ELLIPTIC GENUS AND MODULAR DIFFERENTIAL EQUATIONS

DMITRII ADLER AND VALERY GRITSENKO

Abstract. We study modular differential equations for the basic weak Jacobi forms in one abelian variable with applications to the elliptic genus of Calabi–Yau varieties. We show that the elliptic genus of any CY$_3$ satisfies a differential equation of degree one with respect to the heat operator. For a K3 surface or any CY$_5$ the degree of the differential equation is 3. We prove that for a general CY$_4$ its elliptic genus satisfies a modular differential equation of degree 5. We give examples of differential equations of degree two with respect to the heat operator similar to the Kaneko–Zagier equation for modular forms in one variable. We find modular differential equations of Kaneko–Zagier type of degree 2 or 3 for the second, third and fourth powers of the Jacobi theta-series.

1. Elliptic genus and Jacobi modular forms

1.1. Elliptic genus of complex varieties with \( c_1 = 0 \). Jacobi modular forms appear in geometry and physics as special partition functions. For example, the elliptic genus of any complex compact variety of dimension \( d \) with trivial first Chern class is a weak Jacobi form of weight 0 and index \( d/2 \) with integral Fourier coefficients (see [18] in the context of \( N = 2 \) superconformal field theory, [8] in the context of Jacobi forms, [22] in the context of elliptic homology).

Differential equations for modular forms in one variable \( \tau \in \mathbb{H} \) were considered in the first half of the 20th century by Ramanujan and Rankin (see [23]). In this paper we find differential equations for the elliptic genera of Calabi–Yau varieties of small dimensions. For a Jacobi form of index \( d/2 \) one has to consider the heat operator \( H^{(d)} = 4\pi i d \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \) instead of the differentiation \( \frac{d}{d\tau} \) where \( \tau \in \mathbb{H} \) and \( z \in \mathbb{C} \). We prove that the elliptic genus of any CY$_3$ satisfies a differential equation of degree one with respect to the heat operator. We find a modular differential equation of degree 3 in \( H^{(1)} \) for a K3 surface and in \( H^{(\frac{5}{2})} \) for any CY$_5$. We prove that for a general CY$_4$ there exists a modular differential equation of degree 5 with respect to the heat operator.

We note that modular differential equations are important in the description of the characters of a conformal field theory. For \( N = 2 \) superconformal
field theories elliptic genera are a natural generalization of the chiral characters (see \[5\]). Moreover elliptic genera can be computed geometrically in terms of Gromov–Witten invariants in dual string compactifications (see recent papers \[21\], \[14\]). The target ring for the elliptic genus of Calabi–Yau varieties is the graded ring of the weak Jacobi forms $J_{0,+2}$ of weight 0 and half-integral index with integral Fourier coefficients. In this paper we find modular differential equations for all generators of $J_{0,+2}$. We find also modular differential equations for the square, the cube, and the fourth power of the Jacobi theta-series $\vartheta(\tau, z)$. In \[1\], we found modular differential equations for generators of the rings of the weak Jacobi forms for the root lattices $D_2 < D_3 < \ldots < D_8$. In these terms the subject of this paper is Jacobi forms for the simplest root lattice $A_1$.

Let $M = M_d$ be an (almost) complex compact manifold of (complex) dimension $d$ and $T_X$ be its tangent bundle. Let $\tau \in \mathbb{H}$ be a variable in the upper half-plane and $z \in \mathbb{C}$. We set $q = \exp(2\pi i \tau)$ and $\zeta = \exp(2\pi iz)$. We define the formal series

$$E_{q, \zeta} = \bigotimes_{n=0}^{\infty} (-\zeta q^n) T_M^k \otimes \bigotimes_{n=1}^{\infty} (-\zeta q^n) T_M \otimes \bigotimes_{n=1}^{\infty} S_q^n T_M \otimes \bigotimes_{n=1}^{\infty} S_{\zeta}^n T_M,$$

where $\wedge^k$ is the $k^{th}$ exterior power, $S^k$ is the $k^{th}$ symmetric product, and

$$\bigotimes_x E = \sum_{k \geq 0} (\wedge^k E) x^k, \quad S_x E = \sum_{k \geq 0} (S^k E) x^k.$$

The elliptic genus (see \[18\], \[9\]) of $M$ is a function of two variables $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$

$$\chi(M; \tau, z) = \zeta^{d/2} \int_M \text{ch}(E_{q, \zeta}) \text{td}(T_M),$$

where $\text{td}$ is the Todd class, $\text{ch}(E_{q, \zeta})$ is the Chern character applied to each coefficient of the formal power series, and $\int_M$ denotes the evaluation of the top degree differential form on the fundamental cycle of the manifold. The coefficient $a(n, l)$ of the elliptic genus

$$\chi(M; \tau, z) = \sum_{n\geq 0, l \in \mathbb{Z}} a(n, l) q^n \zeta^l$$

is equal to the index of the Dirac operator twisted with the vector bundle $E_{n,l}$ where $E_{n,l} = \bigoplus_{q \in \mathbb{Z}} q^n \zeta^l$. The $q^0$-term of the elliptic genus of $M$ is equal to the Hirzebruch $\chi_y$-genus (with a small renormalization)

$$\chi(M_d; \tau, z) = \sum_{p=0}^{d} (-1)^p \chi_p(M) \zeta^{d/2-p} + q(\ldots)$$

where $\chi_p(M_d) = \sum_{q=0}^{d} (-1)^q h^{p,q}(M_d)$. 

Let $M = M_d$ be an (almost) complex compact manifold of (complex) dimension $d$ and $T_X$ be its tangent bundle. Let $\tau \in \mathbb{H}$ be a variable in the upper half-plane and $z \in \mathbb{C}$. We set $q = \exp(2\pi i \tau)$ and $\zeta = \exp(2\pi iz)$. We define the formal series

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Theorem 1.1. (See \cite{18}, \cite{8}.) If $M_d$ is a compact complex manifold of dimension $d$ with $c_1(M) = 0$ (over $\mathbb{R}$), then its elliptic genus $\chi(M_d; \tau, z)$ is a weak Jacobi form of weight 0 and index $\frac{d}{2}$ with integral Fourier coefficients.

One can define $\chi(M; \tau, z)$ for any compact complex variety. It has nice automorphic property only if $c_1(M) = 0$. If $c_1(M) \neq 0$ then one can define an appropriate quasi-modular correction of $\chi(M; \tau, z)$ in order to obtain a generalized elliptic genus $\chi(M, E; \tau, z)$ for any vector bundle over $M$ (see \cite{9} and \cite{10}). This function is a meromorphic function of Jacobi type. In particular, one gets in this way the elliptic genera of (0, 2)-sigma models.

