FINITE GRÖBNER BASIS ALGEBRAS WITH UNSOLVABLE
NILPOTENCY PROBLEM AND ZERO DIVISORS PROBLEM

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Abstract. This work presents a sample constructions of two algebras both with
the ideal of relations defined by a finite Gröbner basis. For the first algebra the
question whether a given element is nilpotent is algorithmically unsolvable, for the
second one the question whether a given element is a zero divisor is algorithmically
unsolvable. This gives a negative answer to questions raised by Latyshev. for which
the question whether a given element is nilpotent is algorithmically unsolvable. This
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1. Introduction

The word equality problem in finitely presented semigroups (and in algebras) cannot
be algorithmically solved. This was proved in 1947 by Markov (Ma) and independently
by Post (Po). In 1952 Novikov constructed the first example of the group with unsolvable
problem of word equality (see N1 and N2).

In 1962 Shirshov proved solvability of the equality problem for Lie algebras with one
relation and raised a question about finitely defined Lie algebras (see Sh).

In 1972 Bokut settled this problem. In particular, he showed the existence of a finitely
defined Lie algebra over an arbitrary field with algorithmically unsolvable identity problem
(Bo).

A detailed overview of algorithmically unsolvable problems can be found in BK.

Otherwise, some problems become decidable if a finite Gröbner basis defines a relations
ideal. In this case it is easy to determine whether two elements of the algebra are equal
or not (see Be).

Gröbner bases for various structures are investigated by the Bokut school in Guangzhou
(BC).

In his work, Piontkovsky extended the concept of obstruction, introduced by Latyshev
(see Pi1, Pi2, Pi3, Pi4).

Latyshev raised the question concerning the existence of an algorithm that can find out
if a given element is either a zero divisor or a nilpotent element when the ideal of relations
in the algebra is defined by a finite Gröbner basis.

Similar questions for monomial automaton algebras can be solved. In this case the
existence of an algorithm for nilpotent element or a zero divisor was proved by Kanel-
Belov, Borisenko and Latyshev (KBBL). Note that these algebras are not Noetherian and
not weak Noetherian. Iyudu showed that the element property of being one-sided zero
divisor is recognizable in the class of algebras with a one-sided limited processing (see I1,
I2). It also follows from a solvability of a linear recurrence relations system on a tree
(see KBII).

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An example of an algebra with a finite Gröbner basis and algorithmically unsolvable problem of zero divisor is constructed in [IP].

A notion of Gröbner basis (better to say Gröbner-Shirshov basis) first appeared in the context of noncommutative (and not Noetherian) algebra. Note also that Poincaré-Birkhoff-Witt theorem can be canonically proved using Gröbner bases. More detailed discussions of these questions see in [Bo], [U], [KBBL].

In the present paper we construct an algebra with a finite Gröbner basis and algorithmically unsolvable problem of nilpotency. We also provide a shorter construction for the zero divisors question.

For these constructions we simulate a universal Turing machine, each step of which corresponds to a multiplication from the left by a chosen letter.

Thus, to determine whether an element is a zero divisor or is a nilpotent, it is not enough for an algebra to have a finite Gröbner basis.

2. The plan of construction

Let $A$ be an algebra over a field $\mathbb{K}$. Fix a finite alphabet of generators $\{a_1, \ldots, a_N\}$. A word in the alphabet of generators is called a word in algebra.

The set of all words in the alphabet is a semigroup. The main idea of the construction is a realization of a universal Turing machine in the semigroup. We use the universal Turing machine constructed by Marvin Minsky in [Mi]. This machine has 7 states and 4-color tape. The machine can be completely defined by 28 instructions. Note that 27 of them have a form

$$(i, j) \to (L, q(i, j), p(i, j)) \text{ or } (i, j) \to (R, q(i, j), p(i, j)),$$

where $0 \leq i \leq 6$ is the current machine state, $0 \leq j \leq 3$ is the current cell color, $L$ or $R$ (left or right) is the direction of a head moving after execution of the current instruction, $q(i, j)$ is the state after current instruction, $p(i, j)$ is the new color of the current cell.

Thus, the instruction $(2, 3) \to (L, 3, 1)$ means the following: “If the color of the current cell is 3 and the state is 2, then the cell changes the color to 1, the head moves one cell to the left, the machine changes the state to 3.

The last instruction is $(4, 3) \to \text{STOP}$. Hence, if the machine is in state 4 and the current cell has color 3, then the machine halts.

Letters. By $Q_i, 0 \leq i \leq 6$ denote the current state of the machine. By $P_j, 0 \leq j \leq 3$ denote the color of the current cell.

