Codimension Two Holography

Enrique Álvarez, Jorge Conde and Lorenzo Hernández

Instituto de Física Teórica UAM/CSIC, C-XVI, and Departamento de Física Teórica, C-XI,
Universidad Autónoma de Madrid E-28049-Madrid, Spain

Abstract A holographic interpretation for some specific Ricci flat string backgrounds of the form $A_6 \times C_4$ is proposed. The conjecture is that there is a Four-dimensional Euclidean Conformal Field Theory (ECFT) defined on a codimension two submanifold of the manifold $A_6$ (where one of the two remaining holographic coordinates of $A_6$ is timelike, and the other one spacelike), with central charge proportional to the radius of curvature of the six-dimensional manifold, $c \sim l^4$. 
1 Introduction

Let us fix our attention on a string background described as a Ricci flat manifold of the form

\[ M_{10} \equiv A_6 \times C_4 \]  

where \( C_4 \) is an internal compact manifold, and \( A_6 \) will be denoted by the name ambient space. There are no Ramond-Ramond backgrounds excited, so that this background is valid for all types of strings. In this ambient space lives a codimension two euclidean four-manifold, which will be interpreted as the spacetime \( M_4 \subset A_6 \). The spacetime coordinates will be denoted by \( x^i \equiv \vec{x} \) and its metric by \( g_{ij}dx^i dx^j \); whereas the extra two coordinates of the ambient space by \( \rho \in \mathbb{R}^+ \) and \( t \in \mathbb{R}^+ \), where \( \rho \) is spacelike and \( t \) timelike. There is then a natural boundary defined in this patch by

\[ \partial A_6 \equiv \{ \rho = 0 \} \]  

It will be moreover assumed that the ambient space metric (cf. Appendix) can be written as:

\[ ds^2 = \frac{t^2}{l^2}ds^2(x, \rho) + \rho dt^2 + t d\rho dt \]  

where the metric induced on the hypersurfaces \( \Sigma \equiv \{ t = \text{const.} \} \), namely \( ds^2(x, \rho) \equiv h_{ij}(x, \rho)dx^i dx^j \) is such that it reduces to the (euclidean) spacetime metric on \( \rho = 0 \):

\[ ds^2(x, \rho = 0) = g_{ij}(x)dx^i dx^j. \]  

The set of spaces obeying these restrictions is a non-empty set, and we provide several examples in the appendices.

The purpose of the present paper is to show, first of all, that there are diffeomorphisms on \( A_6 \) that reduce to Weyl transformations on the boundary. This strongly suggests that there is some conformal theory associated to the said boundary.
We then further discuss the boundary energy momentum tensor, and show, by discarding a divergent part, that it is proportional to the four-dimensional conformal anomaly for conformally invariant matter.

We then conclude with some comments on the relationship of this approach with the usual AdS/CFT of Maldacena’s.

2 PBH Diffeomorphisms

One way of understanding the fact that diffeomorphism invariants on some manifold give rise to conformal invariants in some other manifold which is in a precise sense the boundary of the former one, is by establishing the existence of the so-called Penrose-Brown-Henneaux (PBH) ([19][8]) diffeomorphisms; that is, diffeomorphisms that reduce to conformal transformations on the boundary.

This approach has been pioneered in a related context, namely for the study of the bulk space in AdS/CFT in ([15][21]). Let us examine it in the present context. We shall perform the computations for arbitrary $A_{n+2}$ in the sequel, although we shall be mostly interested in the case $n = 4$.

The most general diffeomorphism that maintains the coordinate gauge (that is, such that $\delta g_{tt} = \delta g_{\rho t} = \delta g_{\rho\rho} = \delta g_{\rho\rho} = 0$) is generated by the vector

$$\xi = (-a(x)t + b(x)) \partial_t + \left(2a(x)\rho - \frac{b(x)\rho + c(x)}{t}\right) \partial_\rho + \xi^i(t, \rho, x) \partial_i$$  \hspace{1cm} (5)

where

$$\frac{2t^2}{l^2} h_{ij} \frac{\partial}{\partial \rho} \xi^j - t^2 \partial_t a + t \partial_t b = 0$$

$$\frac{2t^2}{l^2} h_{ij} \frac{\partial}{\partial t} \xi^j + \rho \partial_t b - \partial_t c = 0$$  \hspace{1cm} (6)

\(^2\text{We are referring here to the components of the } (n+2)\text{-dimensional ambient metric, not to be confused with the zero mode at the boundary}\)
and, besides, $\xi|_{\rho=0} = 0$.

