Existence of Metrics with Prescribed Scalar Curvature on the Volume Element Preserving Deformation

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Abstract

In this paper, we obtain two results on closed Riemannian manifold $M \times [0, T]$. When $T$ is small enough, to any prescribed scalar curvature, the existence and uniqueness of metrics are obtained on the volume element preserving deformation. When $T$ is large and the given scalar curvature is small enough, the same result holds.

1 Introduction

This article concerns the following question: Suppose $(M, g), (N, h)$ are $m$-dimension and $n$-dimension Riemannian manifolds, on which $g, h$ are corresponded metrics. Then $g + h$ is a metric on $M \times N$. (Here, $(g + h)(X_P, Y_P) = g((X_1)_P, (Y_1)_P) + h((X_2)_P, (Y_2)_P), X_P = (X_1)_P + (X_2)_P, Y_P = (Y_1)_P + (Y_2)_P, X_P, Y_P$ are tangent vectors on $M \times N$ at point $P$. They can be decomposed into $(X_1)_P, (Y_1)_P$ and $(X_2)_P, (Y_2)_P$, which are tangent vectors on $M$ and $N$.) The author considers a class of metric deformation: in fact, one can construct a metric $K$ on $M \times N$, such that $K = \rho^n g + \rho^{-m} h$, here $\rho$ is a smooth enough positive function defined on $M \times N$. Obvious, $det(K) = det(\rho^n g)det(\rho^{-m} h) = det(g)det(h) = det(g + h)$, then the volume forms of $g + h$ and $K$ are same at every point. So we call this deformation as the volume element preserving deformation. And the question is that with what

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condition, a smooth enough function $\tilde{R}$ defined on $M \times N$ can be viewed as the scalar curvature of this kind of metrics $K$.

This question can induce a partial differential equation namely,

$$\tilde{R} = \rho^{-n}R_g + \rho^nR_h + \rho^{-n}\triangle_g\ln\rho^n + \rho^n\triangle_h\ln\rho^{-m} - \frac{n}{4}(nm + 2n + m^2)\rho^{-n}\vert\nabla_g\ln\rho\vert^2_g$$

$$- \frac{m}{4}(nm + 2m + n^2)\rho^m\vert\nabla_h\ln\rho\vert^2_h \quad (1.1)$$

It is too complicated, the author only study this equation when $n = 1$. Do a transformation, this equation becomes,

$$\tilde{R} = e^{-2u}R_g + 2e^{-2u}\triangle u - 2me^{2mu}\frac{\partial^2 u}{\partial t^2} - (m^2 + m + 2)e^{-2u}\vert\nabla u\vert^2 - m(3m + 1)e^{2mu}u_t^2 \quad (1.2)$$

It is a degenerate quasi-linear hyperbolic equation.

This article is composed by 5 sections. In section 2, the author introduces the notations of some spaces and corresponded norms, refers some Lemmas which are useful to the rest part of this paper. Section 3 gives the details to obtain equation (1.1), (1.2) from the above proposed question. In section 4, using the energy estimate, the prior estimate of equation (1.2) obtained on $M \times I$, here $I = [0, T]$ is a interval of length $T$. Section 5 studies the linear equation of the following form:

$$\frac{\partial^2 u}{\partial t^2} + a(x, t)\frac{\partial u}{\partial t} - (a(x, t)\triangle u + <\nabla \beta(x, t),\nabla u > + \gamma(x, t)u) = f(x, t) \quad (1.3)$$

Using the estimate of section 4 and Banach contraction principle, the last section gives two results.

In a word, the paper mainly obtains the following two theorems.

**Theorem 1.1** Suppose $(M, g)$ be $m$-dimensional closed Riemannian manifold, $I = [0, T]$ be a interval of length $T$ with standard Euclidean metric $h$, and $\tilde{R} \in C^\infty(M \times I)$. To any integer $k > \frac{m}{2} + 3$, exist $0 < t_0 \leq T$, then exist a unique volume element preserving deformation $e^{2u}g + e^{-2mu}h$ such that the scalar curvature of this metric is $\tilde{R}$ on $M \times [0, t_0]$. Here, $u \in C^k(M \times [0, t_0])$, and satisfies $u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x)$ and $\varphi(x), \psi(x) \in C^\infty(M)$ are two prescribed functions. And $t_0$ relates to manifold $M, k, \tilde{R}, \varphi, \psi$. 

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Theorem 1.2 Suppose \((M, g)\) be \(m\)-dimensional closed Riemannian manifold which is scalar curvature flat \(R_g = 0\). And \(I = [0, T]\) be a interval of length \(T\) with standard Euclidean metric \(h\), and \(\bar{R} \in C^{\infty}(M \times I)\). To any integer \(k > \frac{m}{2} + 3\), exist \(\varepsilon > 0\), and when \(\|\bar{R}\|_{H^{\frac{m}{2}+k}(M)} \leq \varepsilon\) then exist a unique volume element preserving deformation such that the scalar curvature of the metric \(e^{2u}g + e^{-2mu}h\) is \(\bar{R}\) on \(M \times I\). Here, \(u \in C^k(M \times I)\), and \(u\) satisfies \(u(x, 0) = 0, u_t(x, 0) = 0\), and \(\varepsilon\) only relates to manifold \(M, k, I\).

Remark 1.3 Here, closed manifold means compact oriented and without boundary manifold.

At last, the author should thank to his supervisor Professor Huang Xuananguo. This question is proposed originally by him. And he induced the correspond equation in some special cases. The section 3 is the generalize of his computation. He carefully checks my paper, and gives me much valuable suggestion. This paper can not be completed without his continual encouragement. And the author also wants to thank to his classmates: Du Yi, Hu Junqi, Yang Yongfu. They give me a lot of help during the paper finished.

2 Preliminary

Suppose \((M, g)\) be a closed \(m\)-dimensional Riemannian manifold. \(\{x_i\}_{i=1}^m\) is its local coordinate. \((g^{ij})\) is the inverse matrix of \((g_{ij})\). Let \(u\) be a smooth enough function on \(M\). Using \(\nabla_{i_1} \cdots \nabla_{i_s} u\) as \(u\)’s the \(s\)-order covariant derivation, here \(\nabla\) the Levi-Civita connection to \(g\). We use the following notations:

\[
|\nabla^s u|_g^2 = \sum_{i_1, \cdots, i_s; j_1, \cdots, j_s=1}^m (g^{i_1j_1} \cdots g^{i_sj_s})(\nabla_{i_1} \cdots \nabla_{i_s} u)(\nabla_{j_1} \cdots \nabla_{j_s} u);
\]

\[
\|u\|_{L^p(M)} = \left( \int_M |u|^p dV_g \right)^{1/p}; \|\nabla^s u\|_{L^p(M)} = \left( \int_M |\nabla^s u|_g^p dV_g \right)^{1/p};
\]

\[
\|u\|_{W^{s,p}(M)} = \left( \sum_{j=0}^s \|\nabla^j u\|_{L^p(M)}^p \right)^{1/p}; \|u\|_{C^0(M)} = \max_M |u|;
\]

\[
\|\nabla^k u\|_{C^0(M)} = \max_M |\nabla^k u|_g; \|u\|_{C^k(M)} = \left( \sum_{j=0}^k \|\nabla^j u\|_{C^0(M)}^2 \right)^{1/2} \quad (2.1)
\]
here, \(1 \leq p, s, k < \infty\). Denote \(W^{s,p}(M), C^k(M)\) as the Banach space with the corresponded norms. Especially, denote \(H^s(M)\) as \(W^{s,2}(M)\).

Let \(I = [0, T]\) be a interval of length \(T\), with the metric \(h\) and the arch length coordinate \(\{t\}\). Then on \(M \times I\) one can choose a local coordinate \(\{x_i\}_{i=1}^{m+1}\) where \(x_{m+1} = t\). Then direct computation shows \(\Gamma^C_{AB} = 0, R^D_{ABC} = 0\), if some of \(A, B, C, D\) \(\in \{1, \cdots , m+1\}\) are \(m+1\). Here, \(\Gamma, R\) are the christoffel symbol and curvature operator of product metric \(\bar{g} := g + h\). Using this result and Ricci identity, one can finds, to any smooth enough \((r, s)\) tensor field \(A\) on \(M \times I\) which has the form in local \(A = \sum_{i_1, \cdots , i_k; j_1, \cdots , j_s} A^{i_1, \cdots , i_k | j_1, \cdots , j_s}(x, t) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}\), \(\nabla_i A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s} = \nabla_i A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s}, \nabla \frac{\partial}{\partial t} A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s} = \frac{\partial}{\partial t} A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s}, i \in \{1, \cdots , m\}\), and

\[
\frac{\partial}{\partial t} \nabla^l A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s} = \nabla^l \frac{\partial}{\partial t} A^{i_1, \cdots , i_k}_{j_1, \cdots , j_s}\tag{2.2}
\]

Here, \(\nabla\) is the corresponded connection of \(\bar{g}\). Then we introduce the following norms:

\[
\|\partial^k_t u\|_{L^p([0, t], W^{s,q}(M))} = \|\nabla^k_{\partial_t} u\|_{L^p([0, t], W^{s,q}(M))} = \|\left\|\frac{\partial^k_t u}{\partial t}\right\|_{W^{s,q}(M)}\|_{L^p([0, t])}
\]

\[
\|u\|_{W^{k,p}(0, t), W^{s,q}(M))} = \left(\sum_{i=0}^{k} \|\partial^i_t u\|_{L^p([0, t], W^{s,q}(M))}^p\right)^{1/p}
\]

\[
\|u\|_{C^k([0, t], C^s(M))} = \left(\sum_{i=0}^{k} \|\partial^i_t u\|^2_{C^s(M)}\right)^{1/2}
\]

(2.3)

and the space \(L^p([0, t], W^{s,q}(M))\) is the completion of the \(C^\infty(M \times [0, t])\) with this norm, \(0 \leq k, s < \infty, 1 \leq p \leq \infty, 1 \leq q < \infty\), here \(u\) is a smooth enough function on \(M \times I\) and \(0 < t < T\). It is weakly star compact.

The following three Lemmas are classical in partial differential equation, we refer [Hörmander] p106-108 [Nirenberg] and [Zheng] p10-11, p186-187 for the detail proof.

**Lemma 2.1 (Gagliardo-Nirenberg Inequality)** Let \(j, n \in \mathbb{Z}\) and \(0 \leq j < n\). Let \(1 \leq q, r \leq +\infty, p \in \mathbb{R}, \frac{1}{p} - \frac{1}{r} = a\) \(\leq 1\), such that \(\frac{1}{p} - \frac{1}{r} = a(\frac{1}{r} - \frac{1}{m}) + (1-a)\frac{1}{q}\). For any \(u \in W^{n,r}(\mathbb{R}^m) \cap L^q(\mathbb{R}^m)\), there is a positive constant \(C\) depending on \(n, m, j, q, r, a\) such that

\[
\|\nabla^j u\|_{L^p(\mathbb{R}^m)} \leq C\|\nabla^n u\|_{L^r(\mathbb{R}^m)}\|u\|^{1-a}_{L^q(\mathbb{R}^m)}
\]

(2.4)

with a exception: if \(1 < r < \infty, n - j - \frac{m}{r}\) is a nonnegative integer then the inequality hold only \(\frac{j}{n} \leq a < 1\).
Lemma 2.2  suppose \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, 1 \leq r, p, q \leq \infty \), and suppose that all norms appearing what follows are bounded. Then for any integer \( s \geq 0 \) one has

\[
\| \nabla^s (fg) \|_{L^r(\mathbb{R}^m)} \leq C(\| f \|_{L^p(\mathbb{R}^m)} \| \nabla^s g \|_{L^q(\mathbb{R}^m)} + \| \nabla^s f \|_{L^q(\mathbb{R}^m)} \| g \|_{L^p(\mathbb{R}^m)})
\]

(2.5)

here \( C \) is not depend on \( f, g \).

