Multisymplectic Geometry Method for Maxwell’s Equations and Multisymplectic Scheme

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March 28, 2022

Abstract

In this paper we discussed the self-adjointness of the Maxwell’s equations with variable coefficients $\varepsilon$ and $\mu$. Three different Lagrangian are attained. By the Legendre transformation, a multisymplectic Bridge’s (Hamilton) form is obtained. Based on the multisymplectic structure, the multisymplectic conservation law of the system is derived and a nine-point Preissman multisymplectic scheme which preserve the multisymplectic conservation law is given for the Maxwell’s equations in an inhomogeneous, isotropic and lossless medium. At last a numerical example is illustrated.

1 Introduction

Transient electromagnetic field problems are important problems in many modern technology applications. M. J. Gotay and J. E. Marsden et al[1] discussed the Lagrangian density for electromagnetism on a fixed background space-time $X$ with metric $g$, and discussed the existence of it’s Euler-Lagrange equation. In this paper we will discuss this problem from the point of inverse variational problem. As a complete depiction of the behavior of the electromagnetic field, Maxwell’s equations have very interesting structure. we will discuss its Lagrangian density and give its Multisymplectic Bridge’s(Hamilton) form.

* Supported by the Special Funds for Major State Basic Research Projects , G 1999, 032800
The multisymplectic method is used to phrase the dynamics, which is described by a Euler-Lagrange system, in the terms of the finite-dimensional space of fields at a given event in space-time. Some Euler-Lagrange system can be formalized in a multisymplectic Hamilton form, which manifest some characteristics of original systems, so we discretize multisymplectic Hamilton form directly in order to simulate original system intrinsically.

Determining whether a given nonlinear partial differential equation (PDE) is an Euler-Lagrange equation of some variational problem is fundamental work in finite-element method and nonlinear wave theory and etc. In this paper we will use the straightforward result of this problem given by Atherton and Homsy to discuss self-adjointness of Maxwell’s equations in different representations. We show that the same equations in different forms shows different potentialness, consequently it can be expressed as different variational problems. So we choose an appropriate form of the Maxwell’s equations to derive it’s Lagrangian and then rewrite it in a multisymplectic Hamilton form directly in order to simulate original system intrinsically.

This paper falls into 4 parts. In Sec. 2, we obtain the necessary and sufficient conditions of self-adjointness for a 1-st order form of the Maxwell’s equations in an inhomogeneous, isotropic and lossless medium. In Sec. 3, we rewrite the equations in a 2-ed order form by introducing two vector potential functions. A complex multisymplectic Hamilton form and a multisymplectic conservation law are given as well as a ordinary conservation law. In Sec. 4, as application of the multisymplectic method, a multisymplectic algorithm is constructed and numerical simulation is shown.

2 Lagrangian Formalisms for Maxwell’s Equations

First we shall introduce some basic concepts in variational problems. Let \( X \subset \mathbb{R}^m \), an open subset with smooth boundary \( \partial X \). \( M \) is some function space. Given a vector differential equation with a vector of dependent variables \( u_i = u_i(x_1, x_2, \cdots, x_m) \in M \), \( i = 1, 2, \cdots, n \), and a vector of differential operators \( N^j(u), \quad j = 1, 2, \cdots, n \), such that

\[
N(u) = 0. \tag{1}
\]
We assume the Frechét derivative of $N$ exist,

$$N'_u\phi = \lim_{h \to 0} \frac{N(u + h\phi) - N(u)}{h}. \quad (2)$$

The arbitrary function $\phi$ now belongs to $M$. The Frechét derivative derives a matrix $N'_u$ with operator elements $N'(i, j)$ which are the derivatives of the operators $N^i$ with respect to $u^j$, so

$$N'_u = N'(i, j)_{i,j=1}^n. \quad (3)$$

It is called as differential operator matrix of operator matrix $N$. Then the self-adjoint test functional is given as

$$S(\psi, \phi) = \int_X \psi N'_u \phi d\nu = \int_X \psi_i N'(i, j) \phi_j d\nu. \quad (4)$$

From Vainberg’s theorem the condition of self-adjointness can be given as

$$S(\psi, \phi) = S(\phi, \psi) + \int_X \text{div} P d\nu, \quad (5)$$

if we set the boundary condition to be free, i.e. $P$ vanishes on $\partial X$, then

$$N'(i, j) = N'(j, i)^*, \quad (6)$$

where $N'(j, i)^*$ is denoted as the adjoint operator of $N'(j, i)$.