This means that the induced variation on the metric $h_{ij}$ is given by:

$$\delta h_{ij}(\rho, x) = \left(2a(x) - \frac{b(x)}{t}\right) h_{ij} + \left(2a(x)\rho - \frac{b(x)\rho + c(x)}{t}\right) \partial_{\rho} h_{ij} + \nabla_i \xi_j + \nabla_j \xi_i \quad (7)$$

Please note that we have defined here

$$\frac{t^2}{l^2} h_{ij}'(\rho', x') = \frac{\partial x_k}{\partial x'^i} \frac{\partial x^i}{\partial x'^j} h_{kl}(\rho, x) + O(\xi^2) \quad (8)$$

If we compute the action of such a transformation on the metric at the boundary $\rho = 0$, it results in

$$\delta g_{ij} = (2a(x) - \frac{b(x)}{t}) g_{ij} + \left(\frac{c(x)}{t}\right) h_{ij}^{(1)} + 2 \nabla(\xi j) |_{\rho=0} \quad (9)$$

where

$$g_{ij} = h_{ij}(\rho = 0)$$

$$h_{ij}^{(1)} = \frac{dh_{ij}}{d\rho} |_{\rho=0} \quad (10)$$

and the covariant derivative is with respect to the metric $g$. In order to obtain a pure Weyl transformation on the boundary we have to choose

$$b(x) = c(x) = 0$$

$$\xi(\rho, x)|_{\rho=0} = 0 \quad (11)$$

We have not analyzed the interesting possibility of keeping $b \neq 0$, which leads to time-dependent Weyl transformations at the boundary.

To summarize, the most general PBH is given by:

$$\xi = \epsilon(x)[-t \frac{\partial}{\partial t} + 2 \rho \frac{\partial}{\partial \rho}] + \xi^i(\rho, x) \frac{\partial}{\partial x^i} \quad (12)$$

It is curious to notice that the boundary $B_{n+1} \equiv \partial A_{n+2}$ is a $n + 1$-dimensional theory with Lorentzian signature, involving the coordinates $x^i$ as well as the time $t$. The metric on this boundary is degenerate, namely

$$ds^2 = \frac{t^2}{l^2} h_{ij}(x, \rho) dx^i dx^j \quad (13)$$
The timelike coordinate only appears in this metric as a multiplicative factor, and never in the metric $h_{ij}$ itself, so that its zero mode, namely $g_{ij}(x) \equiv h_{ij}(x, \rho = 0)$ is also a fortiori time independent. Besides, the conformal transformations do not depend on time at all. For all that matters at the boundary, time is just an external parameter.

The essential part of the PBH diffeomorphism\(^3\) however, is

$$\xi = \epsilon(x)\left[−t \frac{∂}{∂t} + 2ρ \frac{∂}{∂ρ}\right] \quad (14)$$

which mixes the two holographic coordinates $(t, \rho)$ in a particular combination. This is the root of many properties of this construction.

In conclusion, any covariantly defined theory in the ambient space generates a Weyl invariant one on the boundary with the qualifications as above.

### 3 The Regularized Boundary

The true $n+1$-dimensional boundary, $B_{n+1} \equiv \{\rho = 0\}$ is a null surface. The null character of the normal vector is however an isolated fact of the normal vector field. This suggests the consideration of the surface $\rho = \epsilon$, which we shall call the regularized boundary $B_\epsilon$.

