CHERN CLASSES OF THE TANGENT BUNDLE ON THE
HILBERT SCHEME OF POINTS ON THE AFFINE PLANE

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Abstract. The cohomology of the Hilbert schemes of points on smooth pro-
jective surfaces can be approached both with vertex algebra tools and equi-
variant tools. Using the first tool, we study the existence and the structure of
universal formulas for the Chern classes of the tangent bundle over the Hilbert
scheme of points on a projective surface. The second tool leads then to nice
generating formulas in the particular case of the Hilbert scheme of points on
the affine plane.

1. Introduction

Let $S$ be a smooth projective complex surface and $n$ a non-negative integer.
The Hilbert scheme of $n$ points on $S$, denoted by $\text{Hilb}^n(S)$ or $S^{[n]}$, parameterizing
generalized $n$-tuples on $S$, i.e. zero-dimensional subschemes of length $n$ of $S$ is, by
a result of Fogarty [10], a smooth projective surface of complex dimension $2n$. In
the study of his rational cohomology, initiated by Göttsche [12], Grojnowski [13]
and Nakajima [31], several vertex algebra tools have been developed by Lehn [20]
and after by Li-Qin-Wang [22, 23, 24], providing a better understanding of the
universality of the ring structure of the cohomology as such as universal formulas
for the Chern classes of tautological bundles on the Hilbert scheme.

In a different flavor, the equivariant structure of the Hilbert scheme of points on
the projective plane for a natural action of the torus has been used by Ellingsrud-
Strømme [9], Nakajima [29] and Vasserot [32] to get a better understanding of
the tangent bundle and the cohomology ring of $\text{Hilb}^n(\mathbb{C}^2)$. In this context, the
equivariant cohomology of the Hilbert scheme of points on the affine plane was the
main tool in our preceding work [2] to get general combinatorial formulas for the
Chern classes of linearized vector bundles over $\text{Hilb}^n(\mathbb{C}^2)$.

In view of the existing generating formulas for the Chern classes of tautological
bundles discovered by Lehn [20], it is natural (as we learned with Lehn [19]) to ask
for a similar result in the case of the tangent bundle over the Hilbert scheme of points
on the affine plane. The total cohomology space $\bigoplus_n H^*(\text{Hilb}^n(\mathbb{C}^2))$ is naturally
isomorphic, as a vertex algebra, to the space of polynomials $\Lambda := \mathbb{Q}[p_1, p_2, p_3, \ldots]$. In
this paper, we prove the following generating formulas for the Chern classes and
the Chern characters of the tangent bundles $T \text{Hilb}^n(\mathbb{C}^2)$, expressed in $\Lambda$:

Theorem 1.1. The Chern classes of the tangent bundle over the Hilbert scheme
$\text{Hilb}^n(\mathbb{C}^2)$ are given by the following generating series:

$$
\sum_{n \geq 0} c_*(T \text{Hilb}^n(\mathbb{C}^2)) = \exp \left( \sum_{k \geq 0} (-1)^k C_k \frac{p_{2k+1}}{2k+1} \right).
$$

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vertex algebras.
where \( C_k := \frac{1}{k+1} \binom{2k}{k} \) is the \( k \)-th Catalan number.

The Chern characters are given by the following generating series:

\[
\sum_{n \geq 0} \chi \left( T \text{Hilb}^n(\mathbb{C}^2) \right) = 2e^{p_1} \sum_{k \geq 0} \frac{p_{2k+1}}{(2k+1)!}.
\]

The strategy of the proof is as follows. We first have to understand the common features of such generating series of Chern classes for the tangent bundle over \( S^{[n]} \).

To do so, we develop the notion of “universal formula” in the same spirit as in [23], and we prove a refined version of the theorem of Ellingsrud-Göttsche-Lehn [8] on the Chern numbers of the tangent bundles \( TS^{[n]} \) by establishing the universality of the Chern classes ([3,4]). Applied properly to the case of the affine plane, we see that the formulas are then much easier and we finish the computation by means of our general formulas [2] and some combinatorial tricks.

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2. Cohomology of Hilbert schemes of points

In this section, we recall some classical results related to the vertex algebra structure of the total cohomology space of Hilbert schemes of points on surfaces.

Conventions. For a smooth quasi-projective variety \( X \), we denote by \( H^*(X) \) the singular cohomology with rational coefficients of the underlying real manifold, by \( K(X) \) the rational Grothendieck group generated by locally free sheaves, or equivalently by arbitrary coherent sheaves and by \( \chi : K(X) \to H^{2*}(X) \) the Chern character. Our conventions and notations for the various operations in both theories follow [11]. For any continuous map \( f : X \to Y \) between two smooth oriented compact manifolds, we define a cohomological push-forward by \( f_! := D_Y^{-1} \circ f_* \circ D_X \) where \( D \) stands for the Poincaré duality and \( f_* \) for the homological push-forward.

2.1. The Fock space. Let \( S \) be a smooth complex projective surface with canonical class \( K_S \) and Euler class \( e_S \). For any integer \( n \geq 0 \), define the Hilbert scheme of points \( S^{[n]} := \text{Hilb}^n(S) \) as the scheme representing the functor of flat families of length \( n \) zero-dimensional closed subschemes on \( S \). By a result of Grothendieck ([12]), it has a natural structure of projective scheme and is equipped with a universal family \( \Xi^n_S \subset S^{[n]} \times S \). By a theorem of Fogarty ([13]), \( S^{[n]} \) is in fact a smooth manifold of complex dimension \( 2n \). We study his singular cohomology with rational coefficients:

\[
\mathbb{H}^S := \bigoplus_{n \geq 0} H^*(S^{[n]}), \quad \mathbb{H}^S := \bigoplus_{n \geq 0} \mathbb{H}^S_n.
\]

The unit in \( \mathbb{H}_0^S \cong \mathbb{Q} \) is called vacuum vector and denoted by \( |0\rangle \) (or \( |0\rangle_S \) if necessary). The Fock space \( \mathbb{H}^S \) is double graded by \((n, i)\): the integer \( n \) is called the conformal weight and the integer \( i \) the cohomological degree, also denoted by \(|\cdot|\).

A linear operator \( f \in \text{End}(\mathbb{H}^S) \) is homogeneous of bidegree \((u,v)\) if for any \( n \) we have \( f(H^*(S^{[n]})) \subset H^{i+v}(S^{[n+u]}) \). The super-commutator of two homogeneous operators \( f, g \) is defined by:

\[
[f, g] := f \circ g - (-1)^{|f||g|} g \circ f.
\]
Lemma 3 

We shall make use of the following technical formulas, of the same spirit as in [23, τ]

Then:

For \( n \geq 0 \) and \( k > 0 \), let \( S^{[n,n+k]} \subset S^{[n]} \times S \times S^{[n+k]} \) be the subvariety defined

set-theoretically by:

\[
S^{[n,n+k]} := \{ (\xi, x, \xi') \mid \xi \subset \xi' \And \text{Supp}(\mathcal{I}_\xi/\mathcal{I}_{\xi'}) = x \},
\]

where \( \mathcal{I}_\xi \) denotes the ideal sheaf of the subscheme \( \xi \) (with \( S^{[n,n]} = \emptyset \)). We denote the projections by:

\[
\begin{array}{rcl}
S^{[n]} \times S \times S^{[n+k]} & \xrightarrow{\varphi} & S^{[n]} \\
& \xrightarrow{\rho} & S \\
& \xrightarrow{\psi} & S^{[n+k]}
\end{array}
\]

The Heisenberg operators are the linear operators

\[
q_k : H^*(S) \to \text{End}(H^S), \quad k \in \mathbb{Z}
\]
defined as follows. If \( k \geq 0 \), for \( \alpha \in H^*(S) \) and \( x \in H^*(S^{[n]}) \) we set

\[
q_k(\alpha)(x) := \psi_1 \left( \left[ S^{[n,n+k]} \right] \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right)
\]

and the operators for negative indices are defined by adjonction:

\[
q_{-k}(\alpha) := (-1)^k q_k(\alpha)^\dagger, \quad \forall k \geq 0.
\]

By convention, \( q_0 = 0 \). The operators \( q_k \) are called creation operators if \( k \geq 1 \) and annihilation operators if \( k \leq -1 \).

**Theorem 2.1 (Nakajima).** ([50, 31]) The operators \( q \) satisfy the following commutation formula:

\[
[q_i(\alpha), q_j(\beta)] = i \cdot \delta_{i+j,0} \cdot \int_S \alpha \beta \cdot \text{id}_{H^S}.
\]

In particular, the total cohomology space \( H^S \) admits a basis of vectors

\[
q_{n_1}(u_1^S) \cdots q_{n_k}(u_k^S) |0\rangle
\]

for \( n_i \geq 1 \), where the classes \( u_i^S \) run over a basis of \( H^*(S) \).

For \( k \geq 0 \), we denote by \( \tau_k : H^*(S) \to H^*(S^k) \) the push-forward map induced by the diagonal inclusion \( \tau_k : S \to S^k \). For \( k = 0 \), \( \tau_k \alpha \) is understood to be \( \int_X \alpha \).

