Upper topology and its relation with the projective modules

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Abstract
In this article, first we obtain a number of new results on projective modules and on the upper topology of an ordinal number. Then it is shown that the rank map of a locally finite type projective module is continuous with respect to the upper topology (by contrast, it is well known that this map is not necessarily continuous with respect to the discrete topology).

Keywords Kaplansky theorem · Projective module · Locally free module · Locally finite type module · Upper topology · Exterior power of a module

Mathematics Subject Classification 14A05 · 13C10 · 19A13 · 03E10 · 13A15

1 Introduction
This article grew out of an attempt to understand when the rank map of an infinitely generated projective module is continuous. It is well known that the rank map of a finitely generated projective $R$-module is continuous whenever $\text{Spec}(R)$ is equipped with the Zariski topology and the set of natural numbers with the “discrete” topology. In Theorem 3.6, this fact is also reproved by a new method, see e.g. [14, Corollary 2.2.2] for another proof of it. It is important to notice that in this result the “finitely generated” assumption of the module is a crucial point in the continuity of its rank map. If we drop this hypothesis, then the rank map is no longer continuous. For example, there are locally finite type projective modules whose rank maps are not necessarily continuous with respect to the discrete topology, see e.g. [14, Ex. 2.15]. Therefore it is natural to ask, under what conditions, then the rank map of a locally finite type projective module will be continuous. One of the main aims of the present article is to realize this goal. In order to do this we need a suitable topology to replace instead of the “discrete” topology of the natural numbers $\omega = \{0, 1, 2, \ldots\}$. Finding such a topology requires familiarity with the structure of the natural numbers. The set $\omega$ is an ordinal number and after some effort then we realized that the desired topology is the upper topology.

With this in mind, in this article first we study the upper topology over an ordinal number and several new results are obtained, see Theorems 2.1, 2.2 and 2.4. Then, in Theorem 4.1,
it is shown that if the “discrete” topology of the natural numbers $\omega$ is replaced by the upper topology then the rank map remains continuous even if the projective module is infinitely generated. To prove its continuity the main ingredients which are used, in addition to the Kaplansky theorem, are some basic properties of the exterior powers of a module.

In this article, we are also interested in investigating the continuity of the rank map of some modules. It should be noted that the projectivity of modules has a very close connection with the continuity of their rank maps. For instance, a finitely generated flat module is projective if and only if its rank map is continuous with respect to the discrete topology. An analogue of this result is also proved for the upper topology, see Theorem 4.2.

2 Preliminaries and the upper topology

We collect in this section some basic background for the reader’s convenience. This section also contains our results on the upper topology.

In this article, all rings are commutative. Let $(R, m)$ be a local ring with the residue field $\kappa = R/m$ and let $M$ be a finitely generated $R$-module. Then a finite subset $\{x_1, \ldots, x_n\} \subseteq M$ is a minimal generating set of the $R$-module $M$ if and only if $\{x_i + mM : 1 \leq i \leq n\}$ is a $\kappa$-basis of $M/mM$. Hence, any two minimal generating sets of a finitely generated module over a local ring have the same numbers of elements. Note that this does not hold in general. As a specific example, $\{1\}$ and $\{2, 3\}$ are two minimal generating sets of $\mathbb{Z}$ as a module over itself with different numbers of elements.

A celebrated theorem due to Kaplansky states that every projective module over a local ring is free, see [5, Theorem 2] or [4, Tag 00NZ]. It is also well known that every finitely generated flat module over a local ring is free, see [7, Theorem 7.10] or [4, Tag 00NZ]. Bass’ result [2], is another major result in this realm which (in roughly speaking) asserts that big projective modules are always free, where “big” means essentially infinitely generated in a certain technical sense. If $M$ is a free $R$-module, then $M_p$ is a free $R_p$-module for all $p \in \text{Spec}(R)$. The same assertion holds for projective (resp. flat) modules. In spite of these facts, it is very important to notice that in the theory of modules, only the flatness is a local property (that is, a given $R$-module $M$ is $R$-flat if and only if each $M_p$ is $R_p$-flat); the freeness and projectiveness are not local properties. (If freeness was a local property, then every projective module would be a free module. Similarly, if projectiveness was a local property, then every finitely generated flat module would be a projective module).

