Posteriori Error bound For Fullydiscrete Semilinear Parabolic Integro-Differential equations

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Abstract. The main goal of this paper is to obtain error bounds for parabolic integro-differential equation. The derivation of these bounds is based elliptic and Ritz-Volterra reconstructions introduced by Makridakis and Nochetto 2003 and then extended to Ritz-Volterra reconstruction in case of integro-parabolic differential problems by Reddy and Sinha 2015. We proved optimal order bounds for certain classes semilinear parabolic integro-differential equations. The key points for these estimators can be reduced the numbers of iterations.

1. Introduction
Numerous areas of chemical reactions, ecological interactions, biological process, can be modelled as semilinear parabolic integro-differential equations (SPIDEs). A posteriori error estimates for linear and nonlinear parabolic problems have been proved in the literature [1-17]. The key point for deriving these errors is the elliptic reconstruction introduced by [2, 4] and modified for dG methods in [8].

However, in terms of proving of a posteriori error for SPIDEs see, [18, 19, 20, 21]. Jingtang and bernner [18] derived a posterior error estimators for nonstandard Volterra integro-differential equations. They established a posteriori error estimates for the (dG) time. Reddy and Sinha [19, 20] derived posterior error bound for linear parabolic integro equations. This is inspired by using Ritz-Volterra reconstruction to arrive optimal bound in terms of \( L_\infty (L_2) \) and \( L_\infty (H^1) \) norms. Sabawi [21] have proved a posterior error bound for this problem. He applied Ritz-Volterra techniques in his analysis.

The purpose of this paper is to extend the work [21] to the case of fully discrete. We have derived first optimal order a posteriori bounds for SPIDEs in terms of \( L_\infty (L_2) \) and \( L_\infty (H^1) \) -norms for the backward Euler fully discrete scheme. The proof of a posteriori bounds for fully discrete analysis necessitates the careful introduction of a novel space–time reconstruction operator, and we call it as Ritz–Volterra [19, 20].

The remaining of this work is organised as follows. In Section 2, the model problem is introduced. The backward Euler fully method is discussed in Section 3. Reconstructions are given in Section 4. A posteriori error estimate for fully discrete are presented in Section 5. Finally, conclusions are given in Section 6.

2. Model Problem
Let \( \Omega \subset R^d \), \( d = 2 \) or \( d = 3 \), be a convex bounded with smooth boundary \( \partial \Omega \). For some final time \( T > 0 \), consider the linear parabolic integro-differential problem: find \( u : \Omega \times (0,T] \rightarrow R \) such that

\[
\frac{du}{dt} + Au - \int_0^t B u(x,s)ds = f(u) \tag{1}
\]
Here, $\mathcal{A}$ and $\mathcal{B}$ are linear elliptic operator defined by $\mathcal{A} = -\nabla \cdot (A \nabla u)$, $\mathcal{B} = -\nabla \cdot (B(t, s) \nabla u)$, respectively.

To define weak form for (1), we multiply (1) by a test function $v \in H^1_0$ and then integrate by parts, gives
\[
\left( \frac{du}{dt}, v \right) + D_1(u, v) = \int_0^t D_2(u(s), v)ds = (f(u), v),
\]
with
\[
D_1(u, v) = \int_{\Omega} \mathcal{A} u \cdot \nabla v \, dx, \quad D_2(u(s), v) = \int_{\Omega} \mathcal{B} u(x, s) \cdot \nabla v \, dx.
\]
Note that the convercitivy and continuity of the bilinear $D_1$ and $D_2$ by
\[
a(u, u) \geq C_{covr} \| \nabla u \|^2, \text{ for all } v \in H^1_0(\Omega) \tag{3}
\]
\[
a(u, u) \leq C_{cont} \| \nabla u \| \| \nabla v \|, \text{ for all } u, v \in H^1_0(\Omega) \tag{4}
\]
\[
a(u, v) \leq C_{cont} \| \nabla u(s) \| \| \nabla v \|, \text{ for all } u(s), v \in H^1_0(\Omega) \tag{5}
\]

3. Fully discrete Backward Euler Formula

To define the fully discrete backward Euler Galerkin method approximation for equation (2), we will discrete the time interval $[0, T]$ into $N$ subintervals $I_m = (t_{m-1}, t_m]$, $m = 1, ..., N$, and $\tau_m := t_m - t_{m-1}$ is the local time step.