Lemma 2.3  Suppose \( F : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) is a smooth function satisfies \( F(0) = 0 \). For any integer \( s \geq 0 \), if for \( w \in W^{s,p}(\mathbb{R}^m), 1 \leq p \leq \infty \) and \( \| w \|_{L^\infty(\mathbb{R}^m)} \leq \nu_0 \), then \( F(w) \in W^{s,p}(\mathbb{R}^m) \) and

\[
\| F(w) \|_{W^{s,p}(\mathbb{R}^m)} \leq C \| F \|_{C_s([-\nu_0, \nu_0])} (1 + \| w \|_{L^\infty(\mathbb{R}^m)}) \| w \|_{W^{s,p}(\mathbb{R}^m)}
\]

(2.6)

here, \( C \) is not depend on \( F, w \).

Corollary 2.4  On a closed Riemannian manifold \( (M, g) \), replace \( \mathbb{R}^m \) by \( M \) in the hypothesis of Lemma 2.1-2.3, then the corresponded result becomes:

(1) \( \| \nabla^j u \|_{L^p(M)} \leq C \| u \|_{W^{n,r}(M)} \| u \|_{L^q(M)}^{1-a} \)

(2) \( \| \nabla^s (fg) \|_{L^r(M)} \leq C(\| f \|_{L^p(M)} \| g \|_{W^{s,q}(M)} + \| f \|_{W^{s,q}(M)} \| g \|_{L^p(M)}) \)

\( \| \nabla^s \phi \|_{L^r(M)} \leq C(\| f \|_{W^{1,p}(M)} \| g \|_{W^{s+1,q}(M)}) + \| \nabla \|_{W^{s+1,q}(M)} \| g \|_{W^{1,p}(M)}) \)

(3) \( \| F(w) \|_{W^{s,p}(M)} \leq C \| F \|_{C_s([-\nu_0, \nu_0])} (1 + \| w \|_{C_0(M)}) \| w \|_{W^{s,p}(M)} \)

(2.7)

here \( C \) also relates manifold \( M \).

Convention 2.5  Here, \( W^{k,\infty}(M) \) which appears in the above understood as \( C^k(M) \), namely \( \| \cdot \|_{W^{k,\infty}(M)} = \| \cdot \|_{C^k(M)} \).

Proof of Corollary 2.4  The proof of (1)(2) are similar, we only prove (1). Since \( M \) is compact, \( M \) has finite partition of unit, namely \( \{(U_i, \varphi_i)\}_{i=1}^N \), \( \sum_{i=1}^N \varphi_i = 1, \varphi \geq 0, \varphi_i \in C^\infty(M), \text{supp} \varphi_i \subset U_i \). Since

\[
\| \varphi_i u \|_{W^{n,r}(U_i)} \leq C(\sum_{l=0}^n \sum_{k=0}^l \| \nabla^k u \|_{L^r(U_i)}^r)^{1/r} \leq C(\sum_{k=0}^n \| \nabla^k u \|_{L^r(U_i)}^r)^{1/r} \leq C \| u \|_{W^{n,r}(M)}
\]

(2.8)
by Lemma 2.1 and the norm equivalence on $U_i$, namely \( \| \cdot \|_{W^{s,p}(U_i)} \) with the metric $g$ equivalence to \( \| \cdot \|_{W^{s,p}(U_i)} \) with the standard Euclidean metric. One has

$$
\| \nabla^j (\varphi_i u) \|_{L^p(U_i)} \leq C \| \varphi_i u \|_{W^{n,r}(U_i)}^a \| \varphi_i u \|_{L^q(U_i)}^{1-a}
$$

(2.9)

dtherefore, using (2.8)(2.9),

$$
\| \nabla^j u \|_{L^p(M)} \leq \sum_{i=1}^N \| \nabla^j (\varphi_i u) \|_{L^p(U_i)}
$$

$$
\leq C \sum_{i=1}^N \| u \|_{W^{n,r}(M)}^a \| u \|_{L^q(M)}^{1-a}
$$

$$
\leq C \| u \|_{W^{n,r}(M)}^a \| u \|_{L^q(M)}^{1-a}
$$

(2.10)

This gives the result. To (3) the proof is similar to Lemma 2.3 since (1) is right.

**Remark 2.6** If the hypothesis of Corollary 2.4(1) becomes $1 \leq j \leq n$, $1 \leq q,r \leq \infty$, $p \in \mathbb{R}$, $\frac{j}{n-1} \leq a \leq 1$, $\frac{1}{p} - \frac{j}{n} = a(\frac{1}{r} - \frac{a}{m}) + (1-a)\frac{1}{q}$, then (2.7)(1) becomes,

$$
\| \nabla^j u \|_{L^p(M)} \leq C \| u \|_{W^{n,r}(M)}^a \| u \|_{W^{n,r}(M)}^{1-a}
$$

(2.11)

and with the similar exception: if $1 < r < \infty$, $n - j - \frac{m}{r}$ is a nonnegative integer then the inequality hold only $\frac{j}{n-1} \leq a < 1$.

At last, we refer the following Lemma which is need in section 5. The detail proof can be found in [Zheng] chapter 3 p103.

**Lemma 2.7** Let $B, B^*$ be both Banach space, and $B$ is the dual of $B^*$. Suppose $1 < p \leq \infty$ and $u_n \rightharpoonup u$ weakly star in $L^p([0,t], B), u'_n \rightharpoonup u'$ weakly star in $L^p([0,t], B)$, then $u_n(0) \rightharpoonup u(0)$ weakly star in $B$. Here, $u'_n, u'$ are the derivation in the meaning of distribution (or current).

### 3 Induced Equation

The hypothesis of $(M, g), (N, h), \rho$ as in section 1, $\Sigma = M \times N$, then $g + h$ is the product metric on $\Sigma$. Then one can construct a new metric $K = \rho^\alpha g +$
\[ g_{ij}(P) = \delta_{ij}; \Gamma^k_{ij}(P) = 0; h_{\alpha\beta}(P) = \delta_{\alpha\beta}; \Gamma^\gamma_{\alpha\beta}(P) = 0 \] (3.1)

with this coordinate,

\[ K_{ij} = \rho^n g_{ij}; K_{\alpha\beta} = \rho^{-m} h_{\alpha\beta}; K_{i\alpha} = 0 \] (3.2)

Denote \((g^{ij}), (h^{\alpha\beta}), (K^{AB})\) the inverse matrix of \((g_{ij}), (h_{\alpha\beta}), (K_{AB})\) respectively. And denote \(\tilde{\Gamma}^A_{BC}, \tilde{\Gamma}^k_{ij}, \tilde{\Gamma}^\gamma_{\alpha\beta}\) the christoffel symbols to \(K, g, h\) respectively. At point \(P\),

\[
\tilde{\Gamma}^A_{ABC} = \frac{1}{2} \left( \frac{\partial K^A_{BC}}{\partial x^B} + \frac{\partial K^A_{BC}}{\partial x^C} - \frac{\partial K^A_{BC}}{\partial x^A} \right) \\
\tilde{\Gamma}^k_{ijk} = \frac{1}{2} \left( \frac{\partial K^k_{ij}}{\partial x^j} + \frac{\partial K^k_{ij}}{\partial x^i} - \frac{\partial K^k_{ij}}{\partial x^k} \right) \\
= \rho^n \Gamma^k_{ijk} + \frac{1}{2} \left( \partial \ln \rho^n \right) g_{ij} \left( \frac{\partial}{\partial x^j} g_{ik} + \frac{\partial}{\partial x^i} g_{jk} - \frac{\partial}{\partial x^k} g_{ij} \right) \\
= \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} g_{ij} \right) + \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} \right) \left( \frac{\partial}{\partial y^\beta} \right) g_{ij} \\
= \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} \right) \left( \frac{\partial}{\partial y^\beta} \right) g_{ij} \\
\tilde{\Gamma}^\alpha_{ij} = \sum_{\beta=m+1}^{m+n} K^{\alpha\beta} \tilde{\Gamma}_{ij\beta} = \rho^n \sum_{\beta=m+1}^{m+n} h^{\alpha\beta} \left( -\frac{1}{2} \frac{\partial}{\partial y^\beta} \right) g_{ij} \\
\tilde{\Gamma}^k_{\alpha j} = \frac{1}{2} \left( \frac{\partial K^k_{\alpha j}}{\partial y^\alpha} \right) = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} \right) g_{jk} \\
\tilde{\Gamma}^k_{\alpha j} = \sum_{l=1}^{m} K^{kl} \tilde{\Gamma}_{\alpha jl} = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} \right) \delta_{jl} \\
\tilde{\Gamma}^k_{ij} = \sum_{l=1}^{m} K^{kl} \tilde{\Gamma}_{ijl} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \right) \delta_{kj} \\n\tilde{\Gamma}^k_{i\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \right) \left( \frac{\partial}{\partial y^\alpha} \right) = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \right) h_{\beta\gamma} \\
= \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} \right) \left( \frac{\partial}{\partial y^\beta} \right) h_{\beta\gamma} \\
(3.3) \quad (3.4) \quad (3.5) \quad (3.6) \quad (3.7) \quad (3.8) \quad (3.9) \quad (3.10)
\[ \tilde{\Gamma}^\gamma_{i\beta} = \sum_{\delta=m+1}^{m+n} K^{\gamma\delta} \tilde{\Gamma}^\delta_{i\beta} = \frac{1}{2} \frac{\partial \ln \rho^{-m}}{\partial x_i} \delta_{\gamma\beta} \] (3.11)

\[ \tilde{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} \left( \frac{\partial \ln \rho^{-m}}{\partial y^\beta} \delta_{\alpha\gamma} + \frac{\partial \ln \rho^{-m}}{\partial y^\alpha} \delta_{\beta\gamma} - \sum_{\delta=m+1}^{m+n} \frac{\partial \ln \rho^{-m}}{\partial y^\delta} h_{\alpha\beta} h^{\gamma\delta} \right) \] (3.12)

\[ \tilde{\Gamma}^k_{\alpha\beta} = \frac{1}{2} \left( - \frac{\partial K_{\alpha\beta}}{\partial x^k} \right) = \frac{1}{2} \frac{\partial \rho^{-m}}{\partial x^k} h_{\alpha\beta} \] (3.13)

\[ \tilde{\Gamma}^k_{\alpha\beta} = \sum_{l=1}^m K^{kl} \Gamma_{\alpha\beta l} = \rho^{-m} \sum_{l=1}^m g^{kl} \left( - \frac{1}{2} \frac{\partial \rho^{-m}}{\partial x^l} \right) h_{\alpha\beta} \] (3.14)

Using (3.5) and (3.11), at point \( P \),

\[ \sum_{A=1}^{m+n} \tilde{\Gamma}^A_{iA} = \sum_{k=1}^m \tilde{\Gamma}^k_{ik} + \sum_{\beta=m+1}^{m+n} \tilde{\Gamma}^\beta_{i\beta} = \sum_{k=1}^m \Gamma^k_{ik} = 0 \] (3.15)

Similar, by (3.9) and (3.12), at point \( P \),

\[ \sum_{A=1}^{m+1} \tilde{\Gamma}^A_{\alpha A} = \sum_{\beta=m+1}^{m+n} \Gamma^\beta_{\alpha\beta} = 0 \] (3.16)

In what follows, compute the Ricci curvature of point \( P \). Denote \( \tilde{R}_{AB}, R_{ij}, R_{\alpha\beta} \) the Ricci curvature of \( K, g, h \),

\[ \tilde{R}_{AB} = \sum_{C=1}^{m+n} \frac{\partial \tilde{\Gamma}^C_{AB}}{\partial x^C} - \sum_{C=1}^{m+n} \frac{\partial \tilde{\Gamma}^C_{AC}}{\partial x^B} + \sum_{C,D=1}^{m+n} \tilde{\Gamma}^D_{AB} \tilde{\Gamma}^C_{DC} - \sum_{C,D=1}^{m+n} \tilde{\Gamma}^D_{AC} \tilde{\Gamma}^C_{BD} \] (3.17)