The action of the machine depends on the current state $Q_i$ and current cell color $P_j$. Thus every pair $Q_i$ and $P_j$ corresponds to one instruction of the machine.

The instructions moving the head to the left (right) are called left (right) ones. Therefore there are left pairs $(i, j)$ for the left instructions, right pairs for the right ones and instruction STOP for the pair $(4, 3)$.

All cells with nonzero color are said to be non-empty cells. We shall use letters $a_1$, $a_2$, $a_3$ for nonzero colors and letter $a_0$ for color zero. Also, we use $R$ for edges of colored area. Hence, the word $Ra_{a_1}a_{a_2} \ldots a_{a_k}Q_iP_ja_{a_1}a_{a_2} \ldots a_{a_k}R$ presents a full state of Turing machine.

We model head moving and cell painting using computations with powers of $a_i$ (cells) and $P_i$ and $Q_i$ (current cell and state of the machine’s head).

3. Universal Turing machine

We use the universal Turing machine constructed by Minsky. This machine is defined by the following instructions:

$$(0, 0) \to (L, 4, 1) (0, 1) \to (L, 1, 3) (0, 2) \to (R, 0, 0) (0, 3) \to (R, 0, 1)$$

$$(1, 0) \to (L, 1, 2) (1, 1) \to (L, 1, 3) (1, 2) \to (R, 0, 0) (1, 3) \to (L, 1, 3)$$
The relations \( i, a, . . . a_{3}, Q_{0}, . . . Q_{6}, P_{0}, . . . P_{6}, R \) are used to move \( t \) from the left edge to the last letter \( a_{i} \) standing before \( Q_{i}P_{j} \), which represent the head of the machine. The relations \( 4.1 - 4.11 \) represent the computation process. The relation \( 4.2 \) is used to move \( t \) through the finishing letter \( R \).

Finally, the relation \( 4.12 \) halts the machine.

5. Nilpotency of the fixed word and machine halt

Let us call the word \( tR_{a_{1}}a_{u_{2}} . . . a_{u_{k}}Q_{i}P_{j}a_{v_{1}}a_{v_{2}} . . . a_{v_{l}}R \) the main word. The main goal is to prove the following theorem:

**Theorem 5.1.** The machine halts if and only if the main word is nilpotent in the algebra presented by the defining relations \( 4.1 - 4.12 \).

First, we prove some propositions.

**Remark.** We use sign \( \equiv \) for lexicographical equality and sign \( = \) for equality in algebra.

Consider a full state of our Turing machine represented by the word

\[ R_{a_{1}}a_{u_{2}} . . . a_{u_{k}}Q_{i}P_{j}a_{v_{1}}a_{v_{2}} . . . a_{v_{l}}R. \]

Suppose that \( U \equiv a_{u_{1}}a_{u_{2}} . . . a_{u_{k}} \) and \( V \equiv a_{v_{1}}a_{v_{2}} . . . a_{v_{l}} \). Therefore \( U \) and \( V \) represent the colors of all cells on the Turing machine tape. We denote the full state of this machine as \( M(i, j, U, V) \). Suppose that \( M(i', j', U', V') \) is the next state \( (M(i, j, U, V) \rightarrow M(i', j', U', V')) \).
Consider a semigroup $G$ presented by the defining relations \( \{1\} - \{12\} \). Suppose that $W(i, j, U, V)$ is a word in $G$ corresponding to machine state $M(i, j, U, V)$. (Actually $W(i, j, U, V) \equiv R a_u a_{u_2} \ldots a_{u_k} Q_i P_i a_{v_1} a_{v_2} \ldots a_{v_l} R_i$.)

**Proposition 5.1.** Let us move all the words from the relations \( \{1\} - \{12\} \) to the left-hand side. There exists a reduction order on the free monoid generated by alphabet $\Phi = \{t, a_0, \ldots , a_3, Q_0, \ldots , Q_6, P_0, \ldots , P_3, R\}$, such that the left-hand sides of the obtained equalities comprise a Gröbner basis in the ideal generated by them.

**Proof.** Recall, that by reduction order on the free monoid $\Phi^*$ we mean a well order such that the empty word is the minimal one, and for any $a, b, s_1, s_2 \in \Phi^*$, if $s_1 \prec s_2$ then $as_1b < as_2b$.

Any word $w$ from $\Phi^*$ can be uniquely written as $X_0tX_1t \cdots tX_n$, where $X_i \in \Phi^*$ are free from the letter $t$. Each $X_i$ can be empty, even all of them (if the word is $t^n$). By *height* of this word we call

$$h(w) = \sum_{i=0}^{n} 2^i \deg X_i.$$  

We define the following order. Given two words $w_1$ and $w_2$, we compare them with respect to the degree of $t$. If $\deg_t(w_1) < \deg_t(w_2)$ then $w_1 < w_2$. If $\deg_t(w_1) = \deg_t(w_2)$ then we compare them with respect to the height. If $h(w_1) < h(w_2)$ then $w_1 < w_2$. If their heights are also equal then we use a deglex order to compare them.