Let $U^m \in S^m$ is an approximate for $u^m = u(t^m)$ such that
\[
\left( \frac{U^m - U^{m-1}}{\tau_m}, v^n \right) + D_1(U^m, v^n) = \Theta^m(D_2(t, U(s), v^n) + (f(U^m), v^n)), \tag{6}
\]
\[
\Theta^m(y) = \sum_{i=1}^{n-1} w_{i+1} y(t_i) \approx \int_0^t y(s)ds,
\]
and
\[
\Theta^m(b(t_m, U(s), v)) = \Theta^m(B(t_m, U(s), v)) = \sum_{i=1}^{n-1} w_{i+1} B(t_m, t_i) \nabla v(t_i), \nabla v.
\]

4. Reconstructions

We now discuss the elliptic reconstruction technique for linear parabolic problems [2], which is a combination of the concepts of Ritz-Volterra reconstruction [19, 20] for the spatial discretisation for the integro-differential equations to arrive optimal order in terms of $L_\infty(L_2)$ and $L_\infty(H^1)$.

**Definition 4.1.** Let $R^m(t) \in H^1_0(\Omega)$ is reconstruction techniques for unique solution of the elliptic problem $U^m$, so that
\[
D_1(R^m(t), \Phi) = (A^m U^m, \Phi), \tag{7}
\]
\[ \mathcal{A}_m U^m = \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^{m-1}}{\tau_m}. \]

**Definition 4.2.** Let \( \mathcal{R}_r, \mathcal{R}_r^m : S^m \to H_0^1(\Omega) \) is Ritz-Volterra reconstruction which plays an important role in the error analysis, can be defined by

\[
D_r(\mathcal{R}_r(t)w, \chi) - \int_0^{\tau_m} b(t, s, w(s), \chi(s)) \, ds = (\mathcal{A}^m w, \chi) - \left( \int_0^{\tau_m} \mathcal{B}^m(s)w(s) \, ds, \chi \right), \tag{8}
\]

for all \( \chi \in H_0^1(\Omega) \). For all \( t \in [0, T] \), \( W_T(t) \) are of \( U(t) \) are defined as

\[
U(t) = \ell_m U^m + \ell_{m-1} U^{m-1}, \tag{9}
\]

\[
W_T(t) = \ell_{m-1} W_T(t) + \ell_{m-1} W_T(t) = \ell_m \mathcal{R}_r^m(t) U^m + \ell_{m-1} \mathcal{R}_r^{m-1}(t) U^{m-1} \tag{10}
\]

where

\[
\ell_m = \frac{t - \tau_{m-1}}{\tau_m}, \quad \ell_{m-1} = \frac{t - \tau_m}{\tau_m}.
\]

We introduce

\[
\mathcal{Y}(t) = \int_0^t B(t, s)\nabla Y(s) \, ds,
\]

for \( t \in I_n \), let \( \mathcal{Y}_i(t) \) be the linear interpolant associated with the vectors \( \mathcal{Y}_i(t_{m-1}) \) and \( \mathcal{Y}_i(t_m) \), and is defined as

\[
\mathcal{Y}_i(t) = \ell_{m-1} \mathcal{Y}_i(t_{m-1}) + \ell_m \mathcal{Y}_i(t_m).
\]

Note that \( \int_0^t \mathcal{B}u(x, s) \, ds = 0 \) in (1) the above definition will be equivalent to (7).

**Lemma 4.3.** (Ritz Volterra Reconstruction error bounds). For any \( v \in S^m \), then the error below hold

\[
\|\nabla (\mathcal{R}_r^m U^m - U^m)\| \leq C_2 \Phi_{m, H^1}^2 \tag{11}
\]

\[
\|\mathcal{R}_r^m U^m - U^m\| \leq C_2 \Phi_{m, L^2} \tag{12}
\]

where

\[
\Phi_{m, H^1}^2 := \|h_m Z^m\| + \|\frac{1}{h_m^2} J^m\|_{\Sigma_m} + h_m Q_{1m}^m(U),
\]

\[
\Phi_{m, L^2}^2 := \|h_m^2 Z^m\| + \|h_m^2 J^m\|_{\Sigma_m} + h_m Q_{1m}^m(U) + h_m Q_{2m}^m(U),
\]

and

\[
Z^m := \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^{m-1}}{\tau_m} + \mathcal{A}_{el} U^m - \Theta_m (\mathcal{B}_{el} U^m),
\]

\[
J^m := \frac{\tau_m}{\tau_m} \Pi^m U^m - \Theta_m (\mathcal{B}_{el} U^m),
\]

\[
Q_{1m}^m(U) = \tau_m \left\{ \sum_{j=0}^m \tau_j \|\nabla U^j\| + \sum_{j=0}^m \tau_j \|\nabla (\nabla U^j)\| \right\},
\]

\[
Q_{2m}^m(U) = h_m \tau_m \left\{ \sum_{j=0}^m \tau_j \|U^j\| + \sum_{j=0}^m \tau_j \|\nabla U^j\| + \sum_{j=0}^m \tau_j \|\nabla (\nabla U^j)\| \right\}.
\]

Proof. See [19].

