Abstract. Let \( R \) be a real closed field, \( \mathbb{Q} \subset R[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k] \), with \( \deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in \mathbb{Q}, \#(Q) = m \), and \( \mathcal{P} \subset R[X_1, \ldots, X_k] \) with \( \deg_X(P) \leq d, P \in \mathcal{P}, \#(P) = s \), and \( S \subset R^{\ell+k} \) a semi-algebraic set defined by a Boolean formula without negations, with atoms \( P = 0, P \leq 0, P \geq 0, P \in \mathcal{P} \cup Q \). We prove that the sum of the Betti numbers of \( S \) is bounded by \( (\ell s m d)^{O(m+k)} \). This is a common generalization of previous results in [7] and [2] on bounding the Betti numbers of closed semi-algebraic sets defined by polynomials of degree \( d \) and 2, respectively.

We also describe algorithms for computing the Euler-Poincaré characteristic, as well as all the Betti numbers of such sets, generalizing similar algorithms described in [7, 4] and [5]. The complexity of the first algorithm is bounded by \( (\ell s m d)^{O(m(m+k))} \), while that of the second is bounded by \( (\ell s m d)^{2O(m+k)} \).

1. Introduction and Main Results

Let \( R \) be a real closed field and \( S \subset R^k \) a semi-algebraic set defined by a Boolean formula with atoms of the form \( P > 0, P < 0, P = 0 \) for \( P \in \mathcal{P} \subset R[X_1, \ldots, X_k] \). We call \( S \) a \( \mathcal{P} \)-semi-algebraic set and the Boolean formula defining \( S \) a \( \mathcal{P} \)-formula.

If, instead, the Boolean formula has atoms of the form \( P = 0, P \leq 0, P \geq 0, P \in \mathcal{P} \), and additionally contains no negation, then we will call \( S \) a \( \mathcal{P} \)-closed semi-algebraic set, and the formula defining \( S \) a \( \mathcal{P} \)-closed formula.

For any closed semi-algebraic set \( X \subset R^k \), and any field of coefficients \( K \), we denote by \( b_i(X, K) \) the dimension of the \( K \)-vector space, \( H_i(X, K) \), which is the \( i \)-th homology group of \( X \) with coefficients in \( K \). We refer to [10] for the definition of homology in the case of \( R \) being an arbitrary real closed field, not necessarily the field of real numbers, and \( K = \mathbb{Q} \). The definition for a more general \( K \) is similar.

We denote by \( b(X, K) \) the sum \( \sum_{i \geq 0} b_i(X, K) \). We write \( b_i(X) \) for \( b_i(X, \mathbb{Z}/2\mathbb{Z}) \) and \( b(X) \) for \( b(X, \mathbb{Z}/2\mathbb{Z}) \). Note that the mod-2 Betti numbers \( b_i(X) \) are an upper bound on the Betti numbers \( b_i(X, \mathbb{Q}) \) (as a consequence of the Universal Coefficient Theorem for homology (see [18] for example)).

The following result appeared in [7].

Theorem 1.1. [7] For a \( \mathcal{P} \)-closed semi-algebraic set \( S \subset R^k \), \( b(S, K) \) is bounded by \( (O(sd))^s \), where \( s = \#(\mathcal{P}) \), and \( d = \max_{P \in \mathcal{P}} \deg(P) \). \( \square \)

It is a generalization of the results due to Oleinik, Petrovsky [21], Thom [23], and Milnor [19] on bounding the Betti numbers of real varieties. It provides an upper
bound on the sum of the Betti numbers of $P$-closed semi-algebraic sets in terms of the number and degrees of the polynomials in $P$ (see also [9] for a slightly more precise bound, and [14] for an extension of this result to arbitrary semi-algebraic sets with a slight worsening of the bound). Notice that this upper bound has singly exponential dependence on $k$, and this dependence is unavoidable (see Example 1.2 below).

In another direction, a restricted class of semi-algebraic sets - namely, semi-algebraic sets defined by quadratic inequalities - have been considered by several researchers [3, 2, 15, 8]. As in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large in the number of variables, as can be seen in the following example.

**Example 1.2.** The set $S \subset \mathbb{R}^\ell$ defined by

$$Y_1(Y_1 - 1) \geq 0, \ldots, Y_\ell(Y_\ell - 1) \geq 0$$

satisfies $b_0(S) = 2^\ell$.

However, it turns out that for a semi-algebraic set $S \subset \mathbb{R}^\ell$ defined by $m$ quadratic inequalities, it is possible to obtain upper bounds on the Betti numbers of $S$ which are polynomial in $\ell$ and exponential only in $m$. The first such result is due to Barvinok [2], who proved the following theorem.

**Theorem 1.3.** [2] Let $S \subset \mathbb{R}^\ell$ be defined by $Q_1 \geq 0, \ldots, Q_m \geq 0$, $\deg(Q_i) \leq 2, 1 \leq i \leq m$. Then, $b(S, K) \leq \ell^{O(m)}$.

A tighter bound appears in [8].

Even though Theorem 1.1 [7] and Theorem 1.3 [2] are stated and proved in the case $K = \mathbb{Q}$ in the original papers, the proofs can be extended without any difficulty to a general $K$.

**Remark 1.4.** Notice that the bound in Theorem 1.3 is polynomial in the dimension $\ell$ for fixed $m$, and this fact depends crucially on the assumption that the degrees of the polynomials $Q_1, \ldots, Q_m$ are at most two. For instance, the semi-algebraic set defined by a single polynomial of degree 4 can have Betti numbers exponentially large in $\ell$, as exhibited by the semi-algebraic subset of $\mathbb{R}^\ell$ defined by

$$\sum_{i=0}^{\ell} Y_i^2(Y_i - 1)^2 \leq 0.$$  

The above example illustrates the delicate nature of the bound in Theorem 1.3 since a single inequality of degree 4 is enough to destroy the polynomial nature of the bound. In contrast to this, we show in this paper (see Theorem 1.5 below) that a polynomial bound on the Betti numbers of $S$ continues to hold, even if we allow a few (meaning any constant number) of the variables to occur with degrees larger than two in the polynomials used to describe the set $S$.

We now state the main results of this paper.

1.1. **Bounds on the Betti Numbers.** We consider semi-algebraic sets defined by polynomial inequalities, in which the dependence of the polynomials on a subset of the variables is at most quadratic. As a result we obtain common generalizations of the bounds stated in Theorems 1.1 and 1.3. Given any polynomial $P \in \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$, we will denote by $\deg_X(P)$ (resp. $\deg_Y(P)$) the total degree of $P$ with respect to the variables $X_1, \ldots, X_k$ (resp. $Y_1, \ldots, Y_\ell$).
Notation 1. Throughout the paper we fix a real closed field \( R \), and denote by

- \( Q \subset R[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k] \), a family of \( m \) polynomials, with \( \deg Y(Q) \leq 2, \deg X(Q) \leq d, Q \in Q, \#(Q) = m \),
- \( P \subset R[X_1, \ldots, X_k] \) a family of polynomials of \( s \) polynomials, with \( \deg X(P) \leq d, P \in P, \#(P) = s \).

We prove the following.

**Theorem 1.5.** Let \( S \subset R^{\ell+k} \) be a \((P \cup Q)\)-closed semi-algebraic set. Then,

\[
\beta(S) \leq (\ell s m d)^{O(m+k)}.
\]

In particular, for \( m \leq \ell \), we have \( \beta(S) \leq (\ell s d)^{O(m+k)} \).

Notice that Theorem 1.5 can be seen as a common generalization of Theorems 1.1 and 1.3 in the sense that we recover similar bounds (that is bounds having the same shape) as in Theorem 1.1 (respectively, Theorem 1.3) by setting \( \ell \) and \( m \) (respectively, \( s, d \) and \( k \)) to \( O(1) \). Since we use Theorem 1.1 in the proof of Theorem 1.5, our proof does not give a new proof of Theorem 1.1. However, our methods do give a new proof of the known bound on Betti numbers in the quadratic case (Theorem 1.3), and this new proof is quite different from those given in [2, 8, 16].

The underlying technique in [2] and [16] is derived from Morse theory, while that in [8] uses complex algebraic geometry, and none of these techniques appear to generalize easily to the parametrized situation considered in this paper.

Note also that as a special case of Theorem 1.5 we obtain a bound on the sum of the Betti numbers of a semi-algebraic set defined over a quadratic map. Such sets have been considered from an algorithmic point of view in [15], where an efficient algorithm is described for computing sample points in every connected component, as well as testing emptiness, of such sets.

More precisely, we show the following.

**Corollary 1.6.** Let \( Q = (Q_1, \ldots, Q_k) : R^\ell \to R^k \) be a map where each \( Q_i \in R[Y_1, \ldots, Y_\ell] \) and \( \deg(Q_i) \leq 2 \). Let \( V \subset R^k \) be a \( P \)-closed semi-algebraic set for some family \( P \subset R[X_1, \ldots, X_k] \), with \( \#(P) = s \) and \( \deg(P) \leq d, P \in P \). Let \( S = Q^{-1}(V) \). Then,

\[
\beta(S) \leq (\ell s d)^{O(k)}.
\]

**Remark 1.7.** Note that the Morse theoretic techniques developed in [16] give a possible alternative approach for proving Corollary 1.6.

1.2. **Algorithmic Results.** We describe two main algorithms.

The first algorithm (Algorithm [3] below) computes the Euler-Poincaré characteristic \( \chi(S) \) of a \((P \cup Q)\)-closed semi-algebraic set \( S \). The complexity of this algorithm is bounded by \( (\ell s m d)^{O(m(m+k))} \). In the case of basic semi-algebraic sets, we obtain a better bound on the complexity, namely \( (\ell s m d)^{O(m+k)} \).

The second algorithm (Algorithm [10] below) computes all the Betti numbers of \( S \). The complexity of this algorithm is bounded by \( (\ell s m d)^{O(m+k)} \).

While the complexity of both the algorithms stated above is polynomial for fixed \( m \) and \( k \), the complexity of the algorithm for computing the Euler-Poincaré characteristic is significantly better than that of the algorithm for computing all the Betti numbers. Also note that our algorithmic results are obtained by modifying similar algorithms developed for the unparametrized situation in [15], but the algorithms
in the current paper are more general in several aspects. Only a restricted class consisting of basic closed semi-algebraic sets was treated in [4, 5].

1.3. **Significance from the complexity theory viewpoint.** The problem of computing the Betti numbers of semi-algebraic sets in general is a PSPACE-hard problem. We refer the reader to [5] and the references contained therein, for a detailed discussion of these hardness results. In particular, the problem of computing the Betti numbers of a real algebraic variety defined by a single quartic equation is also PSPACE-hard, and the same is true for semi-algebraic sets defined by many quadratic inequalities. On the other hand, as shown in [3], the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of quadratic inequalities is solvable in polynomial time. We show in this paper that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of polynomial inequalities is solvable in polynomial time, even if we allow a small (constant sized) subset of the variables to occur with degrees larger than two in the polynomials defining the given set. Note that such a result is not obtainable directly from the results in [5] by the naive method of replacing the monomials having degrees larger than two by a larger set of quadratic ones (introducing new variables and equations in the process).

For general semi-algebraic sets, the algorithmic problem of computing all the Betti numbers is notoriously difficult and only doubly exponential time algorithm is known for this problem. Very recently, singly exponential time algorithms [11, 6] have been found for computing the first few Betti numbers of such sets, but the problem of designing singly exponential time algorithm for computing all the Betti numbers remains open. Singular exponential time algorithm is also known for computing the Euler-Poincaré characteristic of general semi-algebraic sets [7].

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.5. In Section 3 we describe our algorithm for computing the Euler-Poincaré characteristic, and in Section 4 we describe our algorithm for computing all the Betti numbers of sets defined by partly quadratic system of polynomials.

## 2. PROOF OF THEOREM 1.5

One of the main ideas behind our proof of Theorem 1.5 is to parametrize a construction introduced by Agrachev in [1] while studying the topology of sets defined by (purely) quadratic inequalities (that is without the parameters $X_1, \ldots, X_k$ in our notation). However, we do not make any non-degeneracy assumptions on our polynomial, and we also avoid construction of Leray spectral sequences as done in [1].

We first need to fix some notations and a few preliminary results needed later in the proof.

### 2.1. Mathematical Preliminaries.

#### 2.1.1. Some Notations.

Let $A$ be a finite subset of $R[X_1, \ldots, X_k]$. A **sign condition** on $A$ is an element of $\{0, 1, -1\}^A$. The **realization of the sign condition** $\sigma$, $R(\sigma, R^k)$, is the basic semi-algebraic set

$$\{x \in R^k \mid \bigwedge_{P \in A} \text{sign}(P(x)) = \sigma(P)\}.$$
A weak sign condition on $A$ is an element of $\{\{0\}, \{0, 1\}, \{0, -1\}\}^A$. The realization of the weak sign condition $\rho$, $R(\rho, R^k)$, is the basic semi-algebraic set

$$\{x \in R^k \mid \bigwedge_{P \in A} \text{sign}(P(x)) \in \rho(P)\}.$$ 

Given weak sign conditions $\rho_1, \ldots, \rho_N$ on $A$, their common refinement $\rho$ is the weak sign condition defined by

$$\rho(P) = \bigcap_{i=0}^N \rho_i(P)$$

for each $P \in A$.

We often abbreviate $R(\sigma, R^k)$ by $R(\sigma)$, and we denote by $\text{Sign}(A)$ the set of realizable sign conditions $\text{Sign}(A) = \{\sigma \in \{0, 1, -1\}^A \mid R(\sigma) \neq \emptyset\}$.

More generally, for any $A \subset R[X_1, \ldots, X_k]$ and an $A$-formula $\Phi$, we denote by $R(\Phi, R^k)$, or simply $R(\Phi)$, the semi-algebraic set defined by $\Phi$ in $R^k$.

2.1.2. Use of Infinitesimals. Later in the paper, we extend the ground field $R$ by infinitesimal elements. We denote by $R(\zeta)$ the real closed field of algebraic Puiseux series in $\zeta$ with coefficients in $R$ (see [10] for more details). The sign of a Puiseux series in $R(\zeta)$ agrees with the sign of the coefficient of the lowest degree term in $\zeta$. This induces a unique order on $R(\zeta)$ which makes $\zeta$ infinitesimal: $\zeta$ is positive and smaller than any positive element of $R$. When $a \in R(\zeta)$ is bounded from above and below by some elements of $R$, $\lim_{\zeta}(a)$ is the constant term of $a$, obtained by substituting 0 for $\zeta$ in $a$.

Let $R'$ be a real closed field containing $R$. Given a semi-algebraic set $S$ in $R^k$, the extension of $S$ to $R'$, denoted $\text{Ext}(S, R')$, is the semi-algebraic subset of $R'^k$ defined by the same quantifier free formula that defines $S$. The set $\text{Ext}(S, R')$ is well defined (i.e. it only depends on the set $S$ and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle (see for instance [10]).

We will need a few results from algebraic topology, which we state here without proof, referring the reader to papers where the proofs appear.