For \( k \geq 1 \), by Künneth decomposition we can set:

\[
\tau_k \alpha = \sum_i \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k} \in H^*(S) \otimes \cdots \otimes H^*(S).
\]

We shall make use of the following technical formulas, of the same spirit as in [23, Lemma 3.1]:

**Lemma 2.2.** Let \( \alpha, \beta, \gamma \in H^*(S) \) and \( p, q \geq 1 \). Assume that \( \tau_{2l} \gamma = \sum_i \gamma_{i,1} \otimes \gamma_{i,2} \).

Then:

(a) \( \tau_{(k-1)} \alpha = \sum_i \int_S \alpha_{i,k} \cdot \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k-1} \)

(b) \( \tau_{(p+q)}(\alpha \beta \gamma) = \sum_i \tau_p(\alpha \gamma_{i,1}) \otimes \tau_q(\beta \gamma_{i,2}) \)
The Virasoro operators defined by the normally ordered product \( L \) are the elementary operators \( \tau \) and then since for formula (b), notice that:

\[
\tau_{(k-1)!}\alpha = \sum_{i} p_{[1,k-1]}^{i} (\alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k})
\]

\[
= \sum_{i} p_{[1,k-1]}^{i} \left( p_{[1,k]}^{i} (\alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k-1}) \cdot p_{k}^{i} \alpha_{i,k} \right)
\]

\[
= \sum_{i} \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k-1} \cdot p_{[1,k-1]}^{i} p_{k}^{i} \alpha_{i,k}
\]

\[
= \sum_{i} \int_{S} \alpha_{i,k} \cdot \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k-1}
\]

For formula (b), notice that:

\[
\tau_{2!}(\alpha \beta \gamma) = \tau_{2!}(\tau_{2}^{\ast}(\alpha \otimes \beta) \cdot \gamma)
\]

\[
= (\alpha \otimes \beta) \cdot \tau_{2!}(\gamma)
\]

\[
= \sum_{i} (\alpha \gamma_{i,1}) \otimes (\beta \gamma_{i,2})
\]

and then since \( \tau_{p+q} = (\tau_{p} \times \tau_{q}) \circ \tau_{2} \) one gets the result. \( \square \)

For \( k \geq 1 \) and \( \alpha \in H^{*}(S) \), by use of the Künneth decomposition we define the elementary operators as the operators:

\[
q_{n_{1}} \cdots q_{n_{k}}(\tau_{k!}\alpha) := \sum_{i} q_{n_{1}}(\alpha_{i,1}) \circ \cdots \circ q_{n_{k}}(\alpha_{i,k}).
\]

The normally ordered product of two operators \( q \) is defined by the convention:

\[
: q_{n} q_{m} : = \begin{cases} q_{n} q_{m} & \text{if } n \geq m \\ q_{m} q_{n} & \text{if } n \leq m \end{cases}
\]

The Virasoro operators are the linear operators

\[
\mathcal{L}_{n} : H^{*}(S) \to \text{End}(H^{S}), \quad n \in \mathbb{Z}
\]

defined by

\[
\mathcal{L}_{n} := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} q_{\nu} q_{n-\nu} : \tau_{2!}.
\]

**Theorem 2.3** (Lehn). ([20 Theorem 3.3]) The operators \( \mathcal{L} \) satisfy the following commutation formulas:

\[
[\mathcal{L}_{n}(\alpha), q_{m}(\beta)] = -m \cdot q_{n+m}(\alpha \beta);
\]

\[
[\mathcal{L}_{n}(\alpha), \mathcal{L}_{m}(\beta)] = (n - m) \cdot \mathcal{L}_{n+m}(\alpha \beta) - \frac{n^{3} - n}{12} \cdot \delta_{n+m,0} \cdot \int_{S} e_{S} \alpha \beta \cdot \text{id}_{H^{S}}.
\]

Denote the canonical projection of the universal family on the Hilbert scheme by \( p : \Xi_{n}^{S} \to S^{[n]} \) and let \( B_{n}^{S} := p_{*} \mathcal{O}_{\Xi_{n}^{S}} \) be the rank \( n \) tautological bundle on \( S^{[n]} \). Let \( \delta \in \text{End}(H^{S}) \) be the linear operator defined by:

\[
\delta(x) := e_{1}(B_{n}^{S}) \cdot x \quad \forall x \in H^{*}(S^{[n]}).
\]
The derivative of a linear operator $\mathfrak{f} \in \text{End}(\mathbb{H}^S)$ is defined by $\mathfrak{f}' := [\partial, \mathfrak{f}]$ and the higher derivatives are $\mathfrak{f}^{(n)} := (\text{ad} \partial)^n(\mathfrak{f})$.

**Theorem 2.4 (Lehn).** ([20] Main Theorem 3.10) The derivatives of the operators $q$ satisfy the formulas:

\[
[q_n(\alpha), q_m(\beta)] = -nm \cdot \left( q_{n+m}(\alpha \beta) + \frac{n-1}{2} \delta_{n+m,0} \cdot \int_K S \alpha \beta \cdot \text{id}_{\mathbb{H}^S} \right);
\]

\[
q'_n(\alpha) = n \cdot \mathcal{L}_n(\alpha) + q_n(K S \alpha).
\]

### 2.2. Tautological bundles

Consider the following diagram:

\[
\begin{array}{ccc}
\Xi_n^S & \longrightarrow & S[n] \times S \\
\downarrow p & & \downarrow q \\
S^n & & S
\end{array}
\]

Let $F$ be a locally free sheaf on $S$. For any $n \geq 0$, the associated *tautological bundle* on $S^{[n]}$ is defined as:

\[
F^{[n]} := p_* \left( \mathcal{O}_{\Xi_n^S} \otimes q^* F \right).
\]

Since the projection $p$ is flat and finite of degree $n$, $F^{[n]}$ is a fibre bundle of rank $n \cdot \text{rk}(F)$ (by convention, $F^{[0]} = 0$). This construction extends naturally to a well-defined group homomorphism:

\[
-^{[n]} : K(S) \to K(S^{[n]}).
\]

For $u \in K(S)$, let $c(u)$ and $\mathcal{C}(u)$ be the linear operators acting for any $n \geq 0$ on $H^*(S^{[n]})$ by multiplication by the total Chern class $c(u^{[n]})$ and the total Chern character $\mathcal{C}(u^{[n]})$ respectively.

**Theorem 2.5 (Lehn).** ([20] Theorem 4.2) Let $u \in K(S)$ be the class of a vector bundle of rank $r$ and $\alpha \in H^*(S)$. Then:

\[
[c(\mathcal{C}(u)), q_1(\alpha)] = \exp(\text{ad} \partial)(q_1(\mathcal{C}(u)\alpha));
\]

\[
c(u) \circ q_1(\alpha) \circ (c(u))^{-1} = \sum_{\nu,k \geq 0} \binom{r-k}{\nu} q_1^\nu(c_k(\alpha)).
\]

By analogy with the construction of tautological bundles, one defines a linear operation $-^{[n]} : H^*(S) \to H^*(S^{[n]})$ as follows. For any cohomology class $\gamma \in H^*(S)$ we set

\[
\gamma^{[n]} := p_* \left( \mathcal{C}(\mathcal{O}_{\Xi_n^S}) \cdot q^* \text{td}(S) \cdot q^* \gamma \right)
\]

where $\text{td}(S)$ denotes the Todd class of the tangent bundle $TS$ and we define a linear operator $\mathfrak{S}(\gamma) \in \text{End}(\mathbb{H}^S)$ acting on $H^*(S^{[n]})$ by multiplication by $\gamma^{[n]}$. This definition is such that, by the Riemann-Roch-Grothendieck theorem, the following diagram is commutative:

\[
\begin{array}{ccc}
H^*(S) & \longrightarrow & H^*(S^{[n]}) \\
\downarrow c(h) & & \downarrow c(h) \\
K(S) & \longrightarrow & K(S^{[n]})
\end{array}
\]

**Theorem 2.6 (Li-Qin-Wang).** ([22] Lemma 5.8) Let $\gamma, \alpha \in H^*(S)$. Then:

\[
[\mathfrak{S}(\gamma), q_1(\alpha)] = \exp(\text{ad} \partial)(q_1(\gamma \alpha)).
\]
3. Universal formulas

In this section, we develop the notion of universal formula and we prove some general results about the existence and the structure of universal formulas for the characteristic classes of natural bundles on Hilbert schemes of points.

3.1. Definition of a universal formula. For any projective variety $X$, let $U^X$ be a cohomology class in $H^*(X)$ which is functorial with respect to pull-backs. For any smooth projective surface $S$, we set $U_n^S := U^{S[n]}$ and $U^S := \sum_{n \geq 0} U_n^S \in H^S$.

