TORIC VARIETIES AND MINIMAL COMPLEXES

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1. Introduction

Let $T = (\mathbb{C}^*)^n$ and $N := \text{Lie}_\mathbb{R} (T) \simeq \mathbb{R}^n$. Let $A = \text{Sym}^\bullet N^*$ be the graded polynomial ring of functions on $N$ where linear functions have degree 2. Let $BT$ be the classifying space for $T$. There exists a natural (and well known) isomorphism of graded algebras

$$A \simeq H(BT).$$

Let $X$ be a $(T)$-toric variety. The above isomorphism suggests that the equivariant geometry on $X$ can be described in terms of coherent objects on $N$, or more precisely, on the fan $\Phi$ corresponding to $X$.

In fact, in [BL], ch.15 a certain natural ”minimal” complex $K_\Phi$ of $A$-modules was defined on $\Phi$. In case $\Phi$ is a complete fan, $K_\Phi$ has remarkable properties: it is acyclic, except in degree $-n$ and $H^{-n} (K_\Phi) \simeq IH_{T,c}(X)$ - the equivariant intersection cohomology with compact supports of $X$ (see [BL]). In a sense, the coherent complex $K_\Phi$ on $\Phi$ represents the equivariant intersection cohomology complex $IC_T(X)$ on $X$.

Let $\pi : X \to Y$ be a proper morphism of toric varieties. Denote also by $\pi : \Phi \to \Psi$ the subdivision of corresponding fans ($\Phi$ subdivides $\Psi$). By the equivariant decomposition theorem ([BL]) the direct image $\pi_* IC_T(X)$ is isomorphic to a direct sum of simple equivariant perverse sheaves on $Y$. In this work we translate this theorem into a statement about the semisimplicity of the direct image $\pi_* K_\Phi$ (which we define first) (Theorem 3.3.1).

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2. Complexes associated to fans

2.1. Fans and toric varieties. Let $T = (\mathbb{C}^*)^n$ be a complex torus of dimension $n$, $\Lambda = \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$, $N = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. The space $N \otimes_{\mathbb{R}} \mathbb{C}$ is naturally isomorphic to the Lie algebra of $T$.

A convex polyhedral cone in $N$ (or, simply a cone) is a closed subset of $N$ which is the collection of solutions of a finite system of linear homogeneous equations and inequalities. The term ”convex” means ”strictly convex”, i.e. we assume that a

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cone does not contain a nonzero linear subspace. A cone is called rational if the linear functionals in the systems take rational values on \( \Lambda \). Let \( \mathfrak{o} \) denote the cone which consists of the origin of \( N \).

A fan \( \Phi \) in \( N \) is a (finite) collection of convex polyhedral cones in \( N \), such that, if \( \sigma \in \Phi \) and \( \tau \) is a face of \( \sigma \), then \( \tau \in \Phi \). Also any two cones in \( \Phi \) intersect along a common face. In particular \( \mathfrak{o} \) belongs to every fan.

For a cone \( \sigma \subset N \) we denote by \( \langle \sigma \rangle \) the fan generated by \( \sigma \) (i.e. consisting of \( \sigma \) and its faces), and by \( \partial \sigma \) the fan which consists of the proper faces of \( \sigma \).

A fan \( \Phi \) is called rational is every cone of \( \Phi \) is rational. A fan \( \Phi \) is called complete if the union of all cones of \( \Phi \) is equal to \( N \).

Let \( A = \mathbb{R}[x_1, \ldots, x_n] \) denote the ring of polynomial functions on \( N \) considered as a graded \( \mathbb{R} \)-algebra by \( \deg(x_i) = 2 \). Let \( \mathfrak{m} \) denote the maximal ideal of elements of positive degree.

For a cone \( \sigma \subset N \) let \( A_\sigma \) denote the graded ring of polynomial functions on \( \sigma \). An inclusion of cones \( \tau \subset \sigma \) induces the restriction homomorphism of rings \( A_\sigma \to A_\tau \). In particular there are canonical restriction maps \( A \to A_\sigma \) for all cones \( \sigma \).

An irreducible algebraic varietiy \( X \) is called \( T \)-toric (or, simply, toric), if \( T \) acts on \( X \) and \( X \) contains a dense \( T \)-orbit isomorphic to \( T \) (in which case the number of \( T \)-orbits is finite).

There is a natural one-to-one correspondence between normal toric varieties and rational fans in \( N \). We refer the reader to [KKMS-D] for detail of this construction.

In what follows we will denote by \( X_\Phi \) the normal toric variety which corresponds to the rational fan \( \Phi \) and by \( \Phi_X \) the fan, corresponding to the toric variety \( X \). We will need the following facts and notations.