Recall that for a given ring $R$, the collection of subsets $D(f) \cap V(I)$ with $f \in R$ and $I$ runs through the set of finitely generated ideals of $R$ forms a basis for the opens of the patch topology over $\text{Spec}(R)$. Moreover, the collection of subsets $V(I)$ where $I$ runs through the set of finitely generated ideals of $R$ forms a basis for the opens of the flat topology over $\text{Spec}(R)$. The patch topology is clearly finer than the Zariski and flat topologies. The flat topology is the dual of the Zariski topology. For more details on the patch and flat topologies please consider e.g. [9].

An $R$-module $M$ is called locally finite type if $M_p$ is a finitely generated $R_p$-module for all $p \in \text{Spec}(R)$. If $p$ is a prime ideal of $R$ then we define $\text{rank}_{R_p}(M_p)$ as the number of elements of a minimal generating set of the $R_p$-module $M_p$. It is well-defined. This leads us to a map from $\text{Spec}(R)$ into the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$ given by $p \rightsquigarrow \text{rank}_{R_p}(M_p)$. It is called the rank map of $M$. Note that $\text{rank}_{R_p}(M_p)$ is equal to the dimension of the $\kappa(p)$-vector space $M \otimes_R \kappa(p)$ where $\kappa(p)$ is the residue field of $R$ at $p$. The rank map is said to be Zariski (resp. flat, patch) continuous if it is continuous whenever $\text{Spec}(R)$ is equipped with...
the Zariski (resp, flat, patch) topology and \( \omega \) with the discrete topology. It is easy to see that the rank map is locally constant if and only if it is Zariski continuous. The same statement is true for flat and patch topologies. For further information on the rank map we refer the interested reader to algebraic K-theory books, see e.g. [1, Chap III, §7].

The theory of exterior powers of a module is quite well known and can be found easily in the literature, (for a rapid review of this topic we refer the reader to [10, §2]).

Recall that by the basic set theory, each member of a set is itself a set. A set \( T \) is called a transitive set if each element of \( T \) is a proper subset of \( T \). A relation \( \prec \) on a set \( S \) is called linear (or, totally ordered) if it is transitive (i.e., if \( x \prec y \) and \( y \prec z \) then \( x \prec z \)) and satisfies the trichotomy law (i.e., for each pair \((x, y)\) of elements of \( S \) then exactly one of the following conditions hold: \( x \prec y \) or \( x = y \) or \( y \prec x \)). We also say that \( x \leq y \) if either \( x = y \) or \( x \prec y \).

A linear relation \( \prec \) on a set \( S \) is said to be well-ordered if every non-empty subset of \( S \) has a least element with respect to \( \prec \). By an ordinal (or, ordinal number) we mean a transitive set such that it is well-ordered with respect to the membership relation \( \in \). Let \( \alpha \) and \( \beta \) be two ordinals. Then \( \beta \in \alpha \) if and only if \( \beta \) is a proper subset of \( \alpha \). If \( \alpha \) and \( \beta \) are ordinals then we say that \( \beta < \alpha \) if \( \beta \in \alpha \) or equivalently \( \beta \subset \alpha \), i.e., \( \beta \) is a proper subset of \( \alpha \). Hence, each ordinal is the set of the ordinals smaller than itself.

Let \( (P, \prec) \) be a poset. Then the collection of \( d(x) := P \setminus \{ y \in P : y \leq x \} \) with \( x \in P \) forms a sub-basis for the opens of the upper topology on \( P \). Note that \( \{ y \in P : x < y \} \subseteq d(x) \), if the relation \( \prec \) is linear (totally ordered) then the equality holds. The dual of the upper topology is called the lower topology. Thus the collection of \( d'(x) = P \setminus \{ y \in P : x \leq y \} \) with \( x \in P \) is a sub-basis for the opens of the lower topology on \( P \). Finally, the collection of \( d(x) \cap d'(y) \) with \( x, y \in P \) is a sub-basis for the opens of a topology on \( P \). We call it the patch topology.

Let \( R \) be a commutative ring and consider \( \text{Spec}(R) \) as a poset with respect to the strict inclusion. Then the lower topology over the poset \( \text{Spec}(R) \) is coarser than the Zariski topology. Similarly, the upper topology on the poset \( \text{Spec}(R) \) is coarser than the flat topology.

If \( \mathbb{R} \) denotes the set of real numbers, then the patch topology over the usual poset \( (\mathbb{R}, \prec) \) coincides with the Euclidean topology.

