Global Well-posedness of Incompressible Inhomogeneous Fluid Systems with Bounded Density or Non-Lipschitz Velocity

JINGCHI HUANG, MARIUS PAICU, & PING ZHANG

Communicated by F. LIN

Abstract

In this paper, we first prove the global existence of weak solutions to the $d$-dimensional incompressible inhomogeneous Navier–Stokes equations with initial data $a_0 \in L^\infty(\mathbb{R}^d), u_0 = (u^h_0, u^d_0) \in \dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d)$, which satisfy $(\mu \|a_0\|_L^\infty + \|u^h_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}}) \exp(C_r \mu^{-2r} \|u^d_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}}^2) \leq c_0 \mu$ for some positive constants $c_0, C_r$ and $1 < p < d, 1 < r < \infty$. The regularity of the initial velocity is critical to the scaling of this system and is general enough to generate non-Lipschitz velocity fields. Furthermore, with additional regularity assumptions on the initial velocity or on the initial density, we can also prove the uniqueness of such a solution. We should mention that the classical maximal $L^p(L^q)$ regularity theorem for the heat kernel plays an essential role in this context.

1. Introduction

In this paper, we consider the global well-posedness of the following $d$-dimensional incompressible inhomogeneous Navier–Stokes equations with the regularity of the initial velocity being almost critical and the initial density being a bounded positive function, which satisfy a nonlinear smallness condition,

$$
\begin{cases}
  \partial_t \rho + \text{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,
  \\
  \partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = 0,
  \\
  \text{div } u = 0,
  \\
  \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0,
\end{cases}
$$

(1.1)

where $\rho, u = (u^h, u^d)$ stands for the density and velocity of the fluid, $\Pi$ is a scalar pressure function, and $\mu$ the viscosity coefficient. This system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that
have different densities. It can also describe a fluid containing a melted substance. We remark that our hypothesis on the density is of physical interest, corresponding to the case of a mixture of immiscible fluids with different and bounded densities.

In particular, we shall focus on the global well-posedness of (1.1) with small homogeneity for the initial density function in \(L^\infty(\mathbb{R}^d)\) and small horizontal components of the velocity compared with its vertical component. This approach was already applied by Paicu and Zhang [26,27] for three-dimensional anisotropic Navier–Stokes equations and for inhomogeneous Navier–Stokes systems in the framework of Besov spaces. The main contribution of the present paper is to consider the initial density function in \(L^\infty(\mathbb{R}^d)\), which is close enough to some positive constant. Then, in order to handle the nonlinear terms appearing in (1.1), we need to use the maximal regularity effect for the classical heat equation. We should mention that the initial data have scaling invariant regularities and the global weak solutions obtained here, under a nonlinear-type smallness condition, also belong to the critical spaces. Moreover, the regularity of the velocity field obtained in this paper is general enough to include the case of non-Lipschitz vector fields.

Kazhikov [22] proved that when \(\rho_0\) is bounded away from 0, the inhomogeneous Navier–Stokes equations (1.1) have at least one global weak solution in the energy space. In addition, he also proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. However, the uniqueness of both types of weak solutions has not been solved. Ladyženskaja and Solonnikov [23] first addressed the question of unique resolvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain \(\Omega\) with a homogeneous Dirichlet boundary condition for \(u\). Under the assumption that \(u_0 \in W^{2,\frac{2}{d}}(\Omega)(p > d)\) is divergence free and vanishes on \(\partial\Omega\) and that \(\rho_0 \in C^1(\Omega)\) is bounded away from zero, they then proved [23]

\begin{itemize}
\item Global well-posedness in dimension \(d = 2\);
\item Local well-posedness in dimension \(d = 3\). If, in addition, \(u_0\) is small in \(W^{2,\frac{2}{d}}(\Omega)\), then global well-posedness holds true.
\end{itemize}

Similar results were obtained by Danchin [11] in \(\mathbb{R}^d\) with initial data in almost-critical Sobolev spaces. Abidi et al. [3] investigated large time decay and stability of any given global smooth solutions of (1.1), which, in particular, implies the global well-posedness of three-dimensional inhomogeneous Navier–Stokes equations with axi-symmetric initial data, provided that there is no swirl part for the initial velocity field and the initial density is close enough to a positive constant. In general, when the viscosity coefficient, \(\mu(\rho)\), depends on \(\rho\), Lions [24] proved the global existence of weak solutions to (1.1) in any space dimensions.

In the case when the density function \(\rho\) is away from zero, which we denote by \(a \overset{\text{def}}{=} 1/\rho - 1\), then the system (1.1) can be equivalently reformulated as

\[
\begin{align*}
\partial_t a + u \cdot \nabla a &= 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \mu \Delta u) &= 0, \\
\text{div} u &= 0, \\
(a, u)|_{t=0} &= (a_0, u_0).
\end{align*}
\] (1.2)
Note that just like the classical Navier–Stokes system, the inhomogeneous Navier–Stokes system (1.2) also has a scaling. More precisely, if \((a, u)\) solves (1.2) with initial data \((a_0, u_0)\), then for \(\forall \ell > 0\),

\[
(a, u)_\ell \overset{\text{def}}{=} (a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot)) \quad \text{and} \quad (a_0, u_0)_\ell \overset{\text{def}}{=} (a_0(\ell \cdot), \ell u_0(\ell \cdot)),
\]

so \((a, u)_\ell\) is also a solution of (1.2) with initial data \((a_0, u_0)_\ell\).

In [10], DANCHIN studied, in general space dimension \(d\), the unique solvability of the system (1.2) in scaling invariant homogeneous Besov spaces, which generalized the celebrated results by Fujita and Kato [16] devoted to the classical Navier–Stokes system. In particular, the norm of \((a, u)\) \(\in \dot{B}_{2,\infty}^d(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times \dot{B}_{2,1}^{d-1}(\mathbb{R}^d)\) is scaling invariant under the change of scale of (1.3). In this case, Danchin proved that if the initial data \((a_0, u_0)\) \(\in \dot{B}_{2,\infty}^d(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times \dot{B}_{2,1}^{d-1}(\mathbb{R}^d)\) with \(a_0\) sufficiently small in \(\dot{B}_{2,\infty}^d(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\), then the system (1.2) has a unique local-in-time solution. Abidi [1] proved that if \(1 < p < 2d\), \(0 < \mu < \tilde{\mu}(a)\), \(u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^d)\) and \(a_0 \in \dot{B}_{p,1}^d(\mathbb{R}^d)\), then (1.2) has a global solution, provided that \(\|a_0\|_{\dot{B}_{p,1}^d} + \|u_0\|_{\dot{B}_{p,1}^d} \leq c_0\) for some \(c_0\) sufficiently small. Furthermore, such a solution is unique if \(1 < p \leq d\). This result generalized the corresponding results in [10, 11] and was improved by Abidi and Paicu [2] when \(\tilde{\mu}(a)\) is a positive constant by using different Lebesgue indices for the density and for the velocity.

More precisely, for \(a_0 \in \dot{B}_{q,1}^d(\mathbb{R}^d)\) and \(u_0 \in \dot{B}_{p,1}^{-1+d/p}(\mathbb{R}^d)\) with \(|\frac{1}{p} - \frac{1}{q}| < \frac{1}{d}\) and \(\frac{1}{p} + \frac{1}{q} > \frac{1}{d}\), they obtained the existence of solutions to (1.2) and, under a more restrictive condition, \(\frac{1}{p} + \frac{1}{q} \geq \frac{2}{d}\), they proved the uniqueness of this solution. In particular, with a well prepared regularity for the density function, this result implies the global existence of solutions to (1.2) for any \(1 < p < \infty\) and the uniqueness of such solutions when \(1 < p < 2d\). Very recently, Danchin and Mucha [13] filled the gap for the uniqueness result in [1] with \(p \in (d, 2d)\) through a Langrange approach, and Abidi, Gui and Zhang relaxed the smallness condition for \(a_0\) in [4, 5].

On the other hand, when the initial density \(\rho_0 \in L^\infty(\Omega)\) with positive lower bound and \(u_0 \in H^2(\Omega)\), Danchin and Mucha [14] proved the local well-posedness of (1.1). They also proved the global well-posedness result, provided that the fluctuation of the initial density is sufficiently small, and initial velocity is small in \(\dot{B}_{q,p}^{2-\frac{2}{p}}(\Omega)\) for \(1 < p < \infty\), \(d < q < \infty\) in three dimensions and any velocity in \(\dot{B}_{4,2}^d(\Omega) \cap L^2(\Omega)\) in two dimensions. Motivated by Proposition 2.1, below, concerning the alternative definition of Besov spaces (see Definition A.1) with negative indices and [17], where Kato solved the local (respectively global) well-posedness of the three-dimensional classical Navier–Stokes system through an elementary \(L^p\) approach, we shall investigate the global existence of weak solutions to (1.2) with initial data \(a_0 \in L^\infty(\mathbb{R}^d)\) and \(u_0 \in \dot{B}_{p,r}^{-1+d/p}(\mathbb{R}^d)\) for \(p \in (1, d)\) and \(r \in (1, \infty)\), which satisfy the nonlinear smallness condition (1.6). Furthermore, if we assume,
in addition, \( u_0 \in B_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d) \) for some sufficiently small \( \varepsilon > 0 \), we can also prove the uniqueness of such solutions.

**Definition 1.1.** We call \((a, u, \nabla \Pi)\) a global weak solution of \((1.2)\) if

- for any test function \( \phi \in C_c^\infty([0, \infty) \times \mathbb{R}^d) \), there holds

\[
\int_0^\infty \int_{\mathbb{R}^d} a(\partial_t \phi + u \cdot \nabla \phi) \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^d} \phi(0, x) a_0(x) \, dx = 0,
\]

\[
\int_0^\infty \int_{\mathbb{R}^d} \text{div} \, u \phi \, dx \, dt = 0,
\]  \hspace{1cm} (1.4)

- for any vector valued function \( \Phi = (\Phi^1, \ldots, \Phi^d) \in C_c^\infty([0, \infty) \times \mathbb{R}^d) \), one has

\[
\int_0^\infty \int_{\mathbb{R}^d} \{ u \cdot \partial_t \Phi - (u \cdot \nabla u) \cdot \Phi + (1 + a)(\mu \Delta u - \nabla \Pi) \cdot \Phi \} \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^d} u_0 \cdot \Phi(0, x) \, dx = 0.
\]  \hspace{1cm} (1.5)

We denote the vector field by \( u = (u^h, u^d) \), where \( u^h = (u^1, u^2, \ldots, u^{d-1}) \). Our first main result in this paper is as follows:

**Theorem 1.1.** Let \( p \in (1, d) \) and \( r \in (1, \infty) \). Let \( a_0 \in L^\infty(\mathbb{R}^d) \) and \( u_0 \in B_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d) \). Then there exist positive constants \( c_0, C_r \), such that if

\[
\eta \overset{\text{def}}{=} \mu \| a_0 \|_{L^\infty} + \| u^h_0 \|_{B_{p,r}^{-1+\frac{d}{p}}} \exp \left\{ C_r \mu^{-2r} \| u^d_0 \|_{B_{p,r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq c_0 \mu, \tag{1.6}
\]

then \((1.2)\) has a global weak solution \((a, u)\) in the sense of Definition 1.1, which satisfies

(1) when \( p \in (1, \frac{dr}{3r-2}] \),

\[
\mu^\frac{1}{r} \| \Delta u^h \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} + \mu^\frac{1}{r} \| \nabla u^h \|_{L^{2r}(\mathbb{R}^+; L^{\frac{2dr}{3r-1}})} \leq C \eta,
\]

\[
\mu^\frac{1}{r} \| \Delta u^d \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} + \mu^\frac{1}{r} \| \nabla u^d \|_{L^{2r}(\mathbb{R}^+; L^{\frac{2dr}{3r-1}})} \leq C \| u^d_0 \|_{B_{p,r}^{-1+\frac{d}{p}}} + c \mu,
\]  \hspace{1cm} (1.7)

\[
\mu^\frac{1}{r} \| \nabla \Pi \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} \leq C \eta \left( \| u^d_0 \|_{B_{p,r}^{-1+\frac{d}{p}}} + c \mu \right);
\]
(2) when \( p \in (\frac{dr}{3r-2}, d) \),

\[
\mu^{\frac{1}{4}}(3 - \frac{d}{p_1}) \| \tau^\alpha_1 \Delta u^h \|_{L^{2r}(\mathbb{R}^+; L^p_1)} + \mu^{\frac{1}{2} - \frac{d}{p_2}} (\| \tau^\beta_1 \nabla u^h \|_{L^{2r}(\mathbb{R}^+; L^p_2)} + \| \tau^\gamma_1 u^h \|_{L^\infty(\mathbb{R}^+; L^p_3)}) + \| \tau^\delta_1 u^h \|_{L^\infty(\mathbb{R}^+; L^p_3)} \right) \leq C \eta,
\]

(1.8)

and

\[
\mu^{\frac{1}{4}}(3 - \frac{d}{p_1}) \| \tau^\alpha_2 \Delta u \|_{L^r(\mathbb{R}^+; L^{p_1})} \leq C \left( \| u_0 \|_{\dot{B}^{1 + \frac{d}{p}}_{p,r}} + \eta \left( \| u_0 \|_{\dot{B}^{1 + \frac{d}{p}}_{p,r}} + c \mu + \eta \right) \right),
\]

(1.9)

for some small enough constant \( c \), where \( p_1, p_2, p_3 \) satisfy \( \max(p, \frac{dr}{2r-1}) < p_1 < d \), and \( \frac{dr}{r-1} < p_3 < \infty \) so that \( \frac{1}{p} + \frac{1}{p_1} = \frac{1}{p_3} \), the indices \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) are determined by

\[
\begin{align*}
\alpha_1 &= \frac{1}{2} \left( 3 - \frac{d}{p_1} \right) - \frac{1}{2r}, & \beta_1 &= \frac{1}{2} \left( 2 - \frac{d}{p_2} \right) - \frac{1}{2r}, \\
\gamma_1 &= \frac{1}{2} \left( 1 - \frac{d}{p_3} \right), & \alpha_2 &= \frac{1}{2} \left( 3 - \frac{d}{p_1} \right) - \frac{1}{r}, \\
\beta_2 &= \frac{1}{2} \left( 2 - \frac{d}{p_2} \right) & \gamma_2 &= \frac{1}{2} \left( 1 - \frac{d}{p_3} \right) - \frac{1}{2r}.
\end{align*}
\]

(1.10)

Furthermore, if we assume, in addition, that \( u_0 \in \dot{B}^{1 + \frac{d+\varepsilon}{p}}_{p,r}(\mathbb{R}^d) \) for \( 0 < \varepsilon < \min\left\{ \frac{1}{r}, 1 - \frac{d}{p}, \frac{d}{p} - 1 \right\} \), then this global solution is unique.

**Remark 1.1.** The main idea for proving Theorem 1.1 is to use the maximal \( L^p(L^q) \) regularizing effect for a heat kernel (see Lemma 2.1). In fact, similar to the classical Navier–Stokes equations [17], we first reformulate (1.2) as

\[
u = e^{\mu t \Delta} u_0 + \int_0^t e^{\mu(t-s) \Delta} \left\{ -u \cdot \nabla u + \mu a \Delta u - (1 + a) \nabla \Pi \right\} ds.
\]

(1.11)
Then we can prove that appropriate approximate solutions to (1.11) satisfy the uniform estimates (1.7–1.9). With these estimates, the existence part of Theorem 1.1 follows by a compactness argument.

We remark that given initial data \( u_0 \in \dot{B}_{p,r}^{-1 + \frac{d}{p}}(\mathbb{R}^d) \), the maximal regularity we can expect for \( u \) is \( \dot{L}^1(B_{p,r}^{-1 + \frac{d}{p}}) \). With this regularity for \( u \) and \( a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \), we do not know how to define the product \( a \Delta u \) in the sense of distribution if \( p < d \). This explains in some sense why we can prove Theorem 1.1 only for \( p \in (1, d) \).

**Remark 1.2.** The smallness condition (1.6) is motivated by the one in [27] (see also [18, 26, 29] for related works on three-dimensional incompressible anisotropic Navier–Stokes systems), where we prove that for \( 1 < q \leq p < 6 \) with \( \frac{1}{q} - \frac{1}{p} \leq \frac{1}{3} \), given any data \( a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3) \) and \( u_0 = (u^0_0, u_0^1) \in \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3) \) verifying

\[
\eta \overset{\text{def}}{=} \left( \mu \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u_0^h\|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}} \right) \exp \left\{ C_0 \|u_0^3\|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}}^2 / \mu^2 \right\} \leq c_0 \mu , \tag{1.12}
\]

for some positive constants \( c_0 \) and \( C_0 \), system (1.2) has a unique global solution \( a \in C([0, \infty); \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)) \) and \( u \in C([0, \infty); \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)) \). Similar well-posedness results [20] hold with \( \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \) in (1.12) being replaced by \( \|a_0\|_{\dot{M}(\dot{B}_{p,1}^{-1 + \frac{3}{p}})} \), the norm to the multiplier space of \( \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3) \). We emphasize that our proof in [27, 20] uses the algebraical structure of (1.2) in a fundamental way, namely, \( \Delta u = 0 \), which will also be one of the key ingredients in the proof of Theorem 1.1 and Theorem 1.2, below.

**Remark 1.3.** We should also mention the recent interesting result by DANCHIN and MUCHA [14]; with more regularity assumptions on the initial velocity field, namely, \( m \leq \rho_0 < M \) and \( u_0 \in H^2(\mathbb{R}^d) \) for \( d = 2, 3 \), they can prove the local well-posedness of (1.1) for large data and global well-posedness for small data. We emphasize that here we work with our initial velocity field in the critical space \( \dot{B}_{p,r}^{-1 + \frac{d}{p}}(\mathbb{R}^d) \) for \( p \in (1, d) \) and \( r \in (1, \infty) \). In addition, we note the fact that Theorem 1.1 remains to be validated in the case of bounded smooth domain with Dirichlet boundary conditions for the velocity field. Moreover, our uniqueness result in Theorem 1.1 is strongly inspired by the Lagrangian approach in [14], but with an almost-critical regularity for the velocity, the proof here will be much more complicated. In fact, the small extra regularity compared to the scaling (1.3), namely, \( u_0 \in \dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}(\mathbb{R}^d) \) for some small \( \varepsilon > 0 \), is useful in obtaining that \( \Delta u \in L^1_{loc}(\mathbb{R}^+; L^{d+\eta}(\mathbb{R}^d)) \) for some \( \eta > 0 \), which, combined with \( \Delta u \in L^1_{loc}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d)) \) with \( p_1 < d \) (see (1.8)) implies that \( u \in L^1_{loc}(\mathbb{R}^+; Lip(\mathbb{R}^d)) \). This allows us to reformulate (1.2) in Lagrangian coordinates. One can check Theorem 4.1 below for more information about this solution.
A different approach for recovering the uniqueness of the solution is to impose more regularity on the density function. Indeed, if the density is such that \(a_0 \in B_{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)\), for some small positive \(\varepsilon\), we can also prove the global well-posedness of (1.2) under the nonlinear smallness condition (1.13):

**Theorem 1.2.** Let \(r \in (1, \infty)\), \(1 < q \leq p < 2d\) with \(\frac{1}{q} - \frac{1}{p} \leq \frac{1}{d}\) and \(\varepsilon \in (0, \frac{2d}{p} - 1)\) be any positive real number. Let \(a_0 \in L^\infty(\mathbb{R}^d) \cap B_{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)\) and \(u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)\). There exist positive constants \(c_0, C_{r,\varepsilon}\) such that if

\[
\delta \overset{\text{def}}{=} \left( \mu \|a_0\|_{L^\infty} + \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}} \right) \times \exp \left\{ C_{r,\varepsilon} \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}} \right\} \leq c_0 \mu,
\]

then system (1.2) has a global solution \((a, u)\) such that

\[
a \in C([0, \infty); L^\infty(\mathbb{R}^d) \cap B_{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)),
\]

\[
u \in C([0, \infty); \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d))
\]

and there holds

\[
\|u^h\|_{L^\infty(\mathbb{R}^d; \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u_0^h\|_{L^\infty(\mathbb{R}^d; \dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu \left( \|a\|_{L^\infty(\mathbb{R}^d; B_{\frac{d}{q}+\varepsilon})} \right) \leq C \delta,
\]

\[
\|u^d\|_{L^1(\mathbb{R}^d; \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u_0^d\|_{L^1(\mathbb{R}^d; \dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu \left( \|u^d\|_{L^1(\mathbb{R}^d; B_{\frac{d}{q}+\varepsilon})} \right) \leq 2 \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}} + c_2 \mu,
\]

for some \(c_2\) sufficiently small, and the norm \(\| \cdot \|_{X \cap Y} = \| \cdot \|_X + \| \cdot \|_Y\). Furthermore, this solution is unique if \(\frac{1}{q} + \frac{1}{p} \geq \frac{2}{d}\).

**Remark 1.4.** We point out that Haspot [19] proved the local well-posedness of (1.2) under conditions similar to Theorem 1.2. Our insight here is the global existence of solutions to (1.2) under the smallness condition (1.13). We should also mention that to overcome the difficulty that one cannot use Gronwall’s inequality in the framework of Chemin–Lerner spaces, motivated by [26, 27], we introduced the weighted Chemin–Lerner type Besov norms in Definition A.3, which will be one of key ingredients used in the proof of Theorem 1.2.
Remark 1.5. We remark that in the previous works on the global well-posedness of (1.2), the third index, $r$, of the Besov spaces, to which the initial data belong, always equals 1. In this case, the regularizing effect of the heat equation allows the velocity field to be in $L^1(\mathbb{R}^+; Lip(\mathbb{R}^d))$, which is very useful in solving the transport equation without losing any derivatives of the initial data. In both Theorem 1.1 and Theorem 1.2, the regularity of the velocity field is general enough to include non-Lipschitz vector-fields.

The organization of the paper. In the second section, we present the proof of the existence part of Theorem 1.1 in the case when $1 < p \leq \frac{dr}{s-2}$. In Section 3, we present the proof of the existence part of Theorem 1.1 for the remaining case, $\frac{dr}{s-2} < p < d$. In Section 4, we present the proof to the uniqueness part of Theorem 1.1 with an extra regularity on the initial velocity. In Section 5, we prove Theorem 1.2, which gives us uniqueness in the case where we have an additional regularity on the density. Finally, in the appendix, we collect some basic facts on Littlewood–Paley theory and Besov spaces that have been used throughout this paper.

Let us complete this section by describing the notation we shall use in this context:

Notation. For $X$, a Banach space, and $I$, an interval of $\mathbb{R}$, we denote by $C^0(I; X)$ the set of continuous functions on $I$ with values in $X$. $L^q(I; X)$ stands for the set of measurable functions on $I$ with values in $X$, such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$. By $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(c_j, r)_{j \in \mathbb{Z}}$ a generic element of $\ell^r(\mathbb{Z})$ such that $c_j, r \geq 0$ and $\sum_{j \in \mathbb{Z}} c_j, r = 1$.

2. Proof of the Existence Part of Theorem 1.1 for $1 < p \leq \frac{dr}{s-2}$

One of the key ingredients used in the proof of the existence part of Theorem 1.1 lies in Proposition 2.1:

Proposition 2.1. (Theorem 2.34 of [6]) Let $s$ be a negative real number and let $(p, r) \in [1, \infty]^2$. A constant $C$ exists such that

$$C^{-1} \mu^{\frac{s}{2}} \|f\|_{\dot{B}_p^s} \leq \|t^{-\frac{s}{2}} e^{\mu t \Delta} f\|_{L^p(\mathbb{R}^d)} \leq C \mu^{\frac{s}{2}} \|f\|_{\dot{B}_p^s}.$$

In particular, for $r \geq 1$, we deduce from Proposition 2.1 that $f \in \dot{B}^{-\frac{3}{2}}_{p, r}(\mathbb{R}^d)$ is equivalent to $\Delta u_0 \in \dot{B}^{-\frac{3}{p}}_{p, r}(\mathbb{R}^d)$ with $1 < p \leq \frac{dr}{s-2}$ and $r \in (1, \infty)$.

