Abstract

We prove that the exponential growth rate of the regular language of penetration sequences is smaller than the growth rate of the regular language of normal form words, if the acceptor of the regular language of normal form words is strongly connected. Moreover, we show that the latter property is satisfied for all irreducible Artin monoids of spherical type, extending a result by Caruso.

Apart from establishing that the expected value of the penetration distance $pd(x, y)$ in irreducible Artin monoids of spherical type is bounded independently of the length of $x$, if $x$ is chosen uniformly among all elements of given canonical length and $y$ is chosen uniformly among all atoms, our results also give an affirmative answer to a question posed by Dehornoy.

1 Introduction

Random walks on discrete infinite groups in general are a subject that has received substantial interest; see for instance [KV83, Woe00] and the references therein. Random walks on braid groups in particular have many applications in areas as diverse as organic chemistry, nanotechnology, fluid dynamics or astrophysics [Woe94, Nec96, Woe00, BBP02].

Random walks on the 3-strand braid group were analysed in [MM07]. A complete understanding of random walks on braid groups on a larger number of strands, let alone more general classes of groups such as Artin–Tits groups or Garside groups, has not been achieved yet.

A topic closely related to random walks on Garside groups is the behaviour of the Garside normal forms of random elements: Fixing a position in the normal form, one obtains an induced distribution on the set of simple elements (that is, on the symmetric group in the case of braids) which can be studied. In [GT13], the authors studied these induced distributions and observed convergence. More precisely, experimental data suggests that, except for an initial and a final region whose lengths are uniformly bounded, the distributions of the factors of the normal form of sufficiently long random braids depend neither on the position in the normal form nor on the lengths of the random braids.
Another, related, observation made for random braids in \cite{GT13} was the fact that the expected value of the \textit{penetration distance} $\text{pd}(x,y)$, that is, the number of factors in the normal form of $x$ that are modified when computing the normal form of the product $x \cdot y$, is bounded above independently of the length of $x$.

For specific distributions of $x$ and $y$, we related the behaviour of the expected value $\mathbb{E}[\text{pd}]$ of the penetration distance to the growth rates $\alpha$ and $\beta$ of two regular languages (cf. Section 2.2). More precisely, we showed that, if $x$ is chosen uniformly among all elements of canonical length $k$ and $y$ is chosen uniformly among the atoms, then $\mathbb{E}[\text{pd}]$ is bounded if $\alpha < \beta$ holds \cite[Theorem 4.7]{GT13}. Experimental data strongly suggested that the latter condition is satisfied for all irreducible Artin monoids of spherical type.

In this paper, we relate the condition $\alpha < \beta$ to certain structural properties of the lattice of simple elements (more precisely, to the connectedness of the acceptor of the regular language of normal forms) and prove that this condition is satisfied for all irreducible Artin monoids of spherical type.

Apart from establishing that the expected value of the penetration distance is bounded in the sense of \cite[Theorem 4.7]{GT13} for irreducible Artin monoids of spherical type, our results also answer a question posed by Dehornoy in \cite{Deh07} in the affirmative.

We also prove that the expected value of the penetration distance is unbounded for Zappa–Szép products of irreducible Artin monoids of spherical type.

The structure of the paper is as follows. In Section 2.1, we recall the basic concepts of normal forms in Garside groups; experts may skip this section. In Section 2.2, we recall the regular languages defined in \cite{GT13} to study the expected value of the penetration distance. Section 2.3 recalls the notion of Zappa–Szép products.

In Section 3 we define the notions of \textit{essential} simple elements and \textit{essential transitivity}, which will be used in Section 4 to compare the growth rates $\alpha$ and $\beta$. Section 5 establishes that irreducible Artin monoids of spherical type are essentially transitive and have the property that all proper simple elements are essential, and thus that the results of the preceding sections can be applied to this class of monoids.

\section{Background}

In order to fix notation, we briefly recall the main concepts used in the paper. The material in Section 2.1 is rather well-known, so experts may skip that section.

