Research Article

Theta Omega Topological Operators and Some Product Theorems

Samer Al Ghour and Salma El-Issa

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Correspondence should be addressed to Samer Al Ghour; algore@just.edu.jo

Received 6 July 2021; Revised 1 August 2021; Accepted 27 October 2021; Published 16 November 2021

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We introduce and investigate the concepts of $\theta_\omega$-limit points and $\theta_\omega$-interior points, and we use them to introduce two new topological operators. For a subset $B$ of a topological space $(Y, \sigma)$, denote the set of all limit points of $B$ (resp. $\theta$-limit points of $B$), $\theta_\omega$-limit points of $B$, interior points of $B$, $\theta$-interior points of $B$, and $\theta_\omega$-interior points of $B$ by $D(B)$ (resp. $D_\theta(B)$, $D_\omega(B)$, $\text{Int}(B)$, $\text{Int}_\theta(B)$, and $\text{Int}_\omega(B)$). Several results regarding the two new topological operators are given. In particular, we show that $D_\omega(B)$ lies strictly between $D(B)$ and $D_\theta(B)$ and $\text{Int}_\omega(B)$ lies strictly between $\text{Int}_{\theta}(B)$ and $\text{Int}(B)$. We show that $D(B) = D_\omega(B)$ (resp. $\text{Cl}_\omega(B) = \text{Cl}_\omega(B)$ and $D(B) = D_\omega(B) = D_\theta(B)$) for locally countable topological spaces (resp. antilocally countable topological spaces and regular topological spaces). In addition to these, we introduce several product theorems concerning metacompactness.

1. Introduction

In 1943, Fomin [1] introduced the notion of $\theta$-continuity. For the purpose of studying the important class of $H$-closed spaces in terms of arbitrary filterbases, the notions of $\theta$-open subsets, $\theta$-closed subsets, and $\theta$-closure were introduced by Velicko [2] in 1966, in which he showed that the family of $\theta$-open sets in a topological space $(Y, \sigma)$ forms a topology on $Y$ denoted by $\sigma_\theta$ (see also [3]). The work of Velicko is continued by [3–26] and others. Hdeib [27] introduced the class of $\omega$-closed sets by which he introduced and investigated the notion of $\omega$-continuity. The family of all $\omega$-open sets in $(Y, \sigma)$ is denoted by $\sigma_\omega$. It is known that $\sigma_\omega$ is a topology on $Y$ which is finer than $\sigma$. Research related to $\omega$-open sets is still a hot area of research [28–36]. In 2017, Al Ghour and Irshidat [37] introduced $\theta_\omega$-open subsets, $\theta_\omega$-closed subsets, and $\theta_\omega$-closure utilizing the topological spaces $(Y, \sigma_\theta)$ and $(Y, \sigma_\omega)$. It is proved in [37] that $\sigma_\theta$ forms a topology on $Y$ which lies between $\sigma_\theta$ and $\sigma$, and that $\sigma_\omega = \sigma$ if and only if $(Y, \sigma)$ is $\omega$-regular. Also, in [37], $\omega - T_2$ topological spaces were characterized via $\theta_\omega$-open sets. Authors in [35] introduced $\theta_\omega$-connectedness and some new separation axioms. Also, research in [37] was continued by various researchers in [28–31]. The notion of interior operators is important in the axiomatization of modal logics. Judging from the importance of limit points in mathematical analysis, introducing a new limit point notion in any topological structure is still a hot area of research. The first goal of this paper is to introduce and investigate the concepts of $\theta_\omega$-limit points and $\theta_\omega$-interior points.

In general topology, several topological properties are not finitely productive, such as paracompactness, strong paracompactness, Lindelöfness, and metacompactness. The area of research regarding the problem “What conditions on $(Y, \sigma)$ and $(Z, \delta)$ to insure that their product has property $\mathcal{P}$” is still hot [38–45]. The second goal of this paper is to introduce several product theorems concerning metacompactness.