First, compute \( \tilde{R}_{ij} \), by (3.5), (3.7), at point \( P \),

\[ \sum_{C=1}^{m+n} \frac{\partial \tilde{\Gamma}^C_{ij}}{\partial x^C} = \sum_{k=1}^m \frac{\partial \tilde{\Gamma}^k_{ij}}{\partial x^k} + \sum_{\alpha=m+1}^{m+n} \frac{\partial \tilde{\Gamma}^\alpha_{ij}}{\partial y^\alpha} \]
\[
\sum_{k=1}^{m} \frac{\partial^2 \ln \rho^n}{\partial x_i \partial x_j} = \sum_{k=1}^{m} \frac{\partial^2 \ln \rho^n}{\partial x_i \partial x_j} - \frac{1}{2} g_{ij} \Delta \ln \rho^n
\]

\[
+ \sum_{\alpha, \gamma = m+1}^{m+n} \left( - \frac{1}{2} h^{\alpha \gamma} g_{ij} \right) \left( \frac{\partial \rho^{m+n}}{\partial y_\alpha} \frac{\partial \rho^{m+n}}{\partial y_\gamma} + \rho^{m+n} \frac{\partial^2 \ln \rho^n}{\partial y_\alpha \partial y_\gamma} \right)
\]

\[
= \sum_{k=1}^{m} \frac{\partial \Gamma^k_{ij}}{\partial x_k} + \frac{\partial^2 \ln \rho^n}{\partial x_i \partial x_j} - \frac{1}{2} g_{ij} \Delta \ln \rho^n - \frac{1}{2} g_{ij} \rho^{m+n} \Delta \ln \rho^n
\]

\[
- \frac{1}{2} g_{ij} < \nabla_h \rho^{m+n}, \nabla_h \ln \rho^n >_h
\]

(3.18)

Here, denote \( \Delta_g, \Delta_h \) the Beltrami-Laplace operators to \((M, g), (N, h)\) respectively. Use \( \nabla_g, \nabla_h \) as the covariant derivation to \( M, N \), and \( < \cdot, \cdot >_g, < \cdot, \cdot >_h \) as the metric \( g, h \). And using (3.15),

\[
\sum_{C=1}^{m+n} \frac{\partial \Gamma^C_{iC}}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial \Gamma^k_{ik}}{\partial x_j}
\]

(3.19)

By (3.15) and (3.16), at point \( P \),

\[
\sum_{C,D=1}^{m+n} \tilde{\Gamma}^D_{ij} \tilde{\Gamma}^C_{DC} = \sum_{k=1}^{m} \sum_{C=1}^{m+n} \tilde{\Gamma}^k_{ij} \tilde{\Gamma}^C_{kC} + \sum_{\alpha=m+1}^{m+n} \sum_{C=1}^{m+n} \tilde{\Gamma}^\alpha_{ij} \tilde{\Gamma}^C_{\alpha C} = 0
\]

(3.20)

\[
\sum_{C,D=1}^{m+n} \tilde{\Gamma}^D_{iC} \tilde{\Gamma}^C_{jD} = \sum_{k,l=1}^{m} \sum_{l=1}^{m+n} \tilde{\Gamma}^k_{iL} \tilde{\Gamma}^l_{jK} + \sum_{l=1}^{m+n} \sum_{\alpha=m+1}^{m+n} \tilde{\Gamma}^\alpha_{iL} \tilde{\Gamma}^\alpha_{jK}
\]

\[
+ \sum_{\alpha,m+1}^{m+n} \tilde{\Gamma}^\beta_{i \alpha} \tilde{\Gamma}^\alpha_{j \beta}
\]

(3.21)

Using (3.5), at point \( P \) one has,

\[
\sum_{k,l=1}^{m} \tilde{\Gamma}^k_{il} \tilde{\Gamma}^l_{jk} = \sum_{k,l=1}^{m} \left( \frac{\partial \ln \rho^n}{\partial x_i} \delta_{kj} + \frac{\partial \ln \rho^n}{\partial x_j} \delta_{ik} - \sum_{s=1}^{m} \frac{\partial \ln \rho^n}{\partial x_s} \delta_{sk} \delta_{il} \right) \times
\]

\[
\frac{1}{2} \left( \frac{\partial \ln \rho^n}{\partial x_k} \delta_{ij} + \frac{\partial \ln \rho^n}{\partial x_l} \delta_{ij} - \sum_{t=1}^{m} \frac{\partial \ln \rho^n}{\partial x_t} \delta_{it} \delta_{jk} \right)
\]
\[
\frac{1}{4} \sum_{k,l=1}^{m} \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{kl} \delta_{kl} + \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{kl} - \frac{\partial \ln \rho^{n}}{\partial x_{k}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{kl} \\
+ \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jl} + \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jl} - \frac{\partial \ln \rho^{n}}{\partial x_{k}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jl} \\
- \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jk} + \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jk} + \frac{\partial \ln \rho^{n}}{\partial x_{k}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} \delta_{ik} \delta_{jk}
\]

= \frac{1}{4} \left[ (m + 2) \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} - 2 |\nabla_{g} \ln \rho^{n}|_{g}^{2} \delta_{ij} \right] \quad (3.22)

Here, to a smooth enough function \(u\), \(|\nabla_{g} u|^{2} = \sum_{i,j=1}^{m} g^{ij} \nabla_{i} u \nabla_{j} u\). Using (3.7) and (3.9), at point \(P\),

\[
\sum_{l=1}^{m} \sum_{\alpha=m+1}^{m+n} \tilde{\Gamma}_{i l}^{\alpha} \tilde{\Gamma}_{j \alpha} = \sum_{l=1}^{m} \sum_{\alpha, \beta=m+1}^{m+n} \rho^{m} h^{\alpha \beta} (-\frac{1}{2}) \frac{\partial \rho^{n}}{\partial y_{\beta}} g^{il} \frac{1}{2} \frac{\partial \ln \rho^{n}}{\partial y_{\alpha}}
\]

= \(-\frac{1}{4} \rho^{m+n} \delta_{ij} |\nabla_{h} \ln \rho^{n}|_{h}^{2} \) \quad (3.23)

Here, \(|\nabla_{h} u|^{2} = \sum_{\alpha, \beta=m+1}^{m+n} h^{\alpha \beta} \nabla_{\alpha} u \nabla_{\beta} u\), so, at point \(P\),

\[
\sum_{k=1}^{m} \sum_{\alpha=m+1}^{m+n} \tilde{\Gamma}_{i k}^{\alpha} \tilde{\Gamma}_{j \alpha} = -\frac{1}{4} \rho^{m+n} \delta_{ij} |\nabla_{h} \ln \rho^{n}|_{h}^{2} \quad (3.24)
\]

By (3.11), at point \(P\),

\[
\sum_{\alpha, \beta=m+1}^{m+n} \tilde{\Gamma}_{i \alpha}^{\beta} \tilde{\Gamma}_{j \beta} = \sum_{\alpha, \beta=m+1}^{m+n} \frac{1}{2} \delta_{\alpha \beta} \frac{\partial \ln \rho^{m}}{\partial x_{i}} \frac{1}{2} \delta_{\alpha \beta} \frac{\partial \ln \rho^{m}}{\partial x_{j}}
\]

= \frac{n}{4} \frac{\partial \ln \rho^{m}}{\partial x_{i}} \frac{\partial \ln \rho^{m}}{\partial x_{j}} \quad (3.25)

Insert (3.22)-(3.25) to (3.21), at point \(P\),

\[
\sum_{C, D=1}^{m+n} \tilde{\Gamma}_{i C}^{D} \tilde{\Gamma}_{j D} = \frac{1}{4} \left\{ (m + 2) \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} - 2 |\nabla_{g} \ln \rho^{n}|_{g}^{2} \delta_{ij} \right\}
\]

\[
-2 \rho^{m+n} \delta_{ij} \nabla_{h} \ln \rho^{n}|_{h}^{2} + n \frac{\partial \ln \rho^{m}}{\partial x_{i}} \frac{\partial \ln \rho^{m}}{\partial x_{j}} \right\}
\]

= \frac{1}{4} \left\{ n(m + 2n + m^{2}) \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} - 2 |\nabla_{g} \ln \rho^{n}|_{g}^{2} \delta_{ij} \right\}
\]

= \frac{1}{4} \left\{ n(m + 2n + m^{2}) \frac{\partial \ln \rho^{n}}{\partial x_{i}} \frac{\partial \ln \rho^{n}}{\partial x_{j}} - 2 |\nabla_{g} \ln \rho^{n}|_{g}^{2} \delta_{ij} \right\} \quad (3.26)
By (3.17)-(3.20) and (3.26), at point $P$
\[ \tilde{R}_{ij} = \sum_{k=1}^{m} \frac{\partial \Gamma^{k}_{ij}}{\partial x_{k}} - \sum_{k=1}^{m} \frac{\partial \Gamma^{k}_{ik}}{\partial x_{j}} + \frac{\partial^{2} \ln \rho_{n}}{\partial x_{i} \partial x_{j}} - \frac{1}{2} g_{ij} \Delta_{g} \ln \rho_{n} - \frac{1}{2} g_{ij} \rho^{m+n} \Delta_{h} \ln \rho_{n} \]
\[ - \frac{1}{2} g_{ij} \nabla_{h} \rho^{m+n}, \nabla_{h} \ln \rho_{n} >_{h} - \frac{1}{4} \{ n(m + 2n + m^{2}) \frac{\partial \ln \rho}{\partial x_{i}} \frac{\partial \ln \rho}{\partial x_{j}} \}
\[ - 2|\nabla_{g} \ln \rho_{n}|_{g}^{2} \delta_{ij} - \rho^{m+n} |\nabla_{h} \ln \rho_{n}|_{h}^{2} \delta_{ij} \}
\[ = R_{ij} + \frac{\partial^{2} \ln \rho_{n}}{\partial x_{i} \partial x_{j}} - \frac{1}{2} g_{ij} \Delta_{g} \ln \rho_{n} - \frac{1}{2} g_{ij} \rho^{m+n} \Delta_{h} \ln \rho_{n} \]
\[ - \frac{m+n}{2} g_{ij} \rho^{m+n} |\nabla_{h} \ln \rho_{n}|_{h}^{2} - \frac{1}{4} n(m + 2n + m^{2}) \frac{\partial \ln \rho}{\partial x_{i}} \frac{\partial \ln \rho}{\partial x_{j}} \]
\[ + \frac{n^{2}}{2} |\nabla_{g} \ln \rho_{n}|_{g}^{2} \delta_{ij} + \frac{n^{2}}{2} |\nabla_{h} \ln \rho_{n}|_{h}^{2} \delta_{ij} \rho^{m+n} \] (3.27)

Using (3.2) and (3.27), at point $P$ one has,
\[ \sum_{i,j=1}^{m} K^{ij}_{i,j} \tilde{R}_{ij} = \rho^{-n} R_{g} + \rho^{-n} \Delta_{g} \ln \rho_{n} - \frac{m}{2} \rho^{-n} \Delta_{g} \ln \rho_{n} - \frac{m}{2} \rho^{m} \Delta_{h} \ln \rho_{n} \]
\[ - \frac{mn}{2} m \rho^{m} |\nabla_{h} \ln \rho_{n}|_{h}^{2} + \frac{n^{2}}{4} \rho^{-n} (nm + 2n + m^{2}) |\nabla_{g} \ln \rho_{n}|_{g}^{2} \]
\[ + \frac{mn^{2}}{2} |\nabla_{g} \ln \rho_{n}|_{g}^{2} \rho^{m-n} \]
\[ = \rho^{-n} R_{g} + \frac{2 - m}{2} \rho^{-n} \Delta_{g} \ln \rho_{n} - \frac{m}{2} \rho^{m} \Delta_{h} \ln \rho_{n} \]
\[ - \frac{n}{4} (-nm + 2n + m^{2}) \rho^{-n} |\nabla_{g} \ln \rho_{n}|_{h}^{2} \]
\[ - \frac{m^{2} n}{2} \rho^{m} |\nabla_{h} \ln \rho_{n}|_{h}^{2} \] (3.28)