**Definition 3.1.** A class $U_n^S \in H^*(S[n])$ admits a universal formula if there exists a polynomial $P \in \mathbb{Q}[Z_1, \ldots, Z_p]$ independent of $S$, integers $k_1, \ldots, k_p \geq 1$, indices $n_{i,j} \geq 1$ for $1 \leq i \leq p$ and $1 \leq j \leq k_i$ together with cohomology classes $u_{i,j}^S \in H^*(S)$ in the sub-algebra generated by $1_S, K_S, e_S$ (which could also depend on some additional data constructing $U_n^S$) such that one has:

$$U_n^S = P \left( q_{n_{1,1}} \cdots q_{n_{1,k_1}} (\tau_{k_1} u_{1}^S), \ldots, q_{n_{p,1}} \cdots q_{n_{p,k_p}} (\tau_{k_p} u_{p}^S) \right) [0].$$

If $P$ is homogeneous of degree 1, we shall say that the universal formula for $U_n^S$ is a universal linear combination.

Our definition is inspired by the notion of universal linear combination defined by Li-Qin-Wang [23, Definition 3.1], but our definition is more restrictive since we only consider creation operators (for some reasons that will appear soon) and we restrict the cohomology classes in $H^*(S)$ to the natural sub-algebra generated by the canonical class and the Euler class.

3.2. General results on universality. Let $S_1, S_2$ be two smooth projective surfaces and denote by $S_1 \amalg S_2$ their disjoint union. The Hilbert scheme of points decomposes as follows (see [5, Formula (0.1)]):

$$(S_1 \amalg S_2)[n] = \coprod_{n_1 + n_2 = n} S_1^{[n_1]} \times S_2^{[n_2]},$$

inducing the decomposition of the total cohomology:

$$H^* \left( (S_1 \amalg S_2)[n] \right) \cong \bigoplus_{n_1 + n_2 = n} H^* \left( S_1^{[n_1]} \right) \otimes H^* \left( S_2^{[n_2]} \right),$$

and the corresponding decomposition of the Fock space:

$$H^{S_1 \amalg S_2} \cong H^{S_1} \otimes H^{S_2}.$$

In particular, there is a double graded inclusion $H^{S_1} \otimes H^{S_2} \subset H^{S_1 \amalg S_2}$.

**Lemma 3.2.** Suppose that a class $U^S \in H^S$ admits a universal formula such that:

$$U^{S_1} \otimes U^{S_2} = U^{S_1 \amalg S_2}$$

for any smooth projective surfaces $S_1, S_2$. Then the universal formula is an (infinite) universal linear combination.

**Proof.** Suppose a composition $q_{m_1} \cdots q_{m_p} (\tau_{p} u_{p}^S) \circ q_{m_{p-1}} \cdots q_{m_1} (\tau_{1} u_{1}^S)$ occurs in the universal formula of $U^S$. Then by the decompositions $u_{i}^{S_1 \amalg S_2} = u_{i}^{S_1} + u_{i}^{S_2}$ (this is the case for all classes in the sub-algebra generated by $1_S, K_S, e_S$, and if these classes use additional data, we suppose that this decomposition holds, as will always be the case in the sequel), the composition decomposes and extra non-zero terms arise. So the polynomial defining the universal formula is necessarily of degree 1. \qed
Lemma 3.3. Suppose that a class $U^S \in \mathbb{H}^S$ admits a universal formula such that $U^S_0 = |0\rangle$ and:

$$U^{S_1} \otimes U^{S_2} = U^{S_1 U S_2}$$

for any smooth projective surfaces $S_1, S_2$. Then the universal formula is an exponential of an (infinite) universal linear combination.

Proof. Denote by $U^S \in \text{End}(\mathbb{H}^S)$ the operator defined by the universal formula of $U^S$. By construction, $U^S |0\rangle = U^S$ and with $|0\rangle_{S_1 U S_2} = |0\rangle_{S_1} \otimes |0\rangle_{S_2}$ one gets:

$$U^{S_1 U S_2} |0\rangle_{S_1 U S_2} = (U^{S_1} |0\rangle_{S_1}) \otimes (U^{S_2} |0\rangle_{S_2}) = (U^{S_1} \otimes U^{S_2}) (|0\rangle_{S_1} \otimes |0\rangle_{S_2}).$$

Since the operator $U^S$ contains only creation operators, this equation implies the equality of the operators:

$$U^{S_1 U S_2} = U^{S_1} \otimes U^{S_2},$$

and since $U^S = \text{id}_{\mathbb{H}^S} + \cdots$, it admits a logarithm and:

$$\log U^{S_1 U S_2} = \log U^{S_1} \oplus \log U^{S_2}.$$

Applying the lemma to $\log U^S |0\rangle_S$, one gets the result. \hfill \Box

We dress now a list of some technical results needed to compute with elementary operators; the following statements deal with a weaker form of universality (see §3.1).

Lemma 3.4 (Li-Qin-Wang). (\cite{LiQinWang})

1. Any commutator $[q_{n_1} \cdots q_{m_k} (\tau_{p_1} \alpha), q_{m_1} \cdots q_{m_i} (\tau_{q_1} \beta)]$ can be expressed as a linear combination of elementary operators $q_{n_1} \cdots q_{n_i} (\tau_{\alpha})$, whose coefficients do not depend on $S$ (with $n_j, m_j, i_j \in \mathbb{Z}$).

2. Any derived operator $q_{n}^{(w)} (\alpha)$ can be expressed as a linear combination of elementary operators $q_{n_1} \cdots q_{n_k} (\tau_{e} (K^S_\alpha))$ for $0 \leq r \leq 2$ and $n_i \in \mathbb{Z}$, whose coefficients do not depend on $S$.

3. For any $\alpha \in H^*(S)$, $n_j \in \mathbb{Z}$, $k \geq 2$ and $1 \leq j < k$ one has:

$$q_{n_1} \cdots q_{n_j} q_{n_{j+1}} \cdots q_{n_k} (\tau_{e} (\alpha)) = q_{n_1} \cdots q_{n_{j+1}} q_{n_{j+1}} \cdots q_{n_k} (\tau_{e} (\alpha))$$

$$= n_j \delta_{2n_j+1,0} q_{n_1} \cdots q_{n_{j-1}} q_{n_{j+2}} \cdots q_{n_k} (\tau_{e} (\alpha))$$

4. Any commutator $[\mathfrak{g}(\alpha), q_{n_1} \cdots q_{n_k} (\tau_{e} \beta)]$ can be expressed as a linear combination of operators $q_{m_1} \cdots q_{m_r} (\tau_{e} (K^S_\alpha \beta))$ with $0 \leq r \leq 2$, whose coefficients do not depend on $S$ (with $n_j, m_j \in \mathbb{Z}$).

5. The cup-product of two classes $q_{n_1} \cdots q_{n_r} (\tau_{e} \alpha) |0\rangle$ and $q_{m_1} \cdots q_{m_s} (\tau_{e} \beta) |0\rangle$ can be expressed as a linear combination of classes $q_{i_1} \cdots q_{i_t} (\tau_{e} \gamma) |0\rangle$ where $\gamma$ depends on $\alpha, \beta, K^S, e, S$, whose coefficients do not depend on $S$ (with $n_j, m_i, i_j \in \mathbb{N}$).

3.3. Universal formulas for tautological bundles.

Proposition 3.5. Let $u \in K(S)$. The Chern characters $\text{ch}(u^{[n]})$ enter in a universal generating series of the kind:

$$\sum_{n \geq 0} \text{ch}(u^{[n]}) = \exp(q_1 (1_S) \mathfrak{g}(u) |0\rangle),$$

where $\mathfrak{g}(u) |0\rangle$ is an (infinite) universal linear combination depending on $K^S, e, S, \text{ch}(u)$.\hfill \Box
Proof. Start from the commutation formula given by the theorem \[ [\text{ch}(u), q_1(1_S)] = \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(\text{ch}(u)), \]
evaluated in $1_{S^{[n-1]}}$ for $n \geq 1$. Since $1_{S^{[n]}} = \frac{1}{n!} q_1(1_S)^n |0\rangle$, we get:

\[
n \cdot \text{ch} \left( u^n \right) = q_1(1_S) \text{ch} \left( u^{[n-1]} \right) + \sum_{\nu \geq 0} \frac{1}{\nu!(n-1)!} q_1^{(\nu)}(\text{ch}(u)) q_1(1_S)^{n-1} |0\rangle.
\]

Set $F(t) := \sum_{n \geq 1} c_h(u^n) t^n$ (the sum begins in $n = 1$ since $u^0 = 0$). Summing up the preceding formula we get:

\[
F'(t) - q_1(1_S) F(t) = \left( \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(\text{ch}(u)) \right) \exp(q_1(1_S)t) |0\rangle.
\]

We have to show that the exponential can be pushed to the left of the formula, in such a way that the remaining operators form a linear combination of elementary operators that, applied on the vaccum, can be simplified to a universal linear combination. This last step is performed with the lemma 3.4(3) which explains how one can push to the right all annihilation operators occurring in an elementary operator. Then, any annihilation operator vanishes on the vaccum, so one gets a universal linear combination (with only creation operators). So it only remains to understand how the exponential can be pushed to the left.

