EIT IN A LAYERED ANISOTROPIC MEDIUM

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(Communicated by Mikko Salo)

ABSTRACT. We consider the inverse problem in geophysics of imaging the subsurface of the Earth in cases where a region below the surface is known to be formed by strata of different materials and the depths and thicknesses of the strata and the (possibly anisotropic) conductivity of each of them need to be identified simultaneously. This problem is treated as a special case of the inverse problem of determining a family of nested inclusions in a medium $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. Introduction. We consider the inverse Calderón problem [9], also known as Electrical Impedance Tomography (EIT) or, in geophysics, as Direct Current (DC) method, of determining a matrix-valued conductivity $\sigma(x)$ of a body $\Omega$ in which electrostatic equilibrium is modelled by the elliptic equation

$$\text{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega,$$

where $u$ represents the electrostatic potential and the available measurements are all possible pairs of current fluxes $\sigma \nabla u \cdot \nu|_{\Sigma}$ and boundary voltages $u|_{\Sigma}$ collected on a given open portion $\Sigma$ of $\partial \Omega$.

When $\sigma$ is isotropic, that is $\sigma = \gamma I$, $I$ denotes the identity matrix and $\gamma$ is a scalar function on $\Omega$, a vast literature is available and the theory has achieved a substantial level of completeness, see, for instance [29]. With the celebrated counterexample by Tartar [19], the general anisotropic problem still poses several open issues. A principal line of investigation concerning anisotropy in EIT has been of proving uniqueness modulo a change of variables which fixes the boundary [22, 28, 26, 20, 21, 7, 6]. In most applications, however, knowledge of position and, hence, coordinates are important. In this direction, certain, diverse results are available [19, 2, 4, 5, 13, 14, 23, 18]. In [3] a uniqueness result was obtained.
when the unknown anisotropic conductivity is assumed to be piecewise constant on a given domain partition, or segmentation, with non-flat interfaces. Non-flatness shall be rigorously defined in Section 2.1 where our other definitions are given as well. We recall that in [3] we also specialized Tartar’s counterexample to the case of a half space and a constant conductivity thus demonstrating that the non-flatness condition on boundary and interfaces is necessary.

Here we address the more general problem when also the interfaces defining the domain partition are unknown. In this respect, in the context of elastostatics, Cărstea, Honda and Nakamura [10] obtained uniqueness from a local boundary map of a piecewise constant anisotropic elasticity tensor where the partitioning is allowed to be unknown, provided it is formed by subanalytic sets [8]. Here, for EIT, we significantly relax the regularity requirements on the domain partition on the one hand, but impose stricter conditions on the configuration on the other hand. More precisely, we treat the case in which the interfaces are the (non-flat) \( C^{1,\alpha} \) boundaries of a nested family of subdomains \( \Omega_k, \Omega_{k+1} \subset \subset \Omega_k \subset \subset \Omega, \ k = 1, \ldots, K \). Within this setting, assuming that the unknown conductivity \( \sigma \) has the structure

\[
\sigma = \sum_{k=1}^{K+1} \sigma_k \chi(\Omega_{k-1} \setminus \pi_k),
\]

where we understand that \( \Omega_0 = \Omega, \Omega_{K+1} = \emptyset \) (so that the innermost layer consists of all of \( \Omega_K \)) and \( \sigma_k \) are positive definite constant matrices satisfying the jump or visibility conditions

\[
\sigma_k \neq \sigma_{k+1}, \text{ for all } k = 1, \ldots, K,
\]

we prove (Theorem 2.6 below) that \( \sigma \) is uniquely determined from the knowledge of the local Neumann-to-Dirichlet (N-D) map

\[
N_{\sigma}^\Sigma : \sigma \nabla u \cdot \nu|_{\Sigma} \longrightarrow u|_{\Sigma},
\]

for all solutions \( u \in H^1(\Omega) \) to (1). Here \( \Sigma \) is an open (non-flat) portion of \( \partial \Omega \).