Note that given $u_0 \in \dot{B}^{-\frac{3}{r}+\frac{d}{p}}_{p, r}(\mathbb{R}^d)$, we can always find some $q_1 \geq p$ and $r_1 \geq r$ such that

$$-3 + \frac{d}{p} \geq -3 + \frac{d}{q_1} = -\frac{2}{r_1} \geq -\frac{2}{r}.$$
Choosing \( r_1 = r \) in the above inequality leads to \( q_1 = \frac{dr}{3r-2} \). Then \( \triangle u_0 \in \mathcal{B}_{\frac{dr}{3r-2}, r}(\mathbb{R}^d) \), and we infer that \( \triangle e^{it\triangle} u_0 \in L^r(\mathbb{R}^d; L^{\frac{dr}{3r-2}}(\mathbb{R}^d)) \). Similarly, we can choose some \( q_2 \geq p \) and \( r_2 \geq r \) with \(-2 + \frac{dr}{q_2} = -\frac{2}{r_2^2}\), so that \( \nabla u_0 \in \mathcal{B}_{p, r}^{\frac{-2 + \frac{dr}{q_2}}{2}}(\mathbb{R}^d) \). Choosing \( r_2 = 2r \) gives rise to \( q_2 = \frac{dr}{2r-1} \geq \frac{dr}{3r-2} \) for \( p \) and \( \nabla e^{it\triangle} u_0 \in L^{2r}(\mathbb{R}^d; L^{\frac{dr}{2r-1}}(\mathbb{R}^d)) \). Furthermore, Sobolev embedding ensures that \( e^{it\triangle} u_0 \in L^{2r}(\mathbb{R}^d; L^{\frac{dr}{2r-1}}(\mathbb{R}^d)) \), in this case.

The other key ingredient used in the proof of Theorem 1.1 is the following lemma (see [25] for instance), which is called maximal \( L^p(L^q) \) regularity for the heat kernel.

**Lemma 2.1.** (Lemma 7.3 of [25]) The operator \( \mathcal{A} \) defined by \( f(t, x) \mapsto \int_0^t \nabla e^{\mu(t-s)\triangle} f \, ds \) is bounded from \( L^p((0, T); L^q(\mathbb{R}^d)) \) to \( L^p((0, T); L^q(\mathbb{R}^d)) \) for every \( T \in (0, \infty) \) and \( 1 < p, q < \infty \). Moreover, there holds

\[
\| \mathcal{A} f \|_{L^p_T(L^q)} \leq \frac{C}{\mu} \| f \|_{L^p_T(L^q)}.
\]

**Lemma 2.2.** Let \( 1 < r < \infty \). The operator \( \mathcal{B} \) defined by

\[
f(t, x) \mapsto \int_0^t \nabla e^{\mu(t-s)\triangle} f \, ds
\]

is bounded from \( L^{2r}((0, T); L^{\frac{dr}{2r-1}}(\mathbb{R}^d)) \) to \( L^r((0, T); L^{\frac{dr}{3r-2}}(\mathbb{R}^d)) \) for every \( T \in (0, \infty] \), and there holds

\[
\left\| \int_0^t \nabla e^{\mu(t-s)\triangle} f \, ds \right\|_{L^r_T(L^{\frac{dr}{3r-2}})} \leq \frac{C}{\mu} \left( \frac{2r}{3r-2} \right)^{\frac{d+1}{2}} \| f \|_{L^r_T(L^{\frac{dr}{3r-2}})}.
\]  \hspace{1cm} (2.1)

**Proof.** Note that

\[
\nabla e^{\mu(t-s)\triangle} f(s, x) = \frac{\sqrt{\pi}}{(4\pi \mu(t-s))^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \frac{(x - y)}{2\sqrt{\mu(t-s)}} \exp \left\{ -\frac{|x - y|^2}{4\mu(t-s)} \right\} f(s, y) \, dy
\]

\[
\text{def} \quad = \frac{\sqrt{\pi}}{(4\pi \mu(t-s))^{\frac{d+1}{2}}} K \left( \frac{\cdot}{4\mu(t-s)} \right) * f(s, x).
\]  \hspace{1cm} (2.2)

Applying Young’s inequality in the space variables yields

\[
\left\| \nabla e^{\mu(t-s)\triangle} \left( 1_{[0,T]}(s) f(s, \cdot) \right) \right\|_{L^{\frac{dr}{2r-1}}} \leq C \left( \mu(t-s) \right)^{\frac{d+1}{2}} \left\| K \left( \frac{\cdot}{4\pi \mu(t-s)} \right) \right\|_{L^{\frac{dr}{2r-1}}} \left\| 1_{[0,T]}(s) f(s) \right\|_{L^{\frac{dr}{2r-1}}}
\]

\[
\leq C \left( \mu(t-s) \right)^{\frac{d+1}{2}} \left\| 1_{[0,T]}(s) f(s) \right\|_{L^{\frac{dr}{3r-2}}},
\]

where \( 1_{[0,t]}(s) \) denotes the characteristic function on \([0, t] \). From this and the Hardy–Littlewood–Sobolev inequality, we conclude the proof of (2.1). □
In what follows, we shall seek a solution \((a, u)\) of \((1.1)\) in the following space:

\[
X \equiv \left\{ (a, u) : a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \nabla u \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{r-2}}(\mathbb{R}^d)), \right. \\
\triangle u \in L^r(\mathbb{R}^+; L^{\frac{dr}{r-2}}(\mathbb{R}^d)) \left. \right\}. 
\]

(2.3)

We first mollify the initial data \((a_0, u_0)\), and then construct the approximate solutions \((a_n, u_n)\) via

\[
\begin{aligned}
\partial_t a_n + u_n \cdot \nabla a_n &= 0, \\
\partial_t u_n + u_n \cdot \nabla u_n - \mu \Delta u_n + \nabla \Pi_n &= a_n(\mu \Delta u_n - \nabla \Pi_n) \\
\text{div} u_n &= 0 \\
\left( a_n, u_n \right)_{t=0} &= (S_{N+n} a_0, S_{N+n} u_0),
\end{aligned}
\]

(2.4)

where \(N\) is a large enough positive integer, and \(S_{N+n} a_0\) denotes the partial sum of \(a_0\) (see the Appendix for its definition).

We have the following proposition concerning the uniform bounds of \((a_n, u_n)\).

**Proposition 2.2.** Under the assumptions of Theorem 1.1, \((2.4)\) has a unique global smooth solution \((a_n, u_n, \nabla \Pi_n)\), which satisfies

\[
\begin{aligned}
\mu \frac{1}{r} \| \Delta u^h_n \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{r-2}})} + \mu \frac{1}{r} \| \nabla u^h_n \|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{r-2}})} &\leq C \eta, \\
\mu \frac{1}{r} \| \Delta u^d_n \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{r-2}})} + \mu \frac{1}{r} \| \nabla u^d_n \|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{r-2}})} &\leq C \| u^d_0 \|_{B^{-1+\frac{d}{p}}_{p,r}} + c \mu,
\end{aligned}
\]

(2.5)

and

\[
\mu \frac{1}{r} \| \nabla \Pi_n \|_{L^r(\mathbb{R}^+; L^{\frac{dr}{r-2}})} \leq C \eta \left( \| u^d_0 \|_{B^{-1+\frac{d}{p}}_{p,r}} + c \mu \right).
\]

(2.6)

for some small enough constant \(c\) and \(\eta\) given by \((1.6)\).

**Proof.** For \(N\) large enough, it is easy to prove that \((2.4)\) has a unique local smooth solution \((a_n, u_n, \nabla \Pi_n)\) on \([0, T^*_n)\) for some positive time \(T^*_n\). Without loss of generality, we may assume that \(T^*_n\) is the lifespan to this solution. It is easy to observe that

\[
\| a_n \|_{L^\infty([0, T^*_n) \times \mathbb{R}^d)} \leq \| a_0 \|_{L^\infty}. 
\]

(2.7)

Next, for \(\lambda_1, \lambda_2 > 0\), we denote

\[
\begin{aligned}
f_{1,n}(t) &\equiv \| \nabla u^d_n(t) \|^{2r}_{L^{\frac{dr}{r-2}}}, & f_{2,n}(t) &\equiv \| \Delta u^d_n(t) \|^{r}_{L^{\frac{dr}{r-2}}}, \\
u_{\lambda_1,n}(t, x) &\equiv u_n(t, x) \exp \left\{ - \int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) \, dt' \right\},
\end{aligned}
\]

(2.8)
and similar notation for $\Pi_{\lambda,n}(t, x)$. To deal with the pressure function $\Pi_n$ in (2.4) we get, by taking the divergence of the momentum equation of (2.4), that

$$-\triangle \Pi_{\lambda,n} = \text{div}(a_n \triangledown \Pi_{\lambda,n}) - \mu \text{div}(a_n \triangle u_{\lambda,n}) + \text{div}(u_n \cdot \triangledown u_{\lambda,n}),$$

from which, (2.7), and the fact $\text{div} u_n = 0$, so that

$$\text{div}(u_n \cdot \triangledown u_{\lambda,n}) = \text{div}_h(u_n^h \cdot \triangledown u_{\lambda,n}^h) + \text{div}_h(u_n^d \partial_d u_{\lambda,n}^h) + \partial_d(u_{\lambda,n}^h \cdot \triangledown u_n^d) - \partial_d(u_n^h \text{div}_h u_{\lambda,n}^d),$$

we deduce that

$$\|\triangledown \Pi_{\lambda,n}(t)\|_{L^\frac{dr}{2}} \leq C \left\{ \|a_0\|_{L^\infty} \|\triangledown \Pi_{\lambda,n}(t)\|_{L^\frac{dr}{2}} + \mu \|a_0\|_{L^\infty} \|\triangle u_{\lambda,n}\|_{L^\frac{dr}{2}} + \left( \|u_n^h\|_{L^\frac{dr}{2}} + \|u_n^d\|_{L^\frac{dr}{2}} \right) \|\triangledown u_{\lambda,n}^h\|_{L^\frac{dr}{2}} + \|u_{\lambda,n}^h\|_{L^\frac{dr}{2}} \|\triangledown u_n^d\|_{L^\frac{dr}{2}} \right\}.$$  \hspace{1cm} (2.9)

In particular, if $\eta$ in (1.6) is so small that $C \|a_0\|_{L^\infty} \leq \frac{1}{2}$, we infer from (2.9) that

$$\|\triangledown \Pi_{\lambda,n}(t)\|_{L^\frac{dr}{2}} \leq C \left\{ \mu \|a_0\|_{L^\infty} \|\triangle u_{\lambda,n}\|_{L^\frac{dr}{2}} + \|u_n^h\|_{L^\frac{dr}{2}} \|\triangledown u_{\lambda,n}^h\|_{L^\frac{dr}{2}} + \|u_n^d\|_{L^\frac{dr}{2}} \|\triangledown u_{\lambda,n}^h\|_{L^\frac{dr}{2}} \right\}.$$  \hspace{1cm} (2.10)

Notice, on the other hand, that we can also equivalently reformulate the momentum equation of (2.4) as

$$u_n^i = u_{n,L}^i + \int_0^t e^{\mu(t-s)\triangle} \left\{ -u_n \cdot \triangledown u_n^i + \mu a_n \triangle u_n^i - (1 + a_n)\partial_t \Pi_n \right\} ds,$$  \hspace{1cm} (2.11)

for $i = 1, 2, 3$, and $u_{n,L} \overset{\text{def}}{=} e^{\mu t} S_{n+1} u_0$, from which and (2.8), we write

$$u_{\lambda,n}^h = u_{n,L}^h \exp \left\{ - \int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) \, dt' \right\} + \int_0^t e^{\mu(t-s)\triangle} \left\{ - \int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) \, dt' \right\} \times \left\{ -u_n \triangledown u_{\lambda,n}^h + \mu a_n \triangle u_{\lambda,n}^h - (1 + a_n)\triangledown_h \Pi_{\lambda,n} \right\} ds.$$  \hspace{1cm} (2.12)

Applying Lemma 2.1, Lemma 2.2, (2.7), and (2.9) leads to

$$\mu \frac{1}{r} \|\triangledown u_{\lambda,n}^h\|_{L^\frac{dr}{r}}(L^\frac{dr}{2}) + \mu \|\triangle u_{\lambda,n}^h\|_{L^\frac{dr}{r}}(L^\frac{dr}{2}) \leq \mu \frac{1}{r} \|\triangledown u_{n,L}^h\|_{L^\frac{dr}{r}}(L^\frac{dr}{2}) + \mu \|\triangle u_{n,L}^h\|_{L^\frac{dr}{r}}(L^\frac{dr}{2}) + C \left\{ \mu \|a_0\|_{L^\infty} \|\triangle u_{\lambda,n}^h\|_{L^\frac{dr}{r}}(L^\frac{dr}{2}) \right\}.$$
\[ + \mu \| a_0 \| L^\infty \| \Delta u^d_{\lambda,n} \| L^1_t \left( L^{\frac{dr}{3-r}} \right) + \| u^h_n \| L^2_t \left( L^{\frac{dr}{r-1}} \right) \| \nabla u^h_{\lambda,n} \| L^2_r \left( L^{\frac{dr}{r-1}} \right) + \left( \int_0^t e^{-\lambda_1 r} \int_s^r (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) \, dt' \right) \left( \left\| u^d_n(s) \right\| L^r_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n}(s) \| L^r_r \left( L^{\frac{dr}{r-1}} \right) + \| u^h_n \| L^r_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n}(s) \| L^r_r \left( L^{\frac{dr}{r-1}} \right) \right) \, ds \right)^{\frac{1}{r}} \right\} \text{ for } t \leq T^*_n. \text{ (2.13)}

Then, as \( C \| a_0 \| L^\infty \leq \frac{1}{2} \), and

\[ \| u_n \| L^2_t \left( L^{\frac{dr}{3-r}} \right) \leq C \| \nabla u_n \| L^2_t \left( L^{\frac{dr}{3-r}} \right), \quad \| \Delta u^d_{\lambda,n} \| L^1_t \left( L^{\frac{dr}{3-r}} \right) \leq \frac{1}{(\lambda_2 r)^{-\frac{1}{r}}}, \quad \left( \int_0^t e^{-\lambda_1 r} \int_s^r (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) \, dt' \right) \left( \left\| u^d_n(s) \right\| L^r_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n}(s) \| L^r_r \left( L^{\frac{dr}{r-1}} \right) + \| u^h_n \| L^r_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n}(s) \| L^r_r \left( L^{\frac{dr}{r-1}} \right) \right) \, ds \right)^{\frac{1}{r}} \text{ for } t \leq T^*_n. \text{ (2.13)} \]

we can infer from (2.13) that

\[ \mu^{-\frac{1}{r}} \| \nabla u^h_{\lambda,n} \| L^2_t \left( L^{\frac{dr}{3-r}} \right) + \mu \| \Delta u^h_{\lambda,n} \| L^1_t \left( L^{\frac{dr}{3-r}} \right) \leq C \mu^{-\frac{1}{r}} \| u_0^h \| B_{p,r} + C \left\{ \frac{\mu}{(\lambda_2 r)^{-\frac{1}{r}}} \| a_0 \| L^\infty + \| u^h_n \| L^2_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n} \| L^2_r \left( L^{\frac{dr}{r-1}} \right) + \frac{1}{(\lambda_1 r)^{-\frac{1}{r}}} \| \nabla u^h_{\lambda,n}(s) \| L^2_r \left( L^{\frac{dr}{r-1}} \right) \right\}. \]

Taking \( \lambda_1 = \frac{(4C)^2}{\mu^2 - 1} \) and \( \lambda_2 = \frac{C}{r \mu^{-\frac{1}{r}}} \) in the above inequality, we obtain

\[ \frac{3}{4} \mu^{-\frac{1}{r}} \| \nabla u^h_{\lambda,n} \| L^2_t \left( L^{\frac{dr}{3-r}} \right) + \mu \| \Delta u^h_{\lambda,n} \| L^1_t \left( L^{\frac{dr}{3-r}} \right) \leq C \mu^{-\frac{1}{r}} \| u_0^h \| B_{p,r} + \mu^{2-\frac{1}{r}} \| a_0 \| L^\infty \]

\[ + C \| u_n^h \| L^2_t \left( L^{\frac{dr}{3-r}} \right) \| \nabla u^h_{\lambda,n} \| L^2_r \left( L^{\frac{dr}{r-1}} \right). \text{ (2.14)} \]

Letting \( c_1 \) be a small enough positive constant, which will be determined later, we denote

\[ \tilde{T}_n \overset{\text{def}}{=} \sup \left\{ T \leq T^*_n ; \mu^{-\frac{1}{r}} \| \nabla u^h_n \| L^2_t \left( L^{\frac{dr}{3-r}} \right) + \mu^{-\frac{1}{r}} \| \Delta u^h_n \| L^1_t \left( L^{\frac{dr}{3-r}} \right) \leq c_1 \mu \right\}. \text{ (2.15)} \]
Then it follows from (2.8) and (2.14) that for $t \leq \tilde{T}_n$

$$
\frac{1}{2} \mu \hat{\pi} \left\| \nabla u^h_n \right\|_{L^r_t(L^{dr/(dr-1)})}^2 + \mu \hat{\pi} \left\| \Delta u^h_n \right\|_{L^r_t(L^{dr/(dr-2)})}^2 \leq C \left( \left\| u^h_0 \right\|_{B^{-1+\frac{4}{n}}_{p,r}} + \mu \|a_0\|_{L^\infty} \right) \times \exp \left\{ C_r \int_0^t \left( \mu^{-1-\frac{1}{r}} \left\| \nabla u^d(s) \right\|_{L^{dr/(dr-1)}}^2 r^{dr/(dr-2)} + \mu \left\| \Delta u^d(s) \right\|_{L^{dr/(dr-2)}}^r dr \right) ds \right\}.
$$

(2.16)

On the other hand, it follows from a similar derivation of (2.13) that

$$
\mu^{-1-\frac{1}{r}} \left\| \nabla u^d_n \right\|_{L^r_t(L^{dr/(dr-1)})}^r + \mu \left\| \Delta u^d_n \right\|_{L^r_t(L^{dr/(dr-2)})}^r \leq \mu^{-1-\frac{1}{r}} \left\| \nabla u^d_n,L \right\|_{L^r_t(L^{dr/(dr-1)})}^r + \mu \left\| \Delta u^d_n,L \right\|_{L^r_t(L^{dr/(dr-2)})}^r + C \left\{ \mu \|a_0\|_{L^\infty} \left\| \Delta u_n \right\|_{L^r_t(L^{dr/(dr-2)})}^r \right\} + \left( \left\| u^h_n \right\|_{L^2_t(L^{dr/(dr-1)})}^2 + \left\| u^d_n \right\|_{L^2_t(L^{dr/(dr-1)})}^2 \right) \left\| \nabla u^h_n \right\|_{L^2_t(L^{dr/(dr-1)})}^r + \left\| u^h_n \right\|_{L^2_t(L^{dr/(dr-1)})} \left\| \nabla u^d_n \right\|_{L^2_t(L^{dr/(dr-1)})}^r \right\} \text{ for } t \leq T_n^*,
$$

from which and (2.15), we deduce that

$$
\mu \hat{\pi} \left\| \Delta u^d_n \right\|_{L^r_t(L^{dr/(dr-2)})}^r + \mu \hat{\pi} \left\| \nabla u^d_n \right\|_{L^r_t(L^{dr/(dr-1)})}^r \leq 2C \left\| u^h_0 \right\|_{B^{-1+\frac{4}{n}}_{p,r}} + 2C_1 \mu \left( 1 + \|a_0\|_{L^\infty} \right) \text{ for } t \leq \tilde{T}_n.
$$

(2.17)

Substituting (2.17) into (2.16) leads to

$$
\mu \hat{\pi} \left\| \Delta u^d_n \right\|_{L^r_t(L^{dr/(dr-2)})}^r + \mu \hat{\pi} \left\| \nabla u^d_n \right\|_{L^r_t(L^{dr/(dr-1)})}^r \leq C \left( \left\| u^h_0 \right\|_{B^{-1+\frac{4}{n}}_{p,r}} + \mu \|a_0\|_{L^\infty} \right) \exp \left\{ C_r \mu^{-2r} \left\| u^d_0 \right\|_{B^{-1+\frac{4}{n}}_{p,r}} r^{2r} \right\} \leq \frac{c_1}{2} \mu
$$

(2.18)

for $t \leq \tilde{T}_n$, as long as $C_r$ is sufficiently large and $c_0$ is small enough in (1.6). This shows that $\tilde{T}_n = T_n^*$. Then, thanks to (2.17), (2.18) and Theorem 1.3 in [21], we conclude that $T_n^* = \infty$ and (2.5) holds. By virtue of (2.5) and (2.9), we infer

$$
\left\| \nabla \Pi_n \right\|_{L^r_t(L^{dr/(dr-2)})} \leq C \left\{ \mu \|a_0\|_{L^\infty} \left\| \Delta u_n \right\|_{L^r_t(L^{dr/(dr-2)})} + \left\| \nabla u^h_n \right\|_{L^2_t(L^{dr/(dr-1)})}^2 \right\} \left\| \nabla u^d_n \right\|_{L^2_t(L^{dr/(dr-1)})}^r \right\} \leq C \mu^{-\frac{1}{r}} \eta \left( \left\| u^d_0 \right\|_{B^{-1+\frac{4}{n}}_{p,r}} + \eta + c \mu \right),
$$

where $\eta$ is sufficiently small.
for some small constant $c$, which leads to (2.6). This completes the proof of the proposition. \[\square\]

We now present the proof of the existence part of Theorem 1.1 in the case when $1 < p \leq \frac{dr}{3r-2}$.

**Proof of the existence part of Theorem 1.1** for $1 < p \leq \frac{dr}{3r-2}$. Indeed, thanks to (2.4) and (2.5), (2.6), we infer that $\{\partial_t u_n\}$ is uniformly bounded in $L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$, from which, (2.5), (2.6), and the Ascoli-Arzela Theorem, we conclude that there exists a subsequence of $\{a_n, u_n, \nabla \Pi_n\}$, which we still denote by $\{a_n, u_n, \nabla \Pi_n\}$ and some $(a, u, \nabla \Pi)$ with $a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, $\nabla u \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$ and $\Delta u, \nabla \Pi \in L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$, such that

\[
a_n \to a \text{ in } L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^d),
\]

\[
u_n \to u \text{ strongly in } L^{2r}_{loc}(\mathbb{R}^+; L^{\frac{dr}{3r-2}}_{loc}(\mathbb{R}^d)),
\]

\[
\nabla \nu_n \to \nabla u \text{ strongly in } L^2_{loc}(\mathbb{R}^+; L^{\frac{dr}{3r-2}}_{loc}(\mathbb{R}^d)),
\]

\[
\Delta \nu_n \to \Delta u \text{ and } \nabla \Pi_n \to \nabla \Pi \text{ weakly in } L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d)).
\]

(2.19)

Obviously, $(a_n, u_n, \nabla \Pi_n)$ satisfies

\[
\int_0^\infty \int_{\mathbb{R}^d} a_n(\partial_t \phi + u_n \cdot \nabla \phi) \, dx \, dt + \int_{\mathbb{R}^d} S_{n+N} a_0(x) \phi(0, x) \, dx = 0,
\]

\[
\int_0^\infty \int_{\mathbb{R}^d} \text{div} \, u_n \phi \, dx \, dt = 0 \quad \text{and}
\]

\[
\int_0^\infty \int_{\mathbb{R}^d} \{u_n \cdot \partial_t \Phi - (u_n \cdot \nabla u_n) \cdot \Phi + (1 + a_n) (\mu \Delta u_n)
\]

\[
- \nabla \Pi_n \cdot \Phi\} \, dx \, dt + \int_{\mathbb{R}^d} S_{n+N} a_0 \cdot \Phi(0, x) \, dx = 0,
\]

(2.20)

for all test functions $\phi, \Phi$ given by Definition 1.1.

