Automorphisms of crepant resolutions for quotient spaces

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1. Introduction

Let $X$ be a Kahler manifold and $G$ be a finite group of biholomorphic volume preserving automorphisms of $X$. Let $\pi : \widetilde{X}/G \rightarrow X/G$ be a crepant resolution of the orbit space $X/G$ (i.e. the pullback of the dualizing sheaf of $X/G$ is the canonical bundle of $\widetilde{X}/G$). The euler characteristic of $\widetilde{X}/G$ can be found as follows: (cf. [R],[BD],[HH]):

$$e(\widetilde{X}/G) = \frac{1}{|G|} \sum_{hg=gh} e(X^g \cap X^h) = \sum_{[g]} e(X^g/C(g)) \quad (1.0)$$

Here $X^g$, for $g \in G$, is the fixed point set of an automorphism and $C(g)$ is the centralizer. The summation in (1.0) is over all conjugacy classes $[g]$ of elements $g \in G$. This formula plays a key role in validating that pairs of Calabi Yau manifolds constitute mirror pairs (cf. [COGP]).

In this paper we shall propose a formula designed for the study of manifolds for which the pair $(X, G)$ admits a symmetry $h$. This formula allows one to calculate the Lefschetz number of an automorphism acting on a crepant resolution of the quotient (for $h = 1$ this formula becomes (1.0)). We show that it can be derived from a certain equivariant form of McKay correspondence between a data from a crepant resolution of quotient singularity $\mathbb{C}^n/G$, where $G$ admits certain automorphism $h$ and the action of $h$ on the conjugacy classes of $G$. This version of McKay correspondence is valid in dimension $n = 2$ (cf. section 4) and follows from the existence of certain triangulations for abelian $G$ for $n \geq 3$ (such triangulation can always be constructed in dimension $n = 3$ (cf. section 5). In particular this can be used to compare the Lefschetz numbers of certain involutions on some Calabi Yau threefolds (considered in [COGP] and [LT]) and their mirrors (section 3). Note that detailed investigation of connections between McKay correspondence and crepant resolutions was carried out in [BD].

More precisely we have the following (here $L(h, Z)$ denotes the Lefschetz number $\Sigma_i(-1)^i tr(h, H^i(Z))$ of a transformation $h$ acting on a topological space $Z$):

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Theorem 1. Let $X$ be a Kahler manifold on which a finite group $G$ of volume preserving automorphisms acts holomorphically and let $h$ be a biholomorphic automorphism of $X$ such that the group $H$ generated by $G$ and $h$ is finite and contains $G$ as a normal subgroup. Let $X/G$ be a crepant resolution of $X/G$. Let us consider the class $C(h)$ of elements in $G$ with the property:

(*) In any $h$-invariant stabilizer $S_i$ of any point of $X^g$, the conjugacy class of $g$ is $h$-invariant.

Let us assume that:
1. For $g \in C(h)$ $h$ leaves invariant the set of stabilizers containing $g$.
2. $h$ normalizes the centralizer of any element from $C(h)$.

Then

$$L(h, \tilde{X}/G) = \Sigma_{[g]. g \in C(h)} L(h, X^g/C(g))$$  \hspace{0.5cm} (1.1)

Remarks. 1. Let $S$ be the collection of subgroups of $G$ which are stabilizers of points of $X$. We have: $X^g = \bigcup_{S_i \in S, g \in S_i} X^{S_i}$. Therefore condition 1 implies that if $g \in C(h)$ then $h$ preserves $X^g$. Also $h$ acts on the quotient $X^g/C(g)$ as a consequence of condition 2.

2. Throughout the paper we use the following additivity of the Lefschetz numbers: if $Z = Z_1 \cup Z_2, Z_1 \cap Z_2 = \emptyset$ is an $h$-equivariant decomposition then $L(h, Z) = L(h, Z_1) + L(h, Z_2)$ (cf. [Br]).

3. In what follows by stratification we mean just a partition into a union of disjoint sets.

4. The proof is based on the following statement (an equivariant form of weak McKay correspondence):

(**) Let $H$ be a finite subgroup of $GL_n(\mathbb{C})$, $G$ a normal subgroup of $H$ which belongs to $SL_n(\mathbb{C})$ and such that $H/G$ is a cyclic group with a generator $h$. (In particular $h$ induces automorphisms of $\mathbb{C}^n/G$ and $G$). Then the Lefschetz number of $h$ acting on an $h$-equivariant crepant resolution of $\mathbb{C}^n/G$ is equal to the number of $h$-invariant conjugacy classes in $G$.

