty e^{-\lambda t} f_T(t) dt$$

where $\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) \rho(ds)$ and $\rho$ denotes its unique Lévy density. Let $H(\cdot)$ denote a probability measure on a Polish space $\mathcal{X}$. For the moment we shall assume that $H$ is fixed and non-atomic. We will later relax this assumption. It is known that for each $T$ and fixed $H$ one may construct a finite completely random measure, say $\mu$, on a Polish space $\mathcal{X}$, characterized by its Laplace functional for every positive measurable function $g$ on $\mathcal{X}$ as

$$\mathbb{E}[e^{-\mu(g)}] = e^{-\int_{\mathcal{X}} \psi(g(x)) H(dx)}$$

where $\mu(g) = \int_{\mathcal{X}} g(x) \mu(dx)$. It is evident that $T = \mu(\mathcal{X}) := \int_{\mathcal{X}} \mu(dx) = \int_{\mathcal{X}} I\{x \in \mathcal{X}\} \mu(dx)$. We denote the law of $\mu$ as $\mathbb{P}(d\mu|\rho H)$, where $\mathbb{P}(\cdot|\rho H)$ is a probability measure on a suitably measurable space of finite measures, say $\mathcal{M}$. Harkening back to Kingman (1975) one may describe a class of random probability measures on $\mathcal{X}$ by the normalization

$$P_K(\cdot) = \frac{\mu(\cdot)}{T} = \sum_{i=1}^{\infty} P_i \delta_{Z_i}(\cdot)$$

where $(P_i) \in \mathcal{P}_1^+$ has some law denoted as $Q = PK(\rho)$ and independent of $P_i$ the $(Z_i)$ are iid $H$. That is to say the $P_K$ constitute a class of species sampling random probability models. The construction of the $(P_i)$ equates with the basic Poisson-Kingman models discussed in Pitman (2003). Pitman (2003) provides a thorough characterization of the laws $Q = PK(\rho)$ on $\mathcal{P}_1^+$ via their corresponding exchangeable partition probability function (EPPF). Specifically, according to Corollary 6 of Pitman (2003), for some random partition of the integers $1, \ldots, n$, $(A_1, \ldots, A_k)$, with block sizes $|A_i| = n_i$ for $i = 1, \ldots, k \leq n$ blocks, the EPPF associated with each $Q$ is given by

$$p_K(n_1, \ldots, n_k) := \frac{(-1)^{n-k}}{\Gamma(n)} \int_0^\infty \lambda^{n-1} e^{-\psi(\lambda)} \prod_{i=1}^k \psi(n_i(\lambda)) d\lambda$$
where for \( m = 1, \ldots, n, \)
\begin{equation}
\psi_m(\lambda) := \frac{d^m}{d\lambda^m} \psi(\lambda) = (-1)^{m-1} \kappa_m(\lambda),
\end{equation}
and,
\begin{equation}
\kappa_m(\lambda) = \int_0^\infty s^m e^{-\lambda s} \rho(ds)
\end{equation}
represents the \( m \)-th cumulant of a random variable with tilted density \( e^{\psi(\lambda)} e^{-\lambda f_T(t)} \). As discussed in Pitman (1996), the EPPF \( p_K \) along with specific knowledge of \( H \) determines the law of \( P_K \) which is governed by \( \mathbb{P}(d\mu|\rho H) \). The basic Poisson-Kingman laws generate a much larger class of laws by first conditioning \( (P_i)|T = t \) or equivalently \( P_K|T = t \) and substituting \( f_T(t)dt \) by another probability measure on \((0, \infty), \gamma(dt)\). Pitman (2003) denotes these laws as \( PK(\rho, \gamma) = \int_0^\infty PK(\rho(t)\gamma(dt) \). The \( PK(\rho, \gamma) \) is referred to as a Poisson-Kingman distribution with Lévy density \( \rho \) and mixing distribution \( \gamma \).

**Remark 1.** It is obvious that if the \( (Z_i) \) are replaced by \( (U_i) \) then the composition of such a \( P_K(\cdot) = \sum_{i=1}^\infty P_i \delta_{Z_i}(\cdot) \) random probability measure with an \( H, P_K \circ H \), is equivalent to a random probability measure determined by \( \int_X \psi(g(x))H(dx) \) as above. Equivalently, any \( P_K \) in (6), can be represented as
\begin{equation}
\sum_{k=1}^\infty P_i \delta_{Z_i}(\cdot) \overset{d}{=} \frac{\hat{\mu}(H(\cdot))}{\hat{\mu}(H(\mathcal{X}))}
\end{equation}
where the law of \( \hat{\mu} \) on \([0, 1]\) is specified by its Laplace functional with \( \mathbb{E}[e^{-\hat{\mu}(g)}] := e^{-\int_0^1 \psi(g(x))dx} \). That is \( \mu = \hat{\mu} \circ H \), with distribution characterized by (4). Importantly these result hold for any fixed \( H \), whether it possesses atoms or not.