Surely a certain variational problem can lead to more than one Euler-Lagrange equation and different boundary conditions, responsibly the Lagrangian and the boundary condition are not unique in the inverse problems of finding corresponding functionals for a Euler-Lagrange equation, so the requirement for free boundary conditions in Ref. [5] is too strict for some applications such as Multi-symplectic method [3, 4] which needs to verify self-adjointness of a given system to find one of its Lagrangians at first step of the method. For example, the wave equation such as

$$u_{tt} - u_{xx} = 0, \quad (7)$$

according to Ref. [5] its self-adjointness condition is

$$\frac{\partial N(u)}{\partial u^2} \bigg|_{\partial X} = 0, \quad \text{where} \quad (t, x) \in X, u_2 = u_{xx}, u_{tt}, \quad (8)$$

so it does not satisfy this condition whatever boundary condition is given, but it actually has Lagrangian, therefor we just set self-adjointness condition only to be condition [6] without free boundary condition.
Let us consider a $k$th order functional on maps $u : \mathbb{R}^{1,3} \to \mathbb{R}^6$, for Maxwell’s equation, $u = [H_1, H_2, H_3, E_1, E_2, E_3]$. Here $\mathbb{R}^{1,3}$ is a $(1+3)$-dimensional time-space domain, denoted by coordinates $x_j, j = 1, \cdots, 4$. $\mathbb{R}^6$ is the target space, denoted by coordinates $y^i, i = 1, \cdots, 6$, additionally we denote the space $T^*_y \mathbb{R}^6 \otimes T_x \mathbb{R}^{1,3}$ by $\mathcal{R}$. The functional is defined by means of a Lagrangian $L$; it is a function on the space $\mathbb{R}^{1,3} \times \mathbb{R}^6 \times \mathcal{R}$, i.e. $L$ is a smooth function defined on the bundle over $\mathbb{R}^{1,3} \times \mathbb{R}^6$ with fiber over $(x,y)$ equal to $\mathcal{R}$. We denote this bundle by $T \mathbb{R}^6 \otimes_{\mathbb{R}^{1,3}\times \mathbb{R}^6} T^* \mathbb{R}^{1,3}$. Hence the functional is defined as

$$
\mathcal{L} = \int_{\mathbb{R}^{1,3}} L(x_j, u^i(x), u^\nu_i(x)) \omega,
$$

where $\omega$ is some volume 4-form on $\mathbb{R}^{1,3}$, $u^\nu_i = \frac{\partial u_i}{\partial x_1^{\nu_1} \cdots \partial x_4^{\nu_k}}, \nu = 1, 2, \cdots, k$. We define Legendre transformations as

$$
\pi^\nu_i := \frac{\partial L}{\partial u^\nu_i}(x_j, u^i(x), u^\nu_i(x)),
$$

it leads to covariant Hamiltonian function $\mathcal{H}$ on the multisymplectic manifold $T^* \mathbb{R}^6 \otimes_{\mathbb{R}^{1,3}\times \mathbb{R}^6} T \mathbb{R}^{1,3}$, as

$$
\mathcal{H}(x_j, u^i(x), \pi^\nu_i) = \pi^\nu_i u^i - L(x_j, u^i(x), u^\nu_i(x)).
$$

Now we consider the Maxwell’s equations in an inhomogeneous, isotropic, and lossless medium

$$
\mu \frac{\partial H}{\partial t} + \nabla \times E = -K \\
-\nabla \times H + \varepsilon \frac{\partial E}{\partial t} = -J,
$$

$E = [E_1, E_2, E_3]^T$ is the electric field strength, $H = [H_1, H_2, H_3]^T$ is the magnetic field strength with superscript T denoting transpose. $\varepsilon$ is the permittivity, $\mu$ is the permeability, both $\varepsilon$ and $\mu$ are scalar functions of space and time variables. $J$ is the external electric-current density, and $K$ is the external magnetic-current density. System (12) can be arranged in the form

$$
\mathcal{G} \cdot \mathcal{Z} = \mathcal{F},
$$

where $\mathcal{Z} = \mathcal{Z}(x,y,z,t)$ is the field vector consisting of the components of the electric field strength $E$ and the magnetic field strength $H$ as

