Bianchi type-I transit cosmological models with time dependent gravitational and cosmological constants

Anirudh Pradhan\textsuperscript{1}, Bijan Saha\textsuperscript{2}*, Victor Rikhvitsky\textsuperscript{2} †

\textsuperscript{1}Department of Mathematics, Hindu Post-graduate College, Zamania-232 331, Ghazipur, India
E-mail: pradhan@iucaa.ernet.in

\textsuperscript{2}Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia
*E-mail: bijan@jinr.ru
†E-mail: rqvtsk@jinr.ru

Abstract

The present study deals with the exact solutions of the Einstein’s field equations with variable gravitational and cosmological “constants” for a spatially homogeneous and anisotropic Bianchi type-I space-time. To study the transit behavior of Universe, we consider a law of variation of scale factor $a(t) = (t^k e^t)^{1/n}$ which yields a time dependent deceleration parameter (DP) $q = -1 + nk/(k+1)$, comprising a class of models that depicts a transition of the universe from the early decelerated phase to the recent accelerating phase. We find that the time dependent DP is reasonable for the present day Universe and give an appropriate description of the evolution of the universe. For $n = 0.27k$, we obtain $q_0 = -0.73$ which is similar to observed value of DP at present epoch. It is also observed that for $n \geq 2$ and $k = 1$, we obtain a class of transit models of the universe from early decelerating to present accelerating phase.

For $k = 0$, the universe has non-singular origin. The gravitational constant $G(t)$ is permitted to follow a power-law expansion which is suitable for the present evolution of the universe. The cosmological constant $\Lambda(t)$ is obtained as a decreasing function of time and approaching a small positive value at present epoch which is corroborated by consequences from recent supernovae Ia observations. We also observed that our solutions approach to the $\Lambda$-CDM model. The physical significance of the cosmological models have also been discussed.

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1 Introduction

The Einstein field equation has two parameters, the cosmological constant $\Lambda$ and the gravitational constant $G$. In 1998, the discovery that the accelerated expansion of the Universe is driven by the dark energy (DE) from the type Ia supernovae (SN Ia) observations (Riess et al. 1998; Perlmutter et al. 1999) greatly astonished the world. The Wilkinson Microwave Anisotropy Probe (Peiris et al. 2003; Spergel et al. 2007), combined with more accurate SN Ia data (Riess et al. 2004) indicates that the Universe is almost spatially flat and the dark energy accounts for about 70% of the total content of the Universe. However, we know little about the nature of dark energy except for its negative pressure. Observations strongly favour a small and positive value of the effective cosmological constant at the present epoch. Among many possible alternatives, the simplest and theoretically appealing possibility of dark energy is the energy density stored on the vacuum state of all existing fields in the universe i. e., $\rho_v = \frac{\Lambda}{3H^2}$. The variable cosmological constant (Overduin and Cooperstock 1998; Sahni and Starobinsky 2000; Peebles and Ratra 2003; Padmanabhan 2003, 2008; Freese 2005) is one of the phenomenological ways to explain the dark energy problem, because it is a straightforward modification of the cosmological constant $\Lambda$ which enable itself to be compatible with observations. The problem in this approach is to determine the right dependence of $\Lambda$ upon scale factor $R$ or $t$. Motivated by
dimensional grounds with quantum cosmology, the variation of cosmological term as $\Lambda \propto R^{-2}$ is considered by Chen and Wu (1990). However, several ansatz have been proposed in which the $\Lambda$-term decays with time (Gasperini 1988; Berman 1990, 1991, 2001; Ratra and Peebles 1988; Abdussattar and Vishwakarma 1996). Several authors have recently studied the time dependent cosmological constant in different contexts (Singh et al. 2007; Pradhan and Kumhar 2009; Pradhan and Jotania 2010, 2011; Pradhan 2011; Amirhashchi et al. 2011a,b; Pradhan et al. 2011, 2013a,b,c).

The other constant of Einstein’s field equations i.e. the gravitational constant ($G$) plays the role of coupling constant between geometry and matter. Recent observations also incertitude the stability of fundamental constants and “Equivalence Principle” of general relativity. Dirac (1937a,b) was first to introduce the time variation of the gravitational constant $G$ in his large number hypothesis and since then it has been used frequently in numerous modifications of general theory of relativity. $G$ has many interesting consequences in astrophysics. It is shown that $G$-varying cosmology is consistent with whatsoever cosmological observations available at present (Canuto and Narlikar 1980). Several authors have recently investigated and discussed the time dependent $\Lambda$ and $G$ in different contexts (Saha 2001a; Singh et al. 2008; Singh and Kale 2009; Amirhashchi et al. 2012; Yadav et al. 2012; Chawla et al. 2012). Recently, Yadav and Sharma (2013) and Yadav (2012) have discussed about transit universe in Bianchi type-V space-time with variable $G$ and $\Lambda$.

