IRREDUCIBLE MODULES OF TOROIDAL LIE ALGEBRAS ARISING
FROM $\phi$-COORDINATED MODULES OF VERTEX ALGEBRAS

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Abstract. In this paper, for every $\epsilon \in \mathbb{Z}$, we introduce an extension of the 2-toroidal Lie algebra by certain derivations. Based on the $\phi$-coordinated modules theory for vertex algebras, we give an explicit realization of a class of irreducible highest weight modules for this extended toroidal Lie algebra. When $\epsilon = 1$, this affords a realization of certain irreducible modules for the toroidal extended affine Lie algebras first constructed by Billig.

1. Introduction

Toroidal Lie algebras are natural multi-variable generalizations of affine Lie algebras. One starts with a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and forms the multi-loop algebra $R \otimes \mathfrak{g}$, where $R = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$ is the ring of Laurent polynomials in the variables $t_0, t_1, \ldots, t_N$. The $(N + 1)$-toroidal Lie algebra $\mathfrak{t}(\mathfrak{g})$ is by definition the universal central extension of $R \otimes \mathfrak{g}$ [FM, MR, Y]. When $N = 0$, this yields the usual affine Lie algebra. In the literature, various extensions of $\mathfrak{t}(\mathfrak{g})$ obtained by adding certain subalgebras of $\text{Der}(R)$ (the Lie algebra of derivations over $R$) have been studied extensively. For examples, the subalgebra $\sum_{i=0}^{N} \mathbb{C}d_i := \left\{ t_i \frac{\partial}{\partial t_i} \right\}$ of degree-zero derivations [E1, E2], the subalgebra $\mathbb{C}d_0 \oplus \cdots \oplus \mathbb{C}d_N$ [EM, FM, BB, JM], the subalgebra of divergence zero (or skew) derivations [B2, CLT1, ESB, CLT2], and the algebra $\text{Der}(R)$ itself [B1, EJ].

In this paper we consider a sequence of new extensions

$$\mathfrak{t}(\mathfrak{g})^\epsilon = \mathfrak{t}(\mathfrak{g}) \oplus \mathfrak{D}^\epsilon \quad (\epsilon \in \mathbb{Z})$$

of the 2-toroidal Lie algebra $\mathfrak{t}(\mathfrak{g})$, where

$$\mathfrak{D}^\epsilon = \left\{ f_0 t_0 \frac{\partial}{\partial t_0} + f_1 d_1 \mid f_0, f_1 \in \mathcal{R}, \; t_0 \frac{\partial}{\partial t_0} (f_0) + d_1 (f_1) = 0 \right\}$$

is a Lie subalgebra of $\text{Der}(\mathcal{R})$. The Lie algebras $\mathfrak{D}^\epsilon$ are closely related to the Lie algebras $\mathfrak{B}(q)$, $q \in \mathbb{C}$ of Block type studied in [CGZ, SXX1, SXX2, XZ]. In fact, when $q = \epsilon$, $\mathfrak{B}(q)$ is a “half” of $\mathfrak{D}^\epsilon$, which consists of those derivations in (1.1) satisfying the condition that $f_0, f_1$ are spanned by the monomials $t_0^{m_0} t_1^{m_1}$ with $m_0, m_1 \in \mathbb{Z}$ and $m_1 \geq 0$. In particular, when $\epsilon = 1$, $\mathfrak{D}^\epsilon$ is the algebra of divergence zero derivations and $\mathfrak{t}(\mathfrak{g})^\epsilon$ is a nullity 2 extended affine Lie algebra in the sense of [AABGP], which is commonly called toroidal extended affine Lie algebras [B2].

One of the most important classes of modules for affine Kac-Moody algebras is the highest weight modules. Their analogs for various extensions of toroidal Lie algebras have been extensively studied in [B1, B2, BB, BBS, CLT1, CLT2, EM, EJ, E], [ESB, FM, JM]. In this paper we will give an explicit realization of a class of irreducible highest weight $\mathfrak{t}(\mathfrak{g})^\epsilon$-modules. When $\epsilon = 1$, this construction have been previously obtained by Billig in [B2].

The key ingredient in our realization is the theory of $\phi$-coordinated modules for vertex algebras, which was first introduced in [Li] for the purpose of associating quantum affine algebras with
Proposition 4.9, we deduce in Section 5.1 the following three correspondences:

\[ \phi(w,0) = w \quad \text{and} \quad \phi(w, z_1), z_2) = \phi(w, z_1 + z_2). \]

It was proved therein that for every \( p(w) \in \mathbb{C}(w) \), \( \phi(w, z) = e^{z(p(w)\Theta)}(w) \) is an associate of \( F(z, w) \) and every associate is of this form. When \( p(w) = 1 \), the \( \phi \)-coordinated modules are just the usual modules. When \( p(w) = w^t \) with \( t \in \mathbb{Z} \), set

\[ \phi_t(w, z) = e^{z(w^t \Theta)}(w). \]

The structure of \( \phi_t \)-coordinated modules for vertex algebras was further studied in [BLP]. In particular, a Jacobi-like identity for \( \phi_t \)-coordinated modules was established therein.

Now we give an outline of the structure of this paper. In Section 2, we first introduce the extended toroidal Lie algebra \( \hat{\mathfrak{g}}^{(t)} \) and give the commutators in terms of certain generating functions on \( \hat{\mathfrak{g}}^{(t)} \) (see Proposition 2.3). We will work in a more general framework with \( \mathfrak{g} \) an arbitrary Lie algebra equipped with an invariant symmetric bilinear form. Meanwhile, in the process of adding \( \mathcal{D}^t \) to \( \mathfrak{g} \), an abelian two-cocycle depending on any complex number \( \mu \) will be involved.

In Section 3, we first recall some facts about affine-Virasoro algebras \( \hat{\mathfrak{g}} \times \mathfrak{Vir} \) associated to \( \mathfrak{g} \). Then we give the first main result of this paper in Theorem 3.2. Namely, we construct a class of irreducible \( \hat{\mathfrak{g}}^{(t)} \)-modules

\[ L_{\mathfrak{g}\mathfrak{Vir}}(\ell, 24\mu t - 2, U, \beta) \otimes V_{\mathfrak{g}}^\mathfrak{Vir}(\ell, e^{\alpha \mathfrak{C}}[L]) \]

where \( \alpha, \beta, \ell \in \mathbb{C} \) with \( \ell \neq 0 \), \( U \) is an irreducible \( \mathfrak{g} \)-module, \( L_{\mathfrak{g}\mathfrak{Vir}}(\ell, 24\mu t - 2, U, \beta) \) is an irreducible highest weight module for the affine-Virasoro algebra \( \hat{\mathfrak{g}} \times \mathfrak{Vir} \), and \( V_{\mathfrak{g}}^\mathfrak{Vir}(\ell, e^{\alpha \mathfrak{C}}[L]) \) is a highest weight module for the Heisenberg algebra \( \hat{\mathfrak{h}} \) associated to \( \mathfrak{g} \) and \( \mathfrak{Vir} \). The second main result of this paper is a characterization of these irreducible \( \hat{\mathfrak{g}}^{(t)} \)-modules in terms of the highest weight modules (see Theorem 3.4).

Sections 4 and 5 are devoted to the proof of Theorem 3.2. In Section 4 we first recall some results on \( \phi_t \)-coordinated modules. Let \( V \) be a vertex algebra. A fundamental property for a \( V \)-module \((W, Y_W)\) that for \( u, v \in V \), \( Y_W(u, v, z) \) is just the normally ordered product \( \frac{d}{dz}Y_W(u, z)Y_W(v, z) \) (see [LL] for example). We prove in Proposition 4.6 that for a \( \phi_t \)-coordinated \( V \)-module \((W, Y_W^t)\),

\[ Y_W(u_{-1}v, z) = Y_W^t(u, z)Y_W^t(v, z) + \sum_{n \geq 0} c_n z^{(t-1)(n+1)}Y_W^t(u, v, z), \]

where \( c_n \) are some explicitly determined complex numbers. We also determine the \( \phi_t \)-coordinated modules for vertex algebras arising from vertex Lie algebras introduced in [DLM] [K2] [P] (see Proposition 4.9).

In Section 5, we first introduce a subalgebra \( \hat{\mathfrak{g}}^{(t)} \) of \( \hat{\mathfrak{g}}^{(t)} \) such that \( \hat{\mathfrak{g}}^{(t)} = \hat{\mathfrak{g}}^{(t)} \otimes \mathfrak{C}_\ell^{-1} d_1 \). When \( \ell = 0 \), \( \hat{\mathfrak{g}}^{(0)} \) is a vertex Lie algebra and therefore there is a vertex algebra \( V_{\hat{\mathfrak{g}}^{(t)}}(\gamma \ell) \) associated to \( \hat{\mathfrak{g}}^{(t)} \). Based on a result of [B1], it was proved in [CLT] that there is a surjective vertex algebra homomorphism

\[ \Theta : V_{\hat{\mathfrak{g}}^{(t)}}(\gamma \ell) \rightarrow V_{\hat{\mathfrak{g}}\mathfrak{Vir}}(\ell, 24\mu t - 2) \otimes V_{(\mathfrak{h}, L)} \]

where \( V_{\hat{\mathfrak{g}}\mathfrak{Vir}}(\ell, 24\mu t - 2) \) is the affine-Virasoro vertex algebra and \( V_{(\mathfrak{h}, L)} \) is the lattice vertex algebra associated to the pair \((\mathfrak{h}, L := 2\mathfrak{k}) \) [B1] [LW]. Note that \( V_{(\mathfrak{h}, L)} \) can be realized as a quotient of the affine vertex algebra associated to an affine Lie algebra \( \hat{\mathfrak{p}} \) [LW]. By applying Proposition 4.9 we deduce in Section 5.1 the following three correspondences:
The second correspondence implies that there is an irreducible \( \phi_\ell \)-coordinated \( \hat{V}_{\hat{g}} \times \hat{\text{Vir}}(\ell,24\mu_\ell-2) \)-module structure on the \( \hat{g} \times \hat{\text{Vir}} \)-module \( L_{\hat{g} \times \hat{\text{Vir}}}(\ell,24\mu_\ell-2,U,\beta) \) (see Proposition 5.3.2). Meanwhile, the third correspondence allows us to give an irreducible \( \phi_\ell \)-coordinated \( V_{(b,L)} \)-module structure on the \( \hat{h} \)-module \( \hat{V}_h(\ell,e^{ak}C[L]) \) (see Proposition 5.3.9). Thus, the tensor product space becomes an irreducible \( \phi_\ell \)-coordinated \( \hat{V}_{\hat{g}} \times \hat{\text{Vir}}(\ell,24\mu_\ell-2) \otimes V_{(b,L)} \)-module. Via the homomorphism \( \Theta \), this \( \phi_\ell \)-coordinated \( \hat{V}_{\hat{g}} \times \hat{\text{Vir}}(\ell,24\mu_\ell-2) \otimes V_{(b,L)} \)-module carries a canonical (irreducible) \( \phi_\ell \)-coordinated \( \hat{V}_{\hat{g}} \times \hat{\text{Vir}}(\ell,24\mu_\ell-2) \)-module structure. Then by applying the first correspondence above, the space \( \hat{V}(g)^{\ell} \)-module structure on the \( \hat{g} \)-module \( \hat{V}_{\hat{g}}(\ell,\beta) \) becomes an irreducible \( \hat{V}(g)^{\ell} \)-module structure. This irreducible \( \hat{V}(g)^{\ell} \)-module can be extended to a \( \hat{V}(g)^{\ell} \)-module (see Theorem 5.7). And, based on the formula (1.3), we are able to compute the module action explicitly, which is exactly what we state in Theorem 3.2.

In the rest of this paper, we let \( \mathbb{Z}, \mathbb{Z}^\times, \mathbb{N}, \mathbb{C} \) and \( \mathbb{C}^\times \) be the sets of integers, nonzero integers, nonnegative integers, complex numbers and nonzero complex numbers, respectively. And, let \( z, w, z_0, z_1, z_2, \ldots \) be mutually commuting independent formal variables. For a vector space \( W, W([z_1, z_2, \ldots, z_r]) \) denotes the space of formal (possibly doubly infinite) power series in \( z_1, z_2, \ldots, z_r \) with coefficients in \( W \), and \( W((z_1, z_2, \ldots, z_r)) \) denotes the space of lower truncated Laurent power series in \( z_1, z_2, \ldots, z_r \) with coefficients in \( W \).

2. Toroidal Lie algebras

In this section we introduce the extended toroidal Lie algebras we concern about in this paper.

2.1. Toroidal Lie algebras \( \hat{V}(g)^{\ell} \). Let \( \mathcal{R} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}] \) be the Laurent polynomial ring in the variables \( t_0 \) and \( t_1 \). Denote by

\[ \Omega^1_{\mathcal{R}} = \mathcal{R}dt_0 \oplus \mathcal{R}dt_1 = \mathcal{R}k_0 \oplus \mathcal{R}k_1 \]

the space of 1-forms on \( \mathcal{R} \), where \( k_0 := t_0^{-1}dt_0 \) and \( k_1 := t_1^{-1}dt_1 \). Set

\[ d(\mathcal{R}) = \{ d(f) \mid f \in \mathcal{R} \}, \]

the space of exact 1-forms, where

\[ d(f) := \frac{\partial f}{\partial t_0} dt_0 + \frac{\partial f}{\partial t_1} dt_1 = t_0 \frac{\partial f}{\partial t_0} k_0 + t_1 \frac{\partial f}{\partial t_1} k_1 \in \Omega^1_{\mathcal{R}} \]

is the differential of \( f \). Form the quotient vector space

\[ \mathcal{K} = \Omega^1_{\mathcal{R}}/d(\mathcal{R}). \]

For \( m_0, m_1 \in \mathbb{Z} \), set

\[ k_{m_0, m_1} = \begin{cases} \frac{1}{m_1} t_0^{m_0} t_1^{m_1} k_0, & \text{if } m_1 \neq 0 \\ -\frac{1}{m_0} t_0^{m_0} t_1^{m_1} k_1, & \text{if } m_1 = 0, m_0 \neq 0 \\ 0, & \text{if } m_0 = m_1 = 0. \end{cases} \]

Then the set

\[ \{ k_0, k_1 \} \cup \{ k_{m_0, m_1} \mid m_0, m_1 \in \mathbb{Z} \text{ with } (m_0, m_1) \neq (0,0) \} \]

is a basis of \( \mathcal{K} \), noting that these elements in \( \Omega^1_{\mathcal{R}} \) should be understood as their images in \( \mathcal{K} \).
Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Form a central extension of the double loop algebra $\mathcal{R} \otimes \mathfrak{g}$:

$$t(\mathfrak{g}) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K},$$

called the 2-toroidal Lie algebra, where $\mathcal{K}$ is central and

$$\left[t_0^{m_0}t_1^{n_1}x, t_0^{m_0}t_1^{n_1}v\right] = \sum_{r=0,1} m_r t_0^{m_0}t_1^{n_1+k_r} \cdot \cdot \cdot $$

for $u, v \in \mathfrak{g}$, $m_0, n_0, m_1, n_1 \in \mathbb{Z}$. If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and $\langle \cdot, \cdot \rangle$ is nondegenerate, $t(\mathfrak{g})$ is known as a universal central extension of $\mathcal{R} \otimes \mathfrak{g}$ (see [MIN1]).