**Lemma 4.4.** (error equation) For each \( \tau_m \in [1: N] \), the following semilinear parabolic integro-differential problems error bound held
Proof. The fully discrete scheme (6) can be expressed in distributional form as

\[
\int \left( \frac{\partial \xi}{\partial t}, \chi \right) + D_1(\xi, \chi) \int_0^t D_2(t, s, \xi(s), \chi) ds + D_1(W_T(t) - W_T^m, \chi) - \int_0^t D_2(t, s, \xi(s), \chi) ds \wedge \Theta^m(B^m(U), \chi) - (f(u) - f^m(U_m), \chi) \\
- \left( f^m(U_m) - \Pi_0^m f^m(U_m) - \frac{U^m - \Pi_0^m U^{m-1}}{\tau_m}, \chi \right) \tag{13}
\]

Subtracting above equation from (2), this becomes

\[
\int \left( \frac{\partial \xi}{\partial t}, \chi \right) + D_1(\xi, \chi) - \int_0^t D_2(t, s, \xi(s), v) ds = \left( \frac{\partial W_T}{\partial t}, \chi \right) + D_1(W_T(t), \chi) \\
- \int_0^t D_2(t, s, W_T(s), \chi) ds - \left( \int_0^t B^m(s) v(s) ds - \Theta^m(B^m(U), \chi) \right) \\
- (f(u), \chi) - \left( \Pi_0^m U^{m-1} - \Pi_0^m U^{m-1}, \chi \right) + (\Pi_0^m f^m(U_m), \chi).
\]

By applying (8), imply

\[
\int \left( \frac{\partial \xi}{\partial t}, \chi \right) + D_1(\xi, \chi) - \int_0^t D_2(t, s, \xi(s), v) ds = \left( \frac{\partial \Phi}{\partial t}, \chi \right) + D_1(W_T(t) - W_T^m, \chi) \\
- \int_0^t D_2(t, s, \xi(s), \chi) ds - \left( \int_0^t B^m(s) v(s) ds - \Theta^m(B^m(U), \chi) \right) - (f(u), \chi) \\
- \left( f^m(U_m) - \Pi_0^m f^m(U_m) - \frac{U^m - \Pi_0^m U^{m-1}}{\tau_m}, \chi \right).
\]

By adding and subtracting \( f^m(U_m) \) on the above equation, the proof of lemma is completed.

Next, Lemmas 4.5 to 4.7 are given in details [4, 19].

**Lemma 4.5.** Let \( Z_{m,1} \) and \( Z_{m,2} \) are spatial error estimates for \( m \in [1, N] \), so that

\[
Z_{m,1} = \int_{t_{m-1}}^{t_m} \frac{\partial \Phi}{\partial t}, \xi \right) dt, \\
Z_{m,2} = \int_{t_{m-1}}^{t_m} \frac{\partial \Phi}{\partial t} \frac{\partial \xi}{\partial t} \right) dt,
\]

we have

\[
Z_{m,1} \leq \tau_m \max_{t \in [0, t_m]} \| \xi \| F_{m,1}, \\
Z_{m,2} \leq \left( \int_{t_{m-1}}^{t_m} \frac{\partial \xi}{\partial t} \frac{\partial \xi}{\partial t} \right)^{1/2} \right) \left( \tau_m \right)^{1/2} F_{m,1},
\]

where

\[
F_{m,1} := \frac{1}{\tau_m} \left\{ \left\| h_m^2 Z_{m,1} \right\| + \left\| h_m^{3/2} J_m \right\| \sum_{m}^{m-1} J_m + \sum_{j=0}^{m-1} \psi_j + h_m Q_{m}^1(U) + h_m Q_{m}^{m-1}(U) + h_m Q_{m}^{m}(U) \\
+ h_m Q_{m}^{m-1}(U) \right\}
\]
and
\[ \psi_j = \| h^j \mathcal{Z}^j \| + \left\| \frac{3}{2} \frac{\partial j^j}{\partial t} \right\|_{\Sigma_m} + h_j \mathcal{Q}_1^j(U) + h_j j(U), \]
where \( \mathcal{Q}_1^m(U) \) and \( \mathcal{Q}_2^m(U) \) are defined in lemma 4.3.

