Geometric involutive bases for positive dimensional polynomial ideals and SDP methods

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Abstract

Geometric involutive bases for polynomial systems of equations have their origin in the prolongation and projection methods of the geometers Cartan and Kuranishi for systems of PDE. They are useful for numerical ideal membership testing and the solution of polynomial systems. In this paper we further develop our symbolic-numeric methods for such bases. We give methods to explicitly extract and decrease the degree of intermediate systems and the output basis. Algorithms for the numerical computation of involutivity criteria for positive dimensional ideals are also discussed.

We were also motivated by some remarkable recent work by Lasserre and collaborators who employed our prolongation projection involutive criteria as a part of their semi-definite based programming (SDP) method for identifying the real radical of zero dimensional polynomial ideals. Consequently in this paper we begin an exploration of the interaction between geometric involutive bases and these methods particularly in the positive dimensional case. Motivated by the extension of these methods to the positive dimensional case we explore the interplay between geometric involutive bases and the new SDP methods.

1 Introduction

This paper is part of a stream devoted to developing symbolic-numeric prolongation projection algorithms for general systems of partial and differential algebraic equations. Such

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algorithms prolong (differentiate) such systems and project the prolonged systems to determine obstructions or missing constraints to their integrability. See Kuranishi [18] for proof of termination of such methods using Cartan’s geometric involutivity criteria. A by-product of these methods has been their implementation for linear homogeneous partial differential equations with constant coefficients, and consequently for polynomial algebraic systems. See [13] for applications and symbolic algorithms for polynomial systems. The symbolic-numeric version of a geometric involutive form was first described and implemented in Wittkopf and Reid [32]. It was applied to approximate symmetries of differential equations in [6] and to polynomial solving in [35, 34, 30]. See [11] where it is applied to the deflation of multiplicities in multivariate polynomial solving.

The current paper is focused on further development of our geometric involutive basis algorithm particularly in the positive dimensional case, and also in relation to real solving. It is especially motivated by remarkable recent developments concerning real solution of such systems by Lasserre, Laurent and Rostalski [19] and their use of aspects of our prolongation projection algorithm in the paper “A prolongation-projection algorithm for computing the finite real variety of an ideal”. They developed a new approach for computing the real radical of zero dimensional polynomial systems using semi-definite programming (SDP) techniques. See [10] for early fundamental work on such problems. Zero dimensional systems are those having finitely many real solutions, and the real radical is the set of polynomials which vanish on these solutions. In contrast to the input systems the output radical systems from their approach are multiplicity free and so are better conditioned for numerical solution techniques. The output radical systems only have real roots and no complex roots. This leads to possibility of lower complexity methods, since current methods for finding real solutions, mostly explicitly, or implicitly pass through complex root formulations. Given the widespread popularity of linear programming (and by implication) SDP methods, the surprising links between this area also open interesting research possibilities. See [5] for a recent book on the connections between semi-definite optimization and convex algebraic geometry.

We briefly list some background references. There have been considerable recent advances in numerical complex geometry. See especially the books [38, 3] and the references therein. In approaches based on homotopy continuation, positive dimensional components characterize the variety over C by certain witness points cut out by intersections of the components with random linear spaces. For a modern text with many references on computational real algebraic geometry see [2]. Real algebraic geometry is a vast subject with many applications. Sturm’s ancient method on counting real roots of a polynomial in an interval is central to Tarski’s real quantifier elimination [10] and was further developed by Seidenberg [36]. One of the most important algorithms of real algebraic geometry is cylindrical algebraic decomposition. CAD was introduced by Collins [9] and improved by Hong [17] who made Tarski’s quantifier elimination algorithmic. This algorithm decomposes \( \mathbb{R}^n \) into cells on which each polynomial of a given system has constant sign. The projections of two cells in \( \mathbb{R}^n \) to \( \mathbb{R}^k \) with \( k < n \) either don’t intersect or are equal. The computational cost of this algorithm, which is doubly exponential [11], is a major barrier to its application. See [8] and [7] for modern improvements using triangular decompositions. For approaches based on obtaining
witness points for the real positive dimensional case see [29, 15, 16, 42]. Homotopy methods are used in [20] and [4] for real algebraic geometry. Recently such moment matrix completion techniques are explored by Zhi et al in [22] for finding at least one real root of a given semi-algebraic system. Furthermore, based on critical point technique and moment matrix completion, they studied the computation of verified real solutions on positive dimensional system in [43].

As part of our initial exploration of this area, in this paper, we make some improvements in our geometric involutive bases, by enabling the explicit extraction of projected systems and hence reducing the size of matrices that can appear in intermediate computations. Similarly motivated by the extension of these methods to the positive dimensional case we explore the interplay between geometric involutive bases and the new SDP methods. The symbol space of a polynomial system or kernel of the matrix of its highest coefficients is the geometric generalization of the highest coefficient of a polynomial. Certain projections within the symbol space encode a geometric test - an analogue of the S-polynomials in Gröbner basis approaches - for new members of the polynomial ideal. We provide details and example of this in the numerical case. An attempt in this paper is made to minimize use of terminology from the jet geometry of partial differential equations, in order to make this accessible to a wider audience.

2 Brief background on ideals and varieties

In this section we briefly sketch some basic objects from real and complex algebraic geometry and introduce some notation for our paper.

2.1 Some basic objects in complex algebraic geometry

Consider the set \( \mathbb{C}[x_1, x_2, ..., x_n] \) of multivariate polynomials with complex coefficients in the complex variables \( x = (x_1, x_2, ..., x_n) \in \mathbb{C}^n \). Then \( \mathbb{C}[x_1, x_2, ..., x_n] \) is a ring. Given \( P = \{p_1(x), p_2(x), ..., p_m(x)\} \subseteq \mathbb{C}[x_1, x_2, ..., x_n] = \mathbb{C}[x] \) its solution set or variety is:

\[
V_{\mathbb{C}}(P_{1}, P_{2}, ..., P_{m}) = \{ x \in \mathbb{C}^n : p_j(x) = 0, 1 \leq j \leq m \} \tag{1}
\]

For brevity we sometimes write \( V_{\mathbb{C}}(P) = \{ x \in \mathbb{C}^n : P(x) = 0 \} \). Upper case letters \( P, Q, R, \) etc will denote sets of polynomials and lower case letters \( p, q \) etc will denote individual polynomials.