The orbits of \( T \) on \( X_\Phi \) are in one-to-one correspondence with the cones of \( \Phi \). Let \( O_\sigma \) denote the orbit associated with the cone \( \sigma \), and let \( \sigma_O \) denote the cone associated with the orbit \( O \). The subspace \( \text{span}(\sigma_O) \otimes \mathbb{C} \subset N \otimes \mathbb{C} \) is identified with the Lie algebra of the stabilizer \( \text{Stab}(O) \subset T \) of \( O \) and \( \dim_{\mathbb{C}} O = n - \dim_{\mathbb{R}} \sigma \).

The ring \( A \) is canonically isomorphic to the (graded) cohomology ring \( H(BT; \mathbb{R}) \) (where \( BT \) is the classifying space of \( T \)). Under the above correspondence the ring \( A_\sigma \) is canonically isomorphic to \( H(B\text{Stab}(O); \mathbb{R}) \).

A toric variety \( X \) is affine (resp. complete), if \( \Phi_X = \langle \sigma \rangle \) for some cone \( \sigma \) (resp. the union of cones in \( \Phi_X \) is the whole space \( N \)).

Let \( \Phi, \Psi \) be two fans in \( N \). We say that \( \Phi \) maps to \( \Psi \), if for every \( \sigma \in \Phi \) there exists \( \tau \in \Psi \), such that \( \sigma \subset \tau \). In case the fans are rational, \( \Phi \) maps to \( \Psi \) iff there exists a morphism of toric varieties \( X_\Phi \to X_\Psi \) (this morphism is unique). Moreover, this morphism is proper iff

\[
\bigcup_{\sigma \in \Phi} \sigma = \bigcup_{\tau \in \Psi} \tau.
\]

We call \( \Phi \) a subfan of \( \Psi \) iff every cone in \( \Phi \) belongs to \( \Psi \). This corresponds to an open embedding of corresponding toric varieties.

2.2. Complexes on fans. An ”\( A \)-module” will mean ”a graded \( A \)-module”. Morphisms between \( A \)-modules preserve the grading. The tensor product \( \otimes \) means
⊗_R unless specified otherwise. Given an A-module M we denote by \( \overline{M} \) the graded vector space \( M/mM \).

Fix a fan \( \Phi \) in \( N \), not necessarily rational.

**Definition 2.2.1.** A complex \( M^\bullet \) on the fan \( \Phi \) is a complex of \( A \)-modules

\[
0 \longrightarrow M^{-n} \xrightarrow{d^{-n}} M^{-n+1} \xrightarrow{d^{-n+1}} \cdots \xrightarrow{d^{-1}} M^0 \longrightarrow 0
\]

such that

1. for each \( i \) the module \( M^i \) has a direct sum decomposition

\[
M^{-i} = \bigoplus_{\sigma \in \Phi, \dim \sigma = i} M_\sigma
\]

2. each summand \( M_\sigma \) is a finitely generated \( A_\sigma \)-module;
3. the differential \( d : M^\bullet \to M^\bullet+1 \) restricts to a morphism of modules

\[
d : M_\sigma \to \bigoplus_{\tau \in \partial \sigma, \dim \tau = \dim \sigma - 1} M_\tau
\]

compatible with the restriction maps of rings \( A_\sigma \to A_\tau \).

Suppose that \( \Psi \) is a subfan of \( \Phi \). Then, for a complex \( M^\bullet \) on \( \Phi \), the submodule \( \bigoplus_{\sigma \in \Psi} M_\sigma \subset \bigoplus_i M^{-i} \) is preserved by the differential.

**Definition 2.2.2.** The submodule \( \bigoplus_{\sigma \in \Psi} M_\sigma \) endowed with the restriction of the differential in \( M^\bullet \) is called the restriction of \( M^\bullet \) to \( \Psi \) and will be denoted by \( M^\bullet |_\Psi \).

**Definition 2.2.3.** A complex \( M^\bullet \) on the fan \( \Phi \) is called locally free if for every \( \sigma \in \Phi \) the \( A_\sigma \)-module \( M_\sigma \) is free.

**Definition 2.2.4.** A complex \( M^\bullet \) on the fan \( \Phi \) is called locally exact if for every \( \tau \in \Phi \) of positive dimension the complex \( M^\bullet |_\langle \tau \rangle \) is acyclic in degree \( -\dim \tau + 1 \), i.e. the differential

\[
(2.1) \quad M_\tau \xrightarrow{d} \ker \left( \left( M^\bullet |_\langle \tau \rangle \right)^{-\dim \tau + 1} \xrightarrow{d} \left( M^\bullet |_\langle \tau \rangle \right)^{-\dim \tau + 2} \right)
\]

is a surjection.