2.1.3. Mayer-Vietoris Inequalities. The following inequalities are consequences of the Mayer-Vietoris exact sequence.

**Proposition 2.1** (Mayer-Vietoris inequalities). Let the subsets $W_1, \ldots, W_t \subset R^n$ be all closed. Then for each $i \geq 0$ we have

$$b_i \left( \bigcup_{1 \leq j \leq t} W_j \right) \leq \sum_{J \subset \{1, \ldots, t\}} b_{i-\#(J)+1} \left( \bigcap_{j \in J} W_j \right),$$

$$b_i \left( \bigcap_{1 \leq j \leq t} W_j \right) \leq \sum_{J \subset \{1, \ldots, t\}} b_{i+\#(J)-1} \left( \bigcup_{j \in J} W_j \right).$$

**Proof.** See [10].
2.1.4. Topology of Sphere Bundles. Given a closed and bounded semi-algebraic set $B$, a semi-algebraic $\ell$-sphere bundle over $B$ is given by a continuous semi-algebraic map $\pi : E \to B$, such that for each $b \in B$, $\pi^{-1}(b)$ is homeomorphic to $S^\ell$ (the $\ell$-dimensional unit sphere in $R^{\ell+1}$).

We need the following proposition that relates the Betti numbers of $B$ with that of $E$.

**Proposition 2.2.** Let $B \subset R^k$ be a closed and bounded semi-algebraic set and let $\pi : E \to B$ be a semi-algebraic $\ell$-sphere bundle with base $B$. Then

$$b(E) \leq 2 \cdot b(B). \quad (2.4)$$

*Proof.* In case $\ell > 0$, the proposition follows from the inequality \[P_E(t) \leq P_{S^\ell}(t)P_B(t),\] proved in [12, page 252 (4.1)]

where $P_X(t) = \sum_{i \geq 0} b_i(X)t^i$ denotes the Poincaré polynomial of a topological space $X$, and the inequality holds coefficient-wise. The inequality \[b(E) \leq 2 \cdot b(B),\] holds for the Betti numbers with coefficients in $\mathbb{Q}$, as well.

For $\ell = 0$, inequality \[P_E(t) \leq P_{S^\ell}(t)P_B(t),\] is no longer true for the ordinary Betti numbers, as can be observed from the example of the two-dimensional torus, which is a double cover of the Klein bottle. But inequality \[b(E) \leq 2 \cdot b(B),\] holds for Betti numbers with $\mathbb{Z}/2\mathbb{Z}$-coefficients. This is an interesting exercise in covering space theory, and a proof can be found in [8].

We now return to the proof of Theorem 1.5.

2.2. Homogeneous Case.

**Notation 2.** We denote by

- $Q^h$ the family of polynomials obtained by homogenizing $Q$ with respect to the variables $Y$, i.e.

  $Q^h = \{Q^h \mid Q \in Q\} \subset R[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$,

  where $Q^h = Y_0^2Q(Y_1/Y_0, \ldots, Y_\ell/Y_0, X_1, \ldots, X_k)$,

- $\Phi$ a formula defining a $P$-closed semi-algebraic set $V$,

- $A^h$ the semi-algebraic set

$$A^h = \bigcup_{Q \in Q^h} \{(y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\}, \quad (2.6)$$

- $W^h$ the semi-algebraic set

$$W^h = \bigcap_{Q \in Q^h} \{(y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\}. \quad (2.7)$$

We are going to prove

**Theorem 2.3.**

$$b(A^h) \leq (\ell\text{md})^{O(m+k)}. \quad (2.8)$$

and
Theorem 2.4.  
\begin{equation}  
\text{b}(W^h) \leq (\ell\text{msd})^{O(m+k)}.  
\end{equation}  

Before proving Theorem 2.3 and Theorem 2.4 we need a few preliminary results.

Notation 3. Let \( Q = \{Q_1, \ldots, Q_m\} \) and \( Q^h = \{Q^h_1, \ldots, Q^h_m\} \).

We denote by  
\[ Q^h : \mathbb{R}^{\ell+1} \times \mathbb{R}^k \rightarrow \mathbb{R}^m, \]
the map defined by the polynomials \( Q^h_1, \ldots, Q^h_m \) of \( Q^h \).

Let  
\begin{equation}  
\Omega = \{ \omega \in \mathbb{R}^m \mid ||\omega|| = 1, \omega_i \leq 0, 1 \leq i \leq m \}.  
\end{equation}  

For \( \omega \in \Omega \) we denote by  
\[ \langle \omega, Q^h \rangle \in \mathbb{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k] \]
the polynomial defined by  
\begin{equation}  
\langle \omega, Q^h \rangle = m \sum_{i=1}^m \omega_i Q^h_i.  
\end{equation}  

For \( (\omega, x) \in \Omega \times V \), we denote by \( \langle \omega, Q^h \rangle(\cdot, x) \) the quadratic form in \( Y_0, \ldots, Y_\ell \)
obtained from \( \langle \omega, Q^h \rangle \) by specializing \( X_i = x_i, 1 \leq i \leq k \).

Let  
\begin{equation}  
B = \{(\omega, y, x) \mid \omega \in \Omega, y \in S^\ell, x \in V, \langle \omega, Q^h \rangle(y, x) \geq 0 \}.  
\end{equation}  

We denote by \( \varphi_1 : B \rightarrow F \) and \( \varphi_2 : B \rightarrow S^\ell \times V \) the two projection maps (see diagram below).

The following key proposition was proved by Agrachev [1] in the unparametrized situation, but as we see below it works in the parametrized case as well.

**Proposition 2.5.** The semi-algebraic set \( B \) is homotopy equivalent to \( A^h \).

**Proof.** We first prove that \( \varphi_2(B) = A^h \). If \((y, x) \in A^h\), then there exists some \( i, 1 \leq i \leq m \), such that \( (Q^h_i(y, x) \leq 0) \land \Phi(x) \). Then for \( \omega = (-\delta_{1,i}, \ldots, -\delta_{m,i}) \) (where \( \delta_{ij} = 1 \) if \( i = j \), and 0 otherwise), we see that \( (\omega, y, x) \in B \). Conversely, if \((y, x) \in \varphi_2(B)\), then there exists \( \omega \in \Omega \) such that \( \langle \omega, Q^h \rangle(y, x) \geq 0 \). Since \( \omega \leq 0 \) and \( \omega \neq 0 \), we have that \( (Q^h_i(y, x) \leq 0) \land \Phi(x) \) for some \( i, 1 \leq i \leq m \). This shows that \((y, x) \in A^h\).

For \((y, x) \in \varphi_2(B)\), the fibre  
\[ \varphi_2^{-1}(y, x) = \{ (\omega, y, x) \mid \omega \in \Omega \text{ such that } \langle \omega, Q^h \rangle(y, x) \geq 0 \}, \]
is a non-empty subset of \( \Omega \) defined by a single linear inequality. Thus, each non-empty fiber is an intersection of a convex cone with \( S^{m-1} \). The proposition now follows from the well-known Vietoris-Smale theorem [22].  

We will use the following notation.
Notation 4. For a quadratic form $Q \in \mathbb{R}[Y_0, \ldots, Y_\ell]$, we denote by $\text{index}(Q)$ the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form, i.e. of the matrix $M$ such that $Q(y) = \langle My, y \rangle$ for all $y \in \mathbb{R}^{\ell+1}$ (here $\langle \cdot, \cdot \rangle$ denotes the usual inner product). We also denote by $\lambda_i(Q), 0 \leq i \leq \ell$ the eigenvalues of $Q$ in non-decreasing order, i.e.
$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_\ell(Q).$$

For $F = \Omega \times V$ as above we denote
$$F_j = \{ (\omega, x) \in F \mid \text{index}(\omega, Q^h)(\cdot, x)) \leq j \}.$$ 

It is clear that each $F_j$ is a closed semi-algebraic subset of $F$ and we get a filtration of the space $F$ given by
$$F_0 \subset F_1 \subset \cdots \subset F_{\ell+1} = F.$$

Lemma 2.6. The fibre of the map $\varphi_1$ over a point $(\omega, x) \in F_j \setminus F_{j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.

Proof. Denote $\lambda_i(\omega, x) = \lambda_i(\omega, Q^h)(\cdot, x))$ the eigenvalues of $(\omega, Q^h)(\cdot, x)$ in increasing order. First notice that for $(\omega, x) \in F_j \setminus F_{j-1},$
$$\lambda_0(\omega, x) \leq \cdots \leq \lambda_j(\omega, x) < 0.$$ 

Moreover, letting $W_0(\omega, Q^h)(\cdot, x)), \ldots, W_\ell(\omega, Q^h)(\cdot, x))$ be the co-ordinates with respect to an orthonormal basis consisting of eigenvectors of $(\omega, Q^h)(\cdot, x)$, we have that $\varphi^{-1}_1(\omega, x)$ is the subset of $S^\ell = \{ \omega \} \times S^\ell \times \{ x \}$ defined by
$$\sum_{i=0}^\ell \lambda_i(\omega, x)W_i(\omega, Q^h)(\cdot, x))^2 \geq 0,$$
$$\sum_{i=0}^\ell W_i(\omega, Q^h)(\cdot, x))^2 = 1.$$ 

Since $\lambda_i(\omega, x) < 0$ for all $0 \leq i < j$, it follows that for $(\omega, x) \in F_j \setminus F_{j-1}$, the fiber $\varphi^{-1}_1(\omega, x)$ is homotopy equivalent to the $(k - j)$-dimensional sphere defined by setting
$$W_0(\omega, Q^h)(\cdot, x)) = \cdots = W_{j-1}(\omega, Q^h)(\cdot, x)) = 0$$ 
on the sphere defined by
$$\sum_{i=0}^\ell W_i(\omega, Q^h)(\cdot, x))^2 = 1.$$ 

□

For each $(\omega, x) \in F_j \setminus F_{j-1}$, let $L_j^+(\omega, x) \subset \mathbb{R}^{\ell+1}$ denote the sum of the non-negative eigenspaces of $(\omega, Q^h)(\cdot, x)$. Since $\text{index}(\omega, Q^h)(\cdot, x)) = j$ stays invariant as $(\omega, x)$ varies over $F_j \setminus F_{j-1}$, $L_j^+(\omega, x)$ varies continuously with $(\omega, x)$.

We denote by $C$ the semi-algebraic set defined by the following. We first define for $0 \leq j \leq \ell + 1$
$$C_j = \{ (\omega, y, x) \mid (\omega, x) \in F_j \setminus F_{j-1}, y \in L_j^+(\omega, x), |y| = 1 \},$$

(2.13)
and finally we define

\[(2.14)\]

\[C = \bigcup_{j=0}^{\ell+1} C_j.\]

The following proposition relates the homotopy type of \(B\) to that of \(C\).

**Proposition 2.7.** The semi-algebraic set \(C\) defined by (2.14) is homotopy equivalent to \(B\).

Before proving the Proposition we give an illustrative example.

**Example 2.8.** In this example \(m = 2, \ell = 3, k = 0,\) and \(Q^h = \{Q^h_1, Q^h_2\}\) with

\[Q^h_1 = -Y_0^2 - Y_1^2 - Y_2^2,\]
\[Q^h_2 = Y_0^2 + 2Y_1^2 + 3Y_2^2.\]

The set \(\Omega\) is the part of the unit circle in the third quadrant of the plane, and \(F = \Omega\) in this case (since \(k = 0\)). In the following Figure 1, we display the fibers of the map \(\varphi_1^{-1}(\omega) \subset B\) for a sequence of values of \(\omega\) starting from \((-1,0)\) and ending at \((0,-1)\). We also show the spheres, \(C \cap \varphi_1^{-1}(\omega)\), of dimensions 0, 1, and 2, that these fibers retract to. At \(\omega = (-1,0)\), it is easy to verify that \(\text{index}(\omega, Q^h) = 3\), and the fiber \(\varphi_1^{-1}(\omega) \subset B\) is empty. Starting from \(\omega = (-\cos(\arctan(1)), -\sin(\arctan(1)))\) we have \(\text{index}(\omega, Q^h) = 2\), and the fiber \(\varphi_1^{-1}(\omega)\) consists of the union of two spherical caps, homotopy equivalent to \(S^0\). Starting from \(\omega = (-\cos(\arctan(1/2)), -\sin(\arctan(1/2)))\) we have \(\text{index}(\omega, Q^h) = 1\), and the fiber \(\varphi_1^{-1}(\omega)\) is homotopy equivalent to \(S^1\). Finally, starting from \(\omega = (-\cos(\arctan(1/3)), -\sin(\arctan(1/3)))\), \(\text{index}(\omega, Q^h) = 0\), and the fiber \(\varphi_1^{-1}(\omega)\) stays equal to \(S^2\).

![Figure 1. Type change: $\emptyset \to S^0 \to S^1 \to S^2$. $\emptyset$ is not shown.](image)

**Proof of Proposition 2.7.** We construct a deformation retraction of \(B\) to \(C\) as follows. Let

\[(2.15)\]

\[B_j = \bigcup_{i=j}^{\ell+1} C_i \cup \varphi_1^{-1}(F_{j-1}),\]

and note that \(B_{\ell+1} = B, \ldots, B_0 = C\).

We construct a sequence of homotopy equivalences from \(B_{j+1}\) to \(B_j\) for every \(j = \ell, \ldots, 0\) as follows.

Let \(0 \leq j \leq \ell\). For each \((\omega, x) \in F_j \setminus F_{j-1}\), we retract the fiber \(\varphi_1^{-1}(\omega, x)\) to the \((\ell-j)\)-dimensional sphere, \(L^+_j(\omega, x) \cap S^\ell\) as follows. Let

\[W_0(\omega, Q^h)(\cdot, x), \ldots, W_2(\omega, Q^h)(\cdot, x)\]
be the co-ordinates with respect to an orthonormal basis consisting of eigenvectors
e_0(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle), \ldots, e_\ell(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle)
of \langle \omega, \mathcal{Q}^h(\cdot, x) \rangle$ corresponding to the non-decreasing sequence of eigenvalues of 
$\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle$. Then, $\varphi_1^{-1}(\omega, x)$ is the subset of $S^\ell$ defined by
$$
\sum_{i=0}^\ell \lambda_i(\omega, x)W_i(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle)^2 \geq 0,
$$
and $L_j^\ell(\omega, x)$ is defined by $W_0(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle) = \cdots = W_{j-1}(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle) = 0$. We 
retract $\varphi_1^{-1}(\omega, x)$ to the $(\ell - j)$-dimensional sphere, $L_j^\ell(\omega, x) \cap S^\ell$ by the retraction sending,
$(w_0, \ldots, w_\ell) \in \varphi_1^{-1}(\omega, x)$, at time $t$ to $(tw_0, \ldots, tw_{j-1}, t^jw_j, \ldots, t^\ell w_\ell)$,
where $0 \leq t \leq 1$, and $t' = \left(1 - t^2 \sum_{i=j}^{\ell-1} w_i^2 \right)^{1/2}$. Notice that even though the local 
co-ordinates $W_0(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle), \ldots, W_\ell(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle)$ in $\mathbb{R}^{\ell+1}$ with respect to the orthonormal basis $(e_0(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle), \ldots, e_\ell(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle))$ of eigenvectors may not be uniquely defined at the point $(\omega, x)$ (for instance, if the quadratic form $\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle$ has multiple eigen-values), the retraction is still well-defined since it only depends on the decomposition of $\mathbb{R}^{\ell+1}$ into orthogonal complements $\text{span}(e_0, \ldots, e_{j-1})$ and $\text{span}(e_j, \ldots, e_\ell)$ which is well defined.