**Lemma 4.6.** (Time error estimates). Let \( Z_{m,3}, Z_{m,4} \) for \( m \in [1, N] \), then
\[
Z_{m,3} := \left| \int_0^t D_2 (t, s, \nu(s), \tau) \, ds - D_4 (W_1(t) - W_1^m, \tau) \right|
\]
\[
Z_{m,4} := \left| \int_0^t D_2 \left( t, s, \nu(s), \frac{\partial \xi}{\partial t} \right) \, ds - D_1 \left( W_1(t) - W_1^m, \frac{\partial \xi}{\partial t} \right) \right|
\]
then
\[
Z_{m,3} \leq \tau_m \max_{t \in [0, t_m]} \| \xi \| \sum_{m=1}^n \tau_m F_{m,2} + \sum_{m=1}^n \left( \int_0^{t_m} \| \nabla \xi \|^2 \, ds \right)^{\frac{1}{2}} \tau_m^{\frac{1}{2}} F_{m,3},
\]
\[
Z_{m,4} \leq \left( \int_{t_{m-1}}^{t_m} \left( \frac{\partial \xi}{\partial t} \right)^2 \, dt \right)^{\frac{1}{2}} \tau_m^{\frac{1}{2}} F_{m,2} + \left( \int_{t_{m-1}}^{t_m} \| \frac{\partial \xi}{\partial t} \|^2 \, dt \right)^{\frac{1}{2}} \tau_m^{\frac{1}{2}} F_{m,3}
\]
where
\[ F_{m,2} = \mathcal{F}_{m,2} + Q_2^{m-1}(U) + \mathcal{Q}_2^m(U) \]
and
\[
\mathcal{F}_{m,2} = \begin{cases} \frac{1}{2} \left\| \Pi_0 f^1(U^1) - \frac{U^1 - \Pi_0 U^0}{\tau m} - \mathcal{A} U^0 \right\|, m = 1 \\ \frac{1}{2} \tau m \left\| \frac{\partial}{\partial t} \left( \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^m}{\tau m} \right) \right\|, m \in [2, N] \end{cases}
\]
where \( \mathcal{Q}_2^m(U) \) is given by lemma 4.3 and
\[
F_{m,3} = \left( \frac{1}{\tau m} \int_{t_{m-1}}^{t_m} \| \mathcal{Y}_j(t) - \mathcal{Y}_j(t) \|^2 \, dt \right)^{\frac{1}{2}}.
\]

**Lemma 4.7.** (Mesh change estimates). Let \( Z_{m,5}, Z_{m,6} \) for \( m \in [1, N] \), gives
\[
Z_{m,5} := \int_0^{t_m} \left( \int_{t_{m-1}}^{t_m} \left( f^m(U^m) - \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^m}{\tau m} \right), \xi \right) \right),
\]
\[
Z_{m,6} := \int_0^{t_m} \left( \int_{t_{m-1}}^{t_m} \left( f^m(U^m) - \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^m}{\tau m} \right), \frac{\partial \xi}{\partial t} \right) \right),
\]
such that
\[
Z_{m,5} \leq \sum_{m=1}^n \left( \int_0^{t_m} \| \nabla \xi \|^2 \, ds \right)^{\frac{1}{2}} \tau_m \frac{1}{2} F_{m,4}
\]
\[
Z_{m,6} \leq \tau_m \max_{t \in [0, t_m]} \| \nabla \xi \| \left( F_{m,\infty} + \sum_{m=1}^{n} F_{m,5} + \tau_m F_{m,\infty,1} \right) \\
F_{m,5} := \| \tilde{h}_m \partial (\Pi_0^m - I) f^m (U^m) - \tau_m U^{m-1} \|, \\
F_{m,\infty} := \| h_m (\Pi_0^m - I) f^m (U^m) - \tau_m U^{m-1} \|.
\]

**Error analysis**

In this section, we will derive error bounds in terms of $L_\infty (L_2)$ and $L_\infty (H^1)$ norms.

