Uniform boundedness and continuity at the Cauchy horizon for linear waves on Reissner–Nordström–AdS black holes

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Abstract

Motivated by the Strong Cosmic Censorship Conjecture for asymptotically AdS spacetimes, we initiate the study of massive scalar waves satisfying $\Box_g \psi - \mu \psi = 0$ on the interior of Anti-de Sitter (AdS) black holes. We prescribe initial data on a spacelike hypersurface of a Reissner–Nordström–AdS black hole and impose Dirichlet (reflecting) boundary conditions at infinity. It was known previously that such waves only decay at a sharp logarithmic rate (in contrast to a polynomial rate as in the asymptotically flat regime) in the black hole exterior. In view of this slow decay, the question of uniform boundedness in the black hole interior and continuity at the Cauchy horizon has remained up to now open. We answer this question in the affirmative.

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1 Introduction

We initiate the study of (massive) linear waves satisfying
\[ \Box g \psi - \mu \psi = 0 \] 
(1.1)
on the interior of asymptotically Anti-de Sitter (AdS) black holes \((\mathcal{M}, g)\). In the context of asymptotically AdS spacetimes it is natural to consider (possibly negative) mass parameters \(\mu\) satisfying the Breitenlohner–Freedman [6] bound \(\mu > \frac{3}{2} \Lambda\), where \(\Lambda < 0\) is the cosmological constant of the underlying spacetime. In particular, this covers the conformally invariant operator with \(\mu = \frac{3}{2} \Lambda\). We will consider Reissner–Nordström–AdS (RN–AdS) black holes [7] which can be viewed as the simplest model in the context of the question of stability of the Cauchy horizon. These spacetimes are spherically symmetric solutions of the Einstein equations
\[ \text{Ric}_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} = 8 \pi T_{\mu \nu} \] 
(EE)
coupled to the Maxwell equations via the energy momentum tensor \(T_{\mu \nu}\). Our main result Theorem 1 (see Theorem 3.1 in Section 3 for its precise formulation) is the statement of uniform boundedness in the black hole interior and continuity at the Cauchy horizon of solutions to (1.1) arising from initial data on a spacelike hypersurface on RN–AdS. We moreover assume Dirichlet (reflecting) boundary conditions at infinity. Our result is surprising because in contrast to black hole backgrounds with non-negative cosmological constants \((\Lambda \geq 0)\), the decay of \(\psi \) in the exterior region for asymptotically AdS black holes \((\Lambda < 0)\) is only logarithmic as shown by Holzegel–Smulevici [38] (cf. polynomial [58, 18, 1] \((\Lambda = 0)\) and exponential [5, 24] \((\Lambda > 0)\)). Indeed, the logarithmic decay is too slow to adapt the mechanism exploited in previous studies of black hole interiors [13, 25, 16]. The proof of our main theorem will now follow a new approach, combining physical space estimates with Fourier based estimates exploited in the scattering theory developed in [42].

In the rest of the introduction we will give some background on the problem and formulate our main result Theorem 1.

The Cauchy horizon and the Strong Cosmic Censorship Conjecture. The main motivation for studying linear waves on black hole interiors is to shed light on one of the most fundamental puzzles in general relativity: The Kerr(–de Sitter or –Anti-de Sitter) and Reissner–Nordström(–de Sitter or –Anti-de Sitter) black holes share the property that in addition to the event horizon \(\mathcal{H}\), they hide another horizon, the so-called Cauchy horizon \(\mathcal{CH}\), in their interiors. This Cauchy horizon defines the boundary beyond which initial data on a spacelike hypersurface (together with boundary conditions at infinity in the asymptotically AdS case) no longer uniquely determine the spacetime as a solution of (EE). In particular, these spacetimes admit infinitely many smooth extensions beyond their Cauchy horizons solving (EE). This severe violation of determinism is conjectured to be an artifact of the high degree of symmetry in these explicit spacetimes and generically, due to blue-shift instabilities, it is expected that a singularity ought to form at or before the Cauchy horizon. This is known as the Strong Cosmic Censorship Conjecture (SCC) [56, 9]. A full resolution of the SCC conjecture would also include a precise description of the breakdown of regularity at or before the Cauchy horizon.

We first present the \(C^0\) formulation of SCC (see [9, 16]), which can be seen as the strongest inextendibility statement in this context.

Conjecture 1 \((C^0\) formulation of strong cosmic censorship). For generic compact or asymptotically flat (asymptotically Anti-de Sitter) vacuum initial data, the maximal Cauchy development of (EE) is inextendible as a Lorentzian manifold with \(C^0\) (continuous) metric.

Surprisingly, the \(C^0\) formulation (Conjecture 1) was recently proved to be false for both cases \(\Lambda = 0\) and \(\Lambda > 0\) (see discussion later, [16]). However, the following weaker, yet well-motivated, formulation introduced by Christodoulou in [9] is still expected to hold true (at least) in the asymptotically flat case \((\Lambda = 0)\).

Conjecture 2 \((C^0\) formulation of strong cosmic censorship). For generic asymptotically flat vacuum initial data, the maximal Cauchy development of (EE) is inextendible as a Lorentzian manifold with \(C^0\) (continuous) metric and locally square integrable Christoffel symbols.

In order to gain insight about SCC, the most naive approach (often referred to as “poor man’s linearization”) is to study solutions of (1.1) with \(\mu = 0\) on a fixed explicit black hole spacetime (e.g. Kerr or Reissner–Nordström). This can be considered as the most naive toy model for (EE) with initial data close to Kerr or
Reissner–Nordström data, for which many features of (EE) including the non-linear terms and the tensorial structure are neglected; see the pioneering works for asymptotically flat (Λ = 0) black holes [59, 47, 48, 8]. Under the identification ψ ∼ g and ∂ψ ∼ Γ, where ψ is a solution to (1.1), Conjecture 1 corresponds to a failure of ψ to be continuous (C0) at the Cauchy horizon. Similarly, Conjecture 2 corresponds to a failure of ψ to lie in Hµ loc at the Cauchy horizon.

The state of the art for Λ = 0 and Λ > 0. The definitive disproof [16] of Conjecture 1 was preceded by corresponding results on the level of (1.1).

Linear level for Λ = 0. In the asymptotically flat case (Λ = 0) it was shown in [25, 26] (see also [33]) that solutions of (1.1) with µ = 0 arising from data on a spacelike hypersurface remain continuous and uniformly bounded (no C0 blow-up) at the Cauchy horizon of general subextremal Kerr or Reissner–Nordström black hole interiors. (For the extremal case see [29, 30].) The key method for the proof is to use the polynomial decay on the event horizon proved in [18] (with rate |ψ| ≲ v−p and p > 1) and propagate it into the interior. The boundedness and continuity of ψ at the Cauchy horizon was then concluded from red-shift estimates, energy estimates associated to the novel vector field

\[ S = |u|^p \partial_u + |v|^p \partial_v \]  (1.2)

and commuting with angular momentum operators followed by Sobolev embeddings. Here \( u, v \) are Eddington–Finkelstein-type null coordinates in the interior.

Besides the above C0 boundedness, it was proved that the (non-degenerate) local energy at the Cauchy horizon blows up for a generic set of solutions ψ in Reissner–Nordström [43] and Kerr [19] black holes. (Note that this blow-up is compatible with the finiteness of the flux associated to (1.2) because \( \partial_t \) and \( \partial_\theta \) degenerate at the Cauchy horizons \( CH_A \) and \( CH_B \), respectively.) A similar blow-up behavior was obtained for Kerr in [16] assuming lower bounds on the energy decay rate of a solution along the event horizon. These results support Conjecture 2 at least on the level of (1.1).

Another type of result that has been shown in [42] is a finite energy scattering theory for solutions of (1.1) (with \( \mu = 0 \)) from the event horizon \( H_A^+ \cup H_B^+ \) to the Cauchy horizon \( CH_A \cup CH_B \) in the interior of Reissner–Nordström black holes. In this scattering theory a linear isomorphism between the degenerate energy spaces (associated to the Killing field \( T = \partial_v - \partial_u \)) corresponding to the event and Cauchy horizon was established. The question reduced to obtaining uniform control over transmission and reflection coefficients \( \Sigma(\omega, \ell) \) and \( \mathfrak{R}(\omega, \ell) \) corresponding to fixed frequency solutions. Intuitively, for a purely incoming wave at the event horizon \( H_A^+ \), the transmission and reflection coefficients correspond to the amount of T-energy scattered to \( CH_B \) and \( CH_A \), respectively. Indeed, the theory also carries over to \( \Lambda \neq 0 \) and \( \mu \neq 0 \) except for the \( \omega = 0 \) frequency. This will turn out to be important for the present paper.

Linear level for Λ > 0. For Kerr (and Reissner–Nordström)–de Sitter (\( \Lambda > 0 \)) it was shown in [35] that solutions of (1.1) (with \( \mu = 0 \)) also remain bounded up to and including the Cauchy horizon. Note that in both cases, \( \Lambda = 0 \) and \( \Lambda > 0 \), the proofs rely crucially on quantitative decay along the event horizon (polynomial for \( \Lambda = 0 \) and exponential for \( \Lambda > 0 \)).

On the other hand the exponential convergence on the event horizon of a Kerr–de Sitter black hole is in direct competition with the exponential blue-shift instability and the question of local energy blow-up at the Cauchy horizon for (1.1) is more subtle, see the conjecture in [14] and the more recent [20, 22, 21].

Nonlinear level for Λ = 0 and Λ > 0. Now we turn to the full nonlinear problem for (EE). As mentioned before, for the Einstein vacuum equations Dafermos–Luk showed that the Kerr Cauchy horizon is C0 stable [16], i.e. the spacetime is extendible as a \( C^0 \) Lorentzian manifold. Note that this definitively falsifies Conjecture 1 for \( \Lambda = 0 \) (subject only to the completion of a proof of the nonlinear stability of the Kerr exterior). In principle, their proof of \( C^0 \) extendibility also applies to the interior of Kerr–de Sitter black holes, where the exterior has been proved to be stable for slowly rotating Kerr–de Sitter black holes [34], thus falsifying Conjecture 1 for \( \Lambda > 0 \).

Nonlinear extendibility results at the Cauchy horizon have been proved only in spherical symmetry: Coupling the Einstein equation (EE) to a Maxwell–Scalar field system, it is proved in [13] that the Cauchy horizon is \( C^0 \) stable, yet \( C^2 \) unstable [14, 45, 13] for a generic set of spherically symmetric initial data. See also the pioneering work in [57, 55]. This shows the \( C^2 \) formulation of SCC (but not yet Conjecture 2) in spherical symmetry. See [14, 12] for work in the \( \Lambda > 0 \) case. The question of any type of nonlinear instability
of the Cauchy horizon without symmetry assumptions and the validity of Conjecture 2 (even restricted to a neighborhood of Kerr) have yet to be understood.

**Linear waves and SCC for asymptotically AdS black holes.** The situation is changed radically if one considers asymptotically Anti-de Sitter ($\Lambda < 0$) spacetimes. Due to the timelike nature of null infinity $I = I_A \cup I_B$, see for example Fig. 1, these spacetimes are not globally hyperbolic. For well-posedness of (EE) and (1.1) it is required to impose also boundary conditions at infinity. The most natural conditions are Dirichlet (reflecting) boundary conditions, see [28]. Before we address the question of stability of the Cauchy horizon, it is essential to understand the behavior in the exterior region of Kerr–AdS or Reissner–Nordström–AdS.

**Logarithmic decay for linear waves on the exterior of Kerr–AdS and Reissner–Nordström–AdS.** For the massive linear wave equation (1.1) on Kerr–AdS and Reissner–Nordström–AdS, Holzegel–Smulevici showed in [38] stability in the exterior region. Indeed, they proved that solutions decay at least at logarithmic rate towards $i^+$ (cf. polynomial ($\Lambda = 0$) and exponential ($\Lambda > 0$)) assuming the Hawking–Reall [32] bound $1/r + |a|/l$ and the Breitenlohner–Freedman [6] bound $\mu > 3/4 \Lambda$. Moreover, they showed that solutions of (1.1) with fixed angular momentum actually decay exponentially on the exterior of Reissner–Nordström–AdS. (This is in contrast to the asymptotically flat case, in which fixed angular momentum solutions of (1.1) decay polynomially on the exterior of Reissner–Nordström.) However, their main insight was that a suitable infinite sum of such rapidly decaying fixed angular momentum solutions, possessing finite energy in some weighted norm, indeed achieves the logarithmic decay rate [40]. This is due to the presence of stable trapping. Note that this sharpness can also be concluded from later work showing the existence of quasinormal modes converging to the real axis at an exponential rate as the real part of the frequency and angular momentum tend to infinity [63, 31]. (For some asymptotically flat five dimensional black holes a similar inverse logarithmic lower bound was shown in [2].)

**Strong Cosmic Censorship for AdS black holes.** With the logarithmic decay on the exterior in hand, we turn to the question of the stability of the Cauchy horizon. Indeed, the logarithmic decay rate on the exterior is too slow to follow the methods involving the red-shift vector field and the vector field $S$ as in (1.2) (see discussion before) to prove uniform boundedness and $C^0$ (continuous) extendibility at the Cauchy horizon of solutions to (1.1). More specifically, after propagating the logarithmic decay through the red-shift region, the energy flux associated to $S$ is infinite on a $\{r = \text{const.}\}$ hypersurface in the black hole interior due to the

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1 Note that otherwise exponentially growing mode solutions can be constructed as shown in [23].
slow logarithmic decay towards \( i^+ \). Thus, the question of whether to expect the validity of Conjecture 1 for asymptotically AdS black holes appears to be completely open.

The present paper is an attempt to shed some first light on SCC in the asymptotically AdS case: We will show that, despite the slow decay on the exterior, boundedness in the interior and continuous extendibility to the Cauchy horizon still holds for solutions of (1.1) on Reissner–Nordström–AdS black holes (cf. Theorem 1). The additional phenomenon which we exploit to prove boundedness is that the trapped frequencies responsible for slow decay have high energy with respect to the \( T \) vector field and can be bounded using the scattering theory developed in [42]. Thus, for Reissner–Nordström–AdS, the analog of Conjecture 1 is false on the linear level, just as in the \( \Lambda \geq 0 \) cases. See however our remarks on Kerr–AdS later in the introduction.

The massive linear wave equation on Reissner–Nordström–AdS. As mentioned above, we will consider the massive linear wave equation

\[
\Box_{\text{GRNA} \text{-AdS}} \psi + \frac{\alpha}{T^2} \psi = 0 \tag{1.3}
\]

for AdS radius \( l^2 := -\frac{3}{4} \) on a fixed subextremal Reissner–Nordström–AdS black hole. Moreover, we assume the so-called Breitenlohner–Freedman bound [6] for the Klein–Gordon mass parameter \( \alpha < \frac{l^2}{2} \), which includes the conformally invariant case \( \alpha = 2 \). This bound is required to obtain well-posedness [37, 62, 61] of (1.3).

Recall from the discussion above that solutions with fixed angular momentum \( \ell \) actually decay exponentially in the exterior region. For such solutions with fixed \( \ell \), uniform boundedness with upper bound \( C = C_\ell \) in the interior and continuity at the Cauchy horizon can be shown using the methods involving the vector field \( S \) as in [12]. Note however that this does not imply that a general solution remains bounded in the interior as the constant \( C_\ell \) is not summable: \( \sum_{\ell=0}^{L-1} C_\ell \sim e^{\ell} \to +\infty \) as \( L \to \infty \). Note in particular that, as a result of this, one cannot study the new non-trivial aspect of this problem restricted to spherical symmetry. (Nevertheless, see [3] for a discussion of the Ori model for RN–AdS black holes.)

Main theorem: Uniform boundedness and continuity at the Cauchy horizon. We now state a rough version of our main result. See Theorem 3.1 for the precise statement.

**Theorem 1** (Rough version of Theorem 3.1). Let \( \psi \) be a solution to (1.3) arising from smooth and compactly supported initial data \( (\psi_0, \psi_1) \) posed on a spacelike hypersurface \( \Sigma_0 \) as depicted in Fig. 1. Then, \( \psi \) remains uniformly bounded in the black hole interior

\[
|\psi| \leq C,
\]

where \( C \) is constant depending on the parameters \( M, Q, l, \alpha \), the choice of \( \Sigma_0 \) and on some higher order Sobolev norm of the initial data \( (\psi_0, \psi_1) \). Moreover, \( \psi \) can be extended continuously across the Cauchy horizon.

As we have explained above, the main difficulty compared to the asymptotically flat case, where the analysis was carried out entirely in physical space and requires inverse polynomial decay in the exterior [25], is the slow decay of \( \psi \) along the event horizon. Our strategy is to decompose the solution \( \psi \) into a low and high frequency part \( \psi = \psi_\ell + \psi_\flat \) with respect to the Killing field \( T = \partial_T \) and treat each term separately.

For the low frequency part \( \psi_\ell \), we will show a superpolynomial decay rate in the exterior, see already Proposition 4.8. This decay is sufficient so as to follow the method of [25] with vector fields of the form (1.2) to show boundedness and continuity at the Cauchy horizon, up to the additional difficulty caused by the fact that we allow a possibly negative Klein–Gordon mass parameter. The violation of the dominant energy condition due to the presence of a negative mass term can be overcome with twisted derivatives [62, 41], which provide a useful framework to replace Hardy inequalities for the lower order terms in this context.

For the high frequency part \( \psi_\flat \), which is exposed to stable trapping and does in general only decay at a sharp logarithmic rate in the exterior, the key ingredient is the scattering theory developed in [12] (see discussion above). More specifically, the uniform bounds for the transmission and reflections coefficients \( \Sigma \) and \( \Re \) for \( |\psi| \geq \omega_0 \) proved in [12] turn out to be useful for the high frequency part \( \psi_\flat \). These bounds allow us to control \( |\psi_\flat| \) at the Cauchy horizon by the \( T \)-energy norm on the event horizon commuted with angular derivatives. The \( T \)-energy flux on the event horizon is in turn bounded from initial data by a simple application of the \( T \)-energy identity in the exterior. In particular, no quantitative decay along the event horizon is used for the high frequency part \( \psi_\flat \). This is what allows us to overcome the problem of slow logarithmic decay.
Outlook on Kerr–AdS. We strongly believe that our arguments also apply to axially symmetric solutions $\psi$ of (1.3) on a Kerr–AdS black hole. For general non-axisymmetric solutions, however, the question of uniform boundedness and continuity at the Cauchy horizon is less clear. Indeed, specific high frequency solutions which decay at a logarithmic decay rate can be considered as “low frequency” solutions when frequency is measured with respect to the Killing generator of the Cauchy horizon. In fact, it might well be the case that for solutions of (1.3) on Kerr–AdS there is $C^0$ blow-up at the Cauchy horizon, supporting the validity of Conjecture 1 after all in this context!

Instability of asymptotically AdS spacetimes? Turning to the fully nonlinear dynamics, there is another scenario which could happen. Recall that Minkowski space ($\Lambda = 0$) and de Sitter space ($\Lambda > 0$) are proved to be nonlinearly stable [27, 10]. Anti-de Sitter space ($\Lambda < 0$), however, is expected to be nonlinearly unstable with Dirichlet conditions imposed at infinity. This was recently proved in [50, 49, 52, 51] for appropriate matter models. See also the original conjecture in [15] and the numerical results in [4]. Similarly, for Kerr–AdS (or Reissner–Nordström–AdS), the slow logarithmic decay on the linear level proved in [40] could in fact give rise to nonlinear instabilities in the exterior. If indeed the exterior of Kerr–AdS was nonlinearly unstable, linear analysis like that in the present paper would be manifestly inadequate and the question of the validity of Strong Cosmic Censorship would be thrown even more open! Refer to the introduction of [16] for a more elaborate discussion.

Outline. This paper is organized as follows. In Section 2 we set up the spacetime and summarize relevant previous work. In Section 3 we state and prove our main result Theorem 3.1. Parts of the proof require a separate analysis which are treated in Section 4 and Section 5.

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2 Preliminaries

We start by setting up the Reissner–Nordström–AdS spacetime (see [7]) and defining relevant norms and energies. We will also introduce useful coordinate systems.

2.1 The Reissner–Nordström–AdS black hole

We are ultimately interested in the behavior of solutions to (1.3) to the future of a spacelike hypersurface $\Sigma_0$ as depicted in Fig. 1. For technical reasons (Fourier space decompositions are non-local operations) we will however construct also parts to the past of $\Sigma_0$. In the following will define the spacetime pictured in Fig. 2.

2.1.1 Construction of the spacetime $(M_{RNAdS}, g_{RNAdS})$

First, for black hole parameters $M > 0, Q \neq 0, l^2 \neq 0$ define the polynomial

$$\Delta_{M,Q,l}(r) := r^2 - 2Mr + \frac{r^4}{l^2} + Q^2$$

and define the non-degenerate set

$$\mathcal{P} := \{(M, Q, l) \in (0, \infty) \times \mathbb{R} \times (0, \infty) : \Delta_{M,Q,l}(r) \text{ has two positive roots satisfying } 0 < r_- < r_+\}.$$  \hspace{1cm} (2.2)

Note that $\mathcal{P}$ defines black hole parameters in the subextremal range. From now on, we will consider fixed parameters $M, Q, l, \alpha$, where

$$(M, Q, l) \in \mathcal{P} \text{ and } \alpha < \frac{9}{4}. \hspace{1cm} (2.3)$$

Note that in contrast, nonlinear stability for spherically symmetric perturbations of Schwarzschild–AdS was shown for Einstein–Klein–Gordon systems [39].
Figure 2: Penrose diagram of the constructed spacetime \((\mathcal{M}_{\text{RNAdS}}, g_{\text{RNAdS}})\)

Note that \(M\) is the mass parameter, \(Q\) the charge parameter of the black hole and \(l = \sqrt{-\frac{3}{\Lambda}}\) is the Anti-de Sitter radius. For this specific choice of parameters we will also write \(\Delta(r) := \Delta_{M,Q,l}(r)\) and denote by \(0 < r_- < r_+\) the positive roots of \(\Delta\).

