On The UNRECOGNIZABILITY BY PRIME GRAPH FOR THE ALMOST SIMPLE GROUP PGL(2, 9)

Ali Mahmoudifar
Department of Mathematics, Tehran North Branch,
Islamic Azad University, Tehran, Iran
e-mail: alimahmoudifar@gmail.com

Abstract

The prime graph of a finite group $G$ is denoted by $\Gamma(G)$. Also $G$ is called recognizable by prime graph if and only if each finite group $H$ with $\Gamma(H) = \Gamma(G)$, is isomorphic to $G$. In this paper, we classify all finite groups with the same prime graph as PGL(2, 9). In particular, we present some solvable groups with the same prime graph as PGL(2, 9).

2000 AMS Subject Classification: 20D05, 20D60, 20D08.

Keywords: Projective general linear group, prime graph, recognition.

1 Introduction

Let $n$ be a natural number. We denote by $\pi(n)$, the set of all prime divisors of $n$. Also Let $G$ be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. The set of element orders of $G$ is denoted by $\pi_e(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The prime graph of $G$ is a graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (and we write $p \sim q$), whenever $G$ contains an element of order $pq$. The prime graph of $G$ is denoted by $\Gamma(G)$. A finite group $G$ is called recognizable by prime graph if for every finite group $H$ such that $\Gamma(G) = \Gamma(H)$, then we have $G \cong H$. So $G$ is recognizable by prime graph whenever there exists a finite group $K$ such that $\Gamma(K) = \Gamma(G)$ in while $K$ is not isomorphic to $G$.

For the almost simple group PGL(2, $q$), there are a lot of results about the recognition by prime graph. In [8], it is proved that if $p$ is a prime number which is not a Mersenne or Fermat prime and $p \neq 11, 19$ and $\Gamma(G) = \Gamma(PGL(2, p))$, then $G$ has a unique nonabelian composition factor which is isomorphic to PSL(2, $p$) and if $p = 13$, then $G$ has a unique nonabelian composition factor which is isomorphic to PSL(2, $13$) or PSL(2, $27$). We know that PGL(2, $2^\alpha$) $\cong$ PSL(2, $2^\alpha$). For the characterization of such simple groups we refer to [9, 10]. In [11], it is proved that if $q = p^\alpha$, where $p$ is an odd prime and $\alpha$ is an odd natural number, then PGL(2, $q$) is uniquely determined by its prime graph.

By the above description, we get that the characterization by prime graph of PGL(2, $p^k$), where $p$ is an odd prime number and $k$ is even, is an open problem. In this paper as the main result we consider the recognition by prime graph of the almost simple groups PGL(2, $3^2$). Moreover, we construct some solvable group with the same prime graph as PGL(2, $3^2$).
2 Preliminary Results

Lemma 2.1. Let $G$ be a finite group and $N \trianglelefteq G$ such that $G/N$ is a Frobenius group with kernel $F$ and cyclic complement $C$. If $(|F|, |N|) = 1$ and $F$ is not contained in $NC_G(N)/N$, then $p|C| = \pi_e(G)$ for some prime divisor $p$ of $|N|$.

Lemma 2.2. Let $G$ be a Frobenius group with kernel $F$ and complement $C$. Then the following assertions hold:

(a) $F$ is a nilpotent group.
(b) $|F| \equiv 1 \pmod{|C|}$.
(c) Every subgroup of $C$ of order $pq$, with $p$, $q$ (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of $C$ of odd order is cyclic and a Sylow 2-subgroup of $C$ is either cyclic or generalized quaternion group. If $C$ is a non-solvable group, then $C$ has a subgroup of index at most 2 isomorphic to $S_3 \times M_2$, where $M$ has cyclic Sylow $p$-subgroups and $(|M|, 30) = 1$.

By using [13, Theorem A] we have the following result:

Lemma 3.1. There exists a Frobenius group $C = K : C$, where $K$ is an abelian 3-group and $\pi(C) = \{2, 5\}$, such that $\Gamma(G) = \Gamma(\text{PGL}(2, 9))$.