We denote by

$$\text{Der}(\mathcal{R}) = \mathcal{R} \frac{\partial}{\partial t_0} \oplus \mathcal{R} \frac{\partial}{\partial t_1} \oplus \mathcal{R} \frac{\partial}{\partial t_0} \oplus \mathcal{R} \frac{\partial}{\partial t_1}$$

the derivation Lie algebra on $\mathcal{R}$, where $d_0 := t_0 \frac{\partial}{\partial t_0}$ and $d_1 := t_1 \frac{\partial}{\partial t_1}$. By adding $\text{Der}(\mathcal{R})$ to $t(\mathfrak{g})$, we obtain the full 2-toroidal Lie algebra (see [B1])

$$\mathcal{T}(\mathfrak{g}) = t(\mathfrak{g}) \oplus \text{Der}(\mathcal{R}) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \text{Der}(\mathcal{R}),$$

with Lie relations

$$\left[t_0^{m_0}t_1^{n_1}d_i, t_0^{m_0}t_1^{n_1}x\right] = n_i \left[t_0^{m_0}t_1^{n_1}x, t_0^{m_0}t_1^{n_1}d_i\right],$$

$$\left[t_0^{m_0}t_1^{n_1}d_i, t_0^{m_0}t_1^{n_1}d_j\right] = n_i n_j \left[t_0^{m_0}t_1^{n_1}x, t_0^{m_0}t_1^{n_1}d_i\right].$$

for $x \in \mathfrak{g}$, $m_0, n_0, m_1, n_1 \in \mathbb{Z}$ and $i, j \in \{0, 1\}$, where $\mu$ is a fixed complex number throughout this paper. The algebra $\mathcal{T}(\mathfrak{g})$ is $\mathbb{Z}$-graded with respect to the adjoint action of $-d_0$, i.e.,

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(\mathfrak{g})(n),$$

where

$$\mathcal{T}(\mathfrak{g})(n) = \{x \in \mathcal{T}(\mathfrak{g}) \mid [d_0, x] = -nx\}.$$

For a given integer $\epsilon$, set

$$\mathcal{D}^\epsilon = \left\{f_0, f_1 \frac{\partial}{\partial t_0} + f_1 d_1 : f_0, f_1 \in \mathcal{R}, t_0 \frac{\partial}{\partial t_0} (f_0) + d_1 (f_1) = 0\right\} \subset \text{Der}(\mathcal{R}).$$

For $m_0, m_1 \in \mathbb{Z}$, write

$$\tilde{t}_0^{m_0}t_1^{m_1} = (m_0 - \epsilon + 1) t_0^{m_0}t_1^{m_1} d_0 - m_0 t_0^{m_0}t_1^{m_1} d_0,$$

which is an element of $\mathcal{D}^\epsilon$. A simple fact is that the set

$$\{t_0^{-1}d_0, t_0^{-1}d_1\} \cup \{\tilde{t}_0^{m_0}t_1^{m_1} \mid (m_0, m_1) \in (\mathbb{Z} \times \mathbb{Z}) \setminus \{(\epsilon - 1, 0)\}\}$$

is a basis of $\mathcal{D}^\epsilon$. Furthermore, $\mathcal{D}^\epsilon$ is a Lie subalgebra of $\text{Der}(\mathcal{R})$ with the relations

$$[t_0^{-1}d_0, \tilde{t}_0^{m_0}t_1^{m_1}] = (m_0 - \epsilon + 1) \tilde{t}_0^{m_0+\epsilon-1}t_1^{m_1}, \quad [t_0^{-1}d_1, \tilde{t}_0^{m_0}t_1^{m_1}] = m_1 \tilde{t}_0^{m_0+\epsilon-1}t_1^{m_1},$$

$$[\tilde{t}_0^{m_0}t_1^{m_1}, \tilde{t}_0^{m_0}t_1^{m_1}] = (m_0 - \epsilon + 1) \tilde{t}_0^{m_0+\epsilon-1}t_1^{m_1},$$

where $m_0, m_1, n_0, n_1 \in \mathbb{Z}$. Set

$$\tilde{t}(\mathfrak{g}) = t(\mathfrak{g}) \oplus \mathcal{D}^\epsilon \subset \mathcal{T}(\mathfrak{g}),$$

which is the Lie algebra we study in this paper.
In addition, for $m,n \geq 2$, the toroidal extended affine Lie algebra $[B_2, CLT_1]$ is an extended affine Lie algebra in the sense of $[AABGP]$, and is often referred as the nullity-2 finite-dimensional simple Lie algebra equipped with a Cartan subalgebra $H$. Let $u,v \in g, m_0, m_1, n_0, n_1 \in \mathbb{Z}$ with $(n_0, n_1) \neq (0,0)$ and $i = 0,1$. Furthermore, if $g$ is a finite-dimensional simple Lie algebra equipped with a Cartan subalgebra $H$, then the triple 
\[
tilde{(g)}^1, \tilde{H} := H + \mathbb{C}k_0 + \mathbb{C}k_1 + \mathbb{C}d_0 + \mathbb{C}d_1, (\langle \cdot, \cdot \rangle)
\]
is an extended affine Lie algebra in the sense of $[AABGP]$, and is often referred as the nullity-2 toroidal extended affine Lie algebra $[B_2, CLT_1]$.

2.2. Relations of generating functions on $\tilde{(g)}^\mu$. Consider the following subalgebra of $\tilde{(g)}^\mu$:
\[
\tilde{g}_1 = (\mathbb{C}[t_1, t_1^{-1}] \otimes \tilde{g}) \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_1,
\]
which is isomorphic to the affine Kac-Moody algebra associated to $g$ $[K1]$. For any $m \in \mathbb{Z}$ and $u = t_0^m \otimes x + bk_1 + cd_1 \in \tilde{g}_1$ with $x \in g$, $n \in \mathbb{Z}$ and $b,c \in \mathbb{C}$, we will often write
\[
t_0^n u := t_0^n t_1^m \otimes x + bt_0^nk_1 + ct_0^nd_1 \in \tilde{(g)}^\mu.
\]
In addition, for $m,n \in \mathbb{Z}$, set
\[
d_{n,m}^\mu := \tilde{d}_{n,m} + \mu(1 - \epsilon) \left( n + \frac{1}{2} (1 - \epsilon) \right) m^2 k_{n,m}.
\]
Let $B$ be a vector space with a designated basis $\{K_n, D_n \mid n \in \mathbb{Z}^\times\}$. Set
\[
B_g := \tilde{g}_1 \oplus B = (\mathbb{C}[t_1, t_1^{-1}] \otimes \tilde{g}) \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_1 \oplus \sum_{n \in \mathbb{Z}^\times} (\mathbb{C}K_n \oplus \mathbb{C}D_n).
\]

Define a linear map
\[
\psi^\epsilon_g : B_g \rightarrow \tilde{(g)}^\mu[[z, z^{-1}]], \quad a \mapsto a^\mu(z)
\]
by letting
\[
u^\mu(z) = \sum_{n \in \mathbb{Z}} (t_0^m u)z^{\epsilon - n - 1}, \quad D_{n}^\mu(z) = \sum_{n \in \mathbb{Z}} d_{n,m}^\mu z^{2\epsilon - n - 2}, \quad K_{n}^\mu(z) = \sum_{n \in \mathbb{Z}} k_{n,m} z^{-n},
\]
where $u \in \tilde{g}_1$, $m \in \mathbb{Z}^\times$. Recall that $k_{m,0} = -\frac{1}{m} t_0^m k_1$ for $m \in \mathbb{Z}^\times$ and $d_{n,0}^\mu = d_{n,0}^\mu = (n - \epsilon + 1)t_0^nd_1$ for $n \in \mathbb{Z}$. It follows from $[B1]$ and $[B2]$ that the coefficients of $a^\mu(z)$ for $a \in B_g$ together with the elements $k_0$ and $t_0^{-1}d_1$ linearly span the algebra $\tilde{(g)}^\mu$.

For $a \in \mathbb{C}$ and $r \in \mathbb{N}$, set
\[
a^{(r)}_\epsilon = \prod_{s=0}^{r-1} (a + s(\epsilon - 1)).
\]
The following result will be used later:

**Lemma 2.2.** Let $p \in \mathbb{N}$, $a, b, \alpha, \beta \in \mathbb{C}$. Then
\[
\sum_{r=0}^{p} \binom{p}{r} a^{p-r} a_c^{(p-r)}(-\beta)^r b_c^{(r)} = \sum_{s=0}^{p} \binom{p}{s} (a + \beta)^{p-s} a_c^{(p-s)}(-a - b - (p - 1)(\epsilon - 1))^{(s)}.
\]

**Proof.** Note that we have the following identity
\[
(a + b)^{(p)} = \sum_{i=0}^{p} \binom{p}{i} a_i^{(p)} b^{(p-i)},
\]

(2.6)
which can be proved by induction on \( p \). Using (5) and the facts
\[
\begin{align*}
\left( \frac{p-r}{t} \right) \left( \frac{p}{r} \right) & = \left( \frac{p}{r+t} \right) \left( \frac{r+t}{r} \right) \quad (0 \leq r \leq p, \ 0 \leq t \leq p-r), \\
\end{align*}
\]
and
\[
\begin{align*}
 a_{\delta}^{(p-r)} & = a_{\delta}^{(p-s)} (a + (p-s)(\epsilon - 1))^{(s-r)} \quad (0 \leq r \leq s \leq p),
\end{align*}
\]
we get
\[
\begin{align*}
\sum_{r=0}^{p} \left( \frac{p}{r} \right) a_{\delta}^{p-r} a_{\delta}^{(p-r)} (-\beta)^{r} b_{\delta}^{(r)} \\
= \sum_{r=0}^{p} \sum_{t=0}^{p-r} \left( \frac{p-r}{t} \right) \left( \frac{p}{r} \right) (a + \beta)^{p-r-t} a_{\delta}^{(p-r)} (-\beta)^{r+t} b_{\delta}^{(r)} \\
= \sum_{r=0}^{p} \sum_{t=0}^{p-r} \left( \frac{p}{r+t} \right) \left( \frac{r+t}{r} \right) (a + \beta)^{p-r-t} a_{\delta}^{(p-r)} (-\beta)^{r+t} b_{\delta}^{(r)} \\
= \sum_{s=0}^{p} \sum_{r=0}^{s} \left( \frac{p}{s} \right) \left( \frac{s}{r} \right) (a + \beta)^{p-s} a_{\delta}^{(p-s)} (-\beta)^{s} b_{\delta}^{(s)} \\
= \sum_{s=0}^{p} (a + \beta)^{p-s} a_{\delta}^{(p-s)} (a + (p-s)(\epsilon - 1))^{(s-r)} (-\beta)^{s} b_{\delta}^{(s)} \\
= \sum_{s=0}^{p} (a + \beta)^{p-s} a_{\delta}^{(p-s)} (-a - (p-1)(\epsilon - 1))^{(s)} b_{\delta}^{(s)}.
\end{align*}
\]

The following proposition collects the Lie relations among these generating functions.

**Proposition 2.3.** For \( u, v \in \mathfrak{g} \), \( a \in B_{\mathfrak{g}}, m, n \in \mathbb{Z} \) and \( k, l \in \mathbb{Z}^{\times} \), we have

1. \([t_{1}^{m} \otimes u \delta^{r} (z), (t_{1}^{n} \otimes v) \delta^{s} (w)] = (t_{1}^{m+n} \otimes [u, v]) \delta^{r+s} (z) z^{-1} \delta \left( \frac{w}{z} \right) + \delta_{r} \delta_{s} \delta_{m+n} \delta_{[u, v]} [u, v] m \left( w \frac{\partial}{\partial w} K_{m+n} (w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + \delta_{r} \delta_{s} \delta_{m+n} \delta_{[u, v]} [u, v] m \left( w \frac{\partial}{\partial w} K_{m+n} (w) \right) z^{-1} \delta \left( \frac{w}{z} \right)
\]
2. \([t_{1}^{m} \otimes u, t_{1}^{n} \otimes v] = 0 = [K_{m}^{\delta}, [t_{1}^{m} \otimes u, t_{1}^{n} \otimes v]],
\]
3. \([D_{k}^{\delta}, [t_{1}^{m} \otimes u, t_{1}^{n} \otimes v]] = k \left( w \frac{\partial}{\partial w} (t_{1}^{m+n} \otimes u) \delta^{r} (w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + (k + n) (t_{1}^{m+n} \otimes u) \delta^{r} (w) w \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right)
\]
4. \([D_{k}^{\delta}, [t_{1}^{m} \otimes u, t_{1}^{n} \otimes v]] = n(t_{1}^{m+n} \otimes u) \delta^{r} (w) z^{-1} \delta \left( \frac{w}{z} \right)
\]
5. \([k_{1}^{\delta}, k_{1}^{\delta} (w)] = [K_{m}^{\delta}, K_{n}^{\delta} (w)] = [k_{1}^{\delta}, K_{m}^{\delta} (w)] = 0,
\]
6. \([D_{k}^{\delta}, K_{k+l}^{\delta} (w)] = k \left( w \frac{\partial}{\partial w} K_{k+l}^{\delta} (w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + (k + l) K_{k+l}^{\delta} (w) w \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right)
\]
\[+ \delta_{k+1,0} \left( k k'_{i}(z) z^{-1} \delta \left( \frac{w}{z} \right) + k_0 w' \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right) \right),\]

(7) \[\left[ D'_{k}(z), k'_{i}(w) \right] = k w' \frac{\partial}{\partial w} \left( \left( w' \frac{\partial}{\partial w} K_k(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + K'_k(w) w' \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right) \right),\]

(8) \[\left[ d'_{i}(z), K'_n(w) \right] = nK'_n(w) z^{-1} \delta \left( \frac{w}{z} \right),\]

(9) \[\left[ d'_{i}(z), k'_i(w) \right] = k_0 z^{-1} \delta \left( \frac{w}{z} \right),\quad \left[ d'_i(z), d'_i(w) \right] = 0,

(10) \[\left[ D'_{k}(z), D'_{l}(w) \right] = k \left( \left( w' \frac{\partial}{\partial w} \right)^2 d'_{i}(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + (k + l) D'_{k+l}(w) u' \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right) \]

\[- k \delta_{k+l,0} \left( \left( w' \frac{\partial}{\partial w} \right)^2 d'_{i}(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + \mu \sum_{r=0}^{3} \delta \left( r \right) \left( \left( k w' \frac{\partial}{\partial w} \right)^r K'_{k+l}(w) \right) \left( \left( (k + l) w' \frac{\partial}{\partial w} \right)^{3-r} \right) z^{-1} \delta \left( \frac{w}{z} \right),\]

(11) \[\left[ d'_{i}(z), D'_{j}(w) \right] = lD'_{j}(w) z^{-1} \delta \left( \frac{w}{z} \right) + \mu k^3 K'_i(w) \left( w' \frac{\partial}{\partial w} \right)^2 z^{-1} \delta \left( \frac{w}{z} \right),\]

(12) \[\left[ t_{0}^{i-1}d_{0}, o'(z) \right] = -\varepsilon' \frac{d}{dz} o'(z),\]

where \( K'_{0}(z) \) and \( D'_{0}(z) \) are understood as the zero formal series.

**Proof.** We will only verify relation (10), which is the most complicated one. The verification of other relations are straightforward. Let \( i, j, k, l \in \mathbb{Z} \). Firstly, from (10) we have

\[
\left[ d'_{i, k}, d'_{j, l} \right] = ([i - \epsilon + 1] t_{0}^{i} t_{i}^{k} d_{1} - k t_{0}^{i} t_{i}^{k} d_{0}, (j - \epsilon + 1) t_{0}^{j} t_{j}^{l} d_{1} - l t_{0}^{j} t_{j}^{l} d_{0}] = ((i - \epsilon + 1) l - k(j - \epsilon + 1))(1 - \delta_{k+l,0})\tilde{d}_{i, j, k+l} - k \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2)t_{0}^{i+j} d_{1}
\]

\[+ \mu ((i - \epsilon + 1) l - k)(i - \epsilon + 1)k(i - j - 2 \epsilon + 2) \delta_{k+l,0}(k_{i+j,k+l} + 1) + \delta_{k+l,0}(k_{i+j,k+l} + 1),\]

\[= ((i - \epsilon + 1) l - k(j - \epsilon + 1))(1 - \delta_{k+l,0})\tilde{d}_{i, j, k+l} - \mu (1 - \epsilon)(i + j + \frac{1}{2}(1 - \epsilon))(k + l)^2 k_{i+j,k+l} + \frac{3}{\mu} \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2)\tilde{d}_{i, j, k+l}
\]

\[- k \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2)\tilde{d}_{i, j, k+l} + \mu (i - \epsilon + 1) l - k)(i - \epsilon + 1)k(i - j - 2 \epsilon + 2) \delta_{k+l,0}(k_{i+j,k+l} + 1) + \mu (i - \epsilon + 1) l - k)(i - \epsilon + 1)k(i - j - 2 \epsilon + 2) \delta_{k+l,0}(k_{i+j,k+l} + 1),\]

\[= ((i - \epsilon + 1) l - k(j - \epsilon + 1))(1 - \delta_{k+l,0})\tilde{d}_{i, j, k+l} - k \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2)\tilde{d}_{i, j, k+l}
\]

\[+ \mu (1 - \delta_{k+l,0}) C_{k, j}^{i, l} k_{i+j,k+l} + \mu \delta_{k+l,0}(i + j - \epsilon + 1)2 k^{2} i_{0}^{i+j} k_{1} + (1 - \epsilon)^{2} k^{2} i_{0}^{i+j} k_{0},\]

where \( C_{k, j}^{i, l} \) stands for the following number

\[
l^{3}(i - \epsilon + 1) \left( i^{2} - (1 - \epsilon)(i + j + \frac{1}{2}(1 - \epsilon)) \right)
\]

\[- k^{3}(j - \epsilon + 1) \left( j^{2} - (1 - \epsilon)(i + j + \frac{1}{2}(1 - \epsilon)) \right)
\]

\[- k l^{2} \left( 3(i - \epsilon + 1)i(j - \epsilon + 1) - (j + \frac{1}{2}(1 - \epsilon))(j + \epsilon - 1) \right)\]
+ k^2 l \left( 3(i - \epsilon + 1) i(j - \epsilon + 1) - (i + \frac{1}{2}(1 - \epsilon))(i + \epsilon - 1) \right).