Therefore, thanks to (2.19), in order to prove that $(a, u, \nabla \Pi)$ obtained in (2.19) is indeed a global weak solution of (1.2) in the sense of Definition 1.1, we need to show only that $\frac{1}{1+a_n} \to \frac{1}{1+a}$ almost everywhere in $\mathbb{R}^+ \times \mathbb{R}^d$ as $n \to \infty$. Toward this end, we shall follow the compactness argument in [24] to prove that $\{a_n\}$ strongly converges to $a$ in $L^m_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ for any $m < \infty$. In fact, it is easy to observe from the transport equation of (2.4) that

\[
\partial_t a_n^2 + \text{div}(u_n a_n^2) = 0,
\]

from which we deduce that

\[
\partial_t \overline{a^2} + \text{div}(u \overline{a^2}) = 0,
\]

(2.21)

where we denote $\overline{a^2}$ to be the weak * limit of $\{a_n^2\}$.

Thanks to (2.19) and (2.20), there holds

\[
\partial_t a + \text{div}(u a) = 0
\]
in the sense of distributions. Moreover, as $\nabla u \in L^r(\mathbb{R}^d; L^{\frac{dr}{d-2}}(\mathbb{R}^d))$, we infer by a mollifying argument like that in [15] that
\[
\partial_t a^2 + \text{div}(ua^2) = 0. \tag{2.22}
\]
Subtracting (2.22) from (2.21), we obtain
\[
\partial_t (a^2 - a^2) + \text{div}(u(a^2 - a^2)) = 0. \tag{2.23}
\]
Notice that $\{S_{n+N}a_0\}$ converges to $a_0$ almost everywhere in $\mathbb{R}^d$, which implies $(a^2 - a^2)(0, x) = 0$ for almost everywhere $x \in \mathbb{R}^d$. It follows from the uniqueness theorem for the transport equation in [15] that
\[
(a^2 - a^2)(t, x) = 0 \quad \text{for almost everywhere} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,
\]
which, together with $\|a_n\|_{L^\infty} \leq \|a_0\|_{L^\infty}$, implies that
\[
a_n \to a \quad \text{strongly in} \quad L^m_{loc}(\mathbb{R}^+ \times \mathbb{R}^d) \quad \text{for any} \quad m < \infty. \tag{2.24}
\]

Thanks to (2.19) and (2.24), we can take $n \to \infty$ in (2.20) to verify that $(a, u, \nabla \Pi)$ obtained in (2.19) satisfies (1.4) and (1.5). Moreover, thanks to (2.5) and (2.6), there holds (1.7). This completes the proof of the existence part of Theorem 1.1 for $p \in (1, \frac{dr}{d-2}]$. \hfill \Box

3. Proof of the Existence Part of Theorem 1.1 for $\frac{dr}{d-2} < p < d$

In this case, as $-3 + \frac{d}{p} < -\frac{2}{r}$, it is impossible to find some $p_1 \geq p$ and $r_1 \geq r$ such that $-3 + \frac{d}{p_1} = -\frac{2}{r_1}$. However, because for all $p_1 \geq p$ and $r_1 \geq r$, $\Delta u_0 \in \mathcal{B}_{p_1,r_1}(\mathbb{R}^d)$, we can then deduce from Proposition 2.1 that $t^{\frac{1}{2}(3-\frac{d}{p_1})-\frac{1}{r_1}} \Delta e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d))$. Similarly, we choose $p_2 > \frac{d}{2}$ and $p_3 > d$ with $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$, there holds $t^{\frac{1}{2}(2-\frac{d}{p_2})-\frac{1}{r_1}} \nabla e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d))$ and $t^{\frac{1}{2}(1-\frac{d}{p_3})} e^{t\Delta} u_0 \in L^{\infty}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d))$. With these time weights before $e^{t\Delta} u_0$, $\nabla e^{t\Delta} u_0$ and $\Delta e^{t\Delta} u_0$, in order to prove the existence part of Theorem 1.1 for $\frac{dr}{d-2} < p < d$, we need to use the following time-weighted version of maximal $L^p(L^q)$ regularizing effect for the heat kernel:

Lemma 3.1. Let $1 < p, q < \infty$ and let $\alpha$ be a non-negative real number satisfying $\alpha + \frac{1}{p} < 1$. Let $\mathcal{A}$ be the operator defined by Lemma 2.1. Then if $t^\alpha f \in L^p((0, T); L^q(\mathbb{R}^d))$ for some $T \in (0, \infty]$, $t^\alpha \mathcal{A} f \in L^p((0, T); L^q(\mathbb{R}^d))$, then there holds
\[
\|t^\alpha \mathcal{A} f\|_{L^p_t(L^q_x)} \leq C \frac{t^\alpha}{\mu} \|t^\alpha f\|_{L^p_t(L^q_x)}, \tag{3.1}
\]
Proof. We first split the operator $A$ as

$$
A f = \left( \int_0^{t/2} + \int_{t/2}^t \right) \Delta e^{\mu(t-s)\Delta} f \, ds \overset{\text{def}}{=} A_1 f + A_2 f. \tag{3.2}
$$

Note that when $s \in [t/2, t]$, $t^\alpha$ is comparable to $s^\alpha$, so that it follows from the proof of Lemma 7.3 in [25] that

$$
\|t^\alpha A_2 f\|_{L^p_t(L^q)} \leq C_\mu \|t^\alpha f\|_{L^p_t(L^q)}.
$$

To handle $A_1 f$ in (3.2), we write

$$
t^\alpha A_1 f(t) = \int_0^{t/2} \frac{t^\alpha}{\mu(t-s)s^\alpha} \mu(t-s) \Delta e^{\mu(t-s)\Delta} (s^\alpha f) \, ds.
$$

As $\mu t \Delta e^{\mu t \Delta}$ is a bounded operator from $L^q(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, we have

$$
\|t^\alpha A_1 f(t)\|_{L^q(\mathbb{R}^d)} \leq C_\mu \int_0^{t/2} \frac{1}{(1-\tau)\tau^\alpha} F_\alpha(t \tau) \, d\tau,
$$

from which, along with the Minkowski inequality, we infer

$$
\|t^\alpha A_1 f\|_{L^p_t(L^q)} \leq C_\mu \int_0^{t/2} \frac{1}{(1-\tau)\tau^\alpha + \frac{1}{\frac{1}{p}} \delta^\alpha} \|F_\alpha\|_{L^p_t}.
$$

which, along with the fact that $\alpha + \frac{1}{p} < 1$, concludes the proof of (3.1). \hfill \Box

In order to deal with the estimate of $u$ and $\nabla u$, we need the following lemmas, which will be used in the proof of both the existence part and uniqueness part of Theorem 1.1.

**Lemma 3.2.** Let the operator $B$ be defined by $f(t, x) \mapsto \int_0^t \nabla e^{\mu(t-s)\Delta} f \, ds$ and let $C$ be defined by $f(t, x) \mapsto \int_0^t \mu(t-s) \Delta e^{\mu(t-s)\Delta} f \, ds$. Let $\varepsilon \geq 0$ be a small enough number, $r_1 > 1$, and let $q_1, q_2, q_3$ satisfy $\frac{d r_1}{2(1-\varepsilon)} < q_1 < \frac{d}{1-\varepsilon}$, $d < q_3 < \infty$ and $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$. We denote

$$
\tilde{\alpha}^\varepsilon = \frac{1}{2}\left(3 - \frac{d}{q_1} - \varepsilon\right) - \frac{1}{r_1}, \quad \tilde{\beta}^\varepsilon = \frac{1}{2}\left(2 - \frac{d}{q_2} - \varepsilon\right) - \frac{1}{r_1} \quad \text{and} \quad \tilde{\gamma}_1^\varepsilon = \frac{1}{2}\left(1 - \frac{d}{q_3} - \varepsilon\right).
$$

(3.3)
Then if \( \tilde{\alpha} \) \( f \in L^r((0, T); L^{q_1}(\mathbb{R}^d)) \) for some \( T \in (0, \infty) \), \( \tilde{\beta} \) \( Bf \in L^r((0, T); L^{q_2}(\mathbb{R}^d)) \) and \( \tilde{\gamma} \) \( C \) \( f \in L^\infty((0, T); L^{q_3}(\mathbb{R}^d)) \), and there holds

\[
\| t^{\frac{\tilde{\beta}}{r}} Bf \|_{L^r_T(L^{q_2})} \leq \frac{C_\epsilon}{\mu^{\frac{1}{r} + \frac{d}{2q_3}}} \| t^{\tilde{\alpha}} f \|_{L^r_T(L^{q_1})},
\]  
(3.4)

and

\[
\| t^{\frac{\tilde{\gamma}}{r}} Cf \|_{L^\infty_r(L^{q_3})} \leq \frac{C_\epsilon}{\mu^{\frac{1}{r} + \frac{d}{2q_2}}} \| t^{\tilde{\alpha}} f \|_{L^r_T(L^{q_1})}.
\]  
(3.5)

**Proof.** We first get, by applying Young’s inequality to (2.2), that

\[
\| \nabla e^{\mu(t-s)\Delta} f(s, \cdot) \|_{L^{q_1}} \leq C(\mu(t-s))^{-\frac{d}{2q_1}} \| K \left( \frac{\cdot}{\sqrt{4\mu(t-s)}} \right) \|_{L^{q_3/(q_3-1)}} \| f(s) \|_{L^{q_1}}
\]  
(3.6)

Then, letting \( F^\epsilon(s) \) \( \equiv \| s^{\tilde{\alpha}} f(s) \|_{L^{q_1}} \), we have

\[
t^{\tilde{\alpha}} \| Bf(t) \|_{L^{q_2}} \leq C \mu^{-(\frac{1}{r} + \frac{d}{2q_3})} t^{\tilde{\beta}} \int_0^t (t-s)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} s^{\tilde{\alpha}} F^\epsilon(s) \, ds.
\]

Using the change of variables \( s = \tau \) and the fact that \( \tilde{\alpha} = \left(\frac{1}{2} + \frac{d}{2q_3}\right) \tilde{\alpha} + 1 = 0 \), we obtain

\[
t^{\tilde{\beta}} \| Bf(t) \|_{L^{q_2}} \leq C \mu^{-(\frac{1}{r} + \frac{d}{2q_3})} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\tilde{\alpha}} F^\epsilon(\tau) \, d\tau.
\]

Applying Minkowski’s inequality leads to

\[
\| t^{\tilde{\beta}} Bf \|_{L^r_T(L^{q_2})} \leq C \mu^{-(\frac{1}{r} + \frac{d}{2q_3})} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\tilde{\alpha}-\frac{1}{r}} t \| F^\epsilon \|_{L^r_T} \leq C \mu^{-(\frac{1}{r} + \frac{d}{2q_3})} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\frac{1}{2}(3-\frac{d}{q_1}-\epsilon)} \, d\tau \| t^{\tilde{\alpha}} f \|_{L^r_T(L^{q_1})}.
\]

Note that the assumption \( q_3 > d \), together with \( q_1 < \frac{d}{1-\epsilon} \) implies that \( 0 < \frac{1}{2} + \frac{d}{2q_3}, \frac{1}{2}(3-\frac{d}{q_1}-\epsilon) < 1 \). This proves (3.4).

To deal with \( Cf \), we write

\[
e^{\mu(t-s)\Delta} f = (\mu(t-s))^{-\frac{d}{2q_2}} e^{-\mu(t-s)\Delta} (-\mu(t-s)\Delta)^{\frac{d}{2q_2}} (-\Delta)^{-\frac{d}{2q_2}} f.
\]

Observing that for any \( \delta > 0 \), \( e^{\Delta(-\Delta)^\delta} \) is a bounded linear operator from \( L^{q_3}(\mathbb{R}^d) \) to \( L^{q_3}(\mathbb{R}^d) \), and \( (-\Delta)^{-\delta} f = | \cdot |^{-(d-2\delta)} * f \) for \( 0 < 2\delta < d \), we see that applying Hardy–Littlewood–Sobolev inequality gives rise to

\[
\| e^{\mu(t-s)\Delta} f(s) \|_{L^{q_3}} \leq C(\mu(t-s))^{-\frac{d}{2q_2}} \| (-\Delta)^{-\frac{d}{2q_2}} f(s) \|_{L^{q_3}} \leq C(\mu(t-s))^{-\frac{d}{2q_2}} \| f(s) \|_{L^{q_1}}.
\]  
(3.7)
Let \( r_1' \) be the conjugate index of \( r_1 \); we get, by applying Hölder’s inequality, that
\[
\| \mathcal{C} f(t) \|_{L^{q_3}} \leq C \mu^{-\frac{d}{2q_2}} \int_0^t (t-s)^{-\frac{d}{2q_2} - \frac{r_1}{q_3}} F^e(s) \, ds \\
\leq C \mu^{-\frac{d}{2q_2}} \left( \int_0^t (t-s)^{-\frac{d}{2q_2} r_1'} s^{-\alpha r_1'} \, ds \right)^{\frac{1}{r_1'}} \| t^\alpha f \|_{L_{r_1}^1(L^{q_1})}.
\] (3.8)

While once again using the change of variables \( s = t \tau \), we obtain
\[
\int_0^t (t-s)^{-\frac{d}{2q_2} r_1'} s^{-\alpha r_1'} \, ds = t^{1-\frac{d}{2q_2} r_1'} \int_0^1 (1-\tau)^{-\frac{d}{2q_2} r_1'} \tau^{-\alpha r_1'} \, d\tau \\
= t^{-\gamma_1 r_1'} \int_0^1 (1-\tau)^{-\frac{d}{2q_2} r_1'} \, d\tau.
\]

Now recall the assumptions that \( q_2 > q_1 > \frac{dr_1}{2q_1-2} \) and \( q_1 < \frac{d}{1-\varepsilon} \), which imply \( \frac{d}{2q_2} r_1' < 1 \) and \( \alpha r_1' < 1 \). This, together with (3.8), ensures that
\[
\| \mathcal{C} f(t) \|_{L^{q_3}} \leq C \mu^{-\frac{d}{2q_2}} t^{-\gamma_1} \| t^\alpha f \|_{L_{r_1}^1(L^{q_1})}.
\]

This concludes the proof of (3.5) and Lemma 3.2. \( \square \)

In order to deal with the term \( u^d \partial_d u \) in (2.11), we need the following lemma.

**Lemma 3.3.** Let \( \varepsilon, r_1, q_1, \alpha \) and the operators \( \mathcal{B}, \mathcal{C} \) be given by Lemma 3.2. Letting \( r_1 > 2 \) and \( q_2, q_3 \) satisfy \( \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1} \), \( r_1' > q_3 \), \( q_1 < \infty \), we denote
\[
\tilde{\beta}_2^\varepsilon = \frac{1}{2} \left( 2 - \frac{d}{q_2} - \varepsilon \right) \quad \text{and} \quad \tilde{\gamma}_2^\varepsilon = \frac{1}{2} \left( 1 - \frac{d}{q_3} - \varepsilon \right) - \frac{1}{r_1}.
\] (3.9)

Then, if \( t^\alpha f \in L^{r_1}((0, T); L^{q_1}(\mathbb{R}^d)) \) for some \( T \in (0, \infty) \), \( t^\tilde{\beta}_2^\varepsilon \mathcal{B} f \in L^\infty((0, T); L^{q_2}(\mathbb{R}^d)) \), \( t^\tilde{\gamma}_2^\varepsilon \mathcal{C} f \in L^{r_1}((0, T); L^{q_3}(\mathbb{R}^d)) \), then there holds
\[
\| t^\tilde{\beta}_2^\varepsilon \mathcal{B} f \|_{L_{r_1}^1(L^{q_2})} \leq \frac{C_\varepsilon}{\mu^{\frac{1}{2} + \frac{d}{2q_3}}} \| t^\alpha f \|_{L_{r_1}^1(L^{q_1})},
\] (3.10)

and
\[
\| t^\tilde{\gamma}_2^\varepsilon \mathcal{C} f \|_{L_{r_1}^1(L^{q_3})} \leq \frac{C_\varepsilon}{\mu^{-\frac{d}{2q_2}}} \| t^\alpha f \|_{L_{r_1}^1(L^{q_1})}.
\] (3.11)

**Proof.** We first get, by a derivation similar to (3.6), that
\[
\| \mathcal{B} f(t) \|_{L^{q_2}} \leq C \mu^{-(\frac{1}{2} + \frac{d}{2q_3})} \int_0^t (t-s)^{-(\frac{1}{2} + \frac{d}{2q_3})} s^{-\alpha r_1'} F^e(s) \, ds \\
\leq C \mu^{-(\frac{1}{2} + \frac{d}{2q_3})} \left( \int_0^t (t-s)^{-(\frac{1}{2} + \frac{d}{2q_3})} r_1' s^{-\alpha r_1'} \, ds \right)^{\frac{1}{r_1'}} \| t^\alpha f \|_{L_{r_1}^1(L^{q_1})}.
\]
Whereas, using the change of variables \( s = t \tau \) leads to
\[
\int_0^t (t - s)^{-\left(1 + \frac{d}{2q_3}\right)} \alpha^\varepsilon r_1^\varepsilon s^{\alpha^\varepsilon r_1^\varepsilon} ds = t^{1 - \left(1 + \frac{d}{2q_3}\right)} \alpha^\varepsilon r_1^\varepsilon \int_0^1 (1 - \tau)^{-\left(1 + \frac{d}{2q_3}\right)} \tau^{\alpha^\varepsilon r_1^\varepsilon} d\tau = t^{-\frac{\beta_2}{r_1^\varepsilon}} \int_0^1 (1 - \tau)^{-\left(1 + \frac{d}{2q_3}\right)} \tau^{\alpha^\varepsilon r_1^\varepsilon} d\tau.
\]
By virtue of the assumptions \( q_3 > \frac{d r_1}{r_1 - 2} \) and \( q_1 < \frac{d}{1 - \varepsilon} \), we have \( 0 < \left(\frac{1}{2} + \frac{d}{2q_3}\right) r_1^\varepsilon \), \( \alpha^\varepsilon r_1^\varepsilon < 1 \). As a consequence, we obtain
\[
\|B f(t)\|_{L^{q_2}} \leq C_\varepsilon \mu^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} t^{-\frac{\beta_2}{r_1^\varepsilon}} \|T^\varepsilon f\|_{L^{q_2}_T(L^{q_1})},
\]
which yields (3.10).

On the other hand, it follows the same line as (3.7) that
\[
\|e^{\mu(t-s)\Delta} f(s)\|_{L^{q_3}} \leq C(\mu(t-s))^{-\frac{d}{2q_2}} \|f(s)\|_{L^{q_1}},
\]
which implies
\[
t^{\frac{\gamma_1^\varepsilon}{r_1^\varepsilon}} \|C f(t)\|_{L^{q_3}} \leq C \mu^{-\frac{d}{2q_2}} t^{\frac{\gamma_1^\varepsilon}{r_1^\varepsilon}} \int_0^t (t - s)^{-\frac{d}{2q_2}} s^{\alpha^\varepsilon F^\varepsilon(s)} ds.
\]
Using the change of variables \( s = t \tau \) and the fact that \( \gamma_1^\varepsilon = \frac{d}{2q_2} - \alpha^\varepsilon + 1 = 0 \), we obtain
\[
t^{\frac{\gamma_1^\varepsilon}{r_1^\varepsilon}} \|C f(t)\|_{L^{q_3}} \leq C \mu^{-\frac{d}{2q_2}} \int_0^1 (1 - \tau)^{-\frac{d}{2q_2}} \tau^{\alpha^\varepsilon H^\varepsilon(t \tau)} d\tau,
\]
from which we deduce
\[
\|t^{\frac{\gamma_1^\varepsilon}{r_1^\varepsilon}} C f\|_{L^{q_2}_T(L^{q_3})} \leq C \mu^{-\frac{d}{2q_2}} \int_0^1 (1 - \tau)^{-\frac{d}{2q_2}} \tau^{\alpha^\varepsilon - \frac{1}{r_1}} d\tau \|H^\varepsilon\|_{L^{q_1}_T} \leq C \mu^{-\frac{d}{2q_2}} \int_0^1 (1 - \tau)^{-\frac{d}{2q_2}} \tau^{\frac{3}{2} - \frac{d}{2q_1}} d\tau \|t^{\alpha^\varepsilon H^\varepsilon}\|_{L^{q_1}_T(L^{q_1})}.
\]
This, along with the facts \( q_2 > q_1 > \frac{d r_1}{r_1 - 2} > 0 \), ensures the integral above is finite. This gives (3.11) and we complete the proof of the lemma.

In what follows, we take \( r_1 = 2r > 2 \) and \( p_1, p_2 \) and \( p_3 \) satisfying \( \max\left(p, \frac{dr_1}{r_1 - 2}\right) < p_1 < d, \) and \( \frac{dr_1}{r_1 - 2} < p_3 < \infty \), so that \( \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1} \). We shall seek a solution \( \left(a, u, \nabla \Pi\right) \) of (1.2) in the following functional space:
\[
X = \left\{ u : a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \ t^{r_1} u \in L^\infty(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d)), \right. \\
\left. t^{\gamma_1^\varepsilon} u \in L^{2r}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d)), \ t^\beta_1 \nabla u \in L^{2r}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d)), \right. \\
\left. t^{\beta_2} \nabla u \in L^{\infty}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d)) \right\},
\]
with the indices \( \alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2 \) being determined by (1.10).
We construct the approximate solution \((a_n, u_n, \nabla \Pi_n)\) through (2.4). Similar to Proposition 2.2, we have the following proposition concerning the uniform time-weighted bounds of \((a_n, u_n)\).

**Proposition 3.1.** Under the assumptions of Theorem 1.1, (2.4) has a unique global smooth solution \((a_n, u_n, \nabla \Pi_n)\) which satisfies

\[
\mu^{\frac{1}{2}(3 - \frac{d}{p_1})} \| t^{\alpha_1} \Delta u_n^h \|_{L^2(\mathbb{R}^+; L^{p_1})} + \mu^{\frac{1}{2}(1 - \frac{d}{p_3})} \| t^{\alpha_2} \nabla u_n^h \|_{L^2(\mathbb{R}^+; L^{p_3})} + \| t^{\alpha_2} \nabla u_n^d \|_{L^2(\mathbb{R}^+; L^{p_3})} \leq C \eta, \tag{3.13}
\]

\[
\mu^{\frac{1}{2}(3 - \frac{d}{p_1})} \| t^{\alpha_1} \Delta u_n^d \|_{L^2(\mathbb{R}^+; L^{p_1})} + \mu^{\frac{1}{2}(1 - \frac{d}{p_3})} \| t^{\alpha_2} \nabla u_n^d \|_{L^2(\mathbb{R}^+; L^{p_3})} + \| t^{\alpha_2} \nabla u_n^d \|_{L^2(\mathbb{R}^+; L^{p_3})} \leq C \| u_0^d \|_{B_{p,r}} + c \mu, \tag{3.14}
\]

for some small enough constant \(c, \eta\) being given by (1.6) and \(\alpha_2\) by (1.10).