Anisotropic Bianchi type-I universe, which is more general than FRW universe, plays a significant role to understand the phenomenon like formation of galaxies in early universe. Theoretical arguments as well as the recent observations of cosmic microwave background radiation (CMBR) support the existence of anisotropic phase that approaches an isotropic one. Motivated by the above discussions, in this paper, we propose to study homogeneous and anisotropic Bianchi type-I transit cosmological models with time dependent gravitational and cosmological “constants”. The paper is organized as follows. In Sect. 2, the metric and basic equations have been presented. Section 3 deals with solutions of field equations. In Sect. 4, the results and discussions are described. The Statefinder diagnostic has been discussed in Sect. 5. Finally, conclusions are summarized in the last Sect. 6.

# 2 The Metric and Basic Equations

We consider the space-time metric of the spatially homogeneous and anisotropic Bianchi-I of the form

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2. \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ are the metric functions of cosmic time $t$.

Einstein field equations with time-dependent $G$ and $\Lambda$ are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G T_{ij} + \Lambda g_{ij}, \tag{2}$$

where the symbols have their usual meaning.

For a perfect fluid, the stress-energy-momentum tensor $T_{ij}$ is given by

$$T_{ij} = (\rho + p)u_iu_j - p g_{ij}, \tag{3}$$

where $\rho$ is the matter density, $p$ is the thermodynamics pressure and $u^i$ is the fluid four-velocity vector of the fluid satisfying the condition

$$u^iu_i = 1. \tag{4}$$

In the field equations (2), $\Lambda$ accounts for vacuum energy with its energy density $\rho_v$ and pressure $p_v$ satisfying the equation of state

$$p_v = -\rho_v = -\frac{\Lambda}{8\pi G}. \tag{5}$$

The critical density and the density parameters for matter and cosmological constant are, respectively, defined as

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{6}$$
\[ \Omega_M = \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2}, \quad (7) \]
\[ \Omega_\Lambda = \frac{\rho_v}{\rho_c} = \frac{\Lambda}{3H^2}. \quad (8) \]

We observe that the density parameters \( \Omega_M \) and \( \Omega_\Lambda \) are singular when \( H = 0 \).

In a comoving system of coordinates, the field Eqs. (2) for the metric (1) with (3) read as

\[ \ddot{A} \frac{A}{A} + \ddot{B} \frac{B}{B} + \ddot{AB} \frac{AB}{AB} = -8\pi G \rho + \Lambda, \quad (9) \]
\[ \ddot{A} \frac{A}{A} + \ddot{C} \frac{C}{C} + \ddot{AC} \frac{AC}{AC} = -8\pi G \rho + \Lambda, \quad (10) \]
\[ \ddot{B} \frac{B}{B} + \ddot{C} \frac{C}{C} + \ddot{BC} \frac{BC}{BC} = -8\pi G \rho + \Lambda, \quad (11) \]
\[ \ddot{AB} \frac{AB}{AB} + \ddot{BC} \frac{BC}{BC} + \ddot{CA} \frac{CA}{CA} = 8\pi G \rho + \Lambda. \quad (12) \]

The covariant divergence of Eq. (2) yields

\[ \dot{\rho} + 3(\rho + p)H + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \quad (13) \]

Spatial volume for the model given by Eq. (1) reads as

\[ V = ABC \quad (14) \]

We define average scale factor \( a \) of anisotropic model as

\[ a = (ABC)^\frac{1}{3} = V^\frac{1}{3}. \quad (15) \]

So that generalized mean Hubble parameter \( H \) is given by

\[ H = \frac{1}{3}(H_x + H_y + H_z), \quad (16) \]

where \( H_x = \frac{\dot{A}}{A}, H_y = \frac{\dot{B}}{B}, H_z = \frac{\dot{C}}{C} \) are the directional Hubble parameters in direction of \( x, y \) and \( z \) respectively and a dot denotes differentiation with respect to cosmic time \( t \).