2. Preliminaries

From now on TS will denote topological space for simplicity. Let $(Y, \sigma)$ and $(Z, \delta)$ be TSs and let $B \subseteq C \subseteq Y$ with $C$ as nonempty. Then, $B$ is called $\omega$-open set in $(Y, \sigma)$ [27] if for each $y \in B$, there is $M \in \sigma$ and a countable set $F \subseteq Y$ such that $y \in M - F \subseteq B$. The relative topology on $C$ is denoted by $\sigma_\delta$, and the product topology on $Y \times Z$ is denoted by $\sigma \times \delta$. The closure of $B$ in $(Y, \sigma)$ (resp. $(C, \sigma_\delta)$, $(Y, \sigma_\omega)$) is denoted by $\overline{B}$ (resp. $\overline{B}_\delta$, $\overline{B}_\omega$). A point $y \in Y$ is in $\theta$-closure of $B$ [2] ($y \in \text{Cl}_\theta(B)$) if for every $G \in \sigma$ with $y \in G$, $\overline{G} \cap B \neq \emptyset$. $B$ is...
called θ-closed [2] if Clθ(B) = B. The complement of a
θ-closed set is called a θ-open set. It is known that σθ = σ if
and only if (Y, σ) is regular. A TS (Y, σ) is called ω-regular
[37] if for each closed set C in (Y, σ) and y ∈ Y \ C, there
exist G ∈ σ and H ∈ σω such that y ∈ G, C ⊆ H, and
G \ H = Ø. In [37], the author defined θω-closure operator
as follows: a point y ∈ Y is in θω-closure of B (y ∈ Clθω(B)) if
for any G ∈ σ with y ∈ G we have $G ∩ B \neq Ø$. G is called
θω-closed if Clθω(G) = G. The complement of a θω-closed
set is called a θω-open set. A TS (Y, σ) is called metacompact
[46] if every open cover of (Y, σ) has a point-finite open
refinement.

The following sequence of definitions and theorems will
be used in the sequel.

Definition 1 (see [47]). A TS (Y, σ) is called locally
countable if for each y ∈ Y, there is G ∈ σ such that G is
countable and y ∈ G.

Definition 2 (see [48]). A TS (Y, σ) is called antilocally
countable if each G ∈ σ − [Ø] is uncountable.

Definition 3 (see [9]). Let (Y, σ) be a TS BcY. A point y ∈ Y
is called θ-limit point of B if for each G ∈ σy with y ∈ G,
G ∩ (B − {y}) ≠ Ø. The set of all θ-limit points of B is called
the θ-derived set of B and is denoted by Dθ(B).

Theorem 1 (see [37]). If (Y, σ) is locally countable and BcY,
then $\overline{B} = Cl_{\omega}(B)$.

Theorem 2 (see [37]). If (Y, σ) is antilocally countable and
BcY, then Clθ(B) = Clω(B).

Theorem 3 (see [37]). For any TS (Y, σ), a ∈ σ, b ∈ σy, then
σy ≠ σy.

Theorem 4 (see [2]). A TS (Y, σ) is regular if and only if
σ = σσ.

Theorem 5 (see [37]). Let (Y, σ) be a TS and BcY. Then,
B is θω-open set if and only if for each y ∈ B, there exists G ∈ σ
such that

Definition 5. Let (Y, σ) and (Z, δ) be TSs and let BcY. Then,

(a) (Y, σ) is called C-scattered if every $B ∈ σ − [Ø]$, there
is b ∈ B and a compact set K such that $b ∈ Int(K) \subseteq K \subseteq B$ [49]

(b) B is called strongly placed in Y × Z if for every z ∈ Z
and $H ∈ σ \times δ$ with $B × [z] ∈ H$, there are $V ∈ σ$ and
W ∈ δ such that $B × [z] \subseteq V × W \subseteq H$ [50]

(c) Y is called scattered relative to Y × Z if for each
B ∈ σ, there exists b ∈ B and $V ∈ σy$ such that $\overline{V}$ is
Lindelöf and strongly placed in Y × Z [50]

It is well known that if (Y, σ) and (Z, δ) are TSs and Y is
C-scattered, then Y is scattered relative to Y × Z but not
conversely.

Definition 6 (see [51]). A Hausdorff TS (Y, σ) is called
ultraparacompact if every open cover of Y has a locally finite
clopen refinement.