**Lemma 5.1.** Let $f$ satisfying $|f (w_1) - f (w_2)| \leq C_f |w_1 - w_2|$, $C_f$ is a constant of Lipschitz continuous so that

\[
Z_{m,7} := \int_{t_{m-1}}^{t_m} \| (f (u) - f^m (U^m), \xi ) \| dt \leq \frac{\sqrt{C_f}}{2Y} \tau_m \max_{t \in [0, t_m]} \| \xi \|^2 \\
+ \frac{\sqrt{C_f}}{2Y} \int_{t_{m-1}}^{t_m} \| \nabla \xi \|^2 + \tau_m \delta_{m,1} \max_{t \in [0, t_m]} \| \xi \| + \tau_m \delta_{m,2} \max_{t \in [0, t_m]} \| \xi \|, \\
Z_{m,8} := \int_{t_{m-1}}^{t_m} \left\| \left( f (u) - f^m (U^m), \frac{\partial \xi}{\partial t} \right) \right\| dt \leq \frac{\sqrt{C_f}}{2Y} \tau_m \max_{t \in [0, t_m]} \| \nabla \xi \|^2 \\
+ \frac{\sqrt{C_f}}{2Y} \int_{t_{m-1}}^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 dt + \tau_m \delta_{m,1} \left( \int_{t_{m-1}}^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 ds \right)^{\frac{1}{2}} + \tau_m \delta_{m,2} \left( \int_{t_{m-1}}^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 ds \right)^{\frac{1}{2}},
\]

where

\[
\delta_{m,1} = \sqrt{C_f \{ \| \Phi^m \|, \Phi^{m-1} \}} \\
\delta_{m,2} = \frac{1}{\tau_m} \int_{t_{m-1}}^{t_m} \| f (U) - f^m (U^m) \|^2.
\]

**Proof.** To start with, $Z_{m,7}$ can be defined as

\[
Z_{m,7} = \int_{t_{m-1}}^{t_m} \| (f (u) - f^m (U^m), \xi ) \| dt \leq \int_{t_{m-1}}^{t_m} \| (f (u) - f (W_R), \xi ) \| dt \\
+ \int_{t_{m-1}}^{t_m} \| (f (W_R) - f (u), \xi ) \| dt + \int_{t_{m-1}}^{t_m} \| (f (U) - f^m (U^m), \xi ) \| dt \\
\leq G_{m,1} + G_{m,2} + G_{m,3}.
\]

Recalling Cauchy–Schwarz and Young’s inequalities, $G_{m,1}$ gives

\[
G_{m,1} := \int_{t_{m-1}}^{t_m} \| (f (u) - f (W_R), \xi ) \| dt \leq \int_{t_{m-1}}^{t_m} \| f (U) - f^m (U^m) \| \| \xi \| dt \\
\leq \frac{\sqrt{C_f}}{2Y} \tau_m \max_{t \in [0, t_m]} \| \xi \|^2 + \frac{\sqrt{C_f}}{2Y} \int_{t_{m-1}}^{t_m} \| \nabla \xi \|^2.
\]
The second term $G_{m,2}$ reads

$$G_{m,2} = \int_{t_{m-1}}^{t_m} \| (f(W_T) - f(U), \xi) \| dt \leq \int_{t_{m-1}}^{t_m} \| W_T - U \| \| \xi \| dt$$

$$\leq \sqrt{C_f} \int_{t_{m-1}}^{t_m} \left( \frac{t_m - t}{t_m} \| \phi^m \| + \frac{t - t_{m-1}}{t_{m-1}} \| \phi^{m-1} \| \right) \| \xi \| dt$$

$$\leq \tau_m \delta_{m,1} \max_{t \in [0,t_m]} \| \xi \|$$

$$G_{m,3} = \int_{t_{m-1}}^{t_m} \| f(U) - f^m(U^m), \xi \| dt \leq \tau_m \delta_{m,2} \max_{t \in [0,t_m]} \| \xi \|$$

for proving $Z_{m,8}$ will follow the same as $Z_{m,7}$.

**Lemma 5.2.** $L_2$-norm. Let $u$ and $U^m$ be the exact and approximation solutions for (2) and (6), respectively. Then, we have

$$\left( \max_{t \in [0,t_m]} \| e \|^2 + \int_{t_{m-1}}^{t_m} \| V e \|^2 dt \right)^{\frac{1}{2}} \leq 4M_C(m)\| z^0_r U^0 - U^0 \| + \left\{ 2M_C(m) \max_{t \in [0,t_m]} \| \phi^2_{m,L_2}(0) \| \right\}^2$$

$$+ 2M_C(m) (Z_{m,1} + Z_{m,3} + Z_{m,5} + Z_{m,7}) + 2C_{e} \max_{t \in [0,t_m]} \| \phi^2_{m,L_2} \|.$$