Note that the last two definitions are local, i.e. stable under restriction to a subfan.

**Definition 2.2.5.** For a complex \( M^\bullet \) on the fan \( \Phi \) and a cone \( \sigma \in \Phi \) we call \( M_\sigma \) the \( \sigma \)-component of \( M^\bullet \). The support of \( M^\bullet \) is the union of the cones \( \sigma \in \Phi \), such that \( M_\sigma \neq 0 \).
2.3. Minimal complexes. Recall that \( \mathfrak{o} \) denotes the origin in \( N \) viewed as a zero-dimensional cone. Thus \( A_{\mathfrak{o}} \cong \mathbb{R} \). Let \( \mathbb{R}(n) \) denote the free \( A_{\mathfrak{o}} \)-module with a unique generator of degree \( -n = -\dim C_T \). Recall that \( \mathfrak{m} \) denotes the ideal of elements of positive degree in \( A \).

**Definition 2.3.1.** A complex \( M^\bullet \) on the fan \( \Phi \) is called minimal if it satisfies the following:

1. \( M^\mathfrak{o} = \mathbb{R}(n) \);
2. it is locally free and locally exact;
3. for every \( \tau \in \Phi \) of dimension \( \dim \tau > 0 \) the reduction of the map (2.1) modulo \( \mathfrak{m} \) is an isomorphism.

For any fan \( \Phi \) one can construct a minimal complex on \( \Phi \). Moreover, one can show that any two minimal complexes are non-canonically isomorphic. In what follows we will denote a minimal complex on \( \Phi \) by \( K_\Phi \) or simply by \( K \). The definition of a minimal complex is local, i.e. the restriction of a minimal complex to a subfan is also minimal.

The remarkable properties of minimal complexes are summarized in the following theorem (Theorem 15.7 of [BL]).

**Theorem 2.3.2.** Suppose that \( \Phi \) is a rational fan in \( N \) which is either complete or generated by a single cone of dimension \( n \). Then

1. the minimal complex \( K_\Phi \) is acyclic except in the lowest degree (which is equal to \(-n\));
2. there is an isomorphism of graded \( A \)-modules \( H^{-n}K_\Phi \cong IH_{T,c}(X_\Phi; \mathbb{R}) \);
3. for any cone \( \sigma \in \Phi \) there is an isomorphism of graded \( A_\sigma \)-modules \( K_\sigma \cong H(IC(X_\Phi; \mathbb{R})_{\mathcal{O}_\sigma}) \otimes A_\sigma \).

The notations in 2.3.2 are as follows: \( X_\Phi \) is the normal toric variety which corresponds to the rational fan \( \Phi \); \( IH_{T,c}(X_\Phi; \mathbb{R}) \) is the \( T \)-equivariant intersection cohomology of \( X_\Phi \) with compact supports ([BL],13.4); \( IC(X_\Phi; \mathbb{R})_{\mathcal{O}_\sigma} \) is the stalk of the intersection complex of \( X_\Phi \) at a point of the orbit \( \mathcal{O}_\sigma \) which corresponds to the cone \( \sigma \) — this is a complex of vector spaces with naturally graded (by cohomological degree) finite dimensional cohomology.

**Remark 2.3.3.** We know of no ”elementary” proof of the above theorem. In particular, we cannot prove the first statement for fans, which are not rational. The proof of Theorem 2.3.2 depends on the following fact: if \( \Phi \) is a rational fan in \( N \) which is either complete or generated by a single cone of dimension \( n \), then \( IH_{T,c}(X_\Phi; \mathbb{R}) \) is a free module over \( A \). The same proof shows that the conclusions of 2.3.2 hold for any rational fan \( \Phi \) such that \( IH_{T,c}(X_\Phi; \mathbb{R}) \) is a free \( A \)-module.

**Proposition 2.3.4.** Suppose that \( \Phi \) is a rational fan in \( N \) generated by several \( n \)-dimensional cones whose union is convex (therefore a convex cone). Then \( IH_{T,c}(X_\Phi; \mathbb{R}) \) is a free module over \( A \).