Ellis [51] showed that a Hausdorff space (Y, σ) is
ultraparacompact if every open cover has a pairwise disjoint
open refinement.

Theorem 6 (see [52]). Let $f : (Y, \sigma) → (Z, \delta)$ be closed
and continuous with (Y, σ) regular. If (Z, δ) is metacomparct
and $f^{-1}(z)$ is Lindelöf for each $z ∈ Z$, then (Y, σ) is
metacompact.

Theorem 7 (see [50]). For any two TSs (Y, σ) and (Z, δ), Y
is strongly placed in $Y × Z$ if and only if the projection $\pi : Y ×
Z, σ × δ → (Z, δ)$ is closed.

Theorem 8 (see [35]). Let (Y, σ) and (Z, δ) be TSs and let
BcY. If B is strongly placed in $Y × Z$ and $C ∈ σ \cap C$, then
B ∩ C is strongly placed in $Y × Z$.

3. Theta Omega Limit Points

In this section, we explore the concept of $θω$-limit points of a
set and study its fundamental properties.

Definition 7. Let (Y, σ) be a TS and BcY. A point $y ∈ Y$
is called $θω$-limit point of B if for each $G ∈ σy$ with $y ∈ G,
G ∩ (B − {y}) \neq Ø$.

The set of all $θω$-limit points of B is called the $θω$-derived
set of B and is denoted by $Dθ(B)$.

The following result shows that $θω$-derived set of a set B
contains the derived set of B and contained in the $θω$-derived
set of B.

Theorem 9. Let (Y, σ) be a TS BcY. The derived set of B is
denoted by $D(B)$. Then, $D(B) ⊆ Dθ(B) ⊆ Dθ(B)$.

Proof. To see that $D(B) ⊆ Dθ(B)$, let $y \notin Dθ(B)$, then there
exists $G ∈ σy$ such that $y ∈ G$ and $G ∩ (B − {y}) = Ø$. By
Theorem 3, $G ∈ σ$ and so $y \notin D(B)$. Therefore, we have
$D(B) ⊆ Dθ(B)$. To see that $Dθ(B) ⊆ Dθ(B)$, let $y \notin Dθ(B)$,
then there exists $G ∈ σy$ such that $y ∈ G$ and $G ∩ (B − {y}) = Ø$. By
Theorem 3, $G ∈ σy$ and so $y \notin Dθ(B)$. Therefore, we have
$Dθ(B) ⊆ Dθ(B)$.

The following example shows that the equality of each of
the inclusions in Theorem 9 does not hold in general.

Example 1 (Example 2.26 of [37]). Let $X = \mathbb{R}$ and let
$A = \{0, 1\}$. It is proved in [37] that $σy = \{0, 1\}$ and
$σθ = \{0, 1\}$. Let $B = \{-n : n ∈ \mathbb{N}\}$. Then,$Dθ(B) = \mathbb{R} \setminus \mathbb{N} \setminus \{1\}$, and $D(B) = \mathbb{R} \setminus \{1\}$.

Under the condition “regularity,” the $θω$-derived set, the
derived set, and the $θ$-derived set are all equal.
Theorem 10. Let \((Y, \sigma)\) be a regular TS and \(B \subseteq Y\). Then, \(D(B) = D_{\sigma_B}(B) = D_\sigma(B)\).

Proof. It follows from Theorems 3, 4, and 9.

“Local countability” is a sufficient condition for the \(\theta_{\omega}\)-derived set and the derived set to be equal to each other. \(\Box\)

Theorem 11. Let \((Y, \sigma)\) be a locally countable TS and \(B \subseteq Y\). Then, \(D(B) = D_{\sigma_B}(B)\).

Proof. By Theorem 9, we have \(D(B) = D_{\sigma_B}(B)\). To see that \(D(B) = D_{\sigma_B}(B)\), suppose to the contrary that there is \(y \in D_{\sigma_B}(B) - D(B)\). Since \(x \notin D(B)\), there is \(G \in \sigma\) such that \(G \cap (B - \{y\}) = \emptyset\). By Theorem 1, \(\text{Cl}_{\sigma_B}(Y - G) = Y - G\) and so \(G \in \sigma_{\omega}\). We conclude that \(y \notin D_{\sigma_B}(B)\), a contradiction.

