The Homotopy Class of twisted $L_\infty$-morphisms

Andreas Kraft;

Dipartimento di Matematica
Università degli Studi di Salerno
via Giovanni Paolo II, 123
84084 Fisciano (SA)
Italy

Jonas Schnitzer†

Department of Mathematics
University of Freiburg
Ernst-Zermelo-Straße, 1
D-79104 Freiburg
Germany

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Abstract
The global formality of Dolgushev depends on the choice of a torsion-free covariant derivative. We prove that the globalized formalities with respect to two different covariant derivatives are homotopic. More explicitly, we derive the statement by proving a more general homotopy equivalence between $L_\infty$-morphisms that are twisted with gauge equivalent Maurer-Cartan elements.

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†jonas.schnitzer@math.uni-freiburg.de
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1 Introduction

The celebrated formality theorem by Kontsevich [15] provides the existence of an \( L_\infty \)-quasi-isomorphism from the differential graded Lie algebra (DGLA) of polyvector fields \( T_{\text{poly}}(\mathbb{R}^d) \) to the DGLA of polydifferential operators \( D_{\text{poly}}(\mathbb{R}^d) \). In \[6,7\] Dolgushev globalized this result to general smooth manifolds \( M \) using a geometric approach. Being a quasi-isomorphism, this formality induces a bijective correspondence

\[
U : \text{Def}(T_{\text{poly}}(M)[[\hbar]]) \rightarrow \text{Def}(D_{\text{poly}}(M)[[\hbar]])
\]

between equivalence classes \( \text{Def}(T_{\text{poly}}(M)[[\hbar]]) \) of formal Poisson structures \( \hbar \pi \in \Gamma^\infty(\Lambda^2TM)[[\hbar]] \) on \( M \) and equivalence classes \( \text{Def}(D_{\text{poly}}(M)[[\hbar]]) \) of star products \( \ast \) on \( M \), see also \[14,15\] for more details on deformation theory. In particular, this associates to a classical Poisson structure \( \pi_{\text{cl}} \) a class of deformation quantizations \( U([\hbar \pi_{\text{cl}}]) \) in the sense of the seminal paper \[1\]. On the other hand, it also gives a way to assign to each star product a class of formal Poisson structures, the so-called Kontsevich class of the star product.

However, the above mentioned globalization procedure of the Kontsevich formality from \( \mathbb{R}^d \) to a general manifold \( M \) discussed in \[6\] depends on the choice of a torsion-free covariant derivative. More explicitly, it uses the covariant derivative to obtain Fedosov resolutions of the polyvector fields and polydifferential operators between which one has a fiberwise Kontsevich formality. Recently, in \[2\] Theorem 2.6 it has been shown that the map \( U \) from (1.1) does not depend on the choice of the connection. In this paper we investigate the role of the covariant derivative at the level of the formality and not at the level of equivalence classes of Maurer-Cartan elements.

The key point is that changing the covariant derivative corresponds to twisting by a Maurer-Cartan element that is equivalent to zero, see \[2\] Appendix C for this observation and \[9,10,11,13\] for more details on the twisting procedure. This corresponds to a more general observation: Let \( F : (g,d,[\cdot,\cdot]) \rightarrow (g',d',[\cdot,\cdot]) \) be an \( L_\infty \)-morphism between DGLAs with complete descending and exhaustive filtrations \( F^*g \) resp. \( F^*g' \). Moreover, let \( \pi \in F^1g^1 \) be a Maurer-Cartan element equivalent to zero via \( \pi = \exp([g,\cdot]) \circ 0 \) with \( g \in F^1g^0 \). The element \( \pi' = \sum_{k=1}^\infty \frac{1}{k!} F_k^1(\pi \vee \cdots \vee \pi) \in F^1g^1 \) is a Maurer-Cartan element in \( g' \) equivalent to zero. Let the equivalence be given by \( g' \in F^1g^0 \), then one obtains Proposition 3.10.

**Proposition** The \( L_\infty \)-morphisms \( F \) and \( e^{1-\delta g \cdot \cdot} \circ F^\pi \circ e^{1-\delta g' \cdot \cdot} \) from \( (g,d,[\cdot,\cdot]) \) to \( (g',d',[\cdot,\cdot]) \) are homotopic, where \( F^\pi \) denotes the \( L_\infty \)-morphism \( F \) twisted by \( \pi \).

By homotopic we mean here that the two \( L_\infty \)-morphisms are equivalent Maurer-Cartan elements in the convolution DGLA, compare \[5\] Definition 3], see also \[9\] for a comparison of different notions of homotopies between \( L_\infty \)-morphisms.

This general statement can be applied to the globalization of the Kontsevich formality. Our main result here is the following theorem, see Theorem 4.12.

**Theorem** Let \( \nabla \) and \( \nabla' \) be two different torsion-free covariant derivatives. Then the two global formalities constructed via Dolgushev’s globalization procedure are homotopic.

This immediately implies that they induce the same map on the equivalence classes of formal Maurer-Cartan elements, i.e. \[2\] Theorem 2.6).

Note that there are many other similar globalization procedures of formalities based on Dolgushev’s globalization of the Kontsevich formality \[5,7\], e.g. \[3\] for Lie algebroids, \[16\] for differential graded manifolds and \[5\] for Hochschild chains. The above technique can be adapted to these cases and we plan to pursue them in further works.

Finally, we want to mention that in \[13\] Section 7 there is also a globalization procedure explained, using the language of \( \infty \)-jet spaces of polyvector fields and polydifferential operators, respectively. However, these \( \infty \)-jet spaces are (non-canonically) isomorphic as vector bundles to...
the formally completed fiberwise polyvector fields and polydifferential operators, respectively. The corresponding isomorphisms are constructed by the choice of a connection. We strongly believe that the globalization procedure proposed by Kontsevich in [15] is homotopy to the globalization from Dolgushev [5, 6] we are using in this note.

The paper is organized as follows: In Section 2 we recall the basics concerning Maurer-Cartan elements in DGLAs and \( L_\infty \)-algebras, the notions of gauge and homotopy equivalence as well as the twisting procedure. Then we recall in Section 3 the interpretation of \( L_\infty \)-morphisms as Maurer-Cartan elements and the notion of homotopic \( L_\infty \)-morphisms. We show that pre- and post-compositions of homotopic \( L_\infty \)-morphisms with an \( L_\infty \)-morphism are again homotopic, a statement that is probably well-known to the experts, but that we could not find in the literature. Moreover, we prove here Proposition 3.10, i.e. that the twisted \( L_\infty \)-morphisms are homotopic for equivalent Maurer-Cartan elements. Finally, we apply these general results to the globalization of Kontsevich’s formality theorem, proving Theorem 4.12 and also an equivariant version for Lie group actions with invariant covariant derivatives.

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2 Preliminaries: Maurer-Cartan Elements and Twisting

2.1 Maurer-Cartan Elements in DGLAs

We want to recall the basics concerning differential graded Lie algebras (DGLAs), Maurer-Cartan elements and their equivalence classes. In order to make sense of the gauge equivalence we consider in this context DGLAs \((\mathfrak{g}^*, d, [\cdot, \cdot])\) with complete descending filtrations

\[
\cdots \supseteq \mathcal{F}^{-2}\mathfrak{g} \supseteq \mathcal{F}^{-1}\mathfrak{g} \supseteq \mathcal{F}^{0}\mathfrak{g} \supseteq \mathcal{F}^{1}\mathfrak{g} \supseteq \cdots, \quad \mathfrak{g} \cong \varprojlim_{n} \mathfrak{g}/\mathcal{F}^{n}\mathfrak{g}
\]

(2.1)

and

\[
d(\mathcal{F}^{k}\mathfrak{g}) \subseteq \mathcal{F}^{k+1}\mathfrak{g} \quad \text{and} \quad [\mathcal{F}^{k}\mathfrak{g}, \mathcal{F}^{\ell}\mathfrak{g}] \subseteq \mathcal{F}^{k+\ell}\mathfrak{g}.
\]

(2.2)

In particular, \(\mathcal{F}^{1}\mathfrak{g}\) is a projective limit of nilpotent DGLAs. In most cases the filtration will be bounded below, i.e. bounded from the left with \(\mathfrak{g} = \mathcal{F}^{k}\mathfrak{g}\) for some \(k \in \mathbb{Z}\). If the filtration is unbounded, then we assume always that it is in addition exhaustive, i.e. that

\[
\mathfrak{g} = \bigcup_{n} \mathcal{F}^{n}\mathfrak{g},
\]

(2.3)

even if we do not mention it explicitly. Moreover, we assume that the DGLA morphisms are compatible with the filtrations.

Example 2.1 One motivation to consider the case of filtered DGLAs are formal power series \(\mathfrak{g}[[\hbar]]\) of a DGLA \(\mathfrak{g}\) with filtration \(\mathcal{F}^{k}(\mathfrak{g}[[\hbar]]) = \hbar^{k}(\mathfrak{g}[[\hbar]])\).

Definition 2.2 (Maurer-Cartan elements) Let \((\mathfrak{g}, d, [\cdot, \cdot])\) be a DGLA with complete descending filtration. Then \(\pi \in \mathcal{F}^{1}\mathfrak{g}^{1}\) is called Maurer-Cartan element if it satisfies the Maurer-Cartan equation

\[
d\pi + \frac{1}{2}[\pi, \pi] = 0.
\]

(2.4)

The set of Maurer-Cartan elements is denoted by \(\text{MC}(\mathfrak{g})\).

Maurer-Cartan elements \(\pi\) lead to twisted DGLA structures \((\mathfrak{g}, d + [\pi, \cdot], [\cdot, \cdot])\) and one has a gauge action on the set of Maurer-Cartan elements.
Proposition 2.3 (Gauge action) Let \((g, d, \cdot, \cdot)\) be a DGLA with complete descending filtration. The gauge group \(G^0(g) = \{ \Phi = e^{g \cdot} : g \longrightarrow g \mid g \in \mathfrak{g}^0 \}\) defines an action on \(\text{MC}(g)\) via
\[
\exp([g, \cdot]) \circ \pi = \sum_{n=0}^{\infty} \left( \frac{([g, \cdot])^n}{n!} \right) (\pi) - \sum_{n=0}^{\infty} \left( \frac{([g, \cdot])^n}{(n+1)!} \right) (dg) = \pi - \frac{\exp([g, \cdot]) - \text{id}}{[g, \cdot]} (dg + [\pi, g]).
\] (2.5)

The set of equivalence classes of Maurer-Cartan elements in \(g\) is denoted by
\[
\text{Def}(g) = \frac{\text{MC}(g)}{G^0(g)}.
\] (2.6)

Note that the gauge action is well-defined since \(g \in \mathfrak{g}^0\) and as the filtration is complete. \(\text{Def}(g)\) is the transformation groupoid of the gauge action and also called \(\text{Goldman-Millson groupoid} \ [14]\). It plays an important role in deformation theory \([18]\). In particular, the definition implies that twisting with gauge equivalent Maurer-Cartan elements leads to isomorphic DGLAs.

