Kolmogorov Numbers of Embeddings of Besov Spaces of Dominating Mixed Smoothness into $L_\infty$

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Abstract

In this paper we shall give two-sided sharp estimates of Kolmogorov numbers of embeddings of the Besov spaces with dominating mixed smoothness $S_{t}^{p,q}B((0,1)^d)$ into $L_\infty((0,1)^d)$.

1 Introduction and main results

In this paper we investigate the asymptotic behaviour of the Kolmogorov numbers of the embedding

$$id: S_{p,q}^{t}B(\Omega) \to L_\infty(\Omega), \quad 0 < p, q \leq \infty \quad \text{and} \quad t > \frac{1}{p},$$

where $S_{p,q}^{t}B(\Omega)$ is the Besov space of dominating mixed smoothness on the unit cube $\Omega = (0,1)^d$. Let $T : X \to Y$ be a bounded linear operator between complex quasi-Banach spaces. Then the $n$-th Kolmogorov number of $T$ is defined by

$$d_n(T : X \to Y) = \inf\{\|Q_N T\| : \dim(N) < n\},$$

where $N$ is a subspace of $Y$ and $Q_N$ is a canonical surjection from $Y$ onto the quotient space $Y/N$. The problem of estimating the behaviour of Kolmogorov numbers of the embedding $id : S_{p,q}^{t}B(\Omega) \to L_{p_0}(\Omega)$, $1 \leq p_0 \leq \infty$, has been considered at various place over the last thirty years, we refer to Belinskii [3], Galeev [8], Romanyuk [17, 18, 19, 20, 21] and Temlyakov [26, 27, 28]. There exists fairly complete descriptions of the asymptotic behaviour of Kolmogorov numbers in the situation $1 < p_0 < \infty$, we refer to Bazarkhanov [2] for the most recent publication in this direction. However in the extreme cases, that is $p_0 = 1$ and $p_0 = \infty$, the picture is less satisfactory. There are a few results where the exact order of Kolmogorov numbers of the embeddings $S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)$ and $S_{p,q}^{t}B(\Omega) \to L_{1}(\Omega)$ have been found, e.g. in case $S_{\infty,1}^{t}B(\Omega) \to L_{\infty}(\Omega)$, see [21] or in case $S_{\infty,\infty}^{t}B(\Omega) \to L_{\infty}(\Omega)$.
(consequence of [27]), see Lemma 3.3 below. Romanyuk [21] has attempted to investigate the general case. However, his estimate from above has the additional factor $(\log n)^{1/2}$, see Proposition 3.1.

In the literature many times people prefer to work with the notion of Kolmogorov widths. If $W$ is a subset of a normed space $X$ then the Kolmogorov $n$-width $d_n(W, X)$ is defined by

$$d_n(W, X) := \inf_{L \in \mathcal{L}_{n}} \sup_{f \in W} \inf_{g \in L} \|f - g\|_X,$$

where outer infimum is taken over all linear subspaces in $X$ of dimension at most $n$. It is not difficult to see that the Kolmogorov $(n - 1)$-width of the set $T(B_X)$ in the space $Y$ coincide with the $n$-th Kolmogorov number of the operator $T : X \to Y$. Here $B_X$ is the closed unit ball in $X$, that is

$$d_n(T : X \to Y) = \inf_{L \in \mathcal{L}_{n-1}} \sup_{\|x\| \leq 1} \inf_{y \in L} \|T x - y\|_Y.$$  \hfill (1.1)

Kolmogorov numbers are particular examples of so-called $s$-numbers, see [16, Chapter 2]. Let $X, X_0, Y_0$ be quasi-Banach spaces and let $Y$ be a Banach space. Then an $s$-number of the operator $T \in \mathcal{L}(X, Y)$ is a rule $s : T \to \{s_n(T)\}_{n \in \mathbb{N}}$ satisfying

s1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in \mathcal{L}(X, Y)$;

s2. $s_{n+m-1}(S + T) \leq s_n(S) + s_m(T)$ for $S, T \in \mathcal{L}(X, Y)$ and $m, n = 1, 2, ...$;

s3. $s_n(B T A) \leq \|B\| \cdot s_n(T) \cdot \|A\|$ for $A \in \mathcal{L}(X_0, X), T \in \mathcal{L}(X, Y), B \in \mathcal{L}(Y, Y_0)$;

s4. $s_n(T) = 0$ if rank$(T) < n$ for all $n \in \mathbb{N}$;

s5. $s_n(id : \ell^n_2 \to \ell^n_2) = 1$ for all $n \in \mathbb{N}$.

Another example of $s$-numbers are the approximation numbers defined by the formula

$$a_n(T) := \inf\{\|T - A\| : A \in \mathcal{L}(X, Y), \text{ rank}(A) < n\}.$$ 

The approximation numbers are the largest $s$-numbers, see [16, Theorem 2.3.4]. Kolmogorov and approximation numbers have some further interesting property namely multiplicativity, see [15, Theorem 11.9.2]. That is, if $Z$ is a further Banach space then

s6. $s_{n+m-1}(S T) \leq s_n(S) s_m(T)$ for $T \in \mathcal{L}(X, Y), S \in \mathcal{L}(Y, Z)$ and $m, n = 1, 2, ...$.

By using abstract properties of the Kolmogorov numbers and results obtained by Belinskii, Romanyuk and Temlyakov, see Proposition 3.1 for details, we are able to derive the following quite satisfactory results.

**Theorem 1.1.** Let $0 < q \leq \infty$. 


(i) If $0 < p \leq 2$ and $t > \frac{1}{p}$ we have
\[
d_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t + \frac{1}{p} - \frac{1}{2}(\log n)^{(d-1)(t-\frac{1}{2} + \frac{1}{2} - \frac{1}{q} + \frac{1}{2})}},
\tag{1.2}
\]
for all $n \geq 2$.

(ii) If $2 < p \leq \infty$ and $t > \frac{1}{p}$ we have
\[
d_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{q})},
\tag{1.3}
\]
for all $n \geq 2$. In case $2 < p \leq \infty$ and $t > \frac{1}{p}$ we have
\[
d_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{q})}, \quad n \geq 2.
\tag{1.4}
\]

**Remark 1.2.**

(i) Recall that the embedding $id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)$ is compact if and only if $t > \frac{1}{p}$, see [37, Theorem 3.17].

(ii) The most interesting case is given by $p = q$. It follows that if $0 < p \leq \infty$ and $t > \frac{1}{p}$, we have
\[
d_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp \begin{cases} 
    n^{-t + \frac{1}{p} - \frac{1}{2}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p})}} & \text{if } p \leq 2, \\
    n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p})} & \text{if } 2 < p,
\end{cases}
\]
for all $n \geq 2$.

(iii) Also the case $q = \infty$ is of particular interest. That is if $0 < p \leq \infty$ and $t > \frac{1}{p}$, it holds
\[
d_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp \begin{cases} 
    n^{-t + \frac{1}{p} - \frac{1}{2}(\log n)^{(d-1)(t-\frac{1}{2} + \frac{1}{2} + \frac{1}{2})}} & \text{if } p \leq 2, \\
    n^{-t}(\log n)^{(d-1)(t+\frac{1}{2})} & \text{if } 2 < p,
\end{cases}
\]
for all $n \geq 2$.

Fortunately our methods allow a step aside to approximation numbers.

**Theorem 1.3.** Let $0 < p \leq 1$ and $t > \frac{1}{p}$. Then we have
\[
a_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t + \frac{1}{p} - \frac{1}{2}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p})}},
\tag{1.5}
\]
for all $n \geq 2$.

**Remark 1.4.** Theorem 1.3 is a supplement of a result obtained by Nguyen and Sickel [41, Theorem 2.8], that is
\[
a_{n}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega)) \asymp \begin{cases} 
    n^{-t + \frac{1}{2}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p})}} & \text{if } 1 \leq p < 2, t > 1, \\
    n^{-t + \frac{1}{p}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p})}} & \text{if } 2 \leq p \leq \infty, t > \frac{1}{p},
\end{cases}
\]
for all $n \geq 2$. 

3
The plan of the paper is as follows. In Section 2 we recall the definition of Besov spaces of dominating mixed smoothness and discuss their interpolation properties. The next Section 3 is devoted to the investigation of the Kolmogorov numbers of embeddings of certain sequence spaces associated to Besov spaces of dominating mixed smoothness. There we also prove our main results.