“Antilocality countability” is a sufficient condition for the \(\theta_{\omega}\)-derived set and the \(\theta\)-derived set to be equal to each other. \(\Box\)

Theorem 12. Let \((Y, \sigma)\) be an antilocally countable TS and \(B \subseteq Y\). Then, \(D(B) = D_{\sigma_B}(B)\).

Proof. By Theorem 9, we have \(D(B) \subseteq D_{\sigma_B}(B)\). To see that \(D(B) = D_{\sigma_B}(B)\), suppose to the contrary that there is \(y \in D_{\sigma_B}(B) - D(B)\). Since \(x \notin D(B)\), there is \(G \in \sigma\) such that \(G \cap (B - \{y\}) = \emptyset\). By Theorem 2, \(\text{Cl}_\sigma(Y - G) = \text{Cl}_\sigma(Y - G) = Y - G\) and so \(G \in \sigma_{\omega}\). We conclude that \(y \notin D_{\sigma_B}(B)\), a contradiction.

In Theorems 13–16, we give some natural properties for \(\theta_{\omega}\)-derived set.

Theorem 13. Let \((Y, \sigma)\) be a TS. If \(B \subseteq C \subseteq Y\), then \(D_{\sigma_B}(B) \subseteq D_{\sigma_C}(C)\).

Proof. Let \(y \notin D_{\sigma_B}(C)\), there exists \(G \in \sigma_{\omega}\) such that \(y \in G\) and \(G \cap (C - \{y\}) = \emptyset\). Since \(B \subseteq C\), then \(G \cap (B - \{x\}) = \emptyset\) and hence \(y \notin D_{\sigma_B}(B)\). It follows that \(D_{\sigma_B}(B) \subseteq D_{\sigma_C}(C)\). \(\Box\)

Theorem 14. Let \((Y, \sigma)\) be a TS, and let \(A\) and \(B\) be subsets of \(Y\). Then, \(D_{\sigma_A}(A) \cup D_{\sigma_B}(B) = D_{\sigma_A}(A \cup B)\).

Proof. By Theorem 13, \(D_{\sigma_B}(B) \subseteq D_{\sigma_A}(A \cup B)\) and \(D_{\sigma_B}(B) \subseteq D_{\sigma_A}(A \cup B)\). Therefore, \(D_{\sigma_B}(B) \subseteq D_{\sigma_A}(A \cup B)\). Now, let \(y \notin (D_{\sigma_A}(A) \cup D_{\sigma_B}(B))\), then there exist \(\theta_{\omega}\)-open sets \(G, H \in \sigma_{\omega}\) such that \(y \in G \cap H\), \((G - \{y\}) \cap A = \emptyset\), and \((H - \{y\}) \cap B = \emptyset\). Let \(W = G \cap H\). Then, \(W \in \sigma_{\omega}\) and \((W - \{y\}) \cap (A \cup B) = ((W - \{y\}) \cap A) \cup ((W - \{y\}) \cap B) = \emptyset \cup \emptyset = \emptyset\).

Thus, \(y \notin D_{\sigma_A}(A \cup B)\). \(\Box\)

Theorem 15. Let \((Y, \sigma)\) be a TS, and let \(A\) and \(B\) be subsets of \(Y\). Then, \(D_{\sigma_A}(A \cap B) \subseteq D_{\sigma_A}(A) \cap D_{\sigma_B}(B)\).

Proof. By Theorem 13, \(D_{\sigma_B}(A \cap B) \subseteq D_{\sigma_B}(A) \cap D_{\sigma_B}(B)\). Then, \(D_{\sigma_B}(A \cap B) \subseteq D_{\sigma_A}(A) \cap D_{\sigma_B}(B)\). The following example shows that the inclusion in Theorem 15 cannot be replaced by equality in general. \(\Box\)

Example 2 (Example 2.26 of [37]). Let \(Y = \mathbb{R}\) and \(\sigma = \{\mathbb{R}, \emptyset, \mathbb{N}, \mathbb{Q}, \mathbb{N} \cup \mathbb{Q}\}\). It is proved in [37] that \(\sigma_{\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}\). Let \(A = (3/4, (3/2))\) and \(B = ((5/4), (9/4))\). Then, \(D_{\sigma_A}(A) = \mathbb{R} - \{1\}\) and \(D_{\sigma_B}(B) = \mathbb{R} - \{2\}\). On the other hand, \(D_{\sigma_A}(A \cap B) = D_{\sigma_A}((5/4), (3/2))) = \mathbb{R} - \mathbb{N}\).

