On the possible distributions of temperature in nonequilibrium steady states

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Abstract
Superstatistics is a framework in nonequilibrium statistical mechanics that successfully describes a wide variety of complex systems, including hydrodynamic turbulence, weakly-collisional plasmas, cosmic rays, power grid fluctuations, among several others. In this work we analyze the class of nonequilibrium steady-state systems consisting of a subsystem and its environment, and where the subsystem is described by the superstatistical framework. In this case we provide an answer to the mechanism by which a broad distribution of temperature arises, namely due to correlation between subsystem and environment. We prove that there is a unique microscopic definition $B$ of inverse temperature compatible with superstatistics, in the sense that all moments of $B$ and $\beta = 1/(k_B T)$ coincide. The function $B$ however, cannot depend on the degrees of freedom of the system itself, only on the environment, in full agreement with our previous impossibility theorem (Davis and Gutiérrez 2018 Physica A 505 864–70). The present results also constrain the possible joint ensembles of system and environment compatible with superstatistics.

Keywords: superstatistics, temperature fluctuations, nonequilibrium

1. Introduction

An increasing number of complex systems out of equilibrium are described by non-canonical statistics, most remarkably ensembles described by power laws instead of exponential distributions. Currently several theoretical frameworks employed to explain such non-canonical states are used, among them Tsallis’ nonextensive statistics [1, 2], superstatistics [3–6], Kaniadakis statistics [7] and incomplete statistics [8]. While Tsallis statistics postulates a generalization...
of the Boltzmann–Gibbs entropy, superstatistics aims to describe the same steady states and others without the need for redefining the entropy. It has been successfully applied to the description of turbulence [9, 10], space and laboratory plasmas [11], solar flares [12], fluctuations in electrical power grids [13], cosmology [14], among several others [15–18].

Fundamentally, superstatistics replaces the single value of the inverse temperature parameter \( \beta = 1/(k_B T) \) in the canonical ensemble by a probability distribution of (inverse) temperatures, that we will express through the Bayesian notation \( P(\beta|S) \), where \( S \) denotes a steady state. More precisely, we will define superstatistics for a system with microstates \( x = (x_1, \ldots, x_N) \) through the joint distribution

\[
P(x, \beta|S) = P(x|\beta)P(\beta|S) = \left[ \frac{\exp(-\beta H(x))}{Z(\beta)} \right] P(\beta|S),
\]

where \( H(x) \) is the Hamiltonian of the system, \( Z(\beta) \) the partition function,

\[
Z(\beta) = \int dx \exp(-\beta H(x)) = \int dE \Omega(E) \exp(-\beta E),
\]

and \( \Omega(E) = \int dx \delta(E - H(x)) \) the density of states of \( H \). The probability density of microstates is then expressed through integration over all possible values of \( \beta \) as [19]

\[
P(x|S) = \int d\beta \left[ \frac{\exp(-\beta H(x))}{Z(\beta)} \right] P(\beta|S).
\]

From this is clear that the probability of the microstate only depends on its energy, and we can write this dependence as \( P(x|S) = \rho(H(x)) \), where we have defined the ensemble function

\[
\rho(E) := \int d\beta \exp(-\beta E) f(\beta),
\]

and the superstatistical weight function

\[
f(\beta) := P(\beta|S)/Z(\beta)
\]

for convenience.

Before presenting our problem, we will give first a brief note on our notation for expectations. An expectation \( \langle A \rangle_I \) of a quantity \( A \) under the state of knowledge \( I \) has an underlying integral,

\[
\langle A \rangle_I = \int du P(u|I)A(u),
\]

with the appropriate distribution function \( P(u|S) \), where \( u \) represents any set of variables large enough that includes everything needed to compute \( A \). For instance, if \( A = A(x) \) then we can take \( u = x \) and write

\[
\langle A \rangle_I = \int dx P(x|I)A(x),
\]

while we can also take \( u = A \) and write

\[
\langle A \rangle_I = \int dA P(A|I)A,
\]

in the understanding that the functions represented formally by the notation \( P(\cdot|I) \) are different functions for different sets of variables.
2. A microscopic definition of temperature

The fact that superstatistics assumes a statistical distribution of temperature raises several questions, most important of which is the existence and nature of temperature fluctuations [20]. When $\beta$ is broadly distributed and we can assign a variance $\langle (\delta \beta)^2 \rangle_S$, is it just statistical uncertainty or is there a fluctuating quantity?