Let \( (P, \prec) \) be a poset and consider the upper topology over it. Then for each \( x \in P \) we have \( [x] = \{ y \in P : y \leq x \} \). If \( S \subseteq P \) is a subset then the upper and subspace topologies over \( S \) are the same.

Let \( \alpha \) be an ordinal number. By the upper topology over \( \alpha \) we mean the upper topology over the poset \( (\alpha, \in) \).

**Theorem 2.1** Let \( \alpha \) be an ordinal and consider the upper topology over it. Then \( \beta \) is a closed subset of \( \alpha \) if and only if \( \beta \) is an ordinal number with \( \beta \leq \alpha \).

**Proof** If \( F \) is a closed subset of \( \alpha \) then to prove that \( F \) is an ordinal it will be enough to show that it is a transitive set. If \( x \in F \) then \( [x] \subseteq F \). We have \( y \in [x] \) if and only if either \( y \in x \) or \( y = x \) (note that \( x \) is an ordinal and hence it is a set). Hence, \( F \) is a transitive set. Conversely, if \( \beta \) is an ordinal with \( \beta \leq \alpha \) then \( \beta \subseteq \alpha \). If \( x \in \beta \) then \( [x] \subseteq \beta \) because \( \beta \) is a transitive set. Therefore \( \beta \) is closed. \( \square \)

Recall that if \( \alpha \) and \( \beta \) are two ordinal numbers then \( \alpha \) is called the successor of \( \beta \) if \( \alpha = \beta^+ = \beta \cup \{ \beta \} \). An ordinal number is said to be a limit ordinal if it is not the successor of an ordinal number.

**Theorem 2.2** Let \( \alpha \) be an ordinal and consider the upper topology over it. Then the following assertions hold.
(i) If $\beta \in \alpha$ then $[\beta] = \beta^+$.
(ii) $\alpha$ has a generic point if and only if it is not a limit ordinal. Moreover the generic point, if it exists, is unique.
(iii) If $\alpha \neq 0$ then $\alpha$ is an irreducible space.
(iv) The closed subsets of $\alpha$ are stable under the arbitrary unions.
(v) Every open subset of $\alpha$ is quasi-compact.
(vi) $\alpha$ is a Noetherian space.

**Proof** The assertions (i) and (ii) follow from the fact that an ordinal number is a transitive set.
(iii): It follows from Theorem 2.1 and the fact that any two ordinal numbers are comparable.
(iv): It implies from Theorem 2.1 and the fact that if $\{\beta_i\}$ is a family of ordinal numbers then $\bigcup_i \beta_i$ is also an ordinal number, see e.g. [3, Corollary 7N (d)].
(v): Let $U$ be an open subset of $\alpha$ and let $\{U_i\}_{i \in I}$ be an open covering of it. We may assume that $I \neq \emptyset$. For each $i$ there exists an ordinal number $\beta_i$ with $\beta_i \leq \alpha$ such that $U_i^c = \beta_i$. Let $\beta_k$ be the least element of the set $\{\beta_i : i \in I\}$ because it is well known that any non-empty set of ordinals has the least element. It follows that $U = U_k$.
(vi): It follows from (v). 

**Remark 2.3** By [3, Corollary 7N, (b) and (c)] and [3, Theorem 4I], every natural number is an ordinal number where $0 = \emptyset$, $1 = 0^+ = \{0\}$ and $n + 1 = n^+ = \{0, 1, \ldots, n\}$ for all $n \geq 1$. Moreover, by [3, Corollary 7N, (a)] and [3, Theorem 4G], the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$ is also an ordinal number. In fact, $\omega$ is the first non-zero limit ordinal.

The upper topology over an ordinal number $\alpha$ is discrete if and only if either $\alpha = 0$ or $\alpha = 1$. The upper topology over $2 = \{0, 1\}$ is just the Sierpiński topology.