Here, $R_{g}, R_{h}$ denote the scalar curvature of $(M, g), (N, h)$. Since the position of $g$ and $h$ are symmetry, interchange $g$ and $h$ and $\rho^{-1}, m$ and $n$ of (3.28) at point $P$, one has,
\[ \sum_{\alpha, \beta=m+1}^{m+n} K^{\alpha \beta} \tilde{R}_{\alpha \beta} = \rho^{m} R_{h} + \frac{2 - n}{2} (\Delta_{h} \ln \rho^{-m}) \rho^{m} - \frac{n}{2} \rho^{-n} \Delta_{g} \ln \rho^{-m} \]
\[ - \frac{m}{4} (-nm + 2m + n^{2}) \rho^{m} |\nabla_{g} \ln \rho_{n}|_{g}^{2} \]
\[ - \frac{n^{2} m}{2} \rho^{-n} |\nabla_{g} \ln \rho_{n}|_{g}^{2} \] (3.29)
Take $\tilde{R}$ as the scalar curvature of $(\Sigma, K)$, then at point $P$, by (3.28), (3.29) and
\[\tilde{R} = \sum_{i,j=1}^{m} K^{ij} \tilde{R}_{ij} + \sum_{\alpha,\beta=m+1}^{m+n} K^{\alpha\beta} \tilde{R}_{\alpha\beta}\] (3.30)
one has (1.1), and let $\rho = e^{2u}$, where $u$ is a smooth enough function on $\Sigma$. Then one has,
\[\tilde{R} = e^{-2nu} R_g + e^{2mu} R_h + 2ne^{-2nu} \Delta_g u - 2me^{2mu} \Delta_h u\]
\[-n(nm + 2m + n^2)e^{-2nu} \nabla_g u|_g^2\]
\[-m(nm + 2m + n^2)e^{2mu} \nabla_h u|_h^2\] (3.31)
Since, the left and right hand sides of the above equation both are not depended on the choice of local coordinates on $\Sigma$, then it is a equation on manifold. If $n = 1$, simplify it, then one obtains
\[\tilde{R} = e^{-2u} R_g + e^{2mu} R_h + 2e^{-2u} \Delta_g u - 2me^{2mu} \Delta_h u\]
\[-(m^2 + m + 2)e^{-2u} \nabla_g u|_g^2 - m(3m + 1)e^{2mu} \nabla_h u|_h^2\] (3.32)
Since $n = 1$, $N$ becomes a curve. Suppose $t$ be $N$’s arch length parameter, then write $u$ as $u(x, t)$. So with this parameter, one has,
\[h(\partial, \partial) = 1; \Delta_h u = \frac{\partial^2 u}{\partial t^2}; \nabla_h u = \frac{\partial u}{\partial t}; R_h = 0\] (3.33)
By (3.32)(3.33), one obtains (1.2). Here, we take the conservation that $\Delta = \Delta_g$, $\nabla u = \nabla_g u$, $|\nabla u|^2 = |\nabla_g u|^2$, $<\cdot, \cdot> = <\cdot, \cdot>_g$. And the following paragraph holds the conservation.

4 Prior Estimate: Energy Estimate

The idea of the energy estimate used here is inspired by [Smith&Tataru]. Let
\[F(u, v) = \frac{1}{2} \left[ -e^{-2nu} \tilde{R} + e^{-2(m+1)v} R_g\right.\]
\[\left. - (m^2 + m + 2)e^{-2(m+1)v} <\nabla v, \nabla u> - m(3m + 1)v v_t u_t \right]\] (4.1)
In this section, consider the prior estimate of the following equation,
\[mu_{tt} - e^{-2(m+1)v} \Delta u = F(u, v)\] (4.2)
when \( u = v \), (4.2) becomes to (1.2). Here, suppose \((M, g)\) be a closed Riemannian manifold of \( m \)-dimension. \( I = [0, T] \) is an interval of length \( T \). And, \( u, v \in C^{s+1}(M \times I) \), \( s > \frac{m}{2} + 1 \). First estimate \( \|me^{2(m+1)v}\nabla^{l-1}u_{tt} - \Delta \nabla^{l-1}u\|_{L^2(M)} \), where \( 1 \leq l \leq s \). Then, by (4.2)

\[
me^{2(m+1)v}\nabla^{l-1}u_{tt} - \Delta \nabla^{l-1}u = [e^{2(m+1)v}\nabla^{l-1}F(u, v)] + [e^{2(m+1)v}\nabla^{l-1}(e^{-2(m+1)v} \Delta u) - \nabla^{l-1} \Delta u] + [\nabla^{l-1} \Delta u - \Delta \nabla^{l-1}u]
\]

\[= I + II + III \tag{4.3}\]

In what follows, estimate \( I, II, III \) respectively.

\[
\|I\|_{L^2(M)} \leq Ce^{C\|v\|_{C^0(M)}}\|\nabla^{l-1}[\eta - e^{-2mv} \tilde{R} + e^{-2(m+1)v} R_g - (m^2 + m + 2)e^{-2(m+1)v} < \nabla v, \nabla u > -m(3m + 1)v]\|_{L^2(M)} \tag{4.4}\]

By Corollary 2.4 and Sobolev embedding theorem, \( H^{s-1}(M) \hookrightarrow C^0(M) \) one has,

\[
\|\nabla^{l-1}(e^{-2mv} \tilde{R})\|_{L^2(M)}
\leq \|\nabla^{l-1}[e^{-2mv} - 1] \tilde{R}\|_{L^2(M)} + \|\nabla^{l-1} \tilde{R}\|_{L^2(M)}
\leq C(\|e^{-2mv} - 1\|_{C^0(M)}\|\tilde{R}\|_{H^{l-1}(M)} + \|e^{-2mv} - 1\|_{H^{l-1}(M)}\|\tilde{R}\|_{C^0(M)}) + \|\nabla^{l-1} \tilde{R}\|_{L^2(M)}
\leq C\|e^{-2mv} - 1\|_{H^{s-1}(M)}\|\tilde{R}\|_{H^{s-1}(M)} + \|\nabla^{l-1} \tilde{R}\|_{L^2(M)}
\leq C\|e^{-2mv} - 1\|_{C^0(M)}\|\tilde{R}\|_{H^{s-1}(M)} + \|\nabla^{l-1} \tilde{R}\|_{L^2(M)}
\leq Ce^{C\|v\|_{C^0(M)}(1 + \|v\|_{H^{s-1}(M)})\|\tilde{R}\|_{H^{s-1}(M)}} \tag{4.5}\]

Similar,

\[
\|\nabla^{l-1}(e^{-2(m+1)v} R_g)\|_{L^2(M)} \leq Ce^{C\|v\|_{H^{s-1}(M)}\|\tilde{R}\|_{H^{s-1}(M)}} \tag{4.6}\]
Again, by Corollary 2.4 and Sobolev embedding theorem, $H^{s-1}(M) \hookrightarrow C^0(M)$, $\mathcal{H}^s(M) \hookrightarrow C^1(M)$ one has,
\[
\left\| \nabla^{l-1}(e^{-2(m+1)v} < \nabla v, \nabla u >) \right\|_{L^2(M)} = C\left\| \nabla^{l-1} < \nabla(e^{-2(m+1)v} - 1), \nabla u > \right\|_{L^2(M)} \leq C(\|e^{-2(m+1)v} - 1\|_{C^1(M)}\|u\|_{H^i(M)} + \|e^{-2(m+1)v} - 1\|_{H^i(M)}\|u\|_{C^1(M)}) \leq C\|e^{-2(m+1)v} - 1\|_{H^s(M)}\|u\|_{H^s(M)} \leq Ce^{\|v\|_{H^s(M)}}\|u\|_{H^s(M)} \tag{4.7}
\]
the last step is similar to (4.5). And
\[
\left\| \nabla^{l-1}(v_i u_i) \right\|_{L^2(M)} \leq C(\|v_i\|_{H^{i-1}(M)}\|u_i\|_{C^0(M)} + \|v_i\|_{C^0(M)}\|u_i\|_{H^{i-1}(M)}) \leq C\|v_i\|_{H^{i-1}(M)}\|u_i\|_{H^{i-1}(M)} \tag{4.8}
\]
Conservation that,
\[
A_s = \|\tilde{R}\|_{H^{s-1}(M)} + \|R_g\|_{H^{s-1}(M)} \tag{4.9}
\]
Then, by (4.4)-(4.9) and $H^{s-1}(M) \hookrightarrow C^0(M)$, one has,
\[
\left\| I \right\|_{L^2(M)} \leq Ce^{\|v\|_{H^s(M)} + \|v\|_{H^{s-1}(M)}}(A_s + \|u\|_{H^s(M)} + \|u_i\|_{H^{s-1}(M)}) \tag{4.10}
\]
For $II$,
\[
\left\| II \right\|_{L^2(M)} = \|e^{2(m+1)v}\nabla^{l-1}(e^{-2(m+1)v}\nabla^2 u) - \nabla^{l-1}\nabla^2 u\|_{L^2(M)} = \|e^{2(m+1)v}\sum_{i+j=l-1, i \geq 1} \nabla^i(e^{-2(m+1)v})\nabla^j\nabla^2 u\|_{L^2(M)} \leq Ce^{\|v\|_{C^0(M)}}\sum_{i+j=l-1, i \geq 1} \left\| \nabla^i(e^{-2(m+1)v} - 1)\nabla^j\nabla^2 u \right\|_{L^2(M)} \tag{4.11}
\]
For fixed $i, j$, let,
\[
p = \frac{i - 1}{i - 1}, q = \frac{j + 1}{j + 1}, \frac{1}{2p} + \frac{1}{2q} = \frac{1}{2} \tag{4.12}
\]
Then,
\[
\frac{1}{2p} - \frac{i - 1}{m} = \frac{i - 1}{l - 1} \left( \frac{1}{2} - \frac{l - 1}{m} \right), \frac{1}{2q} - \frac{j + 1}{m} = \frac{j + 1}{l - 1} \left( \frac{1}{2} - \frac{l - 1}{m} \right) \tag{4.13}
\]
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Using Hölder inequality, Corollary 2.4, Remark 2.6 and Sobolev embedding theorem, one obtains,

\[
\|\nabla^i (e^{-2(m+1)v} - 1) \nabla^j \nabla^k u\|_{L^2(M)} \\
\leq \|\nabla^i (e^{-2(m+1)v} - 1)\|_{L^2(M)} \|\nabla^j \nabla^k u\|_{L^2(M)} \\
\leq C \|\nabla^i (e^{-2(m+1)v} - 1)\|_{L^2(M)} \|\nabla^j u\|_{L^2(M)} \\
\leq C \|e^{-2(m+1)v} - 1\|_{H^{\frac{1}{2}}(M)} \|e^{-2(m+1)v} - 1\|_{H^{\frac{1}{2}}(M)} \|u\|_{C^{\frac{1}{2}}(M)} \\
\leq C \|e^{-2(m+1)v} - 1\|_{H^s(M)} \|u\|_{H^s(M)} \\
\leq C e^{C\|u\|_{C^0(M)}} \|u\|_{H^s(M)} \|u\|_{H^s(M)}
\]
(4.14)

Then,

\[
\|II\|_{L^2(M)} \leq C e^{C\|u\|_{C^0(M)}} \|u\|_{H^s(M)} \|u\|_{H^s(M)}
\]
(4.15)

For III, denote \(B_{i_1 \cdots i_k} = \nabla_{i_1} \cdots \nabla_{i_k} u\), by the Ricci identity,

\[
(\nabla_p \nabla_q - \nabla_q \nabla_p) B_{i_1 \cdots i_k} = - \sum_{t=1}^{m} \sum_{n=1}^{k} R^n_{pqit} B_{i_1 \cdots i_{t-1} m t+1 \cdots i_k}
\]
(4.16)

here, all index take values in \(\{1, \cdots, m\}\), then direct computation gives,

\[
\nabla_j \nabla^k B_{i_1 \cdots i_k}
\]

\[
= \nabla_j B_{i_1 \cdots i_k} - \sum_{p,q,n=1}^{m} \sum_{t=1}^{k} g^{pq} \nabla_p R^n_{jqt} B_{i_1 \cdots i_{t-1} n t+1 \cdots i_k} \\
- \sum_{p,q,n=1}^{m} \sum_{t=1}^{k} g^{pq} R^n_{jqt} \nabla_p B_{i_1 \cdots i_{t-1} n t+1 \cdots i_k} - \sum_{p,q,n=1}^{m} \sum_{t=1}^{k} g^{pq} R^n_{jqt} \nabla_q B_{i_1 \cdots i_{t-1} n t+1 \cdots i_k} \\
- \sum_{p,q,n=1}^{m} g^{pq} R^n_{jpq} \nabla_n B_{i_1 \cdots i_k}
\]
(4.17)