**Proof.** Let \( \sigma \) denote the union of the \( n \)-dimensional cones of \( \Phi \). Then \( \sigma \) is a convex cone in \( N \) and \( \Phi \) is a subdivision of \( \langle \sigma \rangle \), hence there is a unique morphism of toric
varieties
\[ \pi : X_\Phi \to X_{\langle \sigma \rangle} \]
which is \( T \)-equivariant and proper. Therefore the direct image functors
\[ \pi_*, \pi! : D_{T,c}(X_\Phi) \to D_{T,c}(X_{\langle \sigma \rangle}) \]
are defined and \( \pi_* = \pi! \). Here \( D_{T,c} \) denotes the bounded constructible equivariant derived category of sheaves as defined in [BL]. By the equivariant decomposition theorem ([BL], 5.3) the object \( \pi! IC_T(X_\Phi) \) is isomorphic to a direct sum of shifted simple equivariant perverse sheaves on \( X_{\langle \sigma \rangle} \). By [BL], 5.2, simple equivariant perverse sheaves on \( X_{\langle \sigma \rangle} \) are of the form \( IC_T(Z) \), where \( Z \) is a closed \( T \)-invariant irreducible subvariety of \( X_{\langle \sigma \rangle} \). (Here we use the normality of \( X_{\langle \sigma \rangle} \) to conclude that the stabilizers of all points in \( X_{\langle \sigma \rangle} \) are connected, hence the only \( T \) equivariant local systems on \( T \)-invariant subvarieties are the trivial ones.)

Consider a closed irreducible \( T \)-invariant subvariety \( Z \) of \( X_{\langle \sigma \rangle} \). Let \( T_0 \) denote the stabilizer of a general point of \( Z \), and let \( T_1 = T/T_0 \). Then, \( Z \) is a \( T_1 \)-toric variety. Since \( X_{\langle \sigma \rangle} \) is affine with a unique \( T \)-fixed point \( Z \) is also affine with a unique \( T_1 \)-fixed point.

Therefore, by [BL] 14.3 (ii'), \( IH_{T_1,c}(Z) \) is a free module over \( H(BT_1; \mathbb{R}) \). Since \( A = H(BT; \mathbb{R}) = H(BT_0; \mathbb{R}) \otimes H(BT_1; \mathbb{R}) \) and \( T_0 \) acts trivially on \( Z \), it follows that \( IH_{T,c}(Z) \) is a free module over \( A \).

Hence, \( IH_{T,c}(X_\Phi; \mathbb{R}) \simeq H_{T,c}(X_{\langle \sigma \rangle}; \pi! IC_T(X_\Phi)) \) is a free module over \( A \).

Corollary 2.3.5. Suppose that \( \Phi \) is a rational fan in \( N \) generated by several \( n \)-dimensional cones whose union is convex (therefore a convex cone). Then the conclusions of Theorem 2.3.2 hold for \( \Phi \).

Proof. Follows from Proposition 2.3.4 by Remark 2.3.3. \( \square \)

2.4. Shifted minimal complexes. Let \( \Phi \) be a fan in \( N \) (not necessarily rational) and let \( \sigma \) be a cone in \( \Phi \). We have the following generalization of the minimal complex \( K_\Phi \).

Definition 2.4.1. A shifted (by \( k \)) minimal complex based on \( \sigma \), denoted by \( K_\Phi[\sigma](k) \), is a complex on \( \Phi \) which satisfies the following conditions:

(1) \( K_\Phi[\sigma](k) \) is locally free and locally exact;
(2) \( K_\Phi[\sigma](k)^{-i} = 0 \) for \( i < \dim \sigma \);
(3) \( K_\Phi[\sigma](k)^{-\dim \sigma} = (K_\Phi[\sigma](k))_\sigma = A_\sigma(n - \dim \sigma + k) \) (where \( A_\sigma(n - \dim \sigma + k) \) is the free graded \( A_\sigma \)-module on one generator of degree \( n - \dim \sigma + k \));
(4) for every \( \tau \in \Phi \) of dimension \( \dim \tau > \dim \sigma \) the reduction of the map 2.1 (with \( M^\bullet = K_\Phi[\sigma](k) \)) modulo \( m \) is an isomorphism.

Remark 2.4.2.

(1) It is easy to show that \( \sigma \) and \( k \) determine \( K_\Phi[\sigma](k) \) uniquely up to a non-canonical isomorphism.
(2) \( K_\Phi = K_\Phi[\emptyset](0) \).
Recall that, for a cone \( \sigma \) of \( \Phi \) the star of \( \sigma \) is defined as
\[
\text{Star}(\sigma) \overset{\text{def}}{=} \{ \tau \in \Phi \mid \sigma \in \langle \tau \rangle \}.
\]
As is easy to see, the shifted minimal complex \( K_\Phi[\sigma](k) \) is supported on Star(\( \sigma \)).

