The effect of Pressure in Higher Dimensional Quasi-Spherical Gravitational Collapse

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We study gravitational collapse in higher dimensional quasi-spherical Szekeres space-time for matter with anisotropic pressure. Both local and global visibility of central curvature singularity has been studied and it is found that with proper choice of initial data it is possible to show the validity of CCC for six and higher dimensions. Also the role of pressure in the collapsing process has been discussed.

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I. INTRODUCTION

Usually, for cosmological phenomena over galactic scale or in the smaller scale, it is reasonable to consider inhomogeneous solutions to Einstein equations. Szekeres' \cite{1} in 1975, gave a class of inhomogeneous solutions representing irrotational dust. The space-time represented by these solutions has no killing vectors and it has invariant family of spherical hypersurfaces. Hence this space-time is referred as quasi-spherical space-time. Recently, Chakraborty et al \cite{2} have extended the Szekeres solution to (n + 2) dimensional space-time and generalized it for matter containing heat flux \cite{3}.

In classical general relativity, one of the challenging issues is gravitational collapse. This problem became important after the formulation of famous singularity theorems \cite{4} and Cosmic Censorship Conjecture (CCC)\cite{5}. Also in the perspective of black hole physics and its astrophysical implications, the end state of collapse (black hole or naked singularity) is interesting. As there exists no formal method to address this problem so it is natural to study various examples of collapsing system (namely, Tolman-Bondi-Lemaître (TBL) spherically symmetric model \cite{6-15} or quasi-spherical Szekeres' model \cite{16-18}) with a view to gain some insight. In general, these studies conclude that the local or global visibility of the central curvature singularity depends on the initial data.

The above studies are mostly confined to the dust model, there are very few works on anisotropic stress \cite{19} (confined to TBL model). For the last few years, there are attempts to study collapse dynamics in TBL model with anisotropic pressure to address the question “can non-zero pressures within a collapsing matter cloud avoid a naked singularity forming as the end state of a continual gravitational collapse?” But so far, the actual role the pressures play in determining the end state of collapse is not yet clearly understood. Due to complicated nature of the space-time geometry there are no works on gravitational collapse with anisotropic stresses in quasi-spherical model except recently by Chakraborty et al \cite{20} where they have shown the role of pressure in 4 dimension.

In this work, we extend this study to (n + 2) dimensional Szekeres’ model and examine the role of the dimension on collapse dynamics. The paper is organized as follows: Higher dimensional Szekeres’ model is described in section II, while collapse dynamics including the study of geodesic is presented in section III. In section IV, there are discussion and concluding remarks. Finally at the end there are two appendix dealing with detailed calculations.

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II. HIGHER DIMENSIONAL SZEKERES’ MODEL

The metric ansatz for (n+2)dimensional Szekeres’ space-time is of the form

\[ ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^{n} dx_i^2 \]  

(1)

where the metric coefficients \( \alpha \) and \( \beta \) are functions of all space-time co-ordinates i.e.,

\[ \alpha = \alpha(t, r, x_1, ..., x_n), \quad \beta = \beta(t, r, x_1, ..., x_n). \]

Now considering both radial and transverse stresses the energy momentum tensor has the structure

\[ T^\nu_\mu = \text{diag}(\rho, -p_r, -p_T, -p_T) \]

and the compact form of the Einstein equations are

\[ \rho = \frac{F'}{\zeta n} \]

\[ p_r = -\frac{F}{\zeta n} \]

\[ p_T = p_r + \frac{\zeta F'}{n\zeta} \]

(2)

where \( F(r, t) = \frac{n}{2} R^{n-1} e^{(n+1)\nu}(\dot{R}^2 - f(r)) \) and \( \zeta = e^\beta \).