### 2.1 Two-parameter Poisson-Dirichlet models

The \( PD(\alpha, \theta) \) models for \( 0 \leq \alpha < 1 \) and \( \theta > -\alpha \) are special cases of the above construction, that is \( PK(\rho, \gamma) \) models. The two-parameter \((\alpha, \theta)\) Poisson-Dirichlet random probability measure, with parameters \( 0 \leq \alpha < 1 \) and \( \theta > -\alpha \) has the known representation,
\begin{equation}
P_{\alpha, \theta}(dx) = \frac{\mu_{\alpha, \theta}(dx)}{T_{\alpha, \theta}}
\end{equation}
where \( \mu_{\alpha, \theta} \) is a finite random measure on \( \mathcal{X} \) with law denoted as \( \mathbb{P}_{\alpha, \theta}(\cdot|H) \), and \( T_{\alpha, \theta} = \mu_{\alpha, \theta}(\mathcal{X}) \) is a random variable. The law of the random measure \( \mu_{\alpha, \theta} \) can be described as follows. When \( \alpha = 0 \), \( \mu_{0, \theta} \) is a Gamma process with shape \( \theta H \), hence \( P_{0, \theta} \) is a Dirichlet process with shape \( \theta H \). That is a \( PD(0, \theta) \) model for \( (P_i) \) coupled with a specification for \( H \) yields a Dirichlet process with shape parameter \( \theta H \), for \( \theta > 0 \). In this case of \( \mu_{0, \theta} \), the \( \psi(g(x)) \) is expressed for any positive measureable function \( g \) as
\begin{equation}
d_\theta(g(x)) := \theta \ln(1 + g(x)),
\end{equation}
and its Lévy density is \( \rho_{0, \theta}(ds) = \theta s^{-1} e^{-s} ds \). Hence its law is \( \mathbb{P}_{0, \theta}(\cdot|H) := \mathbb{P}(\cdot|\rho_{0, \theta} H) \). The total random mass, say \( T_{0, \theta} = \mu_{0, \theta}(\mathcal{X}) \), is a Gamma random variable with shape parameter \( \theta \). For the \( PD(\alpha, 0) \) model, recall that
\begin{equation}
\rho_{\alpha, 0}(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} s^{-\alpha-1} ds
\end{equation}
is the Lévy density corresponding to a stable law of index \( 0 < \alpha < 1 \). Equivalently, \( T_{\alpha, 0} \) is a stable random variable determined by \( \rho_{\alpha, 0} \), with density denoted as \( f_\alpha(t) = f_{T_{\alpha, 0}}(t) \), its Laplace transform is given by
\begin{equation}
\mathbb{E}[e^{-\lambda T_{\alpha, 0}}] = e^{-\lambda^\alpha}.
\end{equation}
This shows that $\mu_{\alpha,0}$ is a completely random measure based on a stable law. The law of $\mu_{\alpha,0}$ is $P_{\alpha,0}(\cdot|H) := P_{\alpha}(\cdot|\rho_{\alpha,0}H)$. In the cases above both $\mu_{\alpha,0}$ and $\mu_{0,\theta}$ are completely random measures and $PD(\alpha,0) = PK(\rho_{\alpha,0})$ and $PD(0,\theta) := PK(\rho_{0,\theta})$. This is not the case for $PD(\alpha,\theta)$ models for the range $0 < \alpha < 1$ and $\theta \neq 0$, $\theta > -\alpha$. The two-parameter Poisson-Dirichlet model with $0 < \alpha < 1$ and $\theta \neq 0$, $\theta > -\alpha$ is obtained by the specification $PD(\alpha,\theta) := PK(\rho_{\alpha,0}, f_{\alpha,\theta})$ where

$$f_{\alpha,\theta}(t)dt = c_{\alpha,\theta}t^{-\theta}f_{\alpha}(t)dt,$$

with $c_{\alpha,\theta} = 1/E[T_{\alpha,0}^{-\theta}] = \Gamma(\theta + 1)/\Gamma(\theta + 1)$. At the level of the random measure $\mu_{\alpha,\theta}$ one has the following absolute continuity relationship, for every measurable function $h$,

$$(8) \quad E[h(\mu_{\alpha,\theta})|H] = c_{\alpha,\theta}E[T_{\alpha,0}^{-\theta}h(\mu_{\alpha,\theta})|H] = c_{\alpha,\theta}\int_{\mathcal{M}} T^{-\theta}h(\mu)d\mu|\mu_{\alpha,0}H$$

where the first expectation is taken with respect to the law $P_{\alpha,\theta}(\cdot|H)$. The $PD(\alpha,\theta)$ model is defined in general for two ranges $0 \leq \alpha < 1$ and $\theta > -\alpha$ or $\alpha = -\kappa < 0$, and $\theta = m\kappa$ for $m = 1, 2, \ldots$. In any case the EPPF is given by

$$p_{\alpha,\theta}(n_1, \ldots, n_k) = \frac{(\theta + \alpha)_{k-1+\alpha} \prod_{i=1}^{k} (1 - \alpha)_{n_i-1+\alpha}}{\theta + 1}_{n_1+1}$$

where $(x)_{n+1} := \prod_{i=0}^{n-1}(x + i\alpha)$.