**Lemma 2.4.3.** Suppose that \( \Phi \) is a rational fan which is either complete or is generated by a single cone of dimension \( n \), \( \sigma \) is a cone in \( \Phi \), and \( k \) is an integer. Then the shifted minimal complex \( K_\Phi[\sigma](k) \) is acyclic except in the lowest degree (which is equal to \(-n\)).

**Proof.** Let \( p : N \to N' := N/\text{span}(\sigma) \) denote the quotient map. The image \( \Psi \) of Star(\( \sigma \)) under \( p \) is a rational fan in \( N' \) which is complete (respectively generated by a single cone of dimension \( n - \dim \sigma \)) if \( \Phi \) is complete (respectively generated by a single cone of dimension \( n \)).

Let \( A' \) denote the ring of polynomial functions on \( N' \). The map \( p \) induces on \( A \) a structure of an algebra over \( A' \). Let \( o' \) be the origin in \( N' \). It is easy to check that \( K_\Phi[\sigma](k) = (A \otimes_{A'} K_\Psi[o'][k]) \otimes_{\text{dim } \sigma} \) as complexes of \( A \)-modules. By Theorem 2.3.2 the complex \( K_\Psi[o'][k] \) is acyclic except in degree \(-n + \dim \sigma \). Hence \( K_\Phi[\sigma](k) \) is acyclic except in degree \(-n \). \( \square \)

### 2.5. A Criterion for semi-simplicity.

**Lemma 2.5.1.** Suppose that \( \Phi \) is a fan in \( N \) (not necessarily rational) and \( M^\bullet \) is a locally free, locally exact complex on \( \Phi \), such that \( M^\bullet = 0 \) for \( p > -i \). Let \( \tau \in \Phi \) be of dimension \( i \). Then \( M^\bullet \) contains a direct summand isomorphic to \( K_\Phi[\tau][i-n] \otimes M^\tau \).

**Proof.** For a complex \( L^\bullet \) denote by \( L^{>k} \) the truncated complex
\[
0 \to L^{k+1} \to L^{k+2} \to \cdots .
\]

We will construct a splitting
\[
M^\bullet \simeq (K_\Phi[\tau][i-n] \otimes \overline{M}_\tau) \oplus N^\bullet
\]
by induction on the dimension of cones working from right to left. Since \( K_{\Phi}^{-i}[\tau](i-n) \otimes \overline{M}_\tau \simeq \overline{M}_\tau \), the decomposition
\[
M^{-i} = M_{\tau} \oplus \bigoplus_{\dim(\xi)=i} M_{\xi}
\]
determines the desired decomposition
\[
M^{>i} = (K_\Phi[\tau][i-n] \otimes \overline{M}_\tau)^{>i} \oplus N^{>i}.
\]

Assume that we have constucted a splitting
\[
M^{>-j} = (K_\Phi[\tau][i-n] \otimes \overline{M}_\tau)^{>-j} \oplus N^{>-j}
\]
for some \( j > i \). Let \( \sigma \in \Phi \) be of dimension \( j \). It suffices to show that we can extend the splitting to the subfan \( \langle \sigma \rangle \). The following claim ensures that we can do so.
Claim 2.5.2. Let 
\[ d : L \rightarrow Z \]
be a surjection of finitely generated (graded) \(A_\sigma\)-modules, where the module \(L\) is free. Given a decomposition \(Z = Z_K \oplus Z_N\), there exists a decomposition \(L = L_K \oplus L_N\) with the properties

a) \(d(L_K) = Z_K\), \(d(L_N) = Z_N\).

b) \(d\) induces an isomorphism of residues \(\bar{d} : \bar{L}_K \simrightarrow \bar{Z}_K\).

Proof of claim. Choose a subspace \(\bar{Z}_K \subset Z_K\), which maps isomorphically onto \(Z_K\) under the residue map. Choose a subspace \(S_K \subset L\) so that \(d(S_K) \simrightarrow \bar{Z}_K\). Since \(S_K \cap mL = 0\), we can choose a subspace \(S_N \subset L\), such that \(S_K \cap S_N = 0\) and \(S_K \oplus S_N\) generates \(L\) freely. Subtracting, if necessary, elements of \(S_K\) from elements of \(S_N\) we may assume that \(d(S_N) \subset N\). Thus we may take \(L_K := A_\sigma S_K\), \(L_N := A_\sigma S_N\). This proves the claim and the lemma. \(\square\)

Proposition 2.5.3. Suppose that \(\Phi\) is a fan in \(N\) (not necessarily rational) and \(M^\bullet\) is a locally free, locally exact complex on \(\Phi\). Then \(M^\bullet\) is noncanonically isomorphic to a direct sum of shifted minimal complexes.