Proof. Let $F$ be a finite field with $3^4$ elements. Also let $V$ be the additive group of $F$ and $H$ be the multiplicative group of $F \setminus \{0\}$. We know that $H$ acts on $V$ by right product. So $G := V \times H$ is a finite group such that $\pi(G) = \{2, 3, 5\}$, since $|V| = 3^4$ and $|H| = 80$. On the other hand $H$ acts fixed point freely on $V$, so $G$ is a Frobenius group with kernel $V$ and complement $H$. Since the multiplicative group $F \setminus \{0\}$ is cyclic, $H$ is cyclic too. Therefore, $G$ has an element of order 10, and so the prime graph of $G$ consists just one edge, which is the edge between 2 and 5. This implies that $\Gamma(G) = \Gamma(\text{PGL}(2, 9))$, as desired. □

Lemma 3.2. There exists a Frobenius group $G = K : C$, where $\pi(K) = \{2, 5\}$ and $C$ is a cyclic 3-group such that $\Gamma(G) = \Gamma(\text{PGL}(2, 9))$.

Proof. Let $F_1$ and $F_2$ be two fields with $2^2$ and $5^2$ elements, respectively. Let $V := F_1 \times F_2$ be the direct product of the additive groups $F_1$ and $F_2$. Also let $H := H_1 \times H_2$, be the direct product of $H_1$ and $H_2$, which are the multiplicative groups $F_1 \setminus \{0\}$ and $F_2 \setminus \{0\}$, respectively. We know that $H_i$ acts fixed point freely on $F_i$, where $1 \leq i \leq 2$. So we define an action of $H$ on $V$ as follows: for each $(h, h') \in H$ and $(g, g') \in V$, we define $(g, g')(h, h') := (hg, h'g')$. It is clear that this definition is well defined. So we may construct a finite group $G = V \times H$. On the other hand $H$ acts fixed point freely on $V$. So $G$ is a Frobenius group with kernel $V$ and complement $H$ such that $\pi(V) = \{2, 5\}$ and $\pi(H) = \{3\}$. Finally, since $V$ is nilpotent, we get that $\Gamma(G)$ contains an edge between 2 and 5, so $\Gamma(G) = \Gamma(\text{PGL}(2, 9))$. □

Lemma 3.3. There exists a 2-Frobenius group $G$ with normal series $1 \trianglel H \trianglel K \trianglel G$, such that $\pi(H) = \{3\}$, $\pi(G/K) = \{2\}$ and $\pi(K/H) = \{3\}$, such that $\Gamma(G) = \Gamma(\text{PGL}(2, 9))$.  

2
Proof. Let $F$ be a field with $5^2$ elements and $V$ be its additive group. We know that $F = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{23}\}$, where $\alpha$ is a generator of the multiplicative group $F \setminus \{0\}$. Hence $|\alpha| = 24$, and so $\beta := \alpha^8$ has order 3. Also $\langle \beta \rangle \cong Z_3$ and $\text{Aut}(Z_3) \cong Z_2$. This argument implies that we may construct a Frobenius group $T := \langle \beta \rangle : \langle \gamma \rangle$, where $\gamma$ is an involution.

Now we define an action of $T$ on $V$ as follows: for each $\beta^x\gamma^y \in T$ and $v \in V$, $v^{\beta^x\gamma^y} := \beta^x v$, where $1 \leq x \leq 3$ and $1 \leq y \leq 2$. Therefore $G := V : T$ is a 2 Frobenius group with desired properties. \qed

Theorem 3.4. Let $G$ be a finite group. Then $\Gamma(G) = \Gamma(\text{PGL}(2,3^2))$ if and only if $G$ is isomorphic to one of the following groups:

1. A Frobenius group $K : C$, where $K$ is an abelian 3-group and $\pi(C) = \{2, 5\}$,
2. A Frobenius group $K : C$, where $\pi(K) = \{2, 5\}$ and $C$ is a cyclic 3-group,
3. A 2-Frobenius group with normal series $1 \triangleleft H \triangleleft K \triangleleft G$, such that $\pi(H) \subseteq \{2, 5\}$, $\pi(G/K) = \{2\}$ and $\pi(K/H) = \{3\}$,
4. Almost simple group $\text{PGL}(2,3^2)$.

Proof. Throughout the proof, we assume that $G$ is a finite group with the same prime graph as the almost simple group $\text{PGL}(2,3^2)$. First we note that by [16, Lemma 7], we have:

$$\mu(\text{PGL}(2,9)) = \{3, 8, 10\}.$$

Hence in $\Gamma(\text{PGL}(2,3^2))$ (and so in $\Gamma(G)$), there is only one edge which is the edge between 2 and 5 and so 3 is an isolated vertex. This implies that $\Gamma(G)$ has two connected components $\{3\}$ and $\{2, 5\}$.

Thus by Lemma 2.3 we get that $G$ is a Frobenius group or 2-Frobenius group or there exists a nonabelian simple group $S$ such that $S \leq G/\text{Fit}(G) \leq \text{Aut}(S)$. We consider each possibility for $G$.