We can thus retract simultaneously all fibers over $F_j \setminus F_{j-1}$ continuously, to obtain $B_j \subset B$, which is moreover homotopy equivalent to $B_{j+1}$. \hfill \Box

Notice that the semi-algebraic set $\varphi_1^{-1}(F_j \setminus F_{j-1}) \cap C$ is a $S^{\ell-j}$-bundle over $F_j \setminus F_{j-1}$ under the map $\varphi_1$, and $C$ is a union of these sphere bundles. Since we have good control over the bases, $F_j \setminus F_{j-1}$, of these bundles, i. e. we have good bounds on the number as well as the degrees of polynomials used to define them, we can bound the Betti numbers of each of these bundles using Proposition 2.2. However, these bundles could be possibly glued to each other in complicated ways, and thus knowing upper bounds on the Betti numbers of each of these bundles does not immediately produce a bound on Betti numbers of $C$. In order to get around this difficulty, we consider certain closed subsets, $F_j'$ of $F_j$, where each $F_j'$ is an infinitesimal deformation of $F_j \setminus F_{j-1}$, and form the base of a $S^{\ell-j}$-bundle. Additionally, these new sphere bundles are glued to each other along sphere bundles over $F_j' \cap F_{j-1}'$, and their union, $C'$, is homotopy equivalent to $C$. Since these new bundles are closed and bounded semi-algebraic sets, and we have good bounds on their Betti numbers as well as the Betti numbers of their non-empty intersections, we can use Mayer-Vietoris inequalities (Proposition 2.1) to bound the Betti numbers of $C'$, which in turn are equal to the Betti numbers of $C$.

We now make precise the argument outlined above.

Let $\Lambda \in \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k, T]$ be the polynomial defined by

$$
\Lambda = \det(T \cdot \text{Id}_{\ell+1} - M_{\mathcal{Q}^h}),
= T^{\ell+1} + C_\ell T^\ell + \cdots + C_0,
$$
where $Z \cdot Q^h = \sum_{i=1}^{m} Z_i Q_i^h$, and each $C_i \in \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

Note that for $(\omega, x) \in \Omega \times \mathbb{R}^k$, the polynomial $\Lambda(\omega, x, T)$, being the characteristic polynomial of a real symmetric matrix, has all its roots real. It then follows from Descartes’ rule of signs (see for instance [10]), that for each $(\omega, x) \in \Omega \times \mathbb{R}^k$, $\text{index}(\langle \omega, Q^h \rangle(\cdot, x))$ is determined by the sign vector

$$(\text{sign}(C_2(\omega, x)), \ldots, \text{sign}(C_0(\omega, x))).$$

More precisely, the number of sign variations in the sequence

$$\text{sign}(C_0(\omega, x)), \ldots, (-1)^i\text{sign}(C_i(\omega, x)), \ldots, (-1)^\ell\text{sign}(C_\ell(\omega, x)), +1.$$ 

is $\text{index}(\langle \omega, Q^h \rangle(\cdot, x))$. Hence, denoting

$$C = \{C_0, \ldots, C_\ell\} \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k],$$

we have

**Lemma 2.9.** $F_j$ is the intersection of $F$ with a $C$-closed semi-algebraic set $D_j \subset \mathbb{R}^{m+k}$, for each $0 \leq j \leq \ell + 1$. \hfill $\square$

**Notation 5.** Let

$$0 < \varepsilon_0 \ll \cdots \ll \varepsilon_{\ell+1} \ll 1$$

be infinitesimals. For $0 \leq j \leq \ell + 1$, we denote by $R_j$ the field $\mathbb{R}\langle \varepsilon_{\ell+1}, \ldots, \varepsilon_j \rangle$.

Let

$$C_j' = \{P \pm \varepsilon_j, P \in C\}.$$

Given $\rho \in \text{Sign}(C)$, and $0 \leq j \leq \ell + 1$, we denote by $\mathcal{R}(\rho^j_j) \subset \mathbb{R}^{m+k}$ the $C_j'$-semi-algebraic set defined by the formula $\rho^j_j$ obtained by taking the conjunction of

$$-\varepsilon_j \leq P \leq \varepsilon_j \text{ for each } P \in C \text{ such that } \rho(P) = 0,$$
$$P \geq -\varepsilon_j \text{, for each } P \in C \text{ such that } \rho(P) = 1,$$
$$P \leq \varepsilon_j \text{, for each } P \in C \text{ such that } \rho(P) = -1.$$ 

Similarly, we denote by $\mathcal{R}(\rho_{-j}^j) \subset \mathbb{R}^{m+k}$ the $C_j'$-semi-algebraic set defined by the formula $\rho_{-j}^j$ obtained by taking the conjunction of

$$-\varepsilon_j < P < \varepsilon_j \text{ for each } P \in C \text{ such that } \rho(P) = 0,$$
$$P > -\varepsilon_j \text{, for each } P \in C \text{ such that } \rho(P) = 1,$$
$$P < \varepsilon_j \text{, for each } P \in C \text{ such that } \rho(P) = -1.$$ 

Since the semi-algebraic sets $D_j$ defined above in Lemma 2.9 are $C$-semi-algebraic sets, each $D_j$ is defined by a disjunction of sign conditions on $C$. More precisely, for each $0 \leq j \leq \ell + 1$ let $D_j$ be defined by the formula

$$D_j = \bigcup_{\rho \in \Sigma_j} \mathcal{R}(\rho),$$

where $\Sigma_j \subset \text{Sign}(C)$.

For each $j, 0 \leq j \leq \ell + 1$, let
where we denote by $D_{j-1}^o = \emptyset$.

**Lemma 2.10.** For $0 \leq j + 1 < i \leq \ell + 1$,

$$\text{Ext}(D_i^o, R_{j-1}) \cap D_j^o = \emptyset.$$  

**Proof.** The inclusions

$$D_{j-1} \subset D_j \subset D_{i-1} \subset D_i,$$

$$D_{j-1}^o \subset \text{Ext}(D_i^o, R_{j-1}) \subset \text{Ext}(D_{i-1}^o, R_{j-1}) \subset \text{Ext}(D_i^o, R_{j-1})$$

follow directly from the definitions of the sets

$$D_i, D_j, D_{j-1}, D_i^o, D_j^o, D_{i-1}^o, D_{j-1}^o,$$

and the fact that

$$\epsilon_i \gg \epsilon_{i-1} \gg \epsilon_j \gg \epsilon_{j-1}.$$  

It follows immediately that

$$D_i^o = \text{Ext}(D_i^o, R_{j-1}) \setminus \text{Ext}(D_{i-1}^o, R_{j-1})$$

is disjoint from $\text{Ext}(D_i^o, R_{j-1})$, and hence also from $D_j^o$. \hfill \Box

We now associate to each $F_j'$ an $S^{\ell-j}$-bundle as follows.

For each $(\omega, x) \in F_j'' = \text{Ext}(F_j', R_{j-2}) \setminus F_j''_{j-1}$, let $L_j^+(\omega, x) \subset R_j^{\ell+1}$ denote the sum of the non-negative eigenspaces of $(\omega, Q^h)(\cdot, x)$ (i.e. $L_j^+(\omega, x)$ is the largest linear subspace of $R_j^{\ell+1}$ on which $(\omega, Q^h)(\cdot, x)$ is positive semi-definite). Since $\text{index}((\omega, Q^h)(\cdot, x)) = j$ stays invariant as $(\omega, x)$ varies over $F_j''$, $L_j^+(\omega, x)$ varies continuously with $(\omega, x)$.

Let

$$\lambda_0(\omega, x) \leq \cdots \leq \lambda_{j-1}(\omega, x) < 0 \leq \lambda_j(\omega, x) \leq \cdots \leq \lambda_{\ell}(\omega, x)$$

be the eigenvalues of $(\omega, Q^h)(\cdot, x)$ for $(\omega, x) \in F_j''$. There is a continuous extension of the map sending $(\omega, x) \mapsto L_j^+(\omega, x)$ to $(\omega, x) \in \text{Ext}(F_j', R_{j-2})$. To see this observe that for $(\omega, x) \in F_j''$ the block of the first $j$ (negative) eigenvalues, $\lambda_0(\omega, x) \leq \cdots \leq \lambda_{j-1}(\omega, x)$, and hence the sum of the eigenspaces corresponding to them can be extended continuously to any infinitesimal neighborhood of $F_j''$, and in particular to $\text{Ext}(F_j', R_{j-2})$. Now $L_j^+(\omega, x)$ is the orthogonal complement of the sum of the eigenspaces corresponding to the block of negative eigenvalues, $\lambda_0(\omega, x) \leq \cdots \leq \lambda_{j-1}(\omega, x)$.

We denote by $C_j' \subset F_j' \times R_j^{\ell+1}$ the semi-algebraic set defined by

$$C_j' = \{ (\omega, x, y) \mid (\omega, x) \in F_j', y \in L_j^+(\omega, x), |y| = 1 \}.$$
Proposition 2.11. For every $j$ from $\ell$ to 1,
\[ C'_{j-1} \cap \text{Ext}(C'_j, R_{j-2}) = \pi_j^{-1}(\text{Ext}(F'_j, R_{j-2}) \cap F'_{j-1}), \]
and
\[ \pi_j|_{C'_{j-1} \cap \text{Ext}(C'_j, R_{j-2})} : C'_{j-1} \cap \text{Ext}(C'_j, R_{j-2}) \to \text{Ext}(F'_j, R_{j-2}) \cap F'_{j-1} \]
is a $S^{\ell-j}$-bundle over $\text{Ext}(F'_j, R_{j-2}) \cap F'_{j-1}$. □

We have the following.

Proposition 2.12. The semi-algebraic set
\[ C' = \bigcup_{j=0}^{\ell+1} \text{Ext}(C'_j, R_0) \]
is homotopy equivalent to $\text{Ext}(C, R_0)$.

Proof. First observe that $C = \lim_{\varepsilon \to 1} C'$ where $C$ is the semi-algebraic set defined in (2.14) above.

Now let,
\[ C_0 = \lim_{\varepsilon \to 0} C', \]
\[ C_i = \lim_{\varepsilon \to \varepsilon_i} C_{i-1}, 1 \leq i \leq \ell + 1. \]

Note that each $C_i$ is a closed and bounded semi-algebraic set. Also, let $C_{i-1, t} \subseteq R^{\ell-k}_{t-1}$ be the semi-algebraic set obtained by replacing $\varepsilon_i$ in the definition of $C_{i-1}$ by the variable $t$. Then, there exists $t_0 > 0$, such that for all $0 < t_1 < t_2 \leq t_0$, $C_{i-1, t_1} \subseteq C_{i-1, t_2}$.

It follows (see Lemma 16.17 in [10]) that for each $i$, $0 \leq i \leq \ell + 1$, $\text{Ext}(C_i, R_t)$ is homotopy equivalent to $C_{i-1}$ where $C_{-1} = C'$.

The proposition is now a consequence of Proposition 2.7. □

Proof of Theorem 2.3. In light of Propositions 2.7 and 2.12 it suffices to bound the Betti numbers of the semi-algebraic set $C'$. Now,
\[ C' = \bigcup_{j=0}^{\ell+1} \text{Ext}(C'_j, R_0). \]

By (2.2) it suffices to bound the Betti numbers of the various intersections amongst the sets $\text{Ext}(C'_j, R_0)$'s. However, by Lemma 2.10 the only non-empty intersections among $\text{Ext}(C'_j, R_0)$'s are of the form $\text{Ext}(C'_j, R_0) \cap \text{Ext}(C'_{j+1}, R_0)$. Using Proposition 2.2 and Proposition 2.11 we have that $b(C'_j)$ (resp. $b(C'_j \cap C'_{j+1})$) is bounded by $2b(F'_j)$ (resp. $2b(\text{Ext}(F'_j, R_0) \cap \text{Ext}(F'_{j+1}, R_0))$).

Finally, each $F'_j$ (resp. $\text{Ext}(F'_j, R_0) \cap \text{Ext}(F'_{j+1}, R_0)$) is a bounded $\mathcal{P}^c$-closed semi-algebraic set, where $\mathcal{P}^c_j = R[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$ is defined by
\[ \mathcal{P}^c_j = \mathcal{P} \cup C'_j \cup \bigcup_{i=1}^{m} \{ Z_i \}. \]
Note that
\[
\deg(P) \leq d, \quad P \in \mathcal{P},
\]
\[
\deg(P) \leq d(\ell + 1), \quad P \in \mathcal{C}_j',
\]
\[
\#(P) = s,
\]
\[
\#(\mathcal{C}_j') = 2(\ell + 1).
\]
Now applying Theorem 1.1 we obtain that
\[
b(F_j'), b(\text{Ext}(F_j', R_0) \cap \text{Ext}(F_{j+1}', R_0)) \leq (\ell s \text{md})^O(m+k).
\]
The theorem follows immediately. \(\square\)

Proof of Theorem 2.4. Apply (2.3) together with Theorem 2.3. \(\square\)

2.3. General Case. We now prove the general version of Theorem 2.3. We follow Notation 2.

Theorem 2.13. Let \(W \subset \mathbb{R}^\ell \times \mathbb{R}^k\) be semi-algebraic set defined by
\[
W = \bigcap_{Q \in \mathcal{Q}} \{(y, x) \mid Q(y, x) \leq 0 \land \Phi(x)\},
\]
where \(\Phi(x)\) is a \(\mathcal{P}\)-closed formula defining a bounded \(\mathcal{P}\)-closed semi-algebraic set \(V \subset \mathbb{R}^k\).
Then,
\[
b(W) \leq (\ell s \text{md})^O(m+k).
\]

Proof. Let \(1 \gg \varepsilon > 0\) be an infinitesimal and let \(B_0(0, 1/\varepsilon)\) denote the closed ball
in \(\mathbb{R}^{\ell + k}\) centered at the origin and of radius \(1/\varepsilon\).
Let \(W_b \subset \mathbb{R}^{\ell + k}\) be the set defined by
\[
W_b = W \cap \left(B_0(0, 1/\varepsilon) \times \mathbb{R}^k\right).
\]
It follows from the local conical structure of semi-algebraic sets at infinity \[13, Theorem 9.3.6\] that \(W_b\) has the same homotopy type as \(\text{Ext}(W, \mathbb{R}(\varepsilon))\).
Let \(Q_0 = \varepsilon^2(Y_1^2 + \cdots + Y_\ell^2) - 1\),
and \(W_b^h \subset \mathbb{S}^\ell \times \mathbb{R}(\varepsilon)^k\) be the semi-algebraic set defined by
\[
W_b^h = \bigcap_{i=0}^m \{(y, x) \mid |y| = 1 \land Q_i^h(y, x) \leq 0 \land \Phi(x)\}.
\]
It is clear that \(W_b^h\) is a union of two disjoint, closed and bounded semi-algebraic sets, each homeomorphic to \(W_b\). Hence, for every \(i = 0, \ldots, k + \ell - 1\)
\[
b_i(W_b^h) = 2b_i(W_b) = 2b_i(W).
\]
The theorem is proved by applying Theorem 2.4 to \(W_b^h\). \(\square\)
2.4. Proof of Theorem [1.5]. We are now in a position to prove Theorem [1.5]. We first need a few preliminary results.

Given a list of polynomials \( A = \{ A_1, \ldots, A_t \} \) with coefficients in \( \mathbb{R} \), we introduce \( t \) infinitesimals, \( 1 \gg \delta_1 \gg \cdots \gg \delta_t > 0 \).

We define \( A_{i+1} = \{ A_{i+1}, \ldots, A_t \} \) and

\[
\Sigma_i = \{ A_i = 0, A_i = \delta_i, A_i = -\delta_i, A_i \geq 2\delta_i, A_i \leq -2\delta_i \},
\]

\[
\Sigma_{\leq i} = \{ \Psi \mid \Psi = \bigwedge_{j=1, \ldots, i} \Psi_j, \Psi_j \in \Sigma_i \}.
\]

If \( \Phi \) is any \( A \)-closed formula, we denote by \( R_i(\Phi) \) the extension of \( R(\Phi) \) to \( R\langle \delta_1, \ldots, \delta_i \rangle \).