The first thing to notice is that this surface is now timelike, with normal vector

$$n_\epsilon = \epsilon^{-1/2} \frac{∂}{∂t} - \frac{2\epsilon^{1/2}}{t} \frac{∂}{∂ρ} \quad (15)$$

Only in the strict limit $\epsilon = 0$ the surface becomes null. This is however the most natural extension of the normal to a vector field, and, as we shall see, it is essential to regularize the action in some way in order to properly define physical quantities. Correspondingly, the induced metric is now non-degenerate, and given by:

$$ds^2_{B_\epsilon} = \epsilon dt^2 + \frac{t^2}{l^2} h_{ij} dx^i dx^j. \quad (16)$$

\(^3\)That is, suppressing purely spatial diffeomorphisms generated by $\xi^i \partial_i$.  

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4 The Brown-York quasilocal energy

There is a convenient definition of a quasilocal gravitational energy due to Brown and York (BY), \[9\] which, although not conserved in general, embodies a large fraction of the asymptotic symmetries of the gravitational field. As will be shown in the sequel, this quantity is precisely defined on codimension two surfaces.

This general idea has been successfully exploited by Brown and Henneaux in \[8\] to associate a CFT to $AdS_3$, and by Balasubramanian and Kraus \[5\] to introduce a boundary energy-momentum tensor.

This definition of energy shares with the usual Arnowitt, Deser and Misner (ADM) definition, which is valid for asymptotically flat spacetimes, the fact that it is defined with respect to a foliation by a family of spacelike surfaces $\Sigma$ and then it is expressed as an integral over the boundary $\partial \Sigma$, but it differs in the detail, and can be also applied to non asymptotically flat situations.

In order to properly define a variational principle, it is convenient to consider a region of the total spacetime $A_{n+2}$, say $M$, bounded by two initial and final spacelike hypersurfaces, $\Sigma_i$ and $\Sigma_f$, which in our case are $(n+1)$-dimensional, and a timelike boundary, $B_{n+1} = \partial M$. Please refer to the included figure for the geometrical setup.

Actually, instead of considering the $(n+1)$-dimensional boundary $B_{n+1} \equiv \{ \rho = 0 \}$ which is, as we have seen, a null surface, we shall consider previously introduced regularized boundary. The extrinsic curvature $\Theta_{ij} \equiv \frac{1}{2} L_n g_{ij}$ is given by:

$$\Theta_{ij} = \frac{t}{l^2 e^{1/2}} [h_{ij} - \epsilon h'_{ij}]$$ (17)

where $h'_{ij} \equiv \frac{d}{d\rho} h_{ij}$. The corresponding boundary energy-momentum tensor, defined as

$$\tau_{ab} \equiv \frac{1}{\kappa^2} [\Theta_{ab} - \Theta g_{ab}]$$ (18)

has got components

$$\tau_{tt} = -\frac{1}{\kappa^2} e^{3/2} h^{kl} h'_{kl} - n e^{1/2}$$
Figure 1: *The hypersurfaces* $\Sigma_\perp$, *contrary to other foliations of the spacetime* $\Sigma$ (*such as* $t = \text{constant}$), *enjoy an orthogonal intersection with the spacetime boundary* $B$

$$\tau_{ij} = -\frac{1}{\kappa^2} \left[ \frac{(n-1)t}{\epsilon^{1/2} l^2} h_{ij} + \frac{\epsilon^{1/2} t}{l^2} (h'_{ij} - (h_{kl} h'_{kl}) h_{ij}) \right]$$ (19)

It is to be remarked that in spite of its name, this boundary energy-momentum tensor is a quantity that refers to a manifold such as the regularized boundary, of Lorentzian signature. In order to define a energy in the BY sense, we still need to foliate the complete ambient spacetime with a family of spacelike surfaces $\Sigma$, and the energy so defined depends on the foliation in a nontrivial way.

The choice $\Sigma \equiv \{ t = \text{const} \}$ is not adequate, first of all, because these surfaces are null and in addition because they do not enjoy an orthogonal intersection with the boundary of the spacetime. This last point, although technical, greatly complicates the analysis, and makes the definition of energy less useful.