This will be proven below in the case $n = 2$ and $n = 3$ and $G$ abelian.

2. Proof of Theorem 1.

Let $S_i \in S$ ($i = 1, \ldots, \text{card} S$) be collection of subgroups of $G$ which are stabilizers of points of $X$. The manifold $X$ admits the stratification $X = \bigcup X^{S_i}$ such that each stratum consists of points having the same stabilizer $S_i$. Let $X^{[S_i]}$ be the $G$-orbit of the strata $X^{S_i}$'s i.e. $X^{[S_i]}$ is the unions of $X^{S_i}$ with $G$-conjugate $S_i$'s ($[S_i] \in S/G$ where $G$ acts on $S$ by conjugation). The sets $X^{[S_i]}$ form a stratification of $X$ with $G$-invariant strata and the $G$-quotients $X^{[S_i]}/G$ of these strata provide a stratification of $X/G$. Let $\pi^{-1}(X^{[S_i]}/G) \subset \tilde{X}/G$
be the preimages of strata $X^{[S_i]}/G$ in a chosen crepant resolution $\pi : \tilde{X}/G \to X/G$ of $\tilde{X}/G$ and $X^{[S_i]}/G(h)$ be the $h$-orbit of $\pi^{-1}(X^{[S_i]}/G)$. The automorphism $h$ acts on $X^{S_i}/G(h)$ and we have:

$$L(h, \tilde{X}/G)) = \sum_{X^{[S_i]}/G(h)} L(h, X^{[S_i]}/G(h))$$  \hspace{1cm} (2.0)

where the summation is over all $h$ orbits of preimages of the strata $X^{[S_i]}/G$.

Next we have:

$$\sum_{X^{[S_i]}/G(h)} L(h, X^{[S_i]}/G(h)) = \sum_{[S_i] \in \mathcal{S}, h([S_i]) = [S_i]} L(h, X^{S_i}/G) =$$

$$= \sum_{[S_i] \in \mathcal{S}, h([S_i]) = [S_i]} L(h, X^{S_i}/G) \cdot \text{con}(h, S_i)$$  \hspace{1cm} (2.1)

where $\text{con}(h, S_i)$ is the number of $h$ invariant conjugacy classes in a subgroup $S_i$ and the summation on the last two sums is over $h$ invariant conjugacy classes subgroups $[S_i]$. (Note that $h([S_i]) = [S_i]$ implies that $h$ acts on corresponding stratum $X^{[S_i]}$). The first equality takes place since the Lefschetz number of $h$ acting on an orbit of $h$ of a conjugacy class for which $h([S_i]) \neq [S_i]$ is zero due to the absence of fixed points. The second equality is a consequence of the equivariant McKay correspondence as stated in Remark 4. Note that this remark also implies that the number $\text{con}(h, S)$ is the same for the conjugate subgroups (since both singularities $X/S$ and $X/gSg^{-1}$ are $h$-equivariantly equivalent to the same singularity of $X/G$). The last term can be rewritten as:

$$\sum_{[S_i], S_i \in \mathcal{S}, h([S_i]) = [S_i]} L(h, X^{[S_i]}) \cdot \frac{|S_i|}{|G|} \cdot \text{con}(h, S_i)$$  \hspace{1cm} (2.2)

where the summation is over all $h$-invariant conjugacy classes of stabilizers. The latter can be replaced by the sum:

$$\sum_{[S_i]} L(h, X^{[S_i]}) \cdot \frac{|S_i|}{|G|}$$  \hspace{1cm} (2.3)

where the summation is over all pairs $([g], [S_i])$ and where $g$ is an element in $S_i$ with $h$-invariant conjugacy class. Finally the number of elements is a conjugacy class of $g$ in $S_i$ is $\frac{|C(g) \cap S_i|}{|S_i|}$. Therefore, after splitting $X^{[S_i]}$ into a disjoint union of $X^{S_i}$ and omitting terms corresponding to $S_i$ for which $h(S_i) \neq S_i$ (hence giving zero contribution due to the vanishing of the Lefschetz number), the latter sum can be replaced by the following sum over the pairs $(g, S_i), h(S_i) = S_i, g \in S_i$:

$$\sum_{S_i} \frac{L(h, X^{[S_i]}) |S_i| \cdot |C(g) \cap S_i|}{|G| \cdot |S_i|} = \sum_{(g, S_i), g \in S_i, h(S_i) = S_i} L(h, X^{S_i}) \frac{|C(g) \cap S_i| \cdot |C(g)|}{|C(g)| \cdot |G|}$$  \hspace{1cm} (2.4)

