HOPF COACTIONS ON ODD SPHERES

SUVR AJIT BHATTACHARJEE AND DEBASHISH GOSWAMI

Abstract. We prove that the q-deformed unitary group, i.e., $U_q(N)$, is the universal compact quantum group in the category of (compact) quantum groups which coact on the q-deformed odd sphere $S^{2N-1}_q$ leaving the space spanned by the natural set of generators invariant and preserving the unique $SU_q(N)$ invariant functional on $S^{2N-1}_q$. Using this, we identify $U_q(N)$ as the quantum group of orientation preserving isometries (in the sense of Bhowmick and Goswami [BG09]) for a natural spectral triple associated with $S^{2N-1}_q$ constructed by Chakraborty and Pal [CP08].

1. Introduction

Symmetry is one of the most fundamental and ubiquitous concepts in mathematics and other areas of science. Classically, symmetry of some mathematical structure is understood as the group of automorphisms or isomorphisms (in a suitable sense) of the structure. Generalizing the concept of group, Drinfel’d and Jimbo (Dri87, Jim85, Jim86) introduced the notion of quantum groups. Later on, Woronowicz (Wor87, Wor98) formulated an analogue of this notion in the analytical framework of $C^*$-algebras.

It is natural to conceive of quantum groups as ‘symmetry objects’ for certain mathematical structures, e.g., rings or $C^*$-algebras. Indeed, Manin (Man88) pioneered the idea of viewing q-deformation of classical Lie groups (e.g. $SL_q(N)$) as some kind of ‘quantum automorphism group’.

A similar approach was taken by Wang (Wan98) who defined quantum permutation group of finite sets and quantum automorphism group of finite dimensional matrix algebras. This was followed by flurry of work by several other mathematicians including Banica and Bichon (just to name a few). The second author of the present article and his collaborators (including Bhowmick, Skalski and others) approached the problem from a geometric perspective and formulated an analogue of the Riemannian isometry groups in the framework of (compact) quantum groups acting on $C^*$-algebras. We refer the reader to [GB16] and the references therein for a comprehensive account of the theory of quantum isometry groups.

A useful procedure of producing genuine examples of ‘non-commutative spaces’ is to deform the coordinate algebra or some other suitable function algebra underlying a classical space. In this context, it is natural to ask the following: what is the quantum isometry group of a non-commutative space obtained by deforming a classical space? It is expected that under mild assumptions, it should be isomorphic with a deformation of the isometry group of the classical space, at least when the
classical space is connected. Indeed, for a quite general class of cocycle deformation (called the Rieffel deformation), such a result has been proved by Bhowmick, Goswami and Joardar ([GJ14]).

However, no such general result has yet been achieved for the Drinfel’d-Jimbo type q-deformation of semisimple Lie groups and the corresponding homogeneous spaces. The goal of the present paper is to make some progress in this direction. We have been able to prove the above result for q-deformed odd spheres, i.e., \( S^q_{2N-1} \) ([VS90]). Classically (for \( q = 1 \)), these are nothing but the complex \( N - 1 \) dimensional spheres \( \{ (z_1, \cdots, z_N) \mid \sum_i |z_i|^2 = 1 \} \) inside \( \mathbb{C}^N \). The universal group that can act ‘linearly’, i.e., leaves the span of the complex coordinates \( z_1, \cdots, z_N \) invariant and also preserves the canonical inner-product on \( \text{span} \{ z_1, \cdots, z_n \} \) coming from the standard inner-product of \( \mathbb{C}^N \), is the unitary group \( U(N) \).

We have proved in Section 6 (Theorem 6.6) a q-analogue of this result. More precisely, we have proved

**Theorem.** Let \( Q \) be a Hopf \(*\)-algebra coacting on \( \mathcal{O}(S^q_{2N-1}) \) by \( \rho \) making it a \(*\)-comodule algebra. We view \( \mathcal{O}(S^q_{2N-1}) \) as a \(*\)-coideal subalgebra of \( \mathcal{O}(SU_q(N)) \). Moreover, suppose that \( Q \) leaves invariant the span \( V \) of \( z_1, \cdots, z_N \), \( q = (q^j_i) \) being the matrix of coefficients and preserves the inner product on \( V \) induced by the Haar functional. Then there is a unique map \( \Phi : U_q(N) \to Q \) such that \( (id \otimes \Phi)\rho_a = \rho \).

Using this, we have also identified (Theorem 7.12) \( U_q(N) \) with the (orientation preserving) quantum isometry group of a natural spectral triple on \( S^q_{2N-1} \) constructed in [CP08].

## 2. Coquasitriangular Hopf Algebras

In this and the following sections we introduce some well known material on cosemisimple Hopf algebras and coquasitriangular Hopf algebras. The main object of investigation, the Vaksman-Soibelman (also called quantum or odd) sphere, is also introduced.

Let \( H \) be a Hopf algebra with comultiplication \( \Delta \), counit \( \epsilon \), antipode \( S \), unit 1 and multiplication \( m \). We use Sweedler notation throughout, i.e., for the coproduct we write \( \Delta(a) = a_{12} \otimes a_{2} \) and for a right coaction \( \rho \), we write \( \rho(x) = x_0 \otimes x_1 \).

**Definition 2.1.** [KS97] page 331] A coquasitriangular Hopf algebra is a Hopf algebra \( H \) equipped with a linear form \( \varphi : H \otimes H \to \mathbb{C} \) such that the following conditions hold:

i) \( \varphi \) is invertible with respect to the convolution, that is, there exists another linear form \( \varphi : H \otimes H \to \mathbb{C} \) such that \( \varphi \rho = \rho \varphi = \epsilon \otimes \epsilon \) on \( H \otimes H \);

ii) \( m_{\varphi \rho} = \varphi \star m_H \star \varphi \) on \( H \otimes H \);

iii) \( \varphi (m_H \otimes id) = r_{13}r_{23} \) and \( \varphi (id \otimes m_H) = r_{13}r_{12} \) on \( H \otimes H \otimes H \),

where \( r_{i2}(a \otimes b \otimes c) = \varphi(a \otimes b)\epsilon(c), r_{23}(a \otimes b \otimes c) = \varphi(a)\varphi(b \otimes c) \) and \( r_{13}(a \otimes b \otimes c) = \varphi(b)\varphi(a \otimes c) \), \( a, b, c \in H \).