From Eqs. (15) and (16), we obtain an important relation

\[ H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \quad (17) \]

Expressions for the dynamical scalars such as the expansion scalar (\( \theta \)), anisotropy parameter (\( A_m \)) and the shear scalar (\( \sigma \)) are defined as usual:

\[ \theta = u^i_{;i} = \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \quad (18) \]
\[ A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2, \quad (19) \]
\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left[ \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 \right] - \frac{\theta^2}{6}. \quad (20) \]

We define deceleration parameter (DP) \( q \) as

\[ q = -\frac{a\ddot{a}}{a^2} = -\left( \frac{\dot{H} + H^2}{H^2} \right). \quad (21) \]
3 Solution of field equations

The field Eqs. (9)−(12) are a system of four equations with seven unknown parameters $A, B, C, G, p, \rho$ and $\Lambda$. Hence, three additional constraints relating these parameters are required to obtain explicit solution of the system.

So firstly, we assume a power-law form of the gravitational constant ($G$) with scale factor $a$ as proposed by Singh and Kumar (2009) and Chawla et al. (2012)

$$G \propto a^m,$$

where $m$ is a constant. For sake of mathematical simplicity, Eq. (22) may be written as

$$G = G_0 a^m,$$

where $G_0$ is a positive constant.

Secondly, we assume equation of state for perfect fluid as

$$p = \gamma \rho,$$

where $\gamma \ (0 \leq \gamma \leq 1)$ is constant.

Following the technique (Pradhan 2013; Saha 2001a,b; Saha et al. 2012; Pradhan and Amirhashchi 2011), we get three equations from the field Eqs. (9)−(11)

$$\frac{A}{B} = d_1 \exp \left( k_1 \int a^{-3} dt \right),$$

$$\frac{B}{C} = d_2 \exp \left( k_2 \int a^{-3} dt \right),$$

$$\frac{C}{A} = d_3 \exp \left( k_3 \int a^{-3} dt \right),$$

where $d_1, d_2, d_3$ and $k_1, k_2, k_3$ are constants of integration. Finally, using $a = (ABC)^{\frac{1}{3}}$, we write the metric functions from Eqs. (25)−(27) in explicit form as

$$A(t) = l_1 a \exp \left( m_1 \int a^{-3} dt \right),$$

$$B(t) = l_2 a \exp \left( m_2 \int a^{-3} dt \right),$$

$$C(t) = l_3 a \exp \left( m_3 \int a^{-3} dt \right),$$

where constants $m_1, m_2, m_3$ and $l_1, l_2, l_3$ satisfy the following two relations:

$$m_1 + m_2 + m_3 = 0, \quad l_1 l_2 l_3 = 1.$$ 

in the particular case

$$l_1 = \sqrt[d_1]{d_2}, \quad l_2 = \sqrt[d_1^{-1}]{d_3}, \quad l_3 = \sqrt[(d_2 d_3)^{-1}]{},$$

and

$$m_1 = \frac{k_1 + k_2}{3}, \quad m_2 = \frac{k_3 - k_1}{3}, \quad m_3 = \frac{-(k_2 + k_3)}{3}.$$ 

Now, the metric functions can be determined as functions of cosmic time $t$ if the average scale factor is known. Hence, following (Yadav and Sharma 2013; Yadav 2012; Pradhan et al. 2013d) we consider the ansatz for the scale factor, where increase in term of time evolution is

$$a = (t^k e^l)^{\frac{1}{3}}.$$
This ansatz generalizes the one proposed in (Pradhan 2013; Saha et al. 2012; Pradhan and Amirhashchi 2011). This choice of scale factor yields a time-dependent deceleration parameter (see Eq. (47)) such that before dark energy era (cosmological constant), the corresponding solution gives inflation and radiation/matter dominance era with subsequent transition from deceleration to acceleration. Thus, our choice of scale factor is physically acceptable. The motivation for such consideration is due to observational fact that the universe has accelerated expansion at present as observed in recent observations of Type Ia supernova (Riess et al. 1998; Perlmutter et al. 1999; Tonry et al. 2003; Clocchiatti et al. 2006) and CMB anisotropies (De Bernardis 1998; Bennett et al. 2003; Hanany et al. 2000) and decelerated expansion in the past. It is well established the transition redshift from decelerating expansion to accelerated expansion is about 0.5. Now for a Universe which was decelerating in past and accelerating at the present time, the DP must show signature flipping (Padmanabhan and Roychowdhury 2003; Amendola 2003; Riess et al. 2001). This theme motivates to choose such scale factor (34) that yields a time dependent DP (47).