**Notation.** As usual, \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{Z} \) the integers and \( \mathbb{R} \) the real numbers. If \( \vec{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d \) then we put \( |\vec{j}|_1 := j_1 + \cdots + j_d \). The symbol \( c \) denotes positive constants which depend only on the fixed parameters \( t, p, q \) and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. The meaning of \( \lesssim \) is given by: there exists a constant \( c > 0 \) such that \( A \leq cB \). Similarly \( \gtrsim \) is defined. The symbol \( \mathcal{A} \sim \mathcal{B} \) will be used as an abbreviation of \( \mathcal{A} \lesssim \mathcal{B} \lesssim \mathcal{A} \). If \( \mathcal{X} \) and \( \mathcal{Y} \) are two quasi-Banach spaces, then the symbol \( \mathcal{X} \hookrightarrow \mathcal{Y} \) indicates that the embedding is continuous. For a discrete set \( \nabla \) the symbol \( |\nabla| \) denotes the cardinality of this set.

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### 2 Besov spaces of dominating mixed smoothness

Let us recall the definition of Besov spaces of dominating mixed smoothness in Fourier-analytic terms, we refer to [24, 2.2] and [25]. Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be a function such that \( \varphi(t) = 1 \) in an open set containing the origin. Then by means of

\[
\varphi_0(t) = \varphi(t), \quad \varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N},
\]

we get a smooth dyadic decomposition of unity, i.e.,

\[
\sum_{j=0}^\infty \varphi_j(t) = 1 \quad \text{for all} \quad t \in \mathbb{R},
\]

and \( \text{supp} \varphi_j \) is contained in the dyadic annulus \( \{ t \in \mathbb{R} : \quad a2^j \leq |t| \leq b2^j \} \) with \( 0 < a < b < \infty \) independent of \( j \in \mathbb{N} \). By means of

\[
\varphi_{\vec{j}} := \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_d}, \quad \vec{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d, \quad (2.2)
\]

we obtain a smooth decomposition of unity on \( \mathbb{R}^d \).

**Definition 2.1.** Let \( 0 < p, q \leq \infty \) and \( t \in \mathbb{R} \). The Besov space of dominating mixed smoothness \( S^t_{p,q}B(\mathbb{R}^d) \) is the collection of all tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that

\[
\| f \|_{S^t_{p,q}B(\mathbb{R}^d)} := \left( \sum_{\vec{j} \in \mathbb{N}_0^d} 2^{j_1tq} \| F^{-1}[\varphi_{\vec{j}} Ff](\cdot) \|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}
\]

is finite.
Next we will describe the wavelet decomposition for Besov spaces of dominating mixed smoothness. We are going to recall a few results from [37], see also Bazarkhanov [1]. We have to introduce some sequence spaces.

**Definition 2.2.** If $0 < p, q \leq \infty$, $t \in \mathbb{R}$ and $\lambda := \{\lambda_{\bar{\nu}, \bar{m}} \in \mathbb{C} : \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d\}$, then we define

$$s_{p,q}^t b := \left\{ \lambda : ||\lambda||_{s_{p,q}^t b} = \left( \sum_{\nu \in \mathbb{N}_0^d} 2^{\nu \cdot |\frac{1}{2} t|} \left( \sum_{m \in \mathbb{Z}^d} |\lambda_{\bar{\nu}, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}.$$ 

We continue with wavelet bases of Besov spaces of dominating mixed smoothness. Let $N \in \mathbb{N}$. Then there exist $\psi_0, \psi_1 \in C^N(\mathbb{R})$, compactly supported,

$$\int_{-\infty}^{\infty} t^m \psi_1(t) \, dt = 0, \quad m = 0, 1, \ldots, N,$$

such that $\{2^{j/2} \psi_{j,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}\}$, where

$$\psi_{j,m}(t) := \begin{cases} \psi_0(t - m) & \text{if } j = 0, m \in \mathbb{Z}, \\ \sqrt{1/2} \psi_1(2^{j-1} t - m) & \text{if } j \in \mathbb{N}, m \in \mathbb{Z}, \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$. We put

$$\Psi_{\bar{\nu}, \bar{m}}(x) := \prod_{\ell=1}^d \psi_{\nu_\ell, m_\ell}(x_\ell).$$

Then

$$\Psi_{\bar{\nu}, \bar{m}}, \quad \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d,$$

is a tensor product wavelet basis of $L_2(\mathbb{R}^d)$. Vybiral [37] has proved the following, see also Bazarkhanov [1].

**Lemma 2.3.** Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then there exists $N = N(t,p) \in \mathbb{N}$ s.t. the mapping

$$W : f \mapsto (2^{|\bar{\nu}|_1 (f, \Psi_{\bar{\nu}, \bar{m}})})_{\bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d} \quad (2.3)$$

is an isomorphism of $S_{p,q}^t B(\mathbb{R}^d)$ onto $s_{p,q}^t b$.

### Spaces on $\Omega$

**Definition 2.4.** Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then $S_{p,q}^t B(\Omega)$ is the space of all complex-valued distributions $f$ on $\Omega$ such that there exists a distribution $g \in S_{p,q}^t B(\mathbb{R}^d)$ satisfying $f = g|_{\Omega}$. It is endowed with the quotient norm

$$\| f |_{S_{p,q}^t B(\Omega)} \| = \inf \left\{ \| g |_{S_{p,q}^t B(\mathbb{R}^d)} \| : g|_{\Omega} = f \right\}.$$
Next, we define the sequence spaces associated to \( S^t_{p,q}B(\Omega) \). Let \( t, p \) and \( q \) be fixed. Let the wavelet basis \( \{\Psi_{\check{\nu},\check{m}}\}_{\check{\nu},\check{m}} \) be admissible in the sense of Lemma 2.3. We put

\[
A^{\Omega}_\check{\nu} := \left\{ \check{m} \in \mathbb{Z}^d : \supp \Psi_{\check{\nu},\check{m}} \cap \Omega \neq \emptyset \right\}, \quad \check{\nu} \in \mathbb{N}_0^d.
\]

For given \( f \in S^t_{p,q}B(\Omega) \), let \( E f \) be an element of \( S^t_{p,q}B(\mathbb{R}^d) \) s.t.

\[
\|E f | S^t_{p,q}B(\mathbb{R}^d)\| \leq 2 \|f | S^t_{p,q}B(\Omega)\| \quad \text{and} \quad (Ef)|_\Omega = f.
\]

We define

\[
g := \sum_{\check{\nu} \in \mathbb{N}_0^d} \sum_{\check{m} \in A^{\Omega}_\check{\nu}} 2^{||\check{\nu}||_1} \langle E f, \Psi_{\check{\nu},\check{m}} \rangle \Psi_{\check{\nu},\check{m}}.
\]

Then it follows that \( g \in S^t_{p,q}B(\mathbb{R}^d) \), \( g|_\Omega = f \),

\[\text{supp} g \subset \{ x \in \mathbb{R}^d : \max_{j=1,\ldots,d} |x_j| \leq c_1 \} \quad \text{and} \quad \|g | S^t_{p,q}B(\mathbb{R}^d)\| \leq c_2 \|f | S^t_{p,q}B(\Omega)\|.
\]

Here \( c_1, c_2 \) are independent of \( f \). For that reason the sequence spaces associated with \( \Omega \) are defined as follows.

**Definition 2.5.** If \( 0 < p, q \leq \infty \), \( t \in \mathbb{R} \) and \( \lambda = \{\lambda_{\check{\nu},\check{m}} \in \mathbb{C} : \check{\nu} \in \mathbb{N}_0^d, \check{m} \in A^{\Omega}_{\check{\nu}}\} \), then we define

\[
s^t_{p,q}b := \left\{ \lambda : \|\lambda | s^t_{p,q}b\| = \left( \sum_{\check{\nu} \in \mathbb{N}_0^d} 2^{||\check{\nu}||_1(t-\frac{1}{p})q} \left( \sum_{\check{m} \in A^{\Omega}_{\check{\nu}}} |\lambda_{\check{\nu},\check{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}
\]

and corresponding building blocks

\[
(s^t_{p,q}b)_\mu = \left\{ \lambda : \|\lambda | (s^t_{p,q}b)_\mu\| = \left( \sum_{|\check{\nu}|=\mu} 2^{||\check{\nu}||_1(t-\frac{1}{p})q} \left( \sum_{\check{m} \in A^{\Omega}_\check{\nu}} |\lambda_{\check{\nu},\check{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}, \quad \mu \in \mathbb{N}_0.
\]

Later on we shall need the following Lemma, see [10, 37].