$$
\mathcal{Z} = [H_1, H_2, H_3, E_1, E_2, E_3]^T
$$

}\]
simplified into \( Z = [H, E]^T \), similarly \( F = F(x, y, z, t) \) is the source vector composed of the external electric current source \( J \) and the external magnetic current source \( K \). \( G \) is the operator matrix given by

\[
G = \begin{pmatrix}
\nabla \times & -\varepsilon \partial_t \\
\mu \partial_t & \nabla \times
\end{pmatrix},
\]

where each element of \( G \) is a 3 \(-rd\) order matrix such as

\[
\nabla \times = \begin{pmatrix}
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{pmatrix}
\]

\[
\varepsilon(\mu) \partial_t = \varepsilon(\mu) \cdot \begin{pmatrix}
\partial_t & 0 & 0 \\
0 & \partial_t & 0 \\
0 & 0 & \partial_t
\end{pmatrix}.
\]

Because the operator \( G \) is a linear operator matrix, i.e. each element of \( G \) is a linear operator, its differential operator matrix denoted as \( G' \) equals \( G \) itself. So the adjoint operator matrix of \( G' \) is as

\[
G'^* = \begin{pmatrix}
(\nabla \times)^T & \varepsilon \partial_t + \varepsilon_t \\
-\mu \partial_t - \mu_t (\nabla \times)^T
\end{pmatrix},
\]

It is obvious that the self-adjoint condition \( \textbf{6} \) is that \( G'^T = G'^* \), so the equations have to satisfy \( \varepsilon = \mu \), and the two coefficients are independent on time variable \( t \). The system in vacuum of course satisfies \( G'^T = G'^* \). The systems with such self-adjointness condition have one Lagrangian as

\[
L = \frac{1}{2} < H, \nabla \times H > + \frac{1}{2} < E, \nabla \times E > - \mu < H, E_t > - < H, J > + < E, K >,
\]

where \( \mu = \mu(x, y, z) = \varepsilon(x, y, z) \). How to calculate Lagrangian density form Euler-Lagrange equation is to be introduced in the next section.

Consider another form of operator matrix \( G \)

\[
G_1 = \begin{pmatrix}
\frac{1}{\varepsilon} \nabla \times & \frac{1}{\mu} \partial_t \\
-\partial_t & \frac{1}{\mu} \nabla \times
\end{pmatrix},
\]

with the source function \( F \) changed in

\[
F_1 = -[\frac{1}{\varepsilon} J, \frac{1}{\mu} K]^T,
\]

the dependent variable \( Z \) is unchanged. But the differential operator matrix of \( G_1 \) is not a self-adjoint operator matrix, since that

\[
G_1'^* = \begin{pmatrix}
A + \frac{1}{\varepsilon} (\nabla \times)^T & -\partial_t \\
\partial_t & B + \frac{1}{\mu} (\nabla \times)^T
\end{pmatrix} \neq G_1,
\]
where \( A \) and \( B \) are two 3-rd order matrix as
\[
A = \begin{pmatrix}
0 & \left(\frac{1}{\varepsilon}\right)_z & -\left(\frac{1}{\varepsilon}\right)_y \\
-(\frac{1}{\varepsilon})_z & 0 & -(\frac{1}{\varepsilon})_x \\
\left(\frac{1}{\varepsilon}\right)_y & -(\frac{1}{\varepsilon})_x & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & \left(\frac{1}{\mu}\right)_z & -\left(\frac{1}{\mu}\right)_y \\
-(\frac{1}{\mu})_z & 0 & -(\frac{1}{\mu})_x \\
\left(\frac{1}{\mu}\right)_y & -(\frac{1}{\mu})_x & 0
\end{pmatrix},
\]
(22)

obviously if \( G_1^T = G_1^* \), \( \varepsilon \) and \( \mu \) are free of space variables, additionally such form excludes the condition of \( \varepsilon, \mu = 0 \).

So we get another Lagrangian of system (12)
\[
L = \frac{1}{2\varepsilon} <H, \nabla \times H> + \frac{1}{2\mu} <E, \nabla \times E> - <H, E_t> - \frac{1}{\varepsilon} <H, J>
+ \frac{1}{\mu} <E, K>.
\]
(23)

This Lagrangian seems equivalent to the Lagrangian (18), but in fact the above cases actually dealt with two different systems because of different restrictions for the coefficients \( \varepsilon \) and \( \mu \) in their self-adjointness conditions.