From (4.17), one can obtain, if \(0 \leq \tilde{k}\),

\[
\|\nabla^\tilde{k} (\nabla \Delta B - \Delta \nabla B)\|_{L^2(M)} \leq C \|u\|_{H^{k+\tilde{k}+1}(M)}
\]
(4.18)
Here, \( C \) is a positive constant depended only on manifold \( M \), then iterate (4.18), one has,

\[
\| \nabla \Delta \nabla^{l-2} u - \Delta \nabla^{l-1} u \|_{L^2(M)} \leq C \| u \|_{H^{l-1}(M)}
\]

\[
\| \nabla^2 \Delta \nabla^{l-3} u - \nabla \nabla^{l-2} u \|_{L^2(M)} \leq C \| u \|_{H^{l-1}(M)}
\]

\[
\cdots \cdots
\]

\[
\| \nabla^{l-1} \nabla u - \nabla^{l-2} \nabla u \|_{L^2(M)} \leq C \| u \|_{H^{l-1}(M)}
\]

Adding the left and right hand sides of (4.19), then one obtains

\[
\| II \|_{L^2(M)} \leq C \| u \|_{H^s(M)}
\]

By (4.3)(4.9)(4.10)(4.15)(4.20) and Sobolev embedding theorem one obtains,

\[
\| me^{2(m+1)v} \nabla^{l-1} u_{tt} - \Delta \nabla^{l-1} u \|_{L^2(M)} \leq C_1 e^{C_2} \| u \|_{H^s(M)} \| u_t \|_{H^{s+1}(M)}
\]

(4.21)

Here, \( C_1, C_2 \) are positive constants depended on \( s \), manifold \( M \), not depended on \( R, u \). Then we do energy estimate. One can introduce the following energy functions:

\[
E_1^{(l)}(t) = \frac{1}{2} \int_M [ \| \nabla^l u \|^2 + me^{2(m+1)v} \| \nabla^{l-1} u_t \|^2 ]dV
\]

(4.22)

\[
E_2^{(l)}(t) = \frac{1}{2} \int_M [ e^{-2(m+1)v} \| \nabla^l u \|^2 + m \| \nabla^{l-1} u_t \|^2 ]dV
\]

(4.23)

And let,

\[
E_s(t) = \sum_{i=1}^s (E_1^{(l)}(t) + E_2^{(l)}(t)) + \frac{1}{2} \int_M \| u \|^2 dV
\]

(4.24)

It is obvious,

\[
\| u \|_{H^s(M)} + \| u_t \|_{H^{s+1}(M)} \leq C (E_s(t))^{\frac{1}{2}}
\]

(4.25)

To (4.22), take derivation respect to \( t \), by (2.2) one has,

\[
\frac{d}{dt} E_1^{(l)}(t) = \int_M [ \langle \nabla^l u_t, \nabla^l u - m(m+1)e^{2(m+1)v} \| \nabla^{l-1} u_t \|^2 ] \langle \nabla^l u_t \rangle^2
\]

\[
+ me^{2(m+1)v} \langle \nabla^{l-1} u_t, \nabla^{l-1} u_{tt} \rangle ]dV
\]

\[
\leq C \| v_t \|_{L^2(M)} E_1^{(l)}(t) + \int_M [ \langle \nabla^l u_t, \nabla^l u \rangle
\]

\[
+ \langle \nabla^{l-1} u_t, me^{2(m+1)v} \nabla^{l-1} u_{tt} \rangle ]dV
\]

(4.26)
Since $M$ is closed, $\int_M \text{div} <\nabla^{l-1} u, \nabla \nabla^{l-1} u > dV = 0$. This implies,

$$\int_M <\nabla^{l} u, \nabla^{l} u > dV = - \int_M <\nabla^{l-1} u, \triangle \nabla^{l-1} u > dV \quad (4.27)$$

then, by (4.26), and Hölder inequality,

$$\frac{d}{dt} E_1^{(l)}(t)$$

\begin{align*}
\leq & \ C \|v_t\|_{C^0(M)} E_1^{(l)}(t) + \int_M <\nabla^{l-1} u_t, m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u > dV \\
\leq & \ C \|v_t\|_{C^0(M)} E_1^{(l)}(t) + \|
abla^{l-1} u_t\|_{L^2(M)}\|m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u\|_{L^2(M)} \\
\leq & \ C \|v_t\|_{C^0(M)} E_1^{(l)}(t) + C e^{\|v\|_{C^0(M)}(4.29)} m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u\|_{L^2(M)}(E_1^{(l)}(t))^{\frac{1}{2}} \\
\quad + C e^{\|v\|_{C^0(M)}(4.29)} m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u\|_{L^2(M)}(E_1^{(l)}(t))^{\frac{1}{2}}
\end{align*}

Similar,

$$\frac{d}{dt} E_2^{(l)}(t)$$

\begin{align*}
= & \int_M [-e^{-2(m+1)v}(m+1)v_t |\nabla^{l} u|^2 + <e^{-2(m+1)v} \nabla^{l} u, \nabla^{l} u_t > \\
+ <\nabla^{l-1} u_t, m \nabla^{l-1} u_{tt} >]dV \\
\leq & \ C \|v_t\|_{C^0(M)} E_2^{(l)}(t) + \|
abla^{l-1} u_t\|_{L^2(M)}\|m \nabla^{l-1} u_{tt} - e^{-2(m+1)v} \triangle \nabla^{l-1} u\|_{L^2(M)} \\
+ & \ C e^{\|v\|_{C^0(M)}m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u\|_{L^2(M)}(E_2^{(l)}(t))^{\frac{1}{2}} \\
\quad + C e^{\|v\|_{C^0(M)}m e^{2(m+1)v} \nabla^{l-1} u_{tt} - \triangle \nabla^{l-1} u\|_{L^2(M)}(E_2^{(l)}(t))^{\frac{1}{2}}
\end{align*}

(4.28)
Using (4.28)(4.29), one has

\[
\frac{d}{dt} E_s(t) \leq Ce^{\|v\|C^0(M)} (\|v_t\|C^0(M) + \|\nabla v\|C^0(M)) E_s(t) \\
+ Ce^{\|v\|C^0(M)} \sum_{l=1}^{s} (m_{l+1}v^l - \Delta v^{l-1} u_t)_{L^2(M)} (E_s(t))^{\frac{1}{2}} \\
+ \int_M uu_t dV
\]  

(4.30)

By Hölder inequality,

\[
\int_M uu_t dV \leq CE_s(t) 
\]  

(4.31)

then by (4.21)(4.25)(4.30)(4.31) and Sobolev embedding theorem, one obtains

\[
\frac{d}{dt} E_s(t) \\
\leq Ce^{\|v\|C^0(M)} (1 + \|v_t\|C^0(M) + \|\nabla v\|C^0(M)) E_s(t) \\
+ Ce^{\|v\|H^s(M) + \|v_t\|H^{s-1}(M)} (A_s + \|u\|H^s(M) + \|u_t\|H^{s-1}(M)) (E_s(t))^{\frac{1}{2}} \\
\leq Ce^{\|v\|H^s(M) + \|v_t\|H^{s-1}(M)} (A_s + \|u\|H^s(M) + \|u_t\|H^{s-1}(M) + (E_s(t))^{\frac{1}{2}}) (E_s(t))^{\frac{1}{2}} \\
\leq Ce^{\|v\|H^s(M) + \|v_t\|H^{s-1}(M)} (A_s + (E_s(t))^{\frac{1}{2}}) (E_s(t))^{\frac{1}{2}} 
\]  

(4.32)

Then obvious,

\[
\frac{d}{dt} (E_s(t))^{\frac{1}{2}} \leq C_1 e^{C_2 (\|v\|H^s(M) + \|v_t\|H^{s-1}(M))} (A_s + (E_s(t))^{\frac{1}{2}}) 
\]  

(4.33)

here, \(C_1, C_2\) related to manifold \(M\) and \(s\). Then using (4.33), one has the following Proposition.

**Proposition 4.1** Suppose \((M,g)\) be \(m\)-dimensional closed Riemannian manifold, and \(I = [0,T]\) be an interval of length \(T\). Let \(u, v \in C^{s+1}(M \times I), s > \frac{m}{2} + 1\), and \(u, v\) satisfy (4.2) then:

(i) If exist a positive constant \(D > 2\sqrt{2} [1 + (E_s(0))^{\frac{1}{2}}]\). There is \(t_0 > 0\) which is related to manifold \(M, s, \|\bar{R}\|C^0(I,H^{s-1}(M)), D, E_s(0)\), such that

\[
\|v\|_{H^s(M)} + \|v_t\|_{H^{s-1}(M)} \leq D
\]  

(4.34)
held in $0 \leq t \leq t_0$ implies
\[
\|u\|_{H^s(M)} + \|u_t\|_{H^{s-1}(M)} \leq D
\] (4.35)

held in $0 \leq t \leq t_0$

(ii) If $u = v$ which means $u$ satisfy (1.2), then exist $t_0, \bar{C} > 0$ such that
\[
\|u\|_{H^s(M)} + \|u_t\|_{H^{s-1}(M)} \leq \bar{C}
\] (4.36)
holds in $0 \leq t \leq t_0$. Here, $t_0$, $\bar{C}$ relates to manifold $M$, $s$, $\|\tilde{R}\|_{C^0(I,H^{s-1}(M))}$, $E_s(0)$, and $\bar{C}$ also relates to $t_0$.

**Proof**

(i) Using (4.9) and (4.33),
\[
\frac{d}{dt}(E_s(t))^{\frac{1}{2}} \leq C_3 e^{C_2(\|v\|_{H^s(M)} + \|u_t\|_{H^{s-1}(M)})} (1 + (E_s(t))^{\frac{1}{2}})
\] (4.37)
here, $C_3$ related to manifold $M$, $s$, and extra $\|\tilde{R}\|_{C^0(I,H^{s-1}(M))}$. Take $t_0$, such that
\[
2\sqrt{2}[1 + (E_s(0))^{\frac{1}{2}}] e^{C_3 e^{C_2 D}} \leq D
\] (4.38)

Then in $0 \leq t \leq t_0$,
\[
\frac{d}{dt}(E_s(t))^{\frac{1}{2}} \leq C_3 e^{C_2 D} (1 + (E_s(t))^{\frac{1}{2}})
\] (4.39)

Integrate (4.39) one has,
\[
(E_s(t))^{\frac{1}{2}} \leq [1 + (E_s(0))^{\frac{1}{2}}] e^{C_3 e^{C_2 D} t} \leq \frac{D}{2\sqrt{2}}
\] (4.40)
This implies the result by (4.25).

(ii) Let $v = u$ in (4.37), then
\[
\frac{d}{dt}(E_s(t))^{\frac{1}{2}} \leq C_3 e^{C_2 (E_s(t))^{\frac{1}{2}}}
\] (4.41)
Integrate it,
\[
e^{-C_2 (E_s(t))^{\frac{1}{2}}} \geq e^{-C_2 (E_s(0))^{\frac{1}{2}}} - C_3 C_2 t
\] (4.42)
Hence, when \( t_0 < \frac{e^{-C_2(E_s(0))^\frac{1}{2}}}{C_3C_2} \), one has,

\[
\left( E_s(t) \right)^\frac{1}{2} \leq \frac{1}{C_2} \frac{\ln 1}{e^{-C_2(E_s(0))^\frac{1}{2}} - C_3C_2t_0} \] (4.43)

This gives what we want.