**Lemma 2.6.** (i) We have

\[
|A^{\Omega}_{\check{\nu}}| \asymp 2^{||\check{\nu}||_1} \quad \text{and} \quad D_\mu := \sum_{|\check{\nu}|=\mu} |A^{\Omega}_{\check{\nu}}| \asymp \mu^{d-1} 2^\mu
\]

with equivalence constants independent of \( \check{\nu} \in \mathbb{N}_0^d \) and \( \mu \in \mathbb{N}_0 \).

(ii) Let \( 0 < p \leq \infty \) and \( t \in \mathbb{R} \). Then

\[
(s^t_{p,q}b)_\mu = 2^{\mu(t-\frac{1}{p})} f^t_{\mu}, \quad \mu \in \mathbb{N}_0.
\]

(iii) Let \( 0 < p, p_1, q_1 \leq \infty \) and \( t, r \in \mathbb{R} \). Then

\[
\|id^r_\mu : (s^t_{p,q}b)_\mu \rightarrow (s^{t}_{p_1,q_1}b)_\mu\| \lesssim 2^\mu(-t+r+(\frac{1}{p}-\frac{1}{p_1})+) \mu^{(d-1)(\frac{1}{q_1}-\frac{1}{q})+},
\]

with a constant behind \( \lesssim \) independent of \( \mu \).
Pointwise multipliers

This subsection deals with multiplication properties of Besov spaces of dominating mixed smoothness. The multiplication properties for isotropic Besov spaces have been studied in various places, see Triebel [30, 2.6], [32, 2.8], [33, 4.2] and Runst, Sickel [22, Chapter 4]. For the most recent publication in this direction we refer to Scharf [23]. Let \( \ell \in \mathbb{N}_0 \) and \( f : \mathbb{R} \to \mathbb{C} \). We define

\[
\Delta_\ell^h(f,x) := \sum_{j=0}^{\ell} (-1)^{\ell-j} \left( \begin{array}{c} \ell \\ j \end{array} \right) f(x + jh), \quad x \in \mathbb{R}, \ h \in [0,1].
\]

Let \( e \subset [d] := \{1,2,...,d\} \), \( \bar{\ell} = (\ell_1,...,\ell_d) \in \mathbb{N}_0^d \) and \( f : \mathbb{R}^d \to \mathbb{C} \). We define

\[
\Delta_{\bar{\ell}}^\bar{h} = \prod_{i \in e} \Delta_{\ell_i}^{h_i,i}, \quad \Delta_{\bar{\ell}}^\emptyset = \text{Id}, \quad \bar{h} \in [0,1]^d,
\]

and

\[
\omega_{\bar{\ell}}^e(f,t) = \sup_{|h_i| \leq t_i, i \in e} \|\Delta_{\bar{\ell}}^\bar{h}(f,\cdot)|L_p(\mathbb{R}^d)\|
\]

The following theorem was proved by Ullrich [35] for general \( d \), for the case \( d = 2 \) we refer to [24, 2.3.4].

**Proposition 2.7.** Let \( 0 < p,q \leq \infty \), \( r \in \mathbb{R} \) and \( \bar{m} = (m,...,m) \in \mathbb{N}^d \) with \( m > r > \frac{1}{p} \). Then

\[
\|f|S_{p,q}^r B(\mathbb{R}^d)\| \asymp \|f|S_{p,q}^r B(\mathbb{R}^d)\|_m = \sum_{e \subset [d]} \left( \int_{(0,1)^{\bar{e}}} \left[ \left( \prod_{i \in e} t_i^{-r} \right) \omega_{\bar{m}}^e(f,t)_p \right]^q \prod_{i \in e} dt_i \right)^{1/q}
\]

for all \( f \in S_{p,q}^r B(\mathbb{R}^d) \).

**Remark 2.8.** In the literature many times people prefer to work with Besov spaces of dominating mixed smoothness defined by the modulus of smoothness, for example see Dinh Dũng [5, 6].

**Theorem 2.9.** Let \( 0 < p,q \leq \infty \), \( r > \frac{1}{p} \) and \( \varphi \in \mathcal{D}(\mathbb{R}^d) \). Then there exists a constant \( C > 0 \) such that

\[
\|\varphi f|S_{p,q}^r B(\mathbb{R}^d)\| \leq C\|f|S_{p,q}^r B(\mathbb{R}^d)\|
\]

holds for all \( f \in S_{p,q}^r B(\mathbb{R}^d) \).

**Proof.** We fix \( m \in \mathbb{N} \), \( m > r \) and put \( \bar{m} = (m,...,m) \in \mathbb{N}^d \).

**Step 1.** Preparation. Let \( g : \mathbb{R} \to \mathbb{C} \) and \( \psi \in \mathcal{D}(\mathbb{R}) \). We have

\[
\Delta_{2m}^h(\psi g)(x) = \sum_{j=0}^{2m} c_j \Delta_{2m-j}^h \psi(x + jh) \Delta_j^h g(x), \quad h \in [0,1].
\]
for some real numbers $c_j$ and

$$|\Delta_h^{2m}(\psi g)(x)| \leq C \left( |h|^m \sum_{j=0}^{m} |g(x + jh)| + \sum_{j=m+1}^{2m} |\Delta_h^j g(x)| \right),$$

see [33 3.5.3/(17,18)].

**Step 2.** For $a \subset [d]$ and $\vec{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}_0^d$ we put

$$\vec{a} = [d] \setminus a, \quad \vec{\ell}_a = (b_1, \ldots, b_d)$$

with $b_i = \ell_i$ if $i \in a$ and $b_i = 0$ if $i \notin \vec{a}$. Furthermore for $\vec{h} \in [0,1]^d$ we define

$$\vec{\ell} \ast \vec{h} = (\ell_1 h_1, \ldots, \ell_d h_d).$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $f \in S^r_{p,q} B(\mathbb{R}^d)$. Applying step 1 we have

$$\Delta_h^{2m,|d|}(\varphi f)(x) = \sum_{\vec{\ell} = 0}^{2m} c_{\vec{\ell}} (\Delta_h^{2m-\vec{\ell},|d|}\varphi)(x + \vec{\ell} \ast \vec{h})(\Delta_h^{\vec{\ell},d} f)(x)$$

for some $c_{\vec{\ell}}$ and

$$|\Delta_h^{2m,|d|}(\varphi f)(x)| \leq C_1 \left( \sum_{a \subset [d]} \left[ \prod_{i \in a} |h_i|^m \sum_{\ell_j = 0}^{m} \sum_{\kappa \in \vec{\ell}_a} \Delta_h^{\ell \ast \vec{a}} f(x + \ell \ast \vec{h}) \right] \right).$$

Consequently

$$\|\Delta_h^{2m,|d|}(\varphi f)|L_p(\mathbb{R}^d)\| \leq C_2 \left( \sum_{a \subset [d]} \left[ \prod_{i \in a} |h_i|^m \sum_{\ell_j = 0}^{m} \sum_{\kappa \in \vec{\ell}_a} \|\Delta_h^{\ell \ast \vec{a}} f|L_p(\mathbb{R}^d)\| \right] \right).$$

Because differences with order $\ell_i > m$ can be reduced to differences of order $m$ and translation does not change the value of integral in $L_p(\mathbb{R}^d)$ we have

$$\|\Delta_h^{2m,|d|}(\varphi f)|L_p(\mathbb{R}^d)\| \leq C_3 \left( \sum_{a \subset [d]} \left[ \prod_{i \in a} |t_i|^m \sup_{|h_i| \leq t_i, i \in [d]} \|\Delta_h^{m \ast \vec{a}} f|L_p(\mathbb{R}^d)\| \right] \right).$$

This leads to

$$\sup_{|h_i| \leq t_i, i \in [d]} \|\Delta_h^{2m,|d|}(\varphi f)|L_p(\mathbb{R}^d)\| \leq C_3 \left( \sum_{a \subset [d]} \left[ \prod_{i \in a} |t_i|^m \sup_{|h_i| \leq t_i, i \in \vec{a}} \|\Delta_h^{m \ast \vec{a}} f|L_p(\mathbb{R}^d)\| \right] \right).$$