Both problems could be remedied at one fell swoop if we consider instead the surfaces $\Sigma_\perp$ generated by the vector.

$$u \equiv \rho^{-1/2} \frac{\partial}{\partial t}$$ (20)
which are easily found to correspond to
\[ \Sigma_\perp \equiv \{ t\rho^{1/2} = L, \rho > 0 \}. \] (21)
(\text{where } L \text{ is an arbitrary constant}). The quasilocal energy is then defined in the \( n \)-dimensional surface \( B \cap \Sigma_\perp \), a codimension two submanifold, whose metric is
\[ ds^2 = \frac{L^2}{\epsilon l^2} h_{ij} dx^i dx^j \] (22)
and is given by the integral
\[ E(B \cap \Sigma_\perp) = -\frac{L^{n-1}}{\kappa^2} \int_{B \cap \Sigma_\perp} \frac{1}{l^n \epsilon^{n/2}} \sqrt{h} d^n x (-n + \epsilon h^k h'_k) \] (23)
The divergences appearing in this expression have to be taken care of before physical results can be obtained (cf. \[5\] \[17\]). We shall come back to this basic point in the next section.

5 The Conformal Anomaly

The expression for the Brown-York quasilocal energy reads, after expanding \( h_{ij} \) explicitly as a powers of \( \epsilon \), \( h_{ij} = g_{ij} + \epsilon h^{(1)}_{ij} + \epsilon^2 h^{(2)}_{ij} + O(\epsilon^3) \)
\[ E = \frac{1}{\kappa^2} \int d^n x \frac{L^{n-1}}{l^n \epsilon^{n/2}} \sqrt{|g|} \left[ n + \epsilon \frac{n-2}{2} h^{(1)} + \epsilon^2 \frac{n-4}{4} (2h^{(2)} - h^{(1)} h^{(1)} + \frac{1}{2} h^{(1)}^2) + O(\epsilon^3) \right] \] (24)
Expanding also the formulas in the Appendix \[B\] embodying the Ricci-flatness condition on the ambient metric leads to the relations
\[ (n - 2) h^{(1)}_{ij} + h^{(1)}_{ij} g_{ij} - l^2 R_{ij}(g) = 0 \]
\[ h^{(1)} = \frac{l^2}{2(n-1)} R \]
\[ h^{(2)} = \frac{1}{4} h^{(1)}_{ij} h^{(1)}_{ij} \] (25)
and so on for higher dimensions.
For $d \in 2\mathbb{Z}$ the term independent of $\epsilon$ gives zero (in the form $(n - d)$) times the corresponding conformal anomaly. To be specific, in the two dimensional case,

$$a_{(2)} = (n - 2) \frac{L}{2 l^2 \kappa_4^2} \epsilon^{(1)} = (n - 2) \frac{L}{4 \kappa_4^2} \epsilon R \sim (n - 2) E_2$$

(26)

whereas in four dimensions

$$a_{(4)} = (n - 4) \frac{L^3}{4 l^4 \kappa_6^2} (2 \epsilon^{(2)} - \epsilon^{(1)}_{ij} \epsilon \epsilon^{(1)}_{ij} + \frac{1}{2} \epsilon^{(1)}_{ij}^2) =$$

$$= \frac{-L^3}{32 \kappa_6^2} (n - 4) (R_{ij} R_{ij} - \frac{1}{3} R^2) \sim (n - 4) (E_4 + W_4)$$

(27)

where $E_n$ is the integrand of the Euler character in dimension $d = n$ and $W_n$ is the quadratic Weyl invariant. On the other hand, the quasilocal energy has to be refered to a particular template, which is to be attributed the zero of energy. In our case this would mean to substract the energy of the flat six dimensional space, and stay with

$$E = -\frac{1}{\kappa^2} \int d^n x \frac{L^{n-1}}{l^n \epsilon^2} \sqrt{h} \epsilon h^{ij} h^{ij}_l$$

(28)

which is such that its finite part is proportional to $E_4 + W_4$ with non-zero coefficient.