### 2.2 $PK(\rho, \gamma_\theta)$ models

James (2002, section 6), influenced by the power tempering construction of the $PD(\alpha,\theta)$ models, discussed and analyzed various features of a natural extension to more general $PK(\rho, \gamma)$ models of this type. Suppose that for a $PK(\rho)$ model there exists $-\infty < \theta < \infty$ such that

$$\frac{1}{m_\theta(\rho)} = \int_{0}^{\infty} t^{-\theta}f_T(t)dt < \infty,$$

then one may define a class of power tempered PK models by specifying $PK(\rho, \gamma_\theta)$ with

$$\gamma_\theta(dt) = m_\theta(\rho)t^{-\theta}f_T(t)dt.$$

Hereafter, we shall only consider the range $\theta > -\alpha$. The corresponding random probability measures on $\mathcal{X}$ are denoted as

$$(9) \quad P_{K,\theta}(\cdot) = \frac{\mu(\cdot)}{T} = \sum_{i=1}^{\infty} \nu_i \delta_{Z_i}(\cdot)$$

where $(P_i) \in \mathcal{P}_1^+$ has law $PK(\rho, \gamma_\theta)$ and $(Z_i)$ are iid $H$. Equivalently if $P(d\mu|\rho H)$ denotes the distribution of $\mu$ under the the $PK(\rho)$ model, for a specific $H$, then we say that

$$P(d\mu|\rho H, \gamma_\theta) = m_\theta(\rho)T^{-\theta}P(d\mu|\rho H)$$

is the distribution of $\mu$ under the $PK(\rho, \gamma_\theta)$ laws. That is, similar to $\mathbb{E}$, if $\mu_{K,\theta}$ denotes a version of $\mu$ with law $P(\cdot|\rho H, \gamma_\theta)$, then one has the following absolute continuity relationship, for every measurable function $h$,

$$(10) \quad \mathbb{E}[h(\mu_{K,\theta})|H] = m_\theta(\rho)\mathbb{E}[T^{-\theta}h(\mu)|H] = m_\theta(\rho)\int_{\mathcal{M}} T^{-\theta}h(\mu)d\mu|\mu_{\alpha,0}H).$$
Hence we see that (10) is a generalization of (8). It follows from Pitman (2003) that the EPPF of these models may be described as

\[ p_{K,\theta}(n_1, \ldots, n_k) := \frac{(-1)^{n-k} m_\theta(\rho)}{\Gamma(\theta + n)} \int_0^\infty \lambda^{\theta + n - 1} e^{-\psi(\lambda)} \prod_{i=1}^k \psi_{n_i}(\lambda) d\lambda, \]

where \( \psi_{n_i}(\lambda) \) is defined as in (6).

### 3 Cauchy-Stieltjes Transforms

We now proceed to show how one may describe the laws of \( \tilde{P}_{\alpha,\theta,Q} := P_{\alpha,\theta} \circ \tau_Q \) and related expressions.

In view of Remark 1 we will always assume that \( P_{\alpha,\theta} \) is defined by uniform atoms \((U_i)\), but allow \( \tau_Q \) to be based on atoms with a more general distribution. As mentioned previously there are various techniques that can be used to identify the laws of random probability measures. For instance one may calculate its EPPF, identify its finite dimensional distribution by direct means, or its Laplace functional. The idea of using Laplace functionals is intuitively appealing, however it is not particularly suited to handle random probability measures. It turns out that a more appropriate tool are generalized Cauchy-Stieltjes transforms (CS) defined for some generic random probability measure \( \tau \), positive measurable \( g \), positive \( z \) and real valued \( q \) as

\[ \mathbb{E}[(1 + z \tau(g))^{-q}]. \]

Not many results for specific \( \tau \) are widely known. Fortunately there are useful results for the \( PD(\alpha, \theta) \) class. Specifically we shall use the results of Cifarelli and Regazzini (1990) and Vershik, Yor and Tsilevich (2004). Somewhat less known are the results of James (2002) who obtains specific transforms for \( PK(\rho) \) and \( PK(\rho, \gamma_\theta) \) models. These results are extended to larger classes, in a manuscript in preparation of James, Lijoi and Prünster (2005). We first describe what is known for the \( PD(\alpha, \theta) \) models. We then describe the results for the general \( PK(\rho, \gamma_\theta) \) for \( \theta > -\alpha \). This large class of models turn out to be particularly well suited for coagulation with \( PD(\alpha, \theta) \).

#### 3.1 CS for \( PD(\alpha, \theta) \)

Note, as can be seen from Remark 1, that given \( \tau_Q \), \( \tilde{P}_{\alpha,\theta,Q} \) is a \( PD(\alpha, \theta) \) model with \( H = \tau_Q \) fixed. Probably the most widely known CS result is for the Dirichlet process with shape \( \theta H \) where it was shown by Cifarelli and Regazzini (1990) that quite remarkably

\[ \mathbb{E}[(1 + z P_{\alpha,\theta}(g))^{-\theta}| H] = e^{-\int_x d\theta(zg(x))H(dx)}, \]

where \( d\theta \) is defined in (11). Now, key to our exposition is the following elegant result of Vershik, Yor and Tsilevich (2004) for the \( PD(\alpha, \theta) \) model with the range \( 0 < \alpha < 1, \theta \neq 0 \) and otherwise \( \theta > -\alpha \) we have for any \( H \) that

\[ \mathbb{E}[(1 + z P_{\alpha,\theta}(g))^{-\theta}| H] = \left[ \int_\mathcal{X} (1 + zg(x))^\theta H(dx) \right]^{-\frac{\alpha}{\theta}}. \]