Further, the expressions for the metric functions are

\[ e^\beta = R(t, r) e^{\nu(r, x_1, ..., x_n)}, \]

\[ e^{\alpha} = R' + R \nu \]

(3)

and the evolution equation for \( R \) gives

\[ R\ddot{R} + \frac{1}{2}(n-1)\dot{R}^2 + \frac{p_r}{n} R^2 = \frac{n-1}{2} f(r), \quad (f(r) = \text{arbitrary separation function}) \]

(4)

Also the function \( \nu \) satisfies

\[ e^{-2\nu} \sum_{i=1}^{n} [(n-2)\nu_{x_i}^2 + 2\nu x_i] = n(f(r) - 1) \]

(5)

which has a solution of the form

\[ e^{-\nu} = A(r) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r) x_i + C(r) \]

(6)

with the restriction,

\[ \sum_{i=1}^{n} B_i^2 - 4AC = f(r) - 1 \]

(7)

for the arbitrary functions \( A(r), B_i(r), (i = 1, 2, ..., n) \) and \( C(r) \).
As we are considering quasi-spherical gravitational collapse so it is natural to assume the initial configuration (from which the collapse has started) to be smooth everywhere. Thus \( p_r \) should be regular initially at the center and blows up at the singularity. So a natural choice for \( p_r \) is

\[
p_r = \frac{g(r)}{R^l} \tag{8}
\]

where the arbitrary function \( g(r) \) has the form \( r^l \) near \( r = 0 \) to make initial \( p_r \) finite (non-zero) at the centre \( r = 0 \) and \( l \) is any constant. Hence, the expressions for matter density and tangential stress become

\[
\rho = \frac{H' + (n + 1)H
}{R^n(R' + R
)} \tag{9}
\]

and

\[
p_r = \frac{g(r)}{R^l} \left[ 1 - \frac{lR'}{n(R' + R
)} \right] + \frac{g'(r)}{nR^{l-1}(R' + R
)} \tag{10}
\]

where \( H(R, t) = D(r) - \frac{g(r)}{(n-l+1)} R^{n-l+1} \), \( (l \neq (n + 1)) \) and \( D(r) \), an arbitrary integration function.

Now, due to this choice of \( p_r \) (see eq (8)) the evolution eq (4) for \( R \) can be integrated once and the radial velocity of collapsing shell at a distance \( r \) from the centre is given by

\[
\dot{R}^2 = f(r) + \frac{2H(R, t)}{nR^{n-1}} \tag{11}
\]

This is termed as the equation of the collapsing process.

### III. COLLAPSE DYNAMICS

To characterize the nature of the singularity (black hole or naked singularity), the event horizon of observers at infinity plays an important role. But formation of event horizon depends greatly on the computation of null geodesics whose computation are almost impracticable for the present space-time geometry. So a closely related concept of a trapped surface (a space-like 2-surface whose normals on both sides are future pointing converging null geodesic families) will be considered. Thus, if the 2-surface \( S_{r,t} \) \((r = \text{constant}, t = \text{constant})\) is a trapped surface then it and its entire future development lie behind the event horizon provided the density falls off fast enough at infinity. Hence mathematically, if \( K^\mu \) denotes the tangent vector field to the null geodesics which is normal to \( S_{r,t} \) then we have

\[
K_\mu K^\mu = 0, \quad K_\mu ;^\mu K^\nu = 0 .
\]

Now the null geodesics will converge (or diverge) if the invariant \( K_\mu ;^\mu < 0 \) (or \( K_\mu ;^\mu > 0 \)) on the surface \( S_{r,t} = 0. \)

As a consequence, it is easy to show that the inward geodesics converges initially and throughout the collapsing process while the outward geodesics diverges initially but becomes convergent after a time \( t_{ah}(r) \) (time of formation of apparent horizon) given by

\[
\dot{R}(t_{ah}(r), r) = -\sqrt{1 + f(r)}
\]

Now using equations (8) and (11) we have

\[
g(r)R^{n+1-l}(t_{ah}(r), r) + \frac{n}{2}(n + 1 - l)R^{n-1}(t_{ah}(r), r) - (n + 1 - l)D(r) = 0 \tag{12}
\]
From Appendix II, it is to be noted that the central singularity (at \( r = 0 \)) forms at time \( t_0 \) while a trapped surface is formed at a distance \( r \) at time \( t_{ah} \) and their difference is given by the equation (37). Thus if the trapped surface is formed at a later instant than \( t_0 \) then it is possible for light signals from the singularity to reach a distant observer. Hence, \( t_{ah} > t_0 \) is the necessary condition for formation of naked singularity and on the other hand, \( t_{ah} \leq t_0 \) is the sufficient condition for black hole formation. Also it should be mentioned that this criterion for naked singularity is purely local.