It is indeed remarkable that this is the correct form (up to normalization) for the conformal anomaly for conformal invariant matter; this fact allows for an identification of the central function of the CFT, namely $c$,

$$c \sim \frac{l^4}{\kappa^6}$$

(29)

It could be thought that the scale $l$ is arbitrary in our problem, because the background is Ricci-flat; this is an illusion, however, because by dimensional analysis, the Riemann squared scalar (which determines, for example, the geodesic deviation equation) is proportional to $\frac{1}{l^4}$.

$$R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta} \sim \frac{1}{l^4}$$

(30)

\footnote{Choosing $L = l$ in order not to introduce an extra arbitrary scale.}
We can then still refer to $l$ as the *radius of curvature* albeit in a generalized\(^5\) sense. What is physically important is that string corrections are proportional to the curvature invariants, so that in order for them to be small $l$ has to be large in string units: $l >> l_s$.

If we assume that (up to factors of order unity)

$$\frac{1}{\kappa_6^2} = \frac{V_4}{g_s^{2}l_s^8}$$

(31)

where $V_4$ is the volume of the compact manifold $C_4$, and we assume that the boundary CFT is a gauge theory in the large $N$ limit, (so that the central charge scales as $c \sim N^2$) then this implies that

$$l^4 = \frac{g_s^{2}l_s^8N^2}{V_4}$$

(32)

so that $l \sim N^{1/2}$, which is different to AdS/CFT, in which $l \sim N^{1/4}$; the difference is explained by dimensional analysis, owing to the fact that we now have to employ the six-dimensional Newton’s constant instead of the five-dimensional one.

### 6 Conclusions

Some novel string backgrounds have been presented which seem to embody holographic behavior, at least in the semiclassical regime. This behavior is of a different kind from the one involved in the usual AdS/CFT duality in that there are two holographic coordinates, of which one is timelike and the other spacelike. This fact gives the regulated five-dimensional boundary a dynamic character which we have only partially explored.

In order for the system to retain some supersymmetry, then in the simplest case in which $A_6$ is flat the compact space $C_4$ must be a Calabi-Yau twofold. The likely candidate for a ECFT is then a finite ($\beta(g) = 0$) euclidean super Yang-Mills theory with $SU(N)$ gauge group, $\mathcal{N} = 2$ supersymmetries and broken $R$-symmetry. We indeed know how to

\(^5\)This argument fails for the flat background; then $l$ is really arbitrary.
write down the central charge of this CFT on the gravitational side, namely $c = \frac{l^4}{\kappa_6}$. This would mean that $l \sim N^{1/2}$.

The relationship of the backgrounds considered in this paper to the ones related to the Maldacena AdS/CFT conjecture (and, in particular, the origin of the Ramond-Ramond fields from this point of view) still eludes us. Besides, the situation for the proposed duality has also some similarities with the dS/CFT duality of Strominger’s (23), in the sense that here also, at least one of the holographic coordinates is timelike.

Further work is needed to clarify this relationship, as well as to further expand the operator mapping.

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**A AdS\( (p,q) \) and its horospheres**

Let us recall some elementary facts on horospheric coordinates (3). For arbitrary ± signs, denoted by $\epsilon_\mu = \pm 1$, the metric induced on the surface

$$\sum_{\mu=1}^{n+1} \epsilon_\mu x_\mu^2 = \pm l^2$$  \(33\)

by the imbedding on the flat space with metric

$$ds^2 = \sum_{\mu=1}^{n+1} \epsilon_\mu dx_\mu^2$$  \(34\)

can easily be reduced to a generalization of Poincaré’s metric for the half-plane by introducing the coordinates

$$\frac{l}{z} \equiv x^-$$

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\[ y^i \equiv z \ x^i \] (35)

where we have chosen the two last coordinates, \( x^n \) and \( x^{n+1} \) in such a way that their contribution to the metric is \( dx^2_n - dx^2_{n+1} \) (this is always possible if we have at least one timelike coordinate); and we define \( x^- \equiv x^{n+1} - x^n \). \( 1 \leq i, j \ldots \leq n-1 \). The generalization of the Poincaré metric is:

\[ ds^2 = \sum \epsilon_i dy_i^2 \mp l^2 dz^2 \] (36)

(where the signs are correlated with the ones defined in (33), and the surfaces \( z = \text{const} \) are called horospheres in the mathematical literature.