Theorem 1.1 means that $\chi(M_d; \tau, z)$ satisfies two functional equations with respect to the modular group $SL_2(\mathbb{Z})$ and the Heisenberg group $H(\mathbb{Z})$. We note that the definition of $\chi(M_d; \tau, z)$ is a geometric realization of the Jacobi triple product formula for the anti-invariant Jacobi theta-series of characteristic two

$$\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1} (1 - q^n\zeta)(1 - q^n\zeta^{-1})(1 - q^n)$$

(1.2)

$$= q^{i\pi \frac{k}{2}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(n+1)}{2}} \zeta^n.$$

We note that $\vartheta(\tau, -z) = -\vartheta(\tau, z)$ and $\vartheta(\tau, z) = -i\vartheta_{11}(z, \tau)$ in the notation of \cite{20} Chapter 1. The theta-series $\vartheta(\tau, z)$ is a Jacobi modular form of index $\frac{1}{2}$ and it is one of the main instruments of our study.

1.2. Definition of Jacobi forms. The natural model of the Jacobi modular group $\Gamma^J(\mathbb{Z})$ is the quotient $\Gamma_\infty/\{ \pm I \}$ of the integral maximal parabolic subgroup of the Siegel modular group of genus 2 fixing an isotropic line. We refer to \cite{12} §1 and \cite{3} for this point of view and for the definition below. The Jacobi group $\Gamma^J(\mathbb{Z})$ is the semidirect product of $SL_2(\mathbb{Z})$ with the integral Heisenberg group $H(\mathbb{Z})$ which is the central extension of $\mathbb{Z} \times \mathbb{Z}$. ($H(\mathbb{Z})$ is the unipotent subgroup of $\Gamma_\infty(\mathbb{Z})$.) One can define a binary character of $H(\mathbb{Z})$

$$v_H([x, y; r]) = (-1)^{x+y+xy+r}.$$

Definition 1.2. Let $k \in \mathbb{Z}$ and $m \in \frac{1}{2} \mathbb{N}$. A holomorphic function

$\varphi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ is called a weak Jacobi form of weight $k$ and index $m$ if it satisfies

$$\varphi \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = (c \tau + d)^k e^{2\pi im \frac{z^2}{c \tau + d}} \varphi(\tau, z),$$

$$\varphi(\tau, z + x \tau + y) = v_H([x, y; 0])^{2m} e^{-2\pi im(x^2 \tau + 2xz)} \varphi(\tau, z),$$

for any $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z})$ and $x, y \in \mathbb{Z}$, and has a Fourier expansion of the form

(1.3)

$$\varphi(\tau, z) = \sum_{n \geq 0, \, l \in \frac{1}{2} \mathbb{Z}} a(n, l) \exp(2\pi i(n\tau +lz)).$$
where \( l \) is half-integral for half-integral index \( m \). If \( \varphi \) satisfies the additional condition \( a(n,l) = 0 \) for \( 4nm - l^2 < 0 \) then it is called a holomorphic (at infinity) Jacobi form. Moreover, if \( \varphi \) satisfies the stronger condition \( f(n,l) = 0 \) for \( 4nt - l^2 \leq 0 \) then it is called a Jacobi cusp form.

In this paper we denote by \( J_{k,m} \) the vector space of weak Jacobi forms of weight \( k \) and index \( m \). It is easy to see that \( J_{k,0} = M_k(SL_2(\mathbb{Z})) \) is the space of classical modular forms with respect to the modular group.

We note that the Jacobi theta-series \( \vartheta(\tau,z) \) introduced above is a holomorphic Jacobi form of weight \( \frac{1}{2} \) and index \( \frac{1}{2} \) in \( J_{1,\frac{1}{2}}(v_0^3 \times v_H) \) where \( v_0 \) is the multiplier system of the Dedekind eta-function

\[
(1.4) \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \in M_\frac{1}{2}(SL_2(\mathbb{Z}), v_0).
\]

According to [4, Theorem 9.3] the bigraded ring of the weak Jacobi forms of even weight and integral index is a polynomial ring with four generators

\[
(1.5) \quad J_{2*,*} = \bigoplus_{k \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}} J_{2k,m} = \mathbb{C}[E_4, E_6, \varphi_{-2,1}, \varphi_{0,1}],
\]

where \( E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n \) and \( E_6(\tau) = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n \) are the classical Eisenstein series of weight 4 and 6, and

\[
(1.6) \quad \varphi_{-2,1} = \frac{\partial^2(\varphi, z)^2}{\eta(\tau)^6} = \zeta^{\pm 1} - 2 + q(-2\zeta^{\pm 2} + 8\zeta^{\pm 1} - 12) + O(q^2),
\]

\[
(1.7) \quad \varphi_{0,1} = -\frac{3}{\pi^2} \varphi(\tau,z)\varphi_{-2,1} = \zeta^{\pm 1} + 10 + q(10\zeta^{\pm 2} - 64\zeta^{\pm 1} + 108) + O(q^2),
\]

where \( \varphi(\tau,z) \) is the Weierstrass \( \wp \)-function. We note that

\[
\frac{\partial \varphi(\tau,z)}{\partial z} \bigg|_{z=0} = 2\pi i \eta(\tau)^3, \quad \frac{\partial^2 \log \varphi(\tau,z)}{\partial z^2} = -\varphi(\tau,z) + 8\pi^2 G_2(\tau)
\]

where \( G_2(\tau) = -\frac{1}{\tau^2} + \sum_{n \geq 1} \sigma_1(n)q^n \) is the quasi-modular Eisenstein series.

For Jacobi forms of half-integral weights and indices and for the interpretation of \( \vartheta(\tau,z) \) as a Jacobi modular form see [12] and [8]. (See also [7] and [3] for general case of Jacobi forms for orthogonal groups.) We have

\[
(1.8) \quad J_{-1,\frac{1}{2}} = \mathbb{C}\varphi_{-1,\frac{1}{2}}, \quad J_{-1,2} = \mathbb{C}\varphi_{-1,2}, \quad J_{0,\frac{3}{2}} = \mathbb{C}\varphi_{0,\frac{3}{2}}
\]

where

\[
(1.9) \quad \varphi_{-1,\frac{1}{2}}(\tau,z) = \frac{\vartheta(\tau,z)}{\eta^3(\tau)}, \quad \varphi_{-1,2}(\tau,z) = \frac{\vartheta(\tau,2z)}{\eta^3(\tau)}, \quad \varphi_{0,\frac{3}{2}}(\tau,z) = \frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)}.
\]

Moreover the following relations are true for any \( k \in \mathbb{Z} \) and \( m \in \mathbb{N} \)

\[
(1.10) \quad J_{2k+1,m} = \varphi_{-1,2} \cdot J_{2k+2,m-2}, \quad J_{2k+1,m+\frac{1}{2}} = \varphi_{-1,\frac{1}{2}} \cdot J_{2k+2,m},
\]

\[
(1.11) \quad J_{2k,m+\frac{1}{2}} = \varphi_{0,\frac{1}{2}} \cdot J_{2k,m-1}.
\]
Since the functions $\varphi_{-1,2}$, $\varphi_{0,\frac{1}{2}}$ and $\varphi_{-1,\frac{1}{2}}$ have infinite product expansions, the relations (1.10) and (1.11) hold for the corresponding spaces of Jacobi forms with integral Fourier coefficients.