Proposition 2.14. For every \( A \)-closed formula \( \Phi \),

\[
b(\Phi) \leq \sum_{\Psi \in \Sigma_{\leq i}} b(\Psi).
\]

Proof. See [10].

Proof of Theorem [1.5]. First note that we can assume (if necessary by adding to \( Q \) an extra degree 2 inequality) that the set \( S \) is bounded.

Denoting \( P = \{ P_1, \ldots, P_s \} \), define \( B = \{ B_1, \ldots, B_{s+m} \} \), where

\[
B_i = \begin{cases} Q_i, & 1 \leq i \leq m, \\ P_{i-m}, & m + 1 \leq i \leq m + s. \end{cases}
\]

It follows from Proposition 2.14 that in order to bound \( b(S) \), it suffices to bound \( b(T) \), where \( T \) is defined by

\[
\bigwedge_{i=1}^{s+m} B_i^2(B_i^2 - \delta_i^2)^2(B_i^2 - 4\delta_i^2) \geq 0
\]

We now introduce \( m \) new variables, \( Z_1, \ldots, Z_m \) and let

\( A = \{ A_1, \ldots, A_{s+m} \} \subset \mathbb{R}[Y_1, \ldots, Y_t, X_1, \ldots, X_k, Z_1, \ldots, Z_m] \)

be defined by

\[
A_i = \begin{cases} Z_i, & 1 \leq i \leq m, \\ P_{i-m}, & m + 1 \leq i \leq m + s. \end{cases}
\]

Consider the semi-algebraic set \( T' \subset \mathbb{R}^{m+k+l} \) defined by

\[
\bigwedge_{i=1}^{s+m} A_i^2(A_i^2 - \delta_i^2)^2(A_i^2 - 4\delta_i^2) \geq 0 \land \bigwedge_{i=1}^{m} (Z_i - Q_i = 0),
\]

Clearly, \( T \) is homeomorphic to \( T' \). Notice that the number of polynomials in the definition of \( T' \), which depend only on \( X \) and \( Z \) is \( s + m \), and the degrees of these polynomials are bounded by \( 6d \). The number of polynomials depending on \( X, Y \) and \( Z \) is \( m \) and these are of degree at most 2 in \( Y \) and at most \( d \) in the remaining variables. Thus, we are in a position to apply Theorem 2.13 to obtain that

\[
b(S) \leq b(T') \leq (sfdm)^{O(m+k)}.
\]

This proves the theorem. 

It is clear that ˜S is semi-algebraically homeomorphic to S. Applying Theorem 1.5 to ˜S, we obtain the desired bound. □

3. Algorithm for Computing the Euler-Poincaré Characteristic

We first need a few preliminary definitions and results.

3.1. Some Algorithmic and Mathematical Preliminaries. Recall that for a locally closed semi-algebraic set S, the Borel-Moore Euler-Poincaré characteristic of S, denoted by χBM(S), is defined by

\[ \chi_{BM}(S) = \sum_{i=0}^{k} (-1)^i b_i^{BM}(S), \]

where \( b_i^{BM}(S) \) denotes the dimension of the \( i \)-th Borel-Moore homology group \( H_i^{BM}(S, \mathbb{Z}/2\mathbb{Z}) \) of S. Note that \( \chi_{BM}(S) = \chi(S) \) for S closed and bounded.

Note that \( \chi_{BM}(S) \) has the following classically known (see e.g. [10] for a proof) additivity property.

**Proposition 3.1.** Let \( X_1 \) and \( X_2 \) be locally closed semi-algebraic sets such that \( X_1 \cap X_2 = \emptyset \). Then

\[ \chi_{BM}(X_1 \cup X_2) = \chi_{BM}(X_1) + \chi_{BM}(X_2), \]

provided that \( X_1 \cup X_2 \) is locally closed, as well. □

Let \( Z \subset \mathbb{R}^k \) and \( Q \in \mathbb{R}[X_1, \ldots, X_k] \). We define

\[ \mathcal{R}(Q = 0, Z) = \{ x \in Z \mid Q(x) = 0 \}, \]
\[ \mathcal{R}(Q > 0, Z) = \{ x \in Z \mid Q(x) > 0 \}, \]
\[ \mathcal{R}(Q < 0, Z) = \{ x \in Z \mid Q(x) < 0 \}. \]

**Corollary 3.2.** Let \( Z \subset \mathbb{R}^k \) be a locally closed semi-algebraic set. Then

\[ \chi_{BM}(Z) = \chi_{BM}(\mathcal{R}(Q = 0, Z)) + \chi_{BM}(\mathcal{R}(Q > 0, Z)) + \chi_{BM}(\mathcal{R}(Q < 0, Z)). \]

**Notation 6.** Let \( Z \subset \mathbb{R}^k \) be a locally closed semi-algebraic set and let \( \mathcal{A} \) be a finite subset of \( \mathbb{R}[X_1, \ldots, X_k] \).

The realization of the sign condition \( \rho \in \{0, 1, -1\}^\mathcal{A} \) on \( Z \) is

\[ \mathcal{R}(\rho, Z) = \{ x \in Z \mid \bigwedge_{A \in \mathcal{A}} \text{sign}(A(x)) = \rho(A) \}, \]

and its Borel-Moore Euler-Poincaré characteristic is denoted \( \chi_{BM}(\rho, Z) \).

We denote by \( \text{Sign}(\mathcal{A}, Z) \) the list of \( \rho \in \{0, 1, -1\}^\mathcal{A} \) such that \( \mathcal{R}(\rho, Z) \) is non-empty. We denote by \( \chi_{BM}(\mathcal{A}, Z) \) the list of Euler-Poincaré characteristics \( \chi_{BM}(\rho, Z) = \chi_{BM}(\mathcal{R}(\rho, Z)) \) for \( \rho \in \text{Sign}(\mathcal{A}, Z) \).

Finally, given two finite families of polynomials, \( \mathcal{A} \subset \mathcal{A}' \), and \( \rho \in \{0, 1, -1\}^\mathcal{A}, \rho' \in \{0, 1, -1\}^{\mathcal{A}'} \), we define \( \rho \prec \rho' \) by: for all \( P \in \mathcal{A} \), \( \rho(P) = \rho'(P) \).
We will use the following algorithm for computing the list \( \chi^{BM}(A, Z) \) described in [10]. We describe here the input, output and complexity of the algorithm.

**Algorithm 1 (Euler-Poincaré Characteristic of Sign Conditions).**

**Input** A finite list \( A = \{A_1, \ldots, A_i\} \) of polynomials in \( R[X_1, \ldots, X_k] \).

**Output** The list \( \chi^{BM}(A) \).

**Complexity:** Let \( d \) be a bound on the degrees of the polynomials in \( A \), and \( t = \#(A) \). The number of arithmetic operations is bounded by

\[
t^{k+1}O(d)^k + t^k((k \log_2(s) + k \log_2(d))d)^{O(k)}.
\]

The algorithm also involves the inversion of matrices of size \( t^kO(d)^k \) with integer coefficients.

**3.2. Algorithms for the Euler-Poincaré characteristic.** We first deal with the special case of polynomials which are homogeneous and of degree two in the variables \( y_0, \ldots, y_t \), and in this case we describe algorithms (Algorithms 2 and 3 below) for computing the Euler-Poincaré characteristic of the sets \( A^h \) and \( W^h \) respectively. We then use Algorithm 3 to derive algorithms for computing the Euler-Poincaré characteristic in the general case (Algorithms 4 and 5 below).

**3.2.1. Homogeneous quadratic polynomials.**

**Algorithm 2 (Euler-Poincaré characteristic, homogeneous union case).**

**Input**
- A family of polynomials, \( Q^h \subset R[y_0, \ldots, y_t, x_1, \ldots, x_k] \), with \( \deg_y(Q) \leq 2, \deg_x(Q) \leq d, Q \in Q^h, \#(Q^h) = m \), homogeneous with respect to \( Y \);
- another family, \( P \subset R[x_1, \ldots, x_k] \) with \( \deg_x(P) \leq d, P \in P, \#(P) = s \);
- a formula \( \Phi \) defining a bounded \( P \)-closed semi-algebraic set \( V \);
- the semi-algebraic set defined by

\[
A^h = \bigcup_{Q \in Q^h} \{(y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\}.
\]

**Output** the Euler-Poincaré characteristic \( \chi(A^h) \).

**Procedure**

Step 1 Let \( Z = (z_1, \ldots, z_m) \) be variables and let \( M \) be the symmetric matrix with entries in \( R[Z_1, \ldots, Z_m, X_1, \ldots, X_k] \) associated to the quadratic form \((Z, Q^h)\). Obtain \( C_i \in R[Z_1, \ldots, Z_m, X_1, \ldots, X_k] \) by computing the following determinant.

\[
\det(T \cdot \text{Id}_{\ell+1} - M) = T^{\ell+1} + C_1 T^\ell + \cdots + C_0.
\]

Step 2 Compute \( \chi^{BM}(C, F) \) as follows. Call Algorithm 4 with input \( C' = C \cup P \). Compute from the output the list \( \chi^{BM}(C, F) \), using the additivity property of the Borel-Moore Euler-Poincaré characteristic (Proposition 3.1). For each \( \rho \in \{0, +1, -1\}^C \), such that there exists \( \rho' \in \text{Sign}(C', F) \) with \( \rho \prec \rho' \) (see Notation 6) and \( \rho'(Z_j) \in \{0, -1\} \) for \( 1 \leq j \leq m \), compute

\[
\chi^{BM}(\rho, F) = \sum_{\rho' : \rho \prec \rho'} \chi^{BM}(\rho', F).
\]

\[
\rho'(Z_j) \in \{0, -1\}, 1 \leq j \leq s
\]
Step 3 Output

\[ \chi(A_h) = \sum_{\rho \in \text{Sign}(C, F)} \chi_{BM}(R(\rho, F)) \cdot (1 + (-1)^{(k-n(\rho))}), \]

\( n(\rho) \) denote the number of sign variations in the sequence, \( \rho(C_0), \ldots, (-1)^i\rho(C_i), \ldots, (-1)^\ell\rho(C_\ell), +1 \).

**Proof of Correctness:** It follows from Lemma 2.6 that for any \( \rho \in \text{Sign}(C, F) \),

\[ \chi_{BM}(\varphi_1^{-1}(R(\rho))) = \chi_{BM}(R(\rho)) \cdot (1 + (-1)^{(k-n(\rho))}). \]

Also, by virtue of Proposition 2.5 we have that

\[ \chi_{BM}(B) = \chi(A), \quad \text{and} \quad B = \bigcup_{\rho \in \text{Sign}(C, F)} \varphi_1^{-1}(R(\rho)). \]

The correctness of the algorithm is now a consequence of the additivity property of the Borel-Moore Euler Poincaré characteristic (Proposition 3.1) and the correctness of Algorithm 1.

**Complexity Analysis:** The complexity of the algorithm is \( (\ell smd)^{O(m+k)} \) using the complexity of Algorithm 1.

We are now in a position to describe the algorithm for computing the Euler-Poincaré characteristic in the homogeneous intersection case.

**Algorithm 3** (Euler-Poincaré characteristic, homogeneous intersection case).

**Input**
- A family of polynomials, \( Q_h = \{Q^1_h, \ldots, Q^m_h\} \subset \mathbb{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k] \), with \( \deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in Q_h \), homogeneous with respect to \( Y \),
- another family, \( P \subset \mathbb{R}[X_1, \ldots, X_k] \) with \( \deg_P \leq d, P \in P, \#(P) = s \),
- a formula \( \Phi \) defining a bounded \( P \)-closed semi-algebraic set \( V \),
- the semi-algebraic set

\[ W_h = \bigcap_{Q \in Q_h} \{(y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\}. \]

**Output**
the Euler-Poincaré characteristic \( \chi(W_h) \).

**Procedure**

Step 1 For each subset \( J \subset [m] \) do the following.

Compute \( \chi(A^J) \) using Algorithm 2, where

\[ A^J = \bigcup_{Q \in J} \{(y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\}. \]

Step 2 Output

\[ \chi(W_h) = \sum_{J \subset Q} (-1)^{\#(J)+1} \chi(A^J). \]

**Proof of Correctness:** The correctness of the algorithm is a consequence of the additivity of Euler-Poincaré characteristic (Proposition 3.1) and the correctness of Algorithm 2.
Complexity Analysis: There are $2^m$ calls to Algorithm 2. Using the complexity analysis of Algorithm 2, the complexity of the algorithm is bounded by $(\ell \text{smd})^{O(m+k)}$.

3.2.2. The Case of Intersections.

Algorithm 4 (Euler-Poincaré Characteristic, Intersection Case).

**Input**
- A family of polynomials, $Q \subset \mathbb{R}[Y_1, \ldots, Y_{\ell}, X_1, \ldots, X_k]$, with $\deg_Y(Q) \leq 2$, $\deg_X(Q) \leq d$, $Q \in Q$, $\#(Q) = m$
- another family of polynomials, $P \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\deg_X(Q) \leq d$, $P \in P$, $\#(P) = s$,
- a $P$-closed formula $\Phi(x)$ $P$-closed semi-algebraic set $V \subset \mathbb{R}^k$,
- the semi-algebraic set $W = \bigcap_{Q \in Q} \{ (y, x) \mid Q(y, x) \leq 0 \land \Phi(x) \}$.

**Output** the Euler-Poincaré characteristic $\chi(W)$.

**Procedure**

Step 1 Replace $Q^h$ by $Q^h \cup \{Q^b_0\}$, with $Q_0 = \varepsilon^2(Y_1^2 + \ldots + Y_\ell^2) - 1$. Define

$$W^h_b = \bigcap_{Q^b \in Q^b} \{ (y, x) \mid |y| = 1 \land Q^b(y, x) \leq 0 \land \Phi(x) \}.$$

Step 2 Using Algorithm 3 compute $\chi(W^h_b)$.

Step 3 Output $\chi(W) = \frac{1}{2} \chi(W^h_b)$.

**Proof of Correctness:** The correctness of Algorithm 4 follows from (2.18) and the correctness of Algorithm 3.

**Complexity Analysis:** The complexity of the algorithm is clearly $(\ell \text{smd})^{O(m+k)}$ arithmetic operations in $\mathbb{R}(\varepsilon)$ from the complexity analysis of Algorithm 3. Moreover, the maximum degree in $\varepsilon$ is bounded by $(\ell \text{md})^{O(m+k)}$. Finally, the complexity of the algorithm is $(\ell \text{smd})^{O(m+k)}$ arithmetic operations in $\mathbb{R}$.

3.2.3. The case of a $Q \cup P$-closed semi-algebraic set. Since we want to deal with a general $Q \cup P$-closed semi-algebraic set, we shall need a property similar to Corollary 3.2 in a context where all the sets considered are closed and bounded.

We need a few preliminary definitions and results. Let $Q = \{Q_1, \ldots, Q_m\}$ and $0 < \varepsilon_m \ll \ldots \ll \varepsilon_1 \ll \varepsilon_0 \ll 1$ be infinitesimals. For every $j \in [m] = \{1, \ldots, m\}$, denote $R_j = \mathbb{R}(\varepsilon_0, \ldots, \varepsilon_j)$. Let

$$\Psi^0_i = (Q_i = 0),$$  
$$\Psi^1_i = (Q_i \geq \varepsilon_i),$$  
$$\Psi^{-1}_i = (Q_i \leq -\varepsilon_i),$$  
$$\Psi^2_i = (Q_i = \varepsilon_i),$$  
$$\Psi^{-2}_i = (Q_i = -\varepsilon_i).$$

The following Lemma 3.3 plays a role similar to Corollary 3.2.