Using Eq. (34) in Eqs. (28)−(30), we obtain

\[ A(t) = l_1(t^k e^t)^{\frac{2}{n}} \exp [m_1 F(t)], \]  

\[ B(t) = l_2(t^k e^t)^{\frac{2}{n}} \exp [m_2 F(t)], \]  

\[ C(t) = l_3(t^k e^t)^{\frac{2}{n}} \exp [m_3 F(t)], \]

where

\[ F(t) = \int (t^k e^t)^{-\frac{2}{n}} dt = \sum_{i=1}^{\infty} \frac{(-3)^{i-1} t^i - \frac{4}{3}}{n^i - 2(n - 3k)(i - 1)!}. \]

From Eqs. (35)−(37), we obtain

\[ \frac{\dot{A}}{A} = \frac{1}{n} \left( \frac{k}{t} + 1 \right) + m_1 (t^k e^t)^{-\frac{2}{n}}, \]  

\[ \frac{\dot{B}}{B} = \frac{1}{n} \left( \frac{k}{t} + 1 \right) + m_2 (t^k e^t)^{-\frac{2}{n}}, \]  

\[ \frac{\dot{C}}{C} = \frac{1}{n} \left( \frac{k}{t} + 1 \right) + m_3 (t^k e^t)^{-\frac{2}{n}}, \]

and

\[ \frac{\ddot{A}}{A} = \frac{1}{n^2} \left( \frac{k}{t} + 1 \right)^2 - \frac{k}{n t^2} + m_1^2 (t^k e^t)^{-\frac{2}{n}} - \frac{m_1}{n} (t^k e^t)^{-\frac{2}{n}} \left( \frac{k}{t} + 1 \right), \]  

\[ \frac{\ddot{B}}{B} = \frac{1}{n^2} \left( \frac{k}{t} + 1 \right)^2 - \frac{k}{n t^2} + m_2^2 (t^k e^t)^{-\frac{2}{n}} - \frac{m_2}{n} (t^k e^t)^{-\frac{2}{n}} \left( \frac{k}{t} + 1 \right), \]  

\[ \frac{\ddot{C}}{C} = \frac{1}{n^2} \left( \frac{k}{t} + 1 \right)^2 - \frac{k}{n t^2} + m_3^2 (t^k e^t)^{-\frac{2}{n}} - \frac{m_3}{n} (t^k e^t)^{-\frac{2}{n}} \left( \frac{k}{t} + 1 \right). \]

Hence the geometry of the universe (41) is reduced to

\[ ds^2 = -dt^2 + (t^k e^t)^{\frac{2}{n}} \left[ l_1^2 \exp \{2m_1 F(t)\} dx^2 + l_2^2 \exp \{2m_2 F(t)\} dy^2 + l_3^2 \exp \{2m_3 F(t)\} dz^2 \right]. \]
4 Results and discussion

Expressions for physical parameters such as spatial volume \( V \), mean Hubble’s parameter \( H \), expansion scalar \( \theta \), shear scalar \( \sigma \) and anisotropy parameter \( A_m \) for model (41) are given by

\[
V = (t^k e^t)^{\frac{n}{k}},
\]

\[
H = \frac{1}{n} \left( \frac{k}{t} + 1 \right),
\]

\[
\theta = \frac{3}{n} \left( \frac{k}{t} + 1 \right),
\]

\[
A_m = \frac{\beta_1}{3n^2} \left( \frac{k}{t} + 1 \right)^{-2} (t^k e^t)^{\frac{n}{k}},
\]

\[
\sigma^2 = \frac{\beta_1}{2} (t^k e^t)^{\frac{n}{k}}.
\]

where

\[
\beta_1 = m_1^2 + m_2^2 + m_3^2.
\]

From Eq. (21), the deceleration parameter is computed as

\[
q = -1 + \frac{nk}{(k + t)^2}.
\]

From Eq. (47), we observe that \( q > 0 \) for \( t < \sqrt{nk} - k \) and \( q < 0 \) for \( t > \sqrt{nk} - k \). It is observed that for \( n \geq 3 \& k = 1 \), our model is evolving from decelerating phase to accelerating phase. Also, recent observations of SNe Ia, expose that the present universe is accelerating and the value of DP lies on some place in the range \(-1 < q < 0\). It follows that in our derived model, one can choose the value of DP consistent with the observation. Figure 1 depicts the variation of deceleration parameter \( q \) versus cosmic time which gives the behaviour of \( q \) as in accelerating phase at present epoch for different values of \( (n, k) \) which is consistent with recent observations of Type Ia supernovae (Riess et al. 1998, 2004; Perlmutter et al. 1999; Tonry et al. 2003; Clocchiatti et al. 2006).