**Proof.** Now, setting $v = \xi$ in (13), gives

$$\left( \frac{\partial \xi}{\partial t}, \xi \right) + D_1(t, \xi(s), s) ds = D_1(W_T(t) - W^m_T, \xi)$$

$$- \int_{t_{m-1}}^{t_m} D_2(t, s, \xi(s), \xi) ds - \left( \int_{t_{m-1}}^{t_m} B^m(s) v(s) ds - \Theta^m(B^m(U^m), \xi) - (f(u) - f^m(U^m), \xi) \right)$$

$$- \left( f^m(U^m) - \Pi^m f^m(U^m) - \frac{U^m - \Pi^m U^m-1}{t_m}, \xi \right).$$

By Integrating the above from $t_m$ to $t_{m-1}$, gives

$$\left( \frac{1}{2} \| \xi(t^*_m) \|^2 - \frac{1}{2} \| \xi(t_{m-1}) \|^2 \right) + \frac{C_{cont}}{2C_{covr}} \int_{t_{m-1}}^{t_m} \| \nabla \xi \|^2 dt \leq \frac{C_{cont}}{2C_{covr}} \int_{t_{m-1}}^{t_m} \left( \int_{0}^{t} \| \nabla \xi(s) \|^2 ds \right)^{\frac{1}{2}}$$

$$+ Z_{m,1} + Z_{m,3} + Z_{m,5} + Z_{m,7},$$

where $Z_{m,i}, i = 1, 2, 3, 4$ defined in Lemmas 4.5, 4.6, 4.7 and 5.1, respectively. Summing up over $m = 1 : N$

$$\| \xi(t^*_m) \|^2 + C_{covr} \int_{0}^{t_m} \| \nabla \xi \|^2 dt \leq \frac{C_{cont}}{C_{covr}} \sum_{m=0}^{n} \frac{C_{cont}}{C_{covr}} \int_{t_{m-1}}^{t_m} \left( \int_{0}^{t} \| \nabla \xi(s) \|^2 ds \right)^{\frac{1}{2}}$$

$$+ Z_{m,1} + Z_{m,3} + Z_{m,5} + Z_{m,7}.$$

Setting $\| \xi^*_m \| = \| \xi(t^*_m) \| = \max_{t \in [0,t_m]} \| \xi \|$, therefore

$$\max_{t \in [0,t_m]} \| \xi(t^*_m) \|^2 + C_{covr} \int_{0}^{t_m} \| \nabla \xi \|^2 dt \leq \| \xi(0) \|^2 + \frac{C_{cont}}{C_{covr}} \sum_{m=0}^{n} \frac{C_{cont}}{C_{covr}} \int_{t_{m-1}}^{t_m} \left( \int_{0}^{t} \| \nabla \xi(s) \|^2 ds \right)^{\frac{1}{2}}$$
Recalling Lemmas 4.5, 4.6, 4.7 and 5.1, leads to

\[
\begin{align*}
\sum_{m=0}^{n} (Z_{m,1} + Z_{m,3} + Z_{m,5} + Z_{m,7}).
\end{align*}
\]

Now, setting \(2C_{cont}^2 \sqrt{\epsilon_f} - C_{coer} > 0\), and using Gronwall’s inequality, gives

\[
\begin{align*}
\max_{t \in [0,t_m]} \|\xi(t_m)\|^2 &\leq 2\|\xi(0)\|^2 + \left(2\gamma \sqrt{\epsilon_f - C_{coer}}\right) \int_0^{t_m} \|\nabla \xi\|^2 dt + \frac{C_{cont}^2}{C_{coer}} \sum_{m=0}^{n} \int_0^{t_m} \left(\int_0^t \|\nabla \xi(s)\|^2 ds\right)^{1/2} \\
&+ \frac{\sqrt{\epsilon_f}}{2\gamma} \max_{t \in [0,t_m]} \|\xi\|^2 + \max_{t \in [0,t_m]} \|\xi\| \sum_{m=0}^{n} \left(\epsilon_{m,1} + \epsilon_{m,2} + \delta_{m,1} + \delta_{m,2}\right) \\
&+ \left(\int_0^{t_m} \|\nabla \xi(s)\|^2 ds\right)^{1/2} \left(\epsilon_{m,3} + \epsilon_{m,4}\right).
\end{align*}
\]

where

\[
N_G(m) = \max_{t \in [0,t_m]} \left\{1, \sum_{m=1}^{N} \frac{2\sqrt{\epsilon_f}}{\gamma} e^{\frac{2\sqrt{\epsilon_f}}{\gamma} \sum_{m<j<n} t_m}\right\},
\]

and by taking

\[
|r|^2 \leq c^2 + rs,
\]

then

\[
|r| \leq |c| + |s|,
\]

and by taking

\[
r_0 = \max_{t \in [0,t_m]} \|\xi(t_m)\|, r_m = \left(N_G(m) \int_0^{t_m} \|\nabla \xi\|^2 dt\right)^{1/2}, c = \left(2N_G(m)\|\xi(0)\|^2\right)^{1/2}
\]

\[
s_0 = 2N_G(m) \sum_{m=0}^{n} \int_0^{t_m} \tau_m(\epsilon_{m,1} + \epsilon_{m,2} + \delta_{m,1} + \delta_{m,2}), s_n = (\tau_m)^{1/2}(\epsilon_{m,3} + \epsilon_{m,4})^{1/2}.
\]