**Lemma 3.3.** Let $S$ be a $Q \cup P$-closed bounded semi-algebraic set. For every $j \in [m]$ 

$$\chi(S) = \chi(R(\Psi^j_0, S)) + \chi(R(\Psi^1_0, S)) + \chi(R(\Psi^{-1}_0, S)) - \chi(R(\Psi^2_0, S)) - \chi(R(\Psi^{-2}_0, S)).$$
The claims follow from the additivity property of the Euler-Poincaré characteristic, and the fact that
\[ \chi(R(\Psi_j^0, S)) = \chi(\{(x, y) \in S \mid -\varepsilon_j \leq Q_j(x, y) \leq \varepsilon_j\}) , \]
since \( R(\Psi_j^0, S) \) is a deformation retract of \( \{(x, y) \in S \mid -\varepsilon_j \leq Q_j(x, y) \leq \varepsilon_j\} \). □

We define \( \Sigma_m = \{-2, -1, 0, 1, 2\}^{[m]} \) and to every \( \rho \in \Sigma_m \) we associate its realization \( R(\rho) \) defined by
\[
R(\rho) = \bigcap_{i=1}^{m} \{ (x, y) \in R^{d+k} \mid \Psi_i^{\rho(i)}(x, y) \}.
\]

To any \( \rho \in \Sigma_m \) we associate a sign condition \( \rho' \in \{-1, 0, 1\}^{[m]} \) defined by
\[
\rho'(i) = 0 \text{ if } \rho(i) = 0, \\
\rho'(i) = 1 \text{ if } \rho(i) > 0, \\
\rho'(i) = -1 \text{ if } \rho(i) < 0.
\]

Algorithm 5 (Euler-Poincaré, the general case).

**INPUT**
- A family of polynomials, \( Q = \{Q_1, \ldots, Q_m\} \subset R[Y_1, \ldots, Y_t, X_1, \ldots, X_k] \), with \( \deg_x(Q) \leq 2, \deg_x(Q) \leq d \),
- another family of polynomials, \( P \subset R[X_1, \ldots, X_k] \) with \( \deg_x(P) \leq d, P \in P, \#(P) = s \),
- a \( Q \cup P \)-closed semi-algebraic set \( S \).

**OUTPUT** the Euler-Poincaré characteristic \( \chi(S) \).

**PROCEDURE**

Step 1 Define \( Q_0 = \varepsilon_0^2(1^2 + \ldots + Y_t^2) - 1, P_0 = \varepsilon_0^2(1^2 + \ldots + X_k^2) - 1 \). Replace \( P \) by \( P \cup \{P_0\} \) and \( S \) by \( R(S, R(\varepsilon)) \cap (R(Q_0 \leq 0) \times R(P_0 \leq 0)) \).

Step 2 Write the formula defining \( S \) as
\[
\bigvee_{\rho' \in \{-1, 0, 1\}^{[m]}} \left( \bigwedge_{i=1}^{m} \text{sign}(Q_i(x, y)) = \rho'(i) \right) \land \Phi_{\rho'}(x) \land (Q_0(y) \leq 0)
\]
where each \( \Phi_{\rho'} \) is a \( P \)-closed formula.

Step 3 Using Algorithm 4 for every generalized sign condition \( \rho \in \Sigma_m \), compute \( \chi(R(\rho, S)) \) where
\[
R(\rho, S) = \{(x, y) \in \text{Ext}(S, R_m) \mid (x, y) \in R(\rho) \land \Phi_{\rho'}(x) \land (Q_0(y) \leq 0)\}
\]

Step 4 Denoting by \( n(\rho) = \#(\{i \in [m] \mid |\rho(i)| = 2\}) \), output
\[
\chi(S) = \sum_{\rho} (-1)^{n(\rho)} \chi(R(\rho, S)).
\]

**Proof of Correctness:** It follows from the local conic structure of semi-algebraic sets at infinity [13, Theorem 9.3.6] that replacing \( S \) by \( R(S, R(\varepsilon)) \cap (R(Q_0 \leq 0) \times R(P_0 \leq 0)) \) does not modify the Euler-Poincaré characteristic. The proof is now based on the following two lemmas.
Lemma 3.4. Let $S$ be a $Q \cup P$-closed and bounded semi-algebraic set. Denoting by $n(\rho) = \#(\{i \in [m] \mid |\rho(i)| = 2\})$, for $\rho \in \Sigma_m$,

\[
\chi(S) = \sum_{\rho \in \Sigma_m} (-1)^{n(\rho)} \chi(\mathcal{R}(\rho, S)).
\]

Proof. The proof is by induction on $m$. The induction hypothesis $H_j$ states that denoting by $n(\rho) = \#(\{i \in [j] \mid |\rho(i)| = 2\})$ for $\rho \in \Sigma_j$,

\[
\chi(S) = \sum_{\rho \in \Sigma_j} (-1)^{n(\rho)} \chi(\mathcal{R}(\rho, S)).
\]

The base case $H_1$ is exactly Lemma 3.3 applied to $S$. Suppose now that $H_{j-1}$ holds for some $1 < j \leq m$, i.e.

\[
(3.1) \quad \chi(S) = \sum_{\rho \in \Sigma_{j-1}} (-1)^{n(\rho)} \chi(\mathcal{R}(\rho, S))
\]

and let us prove $H_j$. Define $Q_j = Q \cup \{Q_i \pm \varepsilon_i, i = 1, \ldots, j\}$.

For every $\rho \in \Sigma_{j-1}$, $\mathcal{R}(\rho, S)$ is a $Q_{j-1} \cup P$-closed semi-algebraic set. Denoting by $\rho_i \in \Sigma_j$, for $\rho \in \Sigma_{j-1}$, $i \in \{-2, -1, 0, 1, 2\}$, the generalized sign condition defined by $\rho_i(u) = \rho(u)$, $u = 1, \ldots, j - 1$, $\rho_i(j) = i$, notice that $\mathcal{R}(\rho_i, S) = \mathcal{R}(\Psi_j^i, \mathcal{R}(\rho, S))$.

Using Lemma 3.3 applied to $\mathcal{R}(\rho, S)$, we obtain

\[
\chi(\mathcal{R}(\rho, S)) = \chi(\mathcal{R}(\rho_0, S)) + \chi(\mathcal{R}(\rho_1, S)) + \chi(\mathcal{R}(\rho_{-1}, S)) - \chi(\mathcal{R}(\rho_2, S)) - \chi(\mathcal{R}(\rho_{-2}, S)).
\]

Substituting each $\chi(\mathcal{R}(\rho, S))$ by its value in (3.1) one gets $H_j$, since every element of $\Sigma_j$ is of the form $\rho_i$ for some $\rho \in \Sigma_{j-1}$, $i \in \{-2, -1, 0, 1, 2\}$.

\[\Box\]

Complexity Analysis: There are $5^m$ calls to Algorithm 4. The complexity of the algorithm is clearly $(\ell smd)^O(m+k)$ arithmetic operations in $\mathbb{R}_m$ from the complexity analysis of Algorithm 4. Moreover the maximum degree in $\varepsilon_0, \ldots, \varepsilon_m$ is bounded by $(\ell smd)^O(m+k)$. Finally the complexity of the algorithm is $(\ell smd)^O(m(m+k))$ arithmetic operations in $\mathbb{R}$.

4. Computing all Betti Numbers

We now consider the algorithmic problem of computing all the Betti numbers of a semi-algebraic set defined by a partly quadratic system of polynomials. The same problem for a fully quadratic system of polynomials was considered in [5], where an algorithm is described for computing the Betti numbers of a set defined by the intersection of quadratic inequalities, whose complexity is doubly exponential in the number of inequalities, but polynomial in the number of variables for a fixed number of inequalities. The algorithm we describe here is an adaptation of the algorithm in [5], to the case where there are parameters, and the degrees with respect to these parameters could be larger than two. In this paper we treat the case of general $P \cup Q$-closed sets, not just basic closed ones. We also provide more details and analyze the complexity of the algorithm more carefully, in order to take into account the dependence on the additional parameters.
4.1. Summary of the main idea. The main idea behind the algorithm can be summarized as follows.

By virtue of Proposition 2.5, in order to compute the Betti numbers of \( A^h \), it suffices to construct a cell complex, \( K(B,V) \), whose associated space is homotopy equivalent to the set \( B \) defined by \( \omega, x \). In order to do so, we first compute a semi-algebraic triangulation, \( h : \Delta \rightarrow F \), such that as \( (\omega, x) \) varies over the image of any simplex \( \sigma \in \Delta \), the index of \( (\omega, Q^h(\cdot, x)) \) stays fixed, and we have a continuous choice of an orthonormal basis,

\[
\{ e_0(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x) \}
\]

consisting of eigenvectors of the symmetric matrix associated to the quadratic form \( \langle \omega, Q^h(\cdot, x) \rangle \). Moreover, if index(\( (\omega, Q^h(\cdot, x)) \)) = \( j \) for \( (\omega, x) \in h(\sigma) \), then \( \varphi^{-1}_1(\omega, x) \) can be retracted to \( S^f \cap \text{span}(e_j(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x)) \), and the flag of subspaces defined by the orthonormal basis, \( \{ e_0(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x) \} \), gives an efficient regular cell decomposition of the sphere \( S^f \cap \text{span}(e_j(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x)) \) into \( 2(\ell - j + 1) \) cells, having two cells of each dimension from 0 to \( \ell - j \) (see Definition 4.3).

Now consider a pair of simplices, \( \sigma, \tau \in \Delta \), with \( \sigma \prec \tau \). The orthonormal basis \( \{ e_0(\tau, \omega, x), \ldots, e_\ell(\tau, \omega, x) \} \), defined for \( (\omega, x) \in h(\tau) \) might not have a continuous extension to \( h(\sigma) \) on the boundary of \( h(\tau) \). In particular, the cell decompositions of the fibers, \( S^f \cap \text{span}(e_j(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x)) \), over points in \( (\omega, x) \in h(\sigma) \) might not be compatible with those over neighboring points in \( h(\tau) \). In order to obtain a proper cell complex we need to compute a common refinement of the cell decomposition of the sphere over each point in \( (\omega, x) \in h(\sigma) \) induced by the basis \( \{ e_0(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x) \} \), and the one obtained as a limit of those over certain points in \( h(\tau) \) converging to \( (\omega, x) \). We need to further subdivide \( h(\sigma) \) to ensure that over each cell of this subdivision the combinatorial type of the above refinements stays the same.

Before describing the construction of \( K(B,V) \) in more detail, we need some preliminaries on triangulations.

4.2. Triangulations. We first need to recall a fact from semi-algebraic geometry about triangulations of semi-algebraic sets, and then we define the notion of an Index Invariant Triangulation and give an algorithm for computing it.

4.2.1. Triangulations of semi-algebraic sets. A triangulation of a closed and bounded semi-algebraic set \( S \) is a simplicial complex \( \Delta \) together with a semi-algebraic homeomorphism from \( \Delta \) to \( S \). We always assume that the simplices in \( \Delta \) are open. Given such a triangulation we will often identify the simplices in \( \Delta \) with their images in \( S \) under the given homeomorphism, and will refer to the triangulation by \( \Delta \).

Given a triangulation \( \Delta \), the cohomology groups \( H^i(S) \) are isomorphic to the simplicial cohomology groups \( H^i(\Delta) \) of the simplicial complex \( \Delta \) and are in fact independent of the triangulation \( \Delta \) (this fact is classical over \( \mathbb{R} \); see for instance [10] for a self-contained proof in the category of semi-algebraic sets).

We call a triangulation \( h_1 : \Delta_1 \rightarrow S \) of a semi-algebraic set \( S \), to be a refinement of a triangulation \( h_2 : \Delta_2 \rightarrow S \) if for every simplex \( \sigma_1 \in \Delta_1 \), there exists a simplex \( \sigma_2 \in \Delta_2 \) such that \( h_1(\sigma_1) \subset h_2(\sigma_2) \).

Let \( S_1 \subset S_2 \) be two compact semi-algebraic subsets of \( \mathbb{R}^k \). We say that a semi-algebraic triangulation \( h : \Delta \rightarrow S_2 \) of \( S_2 \) respects \( S_1 \) if for every simplex \( \sigma \in \Delta \),
Note that for every \((\omega, x)\) (4.1) \(M_{1}\) is identified with a sub-complex of \(\Delta\) and \(h_{1}^{-1}(S_{1}) : h^{-1}(S_{1}) \rightarrow S_{1}\) is a semi-algebraic triangulation of \(S_{1}\). We will refer to this sub-complex by \(\Delta|_{S_{1}}\).

We will need the following theorem which can be deduced from Section 9.2 in [13] (see also [10]).

**Theorem 4.1.** Let \(S_{1} \subset S_{2} \subset \mathbb{R}^{k}\) be closed and bounded semi-algebraic sets, and let \(h_{i} : \Delta_{i} \rightarrow S_{i}, i = 1, 2\) be semi-algebraic triangulations of \(S_{1}, S_{2}\). Then, there exists a semi-algebraic triangulation \(h : \Delta \rightarrow S_{2}\) of \(S_{2}\) such that \(\Delta\) respects \(S_{1}\), \(\Delta\) is a refinement of \(\Delta_{2}\), and \(\Delta|_{S_{1}}\) is a refinement of \(\Delta_{1}\).

Moreover, there exists an algorithm which computes such a triangulation with complexity bound \((sd)^{O(1)}\epsilon\), where \(s\) is the number of polynomials used in the definition of \(S_{1}\) and \(S_{2}\), and \(d\) is a bound on their degrees.

**4.2.2. Parametrized eigenvector basis.** Let \(M(\omega, x)\) be the symmetric matrix associated to the quadratic form \((\omega, Q^h)(\cdot, x)\) defined by (2.11). When \(M(\omega, x)\) has simple eigenvalues for all possible choice of \(\omega, x\) in some domain, there is a finite choice of orthonormal bases consisting of eigenvectors of \(M(\omega, x)\). However, when \(M(\omega, x)\) has multiple eigenvalues, the number of choices of orthonormal basis of eigenvectors is infinite. In order to avoid the problem caused by the latter situation we are going to use an infinitesimal deformation as follows.

Let \(0 < \epsilon < 1\) be an infinitesimal and

\[
(4.1) \quad M_\epsilon(\omega, x) = (1 - \epsilon) M(\omega, x) + \epsilon \text{diag}(0, 1, 2, \ldots, \ell).
\]

Note that for every \((\omega, x) \in \Omega \times \mathbb{R}^{k}\) the eigenvalues of \(M_\epsilon(\omega, x)\) in \(\mathbb{R}^{\ell}\) are distinct and nonzero. Indeed, replace \(\epsilon\) by \(t\) in the definition of \(M_\epsilon(\omega, x)\) and obtain \(M_t(\omega, x)\). Observe that the statement is true if \(t = 1\), since the matrix \(M_1(\omega, x)\) has distinct eigenvalues. Thus, the set of \(t\)'s in the algebraically closed field \(\mathbb{R}[i]\) for which \(M_1(\omega, x)\) has \(\ell + 1\) distinct eigenvalues is non-empty, constructible and contains an open subset, since the condition of having distinct eigenvalues is a stable condition. Thus, there exists \(\epsilon_0 > 0\) such that for all \(\epsilon \in (0, \epsilon_0)\), \(M_\epsilon(\omega, x)\) has \(\ell + 1\) distinct eigenvalues, and hence it is also the case for the infinitesimal \(\epsilon\).