Corollary 2.4 Let \((g, d, \cdot, \cdot)\) be a DGLA with complete descending filtration and with gauge equivalent Maurer-Cartan elements \(\pi', \pi\) via \(g \in G^0(g)\). Then one has
\[
d + [\pi', \cdot] = \exp([g, \cdot]) \circ (d + [\pi, \cdot]) \circ \exp([-g, \cdot]).
\] (2.7)

In other words, \(\exp([g, \cdot]) : (g, d + [\pi, \cdot], \cdot, \cdot) \rightarrow (g, d + [\pi', \cdot], \cdot, \cdot)\) is an isomorphism of DGLAs.

2.2 Maurer-Cartan Elements in \(L_\infty\)-algebras

Let us recall the basics of \(L_\infty\)-algebras and \(L_\infty\)-morphisms. Proofs and further details can be found in \([3],[7],[11]\). Note that in this work we only consider \(L_\infty\)-morphisms between DGLAs.

An \(L_\infty\)-algebra \((L, Q)\) is a graded vector space \(L\) together with a degree \(+1\) codifferential \(Q\) on the graded cocommutative cofree coalgebra \((S(L[1]), \Delta)\) without counit cogenerated by \(L[1]\). We always consider a vector space over a field \(K\) of characteristic zero. The codifferential \(Q\) is uniquely determined by the Taylor components \(Q_n : S^n(L[1]) \rightarrow L[2]\) for \(n \geq 1\). Sometimes we also write \(Q_n = Q_n^L\) and following \([11]\) we denote by \(Q_n^r\) the component of \(Q_n^S : S^n(L[1]) \rightarrow S^n(L[1])[1]\) of \(Q\). The property \(Q^2 = 0\) implies in particular that \(Q_1^n : L \rightarrow L[1]\) is a cochain differential. Let us consider two \(L_\infty\)-algebras \((L, Q)\) and \((L', Q')\). A degree 0 coalgebra morphism \(F : S(L[1]) \rightarrow S(L'[1])\) such that \(FQ = Q'F\) is called \(L_\infty\)-morphism. Just like the codifferential also the morphism \(F\) is also uniquely determined by its Taylor components \(F_n : S^n(L[1]) \rightarrow S^n(L'[1])\), where \(n \geq 1\). We write again \(F_k = F_k^L\) and we get coefficients \(F_k^S : S^k(L[1]) \rightarrow S^k(L'[1])\) of \(F\). Note that \(F^S_k\) depends only on \(F^L_k\) for \(k \leq n - j + 1\). In particular, the first structure map of \(F\) is a map of complexes \(F_1 : (L, Q_1^L) \rightarrow (L', Q_1^L)\) and one calls \(F\) \(L_\infty\)-quasi-isomorphism if \(F_1^S\) is a quasi-iso morphism of complexes.

Example 2.5 (DGLA) A DGLA \((g, d, \cdot, \cdot)\) is an \(L_\infty\)-algebra with \(Q_1 = -d\) and \(Q_2(\gamma \vee \mu) = -(-1)^{\gamma|\mu} (\gamma, \mu)\), where \(|\gamma|\) denotes the degree in \(g[1]\).

In order to generalize the definition of Maurer-Cartan elements we consider again \(L_\infty\)-algebras with complete descending and exhaustive filtrations on \(L\). We assume again that \(L_\infty\)-morphisms are compatible with the filtrations.

Definition 2.6 (Maurer-Cartan elements II) Let \((L, Q)\) be an \(L_\infty\)-algebra with compatible complete descending filtration. Then \(\pi \in \mathfrak{g}^1 L[1]^0\) is called Maurer-Cartan element if it satisfies the Maurer-Cartan equation
\[
\sum_{n>0} \frac{1}{n!} Q_n(\pi \vee \cdots \vee \pi) = 0.
\] (2.8)

The set of Maurer-Cartan elements is again denoted by \(\text{MC}(L)\).

Note that the sum in (2.8) is well-defined for \(x \in \mathfrak{g}^1 L[1]\) because of the completeness of \(L\). We recall some useful properties from \([7]\) Prop. 1:
Lemma 2.7 Let $F: (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be an $L_\infty$-morphism of DGLAs and $\pi \in \mathfrak{F}^1\mathfrak{g}'$.

i.) $d\pi + \frac{1}{2} [\pi, \pi] = 0$ is equivalent to $Q(\exp(\pi)) = 0$, where $\exp(\pi) = \sum_{k=1}^\infty \frac{1}{k!} \pi^k$.

ii.) $F(\exp(\pi)) = \exp(\pi)$ with $S = F^1(\exp(\pi)) = \sum_{n>0} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi)$.

iii.) If $\pi$ is a Maurer-Cartan element, then so is $S$.

We recall the generalization of the gauge action to an equivalence relation on the set of Maurer-Cartan elements of $L_\infty$-algebras. We follow [4] Section 4 but adapt the definitions to the case of $L_\infty$-algebras with complete descending and exhaustive filtrations as in [9]. Let therefore $(L, Q)$ be such an $L_\infty$-algebra with complete descending and exhaustive filtration and consider $L[t] = L \otimes \mathbb{K}[t]$ which has again a descending and exhaustive filtration

$$\mathcal{F}^k L[t] = \mathcal{F}^k L \otimes \mathbb{K}[t].$$

We denote its completion by $\hat{L}[t]$ and note that since $Q$ is compatible with the filtration it extends to $\hat{L}[t]$. Similarly, $L_\infty$-morphisms extend to these completed spaces.

Remark 2.8 Note that one can define the completion as space of equivalence classes of Cauchy sequences with respect to the filtration topology. Alternatively, the completion can be identified with

$$\lim_{\xi} L[t]/\mathcal{F}^n L[t] = \prod_n L[t]/\mathcal{F}^n L[t] \cong \prod_n L/\mathcal{F}^n L \otimes \mathbb{K}[t]$$

consisting of all coherent tuples $X = (x_n)_n \in \prod_n L[t]/\mathcal{F}^n L[t]$, where

$$L[t]/\mathcal{F}^n L[t] \ni x_{n+1} \mapsto x_n \in L[t]/\mathcal{F}^n L[t]$$

under the obvious surjections. Moreover, $\mathcal{F}^n \hat{L}[t]$ corresponds to the kernel of $\lim_{\xi} L[t]/\mathcal{F}^n L[t] \to L[t]/\mathcal{F}^n L[t]$ and thus

$$\hat{L}[t]/\mathcal{F}^n \hat{L}[t] \cong L[t]/\mathcal{F}^n L[t].$$

Since $L$ is complete, we can also interpret $\hat{L}[t]$ as the subspace of $L[[t]]$ such that $X \mod \mathcal{F}^n L[[t]]$ is polynomial in $t$. In particular, $\mathcal{F}^n \hat{L}[t]$ is the subspace of elements in $\mathcal{F}^n L[[t]]$ that are polynomial in $t$ modulo $\mathcal{F}^m L[[t]]$ for all $m$.

By the above construction of $\hat{L}[t]$ it is clear that differentiation $\frac{d}{dt}$ and integration with respect to $t$ extend to it since they do not change the filtration. Sometimes we write also $X$ instead of $\frac{d}{dt} X$ and, moreover, the evaluation

$$\delta_s: \hat{L}[t] \ni X \mapsto X(s) = X|_{t=s} \in L$$

is well-defined for all $s \in \mathbb{K}$ since $L$ is complete.

Example 2.9 In the case that the filtration of $L$ comes from a grading $L^*$, the completion is given by $\hat{L}[t] \cong \prod_i L^i[t]$, i.e. by polynomials in each degree. A special case is here the case of formal power series $L = V[[h]]$ with $\hat{L}[t] \cong (V[t])[h]$ as in [2] Appendix A].

Now we can introduce a general equivalence relation between Maurer-Cartan elements of $L_\infty$-algebras.

Definition 2.10 (Homotopy equivalence) Let $(L, Q)$ be a $L_\infty$-algebra with a complete descending filtration. The homotopy equivalence relation on the set $MC(L)$ is the transitive closure of the relation $\sim$ defined by: $\pi_0 \sim \pi_1$ if and only if there exist $\pi(t) \in \mathfrak{F}^1\hat{L}[t]$ and $\lambda(t) \in \mathfrak{F}^1\hat{L}^0[t]$ such that

$$\frac{d}{dt} \pi(t) = Q^1(\lambda(t) \vee \exp(\pi(t))) = \sum_{n=0}^\infty \frac{1}{n!} Q^n(\lambda(t) \vee \pi(t) \vee \cdots \vee \pi(t)), \quad \pi(0) = \pi_0, \quad \text{and} \quad \pi(1) = \pi_1.$$  \tag{2.9}

The set of equivalence classes of Maurer-Cartan elements of $L$ is denoted by $Def(L) = MC(L)/\sim$.  

5
Note that in the case of nilpotent $L_\infty$-algebras it suffices to consider polynomials in $t$ as there is no need to complete $L[t]$, compare \[13\]. We check now that this is well-defined and even yields a curve $\pi(t)$ of Maurer-Cartan elements.

**Proposition 2.11** For every $\pi_0 \in F^1 L^1$ and $\lambda(t) \in F^1 L^0[t]$ there exists a unique $\pi(t) \in F^1 L^1[t]$ such that $\frac{d}{dt} \pi(t) = Q^1(\lambda(t) \lor \exp(\pi(t)))$ and $\pi(0) = \pi_0$. If $\pi_0 \in \mathrm{MC}(L)$, then $\pi(s) \in \mathrm{MC}(L)$ for all $s \in \mathbb{K}$.

**Proof:** The proof for the nilpotent case can be found in [4, Prop. 4.8]. In our setting of complete filtrations we only have to show that the solution $\pi(t) = \sum_{k=0}^{\infty} \pi_k t^k$ in the formal power series $F^1 L^1 \otimes \mathbb{K}[t]$ is an element of $F^1 L^1[t]$. By Remark 2.8 this is equivalent to $\pi(t) \bmod F^0 L^1[t] \in L^1[t]$ for all $n$. Indeed, we have inductively

$$
\frac{d}{dt} \pi(t) \bmod F^2 L^1[[t]] = Q^1(\lambda(t)) \bmod F^2 L^1[[t]] \in L^1[1].
$$

For the higher orders we get

$$
\frac{d}{dt} \pi(t) \bmod F^n L^1[[t]] = \sum_{k=0}^{n-1} \frac{1}{k!} Q^1_k(\lambda(t) \lor (\pi(t) + F^0)^{n-1}) \lor \cdots \lor (\pi(t) + F^0)^{n-1}) \mod F^n L^1[[t]]
$$

and thus $\pi(t) \bmod F^n L^1[[t]] \in L^1[t]$. \hfill \Box

One can show that for DGLAs with complete filtrations the two notions of equivalences are equivalent, see e.g. [13, Thm. 5.5].