Note that a centralizer $C(g)$ acts on the set of stabilizers containing $g$. $(g \in S_i, c \in C(g)$ implies $g = cg^{-1} \in cS_i c^{-1})$. The latter sum can be rewritten as a sum over $C(g)$ orbits $C(g)(S_i)$ of $S_i$:

$$\sum_{L(h, \prod X^{S_i}) \frac{|C(g) \cap S_i| \cdot |C(g)|}{|G|}$$  \hspace{1cm} (2.5)
since the centralizers of $g$ in each subgroup in $C(g)(S_i)$ are the same. This is equal to:

$$\Sigma_{g,C(g)(S_i)} L(h, X^{C(g)(S_i)} / C(g)) \frac{|C(g)|}{|G|} = \Sigma L(h, X^g / C(g)) \frac{|C(g)|}{|G|}$$

(2.6)

where the summation is over all elements $g$ which have $h$ invariant conjugacy class in any $h$ invariant stabilizer to which it belongs. The last sum finally is equal to

$$\Sigma_{g \in C(h)} L(h, X^g / C(g))$$

(2.7)

where the summation is over all conjugacy classes in $G$ of elements with the specified property.

3. Applications to the calculation of the Lefschetz number of the actions on mirror manifolds.

Here, using the formula from preceding section, we shall calculate the Lefschetz number of the involution induced on the mirror of quintic $V$ in $\mathbb{P}^4$ considered in [COGP]:

$$Q(\lambda x_0, x_1, x_2, x_3, x_4) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\lambda x_0 x_1 x_2 x_3 x_4 = 0$$

(3.1)

when the involution $h : (x_0, x_1, x_2, x_3, x_4) \rightarrow (x_1, x_0, x_2, x_3, x_4)$ is induced by the interchange of first two coordinates.

Recall that the mirror is a crepant resolution of $\widetilde{V}/G$ where $G = \mathbb{Z}_5^3$ is (the maximal) faithfully-acting quotient of the group of automorphisms of $V$ given by

$$(x_0, x_1, x_2, x_3, x_4) \rightarrow (\omega_5^{\alpha_0} x_0, \omega_5^{\alpha_1} x_1, \omega_5^{\alpha_2} x_2, \omega_5^{\alpha_3} x_3, \omega_5^{\alpha_4} x_4)$$

(3.2)

where $\Sigma \alpha_i \equiv 0 \mod 5$ and $\omega_5$ is a non-trivial root of unity of degree 5. First note that the condition $g \in C(h)$ implies that $\alpha_0 = \alpha_1$ in (3.2). Each 1-dimensional fixed point set is the intersection of $V$ with the subspace given by vanishing of two coordinates and is fixed by elements (3.2) for which three exponents $\alpha_i$ coincide. Since there are 3 possibility to choose $\alpha_i$ coinciding with chosen $\alpha_0 = \alpha_1$ we obtain 12 elements (different from identity) in $C(h)$ having 1-dimensional fixed point set. The Lefschetz number of $h$ acting on the quotient of each of these fixed point sets is equal to 2.

Similarly one can consider the 0-dimensional fixed point sets. Each is the fixed point set of an element (3.1) with two pairs of equal components ($\alpha_i = \alpha_j$ for two pairs of indices $(i,j)$). Since $\alpha_0 = \alpha_1$ we see that there are 12 elements in $C(h)$ having 0 dimensional fixed point set. Since the quotient in case of 0-dimensional fixed point sets has euler characteristic 2 and $h$ acts trivially on it we see that the contribution of $g \neq id$ in the expression (1.1) for $L(h, V/G)$ is $2 \times 12 + 2 \times 12 = 48$. Let us consider the contribution of the identity element. The cohomology of $V/G$ can be identified with the $G$-invariant part of the cohomology of $V$. Hence the even dimensional cohomology of $V/G$ in each
dimension has rank 1. In dimension 3 the $G$-invariant part of $H^3(V,\mathbb{C})$ is generated by the residues of meromorphic forms

$$\omega_\lambda = \frac{\Sigma(-1)^i x_i dx_0 \wedge \ldots \wedge dx_i}{Q(\lambda, x_0, x_1, x_2, x_3, x_4)} , \frac{d\omega_\lambda}{d\lambda} , \frac{d^2\omega_\lambda}{d\lambda^2} , \frac{d^3\omega_\lambda}{d\lambda^3}$$