By lemma 3.4(2), any derived operator $q_1^{(\nu)}(-)$ is a linear combination of elementary operators $q_{n_1} \cdots q_{n_k}(\tau_k; \alpha)$ with coefficients independent of $S$ so it is enough to study the situation:

\[
q_{n_1} \cdots q_{n_k}(\tau_k; \alpha) \exp(q_1(1_S)t) |0\rangle.
\]

If no index $n_i$ is equal to $-1$, there is no problem since the exponential commutes with $q_{n_1} \cdots q_{n_k}(\tau_k; \alpha)$. Else, by use of the lemma 3.4(2) we can assume that $n_k = -1$ and we observe the following lemma:

**Lemma 3.6.** For $\alpha \in H^*(S)$, $x \in \mathbb{R}^S$ and $n \geq 0$ one has:

\[
q_{-1}(\alpha) q_1(1_S)^n x = -n \int_S \alpha \cdot q_1(1_S)^{n-1} x + q_1(1_S)^n q_{-1}(\alpha)x;
\]

\[
q_{-1}(\alpha) \exp(q_1(1_S)x) = - \int_S \alpha \cdot \exp(q_1(1_S)x) + \exp(q_1(1_S))q_{-1}(\alpha)x.
\]

**Proof of the lemma.** The first assertion results on an easy induction. The second is then straightforward. \[\square\]

Set the decomposition $\tau_k; \alpha = \sum_i \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k}$. Then by the preceding lemma applied to $x = |0\rangle$:

\[
q_{n_1} \cdots q_{n_k}(\tau_k; \alpha) \circ \exp(q_1(1_S)) |0\rangle = \sum_i q_{n_1}(\alpha_{i,1}) \circ \cdots \circ q_{n_k}(\alpha_{i,k}) \circ \exp(q_1(1_S)) |0\rangle
\]

\[=- \sum_i \int_S \alpha_{i,k} \cdot q_{n_1}(\alpha_{i,1}) \circ \cdots \circ q_{n_k-1}(\alpha_{i,k-1}) \circ \exp(q_1(1_S)) |0\rangle.
\]

Use now the formula (a) of lemma 3.2

\[
\tau_{(k-1); \alpha} = \sum_i \int_S \alpha_{i,k} \cdot \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k-1},
\]
to get that for $n_k = -1$ one has:

$$q_{n_1} \cdots q_{n_k}(\tau_k \alpha) \circ \exp(q_1(1_S)) \cdot 0 = -q_{n_1} \cdots q_{n_{k-1}}(\tau_{(k-1)} \alpha) \circ \exp(q_1(1_S)) \cdot 0.$$ 

Repeating this process if necessary, one can push the exponential to the left.

We are eventually lead to the following differential equation:

$$F'(t) - q_1(1_S)F(t) = \exp(q_1(1_S)t) \mathfrak{F}(u) \cdot 0$$

where $\mathfrak{F}(u) \cdot 0$ is an universal linear combination. Resolving this equation we get:

$$F(t) = \exp(q_1(1_S)t) \mathfrak{F}(u)t \cdot 0,$$

hence the proposition.

**Proposition 3.7.** Let $u \in K(S)$. The Chern classes $c(u[n])$ enter in a universal generating series of the kind:

$$\sum_{n \geq 0} c(u[n]) = \exp(\mathfrak{F}(u)) \cdot 0,$$

where $\mathfrak{F}(u) \cdot 0$ is an (infinite) universal linear combination depending on $K_S$, e.g., $c(u)$. 

**Proof.** The Chern classes are polynomials in the Chern characters. By the proposition 3.5 and the lemma 3.4, which says that cup-products of universal formulas are universal, we see that there exists a universal formula for the Chern classes. Denote by $U^S := c(u[n])$ this universal formula. Let $S_1, S_2$ be two smooth projective surfaces, $n_1, n_2$ two integers and $pr_i : S_1^{[n_1]} \times S_2^{[n_2]} \to S_1^{[n_1]}$ the projections. For $u_1 \oplus u_2 \in K(S_1 \amalg S_2) = K(S_1) \oplus K(S_2)$ one has the following decomposition (see [8, Theorem 4.2]):

$$(u_1 \oplus u_2)^{[n_1+n_2]}_{S_1^{[n_1]} \times S_2^{[n_2]}} = pr_1^* \left(u_1^{[n_1]}\right) \oplus pr_2^* \left(u_2^{[n_2]}\right),$$

Hence:

$$U^{S_1 \amalg S_2} = U^{S_1} \otimes U^{S_2}.$$ 

Applying the lemma 3.4 one gets the result. $\Box$

As an example of the formulas we have in mind, let us recall the following explicit result:

**Theorem 3.8 (Lehn).** [20, Theorem 4.6] Let $L$ be a line bundle on $S$. Then:

$$\sum_{n \geq 0} c(L[n]) = \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} q_m(c(L)) \right) \cdot 0.$$ 

3.4. Universal formulas for the tangent bundle. The first result about the universality of the Chern classes of the tangent bundle $TS^{[n]}$ is the following, where for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $2n$, the Chern numbers of $S$ are defined as:

$$c^S(TS^{[n]} := c_{\lambda_1}(TS^{[n]}), \ldots, c_{\lambda_k}(TS^{[n]})) \in H^{4\lambda}(S^{[n]}) \cong \mathbb{Q}.$$ 

**Theorem 3.9 (Ellingsrud-Göttsche-Lehn).** [8, Proposition 0.5] For any integer $n$ and any partition $\lambda$ of $2n$, there exists a universal polynomial $P_\lambda \in \mathbb{Q}[z_1, z_2]$ such that for any projective surface $S$ one has:

$$c_\lambda(TS^{[n]}) = P_\lambda \left(c_1(S)^2, c_2(S)\right).$$

We shall prove more precise statements for the Chern classes themselves.

---

1There is an inaccuracy in this formula: instead of a product $\cdot$ one should read a sum $\oplus$. 

Proposition 3.10. The Chern characters \( \text{ch}(TS^{[n]}) \) enter in a universal generating series of the kind:
\[
\sum_{n \geq 0} \text{ch}(TS^{[n]}) = \exp(q_1(1_S)) \mathfrak{g} [0],
\]
where \( \mathfrak{g} [0] \) is an (infinite) universal linear combination depending on \( K_S, e_S \).

**Proof.** Recall some geometric results (see \[8, Proposition 2.3\]). Consider the following commutative diagram:

\[
\begin{array}{c}
S^{[n,n+1]} \xrightarrow{\psi} S^{[n+1]} \\
\downarrow \sigma \quad \downarrow \rho \\
S^{[n]} \times S \quad S
\end{array}
\]

Then \( \sigma = (\varphi, \rho) : S^{[n,n+1]} \to S^{[n]} \times S \) is the blow-up of \( S^{[n]} \times S \) along the universal family \( \Xi_n \), i.e. \( S^{[n,n+1]} \cong \text{Bl}_{\Xi_n} (S^{[n]} \times S) \). Denote by \( E \) the exceptional divisor, set \( \mathcal{L} := \mathcal{O}_{S^{[n+1]}}(-E) \) and denote the class of the tangent bundle \( TS^{[n]} \) in \( K(S^{[n]}) \) by \( T_n \). Recall the following result:

**Proposition 3.11** (Ellingsrud-Göttsche-Lehn). \[8, Proposition 2.3\] The following relation holds in \( K(S^{[n,n+1]}) \):
\[
\psi^! T_{n+1} = \varphi^! T_n + \mathcal{L} \cdot \sigma^!(\mathcal{O}_{\Xi_n}) + \mathcal{L}^\vee \cdot \rho^! \omega_S^\vee - \mathcal{L}^\vee \cdot \sigma^!(\mathcal{O}_{\Xi_n}) \cdot \rho^! \omega_S^\vee - \rho^!(\mathcal{O}_S - T_n + \omega_S^\vee).
\]