Denote by \(\Lambda(M_\epsilon(\omega, x), T) = \det(T \cdot \text{Id}_{\ell+1} - M_\epsilon(\omega, x))\) the characteristic polynomial of \(M_\epsilon(\omega, x)\). Let \(A \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]\) be a set of polynomials containing \(C\) (see (2.16)) and such that for every sign condition \(\rho \in \{0, 1, -1\}^{A}\) and every \((\omega, x) \in \mathcal{R}(\rho, \Omega \times \mathbb{R}^{k})\), the Thom encodings of the roots of \(\Lambda(M_\epsilon(\omega, x), T)\) stay fixed, as well as the list of the non-singular minors of size \(\ell\) in \(M_\epsilon(\omega, x), T\) at each root of \(\Lambda(M(\omega, x), T)\).

Then choosing a non-vanishing minor and using Cramer’s rule, we find \((\ell + 1)^2\) rational functions in the variables \(u, \omega, x, T\) which give for every \((u, \omega, x) \in \mathcal{R}(\rho, \Omega \times \mathbb{R}^{k+1})\) the coordinates of an eigenvector \(v_\tau(u, \omega, x, t_\tau)\) associated to the eigenvalue \(t_\tau\) (where \(u\) denotes the co-ordinate left out in the non-singular \(\ell \times \ell\) minor chosen for this eigenvalue in the application of Cramer’s rule). We denote by \(e_\tau(\omega, x, t_\tau)\) the unit eigenvector \(v_\tau(1, \omega, x, t_\tau) / ||v_\tau(1, \omega, x, t_\tau)||\) when \(t_\tau\) is an eigenvalue.

If the eigenvalues \(\lambda_{0, \ell} < \ldots < \lambda_{\ell, \ell}\) are in increasing order, we define

\[
e_{\tau, i}(\omega, x) = e_\tau(\omega, x, \lambda_{\tau, i}).
\]

Note that for every \((\omega, x) \in \Omega \times \mathbb{R}^{k}\)

\[
(\lim_\tau e_{\tau, 0}(\omega, x)), \ldots, (\lim_\tau e_{\tau, \ell}(\omega, x))
\]
is an orthonormal basis consisting of eigenvectors of $M(\omega, x)$.

4.2.3. *Index Invariant Triangulations.* We now define a certain special kind of semi-algebraic triangulation of $F$ that will play an important role in our algorithm.

**Definition 4.2.** (Index Invariant Triangulation) An *index invariant triangulation* of $F$ is a triangulation

$$h : \Delta \to F$$

of $F$, which respects all the realization of the weak sign conditions on $P$ and $A$ (see definition in [4.2.2]). As a consequence, $h$ respects the subsets $F_I$ for every $I \subset Q$. Moreover, $\text{index}(\langle \omega, Q^h \rangle(x), x)$, stays invariant as $(\omega, x)$ varies over $h(\sigma)$, and the maps $e_{\varepsilon,0}(\sigma), \ldots, e_{\varepsilon, \ell}$ sending $(\omega, x) \in h(\sigma)$ to the orthonormal basis $e_{\varepsilon,0}(\omega, x), \ldots, e_{\varepsilon, \ell}(\omega, x)$, are uniformly defined. Note also that for every $(\omega, x) \in h(\sigma)$,

$$\{e_j(\sigma, \omega, x), \ldots, e_{\ell}(\sigma, \omega, x)\} = \lim_{\varepsilon \to 0} \{e_{\varepsilon,j}(\omega, x), \ldots, e_{\varepsilon, \ell}(\omega, x)\}$$

is a basis for the linear subspace $L^+(\omega, x) \subset \mathbb{R}^{\ell+1}$, (which is the orthogonal complement to the sum of the eigenspaces corresponding to the first $j$ eigenvalues of $\langle \omega, Q^h \rangle(x)$).

We now describe an algorithm for computing index invariant triangulations.

**Algorithm 6 (Index Invariant Triangulation).**

**Input**

- A family of polynomials, $Q^h = \{Q_1^h, \ldots, Q_m^h\} \subset \mathbb{R}[Y_0, \ldots, Y_{\ell}, X_1, \ldots, X_k]$, where each $Q_i^h$ is homogeneous of degree 2 in the variables $Y_0, \ldots, Y_{\ell}$, and of degree at most $d$ in $X_1, \ldots, X_k$,
- another family of polynomials, $P \subset \mathbb{R}[X_1, \ldots, X_k]$, with $\deg(P) \leq d, P \in P, \#(P) = s$,
- a $P$-closed formula $\Phi$ defining a bounded $P$-closed semi-algebraic set $V \subset \mathbb{R}^k$.

**Output** : an index invariant triangulation, $h : \Delta \to F$ of $F$ and for each simplex $\sigma$ of $\Delta$, the rational functions $e_{\varepsilon,0}(\sigma), \ldots, e_{\varepsilon, \ell}(\sigma)$.

**Procedure**

Step 1 Let $\varepsilon > 0$ be an infinitesimal and let $Z = (Z_1, \ldots, Z_m)$. Let $M_{\varepsilon}$ be the symmetric matrix corresponding to the quadratic form (in $Y_0, \ldots, Y_{\ell}$) defined by

$$M_{\varepsilon}(X, Z) = (1 - \varepsilon)(Z_1Q_1^h + \cdots + Z_mQ_m^h) + \varepsilon \bar{Q},$$

where $\bar{Q} = \sum_{i=0}^{\ell} Y_i^2$. Compute the polynomials

$$(4.2) \quad \lambda(Z, X, T) = \det(T \cdot \text{Id}_{\ell+1} - M_{\varepsilon}) = T^{\ell+1} + C_{\ell}T^{\ell} + \cdots + C_0.$$ 

Step 2 Using Algorithm 11.19 in [10] (Restricted Elimination), compute a family of polynomials $A' \subset \mathbb{R}[\varepsilon][Z_1, \ldots, Z_m, X_1, \ldots, X_k]$ such that for each $\rho \in \text{Sign}(A')$, and $(\omega, x) \in \mathbb{R}(\rho \times \mathbb{R}^k) \cap F$ the Thom encodings of the roots of $\lambda(\omega, x, T)$ in $\mathbb{R}(\varepsilon)$ and the number of non-negative roots of $\lambda(\omega, x, T)$ in $\mathbb{R}(\varepsilon)$ stay fixed, as well as the list of the non singular minors of size $\ell$ in $M_{\varepsilon}(\omega, x, T)$ at each root of $\lambda(M_{\varepsilon}(\omega, x, T)$.
Let $A \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$ be the set of all coefficients of the polynomials in $A'$, when each of them is written as a polynomial in $\varepsilon$.

**Step 3** Using the algorithm implicit in Theorem 4.1 (Triangulation), compute a semi-algebraic triangulation, $h : \Delta \to F$, respecting all the realizations of the weak sign conditions on $A \cup P$.

**Step 4** For each simplex $\sigma$ of $\Delta$, output the maps $e_{\varepsilon,0}(\sigma), \ldots, e_{\varepsilon,\ell}(\sigma)$.

**Complexity Analysis:** The complexity of the algorithm is dominated by the complexity of Step 3, which is $(s\ell m d)^{2O(m+k)}$.

**Proof of Correctness:** It follows from the fact that the triangulation respects all weak sign conditions on $A$ that $\text{index}(\langle \omega, Q^h(\cdot, x) \rangle)$ is constant for $(\omega, x) \in h(\sigma)$ for any simplex $\sigma$ of $\Delta$.

Since $e_{\varepsilon,0}(\sigma, \omega, x), \ldots, e_{\varepsilon,\ell}(\sigma, \omega, x)$ are orthonormal, so are $e_0(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x)$ for every $(\omega, x) \in h(\sigma)$. Moreover, letting $j = \text{index}(\langle \omega, Q^h(\cdot, x) \rangle)$ for $(\omega, x) \in h(\sigma)$, we have that $e_{\varepsilon,j}(\sigma, \omega, x), \ldots, e_{\varepsilon,\ell}(\sigma, \omega, x)$ span the sum of the non-negative eigenspaces of $M_\omega(x, x)$, their images under the $\lim_\varepsilon$ map will span the sum of the non-negative eigenspaces of $M(\omega, x)$.

**4.3. Computing Betti numbers in the homogeneous union case.** Now that we obtained an Index Invariant Triangulation $\Delta$, our next goal is to construct a cell complex $K(B, V)$ homotopy equivalent to $B$ (see Notation 2.12) that will be used to compute the Betti numbers of $A^h$ (see Notation 2). The cell complex $K(B, V)$ is obtained by glueing together certain regular cell complexes, $K(\sigma)$, where $\sigma \in \Delta$.

**Figure 2. The complex $\Delta$.**

**4.3.1. Definition of $\mathcal{C}(\Delta)$.** Let $1 \gg \varepsilon_0 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_{m+k} > 0$ be infinitesimals. For $\tau \in \Delta$, we denote by $D_\tau$ the subset of $\bar{\tau}$ defined by

$$D_\tau = \{ v \in \bar{\tau} \mid \text{dist}(v, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \sigma \},$$

where dist refers to the ordinary Euclidean distance. Now, let $\sigma \prec \tau$ be two simplices of $\Delta$. We denote by $D_{\sigma, \tau}$ the subset of $\bar{\tau}$ defined by

$$D_{\sigma, \tau} = \{ v \in \bar{\tau} \mid \text{dist}(v, \sigma) \leq \varepsilon_{\dim(\sigma)}, \text{ and } \text{dist}(v, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \sigma \}.$$
Note that

\[ |\Delta| = \bigcup_{\sigma \in \Delta} D_\sigma \cup \bigcup_{\sigma, \tau \in \Delta, \sigma < \tau} D_{\sigma, \tau}. \]

Also, observe that the various \( D_\tau \)'s and \( D_{\sigma, \tau} \)'s are all homeomorphic to closed balls, and moreover all non-empty intersections between them also have the same property. Thus, the union of the \( D_\tau \)'s and \( D_{\sigma, \tau} \)'s together with the non-empty intersections between them form a regular cell complex, \( C(\Delta) \), whose underlying topological space is \( |\Delta| \) (see Figures 2 and 3).

4.3.2. Definition of \( K(\sigma) \) and \( K(\sigma, \tau) \) where \( \sigma, \tau \) are simplices of \( \Delta \). We now associate to each \( D_\sigma \) (respectively, \( D_{\sigma, \tau} \)) a regular cell complex, \( K(\sigma) \) (respectively, \( K(\sigma, \tau) \)) homotopy equivalent to \( \varphi^{-1}_1(h(D_\sigma)) \) (respectively, \( \varphi^{-1}_1(h(D_{\sigma, \tau})) \)).

For each \( \sigma \in \Delta \), and \( (\omega, x) \in h(\sigma) \), the orthonormal basis

\[ \{e_0(\sigma, \omega, x), \ldots, e_\ell(\sigma, \omega, x)\} \]

determines a complete flag of subspaces, \( F(\sigma, \omega, x) \), consisting of

\[
F^0(\sigma, \omega, x) = 0, \\
F^1(\sigma, \omega, x) = \text{span}(e_\ell(\sigma, \omega, x)), \\
F^2(\sigma, \omega, x) = \text{span}(e_\ell(\sigma, \omega, x), e_{\ell-1}(\sigma, \omega, x)), \\
\vdots \\
F^{\ell+1}(\sigma, \omega, x) = \mathbb{R}^{\ell+1}.
\]

**Definition 4.3.** For \( 0 \leq j \leq \ell \), let \( c_j^+(\sigma, \omega, x) \) (respectively, \( c_j^-\omega, \sigma, x) \)) denote the \((\ell - j)\)-dimensional cell consisting of the intersection of the \( F^{\ell-j+1}(\sigma, \omega, x) \) with the unit hemisphere in \( \mathbb{R}^{\ell+1} \) defined by

\[ \{y \in \mathbb{S}^\ell \mid \langle y, e_j(\sigma, \omega, x) \rangle \geq 0 \} \]

(respectively, \( \{y \in \mathbb{S}^\ell \mid \langle y, e_j(\sigma, \omega, x) \rangle \leq 0 \} \)).
The regular cell complex $K(\sigma)$ (as well as $K(\sigma, \tau)$) is defined as follows.

For each $v \in |\Delta|$ and $\sigma \in \Delta$, let $v(\sigma) \in |\sigma|$ denote the point of $|\sigma|$ closest to $v$. The cells of $K(\sigma)$ are

$$\{(y, \omega, x) \mid y \in c_\sigma^+(\sigma, \omega, x), (\omega, x) \in h(c)\},$$

where $\text{index}(\omega, Q^h(\cdot, x)) \leq j \leq \ell$, and $c \in C(\Delta)$ is either $D_\sigma$ itself, or a cell contained in the boundary of $D_\sigma$.

Similarly, the cells of $K(\sigma, \tau)$ are

$$\{(y, \omega, x) \mid y \in c_\tau^-(\sigma, h(v(\sigma)), v = h^{-1}(\omega, x) \in c)\},$$

where $\text{index}(\omega, Q^h(\cdot, x)) \leq j \leq \ell$, $c \in C(\Delta)$ is either $D_{\sigma, \tau}$ itself, or a cell contained in the boundary of $D_{\sigma, \tau}$.

4.3.3. **Definition of $K(D)$, where $D$ is a cell of $C(\Delta)$.** Our next step is to obtain cellular subdivisions of each non-empty intersection amongst the spaces associated to the complexes constructed above, and thus obtain a regular cell complex, $K(B, V)$, whose associated space, $|K(B, V)|$, will be shown to be homotopy equivalent to $B$ (Proposition 4.6 below).

First notice that $|K(\sigma', \tau')|$ (respectively, $|K(\sigma)|$) has a non-empty intersection with $|K(\sigma, \tau)|$ only if $D_{\sigma', \tau'}$ (respectively, $D_{\sigma, \tau}$) intersects $D_{\sigma, \tau}$.

Let $D$ be some non-empty intersection amongst the $D_\sigma$’s and $D_{\sigma, \tau}$’s, that is $D$ is a cell of $C(\Delta)$. Then, $D \subset |\tau|$ for a unique simplex $\tau \in \Delta$, and

$$D = D_{\sigma_1, \tau} \cap \cdots \cap D_{\sigma_p, \tau} \cap D_{\tau},$$

with $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_p \prec \sigma_{p+1} = \tau$ and $p \leq m + k$.

For each $i, 1 \leq i \leq p + 1$, let $\{f_0(\sigma_i, v), \ldots, f_{\ell}(\sigma_i, v)\}$ denote an orthonormal basis of $\mathbb{R}^{\ell+1}$ where

$$f_j(\sigma_i, v) = \lim_{t \to 0} e_j(\sigma_i, h(t v(\sigma_i) + (1 - t)v(\sigma_i))), 0 \leq j \leq \ell,$$

and let $F(\sigma_i, v)$ denote the corresponding flag, consisting of

$$F^0(\sigma_i, v) = 0,$$
$$F^1(\sigma_i, v) = \text{span}(f_0(\sigma_i, v)),$$
$$F^2(\sigma_i, v) = \text{span}(f_0(\sigma_i, v), f_{\ell-1}(\sigma_i, v)),$$
$$\vdots$$

$$F^{\ell+1}(\sigma_i, v) = \mathbb{R}^{\ell+1}.$$

We thus have $p + 1$ different flags,

$$F(\sigma_1, v), \ldots, F(\sigma_p, v),$$

and these give rise to $p + 1$ different regular cell decompositions of $S^\ell$.

