On the Łojasiewicz Exponent of the Quadratic Sphere Constrained Optimization Problem

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Abstract. In this paper, we prove that the global version of the Łojasiewicz gradient inequality holds for the quadratic sphere constrained optimization problem with exponent \( \theta = \frac{3}{4} \). An example from [29] inspired by [4] shows that \( \theta = \frac{3}{4} \) is tight. This is the first Łojasiewicz gradient inequality established for the sphere constrained optimization problem with a linear term.

1 Introduction

In the pioneering work 1965 [25], Łojasiewicz analyzed the relationship between function value and its zeros of real-analytic functions, and first proposed the following Łojasiewicz inequality.

**Definition 1.1.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a real function, \( f(x^*) = 0 \). Suppose there exist constants \( \eta > 0, C, \epsilon > 0 \) satisfying the following inequality:

\[
|f(x)| \geq C \text{dist}(x, L_f(x^*))^\eta, \quad \forall x \in B(x^*, \epsilon),
\]

where \( \text{dist}(x, \Omega) := \inf_{z \in \Omega} ||x - z||_2 \), \( L_f(x^*) = \{ x \mid f(x) = f(x^*) \} \) and \( B(x^*, \epsilon) := \{ x \in \mathbb{R}^n \mid ||x - x^*||_2 \leq \epsilon \} \). Then we call inequality (1) the Łojasiewicz inequality.

It was showed in [25] that for any real analytic function \( f \), the Łojasiewicz inequality holds. But the Łojasiewicz inequality does not necessarily hold even for \( C^\infty \) function

\[
f(x) := \begin{cases} 
  e^{-1/|x|^2}, & \text{if } x \neq 0; \\
  0, & \text{if } x = 0.
\end{cases}
\]

Another version of the Łojasiewicz inequality is like following.

**Definition 1.2.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a real differentiable function, \( f(x^*) = 0 \) and \( x^* \) be a stationary point of \( f \). Suppose there exist constants \( \theta \in [0, 1), C, \epsilon > 0 \) such that

\[
|\nabla f(x)| \geq C |f(x)|^\theta, \quad \forall x \in B(x^*, \epsilon),
\]

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Then we call inequality (2) the gradient form of the Łojasiewicz inequality or Łojasiewicz gradient inequality.

Usually, the smallest $\eta$ satisfying inequality (1) is called the Łojasiewicz exponent of $f$ at $x^*$. Similarly, the smallest $\theta$ satisfying (2) is called the Łojasiewicz exponent of $f$ at $x^*$ in the gradient inequality. The relationship between the Łojasiewicz exponents $\eta$ and $\theta$ can be found in [14] and [20].

If the Łojasiewicz gradient inequality holds for any $x^* \in \Omega_f := \{ x \mid \nabla f(x) = 0 \}$, namely,

$$|\nabla f(x)| \geq C|f(x) - f(x^*)|^\theta, \quad \forall x \in B(x^*, \epsilon), \quad \forall x^* \in \Omega_f,$$

we call (3) the global version of the Łojasiewicz gradient inequality.

In this paper, We focus on the following quadratic sphere constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2} x^T A x + g^T x, \quad \text{s. t.} \quad x^T x = 1.$$  (4)

We are only interested in investigating the global version of its Łojasiewicz gradient inequality for (4). Therefore, we omit “global version” and “gradient” and simply call (3) the Łojasiewicz inequality if it does not cause any confusion.

### 1.1 Łojasiewicz exponent

For the real-analytic functions, Łojasiewicz [25] established the Łojasiewicz inequality with some $\theta \in \left[\frac{1}{2}, 1\right)$. If $f$ is an analytic function in a neighborhood of the origin in $\mathbb{R}^n$ and assume $f(0) = 0$ and $\nabla f(0) = 0$, Bochnak [5] proved that the Łojasiewicz exponent $\theta$ of $f$ at the origin is a rational number.