We need to prove that this order is a reduction order.

Note that an empty word is the minimal (it has a zero degree of $t$, a zero height and a zero degree).

Assume $a, b, s_1, s_2 \in \Phi^*$ and $s_1 \prec s_2$.

If $\deg_t(s_1) < \deg_t(s_2)$, then $\deg_t(as_1b) < \deg_t(as_2b)$, therefore $as_1b < as_2b$.

Assume $\deg_t(s_1) = \deg_t(s_2) = n$ and $h(s_1) < h(s_2)$. In this case we will show that multiplication of inequality by one symbol does not change it. In other words, we will show for any symbol $x \in \Phi$ that $xs_1 \prec xs_2$ and $s_1x < s_2x$. First assume that $x \neq t$. Then multiplication by $x$ from the left increases a height by 1 of both sides, thus an inequality remains. Note that multiplication by $x$ from the right increases a height by $2^n$ of both sides, and an inequality remains also. The multiplication by $t$ from the left multiplies both heights by 2 and multiplication by $t$ from the right does not change it.

Now assume that $\deg_t(s_1) = \deg_t(s_2)$ and $h(s_1) = h(s_2)$. Hence $\deg_t(as_1b) = \deg_t(as_2b)$ and $h(as_1b) = h(as_2b)$. In this case we compare both pairs $(s_1, s_2)$ and $(as_1b, as_2b)$ by deglex order which is a reduction order.

Note that every left-hand side contains a leading monomial. There is no such word that begins some leading monomial in the basis and ends some other leading monomial. □

**Lemma 5.1.** For any nonempty word $U \equiv a_{i_1} \cdots a_{i_l}$ we have $tUR = UTr$.

**Proof.** We can use the relation \( \{8\} \) $(l-1)$ times and transform $tUR$ to $a_{i_1} \cdots a_{i_{l-1}} ta_{i_l} R$. After that we use the relation \( \{12\} \). □

**Proposition 5.2.** If $i = 4$ and $j = 3$ then $tW(i, j, U, V) = 0$. Otherwise, the following condition holds: $tW(i, j, U, V) = W(i', j', U', V')t$.

**Proof.** Consider the word $tW(i, j, U, V) = tRUQ_iP_j VR$. If $i = 4$ and $j = 3$ then we can apply relation \( \{12\} \). Otherwise, suppose that $(i, j)$ is a left pair.

If $U$ is not empty word, then we can write $U = \tilde{U}a_k$ for some $0 \leq k \leq 3$. In this case we have the word $tRUa_kQ_iP_j VR$. We can use the relation \( \{3\} \) to transform it to $RtUa_kQ_iP_j VR$. Now we use the relation \( \{3\} \) the degree of $\tilde{U}$ times: our words transforms to $R\tilde{U}ta_kQ_iP_j VR$. After that we use the relation \( \{4\} \) and our word transforms to $R\tilde{U}Q_iP_j VR$. Now we use Lemma \( \{5\} \).
If \( U \) is empty, then \( tW(i,j,U,V) \equiv tRQ_iP_jVR \). In this case we will start our chain with using relation (4.5): \( tRQ_iP_jVR = RQ_iP_jta_{p(i,j)}VR \). After that we use Lemma 5.3.

Suppose that \((i,j)\) is a right pair. In this situation we will have six cases:

**Case 1** \( U \) and \( V \) are empty words. In this case our word is \( tRQ_iP_jR \) and we use the relation (4.14).

**Case 2** \( U \) is empty and \( V = a_k \) is a word of degree 1. In this case our word is \( tRQ_iP_ja_kVR \). We can use a relation (4.3) to transform it to \( Ra_{p(i,j)}Q_jP_jVR \), where \( V \) is not empty. Thus we can use Lemma 5.1 to complete the chain.

**Case 3** \( U \) is empty and \( V = a_k\tilde{V} \) is a word of degree greater than 1. In this case our word is \( tRQ_iP_ja_k\tilde{V}R \). We can use a relation (4.3) to transform it to \( Ra_{p(i,j)}Q_jP_jt\tilde{V}R \), where \( \tilde{V} \) is not empty. Hence we can use Lemma 5.1 to complete the chain.