Now, let the two exterior regions \(\mathcal{R}_A, \mathcal{R}_B\) and the black hole region \(\mathcal{B}\) be smooth four dimensional manifolds diffeomorphic to \(\mathbb{R}^2 \times S^2\). On \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\) we introduce global coordinate charts:

\[
\begin{align*}
(r_{\mathcal{R}_A}, t_{\mathcal{R}_A}, \theta_{\mathcal{R}_A}, \varphi_{\mathcal{R}_A}) &\in (r_+, \infty) \times \mathbb{R} \times S^2, \\
(r_{\mathcal{R}_B}, t_{\mathcal{R}_B}, \theta_{\mathcal{R}_B}, \varphi_{\mathcal{R}_B}) &\in (r_+, \infty) \times \mathbb{R} \times S^2, \\
(r_{\mathcal{B}}, t_{\mathcal{B}}, \theta_{\mathcal{B}}, \varphi_{\mathcal{B}}) &\in (r_-, r_+) \times \mathbb{R} \times S^2.
\end{align*}
\]

If it is clear from the context which coordinates are being used, we will omit their subscripts throughout the paper. Again, on the manifolds \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\) we define—using the coordinates \((t, r, \theta, \varphi)\) on each of the patches—the Reissner–Nordström–Anti-de Sitter metric

\[
g := -\frac{\Delta(r)}{r^2} dt \otimes dt + \frac{\Delta(r)}{r^2} dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi). \tag{2.5}
\]

On each of \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\), we define time orientations using the vector field \(\partial_{t_{\mathcal{R}_A}}\) on \(\mathcal{R}_A\), \(-\partial_{t_{\mathcal{R}_B}}\) on \(\mathcal{R}_B\) and \(-\partial_{r_{\mathcal{B}}}\) on \(\mathcal{B}\).

We will also define the tortoise coordinate \(r_*\) by

\[
\frac{dr_*}{dr} := \frac{r^2}{\Delta} \tag{2.6}
\]

in \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\) independently. This defines \(r_*\) up to an unimportant constant. Then, in each of the regions \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\), we define null coordinates by

\[
v = r_* + t \quad \text{and} \quad u = r_* - t, \tag{2.7}
\]

where for example for the \(v\) coordinate on \(\mathcal{R}_A\), we will use the notation \(v_{\mathcal{R}_A}\) and analogously for the other regions. Note that throughout the paper we will use the notation \('\) for derivatives \(\frac{\partial}{dr_*}\).

\(^3\)Up to the known degeneracy of spherical coordinates at the poles of the sphere.
**Patching the regions $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$ together.** Now, we patch the regions $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$ together. We begin by attaching the future (resp. past) event horizon $\mathcal{H}_A^+$ (resp. $\mathcal{H}_A^-$) to $\mathcal{R}_A$ by formally\footnote{This can be made rigorous using ingoing Eddington–Finkelstein coordinates $(r, v, \varphi, \theta)$ adapted to the event horizon. Since this is well-known, we avoid introducing yet another coordinate system.} setting

$$\mathcal{H}_A := \{ u_{\mathcal{R}_A} = -\infty \} \text{ and } \mathcal{H}_A^- := \{ v_{\mathcal{R}_A} = -\infty \}. \quad (2.8)$$

Similarly, we attach $\mathcal{H}_B^+ := \{ v_{\mathcal{R}_B} = -\infty \}$ and $\mathcal{H}_B^- := \{ u_{\mathcal{R}_B} = -\infty \}$ to $\mathcal{R}_B$. In the $(u_B, v_B)$ coordinates associated to $\mathcal{B}$ we make the identifications $\mathcal{H}_A^+ = \{ u_B = -\infty \}$ and $\mathcal{H}_B^+ = \{ v_B = -\infty \}$. Then, we attach the Cauchy horizon $\mathcal{C}_A := \{ v_B = +\infty \}$ and $\mathcal{C}_B := \{ u_B = +\infty \}$ to $\mathcal{B}$.

Finally, we attach the past (resp. future) bifurcation sphere $\mathcal{B}_-$ (resp. $\mathcal{B}_+$) to $\mathcal{B}$ as

$$\mathcal{B}_- := \{ u_B = -\infty, v_B = -\infty \} \text{ and } \mathcal{B}_+ := \{ u_B = +\infty, v_B = +\infty \}. \quad (2.9)$$

We shall also set $\mathcal{C}_B := \mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{B}_+$. Note that all horizons $\mathcal{H}_A^+, \mathcal{H}_A^-, \mathcal{H}_B^+, \mathcal{H}_B^-, \mathcal{C}_A$, and $\mathcal{C}_B$ are diffeomorphic to $\mathbb{R} \times S^2$ and the past (future) bifurcation sphere $\mathcal{B}_- (\mathcal{B}_+)$ is diffeomorphic to $S^2$. Moreover, we identify $\mathcal{B}_-$ with $\{ u_{\mathcal{R}_B} = -\infty, v_{\mathcal{R}_A} = -\infty \}$ and also with $\{ u_{\mathcal{R}_B} = -\infty, v_{\mathcal{R}_B} = -\infty \}$. The resulting manifold will be called $\mathcal{M}_{\text{RNAAS}}$. Note that, $g$ extends to a smooth Lorentzian metric on $\mathcal{M}_{\text{RNAAS}}$ which we will call $g_{\text{RNAAS}}$ and in particular, $(\mathcal{M}_{\text{RNAAS}}, g_{\text{RNAAS}})$ is a time oriented smooth Lorentzian manifold with corners. We illustrate the constructed spacetime as a Penrose diagram in Fig. 2. Note that the vector field $\partial_\varphi$ defined on $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$, respectively, extends to a smooth Killing field on $\mathcal{M}_{\text{RNAAS}}$, which we will from now on call $T$. Moreover, the standard angular momentum operators $\mathcal{W}_i$ for $i = 1, 2, 3$, the generators of $so(3)$ defined as

$$\mathcal{W}_1 = \sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi, \quad \mathcal{W}_2 = -\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi, \quad \mathcal{W}_3 = -\partial_\varphi \quad (2.10)$$

are Killing vector fields. It shall be noted that $\mathcal{W}_i$ for $i = 1, 2, 3$ are spacelike everywhere, whereas $T$ is future-directed timelike on $\mathcal{R}_A$, spacelike on $\mathcal{B}$ and past-directed timelike on $\mathcal{R}_B$. Moreover, $T$ is future-directed null on $\mathcal{H}_A^-, \mathcal{H}_A^+, \mathcal{C}_A$, past-directed null on $\mathcal{H}_B^-, \mathcal{H}_B^+, \mathcal{C}_A$ and vanishes on $\mathcal{B}_-, \mathcal{B}_+$. Finally, note that one can attach conformal timelike boundaries $\mathcal{I}_A$ and $\mathcal{I}_B$ corresponding to $\{ r_{\mathcal{R}_A} = +\infty \}$ and $\{ r_{\mathcal{R}_B} = +\infty \}$, respectively\footnote{Note that $\mathcal{I}_A$ and $\mathcal{I}_B$ are not contained in $\mathcal{M}_{\text{RNAAS}}$.}

### 2.1.2 Initial hypersurface $\Sigma_0$

We will impose initial data on a spacelike hypersurface $\Sigma_0$ to be made precise in the following. Note that we can choose for convenience that the spacelike hypersurface $\Sigma_0$ lies to the future of the past bifurcation sphere $\mathcal{B}_-$. Indeed, by general theory (an energy estimate in a compact region) this can be assumed without loss of generality.\footnote{This can be made rigorous using ingoing Eddington–Finkelstein coordinates $(r, v, \varphi, \theta)$ adapted to the event horizon. Since this is well-known, we avoid introducing yet another coordinate system.} More precisely, let $\Sigma_0$ be a 3 dimensional connected, complete and spherically symmetric spacelike hypersurface extending to the conformal infinity $\mathcal{I} = \mathcal{I}_A \cup \mathcal{I}_B$. Moreover, assume that $\mathcal{B}_- \subset J^- (\Sigma_0) \setminus \Sigma_0$.

A possible choice of $\Sigma_0$ is denoted in Fig. 2. We are ultimately interested in the shaded region to the future of $\Sigma_0$. For the rest of the paper, we will consider such a $\Sigma_0$ to be fixed.

### 2.2 Conventions

With $a \leq b$ for $a \in \mathbb{R}$ and $b \geq 0$ we mean that there exists a constant $C(M, Q, l, \alpha, \Sigma_0)$ with $a \leq Cb$. If $C(M, Q, l, \alpha, \Sigma_0)$ depends on an additional parameter, say $\ell$, we will write $a \lesssim b$. We also use $a \sim b$ for some $a, b \geq 0$ if there exist constants $C_1(M, Q, l, \alpha, \Sigma_0), C_2(M, Q, l, \alpha, \Sigma_0) > 0$ with $C_1a \leq b \leq C_2a$. We shall also make use of the standard Landau notation $O$ and $o$. To be more precise, let $X$ be a point set (e.g. $X = \mathbb{R}, [a, b], \mathbb{C}$) with limit point $c$. As $x \to c$ in $X$, $f(x) = O(g(x))$ means $\frac{|f(x)|}{|g(x)|} \leq C(M, Q, l, \alpha)$. We write $O(f(x))$ if the constant $C$ depends on an additional parameter $\ell$. For the standard volume form in spherical coordinates $(\varphi, \theta)$ on the sphere $S^2$ we will use the notation $d\sigma_{S^2} := \sin \theta d\varphi d\theta$. Finally, let the Japanese symbol be defined as $\langle x \rangle := \sqrt{1 + x^2}$ for $x \in \mathbb{R}$. 
2.3 Wave equation and mixed boundary value Cauchy problem

We are interested in solutions to the massive wave equation (1.3) associated to the metric $g_{\text{RNAdS}}$ on a subextremal Reissner–Nordström AdS black hole with black hole parameters $M, Q, l$ as in (2.3). In view of the timelike boundaries $\mathcal{I}_A$ and $\mathcal{I}_B$, we need to specify boundary conditions on $\mathcal{I}_A$ and $\mathcal{I}_B$ in addition to prescribing data on the spacelike hypersurface $\Sigma_0$, cf. Fig. 3. We will use Dirichlet (reflecting) boundary conditions which can be viewed as the most natural conditions in the context of stability of the Cauchy horizon. In principle, however, in view of [62], we could also use more general boundary conditions like Neumann or Robin conditions.

We will now introduce an appropriate foliation and norms in order to state the well-posedness statement.

We will foliate $\mathcal{R}_A \cup \mathcal{R}_B \cup \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \cup B$ with spacelike hypersurfaces. To do so, we let $\mathcal{T}$ be a smooth future-directed causal vector field on $\mathcal{R}_A \cup \mathcal{R}_B \cup \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \cup B$ with the properties that

$$\mathcal{T} = \begin{cases} T & \text{on } \mathcal{R}_A \cup \mathcal{H}_A^+ \\ -T & \text{on } \mathcal{R}_B \cup \mathcal{H}_B^+ \end{cases}$$

and that $\mathcal{T}$ is a future-directed timelike vector field on $B$. Now, define the leaves

$$\Sigma_{t^*} := \Phi^\mathcal{T}(t^*)[\Sigma_0],$$

where $\Phi^\mathcal{T}$ is the flow generated by $\mathcal{T}$ and $t^* \in \mathbb{R}$ is its affine parameter. We have illustrated some leaves in Fig. 4.

2.3.1 Further coordinates in the exterior region

In the region $\mathcal{R}_A \cup \mathcal{H}_A^+$, we moreover define a global (up to the well-known degeneracy on $S^2$) coordinate system $(t^*, r, \varphi, \theta)$, where $t^*$ is the affine parameter of the flow generated by $\mathcal{T}$. Note that on $\mathcal{R}_A \cup \mathcal{H}_A^+$ we have $\partial_{t^*} = T$ such that $t^*(t_2, r) - t^*(t_1, r) = t_2 - t_1$ and $t(t_2^*, r) - t(t_1^*, r) = t_2^* - t_1^*$. Similarly, we can define such a coordinate system on $\mathcal{R}_B$. 
2.3.2 Norms on hypersurfaces $\Sigma_{t^*}$

By construction $\Sigma_{t^*}$ intersects $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$. We will now define norms on $\Sigma_{t^*}$ which are adaptations of the norms introduced in [37]. We define

$$
\|\psi\|_{H^k_{RNAdS}(\Sigma_{t^*})}^2 := \|\psi\|_{L^2(\Sigma_{t^*} \cap \mathcal{B})}^2 + \|\psi\|_{H^k_{RNAdS}(\Sigma_{t^*} \cap (\mathcal{R}_A \cup \mathcal{H}_{A}^+))}^2 + \|\psi\|_{H^k_{RNAdS}(\Sigma_{t^*} \cap (\mathcal{R}_B \cup \mathcal{H}_{B}^+))}^2,
$$

where each of the terms appearing in (2.12) will be defined in the following.

**Norms in the interior region.** We begin by defining the first term in (2.12). We define

$$
\|\psi\|_{H^0_{RNAdS}(\Sigma_{t^*} \cap \mathcal{B})}^2 := \int_{\Sigma_{t^*} \cap \mathcal{B}} r^s |\psi|^2 r^2 \sin \theta d\theta d\varphi
$$

and similarly for higher order norms. Here and in the following we denote with $\nabla$ and $\gamma$ the induced covariant derivative and the induced metric, respectively, on spheres of constant $(t^*, r)$. We will also use the notation $|\nabla \psi|^2 := g(\nabla \psi, \nabla \psi)$. Now having defined (2.12), we will define energies in the following.

2.3.3 Energies on hypersurfaces $\Sigma_{t^*}$

We set

$$
E_i[\psi](t^*) := E_i^A[\psi](t^*) + E_i^B[\psi](t^*) + E_i^R[\psi](t^*)
$$

for $i = 1, 2$, where all terms in (2.13) will be defined in the following.
Energies in the interior region. In the interior region we are not concerned with \( r \)-weights and define the energies as
\[
E^\mathcal{B}_i[\psi](t^*) := \|\psi\|^2_{H^1_0(\Sigma_t \cap \mathcal{B})} + \|\partial_t \psi\|^2_{L^2(\Sigma_t \cap \mathcal{B})},
\]
\[
E^\mathcal{H}_i[\psi](t^*) := \|\psi\|^2_{L^2(\Sigma_t \cap \mathcal{H})} + \|\partial_t \psi\|^2_{L^1(\Sigma_t \cap \mathcal{H})} + \|\partial^\alpha_\psi\|^2_{L^2(\Sigma_t \cap \mathcal{H})}.
\]

Energies in the exterior region. To define the energies in the exterior region, it is convenient to start with defining the following energy densities
\[
e_1[\psi] := \frac{1}{r^2} |\partial_t \psi|^2 + r^2 |\partial_r \psi|^2 + |\nabla \psi|^2 + |\psi|^2,
\]
\[
e_2[\psi] := e_1[\psi] + \sum_{i=1}^3 e_1[\mathcal{W}_i \psi] + r^4 |\partial_r \psi|^2 + r^2 |\nabla \partial_r \psi|^2 + |\nabla \psi|^2
\]
and their integrals as
\[
E^\mathcal{A}_i[\psi](t^*) := \int_{\Sigma_t \cap (\mathcal{R}_A \cup \mathcal{H}_A)} e_i[\psi] r^2 \sin \theta d\theta d\varphi
\]
for \( i = 1, 2 \). Note that we will write \( E^\mathcal{B}_i \) for the analogous energy restricted to \( \mathcal{R}_B \).

Also remark the following relation between the norms and energies defined above
\[
E^\mathcal{A}_i[\psi] = \|\psi\|^2_{H^1_0(\Sigma_t \cap \mathcal{R}_A)} + \|\partial_t \psi\|^2_{H^{-1}_0(\Sigma_t \cap \mathcal{R}_A)},
\]
\[
E^\mathcal{B}_2[\psi] = \sum_{i} \|\mathcal{W}_i \psi\|^2_{H^1_0(\Sigma_t \cap \mathcal{R}_A)} + \|\partial_r \psi\|^2_{H^{-1}_0(\Sigma_t \cap \mathcal{R}_A)} + \|\psi\|^2_{H^2_0(\Sigma_t \cap \mathcal{R}_A)} + \|\partial^\alpha_\psi\|^2_{H^{2-2}_0(\Sigma_t \cap \mathcal{R}_A)}.
\]

Energy-momentum tensor. Our energies are based on the energy momentum tensor associated to \( (1.3) \) which is defined as
\[
T_{\mu\nu}[\phi] := \text{Re}(\partial_\mu \phi \overline{\partial_\nu \phi}) - \frac{1}{2} g_{\mu\nu} \left( \overline{\partial_\alpha \phi} \partial^\alpha \phi - \frac{\alpha}{l^2} |\phi|^2 \right).
\]

For a smooth vector field \( X \) we also define
\[
J^X[\phi] := T[\phi](X, \cdot) \text{ and } K^X := X_{\pi\mu} T^{\mu\nu}[\phi].
\]

The term \( K^X \) is often referred to as the “bulk term” and satisfies
\[
K^X[\phi] = \nabla^\mu J^X_{\mu}[\phi]
\]
if \( \phi \) is a solution to \( (1.3) \). Note that if \( X \) is Killing, then \( K^X \) vanishes. More generally, integrating \( (2.19) \) one obtains an energy identity relating boundary and bulk terms. For more details about the energy-momentum tensor and its usage for standard energy estimates we refer to [17].

2.3.4 Well-posedness

Having set up the spacetime and the norms, we will restate the well-posedness result for \( (1.3) \) as a mixed boundary value-Cauchy problem. For asymptotically AdS spacetimes, well-posedness was first proved in [37].

**Theorem 2.1** ([37]). Let the Reissner–Nordström–AdS parameters \( (M, Q, l) \) and the Klein–Gordon mass \( \alpha < \frac{2}{l} \) be as in [2.3]. Let initial data \( (\psi_0, \psi_1) \in C^\infty(\Sigma_0) \times C^\infty(\Sigma_0) \) be prescribed on the spacelike hypersurface \( \Sigma_0 \) and impose Dirichlet (reflecting) boundary conditions on \( \mathcal{I} = \mathcal{I}_A \cup \mathcal{I}_B \).

Then, there exists a smooth solution \( \psi \in C^\infty(\mathcal{R}_{\text{AdS}}) \cap \mathcal{C}^\infty(\mathcal{S} \cup \mathcal{H}) \) of \( (1.3) \) such that \( \psi |_{\Sigma_0} = \psi_0, \mathcal{T}^\psi |_{\Sigma_0} = \psi_1 \). The solution \( \psi \) is also unique in the class \( C(\mathcal{R}_t^{\text{AdS}}, \mathcal{H}^1_{\text{AdS}}(\Sigma_t)) \) and satisfies the energy identity
\[
\int_{\Sigma_t^{\cap \mathcal{R}_A}} J^\mathcal{A}_i[\psi] n^\mu_{\Sigma_t} dvol_{\Sigma_t} + \int_{\mathcal{H}_A^{\cap \mathcal{R}_A}} J^\mathcal{H}_i[\psi] n^\mu_{\mathcal{H}_A} dvol_{\mathcal{H}_A} = \int_{\Sigma_t^{\cap \mathcal{R}_A}} J^T_i[\psi] n^\mu_{\Sigma_t} dvol_{\Sigma_t},
\]

(2.20)
where \( t_1^* \leq t_2^* \) and \( \mathcal{H}_A^+(t_1^*, t_2^*) := \mathcal{H}_A^+ \cap \{ t_1^* \leq t^* \leq t_2^* \} \). The analogous energy identity holds in \( \mathcal{R}_B \). In particular, (2.20) shows that the \( T \)-energy flux through \( I = I_A \cup I_B \) vanishes.

Moreover, the \( T \)-energy flux through the event horizon is bounded by initial data

\[
\int_{\mathcal{H}_A^{-}} J^T_\mu [\psi] n^\mu_{\mathcal{H}_A^+} \, d\text{vol}_{\mathcal{H}_A^+} + \int_{\mathcal{H}_B^-} J^T_\mu [\psi] n^\mu_{\mathcal{H}_B^-} \, d\text{vol}_{\mathcal{H}_B^-} \lesssim E_1[\psi](0). \tag{2.21}
\]

**Remark 2.2.** The well-posedness statement in Theorem 2.1 holds true for a more general class of initial data, called a \( H^2_{AdS} \) initial data triplet, see [37].

### 2.3.5 Boundedness and decay in the exterior region

In the exterior regions \( \mathcal{R}_A \) and \( \mathcal{R}_B \) we have energy decay and boundedness results which have been proved in [37, 36, 38, 40]. We summarize these results in

**Theorem 2.3** ([40, Theorem 1.1], [38, Section 12]). A solution \( \psi \) to (1.3) arising from smooth and compactly supported data on \( \Sigma_0 \) as in Theorem 2.1 satisfies

\[
\int_{\Sigma \cap \mathcal{R}_A} e_1[\psi] r^2 \sin \theta \, d\theta \, d\varphi \lesssim \int_{\Sigma \cap \mathcal{R}_A} e_1[\psi] r^2 \sin \theta \, d\theta \, d\varphi, \tag{2.22}
\]

\[
\int_{\Sigma \cap \mathcal{R}_A} e_2[\psi] r^2 \sin \theta \, d\theta \, d\varphi \lesssim \int_{\Sigma \cap \mathcal{R}_A} e_2[\psi] r^2 \sin \theta \, d\theta \, d\varphi, \tag{2.23}
\]

and similarly for higher order norms. Moreover, we have the energy decay statements

\[
\int_{\Sigma \cap \mathcal{R}_A} e_1[\psi] r^2 \sin \theta \, d\theta \, d\varphi \lesssim \frac{1}{(\log(2 + t^*))^2} \int_{\Sigma \cap \mathcal{R}_A} e_2[\psi] r^2 \sin \theta \, d\theta \, d\varphi \tag{2.24}
\]

for \( t^* \geq 0 \) and the pointwise decay

\[
\sup_{\Sigma \cap \mathcal{R}_A} |\psi|^2 \lesssim \frac{1}{(\log(2 + t^*))^2} \int_{\Sigma \cap \mathcal{R}_A} (e_2[\psi] + e_2[\partial_t \psi]) r^2 \sin \theta \, d\theta \, d\varphi \tag{2.25}
\]

for \( t^* \geq 0 \) in the exterior region \( \mathcal{R}_A \) and similarly in \( \mathcal{R}_B \). Moreover, just like for Schwarzschild–AdS (cf. [38]), fixed angular frequencies decay exponentially. More precisely, let \( Y_{lm} \) denote the spherical harmonics and let \( \psi \) be a solution to (1.3) arising from smooth and compactly supported data on \( \Sigma_0 \). If there exists an \( L \in \mathbb{N} \) with \( (\psi, Y_{lm})_{L^2(\mathbb{S}^2)} = 0 \) for \( \ell \geq L \), then

\[
\int_{\Sigma \cap \mathcal{R}_A} e_1[\psi] r^2 \sin \theta \, d\theta \, d\varphi \lesssim \exp \left( -e^{-C(M,Q,l,\alpha)L^1} \right) \int_{\Sigma \cap \mathcal{R}_A} e_1[\psi] r^2 \sin \theta \, d\theta \, d\varphi \tag{2.26}
\]

for \( t^* \geq 0 \) and a constant \( C(M,Q,l,\alpha) > 0 \) only depending on the parameters \( M, Q, l, \alpha \).