Further, as the time difference equation (37) is complicated, so to make a comparative study between \( t_{ah} \) and \( t_0 \) one can choose for simplicity \( l = (n+1)/2 \) and equation (37) takes the form

\[
t_{ah}(r) - t_0 = \sqrt{2n} \left( D_0 g_1 - D_1 g_0 - g_1 \sqrt{D_0} \sqrt{D_0 - \frac{2}{n+1} g_0} \right) / (n+1)g_0 \sqrt{D_0} \sqrt{D_0 - \frac{2}{n+1} g_0} \left( \sqrt{D_0} + \sqrt{D_0 - \frac{2}{n+1} g_0} \right) + O(r^2)
\]  

(13)

The following table shows the possibility of naked singularity or a black hole under different conditions:

| Choice of the parameters | Naked Singularity | Black hole |
|--------------------------|-------------------|------------|
| (i) \( g_1 > 0, D_1 < 0 \) | Always possible   | Not possible |
| (ii) \( g_1 < 0, D_1 > 0 \) | Not possible      | Always possible |
| (iii) \( g_1 > 0, D_1 > 0 \) | \( g_1 > \frac{n+1}{D_1} \left( 1 + \sqrt{1 - \frac{2}{n+1} \frac{g_0}{D_0}} \right) \) | \( g_1 > \frac{n+1}{D_1} \left( 1 + \sqrt{1 - \frac{2}{n+1} \frac{g_0}{D_0}} \right) \) |
| (IV) \( g_1 < 0, D_1 < 0 \) | \( \left| \frac{g_1}{D_1} \right| < \frac{n+1}{2} \left( 1 + \sqrt{1 - \frac{2}{n+1} \frac{g_0}{D_0}} \right) \) | \( \left| \frac{g_1}{D_1} \right| > \frac{n+1}{2} \left( 1 + \sqrt{1 - \frac{2}{n+1} \frac{g_0}{D_0}} \right) \) |

From the table, to make the initial density gradient to be negative at the centre (i.e., \( \rho_1 < 0 \)) one must have \( (D_1 - \frac{2g_1}{n+1}) < 0 \) (for \( \nu_1 > -1 \)). For the first case (i.e., \( g_1 > 0, D_1 < 0 \)) \( \rho_1 \) is negative definite and there is always naked singularity as in the dust model. Similarly, \( \rho_1 \) is positive definite in the second case which leads to black hole solution same as dust model. For the third and fourth cases (when \( g_1 \) and \( D_1 \) have same sign) either NS or BH is possible depending on the restrictions given in the table I (see also figs 1 - 6). However, in the last two cases, for \( \rho_1 > 0 \) only black hole solution is possible but for \( \rho_1 < 0 \), both NS and BH are possible.

Further, if we assume that \( D_1 = 0 = g_1 \) then the time difference in eq (37) becomes

\[
t_{ah}(r) - t_0 = \sqrt{\frac{n}{2}} \left[ -\frac{2n-1}{n+1} \frac{n+1}{2} F_1 \left( \frac{1}{2}, b, b + 1, \frac{2D_0}{n} \right) \right]^{\frac{n+1}{n+1}} D_0^{\frac{1}{n+1}} \left( r^{\frac{3n+3-2l}{n+1}} + \ldots \right)
\]  

(14)

We see that if \( n > 3 \) (with \( l < n + 1 \)) then the first term on the right side will be dominating compare to other terms and hence we always have \( t_{ah} < t_0 \). Thus in this case black hole is the only final state collapse for six and higher dimensional space-times. This distinctive result is similar to dust collapse [13, 16] and we conclude that CCC is valid in this case for six and higher dimension with anisotropic pressure.

### A. Study of Geodesics

In this section, the nature of the singularity (NS or BH) is examined by studying the geodesics from the singularity. In particular, it will be investigated whether there exist one or more radial outgoing null geodesics which terminate in the past at the central singularity. For simplicity of calculation only marginally bound case
Figs. 1 - 6 show variation of $t_{ah} - t_0$ of eq.(13) for the variation of $k_0 (= g_0 / D_0)$ and $k_1 (= g_1 / D_1)$. Figs. 1, 3 and 5 correspond to $D_1 > 0$ for $n = 4, 12$ and 25 respectively while Figs. 2, 4 and 6 correspond to $D_1 < 0$ for $n = 4, 12$ and 25 respectively.