From a geological perspective, the mentioned stratification arises naturally in sedimentary basins, containing hydrocarbon reservoirs. Through the electrical conductivity, indeed, a geological image can be obtained from boundary data. This is because the electrical conductivity of Earth’s materials varies over many orders of magnitude while it depends upon many factors, including rock type, porosity, connectivity of pores or permeability, nature of fluid, and metallic content of a solid matrix. The representation of conductivity used in this paper was motivated by the work of Loke, Acworth and Dahlin [24] and Farquharson [12]. In the DC inverse problem, the location of the boundaries or interfaces and the conductivities are unknown, though the occurrence of stratification might be inferred from independent or joint imaging of seismic data [17, 15]. In reality, the stratification will have a finite extent, that is, appear in some cylindrical cut of the domain’s interior or subsurface. We adapt our analysis to this case, in a variation of Theorem 2.6 in which we assume that \( \sigma \) satisfies the layered structure assumption only on a subdomain \( C \) of which the “top” boundary, contained in \( \partial \Omega \), is an appropriate neighborhood of \( \Sigma \) where measurements are collected. We then uniquely determine \( \sigma \) in \( C \) only. In the absence of a uniqueness result, this inverse problem has been extensively studied in geophysics, mostly through experimenting with optimization and sometimes motivated by a statistics framework [25, 11, 30, 16]. Most recently,
a regularization emphasizing sharp boundaries has been investigated by Paré and Li [27]; this strategy is closely aligned with our analysis.

2. Main result.

2.1. Notation and definition. In several places in this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points \( x \in \mathbb{R}^n \) will be adopted. For \( n \geq 3 \), a point \( x \in \mathbb{R}^n \) will be denoted by \( x = (x', x_n) \), where \( x' \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R} \). Moreover, given a point \( x \in \mathbb{R}^n \), we shall denote with \( B_r(x), B'_r(x) \) the open balls in \( \mathbb{R}^n, \mathbb{R}^{n-1} \) respectively centred at \( x \) with radius \( r \) and by \( Q_r(x) \) the cylinder \( B'_r(x') \times (x_n - r, x_n + r) \). We shall denote \( \mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\} \), \( B_r^+ = B_r \cap \mathbb{R}_+^n \), where we understand \( B_r = B_r(0) \) and \( Q_r = Q_r(0) \).

We shall assume throughout that \( \Omega \) is a bounded domain with Lipschitz boundary, see e.g. [1, 4.9].

**Definition 2.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Given \( \alpha, \alpha \in (0,1) \), we say that a portion \( \Sigma \) of \( \partial \Omega \) is of class \( C^{1,\alpha} \) if for any \( P \in \Sigma \) there exists a rigid transformation of \( \mathbb{R}^n \) under which we have \( P = 0 \) and

\[
\Omega \cap Q_{r_0} = \{ x \in Q_{r_0} | x_n > \varphi(x') \},
\]

where \( \varphi \) is a \( C^{1,\alpha} \) function on \( B_{r_0}^+ \) satisfying

\[
\varphi(0) = |\nabla_{x'} \varphi(0)| = 0.
\]

**Definition 2.2.** Given \( \Sigma \) as above, we shall say that such a portion of a surface is non-flat (and equivalently the function \( \varphi \)) at a point \( P \in \Sigma \) if, considering the reference system and the function \( \varphi \) as above, we have that \( \varphi \) is not identically zero in any open neighborhood of \( P = 0 \).

**Definition 2.3.** We shall say that a whole boundary \( \partial \Omega \) is non-flat if for each \( P \in \partial \Omega \) there exists an open portion \( \Sigma \) of \( \partial \Omega \) such that \( P \in \Sigma, \Sigma \) is of class \( C^{1,\alpha} \) and it is non-flat at \( P \).

The Neumann-to-Dirichlet map. We denote by \( \text{Sym}_n \) the class of \( n \times n \) symmetric real valued matrices. Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega \) and assume that \( \sigma \in L^\infty(\Omega, \text{Sym}_n) \) satisfies the ellipticity condition

\[
\lambda^{-1} |\xi|^2 \leq \sigma(x) \xi \cdot \xi \leq \lambda |\xi|^2, \quad \text{for almost every } x \in \Omega,
\]

(5)

for every \( \xi \in \mathbb{R}^n \).

We shall also denote by \( \langle \cdot, \cdot \rangle \) the \( L^2(\partial \Omega) \)-pairing between \( H^{\frac{1}{2}}(\partial \Omega) \) and its dual \( H^{-\frac{1}{2}}(\partial \Omega) \).