**Theorem 2.4** The upper topology over an ordinal number is spectral if and only if it is a natural number.

**Proof** First assume that the upper topology over an ordinal number $\alpha$ is spectral. If $\alpha \geq \omega$ then by Theorem 2.1, $\omega$ is a closed subset of $\alpha$ and so by Theorem 2.2 (iii), it is irreducible. But $\omega$ does not have any generic point. Because, suppose there is some $m \in \omega$ such that $\omega = \{m\} = m^+$. By [3, Theorem 4I], $m^+ = m + 1$ is a natural number, a contradiction. Hence $\alpha < \omega$, i.e., $\alpha$ is a natural number. Conversely, let $n$ be a natural number. To prove the assertion, by Theorem 2.2, it suffices to show that every closed and irreducible subset of $n$ has a generic point. We may assume that $n > 0$. If $m$ is a closed and irreducible subset of $n$ then $m$ is a natural number with $0 < m \leq n$. By [3, Theorem 4C], there exists a natural number $k$ such that $m = k^+$. Therefore $m = \lceil k \rceil$. 

**3 Locally free modules**

By a locally free (or, stalkwise free) module we mean an $R$-module $M$ such that $M_p$ is a free $R_p$-module for all $p \in \text{Spec}(R)$. It is easy to see that an $R$-module $M$ is locally free if and only if $M_p$ is a free $R_p$-module for all $p \in \text{Max}(R)$. Every locally free $R$-module is a flat $R$-module, since the flatness is a local property. But the converse is not necessarily true. As an example, the field of rational numbers $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not locally free. It can be shown that a finitely generated module over a ring $R$ is locally free if and only if it is a flat $R$-module. Note that projective modules and locally finite type flat modules are typical modules.
examples of locally free modules. There are locally free modules which are not projective. For example, let \( R = \prod_{i \geq 1} R_i \) where each \( R_i \) is a nonzero ring and let \( I = \bigoplus_{i \geq 1} R_i \) which is an ideal of \( R \). If \( f \in I \) then \( f = fg \) for some \( g \in I \). Thus \( R/I \) is a flat \( R \)-module and so it is locally free. But \( R/I \) is not a projective \( R \)-module, because the annihilator of every finitely generated projective module is generated by an idempotent element (see e.g. [11, Corollary 3.2]), but \( I \) is not a finitely generated ideal.

If \( M \) is a locally free \( R \)-module then we may define \( \text{rank}_{R_p}(M_p) \) as the cardinality of a \( R_p \)-basis of \( M_p \). It is well-defined, since over a commutative ring any two bases of a free module have the same cardinality.

**Lemma 3.1** Let \( M \) be an \( R \)-module. Then \( \text{Supp}(M) \) is stable under the specialization. If moreover, \( M \) is locally free then \( \text{Supp}(M) \) is stable under the generalization.

**Proof** Let \( p \subseteq q \) be prime ideals of \( R \). If \( p \in \text{Supp}(M) \) then there is some \( x \in M \) such that \( \text{Ann}_R(x) \subseteq p \). Thus \( x/1 \) is a nonzero element of \( M_q \). Hence \( \text{Supp}(M) \) is stable under the specialization. Now assume that \( q \in \text{Supp}(M) \). If \( M \) is locally free, then \( \text{rank}_{R_q}(M_p) = \text{rank}_{R_p}(M_q) \) and so \( M_p \neq 0 \).

**Lemma 3.2** Let \( M \) be a projective \( R \)-module and let \( (I_k) \) be a family of ideals of \( R \). Then \( \bigcap_k (I_k M) = \left( \bigcap_k I_k \right) M \).

**Proof** There is a free \( R \)-module \( F \) such that \( M \) is a direct summand of it. Thus there exists an \( R \)-submodule \( N \) of \( F \) such that \( F = M + N \) and \( M \cap N = 0 \). Let \( \{x_i\} \) be a \( R \)-basis of \( F \). Consider the isomorphism \( \psi : F \to \bigoplus_i R \) given by \( x \mapsto (r_i) \) where \( x = \sum_i r_i x_i \). If \( I \) is an ideal of \( R \) then \( \psi(I F) = \bigoplus_i I \). Moreover \( \bigcap_k \left( \bigoplus_i I_k \right) = \bigoplus_i \left( \bigcap_k I_k \right) \). It follows that \( \psi\left( \bigcap_k \left( I_k F \right) \right) = \psi\left( \bigcap_k \left( I_k M \right) \right) \). Thus \( \bigcap_k \left( I_k F \right) = \bigcap_k \left( I_k M \right) \). We also have \( IF = IM + IN \) and \( \bigcap_k \left( I_k M + I_k N \right) = \bigcap_k \left( I_k M \right) + \bigcap_k \left( I_k N \right) \). Therefore \( \bigcap_k \left( I_k M \right) + \bigcap_k \left( I_k N \right) \) is a free \( R \)-module.