Next, using the fact

\[ \tilde{d}_{i,k}^r, k_{j,l} = ((i - \epsilon + 1) l - k(j + \epsilon - 1)) k_{i+j,k+l}, \]

we obtain

\[
\begin{align*}
\tilde{d}_{i,k}^r, \mu(1 - \epsilon)(j + \frac{1}{2}(1 - \epsilon))k_{j,l} - \tilde{d}_{i,l}^r, \mu(1 - \epsilon)(i + \frac{1}{2}(1 - \epsilon))k^2 k_i,k & = \mu(1 - \epsilon)(1 - \delta_{k+l,0}) \left( (i - \epsilon + 1) l - k(j + \epsilon - 1) \right) (j + \frac{1}{2}(1 - \epsilon)) l^2 \\
- ((j - \epsilon + 1) k - (i + \epsilon - 1) l)(i + \frac{1}{2}(1 - \epsilon)) k_i+j,k+l & + \mu(1 - \epsilon) \delta_{k+l,0} \left( (j + \frac{1}{2}(1 - \epsilon)) l^2 ((i - \epsilon + 1) \delta_{i+j,0} k_0 + k t_{0}^{i+j} k_1) \right) \\
- (i + \frac{1}{2}(1 - \epsilon)) k^2 ((j - \epsilon + 1) \delta_{i+j,0} k_0 + l t_{0}^{i+j} k_1) & = \mu(1 - \epsilon)(1 - \delta_{k+l,0}) \left( (i - \epsilon + 1) l - k(j + \epsilon - 1) \right) (j + \frac{1}{2}(1 - \epsilon)) l^2 \\
- ((j - \epsilon + 1) k - (i + \epsilon - 1) l)(i + \frac{1}{2}(1 - \epsilon)) k_i+j,k+l & + \mu(1 - \epsilon) \delta_{k+l,0} (k^3(i + j - \epsilon + 1) t_{0}^{i+j} k_1 - k^2 i(l - \epsilon) \delta_{i+j,0} k_0) \right).
\end{align*}
\]

By Lemma 2.2 with \( p = 3 \), we have

\[
\sum_{r=0}^{3} \left( \begin{array}{c} 3 \\ r \end{array} \right) \beta^{-r} (i - \epsilon + 1) \epsilon^{3-r} (-k)^r (j - \epsilon + 1) \epsilon^{r} \]

\[
= \sum_{s=0}^{3} \left( \begin{array}{c} 3 \\ s \end{array} \right) (k + l)^{3-s} (i - \epsilon + 1) \epsilon^{3-s} k^s (-i - j) \epsilon^{s}.
\]

Now by summarizing the above results, we find

\[
\begin{align*}
\tilde{d}_{i,k}^r, d_{j,l}^r, & = \tilde{d}_{i,k}^r + \mu(1 - \epsilon) \left( i + \frac{1}{2}(1 - \epsilon) \right) k^2 k_i,k, \tilde{d}_{i,l}^r, + \mu(1 - \epsilon) \left( j + \frac{1}{2}(1 - \epsilon) \right) l^2 k_j,l & = (1 - \delta_{k+l,0})((i - \epsilon + 1) l - k(j - \epsilon + 1)) d_{i+k,l}^{r} \\
+ \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2) \left( -k t_{0}^{i+j} d_1 + \mu k^3 t_{0}^{i+j} k_1 \right) & + \mu(1 - \delta_{k+l,0}) \sum_{r=0}^{3} \left( \begin{array}{c} 3 \\ r \end{array} \right) \beta^{-r} (i - \epsilon + 1) \epsilon^{3-r} (-k)^r (j - \epsilon + 1) \epsilon^{r} k_{i+j,k+l} \\
& = \tilde{d}_{i,k}^r + \mu(1 - \epsilon) \left( i + \frac{1}{2}(1 - \epsilon) \right) k^2 k_i,k, \tilde{d}_{i,l}^r, + \mu(1 - \epsilon) \left( j + \frac{1}{2}(1 - \epsilon) \right) l^2 k_j,l \\
+ \delta_{k+l,0}(i + j - \epsilon + 1)(i + j - 2 \epsilon + 2) \left( -k t_{0}^{i+j} d_1 + \mu k^3 t_{0}^{i+j} k_1 \right) & + \mu(1 - \delta_{k+l,0}) \sum_{s=0}^{3} \left( \begin{array}{c} 3 \\ s \end{array} \right) (k + l)^{3-s} (i - \epsilon + 1) \epsilon^{3-s} k^s (-i - j) \epsilon^{s} k_{i+j,k+l}.
\end{align*}
\]
According to the above commutator relation and the fact that
\[(i - \epsilon + 1)l - k(j - \epsilon + 1) = k(2\epsilon - i - j - 2) + (i - \epsilon + 1)(k + l),\]
we have
\[
\begin{align*}
[D^r_k(z), D^r_l(w)] &= \sum_{i,j \in \mathbb{Z}} [d^r_{i,k}, d^r_{j,l}] z^{2r-i-2} w^{2r-j-2} \\
&= k(1 - \delta_{k+l,0}) \sum_{i,j \in \mathbb{Z}} (2\epsilon - i - j - 2)d^r_{i+j,k+l} w^{2r-i-j-2} z^{2r-i-2} w^{i+1} \\
&\quad + (k + l) \sum_{i,j \in \mathbb{Z}} d^r_{i+j,k+l} w^{2r-i-j-2} (i + \epsilon + 1) z^{-i+2r-2} w^i \\
&\quad - k\delta_{k+l,0} \sum_{i,j \in \mathbb{Z}} (\epsilon - i - j - 1)^{2}\sum_{t} t^{t+j} d^r_{i,j} w^{2r-i-j-3} z^{2r-i-2} w^{i+1} \\
&\quad + \mu(1 - \delta_{k+l,0}) \sum_{d, t} \sum_{i,j \in \mathbb{Z}} (3r)_3 k^r(-i - j)^{t} k_{i+j,k+l} w^{i-r} z^{2r-i-j-2} w^{i+1} \\
&\quad + \mu k^{3}\delta_{k+l,0} \sum_{i,j \in \mathbb{Z}} (\epsilon - i - j - 1)^{2}\sum_{t} t^{t+j} k_{i,j} w^{2r-i-j-3} z^{2r-i-2} w^{i+1} \\
&= kw^r \frac{\partial}{\partial w} D^r_{k+l}(w) z^{r-1} \delta \left( \frac{w}{z} \right) + (k + l) D^r_{k+l}(w) w^r \frac{\partial}{\partial w} z^{r-1} \delta \left( \frac{w}{z} \right) \\
&\quad - k\delta_{k+l,0} w^r \frac{\partial}{\partial w} d^r_{i,j} w \delta \left( \frac{w}{z} \right) \\
&\quad + \mu \sum_{d, t} \sum_{i,j \in \mathbb{Z}} (3r)_3 kw^r \frac{\partial}{\partial w} k^r_{i+j,k+l} w^r \frac{\partial}{\partial w} z^{r} \delta \left( \frac{w}{z} \right) \\
&\quad + \mu k^{3}\delta_{k+l,0} w^r \frac{\partial}{\partial w} k^r_{i,j} w \delta \left( \frac{w}{z} \right),
\end{align*}
\]
proving (10). \(\square\)

3. Realization of irreducible highest weight \(\tilde{\mathfrak{u}}(\mathfrak{g})^r\)-modules

In this section, as the main result of the paper, we present an explicit realization of certain irreducible highest weight \(\tilde{\mathfrak{u}}(\mathfrak{g})^r\)-modules.

3.1. Highest weight modules for affine-Virasoro algebras. Let \(\mathfrak{b}\) be a Lie algebra equipped with an invariant symmetric bilinear form \(\langle \cdot, \cdot \rangle\). We denote by
\[
\mathfrak{b} \rtimes \mathfrak{Vir} = (\text{Der } \mathbb{C}[t, t^{-1}]) \ltimes (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{b}) \oplus \mathbb{C}k \oplus \mathbb{C}k_{\mathfrak{Vir}},
\]
the affine-Virasoro algebra associated to \(\mathfrak{b}\), where \(k\) and \(k_{\mathfrak{Vir}}\) are central elements and
\[
\begin{align*}
[L(m), L(n)] &= (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m + n, 0} k_{\mathfrak{Vir}}, \\
[u(m), v(n)] &= [u, v](m + n) + m \delta_{m + n, 0} \langle u, v \rangle k, \\
[L(m), u(n)] &= -nu(m + n) \quad (3.1)
\end{align*}
\]
for \( u, v \in \mathfrak{b}, m, n \in \mathbb{Z} \) and
\[
L(m) := -t^{m+1} \frac{d}{dt}, \quad u(m) := t^m \otimes u \quad \text{for } m \in \mathbb{Z}, \ u \in \mathfrak{b}.
\]

Note that \( \hat{\mathfrak{b}} \rtimes \mathfrak{Vir} \) contains the Virasoro algebra
\[
\mathfrak{Vir} = \text{Der } \mathbb{C}[t, t^{-1}] + \mathfrak{Ck}_{\mathfrak{Vir}}
\]
and the affine algebras
\[
\hat{\mathfrak{b}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{b}) \oplus \mathfrak{Ck}, \quad \tilde{\mathfrak{b}} = \hat{\mathfrak{b}} \oplus \mathbb{C} \frac{d}{dt}
\]
as subalgebras. For any \( \epsilon \in \mathbb{Z} \), set
\[
\tag{3.2}
a^\epsilon(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) z^{\epsilon-n-1}, \quad \forall a \in \mathfrak{b},
\]
\[
\tag{3.3}
L^\epsilon(z) = \sum_{n \in \mathbb{Z}} L(n) z^{2\epsilon-n-2} + \frac{\epsilon^2 - 2\epsilon}{24} z^{2(\epsilon-1)}k_{\mathfrak{Vir}}^2.
\]

When \( \epsilon = 0 \), we also write \( a(z) = a^0(z) \) and \( L(z) = L^0(z) \). To emphasize the dependence on \( \mathfrak{b} \), sometimes we will write \( L_{\mathfrak{b}}(n) \), \( L^\epsilon(z) \) and \( L_{\mathfrak{b}}(z) \), instead of \( L(n) \), \( L^\epsilon(z) \) and \( L(z) \), respectively.

We fix a \( \mathbb{Z} \)-gradation on \( \hat{\mathfrak{b}} \rtimes \mathfrak{Vir} \) with respect to the adjoint action of \( L(0) \):
\[
\hat{\mathfrak{b}} \rtimes \mathfrak{Vir} = \oplus_{n \in \mathbb{Z}} (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})(n).
\]

This \( \mathbb{Z} \)-gradation naturally induces a triangular decomposition of \( \hat{\mathfrak{b}} \rtimes \mathfrak{Vir} \):
\[
\hat{\mathfrak{b}} \rtimes \mathfrak{Vir} = (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(+)} \oplus (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(0)} \oplus (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(-)},
\]
where \( (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(0)} = \mathfrak{b} \oplus \mathfrak{C}L(0) \oplus \mathfrak{Ck} \oplus \mathfrak{Ck}_{\mathfrak{Vir}} \), and
\[
(\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(\pm)} = \oplus_{n>0} (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})(\mp n) = \text{Span}\{u(\pm n), \ L(\pm n) \mid u \in \mathfrak{b}, \ n > 0\}.
\]

Let \( U \) be a \( \mathfrak{b} \)-module and \( \beta, \ell, c \in \mathfrak{C} \). View \( U \) as a \( (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(0)} \)-module with \( L(0) \), \( k \) and \( k_{\mathfrak{Vir}}^2 \) act respectively as the scalars \( \beta \), \( \ell \) and \( c \), and extend it to a \( ((\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(+)} + (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(0)}) \)-module by letting \( (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(+)} \) acts trivially. Then we have the induced \( \hat{\mathfrak{b}} \rtimes \mathfrak{Vir} \)-module:
\[
\tag{3.4}
V_{\hat{\mathfrak{b}} \rtimes \mathfrak{Vir}}(\ell, c, U, \beta) = U((\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(+)} + (\hat{\mathfrak{b}} \rtimes \mathfrak{Vir})_{(0)}) U.
\]

When the \( \mathfrak{b} \)-module \( U \) is irreducible, \( V_{\hat{\mathfrak{b}} \rtimes \mathfrak{Vir}}(\ell, c, U, \beta) \) has a unique irreducible quotient, called \( L_{\hat{\mathfrak{b}} \rtimes \mathfrak{Vir}}(\ell, c, U, \beta) \). Note that the \( \mathbb{Z} \)-grading on \( \hat{\mathfrak{b}} \rtimes \mathfrak{Vir} \) affords naturally \( \mathbb{N} \)-grading structures on \( V_{\hat{\mathfrak{b}} \rtimes \mathfrak{Vir}}(\ell, c, U, \beta) \) and \( L_{\hat{\mathfrak{b}} \rtimes \mathfrak{Vir}}(\ell, c, U, \beta) \) with deg \( U = 0 \).

Similarly, we have a \( \mathbb{Z} \)-gradation \( \hat{\mathfrak{b}} = \bigoplus \hat{b}(n) \) and a triangular decompositions \( \hat{\mathfrak{b}} = \hat{b}(+ \oplus \hat{b}(0) \oplus \hat{b}(-) \) of the affine Lie algebra \( \hat{\mathfrak{b}} \), where \( b(n) = t^{-n} \otimes b + \delta_{n,0} \mathfrak{Ck} \) and \( b(\pm) = t^{\pm 1} \mathfrak{C}[t^{\pm 1}] \otimes b \). Extend the \( \mathfrak{b} \)-module \( U \) to a \( (\hat{b}(+) \oplus \hat{b}(0)) \)-module such that \( \hat{b}(+) \) acts trivially and \( k \) acts as the scalar \( \ell \). Then we have the Verma type highest weight \( \mathfrak{b} \)-module
\[
\tag{3.5}
V_{\hat{b}}(\ell, U) = U(\hat{b}) \otimes U(\hat{b}(+) \oplus \hat{b}(0)) U
\]
and its irreducible quotient \( L_{\hat{b}}(\ell, U) \) provided that \( U \) is irreducible. Note that both \( V_{\hat{b}}(\ell, U) \) and \( L_{\hat{b}}(\ell, U) \) are naturally \( \mathbb{N} \)-graded with \( U \) as the degree 0 subspace.
We also have a triangular decomposition $\text{Vir} = \text{Vir}_{(+)} \oplus \text{Vir}_{(0)} \oplus \text{Vir}_{(-)}$ of the Virasoro algebra $\text{Vir}$, where $\text{Vir}_{(\pm)} = \oplus_{m \geq 0} \mathbb{C}L(\pm m)$ and $\text{Vir}_{(0)} = \mathbb{C}L(0) + \mathbb{C}k_{\text{Vir}}$. View $\mathbb{C}$ as a $(\text{Vir}_{(+)} \oplus \text{Vir}_{(0)})$-module such that $\text{Vir}_{(\pm)}$ acts trivially, $L(0)$ acts as the scalar $\beta$ and $k_{\text{Vir}}$ acts as the scalar $c$. 

Then we have the Verma type highest weight module 

$$V_{\text{Vir}}(c, \beta) = \mathcal{U}(\text{Vir}) \otimes_{\mathcal{U}(\text{Vir}_{(+)} + \text{Vir}_{(0)})} \mathbb{C}$$

and its irreducible quotient $L_{\text{Vir}}(c, \beta)$.

**Remark 3.1.** Assume that $\mathfrak{b}$ is finite-dimensional simple, $(\cdot, \cdot)$ is the normalized form, and $U$ is finite dimensional. By the Sugawara construction (see [K1]), when $\ell$ is not the negative dual Coxeter number, the irreducible $\hat{\mathfrak{b}} \rtimes \text{Vir}$-module $L_{\mathfrak{b} \rtimes \text{Vir}}(\ell, c, U, \beta)$ can be realized as the tensor product of the $\mathfrak{g}$-module $L_{\mathfrak{g}}(\ell, U)$ and the Vir-module $L_{\text{Vir}}(\ell, c')$ for some suitable $c', \beta' \in \mathbb{C}$.

### 3.2. The main construction.

In this subsection we construct a class of irreducible $\tilde{\mathfrak{g}}^\gamma$-modules.