The ideal over \( \mathbb{C} \) generated by \( P = \{p_1, ..., p_k\} \) is:

\[
\langle P \rangle_{\mathbb{C}} = \langle p_1, ..., p_k \rangle_{\mathbb{C}} = \{ f_1p_1 + ... + f_kp_k : f_j \in \mathbb{C}[x], 1 \leq j \leq k \} \tag{2}
\]

and its associated radical ideal over \( \mathbb{C} \) is

\[
\sqrt{\langle P \rangle} = \{ f \in \mathbb{C}[x] : f(x) = 0 \ \text{for all} \ x \in V_{\mathbb{C}}(P) \} = \{ f \in \mathbb{C}[x] : f^m \in \langle P \rangle \ \text{for some} \ m \in \mathbb{N} \} \tag{3}
\]
where \( \mathbb{N} \) is the set of non-negative integers.

**Example 2.1** To make this paper accessible to a wide audience we illustrate first some of the main ideas on the simple and well-known case of systems of univariate polynomials. Given a system of \( k \) univariate polynomials \( P = \{ p_1, \ldots, p_k \} \) with coefficients from some computable field (e.g. \( \mathbb{Q} \)), a Gröbner basis (or gcd) computation returns a single polynomial \( q(x) \):

\[
\langle q \rangle_{\mathbb{C}} = \langle p_1, \ldots, p_k \rangle_{\mathbb{C}}
\]

The factorization of \( q(x) \) over \( \mathbb{C} \) has form:

\[
q(x) = a(x - a_1)^{n_1} \cdots (x - a_\ell)^{n_\ell}
\]

where the roots \( a_j \in \mathbb{C} \) of \( q(x) \) are distinct. Though the \( a_j \) can’t be found in general by finitely many rational operations the so-called square-free factorization can be found by such operations yielding:

\[
\tilde{q}(x) = \frac{q(x)}{\gcd(q(x), q'(x))} = a(x - a_1) \cdots (x - a_\ell)
\]

For this example the ideal, variety and radical ideal over \( \mathbb{C} \) are:

\[
\langle P \rangle_{\mathbb{C}} = \{ g(x) \cdot (x - a_1)^{n_1} \cdots (x - a_\ell)^{n_\ell} : g(x) \in \mathbb{C}[x] \} \\
V_{\mathbb{C}}(P) = \{ a_1, a_2, \ldots, a_\ell \} \\
\sqrt{\langle P \rangle} = \{ g(x) \cdot (x - a_1) \cdots (x - a_\ell) : g(x) \in \mathbb{C}[x] \}
\]

For sophisticated generalizations to primary decomposition for multivariate systems see Gianni et al. [14].

### 2.2 Some basic objects in real algebraic geometry

Suppose that \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and consider a system of \( k \) multivariate polynomials \( P = \{ p_1(x), p_2(x), \ldots, p_k(x) \} \subseteq \mathbb{R}[x_1, x_2, \ldots, x_n] \) with real coefficients. Its solution set or variety is

\[
V_{\mathbb{R}}(p_1, \ldots, p_k) = \{ x \in \mathbb{R}^n : p_j(x) = 0, \ 1 \leq j \leq k \}
\]

The ideal generated by \( P = \{ p_1, \ldots, p_k \} \subseteq \mathbb{R} \) is:

\[
\langle P \rangle_{\mathbb{R}} = \langle p_1, \ldots, p_k \rangle_{\mathbb{R}} = \{ f_1 p_1 + \cdots + f_k p_k : f_j \in \mathbb{R}[x], 1 \leq j \leq k \}
\]

and its associated radical ideal over \( \mathbb{R} \) is defined as

\[
\sqrt{\langle P \rangle} = \{ f \in \mathbb{R}[x] : f^{2m} + \sum_{j=1}^{s} q_j^2 \in \langle P \rangle \text{ for some } q_j \in \mathbb{R}[x], m \in \mathbb{N} \setminus \{0\} \}
\]

A fundamental result [2] is:
Theorem 2.1  [Real Nullstellensatz] For any ideal $I \subseteq \mathbb{R}[x]$ we have $\sqrt[\mathbb{R}]{I} = I(V_\mathbb{R}(I))$. Consequently

\[ \sqrt[\mathbb{R}]{\langle P \rangle} = \{ f(x) \in \mathbb{R}[x] : f(x) = 0 \text{ for all } x \in V_\mathbb{R}(P) \} \]  \hspace{1cm} (11)

Remark 2.2  An ideal $I \subseteq \mathbb{R}[x]$ is real radical if and only if for all $p_1, \ldots, p_k \in R[x]$:  

\[ p_1^2 + \cdots + p_k^2 \in I \implies p_1, \ldots, p_k \in I. \]  \hspace{1cm} (12)

For these and many other results see [2] and the references cited therein.

Example 2.2  Consider the simplest case of a system of $k$ univariate polynomials in some computable subfield of $\mathbb{R}$ (e.g. $\mathbb{Q}$). Then as in the complex case a Gröbner basis of such a system yields a single polynomial $q(x)$ having the same roots. Discarding the factors with complex roots with nonzero imaginary parts yields a polynomial of form:

\[ \tilde{q}(x) = b(x - b_1)^{m_1} \cdots (x - b_j)^{m_j} \]  \hspace{1cm} (13)

where $b_1, b_2, \ldots, b_j$ are the real roots and $m_1, \ldots, m_j$ their corresponding multiplicities. Then  

\[ \langle P \rangle_\mathbb{R} = \{ f(x) \cdot (x - b_1)^{m_1} \cdots (x - b_j)^{m_j} : f(x) \in \mathbb{R}[x] \} \]

\[ V_\mathbb{R}(P) = \{ b_1, b_2, \ldots, b_j \} \]

\[ \sqrt[\mathbb{R}]{\langle P \rangle} = \{ g(x) \cdot (x - b_1) \cdots (x - b_j) : g(x) \in \mathbb{R}[x] \} \]  \hspace{1cm} (14)

3  Geometric prolongation and projection for polynomial systems

In this section we give a brief description the well-known presentation of polynomial systems as linear functions of their monomials and the related coefficient matrix and its kernel and rowspace [39, 24, 25, 23] and historical work by Macaulay [21]. We describe a type of elimination called geometric projection and then describe geometric prolongation resulting from multiplying polynomials by monomials.