(3.3)

via Griffiths theory (cf.[COGP], [M]). These forms are clearly $h$-anti-invariant. Hence $L(h, V^{id}/G) = 8$. Therefore $L(h, \tilde{V}/G) = 8 + 48 = 56$. On the other hand $L(h, V) = e(V^h)$. The fixed point set of $h$ acting in $V$ consists of the point $(1, -1, 0, 0, 0)$ and the points of the quintic in the hyperplane $x_0 = x_1$. i.e. a non singular quintic surface. Its euler characteristic is $(3 - (-10)) \times 5 + (-10) = 55$ (since quintic surface in $\mathbb{P}^3$ is a 5-fold cyclic cover of projective plane branched over a plane quintic having genus 6). Hence we obtain: $L(h, V) = L(h, \tilde{V}/G)(= 56)$.

Now let us consider the action on the quintic (3.1) of the transformation:

$$h : (x_0, x_1, x_2, x_3, x_4) \rightarrow (x_1, x_0, x_2, x_4, x_3)$$

(3.4)

Here $C(h)$ consists of elements (3.2) for which $\alpha_0 = \alpha_1, \alpha_3 = \alpha_4$. There are 4 such elements different from identity and the Lefschetz number of $h$ acting on the quotient of their fixed point sets is equal to 2. The contribution of the identity element now is zero since now the action of $h$ on forms (3.3) is trivial. On the other hand the fixed point set of (3.4) consists of the union of the line $(x_0, -x_0, 0, x_3, -x_3)$ and the plane quintic curve which is the intersection of (3.1) and $x_0 = x_1, x_3 = x_4$ (which has genus 6). Hence $L(h, V) = -L(h, \tilde{V}/G)(= -8)$.

Finally let us consider example from [LT] in which $V$ is a complete intersection on $\mathbb{P}^5$ given by equations:

$$x_1^3 + x_2^3 + x_3^3 = 3x_4x_5x_6,$$

$$x_4^3 + x_5^3 + x_6^3 = 3x_1x_2x_3$$

(3.5)

with the group $G_{81}$ of order 81 acting on (3.5) as follows:

$$(x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (\zeta_3^{\alpha_1}\zeta_9^\mu x_1, \zeta_3^{\alpha_1}\zeta_9^\mu x_2, \zeta_9 x_3, \zeta_3^{\alpha_2}\zeta_9^{-\mu} x_4, \zeta_3^{\alpha_3}\zeta_9^{-\mu} x_5, \zeta_9^{-\mu} x_6)$$

(3.6)

where $\alpha_i \in \mathbb{Z}_3, i = 1, 2, 3, 4, \mu \in \mathbb{Z}_3$ and $\mu \equiv \alpha_1 + \alpha_2 \equiv \alpha_4 + \alpha_5 \mod 3$. Let us consider the following involution:

$$h : (x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (x_2, x_1, x_3, x_5, x_4, x_6)$$

(3.7)

The condition $g \in C(h)$ implies that in (3.6) we have $\alpha_1 = \alpha_2, \alpha_4 = \alpha_5$ i.e. $g \in C(h)$ must have the form:

$$(x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (\zeta_3^{\alpha_1}\zeta_9^\mu x_1, \zeta_3^{\alpha_1}\zeta_9^\mu x_2, \zeta_9 x_3, \zeta_3^{-\alpha_2}\zeta_9^{-\mu} x_4, \zeta_3^{-\alpha_3}\zeta_9^{-\mu} x_5, \zeta_9^{-\mu} x_6)$$

(3.8)

where $\alpha \in \mathbb{Z}_3, \mu \in \mathbb{Z}_9$ and $2\alpha = \mu \mod 3$. Hence we have 9 elements in $C(h)$ out of which 8 non identity elements have zero dimensional fixed point set. The Lefschetz number of
on the quotient of each of these zero dimensional fixed point sets by $G_{81}$ is equal to 2. For $g = id$ the contribution in (1.1) is the Lefschetz number $L(h, V^{id}/G_{81})$ which is equal to zero. This follows from explicit expression for the forms representing $G_{81}$-invariant cohomology classes on complete intersection (3.5) (cf. [LT] (8) on p.32) as was done above in the case of quintic. Hence $L(h, V/\tilde{G}_{81}) = 0 + 2 \times 8 = 16$. On the other hand the fixed point set of $h$ acting on (3.5) consists of the line: $(x, -x, 0, y, -y, 0)$ and the intersection of complete intersection (3.5) with the $x_1 = x_2, x_4 = x_5$ has the Euler characteristic $-18$. Hence $L(h, V) = -L(h, V/\tilde{G}_{81})$.