We consider the following function spaces

\[
0H^{\frac{1}{2}}(\partial \Omega) = \left\{ f \in H^{\frac{1}{2}}(\partial \Omega) | \int_{\partial \Omega} f = 0 \right\},
\]

\[
0H^{-\frac{1}{2}}(\partial \Omega) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial \Omega) | \langle \psi, 1 \rangle = 0 \right\}.
\]

We define the global Neumann-to-Dirichlet map as follows.

**Definition 2.4.** The Neumann-to-Dirichlet (N-D) map associated with \( \sigma \),

\[
\mathcal{N}_\sigma : 0H^{-\frac{1}{2}}(\partial \Omega) \rightarrow 0H^{\frac{1}{2}}(\partial \Omega)
\]
is given by the selfadjoint operator satisfying

\[ \langle \psi, N_{\sigma} \psi \rangle = \int_{\Omega} \sigma(x) \nabla u(x) \cdot \nabla u(x) \, dx, \]

for every \( \psi \in \_0^H H^{-\frac{1}{2}}(\partial \Omega) \), where \( u \in H^1(\Omega) \) is the weak solution to the Neumann problem

\[
\begin{cases}
\text{div}(\sigma \nabla u) = 0, & \text{in } \Omega, \\
\sigma \nabla u \cdot \nu|_{\partial \Omega} = \psi, & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u = 0.
\end{cases}
\]

Note that from (6) the bilinear form

\[ \langle \varphi, N_{\sigma} \psi \rangle = \int_{\Omega} \sigma(x) \nabla u_{i}(x) \cdot \nabla u_{j}(x) \, dx, \]

can be defined by polarization in a straightforward fashion. Given \( \sigma^{(i)} \in L^{\infty}(\Omega, Sym_n) \), satisfying (5), for \( i = 1, 2 \), the following identity can be recovered from Alessandrini’s identity (see [2, (b), p. 253]) and the (obvious) equality

\[ N_{\sigma^{(i)}}^{-1} - N_{\sigma^{(2)}}^{-1} = N_{\sigma^{(2)}} (N_{\sigma^{(2)}} - N_{\sigma^{(1)}}) N_{\sigma^{(1)}}^{-1}, \]

that is,

\[ \langle \sigma^{(1)} \nabla u_{1} \cdot \nu, (N_{\sigma^{(2)}} - N_{\sigma^{(1)}}) \sigma^{(2)} \nabla u_{2} \cdot \nu \rangle \]

\[ = \int_{\Omega} \left( \langle \sigma^{(1)}(x) - \sigma^{(2)}(x) \rangle \nabla u_{1}(x) \cdot \nabla u_{2}(x) \right), \]

for any \( u_{i} \in H^1(\Omega) \) being the weak solution to

\[ \text{div}(\sigma^{(i)}(x) \nabla u_{i}(x)) = 0, \quad \text{in } \Omega, \]

for \( i = 1, 2 \).

Now we introduce the local version of the N-D map. Let \( \Sigma \) be an open portion of \( \partial \Omega \) and let \( \Delta = \partial \Omega \setminus \Sigma \). We introduce the subspace of \( H^{\frac{1}{2}}(\partial \Omega) \),

\[ H_{\Delta}^{\frac{1}{2}}(\Delta) = \left\{ f \in H^{\frac{1}{2}}(\partial \Omega) \mid \text{supp}(f) \subset \Delta \right\}. \]

We denote by \( H_{\Delta 00}^{\frac{1}{2}}(\Delta) \) the closure in \( H^{\frac{1}{2}}(\Delta) \) of the space \( H_{\Delta 0}^{\frac{1}{2}}(\Delta) \) and we introduce

\[ _0H^{-\frac{1}{2}}(\partial \Omega) = \left\{ \psi \in _0H^{-\frac{1}{2}}(\partial \Omega) \mid \langle \psi, f \rangle = 0, \quad \text{for any } f \in H_{\Delta 00}^{\frac{1}{2}}(\Delta) \right\}, \]

that is, the space of distributions \( \psi \in H^{-\frac{1}{2}}(\partial \Omega) \) which are supported in \( \Sigma \) and have zero average on \( \partial \Omega \). The local N-D map is then defined as follows.