There is a unique smallest regular cell complex, $K'(D, v)$, that refines all these cell decompositions, whose cells are the following. Let $L \subset \mathbb{R}^{\ell+1}$ be any $j$-dimensional linear subspace, $0 \leq j \leq \ell + 1$, which is an intersection of linear subspaces $L_1, \ldots, L_{p+1}$, where $L_i \in F(\sigma_i, v), 1 \leq i \leq p + 1 \leq m + k + 1$. The elements of the flags, $F(\sigma_1, v), \ldots, F(\sigma_{p+1}, v)$ of dimensions $j + 1$, partition $L$ into polyhedral cones of various dimensions. The intersections of these cones with $S^\ell$, over all such subspaces $L \subset \mathbb{R}^{\ell+1}$, are the cells of $K'(D, v)$.

Figure 4 illustrates the refinement
described above in case of two flags in $\mathbb{R}^3$. We denote by $\mathcal{K}(D,v)$ the sub-complex of $\mathcal{K}'(D,v)$ consisting of only those cells included in $L^+(\sigma_1, h(v(\sigma_1))) \cap S^\ell$.

We now triangulate $h(D)$ using the algorithm implicit in Theorem 4.1 (Triangulation), so that the combinatorial type of the arrangement of flags, $\mathcal{F}(\sigma_1, v), \ldots, \mathcal{F}(\sigma_{p+1}, v)$ and hence the combinatorial type of the cell decomposition $\mathcal{K}'(D,v)$, stays invariant over the image, $h_D(\theta)$, of each simplex, $\theta$, of this triangulation. Notice that the combinatorial type of the cell decomposition $\mathcal{K}'(D,v)$, is determined by the signs of the inner products, $\langle f_j(\sigma_i, v), f_{j'}(\sigma_{i'}, v) \rangle$ where $0 \leq j, j' \leq \ell, 1 \leq i, i' \leq p + 1$.

Introducing an infinitesimal $\delta$ such that $1 \gg \delta \gg \varepsilon > 0$, we note that for each $0 \leq j \leq \ell$,

$$f_j(\sigma_i, h^{-1}(\omega, x)) = \lim_{\delta \to 0} e_{\varepsilon,j}(h(\delta v(\sigma_i) + (1 - \delta)v(\sigma_1)))$$

Using the uniform formula defining $e_{\varepsilon,j}(\sigma_i)$ and Proposition 14.7 of [10], the vanishing or non-vanishing of the inner products

$$\langle f_j(\sigma_i, h^{-1}(\omega, x)), f_{j'}(\sigma_{i'}, h^{-1}(\omega, x)) \rangle, 0 \leq j, j' \leq \ell, 1 \leq i, i' \leq p + 1.$$ are determined by the signs of the family of polynomials

$$\mathcal{A}_D \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$$ obtained by taking the coefficients of the inner products

$$\langle e_{\varepsilon,j}(h(\delta v(\sigma_i) + (1 - \delta)v(\sigma_1))), e_{\varepsilon,j'}(h(\delta v(\sigma_{i'}) + (1 - \delta)v(\sigma_1))) \rangle$$
expressed as polynomials in $\varepsilon$ and $\delta$. The combinatorial type of the cell decomposition $\mathcal{K}'(D,v)$ will stay invariant as $(\omega,x)$ varies over each connected component of any realizable sign condition on $\mathcal{A}_D \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

Given the degree bounds on the rational functions defining $\{e_{\varepsilon,0}(\sigma), \ldots, e_{\varepsilon,\ell}(\sigma)\}$, $(\omega,x) \in h(\sigma)$, it is clear that the number and degrees of the polynomials in the family $\mathcal{A}_D$ are bounded by $(stmd)^{2O(m+k)}$. We then use the algorithm implicit in Theorem \ref{trip} (Triangulation), with $\mathcal{A}_D$ as input, to obtain the required triangulation.

The closures of the sets

$$\{(y,\omega,x) \mid y \in c \in \mathcal{K}(D, h^{-1}(\omega,x)), (\omega,x) \in h(h_D(\theta))\}$$

form a regular cell complex which we denote by $\mathcal{K}(D)$.

The following proposition gives an upper bound on the size of the complex $\mathcal{K}(D)$. We use the notation introduced in the previous paragraph.

**Proposition 4.4.** For each $(\omega,x) \in h(D)$, the number of cells in $\mathcal{K}(D, h^{-1}(\omega,x))$ is bounded by $\ell^{O(m+k)}$. Moreover, the number of cells in the complex $\mathcal{K}(D)$ is bounded by $(stmd)^{2O(m+k)}$.

**Proof.** The first part of the proposition follows from the fact that there are at most $(\ell+1)^{m+k+1}$ choices for the linear space $L$ and the number of $(j-1)$ dimensional cells contained in $L$ is bounded by $2^{m+k}$ (which is an upper bound on the number of full dimensional cells in an arrangement of at most $m+k$ hyperplanes). The second part is a consequence of the complexity estimate in Theorem \ref{trip} (Triangulation) and the bounds on number and degrees of polynomials in the family $\mathcal{A}_D$ stated above. \hfill $\Box$

4.3.4. **Definition of $\mathcal{K}(B,V)$.** Note that there is a homeomorphism

$$i_{D,\sigma_i} : |\mathcal{K}(\sigma_i, \tau)| \cap \varphi_1^{-1}(h(D)) \to |\mathcal{K}(D)|$$

which takes each cell of $|\mathcal{K}(\sigma_i, \tau)| \cap \varphi_1^{-1}(h(D))$ to a union of cells in $\mathcal{K}(D)$. We use these homeomorphisms to glue the cell complexes $\mathcal{K}(\sigma_i, \tau)$ together to form the cell complex $\mathcal{K}(B,V)$.

**Definition 4.5.** The complex $\mathcal{K}(B,V)$ is the union of all the complexes $\mathcal{K}(D)$ constructed above, where we use the maps $i_{D,\sigma_i}$ to make the obvious identifications. It is clear that $\mathcal{K}(B,V)$ so defined is a regular cell complex.

We have

**Proposition 4.6.** $|\mathcal{K}(B,V)|$ is homotopy equivalent to $B$.

**Proof.** We have from Proposition \ref{align} that the semi-algebraic set $C \subset B$ (see \ref{align} for definition) is homotopy equivalent to $B$. We now prove that $|\mathcal{K}(B,V)|$ is homotopy equivalent to $C$ which will prove the proposition.

Let $X_{m+k} = |\mathcal{K}(B,V)|$ and for $0 \leq j \leq m + k - 1$, let $X_j = \lim_{\varepsilon_j} X_{j+1}$.

It follows from an application of the Vietoris-Smale theorem \ref{trip} that for each $j, 0 \leq j \leq m + k - 1$, $\text{Ext}(X_j, R\langle \varepsilon_0, \ldots, \varepsilon_j \rangle)$ is homotopy equivalent to $X_{j+1}$. Also, by construction of $\mathcal{K}(B,V)$, we have that $X_0 = \lim_{\varepsilon_0} |\mathcal{K}(B,V)| = C$, which proves the proposition. \hfill $\Box$

We also have
Proposition 4.7. The number of cells in the cell complex $K(B, V)$ is bounded by $(s\ell m d)^{2O(m+k)}$.

Proof. The proposition is a consequence of Proposition 4.4 and the fact that the number of cells in the complex $C(\Delta)$ is bounded by $(s\ell m d)^{2O(m+k)}$. □

4.3.5. Algorithm for computing the Betti numbers in the homogeneous union case.

We now describe formally an algorithm for computing the Betti numbers of $A^h$ using the complex $K(B, V)$ described above.

Algorithm 7 (Betti numbers, homogeneous union case).

Input
- A family of polynomials, $Q^h \subset R[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$, homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$, $\deg_X(Q^h) \leq d, Q^h \in Q^h, \#(Q^h) = m$,
- another family of polynomials, $P \subset R[X_1, \ldots, X_k]$, with $\deg_X(P) \leq d, P \in P, \#(P) = s$,
- a $P$-closed formula $\Phi(x)$ defining a bounded $P$-closed semi-algebraic set $V \subset R^k$,
- the semi-algebraic set $A^h = \bigcup_{Q^h \in Q^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\}$.  

Output
- a description of the cell complex $K(B, V)$,
- the Betti numbers of $A^h$.

Procedure

Step 1 Call Algorithm 6 (Index Invariant Triangulation) with input $Q^h, P$ and $\Phi$ and compute $h$ and $\Delta$.

Step 2 Construct the cell complex $C(\Delta)$ (following its definition given in Section 4.3).

Step 3 For each cell $D \in C(\Delta)$, compute, using the algorithm implicit in Theorem 4.1 (Triangulation), the cell complex $K(D)$.

Step 4 Compute a description of $K(B, V)$, including the matrices corresponding to the differentials in the complex $C_*(K(B, V))$.

Step 5 Compute the Betti numbers of the complex $C_*(K(B, V))$ using linear algebra.

Complexity Analysis: The complexity of the algorithm is $(s\ell m d)^{2O(m+k)}$, using the complexity of Algorithm 6. □

Proof of Correctness: The correctness of the algorithm is a consequence of the correctness of Algorithm 6 and Proposition 4.4. □

4.4. Computing Betti numbers in the homogeneous intersection case.

4.4.1. Definition of $K(B_I, V)$. We now define a subcomplex of $K(B, V)$ corresponding to a subset $I \subset [m]$.

We first extend a few definitions from Section 2.
For each subset $I \subset [m]$, we denote by $\mathcal{Q}_I$ the subset of $\mathcal{Q}$ of polynomials with indices in $I$ and by $\Omega_I$ the subset of
\[
\Omega = \{ \omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq m \},
\]
on obtained by setting the coordinates corresponding to the elements of $[m] \setminus I$ to 0. More precisely,
\[
\Omega_I = \{ \omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, \text{ for } i \in I, \text{ and } \omega_i = 0 \text{ for } i \in [m] \setminus I \}.
\]

Note that we have a natural inclusion $\Omega_I \hookrightarrow \Omega_{[m]} = \Omega$.

Similarly, we denote by $F_I \subset F_{[m]}$, the set $\Omega_I \times V$, and denote by $B_I \subset \Omega_I \times S^t \times V$ the semi-algebraic set defined by
\[
B_I = \{ (\omega, y, x) \mid \omega \in \Omega_I, y \in S^t, x \in V, \langle \omega, Q \rangle(y, x) \geq 0 \}.
\]

We denote by $\varphi_{1,I} : B_I \rightarrow F_I$ and $\varphi_{2,I} : B_I \rightarrow S^t \times V$ the two projection maps.

Now we define $\mathcal{K}(B_I, V)$ for every $I \subset [m]$.

**Definition 4.8.** The complex $\mathcal{K}(B_I, V)$ is the union of all the complexes $\mathcal{K}(D)$ in $\mathcal{C}(\Delta_I)$, where $\mathcal{C}(\Delta_I)$ is the subcomplex of $\mathcal{C}(\Delta)$ consisting of cells contained in $\Delta_I = h^{-1}(F_I)$.

Using proofs similar to the ones give for $\mathcal{K}(B, V)$, we have

**Proposition 4.9.** $|\mathcal{K}(B_I, V)|$ is homotopy equivalent to $B_I$. \qed

**Algorithm 8** (Computing the collection of $\mathcal{K}(B_I, V)$, $I \subset [1 \ldots , m]$).

**INPUT**
- $\mathcal{Q} = \{Q_0^h, \ldots , Q_m^h\} \subset \mathbb{R}[Y_0, \ldots , Y_t, X_1, \ldots , X_k]$, where each $Q_i^h$ is homogeneous of degree 2 in the variables $Y_0, \ldots , Y_t$, and of degree at most $d$ in $X_1, \ldots , X_k$,
- $\mathcal{P} \subset \mathbb{R}[X_1, \ldots , X_k]$, with $\deg(P) \leq d, P \in \mathcal{P}$,
- a $\mathcal{P}$-closed formula $\Phi(x)$ defining a bounded $\mathcal{P}$-closed semi-algebraic set $V \subset \mathbb{R}^k$.

**OUTPUT**
- For each subset $I \subset [m]$, a description of the cell complex $\mathcal{K}(B_I, V)$.
- For each $I \subset J \subset [m]$, a homomorphism
  \[
i_{I,J} : \mathbb{C}_*(B_I, V) \rightarrow \mathbb{C}_*(B_J, V)
  \]
  inducing the inclusion homomorphism $i_{I,J,*} : H_*(B_I, V) \rightarrow H_*(B_J, V)$.

**PROCEDURE**

Step 1 Call Algorithm 7 to compute $\mathcal{K}(B, V)$.
Step 2 Give a description of $\mathcal{K}(B_I, V)$ for each $I \subset [m]$ and compute the matrices corresponding to the differentials in the complex $\mathbb{C}_*(\mathcal{K}(B_I, V))$.
Step 3 For $I \subset J \subset [m]$ with compute the matrices for the homomorphisms of complexes,
\[
i_{I,J} : \mathbb{C}_*(\mathcal{K}(B_I, V)) \rightarrow \mathbb{C}_*(\mathcal{K}(B_J, V))
\]
in the following way.
The complex $K(B_I, V)$ is a subcomplex of $K(B_J, V)$ by construction. Compute the matrix for the inclusion homomorphism,

$$i_{I,J} : C_\bullet(K(B_I, V)) \to C_\bullet(K(B_J, V)).$$

and output the matrix for the homomorphism.

**Complexity Analysis:** The complexity of the algorithm is $O(s \cdot \ell \cdot m d)$, using the complexity of Algorithm 6.

**Proof of Correctness:** The correctness of the algorithm is a consequence of the correctness of Algorithm 6 and Proposition 4.6.

4.4.2. Algorithm for computing the Betti numbers in the homogeneous intersection case. Let $W^h \subset S^\ell \times R^k$ be the semi-algebraic set defined by

$$W^h = \bigcap_{Q \in \mathcal{Q}^h} \{(y, x) \mid ||y|| = 1 \land Q(y, x) \leq 0 \land \Phi(x)\},$$

using Notation 2.

Then,

$$H_\bullet(W^h) \cong H^\bullet(Tot_\bullet(N_{\bullet,\bullet}(K(B, V)))),$$

where $N_{\bullet,\bullet}(K(B, V))$ is the bi-complex

$$N_{p,q}(K(B, V)) = \bigoplus_{J \subset [m], \#(J) = p+1} C_q(K(B_J, V)),$$

with the horizontal and vertical differentials defined as follows. The vertical differentials,

$$d_{p,q} : N_{p,q}(K(B, V)) \to N_{p,q-1}(K(B, V)),$$

are induced by the boundary homomorphisms,

$$\partial_q : C_q(K(B_I, V)) \to C_{q-1}(K(B_J, V)),$$

and the horizontal differentials,

$$\delta_{p,q} : N_{p,q}(K(B, V)) \to N_{p+1,q}(K(B, V))$$

are defined by

$$(\delta_{p,q}(\varphi))_J = \sum_{j \in J} i_{J \setminus \{j\}, J}(\varphi_{J \setminus \{j\}}),$$

where $J \subset [m], \#(J) = p+1,$

$$\varphi \in N_{p,q}(K(B, V)) = \bigoplus_{J \subset [m], \#(J) = p+1} C_\bullet(K(B_J, V))$$

and for $I \subset J \subset [m],$ and

$$i_{I,J} : C_\bullet(K(B_I, V)) \to C_\bullet(K(B_J, V))$$

denotes the homomorphism induced by inclusion.

For a proof of (4.3) see [5].