As a consequence we obtain

$$\left( \int_{(0,1)^d} \left[ \left( \prod_{i \in [d]} t_i^{\vec{r}_i} \right) \sup_{|h_i| \leq t_i, i \in [d]} \|\Delta_h^{2m,|d|}(\varphi f)|L_p(\mathbb{R}^d)\| \right]^q \prod_{i \in [d]} \frac{dt_i}{t_i} \right)^{1/q} \leq \sum_{a \subset [d]} \left( \int_{(0,1)^d} \left[ \left( \prod_{i \in [d]} t_i^{\vec{r}_i} \right) \prod_{i \in a} |t_i|^m \sup_{|h_i| \leq t_i, i \in \vec{a}} \|\Delta_h^{m \ast \vec{a}} f|L_p(\mathbb{R}^d)\| \right]^q \prod_{i \in [d]} \frac{dt_i}{t_i} \right)^{1/q} \leq \sum_{a \subset [d]} \left( \int_{(0,1)^d} \left[ \left( \prod_{i \in [a]} t_i^{\vec{r}_i} \right) \sup_{|h_i| \leq t_i, i \in \vec{a}} \|\Delta_h^{m \ast \vec{a}} f|L_p(\mathbb{R}^d)\| \right]^q \prod_{i \in [a]} \frac{dt_i}{t_i} \right)^{1/q}.$$
The last inequality comes from the fact that $1 - (m - r)q < 1$. Similar estimates hold for

$$\left( \int_{(0,1)^e} \left[ \left( \prod_{i \in e} t_i^{-r} \right) \sup_{|h_i| \leq t_i, i \in e} \| \Delta_{h_i}^{m,e}(\varphi f) \|_{L_p(\mathbb{R}^d)} \right] \prod_{i \in e} \frac{dt_i}{t_i} \right)^{1/q}$$

for all $e \subset [d]$.

**Step 3.** From the estimates in Step 2 we have

$$\sum_{e \subset [d]} \left( \int_{(0,1)^e} \left[ \left( \prod_{i \in e} t_i^{-r} \right) \sup_{|h_i| \leq t_i, i \in e} \| \Delta_{h_i}^{m,e}(\varphi f) \|_{L_p(\mathbb{R}^d)} \right] \prod_{i \in e} \frac{dt_i}{t_i} \right)^{1/q} \leq C_6 \sum_{e \subset [d]} \left( \int_{(0,1)^e} \left[ \left( \prod_{i \in e} t_i^{-r} \right) \sup_{|h_i| \leq t_i, i \in e} \| \Delta_{h_i}^{m,e}(\varphi f) \|_{L_p(\mathbb{R}^d)} \right] \prod_{i \in e} \frac{dt_i}{t_i} \right)^{1/q}$$

which means

$$\| \varphi f \|_{S^r_{p,q} B(\mathbb{R}^d)} \|_{2m} \leq C_6 \| f \|_{S^r_{p,q} B(\mathbb{R}^d)} \|_{m}.$$ 

In view of Proposition 2.7, the proof is complete. 

**Extension operators**

In this part we will discuss extension operators from $S^t_{p,q} B(\Omega)$ to $S^t_{p,q} B(\mathbb{R}^d)$. Extension operators on Besov spaces of dominating mixed smoothness have been previously considered in [7, 14, 34, 36]. Here we are going to recall a result from Ullrich [36]. A domain $D$ is called a rectangular domain at the point $a = (a_1, \ldots, a_d)$, $a_i \in \mathbb{R}$, $i = 1, \ldots, d$, if

$$D = M_1 \times \ldots \times M_d,$$

where $M_i = (a_i, \infty)$ or $M_i = (-\infty, a_i)$, $i = 1, \ldots, d$. Ullrich [36, Theorem 3.4] has proved the following.

**Proposition 2.10.** Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Let $D$ be a rectangular domain. Then there exist a linear bounded extension operator $\mathcal{E}$

$$\mathcal{E} : S^t_{p,q} B(D) \rightarrow S^t_{p,q} B(\mathbb{R}^d).$$

Now we consider the unit cube $\Omega$. We associate to each vertex $e = (e_1, \ldots, e_d)$, $e_i \in \{0,1\}$, $i = 1, \ldots, d$, the rectangular domain $D_e$ at the point $e$ such that $\Omega \subset D_e$. We cover the cube $[0,1]^d$ with $2^d$ cubes $Q_e$ centering at $e$ with sides parallel to the axes and side-length $\frac{2}{3}$. Then there exist $2^d$ functions $\varphi_e$, $e \in \{0,1\}^d$, such that $\varphi_e \in \mathcal{D}(Q_e)$ and make up a decomposition of unity on $\Omega$. We use the same notation for the function extended by zero to all of $\mathbb{R}^d$. Let $f \in S^t_{p,q} B(\Omega)$ and $g \in S^t_{p,q} B(\mathbb{R}^d)$ such that

$$g|\Omega = f.$$
Then \( \varphi_e g \) belongs to \( S^t_{p,q}(\mathbb{R}^d) \), \( t > \frac{1}{p} \), as well and
\[
\| \varphi_e g |_{S^t_{p,q}(\mathbb{R}^d)} \| \leq C \| g |_{S^t_{p,q}(\mathbb{R}^d)} \|
\]
with \( C \) independent of \( e \) and \( f \), see Theorem 2.9. Let \( \psi \in \mathcal{D}(D_e) \). Then there exists a splitting
\[
\psi = \psi_1 + \psi_2
\]
where \( \psi_1 \in \mathcal{D}(\Omega) \), \( \psi_2 \in \mathcal{D}(\tilde{\Omega}) \) and
\[
\tilde{\Omega} = \{ x \in D_e : \text{dist}(x, \text{supp} \varphi_e) > \tau \}
\]
for some sufficiently small \( \tau > 0 \). It follows
\[
(\varphi_e g)(\psi) = (\varphi_e g)(\psi_1) + (\varphi_e g)(\psi_2) = (\varphi_e g)(\psi_1) = (\varphi_e f)(\psi_1).
\]
With other words, \( \varphi_e g |_{D_e} \) is the extension of \( \varphi_e f \) to \( D_e \) by zero. It is independent of the chosen particular \( f \). Clearly, by definition of \( S^t_{p,q}(D_e) \), \( \varphi_e g |_{D_e} \) belongs to \( S^t_{p,q}(D_e) \). Let \( E_e \) be the linear and continuous extension operator with respect to \( D_e \). Then we define
\[
E_f = \sum_{e \in \{0,1\}^d} E_e(\varphi_e g |_{D_e})
\]
where \( g \in S^t_{p,q}(\mathbb{R}^d) \) is an arbitrary extension of \( f \). Since
\[
\sum_{e \in \{0,1\}^d} \varphi_e = 1 \quad \text{on} \quad \Omega
\]
we have for \( \psi \in \mathcal{D}(\Omega) \)
\[
E_f(\psi) = \sum_{e \in \{0,1\}^d} E_e(\varphi_e g |_{D_e})(\psi) = g(\sum_{e \in \{0,1\}^d} \varphi_e \psi) = f(\psi)
\]
which means
\[
E_f |_{\Omega} = f \quad \text{in} \quad \mathcal{D}'(\Omega).
\]
**Theorem 2.11.** Let \( 0 < p, q \leq \infty \) and \( t > \frac{1}{p} \). Then there exists a linear and continuous extension operator
\[
E : S^t_{p,q}(\Omega) \rightarrow S^t_{p,q}(\mathbb{R}^d).
\]

**Interpolation properties**

In this Subsection we discuss the interpolation properties of Besov spaces of dominating mixed smoothness. For the basics in interpolation theory we refer to the monographs \[11, 12, 31\]. Let \((X_1, X_2)\) be an interpolation couple of quasi-Banach spaces. Let \( x \in X_1 + X_2 \) and \( t \in (0, \infty) \). Then Peetre’s \( K \)-functional is defined as
\[
K(t, x, X_1, X_2) = \inf \{ \| x_1 | X_1 \| + t \| x_2 | X_2 \| : x = x_1 + x_2 \}.
\]
By \([X_1, X_2]_\theta, \theta \in (0, 1)\), we denote the classical complex method of Calderón, here \(X_1\) and \(X_2\) are Banach spaces. Under some restrictions this method can be extended to quasi-Banach spaces, see \[11, 13\]. The following proposition was proved by Vybiral \[37, \text{Theorem 4.6, Theorem 4.8}\].