Theorem 16. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(D_{\sigma_A}(D_{\sigma_A}(B) - B) \subseteq D_{\sigma_B}((B - \{y\}) \cap \sigma_{\omega})\).

Proof. Let \(y \in D_{\sigma_A}(D_{\sigma_A}(B) - B)\). Let \(G \in \sigma_{\omega}\) with \(y \in G\). Since \(y \in D_{\sigma_A}(D_{\sigma_A}(B) - \{y\})\), \(G \cap (D_{\sigma_A}(B) - \{y\}) \neq \emptyset\). Choose \(z \in G \cap (D_{\sigma_A}(B) - \{y\})\). Since \(z \in D_{\sigma_B}(B)\) and \(z \in G \in \sigma_{\omega}\), then \(G \cap (B - \{z\}) \neq \emptyset\). Choose \(w \in G \cap (B - \{z\})\). Since \(w \in B\) and \(y \notin B\), \(w \neq y\). Thus, \(G \cap (B - \{y\}) \neq \emptyset\) and hence \(y \in D_{\sigma_A}(B)\).

The following example shows that the inclusion in Theorem 16 cannot be replaced by equality in general. \(\Box\)

Example 3. Let \(Y = \mathbb{R}\) and \(\sigma = \{\mathbb{R}, \emptyset, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}\). Let \(B = ((3/4), (3/2))\). It is proved in [37] that \(\sigma_{\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}\). By Example 2, \(D_{\sigma_A}(\mathbb{R}) = \mathbb{R} - \{1\}\). On the other hand, \(D_{\sigma_A}(D_{\sigma_A}(B) - B) = D_{\sigma_A}((\mathbb{R} - \{1\}) - \left(\frac{3}{4}, \frac{3}{2}\right)) = \mathbb{R} - \left(\frac{3}{4}, \frac{3}{2}\right)\).

4. Theta Omega Interior Points

In this section, we explore the concept of \(\theta_{\omega}\)-interior points of a set and study its fundamental properties.

Definition 8. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). A point \(y \in Y\) is called a \(\theta_{\omega}\)-interior point of \(B\) if there exists \(G \in \sigma\) such that \(y \in G \subseteq B\). The set of all \(\theta_{\omega}\)-interior points of \(B\) is called the \(\theta_{\omega}\)-interior of \(B\) and is denoted by \(\text{Int}_{\theta_{\omega}}(B)\).

The following result shows that the \(\theta_{\omega}\)-interior of a set \(B\) contains the \(\theta\)-interior \(B\) and contained in the \(\theta_{\omega}\)-interior of \(B\).

Theorem 17. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(\text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}(B)\).

The following example shows that each of the two inclusions in Theorem 17 cannot be replaced by equality in general.

Example 4. Example 2.26 of [37]. Let \(Y = \mathbb{R}\) and let \(\sigma = \{\mathbb{R}, \emptyset, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}\). Let \(A = \mathbb{N}\) and \(B = \mathbb{Q}^c\). It is proved in [37] that \(\sigma_{\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}\) and \(\sigma_B = \{\emptyset, \mathbb{R}\}\). We have
Theorem 18. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). If \(G \in \sigma\) such that \(G \subseteq G \subseteq B\), then \(G \subseteq \text{Int}_{\theta_{\omega}}(B)\).

Proof. It follows directly from the definition of \(\text{Int}_{\theta_{\omega}}(B)\).

Theorem 19. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(\text{Int}_{\theta_{\omega}}(B)\) is \(\theta_{\omega}\)-open.

Proof. By the definition of \(\text{Int}_{\theta_{\omega}}(B)\) and Theorem 18, for every \(y \in \text{Int}_{\theta_{\omega}}(B)\), there exists \(G_y \in \sigma\) such that \(y \in G_y \subseteq G \subseteq B\). By Theorem 5, it follows that \(\text{Int}_{\theta_{\omega}}(B)\) is \(\theta_{\omega}\)-open.