Figure 1: Plots of deceleration parameter \( q \) vs time \( t \)

Figure 2: Plots of anisotropic parameter \( A_m \) vs time \( t \) for \( m_1 = 0.25, m_2 = 0.75, m_3 = -1 \)

From above Eq. (47), present value of deceleration parameter can be estimated as

\[
q_0 = -1 + \frac{k}{nH_0^2 t_0^2}.
\]
where $H_0$ is present value of Hubble’s parameter and $t_0$ is the age of universe at present epoch. Recent observations show that the deceleration parameter of the universe is in the range $-1 \leq q \leq 0$ i.e $q_0 \approx -0.77$. For $n = 0.27k$, we obtain $q_0 = -0.73$ which is similar to the observed value of DP at present epoch (Cunha et al. 2009). Therefore, we restrict the values of $n$ and $k$ such that the condition $n = 0.27k$ is satisfied for graphical representations of the physical parameters. As a representative case, we have considered four values of $(n, k)$ as $(0.25, 0.9259259260)$, $(0.50, 0.1851851852)$, $(0.75, 2.77777778)$ and $(3, 11.11111111)$ respectively for graphic presentation of Figures 1–8.

From Eqs. (42) and (44) we observe that the spatial volume is zero at $t = 0$ and the expansion scalar is infinite, which show that the universe starts evolving with zero volume at $t = 0$ which is big bang scenario. From Eqs. (35)–(37), we observe that the spatial scale factors are zero at the initial epoch $t = 0$ and hence the model has a point type singularity (MacCallum 1971). We observe that proper volume increases with time.

From Eq. (45), we observe that at late time when $t \to \infty$, $A_m \to 0$. Thus, our model has transition from initial anisotropy to isotropy at present epoch which is in good harmony with current observations. Figure 2 depicts the variation of anisotropic parameter ($A_m$) versus cosmic time $t$. From the figure, we observe that $A_m$ decreases with time and tends to zero as $t \to \infty$. Thus, the observed isotropy of the universe can be achieved in our model at present epoch.

It is important to note here that $\lim_{t \to 0} (\rho'')$ spread out to be constant. Therefore the model of the universe goes up homogeneity and matter is dynamically negligible near the origin. This is in good agreement with the result already given by Collins (1977).

Using Eq. (34) into (23), the gravitational constant is obtained as

$$G = G_0 (t^k e^t)^n. \tag{49}$$

From Eq. (49), we observe that $G$ is an increasing function of time i.e., $G \to 0$ as $t \to 0$ whereas for $t \to \infty$, $G \to \infty$. This nature of variation of $G$ with cosmic time is shown in Figure 3 for three values of $n = 0.25, 0.50$ and 0.75. When the universe is required to have expanded from a finite minimum volume, the critical density assumption and conservation of energy-momentum tensor dictate that $G$ increases in a perpetually expanding universe (Abdel-Rahaman 1990). In most variable $G$ cosmologies (Weinberg 1972; Norman 1986), $G$ is a decreasing function of time. But the possibility of an increasing $G$ has also been suggested by several authors (Pradhan et al. 2007; Singh et al. 2008; Singh and Kale 2009; Singh et al. 1998). An appealing feature of this modification in $G$ is that it leaves the form of Einstein’s equations formally unchanged by allowing the variation of $G$ to be accompanied by a change in $\Lambda$ and enables us to solve many cosmological problems such
as the cosmological constant problem, inflationary scenario etc. (Sistero 2000).

Using Eqs. (24), (39) and (40) and solving the field Eqs. (9)−(12), we get the expressions for energy density, pressure and cosmological constant for universe (41) as

\[
\rho = \frac{1}{8\pi G_0 (1 + \gamma)} \left[ \frac{2k}{n t^2} (t^k e^t)^{-\frac{n}{\gamma}} - \beta_1 (t^k e^t)^{\frac{(m+6)}{n}} \right], \\
p = \frac{\gamma}{8\pi G_0 (1 + \gamma)} \left[ \frac{2k}{n t^2} (t^k e^t)^{-\frac{n}{\gamma}} - \beta_1 (t^k e^t)^{\frac{(m+6)}{n}} \right], \\
\Lambda = \frac{3}{n^2} \left( \frac{k}{t} + 1 \right)^2 + \frac{1}{(1 + \gamma)} \left[ \beta_2 (t^k e^t)^{-\frac{n}{\gamma}} - \frac{2k}{n t^2} \right],
\]

where \( \beta_2 = m_1^2 + m_2^2 + m_1 m_2 + \gamma (m_1 m_2 + m_2 m_3 + m_3 m_1) \).

We find that the above solutions satisfy Eq. (13) identically and hence represent exact solution of Einstein’s field equations (9)−(12).