**Proposition 4.2**  Suppose \((M, g)\) be \(m\)-dimensional closed Reimainnian manifold, whose scalar curvature is zero, and \(I = [0, T] \) be a interval of length \( T \). Let \( u, v \in C^{s+1}(M \times I), \ s > \frac{m}{2} + 1 \). \( u, v \) satisfy (4.2), and \( u(x, 0) = u_t(x, 0) = 0 \) to any \( x \in M \). Then to any given positive constant \( D \), exist constant \( \varepsilon > 0 \) which depends on manifold \( M, s, D \), such that

\[
\|v\|_{H^s(M)} + \|v_t\|_{H^{s-1}(M)} \leq D \] (4.44)

held in \( I \) implies

\[
\|u\|_{H^s(M)} + \|u_t\|_{H^{s-1}(M)} \leq D \] (4.45)

held in \( I \) when \( \|\tilde{R}\|_{H^{s-1}(M)} \leq \varepsilon \).

**Proof**  Take \( \varepsilon \) satisfying

\[
\varepsilon e^{C_1e^{C_2DT}} \leq \frac{D}{2\sqrt{2}} \] (4.46)

here, \( C_1, C_2 \) are constant of (4.33). Since \( R_g = 0, A_s = \|\tilde{R}\|_{H^{s-1}(M)} \), so by (4.33) and (4.44),

\[
\frac{d}{dt}(E_s(t))^\frac{1}{2} \leq C_1 e^{C_2D}(\varepsilon + (E_s(t))^\frac{1}{2}) \] (4.47)

Integrate (4.47)

\[
(E_s(t))^\frac{1}{2} \leq \varepsilon e^{C_1e^{C_2DT}} \leq \frac{D}{2\sqrt{2}} \] (4.48)

and using (4.25), then one gets the result.
5 Linear Equation

In this section we consider the linear equation (1.3) on the condition:

\[ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) \] (5.1)

here, \((M, g)\) the \(m\)-dimensional closed Riemannian manifold, \(I = [0, T]\) interval of length \(T\), and \(a, \alpha, \beta, \gamma, f \in C^\infty(M \times I), \varphi, \psi \in C^\infty(M)\), and exist positive constant \(L\), such that \(\frac{1}{L} \leq \alpha \leq L\). In what follows, we discuss the existence and property of the solution. If \(M\) is a domain of \(\mathbb{R}^n\), then the following results are classical in linear hyperbolic equation (c.f. [H"ormander] and [Xin]). We follow their method. First, give some prior estimates. Multiple \(u_t\) in (1.3), and integrate on \(M\). Since \(M\) is closed, one has,

\[
\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2(M)} + \int_M \alpha |\nabla u|^2 dV \\
\leq \|f\|_{L^2(M)} \|u_t\|_{L^2(M)} - \int_M u_t < \nabla (\alpha - \beta), \nabla u > dV + C \int_M |\nabla u|^2 dV \\
+ C\|u\|_{L^2(M)} \|u_t\|_{L^2(M)} + C\|u_t\|^2_{L^2(M)}
\] (5.2)

Using,

\[
\|u\|^2_{L^2(M)} \leq \int_0^t \|u_t\|^2_{L^2(M)} d\tau + \|\varphi\|^2_{L^2(M)}
\] (5.3)

one has,

\[
\|u_t\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)} \\
\leq C[\|\varphi\|^2_{H^1(M)} + \|\psi\|^2_{L^2(M)} + \int_0^t (\|u_t\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)} + \|f\|^2_{L^2(M)}) d\tau]
\] (5.4)

Integrate it, one has

\[
\|u_t\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)} \leq C[\|\varphi\|^2_{H^1(M)} + \|\psi\|^2_{L^2(M)} + \int_0^t \|f\|^2_{L^2(M)} d\tau]
\] (5.5)
Multiple $-\Delta u_t$ in (1.3), integrate on $M$, and using (5.3)(5.5) $W^{k,p}$ estimate of elliptic equation, one has,
\[
\|\nabla u_t\|^2_{L^2(M)} + \frac{1}{L}\|\Delta u\|^2_{L^2(M)} + 2\int_M (\Delta u) f dV \\
\leq C[\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)} + \int_0^t (\|f_t\|^2_{L^2(M)} + \|f\|^2_{L^2(M)}) \\
+ \|\Delta u\|^2_{L^2(M)} + \|\nabla u_t\|^2_{L^2(M)}] d\tau \tag{5.6}
\]
Using the inequality $2 \int_M (\Delta u) f \leq \frac{1}{2L} \|\Delta u\|^2_{L^2(M)} + C \|f\|^2_{L^2(M)}$ and,
\[
\|f(t)\|^2_{L^2(M)} \leq \int_0^t \|f_t(\tau)\|^2_{L^2(M)} d\tau + \|f(0)\|^2_{L^2(M)} \tag{5.7}
\]
One has,
\[
\|\nabla u_t\|^2_{L^2(M)} + \|\Delta u\|^2_{L^2(M)} \\
\leq C[\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)} + \int_0^t (\|f_t\|^2_{L^2(M)} + \|\Delta u\|^2_{L^2(M)}) \\
+ \|\nabla u_t\|^2_{L^2(M)}] d\tau \tag{5.8}
\]
Integrate it,
\[
\|\nabla u_t\|^2_{L^2(M)} + \|\Delta u\|^2_{L^2(M)} \leq C[\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)} \\
+ \int_0^t \|f_t\|^2_{L^2(M)} d\tau] \tag{5.9}
\]
Take derivation to $t$ in (1.3), and multiple $u_{tt}$ in it, integrate on $M$, using (5.3)(5.5)(5.7) one has,
\[
\|u_{tt}\|^2_{L^2(M)} + \|\nabla u_t\|^2_{L^2(M)} \leq C[\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)} \\
+ \int_0^t (\|f_t\|^2_{L^2(M)} + \|\Delta u\|^2_{L^2(M)}) d\tau] \tag{5.10}
\]
Using (5.3)(5.5)(5.9)(5.10), one has
\[
\sum_{i=0}^2 \|u\|_{W^{i,\infty}(I,H^{2-i}(M))} \leq C[\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)} \\
+ \int_0^T \|f_t\|^2_{L^2(M)} d\tau] \tag{5.11}
\]
Here, \( C \) is a constant depended only on manifold \( M, L, T, a, \alpha, \beta, \gamma \).

Then, use the Faedo-Galerkin method to give the existence of (1.3). Let \( \lambda_0, \lambda_1, \cdots \) be eigenvalues of laplace operator on Riemannian manifold \( M \), and \( w_i \) be corresponded eigenvectors, namely \( \triangle w_i + \lambda_i w_i = 0, i \in \{0, 1, \cdots \} \). By Chapter 3 p87 of [Schoen&Yau], one knows that \( w_i \) which are smooth functions on \( M \) are complete standard orthonormal basis of \( L^2(M) \) and also are complete orthogonal basis of \( L^2(M) \), and \( \lambda_i \geq 0 \). Then we give approximate sequence \( u_n(x, t) = \sum_{i=0}^{n} \eta_i(t) w_i(x) \). In what follows, we denote \((, )\) the inner product of \( L^2(M) \). Consider

\[
((u_n)_{tt}, w_j) + (a(u_n)_t, w_j) - (\alpha \triangle u_n + < \nabla \beta, \nabla u_n > + \gamma u_n, w_j) = (f, w_j), \quad 0 \leq j \leq n \tag{5.12}
\]

By the expansion of \( u_n \),

\[
\eta''_j(t) + \sum_{i=0}^{n} (aw_i, w_j) \eta'_i(t) + \sum_{i=0}^{n} (\alpha \lambda_i w_i - < \nabla \beta, \nabla w_i > - \gamma w_i, w_j) \eta_i(t) = f_j := (f, w_j), \quad 0 \leq j \leq n \tag{5.13}
\]

and on the condition,

\[
\eta_j(0) = \varphi_j := (\varphi, w_j), \quad \eta'_j(0) = \psi_j := (\psi, w_j) \tag{5.14}
\]

This means

\[
u_n(0) = \sum_{i=0}^{n} \eta_i(0) w_i = \sum_{i=0}^{n} \varphi_i w_i, \quad u'_n(0) = \sum_{i=0}^{n} \psi_i w_i \tag{5.15}
\]

So, let \( A_{ij}(t) = (\alpha \lambda_i w_i - < \nabla \beta, \nabla w_i > - \gamma w_i, w_j) \), then \( (5.13)(5.14) \) give a linear ordinary differential system with initial data. By the theory of ordinary differential equation, \( (5.13)(5.14) \) have a unique smooth solution \((\eta_0, \cdots, \eta_n)\).

Obvious, \( u_n(0) \to \varphi \) in \( L^2(M) \), \( u'_n(0) \to \psi \) in \( L^2(M) \). Then using the closeness of \( M \),

\[
\| \varphi \|^2_{H^1(M)} = (\varphi - \triangle \varphi, \varphi) = \sum_{i=0}^{\infty} (1 + \lambda_i)|\varphi_i|^2 < +\infty \tag{5.16}
\]

This implies \( u_n(0) \to \varphi \) in \( H^1(M) \). Similar \( u'_n(0) \to \psi \) in \( H^1(M) \). And

\[
\| \varphi \|^2_{H^2(M)} = \sum_{i=0}^{\infty} (1 + \lambda_i)|\varphi_i|^2 + \int_M < -\triangle \varphi, \nabla \varphi > dV \tag{5.17}
\]

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By the Ricci identity,

\[
\int_M < -\Delta \nabla \varphi, \nabla \varphi > dV = \int_M \left[ < -\nabla \Delta \varphi, \nabla \varphi > - \sum_{i,j,p,q=1}^{m} g^{ij} g^{pq} R_{ip} \nabla_j \varphi \nabla_q \varphi \right] dV \\
\geq (\Delta^2 \varphi, \varphi) - C\|\varphi\|^2_{H^1(M)}
\]

(5.18)

Hence,\(\sum_{i=0}^{\infty} (1 + \lambda_i + \lambda_i^2) |\varphi_i|^2 \leq C\|\varphi\|^2_{H^2(M)}\). And, similar

\[
\|u_n(0) - \varphi\|^2_{H^2(M)} \leq \sum_{i=n+1}^{\infty} (1 + \lambda_i + \lambda_i^2) |\varphi_i|^2 + C\|u_n(0) - \varphi\|^2_{H^1(M)} \rightarrow 0
\]

(5.19)

when \(n \rightarrow \infty\). Then \(u_n(0) \rightarrow \varphi\) in \(H^2(M)\). In (5.12), multiple \(\eta'_j\), and sum up \(j\) from 0 to \(n\); multiple \(\lambda_j \eta'_j\), and sum up \(j\) from 0 to \(n\); take derivation to \(t\) then multiple \(\eta''_j\) and sum up \(j\) from 0 to \(n\), similar to (5.11) one obtains,

\[
\sum_{i=0}^{2} \|u_n\|_{W^{i,\infty}(I,H^{2-i}(M))} \leq C\|\varphi\|^2_{H^2(M)} + \|\psi\|^2_{H^1(M)} + \|f(0)\|^2_{L^2(M)}
\]

\[
+ \int_0^T \|f_t\|^2_{L^2(M)} d\tau
\]

(5.20)

This means \(u_n\) are uniformly bounded in \(\bigcap_{i=0}^{2} W^{i,\infty}(I,H^{2-i}(M))\). So exist some \(u \in \bigcap_{i=0}^{2} W^{i,\infty}(I,H^{2-i}(M))\), and \(u_n \rightarrow u\) weakly star in \(L^\infty(I,H^1(M))\); \((u_n)_t \rightarrow u_t\) weakly star in \(L^\infty(I,H^1(M))\); \((u_n)_{tt} \rightarrow u_{tt}\) weakly star in \(L^\infty(I,L^2(M))\). This means \((u_{nt}, w_j)+(a(u_n)_t, w_j) - (\alpha \Delta u_n + < \nabla \beta, \nabla u_n > + \gamma u_n, w_j) \rightarrow ((u)_t, w_j) + (au_t, w_j) - (\alpha \Delta u + < \nabla \beta, \nabla u > + \gamma u, w_j)\) weakly star in \(L^\infty[0,T]\). Hence \(u\) is the solution of (1.3). And by Lemma 2.7, \(u\) satisfies (5.1). So one has

**Lemma 5.1** Suppose \(f \in W^{1,\infty}(I,L^2(M))\), \(a, \alpha, \beta, \gamma \in C^\infty(M \times I), \varphi, \psi \in C^\infty(M)\), then there is a unique solution of (1.3) (5.1) in \(\bigcap_{i=0}^{2} W^{i,\infty}(I,H^{2-i}(M))\) and it has prior estimate (5.11). Here, \((M, g)\) is a m-dimensional closed Riemannian manifold, \(I = [0,T]\) is an interval of length \(T\).
In what follows, we discuss the regularity of the solution of (1.3). This comes from the following Lemma.