**Definition 2.5.** The local Neumann-to-Dirichlet map associated with \( \sigma, \Sigma \) is the operator \( N_{\sigma}^{\Sigma} : _0H^{-\frac{1}{2}}(\partial \Omega) \rightarrow (\_0H^{-\frac{1}{2}}(\Sigma))^\ast \subset _0H^{\frac{1}{2}}(\partial \Omega) \) given by

\[ \langle \varphi, N_{\sigma}^{\Sigma} \psi \rangle = \langle \varphi, N_{\sigma} \psi \rangle, \]

for every \( \varphi, \psi \in _0H^{-\frac{1}{2}}(\Sigma) \).

Given \( \sigma^{(i)} \in L^{\infty}(\Omega, Sym_n) \), satisfying (5), for \( i = 1, 2 \), we also recover from (8)

\[ \langle \psi_{1}, (N_{\sigma^{(2)}}^{\Sigma} - N_{\sigma^{(1)}}^{\Sigma}) \psi_{2} \rangle = \int_{\Omega} \left( \sigma^{(1)}(x) - \sigma^{(2)}(x) \right) \nabla u_{1}(x) \cdot \nabla u_{2}(x), \]
for any $\psi_i \in \alpha H^{-\frac{1}{2}}(\Sigma)$, for $i = 1, 2$ and $u_i \in H^1(\Omega)$ being the unique weak solution to the Neumann problem
\begin{align}
\begin{cases}
\text{div}(\sigma^{(i)} \nabla u_i) = 0, & \text{in } \Omega, \\
\sigma^{(i)} \nabla u_i \cdot \nu |_{\partial \Omega} = \psi_i, & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_i = 0.
\end{cases}
\end{align}
(13)

2.2. The a-priori assumptions. The assumptions pertaining to the domain partition are

1. $\Omega \subset \mathbb{R}^n$ is a bounded domain, with $n \geq 3$.
2. $\partial \Omega$ is of Lipschitz class.
3. We fix a connected open non-empty subset $\Sigma$ of $\partial \Omega$ (where the measurements in terms of the local N-D map are taken) and assume there exists $\alpha, \alpha \in (0, 1)$ such that $\Sigma$ is $C^{1, \alpha}$ and non-flat.

More specifically we assume that there exists $P_0 \in \Sigma$ and a rigid transformation of coordinates under which we have $P_0 = 0$ and
\begin{align}
\Sigma \cap Q_{r_0/3} &= \{ x \in Q_{r_0/3} | x_n = \varphi_0(x') \} \\
(\mathbb{R}^n \setminus \Omega) \cap Q_{r_0/3} &= \{ x \in Q_{r_0/3} | x_n < \varphi_0(x') \} \\
\Omega \cap Q_{r_0/3} &= \{ x \in Q_{r_0/3} | x_n > \varphi_0(x') \},
\end{align}
(14)
where $\varphi_0$ is a non-flat $C^{1, \alpha}$ function on $B'_{r_0/3}$ satisfying
$$\varphi_0(0) = |\nabla \varphi_0(0)| = 0.$$

4. Let $K$ be a positive integer and let $\Omega_0, \Omega_1, \ldots, \Omega_K$ be nested domains
$$\Omega_K \subset \subset \Omega_{K-1} \subset \subset \ldots \Omega_0 = \Omega.$$

For $k = 1, \ldots, K$ we denote
\begin{align}
D_k &= \Omega_{k-1} \setminus \overline{\Omega}_k,
\end{align}
(15)
where for $k = K + 1$ we set
$$D_{K+1} = \Omega_K$$
and we assume that all $D_k$ are connected.

5. We assume that $\partial \Omega_k$ is $C^{1, \alpha}$ and it is non-flat according to Definition 2.3, for every $k = 1, \ldots, K$.

We then assume that the conductivity $\sigma \in L^\infty(\Omega, \text{Sym}_n)$, satisfies the uniform ellipticity condition (5) and is of type
\begin{align}
\sigma(x) = \sum_{k=1}^{K+1} \sigma_k \chi_{D_k}(x), \quad x \in \Omega,
\end{align}
(16)
where the $\sigma_k$ are positive definite constant matrices, for $k = 1, \ldots, K + 1$.