**Remark 2.4.** Note that (2.26) also implies pointwise exponential decay for \( \psi \) (assuming \( (\psi, Y_{lm})_{L^2(\mathbb{S}^2)} = 0 \) for \( \ell \geq L \)) and all higher derivatives of \( \psi \) using standard techniques like commuting with \( T \) and \( W_t \), elliptic estimates as well as applying a Sobolev embedding.

**Remark 2.5.** The previous decay estimates have only been stated to the future of \( \Sigma_0 \) in the region \( \mathcal{R}_A \), nevertheless, they also hold in \( \mathcal{R}_B \). Moreover, they also hold true to the past of \( \Sigma_0 \) for an appropriate foliation for which the leaves intersect \( \mathcal{H}_A \) and \( \mathcal{H}_B \), and are transported along the flow of \( -T \) for \( \mathcal{R}_A \cup \mathcal{H}_A \) and along the flow of \( T \) for \( \mathcal{R}_B \cup \mathcal{H}_B \).

**Notation.** In the main part of the paper we will make use of the Fourier transform and convolution associated to the coordinate \( t \) in \((t, r, \theta, \varphi)\) coordinates as in (2.4). We denote \( \mathcal{F}_T \) as the Fourier transform (and \( \mathcal{F}_T^{-1} \) as its inverse) defined as

\[
\mathcal{F}_T[f](\omega, r, \theta, \varphi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} f(t, r, \theta, \varphi) \, dt \tag{2.27}
\]

Strictly speaking, in [38] this has been only explicitly proved for Kerr–AdS which includes Schwarzschild–AdS. However, the same proof as for Schwarzschild–AdS works completely analogously for Reissner–Nordström–AdS and we shall not repeat these arguments here.
in the coordinates \((t, r, \varphi, \theta)\) of \(\mathcal{R}_A, \mathcal{R}_B\) and \(\mathcal{B}\), respectively. Here, we assume that \(t \mapsto f(t, r, \theta, \varphi)\) is (at least) a tempered distribution and \((2.27)\), in general, is to be understood in the distributional sense. Moreover, the convolution \(*\) associated to the coordinate \(t\) is defined as

\[
(f * g)(t, r, \theta, \varphi) := \int_{\mathbb{R}} f(t - s, r, \theta, \varphi) g(s, r, \theta, \varphi) ds,
\]

where we again assume that \(t \mapsto f(t, r, \theta, \varphi)\) is a tempered distribution and \((2.28)\), in general, is to be understood in the distributional sense.

### 3 Main theorem and frequency decomposition

Now, we are in the position to state our main result

**Theorem 3.1.** Let the Reissner–Nordström–AdS parameters \((M, Q, l)\) and the Klein–Gordon mass \(\alpha < \frac{9}{4}\) be as in \((2.3)\). Let \(\psi \in C^\infty(\mathcal{M}_{\text{RNAdS}} \setminus \mathcal{CH})\) be a solution to \((1.3)\) arising from smooth and compactly supported initial data \((\psi, T\psi) |_{\Sigma_0} = (\psi_0, \psi_1) \in C^\infty(\Sigma_0) \times C^\infty(\Sigma_0)\) on \(\Sigma_0\) with Dirichlet (reflecting) boundary conditions imposed at \(\mathcal{I}_A\) and \(\mathcal{I}_B\) (cf. Theorem 2.1). Then, \(\psi\) is uniformly bounded in the interior region \(\mathcal{B}\) satisfying

\[
\sup_{\mathcal{B}} |\psi| \lesssim D[\psi]^{\frac{1}{2}},
\]

where \(D[\psi]\) is defined as

\[
D[\psi] := E_1[\psi](0) + \sum_{i,j=1}^{3} E_1[\mathcal{W}_i \mathcal{W}_j \psi](0).
\]

Moreover, \(\psi\) extends continuously to the Cauchy horizon, i.e. \(\psi \in C^0(\mathcal{M}_{\text{RNAdS}})\).

**Remark 3.2.** The data term \(D[\psi]\) in \((3.2)\) can be controlled by the initial data \((\psi_0, \psi_1)\) such that \((3.1)\) can be written in terms of initial data as

\[
\sup_{\mathcal{B}} |\psi| \leq C(M, Q, l, \alpha, \Sigma_0) \left( \|\psi_0\|_{H^{1,0}_{\text{RNAdS}}(\Sigma_0)} + \|\psi_1\|_{H^{0,-2}_{\text{RNAdS}}(\Sigma_0)} + \sum_{i,j=1}^{3} \|\mathcal{W}_i \mathcal{W}_j \psi_0\|_{H^{1,0}_{\text{RNAdS}}(\Sigma_0)} + \sum_{i,j=1}^{3} \|\mathcal{W}_i \mathcal{W}_j \psi_1\|_{H^{0,-2}_{\text{RNAdS}}(\Sigma_0)} \right)
\]

for a constant \(C(M, Q, l, \alpha, \Sigma_0)\) only depending on the parameters \(M, Q, l, \alpha\) and the choice of initial hypersurface \(\Sigma_0\).

**Remark 3.3.** Theorem 3.1 can be extended to a more general class of initial data using standard density arguments. In the context of uniform boundedness and continuity at the Cauchy horizon, it is enough to consider smooth and localized initial data. Nevertheless, note that for more general initial data in appropriate Sobolev spaces, already well-posedness becomes more delicate.

**Proof of Theorem 3.1.** We split up the proof in four steps, where Step 3 and Step 4 are the main parts relying on Section 4 and Section 5.

**Step 1: Decomposition into low and high frequencies.** Let

\[
\psi \in C^\infty(\mathcal{M}_{\text{RNAdS}} \setminus \mathcal{CH})
\]

be as in the assumption of Theorem 3.1. Now, in \(\mathcal{R}_A, \mathcal{R}_B\) and in \(\mathcal{B}\), define the low frequency part \(\psi_\delta\) and the high frequency part \(\psi_\sharp\) as

\[
\psi_\delta := \frac{1}{\sqrt{2\pi}} \mathcal{F}_{T}^{-1} [\chi_\omega] * \psi \quad \text{and} \quad \psi_\sharp := \psi - \psi_\delta,
\]
where
\[ \chi_{\omega_0} \in C_c^\infty(\mathbb{R}) \text{ such that } \chi_{\omega_0}(\omega) = 0 \text{ for } |\omega| \geq \omega_0 \text{ and } \chi_{\omega_0}(\omega) = 1 \text{ for } |\omega| \leq \frac{1}{2}\omega_0. \] (3.6)

From Proposition \[A.4\] in the appendix we know that the low and high frequency parts \( \psi_b \) and \( \psi_t \) in (3.3) are well-defined and \( \psi_b \) and \( \psi_t \) extend to smooth solutions of (1.3) on \( M_{\text{AdS}} \setminus CH \). The cut-off frequency \( \omega_0 = \omega_0(M, Q, l, \alpha) > 0 \) will be chosen in the proof of Proposition 4.5 only depending on \( M, Q, l, \alpha \). For convenience we can also assume that \( \chi_{\omega_0} \) is a symmetric function which implies that \( \psi_b \) and \( \psi_t \) will be real-valued as long as \( \psi \) was real valued. This concludes Step 1.

Having decomposed the solution in low and high frequency parts \( \psi_b \) and \( \psi_t \), we shall now see how the initial data \( D[\psi_b] \text{ and } D[\psi_t] \), respectively, can be bounded by the initial data \( D[\psi] \) of \( \psi \).

**Step 2: Estimating the initial data of the decomposed solution.** This step is the content of the following proposition.

**Proposition 3.4.** Let \( \psi \) be as in (3.4) and \( \psi_b, \psi_t \) be as in (3.5) and recall the definition of \( D[\cdot] \) from (3.2). Then,
\[ D[\psi_b] \lesssim D[\psi] \text{ and } D[\psi_t] \lesssim D[\psi]. \] (3.7)

**Proof.** Since \( \psi = \psi_b + \psi_t \), it suffices to obtain a bound of the type \( D[\psi_b] \lesssim D[\psi] \), where \( D[\cdot] \) is defined in (3.2). Because of the Dirichlet conditions imposed at infinity, the energy fluxes through \( I_A \) and \( I_B \) vanish (see (2.20)), and we estimate
\[ D[\psi_b] \lesssim \tilde{D}[\psi_b], \]
where \( \tilde{D}[\psi_b] \) is a higher order energy on the hypersurface
\[ \tilde{\Sigma}_0 := (\mathcal{R}_A \cap \{ t_{\mathcal{R}_A} = 0 \}) \cup B_- \cup (\mathcal{R}_B \cap \{ t_{\mathcal{R}_B} = 0 \}) \]
to be made precise in the following. Note also that the normal vector field on \( \mathcal{R}_A \cap \tilde{\Sigma}_0 \) is \( n_{\tilde{\Sigma}_0} = \frac{1}{r^2} \partial_r \).

More precisely, due to the support properties of the initial data, there exists a spherically symmetric compact set \( K := \tilde{\Sigma}_0 \cap \tilde{J} (\tilde{\Sigma}_0 \cap B) \) such that
\[ D[\psi_b] \lesssim \tilde{D}[\psi_b] := ||\psi_b||^2_{L^2(K)} + ||n_{\tilde{\Sigma}_0} \psi_b||^2_{L^2(K)} + \sum_{i,j=1}^3 ||W_i W_j \psi_b||^2_{H^1(K)} + \sum_{i,j=1}^3 ||W_i W_j n_{\tilde{\Sigma}_0} \psi_b||^2_{L^2(K)} \]
\[ + \int_{\tilde{\Sigma}_0 \cap \mathcal{R}_A \setminus K} \left( e_1[\psi_b] + \sum_{i,j=1}^3 e_1[W_i W_j \psi_b] \right) r^2 \sin \theta dr d\theta d\varphi \]
\[ + \int_{\tilde{\Sigma}_0 \cap \mathcal{R}_B \setminus K} \left( e_1[\psi_b] + \sum_{i,j=1}^3 e_1[W_i W_j \psi_b] \right) r^2 \sin \theta dr d\theta d\varphi. \] (3.8)

Estimate (3.8) follows from general theory \[17\], that is a (higher order) energy estimate followed by an application of Grönwall’s lemma. In order to estimate the energy on the compact hypersurface \( K \) we decompose \( K \cap \mathcal{R}_A \) and \( K \cap \mathcal{R}_B \) and estimate the energy on each of those slices independently. Again, in view of the fact that \( \mathcal{R}_A \) and \( \mathcal{R}_B \) can be treated analogously, we only show the estimate in \( \mathcal{R}_A \). Note that all the terms of
\[ ||\psi_b||^2_{L^2(K \cap \mathcal{R}_A)} + ||n_{\tilde{\Sigma}_0} \psi_b||^2_{L^2(K \cap \mathcal{R}_A)} + \sum_{i,j=1}^3 ||W_i W_j \psi_b||^2_{H^1(K \cap \mathcal{R}_A)} + \sum_{i,j=1}^3 ||W_i W_j n_{\tilde{\Sigma}_0} \psi_b||^2_{L^2(K \cap \mathcal{R}_A)} \]
\[ + \int_{\tilde{\Sigma}_0 \cap \mathcal{R}_A} \left( e_1[\psi_b] + \sum_{i,j=1}^3 e_1[W_i W_j \psi_b] \right) r^2 \sin \theta dr d\theta d\varphi \]
are of the form
\[ \int_{\{t=0\}\cap R_A} f|\partial^k \psi|^2 \sin \theta drd\theta d\varphi \]
for appropriate \( T \) invariant weight functions \( f \geq 0 \) and \( T \) invariant coordinate derivatives \( \partial \in \{ \partial_t, \partial_r, \partial_\theta, \partial_\varphi \} \) of order \( k = 0, 1, 2, 3 \). Using that
\[ \psi_s = \frac{1}{\sqrt{2\pi}} F_T^{-1}[\chi t] \ast \psi, \]
where \( F_T^{-1}[\chi t] =: \eta \) is a fixed Schwartz function, we conclude—again since \( T \) is Killing—that
\[
\int_{\{t=0\}\cap R_A} f(r)|\partial^k \psi|^2(0, r, \varphi, \theta) drd\sigma_{S^2} = \int_{\{r \geq r_+\} \times S^2} f(r)|\eta * \partial^k \psi|^2(0, r, \varphi, \theta) drd\sigma_{S^2}
\]
\[
= \int_{\{r \geq r_+\} \times S^2} f(r) \left| \int_{\mathbb{R}} \eta(-s) \partial^k \psi(s, r, \varphi, \theta) ds \right|^2 drd\sigma_{S^2}
\]
\[
\leq \int_{\mathbb{R}} |\eta(s)| ds \int_{\mathbb{R}} |\eta(-s)| \int_{\{r \geq r_+\} \times S^2} f(r)|\partial^k \psi(s, r, \varphi, \theta)|^2 drd\sigma_{S^2} ds
\]
\[
\lesssim \sup_{s \in \mathbb{R}} \int_{\mathbb{R}} f(r)|\partial^k \psi(s, r, \varphi, \theta)|^2 drd\sigma_{S^2}
\]
\[
\lesssim \int_{\{t=0\}\cap R_A} f(r)|\partial^k \psi|^2(0, r, \varphi, \theta) drd\sigma_{S^2} \lesssim \tilde{D}[\psi],
\]
where we have used boundedness of higher order energies in the exterior which are proved in [36, 38] and restated in Theorem 2.3. Also note that we can interchange the derivatives with the convolution since \( \tilde{T} \) is a Killing vector field. Thus, we conclude that \( \tilde{D}[\psi_s] \lesssim \tilde{D}[\psi] \) and again by Cauchy stability and the vanishing of the energy flux at \( \partial \) (see (2.20)), we can bound \( \tilde{D}[\psi] \lesssim \tilde{D}[\psi] \) which finally shows \( D[\psi_s] \lesssim D[\psi] \). Hence, \( D[\psi_s] \lesssim D[\psi] \) also holds true.

The previous analysis in Step 1 and Step 2 allows us to treat the low and high frequency parts \( \psi_s \) and \( \psi_t \) completely independently.

**Step 3: Uniform boundedness for \( \psi_s \) and \( \psi_t \).** This step is at the heart of the paper and will be proved in Section 4 and Section 5. According to Proposition 4.22 and Proposition 5.3
\[
\sup_{B} |\psi_s|^2 \lesssim D[\psi_s] \tag{3.9}
\]
and
\[
\sup_{B} |\psi_t|^2 \lesssim D[\psi_t]. \tag{3.10}
\]
Thus, in view of Step 2, we conclude
\[
\sup_{B} |\psi|^2 \lesssim \sup_{B} |\psi_s|^2 + \sup_{B} |\psi_t|^2 \lesssim D[\psi_s] + D[\psi_t] \lesssim D[\psi] \tag{3.11}
\]
which shows (3.1).

**Step 4: Continuous extendibility beyond the Cauchy horizon.** Again, this is proved Section 4 and Section 5. In particular, in Proposition 4.23 and Proposition 5.4 it is proved that \( \psi_s \) and \( \psi_t \), respectively, are continuously extendible beyond the Cauchy horizon. Thus, \( \psi = \psi_s + \psi_t \) can be continuously extended beyond the Cauchy horizon which concludes the proof. \( \square \)

### 4 Low frequency part \( \psi_s \)

We will begin this section by showing that \( \psi_s \) decays superpolynomially in the exterior regions \( R_A \) and \( R_B \) (Section 4.1). This strong decay in the exterior regions then leads to uniform boundedness of \( \psi_s \) in the interior \( B \) and continuous extendibility of \( \psi_s \) beyond the Cauchy horizon. This will be shown in Section 4.2. In the following, it suffices to only consider \( R_A \) because the region \( R_B \) can be treated completely analogously.
4.1 Exterior estimates

We will now consider $\psi_3$ in the exterior region $\mathcal{R}_A$ and show an integrated energy decay estimate which will eventually lead to the superpolynomial decay for $\psi_3$. First, however, we review the separation of variables for solutions to (1.3).

**Definition 4.1.** Let $\phi \in C(\mathbb{R}^4; H^1_{\text{RNAdS}}(\Sigma_{t^*})) \cap C^1(\mathbb{R}^4; H^0_{\text{RNAdS}}(\Sigma_{t^*}))$ be a solution to (1.3) satisfying

$$\int_{\mathbb{R}} |(\phi, Y_{l,m})|^{2} dt < \infty.$$  \hspace{1cm} (4.1)

for $r \in (r_-, r_+)$ and $r \in (r_+, \infty)$. In the regions $B$ and $\mathcal{R}_A$, respectively, set

$$u[\phi](r, \omega, \ell, m) := \frac{r}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} (\phi, Y_{l,m})_{L^2(\mathbb{S}^2)} dt,$$  \hspace{1cm} (4.2)

where $(Y_{l,m})_{|m| \leq \ell}$ are the standard spherical harmonics.

**Proposition 4.2.** Let $\psi$ be as in (4.1) and $\psi_2$, $\psi_3$ be as in (3.5). Then, $u[\psi](r, \omega, \ell, m)$ and $u[\psi_2](r, \omega, \ell, m)$ as in Definition 4.1 are well-defined and smooth functions of $r, \omega$ in $\mathcal{R}_A$ and $B$.

**Proof.** First, note that $\psi_{l,m} := (\psi, Y_{l,m})Y_{l,m}$ is a solution to (1.3), supported on the fixed angular parameter tuple $(\ell, m)$. Thus, in view of Theorem 2.3 and Proposition A.5, $\psi_{l,m}(t, r, \theta, \phi)$ decays exponentially in $t$ in $\mathcal{R}_A$ and $B$ on any $\{r = \text{const.}\}$ slice. \hfill $\square$

**Proposition 4.3.** Let $\phi \in C(\mathbb{R}^4; H^1_{\text{RNAdS}}(\Sigma_{t^*})) \cap C^1(\mathbb{R}^4; H^0_{\text{RNAdS}}(\Sigma_{t^*}))$ be a solution to (1.3) satisfying (4.1). Let $u[\phi]$ be defined as in (4.2). Then, in the exterior region $\mathcal{R}_A$ we have $\lim_{r \to \infty} |r^2 u[\phi]| = 0$, $\lim_{r \to \infty} |r^{-\frac{1}{2}} u[\phi] | = 0$. Moreover, $u[\phi]$ solves the radial o.d.e. (in $B$ and $\mathcal{R}_A$)

$$-u'' + (V_t - \omega^2)u = 0,$$  \hspace{1cm} (4.3)

where $t = \frac{d}{dr} t$.

$$V_t(r) = h \left( \frac{d}{dr} \frac{\ell}{r} + \frac{\ell(\ell + 1)}{r^2} - \frac{a}{r^2} \right),$$  \hspace{1cm} (4.4)

and

$$h = \frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + \frac{Q^2}{r^2}.$$  \hspace{1cm} (4.5)

Finally, note that

$$\frac{dV_t}{dr} = \frac{d}{dr} \left( \frac{d}{dr} \frac{\ell}{r} + \frac{\ell(\ell + 1)}{r^2} - \frac{a}{r^2} \right) + h \left( - \frac{d}{dr} \frac{\ell}{r^2} + \frac{d^2}{dr^2} \frac{\ell}{r^2} - \frac{2 \ell(\ell + 1)}{r^3} \right).$$  \hspace{1cm} (4.6)

**Proof.** We shall begin by showing the decay statement for $u[\phi]$ in the exterior. Note that $\phi(t^*) \in H^1_{\text{RNAdS}}(\Sigma_{t^*})$ for every fixed $t^*$. In particular, the solution $\phi$ also satisfies

$$\int_{t^*_+}^{\infty} |\phi(r, t, \theta, \phi)|^2 r^2 dt < \infty$$

and

$$\int_{t^*_+}^{\infty} r^4 |\partial_r \phi(r, t, \theta, \phi)|^2 dt < \infty$$

which implies $\lim_{r \to \infty} |r^2 \phi| = 0$, $\lim_{r \to \infty} |r^2 \phi'| = 0$ and hence, $\lim_{r \to \infty} |r^2 u[\phi]| = 0$, $\lim_{r \to \infty} |r^{-\frac{1}{2}} u[\phi]| = 0$. The rest of the proof is a direct computation. \hfill $\square$

Next, we prove that the potential $V_t$ has a local maximum for large enough angular parameter $l_0$.

**Proposition 4.4.** There exists an $\ell_0(M, Q, l, \alpha) \in \mathbb{N}$ such that for all $\ell \geq \ell_0$, the potential $V_t$ has a local maximum $r_{l, \text{max}} > r_+$ and $V'_t \geq 0$ for $r_+ \leq r \leq r_{l, \text{max}}$. Moreover, $r_{l, \text{max}} \to r_{\text{max}} := \frac{3}{4} M + \sqrt{\frac{9}{4} M^2 - 2Q^2}$ as $\ell \to \infty$.  


Proof. Note that for $\ell$ large enough, $V_\ell$ is non-negative in a neighborhood of $r_+$ with $r \geq r_+$. Also, $V_\ell$ vanishes at $r = r_+$. Hence, it suffices to show that $\frac{dV_\ell}{dr}$ is negative somewhere for $r \geq r_+$. But note that

$$\frac{dV_\ell}{dr} = F(r) + r^3 \ell(\ell + 1) \left( \frac{dh}{dr} - 2h \right) = F(r) + 2r^3 \ell(\ell + 1) \left( \frac{3M}{r} - 1 - \frac{2Q^2}{r^2} \right)$$

(4.7)

for some function $F(r)$ which is independent of $\ell$. Now, first choose $r > r_+$ large enough only depending on $M,Q$ such that the last term is negative. Then, choose $\ell$ large enough such that it dominates the first term which proves that a $r_{\ell,\text{max}}$ as in the statement exists. The limiting behavior $r_{\ell,\text{max}} \to \frac{3}{2}M + \sqrt{\frac{2}{4}M^2 - 2Q^2}$ as $\ell \to \infty$ also follows from (4.7). This concludes the proof. $\square$

Now, we are in the position to prove a frequency localized integrated decay estimate in the exterior region for the bounded frequencies $|\omega| \leq 2\omega_0$.

