Shimura’s Vector-Valued Modular Forms, Weight Changing Operators, and Laplacians

Shaul Zemel

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Introduction

The systematic study of quasi-modular forms has started with the paper [KZ], which determined the structure of the ring of holomorphic scalar-valued quasi-modular forms on \( SL_2(\mathbb{Z}) \). This combines the usual modular forms and the classical example of a quasi-modular form—the holomorphic Eisenstein series of weight 2 on \( SL_2(\mathbb{Z}) \). The ring of quasi-modular forms is the smallest ring containing the ring of modular forms on \( SL_2(\mathbb{Z}) \) that is closed under differentiation (see, e.g., [MR1]), and [A] obtained some results for modular and quasi-modular groups on a larger class of Fuchsian groups. The prequel [Ze] to the current paper then showed how quasi-modular forms are related to the vector-valued modular forms defined in [Sh] (and previously, in a different language, in [E]), that involve symmetric powers of the standard representation, and established some properties of these vector-valued modular forms.

Classical modular forms admit are operated on by the non-holomorphic weight changing operators, named after Shimura and Maaß. Explicitly, the operator \( \delta_k = \frac{\partial}{\partial \tau} + \frac{k}{2y} \) sends modular forms of weight \( k \) to modular forms of weight \( k + 2 \), and \( 4y^2 \frac{\partial}{\partial \tau} \) decreases the weight by 2 (but annihilates holomorphic and meromorphic functions). Their Lie-theoretic origin is explained in [V]. On the other hand, the usual differentiation preserves quasi-modularity and increases the weight again by 2, but also increases the depth by 1 (see [MR1]). The first goal of this paper is to interpret these results in terms of the vector-valued modular forms of Shimura, and deduce some of their properties. In particular we show that while every weight raising operator \( \delta_l \) increases the weight and depth of a quasi-modular form of weight \( k \) and depth \( d \) by 2 and 1 respectively, the operator \( \delta_{k-d} \) leaves the depth unchanged. As a corollary we simplify the proof of the uniqueness and existence of the Rankin–Cohen brackets for quasi-modular forms appearing in [MR2]. In fact, the case \( d = 0 \) of classical modular forms of this argument can be interpreted as defining the classical Rankin–Cohen brackets as the combination of weight raising operators that preserves holomorphicity, rather than the combination of the differentiations that preserve modularity. This approach may be more intuitive, since modularity is harder to preserve and holomorphicity is easier to check.
As already mentioned in [V], the weight changing operators form, together with the multiplication-by-weight operator, an $\mathfrak{sl}_2$-triple. Another $\mathfrak{sl}_2$-triple appears implicitly in [A] for holomorphic quasi-modular forms. Every such $\mathfrak{sl}_2$-triple produces naturally an invariant operator, namely the Casimir or Laplacian operator (though the normalization of the latter usually differs from that of the former). We prove the existence of a 2-dimensional family of $\mathfrak{sl}_2$-triples (hence of Laplacians) in our case, and work out their eigenspaces. We remark that the results for the eigenspaces in depth $d$ become more difficult when the weight is an integer between $d+1$ and $2d$, a case which was shown in [Ze] to be more delicate (e.g., this is the case where the dimension formulae in that reference depend on whether the Fuchsian group has cusps or not). We find that unless a certain parameter is an integer between 0 and the depth $d$ and another parameter does not vanish, only finitely many eigenvalues admit non-trivial eigenspaces for any given depth.

This paper is divided into 2 sections. Section 1 describes our differential operators in the various settings, and determines the relevant $\mathfrak{sl}_2$-triples. In Section 2 we present the action of the resulting Laplacians, and determine, in most cases, their eigenspaces.

I would like to thank M. Neururer, during the conversation with whom I realized that this project could be carried out.

1 Operators on Quasi-Modular Forms and on Vector-Valued Modular Forms

In this Section we describe the weight changing operators on the various spaces of modular and quasi-modular forms considered in [Ze].

1.1 Operators on Quasi-Modular Forms

The Lie group $SL_2(\mathbb{R})$ has a well-known operation on the Poincaré upper half-plane $\mathcal{H} = \{\tau = x + iy \in \mathbb{C} | y > 0\}$ via fractional linear transformations: The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes $\tau$ to $\gamma \tau = \frac{a\tau + b}{c\tau + d}$, with the factor of automorphy $j(\gamma, \tau) = c\tau + d$. We shall also use the notation $j_\gamma(\tau)$ for $j(\gamma, \tau)$, so that the lower left entry of $\gamma$ is just the derivative $j'_\gamma$ (independently of $\tau$) of $j_\gamma$. We shall work, as in [Ze], with arbitrary Fuchsian groups $\Gamma \leq SL_2(\mathbb{R})$, namely discrete subgroups such that the quotient $\Gamma \backslash \mathcal{H}$ has finite volume in the $SL_2(\mathbb{R})$-invariant measure $\frac{dxdy}{y^2}$.

Let $k \in \mathbb{Z}$ be a weight, and let $\rho$ be a representation of a Fuchsian subgroup $\Gamma$ of $SL_2(\mathbb{R})$ on some finite-dimensional complex vector space $V_\rho$. In fact, all of our definitions and results will hold in the general context: $k$ may be in $\frac{1}{2}\mathbb{Z}$ if $\Gamma$ is a subgroup of the metaplectic double cover of $SL_2(\mathbb{R})$, or alternatively $k$ can be an arbitrary real (or even complex) number and $\rho$ is a (possibly vector-valued) multiplier system of weight $k$. For the definitions of the few notions required for these generalizations see, e.g., Subsection 1.1 of [Ze]. We do remark that an
equivalent approach to multiplier systems in general weights can be obtained by considering subgroups $\Gamma$ of the universal covering group $SL_2(\mathbb{R})$ (a realization of which replaces the metaplectic data $\sqrt{j(\gamma, \tau)}$ attached to some $\gamma \in SL_2(\mathbb{R})$ by a choice of $\log j(\gamma, \tau)$), and taking representations of such groups. The previous paper [Z] has defined, extending previous definitions of [KZ], [MR1], and others, a quasi-modular form of weight $k$, depth $d$, and representation (or multiplier system) $\rho$ with respect to $\Gamma$ to be a function $f : \mathcal{H} \to V_\rho$ for which there exist functions $f_r$, $0 \leq r \leq d$ with $f_0 \neq 0$ (otherwise the depth is smaller than $d$) such that the functional equation

$$f(\gamma \tau) = \sum_{r=0}^d j_\gamma(\tau)^{k-r}(j'_\gamma(\tau))^r \rho(\gamma) f_r(\tau)$$  \hspace{1cm} (1)

holds for every $\gamma \in \Gamma$ and $\tau \in \mathcal{H}$. Setting $\gamma = I$ in Equation (1) shows that $f_0 = f$. A modular form of weight $k$ and representation (or multiplier system) $\rho$ with respect to $\Gamma$, satisfying just $f(\gamma \tau) = j_\gamma(\tau)^k \rho(\gamma) f(\tau)$ for every such $\tau$ and $\gamma$, is just a quasi-modular form of depth 0. Unlike [Z], we only consider holomorphic weights here in Equation (1), since powers of $y$ can always be used to avoid the anti-holomorphic weights. We adopt the notation $\mathcal{M}_k^{\text{hol}}(\rho)$, $\mathcal{M}_k^{\text{sing}}(\rho)$, $\mathcal{M}_k^{\text{hol}}(\rho)$, and $\mathcal{M}_k^{\text{mer}}(\rho)$ from [Z] for the space of modular forms of weight $k$ with the respective differential properties (namely real-analytic, real-analytic except for discrete singularities, holomorphic, and meromorphic), together with the additional spaces $\mathcal{M}_k^{\text{cusp}}(\rho)$, and $\mathcal{M}_k^{\text{wh}}(\rho)$ (for cusp forms and weakly holomorphic modular forms respectively) in case $\Gamma$ has cusps. For quasi-modular forms (of arbitrary depth) we replace every $M$ by $\tilde{M}$, and in case we wish to bound the depth by $d$ we replace $\tilde{M}_k^*$ by $\tilde{M}_k^{* \leq d}$ (for $*$ being any of the superscripts indicating differential properties as above).

We recall from [MR1] that the ring $\bigoplus_{k \in \mathbb{Z}} \tilde{M}_k^{\text{hol}}(SL_2(\mathbb{Z}))$ of (holomorphic) quasi-modular forms on $SL_2(\mathbb{Z})$ is closed under the holomorphic differentiation $\bar{\partial}_\tau = \frac{\partial}{\partial \tau}$, an operation that increases the weight by 2 (and the depth by 1). On the other hand, an operation that preserves modularity, but not holomorphicity (only nearly holomorphicity), is the weight raising operator $\delta_k = \bar{\partial}_\tau + \frac{k}{2y}$, also increasing the weight by 2. The ring of nearly holomorphic modular forms on $SL_2(\mathbb{Z})$ is closed under the appropriate $\delta_k$ operations, and is canonically isomorphic to $\bigoplus_{k \in \mathbb{Z}} \tilde{M}_k^{\text{hol}}(SL_2(\mathbb{Z}))$ (in fact, via the restriction to $f_0$ and $F_0$ of the maps from Theorem 133 as $\mathbb{C}$-algebras that are graded (by the weight), filtered (by the depth), and differential (with the usual derivative and the $\delta_k$s). For the proof, which we shall now generalize to arbitrary Fuchsian groups and representations or multiplier systems (scalar or vector-valued), see Lemma 118 and Propositions 131, 132, and 135 of [MR1]. In addition, when we leave the holomorphic/meromorphic world, the lowering operator $y^2 \bar{\partial}_\tau$ (where $\bar{\partial}_\tau = \frac{\partial}{\partial \tau}$) also becomes of interest, as it decreases the weight of modular forms by 2. The paper [A] considers an operator on (holomorphic) quasi-modular forms that lowers the weight by 2 (and the depth by 1), namely the one sending a quasi-modular form $f$, with functions $f_r$, $0 \leq r \leq d$ as in Equation (1), to $f_1$. Indeed,
Lemma 1.1 of [Ze] (generalizing Lemma 119 of [MR1] and Proposition 2 of [A]) shows that if \( f \in \tilde{M}_k^{*,\leq d}(\rho) \) then \( f_r \in \tilde{M}_k^{*,\leq d-r}(\rho) \) for every \( 0 \leq r \leq d \), and that the associated function with index \( 0 \leq h \leq d - r \) is just \( (\frac{r+h}{r})f_{r+h} \). We summarize these assertions as follows.

**Proposition 1.1.** Let \( k, \Gamma, \) and \( \rho \) be as above, let \( d \) be a depth bound, and take \( f \in \tilde{M}_k^{*,\leq d}(\rho) \) for \( * \) being an or sing, with associated functions \( f_r, 0 \leq r \leq d \).

(i) The derivative \( \partial_\tau f \) of \( f \) lies in \( \tilde{M}_k^{*,\leq d+1}(\rho) \). The associated \( r \)th function is \( \partial_\tau f_r + (k + 1 - r)f_{r-1} \) (where \( f_{-1} \) and \( f_{d+1} \) are understood as 0).

(ii) The function \( \frac{f}{2iy} \) is also in \( \tilde{M}_k^{*,\leq d+1}(\rho) \). The associated \( r \)th function here is \( \frac{\partial_\tau f}{2iy} + f_{r-1} \).

(iii) The combination \( \delta_k - af \), with respective functions \( \delta_k - af_r + (d+1-r)f_{r-1} \), has the stronger property of being in \( \tilde{M}_k^{*,\leq d}(\rho) \).

(iv) The image of \( f \) under \( y^2\partial_\tau \) is in \( \tilde{M}_k^{*,\leq d}(\rho) \), with the \( r \)th function being just \( y^2\partial_\tau f_r \).

(v) The map \( f \mapsto f_1 \) takes elements of our space \( \tilde{M}_k^{*,\leq d}(\rho) \) to \( \tilde{M}_k^{*,\leq d-1}(\rho) \).

**Proof.** Part (i) follows, as in Lemma 118 of [MR1], from the simple observation that \( \partial_\tau(\gamma\tau) = \frac{1}{j(\gamma, \tau)} \), after differentiating Equation (1) with respect to \( \tau \). Recalling that \( \Im(\gamma\tau) = \frac{y}{|j(\gamma, \tau)|^2} \), we multiply Equation (1) by \( \frac{|j(\gamma, \tau)|^2}{2iy} \) and observe that \( j(\gamma, \tau) = j(\gamma, \tau) - 2iyj' \), to deduce part (ii). Part (iii) is a consequence of parts (i) and (ii), since the coefficient appearing in the function representing the depth \( d + 1 \) vanishes. Part (iv) follows simply by applying \( L \) to Equation (1), since \( j_\gamma \) is holomorphic (and \( j'_\gamma \), as well as \( \rho(\gamma) \), are constants that are independent of \( \tau \)). Part (v) was already seen above to be contained in Lemma 1.1 of [Ze]. This proves the proposition.

In particular, the fact that \( \delta_k \) is a map from \( M_k^*(\rho) \) to \( M_{k+2}^*(\rho) \) and \( y^2\partial_\tau \) sends \( M_k^*(\rho) \) to \( M_{k-2}^*(\rho) \) (again with \( * \) being just an or sing) is a special case of parts (iii) and (iv) of Proposition 1.1.

Before we turn to the other objects appearing in [Ze], we show how Proposition 1.1 can be used for obtaining a short and direct proof of the construction of Rankin–Cohen brackets on quasi-modular forms appearing in [MR2]. Moreover, our argument generalizes the result of [MR2] to a much broader context, since only the scalar-valued case with \( \Gamma \) a congruence subgroup is considered in that reference (though the proof does not use these facts). In particular, this argument simplifies (and generalizes) the proof of the properties of the classical Rankin–Cohen brackets, operating on modular forms (i.e., the depth 0 case).

**Theorem 1.2.** Consider two weights \( k \) and \( l \), two (natural) depths \( d \) and \( e \), and a natural parameter \( n \). We are excluding the situation in which both \( d - k \) and
e − l are non-negative integers and n is larger than both d − k and e − l but does not exceed d + e − k − l + 1. Then there is only one linear combination \([\cdot, \cdot]_{n, k, d, l, e}\), up to global scalar multiplication, of the bilinear operators sending two functions \(f\) and \(g\) to \(\partial_l f \otimes \partial_r^\ast g\) (for fixed \(n\)), for which if \(f, g \in \mathcal{M}_{k}^{\ast, \leq d}(\rho)\) and \(g\) is in \(\mathcal{M}_{l}^{\ast, \leq e}(\eta)\) (for another weight \(l\), another depth \(e\), and another representation \(\eta\) of the same group \(\Gamma\)) then \([\cdot, \cdot]_{n, k, d, l, e} \in \mathcal{M}_{k + l + 2n}^{\ast, \leq d + e}(\rho \otimes \eta)\). One normalization of \([f, g]_{n, k, d, l, e}\) involves each of the terms \(\partial_l f \otimes \partial_r^\ast \eta\) coming with the coefficient \((-1)^r \prod_{r=1}^{n-1} (k - d + j) \prod_{i=1}^{n-1} (l - e + q)\). Here \(*\) can be any of the types \(a, s, n, l, \ast\), and \(\ast = \otimes\) then the resulting function \([f, g]_{n, k, d, l, e}\) in \(\mathcal{M}_{k + l + 2n}^{\ast, \leq d + e}(\rho \otimes \eta)\).

Proof. We recall that the weights and depths of quasi-modular forms are additive with respect to tensor products. Now, part (iii) of Proposition 1.1 shows that \(\delta_{k - d}\) does not increase the depth bound on elements of \(\mathcal{M}_{k}^{\ast, \leq d}(\rho)\), while changing the index \(k - d\) of this operator (i.e., adding a multiple of the “division by \(-2iy\)” operator) will increase the depth of elements not lying in \(\mathcal{M}_{k}^{\ast, \leq d - 1}(\rho)\). Therefore the combinations of images of \(\sum_{
aturals} s\}_{n, k, d, l, e}\) and \(\sum_{\naturals} s\}_{n, k, d, l, e}\) and another representation \(\eta\) of \(\Gamma\) with cusps \(n \geq 1\), and \(* = \otimes\) then the resulting function \([f, g]_{n, k, d, l, e}\) is in \(\mathcal{M}_{k + l + 2n}^{\ast, \leq d + e}(\rho \otimes \eta)\).