Suppose $f$ is a polynomial of degree $d$. Gwoździewicz [14] proved that the Łojasiewicz inequality holds with

$$\theta \leq 1 - \frac{1}{(d-1)^n + 1}$$

if $f$ has an isolated stationary point at the origin. Later on, Kurdyka [19] considered the case that $f$ is of non-isolated stationary point, and estimated an upper bound for the Łojasiewicz exponent as follows

$$\theta \leq 1 - \frac{1}{d(3d-3)^{n-1}},$$

for $d \geq 2$. Yang [35] proved that the Łojasiewicz inequality holds with

$$\theta = 1 - \frac{1}{d}$$  (5)

if polynomial $f$ is convex. The Łojasiewicz exponent (5) does not work for nonconvex case. For instance, for function

$$f(x) = x_n^2 + \sum_{i=1}^{n-1} (x_i^2 - x_{i+1})^2, \quad x \in \mathbb{R}^n (n \geq 3),$$  (6)

the Łojasiewicz exponent has a lower bound

$$\theta \geq 1 - \frac{1}{2n},$$

which is in the exponent order of $n$ and can be much greater than the Łojasiewicz exponent (5) of the convex case.

2
1.2 Relationship to Optimization

To prove the local linear convergence rate of algorithms for solving linear variational inequalities or convex optimization problems with weaker condition than strongly convexity, certain local error bound conditions are established. The early works refer to \[28, 27, 32\] and recent progresses refer to \[22, 36, 11\]. A local error bound condition usually measures the relationship between certain norm of the subgradient at a point in the neighborhood of the set of minimizers and its distance to this set. To study the nonconvex case, we need to use the Łojasiewicz inequality instead.

The authors of \[6\] extended the Łojasiewicz inequality to a wide class of nonsmooth functions which are lower semicontinuous convex subanalytic or continuous subanalytic. They analyzed the convergence with respect to subgradient-type algorithms, and gave the iterate convergence rate with different Łojasiewicz exponents.

Later on, the authors of \[2\] extended the Łojasiewicz property to proper lower semicontinuous functions as follows.

**Definition 1.3.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous function satisfying the restriction of \( f \) to its domain is a continuous function. The function \( f \) is said to have the Łojasiewicz property if there exist constants \( C, \epsilon > 0 \) and \( \theta \in [0, 1) \) satisfying

\[
|f(x) - f(x^*)| \leq C\|y\|_2, \quad \forall x \in B(x^*, \epsilon), \quad \forall y \in \partial f(x), \quad \forall x^* \in \tilde{\Omega}_f,
\]

where \( \tilde{\Omega}_f : \{x \mid 0 \in \partial f(x)\} \) and \( \partial f(x) \) stands for the subdifferential (the set containing all the subgradients) of \( f \) at \( x \).

They also showed that the iterate convergence and local convergence rate hold for any approach satisfying certain sufficient function value reduction and asymptotic small stepsize safe-guard conditions in solving nonsmooth nonconvex optimization problems satisfying the Łojasiewicz inequality. Their main result can be described as follows.

**Proposition 1.1.** \[2\] Suppose that \( f \) satisfies the Łojasiewicz inequality (7). Let \( \{x^k\}_{k \in \mathbb{N}} \) be a bounded iterate sequence satisfying

1. **sufficient function value reduction:** \( f(x^k) - f(x^{k+1}) \geq C_1 \cdot \|x^k - x^{k+1}\|_2^2 \);  
2. **asymptotic small stepsize safe-guard:** \( \|y\|_2 \leq C_2 \cdot \|x^{k-1} - x_k\|_2 \), for all \( y \in \partial f(x^k) \),

where \( C_1 \) and \( C_2 \) are positive constants. Then the sequence \( \{x^k\}_{k \in \mathbb{N}} \) converges. Furthermore, let \( x^* = \lim_{k \to +\infty} x^k \), we have

1. If \( \theta = 0 \), the sequence \( \{x^k\}_{k \in \mathbb{N}} \) converges in a finite number of steps;  
2. If \( \theta \in (0, \frac{1}{2}] \), then there exist \( c > 0 \) and \( Q \in [0, 1) \) such that \( \|x^k - x^*\|_2 \leq c \cdot Q^k \);  
3. If \( \theta \in (\frac{1}{2}, 1) \), then there exists \( c > 0 \) such that \( \|x^k - x^*\|_2 \leq c \cdot k^{-\frac{\theta}{1-\theta}} \).

We clearly observe the close relationship between the Łojasiewicz inequality, especially the Łojasiewicz exponent and convergence properties of related optimization algorithms.