Both the two Lagrangian are of first order functions, and the Legendre transformations are unnecessary in constructing their multisymplectic Hamilton form, because the conjugate momentums are \( H \) and \( E \) multiplying some coefficients, for example in \( L_2 \)
\[
\frac{\partial L_2}{\partial \nabla \times H} = \frac{1}{2\varepsilon} H, \quad \frac{\partial L_2}{\partial \nabla \times E} = \frac{1}{2\mu} E, \quad \frac{\partial L_2}{\partial E_t} = -H.
\]
(24)

In fact, for linear 1-st order system which is self-adjoint, if its coefficients satisfy some conditions, then it can be written in multisymplectic Hamilton form directly.

### 3 Multisymplectic Hamilton Forms for Maxwell’s Equations

Introducing two vector functions \( U \) and \( V \) satisfying \( U_t = E \) and \( V_t = H \), so the system can be rewritten in form (2) with different operator matrix \( G \),
different dependent variable \( Z \) and source function \( F \)

\[
\mathcal{G} = \left( \begin{array}{cc}
\mu \partial_t^2 & \nabla \times \partial_t \\
-\nabla \times \partial_t & \varepsilon \partial_t^2
\end{array} \right), \quad Z = [V_1, V_2, V_3, U_1, U_2, U_3]^T, \quad F = -[K, J]^T.
\]

(25)

The differential operator matrix \( \mathcal{G}' \) equals \( \mathcal{G} \) itself. If \( \varepsilon \) and \( \mu \) are not dependent on spatial variables, then it is easy to demonstrate that \( \mathcal{G}' \) is self-adjoint, so the self-adjointness of the form (13) with sketch (25) is verified, and the potential functional is given by

\[
F(V, U) = \int_\Omega \int_0^1 [V, U]G(\lambda Z)\,d\lambda \,dv
\]

\[
= \int_\Omega \left( \frac{1}{2} \mu < V, V_{tt} > + \frac{1}{2} < V, \nabla \times U_t > + \frac{1}{2} \varepsilon < U, U_{tt} > \right.
\]

\[
- \frac{1}{2} < U, \nabla \times V_t > + UJ + VK \big)dv
\]

\[
= \int_\Omega \left( -\frac{1}{2} \mu < V_t, V_t > - \frac{1}{2} < V_t, \nabla \times U > - \frac{1}{2} \varepsilon < U_t, U_t > \right.
\]

\[
+ \frac{1}{2} < U_t, \nabla \times V > + UJ + VK + \text{div}P \big)dv,
\]

(26)

where \( < \cdot, \cdot > \) represents the inner production of vectors. We know the Lagrangian is unique for a given Euler-Lagrange equation under difference of a term \( \text{div}P \). Here for simplicity we consider the Lagrangian as

\[
L = \frac{1}{2} \mu < V_t, V_t > + \frac{1}{2} < V_t, \nabla \times U >
\]

\[
+ \frac{1}{2} \varepsilon < U_t, U_t > - \frac{1}{2} < U_t, \nabla \times V > - UJ - VK,
\]

(27)

correspondingly the generalized conjugate momentums are

\[
P = \frac{\partial L}{\partial V_t} = \mu V_t + \frac{1}{2} \nabla \times U, \quad \frac{\partial L}{\partial \nabla \times V} = - \frac{1}{2} U_t,
\]

\[
Q = \frac{\partial L}{\partial U_t} = \varepsilon U_t - \frac{1}{2} \nabla \times V, \quad \frac{\partial L}{\partial \nabla \times U} = \frac{1}{2} V_t.
\]

(28)

We get the covariant Hamiltonian by

\[
S = < P, V_t > + < Q, U_t > + < \frac{\partial L}{\partial \nabla \times V}, \nabla \times V >
\]

\[
+ < \frac{\partial L}{\partial \nabla \times U}, \nabla \times U > - L
\]

\[
= < P, H > + < Q, E > - \frac{1}{2} \mu < H, H > - \frac{1}{2} \varepsilon < E, E > + UJ + VK,
\]

(29)
here $Z = [H, E, V, U, P, Q]^T$. So the Maxwell’s equations are transformed into the following form

$$
\begin{align*}
\frac{1}{2} \nabla \times U &= P - \mu H, \\
- \frac{1}{2} \nabla \times V &= Q - \varepsilon E, \\
- \frac{1}{2} \nabla \times E &= K, \\
- Q_t - \frac{1}{2} \nabla \times H &= J, \\
V_t &= H, \\
U_t &= E.
\end{align*}
\tag{30}
$$