**Theorem 2.12** Two Maurer-Cartan elements in $(g, d, [\cdot, \cdot])$ are homotopy equivalent if and only if they are gauge equivalent.

This theorem can be rephrased in a more explicit manner in the following proposition.

**Proposition 2.13** Let $(g, d, [\cdot, \cdot])$ be a DGLA with complete descending filtration. Consider $\pi_0 \sim \pi_1$ with equivalence given by $\pi(t) \in F^1 g^0[t]$ and $\lambda(t) \in F^1 g^0[t].$ The formal solution of

$$
\lambda(t) = \exp([A(t), \cdot]) - \frac{\text{id} A(t)}{[A(t), \cdot]} \frac{d}{dt}, \quad A(0) = 0
$$

is an element $A(t) \in F^1 g^0[t]$ and satisfies

$$
\pi(t) = e^{[A(t), \cdot]} \pi_0 - \frac{\exp([A(t), \cdot]) - \text{id}}{[A(t), \cdot]} \frac{d}{dt} A(t).
$$

In particular, for $g = A(1) \in F^1 g^0$ one has

$$
\pi_1 = \exp([g, \cdot]) \circ \pi_0.
$$

**Proof:** As formal power series in $t$ Equation \[2.10\] has a unique solution $A(t) \in F^1 g^0 \otimes \mathbb{K}[[t]].$ But one has even $A(t) \in F^1 g^0[t]$ since

$$
\frac{dA(t)}{dt} \equiv \lambda(t) - \sum_{k=1}^{n-2} \frac{1}{(k+1)!} [A(t), \cdot]^k \frac{dA(t)}{dt} \mod F^n g[[t]]
$$

$$
\equiv \lambda(t) - \sum_{k=1}^{n-2} \frac{1}{(k+1)!} [A(t) \mod F^n g[[t]], \cdot]^k \left( \frac{dA(t)}{dt} \mod F^{n-1} g[[t]] \right) \mod F^n g[[t]]
$$

is by induction polynomial in $t$. Note that one has

$$
\frac{d}{dt} e^{[A(t), \cdot]} = \left[ \frac{\exp([A(t), \cdot]) - \text{id} A(t)}{[A(t), \cdot]} \frac{d}{dt}, \cdot \right] \circ \exp([A(t), \cdot]).
$$

\[\ast\]
Our aim is now to show that \( \pi'(t) = e^{[A(t), \cdot]} \pi_0 - \frac{\exp([A(t), \cdot]) - \text{id}}{[A(t), \cdot]} dA(t) \) satisfies

\[
\frac{d\pi'(t)}{dt} = -d\lambda(t) + \left[ \lambda(t), e^{[A(t), \cdot]} \pi_0 - \frac{\exp([A(t), \cdot]) - \text{id}}{[A(t), \cdot]} dA(t) \right].
\]

Then we know \( \pi'(t) = \pi(t) \in \mathcal{F} g^1[t] \) since the solution \( \pi(t) \) is unique by Proposition 2.11 which immediately gives \( \pi'(1) = \pi_1 \). At first we compute

\[
d\lambda(t) = \exp([A(t), \cdot]) - \text{id} \frac{dA(t)}{dt} + \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} \frac{1}{(k+1)!} \left( \sum_{j=0}^{k} \exp([A(t), \cdot]) \frac{dA(t)}{dt} \right)
\]

and using \([3]\) we get

\[
\frac{d\pi'(t)}{dt} = \left[ \exp([A(t), \cdot]) - \text{id} \frac{dA(t)}{dt}, \exp([A(t), \cdot]) \pi_0 \right] - \frac{\exp([A(t), \cdot]) - \text{id}}{[A(t), \cdot]} \frac{dA(t)}{dt}
\]

\[
= -d\lambda(t) + \left[ \lambda(t), e^{[A(t), \cdot]} \pi_0 - \frac{\exp([A(t), \cdot]) - \text{id}}{[A(t), \cdot]} dA(t) \right]
\]

and the proposition is proven. \( \square \)

**Remark 2.14** There are also different notions of homotopy resp. gauge equivalences for Maurer-Cartan elements in \( L_\infty \)-algebras: e.g. the above definition, sometimes also called Quillen homotopy, and the gauge homotopy where one requires \( \lambda(t) = \lambda \) to be constant, compare [8]. In [7] it is shown that these notions are also equivalent for complete \( L_\infty \)-algebras, extending the result for DGLAs.

One important property is that \( L_\infty \)-morphisms map equivalence classes of Maurer-Cartan elements to equivalence classes, see [3, Prop. 4.9].

**Proposition 2.15** Let \( F : (L, Q) \to (L', Q') \) be an \( L_\infty \)-morphism between \( L_\infty \)-algebras with complete filtrations, and \( \pi_0, \pi_1 \in \text{MC}(L) \) with \( \pi_0 \sim \pi_1 \) via \( \pi(t) \in \mathcal{F} g^1[t] \) and \( \lambda(t) \in \mathcal{F} g^0[t] \). Then \( F \) is compatible with the homotopy equivalence relation, i.e. one has \( F^1(\exp(\pi_0)) \sim F^1(\exp(\pi_1)) \) via

\[
\pi'(t) = F^1(\exp(\pi(t))) \quad \text{and} \quad \lambda'(t) = F^1(\lambda(t) \vee \exp(\pi(t))).
\]

If \( F \) is an \( L_\infty \)-quasi-isomorphism, then it is well-known that it induces a bijection on the equivalence classes of Maurer-Cartan elements. Finally, recall that also the twisting with Maurer-Cartan elements can be generalized to \( L_\infty \)-algebras, see e.g. [3, Section 2.3].

**Lemma 2.16** Let \( (L, Q) \) be an \( L_\infty \)-algebra and \( \pi \in \mathcal{F} L[1]^0 \) a Maurer-Cartan element. Then the map \( Q^\pi \) given by

\[
Q^\pi(X) = \exp((-\pi) \vee Q(\exp(\pi) \vee X)) \quad X \in \mathcal{F}(L[1])
\]

(2.13) defines a codifferential on \( \mathcal{F}(L[1]) \).

One can not only twist the DGLAs resp. \( L_\infty \)-algebras, but also the \( L_\infty \)-morphisms between them. Below we need the following result, see [3, Prop. 2] and [2, Prop. 1].

**Proposition 2.17** Let \( F : (g, Q) \to (g', Q') \) be an \( L_\infty \)-morphism of DGLAs, \( \pi \in \mathcal{F} g^1 \) a Maurer-Cartan element and \( S = F^1(\exp(\pi)) \in \mathcal{F} g^1 \).

i. The map

\[
F^v = \exp(-S v) \exp(\pi v) : \mathcal{F}(g[1]) \to \mathcal{F}(g'[1])
\]

defines an \( L_\infty \)-morphism between the DGLAs \( (g, d + [\pi, \cdot]) \) and \( (g', d + [S, \cdot]). \)
\[ F^\pi_n (x_1, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F^\pi_{n+k} (\pi, \ldots, \pi, x_1, \ldots, x_n). \quad (2.14) \]

iii.) Let \( F \) be an \( L_\infty \)-quasi-isomorphism such that \( F^1 \) is not only a quasi-isomorphism of filtered complexes \( L \to L' \), but even induces a quasi-isomorphism

\[ F^1 : \mathcal{F}^k L \to \mathcal{F}^k L' \]

for each \( k \). Then \( F^\pi \) is an \( L_\infty \)-quasi-isomorphism.

3 Relation between Twisted Morhpisms

Here we prove the main results about the relation between twisted \( L_\infty \)-morphisms. More explicitly, consider an \( L_\infty \)-morphism \( F : (g, Q) \to (g', Q') \) between DGLAs and let \( \pi_0, \pi_1 \in \mathcal{F}^1 g \) be two equivalent Maurer-Cartan elements via \( \pi_1 = \exp([g, \cdot]) \circ \pi_0 \). We show that \( F^\pi_0 \) and \( F^\pi_1 \) can be interpreted as homotopic in the sense of [8, Definition 3].

3.1 \( L_\infty \)-morphisms as Maurer-Cartan Elements

At first, recall that we can interpret \( L_\infty \)-morphisms as Maurer-Cartan elements in the convolution algebra. More explicitly, let \( (L, Q), (L', Q') \) be two \( L_\infty \)-algebras and denote the graded linear maps by \( \text{Hom}(\mathcal{S}(L[1]), L') \). If \( L \) and \( L' \) are equipped with complete descending filtrations, then we require the maps to be compatible with the filtration. The \( L_\infty \)-structures on \( L \) and \( L' \) lead to an \( L_\infty \)-structure on this vector space of maps, see [8] Proposition 1 and Proposition 2] and also [2] for the case of DGLAs.

Proposition 3.1 The coalgebra \( \mathcal{S}(\text{Hom}(\mathcal{S}(L[1]), L')[1]) \) can be equipped with a codifferential \( \check{Q} \) with structure maps

\[ \check{Q}^1 F = Q^1 F - (-1)^{|F|} F \circ Q \]

and

\[ \check{Q}^n (F_1 \vee \cdots \vee F_n) = (Q')^{n-1} \circ (F_1 \circ F_2 \circ \cdots \circ F_n) \circ \overline{\Delta}^{n-1}. \]

It is called convolution \( L_\infty \)-algebra and its Maurer-Cartan elements are identified with \( L_\infty \)-morphisms. Here \( |F| \) denotes the degree in \( \text{Hom}(\mathcal{S}(L[1]), L')[1]) \).

Example 3.2 Let \( g, g' \) be two DGLAs. Then \( \text{Hom}(\mathcal{S}(g[1]), g') \) is in fact a DGLA with differential

\[ \partial F = d' \circ F + (-1)^{|F|} F \circ Q \]

and Lie bracket

\[ [F, G] = -(1)^{|F|}(Q')^1 \circ (F \circ G) \circ \overline{\Delta}. \]

Here \( |F| \) denotes again the degree in \( \text{Hom}(\mathcal{S}(g[1]), g')[1]) \). This DGLA is also called convolution DGLA.