$f(r) = 0$ with $l = (n + 1)/2$ will be considered (as in the previous section). Now choosing the initial time $t_i = 0$, the explicit solution for $R(t, r)$ can be written as

$$t(r) = \frac{\sqrt{2n}}{g(r)} \left[ \sqrt{D(r) - \frac{2}{n+1} g(r) R^{(n+1)/2}} - \sqrt{D(r) - \frac{2}{n+1} g(r) r^{(n+1)/2}} \right]$$

(15)

Thus the expression for the singularity time for the shell of radius $R$ is given by $(R(t_s(r), r) = 0)$
and consequently the time for central singularity is

\[ t_0 = \frac{\sqrt{2n}}{g^0} \left( \sqrt{D_0} - \frac{2}{n+1} g^0 r^{(n+1)/2} \right) \]  

(17)

Here the polynomial form of \( D(r) \) and \( g(r) \) are taken in the form

\[ D(r) = D_0 r^{n+1} + D_k r^{k+n+1} \]
\[ g(r) = g_0 r^{(n+1)/2} + g_j r^{j+(n+1)/2} \]  

(18)

where \( D_0, g_0 \) are constants and \( D_k (\leq 0) \) and \( g_j (\leq 0) \) are the first non-vanishing term beyond \( D_0 \) and \( g_0 \) respectively. Now using these expression for \( D(r) \) and \( g(r) \) the time of collapse of a typical shell of radius \( R \) (i.e., \( t_s(r) \)) becomes (see eq (16))

\[ t_s(r) = t_0 + \sqrt{\frac{n D_k}{2 g_0}} \left( \frac{1}{\sqrt{D_0}} - \frac{1}{\sqrt{D_0 - \frac{2}{n+1} g_0}} \right) r^k + \frac{\sqrt{2n} g_j}{g_0} \left( \frac{1}{(n+1)\sqrt{D_0 - \frac{2}{n+1} g_0}} - \frac{\sqrt{D_0} - \sqrt{D_0 - \frac{2}{n+1} g_0}}{g_0} \right) r^j + ... \]  

(19)

Now the equation of the outgoing radial null geodesic (ORNG) which passes through the central singularity, can be chosen to be (near \( r = 0 \))

\[ t_{ORNG} = t_0 + a r^\xi \]  

(20)

where \( a > 0 \) and \( \xi > 0 \) are constants.

In the polynomial form for \( D(r) \) and \( g(r) \) in the equation (18) one can choose two possibilities:

(i) \( k < j \),  
(ii) \( k > j \)

**Case I : \( k < j \)**

Here near \( r = 0 \), the expression for \( t_s(r) \) can be written as (see eq (19))

\[ t_s(r) = t_0 + \sqrt{\frac{n D_k}{2 g_0}} \left( \frac{1}{\sqrt{D_0}} - \frac{1}{\sqrt{D_0 - \frac{2}{n+1} g_0}} \right) r^k, \quad (D_k < 0) \]  

(21)

In order that null geodesic passes through the shell of radius \( R \) before the trapped surface is formed there, comparing (20) and (21) one gets \( (t_{ORNG} < t_s(r)) \)

(a) \( \xi > k \) or  
(b) \( \xi = k \) and \( a < -\frac{D_k}{g_0} \sqrt{\frac{n}{2}} \left( \frac{1}{\sqrt{D_0 - \frac{2}{n+1} g_0}} - \frac{1}{\sqrt{D_0}} \right) \)  

(22)

When \( \xi > k \) then near \( r = 0 \) the solution for \( R \) simplifies to

\[ R = r \left[ 1 - \frac{n+1}{4n} g_0 t^2 - \frac{n+1}{2n^2} \left( \sqrt{D_0 - \frac{2}{n+1} g_0} + \frac{D_k r^k}{2\sqrt{D_0 - \frac{2}{n+1} g_0}} \right) \right]^{2/n+1} \]  

(23)
Further for the given metric an ORNG should satisfy

$$\frac{dt}{dr} = R' + R \nu'$$

(24)