Recently, Kurdyka \[18\] extended the Łojasiewicz inequality to definable functions and the corresponding inequality is called the Kurdyka-Łojasiewicz (KL) inequality. With the help of KL inequality one can magically show the convergence of plenty of first-order algorithms for solving a large variety of difficult problems, see \[1, 3, 7, 9, 33, 30, 34, 23\]. On the other hand, in \[8, 22, 11\], the authors pointed out the relationship between KL inequality and the local error bound conditions. More specifically, the KL inequality with \( \theta = \frac{1}{2} \) is equivalent to the Luo-Tseng error bound \[21\]. The Łojasiewicz inequality with exponent \( \theta = \frac{1}{2} \) is also known as Polyak-Łojasiewicz (PL) inequality. When
\( f \) is the summation of a convex function with Lipschitz continuous gradient and a proper closed convex function, one can consider a so-called proximal PL inequality \([16, 29]\) which helps the convergence analysis for proximal gradient methods for such kind of structured convex optimization problems.

## 1.3 Sphere Constrained Problem

For problems with nonconvex constraints such as sphere constraint or orthogonal constraint, the study of the Łojasiewicz property becomes much more complicated.

For orthogonal constrained problems, the authors of \([24]\) proved that the Łojasiewicz inequality with exponent \( \theta = \frac{1}{2} \) (Theorem 1 of \([24]\)) holds only for the case

\[
\min_{X \in \mathbb{R}^{n \times p}} \quad \text{tr}(X^\top AX) \\
\text{s. t.} \quad X^\top X = I_p, \tag{8}
\]

where \( I_p \) is the \( p \times p \) identity matrix. A special case of problem (8) with \( p = 1 \) is a special case of problem (4) with \( g = 0 \).

For sphere constrained problems, if the objective is a quartic function in the following form

\[
f(x) = x_{n-1}^2 + \sum_{i=1}^{n-2} (x_i^2 - x_{i+1})^2, \quad x \in \mathbb{R}^n \ (n \geq 4),
\]

we can easily show its Łojasiewicz exponent cannot be smaller than \( 1 - \frac{1}{2^{n-1}} \) due to the instance (6). For a quadratic sphere constrained problem, locally its characteristic is similar to a quartic function. Therefore, to show that it is of a constant Łojasiewicz exponent, which is not related to dimension of variable \( n \), is not trivial.

The main contribution of this paper is to establish the Łojasiewicz inequality for the quadratic minimization with sphere constraint, and \( \theta = \frac{3}{4} \) is the Łojasiewicz exponent. Such result guarantees the convergence property of first-order algorithms for solving this type of problems and also gives insights for further study of the quadratic minimization with orthogonal constraint.

The rest of this paper is organized as follows. In the next section, we deliver the detailed proof of our main result. In the last section, remarks and discussions are presented.

## 2 The Proof of Our Main Result

In this section, we first give a specific formulation of the Łojasiewicz inequality in the sphere constrained case, and then prove the main result. Finally, we put forward an example from \([29]\) showing that \( \theta = \frac{3}{4} \) is the Łojasiewicz exponent.

### 2.1 Preliminary

By penalizing the sphere constraint, we obtain the following unconstrained problem

\[
\min_{x \in \mathbb{R}^n} \tilde{f}(x) := f(x) + \delta_S(x), \tag{9}
\]
where \( \delta_S(x) := \begin{cases} 0, & \text{if } x \in S; \\
+\infty, & \text{otherwise,} \end{cases} \) and \( S := \{ x \mid x^\top x = 1 \} \) denotes the feasible region of (4). It is clear that (9) is equivalent to (4). Hence, the Łojasiewicz inequality (7) for (9) can be described as following.

\[
|f(x) - f(x^*)|^\theta \leq C \cdot \|y\|_2, \quad \forall x \in B(x^*, \epsilon) \cap S, \quad \forall y \in \partial f(x), \quad \forall x^* \in \tilde{\Omega}_f. \quad (10)
\]

We can easily verify that

\[
\arg \min_{y \in \partial f(x)} \|y\|_2^2 = \begin{cases} (I - xx^\top) \nabla f(x), & \text{if } x^\top \nabla f(x) \leq 0; \\
\nabla f(x), & \text{otherwise}, \end{cases} \quad \forall x \in S.
\]

Hence, the Łojasiewicz inequality (10) is implied by

\[
|f(x) - f(x^*)|^\theta \leq C \cdot \| (I - xx^\top) \nabla f(x) \|_2, \quad \forall x \in B(x^*, \epsilon) \cap S, \quad \forall x^* \in \tilde{\Omega}_f. \quad (11)
\]

### 2.2 Proof

**Theorem 2.1.** There exist constants \( C, \epsilon > 0 \) satisfying the Łojasiewicz inequality (10) for problem (4) with exponent \( \frac{3}{4} \).