According to Theorem 1.1 the target ring for the elliptic genus is the graded ring $J_{0,\ast} \cong \oplus_{m \in \mathbb{N}} J_{0, m}$. To describe its structure we have to introduce three additional Jacobi forms of weight 0 and index 2, 3 and 4 with integral Fourier coefficients.

(1.12)

\[
\begin{align*}
\varphi_{0,2}(\tau, z) &= \frac{\varphi_{0,1}^2 - E_4 \varphi_{2,1}^2}{24} = \zeta^{\pm 1} + 4 + \eta(\zeta^3 - 8\zeta^2 - \zeta^{\pm 1} + 16) + \ldots, \\
\varphi_{0,3}(\tau, z) &= \varphi_{0,\frac{3}{2}}(\tau, z)^2 = \zeta^{\pm 1} + 2 + 2q(\zeta^3 - \zeta^2 + \zeta^{\pm 1}) + \ldots, \\
\varphi_{0,4}(\tau, z) &= \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)} = \zeta^{\pm 1} + 1 - q(\zeta^4 + \zeta^3 - \zeta^{\pm 1}) + \ldots.
\end{align*}
\]

**Theorem 1.3.** (See [3].) 1. $J_{0,\ast/2}^Z = J_{0,\ast}^Z[\varphi_{0,\frac{3}{2}}]$. 2. The graded ring $J_{0,\ast}^Z$ of the weak Jacobi forms of weight 0 and integral index with integral Fourier coefficients is finitely generated over $\mathbb{Z}$. More precisely,

\[
J_{0,\ast}^Z = \mathbb{Z}[\varphi_{0,1}, \varphi_{0,2}, \varphi_{0,3}, \varphi_{0,4}].
\]

The functions $\varphi_{0,1}$, $\varphi_{0,2}$, $\varphi_{0,3}$ are algebraically independent over $\mathbb{C}$ and $4\varphi_{0,4} = \varphi_{0,1}\varphi_{0,3} = \varphi_{0,2}^2$.

2. **Modular differential equations for the elliptic genus of Calabi–Yau varieties of dimension 2, 3 and 5**

Modular differential operators appear naturally in the geometric and physics context of elliptic genera (see [5], [6], [21]).

2.1. **The modular differential operator $D_k$.** Let

\[
D = \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{d\tau}.
\]

For any $f(\tau) = \sum_{n \geq 0, a \in \mathbb{Z}} a(n)q^n$ we have $D(f) = \sum_{n \geq 0, a \in \mathbb{Z}} na(n)q^n$. Moreover, for any meromorphic automorphic form $f(\tau)$ of weight 0 we have

\[
D(f) \in M_{2}(\text{mer})(SL_2(\mathbb{Z})).
\]

It gives the following modular differential operator

\[
D_k : M_k(SL_2(\mathbb{Z})) \to M_{k+2}(SL_2(\mathbb{Z})),
\]

\[
D_k(f) = \eta^{2k} D(\frac{f}{\eta^{2k}}) = D(f) - 2k \frac{D(\eta)}{\eta} f = D(f) - \frac{k}{12} E_2 \cdot f
\]

where

\[
\frac{D(\eta)}{\eta} = -G_2(\tau) = \frac{1}{24} - \sum_{n \geq 1} \sigma_1(n)q^n = \frac{1}{24} E_2(\tau)
\]

is the quasi-modular Eisenstein series of weight 2.
Directly from the definition of $D_k$ we obtain

\begin{equation}
D_4(\eta^k) = 0, \quad D_4(E_4) = -\frac{1}{3}E_6, \quad D_6(E_6) = -\frac{1}{2}E_4^2.
\end{equation}

In particular, $D_{12}(\Delta(\tau)) = 0$ where $\Delta(\tau) = \eta^{24}(\tau)$ is the Ramanujan cusp form of weight 12.

### 2.2. The heat operator and its modular correction.

The analogue of the derivation $\frac{d}{d\tau}$ in the case of Jacobi modular forms of index $m$ is the heat operator

\begin{equation}
H^m = \frac{3}{m(2\pi i)^2} \left( 8\pi im \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right) = 12\frac{d}{dq} - \frac{3}{m} \left( \zeta \frac{d}{d\zeta} \right)^2.
\end{equation}

We have

\[
H^m(q^n \zeta^l) = \frac{3}{m} (4nm - l^2)q^n \zeta^l,
\]

where $4nm - l^2$ is the hyperbolic norm of the index $(n, l)$ of the Fourier coefficient $a(n, l)$ (see [1.3]). This normalization of the heat operator with operator $D$ multiplied by 12 is useful in the context of the differential equations. We firstly used it in [1]. To simplify the notation we write $H = H^m$ in all cases when the exact index of Jacobi modular forms is clear.

Any holomorphic Jacobi forms of weight $\frac{1}{2}$ (this is the so-called singular weight for Jacobi modular forms of Eichler–Zagier type) lies in the kernel of the heat operator. In particular, $H^{\frac{1}{2}}(\vartheta(\tau, z)) = 0$. See [7] where the general case of Jacobi forms of orthogonal type was considered. We know that the heat operator $H^m$ transforms a (meromorphic) Jacobi form of the singular weight (i.e. weight $\frac{1}{2}$ in our case) and index $m$ into a Jacobi form of the weight $\frac{1}{2} + 2$ and the same index (see [11, Lecture 11] for more details). Similarly to the operator $D_k$ above one can define the modular differential operator for Jacobi forms

\begin{equation}
H_k : J_{k,m} \to J_{k+2,m}, \quad H_k(\varphi_{k,m}) = H(\varphi_{k,m}) - \frac{(2k-1)}{2}E_2 \cdot \varphi_{k,m}
\end{equation}

with the quasi-modular Eisenstein series $E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$. This differential operator transforms weak (holomorphic or cusp) Jacobi forms into Jacobi forms of the same type.

### 2.3. The elliptic genus of varieties of dimension 2, 3 and 5.

There are three cases when the space $J_{0,m}$ is one-dimensional. They are $m = 1, \frac{3}{2}$ and $\frac{5}{2}$. According to Theorem [11] and [11] the elliptic genus of a Calabi–Yau variety of dimension 2, 3 or 5 depends only on its Euler characteristic.