The following is a characterization of \(\theta_{\omega}\)-open via \(\text{Int}_{\theta_{\omega}}\).

Theorem 20. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(B\) is \(\theta_{\omega}\)-open if and only if \(B = \text{Int}_{\theta_{\omega}}(B)\).

Proof. Necessity: suppose that \(B\) is a \(\theta_{\omega}\)-open set. By the definition, we have \(\text{Int}_{\theta_{\omega}}(B)\) is \(\theta_{\omega}\)-open. To see that \(B \subseteq \text{Int}_{\theta_{\omega}}(B)\), let \(y \in B\). By Theorem 5, there exists \(G \in \sigma\) such that \(y \in G \subseteq G \subseteq B\). Then, \(y \in \text{Int}_{\theta_{\omega}}(B)\).

Sufficiency: suppose that \(B = \text{Int}_{\theta_{\omega}}(B)\). Then, by Theorem 19, \(B\) is \(\theta_{\omega}\)-open.

The results in the rest of this section are some natural properties of \(\text{Int}_{\theta_{\omega}}\).

Theorem 21. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(\text{Int}_{\theta_{\omega}}[\text{Int}_{\theta_{\omega}}(B)] = \text{Int}_{\theta_{\omega}}(B)\).

Proof. Follows from Theorem 20.

Theorem 22. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(Y - \text{Int}_{\theta_{\omega}}(B) = \text{Cl}_{\theta_{\omega}}(Y - B)\).

Proof. To see that \(Y - \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Cl}_{\theta_{\omega}}(Y - B)\), let \(y \notin \text{Cl}_{\theta_{\omega}}(Y - B)\). Then, there is \(G \in \sigma\) such that \(y \in G \subseteq G \subseteq Y - B\). So, we have \(y \in G \subseteq G \subseteq B\). This shows that \(y \notin Y - \text{Int}_{\theta_{\omega}}(B)\). To see that \(\text{Cl}_{\theta_{\omega}}(Y - B) \subseteq Y - \text{Int}_{\theta_{\omega}}(B)\), let \(y \notin Y - \text{Int}_{\theta_{\omega}}(B)\). Then, \(y \in \text{Int}_{\theta_{\omega}}(B)\), and so there is \(G \in \sigma\) such that \(y \in G \subseteq G \subseteq B\). Therefore, we have \(G \cap (Y - B) = \emptyset\), and so \(y \notin \text{Cl}_{\theta_{\omega}}(Y - B)\).

Theorem 23. Let \((Y, \sigma)\) be a TS and \(B \subseteq Y\). Then, \(Y - \text{Cl}_{\theta_{\omega}}(B) = \text{Int}_{\theta_{\omega}}(Y - B)\).

Proof. By Theorem 22,

\[
Y - \text{Cl}_{\theta_{\omega}}(B) = Y - (Y - \text{Int}_{\theta_{\omega}}(Y - B)) = \text{Int}_{\theta_{\omega}}(Y - B).
\]

Theorem 24. Let \((Y, \sigma)\) be a TS and let \(A \subseteq B \subseteq Y\). Then, \(\text{Int}_{\theta_{\omega}}(A) \subseteq \text{Int}_{\theta_{\omega}}(B)\).

Proof. Let \(y \in \text{Int}_{\theta_{\omega}}(A)\). Then, there exists \(G \in \sigma\) such that \(y \in G \subseteq G \subseteq A\). Since \(A \subseteq B\), then \(G \subseteq U \subseteq B\). Thus, \(y \in \text{Int}_{\theta_{\omega}}(B)\).

Theorem 25. Let \((Y, \sigma)\) be a TS and let \(A\) and \(B\) be subsets of \(Y\). Then, \(\text{Int}_{\theta_{\omega}}(A) \cup \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(A \cup B)\).

Proof. By Theorem 24, we have \(\text{Int}_{\theta_{\omega}}(A) \subseteq \text{Int}_{\theta_{\omega}}(A \cup B)\) and \(\text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(A \cup B)\). Thus, \(\text{Int}_{\theta_{\omega}}(A) \cup \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(A \cup B)\).