**Case 4** \( U = Ua_l \) is not empty and \( V = a_k\tilde{V} \) is a word of degree 1. In this case our word is \( tRUa_lQ_jP_jR \). We use a relation (4.14) and transform it to \( RRUa_lQ_jP_jVR \). Using relation (4.3) the degree of \( U \) times will transform our word to \( RRUa_lQ_jP_jR \). A relation (4.14) completes a chain.

**Case 5** \( U = Ua_l \) is not empty and \( V = a_k \) is a word of degree 1. In this case our word is \( tRUa_lQ_jP_ja_kVR \). Similar to Case 4 we can transform our word to \( RRUa_lQ_jP_ja_kVR \). A relation (4.14) completes a chain.

**Case 6** \( U = Ua_l \) is not empty and \( V = a_k\tilde{V} \) is a word of degree greater than 1. In this case our word is \( tRUa_lQ_jP_ja_k\tilde{V}R \). Similar to Case 4 we can transform our word to \( RRUa_lQ_jP_ja_k\tilde{V}R \). A relation (4.14) transforms it to \( RRUa_lQ_jP_ja_k\tilde{V}R \). Now we use lemma 5.1 to complete our chain.

\[ \square \]

**Proposition 5.3.** The following statements are equivalent:

(i) The Turing machine described above begins with the state \( M(i,j,U,V) \) and halts in several steps.

(ii) There exists a positive integer \( N \) such that \( t^N RUQ_iP_jVR = 0 \).

**Proof.** First, prove that second statement is a consequence of the first one.

Suppose that \( M(i,j,U,V) \) transforms to \( M(4.3,':',V') \) in one step. According to Proposition 5.2 \( tW(i,j,U,V) = W(4.3,':',V')t \). Then we can apply \( Q_4P_3 = 0 \) by (4.14) and obtain zero.

Suppose that the statement is true for \( m \) (and fewer) steps. Let the machine begin with state \( M(i,j,k,n) \) and halt after \( m+1 \) step. Consider the first step in the chain. Let it be the step from \( M(i,j,k,n) \) to \( M(4.3,':',V') \). Apply Proposition 5.2 for this step. Hence \( tRUQ_iP_jVR = R'U'Q_jP_jVR' \).

The machine started in the state \( M(4.3,':',V') \) halts in \( m \) steps. Using induction we complete the proof.

Now let us prove that the first statement is a consequence of the second one.

If \( t^N RUQ_iP_jVR = 0 \), then there exists a chain of equivalent words, starting with \( t^N RUQ_iP_jVR \) and finishing with 0. The only way to obtain 0 is to use a relation \( Q_4P_3 = 0 \). Therefore the word before 0 in the chain contains \( Q_4P_3 \).

By structure of the word \( W \), \( S(W) \) let us denote the word \( W_Q \), where all letters \( t \) will be deleted. Each word in the chain will have a structure \( RU_{i,t}Q_{i,t}P_{j,t}V_{k,t}R \) because the only relation that breaks this structure is \( Q_4P_3 = 0 \), and it will be used only one time, in the end of the chain. Note that each structure corresponds to the Turing machine. The only way to obtain 0 in this chain is to change indices of \( Q \) and \( P \) in the structure. This can be done by moving \( t \).

According to the Proposition 5.2 moving \( t \) from the left to the right corresponds to the Turing Machine’s one step to the future, and moving \( t \) from the right to the left corresponds to the Turing Machine’s one step to the past (note that this is not always possible). There is a Gröbner basis of relations in our algebra, thus we can assume that in
our chain words decrease (each word is lower than the previous with respect to the order on the free monoid \( \Phi^* \)). Therefore letters \( t \) move only from the left to the right. Hence there exists \( k \leq N \) such that 
\[
t^N R \mathcal{U} Q_i P_j V R = t^{N-k} R \mathcal{U} Q_i P_j V R t^k.
\]
Therefore the machine halts after \( k \) steps. \( \Box \)

Now we are ready to prove the theorem above.

**Theorem 5.2.** Consider an algebra \( A \) presented by the defining relations \((4.1)-(4.12)\). The word \( t \mathcal{U} Q_i P_j V R \) is nilpotent in \( A \) if and only if machine \( M(i, j, U, V) \) halts.

**Proof.** Suppose that \( (t \mathcal{U} Q_i P_j V R)^n = 0 \). The structure of this word corresponds to a row of \( n \) separate machines. Using relations we can transform some machine to the next state (note that we have a Gröbner basis in the algebra, therefore we can assume that words in the chain will decrease). Thus if we obtain \( Q_4 P_3 \) for some machine, we can conclude that this machine halts after several steps. Therefore \( M(i, j, U, V) \) halts.

