Some notes about matrices, 2

Stephen William Semmes
Rice University
Houston, Texas

Contents
1 Lattices in \(\mathbb{R}^n\) 1
2 Lattices in \(\mathbb{C}^n\) 4
3 Spaces of lattices 5

1 Lattices in \(\mathbb{R}^n\)

As usual, \(\mathbb{R}\) denotes the real numbers, \(\mathbb{Z}\) denotes the integers, and \(\mathbb{R}^n, \mathbb{Z}^n\) consist of \(n\)-tuples of real numbers and integers, respectively. Sometimes we might refer to \(\mathbb{Z}^n\) as the standard integer lattice in \(\mathbb{R}^n\). If we say that \(L\) is a lattice in \(\mathbb{R}^n\), then we mean that there is an invertible linear transformation \(A\) on \(\mathbb{R}^n\) such that
\[(1.1) \quad L = A(\mathbb{Z}^n).\]

If \(L\) is a lattice in \(\mathbb{R}^n\), then we can form the quotient \(\mathbb{R}^n/L\). That is, two vectors \(x, y\) in \(\mathbb{R}^n\) are identified in the quotient if their difference \(x - y\) lies in \(L\). In particular, we get a canonical quotient mapping
\[(1.2) \quad \rho : \mathbb{R}^n \to \mathbb{R}^n/L\]
which sends a vector \(x\) in \(\mathbb{R}^n\) to the corresponding element of the quotient.

Now, with respect to ordinary vector addition, \(\mathbb{R}^n\) is an abelian group, and a lattice \(L\) is a subgroup of \(\mathbb{R}^n\). We can think of the quotient \(\mathbb{R}^n/L\) as a
quotient in the sense of group theory. The quotient is an abelian group under addition, and the canonical quotient mapping is a group homomorphism.

We can also look at the quotient $\mathbb{R}^n/L$ in terms of topology. Namely, it inherits a topology from the one on $\mathbb{R}^n$ so that the canonical quotient mapping is an open continuous mapping, which means that both images and inverse images of open sets are open sets, and indeed the canonical quotient mapping is a nice covering mapping, so that for every point $x$ in $\mathbb{R}^n$ there is a neighborhood $U$ of $x$ in $\mathbb{R}^n$ such that the restriction of $\rho$ to $U$ is a homeomorphism from $U$ onto the open set $\rho(U)$ in $\mathbb{R}^n/L$. For that matter we can think of $\mathbb{R}^n/L$ as a smooth manifold, with the quotient mapping $\rho$ as a smooth mapping which is a local diffeomorphism.

Suppose that $L_1$, $L_2$ are lattices in $\mathbb{R}^n$, and let

$$\rho_1 : \mathbb{R}^n \to \mathbb{R}^n/L_1, \quad \rho_2 : \mathbb{R}^n \to \mathbb{R}^n/L_2$$

be the corresponding canonical quotient mappings. If $A$ is an invertible linear transformation on $\mathbb{R}^n$ such that

$$A(L_1) = L_2,$$

then we get an induced mapping

$$\hat{A} : \mathbb{R}^n/L_1 \to \mathbb{R}^n/L_2.$$

This mapping is a group isomorphism and a homeomorphism, and even a diffeomorphism, which satisfies the obvious compatibility condition with the corresponding canonical quotient mappings $\rho_1$, $\rho_2$, namely $\rho_1 \circ A = \hat{A} \circ \rho_2$.

When $n = 1$, one can consider the lattice $2\pi\mathbb{Z}$ consisting of integer multiples of $2\pi$, and it is customary to identify $\mathbb{R}/2\pi\mathbb{Z}$ with the unit circle $\mathbb{T}$ in the complex numbers $\mathbb{C}$,

$$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \},$$

where $|z|$ denotes the usual modulus of $z \in \mathbb{C}$, $|z| = (x^2 + y^2)^{1/2}$ when $z = x + iy$, $x, y \in \mathbb{R}$. More precisely, $\exp(it)$ is an explicit version of the canonical quotient mapping from $\mathbb{R}/2\pi\mathbb{Z}$ onto $\mathbb{T}$ with respect to this identification, which is a local diffeomorphism and a group homomorphism using the group structure of multiplication on $\mathbb{T}$. In general, we can identify $\mathbb{R}^n/2\pi\mathbb{Z}^n$ with $\mathbb{T}^n$, the $n$-fold Cartesian product of $\mathbb{T}$, where $2\pi\mathbb{Z}^n$ denotes the lattice of points whose coordinates are all integer multiples of $2\pi$. 
Suppose that $L$ is a lattice in $\mathbb{R}^n$. Also let $A$ be an invertible linear mapping on $\mathbb{R}^n$ such that $A(2\pi \mathbb{Z}^n) = L$. Thus $\hat{A}$ is a group isomorphism and a diffeomorphism from $\mathbb{R}^n/2\pi \mathbb{Z}^n \cong T^n$ onto $\mathbb{R}^n/L$.