Proof. The proposition follows by induction on \(i = \min \{p | M^p \neq 0\}\) and the number of cones of dimension \(i\) using Lemma 2.5.1. \(\square\)

Corollary 2.5.4. Suppose that \(\Phi\) is a rational fan which is either complete or is generated by a single cone of dimension \(n\) and \(M^\bullet\) is a locally free, locally exact complex on \(\Phi\). Then \(M^\bullet\) is acyclic except in the lowest degree (which is equal to \(-n\)).

Proof. By Proposition 2.5.3 \(M^\bullet\) is (isomorphic to) a direct sum of shifted minimal complexes. The conclusion now follows from Lemma 2.4.3. \(\square\)

3. Direct image complexes and a decomposition theorem

3.0. In this section we consider normal toric varieties \(X\) and \(Y\) with corresponding rational fans \(\Phi\) and \(\Psi\) in \(N\) and a proper morphism of toric varieties

\[ \pi : X \rightarrow Y.\]

Thus, \(\Phi\) is a subdivision of \(\Psi\). Put \(K^\bullet_X = K^\bullet_\Phi\) and \(K^\bullet_Y = K^\bullet_\Psi\)

By the equivariant decomposition theorem ([BL], 5.3) the object \(\pi_*IC_T(X)\) is (noncanonically isomorphic to) a direct sum of shifted objects \(IC_T(Z)\) for various closed \(T\)-invariant irreducible subvarieties of \(Y\) (see the proof of Proposition 2.3.4).

We will translate this into the language of minimal complexes. Namely, we will

1) construct a direct image complex \(\pi_*K^\bullet\) on \(\Psi\): we define \(\pi_*K^\bullet\) as a certain subcomplex of \(K^\bullet_X\);

2) show that the inclusion \(\pi_*K^\bullet \hookrightarrow K^\bullet_X\) is a quasiisomorphism;

3) show that \(\pi_*K^\bullet\) is (noncanonically isomorphic to) a direct sum of shifted minimal complexes on \(\Psi\);

4) in particular \(K^\bullet_Y\) is a direct summand in \(\pi_*K^\bullet\).
3.1. **Construction of the direct image complexes.** Suppose for a moment that \( \Phi \) and \( \Psi \) are fans in \( N \) (not necessarily rational) and \( \Phi \) maps to \( \Psi \). Call this map of fans \( \pi : \Phi \to \Psi \). Let \( \mathcal{M}^\bullet \) be a complex on \( \Phi \). Let us define its direct image \( \pi_* \mathcal{M}^\bullet \) on \( \Psi \). It will be defined as a subcomplex of \( \mathcal{M}^\bullet \). The terms of \( \pi_* \mathcal{M}^\bullet \) are defined inductively as follows.

Suppose that \( (\pi_* \mathcal{M})^{>-i} \) has been defined (as a subcomplex of \( \mathcal{M}^{>-i} \)).

For \( \sigma \in \Psi \) let \( \pi^{-1}(\sigma) \) denote the collection of cones \( \tau \) of \( \Phi \) of the same dimension as \( \sigma \) which are contained in \( \sigma \).

Consider a cone \( \sigma \) of dimension \( i \). The requirement that the square

\[
\begin{array}{ccc}
(\pi_* \mathcal{M})_\sigma & \xrightarrow{d(\pi_* \mathcal{M})_\sigma} & (\pi_* \mathcal{M})^{-i+1} \\
\bigoplus_{\tau \in \pi^{-1}(\sigma)} \mathcal{M}_\tau & \xrightarrow{d} & \mathcal{M}^{-i+1}
\end{array}
\]

is Cartesian determines \( \pi_* \mathcal{M}_\sigma \) and \( d(\pi_* \mathcal{M})_\sigma \) uniquely. Let

\[
(\pi_* \mathcal{M})^{-i} = \bigoplus_{\sigma \in \Psi, \dim \sigma = i} (\pi_* \mathcal{M})_\sigma, \quad d = \sum_{\sigma} d(\pi_* \mathcal{M})_\sigma.
\]

This defines \( \pi_* \mathcal{M}^\bullet \) as a complex on the fan \( \Psi \).

Note, that the construction of \( \pi_* \mathcal{M} \) is local, i.e. compatible with the restriction to a subcomplex \( \Psi' \subset \Psi \) (and the corresponding subcomplex \( \Phi' = \Phi \cap \Psi' \)).

In the remainder of this section we will investigate the properties of the direct image of the minimal complex under a proper morphism of normal toric varieties.