We exploit the well-known correspondence between polynomial systems and systems of constant coefficient linear homogeneous PDE. This equivalence has been extensively studied and exploited in the exact case by Gerdt [13] and his co-workers in their development of involutive bases. Our geometric involutive bases are involutive by the geometric criteria in [18, 28, 37] and more distantly related to that of [13] which are closer relatives of Gröbner bases.
Consider a system of \( \ell \) polynomials \( P \subseteq \mathbb{K}[x] \) of degree \( d \) in the variables \( x = (x_1, \ldots, x_n) \) where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Monomials are denoted by \( x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n} \) where \( \alpha \in \mathbb{N} \) and the degree of \( x^\alpha \) is \( |\alpha| = \alpha_1 + \ldots + \alpha_n \). Then the system \( P \) can be written as:

\[
P = \left\{ \sum_{|\alpha| \leq d} a_{k,\alpha} x^\alpha : k = 1 \ldots \ell \right\}
\]

(15)

To apply the methods of numerical linear algebra the system is converted into matrix form [39, 24, 25, 23].

**Definition 3.1 (Coefficient Matrix \( C(P) \), \( J^d \) and vector of monomials)** Denote the coefficient matrix of \( P \) in (15) by \( C(P) \). Let \( x^{(\leq d)} \) be the column vector of monomials \( x^\alpha \) with \( 0 \leq \alpha \leq d \) sorted by total degree. We suppose that the columns of \( C(P) \) are sorted in the same order. Then \( P = C(P)x^{(\leq d)} \) where \( C(P) \in \mathbb{R}^{\ell \times N(n,d)} \) and \( N(n,d) := \binom{d+n}{d} \) is the number of monomials in \( x^{(\leq d)} \). Polynomials can be equivalently represented by the row vectors of \( C(P) \), that is as vectors in \( J^d := \mathbb{R}^{N(n,d)} \).

Prolonging polynomials by multiplying them by monomials is an essential geometric operation in this paper.

**Definition 3.2 (prolongations \( \hat{D} \) and \( \hat{D} \))** Consider a system of polynomials \( P \) of degree \( d \). Let \( p \in P \) have degree \( \hat{d} \). Then the prolongation of \( p \) written \( \hat{D}(p) \) is defined as \( \hat{D}(p) = \{p\} \cup \{x_j p : 1 \leq j \leq n\} \). The prolongation of the system \( P \) is defined as \( \hat{D}^k(P) = \{x^\alpha p : 0 \leq \deg(x^\alpha p) \leq d+k, \alpha \in \mathbb{N}^n, p \in P\} \). Equivalently we can represent the prolongation geometrically as the span of the corresponding row vectors of \( C(\hat{D}^k P) \), which we denote by \( \hat{D}^k(P) := \text{rowsp}(C(\hat{D}^k P)) \) which is a subspace of \( J^{d+k} \).

**Example 3.1** Suppose \( x = (y, z) \) and \( P = \{2, 2y+z\} \). Then \( \hat{D}(P) = \{2, 2y, 2z, 2y^2, 2yz, 2z^2, 2y^2 + yz, 2yz + z^2\} \).

**Definition 3.3 (projections \( \hat{\pi} \) and \( \hat{\pi} \))** Consider a polynomial system of degree \( d \geq 1 \) written in the form \( P = C(P)x^{(\leq d)} \) with the columns of \( C(P) \) sorted in descending order by degree. The rows in the Gauss echelon form of \( C(P) \) with pivots of degree less than \( d \) span a subspace of \( J^{d-1} \) which we denote by \( \hat{\pi}(P) \). We denote the set of polynomials of degree \( \leq d-1 \) corresponding to the row vectors by \( \hat{\pi}(P) \). Iterations of projections \( \hat{\pi}^\ell(P) \subset \mathbb{R}[x] \) and equivalently \( \hat{\pi}^\ell(P) \subset J^{d-\ell} \) are defined similarly.

We have adopted an abbreviated notation for prolongation and projection here to avoid cumbersome indices indicating the spaces on which these operators act. We will also need to prolong and project kernels of the coefficient matrices of polynomial systems.
Definition 3.4 (prolongation \( D \) and projection \( \pi \) on the kernel) Consider a polynomial system \( P \subset \mathbb{R}[x] \) of degree \( d \). Given a subspace \( V \) of \( J^d \) and \( \ell \leq d \) define \( \pi^{\ell}(V) \) as the vectors of \( V \) with the components of degree \( \geq d - \ell \) discarded. To abbreviate notation we will write \( \pi^{\ell}(P) := \pi^{\ell} \ker C(P) \). The \( k \)-th prolongation of the kernel is \( D^k(P) := \ker C(\hat{D}^k P) \).

In summary we have presented three (!) notations for prolongation and projection since we need to work directly with them sometimes as polynomial systems, and sometimes row spaces or kernels. The row space and kernel are orthogonal to each other in \( J^d \). Projection in the kernel is the usual projection operator \( \pi^{\ell} \). Geometrically the corresponding projection in the row space can be obtained as the orthogonal complement of \( \pi^{\ell}(P) \). Alternatively it can be obtained by first considering \( J^{d-\ell} \) as a subspace in \( J^d \) and then intersecting the subspace \( J^{d-\ell} \) with rowsp\((P)\).

Suppose that \( A = C(P) \) is the coefficient matrix of a system of polynomials \( P \). To numerically implement an approximate involutive form method, we proposed in [6, 32, 34] a numeric version of the projection operator based on singular value decomposition (SVD). We first find the SVD of \( A \) given by \( A = U \cdot \Sigma \cdot V \) where \( U \) and \( V \) are unitary matrices and \( \Sigma \) is a diagonal matrix whose diagonal entries are real decreasing non-negative numbers. The approximate rank \( r \) is the number of singular values bigger than a fixed tolerance. Deleting the first \( r \) rows of \( V \) yields an approximate basis for \( \ker A \) and an estimate for \( \dim \ker A \). Deleting highest degree components of the vectors in this basis, yields an approximate spanning set for \( \pi \ker A \) and an estimate for \( \dim \pi \ker A \). If desired further computation yields bases for \( \pi \ker A \). Then we compute the kernel of the spanning set of \( \pi \ker A \). Similarly we can compute approximate spanning sets and if desired bases of the prolongations and projections of the system.