**Remark 2.2.** A linear form \( \varphi \) on \( H \otimes H \) with the properties (i) – (iii) is called a universal \( \varphi \)-form on \( H \).
Since linear forms on $H \otimes H$ correspond to bilinear forms on $H \times H$, we can consider any linear form $r : H \otimes H \to \mathbb{C}$ as a bilinear form on $H \times H$ and write $r(a, b) := r(a \otimes b), a, b \in H$. Then the above conditions (i) – (iii) read as

(1) $r(a_1, b_1) r(a_2, b_2) = r(a_1, b_1) r(a_2, b_2) = \epsilon(a) \epsilon(b),$

(2) $ba = r(a_1, b_1) a_2 b_2 r(a_3, b_3),$

(3) $r(ab, c) = r(a, c_1) r(b, c_2),$

(4) $r(a, bc) = r(a_1, c) r(a_2, b),$

with $a, b, c \in H$.

**Remark 2.3.** [KS97, page 334] It can be shown that $r(S(a), S(b)) = r(a, b)$.

Let $H$ be a coquasitriangular Hopf algebra with universal r-form $r$. For right $H$-comodules $V$ and $W$ we define a linear mapping $r_{V,W} : V \otimes W \to W \otimes V$ by

(5) $r_{V,W}(v \otimes w) = r(v_1, w_1) w_0 \otimes v_0,$

$v \in V, w \in W$.

**Remark 2.4.** [KS97, page 333] It can be shown that $r_{V,W}$ is an isomorphism of the right $H$-comodules $V \otimes W$ and $W \otimes V$.

The compatibility of a universal r-form and a $\ast$ structure is described in

**Definition 2.5.** [KS97, page 336] A universal r-form $r$ of a Hopf $\ast$-algebra $H$ is called real if $r(a \otimes b) = r(b^* \otimes a^*)$.

### 3. The Quantum Semigroup $M_q(N)$

Let $q$ be a positive real number. We now introduce some of the well known deformations of classical objects.

**Definition 3.1.** [KS97, page 310; RTF89] The FRT bialgebra, also called the coordinate algebra of the quantum matrix space $M_q(N)$, denoted $O(M_q(N))$, is the free unital $\mathbb{C}$-algebra with a set of $N^2$ generators $\{u_{ij} \mid i, j = 1, \cdots, N\}$ and defining relations

(6) $u_{ij}^l u_{ij}^k = qu_{ij}^l u_{ij}^k, \quad u_{ij}^k u_{ij}^k = qu_{ij}^k u_{ij}^k, \quad i < j,$

(7) $u_{ij}^l u_{ij}^k = u_{ij}^k u_{ij}^l, \quad i < j, \quad k < l,$

(8) $u_{ij}^l u_{ij}^k - u_{ij}^k u_{ij}^l = (q - q^{-1}) u_{ij}^l u_{ij}^k, \quad i < j, \quad k < l.$

We have,

**Proposition 3.2.** There is a unique bialgebra structure on the algebra $O(M_q(N))$ such that

(9) $\Delta(u_{ij}^l) = \sum_k u_{ik}^l \otimes u_{kj}^l, \quad$ and $\epsilon(u_{ij}^l) = \delta_{ij}, \quad i, j = 1, \cdots, N.$
The above construction can be realized more conceptually as follows. Let \( \hat{R} : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N \) be the linear operator whose matrix with respect to the standard basis of \( \mathbb{C}^N \) is given by
\[
\hat{R}^i_j = q^{\delta_{ij}} \delta_{ij} + (q - q^{-1}) \delta_{im} \delta_{jm} \theta(j - i),
\]
where \( \theta \) is the Heaviside symbol, that is, \( \theta(k) = 1 \) if \( k > 0 \) and \( \theta(k) = 0 \) if \( k \leq 0 \). Let the inverse of \( \hat{R} \) be \( \hat{R}^{-1} \). Also, let \( \hat{R} \) be the “dual” operator defined by \( \hat{R}^i_j = \hat{R}^{ji} \) and \( \hat{R}^{-i} = \hat{R}^{-ji} \).

**Remark 3.3.** [KS97] page 309 It is known that \( \hat{R} \) satisfies
\[
(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0,
\]
where \( I \) is the identity operator. Then \( \hat{R} \) becomes what is known as a Hecke operator.

The following shows that \( M_q(N) \) is universal in a sense.

**Proposition 3.4.** [KS97] page 305 There is a linear map \( \phi : \mathbb{C}^N \to \mathbb{C}^N \otimes O(M_q(N)) \) such that \( \mathbb{C}^N \) is a right comodule of \( O(M_q(N)) \) with coaction \( \phi \) and \( \hat{R} \) is a comodule morphism. If \( A \) is any other bialgebra and \( \psi : \mathbb{C}^N \to \mathbb{C}^N \otimes A \) is a right coaction of \( A \) on \( \mathbb{C}^N \) such that \( \hat{R} \) is a comodule morphism then there exists a unique bialgebra morphism \( \Theta : O(M_q(N)) \to A \) such that \( (id \otimes \Theta)\phi = \psi \).

Since \( O(M_q(N)) \) is only a bialgebra, we want to construct a Hopf algebra out of it. For that we need,

**Definition 3.5.** [KS97] page 312 The quantum determinant, denoted \( D_q \), is the element of \( O(M_q(N)) \) defined by
\[
\sum_{\pi \in S_N} (-q)^{\ell(\pi)} u_{\pi(1)}^1 \cdots u_{\pi(N)}^N,
\]
where \( S_N \) is the symmetric group on \( N \) letters and \( \ell(\pi) \) is the number of inversions in \( \pi \).

**Remark 3.6.** It is an important fact that \( D_q \) is a central element of \( O(M_q(N)) \).

4. The Quantum Group \( SU_q(N) \)

The deformation of the special linear group is realized as,

**Definition 4.1.** [KS97] page 314 The coordinate algebra of the quantum special linear group \( SL_q(N) \) is defined to be the quotient
\[
O(SL_q(N)) = O(M_q(N))/\langle D_q - 1 \rangle
\]
of the algebra \( O(M_q(N)) \) by the two-sided ideal generated by the element \( D_q - 1 \).