**Proof.** For \(N\) large enough, it is easy to prove that (2.4) has a unique local smooth solution \((a_n, u_n)\) on \([0, T_n^*)\). Without loss of generality, we may assume that \(T_n^*\) is the lifespan to this solution. For \(\lambda_1, \lambda_2, \lambda_3 > 0\), we denote

\[
f_{1,n}(t) \overset{\text{def}}{=} \| t^{\beta_1} \nabla u_n^d (t) \|_{L^{p_2}}, \quad f_{2,n}(t) \overset{\text{def}}{=} \| t^{\gamma_2} u_n^d (t) \|_{L^{p_3}},
\]

\[
f_{3,n}(t) \overset{\text{def}}{=} \| t^{\alpha_1} \Delta u_n^d (t) \|_{L^{p_1}}, \quad u_{\lambda,n}(t, x) \overset{\text{def}}{=} u_n(t, x)
\]

\[
\times \exp \left\{ - \int_0^t \left( \lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t') \right) \, dt' \right\}, \tag{3.15}
\]

and similar notation for \(\nabla \Pi_{\lambda,n}(t, x)\). We then we deduce by a derivation similar to (2.9) that

\[
\| \nabla \Pi_{\lambda,n}(t) \|_{L^{p_1}} \leq C \left\{ a_0 \| L^\infty \| \nabla \Pi_{\lambda,n}(t) \|_{L^{p_1}} + \mu \| a_0 \|_{L^\infty} \| \Delta u_{\lambda,n} \|_{L^{p_1}} + \| u_n^h \|_{L^{p_3}} \| \nabla u_n^h \|_{L^{p_2}} + \| u_n^d \|_{L^{p_3}} \| \nabla u_n^d \|_{L^{p_2}} \right\}.
\]
In particular, if \( \eta \) in (1.6) is so small that \( C \| a_0 \|_{L^\infty} \leq \frac{1}{2} \), we obtain
\[
\| \nabla \Pi_{\lambda,n}(t) \|_{L^p} \leq C \left\{ \mu \| a_0 \|_{L^\infty} \| \Delta u_{\lambda,n} \|_{L^p} + \| u_h^h \|_{L^p} \| \nabla u_{\lambda,n}^h \|_{L^p} \right. \\
+ \left. \| u_n^d \|_{L^p} \| \nabla u_{\lambda,n}^h \|_{L^p} + \| u_h^h \|_{L^p} \| \nabla u_n^d \|_{L^p} \right\}.
\] (3.16)

It follows the form of (2.11) and (3.15) that
\[
u_{\lambda,n}^h = u_{n,L}^h \exp \left\{ - \int_0^t \left( \lambda_1 f_1,n(t') + \lambda_2 f_2,n(t') + \lambda_3 f_3,n(t') \right) dt' \right\} \\
+ \int_0^t e^{\mu(t-s)A} \exp \left\{ - \int_s^t \left( \lambda_1 f_1,n(t') + \lambda_2 f_2,n(t') + \lambda_3 f_3,n(t') \right) dt' \right\} \\
\times \left( -u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla \Pi_{\lambda,n} \right) \, ds.
\] (3.17)

Applying Lemma 3.2 for \( \epsilon = 0 \) and \( r_1 = 2r \) gives
\[
\mu \frac{d}{dt} \| t^{\gamma_1} u_{\lambda,n}^h \|_{L^\infty(L^P)} + \mu \frac{1}{2} + \frac{d}{dt} \| t^{\beta_1} \nabla u_{\lambda,n}^h \|_{L^2(L^P)} \\
\leq \mu \frac{d}{dt} \| t^{\gamma_1} u_{\lambda,n,L}^h \|_{L^\infty(L^P)} + \mu \frac{1}{2} + \frac{d}{dt} \| t^{\beta_1} \nabla u_{\lambda,n,L}^h \|_{L^2(L^P)} \\
+ \| \exp \left\{ - \int_s^t \left( \lambda_1 f_1,n(t') + \lambda_2 f_2,n(t') + \lambda_3 f_3,n(t') \right) dt' \right\} \\
\times s^{\alpha_1} \left( -u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla \Pi_{\lambda,n} \right) \|_{L^2(L^P)},
\]
for indices \( \beta_1, \gamma_1 \) given by (1.10).

Similarly, it follows from Lemma 3.1 and Lemma 3.3 for \( \epsilon = 0 \) and \( r_1 = 2r \) that
\[
\mu \frac{d}{dt} \| t^{\gamma_2} u_{\lambda,n}^h \|_{L^\infty(L^P)} + \mu \frac{1}{2} + \frac{d}{dt} \| t^{\beta_2} \nabla u_{\lambda,n}^h \|_{L^2(L^P)} + \mu \| t^{\alpha_1} \Delta u_{\lambda,n}^h \|_{L^2(L^P)} \\
\leq \mu \frac{d}{dt} \| t^{\gamma_2} u_{\lambda,n,L}^h \|_{L^\infty(L^P)} + \mu \frac{1}{2} + \frac{d}{dt} \| t^{\beta_2} \nabla u_{\lambda,n,L}^h \|_{L^2(L^P)} + \mu \| t^{\alpha_1} \Delta u_{\lambda,n,L}^h \|_{L^2(L^P)} \\
+ \| \exp \left\{ - \int_s^t \left( \lambda_1 f_1,n(t') + \lambda_2 f_2,n(t') + \lambda_3 f_3,n(t') \right) dt' \right\} \\
\times s^{\alpha_1} \left( -u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla \Pi_{\lambda,n} \right) \|_{L^2(L^P)},
\]
for indices \( \alpha_1, \beta_2, \gamma_2 \) given by (1.10).

As a consequence, we obtain, by using (2.7), that
\[
\mu \frac{d}{dt} \left( \| t^{\gamma_1} u_{\lambda,n}^h \|_{L^\infty(L^P)} + \| t^{\gamma_2} u_{\lambda,n}^h \|_{L^\infty(L^P)} \right) \\
+ \mu \frac{1}{2} + \frac{d}{dt} \left( \| t^{\beta_1} \nabla u_{\lambda,n}^h \|_{L^2(L^P)} + \| t^{\beta_2} \nabla u_{\lambda,n}^h \|_{L^2(L^P)} \right) \\
+ \mu \| t^{\alpha_1} \Delta u_{\lambda,n}^h \|_{L^2(L^P)} \\
\leq \mu \frac{1}{2} (1 - \frac{d}{dt}) \| u_{\lambda,n}^0 \|_{L^2(L^P)} + C \left\{ \mu \| a_0 \|_{L^\infty} \left( \| t^{\alpha_1} \Delta u_{\lambda,n}^h \|_{L^2(L^P)} \right) \right\}
\]
\[
+ \| t^{a_1} \Delta u^d_{\lambda, n} \|_{L^r_t(L^p_1)} + \| t^{\gamma_1} u^h_{\lambda, n} \|_{L^\infty_t(L^p_3)} \| t^{\beta_1} \nabla u^h_{\lambda, n} \|_{L^r_t(L^p_2)} \\
+ \left( \int_0^r e^{-2\lambda r} \int_s^{t_s} (\lambda_1 f_1(\lambda^2) + \lambda_2 f_2 (\lambda^2)) \, dt' \right) \left( \| s^{a_2} u^d_n (s) \|_{L^p_2}^2 \right) \left( \| s^{\gamma_2} u^h_{\lambda, n} \|_{L^p_3}^2 \| s^{\beta_1} \nabla u^h_{\lambda, n} (s) \|_{L^p_2}^2 \right) \, ds \right) \frac{1}{\gamma r^r} \right) .
\] 

(3.18)

Then, as \( C \| a_0 \|_{L^\infty} \leq \frac{1}{2} \), and

\[
\| s^{a_1} \Delta u^d_{\lambda, n} \|_{L^r_t(L^p_1)} \leq \frac{1}{(2\lambda_3 r)^{\frac{1}{\gamma r}}},
\]

\[
\left( \int_0^r e^{-2\lambda_1 r} \int_s^{t_s} f_1 (\lambda^2) \, dt' \right) \left( \| s^{\gamma_1} u^h_{\lambda, n} \|_{L^p_3}^2 \| t^{\beta_1} \nabla u^h_{\lambda, n} (s) \|_{L^p_2}^2 \right) \frac{1}{\gamma r^r}
\]

we infer from (3.18) that

\[
\mu \frac{d}{r^2} \left( \| t^{\gamma_1} u^h_{\lambda, n} \|_{L^\infty_t(L^p_3)} + \| t^{\gamma_2} u^h_{\lambda, n} \|_{L^r_t(L^p_3)} \right) + \frac{1}{2} \mu \frac{d}{r^2} \left( \| t^{\beta_1} \nabla u^h_{\lambda, n} \|_{L^r_t(L^p_2)} \right) + \mu \| t^{a_1} \Delta u^h_{\lambda, n} \|_{L^r_t(L^p_1)}
\]

\[
\leq \mu \frac{d}{r^2} \left( \| a_0 \|_{L^\infty} \right) \left( \int_0^r e^{-2\lambda_1 r} \int_s^{t_s} f_1 (\lambda^2) \, dt' \right) \left( \| s^{\gamma_1} u^h_{\lambda, n} \|_{L^p_3}^2 \| t^{\beta_1} \nabla u^h_{\lambda, n} (s) \|_{L^p_2}^2 \right) \frac{1}{\gamma r^r}
\]

\[
\leq \mu \frac{d}{r^2} \left( \| a_0 \|_{L^\infty} \right) \left( \int_0^r e^{-2\lambda_1 r} \int_s^{t_s} f_1 (\lambda^2) \, dt' \right) \left( \| s^{\gamma_1} u^h_{\lambda, n} \|_{L^p_3}^2 \| t^{\beta_1} \nabla u^h_{\lambda, n} (s) \|_{L^p_2}^2 \right) \frac{1}{\gamma r^r}
\]

Taking \( \lambda_1 = \frac{(4C)^{2r}}{2r \mu \frac{d}{r^2}} \), \( \lambda_2 = \frac{(4C)^{2r}}{2r \mu (1 + \frac{d}{r^2})^r} \) and \( \lambda_3 = \frac{C^{2r}}{2r \mu (\frac{d}{r^2} - 1)^r} \) in the above inequality results in

\[
\mu \frac{d}{r^2} \left( \frac{3}{4} \| t^{\gamma_1} u^h_{\lambda, n} \|_{L^\infty_t(L^p_3)} + \| t^{\gamma_2} u^h_{\lambda, n} \|_{L^r_t(L^p_3)} \right) + \mu \frac{1}{2} \frac{d}{r^2} \left( \| t^{\beta_1} \nabla u^h_{\lambda, n} \|_{L^r_t(L^p_2)} \right) + \frac{3}{4} \mu \| t^{\beta_1} \nabla u^h_{\lambda, n} \|_{L^\infty_t(L^p_2)}
\]

\[
+ \mu \| t^{a_1} \Delta u^h_{\lambda, n} \|_{L^r_t(L^p_1)}
\]
Let $c_1$ be a small enough positive constant, which will be determined later. We denote

\[
\tilde{T}_n = \sup \left\{ T \leq T_n^*; \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_1 \Delta u_n \|_{L_t^\infty(L^3)} \right) + \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) \right. 
\]

\[
\left. + \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_1 \Delta u_n \|_{L_t^\infty(L^3)} \right) \right\}. 
\]

We shall prove that $\tilde{T}_n = T_n^* = \infty$, as long as we take $C_r$ sufficiently large and $c_0$ sufficiently small in (1.6). In fact, if $\tilde{T}_n < T_n^*$, it follows from (3.15) and (3.19) that

\[
\mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_1 \Delta u_n \|_{L_t^\infty(L^3)} \right) + \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) \leq C \left( \| u_0 \|_{B_{p,r}} + \mu \| a_0 \|_{L^\infty} \right) \exp \left\{ C_r \left( \int_0^t \left( \mu \frac{1}{2} \| \nabla u_n(s) \|_{L_t^\infty(L^3)} + \mu \frac{1}{2} \| \nabla u_n(s) \|_{L_t^\infty(L^3)} \right) ds \right\} 
\]

On the other hand, it follows from a derivation similar to (3.18) that

\[
\mu \frac{d}{dt} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) 
\]

\[
+ \mu \frac{d}{dt} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) \leq \mu \frac{1}{2} \left( \| u_0 \|_{B_{p,r}} + \mu \| a_0 \|_{L^\infty} \right) \exp \left\{ C \left( \| u_0 \|_{L^\infty} + \mu \| a_0 \|_{L^\infty} \right) \right\} 
\]

From this, (3.20) and taking $c_0$ small enough in (1.6) so that $C \| a_0 \|_{L^\infty} \leq \frac{1}{2}$, we deduce

\[
\mu \frac{d}{dt} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) + \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) \leq 2C \mu \frac{1}{2} \left( \| u_0 \|_{B_{p,r}} + \mu \| a_0 \|_{L^\infty} \right), 
\]

for $t < T_n^*$. From this, (3.20) and taking $c_0$ small enough in (1.6) so that $C \| a_0 \|_{L^\infty} \leq \frac{1}{2}$, we deduce

\[
\mu \frac{d}{dt} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) + \mu \frac{1}{2} \left( \| \nabla u_n \|_{L_t^\infty(L^3)} + \| t^\beta_2 \Delta u_n \|_{L_t^\infty(L^3)} \right) \leq 2C \mu \frac{1}{2} \left( \| u_0 \|_{B_{p,r}} + \mu \| a_0 \|_{L^\infty} \right), 
\]
for \( t \leq \tilde{T}_n \), which implies
\[
\mu^{\frac{1}{2} \left( 1 - \frac{d}{p_3} \right)} \left( \| t^{\gamma_1} u_n^{d} \|_{L^\infty_t(L^{p_3})} + \| t^{\gamma_2} u_n^{d} \|_{L^2_t(L^{p_3})} \right)
+ \mu^{\frac{1}{2} \left( 1 - \frac{d}{p_2} \right)} \left( \| t^{\beta_1} \Delta u_n \|_{L^2_t(L^{p_2})} + \| t^{\beta_2} \Delta u_n \|_{L^2_t(L^{p_2})} \right)
+ \mu^{\frac{1}{2} \left( 3 - \frac{d}{p_1} \right)} \| t^{a_1} \Delta u_n \|_{L^2_t(L^{p_1})} \leq C \| u_0^d \|_{B^{-1+\frac{d}{p}}_{p,r}} + c \mu, \tag{3.23}
\]
for \( t \leq \tilde{T}_n \). Substituting (3.23) into (3.21) leads to
\[
\mu^{\frac{1}{2} \left( 1 - \frac{d}{p_3} \right)} \left( \| t^{\gamma_1} u_n^{h} \|_{L^\infty_t(L^{p_3})} + \| t^{\gamma_2} u_n^{h} \|_{L^2_t(L^{p_3})} \right)
+ \mu^{\frac{1}{2} \left( 1 - \frac{d}{p_2} \right)} \left( \| t^{\beta_1} \Delta u_n \|_{L^2_t(L^{p_2})} + \| t^{\beta_2} \Delta u_n \|_{L^2_t(L^{p_2})} \right)
+ \mu^{\frac{1}{2} \left( 3 - \frac{d}{p_1} \right)} \| t^{a_1} \Delta u_n \|_{L^2_t(L^{p_1})} \leq C \left( \| u_0^h \|_{B^{-1+\frac{d}{p}}_{p,r}} + \mu \| a_0 \|_{L^\infty} \right) \exp \left\{ C_r \mu^{-2r} \| u_0^d \|_{B^{-1+\frac{d}{p}}_{p,r}} \right\} \leq \frac{c_1}{2} \mu, \tag{3.24}
\]
for \( t \leq \tilde{T}_n \), as long as \( C_r \) is sufficiently large and \( c_0 \) small enough in (1.6). This contradicts (3.20), which, in turn, shows that \( \tilde{T}_n = T_n^* \). Then, thanks to (3.23), (3.24) and Theorem 1.3 in [21], we conclude that \( T_n^* = \infty \), and there holds (3.13). It remains to prove (3.14). Indeed, similar to (3.22), we get, by applying Lemma 3.1, that
\[
\mu \| t^{\alpha_2} \Delta u_n \|_{L^{r}(\mathbb{R}^+,L^{p_1})} \leq \mu^{-\frac{1}{2} \left( 1 - \frac{d}{p_1} \right)} \| u_0 \|_{B^{-1+\frac{d}{p}}_{p,r}}
+ C \left\{ \mu \| a_0 \|_{L^\infty} \| t^{\alpha_2} \Delta u_n \|_{L^{r}(\mathbb{R}^+,L^{p_1})} \right.
+ \left( \| t^{\gamma_2} u_n^h \|_{L^2_t(L^{r}(\mathbb{R}^+,L^{p_3}))} + \| t^{\gamma_2} u_n^d \|_{L^2_t(L^{r}(\mathbb{R}^+,L^{p_3}))} \right) \| t^{\beta_1} \Delta u_n \|_{L^2_t(L^{r}(\mathbb{R}^+,L^{p_2})})
+ \left. \| t^{\gamma_2} u_n^h \|_{L^2_t(L^{r}(\mathbb{R}^+,L^{p_3}))} \| t^{\beta_1} \Delta u_n \|_{L^2_t(L^{r}(\mathbb{R}^+,L^{p_2}))} \right),
\]
which, along with (3.13) and the fact that \( C \| a_0 \|_{L^\infty} \leq \frac{1}{2} \), gives rise to the first inequality of (3.14). Along the same lines, one gets the estimate of \( \| t^{\alpha_2} \nabla \Pi_n \|_{L^{2r}(\mathbb{R}^+,L^{p_1})} \) in (3.14). It follows from this and (3.16) that
\[
\| t^{\alpha_1} \nabla \Pi_n \|_{L^{2r}(\mathbb{R}^+,L^{p_1})} \leq C \left\{ \mu \| a_0 \|_{L^\infty} \| t^{\alpha_1} \Delta u_n \|_{L^{2r}(\mathbb{R}^+,L^{p_1})} \right.
+ \left( \| t^{\gamma_1} u_n^h \|_{L^\infty(\mathbb{R}^+,L^{p_3})} \| t^{\beta_1} \Delta u_n \|_{L^{2r}(\mathbb{R}^+,L^{p_2})} \right.
+ \left. \| t^{\gamma_2} u_n^d \|_{L^2(\mathbb{R}^+,L^{p_3})} \| t^{\beta_2} \Delta u_n \|_{L^\infty(\mathbb{R}^+,L^{p_2})} \right),
\]
from which and (3.13), we obtain the estimate of \( \| t^{\alpha_1} \nabla \Pi_n \|_{L^{2r}(\mathbb{R}^+,L^{p_1})} \) in (3.14). This completes the proof of the proposition. \( \Box \)
Now we are in a position to complete the proof of the existence part of Theorem 1.1 for the remaining case.

**Proof of the existence part of Theorem 1.1 for** $p \in \left( \frac{dr}{3r-2}, d \right)$. Note that $p_1 < d$ ensures $\alpha r^r < 1$, so that for any $T > 0$, we deduce from (3.14) that $\Delta u_n = t^{-\alpha r^r} (t^{\alpha r^r} \Delta u_n)$ is uniformly bounded in $L^{r_1}((0, T); L^{p_1}(\mathbb{R}^d))$ for some $r_1 \in (1, \infty)$. Similarly, we infer from (1.10) and (3.13) that $\{\nabla u_n\}$ is uniformly bounded in $L^{r_2}((0, T); L^{p_2}(\mathbb{R}^d))$ for any $r_2 < \frac{2p_2}{2p_2 - d}$, and $\{u_n\}$ is uniformly bounded in $L^{r_3}((0, T); L^{p_3}(\mathbb{R}^d))$ for any $r_3 < \frac{2p_3}{p_3 - d}$. Moreover, as $\frac{2p_2 - d}{2p_2} + \frac{p_3 - d}{2p_3} = \frac{3}{2} - \frac{d}{2p_3}$ and $p_1 < d$, we can choose $r_2$ and $r_3$ so that $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_4} < 1$. This, together with (2.4), implies that $\{\partial_t u_n\}$ is uniformly bounded in $L^{r_1}((0, T); L^{p_1}(\mathbb{R}^d)) + L^{r_2}((0, T); L^{p_1}(\mathbb{R}^d))$ for any $T < \infty$, from which, with the Ascoli-Arzela Theorem and $p_2 < \frac{d}{d - p_1}$, we conclude that there exists a subsequence of $\{a_n, u_n, \nabla \Pi_n\}$, which we still denote by $\{a_n, u_n, \nabla \Pi_n\}$ and some $(a, u, \nabla \Pi)$ with $a \in L^{\infty}(\mathbb{R}^+ \times \mathbb{R}^d)$, $u \in L^{r_3}_{loc}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d))$ with $\nabla u \in L^{r_2}_{loc}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d))$ and $\Delta u$, $\nabla \Pi \in L^{r_1}_{loc}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d))$, such that

$$a_n \rightarrow a \text{ weak * in } L^{\infty}_{loc}(\mathbb{R}^+ \times \mathbb{R}^d),$$

$$u_n \rightarrow u \text{ strongly in } L^{r_3}_{loc}(\mathbb{R}^+; L^{p_3}_{loc}(\mathbb{R}^d)),$$

$$\nabla u_n \rightarrow \nabla u \text{ strongly in } L^{r_2}_{loc}(\mathbb{R}^+; L^{p_2}_{loc}(\mathbb{R}^d)),$$

$$\Delta u_n \rightarrow \Delta u \text{ and } \nabla \Pi_n \rightharpoonup \nabla \Pi \text{ weakly in } L^{r_1}_{loc}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d)).$$

(3.25)

With (3.13), (3.14) and (3.25), we can repeat the argument at the end of Section 2 to complete the proof of the existence part of Theorem 1.1 for the case when $\frac{dr}{3r-2} < p < d$. □

### 4. The Uniqueness Part of Theorem 1.1

With a little bit more regularity on the initial velocity, namely $u_0 \in B^{-1 + \frac{d}{p} + \varepsilon}_{p, r}(\mathbb{R}^d)$ for some small enough $\varepsilon > 0$, we can prove the uniqueness of the solution constructed in the last two sections. This result is strongly inspired by the Lagrangian approach in [13,14]. Nevertheless, with an almost critical regularity for the initial velocity field, the proof here is more challenging. The main result reads as follows:

**Theorem 4.1.** Let $r \in (1, \infty)$, $p \in (1, d)$ and $0 < \varepsilon < \min\{\frac{1}{r}, 1 - \frac{1}{r}, \frac{d}{p} - 1\}$. Let $a_0 \in L^{\infty}(\mathbb{R}^d)$ and $u_0 \in B^{-1 + \frac{d}{p} + \varepsilon}_{p, r}(\mathbb{R}^d) \cap B^{-1 + \frac{d}{p}}_{p, r}(\mathbb{R}^d)$, which satisfies the nonlinear smallness condition (1.6). Then (1.2) has a unique global weak solution $(a, u)$, which satisfies (1.7–1.9) and

$$\mu \frac{1}{2} (3 - \frac{d}{q_1} - \varepsilon) \|t^{\alpha r^r} \Delta u\|_{L^{2r}(\mathbb{R}^+; L^{q_1})}$$

$$+ \mu \frac{1}{2} (3 - \frac{d}{q_2} - \varepsilon) \left( \|t^{\beta_1} \nabla u\|_{L^{2r}(\mathbb{R}^+; L^{q_2})} + \|t^{\beta_2} \nabla u\|_{L^{\infty}(\mathbb{R}^+; L^{q_2})} \right)$$
where \( q_1, q_2, q_3 \) satisfy \( \max(p, \frac{dr}{r-1}) < q_1 < \frac{d}{1-\varepsilon} \), and \( \frac{dr}{r-1} < q_3 < \infty \) so that \( \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1} \), the indices \( \alpha^e_1, \alpha^e_2, \beta^e_1, \beta^e_2, \gamma^e_1, \gamma^e_2 \) are determined by

\[
\begin{align*}
\alpha^e_1 &= \frac{1}{2} \left( 3 - \frac{d}{q_1} - \varepsilon \right) - \frac{1}{2r}, \\
\beta^e_1 &= \frac{1}{2} \left( 2 - \frac{d}{q_2} - \varepsilon \right) - \frac{1}{2r}, \\
\gamma^e_1 &= \frac{1}{2} \left( 1 - \frac{d}{q_3} - \varepsilon \right), \\
\alpha^e_2 &= \frac{1}{2} \left( 3 - \frac{d}{q_1} - \varepsilon \right) - \frac{1}{r}, \\
\beta^e_2 &= \frac{1}{2} \left( 2 - \frac{d}{q_2} - \varepsilon \right), \\
\gamma^e_2 &= \frac{1}{2} \left( 1 - \frac{d}{q_3} - \varepsilon \right) - \frac{1}{2r}. 
\end{align*}
\]  

**Remark 4.1.** To prove the uniqueness part of Theorem 4.1, we shall choose \( p_1 = \frac{d}{1+\varepsilon} \) in (1.10). This choice of \( p_1 \) satisfies \( \max(p, \frac{dr}{r-1}) < p_1 < d \) with \( 0 < \varepsilon < \min(1 - \frac{1}{r}, \frac{d}{p} - 1) \). Then by virtue of (1.8), \( \Delta u = t^{-\alpha^e_2}(t^{\alpha^e_2} \Delta u) \in L^{\tau_1}(0, T); L^{p_1}(\mathbb{R}^d) \) for any \( T < \infty \) and \( \tau_1 \) satisfying \( \alpha^e_2 < \frac{1}{\tau_1} < \frac{1}{r} \), which implies \( \tau_1 < \frac{2}{2-\varepsilon} \), we thus take \( \tau_1 = \frac{8}{8-\varepsilon} \). As \( q_1 < \frac{d}{1+\varepsilon} \) in Theorem 4.1, we can choose \( q_1 > d \) in order to get \( \nabla u \in L^1_{loc}(\mathbb{R}^\infty) \) (see Lemma 4.1 below), and this is the reason we need an additional regularity on \( u_0 \) to prove the uniqueness part of Theorem 1.1. We shall choose \( q_1 = \frac{2d}{2-\varepsilon} < \frac{d}{1-\varepsilon} \) in (4.3) later. Then, in order that \( \Delta u \in L^{r_1}(0, T); L^{q_1}(\mathbb{R}^d) \), we have \( \alpha^e_2 < \frac{1}{r_1} < \frac{1}{r} \), and hence we take \( \tau_1^e = \frac{8}{8-\varepsilon} < \frac{4}{4-\varepsilon} \).