Remark 3.5 (Alternative representations and extraction of intermediate systems)

In summary prolongation and projection can equivalently be computed in either the kernel, the rowspace, and at any time polynomial generators can be extracted. Underlying this is a 1 to 1 correspondence between vector spaces (not elements): in particular between the row spaces and its orthogonal complement, the kernel.

Example 3.2 Consider

\[
P = \{x^8 - x^4 - 2, x^8 - 3x^4 + 2\} \subseteq \mathbb{R}[x]
\]  

(16)

Here the coefficient matrix is given by \( C(P) \) below:

\[
C(P) \cdot x^{(\leq 8)} = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x^8 \\
x^7 \\
x^6 \\
x^5 \\
x^4 \\
x^3 \\
x^2 \\
x^1 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]  

(17)
\[ k = 0 \quad k = 1 \quad k = 2 \quad k = 3 \]
\[
\begin{array}{c|cccc}
\ell = 0 & 7 & 6 & 5 & 4 \\
\ell = 1 & 7 & 6 & 5 & 4 \\
\ell = 2 & 6 & 6 & 5 & 4 \\
\ell = 3 & 5 & 5 & 5 & 4 \\
\ell = 4 & 4 & 4 & 4 & 4 \\
\ell = 5 & 4 & 4 & 4 & 4 \\
\ell = 6 & 3 & 4 & 4 & 4 \\
\ell = 7 & 2 & 3 & 4 & 4 \\
\ell = 8 & 1 & 2 & 3 & 4 \\
\ell = 9 & 1 & 2 & 3 & 4 \\
\ell = 10 & 1 & 2 & 3 & 4 \\
\ell = 11 & & & & 1 \\
\end{array}
\]

Figure 1: Table of \( \dim \pi^k \ell D_k \) for (17) for Example 3.2. The (red) boxed 4 in the first column corresponds to \( \pi^4 P \) and a geometric involutive basis for \( P \) as found by Algorithm 4.1. The blue and black boxed 4’s in the fourth column correspond to geometric involutive bases for \( P \).

The most familiar computation for most readers is to eliminate the polynomials as in a Gröbner basis calculation: \( x^8 - x^4 - 2 - (x^8 - 3x^4 + 2) = 2x^4 - 4 \). This can also be done as a computation on the row space of \( C(P) \), yielding the result as the generator of \( \bar{\pi}^4 P \). Equivalently by Remark 3.5 we can compute the result by projecting basis vectors of the kernel of \( C(P) \) obtaining \( \pi^4 P \) and then recover the generator \( 2x^4 - 4 \). The original 8 degree polynomials can be discarded since they are consequences of \( 2x^4 - 4 \). In particular a Gröbner basis for the ideal generated by \( P \) is

\[ x^4 - 2 \quad (18) \]

The kernel of \( C(P) \) is easily calculated numerically by the SVD. We obtain the table of dimensions for the projections of \( \ker C(P) \) in Figure 7. We use singular value decomposition to compute its kernel and then project its vectors to \( \pi^4 P \). The generator corresponding to this projection is:

\[ 0.4472136 x^4 - 0.8944272. \quad (19) \]

where the coefficients here and elsewhere in the paper have been truncated from 15 digits to 7 digits. After normalization, we get the generator \( x^4 - 2 \).

4 Geometric involutive bases

In this section we describe geometric involutive form. For a more detailed description see [6, 33, 32, 34].

Exact elimination methods for exactly given polynomial systems (e.g. Gröbner Bases),
usually employ Gaussian Elimination (e.g. linear elimination of monomials). Such exact methods usually depend on the ordering of input (e.g. term ordering in the case of Gröbner Bases), and so are coordinate dependent. Since the order of elimination can force division by small leading entries, such methods are generally unstable, when used on approximate systems. In contrast, exact elimination methods from the geometric theory of PDE are coordinate independent \[18, 28\] and this motivated our study of numerical versions of such methods which is continued in this paper.

4.1 Symbol, class and Cartan involution test

Definition 4.1 (Symbol matrix and class of a monomial) Given a polynomial system of degree \(d\), its symbol matrix, denoted \(S(P)\) is the submatrix of \(C(P)\) corresponding to its degree \(d\) monomials. Consider a monomial \(x^\alpha\) where \(\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n\). Then the class of \(x^\alpha\) is the least \(j\) such that \(\alpha_j \neq 0\).

For Example 3.2 the symbol matrix is the submatrix \(S(P) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) of \(C(P)\) given in (17). Consider the system

\[
P = \{x_2^2 - 1, 2x_1x_2 - 3x_1\}\]

(20)

For what follows we sort the columns of the symbol matrix in descending order according to class. The degree two monomials are \(x^{(0,2)} = x_2^2, x^{(1,1)} = x_1x_2, x^{(2,0)} = x_1^2\). Here \(x_2^2\) is class 2. Monomials \(x_1x_2\) and \(x_1^2\) are class 1. Then the symbol matrix is:

\[
S(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}
\]

(21)

Definition 4.2 (Cartan test for involutivity of the Symbol) Suppose that the columns of the symbol matrix for a system of degree \(d\) are sorted in descending order by class and that it is reduced to Gauss echelon form. For \(k = 1, 2, ..., n\) define the quantities \(\beta_k\) as the number of pivots in this reduced matrix of class \(k\). Then in a generic system of coordinates the symbol is involutive if:

\[
\sum_{k=1}^{k=n} k\beta_d^{(k)} = \text{rank } S(\hat{D}P)
\]

(22)

The following combinatorial quantities will be useful in our numerical determination of involutivity of symbol matrices. Consider systems in \(n\) variables of degree \(d\).
Denote:

\[ N(n, d) = \binom{n+d}{d} = \text{Number of monomials of degree } \leq d \]
\[ N_{\text{deg}}(n, d) = \binom{n+d-1}{d} = \text{Number of monomials of degree } d \]
\[ N_c(n, d, k) = \binom{n+d-k-1}{d-1} = \text{Number of class } k \text{ monomials of degree } d \]

**Example 4.1** For system \( P \) given in (24):

\[ N(2, 2) = 6, N_{\text{deg}}(2, 2) = 2, N_c(2, 2, 1) = 2, N_c(2, 2, 2) = 1 \]