Using (4.3), we are able to compute the Betti numbers of $W^h$ using only linear algebra, once we have computed the various complexes $K(B_I, V)$, as well as the homomorphisms $i_{I,J}$ for all $I \subset J \subset [m]$ using Algorithm 8. Moreover, the complexity of this algorithm is asymptotically the same as that of Algorithm 2.
We now formally describe this algorithm.

**Algorithm 9 (Betti numbers, homogeneous intersection case).**

**Input**
- A family of polynomials, \( Q^h = \{ Q^h_1, \ldots, Q^h_m \} \subset R[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k] \), homogeneous of degree 2 with respect to \( Y_0, \ldots, Y_\ell \), \( \deg_X(Q^h) \leq d, Q^h \in Q^h \),
- another family, \( P \subset R[X_1, \ldots, X_k] \) with \( \deg_X(P) \leq d, P \in P, \#(P) = s \),
- a formula \( \Phi \) defining a bounded \( P \)-closed semi-algebraic set \( V \),
- the semi-algebraic set \( W^h = \bigcap_{Q^h \in Q^h} \{ (y, x) \mid |y| = 1 \land Q^h(y, x) \leq 0 \land \Phi(x) \} \).

**Output** the Betti numbers \( b_i(W^h) \).

**Procedure**

Step 1 Call Algorithm 8 (Computing the collection of \( K(B_I, V) \)) to compute for each \( I \subset J \subset [m] \), the complex \( C_\bullet(K(B_I, V)) \) using the natural basis consisting of the cells of \( K(B_I, V) \) of various dimensions, as well as the matrices in this basis for the inclusion homomorphisms,

\[ i_{I,J} : C_\bullet(K(B_I, V)) \rightarrow C_\bullet(K(B_J, V)) \].

Step 2 Using the data from the previous step, compute matrices corresponding to the differentials in the complex, \( \text{Tot}_\bullet(N_\bullet_\bullet(K(B, V))) \), where \( N_\bullet_\bullet(K(B, V)) \) is the bi-complex described by (4.4)-(4.6).

Step 3 Compute, using linear algebra subroutines,

\[ b_i(W^h) = H_i((\text{Tot}_\bullet(N_\bullet_\bullet(K(B, V))))). \]

**Complexity Analysis:** The complexity of the algorithm is dominated by the first step, whose complexity is \( (s\ell md)^2 O(m+k) \), using the complexity of Algorithm 8 \( \square \)

**Proof of Correctness:** The correctness of the algorithm is a consequence of the correctness of Algorithm 8 and (4.3). \( \square \)

### 4.5. Computing Betti numbers of general \( P \cup Q \)-closed sets.

Let \( S \subset R^{\ell+k} \) be a semi-algebraic set defined by a \( P \cup Q \) closed formula \( \Phi \).

Let \( \Sigma_Q \) denote the set of all possible weak sign conditions on the family \( Q \), i.e.

\[ \Sigma_Q = \{0, \{0, 1\}, \{0, -1\}\}^Q. \]

In the last section we defined a bi-complex \( N_\bullet_\bullet(K(B, V)) \) whose total complex has homology groups isomorphic to these of the semi-algebraic set \( W^h = R(\rho \cap \phi) \), where \( \rho \in \Sigma_Q \) is given by \( \rho(Q_i) = \{0, -1\} \) for each \( i, 1 \leq i \leq m \), and \( \rho^h \) is obtained from \( \rho \) by replacing each \( Q_i \in Q \) by \( Q_i^h \). We now generalize this definition to the case of multiple weak sign conditions. More precisely, given a set \( \Sigma = \{\rho_1, \ldots, \rho_N\} \subset \Sigma_Q \), we define a corresponding bi-complex having properties similar to that of \( N_\bullet_\bullet(K(B, V)) \), but now with respect to \( \Sigma \) instead of a single weak sign condition \( \rho \).
Without loss of generality we can write $\Phi$ in the form

$$\Phi = \bigvee_{\rho \in \Sigma} \rho \land \phi_\rho,$$

where each $\phi_\rho$ is a $\mathcal{P}$-closed formula.

Let

$$W_\rho = \mathcal{R}(\rho \land \phi_\rho, R^{\ell+k}),$$

$$V_\rho = \mathcal{R}(\phi_\rho, R^k).$$

Let $1 \gg \varepsilon > 0$ be an infinitesimal, and let

$$Q_0 = \varepsilon^2 (Y_1^2 + \cdots + Y_\ell^2) - 1,$$

$$P_0 = \varepsilon^2 (X_1^2 + \cdots + X_\ell^2) - 1,$$

and $S_b \subset \mathcal{R}(\varepsilon)^{\ell+k}$ be the semi-algebraic set defined by

$$S_b = \bigcap_{i=0}^{m} \{(y, x) \mid Q_0(y) \leq 0 \land P_0(x) \leq 0 \land \Phi(x)\}.$$

We denote by $\Phi_b$ (resp. $\phi_{\rho,b}$) the formula $(Q_0(y) \leq 0) \land (P_0(x) \leq 0) \land \Phi$ (resp. $(P_0(x) \leq 0) \land \phi_\rho$).

Let $S^h_{\rho,b}, W^h_{\rho,b} \subset S^\ell \times \mathcal{R}(\varepsilon)^k$ be the sets defined by $\Phi_b$ and $\rho \land (Q_0^h \leq 0) \land \phi_{\rho,b}$ respectively on $S^\ell \times \mathcal{R}(\varepsilon)^k$ after replacing each $Q_i \in \mathcal{Q}$ by $Q_i^h$ in the formulas $\Phi_b$ and $\rho$.

Let

$$V_{\rho,b} = \mathcal{R}(\phi_{\rho,b}, R(\varepsilon)^k).$$

Let $Q^h_\pm = \{\pm Q^h_Y \mid Q^h \in \mathcal{Q}^h\}$, and let $\mathcal{K}(B, V)$ denote the complex constructed by Algorithm 9 [13] with input the families of polynomials $Q^h_\pm$, $\mathcal{P}_b = \mathcal{P} \cup \{P_0\}$, and the semi-algebraic subset $V = B_\ell(0, 1/\varepsilon)$.

It follows from the correctness of Algorithm 9 [13] that for each $\rho \in \tilde{\Sigma}_\mathcal{Q}$, there exists $J_\rho \subset \mathcal{Q}_\pm^h$, and a subcomplex, $\mathcal{K}(B_{J_\rho}, V_{\rho,b}) \subset \mathcal{K}(B, V)$, such that

$$H_*(\text{Tot}^*(\mathcal{N}_{\cdot,\cdot}(\mathcal{K}(B_{J_\rho}, V_{\rho,b}))) \cong H_*(W^h_{\rho,b}).$$

More generally, for any $\Sigma = \{\rho_1, \ldots, \rho_N\} \subset \tilde{\Sigma}_\mathcal{Q}$, there exists a subcomplex, $\mathcal{K}(B_{J_\rho}, V_{\rho_1,b} \cap \cdots \cap V_{\rho_N,b}) \subset \mathcal{K}(B, V)$, such that the homology groups of the complex

(4.7) 

$$C_{\Sigma,\cdot} = \text{Tot}^*(\mathcal{N}_{\cdot,\cdot}(\mathcal{K}(B_{J_\rho}, V_{\rho_1,b} \cap \cdots \cap V_{\rho_N,b}))$$

are naturally isomorphic to those of $W^h_{\rho,b}$, where $\rho$ is the common refinement of $\rho_1, \ldots, \rho_N$ (see (2.1) above for the definition of refinements of weak sign conditions).

Moreover, for $\Sigma \subset \Sigma' \subset \tilde{\Sigma}_\mathcal{Q}$, there exists a natural homomorphism,

$$i_{\Sigma',\Sigma} : C_{\Sigma',\cdot} \to C_{\Sigma,\cdot},$$

such that the induced homomorphism,

$$i_{\Sigma',\Sigma} : C_*(C_{\Sigma',\cdot}) \to C_*(C_{\Sigma,\cdot})$$

is the one induced by the inclusion

$$\bigcap_{\rho \in \Sigma'} W^h_{\rho,b} \hookrightarrow \bigcap_{\rho \in \Sigma} W^h_{\rho,b}.$$
Definition 4.10. Let $C_\bullet(\Phi)$ denote the complex defined by

\begin{equation}
C_\bullet(\Phi) = \text{Tot}_\bullet(N_\bullet,\bullet(\Phi)),
\end{equation}

where

\begin{equation}
N_{p,q}(\Phi) = \bigoplus_{\Sigma \subseteq \bar{\Sigma}_Q, \#(\Sigma) = p+1} C_{\Sigma,q}.
\end{equation}

The vertical and horizontal homomorphisms in the complex $N_\bullet,\bullet(\Phi)$ are induced by the differentials in the individual complexes $C_{\Sigma,\bullet}$ and the inclusion homomorphisms $i_{\Sigma,\Sigma'}$ respectively.

By the properties of the complexes $C_{\Sigma,\bullet}$ stated above and the exactness of the generalized Mayer-Vietoris sequence, we obtain

**Theorem 4.11.**

\[ H_\bullet(S^h_b) \cong H_\bullet(C_\bullet(\Phi)). \]

We are now in a position to describe formally the algorithm for computing all the Betti numbers of a given $P \cup Q$-closed set $S$.

4.5.1. **Description of the algorithm in the general case.**

**Algorithm 10** (Betti numbers, general case).

**Input**
- A family of polynomials $Q = \{Q_1, \ldots, Q_m\} \subset R[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$, with $\deg_Y(Q_i) \leq 2$, $\deg_X(Q_i) \leq d$, $1 \leq i \leq \ell$,
- another family of polynomials $P \subset R[X_1, \ldots, X_k]$ with $\deg(P) \leq d$, $P \in P$,
- a $Q \cup P$-closed semi-algebraic set $S$ defined by a $Q \cup P$-closed formula $\Phi$.

**Output** the Betti numbers $b_0(S), \ldots, b_{k+\ell-1}(S)$.

**Procedure**

Step 1 Define $Q_0 = \varepsilon^2_0(Y^2_1 + \ldots + Y^2_\ell) - 1$, $P_0 = \varepsilon^2_0(X^2_1 + \ldots + X^2_k) - 1$. Replace $S$ by $R(S, R(\varepsilon)) \cap (R(P_0 \leq 0) \times R(Q_0 \leq 0))$.

Step 2 Define $Q^h_0 = \{Q^h \mid Q^h \in Q^h\} \cup (Q^h_0)$, and let $K(B, V)$ denote the complex constructed by Algorithm 8, with input the families of polynomials $Q^h_0, P_b$, and the semi-algebraic set $V = B_0(0,1/\varepsilon) \subset R(\varepsilon)^k$.

Step 3 Compute, using the definitions given above, the matrices corresponding to the differentials in the complex $C_\bullet(\Phi)$.

Step 4 Compute, using linear algebra subroutines, for each $i, 0 \leq i \leq k + \ell - 1$

\[ b_i(S^h_b) = H_i(C_\bullet(\Phi)). \]

Step 5 Output for each $i, 0 \leq i \leq k + \ell - 1,

\[ b_i(S) = \frac{1}{2} b_i(S^h_b). \]

**Proof of Correctness:** The correctness of the algorithm follows from Theorem 4.11 and the correctness of Algorithm 8.

**Complexity Analysis:** Since $\#(\Sigma_Q) = 3^m$, the number of subsets that enters in the definition of $N_\bullet,\bullet(\Phi)$ (cf. (4.9)) is at most $23^m$. The complexity of the algorithm is now seen to be $(s\ell md)^{2O(m+\ell)}$, using the complexity of Algorithm 8.
References

[1] A.A. Agrachev, *Topology of quadratic maps and Hessians of smooth maps*, Algebra, Topology, Geometry, Vol. 26 (Russian), 85-124, 162, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988. Translated in J. Soviet Mathematics, 49 (1990), no. 3, 990-1013.

[2] A. I. Barvinok, *On the Betti numbers of semi-algebraic sets defined by few quadratic inequalities*, Mathematische Zeitschrift, 225, 231-244 (1997).

[3] A. I. Barvinok, *Feasibility Testing for Systems of Real Quadratic Equations*, Discrete and Computational Geometry, 10:1-13 (1993).

[4] S. Basu, *Efficient algorithm for computing the Euler-Poincaré characteristic of semi-algebraic sets defined by few quadratic inequalities*, Computational Complexity, 15 (2006), 236-251.

[5] S. Basu, *Computing the top few Betti numbers of semi-algebraic sets defined by quadratic inequalities in polynomial time*, Foundations of Computational Mathematics, to appear. Available at [arxiv:math.AG/0603262](http://arxiv.org/abs/math.AG/0603262).

[6] S. Basu, *Computing the first few Betti numbers of semi-algebraic sets in single exponential time*, Journal of Symbolic Computation, Volume 41, Issue 10, October 2006, 1125-1154.

[7] S. Basu, *On Bounding the Betti Numbers and Computing the Euler Characteristics of Semi-algebraic Sets*, Discrete and Computational Geometry, 22:1-18 (1999).

[8] S. Basu, M. Kettner, *A sharper estimate on the Betti numbers of sets defined by quadratic inequalities*, Discrete and Computational Geometry, to appear.

[9] S. Basu, R. Pollack, M.-F. Roy, *On the Betti numbers of sign conditions*, Proc. Amer. Math. Soc. 133 (2005), 965-974.

[10] S. Basu, R. Pollack, M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Second Edition, Springer-Verlag, 2006.

[11] S. Basu, R. Pollack, M.-F. Roy, *Computing the first Betti number and the connected components of semi-algebraic sets*, to appear in Foundations of Computational Mathematics and available at [arxiv:math.AG/0603248](http://arxiv.org/abs/math.AG/0603248).

[12] S.S. Chern, E. Spanier, *The homology structure of sphere bundles*, PNAS, 36: 248-255 (1950).

[13] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, Springer-Verlag (1987). Real algebraic geometry, Springer-Verlag (1998).

[14] A. Gabrielov, N. Vorobjov, *Betti Numbers for Quantifier-free Formulae*, Discrete and Computational Geometry, 33:395–401, 2005.

[15] D. Grigoriev, D.V. Pasechnik, *Polynomial time computing over quadratic maps I. Sampling in real algebraic sets*, Computational Complexity, 14:20-52 (2005).

[16] D. Grigoriev, D.V. Pasechnik, *On Betti numbers of semi-algebraic sets over quadratic maps*, Manuscript, 2004.

[17] R. M. Hardt, *Semi-algebraic Local Triviality in Semi-algebraic Mappings*, Am. J. Math. 102, 291-302 (1980).

[18] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[19] J. Milnor, *On the Betti numbers of real varieties*, Proc. AMS 15, 275-280 (1964).

[20] O. A. Oleënik, *Estimates of the Betti numbers of real algebraic hypersurfaces*, Mat. Sb. (N.S.), 28 (70): 635-640 (Russian) (1951).

[21] O. A. Oleënik, I. B. Petrovskii, *On the topology of real algebraic surfaces*, Izv. Akad. Nauk SSSR 13, 389-402 (1949).

[22] S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. 8:3, 604-610 (1957).

[23] R. Thom, *Sur l’homologie des variétés algébriques réelles*, Differential and Combinatorial Topology, 255-265. Princeton University Press, Princeton (1965).
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.  
E-mail address: saugata.basu@math.gatech.edu

MAS Division, School of Physical and Mathematical Sciences, Nanyang Technological University, 1 Nanyang Walk, Blk 5, Singapore 637616.  
E-mail address: dima@ntu.edu.sg

IRMAR (URA CNRS 305), Université de Rennes I, Campus de Beaulieu 35042 Rennes Cedex FRANCE.  
E-mail address: marie-francoise.roy@univ-rennes1.fr