There is a more precise way to look at the quotient of $\mathbb{R}^n$ by a lattice, which is to say that the quotient space has a kind of local affine structure. That is, there is a local affine structure in which the canonical quotient mapping is considered to be locally affine, and which permits one to say when a curve in the quotient is locally a straight line segment, like an arc on a line, and when it has locally constant speed, etc. If $L_1, L_2$ are lattices in $\mathbb{R}^n$ and $A$ is an invertible linear mapping on $\mathbb{R}^n$ such that $A(L_1) = L_2$, then the induced mapping $\hat{A}$ from $\mathbb{R}^n/L_1$ onto $\mathbb{R}^n/L_2$ preserves this local affine structure on the quotient spaces.

There is an even more precise way to look at the quotient $\mathbb{R}^n/L$ of $\mathbb{R}^n$ by a lattice $L$, which is that it has a local flat geometric structure, induced from the one on $\mathbb{R}^n$. With respect to this structure one can make local measurements of lengths, volumes, and angles, like the length of a curve, the angle at which two curves meet at a point, or the volume of a nice subset. In technical terms this can be seen as a special case of a Riemannian metric.

In particular, one can define the volume of such a quotient $\mathbb{R}^n/L$, where the volume of $\mathbb{R}^n/\mathbb{Z}^n$ is equal to 1, and the volume of $\mathbb{R}^n/2\pi \mathbb{Z}^n$ is equal to $(2\pi)^n$. In general, if $L_1, L_2$ are lattices in $\mathbb{R}^n$ and $A$ is an invertible linear transformation on $\mathbb{R}^n$ such that $A(L_1) = L_2$, then the volume of $\mathbb{R}^n/L_2$ is equal to $|\det A|$ times the volume of $\mathbb{R}^n/L_1$, and more generally if $E$ is a nice subset of $\mathbb{R}^n/L_1$, then the volume of $\hat{A}(E)$ in $\mathbb{R}^n/L_2$ is equal to $|\det A|$ times the volume of $A$ in $\mathbb{R}^n/L_1$. This is a variant of the fact that on $\mathbb{R}^n$ a linear transformation $A$ distorts volumes by a factor of $|\det A|$, where $\det A$ denotes the determinant of $A$.

Suppose that $L_1, L_2$ are lattices in $\mathbb{R}^n$, and that $T$ is a linear transformation on $\mathbb{R}^n$ such that $T(L_1) = L_2$. Recall that $T$ is an orthogonal transformation on $\mathbb{R}^n$ if $T$ is invertible with inverse given by the adjoint, also known as the transpose, of $T$, and that this is equivalent to saying that $T$ preserves the standard norm of vectors in $\mathbb{R}^n$, and the standard inner product of vectors in $\mathbb{R}^n$. In other words, orthogonal transformations on $\mathbb{R}^n$ are linear mappings which preserve the geometry in $\mathbb{R}^n$, and for the lattices $L_1, L_2$ and the quotients of $\mathbb{R}^n$ by them we have that the induced mapping $\hat{T}$ from $\mathbb{R}^n/L_1$ onto $\mathbb{R}^n/L_2$ preserves the geometry as well.

In short, quotients of $\mathbb{R}^n$ by lattices are the same in terms of group structure, topological and even smooth structure, and affine structure, and
not in general for more precise geometry. The volume of the quotient space is one basic parameter that one can consider. It is also interesting to look at closed curves in the quotient which are locally flat, their lengths, the angles at which they meet, and so on.

2 Lattices in \( \mathbb{C}^n \)

As before we write \( \mathbb{C}^n \) for the \( n \)-tuples of real numbers. We can identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) in the usual manner, so that the real and imaginary parts of the \( n \) components of an element of \( \mathbb{C}^n \) give rise to the \( 2n \) components of an element of \( \mathbb{R}^{2n} \). By a lattice in \( \mathbb{C}^n \) we mean a lattice in \( \mathbb{R}^{2n} \) which is then identified with \( \mathbb{C}^n \).

Let us write \( \mathbb{Z}[i] \) for the Gaussian integers, which are complex numbers of the form \( a + ib \), where \( a, b \) are integers. We also write \( (\mathbb{Z}[i])^n \) for the lattice in \( \mathbb{C}^n \) consisting of \( n \)-tuples of Gaussian integers. We call this the standard integer lattice in \( \mathbb{C}^n \).