The above equations can be organized in the universal form

$$
\mathcal{M} Z_t + \mathcal{K} \nabla \times Z = \nabla_Z S(Z),
\tag{31}
$$

where the rotation action $\nabla \times Z$ denotes $[\nabla \times H, \nabla \times E, \nabla \times V, \nabla \times U, \nabla \times P, \nabla \times Q]^T$, and $\mathcal{K} \nabla \times Z$ consists of three components such that

$$
\mathcal{K} \nabla \times Z = K_1 Z_x + K_2 Z_y + K_3 Z_z.
\tag{32}
$$

So

$$
\mathcal{M} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & -I \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0
\end{pmatrix}, \quad
\mathcal{K}_i = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} \mathcal{R}_i \\
0 & 0 & -\frac{1}{2} \mathcal{R}_i & 0 \\
\frac{1}{2} \mathcal{R}_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{33}
$$

where $I$ is the identity element belonging to $R^{3 \times 3}$. $\mathcal{M}$ is obviously anti-symmetric, and

$$
R_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad R_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad R_3 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\tag{34}
$$

ultimately $\mathcal{K}_i$ belongs to $R^{18 \times 18}$ and it is anti-symmetric. Additionally the rotation operator $\nabla \times = R_1 \frac{\partial}{\partial x} + R_2 \frac{\partial}{\partial y} + R_3 \frac{\partial}{\partial z}$.

The representation (31) is simplified expression in vector form, its complete extension is the multisymplectic Hamiltonian system which was first
introduced by Bridges and Derks in [4]. It has the multisymplectic conservative law
\[ \frac{\partial}{\partial t} \omega + \nabla \times \kappa = 0, \] (35)
\( \omega \) and \( \kappa \) are the presymplectic forms
\[ \omega = \frac{1}{2} dZ \dot{\lambda} M dZ, \quad \kappa = \frac{1}{2} dZ \dot{\lambda} K dZ, \] (36)
where \( dZ = (dZ_1, dZ_2, \cdots)^T \), and for two \( n \)-order vector functions \( p = (p_1, \cdots, p_n)^T \) and \( q = (q_1, \cdots, q_n)^T \), notation \( \dot{\lambda} \) is defined as
\[ dp \dot{\lambda} dq = \sum_{i=1}^{n} dp_i \wedge dq_i. \] (37)

The system (31) also has the local multisymplectic conservation law
\[ \frac{1}{2} \nabla \times [dU \dot{\lambda} dH - dV \dot{\lambda} dE] + \frac{\partial}{\partial t} (dV \dot{\lambda} dP + dU \dot{\lambda} dQ) = 0. \] (38)

Now we return to the second case introduced in Sec. 2. We can rewrite it in the multisymplectic Hamilton form directly
\[ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} H \\ E \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} R_1 & 0 \\ 0 & \frac{1}{\mu} R_1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} H \\ E \end{pmatrix} = \begin{pmatrix} -J/\varepsilon \\ -K/\mu \end{pmatrix}, \] (39)
here the covariant Hamiltonian is \( S = -\frac{1}{\varepsilon} < J, H > -\frac{1}{\mu} < K, E > \). It has the multisymplectic conservation law
\[ \nabla \times \left[ \frac{1}{\varepsilon} dH \dot{\lambda} dH + \frac{1}{\mu} dE \dot{\lambda} dE \right] + \frac{\partial}{\partial t} (dE \dot{\lambda} dH) = 0. \] (40)

4 A Multisymplectic Scheme for the Maxwell’s Equations

Maxwell’s equations are simulated by many methods. Its first numerical simulation is given by Yee[6] in 1966, which was developed in many more efficient algorithms. In this section, we adopt a multisymplectic scheme to approximate the system.
For simplicity, we consider 1 + 1-dimensional multisymplectic form of (30), and suppose that the external magnetic-current vanished, so \( K = 0 \), and the rotation operator becomes
\[
\nabla \times Z = R_1 Z_x. \tag{41}
\]