2.3. Global uniqueness. Our main result is stated below.

**Theorem 2.6.** Let $K_i \in \mathbb{N} \setminus \{0\}$ and $\Omega, \Sigma, \Omega_k^{(i)}, k = 0, \ldots, K_i, D_k^{(i)}, k = 1, \ldots, K_i + 1$, for $i = 1, 2$ satisfy assumptions 1. – 5. of subsection 2.2. If $\sigma^{(i)}$, $i = 1, 2$ are two conductivities of type
\begin{align}
\sigma^{(i)}(x) = \sum_{k=1}^{K_i+1} \sigma_k^{(i)} \chi_{D_k^{(i)}}(x), \quad x \in \Omega, \ i = 1, 2,
\end{align}
(17)
where \( \sigma_k^{(i)} \in \text{Sym}_n \) are positive definite constant matrices satisfying
\[
\sigma_k^{(i)} \neq \sigma_k^{(i+1)} \quad k = 1, \ldots, K_i
\]
and the uniform ellipticity condition (5), for \( k = 1, \ldots, K_i + 1 \) and
\[
\mathcal{N}_\sigma^{\Sigma(1)} = \mathcal{N}_\sigma^{\Sigma(2)},
\]
then
\[
K_1 = K_2 := K,
\]
(20) \( \Omega_k^{(1)} = \Omega_k^{(2)} \) and \( \sigma_k^{(1)} = \sigma_k^{(2)} \), for any \( k = 0, \ldots, K \).

3. Proof of the main result.

Proof of Theorem 2.6. We assume without loss of generality that \( K_1 = \min \{K_1, K_2\} \). First, we prove (20) for \( k = 0, \ldots, K_1 \). We proceed by induction on \( k \), \( 0 \leq k \leq K_1 \).

For the case \( k = 0 \), \( \Omega_0^{(1)} = \Omega_0^{(2)} \) trivially holds true. By rephrasing the arguments used in [3, Theorem 2.1] we obtain that the equality of the maps
\[
\mathcal{N}_\sigma^{\Sigma(1)} = \mathcal{N}_\sigma^{\Sigma(2)}
\]
implies that, denoting by \( E_1 \) the connected component of \( \Omega \setminus (\Omega_1^{(1)} \cup \Omega_1^{(2)}) \) such that \( \Sigma \subset \partial E_1 \), we have
\[
\sigma_1^{(1)} = \sigma_1^{(2)} \quad \text{in} \quad E_1.
\]
(21)

In fact (21) is obtained as follows.

- The knowledge of \( \mathcal{N}_\sigma^{\Sigma} \) enables us to determine the tangential asymptotics near the singularity of the Neumann kernel \( N_\sigma(\cdot, y) \) for each \( y \in \Sigma \) [3, Lemma 3.8]. Here \( N_\sigma(\cdot, y) \) is defined as the distributional solution of the following boundary value problem
\[
\begin{align*}
\text{div}(\sigma \nabla N_\sigma(\cdot, y)) &= 0, & \text{in} \quad \Omega \\
\sigma \nabla N_\sigma(\cdot, y) \cdot \nu &= \delta(\cdot - y) - \frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega.
\end{align*}
\]

- The tangential asymptotics of \( N_\sigma(\cdot, y) \) allows us to identify the tangential \( (n - 1) \times (n - 1) \) submatrices \( g_{n-1}(y) \) of the metric
\[
g = (\det \sigma)^{\frac{1}{n-2}} \sigma^{-1}
\]
associated to the elliptic operator \( \text{div}(\sigma \nabla \cdot) \) [3, Lemma 3.5].

- The non-flatness assumptions of \( \Sigma \) permits us to find enough independent tangent planes so to determine all of \( g \) (hence \( \sigma \)) provided \( \sigma \) is locally constant [3, Lemma 3.6].

Next we prove the induction step. Let \( 1 \leq k \leq K_1 \). We assume that for every \( j = 0, \ldots, k-1 \)
(22) \( \Omega_j^{(1)} = \Omega_j^{(2)} \) and \( \sigma_j^{(1)} = \sigma_j^{(2)} \)
and suppose by contradiction that
(23) \( \partial\Omega_k^{(1)} \setminus \Omega_k^{(2)} \neq \emptyset \)
(the symmetric case \( \partial\Omega_k^{(2)} \setminus \Omega_k^{(1)} \neq \emptyset \) being equivalent). We denote by \( E_k \) the connected component of \( \Omega \setminus (\Omega_k^{(1)} \cup \Omega_k^{(2)}) \) such that \( \Sigma \subset \partial E_k \). Let us fix an open
portion of \((\partial \Omega_k^{(1)} \setminus \overline{\Omega_k^{(2)}}) \cap \partial E_k\), which we denote by \(\Sigma_k\) and let us select a subdomain \(E_k \subset E_k\) such that \(E_k\) and \(F_k = \Omega \setminus E_k\) have both Lipschitz boundary and such that
\[
\Sigma \cup \Sigma_k \subset \partial E_k.
\]