The following shows that \( O(SL_q(N)) \) is indeed a Hopf algebra.

**Proposition 4.2.** There is a unique Hopf algebra structure on the algebra \( O(SL_q(N)) \) with comultiplication \( \Delta \) and counit \( \epsilon \) such that
\[
\Delta(u^i_j) = \sum_k u^i_k \otimes u^j_k \quad \text{and} \quad \epsilon(u^i_j) = \delta_{ij},
\]
The antipode \( S \) of the Hopf algebra is given by
\[
S(u^i_j) = (-q)^{i-j} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u^{k_1}_{\pi(l_1)} \cdots u^{k_{N-1}}_{\pi(l_{N-1})}.
\]
where \( \{k_1, \ldots, k_{N-1}\} := \{1, \ldots, N\} \setminus \{j\} \) and \( \{l_1, \ldots, l_{N-1}\} := \{1, \ldots, N\} \setminus \{i\} \) as ordered sets.

The composite \( \mathbb{C}^N \xrightarrow{\varphi} \mathbb{C}^N \otimes \mathcal{O}(M_q(N)) \to \mathbb{C}^N \otimes \mathcal{O}(SL_q(N)) \) gives the natural coaction of \( \mathcal{O}(SL_q(N)) \) on \( \mathbb{C}^N \).

As quantum groups are understood to be quasitriangular Hopf algebras, quantum function algebras are assumed to be coquasitriangular Hopf algebras. We want to think of \( \mathcal{O}(SL_q(N)) \) as the quantum function algebra of \( SL(N) \).

**Theorem 4.3.** [KS97, page 339] \( \mathcal{O}(SL_q(N)) \) is a coquasitriangular Hopf algebra with universal \( r \)-form \( r_t \) uniquely determined by

\[
(15) \quad r_t(u^j_i \otimes u^k_l) = t \hat{R}_{jl}^{ki},
\]

where \( t \) is the unique positive real number such that \( t^N = q^{-1} \).

**Remark 4.4.** It can be shown that the morphism \( r_{\mathbb{C}^N, \mathbb{C}^N} \) induced by the universal \( r \)-form \( r_t \) of \( \mathcal{O}(SL_q(N)) \), equals \( t \hat{R} \). Let us denote it by \( \sigma \).

The following resembles complex conjugation.

**Proposition 4.5.** [KS97, page 316] There is a unique \( \ast \)-structure on the Hopf algebra \( \mathcal{O}(SL_q(N)) \) given by \( (u^j_i)^\ast = S(u^j_i) \), making it into a Hopf \( \ast \)-algebra.

Let us now introduce the quantum version of the real form \( SU(N) \) of \( SL(N) \).

**Definition 4.6.** The coordinate algebra of the quantum special unitary group \( SU_q(N) \) is the Hopf \( \ast \)-algebra \( \mathcal{O}(SL_q(N)) \) of the above proposition.

We have,\n
**Theorem 4.7.** [KS97, page 340] The universal \( r \)-form \( r_t \) of \( \mathcal{O}(SU_q(N)) \) is real.

The algebraic counterpart of classical Peter-Weyl theorem for compact groups is contained in the following:

**Definition 4.8.** [KS97, page 403] A Hopf algebra \( H \) is said to be cosemisimple if there exists a unique linear form \( h : A \to \mathbb{C} \), which we call the Haar functional, such that \( h(1) = 1 \), and for all \( a \) in \( H \)

\[
(id \otimes h)\Delta(a) = h(a)1, \quad (h \otimes id)\Delta(a) = h(a)1.
\]

The concept of cosemisimple Hopf algebras is a very general one. We need some faithfulness assumption on the Haar functional.

**Definition 4.9.** [KS97, page 416] A compact quantum group (CQG) algebra is a cosemisimple Hopf \( \ast \)-algebra such that \( h(a^\ast a) > 0 \), for all \( a \neq 0 \).

The condition of \( H \) being a CQG algebra is equivalent to it being the dense Hopf \( \ast \)-algebra of a compact quantum group, in the sense of Woronowicz [Wor98]. The following captures the compactness of the real form \( SU(N) \).

**Theorem 4.10.** [KS97, page 418] \( \mathcal{O}(SU_q(N)) \) is a CQG algebra.

For any CQG algebra, an inner product is given by \( \langle a, b \rangle = h(a^\ast b) \).
5. The odd sphere

We now introduce the main example to be studied.

**Definition 5.1.** [Wel98 page 2; VS90] The coordinate algebra \( O(S^{2N-1}_q) \) of the quantum sphere \( S^{2N-1}_q \) is the free unital \( \mathbb{C} \)-algebra with a set of \( 2N \) generators \( \{ z_i, z_i^* \mid i = 1, \ldots, N \} \) and defining relations

\[
(16) \quad z_iz_j = qz_jz_i, \quad z_i^*z_j^* = q^{-1}z_j^*z_i^*, \quad i < j
\]

\[
(17) \quad z_iz_j^* = qz_j^*z_i, \quad i \neq j
\]

\[
(18) \quad z_iz_i^* - z_i^*z_i + q^{-1}(q - q^{-1}) \sum_{k>i} z_kz_k^* = 0,
\]

\[
(19) \quad \sum_{i=1}^N z_iz_i^* = 1,
\]

**Proposition 5.2.** Putting \( z_i = u_i^1 \) and \( z_i^* = (u_i^1)^* = S(u_i^1) \) gives an embedding of \( O(S^{2N-1}_q) \) into \( O(SU_q(N)) \) making \( S^{2N-1}_q \) into a quantum homogeneous space for \( SU_q(N) \) with the coaction

\[
\Delta_R(z_i) = \sum_j z_j \otimes u_i^1, \quad \Delta_R(z_i^*) = \sum_j z_j^* \otimes S(u_i^j).
\]

Classically, the sphere \( S^{2N-1}_q \) can be described as a homogeneous space for \( SU(N) \). The above proposition states the quantum version of it. Moreover, one can view the sphere as a homogeneous space for \( U(N) \) also. As expected, the last statement continues to hold in the quantum world too.

We need the following.