Now, the power \(\delta_{n, k, d, l, e}\) has an explicit expression, given in, e.g., Equation (56) of [Za]. We have \(\delta_{n, k, d, l, e} = \sum_{p=0}^{s} \binom{s}{p} \prod_{j=s-p}^{s-1} \left(\frac{\partial_j^{-1} f \otimes \partial_l^\ast \eta}{(2iy)^p}\right)\) with the empty product appearing for \(s = 0\) being just 1 (this is easily proved by induction on \(s\)). Therefore if \(\{a_{r, n}\}_{n=0}^{\infty}\) are complex coefficients and \(f\) and \(g\) are as above then the element \(\sum_{r=0}^{n} \sum_{r=0}^{n} \sum_{p=0}^{s} \binom{s}{p} \prod_{j=s-p}^{s-1} \left(\frac{\partial_j^{-1} f \otimes \partial_l^\ast \eta}{(2iy)^p}\right)\) is explicitly,

\[
\sum_{r=0}^{n} \sum_{r=0}^{n} \sum_{p=0}^{s} \binom{s}{p} \prod_{j=s-p}^{s-1} \left(\frac{\partial_j^{-1} f \otimes \partial_l^\ast \eta}{(2iy)^p}\right).
\]

After the summation index changes \(t = i + p\) and \(s = r - i\) (i.e., \(r = s + i\) and \(p = t - i\)) and simple manipulations of the binomial coefficients, the latter expression becomes

\[
\sum_{t=0}^{n} \sum_{s=0}^{n-t} \frac{n!}{s!(n-s-t)!} \binom{t}{i} \prod_{j=s}^{t} \frac{a_{s+i} \prod_{j=s}^{t} (k-d+j) \prod_{j=s-t}^{n-s-1} (l-e+q)}{(2iy)^t} \delta_{n, k, d, l, e}^\ast f \otimes \partial_l^\ast \eta\]

The terms with \(t = 0\) give just \(\sum_{n=0}^{n} \binom{n}{i} a_{s+i} \prod_{j=s}^{n} (k-d+j) \prod_{j=s-t}^{n-s-1} (l-e+q) \delta_{n, k, d, l, e}^\ast f \otimes \partial_l^\ast \eta\). The vanishing of the terms with \(t = 1\) yields \(a_{s}(l-e+n-s-1) + a_{s+1}(k-d+s) = 0\) for each
0 \leq s < n$, and we claim that except in the excluded case, these equations already determine all the coefficients $a_s$ up to a global constant. This is clear if either $d-k$ or $e-l$ is not an integer between 0 and $n-1$, but also holds if both of these numbers are integers between 0 and $n-1$, provided that $n \geq d+e-k-l+2$ (indeed, we get vanishing of $a_s$ for any $s \leq k-d$ and for any $s \geq n-e+l$ in this case, but the remaining coefficients still satisfy linear equations of co-dimension 1). Setting $a_r$ to be the asserted value is easily seen to satisfy the desired equalities, and not to vanish for some $r$ (by the same considerations). Plugging this expression for $a_{s+1}$ into the coefficient of \( \partial_{f,g}^{n-1-s} \) above, we find that the total sum becomes the constant \( \prod_{j=0}^{n-1} (k-d+j) \prod_{q=n-s}^{n-1} (l-e+q) \) times the sum \( \sum_{i=0}^{\tau} \binom{\tau}{i} (-1)^{i+1} \), which is known to vanish for $t > 0$. Hence the combination with these $a_s$ yields an operator of the desired form (i.e., involving only powers of $\partial_\tau$), but recall that the coefficient of $\partial_{f,g}^n \partial_{\tau}^{n-r}$ is $\binom{n}{r} a_r$. The fact that all the types $*a$ are preserved is now immediate (since $\partial_r$ preserves them), and the fact that for $n > 0$ the Rankin–Cohen brackets map \( \mathcal{M}_{k,\rho}^{hol,\leq d} \otimes \mathcal{M}_{n,\eta}^{hol,\leq e} \) into \( \mathcal{M}_{k+l+2n,\rho \otimes \eta}^{\text{cusp},\leq e+r} \) is clear since derivatives annihilate constant coefficients at the cusps. This proves the theorem.

Looking at the case excluded in Theorem 1.2 one can show that the space of solutions to the equations involving the coefficients $a_s$ then has dimension 2. One solution has $a_r = \frac{(n-1-d+k)! (e-l+r-n)!}{(r-1-d+k)! (e-r)!}$ for $r \geq d-k+1$ and $a_r = 0$ otherwise, and another solution is defined by taking $a_r = \frac{(n-1-e+l)! (d-k-r)!}{(n-1-e+r)! (d-k)!}$ if $r \leq n-e+l-1$ and $a_r = 0$ otherwise. These solutions are clearly non-zero, and their independence follows easily from the fact that $d-k+1 > n-e+l-1$. One can verify that both these two solutions indeed do give rise to holomorphic Rankin–Cohen brackets. But this has a simple explanation: As implied by Bol’s identity (which is a special case of Equation (56) of [Za] mentioned above), when $d-k$ is an integer and $n \geq s > d-k$, the operator $\delta_{d-k}$ is just $\delta_{d-k}^{n-s-d-k-1} \circ \partial_{\tau}^{d-k+1}$. Similarly, if $e-l \in \mathbb{N}$ then $\delta_{e-l}$ is just $\delta_{e-l}^{n-s-e+l-1} \circ \partial_{\tau}^{e-l+1}$ whenever $n \geq s > e-l$ (this explains the vanishing of $a_s$ for $r \leq d-k$ in the former case and for $r \geq n-e+l-1$ in the latter one). Hence if $d-k$ is an integer between 0 and $n-1$ then $[f,g]_{n,k,d,t,e}$ is a constant multiple of $[\delta_{\tau}^{d-k+1} f, g]_{n-d+k-1,2d-,k+2,2d,1,e}$, and in case $e-l$ is an integer in that interval we have $[f,g]_{n,k,d,t,e} = [f, \partial_{\tau}^{e-l+1} g]_{n-e+l-1; k, d; 2e-2l+2}$ up to scalar multiples (the fact that these derivatives increase the weight in this way and leaves the depth invariant is a consequence of part (iii) of Proposition 1.6 and Bol’s identity). Combining these cases, if both $d-k$ and $e-l$ are integers and $n \geq d+e-k-l+2$ then $[f,g]_{n,k,d,t,e}$ becomes a multiple of $[\delta_{\tau}^{d-k+1} f, \partial_{\tau}^{e-l+1} g]_{n-d+e+k+1,2d-k+2,2d,2e-2l+2}$ (one can also verify this using the formulae for the coefficients). But when $n$ is in the domain excluded from Theorem 1.2 the two bilinear operations sending $f$ and $g$ either to $[\delta_{\tau}^{d-k+1} f, g]_{n-d+k-1,2d-,k+2,2d,1,e}$ or to $[f, \partial_{\tau}^{e-l+1} g]_{n-e+l-1; k, d; 2e-2l+2}$ satisfy the desired properties, and they are linearly independent.

We remark that the Leibnitz rule for Rankin–Cohen brackets appearing in
Theorem 2 of [MR2] can also be established using the non-holomorphic operators \( \delta_m \). However, for this assertion the original proof from [MR2] is much simpler.

1.2 Operators on Vector–Valued Modular Forms

Let \( V_m \) be the symmetric power of the standard representation of \( SL_2(\mathbb{R}) \) on \( \mathbb{C}^2 \) (vectors of the representation space of which, also denoted by \( V_m \), are written as products of elements of \( \mathbb{C}^2 \)), and let \( k, \Gamma, \) and \( \rho \) be as above. The paper [Ze] investigated the spaces \( \mathcal{M}^\ast_k(V_m \otimes \rho) \) for various differential conditions \( \ast \), and proved the following result (see Proposition 1.2 and Theorem 2.3 of that reference), which in particular extends Propositions 132 and 133 of [MR1]:

**Theorem 1.3.** For \( \ast = \text{an or } \ast = \text{sing} \) the spaces \( \tilde{\mathcal{M}}^\ast_k(V_m \otimes \rho) \), \( \bigoplus_{s=0}^m \mathcal{M}^\ast_{k-2s}(\rho) \), and \( \mathcal{M}^\ast_{k-m}(V_m \otimes \rho) \) are canonically isomorphic. The maps between the first two spaces take, in one direction, a quasi-modular form \( f \) with functions \( f_r, 0 \leq r \leq m \) to \( F_s : \tau \mapsto \sum_{r=s}^m \binom{r}{s} \frac{f_r(\tau)}{(2iy)^{r-s}} \in \mathcal{M}^\ast_{k-2s}(\rho), \ 0 \leq s \leq m, \)

with the inverse map sending \( (F_s)_{s=0}^m \) to the quasi-modular form \( f = f_0 \) with functions \( f_r : \tau \mapsto \sum_{s=r}^m \binom{s}{r} \frac{F_s(\tau)}{(-2iy)^{s-r}}. \)

The element of \( \mathcal{M}^\ast_{k-m}(V_m \otimes \rho) \) that is associated to \( f \) and to \( (F_s)_{s=0}^m \) related in this manner is the vector-valued modular form \( F : \mathcal{H} \to V_m \otimes \rho \) defined by

\[
F(\tau) = \sum_{s=0}^m \frac{F_s(\tau)}{(-2iy)^s} \left( \begin{array}{c} \tau \vspace{1mm} \\ 1 \end{array} \right)^{m-s} \left( \begin{array}{c} \tau \vspace{1mm} \\ 1 \end{array} \right)^s = \sum_{r=0}^m f_r(\tau) \left( \begin{array}{c} \tau \vspace{1mm} \\ 1 \end{array} \right)^{m-r} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^r,
\]

the inverse maps being just taking components in bases of \( V_m \). Moreover, \( f \) is holomorphic (or meromorphic) if and only if the \( f_r \)s and \( F \) have the same property.

The modular forms \( (F_s)_{s=0}^m \) associated with a holomorphic quasi-modular form are not holomorphic, but are rather nearly holomorphic, i.e., are polynomials in \( \frac{1}{y} \) in holomorphic functions on \( \mathcal{H} \). The depth of such a function is the maximal degree of \( \frac{1}{y} \) appearing in it. The same statement holds when replacing every instance of holomorphic by meromorphic.

For every \( m \) there is a map \( i_m : V_m \to V_{m+1} \) defined via multiplication by \( \binom{1}{m} \), and Corollary 2.2 of [Ze] shows that applying it to modular forms yields a map, also denoted \( i_m \), from \( \mathcal{M}^\ast_{k-m}(V_m \otimes \rho) \) to \( \mathcal{M}^\ast_{k-m-1}(V_{m+1} \otimes \rho) \). The following immediate consequence of Theorem 1.3 is also implicitly contained in [Ze].
Corollary 1.4. The map \( i_m : \mathcal{M}_{k-m}^*(V_m \otimes \rho) \to \mathcal{M}_{k-m-1}^*(V_{m+1} \otimes \rho) \) commutes with the trivial inclusion of \( \mathcal{M}_{k-m}^*(\rho) \) into \( \mathcal{M}_{k-m-1}^*(\rho) \) and of the natural embedding of \( \bigoplus_{s=0}^{m} \mathcal{M}_{k-2s}^*(\rho) \) into \( \bigoplus_{s=0}^{m+1} \mathcal{M}_{k-2s}^*(\rho) \). The image of the latter map consists precisely of those sequences whose last coordinate vanishes.

We now turn to operations on direct sums like \( \bigoplus_{s=0}^{m} \mathcal{M}_{k-2s}^*(\rho) \) that change the weight. One such operation that increases the weight by 2 is letting, for each \( s \), a multiple of \( \delta_{k-2s} \) operate on the \( s \)th component. Another one is replacing \( F_s \) by a multiple of \( F_{s-1} \) (with \( F_{s-1} \) being 0). Any linear combination of these two operations will also do. For lowering the weight, we have scalar multiples of \( \rho \) on every component, replacing \( F_s \) by \( F_{s+1} \) (where \( F_{m+1} = 0 \)), and their linear combinations. We now find what are the operations on this space that correspond to those from Proposition 1.1 under the isomorphism from Theorem 1.3.

Proposition 1.5. Let \( f \in \widetilde{\mathcal{M}}_{k}^{*} \leq m(\rho) \) as above, and denote the \( m \)-tuple of modular forms that is associated to \( f \) in Theorem 1.3 by \( (F_s)_{s=0}^{m} \in \bigoplus_{s=0}^{m} \mathcal{M}_{k-2s}^*(\rho) \).

In parts (i) and (ii) assume further that the depth of \( f \) is at most \( m-1 \).

(i) The \( m \)-tuple associated with \( \partial_{\tau} f \) has \( \delta_{k-2s} F_s + (k + 1 - s) F_{s-1} \) as its \( s \)th function (with 0th component \( \delta_k F_0 \)).

(ii) For \( \frac{df}{d\tau} \) we get \( F_{s+1} \) as the \( s \)th function (and no 0th function).

(iii) Applying the combination \( \delta_{k-m} \) to \( f \) corresponds to getting the \( s \)th function \( \delta_{k-2s} F_s + (m + 1 - s) F_{s+1} \), again with the 0th function being just \( \delta_k F_0 \). This assertion holds also for quasi-modular forms \( f \) of depth precisely \( m \).

(iv) The image of the operation of \( y^2 \partial_{\tau} \) on \( f \) yields \( y^2 \partial_{\tau} F_s(\tau) + \frac{s+1}{s} F_{s+1} \) as the \( s \)th function (where the \( m \)th function is just \( y^2 \partial_{\tau} F_m \)).

(v) Replacing \( f = f_0 \) by \( f_1 \) commutes with the isomorphisms from Theorem 1.3 and the operation sending \( (F_s)_{s=0}^{m} \) to \( ((s+1) F_{s+1})_{s=0}^{m} \) (with vanishing \( m \)th function).

Proof. We recall from Theorem 1.3 that \( F_s(\tau) = \sum_{r=s}^{m} \binom{r}{s} \frac{f_{r}(\tau)}{(2iy)^{r-s}} \) (showing indeed that \( F_m = 0 \) if and only if \( f \in \mathcal{M}_{k}^{*} \leq m (\rho) \)). We do the same transformation on the sequences of functions appearing in the various parts of Proposition 1.1 and express the results in terms of the original functions \( F_s \). In part (i), with \( F_m = 0 \), we consider \( \sum_{r=s}^{m} \binom{r}{s} \frac{\partial \theta_{r+k+1-r} f_{r}(\tau)}{(2iy)^{r-s}} \). Comparing the first term with index \( r = s \) with the definition of \( F_s \), the effect of the derivative should be related to \( \delta_{k-2s} F_s \). Recalling that \( \delta_i = y^{-i} \partial_{\tau} y^i \), we find that applying \( \delta_i \) to an expression of the form \( \frac{\partial^i}{\partial_{\tau} y^i} \) yields \( \frac{\delta_{i+s} g(\tau)}{y^i} \). It follows that

\[
\delta_{k-2s} F_s(\tau) = \sum_{r=s}^{m} \binom{r}{s} \delta_{k-r-s} f_{r}(\tau) (2iy)^{r-s} = \sum_{r=s}^{m} \binom{r}{s} \frac{\partial_{\tau} f_{r}(\tau) + \frac{k-r-s}{2iy} f_{r}(\tau)}{(2iy)^{r-s}}.
\]
Therefore subtracting $\delta_{k-2s}F_s$ from the desired function indeed cancels the terms involving $\partial_r f_r$, where replacing the summation index in the remaining terms of the desired function shows that the difference becomes just

$$\tau \mapsto \sum_{r=s-1}^{m-1} \left[ (k-r) \left( \frac{r+1}{s} \right) \right] \frac{f_r(\tau)}{(2iy)^{r-s-1}}$$

(recall that in the part coming from $\delta_{k-2s}F_s$, the term with $s-1$ vanishes since $\binom{s-1}{s} = 0$, and as we also assume that $f_m = 0$ here, replacing $\sum_{r=s}^m$ by $\sum_{r=s-1}^{m-1}$ leaves the sum unaffected). The classical binomial identity simplifies the coefficient to $(k-r)\binom{r}{s-1} + s\binom{r}{s}$. Moreover, the second term $\binom{r}{s-1}$ also equals $(r+1-s)\binom{r}{s-1}$, so that we get a global coefficient of $k + 1 - s$ times $\sum_{r=s-1}^{m-1} \binom{r}{s-1} f_r(\tau)$, which proves part (i) since the latter sum is $F_{s-1}$ by definition.