It follows from the existence proof of Theorem 1.1 that, in order to prove that the solution constructed in the last two sections satisfies (4.1) and (4.2), we need to prove only that the same inequalities hold for the approximate solutions \((a_n, u_n, \nabla \Pi_n)\) determined by (2.4).

We now turn to the uniform estimate of \((u_n, \nabla \Pi_n)_{n \in \mathbb{N}}\) when the initial velocity \( u_0 \in \dot{B}^{-3+\frac{d}{q_1}+\varepsilon}_{p, r} \) for some \( \varepsilon \in (0, \min(\frac{1}{r}, 1 - \frac{1}{r}, \frac{d}{p} - 1)) \). Note that for all \( q_1 \geq p \) and \( r_1 \geq r \), \( \Delta u_0 \in \dot{B}^{-3+\frac{d}{q_1}+\varepsilon}_{q_1, r_1} \), so we deduce from Proposition 2.1
that \( t^{\frac{1}{2}}(3 - \frac{d}{q_1 - \varepsilon}) - \frac{1}{q_1} \Delta e^t u_0 \in L^{r_1} (\mathbb{R}^+; L^{q_1} (\mathbb{R}^d)) \). Similarly, if we choose \( q_2 > \frac{d}{2 - \varepsilon} \) and \( q_3 > \frac{d}{1 - \varepsilon} \) with \( \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1} \), then there holds \( t^{\frac{1}{2}}(3 - \frac{d}{q_2 - \varepsilon}) - \frac{1}{q_2} \nabla e^t u_0 \in L^{r_1} (\mathbb{R}^+; L^{q_2} (\mathbb{R}^d)) \) and \( t^{\frac{1}{2}}(1 - \frac{d}{q_3 - \varepsilon}) e^t u_0 \in L^{r_{\infty}} (\mathbb{R}^+; L^{q_3} (\mathbb{R}^d)) \). We thus take \( r_1 = 2r > 2 \) and \( q_1, q_2 \) and \( q_3 \) satisfying \( \max(p, \frac{dr}{r - 1}) < q_1 < \frac{d}{2 - \varepsilon}, \frac{dr}{r - 1} < q_3 < \infty \), and \( \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1} \). We shall investigate the uniform estimate to the solutions \((a_n, u_n, \nabla \Pi_n)\) of \((2.4)\) in the following functional space:

\[
X = \left\{ u : a \in L^{\infty} (\mathbb{R}^+ \times \mathbb{R}^d), \quad t^{\gamma^2} u \in L^{2r} (\mathbb{R}^+; L^{q_1} (\mathbb{R}^d)), \quad t^{\gamma^1} u \in L^{2r} (\mathbb{R}^+; L^{q_2} (\mathbb{R}^d)), \quad t^{\gamma^3} \nabla u \in L^{2r} (\mathbb{R}^+; L^{q_2} (\mathbb{R}^d)) \right\},
\]

with the indices \( \alpha^i, \beta^i, \gamma^i \) being determined by \((4.3)\). Note that \( \alpha^i \neq \beta^i + \gamma^i \), but \( \alpha^i = \beta_1 + \gamma^i = \beta^i + \gamma^2 \) with \( \beta_1 = \frac{1}{2} (1 - \frac{d}{q_3}) - \frac{1}{2r} \) and \( \gamma^2 = \frac{1}{2} (2 - \frac{d}{q_2}) - \frac{1}{2r} \). Then we can apply Proposition 3.1 to prove the following proposition concerning the uniform time-weighted bounds of \((a_n, u_n, \nabla \Pi_n)\) of \((2.4)\).

**Proposition 4.1.** Under the assumptions of Theorem 4.1, system \((2.4)\) has a unique global smooth solution \((a_n, u_n, \nabla \Pi_n)\) which satisfies

\[
\mu^{\frac{1}{2}} (3 - \frac{d}{q_1 - \varepsilon}) \left\| t^{\alpha^1} \Delta u_n \right\|_{L^{2r} (\mathbb{R}^+; L^{q_1})} + \mu^{\frac{1}{2}} (1 - \frac{d}{q_2 - \varepsilon}) \left\| t^{\beta^1} \nabla u_n \right\|_{L^{2r} (\mathbb{R}^+; L^{q_2})} + \left\| t^{\beta^2} \nabla u_n \right\|_{L^{\infty} (\mathbb{R}^+; L^{q_2})}
\]

\[
\leq C \| u_0 \|_{B^{-1 + \frac{d}{q_1} + \varepsilon}} \exp \left( C_r \mu^{-2r} \| u_0 \|_{B^{-1 + \frac{d}{q_1} + \varepsilon}^{2r}} \right),
\]

and

\[
\mu^{\frac{1}{2}} (3 - \frac{d}{q_1 - \varepsilon}) \left\| t^{\alpha^2} \Delta u_n \right\|_{L^{r} (\mathbb{R}^+; L^{q_1})} + \left\| t^{\alpha^3} \nabla u_n \right\|_{L^{2r} (\mathbb{R}^+; L^{q_1})} + \left\| t^{\alpha^4} \nabla u_n \right\|_{L^{2r} (\mathbb{R}^+; L^{q_1})}
\]

\[
\leq C \eta \| u_0 \|_{B^{-1 + \frac{d}{q_1} + \varepsilon}} \exp \left( C_r \mu^{-2r} \| u_0 \|_{B^{-1 + \frac{d}{q_1} + \varepsilon}^{2r}} \right),
\]

for some constant \( C \) and \( \alpha^i, \beta^i, \gamma^i, i = 1, 2 \), given by \((4.3)\).

**Proof.** For \( N \) large enough, we have already proved in Proposition 3.1 that \((2.4)\) has a unique global smooth solution \((a_n, u_n, \nabla \Pi_n)\). It remains to prove \((4.5)\) and
(4.6). In order to do so, for \( \lambda_1, \lambda_2 > 0 \), we denote

\[
\begin{align*}
g_{1,n}(t) & \overset{\text{def}}{=} \| t^{\beta_1} \nabla u_n^d(t) \|_{L^q_{2r}}, \quad g_{2,n}(t) \overset{\text{def}}{=} \| t^{\beta_2} u_n^d(t) \|_{L^q_{2r}}, \\
u_{\lambda,n}(t, x) & \overset{\text{def}}{=} u_n(t, x) \exp \left\{ - \int_0^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) \, dt' \right\},
\end{align*}
\]

and similar notation for \( \Pi_{\lambda,n}(t, x) \). Then it follows from a derivation similar to (2.10) that

\[
\| \nabla \Pi_{\lambda,n}(t) \|_{L^{q_1}} \leq C \left\{ \mu \| a_0 \|_{L^\infty} \| \Delta u_{\lambda,n} \|_{L^{q_1}} + \| u_{\lambda,n}^h \|_{L^{q_1}} \| \nabla u_{\lambda,n}^h \|_{L^{q_2}} + \| u_n^d \|_{L^{q_2}} \| \nabla u_{\lambda,n}^h \|_{L^{q_2}} + \| u_n^h \|_{L^{q_2}} \| \nabla u_n^d \|_{L^{q_2}} \right\}.
\]

By virtue of (2.11) and (4.7), we write

\[
u_{\lambda,n} = u_{n,L} \exp \left\{ - \int_0^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) \, dt' \right\} + \int_0^t e^{\mu(t-s)\Delta} \exp \left\{ - \int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) \, dt' \right\} \\
\times (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n}) \, ds.
\]

For \( \gamma_1^\varepsilon, \beta_1^\varepsilon \) given by (4.3), we get, by applying Lemma 3.2 for \( r_1 = 2r \), that

\[
\mu \frac{d}{dt} \| t^{\gamma_1^\varepsilon} u_{\lambda,n} \|_{L^\infty_{r_1}(L^{q_3})} + \mu \frac{1}{2} + \frac{d}{q_3} \| t^{\beta_1^\varepsilon} \nabla u_{\lambda,n} \|_{L^2_{r_1}(L^{q_2})} \\
\leq \mu \frac{d}{dt} \| t^{\gamma_1^\varepsilon} u_{n,L} \|_{L^\infty_{r_1}(L^{q_3})} + \mu \frac{1}{2} + \frac{d}{q_3} \| t^{\beta_1^\varepsilon} \nabla u_{n,L} \|_{L^2_{r_1}(L^{q_2})} + \left\| \exp \left\{ - \int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) \, dt' \right\} \\
\times s^{\alpha_1^\varepsilon} (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n}) \right\|_{L^2_{r_1}(L^{q_1})}.
\]

Similarly for \( \gamma_2^\varepsilon, \beta_2^\varepsilon \) and \( \alpha_1^\varepsilon \), we deduce from Lemma 2.1 and Lemma 3.3 for \( r_1 = 2r \) that

\[
\mu \frac{d}{dt} \| t^{\gamma_2^\varepsilon} u_{\lambda,n} \|_{L^\infty_{r_1}(L^{q_3})} + \mu \frac{1}{2} + \frac{d}{q_3} \| t^{\beta_2^\varepsilon} \nabla u_{\lambda,n} \|_{L^2_{r_1}(L^{q_2})} + \mu \| t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n} \|_{L^2_{r_1}(L^{q_1})} \\
\leq \mu \frac{d}{dt} \| t^{\gamma_2^\varepsilon} u_{n,L} \|_{L^\infty_{r_1}(L^{q_3})} + \mu \frac{1}{2} + \frac{d}{q_3} \| t^{\beta_2^\varepsilon} \nabla u_{n,L} \|_{L^2_{r_1}(L^{q_2})} + \left\| \exp \left\{ - \int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) \, dt' \right\} \\
\times s^{\alpha_1^\varepsilon} (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n}) \right\|_{L^2_{r_1}(L^{q_1})}.
\]
Hence, by using (2.7), Proposition 2.1 and (4.8), we obtain
\[
\mu \frac{d}{dt} \left( \| t^\gamma u_{\lambda,n} \|_{L_\infty^\gamma (L^q_2)} + \| t^{\gamma_2} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right) \\
+ \mu \frac{1}{2} + \frac{d}{dt} \left( \| t^\beta \nabla u_{\lambda,n} \|_{L_2^\gamma (L^q_2)} + \| t^{\beta_2} \nabla u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right) \\
+ \mu \| t^{\alpha_1} \Delta u_{\lambda,n} \|_{L_2^\gamma (L^q_1)} \leq C \left\{ \mu^{\frac{1}{2}} \left( \frac{1 - \frac{d}{q_1} - \epsilon}{q_1} \right) \| u_0 \|_{B^1_{q_1,\frac{d}{q_1} + \epsilon}} \right\}.
\]
(4.10)

Then as \( C \| a_0 \|_{L^\infty} \leq \frac{1}{2} \), and
\[
\left( \int_0^t e^{-2\lambda_1 r} \int_{\gamma_1,n(t')} \| s^{\gamma_1} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right)^{\frac{1}{2}} \\
\leq \frac{1}{(2\lambda_1 r)^{\frac{1}{d}}} \| s^{\gamma_1} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)},
\]
\[
\left( \int_0^t e^{-2\lambda_2 r} \int_{\gamma_2,n(t')} \| s^{\gamma_2} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right)^{\frac{1}{2}} \\
\leq \frac{1}{(2\lambda_2 r)^{\frac{1}{d}}} \| s^{\gamma_2} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)},
\]
we infer from (4.10) that
\[
\mu \frac{d}{dt} \left( \| t^\gamma u_{\lambda,n} \|_{L_\infty^\gamma (L^q_2)} + \| t^{\gamma_2} u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right) \\
+ \mu \frac{1}{2} + \frac{d}{dt} \left( \| t^\beta \nabla u_{\lambda,n} \|_{L_2^\gamma (L^q_2)} + \| t^{\beta_2} \nabla u_{\lambda,n} \|_{L_2^\gamma (L^q_3)} \right) \\
+ \mu \| t^{\alpha_1} \Delta u_{\lambda,n} \|_{L_2^\gamma (L^q_1)} \leq C \left\{ \mu^{\frac{1}{2}} \left( \frac{1 - \frac{d}{q_1} - \epsilon}{q_1} \right) \| u_0 \|_{B^1_{q_1,\frac{d}{q_1} + \epsilon}} \right\}.
\]
(4.11)

Recall from (3.13) that
\[
\| t^{\tilde{\beta}_1} \nabla u_{\lambda,n} \|_{L_2^\gamma (L^q_2)} \leq C_0 \mu \frac{d}{dt},
\]
so that as long as $c_0$ is small enough in (1.6), taking $\lambda_1 = \frac{(4C)^2r}{2r\mu_{\frac{\eta}{2r}}}$, $\lambda_2 = \frac{(4C)^2r}{2r\mu_{\frac{(1+\frac{\eta}{2r})}r}}$
in (4.11) results in

\[
\mu_{\frac{\eta}{2r}} \left( \frac{1}{2} \| \tau_1^\varepsilon u_{n,\alpha}^h \|_{L^\infty_t(L^q_\gamma)} + \| \tau_2^\varepsilon u_{n,\alpha}^h \|_{L^2_t(L^q_\gamma)} \right) \\
+ \mu^{\frac{1}{2} + \frac{d}{2q_3}} \left( \| \tau_1^\varepsilon \nabla u_{n,\alpha}^h \|_{L^2_t(L^{q_2})} + \frac{3}{4} \| \tau_2^\varepsilon \nabla u_{n,\alpha}^h \|_{L^\infty_t(L^{q_2})} \right) \\
+ \mu \| \tau_1^\varepsilon \Delta u_{n,\alpha}^h \|_{L^2_t(L^{q_1})} \leq C \mu^{\frac{1}{2} \left( \frac{d}{q_1} - 1 - \varepsilon \right)} \| u_0 \|_{B_{\mu,1}^{-1 + \frac{d}{q} + \varepsilon}}.
\]

From (3.13), (4.7), and (4.12), we infer

\[
\mu_{\frac{\eta}{2r}} \left( \| \tau_1^\varepsilon u_n \|_{L^\infty_t(L^q_\gamma)} + \| \tau_2^\varepsilon u_n \|_{L^2_t(L^q_\gamma)} \right) + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \left( \| \tau_1^\varepsilon \nabla u_n \|_{L^2_t(L^{q_2})} \right) \\
+ \| \tau_2^\varepsilon \nabla u_n \|_{L^\infty_t(L^{q_2})} \right) + \mu \| \tau_1^\varepsilon \Delta u_n \|_{L^2_t(L^{q_1})} \\
\leq C \mu^{\frac{1}{2} \left( \frac{d}{q_1} - 1 - \varepsilon \right)} \| u_0 \|_{B_{\mu,1}^{-1 + \frac{d}{q} + \varepsilon}} \exp \left\{ C_r \int_0^t \mu^{\frac{-d}{q_3}} \| s^{\tilde{\gamma}_1} \nabla u_n^d(s) \|_{L^{q_2}}^{2r} ds \right\} \\
+ \mu^{\frac{1}{2} \left( \frac{d}{q_1} - 1 - \varepsilon \right)} \| u_0 \|_{B_{\mu,1}^{-1 + \frac{d}{q} + \varepsilon}} \exp \left\{ C_r \mu^{-2r} \| u_0^d \|_{B_{\mu,1}^{-1 + \frac{d}{q}}}^{2r} \right\},
\]

which implies (4.5). It remains to prove (4.6). In fact, we get, by applying Lemma 2.1 and (4.8), that

\[
\mu \| \tau_1^\varepsilon \Delta u_n \|_{L^r_t(\mathbb{R}^d; L^{q_1})} \leq \mu^{\frac{1}{2} \left( 1 - \frac{d}{q_1} - \varepsilon \right)} \| u_0 \|_{B_{\mu,1}^{-1 + \frac{d}{q} + \varepsilon}} \\
+ C \left\{ \mu \| a_0 \|_{L^\infty} \| \tau_1^\varepsilon \Delta u_n \|_{L^r_t(\mathbb{R}^d; L^{q_1})} \\
+ \| \tau_2^\varepsilon u_n \|_{L^2_t(\mathbb{R}^d; L^{q_1})} \| \tau_1^\varepsilon \nabla u_n^h \|_{L^2_t(\mathbb{R}^d; L^{q_2})} \\
+ \| \tau_2^\varepsilon u_n \|_{L^2_t(\mathbb{R}^d; L^{q_1})} \| \tau_1^\varepsilon \nabla u_n^d \|_{L^2_t(\mathbb{R}^d; L^{q_2})} \right\},
\]

which along with (3.13), (4.5) and the fact that $C \| a_0 \|_{L^\infty} \leq \frac{1}{2}$, gives rise to the first inequality of (4.6). The second inequality of (4.6) follows along the same lines. This completes the proof of Proposition 3.1.

To prove the uniqueness part of Theorem 4.1, we need the following lemma:

**Proposition 4.2.** Let $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2$, and $p_1, p_2, p_3$ be given by Theorem 1.1, if $t^{\alpha_1} f, t^{\alpha_1} \nabla g, t^{\alpha_1} R \in L^{2r}$ ($0, T; L^{p_1}(\mathbb{R}^d)$). Then the system

\[
\begin{align*}
\partial_t v - \Delta v + \nabla P &= f, \\
\text{div } v &= g, \\
\partial_t g &= \text{div } R, \\
v|_{t=0} &= 0,
\end{align*}
\]
has a unique solution \((v, \nabla P)\) such that

\[
\|t^\gamma v\|_{L_T^\infty(L^p)} + \|t^\gamma v\|_{L_T^\infty(L^p)} + \|t^{\beta_1} \nabla v\|_{L_T^\infty(L^p)} + \|t^{\beta_2} \nabla v\|_{L_T^\infty(L^p)} + \|(t^{\alpha_1} \partial_t v, t^{\alpha_1} \nabla^2 v, t^{\alpha_1} \nabla P)\|_{L_T^\infty(L^p)} \leq C\|(t^{\alpha_1} f, t^{\alpha_1} \nabla g, t^{\alpha_1} R)\|_{L_T^\infty(L^p)}.
\]

(4.14)

**Proof.** We first get, by taking space divergence on (4.13), that

\[
\Delta P = \text{div}(f + \nabla g - R),
\]

which implies

\[
\|\nabla P\|_{L_q(\mathbb{R}^d)} \leq C\|(f, \nabla g, R)\|_{L_q(\mathbb{R}^d)},
\]

for any \(q \in (1, \infty)\). On the other hand, we have

\[
v = \int_0^t e^{(t-s)\Delta} (f - \nabla P) \, ds,
\]

(4.15)

from which, (4.13), Lemma 3.1, and Lemma 3.2 and Lemma 3.3 for \(\varepsilon = 0\) and \(r_1 = 2r\), we deduce (4.14). \(\Box\)

**Lemma 4.1.** Let \((a, u)\) be a global weak solution of (1.2) that satisfies (1.8–1.9) and (4.1–4.2). Then, for any \(\varepsilon \in (0, \min\{1 - \frac{1}{r}, \frac{d}{p} - 1\})\), one has

\[
\|\nabla^2 u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^d)} + \|\nabla u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^\infty)} \leq C_{T, \varepsilon},
\]

\[
\|t^{\alpha_1} \nabla u\|_{L_T^{\infty}(L^\infty)} + \|t^{\alpha_1} \nabla^2 u\|_{L_T^{\infty}(L^d)} \leq C_{T, \varepsilon} \quad \text{for any} \ T < \infty.
\]

(4.16)

**Proof.** We first deduce from Remark 4.1 that \(\nabla^2 u \in L^{\frac{8}{\varepsilon^2}}((0, T); L^{\frac{d}{1+\varepsilon}}(\mathbb{R}^d)) \cap L^{\frac{8}{\varepsilon^2}}((0, T); L^{\frac{2d}{1+\varepsilon}}(\mathbb{R}^d))\) for any \(T < \infty\). Then, applying Hölder’s inequality yields

\[
\|\nabla^2 u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^d)} \leq \|\nabla^2 u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^{\frac{d}{1+\varepsilon}})} \cdot \|\nabla^2 u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^{\frac{2d}{1+\varepsilon}})} \leq C_{T, \varepsilon}.
\]

(4.17)

Whereas by virtue of Lemma A.1, one has

\[
\|\nabla u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^\infty)} \leq \sum_{j \leq 0} \|\nabla \hat{\Delta} j u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^\infty)} + \sum_{j > 0} \|\nabla \hat{\Delta} j u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^\infty)} \leq \sum_{j \leq 0} 2^j \|\nabla^2 \hat{\Delta} j u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^{\frac{d}{1+\varepsilon}})} + \sum_{j > 0} 2^{-j} \|\nabla^2 \hat{\Delta} j u\|_{L_T^{\frac{8}{\varepsilon^2}}(L^{\frac{2d}{1+\varepsilon}})} \leq C_{T, \varepsilon},
\]

which together with (4.17) proves the first line of (4.16). The second line of (4.17) follows along the same lines. \(\Box\)
Thanks to Lemma 4.1, we can taking $T$ small enough so that

$$\int_0^T \| \nabla u(t) \|_{L^\infty} \, dt \leq \frac{1}{2}. \quad (4.18)$$

As in [13,14], we shall prove the uniqueness part of Theorem 4.1 by using the Lagrangian formulation of (1.2). Toward this end, we first recall some basic facts concerning Lagrangian coordinates from [13,14]. By virtue of (4.18), for any $y \in \mathbb{R}^d$, the following ordinary differential equation has a unique solution on $[0, T]$: \[ \frac{dX(t, y)}{dt} = u(t, X(t, y)) \overset{\text{def}}{=} v(t, y), \quad X(t, y)|_{t=0} = y. \quad (4.19) \]

This leads to the following relation between the Eulerian coordinates $x$ and the Lagrangian coordinates $y$:

$$x = X(t, y) = y + \int_0^t v(\tau, y) \, d\tau. \quad (4.20)$$

Let $Y(t, \cdot)$ be the inverse mapping of $X(t, \cdot)$. Then $D_x Y = (D_{\cdot} X - Id)$ is small enough, we have

$$D_x Y = (Id + (D_{\cdot} Y) - Id)^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \int_0^t D_{\cdot} v(\tau, y) \, d\tau \right)^k. \quad (4.21)$$

We denote $A(t, y) \overset{\text{def}}{=} \nabla X(t, y)^{-1} = \nabla_x Y(t, x)$, then we have

$$\nabla_x u(t, x) = T A(t, x) \nabla_y v(t, y), \quad \text{div} u(t, x) = \text{div}(A(t, y) v(t, y)). \quad (4.22)$$

By the chain rule, we also have

$$\text{div}_y (A \cdot) = T A : \nabla_y. \quad (4.23)$$

Here and in what follows, we always denote $^T A$ the transpose matrix of $A$.