A Calabi–Yau variety of dimension 2 is a $K3$ surface. Since $e(K3) = 24$, we have

\[
\chi(K3; \tau, z) = 2\varphi_{0,1}(\tau, z) = 2\zeta + 20 + 2\zeta^{-1} + q(\ldots)
\]

where $\varphi_{0,1} \in J_{0,1}$ is one of the two generators of the bigraded ring of weak Jacobi forms defined in [17]. We note that the Jacobi form $\varphi_{0,1}$ is the
elliptic genus of an Enriques surface. For Calabi–Yau varieties of dimension 3 and 5 we have
\[
\chi(CY_3; \tau, z) = \frac{e(CY_3)}{2} \varphi_{0, \frac{3}{2}} = \frac{e(CY_3)}{2} \left( \zeta^{-\frac{1}{2}} + \zeta^{\frac{1}{2}} + q(\ldots) \right),
\]
\[
\chi(CY_5; \tau, z) = \frac{e(CY_5)}{24} \varphi_{0, \frac{5}{2}} \cdot \varphi_{0, 1} = \frac{e(CY_5)}{24} \left( \zeta^{\frac{5}{2}} + 11\zeta^{\frac{1}{2}} + q(\ldots) \right)
\]
where \(e(M_d)\) is the Euler number of \(M_d\). The last identity shows that the Euler number of a Calabi–Yau variety of dimension 5 is divisible by 24 (see [8]).

**Theorem 2.1.** Let \(M_3\) be a Calabi–Yau variety of dimension 3 with \(e(M_3) \neq 0\). Its elliptic genus \(\chi(M_d; \tau, z)\) satisfies a modular differential equation of degree 1 with respect to the heat operator \(H\). More exactly
\[
H(\varphi_{0, \frac{3}{2}}) + \frac{1}{2} E_2 \cdot \varphi_{0, \frac{3}{2}} = 0.
\]

**Proof.** We note that \(J_{2, \frac{3}{2}} = M_2(SL_2(\mathbb{Z}))\varphi_{0, \frac{3}{2}} = \{0\}\). Therefore (2.7) follows directly from the equality \(H_0(\phi_{0, \frac{3}{2}}) = 0\). \(\square\)

**Remark.** We can give another explanation of (2.7). According to the quintuple product formula (see [12] Lemma 1.6), we have that
\[
\eta(\tau) \varphi_{0, \frac{3}{2}}(\tau, z) = \eta(\tau) \frac{\vartheta(2z)}{\vartheta(z)} = \sum_{n \in \mathbb{Z}} \left( \frac{n}{12} \right) q^{n^2/24} z^{n/2} \in J_{2, \frac{3}{2}}^{hol}(\mathfrak{h}_\eta \cdot \mathfrak{v}_H),
\]
where \(\left( \frac{n}{12} \right) = \pm 1\) or 0 is the quadratic Legendre symbol, is a holomorphic Jacobi form of singular weight \(\frac{1}{2}\). Therefore
\[
H(\eta(\tau) \varphi_{0, \frac{3}{2}}(\tau, z)) = 0.
\]
Using the relation \(\frac{12D(n)}{\eta} = \frac{1}{2} E_2(\tau)\) we rewrite this identity as (2.7).

We formalize computation used in the second proof of (2.7) above.

**Proposition 2.2.** Let \(f_{k_1}(\tau) \in M_{k_1}(SL_2(\mathbb{Z}))\) and \(\varphi_{k_2,m}(\tau, z) \in J_{k_2,m}\) where \(m \in \frac{1}{2}\mathbb{Z}\). We have
\[
H_{k_1+k_2}(f_{k_1} \varphi_{k_2,m}) = 12D_{k_1}(f_{k_1}) \varphi_{k_2,m} + f_{k_1} H_{k_2}(\varphi_{k_2,m}).
\]
In particular, \(H_{\frac{2}{3}+k}(\eta^n \varphi_{k,m}) = \eta^n H_k(\varphi_{k,m})\).

The cases of Calabi–Yau varieties of dimensions two and five are more difficult, but quite similar.

**Theorem 2.3.** Let \(M_d\) be a Calabi–Yau variety of dimension 2 or 5 and \(e(M_5) \neq 0\). Then the elliptic genus \(\chi(M_d; \tau, z)\) satisfies a modular differential equation of degree 3 with respect to the heat operator \(H\). More exactly we have
\[
H_4 H_2 H_0(\varphi_{0,1}) - \frac{101}{4} E_4 H_0(\varphi_{0,1}) + 10 E_6 \varphi_{0,1} = 0,
\]
(2.9) \[ H_4 H_2 H_0(\varphi_{0,\frac{3}{2}}) - \frac{611}{25} E_4 H_0(\varphi_{0,\frac{3}{2}}) + \frac{88}{25} E_6 \varphi_{0,\frac{3}{2}} = 0. \]

We can rewrite the last two equations using only the heat operator and the Eisenstein series \( E_2, E_4 \) and \( E_6 \):

\[
H^3(\varphi_{0,1}) - \frac{9}{2} E_2 H^2(\varphi_{0,1}) + \left( \frac{9}{4} E_2^2 - \frac{99}{4} E_4 \right) H(\varphi_{0,1}) + \left( \frac{3}{8} E_2^3 - \frac{99}{8} E_2 E_4 + 12 E_6 \right) \varphi_{0,1} = 0
\]

and

\[
H^3(\varphi_{0,\frac{3}{2}}) - \frac{9}{2} E_2 H^2(\varphi_{0,\frac{3}{2}}) + \left( \frac{9}{4} E_2^2 - \frac{1997}{50} E_4 \right) H(\varphi_{0,\frac{3}{2}}) + \left( \frac{3}{8} E_2^3 - \frac{1997}{100} E_2 E_4 + \frac{138}{25} E_6 \right) \varphi_{0,\frac{3}{2}} = 0.
\]

By \( H^3(\phi) \) we denote as usual the operator \( H(H(H(\phi))) \). We note that we use the heat operator \( H^{(1)} \) for \( \varphi_{0,1} \) and the heat operator \( H^{(2)} \) for \( \varphi_{5/2} \) (see (2.7)).

**Proof.** Let \( \varphi_{k,m} \in J_{k,m} \). By \( J_{k,m}(q) \) we denote the subspace of the weak Jacobi forms in \( J_{k,m} \) whose \( q^0 \)-Fourier coefficient is equal to 0.