Let $G$ be a Frobenius group with kernel $K$ and complement $C$. We know that $K$ is nilpotent and $C$ is a connected component of the prime graph of $G$. Also by the above description, 3 is not adjacent to 2 and 5 in $\Gamma(G)$.

This shows that either $\pi(K) = \{3\}$ or $\pi(C) = \{3\}$. We consider these cases, separately.

Case 1. Let $\pi(K) = \{3\}$. Hence the order of complement $C$, is even and so $K$ is an abelian subgroup of $G$. Also by the above description, $\pi(C) = \{2, 5\}$. Since $C$ is a connected component of $\Gamma(G)$, there is an edge between 2 and 5.

So it follows that $\Gamma(G) = \Gamma(\text{PGL}(2,9))$, which implies groups satisfying in (1).

Case 2. Let $\pi(C) = \{3\}$. Hence $\pi(K) = \{2, 5\}$. Since $K$ is nilpotent, we get that 2 and 5 are adjacent in $\Gamma(G)$. This means $\Gamma(G) = \Gamma(\text{PGL}(2,9))$ and so we get (2).

Case 3. Let $G$ be a 2-Frobenius group with normal series $1 \triangleleft H \triangleleft K \triangleleft G$. Since $\pi(K/H)$ and $\pi(H) \cup \pi(G/K)$ are the connected components of $\Gamma(G)$, we get that $\pi(K/H) = \{3\}$ and $\pi(H) \cup \pi(G/K) = \{2, 5\}$. This implies (3).

Case 4. Let there exist a nonabelian simple group $S$, such that $S \leq \bar{G} := G/\text{Fit}(G) \leq \text{Aut}(S)$. Since $\pi(S) \subseteq \pi(G)$, $\pi(S) = \{2, 3, 5\}$. The finite simple groups with this property are classified in [12, Table 8]. So we get that $S$ is isomorphic to one of the simple groups $A_5$, $\text{PSU}(4, 2)$ and $\text{PSL}(2,9)(\cong A_6)$.

Subcase 4.1. Let $S \cong A_5$. We know that $\text{Aut}(A_5) = S_5$. So $\bar{G}$ is isomorphic to the alternating group $A_5$ or the symmetric group $S_5$. Since in the prime graph of $S_5$, there is an edge between 2 and 3, hence we get that $\bar{G}$ is not isomorphic to $S_5$. Thus, $G/\text{Fit}(G) = A_5$. In the prime graph of $A_5$, 2 and 5 are nonadjacent. Then at least one of the prime numbers 2 or 5, belongs to $\pi(\text{Fit}(G))$. 

3
Let $5 \in \pi(\text{Fit}(G))$. Let $F_5$ be a Sylow 5-subgroup of $\text{Fit}(G)$. Since $F_5$ is a characteristic subgroup of $\text{Fit}(G)$ and $\text{Fit}(G)$ is a normal subgroup of $G$, $F_5 \trianglelefteq G$. On the other hand in alternating group $A_5$, the subgroup $\langle (12)(34), (13)(24) \rangle : \langle (123) \rangle$ is a Frobenius subgroup isomorphic to $2^2 : 3$. We recall that by the previous argument, $F_5 \trianglelefteq \text{Fit}(G)$, and so $G$ has a subgroup isomorphic to $5^a : (2^2 : 3)$. So by Lemma 2.1 we get that $3$ is adjacent to $5$, a contradiction.

**Subcase 4.2.** Let $S \cong \text{PSU}(4, 2)$. By [4], there is an edge between $3$ and $2$ which is a contradiction.

**Subcase 4.3.** Let $S \cong \text{PSL}(2, 9)$. Then $\bar{G}$ is isomorphic to $\text{PSL}(2, 9)$ or $\text{PSL}(2, 9) : \langle \theta \rangle$, where $\theta$ is a diagonal, field or diagonal-field automorphism of $\text{PSL}(2, 9)$. In particular $\theta$ is an involution. If $\theta$ is a field or diagonal-field automorphism of $\text{PSL}(2, 9)$, then the semidirect product $\text{PSL}(2, 9) : \langle \theta \rangle$ contains an element of order $6$. Hence $\theta$ is neither a field automorphism nor a diagonal-field automorphism. Therefore $\theta$ is a diagonal automorphism and so $\bar{G} \cong \text{PGL}(2, 9)$.

