Topological order and reflection positivity

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received 19 December 2013; accepted in final form 11 February 2014
published online 3 March 2014

PACS 03.65.-w – Quantum mechanics
PACS 05.30.Fk – Fermion systems and electron gas
PACS 05.30.Pr – Fractional statistics systems (anyons, etc.)

Abstract – The focus of this paper is twofold. First, we observe that Hamiltonians displaying both topological order and reflection positivity have an interesting property: expectations in different ground-state vectors of a given local operator $W_A$ have the same sign. Secondly, we illustrate this result with a specific Majorana Hamiltonian, related to the toric code which is widely studied in quantum information theory. We show that expectations of reflection-symmetric loops in ground states of this Hamiltonian are vortex-free or vortex-full.

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Introduction. – Topologically ordered systems have attracted much attention, since they represent promising candidates for the realization of a fault-tolerant quantum computing architecture. Intuitively, topological order for a Hamiltonian $H$ is the property that perturbation by a local operator cannot cause transitions between the different degenerate ground states. Such transitions require perturbations by a global operator. Therefore the subspace of topologically ordered ground states seems to be a good place to store and process quantum information [1–4].

Another fundamental concept that we use is reflection positivity. This notion originally arose in the theory of random fields, as the property that justifies inverse Wick rotation from random fields to quantum fields [5], both at zero and at positive temperature [6]. This property has also played an important role in the analysis of phase transitions and ground states in statistical mechanical systems [7–9].

In this letter we point out that combining these two concepts yields insight into the structure of ground states. We show that the expectation value of certain operators is non-negative. This is our main result.

Finally, in the third part we apply our method to a Hamiltonian introduced in [10] which is a quartic polynomial in Majoranas (Majorana operators) defined on a planar lattice.

We focus on this interaction because, in lowest-order perturbation theory, it is equivalent to the toric code model, the archetypical model for a topologically ordered quantum memory [1]. We show in this example that, with our choice of the signs of the coupling constants, all the ground states are free of vortices.

Understanding the ground-state properties of topologically ordered systems is not only interesting from a fundamental point of view, but is also relevant to topological quantum computation: the ground states encode the logical-qubit states.

Loops and homotopy on the lattice. – Curves and loops. Consider a finite lattice $\Lambda$ embedded in a manifold $\mathcal{M}$ of arbitrary dimension and containing $|\Lambda|$ sites. We assign a distance $d_{i,i'}$ between each pair of sites $i, i'$, which is the length of the geodesic $(i, i')$ in $\mathcal{M}$ that connects these sites. We also refer to $(i, i')$ as the directed bond connecting site $i$ to the site $i'$.

Define a curve $C$ in $\Lambda$ as an ordered sequence of sites $\{i_1, i_2, \ldots, i_\ell\}$ starting at site $i_1$ and ending at site $i_\ell$. (One could also define the curve as an ordered sequence of its bonds.) The reverse curve $C^{-1}$ contains the same sites as $C$ but in reverse direction, $\{i_\ell, i_{\ell-1}, \ldots, i_1\}$. The length
\[ \mathcal{L}(C) \text{ of the curve } C \text{ is} \]
\[ \mathcal{L}(C) = \sum_{j=1}^{\ell - 1} d_{ij, i_{j+1}}. \] (1)

Furthermore \( \mathcal{L}(C) = \mathcal{L}(C^{-1}) \).

A curve \( C \) that starts and ends at the same site is a loop, \( \{i_1, i_2, \ldots, i_k, i_1\} \). The trivial loop \( \{i\} \) is composed of a single site and has length \( \mathcal{L}(\{i\}) = 0 \).

**Homotopy.** We define a standard way to deform a curve \( C \) by adding a site to the curve or by deleting a site. In the case of addition, the result depends upon choosing a bond \( (i_{j-1}, i_j) \) to break by the addition of the new site. Replace this bond of \( C \) by the two new bonds \((i_{j-1}, i_k)\) and \((i_k, i_j)\). The resulting curve has length \( \mathcal{L}(C') = \mathcal{L}(C) - d_{i_{j-1}, i_k} + d_{i_{j-1}, i_j} + d_{i_k, i_j} \).

Similarly, the result of an elementary subtraction of a site \( i_j \) from a curve \( C \) is a curve \( C'' = C - \{i_j\} \). One obtains \( C'' \) by replacing two bonds of \( C \) by one. We remove \((i_{j-1}, i_j)\) and \((i_j, i_{j+1})\) and substitute the single bond \( (i_{j-1}, i_{j+1}) \). The resulting curve has length \( \mathcal{L}(C'') = \mathcal{L}(C) - d_{i_{j-1}, i_j} - d_{i_j, i_{j+1}} + d_{i_{j-1}, i_{j+1}} \).

The operations of “adding” or “subtracting” a site are elementary homotopies. Two curves are homotopic if a finite number of elementary homotopies transform one curve into the other. A loop is contractible, if it is homotopic to a loop of length zero. We also say that a contractible loop is localized.