We note that \(\sigma^{(1)} = \sigma^{(2)}\) in \(E_k\). Let us denote by \(N_{\sigma(i)}^{\Sigma_k}\) the local N-D maps for \(\sigma^{(i)}\) in \(F_k\). Then [3, Claim 4.1] implies that
\[
(24) \quad N_{\sigma(i)}^{\Sigma_k} = N_{\sigma(i)}^{\Sigma_k}.
\]

Note that the set \(D\) appearing in [3, Claim 4.1] needs to be replaced by \(E_k\).

Let us fix \(y_k \in \Sigma_k\) and a neighborhood \(U_k\) of \(y_k\) in \(F_k\) such that \(U_k \cap \Omega^{(2)}_k = \emptyset\).

We can choose \(U_k\) small enough so that \(\sigma^{(1)}\) and \(\sigma^{(2)}\) are both constant in \(U_k\). Using once more [3, Lemma 3.6] we obtain
\[
(25) \quad \sigma^{(1)}_{k+1} = \sigma^{(2)}_k,
\]
which, combined with (22), implies that
\[
(26) \quad \sigma^{(1)}_{k+1} = \sigma^{(1)}_k.
\]

Hence by (18) we have reached a contradiction with the assumption (23), and therefore
\[
(27) \quad \Omega^{(1)}_k = \Omega^{(2)}_k.
\]

Once we know that such domains coincide, (24) in combination with [3, Lemma 3.6] implies
\[
(28) \quad \sigma^{(1)}_{k+1} = \sigma^{(2)}_{k+1};
\]
thus, the induction step is proven and (20) holds true for \(k = 0, \ldots, K_1\). In particular, this implies that
\[
(29) \quad \Omega^{(1)}_{K_1} = \Omega^{(2)}_{K_1} := \Omega_{K_1} \quad \text{and} \quad \sigma^{(1)}_{K_1+1} = \sigma^{(2)}_{K_1+1} \quad \text{on} \quad D^{(2)}_{K_1+1}.
\]

Next, we show that \(K_1 = K_2 := K\). Suppose, on the contrary, \(K_2 > K_1\) and denote
\[
(30) \quad \tilde{\Omega}^{(1)}_k = \Omega^{(2)}_k, \quad \text{for} \quad k = K_1 + 1, \ldots, K_2;
\]
\[
(31) \quad \tilde{D}^{(1)}_k = D^{(2)}_k, \quad \text{for} \quad k = K_1 + 1, \ldots, K_2
\]
and
\[
(32) \quad \tilde{\sigma}^{(1)}_k = \sigma^{(1)}_{K_1+1}, \quad \text{for} \quad k = K_1 + 1, \ldots, K_2.
\]

Let \(\Sigma_{K_1+1}\) be a non-empty portion of \(\partial \tilde{\Omega}^{(1)}_{K_1+1} = \partial \Omega^{(2)}_{K_1+1}\). By the same argument adopted above, we have that
\[
\sigma^{(1)} = \sigma^{(2)} \quad \text{on} \quad \Omega \setminus \tilde{\Omega}^{(1)}_{K_1+1} = \Omega \setminus \overline{\Omega^{(2)}_{K_1+1}},
\]
which combined with [3, Claim 4.1] leads to
\[
N_{\sigma^{(1)}}^{\Sigma_{K_1+1}} = N_{\sigma^{(2)}}^{\Sigma_{K_1+1}};
\]
whence
\[
(33) \quad \sigma^{(1)}_{K_1+2} = \sigma^{(2)}_{K_1+2}.
\]
Moreover, by (32) we have that
\[ \sigma_{K_1+1}^{(1)} = \sigma_{K_1+2}^{(1)}, \]
which combined with (29) and (33), implies that
\[ \sigma_{K_1+1}^{(2)} = \sigma_{K_1+2}^{(2)}. \]

The latter contradicts (18), therefore \( K_1 = K_2 \) which concludes the proof. \( \square \)

We conclude with presenting a variation of the result obtained in Theorem 2.6 which, we believe, is of interest in the context of imaging materials with a structure that is layered locally only. Let \( \Sigma \) and \( \varphi \) be the portion (where the measurements are collected) and the function respectively introduced in Definitions 2.1, 2.2.