**Proposition 2.12.** Let \(t_i \in \mathbb{R}, t_i > \frac{1}{p_i}, 0 < p_i, q_i \leq \infty, i = 1, 2, \text{ and } \min(q_1, q_2) < \infty\). Let \(0 < \theta < 1\). If \(t_0, p_0\) and \(q_0\) are given by

\[
\begin{align*}
\frac{1}{p_0} &= 1 - \theta + \frac{\theta}{p_2}, & \frac{1}{q_0} &= 1 - \theta + \frac{\theta}{q_2}, & t_0 &= (1 - \theta)t_1 + \theta t_2,
\end{align*}
\]

then

\[
[s_{p_1, q_1}^t b, s_{p_2, q_2}^t b]_\theta = s_{p_0, q_0}^t b
\]

and

\[
t^{-\theta}K(t, \lambda, s_{p_1, q_1}^t b, s_{p_2, q_2}^t b) \leq \|\lambda s_{p_0, q_0}^t b\|,
\]

for all \(\lambda \in s_{p_0, q_0}^t b\) and all \(0 < t < \infty\).

By making use of the Lemma 2.3 we can turn Proposition 2.12 to the situation of function spaces.

**Lemma 2.13.** Let \(0 < \theta < 1\) and \(t_j, p_j, q_j, j = 0, 1, 2\), as in Proposition 2.12. Then

\[
S_{p_0, q_0}^t B(\Omega) \hookrightarrow S_{p_1, q_1}^t B(\Omega) + S_{p_2, q_2}^t B(\Omega)
\]

and there exists a constant \(C\) such that

\[
t^{-\theta}K(t_0, f_0, S_{p_1, q_1}^t B(\Omega), S_{p_2, q_2}^t B(\Omega)) \leq C\|f_0\| S_{p_0, q_0}^t B(\Omega),
\]

for all \(f_0 \in S_{p_0, q_0}^t B(\Omega)\) and all \(t > 0\).

**Proof.** Step 1. Preparation. Let \(f \in S_{p,q}^t B(\Omega)\). Then \(g := \mathcal{E}f\) belongs to \(S_{p,q}^t B(\mathbb{R}^d)\). The associated wavelet expansions given by

\[
g := \sum_{\nu \in \mathbb{N}_0^d} \sum_{\bar{m} \in \mathbb{Z}^d} 2^{|
u|_1} \langle \mathcal{E}f, \Psi_{\nu, \bar{m}} \rangle \Psi_{\nu, \bar{m}}.
\]

We define

\[
g^\Omega = \sum_{\nu \in \mathbb{N}_0^d} \sum_{\bar{m} \in A_0^d} 2^{|
u|_1} \langle g, \Psi_{\nu, \bar{m}} \rangle \Psi_{\nu, \bar{m}}
\]

and

\[
\lambda = \mathcal{W}g = (2^{|
u|_1} \langle g, \Psi_{\nu, m} \rangle)_{\nu \in \mathbb{N}_0^d, m \in A_0^d}.
\]

Since \(g^\Omega\) is an extension of \(f\) as well it follows

\[
\|f| S_{p,q}^t B(\Omega)\| \leq \|g^\Omega| S_{p,q}^t B(\mathbb{R}^d)\| \leq c_1 \|\lambda| s_{p,q}^t b\|
\]
where $c_1$ is independent of $\lambda$, see Lemma 2.3. Employing once again Lemma 2.3 we find

$$
\|\lambda|s^t_{p,q}b\| = \|\lambda|s^t_{p,q}b\| \leq c_2 \|E f|S^t_{p,q}B(\mathbb{R}^d)\| \leq c_2 \|E\| \|f|S^t_{p,q}B(\Omega)\|
$$

where $c_2$ is independent of $f$.

**Step 2.** Let $f_0 \in S^p_{0^q}B(\Omega)$. We define $g_0 = E f_0$ and $\lambda_0 = W g_0$. Because of $\lambda_0 \in s^t_{p,q}b$ we conclude that for any $\epsilon > 0$ there exists $\lambda \in \lambda_{p,q}b$, $i = 1, 2$, such that $\lambda_0 = \lambda_1 + \lambda_2$ and

$$
\|\lambda_1|s^t_{1,\Omega}b\| + \|\lambda_2|s^t_{2,\Omega}b\| \leq K(t, \lambda_0, s^t_{1,\Omega}b, s^t_{2,\Omega}b) + \epsilon. \quad (2.5)
$$

We define

$$
W^*\lambda := \sum_{\nu \in B_{p,q}} \sum_{\bar{m} \in \mathcal{A}_p} \lambda_{\nu, \bar{m}} \Psi_{\nu, \bar{m}},
$$

and put $f_i = W^*\lambda_i$, $i = 1, 2$. It follows

$$
f_1 + f_2 = W^*(\lambda_1 + \lambda_2) = W^*\lambda_0 = (W^*W^E)f_0
$$

Clearly $\tilde{f}_0 := (W^*W^E)f_0$ is an extension of $f_0$. Hence

$$
R^\Omega(f_1 + f_2) = R^\Omega \tilde{f}_0 = f_0
$$

where $R^\Omega$ denotes the restriction to $\Omega$. We conclude

$$
K(t, f_0, S^t_{p,1,q}B(\Omega), S^t_{p,2,q}B(\Omega)) \leq \|R^\Omega f_1|S^t_{p,1,q}B(\Omega)\| + t\|R^\Omega f_2|S^t_{p,2,q}B(\Omega)\|
$$

$$
\leq \|f_1|s^t_{1,\Omega}B(\mathbb{R}^d)\| + t\|f_2|s^t_{2,\Omega}B(\mathbb{R}^d)\|
$$

$$
\leq c_1 \left[\|\lambda_1|s^t_{1,\Omega}b\| + t\|\lambda_2|s^t_{2,\Omega}b\|\right].
$$

Because of

$$
\|\lambda_i|s^t_{p,q}b\| = \|\lambda_i|s^t_{p,q}b\|, \quad i = 1, 2,
$$

and (2.5) we obtain

$$
K(t, f_0, S^t_{p,1,q}B(\Omega), S^t_{p,2,q}B(\Omega)) \leq c_1 \left[K(t, \lambda_0, s^t_{1,\Omega}b, s^t_{2,\Omega}b) + \epsilon\right].
$$

Employing Proposition 2.12 this leads to

$$
K(t, f_0, S^t_{p,1,q}B(\Omega), S^t_{p,2,q}B(\Omega)) \leq c_1 \left[t^\theta\|\lambda_0|s^t_{p,0}b\| + \epsilon\right]
$$

$$
\leq c_1c_2 t^\theta\|f|S^t_{p,0}B(\Omega)\| + c_1 \epsilon
$$

with $c_1, c_2$ independent of $f$ and $\epsilon$. This proves the claim. \[\square\]
3 Proofs

First, let us recall some results obtained by Belinskii [3], Romanyuk [21] and Temlyakov [26].

Proposition 3.1. Let $1 \leq q \leq \infty$.

(i) If $1 \leq p \leq 2$ and $t > \frac{1}{p}$ we have

\[
\begin{aligned}
    n^{-t+\frac{1}{p}-\frac{1}{2}}(\log n)^{(d-1)\left(t-\frac{1}{p}+\frac{1}{2}+\left(\frac{1}{q} - \frac{1}{2}\right)_{+}\right)} & \lesssim d_n(id : S^t_{p,q}B(\Omega) \to L_{\infty}(\Omega)) \\
    & \lesssim n^{-t+\frac{1}{p}-\frac{1}{2}}(\log n)^{(d-1)\left(t-\frac{1}{p}+\frac{1}{2}+\left(\frac{1}{q} - \frac{1}{2}\right)_{+}\right)}(\log n)^{1/2},
\end{aligned}
\]

for all $n \geq 2$.

(ii) If $2 < p \leq \infty$ and $t > \frac{1}{2}$ we have

\[
\begin{aligned}
    n^{-t}(\log n)^{(d-1)\left(t+\frac{1}{2} - \frac{1}{q}\right)_{+}} & \lesssim d_n(id : S^t_{p,q}B(\Omega) \to L_{\infty}(\Omega)) \\
    & \lesssim n^{-t}(\log n)^{(d-1)\left(t+\frac{1}{2} - \frac{1}{q}\right)_{+}}(\log n)^{1/2},
\end{aligned}
\]

for all $n \geq 2$.

Remark 3.2. Romanyuk [21] has considered the case $1 \leq q < \infty$ in Proposition 3.1. The upper bound in case $q = \infty$ was obtained by Belinskii [3] and the lower bound is derived from the estimate $d_n(id : S^t_{p,\infty}B(\Omega) \to L_2(\Omega))$ by Temlyakov [26]. Note that in the literature many times the notations $H^t_p(\Omega)$ and $MH^t_p(\Omega)$ are used instead of $S^t_{p,\infty}B(\Omega)$.