Let \( \{b_i\} \) be a basis of \( H^*(S) \) such that \( \int_S b_i b_j td(S) = \delta_{i,j} \). In the Künneth decomposition \( H^*(S^{[n]} \times S) \cong H^*(S^{[n]}) \otimes H^*(S) \) we can write:
\[
\text{ch}(\mathcal{O}_{\Xi_n}) = \sum_i \alpha_i \otimes b_i = \sum_i p^* \alpha_i \cdot q^* b_i
\]
for some classes \( \alpha_i \). By definition of the tautological classes and by use of the projection formula we find:
\[
b_j^{[n]} = p_!(\text{ch}(\mathcal{O}_{\Xi_n}) \cdot q^* b_j \cdot q^* td(S))
= \sum_i p_!(p^* \alpha_i \cdot q^* b_i \cdot q^* b_j \cdot q^* td(S))
= \sum_i \alpha_i \cdot p q^* (b_i b_j td(S))
= \alpha_j,
\]
Hence:
\[
\text{ch}(\mathcal{O}_{\Xi_n}) = \sum_i b_j^{[n]} \otimes b_i = \sum_i p^* b_i^{[n]} \cdot q^* b_i.
\]
Define “dual” tautological classes by:
\[
\gamma_j^{[n]} := p_!(\text{ch}(\mathcal{O}_{\Xi_n}) ^\vee \cdot q^* \gamma \cdot q^* td(S)).
\]
One finds similarly:
\[
\text{ch}(\mathcal{O}_{\Xi_n}) = \sum_i b_i^{[n]} \otimes b_i = \sum_i p^* b_i^{[n]} \cdot q^* b_i.
\]
Denote by $\text{ch}T \in \text{End}(\mathbb{H}^S)$ the operator acting by multiplication by $\text{ch}(T_n)$ on each conformal weight $n$, and by $\mathcal{G}^\vee(\gamma)$ the operator multiplying by $\gamma^{(n)}$. Note the formulas:

$$\sigma^* \text{ch}(\mathcal{O}_n) = \sum_i \varphi^* b_i^{[n]} \cdot \rho^* b_i$$

$$\sigma^* \text{ch}(\mathcal{O}_n^\vee) = \sum_i \varphi^* b_i^{(n)} \cdot \rho^* b_i,$$

We prove the following commutation relation:

**Lemma 3.12.**

$$[\text{ch}T, q_1(\alpha)] = \sum_{\nu} \frac{1}{\nu!} q_1^{(\nu)}(\alpha)$$

$$- \sum_{i,\nu} \frac{1}{\nu!} q_1^{(\nu)}(b_i \alpha) \circ \mathcal{G}^\vee(b_i)$$

$$+ \sum_{\nu} \frac{(-1)^\nu}{\nu!} q_1^{(\nu)}(\text{ch}(\omega_S^\vee) \alpha)$$

$$- \sum_{i,\nu} \frac{(-1)^\nu}{\nu!} q_1^{(\nu)}(b_i \text{ch}(\omega_S^\vee) \alpha) \circ \mathcal{G}(b_i)$$

$$- q_1(\text{ch}(\mathcal{O}_S - T_S + \omega_S^\vee) \alpha).$$

**Proof of the lemma.** The computation is similar to [20, Theorem 4.2], with more terms. For any $x \in H^*(S^{[n]})$, by the projection formula one gets:

$$\text{ch}T \circ q_1(\alpha)(x) = \text{ch}(T_{n+1}) \cdot \psi_1 \left( [S^{[n,n+1]}] \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right)$$

$$= \psi_1 \left( [S^{[n,n+1]}] \cdot \psi^*(\text{ch}(T_{n+1})) \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right).$$

The proposition 3.11 gives then:

$$\text{ch}T \circ q_1(\alpha)(x) = \psi_1 \left( [S^{[n,n+1]}] \cdot \varphi^*(\text{ch}(T_n) \cdot x) \cdot \rho^*(\alpha) \right)$$

$$+ \psi_1 \left( [S^{[n,n+1]}] \cdot \text{ch}(\mathcal{L}) \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right)$$

$$- \psi_1 \left( [S^{[n,n+1]}] \cdot \text{ch}(\mathcal{L}) \cdot \sigma^* \text{ch}(\mathcal{O}_n^\vee) \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right)$$

$$+ \psi_1 \left( [S^{[n,n+1]}] \cdot \text{ch}(\mathcal{L}^\vee) \cdot \varphi^*(x) \cdot \rho^*(\text{ch}(\omega_S^\vee) \cdot \alpha) \right)$$

$$- \psi_1 \left( [S^{[n,n+1]}] \cdot \text{ch}(\mathcal{L}^\vee) \cdot \sigma^* \text{ch}(\mathcal{O}_n) \cdot \varphi^*(x) \cdot \rho^*(\text{ch}(\omega_S^\vee) \cdot \alpha) \right)$$

$$- \psi_1 \left( [S^{[n,n+1]}] \cdot \varphi^*(x) \cdot \rho^*(\text{ch}(\mathcal{O}_S - T_S + \omega_S^\vee) \alpha) \right)$$
Set \( \lambda := c_1(\mathcal{L}) \). Then \( \text{ch}(\mathcal{L}) = \sum_{\nu \geq 0} \frac{1}{\nu!} \lambda^\nu \) and using the preceding decompositions we find:

\[
\text{ch}T \circ q_1(\alpha)(x) = q_1(\alpha) \circ \text{ch}T(x) \\
+ \sum_{\nu} \frac{1}{\nu!} \psi_1 \left( [S^{[n,n+1]}] \cdot \lambda^\nu \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right) \\
- \sum_{i,\nu} \frac{(-1)^\nu}{\nu!} \psi_1 \left( [S^{[n,n+1]}] \cdot \lambda^\nu \cdot \varphi^*(b_i) \cdot \rho^*(b_i \alpha) \right) \\
+ \sum_{\nu} \frac{(-1)^\nu}{\nu!} \psi_1 \left( [S^{[n,n+1]}] \cdot \lambda^\nu \cdot \varphi^*(x) \cdot \rho^*(\text{ch}(\omega_S^\nu)) \cdot \alpha \right) \\
- \sum_{i,\nu} \frac{(-1)^\nu}{\nu!} \psi_1 \left( [S^{[n,n+1]}] \cdot \lambda^\nu \cdot \varphi^*(b_i) \cdot \rho^*(b_i \text{ch}(\omega_S^\nu) \alpha) \right) \\
- \psi_1 \left( [S^{[n,n+1]}] \cdot \varphi^*(x) \cdot \rho^*(\text{ch}(\mathcal{O}_S - T_S + \omega_S^\nu) \alpha) \right)
\]

As explained in the proof of [20, Theorem 4.2] (or [20, Lemma 3.9]), a cycle \([S^{[n,n+1]}] \cdot \lambda^\nu\) induces the operator \(q_1^{(\nu)}\) and with our notations we get:

\[
\text{ch}T \circ q_1(\alpha)(x) = q_1(\alpha) \circ \text{ch}T(x) \\
+ \sum_{\nu} \frac{1}{\nu!} q_1^{(\nu)}(\alpha)(x) \\
- \sum_{i,\nu} \frac{1}{\nu!} q_1^{(\nu)}(b_i \alpha) \circ \Theta^\nu(b_i)(x) \\
+ \sum_{\nu} \frac{(-1)^\nu}{\nu!} q_1^{(\nu)}(\text{ch}(\omega_S^\nu) \alpha)(x) \\
- \sum_{i,\nu} \frac{(-1)^\nu}{\nu!} q_1^{(\nu)}(b_i \text{ch}(\omega_S^\nu) \alpha) \circ \Theta(b_i)(x) \\
- q_1(\text{ch}(\mathcal{O}_S - T_S + \omega_S^\nu) \alpha)(x).
\]

Since \( c_1(T_S) = -K_S, c_2(T_S) = e_S, c_1(\omega_S) = K_S \), we deduce \( \text{ch}(\omega_S) = \exp(K_S) \) and \( \text{ch}(\mathcal{O}_S - T_S + \omega_S^\nu) = e_S + K_S^\nu \). We follow now the same argument as for the proposition 4.5. We have to show that the following classes are universal formulas for which the exponential can be pushed to the left:

\[
\begin{align*}
(2) \quad & q_1^{(\nu)}(1_S) \exp(q_1(1_S)) |0\rangle, \\
(3) \quad & \sum_i q_1^{(\nu)}(b_i) \Theta^\nu(b_i) \exp(q_1(1_S)) |0\rangle, \\
(4) \quad & q_1^{(\nu)}(\exp(-K_S)) \exp(q_1(1_S)) |0\rangle, \\
(5) \quad & \sum_i q_1^{(\nu)}(b_i \exp(-K_S)) \Theta(b_i) \exp(q_1(1_S)) |0\rangle, \\
(6) \quad & q_1(e_S + K_S^\nu) \exp(q_1(1_S)) |0\rangle.
\end{align*}
\]

Formulas (2) and (5) have been studied before, and formula (6) is obvious. We shall study in details the formula (4), and then explain how one deduces the result for the formula (3).
By lemma 3.12, we can assume that $q^{(\nu)}_1(b_1 \exp(-K_S))$ is only an elementary operator $q_{n_1} \cdots q_{n_k}(\tau \ell(b_1 \alpha))$ where $\alpha$ is a polynomial in $K_S$. Then:

$$q_{n_1} \cdots q_{n_k}(\tau \ell(b_1 \alpha)) \mathcal{G}(b_1) \exp(q_1(1_S)) |0\rangle = q_{n_1} \cdots q_{n_k}(\tau \ell(b_1 \alpha)) \sum_{n \geq 0} b_i^{[n]}.$$ 