For part (ii) we have to evaluate the expression $\sum_{r=s}^m \binom{r}{s} \frac{f_{r+1}(\tau) - (2iy)^{r+s-1}}{(2iy)^{r-1-s}}$ arising from part (ii) of Proposition 1.1, which after a similar summation index change becomes $\sum_{r=s-1}^{m-1} \binom{r+1}{s} \frac{f_r(\tau)}{(2iy)^{r-s-1}}$. The required assertion is again a consequence of the classical binomial identity and the definition of $F_{s-1}$. Part (iii) follows from parts (i) and (ii), and the extension to quasi-modular forms of depth $m$ is well-defined since the undesired function with index $m+1$ vanishes because of the coefficient $m+1-s$ (in correspondence with the depth assertion in part (iii) of Proposition 1.1). We now evaluate the difference between the expansion $\sum_{r=s}^m \binom{r}{s} \frac{y^2 \partial_{\tau} f_r(\tau)}{(2iy)^{r+s-1}}$ from part (iv) of Proposition 1.1 and $y^2 \partial_{\tau} F_s(\tau)$ as

$$- \sum_{r=s}^m \binom{r}{s} f_r(\tau) \cdot y^2 \frac{\partial(\tau - \tau)^{s-r}}{\partial \tau} = - \sum_{r=s}^m \binom{r}{s} (r-s) f_r(\tau) \frac{y^2}{(2iy)^{r+1-s}}.$$

Since this simplifies to $+\frac{s+1}{4} \sum_{r=s}^m \binom{r+1}{s+1} \frac{f_r(\tau)}{(2iy)^{r-s}} = +\frac{s+1}{4} F_{s+1}$ by considerations as above (since $(2i)^2 = -4$), this establishes part (iv) as well. Finally, part (v) of Proposition 1.1 directs us to consider the sum $\sum_{r=s}^m \binom{r}{s} \frac{(r+1)f_{r+1}(\tau)}{(2iy)^{r+s-1}}$ for part (v) here, where we can omit the term with $r = m$ since $f_{m+1} = 0$. The coefficient coincides with $(s+1)\binom{r+1}{s+1}$, and a summation index change identifies the rest of this expression (with $\binom{r+1}{s+1}$ alone) as the desired $F_{s+1}$. This completes the proof of the proposition. □

We now turn to the equivalent operators on the space $\mathcal{M}^k_{-m}(V_m \otimes \rho)$ of vector-valued modular forms. The operators $\delta_{k-m}$ and $L$ would again take elements of this space to $\mathcal{M}^k_{n-m+2}(V_m \otimes \rho)$ and $\mathcal{M}^k_{n-m-2}(V_m \otimes \rho)$ respectively. But also here there are other weight changing operators, defined on appropriate subspaces of $\mathcal{M}^k_{-m}(V_m \otimes \rho)$, that do not involve differentiation. Indeed, apart from the map $\iota_m$ from Corollary 2.2 of [Ze] (or Corollary 1.4 here), one can define the complex conjugate map $\iota_m : V_m \to V_{m+1}$ using multiplication by the complex conjugate vector ($\bar{r}$). In addition, the same (simple) argument from
Corollary 2.2 of [Ze] also proves that \( \frac{\tau}{-2iy} \) takes elements of \( M_{1-k}^r(V_m \otimes \rho) \) injectively to elements of \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \), and the image consists of those modular forms

\[
\tilde{F} \in M_{1-k+1}^r(V_{m+1} \otimes \rho), \quad F(\tau) = \sum_{m_+ + m_- = m+1} \tilde{f}^{m_+,m_-}(\tau) \binom{m_+}{1} \binom{m_-}{1},
\]

in which the coefficient \( \tilde{f}^{m_+,0} \) takes values in \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \) if \( m_+ \geq 0 \), and \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \) if \( m_+ = 0 \). It follows that given an element \( F \) in \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \), the map \( \frac{\tau}{-2iy} \circ i_{m+1}^{-1} \) is well-defined since \( i_{m+1} \) is injective. Similarly, any element \( F \) in \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \) has a well-defined image under \( i_{m+1} \circ (-2iy) \), and this image lies in \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \). One way to get such an \( F \) is by taking an arbitrary element in \( M_{1-k+1}^r(V_{m+1} \otimes \rho) \) and multiplying the associated coefficient \( \tilde{f}^{m_+,m_-} \) of \( \binom{m_+}{1} \binom{m_-}{1} \) by a constant \( c_{m_+} \), such that \( c_{m+0} = 0 \). We choose \( c_{m_+} = m_- \), and the composition \( D_m \) of this multiplication and followed by \( -2iy \) looks like “differentiating the vectors \( \binom{m_+}{1} \binom{m_-}{1} \) with respect to \( \frac{\tau}{-2iy} \).

The relations between these operations and the ones appearing in Propositions 1.11 and 1.13 are as follows.

**Theorem 1.6.** Let \( F \in \tilde{M}_{1-k}^r(V_m \otimes \rho) \) be associated with the quasi-modular form \( f \in \tilde{M}_{1-k}^{\leq m}(\rho) \) as in Theorem 1.3.

(i) The operator \( \delta_{k-m} \) on \( F \) is compatible with the same operator on \( f \).

(ii) On \( i_{m-1} \) images (or equivalently elements of \( \tilde{M}_{1-k}^{\leq m-1}(\rho) \)), the operation \( \frac{\tau}{-2iy} \circ i_{m-1}^{-1} \) corresponds to dividing \( f \) by \( -2iy \).

(iii) Under the restriction from part (ii), any operation \( \delta_l \) on \( f \) (in particular \( \delta_r = \delta_0 \)) has the counterpart \( \delta_{k-m} + (k-m-l)(\frac{\tau}{-2iy} \circ i_{m-1}^{-1}) \) on \( F \).

(iv) The operation of \( y^2 \partial_y \) on the two functions \( f \) and \( F \) is also compatible.

(v) The operator on \( F \) that commutes with \( f \mapsto f_1 \) is \( i_{m-1} \circ D_m \).

**Proof.** We denote the element of \( \bigoplus_{i=0}^m \tilde{M}_{1-k}^{\leq 2i}(\rho) \) associated with \( f \) and \( F \) by \( (F)_{i=0}^m \), and recall the two presentations of \( F \) from Theorem 1.3. We begin by evaluating \( \delta_{k-m} F \), using the presentation of \( F \) in terms of the functions \( f_r \).

As the part \( \partial_r \) of \( \delta_{k-m} \) has to operate also on the vectors \( \binom{r}{1} \binom{m-r-1}{0} \), yielding a contribution involving the next vector \( (m-r)(\binom{r}{1} \binom{m-r-1}{0} \partial_r) \), the function \( \delta_{k-m} F \) a simple summation index change shows that is easily seen (by a simple summation index change) to be associated with the quasi-modular form for which the \( r \)th function appearing in Equation (1) is \( \delta_{k-m} f_r + (m+1-r)f_{r-1} \) (unless \( r = 0 \), where the second term does not appear). Part (i) hence follows.
from part (iii) of Proposition 1.1. For part (ii) we use the expression for \( F \) involving the functions \( F_s \) in Theorem 1.3. It is clear from the definition that if \( F \) is an \( i_{m-1} \)-image (i.e., if \( F_{m_0} = 0 \)) then \( \frac{i_{m-1}}{2iy}(i_{m-1}F) \) takes \( \tau \), after a summation index change, to \( \sum_{s=1}^{m} \frac{F_{s-1}(\tau)}{(2iy)^s} \tau^m \) (with no coefficient associated with \( s = 0 \)). Part (iii) is therefore a consequence of part (ii) of Proposition 1.5 and part (iii) is an immediate linear combination of parts (i) and (ii). Considering the presentation of \( F \) using the functions \( f_r \) once again, and observing that the vectors \( \tau^{m-r}(1)^r \) are holomorphic, we deduce that the operation of \( y^2 \partial_{\tau} \) on \( F \) is just through its operation on the components \( f_r \), \( 0 \leq r \leq m \). Part (iv) therefore follows from part (iv) of Proposition 1.1. Finally, we apply \( D_m \) on the presentation of \( F \) involving the \( F_s \)s in Theorem 1.3 and it is clear (after another summation index change) that \( i_{m-1}(D_mF) \) sends \( \tau \) to \( \sum_{s=0}^{m-1} \frac{(s+1)F_{s+1}(\tau)}{(2iy)^{s+1}} \tau^{m-s} \). Part (v) is then established by an application of part (v) of Proposition 1.5. This completes the proof of the theorem.

We remark that parts (ii) and (v) of Theorem 1.6 could have also been proved using Proposition 1.1 alone, by applying the equality \( \tau^{m-r}(1)^r \) (indeed, \( i_{m-1} \circ D_m \) takes \( \tau^{m-r}(1)^r \) to \( \tau^{m+1-r}(1)^{r-1} \) for every \( 0 \leq r \leq m \)). Similarly, differentiating the expression for \( F \) involving the functions \( F_s \) in Theorem 1.3 and using the same equality provides an alternative proof for parts (i) and (iv) as well, using just Proposition 1.5. However, the proof we chose for each part is the simpler one. In addition, the fact that any modular or quasi-modular form is meromorphic on \( \mathcal{H} \) if and only if it is annihilated by the operator \( y^2 \partial_{\tau} \) (and has no essential singularities, which we exclude in \( * = \text{sing} \) and \( * = \text{mer} \) in any case) suggests another proof of the last assertion in Theorem 1.3. Just use part (iv) of Theorem 1.6.

We recall that the multiplicative structure of the ring of quasi-modular forms was adapted not to a single representation \( V_m \), but rather to their direct limit using the maps \( i_m \). Therefore we investigate the relations between the operators from Theorem 1.3 and \( i_m \).

**Corollary 1.7.** The operators \( y^2 \partial_{\tau} \) and \( i_{m-1} \circ D_m \) on \( \mathcal{M}^*_{k-m}(V_m \otimes \rho) \) commute with \( i_m \). On \( i_{m-1} \)-images the same assertion holds for \( \frac{i_{m-1}}{2iy} \circ i_{m-1}^{-1} \). As for \( \delta \) operators, the operator on \( \mathcal{M}^*_{k-m}(V_{m+1} \otimes \rho) \) commuting with \( \delta_{k-m} \) on \( \mathcal{M}^*_{k-m}(V_m \otimes \rho) \) is \( \delta_{k-m-1} - \frac{1}{2iy} \circ i_{m-1}^{-1} \). In particular, we have the following respective limit operations on the limit space \( \mathcal{M}^*_{k-\infty}(V_{\infty} \otimes \rho) \): \( y^2 \partial_{\tau} \) remains the same, the next two become just \( D \) and \( \frac{\tau}{2iy} \), and the last one is \( \delta_l = \lim_{s \to \infty} (\delta_{k-m} - (l + m - k)) \frac{-\tau}{2iy} \).

Here \( D \) and \( \tau \) have the obvious limiting meaning, sending \( \frac{1}{-2iy} \tau(1)^{s+1} \) to \( \frac{1}{-2iy} \tau(1)^{s+1} \) respectively, and observe that the maps taking \( f \in \mathcal{M}^*_{k-\infty}(\rho) \) to \( y^2 \partial_{\tau}f \), \( f_1, \frac{\tau}{2iy} \) in case \( f \)
is in \( \tilde{M}_k^{\ast \leq m-1}(\rho) \), and \( \delta_k-m f \) do not distinguish between \( f \) as an element of \( \tilde{M}_k^{\ast \leq m}(\rho) \) or of \( \tilde{M}_k^{\ast \leq m+1}(\rho) \). The expressions for the limit operators are immediate, with only a small linear combination argument for the one arising from \( \delta_l \). This proves the corollary.

In fact, all the commutativity assertions of Corollary 1.7 can easily be proved directly: Recalling that \( i_m F = F \cdot (\cdot) \), the anti-holomorphic differentiation in \( y^2 \partial_\varphi \) does not operate on \( (\cdot) \), and adding another multiplier of \( (\cdot) \) clearly commutes with adding or removing multipliers like \( (\cdot) \) or \( (\cdot) \), multiplication by \( 2iy \), and multiplication of the components \( (\cdot)^{m_+} (\cdot)^{m_-} \) by constants \( c_{m_+,m_-} \) that depend only on \( m_- \). Finally, the Leibnitz rule for \( \delta \) operators (which is a simple consequence of the usual Leibnitz rule for \( \partial_\tau \)) allows us to decompose \( \delta_{k-m-1}(i_m F) \) as the sum of \( i_m(\delta_{k-m} F) \) and \( F \cdot \delta_{-1}(\cdot) \), where the latter multiplier of \( F \) is just \(- (\cdot)/2iy \). The operators involving \( D_m \) and \( \tilde{m}_{-2iy} \) simplify in the limit because we do not have to worry about landing in the space with the same \( V_m \) anymore.

We recall from [Ze] that the modular forms with representations involving \( V_m \) admit a multiplicative structure, arising from the tensor product and the natural projection from \( V_m \otimes V_p \) onto \( V_{m+p} \). This multiplication corresponds to the usual (tensor) product of quasi-modular forms via Theorem 1.3. It also behaves well with respect to the embeddings \( i_m \). On the other hand, the operators appearing in the Rankin–Cohen brackets in Theorem 1.2 do not commute well with the inclusions \( i_m \). Therefore the only assertion about Rankin–Cohen brackets for modular forms involving \( V_m \) that we can make at this point is the following.

**Corollary 1.8.** Composing the combinations defining the Rankin–Cohen brackets in Theorem 1.2 with the projection \( V_m \otimes V_p \rightarrow V_{m+p} \) yields bilinear operators from \( M_{k-m}(V_m \otimes \rho) \times M_{l-p}(V_p \otimes \eta) \) to \( M_{k+l+2n-m-p}(V_{m+p} \otimes \rho \otimes \eta) \) for every type *.

**Proof.** The corollary follows directly from part (i) of Theorem 1.6 and the proof of Theorem 1.2.

Note that unlike Corollary 1.7, the Rankin–Cohen brackets from Corollary 1.8 do not go naturally over to \( V_{m+p+1} \) (or to the limit space with \( V_\infty \)). This is not surprising, since the Rankin–Cohen brackets defined in Theorem 1.2 depend on the depth of the quasi-modular forms on which they operate.

### 1.3 \( \mathfrak{sl}_2 \)-Triples

Next, we turn to the commutation relations between our operators.

**Proposition 1.9.** The following assertions holds for our operators:

(i) The operators \( y^2 \partial_\varphi \) and \( i_{m-1} \circ D_m \) on \( M^\text{sing}_{k-m}(V_m \otimes \rho) \) commute with one another. Similarly for \( y^2 \partial_\varphi \) and \( D \) on \( M^\text{sing}_{k-\infty}(V_\infty \otimes \rho) \).
(ii) Applying $\delta_{k+2-m}$ to the $\frac{m}{-2iy} \circ i_{m-1}$-image of an element $F$ of the subspace $i_{m-1}(M_{k+1-m}(V_{m-1} \otimes \rho))$ of $M_{k+1}^{\text{sing}}(V_m \otimes \rho)$ yields the same result as the operation of $\frac{i_{m-1}}{-2iy}$ on the image of $F$ under $\delta_{k-m} - \left(\frac{i_{m-1}}{-2iy} \circ i_{m-1}\right)$. On $M_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$ we obtain the commutativity relation $\delta_{l+2} \circ \frac{i}{-2iy} = \frac{i}{-2iy} \circ \delta_l$ for every $l$.

(iii) If $F \in M_{k-m}^{\text{sing}}(V_m \otimes \rho)$ in an $i_{m-1}$-image then letting $i_{m-1} \circ D_m$ act on the $\left(\frac{i_{m-1}}{-2iy} \circ i_{m-1}\right)F$ yields the same result as when $\frac{i_{m-1}}{-2iy} \circ i_{m-1}$ operates on $(i_{m-1} \circ D_m)F$, plus the original function $F$. On $M_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$, the commutator $D \circ \frac{i}{-2iy} = \frac{i}{-2iy} \circ D$ is just the identity operator.

(iv) The operator with which we compare the composition $(i_{m-1} \circ D_m) \circ \delta_{k-m}$ on $M_{k-m}^{\text{sing}}(V_m \otimes \rho)$ is the composition of $\delta_{k-2-m} - \frac{2}{-2iy} \circ i_{m-1}$ with $i_{m-1} \circ D_m$. The resulting commutator is just $m$ times the identity operator. When restricting to $i_{m-1}$-images, subtracting $(l + m - k) \frac{i_{m-1}}{-2iy} \circ i_{m-1}$ from both operators yields a commutator of $k - l$ times the identity operator, which is also the commutator of $D$ and $\delta_l$ on $M_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$.

(v) On elements of $M_{k-m}^{\text{sing}}(V_m \otimes \rho)$ that are $i_{m-1}$-images, the commutator $y^2 \partial_{\tau} \circ \left(\frac{i_{m-1}}{-2iy} \circ i_{m-1}\right) - \left(\frac{i_{m-1}}{-2iy} \circ i_{m-1}\right) \circ y^2 \partial_{\tau}$ is just $\frac{k}{4}$ times the identity operator. So is $y^2 \partial_{\tau} \circ \frac{i}{-2iy} - \frac{i}{-2iy} \circ y^2 \partial_{\tau}$ on $M_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$.

(vi) The commutation relation $y^2 \partial_{\tau} \circ \delta_{k-m} - \delta_{k-2-m} \circ y^2 \partial_{\tau}$ on $M_{k-m}^{\text{sing}}(V_m \otimes \rho)$ is $-\frac{k}{4}$ times the identity operator. On $i_{m-1}$-images we can subtract $(l + m - k) \frac{i_{m-1}}{-2iy}$ from each $\delta$ operator, and obtain a commutator of $-\frac{l}{4}$ times the identity operator. The latter assertion extends to the commutator $y^2 \partial_{\tau} \circ \delta_l - \delta_{l-2} \circ y^2 \partial_{\tau}$ on $M_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$.