By the above discussion, $G/\text{Fit}(G) \cong \text{PGL}(2, 9)$. It is enough to prove that $\text{Fit}(G) = 1$. On the contrary, let $r \in \pi(\text{Fit}(G))$. Also let $F_r$ be the Sylow $r$-subgroup of $\text{Fit}(G)$. Since $\text{Fit}(G)$ is nilpotent, we can write $\text{Fit}(G) = O_{r'}(\text{Fit}(G)) \times F_r$. So if we put $\tilde{G} = G/O_{r'}(\text{Fit}(G))$, then we get that:

$$\text{PGL}(2, 9) \cong \frac{G}{\text{Fit}(G)} \cong \frac{\tilde{G}}{F_r} \cong \frac{\tilde{G}/\Phi(F_r)}{F_r/\Phi(F_r)}.$$  

Since $F_r/\Phi(F_r)$ is an elementary abelian group, without loose of generality we may assume that $F := \text{Fit}(G)$ is an elementary abelian $r$-group and $G/F \cong \text{PGL}(2, 9)$.

If $r = 3$, then by , we conclude that in $\Gamma(G)$, $2$ and $3$ are adjacent, which is a contradiction. So let $r \neq 3$. Also let $S_3$ be a Sylow $3$-subgroup of $\text{PGL}(2, 9)$. We know that $S_3$ is not cyclic. On the other hand $F \rtimes S_3$ is a Frobenius group since $3$ is an isolated vertex of $\Gamma(G)$. This follows that $S_3$ is cyclic which is impossible. Therefore $F = 1$ and so $G \cong \text{PGL}(2, 9)$, which completes the proof.

\[\square\]

**References**

[1] Z. Akhlaghi, M. Khatami and B. Khosravi, Characterization by prime graph of PGL(2, $p^k$) where $p$ and $k$ are odd, International Journal of Algebra and Computation 20 (7) (2010) 847-873.

[2] S. Shariati Beynekalae, A. Iranmanesh and M. Foroudi Ghasemabadi, Quasirecognition by prime graph of the simple group, Southeast Asian Bull. Math., Bn(2), 39 (2015) 181–193.

[3] L. Wang, Quasirecognition of $L_{22}(2)$ by Its prime graph, Southeast Asian Bull. Math., 35 (2015) 883–889.

[4] A. A. Buturlakin, Spectra of finite linear and unitary groups, Algebra and Logic 47(2) (2008) 91–99.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups (Oxford University Press, Oxford, 1985).

[6] K. W. Gruenberg and K. W. Roggenkamp, Decomposition of the augmentation ideal and of the relation modules of a finite group, Proc. London Math. Soc. 31(2) (1975) 149–166.

[7] M. Hagie, The prime graph of a sporadic simple group, Comm. Algebra 31(9) (2003) 4405–4424.
[8] M. Khatami, B. Khosravi and Z. Akhlaghi, NCF-distinguishability by prime graph of $\text{PGL}(2,p)$, where $p$ is a prime, Rocky Mountain J. Math., to appear.

[9] B. Khosravi, n-Recognition by prime graph of the simple group $\text{PSL}(2,q)$, J. Algebra Appl. 7(6) (2008) 735–748.

[10] B. Khosravi, B. Khosravi and B. Khosravi, 2-Recognizability of $\text{PSL}(2,p^2)$ by the prime graph, Siberian Math. J. 49(4) (2008) 749.757.

[11] B. Khosravi, B. Khosravi and B. Khosravi, On the prime graph of $\text{PSL}(2,p)$ where $p > 3$ is a prime number, Acta. Math. Hungarica 116(4) (2007) 295–307.

[12] R. Kogani-Moghadam and A. R. Moghaddamfar, Groups with the same order and degree pattern, Sci. China Math., 55 (4) (2012), 701–720.

[13] A. S. Kondrat’ev, Prime graph components of finite simple groups, Math. USSR-SB. 67(1) (1990) 235–247.

[14] V. D. Mazurov, Characterizations of groups by arithmetic properties, Proceedings of the International Conference on Algebra, Algebra Colloq. 11(1) (2004) 129–140.

[15] V. D. Mazurov, Characterizations of finite groups by sets of their element orders, Algebra Logic 36(1) (1997) 23–32.

[16] A. R. Moghaddamfar and W. J. Shi, The number of finite groups whose element orders is given., Beitrage Algebra Geom., 47(2) (2006) 463–479.

[17] J. S. Williams, Prime graph components of finite groups, J. Algebra 69(2) (1981) 487–513.

[18] A. V. Zavarnitsin, Recognition of finite groups by the prime graph, Algebra Logic 43(4) (2006) 220–231.

[19] K. Zsigmondy, Zur theorie der potenzreste, Monatsh. Math. Phys. 3 (1892) 265–284.