Observe that $b_i^{[n]} = ch((ch^{-1}b_i)^{[n]})$ so by the proposition 3.5 $\sum_{n \geq 0} b_i^{[n]}$ admits a universal formula of the kind $\exp(q_1(1_S)\mathcal{G}(b_1) |0\rangle$ where $\mathcal{G}(b_1) |0\rangle$ is a universal linear combination depending linearly on $b_i$. The lemma 3.6 applied to $x = \mathcal{G}(b_1) |0\rangle$ shows, as in the proof of the proposition 3.5, that:

$$q_{n_1} \cdots q_{n_k}(\tau \ell(b_1 \alpha)) \exp(q_1(1_S)\mathcal{G}(b_1) |0\rangle$$

can be expressed in a similar form with the exponential on the left, followed by a linear combination of operators:

$$q_{i_1} \cdots q_{i_p}(\tau \ell(b_i \beta)) \mathcal{G}(b_i).$$

By the linearity of the universal formula $\mathcal{G}(b_i)$, we assume that it consists only on an elementary operator $q_{j_1} \cdots q_{j_q}(\tau \ell(b_i \gamma))$. In order to get a universal formula, we have to get rid of the classes $b_i$. To do so, we call back the summation over the indices $i$:

$$\sum_i (q_{i_1} \cdots q_{i_p}(\tau \ell(b_i \beta))) (q_{j_1} \cdots q_{j_q}(\tau \ell(b_i \gamma)))$$

$$= q_{i_1} \cdots q_{i_p} q_{j_1} \cdots q_{j_q} \left( \sum_i \tau \ell(b_i \beta) \otimes \tau \ell(b_i \gamma) \right)$$

Since $\tau \ell td(S) = \sum_i b_i \otimes b_i$, formula (b) of lemma 3.2 gives

$$\tau (\ell(td(S) + \gamma) = \sum_i \tau \ell(b_i \beta) \otimes \tau \ell(b_i \gamma)$$

hence:

$$\sum_i (q_{i_1} \cdots q_{i_p}(\tau \ell(b_i \beta))) (q_{j_1} \cdots q_{j_q}(\tau \ell(b_i \gamma)))$$

$$= q_{i_1} \cdots q_{i_p} q_{j_1} \cdots q_{j_q} (\tau (\ell(td(S) + \gamma)),$$

which shows that we have an universal formula.

The case of the formula 3.8 is similar since $ch(O_{\Xi^n})$ and $ch(O_{\Xi^n}^\vee)$ differ only in some signs for some cohomological degrees, so the operators $\mathcal{G}(\gamma)$ et $\mathcal{G}(\gamma)^\vee$ behave similarly for all the results we have used: in the universal formulas, only some signs are different.

**Proposition 3.13.** The Chern classes $c(TS^{[n]})$ enter in a universal generating series of the kind:

$$\sum_{n \geq 0} c(u^{[n]}) = \exp(\mathcal{F}) |0\rangle,$$

where $\mathcal{F} |0\rangle$ is an (infinite) universal linear combination depending on $K_S, e_S$.

**Proof.** The proof is similar to the proof of the proposition 3.12 since if $S_1, S_2$ are two smooth projective surfaces, we have the following decomposition:

$$T(S_1 \amalg S_2)^{[n_1+n_2]} |_{S_1^{[n_1]} \times S_2^{[n_2]}} = p^{*}_1 \left( T S_1^{[n_1]} \right) \oplus p^{*}_2 \left( T S_2^{[n_2]} \right).$$

\[\square\]

\[\text{See also} \ [24, \text{Theorem} \ 4.1].\]
4. Hilbert scheme of points in the affine plane

4.1. Cohomology via symmetric functions. The ring of symmetric functions is the polynomial ring $\Lambda := \mathbb{Q}[p_1, p_2, \cdots]$ in a countably infinite number of indeterminates. This space is given a double grading by letting $p_i$ have conformal weight $i$ and cohomological degree $i - 1$. We denote by $\Lambda^n$ the subspace of polynomials of conformal weight $n$.

The manifold $\text{Hilb}^n(\mathbb{C}^2)$ has no odd cohomology; his even cohomology has no torsion and is generated by algebraic cycles (see [30, 31]). The preceding construction of Heisenberg operators (naturally extended to the quasi-projective setup by use of the Borel-Moore homology, see [30, 31]) induces a natural isomorphism

$$\Lambda^n \cong H^*(\text{Hilb}^n(\mathbb{C}^2)).$$

In the sequel, we shall study Chern classes of vector bundles on $\text{Hilb}^n(\mathbb{C}^2)$. All the formulas will be written in the space $\Lambda^n$, since this space provides a powerful tool for various computations.

A partition of an integer $n$ is a decreasing sequence $\lambda := (\lambda_1, \ldots, \lambda_k)$ of non-negative integers such that $\sum_{i=1}^{k} \lambda_i = n$ (denoted by $\lambda \vdash n$). The $\lambda_i$ are the parts of the partition. The number $l(\lambda)$ of non-zero parts is the length of the partition and the sum $|\lambda|$ of the parts is the weight. If a partition $\lambda$ has $\alpha_1$ parts equal to 1, $\alpha_2$ parts equal to 2, $\ldots$ we shall also denote it by $\lambda := (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and we set $z_\lambda := \prod_{r \geq 1} \alpha_r! r^{\alpha_r}$. A natural basis of $\Lambda^n$ is given by the Newton functions $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k}$ for all partitions $\lambda$ of $n$.

The Young diagram of a partition $\lambda$ is defined by:

$$D(\lambda) := \{(i, j) \in \mathbb{N} \times \mathbb{N} | j < \lambda_{i+1}\}.$$

In the representation of such a diagram, we follow a matrix convention:

```
| x | h | h | h |
|---|---|---|---|
| h |
| h |
| l(\lambda) = 3 |
```

where for each cell $x \in D(\lambda)$, the hook length $h(x)$ at $x$ is the number of cells on the right and below $x$ (including the cell $x$ itself) and we set $h(\lambda) := \prod_{x \in D(\lambda)} h(x)$.

Let $C(S_n)$ be the $\mathbb{Q}$-vector space of class functions on $S_n$. Since conjugacy classes in $S_n$ are indexed by partitions, the functions $\chi_\lambda$ taking the value 1 on the conjugacy class $\lambda$ and 0 else form a basis of $C(S_n)$. Let $R(S_n)$ be the $\mathbb{Q}$-vector space of representations of $S_n$. By associating to each representation of $S_n$ his character, one gets an isomorphism $\chi : R(S_n) \rightarrow C(S_n)$. The Frobenius morphism is the isomorphism $\Phi : C(S_n) \rightarrow \Lambda^n$ characterized by $\Phi(\chi_\lambda) = z_\lambda^{-1} p_\lambda$. Denote by $\chi^\lambda$ the class function such that $\Phi(\chi^\lambda) = z_\lambda$ and by $\chi^\lambda_\mu$ the value of $\chi^\lambda$ at the conjugacy class $\mu$. Then the representations of characters $\chi^\lambda$ are the irreducible representations of $S_n$.

4.2. Generating formulas for the tautological bundle on $\text{Hilb}^n(\mathbb{C}^2)$. The following formulas for the total Chern class and the total Chern character provide basic examples of the results we have proved about the structure of universal formulas.

The generating formula for the total Chern class of the tautological bundle $B_n$ on $\text{Hilb}^n(\mathbb{C}^2)$ is well-known:
Proposition 4.1 (Lehn). ([20, Theorem 4.6]) The total Chern class of the tautological bundle on $\text{Hilb}^n(\mathbb{C}^2)$ is given in $\Lambda$ by the following generating formula:

$$
\sum_{n \geq 0} c(B_n) = \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{p_m}{m} \right).
$$

The generating formula for the total Chern character is an easy consequence of a well-known result:

Proposition 4.2. The total Chern character of the tautological bundle on $\text{Hilb}^n(\mathbb{C}^2)$ is given in $\Lambda$ by the following generating formula:

$$
\sum_{n \geq 0} \text{ch}(B_n) = e^{n!} \sum_{k \geq 1} (-1)^{k-1} \frac{p_k}{k!}
$$

Proof. By Lehn’s theorem (see [21, Theorem 4.1]), the Chern character of $B_n$ is given by:

$$
\text{ch}(B_n) = \mathcal{D} \left( \frac{1}{n!} p_1^n \right),
$$

where the operator $\mathcal{D}$ is defined by:

$$
\mathcal{D} = \left( -\sum_{r \geq 1} p_r t^r \right) \exp \left( -\sum_{r \geq 1} r \frac{\partial}{\partial p_r} t^{-r} \right) \bigg|_{t^0}.
$$

Developing the expression we get:

$$
\mathcal{D} = \left( -\sum_{r \geq 1} p_r t^r \right) \left( 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} \left( \sum_{r \geq 1} r \frac{\partial}{\partial p_r} t^{-r} \right)^k \right) \bigg|_{t^0}
$$

$$
= \left( -\sum_{r \geq 1} p_r t^r \right) \left( 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} \sum_{n_1, \ldots, n_k \geq 1} n_1 \cdot \ldots \cdot n_k \frac{\partial}{\partial p_{n_1}} \cdot \ldots \cdot \frac{\partial}{\partial p_{n_k}} t^{-(n_1 + \ldots + n_k)} \right) \bigg|_{t^0}
$$

$$
= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k!} \sum_{n_1, \ldots, n_k \geq 1} n_1 \cdot \ldots \cdot n_k p_{n_1} \cdot \ldots \cdot p_{n_k}.
$$

Hence:

$$
\mathcal{D}(p_1^n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} p_1^{n-k} p_k,
$$

so:

$$
\text{ch}(B_n) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!(n-k)!} p_1^{n-k} p_k,
$$

hence the formula. \qed

4.3. Chern classes of linearized bundles. The torus $T = \mathbb{C}^*$ acts on $\mathbb{C}[x, y]$ by $s \cdot x = sx, s \cdot y = s^{-1} y$ for $s \in T$. This induces a natural action on $\text{Hilb}^n(\mathbb{C}^2)$ with finitely many fixed points $\xi_\lambda$, parameterized by the partitions $\lambda$ of $n$.

