Surface growth approach for bulk reconstruction

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We propose a novel and natural surface growth approach to reconstruct the bulk spacetime. The picture is similar to Huygens’ principle of wave propagation, in which each subregion of the old entanglement surface becomes a “wave source” of the new one, then layer by layer, the newly generated entanglement surfaces can reach bulk region far away from the boundary. We show that this picture can be described reasonably and quantitatively, and thus provides a concrete and intuitive way for the entanglement wedge reconstruction.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence has established a bridge between the boundary CFT and the gravity in the bulk asymptotically AdS spacetime \cite{1,2,3}. The correspondence also indicates an emergent picture of gravity, namely, the geometry and gravitational dynamics of bulk spacetime should in principle be constructed from information of the boundary CFT, which is called the bulk reconstruction \cite{18,19,20,21,22,23,24}. In the reconstruction of bulk gravitational theory, the notion of holographic entanglement entropy plays a key role \cite{4,5}, which states that the entanglement entropy of a boundary subregion $A$ is a quarter of the area of a co-dimensional-2 minimal surface $\gamma_A$ growing into the bulk from the boundary of $A$ (to leading order in the gravitational coupling constant $G$), i.e.,

$$S_A = \frac{\text{Area}(\gamma_A)}{4G}.$$ \hspace{1cm} (1)

It was shown that field theory information contained in the subregion $A$ can determine the information in spatial region bounded by $A$ and the bulk extremal surface, which is called the entanglement wedge \cite{7,8,18}. Subsequently, much progresses have been made along this direction, such as reconstruction of bulk operators from the boundary CFT operators in subregions \cite{18,19,20,21,22,23,24}, generating the bulk AdS geometry from entanglement renormalization of the tensor networks \cite{11,12,13,16} and furthermore, investigating the emergence of gravitational dynamics from the geometry generated from the tensor networks \cite{25}. Obviously, according to RT formula (1), a boundary subregion $A$ can at least detect the nonlocal information about the global configuration of the extremal surface $\gamma_A$ by reading out its classical area. However, how can the information in $A$ detect (or reconstruct) the information in the region inside the RT surface $\gamma_A$ (i.e., the entanglement wedge of $A$) is not apparent and direct.

In the present paper, we will propose a concrete and very natural approach to reconstruct the bulk geometry in the entanglement wedge from a surface growth procedure, motivated by the Huygens’ principle of wave propagation. The basic ideas of our approach can be briefly described as follows \textsuperscript{1}. Firstly, considering there are many minimal surfaces “growing” out from a set of small boundary subregions side by side. Secondly, we can regard these minimal surface as the new boundaries and further take the points on them as the anchor points for the new minimal surfaces to grow into the deeper bulk regions. Then repeating the procedure layer by layer such that the new minimal surfaces can probe arbitrary region in the entanglement wedge. Consequently, the information of these bulk regions can be detected and reconstructed from the initial boundary regions in this way, see FIG. 1. Moreover, as long as the exten-

\textsuperscript{1} Let us take asymptotical AdS$_3$ spacetime as an example, the approach is also workable for higher dimensions, though the calculations are more complicated.

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FIG. 1: An example of surface growth scheme
sion of the minimal surface of concern is small enough, this detection is approximately “local”. This paper aims to argue that the above picture can be described reasonably and quantitatively, and thus provides a concrete and intuitive way for the entanglement wedge reconstruction [7,8,18]. To achieve this purpose, we utilize and generalize the so-called one-shot entanglement distillation (OSED) method [13]. We first investigate a special surface growth picture with spherical symmetry and fractal property. Interestingly, it turns out that this surface growth picture can be exactly identified with the well-known MERA-like tensor network [14-16]. This consistency enhances our confidence in two aspects. On the one hand, since it has been proved in [13] that the tensor network constructed by the systematical OSED method can reproduce the correct boundary state with high fidelity, and have a bulk geometry that matches the bulk spacetime perfectly, to some extent, we have obtained a proof that the MERA-like tensor network is indeed a discretized version of the time slice of AdS spacetime, rather than just an analogy. On the other hand, this well-known credible tensor network model turns to provide a satisfactory example for our generalization of [13] later.