Although it would be natural to assume the existence of one, or even a family of microscopic definitions of temperature, it has been recently shown [21] that no function $B(x)$ can be identified one-to-one with the inverse temperature $\beta$ in superstatistics. In other words, for a superstatistical ensemble $S$ and an arbitrary function $g$ of the inverse temperature, no function $B(x)$ can be constructed such that

$$\langle g(\beta) \rangle_S = \langle g(B) \rangle_S,$$

(9)

where the left-hand side is the expectation of $g(\beta)$ under $P(\beta|S)$,

$$\langle g(\beta) \rangle_S = \int d\beta P(\beta|S) g(\beta),$$

(10)

and the right-hand side is the expectation of the quantity $g(B(x))$ under $P(x|S)$,

$$\langle g(B) \rangle_S = \int dx P(x|S) g(B(x)).$$

(11)

If a function $B$ existed such that equation (9) holds, we would be able to measure $\beta$ and its statistical properties (e.g. all its moments) from within the system, by instead collecting statistics of $B$, because by assuming the ergodic hypothesis the expectation in equation (11) could in principle be approximated by the dynamical average

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \, g(B(x(t)))$$

over a trajectory $x(t)$ consistent with $S$, while the expectation in equation (10) has no such dynamical interpretation. In particular, a histogram of $B$ would converge to the superstatistical distribution $P(\beta|S)$, because then

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \, \delta(B(x(t)) - \beta_0) = \langle \delta(\beta - \beta_0) \rangle_S = P(\beta = \beta_0|S).$$

(12)

Despite the fact that there is no function $B(x)$ such that equation (9) holds, there is still hope. In this work it is shown that by considering an extended setup of system $x$ and environment $y$ with a joint steady state probability

$$P(x,y|S) = p(H(x), G(y)),$$

(13)

where, as before, $H(x)$ is the Hamiltonian of the system of interest but we have defined a new Hamiltonian $G(y)$ describing the environment, then it is possible to define a function $B$ such that

$$\langle g(\beta) \rangle_S = \langle g(B) \rangle_S$$

(14)

for any $g(\beta)$. Moreover, the function $B$ only depends on the environment $y$, and is uniquely defined by

\footnote{The original proof only ruled out functions $B(x)$ which are independent of the ensemble function $\rho$. As described in the main text, the current derivation is stronger, ruling out all functions $B(x)$ except the constant function.}
\[ B = -\frac{\partial}{\partial E} \ln p(E, G) = B(G). \]  

Equation (15) is the main new result of the present work, whose proof we delay until section 5 in order to first discuss its implications in sections 3 and 4.

First, let us motivate the assumption of equation (13). We require the dependence on \( x \) to be only through \( H(x) \), otherwise we cannot have a marginal distribution \( P(x|S) \) given by equation (3). Given this, the broadest assumption is a dependence on \( y \) through another function \( G(y) \) in which the interaction between both systems is left unspecified. It is instructive to notice that because

\[ P(x|S) = \rho(H(x)) = \int dy p(H(x), G(y)) \]

the superstatistical ensemble for \( x \) and therefore its possible nonequilibrium states, are fully determined by properties of the environment.

The fact that the inverse temperature function \( B(G) \) cannot depend on \( x \), directly implies an even stronger version of the original impossibility theorem of [21], that also rules out the definitions \( B(x) \) which are dependent of the ensemble function \( \rho(E) \). Note also that, if the number of degrees of freedom of \( y \) is large enough (i.e. the system \( x \) is in contact with an infinite reservoir), by the asymptotic equipartition property (AEP) [22] the fluctuations of \( G \) vanish and \( B(G) \) becomes constant, hence the system \( x \) approaches the canonical ensemble. In this way, only a system \( y \) with finite number of degrees of freedom (equivalently, a finite heat capacity) can induce a non-canonical distribution on \( x \), which is a well-known and robust result in nonextensive statistical mechanics [23, 24, 25, 26].

From integration of equation (15) and proper normalization, we see that the only ensembles leading to superstatistics for \( x \) are of the form

\[
P(x,y|S) = \rho_G(G(y)) \left[ \frac{\exp(-\beta H(x))}{Z(\beta)} \right] \bigg|_{\beta=B(G(y))} = P(y|S) \times P(x|\beta = B(G(y))),
\]

which directly implies that

\[
P(x|y, S) = P(x|\beta = B(G(y))).
\]  

In fact, equation (17) can be understood as the defining condition for superstatistics, namely that the statistical ensemble of the system \( x \) under a ‘frozen’ environment \( y \) must be completely equivalent to a canonical ensemble with inverse temperature \( B(G(y)) \). In other words, the system and its environment are coupled through the fluctuating inverse temperature \( B \), as \( P(x,y|S) \) is only separable into a product \( P(x|S) \times P(y|S) \) when \( B \) is a constant (the canonical ensemble). We arrive at the following conclusion: correlation between system and environment implies non-canonical superstatistics.