**Definition 5.3.** [KS97 page 313] The coordinate algebra of the quantum general linear group \( GL_q(N) \) is defined to be the quotient

\[
O(GL_q(N)) = O(M_q(N))[t]/(tD_q - 1)
\]

of the polynomial algebra \( O(M_q(N))[t] \) in \( t \) over \( O(M_q(N)) \) by the two-sided ideal generated by the element \( tD_q - 1 \).

**Proposition 5.4.** There is a unique Hopf algebra structure on the algebra \( O(SL_q(N)) \) with comultiplication \( \Delta \) and counit \( \epsilon \) such that

\[
\Delta(u_i^j) = \sum_k u_k^i \otimes u_k^j \quad \text{and} \quad \epsilon(u_i^j) = \delta_{ij},
\]

The antipode \( S \) of the Hopf algebra is given by

\[
S(u_i^j) = (-q)^{i-j}D_q^{-1} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u_{\pi(l_1)}^{k_1} \cdots u_{\pi(l_{N-1})}^{k_{N-1}} \quad \text{and} \quad S(D_q) = D_q^{-1},
\]

where \( \{ k_1, \ldots, k_{N-1} \} := \{ 1, \ldots, N \} \setminus \{ j \} \) and \( \{ l_1, \ldots, l_{N-1} \} := \{ 1, \ldots, N \} \setminus \{ i \} \) as ordered sets.

We have the following analogue of the real form \( U(N) \),
Proposition 5.5. [KS97, page 316] There is a unique $^\ast$-structure on the Hopf algebra $O(GL_q(N))$ given by $(u_j^i)^\ast = S(u_j^i)$, making it into a Hopf $^\ast$-algebra.

Let us make

Definition 5.6. The coordinate algebra of the quantum unitary group $U_q(N)$ is the Hopf $^\ast$-algebra $O(GL_q(N))$ of the above proposition. In this $^\ast$-algebra, the quantum determinant $D_\varphi$ becomes a unitary element.

The fact that $U(N)$ is compact is reflected in,

Theorem 5.7. [KS97, page 418] $O(U_q(N))$ is a CQG algebra.

Thus we have the analogues of Proposition 4.5 and Theorem 4.10. Moreover, Proposition 5.2 remains true in this case and makes $S_q^{2N-1}$ a quantum homogeneous space for $U_q(N)$.

6. Main results

We start with a basic observation.

Lemma 6.1. Let $H$ be any cosemisimple Hopf $^\ast$-algebra with the Haar functional $h$. Then for $a,b \in H$, $h(a_1^ib)_1^2h = h(a^ib_1)S(b_2)$.

Proof. By definition, $h(x)1 = h(x_1)x_2$ for $x \in H$. Applying the antipode $S$, we get $h(x)1 = h(x_1)S(x_2)$. Now,

$$h(a_1^ib)_1^2h = h(a_1^ib_1)S(a_2^ib_2)a_3^2 = h(a_1^ib_1)S(b_2)S(a_2^ib_3) = h(a_1^ib_1)S(b_2)e(a_3^2) = h(a_1^ib_1)e(a_3^2)b_1S(b_2) = h(a^ib_1)S(b_2)$$

$\Box$

The following lemma exploits the relation between the two apparently different concepts, namely coquasitriangularity and faithfulness of the Haar functional.

Lemma 6.2. Let $H$ be a coquasitriangular CQG algebra with Haar functional $h$ and real universal $r$-form $r$. Let the induced inner-product be $\langle , \rangle$. Let $V$ be any submodule of $H$. Let $r_{V,V}$ be the induced morphism on $V \otimes V$. Then $r_{V,V}$ is hermitian with respect to the restricted inner-product.

Proof. We have

$$\langle r_{V,V}(v \otimes w), v' \otimes w' \rangle = r(v_1, w_1)\langle w_0 \otimes v_0, v' \otimes w' \rangle = r(w_1^*, v_1^*)\langle w_0, v' \rangle \langle v_0, w' \rangle \quad \text{(we use reality of $r$)}$$

$$= r(h(w_0^*v')w_1^*, h(v_0^*w')v_1^*)$$

$$= r(h(w^*v_0)S(v_1'), h(v^*w_0)S(w_1')) \quad \text{(using Lemma 6.1)}$$

$$= r(S(v_1'), S(w_1'))\langle w, v_0 \rangle \langle v, w_0' \rangle$$

$$= r(v_1', w_1')\langle v \otimes w, w_0' \otimes v_0' \rangle \quad \text{(by Remark 2.3)}$$

$$= \langle v \otimes w, r_{V,V}(v' \otimes w') \rangle$$

$\Box$
Lemma 6.3. Let $A$ and $Q$ be Hopf $*$-algebras, $B \subset A$ a $*$-coideal subalgebra of $A$, $A$ be cosemisimple. Let $B$ be a comodule algebra over $Q$ with coaction $\rho : B \to B \otimes Q$ such that $\rho(b^*) = \rho(b)^*$ for all $b \in B$. Suppose that $Q$ preserves the restriction of the Haar functional to $B$ i.e., $(h \otimes \text{id})\rho(b) = h(b)1$. Then $Q$ preserves the induced inner-product on $B$.

Proof. We use Sweedler notation for $\rho$: $\rho(b) = b_0 \otimes b_1$. So $Q$ preserves the restriction of Haar functional on $B$ implies $h(b)1 = h(b_0)b_1$ for all $b \in B$. Then

\[
(a, b)1 = h(a^*b)1 = h((a^*b_0)(a^*b_1) = h(a_0^*b_0)a_1^*b_1 = (a_0, b_0)a_1^*b_1
\]

where $\rho(a^*b) = \rho(a^*)\rho(b) = \rho(a)^*\rho(b) = a_0^*b_0 \otimes a_1^*b_1$. \qed

Proposition 6.4. Let $\pi : U_q(N) \to SU_q(N)$ be the quotient homomorphism and $\rho_u$, $\rho_{su}$, respectively, be the corresponding coactions on $S^{2N-1}_q$ so that $(\text{id} \otimes \pi)\rho_u = \rho_{su}$. Then $\rho_u$ preserves the restriction of the Haar functional $h$ on $O(S^{2N-1}_q)$.