**Lemma 5.2** The hypothesis of $M, I$ are same to the above Lemma. And let $\forall N \in \mathbb{N}, N \geq 2, a, \alpha, \beta, \gamma \in C^\infty(M \times I), \varphi, \psi \in C^\infty(M), f \in \bigcap_{i=1}^{N-1} W^{i,\infty}(I, H^{N-1-i}(M))$, then the solution of (1.3)(5.1) $u \in \bigcap_{i=0}^{N} W^{i,\infty}(I, H^{N-1}(M))$, and $u$ satisfy the following inequality:

$$
\sum_{i=0}^{N} \| \partial_i^2 u \|_{H^{N-i}(M)}^2 \leq C_N (\| \varphi \|_{H^{N}(M)}^2 + \| \psi \|_{H^{N-1}(M)}^2 + \| f(0) \|_{H^{N-2}(M)}^2 \\
+ \int_0^t \sum_{i=1}^{N-1} \| \partial_i f \|_{H^{N-i-1}(M)}^2 d\tau) 
$$

(5.21)

Here, $C_N$ depends only on $N$ and manifold $M, L, T, a, \alpha, \beta, \gamma$.

**Proof** We use induction to prove this Lemma. When $N = 2$, this is the result of Lemma 5.1. Then one suppose this is true of case $N$, and consider the Lemma of case $N+1$. Take derivation to $t$ in (1.3), and denote $v = u_t$, then

$$
v_t + av_t - (\alpha \triangle v + < \nabla \beta, \nabla v > + \gamma v) = f_t - a_t v + \alpha_t \triangle (\varphi + \int_0^t v d\tau) + < \nabla \beta_t, \nabla (\varphi + \int_0^t v d\tau) > \\
+ \gamma_t (\varphi + \int_0^t v d\tau) 
$$

(5.22)

Introduce the following approximate sequence $\{ v_q \}_{q=0}^\infty$.

$$
(v_{q+1})_t + a(v_{q+1})_t - (\alpha \triangle v_{q+1} + < \nabla \beta, \nabla v_{q+1} > + \gamma v_{q+1}) = f_t - a_t v_q + \alpha_t \triangle (\varphi + \int_0^t v_q d\tau) + < \nabla \beta_t, \nabla (\varphi + \int_0^t v_q d\tau) > \\
+ \gamma_t (\varphi + \int_0^t v_q d\tau) 
$$

(5.23)

on the condition that $v_0 = 0$, $v_{q+1}(x, 0) = \psi(x)$, $v_{q+1}'(x, 0) = f(x, 0) - a(x, 0) \psi(x) + \alpha(x, 0) \varphi(x) + < \nabla \beta(x, 0), \nabla \varphi(x) > + \gamma(x, 0) \varphi(x)$. Then using
(5.21)(5.23), one has
\[
\sum_{i=0}^{N} \| \partial_t^i (v_{q+1} - v_q) \|^2_{H^{N-i}(M)} \\
\leq C_N \int_0^t \sum_{i=1}^{N-1} \| \partial_t^i [-a_t (v_q - v_{q-1}) + \alpha_t \Delta \int_0^t (v_q - v_{q-1}) d\tau'] \\
+ < \nabla \beta_t, \nabla \int_0^t (v_q - v_{q-1}) d\tau' > + \gamma_t \int_0^t (v_q - v_{q-1}) d\tau'] \|_{H^{N-i-1}(M)}^2 d\tau \tag{5.24}
\]

Since, when \(i \geq 1, q \geq 2\),
\[
\| \partial_t^i [\alpha_t \Delta \int_0^t (v_q - v_{q-1}) d\tau'] \|_{H^{N-i-1}(M)}^2 \\
\leq \sum_{j=0}^{i} \| \partial_t^j \int_0^t (v_q - v_{q-1}) d\tau' \|_{H^{N-i+1}(M)}^2 \\
\leq \sum_{j=0}^{i} \int_0^t \cdots \int_0^{t_{i-j}} \| \partial_t^j \int_0^t (v_q - v_{q-1}) d\tau' \|_{H^{N-i+1}(M)}^2 d\tau_1 \cdots d\tau_{i-j} \\
\leq C_N \int_0^t \| \partial_t^{i-1} (v_q - v_{q-1}) \|_{H^{N-i+1}(M)}^2 d\tau \tag{5.25}
\]

Similar estimate is right to the other three terms of (5.24). Then using (5.24)(5.25), one has,
\[
\sum_{i=0}^{N} \| \partial_t^i (v_{q+1} - v_q) \|^2_{H^{N-i}(M)} \leq C_N \int_0^t \sum_{i=0}^{N-1} \| \partial_t^i (v_q - v_{q-1}) \|^2_{H^{N-i}(M)} \tag{5.26}
\]

Denote \(h_{q+1} = \sum_{i=0}^{N-1} \| \partial_t^i (v_{q+1} - v_q) \|^2_{H^{N-i}(M)}\), the above equation gives \(h_{q+1}(t) \leq C_N \int_0^t h_q(t) d\tau\), and denote \(C(T) = \| h_2 \|_{L^\infty[0,T]}\). Then by iteration, one can obtain \(h_{q+1}(t) \leq C(T) \frac{C_N t^{q-1}}{(q-1)!}\). This implies:
\[
\sum_{i=0}^{N} \| \partial_t^i (v_{q+1} - v_q) \|^2_{H^{N-i}(M)} \leq C(T) \frac{(C_N t)^{q-1}}{(q-1)!} \tag{5.27}
\]
Then, \( \{v_\gamma\}_{\gamma=0}^\infty \) convergence to some \( v \) in \( \bigcap_{i=0}^N W^{i,\infty}(I, H^{N-i}(M)) \), and \( v \) satisfy (5.22) with the condition \( v(0) = \psi, v'(0) = f(0) - a(0)\psi + \alpha(0)\Delta \varphi + < \nabla \beta(0), \nabla \varphi > + \gamma(0)\varphi \). Hence, \( u \in \bigcap_{i=0}^N W^{i+1,\infty}(I, H^{N-i}(M)) \). By the \( W^{k,p} \) estimate of elliptic equation, and (1.3), one has \( u \in \bigcap_{i=0}^{N+1} W^{i,\infty}(I, H^{N+1-i}(M)) \), and \( v \) satisfies,

\[
\sum_{i=0}^N \| \partial_t^i v \|^2_{H^{N-i}(M)} \leq C_N[\| \varphi \|^2_{H^{N+1}(M)} + \| \psi \|^2_{H^N(M)} + \| f(0) \|^2_{H^{N-1}(M)} + \int_0^t \sum_{i=1}^{N-1} \| \partial_t^i f \|^2_{H^{N-i-1}(M)} d\tau + \int_0^t \sum_{i=1}^{N-1} \sum_{j=0}^i (\| \partial_t^j v \|^2_{H^{N-i-1}(M)} + \| \partial_t^j (\varphi + \int_0^t v d\tau') \|^2_{H^{N-i+1}(M)}) d\tau] 
\]

(5.28)

Integrate it, and \( v = u_t \), one has,

\[
\sum_{i=0}^N \| \partial_t^{i+1} u \|^2_{H^{N-i}(M)} \leq C_N[\| \varphi \|^2_{H^{N+1}(M)} + \| \psi \|^2_{H^N(M)} + \| f(0) \|^2_{H^{N-1}(M)} + \int_0^t \sum_{i=1}^N \| \partial_t^i f \|^2_{H^{N-i}(M)} d\tau] 
\]

(5.29)

and,

\[
\| f \|^2_{H^{N-1}(M)} \leq \int_0^t \| f_t \|^2_{H^{N-1}(M)} d\tau + \| f(0) \|^2_{H^{N-1}(M)} 
\]

(5.30)

Using (1.3), the \( W^{k,p} \) estimate of elliptic equation, (5.29)(5.30), one can obtains the inequality (5.21) in the case of \( N + 1 \). This complete the proof.

**Proposition 5.3** The hypothesis of \( M, I \) as above. Suppose \( f, a, \alpha, \beta, \gamma \in C^\infty(M \times I) \), \( \varphi, \psi \in C^\infty(M) \). Then (1.3)(5.1) have a unique smooth solution and it satisfies (5.21).

**Proof** It is obvious from Lemma 5.1 and 5.2.
6 Solution of Equation in Some Cases

The idea of this section can be found in [H"{o}mander],[John],[Klainerman]. Suppose \((M, g)\) be \(m\)-dimensional closed Riemannian manifold, \(I = [0, T]\) is an interval of length \(T\). By Proposition 5.3, one can introduce the Picard iterate sequence \(\{u_n\}_{n=1}^{\infty}\) as follows,

\[
m(u_{n+1})_{tt} - e^{-2(m+1)u_n} \Delta u_{n+1} = F(u_{n+1}, u_n)
\]  \(  \tag{6.1} \)

with the condition \(u_{n+1}(x, 0) = \varphi(x), (u_{n+1})_t(x, 0) = \psi(x), \varphi, \psi \in C^\infty(M), u_1 = 0\). Here, \(F\) is according to (4.1). Let \(s > \frac{m}{2} + 1\) and we suppose that exist \(\tilde{T} \leq T\) such that in \([0, \tilde{T}]\) exist positive constant \(D\) satisfying, to any \(n\),

\[
\|u_n\|_{H^{s+1}(M)} + \|(u_n)_t\|_{H^{s}(M)} \leq D
\]  \(  \tag{6.2} \)

Let

\[
G(u, v) = [1 - e^{2(m+1)(u-v)}](R_g + 2\Delta u) - (m^2 + m + 2)\nabla(u - v), \nabla u > + (1 - e^{2(m+1)(u-v)}) < \nabla v, \nabla u > - m(3m + 1)e^{2(m+1)u}u_t(u - v)_t
\]  \(  \tag{6.3} \)

Then,

\[
m(u_{n+1} - u_n)_{tt} - e^{-2(m+1)u_n} \Delta (u_{n+1} - u_n)
= F(u_{n+1}, u_n) - F(u_n, u_{n-1}) + e^{-2(m+1)u_n} - e^{-2(m+1)u_{n-1}} \Delta u_n
= \frac{1}{2}\{ -e^{-2mu_n} (1 - e^{2m(u_n - u_{n-1})}) \tilde{R} + e^{-2(m+1)u_n} G(u_n, u_{n-1})
- (m^2 + m + 2)e^{-2(m+1)u_n} < \nabla u_n, \nabla (u_{n+1} - u_n) > \\
- m(3m + 1)(u_n)_t(u_{n+1} - u_n)_t \tag{6.4} \)

with the condition that \((u_{n+1} - u_n)(x, 0) = (u_{n+1} - u_n)_t(x, 0) = 0\). So, we replace \(u, v, \tilde{R}, R_g\) by \(u_{n+1} - u_n, u_n, (1-e^{2m(u_n - u_{n-1})}) \tilde{R}, G(u_n, u_{n-1})\) respectively in (4.1), using (4.9), one has,

\[
A_s = \|(1 - e^{2m(u_n - u_{n-1})}) \tilde{R}\|_{H^{s-1}(M)} + \|G(u_n, u_{n-1})\|_{H^{s-1}(M)}
\]  \(  \tag{6.5} \)

and using (6.2), and Corollary 2.4, and Sobolev embedding theorems,

\[
\|(1 - e^{2m(u_n - u_{n-1})}) \tilde{R}\|_{H^{s-1}(M)} \\
\leq C\|1 - e^{2m(u_n - u_{n-1})}\|_{H^{s-1}(M)} \|\tilde{R}\|_{H^{s-1}(M)} \\
\leq Ce^{C\|u_n - u_{n-1}\|_{H^{s-1}(M)}} \|u_n - u_{n-1}\|_{H^{s-1}(M)} \|\tilde{R}\|_{H^{s-1}(M)} \\
\leq Ce^{CD}\|u_n - u_{n-1}\|_{H^{s-1}(M)} \|\tilde{R}\|_{H^{s-1}(M)}
\]  \(  \tag{6.6} \)
and similar,
\[
\|G(u_n, u_{n-1})\|_{H^{s-1}(M)} \\
\leq C e^{CD} \|u_n - u_{n-1}\|_{H^{s-1}(M)} (\|R_g\|_{H^{s-1}(M)} + D) + CD \|u_n - u_{n-1}\|_{H^s(M)} \\
+ C e^{CD} \|u_n - u_{n-1}\|_{H^{s-1}(M)} D^2 + CD(1 + e^{CD}) \|(u_n - u_{n-1})_t\|_{H^{s-1}(M)}
\]
(6.7)