We discretize the equations (30) by using midpoint scheme in both \( t \) and \( x \) directions,
\[
\begin{align*}
R_1 \frac{U_{i+1,j+\frac{1}{2}} - U_{ij+\frac{1}{2}}}{2\Delta x} &= P_{i+\frac{1}{2},j+\frac{1}{2}} - \mu_{i+\frac{1}{2},j+\frac{1}{2}} H_{i+\frac{1}{2},j+\frac{1}{2}}, \\
- R_1 \frac{V_{i+1,j+\frac{1}{2}} - V_{ij+\frac{1}{2}}}{2\Delta x} &= Q_{i+\frac{1}{2},j+\frac{1}{2}} - \epsilon_{i+\frac{1}{2},j+\frac{1}{2}} E_{i+\frac{1}{2},j+\frac{1}{2}}, \\
- \frac{P_{i+\frac{1}{2},j+1} - P_{i+\frac{1}{2},j}}{\Delta t} - R_1 \frac{E_{i+1,j+\frac{1}{2}} - E_{ij+\frac{1}{2}}}{2\Delta x} &= 0, \\
- \frac{Q_{i+\frac{1}{2},j+1} - Q_{i+\frac{1}{2},j}}{\Delta t} + R_1 \frac{H_{i+1,j+\frac{1}{2}} - H_{ij+\frac{1}{2}}}{2\Delta x} &= J_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\frac{V_{i+\frac{1}{2},j+1} - V_{i+\frac{1}{2},j}}{\Delta t} &= H_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\frac{U_{i+\frac{1}{2},j+1} - U_{i+\frac{1}{2},j}}{\Delta t} &= E_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{42}
\end{align*}
\]

Here the subscript \( i \) and \( j \) denote respectively \( x \) and \( t \) directions. The above scheme is called the Preissman multisymplectic scheme which preserves the multisymplectic conservative law (35) in discrete space, such as
\[
\frac{\kappa_{i+1,j+\frac{1}{2}} - \kappa_{ij+\frac{1}{2}}}{2\Delta x} + \frac{\omega_{i+\frac{1}{2},j+1} - \omega_{i+\frac{1}{2},j}}{\Delta t} = 0, \tag{43}
\]
where
\[
\begin{align*}
\kappa_{ij+\frac{1}{2}} &= R_1 dU_{ij+\frac{1}{2}} \hat{d}H_{ij+\frac{1}{2}} - R_1 dV_{ij+\frac{1}{2}} \hat{d}E_{ij+\frac{1}{2}}, \\
\omega_{i+\frac{1}{2},j} &= dV_{i+\frac{1}{2},j+1} \hat{d}P_{i+\frac{1}{2},j+1} + dU_{i+\frac{1}{2},j+1} \hat{d}Q_{i+\frac{1}{2},j+1}. \tag{44}
\end{align*}
\]
Eliminating the auxiliary variables \( V, U, P \) and \( Q \) from the Preissman scheme
we get a nine-point multisymplectic integrator as

\[
\mathcal{R}_1\frac{E_{i+2j+2} + 2E_{i+2j+1} + E_{i+2j} - E_{ij+2} - 2E_{ij+1} - E_{ij}}{4\Delta x} = -\mu_{i+3/2}\frac{H_{i+2j+2} + H_{i+1j+2} - H_{i+2j} - H_{i+1j}}{4\Delta t} - \mu_{i+1/2}\frac{H_{i+1j+2} + H_{ij+2} - H_{i+1j} - H_{ij}}{4\Delta t}, \\
\]

\[
\mathcal{R}_1\frac{H_{i+2j+2} + 2H_{i+2j+1} + H_{i+2j} - H_{ij+2} - 2H_{ij+1} - H_{ij}}{4\Delta x} = \varepsilon_{i+3/2}\frac{E_{i+2j+2} + E_{i+1j+2} - E_{i+2j} - E_{i+1j}}{4\Delta t} + \varepsilon_{i+1/2}\frac{E_{i+1j+2} + E_{ij+2} - E_{i+1j} - E_{ij}}{4\Delta t} + (J_{i+3/2j+3/2} + J_{i+3/2j+1/2} + J_{i+1/2j+3/2} + J_{i+1/2j+1/2})/2. \\
\]