One can wonder if there is a physical reason for these simple equalities $L(h, V) = -\text{sign}(h)L(h, \tilde{V}/G)$ between Lefschetz numbers of automorphisms of a Calabi Yau manifold and its mirror which came out in these examples.

### 4. Actions on resolutions of 2-dimensional singularities

First we shall consider several explicit examples (cf. [Sl]). Let $G = \mathbb{Z}_n$ be a cyclic subgroup of the torus of $SL_2(\mathbb{C})$ consisting of the matrices of the form:

$$
\begin{pmatrix}
\omega^a_n & 0 \\
0 & \omega^b_n
\end{pmatrix}
$$

(4.1)

where $a + b \equiv 0 \mod n$. The matrix:

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

(4.2)

acts on $G$ by conjugation provided $n$ is even. The number of invariant conjugacy classes is 2 while only one component of the exceptional set of the resolution is $h$ invariant (cf. [Sl] p. 76). Hence the corresponding Lefschetz number is 2 and so $C^h = L(E, h)$.

If $\tilde{h}$ is given by

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

(4.3)

then the number of $\tilde{h}$-invariant elements of $G$ is 1 (resp. 2) if $n$ is even (resp. odd). The action on the resolution has 1 or 2 fixed point in respective cases.

Let us consider the case of binary dihedral group corresponding to the Dynkin diagram $D_r$. Recall that $D_r$ is a $\mathbb{Z}_2$ extension of $\mathbb{Z}_{2(r-2)}$ which is the subgroup of $SL_2(\mathbb{C})$ generated by the cyclic group $G$ given by matrices (4.1) with $n = 2(r-2)$ and (4.2). Then $D_{2r-2}$ contains $D_r$ as a normal subgroup of index 2. Let $h$ be the nontrivial element of the quotient and let $C$ be $h$-invariant subgroup of $G$ (CardC = 2). $D_r$ has CardC conjugacy classes having 1 element (i.e., the elements of $C$), Card$G$ – Card$C$/2 classes containing 2 elements (of set $G - C$) and 2 conjugacy classes union of which forms the coset of $G$ in $D_r$. Among these $r + 1$ conjugacy classes $r - 1$, corresponding to those in first two groups are, invariant under the action of the group $\mathbb{Z}_{4(r-2)}/\mathbb{Z}_{2(r-2)}$ (all cyclic groups are groups of matrices as above) i.e., we obtain $r - 1$ invariant conjugacy classes. The action of the non
trivial element \( \tilde{h} \) of the latter on the resolution of \( C^2/D_r \) fixes the chain on \( r-2 \) rational curves (cf. [Sl] p.76). Hence the Lefschetz number is equal to \( r-1 \) i.e. the number of invariant conjugacy classes is \( L(h) \).

In the case of binary tetrahedral group (i.e. the extension of quaternionic group by \( \mathbb{Z}_3 \)) out of 7 conjugacy classes 3 are invariant. In the case \( D_4 \) one has additional automorphisms of order 3. It leaves invariant 2 conjugacy classes and acts on the resolution leaving fixed one component of the Dynkin diagram.

In general one can deduce the equivariant version of McKay correspondence (**) from the geometric description of McKay correspondence due to Gonzalez-Springer and Verdier (cf. [GSV]). The number of \( h \)-invariant conjugacy classes can be identified with the number of \( h \)-invariant representations of each of binary polyhedral group \( B \). Since the extension of the bundle on the minimal resolution of \( C^2/B \) corresponding to an \( h \)-invariant representation of \( B \) will be \( h \)-invariant, its first Chern class will be invariant i.e. corresponding exceptional set will be invariant and the conclusion follows since the traces on \( H^0 \) and \( H^2 \) of the resolution are obvious.

5. Symmetries of resolution of abelian quotient singularities

Let \( X \) be a toric variety i.e. a torus \( T \) acts on \( X \) and action of \( T \) on one of the orbits (which we shall denote \( T \)) is transitive. Let \( h \) be a biregular automorphism of \( X \) normalizing \( T \). Then \( h \) acts on the torus \( T \) as follows: \( h(t) \) is the unique element of the torus which takes any point of \( v \) of \( T \) into \( h^{-1}(v) \).