The curvature scalar is given by:

\[ R = \pm \frac{n(n-1)}{l^2} \] (37)

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra \( SO(n+1) \). The Killing vector fields are explicitly given (no sum in the definition) by

\[ L_{\mu\nu} \equiv \epsilon_{\mu} x^{\mu} \partial_\nu - \epsilon_{\nu} x^{\nu} \partial_\mu \] (38)

To be specific, when the metric is given by:

\[ ds^2 = \delta_{ij} dx^i dx^j \mp l^2 dz^2 \] (39)

then the isometry group is \( SO(n,1) \). This is the case for what could be called euclidean de Sitter, \( EdS_n \), which in our conventions has got all coordinates timelike, and negative curvature.

The symmetric situation where

\[ ds^2 = -\delta_{ij} dx^i dx^j \mp l^2 dz^2 \] (40)

enjoys \( SO(1,n) \) as isometry group, and includes the ordinary de Sitter space, \( dS_n \). What one would want to call Euclidean anti de Sitter , \( EAdS_n \), has got all its coordinates spacelike, and positive curvature.
Finally, when the metric is given by
\[ ds^2 = \frac{\eta_{ij} dx^i dx^j \mp l^2 dz^2}{z^2} \] (41)
(where as usual, \( \eta_{ij} \equiv \text{diag}(1, (-1)^{n-2}) \)), then the isometry group is \( SO(2, n-1) \). This includes the regular Anti de Sitter, \( AdS_n \).

**B The Low Energy Limit**

If we characterize the metric in the \( A_{n+2} \) ambient space as
\[ ds^2 = \frac{t^2}{l^2} h_{ij}(x, \rho) dx^i dx^j + \rho dt^2 + tdtd\rho \] (42)
then its Ricci tensor reads
\[ l^2 R^A_{ij} = \rho [2h''_{ij} - 2h''_{il} h_{mj} + h^{kl} h''_{ij}] + l^2 R_{ij}[h] - (d - 2)h''_{ij} - tr(h^{kl} h''_{kl}) h_{ij} \]
\[ = l^2 R_{ij}^{\text{Bulk}} - \frac{d}{\rho} h_{ij} \]
\[ R^A_{it} \equiv 0 \]
\[ R^A_{ip} = \frac{1}{2} [h^{ji} (\nabla_j h'_{il} - \nabla_i h'_{jl})] = R_{ip}^{\text{Bulk}} \]
\[ R^A_{\rho \rho} = -\frac{1}{2} [(h^{jk} h''_{kj}) - \frac{1}{2} (h^{il} h_{tm} h^{mn} h''_{ni})] = R_{\rho \rho}^{\text{Bulk}} + \frac{d}{4\rho^2} \]
\[ R^A_{\rho t} = 0 \]
\[ R^A_{tt} = 0 \] (43)

where a prime means \( \frac{d}{d\rho} \), and \( \nabla_i \) is the covariant derivative of the Levi-Civita connection of the metric \( h_{ij} \). Demanding that this \( (n+2) \)-dimensional Ricci tensor vanishes reproduces the \( (n+1) \)-dimensional Einstein’s equations corresponding to a fixed cosmological constant \( \lambda = \frac{n(n-1)}{2l^2} \) (cf. for example eq. (191) in [3]).

We have indeed represented by a superscript the corresponding quantities in the five dimensional bulk space, which is defined as the manifold endowed with a metric:
\[ ds^2 = \frac{-l^2 d\rho^2 + 1}{4\rho^2} h_{ij} dx^i dx^j \] (44)
Please refer to [3] for an expansion in powers of $\epsilon$ of this set of equations.