Proof. Part (i) follows, via parts (iv) and (v) of Theorem 1.6 from part (iv) of Proposition 1.1 (since applying $y^2 \partial_{\tau}$ to $f_1$ is the same as taking the corresponding function associated with $y^2 \partial_{\tau} f$). For the first assertion in part (ii) we apply parts (i) and (ii) of Theorem 1.6 Corollary 1.7 and the simple assertion that the equality $\delta_{k+2-m} \frac{f}{-2iy} = \frac{\delta_{k+1-m}f}{-2iy}$ holds for any $f \in \hat{\mathcal{M}}_{k}^{\text{sing}}(\rho)$. In case $F$ is an $(i_{m-1} \circ i_{m-2})$-image we can subtract any multiple of $\left(\frac{i_{m-1}}{-2iy} \circ i_{m-1}\right)^2 F$ from both sides, and the second assertion follows from taking the limit of the appropriate combination. Next, we recall from part (ii) of Proposition 1.1 that dividing an element $f \in \hat{\mathcal{M}}_{k}^{\text{sing}}(\rho)$ by $-2iy$ replaces the function $f_1$ by $\frac{f}{-2iy}$ plus the original function $f_0 = f$. Using parts (ii) and (v) of Theorem 1.6 the resulting commutation relation transforms to the assertion of part (iii). Similarly, part (iii) of Proposition 1.1 shows that the $f_1$-function associated with $\delta_{k-m} f$ is $\delta_{k-m} f_1$ plus $m f_0 = m f$. The first assertion of part (iv) therefore follows from part (iii) of Theorem 1.6 (which is applicable since $i_{m-1}(D_m F)$ is
an $i_{m-1}$-image, the resulting composition being just $(i_m \delta_{k-1-m} - \frac{y}{2iy}) \circ D_m$ by Corollary [1.7], and applying parts (i) and (v) of that theorem allows us to deduce the second assertion there as well from these evaluations. The third assertion is a consequence of part (iii) here, and the result about the limit space follow immediately from the limit process in Corollary [1.7]. Next, we recall from the commutativity of $y^2 \partial_\tau$ and $i_{m-1}$ in Corollary [1.7] that the former operator preserves $i_{m-1}$-images in $M^{sing}_{k-\infty}(V_m \otimes \rho)$, and part (v) here is established using parts (ii) and (iv) of Theorem [1.6] and the fact that $y^2 \partial_\tau \left( \frac{k}{2iy} \right) = -\frac{y^2 \partial_\tau}{2iy}$ equals $-\frac{y^2 f}{(1-y^2)} = +\frac{f}{y}$. Finally, the commutation relation in the first assertion of part (vi) holds because $\delta_{k-2-m} \circ y^2 \partial_\tau = y^2 \delta_{k-m} \partial_\tau$, and the difference between $y^2 \partial_\tau (\delta_{k-m} F)$ and the image of $F$ under the latter combination is just $y^2 F \partial_\tau (\frac{k}{2iy} m) = -\frac{k}{2iy} m F$ (equivalently, we could apply the same considerations to a scalar-valued quasi-modular form $f$ using parts (i) and (iv) of Theorem [1.6] for getting this result). The second assertion is now a consequence of part (v), and for the limit space we just use the definition of the $\delta$ operators from Corollary [1.7]. This proves the proposition.}

We recall that the algebra $\mathfrak{sl}_2(\mathbb{R})$, of real traceless $2 \times 2$ matrices, is 3-dimensional, and the natural basis for it consists of the elements $H = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$, $E = \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$, and $F = \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right)$. The commutation relations are $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. Three operators satisfying these commutation relations are known as a $\mathfrak{sl}_2$-triple. An action of an $\mathfrak{sl}_2$-triple on a space would thus decompose according to the eigenvalue of $H$, where $E$ and $F$ would send an element with a certain eigenvalue $\alpha$ to elements of eigenvalue $\alpha \pm 2$. We are interested in $\mathfrak{sl}_2$-triples acting on (quasi-)modular forms in which $H$ is the operator $W$ multiplying every (quasi-)modular form by its weight. Then $E$ and $F$ must correspond to a weight raising and weight lowering operator respectively. Indeed, when working with real-analytic functions on $H$ or on $SL_2(\mathbb{R})$, [V] showed that $W$ forms with the operators $\delta_k$ and $4y^2 \partial_\tau$ (or, more precisely, $2i \delta_k$ and $-2iy^2 \partial_\tau$) such an $\mathfrak{sl}_2$-triple. As an additional example, Equation (8) of [A] implies that $W$ and the (holomorphic) operators $\partial$, and $f \mapsto f_1$ become another such $\mathfrak{sl}_2$-triple after inverting the sign of one of the latter two operators, now on (holomorphic) quasi-modular forms.

We therefore look for $\mathfrak{sl}_2$-triples arising from $W$ and the various weight changing operators on the spaces $M^{sing}_{k-\infty}(V_\infty \otimes \rho)$ (or their finite-dimensional counterparts). We present a 2-parameter family of such $\mathfrak{sl}_2$-triples acting on the spaces $M^{sing}_{k-\infty}(V_\infty \otimes \rho)$, in which the weight raising operator is assumed to be the usual holomorphic differentiation on the image of $M^{mtr}_{k-\infty}(\rho)$, and of which the $\mathfrak{sl}_2$-triples from [V] and [A] are special cases.

**Corollary 1.10.** Fix any number $a$, as well as two numbers $b$ and $c$ that satisfy $ab + (1 - a)c = 1$, and consider the space $\bigoplus_k M^{sing}_{k-\infty}(V_\infty \otimes \rho)$. The operators $W$, $\delta$ which is $\tilde{\delta}_{ak}$ on $M^{sing}_{k-\infty}(V_\infty \otimes \rho)$, and $\tilde{\delta} = b \cdot 4y^2 \partial_\tau - cD$ then form an $\mathfrak{sl}_2$-triple.
Proof. The two commutation relations with \( W \) simply express the fact that \( \delta \) always increases the weight by 2, while \( \overrightarrow{\delta} \) decreases it by 2. The commutation relations of \( \delta_{ab} \) with the constituents \( 4y^2\partial_\tau \) and \( -D \) of \( \delta \) on \( \mathcal{M}^{\text{sing}}_{k-\infty}(V_\infty \otimes \rho) \) are determined by parts (vi) and (iv) of Proposition 1.9 as \( ak \) and \( (1 - a)k \) respectively (times the identity operator on that space), and the condition on \( b \) and \( c \) makes sure that the commutation relation with \( \overrightarrow{\delta} \) itself would be \( k \) like the operation of \( W \). This proves the corollary. \( \square \)

Note that in Corollary 1.10 we have a 2-parameter family, since \( a \) is free but \( b \) and \( c \) are related as being on an affine line depending on \( a \). The \( \mathfrak{sl}_2 \)-triples from \( \mathfrak{V} \) is the case \( a = 1, b = 1, \) and \( c = 0 \) in Corollary 1.10, while \( \mathfrak{A} \) considers the case with \( a = 0, b = 0 \) (so that \( \overrightarrow{\delta} \) becomes linear over \( \mathcal{M}^{\text{an}}_0(\Gamma) \) or \( \mathcal{M}^{\text{sing}}_0(\Gamma) \) since it does not involve differentiation anymore), and \( c = 1 \). Such a linear choice for \( \overrightarrow{\delta} \) is possible wherever \( a \neq 1 \), with \( b = 0 \) and \( c = \frac{1}{1-a} \). Choosing just a multiple of \( 4y^2\partial_\tau \) for \( \overrightarrow{\delta} \), i.e., with \( c = 0 \), is an option only if \( a \neq 0 \), with \( b = \frac{1}{a} \). It is also possible to make \( \delta \) linear over \( \mathcal{M}^{\text{an}}_0(\Gamma) \) or \( \mathcal{M}^{\text{sing}}_0(\Gamma) \), by completing the \( a \)-line to a projective line, through allowing multiplication of \( b \) and \( c \) by a non-zero scalar \( \tau \) and dividing \( \delta \) by \( \tau \). By setting \( \tau = a \) and taking the limit \( a \to \infty \) we obtain the commutation relations of \( k \cdot \frac{\tau^2}{2y} \) with a difference of the sort \( b \cdot 4y^2\partial_\tau - \epsilon D \) with \( b - c = 1 \). However, the choices from \( \mathfrak{V} \) and \( \mathfrak{A} \) seem like the most natural choices for specific applications.

We conclude this section by presenting the geometric origin of our operators. The representations \( V_m \) can be seen as complex local systems on \( \mathcal{H} \) and its quotients by discrete groups. There is a natural connection \( \nabla \), called the Gauss–Manin connection, which takes sections of a vector bundle \( \mathcal{E} \) involving \( V_m \) (such as those vector bundles having \( \mathcal{M}^{\text{an}}_{k-m}(V_m \otimes \rho) \) as their real-analytic global section of \( \Gamma \backslash \mathcal{H} \) for a discrete subgroup \( \Gamma \)) to sections of \( \mathcal{A} \otimes \mathcal{E} \), where \( \mathcal{A} \) represents real-analytic differential forms on the quotient \( \Gamma \backslash \mathcal{H} \). The decomposition of differential forms in \( \mathcal{A} \) to those involving \( d\tau \) and those with \( d\tau \) decomposes \( \nabla \) naturally as the sum of \( \nabla^h \) (resulting in expressions involving \( d\tau \)) and \( \nabla^k \) (attaining differential forms with \( d\tau \)). Now, \( V_m \) carries a natural Hodge filtration, corresponding to vanishing of the components not involving \( \tau \) to a high enough power (i.e., sections of \( F^p\mathcal{E} \) are precisely those vector-valued modular forms that are associated to the subspace \( \mathcal{M}^{\ast \leq m-p}(\rho) \) of \( \mathcal{M}^{\ast \leq m}(\rho) \) in Theorem 1.6), putting a structure of a variation of Hodge structures on the associated vector bundle. Then each of \( \nabla^h \) and \( \nabla^k \) decomposes again into two components, according to whether the Hodge weight remains the same or is shifted by 1. The parts shifting the Hodge weight are \( C^\infty \)-linear. These components are evaluated, in this case of the VHS arising from \( V_m \), by \( \mathbb{Z}_n \), which is a good reference for many details of this construction. It is easily seen that \( \nabla^h \) increases the weight by 2, while \( \nabla^k \) decreases it by 2, and that they come from the operators \( \delta_{k-m} \) and \( y^2\partial_\tau \) on \( \mathcal{M}^{\ast \leq m}_{k-m}(V_m \otimes \rho) \). The evaluations leading to the alternative proofs for parts (i) and (iv) of Theorem 1.6 namely those in which these operators act on elements of \( \mathcal{M}^{\ast \leq m}_{k-m}(V_m \otimes \rho) \) presented as in Theorem 1.3 with the functions \( F_s \), thus imply that the \( C^\infty \)-linear, Hodge weight changing
part of $\nabla^h$ is just our operator $i_{m-1} \circ D_m$ (up to some normalizing scalars), and the complementary part of $\nabla^h$ operates just as $y^2 \partial_\tau$ on the components $F_s$, $0 \leq s \leq m$. For the decomposition of $\nabla^h$ we get an analogous picture, involving some multiple of the operator $i_{m-1} \circ D_m$ (up to some normalizing scalars), as well as an operator that is “complex conjugate” to $i_{m-1} \circ D_m$ (up to powers of $2iy$).

2 Laplacians and Eigenfunctions

In this Section we present a 2-parameter family of Laplacian operators on the space $M^{\text{sing}}_{k-\infty}(V_\infty \otimes \rho)$, and determine the corresponding eigenspaces in most cases.

2.1 Laplacians and Lifts

The center of the universal enveloping algebra of $\mathfrak{sl}_2$ is generated, as a polynomial ring, by the Casimir element $C = H^2 + 2EF + 2FE$. Given a space on which $\mathfrak{sl}_2(\mathbb{R})$ acts, the Casimir element operates as a central operator, and the Laplacian of the action is a suitable normalization of this operator. We wish to normalize our Laplacians such that they will always annihilate holomorphic (and meromorphic) modular forms of depth 0. We therefore write $C$ as $4EF + H^2 - 2H$, and recall that $F = \delta$ (in the notation of Corollary 1.10) annihilates the required functions, $C$ preserves the weights (since it commutes, in particular, with $H$), and that on the space of modular or quasi-modular forms of a fixed weight, $H = W$ is just multiplication by a scalar (this is the weight $k$ also on the spaces $M^\text{sing}_{k-\infty}(V_\infty \otimes \rho)$ or $M^\text{sing}_{k-m}(V_m \otimes \rho)$, even though the weight of modular forms in the latter space is $k - m$). We shall therefore define the Laplacian operator to be the one corresponding to the action of $\frac{C - W^2 + 2W}{2}$, i.e., we consider just the combination $\delta \circ \overline{\delta}$, which we shall denote by $\Delta^{(a,b,c)}_{k-\infty}$. Here $a$, $b$, and $c$ are assumed to be three numbers satisfying the condition from Corollary 1.10. We recall that in the scalar-valued case considered in [V] this operation yields just the usual modular Laplacian $\Delta_k = 4\delta_{k-2y^2}\partial_\tau$, written in coordinates more neatly as $4y^2\delta_k\partial_\tau$.

We begin our analysis of eigenfunctions by evaluating the action of the composition $\Delta^{(a,b,c)}_{k-\infty}$ of the operators $\delta$ and $\overline{\delta}$ defined in Corollary 1.10 (with those $a$, $b$, and $c$) on elements of $M^\text{sing}_{k-\infty}(V_\infty \otimes \rho)$. Recall that we consider only elements that are the limit images of elements $M^\text{sing}_{k-m}(V_m \otimes \rho)$ (for some $m$), and we present the former modular forms as in Theorem 1.3 with the functions $F_s$. In fact, some of the calculations might be shortened if we had used the presentation with the $f_s$s there, but as the functions $F_s$ are modular while the $f_s$s are quasi-modular, we prefer to apply the more well-known theory of eigenfunctions that are modular.
Lemma 2.1. Let the numbers a, b, and c and the operators δ and \( \vec{\delta} \) be as in Corollary 1.10. The image of the element \( F \in \mathcal{M}_{k-\infty}^\text{sing} (V_\infty \otimes \rho) \) sending \( \tau \) to 
\[
\sum_{s=0}^{d} \frac{F_s(\tau)}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s
\]
for some depth \( d \) under \( \Delta_{k-\infty} = \delta \circ \vec{\delta} \) is the function sending \( \tau \) to 
\[
\sum_{s=0}^{d+1} \frac{F_{s+1}(\tau)}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s,
\]
where \( F_{s,\Delta} \) is 
\[
\delta \Delta_{k-2s}F_s + (b-c)(s+1)\delta_{k-2s}F_{s+1}\partial_{\tau}[1-a](k-2)-s+1][s(b-c)F_s + 4by^2 \partial_{\tau}F_{s-1}]
\]
and in the evaluation of \( F_{0,\Delta} \), \( F_{d,\Delta} \), and \( F_{d+1,\Delta} \) we substitute just 0 for \( F_{-1}, F_{d+1}, \text{ or } F_{d+2} \).

Proof. Take some \( m > d \), and we consider our element \( F \in \mathcal{M}_{k-\infty}^\text{sing} (V_\infty \otimes \rho) \) as the image of the modular form \( F^{(m)} : \tau \rightarrow \sum_{s=0}^{d} \frac{F_s(\tau)}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s \) from \( \mathcal{M}_{k-m}^\text{sing} (V_m \otimes \rho) \) under the limit map. As \( F^{(m)} \) is an \( i_{m-1} \)-image (since \( m > d \)), we can apply all the operators from Theorem 1.6 to it. Corollary 1.7 shows that for \( \vec{\delta} \) we can let \( 4by^2 \partial_{\tau} - c(i_{m-1} \circ D_m) \) operator on \( F^{(m)} \), and applying the maps from Theorem 1.3 together with the formulae for transferring operators in Theorem 1.6 and Proposition 1.3, we find that the resulting element of \( \mathcal{M}_{k-2m}^\text{sing} (V_m \otimes \rho) \) takes \( \tau \) to 
\[
\sum_{s=0}^{d} \frac{4by^2a_s \partial_{\tau}F_s(\tau) + (b-c)(s+1)\partial_{\tau}[1-a](k-2)-s+1][s(b-c)F_s + 4by^2 \partial_{\tau}F_{s-1}}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s
\]
(this can also be proved via a direct evaluation). The operator corresponding to \( \delta = \delta_{\text{odd}} \) on this space is \( \delta_{k-2m} - (m + (a-1)(k-2)) (\mathbb{1}_{m+1} - i_{m-1}) \) (Corollary 1.7 again), and if we denote the latter element of \( \mathcal{M}_{k-2m}^\text{sing} (V_m \otimes \rho) \) as \( \tau \rightarrow \sum_{s=0}^{d} \frac{G_s(\tau)}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s \) then a similar argument using Theorems 1.3 and 1.6 and Proposition 1.5 (or another direct evaluation) produces the function sending \( \tau \) to 
\[
\sum_{s=0}^{d} \frac{\delta_{k-2s}G_s(\tau) + (1-a)(k-2)-s+1][s(b-c)F_s + 4by^2 \partial_{\tau}F_{s-1}}{V_{2y}^s} (\mathbb{1}^\infty - s \mathbb{1})^s
\]
Plugging in the expressions defining the functions \( G_s \), observing the formula for \( \Delta_{k-2s} \), and taking back the limit map from \( \mathcal{M}_{k-2m}^\text{sing} (V_m \otimes \rho) \) to \( \mathcal{M}_{k-\infty}^\text{sing} (V_\infty \otimes \rho) \), we obtain the desired expression for \( F_{0,\Delta} \). This proves the lemma.

We adopt henceforth the usual convention in the spectral theory of modular forms, in which the eigenvalues are with respect to minus the Laplacian operator. The solutions of the equation \( \Delta^{(a,b,c)}_{k-\infty} = (\delta \circ \vec{\delta})F = -\lambda F \) have, in most cases, a component \( F_0 \) that is an eigenfunction with respect to the usual Laplacian \( \Delta_k \). We therefore begin our investigation of these solutions under the assumption that \( \Delta_k F_0 = -\mu F_0 \) for some complex number \( \mu \). We shall be needing the following lemma.