Suppose that \( M(i, j, U, V) \) halts. Then \( t^n \mathcal{U} Q_i P_j V R = 0 \) for some minimal \( n \). We can obtain \( (t \mathcal{U} Q_i P_j V R)^n = A^n \mathcal{U} Q_i P_j V R \) (for some word \( A \)) by using Proposition \( \Box \) several times. Therefore \( (t \mathcal{U} Q_i P_j V R)^n = 0 \). \( \Box \)

Since the halting problem cannot be algorithmically solved, the nilpotency problem in algebra \( A \) is algorithmically unsolvable.

6. DEFINING RELATIONS FOR A ZERO DIVISORS QUESTION

We use the following alphabet:
\[
\Psi = \{ t, s, a_0, \ldots, a_3, Q_0, \ldots, Q_6, P_0, \ldots, P_3, L, R \}.
\]

For every pair except \((4,3)\) the following functions are defined: \( q(i,j) \) is a new state, \( p(i,j) \) is a new color of the current cell (the head leaves it).

Consider the following defining relations:

\[
t L a_k = L t a_k; \quad 0 \leq k \leq 3 \quad (6.1)
\]
\[
t a_k a_l = a_k t a_l; \quad 0 \leq k, l \leq 3 \quad (6.2)
\]
\[
s R = R s; \quad (6.3)
\]
\[
s a_k = a_k s; \quad 0 \leq k \leq 3 \quad (6.4)
\]
\[
t a_k Q_i P_j = Q_{q(i,j)} P_k a_{p(i,j)} s; \quad \text{for left pairs } (i,j) \text{ and } 0 \leq k \leq 3 \quad (6.5)
\]
\[
t L Q_i P_j = L Q_{q(i,j)} P_0 a_{p(i,j)} s; \quad \text{for left pairs } (i,j) \quad (6.6)
\]
\[
t a_k Q_i P_j a_k = a_k a_{p(i,j)} Q_{q(i,j)} P_k s; \quad \text{for left pairs } (i,j) \text{ and } 0 \leq k, l \leq 3 \quad (6.7)
\]
\[
t L Q_i P_j a_k = L a_{p(i,j)} Q_{q(i,j)} P_0 s; \quad \text{for right pairs } (i,j) \text{ and } 0 \leq k \leq 3 \quad (6.8)
\]
\[
t a_k Q_i P_j R = a_k a_{p(i,j)} Q_{q(i,j)} P_0 R s; \quad \text{for right pairs } (i,j) \text{ and } 0 \leq l \leq 3 \quad (6.9)
\]
\[
t L Q_i P_j R = L a_{p(i,j)} Q_{q(i,j)} P_0 R s; \quad \text{for right pairs } (i,j) \quad (6.10)
\]
\[
Q_4 P_3 = 0; \quad (6.11)
\]

The relations \( (6.1)-(6.2) \) are used to move \( t \) from the left edge to the letters \( Q_i, P_j \) which present the head of the machine. The relations \( (6.3)-(6.4) \) are used to move \( s \) from the letter \( Q_i, P_j \) to the right edge. The relations \( (6.5)-(6.6) \) represent the computation process. Here we use relations of the form \( t U = V s \).

Finally, the relation \( (6.11) \) halts the machine.
7. Zero divisors and machine halt

Let us call the word \(La_{u_1}a_{u_2} \ldots a_{u_k}QjPja_va_2 \ldots a_vR\) the main word. The main goal is to prove the following theorem:

**Theorem 7.1.** The machine halts if and only if the main word is a zero divisor in the algebra presented by the defining relations \(6.1 - 6.11\).

Consider a full state of our Turing machine represented by the word

\[La_{u_1}a_{u_2} \ldots a_{u_k}QjPja_va_2 \ldots a_vR.\]

Suppose that \(U = a_{u_1}a_{u_2} \ldots a_{u_k}\) and \(V = a_va_2 \ldots a_v\). Therefore \(U\) and \(V\) represent the colors of all cells on the Turing machine tape. We denote the full state of this machine as \(T(i, j, U, V)\). Suppose that \(T(i', j', U', V')\) is the next state \((T(i, j, U, V) \rightarrow T(i', j', U', V'))\).

Consider a semigroup \(S\) presented by the defining relations \(6.1 - 6.11\). Suppose that \(F(i, j, U, V)\) is a word in \(S\) corresponding to machine state \(T(i, j, U, V)\).

**Proposition 7.1.** Let us move all the words from relations \(6.1 - 6.11\) to the left-hand side. Consider the semi-DEGLEX order: \(\{1, s, a_0, \ldots, a_3, Q_0, \ldots, Q_6, P_0, \ldots, P_3, L, R\}\). The left-hand sides of the obtained equalities comprise a Gröbner basis in the ideal generated by them.