Proof. Let $f$ be any linear functional on $O(S^{2N-1}_q)$ such that $f(1) = 1$. Let $f'$ be defined as $(f \otimes h_u)\rho_u$, where $h_u$ is the Haar functional of $U_q(N)$. Then $f'$ is $\rho_u$ invariant. By $(\text{id} \otimes \pi)\rho_u = \rho_{su}$, we get that $f'$ is $\rho_{su}$ invariant too. It is well known that the restriction of $h$ is the only functional with this property [KS97]. Hence, the conclusion follows. \qed

We think the following is well known. We included it because we couldn’t find it in the literature.

Proposition 6.5. The set \{ $z^k_1 \cdots z^k_N (z^l_{N-1})^{l_{N-1}} \cdots (z^l_1)^l_1 | k_1, \cdots, k_N, l_{N-1}, \cdots, l_1 \in \mathbb{N}_0, \ l_N \in \mathbb{N}$ \} is a vector space basis of $O(S^{2N-1}_q)$.

Proof. It is a simple application of Bergman’s Diamond lemma [Ber78]. \qed

We state and prove below the main result concerning Hopf coactions on the quantum spheres satisfying suitable conditions.

Theorem 6.6. Let $Q$ be a Hopf $*$-algebra coacting on $O(S^{2N-1}_q)$ by $\rho$ making it a $*$-comodule algebra. We view $O(S^{2N-1}_q)$ as a $*$-coideal subalgebra of $O(SU_q(N))$. Moreover, suppose that $Q$ leaves invariant the span $V$ of $z_1, \cdots, z_N$, $q = (q_i^j)$ being the matrix of coefficients and preserves the inner product on $V$ induced by the Haar functional. Then there is a unique map $\Phi : U_q(N) \to Q$ such that $(\text{id} \otimes \Phi)\rho_u = \rho$.

Proof. By hypotheses, $V$ is $Q$-comodule and the coaction $\rho : V \to V \otimes Q$ is given by $\rho(z_i) = \sum_j z_j \otimes q_i^j$. Recall the map $\sigma$ from Remark 6.4. By Lemma 6.2, $\sigma$ is hermitian. Since $\sigma = tR$ and $t$ is real, $\tilde{R}$ is also hermitian. Now $Q$ preserves the image of $\tilde{R} - q$ (which follows from the relation $z_i z_j = q z_j z_i$, $i < j$) whose orthogonal complement is the image of $\tilde{R} + q^{-1}$. Hence $Q$ also preserves the image of $\tilde{R} + q^{-1}$. And we get that $\tilde{R}$ becomes a $Q$-comodule morphism. So by Proposition 6.4, $q$ satisfies the FRT relations (6), (7), (8).

\[Q \text{ preserves the relation } \sum_{i=1}^N z_i z_i^* = 1.\] Applying $\rho$ to both sides, comparing coefficients and using Proposition 6.5, we get that $qq^* = I_n$, where $q^* = \overline{q}$.\]
Definition 7.1. [BG09] \( q = ((q_i^j)^*). \) Since \( V \) is a comodule, we have that \( \Delta(q^i_j) = \sum_k q^i_k \otimes q^j_k \) or in matrix notation, \( \Delta(q) = q \otimes q. \) The antipode \( S \) satisfies \( S(q)q = I_n. \) Hence \( S(q) = q^* \) and \( q^*q = I_n, \) using \( qS(q) = I_n. \) So \( q \) is unitary matrix.

Since the coaction preserves the image of \( \hat{R} + q^{-1}, \) the quantum exterior algebra becomes a comodule algebra [KS97, page 310 & Proposition 4, page 308]. So it makes sense to talk about the quantum determinant of the matrix \( q. \) It is a group like element and hence invertible in \( Q. \)

Now, defining \( \Phi(u^i_j) = q^i_j \) yields the conclusion. \( \square \)

Remark 6.7. The equation (19) can also be written as \( \sum_{i=1}^{N} q^{-2i} z_i z_i = q^{-2}. \) Now applying \( \rho \) to both sides, comparing coefficients and using Proposition 6.4, we see that \( q \) satisfies \( E\Phi^{-1} q^i = q^i E\Phi^{-1} = I_n, \) where \( E \) is the matrix
\[
\frac{1}{q^{n-1}[n]_q} \text{diag}(1, q^2, q^4, \ldots, q^{2(n-1)}), \quad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.
\]
See [BDDD14, VDW96].

We finally have,

Theorem 6.8. Consider the category \( C \) consisting of Hopf \( *-\)algebras satisfying the hypotheses of Theorem 6.3 as objects and Hopf \( *-\)algebras as morphisms. Then \( U_q(N) \) is a universal object in this category.

Proof. By Proposition 6.3 and Lemma 6.3 \( U_q(N) \) is an object in this category. Then Theorem 6.6 shows that \( U_q(N) \) is universal with that property. \( \square \)

7. An application

We provide an application of the main result of the previous section, namely, that of determining the quantum isometry group of the odd sphere, in the sense of [BG09].

Definition 7.1. [Wor98] A compact quantum group (CQG) is given by a pair \((S, \Delta), \) where \( S \) is a unital \( C^* \) algebra and \( \Delta \) is a unital \( C^* \) homomorphism \( \Delta : S \rightarrow S \otimes S \) (\( \otimes \) is \( C^* \)-algebraic tensor product) satisfying

i) \((\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta; \)

ii) the linear span of \( \Delta(S)(S \otimes 1) \) and \( \Delta(S)(1 \otimes S) \) are norm-dense in \( S \otimes S. \)

We say that the compact quantum group \((S, \Delta)\) acts on a unital \( C^* \)-algebra \( B, \) if there is a unital \( C^* \)-homomorphism \( \alpha : B \rightarrow B \otimes S \) satisfying

i) \((\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \Delta) \circ \alpha; \)

ii) the linear span of \( \alpha(B)(1 \otimes S) \) is norm-dense in \( B \otimes S. \)