Probably the most important case in which superstatistics is obtained in this way, is the case where the combined system with Hamiltonian

\[ H_{XY}(x,y) = H(x) + G(y) \]
such that $H \ll G$ is placed in a steady state ensemble with ensemble function $\rho$. This means that

$$P(x,y|S) = \rho(H(x) + G(y))$$

(18)

and we have, by expanding $\ln \rho(H + G)$ to first order around $H = 0$, that

$$P(x,y|S) \approx \rho(G(y)) \left[ \exp(-B(G(y))H(x)) \right]$$

(19)

which we can recognize as equation (16) with

$$B(G) = -\frac{\partial}{\partial G} \ln \rho(G).$$

(20)

This value of $B$ is consistent with equation (15) because in this case $\rho(E, G) = \rho(E + G)$ and we have

$$-\frac{\partial}{\partial E} \ln \rho(E, G) = -\frac{\partial}{\partial E} \ln \rho(E + G)$$

$$= -\frac{\partial}{\partial G} \ln \rho(E + G)$$

$$\approx -\frac{\partial}{\partial G} \ln \rho(G)$$

(21)

when $E \ll G$. That is, in this case the temperature function matches the fundamental temperature [27] of the whole (system plus environment), evaluated at the energy of the environment. This is precisely the result recently shown in [28] in the context of collisionless plasmas, but now understood in a more general way. This case is also connected with superstatistics in small, correlated systems, see for instance [29].

As an example, let us consider the Gaussian ensemble [30], defined by the ensemble function

$$\rho(E + G) = \frac{1}{\zeta(E_0, A)} \exp(-A(E + G - E_0)^2),$$

(22)

and which reduces to the microcanonical ensemble in the limit $A \to \infty$,

$$\lim_{A \to \infty} \rho(E + G) = \frac{\delta(E + G - E_0)}{D(E_0)},$$

(23)

with $D$ the density of states of the full system. For a subsystem with $H(x) = E$ and density of states given by

$$\Omega(E) = \Omega_0 E^\alpha,$$

we have that the inverse temperature is, following equation (20),

$$B(G) = -\frac{\partial}{\partial G} \ln \rho(G) = 2A(G - E_0),$$

(24)

which then leads to the inverse temperature distribution

S Davis
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\[
P(\beta|E_0, A) = \langle \delta(2A(G - E_0) - \beta) \rangle_{E,A}
\]
\[
= \frac{1}{2A} \delta(G - \left[ E_0 + \beta/2A \right]) \in_{E,A}
\]
\[
= \frac{1}{2A} \int dE G(E) W(G) \exp \left( -A(E + G - E_0)^2 \right) \delta(G - \left[ E_0 + \beta/2A \right])
\]
\[
= \frac{\Omega_0 W(E_0 + \frac{\beta}{2A})}{2A(z_{E_0, A})} \int_0^\infty dE E^\nu \exp \left( -A \left( E + \frac{\beta}{2A} \right)^2 \right)
\]
\[
= \frac{W(E_0 + \frac{\beta}{2A})}{\eta(E_0, A)} G(c) \left( \frac{\alpha + 1}{2} \right)^{1/2} \Gamma \left( \frac{\alpha + 1}{2} \right) \beta \left( -\frac{\alpha + 1}{2} - \frac{\beta^2}{4A} \right) - \beta T \left( \frac{\alpha + 1}{2} + 1 \right) i F_1 \left( \begin{array}{c}
\frac{1}{2} - \frac{3}{2} \frac{\beta^2}{4A}
\end{array} \right).
\]

}\] where \( _1 F_1(a, b, z) \) is the Kummer confluent hypergeometric function. In general, this distribution allows both positive and negative values of \( \beta \), and we explicitly see the dependence on the shape of the density of states of the environment \( W(G) \).