Let $\mathfrak{h} = \mathbb{C}k + \mathbb{C}d$ be a vector space equipped with a designated basis $\{k, d\}$ and a symmetric bilinear form $(\cdot, \cdot)$ determined by

$$(k, k) = 0 = (d, d), \quad (k, d) = 1.$$ 

Set $L = Zk \subset \mathfrak{h}$ and $\mathfrak{h}_k = \mathbb{C} \otimes Z \subset \mathfrak{h}$. Denote by

$$\mathbb{C}[L] = \oplus_{\gamma \in L} \mathbb{C}e^\gamma \quad \text{and} \quad \mathbb{C}[h_k] = \oplus_{h \in h_k} \mathbb{C}e^h$$

the group algebras of $L$ and $\mathfrak{h}_k$, respectively. We endow $\mathbb{C}[h_k]$ with an $\mathfrak{h}$-module structure: $he^h' = (h, h') e^{h'}$ for $h \in \mathfrak{h}, h' \in \mathfrak{h}_k$. Note that for each $\alpha \in \mathbb{C}$, $e^{\alpha k} \mathbb{C}[L]$ is an $\mathfrak{h}$-submodule of $\mathbb{C}[h_k]$. Then for any $\ell \in \mathbb{C}^\times$, we have a Verma type highest weight $\mathfrak{h}$-module $V^\ell_{\mathfrak{h}}(\ell, e^{\alpha k} \mathbb{C}[L])$ (see [LL]). Note that we have the following vector spaces identification:

$$V^\ell_{\mathfrak{h}}(\ell, e^{\alpha k} \mathbb{C}[L]) = V^\ell_\mathfrak{h}(\ell, 0) \otimes e^{\alpha k} \mathbb{C}[L].$$

It is known (see [LL]) that the highest weight $\mathfrak{h}$-module $V^\ell_{\mathfrak{h}}(\ell, e^{\alpha k} \mathbb{C}[L])$ can be extended to an $\hat{\mathfrak{h}} \rtimes \text{Vir}$-module with $k_{\text{Vir}} = 2$ and

$$L_{\mathfrak{h}}(z) = \frac{1}{\ell} \partial k(z) d(z).$$

Let $W$ be a vector space and $u, v \in W$. For any two formal series $u(z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$ and $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$ in $W[[z, z^{-1}]]$, we denote by

$$\partial u(z_1) v(z_2) = u^+(z_1) v(z_2) + v(z_2) u^-(z_1)$$

the normally ordered product of $u(z_1)$ and $v(z_2)$, where

$$u^+(z) = \sum_{n < 0} u_n z^{-n-1} \quad \text{and} \quad u^-(z) = \sum_{n \geq 0} u_n z^{-n-1}.$$ 

Form the direct sum

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}$$

of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $U$ be an irreducible $\mathfrak{g}$-module and $\alpha, \beta, \ell \in \mathbb{C}$ with $\ell \neq 0$. We define an $\hat{\mathfrak{f}} \rtimes \text{Vir}$-module structure on the tensor space

$$L_{\mathfrak{g} \rtimes \text{Vir}}(\ell, 24\mu \ell - 2, U, \beta) \otimes V^\ell_{\mathfrak{h}}(\ell, e^{\alpha k} \mathbb{C}[L])$$

by letting $k = \ell$, $k_{\text{Vir}} = 24\mu \ell$ and

$$(3.9) \quad u(z) = u(z) \otimes 1, \quad h(z) = 1 \otimes h(z), \quad L_{\mathfrak{g}}(z) = L_{\mathfrak{h}}(z) \otimes 1 + 1 \otimes L_{\mathfrak{h}}(z),$$

for $u \in \mathfrak{g}$ and $h \in \mathfrak{h}$. 
Let $C[L]$ act on $V_{\ell}^e(\ell, e\alpha k C[L])$ by
\[
e^\gamma (v \otimes e\alpha k + \gamma') = v \otimes e\alpha k + \gamma, \quad \forall \gamma, \gamma' \in L, v \in V_{\ell}^e(\ell, 0).
\]
For $\gamma \in L$, introduce the following operator in $\text{End} \left(V_{\ell}^e(\ell, e\alpha k C[L])\right) [[z, z^{-1}]]$:
\[
E^\gamma(z) = \exp \left(\frac{1}{\ell} \sum_{m=1}^{\infty} \frac{\gamma(-m)}{m} z^m\right) \exp \left(-\frac{1}{\ell} \sum_{m=1}^{\infty} \frac{\gamma(m)}{m} z^{-m}\right) e^\gamma.
\]
For convenience, we also set
\[
(3.11) \quad E^\gamma(z) = 1 \otimes E^\gamma(z) \in \text{End} \left(L_{\hat{\alpha} e\alpha V_{\ell}(\ell, 24\mu \ell - 2, U, \beta) \otimes V_{\ell}^e(\ell, e\alpha k C[L])}[[z, z^{-1}]].
\]

The following theorem is the first main result of this paper. We will prove this theorem in Section 5.

**Theorem 3.2.** Let $\ell, \alpha, \beta \in C$ with $\ell \neq 0$ and let $U$ be an irreducible $\mathfrak{g}$-module. Then there is an irreducible $(\mathfrak{g})^{\mu}$-module structure on the $\hat{\mathfrak{f}} \rtimes \text{Vir}$-module
\[
L_{\hat{\alpha} e\alpha V_{\ell}(\ell, 24\mu \ell - 2, U, \beta) \otimes V_{\ell}^e(\ell, e\alpha k C[L])},
\]
where $k_0 = \ell$, $t_0^{-1} d_0 = -L_1(\epsilon - 1) + \delta_{\epsilon, \mu} \ell$, and
\[
(3.12) \quad \langle t_1^m \otimes u \rangle(z) = u^t(z) E^{nk}(z), \quad k_1^t(z) = k^t(z), \quad d_1^t(z) = d^t(z), \quad K^t_{\alpha}(z) = \frac{\ell}{n} E^{nk}(z),
\]
for $u \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^\times$.

**Remark 3.3.** When $\epsilon = 1$, the $(\hat{\mathfrak{g}})^{\mu}$-module $L_{\hat{\alpha} e\alpha V_{\ell}(\ell, 24\mu \ell - 2, U, \beta) \otimes V_{\ell}^e(\ell, e\alpha k C[L])}$ was first constructed by Billig in [12] (see also [CLIT]). And, when $\epsilon = 0$, this $(\hat{\mathfrak{g}})^{\mu}$-module was previously constructed in [CLIT].

### 3.3. Realization of irreducible highest weight $(\hat{\mathfrak{g}})^{\mu}$-modules

In this subsection we give a characterization of the irreducible $(\hat{\mathfrak{g}})^{\mu}$-modules constructed in Theorem 3.2.

Note that $(\hat{\mathfrak{g}})^{\mu} = \oplus_{n \in \mathbb{Z}} (\hat{\mathfrak{g}})^{\mu}_n$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{T}^{\mu}(\mathfrak{g})$, where $\mathcal{T}^{\mu}(\mathfrak{g}) = (\hat{\mathfrak{g}})^{\mu} \cap \mathcal{T}(\mathfrak{g})$. Explicitly, we have
\[
(\hat{\mathfrak{g}})^{\mu}_n = \text{Span} \{ t_{0}^{-m} u, d_{-m, n} , k_{-m, n} | m \in \mathbb{Z}, u \in \hat{g}_1 \} \oplus \mathbb{C} \delta_{n, -1} t_{0}^{-1} d_0,
\]
if $n \neq 0$, and
\[
(\hat{\mathfrak{g}})^{\mu}_0 = \hat{g}_1 + \sum_{n \in \mathbb{Z}^\times} (\mathbb{C} d_{0, n} + \mathbb{C} k_{0, n}) + \mathbb{C} k_0 + \mathbb{C} \delta_{0, -1} d_0.
\]
This $\mathbb{Z}$-grading induces a natural triangular decomposition of $(\hat{\mathfrak{g}})^{\mu}$:
\[
(\hat{\mathfrak{g}})^{\mu} = (\hat{\mathfrak{g}})^{\mu}_{-} \oplus (\hat{\mathfrak{g}})^{\mu}_{0} \oplus (\hat{\mathfrak{g}})^{\mu}_{+},
\]
where
\[
(\hat{\mathfrak{g}})^{\mu}_{\pm} = \oplus_{n > 0} (\hat{\mathfrak{g}})^{\mu}_n.
\]
where $k$ is an irreducible $L$-module determined by $\deg T = 0$. Denote by $L_{\tilde{\mathfrak{g}}}^{(1)}(T)$ the quotient of $V_{\tilde{\mathfrak{g}}}^{(+)}(T)$ modulo by its maximal graded submodule. Note that $L_{\tilde{\mathfrak{g}}}^{(+)}(T)$ is an irreducible highest weight $\tilde{\mathfrak{g}}$-module, and conversely any irreducible highest weight $\tilde{\mathfrak{g}}$-module is of this form.

Let $U$ be an irreducible $\mathfrak{g}$-module and $\ell, a, b \in \mathbb{C}$. Associated to these data, we will construct an irreducible $\tilde{\mathfrak{g}}(t^{(o)})$-module structure on the loop space $\mathbb{C}[t, t^{-1}] \otimes U$. Assume first that $\epsilon \neq 1$. One can check that in this case

$$\tilde{\mathfrak{g}}(t^{(o)}) = \mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g} + \mathbb{C}k_1 + \sum_{m \in \mathbb{Z}} (\mathbb{C}d_{0,m} + \mathbb{C}e_{1,m}k_0),$$

where $k_1$ is central and

$$[t_1^m \otimes u, t_1^n \otimes v] = t_1^{m+n} \otimes [u, v] + m\delta_{m+n,0} \langle u, v \rangle k_1, \quad [t_1^m k_0, t_1^n \otimes u] = 0, \quad [t_1^m k_0, t_1^n \otimes u] = 0,$$

$$[d_{0,m}^0, t_1^n k_0] = (1-\epsilon)nt_1^{m+n}k_0 - m^2\delta_{m+n,0}k_1, \quad [d_{0,m}^0, t_1^n \otimes u] = (1-\epsilon)nt_1^{m+n}u,$$

$$[d_{0,m}^0, d_{0,n}^0] = (1-\epsilon)(n-m)d_{0,m+n}^0 + 2\mu m^3(\epsilon-1)\delta_{m+n,0}k_1$$

for $m, n, k, l \in \mathbb{Z}$ and $u, v \in \mathfrak{g}$. This implies that the assignment $(m \in \mathbb{Z}, u \in \mathfrak{g})$

$$(3.13) \quad k_1 \mapsto 0, \quad d_{0,m}^0 \mapsto (1-\epsilon) t_1^{m+1} \frac{d}{dt}, \quad t_1^m \otimes u \mapsto t_1^m \otimes u, \quad t_1^m k_0 \mapsto t_1^m \otimes k_0$$

determines a surjective homomorphism from $\tilde{\mathfrak{g}}(t^{(o)})$ to the centerless affine-Virasoro algebra

$$\mathcal{W}_\mathfrak{g} := (\text{Der } \mathbb{C}[t, t^{-1}]) \times (\mathbb{C}[t, t^{-1}] \otimes (\mathfrak{g} + \mathbb{C}k_0))$$

associated to $\mathfrak{g} \oplus \mathbb{C}k_0$. Extend $U$ to a $(\mathfrak{g} \oplus \mathbb{C}k_0)$-module such that $k_0$ acts as a scalar $\ell$. Following [33], there is an irreducible $\mathcal{W}_\mathfrak{g}$-module structure on the loop space $\mathbb{C}[t, t^{-1}] \otimes U$ with

$$(m \otimes v)(t^n \otimes u) = t^{m+n} \otimes (vu), \quad \left(t^{m+1} \frac{d}{dt}\right) (t^n \otimes u) = (n + a + bm)t^{m+n} \otimes u$$

for $x \in \mathfrak{g} \oplus \mathbb{C}k_0$, $m, n \in \mathbb{Z}$ and $u \in U$. Via the homomorphism $\mathcal{W}_\mathfrak{g}$, $\mathbb{C}[t, t^{-1}] \otimes U$ becomes a $\tilde{\mathfrak{g}}(t^{(o)})$-module, which we denote by $\tilde{T}_{U, t^{(a,b)}}$.

When $\epsilon = 1$ we have

$$\tilde{\mathfrak{g}}(t^{(1)}) = \mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g} + \sum_{m \in \mathbb{Z}} (\mathbb{C}d_{1,m} + \mathbb{C}e_{1,m}k_0),$$

which is isomorphic to the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ with

$$t_1^m \otimes u \mapsto u(m), \quad t_1^m k_0 \mapsto k(m), \quad t_1^m d_0 \mapsto d_0(m), \quad k_1 \mapsto k, \quad d_1 \mapsto t \frac{d}{dt}$$

for $m \in \mathbb{Z}, u \in \mathfrak{g}$. Extend $U$ to a $\hat{\mathfrak{g}}$-module such that $d_0$ and $k$ act as the scalars $b$ and $\ell$, respectively. Following [CP], we have the loop module $\mathbb{C}[t, t^{-1}] \otimes U$ for the affine Lie algebra $\hat{\mathfrak{g}}$ with the actions given by

$$x(m)(t^n \otimes u) = t^{m+n} \otimes (xu), \quad \frac{d}{dt}(t^n \otimes u) = (n + a)t^n \otimes u, \quad k = 0,$$
for $x \in \mathfrak{f}$, $m, n \in \mathbb{Z}$ and $u \in U$. Since $\tilde{\mathfrak{z}}(g)_{(0)}^1$ is isomorphism to $\tilde{\mathfrak{f}}$, the loop $\tilde{\mathfrak{f}}$-module $\mathbb{C}[t, t^{-1}] \otimes U$ becomes a $\tilde{\mathfrak{g}}(g)_{(0)}^1$-module, which is also denoted by $T_{U, \ell, a, b}$.

Now, as the second main result of this paper, we give a characterization of the irreducible $\tilde{\mathfrak{g}}(g)$-modules constructed in Theorem 3.2.

**Theorem 3.4.** Let $\ell, \alpha, \beta \in \mathbb{C}$ with $\ell \neq 0$ and let $U$ be an irreducible $g$-module. Then the $\tilde{\mathfrak{g}}(g)$-module $L_{\tilde{\bar{g}} \times \mathfrak{vir}}(\ell, 24\mu \ell - 2, U, \beta) \otimes V_{\delta}(\ell, e^{\alpha k}\mathbb{C}[L])$ is isomorphic to the irreducible highest weight module $L_{\tilde{\mathfrak{g}}}(t, U, \alpha, b)$, where $b = \frac{\beta + (e^{-2} - 2)e\mu \ell}{1 - e}$ if $\epsilon \neq 1$ and $b = \beta - \mu \ell$ if $\epsilon = 1$.

**Proof.** Write

$$W = L_{\tilde{\bar{g}} \times \mathfrak{vir}}(\ell, 24\mu \ell - 2, U, \beta) \otimes V_{\delta}(\ell, e^{\alpha k}\mathbb{C}[L]).$$

It is clear that $W = \oplus_{k \in \mathbb{Z}} W_{(k)}$ is a $Z$-graded $\tilde{\mathfrak{f}} \times \mathfrak{vir}$-module such that $W_{(0)} = U \otimes e^{\alpha k}\mathbb{C}[L]$ and $W_{(k)} = 0$ for $k < 0$. For $n \in \mathbb{Z}$, set

$$E_{n k}(z) = \sum_{m \in \mathbb{Z}} E_{n k}(m) z^{-m}.$$

Then we have $E_{n k}(m) W_{(k)} \subset W_{(m+k)}$ for $n, m, k \in \mathbb{Z}$. This together with (3.12) implies that $\tilde{\mathfrak{g}}(g)_{(m)} W_{(k)} = W_{(m+k)}$ for $m, k \in \mathbb{Z}$. That is, $W$ is an irreducible $\mathbb{N}$-graded $\tilde{\mathfrak{g}}(g)$-module. Thus, $W_{(0)}$ is an irreducible $\tilde{\mathfrak{g}}(g)_{(0)}^1$-module and we need to prove that

$$W_{(0)}(= U \otimes e^{\alpha k}\mathbb{C}[L])$$

is isomorphic to $T_{U, l, a, b}(= \mathbb{C}[t, t^{-1}] \otimes U)$ as $\tilde{\mathfrak{g}}(g)_{(0)}^1$-modules, where $l, U, \alpha, b$ are as given in the theorem.