Now let \( T \) be the maximal torus of \( SL_n(\mathbb{C}) \) consisting of diagonal matrices having traces equal to 1. Let \( H \) be an abelian subgroup of \( T \). Let \( h \) be an element of \( GL_n(\mathbb{C}) \) which normalizes \( T \) and \( H \). \( h \) can be viewed as an automorphism of both \( T \) and \( H \). \( h \) also acts on the quotient \( C^n/H \). The latter is a toric variety in a natural way and the lattices of 1-parameter subgroups \( M \) and \( N \) of the dense tori of \( C^n \) and \( C^n/H \) are related by the following sequence:

\[
0 \to M \to N \to H \to 0 \tag{5.1}
\]

Let \( l \) be the order of \( h \) acting of the lattice \( M \) (or \( N \)). The quotient of the normalizer of \( T \) in \( GL_n(\mathbb{C}) \) is the symmetric group \( S_n \). Let \((l_1, \ldots, l_k)\) be the sequence of lengths of cycles of the permutation (of characters given by coordinates) defined by \( h \). One has \( l = \text{l.c.m.}(l_1, \ldots, l_k) \).

Let \( \Delta \) be the unit simplex \( \{(x_1, \ldots, x_n) \subset M \otimes \mathbb{R} | x_1 + \ldots + x_n \leq 1 \} \). The only face of this simplex which does not belong to any coordinate plane will be called the base of this unit simplex.

We are going to construct standard triangulations of certain simplices which will be used below. Let \( M = \mathbb{Z}^{l_1} \oplus \ldots \oplus \mathbb{Z}^{l_p} \oplus \mathbb{Z}^{m_1} \oplus \ldots \oplus \mathbb{Z}^{m_q} \) and let \( e^i_j \) \((i = 1, \ldots, p, j = 1, \ldots, l_i \) or \( i = 1, \ldots, q, j = 1, \ldots, m_j)\) be the standard generators of each direct summands. Let the action of \( g \) on \( M \) be given by the cyclic permutation of the vectors of the standard basis of each summand: \( g(e^i_j) = e^{i}_{s_i(j)} \) where \( s_i \) is the cyclic permutation of the integers...
1, ..., l_i or 1, ..., m_i. Let \( L \) be the \( k + \sum_{i=1}^{t} m_i \)-dimensional subspace given by the equations \( x_a^i = x_b^i \) (i = 1, ..., k, 1 ≤ a, b ≤ l_i). The volume of the simplex \( L \cap \Delta \) is \( \frac{1}{l_1 \cdots l_k} \cdot \frac{1}{\dim L!} \).

Let us consider the following triangulation of \( \Delta \) by simplices of the form \( \Delta(r_1, ..., r_k) = (\ldots, \hat{X}_j(r_i), \ldots, a_0, \ldots, a_k) \) where \( \hat{X}_j(r_i) \) is the collection of points all coordinates of which but one are zeroes and nonzero coordinate is 1 corresponds a vector \( e_j \) where \( j \neq r_i \), \( a_0 \) is the origin and \( a_i \) (1 ≤ i ≤ k) are the vertices of \( \Delta \cap L \). The action of \( g \) on this triangulation is given by \( g((\ldots, \hat{X}_j(r_i), \ldots, a_0, \ldots, a_k)) = (\ldots, \hat{X}_j(s_i(r_i)), \ldots, a_0, \ldots, a_k) \). The pair \( (\Delta, \Delta \cap L) \) will be called the standard pair of type \( (l_1, \ldots, l_s | m_1, \ldots, m_t) \).

**Definition.** Let \( \Sigma \) be a simplex with the vertices of a lattice \( M \) on which \( g \in Aut M \) acts simplicially, \( T \) is an \( g \)-invariant triangulation of \( \Sigma \) and \( \Sigma^g \) be the fixed point set of \( g \). \( T \) is called standard if the pair \( (\Sigma, \Sigma^g) \) is isomorphic to \( (\Delta, \Delta \cap L) \) by an isomorphism preserving volume of simplices of dimension \( \dim \Sigma^g + 1 \).

**Remarks.** 1. The vertices of simplex \( \Sigma^g \) are not necessarily in \( M \).

2. The volume of each simplex is calculated according to the measure induced by the lattice in the linear subspace supporting this simplex.

The properties of \( g \)-standard simplices which will be used later are the following:

a) \( g \) acts on the vertices of a \( g \)-standard simplices by permutation with cycles of length \( (l_1, \ldots, l_s) \).

b) Codimension of \( \Sigma \cap L \) in \( \Sigma \) is \( \Sigma(m_1 - 1) \).

c) \( g \) acts on simplices of dimension \( \dim \Sigma \cap L + 1 \) as permutation having the lengths of cycles \( (m_1, \ldots, m_t) \).

d) The volume of \( \Sigma \cap L \) is \( \frac{1}{l_1 \cdots l_s} \cdot \frac{1}{\dim \Sigma^g!} \).