There are natural \( C^* \)-norms on \( \mathcal{O}(SU_q(N)), \) \( \mathcal{O}(S^q_{2N-1}) \) and on \( \mathcal{O}(U_q(N)). \) Upon completion with respect to these norms, one gets unital \( C^* \)-algebras \( C(SU_q(N)), \) \( C(S^q_{2N-1}) \) and \( C(U_q(N)) \) that play the role of algebras of continuous functions on \( SU_q(N), \) \( S^q_{2N-1} \) and on \( U_q(N) \) respectively. The \( C^* \)-algebras \( C(SU_q(N)) \) and \( C(U_q(N)) \) are the function algebras of the compact quantum groups \( SU_q(N) \) and \( U_q(N), \) respectively, in the above sense. And these compact quantum groups act on \( C(S^q_{2N-1}). \)
Definition 7.2. A unitary representation of a compact quantum group \((S, \Delta)\) on a Hilbert space \(H\) is a map \(u\) from \(H\) to \(H \otimes S\) such that the element \(\tilde{u}\in M(K(H)\otimes S)\) given by \(\tilde{u}(\xi \otimes b) = u(\xi)(1 \otimes b)\) (\(\xi \in H, b \in S\)) is a unitary satisfying \((id \otimes \Delta)\tilde{u} = \tilde{u}_{(12)}\tilde{u}_{(13)}\), where for an operator \(X \in B(H_1 \otimes H_2)\) we have denoted by \(X_{(12)}\) and \(X_{(13)}\) the operators \(X \otimes I_{H_2} \in B(H_1 \otimes H_2 \otimes H_2)\) and \(\Sigma_{23}X_{(12)}\Sigma_{23}\) respectively (where \(\Sigma_{23}\) being the unitary on \(H_1 \otimes H_2 \otimes H_2\) that the switches the two copies of \(H_2\)).

Irreducible unitary representations of the quantum group \(SU_q(N)\) are indexed by Young tableaux \(\lambda = (\lambda_1, \cdots, \lambda_N)\), where \(\lambda_i\)'s are nonnegative integers \(\lambda_1 \geq \cdots \geq \lambda_N\) \([\text{Wor88}, \text{page 42}]\). Let \(H_\lambda\) be the carrier Hilbert space corresponding to \(\lambda\) whose basis elements are parametrized by arrays of the form

\[
\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1,N-1} & r_{1,N} \\ r_{21} & r_{22} & \cdots & r_{2,N-1} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ r_{N-1,1} & r_{N-1,2} & \cdots & r_{N,1} \\ r_{N,1} \end{bmatrix},
\]

where \(r_{ij}\)'s are integers satisfying \(r_{ij} = \lambda_j\) for \(j = 1, \cdots, N\), \(r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0\) for all \(i, j\). Such arrays are known as Gelfand-Tsetlin (GT) tableaux (see \([\text{CP08}, \text{page 29}]\) for details). For a GT tableaux \(\mathbf{r}\), \(\mathbf{r}_i\) will denote its \(i\)th row.

It was shown by Woronowicz that any compact quantum group possesses a Haar functional in the sense of Definition 4.8. Let us denote the Haar functional of \(SU_q(N)\) again by \(h\). Let \(L_2(SU_q(N))\) denote the corresponding G.N.S space and \(L_2(S_q^{2N-1})\) denote the closure of \(C(S_q^{2N-1})\) in \(L_2(SU_q(N))\).

Proposition 7.3. \([\text{CP08}, \text{page 34}]\) Assume \(N > 2\). The restriction of the right regular representation of \(SU_q(N)\) to \(L_2(S_q^{2N-1})\) decomposes as a direct sum of the irreducibles, with each copy occurring exactly once, given by the Young tableau \(\lambda_{n,k} = (n+k,k,k,\cdots,k,0)\) with \(n,k \in \mathbb{N}\).

The following will be useful in constructing a non-commutative structure on the sphere.

Proposition 7.4. \([\text{CP08}, \text{page 35}]\) Let \(\Gamma_0\) be the set of all GT tableaux \(\mathbf{r}_{nk}\) given by

\[
r_{nk}^{ij} = \begin{cases} n+k & \text{if } i = j = 1 \\ 0 & \text{if } i = 1, j = N \\ k & \text{otherwise,} \end{cases}
\]

for some \(n,k \in \mathbb{N}\). Let \(\Gamma_0^{nk}\) be the set of all GT tableaux with top row \((n+k,k,k,\cdots,k,0)\). Then the family of vectors

\[
\{e_{r_{nk},s} \mid n,k \in \mathbb{N}, s \in \Gamma_0^{nk}\}
\]

form a complete orthonormal basis for \(L_2(S_q^{2N-1})\).

We recall the following,

Definition 7.5. \([\text{Lan97}, \text{page 93}]\); \([\text{Con94}]\) A spectral triple \((A,H,D)\) is given by a \(*\)-algebra with a faithful representation \(\pi : A \to B(H)\) on the Hilbert space \(H\) together with a self-adjoint operator \(D = D^*\) on \(H\) with the following properties:
The resolvent \((D - \lambda)^{-1}, \lambda \not\in \mathbb{R}\), is a compact operator on \(H\);

ii) \([D, a] := D\pi(a) - \pi(a)D \in B(H), \) for any \(a \in A\).

Let us now put a non-commutative structure on the quantum sphere.

**Theorem 7.6.** \([CP08, \text{ page 39}]\) Let \(A = \mathcal{O}(S_q^{2N - 1})\) and \(H\) be \(L_2(S_q^{2N - 1})\). Take \(\pi\) to be the inclusion. Finally, define the operator \(D : e_{r,s} \mapsto d(r)e_{r,s}\) on \(L_2(S_q^{2N - 1})\) where the \(d(r)\)'s are given by

\[
d(r^{nk}) = \begin{cases} 
-k & \text{if } n = 0 \\
+nk & \text{if } n > 0.
\end{cases}
\]

Then \((A, H, D),\) as constructed above, is a spectral triple on \(\mathcal{O}(S_q^{2N - 1})\).

Now let us recall the notion of the quantum isometry group of a non-commutative manifold.