We denote
\[ C = \{ x \in \mathbb{R}^n \mid |x'| \leq R, \varphi \leq x_n \leq M \}, \]
for some positive numbers \( R \) and \( M \). Suppose \( C \subset \Omega \) and also that
\[ \partial C \cap \partial \Omega = \{ x \in \mathbb{R}^n \mid |x'| \leq R, \quad x_n = \varphi(x') \} \supset \Sigma. \]

Let \( \varphi_1, \ldots, \varphi_K : B_R^1 \to \mathbb{R} \) be \( C^1, \alpha \) functions, non-flat at every point as in Definitions 2.1, 2.2 which satisfy
\[ \varphi(x') \equiv \varphi_0(x') < \varphi_1(x') < \cdots < \varphi_K(x') < M, \quad \text{for all} \quad x' \in B_R^1. \]

For \( k = 1, \ldots, K \) denote
\[ D_k = \{ x \in C \mid \varphi_{k-1}(x') < x_n < \varphi_k(x') \} \]
and assume that \( \sigma \in L^\infty(\Omega, Sym_n) \) satisfies the uniform ellipticity condition (5) and
\[ \sigma(x) = \sum_{k=1}^{K} \sigma_k \chi_{D_k}(x), \quad x \in C, \]
where each \( \sigma_k \) is a positive definite constant matrix and
\[ \sigma_k \neq \sigma_{k+1}, \quad \text{for all} \quad k = 1, \ldots, K - 1. \]

With the above setting, we have the following uniqueness result confined to a subdomain \( C \) of \( \Omega \)

**Theorem 3.1.** \( N_{\sigma}^\Sigma \) uniquely determines \( \sigma \) within \( C \).

**Proof.** The proof follows the same line of the proof of Theorem 2.6. Let \( \sigma^{(1)}, \sigma^{(2)} \) satisfy the above structure conditions, that is,
\[ \sigma^{(i)}(x) = \sum_{k=1}^{K_i} \sigma_k^{(i)} \chi_{D_k^{(i)}}(x), \quad x \in \Omega, \quad i = 1, 2, \]
where \( \sigma_k^{(i)} \in Sym_n \) are constant matrices and the layers \( D_k^{(i)} \) are described by the functions \( \varphi_j^{(i)}, i = 1, 2 \). The fact that, within \( C \), the various interfaces are graphs with respect to the same reference system, enables us to select the inner boundary portions \( \Sigma_k \) in such a way that they are all contained in \( C \). The sets \( E_k \) can be explicitly expressed as
\[ \{ x \in \mathbb{R}^n : |x'| < R, \quad \varphi \leq x_n \leq \min\{ \varphi_1^{(1)}(x'), \varphi_2^{(2)}(x') \} \}. \]

Hence \( N_{\sigma^{(1)}} = N_{\sigma^{(2)}} \) leads to \( \sigma^{(1)} = \sigma^{(2)} \) within the set \( C \). \( \square \)
Acknowledgments. The research carried out by G. Alessandrini and E. Sincich for the preparation of this paper has been supported by FRA 2016 “Problemi inversi, dalla stabilità alla ricostruzione” funded by Università degli Studi di Trieste. M.V de Hoop was partially supported by the Simons Foundation under the MATH + X program, the National Science Foundation under grant DMS-1559587, and by the members of the Geo-Mathematical Group at Rice University. R. Gaburro acknowledges the support of MACSI, the Mathematics Applications Consortium for Science and Industry (http://www.macsi.ul.ie), funded by the Science Foundation Ireland Investigator Award 12/IA/1683. E. Sincich has also been supported by Gruppo Nazionale per l’ Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) by the grant “Analisi di problemi inversi: stabilità e ricostruzione”.

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Received August 2017; 1st revision September 2017; 2nd revision December 2017.

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