Lemma 2.2. Let \( g \) and \( \psi \) be two elements of \( \mathcal{M}_{k}^\text{sing}(\rho) \), let \( h \) and \( \varphi \) be two elements of \( \mathcal{M}_{k-2}^\text{sing}(\rho) \), and assume that \( h = \delta g \) and \( \psi = 4y^2 \partial_{\tau} \varphi \). Then the following assertions hold:

(i) If \( g \) has eigenvalue \( \nu \) with respect to (minus) \( \Delta_k \) then \( h \) is also an eigenfunction, the eigenvalue being \( \nu +1 \). In case \( \varphi \) is an eigenfunction belonging to the eigenvalue \( \kappa \) then the eigenvalue of the eigenfunction \( \psi \) is \( \kappa -1 \).
(ii) Assuming now that \( h \) is an eigenfunction, and its eigenvalue \( \kappa \) does not vanish, the function \( g \) equals \( \frac{4y^2\partial_jh}{\kappa} \) plus the complex conjugate of an element of \( \mathcal{M}_{l+2}^{\text{mer}}(\mathcal{F}) \), divided by \( y^l \). If \( \psi \) has eigenvalue \( \nu \neq -l \) then \( \varphi \) is \( \frac{4y^2\partial_jh}{\kappa} \) plus an element of \( \mathcal{M}_{l+2}^{\text{mer}}(\rho) \).

(iii) Under the assumptions of part (ii), if \( g \) (resp. \( \varphi \)) is also an eigenfunction and \( h \neq 0 \) (resp. \( \psi \neq 0 \)) then \( g \) equals precisely \( \frac{4y^2\partial_jh}{\kappa} \) (resp. \( \varphi = \frac{4y^2\partial_j\psi}{\kappa} \)), with eigenvalue \( \kappa - l \) (resp. \( \nu + l \)).

(iv) If \( h \) is harmonic, then it is already meromorphic, and \( g \) has eigenvalue \( -l \). Similarly, in a situation where \( \psi \) has the eigenvalue \( -l \) then it is the complex conjugate of an element of \( \mathcal{M}_{l+2}^{\text{mer}}(\mathcal{F}) \) divided by \( y^l \), and \( \varphi \) is harmonic.

Proof. Part (i) follows directly from the commutation relations between the Laplacians \( \Delta_l \) or \( \Delta_{l+2} \) and the operators \( \delta_l \) and \( 4y^2\partial_j \). For part (ii) we use the definition of the Laplacian \( \Delta_l \), or apply the commutation relation between \( 4y^2\partial_j \) and \( \delta_l \) and the definition of \( \Delta_{l+2} \), and obtain that applying \( \delta_l \) to \( \frac{4y^2\partial_jh}{\kappa} \) or \( 4y^2\partial_j \) to \( \frac{4y^2\partial_j\psi}{\kappa} \) yields again \( h \) or \( \psi \). Therefore the difference between \( g \) or \( \varphi \) and these functions must be in the kernel of the asserted operators. For \( 4y^2\partial_j \) this kernel consists just of meromorphic functions, and for \( \delta_l \), i.e., the conjugation of \( \partial_j \) by \( y^l \), it is indeed the asserted type of functions. This proves part (ii). Now, the non-vanishing assumption on \( h \) and \( \psi \) and the eigenvalue assumption show that the asserted functions do not vanish as well, and their eigenvalues are also the asserted ones by part (i). But dividing the complex conjugate of an element of \( \mathcal{M}_{l+2}^{\text{mer}}(\mathcal{F}) \) by \( y^l \) gives an element of \( \mathcal{M}_{l}^{\text{sing}}(\rho) \) with eigenvalue \( -l \), which is different from \( \kappa - l \) since \( \kappa \neq 0 \). Moreover, the eigenvalue 0 of meromorphic function is different from \( \nu + l \) (as \( \nu \neq -l \)). This implies that if \( g \) and \( \varphi \) are eigenfunctions then these additional parts cannot appear, establishing part (iii). Finally, if \( h \) is harmonic or \( \psi \) has eigenvalue \( -l \) then \( 4y^2\partial_j\delta_lg \) is annihilated by \( \delta_l \) (so that \( g \) has eigenvalue \( -l \)) and \( \Delta_l\varphi \) lies in the kernel of \( 4y^2\partial_j \) (hence it is meromorphic and therefore harmonic). If \( g \) has a constituent of eigenvalue different from \( -l \), this constituent would appear with a non-zero multiplier in \( 4y^2\partial_j\delta_lg \) (which is \( -(\Delta_l+l)g \)), contradicting the fact that the latter function has eigenvalue \( -l \). Similarly, if \( \varphi \) has an eigenpart that is not harmonic then so does \( \Delta_l\varphi \), and the latter function cannot be harmonic. Therefore \( g \) has eigenvalue \( -l \), and \( \varphi \) is harmonic. But then \( h = \delta_lg \) must be annihilated by \( 4y^2\partial_j \) (i.e., be meromorphic), and \( \psi = 4y^2\partial_j\varphi \) is in the kernel of \( \delta_l \) described above. This completes the proof of part (iv), and with it the proof of the lemma.

Recall that an element of \( \mathcal{M}_{k-2d}^{\text{sing}}(\rho) \) has singularities if and only one (or all) of its images under \( \delta_{k-2d}^l \) for some \( l \) has singularities (unless the \( \delta_{k-2d}^l \)-image vanishes). We could therefore replace in Lemma 2.2 as well as in all the results below, the superscripts \( \text{sing} \) by \( \text{an} \) and \( \text{mer} \) by \( \text{hol} \) (or perhaps by \( \text{wh} \) if \( \Gamma \) has cusps, depending on the growth conditions we put on \( \mathcal{M}_{k-\infty}^{\text{an}}(V_\infty \otimes \rho) \)), and still get valid statements.
Lemma 2.2 then allow us to replace \( 4 \mu \lambda F \) to the four functions hypothesis on \( b \) then part (i) of Lemma 1.10) we get \( 4 \) equation with \( 0 \), not including the latter term, yields the equality between

Definition 2.3. Let \( F \in \mathcal{M}_{k-\infty}^{\text{sing}}(V_\infty \otimes \rho) \) be of depth at most \( d \) (i.e., it is the image of an element \( F^{(d)} \in \mathcal{M}^{\text{sing}}(V_d \otimes \rho) \) in the limit map), whose expansion from in Theorem 1.3 is based on the functions \( F_s \). Assume that \( F \) is an eigenfunction with some eigenvalue \( \lambda \) with respect to \( \Delta_{k-\infty}^{(a,b,c)} \), and that each function \( F_s \) is an eigenfunction with respect to \( \Delta_{k-2s} \), where the eigenvalue of \( F_0 \) is some complex number \( \mu \). Then the following assertions hold:

(i) If \( b \neq 0, b \neq c \) and \( \mu \) does not equal any value \( j(k-1-j) \) for \( 0 \leq j < d \), then the eigenvalue of \( F_s \) is \( \mu + s(s+1-k) \) for any \( 0 \leq s \leq d \), and there are explicit scalars \( \xi_s, 1 \leq s \leq d \) such that \( F_{s-1} = \xi_{s-1} \delta_{k-2s} F_s \) for every \( s \) with \( F_s \neq 0 \).

(ii) In the case where \( b = c \) (so that both equal \( 1 \) by the equality from Corollary 1.11) we get \( \lambda = \mu \), and each function \( F_s \) is an explicit multiple of \( 4y^2 \partial_{F_{s-1}} \), with the eigenvalue \( \mu + s(s+1-k) \), unless either \( F_{s-1} \) is harmonic or \( (1-a)(k-2) = s-1 \).

(iii) Assuming that \( b = 0 \) (so that \( a \neq 1 \) and \( c = \frac{1}{1-a} \)), each function \( F_s \) is an explicit multiple of \( \delta_{k-2s} F_{s+1} \), unless \( F_{s+1} \in \ker \delta_{k-2s} \). This assertion holds also without the assumption that each \( F_s \) is an eigenfunction for the Laplacian of the appropriate weight.

Proof. In part (i) we work by induction on \( s \). The basis of our induction is the fact that \( F_{-1} = 0 \) is 0 times \( \delta k F_0 \), and the eigenvalue \( \mu \) of \( F_0 \) is \( \mu + s(s+1-k) \) with \( s = 0 \). Assuming now that our assumption holds for \( s \), and consider the equation associated with \( s \) in Lemma 2.1. It states that adding \( (s+1)(b-c)\delta_{k-2s} F_{s+1} \) to the four functions \( \lambda F_s, b \Delta_{k-2s} F_s, s[(1-a)(k-2) + 1 - s](b - c) F_s \), and \( b[(1-a)(k-2) + 1 - s] \cdot 4y^2 \partial_{F_{s-1}} \) yields the function 0. But the induction hypothesis allows us to replace \( \Delta_{k-2s} \) by the scalar \( -\mu - s(s+1-k) \), and write \( F_{s-1} \) as the \( \delta_{k-2s} \)-image of the eigenfunction \( \xi_{s-1} F_s \). Hence \( F_{s-1} \) has the eigenvalue \( \mu + (s-1)(s-k) \) (either by part (i) of Lemma 2.2 or by the induction hypothesis on \( s-1 \), which is non-zero by our assumption). Parts (ii) and (iii) of Lemma 2.2 then allow us to replace \( 4y^2 \partial_{F_{s-1}} \) by \( -\alpha_{s-1} \xi_{s-1} F_{s-1} \). This means that the function \( (s+1)(b-c)\delta_{k-2s} F_{s+1} \) is a multiple of \( F_s \), and since our assumption is that \( F_s \) is not harmonic, part (iv) of Lemma 2.2 shows that if \( F_{s+1} \neq 0 \) then the constant multiplying \( F_s \) is non-zero. We thus obtain the equality \( F_s = \alpha_s \delta_{k-2s} F_{s+1} \) for an appropriate non-zero constant \( \alpha_s \). Now, \( F_s \) is known to be an eigenfunction, and if we assume the same about \( F_{s+1} \), then part (iii) of Lemma 2.2 shows that the eigenvalue of \( F_{s+1} \) is obtained by subtracting \( l = k - 2s \) from the eigenvalue \( \mu + s(s+1-k) \) of \( F_s \). As the resulting eigenvalue is indeed \( \mu + (s+1)(s+2-k) \), the required assertion holds also for \( s+1 \) (note that this would have been the eigenvalue of \( F_{s+1} \) also if this function was a non-zero element of \( \ker \delta_{k-2s} \)). This proves part (i).

In the case considered in part (ii) the equation from Lemma 2.1 shows that \( -\lambda F_s \) is the sum of \( \Delta_{k-2s} F_s \) and \( [(1-a)(k-2) + 1 - s] \cdot 4y^2 \partial_{F_{s-1}} \). The equation with \( s = 0 \), not including the latter term, yields the equality between
λ and μ. Assume, by induction, that \( F_{s-1} \) has the asserted eigenvalue. Since \( F_s \) is assumed to be an eigenfunction, it follows from the 3rd equation that unless \( F_{s-1} \) is meromorphic or the constant \((1-a)(k-2)\) equals \( s-1 \), some constant multiple of \( F_s \) equals the non-zero function \([(1-a)(k-2)+1-s]\cdot\frac{y^s}{\tau F_{s-1}}\). As the latter function has eigenvalue \( k-2s \) less than the eigenvalue \( \mu+(s-1)(s-k) \) of \( F_{s-1} \), we deduce that \( F_s \) has the desired eigenvalue \( \mu+s(s+1-k) \). This establishes part (ii) as well.

For part (iii) we observe that again two expressions vanish in the equation from Lemma 2.2, leaving only the equality of \(-s\). As we consider functions of a given depth \( d \), we shall begin with a specific \( F_s \) that establishes part (ii) of that proposition can be transferred, except for some specific values of \( \lambda \) (or equivalently \( \mu \)) to the same condition. As we consider functions of a given depth \( d \), we shall begin with a specific element \( \varphi \in \mathcal{M}^{sing}_{k-\infty}(\rho) \), and consider the element of \( \mathcal{M}^{sing}_{k-\infty}(V_\infty \otimes \rho) \) resulting from starting with \( F_d \) as \( \varphi \) and getting the other functions \( F_s \) according to that rule the lift of \( \varphi \) (using the parameters \( a, b, c, \lambda \), the weight \( k \), and the depth \( d \)).

We shall denote, for this set of parameters, by \( \mathcal{M}^{sing,d,\lambda}_{k-\infty}(\rho) \) the set of elements of \( \mathcal{M}^{sing}_{k-\infty}(V_\infty \otimes \rho) \) that arise as lifts of modular forms in \( \mathcal{M}^{sing}_{k-\infty}(\rho) \). We shall now restrict attention to lifts, since Corollary 2.8 below shows that except in very few special cases, the space of elements of \( \mathcal{M}^{sing}_{k-\infty}(V_\infty \otimes \rho) \) that are eigenfunctions with eigenvalue \( \lambda \) with respect to (minus) the Laplacian \( \Delta_k(\rho) \) is just the direct sum over \( d \) of the spaces \( \mathcal{M}^{sing,d,\lambda}_{k-\infty}(V_\infty \otimes \rho) \).

### 2.2 Lifts of Meromorphic Modular Forms

The eigenfunctions that are obtained as lifts of meromorphic modular forms are as follows.

**Theorem 2.4.** Fix \( a, b, c, \Gamma, \rho, \) a depth \( d \), and a weight \( k \) as above.

(i) Assume that \( b \neq 0, (1-a)(k-2) \) is not an integer between 0 and \( d \), and \( k \) is not an integer between \( d+1 \) and \( 2d \) (the condition on \( k \) is trivial if \( d = 0 \)). Then there are at most \( d+1 \) eigenvalues \( \lambda \) for which the space \( \mathcal{M}^{sing,d,\lambda}_{k-\infty}(V_\infty \otimes \rho) \) is non-zero, and for each such \( \lambda \) this space consists precisely of lifts of elements of \( \mathcal{M}^{mer,d}_{k-\infty}(\rho) \).

(ii) Assume that \( b \neq 0, d > 0, (1-a)(k-2) \) is an integer \( 0 \leq j-1 < d \), and \( k \) is not an integer between \( d+j+1 \) and \( 2d \) (the latter condition is empty if \( j = d \)). In this case the number of possible \( \lambda_s \) is \( d+1-j \), and for each
such \( \lambda \) the space \( \mathcal{M}_{k-\infty}^{\text{sing.d.}(a,b,c)}(V_{\infty} \otimes \rho) \) consists again, except in some special cases, of uniquely defined lifts of arbitrary elements of \( \mathcal{M}_{k-2d}^{\text{mer}}(\rho) \). Similarly, if \( b \neq 0 \) and \( d > 0 \), \( (1-a)(k-2) \) is not an integer between 0 and \( d \), but the weight \( k \) is an integer \( d+p \) for some \( 1 \leq p \leq d \), then the same assertion holds with \( j \) replaced by \( p \).

(iii) Assuming that both \( j = (1-a)(k-2) + 1 \) and \( p = k-d \) are integers between 0 and \( d-1 \), the same result from part (ii) holds, with the number of possible values of \( \lambda \) being bounded by \( d+1-\max\{p,j\} \).

Proof. Let \( F \in \mathcal{M}_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho) \) be an element of depth precisely \( d \) that is an eigenfunction associated with some eigenvalue \( \lambda \) and is also a lift, and consider the equation associated with \( s = d+1 \) in Lemma 2.1. Since \( F_{d+1} \) and \( F_{d+2} \) vanish, but \( b \) and the coefficient \( (1-a)(k-2)-d \) are assumed not to vanish, we obtain the vanishing of \( 4y^2\partial_\tau F_d \). Hence \( F_d \) is an element \( \varphi \in \mathcal{M}_{k-2d}^{\text{mer}}(\rho) \), and \( F_d = \alpha_{d,s} \delta_{k-2d}^{d-s} \) for some coefficient \( \alpha_{d,s} \) (because we assume that \( F \) is a lift), with \( \alpha_{d,d} = 1 \). A repeated application of part (i) of Lemma 2.2 starting with the eigenvalue 0 of the meromorphic function \( \varphi \), shows that \( F_s \) has eigenvalue \( (d-s)(k-d-s-1) \). Moreover, parts (ii) and (iii) of Lemma 2.2 imply (under our assumption on \( k \)) that the term \( 4y^2\partial_\tau F_{s-1} \) appearing in the 3th equation from Lemma 2.1 (with \( s > 0 \)) equals \( -(d-s+1)(k-s-d)\alpha_{d,s-1} \delta_{k-2d}^{d-s} \). Substituting all these values into the 3th equation from Lemma 2.1 we obtain a single function \( \delta_{k-2d}^{d-s} \) (which is clearly non-zero since \( \varphi \) is meromorphic and if \( d > 0 \) then it is non-constant), that must vanish after being multiplied by the coefficient

\[
\begin{align*}
\{\lambda + s(b-c)[(1-a)(k-2) - s + 1] - b(d-s)(k-d-s-1)\} & \alpha_{d,s} + \\
+ (b-c)(s+1) & \alpha_{d,s+1} - b[(1-a)(k-2) - s + 1](d-s+1)(k-s-d) & \alpha_{d,s-1}.
\end{align*}
\]

(2)

Starting with the values \( \alpha_{d,d+1} = 0 \) (since the term with \( F_{d+1} \) vanishes in the equation associated with \( s = d \)) and \( \alpha_{d,d} = 1 \), and recalling that our assumptions on \( b \), \( a \), and \( k \) imply that the coefficient of \( \alpha_{d,s-1} \) in Equation (2) does not vanish for any \( 1 \leq s \leq d \), we obtain an expression for \( \alpha_{d,s-1} \) using the previous two coefficients \( \alpha_{d,s} \) and \( \alpha_{d,s+1} \). Finally, setting \( s = 0 \) in Equation (2) yields the vanishing of \( \lambda \) if and only if \( F_s = \alpha_{d,0} \delta_{k-2d}^{d-s} \) with these values of \( \alpha_{d,s} \) and the two coefficients \( \alpha_{d,0} \) and \( \alpha_{d,1} \) satisfy the latter equality. Moreover, this assertion is independent of the choice of the function \( \varphi \in \mathcal{M}_{k-2d}^{\text{mer}}(\rho) \).