**Proof.** According to the preliminaries showed above, it suffices to prove (11) holds for \( \theta = \frac{3}{4} \).

The first-order optimality condition gives us

\[
(I - xx^\top) \nabla f(x^*) = 0.
\]

According to the first-order optimality condition, there exists a Lagrangian multiplier \( \lambda^* \) satisfying

\[
\nabla f(x^*)(= Ax^* + g) = \lambda^* x^*.
\]

Equality (12) implies \( \lambda^* = \nabla f(x^*)^\top x^* \).

Let \( \Delta := x - x^* \), then we have

\[
\Delta^\top x = -\Delta^\top x^* = \frac{1}{2} \| \Delta \|_2^2.
\]

Consequently, we have

\[
\Delta = P_{\perp x^*} \Delta + x^* x^\top \Delta = P_{\perp x^*} \Delta - \frac{1}{2} \| \Delta \|_2^2 \cdot x^*, \quad (14)
\]

where \( P_{\perp x^*} := (I - xx^\top) \).

Now we estimate the left hand side of (11),

\[
f(x) - f(x^*) = \nabla f(x^*)^\top \Delta + \frac{1}{2} \Delta^\top A \Delta = \lambda^* \Delta^\top x^* + \frac{1}{2} \Delta^\top A \Delta = \frac{1}{2} \| \Delta \|_2^2 \Phi^* \Delta
\]

\[
= \frac{1}{2} (P_{\perp x^*} \Delta)^\top \Phi^* (P_{\perp x^*} \Delta) - \frac{1}{2} \| \Delta \|_2^2 \cdot (P_{\perp x^*} \Delta)^\top \Phi^* x^* + \frac{1}{8} \| \Delta \|_4^4 \cdot x^\top \Phi^* x^*,
\]

where \( \Phi^* := A - \lambda^* I \).

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1. The subgradient of indicator function at \( x \notin S \) is the emptyset by convention, and hence this situation is out of our consideration.
2. By slightly abusing of the notation, we use \( I \) to denote \( I_m \) hereinafter.
The right hand side of (11) can be estimated as the following

\[
(I - xx^\top)\nabla f(x) = (I - xx^\top)\nabla f(x) - (I - xx^*\top)\nabla f(x^*) \\
\quad = \nabla f(x) - \nabla f(x^*) - xx^\top \nabla f(x) + xx^\top \nabla f(x^*) - xx^T \nabla f(x^*) \\
\quad + xx^\top \nabla f(x^*) - xx^*\top \nabla f(x^*) + xx^*\top \nabla f(x^*) \\
\quad = A\Delta - xx^\top A\Delta - \lambda^* \cdot x\Delta^\top x - \lambda^\top \Delta \\
= (I - xx^\top)A\Delta + \lambda^* \cdot xx^\top \Delta - \lambda^\top \Delta = (I - xx^\top)\Phi^* \Delta \\
= P^{-1}_x \Phi^* \Delta - xx^\top \Phi^* \Delta - xx^\top \Phi^* \Delta - \Delta \Delta^\top \Phi^* \Delta \\
= P^{-1}_x \Phi^* P^{-1}_x \Delta - \frac{1}{2}\|\Delta\|^2 P^{-1}_x \Phi^* xx^* - xx^\top \Phi^* \Delta - xx^\top \Phi^* \Delta - \Delta \Delta^\top \Phi^* \Delta. \quad (16)
\]

Next, we will discuss the relationship between (16) and (15) through the following three cases.

**Case I.** \(\Phi^*\) is nonsingular. Denote \(B := P^{-1}_x \Phi^* P^{-1}_x\). Clearly, \(\text{rank}(B) = n - 1\), and \(\text{span}\{x^*\}\) is the null space of \(B\). Denote \(\sigma_+(B)\) as the smallest nonzero singular value of \(B\). We have

\[
|f(x) - f(x^*)| = \frac{1}{2}\|\Delta^\top B\Delta\| + o(\|\Delta\|^2); \\
\|\nabla f(x)|^2 \geq \sigma_+(B) \cdot \|\Delta^\top B\Delta\| + o(\|\Delta\|^2),
\]

which implies (11) holds with exponent \(\frac{1}{2}\). Here the last inequality holds because

\[
\|\Delta^\top B^2 \Delta\| = \|\Delta^\top B^\top BB^\top B^\top \Delta\| \geq \sigma_+(B) \cdot \|B^\top \Delta\|^2,
\]

which uses the fact that \(B^\top \Delta\) lies in the range space of \(B\).