**Proof.** We will use a weighted degree instead of the usual: each letter from the alphabet (except for \(t\)) will have degree 1, however the degree of \(t\) equals 2. (For example, \(\text{deg}(tRL) = 4\))

This order is a reduction order.

Note that every left-hand side contains a leading monomial. There is no such word that begins some leading monomial in the basis and ends some other leading monomial. \(\square\)

**Proposition 7.2.** If \(i = 4\) and \(j = 3\) then \(tF(i, j, U, V) = 0\). Otherwise, the following condition holds: \(tF(i, j, U, V) = F(i', j', U', V')s\).

**Proof.** Consider the word \(tF(i, j, U, V) = tLUQjPjVR\). If \(i = 4\) and \(j = 3\) then we can apply relation \(6.11\). Otherwise, suppose that \((i, j)\) is a left pair.

If \(U\) is an empty word then \(tF(i, j, U, V) = tLUQjPjVR\). Hence we can apply relation \(6.0\) to obtain \(tLUQjPjVR = Q_jP_ja_{i+j}sVR\). Using \(6.3\) and \(6.4\) we finally have

\[tLUQjPjVR = Q_jP_ja_{i+j}sVR = Q_jP_ja_{i+j}VRs.\]

According to the definition of \(q(i, j)\) and \(p(i, j)\), the word \(Q_jP_ja_{i+j}VR\) corresponds to the next state of the machine.

If \(U\) is not an empty word, we can write \(U = U_1a_k\) for some \(k\). We use the relations \(6.1\) and \(6.2\) and obtain that \(tLUQjPjVR = U_1tQja_kPjVR\). Further, we use relation \(6.3\):

\[LU_1tQja_kPjVR = LU_1Qja_{q(i,j)}P_ka_{p(i,j)}VRs.\]

The word \(LU_1Qja_{q(i,j)}P_ka_{p(i,j)}VR\) corresponds to the next state of the machine.

Assume that \((i, j)\) is a right pair. If \(U\) and \(V\) are empty words, then we use relation \(6.10\).

If \(U\) is empty, and \(V = a_kV\) is not, then we use the relation \(6.5\) and obtain \(tLUQjPja_kV = P_ka_{p(i,j)}Qja_{q(i,j)}a_{p(i,j)}VR\). After that we use relations \(6.2\) and \(6.3\) and move \(s\) to the right.

Assume \(U = Ua_k\) is not empty. In this case we use the relation \(6.2\) the length of \(U\) times and obtain \(tLUa_k = UtQja_k\). If \(V\) is empty we can use the relation \(6.3\). If \(V = a_kV\) is not empty then we can use the relation \(6.4\), after that we will use relations \(6.4\) and \(6.3\) and move \(s\) to the right. \(\square\)
Proposition 7.3. The following statements are equivalent:

(i) The Turing machine described above begins with the state \(T(i, j, U, V)\) and halts in several steps.

(ii) There exists a positive integer \(N\) such that \(t^N LUQ_i P_j VR = 0\).

Proof. First, prove that second statement is a consequence of the first one.

Suppose that \(T(i, j, U, V)\) transforms to \(T(4, 3, U', V')\) in one step. According to Proposition 6.1, \(tF(i, j, U, V) = F(4, 3, U', V')\). Then we can apply \(Q_i P_j = 0\) by 6.11 and obtain zero.

Suppose that the statement is true for \(m\) (and fewer) steps. Let the machine begin with state \(T(i, j, k, n)\) and halt after \(m + 1\) steps. Consider the first step in the chain. Let it be the step from \(T(i, j, U, V)\) to \(T(i', j', U', V')\). Apply Proposition 7.4 for this step. Hence \(tLUQ_i P_j VR = LU'Q_i P_j V'R\).

The machine started in the state \(T(i', j', U', V')\) halts in \(m\) steps. Using induction we complete the proof.

Now let us prove that the first statement is a consequence of the second one.

If \(t^N LUQ_i P_j VR = 0\), then there exists a chain of equivalent words, starting with \(t^N LUQ_i P_j VR\) and finishing with 0. The only way to obtain 0 is to use a relation \(Q_i P_3 = 0\). Therefore the word before 0 in the chain contains \(Q_i P_3\).

By structure of the word \(W, S(W)\) let us denote the word \(W\), where all letters \(t\) and \(s\) will be deleted. Each word in the chain will have a structure \(LU_0 Q_k P_j V_k R\) because the only relation that breaks this structure is \(Q_i P_3 = 0\), and it will be used only once in time, in the end of the chain. Note that each structure corresponds to the Turing machine. The only way to obtain 0 in this chain is to change indices of \(Q\) and \(P\) in the structure. This can be done by moving \(t\).