The symbol matrix (21) is already in Gauss echelon form with respect to class. There is one pivot of class 2 so \( \beta_2^{(2)} = 1 \) and one pivot of class 1 so \( \beta_2^{(1)} = 1 \). Also an easy calculation gives \( \text{rank } S(\hat{D}P) = 3 \). So

\[ \sum_{k=2}^{n} k \beta_d^{(k)} = 3 = \text{rank } S(\hat{D}P) \]

and the symbol is involutive. In all cases \( \sum_{k=1}^{n} k \beta_d^{(k)} \leq \text{rank } S(\hat{D}P) \). Indeed in our example if we reverse the order of the coordinates and recalculate we get \( S(P) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Then \( \beta_2^{(2)} = 0, \beta_2^{(1)} = 2 \) and \( \sum_{k=2}^{n} k \beta_d^{(k)} = 2 < \text{rank } S(\hat{D}P) \) so the test indicates a non-involutive symbol however the result may be due to the coordinates being nongeneric which is indeed the case here. A generic linear change of coordinates by a random \( 2 \times 2 \) matrix then shows the symbol is involutive.

To extract a matrix for the symbol space of the variables of degree \( d \) we proceed as follows for a system \( P \) of degree \( d' \geq d \). Suppose that vectors that are a basis for the kernel of \( C(P) \) form the rows of a matrix \( \tilde{B} \). First numerically project the kernel of the system \( P \) onto the subspace \( J^d \) via \( \pi^{d'-d}P \) by deleting the coordinates in the basis of degree \( > d \) to obtain for \( \pi^{d'-d}P \) a spanning set \( \tilde{B} \) given by the remaining rows of \( B \). Then delete the columns in \( \tilde{B} \) corresponding to variables of degree \( < d \) to obtain a matrix \( A_d \) corresponding to the orthogonal complement of the symbol for degree \( d \). Let \( A_d^{(k)} \) be the submatrix of \( \tilde{B} \) with columns corresponding to class \( k \) or less deleted. In generic coordinates

\[ \beta_d^{(k)} = N_c(n, d, k) - \left( \text{rank } A_d^{(k-1)} - \text{rank } A_d^{(k)} \right), \quad k = 1 \ldots n. \]

Then the SVD can approximate the ranks in this equation for carrying out the Cartan Test (22).

**Definition 4.3 (Involutive System)** A system of polynomials \( P \in \mathbb{R}[x] \) is involutive if \( \dim \pi DP = \dim P \) and the symbol of \( P \) is involutive.
Definition 4.4 (Projected Involutive System) Consider a system of polynomials $P \in \mathbb{R}[x]$ of degree $d$. Suppose that $k, \ell$ are integers with $k \geq 0$ and $0 \leq \ell \leq k + d$. Then $\pi^\ell D^k P$ is projectively involutive at prolongation order $k$ and projected order $\ell$, if $\pi^\ell D^k P$ satisfies the projected elimination test

$$\dim \pi^\ell D^k P = \dim \pi^{\ell+1} D^{k+1} P$$

and the symbol of $\pi^\ell D^k P$ is involutive.

In [6] we prove:

**Theorem 4.5** A system is projectively involutive if and only if it is involutive.

**Theorem 4.6 (Criterion for zero dimensional involutive system)** A zero dimensional system of polynomials $P \in \mathbb{R}[x]$ is projectively involutive at order $k$ and projected order $\ell$ if and only if $\pi^\ell D^k P$ satisfies the projected elimination test (27) and

$$\dim \pi^\ell D^k P = \dim \pi^{\ell+1} D^k P$$

This criterion is used by Lasserre et al [1] in their prolongation projection algorithm to determine the finite real radical. When there are 2 variables then it is easily shown that:

$$S \pi^\ell D^k P \text{ is involutive } \iff \dim S \pi^\ell D^{k+1} P = \dim S \pi^\ell D^k P$$

and this gives a computationally easy characterization by using

$$\dim S \pi^\ell D^k P = \dim \pi^\ell D^k P - \dim \pi^{\ell+1} D^k P$$

The criteria in (27) applies to both zero and positive dimensional bivariate systems.

### 4.2 Projected involutive form algorithm

The following method completes systems to approximate involutive form. We seek the smallest $k$ such that there exists an $\ell$ with $\pi^\ell D^k P$ approximately involutive, and generates the same ideal as the input system. We choose the system corresponding to the largest such $\ell \leq k$ if there are several such values for the given $k$.

**Algorithm 4.1 (Projected involutive basis)**

**Input:** $Q \subseteq \mathbb{R}[x_1, \ldots, x_n]$. A tolerance $\epsilon$.

Set $k := 0$, $d := \deg(Q)$ and $P := \ker C(Q)$
repeat
  Compute $D^k(P)$
  Initialize set of involutive systems $I := \{\}$
  for $\ell = 0 \cdots (d + k)$ do
    Compute $R := \pi^\ell D^k(P)$
    if $R$ involutive then $I := I \cup \{R\}$ end if
  end do
  Remove systems $\bar{R}$ from $I$ not satisfying $D^{d+k-\bar{d}} \bar{R} \subseteq D^k(P)$
  $k := k + 1$
until $I \neq \{\}$

Output: Return the polynomial generators of the involutive system $\bar{R}$ in $I$
of lowest degree $\bar{d}$.

Note that this algorithm works on kernels, but could by Remark 3.5 equivalently work on
their orthogonal complements - the associated row spaces. The condition $D^{d+k-\bar{d}} \bar{R} \subseteq D^k(P)$
is a standard subspace inclusion test for the prolonged kernels. It ensures that the output
system generates the same ideal as the input system and has the same solutions.

Remark 4.7 (Decreasing degrees by extracting involutive projections)
We note that a simple illustration of Algorithm 4.1 is Example 3.2 where all univariate
polynomials are involutive. This algorithm is an improvement on that published in [30] where
to ensure the inclusion conditions for positive dimensional ideals, the number of projections
was limited to $0 \leq \ell \leq k$.

5 Moment matrices and SDP

5.1 Moment Matrices

Here we focus just on the construction of moment matrices. For the theoretical background
the reader is directed to [1].