From above relations (50)−(52), we can obtain the expressions of energy density, pressure and cosmological constant for four types of models:

- When \( \gamma = 0 \), we obtain empty model.
- When \( \gamma = \frac{1}{3} \), we obtain radiation dominated model.
- When \( \gamma = -1 \), we have the degenerate vacuum or false vacuum or \( \rho \) vacuum model (Cho 1992).
- When \( \gamma = 1 \), the fluid distribution corresponds with the equation of state \( \rho = p \) which is known as Zeldovich fluid or stiff fluid model (Zeldovich 1962; Barrow 1978).

From Eq. (50), it is observed that the energy density \( \rho \) is a decreasing function of time and \( \rho > 0 \) under condition \( t^{3k-1} e^{\frac{nt^k}{m}} > \frac{n\beta_1}{2k} \). The energy density has been graphed versus time in Figure 4 for \( \gamma = 0, \frac{1}{3} \) and 1. It is evident that the energy density remains positive in all three types of models under appropriate condition. However, it decreases more sharply with the cosmic time in Zeldovich universe, compare to radiation dominated and empty fluid universes.
The behavior of the universe in this model will be determined by the cosmological term $\Lambda$, this term has the same effect as a uniform mass density $\rho_{\text{eff}} = -\frac{\Lambda}{4\pi G}$, which is constant in time. A positive value of $\Lambda$ corresponds to a negative effective mass density (repulsion). Hence, we expect that in the universe with a positive value of $\Lambda$ the expansion will tend to accelerate whereas in the universe with negative value of $\Lambda$ the expansion will slow down, stop and reverse. Figure 5 is the plot of cosmological term $\Lambda$ versus time for $\gamma = 0, \frac{1}{3}$ and 1. In all three types of models, we observe that $\Lambda$ is decreasing function of time $t$ and it approaches a small positive value at late time (i.e. at present epoch). However, it decreases more sharply with the cosmic time in empty universe, compare to radiation dominated and stiff fluid universes. Recent cosmological observations (Riess et al. 1998, 2004; Perlmutter et al. 1999; Tonry et al. 2003; Clocchiatti et al. 2006) suggest the existence of a positive cosmological constant $\Lambda$ with the magnitude $\Lambda(\text{Gh}/c^3) \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological $\Lambda$-term. Thus, the nature of $\Lambda$ in our derived models are supported by recent observations.

The vacuum energy density ($\rho_v$), critical density ($\rho_c$) and the density parameters ($\Omega_M, \Omega_\Lambda$) for model (11) read as

$$\rho_v = \frac{1}{8\pi G_0} \left[ \frac{3}{n^2} \left( \frac{k}{t} + 1 \right)^2 \left( t^k e^t \right)^{-\frac{n}{n+1}} + \frac{\beta_2}{(1+\gamma)} \left( t^k e^t \right)^{-\frac{(m+6)}{n}} \right] - \frac{2k}{n(1+\gamma)} t^{-\frac{m+n}{n}} e^{-\frac{m}{n}}, \tag{53}$$

$$\rho_c = \frac{3}{8\pi G_0 n^2} \left( \frac{k}{t} + 1 \right)^2 \left( t^k e^t \right)^{-\frac{n}{n+1}}, \tag{54}$$

$$\Omega_M = \frac{n^2 \left[ 2k \frac{t^n}{n^2} - \beta_1 \left( t^k e^t \right)^{-\frac{n}{n+1}} \right]}{3(1+\gamma) \left( \frac{k}{t} + 1 \right)^2}, \tag{55}$$

$$\Omega_\Lambda = 1 + \frac{n^2 \left[ \beta_2 \left( t^k e^t \right)^{-\frac{n}{n+1}} - \frac{2k}{n^2} \right]}{3(1+\gamma) \left( \frac{k}{t} + 1 \right)^2}. \tag{56}$$

Adding Eqs. (55) and (56), we get

$$\Omega = \Omega_M + \Omega_\Lambda = 1 + \frac{\beta n^2 \left( t^k e^t \right)^{-\frac{n}{n+1}}}{3 \left( \frac{k}{t} + 1 \right)^2}, \tag{57}$$
where $\beta = m_1 m_2 + m_2 m_3 + m_3 m_1$. For $\beta = 0$, we have $\Omega = 1$. We also observe from Eq. (57) that $\Omega$ approaches to one for sufficiently large time independent to $\beta$. Figures 6 and 7 plot the variation of density parameters for matter ($\Omega_M$) and cosmological constant ($\Omega_\Lambda$) versus $t$ respectively. From these figures it is clear that the universe is dominated by matter in early stage of evolution whereas the universe is dominated by dark energy (cosmological constant $\Lambda$) at present epoch. Figure 8 plots the variation of total energy parameter ($\Omega$) versus cosmic time $t$. From the Fig. 8, we observe that $\Omega \to 1$ at late time for arbitrary value of $\beta$. This is in good agreement with the observational results (Spergel 2003).