The following lemma is a consequence of a result obtained by Temlyakov [27].

Lemma 3.3. Let $t > 0$. Then we have

\[
d_n(id : S^t_{\infty,\infty}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2})},
\]

for all $n \geq 2$.

Proof. The upper bound is a direct consequence of the inequality $d_n \leq a_n$, see [16, Theorem 2.3.4] and

\[
a_n(id : S^t_{\infty,\infty}B(\Omega) \to L_{\infty}(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2})}, \quad n \geq 2,
\]

see [27]. Next we consider the following diagram

\[
\begin{array}{ccc}
S^t_{\infty,\infty}B(\Omega) & \xrightarrow{id_1} & L_s(\Omega) \\
\downarrow{id} & & \downarrow{id_2} \\
L_{\infty}(\Omega) & & 
\end{array}
\]
The property (s3) of the Kolmogorov numbers yields

\[ d_n(id_1) \leq d_n(id)\|id\|. \]

From

\[ d_n(id : S^t_{\infty,\infty}B(\Omega) \to L_s(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2})}, \quad t > 0, \quad 1 < s < \infty, \]

for all \( n \geq 2 \), see again [27], we obtain the claimed estimate from below. \( \blacksquare \)

By making use of the chain of embeddings

\[ S^t_{p,q}B(\Omega) \hookrightarrow S^r_{\infty,\infty}B(\Omega) \hookrightarrow L_\infty(\Omega), \quad t > r + \frac{1}{p}, \]

\( r > 0 \), and the multiplicativity of the Kolmogorov numbers, we can derive sharp estimates for upper bounds in the Proposition 3.1. To do so, first we investigate the Kolmogorov numbers of embedding of sequence spaces related to Besov spaces of dominating mixed smoothness.

**Proposition 3.4.** Let \( 0 < p, q \leq \infty \) and \( t > r + \frac{1}{p} \). Then we have

\[ d_n(id^* : s^t_{p,q}b \to s^r_{\infty,\infty}b) \lesssim \begin{cases} n^{-t+r+\frac{1}{p} - \frac{1}{2}}(\log n)^{(d-1)(t-r \frac{1}{p} + \frac{1}{q} - \frac{1}{2})} & \text{if } p \leq 2, \\ n^{-t+r}(\log n)^{(d-1)(t-r \frac{1}{p} + \frac{1}{q} - \frac{1}{2})} & \text{if } 2 < p, \end{cases} \]

for all \( n \geq 2 \).

**Proof.** Step 1. We define the operators

\[ id^* = \sum_{\mu=0}^{\infty} id^*_{\mu}, \quad id^*_{\mu} : s^t_{p,q} \Omega b \to s^r_{\infty,\infty} b, \quad (3.4) \]

where

\[ (id^*_{\mu})_{\bar{\nu}, \bar{m}} := \begin{cases} \lambda_{\bar{\nu}, \bar{m}} & \text{if } |\bar{\nu}| = \mu, \\ 0 & \text{otherwise.} \end{cases} \]

The additivity and the monotonicity of the Kolmologorov numbers yield

\[ d_n(id^*) \leq \sum_{\mu=0}^{J} d_n(id^*_{\mu}) + \sum_{\mu=J+1}^{L} d_n(id^*_{\mu}) + \sum_{\mu=L+1}^{\infty} \|id^*_{\mu}\|, \quad (3.5) \]

where these numbers \( J \) and \( L \) will be chosen in dependence on the parameters and \( n - 1 = \sum_{\mu=0}^{L} (n_{\mu} - 1) \). By Lemma [26](iii) and \( t > r + \frac{1}{p} \) we obtain the estimate

\[ \sum_{\mu=L+1}^{\infty} \|id^*_{\mu}\| \leq \sum_{\mu=L+1}^{\infty} 2^{\mu(r-t+\frac{1}{2})} \lesssim 2^{L(r-t+\frac{1}{2})}. \quad (3.6) \]

Now we choose \( n_{\mu} \) as

\[ n_{\mu} := \begin{cases} D_{\mu} + 1 & \text{if } 0 \leq \mu \leq J, \\ D_{\mu} 2^{(J-\mu)\lambda} & \text{if } J + 1 \leq \mu \leq L, \end{cases} \quad (3.7) \]
for some $\lambda > 1$, which will be chosen later on. Then we get

$$n = \sum_{\mu=0}^{L} (n_{\mu} - 1) + 1 \approx \sum_{\mu=0}^{J} \mu^{(J-\mu)\lambda} + 2^{J+1} d_{\mu} \lambda \approx \sum_{\mu=J+1}^{L} \mu^{(J-\mu)\lambda} \times J^{d-1} \lambda \quad \tag{3.8}$$

and $d_{n_{\mu}}(id_{\mu}^{*}) = 0$, $0 \leq \mu \leq J$, see (s4), which implies

$$\sum_{\mu=0}^{J} d_{n_{\mu}}(id_{\mu}^{*}) = 0. \quad \tag{3.9}$$

**Step 2.** Denote $\delta = \max(p, 2)$. We consider the following diagram:

Using property (s3) of the Kolmogorov numbers we conclude

$$d_{n_{\mu}}(id_{\mu}^{*}) \leq \|id^{2}\| d_{n_{\mu}}(id^{1}).$$

From Lemma 2.6 we get

$$d_{n_{\mu}}(id^{1}) \times d_{n_{\mu}}(id : 2^{\mu(t+\frac{1}{2})} \ell_{\delta}^{P_{\mu}} \rightarrow 2^{\mu(r+\frac{1}{2})} \ell_{\infty}^{P_{\mu}}) \times 2^{\mu(r-t+\frac{1}{2})} d_{n_{\mu}}(id : \ell_{\delta}^{P_{\mu}} \rightarrow \ell_{\infty}^{P_{\mu}})$$

and

$$\|id^{2}\| \lesssim 2^{\mu(t+\frac{1}{2})} \mu^{(d-1)(\frac{1}{2}-\frac{1}{q})+}.$$

Hence

$$d_{n_{\mu}}(id_{\mu}^{*}) \lesssim 2^{\mu(r-t+\frac{1}{2})} \mu^{\lambda} \mu^{(d-1)(\frac{1}{2}-\frac{1}{q})+} d_{n_{\mu}}(id : \ell_{\delta}^{P_{\mu}} \rightarrow \ell_{\infty}^{P_{\mu}}).$$

From

$$d_{n}(id : \ell_{\delta}^{m} \rightarrow \ell_{\infty}^{m}) \approx n^{-\frac{1}{p}}, \quad m, n \in \mathbb{N}, \quad n \leq m,$$

see [9] or [38], we obtain

$$d_{n_{\mu}}(id_{\mu}^{*}) \lesssim 2^{\mu(r-t+\frac{1}{2})} \mu^{\lambda} \mu^{(d-1)(\frac{1}{2}-\frac{1}{q})+} (D_{\mu} 2^{(J-\mu)\lambda})^{-\frac{1}{p}}$$

$$\lesssim 2^{\mu(r-t+\frac{1}{2})} 2^{-\frac{1}{p} (J-\mu) \lambda} \mu^{(d-1)(\frac{1}{2}-\frac{1}{q})+} \left(\frac{1}{\lambda} + \frac{1}{\lambda} \right).$$

This yields

$$\sum_{\mu=J+1}^{L} d_{n_{\mu}}(id_{\mu}^{*}) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu(r-t+\frac{1}{2})} 2^{-\frac{1}{p} (J-\mu) \lambda} \mu^{(d-1)(\frac{1}{2}-\frac{1}{q})+}. \quad \tag{3.10}$$
Because of $t > r + \frac{1}{p}$ we can choose $\lambda > 1$ such that

$$r - t + \frac{1}{p} - \frac{1}{\delta} + \frac{1}{\delta} \lambda < 0.$$  

Consequently, we obtain

$$\sum_{\mu = J + 1}^{L} d_{n_{\mu}}(id_{\mu}) \lesssim 2^{J(r - t + \frac{1}{p} - \frac{1}{\delta} + \frac{1}{\delta} \lambda + (\frac{1}{\delta} - \frac{1}{\delta}))} J^{(d - 1)(-\frac{1}{\delta} + (\frac{1}{\delta} - \frac{1}{\delta}))}$$  

for all $L > J$.  

(3.10)  

Step 3. We choose $L$ large enough such that

$$2^{L(r - t + \frac{1}{p})} \lesssim 2^{J(r - t + \frac{1}{p} - \frac{1}{\delta} + (\frac{1}{\delta} - \frac{1}{\delta}))}.  

