A Comment on the Propagator
of the Radial Oscillator

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Using a hybrid approach, based on the recursion relations for shape invariant potentials developed by Das and Huang and a time-dependent transformation of variables, we derive the propagator for a radial oscillator. Although this is not a new result, we explicitly show that time-dependent transformations are very beneficial even within the context of time-independent Hamiltonians in quantum mechanics.

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After the introduction of Feynman path integrals in quantum mechanics [1] a lot of work has been undertaken to evaluate them for many systems. Today we know the path integrals for all soluble potentials of quantum mechanics. (For a recent review see reference [2]). In fact, more than one methods have been proposed for most of these potentials in order to calculate the corresponding path integrals. The purpose of this note is to present one more method for the potential of the radial oscillator

\[ V(x) = \frac{1}{2} m \omega^2 x^2 \frac{\hbar^2}{2m} \frac{n(n+1)}{x^2}, \quad n = 1, 2, 3, \ldots. \] (0.1)

(Be aware of the terminology: we work in one space dimension). This potential is known to be soluble by shape invariance [3] with superpotential:

\[ W(x) = \sqrt{m\omega} x \frac{\hbar}{\sqrt{m}} \frac{n}{x}. \]

The \( n \)-th hamiltonian of the corresponding series of shape invariant potentials is a simple harmonic oscillator

\[ H^{(n)} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \frac{1}{2} m \omega^2 x^2 \hbar \omega \left( \frac{2n}{2} - \frac{1}{2} \right), \]

for which the propagator is well-known:

\[ K^{(n)}(x, x'; t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} e^{-i\omega t(2n-\frac{1}{2})} \left\{ \exp \left[ -\frac{i\hbar}{2} \left( (x^2 + x'^2) \cot \omega t - \frac{2xx'}{\sin \omega t} \right) \right] - \exp \left[ -\frac{i\hbar}{2} \left( (x^2 + x'^2) \cot \omega t + \frac{2xx'}{\sin \omega t} \right) \right] \right\}. \] (0.2)

Notice that because the radial oscillator potential is singular, we must restrict the motion on the half line \( 0 \leq x < +\infty \); this induces to the propagator the additional term due to the reflection at the origin [4].

One way then to derive the corresponding propagator of the initial potential (0.1) is to use the recursion relations of Das and Huang [5]

\[ \left[ i\hbar \frac{\partial}{\partial t} - 2s\omega \hbar \right] K^{(s)}(x, x'; t) = \frac{1}{2} \left[ \frac{\hbar}{\sqrt{m}} \frac{\partial}{\partial x} - W(x) \right] \left[ \frac{\hbar}{\sqrt{m}} \frac{\partial}{\partial x'} - W(x') \right] K^{(s+1)}(x, x'; t). \] (0.2)
Notice that in order to derive the $s$-th propagator from the $(s+1)$-th one, we must perform an integration over the time variable; although this integration is straightforward, because of the trigonometric functions involved, it is a little cumbersome. In [3], [4], using the recursion relation (0.2) for $s = 0$, the exact propagator for the radial oscillator in the case $n = 1$ has been found although it is not written in the usual standard form (in terms of a Bessel function) but it is expressed in an equivalent form.

In order to derive the propagator for the radial oscillator for a general $n$, we propose the following alternative way. Using the phase transformation

$$\Psi(x, t) = \frac{1}{\sqrt{\cos \omega t}} \exp \left( -\frac{im\omega}{2\hbar} \tan \omega t x^2 \right) \Phi(x, t),$$

and the change of variables

$$\tau = \frac{\tan \omega t}{\omega}, \quad y = \frac{x}{\cos \omega t},$$

the Schrödinger equation of the radial oscillator

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \left( \frac{1}{2} m \omega^2 x^2 + \frac{\hbar^2}{2m} \frac{n(n+1)}{x^2} \right) \Psi,$$

simplifies to

$$i\hbar \frac{\partial \Phi}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial y^2} + \frac{\hbar^2}{2m} \frac{n(n+1)}{y^2} \Phi.$$