**Case II.** \(\Phi^*\) is singular and \(\Phi^* \Delta = 0\). The \(\dot{\text{Lojasiewicz inequality}}\) (11) holds immediately.

**Case III.** \(\Phi^*\) is singular and \(\Phi^* \Delta \neq 0\). We decompose \(\Delta\) into two parts, one in the null space of \(\Phi^*\) and one in the range space of \(\Phi^*\). Namely, \(\Delta = \delta + \eta\), where

\[
\Phi^* \delta = 0, \quad \eta^\top \delta = 0, \quad \|\Phi^* \eta\|_2 \geq \sigma_+ \|\eta\|_2. \quad (17)
\]

By slightly abusing the notation, \(\sigma_+\) stands for \(\sigma_+(\Phi^*)\).

By simple calculation, we have

\[
f(x) - f(x^*) = \frac{1}{2} \eta^\top \Phi^* \eta \quad (18) \\
(I - xx^\top)\nabla f(x) = P^{-1}_x \Phi^* \eta - \eta^\top \Phi^* \eta \cdot x^* - \Delta xx^\top \Phi^* \eta - \eta^\top \Phi^* \eta \cdot \Delta. \quad (19)
\]

We then consider two situations.

**Case III-1.** \(P^{-1}_x \Phi^* \eta \neq 0\). In this situation, we can prove the \(\dot{\text{Lojasiewicz inequality}}\) (11) holds with exponent \(\frac{1}{2}\) in the same manner as Case I.

**Case III-2.** \(P^{-1}_x \Phi^* \eta = 0\). In this situation, we notice that \(\Phi^* \eta\) lies in the range space of \(\text{span}\{x^*\}\). Namely, we have

\[
\Phi^* \eta = \xi x^* \quad (20)
\]
and $\xi \neq 0$ due to (17). Left multiplying both sides of (20) by $\delta^T$ and using (17), we have

$$0 = \delta^T \Phi^* \eta = \xi \delta^T x^*,$$

(21)

which implies $\delta^T x^* = 0$. Using the sphere constraint, we obtain

$$1 = \|x^* + \delta + \eta\|_2^2 = 1 + \|\delta\|_2^2 + \|\eta\|_2^2 + 2\delta^T x^* + 2\eta^T x^*.$$

Together with $\delta^T x^* = 0$, we have

$$-2\eta^T x^* = \|\delta\|_2^2 + \|\eta\|_2^2,$$

(22)

which implies

$$\|\Delta\|_2^2 \leq 2\|\eta\|_2.$$

(23)

On the other hand, by using (17), $P_{x^*}^1 \Phi^* \eta = 0$ and (20), we have

$$\sigma_{\text{max}} \|\eta\|_2 \geq \|\Phi^* \eta\|_2 = \|x^* (x^* + \Phi^* \eta)\|_2 = \|x^* \Phi^* \eta\|_2 = \|x^* \Phi^* \eta\|_2 = \|\xi\| \geq \sigma_+ \|\eta\|_2,$$

where $\sigma_{\text{max}}$ is the largest singular value of $\Phi^*$. Left multiplying (20) by $\eta^T$, we have

$$\sigma_+ \|\eta\|_2 \leq \eta^T \Phi^* \eta = \xi \eta^T x^* = \frac{1}{2} \|\eta\|_2 \|\Delta\|_2^2 \leq \frac{\sigma_{\text{max}}}{2} \|\eta\|_2 \|\Delta\|_2^2,$$

which implies

$$\|\Delta\|_2^2 \geq \frac{2\sigma_+}{\sigma_{\text{max}}} \|\eta\|_2^2.$$

(24)

Substituting (23) and (24) into (18)-(19), we have

$$|f(x) - f(x^*)| \leq \frac{\sigma_{\text{max}}}{2} \|\eta\|_2^2$$

and

$$\|(I - xx^T) \nabla f(x)\|_2 \geq \sqrt{2\sigma_{\text{max}}^2 \|\eta\|_2^2} + o(\|\eta\|_2^2).$$

This completes the proof.

\[\square\]

### 2.3 Example

Theorem 2.1 shows that the Łojasiewicz inequality with $\theta = \frac{3}{4}$ holds. The following example from Pong [29] which is inspired by [4] shows that $\frac{3}{4}$ is a lower bound of $\theta$. Combining this lower bound with Theorem 2.1, $\theta = \frac{3}{4}$ turns to be the Łojasiewicz exponent for problem (4).