**Theorem 3.3** If \( M \) is a locally free \( R \)-module, then \( \text{Supp}(M) = \{ p \in \text{Spec}(R) : pM \neq M \} \).

**Proof** First assume that \( M_p = 0 \). From the exact sequence:

\[
0 \longrightarrow R/p \longrightarrow \kappa(p) \longrightarrow 0
\]

we obtain the following exact sequence:

\[
0 \longrightarrow R/p \otimes_R M \longrightarrow \kappa(p) \otimes_R M.
\]

But \( \kappa(p) \otimes_R M \simeq R/p \otimes_R M_p = 0 \). It follows that \( M = pM \). Conversely, if \( M = pM \) then \( M_p \otimes_R \kappa(p) \simeq M_p \otimes_R R/p \simeq M/pM \otimes_R R_p = 0 \). It follows that \( M_p = 0 \).

The following result is well known, see [6, Lemma 6.2] or [13, Lemma 1.1]. We prove it by a new approach.

**Corollary 3.4** The support of a projective module is Zariski (and flat) open.

**Proof** Let \( M \) be a projective \( R \)-module, let \( X = \text{Spec}(R) \setminus \text{Supp}(M) \) and let \( I = \bigcap_{p \in X} p \). By Theorem 3.3 and Lemma 3.2, \( IM = M \). Clearly \( X \subseteq V(I) \). Conversely, assume that \( I \subseteq p \). Then \( M = IM \subseteq pM \subseteq M \). Thus \( pM = M \) and so by Theorem 3.3, \( p \in X \). Therefore \( X = V(I) \). By Lemma 3.1 and [9, Theorem 3.11], we conclude that \( X \) is also flat closed.
Lemma 3.5 Let $M$ be a locally free $R$-module. Then for each natural number $n$, \( \text{Supp} \left( \Lambda^n(M) \right) = \{ p \in \text{Spec}(R) : \text{rank}_{R_p}(M_p) \geq n \} \) where $\Lambda^n(M)$ is the $n$-th exterior power of $M$.

Proof We have $\Lambda^n(M) \otimes_R R_p \simeq \Lambda^n_{R_p}(M_p)$. Therefore $p \in \text{Supp} \left( \Lambda^n(M) \right)$ if and only if $\text{rank}_{R_p}(M_p) \geq n$. \( \square \)

The following result is also well known, see e.g. [14, Corollary 2.2.2]. We provide a new proof for it.

Theorem 3.6 The rank map of a finitely generated projective module is Zariski (and flat) continuous.

Proof Let $M$ be a finitely generated projective module over a ring $R$. For each natural number $n$, then by Lemma 3.5, $\psi^{-1}(\{n\}) = \text{Supp} \cap \left( \text{Spec}(R) \setminus \text{Supp} N' \right)$ where $\psi$ is the rank map of $M$, $N = \Lambda^n(M)$ and $N' = \Lambda^{n+1}(M)$. But $N$ and $N'$ are finitely generated projective $R$-modules. Thus there are idempotents $e, e' \in R$ such that $\text{Ann}_R(N) = Re$ and $\text{Ann}_R(N') = Re'$, because it is well known that the annihilator of every finitely generated projective module is generated by an idempotent element (see e.g. [11, Corollary 3.2]). It follows that $\psi^{-1}(\{n\}) = V(e) \cap V(1 - e') = D(e'(1 - e))$. Therefore $\psi^{-1}(\{n\})$ is both Zariski and flat open. \( \square \)

Proposition 3.7 Let $M$ be a locally finite type flat $R$-module. Then the following assertions are equivalent.

(i) The rank map of $M$ is patch continuous.

(ii) The rank map of $M$ is Zariski continuous.

(iii) The rank map of $M$ is flat continuous.

Proof For each natural number $n$, then $\psi^{-1}(\{n\})$ is stable under the generalization and specialization where $\psi$ is the rank map of $M$. Now if $\psi$ is patch continuous then, by [9, Theorem 3.11], it is both Zariski and flat continuous. The reverse is easy since the patch topology is finer than the Zariski (and flat) topology. \( \square \)

4 Some applications of the upper topology to projective modules

As stated in the Introduction, the rank map of a locally finite type projective module is not necessarily continuous with respect to the discrete topology. In the following result we will observe that its continuity will be recovered if the discrete topology is replaced by the upper topology.