**Topological order and reflection positivity.**

**Reflection positivity.** Consider a finite lattice \( \Lambda \) which is divided in two parts \( \Lambda_\pm \) mapped into each other by reflection in a plane \( \Pi \). We represent this reflection as an anti-unitary operator \( \vartheta \) on the Hilbert space \( \mathcal{H} \) of our model. The reflection maps each operator \( O_i \) into
\[ \vartheta(O_i) = \vartheta O_i \vartheta^{-1} = O_i^\dagger, \] (2)
where site \( \vartheta i \) is the reflection of site \( i \).

Let \( \mathfrak{A} \) denote the set of operators that are sums of products of those \( O_i \)'s, where \( i \) is in \( \Lambda \). The reflection positivity property for the pair \( H \) and \( \mathfrak{A} \) means: for every \( A \in \mathfrak{A} \) and every \( 0 \leq \beta \),
\[ 0 \leq \text{Tr}(A \vartheta(A)e^{-\beta H}). \] (3)

We use the notation \( W_A = A \vartheta(A) \).

**Vortex loops.** To each vortex of the lattice, we associate one or more operators \( O_i \). Associated to a curve \( C = \{i_1, i_2, \ldots, i_k\} \), define a vortex loop \( \mathfrak{B}(C) \) as a product of the \( O_i \)'s,
\[ \mathfrak{B}(C) = O_{i_1}O_{i_2} \cdots O_{i_k}. \] (4)

We choose operators \( W_A \) that equal a reflection-symmetric vortex loop, namely \( W_A = \mathfrak{B}(C) = A \vartheta(A) \), and \( k = 2l \). In other words, let \( A = O_{1i_1} \cdots O_{k_{i_k}} \) be the product of operators along half of the loop and \( \vartheta(A) = O_{1i_1}O_{i_2i_3} \cdots O_{i_{2l}i_{2l+1}} = O_{1i_1}O_{\vartheta i_2} \cdots O_{\vartheta i_{2l}} \) be the product of operators along the other half of the loop.

**W-order.** Consider a Hamiltonian \( H \), with a ground-state subspace \( \mathcal{P} \). (We also use the symbol \( \mathcal{P} \) for the orthogonal projection onto the ground-state subspace.) Define \( H \) to have \( W \)-order, if the operator \( W \) applied to any vector \( |\Omega\rangle \in \mathcal{P} \) has no component in \( \mathcal{P} \) that is orthogonal to \( |\Omega\rangle \). In other words, \( \mathcal{P} W \mathcal{P} \) is a scalar multiple of \( \mathcal{P} \), and \( W \) does not cause transitions between different ground states.

At this point we have not yet considered the localization of \( W \). The notion of \( W \)-topological order is a special case of \( W \)-order. Topological order involves the additional assumption that \( W \) is localized, which we now define.

**W-topological order.** Below we consider vortex loop operators \( \mathfrak{B}(C) \) defined in (4). In case that the loop \( C \) is contractible, we define the vortex loop operator \( \mathfrak{B}(C) \) to be local. \( W \)-topological order is \( W \)-order for localized operators \( W = \mathfrak{B}(C) \). This definition is a special case of the general concept of topological order defined in [2,3]. Our definition coincides with eq. (1) of ref. [2] for \( W \) a local operator and \( e = 0 \). In the sense of ref. [3], one allows \( W \) to be any operator localized in a square.

**Main result: W-order and positivity.** Consider a Hamiltonian \( H \) with the reflection positivity property (3). Assume that the ground-state subspace has \( W \)-topological order, where \( W_A \) is \( W \)-order. Then we claim that
\[ 0 \leq \langle \Omega, W_A \Omega \rangle, \] (5)
for any ground-state vector \( \Omega \) of \( H \).

**Explanation.** Add a constant to \( H \) to ensure that the ground-state energy is zero. Suppose \( H \) has \( N \) orthonormal ground states denoted by \( \Omega^\mu \), with \( \mu = 1, \ldots, N \). As we are assuming reflection positivity (3), the \( \beta \to \infty \) limit leads to
\[ 0 \leq \sum_{\mu=1}^{N} \langle \Omega^\mu, W_A \Omega^\mu \rangle. \] (6)
Since the ground states are \( \mathcal{W}_A \)-ordered,
\[ \langle \Omega^\mu, W_A \Omega^\mu \rangle = \alpha \langle \Omega^\mu, \Omega^\mu \rangle = \alpha, \] (7)
with \( \alpha \) independent of \( \mu \). Thus by (6)
\[ 0 \leq \sum_{\mu=1}^{N} \langle \Omega^\mu, W_A \Omega^\mu \rangle = N \alpha. \] (8)
Hence,
\[ 0 \leq \langle \Omega^\mu, W_A \Omega^\mu \rangle \quad \text{for} \quad \mu = 1, \ldots, N. \] (9)