\subsection{Garside monoids and Garside normal form}

We refer to \cite{DP99, Deh02, DDG+14} for details.

Let $M$ be a monoid. The monoid $M$ is called \textit{left-cancellative} if for any $x, y, y'$ in $M$, the equality $xy = xy'$ implies $y = y'$. Similarly, $M$ is called \textit{right-cancellative} if for any $x, y, y'$ in $M$, the equality $yx = y'x$ implies $y = y'$. 
For \( x, y \in M \), we say that \( x \) is a left-divisor or prefix of \( y \), writing \( x \preceq_M y \), if there exists an element \( u \in M \) such that \( y = xu \). If the monoid is obvious, we simply write \( x \preceq y \) to reduce clutter. Similarly, we say that \( x \) is a right-divisor or suffix of \( y \), writing \( y \succeq_M x \) or \( y \succeq x \), if there exists \( u \in M \) such that \( y = ux \).

If \( M \) does not contain any non-trivial invertible elements, then the relation \( \preceq \) is a partial order if \( M \) is left-cancellative, and the relation \( \succeq \) is a partial order if \( M \) is right-cancellative.

An element \( a \in M \setminus \{1\} \) is called an atom if whenever \( a = uv \) for \( u, v \in M \), either \( u = 1 \) or \( v = 1 \) holds. The existence of atoms implies that \( M \) does not contain any non-trivial invertible elements. The monoid \( M \) is said to be atomic if it is generated by its set \( A \) of atoms and if for every element \( x \in M \) there is an upper bound on the length of decompositions of \( x \) as a product of atoms, that is, if \( |x|_A := \sup\{k \in \mathbb{N} : x = a_1 \cdots a_k \text{ with } a_1, \ldots, a_k \in A\} < \infty \).

An element \( d \in M \) is called balanced, if the set of its left-divisors is equal to the set of its right-divisors. In this case, we write \( \text{Div}(d) \) for the set of (left- and right-) divisors of \( d \).

**Definition 1.** A quasi-Garside structure is a pair \( (M, \Delta) \) where \( M \) is a monoid and \( \Delta \) is an element of \( M \) such that

(a) \( M \) is cancellative and atomic,
(b) the prefix and suffix relations are lattice orders, that is, for any pair of elements there exist unique least common upper bounds and unique greatest common lower bounds with respect to \( \preceq \) respectively \( \succeq \),
(c) \( \Delta \) is balanced, and
(d) \( M \) is generated by the divisors of \( \Delta \).

If the set of divisors of \( \Delta \) is finite then we say that \( (M, \Delta) \) is a Garside structure.

A monoid \( M \) is a (quasi)-Garside monoid if there exists a (quasi)-Garside element \( \Delta \in M \) such that \( (M, \Delta) \) is a (quasi)-Garside structure. The elements of \( \text{Div}(\Delta) \) are called simple elements. (Note that the set of simple elements depends on the choice of the Garside element.)

**Remark.** If \( (M, \Delta) \) is a quasi-Garside structure in the above sense, then in the terminology of [DDG+14], the set \( \text{Div}(\Delta) \) forms a bounded Garside family for the monoid \( M \).

**Notation 2.** If \( M \) is a left-cancellative atomic monoid, then least common upper bounds and greatest common lower bounds are unique if they exist. In this situation, we will write \( x \lor y \) for the \( \preceq \)-least common upper bound of \( x, y \in M \) if it exists, and we write \( x \land y \) for their \( \preceq \)-greatest common lower bound if it exists. If \( x, y \in M \) admit a \( \preceq \)-least common upper bound, we define \( x \setminus y \) as the unique element of \( M \) satisfying \( x \setminus y = x \lor y \).