**Proposition 4.5.** Let $u(r_\ast) = u(\omega,m,\ell)(r_\ast)$ solve the radial o.d.e. (4.3) and assume that $\lim_{r \to \infty} |r^2 u'| = 0$ and $\lim_{r \to \infty} |r^{\frac{1}{2}} u'| = 0$. Moreover, let $|\omega| \leq 2\omega_0$, where $\omega_0(M,Q,\ell,\alpha) > 0$ small enough will be fixed in the following proof. Then, we have

$$\int_{R_{\ast}^{-\infty}}^{r_{\ast}^{-\infty}} \frac{1}{r^4} \left( |u'|^2 + |u|^2(\ell(\ell + 1) + r^2) \right) \, dr_\ast \lesssim -\hat{Q}(R_{\ast}^{-\infty})$$

(4.8)

for all $R_{\ast}^{-\infty}$ small enough such that $r(R_{\ast}^{-\infty}) < r_0$, where $r_0 = r_0(M,Q,\ell,\alpha) > r_+$ is determined in the following proof. Here, the boundary term $\hat{Q}(R_{\ast}^{-\infty})$ satisfies

$$|\hat{Q}(R_{\ast}^{-\infty})| \lesssim (|\omega|^2 |u|^2 + |u|^2)(1 + O(e(r - r_+))) \text{ as } R_{\ast}^{-\infty} \to -\infty.$$  

(4.9)

**Proof.** In view of (3.8) and Proposition 7.4 and Section 9.3 it suffices to prove (4.8) for $r \geq r_0(M,Q,\ell,\alpha)$ for some fixed $\ell_0(M,Q,\ell,\alpha) \in \mathbb{N}_0$. Let $r_0, r_1$ depending only on $M,Q,\ell,\alpha$ be such that $r_+ < r_0 < r_1 < r_{\text{max}} - \delta$, where $r_{\text{max}}$ is defined in Proposition 4.4. Here, $\delta = \delta(\ell_0) > 0$ is such that $V' \geq 0$ for all $r_+ \leq r \leq r_{\text{max}} - \delta$, cf. Proposition 4.4. We can make $\delta(\ell_0)$ as small as we want by choosing $\ell_0$ sufficiently large. Now, we choose $\omega_0(M,Q,\ell,\alpha) > 0$ small enough and $\ell_0$ large enough such that

$$V - \omega^2 + \frac{\Delta}{4r^2} \gtrsim \ell(\ell + 1) + \frac{\Delta}{r^2} \text{ for } r \geq r_0,$$

(4.10)

$$V - \omega^2 \geq 0 \text{ for } r_0 \leq r \leq r_1,$$  

(4.11)

and for all $|\omega| \leq 2\omega_0$, $\ell \geq \ell_0$. For smooth $f(r_\ast)$ and $\tilde{h}(r_\ast)$, we define the currents

$$Q^f := f \left( |u'|^2 + (\omega^2 - V)|u|^2 \right) + f' \text{ Re}(u'\bar{u}) - \frac{1}{2} f'' |u|^2,$$

(4.12)

$$Q^{\tilde{h}} := \tilde{h} \text{ Re}(\bar{u}u') - \frac{1}{2} \tilde{h}' |u|^2$$

(4.13)

with

$$Q^{f'} = 2f'|u'|^2 - fV'|u|^2 - \frac{1}{2} f''' |u|^2,$$

(4.14)

$$Q^{\tilde{h}'} = \tilde{h} \left( |u'|^2 + (V - \omega^2)|u|^2 \right) - \frac{1}{2} \tilde{h}'' |u|^2.$$  

(4.15)

Thus,

$$Q^{f'} + Q^{\tilde{h}'} = |u'|^2(2f' + \tilde{h}) + |u|^2 \left( -fV' - \frac{1}{2} f''' + \tilde{h}(V - \omega^2) - \frac{1}{2} \tilde{h}'' \right).$$

We choose a smooth $f \leq 0$ such that

- $f$ is monotonically increasing,

- $f \leq 0$.

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Then, we have

\[ |r| \leq c_1 \text{ for } r_+ \leq r \leq r_1 \text{ and some } c_1(M,Q,l) > 0, \]

\[ \Delta \leq f' \leq \Delta \text{ for } r_+ \leq r \leq r_1, \]

\[ |f'''| \leq \Delta, \]

\[ f = 0 \text{ for } r \geq r_{\max} - \delta. \]

and a smooth \( \hat{h} \geq 0 \) such that

\[ \hat{h} = 0 \text{ for } r \leq r_0, \]

\[ |\hat{h}'| \leq 1 \text{ for } r_0 < r_1, \]

\[ \hat{h} = 1 \text{ for } r \geq r_1. \]

Then, we have

\[
Qf' + \hat{Q}h' \geq \begin{cases} 
2f'|u'|^2 + |u|^2(-fV' - \frac{1}{2}f''') & \text{for } r_+ \leq r \leq r_0, \\
2f'|u'|^2 + |u|^2(-fV' - \frac{1}{2}f''' - \frac{1}{2}\hat{h}'') & \text{for } r_0 \leq r \leq r_1, \\
|u|^2 + |u|^2(-\frac{1}{2}f''' + (V - \omega^2)) & \text{for } r \geq r_1.
\end{cases}
\]

(4.16)

Thus, choosing \( \ell_0(M,Q,l,\alpha) \) large enough (and \( \omega_0(M,Q,l,\alpha) > 0 \) possibly smaller), we have

\[
Qf' + \hat{Q}h' \geq \frac{\Delta}{r^4} \left(|u|^2 + |u|^2(\ell(\ell + 1) + \omega^2)\right)
\]

(4.17)

for \( r_+ \leq r \leq r_{\max} - \delta \) and

\[
Qf' + \hat{Q}h' \geq |u|^2 + (V - \omega^2)|u|^2 \geq |u|^2 - |u|^2 \frac{\Delta}{4l^2r^2} + \hat{c}(\ell(\ell + 1) + \frac{\Delta}{r^2}) |u|^2
\]

(4.18)

for \( r \geq r_{\max} - \delta \) and some \( \hat{c}(M, Q, l, \alpha) > 0 \), where we have used (4.10). Integrating \( Qf' + \hat{Q}h' \) in the region \( r_+ \leq r_\max - \delta \) and applying the following Hardy inequality (see [38, Lemma 7.1])

\[
\int_{r=r_{\max}-\delta}^{r=\infty} |u|^2 dr \geq \int_{r=r_{\max}-\delta}^{r=\infty} \frac{\Delta}{4l^2r^2} |u|^2 dr
\]

(4.19)

to control the negative signed term in (4.18), yields

\[
\int_{R_+}^{r=\infty} \frac{\Delta}{r^4} \left(|u|^2 \chi(r \leq r_{\max}-\delta) + |u|^2(\ell(\ell + 1) + r^2)\right) dr \leq -Qf(R_+(-\infty)).
\]

(4.20)

Note that we use \( \lim_{r \to \infty} |r^{\frac{1}{2}}u| = 0 \) and \( \lim_{r \to \infty} |r^{-\frac{3}{2}}u'| = 0 \) to apply the Hardy inequality. To obtain control of \( |u|^2 \) in the region \( r \geq r_{\max} - \delta \) in (4.20) we just add a small portion of the integral over (4.18). This proves

\[
\int_{R_+}^{r=\infty} \frac{\Delta}{r^4} \left(|u|^2 + |u|^2(\ell(\ell + 1) + r^2)\right) dr \leq -Qf(R_+(-\infty)),
\]

(4.21)

where \( |Qf(R_+\infty)| \leq (|u|^2|u|^2 + |u|^2)(1 + O_{\ell}(r - r_+)) \) as \( R_+\infty \to -\infty \) is satisfied by the construction of \( f \).

With the frequency localized integrated energy decay estimate of Proposition [15] we will now prove a local integrated energy decay estimate in physical space. Indeed, a naive application of Plancherel’s theorem to (4.15) gives a global integrated energy estimate. However, localizing this energy decay requires some sort of cut-off which does not respect the compact frequency support. Nevertheless, by carefully choosing a localization, we can show that the error term decays superpolynomially in time. At this point we shall remark that we do expect \( \psi_\lambda \) to decay exponentially. However, for our problem, superpolynomial decay in the exterior is (more than) sufficient.
Proposition 4.6. Let $\psi_\beta$ be as in (3.5). Then, for any $q > 1$ and $\tau_2 \geq 2\tau_1 \geq 0$, we have the integrated energy decay estimate

$$\int_{2\tau_1}^{\infty} \int_{\Sigma_{t^*} \cap \mathcal{R}_A} J^T_\mu [\psi_\beta] n^\mu_{\Sigma_{t^*}} \, d\text{vol}_{\Sigma_{t^*}} \, dt^* \sim \int_{\mathcal{R}_A \cap \{ t^* \geq 2\tau_1 \}} \left[ r^{-2} |\partial_{r^*} \psi_\beta|^2 + r^{-2} |\partial_{r} \psi_\beta|^2 + |\nabla \psi_\beta|^2 + |\psi_\beta|^2 \right] r^2 \, dt^* \, \sin \theta \, d\theta \, d\varphi$$

$$\lesssim \int_{\Sigma_{\tau_1} \cap \mathcal{R}_A} J^T_\mu [\psi_\beta] n^\mu_{\Sigma_{\tau_1}} \, d\text{vol}_{\Sigma_{\tau_1}} \quad (4.22)$$

where $C(q) > 0$ is a constant only depending on $q$. Moreover, this directly implies

$$\int_{\Sigma_{\tau_2} \cap \mathcal{R}_A} J^T_\mu [\psi_\beta] n^\mu_{\Sigma_{\tau_2}} \, d\text{vol}_{\Sigma_{\tau_2}} + \int_{2\tau_1}^{\tau_2} \int_{\mathcal{R}_A \cap \{ t^* \geq \tau_1 \}} J^T_\mu [\psi_\beta] n^\mu_{\Sigma_{t^*}} \, d\text{vol}_{\Sigma_{t^*}} \, dt^* \lesssim \int_{\Sigma_{\tau_1} \cap \mathcal{R}_A} J^T_\mu [\psi_\beta] n^\mu_{\Sigma_{\tau_1}} \, d\text{vol}_{\Sigma_{\tau_1}} \quad (4.23)$$

for the $T$-energy.

**Proof.** In order to show (4.22) we will first construct an auxiliary solution $\Psi$ of (1.3). We set initial data for $\Psi$ on $\Sigma_{\tau_1}$ as $(\Psi_0, \Psi_1) := (\psi_{\beta}, T \psi_{\beta}) |_{\Sigma_{\tau_1} \cap \mathcal{R}_A}$. Then, we will define data $\Psi_2$ on $\mathcal{H}_A^+ \cap \{ t^* \leq \tau_1 \}$ such that the data can be extended to a $C^k$ function in a neighborhood of $\mathcal{H}_A^+ \cap \{ t^* = \tau_1 \}$ for some finite regularity $k$. Choosing the regularity $k$ large enough will guarantee well-posedness. More precisely, in local coordinates $(t^*, r, \varphi, \theta)$ and for $r = r_+$, we define

$$\Psi_2(t^*, r_+, \varphi, \theta) := \sum_{j=1}^k \lambda_j \psi_{\beta} |_{\{ t^* \geq \tau_1 \}} \left( -j(t^* - \tau_1) + \tau_1, r_+, \varphi, \theta \right) \quad (4.24)$$

for $t^* \leq \tau_1$ and some uniquely determined $(\lambda_j)_{1 \leq j \leq k}$ such that

$$\mathbb{R} \times \mathbb{S}^2 \ni (t^*, \varphi, \theta) \mapsto \left\{ \begin{array}{ll}
\Psi_2(t^*, r_+, \varphi, \theta) & \text{for } t^* \leq \tau_1 \\
\psi_{\beta}(t^*, r_+, \varphi, \theta) & \text{for } t^* > \tau_1 
\end{array} \right. \quad (4.25)$$

is $C^k$. Indeed, the function is smooth everywhere except at $t^* = \tau_1$. Now, we consider the mixed boundary value-Cauchy-characteristic problem, where we impose data as follows. On the null hypersurface $\mathcal{H}_A^+ \cap \{ t^* \leq \tau_1 \}$

Figure 5: In the darker shaded region $J^+ (\Sigma_{\tau_1}) \cap \mathcal{R}_A$ we have that $\Psi = \psi_\beta$, and in the lighter shaded region we can estimate the energy of $\Psi$ in terms of $\psi_\beta$. This holds true as $\Psi_2$ is the $C^k$ reflection of $\psi_\beta$ from $\mathcal{H}_A^+ \cap \{ t^* \geq \tau_1 \}$ to $\mathcal{H}_A^+ \cap \{ t^* < \tau_1 \}$.
we impose $\Psi_2$. This null cone intersects the spacelike hypersurface $\Sigma_{t_1}$ on which we have prescribed $(\Psi_0, \Psi_1)$ as data. As before, we assume Dirichlet condition on $I_A$. For fixed $k > 0$ large enough, this is a well-posed problem and can be solved backwards and forwards in $\mathcal{R}_A$ [53 Theorem 2]. We will call the arising solution $\Psi$ and by uniqueness note that $\Psi = \psi_0$ on $(\mathcal{R}_A \cup \mathcal{H}_{\omega}^+) \cap J^+(\Sigma_{t_1})$. Indeed, analogously to $\psi_0$, we have $\Psi \in C(\mathcal{R}_A; H^1_{\text{AdS}}(\Sigma_{t_1} \cap \mathcal{R}_A)) \cap C^1(\mathcal{R}_A; H^{0,2}_{\text{AdS}}(\Sigma_{t_1} \cap \mathcal{R}_A))$. Moreover, $\Psi$ decays logarithmically and $(\Psi, Y_{\ell m})Y_{\ell m}$ decays exponentially towards $+i\epsilon$ and $-i\epsilon$ on a $\{r = \text{const.}\}$ hypersurface. Refer to Fig. 5 for a visualization of the Cauchy-characteristic problem with Dirichlet boundary conditions.

Analogously to $\psi = \psi_0 + \psi_1$, we decompose the new solution $\Psi$ in low and high frequencies $\Psi = \Psi_\delta + \Psi_\mu$.

We define

$$\Psi_\delta := \frac{1}{\sqrt{2\pi}} \mathcal{F}_\delta^{-1}[\chi_{2\omega_0}] \ast \Psi,$$

$$\Psi_\mu := \Psi - \Psi_\delta,$$

(4.26)

where $\chi_{2\omega_0}$ is a smooth cutoff function such that $\chi_{2\omega_0} = 1$ for $|\omega| \leq \omega_0$ and $\chi_{2\omega_0} = 0$ for $|\omega| \geq 2\omega_0$. Now, note that from the $T$-energy identity in (2.21) we have

$$\int_{\mathcal{H}_\omega^+(\tau_1, \infty)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H = \int_{\Sigma_{t_1} \cap \mathcal{R}_A} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_{\Sigma_{t_1}}$$

as the flux through $I_A$ vanishes in view of the Dirichlet boundary condition at $I_A$. Here, we use the notation $\mathcal{H}_\omega^+(a, b) := \mathcal{H}_\omega^+ \cap \{a < t < b\}$. Moreover, from the $T$ energy identity, we have

$$\int_{\mathcal{H}_\omega^+(\tau_1, \infty)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H = \int_{\Sigma_{t_1} \cap \mathcal{R}_A} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_{\Sigma_{t_1}} + \int_{\mathcal{H}_\omega^+(-\infty, \tau_1)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H$$

$$\lesssim \int_{\Sigma_{t_1} \cap \mathcal{R}_A} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_{\Sigma_{t_1}} + \int_{\mathcal{H}_\omega^+(-\infty, \tau_1)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H$$

$$\lesssim \int_{\Sigma_{t_1} \cap \mathcal{R}_A} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_{\Sigma_{t_1}}$$

(4.28)

We have used the estimate

$$\int_{\mathcal{H}_\omega^+(-\infty, \tau_1)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H \lesssim \int_{\mathcal{H}_\omega^+(\tau_1, \infty)} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H$$

which follows from our construction of the initial data. Thus,

$$\int_{\mathcal{H}_\omega^+} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H + \int_{\mathcal{H}_\omega^+} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_H \lesssim \int_{\Sigma_{t_1} \cap \mathcal{R}_A} J_T^{\mu} [\psi_1] n_H^\mu d\text{vol}_{\Sigma_{t_1}}.$$

(4.29)

Now, note that $u[\Psi_\delta]$ defined as

$$u[\Psi_\delta](r, \omega, \ell, m) = \frac{r}{\sqrt{2\pi}} \int e^{-i\omega t} \langle \Psi_\delta, Y_{\ell m} \rangle_{L^2(S^2)} dt$$

(4.30)

satisfies the assumptions of Proposition 4.6 such that (4.8) holds true for $u[\Psi_\delta]$. We now integrate the frequency localized energy estimate (4.8) associated to $u[\Psi_\delta]$ in $\omega$ and sum over all spherical harmonics. There are two main terms appearing and we will estimate them in the following. This step is similar to [38 Sections 9.1 and 9.3] so we will be rather brief. An application of Plancherel’s theorem for the integrated left hand side of (4.8) yields

$$\int_{\mathcal{R}_A} \left[ |\partial_\omega \Psi_\delta|^2 + |\partial_\omega \Psi_\delta|^2 + r^2 |\nabla \Psi_\delta|^2 + r^2 |\Psi_\delta|^2 \right] dt \, d\omega \sin \theta \, d\theta \, d\phi$$

$$\lesssim \lim_{R^{-} \to -\infty} \sum_{\ell m} \int_{\omega} \int_{R^{-} \to -\infty} \int_{R^{-} \to -\infty} \int_{R^{-} \to -\infty} \frac{\Delta}{r^4} \left[ \omega^2 |u[\Psi_\delta]|^2 + |u[\Psi_\delta]|^2 + (\ell + 1) |u[\Psi_\delta]|^2 + r^2 |u[\Psi_\delta]|^2 \right].$$

(4.31)

We will use this statement only in a qualitative way such that $u[\Psi_\delta]$ is well-defined in (4.30) and satisfies (4.8).
To estimate the boundary term on the right hand side of (4.8), we first decompose $u[\Psi]$ as $u[\Psi] = a(\omega, m, \ell)u_1 + b(\omega, m, \ell)u_2$, where $u_1, u_2$ are defined as the unique solutions to the radial o.d.e. (4.3) in the exterior satisfying $u_1 = e^{i(\omega r_+ + \ell \tau)} + O_\ell (r-r_+)$ and $u_2 = e^{-i(\omega r_+ + \ell \tau)} + O_\ell (r-r_+)$ as $r \to r_+$. Here, $a = a(\omega, \ell, m)$ and $b = b(\omega, \ell, m)$ are the unique coefficients of the decomposition. Then, in view of (4.9) and $u'_1 = i\omega u_1 + O_\ell (r-r_+)$, $u'_2 = -i\omega u_2 + O_\ell (r-r_+)$, we estimate

$$\int \frac{1}{r^2} |\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\tau \Psi|^2 \leq \int \frac{1}{r^2} |\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\tau \Psi|^2 + \frac{1}{r^2} \left| |\Psi|^2 + |\bar{\Psi}|^2 \right|^2 \, d\Omega$$

and in particular,

$$\int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu \leq \int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu.$$

Hence, in view of $\psi = \Psi$ in $\{t^* \geq \tau_1\} \cap R_A$, we have

$$\int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\psi] n_{\Sigma_{t^*}}^\mu \, d\nu \leq \int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu.$$

Thus, we conclude the global integrated energy decay statement

$$\int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu \leq \int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu.$$

Finally, we are left with the term $\int_{t^* \geq \tau} \int_{\Sigma_{t^*} \cap R_A} J_\mu^T[\Psi] n_{\Sigma_{t^*}}^\mu \, d\nu$. We will show that this term decays at a superpolynomial rate. First, introduce the notation $\chi_2 := 1 - \chi_{2\omega_0}$ and set $\bar{\chi}_2 := F_T^{-1}(\chi_{2\omega_0})$, $\tilde{\chi}_2 := F_T^{-1}(\chi_2)$, which are well-defined in the distributional sense. Then,

$$\tilde{\Psi} = \frac{1}{\sqrt{2\pi}} \tilde{\chi}_2 \ast \bar{\Psi} = \frac{1}{\sqrt{2\pi}} \tilde{\chi}_2 \ast (\Psi - \psi)$$

since $\tilde{\Psi} = \Psi$ in view of their disjoint Fourier support. In particular, for $t^* \geq \tau_1$ we have

$$\tilde{\Psi} = \frac{1}{\sqrt{2\pi}} \tilde{\chi}_2 \ast (\Psi - \psi) = \frac{1}{\sqrt{2\pi}} (\sqrt{2\pi} \delta - \chi_{2\omega_0}) \ast (\Psi - \psi) = -\frac{1}{\sqrt{2\pi}} \chi_{2\omega_0} \ast (\Psi - \psi)$$
as $\delta * (\Psi - \psi_b) = \Psi - \psi_b = 0$ for $t^* \geq \tau_1$. To make notation easier we define $\phi := -\frac{1}{\sqrt{2\pi}}(\Psi - \psi_b)$ which is only supported for $t^* \leq \tau_1$ and satisfies $\Psi_b = \chi_{\Sigma_0} \ast \phi$. Now, as a result of the $T$ invariance of $\text{dvol}_{\Sigma_*}$ and $J^T_\mu [\nabla] n^\mu_{\Sigma_*}$, we have that

$$\int_{t^* \geq 2\tau_1} \int_{(r_0, \infty) \times \mathbb{S}^2} \frac{1}{r^2} [\partial_t, \Psi_b]^2 + \frac{\Delta}{r^2} [\partial_r, \Psi_b]^2 + |\nabla \Psi_b|^2] r^2 \text{d}\sigma_{S^2} \text{d}t^*$$

$$\leq \int_{t^* \geq 2\tau_1} \int_{(r_0, \infty) \times \mathbb{S}^2} \left[ r^{-2} \int_{-\infty}^{t(\tau_1, r)} \chi_{\Sigma_0} (t(t^*, r) - s)(\partial_r \phi)(s) \text{d}s \right]^2$$

$$+ \frac{\Delta}{r^2} \left[ \int_{-\infty}^{\tau_1} \chi_{\Sigma_0} (t(t^*, r) - s)(\partial_r \phi)(s) \text{d}s \right]^2 \left[ \int_{-\infty}^{\tau_1} |\nabla \phi|(s) \text{d}s \right]^2 r^2 \text{d}\sigma_{S^2} \text{d}t^*$$

$$\leq \int_{-\infty}^{\tau_1} |\chi_{\Sigma_0} (t - s)| \int_{t^* \geq 2\tau_1} \int_{(r_0, \infty) \times \mathbb{S}^2} \left[ \int_{-\infty}^{\tau_1} |\chi_{\Sigma_0} (t - s)| r^{-2} |\partial_r \phi|^2(s) \text{d}s \right]^2$$

$$+ \frac{\Delta}{r^2} \left[ \int_{-\infty}^{\tau_1} |\chi_{\Sigma_0} (t - s)| \int_{t^* \geq 2\tau_1} \int_{(r_0, \infty) \times \mathbb{S}^2} \left[ r^{-2} |\partial_r \phi|^2(s) + \frac{\Delta}{r^2} |\partial_r \phi|^2(s) + |\nabla \phi|^2(s) \right] r^2 \text{d}\sigma_{S^2} \text{d}t^* \right] \text{d}s \text{d}t^*$$

$$\leq \int_{(r_0, \infty) \times \mathbb{S}^2} \left[ \int_{-\infty}^{\tau_1} |\chi_{\Sigma_0} (t - s)| \int_{t^* \geq 2\tau_1} \int_{-\infty}^{\tau_1} |\chi_{\Sigma_0} (t - s)| \text{d}s \text{d}t^* \right] \text{d}s \text{d}t^*$$

$$\leq \int_{\Sigma_0 \cap \mathcal{R}_A} J^T_\mu [\phi] n^\mu_{\Sigma_0} \text{dvol}_{\Sigma_0} \int_{t^* \geq 2\tau_1} \int_{\Sigma_0 \cap \mathcal{R}_A} \frac{1}{1 + \tau_1^2} \text{d}s \text{d}t^*$$

Here, we have used the boundedness of the $T$-energy (cf. (2.21)), i.e.

$$\int_{\Sigma_* \cap \mathcal{R}_A} J^T_\mu [\phi] n^\mu_{\Sigma_*} \text{dvol}_{\Sigma_*} \leq \int_{\Sigma_0 \cap \mathcal{R}_A} J^T_\mu [\phi] n^\mu_{\Sigma_0} \text{dvol}_{\Sigma_0} \leq \int_{\Sigma_0 \cap \mathcal{R}_A} J^T_\mu [\psi_b] n^\mu_{\Sigma_0} \text{dvol}_{\Sigma_0}.$$ (4.38)

Finally, we have also used that the Schwartz function $\chi_{\Sigma_0}$ decays superpolynomially at any power $q > 1$. This concludes the proof in view of (4.35).