We note that the convolution \( L_\infty \)-algebra \( \mathcal{H} = \text{Hom}(\mathcal{S}(L[1]), L') \) is equipped with the following complete descending filtration:

\[ \mathcal{F}^1 \mathcal{H} \supset \mathcal{F}^2 \mathcal{H} \supset \cdots \supset \mathcal{F}^k \mathcal{H} \supset \cdots \]

\[ \mathcal{F}^k \mathcal{H} = \left\{ f \in \text{Hom}(\mathcal{S}(L[1]), L') \mid f|_{S \leq k(L[1])} = 0 \right\}. \]

Thus all twisting procedures are well-defined and one can define a notion of homotopic \( L_\infty \)-morphisms.

Definition 3.3 Two \( L_\infty \)-morphisms \( F, F' \) from \( (L, Q) \) to \( (L', Q') \) are called homotopic if they are homotopy equivalent Maurer-Cartan elements in the convolution \( L_\infty \)-algebra \( \mathcal{H} \).
We collect a few immediate consequences:

**Proposition 3.4** Let $F,F'$ be two homotopic $L_\infty$-morphisms from $(L,Q)$ to $(L',Q')$.

i.) $F^i_1$ and $(F')^i_1$ are chain homotopic.

ii.) If $F$ is an $L_\infty$-quasi-isomorphism, then so is $F'$.

iii.) If $L = g, L' = g'$ are two DGLAs equipped with complete descending filtrations, then $F$ and $F'$ induce the same maps from $\text{Def}(g)$ to $\text{Def}(g')$.

iv.) In the case of DGLAs $g,g'$, compositions of homotopic $L_\infty$-morphisms with a DGLA morphism of degree zero are again homotopic.

**Proof:** The first three points are proven in [2] and the last one follows directly. \qed

We now aim to generalize the last point of the previous proposition to compositions with $L_\infty$-morphisms. We start with the post-composition:

**Proposition 3.5** Let $F_0, F_1$ be two homotopic $L_\infty$-morphisms from $(L,Q)$ to $(L',Q')$. Let $H$ be an $L_\infty$-morphism from $(L',Q')$ to $(L'',Q'')$, then $HF_0 \sim HF_1$.

**Proof:** For $F \in \text{Hom}(\mathcal{S}(L[1]),L')$ we define $\hat{H}(F)$ via

$$(\hat{H}(F))_n = (HF)_n^i = \sum_{i=1}^{n} H_i F^i_n = H_i^1 \left( \frac{1}{1!} F^1 \lor \cdots \lor F^1 \right) \circ \overline{\Delta}^{i-1}.$$ 

Here the $\lor$-product of maps is given by $F \lor G = \lor \circ (F \otimes G) : \mathcal{S}(L[1]) \otimes \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$. Writing $\overline{\Delta} = \sum_{\alpha=0}^{\infty} \overline{\Delta}^\alpha$ and defining all maps to be zero on the domains on which they where previously not defined, we can rewrite this as

$$(\hat{H}(F)) = H^1 \circ \exp F \circ \overline{\Delta}.$$ 

Let $F(t) \in (\text{Hom}(\mathcal{S}(L[1]),L')[1])^0[1]$ and $\lambda(t) \in (\text{Hom}(\mathcal{S}(L[1]),L')[1])^{-1}[1]$ describe the homotopy equivalence between $F_0$ and $F_1$. Then $\hat{H}F(t) \in (\text{Hom}(\mathcal{S}(L[1]),L'')[1])^{-1}[1]$ satisfies

$$\frac{d}{dt} \hat{H}F(t) = \sum_{i=1}^{\infty} H_i^1 \frac{d}{dt} \left( \frac{1}{1!} F(t) \lor \cdots \lor F(t) \right) \circ \overline{\Delta}^{i-1}.$$ 

As in [3] Lemma 4.1 one can check

$$\hat{Q}(\lambda(t) \lor \exp(F(t))) = \exp(F(t)) \lor \hat{Q}(\lambda(t) \lor \exp(F(t))) - \lambda(t) \lor \exp(F(t)) \lor \hat{Q}(\exp(F(t)))$$

since $F(t)$ is a Maurer-Cartan element. This allows us to compute

$$\frac{d}{dt} \hat{H}F(t) = H^1 \circ \hat{Q}(\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta}$$

$$= H^1 \circ \hat{Q} \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta} + H^1 \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta} \circ Q$$

$$= (\hat{Q}')^i \circ H \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta} + H^1 \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta} \circ Q$$

$$= (\hat{Q}')^i \left( H^t \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta} \right) + \sum_{i=2}^{\infty} (Q'^i)_{i} \circ H^t \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta}.$$ 

Concerning the last term we have omitting the $t$-dependency since $F$ and $H$ are of degree zero

$$\frac{1}{k!} H^t_{k+1} \circ (\lambda(t) \lor \exp(F(t))) \circ \overline{\Delta}(X)$$
\[
\frac{1}{k!} (H^1 \vee \cdots \vee H^1) \circ \vec{\Sigma}^{-1} (X) \\
= \frac{1}{k!} (H^1 \vee \cdots \vee H^1) \circ \sum_{i_1 + \cdots + i_k = k+1} \sum_{\sigma \in Sh(i_1, \ldots, i_k)} \sigma \circ \left( (\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \vec{\Sigma} (X) \right) \\
= \frac{\ell}{k!} (H^1 \vee \cdots \vee H^1) \circ \sum_{i_1 + \cdots + i_k = k+1} \sum_{\sigma \in Sh(i_1, \ldots, i_k)} \sigma \circ \left( (\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \vec{\Sigma} (X) \right) \\
= \frac{1}{(\ell - 1)!} \sum_{i_1 + \cdots + i_k = k+1, i_1 \geq 1} \left( \frac{1}{(i_1 - 1)!} H^1_{i_1} (\lambda \vee F \cdots \vee F) \circ \vec{\Sigma}^{-1} - 1 \right) \circ \sum_{i_2 \geq 1} \left( \frac{1}{i_2!} H^1_{i_2} (F \vee \cdots \vee F) \circ \vec{\Sigma}^{i_2 - 1} \right) \circ \vec{\Sigma}^{-1} (X).
\]

Here we wrote

\[
\sigma \circ (x_1 \vee \cdots \vee x_{k+1}) = \epsilon(\sigma)x_{\sigma(i_1)} \vee \cdots \vee x_{\sigma(i_1)} \otimes \cdots \otimes x_{\sigma(k+1-i_1+1)} \vee \cdots \vee x_{\sigma(n)}
\]

with Koszul sign \(\epsilon(\sigma)\). Therefore, it follows

\[
\frac{d}{dt} \hat{H}F(t) = (\hat{Q}')_{\frac{1}{\ell}} \left( H^1 \circ (\lambda(t) \vee \exp(F(t))) \circ \vec{\Sigma} \right) \\
+ \sum_{\ell=2}^{\infty} (\hat{Q}')_{\frac{1}{\ell}} \circ \left( (H^1 \circ (\lambda(t) \vee \exp F) \circ \vec{\Sigma}) \vee \exp(\hat{H}F) \right)
\]

and the statement is shown. \(\square\)

Analogously, we have for the pre-composition:

**Proposition 3.6** Let \(F_0, F_1\) be two homotopic \(L_\infty\)-morphisms from \((L, Q)\) to \((L', Q')\). Let \(H\) be an \(L_\infty\)-morphism from \((L'', Q'')\) to \((L, Q)\), then \(F_0 H \sim F_1 H\).

**Proof:** Let \(F(t) \in (\text{Hom}(\mathcal{S}(L[1]), L'[1])[0, t])\) and \(\lambda(t) \in (\text{Hom}(\mathcal{S}(L[1]), L'[1])[1, t^{-1}]\) describe the homotopy equivalence between \(F_0\) and \(F_1\). Then we consider

\[
(F(t)H) = F(t) \circ H = F(t) \circ \exp H^1, \vec{\Sigma} \in (\text{Hom}(\mathcal{S}(L''[1]), L'[1])[0, t])
\]

in the notation of the above proposition. We compute

\[
\frac{d}{dt} (F(t)H) = \hat{Q}^1 (\lambda(t) \vee \exp(F(t))) \circ H \\
= (Q')_{\frac{1}{\ell}} \circ \lambda \circ H + \lambda \circ Q \circ H + \sum_{\ell=2}^{\infty} \frac{1}{(\ell - 1)!} (Q')_{\frac{1}{\ell}} \circ (\lambda \vee F \vee \cdots \vee F) \circ \vec{\Sigma}^{-1} \circ H \\
= (Q')_{\frac{1}{\ell}} \circ \lambda \circ H + \lambda \circ H \circ Q'' + \sum_{\ell=2}^{\infty} \frac{1}{(\ell - 1)!} (Q')_{\frac{1}{\ell}} \circ (\lambda H \vee FH \vee \cdots \vee FH) \circ \vec{\Sigma}^{-1} \\
= \hat{Q}^1 (\lambda(t)H \vee \exp(F(t)H))
\]

since \(H\) is a coalgebra morphism intertwining \(Q''\) and \(Q\) and of degree zero. Finally, since \(\lambda(t)H \in (\text{Hom}(\mathcal{S}(L''[1]), L'[1])[1, t^{-1}]\) the statement follows. \(\square\)

### 3.2 Homotopy Classification of \(L_\infty\)-algebras

The above considerations allow us to understand better the homotopy classification of \(L_\infty\)-algebras from \([11, 15]\), which will help us in the application to the global formality.
Definition 3.7 Two $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ are said to be homotopy equivalent if there are $L_\infty$-morphisms $F: (L, Q) \to (L', Q')$ and $G: (L', Q') \to (L, Q)$ such that $F \circ G \sim \text{id}_L$ and $G \circ F \sim \text{id}_L$. In such case $F$ and $G$ are said to be quasi-inverse to each other.

This definition coincides indeed with the definition of homotopy equivalence via $L_\infty$-quasi-isomorphisms from [1].

Lemma 3.8 Two $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ are homotopy equivalent if and only if there exists an $L_\infty$-quasi-isomorphism between them.