To complete the picture for the \( PD(\alpha, \theta) \), a result for the \( PD(\alpha, 0) \) may be read from proposition 6.2 of James (2002) with \( n = 1 \) as,

\[ \mathbb{E}[(1 + z P_{\alpha,0}(g))^{-1}| H] = \frac{\int_\mathcal{X} x (1 + zg(x))^{-1} H(dx)}{\int_\mathcal{X} x (1 + zg(x))^\alpha H(dx)}. \]
3.2 CS for $PK(\rho, \gamma_0)$ models

Proposition 6.1 of James (2002) shows that $PK(\rho, \gamma_0)$ models for $\theta \neq 0, \theta > -1$ have the transform,

$$E[(1 + zP_{K, \theta}(g))^{-\theta}|H] = \frac{m_\theta(\rho)}{\Gamma(\theta)} \int_0^\infty e^{-\int x \psi(y(1+zg(x)))H(dx)} y^{\theta-1} dy. \tag{14}$$

This result generalizes (12).

Remark 2. The result (14) is actually stated in James (2002) for the range $\theta > 0$. As the result arises from an identity due to the gamma function, it extends to the negative range by the same argument as noted on p. 2309 “added in translation” of Vershik, Yor and Tsilevich (2004). Otherwise, one can see this by first writing,

$$(1 + zP_{K, \theta}(g))^{-\theta} = (1 + zP_{K, \theta}(g))^{-(1+\theta)}(1 + zP_{K, \theta}(g)).$$

Now noting that $1 + \theta > 0$ for $\theta > -1$, the gamma identity applies with $\theta + 1$, one then argues as in James (2002) and concludes the result by integration by parts.

3.3 Generic CS for $\tilde{P}_{\alpha, \theta, Q}$

Setting $H = \tau_Q$ in (12), we obtain the key formula for the range $0 < \alpha < 1, \theta \neq 0$ and otherwise $\theta > -\alpha$

$$E[(1 + z\tilde{P}_{\alpha, \theta, Q}(g))^{-\theta}|\tau_Q] = \left[ \int x (1 + zg(x))^{\alpha} \tau_Q(dx) \right]^{-\frac{\theta}{\alpha}}. \tag{15}$$

It then follows that

$$E[(1 + z\tilde{P}_{\alpha, \theta, Q}(g))^{-\theta}] = E[(1 + \tau_Q(g_{\alpha}))^{-\frac{\theta}{\alpha}}]$$

where $g_{\alpha}(x) = (1 + zg(x))^{\alpha} - 1$. That is to say, the peculiar nature of the $PD(\alpha, \theta)$ model for the range $0 < \alpha < 1, \theta \neq 0, \theta > -\alpha$ yields basically a double Cauchy-Stieltjes formula. Based on the results given in the previous section an alternative proof for the Theorem 12 in Pitman (1999) is almost completely evident. We will describe the details in the next section. We will then show why the choice of $PK(\rho, \gamma_0)$ as $Q$ models is quite desirable.

Remark 3. As we noted earlier explicit CS formula for more general $\tau_Q$ may be obtained from James (2002) and James, Lijoi and Prünster (2005). We note also that the formula for the Dirichlet model, $PD(0, \theta)$ in (11) suggests that one needs to calculate the Laplace functional of a $Q$ which is generally hard. This makes it more difficult to obtain explicit results for $PD(0, \theta)$ coagulated by some $Q$. We note from the results in James (2002) and James, Lijoi and Prünster (2005) that the appearance of Laplace functional calculations is unfortunately true for many possible candidates as replacements for $PD(\alpha, \theta)$. In contrast the stable $PD(\alpha, 0)$ case is exceptionally easy. We shall discuss the $PD(0, \theta)$ and $PD(\alpha, 0)$ separately from the other $PD(\alpha, \theta)$ models.

4 Proof of Pitman’s Diagram

We shall now illustrate how our framework easily yields the diagram (2). First we address the Dirichlet case.
4.1 $PD(\alpha, \theta)$ coagulated by $PD(0, \theta/\alpha)$

First apply (15) with $\tau_Q = P_{0,\theta/\alpha}$, then it is immediate that the formula (11) applies with $\theta/\alpha$ in place of $\theta$. To complete the result notice that

$$d_{\theta/\alpha}(g_\alpha(x)) = \frac{\theta}{\alpha} \ln (1 + zg(x))^\alpha = d_\theta(zg(x)).$$

That is to say the CS transform of order $\theta$ of $P_{\alpha,\theta} \circ P_{0,\theta/\alpha}$ is given by (11) identifying it as a $PD(0,\theta)$ model. The diagram (2) is completed by calculating the now obvious 3 EPPF’s and applying Bayes rule.

4.2 $PD(\alpha, \theta)$ coagulated by $PD(\beta, \theta/\alpha)$, $\beta > 0$

For this case apply (15) with $\tau_Q = P_{\beta,\theta/\alpha}$ for $0 < \beta < 1$, now apply (12) to conclude that, in this case, (15) is equivalent to,

$$\mathbb{E}[(1 + P_{\beta,\theta/\alpha}(g_\alpha))^{-\frac{\theta}{\beta}}|H] = \left[ \int_{\mathcal{X}} (1 + g_\alpha(x))^{\beta} H(dx) \right]^{-\frac{\alpha}{\beta}}.$$}

The result is completed by noting that

$$(1 + g_\alpha(x))^\beta = (1 + zg(x))^\alpha.$$}

Hence the CS transform of order $\theta$ of $P_{\alpha,\theta} \circ P_{\beta,\theta/\alpha}$ is given by the expressions above and now comparing with (12) identifies its law as $PD(\alpha\beta,\theta)$ model.