For this transformed problem, the frequency of the oscillator is zero and the $n$-th hamiltonian of the series of shape invariant potentials is the one of the free particle:

$$H^{(n)} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial y^2}.$$

The propagator for this last hamiltonian is just the free propagator, including an additional term due to reflection at the origin:

$$K(y, y'; \tau) = \sqrt{\frac{m}{2\pi\hbar \tau}} \left[ e^{i\frac{m(y-y')^2}{2\hbar \tau}} - e^{i\frac{m(y+y')^2}{2\hbar \tau}} \right].$$

Starting with the above propagator, the integrations over the time variable in the recursion relations of Das and Huang are much easier to be carried out. They have already been done in [3], [4]. The result is then given by:
\[ K(y, y'; \tau) = \sqrt{\frac{m}{2\pi\hbar\tau}} \left[ e^{\frac{im(y - y')^2}{2\hbar\tau}} \sum_{p=0}^{n} \frac{(n + p)!}{p!(n - p)!} \left( \frac{-i\hbar\tau}{2myy'} \right)^p 
+ (-1)^{n+1} e^{\frac{im(y + y')^2}{2\hbar\tau}} \sum_{p=0}^{n} \frac{(n + p)!}{p!(n - p)!} \left( \frac{i\hbar\tau}{2myy'} \right)^p \right] \tag{0.7} \]

Separating the odd from the even terms in the above summations, we can rewrite this result in the form

\[ K(y, y'; \tau) = \sqrt{\frac{m}{2\pi\hbar\tau}} (-2i^{1-n})e^{\frac{im}{2\hbar\tau}(y^2 + y'^2)} \left\{ \left[ \frac{\hbar}{2} \sum_{p=0}^{n} \frac{(-1)^p (n + 2p)!}{(2p)!(n - 2p)!(2z)^{2p}} \sin \left( z - \frac{n\pi}{2} \right) \right] 
+ \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^p (n + 2p + 1)!}{(2p + 1)!(n - 2p - 1)!(2z)^{2p+1}} \cos \left( z - \frac{n\pi}{2} \right) \right\} \tag{0.8} \]

where \([x]\) means the integer part of \(x\) and we have defined \(z \equiv myy'/\hbar\tau\). Now, using the definition of the Bessel functions

\[ J_n(z) = \left( \frac{z}{2} \right)^n \sum_{p=0}^{+\infty} \frac{(-1)^p z^{2p}}{2^p p!(n + p)!}, \tag{0.9} \]

we can show that

\[ J_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left[ \frac{\hbar}{2} \sum_{p=0}^{n} \frac{(-1)^p (n + 2p)!}{(2p)!(n - 2p)!(2z)^{2p}} \sin \left( z - \frac{n\pi}{2} \right) \right] 
+ \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^p (n + 2p + 1)!}{(2p + 1)!(n - 2p - 1)!(2z)^{2p+1}} \cos \left( z - \frac{n\pi}{2} \right) \right\} \tag{0.10} \]

Comparing equations (0.8) and (0.10) we derive the following standard expression [2] for the propagator of the potential \(V(y) = \frac{k^2}{2m} \frac{n(n+1)}{y^2}y^2\):

\[ K(y, y'; \tau) = \frac{m\sqrt{yy'}}{\hbar\tau} i^{-(n+3/2)} e^{\frac{im}{2\hbar\tau}(y^2 + y'^2)} J_{n+1/2} \left( \frac{myy'}{\hbar\tau} \right). \tag{0.11} \]

We can return to the original problem of constructing the propagator of the radial oscillator potential (0.1) by making use of the phase transformation (0.3) and the substitution (0.4). The final result is

\[ K(x, x'; t) = \frac{m\omega}{{\hbar\sin}\omega t} i^{-(n+3/2)} \exp \left\{ \frac{i\omega}{2\hbar}\cot\omega t \left( x^2 + x'^2 \right) \right\} J_{n+1/2} \left( \frac{m\omega xx'}{\hbar\sin\omega t} \right). \tag{0.12} \]

This is the standard form of the propagator found in literature [2].
Taking into account the method of derivation presented above, it is worthwhile to stress that time-dependent transformations may be very useful even in the context of time-independent quantum mechanics. This raises the question if all propagators of exactly soluble potentials can be obtained by transforming the original problem to another simpler one using time dependent transformations. Incidentally, notice that the harmonic oscillator can be reduced to the problem of a free particle by using the same transformation.

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