A moment matrix is a symmetric matrix $M = (M_{\alpha,\beta})$ indexed by $\mathbb{N}^n$ ($\alpha, \beta \in \mathbb{N}^n$). Here $\alpha$ is
the index for rows, $\beta$ is the index for columns. Without loss $M_{0,0} = 1$.

Given a multivariate polynomial system $P \subseteq \mathbb{R}[x_1, ..., x_n]$. Let $d = \deg(P)$ and $M \in \mathbb{R}^{N(n,d) \times N(n,d)}$ be the truncated moment matrix. The linear constraints imposed by $P$ are constructed as

$$M \cdot A^T = 0; \ A = C(\hat{D}^d(P)),$$

where $C$ is the coefficient matrix function given in Definition 3.1.
5.2 Moment matrix for univariate example

In Example 3.2 a degree 8 input system was reduced to a degree 4 output polynomial \( p = x^4 - 2 \). Then in matrix form the polynomial is

\[
Bv = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0, \ker B = \text{span}_\mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\]

(32)

The moment matrix is the infinite matrix whose \((\alpha, \beta)\) entry is \( u_{\alpha+\beta} \) and \( \alpha, \beta \in \mathbb{N}^n \) given by:

\[
M = \begin{pmatrix}
  u_0 & u_1 & u_2 & u_3 & u_4 & \cdots \\
  u_1 & u_2 & u_3 & u_4 & u_5 & \cdots \\
  u_2 & u_3 & u_4 & u_5 & u_6 & \cdots \\
  u_3 & u_4 & u_5 & u_6 & u_7 & \cdots \\
  u_4 & u_5 & u_6 & u_7 & u_8 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(33)

In the SDP-moment matrix approach the given polynomial system, in this case \( \{x^4 - 2\} \), is first prolonged to twice its degree:

\[
\hat{D}^4 \{x^4 - 2\} = \{x^4 - 2, x^5 - 2x, x^6 - 2x^2, x^7 - 2x^3, x^8 - 2x^4\}
\]

(34)

From (31) the constraint system when we impose \( u_0 = 1 \) is equivalent to the linear system

\[
u_4 - 2 = 0, u_5 - 2u_1 = 0, u_6 - 2u_2 = 0, u_7 - 2u_3 = 0, u_8 - 2u_4 = 0
\]

(35)

which can be regarded as the rewrite rules: \( u_4 \to 2, u_5 \to 2u_1, u_6 \to 2u_2, u_7 \to 2u_3, u_8 \to 2u_4 \to 4 \). Imposing these constraints the truncated moment matrix to degree 8 is

\[
M = \begin{pmatrix}
  1 & u_1 & u_2 & u_3 & 2 \\
  u_1 & u_2 & u_3 & 2 & 2u_1 \\
  u_2 & u_3 & 2 & 2u_1 & 2u_2 \\
  u_3 & 2 & 2u_1 & 2u_2 & 2u_3 \\
  2 & 2u_1 & 2u_2 & 2u_3 & 4
\end{pmatrix}
\]

(36)

The moment matrix (36) is then sent to the SDP solver Yalmip in Matlab to numerically compute a generic point \((u_1, u_2, u_3)\) if possible such that \( M \) is a positive semidefinite matrix with maximum rank. This solver returns an approximation which can be recognized for illustrative convenience as \((u_1, u_2, u_3) = (0, \sqrt{2}, 0)\). Its associated moment matrix and
moment matrix kernel are:

\[
M = \begin{pmatrix}
1 & 0 & \sqrt{2} & 0 & 2 \\
0 & \sqrt{2} & 0 & 2 & 0 \\
\sqrt{2} & 0 & 2 & 0 & 2 \sqrt{2} \\
0 & 2 & 0 & 2 \sqrt{2} & 0 \\
2 & 0 & 2 \sqrt{2} & 0 & 4
\end{pmatrix}, \text{ker } M = \text{span}_{\mathbb{R}} \{ (-2, -\sqrt{2}, 0), (0, 0, 1), (0, 0, 0, 0, 0) \} \]

(37)

The kernel corresponds to the generating set

\[
\{ \sqrt{2} - x^2, 2 - x^4, \sqrt{2}x - x^3 \}
\]

(38)

Applying geometric involutive form algorithm yields a geometric involutive basis

\[
\{ \sqrt{2} - x^2 \}
\]

(39)

The last two polynomials are a consequence of $\sqrt{2} - x^2$ by our inclusion test, so are discarded. By Rostalski [1], this is a basis of the real radical.

6 Combining geometric involutive bases and moment matrix methods

6.1 Geometric involutive form and moment matrix algorithms

In this section we outline algorithms for combining geometric involutive form and moment matrix methods.

Algorithm 6.1 (gif– \( \mathcal{M} \) Method)

Input: \( P = \{p_1, ..., p_k\} \subseteq \mathbb{R}[x_1, \ldots, x_n] \)

\[
Q_0 := P \\
j := 0
\]

do

\[
d := \text{dim ker } \text{gif}(Q_j) \\
Q_{j+1} := \text{gen}(\text{gif}(Q_j)) \\
r := \text{rank}(\mathcal{M}(Q_{j+1})) \\
Q_{j+2} := \text{gen}(\text{ker } \mathcal{M}(Q_{j+1})) \\
j := j + 2
\]

14
until \( r = d \)

Output: \( Q = \{q_1, \ldots, q_r\} \subseteq \mathbb{R}[x_1, \ldots, x_n] \)

\( Q \) is in geometric involutive form

\( \mathbb{R}\sqrt{\langle P \rangle} \supseteq \langle Q \rangle \supseteq \langle P \rangle \).

**Proof of the termination of Algorithm 6.1**: We prove termination of the \texttt{gif–M} Method under the assumption that suitable generic points, if available, are determined at each iteration of the method.

**Rank-Dim-Involutive Stopping Criterion**: A natural termination criterion used in Algorithm 6.1 is that the generators stabilize at some iteration and the system is involutive:

\[
\text{gen}(\text{gif}(Q)) = \text{gen}(\ker \text{M}(Q)) \quad \text{and} \quad Q \text{ involutive} \quad (40)
\]

Since different representations of the rings are involved we will focus on one, that of polynomial generators during the proof.