It is straightforward to check that in this case the $\tilde{\mathfrak{g}}(g)_{(0)}^1$-module actions on $T_{U, l, a, b}$ can be written in the following expressions:

$$(t_1^n \otimes a)(t^n \otimes u) = t^{m+n} \otimes au,$$

$$(t_1^{m+k} \otimes u)(t^n \otimes u) = (\ell t^{m+n} \otimes u),$$

$$d_1(t^n \otimes u) = (n + \alpha)t^n \otimes u, \quad k_1(t^n \otimes u) = 0,$$

$$d_0(t^n \otimes u) = (k(\beta + (e^{-2} - 2)e\mu \ell) + (1 - \epsilon) (\alpha + n + \frac{1}{2}k \ell)) t^{k\ell} \otimes u, \quad d_0 \circ d_0 = (\mu \ell - \beta)(t^n \otimes u) \quad (\text{when } \epsilon = 1),$$

for $m, n \in \mathbb{Z}$, $k \in \mathbb{Z}^+$, $a \in \mathfrak{g}$ and $u \in U$.

Now we are ready to show that $\tilde{\mathfrak{g}}(g)_{(0)}^1$-module actions on $W_{(0)}$ coincide with that of $T_{U, l, a, b}$ given in (3.14). For $u \in U$ and $r \in \mathbb{Z}$, set

$$u(r) := u \otimes e^{(\alpha + r)k}.$$

Then we have

$$k_1(u(r)) = k(0)(u(r)) = u \otimes (k(0)) e^{(\alpha + r)k} = (k, (\alpha + r)k) \otimes e^{(\alpha + r)k} = 0,$$

$$d_1(u(r)) = d(0)(u(r)) = (d, (\alpha + r)k) u(r) = (\alpha + r)u(r).$$

When $\epsilon = 1$, we also have

$$d_0(u(r)) = (-L(0) - \sum_{i \geq 1} k_i d_{-i} - \sum_{i \geq 1} d_{-i} k_i + \mu \ell)(u(r))$$

$$= (-L(0) + \mu \ell)(u(r)) = (-\beta + \mu \ell)(u(r)).$$

On the other hand, for $m \in \mathbb{Z}$, we have

$$E_{n k}(0) u(r) = u \otimes e^{nk} e^{(\alpha + r)k} = u(m + r).$$
In view of this, we have

\begin{align}
(3.18) & \quad (l_1^N k_0)(u(r)) = n k_0, u(r)) = \ell E^{n k}(0)(u(r)) = \ell(u(n + r)), \\
(3.19) & \quad (l_1^N \otimes a)(u(r)) = \sum_{i \in \mathbb{Z}} a(i) u \otimes E^{n k}(-i) E^{(\alpha+r) k} \\
& \quad = a(0) \otimes E^{n k}(0) e^{(\alpha+r) k} = (au)(m + r),
\end{align}

for \( n \in \mathbb{Z}^\times \), \( a \in g \) and \( m \in \mathbb{Z} \).

For the action of \( d_0^n, n \in \mathbb{Z}^\times \) on \( u(r) \), note firstly that

\[
[L_b(i), E^{n k}(-j)] e^{(\alpha+r) k} = \left( \frac{n}{T} \sum_{j \in \mathbb{Z}} k(j) E^{n k}(-j) \right) e^{(\alpha+r) k} = 0 \quad (i \in \mathbb{Z}),
\]

as \( k(i) \) commutes with \( E^{n k}(j) \) for \( i, j \in \mathbb{Z} \), and that \( L_b(i) e^{(\alpha+r) k} = 0 \) for \( i \geq 0 \). Using these and \ref{eq:3.18}, we have

\[
\text{Res}_z z^{1-2\epsilon} \sum_{i \leq 2\epsilon-2} L_i^N(z) E^{n k}(z) u(r)
\]

\[
= \text{Res}_z \left( \sum_{i \leq 2\epsilon-2} L_i^N(z) z^{i-1} E^{n k}(z) + \sum_{i > 2\epsilon-2} L_i^N(z) z^{i-1} + \frac{\epsilon^2 - 2\epsilon}{24} z^{-1} \text{Vir} E^{n k}(z) \right) u(r)
\]

\[
= \left( \sum_{i \leq 2\epsilon-2} L_i^N(z) E^{n k}(-i) + \sum_{i > 2\epsilon-2} E^{n k}(-i) L_i^N(z) + (\epsilon^2 - 2\epsilon) \mu \ell(\text{Vir} E^{n k}(z)) \right) u(r)
\]

\[
= \sum_{i \in \mathbb{Z}} \left( L_b(i) u \otimes E^{n k}(-i) e^{(\alpha+r) k} + u \otimes L_b(i) E^{n k}(-i) e^{(\alpha+r) k} \right) + (\epsilon^2 - 2\epsilon) \mu \ell(u(n + r))
\]

\[
= L_b(0) u \otimes E^{n k}(0) e^{(\alpha+r) k} + (\epsilon^2 - 2\epsilon) \mu \ell(u(n + r)).
\]

Secondly, we have

\[
\text{Res}_z z^{1-2\epsilon} (z^{2\epsilon-2} E^{n k}(z)) u(r) = (E^{n k}(0) u)(r) = u(n + r).
\]

Thirdly, from the fact that \( k(i) \) commutes with \( E^{n k}(j) \) for \( i, j \in \mathbb{Z} \), we have

\[
\text{Res}_z z^{1-2\epsilon} \left( z^{\epsilon} \frac{d}{dz} k'(z) \right) E^{n k}(z) u(r)
\]

\[
= \text{Res}_z \left( \epsilon - i - 1 \right) k(i) z^{-i-1} E^{n k}(z) u(r)
\]

\[
= \sum_{i \in \mathbb{Z}} (\epsilon - i - 1) k(i) E^{n k}(-i) (u \otimes e^{(\alpha+r) k}) = 0.
\]

Fourthly, using the relation \([d(i), E^{n k}(-i)] = n E^{n k}(0)\) for \( i \in \mathbb{Z} \), we have

\[
\text{Res}_z z^{1-2\epsilon} \left( z^{\epsilon} \frac{d}{dz} n \right) E^{n k}(z) u(r)
\]

\[
= \text{Res}_z z^{1-\epsilon} \frac{d}{dz} n E^{n k}(z) u(r)
\]

\[
= \text{Res}_z z^{\epsilon} (\epsilon - 1) n E^{n k}(z) u(r)
\]
\[(\epsilon - 1)\text{Res}_z \left( \sum_{i \leq \epsilon - 1} d(i)z^{-i-1}E^{nk}(z) + \sum_{i > \epsilon - 1} E^{nk}(z)d(i)z^{-i-1} \right) u(r) \]

\[= (\epsilon - 1) \left( \sum_{i \leq \epsilon - 1} d(i)E^{nk}(-i) + \sum_{i > \epsilon - 1} E^{nk}(-i)d(i) \right) u(r) \]

\[= \begin{cases} 
(\epsilon - 1) \sum_{0 \leq i \leq \epsilon - 1} d(i)E^{nk}(-i)u(r), & \text{if } \epsilon \geq 1 \\
(\epsilon - 1) \sum_{\epsilon - 1 < i \leq 0} E^{nk}(-i)d(i)u(r), & \text{if } \epsilon < 1 
\end{cases} \]

\[= (\epsilon - 1)(\alpha + r + \mu e)u(n + r). \]

Recall that
\[d^\ell_{0,n} = \text{Res}_z z^{1-2\ell}D^\ell_{n}(z) \]
\[= \text{Res}_z z^{1-2\ell} \left[ n \circ L^\ell_{i}(z)E^{nk}(z) \circ + \frac{1}{2}n\epsilon(\epsilon - 1)z^{2\ell - 2}E^{nk}(z) \\
- z^\epsilon \frac{d}{dz} \circ d'(z)E^{nk}(z)_{\circ} + n^2(\mu - \frac{1}{\ell})z^\epsilon \frac{d}{dz} k'(z)E^{nk}(z) \right]. \]

Then by combining the above identities together, we obtain
\[(3.20) \quad d^\ell_{0,n}(u(r)) = (n(\beta + (\epsilon - 2\ell)\mu) + (1 - \epsilon)(\alpha + r + \frac{1}{\ell}\epsilon n))(u(n + r)). \]

Finally, by comparing (3.15)–(3.20) with (3.14), we conclude that the map
\[T_{\ell,u,a,b} = C[t,t^{-1}] \otimes U \to W_{(0)} = U \otimes e^{\circ k_c}C[L], \quad t' \otimes u \mapsto u(r) \quad (r \in Z, u \in U) \]
is a \(\hat{\mathfrak{g}}\)\textsuperscript{\circ}(0)-module isomorphism. This completes the proof. \(\square\)

**Remark 3.5.** Assume that \(\mathfrak{g}\) is finite-dimensional and simple. Note that \(d_1\) acts semisimply on any irreducible highest weight \(\hat{\mathfrak{g}}\)-module \(L_{\hat{\mathfrak{g}}\circ}(T)\). We say that \(L_{\hat{\mathfrak{g}}\circ}(T)\) is bounded if every graded subspace of it is a direct sum of finite dimensional \(d_1\)-eigenvalue spaces. For example, if \(U\) is finite-dimensional, then from Theorem 3.4 we find that every \(L_{\hat{\mathfrak{g}}\circ}(T_{\ell,u,a,b})\) is bounded. When \(\epsilon = 0\), it was proved in [CLT4] that every irreducible bounded highest weight \(\hat{\mathfrak{g}}\)-module has such a form. However, when \(\epsilon = 1\), a large class of different irreducible bounded highest weight \(\hat{\mathfrak{g}}\)-modules was constructed in [CLT4]. Thus, when \(\epsilon \neq 0, 1\), it is interesting to classify the irreducible bounded highest weight \(\hat{\mathfrak{g}}\)-modules, which we believe should have the form \(L_{\hat{\mathfrak{g}}\circ}(T_{\ell,u,a,b})\) with \(U\) finite-dimensional.

4. **Basics on \(\phi_\epsilon\)-coordinated modules for vertex algebras**

In this section, we collect some results on \(\phi_\epsilon\)-coordinated modules for later use. We denote a vertex algebra by \(V = (V, Y, \mathbf{1})\) [FHL, LL], where \(\mathbf{1}\) is the vacuum vector, and
\[Y(\cdot, z) : V \to \text{Hom}(V(V(\cdot))), \quad v \mapsto \sum_{n \in Z} v_nz^{-n-1} \]
is the vertex operator.

4.1. **Basics on \(\phi_\epsilon\)-coordinated modules.** As in Introduction, set
\[\phi_\epsilon(z_2, z_0) = e^{-\epsilon} \left( z_2^\epsilon \right) \]
a particular associate of the one-dimensional additive formal group \(F(z, w) = z + w\) as defined in [L3]. Now we recall the notion of \(\phi_\epsilon\)-coordinated module for a vertex algebra (see [L3] [BLT]).
**Definition 4.1.** Let $V$ be a vertex algebra. A \(\phi\)-coordinated \(\mathcal{V}\)-module \((W, Y_W^\phi)\) is a vector space $W$ equipped with a linear map

\[
Y_W^\phi(\cdot, z) : V \to \text{Hom}(W, W((z))), \quad v \mapsto Y_W^\phi(v, z)
\]
satisfying the conditions that $Y_W^\phi(1, z) = 1_W$ and that for $u, v \in V$, there exists $k \in \mathbb{N}$ such that

\[
\begin{align*}
(4.1) & \quad (z_1 - z_2)^k Y_W^\phi(u, z_1) Y_W^\phi(v, z_2) \in \text{Hom}(W, W((z_1, z_2))), \\
(4.2) & \quad (\phi_\epsilon(z_2, z) - z_2)^k Y_W^\phi(Y(u, z_0)v, z_2) = ((z_1 - z_2)^k Y_W^\phi(u, z_1) Y_W^\phi(v, z_2)) |_{z_1 = \phi_\epsilon(z_2, z_0)}.
\end{align*}
\]

We denote by $\mathcal{D}$ the canonical derivation on $V$ defined by $v \mapsto v_{-2}1$ for $v \in V$. The following result was proved in [4].

**Lemma 4.2.** For a \(\phi\)-coordinated \(\mathcal{V}\)-module \((W, Y_W^\phi)\), we have

\[
Y_W^\phi(\mathcal{D}v, z) = z \frac{d}{dz} Y_W^\phi(v, z), \quad \forall v \in V.
\]

The following Borchers formula and Jacobi-type identity were obtained in [BLP].

**Proposition 4.3.** Let \((W, Y_W^\phi)\) be a \(\phi\)-coordinated \(\mathcal{V}\)-module. Then for $u, v \in V$, we have

\[
(4.3) \quad [Y_W^\phi(u, z), Y_W^\phi(v, w)] = \sum_{j \geq 0} \frac{1}{j!} Y_W^\phi(u_j v, w) \left( w^\epsilon \frac{\partial}{\partial w} \right)^j z^{-1} \delta \left( \frac{w}{z} \right),
\]

and

\[
(4.4) \quad (z_2z)^{-1} \delta \left( \frac{z_1 - z_2}{z_2z} \right) Y_W^\phi(u, z_1) Y_W^\phi(v, z_2) - (z_2z)^{-1} \delta \left( \frac{z_2 - z_1}{z_2z} \right) Y_W^\phi(v, z_2) Y_W^\phi(u, z_1)
\]

\[
= z^{-1} \delta \left( \frac{z_2(z_1 + z)}{z_1} \right) Y_W^\phi(Y(u, f_\epsilon(z_2, z))v, z_2),
\]

where

\[
f_\epsilon(z_2, z) = \begin{cases} 
\frac{z_2^\epsilon - 1}{1 - \epsilon}, & \text{for } \epsilon \neq 1 \\
\log(1 + z), & \text{for } \epsilon = 1.
\end{cases}
\]

As in the module case, we have the following result whose proof is straightforward and is omitted (cf. [FHL]).

**Lemma 4.4.** Let $V, V'$ be two vertex algebras, \((W, Y_W^\phi)\) a \(\phi\)-coordinated \(\mathcal{V}\)-module, and \((W', Y_{W'}^\phi)\) a \(\phi\)-coordinated \(V'\)-module. Then \((W \otimes W', Y_{W \otimes W'}^\phi)\) is a \(\phi\)-coordinated module the tensor product vertex algebra $V \otimes V'$.

In what follows, for $u, v \in V$, we consider the expressions of $Y_W^\phi(u_0 v, z)$ and $Y_W^\phi(\cdot_1 v, z)$ in a \(\phi\)-coordinated \(\mathcal{V}\)-module $W, Y_W^\phi$.