**Definition.** A triangulation of a simplex is \( g \)-adjusted if it is a refinement of a triangulation in which each \( g \)-invariant simplex is \( g \)-standard.

**Theorem 2.** If there exist an \( h \)-adjusted triangulation of the unit simplex \( \Delta \) in \( M \) such that each simplex of triangulation has vertices in \( N \cap \Delta \) and such that the volume of each simplex relative to the lattice induced by \( N \) on the linear subspace supporting its simplex is 1 then there exist a \( h \)-invariant crepant resolution of \( \mathbb{C}^n/H \) such that the Lefschetz number of \( h \) acting on the latter is equal to the order of the group of \( h \)-invariant elements of \( H \).

**Proof.** Since any triangulation of \( b\Delta \) with vertices in \( N \) induces a crepant resolution (cf. [R], [BD]), we obtain a crepant resolution from an \( h \)-adjusted triangulation mentioned in the statement. Let \( L(h) \) be the Lefschetz number of \( h \) acting on a chosen resolution, \( CardH^h \) be the number of \( h \)-invariant elements of \( H \) and \( (l_1, \ldots, l_k) \) be the sequence of greater than 1 lengths of cycles of permutation corresponding to \( h \). Let \( L \) be the subspace of \( M \otimes \mathbb{R} \) of elements fixed by \( h \). We claim that one has

1. \( L(h) = l_1 \cdots l_k \cdot vol_{N \cap b\Delta}(L \cap b\Delta) \cdot k! \)
2. \( \text{vol}_{N \cap b\Delta} = \frac{1}{l_1 \cdots l_k} \cdot \frac{1}{k!} CardH^h \).

Clearly the theorem follows from 1 and 2.
Let us first calculate the Lefschetz number of $h$ acting on the resolution. $h$ acts freely outside of the union of tori corresponding to the simplices which intersect $L$, since any simplex fixed by $h$ contains a fixed point of $h$ and $L$ is the total set of fixed points of $h$. Suppose that a simplex $\sigma$ is $h$ invariant and not in the closure of an invariant simplex. Then since triangulation is $h$-adjusted, its vertices permuted by a permutation consisting of cycles of lengths $(l_i, \ldots, l_k)$ and the union $B_\sigma$ of certain simplices represent a collection of simplices of dimension $dim \sigma + 1$ permuted by permutation with cycles with sequence of lengths complementary to $(l_i \cdot \ldots \cdot l_k)$. Let $A_\sigma$ be the union of these simplices and let $A_\sigma$ and $B_\sigma$ be the corresponding toric varieties. The Lefschetz number of $h$ acting on both $A_\sigma$ and $B_\sigma$ is equal to the Lefschetz number of $h$ acting on the torus corresponding to $\sigma$. On the other hand viewing the torus $T_\sigma$ as a subset of $B_\sigma$ shows that the matrix of $h$ acting on the $H_1(T_\sigma, \mathbb{Z})$ is formed by the blocks $A_{l_1 \cdot \ldots \cdot l_k}$ where $A_\sigma$ is the following $s \times s$ matrix:

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \ldots & \ldots & -1
\end{pmatrix}
$$

If $s > 1$ then $det(I - A_\sigma) = s + 1$ (and 0 for $s = 1$). If $\phi$ is transformation of a torus $T$ and $\phi_\sigma$ is the corresponding automorphism on $H_1(T, \mathbb{Z})$ then the Lefschetz number of $\phi$ is equal to $det(I - \phi_\sigma)$ (since $H^*(T, \mathbb{Z}) = \Lambda^*(H_1(T, \mathbb{Z})$ and the eigenvalues of the map induced by $\phi$ on $\Lambda^k H_1(T, \mathbb{Z})$ are the elementary symmetric functions of the eigenvalues of $\phi$ acting on $H_1(T, \mathbb{Z})$). Hence the Lefschetz number of $h$ acting on $T_\sigma$ is $l_1 \cdot \ldots \cdot l_k$. This calculation implies that the total Lefschetz number of $h$ is

$$
\frac{\sum_{l_1 \cdot \ldots \cdot l_k} r_{l_1 \cdot \ldots \cdot l_k}}{
\frac{1}{l_1 \cdot \ldots \cdot l_k} r_{l_1 \cdot \ldots \cdot l_k}
} $$

where $r_{l_1 \cdot \ldots \cdot l_k}$ is the total number of $h$ adjusted simplices of type $(l_i, \ldots, l_s)$.