Let $F$ be a $T$-linearized bundle of rank $r$ on $\text{Hilb}^n(\mathbb{C}^2)$. Each fibre $F(\xi_\lambda)$ has a structure of representation of $T$, uniquely determined by its weights $f_{\lambda}^1, \ldots, f_{\lambda}^r$. These data are enough to recover the Chern classes and the Chern characters of $F$:

**Theorem 4.3** (Boissière). ([2, Theorem 4.2]) Let $F$ be a $T$-linearized vector bundle of rank $r$ on $\text{Hilb}^n(\mathbb{C}^2)$ and $f_{\lambda}^1, \ldots, f_{\lambda}^r$ the weights of the action on the fibre at each
fixed point. Then the Chern classes of $F$ written in $\Lambda^n$ are:

$$c_k(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \text{Coeff} \left( t^k \prod_{i=1}^{r} (1 + f_i^k t) \right) \sum_{\mu \vdash n \atop \ell(\mu) = n-k} z_\mu^{-1} \chi_\mu p_\mu.$$  

The Chern characters of $F$ are:

$$\text{ch}_k(F) = \frac{1}{k!} \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sum_{i=1}^{r} (f_i^k)^{n} \sum_{\mu \vdash n \atop \ell(\mu) = n-k} z_\mu^{-1} \chi_\mu p_\mu.$$  

5. Chern classes of the tangent bundle on $\text{Hilb}^n(\mathbb{C}^2)$

We can now prove the two formulas announced in the introduction.

**Theorem 5.1.** The total Chern class of the tangent bundle on $\text{Hilb}^n(\mathbb{C}^2)$ is given in $\Lambda$ by the following generating formula:

$$\sum_{n \geq 0} c(T \text{Hilb}^n(\mathbb{C}^2)) = \exp \left( \sum_{k \geq 0} (-1)^k C_k \frac{p_{2k+1}}{2k+1} \right),$$

where $C_k := \frac{1}{k+1} \binom{2k}{k}$ is the $k$-th Catalan number.

**Proof.** The proof follows the strategy explained in the introduction.

**Step 1.** The manifold $\text{Hilb}^n(\mathbb{C}^2)$ is not projective, so in order to use our general results on universal formulas, we embed it in $\mathbb{P}^{[n]}_2$. By proposition 3.13, the Chern classes of the tangent space on $\mathbb{P}^{[n]}_2$ are given by a generating formula:

$$\sum_{n \geq 0} c(T \mathbb{P}^{[n]}_2) = \exp (\mathfrak{F}) [0],$$

where $\mathfrak{F}$ is a linear combination of elementary operators $q_{n_1} \cdots q_{n_k} (\tau_{k!} \alpha)$ (with $n_i \geq 1$) whose argument depends on $1_{\mathbb{P}^2_2}, K_{\mathbb{P}^2_2}$ and $e_{\mathbb{P}^2_2}$. The inclusion $\mathbb{C}^2 \subset \mathbb{P}^2_2$ induces an open immersion $\text{Hilb}^n(\mathbb{C}^2) \subset \mathbb{P}^{[n]}_2$ giving a surjection $\mathbb{H}^{[n]}_2 \rightarrow \mathbb{H}^{[n]}_2$. Since the restrictions to the affine plane of the classes $K_{\mathbb{P}^2_2}$ and $e_{\mathbb{P}^2_2}$ are trivial and the morphisms $\tau_{k!}$ are zero if $k \geq 2$, the formula for the Chern class of the tangent bundle is more simple and, denoting $q_m := q_m(1_{\mathbb{C}^2})$ and using the identification between $q_m$ and the Newton function $p_m$, we see that the formula is only:

$$\sum_{n \geq 0} c(T \text{Hilb}^n(\mathbb{C}^2)) = \exp \left( \sum_{m \geq 1} f_m p_m \right).$$

**Step 2.** In the cohomology $H^*(\text{Hilb}^n(\mathbb{C}^2))$, the greater non-zero cohomological degree is $n-1$ and is generated (through the identification with $\Lambda^n$) by the Newton function $p_n$, the only Newton function $p_k$ of conformal weight $n$ and cohomological degree $n-1$. So if we develop the exponential in the preceding formula and compare both conformal weights and cohomological degrees, we see that:

$$\sum_{n \geq 0} c_{n-1}(T \text{Hilb}^n(\mathbb{C}^2)) = \sum_{m \geq 1} f_m p_m.$$  

So we only have to compute explicitly these “maximal” Chern classes.

**Step 3.** In order to compute the Chern classes $c_{n-1}(T \text{Hilb}^n(\mathbb{C}^2))$, we use the theorem 4.3. The weights of the fibre of the tangent bundle on $\text{Hilb}^n(\mathbb{C}^2)$ at a fixed
we get a nicer expression for the polynomial $P$, for example \[30\]), hence:

\[
\int P \chi_{\lambda} = 2^n \mu_{\lambda}^{n-1} p_{\mu}.
\]

We deduce that if $n$ is even, then $c_{n-1}(T \text{Hilb}^n(C^2)) = 0$. Suppose now that $n = 2k + 1$ and set $\alpha_k := c_{2k}(T \text{Hilb}^{2k+1}(C^2))$. The only partition $\mu$ of $2k + 1$ of length 1 is $\mu = (2k + 1)$ and the evaluation of a character $\chi^\lambda$ on a maximal cycle follows the following rule (see [11 Exercise 4.16]):

\[
\chi^{\lambda}_{(2k+1)} = \begin{cases} 
(-1)^s & \text{if } \lambda = (2k+1-s, 1, \ldots, 1), \quad 0 \leq s \leq 2k \\
0 & \text{else}
\end{cases}
\]

For such a partition $\lambda = (2k+1-s, 1, \ldots, 1)$, the hook lengths are the integers $\{1, \ldots, s\}, \{1, \ldots, 2k-s\}$ and $2k+1$ so:

\[
\alpha_k = \sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \text{Coeff} \left( t^{2k} \prod_{i=1}^s (1-i^2 t^2) \prod_{j=1}^{2k-s} (1-j^2 t^2) \right) \frac{2k+1}{2k+1}.
\]

Extract from this formula the following polynomial:

\[
P_k := \sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \prod_{i=1}^s (1-i^2 t^2) \prod_{j=1}^{2k-s} (1-j^2 t^2).
\]

Introducing the Pochhammer symbol:

\[
(a)_r := a(a+1) \cdots (a+r-1)
\]

we get a nicer expression for the polynomial $P_k$:

\[
P_k = t^{4k} \sum_{s=0}^{2k} (-1)^s \frac{1 - \frac{1}{s} \left(1 + \frac{1}{s}\right) s \left(1 - \frac{1}{s}\right)_{2k-s} \left(1 + \frac{1}{s}\right)_{2k-s}}{(2k-s)!}.
\]

Observe then the following lemma:

**Lemma 5.2.** For any $a, b$ and any integer $k \geq 0$ we have the following identity:

\[
\sum_{s=0}^{2k} (-1)^s \frac{a(s)(b)_s (a)_{2k-s} (b)_{2k-s}}{s!(2k-s)!} = \frac{1}{k!} \frac{(a+b)_{2k} (a)_{k} (b)_{k}}{(a+b)_k}.
\]

**Proof of the lemma.** Recall the definition of a generalized hypergeometric function:

\[\genfrac{[}]{0pt}{}{a_1, \ldots, a_p; b_1, \ldots, b_q}{z} := \sum_{n=0}^\infty \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}\]

Our identity is then nothing else than the developed form of the following product identity on hypergeometric functions (see [27 §II.2.9, p. 63]):

\[
\genfrac{}{}{0pt}{}{2F_0(a,b; -z)}{2F_0(a,b; z)} = 4 F_1 \left( a, b, \frac{a+b}{2}, \frac{a+b+1}{2}, a+b; 4z^2 \right).
\]

Applying this lemma with $a = 1 - \frac{1}{t}$ and $b = 1 + \frac{1}{t}$, we get for the polynomial $P_k$ the simple expression:

\[
P_k = (-1)^k (2k+1) C_k t^{2k} \prod_{i=1}^k (1-i^2 t^2),
\]
from which follows immediately:

\[ \alpha_k = (-1)^k C_k \frac{P_{2k+1}}{2k+1} \]

hence the theorem.