**Definition 7.7.** \([BG09, \text{ page 2538}]\) A quantum family of orientation preserving isometries for the spectral triple \((A, H, D)\) is given by a pair \((S, u)\) where \(S\) is a unital \(C^*\)-algebra and \(u\) is a linear map from \(H\) to \(H \otimes S\) such that \(\tilde{u}\) given by

\[
\tilde{u}(\xi \otimes b) = u(\xi)(1 \otimes b),
\]

extends to a unitary element of \(M(K(H) \otimes S)\) satisfying

i) for every state \(\phi\) on \(S,\) \(u_\phi D = Du_\phi\) where \(u_\phi = (id \otimes \phi) \circ \tilde{u};\)

ii) \((id \otimes \phi) \circ \text{ad}_a(a) \in A',\) for all \(a \in A\) and for all state \(\phi\) on \(S,\) where \(\text{ad}_a(x) = \tilde{u}(x \otimes 1)\tilde{u}^*\) for \(x \in B(H)\).

In case, the \(C^*\)-algebra \(S\) has a coproduct \(\Delta\) such that \((S, \Delta)\) is a compact quantum group and \(U\) is a unitary representation of \((S, \Delta)\) on \(H,\) it is said that \((S, \Delta)\) acts by orientation preserving isometries on the spectral triple.

One considers the category \(Q(A, H, D)\) whose objects are triples \((S, \Delta, u)\), where \((S, \Delta)\) is a compact quantum group acting by orientation preserving isometries on the given spectral triple, with \(u\) being the corresponding unitary representation. The morphisms are homomorphisms of compact quantum groups that also “intertwine” (see \([BG09]\)) the unitary representations. If a universal object \((\tilde{S}_0, \Delta_0, u_0)\) (say) exists in this category then the \(C^*\)-algebra \(\tilde{S}_0\) generated by \(\{t_{\xi, \eta} \otimes id)(\text{ad}_a(a)), \xi, \eta \in H, \ q \in A\)\) is a compact quantum group and it is called the quantum group of orientation preserving isometries. Here, \(t_{\xi, \eta} : B(H) \rightarrow \mathbb{C}\) is given by \(t_{\xi, \eta}(X) = \langle \xi, X\eta \rangle.\)

We record some results from \([BG09]\) for the reader’s convenience.

**Theorem 7.8.** Let \((A, H, D)\) be a spectral triple and assume that \(D\) has a one dimensional eigenspace spanned by a unit vector \(\xi,\) which is cyclic and separating for the algebra \(A.\) Moreover, assume that each eigenvector of \(D\) belongs to the dense subspace \(A\xi\) of \(H.\) Then there is a universal object \((\tilde{S}_0, \Delta_0, u_0)\) in the category \(Q(A, H, D)\)

For the spectral triple in Theorem 7.6 the cyclic separating vector \(\xi\) is \(1_{\mathcal{O}(S_q^{2N - 1})}.\)

**Definition 7.9.** Let \((A, H, D)\) be the spectral triple in Theorem 7.6.\(\) Let \(\tilde{Q}(A, H, D)\) be the category with objects \((S, \alpha)\) where \(S\) is a compact quantum group with an action on \(A\) such that

i) \(\alpha\) is \(h\) preserving;
ii) $\alpha$ commutes with $\widehat{D}$, i.e., $\alpha \widehat{D} = (\widehat{D} \otimes \text{id})\alpha$, where $\widehat{D}$ is the operator $A \rightarrow A$ given by $\widehat{D}(a)\xi = D(a\xi)$, $\xi$ as in Theorem 7.8 which is $1_{O(S^2_{a}^N)}$ in our case.

Note that the eigenspaces of $\widehat{D}$ and $D$ are in one-one correspondence. In fact, eigenspaces of $\widehat{D}$ are of the form $\{a \in A \mid a\xi \in V_{\lambda}\}$, where $V_{\lambda}$ is the eigenspace of $D$ with respect to eigenvalue $\lambda$.

**Proposition 7.10.** There exists a universal object $S$ in the category $\tilde{Q}(A,H,D)$ and it is isomorphic to the $C^*$-subalgebra $S_0$ of $\tilde{S}_0$ obtained in Theorem 7.8.

We conclude by describing the quantum isometry group of the sphere. This generalizes [BG09] Theorem 4.13, page 2559.

**Lemma 7.11.** Given a compact quantum group $S$ with an action $\alpha$ on $A$, the following are equivalent:

1. $(S, \alpha)$ is an object of the category $\tilde{Q}(A,H,D)$, $(A,H,D)$ as in Theorem 7.9
2. $\alpha$ is isomorphic to the $C^*$-subalgebra $S_0$ of $\tilde{S}_0$ obtained in Theorem 7.6; and it is isomorphic to the $C^*$-subalgebra $S_0$ of $\tilde{S}_0$ obtained in Theorem 7.8.
3. $\alpha$ preserves each irreducible occurring in Proposition 7.3.

**Proof.** i) $\Rightarrow$ ii) : Since $\alpha$ commutes with $\widehat{D}$, it preserves the eigenspaces of $\widehat{D}$, in particular $V$. And by definition it preserves $h$.

ii) $\Rightarrow$ iii) : Let the irreducible representation corresponding to Young tableau $\lambda_{n,k} = (n+k,k,\ldots,k)$ be denoted by $V_{n,k}$. According to our previous notation $V = V_{1,0}$, the irreducible with Young tableau $\lambda_{1,0} = (1,0,\ldots,0)$. Consider the space $W^{\otimes(n,k)} := V_{1,0} \otimes \cdots \otimes V_{1,0} \otimes V_{0,1} \otimes \cdots \otimes V_{0,1}$.

Let us denote the image of it in the algebra under multiplication by $W^{\bullet(n,k)} := V_{1,0} \bullet \cdots \bullet V_{1,0} \bullet V_{0,1} \bullet \cdots \bullet V_{0,1}$.

Clearly, $\alpha$ preserves $W^{\bullet(n,0)}$ as it preserves $V_{1,0}$ and $\alpha$ preserves $W^{\bullet(0,k)}$ as it preserves $V_{0,1}$. Let $\Lambda^{\otimes(n,k)}$ be the set of the Young tableaux for the irreducible representations occurring in the decomposition of $W^{\otimes(n,k)}$ and $\Lambda^{\bullet(n,k)}$ be the corresponding set for the decomposition of $W^{\bullet(n,k)}$. Clearly, $\Lambda^{\bullet(n,k)} \subseteq \Lambda^{\otimes(n,k)}$. Moreover, it is known from standard representation theory that any $\lambda \in \Lambda^{\otimes(n,k)}$ is dominated by $\lambda_{n,k} = (n+k,k,\ldots,k)$. Note that, from Proposition 7.3 any $\lambda \in \Lambda^{\bullet(n,k)}$ must be of the form $\lambda_{m,l} = (m+l,l,\ldots,l,0)$.