In order to remove the degeneracy of the $T$-energy at the event horizon, we will use the by now standard red-shift vector field [17]. As usual, the red-shift vector field $N$ is a future-directed $T$ invariant timelike vector field which has a positive bulk term $K^N \geq 0$ near the event horizon. In a compact $r$ region bounded away from the event horizon $\mathcal{H}_A$, the bulk term $K^N$ of $N$ is sign-indeterminate but this will be absorbed in the spacetime integral of the $T$ current in Proposition 4.6. Also, note that $N = T$ for large enough $r$. In the negative mass AdS setting, we refer to Section 4.2 for an explicit construction of the red-shift vector field $N$. Note that the red-shift vector field $N$ has the property that

$$\int_{\Sigma_* \cap \mathcal{R}_A} J_\mu^N [\psi_b] n^\mu_{\Sigma_*} \text{dvol}_{\Sigma_*} \sim \int_{\Sigma_* \cap \mathcal{R}_A} e_1(\psi_b) r^2 \sin \theta d\theta d\varphi$$ (4.39)

for $\psi_b$ as in (3.5).

**Proposition 4.7.** Let $\psi_b$ be as in (3.5). Then for any $\tau_2 \geq 2\tau_1 \geq 0$, we have

$$\int_{\Sigma_0 \cap \mathcal{R}_A} J_\mu^N [\psi_b] n^\mu_{\Sigma_0} \text{dvol}_{\Sigma_0} + \int_{\mathcal{H}_A \cap (2\tau_1 \leq t^* \leq \tau_2)} \left[ (|\partial_\tau \psi_b|^2 + |\nabla \psi_b|^2 + |\psi_b|^2) \right] \text{d}t^* \text{d}\sigma_{S^2}$$

$$+ \int_{2\tau_1}^{\tau_2} \int_{\Sigma_* \cap \mathcal{R}_A} J_\mu^N [\psi_b] n^\mu_{\Sigma_*} \text{dvol}_{\Sigma_*} \text{d}t^* \leq q \int_{\Sigma_* \cap \mathcal{R}_A} J_\mu^N [\psi_b] n^\mu_{\Sigma_*} \text{dvol}_{\Sigma_*} + \int_{\Sigma_0 \cap \mathcal{R}_A} J_\mu^T [\psi_b] n^\mu_{\Sigma_0} \text{dvol}_{\Sigma_0}$$ (4.40)
and in particular,
\[
\int_{\Sigma_{t_2} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t_2}} \, \text{dvol}_{\Sigma_{t_2}} + \int_{2t_1}^{t_2} \int_{\Sigma_{t^*} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t^*}} \, \text{dvol}_{\Sigma_{t^*}} \, dt^* \\
\lesssim_q \int_{\Sigma_{t_1} \cap R_A} J^N_{\mu} [\psi_b] n^\mu + \frac{\int_{\Sigma_0 \cap R_A} J^N_{\mu} [\psi_b] n^\mu \, \text{dvol}_{\Sigma_0}}{1 + \tau_1^q},
\]
(4.41)

Proof. We apply the energy identity (the spacetime integral of (2.18)) with the red-shift vector field \(N\) for \(\psi_b\) in the region \(R_A \cap \{ 2 t_1 \leq t^* \leq t_2 \}\), where \(2 t_1 \leq t_2\). After taking care of the negative lower order term via a Hardy inequality and absorbing the sign-indefinite bulk of \(N\) away from the horizon in the spacetime integral of \(J^T\) on the right hand side (see [36, Section 4] for further details), we arrive at
\[
\int_{\Sigma_{t_2} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t_2}} \, \text{dvol}_{\Sigma_{t_2}} + \int_{R_A \cap \{ 2 t_1 \leq t^* \leq t_2 \}} J^N_{\mu} [\psi_b] n^\mu \, \text{dvol}_H + \int_{2 t_1}^{t_2} \int_{\Sigma_{t^*} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t^*}} \, \text{dvol}_{\Sigma_{t^*}} \, dt^* \\
\lesssim \int_{2 t_1}^{t_2} \int_{\Sigma_{t^*} \cap R_A} J^T_{\mu} [\psi_b] n^\mu_{\Sigma_{t^*}} \, \text{dvol}_{\Sigma_{t^*}} \, dt^* + \int_{\Sigma_{2 t_1} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{2 t_1}} \, \text{dvol}_{\Sigma_{2 t_1}}.
\]
(4.42)

Note that the integral along the horizon \(\int_{\mathcal{H}_+ \cap \{ 2 t_1 \leq t^* \leq t_2 \}} J^N_{\mu} [\psi_b] n^\mu_H \, \text{dvol}_H\) is sign-indefinite due to the (possible) negative mass. However, this can be absorbed in the bulk term using an \(\epsilon\) of the integrated bulk term of the red-shift vector field \(N\) and some of the bulk term from the \(T\) vector field, cf. [36, Equation (70)]. Moreover, using the integrated energy decay estimate of the \(T\) vector field from Proposition 4.7, we conclude
\[
\int_{\Sigma_{t_2} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t_2}} \, \text{dvol}_{\Sigma_{t_2}} + \int_{R_A \cap \{ 2 t_1 \leq t^* \leq t_2 \}} (|\partial_{t^*} \psi_b|^2 + |\nabla \psi_b|^2 + |\psi_b|^2) \, dt^* \, d\sigma_{g_{2}} \\
+ \int_{2 t_1}^{t_2} \int_{\Sigma_{t^*} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t^*}} \, \text{dvol}_{\Sigma_{t^*}} \, dt^* \lesssim_q \int_{\Sigma_{t_1} \cap R_A} J^N_{\mu} [\psi_b] n^\mu + \frac{\int_{\Sigma_0 \cap R_A} J^T_{\mu} [\psi_b] n^\mu \, \text{dvol}_{\Sigma_0}}{1 + \tau_1^q}.
\]
(4.43)

Now we obtain

**Proposition 4.8.** Let \(\psi_b\) be defined as in (3.5). Then, for any \(q > 1\) and \(\tau \geq 0\) we have
\[
\int_{\Sigma_{t} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{t}} \lesssim_q \frac{1}{1 + \tau^q} \int_{\Sigma_{0} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{0}} \, \text{dvol}_{\Sigma_{0}} \lesssim_q \frac{1}{1 + \tau^q} E^A_1 [\psi_b] (0),
\]
(4.44)

and
\[
\int_{\mathcal{H} (\tau, + \infty)} |\partial_{t^*} \psi_b|^2 + (|\nabla \psi_b|^2 + |\psi_b|^2) \lesssim_q \frac{1}{1 + \tau^q} \int_{\Sigma_{0} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{0}} \, \text{dvol}_{\Sigma_{0}} \lesssim_q \frac{1}{1 + \tau^q} E^A_1 [\psi_b] (0).
\]
(4.45)

Proof. In view of Proposition 4.7 it suffices to prove (4.44). Upon setting
\[
f(s) := \int_{\Sigma_{s} \cap R_A} J^N_{\mu} [\psi_b] n^\mu_{\Sigma_{s}} \, \text{dvol}_{\Sigma_{s}},
\]
we have from Proposition 4.7 that
\[
f(t_2) + \int_{2 t_1}^{t_2} f(s) \, ds \lesssim_q f(t_1) + \frac{f(0)}{1 + t_1^q}
\]
for any \(t_2 \geq 2 t_1 \geq 0\). The claim follows now from Lemma 4.9 below. \(\square\)
Lemma 4.9. Let \( f : [0, \infty) \to [0, \infty) \) be a continuous function satisfying
\[
f(t_2) + \int_{2t_1}^{t_2} f(s)ds \leq \alpha(q) \left( f(t_1) + \frac{f(0)}{1 + \frac{1}{t_1}} \right)
\]  
(4.46)
for any \( q > 1 \), \( 0 \leq 2t_1 \leq t_2 \) and some \( \alpha(q) > 0 \) only depending on \( q \). Then, for all \( q > 1 \), there exists a constant \( C(\alpha(q), q) > 0 \) only depending on \( \alpha \) and \( q \) such that
\[
f(t) \leq \frac{C(\alpha(q), q)}{1 + t^q} f(0)
\]  
(4.47)
for all \( t \geq 0 \).

Proof. Fix \( q > 1 \). First, note that from (4.46) we have for any \( t_2 > t_1 > 0 \)
\[
f(t_2) \leq \alpha(q) \left( f(t_1) + \frac{f(0)}{1 + \frac{1}{t_1}} \right).
\]
Now, let \( t > 0 \) be arbitrary. Let us assume without loss of generality that \( t > 1 \). Then, take a dyadic sequence \( \tau_{k+1} = 2\tau_k \), where \( \tau_0 = 1 \). Now, there exists a \( n \in \mathbb{N} \) such that \( t \in [\tau_{n+2}, \tau_{n+3}] \). Then, again from (4.46) we have
\[
\int_{\tau_{n+1}}^{\tau_{n+2}} f(s)ds \leq \alpha(q) \left( f(\tau_n) + \frac{f(0)}{1 + \frac{1}{\tau_n}} \right)
\]
from which we conclude that there exists a \( \xi \in [\tau_{n+1}, \tau_{n+2}] \) such that
\[
f(\xi) \leq \alpha(q) \left( \frac{f(\tau_n)}{\tau_{n+1}} + \frac{f(0)}{1 + \frac{1}{\tau_n}} \right).
\]
Hence, since \( \xi \leq \tau_{n+2} \leq t \leq \tau_{n+3} \),
\[
f(t) \leq \alpha(q) \left( f(\xi) + \frac{f(0)}{1 + \frac{1}{\tau_{n+2}}} \right) \leq \alpha(q) \left( \alpha(q) \left( \frac{f(\tau_n)}{\tau_{n+1}} + \frac{f(0)}{1 + \frac{1}{\tau_n}} \right) + \frac{f(0)}{1 + \frac{1}{\tau_{n+2}}} \right)
\]  
(4.48)
Now, note that \( \tau_n \sim t \) and hence, \( f(t) \leq C(1, \alpha(q)) \frac{1}{t^{\frac{q}{2}}} \). This improved decay can now be fed into (4.48) to obtain a decay of the form \( f(t) \leq C(2, \alpha(q)) \frac{1}{1 + t^{\frac{q}{2}}} \). This procedure can be iterated until one obtains
\[
f(t) \leq \frac{C(q, \alpha(q))}{1 + t^q} f(0).
\]  
(4.49)
\[
\square
\]

4.2 Interior estimates

Having obtained the superpolynomial decay for \( \psi_b \) in the exterior and in particular on the event horizon, we will now use this to show uniform boundedness in the black hole interior. The approach we take is similar to [25], however, due to the negative mass and hence the violation of the dominant energy condition, we will use twisted derivatives introduced in [62, 6].

4.2.1 Metric in the interior

In the interior we will also use null coordinates \((u_B, v_B)\) introduced in Section 2.1. Throughout this section we shall drop the index \( B \). Then, setting
\[
\Omega^2(u, v) := - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2} \right),
\]  
(4.50)
where \( r = r(u, v) \), we write the metric in the interior as
\[
g = -\frac{\Omega^2(u, v)}{2}(du \otimes dv + dv \otimes du) + r^2(u, v)d\sigma_2.
\]  
(4.51)
Note that in the interior we have \( r_- < r(u, v) < r_+ \) and \( dr_+ = \frac{r^2}{\Delta} dr \).
4.2.2 Twisted derivatives and twisted energy momentum tensor

**Definition 4.10 (Twisted derivative).** For a smooth and nowhere vanishing function \( f \) we define the twisted derivative

\[
\tilde{\nabla}_\mu := f \nabla_\mu \left( \cdot \overline{f} \right) \tag{4.52}
\]

and its formal adjoint

\[
\tilde{\nabla}_\mu^* := -\frac{1}{f} \nabla_\mu (f \cdot) \tag{4.53}
\]

We shall refer to \( f \) as the twisting function.

**Remark 4.11.** Note that we can rewrite the Klein–Gordon equation (1.3) in terms of the twisted derivatives as

\[
-\tilde{\nabla}_\mu^* \tilde{\nabla}_\mu \psi - \mathcal{V} \psi = 0, \tag{4.54}
\]

where the potential \( \mathcal{V} \) is given by

\[
\mathcal{V} = -\left( \frac{\alpha}{l^2} + \Box g f f \right). \tag{4.55}
\]

Now, we also associate a twisted energy-momentum tensor to the twisted derivatives.

**Definition 4.12 (Twisted energy-momentum tensor).** Let \( f \) be smooth and nowhere vanishing and \( \tilde{\nabla} \) as defined in Definition 4.10. We define the twisted energy-momentum tensor associated to (1.3) and \( f \) as

\[
\tilde{T}_{\mu\nu}[\psi] := \text{Re} \left( \frac{\tilde{\nabla}_\mu \tilde{\nabla}_\nu \psi}{\overline{\psi}} - \frac{1}{2} g_{\mu\nu} (\tilde{\nabla}_\sigma \overline{\psi} \tilde{\nabla}^\sigma \psi + \mathcal{V} |\psi|^2) \right), \tag{4.56}
\]

where \( \mathcal{V} \) is as in (4.55).

We will now compute the divergence of the twisted energy-momentum tensor.

**Proposition 4.13 ([11], Proposition 3).** Let \( \phi \) be a smooth function. Then,

\[
\nabla_\mu \tilde{T}^\mu_\nu[\phi] = \text{Re} \left( \left( -\tilde{\nabla}_\mu^* \tilde{\nabla}_\nu \phi - \mathcal{V} \phi \right) \tilde{\nabla}_\nu \phi \right) + \tilde{S}_\nu[\phi], \tag{4.57}
\]

where

\[
\tilde{S}_\nu[\phi] = \frac{\nabla^\nu (f \mathcal{V})}{2f} |\phi|^2 + \frac{\nabla^\nu f}{2f} \overline{\phi} \tilde{\nabla}_\sigma \phi \phi. \tag{4.58}
\]

Now, assume that \( \phi \) moreover satisfies (1.3) and \( X \) is a smooth vector field. Set

\[
\tilde{j}^X_\mu[\phi] := \tilde{T}_{\mu\nu}[\phi] X^\nu \quad \text{and} \quad \tilde{K}^X[\phi] := \pi_{\mu\nu} \tilde{T}^{\mu\nu}[\phi] + X^\nu \tilde{S}_\nu[\phi]. \tag{4.59}
\]

Then,

\[
\nabla_\mu \tilde{j}^X_\mu[\phi] = \tilde{K}^X[\phi]. \tag{4.60}
\]

Finally, note that if the twisting function \( f \) associated to \( \tilde{\nabla} \) is chosen such that \( \mathcal{V} \geq 0 \), then \( \tilde{T}_{\mu\nu} \) satisfies the dominant energy condition, i.e. if \( X \) is a future pointing causal vector field, then so is \(-\tilde{j}^X\).

In Proposition A.1 in the appendix we have written down the components of the twisted energy-momentum tensor, the twisted 1-jets \( \tilde{j}^X \) and the twisted bulk term \( \tilde{K}^X \) in null components. We will use the notation \( \mathcal{C}_{u_1} := \{ u = u_1 \} \), \( \mathcal{C}_{u_1} := \{ v = v_1 \} \) for null cones and \( \Sigma_{r_1} := \{ r = r_1 \} \) for spacelike hypersurfaces in the interior. Furthermore, we set

\[
\mathcal{C}_{u_1}(v_1, v_2) := \{ u = u_1 \} \cap \{ v_1 \leq v \leq v_2 \} \tag{4.61}
\]
and analogously for $\Sigma$ and $C$. Recall that $u + v = 2r_*$. Will make also use of the following notation. For any $\tilde{r} \in (r_-, r_+)$ we set

$$v_\tilde{r}(u) := 2r_*(\tilde{r}) - u,$$

$$u_\tilde{r}(v) := 2r_*(\tilde{r}) - v$$

and for hypersurfaces with constant $u,v,r$ we denote $n_u, n_v, n_r$ as their normals. Now, we are in the position to propagate the superpolynomial decay on the horizon established in Proposition 4.8 further into the interior. To do so we will make use of the twisted red-shift.

### 4.2.3 Twisted red-shift in the black hole interior

**Proposition 4.14.** There exist a $r_{\text{red}} \in (r_-, r_+)$, a constant $b(M,Q,l,\alpha) > 0$, a nowhere vanishing smooth function $f$ associated to the twisted energy momentum tensor and a future directed timelike vector field $N$ such that

$$0 \leq J^N_\mu [\psi_b] n^\mu_v \leq b \tilde{K}^N [\psi_b]$$

for $R_{\text{red}} := \{r_{\text{red}} \leq r \leq r_+\} \cap \{v \geq 1\}$ and for $\psi_b$ as in (3.5).

**Proof.** We choose the ansatz $N = N^u \partial_u + N^v \partial_v$ for our red-shift vector field. We will first estimate the twisted 1-jet $\tilde{J}$ and then the twisted bulk term $\tilde{K}$.

**$\tilde{J}$ current.** From (A.2), we have

$$J^N_\mu [\psi_b] n^\mu_v = 2\frac{N^u}{\Omega^2} |\nabla_u \psi_b|^2 + \frac{N^v}{2} \left(|\nabla \psi_b|^2 + V|\psi_b|^2\right),$$

where

$$V = -\left(\frac{\Box g J}{f} + \alpha \frac{\partial g}{f^2}\right).$$

First, if $f = f(r)$ we have

$$-\frac{\Box g f}{f} = \frac{\Omega^2}{4} \frac{\dot{f}}{f} + \left(\frac{\Omega^2}{2r} + \frac{\partial_r (\Omega^2)}{4}\right) \frac{\dot{f}}{f},$$

where $\dot{f} := \frac{df}{dr}$. Thus, choosing $f = e^{-r}$ gives

$$-\frac{\Box g f}{f} = \frac{\Omega^2}{4} - \frac{1}{4} \partial_r (\Omega^2) - \frac{1}{2r} \Omega^2.$$

Note that for $r_{\text{red}} < r_+$ close enough to $r_+$, we have

$$-\partial_r \Omega^2 \geq c_{\text{red}}$$

for all $r_{\text{red}} \leq r \leq r_+$ and some constant $c_{\text{red}} > 0$ only depending on the black hole parameters. The constant $c_{\text{red}} > 0$ does not decrease, when we choose $r_{\text{red}}$ even closer to $r_+$. By choosing $r_{\text{red}}$ close enough to $r_+$, we ensure that $V \geq 1$ in $r_{\text{red}} \leq r \leq r_+$. This finally shows that if we take $N$ as a future directed vector field, the 1-jet $J^N_\mu n^\mu_v$ is positive definite. We will construct the explicit form of $N$ in the bulk term estimate.

**Bulk term $\tilde{K}^N$.** Now, we will estimate the bulk term. We will choose the components of the timelike vector field $N = N^u \partial_u + N^v \partial_v$ as

$$N^u := \frac{1}{\Omega^2} + \frac{1}{\partial_1}$$

and

$$N^v := 1 - \frac{\Omega^2}{\partial_2},$$

---

8For null hypersurfaces there does not exist a unit norm normal vector, however, for a fixed volume form, there exists a canonical normal vector which we will choose here. Our choice of volume forms and the corresponding normals can be found in Appendix A.1.
Note that $N$ is smooth in $\mathcal{R}_{\text{red}}$. Moreover, for fixed $\delta_1, \delta_2 > 0$ (only depending on the black hole parameters), we can choose $r_{\text{red}}$ close enough to $r_+$ such that $N$ is future directed in $\mathcal{R}_{\text{red}}$. Then, note that

\begin{equation}
\tilde{K}^N[\psi_b] = (-\partial_r, \Omega^2) \left( \frac{1}{\delta_2} |\tilde{\nabla}_v \psi_b|^2 + \frac{1}{\Omega^2} |\tilde{\nabla}_u \psi_b|^2 \right) - \frac{2}{r} \left( \frac{1}{\Omega^2} + \frac{1}{\delta_1} + 1 - \frac{1}{\delta_2} \Omega^2 \right) \text{Re}(\tilde{\nabla}_u \psi_b \tilde{\nabla}_v \psi_b) \tag{4.69}
\end{equation}

where we have applied an $\epsilon$-weighted Young’s inequality. We have also used that—by choosing $r_{\text{red}}$ closer to $r_+$—we can make $\Omega^2$ uniformly smaller than any constant, in particular smaller than $\delta_1$ and $\delta_2$ once those are fixed. Choosing $\epsilon$ small enough, we absorb the term $\frac{1}{\Omega^2} |\tilde{\nabla}_u \psi_b|^2$ of (4.69) in the first term of (4.69). Then, choosing $\delta_2(\delta_1, \epsilon)$ small enough, we can also absorb the term $\frac{1}{\Omega^2} |\tilde{\nabla}_v \psi_b|^2$ in the first term of (4.69).

Completely analogously and by potentially choosing $\delta_2$ and $\delta_1$ even smaller, we estimate the terms of the form $\frac{1}{\Omega^2} \text{Re}(\tilde{\nabla}_u \psi_b \tilde{\nabla}_v \psi_b)$ arising from (4.72) and (4.73).