According to the Proposition 7.2, moving \(t\) from the left to the right, and transforming it to \(s\) corresponds to the Turing Machine’s one step. Note that there is a Gröbner basis on our algebra, thus we can assume that words in the chain decrease. In particular, moving \(s\) from the left to the right, transforming it to \(t\) is impossible.

Therefore we can obtain \(Q_i P_3\) only by moving \(t\) from the left to \(s\) on the right, and there exists \(k \leq N\) such that \(t^N LUQ_i P_j VR = t^N - k LUQ_i P_3 VR R^k\).

Therefore the machine halts after \(k\) steps. \(\square\)

Proposition 7.4. If \(Xt = 0\) in \(S\), then \(X = 0\). If \(sX = 0\) in \(S\), then \(X = 0\).

Proof. Suppose that we apply some relations and transform \(Xt\) to zero.

We say that the letter \(t\) is almost last if the word has the form \(Y_1 t Y_2\), and \(Y_2\) contains \(a_k\) and \(L\) letters only. Note that if an almost last \(t\)-letter occurs in some relation then this relation is \(6.1\) or \(6.2\). Therefore that \(t\)-letter is always almost last. It is clear that an almost last \(t\)-letter always exists in every word which is equivalent to \(Xt\). Since an almost last \(t\)-letter never participates in relations \(6.3\) - \(6.11\), we can situate it on the right edge of the word \(Xt\) while we use our relations. We did not use the \(t\)-letter, and therefore we can do the same with the word \(X\).

Similarly we can prove that if \(sX = 0\) then \(X = 0\). \(\square\)

Proposition 7.5. If \(Xt^n = 0\) in \(S\), then \(X = 0\). If \(s^n X = 0\) in \(S\), then \(X = 0\).

Proof. We can prove this by induction. \(\square\)

Now we are ready to prove the theorem above.

Theorem 7.2. Consider an algebra \(H\) presented by the defining relations \(6.1\) - \(6.11\).

The word \(LUQ_i P_j VR\) is a zero divisor in the algebra \(H\) if and only if machine \(T(i, j, U, V)\) halts.
Proof. Suppose that machine \( T(i, j, U, V) \) halts. Using Proposition \( \text{(7.3)} \) we have \( t^{\alpha} \text{LUQ}_t P_V R = 0 \) for some positive integer \( N \). Thus, the word \( \text{LUQ}_t P_V R \) is a zero divisor.

Let \( \text{XLUQ}_t P_V R Y = 0 \) for some algebra elements \( X, Y \neq 0 \). Suppose that \( X, Y \) are some words.

Note that \( L \) and \( R \) letters cannot disappear from the word. Hence we can divide our word into three parts: to the left of \( L \), to the right of \( R \), and between \( L \) and \( R \). There is only one relation which can turn the word \( \text{XLUQ}_t P_V R Y \) to zero: \( Q_4 P_3 = 0 \). Thus this subword \( Q_4 P_3 \) can appear in three possible parts of the word. Note that only \( t \) letters can pass through \( L \) and only \( s \) letters can pass through \( R \). Every relation can change nothing in the area to the left side of \( L \) and to the right side of \( R \), except \( t \) and \( s \)-letters occurrences. Therefore if \( Q_4 P_3 \) appears to the left of \( L \), then \( X s^n = 0 \). Using Proposition \( \text{(7.5)} \) we obtain a contradiction: \( X = 0 \). Similarly if \( Q_4 P_3 \) appears to the right of \( R \), then \( Y = 0 \). Thus \( Q_4 P_3 \) appears between \( L \) and \( R \).

Consider the structure of the word \( \text{LUQ}_t P_V R \). For any structure of a word equivalent to \( \text{LUQ}_t P_V R \) there exists a corresponding state of the machine. Since only \( t \) letters can pass through \( L \) and only \( s \) letters can pass through \( R \), we can change the structure of the word \( \text{LUQ}_t P_V R \) by turn to the next or the previous machine state. If \( Q_4 P_3 \) appears between \( L \) and \( R \) then we can obtain a STOP state. Thus the machine \( T(i, j, U, V) \) halts.

Now let us consider the general case: \( X, Y \) are some algebra elements. Suppose that \( X = c_1 X_1 + \ldots c_n X_n \), \( Y = d_1 Y_1 + \ldots d_m Y_m \), where \( X_k, Y_l \) are words, and \( c_k \) and \( d_l \) are elements of the field. Without loss of generality we may assume that \( n \) is the minimal possible, and for this \( n \) \( m \) is the minimal possible. We also may assume that \( X_k, Y_l \) are written in the reduced form. We assume that either \( n > 1 \), or \( m > 1 \).