Now using (20) and (23) in (24) one gets (up to leading order in $r$)

$$a\xi r^{\xi-1} = \left(1 + \nu_{-1} + \frac{2k}{n+1}\right) \left[-\frac{(n+1)D_k t_0}{2\sqrt{2\pi} / \sqrt{D_0 - \frac{2}{n+1} g_0}}\right]^{2/(n+1)} r^{\frac{2k}{n+1}}, \quad (\nu_{-1} \neq 0)$$

(25)

This gives

$$\xi = 1 + \frac{2k}{n+1} > 0 \quad \text{and} \quad a = \frac{1}{\xi} \left(1 + \nu_{-1} + \frac{2k}{n+1}\right) \left[-\frac{(n+1)D_k t_0}{2\sqrt{2\pi} / \sqrt{D_0 - \frac{2}{n+1} g_0}}\right]^{2/(n+1)}$$

(26)

As $\xi > k$, so from the above relations (26)

$$k < \frac{n+1}{n+1} \quad \text{and} \quad \xi < \frac{n+1}{n+1},$$

i.e. one could have $(n > 2)$

$$k = 1, \quad \xi = \frac{n+3}{n+1}$$

(27)

On the other hand for $\xi = k$, as before $k = \frac{n+1}{n+1}$ and

$$a = \frac{n-1}{n+1} \left[-\frac{n+1}{4} \left(\frac{2ag_0 t_0}{n} + 2a\sqrt{\frac{2}{n}} \sqrt{D_0 - \frac{2}{n+1} g_0} + \frac{D_k t_0 \sqrt{\frac{2}{n}}}{\sqrt{D_0 - \frac{2}{n+1} g_0}}\right)\right]^{(1-n)/(n+1)} 	imes \left[\frac{n+1}{4} \left(1 + \nu_{-1}\right) \left(\frac{2ag_0 t_0}{n} + 2a\sqrt{\frac{2}{n}} \sqrt{D_0 - \frac{2}{n+1} g_0} + \frac{1 + \nu_{-1} + \frac{2k}{n+1}}{\sqrt{D_0 - \frac{2}{n+1} g_0}} \sqrt{\frac{2}{n}} D_k t_0\right)\right]$$

(28)

Thus for $\xi > k$, $k$ has only one value (namely $k = 1$) and $n$ can take any value $(>2)$ while for $\xi = k$ the only possible values of $n$ are 2 and 3 only. Therefore, geodesic equations are possible in any dimension for $\xi > k$ but it is only possible up to five dimension for $\xi = k$. In other words, for $\xi = k$, naked singularity is possible only up to five dimension which supports the results in the previous section. Lastly, it should be mentioned that the other choice namely $k > j$ is similar to the above and hence not presented here.

IV. DISCUSSIONS AND CONCLUDING REMARKS

A detailed analysis of the central curvature singularity as the final state of collapse in the $(n+2)$-dimensional quasi-spherical Szekeres’ model has been done for matter with anisotropic pressure (i.e., both radial and tangential pressures are non-zero and distinct). The local visibility of the central singularity has been discussed by comparing the time of formation of trapped surface and the time of formation of central shell focusing singularity while global visibility is examined by considering only the radial null geodesics (for simplicity). Most of the results are very similar to that for four dimension in ref. [13]. It is to be noted that though we have considered quasi-spherical Szekeres model but still these are valid for TBL model. In fact, in Szekeres’ solution if we assume the function $\nu$ to be independent of $r$ (i.e., $\nu' = 0$) then Szekeres’ model can be converted
to TBL model by the following coordinate transformation:

\[
\begin{align*}
    x_1 &= \sin\theta_n \sin\theta_{n-1} \ldots \sin\theta_2 \cot\frac{1}{2}\theta_1 \\
    x_2 &= \cos\theta_n \sin\theta_{n-1} \ldots \sin\theta_2 \cot\frac{1}{2}\theta_1 \\
    x_3 &= \cos\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_2 \cot\frac{1}{2}\theta_1 \\
    \vdots & \vdots & \vdots & \vdots \\
    x_{n-1} &= \cos\theta_3 \sin\theta_2 \cot\frac{1}{2}\theta_1 \\
    x_n &= \cos\theta_2 \cot\frac{1}{2}\theta_1 \\
\end{align*}
\]