Next, note that, in view of $V \gtrsim 1$ and $|\tilde{\nabla}_v \psi_b|^2 \lesssim \Omega^2$, we choose $\delta_1$ small enough such that we absorb error terms coming from (4.72) and (4.73) in the term with the good sign in (4.70). By doing so we also have to make $\delta_2(\epsilon, \delta_1) > 0$ small enough. Finally, once $\delta_1$ and $\delta_2$ are fixed, note that we can make terms involving higher orders of $\Omega^2$ arbitrarily small by choosing $r_{\text{red}}$ close to $r_+$. This finally shows (4.74) and concludes the proof.

With the help of the constructed twisted red-shift current, we obtain

**Proposition 4.15.** Let $r_0 \in [r_{\text{red}}, r_+)$. Let $\psi_b$ defined as in 3.35 and recall that from Proposition 4.8 we have

\begin{equation}
\int_{\mathcal{H}(v_1, v_2)} \tilde{J}_\mu^N[\psi_b] n_{\mu, H^\nu} \text{dvol}_{H^\nu} \lesssim_q \frac{1}{1 + v_1^2} E_1^A[\psi_b](0) \tag{4.76}
\end{equation}

for $1 \leq v_1 \leq v_2$. Then,

\begin{equation}
\int_{\Sigma_{v_1}(r_0, r_+)} \tilde{J}_\mu^N[\psi_b] n_{\nu, \Sigma^\mu} \text{dvol}_{\Sigma^\mu} \sim \int_{u_{r_0}(v_1)}^{u_{r_+}(v_1)} \int_{\mathbb{S}^2} \frac{1}{\Omega^2} |\tilde{\nabla}_u \psi_b|^2 + \Omega^2 (|\nabla \psi_b|^2 + V|\psi_b|^2) \text{d}\sigma_2 \text{d}u \lesssim_q \frac{1}{1 + v_1^2} E_1[\psi_b](0), \tag{4.77}
\end{equation}

\begin{equation}
\int_{\Sigma_{v_1}(v_1, v_2)} \tilde{J}_\mu^N[\psi_b] n_{\nu, \Sigma^\mu} \text{dvol}_{\Sigma^\mu} \sim \int_{v_1}^{v_2} \int_{\mathbb{S}^2} \frac{1}{\Omega^2} |\tilde{\nabla}_u \psi_b|^2 + \sqrt{\Omega^2 (|\nabla \psi_b|^2 + |\nabla \psi_b|^2 + V|\psi_b|^2)} \text{d}u \text{d}\sigma_2 \\
\quad \lesssim_q \frac{E_1[\psi_b](0)}{1 + v_1^2}, \tag{4.78}
\end{equation}

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and
\[
\int_{\mathcal{R}(v_1,v_2)} \tilde{K}^N[\psi] \, d\text{vol} \sim \int_{v_1}^{v_2} \int_{u_{v_0}(v)}^{u_{v_1}(v)} \int_{\mathbb{S}^2} \frac{1}{\Omega^2} |\nabla_u \psi|^2 + \Omega^2 (|\nabla \psi|^2 + \mathcal{V}|\psi|^2) \, d\sigma_{\mathbb{S}^2} \, du \, dv \lesssim \frac{1}{1 + v_1^q} E_1[\psi](0). \tag{4.79}
\]
for any \(1 \leq v_1 \leq v_2\). Here, we use the notation \(\mathcal{R}(v_1,v_2) := \{r_0 \leq r \leq r_+\} \cap \{v_1 \leq v \leq v_2\}\).

Proof. The strategy is very similar to \[25\]. For any \(1 \leq v_1 \leq v_2\) we start by applying the energy identity (spacetime integral of (4.60)) in the region \(\mathcal{R}(v_1,v_2)\) to obtain
\[
\int_{\mathcal{L}_v(r_0,r_+)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma + \int_{\Sigma_0(v_1,v_2)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma + \int_{\mathcal{R}(v_1,v_2)} \tilde{K}^N[\psi] \, d\text{vol} = \int_{\mathcal{L}_{v_1}(r_0,r_+)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma + \int_{\mathcal{H}(v_1,v_2)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma + \int_{\mathcal{R}(v_1,v_2)} \tilde{K}^N[\psi] \, d\text{vol}. \tag{4.80}
\]
From Proposition 4.8 we have that
\[
\int_{\mathcal{H}(v_1,v_2)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma + \lesssim \frac{E_1[\psi](0)}{1 + v_1^q}. \tag{4.81}
\]
Upon defining
\[
\tilde{E}(v) := \int_{\mathcal{L}_v(r_0,r_+)} \tilde{J}_\mu^N[\psi] n_\Sigma^\mu \, d\text{vol}_\Sigma, \tag{4.82}
\]
we obtain
\[
\tilde{E}(v_2) + \int_{v_1}^{v_2} \tilde{E}(v) \, dv \lesssim \tilde{E}(v_1) + \frac{E_1[\psi](0)}{1 + v_1^q}, \tag{4.83}
\]
for any \(1 \leq v_1 \leq v_2\) which implies
\[
\tilde{E}(v) \lesssim \frac{1}{1 + v^q} \tilde{E}(v = 1) + E_1[\psi](0) \tag{4.84}
\]
for any \(v \geq 1\). This follows from an argument very similar to Lemma 4.9. Note that we have by general theory that \(\tilde{E}(v = 1) \lesssim E_1[\psi](0)\). Thus,
\[
\tilde{E}(v) \lesssim E_1[\psi](0) \frac{1}{1 + v^q} \tag{4.85}
\]
for \(v \geq 1\) which proves (4.77). The estimates (4.78) and (4.79) now follow from applying the energy identity again in the region \(\mathcal{R}(v_1,v_2)\).

\subsection{No-shift region}

In this region we propagate the decay towards \(i^+\) from the red-shift region to the blue-shift region using a \(T = \partial_t\) invariant vector field \(X\) and a \(t\)-independent twisting function \(f\). Take \(r_{\text{red}}\) fixed from Proposition 4.14 and let \(r_{\text{blue}}(M,Q,l) > r_\) be close to \(r_-\) which we will fix later in the proof of Proposition 4.21. As our vector field we will choose
\[
X_{\text{ns}} := \partial_u + \partial_v. \tag{4.86}
\]
(Indeed, any future directed and \(T\) invariant vector field would work.) We define our twisting function as
\[
f_{\text{ns}}(r) = e^{\beta_\text{ns} r}. \tag{4.87}
\]
Proposition 4.16. Let $\psi_5$ defined as in \eqref{3.5}. For any $r_0 \in [r_{\text{blue}}, r_{\text{red}}]$, $q > 1$ and any $v_s \geq 1$ we have
\[
\int_{\Sigma_{r_0}(v_s, 2v_s)} j^X_m [\psi_5] n^m_{\Sigma_r} \, dvol_{\Sigma_r} \lesssim_q E_1[\psi_5](0).
\] (4.90)
Moreover, for any $1 < p < q$ we also have
\[
\int_{\Sigma_{r_0}(v_s, +\infty)} (v^p + \langle u \rangle^p) j^X_m [\psi_5] n^m_{\Sigma_r} \, dvol_{\Sigma_r} \lesssim_{q, p} E_1[\psi_5](0).
\] (4.91)

Proof. We apply the energy identity (spacetime integral of \eqref{4.89}) with $X = \partial_u + \partial_v$ (cf. \eqref{4.86}) and $f_{\text{ns}}$ as in \eqref{4.87} in the region $\{ r_0 \leq r \leq r_{\text{red}} \} \cap \{ u < u_{\text{blue}}(v_s) \} \cap \{ v \leq 2v_s \}$. The choice of $f_{\text{ns}}$ guarantees the twisted dominated energy condition for the twisted energy-momentum tensor. Together with the coarea formula as well as the facts that $[r_s(r_0), r_s(r_{\text{red}})]$ is compact and $X$ is $T$ invariant, we conclude
\[
\int_{\Sigma_{r_0}(v_s, 2v_s)} j^X_m [\psi_5] n^m_{\Sigma_r} \, dvol_{\Sigma_r} \leq B_1 \int_{\bar{r} \leq \tilde{r} \leq r_{\text{red}}} j^X_m [\psi_5] n^m_{\Sigma_r} \, dvol_{\Sigma_r} \, d\tilde{r}
\]
\[
+ \int_{\Sigma_{r_{\text{red}}}(v_{\text{red}}(u_{r_0}(v_s)), 2v_s)} j^X_m [\psi_5] n^m_{\Sigma_{r_{\text{red}}}} \, dvol_{\Sigma_{r_{\text{red}}}}
\] (4.92)
for a constant $B_1 = B_1(M, Q, l, \alpha, \Sigma_0)$. Similarly, after setting
\[
E(\bar{r}, \tilde{r}) := \int_{\Sigma_{\tilde{r}}(v_s, 2v_s)} j^X_m [\psi_5] n^m_{\Sigma_r} \, dvol_{\Sigma_r}
\] (4.93)
for $\bar{r} \in [r_0, r_{\text{red}}]$, we also have
\[
E(v_\bar{r}(u_{r_0}(v_s)), \tilde{r}) \leq \tilde{B}_1 \int_{\bar{r} \leq \tilde{r} \leq r_{\text{red}}} E(v_\bar{r}(u_{r_0}(v_s)), \tilde{r}) \, d\tilde{r} + E(v_{\text{red}}(u_{r_0}(v_s)), r_{\text{red}})
\] (4.94)
for a constant $\tilde{B}_1 = \tilde{B}_1(M, Q, l, \alpha, \Sigma_0)$. An application of Grönwall’s inequality yields
\[
E(v_\bar{r}(u_{r_0}(v_s)), \tilde{r}) \lesssim E(v_{\text{red}}(u_{r_0}(v_s)), r_{\text{red}}).
\] (4.95)
Note that $v_s - v_{\text{red}}(u_{r_0}(v_s)) = \text{const.}$ and hence by Proposition 4.15 we have
\[
E(v_{\text{red}}(u_{r_0}(v_s)), r_{\text{red}}) \lesssim_q E_1[\psi_5](0) \frac{1}{1 + v_s^q}.
\] (4.96)

Then, applying \eqref{4.95} and \eqref{4.96} for $\tilde{r} = r_0$ proves \eqref{4.90}. Finally, \eqref{4.91} is a consequence of the fact that $(v)^p \sim (u)^p$ (using $u_{\text{blue}} \leq r \leq r_{\text{red}}$) and the following well-known lemma.

Lemma 4.17. Let $f: [1, \infty) \to \mathbb{R}_{\geq 0}$ be continuous and assume that there exists a $q \in \mathbb{R}$, $q > 1$ such that $f(x)^{\frac{1}{q-1}} ds \leq \frac{1}{x}$ for all $x \geq 1$. Let $1 < p < q$ be fixed. Then, $\int_1^\infty s^p f(s) ds < \infty$.

Proof. Set $x_i := 2^i$. Then, $\int_1^\infty s^p f(s) ds = \sum_{i=0}^{\infty} \int_{x_i}^{x_{i+1}} s^p f(s) ds \leq 2^p \sum_{i=0}^{\infty} 2^{ip-iq} < \infty$.

Remark 4.18. From now on we will consider $p$ and $q$ as fixed and constants appearing in $\zeta$, $\zeta$ and $\sim$ can additionally depend on $1 < p < q$. 

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By doing the analogous analysis in the neighborhood of the left component of \( i^+ \) we obtain

**Proposition 4.19.** Let \( \psi_\flat \) defined as in (3.5). Then, for any \( r_0 \in [r_{\text{blue}}, r_+) \) we have

\[
\int_{\Sigma_{r_0}} \left( \langle v \rangle^p + \langle u \rangle^p \right) \left( |\tilde{\nabla}_u \psi_\flat|^2 + |\tilde{\nabla}_v \psi_\flat|^2 + |
\nabla \psi_\flat|^2 + |\psi_\flat|^2 \right) \, \text{dvol}_{\Sigma} \lesssim E_1[\psi_\flat](0) .
\] (4.97)

Commuting with angular momentum operators \( (W_i)_{1 \leq i \leq 3} \), an application of the Sobolev embedding \( H^2(S^2) \hookrightarrow L^\infty(S^2) \) and using the fact that \( p > 1 \), we also conclude

**Proposition 4.20.** Let \( \psi_\flat \) defined as in (3.5). Then,

\[
\sup_{B \cap \{ r_{\text{blue}} \leq r < r_+ \}} |\psi_\flat|^2 \lesssim E_1[\psi_\flat](0) + \sum_{i,j=1}^3 E_1[W_i W_j \psi_\flat](0) .
\] (4.98)

Finally, we will use the decay towards \( i^+ \) to show uniform boundedness in the interior and continuity all the way up to and including the Cauchy horizon for \( \psi_\flat \).

### 4.2.5 Blue-shift region

In the blue-shift region we shall use a different twisting function. Recall that we would like to have \( V \gtrsim 1 \), where

\[
V = - \left( \frac{\Box_g f}{f} + \frac{\alpha}{l^2} \right).
\] (4.99)

Now, we set \( f := e^r \) and obtain

\[
- \frac{\Box_g f}{f} = \frac{\Omega^2}{4} + \frac{1}{4} \partial_r (\Omega^2) + \frac{1}{2r} \Omega^2
\] (4.100)

Note that for \( r_{\text{blue}} > r_- \) close enough to \( r_- \), we have

\[
\partial_r \Omega^2 \geq c_{\text{blue}}
\] (4.101)

for all \( r_{\text{blue}} \geq r \geq r_- \) and some constant \( c_{\text{blue}} > 0 \) only depending on the black hole parameters. Thus, we obtain \( V \gtrsim 1 \) uniformly in the blue-shift region \( r_{\text{blue}} \geq r \geq r_- \) by choosing \( r_{\text{blue}} \) close enough to \( r_- \). In the blue-shift region we define the vector field

\[
S_N := r^N \langle u \rangle^p \partial_u + \langle v \rangle^p \partial_v
\] (4.102)

for some potentially large \( N > 0 \) and \( p > 1 \) as in Remark 4.18. We will show in the following that \( \sup_{u, \phi} |\psi_\flat(u_0, v_0, \theta, \varphi)| \) is uniformly bounded from initial data \( D[\psi_\flat] \) independently of \( (u_0, v_0) \in J^+(\Sigma r_{\text{blue}}) \cap B \). To do so, we will apply the energy identity (spacetime integral of \( (4.60) \)) in the region

\[
\mathcal{R}_f = \mathcal{R}_f(u_0, v_0) = J^+(\Sigma r_{\text{blue}}) \cap J^-(v_0, u_0) = J^+(\Sigma r_{\text{blue}}) \cap \{ u \leq u_0 \} \cap \{ v \leq v_0 \}
\] (4.103)

which we depict in Fig. [5]
Proof. The general strategy of the proof is to apply (4.104) and to show that for any $\psi$ and $\beta$ we can control $R$ by initial data. This leads to

\[ \int_{\mathcal{C}_{u_0} (u_{\text{r.h.}} (u_0) , u_0)} \tilde{J}^S \mu [\psi] n^\mu_\Sigma_{u_0} \, \text{dvol}_{\Sigma_{u_0}} + \int_{\mathcal{L}_{u_0} (u_{\text{r.h.}} (u_0), u_0)} \tilde{J}^S \nu [\psi] n^\nu_\Sigma_{u_0} \, \text{dvol}_{\Sigma_{u_0}} + \int_{\mathcal{R}_f} \tilde{K}^S \mu [\psi] \, \text{dvol} \]

where $\psi$ is defined in (3.5). In the following we will show, that after choosing $N > 0$ large enough and an appropriate integration by parts to control error terms, we can control the flux terms by initial data. This gives

**Proposition 4.21.** Let $\psi$ defined as in (3.5). Then,

\[
\int_{\mathcal{C}_{u_0} (u_{\text{r.h.}} (u_0) , u_0)} \tilde{J}^S \mu [\psi] n^\mu_\Sigma_{u_0} \, \text{dvol}_{\Sigma_{u_0}} + \int_{\mathcal{L}_{u_0} (u_{\text{r.h.}} (u_0), u_0)} \tilde{J}^S \nu [\psi] n^\nu_\Sigma_{u_0} \, \text{dvol}_{\Sigma_{u_0}} \\
\lesssim \int_{\Sigma_{\text{r.h.}} \cap J^- (u_0, u_0)} \tilde{J}^S \mu [\psi] n^\mu_{\Sigma_{\text{r.h.}}} \, \text{dvol}_{\Sigma_{\text{r.h.}}} \lesssim E_1 [\psi](0) \quad (4.105)
\]

and

\[
\int_{\mathcal{C}_{u_0} (u_{\text{r.h.}} (u_0) , v_0)} \frac{1}{2} \left( \langle v \rangle^2 | \partial v \psi |^2 + (| \nabla \psi |^2 + | \psi |^2) \Omega^2 \right) \, d\sigma_{\Sigma^2} \\
+ \int_{\mathcal{L}_{v_0} (u_{\text{r.h.}} (v_0), v_0)} \frac{1}{2} \left( \langle v \rangle^2 | \partial v \psi |^2 + (| \nabla \psi |^2 + | \psi |^2) \Omega^2 \right) \, d\sigma_{\Sigma^2} \\
\lesssim \int_{\Sigma_{\text{r.h.}} \cap J^- (u_0, u_0)} \tilde{J}^S \mu [\psi] n^\mu_{\Sigma_{\text{r.h.}}} \, \text{dvol}_{\Sigma_{\text{r.h.}}} \lesssim E_1 [\psi](0) \quad (4.106)
\]

for any $(u_0, v_0) \in J^+ (\Sigma_{\text{r.h.}})$. Commuting with the angular momentum operators $(W_i)_{1 \leq i \leq 3}$ also gives

\[
\int_{\mathcal{C}_{u_0} (u_{\text{r.h.}} (u_0) , v_0)} \langle v \rangle \langle \partial v \psi \rangle^2 + \sum_{i,j} | \partial v W_i W_j \psi |^2 \, d\sigma_{\Sigma^2} \lesssim E_1 [\psi](0) + \sum_{i,j=1}^{3} E_1 [W_i W_j \psi](0). \quad (4.107)
\]

**Proof.** The general strategy of the proof is to apply (4.104) and to show that

\[
\int_{\mathcal{R}_f} \tilde{K}^S \mu \, d\sigma \geq 0 + \text{boundary terms}, \quad (4.108)
\]

Figure 6: Illustration of the region $\mathcal{R}_f$ as the darker shaded region in the Penrose diagram of the interior $B$. The lighter shaded region is the blue-shift region.
where the boundary terms are small (lower orders in $\Omega$) and by choosing $r_{\text{blue}}$ closer to $r_-$, can be absorbed in the positive flux terms on the left hand side of (4.104). In the first part, we compute the flux terms for our vector field $S^N$ defined in (4.102). Then, in the second part, we will estimate the bulk term and indeed show (4.108). From this we will then deduce (4.105).

**Part I: Flux terms of $S_N$.** We obtain three flux terms from (4.104). The future flux terms read (cf. Proposition A.1)

\[
\int_{\mathcal{C}_u(\nu_{\text{blue}}(u_0),v_0)} \tilde{J}_\mu^S |\psi|^n \nu_{\mathcal{C}_u(\nu_{\text{blue}}(u_0),v_0)} \, d\nu_{\mathcal{C}_u(\nu_{\text{blue}}(u_0),v_0)} = \int_{\mathcal{C}_u(\nu_{\text{blue}}(u_0),v_0)} \left( \langle v \rangle^p |\nabla_v \psi|_\mu^2 + \Omega^2 \left( \frac{\langle u \rangle^p}{4} (|\nabla_v \psi|^2 + |\mathcal{V} \psi|^2) \right) \right) r^{q+2} d\nu d\sigma_{\mathbb{S}^2} \tag{4.109}
\]

and

\[
\int_{\mathcal{C}_u(\nu_{\text{blue}}(v_0),u_0)} \tilde{J}_\mu^S |\psi|^n \nu_{\mathcal{C}_u(\nu_{\text{blue}}(v_0),u_0)} \, d\nu_{\mathcal{C}_u(\nu_{\text{blue}}(v_0),u_0)} = \int_{\mathcal{C}_u(\nu_{\text{blue}}(v_0),u_0)} \left( \langle u \rangle^p |\nabla_u \psi|_\mu^2 + \Omega^2 \left( \frac{\langle v \rangle^p}{4} (|\nabla_u \psi|^2 + |\mathcal{V} \psi|^2) \right) \right) r^{q+2} d\nu d\sigma_{\mathbb{S}^2}. \tag{4.110}
\]

The past flux term on the spacelike hypersurface $\Sigma_{r_{\text{blue}}}$ is uniformly bounded by initial data from Proposition A.19

\[
\int_{\Sigma_{r_{\text{blue}}} \cap J^- (v_0,u_0)} \tilde{J}_\mu^S |\psi|^n \nu_{\Sigma_{r_{\text{blue}}}} \, d\nu_{\Sigma_{r_{\text{blue}}}} \lesssim E_1[\psi](0). \tag{4.111}
\]

**Part II: Bulk term of $S_N$.** We will now estimate the bulk term

\[
\int_{\mathcal{R}_f} \tilde{K}_S^N d\nu
\]

appearing in the energy identity (4.104). The terms appearing in $\tilde{K}^S_N$ can be read of in (A.4) with $S^N = X^u = r^N \langle v \rangle^p$ and $S_N^u = X^v = r^N \langle u \rangle^p$. To estimate all terms, we will also integrate by parts and substitute terms of the form $\partial_\nu \partial_v \psi$ using the equation $\Box_y \psi = 0$. The boundary terms arising from the integration by parts will then be absorbed in the future flux terms appearing in Part I: Flux terms of $S_N$. In the following we shall treat each terms of $\tilde{K}^S_N$ as in (A.4) with $X = S_N$ individually.

**First term of (A.4).** The first term of (A.4) is non-negative:

\[
- \frac{2}{\Omega^2} \left( \langle v \rangle^p \partial_u (r^p) |\nabla_v \psi|_\mu^2 + \langle u \rangle^p \partial_v (r^N) |\nabla_u \psi|_\mu^2 \right) = Nr^{N-1} \langle (v)^p |\nabla_v \psi|_\mu^2 + \langle u \rangle^p |\nabla_u \psi|_\mu^2 \rangle. \tag{4.112}
\]

This means that—by choosing $N > 0$ large enough—we will be able to absorb sign-indefinite terms of the form $r^{N-1} \langle (v)^p |\nabla_v \psi|_\mu^2$ and $r^{N-1} \langle u \rangle^p |\nabla_u \psi|_\mu^2$. This will be used in the following.

Before we treat the second term appearing in (A.4), which is sign-indefinite, we look at the angular and potential term in the second line of (A.4).