**Remark 5.3.** The preceding theorem was first conjectured, thanks to the inspired impulsion of Manfred Lehn, by some experiments and numerology. We also thank Marc Nieper-Wißkirchen for his “hypergeometric” help.

6. Chern Character of the Tangent Bundle on \( \text{Hilb}^n(\mathbb{C}^2) \)

**Theorem 6.1.** The Chern characters of the tangent bundle on \( \text{Hilb}^n(\mathbb{C}^2) \) are given in \( \Lambda \) by the following generating formula:

\[
\sum_{n \geq 0} \text{ch}(T \text{Hilb}^n(\mathbb{C}^2)) = 2e^{P_1} \sum_{k \geq 0} \frac{P_{2k+1}}{(2k+1)!}.
\]

**Proof.** The proof is very similar to the preceding proof for the Chern classes, so we only mention the main differences.

**Step 1.** As before, we see that there exists an universal formula \( \exp(q_1(1_s)) \exp(0) \) where \( \exp \) is a linear combination which, in the case of the affine plane, reduces to:

\[
\sum_{n \geq 0} \text{ch}(T \text{Hilb}^n(\mathbb{C}^2)) = e^{P_1} \sum_{m \geq 1} f_{mpn}.
\]

**Step 2.** Comparing both conformal weights and cohomological degrees we see that:

\[
\sum_{n \geq 1} \text{ch}_{n-1}(T \text{Hilb}^n(\mathbb{C}^2)) = \sum_{m \geq 1} f_{mpn}.
\]

**Step 3.** As before, we compute the Chern characters \( \text{ch}_{n-1}(T \text{Hilb}^n(\mathbb{C}^2)) \) with the theorem 4.3.

\[
\text{ch}_{n-1}(T \text{Hilb}^n(\mathbb{C}^2)) = \frac{1}{(n-1)!} \sum_{\lambda+n} \frac{1}{n!} \left( \sum_{x \in D(\lambda)} (1 + (-1)^{n-1}h(x)^{n-1}) \sum_{1 \leq \lambda \leq (n)} \chi_{(\alpha)}^\lambda \right) \chi_{(\alpha)}^{n-1}.
\]

We deduce that \( \text{ch}_{n-1}(T \text{Hilb}^n(\mathbb{C}^2)) = 0 \) for \( n \) even. If \( n = 2k+1 \) is odd, we set \( \beta_k := \text{ch}_{2k}(T \text{Hilb}^{2k+1}(\mathbb{C}^2)) \) and get:

\[
\beta_k = 2 \sum_{s=0}^{2k} \frac{(-1)^s}{(2k+1)s!(2k-s)!} \left( \sum_{i=1}^s i^{2k} + \sum_{j=1}^{2k-s} j^{2k} + (2k+1)^{2k} \right) \frac{P_{2k+1}}{(2k+1)!}.
\]

One verifies that:

\[
\sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} = 0,
\]

\[
\sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \sum_{i=1}^s i^{2k} = \frac{1}{2k \cdot (2k)!} \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} i^{2k+1},
\]

\[
\sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \sum_{j=1}^{2k-s} j^{2k} = \frac{1}{2k \cdot (2k)!} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} j^{2k+1},
\]

by use of the elementary formulas \( \sum_{k=0}^{p} (-1)^k \binom{n}{k} = (-1)^p \binom{n-1}{p} \) and \( \binom{n-1}{p-1} = \frac{2}{n} \binom{n}{p} \).

Observe then the following lemma:
Lemma 6.2. For any \( n \geq 0 \), the following identity holds:
\[
\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^{n+1} = \binom{n+1}{2} n!
\]

Proof of the lemma. The Stirling number of the second kind \( S(n, k) \) is the number of partitions in \( k \) blocks of a set of \( n \) elements, and is given by the following formula (see [2]):
\[
S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} (k - j)^n.
\]
In particular:
\[
S(n+1, n) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^{n+1},
\]
and we observe that \( S(n+1, n) = \binom{n+1}{2} \) since putting \( n+1 \) identical objects in \( n \) boxes is the same as choosing 2 objects among \( n+1 \). □

Applying this lemma we get:
\[
\sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \sum_{i=1}^{s} i^{2k} = \frac{2k+1}{2},
\]
\[
\sum_{s=0}^{2k} \frac{(-1)^s}{s!(2k-s)!} \sum_{j=1}^{2k-s} j^{2k-s} = \frac{2k+1}{2},
\]
from which follows immediately:
\[
\beta_k = 2 \frac{p2k+1}{(2k+1)!},
\]
hence the theorem. □

Comparing the generating formulas for the Chern character of \( T_{\text{Hilb}}(\mathbb{C}^2) \) (theorem 6.1) and of \( B_n \) (theorem 4.2), one remarks the following decomposition:

Corollary 6.3. The following identity holds in \( K(\text{Hilb}^n(\mathbb{C}^2)) \):
\[
T_{\text{Hilb}}(\mathbb{C}^2) = B_n + B^*_n.
\]

Remark 6.4. This formula appears as a particular case of the general decomposition formula of Ellingsrud-Göttsche-Lehn for the tangent bundle over \( S^n \) (see the proof of [8, Proposition 2.2]):
\[
T_n = B_n + B^*_n - p_2(O_n^* \cdot O_n).
\]

References
1. D. Ben-Zvi and E. Frenkel, Vertex algebras and algebraic curves, AMS, 2001.
2. S. Boissière, On the McKay correspondences for the Hilbert scheme of points on the affine plane, arXiv:math.AG/0410281.
3. , Sur les correspondances de McKay pour le schéma de Hilbert de points sur le plan affine, Ph.D. thesis, Université de Nantes, 2004.
4. A. Borel and J.-P. Serre, Le théorème de Riemann-Roch, Bull. Soc. Math. Fr. 86 (1958), 97–136.
5. Louis Comtet, Analyse combinatoire, PUF, 1970.
6. Gentiana Danila, Sur la cohomologie d’un fibré tautologique sur le schéma de Hilbert d’une surface, Journal of Algebraic Geometry 10 (2001), 247–280, arXiv:math.AG/9904004.
7. G. Ellingsrud and L. Göttsche, Hilbert schemes of points and Heisenberg algebras, Moduli spaces in Algebraic Geometry, 1999.
8. G. Ellingsrud and L. Göttsche, and M. Lehn, On the cobordism class of the Hilbert scheme of a surface, J. Algebraic Geometry 10 (2001), 81–100, arXiv:math.AG/9904095.
9. G. Ellingsrud and S. A. Strømme, On the homology of the Hilbert scheme of points in the plane, Invent. Math. 87 (1987), 343–372.
10. J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. 10 (1968), 511–521.
11. W. Fulton and J. Harris, *Representation theory*, Springer, 1991.
12. L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), 193–207.
13. I. Grojnowski, *Instantons and affine algebras i: the Hilbert scheme and vertex operators*, Math. Res. Lett. 3 (1996), 275–291.
14. A. Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique, IV : les schémas de Hilbert*, Séminaire Bourbaki 221 (1960-1961).
15. M. Haiman, *t, q-Catalan numbers and the Hilbert scheme*, Discrete Math. 193 (1998), 201–224.
16. R. Hartshorne, *Algebraic geometry*, Springer, 1977.
17. F. Hirzebruch, *Topological methods in Algebraic Geometry*, Springer, 1966.
18. V. Kac, *Vertex algebras for beginners*, AMS, 1997.
19. M. Lehn, private communication.
20. ______, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. 136 (1999), 157–207.
21. M. Lehn and C. Sorger, *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, Duke Math. J. 110 (2001), 345–357.
22. W.-P. Li, Z. Qin, and W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, Math. Ann. 324 (2002), 105–133, arXiv:math.AG/0009132.
23. _______, *Stability of the cohomology rings of Hilbert schemes of points on surfaces*, J. reine angew. Math. 554 (2003), 217–234, arXiv:math.AG/0107139.
24. W.-Ping Li, Z. Qin, and W. Wang, *Generators for the cohomology ring of Hilbert schemes of points on surfaces*, Intern. Math. Res. Notices 20 (2001), 1057–1074, arXiv:math.AG/0009167.
25. I. G. Macdonald, *Symmetric functions and hall polynomials*, Oxford University Press (2nd edition 1995), 1979.
26. ______, *Symmetric functions and orthogonal polynomials*, AMS, 1991.
27. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions in mathematical physics*, third edition, Springer, 1966.
28. L. Manivel, *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, SMF, 1998.
29. H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces*, 1996, arXiv:math.AG/9610021.
30. _______, *Lectures on Hilbert schemes of points on surfaces*, AMS, 1996.
31. _______, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Annals of math. 145 (1997), 379–388.
32. E. Vasserot, *Sur l’anneau de cohomologie du schéma de Hilbert de C²*, C.-R. Acad. Sc. Paris 332 (2001), 7–12.

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