Remark 4. The $P_{\alpha,0} \circ P_{\beta,0}$ case can certainly be obtained easily using (13) twice. However, we do not believe that there is a simpler proof than that exhibited in Pitman (2005). We shall return to this later.

5 $PD(\alpha, \theta)$ Coagulated By $PK(\rho, \gamma/\alpha)$

We now obtain a new result as follows. First denote a class of species sampling random probability measures on $\mathcal{X}$ as

$$S_{\alpha,\theta}() = \frac{L_{\alpha,\theta}()}{T_{L_{\alpha,\theta}}} := \sum_{i=1}^{\infty} W_i \delta_{Z_i}()$$

with $T_{L_{\alpha,\theta}} = L_{\alpha,\theta}(\mathcal{X})$ and where $(W_i) \in P_1^1$ and $(Z_i)$ are iid $H$. Similar to (8) and (10), the law of $L_{\alpha,\theta}$, and hence that of $(W_i)$, is determined by a power tempered probability measure and satisfies the following absolute continuity relationship, for every measurable function $h$,

$$\mathbb{E}[h(L_{\alpha,\theta})|H] = c_{\alpha,\theta} m_{\rho}(\mathcal{P}) \mathbb{E}[T_{L_{\alpha,\theta}^{-\theta}} h(L_{\alpha,0})|H]$$

where the Laplace functional of the random measure $L_{\alpha,\theta}$ is specified by

$$\mathbb{E}[e^{-L_{\alpha,\theta}(g)}|H] = e^{-\int_{\mathcal{X}} \tilde{\psi}_\alpha(g(x))H(dx)}$$

with

$$\tilde{\psi}_\alpha(g(x)) = \psi([g(x)]^\alpha).$$
In other words $L_{\alpha,0} = \mu_{\alpha,0} \circ \mu$, where $\mu_{\alpha,0}$ is a stable completely random measure on $[0, 1]$ with index $0 < \alpha < 1$ with atoms given by the sequence $(U_i)$, and the law of $\mu$ on $\mathcal{X}$ is specified by (4). In particular,

$$T_{\alpha,0} \circ T_{\alpha,0} = T_{\alpha,0} T_{\alpha,0}$$

where $T_{\alpha,0}$ is independent of $T$. The density of $T_{\alpha,0}$ can be expressed as

$$(19) \quad f_{T_{\alpha,0}}(y) = \int_0^\infty f_\alpha(yt^{-1/\alpha}) t^{-1/\alpha} f_T(t) dt = \alpha \int_0^\infty f_T(\theta y) s^{-\alpha} y^{\alpha-1} f_\alpha(s) ds.$$

As a by-product, we obtain

$$c_{\alpha,\theta}(\rho) = 1/E[|T_{\alpha,0}|].$$

Call the family of laws associated with $S_{\alpha,\theta}$, or more specifically the $(W_i)$, as $PS(\rho, \alpha, \theta)$. In view of (18), the EPPF of the $PS(\rho, \alpha, \theta)$ model can be expressed as

$$(20) \quad p_{S,\alpha,\theta}(b_1, \ldots, b_k) := (-1)^{n-k} c_{\alpha,\theta}(\rho) \int_0^\infty \lambda^{\theta+n-\psi(\lambda)} \prod_{i=1}^k \tilde{\psi}_{\alpha,m}(\lambda) d\lambda$$

where $k$ is the number of blocks formed by the integers $\{1, \ldots, n\}$ and $\{b_1, \ldots, b_k\}$ are the sizes of the blocks $\{B_1, \ldots, B_k\}$, respectively. Additionally, similar to (3), for $m = 1, \ldots, n$,

$$\tilde{\psi}_{\alpha,m}(\lambda) := \frac{d^m}{d\lambda^m} \psi(\lambda^n) := (-1)^{m-1} \kappa_{\alpha,m}(\lambda),$$

where $\kappa_{\alpha,m}(\lambda)$ represents the $m$-th cumulant of a random variable with tilted density

$$e^{\tilde{\psi}_\alpha(\lambda)} e^{-\lambda y} f_{T_{\alpha,0}}(y),$$

specified by (19). The cumulants can be expressed in terms of the moments of this density

$$\int_0^\infty y^m e^{\tilde{\psi}_\alpha(\lambda)} e^{-\lambda y} f_{T_{\alpha,0}}(y) dy$$

using Theile’s recursion.

**Remark 5.** Note that $PS(\rho_{\beta,0}, \alpha, \theta) := PD(\alpha, \beta, \theta)$.

**Theorem 5.1** Suppose that for $0 < \alpha < 1$ and $\theta \neq 0$, $\theta > -\alpha$, PD$(\alpha, \theta)$ is coagulated by $PK(\rho, \gamma_{\theta/\alpha})$. Then the following results hold.

(i) PD$(\alpha, \theta)$ coagulated by $PK(\rho, \gamma_{\theta/\alpha})$ is $PS(\rho, \alpha, \theta)$, with EPPF specified in (20).