**Angular and potential term: Second line of (A.4).** Now, we look at the term involving angular derivatives. In the region $\mathcal{R}_f$ we have

\[
- \left( \frac{1}{2} \partial_v (r^N \langle v \rangle^p) + \partial_u (r^N \langle u \rangle^p) - \frac{r^N}{4} (\partial_\nu \Omega^2) \left( \langle v \rangle^p + \langle u \rangle^p \right) \right) (|\nabla \psi|^2 + |\mathcal{V} \psi|^2) \gtrsim r^N \left( \langle v \rangle^p + \langle u \rangle^p \right) (|\nabla \psi|^2 + |\mathcal{V} \psi|^2). \tag{4.113}
\]

Also recall that we have chosen the twisting function such that $\mathcal{V} \gtrsim 1$.

**Second, sign-indefinite term of (A.4).** Now, note that the second term in the first line of (A.4)

\[
-2r^{N-1} \langle (v)^p + \langle u \rangle^p \rangle \text{Re} \left( \nabla_v \psi \nabla_u \psi \right) \tag{4.114}
\]
is sign-indeterminate, however, we can absorb it in other positive terms after integrating by parts in the region \( \mathcal{R}_f \) as we will see in the following. In order to integrate by parts, it is useful to express the twisted derivatives with ordinary derivatives. The integration by parts will generate boundary terms. As mentioned above, we estimate these boundary terms with the fluxes in the energy identity. This will be done later in \( (4.119) \) and we will not write the boundary terms explicitly in the following. We will also have to control (sign-indeterminate) ordinary derivatives by positive terms in \( (4.112) \) and \( (4.113) \). Note that this is possible since

\[
\langle u \rangle^p |\partial_u \psi_b|^2 = \langle u \rangle^p |\nabla_v \psi|^2 + \langle u \rangle^p \Omega^2 \Re \left( \overline{\psi_b \partial_v \psi_b} \right) - \frac{1}{4} \langle u \rangle^p \Omega^4 |\psi|^2,
\]

where the right hand side of \( (4.115) \) is controlled by \( (4.112), (4.113) \) and potentially choosing \( r_{\text{blue}} \) closer to \( r_- \). The analogous statement holds true for \( \langle u \rangle^p |\partial_u \psi_b|^2 \).

The integrated term we have to estimate reads

\[
\int_{\mathcal{R}_f} -2r^{-1} \langle u \rangle^p + \langle u \rangle^p \frac{1}{f^2} \Re \left( \overline{\psi_b (f \psi_b) \partial_u (f \psi_b)} \right) \Omega^2 r^2 dudvds^2.
\]

We first look at

\[
\left| \int_{\mathcal{R}_f} r^{N+1} \langle u \rangle^p \frac{1}{f^2} \Re \left( \overline{\psi_b (f \psi_b) \partial_u (f \psi_b)} \right) \Omega^2 dudvds^2 \right|.
\]

Using the explicit form of \( f \) and noting that we have control over \( \langle u \rangle^p \Omega^4 |\psi|^2 \) from \( (4.113) \), it suffices to estimate

\[
\left| \int_{\mathcal{R}_f} r^{N+1} \langle u \rangle^p \Omega^2 dudvds^2 \right| + \left| \int_{\mathcal{R}_f} \Omega^2 \langle u \rangle^p \Re \left( \overline{\psi_b (\partial_u \psi_b)} \right) \Omega^2 dudvds^2 \right| + \left| \int_{\mathcal{R}_f} \Omega^2 \langle u \rangle^p \Re \left( \overline{\psi_b (\partial_u \psi_b)} \right) \Omega^2 dudvds^2 \right|.
\]

Now, note that the second term from \( (4.116) \) is controlled by \( (4.112) \) and \( (4.113) \) using Cauchy’s inequality. Now, in both terms, the first and third term of \( (4.116) \), we integrate by parts in \( u \). We also use \( \Re \left( \overline{\psi_b \partial_u \psi_b} \right) = \frac{1}{2} \partial_u (|\psi|^2) \). Then, it follows that—up to boundary contributions which will be dealt with below in \( (4.119) \)—we have to control the terms

\[
\left| \int_{\mathcal{R}_f} N r^N \langle u \rangle^p \Re \left( \overline{\psi_b \partial_v \psi_b} \right) \Omega^2 dudvds^2 \right| + \left| \int_{\mathcal{R}_f} r^{N+1} \langle u \rangle^p \Re \left( \overline{\psi_b (\partial_u \psi_b)} \right) \Omega^4 dudvds^2 \right| + \left| \int_{\mathcal{R}_f} \langle u \rangle^p |\psi|^2 \Omega^4 dudvds^2 \right|.
\]

The first and third term of \( (4.117) \) are controlled by \( (4.112) \) and \( (4.113) \) and by potentially choosing \( r_{\text{blue}} \) even closer to \( r_- \). For the second term of \( (4.117) \) we will use \( (1.3) \) which reads

\[
0 = \Box_{\text{gruNAS}} \psi + \frac{\alpha}{2} \psi = -\frac{4}{\Omega^2} (\partial_u \partial_v \psi) + \frac{2}{r} (\partial_u \psi \partial_v \psi) + \frac{1}{r^2} \Delta_{S^2} \psi + \frac{\alpha}{2} \psi
\]

to substitute \( \partial_u \partial_v \psi_b \). Replacing \( \partial_u \partial_v \psi_b \) and integrating by parts on the sphere, we estimate all but one term of \( (4.117) \) using \( (4.113) \) and \( (4.112) \). The term which we cannot control with \( (4.113) \) and \( (4.112) \) is of the form

\[
\left| \int_{\mathcal{R}_f} r^N \langle u \rangle^p \Re \left( \overline{\psi_b (\partial_u \psi_b)} \right) \Omega^6 dudvds^2 \right| = \frac{1}{2} \left| \int_{\mathcal{R}_f} r^{N+1} \langle u \rangle^p |\psi_b|^2 \Omega^6 dudvds^2 \right|.
\]

This is of a similar form as the third term in \( (4.116) \), which we control—as before—via an integration by parts in \( u \). Finally we have controlled all terms except for boundary terms arising from the integration by parts.

The first boundary terms arose from integrating by parts the first term in \( (4.116) \). It is of the form

\[
\left| \int_{\mathcal{C}_{\alpha} \cap \{ v_{\text{blue}} (u_0) \leq v \leq v_0 \}} r^{N+1} \langle u \rangle^p \Re \left( \overline{\psi_b (\partial_u \psi_b)} \right) \Omega^2 dudvds^2 \right|.
\]

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which we control as
\[
\left| \int_{C_{u_0} \cap \{ v_{\text{blue}}(u_0) \leq v \leq v_0 \}} r^{N+1} \langle v \rangle^p \Re (\overline{\psi}_3 (\partial_v \psi_3)) \frac{1}{\Omega^2} dv d\sigma_{S^2} \right|
\]
\[
\leq \left( \int_{C_{u_0} \cap \{ v_{\text{blue}}(u_0) \leq v \leq v_0 \}} r^{N+1} \langle v \rangle^p \left| \partial_v \psi_3 \right|^2 \sqrt{\Omega^2} dv d\sigma_{S^2} \right)
\]
\[
+ \int_{C_{u_0} \cap \{ v_{\text{blue}}(u_0) \leq v \leq v_0 \}} r^{N+1} \langle v \rangle^p \left| \psi_3 \right|^2 \left( \Omega^2 \right)^{\frac{3}{2}} \Omega^2 dv d\sigma_{S^2}. \tag{4.120}
\]

Now, note that
\[
\langle v \rangle^p \left( \Omega^2 \right)^{\frac{3}{2}} \leq \langle r_\ast - u \rangle^p \left( \Omega^2 \right)^{\frac{3}{2}} \leq 1 + \langle u \rangle^p \left( \Omega^2 \right)^{\frac{3}{2}}, \tag{4.121}
\]
where we have used that \( r_\ast^p \left( \Omega^2 \right)^{\frac{3}{2}} \lesssim 1 \) for \( r_\ast \geq r_\ast (r_{\text{blue}}) \). Using \( 4.121 \) we absorb \( 4.120 \) in the flux term \( 4.109 \) by potentially choosing \( r_{\text{blue}} \) closer to \( r_- \) such that \( \Omega^2 \) is uniformly small in the blue-shift region. Completely analogously, we control the other boundary terms which arose from integrating by parts.

Now, we are left with the terms of the last two lines in \( A.4 \).

Terms from last two lines of \( A.4 \). We will only look at the terms with \( v \) weights as the terms involving \( u \) weights are estimated completely analogously. It suffices to estimate the terms
\[
r^N \frac{\Omega^2}{2r} \langle v \rangle^p \left| \psi_3 \right|^2 + r^N \langle v \rangle^p \left| \partial_v \left( \frac{f^2 v^2}{2f^2} \right) \right|^2 \langle \overline{\psi}_3 \rangle + r^N \langle v \rangle^p \left| \partial_v \left( \frac{f^2 v^2}{2f^2} \right) \overline{\psi}_3 \right|^2.
\]
Since \( \left| \partial_v \left( \frac{f^2 v^2}{2f^2} \right) \right| \lesssim \Omega^2 \), we control the first and second term of \( 4.122 \) using \( 4.113 \) by potentially choosing \( r_{\text{blue}} \) closer to \( r_- \). The last term of \( 4.122 \) is be estimated as
\[
r^N \langle v \rangle^p \left| \partial_v \left( \frac{f^2 v^2}{2f^2} \right) \overline{\psi}_3 \right|^2 \lesssim r^N \langle v \rangle^p \left| \overline{\nabla}_u \psi_3 \right|^2 + r^N \langle v \rangle^p \Omega^2 \left| \overline{\nabla} \psi_3 \right|^2. \tag{4.123}
\]

Finally, we have estimated and absorbed all sign-indefinite terms in the energy identity to obtain \( 4.108 \). Thus, we have proved \( 4.105 \), which concludes the first part of the proof.

Part III: Proof of \( 4.106 \) and \( 4.107 \). Now, observe that the estimate \( 4.106 \) follows from \( 4.105 \) and \( 4.115 \). More precisely, the error arising from interchanging the twisted derivatives with partial derivatives on \( C_u \) are estimated as
\[
\left| \langle v \rangle^p \partial_v | \partial_v |^2 \right| = \langle v \rangle^p \left| \overline{\nabla}_v \psi_3 \right|^2 + \langle v \rangle^p \Omega^2 \left( \Re (\overline{\psi}_3 \partial_v \psi_3) - \frac{1}{4} \langle v \rangle^p \Omega^2 \left| \psi_3 \right|^2 \right)
\]
\[
\leq \langle v \rangle^p \left| \overline{\nabla}_v \psi_3 \right|^2 + \langle v \rangle^p \Omega^2 \left( \Re (\overline{\psi}_3 \partial_v | \psi_3 |) \right).
\]
Finally, note that the error term on the right hand side is controlled as in \( 4.119 \). This works for \( C_u \), completely analogously which concludes the proof.

4.2.6 Uniform boundedness and continuity at the Cauchy horizon for bounded frequencies

Now, Proposition \( 4.21 \) allows us to prove the uniform boundedness.

**Proposition 4.22.** Let \( \psi_3 \) be as defined in \( 3.5 \). Then,
\[
\sup_{B \in J^+ (S_0)} \left| \psi_3 \right|^2 \lesssim E_1 [\psi_3] (0) + \sum_{i,j=1}^3 E_1 [W_i W_j] \psi_3 (0) \lesssim D [\psi_3]. \tag{4.124}
\]

**Proof.** In view of Proposition \( 4.20 \) it suffices to prove \( 4.124 \) only in \( J^+ (S_{\text{blue}}) \cap B \). Let \( (u_0, v_0) \in J^+ (S_{\text{blue}}) \cap B \) be arbitrary. Then, by Proposition \( 4.20 \), Proposition \( 4.21 \) and the Sobolev embedding on the sphere
Similarly to (4.125) we have

\[ |\psi_b(u_0, v_0, \varphi, \theta)|^2 \leq \left( \int_{\mathcal{C}_{u_0}(v_{\text{blue}}(u_0), v_0)} |\partial_\varphi \psi_b(u_0, v, \varphi, \theta)| \, dv \right)^2 + |\psi_b(u_0, v_{\text{blue}}(u_0), \varphi, \theta)|^2 \]

\[ \leq \int_{\mathcal{C}_{u_0}(v_{\text{blue}}(u_0), v_0)} \langle v \rangle^p |\partial_\varphi \psi_b(u_0, v, \varphi, \theta)|^2 \, dv \, d\mathcal{S}_2 + \sum_{i,j} \int_{\mathcal{C}_{u_0}(v_{\text{blue}}(u_0), v_0)} \langle v \rangle^p |\partial_v \mathcal{W}_j \psi_b|^2 \, dv \, d\mathcal{S}_2 \]

\[ + E_1[\psi_b](0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \psi_b] \leq E_1[\psi_b](0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \psi_b](0), \quad (4.125) \]

where \((\mathcal{W}_i)_{i=1,2,3}\) are the angular momentum operators. This shows (4.124).

**Proposition 4.23.** Let \(\psi_b\) be as defined in (3.5). Then, \(\psi_b\) is continuously extendible beyond the Cauchy horizon \(\partial H\).

**Proof.** Similarly to (4.125) we have

\[ |\psi_b(u_0, v_2, \varphi, \theta) - \psi_b(u_0, v_1, \varphi, \theta)|^2 \leq \int_{v_1}^{v_2} \langle v \rangle^{-p} \, dv \int_{\mathcal{C}_{u_0}(v_1, v_2)} \langle v \rangle^p |\partial_\varphi \psi_b(u_0, v, \varphi, \theta)|^2 \, dv \]

\[ \leq \int_{v_1}^{v_2} \langle v \rangle^{-p} \, dv \left( E_1[\psi_b] + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \psi_b] \right) \quad (4.126) \]

uniformly in \(u_0, \varphi, \theta\). The same estimate holds after interchanging the roles of \(u\) and \(v\). After commuting the equation with \(\mathcal{W}_3\), we have from (4.124)

\[ \sup_B |\partial_\varphi \psi|^2 \leq E_1[\partial_\varphi \psi_b](0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \partial_\varphi \psi_b](0) < \tilde{C} < \infty \quad (4.127) \]

for some constant \(\tilde{C} < \infty\) depending on the initial data. (Recall that we assumed our initial data to be smooth and compactly supported.) Thus, for \(\varphi_1 \leq \varphi_2\), we have

\[ |\psi_b(u_0, v_0, \varphi_2, \theta) - \psi_b(u_0, v_0, \varphi_1, \theta)|^2 \leq \int_{\varphi_1}^{\varphi_2} \sup_B |\partial_\varphi \psi_b| \leq \tilde{C} |\varphi_2 - \varphi_1| \quad (4.128) \]

uniformly in \(u_0, v_0, \theta_0\). A similar estimate holds true for \(\theta\). Applications of the fundamental theorem of calculus and a triangle inequality finally yield the continuity result for \(\psi_b\). \(\square\)

## 5 High frequency part \(\psi_b^+\)

In the previous section we have shown the uniform boundedness for the low frequency part \(\psi_b\). Now, we turn to \(\psi_b^+\), the high frequency part. The key ingredient for the proof of the uniform boundedness for \(|\psi_b^+|\) is the uniform boundedness of transmission and reflection coefficients associated to the radial o.d.e. (4.13) which is proved in [42] for \(\Lambda = 0\), together with (b) the finiteness of the (commuted) T-energy flux on the event horizon given by (2.21).

Now, recall the radial o.d.e. (4.13) which reads \(-u'' + V_T u = \omega^2 u\) in the interior, where \(V_T\) decays exponentially as \(r_s \to +\infty\) \((r \to r_-)\) and \(r_s \to -\infty\) \((r \to r_+)\). For \(\omega \neq 0\), so in particular for \(|\omega| > \frac{M}{r_+}\), the radial o.d.e. admits the following pairs of mode solutions \((u_1, u_2)\) and \((v_1, v_2)\), where \(u_1\) and \(u_2\) are solutions to (4.13) satisfying \(u_1 = e^{i\omega r_*} + O_T(r - r_+)^\alpha\) and \(u_2 = e^{-i\omega r_*} + O_T(r - r_+)^\alpha\) as \(r_s \to -\infty\). Similarly, \(v_1\) and \(v_2\) satisfy \(v_1 = e^{i\omega r_*} + O_T(r - r_-)^\alpha\) and \(v_2 = e^{-i\omega r_*} + O_T(r - r_-)^\alpha\) as \(r_s \to +\infty\). Now, for \(\omega \neq 0\), the transmission and reflection coefficients \(\Xi(\omega, \ell)\) and \(\mathcal{R}(\omega, \ell)\) are defined as the unique coefficients satisfying

\[ u_1 = \Xi(\omega, \ell) v_1 + \mathcal{R}(\omega, \ell) v_2. \quad (5.1) \]
See [42] for more details. In the following we will state the uniform boundedness of \( \mathfrak{T}(\omega, \ell) \) and \( \Re(\omega, \ell) \) for \( |\omega| \geq \frac{\omega_0}{2} \). In [42] Proposition 4.7, Proposition 4.8 this has been proven for \( \Lambda = 0 \). However, the proof of Proposition 4.7 and Proposition 4.8 in [42] also applies if we include a non-vanishing cosmological constant \( \Lambda \).

**Lemma 5.1** ([42], Proposition 4.7, Proposition 4.8]). Fix subextremal Reissner–Nordström–AdS black hole parameters \((M, Q, l)\), a constant \( \omega_0 > 0 \) and a Klein–Gordon mass parameter \( \alpha < \frac{2}{\omega} \). Then, the scattering coefficients \( \mathfrak{T}(\omega, \ell) \) and \( \Re(\omega, \ell) \) as defined above satisfy

\[
\sup_{|\omega| \geq \frac{\omega_0}{2}, \ell \in \mathbb{N}_0} \left( |\mathfrak{T}(\omega, \ell)| + |\Re(\omega, \ell)| \right) \lesssim_{M, Q, l, \omega_0, \alpha} 1
\]  

(5.2)

and the mode solutions \( u_1, u_2 \) and \( v_1, v_2 \) are uniformly bounded

\[
\sup_{|\omega| \geq \frac{\omega_0}{2}, \ell \in \mathbb{N}_0} \|u_1\|_{L^\infty(\mathbb{R})} \lesssim_{M, Q, l, \omega_0, \alpha} 1, \quad \sup_{|\omega| \geq \frac{\omega_0}{2}, \ell \in \mathbb{N}_0} \|u_2\|_{L^\infty(\mathbb{R})} \lesssim_{M, Q, l, \omega_0, \alpha} 1, \quad \sup_{|\omega| \geq \frac{\omega_0}{2}, \ell \in \mathbb{N}_0} \|v_1\|_{L^\infty(\mathbb{R})} \lesssim_{M, Q, l, \omega_0, \alpha} 1, \quad \sup_{|\omega| \geq \frac{\omega_0}{2}, \ell \in \mathbb{N}_0} \|v_2\|_{L^\infty(\mathbb{R})} \lesssim_{M, Q, l, \omega_0, \alpha} 1.
\]

(5.3) (5.4)

Another result which we will use from [42] is the representation formula for \( \psi_2 \) in the separated picture. It is essential that \( |\omega| \geq \frac{\omega_0}{2} \) to apply the same steps as in [42] Proof of Proposition 5.1.

**Lemma 5.2** ([42], Proof of Proposition 5.1]). Let \( \psi_2 \) as in (3.5). Then, we have

\[
\psi_2(\ell, r, \varphi, \theta) = \frac{1}{\sqrt{2\pi r}} \sum_{\ell \in \mathbb{N}_0} \sum_{|m| \leq \ell} Y_{\ell m}(\theta, \varphi) \int_{|\omega| \geq \frac{\omega_0}{2}} \mathcal{F}_{H_A^+}[\psi_2](\omega, m, \ell) u_1(\omega, \ell, r)e^{i\omega t} d\omega \\
+ \frac{1}{\sqrt{2\pi r}} \sum_{\ell \in \mathbb{N}_0} \sum_{|m| \leq \ell} Y_{\ell m}(\theta, \varphi) \int_{|\omega| \geq \frac{\omega_0}{2}} \mathcal{F}_{H_B^+}[\psi_2](\omega, m, \ell) u_2(\omega, \ell, r)e^{i\omega t} d\omega,
\]

(5.5)

where

\[
\mathcal{F}_{H_A^+}[\phi](\omega, m, \ell) := \frac{r_r}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega v} \langle \phi, Y_{\ell m}\rangle y dv
\]

and

\[
\mathcal{F}_{H_B^+}[\phi](\omega, m, \ell) := \frac{r_r}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega u} \langle \phi, Y_{\ell m}\rangle y du.
\]

(5.6) (5.7)

**Proof of Lemma 5.2.** This proof is very similar to [42], Proof of Proposition 5.1] so we will be rather brief.

Let \( \psi_2 \) as in (3.5). Since the expansion in spherical harmonics converges pointwise, it suffices to prove (5.5) for \( \psi_2^{\ell m} := \langle \psi_2, Y_{\ell m}\rangle y Y_{\ell m} \) for fixed \( m, \ell \). Now, define \( u[\psi_2^{\ell m}] \) as in (4.2) such that

\[
\psi_2^{\ell m} = \frac{1}{\sqrt{2\pi r}} Y_{\ell m} \int_{|\omega| \geq \frac{\omega_0}{2}} u[\psi_2^{\ell m}] e^{i\omega t} d\omega.
\]

(5.8)

This is well-defined in the interior in view of Proposition 4.2. Moreover, \( u[\psi_2^{\ell m}] \) solves the radial o.d.e. and can be expanded in the basis \( u_1 \) and \( u_2 \) (\( |\omega| > \frac{\omega_0}{2} \)):

\[
u[\psi_2^{\ell m}](r_*, \omega, m, \ell) = a(\omega, m, \ell)u_1(r_*, \ell, \omega) + b(\omega, m, \ell)u_2(r_*, \ell, \omega).
\]

(5.9)

Now, first note Proposition A.5 implies that \( \omega \mapsto u[\psi_2^{\ell m}](r, \omega) \) is a Schwartz function for \( r \in (r_-, r_+) \). Since

\[
|a(\omega, m, \ell)| = \frac{\mathcal{M}(u[\psi_2^{\ell m}], u_2)}{\mathcal{M}(u_1, u_2)} = \left| \frac{\mathcal{M}(u[\psi_2^{\ell m}], u_2)}{2\omega} \right| \lesssim \mathcal{M}(u[\psi_2^{\ell m}], u_2)
\]

(5.10)

\footnote{Note that for \( \Lambda \neq 0 \) the scattering coefficients \( \mathfrak{T} \) and \( \mathfrak{S} \) have a pole at \( \omega = 0 \). However, for frequencies bounded away from \( \omega = 0 \), so in particular for \( |\omega| \geq \frac{\omega_0}{2} \) as in the present case, \( \mathfrak{T} \) and \( \mathfrak{S} \) are uniformly bounded for both cases \( \Lambda = 0 \) and \( \Lambda \neq 0 \). See [42] for more details.}
in view of $|\omega| \geq \frac{2 \pi}{m}$, we conclude that $\omega \mapsto a(\omega, m, \ell)$ is in $L^1(\mathbb{R})$ for fixed $\ell, m$. Recall that the Wronskian $\mathcal{W}(f, g) := fg' - fg'$ is independent of $r_*$ for two solutions of the radial o.d.e. (4.3). We have also used that $\|u_2\|_{L^\infty} \lesssim 1$ and $\|u_2'\|_{L^\infty} \lesssim 1 + |\omega|$ for $|\omega| \geq \frac{2 \pi}{m}$ (cf. [42 Proposition 4.7 and Proposition 4.8]). Similarly, we have that $\omega \mapsto b(\omega, m, \ell)$ is in $L^1(\mathbb{R})$. Using

$$\psi_{\ell m} = Y_{\ell m} \frac{1}{\sqrt{2\pi}r} \int_{|\omega| \geq \frac{2 \pi}{m}} (a(\omega, m) u_1(r, \omega, \ell) + b(\omega, m, \ell) u_2(r, \omega, \ell)) e^{i\omega t} d\omega$$

(5.11)

and a direct adaptation of [42 Proof of Proposition 5.1] finally shows $a(\omega, m, \ell) = F_{\chi_{\frac{10}{m}}} [\psi_{\ell m} |_{\mathcal{H}_\phi^+}] (\omega, m, \ell)$, $b(\omega, m, \ell) = F_{\chi_{\frac{1}{m}}} [\psi_{\ell m} |_{\mathcal{H}_\phi^+}] (\omega, m, \ell)$. This shows the representation formula (5.5) for $\psi_{\ell}$.