As in [13,14], we denote

$$\nabla_u \overset{\text{def}}{=} A \cdot \nabla_y, \quad \text{div}_u \overset{\text{def}}{=} \text{div}(A \cdot) \quad \text{and} \quad \Delta_u \overset{\text{def}}{=} \text{div}_u \nabla_u,$nabla_u \overset{\text{def}}{=} A \cdot \nabla_y, \quad \text{div}_u \overset{\text{def}}{=} \text{div}(A \cdot) \quad \text{and} \quad \Delta_u \overset{\text{def}}{=} \text{div}_u \nabla_u,$$

$$b(t, y) \overset{\text{def}}{=} a(t, X(t, y)), \quad v(t, y) \overset{\text{def}}{=} u(t, X(t, y)) \quad \text{and} \quad (4.24)$$

Notice that for any $t > 0$, the solution of (1.2) obtained in Theorem 4.1 satisfies the smoothness assumption of Proposition 2 in [14], so that $(\hat{b}, v, \nabla P)$ defined by (4.23) fulfils

$$\begin{cases}
    \hat{b}_t = 0, \\
    \partial_t v - (1 + b)(\mu \Delta u v - \nabla u P) = 0, \\
    \text{div}_u v = 0, \\
    (\hat{b}, v)|_{t=0} = (a_0, u_0),
\end{cases} \quad (4.24)$$
which is the Lagrangian formulation of (1.2). For the sake of simplicity, we shall take $\mu = 1$ in what follows.

We now present the proof of Theorem 4.1.

**Proof.** (Proof of Theorem 4.1) We first deduce from the proof of the existence part of Theorem 1.1, (4.5) and (4.6) that the global weak solution $(a, u, \nabla \Pi)$ constructed in Theorem 1.1 satisfies (4.1) and (4.2). It remains to prove the uniqueness part of Theorem 4.1.

Let $(a_i, u_i, \Pi_i), i = 1, 2,$ be two solutions of (1.2) which satisfy (1.8–1.9) and (4.1–4.2). Let $X_i, (v_i , P), A_i, i = 1, 2$ be given by (4.20) and (4.23). We denote

$$\delta v \overset{\text{def}}{=} v_2 - v_1, \quad \delta P \overset{\text{def}}{=} P_2 - P_1,$$

then $(\delta v, \delta P)$ solves

$$\begin{cases}
\partial_t \delta v - \triangle \delta v + \nabla \delta P = a_0(\triangle \delta v - \nabla \delta P) + \delta f_1 + \delta f_2,
\text{div } \delta v = \delta g,
\partial_t \delta g = \text{div } \delta R,
\delta v|_{t=0} = 0,
\end{cases}$$

with

$$\begin{align*}
\delta f_1 & \overset{\text{def}}{=} (1 + a_0)[(Id - T A_2)\nabla \delta P - \delta A \nabla P_1], \\
\delta f_2 & \overset{\text{def}}{=} \mu(1 + a_0) \text{div}[(A_2^T A_2 - Id)\nabla \delta v + (A_2^T A_2 - A_1^T A_1)\nabla v_1], \\
\delta g & \overset{\text{def}}{=} (Id - A_2) : D \delta v - \delta A : D v_1, \\
\delta R & \overset{\text{def}}{=} \partial_t[(Id - A_2)\delta v] - \partial_t[\delta A v_1].
\end{align*}$$

In what follows, we will use repeatedly the fact (see [13], for instance) that

$$\delta A(t) = \left( \int_0^t \frac{D\delta v d\tau}{\tau} \right) \left( \sum \sum_{k \geq 10} C_1^j C_2^{k-1-j} \right), \quad C_i(t) \overset{\text{def}}{=} \int_0^t D v_i d\tau. \quad (4.26)$$

Let the indices $\alpha_1, \beta_1, \beta_2, \gamma_1$ and $\gamma_2$ be given by (1.10). As in Remark 4.1, we take $p_1 = \frac{d}{r + \varepsilon}$ and $p_2, p_3$ satisfying $\frac{dr}{r - 1} < p_3 < \infty$ and $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$. We denote

$$\begin{align*}
G(t) & \overset{\text{def}}{=} \|I^{\beta_1}_t \delta v\|_{L^\infty_t(L^{p_3})} + \|I^{\beta_2}_t \delta v\|_{L^{p_2}_t(L^{p_3})} + \|I^{\beta_1}_t \nabla \delta v\|_{L^{p_2}_t(L^{p_3})} + \|I^{\gamma_1}_t \partial_t \delta v, I^{\alpha_1} \nabla^2 \delta v, I^{\alpha_1} \nabla \delta P\|^{\frac{d}{L^p}}_{L^p_t(L^{\frac{d}{1+\varepsilon}})}.
\end{align*}$$

Then we deduce from Proposition 4.2 and (4.25) that

$$\begin{align*}
G(t) & \leq C \left( \|I^{\alpha_1} \delta f_1\|_{L^{\frac{d}{L^p}}_t(L^{\frac{d}{1+\varepsilon}})} + \|I^{\alpha_1} \delta f_2\|_{L^{\frac{d}{L^p}}_t(L^{\frac{d}{1+\varepsilon}})} \\
& \quad + \|I^{\alpha_1} \nabla \delta g\|_{L^{\frac{d}{L^p}}_t(L^{\frac{d}{1+\varepsilon}})} + \|I^{\alpha_1} \delta R\|_{L^{\frac{d}{L^p}}_t(L^{\frac{d}{1+\varepsilon}})} \right), \quad (4.28)
\end{align*}$$

as long as $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$. 

Well-posedness of Incompressible Inhomogeneous NS Equations 663
Let us now estimate term by term on the right-hand side of (4.28). We first get, by using (1.8) and (4.18), that
\[
\|t^{\alpha_1} \delta f_1\|_{L_t^2(L_{x}^{d \perp})} \lesssim \|Id - T A_2\|_{L_t^{\infty}(L^{\infty})} \|t^{\alpha_1} \nabla \delta P\|_{L_t^{2}(L_{x}^{d \perp})} + \|\delta A\|_{L_t^{\infty}(L_{x}^{d \perp})} \|t^{\alpha_1} \nabla P_1\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \|\nabla \delta v\|_{L_t^{1}(L_{x}^{d})} \left( \|t^{\alpha_1} \nabla P_1\|_{L_t^{2}(L_{x}^{p_1})} \right)^{\frac{\theta}{1 - \theta}} \left( \|t^{\alpha_1} \nabla P_1\|_{L_t^{2}(L_{x}^{q_1})} \right)^{1 - \theta} + \|\nabla v_2\|_{L_t^{1}(L^{\infty})} G(t),
\]
for \(\theta\) determined by \(\frac{1}{d} = \frac{\theta}{p_1} + \frac{1 - \theta}{q_1}\). However, by virtue of Sobolev embedding theorem, \(W^{1, \frac{d}{1 + \sigma}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{\sigma}}(\mathbb{R}^d)\), one has
\[
\|\nabla \delta v\|_{L_t^{\frac{d}{1 + \sigma}}(L_{x}^{d \perp})} \lesssim \|\nabla^2 \delta v\|_{L_t^{\frac{d}{1 + \sigma}}(L_{x}^{d \perp})} \lesssim t^{\frac{1}{2}(\frac{d}{p_1} - 1)} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})},
\]
we obtain
\[
\|t^{\alpha_1} \delta f_1\|_{L_t^{2}(L_{x}^{d \perp})} \lesssim \eta(t) G(t),
\]
for some positive continuous function \(\eta(t)\) which tends to 0 as \(t \to 0\). Along the same lines, we deduce
\[
\|t^{\alpha_1} \nabla ((Id - A_2) \cdot D\delta v)\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \|DA_2 \odot t^{\alpha_1} D\delta v\|_{L_t^{2}(L_{x}^{d \perp})} + \|(Id - A_2) \otimes t^{\alpha_1} D^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \|\nabla^2 v_2\|_{L_t^{1}(L^{d})} \|t^{\alpha_1} \nabla \delta v\|_{L_t^{2}(L_{x}^{d \perp})} + \|\nabla v_2\|_{L_t^{1}(L^{\infty})} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})} \left( \|\nabla^2 v_2\|_{L_t^{1}(L^{d})} + \|\nabla v_2\|_{L_t^{1}(L^{\infty})} \right),
\]
and
\[
\|t^{\alpha_1} \nabla (\delta A : Dv_1)\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \left\| t^{\alpha_1} \left| \nabla v_1 \right| \int_0^T |\nabla^2 \delta v| \, dt' \right\|_{L_t^{2}(L_{x}^{d \perp})} + \left\| t^{\alpha_1} \left| \nabla^2 v_1 \right| \int_0^T |\nabla \delta v| \, dt' \right\|_{L_t^{2}(L_{x}^{d \perp})}
\]
\[
\lesssim \|t^{\alpha_1} \nabla v_1\|_{L_t^{2}(L^{\infty})} \|\nabla^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})} + \|t^{\alpha_1} \nabla^2 v_1\|_{L_t^{2}(L^{d})} \|\nabla \delta v\|_{L_t^{1}(L^{d})}
\]
\[
\lesssim t^{\frac{1}{2}(\frac{d}{p_1} - 1)} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2}(L_{x}^{d \perp})} \left( \|t^{\alpha_1} \nabla v_1\|_{L_t^{2}(L^{\infty})} + \|t^{\alpha_1} \nabla^2 v_1\|_{L_t^{2}(L^{d})} \right).
\]
So, for all \(t \in [0, T]\), we get, by using (4.16), that
\[
\|t^{\alpha_1} \nabla \delta g\|_{L_t^{2}(L_{x}^{d \perp})} \lesssim \eta(t) G(t).
\]
The estimate of the term \(\delta f_2\) can be handled along the same lines.
To deal with $\delta R$, we denote $Dv_{1,2}$ to be the components of $Dv_1$ and $Dv_2$. Then we get, by using (4.26) once again, that
\[
\|t^{\alpha_1} \partial_t ((Id - A_2)\delta v)\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \\
\lesssim \|t^{\beta_1} Dv_2 t^{\gamma_1} \delta v\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} + \|(Id - A_2) t^{\alpha_1} \partial_t \delta v\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \\
\lesssim \|t^{\beta_1} \nabla v_2\|_{L^\infty_t(L^{p_2})} \|t^{\gamma_1} \delta v\|_{L^\infty_t(L^{p_3})} + \|\nabla v_2\|_{L^1_t(L^\infty)} \|t^{\alpha_1} \partial_t \delta v\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})},
\]
and
\[
\|t^{\alpha_1} \partial_t (\delta A v_1)\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \\
\lesssim \|t^{\gamma_2} v_1 t^{\beta_2} D\delta v\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} + \|\delta A t^{\alpha_1} \partial_t v_1\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \\
+ \left\| \int_0^T |D\delta v| |dt' \delta_1 |Dv_{1,2}| t^{\gamma_2} |v_1| \right\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \\
\lesssim \|t^{\beta_2} \nabla \delta v\|_{L^\infty_t(L^{p_2})} \|t^{\gamma_2} v_1\|_{L^\infty_t(L^{p_3})} + \|\nabla v_1\|_{L^1_t(L^\infty)} \|t^{\alpha_1} \partial_t v_1\|_{L^\infty_t(L^d)} \\
+ \|\nabla v_1\|_{L^1_t(L^\infty)} \left( \|t^{\beta_2} \nabla v_{1,2}\|_{L^\infty_t(L^{p_2})} \|t^{\gamma_2} v_1\|_{L^\infty_t(L^{p_3})} \right)^\theta \\
\times \left( \|t^{\beta_2} \nabla v_{1,2}\|_{L^\infty_t(L^{p_2})} \|t^{\gamma_2} v_1\|_{L^\infty_t(L^{p_3})} \right)^{1-\theta},
\]
for $\theta$ given by $\frac{1}{d} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}$. Hence it follows from (1.8) and (4.1) that
\[
\|t^{\alpha_1} \delta R\|_{L^\infty_t(L^d_{\frac{d}{1+\varepsilon}})} \lesssim \eta(t)G(t). \tag{4.32}
\]
Substituting (4.30–4.32) into (4.28) results in
\[
G(t) \lesssim \eta(t)G(t), \tag{4.33}
\]
which implies the uniqueness of the solution to (1.2) on a sufficiently small time interval. The uniqueness part of Theorem 4.1 can be completed by a bootstrap method. This concludes the proof of Theorem 4.1. \(\square\)

5. Proof of Theorem 1.2

The goal of this section is to present the proof of Theorem 1.2. Indeed, given $a_0 \in L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{d+\varepsilon}(\mathbb{R}^d)$, $u_0 \in B_{p,r}^{-1+d/p-\varepsilon}(\mathbb{R}^d) \cap B_{p,r}^{-1+d/p-\varepsilon}(\mathbb{R}^d)$, with $\|a_0\|_{L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{d+\varepsilon}}$ being sufficiently small and $p, q, \varepsilon$ satisfying the conditions listed in Theorem 1.2, we deduce from [19] that there exists a positive time $T$ such that (1.2) has a unique solution $(a, u, \nabla \Pi)$ with
\[
\begin{align*}
 a &\in C([0, T]; L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{d+\varepsilon}(\mathbb{R}^d)), \quad u \in \tilde{L}^\infty_T(\tilde{B}_{p,r}^{-1+d/p-\varepsilon} \cap \tilde{L}^{d+\varepsilon}_T(\tilde{B}_{p,r}^{-1+d/p-\varepsilon})) \\
\n &\cap \tilde{L}^\infty_T(\tilde{B}_{p,r}^{-1+d/p-\varepsilon}) \quad \text{and} \quad \nabla \Pi \in \tilde{L}^1_T(\tilde{B}_{p,r}^{-1+d/p-\varepsilon}) \quad \text{for} \quad s = 0 \quad \text{and} \quad \varepsilon.
\end{align*}
\tag{5.1}
\]
We denote $T^*$ to be the lifespan of this solution. Then the proof of Theorem 1.2 is reduced to proving that $T^* = \infty$. 

Well-posedness of Incompressible Inhomogeneous NS Equations 665
5.1. The Estimate of the Density

Note from (5.1) that \( u \in \tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{p}}}) \cap \tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{q}}}) \), which is not Lipschitz in the space variables, so the regularity of the solution \( a \) to (1.2) may be coarsen for positive time. In order to apply the losing derivative estimate in [6], we first need to prove that \( u \in L^1_T(C_\mu) \) for \( \mu(r) = r(-\log r)^{\alpha} \) and some \( \alpha \in (0, 1) \). Indeed, letting \( u \in \tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{p}}}) \cap \tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{q}}}) \) for some \( \varepsilon > 0 \), we denote \( \theta(t, x, y) \defeq |u(t, x) - u(t, y)| \). Then, similar to the proof of Proposition 2.1 in [8], for any positive integer \( N \) and \( \varepsilon_1 > 0 \), one has

\[
\theta(t, x, y) \leq |x - y| (2 + N)^{1 - \frac{1}{\varepsilon_1}} \sum_{-1 \leq j \leq N} \frac{\|\nabla \Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1 - \frac{1}{\varepsilon_1}}}
\]

\[+ 2 \sum_{j > N} 2^{-(2 + j)} (2 + j)^{1 - \frac{1}{\varepsilon_1}} \cdot \frac{\|\nabla \Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1 - \frac{1}{\varepsilon_1}}}.
\]

Note that \( 2^{-x}x^\alpha \) with \( 0 < \alpha < 1 \) is a decreasing function, so we get, by taking \( N = [1 - \log |x - y|] - 2 \) in the above inequality, that

\[
\theta(t, x, y) \leq C |x - y| (1 - \log |x - y|)^{1 - \frac{1}{\varepsilon_1}} \sum_{j \geq -1} \frac{\|\nabla \Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1 - \frac{1}{\varepsilon_1}}}.
\]

from which, we infer

\[
\int_0^T \sup_{0 < |x - y| < 1} \frac{\theta(t, x, y)}{|x - y| (1 - \log |x - y|)^{1 - \frac{1}{\varepsilon_1}}} \, dt
\]

\[\leq C \left( \sum_{j \geq -1} \|\nabla \Delta_j u\|_{L^r_T(L^\infty)}^r \right)^{\frac{1}{r}} \left( \sum_{j \geq -1} \frac{1}{(2 + j)^{(1 - \frac{1}{\varepsilon_1})r'}} \right)^{\frac{1}{r'}}
\]

(5.2)

\[\leq C \varepsilon_1 \|\nabla u\|_{\tilde{L}^1_T(B^0_{\infty, r})},
\]

where \( r' \) denotes the conjugate number of \( r \).

On the other hand, it follows from Lemma A.1 that

\[
\|\nabla \Delta_{-1} u\|_{L^1_T(L^\infty)} \leq C \varepsilon \|u\|_{\tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{p}}})},
\]

so that

\[
\|\nabla u\|_{\tilde{L}^1_T(B^0_{\infty, r})} \leq C \varepsilon \left( \|u\|_{\tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{p}}})} + \|u\|_{\tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{q}}})} \right),
\]

which together with (5.2) and Theorem 3.33 of [6] (see also [12]) implies that

\[
\|a\|_{\tilde{L}^1_T(B^0_{q, \infty})} \leq C \varepsilon \|a_0\|_{B^q_0, \infty}
\]

\[\times \exp \left\{ C \varepsilon \left( \|u\|_{\tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{p}}})} + \|u\|_{\tilde{L}^1_T(B_{p,r}^{\frac{1}{1+\frac{d}{q}}})} \right) \right\},
\]

(5.3)

for any \( t < T, \frac{d}{q} < 1 + \frac{d}{p} \) and \( 0 < \varepsilon < 1 + \frac{d}{p} - \frac{d}{q} \).
5.2. The Estimate of the Pressure

We first get, by taking space divergence of the momentum equation of (1.2), that

\[-\Delta \Pi = \text{div}(a \nabla \Pi) - \mu \text{div}(a \Delta u) + \text{div}_h (u^h \otimes u^h)\]

+ \text{div}_h (u^d \partial_d u^h) + \partial_d (u^h \cdot \nabla_h u^d) + \partial_d^2 (u^d)^2.\]

Thanks to \( \text{div} u = 0 \), one has

\[\partial_d (u^h \cdot \nabla_h u^d) = \partial_d u^h \cdot \nabla_h u^d + u^h \cdot \nabla_d u^d\]

= \text{div}_h (u^d \partial_d u^h) - u^d \partial_d \text{div}_h u^h + \text{div}_h (u^h \partial_d u^d) - \text{div}_h u^h \partial_d u^d\]

= \text{div}_h (u^d \partial_d u^h) - \partial_d (u^d \text{div}_h u^h) - \text{div}_h (u^h \text{div}_h u^h),\]

which gives rise to

\[-\Delta \Pi = \text{div}(a \nabla \Pi) - \mu \text{div}(a \Delta u) + \text{div}_h (u^h \otimes u^h)\]

+ 2 \text{div}_h (u^d \partial_d u^h) - 3 \partial_d (u^d \text{div}_h u^h) - \text{div}_h (u^h \text{div}_h u^h).\]  \(\text{(5.4)}\)

The following proposition concerning the estimate of the pressure will be the key ingredient used in the estimate of the horizontal component of the velocity.

**Proposition 5.1.** Let \( 1 < q \leq p < 2d, r > 1 \) and \( 0 < \varepsilon < \frac{2d}{p} - 1 \). Let \( a \in L^\infty_T (L^\infty) \cap \tilde{L}^\infty_T (B^{\frac{d}{p}, \infty}_q), u \in \tilde{L}^\infty_T (\dot{B}^{1+\frac{d}{p}-s}_{p, r}) \cap \tilde{L}^1_T (\dot{B}^{1+\frac{d}{p}-s}_{p, r}), \) for \( s = 0, \varepsilon \), be the unique local solution of (1.2) given by (5.1). We denote

\[f(t) \overset{\text{def}}{=} \| u^d(t) \|_{\dot{B}^{1+\frac{d}{p}-s}_{p, r}}^{2r} \quad \text{and} \quad \Pi_\lambda \overset{\text{def}}{=} \Pi \exp \left\{ -\lambda \int_0^t f(t') \, dt' \right\}\]

for \( \lambda > 0 \), and similar notation for \( u_\lambda \). Then (5.4) has a unique solution \( \nabla \Pi \in \tilde{L}^1_T (\dot{B}^{1+\frac{d}{p}-s}_{p, r}) \cap \tilde{L}^1_T (\dot{B}^{1+\frac{d}{p}-s}_{p, r}), \) which decays to zero when \( |x| \to \infty \). Thus, for all \( t \in [0, T], \) there holds

\[\| \nabla \Pi_\lambda \|_{\tilde{L}^1_t (\dot{B}^{1+\frac{d}{p}-s}_{p, r})} \leq \frac{C}{1 - \lambda \| a \|_{L^\infty_T (\tilde{L}^\infty (B^{\frac{d}{p}, \infty}_q) \cap \tilde{L}^\infty_T (B^{\frac{d}{p}, \infty}_q))}} \left\{ \| u^h_\lambda \|_{\tilde{L}^1 (\dot{B}^{1+\frac{d}{p}-s}_{p, r})}^{1-\frac{1}{2p}} \| u^h_\lambda \|_{\tilde{L}^1 (\dot{B}^{1+\frac{d}{p}-s}_{p, r})}^{\frac{1}{2p}} + \| u^h_\lambda \|_{\tilde{L}^1 (\dot{B}^{1+\frac{d}{p}-s}_{p, r})} \right\} \]

for \( s = 0, \varepsilon, \)

provided that \( C \| a \|_{L^\infty_T (\tilde{L}^\infty (B^{\frac{d}{p}, \infty}_q) \cap \tilde{L}^\infty_T (B^{\frac{d}{p}, \infty}_q))} \leq \frac{1}{2}, \) and where the norms of \( \tilde{L}^1_t (\dot{B}^{1+\frac{d}{p}-s}_{p, r}) \) and \( \tilde{L}^1_t (\dot{B}^{1+\frac{d}{p}-s}_{p, r}) \) are given by Definitions A.2 and A.3, respectively. \(\square\)
The proof of this proposition will be based mainly on the following lemmas:

**Lemma 5.1.** For \( s < \frac{d}{p} \), one has

\[
\|g h\|_{L^1_t(B^d_{p,r})} \lesssim \|g\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \|h\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})} + \|g\|_{L^1_t(\dot{B}^{1+\frac{d}{p}}_{p,r})} \|h\|_{L^\infty_t(\dot{B}^{-1+\frac{d}{p}}_{p,r})}.
\]