(ii) Equivalently, suppose that $P_{K, \theta/\alpha}$ is defined as in (4), with law determined by $P(\cdot | \rho H, \gamma_{\theta/\alpha})$, satisfying (19). Then the law of the composition $P_{\alpha, \theta} \circ P_{K, \theta/\alpha}$, is equivalent to the law of $S_{\alpha, \theta}$, specified by (19) and (17).

(iii) Suppose there are $K$ blocks $\{A_1, \ldots, A_K\}$ formed by the integers $\{1, \ldots, n\}$, each with size $a_i$, and $K \geq k$. The law of the corresponding $Q - FRAG$ kernel is determined by the (explicit) EPPF

$$p(a_1, \ldots, a_K) = p_{\alpha, \theta}(a_1, \ldots, a_K) \times \frac{\Gamma(\frac{\theta}{\alpha} + 1) \Gamma(\theta + n)}{\Gamma(\frac{\theta}{\alpha} + K) \Gamma(\theta + 1)} \frac{\prod_{i=1}^k \kappa_{\alpha,b_i}(\lambda) d\lambda}{\prod_{i=1}^k \kappa_{\alpha,1}(\lambda) d\lambda}$$

where $j_i, i = 1, \ldots, k$ (with $\sum_{i=1}^k j_i = K$), is defined as $\#\{\ell : A_\ell \subseteq B_i\}$. 

Proof. First apply (15) with $\tau_Q = P_{K,\theta/\alpha}$ and let $C_1$ and $C_2$ denote the appropriate constants. Now apply (14) to get

$$E[(1 + P_{K,\theta/\alpha}(g))^{-\theta/\alpha}|H] = C_1 \int_0^\infty e^{-\int_x \psi(y[(1 + zg(x))]^\alpha)H(dx)}y^{\theta/\alpha-1}dy.$$ 

Now apply the transformation $y = w^\alpha$ to get

$$E[(1 + P_{K,\theta/\alpha}(g))^{-\theta/\alpha}|H] = C_2 \int_0^\infty e^{-\int_x \psi(w^\alpha[(1 + zg(x))]^\alpha)H(dx)}w^{\theta-1}dw.$$ 

Setting $\tilde{\psi}_\alpha(w^\alpha[(1 + zg(x))]^\alpha) = \psi(w^\alpha[(1 + zg(x))]^\alpha)$, we see that the CS transform of order $\theta$ of the composition has the form in (14) with $\tilde{\psi}$ playing the role of $\psi$. This concludes the result. □

5.1 $PD(\alpha, \theta)$ coagulated by $PD(0, \nu)$, Beta Gamma and power tempered normalized Linnik processes

One might be somewhat surprised that Theorem 5.1 contains results for $PD(\alpha, \theta)$ coagulated by $PD(0, \nu)$ when $\nu > \theta/\alpha$. In other words, certain Dirichlet processes are $PK(\rho_0, \gamma_{\theta/\alpha})$ models. To see this, we recall the Beta Gamma process representation of Dirichlet processes given in James (2005). Specifically, if one chooses a parameter $\nu > \theta/\alpha$, then the law of $\mu$ given by

$$\frac{\Gamma(\nu)}{\Gamma(\nu - \theta/\alpha)} T^{-\theta/\alpha} \mathbb{P}(d\mu|\rho_0, \nu H)$$

is well defined. Relative to this law, $\mu$ is a Beta Gamma process with parameters $(\nu H, \theta/\alpha)$ as defined in James (2005). Setting $\theta = 0$ yields the law of a Gamma process with shape $\nu$. In any case, James (2005) shows that normalizing a Beta Gamma process of this type by its total mass yields a Dirichlet process with shape $\nu H$ for every $\nu > \theta/\alpha$. Hence the $PD(0, \nu)$ models are $PK(\rho_0, \nu, \theta/\alpha)$ models for $\nu > \theta/\alpha$. Note this equivalence does not hold for $\nu = \theta/\alpha$.

The corresponding $S_{\alpha, \theta}$ process is obtained by power tempering of a normalized process, where the law of the process is determined by

$$d_\nu(\lambda^\alpha) = \nu \ln(1 + \lambda^\alpha).$$

Now we may arrange to have a further scaling which results in the case where the law of $S_{\alpha, \theta}$ is equivalently obtained by the power tempering of a normalized Linnik process subordinator [see for instance Huillet (2000, 2003)], where its law is determined by

$$\tilde{\psi}_\alpha(\lambda) := d_\nu(\lambda^\alpha/\nu) = \nu \ln(1 + \lambda^\alpha/\nu) = \int_0^\infty (1 - e^{-\lambda s}) l_{\nu, \alpha}(s) ds,$$

and where

$$l_{\nu, \alpha}(s) = \frac{\alpha \nu}{s} \phi_\alpha(\nu s^\alpha)$$

is the Lévy density of the Linnik process. Specifically,

$$\phi_\alpha(q) = \mathbb{E}[e^{-q T_{\alpha, \theta}}] = \sum_{k=0}^\infty \frac{1}{\Gamma(1 + k \alpha)} (-q)^k$$

is the Mittag-Leffler function or equivalently the Laplace transform of the random random $T_{\alpha, \theta} - \alpha$. There, as before, $T_{\alpha, \theta}$ is a stable random variable of index $0 < \alpha < 1$. In other words, $PS(\rho_0, \nu, \alpha, \theta) = PK(l_{\nu, \alpha}, \gamma_{\theta})$ model.
5.1.1 EPPF calculations