Theorem 26. Let \((Y, \sigma)\) be a TS, and let \(A\) and \(B\) be subsets of \(Y\). Then, \(\text{Int}_{\theta_{\omega}}(A \cap B) = \text{Int}_{\theta_{\omega}}(A) \cap \text{Int}_{\theta_{\omega}}(B)\).

Proof. By Theorem 24, we have \(\text{Int}_{\theta_{\omega}}(A \cap B) \subseteq \text{Int}_{\theta_{\omega}}(A) \cap \text{Int}_{\theta_{\omega}}(B)\). Thus, \(\text{Int}_{\theta_{\omega}}(A) \cap \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(A \cap B)\). To see that \(\text{Int}_{\theta_{\omega}}(A) \cap \text{Int}_{\theta_{\omega}}(B) \subseteq \text{Int}_{\theta_{\omega}}(A \cap B)\), let \(y \in \text{Int}_{\theta_{\omega}}(A) \cap \text{Int}_{\theta_{\omega}}(B)\). Then, there exist \(G, H \in \sigma\) such that \(y \in G \subseteq G \subseteq A\) and \(y \in H \subseteq H \subseteq B\). Let \(W = G \cap H\). Then, \(W \in \sigma\) and \(y \in W \subseteq W = G \cap H \subseteq G \cap H \subseteq A \cap B\). It follows that \(y \in \text{Int}_{\theta_{\omega}}(A \cap B)\).

5. Metacompactness Product Theorems

In this section, we introduce several product theorems concerning metacompactness.

The following result will be used in the proof of Theorems 28 and 29.

Theorem 27. Let \((Y, \sigma)\) and \((Z, \delta)\) be metacompact TSs. If for every \(y \in Y\) there exists \(W \in \sigma\) such that \(y \in W \subseteq W \subseteq Z\) is metacompact, then \((Y \times Z, \sigma \times \delta)\) is metacompact.

Proof. Let \(\mathcal{A}\) be an open cover of \((Y \times Z, \sigma \times \delta)\). For every \(y \in Y\), choose \(W_y \in \sigma\) such that \(y \in W_y \subseteq W_y \subseteq Z\) and \((W_y, \sigma_{\chi W_y})\) is metacompact. Since \(\{W_y : y \in Y\}\) is a point-open refinement \(\mathcal{A}\) as an open cover, and hence \(\mathcal{A}\) has a point-open finite refinement \(\mathcal{A}\). It is not difficult to see that \(H \cap (W_y \subseteq Z): H \in \mathcal{A}\) is a point-open finite refinement of \(\mathcal{A}\) as open cover. It follows that \((Y \times Z, \sigma \times \delta)\) is metacompact.

The following two product theorems concerning metacompactness will be used in the proof of Theorem 31 which is the main result of this section:

Theorem 28. Let \((Y, \sigma)\) and \((Z, \delta)\) be regular metacompact TSs. If for every \(y \in Y\) there exists \(W \in \sigma\) such that \(y \in W \subseteq W \subseteq W \subseteq Z\) is strongly placed in \((Y \times Z, \sigma \times \delta)\) is Lindelöf, then \((Y \times Z, \sigma \times \delta)\) is metacompact.