C The six-dimensional Ambient Space associated to flat four-dimensional space

Let us work out the simplest possible example, namely, the $A_6$ ambient space corresponding to a flat four dimensional space. This is, expressed in the canonical coordinates introduced by Fefferman and Graham (FG), (11)

$$ds^2 = -\frac{t^2}{l^2}d\vec{x}^2 + \rho dt^2 + t\rho dt$$

In this case, there is a very simple scale and invariance in the ambient space, namely

$$x \rightarrow \lambda x$$
$$t \rightarrow \frac{1}{\lambda}t$$
$$\rho \rightarrow \lambda^2 \rho$$

(46)

Through the study of its geodesics, it is not difficult to find a change of coordinates which reduces it to flat six-dimensional space, namely,

$$t \equiv \xi_0 - \xi_5$$
$$x^i \equiv \frac{l\xi^i}{\xi_0 - \xi_5}$$
$$\rho \equiv \frac{(\xi^\mu)^2}{(\xi_0 - \xi_5)^2}$$

(47)

where $(\xi^\mu)^2 = \xi_0^2 - \xi_5^2 - \vec{\xi}^2$. The flat space $\mathbb{R}^6$ with coordinates $\xi^\mu \equiv (\xi^0, \vec{\xi}, \xi^5)$, reads

$$ds^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu$$

(48)
(where $\eta_{\mu\nu} = \text{diag}(1, (-1)^5)$). The inverse change of coordinates is given by:

$$
\xi^0 = \frac{t}{2}(1 + \rho + \frac{x^2}{t^2})
$$

$$
\xi^5 = \frac{t}{2}(-1 + \rho + \frac{x^2}{t^2})
$$

$$
\xi^i = \frac{x^i}{t}
$$

(49)

Lorentz transformations enjoy a nonlinear realization in the FG coordinates. The scale invariance in the FG coordinates, for example, corresponds to a boost in the $\xi^5$ direction in the Minkowskian coordinates. If we perform a general Lorentz transformation $\xi^\mu = \Lambda^\mu_\nu \xi^\nu = (\delta^\mu_\nu + \theta^\mu_\nu)\xi^\nu$ where

$$
\theta_{\mu\nu} = \begin{pmatrix} 0 & -\omega & \frac{\bar{\alpha} + \bar{\beta}}{2} \\ \omega & 0 & \frac{\bar{\alpha} - \bar{\beta}}{2} \\ -\frac{\bar{\alpha} + \bar{\beta}}{2} & -\frac{\bar{\alpha} - \bar{\beta}}{2} & \theta_{ij} \end{pmatrix}
$$

(50)

the corresponding change (linearized) in the FG coordinates reads

$$
\delta t = t(\omega + \frac{\bar{\alpha} + \bar{\beta}}{2})
$$

$$
\delta \rho = -2\rho(\omega + \frac{\bar{\alpha} + \bar{\beta}}{2})
$$

$$
\delta x^i = \frac{1}{2}b^i - \omega x^i + \theta^i_j x^j + \frac{1}{2}(\rho + \frac{\bar{\alpha} + \bar{\beta}}{t^2})a^i - \frac{\alpha \bar{\beta}}{t} x^i
$$

(51)

The corresponding jacobian of the change of coordinates (47) reads

$$
| \det \frac{\partial \xi^\mu}{\partial (\rho, y, x^i)} | = \frac{1}{2l^n} |t^{n+1}|
$$

(52)

This means that there is a horizon at $t = 0$, and we are only covering one-half of Minkowski space, namely a Minkowski wedge,

$$
\xi_0 > \xi_{n+1}
$$

(53)

There are then two interesting hypersurfaces in our problem: $B \equiv \{ \rho = 0 \}$ (which we will call the boundary, a null surface, with null normal vector $n^2 = \frac{2\rho}{t^2} = 0$), and $\Sigma \equiv \{ t = 0 \}$ (which is the horizon determining the portion of Minkowski space covered by the standard FG coordinates). The horizon is also a null surface, $n^2 = 0$.  

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Nothing prevents us from assuming that the coordinates $x^i$ live in a torus. The relationship to the Minkowskki wedge is then lost. For example, in the simplest case in which all coordinates live in a circle of radius $L$, there is an equivalent T-dual formulation (13) of the sort

$$ds^2 = -\frac{l^2}{t^2}d\tilde{x}^2 + \rho dt^2 + t d\rho dt$$

and a dilaton

$$\Phi = -\frac{n}{2} \log \frac{t^2}{l^2}$$

Non constant dilatons are notoriously difficult to work with; the original representation of the background will be then usually preferred.

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