**Vortex loops and Majoranas.** We now consider the case where each of the operators \( O_i \) is a Majorana \( c_i \). This means that \( c_i^\dagger = c_i \) and \( \{c_i, c_j\} = 2 \delta_{ij} \). In this case
\[ \mathfrak{B}(C) = c_i c_{i_2} c_{i_3} \cdots c_{i_{2l}} c_{\vartheta i_2} c_{\vartheta i_3} \cdots c_{\vartheta i_{2l}}, \] (10)
and we have \( \mathfrak{B}(C) = (-1)^l \mathfrak{B}(C)^\dagger = (-1)^l \mathfrak{B}(C)^{-1} \).
The operator $\imath^d B(C)$ has eigenvalues $\pm 1$. We say that the loop $C$ is vortex-free when the expectation value of $\imath^d B(C)$ is $+1$ and vortex-full when the expectation value is $-1$. In the intermediate cases, we say that the loop is partially free and partially full, according to the sign of the expectation value of $\imath^d B(C)$.

In the special case that $B(C)$ commutes with $H$, and therefore $B(C)$ is conserved, it is possible to choose an orthonormal basis of ground states of $H$ for which $\imath^d B(C) = \pm 1$. Then the loop $C$ is either vortex-free or vortex-full in each of these ground states.

**An example of topological quantum memory.** – Our methods apply quite generally to any Hamiltonian that exhibits topological order and reflection positivity. It would be interesting to classify which general Hamiltonians have these properties. The example below is very special; we choose a Majorana Hamiltonian, not only because it demonstrates there is a system satisfying the two criteria, but also because this Hamiltonian is relevant to quantum information theory.

We choose here one specific class of Majorana Hamiltonians $H$ and one type of operators $W_A = B(C)$. We consider the Hamiltonian proposed in [10], describing interactions between Majoranas localized on the vertices of a planar lattice,

$$ H = \sum_j H_{0,j} + \lambda \sum_{j<k} V_{(jk)}. \quad (11) $$

Here $j$ labels square islands of the lattice, see fig. 1, and the Hamiltonian $H_{0,j}$ is a product of four independent Majoranas $c_{j_1}, c_{j_2}, c_{j_3}$, and $c_{j_4}$ of the form

$$ H_{0,j} = -c_{j_1}c_{j_2}c_{j_3}c_{j_4}. \quad (12) $$

The constant $\lambda$ in (11) is dimensionless, and $V_{(jk)} = i c_{j}c_{k}$ are quadratic interactions between Majoranas. The authors in [10] show that for small values of the parameter $\lambda$, the model possesses $W$-topological order for local operators $W$.

We illustrate the planar-lattice configuration in fig. 1. Each pair of nearest-neighbor islands $j$ and $k$ defines a directed bond $(jk)$ that characterizes the coupling of the neighboring islands, and which determines $V_{(jk)}$. Each island $j$ consists of a square with independent Majoranas $c_{j_1}$, $c_{j_2}$, $c_{j_3}$, and $c_{j_4}$ which we place on the four corners of the square island as specified in fig. 1 (in particular $a$, $b$ lie on the top of the square and $c$, $d$ lie on the bottom of the square). For four nearest-neighbor squares labeled $i$, $j$, $k$, $l$ on the lattice, define the loop $C_{ijkl} = \{i, j, k, l\}$ and the vortex operator

$$ B(C_{ijkl}) = c_i c_j c_k c_{\bar{l}}. \quad (13) $$

The vortex operators have the form (10) with $l = 4$ and thus eigenvalues $\pm 1$. Furthermore, the vortex operators $B(C_{ijkl})$ are conserved [10], so $H$ and $B(C_{ijkl})$ can be simultaneously diagonalized.

Majoranas and reflection symmetry. The vortex $B(C_{ijkl})$ that is bisected by the plane $\Pi$, illustrated in fig. 1, can be written

$$ B(C_{ijkl}) = W_A = A \vartheta(A), $$

with $A = c_{i} c_{j} c_{k} c_{\bar{l}}$. With our choice of $V_{(jk)}$ the Hamiltonian $H$ in (11) is reflection-symmetric,

$$ \vartheta(H) = H. \quad (15) $$

This property, that the Hamiltonian is reflection-symmetric, is quite usual; one generally encounters such Hamiltonians in case the interactions are homogeneous. In [11], we have studied a class of reflection-symmetric Majorana Hamiltonians that includes the Hamiltonian (11), and we demonstrated that reflection positivity holds.

As a consequence of topological order and reflection positivity we showed that the expectation value (3) of $B(C_{ijkl}) = W_A$ is positive. Therefore, for the values of $\lambda$ where $H$ has topological order, the loop $C_{ijkl} = \{i, j, k, l\}$ is vortex-free in all ground states. On the other hand, when the ground state is non-degenerate, reflection positivity (3) ensures that the loop $C_{ijkl}$ is vortex-free in the ground state. Finally, if the lattice is periodic, any loop $C_{ijkl}$ can be used in this argument and is vortex-free in any ground state.

We thank ZOHAR NUSSSINOV and DIEGO RAJNIS for useful comments on an earlier version of this manuscript. This work was supported by the Swiss NSF and the NCCR QST. AJ thanks D. LOSS and FLP thanks A. JAFFE for hospitality.
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