**Proof.** Applying Bony’s decomposition \([7]\) gives

\[
\hat{\Delta}_j (g h) = \sum_{j' \geq j-N_0} \hat{\Delta}_j \left( \hat{S}_{j'} g \hat{\Delta}_j h + \hat{\Delta}_j g \hat{S}_{j'+1} h \right),
\]

from which and Lemma A.1, we infer

\[
\|\hat{\Delta}_j (g h)\|_{L^1_t(L^p)} \lesssim \sum_{j' \geq j-N_0} \left( \|\hat{S}_{j'} g\|_{L^\infty_t(L^\infty)} \|\hat{\Delta}_j h\|_{L^1_t(L^p)} + \|\hat{\Delta}_j g\|_{L^1_t(L^p)} \|\hat{S}_{j'+1} h\|_{L^\infty_t(L^\infty)} \right)
\]

\[
\lesssim \sum_{j' \geq j-N_0} c_{j',r} 2^{-j'\left(\frac{d}{p} - s\right)} \left( \|g\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \|h\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})} + \|g\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})} \|h\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \right).
\]

where, and in what follows, we always denote \((c_{j',r})_{j \in \mathbb{Z}}\) as a generic element of \(\ell^r(\mathbb{Z})\) so that \(\sum_{j \in \mathbb{Z}} c_{j,r} = 1\). Then, by virtue of Definition A.2, we complete the proof of the lemma. \(\Box\)

**Lemma 5.2.** Let \( 1 \leq p < 2d \), \( s \in (-\frac{1}{q}, \frac{2d}{p} - 1) \) and \( f \) be given by (5.5). Then under the assumptions of Proposition 5.1, one has

\[
\|u^d \nabla u^h\|_{L^1_t(B^{-1+\frac{d}{p}}_{p,r})} \lesssim \|u^h\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \|u^h\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})},
\]

and

\[
\|u^d \nabla u^h\|_{L^1_t(B^{-1+\frac{d}{p}}_{p,r})} \lesssim \|u^h\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \|u^d\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})} + \|u^d\|_{L^\infty_t(B^{-1+\frac{d}{p}}_{p,r})} \|u^h\|_{L^1_t(B^{1+\frac{d}{p}}_{p,r})}.
\]
**Proof.** We first get, by applying Bony’s decomposition \((A.5)\), that

\[ u^d \nabla u^h = \dot{T}_{u^d} \nabla u^h + \dot{T}_{\nabla u^h} u^d + \dot{R}(u^d, \nabla u^h). \tag{5.9} \]

Applying Lemma A.1 gives

\[
\| \dot{\Delta}_j \left( T_{u^d} \nabla u^h \right) (t') \|_{L^p} \lesssim \sum_{|j'| - j \leq 5} 2^{j'} \| \dot{S}_{j'-1} u^d(t') \|_{L^\infty} \| \dot{\Delta}_j u^h(t') \|_{L^p} \lesssim \sum_{|j'| - j \leq 5} 2^{j'(2 - \frac{1}{r})} \| u^d(t') \|_{B_{p,r}^{-1 + \frac{d}{p}, \frac{1}{r}} \dot{L}_{t, j}^{1}} \| \dot{\Delta}_j u^h(t') \|_{L^p}.
\]

Integrating the above inequality over \([0, t]\) and using Definition A.3, one has

\[
\| \dot{\Delta}_j \left( \dot{T}_{u^d} \nabla u^h \right) \|_{L^1_t(L^p)} \lesssim \sum_{|j'| - j \leq 5} 2^{j'(2 - \frac{1}{r})} \left\{ \int_0^t \| u^d(t') \|_{B_{p,r}^{-1 + \frac{d}{p}, \frac{1}{r}} \dot{L}_{t, j}^{1}} \| \dot{\Delta}_j u^h(t') \|_{L^p} \, dt' \right\}^{\frac{1}{r'}} \times \| \dot{\Delta}_j u^h(t') \|_{L^1_t(L^p)} \lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \| u^h \|_{L^1_t(B_{p,r}^{1 + \frac{d}{p} - s})} \| u^h \|_{L^{1}_{t, j}(B_{p,r}^{-1 + \frac{d}{p} - s})}.
\]

It follows from the same line that

\[
\| \dot{\Delta}_j \left( \dot{T}_{\nabla u^h} u^d \right) (t') \|_{L^p} \lesssim \sum_{|j'| - j \leq 5} \| \dot{S}_{j'-1} \nabla u^h(t') \|_{L^\infty} \| \dot{\Delta}_j u^d(t') \|_{L^p} \lesssim 2^{-j(-1 + \frac{d}{p} + \frac{1}{r})} \sum_{\ell \leq j + 4} 2^{\ell(1 + \frac{d}{p})} \| u^d(t') \|_{B_{p,r}^{-1 + \frac{d}{p}, \frac{1}{r}} \dot{L}_{t, \ell}^{1}} \| \dot{\Delta}_\ell u^h(t') \|_{L^p},
\]

from which, with \(s > -\frac{1}{r}\), we infer

\[
\| \dot{\Delta}_j \left( \dot{T}_{\nabla u^h} u^d \right) \|_{L^1_t(L^p)} \lesssim 2^{-j(-1 + \frac{d}{p} + \frac{1}{r})} \sum_{\ell \leq j + 4} 2^{\ell(1 + \frac{d}{p})} \left\{ \int_0^t \| u^d(t') \|_{B_{p,r}^{-1 + \frac{d}{p}, \frac{1}{r}} \dot{L}_{t, \ell}^{1}} \| \dot{\Delta}_\ell u^h(t') \|_{L^p} \, dt' \right\}^{\frac{1}{r'}} \times \| \dot{\Delta}_\ell u^h(t') \|_{L^1_t(L^p)} \lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \| u^h \|_{L^1_t(B_{p,r}^{1 + \frac{d}{p} - s})} \| u^h \|_{L^{1}_{t, j}(B_{p,r}^{-1 + \frac{d}{p} - s})}.
\]
Finally, we deal with the remaining term in (5.9). Firstly for \(2 \leq p < 2d\), as \(s < \frac{2d}{p} - 1\), we get, by applying Lemma A.1, that

\[
\|\hat{\Delta}_j \left( \hat{R}(u^d, \nabla u^h) \right) \|_{L^1_t(L^p)} \\
\lesssim 2^{j \frac{d}{p}} \sum_{j' \geq j-N_0} 2^{j'} \int_0^t \|\hat{\Delta}_{j'} u^d(t')\|_{L^p} \|\hat{\Delta}_{j'} u^h(t')\|_{L^p} \, dt' \\
\lesssim 2^{j \frac{d}{p}} \sum_{j' \geq j-N_0} 2^{-j'(-2 + \frac{d}{p} + \frac{1}{r})} \int_0^t \|u^d(t')\|_{L^p(B_{p,r}^{-1 + \frac{d}{p} + \frac{1}{r}})} \|\hat{\Delta}_{j'} u^h(t')\|_{L^p} \, dt' \\
\lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \|u^h\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})} \|u^h\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})}.
\]

For \(1 \leq p < 2\), we get, by applying Lemma A.1 once again, that

\[
\|\hat{\Delta}_j \left( \hat{R}(u^d, \nabla u^h) \right) \|_{L^1_t(L^p)} \\
\lesssim 2^{j \frac{d}{p}(1 - \frac{1}{p})} \sum_{j' \geq j-N_0} 2^{j''} \int_0^t \|\hat{\Delta}_{j'} u^d(t')\|_{L^\infty(B_{p,r}^{-1 + \frac{d}{p} + \frac{1}{r}})} \|\hat{\Delta}_{j'} u^h(t')\|_{L^p} \, dt' \\
\lesssim 2^{j \frac{d}{p}(1 - \frac{1}{p})} \sum_{j' \geq j-N_0} 2^{j''(2 + \frac{d}{p} - d - \frac{1}{r})} \\
\times \int_0^t \|u^d(t')\|_{L^\infty(B_{p,r}^{-1 + \frac{d}{p} + \frac{1}{r}})} \|\hat{\Delta}_{j'} u^h(t')\|_{L^p} \, dt' \\
\lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \|u^h\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})} \|u^h\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})}.
\]

Whence, thanks to (5.9), we finish the proof of (5.7).

On the other hand, it is easy to observe that

\[
\|\hat{\Delta}_j (\hat{T}_{u^d} \nabla u^h + \hat{T}_{\nabla u^h} u^d)\|_{L^1_t(L^p)} \\
\lesssim \sum_{|j' - j| \leq 5} \left( \|\hat{S}_{j'-1} u^d\|_{L^\infty_t(L^\infty)} \|\hat{\Delta}_{j'} \nabla u^h\|_{L^1_t(L^p)} \right) \\
+ \|\hat{S}_{j'-1} \nabla u^h\|_{L^\infty_t(L^\infty)} \|\hat{\Delta}_{j'} u^d\|_{L^1_t(L^p)} \\
\lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \left( \|u^h\|_{L^\infty_t(B_{p,r}^{-1 + \frac{d}{p} - s})} \|u^d\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})} \right) \\
+ \|u^d\|_{L^\infty_t(B_{p,r}^{-1 + \frac{d}{p} - s})} \|u^h\|_{L^1_t(B_{p,r}^{-1 + \frac{d}{p} - s})}.
\]
As $s < \frac{2d}{p} - 1$, for $2 \leq p < 2d$, we get, by applying Lemma A.1, that
\[
\| \dot{\Delta}_j \left( \dot{R}(u^d, \nabla u^b) \right) \|_{L^1_t(L^p)} \lesssim 2^{j \frac{d}{p}} \sum_{j' \geq j - N_0} 2^{j'} \| \dot{\Delta}_{j'} u^d \|_{L^1_t(L^p)} \| \dot{\Delta}_{j'} u^b \|_{L^\infty_t(L^p)} \\
\lesssim c_{j,r} 2^{-j(-1 + \frac{d}{p} - s)} \| u^b \|_{L^\infty_t(\dot{B}^{1+\frac{d}{p} - s})} \| u^d \|_{L^1_t(B^{1+\frac{d}{p} - s})}.
\]

Along the same lines as the proof of (5.10), we can prove the same estimate holds for $1 \leq p < 2$. This proves (5.8) and Lemma 5.2. \hfill \Box

**Lemma 5.3.** Let $1 < q \leq p < 2d$, $-1 < \frac{d}{q} + \frac{d}{p} - 1$ and $g \in \dot{L}^1_T(\dot{B}^{-1+\frac{d}{p} - s}_{p,q})$. Then under the assumptions of Proposition 5.1, one has
\[
\| a g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})} \lesssim \| a \|_{L^\infty_t(\dot{B}^{-1+\frac{d}{q} - s}_{q,q})} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})}.
\]

**Proof.** Again thanks to Bony’s decomposition (A.5), we have
\[
a g = \dot{T}_a g + \dot{T}_g a + \dot{R}(a, g).
\]

Applying Lemma A.1 gives
\[
\| \dot{\Delta}_j (\dot{T}_a g) \|_{L^1_t(L^p)} \lesssim \sum_{|j' - j| \leq 5} \| \dot{S}_{j'-1} a \|_{L^\infty_t(L^\infty)} \| \dot{\Delta}_{j'} g \|_{L^1_t(L^p)} \\
\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p} - s)} \| a \|_{L^\infty_t(\dot{B}^{-1+\frac{d}{q} - s}_{q,q})} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})}.
\]

As $p \geq q$, applying Lemma A.1 once again gives rise to
\[
\| \dot{\Delta}_j (\dot{T}_g a) \|_{L^1_t(L^p)} \lesssim \sum_{|j' - j| \leq 5} \| \dot{S}_{j'-1} g \|_{L^1_t(L^\infty)} \| \dot{\Delta}_{j'} a \|_{L^\infty_t(L^p)} \\
\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p} - s)} \| a \|_{L^\infty_t(\dot{B}^{-1+\frac{d}{q} - s}_{q,q})} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})},
\]
where we used the fact that $s > -1$, so that
\[
\| \dot{S}_{j'-1} g \|_{L^1_t(L^\infty)} \lesssim c_{j,r} 2^{j(1+s)} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})}.
\]

Finally, as $s < \frac{d}{p} + \frac{d}{q} - 1$, for $\frac{1}{p} + \frac{1}{q} \leq 1$, we get, by applying Lemma A.1, that
\[
\| \dot{\Delta}_j (\dot{R}(a, g)) \|_{L^1_t(L^p)} \\
\lesssim 2^{j \frac{d}{q}} \sum_{j' \geq j - N_0} \| \dot{\Delta}_{j'} a \|_{L^\infty_t(L^q)} \| \dot{\Delta}_{j'} g \|_{L^1_t(L^p)} \\
\lesssim 2^{j \frac{d}{q}} \sum_{j' \geq j - N_0} c_{j',r} 2^{j'(-1+\frac{d}{p} - \frac{d}{q} + s)} \| a \|_{L^\infty_t(\dot{B}^{\frac{d}{q}}_{q,q})} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})} \\
\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p} - s)} \| a \|_{L^\infty_t(\dot{B}^{\frac{d}{q}}_{q,q})} \| g \|_{L^1_t(B^{-1+\frac{d}{p} - s}_{p,q})}.
\]
For the case when $\frac{1}{p} + \frac{1}{q} > 1$, we have

$$
\| \hat{\Delta} j (\hat{R}(a, g)) \|_{L^1_t(L^p)} \lesssim 2^{jd(1 - \frac{1}{p})} \sum_{j' \geq j - N_0} \| \tilde{\Delta} j' a \|_{L^\infty_t(L^{\frac{p}{p-1}})} \| \hat{\Delta} j' g \|_{L^1_t(L^p)}
$$

$$
\lesssim 2^{jd(1 - \frac{1}{p})} \sum_{j' \geq j - N_0} c_{j',r} 2^{j'(1 - d + s)} \| a \|_{\tilde{L}^\infty_t(B^d_{q,\infty})} \| g \|_{\tilde{L}_t^1(B^{-1 + \frac{d}{p} - s})}
$$

$$
\lesssim c_{j,r} 2^{-j(1 - d + \frac{d}{q} - s)} \| a \|_{\tilde{L}^\infty_t(B^d_{q,\infty})} \| g \|_{\tilde{L}_t^1(B^{-1 + \frac{d}{p} - s})}.
$$

Whence thanks to (5.11), we prove Lemma 5.3. □

We now turn to the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Again, both the proof of the existence and uniqueness of solutions to (5.4) essentially follows from the estimates (5.6) for some appropriate approximate solutions. For the sake of simplicity, we just prove (5.6) for smooth enough solutions of (5.4). Indeed, thanks to (5.4) and (5.5), we write

$$
\nabla \Pi_\lambda = \nabla (-\Delta)^{-1} \left[ \text{div}(a \nabla \Pi_\lambda) + \text{div}_h \text{div}_h (u^h \otimes u^h_\lambda) + 2 \text{div}_h (u^d \partial_d u^h_\lambda) - 3 \partial_d (u^d \text{div}_h u^h_\lambda) - 2 \text{div}_h (u^h \text{div}_h u^h_\lambda) - \mu \text{div}_h (a \Delta u^h_\lambda) - \mu \partial_d (a \Delta u^h_\lambda) \right].
$$

Applying $\hat{\Delta} j$ to the above equation and using Lemma A.1 leads to

$$
\| \hat{\Delta} j (\nabla \Pi_\lambda) \|_{L^1_t(L^p)} \lesssim \| \hat{\Delta} j (a \nabla \Pi_\lambda) \|_{L^1_t(L^p)} + 2^j \| \hat{\Delta} j (u^h \otimes u^h_\lambda) \|_{L^1_t(L^p)} + \| \hat{\Delta} j (u^h \text{div}_h u^h_\lambda) \|_{L^1_t(L^p)} + \| \hat{\Delta} j (u^d \nabla u^h_\lambda) \|_{L^1_t(L^p)} + \mu \| \hat{\Delta} j (a \Delta u^h_\lambda) \|_{L^1_t(L^p)} + \mu \| \hat{\Delta} j (a \Delta u^h_\lambda) \|_{L^1_t(L^p)},
$$

(5.12)

from which, and Lemma 5.1 to Lemma 5.3, we deduce that for $s = 0$ and $\varepsilon$,

$$
\| \hat{\Delta} j (\nabla \Pi_\lambda) \|_{L^1_t(L^p)} \lesssim c_{j,r} 2^{-j(1 - 1 + \frac{d}{p} - s)} \left\{ \begin{array}{l}
\| a \|_{\tilde{L}^\infty_t(L^\infty) \cap \tilde{L}^\infty_t(B^d_{q,\infty})} \\
\| u^h \|_{\tilde{L}_t^1(B^{-1 + \frac{d}{p} - s})} \\
\| \nabla \Pi_\lambda \|_{\tilde{L}_t^1(B^{-1 + \frac{d}{p} - s})} \\
\| u^h \|_{\tilde{L}_t^1(B^{-1 + \frac{d}{p} - s})}
\end{array} \right\}.
$$

Therefore (5.6) follows, as long as $C \| a \|_{\tilde{L}^\infty_t(L^\infty) \cap \tilde{L}^\infty_t(B^d_{q,\infty})} \leq \frac{1}{2}$. This finishes the proof of Proposition 5.1. □

To deal with the estimate of $u^d$, we also need the following proposition:
Proposition 5.2. Under the assumptions of Proposition 5.1, one has for \( s = 0 \) and \( \varepsilon \),
\[
\| \nabla \Pi \|_{L_t^1(B_{p,r}^{1+d/p-s})} \leq \frac{C}{1 - C \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)}} \left\{ \| u^h \|_{L_t^1(B_{p,r}^{1+d/p-s})} \right\} \| u^d \|_{L_t^\infty(B_{p,r}^{1+d/p-s})} \\
+ \left( \mu \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)} + \| u^h \|_{L_t^\infty(B_{p,r}^{1+d/p-s})} \right) \times \left( \| u^h \|_{L_t^1(B_{p,r}^{1+d/p-s})} + \| u^d \|_{L_t^1(B_{p,r}^{1+d/p-s})} \right)
\]
for \( t \leq T \), provided that \( C \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)} \leq \frac{1}{2} \),

Proof. The proof of this proposition follows exactly from that of Proposition 5.1. In fact, taking \( \lambda = 0 \) in (5.12), and then applying Lemma 5.1 and Lemma 5.3, we arrive at
\[
\| \nabla \Pi \|_{L_t^1(B_{p,r}^{1+d/p-s})} \leq C \left\{ \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)} \| \nabla \Pi \|_{L_t^1(B_{p,r}^{1+d/p-s})} \\
+ \| u^h \|_{L_t^1(B_{p,r}^{1+d/p-s})} \| u^d \|_{L_t^\infty(B_{p,r}^{1+d/p-s})} \right\} \| u^h \|_{L_t^1(B_{p,r}^{1+d/p-s})} \times \left( \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)} + \| u^h \|_{L_t^\infty(B_{p,r}^{1+d/p-s})} \right) \| u^d \|_{L_t^1(B_{p,r}^{1+d/p-s})}
\]
for \( t \leq T \), from which and the fact that \( C \| a \|_{L_t^\infty(L_\infty) \cap L_t^\infty(B_{q,\infty}^d)} \leq \frac{1}{2} \), we conclude the proof of (5.13). \( \square \)

5.3. The Estimate of \( u^h \)

We first deduce from the transport equation of (1.2) that
\[
\| a \|_{L_t^\infty(L_\infty)} \leq \| a_0 \|_{L_\infty} \quad \text{for} \quad t < T^*. \quad (5.14)
\]
Let \( f(t), u_\lambda, \Pi_\lambda \) be given by (5.5). Then thanks to (1.2), we write
\[
\partial_t u^h_\lambda + \lambda f(t) u^h_\lambda - \mu \Delta u^h_\lambda = -u \cdot \nabla u^h_\lambda - (1 + a) \nabla \Pi_\lambda + \mu a \Delta u^h_\lambda.
\]
Applying the operator \( \hat{\Delta}_j \) to the above equation and then taking the \( L^2 \) inner product of the resulting equation with \( |\hat{\Delta}_j u^h_\lambda|^{p-2} \hat{\Delta}_j u^h_\lambda \) (in the case when \( p \in (1, 2) \),
we need to make some modifications as in [9]), we obtain
\[
\frac{1}{p} \frac{d}{dt} \| \dot{\Delta} j u^h_\lambda(t) \|_{L^p}^p + \lambda f(t) \| \dot{\Delta} j u^h_\lambda(t) \|_{L^p}^p - \mu \int_{\mathbb{R}^d} \Delta \dot{\Delta} j u^h_\lambda \ | \dot{\Delta} j u^h_\lambda |^{p-2} \dot{\Delta} j u^h_\lambda \ dx
\]
\[
= - \int_{\mathbb{R}^d} \left( \dot{\Delta} j (u \cdot \nabla u^h_\lambda) + \dot{\Delta} j ((1+a) \nabla h \Pi_\lambda) - \mu \dot{\Delta} j (a \Delta u^h_\lambda) \right) \ | \dot{\Delta} j u^h_\lambda |^{p-2} \dot{\Delta} j u^h_\lambda \ dx.
\]

However thanks to [9] (see also [28]), there exists a positive constant \( \tilde{c} \) such that
\[
- \int_{\mathbb{R}^d} \Delta \dot{\Delta} j u^h_\lambda \ | \dot{\Delta} j u^h_\lambda |^{p-2} \dot{\Delta} j u^h_\lambda \ dx \geq \tilde{c} 2^{2j} \| \dot{\Delta} j u^h_\lambda \|_{L^p}^p.
\]
From this, an argument similar to that in [9] gives rise to
\[
\| \Delta \dot{\Delta} j u^h_\lambda \|_{L^\infty(L^p)} + \lambda \int_0^t f(t') \left( \| \dot{\Delta} j u^h_\lambda(t') \|_{L^p} \right) \ dt' + \tilde{c} \mu 2^{2j} \| \dot{\Delta} j u^h_\lambda \|_{L^1(L^p)}
\]
\[
\leq \| \Delta \dot{\Delta} j u^h_0 \|_{L^p} + \| \Delta j (u \cdot \nabla u^h_\lambda) \|_{L^1(L^p)} + \| \dot{\Delta} j ((1+a) \nabla h \Pi_\lambda) \|_{L^1(L^p)} \ (5.15)
\]
\[
+ \mu \| \Delta j (a \Delta u^h_\lambda) \|_{L^1(L^p)}.
\]
For \( s = 0 \) and \( \varepsilon \), applying Lemma 5.2 gives
\[
\| \Delta j (u \cdot \nabla u^h_\lambda) \|_{L^1(L^p)} \leq \| \Delta j (u^h \cdot \nabla_h u^h_\lambda) \|_{L^1(L^p)} + \| \Delta j (u^d \partial_d u^h_\lambda) \|_{L^1(L^p)}
\]
\[
\leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left( \| u^h \|_{\tilde{L}^\infty(B_{p,r}^{\frac{d}{p}})} \| u^h \|_{\tilde{L}^{1+\frac{d}{p}}(B_{p,r}^{\frac{d}{p}})} + \| u^h \|_{\tilde{L}^{1+\frac{d}{p}}(B_{p,r}^{\frac{d}{p}})} \| u^h \|_{\tilde{L}^{1+\frac{d}{p}}(B_{p,r}^{\frac{d}{p}})} \right).
\]
Applying Lemma 5.3 and (5.14) yields
\[
\| \Delta j (a \Delta u^h_\lambda) \|_{L^1(L^p)}
\]
\[
\leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left( \| a_0 \|_{L^\infty} + \| a \|_{\tilde{L}^\infty(B_{q,\infty}^{\frac{d}{q}})} \right) \| u^h \|_{\tilde{L}^{1+\frac{d}{p}}(B_{p,r}^{\frac{d}{p}})}.
\]
and
\[
\| \Delta j ((1+a) \nabla_h \Pi_\lambda) \|_{L^1(L^p)}
\]
\[
\leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left( 1 + \| a_0 \|_{L^\infty} + \| a \|_{\tilde{L}^\infty(B_{q,\infty}^{\frac{d}{q}})} \right) \| \nabla_h \Pi_\lambda \|_{\tilde{L}^{1+\frac{d}{p}}(B_{p,r}^{\frac{d}{p}})}.
\]
Now letting $c_1$ be a small enough positive constant, which will be determined later, we define $\Xi$ by

$$\Xi \overset{\text{def}}{=} \max \left\{ t \in [0, T^*) : \| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{q,\infty}^{-1+\frac{d}{p}})} + \mu \left( \| a_0 \parallel \parallel_{L^\infty} + \| a \parallel \parallel_{L_t^\infty(B_{p,\infty}^{-1+\frac{d}{p}})} + \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,r}^{1+\frac{d}{p}})} \right) \right\} \leq c_1 T.$$  \hspace{1cm} (5.16)