Proof: Due to [1] Prop. 2.8 every $L_\infty$-algebra $L$ is isomorphic to the product of a linear contractible one and a minimal one $(L, Q) \cong (V \oplus W, Q)$. This means $L \cong V \oplus W$ as vector spaces, such that $V$ is an acyclic cochain complex with differential $d_V$ and $W$ is an $L_\infty$-algebra with codifferential $Q_W$ with $Q_W(1) = 0$. The codifferential $\tilde{Q}$ on $\Delta((V \oplus W)[1])$ is given on $v_1 \cdots v_m$ with $v_1, \ldots, v_k \in V$ and $v_{k+1}, \ldots, v_m \in W$ by

$$\tilde{Q}^1(v_1 \cdots v_m) = \begin{cases} -d_V(v_1), & \text{for } k = m = 1 \\ Q^1_W(v_1 \cdots v_m), & \text{for } k = 0 \end{cases}$$

This implies in particular that the canonical maps

$$I_W: W \to V \oplus W \quad \text{and} \quad P_W: V \oplus W \to W$$

are $L_\infty$-morphisms. We want to show now that $I_W \circ P_W \sim \text{id}$. Choose a contracting homotopy $h_V: V \to V[-1]$ with $h_V d_V + d_V h_V = \text{id}_V$ and define the maps

$$P(t): V \oplus W \ni (v, w) \mapsto (tv, w) \in V \oplus W$$

and

$$H(t): V \oplus W \ni (v, w) \mapsto (-h_V(v), 0) \in V \oplus W.$$ 

Note that $P(t)$ is a path of $L_\infty$-morphisms by the explicit form of the codifferential. We clearly have

$$\frac{d}{dt} P^1(t) = \text{pr}_V = \tilde{Q}^1 \circ H(t) + H(t) \circ \tilde{Q}^1 = \tilde{Q}^1(\exp(P(t)))$$

since $h_V$ is a contracting homotopy. This implies

$$\frac{d}{dt} P(t) = \tilde{Q}^1(\exp(P(t)))$$

since $\text{im}(H(t)) \subseteq V$ and as the higher brackets of $\tilde{Q}$ vanish on $V$. Since $P(0) = I_W \circ P_W$ and $P(1) = \text{id}$ we conclude that $I_W \circ P_W \sim \text{id}$. We choose a similar splitting for a $L' = V' \oplus W'$ with the same properties and consider an $L_\infty$-quasi-isomorphism $F: L \to L'$. Since $I_W, I_W', P_W$ and $P_W'$ are $L_\infty$-quasi-isomorphisms and we have that

$$F_W = P_W \circ F \circ I_W: W \to W'$$

an $L_\infty$-isomorphism. Hence it invertible and we denote the inverse $G_W$. We define now

$$G = I_W \circ G_W \circ P_W: L' \to L.$$ 

Since by Proposition [2,3] and Proposition [4,5] compositions of homotopic $L_\infty$-morphisms with an $L_\infty$-morphism are again homotopic, we get

$$F \circ G = F \circ I_W \circ G_W \circ P_W \sim I_W \circ P_W \circ F \circ I_W \circ G_W \circ P_W,
= I_W \circ F_W \circ G_W \circ P_W = I_W \circ P_W \sim \text{id}$$

and similarly $G \circ F \sim \text{id}$.

The other direction follows from Proposition [6,7]. Suppose $F \circ G \sim \text{id}$ and $G \circ F \sim \text{id}$, then we know that $F^1 \circ G^1$ and $G^1 \circ F^1$ are both chain homotopic to the identity. Therefore, $F$ and $G$ are $L_\infty$-quasi-isomorphisms.

\[ \square \]
Corollary 3.9 Let $F: (L, Q) \rightarrow (L', Q')$ be an $L_{\infty}$-quasi-isomorphism with two given quasi-inverses $G, G': (L', Q') \rightarrow (L, Q)$ in the sense of Definition 3.6. Then one has $G \sim G'$.

Proof: One has $G \sim G \circ (F \circ G') = (G \circ F) \circ G' \sim G'$.

3.3 Homotopy Equivalence between Twisted Morphisms

Let now $F: (g, Q) \rightarrow (g', Q')$ be an $L_{\infty}$-morphism between DGLAs with complete descending and exhaustive filtrations. Instead of comparing the twisted morphisms $F^\pi$ and $F'^\pi$ with respect to two equivalent Maurer-Cartan elements $\pi$ and $\pi'$, we consider for simplicity just a Maurer-Cartan element $\pi \in T^1 g^1$ equivalent to zero via $\pi = \exp([g, \cdot]) \circ 0$, i.e. $\lambda(t) = g = \dot{A}(t) \in T^1 g^0[\tau]$. Then we know that 0 and $S = F^1(\exp(\pi)) \in T^1 (g')^0$ are equivalent Maurer-Cartan elements in $(g', d')$.

Let the equivalence be implemented by an $A'(t) \in T^1 (g')^0[\tau]$ as in Proposition 3.9. Then we have the diagram

$$
\begin{array}{c}
(g, d) \quad (g', d') \\
\downarrow \quad \downarrow \\
(g, d + [\pi, \cdot]) \quad (g', d' + [S, \cdot])
\end{array}
\xrightarrow{F} 
\xrightarrow{e^{A'(1) \cdot \cdot}}
\xrightarrow{\beta_{A'(1) \cdot \cdot}}
\xrightarrow{F^\pi}
$$

where $\beta_{A'(1) \cdot \cdot}$ and $e^{A'(1) \cdot \cdot}$ are well-defined by the completeness of the filtrations. In the following we show that it commutes up to homotopy, which is indicated by the vertical arrow.

Proposition 3.10 The $L_{\infty}$-morphisms $F$ and $e^{-A'(1) \cdot \cdot} \circ F^\pi \circ e^{A(1) \cdot \cdot}$ are homotopic, i.e. gauge equivalent Maurer-Cartan elements in $\text{Hom}(\mathcal{S}(g[1]), g')$.

The candidate for the path between $F$ and $e^{-A'(1) \cdot \cdot} \circ F^\pi \circ e^{A(1) \cdot \cdot}$ is

$$
F(t) = e^{-A'(1) \cdot \cdot} \circ F^\pi(t) \circ e^{A(1) \cdot \cdot}.
$$

However, $F(t)$ is not necessarily in the completion $(\text{Hom}(\mathcal{S}(g[1]), g')^1[\tau])$ with respect to the filtration from (3.5) since for example

$$
F(t) \mod T^2 \text{Hom}(\mathcal{S}(g[1]), g')[[\tau]] = e^{-A'(1) \cdot \cdot} \circ F^\pi(t) \circ e^{A(1) \cdot \cdot}
$$

is in general no polynomial in $t$. To solve this problem we introduce a new filtration on the convolution DGLA $h = \text{Hom}(\mathcal{S}(g[1]), g')$ that takes into account the filtrations on $\mathcal{S}(g[1])$ and $g'$:

$$
\mathcal{S}(g[1]) 
= \mathcal{S}(g[1]) \supset \mathcal{S}^2(g[1]) \supset \cdots \supset \mathcal{S}^k(g[1]) \supset \cdots
$$

$$
\mathcal{S}^k(g[1]) = \sum_{n+m=k} \left\{ f \in \text{Hom}(\mathcal{S}(g[1]), g') \mid f|_{\mathcal{S}^n(g[1])} = 0 \right\},
$$

where $\mathcal{S}^k(g[1])$ is the product filtration induced by

$$
\mathcal{S}^k(g[1] \otimes g[1]) = \sum_{n+m=k} \text{im}(\mathcal{S}^n g[1] \otimes \mathcal{S}^m g[1] \rightarrow g[1] \otimes g[1]),
$$

see e.g. [10] Section 1].

Proposition 3.11 The above filtration (3.7) is a complete descending filtration on the convolution DGLA $\text{Hom}(\mathcal{S}(g[1]), g')$.

Proof: The filtration is obviously descending and $h = \mathcal{S}^1 h$ since we consider in the convolution DGLA only maps that are compatible with respect to the filtration. It is compatible with the convolution DGLA structure and complete since $g'$ is complete.
Thus we can finally prove Proposition 3.10.

Proof (of Prop. 3.10): The path $F(t) = e^{-A(t) \cdot \cdot} \circ F^\pi(t) \circ e^{|A(t)|} \cdot$ is an element in the completion $(\text{Hom}(S[\mathfrak{g}[1]], \mathfrak{g}'[1])^\Theta[t]$ with respect to the filtration from (3.7). This is clear since $A(t) \in \mathcal{F}\mathfrak{g}$, $A'(t) \in \mathcal{F}\mathfrak{g}'[t]$ and $\pi(t) \in \mathcal{F}\mathfrak{g}$ imply that

$$\sum_{i=1}^{n-1} e^{-A'(t) \cdot \cdot} \circ F^\pi(t) \circ e^{|A(t)|} \cdot \mod \mathfrak{g}^\Theta(\text{Hom}(S[\mathfrak{g}[1]], \mathfrak{g}'[1])[t])$$

is polynomial in $t$. Moreover, $F(t)$ satisfies by (2.10)

$$\frac{dF(t)}{dt} = - \exp([-A(t), \cdot \cdot]) \circ F^\pi(t) \circ e^{|A(t)|} \cdot + e^{-A'(t) \cdot \cdot} \circ \frac{dF^\pi(t)}{dt} \circ e^{|A(t)|} \cdot.$$

But we have

$$\frac{dF^\pi(t)}{dt}(X_1 \vee \cdots \vee X_k) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1}(\lambda(t)) \vee X_1 \vee \cdots \vee X_k) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) + Q_{2}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) + Q_{2}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^\pi(t)^{1+k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)).$$

Setting now $\lambda^F_{1}(t) \cdots = F^\pi_{k+1}(\lambda(t) \vee \cdots)$ we get

$$\frac{dF^\pi(t)}{dt} = \lambda^F_{1}(t) \vee \cdots \vee X_k) = F^\pi_{k+1}(Q_{1}^{1}(\lambda^F_{1}(t)) \vee X_1 \vee \cdots \vee X_k) = F^\pi_{k+1}(Q_{1}^{1}(\lambda^F_{1}(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^{1}(\lambda^F_{1}(t) \vee X_1 \vee \cdots \vee X_k)) = F^\pi_{k+1}(Q_{1}^{1}(\lambda^F_{1}(t) \vee X_1 \vee \cdots \vee X_k)).$$

Thus we get

$$\frac{dF(t)}{dt} = e^{-A'(t) \cdot \cdot} \circ (\lambda^F_{1}(t) \vee \cdots \vee X_k) = e^{-A'(t) \cdot \cdot} \circ (\lambda^F_{1}(t) \vee \cdots \vee X_k) \circ e^{|A(t)|} \cdot.$$
• Analogously, the sections of formal fiberwise differential operators \(D^\infty_{\text{poly}}\) are \(\mathcal{C}^\infty(M)\)-linear operators \(X: \bigotimes^{k+1} \Gamma^\infty(SM) \to \Gamma^\infty(SM)\) of the local form
\[
X = \sum_{\alpha_0, \ldots, \alpha_k} \sum_{p=0}^\infty X^\alpha_{i_1 \ldots i_p}(x) y^{i_1} \cdots y^{i_p} \frac{\partial}{\partial y^{\alpha_k}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\alpha_0}}.
\]
Here \(X^\alpha_{i_1 \ldots i_p}\) are symmetric in the indices \(i_1, \ldots, i_p\) and \(\alpha\) are multi-indices \(\alpha = (j_0, \ldots, j_k)\). Moreover, the sum in the orders of the derivatives is finite.