It is worth mentioned here that data in favor of nonzero cosmological constant involves the estimates of the age of the Universe as compared with the estimates of the Hubble parameter. With taking into account uncertainties in models the best fit to guarantee consensus between all observational constraints (Krauss and Turner 1995; Ostriker and Steinhardt 1995; Bahcall et al. 1999) is

$$H_0 = (70 - 80) \text{km s}^{-1} \text{Mpc}^{-1}, \quad t_0 = (13 - 16) \pm 3 \text{Gy},$$

$$\Omega_M = (0.3 - 0.4), \quad \Omega_\Lambda = (0.6 - 0.7),$$

where $\Omega + \rho_{\text{today}} / \rho_{\text{cr}}$, and the critical density $\rho_{\text{cr}}$, which correspond to $\Omega = 1$, is given by

$$\rho_{\text{today}} \sim 10^{-30} \text{g cm}^{-3}.$$

which is in good agreement with observations as discussed in introduction. Confrontation of models with observations in cosmology as well as the inflationary paradigm, compellingly favor treating the cosmological constant as a variable dynamical quantity.

For different values of $k$ and $n$, we can generate a class of models of the universe in Bianchi type-I space-time with time dependent gravitational and cosmological constants. We observe that for $n \geq 2$ and $k = 1$, we obtain a class of transit models of the universe from early decelerated to present accelerating phase. For $n \leq 1$ and $k = 1$, we obtain accelerating models at present epoch. For examples:

- If we put $n = 2$ in Eq. (34), we obtain $a(t) = \sqrt{(te^t)}$. In this case, we obtain the expressions for different physical and geometric quantities as obtained by Pradhan et al. (2013d). Thus, our investigations generalize the recent results of Pradhan et al. (2013e).
- If we put $n = 2$ and $k = 1$ in Eq. (34), we obtain $a(t) = \sqrt{(te^t)}$. In this case, we obtain the expressions for different physical parameters and geometric quantities by putting $n = 2$ and $k = 1$ in
Eqs. (42) – (57). Figure 9 depicts the variation of DP with cosmic time for different values of \((n, k)\). From Fig. 9, we observe that for \(n = 2\) and \(k = 1\) the model has a transition from very early decelerated phase to the present accelerating phase. We have already mentioned in Sect. 3 that in such type of universe, the DP must show signature interchange (Padmanabhan & Roychowdhury 2003; Amendola 2003; Riess et al. 2001). The variation of energy density, cosmological constant and density parameters versus cosmic time \(t\) have been shown in Figs. 10, 11 and 12 respectively.

- If we put \(n = 3\) and \(k = 1\) in Eq. (34), we obtain \(a(t) = (te^t)^{\frac{1}{3}}\). In this case, we obtain the expressions for different physical parameters and geometric quantities as usual. From Fig. 9, we observe that for \(n = 3\) and \(k = 1\), the model has transition from early decelerated phase to the present accelerating phase. The variation of energy density, cosmological constant and density parameters versus cosmic time \(t\) have been shown in Figs. 13, 14 and 15 respectively. The present value \(q_0\) of the deceleration parameter obtained from observations are \(-1.27 \leq q_0 \leq 2\) (Schuecker et al. 1998). Studies of galaxy from redshift surveys provide a value of \(q_0 \approx -0.77\) (Schuecker et al. 1998). Recent observations show that the deceleration parameter of the universe is in the range \(-1 \leq q \leq 0\) i.e. \(q_0 \approx -0.77\). First, we set \(n = 3\) and \(k = 1\) in Eq. (48), we obtain \(q_0 = -0.67\). This value is very near to the observed value of DP (i.e., \(q_0 \approx -0.77\)) at present epoch (Cunha et al. 2009). Secondly, if we choose \(n = 3\) and \(k = 1\), we observe that all the values of physical and geometric parameters are easily integrable. Hence this case is important from physical aspects.

- If we put \(n = 1\) and \(k = 1\) in Eq. (34), we obtain \(a(t) = te^t\). In this case, we obtain the expressions for different physical parameters and geometric quantities as usual. From Fig. 9, one can see that for \(n = 1\) and \(k = 1\) the model is accelerating at present epoch. The other physical parameters have the same property as already discussed.

- If we put \(k = 0\) in Eq. (34), we obtain \(a(t) = e^{t/n}\). In this case the universe has non-singular origin which seems reasonable to envision the dynamics of future universe. In this case, we found that energy density (\(\rho\)) is always negative and hence it is an unphysical case. We plan to work out a physically viable non-singular model in forthcoming paper.