In terms of generators our termination criterion rank(\text{M}(Q_{j+1})) = \dim \ker \text{gif}(Q_j) is expressed as \( \text{gen}(\text{gif}(Q_j)) = \text{gen}(\ker \text{M}(Q_{j+1})) \).

Then \( \text{gen}(\ker \text{M}(Q_{j+1})) \) and \( \text{gen}(\text{gif}(Q_j)) \) are both ideals of the system \( P \). Since a generator of the geometric involutive form will also be a generator of the ideal in the moment matrix at each iteration we have \( \text{gen}(\text{gif}(Q_j)) \subseteq \text{gen}(\ker \text{M}(Q_{j+1})) \) in our algorithm. Suppose the algorithm never stops, then we will get a infinite ascending chain of ideals with a strict inclusion at each iteration of the form \( Q_j \subset Q_{j+1} \) where \( Q_j = \text{gen}(\text{gif}(Q_{j-1})) \) and \( Q_{j+1} = \text{gen}(\ker \text{M}(Q_j)) \). This is a violation of the ascending chain condition since \( \mathbb{R}[x_1, \ldots, x_n] \) is a Noetherian Ring. Therefore, the generators must stabilize in the end and when stabilized, \( Q \) is also involutive.

\( \square \)

The algorithm above uses the following subroutines.

**Algorithm 6.2 (\texttt{gif})**

Input: \( Q \subseteq \mathbb{R}[x_1, \ldots, x_n] \)

Output: Return a geometric involutive form \( \text{gif}(Q) \).

Note the algorithm \texttt{4.1} is an explicit implementation of \( \text{gif} \).

**Algorithm 6.3 (\texttt{M})**

Input: \( Q \subseteq \mathbb{R}[x_1, \ldots, x_n] \). Set \( d := \deg(Q) \).
1. Construct the general $N(n,d) \times N(n,d)$ moment matrix.
2. Construct the involutive prolongation $D^dQ$.
3. Use SDP methods to numerically solve for a generic point that maximizes the rank of the moment matrix subject to the constraints $D^dQ$.

Output: Return $M(Q) \succeq 0$ the moment matrix evaluated at this generic point.

Algorithm 6.4 (gen)

Input: $\text{gif}(Q)$ or $\text{ker}M(Q)$
Output: Polynomial generators corresponding to $\text{gif}(Q)$ or $\text{ker}M(Q)$

6.2 Two variable example

Consider the polynomial system with two variables $x$ and $y$.

$$P_2 = \{(y^2 - 1)^2, (y^2 - 1)(x^2 - 1)\}$$

First we apply $\text{gif}$ to $P_2$ to compute the involutive form of it. The dimension table is in Figure 2.

| $\ell = 0$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ |
|------------|---------|---------|---------|---------|
| $\ell = 0$ | 13      | 15      | 17      | 19      |
| $\ell = 1$ | 10      | 12      | 14      | 16      |
| $\ell = 2$ | 6       | 9       | 11      | 13      |
| $\ell = 3$ | 3       | 6       | 9       | 11      |
| $\ell = 4$ | 1       | 3       | 6       | 9       |

Figure 2: Table of $\dim \pi^kD^k(P_2)$ for system (41) The (blue) boxed 11 in the third column corresponds to $\pi^2D^2(P_2)$.

Now $\dim \pi^2D^2(P_2) = \dim \pi^3D^3(P_2)$ so $\pi^2D^2(P_2)$ satisfies one of the conditions for an involutive system. The second condition is that the symbol of $\pi^2D^2(P_2)$ is involutive. Applying the symbol test (29) and we find that $\dim S \pi^2D^3(P_2) = \dim S \pi^2D^2(P_2) = 2$, so the symbol of it turns out to be involutive as well. Therefore $\pi^2D^2(P_2)$ is involutive.

Now we apply the subroutine $M$ to $\text{gen}(\pi^2D^2(P_2))$ to compute the moment matrix $M$. We convert $\ker M$ into polynomial generators by subroutine $\text{gen}$. The dimension of $\ker M$ is 6 which means there are 6 generators in $\text{gen}(\ker M)$, which are moderately complicated numerical polynomials.

We again apply $\text{gif}$ to $\text{gen}(\ker M)$ to compute the involutive form. The dimension table is shown in Figure 3 The input system corresponding to the (red) boxed 9 is already involutive.
As mentioned in Remark 4.7 in algorithm 4.1 and more generally in gif algorithm, we can extract projected systems of lower degree than input system. This improves on our previous algorithm [30]. We demonstrate this procedure here. In Figure 3 the system corresponding to the red boxed 9 is involutive and has degree 4. Since \( N(2, 4) = 15 \) there are \( 15 - 9 = 6 \) polynomials in the system. However descending further down the column of the table, we find the system corresponding to the blue boxed 5 is also involutive. In that case \( N(2, 2) = 6 \) so there is only 1 corresponding generator.

![Figure 3: Table of dim \( \pi^\ell D^k \text{gen}(\ker M) \) for the first gif–M iteration in Example 6.2.](image)

We apply \( \text{gen} \) to compute the generator set:

\[
\{0.7071067 \ast y^2 - 0.7071067 + \text{small terms less than } 10^{-11}\}
\] (42)

If we apply \( \text{gif} \) to equation (42), the dimension table is exactly the same as the one in Figure 3. Therefore the projected system is equivalent to the input system. After normalization and ignoring small terms, we get \( y^2 - 1 \) which is a geometric involutive basis for the real radical for \( P_2 \).

### 6.3 Three variable example

In this section we apply the gif–M method to the following trivariate system with gif explicitly implemented by Algorithm 4.1:

\[
P_3 = \begin{cases} 
  x^2y^2 - y^4 + y^2z^2 - x^2 - z^2 + 1 \\
  x^2y^2 - y^4 + y^2z^2 + x^2 - 2y^2 + z^2 - 1 \\
  x^4z + x^2z^3 - 2x^2y^2 - x^2z - z^3 - 2x^2 + 2y^2 + 2 \\
  x^4z + x^2z^3 - 2x^2y^2 + x^2z + z^3 - 2x^2 - 2y^2 - 2
\end{cases}
\] (43)

We first apply subroutine \( \text{gif} \) to \( P_3 \). The dimension table is shown in Figure 4.