• \(D = -\delta + \nabla + [A, \cdot] = d + [B, \cdot]\) is the Fedosov differential, where \(\delta = [dx^i \frac{\partial}{\partial x^i}, \cdot]\), \(\nabla = dx^i \frac{\partial}{\partial x^i} - \frac{\partial R_{ij}}{\partial y^k} \frac{\partial}{\partial y^k}\) with Christoffel symbols \(\Gamma^k_{ij}\) of a torsion-free connection on \(M\) with curvature \(R = -\frac{1}{2}(dx^i dx^j(R_{ij})^l_k(x) y^{l} \frac{\partial}{\partial y^k}\), and \(A \in \Omega^1(M, \mathcal{T}^0_{\text{poly}}) \subseteq \Omega^1(M, D^0_{\text{poly}})\) is the unique solution of
\[
\begin{align*}
\delta(A) &= R + \nabla A + \frac{1}{2}[A, A], \\
\delta^{-1}(A) &= r, \\
\sigma(A) &= 0.
\end{align*}
\]
Here \(r \in \Omega^0(M, \mathcal{T}^0_{\text{poly}})\) is arbitrary but fixed and has vanishing constant and linear term with respect to the \(y\)-variables. We refer to \((\nabla, r)\) as globalization data.

• \(\tau: \Gamma^\infty_\delta(\mathcal{T}_{\text{poly}}) \to Z^0(\Omega(M, \mathcal{T}_{\text{poly}}), D)\) denotes the Fedosov Taylor series, given by
\[
\tau(a) = a + \delta^{-1}(\nabla \tau(a) + [A, \tau(a)]).
\]
Here one has \(\Gamma^\infty_\delta(\mathcal{T}_{\text{poly}}) = \{v = \sum_i v^{ij} \cdot \frac{\partial}{\partial y^j} \wedge \cdots \wedge \frac{\partial}{\partial y^j}\},\) analogously for the polydifferential operators. In addition, \(\partial_M\) denotes the fiberwise Hochschild differential.

• \(\nu: \Gamma^\infty_\delta(\mathcal{T}_{\text{poly}}) \to T^\infty_{\text{poly}}(M)\) is given by
\[
\nu(w)(f_0, \ldots, f_k) = \sigma w(\tau(f_0), \ldots, \tau(f_k)) \quad \text{for} \quad f_1, \ldots, f_k \in \mathcal{C}^\infty(M),
\]
where \(\sigma\) sets the \(dx^i\) and \(y^j\) coordinates to zero, analogously for the polydifferential operators.

• \(\mathcal{U}^B\) is the fiberwise formality of Kontsevich \(\mathcal{U}\) twisted by
\[
B = D - d = -dx^i \frac{\partial}{\partial y^i} - dx^i \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k} + \sum_{p \geq 1} dx^i A^k_{ij_1 \ldots j_p}(x) y^{i_1} \cdots y^{i_p} \frac{\partial}{\partial y^k}.
\]

By the properties of the Kontsevich formality the first two summands do not contribute, i.e.

\[
\mathcal{U}^B = \mathcal{U}^A.
\]

One obtains the diagram
\[
T^\infty_{\text{poly}}(M) \xrightarrow{\tau \circ \nu^{-1}} (\Omega(M, \mathcal{T}_{\text{poly}}), D) \xrightarrow{\mathcal{U}^B} (\Omega(M, D_{\text{poly}}), D + \partial_M) \xrightarrow{\tau \circ \nu^{-1}} D^\infty_{\text{poly}}(M),
\]
where \(\tau \circ \nu^{-1}\) are quasi-isomorphisms of DGLAs and where \(\mathcal{U}^B\) is an \(L^\infty\)-quasi-isomorphism. In a next step, the morphism \(\mathcal{U}^B \circ \tau \circ \nu^{-1}\) is modified to a quasi-isomorphism
\[
U: T^\infty_{\text{poly}}(M) \to (Z^0_D(\Omega(M, D_{\text{poly}})), \partial_M, [\cdot, \cdot]_G),
\]
see [8] Prop. 5. By [8] Lemma 1 we know that \(\mathcal{U}^B \circ \tau \circ \nu^{-1}\) and \(U\) are homotopic. The desired quasi-isomorphism \(U(\nabla, r) = \nu \circ \sigma \circ U: T^\infty_{\text{poly}}(M) \to D^\infty_{\text{poly}}(M)\) is then the composition of \(U\) with the DGLA isomorphism
\[
\nu \circ \sigma: (Z^0_D(\Omega(M, D_{\text{poly}})), \partial_M, [\cdot, \cdot]_G) \to (D^\infty_{\text{poly}}(M), \partial, [\cdot, \cdot]_G).
\]

**Corollary 4.1** The formality \(U(\nabla, r)\) induces a one-to-one correspondence between equivalent formal Poisson structures on \(M\) and equivalent differential star products on \(\mathcal{C}^\infty(M)\), i.e. a bijection
\[
U(\nabla, r): \text{Def}(T^\infty_{\text{poly}}(M)[[h]]) \to \text{Def}(D^\infty_{\text{poly}}(M)[[h]]).
\]
4.2 Explicit Construction of the Projection $L_{\infty}$-morphism

As an alternative to the modification of the formality in [6 Prop. 5] we want to construct the $L_{\infty}$-quasi-inverse of $\tau \circ \nu^{-1}$. We want to use the construction from [12 Prop. 3.2] that gives a formula for the $L_{\infty}$-quasi-inverse of an inclusion of DGLAs, see also [17] for the existence in more general cases. In our setting we have the contraction

\[
(D_{\text{poly}}(M), \partial) \xrightarrow{\tau \circ \nu^{-1}} (\Omega(M, D_{\text{poly}}), \partial_M + D), \quad \xrightarrow{h}
\]

where the homotopy $h$ with respect to $\partial_M + D$ is constructed as follows: As in the Fedosov construction in the symplectic setting one has a homotopy $D^{-1}$ for the differential $D$, see also [6 Thm. 3]:

**Proposition 4.2** The map

\[
D^{-1} = -\delta^{-1} \frac{1}{\text{id} - [\delta^{-1}, \nabla + [A, \cdot]]} = -\frac{1}{\text{id} - [\delta^{-1}, \nabla + [A, \cdot]]} \delta^{-1}
\]

is a homotopy for $D$ on $\Omega(M, D_{\text{poly}})$, i.e. one has

\[
X = DD^{-1}X + D^{-1}DX + \tau \sigma(X).
\]

**Proof:** The proof is the same as in the symplectic setting, see e.g. [20 Prop. 6.4.17].

If this homotopy is also compatible with the Hochschild differential $\partial_M$, then we can indeed apply [12 Prop. 3.2] to describe the $L_{\infty}$-morphism extending $\nu \circ \sigma$. Let us denote by $(D^{-1})_{k+1}$ the extended homotopy on $S^{k+1}(\Omega(M, D_{\text{poly}})[1])$ and let us write $Q_{D_{\text{poly}}}, Q_{D_{\text{poly}}}$ for the induced codifferentials on the symmetric algebras. Then we get:

**Proposition 4.3** The homotopy $D^{-1}$ anticommutes with $\partial_M$, whence it is also a homotopy for $\partial_M + D$. Therefore, one obtains an $L_{\infty}$-quasi-isomorphism $P: S(\Omega(M, D_{\text{poly}})[1]) \to S(D_{\text{poly}}(M)[1])$ with recursively defined structure maps

\[
P^1_1 = \nu \circ \sigma \quad \text{and} \quad P^1_{k+1} = (Q^1_{D_{\text{poly}}, 2} \circ P^2_{k+1} - P^1_k \circ Q^k_{D_{\text{poly}}, k+1}) \circ (D^{-1})_{k+1}.
\]

**Proof:** The fact that $D^{-1}$ anticommutes with $\partial_M$ is clear as $\nabla + [A, \cdot]$ and $\delta^{-1}$ anticommute with $\partial$, and the rest follows directly from [12 Prop. 3.2].

Summarizing, we obtain another global formality:

**Corollary 4.4** Given the globalization data $(\nabla, r)$ there exists an $L_{\infty}$-quasi-isomorphism

\[
F^{(\nabla, r)} = P \circ \underline{U} \circ \tau \circ \nu^{-1}: T_{\text{poly}}(M) \to D_{\text{poly}}(M)
\]

with $F^1_1$ being the Hochschild-Kostant-Rosenberg map.

**Proof:** We immediately get $F^{(\nabla, r), 1} = P^1 \circ (\underline{U}^B) \circ \tau \circ \nu^{-1} = \nu \circ \sigma \circ F^1$, and the statement follows since $\underline{U}^1$ is the fiberwise Hochschild-Kostant-Rosenberg map.

The higher structure maps of $P^1_{k+1}$ of the $L_{\infty}$-projection contain copies of the homotopy $D^{-1}$ that decrease the antisymmetric form degree. Therefore, they vanish on $\Omega^B(M, D_{\text{poly}})$ and are needed to get rid of the form degrees arising from the twisting with $B$, analogously to the modifying of the formality from $\underline{U}^B \circ \tau \circ \nu^{-1}$ to $U$.

As a last point, we want to remark that we can use the $L_{\infty}$-projection $P$ to obtain a splitting of $\Omega(M, D_{\text{poly}})$ similar to the one used in the proof of Lemma 3.8. Instead of splitting into the product of the cohomology as minimal $L_{\infty}$-algebra and a linear contractible one, we can prove in our setting:
Lemma 4.5  One has an $L_\infty$-isomorphism
\[ L : \Omega(M, \mathcal{D}_{\text{poly}}) \to D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}] . \]  

Here the $L_\infty$-structure on $D_{\text{poly}}(M)$ is the usual one consisting of Gerstenhaber bracket and Hochschild differential $\partial$ and on $\text{im}[D, D^{-1}]$ the $L_\infty$-structure is just given by the differential $\partial_M + D$. The $L_\infty$-structure on $D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}]$ is the product $L_\infty$-structure.