Further, the radial pressure is assumed to be a function of ‘r’ and ‘t’ only with the form \( p_r = \frac{g(r)}{R^l} \). Throughout the paper \( l \) is chosen to be less than \( n + 1 \) (i.e., \( l < n + 1 \)) [It is to be noted that for \( l = n + 1 \) due to appearance of a logarithmic term the calculations become very much complicated while for \( l > n + 1 \) the results are not of much interest]. It has been shown that for \( g_1 = 0 = D_1 \) i.e., for \( h_1 = 0 \), naked singularity is only possible up to five dimension—a result identical in dust collapse. The above choice (i.e., \( h_1 = 0 \)) gives \( \rho_0 = (n + 1)h_0 \) for \( \nu_{-1} \geq (-1) \) and \( \rho_1 = 0 \) for \( \nu_{-1} > -1 \). Hence for \( \nu_{-1} > -1 \), if the initial density gradient falls off at the centre (\( r = 0 \)) then one can say that six dimension plays as critical dimension for naked singularity, a distinct result in higher dimension. Further with the choice \( l = (n + 1)/2 \), a detailed comparative study has been done between the time of formation of trapped surface and that of central singularity and Table I shows all possibilities for the parameters involved in the expression. Also in the figures 1 - 6 we have shown graphically the time difference \( t_{ah} - t_0 \) for 6, 14 and 27 dimensions for \( D_1 > 0 \) and \( D_1 < 0 \) respectively.

As in dust collapse, in this case we have definitely a black hole (or naked singularity) if the initial density gradient at the centre is positive definite (or negative definite). But in the indefiniteness in the sign of \( \rho_1 \) one may note that if the initial density and radial pressure has identical behaviour (i.e., increase or decrease simultaneously) then even with initial negative density gradient (at the centre) it is possible to have black hole as the end state of collapse, while if the initial density and pressure have opposite nature (i.e., one increase when other decreases and vice versa) then the behaviour is identical to dust collapse. Therefore one may conclude that pressure tries to resist the formation of naked singularity.

**APPENDIX I**

**Initial hypersurface and the physical parameters:**

Suppose the collapsing process starts on the initial hypersurface (\( t = t_i \)) and we have \( R = r \) there. Then the expressions for energy density, radial pressure and the tangential pressure, at the beginning of the collapse are

\[
\begin{align*}
    \rho_i(r, x_1, \ldots, x_n) &= \rho(t_i, r, x_1, \ldots, x_n) = \frac{h'(r)+(n+1)h(r)\nu'}{r^{-(1+\nu')}} \\
    p_{r_i}(r, x_1, \ldots, x_n) &= p_r(t = t_i) = \frac{g(r)}{r^{1+\nu'}} \left[ 1 - \frac{l}{n(1+r\nu')} \right] + \frac{g'(r)}{nr^{1+\nu'}/(1+r\nu')}
\end{align*}
\]

where \( h(r) = H(r, t_i) = D(r) - \frac{g(r)}{n-\nu_{-1}} r^{n-\nu_{-1}+1} \).

For smooth initial data \( h(r) \) and \( g(r) \) to be \( C^\infty \) functions and hence one can choose the following series expansions
\[ D(r) = \sum_{j=0}^{\infty} D_j r^{n+1+j} \]
\[ g(r) = \sum_{j=0}^{\infty} g_j r^{l+j} \]
\[ \rho_i(r, x_1, ..., x_n) = \sum_{j=0}^{\infty} \rho_j r^j \quad (30) \]
\[ \nu'(r, x_1, ..., x_n) = \sum_{j=-1}^{\infty} \nu_j r^j, \quad (\nu_{-1} \geq -1). \]
\[ p_{\nu_i}(r, x_1, ..., x_n) = \sum_{j=0}^{\infty} p_j r^j \]

It is to be noted that in the above series expansions the coefficients \( D_j \)'s and \( g_j \)'s \((j = 0, 1, ...)\) are purely constants while \( \rho_j \)'s, \( \nu_j \)'s and \( p_j \)'s are functions of \( x_i \)'s \((i = 1, ..., n)\). Also these coefficients are not independent but are related among themselves through the relations in eq (29) as follows:

For \( \nu_{-1} > -1 \):

\[ p_0 = g_0, \quad p_1 = g_1 \left(1 + \frac{1}{n(1+\nu_{-1})}\right), \quad p_2 = g_2 \left(1 + \frac{1}{(1+\nu_{-1})^2}\right) - \frac{g_1 g_2}{n(1+\nu_{-1})}, \quad \ldots \quad \ldots \]
\[ \rho_0 = (n + 1)h_0, \quad \rho_1 = \frac{(n+2)+(n+1)\nu_{-1}}{1+\nu_{-1}} h_1, \quad \rho_2 = \frac{(n+3)+(n+1)\nu_{-1}}{1+\nu_{-1}} h_2 - \frac{g_1 h_1}{(1+\nu_{-1})}, \quad \ldots \quad \ldots \quad (31) \]

For \( \nu_{-1} = -1 \):

\[ p_0 = g_0 + \frac{g_1}{n
u_0}, \quad p_1 = g_1 \left(1 - \frac{\nu_1}{n\nu_0}\right) + \frac{2g_2}{n\nu_0}, \quad p_2 = g_2 \left(1 - \frac{2\nu_1}{n\nu_0}\right) + \frac{(\nu_1^2 - \nu_0\nu_2)}{n\nu_0} g_1 + \frac{3g_3}{n\nu_0}, \quad \ldots \quad \ldots \]
\[ \rho_0 = (n + 1)h_0 + \frac{h_1}{\nu_0}, \quad \rho_1 = \frac{2h_2}{\nu_0} + h_1 \left(n + 1 - \frac{\nu_1}{\nu_0}\right), \quad \rho_2 = \frac{3h_3}{\nu_0} + h_2 \left(n + 1 - \frac{2\nu_1}{\nu_0}\right) + h_1 \frac{(\nu_1^2 - \nu_0\nu_2)}{\nu_0}, \quad \ldots \quad \ldots \quad (32) \]
with \( h_i = D_i - \frac{g_i}{n-1+i}, \quad i = 0, 1, 2, \ldots \)

**APPENDIX II**

Solution of the evolution equation (11) with \( f(r) = 0 \):

For the initial choice \( R = r \) at \( t = t_i \), the explicit solution is

\[ t - t_i = -\frac{\sqrt{2n} r^{n+1}}{(n+1)\sqrt{D(r)}} F_1 \left[\frac{1}{2}, b, b+1, \frac{g(r)r^{n+1-l}}{D(r)(n+1-l)}\right] - \frac{\sqrt{2n} R^{n+1}}{(n+1)\sqrt{D(r)}} F_1 \left[\frac{1}{2}, b, b+1, \frac{g(r)R^{n+1-l}}{D(r)(n+1-l)}\right] \quad (33) \]
where \( b = \frac{n+1}{2-2l/n} \) and \( F_1 \) is the usual hypergeometric function and \( l \neq n + 1 \).

If \( t = t_s(r) \) stands for time of collapse of the \( r \)-th shell i.e., \( R(t_s(r), r) = 0 \) then we have

\[ t_s(r) - t_i = \frac{\sqrt{2n} r^{(n+1)/2}}{(n+1)\sqrt{D(r)}} F_1 \left[\frac{1}{2}, b, b+1, \frac{g(r)r^{n+1-l}}{D(r)(n+1-l)}\right] \quad (34) \]

Note that \( t = t_s(r) \) is a monotonic increasing function of \( r \) (i.e., \( t'_s(r) \geq 0 \)) for the shell focusing singularity. Using equation (12)

\[ t_{ah}(r) - t_i = \frac{\sqrt{2n} r^{(n+1)/2}}{(n+1)\sqrt{D(r)}} F_1 \left[\frac{1}{2}, b, b+1, \frac{g(r)r^{n+1-l}}{D(r)(n+1-l)}\right] - \frac{\sqrt{2n} F_{(n+1)/2}(t_{ah}, r)}{(n+1)\sqrt{D(r)}} F_1 \left[\frac{1}{2}, b, b+1, \frac{g(r)R^{n+1-l}(t_{ah}, r)}{D(r)(n+1-l)}\right] \quad (35) \]
A comparison of equations (34) and (35) shows that the shell focusing singularity that appears at $r > 0$ is in the future of the apparent horizon. However, the time of occurrence of central shell focusing singularity (which is of main interest here) is given by