If \( (M, \Delta) \) is a Garside structure, we write \( \mathcal{D}_M \) for the set of simple elements \( \text{Div}(\Delta) \), and we define the set of proper simple elements as \( \mathcal{D}_M^0 = \mathcal{D}_M \setminus \{1, \Delta\} \), where \( 1 \) is the identity element of \( M \). To avoid clutter, we will usually drop the subscript if there is no danger of confusion. For \( x \in \mathcal{D} \), there exist a unique elements \( \partial x = \partial_M x \in \mathcal{D} \) and \( \partial x = \partial_M x \in \mathcal{D} \) such that \( x \partial x = \Delta = (\partial x)x \).

We define inductively \( \partial^{k+1} x = \partial(\partial^k x) \) and \( \partial^{k+1} x = \partial(\partial^k x) \) for \( k \in \mathbb{N} \). As \( \partial(\partial x) = x = \partial(\partial x) \) for any \( x \in \mathcal{D} \), we can define \( \partial^k = \partial^{-k} \) for any \( k \in \mathbb{Z} \).
Clearly, \( \partial^k x \in \mathcal{D} \) iff \( x \in \mathcal{D} \). Moreover, for any \( x, y \in \mathcal{D} \), one has \( x \leq y \) iff \( \partial x \vdash \partial y \) iff \( \partial^2 x \leq \partial^2 y \).

Given a set \( X \) we will write \( X^* = \bigcup_{i=0}^{\infty} X^i \) for the set of strings of elements of \( X \). We will write \( \varepsilon \) for the empty string and separate the letters of a string with dots, for example we will write \( a.b.a \).

Definition 3. Given a (quasi)-Garside structure \((M, \Delta)\) we can define the left normal form of an element by repeatedly extracting the \( \preceq \)-GCD of the element and \( \Delta \). More precisely, the normal form of \( x \in M \) is the unique word \( NF(x) = x_1.x_2.\cdots.x_\ell \) in \((\mathcal{D} \setminus \{1\})^*\) such that \( x = x_1.x_2.\cdots.x_\ell \) and \( x_i = \Delta^{\ell} x_{i+1} \cdots x_\ell \) for \( i = 1, \ldots, \ell \), or equivalently, \( \partial x_{i-1} \land x_i = 1 \) for \( i = 2, \ldots, \ell \). We write \( x_1|x_2|\cdots|x_\ell \) for the word \( x_1.x_2.\cdots.x_\ell \) together with the proposition that this word is in normal form.

If \( x_1|x_2|\cdots|x_\ell \) is the normal form of \( x \in M \), we define the infimum of \( x \) as \( \inf(x) = \max\{i \in \{1, \ldots, \ell\} : x_i = \Delta\} \), the supremum of \( x \) as \( \sup(x) = \ell \), and the canonical length of \( x \) as \( \ell = \sup(x) - \inf(x) \). Note that \( \inf(x) \) is the largest integer \( i \) such that \( x \preceq \Delta^i \) holds, and \( \sup(x) \) is the smallest integer \( i \) such that \( x \leq \Delta^i \) holds.

The operation \( \partial \) can be extended to all of \( M \) by defining \( \partial x \) to be the unique element such that \( x\partial x = \Delta^{\sup(x)} \). If \( NF(x) = x_1|x_2|\cdots|x_\ell \) is the normal form of \( x \) and \( \inf(x) = k \), then \( NF(\partial x) = x_1|x_2|\cdots|x_\ell \cdot |\partial^{(\ell-k)+1} x_{\ell+1} \).

Let \( L = L_M \) be the language on the set \( \mathcal{D} \) of proper simple elements consisting of all words in normal form, and write \( L^{(n)} = L^{(n)}_M \) for the subset consisting of words of length \( n \):

\[
L := \bigcup_{n \in \mathbb{N}} L^{(n)} \quad \text{where} \quad L^{(n)} := \{x_1.\cdots.x_n \in (\mathcal{D})^* : \forall i, \partial x_i \land x_{i+1} = 1\}
\]

We also define \( \overline{L} := \overline{L}_M := \bigcup_{n \in \mathbb{N}} \overline{L}^{(n)} \), where

\[
\overline{L}^{(n)} := \overline{L}^{(n)}_M := \{x_1.\cdots.x_n \in (\mathcal{D} \setminus \{1\})^* : \forall i, \partial x_i \land x_{i+1} = 1\}.
\]

Definition 3. Given a Garside monoid \( M \) with Garside element \( \Delta \) and an integer \( k \geq 1 \), let \( M(k) \) denote the same monoid but with the Garside structure given by the Garside element \( \Delta^k \).