By splitting the error \(\|e\|^2 \leq 2\|\xi\|^2 + 2\|\Phi\|^2\), then using (9) and (10), to bound \(\Phi\), gives

\[
\|\Phi\|^2 = \|U - WR\|^2 = \|\ell_m U_m + \ell_{m-1} U_{m-1} - \ell_{m-1} R_{m-1} U_{m-1} - \ell_{m-1} R_{m-1} U_{m-1} - U_{m-1} R_{m-1} U_{m-1}\|^2 \\
\leq \ell_m \|U_m - R_{m-1} U_{m-1}\|^2 + \ell_{m-1} \|U_{m-1} - R_{m-1} U_{m-1}\|^2
\]
The result of proof is ended.

\textbf{Lemma 5.4.} \( L_\infty (H^1) \). Suppose that \( u \) and \( U^m \) be its exact and approximation solutions obtained by (2) and (6). Then, the following error bounds satisfy

\[
\left( \max_{t \in [0,T_m]} \left\| \nabla e \right\|^2 + \int_{t_{m-1}}^{t_m} \left\| \frac{\partial e}{\partial t} \right\|^2 \, dt \right)^{\frac{1}{2}} \\
\leq 4M_G(m)\left( \left\| \nabla (R_0^0(t)U^0 - U^0) \right\|^2 \right)^{\frac{1}{2}} + 4\left\{ M_G(m)\left( \varepsilon_{m,1}^2 + \varepsilon_{m,2}^2 \right) \right\}^{\frac{1}{2}} + C_2 \max_{t \in [0,T_m]} \varepsilon_{m,1}^2,
\]

where

\[
M_{m,1} = 2\tau_m \max_{t \in [0,T_m]} \varepsilon_{m,\infty}^2 + \sum_{m=2}^{n} \tau_m \mathcal{F}_{m,4}^2 + \sum_{m=2}^{n} \tau_m \mathcal{F}_{m,5}^2 = \sum_{m=2}^{n} \tau_m \left( \mathcal{F}_{m,1}^2 + \mathcal{F}_{m,2}^2 + \mathcal{F}_{m,3}^2 + \delta_{m,1}^2 + \delta_{m,2}^2 \right).
\]

Proof. Putting \( v = \frac{\partial \xi}{\partial t} \) in (13), gives

\[
\begin{align*}
\left( \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right) + D_1 \left( \frac{\partial \xi}{\partial t} \right) - \int_0^T D_2 \left( t, s, \xi(s), \frac{\partial \xi}{\partial t} \right) \, ds = \left( \frac{\partial \Phi}{\partial t}, \frac{\partial \xi}{\partial t} \right) \\
D_1 \left( W_r(t) - W_r^m, \frac{\partial \xi}{\partial t} \right) - \int_0^T D_2 \left( t, s, \xi(s), \frac{\partial \xi}{\partial t} \right) \, ds - \left( \int_0^T B_m(s)\nu(s) \, ds - \Theta^m(\hat{B}(U)), \frac{\partial \xi}{\partial t} \right) \\
- \left( f(U) - f^m(U^m), \frac{\partial \xi}{\partial t} \right) - \left( f^m(U^m) - \Pi_0^m f^m(U^m) - \frac{U^m - \Pi_0^m U^{m-1}}{\tau m}, \frac{\partial \xi}{\partial t} \right).
\end{align*}
\]

By Integrating the above from \( t_m \) to \( t_{m-1} \) and recalling Lemmas 4.5, 4.6, 4.7 and 5.1, respectively. Summing up over \( m = 1: N \) and \( Z_{m,i}, i = 1, 2, 3, 4 \)

\[
\| \nabla \xi(t_m) \|^2 + \frac{C_{\text{covr}}}{2} \int_0^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 \, dt \leq \frac{C_{\text{cont}}}{C_{\text{covr}}} \sum_{m=0}^{n} \int_{t_{m-1}}^{t_m} \left( \int_0^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 \, ds \right)^{\frac{1}{2}} \\
+ \sum_{m=0}^{n} \int_{t_{m-1}}^{t_m} \left( Z_{m,2} + Z_{m,4} + Z_{m,6} + Z_{m,8} \right)^{\frac{1}{2}}.
\]