Now, a simple analysis of these coefficients as a function of \( \lambda \) (with \( a \), \( b \), \( c \), \( d \), and \( k \) as fixed parameters) shows that \( \alpha_{d,s} \) is a monic polynomial of degree \( d-s \) in \( \lambda \), divided by \( b^{d-s}(d-s)! \prod_{i=s+1}^{d}(1-a)(k-2) - i(k-d-i-1) \). Multiplying the equality involving \( \alpha_{d,0} \) and \( \alpha_{d,1} \) by the latter denominator of \( \alpha_{d,0} \) we obtain a monic polynomial of degree \( d+1 \), of which \( \lambda \) must be a root if it is an eigenvalue. As a polynomial of degree \( d+1 \) has at most \( d+1 \) distinct roots, this proves part (i). For part (ii) we present only the case of integral
(1 − a)(k − 2), since the other case presented there is identical with j replaced by p. Following the same argument yields again the meromorphicity of \( F_d = \varphi \), as well as such expressions for \( \alpha_{d,s} \) for every \( j + 1 \leq s \leq d \). But now Equation (2) with \( s = j \leq d \) contains a vanishing coefficient in front of \( \alpha_{d,j−1} \). Therefore we obtain the vanishing of \( \left[ \lambda - b(d-j)(k-d-j-1) \right] \alpha_{d,j} + (b-c)(j+1)\alpha_{d,j+1}, \) which is a polynomial of degree \( d + 1 \) in \( \lambda \). Equation (2) with \( 1 \leq s < j \) now expresses \( \alpha_{d,s−1} \) (multiplied there by a non-zero coefficient) in terms of \( \alpha_{d,s+1} \) and \( \alpha_{d,s} \). Since \( \alpha_{d,j} \) is already fixed by the previous argument, we may consider \( \alpha_{d,j−1} \) as free and express all the remaining coefficients in terms of \( \alpha_{d,j−1} \). But Equation (2) with the last value \( s = 0 \), which is a relation between \( \alpha_{d,0} \) and \( \alpha_{d,1} \), yields an affine linear equation for \( \alpha_{d,j−1} \). Unless the coefficient of \( \alpha_{d,j−1} \) vanishes, this determines the value of \( \alpha_{d,j−1} \) uniquely, establishing the existence and uniqueness of the lifts also in this case. If this coefficient does vanish then either no lifts exist, or there is another degree of freedom in the definition of the lift. Part (iii) is proved just like part (ii): If \( j = p \) then nothing changes in the proof, while if (without loss of generality) \( j > p \) then the only difference is that \( \alpha_{d,j−1} \) is determined by Equation (2) with \( s = p \), and \( \alpha_{d,p−1} \) becomes a free variable satisfying an affine linear equation arising from \( s = 0 \) in Equation (2) (again up to special cases where the affine linear equations have vanishing coefficients). This proves the theorem.

Before we turn to the cases in which lifts of non-meromorphic modular forms are involved, we deduce explicit formulae for some eigenvalues \( \lambda \) and their associated lifts appearing in Theorem 2.4.

**Proposition 2.5.** Assume that \( b \neq 0 \) and \( (1-a)(k-2) \neq d \) as in Theorem 2.4.

(i) In case \( b \neq c \) there always exist unique harmonic lifts from \( \mathcal{M}_{k-2d}^{\text{mer}}(\rho) \) into \( \mathcal{M}_{k-\infty}^{\text{sing}}(V_\infty \otimes \rho) \), except when \( k = d + p \) for some \( 1 \leq p \leq d \). Explicitly, the space \( \mathcal{M}_{k-\infty}^{\text{sing},d,a,b,c}(V_\infty \otimes \rho) \) consists precisely of the lifts of elements \( \varphi \in \mathcal{M}_{k-2d}^{\text{mer}}(\rho) \) to functions of the form

\[
\tau \mapsto \sum_{s=0}^{d} \binom{d}{s} (b-c)^{d-s} \left( \prod_{l=0}^{d-s} \frac{1}{k-2d+l} \right) \delta_{2d}^{d-s} \varphi(\tau) \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{\infty-2s} \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{s}.
\]

(ii) The eigenspaces in depth 0 are just the single space \( \mathcal{M}_{k-\infty}^{\text{sing},0,a,b,c}(V_\infty \otimes \rho) \) from part (i). This is also the case for \( \mathcal{M}_{k-\infty}^{\text{sing},d,a,b,c}(V_\infty \otimes \rho) \) with general \( d \) in case \( (1-a)(k-2) = d - 1 \), while if \( k = 2d \) then the unique possible eigenvalue to which we may have lifts is \( \lambda = d(c-b)(1-a)(k-2) - d + 1 \). In depth 1, under the assumption that \( k \neq 2 \) and \( a \neq 1 \), the additional space is \( \mathcal{M}_{k-\infty}^{\text{sing},1,a,b,c}(V_\infty \otimes \rho) \), regardless of the values of \( a, b, \) and \( c \). This space consists of the maps \( \tau \mapsto \varphi(\tau) \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{\infty-2} \left( \begin{array}{c} \tau \\ 1 \end{array} \right) + \frac{\delta_{2d}^{d-s} \varphi(\tau)}{(1-a)(k-2)} \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{\infty} \) for \( \varphi \in \mathcal{M}_{k-2}^{\text{mer}}(\rho) \).
(iii) In case \( b = c \) and the conditions of part (i) of Theorem 2.4 hold, the \( d + 1 \) eigenvalues are \( q(k - 2d + q - 1) \), \( 0 \leq q \leq d \), and 0 is not a multiple root. The resulting lifts of a given element \( \varphi \in M_{k-2d}(\rho) \) involve only non-zero functions \( F_s \) with \( d - q \leq s \leq d \). If, on the other hand, we are in the situation described in parts (ii) or (iii) of Theorem 2.4, the same assertion holds but with \( q \) not exceeding \( d - j \) (or \( d - p \), or \( d - \max\{j, p\} \)).

In particular, in all these cases, the harmonic lift of \( \varphi \in M_{k-2d}(\rho) \) is just \( \tau \mapsto \varphi(\tau)(\frac{\tau}{\lambda})^{\infty-d}(\frac{\tau}{\lambda})^d \).

Proof. We prove, by increasing induction on \( s \), that with \( \lambda = 0 \) and \( b \neq c \), the coefficients \( \alpha_{d,s} \) satisfy the equality \( \alpha_{d,s+1} = \frac{b(d-s)(k-d-s-1)}{(s+1)(b-c)} \alpha_{d,s} \) for every \( 0 \leq s < d \). First, Equation (2) with \( s = 0 \) yields the desired assertion for \( s = 0 \), which is the basis for our induction. Now, Equation (2) with \( s > 0 \) is the sum of \( (s+1)(b-c)\alpha_{d,s+1} - b(d-s)(k-d-s-1)\alpha_{d,s} \) (whose vanishing we require) and the same expression with \( s = 1 \), multiplied by the coefficient \((1-a)(k-2)+1-s\). As the expression with \( s = 1 \) vanishes by the induction hypothesis, our claim follows. This would give us the value for \( \alpha_{d,0} \) as a multiple of the free variable \( \alpha_{d,0} \), and this multiple is non-zero if and only if \( k \) is not an integer between \( d + 1 \) and \( 2d \). The equation with \( s = d \) now becomes trivial: The sum of our vanishing expression with \( s = d - 1 \) multiplied by \((1-a)(k-2)+1-d\), a term involving \( d - d \), and another one with \( \alpha_{d,d+1} = 0 \) must indeed vanish (equivalently, setting \( s = d \) in the equality from above yields a valid statement with any value of \( \alpha_{d,d} \) since both \( \alpha_{d,d+1} \) and the numerator vanish). It follows that a lift with \( \lambda = 0 \) exists if and only if an expression with \( \alpha_{d,d} = 1 \) exists, and as this happens if and only if this multiplier does not vanish. This establishes part (i), since the values of \( \alpha_{d,s} \) are easily seen to be the one appearing in our formula for the lift.

The first assertion of part (ii) follows immediately from part (i) and the fact that there are at most \( d + 1 \) lifts of \( M_{k-2d}(\rho) \) to eigenspaces of \( \Delta_{k-\infty}^{(a,b,c)} \). The same happens with the second assertion, since then the number of lifts is bounded by \( d + 1 - j \) or a similar degree with \( j \) replaced by \( p \) or \( \max\{j, p\} \), and our assumption sets either \( j \) or \( p \) to be \( d \) (but observe that if \( p = d \) then the multiplier of \( \alpha_{d,d} \) that must vanish is \( \lambda + d(b-c)[(1-a)(k-2) - d + 1] \)). For the third assertion, we first observe that the condition \( p = 1 \) stands for \( k = 2 \), which also implies \( j = 1 \) (but the latter equality would also be a consequence of \( a \) being equal to 1). Consider now Equation (2) with \( d = 1 \) and \( s = 1 \), and as that under our assumptions the coefficient of \( \alpha_{1,0} \) there does not vanish, we get \( \alpha_{1,0} = \frac{\lambda^2(b-c)[(1-a)(k-2)]}{b(1-a)(k-2)} \) (since \( \alpha_{1,1} = 1 \)). Equation (2) with \( s = 0 \) was seen to state the cancelation of \( \lambda - b(k-2)\alpha_{1,0} \) and \( (b-c) \) (recall again the value of \( \alpha_{1,1} \)). Multiplying the resulting equality by \( b(1-a)(k-2)^2 \) we obtain a monic quadratic equation for \( \lambda \), in which the free coefficient indeed vanishes (in correspondence with part (i) here), and the coefficient of \( -\lambda \) (which is therefore the other root of this quadratic equation) is \( b(k-2) - (b-c)(1-a)(k-2) \). As this becomes \( k - 2 \) times the expression \( ab + (1-a)c \) from Corollary 1.10 the second root is therefore indeed \( k - 2 \) regardless of the values of \( a, b, \) and \( c \) (and it is different from the first root 0 by our assumption). After substituting the
value of \( \alpha_{1,0} \), which by the same argument simplifies to \( \frac{1}{1-a(k-2)} \), this proves part (ii).

For part (iii) we observe that when \( b = c \) the equality from Corollary 1.10 implies that their common value is 1, and the term involving \( \alpha_{d,s+1} \) in Equation (2) no longer appears. As the coefficient appearing next to \( \lambda \) also vanishes, we obtain that if the coefficient multiplying \( \alpha_{d,s-1} \) does not vanish, then this number becomes \( [(d-s)(k-d-s-1) - \lambda] \alpha_{d,s} \) times a non-zero expression that is independent of \( \lambda \) (we have already substituted \( b = 1 \) as well). From this one deduces, by an easy induction, that \( \alpha_{d,s} = \prod_{i=0}^{d-s-1} (\lambda - i(2d + i - 1)) \)

as long as the denominator does not vanish. If we are in the situation described in case (i) of Theorem 2.4, then \( \alpha_{d,0} = \prod_{i=0}^{d-1} [\lambda - i(k - 2d + i - 1)] \) (up to a non-zero multiplier not depending on \( \lambda \)), and Equation (2) with \( s = 0 \) reduces (again, since \( b = c \)) to \( [\lambda - d(k - d - 1)] \alpha_{d,0} = 0 \). The roots of the resulting polynomial equation are clearly the asserted ones, proving the first assertion.

In the cases described by parts (ii) and (iii) of Theorem 2.4 the same argument yields the value of \( \alpha_{d,s} \) for any \( s \) that is larger than or equal to \( j \) (or \( p \), or \( \max\{j, p\} \)), and Equation (2) with \( s = j \) shows that multiplying \( \alpha_{s,j} \) by \( \lambda - (d-j)(k-2d-1+j) \) yields 0 (and the same with \( p \) or \( \max\{j, p\} \)). This establishes the required assertion also in these cases. The last assertion in part (iii) is just the special case \( q = 0 \) here with \( \alpha_{d,d} = 1 \), and is in correspondence with the formula for the lift in part (i) with \( b = c \). This proves the proposition. \( \square \)

We can use the calculations appearing in the proof of Proposition 2.5 to present a case where lifts do not exist. Take \( d = 1 \), \( b \neq c \) (but still \( b \neq 0 \)), and \( k = 2 \). By part (ii) of that proposition, the only possible eigenvalue is 0 (since \((1-a)(k-2) = 0 \) and \( d = 1 \)). But then the proof of part (i) there shows that \( \alpha_{1,1} \) must equal \( \frac{b(k^2-2)}{b-c} \alpha_{1,0} \), and the numerator vanishes. Hence no lifts exist with these parameters.

### 2.3 Lifts of More General Functions

We now turn to eigenfunctions arising from lifts of elements of \( \mathcal{M}_{3k-2d}^{\text{sing}}(\rho) \) that are not necessarily meromorphic. We denote the space of elements of \( \mathcal{M}_{3k-2d}^{\text{sing}}(\rho) \) that are eigenfunctions belonging to the eigenvalue \( \mu \) by \( \mathcal{M}_{3k-2d}^{\text{sing},\mu}(\rho) \).

**Theorem 2.6.** Let again \( \Gamma \) be a group, \( k \) be a weight, \( \rho \) be a representation (or multiplier system of weight \( k \)) of \( \Gamma \), \( d \) be a depth, and \( a, b, \) and \( c \) be parameters satisfying the condition from Corollary 1.10.

(i) Assume that \( b = 0 \), so that \( a \neq 1 \) and \( c = \frac{1}{1-a} \), and that \( (1-a)(k-2) \) is not an integer between \( d - 1 \) and \( 2d - 2 \) (this is an empty condition if \( d = 0 \)). Then the space \( \mathcal{M}_{3k-2d}^{\text{sing},d,\alpha,b,c}(V_{\infty} \otimes \rho) \) is non-empty only for \( \lambda = d(k-2 + \frac{1}{1-a}) \), while for this value it consists of explicit lifts of arbitrary elements of \( \mathcal{M}_{3k-2d}^{\text{sing}}(\rho) \).
(ii) Still in the case with $b = 0$, but now assuming that $(1 - a)(k - 2)$ is an integer $d - 1 + j$ for some $0 \leq j < d$, the only elements of $M^{sing,d}_{k-2d}(\rho)$ that may be lifted are the complex conjugates of nearly meromorphic modular forms in $M^{2d-k}_{\rho}(\mathcal{F})$, of depth smaller than $d - j$, divided by $y^{k-2d}$. These lifts are based only on functions $F_s$ with $j < s \leq d$.

(iii) Consider now the case where $b \neq 0$, but where $(1 - a)(k - 2)$ coincides with the depth $d$. Then for any $\lambda$ the space $M^{sing,d,(a,b,c)}_{k-\infty,\Delta,\lambda}(V_{\infty} \otimes \rho)$ consists of the direct sum of explicit lifts of the spaces $M^{sing,d}_{k-2d,d}(\rho)$ for at most $d + 1$ values $\mu$, the number being precisely $d + 1$ for all but finitely many $\lambda$s. As for the dependence on $\mu$, the space $M^{sing,d}_{k-2d,\Delta}(\rho)$ lifts to $d + 1$ distinct spaces $M^{sing,d,(a,b,c)}_{k-\infty,\Delta,\lambda}(V_{\infty} \otimes \rho)$, except for finitely many values of $\mu$: When $\mu = -(d + 1 - j)(k - d - j)$ for some $1 \leq j \leq d$ the space $M^{sing,\mu}_{k-\infty,\Delta}(\rho)$ lifts to $M^{sing,d,(a,b,c)}_{k-\infty,\Delta,\lambda}(V_{\infty} \otimes \rho)$ for at most $d + 1 - j$ values $\lambda$, and when $\mu$ is a root of a certain discriminant polynomial the number of values $\lambda$ is also smaller than $d + 1$.