$$t_0 = \lim_{r \to 0} t_s(r)$$

$$= t_i + \frac{\sqrt{2n}}{(n+1) \sqrt{D_0}} 2F_1\left[\frac{1}{2}, b, b + 1, z\right], \quad (z = \frac{2r}{D_0(n+1-1)})$$

where the series form of $g(r)$ and $D(r)$ (from eq. (30)) have been used in evaluating the limit. Thus for the restriction $l < n + 1$, the explicit form of the difference between $t_{ah}(r)$ and $t_0$ is

$$t_{ah}(r) - t_0 = \sqrt{\frac{n}{2}} \left\{ \left( -\frac{D_0^{-3/2}D_1}{n+1} 2F_1\left[\frac{1}{2}, b, b + 1, z\right] + \frac{(D_0g_1 - D_1g_0)}{(n+1-1)(3+3n-2l)} D_0^{-5/2} 2F_1\left[\frac{3}{2}, b, b + 2, z\right] \right) \right. $$

$$+ O(r^2) - \frac{2^{1+n} n^{1+n} D_0^{1+n}}{n+1} 2F_1\left[\frac{1}{2}, b, b + 1, z \left(\frac{2D_0}{n}\right)^{\frac{n+1}{2}} \right] \left( \frac{3n+3}{2} \right) + \ldots \ldots \right\} \right.$$ (37)

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**References:**

[1] P. Szekeres, *Commun. Math. Phys.* 41 55 (1975).
[2] S. Chakraborty and U. Debnath, *Int. J. Mod. Phys. D* 13 1085 (2004).
[3] U. Debnath, S. Nath and S. Chakraborty, *Gen. Rel. Grav.* 37 215 (2005).
[4] S.W. Hawking, and G.F.R. Ellis, "The Large Scale Structure of Space-Time", (Cambridge University Press, Cambridge, England, 1973).
[5] R. Penrose, *Riv. Nuovo Cim.* 1 252 (1969); R. Penrose, In General Relativity, an Einstein Centenary Volume, S.W. Hawking and W. Israel (Eds.), (Camb. Univ. Press, Cambridge, 1979).
[6] P. S. Joshi and I. H. Dwivedi, *Commun. Math. Phys.* 166 117 (1994).
[7] P. S. Joshi and I. H. Dwivedi, *Class. Quantum Grav.* 16 41 (1999).
[8] K. Lake, *Phys. Rev. Lett.* 68 3129 (1992).
[9] A. Ori and T. Piran, *Phys. Rev. Lett.* 59 2137 (1987).
[10] T. Harada, *Phys. Rev. D* 58 104015 (1998).
[11] P.S. Joshi, *Global Aspects in Gravitation and Cosmology*, (Oxford Univ. Press, Oxford, 1993).
[12] U. Debnath and S. Chakraborty, *Gen. Rel. Grav.* 36 1243 (2004).
[13] A. Banerjee, U. Debnath and S. Chakraborty, *Int. J. Mod. Phys. D* 12 1255 (2003).
[14] R. Goswami and P.S. Joshi, *gr-qc/0212097* (2002).
[15] S. Schoen and S. T. Yau, *Commun. Math. Phys.* 90 575 (1983).
[16] U. Debnath, S. Chakraborty and J. D. Barrow, *Gen. Rel. Grav.* 36 231 (2004).
[17] U. Debnath and S. Chakraborty, *JCAP* 05 001 (2004).
[18] R. Goswami and P. S. Joshi, *Class. Quantum Grav.* 21 3645 (2004); *Class. Quantum Grav.* 19 5229 (2002).
[19] J. R. Gair, *Class. Quantum Grav.* 18 4897 (2001); R. Goswami and P. S. Joshi, *Class. Quantum Grav.* 19 5229 (2002); P. S. Joshi, R. Goswami and N. Dadhich, *Phys. Rev. D* 70 087502 (2004).
[20] S. Chakraborty, S. Chakraborty and U. Debnath, *Int. J. Mod. Phys. D (In press)* (2005); *gr-qc/0506048.*