At prolongation zero of Algorithm 4.1 we determine if there are any projected involutive systems whose prolongations yield the same ideal as the system (so that the prolongations can be discarded). We find such an involutive system \( \pi^2D^0(P_3) \) which corresponds to the red boxed 15 in column 1 of Figure 4. From the dimension information we can deduce that since
the number of monomials of degree \( \leq 3 \) is \( N(3, 3) = 20 \) there will be \( 20 - 15 = 5 \) polynomials generators corresponding to \( \pi^2 D^0(P_3) \). System \( \pi^3 D^0(P_3) = \bar{R} \) is of lower degree and also easily found to be involutive. However it does not satisfy the inclusion test of Algorithm 4.1 given by \( D^{d+k-d} \bar{R} \subseteq D^k(P_3) \) which shows that it is not equivalent to the original system. We find that \( \pi^2 D^0(P_3) \) does satisfy the inclusion output condition, so we exit gif and apply subroutine \( \mathbb{M} \) to \( \text{gen}(\pi^2 D^0(P_3)) \). In our previously published method we would have first identified the blue boxed 27 corresponding to the involutive system \( \pi^2 D^2(P_3) \). Our approach is a clear improvement, and avoids creating the large degree 5 moment matrix of the previous approach.

We compute the generator set of the moment matrix \( M \) using the subroutine \( \text{gen(ker} M) \). The rank of moment matrix is 7 which means \( \text{gen(ker} M) \) has dimension 13. We apply gif to \( \text{gen(ker} M) \) and the dimension table is given in Figure 5.

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & 0 & 1 & 2 & 3 & 4 \\
\hline
\ell = 0 & 46 & 57 & 66 & 73 & 79 \\
\ell = 1 & 29 & 38 & 46 & 53 & 59 \\
\ell = 2 & 15 & 21 & 27 & 33 & 39 \\
\ell = 3 & 9 & 15 & 21 & 27 & 33 \\
\ell = 4 & 4 & 9 & 15 & 21 & 27 \\
\hline
\end{array}
\]

Figure 4: Table of dim \( \pi^d D^k(P_3) \) for system (43).

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & 0 & 1 & 2 & 3 & 4 \\
\hline
\ell = 0 & 7 & 9 & 11 & 13 \\
\ell = 1 & 5 & 7 & 9 & 11 \\
\ell = 2 & 3 & 5 & 7 & 9 \\
\ell = 3 & 1 & 3 & 5 & 7 \\
\hline
\end{array}
\]

Figure 5: Table of dim \( \pi^d D^k \text{gen(ker} M) \) in the moment matrix calculation for \( \text{gen}(\pi^2 D^0(P_3)) \).

In this iteration of gif three systems are involutive and correspond to the \( \ell = 0, 1, 2 \) entries of column 1 of Figure 5. Corresponding to the elimination of higher order systems by the inclusion test in Algorithm 4.1 we can discard 2 of the 3 systems, which correspond to \( \ell = 0 \) and \( \ell = 2 \) entries in the first column. The output lower degree geometric involutive basis therefore corresponds to the blue boxed entry in the figure.

At the next iteration the generators corresponding to \( \ell = 1 \) are sent to the moment matrix. We find that the termination condition is satisfied, that is \( d = 5 = r \). The algorithm then terminates with an output of \( 10 - 5 = 5 \) generators.

To get more insight into the output we now analyze it further. From the Figure 5 we see that there is a projected system corresponding to \( \ell = 2 \) of dimension 3 and degree 1. When
it is extracted we find a single nice generator:

\[ 0.8944271 - 0.4472135z + 5.5511151 \times 10^{-17} \times y \]  

(44)

After dropping off the small term and normalization, we get \( z - 2 \). Now we consider the other generators of degree 2. We can simplify them by substituting \( z = 2 \) from the projected generator and find

\[ -0.3015113 \times x^2 + 0.3015113 \times y^2 - 0.9045340 + \text{small terms less than } 10^{-15}. \]  

(45)

which is approximately \( x^2 - y^2 + 3 \). Thus our output geometric involutive basis is

\[ \{ z - 2, x(z - 2), y(z - 2), z(z - 2), x^2 - y^2 + 3 \} \]  

(46)

A hand calculation checks that this is a geometric involutive basis for the real radical of the input system.

7 Discussion

In this paper we present improvements of our numerical geometric involutive bases for polynomial systems of equations. We also began an exploration of the interaction of these methods with SPD programming methods and computation of such bases for positive dimensional real radical ideals.

We give methods to extract and decrease the degree of immediate systems and the output basis. One such tool is an inclusion test whereby higher degree redundant systems can be discarded. Prompted by a number of requests we have given more details of our implementation of Cartan’s involutivity test for positive dimensional ideals. Reduction of degree techniques are critical and have been extensively developed in the symbolic case for Gröbner bases \[12\] and triangular decompositions \[7, 8\]. Significant progress has also been made in symbolic-numeric methods such as border bases \[24, 25, 26, 27\] in removing higher degree polynomials. Perhaps the closest objects to geometric involutive bases in the zero dimensional case are H-Bases \[23\].

Moreover, we were motivated by remarkable recent work by Lasserre and collaborators \[1\] using SDP methods for identifying the real radical of zero dimensional polynomial ideals.

The work \[1\] motivated us to combine SDP − moment matrix methods with our geometric involutive bases to approximate positive dimensional real radical ideals. In particular, the termination criterion \( \text{rank}(M(Q)) = \dim \ker \text{gif}(Q) \) in Algorithm \[1.1\] is equivalent to the rank stabilization condition in Lasserre \[1\] for zero dimensional systems. Moreover in our initial explorative experiments we obtained generators for the real radical of positive dimensional ideals for a small set of examples and deserves further study.

In our preliminary study in order to study the interaction between these two methods we focused on an algorithm that cleanly separates the step of taking a geometric involutive basis
at each iteration of the algorithm. An alterative strategy that we will pursue in future work is motivated by the approach of Lasserre et al in the zero dimensional case [19]. Instead of demanding a (projected) involutive form at each iteration, they allowed the iteration and prolongation of moment matrices until the projected criteria for involution were obtained (that is a zero dimensional symbol in that case). This has the advantage that geometric involutive form calculations whose complexity implicitly depends on the total number of complex solutions are avoided until later, when such complex solutions have been discarded as a result of new generators being found in the kernel of the moment matrix.

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