**Lemma 4.5.** Let \((W, Y_W^\phi)\) be a \(\phi\)-coordinated \(\mathcal{V}\)-module. Then for $u, v \in V$,

\[
(4.5) \quad Y_W^\phi(u_0 v, w) = [\text{Res}_z z^{-\epsilon} Y_W^\phi(u, z), Y_W^\phi(v, w)].
\]

**Proof.** By taking $\text{Res}_z z^{-\epsilon}$ in the both sides of (4.3), we have

\[
[\text{Res}_z z^{-\epsilon} Y_W^\phi(u, z), Y_W^\phi(v, w)] = \text{Res}_z z^{-\epsilon} Y_W^\phi(u_0 v, w) z^{-1} \delta \left( \frac{w}{z} \right) = Y_W^\phi(u_0 v, w),
\]

noting that for $j > 0$

\[
\text{Res}_z z^{-\epsilon} \left( w^\epsilon \frac{\partial}{\partial w} \right)^j z^{-1} \delta \left( \frac{w}{z} \right) = \text{Res}_z z^{-\epsilon} \left( -z^\epsilon \frac{\partial}{\partial z} \right)^j z^{-1} \delta \left( \frac{w}{z} \right) = 0.
\]

\qed
Set \( h_0 = 1 \) and
\[
h_n = \sum_{r_1, r_2, \ldots, r_n > 0} (-1)^\frac{r_1^2}{2} \left( \sum_{m=1}^{n} r_m \right) \frac{1}{n!} \prod_{m=1}^{n} \left( \frac{\epsilon^{(m)}}{(m+1)!} \right) \epsilon^{r_m}, \quad \forall n \geq 1,
\]
where \( \epsilon^{(m)} = \prod_{i=0}^{m-1} (\epsilon + s(\epsilon - 1)) \). We have

**Proposition 4.6.** Let \((W, Y_W)\) be a \( \phi \)-coordinated \( V \)-module. Then for \( u, v \in V \),
\[
Y_W^ε(u - 1, v, z) = Y_W^ε(u, z)Y_W^ε(v, z) - \sum_{n \geq 0} \frac{1}{n!} \sum_{i=0}^{n+1} \epsilon^{(i)} h_{n-i+1} z^{(n+1)(\epsilon-1)} Y_W^ε(u, v, z).
\]

**Proof.** Note that we have
\[
\phi_ε(z_2, z_0) - z_2 = \sum_{n \geq 1} \frac{1}{n!} z_0^n \left( z_2^2 \frac{d}{dz_2} \right)^n z_2 = z_0 z_2^2 \phi_ε(z_2, z_0),
\]
where
\[
h(z_2, z_0) = \sum_{n \geq 0} \frac{1}{(n+1)!} (n)_2 z_2^n z_0^n = 1 \in \mathbb{C}((z_2))[\left[ z_0 \right]].
\]
Since \( h(z_2, z_0) = 1 \neq 0 \), the inverse \( h(z_2, z_0)^{-1} \) of \( h(z_2, z_0) \) exists in \( \mathbb{C}((z_2))[\left[ z_0 \right]] \). Explicitly, we have
\[
h(z_2, z_0)^{-1} = \left( 1 + \sum_{i \geq 1} \frac{1}{(i+1)!} z_0^i z_2^{i-1} \right)^{-1} = \sum_{j \geq 0} \frac{1}{(i+1)!} z_0^i z_2^{i+1} = h_n z_2^n z_0^n.
\]
Next we compute the residue \( \text{Res}_{z_2} z^{-1} Y_W^ε(Y(u, f_ε(z_2, z))v, z_2) \). By substituting \( z \) by \( \frac{\phi_ε(z_2, z_0)}{z_2} - 1 \) and using the facts that (see [BLP])
\[
f_ε \left( \frac{\phi_ε(z_2, z_0)}{z_2} - 1 \right) = z_0 \quad \text{and} \quad \frac{\partial}{\partial z_0} \phi_ε(z_2, z_0) = \phi_ε(z_2, z_0)^{ε},
\]
we have
\[
\text{Res}_{z_2} z^{-1} Y_W^ε(Y(u, f_ε(z_2, z))v, z_2) = \text{Res}_{z_0} \left( \frac{\phi_ε(z_2, z_0)}{z_2} - 1 \right)^{-1} \left( \frac{\phi_ε(z_2, z_0)}{z_2} - 1 \right) Y_W^ε(Y(u, z_0)v, z_2) = \text{Res}_{z_0} (\phi_ε(z_2, z_0) - z_2)^{-1} \phi_ε(z_2, z_0) Y_W^ε(Y(u, z_0)v, z_2) = \text{Res}_{z_0} \sum_{n \geq 0} z_0^n z_2^{-n} h(z_2, z_0)^{-1} \phi_ε(z_2, z_0)^{ε} Y_W^ε(Y(u, z_0)v, z_2) = Y_W^ε(u - 1, v, z_2) + \sum_{n \geq 0} \sum_{i=0}^{n+1} \frac{1}{n!} \epsilon^{(i)} h_{n-i+1} z_0^{(n+1)(\epsilon-1)} Y_W^ε(u, v, z_2).
\]

On the other hand, by taking \( \text{Res}_z \text{Res}_{z_2} z^{-1} \) in the Jacobi-type identity (4.4), we find
\[
\text{Res}_z \text{Res}_{z_2} z^{-1} Y_W^ε(Y(u, f_ε(z_2, z))v, z_2) = \text{Res}_z \left( (z_1 - z_2)^{-1} Y_W^ε(u, z_1) Y_W^ε(v, z_2) - (z_2 - z_1)^{-1} Y_W^ε(v, z_2) Y_W^ε(u, z_1) \right) = \phi_ε(Y(u, z_2)) Y_W^ε(v, z_2)^{ε}.
\]
This completes the proof of the proposition. \( \square \)
Remark 4.7. Here we give the first four terms in the expression (4.6) for later use:

\[ Y^\gamma_W(u_{-1}v, z) = Y^\gamma_W(u, z)Y^\gamma_W(v, z) - \frac{1}{2} \varepsilon z^{-1}Y^\gamma_W(u_0 v, z) - \frac{5\varepsilon - 4}{12} \varepsilon z^{2(\varepsilon - 1)}Y^\gamma_W(u_2 v, z) + \cdots. \]

4.2. On \( \epsilon \)-deformation of vertex Lie algebras. We start with a notion of vertex Lie algebra introduced in [DLM] (see also [K2, P]). A vertex Lie algebra is a quadruple \((\mathcal{L}, \mathcal{A}, \mathcal{Z}, \eta)\), where \(\mathcal{L}\) is a Lie algebra, \(\mathcal{A}\) and \(\mathcal{Z}\) are two vector spaces, and

\(\eta : (\mathbb{C}[t, t^{-1}] \otimes \mathcal{A}) \oplus \mathbb{Z} \to \mathcal{L}, \ t^n \otimes a + c \mapsto a(n) + c \ (n \in \mathbb{Z}, a \in \mathcal{A}, c \in \mathcal{Z})\)

is a linear bijection, satisfying the condition that \(\eta(\mathcal{Z})\) is central in \(\mathcal{L}\) and for \(a, b \in \mathcal{A}\), there exist finitely many elements \(a(i,j)b \in \mathcal{A}\) and \(a(j)b \in \mathcal{Z}\) for \(i, j \in \mathbb{N}\) such that

\(\{a(z), b(w)\} = \sum_{i,j \in \mathbb{N}} \left( \frac{\partial}{\partial w} \right)^i (a(i,j)b)(w) + (a(j)b) \left( \frac{\partial}{\partial w} \right)^j z^{-1} \delta \left( \frac{w}{z} \right),\)

where \(a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}\) for \(a \in \mathcal{A}\).

Let \((\mathcal{L}, \mathcal{A}, \mathcal{Z}, \eta)\) be a vertex Lie algebra. Note that \(\mathcal{L}^+ = \eta(\mathbb{C}[t] \otimes \mathcal{A} \oplus \mathcal{Z})\) is a subalgebra of \(\mathcal{L}\). For any \(\gamma \in \mathbb{Z}^\ast\), denote by \(\mathbb{C}_\gamma\) the one dimensional \(\mathcal{L}^+\)-module such that \(\eta(\mathbb{C}[t] \otimes \mathcal{A})\) acts trivially and \(c \in \mathcal{Z}\) act as the scalars \(\gamma(c)\). Form the induced \(\mathcal{L}\)-module

\(V_\mathcal{L}(\gamma) = U(\mathcal{L}) \otimes U(\mathcal{L}) \mathbb{C}_\gamma.\)

Set \(1 = 1 \otimes 1 \in V_\mathcal{L}(\gamma)\) and identify \(\mathcal{A}\) as a subspace of \(V_\mathcal{L}(\gamma)\) with \(a = a(-1)1\) for \(a \in \mathcal{A}\). It was proved in [DLM] that there is a unique vertex algebra structure on \(V_\mathcal{L}(\gamma)\) with \(1\) as the vacuum vector and with \(Y(a, z) = a(z)\) for \(a \in \mathcal{A}\).

In what follows we define an \(\epsilon\)-deformation Lie algebra \(\mathcal{L}_\epsilon\) of \(\mathcal{L}\). As a vector space, \(\mathcal{L}_\epsilon\) is a linear copy of \(\mathcal{L}\) equipped with a linear bijection

\(\eta_\epsilon : (\mathbb{C}[t, t^{-1}] \otimes \mathcal{A}) \oplus \mathbb{Z} \to \mathcal{L}_\epsilon, \ t^n \otimes a + c \mapsto a^{(\epsilon)}(n) + c^{(\epsilon)} \ (n \in \mathbb{Z}, a \in \mathcal{A}, c \in \mathcal{Z}).\)

We define a multiplication on \(\mathcal{L}_\epsilon\) such that \(\eta_\epsilon(\mathcal{Z})\) is central in \(\mathcal{L}_\epsilon\) and

\[ [a^{(\epsilon)}(z), b^{(\epsilon)}(w)] = \sum_{i,j \in \mathbb{N}} \left( w^i \frac{\partial}{\partial w} \right)^i (a(i,j)b^{(\epsilon)})(w) + (a(j)b^{(\epsilon)}) \left( w^i \frac{\partial}{\partial w} \right)^j z^{-1} \delta \left( \frac{w}{z} \right), \]

where \(a^{(\epsilon)}(z) = \sum_{n \in \mathbb{Z}} a^{(\epsilon)}(n)z^{-n-1}\) for \(a \in \mathcal{A}\).

We say that an \(\mathcal{L}_\epsilon\)-module \(W\) is restricted if \(a^{(\epsilon)}(z) \in \text{Hom}(W, W((z)))\) for any \(a \in \mathcal{A}\), and is of level \(\gamma \in \mathbb{Z}^\ast\) if \(a^{(\epsilon)}\) act as the scalars \(\gamma(c)\) for \(c \in \mathcal{Z}\).

The main goal of this subsection is to prove the following two propositions.

Proposition 4.8. Let \((\mathcal{L}, \mathcal{A}, \mathcal{Z}, \eta)\) be a vertex Lie algebra. Then \(\mathcal{L}_\epsilon\) is a Lie algebra.

Proposition 4.9. Let \((\mathcal{L}, \mathcal{A}, \mathcal{Z}, \eta)\) be a vertex Lie algebra and \(\gamma \in \mathbb{Z}^\ast\). Then the restricted \(\mathcal{L}_\epsilon\)-modules \(W\) of level \(\gamma\) are exactly \(\phi_\gamma\)-coordinated \(V_\mathcal{L}(\gamma)\)-modules \((W, Y^\gamma_W)\) with \(Y^\gamma_W(a, z) = a^{(\epsilon)}(z)\) for \(a \in \mathcal{A}\).
For a conformal algebra $\mathcal{C}$, we define a multiplication on its loop space $\mathcal{C}[t, t^{-1}] \otimes \mathcal{C}$ by

$$[a^{(c)}(z), b^{(c)}(w)] = \sum_{i \geq 0} \frac{1}{i!} (a, b)^{(c)}(w) \left( w' \frac{\partial}{\partial w} \right)^i z^{-i-1} \delta \left( \frac{w}{z} \right)$$

where $a, b \in \mathcal{C}$ and $a^{(c)}(z) = \sum_{n \in \mathbb{Z}} (t^n \otimes a) z^{c-n-1}$. One can easily check that $\text{Im} (1 \otimes \partial + t^c \frac{\partial}{\partial t} \otimes 1)$ is a two-sided ideal of $\mathcal{C}[t, t^{-1}] \otimes \mathcal{C}$ under this multiplication. Form the quotient algebra

$$\widehat{\mathcal{C}} := \mathcal{C}[t, t^{-1}] \otimes \mathcal{C}/\text{Im} (1 \otimes \partial + t^c \frac{\partial}{\partial t} \otimes 1),$$

and denote by $\rho_c : \mathcal{C}[t, t^{-1}] \otimes \mathcal{C} \to \widehat{\mathcal{C}}$ the natural homomorphism. For each $a \in \mathcal{C}$, we will still denote the generating function $\sum_{n \in \mathbb{Z}} \rho_c(t^n \otimes a) z^{c-n-1}$ in $\widehat{\mathcal{C}}[[z, z^{-1}]]$ by $a^{(c)}(z)$.

A Lie conformal algebra is a conformal algebra $\mathcal{C}$ such that for $a, b, c \in \mathcal{C} and m, n \in \mathbb{N},$

$$a_n b = -\sum_{j=0}^{\infty} (-1)^{j+n} j! \partial^j (b_{n+j}) \quad \text{and} \quad a_m (b_n c) = \sum_{j=0}^{m} \binom{m}{j} \left( a_j b \right)_{m+n-j} c + b_n (a_m c).$$

The following results are from in [P, K2].

**Lemma 4.10.** Let $\mathcal{C}$ be a conformal algebra. Then $\mathcal{C}$ is a Lie conformal algebra if and only if $\widehat{\mathcal{C}}_0$ is a Lie algebra. Furthermore, in this case each $\widehat{\mathcal{C}}_0$ is also a Lie algebra.

Let $\mathcal{C}$ be a Lie conformal algebra. Set $\widehat{\mathcal{C}}^+_0 = \rho_0(\mathcal{C}[t, t^{-1}] \otimes \mathcal{C})$, a subalgebra of $\widehat{\mathcal{C}}_0$. Let $\mathcal{C}$ be the trivial $\widehat{\mathcal{C}}^+_0$-module and form the induced $\widehat{\mathcal{C}}^+_0$-module

$$V_{\mathcal{C}} = \mathcal{U}(\widehat{\mathcal{C}}_0) \otimes_{\mathcal{U}(\widehat{\mathcal{C}}_0)} \mathcal{C}.$$ 

Set $1 = 1 \otimes 1$ and identify $\mathcal{C}$ as a subspace of $V_{\mathcal{C}}$ with $a \mapsto \rho_0(t^{-1} \otimes a) 1$ for $a \in \mathcal{C}$. It is known that there is a unique vertex algebra structure on $V_{\mathcal{C}}$ with $1$ as the vacuum vector and with $Y(a, z) = a^{(0)}(z)$ for $a \in \mathcal{C}$ [P]. We say that a $\mathcal{C}$-module $W$ is restricted if $a^{(c)}(z) \in \text{Hom}(W, W(\{z\}))$ for any $a \in \mathcal{C}$. The following result was from [CLTW].

**Proposition 4.11.** Let $\mathcal{C}$ be a Lie conformal algebra. Then the restricted $\mathcal{C}$-modules $W$ are exactly the $\phi_c$-coordinated $\mathcal{C}$-modules $(W, Y_W(c))$ with $a^{(c)}(z) = Y_W(c)_{H}(a, z)$ for $a \in \mathcal{C}$.

**Proof of Propositions 4.8 and 4.9.** Associated to the vertex Lie algebra $\mathcal{L}$, we introduce a vector space as follows:

$$\mathcal{CL} = (\mathcal{C}\partial) \otimes \mathcal{A} \otimes \mathcal{Z}.$$ 

Endow $\mathcal{CL}$ with a $\mathcal{C}\partial$-module structure such that $\partial(\partial^m \otimes a) = \partial^{m+1} \otimes a$ and $\partial(c) = 0$ for $m \in \mathbb{N}, a \in \mathcal{A}$ and $c \in \mathcal{Z}$. Furthermore, define $j$-products ($j \in \mathbb{N}$) on $\mathcal{CL}$ such that

$$a_j b = a_{(j)} b + \sum_{i \geq 0} \partial^i (a_{(i, j)} b), \quad a_j c = c_j a = c_j c' = 0$$

for $a, b \in \mathcal{A}$ and $c, c' \in \mathcal{Z}$, and then extend them to the whole space $\mathcal{CL}$ via the rule (4.14). This gives a conformal algebra structure on $\mathcal{CL}$. Note that in $\mathcal{CL}$, we have $(\partial a)^{(c)}(z) = z^c \frac{\partial}{\partial z} a^{(c)}(z)$ for $a \in \mathcal{A}$ and $(\partial c)^{(c)}(z) = z^c \frac{\partial}{\partial z} c^{(c)}(z) = 0$ for $c \in \mathcal{Z}$. This implies that there is a linear isomorphism from $\mathcal{CL}$ to $\mathcal{L}$, such that

$$\rho_c (t^n \otimes a) \mapsto a^{(c)}(n), \quad \rho_c (t^{c-1} \otimes c) \mapsto c^{(c)}$$

for $n \in \mathbb{Z}, a \in \mathcal{A}$ and $c \in \mathcal{Z}$. Furthermore, under this linear isomorphism, the multiplication (4.12) on $\mathcal{CL}$ coincides with the multiplication (4.10) on $\mathcal{L}$. 

Since $L_0 = L$ is a Lie algebra, $\tilde{C}L_0$ is a Lie algebra as well. This together with Lemma 4.10 gives that $C\ell$ is a Lie conformal algebra and $\tilde{C}L_\gamma$ is a Lie algebra. Thus $L_\gamma$ is a Lie algebra, which proves the Proposition 4.8. For the Proposition 4.9, by identifying $\tilde{C}L_0$ with $L$, one can easily check that the vertex algebra $V_\ell(\gamma)$ is isomorphic to the quotient vertex algebra of $V_\ell \ell$ modulo the ideal generated by $c - \gamma(c)$ for $c \in Z$. According to Proposition 4.11, $\phi_c$-coordinated $V_\ell\ell$-modules are exactly restricted $L_{\gamma}$-modules, and hence $\phi_c$-coordinated $V_\ell(\gamma)$-modules are exactly restricted $L_{\gamma}$-modules of level $\gamma$. This finishes the proof of proposition 4.9.

5. Proof of theorem 3.2

5.1. $\phi_c$-coordinated modules for certain vertex algebras. In this subsection we study the $\phi_c$-coordinated modules for three types of vertex algebras.