Proof. The equation associated with $s = d$ in Lemma 2.4 becomes, after substituting $b = 0$ and the value of $c$, just the vanishing of $[\lambda - d(k - 2 + \frac{1}{d,s})]F_d$ (since $F_{d+1} = 0$). As we assume that $F_d$, or the lifted modular form $\varphi$ from $M^{sing}_{k-2d}(\rho)$, is non-zero, this determines the value of $\lambda$. As in the proof of part (iii) of Proposition 2.3 setting $F_s = \beta_{d,s}\delta_{k-2d,d}^{d-s}\varphi$ (with $\beta_{d,d} = 1$ as above) implies that the equation associated with $s$ in Lemma 2.4 (with $b = 0$ etc.) becomes the coincidence of $[\lambda - s(k - 2 + \frac{1}{d,s})]\beta_{d,s}$ with $\frac{1}{\rho}\beta_{d,s+1}$. Substituting the value of $\lambda$, this determines $\beta_{d,s}$ as $(s+1)\beta_{d,s+1}$ for any $0 \leq s < d$, which is well-defined since our assumption implies that the numerator does not vanish for any such $s$. As the fact that all these equations are satisfied is equivalent to our lift being a eigenfunction, this establishes part (i). For part (ii) we write $(1 - a)(k - 2)$ as in our assumption, and again get the same single eigenvalue $\lambda$ (which reduces to just $\frac{1}{d,s}$). Each $\beta_{d,s}$ with $j < s < d$ is obtained, via the same argument, from $\beta_{d,s+1}$ via multiplication by $-\frac{s+1}{(d-s)(s-j)}$, but the equation associated with $s = j$ reduces to the vanishing of $\frac{s+1}{(d-j)(d-s)}\delta_{k-2j-2d}^{d-j}\varphi$, i.e., of $\delta_{k-2d}^{d-j}\varphi$. We recall from the proof of Lemma 2.2 that the kernel of the last operation $\delta_{k-2j-2d}^{d-j}$ consists of the complex conjugates of meromorphic functions, divided by $y^{k-2j-2}$. The one-before-last operator to act in $\delta_{k-2d}^{d-j}$ is $\delta_{k-2j-4}$, and we know that its operation on the complex conjugate of a meromorphic function divided by $y^{k-2j-4}$ yields the same conjugated meromorphic function multiplied by $\frac{1}{y^{k-2j-2d}}$. By adding an element of the kernel of $\delta_{k-2j-4}$, we get two conjugated meromorphic functions, divided by two different powers of $y$. By applying the same argument repeatedly we find that an element of $M^{sing}_{k-2d}(\rho)$ is annihilated by $\delta_{k-2d}^{d-j}$ if and only if it can be written as a sum $\sum_{p=0}^{d-j-1} \frac{p}{y^{k-2d-p}}$ for meromorphic functions $\psi_p$, $0 \leq p < d - j$, which amounts to being the complex conjugate of a nearly meromorphic function (that has to be in $M^{sing}_{k-2d}(\mathcal{F})$).
divided by $y^{k-2d}$. This proves part (ii).

In part (iii) we would like to argue as in the proof of Theorem 2.4 but as the coefficient multiplying $4y^2 \partial_y F_d$ in the equation from Lemma 2.1 with $s = d + 1$ vanishes under our assumption, the lifted modular form $\varphi$ can be, at this point, an arbitrary element of $\mathcal{M}_{k-2d}(\rho)$. Let us begin by assuming that $\varphi$ lies in $\mathcal{M}_{k-2d,\lambda}^{\text{sing},\mu}(\rho)$ for some eigenvalue $\mu$. The same arguments from Theorem 2.4, but with the eigenvalue of $\delta_{k-2d}^\mu \varphi$ (or $F_s$) being $\mu + (d - s)(k - 1 - d - s)$, yield an equality similar to the vanishing of the expression from Equation (2).

Assuming first that $\mu$ is not any of the values $-(d + 1 - j)(k - d - j)$ for $1 \leq j \leq d$, the coefficient in front of $\alpha_{d,s-1}$ in the modified Equation 2 does not vanish for any $s$, and the same argument from the proof of Theorem 2.4 shows that $\alpha_{d,s}$ is a monic polynomial of degree $d - s$ in $\lambda$ (considering $\mu$ as a parameter here) divided by the coefficient $b^{d-s}(d - s)! \prod_{j=s+1}^d [\mu + j(k - 2d - 1 + j)]$.

Continuing with the same argument, the equation arising from $s = 0$ in the modified Equation 2 yields a polynomial of degree $d + 1$ in $\lambda$, and it follows that $\mathcal{M}_{k-2d,\lambda}^{\text{sing},\mu}(\rho)$ lifts to $\mathcal{M}_{k-\infty,\Delta,\lambda}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$ if and only if $\lambda$ is a root of this polynomial (and then the lift is given explicitly in the process). As the coefficients of this polynomial are themselves polynomials in $\mu$, we obtain that there are $d + 1$ distinct roots $\lambda$, unless $\mu$ is a root of the polynomial obtained as the discriminant of the polynomial in $\lambda$. In the latter case the polynomial in $\lambda$ has less than $d + 1$ distinct roots. If $\mu = -(d + 1 - j)(k - d - j)$ for such $j$, the coefficient of $\alpha_{d,1-1}$ in the the equation associated with $s = j$ vanishes. Then we get a polynomial of degree $d + 1 - j$ in $\lambda$ whose roots are those values for which $\mathcal{M}_{k-2d,\lambda}^{\text{sing},\mu}(\rho)$ lifts to $\mathcal{M}_{k-\infty,\Delta,\lambda}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$. In this case we indeed have at most $d + 1 - j$ distinct values for $\lambda$. This proves the second assertion.

For the first assertion, we have to consider the coefficients $\alpha_{d,s}$ as expressions in $\mu$. Assuming again that the values $-(d + 1 - j)(k - d - j)$ with $1 \leq j \leq d$ are excluded, and that now $\lambda$ becomes a fixed parameter, a similar analysis proves that $\alpha_{d,s}$ is a polynomial of degree $d - s$ in $\mu$, with leading coefficient $(-b)^{d-s}$, divided by the same expression from above. The polynomial of degree $d + 1$ in $\mu$ arising from the modified Equation (2) with $s = 0$ is the polynomial whose roots are those values $\mu$ for which $\mathcal{M}_{k-2d,\lambda}^{\text{sing},\mu}(\rho)$ lifts to $\mathcal{M}_{k-\infty,\Delta,\lambda}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$. Excluding those finitely many $\lambda$s for which one of the excluded values of $\mu$ is a root, as well as those $\lambda$s for which the polynomial in $\mu$ has multiple roots (i.e., those $\lambda$s for which the discriminant of the polynomial in $\mu$, which is a polynomial in $\lambda$, vanishes), the first assertion in part (iii) is also established.

This completes the proof of the theorem.

We have the following complement of Proposition 2.5 for part (iii) of Theorem 2.6.

**Proposition 2.7.** Let $a$, $b$, $c$, $d$, $k$, and $\lambda$ be as above, and let $\mu$ be an eigenvalue. Assume that $b \neq 0$ and $(1 - a)(k - 2) = d$.

(i) If $b \neq c$ then harmonic elements of $\mathcal{M}_{k-2d}(\rho)$, i.e., elements of the space $\mathcal{M}_{k-2d,\lambda}^{\text{sing},0}(\rho)$, always lift to $\mathcal{M}_{k-\infty,\Delta,\lambda}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$, unless $k = d + p$ for
some integer $1 \leq p \leq d$. Moreover, only lifts of harmonic elements lie in $\mathcal{M}_{k=\infty,\Delta,0}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$.

(ii) In depth 0 the space $\mathcal{M}_{k=\infty,\Delta}^{\text{sing},\mu}(\rho)$ lifts to $\mathcal{M}_{k=\infty,\Delta,\varphi}^{\text{sing},d,a,b,c}(V_\infty \otimes \rho)$, or equivalently the space with eigenvalue $\lambda$ consists of the functions $\tau \mapsto \varphi(\tau)(\xi)^\lambda$ for $\varphi \in \mathcal{M}_{k=\infty,\Delta}^{\text{sing},\lambda/b}(\rho)$. For general $d$, the space $\mathcal{M}_{k=\infty,\Delta}^{\text{sing},2d-k}(\rho)$ lifts only to $\mathcal{M}_{k=\infty,\Delta,\varphi}^{\text{sing},d,(a,b,c)}(V_\infty \otimes \rho)$, the lift of $\varphi$ in the former space being $\tau \mapsto \varphi(\tau)(\xi)^\lambda(\xi)^d$. In depth 1 with $\mu \neq 2 - k$ we get the possible values $\mu$ and $\mu + k - 2$ for $\lambda$ if $b = 1$, and otherwise there is a parameter $t$ such that $\mu = \frac{t}{b - c} + \frac{b}{b - c}$ and the eigenvalue $\lambda$ is $\frac{t}{b - c} + \frac{b}{b - c}$ (yielding the eigenvalues $\lambda = 0$ and $\lambda = k - 2$ from part (ii) of Proposition 2.5 for the harmonic case $\mu = 0$).

(iii) When $b = c$ the possible values for $\lambda$ are $\mu + q(k - 2d + q - 1)$ with $0 \leq q \leq d$, with the corresponding lifts of $\mathcal{M}_{k=\infty,\Delta}^{\text{sing},\mu}(\rho)$ having non-zero functions $F_s$, only for $d - q \leq s \leq d$. This is the case unless $\mu$ takes the value $(d + 1 - j)(k - d - j)$ for some $1 \leq j \leq d$, a case in which the same assertion holds but only with $0 \leq q \leq d - j$. If $q = 0$ (i.e., $\lambda = \mu$) then the element $\varphi \in \mathcal{M}_{k=\infty,\Delta}^{\text{sing},\mu}(\rho)$ lifts to $\tau \mapsto \varphi(\tau)(\xi)^\lambda(\xi)^d$.

Proof. The proof of part (i) of Proposition 2.5 did not use the fact that the map $\varphi$ was meromorphic there, but only its harmonicity. Hence the same argument proves the first assertion of part (i) here. For the second assertion, we consider the modified Equation (2) with $\lambda = 0$, and recall that we assume $b \neq c$. The coefficients $\alpha_{d,s}$ then satisfy, similarly to part (i) of Proposition 2.5, the equality $\alpha_{d,s+1} = \frac{b[k+(d-s)(k-d-s-1)]}{(s+1)(d-c)} \alpha_{d,s}$ for every $1 \leq s < d$. Once again, the modified Equation (2) with $s = 0$ proves this for $s = 0$, while the equation with $s > 0$ yields the vanishing of $(s+1)(b-c)\alpha_{d,s+1} - b\mu + (d-s)(k-d-s-1)\alpha_{d,s} + d+1-s$ times the same expression but with the index $s-1$. Hence the same inductive argument from part (i) of Proposition 2.5 yields our claim also here. This presents $\alpha_{d,d} = 1$ as the product of the free variable $\alpha_{d,0}$ and a coefficient that vanishes if and only of $\mu$ takes one of the values $(d+1-j)(d+j-k)$ with $1 \leq j \leq d$. But putting $s = d$ (and still $\lambda = 0$) in the modified Equation (2) yields again the vanishing expression from the induction hypothesis, plus $(b-c)(s+1)\alpha_{d,d+1} = 0$, plus the expression $-b\mu \alpha_{d,d}$. As $b \neq 0$ and $\alpha_{d,d}$ is assumed to be 1, the equation in question can only hold if $\mu = 0$, and then we are in the situation described above. Part (i) is therefore established.

The first assertion of part (ii) follows from the fact that the modified Equation (2) with $s = d = 0$ amounts to $(\lambda - b\mu)\alpha_{0,0} = 0$ with non-vanishing $\alpha_{0,0}$. With general $d$ but with $\mu = 2d - k$, the same equation with $s = d$ yields again the vanishing of $\alpha_{d,d} = 1$ times the coefficient $\lambda + d(b-c) - b\mu$, and substituting the value of $\mu$ yields the second assertion. Now, if $d = 1$ and $\mu \neq 2 - k$ then the modified Equation (2) with $s = 1$ implies $\alpha_{1,1} = \frac{b-k}{c} \alpha_{1,1}$. In the modified Equation (2) with $s = 0$ we then get the usual cancelation between the part of
\[-by - b(k - 2)]\alpha_{1,0}\] involving \(b - c\) and \((b - c)\alpha_{1,1}\). Recalling the value 1 of \((1 - a)(k - 2)\) and the equality from Corollary 1.10 we see that multiplying the rest of that equation by the denominator \(b(\mu + k - 2)\) produces the quadratic equation \(\lambda^2 - (2b\mu + k - 2)\lambda + b^2\mu(\mu + k - 2) = 0\). As \(k - 2 = \frac{1}{1-a} \neq 0\), we can write the discriminant of that equation is \((k - 2)^2(1 + \frac{4(1-b)\mu}{k-2})\), yielding the asserted eigenvalues if \(b = 1\) since their sum has to be \(2b\mu + k - 2\) in any case. Assuming that \(b \neq 1\), we write the expression in parentheses as \((1 + \frac{4b}{k-2})^2\), and obtain, since \(k - 2\) can be written as \(\frac{1}{1-a}\) by the argument from above and \(b \neq c\) since \(b \neq 1\), that \(\mu\) has the asserted value. One possibility for the eigenvalue \(\lambda\) can therefore be taken to be \(b\mu - t\), which is easily seen to be the required value.

As for the other possibility, note that replacing \(t\) by \(2 - k - t\) yields the same value for \(\mu\), while for \(\lambda\) it produces the additive complement to \(2b\mu + k - 2\), indeed the other eigenvalue. The values \(t = 0\) and \(t = 2 - k\) both yield \(\mu = 0\), with the respective eigenvalues \(\lambda\) being again 0 and \(k - 2\). This proves part (ii).

For part (iii) we recall again that \(b = c\) implies \(b = 1\), and that the term with \(\alpha_{d,s+1}\) vanishes also in the modified Equation 2 with index \(s\). Once again a simple inductive argument shows that \(\alpha_{d,s} = \prod_{i=0}^{d-s-1} \frac{\lambda - \mu - i(k - 2d + i - 1)}{i(\mu + (i + 1)(k - 2d + i))}\) for any \(0 \leq s \leq d\), unless \(\mu\) takes the value \((d+1-j)(k-d-j)\) for some \(j\), restricting the validity of the latter formula to \(0 \leq s \leq d\). The modified Equation 2 with \(s = 0\) (or with \(s = j\)) yields the vanishing of \(\alpha_{d,0}\) (or \(\alpha_{d,j}\)) times the multiplier \(\lambda - \mu - d(k - d - 1)\) (or \(\lambda - \mu - (d - j)(k - d - j - 1)\) in the special value of \(\mu\)), and the roots of the resulting polynomial in \(\lambda\) are as asserted. Moreover, if \(\lambda\) is the eigenvalue coming from \(q\) then the product defining \(\alpha_{d,s}\) here vanishes whenever \(s \leq d - q - 1\), so that indeed only the functions \(F_s\) with \(s \geq d - q\) are non-zero in such a lift. The result about the special case with \(q = 0\) is obvious, since \(F_d = \varphi\) and the remaining functions \(F_s\) with \(0 \leq s < d\) vanish in the case. This proves the proposition. 

\[2.4 \text{ Eigenfunctions and Quasi-Modular Forms}\]

The spaces \(\mathcal{M}_{k,\infty,\Delta,\lambda}^{\text{sing}}(V_m \otimes \rho)\) of lifts, as useful as they may be, are not the ones arising naturally from \(\mathcal{M}_{k,\infty}^{\text{sing}}(V_m \otimes \rho)\). By putting a depth bound (i.e., concentrating on the image of one of the spaces \(\mathcal{M}_{k,\infty}^{\text{sing}}(V_m \otimes \rho)\) of modular forms with finite dimensional representations), we would like to investigate the space of elements of \(\mathcal{M}_{k,\infty}^{\text{sing}}(V_m \otimes \rho)\), of depth bounded by \(m\), that are eigenfunctions of a fixed eigenvalue \(\lambda\) for some Laplacian \(\Delta_{k,\infty}^{(a,b,c)}\). The proofs of Theorems 2.4 and 2.8 contain the answer to that question.

**Corollary 2.8.** Let \(a, b, c, k, \Gamma, \rho,\) and a depth bound \(m\) be fixed. Then the space of elements of \(\mathcal{M}_{k,\infty}^{\text{sing}}(V_m \otimes \rho)\) that are annihilated by \(\Delta_{k,\infty}^{(a,b,c)} + \lambda\) and whose depth is bounded by \(m\) is as follows.

(i) If \(b \neq 0\) and \((1-a)(k-2)\) is not an integer between 0 and \(m\) then the space in question is the direct sum over \(0 \leq d \leq m\) of the lifts of \(\mathcal{M}_{k,\infty}^{\text{sing}}(\rho)\) for which there exists a lift to the eigenvalue \(\lambda\) with these \(a, b, c,\) and \(k\). The
number of $\lambda$s for which this space is non-trivial is at most $\sum_{d=0}^{m}(d+1)$, with a smaller bound in case $k \in \mathbb{N}$ and $2 \leq k \leq 2m$.

(ii) Still with $b \neq 0$, but now assume that $(1-a)(k-2)$ is an integer $0 \leq e \leq m$. Then we may have lifts from $\mathcal{M}_{k-2d}^{\text{mer}}(\rho)$ for $0 \leq d < e$ as above as well as lifts from $\mathcal{M}_{k-2d}^{\text{sing}}(\rho)$ for $e < d \leq m$ (the latter existing for at most $\sum_{d=e+1}^{m}(d-e)$ values $\lambda$), plus the direct sum of lifts of $\mathcal{M}_{k-2e,\Delta}^{\text{sing}}(\rho)$ for at most $e+1$ values $\mu$. In the generic case, in which wherever $\lambda$ is the root of the polynomial of degree $d+1$ from Theorem 2.4 for $d > e$ there is a lift from $\mathcal{M}_{k-2d}^{\text{mer}}(\rho)$ into the desired space, these lifts generate the entire eigenspace in $\mathcal{M}_{k-\infty}^{\text{sing}}(V_{\infty} \otimes \rho)$.

(iii) In case $b = 0$, if $(1-a)(k-2)$ is not an integer between 1 and $m$ then the space in question contains lifts from $\mathcal{M}_{k-2d}^{\text{sing}}(\rho)$ if and only if the eigenvalue is $\lambda = d(k-2 + \frac{1-a}{1-a})$ (this happens for at most two values of $d$ for integers $0 \leq d \leq m$). Otherwise $(1-a)(k-2)$ is some integer $p$. Then if $\lambda = d(k-2 + \frac{1-a}{1-a})$ for $0 \leq d < p$ then we get the same lifts of the same functions, while if $p \leq d \leq m$ then only lifts of elements of $\mathcal{M}_{k-2d}^{\text{sing}}(\rho)$ that are annihilated by $\delta_{k-2d}^{d+1-p}$ are to be considered. These spaces always generate the eigenspace in question.