3.2. **Properties of \( \pi_* \mathcal{K}_X \).**

We are back to notations of 3.0 above.

**Lemma 3.2.1.** Suppose that \( \pi : X \to Y \) is a proper morphism of normal toric varieties. Then the complex \( \pi_* \mathcal{K}_X^\bullet \) is locally exact.

**Proof.** Fix a cone \( \sigma \in \Psi \). To check local exactness of \( \pi_* \mathcal{K}_X^\bullet \) at \( \sigma \) we may assume that \( \Psi = \langle \sigma \rangle \). Clearly \( \Phi \) satisfies the hypotheses of Corollary 2.3.5 (with \( N \) replaced by \( \text{span}(\sigma) \) and \( n \) replaced by \( \dim \sigma \)) and therefore the minimal complex \( \mathcal{K}_X^\bullet \) is acyclic except in degree \( -\dim \sigma \).

Let \( i = \dim \sigma \). By construction of \( \pi_* \mathcal{K}_X^\bullet \), the square

\[
\begin{array}{ccc}
(\pi_* \mathcal{K}^\bullet)_\sigma & \xrightarrow{d} & (\pi_* \mathcal{K}^\bullet)^{-i+1} \\
\mathcal{K}_X^{-i} & \xrightarrow{d} & \mathcal{K}_X^{-i+1}
\end{array}
\]

is Cartesian. Since \( \mathcal{K}_X^\bullet \) is acyclic in degree \( -i + 1 \), this implies that \( \pi_* \mathcal{K}_X^\bullet \) is acyclic in degree \( -i + 1 \).

As this happens for every cone \( \sigma \in \Psi \), the complex \( \pi_* \mathcal{K}_X^\bullet \) is locally exact. \( \square \)
Proposition 3.2.2. Suppose that \( \pi : X \to Y \) is a proper morphism of normal toric varieties. Then

1. the complex \( \pi_*K^\bullet_X \) is locally free;
2. the inclusion \( \pi_*K^\bullet_X \hookrightarrow K^\bullet_X \) is a quasiisomorphism.

Proof. By induction on dimension of the torus we may assume that the conclusions hold for any proper morphism of normal toric varieties of dimension smaller than \( n = \dim_{\mathbb{R}} N \).

Case 1. The fan \( \Psi \) is generated by a single cone \( \sigma \) of dimension \( i \), i.e. \( \Psi = \langle \sigma \rangle \).

By induction on the dimension of the cone (the case of degree zero being obvious since \( A_{\sigma} = \mathbb{R} \)) we may assume that \( (\pi_*K^\bullet_X)\tau \) is free over \( A_\tau \) for all \( \tau \in \Psi \) of dimension smaller than \( i \).

Denote by \( \partial \Psi \) the subfan of \( \Psi \) consisting of proper faces of \( \sigma \), and by \( \partial \Phi \) its preimage in \( \Phi \). Consider the commutative diagram with rows short exact sequences of complexes (which serves as the definition of the complex \( M^\bullet \)):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_*K^\bullet_X|_{\partial \Psi} & \longrightarrow & \pi_*K^\bullet_X & \longrightarrow & (\pi_*K^\bullet_X)_\sigma[i] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^\bullet_X|_{\partial \Phi} & \longrightarrow & K^\bullet_X & \longrightarrow & M^\bullet & \longrightarrow & 0
\end{array}
\]

Claim 3.2.3.

1. The inclusion \( \pi_*K^\bullet_X|_{\partial \Psi} \hookrightarrow K^\bullet_X|_{\partial \Phi} \) (the left vertical arrow in (3.1)) is a quasiisomorphism.
2. The complexes \( \pi_*K^\bullet_X|_{\partial \Psi} \) and \( K^\bullet_X|_{\partial \Phi} \) are acyclic except in the lowest degree \( -i + 1 \).

Proof. Replacing \( N \) by \( \text{span}(\sigma) \) is necessary we may assume that \( \dim \sigma = n \). Choose an integral vector \( v \) in the interior of \( \sigma \), and consider the projection \( N \to N/\text{span}(v) \cong N' \). Then, \( \partial \Psi \) and \( \partial \Phi \) project (isomorphically) onto complete rational fans in \( N' \) which we denote \( \Psi' \) and \( \Phi' \) respectively. Since \( \Phi' \) is a subdivision of \( \Psi' \) there is a proper morphism of corresponding toric varieties

\[
\pi' : X' \to Y'.
\]

Let \( A' \) denote the ring of polynomial functions on \( N' \), naturally included in \( A \). Then, clearly there are isomorphisms of complexes of \( A' \)-modules \( K^\bullet_X|_{\partial \Phi} \cong K^\bullet_X \), and \( \pi_*K^\bullet_X|_{\partial \Psi} \cong \pi'_*K^\bullet_X \), compatible with inclusions of the latter into the former.