We will now prove the uniform boundedness for $\psi_{\ell}$.

**Proposition 5.3.** Let $\psi_{\ell}$ be as defined in (5.5). Then,

$$\sup_{B \cap J^{-1}(S_0)} |\psi_{\ell}|^2 \lesssim E_1[\psi_{\ell}] (0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \psi_{\ell}] (0) \lesssim D[\psi_{\ell}].$$

(5.12)

**Proof.** We start with the representation of $\psi_{\ell}$ as in (5.5). For convenience, we will only estimate the term involving $F_{\chi_{\frac{10}{m}}} [\phi] (\omega, m, \ell)$ and assume without loss of generality that $F_{\chi_{\frac{1}{m}}} [\phi] (\omega, m, \ell) = 0$. Indeed, the term $F_{\chi_{\frac{1}{m}}} [\phi] (\omega, m, \ell)$ can be treated analogously. Now, in view of (5.3), we conclude

$$|\psi_{\ell}(r, t, \varphi, \theta)|^2 \lesssim \left| \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}, |m| \leq \ell} Y_{\ell m}(\varphi, \theta) \int_{|\omega| \geq \omega_0} F_{\mathcal{H}_\phi^+ \chi_{\frac{1}{m}}} \left[ \psi_{\ell m} \right] (\omega, m, \ell) d\omega \right|^2$$

$$\leq \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}, |m| \leq \ell} \frac{Y_{\ell m}(\varphi, \theta)^2}{(1 + \ell)^3} \frac{1}{|\omega|^3} \int_{|\omega| \geq \omega_0} \frac{1}{\omega} d\omega$$

$$\lesssim \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}, |m| \leq \ell} \int_{|\omega| \geq \omega_0} (1 + \ell)^3 \omega^2 \left| F_{\mathcal{H}_\phi^+ \chi_{\frac{1}{m}}} \left[ \psi_{\ell m} \right] (\omega, m, \ell) \right|^2 d\omega$$

$$\lesssim \int_{\mathcal{H}_\phi^+} |T \psi_{\ell}|^2 d\nu d\sigma_{S_2} + \sum_{i,j=1}^3 \int_{\mathcal{H}_\phi^+} |T \mathcal{W}_i \mathcal{W}_j \psi_{\ell}|^2 d\nu d\sigma_{S_2}. \tag{5.13}$$

Here, we have used that

$$\sum_{m=-\ell}^\ell |Y_{\ell m}(\varphi, \theta)|^2 = \frac{2\ell + 1}{4\pi} \tag{5.14}$$

which is known as Unsöld’s Theorem [60, Eq. (69)].

Finally, on the right hand side of (5.13) we only see the commuted $T$-energy flux. An application of the $T$-energy identity in the exterior and an energy estimate in a compact spacetime region shows that the commuted $T$-energy flux on the event horizon is controlled from the initial data (cf. (2.21) in Theorem 2.1). Thus, in view of (5.13) we conclude

$$|\psi_{\ell}(r, t, \varphi, \theta)|^2 \lesssim E_1[\psi_{\ell}] (0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i \mathcal{W}_j \psi_{\ell}] (0). \tag{5.15}$$

10More precisely, following the lines starting from equation (5.20) in [42 Proof of Propostion 5.1] which contain an application of Lebesgue’s dominated convergence, the Riemann–Lebesgue lemma and the inverse Fourier transform yields the result.
**Proposition 5.4.** Let \( \psi_2 \) be as defined in (3.3). Then, \( \psi_2 \) is continuously extendible across the Cauchy horizon \( CH \).

**Proof.** Let \((u_n,v_n,\theta_n,\varphi_n) \rightarrow (\hat{u},\hat{v},\hat{\theta},\hat{\varphi})\) be a convergent sequence. We will also allow \( \hat{u} = +\infty \) and \( \hat{v} = +\infty \) as limits which correspond to limits to the Cauchy horizon. We represent \( \psi_2 \) again as in (5.3). Similar to the proof of Proposition 5.3, it is enough to consider the case where \( F_{H^+_{B}}[\psi_2 \mid H^+_{B}] \) vanishes. Hence,

\[
\psi_2(t, r, \varphi, \theta) = \frac{1}{\sqrt{2\pi}} \int_{|\omega| \geq \frac{\omega}{r}} F_{H^+_{A}} \left[ \psi_2 \mid H^+_{A} \right] \left( m, \ell, \omega, u_1(\omega, \ell, r) \right)e^{i\omega t}d\omega. \tag{5.16}
\]

First from (5.14) we have \( \sup_{\varphi, \theta} |Y_{\ell m}(\varphi, \theta)| \lesssim 1 + \ell \) and from (5.3) we have that

\[
\sup_{u,v} \left| u_1 e^{i\omega t(u,v)} \right| = \sup_{t, r} \left| u_1 e^{i\omega t} \right| \lesssim 1.
\]

Then, a similar estimate as in (5.13) and an application of Lebesgue’s dominated convergence theorem allow us to interchange the limit \( n \rightarrow \infty \) with the sum \( \sum_{\ell \in \mathbb{N}_0} \sum_{|m| \leq \ell} \). Since \( Y_{\ell m}(\theta_n, \varphi_n) \rightarrow Y_{\ell m}(\hat{\theta}, \hat{\varphi}) \) pointwise as \( n \rightarrow \infty \), it remains to show that

\[
\left| \int_{|\omega| \geq \frac{\omega}{r}} F_{H^+_{A}} \left[ \psi_2 \mid H^+_{A} \right] \left( m, \ell, \omega, u_1(\omega, \ell, r) \right)e^{i\omega t(u_n,v_n)}d\omega \right|
\]

converges as \( n \rightarrow \infty \) for fixed angular parameters \( m, \ell \). But, in view of (5.2), depending on whether \( \hat{v} = +\infty \) or \( \hat{u} = +\infty \), we can deduce the continuity using Lebesgue’s dominated convergence and the Riemann-Lebesgue lemma. Both are justified by a slight adaptation of the estimate (5.13). This concludes the proof. \( \square \)

**A. Appendix**

**A.1 Twisted energy-momentum tensor in null coordinates in the interior**

We will write out the components of the twisted energy-momentum tensor in the interior.

**Proposition A.1.** Consider null coordinates \((u,v,\theta,\varphi)\) in the interior region \( B \). Recall that the metric is given by (1.51). Let \( f \in C^{\infty}(B) \) be a spherically symmetric nowhere vanishing real valued function and \( X \) be a smooth vector field of the form \( X = X^u\partial_u + X^\nu\partial_\nu \).

The components of the twisted energy-momentum tensor (4.56) associated to \( f \) are given by

\[
\begin{align*}
T_{uu} &= |\nabla u \psi_3|^2 = f^2 \left| \partial_u \left( \frac{\psi_3}{f} \right) \right|^2, \\
T_{vv} &= |\nabla v \psi_3|^2 = f^2 \left| \partial_v \left( \frac{\psi_3}{f} \right) \right|^2, \\
T_{uv} &= T_{vu} = f^2 \left| \partial_u \left( \frac{\psi_3}{f} \right) \right|^2 = \frac{\Omega^2}{4} \left( |\nabla \psi_3|^2 + V|\psi_3|^2 \right), \\
T_{\theta\theta} &= |\partial_\theta \psi_3|^2 + \frac{2r^2}{\Omega^2} \text{Re} \left( \overline{\nabla_u \psi_3} \nabla_v \psi_3 \right) - \frac{r^2}{2} \left( |\nabla \psi_3|^2 + V|\psi_3|^2 \right), \\
T_{\varphi\varphi} &= |\partial_\varphi \psi_3|^2 + \frac{2r^2 \sin^2 \theta}{\Omega^2} \text{Re} \left( \overline{\nabla_u \psi_3} \nabla_v \psi_3 \right) - \frac{r^2 \sin^2 \theta}{2} \left( |\nabla \psi_3|^2 + V|\psi_3|^2 \right).
\end{align*}
\]

The deformation tensor \( X \pi := \frac{1}{2} \mathcal{L}_X g \) is given by

\[
\begin{align*}
X_{\nu u} &= -\frac{\Omega^2}{2} \partial_u X^\nu, \quad X_{u u} = -\frac{\Omega^2}{2} \partial_u X^u, \quad X_{\nu u} = -\frac{1}{\Omega^2} \left( \partial_u X^\nu + \partial_\nu X^u \right) - \frac{2}{\Omega^2} \left( \partial_u \sqrt{\frac{\Omega^2}{\Omega^2}} X^\nu + \partial_\nu \sqrt{\frac{\Omega^2}{\Omega^2}} X^u \right), \\
X_{\nu \theta} &= -\frac{\Omega^2}{2r^3} (X^\nu + X^u), \quad X_{\nu \varphi} = -\frac{\Omega^2}{2r^3 \sin^2 \theta} (X^\nu + X^u).
\end{align*}
\]

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In the following we explicitly write down future-directed normals and induced volume forms for hypersurfaces of constant $r$ values $\Sigma_r$ and for null cones $\mathcal{C}_u$ and $\mathcal{C}_v$ of constant $u$ and $v$ values, respectively.

\[ n^u_{\Sigma_r} = \frac{1}{\sqrt{\Omega^2}} (\partial_u + \partial_v), \quad \text{dvol}_{\Sigma_r} = \frac{r^2}{2} \Omega^2 \text{d}\sigma_{\Sigma^2} \text{d}u = \frac{r^2}{2} \Omega^2 \text{d}\sigma_{\Sigma^2} \text{d}v, \]
\[ n^u_{\mathcal{C}_u} = \frac{2}{\Omega^2} \partial_u, \quad \text{dvol}_{\mathcal{C}_u} = \frac{r^2}{2} \Omega^2 \text{d}\sigma_{\Sigma^2} \text{d}u, \]
\[ n^u_{\mathcal{C}_v} = \frac{2}{\Omega^2} \partial_v, \quad \text{dvol}_{\mathcal{C}_v} = \frac{r^2}{2} \Omega^2 \text{d}\sigma_{\Sigma^2} \text{d}v. \]

Then, the fluxes of $X$ are given by
\[ J^X_{\mu} [\psi] n^\mu_{\Sigma_r} = \frac{2X^\nu}{\Omega^2} (\tilde{\nabla}_\nu \psi) |^2 + \frac{X^u}{2} \left( (\nabla \psi)^2 + \mathcal{V} |\psi|^2 \right), \quad (A.1) \]
\[ J^X_{\mu} [\psi] n^\mu_{\mathcal{C}_u} = \frac{2X^u}{\Omega^2} (\tilde{\nabla}_u \psi) |^2 + \frac{X^v}{2} \left( (\nabla \psi)^2 + \mathcal{V} |\psi|^2 \right), \quad (A.2) \]
\[ J^X_{\mu} [\psi] n^\mu_{\mathcal{C}_v} = \frac{1}{\sqrt{\Omega^2}} \left( X^u |\tilde{\nabla}_u \psi|^2 + X^v |\tilde{\nabla}_v \psi|^2 + \frac{\Omega^2}{4} (X^u + X^v) (\nabla \psi)^2 + \mathcal{V} |\psi|^2 \right). \quad (A.3) \]

The twisted bulk term associated to the twisting function $f$ reads (cf. \[62\])
\[ \tilde{K}^X = X_{\pi_{\mu\nu}} T^\mu_{\nu v} + X^\nu \tilde{S}_v, \]

where
\[ \tilde{S}_v = \frac{\tilde{\nabla}_v (f \mathcal{V})}{2f} |\psi|^2 + \frac{\tilde{\nabla}_v f}{2f} \varpi_{\sigma} \psi \tilde{\nabla}^\sigma \psi. \]

In coordinates we have
\[ \tilde{K}^X = -\frac{2}{\Omega^2} \left( \partial_u X^u |\tilde{\nabla}_v \psi|^2 + \partial_v X^u |\tilde{\nabla}_u \psi|^2 \right) - \frac{2}{r} (X^u + X^v) \text{Re}(\tilde{\nabla}_u \psi \tilde{\nabla}_v \psi) \]
\[ - \left( \frac{1}{2} (\partial_v X^v + \partial_u X^u) - \frac{\partial_u \Omega^2}{4} (X^v + X^u) \right) (\nabla \psi)^2 + \mathcal{V} |\psi|^2 \]
\[ + \frac{\Omega^2}{2r} (X^v + X^u) |\psi|^2 + X^v \left( -\frac{\partial_u (f^2 \mathcal{V})}{2f^2} |\psi|^2 - \frac{\partial_v f^2}{2f^2} \varpi_{\sigma} \psi \tilde{\nabla}^\sigma \psi \right) \]
\[ + X^u \left( -\frac{\partial_v (f^2 \mathcal{V})}{2f^2} |\psi|^2 - \frac{\partial_v f^2}{2f^2} \varpi_{\sigma} \psi \tilde{\nabla}^\sigma \psi \right). \quad (A.4) \]

**A.2 Well-definedness of the Fourier projections $\psi_\omega$ and $\psi_2^\omega$**

**Proposition A.2.** Let $\psi \in C^\infty(\mathcal{M}_{\text{RNAdS}} \setminus \mathcal{C}H)$ be as in \[3.4\] and let $r \in (r_-, r_+)$, $(\varphi, \theta) \in S^2$. Then, $t \mapsto \psi(t,r,\varphi,\theta)$ is a smooth tempered distribution. Moreover, higher derivatives $t \mapsto \partial^k \psi(t,r,\varphi,\theta)$, where $\partial \in \{ \partial_t, \partial_r, \partial_\varphi, \partial_\theta \}$ are also smooth tempered distributions.

**Proof.** Fix $r \in (r_-, r_+)$, $(\varphi, \theta) \in S^2$. We first prove that $t \mapsto \psi(t,r,\varphi,\theta)$ is a slowly growing\footnote{With slowly growing we mean that $t \mapsto \psi(t,r,\varphi,\theta)$ and all its $\partial_t$ derivatives have at most polynomial growth as $|t| \to \infty$.} smooth distribution. Since $\psi \in C^\infty(\mathcal{M}_{\text{RNAdS}} \setminus \mathcal{C}H)$ and in view of the facts that $\square_y$ commutes with $T = \partial_t$ and our initial data are smooth and compactly supported, it suffices to obtain a polynomial bound for $\psi(t,r,\varphi,\theta)$. First, note that from an argument very similar to the one in the proof of Proposition \[4.7\] leading to \[4.43\] we have that
\[ \int_{H(v_1,v_2)} J^X_{\mu} [\psi] n^\mu_{\mathcal{H}^+} \text{dvol}_{\mathcal{H}^+} \lesssim D[\psi] \langle v_2 \rangle \]

(A.5)
for \(0 \leq v_1 \leq v_2\), where \(D[\psi]\) is as in (3.2). In the analogous step from (4.42) to (4.43), we estimate the spacetime integral as

\[
\int_{t_1}^{t_2} \int_{\Sigma_s} J^T_\mu [\psi] n^\mu_{\Sigma_s} \, d\text{vol}_{\Sigma_s} \, ds \lesssim D[\psi](t_2).
\]

(A.6)

Then, using the red-shift vector field as in Section 4.2.3 and a completely analogous argument in the no-shift region as in Section 4.2.4, we can propagate this polynomial upper bound to any \(\{r = \text{const.}\}\) hypersurface in the interior and end up with

\[
\int_0^t |\psi(t, r, \varphi, \theta)|^2 + |\partial_\tau \psi(t, r, \varphi, \theta)|^2 \, dt \lesssim D[\psi](t)
\]

(A.7)

from which we can deduce that \(t \mapsto \psi(t, r, \varphi, \theta)\) is slowly growing. Similarly, as \(t \to -\infty\), we obtain the same conclusion. Now, commuting with \(\partial_\tau\), the angular momentum operators \(W_i\) and using elliptic estimates it follows that higher order derivatives are also slowly growing which concludes the proof.

Corollary A.3. The Fourier projections \(\psi_b\) and \(\psi_d\) in the interior \(B\) as in (3.3) are well-defined and are smooth solutions of (1.3).

Proof. From Proposition A.2 we know that \(t \mapsto \psi(t, r, \varphi, \theta)\) is a tempered distribution in the interior. Thus, \(\psi_b\) defined (3.5) is well defined as \(F^{-1}_T[\chi_{\omega_0}]\) is a Schwartz function. Moreover, \(\psi_b\) is smooth because \(\psi\) is smooth and by Proposition A.2 we have that higher derivatives \(t \mapsto \partial^k \psi(t, r, \varphi, \theta)\) are tempered distributions, too. Now, this also implies that \(\psi_b \in C^\infty(B)\) solves (1.3) which concludes the proof in view of what \(\psi = \psi_b + \psi_d\).

Proposition A.4. Let \(\psi \in C^\infty(M_{\text{RNAdS}} \setminus CH)\) be defined as in (3.4). Then, there exist \(\psi_b \in C^\infty(M_{\text{RNAdS}} \setminus CH)\) and \(\psi_d \in C^\infty(M_{\text{RNAdS}} \setminus CH)\), two solutions of (1.3) with

\[
\psi_b = \frac{1}{\sqrt{2\pi}} F^{-1}_T[\chi_{\omega_0}] \ast \psi \text{ and } \psi_d = \psi - \psi_b,
\]

(A.8)

where \(\chi_{\omega_0}\) is defined in (3.6) and

\[
\psi_b(t, r, \varphi, \theta) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} F^{-1}_T[\chi_{\omega_0}](s) \psi(t - s, r, \varphi, \theta) \, ds
\]

(A.9)

in all coordinate patches \((t_{R_A}, r_{R_A}, \theta_{R_A}, \varphi_{R_A}), (t_{R_B}, r_{R_B}, \theta_{R_B}, \varphi_{R_B})\) and \((t_{B}, r_{B}, \theta_{B}, \varphi_{B})\) in the regions \(R_A\), \(R_B\) and \(B\), respectively.

Proof. First, from Theorem 2.3 we know that \(\psi\) and all higher derivatives decay logarithmically on the exterior regions \(R_A\) and \(R_B\) \(\left[\footnote{This decay is only used in a qualitative way.}\right]\). Hence, \(\psi\) and all higher derivatives are smooth tempered distributions in the exterior regions \(R_A\) and \(R_B\) as functions of \(t_{R_A}\) and \(t_{R_B}\), respectively. Thus, the Fourier projections \(\psi_b\) (A.9) is well-defined in \(R_A\) and \(R_B\) and it follows by Lebesgue’s dominated convergence that \(\psi_b\) is a smooth solution of (1.3). Moreover, from Corollary A.3 we deduce that \(\psi_b\) is also a well-defined smooth solution of (1.3) in the interior \(B\).

Finally, \(\psi_b\) defined a priori only in \(R_A\), \(R_B\) and \(B\), extends to a smooth solution of (1.3) on \(M_{\text{RNAdS}} \setminus CH\). This follows from using regular coordinates near the respective event horizons (outgoing Eddington–Finkelstein coordinates \((v, r, \theta, \varphi)\), where \(v(t, r) = t + r, r(t, r) = r, \theta = \theta, \varphi = \varphi) near \(H_A\) and ingoing Eddington–Finkelstein coordinates near \(H_B\) and writing \(\psi_b\) again as a convolution in this coordinate system

\[
\psi_b = \frac{1}{\sqrt{2\pi}} F^{-1}_T[\chi_{\omega_0}] \ast \psi.\]

Note that \(T = \partial_v\) in this coordinate system. This concludes the proof in view of what \(\psi = \psi_b + \psi_d\).

Proposition A.5. Assume that \(\psi \in C^\infty(M_{\text{RNAdS}} \setminus CH)\) is a solution of (1.3) arising from smooth and compactly supported initial data as in Theorem 2.1. Assume further that there exists an \(L \in \mathbb{N}\) with \(\langle \psi, Y_{\ell} \rangle_{L^2(S^2)} = 0\) for \(\ell \geq L\). Then, for every \(r \in (r_-, r_+)\) and \((\theta, \varphi) \in S^2\), the function \(t \mapsto \psi(t, r, \varphi, \theta)\) is a Schwartz function. Moreover, higher derivatives \(t \mapsto \partial^k \psi(t, r, \varphi, \theta)\), where \(\partial \in \{\partial_t, \partial_r, \partial_{\theta}, \partial_{\varphi}\}\) are also Schwartz functions.
Proof. The proof follows the same lines as the proof Proposition A.2 with the difference that we have exponential decay on the event horizon

\[ \int_{\mathcal{V}_1} J^N_\mu [\psi] n^\mu_\lambda \, d\text{vol}_{\mathcal{H}_\lambda} \lesssim D[\psi] \exp \left( -e^{-C(M,Q,l,\omega)L \sqrt{v_1}} \right), \tag{A.10} \]

where \( D[\psi] \) is as in (3.2). Note that (A.10) follows from [38, Section 12]. Analogously to the proof of Proposition A.2 we can propagate this decay to any \( \{ r = \text{const.} \} \) hypersurface in the interior. As before, by commuting with \( \partial_t \) and \( W_i \) as well as using elliptic estimates, we see that on \( \{ r = \text{const.} \} \), \( \psi \) and higher derivatives \( \partial^k \psi \) decay exponentially towards both components of \( i^\pm \). This concludes the proof. \qed

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