In particular, (5.16) implies that $\| a_0 \parallel \parallel_{L^\infty} + \| a \parallel \parallel_{L_t^\infty(B_{q,\infty}^{-1+\frac{d}{p}})} \leq c_1$ for $t \leq \Xi$. Taking $c_1$ so small that $Cc_1 \leq \frac{1}{2}$, (5.6) and (5.14) ensure that

$$\| \check{\Delta}_j (1 + a) \nabla_h \Pi_\lambda \parallel \parallel_{L_t^1(L^p)} \leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left\{ \| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{p,\infty}^{-1+\frac{d}{p}-s})} + \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} \right\} \text{ for } t \leq \Xi.$$  

Substituting the above estimates into (5.15), we obtain for $s = 0$ and $\varepsilon$ that

$$\| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{p,\infty}^{-1+\frac{d}{p}-s})} + \lambda \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} + \bar{\lambda} \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} + \bar{\mu} \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} \leq C \left\{ \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} + \mu \left( \| a_0 \parallel \parallel_{L^\infty} + \| a \parallel \parallel_{L_t^\infty(B_{q,\infty}^{-1+\frac{d}{p}})} + \| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{q,\infty}^{-1+\frac{d}{p}})} \right) \right\} \text{ for } t \leq \Xi.$$  

for $t \leq \Xi$. Taking $\lambda = \frac{2C}{\mu^{s-1}}$ in the above inequality and thanks to (5.16), we get

$$\| u_h^\parallel \parallel_{\bar{L}_t^\infty(B_{p,\infty}^{-1+\frac{d}{p}-s})} + \frac{\bar{\mu}}{2} \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} \leq \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{-1+\frac{d}{p}-s})} + \mu \left( \| a_0 \parallel \parallel_{L^\infty} + \| a \parallel \parallel_{L_t^\infty(B_{q,\infty}^{-1+\frac{d}{p}})} \right) \| u_h^\parallel \parallel_{\bar{L}_t^1(B_{p,\infty}^{1+\frac{d}{p}})} \text{ for } t \leq \Xi,$$  \hspace{1cm} (5.17)
On the other hand, it is easy to observe from (5.5) and Definition A.3 that
\[
\left( \| u^h \|_{\tilde{L}_t^{\infty}(\tilde{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\tilde{c}}{2} \| u^h \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})} \right) \exp \left\{ - \int_0^t \lambda f(t') \, dt' \right\}
\leq \| u^h \|_{\tilde{L}_t^{\infty}(\tilde{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\tilde{c}}{2} t \| u^h \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})},
\]
from which, and (5.3), (5.16), (5.17), we infer that for \( s = 0, \varepsilon, \) and \( t \leq \bar{\varepsilon}, \)
\[
\| u^h \|_{\tilde{L}_t^{\infty}(\tilde{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\mu}{2} \left( \| a \|_{\tilde{L}_t^{\infty}(\tilde{B}_{q,\infty}^{d+\frac{p}{2}})} + \| u^h \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})} \right)
\leq C \left( \mu (\| a_0 \|_{L^\infty} + \| a_0 \|_{\tilde{B}_{q,\infty}^{d+\frac{p}{2}}} + \| u_0^h \|_{\tilde{B}_{p,r}^{1+\frac{d}{p}-s}} \right)
\times \exp \left\{ C_{\varepsilon} \left( \| u^d \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})} + \| u^d \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}})} + \int_0^t \frac{1}{\mu^{2r-1}} \| u^d(t') \|_{\tilde{B}_{p,r}^{-1+\frac{d}{p}+\frac{r}{1+r}}} \, dt' \right) \right\}.
\] (5.18)

4. The Estimate of \( u^d \)

By virtue of (1.2), we get, by a derivation similar to (5.15), that
\[
\| \tilde{J}_j u^d \|_{L_t^{\infty}(L^p)} + \tilde{c} \mu 2^j \| \tilde{J}_j u^d \|_{L_t^{1}(L^p)}
\leq \| \tilde{J}_j u^d_0 \|_{L^p} + C \left( \| \tilde{J}_j (u \cdot \nabla u^d) \|_{L_t^{1}(L^p)} + \| \tilde{J}_j ((1 + a) \partial_d \Pi) \|_{L_t^{1}(L^p)} + \mu \| \tilde{J}_j (a \Delta u^d) \|_{L_t^{1}(L^p)} \right).
\] (5.19)

Applying Lemma 5.1 and Lemma 5.2 gives, for \( s = 0 \) and \( \varepsilon, \) that
\[
\| \Delta_j (u \cdot \nabla u^d) \|_{L_t^{1}(L^p)} \lesssim 2^j \| \Delta_j (u^h u^d) \|_{L_t^{1}(L^p)} + \| \Delta_j (u^d \text{ div}_h u^h) \|_{L_t^{1}(L^p)}
\lesssim c_{j,r} 2^{-j\left(-\frac{1}{2}+\frac{r}{p}-s\right)} \left( \| u^h \|_{\tilde{L}_t^{\infty}(\tilde{B}_{p,r}^{-1+\frac{d}{p}-s})} \| u^d \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})} + \| u^h \|_{\tilde{L}_t^{1}(\tilde{B}_{p,r}^{1+\frac{d}{p}-s})} \| u^d \|_{\tilde{L}_t^{\infty}(\tilde{B}_{p,r}^{-1+\frac{d}{p}-s})} \right).
\]

Thanks to (5.16) and (5.14), we get, by applying Lemma 5.3 and Proposition 5.2, that
\[ \| \tilde{\Delta}_j((1 + a)\partial_d \Pi) \|_{L^1_t(L^P)} \leq C \tilde{c}_j, 2^{-j(-1 + \frac{d}{p} - s)} (1 + \| a_0 \|_{L^\infty} + \| a \|_{L^\infty(B^{q}_{\infty})}) \| \partial_d \Pi \|_{L^1_t(B^{p-1 + \frac{d}{p} - s}_{p,r})} \]

\[ \leq C \tilde{c}_j, 2^{-j(-1 + \frac{d}{p} - s)} \left\{ \| u^h \|_{L^1_t(B^{1 + \frac{d}{p}}_{p,r})} \| u^d \|_{L^\infty(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} + \left\{ \| u^h \|_{L^\infty(B^{q}_{\infty})} \| u^d \|_{L^\infty(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} \times \left\{ \| u^h \|_{L^1_t(B^{1 + \frac{d}{p}}_{p,r})} \| u^d \|_{L^1_t(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} , \]

for \( t \leq \tilde{T} \). Substituting the above estimates into (5.19) leads to

\[ \| u^d \|_{L^\infty(B^{1 + \frac{d}{p}}_{p,r})} \leq \| u_0^d \|_{B^{p-1 + \frac{d}{p}}_{p,r}} + C \left\{ \| u^h \|_{L^1_t(B^{1 + \frac{d}{p}}_{p,r})} \| u^d \|_{L^\infty(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} + \left\{ \| u^h \|_{L^\infty(B^{q}_{\infty})} \| u^d \|_{L^\infty(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} \times \left\{ \| u^h \|_{L^1_t(B^{1 + \frac{d}{p}}_{p,r})} \| u^d \|_{L^1_t(B^{p-1 + \frac{d}{p}}_{p,r})} \right\} \]  

(5.20)

for \( t \leq \tilde{T} \) and \( s = 0, \varepsilon \).

In particular, if we take \( c_1 \leq \min \{ \frac{1}{\tilde{c} T}, \frac{\tilde{c}}{4 T} \} \) in (5.16), we deduce from (5.20) that

\[ \| u^d \|_{L^\infty(B^{1 + \frac{d}{p} - s}_{p,r})} + \mu \tilde{c} \| u^d \|_{L^1_t(B^{p-1 + \frac{d}{p} - s}_{p,r})} \leq 2 \| u_0^d \|_{B^{p-1 + \frac{d}{p} - s}_{p,r}} + c_2 \mu \]  

(5.21)

for \( t \leq \tilde{T} \) and \( s = 0, \varepsilon \).

5.5. The Proof of Theorem 1.2

According to the arguments at the beginning of this section, we need to prove only that \( \tilde{T} = \infty \) under the assumption of (1.13). Otherwise, if \( \tilde{T} < T^* < \infty \), we first deduce from (5.21) that

\[ \| u^d \|_{L^\infty(B^{1 + \frac{d}{p} - s}_{p,r})} \leq C \| u_0^d \|_{B^{p-1 + \frac{d}{p} - s}_{p,r}} + \mu \tilde{c} \| u^d \|_{L^1_t(B^{p-1 + \frac{d}{p} - s}_{p,r})} \]  

(5.22)

for \( t \leq \tilde{T} \).
Substituting (5.21) and (5.22) into (5.18) gives rise to

\[\|u^h\|_{L_t^\infty(B_{p,r}^{−1+\frac{d}{r}−\varepsilon})} + \|u^h\|_{L_t^\infty(B_{p,r}^{−1+\frac{d}{r}})} + \mu \left(\|a\|_{L_t^\infty(B_{q,\infty}^{\frac{d}{r} + \frac{2}{r} + \varepsilon})} + \|u^h\|_{L_t^1(B_{p,r}^{−1+\frac{d}{r}−\varepsilon})} + \|u^h\|_{L_t^1(B_{p,r}^{−1+\frac{d}{r}})}\right)\]

\[\leq \mathcal{C}_1 \left(\mu \|a_0\|_{L_t^\infty(B_{q,\infty}^{\frac{d}{r} + \frac{2}{r} + \varepsilon})} + \|u^h\|_{B_t^{−1+\frac{d}{r}−\varepsilon}} + \|u^h\|_{B_t^{−1+\frac{d}{r}}})\times \exp \left\{\frac{\mathcal{C}_2}{\mu} \|u_0^d\|_{B_t^{−1+\frac{d}{r}−\varepsilon}} + \|u^h\|_{B_t^{−1+\frac{d}{r}}}\right\}\]

for \(t \leq \bar{\tau}\) and some positive constants \(\mathcal{C}_1, \mathcal{C}_2\) which depend on \(\tilde{c}, c_1\) and \(\varepsilon\). In particular, if we take \(C_{r, \varepsilon}\) large enough and \(c_0\) sufficiently small in (1.13), the above inequality implies that, for \(\delta\) given by (1.13),

\[\|u^h\|_{\dot{L}_t^\infty(B_{p,r}^{−1+\frac{d}{r}−\varepsilon})} + \|u^h\|_{\dot{L}_t^\infty(B_{p,r}^{−1+\frac{d}{r}})} + \mu \left(\|a_0\|_{L^\infty} + \|a\|_{L_t^\infty(B_{q,\infty}^{\frac{d}{r} + \frac{2}{r} + \varepsilon})} + \mu \left(\|a_0\|_{L_t^\infty} + \|a\|_{L_t^\infty(B_{q,\infty}^{\frac{d}{r} + \frac{2}{r} + \varepsilon})}\right)\right)\]

\[\leq C\delta \leq \frac{c_1}{2} \mu \quad \text{for} \quad t \leq \bar{\tau},\]

which contradicts (5.16). Thus, we conclude that \(\bar{\tau} = T^* = \infty\), and there holds (1.14). This completes the proof of Theorem 1.2.

\(\square\)

**Acknowledgements.** We would like to thank the anonymous referee for profitable suggestions. Part of this work was done when Marius Paicu was visiting Morningside Center of the Chinese Academy of Sciences in the Spring of 2012. We appreciate the hospitality of MCM and the financial support from the Chinese Academy of Sciences. Ping Zhang is partially supported by NSF of China under Grant 10421101 and 10931007, the one hundred talents’ plan from the Chinese Academy of Sciences under Grant GJHZ200829 and innovation grant from National Center for Mathematics and Interdisciplinary Sciences.

**Appendix A. Littlewood–Paley Analysis**

The proof of Theorem 1.2 requires Littlewood–Paley decomposition. Let us briefly explain how it may be built in the case \(x \in \mathbb{R}^d\) (see, for example, [6]). Let \(\varphi\) be a smooth function supported in the ring \(C \overset{\text{def}}{=} \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\) and let \(\chi\) be a smooth function supported in the ball \(B \overset{\text{def}}{=} \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}\) such that

\[\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^d, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \neq 0.\]
Then for $u \in S'(\mathbb{R}^d)$, we set
\[
\forall j \in \mathbb{Z}, \quad \hat{\Delta}_j u \overset{\text{def}}{=} \varphi(2^{-j}D)u, \quad \hat{S}_j u \overset{\text{def}}{=} \chi(2^{-j}D)u,
\]
\[
\forall q \in \mathbb{N}, \quad \Delta_q u = \varphi(2^{-q}D), \quad S_q u = \sum_{\ell \leq q-1} \Delta_\ell u, \quad (A.1)
\]
\[
\Delta_{-1} u = \chi(D)u.
\]
We have the formal decomposition
\[
u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \forall u \in S'(\mathbb{R}^d)/\mathcal{P}[\mathbb{R}^d] \quad \text{and}
\]
\[
u = \sum_{q \geq -1} \Delta_q u, \quad \forall u \in S'(\mathbb{R}^d), \quad (A.2)
\]
where $\mathcal{P}[\mathbb{R}^d]$ is the set of polynomials. Moreover, the Littlewood–Paley decomposition satisfies the property of almost orthogonality:
\[
\hat{\Delta}_j \hat{\Delta}_\ell u \equiv 0 \quad \text{if} \quad |j - \ell| \geq 2, \quad \hat{\Delta}_j (\hat{S}_{\ell-1} u \hat{\Delta}_\ell u) \equiv 0 \quad \text{if} \quad |j - \ell| \geq 5. \quad (A.3)
\]
We recall now the definition of homogeneous Besov spaces and Bernstein type inequalities from [6].

**Definition A.1.** Letting $(p, r) \in [1, +\infty]^2, s \in \mathbb{R}$ and $u \in S'_h(\mathbb{R}^3)$, which means that $u \in S'(\mathbb{R}^3)$ and $\lim_{j \to -\infty} \|\hat{S}_j u\|_{L^\infty} = 0$, we define
\[
\|u\|_{B^s_{p,r}} \overset{\text{def}}{=} \left(2^q \|\Delta_q u\|_{L^p}\right)_{q \in \mathbb{N}}^{\frac{1}{r}}. \quad \hat{B}^s_{p,r}(\mathbb{R}^d) \overset{\text{def}}{=} \{u \in S'_h(\mathbb{R}^d) \mid \|u\|_{B^s_{p,r}} < \infty\}.
\]

Inhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^d)$ can be defined in a similar way, so that
\[
\|u\|_{B^s_{p,r}} \overset{\text{def}}{=} \left(2^q \|\Delta_q u\|_{L^p}\right)_{q \in \mathbb{N}}^{\frac{1}{r}}.
\]

**Lemma A.1.** Let $B$ be a ball and $C$ a ring of $\mathbb{R}^d$. A constant $C$ exists such that for any positive real number $\delta$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, and any couple of real numbers $(a, b)$ with $b \geq a \geq 1$, there hold
\[
\text{Supp } \hat{u} \subset \delta B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \delta^{k+3\left(\frac{1}{a} - \frac{1}{b}\right)} \|u\|_{L^a},
\]
\[
\text{Supp } \hat{u} \subset \delta C \Rightarrow C^{-1-k} \delta^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{1+k} \delta^k \|u\|_{L^a}, \quad (A.4)
\]
\[
\text{Supp } \hat{u} \subset \delta C \Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \delta^m \delta^{3\left(\frac{1}{a} - \frac{1}{b}\right)} \|u\|_{L^a}.
\]

We shall frequently use Bony’s decomposition from [7] in the homogeneous context:
\[
\hat{u} = \hat{T}_u v + \hat{R}(u, v) = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v), \quad (A.5)
\]
where
\[\hat{T}_u v \equiv \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) \equiv \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{S}_{j+2} v,\]
\[\hat{\Delta}_j u \hat{\Delta}_j v \text{ with } \hat{\Delta}_j v \equiv \sum_{|j' - j| \leq 1} \hat{\Delta}_j' v.\]

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin–Lerner type spaces, \(\tilde{L}_T^{\lambda} (\dot{B}_p^s (\mathbb{R}^d))\).

**Definition A.2.** Let \((r, \lambda, p) \in [1, +\infty]^3\) and \(T \in (0, +\infty]\). We define \(\tilde{L}_T^{\lambda} (\dot{B}_p^s (\mathbb{R}^d))\) as the completion of \(C([0, T]; \dot{S}(\mathbb{R}^d))\) by the norm
\[\|u\|_{\tilde{L}_T^{\lambda} (\dot{B}_p^s)} \equiv \left( \sum_{j \in \mathbb{Z}} 2^{j rs} \left( \int_0^T \| \hat{\Delta}_j u(t) \|_{L_p}^r \, dt \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} < \infty,\]
with the usual change if \(r = \infty\). For short, we just denote this space by \(\tilde{L}_T^{\lambda} (\dot{B}_p^s)\).

As one cannot use Gronwall's inequality in the Chemin–Lerner type spaces, we [26,27] introduced weighted Chemin–Lerner norm in the context of anisotropic Besov spaces. To prove 1.2, we need the following version of weighted Chemin–Lerner in the context of isentropic Besov spaces:

**Definition A.3.** Let \((r, p) \in [1, +\infty]^2\) and \(T \in (0, +\infty]\). Let \(0 \leq f(t) \in L^1(0, T)\). We define the norm \(\tilde{L}_T^{\lambda} (\dot{B}_p^s)\) as
\[\|u\|_{\tilde{L}_T^{\lambda} (\dot{B}_p^s)} \equiv \left( \sum_{j \in \mathbb{Z}} 2^{j rs} \left( \int_0^T \| f(t) \hat{\Delta}_j u(t) \|_{L_p}^r \, dt \right)^{\frac{r}{p}} \right)^{\frac{1}{r}}.\]

**References**

1. ABIDI, H.: Équation de Navier–Stokes avec densité et viscosité variables dans l’espace critique. Rev. Mat. Iberoam. 23(2), 537–586 (2007)
2. ABIDI, H., PAICU, M.: Existence globale pour un fluide inhomogène. Ann. Inst. Fourier (Grenoble) 57, 883–917 (2007)
3. ABIDI, H., GUI, G., ZHANG, P.: On the decay and stability to global solutions of the 3–D inhomogeneous Navier–Stokes equations. Commun. Pure. Appl. Math. 64, 832–881 (2011)
4. ABIDI, H., GUI, G., ZHANG, P.: On the wellposedness of 3–D inhomogeneous Navier–Stokes equations in the critical spaces. Arch. Rational Mech. Anal. 204, 189–230 (2012)
5. ABIDI, H., GUI, G., ZHANG, P.: Well-posedness of 3–D inhomogeneous Navier–Stokes equations with highly oscillating initial velocity field. J. Math. Pures Appl. (2012). http://dx.doi.org/10.1016/j.matpur.2012.10.015
6. BAHOURI, H., CHEMIN, J.Y., DANCHIN, R.: Fourier analysis and nonlinear partial differential equations. Grundlehren der mathematischen Wissenschaften, vol. 343. Springer, Berlin, 2011
7. Bony, J.M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. 14(4), 209–246 (1981)
8. Chemin, J.Y., Lerner, N.: Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes. J. Diff. Equ. 121, 314–328 (1995)
9. Danchin, R.: Local theory in critical spaces for compressible viscous and heat-conducting gases. Commun. P. D. E., 26, 1183–1233 (2001)
10. Danchin, R.: Density-dependent incompressible viscous fluids in critical spaces. Proc. R. Soc. Edinburgh Sect. A 133, 1311–1334 (2003)
11. Danchin, R.: Local and global well-posedness results for flows of inhomogeneous viscous fluids. Adv. Diff. Equ. 9, 353–386 (2004)
12. Danchin, R.: Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients. Revista Matemática Iberoamericana 21, 863–888 (2005)
13. Danchin, R., Mucha, P.B.: A Lagrangian approach for the incompressible Navier–Stokes equations with variable density. Commun. Pure. Appl. Math., 65, 1458–1480 (2012)
14. Danchin, R., Mucha, P.B.: Incompressible flows with piecewise constant density. Arch. Rat. Mech. Anal., 207, 991–1023 (2013)
15. DiPerna, R.J., Lions, P.L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math., 98, 511–547 (1989)
16. Fujita, H., Kato, T.: On the Navier–Stokes initial value problem I. Arch. Ration. Mech. Anal. 16, 269–315 (1964)
17. Kato, T.: Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^d$. J. Funct. Anal. 9, 296–305 (1972)
18. Gui, G., Zhang, P.: Stability to the global solutions of 3-D Navier–Stokes equations. Adv. Math. 225, 1248–1284 (2010)
19. Haspot, B.: Well-posedness for density-dependent incompressible fluids with non-lipschitz velocity (2012, preprint)
20. Huang, J., Paicu, M., Zhang, P.: Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system with rough density. In: Cignani, M., Colombini, J., Del Santo, D. (eds.) Studies in Phase Space Analysis with Applications to PDEs. Progress in Nonlinear Differential Equations and Their Applications, vol. 84, pp. 173–194. Springer, New York (2013)
21. Kim, H.: A blow-up criterion for the nonhomogeneous incompressible Navier–Stokes equations. SIAM J. Math. Anal. 37, 1417–1434 (2006)
22. Kazhikov, A.V.: Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid. (Russian) Dokl. Akad. Nauk SSSR 216, 1008–1010 (1974)
23. Ladyženskaja, O.A., Solonnikov, V.A.: The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. (Russian) Boundary value problems of mathematical physics, and related questions of the theory of functions, 8, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 52, 52–109, 218–219 (1975)
24. Lions, P.L.: Mathematical topics in fluid mechanics, vol. 1. Incompressible models. In: Oxford Lecture Series in Mathematics and its Applications, vol. 3. Oxford Science Publications. Oxford University Press, New York, 1996
25. Lemarié-Rieusset, P.G.: Recent developments in the Navier–Stokes Problem. Chapman & Hall/CRC Research Notes in Mathematics, vol. 431. Chapman & Hall/CRC, Boca Raton, 2002
26. Paicu, M., Zhang, P.: Global solutions to the 3-D incompressible anisotropic Navier–Stokes system in the critical spaces. Commun. Math. Phys. 307, 713–759 (2011)
27. Paicu, M., Zhang, P.: Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system. J. Funct. Anal. 262, 3556–3584 (2012)
28. Planchon, F.: Sur un inégalité de type Poincaré. C. R. Acad. Sci. Paris Sér. I Math. 330, 21–23 (2000)
29. Zhang, T.: Global wellposedness problem for the 3-D incompressible anisotropic Navier–Stokes equations in an anisotropic space. *Commun. Math. Phys.* **287**, 211–224 (2009)

Academy of Mathematics and Systems Science, 
The Chinese Academy of Sciences, 
Beijing 100190, China. 
e-mail: jchuang@amss.ac.cn

and

Institut de Mathématiques de Bordeaux 
Université Bordeaux 1 
33405 Talence Cedex, France. 
e-mail: marius.paicu@math.u-bordeaux1.fr

and

Academy of Mathematics & Systems Science, 
Hua Loo-Keng Key Laboratory of Mathematics, 
The Chinese Academy of Sciences 
Beijing 100190, China. 
e-mail: zp@amss.ac.cn

*(Received March 21, 2012 / Accepted February 21, 2013)*
*Published online April 4, 2013 – © Springer-Verlag Berlin Heidelberg (2013)*