Proof. In case $b \neq 0$ and $(1-a)(k-2)$ is not one of the exceptional integers then the non-vanishing function $F_{d}$ with maximal index $0 \leq d \leq m$ was seen to be a meromorphic element $\varphi \in \mathcal{M}_{k-2d}^{\text{mer}}(\rho)$. When comparing the equations yielding the lift of $\varphi$ with those describing any eigenfunction with eigenvalue $\lambda$ and $F_{d} = \varphi$, we only obtain terms with lower index $s$ that are based on functions that are annihilated by some power of $4y^{2}\partial_{\tau}$. More explicitly, each function $F_{s}$ must be of the form $\sum_{r=s}^{d}a_{r,s}\delta_{k-2r}^{-\frac{s}{d}}\varphi_{r}$, with meromorphic functions $\varphi_{r} \in \mathcal{M}_{k-2r}^{\text{mer}}(\rho)$, and with coefficients $a_{r,s}$ that vanish if $s > r$ and satisfy $a_{r,r} = 1$ for each $r$. Assuming first that $k$ is not an integer between 2 and $2m$, we deduce that the functions $\delta_{k-2r}^{-\frac{s}{d}}\varphi_{r}$ are linearly independent over $\mathbb{C}$ (either since the $r$th function is a polynomial of exact degree $r$ in $\frac{1}{y}$ over meromorphic functions, or since the eigenvalues under $\Delta_{k}$ are different, both assertions following from our assumption on $k$). The same argument from the proof of Theorem 2.4 therefore shows that the coefficients $a_{r,s}$ are those arising from the lifts, and that $\lambda$ must satisfy the same polynomial equation (it thus has to be a simultaneous root of all the polynomials arising from non-vanishing $\varphi_{r}$). The required assertion therefore holds in this case. Assuming now that $k$ is one of the omitted integers, we recall that if $k = d + p$ for some $1 \leq p \leq d$ and $\varphi \in \mathcal{M}_{k-2d}^{\text{mer}}(\rho)$ then $\delta_{k-2d}^{d-p+1}\varphi = \delta_{p-d}^{d-p+1}\varphi$ is meromorphic (since the latter operator is $\partial_{\tau}^{d-p+1}$ by Equation (56) of [Zd1], or its predecessor Bol’s identity). In this case it suffices to take the additional function $\varphi_{p-1}$ from $\mathcal{M}_{k-2p+2}^{\text{mer}}(\rho)$ either to be 0 or to be linearly independent of the this element of $\mathcal{M}_{d-p+2}^{\text{mer}}(\rho)$. Under this assumption, the same considerations lead to the desired conclusion also for these values of $k$. This proves part (i).
Part (ii) follows by similar considerations, except that differences appearing in the function $F_e$ no longer have to be annihilated by $4y^2\partial_\tau$ since the multiplying coefficient $(1 - a)(k - 2) - e + 1$ now vanishes. The fact that we may speak about these differences uses the assumption that the lifts in question exist. Decomposing the function $\varphi_j$ obtained in the process according to Laplacian eigenvalues, we find that the parts of the functions $F_s$ for $0 \leq s \leq j$ that arise from $\varphi_j$ also decompose accordingly (with translated eigenvalues), and therefore the resulting equations for the coefficients multiplying the $\delta_l$-images of these components are independent of one another. This proves that also in this case the subspace of $M^{\text{sing}}_{k-\infty}(V_\infty \otimes \rho)$ that is annihilated by $\Delta^{(a,b,c)}_{k-\infty}$ decomposes as the direct sum of the lifts from Theorem 2.4 and part (iii) of Theorem 2.6, establishing part (ii).

As part (iii) here is an immediate consequence of parts (i) and (ii) of Theorem 2.6, this completes the proof of the corollary.

Under some conditions, especially if $b = c$ (i.e., $b = c = 1$), we can describe explicitly the set of eigenvalues $\lambda$ for which the spaces from part (i) in Corollary 2.8 are a non-trivial, using Proposition 2.5. For part (ii) of that corollary, Proposition 2.7 allows us to give, in some cases, an explicit connection between the eigenvalue $\lambda$ of the image function and the eigenvalues $\mu$ contributing to it. On the other hand, in part (iii) the condition that lifts are defined when $d > e$ and $\lambda$ is a root of the required polynomial is essential, since otherwise the missing lifts may be replaced by twisted lifts involving more complicated functions. As an example we consider the case with $d = 1$, $0 \neq b \neq c$, and $k = 2$ (with $e = 0$) described right after Proposition 2.5. The function $F_1 = \varphi_1$ must be in $M^\text{mer}_0(\rho)$, and the equality from Lemma 2.1 with $s = 1$ yields $\lambda \varphi_1 = 0$. If the depth is exactly 1 (and not 0), then $\lambda$ must vanish, and the equality with $s = 0$ becomes (since $\delta_0 = \partial_\tau$) just the vanishing of $b\Delta_2 F_0 + (b - c)\partial_\tau \varphi_1$. This implies that $\Delta_2 F_0$ is meromorphic and non-vanishing, so that (at least in the holomorphic case) $F_0$ must be a sesqui-harmonic modular form of weight 2 in the terminology of [BDR].

If we remove the depth bound in Corollary 2.8 then we get, at least if $k$ is not a positive integer, an infinite direct sum over $d$ of the same lifts. On the other hand, the results of that corollary also show what happens for eigenforms in the spaces $M^{\text{sing}}_{k-m}(V_m \otimes \rho)$, with finite $m$: The picture for $i_{m-1}$-images is just the same as for $M^{\text{sing}}_{k-\infty}(V_\infty \otimes \rho)$. The problem is that on functions that are not $i_{m-1}$-images the Laplacian (which we denote by $\Delta_{k-\infty}$ here as well) is, in general, not defined. We can avoid this problem in some cases: E.g., when $a = 1$ the weight raising part does not increase the weight. Another case is if $b = 0$, when the weight lowering operator, a multiple of $i_{m-1} \circ D_m$, produces $i_{m-1}$-images for the weight raising one.

We conclude with remarking that the Laplacians $\Delta^{(a,b,c)}_{k-\infty}$ can be translated, using Theorem 1.6, to operators on usual quasi-modular forms. The functions from Corollary 2.8 and in particular the lifts from Theorem 2.4 and 2.6 also correspond to quasi-modular forms via Theorem 1.3. However, the formulae
are not very simple: While $F_*$ would be, in a lift of an element $\varphi \in M^{mer}_{k-2d}(\rho)$ to $M^{ring,d,(a,b,c)}_{k-\infty,\Delta,\lambda}(V_{\infty} \otimes \rho)$ for some $\lambda$ (and $a$, $b$, and $c$ satisfying the equality from Corollary 2.9), of the form $\alpha_{d,s}^{d-s}k_{-2d}^s\varphi$ for well-defined constants $\alpha_{d,s}$, the associated quasi-modular form $f = f_0$ is based on the functions $f_r$ from Theorem 1.3. These functions are given, in terms of that theorem, our expression for the coefficients $\alpha_{d,s}$ are in general difficult. Nonetheless, there are some cases in which explicit expressions for the quasi-modular eigenforms of the translated Laplacians $\Delta_{k-\infty}^{(a,b,c)}$ can be given.

**Corollary 2.9.** For $\Gamma$, $\rho$, and $d$ as above, choose a weight $k$.

(i) Assume that $b \neq 0$, $c \neq b$, and $\varphi \in M^{mer}_{k-2d}(\rho)$, where $d$ is a depth and $k$ is not an integer between $d + 1$ and $2d$. Then the lift of $\varphi$ to the subspace of $M^{ring}_{k-\infty}(V_{\infty} \otimes \rho)$ that is annihilated by $\Delta_{k-\infty}^{(a,b,c)}$ is associated with the quasi-modular form in which

$$f_r(\tau) = \sum_{p=0}^{d-r} \left[ \sum_{s=r}^{d-p} (-1)^{d-s-p} \binom{d-s}{r} \binom{d-s}{p} \alpha_{d,s} \prod_{i=p}^{d-s-1} (k - 2d + i) \right] \frac{\partial^p \varphi(\tau)}{(-2iy)^{d-r-p}}$$

(note the change of sign in the $2iy$ in the denominator). However, the explicit expressions for the coefficients $\alpha_{d,s}$ are in general difficult. Nonetheless, there are some cases in which explicit expressions for the quasi-modular eigenforms of the translated Laplacians $\Delta_{k-\infty}^{(a,b,c)}$ can be given.

In case $(1 - a)(k - 2) = d$ the same assertion holds also for (harmonic) $\varphi \in M^{ring,0}_{k-2d,\Delta}(\rho)$.

(ii) If $b = c = 1$ and $\lambda$ is one of the eigenvalues $q(k-2d+q-1)$ from part (iii) of Proposition 2.2 then the functions $f_r$, associated with the quasi-modular form corresponding to the lift of $\varphi \in M^{ring}_{k-2d}(\rho)$ are given by

$$\sum_{h=0}^{\min\{q,d-r\}} \binom{q}{h} \binom{d-r}{h} \prod_{i=0}^{h-1} \frac{k - 2d + q - 1 + i}{k - 2d + i} \cdot \frac{\delta_{k-2d}^h \varphi}{(-2iy)^{d-r-h}}.$$

In the harmonic case with $q = 0$ the function $f_r$ is just $\binom{d}{h} \frac{\varphi}{(-2iy)^{d-r-h}}$.

(iii) If $b = 0$ then the quasi-modular form of depth $d$ that is associated with the lift of an element of $\varphi \in M^{ring}_{k-2d}(\rho)$ to a function with representation $V_{\infty}$ and eigenvalue $d(k - 2 + \frac{d}{1-a})$ has the associated functions $f_r$, $0 \leq r \leq d$ defined by

$$\binom{d}{h} \sum_{h=0}^{d-r} \binom{d-r}{h} \frac{\delta_{k-2d}^h \varphi}{(-2iy)^{d-r-h}} \cdot \prod_{i=0}^{h-1} \frac{1}{(1-a)(k - 2) + 2 - 2d + i}.$$
provided that the number \((1 - a)(k - 2)\) is not an integer between \(d - 1\) and \(2d - 2\). In case this number equals \(d - 1 + j\) with \(0 \leq j < d\), we get the same expression, but only under the assumption that \(y^{k-2d}\phi\) is the complex conjugate of a nearly meromorphic function of degree smaller than \(d - j\), with the sum over \(h\) taken up to \(\min\{d - r, d - 1 - j\}\).

**Proof.** The proof of part (i) of Proposition 2.3 (or of part (iii) of Proposition 2.7 with \(\mu = 0\)) implies that \(\alpha_{d,s}\) equals \(\binom{d}{s} b^{d-s} / \prod_{i=0}^{d-s-1} (k - 2d + i)\) in the case described in part (i). Indeed, we get 1 for \(s = d\), and the quotient \(\frac{\alpha_{d,s+1}}{\alpha_{d,s}}\) is the required one for any \(0 \leq s < d\). Plugging this expression into Equation (3), we first note that the terms involving \(k\) yield the asserted product in the denominator, which no longer depend on \(s\). Moreover, by canceling the appropriate factorials we find that the product \(\binom{d}{s} \binom{d-r}{p}\) coincides with \(\binom{d}{s-r} b^{s-r} (c-b)^{d-s-p}\), and we can also write the combination \(\binom{d}{s-r} b^{s-r} (c-b)^{d-s-p}\) as the product of \(b^{s-r} (c-b)^{d-s-p}\). The sum \(\sum_{s=r}^{d} \binom{d-r}{p} b^{s-r} (c-b)^{d-s-p}\) then becomes \(c^{d-r-p}\) by the Binomial Theorem, and part (i) follows.

Turning to part (ii), the proof of part (iii) of Proposition 2.3 (also extending to the case \(\mu = 0\) in part (iii) of Proposition 2.7) shows, after substituting the value of \(\lambda\), that \(\alpha_{d,s} = \prod_{i=0}^{d-s-1} \frac{(q-i)(k-2d+i-1)}{(i+1)(k-2d+i)}\), or more succinctly \(\binom{q}{d-s} \prod_{i=0}^{d-s-1} \frac{k-2d+i+q-1}{k-2d+i}\) for \(d - q \leq s \leq d\). Here we just plug this expression into the relations between the functions \(f_r\) and \(F_s\) in Theorem 1.3 and apply the summation index change \(s = d - h\), recalling that the binomial coefficients effectively restrict \(h\) from exceeding either \(q\) or \(d - r\) in the sum. This proves the first assertion, and as with \(q = 0\) we only have \(h = 0\), trivial coefficient \(\binom{q}{d}\), and an empty product, the second assertion follows as well (extending the result of part (i) to the case with \(b = c = 1\)).

Part (iii) is even simpler: The proof of part (i) of Proposition 2.7 determines \(F_s\) as \(\binom{d}{s} b^{d-s} / \prod_{i=0}^{d-s-1} (1-a)(k-2) - 2d + i\) (indeed, the coefficients \(\beta_{d,s}\) here satisfy the initial condition \(\beta_{d,d} = 1\) and the recursive condition appearing in that proof), where now \(\varphi\) is a general element of \(M_{k-2d}(\rho)\). Applying again Theorem 1.3 and simple identities between binomial coefficients, a similar argument allows us to deduce the required result (also in case \(1-a)(k-2) = d-1+j\), since then the bound on \(h\) is effectively \(h \leq d - 1 - j\), and no denominator vanishes). This proves the corollary.

In particular, part (i) of Corollary 2.9 shows that the harmonic lifts with respect to the Laplacian \(\Delta_k^{1,0}\) arising from \(V\) are the quasi-modular forms for which \(f_r, 0 \leq r \leq d\) is the meromorphic functions \(\binom{d}{s} \partial_r^{d-r} \varphi / \prod_{i=0}^{r-1} (k - 2d + i)\), where \(\varphi\) is an element of \(M_{k-2d}^{\text{mer}}(\rho)\) (or, when \(d = 0\), just the modular forms from the space \(M_{k,\Delta}^{\text{sing},\rho}(\rho)\) itself). In the case of the Laplacian \(\Delta_k^{0,1}\) obtained from the map \(\delta = \partial_r\) and \(\overline{\delta} : f \mapsto f_1\) of \(A\), part (iii) of Corollary 2.9 describes the eigenfunctions, all belonging to the unique eigenvalue \(d(k-1-d)\). Such a quasi-modula forms has for its function \(f_r\) the expression
\begin{align*}
\binom{d}{r} \sum_{h=0}^{d-r} \binom{d-r}{h} \frac{\delta^h_{k-2d}}{(-2i\pi)^h} / \prod_{i=0}^{h-1} [k - 2d + i], \end{align*}
where \( \varphi \in \mathcal{M}_k^{\text{sing}}(\rho) \) is arbitrary if \( k \) is not an integer between \( d + 1 \) and \( 2d \), and only with \( \varphi \in \ker \delta_{k-2d}^{d-j} \) if \( k = d + 1 + j \) and \( j \) is an integer between \( 0 \) and \( d - 1 \). In fact, applying Equation (56) of [Za] as for Equation (3) we obtain that the coefficient in front of \( \binom{d}{r} \sum_{h=0}^{d-r} \binom{d-r}{h} \frac{\delta^h_{k-2d}}{(-2i\pi)^h} / \prod_{i=0}^{h-1} [k - 2d + i], \) which is a constant (independent of \( h \)) times \( \binom{d-r}{-r} \frac{\partial_{\tau}(-1)^h/ \prod_{i=0}^{h-1} [k - 2d + i]. \) If the weight \( k \) is not between \( d + 1 \) and \( 2d \) (and \( \varphi \in \mathcal{M}_k^{\text{sing}}(\rho) \) is arbitrary) then there is no additional restriction on \( h \), and this coefficient vanishes for \( p < d - r \) and gives 1 for \( p = d - r. \) Therefore \( f_r \) reduces to simply \( \binom{d}{r} \frac{\partial_{\tau}(-1)^h/ \prod_{i=0}^{h-1} [k - 2d + i]. \) Assuming now that \( k = d + 1 + j \) and \( \varphi \in \ker \delta_{k-2d}^{d-j} \) as above, the index \( h \) becomes restricted to be smaller than \( d - j \), and therefore so is \( p. \) In the functions \( f_r \) with \( r > j \) the initial restriction \( h \leq d - r \) remains unaffected, and we get the same formula for \( f_r \) (which now looks like \( \binom{d}{r} (-1)^d (\frac{\partial_{\tau}(-1)^h}{\prod_{i=0}^{h-1} [k - 2d + i]}). \) On the other hand, for \( j = d - 1 \) the fact that \( h \) and \( p \) must vanish yields the expression \( \binom{d}{r} \frac{\partial_{\tau}(-1)^h}{\prod_{i=0}^{h-1} [k - 2d + i]} \) for \( f_r \). The expression for \( f_r \) with \( r \leq j < d - 1 \) are more complicated, and do not seem to have a simple presentation.

XXXX in pf of Cor 2.8

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Einstein Institute of Mathematics, the Hebrew University of Jerusalem, Edmund Safra Campus, Jerusalem 91904, Israel
E-mail address: zemels@math.huji.ac.il