PROOF: By Proposition 4.3 we already have an $L_\infty$-morphism $P : \Omega(M, \mathcal{D}_{\text{poly}}) \to D_{\text{poly}}(M)$ with first structure map $\nu \circ \sigma$. Now we construct an $L_\infty$-morphism $F : \Omega(M, \mathcal{D}_{\text{poly}}) \to \text{im}[D, D^{-1}]$. We set $F^1_n = DD^{-1}D^{-1}D + D^{-1}D^1_n$ and $F^1_n = -D^{-1}o F^1_{n-1} \circ (Q_{\mathcal{D}_{\text{poly}}})^n_{n-1}$ for $n \geq 2$ and note that in particular $F^1_k = 0$ for $k \geq 3$ by $D^{-1}D^{-1} = 0$. In the following, we denote by $Q_{\mathcal{D}_{\text{poly}}}$ the $L_\infty$-structure on $\Omega(M, \mathcal{D}_{\text{poly}})$ and by $\tilde{Q}$ the $L_\infty$-structure on $\text{im}[D, D^{-1}]$ with $\tilde{Q}^1_1 = - (\partial_M + D)$ as only vanishing component. We have $F^1_n = D^{-1} \circ L_{\infty,n}$ with $L_{\infty,n} = -F^1_{n-1} \circ (Q_{\mathcal{D}_{\text{poly}}})^n_{n-1}$.

By [12] Lemma 3.1 we know that if $F$ is an $L_\infty$-morphism up to order $n$, i.e. if $(\tilde{Q}F)^k_k = (FQ_{\mathcal{D}_{\text{poly}}})^k_k$ for all $k \leq n$, then one has $\tilde{Q}^1_1 \circ L_{\infty,n+1} = -L_{\infty,n+1} \circ (Q_{\mathcal{D}_{\text{poly}}})^n_{n+1}$. By Proposition 4.3 we know that $F$ is an $L_\infty$-morphism up to order one. Assuming it defines an $L_\infty$-morphism up to order $n$, then we get with (4.4)

\[ \tilde{Q}^1_1 \circ F^1_{n+1} = - (\partial_M + D) \circ D^{-1} \circ L_{\infty,n+1} \]
\[ = D^{-1} \circ \partial_M \circ L_{\infty,n+1} + L_{\infty,n+1} + D^{-1} \circ D \circ L_{\infty,n+1} + \tau \circ \sigma \circ L_{\infty,n+1} \]
\[ = -L_{\infty,n+1} - D^{-1} \circ \tilde{Q}^1_1 \circ L_{\infty,n+1} \]
\[ = -L_{\infty,n+1} + F^1_{n+1} \circ (Q_{\mathcal{D}_{\text{poly}}})^n_{n+1} . \]

Thus $F$ is an $L_\infty$-morphism up to order $n + 1$ and therefore an $L_\infty$-morphism.

The universal property of the product gives the desired $L_\infty$-morphism $L = P \oplus F$ which is even an isomorphism since its first structure map $(\nu \circ \sigma) \oplus (DD^{-1} + D^{-1}D)$ is an isomorphism with inverse $(\tau \circ \nu^{-1}) \oplus \text{id}$, see e.g. [3] Prop. 2.2. \[ \square \]

4.3 Homotopic Global Formalities

The above globalization of the Kontsevich Formality depends on the choice of a covariant derivative. We want to show that globalizations with respect to different covariant derivatives are homotopic in the sense of Definition 4.3. The ideas are similar to those in the proof of [2] Theorem 2.6: observe that changing the covariant derivative corresponds to twisting with a Maurer-Cartan element which is equivalent to zero, and apply Proposition 3.10.

Remark 4.6 (Filtrations on Fedosov resolutions) In order to apply Proposition 3.10 we need complete descending and exhaustive filtrations on the Fedosov resolutions. As in [2] Appendix C we assign to $dz^i$ and $y^i$ the degree 1 and to $\frac{\partial}{\partial y^i}$ the degree $-1$ and consider the induced descending filtration. The filtration on $\Omega(M, \mathcal{T}_{\text{poly}})$ is complete and bounded below since

\[ \Omega(M, \mathcal{T}_{\text{poly}}) \cong \lim_{k \to -\infty} \Omega(M, \mathcal{T}_{\text{poly}})/\mathcal{F}^k \Omega(M, \mathcal{T}_{\text{poly}}) \quad \text{and} \quad \Omega(M, \mathcal{T}_{\text{poly}}) = \mathcal{F}^{-d} \Omega(M, \mathcal{T}_{\text{poly}}) , \]

where $d$ is the dimension of $M$. In the case of the differential operators the filtration is unbounded in both directions. Instead of $\mathcal{D}_{\text{poly}}$ we consider from now on its $y$-adic completion without changing the notation. This is the completion with respect to the filtration induced by assigning $y^i$ the degree 1 and $\frac{\partial}{\partial y^i}$ the degree $-1$. The globalization of the formality works just the same and one obtains the desired properties

\[ \Omega(M, \mathcal{D}_{\text{poly}}) \cong \lim_{k \to -\infty} \Omega(M, \mathcal{D}_{\text{poly}})/\mathcal{F}^k \Omega(M, \mathcal{D}_{\text{poly}}) \quad \text{and} \quad \Omega(M, \mathcal{D}_{\text{poly}}) = \bigcup_k \mathcal{F}^k \Omega(M, \mathcal{D}_{\text{poly}}) . \]

Let $(\nabla', \tau')$ be a second pair of globalization data, then there is a second diagram

\[ T_{\text{poly}}(M) \xrightarrow{\tau' \circ \nu'^{-1}} (\Omega(M, \mathcal{T}_{\text{poly}}), D') \xrightarrow{\nu'} (\Omega(M, \mathcal{D}_{\text{poly}}), D' + \partial_M) \xrightarrow{\tau'^{-1} \circ (\nu')^{-1}} D_{\text{poly}}(M) , \]
Thus

\[ D' = -\delta + \nabla' + [A', \cdot] = d + [B', \cdot], \]

and \( A' \) is the unique solution of

\[
\begin{align*}
\delta(A') &= R' + \nabla'A' + \frac{1}{2}[A', A'], \\
\delta^{-1}(A') &= r', \\
\sigma(A') &= 0.
\end{align*}
\]

In the case of polyvector fields one easily sees that \( \nu = \nu' \). Note

\[
\nabla' - \nabla = \left[ -dx^j y^l (\Gamma^k_{ij} - \Gamma^k_{lj}) \frac{\partial}{\partial y^k}, \cdot \right] = [\delta s, \cdot]
\]

for \( s = -\frac{1}{2} y^j y^l (\Gamma^k_{ij} - \Gamma^k_{lj}) \frac{\partial}{\partial y^k}. \) Thus we get

\[
D' = -\delta + \nabla + [A' + \delta s, \cdot] = -\delta + \nabla + [\tilde{A}, \cdot]
\]

and since \( R' = R + \nabla \delta s + \frac{1}{2} [\delta s, \delta s] \) we know that \( \tilde{A} = A' + \delta s \) is the unique solution of

\[
\begin{align*}
\delta(\tilde{A}) &= R + \nabla \tilde{A} + \frac{1}{2} [\tilde{A}, \tilde{A}], \\
\delta^{-1}(\tilde{A}) &= r' + s, \\
\sigma(\tilde{A}) &= 0.
\end{align*}
\]

As in [2] Appendix C] one can now show that \( B \) and \( B' \) can be interpreted as equivalent Maurer-Cartan elements:

**Proposition 4.7** There exists an element

\[ h \in \mathcal{T}^1 \Omega^0(M, \mathcal{T}^0_{\text{poly}}) \] (4.9)

that is at least quadratic in the fiber coordinates \( y \) such that one has

\[ B' - B = \tilde{A} - A = -\frac{\exp([h, \cdot]) - \text{id}}{[h, \cdot]} D h \in \mathcal{T}^1 \Omega^1(M, \mathcal{T}^0_{\text{poly}}) \] (4.10)

and

\[ \exp([h, \cdot]) \circ D \circ \exp([-h, \cdot]) = D'. \] (4.11)

Thus \( B' - B \) is gauge equivalent to zero in \( (\Omega(M, \mathcal{T}_{\text{poly}}), D) \) and \( (\Omega(M, \mathcal{D}_{\text{poly}}), D + \partial_M) \), where \( h \) implements the gauge equivalences.

**Proof:** For the existence of the element \( h \in \mathcal{T}^1 \Omega^0(M, \mathcal{T}^0_{\text{poly}}) \) encoding the gauge equivalence in the polyvector fields see [2] Appendix C]. Thus we have a path

\[ B(t) = -\frac{\exp([th, \cdot]) - \text{id}}{[th, \cdot]} D(th) \in \mathcal{T}^1 \Omega^1(M, \mathcal{T}^0_{\text{poly}})[t] \]

that satisfies \( B(0) = 0, B(1) = B' - B \) and

\[ \frac{dB(t)}{dt} = Q^1(\lambda(t) \lor \exp(B(t))) \quad \text{with} \quad \lambda(t) = h. \]

The formality \( \mathcal{U}^B \) satisfies in the notation of Proposition 2.15

\[ \tilde{B}(t) = \mathcal{U}^{B, 1}(\exp(B(t))) = B(t) \quad \text{and} \quad \tilde{\lambda}(t) = \mathcal{U}^{B, 1}(h \lor \exp(B(t))) = h \]

since the higher orders of the Kontsevich formality vanish if one only inserts vector fields. \( \square \)

Now it follows directly from Proposition 3.10 and \( (\mathcal{U}^B)^{B - B} = \mathcal{U}^{B'} \) that the twisted formalities are homotopic.
Corollary 4.8 The $L_\infty$-morphisms $\mathbb{U}^B$ and $e^{-[h, \cdot]} \circ \mathbb{U}^B \circ e^{[h, \cdot]}$ are homotopic.

Moreover, the Fedosov Taylor series is compatible in the following sense:

Corollary 4.9 For all $X \in T_{\text{poly}}(M)$ one has
\[
e^{[h, \cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ (\nu')^{-1}(X).
\]

Proof: By the above proposition $\exp([h, \cdot])$ maps the kernel of $D$ into the kernel of $D'$. Therefore,
\[
e^{[h, \cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ \sigma \circ e^{[h, \cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ \nu^{-1}(X)
\]
since $h$ is at least quadratic in the $y$ coordinates.