If \( L \) is a lattice in \( \mathbb{C}^n \), then the quotient \( \mathbb{C}^n/L \) inherits a complex structure from \( \mathbb{C}^n \). If \( L_1, L_2 \) are lattices in \( \mathbb{C}^n \) and \( A \) is an invertible complex-linear transformation on \( \mathbb{C}^n \) such that \( A(L_1) = L_2 \), then \( A \) induces a mapping \( \hat{A} \) from \( \mathbb{C}^n/L_1 \) to \( \mathbb{C}^n/L_2 \) which preserves this complex structure. Thus, although lattices in \( \mathbb{C}^n \) can be defined as images of the standard integer lattice in \( \mathbb{C}^n \) under invertible real linear transformations, complex structures and complex linear transformations play an important role.

Recall that a unitary transformation on \( \mathbb{C}^n \) is an invertible complex-linear transformation whose inverse is given by the adjoint. This is the same as saying that the linear transformation preserves the standard Hermitian inner product on \( \mathbb{C}^n \), and it is also equivalent to saying that the transformation is both complex-linear and an orthogonal transformation with respect to the standard identification of \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \). This leads to another and more precise relationship between lattices in \( \mathbb{C}^n \) and their quotients, i.e., having a unitary transformation which takes one lattice to another, and which then induces a nice transformation between the corresponding quotients.
3 Spaces of lattices

Let us write $GL(\mathbb{R}^n)$ for the general linear group on $\mathbb{R}^n$, consisting of the invertible linear transformations on $\mathbb{R}^n$, with composition as the group structure. The special linear group $SL(\mathbb{R}^n)$ is the subgroup of $GL(\mathbb{R}^n)$ of linear transformations with determinant equal to 1. The orthogonal group $O(\mathbb{R}^n)$ is the subgroup of $GL(\mathbb{R}^n)$ of orthogonal linear transformations on $\mathbb{R}^n$, and the special orthogonal group $SO(\mathbb{R}^n)$ consists of orthogonal linear transformations whose determinant is equal to 1.

Consider the quotient space $O(\mathbb{R}^n)\backslash GL(\mathbb{R}^n)$, in which two invertible linear transformations on $\mathbb{R}^n$ are identified if one can be written as an orthogonal linear transformation times the other. We can identify this quotient space with the space of symmetric linear transformations on $\mathbb{R}^n$ which are positive definite, through the mapping

\[(3.1) \quad T \mapsto T^* T.\]

In other words, if $T$ is an invertible linear transformation on $\mathbb{R}^n$, then $T^* T$ is a symmetric linear transformation on $\mathbb{R}^n$ which is positive-definite, $T_1^* T_1 = T_2^* T_2$ for $T_1, T_2 \in GL(\mathbb{R}^n)$ if and only if $T_2 = R T_1$ for some orthogonal transformation $R$, and every symmetric linear transformation on $\mathbb{R}^n$ which is positive-definite can be expressed as $T^* T$ for an invertible linear transformation $T$.

Similarly, the quotient $SO(\mathbb{R}^n)\backslash SL(\mathbb{R}^n)$ can be identified with the space $\mathcal{M}(\mathbb{R}^n)$ of symmetric linear transformations on $\mathbb{R}^n$ which are positive definite and have determinant equal to 1. Let us also write $\Sigma(\mathbb{R}^n)$ for the elements of $SL(\mathbb{R}^n)$ whose matrices with respect to the standard basis have integer entries. The inverse of a linear transformation in $\Sigma(\mathbb{R}^n)$ also lies in $\Sigma(\mathbb{R}^n)$, because Cramer’s rule gives a formula for the matrix of the inverse which shows that it has integer entries when the original matrix has integer entries and determinant equal to 1.

Elements of $\Sigma(\mathbb{R}^n)$ can be described as the invertible linear transformations which take $\mathbb{Z}^n$ onto itself. The quotient $SL(\mathbb{R}^n)/\Sigma(\mathbb{R}^n)$ describes the space of lattices $L$ in $\mathbb{R}^n$ such that the corresponding quotient $\mathbb{R}^n/L$ has volume equal to 1 and for which there is an extra piece of data concerning orientation, and the double quotient $SO(\mathbb{R}^n)\backslash SL(\mathbb{R}^n)/\Sigma(\mathbb{R}^n)$ deals with these lattices up to equivalence under rotation. Of course there are a lot of variants of these themes, which may involve complex structures in particular.