The first assertion now follows by induction on the dimension of the torus. The second assertion follows from Theorem 2.3.2 applied to \( \Phi' \).

Claim 3.2.4. The inclusion \( \pi_*K^\bullet_X \hookrightarrow K^\bullet_X \) (the middle vertical arrow in (3.1)) is a quasiisomorphism.

Proof. By the previous claim the complex \( \pi_*K^\bullet_X|_{\partial \Psi} \) is acyclic, except in the lowest degree \( -i + 1 \). By Lemma 3.2.1 the complex \( \pi_*K^\bullet_X \) is locally exact. Hence \( \pi_*K^\bullet_X \)
is acyclic, except in the lowest degree $-i$. But it follows immediately from the construction of $\pi_*K^•_X$ that $H^{-i}\pi_*K^• \cong H^{-i}K^•_X$. □

It follows from Claims 3.2.3 and 3.2.4 that the map $(\pi_*K^•_X)_\sigma[i] \to M^•$ (the right vertical map in (3.1)) is a quasiisomorphism, in particular the complex $M^•$ is acyclic except in the lowest degree equal to $-i$.

Note that $M^{-j} = \bigoplus M_\tau$, where $\dim \tau = j$ and $M_\tau$ is a free $A_\tau$-module. Therefore $\text{Ext}_{A_\sigma}^p(M^{-j}, A_\sigma) = 0$ for $p \neq i-j$. The standard argument shows that $H^{-i}M^•$ satisfies $\text{Ext}_{A_\sigma}^p(H^{-i}M^•, A_\sigma) = 0$ for $p \neq 0$. This implies that $(\pi_*K^•_X)_\sigma = H^{-i}M^•$ is free over $A_\sigma$. □

Case 2. $\Psi$ is general.

In this case we still know the first assertion of the proposition, since it is a local property and the proof in Case 1 applies. It remains to prove the second assertion.

For a cone $\tau \in \Psi$ let $\Phi \cap \tau := \{\xi \in \Phi | \xi \subset \tau\}$ be the corresponding subfan of $\Phi$. Denote again by $\pi : \Phi \cap \tau \to \langle \tau \rangle$ the corresponding subdivision of fans. The inclusion $\pi_*K^•_X \hookrightarrow K^•_X$ restricts to the inclusion $\pi_*K^•_{\Phi \cap \tau} \hookrightarrow K^•_{\Phi \cap \tau}$.

Consider the Cech resolution of the pair of complexes $\pi_*K^•_X \hookrightarrow K^•_X$

\[ \cdots \to \bigoplus_{\tau, \xi \in \Psi} \pi_*K^•_{\Phi \cap \tau \cap \xi} \to \bigoplus_{\tau \in \Psi} \pi_*K^•_{\Phi \cap \tau} \to \pi_*K^•_X \]

The two rows are exact and each vertical arrow, except the rightmost, is a quasiisomorphism by the proof in Case 1 above. Hence the rightmost arrow is also a quasiisomorphism. This completes the proof of proposition 3.2.2. □

Corollary 3.2.5. Suppose that $\pi_*X \to Y$ is a proper morphism of normal toric varieties. Then the direct image complex $\pi_*K^•_X$ is a direct sum of shifted minimal complexes: $K^•_Y$ is a direct summand of $\pi_*K^•_X$ (in fact the only one based on $\sigma$).

Proof. The first assertion is a direct consequence of Lemma 3.2.1, Proposition 3.2.2.(1), and of Proposition 2.5.3. The second assertion is clear by the construction of $\pi_*K^•_X$. □

3.3. A decomposition theorem for minimal complexes.

Let us summarize our main results.

Theorem 3.3.1. Suppose that $\pi : X \to Y$ is a proper morphism of normal toric varieties. Then the direct image complex $\pi_*K^•_X$ (see (3.1)) has the following properties:

1. $\pi_*K^•_X$ is locally exact and locally free;
2. the inclusion $\pi_*K^•_X \hookrightarrow K^•_X$ is a quasiisomorphism;
3. $\pi_*K^•_X$ is a direct sum of shifted minimal complexes on $\Psi_Y$;
4. $K^•_Y$ is a direct summand of $\pi_*K^•_X$ (in fact the only one based on $\sigma$).

Proof. The first two assertions follow from Lemma 3.2.1 and Proposition 3.2.2. The last two assertions are taken from Corollary 3.2.5. □
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