First note that the derived subalgebra of $D'$ is

$$[D', D'] = \text{Span}_C\{\tilde{d}_{m,n} | m, n \in Z\}.$$ Set

$$\tilde{u}(\mathfrak{g})' = t(\mathfrak{g}) + [D', D'] + C\ell_0^{-1}d_1,$$

which is a subalgebra of $\tilde{u}(\mathfrak{g})'$ such that $\tilde{u}(\mathfrak{g})' = \tilde{u}(\mathfrak{g})' \oplus C\ell_0^{-1}d_0$. Recall the space $B_{\ell}$ defined in (2.5). We define a linear isomorphism $\tilde{\theta}: \tilde{u}(\mathfrak{g})' \rightarrow \tilde{w}(\mathfrak{g})'$ as follows:

$$\tilde{t}^n \otimes u \mapsto \tilde{t}^n u, \quad \tilde{t}^n \otimes D_m \mapsto d_{n+1-m}, \quad \tilde{t}^n \otimes K_m \mapsto K_{n+1-m}, \quad k_0 \mapsto k_0$$

for $u \in \tilde{\mathfrak{g}}_1, n \in Z$ and $m \in Z^\times$.

In view of Proposition 2.3 we see that the quadruple

$$(L = \tilde{u}(\mathfrak{g})^0, A = B_{\ell}, Z = Ck_0, \eta = \theta_0)$$

is a vertex Lie algebra. Furthermore, the map $\eta_c \circ \theta_c^{-1}$ (see (1.10)) is a Lie isomorphism from $\tilde{u}(\mathfrak{g})'$ to $\tilde{u}(\mathfrak{g})^0$. Then for any complex number $\ell$, we have a vertex algebra $V_{\tilde{u}(\mathfrak{g})^0(\gamma_\ell)}$, where $\gamma_\ell \in (Ck_0)\ell$ defined by $\gamma_\ell(k_0) = \ell$. We say that a $\tilde{u}(\mathfrak{g})'$-module $W$ is restricted if for any $a \in B_{\ell}$, $a'(z) \in \text{Hom}(W, W(\ell(z)))$, and is of level $\ell \in C$ if $k_0$ acts as the scalar $\ell$ on $W$. Then by Proposition 1.9 and the Lie isomorphism $\eta_c \circ \theta_c^{-1} : \tilde{u}(\mathfrak{g})' \rightarrow \tilde{u}(\mathfrak{g})^0_c$, we immediately have the following result.

**Proposition 5.1.** For any $\ell \in C$, the restricted $\tilde{u}(\mathfrak{g})'$-modules of level $\ell$ are exactly the $\phi_c$-coordinated $V_{\tilde{u}(\mathfrak{g})^0(\gamma_\ell)}$-modules ($W, Y_{\tilde{\mathfrak{g}}}$) with $Y_{\tilde{\mathfrak{g}}}(a, z) = a'(z)$ for $a \in B_{\ell}$.

Next, recall the affine-Virasoro algebra $\tilde{\mathfrak{g}} \rtimes \mathfrak{g}$ defined in Section 3. Form the vector spaces $A = \mathfrak{g} \oplus C\omega_0$ and $Z = Ck \oplus Ck_{\mathfrak{g}}$. Define a linear bijection

$$\eta : (C[t, t^{-1}] \otimes A) \oplus Z \rightarrow \tilde{\mathfrak{g}} \rtimes \mathfrak{g}$$

by letting

$$t^n \otimes u \mapsto u(n), \quad t^n \otimes \omega \mapsto L(n-1), \quad k \mapsto k, \quad k_{\mathfrak{g}} \mapsto k_{\mathfrak{g}}$$

for $n \in Z, u \in \mathfrak{g}$. Note that in terms of generating functions

$$u(z) = \sum_{n \in Z} u(n)z^{-n-1} \quad (u \in \mathfrak{g}) \quad \text{and} \quad \omega(z) = \sum_{n \in Z} \omega(n)z^{-n-1} = \sum_{n \in Z} L(n)z^{-n-2},$$
the Lie relations in \((5.3)\) can be rewritten as follows:

\[
[\omega_\ell(z), \omega_\ell(w)] = \left( \frac{\partial}{\partial w} \omega_\ell(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + 2 \omega_\ell(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right) + \frac{1}{12} \kappa_{\mathfrak{Vir}} \left( \frac{\partial}{\partial w} \right)^3 z^{-1} \delta \left( \frac{w}{z} \right),
\]

\[
[u(z), v(w)] = [u, v](w) z^{-1} \delta \left( \frac{w}{z} \right) + \langle u, v \rangle k \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right),
\]

\[
[\omega_\ell(z), u(w)] = \left( \frac{\partial}{\partial w} u(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + u(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right)
\]

for \(u, v \in \mathfrak{g}\). This implies that the quadruple

\[(\mathcal{L} = \hat{\mathfrak{g}} \rtimes \mathfrak{Vir}, \mathcal{A} = \mathfrak{g} \oplus \mathbb{C} \omega_\ell, Z = C \mathfrak{g} \oplus C \kappa_{\mathfrak{Vir}}, \eta)\]

is a vertex Lie algebra.

For \(\ell, c \in \mathbb{C}\), denote by \(\gamma_{\ell, c}\) the linear functional on \(\mathcal{Z}^*\) such that \(\gamma_{\ell, c}(k) = \ell\) and \(\gamma_{\ell, c}(\kappa_{\mathfrak{Vir}}) = c\). Then we have the (universal) affine-Virasoro vertex algebra

\[V_{\mathfrak{g} \rtimes \mathfrak{Vir}}(\ell, c) := V_{\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}}(\gamma_{\ell, c}),\]

which is equipped with a conformal vector \(\omega_\ell\). We say that a \(\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}\)-module \(W\) is restricted if for any \(w \in W, a \in \mathfrak{g}\), one has \(a(n)w = L(n)w\) for \(n \gg 0\), is of level \(\ell \in \mathbb{C}\) if \(k\) acts as the scalar \(\ell\), and is of central charge \(c\) if \(\kappa_{\mathfrak{Vir}}\) acts as the scalar \(c\). Recall the generating functions \(a^c(z)\) for \(a \in \mathfrak{g}\) and \(L^c(z)\) defined in \((3.2)\) and \((3.3)\), respectively. Then we have:

**Proposition 5.2.** For \(\ell, c \in \mathbb{C}\), the restricted \(\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}\)-modules \(W\) of level \(\ell\) and central charge \(c\) are exactly the \(\phi_c\)-coordinated \(V_{\mathfrak{g} \rtimes \mathfrak{Vir}}(\ell, c)\)-modules \((W, Y^c_W)\) with

\[
Y^c_W(a(z)) = a^c(z), \forall a \in \mathfrak{g}; \quad Y^c_W(\omega_{\ell,c}, z) = L^c(z).
\]

**Proof.** By Proposition 4.8 we have an \(c\)-deformation Lie algebra

\[\hat{\mathfrak{g}} \rtimes \mathfrak{Vir} = \text{Span}_\mathbb{C}\{u^c(n), \omega_{\ell,c}(n), k^c, \kappa_{\mathfrak{Vir}} \mid u \in \mathfrak{g}, n \in \mathbb{Z}\}\]

of \(\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}\) with the Lie brackets given by

\[
[w^c(z), \omega_{\ell,c}(w)] = \left( w^c \frac{\partial}{\partial w} \omega_{\ell,c}(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + 2 \omega_{\ell,c}(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right) + \frac{1}{12} k_{\mathfrak{Vir}} \left( \frac{\partial}{\partial w} \right)^3 z^{-1} \delta \left( \frac{w}{z} \right),
\]

\[
[u^c(z), v^c(w)] = [u, v]^c(w) z^{-1} \delta \left( \frac{w}{z} \right) + \langle u, v \rangle k^c \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right),
\]

\[
[\omega_{\ell,c}(z), u^c(w)] = \left( w^c \frac{\partial}{\partial w} u^c(w) \right) z^{-1} \delta \left( \frac{w}{z} \right) + u^c(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right)
\]

for \(u, v \in \mathfrak{g}\).

It is straightforward to check that the linear map

\[
u^c(n) \mapsto u(n), \quad \omega_{\ell,c}(n) \mapsto L(n + c - 1) + \delta_{n, c-1} \frac{\ell^2 - 2\ell}{24} k_{\mathfrak{Vir}}, \quad k^c \mapsto k, \quad \kappa_{\mathfrak{Vir}} \mapsto \kappa_{\mathfrak{Vir}}
\]

where \(u \in \mathfrak{g}\) and \(n \in \mathbb{Z}\), is a Lie algebra isomorphism from \((\hat{\mathfrak{g}} \rtimes \mathfrak{Vir})_c\) to \(\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}\). Then the assertion follows immediately from Proposition 4.8. \(\square\)

Finally, recall from Section 3.2 that we have an \(\mathfrak{h}\)-module

\[V_{\mathfrak{g}}(\ell, c) \mathbb{C}[L] = V_{\mathfrak{g}}(\ell, 0) \otimes e^{\alpha c} \mathbb{C}[L].\]
When \( \ell = 1 \) and \( \alpha = 0 \), it is known that there is a vertex algebra structure on

\[
V_{(b,L)} := V_{\hat{b}}(1, C[L]) = V_{\hat{b}}(1, 0) \otimes C[L]
\]

with \( Y(h, z) = h(z) \) for \( h \in \mathfrak{h} \) and \( Y(e^\gamma, z) = E^\gamma(z) \) (see [31] or [LW] for example). Note that \( \omega_b = k_{-1}d \) is a conformal vector of \( V_{(b,L)} \) of rank 2, which is also a conformal vector of the Heisenberg vertex subalgebra \( V_{\hat{b}}(1,0) \).

View \( \mathfrak{h} \) and \( C[L] \) as abelian Lie algebras and let \( \mathfrak{h} \) act on \( C[L] \) by derivations with

\[
he^\gamma = \langle h, \gamma \rangle e^\gamma
\]

for \( h \in H, \gamma \in \mathcal{L} \). Form the semiproduct Lie algebra

\[
\mathfrak{p} = \mathfrak{h} \ltimes C[L],
\]

on which we have

\[
[h,h'] = 0 = [e^\gamma, e^\gamma'], \quad [h, e^\gamma] = \langle h, \gamma \rangle e^\gamma
\]

for \( h, h' \in \mathfrak{h}, \gamma, \gamma' \in \mathcal{L} \). Extend the form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h} \) (see [31]) to \( \mathfrak{p} \) by letting \( \langle \mathfrak{p}, C[L] \rangle = 0 \), which is still symmetric and invariant. Recall that \( \mathfrak{p} \) is the affine Lie algebra associated to the pair \( (\mathfrak{p}, \langle \cdot, \cdot \rangle) \), which is obviously a vertex Lie algebra with \( \mathcal{A} = \mathfrak{p} \) and \( Z = \mathbb{C}k \). Take \( \gamma \in (\mathbb{C}k)^* \) such that \( \gamma(k) = 1 \). Then we have the universal affine vertex algebra

\[
V_{\hat{b}}(1,0) := V_{\hat{b}}(\gamma).
\]

The following relation between the vertex algebras \( V_{(b,L)} \) and \( V_{\hat{b}}(1,0) \) was proved in [LW].

**Lemma 5.3.** The vertex algebra \( V_{(b,L)} \) is isomorphic to the quotient vertex algebra of \( V_{\hat{b}}(1,0) \) modulo the ideal generated by the set

\[
\{e^0 - 1, e^\gamma e^\lambda - e^{\gamma + \lambda}, D e^\lambda - \lambda(-1)e^\lambda \mid \gamma, \lambda \in \mathcal{L} \}.
\]

Recall that \( u'(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1} \) for \( u \in \mathfrak{p} \). In view of the above lemma, we have:

**Proposition 5.4.** The \( \phi_e \)-coordinated \( V_{(b,L)} \)-modules \( (W, Y_W^\gamma) \) are exactly the restricted \( \hat{\mathfrak{p}} \)-modules \( W \) of level 1 satisfying the conditions that

\[
(e^0)^\gamma(z) = 1, \quad (e^\gamma)^\lambda(z) = (e^{\gamma + \lambda})^\gamma(z), \quad z^\gamma d \frac{d}{dz}(e^\lambda)^\gamma(z) = \gamma'(z)(e^\lambda)^\gamma(z)
\]

for \( \gamma, \lambda \in \mathcal{L} \). And the correspondence is given by \( Y_W^\gamma(h,z) = h^\gamma(z) \) and \( Y_W^\gamma(e^\gamma,z) = (e^\gamma)^\gamma(z) \) for \( h \in \mathfrak{h} \) and \( \gamma \in \mathcal{L} \).

Proof. It is routine to see that the \( e \)-deformation \( \hat{\mathfrak{p}}_e \) of \( \hat{\mathfrak{p}} \) is isomorphic to \( \hat{\mathfrak{p}} \) itself with

\[
u^{(e)}(n) \mapsto u(n) \quad \text{and} \quad k^{(e)} \mapsto k
\]

for \( u \in \mathfrak{p}, n \in \mathbb{Z} \). Thus from Proposition 4.9 it follows that the \( \phi_e \)-coordinated \( V_{\hat{b}}(1,0) \)-modules \( (W, Y_W^\gamma) \) are exactly the restricted \( \hat{\mathfrak{p}} \)-modules \( W \) of level 1 with

\[
Y_W^\gamma(u,z) = u^\gamma(z), \quad \forall u \in \mathfrak{p}.
\]

Lemma 5.3 implies that the \( \phi_e \)-coordinated \( V_{(b,L)} \)-modules are exactly those \( \phi_e \)-coordinated \( V_{\hat{b}}(1,0) \)-modules \( (W, Y_W^\gamma) \) such that \( Y_W^\gamma(v,z) = 0 \) for all \( v \) in the set \( (5.5) \). Thus, by the correspondence \( (5.5) \), the \( \phi_e \)-coordinated \( V_{(b,L)} \)-modules are exactly the restricted \( \hat{\mathfrak{p}} \)-modules \( W \) of level 1 such that the following relations hold:

\[
Y_W^\gamma(e^0 - 1, z) = (e^0)^\gamma(z) - 1 = 0,
\]

\[
Y_W^\gamma(e^\gamma e^\lambda - e^{\gamma + \lambda}, z) = Y_W^\gamma(e^\gamma, z)Y_W^\gamma(e^\lambda, z) - Y_W^\gamma(e^{\gamma + \lambda}, z)
\]

\[
= (e^\gamma)^\gamma(z)(e^\lambda)^\gamma(z) - (e^{\gamma + \lambda})^\gamma(z) = 0,
\]
Remark 4.7 we have \( \omega \), and it remains to deduce the action \( \phi \) irreducible. This completes the proof. 

Recall the operator \( E^\gamma(z) \) defined in (3.10). The following result shows that there is a canonical \( \phi_\gamma \)-coordinated \( \hat{V}_{(b,L)} \)-module structure on the \( \hat{V}_b \)-module \( \hat{V}_b(\ell,e^{\alpha k}C[L]) \). Similar results for lattice vertex algebras have been obtained in [JKK].

**Proposition 5.5.** For every \( \alpha \in \mathbb{C} \) and \( \ell \in \mathbb{C}^\times \), the \( \hat{V}_b \)-module \( \hat{V}_b(\ell,e^{\alpha k}C[L]) \) admits an irreducible \( \phi_\gamma \)-coordinated \( \hat{V}_{(b,L)} \)-module structure \( \hat{Y}_W \), which is uniquely determined by

\[
\hat{Y}_W(\kappa, z) = \frac{1}{\ell} \kappa'(z), \quad \hat{Y}_W(\ell, z) = \ell' \quad \text{and} \quad \hat{Y}_W(e^\gamma, z) = E^\gamma(z), \quad \forall \gamma \in L.
\]

Furthermore, the action of the conformal vector \( \omega_b \) is (see (3.3) and (3.5))

\[
\hat{Y}_W(\omega_b, z) = L_b'(z).
\]

**Proof.** We first claim that the assignment

\[
\kappa(z) \mapsto \frac{1}{\ell} \kappa(z), \quad d(z) \mapsto d(z), \quad k \mapsto 1 \quad \text{and} \quad e^\gamma(z) \mapsto z^{-\ell} E^\gamma(z) \quad (\gamma \in L)
\]

gives an irreducible restricted \( \hat{V}_b \)-module structure on the \( \hat{V}_b \)-module \( \hat{V}_b(\ell,e^{\alpha k}C[L]) \) and on which the three conditions stated in (5.5) hold.