Theorem 4.1 Let $M$ be a locally finite type flat $R$-module and consider the upper topology on the set of natural numbers. If $M$ is $R$-projective then the rank map of $M$ is Zariski (and flat) continuous.

Proof Let $n$ be a natural number. By [3, Theorem 4C], $n$ is either the empty set or there exists a natural number $m$ such that $n = m + 1$. Thus either $n = \emptyset$ or $n = \{0, 1, 2, \ldots, m\}$. By applying this observation and Lemma 3.5, we obtain that $\psi^{-1}(\{n\}) = \psi^{-1}(\{0, 1, 2, \ldots, m\}) = \text{Spec}(R) \setminus \text{Supp} \left( \Lambda^n(M) \right)$ where $\psi$ is the rank map of $M$. Now using Corollary 3.4, then the assertion is concluded. \( \square \)
**Theorem 4.2** Let $M$ be a finitely generated flat module over a ring $R$. Then $M$ is $R$-projective if and only if the rank map of $M$ is patch continuous with respect to the upper topology.

**Proof** The implication “$\Rightarrow$” implies from Theorem 3.6 or Theorem 4.1 together with Proposition 3.7. Conversely, let $n$ be a natural number. By Lemma 3.5, $\psi^{-1}(\{n\}) = \text{Supp } N \cap \left( \text{Spec}(R) \setminus \text{Supp } N' \right)$ and $\text{Supp } N = \text{Spec}(R) \setminus \psi^{-1}(n)$ where $\psi$ is the rank map of $M$, $N = \Lambda^n(M)$ and $N' = \Lambda^{n+1}(M)$. Thus, $\psi$ is patch continuous with respect to the discrete topology. Therefore, by [4, Tags 00NZ, 00NX] and Proposition 3.7, $M$ is $R$-projective.

In the proof of Theorem 4.2, do not confuse $\psi^{-1}(\{n\})$ with $\psi^{-1}(n)$.

An $R$-module $M$ is said to be locally of countable rank if for each prime ideal $p$ of $R$ then $M_p$ as $R_p$-module is countably generated (possibly infinite). Note that if $M$ is a projective $R$-module then $M = \bigoplus_{i \in I} M_i$ is a direct sum of countably generated projective $R$-submodules, see [4, Tag 058Y]. If the index set $I$ is countable then $M$ is countably generated and so it is locally of countable rank.

**Corollary 4.3** Let $R$ be a ring such that $\text{Spec}(R)$ is Noetherian with respect to the flat topology. Let $M$ be a projective $R$-module which is locally of countable rank. Consider the upper topology over $\omega^+$ and the Zariski topology over $\text{Spec}(R)$. Then the rank map of $M$ is continuous.

**Proof** We have $\psi^{-1}(\omega) = \bigcup_{n \in \omega} \psi^{-1}(n)$ where $\psi$ is the rank map of $M$. By the proof of Theorem 4.1, $\psi^{-1}(n) = \text{Spec}(R) \setminus \text{Supp } \Lambda^n(M)$. Thus, by Corollary 3.4, it is Zariski closed. Therefore by [8, Theorem 4.2] or by [9, Theorem 4.2], $\psi^{-1}(\omega)$ is Zariski closed.

**Remark 4.4** In order to see more equivalents and characterizations of the Noetherianess of the prime spectrum with respect to the flat topology we refer the interested reader to [8, Theorem 4.2] or [9, Theorem 4.2].

As a dual of Corollary 4.3, we have the following result.

**Corollary 4.5** Let $R$ be a ring such that $\text{Spec}(R)$ is Noetherian with respect to the Zariski topology. Let $M$ be a projective $R$-module which is locally of countable rank. Consider the upper topology over $\omega^+$ and the flat topology over $\text{Spec}(R)$. Then the rank map of $M$ is continuous.

**Proof** We have $\psi^{-1}(\omega) = \bigcup_{n \in \omega} \psi^{-1}(n)$ where $\psi$ is the rank map of $M$. By Corollary 3.4, $\psi^{-1}(n) = \text{Spec}(R) \setminus \text{Supp } \Lambda^n(M)$ is a flat closed. Thus by [12, Theorem 5.1], $\psi^{-1}(\omega)$ is flat closed.

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