According to (2.4), we have a simple formula for the \( q^0 \)-Fourier coefficient of \( H_k(\varphi_{k,m}) \). If \( q^0(\varphi_{k,m}) = \sum_l a(0,l) \zeta^l \) then

(2.10) \[ q^0(H_k(\varphi_{k,m})) = \sum_l \left( -\frac{2}{m} \frac{3l^2 - 2k - 1}{2} \right) a(0,l) \zeta^l. \]

We know that \( J_{0,1} = \mathbb{C} \varphi_{0,1} \) and \( J_{2,1} = \mathbb{C} E_4 \varphi_{-2,1} \). Since \( q^0(H_2(\varphi_{-2,1})) = -\frac{1}{2} \varphi_{-1}^\pm + 5 \) and \( q^0(0(\varphi_{0,1})) = -\frac{5}{2} \varphi_{-1}^\pm + 5 \), we have

(2.11) \[ H_{-2}(\varphi_{-2,1}) = -\frac{1}{2} \varphi_{0,1}, \quad H_0(\varphi_{0,1}) = -\frac{5}{2} E_4 \varphi_{-2,1}. \]

Using (2.3), (2.11) and Proposition 2.2 we obtain

(2.12) \[ H_2 H_0(\varphi_{0,1}) = -\frac{5}{2} H_2(E_4 \varphi_{-2,1}) = 10 E_6 \varphi_{-2,1} + \frac{5}{4} E_4 \varphi_{0,1}. \]

This equation shows that \( \varphi_{0,1} \) does not satisfies a modular differential equation of degree 2 with respect to the heat operator \( H \). We will use the identity (2.12) in §3.

To obtain a differential equation of order 3 for \( \varphi_{0,1} \) one can continue the calculation above, but we propose a more simple method of \( q^0 \)-cancellation. Using (2.10) we obtain

\[ q^0(H_4 H_2 H_0(\varphi_{0,1})) = -\frac{13}{8} \frac{9}{2} \varphi_{0,1}^\pm + \frac{21}{8} \cdot 10. \]

It follows that the \( q^0 \)-Fourier coefficient of the Jacobi form \( \psi_{0,1} \) of weight 6

in the left hand side of the differential equation (2.8) is equal to 0, i.e., \( \psi_{0,1} \in \).
$J_{6,1}(q)$. Therefore $\Delta^{-1}\psi_{6,1} \in J_{-6,1}$. The latter space is trivial according to the structure result [1,5]. Equation (2.8) is proved.

Next we consider a variety $CY_5$ of dimension 5. We know that $J_{2,5} = \mathbb{C}E_4\varphi_{-2,1}\varphi_{0,1}$. Using (2.6) and (2.10) we obtain

$$q^0(H_0(\varphi_{0,\frac{5}{2}})) = -\frac{11}{5}(\zeta^{\pm\frac{3}{2}} - \zeta^{\pm\frac{1}{2}}), \quad q^0(H_{-2}(\varphi_{-2,1}\varphi_{0,\frac{5}{2}})) = -\frac{1}{5}(\zeta^{\pm\frac{3}{2}} + 11\zeta^{\pm\frac{1}{2}}).$$

Therefore

$$H_0(\varphi_{0,\frac{5}{2}}) = -\frac{11}{5}E_4\varphi_{-2,1}\varphi_{0,\frac{5}{2}}, \quad H_{-2}(\varphi_{-2,1}\varphi_{0,\frac{5}{2}}) = -\frac{1}{5}\varphi_{0,1}\varphi_{0,\frac{5}{2}}.$$

By Proposition 2.2 and 2.3 we obtain

$$H_2H_0(\varphi_{0,\frac{5}{2}}) = H_2 \left( -\frac{11}{5}E_4\varphi_{-2,1}\varphi_{0,\frac{5}{2}} \right) = \frac{44}{5}E_6\varphi_{-2,1}\varphi_{0,\frac{5}{2}} + \frac{11}{25}E_4\varphi_{0,\frac{5}{2}}.$$

It follows that $\varphi_{0,\frac{5}{2}}$ does not satisfy a modular differential equation of degree 2 with respect to the heat operator $H$. The third iteration $H_4H_2H_0(\varphi_{0,\frac{5}{2}})$ has weight 6. Since $J_{6,\frac{5}{2}} = \phi_{0,\frac{3}{2}}J_{6,1}$ we can use the method of $q^0$-cancellation as in the case of $K3$ above. We have

$$q^0(H_4H_2H_0(\varphi_{0,\frac{5}{2}})) = -\frac{31}{125}\cdot\frac{21}{125}\cdot\frac{11}{125}\cdot\frac{19}{125}\cdot\frac{9}{125}\cdot\frac{1}{125}\cdot11\zeta^{\pm\frac{3}{2}}.$$

This proves (2.9).

To rewrite equations (2.8) and (2.9) in terms of the heat operator we take into account the identity $\frac{\partial}{\partial \tau}E_2 = E_2^2 - E_4$ and $\frac{\partial}{\partial \tau}E_4 = 4E_2E_4 - 4E_6$. The theorem is proved.

\section{Modular differential equations of degree 2}

\subsection{The Kaneko–Zagier modular differential equation}

The Kaneko–Zagier differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E'_2(\tau)f(\tau) = 0$$

appeared in [17] (see also [15]) in connection with liftings of supersingular $j$-invariants of elliptic curves. For our purpose we use a representation of the Kaneko–Zagier equation in terms of the modular differential operator $D_k$. The equation (3.1) is equivalent to

$$D_{k+2}D_k(f) - \frac{k(k+2)}{144}E_4 \cdot f = 0.$$

It is evident that the Eisenstein series $E_4(\tau)$ of weight $k = 4$ is a solution of the Kaneko–Zagier equation. This equation and its generalization for degree 3 (see [16]) have many applications in number theory and the theory of vertex algebras.
If we change \( D_k \) by the modular differential operator \( H_k \) then the Eisenstein-Jacobi series \( E_{4,1}(\tau, z) \) of weight 4 and index 1 satisfies a similar equation

\[
H_6H_4(E_{4,1}(\tau, z)) - \frac{77}{4}E_4(\tau)E_{4,1}(\tau, z) = 0,
\]

because \( J_{8,1}^{\text{hol}} = \mathbb{C}E_4(\tau)E_{4,1}(\tau, z) \). See [19] for more solutions in index 1 for different \( k \). We recall that we use factor 12 before the differential operator \( D \) in the modular correction \( H_k \) of the heat operator in (2.4).

In this section we show that many basic generators of the bigraded ring \( J_{*,*} \) of weak Jacobi forms satisfy differential equations of Kaneko–Zagier type with the heat operator \( H \) instead of the derivation \( D = \frac{1}{2\pi i} \frac{d}{d\tau} \).

3.2. The Kaneko–Zagier type equation for \( \varphi_{-2,1} \) or \( \vartheta^2(\tau, z) \), and \( \vartheta(\tau, z)\vartheta(\tau, 2z) \). In this subsection we consider two examples of solutions of Kaneko–Zagier type equations for Jacobi modular forms.