Indeed, the commutators among those operators in (5.5) are as follows (see [LL] Proposition 6.5.2) for example

\[
[k(z), d(w)] = \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right), \quad [k(z), k(w)] = 0 = [d(z), d(w)],
\]

\[
[k(z), E^\gamma(w)] = 0, \quad [d(z), w^{-\ell} E^\gamma(w)] = (d, \gamma) w^{-\ell} E^\gamma(w) z^{-1} \delta \left( \frac{w}{z} \right).
\]

This implies that \( \hat{V}_b(\ell,e^{\alpha k}C[L]) \) is a restricted \( \hat{V}_b \)-module of level 1 (with the action (5.9)). Meanwhile, it is known that (see [LL] (6.5.64) for example)

\[
E^\lambda(z) E^\gamma(z) = E^\lambda+\gamma(z), \quad \frac{d}{dz} E^\lambda(z) = \lambda(z) E^\lambda(z)
\]

for \( \gamma, \lambda \in L \). From this and the fact that \( (e^\gamma)'(z) = E^\gamma(z) \), one can conclude that this \( \hat{V}_b \)-module satisfies the conditions in (5.5), as desired.

In view of Proposition 5.5, the above claim implies that the assignment (5.9) determines a \( \phi_\gamma \)-coordinated \( \hat{V}_{(b,L)} \)-module structure on \( \hat{V}_b(\ell,e^{\alpha k}C[L]) \). The irreducibility of this \( \hat{V}_{(b,L)} \)-module is obvious, and it remains to deduce the action \( \omega_b \) on \( W \).

Note that in \( \hat{V}_{b,L} \) we have \( \kappa_n d = 0 \) for \( n \geq 0 \) unless \( n = 1 \), in which case \( \kappa_1 d = 1 \). Then from Remark 4.7 we have

\[
\hat{Y}_W(\omega_b, z) = Y^\gamma(\kappa_1 d, z)
\]

\[
= -\delta_0 Y_W(\kappa, z) Y_W(\ell, z) - \frac{5\ell - 4}{12} \varepsilon z^{2(\ell - 1)}
\]

\[
= \frac{1}{\ell} \delta k'(z) d'(z) - \frac{5\ell - 4}{12} \varepsilon z^{2(\ell - 1)}.
\]
Recall from (3.8) that \( L_\ell(z) = \frac{1}{\ell} \partial \phi(z) d'(z) \), and from (3.3) that
\[
L'_\ell(z) = z^{2\ell} L_\ell(z) + \frac{\ell^2 - 2\ell}{12} z^{2\ell-1} = \frac{1}{\ell} z^{2\ell} \partial \phi(z) d(z) + \frac{\ell^2 - 2\ell}{12} z^{2\ell-1}.
\]

By the definition of normally ordered products, in the \( \hat{\mathfrak{h}} \)-module \( V_\ell(\ell, e^{ak} C[L]) \) we have
\[
\frac{1}{\ell} \partial \phi(z) d'(z) = \frac{1}{\ell} \sum_{n<\epsilon} k(n) z^{n+1} d'(z) + d'(z) \sum_{n\geq \epsilon} k(n) z^{-n-1} = z^{2\ell} \frac{1}{\ell} \sum_{n<\epsilon} k(n) z^{-n-1} d(z) + d(z) \sum_{n\geq \epsilon} k(n) z^{-n-1}.
\]

This implies that
\[
\frac{1}{\ell} \partial \phi(z) d'(z) = z^{2\ell} \frac{1}{\ell} \sum_{n<\epsilon} k(n) z^{-n-1} d(z) + d(z) \sum_{n\geq \epsilon} k(n) z^{-n-1}.
\]

This finishes the proof. \( \square \)

5.2. **Proof of theorem 3.2.** This section is devoted to a proof of Theorem 3.2.

By taking the tensor product of the latter two vertex algebras considered in last subsection, for every nonzero complex number \( \ell \), we have a conformal vertex algebra
\[
V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\ell, 24\mu \ell - 2) \otimes V_{(\mathfrak{h}, L)}
\]
which has a conformal vector
\[
\omega = \omega_0 \otimes 1 + 1 \otimes \omega_\mathfrak{h}.
\]

We will often view \( V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\ell, 24\mu \ell - 2) \) and \( V_{(\mathfrak{h}, L)} \) as subalgebras of \( V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\ell, 24\mu \ell - 2) \otimes V_{(\mathfrak{h}, L)} \) in the canonical way. The following result was obtained in [CLIT, Corollary 5.9]:

**Proposition 5.6.** Let \( \ell \in \mathbb{C}^\times \). Then there exists a vertex algebra epimorphism
\[
\Theta : V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\gamma_\ell) \rightarrow V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\ell, 24\mu \ell - 2) \otimes V_{(\mathfrak{h}, L)}
\]
which is uniquely determined by
\[
\Theta(t^n \otimes u) = u \otimes e^{nk}, \quad \Theta(k) = k, \quad \Theta(d_1) = d, \quad \Theta(K_n) = \frac{\ell}{n} e^{nk},
\]
\[
\Theta(D_n) = n \omega_0 e^{nk} - \omega d_1 e^{nk} + n^2 (\mu \ell - 1) k - 2e^{nk}
\]
for \( u \in \mathfrak{g}, m \in \mathbb{Z} \) and \( n \in \mathbb{Z}^\times \).

The following result asserts that every (irreducible) \( \phi_\ell \)-coordinated \( V_{\hat{\mathfrak{h}} \otimes \mathfrak{vir}}(\ell, 24\mu \ell - 2) \otimes V_{(\mathfrak{h}, L)} \)-module admits an (irreducible) \( \hat{\mathfrak{g}} \)-module structure.
Proof. This implies that for \( j > 0 \).

From Proposition 5.6, it follows that \((W,Y)\) is irreducible, then \( W \) is also irreducible as a \( \hat{\mathfrak{h}}(g)\)-module.

Let \((W,Y)\) be a \( \mathfrak{h}(g)\)-module with \( \ell \neq 0 \). For every \( v \in V_{\hat{\mathfrak{h}}}(\gamma) \), set

\[
Y_W[v, z] := Y_W^\ell(\Theta(v), z) \in \text{Hom}(W, W((z))).
\]

From Proposition 5.6, it follows that \((W,Y_W^\ell[v, z])\) is a \( \mathfrak{h}(g)\)-module. Thus, by applying Proposition 5.1, \( W \) becomes a restricted \( \hat{\mathfrak{h}}(g)_\mu\)-module of level \( \ell \) with

\[
a^\ell(z) = Y_W^\ell[a, z], \quad \forall a \in B_g.
\]

Assume further that \((W,Y_W^\ell[v, z])\) is irreducible. Then the \( \mathfrak{h}(g)_\ell\)-module \((W,Y_W^\ell[v, z])\) is also irreducible as the map \( \Theta \) is surjective. This together with Proposition 5.1 gives that as a \( \hat{\mathfrak{h}}(g)\)-module, \( W \) is still irreducible.

For every \( v \in V_{\hat{\mathfrak{h}}}(\ell, 24\mu \ell - 2) \otimes V_{(h,L)} \), it follows from Lemma 4.5 and Lemma 4.2 that

\[
[\text{Res}_z z^{-\ell} Y_W^\ell(\omega, z), Y_W^\ell(v, w)] = Y_W^\ell(\omega v, w) = Y_W^\ell(Dv, w) = w^\ell d \frac{d}{dw} Y_W^\ell(v, w).
\]

In particular, for \( a \in B_g \), we have

\[
[-\text{Res}_z z^{-\ell} Y_W^\ell(\omega, z), Y_W^\ell[a, w]] = -w^\ell d \frac{d}{dw} Y_W^\ell[a, w].
\]

This together with Proposition 2.3 (12) implies that the \( \hat{\mathfrak{h}}(g)\)-module \( W \) can be extended to a \( \hat{\mathfrak{h}}(g)^{\ell}\)-module with

\[
\ell_0^{-1}d_0 = -\text{Res}_z z^{-\ell} Y_W^\ell(\omega, z).
\]

Thus it remains to prove that the operators \( Y_W(a, z) \), \( a \in B_g \) have the form as given in theorem.

The case that \( a \in \mathfrak{g} \oplus \sum n \in Z \mathfrak{c} K_n \) is obvious, and so we only need to prove that the operators \( Y_W[Dn, z] \), \( n \in Z^\times \) coincide with the right hand side of \((5.14)\).

Indeed, note that for \( i \geq 0 \) and \( n \in Z \), we have

\[
d_i e^{nk} = n \delta_{i,0} e^{nk}, \quad k_i e^{nk} = 0.
\]

This implies that for \( j > 0 \), \( (\omega_h)_j e^{nk} = (k_{-1}d)_j e^{nk} = 0 \) and hence \( \omega_j e^{nk} = 0 \). Thus, by Remark 4.7 and Lemma 4.2, we have

\[
Y_W^\ell(\omega_{-1} e^{nk}, z) = e^\ell Y_W^\ell(e^{nk}, z) - \frac{\ell}{2} z e^{-1} Y_W(\omega_0 e^{nk}, z) = e^\ell Y_W^\ell(e^{nk}, z) - \frac{\ell}{2} z e^{-1} \frac{d}{dz} Y_W(e^{nk}, z).
\]

Similarly, from Remark 4.7 and Lemma 4.2, it follows that

\[
Y_W^\ell(\omega_0 (d_{-1} e^{nk}), z)
\]
= z^\epsilon \frac{d}{dz} W^\epsilon_Y (d_{-1} e^{nk}, z)
= z^\epsilon \frac{d}{dz} \circ W^\epsilon_Y (d, z) W^\epsilon_Y (e^{nk}, z)\circ - \frac{1}{2} z^\epsilon - 1 W^\epsilon_Y (n e^{nk}, z)
= z^\epsilon \frac{d}{dz} \circ W^\epsilon_Y (d, z) W^\epsilon_Y (e^{nk}, z)\circ - \frac{1}{2} n e (\epsilon - 1) z^{2\epsilon - 2} W^\epsilon_Y (e^{nk}, z) - \frac{1}{2} n e z^{2\epsilon - 1} \frac{d}{dz} W^\epsilon_Y (e^{nk}, z),

and that
\[ Y^\epsilon_W (k_{-2} e^{nk}, z) = Y^\epsilon_W ((\omega_0 k)_{-1} e^{nk}, z) = \circ W^\epsilon_Y (\omega_0 k, z) Y^\epsilon_W (e^{nk}, z)\circ \]
\[ = z^\epsilon \frac{d}{dz} W^\epsilon_Y (k, z) Y^\epsilon_W (e^{nk}, z), \]

where we used the fact that \((\omega_0 k) e^{nk}\) for \(i \geq 0\) as \(k, e^{nk} = 0\).

Finally, by summarizing the above together, we obtain
\[ Y^\epsilon_W [D_n, z] = Y^\epsilon_W (\Theta(D_n), z) \]
\[ = Y^\epsilon_W (n \omega_{-1} e^{nk} - \omega_0 d_{-1} e^{nk} + n^2 (\mu \ell - 1) k_{-2} e^{nk}, z) \]
\[ = n \circ Y^\epsilon_W (\omega, z) W^\epsilon_Y (e^{nk}, z)\circ - \frac{1}{2} n e z^{2\epsilon - 1} \frac{d}{dz} W^\epsilon_Y (e^{nk}, z) \]
\[ = z^\epsilon \frac{d}{dz} \circ Y^\epsilon_W (d, z) Y^\epsilon_Y (e^{nk}, z)\circ + \frac{1}{2} n e (\epsilon - 1) z^{2\epsilon - 2} Y^\epsilon_W (e^{nk}, z) \]
\[ + \frac{1}{2} n e z^{2\epsilon - 1} \frac{d}{dz} Y^\epsilon_W (e^{nk}, z) + n^2 (\mu \ell - 1) z^\epsilon \frac{d}{dz} Y^\epsilon_W (k, z) Y^\epsilon_Y (e^{nk}, z) \]
\[ = n \circ Y^\epsilon_W (\omega, z) Y^\epsilon_W (e^{nk}, z)\circ + z^\epsilon \frac{d}{dz} Y^\epsilon_W (d, z) Y^\epsilon_Y (e^{nk}, z)\circ 
\[ + \frac{1}{2} n e (\epsilon - 1) z^{2\epsilon - 2} Y^\epsilon_Y (e^{nk}, z) + n^2 (\mu \ell - 1) z^\epsilon \frac{d}{dz} Y^\epsilon_Y (k, z) Y^\epsilon_Y (e^{nk}, z), \]

as desired.

\[
\textbf{Proof of Theorem 3.2} \quad \text{Now we are ready to complete the proof of Theorem 3.2. Let } U, \ell, \alpha, \beta \in \mathbb{C} \text{ be as in Theorem 3.2. By Proposition 5.2, the irreducible highest weight } \mathfrak{g} \rtimes \text{Vir-module} \]
\[ W_1 := L_{\mathfrak{g} \rtimes \text{Vir}} (\ell, 24 \mu \ell - 2, U, \beta) \]

is naturally an irreducible \(\phi_e\)-coordinated \(V_{\mathfrak{g} \rtimes \text{Vir}} (\ell, 24 \mu \ell - 2)\)-module. Meanwhile, by Proposition 5.6, the \(\hat{h}\)-module
\[ W_2 := V^\epsilon_{\hat{h}} (\ell, e^{nk} \mathbb{C}[L]) \]

is naturally an irreducible \(\phi_e\)-coordinated \(V_{(\ell, L)}\)-module. Thus, by taking tensor product (see Lemma 4.4), we obtain an irreducible \(\phi_e\)-coordinated \(V_{\mathfrak{g} \rtimes \text{Vir}} (\ell, 24 \mu \ell - 2) \otimes V_{(\ell, L)}\)-module
\[ W := W_1 \otimes W_2 = L_{\mathfrak{g} \rtimes \text{Vir}} (\ell, 24 \mu \ell - 2, U, \beta) \otimes V^\epsilon_{\hat{h}} (\ell, e^{nk} \mathbb{C}[L]) \]

with the actions given by
\[ Y^\epsilon_W (u, z) = Y^\epsilon_{W_1} (u, z) \otimes 1 = u^\epsilon (z) \otimes 1 = u^\epsilon (z), \]
\[ Y^\epsilon_W (k, z) = 1 \otimes Y^\epsilon_{W_2} (k, z) = 1 \otimes \frac{1}{\ell} k^\epsilon (z) = \frac{1}{\ell} k^\epsilon (z), \]
\[ Y^\epsilon_W (d, z) = 1 \otimes Y^\epsilon_{W_2} (d, z) = 1 \otimes d^\epsilon (z) = d^\epsilon (z), \]
\[ Y^\epsilon_W (e^{nk}, z) = 1 \otimes Y^\epsilon_{W_2} (e^{nk}, z) = 1 \otimes E^{nk} (z) = E^{nk} (z), \]
\[ Y^\epsilon_W (\omega, z) = Y^\epsilon_{W_1} (\omega_0, z) \otimes 1 + 1 \otimes Y^\epsilon_{W_2} (\omega_1, z) = L^\epsilon_0 (z) \otimes 1 + 1 \otimes L^\epsilon_1 (z) = L^\epsilon_1 (z), \]

\[
\]
for \( u \in g, \, n \in \mathbb{Z} \), where we have used the actions \( \Box \Box \), \( \Box \Box \Box \), the fact \( \Box \Box \Box \), and the conventions \( \Box \Box \Box \), \( \Box \Box \Box \).

According to Theorem \( \Box \Box \Box \), the \( \phi \)-coordinated \( V_{\tilde{g},\omega}(\ell, 24\mu \ell - 2) \otimes V_{(b, L)} \)-module \( W \) is also an irreducible \( \tilde{f}(g)^{\epsilon} \)-module with \( k_0 = \ell \) and

\[
\begin{align*}
\ell_0^{-1} d_0 &= -\text{Res}_z z^{-\gamma} Y_W^{(\ell)}(\omega, z) = -L_1(\epsilon - 1) + \delta_{\epsilon, 1} \mu \ell, \\
(\ell_0^n \otimes u)^{\epsilon}(z) &= Y_W^{(\ell)}(u \otimes e^{m\ell}, z) = (Y_W^{e^{\ell}}(u, z) \otimes 1)(1 \otimes Y_W^{e^{m\ell}}(e^{m\ell}, z)) = u^{\epsilon}(z)E^{m\ell}(z), \\
k_W^{(\ell)}(z) &= \ell Y_W^{(\ell)}(u, z) = k^{(\ell)}(z), \\
K_W^{(\ell)}(z) &= \frac{\ell}{n} Y_W^{(\ell)}(e^{m\ell}, z) = \frac{\ell}{n} E^{m\ell}(z), \\
D_W^{(\ell)}(z) &= n^2 Y_W^{(\ell)}(e^{m\ell}, z) - z^2 \frac{d}{dz} Y_W^{(2)\ell}(\ell, z) + \frac{1}{2} n(\epsilon - 1) z^{2\epsilon - 2} Y_W^{(2)\ell}(\ell, z).
\end{align*}
\]

This finishes the proof of Theorem \( \Box \Box \Box \).

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