On the other hand the volume of $L \cap b\Delta$ is equal to

$$
\frac{1}{k!} \frac{1}{l_1 \cdot \ldots \cdot l_k} r_{l_1 \cdot \ldots \cdot l_k}
$$

since for a $h$ standard simplex $\Sigma$ of type $(l_i, \ldots, l_s)$ for which $dim \Sigma \cap L = k$ is equal to $\frac{1}{l_1 \cdot \ldots \cdot l_k}$. Hence the claim 1 above follows.

The number $Card H^h$ is the volume of the unit parallelepiped of lattice $M \cap L$ measured relative to the lattice $N \cap L$. It is clear from direct calculation that $vol L \cap b\Delta = 1/l_1 \cdot \ldots \cdot l_k vol \Delta_M$. This shows 2.

**Remark.** Let us consider an orbit corresponding to a simplex $\sigma$ in the closure of an invariant simplex $\tau$. We claim that the Lefschetz number of $h$ acting on the orbit corresponding to $\sigma$ is equal to zero. We shall show that the orbit corresponding to $\sigma$ contains one dimensional orbit fixed by $h$ pointwise. We can assume that $dim \sigma = dim \tau - 1$. Let $X_\sigma$ be the closure in $X$ of the orbit corresponding to $\sigma$. Then $h$ acts on $X_\sigma$ and the latter contains $h$ invariant codimension one orbit (the orbit corresponding to $\tau$).
Let us consider the fan corresponding to this toric variety and an orbit \( \phi \) corresponding to the point of the lattice which belong to \( h \)-invariant ray corresponding to \( h \) invariant codimension one orbit. This ray is fixed by \( h \) and hence the lattice point on it is fixed by \( h \). We claim that this one dimensional orbit is fixed pointwise. Indeed \( h \) is a transformation which fixes limit of this orbit when \( t \to 0 \) (note that \( h \) is either the identity or sends \( x \to x^{-1} \)).

**Theorem 3.** An \( h \)-adjusted triangulation of the unit simplex exist in dimension 3. In particular the conclusion of theorem 2 is always true in dimension 3.

**Proof.** The order of a nontrivial automorphism \( h \in S_3 \) is either 2 or 3. If \( ordh = 2 \), \( h \) fixes vertex \( P \) and the intersection of \( L \) with the side \( OP \) of \( b\Delta \) opposite to \( P \) belongs to the lattice \( N \), then we can split \( b\Delta \) by \( L \) into the union of two triangles, \( T \) and \( hT \), take a triangulation of \( T \) in which all vertices are in \( N \cap b\Delta \) and have area 1 and then take \( h \) image of this triangulation in \( hT \). If \( ordh = 2 \) but \( L \cap OP \) is not in \( N \), consider the triangle \( T_1 \) of area one with vertex in the point \( L \cap N \) closest to \( OP \) and two vertices of \( N \) on \( OP \), then triangulate \( T - T \cap T_1 \) by triangles of area 1 with vertices in \( N \) and take its \( h \) image to triangulate \( hT - hT \cap T_1 \). Now any segment along \( L \cap b\Delta \) is \( h \)-standard and \( T_1 \) is \( h \)-standard of type (2). Hence the triangulation is \( h \)-adjusted.

If \( ordh = 3 \) then one takes triangulation of one of the triangles formed by \( L \cap b\Delta \) one then extends it to triangulation of \( b\Delta \) using \( h \).

**Example.** Let us consider the action of subgroup \( \mathbb{Z}_5^2 \) of \( T \) consisting of matrices of the form:

\[
\begin{pmatrix}
\omega_5^a & 0 & 0 \\
0 & \omega_5^b & 0 \\
0 & 0 & \omega_5^c
\end{pmatrix}
\]

where \( a + b + c \equiv 0 \mod 5 \) where \( \omega_5 \) is a root of unity of degree 5. \( h \in GL_3(\mathbb{C}) \) given by

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

induces cyclic permutation of the entries of the matrices of \( T \). The Lefschetz number of the automorphism of the resolution of \( \mathbb{C}^3/\mathbb{Z}_5^2 \) constructed above is equal to 1 (coming from the only invariant simplex which the cone over triangle having \((1/3, 1/3, 1/3)\) in its center and for which the length of intersection with \( L \) is 1/3) and the only \( h \) invariant element of \( \mathbb{Z}_5^2 \) is the identity.

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