### 3. Some new properties of temperature in superstatistics

One useful consequence of equation (17) is

\[
P(x|G, S) = \int dy P(x|y, S) P(y|G, S)
\]
\[
= \int dy P(x|\beta = B(G(y))) \left[ \delta(G(y) - G) \right] \frac{W(G)}{W(G)}
\]
\[
= P(x|\beta = B(G)),
\]

where \( W(G) = \int dy \delta(G(y) - G) \) is the density of states of \( G \). Essentially what equations (17) and (26) tell us is that the states of knowledge \( x, S \) and \( G, S \) regarding features of the system \( x \) can be replaced by canonical states,

\[
\langle \cdot \rangle_{x, S} \rightarrow \langle \cdot \rangle_{\beta = B(G(y))}, \quad \text{(27)}
\]
\[
\langle \cdot \rangle_{G, S} \rightarrow \langle \cdot \rangle_{\beta = B(G)}. \quad \text{(28)}
\]

For instance, the conditional distribution of energy given \( G \) is directly \( P(E|\beta = B(G)) \). Using these rules we can further validate our identification of \( B \) with the thermodynamical inverse temperature of the system by showing that

\[
\langle \beta \rangle_{G, S} = \beta_{G, S} = B(G), \quad \text{(29)}
\]

where

\[
\beta_{G, S} = \frac{\partial}{\partial E} \ln \Omega(E)
\]

is the microcanonical inverse temperature and \( \hat{\beta} \) is the so-called dynamical temperature estimator,

\[
\hat{\beta}(x) = \nabla \cdot \left[ \frac{\omega(x)}{\omega(x) \cdot \nabla H} \right].
\]

Here \( \nabla = \partial/\partial x \) and the estimator \( \hat{\beta}(x) \) is such that Rugh’s identity [31, 32] for the microcanonical ensemble.
\[
\langle \beta \rangle_E = \beta_\Omega(E), \quad (32)
\]
holds. The proof of equation (29) uses the property (proved in appendix),
\[
\left\langle \frac{\partial}{\partial x} \ln P(x|\mathcal{I}) \right\rangle_{\mathcal{I}} = 0 \quad (33)
\]
for a random variable \(x\) and a state of knowledge \(\mathcal{I}\). For the canonical energy distribution we have
\[
\left\langle \frac{\partial}{\partial E} \ln P(E|\beta) \right\rangle_\beta = 0 = -\beta + \langle \beta_\Omega \rangle_\beta, \quad (34)
\]
which, together with equation (32), allows us to write
\[
\langle \beta \rangle_\beta = \int dE P(E|\beta) \langle \hat{\beta} \rangle_E = \int dE P(E|\beta) \beta_\Omega(E) = \langle \beta_\Omega \rangle_\beta. \quad (35)
\]
Therefore, we can write
\[
\langle \beta_\Omega \rangle_\beta = \langle \hat{\beta} \rangle_\beta = \beta, \quad (36)
\]
which after replacing \(\langle \cdot \rangle_{G,S} \iff \langle \cdot \rangle_{\beta=B(G)}\) yields equation (29). Another interesting consequence is the connection between \(B(G)\) and the conditional expectation of energy given \(G\),
\[
\langle H \rangle_{G,S} = \langle H \rangle_{\beta=B(G)} = \left[ -\frac{\partial}{\partial \beta} \ln Z(\beta) \right]_{\beta=B(G)}. \quad (37)
\]
This gives a more intuitive meaning to the superstatistical (inverse) temperature \(B(G)\) as the conjugate quantity (in the thermodynamical sense) to \(\langle H \rangle_{S,G}\).

4. Inverse temperature fluctuations

Despite the fact that the inverse temperature \(B(G)\) cannot be accessed from the system \(x\), its variance (and therefore, that of \(\beta\)) can be computed from within the system, through the microcanonical inverse temperature \(\beta_\Omega(E)\), and is given by
\[
\langle (\delta \beta)^2 \rangle_S = \langle (\delta \beta_\Omega)^2 \rangle_S + \left\langle \frac{\partial \beta_\Omega}{\partial E} \right\rangle_S. \quad (38)
\]
To prove this assertion, we first compute \(P(E,G|S)\),
\[
P(E,G|S) = P(E|G,S) \times P(G|S)
\]
\[
= P(E|\beta = B(G)) \times P(G|S)
\]
\[
= \left[ \frac{\exp(-B(G)E)}{Z(B(G))} \Omega(E) \right] \times P(G|S), \quad (39)
\]
and then, from the conjugate variables theorem [33], obtain the identity
\[
\left\langle \frac{\partial \omega}{\partial E} \right\rangle_S = -\left\langle \omega \frac{\partial}{\partial E} \ln P(E,G|S) \right\rangle_S
\]
\[
= \left\langle \omega \left( B(G) - \beta_\Omega(E) \right) \right\rangle_S, \quad (40)
\]
which is valid for any function \( \omega = \omega(E, G) \) such that \( \partial \omega / \partial E \) exists. Using the constant function \( \omega_1(E, G) = 1 \) we confirm that