(3.11)$$

Summarizing (3.9), (3.10) and (3.11) we find

$$d_{n}(id^{*}) \lesssim 2^{J(r - t + \frac{1}{p} - \frac{1}{\delta} + (\frac{1}{\delta} - \frac{1}{\delta}))}.$$  

Finally, because of $J \approx \log n$ and $2^{J} \approx \frac{n}{\log^{3} n}$, see (3.8), we derive

$$d_{n}(id^{*}) \lesssim n^{r - t + \frac{1}{p} - \frac{1}{\delta} (\log n)} J^{(d - 1)}(-r + t + \frac{1}{\delta} - \frac{1}{\delta})).$$

The proof is complete.  

The following Proposition was proved by Triebel [29] for Banach spaces. However, it can be extended to the situation of quasi-Banach spaces. By $\mathcal{K}(X,Y)$ we denote the collection of all compact linear operators belonging to $L(X,Y)$.

**Proposition 3.5.** Let $(X_{1},X_{2})$ be an interpolation couple of two quasi-Banach spaces. Let $X$ be a quasi-Banach space with $X \hookrightarrow X_{1} + X_{2}$ and such that there exists a positive constant $C$ with

$$t^{-\theta} K(t,x,X_{1},X_{2}) \leq C \| x \| X \quad \text{for all } x \in X \text{ and all } t > 0.  

(3.12)$$

Let $Y$ be a quasi-Banach space such that

$$\| y_{1} + y_{2} \| Y \leq C_{Y} (\| y_{1} \| Y + \| y_{2} \| Y), \quad y_{1}, y_{2} \in Y,  

(3.13)$$

with $C_{Y} \geq 1$. Then, for $T \in \mathcal{K}(X_{1},Y)$, $T \in \mathcal{K}(X_{2},Y)$ and $T \in \mathcal{L}(X,Y)$ we have $T \in \mathcal{K}(X,Y)$ and

$$d_{n+m+1}(T : X \rightarrow Y) \leq 2CC_{Y} d_{n+1}^{1-\theta}(T : X_{1} \rightarrow Y) d_{m+1}^{\theta}(T : X_{2} \rightarrow Y), \quad n,m \in \mathbb{N},$$

where $C,C_{Y}$ are the constants from (3.12) and (3.13).

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Proof. Let $x \in X$, $\|x\| \leq 1$ and $\epsilon, t > 0$. Choose $x_1 \in X_1$, $x_2 \in X_2$ such that $x = x_1 + x_2$ and

$$\|x_1|X_1\| + t\|x_2|X_2\| \leq C(1 + \epsilon)K(t, x, X_1, X_2) \leq C_\epsilon t^\theta,$$

see (3.12), here $C_\epsilon = C(1 + \epsilon)$. Hence

$$\|x_1|X_1\| \leq C_\epsilon t^\theta, \quad \|x_2|X_2\| \leq C_\epsilon t^{\theta - 1}.$$

By definition of the Kolmogorov numbers, see (1.1), we have

$$d_{n+1}(T : X \to Y) = \inf_{L_n, L_m} \sup_{\|x\| \leq 1} \inf_{y_1 \in L_n, y_2 \in L_m} \|T x - y_1 - y_2\|_Y$$

$$\leq C_Y \inf_{L_n, L_m} \left[ \sup_{\|x_1\| \leq C_\epsilon t^\theta} \inf_{y_1 \in L_n} \|T x_1 - y_1\|_Y + \sup_{\|x_2\| \leq C_\epsilon t^{\theta - 1}} \inf_{y_2 \in L_m} \|T x_2 - y_2\|_Y \right]$$

$$\leq C_\epsilon C_Y \left[ t^\theta d_{n+1}(T : X_1 \to Y) + t^{\theta - 1} d_{m+1}(T : X_2 \to Y) \right].$$

Without loss of generality, we suppose that $d_{n+1}(T : X_1 \to Y) \neq 0$. Then we put

$$t = \frac{d_{m+1}(T : X_2 \to Y)}{d_{n+1}(T : X_1 \to Y)}.$$

This yields

$$d_{n+1}(T : X \to Y) \leq 2C_\epsilon C_Y d_{n+1}^{1-\theta}(T : X_1 \to Y) d_{m+1}^\theta(T : X_2 \to Y).$$

The proof is complete.■

Now we are ready to prove the main theorem [1.1]

**Proof of Theorem** [1.1] **Step 1. Estimate from above.**

**Substep 1.1.** We prove that

$$d_n(id : S_{p,q}^t B(\Omega) \to S_{r,\infty}^r B(\Omega)) \leq d_n(id^* : S_{p,q}^t B(\Omega) \to S_{r,\infty}^r B(\Omega)^*), \quad (3.14)$$

for all $n \geq 2$. Indeed, we consider the commutative diagram

$$S_{p,q}^t B(\Omega) \xrightarrow{\mathcal{E}} S_{p,q}^t B(\mathbb{R}^d) \xrightarrow{W} S_{p,q}^{t,\Omega} b \quad \xrightarrow{id^*} \quad S_{r,\infty}^r B(\Omega) \xrightarrow{R^\Omega} S_{r,\infty}^r B(\mathbb{R}^d) \xrightarrow{W^*} S_{r,\infty}^{r,\Omega} b$$

Here the operators $\mathcal{E}, W, W^*$ and $R^\Omega$ are defined as in the proof of Lemma 2.13. From the boundedness of these operators and property (s3) we obtain (3.14).

**Substep 1.2.** Under the given restrictions there always exists some $r$ such that $r > 0$ and $t > r + \frac{1}{p}$. We consider the commutative diagram
The multiplicity of the Kolmogorov numbers yields
\[ d_{2n-1}(id) \leq d_{n}(id)\cdot d_{n}(id). \]

From Lemma 3.3, Proposition 3.4 and inequality (3.14) we obtain the estimates from above in (1.2) and (1.4), respectively.

Substep 1.3. The conditions \( p > 2 \) and \( t > \frac{1}{2} \) guarantee that the following diagram becomes commutative

![Diagram](image)

By the property of Kolmogorov numbers we obtain
\[ d_{n}(id) \leq d_{n}(id)\cdot \|id\|. \]

Inserting the result in (1.2) we derive the estimate from above in (1.3).

Step 2. Estimate from below. For \( 1 \leq p, q \leq \infty \), concerning the lower bounds in (1.2) and (1.3), we refer to Proposition 3.1.

Substep 2.1. We consider the case \( 0 < p, q < 1 \). Because of the restrictions there always exists a triple \((\theta, p_1, q_1)\) such that
\[ 0 < \theta < 1, \quad 1 < p_1, q_1 \leq 2, \quad 1 = \frac{1 - \theta}{p} + \frac{\theta}{p_1} \quad \text{and} \quad 1 = \frac{1 - \theta}{q} + \frac{\theta}{q_1}. \]

From Lemma 2.13 we conclude
\[ K(\alpha, f, S_{p,q}^{t}B(\Omega), S_{p_1,q_1}^{t}B(\Omega)) \lesssim \alpha^{\theta}\|f|S_{1,1}^{t}B(\Omega)\|, \]

for all \( f \in S_{1,1}^{t}B(\Omega) \) and \( 0 < \alpha < \infty \). Now the interpolation property of the Kolmogorov numbers, see Proposition 3.5, yields
\[ d_{2n-1}(id : S_{1,1}^{t}B(\Omega) \to L_{\infty}(\Omega)) \]
\[ \lesssim d_{n}^{-\theta}(id : S_{p,q}^{t}B(\Omega) \to L_{\infty}(\Omega))d_{n}^{\theta}(id : S_{p_1,q_1}^{t}B(\Omega) \to L_{\infty}(\Omega)). \]
In view of Proposition 3.1 and Step 1 we have
\[ d_{2n-1}(id : S_{1,1}^t B(\Omega) \to L_\infty(\Omega)) \asymp n^{-t + \frac{1}{2}} (\log n)^{(d-1)(t-\frac{1}{2} + \frac{1}{2})} \]
and
\[ d_n(id : S_{p_1,q_1}^t B(\Omega) \to L_\infty(\Omega)) \asymp n^{-t + \frac{1}{p_1} - \frac{1}{2}} (\log n)^{(d-1)(t-\frac{1}{p_1} + \frac{1}{2})}, \]
for all \( n \geq 2 \). Consequently we obtain
\[ d_n(id : S_{p,q}^t B(\Omega) \to L_\infty(\Omega)) \asymp n^{-t + \frac{1}{p} - \frac{1}{2}} (\log n)^{(d-1)(t-\frac{1}{p} + \frac{1}{2})}, \quad n \geq 2. \]

Substep 2.2. If \( 0 < q < 1 \leq p \leq \infty \) we can choose \( 0 < \theta < 1 \) such that
\[ 1 = 1 - \frac{\theta}{q} + \frac{\theta}{2} \]
and
\[ K(\alpha, f, S_{p,q}^t B(\Omega), S_{p,2}^t B(\Omega)) \lesssim \alpha^\theta \|f| S_{1,q}^t B(\Omega)\|, \]
for all \( f \in S_{p,1}^t B(\Omega) \) and \( \alpha \in (0, \infty) \). Again the desired result follows from the interpolation property of Kolmogorov numbers.