Similarly, one has on the differential operator side the following identity:

Lemma 4.10 For all $X \in Z^p_D(\Omega(M, \mathbb{D}_{\text{poly}}))$ one has
\[
\nu \circ \sigma \circ e^{-[h, \cdot]}(X) = \nu' \circ \sigma(X).
\]

Proof: Using the definition of $\nu$ we compute for $X \in Z^p_D(\Omega(M, \mathbb{D}_{\text{poly}}))$ and $f_1, \ldots, f_n \in \mathcal{E}_\infty(M)$
\[
(\nu \circ \sigma \circ e^{-[h, \cdot]}X)(f_1, \ldots, f_n) = \sigma((\sigma \circ e^{-[h, \cdot]}X)(\tau f_1), \ldots, \tau(f_n))
\]
\[
= \sigma(e^{-h}(X(e^h \tau(f_1), \ldots, e^h \tau(f_n))))
\]
\[
= \sigma((\sigma \circ X)(\tau(f_1), \ldots, \tau(f_n)))
\]
\[
= (\nu' \circ \sigma X)(f_1, \ldots, f_n)
\]
and the statement is shown.

As a last preparation we want to compare the two different $L_\infty$-projections $P'$ and $P \circ e^{-[h, \cdot]}$ from $(\Omega(M, \mathbb{D}_{\text{poly}}), \partial_M + D')$ to $(D_{\text{poly}}(M), \partial)$.

Lemma 4.11 The $L_\infty$-projections $P'$ and $P \circ e^{-[h, \cdot]}$ are homotopic.

Proof: Since the higher structure maps of $P$ and $P'$ vanish on the zero forms, we have by Lemma 4.10
\[
P \circ e^{-[h, \cdot]} \circ \tau' \circ (\nu')^{-1} = \nu \circ \sigma \circ e^{-[h, \cdot]} \circ \tau' \circ (\nu')^{-1} = \nu' \circ \sigma \circ \tau' \circ (\nu')^{-1} = \text{id}_{D_{\text{poly}}(M)}.
\]
Instead of directly using Lemma 3.8 we recall the splitting from Lemma 4.10 and adapt the proof of Lemma 3.8 Define
\[
M(t): D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}] \ni (D_1, D_2) \mapsto (D_1, tD_2) \in D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}]
\]
which is an $L_\infty$-morphism with respect to the product $L_\infty$-structure. Setting
\[
H(t): D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}] \ni (D_1, D_2) \mapsto (0, -D^{-1}D_2) \in D_{\text{poly}}(M) \oplus \text{im}[D, D^{-1}]
\]
we obtain again
\[
\frac{d}{dt}M(t) = \text{pr}_{\text{im}[D, D^{-1}]} = 0 \oplus (DD^{-1} + D^{-1}D)
\]
\[
= \partial \oplus (\partial_M + D) \circ H(t) - H(t) \circ (\partial \oplus (\partial_M + D))
\]
\[
= \hat{Q}^1(H(t) \vee \exp(M(t))).
\]
Therefore, it follows that
\[
L(t) = L^{-1} \circ M(t) \circ L: \Omega(M, \mathbb{D}_{\text{poly}}) \mapsto \Omega(M, \mathbb{D}_{\text{poly}})
\]
encodes the homotopy between
\[
L(0) = \tau \circ \nu^{-1} \circ P \quad \text{and} \quad L(1) = \text{id}.
\]
But this implies with Proposition 3.5
\[
P \circ e^{-[h, \cdot]} \sim P \circ e^{-[h, \cdot]} \circ \tau' \circ (\nu')^{-1} \circ P' = P'
\]
and the statement is shown.
Combining all the above statements we can show that the globalizations with respect to different covariant derivatives are homotopy equivalent.

**Theorem 4.12** Let \((\nabla, r)\) and \((\nabla', r')\) be two pairs of globalization data. Then the formalities constructed via Dolgushev’s globalization and the globalized formalities via the \(L_\infty\)-projection are all homotopic, i.e. one has

\[
U^{(\nabla, r)} \sim F^{(\nabla, r)} \sim F^{(\nabla', r')} \sim U^{(\nabla', r')}. \tag{4.14}
\]

**Proof:** By Proposition 3.4 we already know that compositions of homotopic \(L_\infty\)-morphisms with DGLA morphisms are homotopic, which yields

\[
U \sim \mathcal{U}^B \circ \tau \circ \nu^{-1} \sim \epsilon^{-|h, \cdot|} \mathcal{U}^B \circ \epsilon^{[h, \cdot]} \circ \tau \circ \nu^{-1} = \epsilon^{-|h, \cdot|} \mathcal{U}^B \circ \tau' \circ (\nu')^{-1} \sim \epsilon^{-|h, \cdot|} \mathcal{U}^B.
\]

It follows with Lemma 4.11, Proposition 3.3 and Proposition 3.9

\[
U^{(\nabla, r)} = \nu \circ \sigma \circ U = P \circ U \sim P \circ \mathcal{U}^B \circ \tau \circ \nu^{-1} \sim P \circ \epsilon^{-|h, \cdot|} \circ \mathcal{U}^B \circ \tau' \circ (\nu')^{-1}
\]

\[
\sim P' \circ \mathcal{U}^B \circ \tau' \circ (\nu')^{-1} \sim P' \circ U' = \nu' \circ \sigma \circ U' = U^{(\nabla', r')}
\]

and the theorem is shown. \(\square\)

**Corollary 4.13** Let \(\mathcal{M}\) be a smooth manifold and let \((\nabla, r)\) be a globalization data. For every coordinate patch \((U, x)\)

\[
F^{(\nabla, r)}|_U \sim K|_U,
\]

holds, where \(K\) denotes the Kontsevich formality on \(\mathbb{R}^d\), and where \(d\) is the dimension of \(\mathcal{M}\).

**Proof:** The formalities themselves are differential operators and can therefore be restricted to open neighbourhoods. Moreover, the Kontsevich formality coincides with the Dolgushev formality on \(\mathbb{R}^d\) for the choice of the canonical flat covariant derivative and \(r = 0\). \(\square\)

This allows us to recover [2, Theorem 2.6], i.e. that the induced maps on equivalence classes of Maurer-Cartan elements are independent of the choice of the covariant derivative. It implies in particular that globalizations with respect to different covariant derivatives lead to equivalent star products.

**Corollary 4.14** The induced map \(\text{Def}(\mathcal{T}_{\text{poly}}(\mathcal{M})[[\hbar]]) \rightarrow \text{Def}(\mathcal{D}_{\text{poly}}(\mathcal{M})[[\hbar]])\) does not depend on the choice of a covariant derivative.

**Proof:** The statement follows directly from Theorem 4.12 and Proposition 3.4. \(\square\)

Finally, note that Theorem 4.12 also holds in the equivariant setting of an action of a Lie group \(G\) on \(\mathcal{M}\) with \(G\)-invariant torsion-free covariant derivatives \(\nabla\) and \(\nabla'\).

**Proposition 4.15** Let \(G\) act on \(\mathcal{M}\) and consider two pairs of globalization data \((\nabla, r)\) and \((\nabla', r')\), where \(\nabla\) and \(\nabla'\) are two \(G\)-invariant torsion-free covariant derivatives and where \(r\) and \(r'\) are \(G\)-invariant. Then the formalities are equivariant and equivariantly homotopic

\[
U^{(\nabla, r)} \sim_G F^{(\nabla, r)} \sim_G F^{(\nabla', r')} \sim_G U^{(\nabla', r')}, \tag{4.15}
\]

i.e. all paths encoding the equivalence relation from (2.10) are \(G\)-equivariant.

**Proof:** The formalities are equivariant since all involved maps are [6, Theorem 5]. Moreover, \(\mathcal{U}^B\) and \(\epsilon^{-|h, \cdot|} \circ \mathcal{U}^B \circ \epsilon^{[h, \cdot]} \circ \tau \circ \nu^{-1} \sim \epsilon^{-|h, \cdot|} \mathcal{U}^B\) are equivariantly homotopic by the explicit form of the homotopy from Proposition 3.10. Moreover, again by [6, Theorem 5] we know that \(U\) and \(\mathcal{U}^B \circ \tau \circ \nu^{-1}\) are equivariantly homotopic, the same holds for the \((\nabla', r')\) case. Thus by Theorem 4.12 it only remains to show that \(P \circ \epsilon^{-|h, \cdot|}\) and \(P'\) are equivariantly homotopic. But this follows directly from Lemma 4.11 since all involved maps are equivariant. \(\square\)

In the case of proper Lie group actions one knows that invariant covariant derivatives always exist and one has even an invariant Hochschild-Kostant-Rosenberg theorem, compare [19, Theorem 5.11]. Thus the \(L_\infty\)-morphisms from (4.15) restrict to the invariant DGLAs and one obtains homotopic formalities from \((\mathcal{T}_{\text{poly}}(\mathcal{M}))^G\) to \((\mathcal{D}_{\text{poly}}(\mathcal{M}))^G\).
5 Final Remarks

In [7, Thm. 6] it is proven that the construction of the formality $U^{(\nabla,0)}$ is functorial for diffeomorphisms of pairs $(M, \nabla)$. More explicitly, the objects of the source category are pairs $(M, \nabla)$ of manifolds with torsion-free covariant derivatives, and a morphism from $(M, \nabla)$ to $(M', \nabla')$ is a diffeomorphism $\phi: M \to M'$ such that

$$\phi_* (\nabla_X Y) = \nabla'_{\phi_* X} \phi_* Y$$

for all $X, Y \in \Gamma^\infty(TM)$. The target category is the category of triples $(A, B, F)$, where $A, B$ are $L_\infty$-algebras and where $F: A \to B$ is an $L_\infty$-quasi-isomorphism. A morphism is a pair $(f, g): (A, B, F) \to (A', B', F')$ consisting of two $L_\infty$-morphisms $f: A \to A'$ and $g: B \to B'$ such that

$$\begin{array}{ccc}
A & \xrightarrow{F} & B \\
f & \downarrow & g \\
A' & \xrightarrow{F'} & B'
\end{array}$$

commutes. The functor from [7] is hence given by

$$(M, \nabla) \mapsto (T_{\text{poly}}(M), D_{\text{poly}}(M), U^{(\nabla,0)}),$$

mapping diffeomorphisms to push-forwards of vectorfields resp. differential operators. Our investigations from above lead now to a functor from the category of manifolds with diffeomorphisms into a category as above but with morphisms being homotopy classes of $L_\infty$-quasi-isomorphisms and $L_\infty$-morphisms, respectively. It is given by

$$M \mapsto ((T_{\text{poly}}(M), D_{\text{poly}}(M), [U^{(\nabla,0)}]),$$

where $[\cdot]$ indicates the homotopy class and where $\nabla$ is an arbitrary torsion-free connection.

Moreover, for a Lie group $G$, we can consider the source category of $G$-manifolds with proper $G$-actions and with equivariant diffeomorphisms to get the functor

$$M \mapsto (T_{\text{poly}}(M)^G, D_{\text{poly}}(M)^G, [U^{(\nabla,0)}]).$$

Here $\nabla$ is an arbitrary $G$-invariant torsion-free connection.

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