\[
\langle B \rangle_S = \langle \beta_1 \rangle_S, \tag{41}
\]

while the choices \( \omega_2(E, G) = B(G) - \beta_1(E) \) and \( \omega_3(E, G) = B(G) \) produce

\[
2 \langle B \beta_1 \rangle_S = \langle B^2 \rangle_S + \langle \beta_1^2 \rangle_S + \left\langle \frac{\partial \beta_1}{\partial E} \right\rangle_S, \tag{42}
\]

and

\[
\langle B^2 \rangle_S = \langle B \beta_1 \rangle_S \tag{43}
\]

respectively. Equation (38) readily follows by combining equations (41)–(43).

5. Proof of the unique definition of temperature

Now the proof of our main result, that equation (15) is the unique solution of the condition given in equation (14), is given. First we establish that \( B(x, y) \) can only depend on \( x \) through \( H(x) \). In order to see why this is so, consider the joint distribution of \( x \) and \( \beta \) in equation (1),

\[
P(x, \beta | S) = \exp(-\beta H(x)) f(\beta), \tag{44}
\]

and impose that \( P(x, \beta | S) = P(x, B = \beta | S) \). We have

\[
\exp(-\beta E) f(\beta) = \int dy p(E, G(y)) \delta(B - \beta), \tag{45}
\]

with \( E := H(x) \). Multiplying by \( \exp(\beta E) \) and using the properties of the Dirac delta to replace \( \exp(\beta E) \) by \( \exp(\beta E) \) inside the integral on the right-hand side, we obtain

\[
f(\beta) = \int dp(E, G(y)) \exp(\beta E) \delta(B - \beta), \tag{46}
\]

hence the integral cannot depend on \( x \) at all. This is only possible if either \( B \) does not depend on \( x \), or if it depends on \( x \) through \( H(x) \). Both cases can be included by replacing the original function \( B(x, y) \) by the new function \( B(H(x), y) \) (but keeping the same name \( B \) for clarity), so that we can now write equation (45) as

\[
\exp(-\beta E) f(\beta) = \int dy p(E, G(y)) \delta(B(E, y) - \beta). \tag{47}
\]

Applying \( \partial^n / \partial E^n \) on both sides, we have

\[
\exp(-\beta E) f(\beta)(-\beta)^n = \int dy \frac{\partial^n}{\partial E^n} \left( \beta \delta(B - \beta) \right), \tag{48}
\]

and now multiplying by \( \Omega(E) \) and integrating in both \( E \) and \( \beta \), we have for the left-hand side

\[
\int d\beta \left[ \int dE \frac{\exp(-\beta E) \Omega(E)}{Z(\beta)} \right] (-\beta)^n P(\beta | S) = \int d\beta (-\beta)^n P(\beta | S) = \langle (-\beta)^n \rangle_S, \tag{49}
\]

while the right-hand side becomes
\[
\int dE \Omega(E) \int d\gamma \frac{\partial p}{\partial En} \left( p \cdot \int d\beta \delta(B - \beta) \right) = \int d\gamma d\gamma p(H, G) \left[ \frac{1}{p} \frac{\partial p}{\partial En} \right]_{n, G} = \left\langle \left[ \frac{1}{p} \frac{\partial p}{\partial En} \right] \right\rangle_S.
\]

We have obtained then a relationship between the moments of \(\beta\) and the derivatives of the ensemble function \(p(E, G)\),

\[
\langle \beta^n \rangle_S = \left\langle \left( -1 \right)^n \left[ \frac{\partial p}{\partial En} \right] \right\rangle_S = \langle B^n \rangle_S,
\]

where the last equality is imposed by equation (14). From this we extract two conclusions. First, \(B(E, y) = B(E, G(y))\), and second, \(B\) is only a functional of the joint ensemble function \(p\), and does not depend on the densities of states \(\Omega(E)\) and \(W(G)\). We can at this point already recognize \(B\) by using \(n = 1\) in equation (51),

\[
B = -\frac{\partial}{\partial E} \ln p(E, G),
\]

however, we can aim for a more exhaustive proof. Using the fact that

\[
-\frac{\partial}{\partial E} \ln \left[ \exp(-\beta E)f(\beta) \right] = \beta,
\]

and replacing equation (47), we obtain

\[
\beta = -\int \frac{dGW(G)}{dGW(G)} \frac{\partial}{\partial E} \left[ p(E, G)\delta(B(E, G) - \beta) \right].
\]

where we have introduced the density of states \(W(G)\) on both integrals. This equation can be rearranged as a functional of \(W\) which is identically zero,

\[
\int dGW(G) \left\{ -\frac{\partial}{\partial E} \left( p\delta(B - \beta) \right) - p\delta(B - \beta)\beta \right\} = 0.
\]