Setting \( \| \nabla \xi(t_m) \| = \| \nabla \xi(t_m) \| = \max_{t \in [0,t_m]} \| \nabla \xi \| \)

therefor

\[
\max_{t \in [0,t_m]} \| \nabla \xi(t_m) \|^2 + C_{\text{covr}} \frac{C_{\text{cont}}}{2C_{\text{covr}}} \sum_{m=0}^{n} \int_{t_{m-1}}^{t_m} \left( \int_0^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 \, ds \right)^{\frac{1}{2}} \\
+ 2 \sum_{m=0}^{n} \int_{t_{m-1}}^{t_m} \left( Z_{m,2} + Z_{m,4} + Z_{m,6} + Z_{m,8} \right)^{\frac{1}{2}}.
\]

Recalling Lemmas 4.5, 4.6, 4.7 and 5.1, leads to
Now, setting \( \left( \frac{C_{cont}}{C_{covr}} \right) \) > 0, and using Gronwall’s inequality, gives

\[
\max_{t \in [0, \tau_m]} \| \nabla \xi(t_m) \|^2 + \mathcal{M}_G(m) \int_0^{t_m} \left\| \frac{\partial \xi}{\partial t} \right\|^2 dt \leq 2\mathcal{M}_G(m)\| \nabla \xi(0) \|^2
\]

where

\[
\mathcal{M}_G(m) = \max_{t \in [0, \tau_m]} \left\{ 1, \sum_{m=1}^{N} \left( \frac{\sqrt{C_f}}{\gamma} e^{\frac{2\sqrt{C_f}}{\gamma} \sum_{m<j<\tau_m} \tau_m} \right) \right\}.
\]

Taking

\[
r_0 = \max_{t \in [0, \tau_m]} \| \nabla e(t_m) \|, r_m = \left\{ \mathcal{M}_G(m) \int_0^{t_m} \left\| \frac{\partial e}{\partial t} \right\|^2 dt \right\}^{\frac{1}{2}}, c = \left\{ 2\mathcal{M}_G(m)\| \nabla e(0) \|^2 \right\}^{\frac{1}{2}}
\]

\[
, s_0 = 2\mathcal{M}_G(m) \sum_{m=0}^{n} \int_{t_{m-1}}^{t_m} \left( \tau_m \| e_{m+1} + e_{m,3} + e_{m,6} \right) \right\}^{\frac{1}{2}},
\]

Now, \( \| \nabla e \|^2 \leq 2\| \nabla \xi \|^2 + 2\| \nabla \Phi \|^2 \).

The second term on the right-hand side of above equation can be obtained by using (9) and (10), gives

\[
\| \Phi \|^2 = \| \nabla (U - W) \|^2 = \| \nabla \left( \ell_m U^m + \ell_{m-1} U^{m-1} - \ell_m R^m(t) U^m + \ell_{m-1} R^{m-1}(t) U^{m-1} \right) \|^2,
\]

\[
\leq \ell_m \| \nabla \left( U^m - R^m(t) \right) \|^2 + \ell_{m-1} \| \nabla \left( U^{m-1} - R^{m-1}(t) \right) \|^2,
\]

\[
\leq \max_{t \in [0, \tau_m]} \left\{ \| \nabla \left( U^m - R^m(t) \right) \|^2 \right\}, \| \nabla \left( U^{m-1} - R^{m-1}(t) \right) \|^2 \right\},
\]

\[
\leq \max_{t \in [0, \tau_m]} \left\{ \| \nabla \left( U^m - R^m(t) \right) \|^2 \right\}.
\]

By combining with \( \| \nabla \xi \| \), the proof will be finished.
5. Conclusion

This paper concerns to obtain an optimal order estimates in term of $L_\infty(L_2)$ and $L_\infty(H^1)$ norms for integro-parabolic problems for Lipschitz case. Ritz -volterra reconstruction techniques introduced in 2015 used in error analysis. This technique enables us to prove optimal order bound for this type of problems. Gronwall’s Lemma is presented with some tools to control of the size of errors. This work can be modified to the case of parabolic interface problems [22, 23, 24, 25]. Another ingesting of this paper is to combine between implicit-explicit Runge Kutta methods with finite element methods [26, 27].

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