**Example 2.1.** Let $n = 2$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $f(x) = \frac{1}{2} (x_2 - 1)^2$. It can be verified that $x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the global minimizer of problem (4). By simple calculation, we can further verify that Łojasiewicz inequality with $\theta < \frac{3}{4}$ does not hold for $f$ at $x^*$. Namely, $\frac{3}{4}$ is a lower bound of the Łojasiewicz exponent $\theta$. 
3 Remarks

By using our main result, Theorem 2.1, we can establish the iteration convergence and local convergence rate of algorithms for solving problem (4) as follows.

**Theorem 3.1.** Let \( \{x^k\} \) be the iterate sequence, generated by any algorithm for solving problem (4) and initiated from a feasible point \( x^0 \), satisfying

\[
\begin{align*}
  f(x^k) - f(x^{k+1}) & \geq C_1 \cdot ||x^k - x^{k+1}||_2^2, \quad x^k \in S, \\
  ||(I - x^k x^k^\top) \nabla f(x^k)||_2 & \leq C_2 \cdot ||x^{k-1} - x^k||_2.
\end{align*}
\]

Then it holds that

\[
\sum_{k=1}^{\infty} ||x^k - x^{k+1}||_2 < +\infty,
\]

which implies the convergence of \( \{x^k\} \). Furthermore, let \( x^* = \lim_{k \to +\infty} x^k \), we have

\[
||x^k - x^*|| \leq C_3 \sqrt{\frac{1}{k}}.
\]

This theorem can be viewed as a corollary of Proposition 1.1, and hence its proof is omitted here. It is worthy of mentioning that GR or GP introduced in [12] satisfy the assumptions in Theorem 3.1, and hence enjoy the convergence result of Theorem 3.1.

If \( g = 0 \) in problem (4), it follows from the main result of [24] that the Łojasiewicz inequality with \( \theta = \frac{1}{2} \) holds in this case. At the mean time, the result stated in Theorem 3.1 can be improved to linear convergence. Naturally, we may ask ourselves under which situations does the Łojasiewicz inequality hold with \( \theta = \frac{1}{2} \)? We can answer this question partly through the following corollary. To completely answer it seems to be extremely difficult.

**Corollary 3.2.** The inequality

\[
|f(x) - f(x^*)|^\theta \leq C \cdot ||(I - xx^\top) \nabla f(x)||_2, \quad \forall x \in B(x^*, \epsilon) \cap S
\]

holds with \( \theta = \frac{1}{2} \) at any first-order stationary point \( x^* \) of problem (4) which satisfies either of the following two statements

(i) \( g = 0 \);

(ii) \( d^\top \Phi^* d \neq 0 \) for any \( d^\top x^* = 0 \).

**Proof.** Suppose in either of the above mentioned two situations, inequality (25) does not hold with \( \theta = \frac{1}{2} \). Let \( \lambda^* \) be the Lagrangian multiplier corresponding to \( x^* \). We denote \( \Phi^* = A - \lambda^* I \). According to the proof of Theorem 2.1, we are dealing with Case III-2. Due to relationships \( \Phi^* \delta = 0 \) and \( \delta^\top x^* = 0 \), Situation (ii) can not happen. Therefore, \( g = 0 \) which implies \( \Phi^* x^* = 0 \). On the other hand, \( \Phi^* \eta = \xi x^* \) implies \( \xi = \eta^\top \Phi^* x^* = 0 \) which contradicts to \( \xi > 0 \). This completes the proof. \( \square \)

A few interesting questions for further development are collected as follows.

- What is the Łojasiewicz exponent for orthogonal constrained optimization problem with \( f(X) = \frac{1}{2} \text{tr}(X^\top AX) + \text{tr}(G^\top X) \)?
Can the Łojasiewicz exponent for general sphere constrained optimization problems be specified?

What is the situation for general quadratic constraints?

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