As neither \(p\) nor \(B\) depend on \(W\), we have

\[
-\frac{\partial}{\partial E} \left( p(E, G)\delta(B - \beta) \right) = p(E, G)\delta(B - \beta)\beta,
\]

which by integrating \(\beta\) on both sides, becomes

\[
-\frac{\partial}{\partial E} p(E, G) = p(E, G)B,
\]

immediately yielding a unique definition of \(B\),

\[
B = -\frac{\partial}{\partial E} \ln p(E, G).
\]

Replacing this value of \(B\) into equation (56), we arrive at

\[
-p(E, G) \frac{\partial}{\partial E} \delta(B - \beta) = 0,
\]

hence \(B(E, G)\) does not depend on \(E\), and
\[ \frac{\partial B}{\partial E} = - \frac{\partial^2}{\partial E^2} \ln p(E, G) = 0. \]  
(60)

Equations (58) and (60) together lead to the solution in equation (15).

We have then, by integrating equation (60) twice, that

\[ p(E, G) = p_0(G) \exp(-B(G)E), \]  
(61)

where the function \( p_0(G) \) remains to be determined. Integrating \( E \) from \( P(E, G|S) \) we have

\[ P(G|S) = \rho_G(G)W(G) = W(G) \int dE \Omega(E) p(E, G), \]  
(62)

thus replacing equation (61) gives

\[ \rho_G(G) = p_0(G) \int dE \Omega(E) \exp(-B(G)E) \]  
(63)

Replacing equation (63) into equation (61) we can finally write

\[ p(E, G) = \rho_G(G) \left[ \frac{\exp(-B(G)E)}{Z(B(G))} \right], \]  
(64)

which gives the full expression for \( P(x, y|S) \) in section 2 (equation (16)). Using this expression for the joint ensemble function \( p(E, G) \) is straightforward to verify that

\[ \frac{(-1)^n}{p} \frac{\partial^n p}{\partial E^n} = B(G)^n, \]  
(65)

as required by equation (51). The proof of equation (14) follows by introducing the series expansion of an arbitrary, analytical function \( g(\beta) = \sum_{n=0}^{\infty} C_n \beta^n \) and taking expectation in \( S \), which yields

\[ \left\langle g(\beta) \right\rangle_S = \sum_{n=0}^{\infty} C_n \left\langle \beta^n \right\rangle_S = \sum_{n=0}^{\infty} C_n \left\langle B^n \right\rangle_S = \left\langle g(B) \right\rangle_S. \]  
(66)

6. Concluding remarks

We have proved that there is a unique microscopic definition of inverse temperature (equation (15)) fully compatible with superstatistics, in the sense that the parameter \( \beta \) can be replaced anywhere by a function \( B(G) \) of the environment. The fact that \( B \) cannot depend on \( x \) seems to rule out a spatial distribution of temperatures in the superstatistical framework. However, it is still possible to maintain a frequentist interpretation of superstatistics with a fluctuating, instantaneous (inverse) temperature \( B(G) \), which is a global property of the environment. We are led to the conclusion that temperature fluctuations essentially map the fluctuations of the energy of the environment, and it is precisely this fluctuating temperature that correlates the system and its environment.

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Appendix. Derivation of equation (33)

We assume that the distribution $P(x|I)$ for $x \in [a,b]$ vanishes at its boundaries, which is the case for the canonical distribution of energy with monotonically increasing $\Omega(E)$. By calling $p(x) := P(x|I)$ we consider constant all parameters included in $I$, and we can write

$$
\left\langle \frac{\partial}{\partial x} \ln p(x) \right\rangle_I = \int_a^b dx p(x) \frac{\partial}{\partial x} \ln p(x) \\
= \int_a^b dx \frac{\partial p(x)}{\partial x} \\
= p(b) - p(a) = 0.
$$

(A.1)

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