Substep 2.3. For \( 0 < p < 1 \leq q < \infty \) we follow argument in Substep 2.2 with \( p \) in replace of \( q \). In case \( 0 < p < 1 \) and \( q = \infty \) we can find a pair \( (\theta, q_1) \) such that
\[ 0 < \theta < 1, \quad 2 < q_1 < \infty, \quad 1 = \frac{1 - \theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q_1} = \frac{1 - \theta}{\infty} + \frac{\theta}{2}. \]
Then we obtain
\[ K(\alpha, f_0, S_{p,q}^t B(\Omega), S_{2,2}^t B(\Omega)) \lesssim \alpha^\theta \|f_0| S_{1,q_1}^t B(\Omega)\|, \]
for all \( f \in S_{1,q_1}^t B(\Omega) \) and \( \alpha \in (0, \infty) \). Now the lower estimate in this case follows from the same argument as used in Substep 2.1.

Substep 2.4. It remains to prove the lower bound in (1.4). In fact, the lower estimate in this case was already proved by Romanyuk, see [21, Proof of Theorem 2.1]. He showed that if \( 2 \leq p \leq \infty, \ 1 \leq q \leq \infty \) and \( t > \frac{1}{p} \) then
\[ d_n(id : S_{p,q}^t B(\Omega) \to L_\infty(\Omega)) \gtrsim n^{-t}(\log n)^{(d-1)(t+(\frac{1}{p} - \frac{1}{q} + \frac{1}{2})}, \]
for all \( n \geq 2 \). However, for a better readability we give a proof here. We need the following result. Let \( 2 \leq p < s < \infty, \ 0 < q \leq \infty \) and \( t > \frac{1}{p} - \frac{1}{2} + \frac{1}{2s} \). Then we have
\[ d_n(id : S_{p,q}^t B(\Omega) \to L_s(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t+(\frac{1}{p} - \frac{1}{q} + \frac{1}{2})}, \quad n \geq 2, \quad (3.15) \]
see [2, 8, 17]. Now, if \( 2 < p < \infty, \ p \leq q \leq \infty \) and \( t > \frac{1}{p} \), there exists \( s \in \mathbb{R} \) such that
\[ p < s < \infty \quad \text{and} \quad t > \frac{1}{p} - \frac{1}{2} + \frac{1}{2s}. \]
We consider the diagram
By the property (s3) of the Kolmogorov numbers we find
\[ d_n(id_1) \leq d_n(id) \|id_2\|. \]

Inserting (3.15) into this we obtain the desired estimate. For the case \( p = q = \infty \), see Lemma 3.3. The proof is complete. ■

As preparation for the proof of Theorem 1.3, let us recall a result obtained by Vybiral [38].

**Lemma 3.6.** Let \( 0 < p \leq 1 \) and \( 0 < \gamma < 1 \). Then there is a number \( c_\gamma > 0 \) such that
\[ a_n(id : \ell_p^m \to \ell_\infty^m) \leq \frac{c_\gamma}{\sqrt{n}} \] (3.16)
holds for all natural numbers \( n \) and \( m \) with \( m^{\gamma} < n \leq m \).

**Proof of Theorem 1.3.** The estimate from below is a direct consequence of the inequality \( d_n \leq a_n \), see [16, Theorem 2.3.4], and Theorem 1.1.

**Step 1.** First we prove that if \( 0 < p \leq 1 \) and \( t > r + 1/p \) then
\[ a_n(id^* : s_{p,p}^{r,t,\Omega} b \to s_{\infty,\infty}^{r,t,\Omega} b) \ll n^{-t + r + 1/p - 1/2} (\log n)^{(d-1)(t-r-1/p)}, \] (3.17)
for all \( n \geq 2 \). Indeed, we follow the argument in Step 1 of the proof of Proposition 3.4 and obtain
\[ a_n(id^*) \leq \sum_{\mu=J+1}^{L} a_n(id_{\mu}^*) + 2^{L(r-t+1/p)}. \] (3.18)

Next we choose
\[ L = \left\lfloor \frac{r-t+1/p-1/2}{r-t+1/p} - \frac{d-1}{2} \frac{\log J}{r-t+1/p} \right\rfloor \]
(here \( \lfloor x \rfloor \) denotes the integer part of \( x \in \mathbb{R} \) and \( \lambda > 1 \) such that
\[ t - r > \frac{1}{p} + \frac{\lambda - 1}{2}. \] (3.19)

This leads to
\[ 2^{L(r-t+1/p)} \ll 2^{J(r-t+1/p - 1/2) J^{-1/2}}. \] (3.20)

Observe that
\[ a_n(id_{\mu}^*) \asymp a_n(id : 2^{\mu(t-1/p)} \ell_{p}^{D_{\mu}} \to 2^{\mu r} \ell_{\infty}^{D_{\mu}}) \ll 2^{\mu(r-t+1/p)} a_n(id : \ell_{p}^{D_{\mu}} \to \ell_{\infty}^{D_{\mu}}), \]
see Lemma 3.6(ii). Using the definition of \( n_\mu \), see (3.7), a simple calculation shows that there is a number 0 < \( \gamma \) < 1 depending on \( t, r \) and \( p \) such that

\[
D_\gamma \mu \leq n_\mu \leq D_\mu, J + 1 \leq \mu \leq L.
\]

Hence

\[
a_n(\mu : \ell_p^{D_\mu} \to \ell_\infty^{D_\mu}) \leq c_\gamma (D_\mu 2^{(J-\mu)\lambda})^{-\frac{1}{2}},
\]

see Lemma 3.6. Consequently

\[
a_n(\mu : \ell_p^{D_\mu} \to \ell_\infty^{D_\mu}) \lesssim 2^{(r-t+\frac{1}{p})/2} (r^2)(J-\mu)^{-\frac{1}{2}}.
\]

The condition (3.19) guarantees

\[
\sum_{\mu=J+1}^{L} a_n(\mu : \ell_p^{D_\mu} \to \ell_\infty^{D_\mu}) \lesssim 2^{(r-t+\frac{1}{p})/2} J^{-\frac{1}{2}}(d-1).
\]

Now (3.18), (3.20) and (3.21) yield

\[
a_n(\mu : s^{t,\Omega} \to s^{r,\Omega}) \lesssim 2^{(r-t+\frac{1}{p})/2} J^{-\frac{1}{2}}(d-1).
\]

Because of \( J \asymp \log n \) and \( 2^J \asymp \frac{n}{\log^{d-1} n} \), see (3.5), we obtain (3.17).

**Step 2.** The diagram in Step 1.1 of the proof of Theorem 1.1 and property (3) lead to

\[
a_n(\mu : S_{p,p}^t B(\Omega) \to S_{\infty,\infty}^r B(\Omega)) \lesssim a_n(\mu : s^{t,\Omega} \to s^{r,\Omega}) \lesssim n^{-t+r+\frac{1}{p}+\frac{1}{2}} (\log n)^{(d-1)(t-r-\frac{1}{p})}, n \geq 2. \quad (3.22)
\]

**Step 3.** We choose \( r > 0 \) such that \( t > r + \frac{1}{p} \). From the commutative diagram

\[
\begin{array}{ccc}
S_{p,p}^t B(\Omega) & \xrightarrow{id} & L_\infty(\Omega) \\
\downarrow{id_1} & & \downarrow{id_2} \\
S_{\infty,\infty}^r B(\Omega)
\end{array}
\]

and the multiplicativity of the approximation numbers we obtain

\[
a_{2n-1}(id) \leq a_n(id_1) a_n(id_2).
\]

Finally, in view of (3.3) and (3.22), we get the estimate from above.

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