The Jacobi theta-series \( \vartheta(\tau, z) \in J_{1/2,1/2}(v, \varpi \cdot \varpi_H) \) lies in the kernel of the heat operator:

\[
\left( 4\pi i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right) \vartheta(\tau, z) = 0.
\]

In the proof of Theorem 2.3 we obtained a formula for \( H_0H_{-2}(-2,1) \) in (2.11) By (2.11),

\[
(3.3) \quad H_0H_{-2}(\varphi_{-2,1}) - \frac{5}{4}E_4\varphi_{-2,1} = 0.
\]

Using Proposition 2.2 we can rewrite the last equation in terms of \( \vartheta^2 \) (or in terms of the first Jacobi cusp form of index one \( \varphi_{10,1} = \eta^{18}(\tau)\vartheta^2(\tau, z) \) as in [19])

\[
(3.4) \quad H_3H_1(\vartheta^2) - \frac{5}{4}E_4\vartheta^2 = 0 \quad \text{or} \quad H_3H_1(\varphi_{10,1}) - \frac{5}{4}E_4\varphi_{10,1} = 0.
\]

Let us compare the last equation with the equation for the elliptic genus of a \( K3 \) surface. (2.8) is equivalent to the equation of degree 3 for the second Jacobi cusp form \( \varphi_{12,1} = \Delta \varphi_{0,1} \) of index 1

\[
H_{16}H_{14}H_{12}(\varphi_{12,1}) - \frac{101}{4}E_4H_{12}(\varphi_{12,1}) + 10E_6\varphi_{12,1} = 0.
\]

Our second example is related to the product of two generators (see (1.9)) of the bigraded ring of weak Jacobi forms

\[
\varphi_{-2,1}(\tau, z) \cdot \varphi_{0,2}(\tau, z) = \frac{\vartheta(\tau, z)\vartheta(\tau, 2z)}{\eta^6(\tau)} \in J_{-2,2}.
\]

According to the structure results (1.10)–(1.11)

\[
J_{2,2} = \varphi_{0,2}J_{2,1} = \mathbb{C}E_4\varphi_{-2,1}\varphi_{0,2}.
\]
We have calculated $H_{-2}(\varphi_{-2,1}\varphi_{0,\frac{1}{2}})$ and $H_0(\varphi_{0,\frac{1}{2}})$ in [2, 13]. It gives us the equation for the product $\varphi_{-2,1}\varphi_{0,\frac{1}{2}}$ which can be rewritten in terms of Jacobi theta-series

\[(3.5) \quad H_2H_1(\vartheta(\tau, z)\vartheta(\tau, 2z)) - \frac{11}{25} E_4(\tau) \cdot (\vartheta(\tau, z)\vartheta(\tau, 2z)) = 0.\]

3.3. Equations for $\vartheta^3(\tau, z)$ and $\vartheta^2(\tau, z)\vartheta(\tau, 2z)$. It is interesting that the cube of the Jacobi theta-series satisfies an equation of Kaneko–Zagier type. To obtain it we consider the product of two weak Jacobi forms $\varphi_{-1,\frac{1}{2}}\varphi_{-2,1} = \vartheta^3/\eta^9 \in J_{-3,\frac{3}{2}}$. We see that

$$J_{1,\frac{3}{2}} = \varphi_{-1,\frac{1}{2}}J_{2,1} = CE_4\varphi_{-1,\frac{1}{2}}\varphi_{-2,1}.$$ 

Then we get the following identities

$$H_{-3}(\varphi_{-1,\frac{1}{2}}\varphi_{-2,1}) = -\varphi_{-1,\frac{1}{2}}\varphi_{0,1}, \quad H_{-1}(\varphi_{-1,\frac{1}{2}}\varphi_{0,1}) = -3E_4\varphi_{-1,\frac{1}{2}}\varphi_{-2,1}.$$ 

Therefore we obtain an equation for the product $\varphi_{-1,\frac{1}{2}}\varphi_{-2,1}$ which we can rewrite as

\[(3.6) \quad H_2^2H_1^2(\vartheta^3(\tau, z)) - 3E_4(\tau)\vartheta^3(\tau, z) = 0.\]

The next example is given by the product

$$\varphi_{-1,2}\varphi_{-2,1} = \vartheta(\tau, 2z)\vartheta^2(\tau, z)/\eta^9(\tau).$$

We have

$$H_{-3}(\varphi_{-1,2}\varphi_{-2,1}) = -\frac{1}{2}\varphi_{-1,2}\varphi_{0,1}, \quad H_{-1}(\varphi_{-1,2}\varphi_{0,1}) = -\frac{5}{2}E_4\varphi_{-1,2}\varphi_{-2,1}.$$ 

Therefore

\[(3.7) \quad H_2^2H_1^2(\vartheta(\tau, 2z)\vartheta^2(\tau, z)) - \frac{5}{4}E_4(\tau) \cdot (\vartheta(\tau, 2z)\vartheta^2(\tau, z)) = 0.\]

3.4. Equation of degree 3 for $\vartheta^4(\tau, z)$. We consider the fourth power of the Jacobi theta-series. To find a modular differential equation of this holomorphic Jacobi form we shall work with the square of the basic weak Jacobi form $\varphi_{-2,1}$. The method of $q^0$-cancellation reduces calculations with Jacobi forms to simple manipulations with reciprocal polynomials.

**Theorem 3.1.** The fourth power $\vartheta^4(\tau, z)$ of the Jacobi theta-series satisfies a modular differential equation of degree 3

\[(3.8) \quad H_6H_4H_1(\vartheta^4) - \frac{23}{4} E_4H_2(\vartheta^4) + \frac{81}{4} E_6\vartheta^4 = 0.\]

**Proof.** We analyze the Jacobi form $\varphi_{2,2}^2 = \vartheta^4/\eta^{12}$. We note that $J_{-2n,n} = \mathbb{C}\varphi_{-2,1}^n$ and $J_{-2n+2,n} = \mathbb{C}\varphi_{-2,1}^{n-1}\varphi_{0,1}$. Using (2.4) we get that for any $n \geq 2$

$$H_{-2n}(\varphi_{-2,1}^n) = (-n + \frac{1}{2})\varphi_{-2,1}^{n-1}\varphi_{0,1}.$$ 

For the third iteration of the modular differential operator we have

$$\psi_{2,2} = H_0H_{-2}H_{-4}(\varphi_{-2,1}^2) \in J_{2,2} = \langle E_4\varphi_{-2,1}^2, \varphi_{-2,1}\varphi_{0,1} \rangle \mathbb{C}. $$
It follows that $H_0H_{-2}H_{-4}(\varphi_{-2,1}^2)$ is a combination of $E_4H_{-4}(\varphi_{-2,1}^2)$ and $E_6\varphi_{-2,1}^2$. To compute the constants in the equation we find two coefficients of the $q^0$-term of $\psi_{2,2}$ using (2.10)

$$q^0(\psi_{2,2}) = -\frac{3}{8} \cdot 7 \cdot 11 \cdot \zeta^2 + \ldots + \frac{9}{8} \cdot 5 \cdot 1 + \ldots.$$  

As a result we obtain

$$H_0H_{-2}H_{-4}(\varphi_{-2,1}^2) = \frac{23}{4} E_4H_{-4}(\varphi_{-2,1}^2) - \frac{81}{4} E_6\varphi_{-2,1}^2.$$ 

To finish the proof we apply Proposition 2.2.  