We use double induction on $(n,k)$. So the statement $P(n,k)$ is “$\alpha$ preserves $V_{n,k}$”.

**P(n,0) :** Observe that $(m+l,l,\ldots,l,0)$ is dominated by $(n,0,\ldots,0,0)$ if and only if $l = 0$ and $m \leq n$. As $(m,0,\ldots,0,0)$ can occur with multiplicity exactly one in $L_2(S^2_{a}^N)$, it is clear that $W^{\bullet(n,0)}$ is the irreducible corresponding to the Young tableau $\lambda_{n,0}$, i.e., $V_{n,0}$. So we are through.

Now we assume that for all $n$ and for all $l < k$, the statement $P(n,l)$ is true and we have to show for all $n$, $P(n,k)$ is true. We use induction on $n$. Again observe
that \((m + l, l, \ldots, l, 0)\) is dominated by \(\lambda_{0,k}\) if and only if \(l \leq k, m + l \leq k\). For \(l = k, m = 0\). As the decomposition of \(W^{(m,k)}\) into irreducibles is orthogonal, we see that the orthogonal complement of \(V_{0,k}\) will include only \(V_{m,l}\) with \(l < k\).

### Theorem 7.12

The quantum group of orientation preserving isometries for the spectral triple in Theorem 7.6 is the compact quantum group \(U_q(N)\).

#### Proof

The result follows from Theorem 6.8, Proposition 7.10 and Lemma 7.11.

#### Acknowledgement

The first author is grateful to Aritra Bhowmick, Jyotishman Bhowmick and Sugato Mukhopadhyay for helpful discussions that led to many improvements of the paper. The second author is partially supported by J.C. Bose National Fellowship and Research Grant awarded by D.S.T. (Govt. of India).

#### References

[Ber78] George M. Bergman, The diamond lemma for ring theory, Adv. in Math. 29 (1978), no. 2, 178–218, DOI 10.1016/0001-8708(78)90010-5. MR506890

[BDDD14] Jyotishman Bhowmick, Francesco D’Andrea, Biswarup Das, and Ludwik Dąbrowski, Quantum gauge symmetries in noncommutative geometry, J. Noncommut. Geom. 8 (2014), no. 2, 433–471, DOI 10.4171/JNCG/161. MR3275038

[BG09] Jyotishman Bhowmick and Debashish Goswami, Quantum group of orientation-preserving Riemannian isometries, J. Funct. Anal. 257 (2009), no. 8, 2530–2572, DOI 10.1016/j.jfa.2009.07.006. MR2555012

[Con94] Alain Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994. MR1303779

[CP08] Partha Sarathi Chakraborty and Arupkumar Pal, Characterization of SU_q(l + 1)-equivariant spectral triples for the odd dimensional quantum spheres, J. Reine Angew. Math. 623 (2008), 25–42, DOI 10.1515/CRELLE.2008.071. MR2458039

[Dri87] V. G. Drinfel’d, Quantum groups, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820. MR934283

[GB16] Debashish Goswami and Jyotishman Bhowmick, Quantum isometry groups, Infosys Science Foundation Series, Springer, New Delhi, 2016. Infosys Science Foundation Series in Mathematical Sciences. MR3559897

[GJ14] Debashish Goswami and Souravjoya Joardar, Quantum isometry groups of noncommutative manifolds obtained by deformation using dual unitary 2-cocycles, SIGMA Symmetry Integrability Geom. Methods Appl. 10 (2014), Paper 076, 18, DOI 10.3842/SIGMA.2014.076. MR3261868

[Jim85] Michio Jimbo, A \(q\)-difference analogue of \(U(g)\) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), no. 1, 63–69, DOI 10.1007/BF00704588. MR797001

[Jim86] _____, A \(q\)-analogue of \(U(g(N + 1))\), Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986), no. 3, 247–252, DOI 10.1007/BF00400222. MR841713

[KS97] Anatoli Klyub and Konrad Schmüdgen, Quantum groups and their representations, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997. MR1492989
[Lan97] Giovanni Landi, *An introduction to noncommutative spaces and their geometries*, Lecture Notes in Physics. New Series m: Monographs, vol. 51, Springer-Verlag, Berlin, 1997. MR1482228

[Man88] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1988. MR1016381

[RTF89] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz 1 (1989), no. 1, 178–206 (Russian); English transl., Leningrad Math. J. 1 (1990), no. 1, 193–225. MR1015339

[VDW96] Alfons Van Daele and Shuzhou Wang, *Universal quantum groups*, Internat. J. Math. 7 (1996), no. 2, 255–263, DOI 10.1142/S0129167X96000153. MR1382726

[VS90] L. L. Vaksman and Ya. S. Soibelman, *Algebra of functions on the quantum group SU(n + 1), and odd-dimensional quantum spheres*, Algebra i Analiz 2 (1990), no. 5, 101–120 (Russian); English transl., Leningrad Math. J. 2 (1991), no. 5, 1023–1042. MR1086447

[Wan98] Shuzhou Wang, *Quantum symmetry groups of finite spaces*, Comm. Math. Phys. 195 (1998), no. 1, 195–211, DOI 10.1007/s002200050385. MR1637425

[Wel98] Martin Welk, *Covariant differential calculus on quantum spheres of odd dimension*, Czechoslovak J. Phys. 48 (1998), no. 11, 1507–1514, DOI 10.1023/A:1021642214226. Quantum groups and integrable systems (Prague, 1998). MR1682006

[Wor87] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), no. 4, 613–665. MR901157

[Wor88] , *Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups*, Invent. Math. 93 (1988), no. 1, 35–76, DOI 10.1007/BF01393687. MR943923

[Wor98] , *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR1616348

Stat-Math Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata-700108
E-mail address: suvra.bhr@isical.ac.in

Stat-Math Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata-700108
E-mail address: goswamid@isical.ac.in