5 Statefinder diagnostic

The various cosmological observational data support the \(\Lambda\) cold dark matter (\(\Lambda\)CDM) model, in which the cosmological constant \(\Lambda\) plays a role of dark energy in general relativity. At the current stage, the \(\Lambda\)CDM model is considered to be a standard cosmological model. Sahni et al. (2003) have introduced a pair of parameters \(\{r, s\}\), called Statefinder parameters. In fact, trajectories in the \(\{r, s\}\) plane corresponding to
different cosmological models demonstrate qualitatively different behavior. The statefinder parameters can effectively differentiate between different form of dark energy and provide simple diagnosis regarding whether a particular model fits into the basic observational data. The above statefinder diagnostic pair has the following form:

\[ r = 1 + 3 \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \quad \text{and} \quad s = \frac{r - 1}{3(q - \frac{1}{2})}, \] (58)

to differentiate among different form of dark energy. Here \( H \) is the Hubble parameter and \( q \) is the deceleration parameter. The two parameters \( \{r, s\} \) are dimensionless and are geometrical since they are derived from the cosmic scale factor \( a(t) \) alone, though one can reproduce them in terms of the parameters of dark energy and dark matter. This pair provides information about dark energy in a model-independent way, that is, it categorizes dark energy in the context of back-ground geometry only which is not dependent on theory of gravity. Hence, geometrical variables are universal.

For our model, the parameters \( \{r, s\} \) can be explicitly written in terms of \( t \) as

\[ r = 1 - \frac{3nk}{(k + t)^2} + \frac{2kn^2}{(k + t)^3}, \] (59)
\[ s = -\frac{3nk}{(k+t)^2} + \frac{2kn^2}{(k+t)^3} \left(\frac{3}{2} + \frac{nk}{(k+t)^2}\right). \] (60)

From Eqs. (59) and (60) we observe that \( s \) is negative when \( r \geq 1 \). It is observed that the universe starts from an Einstein static era \( (r \rightarrow \infty, s \rightarrow -\infty) \) and goes to the \( \Lambda \)CDM model \( (r = 1, s = 0) \).

6 Conclusion

In this paper, we have presented a new class of models of accelerating universe and transit universe with gravitational coupling \( G(t) \) and cosmological term \( \Lambda(t) \) in the framework of general relativity. The models represent expanding, shearing and non-rotating universe. The parameters \( H, \theta, \) and \( \sigma \) diverge at the initial singularity. There is a Point Type singularity (MacCallum 1971) at \( t = 0 \) in the models. The rate of expansion slows down and finally tends to zero at \( t \rightarrow 0 \). The pressure, energy density and cosmological term \( \Lambda \) become negligible whereas the scale factors, gravitational constant \( G \) and spatial volume become infinite as
Figure 15: Plots of $\Omega_M$, $\Omega_\Lambda$, $\Omega$ vs time $t$ for $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = -1$, $n = 3$, $k = 1$

t $\rightarrow \infty$. The nature of decaying vacuum energy density $\Lambda(t)$ in our derived models is supported by recent cosmological observations. We observe that our derived models are isotropic at present epoch which is in good agreement with the current observations.

For different choice of $n$ and $k$, we can generate a class of viable cosmological models of the universe in Bianchi type space-time as well as in FRW universe. For example: if we set $n = 2$ in Eq. (34), we find $a = \sqrt{t^k e^t}$ which is used by Pradhan and Amirhashchi (2011) in studying the accelerating dark energy models in Bianchi type-V space-time and Pradhan et al. (2012a) in studying Bianchi type-I in scalar-tensor theory of gravitation. If we set $k = 1$, $n = 2$ in Eq. (34), we find $a = \sqrt{te^t}$ which is utilized by Amirhashchi et al. (2011b) in studying interacting two-fluid scenario for dark energy in FRW universe. If we set $k = 1$, $n = 1$ in Eq. (34), we find $a = te^t$ which is exercised by Pradhan et al. (2012b) to study the dark energy model in Bianchi type-VI$_0$ universe. It is observed that such models are also in good harmony with current observations. The present work generalizes the the recent works (Pradhan et al. 2013c,e).

We observe that our models approach to $\Lambda$CDM model. So, from the Statefinder parameter $\{r, s\}$ the behaviour of different stages of the evolution of the universe has been generated.

In summary, the solutions described in this paper may be useful for better understanding of the characteristic of Bianchi type-I cosmological models in the evolution of the universe within the framework of time dependent gravitational and cosmological “constants”.

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