Note that the above information allows for several descriptions of the EPPF of the $PK(l_{\nu,\alpha},\gamma\theta)$ model and hence the corresponding $Q-FRAG$. First using [21] one has that the $m$-th cumulant of an exponentially tilted Linnik random variable can be expressed as

$$
\kappa_{\alpha,m}(u) = \alpha u \int_0^\infty s^{m-1} e^{-us} \phi_\alpha(\nu s^\alpha) ds = \alpha u^{-m} \sum_{l=0}^\infty \frac{\Gamma(m+la)}{\Gamma(1+la)} u^{-l\alpha} (-\nu)^l
$$

or

$$
\kappa_{\alpha,m}(u) = \alpha u^{-m} \int_0^\infty \int_0^\infty s^{m-1} e^{-\nu(s/a)^{\alpha} - \alpha} f_\alpha(t) dt ds.
$$

The general EPPF of the $PK(l_{\nu,\alpha},\gamma\theta)$ model can be written as

$$
(22) \quad \frac{c_{\alpha,\theta} \Gamma(\nu)}{\Gamma(\nu-\theta/\alpha)} \frac{\nu^{\theta/\alpha} \alpha^{k-1}}{\Gamma(\theta + n)} \int_0^\infty y^{\theta/\alpha - 1} (1 + y)^{\nu} \prod_{j=1}^k R_{n_j}(y|\alpha) dy,
$$

where

$$
R_{n_j}(y|\alpha) = \sum_{l=0}^\infty \frac{\Gamma(n_j + l\alpha)}{\Gamma(1 + l\alpha)} (-y)^{-l} = \int_0^\infty \int_0^\infty s^{n_j-1} e^{-s - y - s^\alpha} f_\alpha(t) dt ds.
$$

Furthermore in the case of $\alpha = 1/2$, we may follow the results for Brownian excursion in Pitman (2003, Section 8), to obtain

$$
R_{n_j}(y|1/2) = \frac{\Gamma(2n_j)2^{-n_j+1/2}}{\Gamma(1/2)} \int_0^\infty h_{-2n_j}(x/y) e^{-x^2/2} dx
$$

where

$$
h_{-2n_j}(x) = \frac{2^{n_j-1}}{\Gamma(2n_j)} \int_0^\infty s^{n_j-1} e^{-s - x\sqrt{s}} ds
$$

is a Hermite function.

**Proposition 5.1** Suppose that for $0 < \alpha < 1$, $PD(\alpha,\theta)$ is coagulated by $PD(0,\nu)$ for $\nu > \theta/\alpha$, then the following results hold

(i) $PD(\alpha,\theta)$ coagulated by $PD(0,\nu)$ is $PK(l_{\nu,\alpha},\gamma\theta)$, with EPPF specified in [22].

(ii) Suppose there are $K$ blocks $\{A_1,\ldots,A_K\}$ formed by the integers $\{1,\ldots,n\}$, each with size $a_i$, and $K \geq k$. The law of the corresponding $Q-FRAG$ kernel is determined by the EPPF

$$
(23) \quad p_{\alpha,\theta}(a_1,\ldots,a_K) \times \frac{\Gamma(\nu - \theta/\alpha)\Gamma(\theta + n)}{c_{\alpha,\theta}\nu^{\theta/\alpha} \alpha^{k-1} \Gamma(\nu + K)} \int_0^\infty y^{\theta/\alpha - 1} (1 + y)^{\nu} \prod_{j=1}^k R_{n_j}(y|\alpha) dy
$$

where $j_i, i = 1,\ldots,k$ (with $\sum_{i=1}^k j_i = K$), is defined as $\# \{\ell : A_{\ell} \subseteq B_i\}$.

**Remark 6.** It is evident from Proposition 5.1 that it is not easy to obtain the denominator in [23] by summing out appropriately over the numerator. This is despite the fact that both the EPPF’s in the numerator have nice Gibbs form. Hence again this points to the difficulties of a direct combinatorial argument. On the other hand, the results establish some rather peculiar combinatorial identities.
6 PD(α, 0) coagulated by PK(ρ)

In view of the arguments in Bertoin and Le Gall (2003) and Pitman (2005, p. 115) concerning the Bolthausen-Sznitman (1998) coalescent, that is a description of PD(α, 0) coagulated by PD(β, 0), it is easy to extend this to the case of PD(α, 0) coagulated by PK(ρ).

**Theorem 6.1** Suppose that for 0 < α < 1 PD(α, 0) is coagulated by PK(ρ). Then the diagram, according to (5), of this process is described by setting θ = 0 in Theorem 5.1.