II. SURFACE GROWTH PICTURE CORRESPONDS TO THE MERA-LIKE TENSOR NETWORK

We will briefly review the concept of OSED first. In quantum information theory, the direct product of a large number of copies of an arbitrary quantum state can be approximated as a state described by a large number of Bell pairs manifestly representing the entanglement with high fidelity. This operation is called entanglement distillation [17]. It was shown that similar operation can be performed on a single holographic state in the framework of the holographic principle, due to a classical bulk geometric dual. More explicitly, first constructing the so-called “smoothed states” for a holographic CFT full, pure state $|\Psi\rangle$ and the reduced density matrix $\rho_A$ for a certain subregion $A$ of the CFT respectively. Then, rearranging the eigenvalues of the smoothed state $\rho_A^0$ in descending order and dividing them into blocks of size $\Delta = e^{S_\Lambda - O(\sqrt{S_\Lambda})}$. Consequently, one can further approximate the boundary state $|\Psi\rangle$ as the following tensor network representation (see more details in [10,13])

$$\Psi^{A_1A_2} = V^{A_2}_\beta\phi^{\alpha\beta} = (V \otimes W)(|\phi\rangle \otimes |\sigma\rangle),$$

where a maximally entangled state $|\phi\rangle \otimes |\sigma\rangle$ has been constructed (or distilled out), which is defined by

$$|\phi\rangle = \sum_{m=0} e^{S_{\Lambda}/N} |m\tilde{n}\rangle_{\alpha\tilde{\alpha}},$$

$$|\sigma\rangle = \sum_{n=0} \sqrt{\tilde{\beta}_{n\Delta}/N} |n\tilde{m}\rangle_{\beta\tilde{\beta}},$$

where $S$ denotes the entanglement entropy of $A$, and $\tilde{\beta}_{n\Delta}$ is the average eigenvalue of each block. Obviously, the logarithm of the bond dimension, i.e. the Hilbert space dimension of $|\phi\rangle$ matches the entanglement entropy of $A$ exactly, while $|\sigma\rangle$ should be interpreted as the quantum fluctuation. The tensors $W$ and $V$ are isometries, which map the auxiliary states represented by the bonds of $\phi^{\alpha\beta}$ into the eigenstates of the reduced density matrices for $A$ and its complement respectively. Eq.(2) is the OSED for a holographic state. By iterating the OSED procedure on a holographic boundary state, one can construct a tensor network for the state with high fidelity. Moreover, the underlying geometry of the tensor network can be perfectly matched with the discretization of the bulk spacetime by a series of nonintersecting RT surfaces [13].

![FIG. 2: A special surface growth picture with spherical symmetry and fractal feature](image)

Now, using the OSED method, we can construct a corresponding tensor network as shown in FIG. 2 namely, a special surface growth picture with spherical symmetry and fractal feature. To obtain this tensor network structure, we begin with the case where the bulk spacetime is pure AdS$_3$, which corresponds to a dual 2d CFT vacuum state on the boundary. In FIG. 2 we can choose that the boundary is equally divided into $N$ number of small subregions $A_1$, $A_2$, $\cdots$, $A_N$, each small piece can produce a RT surface to detect the bulk information. On the other hand, we can also take every two adjacent pieces as the new subregion, then a new minimal surface can grow into the bulk, and it will surround the original two adjacent minimal surfaces exactly. This can also be considered as taking the points very close to the boundaries of the original surfaces as anchor points to continue producing
new minimal surfaces into deeper bulk regions. Continue repeating this process, we can construct a spherically symmetric and fractal pattern layer by layer.

After the first layer of “surface growth”, we obtain the tensor network representation of the original boundary CFT state as

$$|\Psi\rangle^{1st} = V_N^{1st} \prod_{i=1}^N W_i^{1st} |\phi \rangle^{1st}_i |\sigma \rangle^{1st}_i,$$

where $|\phi \rangle^{1st}_i$ and $|\sigma \rangle^{1st}_i$ represents the distilled state associated with the RT surface corresponding to $A_i$, region, and $W_i^{1st}$ tensor maps the state represented by its $\beta\bar{\alpha}$ legs to the reduced state on $A_i$, and the $V_N^{1st}$ tensor connects all the remaining legs, and the superscript “1st” denotes the first layer. Transferring eq.(4) into another expression as follows will prove to be useful

$$|\Psi\rangle^{1st} = \Psi V_N^{1st} \prod_{i=1}^N W_i^{1st},$$

where the $|\phi \rangle|\sigma \rangle$ states have been absorbed into the $V'$ tensor in each step in order to redefine a new tensor $\Psi V$.  