**Remark.** We note that $\vartheta^3(\tau, z)$ can be considered as a special Jacobi-Eisenstein series. In particular, there is a formula for its Fourier coefficients (see [13]). We can show that $\vartheta^4(\tau, z)$ is a Jacobi-Eisenstein series with a nontrivial character of the full Jacobi group.

4. CY$_4$, CY$_6$ and generating functions of Kac-Moody type

4.1. Modular differential equation for generic Jacobi forms of index 2 and 3. We know that

$$\chi(CY_4; \tau, z) \in J_{0,2}^\mathbb{Z} = \langle E_4\varphi_{-2,1}^2, \varphi_{0,1}^2 \rangle_z = \langle E_4\varphi_{-2,1}^2, \varphi_{0,2} \rangle_z,$$

$$\chi(CY_6; \tau, z) \in J_{0,3}^\mathbb{Z} = \langle E_6\varphi_{-2,1}^3, E_4\varphi_{-2,1}^2\varphi_{0,1}, \varphi_{0,1}^3 \rangle_z.$$ 

We analyze the algorithm of $q^0$-cancellation for weak Jacobi forms of index 2. There are four parameters in an equation of order 5 in $H$

$$(4.1) \quad H_8H_6H_4H_2H_0\phi_{0,2} + aE_4H_4H_2H_0\phi_{0,2}$$

$$+ bE_6H_2H_0\phi_{0,2} + cE_8H_0\phi_{0,2} + dE_{10}\phi_{0,2} = 0$$

for an arbitrary Jacobi form $\phi_{0,2} \in J_{0,2}$ of index 2. The $q^0$-part of the Jacobi form $H_8H_6H_4H_2H_0(\phi_{0,2}) \in J_{0,2}$ has three coefficients $s_2\zeta^{\pm2} + s_1\zeta^{\pm1} + s_0$. Using three parameters in (11) one can obtain a Jacobi form $\xi_{10,2} \in J_{10,2}(q)$ such that $q^0(\xi_{10,2}) = 0$. Therefore $\Delta^{-1}\xi_{10,2} \in J_{-2,2} = \mathbb{C}\varphi_{-2,1}\varphi_{0,1}$. Using the fourth parameter in the differential equation one can annihilate the last term $\xi_{10,2}$. We have proved the following result.

Theorem 4.1. An arbitrary Jacobi form $\phi_{0,2}$ of weight 0 and index 2 (resp. $\phi_{0,3}$ of index 3) satisfies a linear (in the heat operator $H$) modular differential equation of order 5 (resp. 7).

**Proof.** We have to prove the theorem for index 3. Using the same arguments as in the case of index 2 we can construct a differential equation of degree 7 for any weak Jacobi form $\phi_{0,3} \in J_{0,3}$. The $q^0$-Fourier coefficient of the Jacobi form $H_{12}H_{10}\ldots H_0(\phi_{0,3}) \in J_{14,3}$ has four coefficients. A differential equation of degree 7 for any $\phi_{0,3}$ contains the dominant term indicated above and 6 additional summands defined by the corresponding iterations of $H$. The coefficient at $H_0\phi_{0,3}$ is a modular form of weight 12. Therefore, there are 7 free coefficients in the equation at modular factors $E_4, E_6, E_8, E_{10}, E_{12}, \Delta$. 


and $E_{14}$. Using four coefficients one can get a modular form $\xi_{14,3} \in J_{14,3}(q)$ without $q^0$-part. Thus,

$$\Delta^{-1} \xi_{14,3} \in J_{2,3} = \langle E_8 \varphi_{-2,1}^3, E_6 \varphi_{-2,1}^2 \varphi_{0,1}, E_4 \varphi_{-2,1}^2 \varphi_{0,1} \rangle e.$$ 

With three additional parameters in the equation one can annihilate $\xi_{14,3}$. In this way we obtain a modular differential equation of degree 7. \[\square\]

4.2. **Reflective weak Jacobi forms of index 2, 3 and 4.** We will consider high-order modular differential equations in another paper. Now we analyze special differential equations for the generators of the graded ring $J_{2,1}^\mathbb{Z}$, of weak Jacobi forms of weight 0 and integral index. It is the target ring for the elliptic genus of Calabi–Yau varieties of even dimensions according Theorem [11.3]. The ring has four generators $\varphi_{0,1}$, $\varphi_{0,2}$, $\varphi_{0,3}$, $\varphi_{0,4}$ according to Theorem [1.3]. These Jacobi forms have many applications. They are reflective, i.e., they determine Siegel paramodular forms with the simplest divisors which are generating functions of the basic Lorentzian Kac-Moody algebras of hyperbolic rank 3 (see [12]). In particular, these Jacobi forms are generating functions for (super)-multiplicities of the positive roots of the corresponding Lorentzian Kac-Moody algebras. Moreover, the $-1$-power of the automorphic Borcherds product defined by $2\varphi_{0,1}$ is the second quantized elliptic genus of K3 surface (see [2]). The differential equation for $\varphi_{0,1}$ was considered in Theorem [2.3]. It turns out that $\varphi_{0,2}$ and $\varphi_{0,4}$ also satisfy modular differential equations of degree 3. The generator $\varphi_{0,3}$ gives us the first example of weak Jacobi forms of weight 0 whose differential equation has degree 4.

The first step in the construction of a modular differential equation in Theorem [1.1] is the annihilation of the $q^0$-part of a Jacobi form. There exist three Jacobi forms in $J_{2,1}^\mathbb{Z}$ with only two parameters in the $q^0$-Fourier coefficient

$$\varphi_{0,2}(\tau, z) = \zeta^{\pm 1} + 4 + q(\ldots) \quad \text{(see (1.2)),}$$

$$\psi_{0,2}(\tau, z) = \zeta^{\pm 2} + 22 + q(\ldots) = \varphi_{0,1}^2(\tau, z) - 20 \varphi_{0,2}(\tau, z),$$

$$\rho_{0,2}(\tau, z) = 2\psi_{0,2}(\tau, z) - 11 \varphi_{0,2}(\tau, z) = 2\zeta^{\pm 2} - 11\zeta^{\pm 1} + q(\ldots).$$

For them we can expect a special modular differential equation of order less than 5. The Jacobi form $\psi_{0,2}(\tau, z)$ is also reflective and determines a Lorentzian Kac-Moody algebra. We show that each of them satisfies a modular differential equation of order 3.