**Proof.** The proof proceeds along the same lines as Pitman (2005, p. 115) and Bertoin and Le Gall (2003, p. 272). That is,

\[
\frac{\mu_{α,0}(PK(\cdot))}{T_{α,0}} = \frac{\mu_{α,0}(\mu(\cdot))}{T_{α,0}} = \frac{L_{α,0}(\cdot)}{T_{α,0}} := S_{α,0}(\cdot)
\]

where \(L_{α,0}, S_{α,0}\) are as described in the beginning of this section. □

**Remark 7.** The description of PD(α, 0) coagulated by PD(0, ν) for ν > 0 is obtained from Proposition 5.1 with θ = 0.

7 Some comments about PD(0, θ) coagulated by general Q

It was noted earlier that obtaining results for the PD(0, θ) coagulated by some Q does not readily follow from a CS type analysis. We will now briefly describe how one can obtain the finite dimensional distributions of such compositions. Note again that a Dirichlet process coagulated by a Q, i.e., \(P_{0,θ} \circ τ_Q\) given \(τ_Q\), is a Dirichlet process with shape \(θτ_Q\) on \(X\). Hence it follows that for any measurable partition \(C_1, \ldots, C_m\) of \(X\) the finite dimensional distribution of \(P_{0,θ} \circ τ_Q\) given \(τ_Q\) is specified by the joint Dirichlet density of \(Y_i = P_{0,θ}(τ_Q(C_i))\) for \(i = 1, \ldots, m\), which is given by

\[
f(y_1, \ldots, y_m | τ_Q) = \frac{Γ(θ)}{Γ(θz_1)} \prod_{i=1}^{m} y_i^{θz_i-1}
\]

where \(z_i = τ_Q(C_i)\), and \((Y_1, \ldots, Y_m) \in \mathcal{S}_m = \{(a_i)_{i≤m} : 0 < a_i < 1, \sum_{i=1}^{m} a_i = 1\}\). This leads to a general description of the finite dimensional distributions.

**Proposition 7.1** Suppose that \(P_{0,θ}\) denotes a Dirichlet Process on [0, 1] with shape \(θU\) and \(U\) is a uniform distribution. Suppose further that \(τ_Q\) is a random probability measure on \(X\). Then, for a measurable partition \(C_1, \ldots, C_m\) of \(X\), the distribution of \(P_{θ,Q} = P_{0,θ} \circ τ_Q\) is specified by its finite-dimensional distribution

\[
f_{θ,Q}(y_1, \ldots, y_m) = ∫_{\mathcal{S}_m} f_Q(z_1, \ldots, z_m) \frac{Γ(θ)}{Γ(θz_1)} \prod_{i=1}^{m} y_i^{θz_i-1} dz_i
\]

where \(Y_i = P_{θ,Q}(C_i)\), and \(f_Q\) denotes the joint density of \(Z_i = Q(C_i)\) for \(i = 1, \ldots, m\). If \(Q\) is a species sampling model, then this equates to a description of the law of PD(0, θ) coagulated by \(Q\).

Naturally the utility of this result requires knowledge of the finite-dimensional distribution of \(Q\). Below we describe two special cases where \(Q = PD(1/2, η)\) for \(η > -1/2\) and \(Q = PD(0, ν)\) for \(ν > 0\).
Proposition 7.2 The distribution of $PD(0, \theta)$ coagulated by $PD(1/2, \eta)$ for $\eta > -1/2$ is determined by the distribution of $P_{0,\theta} \circ P_{1/2,\eta}$ with finite dimensional distribution,

$$\frac{\prod_{i=1}^m p_i}{\pi^{(m-1)/2} \Gamma(\eta + 1/2)} \int_{\mathcal{M}_m} \prod_{i=1}^m \Gamma(\theta z_i) \prod_{i=1}^m \theta z_i^{\theta z_i - 1} \prod_{i=1}^m \Gamma(p_i) \prod_{i=1}^m p_i^{p_i - 1} d\mathbf{z},$$

where $p_i = H(C_i)$ and otherwise the notation is as in Proposition 7.1.

PROOF. The distribution of $(Q(C_i))$ when $Q$ is $PD(1/2, \eta)$ is given by Theorem 3.1 of Carlton (2002). □

We now describe the Dirichlet case.

Proposition 7.3 The distribution of $PD(0, \theta)$ coagulated by $PD(0, \nu)$ is determined by the distribution of $P_{0,\theta} \circ P_{0,\nu}$ with finite dimensional distribution,

$$\int_{\mathcal{M}_m} \prod_{i=1}^m \Gamma(\theta z_i) \prod_{i=1}^m \theta z_i^{\theta z_i - 1} \prod_{i=1}^m \Gamma(\nu p_i) \prod_{i=1}^m p_i^{p_i - 1} d\mathbf{z},$$

where $p_i = H(C_i)$ and otherwise the notation is as in Proposition 7.1.

Remark 8. It is interesting to note that the dynamics of the coagulation of two or more Dirichlet processes may also be explained, in perhaps a more informative way, as a Chinese restaurant franchise process of Teh, Jordan, Beal and Blei (2006) when there is one franchise. In their setup, one has $F|\tau_Q$ is Dirichlet process with shape $(\theta \tau_Q)$ and $\tau_Q$ is a Dirichlet process with shape, say, $\eta H$. The distribution of $F$ is characterized via a Chinese restaurant franchise with one franchise. It follows from our observations that $F \overset{d}{=} P_{0,\theta} \circ P_{0,\nu}$, leading to an equivalence. Based on this observation, all the processes discussed here lead to some type of Chinese restaurant franchise process, and therefore have potential applications in machine learning and related areas.

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