After the second layer of “surface growth”, we obtain

$$|\Psi\rangle^{2nd} = V_M^{2nd} \left( \prod_{j=1}^{M=N/2} W_j^{2nd} |\phi \rangle^{2nd}_j |\sigma \rangle^{2nd}_j \right) \times \left( \prod_{i=1}^N W_i^{1st} |\phi \rangle^{1st}_i |\sigma \rangle^{1st}_i \right)$$

or

$$|\Psi\rangle^{2nd} = \Psi V_M^{2nd} \prod_{j=1}^{M=N/2} W_j^{2nd} \prod_{i=1}^N W_i^{1st}.$$

Similarly, after the $k$-th layer of “surface growth”, we obtain

$$|\Psi\rangle^{kth} = V'^{kth} \prod_{i=1}^{N/2^{k-1}} \prod_{j=1}^k W_i^{jth} |\phi \rangle^{jth}_i |\sigma \rangle^{jth}_i$$

or

$$|\Psi\rangle^{kth} = \Psi V^{kth} \prod_{i=1}^{N/2^{k-1}} \prod_{j=1}^k W_i^{jth}.$$

The graphical representations for two kinds of expressions eq.(5) and eq.(6) are shown in FIG.3 and FIG.4 respectively, and the difference between them is subtle. Regardless of the quantum fluctuations, it is a pure state $|\phi\rangle = \sum_{m=0}^{e^{-\alpha(S)}} |m\bar{m}\rangle_{\alpha\bar{\alpha}}$ associated with each RT surface in FIG.3 it is constructed by the superposition of $O(e^S)$ basic states $|m\bar{m}\rangle_{\alpha\bar{\alpha}}$ with equal probability amplitudes, and its legs do not intersect with the RT surface. While in FIG.3 since each RT surface is intersecting with a $\bar{\alpha}$ leg, according to the convention of tensor network, we define the state on the RT surface as the state $\{|\bar{m}\rangle, p_m = \frac{1}{M}\}$ represented by $\bar{\alpha}$ leg, which is a mixed state with equal probabilities.

Now we can identify the FIG.3 with a MERA-like tensor network, or more specifically, the Euclidean MERA tensor network on the circle, as shown in the FIG.5. For this purpose, note that the MERA-like tensor network has a subtle property. Considering two adjacent identical boundary subregions $A_1$ and $A_2$ and their union $A_1 A_2$, as shown in the FIG.6 the inner parts of the RT surfaces of $A_1$ and $A_2$ coincide with each other, while their outer parts overlap with the RT surface of the $A_1 A_2$. Therefore, identifying FIG.4 with FIG.5 is straightforward. Each basis $|\bar{m}\rangle$ of the state represented by the $\bar{\alpha}$ leg passing through the RT surface in the OSED tensor network exactly corresponds to the state of an overall configuration in the MERA-like network, which consists of all the legs cut by the RT surface. Then we perform a decomposition of disentangler tensor $u$ as

$$u^{abcd} = t_{abc}^i j_{bd}^i,$$

and obtain

$$W = W_{ab}^{ij} j_{m}^i j_{n}^m e^\epsilon.$$
Therefore, we can conclude that as long as we define the combination of the specific tensors in the MERA-like tensor network just as in FIG. 6(c) (approximately) as the formal $W$ tensor in the systematical OSED method, to some extent, we have obtained a proof that the MERA-like tensor network is indeed a discretized version of the time slice of AdS spacetime.

Interestingly, as shown in FIG. 7 in the MERA-like tensor network, one can either image that the minimal surfaces of the second layer grow directly from the boundary, or from another point of view, the small minimal surface (labeled as $j_mj'_mj_nj'_nc$ in the figure) grows from a part of the adjacent minimal surfaces in the first layer (labeled as $ab$ in FIG. 7). Therefore, we can either think of the isometry tensor $W$ as mapping the states on the RT surfaces of the first layer into the state on the larger RT surface of the second layer, or as mapping the state on the $ab$ region into the state on the minimal surface $j_mj'_mj_nj'_nc$.