The numerical process can be depicted by the following expression

\[
AZ^{j+2} = BZ^{j+1} + CZ^j + \bar{J}, \\
\]

where

\[
A = \begin{pmatrix}
D & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
B_0 & D & A_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & B_1 & D & A_2 & \cdots & 0 & 0 & 0 \\
& \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & B_{s-1} & D & A_s \\
0 & 0 & 0 & 0 & \cdots & 0 & B_s & D \\
& & \ddots & & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}, \\
\]

\[
C = \begin{pmatrix}
-D & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
B_0 & -D & A_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & B_1 & -D & A_2 & \cdots & 0 & 0 & 0 \\
& \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & B_{s-1} & -D & A_s \\
0 & 0 & 0 & 0 & \cdots & 0 & B_s & -D \\
& & \ddots & & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
-2D & -2D & \cdots & -2D \\
\vdots & \ddots & \ddots & \vdots \\
-2D & -2D & \cdots & -2D
\end{pmatrix}, \quad \bar{J} = \begin{pmatrix} a(0) + b(0) \\ a(1) + b(1) \\ \vdots \end{pmatrix}, \quad (49)
\]

where \(a(i) = J_{i + \frac{1}{2}j + \frac{1}{2}} + J_{i + \frac{3}{2}j + \frac{3}{2}}\), \(b(i) = J_{i + \frac{1}{2}j + \frac{1}{2}} + J_{i + \frac{3}{2}j + \frac{3}{2}}\), and

\[
D = \begin{pmatrix}
0 & \Delta t R_1 & \cdots & \cdots & \cdots \\
-\Delta t R_1 & 0 & \Delta t R_1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-\Delta t R_1 & \cdots & \cdots & 0 & \Delta t R_1 \\
\end{pmatrix}, \quad (50)
\]

Set \(r = 1, 2, \cdots\), then for every \(i = 1, 2, \cdots, s, \cdots\), there are

\[
A_i = \begin{pmatrix}
e_{0} + e_1 & e_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
e_1 & e_1 + e_2 & e_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & e_2 & e_2 + e_3 & e_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & e_{r-2} & e_{r-2} + e_{r-1} & e_{r-1} & e_{r-1} + e_r \\
0 & 0 & 0 & \cdots & 0 & e_{r-1} & e_{r-1} & e_{r-1} + e_r
\end{pmatrix},
\]

\[
(51)
\]

here \(e_j = \Delta x \varepsilon_j + i r + \frac{1}{2} I \in R^{3 \times 3}\), \(j = 1, 2, \cdots, r\). \(B_i\) is defined similarly and just change \(e_j\) in \(u_j = \Delta x \mu_{j + i r + \frac{1}{2}} I \in R^{3 \times 3}\).

To check the integrator (45), we consider a simple example. We shall take \(\varepsilon = \mu = 1\) and \(J = 0\). Furthermore, we take exact initial conditions as

\[
H_1(x, 0) = H_3(x, 0) = 0, \quad H_2(x, 0) = -\sqrt{\frac{\varepsilon}{\mu}} \sin(x),
\]
\[
E_1(x, 0) = E_2(x, 0) = 0, \quad E_3(x, 0) = \sin(x), \quad (52)
\]

under this initial condition we can get a exact solution as following

\[
H_1 = H_3 = 0, \quad H_2 = -\sqrt{\frac{\varepsilon}{\mu}} \sin(x - \sqrt{\frac{1}{\varepsilon \mu}} t),
\]
\[
E_1 = E_2 = 0, \quad E_3(x) = \sin(x - \sqrt{\frac{1}{\varepsilon \mu}} t), \quad (53)
\]

\[12\]
We compute $H_2$ in rectangles $[0, 2\pi + 3] \times [0, 0.1]$, $[0, 2\pi + 3] \times [0, 1]$ and choose $\Delta x = (2\pi + 3)/61$, $\Delta t = 0.01$. Fig.1 and Fig.2 shows the numerical solutions respectively. We can see the solution is moving to $x$-direction positively. In the rectangle $[0, 2\pi + 3] \times [0, 10]$, the trend of the solution is more manifest.

The absolute error at t-number=1000 is shown in Fig. 4 by the 'error-line', and the 'exact solution-line' is the graph of solution $0.01 \times H_2$. The former has a very regular route which is so like the solution graph, and their vertexes are almost at the zero points of each other. The regular similitude give a challenge to modify the scheme which is expected in further task.

For the multisymplectic form (39) there are many existing schemes to simulate the system and to preserve the multisymplectic conservation law (40), such as the usual midpoint scheme which doesn’t present here. For cases of more general $\varepsilon$ and $\mu$, the discussion for their multisymplecticity is similar, so omitted here.

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