Proof. For each \(y \in Y\), choose \(W_y \in \sigma\) such that \(y \in W_y \subseteq W_y \subseteq W_y \subseteq Z\) is strongly placed in \((Y \times Z, \sigma \times \delta)\) and \((W_y, \sigma_{\chi W_y})\) is Lindelöf. For every \(y \in Y\), \((W_y, \sigma_{\chi W_y})\) is Lindelöf and so by Theorem 6, the projection function \(\pi_y : (W_y \times Z, (\sigma \times \delta)) \rightarrow (Z, \delta)\) is a closed function. For every \(y \in Y\), \((W_y, \sigma_{\chi W_y})\) is Lindelöf and so \(\pi_y^{-1}(z) = W_y \times \{z\}\), then
Let $(y, \sigma)$ be ultraparacompact and $(Z, \delta)$ be metacompact such that $Y$ is scattered relative to the product $Y \times Z$, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Theorem 31. Let $(Y, \sigma)$ be ultraparacompact and $(Z, \delta)$ be regular and metacompact such that $Y$ is scattered relative to the product $Y \times Z$, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. Denote by $Y^{(0)} = Y$ and $Y^{(1)} = \{y \in Y; \text{there is no } U \in \sigma \text{ such that } y \in U \text{ and } U \text{ is strongly placed in } Y \times Z \}$ and closure $(\bar{U}, \sigma_{\bar{U}})$ is Lindelöf. If there is an ordinal $\alpha > 1$ such that $Y^{(\alpha)}$ has been defined and $\beta = \alpha + 1$, then $Y^{(\beta)} = \bigcup_{\beta_{\alpha}, Y^{(\alpha)}}$. If $\alpha$ is a limit ordinal, then $Y^{(\alpha)} = \bigcap_{\beta_{\alpha}, Y^{(\alpha)}}$. Since $Y$ is scattered relative to $Y \times Z$, then there exists an ordinal $\alpha$ such that $Y^{(\alpha)} = \emptyset$.

The proof proceeds by transfinite induction on $\alpha$. If $Y^{(1)} = \emptyset$, then for every $y \in Y$ there exists $U_{y} \in \sigma$ such that $y \in U$ and $U$ is strongly placed in $Y \times Z$ and closure $(\bar{U}, \sigma_{\bar{U}})$. By Theorem 28, $(Y \times Z, \sigma \times \delta)$ is metacompact. If $Y^{(\alpha+1)} = \emptyset$, then for every $y \in Y^{(\alpha)}$ there exists $y_{\bar{U}} \in \sigma_{\bar{U}}$ such that $y \in U_{y_{\bar{U}}}$, $U_{y_{\bar{U}}}$ is strongly placed in $Y \times Z$, and $(U_{y_{\bar{U}}}, \sigma_{U_{y_{\bar{U}}}})$ is Lindelöf if $y \in Y \setminus Y^{(\alpha)}$. Choose a clopen set $C_{y}$ such that $y \in C_{y} \subset Y \setminus Y^{(\alpha)}$. Since $C_{y} \subset Y^{(\alpha+1)}$ is metacompact, and by the inductive assumption, it follows that $(C_{y}, \sigma \times \delta)_{C_{y}}$ is metacompact.

If $Y^{(\alpha)} = \emptyset$ for the limit ordinal $\alpha$, then the open cover $Y \setminus Y^{(\beta)} \subset \sigma_{\bar{U}}$ has a pairwise disjoint open refinement $\{O_{y}; \, y \in Y^{(\beta)}\}$. For each $y \in Y^{(\beta)}$, choose $\beta < \alpha$ such that $O_{y} = \emptyset$. Therefore, $(O_{y})^{(\emptyset)} = \emptyset$, and hence $(O_{y}), (\sigma \times \delta)_{O_{y}}$ is metacompact. Since $Y \times Z = \bigcup_{y \in Y^{(\emptyset)}}, Y \times Z$, it follows that $(Y \times Z, \sigma \times \delta)$ is metacompact.

Corollary 1. The product of an ultraparacompact $C$-scattered $TS$ with a metacompact regular $TS$ is again metacompact.

By the end of this paper, the authors found it is suitable to raise the following open question.

Question 1. Let $(Y, \sigma)$ be regular and metacompact $TS$s such that $Y$ is scattered relative to the product $Y \times Z$. Is $(Y \times Z, \sigma \times \delta)$ metacompact?

6. Conclusion

In this work, the research via $\theta_{\alpha}$-open sets is continued by introducing the notions of $\theta_{\alpha}$-limit points and $\theta_{\alpha}$-interior points. Several relationships regarding these two notions are introduced. Moreover, several product theorems concerning metacompactness are given. In future studies, the following topics could be considered: (1) define $\theta_{\alpha}$-border, $\theta_{\alpha}$-frontier, and $\theta_{\alpha}$-exterior of a set using $\theta_{\alpha}$-open sets and (2) try to solve Question 1.

Data Availability

No data were used to support this study.
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Conflicts of Interest

The authors declare that they have no conflicts of interest.
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