In fact, the second viewpoint is consistent with the sur-
FIG. 7: A small minimal surface (labeled as $jmjn'jm'n'$c in the figure) grows from a part of the adjacent minimal surfaces in the first layer (labeled as $ab$ in the figure).

face/state correspondence proposal [9], which generalizes the RT formula, it claims that if one considers an arbitrary closed convex surface $\Sigma$ in the bulk spacetime and a subregion $\Sigma_A$ of $\Sigma$, then this closed surface $\Sigma$ and the surface $\Sigma_A$ will correspond to quantum states described by density matrices $\rho(\Sigma)$ and $\rho(\Sigma_A)$ respectively, and the entanglement entropy $S_A^\Sigma$ of subregion $\Sigma_A$ with respect to the quantum state $\rho(\Sigma)$, i.e., the von Neumann entropy of $\rho(\Sigma_A)$ can be calculated by the area of the extremal surface $\gamma_A^\Sigma$ anchored on the boundary of $\Sigma_A$, because the density matrix corresponding to an extremal surface is a direct product of density matrices at each point, i.e. [9],

$$S_A^\Sigma = \frac{\text{Area}(\gamma_A^\Sigma)}{4G}. \quad (12)$$

Therefore, we can go further and assign a mixed state $\{(|\tilde{m}\rangle, p_m = \frac{1}{2})\}$ with equal probabilities to each extremal surface in the bulk, which is exactly consistent with the interpretation for the extremal surface in the surface/state correspondence [10]. Therefore, we can further generalize the OSED procedure of [13] such that it not only can quantitatively describe the RT surface, but also how the general extremal surface in the bulk encodes and decodes the information of some certain part of the extremal surfaces in the former layer in a similar way. More specifically, in the case of figure [7] the minimal surface $jmjn'jm'n'$c encodes and decodes the information of $ab$ surface. More generally, under the framework of surface/state correspondence, we can extend the OSED method to describe any surface growth scheme quantitatively, such as FIG. [8] and hopefully, generalize to other spacetime [3].

Based on the above understanding, we argue that the surfaces growth process is in fact providing a geometric realization of the generalized OSED process. To see this more explicitly, considering a new extremal surface which grows from the endpoints of the $\Gamma_1$ part and $\Gamma_2$ part of two adjacent extremal surfaces in FIG. [8] there is no internal quantum entanglement within $\Gamma_1$ or $\Gamma_2$ itself, however, when one considers their union as a whole surface $\Gamma$, there exists mutual entanglement between the two (which essentially comes from the internal entanglement within the boundary subregion $A$ supporting the initial growing of extremal surfaces). Accordingly, the surface $\Gamma$ can be described by a mixed state $\psi_{\Gamma}$. The process of growing a new extremal surface from $\Gamma$ can then be viewed as “propagating” the information of the mixed state $\psi_{\Gamma}$ to the new extremal surface with the high fidelity. Note that since this information will be “decoded” by calculating the area of the new extremal surface in a classical sense, thus before the propagation, it is necessary to “encode” $\psi_{\Gamma}$ by a “classical smooth” operation. Intuitively, when the quantum correction is neglected, $\psi_{\Gamma}$ can be encoded by this operation into a delicate mixed state, in which each basis is weighted with equal probability, and thus can correspond to a “classical configuration” $|\tilde{m}\rangle_\sigma$ on the new extremal surface. In this way, the surface growth process successfully maps (in other words, “propagates”) the $\psi_{\Gamma}$ state to a mixed state with equal probabilities on a special extremal surface $\gamma(\Gamma)$ in the classical geom-

3 We would like to point out that the basic ideas of our surface growth proposal is independent with the surface/state duality proposal.
etry up to some negligible quantum fluctuation, in the language of holographic principle. For a certain boundary subregion $A$, since the minimal surfaces can grow to any arbitrarily small region in its corresponding entanglement wedge, therefore, we conclude that $A$ can detect the information of arbitrary region in the entanglement wedge, which gives a concrete and efficient realization of the entanglement wedge reconstruction [7, 8, 18].

**IV. CONCLUSIONS AND DISCUSSIONS**

In this paper, we proposed a new surface growth approach for bulk reconstruction in the entanglement wedge, motivated by the Hygens' principle of wave propagation. We showed that the surface growth scheme can be explicitly realized by extending the OSED method, together with the help of the surface/state correspondence.

The proposal provides a concrete and intuitive realization of the subregion duality and an efficient way for bulk reconstruction. In particular, we demonstrate that the well known MERA-like tensor network is a concrete example of our surface growth scheme and accordingly obtain a more specific physical meaning, i.e., it is indeed a discretized version of the time slice of AdS spacetime, rather than just an analogy.

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