Then, by (6.5)-(6.7),
\[
A_s \leq C e^{CD} (D + \|\tilde{R}\|_{H^{s-1}(M)} + \|R_g\|_{H^{s-1}(M)}) \\
\times (\|u_n - u_{n-1}\|_{H^s(M)} + \|(u_n - u_{n-1})_t\|_{H^{s-1}(M)})
\]
(6.8)

Then, using (4.33) and (6.2),
\[
\frac{d}{dt}(E_s(t))^{1/2} \leq C e^{CD} (A_s + (E_s(t))^{1/2})
\]
(6.9)

here, do correspond replacement which is stated above in \(E_s(t)\). Then, integrate (6.9), and using (6.8), one obtains
\[
\|u_n - u_{n-1}\|_{H^s(M)} + \|(u_n - u_{n-1})_t\|_{H^{s-1}(M)} \\
\leq C_4 e^{C_6 D T} (D + \|\tilde{R}\|_{C^0(I, H^{s-1}(M))} + \|R_g\|_{H^{s-1}(M)}) \\
\times \int_0^t (\|u_n - u_{n-1}\|_{H^s(M)} + \|(u_n - u_{n-1})_t\|_{H^{s-1}(M)})
\]
(6.10)

here, \(C_4, C_5, C_6\) only relates to manifold \(M\) and \(s\).

**Proposition 6.1** Suppose \((M, g)\) be \(m\)-dimensional closed Riemannian manifold, \(I = [0, T]\) be an interval of length \(T\), and \(\tilde{R} \in C^\infty(M \times I)\). To any integer \(k \geq 2\), exist \(0 < t_0 \leq T\), such that equation (1.2) has a solution \(u \in C^k(M \times [0, t_0])\), and it satisfies \(u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x)\). Here, \(\varphi(x), \psi(x) \in C^\infty(M)\) are two prescribed functions. And \(t_0\) relates to manifold \(M, k, \tilde{R}, \varphi, \psi\).

**Proof** Take \(s = \left\lfloor \frac{n}{2} + k + 1 \right\rfloor\) (i.e. the biggest integer not bigger than \(\frac{n}{2} + k + 1\)), then \(H^s(M) \hookrightarrow C^k(M)\). By Proposition 4.1(i), to any given \(\varphi, \psi\), take \(D\) to satisfies
\[
D > 2\sqrt{2} [1 + \left( \sum_{l=1}^{s+1} \frac{1}{2} \int_M \|\nabla^l \varphi\|^2 + m e^{2(m+1)\varphi} |\nabla^{l-1} \psi|^2 + e^{2(m+1)\varphi} |\nabla^l \varphi|^2 \\
+ m |\nabla^{l-1} \psi|^2] dV + \frac{1}{2} \int_M |\varphi|^2 dV \]
(6.11)
then exist $t_1$, in $[0, t_1]$, (4.34) implies (4.35). To $\{ u_n \}_{n=1}^\infty$ which is constructed in (6.1), obvious $D \geq \| u_1 \|_{H^{s+1}(M)} + \| (u_1)_t \|_{H^s(M)} = 0$, so by iteration, one has (6.2). Then take $t_0 \leq t_1$ small enough such that,

$$C_4e^{C_5e^{C_6DT}}(D + \| \tilde{R} \|_{H^{s-1}(M)} + \| R_g \|_{H^{s-1}(M)})t_0 < 1 \quad (6.12)$$

$C_4, C_5, C_6$ are constants in (6.10). Then using (6.10),

$$\| u_n - u_{n-1} \|_{C^0([0,t_0], H^s(M))} + \| (u_n - u_{n-1})_t \|_{C^0([0,t_0], H^{s-1}(M))} \leq C_4e^{C_5e^{C_6DT}}(D + \| \tilde{R} \|_{C^0(I, H^{s-1}(M))} + \| R_g \|_{H^{s-1}(M)})t_0$$

$$\times (\| u_n - u_{n-1} \|_{C^0([0,t_0], H^s(M))} + \| (u_n - u_{n-1})_t \|_{C^0([0,t_0], H^{s-1}(M))}) \quad (6.13)$$

This implies $\{ u_n \}_{n=1}^{\infty}$ and $\{ (u_n)_t \}_{n=1}^{\infty}$ are convergence to some $u, u_t$ in $C^0([0, t_0], H^s(M)), C^0([0, t_0], H^{s-1}(M))$ respectively. Obvious $u_n \rightarrow u$ in $C^0([0, t_0], H^s(M)) \cap C^1([0, t_0], H^{s-1}(M))$. By equation (6.1) $(u_n)_t \rightarrow u_{tt}$ in $C^0([0, t_0], C^{k-2}(M))$. Then $u \in C^0([0, t_0], C^{k-2}(M))$ and $u$ satisfies equation (1.2). Take derivation to $t$ of first order, one can finds $u \in C^3([0, t_0], C^{k-3}(M))$. Then it-}

erate this step shows the result holds.

**Proposition 6.2** Suppose $(M, g)$ be $m$-dimensional closed Riemannian manifold which is scalar curvature flat $R_g = 0$. And $I = [0, T]$ be an interval of length $T$, and $\tilde{R} \in C^\infty(M \times I)$. To any integer $k \geq 2$, exist $\varepsilon > 0$, such that equation (1.2) has a solution $u \in C^k(M \times I),$ and it satisfies $u(x, 0) = 0, u_t(x, 0) = 0$ when $\| \tilde{R} \|_{H^{[\frac{m}{2} + k](M)}} \leq \varepsilon$. Here, $\varepsilon$ only relates to manifold $M, k, I$.

**Proof** Similar to Proposition 6.1, take $s = \left[ \frac{m}{2} + k + 1 \right]$ then $H^s(M) \hookrightarrow C^k(M)$. One can take $D, \varepsilon$ small enough such that,

$$C_4e^{C_5e^{C_6DT}}(D + \varepsilon) < 1 \quad (6.14)$$

$C_4, C_5, C_6$ are constants in (6.10) and Proposition 4.2 holds. Then (4.44) implies (4.45) and the construction of $\{ u_n \}_{n=1}^{\infty}$ in (6.1), one finds (6.2). This implies (6.10), namely,

$$\| u_n - u_{n-1} \|_{C^0(I, H^s(M))} + \| (u_n - u_{n-1})_t \|_{C^0(I, H^{s-1}(M))} \leq C_4e^{C_5e^{C_6DT}}(D + \varepsilon)(\| u_n - u_{n-1} \|_{C^0(I, H^s(M))} + \varepsilon)$$

$$\times (\| u_n - u_{n-1} \|_{C^0(I, H^{s-1}(M))} + \| (u_n - u_{n-1})_t \|_{C^0(I, H^{s-1}(M))}) \quad (6.15)$$

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Then, similar argument of Proposition 6.1 gives the result by (6.14)(6.15).

At the end we discuss the uniqueness of solution in the above two Proposition. This is from the next result.

**Proposition 6.3** Suppose \((M, g)\) be m-dimensional closed Riemannian manifold, \(I = [0, T]\) be an interval of length \(T\), and \(\bar{R} \in C^\infty(M \times I)\). Let \(0 \leq \bar{T} \leq T\). If equation (1.2) has a solution \(u \in C^{k+2}(M \times [0, \bar{T}])\), and \(u\) satisfies \(u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x)\), then the solution is unique. Here, \(\varphi(x), \psi(x) \in C^\infty(M)\) are two prescribed functions, and \(k > \frac{m}{2}, k \in \mathbb{Z}\).

**Proof** Suppose \(u, v \in C^{k+1}(M \times [0, \bar{T}])\) are two different solutions of (1.2) with the same initial data. So, exist \(t_{min}, 0 < t_{min} < \bar{T}\) and in \([0, t_{min}], u = v\), but \(u \neq v\) when \(t > t_{min}\). Let \(\bar{u}(x, t) = u(x, t + t_{min}), \bar{v}(x, t) = v(x, t + t_{min})\), then \(\bar{u}(x, 0) = \bar{v}(x, 0), \bar{u}_t(x, 0) = \bar{v}_t(x, 0)\). And \(\bar{u}, \bar{v}\) also satisfy (1.2), differ the corresponding equation, one has

\[
(\bar{u} - \bar{v})_{tt} - e^{-2(m+1)\bar{u}} \Delta (\bar{u} - \bar{v})
= \frac{1}{2} \{-e^{-2m\bar{u}}(1 - e^{2m(\bar{u} - \bar{v})})\bar{R} + e^{-2(m+1)\bar{u}} H(\bar{u}, \bar{v})
- (m^2 + m + 2)e^{-2(m+1)\bar{u}} < \nabla \bar{u}, \nabla (\bar{u} - \bar{v}) > -m(3m + 1)\bar{u}_t(\bar{u} - \bar{v})_t\}
\]

(6.16)

where,

\[
H(\bar{u}, \bar{v}) = [1 - e^{2(m+1)(\bar{u} - \bar{v})}](R_g + 2\triangle \bar{v}) - (m^2 + m + 2)< \nabla (\bar{u} - \bar{v}), \nabla \bar{v} >
+ (1 - e^{2(m+1)(\bar{u} - \bar{v})}) < \nabla \bar{v}, \nabla \bar{v} > -m(3m + 1)e^{2(m+1)\bar{u}} \bar{v}_t(\bar{u} - \bar{v})_t,
\]

(6.17)

Similar to (6.2)-(6.10) and using Proposition 4.1(ii), one obtains,

\[
\|\bar{u} - \bar{v}\|_{H^k(M)} + \|(\bar{u} - \bar{v})_t\|_{H^{k-1}(M)}
\leq C \int_0^t (\|\bar{u} - \bar{v}\|_{H^k(M)} + \|(\bar{u} - \bar{v})_t\|_{H^{k-1}(M)})d\tau
\]

(6.18)

holds in \([0, t_0]\), and \(t_0\) is small enough. Constant \(C\) and \(t_0\) relates to manifold \(M, k, \bar{R}, \bar{u}(0), \bar{u}_t(0)\). (6.18) implies \(u = v\) when \(t_0\) more small such that \(Ct_0 < 1\). This is a contradiction. So, \(t_{min} = \bar{T}\).
Proposition 6.1-6.3 imply Theorem 1.1 and 1.2.

At last, we suggest some questions which maybe is nature to ask:
(1) Replace $I$ by $S^1$ (i.e. the unit cycle), what is the sufficient and necessary condition on which equation (1.2) exists solution?
(2) On what condition, equation (1.2) exists solution on $M \times \mathbb{R}^1$?
(3) How to solve this question of the higher dimension version, namely solve equation (3.31)?
(4) Can this question be generalized to a fiber bundle over a base manifold $M$? What is the corresponded equation?

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