Consider the function \( \tilde{h} : \Psi^* \to \mathbb{N}_0 \): for any word \( w \) \( \tilde{h}(w) = \deg_t(w) + \deg_s(w) \). Note that relations \( \text{(6.1)} - \text{(6.10)} \) do not change value of \( \tilde{h} \), therefore it is invariant under the word reduction. Assume that \( \tilde{h}(X_{k_1}) \neq \tilde{h}(X_{k_2}) \). In this case we will take a subset \( S_x \subseteq \{1, \ldots, n\} \) such that \( \tilde{h} \) takes a maximal value on \( X_k \) for \( k \in S_x \). We also take a subset \( S_y \subseteq \{1, \ldots, m\} \) such that \( \tilde{h} \) takes a maximal value on \( Y_l \) for \( l \in S_y \). We know that \( \sum_{k \in S_x} c_k X_k W(d_1 Y_1 + \ldots d_m Y_m) = \sum_{k,l} c_k d_l X_k \text{LUQ}_t P_V Y_l \). Therefore one can reduce this element to zero. However none of the elements \( X_k \text{LUQ}_t P_V Y_l \) can be reduced to zero. Thus, all elements \( X_k \text{LUQ}_t P_V Y_l \) can be separated to several sets of similar words. Note that all words \( X_k \text{LUQ}_t P_V Y_l \) (where \( k \in S_x \) and \( l \in S_y \)) can be similar only to a word \( X_{k'} \text{LUQ}_t P_V Y_{l'} \) where \( k' \in S_x \) and \( l' \in S_y \). Hence, \( \sum_{k \in S_x} c_k X_k Y_l = 0 \). A contradiction (\( n \) was taken as a minimal possible). Therefore \( \tilde{h}(X_k) \) does not depend on \( k \) and \( \tilde{h}(Y_l) \) does not depend on \( l \).

We have \( \text{XLUQ}_t P_V R Y = \sum_{k,l} c_k d_l X_k \text{LUQ}_t P_V Y_l \). We can consider our defining relations as reductions and use them to find the Gröbner basis of every term \( X_k \text{LUQ}_t P_V Y_l \). Let us fix the \( t \)'s at the end of the \( X_k \) words: \( X_k = X_k t^{\alpha_k} \). These are lexicographical equalities and \( q_k \geq 0 \).

Since \( \sum_{k,l} c_k d_l X_k t^{\alpha_k} \text{LUQ}_t P_V Y_l = 0 \), this sum (in the reduced form) can be separated into several sets of similar monomials. Consider one of these sets: \( X_k t^{\alpha_k} \text{LUQ}_t P_V Y_l \). If these monomials are similar then all \( X_k \) must be also similar. Recall that \( \tilde{h}(X_k) \) does not depend on \( k \), therefore all \( x_k \) must be the same.

Hence, \( n = 1 \), and \( m > 1 \) and we have a situation \( \text{XLUQ}_t P_V R (\sum_{l=1}^m d_l Y_l) = 0 \), where \( X \in \Psi^* \) is a word, and \( m \) is minimal. Therefore, all words \( \text{XLUQ}_t P_V Y_l \) should be equal in the algebra, however \( Y_l \) should be pairwise different. If we will reduce word \( W Y_l \) (for \( W = \text{XLUQ}_t P_V \)), only letter \( s \) can pass through \( R \), therefore the only case to reduce it is to pass letters \( s \) from \( W \) to \( Y_l \). The number of these letters \( s \) depend on \( W \) therefore it will be similar. Therefore we will have an equality \( s^k Y_1 = s^k Y_2 = \cdots = s^k Y_m \).
(for some non negative number $k$) in the algebra. Note that relations with letter $s$ do not change a structure of the word, and one can see that for any two different words $Y$ and $Z$, words $sY$ and $sZ$ must be also different. Therefore $s^kY_1 \neq s^kY_2$ in the algebra, and $m$ cannot be larger than 1.

This contradiction completes the proof. □

Since the halting problem cannot be algorithmically solved, the zero divisors problem in algebra $H$ is algorithmically unsolvable.

**Remark.** We can consider two semigroups corresponding to our algebras: in both algebras each relation is written as an equality of two monomials. The refore the same alphabets together with the same sets of relations define semigroups. I n both semigroups the equality problem is algorithmically solvable, since it is solvable in algebras. However in the first semigroup a nilpotency problem is algorithmically unsolvable, and in the second semigroup a zero divisor problem is algorithmically unsolvable.

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