First, consider the generator $\varphi_{0,2}$. We have $H_0(\varphi_{0,2}) = -\zeta^{\pm 1} + 2 + q(\ldots)$. Then, continuing to apply the modular differential operators, we get

$$H_2 H_0(\varphi_{0,2}) = 3\zeta^{\pm 1} - 3 + q(\ldots), \quad H_4 H_2 H_0(\varphi_{0,2}) = -15\zeta^{\pm 1} + \frac{21}{2} + q(\ldots).$$

From these formulas we obtain

$$\xi_{6,2} = 4H_4 H_2 H_0(\varphi_{0,2}) - 47E_4 H_0(\varphi_{0,2}) + 13E_6 \varphi_{0,2} = q(\ldots) \in J_{6,2}(q).$$
It follows that $\Delta^{-1} \xi_{6,2} \in J_{-6,2} = \{0\}$ according to the structure of the algebra of Jacobi forms (see (1.5)). Therefore $\varphi_{0,2}$ satisfies the equation

$$H_4 H_2 H_0(\varphi_{0,2}) - \frac{47}{4} E_4 H_0(\varphi_{0,2}) + \frac{13}{4} E_6 \varphi_{0,2} = 0.$$  

(4.2) In a similar way we obtain

$$H_0(\psi_{0,2}) = -\frac{11}{2} \zeta^2 + 11 + q(\ldots), \quad H_2 H_0(\psi_{0,2}) = \frac{165}{4} \zeta^2 - \frac{33}{2} + q(\ldots),$$

$$H_4 H_2 H_0(\psi_{0,2}) = -\frac{3135}{8} \zeta^2 + \frac{231}{8} + q(\ldots).$$

Therefore

$$H_4 H_2 H_0(\psi_{0,2}) - \frac{263}{4} E_4 H_0(\psi_{0,2}) + \frac{121}{4} E_6 \psi_{0,2} = 0.$$  

(4.3) For the third Jacobi form $\rho_{0,2}$ we have

$$H_0(\rho_{0,2}) = -11 \zeta^2 + 11 \zeta + q(\ldots), \quad H_2 H_0(\rho_{0,2}) = \frac{165}{2} \zeta^2 - 33 \zeta + q(\ldots)$$

and $H_4 H_2 H_0(\rho_{0,2}) = -\frac{3135}{4} \zeta^2 + 165 \zeta + q(\ldots)$. Therefore

$$H_4 H_2 H_0(\rho_{0,2}) - \frac{335}{4} E_4 H_0(\rho_{0,2}) - \frac{275}{4} E_6 \rho_{0,2} = 0.$$  

(4.4) For the last two generators $\varphi_{0,3}$ and $\varphi_{0,4}$ the situation is more complicated. We have to analyze not only the $q^0$-term, but a part of $q^1$-Fourier coefficients. In particular, there are no modular differential equations of order 3 for $\varphi_{0,3}$.

**Theorem 4.2.** The generators $\varphi_{0,3}, \varphi_{0,4}$ (see (1.12)) satisfy the following modular differential equations of order 4 and 3:

$$H_6 H_4 H_2 H_0(\varphi_{0,3}) - \frac{29}{2} E_4 H_2 H_0(\varphi_{0,3}) + 22 E_6 H_0(\varphi_{0,3}) - \frac{119}{16} E_8 \varphi_{0,3} = 0$$

and

$$H_4 H_2 H_0(\varphi_{0,4}) - \frac{107}{16} E_4 H_0(\varphi_{0,4}) + \frac{23}{32} E_6 \varphi_{0,4} = 0.$$  

(4.5)

**Proof.** In Theorem 4.1 we proved that a general weak Jacobi form in $J_{0,3}$ satisfies a modular differential equation of order 7. The generator $\varphi_{0,3} = \zeta + q(\ldots)$ has the simplest $q^0$-term. To annihilate it one needs two parameters in an equation of order 3. See the proof of the equations (4.2)–(4.4). In the case of index 3 the subspace of the weak Jacobi forms of weight 6 and index 3 without $q^0$-Fourier coefficients is one-dimensional: $J_{6,3}(q) = \mathbb{C} \Delta \varphi^3_{-2,1}$. This is the first case when we need to control the $q^1$-term of
Fourier expansions. We see that
\[
H_0(\varphi_{0,3}) = -\frac{1}{2} \zeta^{\pm 1} + 1 + q \left( -7 \zeta^{\pm 3} + \ldots \right),
\]
\[
H_2 H_0(\varphi_{0,3}) = \frac{5}{4} \zeta^{\pm 1} - \frac{3}{2} + q \left( -\frac{21}{2} \zeta^{\pm 3} + \ldots \right),
\]
\[
H_4 H_2 H_0(\varphi_{0,3}) = -\frac{45}{8} \zeta^{\pm 1} + \frac{21}{4} + q \left( \frac{21}{4} \zeta^{\pm 3} + \ldots \right).
\]
Therefore we obtain the identity
\[
H_4 H_2 H_0(\varphi_{0,3}) - \frac{33}{4} E_4 H_0(\varphi_{0,3}) + \frac{3}{2} E_6 \varphi_{0,3} = 60 \Delta \varphi^{3}_{-2,1}.
\]
In this case we have to analyze equations of order 4 with the dominant term
\[
H_6 H_4 H_2 H_0(\varphi_{0,3}) \in J_{8,3}.
\]
We note that the subspace \(J_{8,3}(q) = \Delta J_{-4,3} = \mathbb{C} \Delta \varphi^{2}_{-2,1} \varphi_{0,1}\) is again one-dimensional. Therefore, it is enough to control only one coefficient in \(q^1\)-terms. We use three formulas above and the fourth identity
\[
H_6 H_4 H_2 H_0(\varphi_{0,3}) = \frac{585}{16} \zeta^{\pm 1} - \frac{231}{8} + q \left( -\frac{105}{8} \zeta^{\pm 3} + \ldots \right)
\]
in order to find the equation of the theorem for \(\varphi_{0,3}\).

Consider the last generator \(\varphi_{0,4} = \varphi(\tau, z)\) of the graded ring \(J^{\mathbb{Z}}_{0,*}\). The direct computation shows that the \(q^0\)-term of the left hand side of (4.5) is equal to zero. The subspace \(J_{6,4}(q) = \Delta J_{-6,4} = \mathbb{C} \Delta \varphi^{3}_{-2,1} \varphi_{0,1}\) is again one-dimensional. But in this case of \(\varphi_{0,4}\) we are lucky. Direct computation shows that the \(q^1\)-part of the Fourier expansion in (4.5) vanishes. Therefore \(\varphi_{0,4}\) satisfies the equation (4.5) of degree 3.

Acknowledgements. The first author was supported by Ministry of Science and Higher Education of the Russian Federation, agreement 075–15–2022–287 and by the Möbius Contest Foundation for Young Scientists. The second author was supported by the HSE University Basic Research Program and by the PRCI SMAGP (ANR-20-CE40-0026-01).

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