As the partial orders \( \preceq \) and \( \succeq \) on \( M \) and \( M(k) \) are identical, so are the lattice operations \( \lor, \land \). It is obvious from the definitions that one has \( \partial_{M(k)} x = \partial_M x \Delta^m \), where \( m = k\left\lceil \frac{\log x}{k} \right\rceil - \sup(x) \).

Lemma 4. If \( w \) is a word in \( M(k) \)-normal form, then replacing each letter of \( w \) with the word for its \( M \)-normal form yields a word in \( M \)-normal form.

Proof. Suppose that we have \( M(k) \)-simple elements \( x, y \in D_{M(k)} \) with \( M \)-normal forms

\[
NF_M(x) = x_1|x_2|\cdots|x_k \\
NF_M(y) = y_1|y_2|\cdots|y_k
\]

It is sufficient to prove that \( x_k|y_1 \) in the original Garside structure of \( M \).

Let \( m = \partial_M x_k \land y_1 \). We have that \( m \preceq y_1 \preceq y \) and \( m \preceq \partial_M x_k \preceq \partial_{M(k)} x \).

Hence \( m \preceq \partial_{M(k)} x \land y = 1 \) and so \( \partial_M x_k \land y_1 = 1 \) as required. \( \square \)
Remark. The map from Lemma 4 does not take $L_{M(k)}$ to $L_M$, as proper simple elements of $M(k)$ may have non-zero infimum in $M$.

Lemma 5. Given a word $x_0|x_1|\cdots|x_l$ in $M$-normal form, let $x_j = 1$ for $j > l$ and define $y_i = x_{ik}x_{ik+1}\cdots x_{(i+1)k-1}$ for $i = 0, \ldots, \lceil \frac{l}{k} \rceil$.

Then $y_0, y_1, \cdots, y_{\lceil \frac{l}{k} \rceil}$ is in $M(k)$-normal form.

Proof. One has $\Delta \land \partial_{M(k)} y_{i-1} \land y_i = \partial_{M} x_{ik-1} \land x_{ik} = 1$ for $i = 1, \ldots, \lceil \frac{l}{k} \rceil$ and $x_{ik} \neq 1$, hence $y_i \neq 1$, for $i = 0, \ldots, \lceil \frac{l}{k} \rceil$.

2.2 Penetration distance and penetration sequences

Throughout this section, let $M$ be a Garside monoid with Garside element $\Delta$, set of atoms $A$, and set of proper simple elements $\mathcal{D}$.

In [GT13], we investigated the penetration distance, that is, the number of factors in the normal form of an element $x \in M$ that undergo a non-trivial change when $x$ is multiplied on the right by an element $y \in M$.

Definition 6 ([GT13, Definition 3.2]). For $x, y \in M$, the penetration distance for the product $xy$ is

$$\text{pd}(x, y) = \text{cl}(x) - \max \{ i \in \{0, \ldots, \text{cl}(x)\} : x\Delta^{-\inf(x)} \land \Delta^i = xy\Delta^{-\inf(xy)} \land \Delta^i \}.$$

For certain probability distributions for $x$ and $y$, we calculated the expected value of $\text{pd}(x, y)$ by analysing the regular languages $\mathcal{L}$ and $\text{PSeq} = \text{PSeq}_M = \bigcup_{k \in \mathbb{N}} \text{PSeq}^{(k)}_M$, where $\text{PSeq}^{(k)}_M = \text{PSeq}^{(k)}_M$ denotes the set of all penetration sequences of length $k$:

Definition 7 ([GT13, Definition 4.2]). A word $(s_k, m_k) \cdots (s_2, m_2) (s_1, m_1)$ in $(\mathcal{D} \times \mathcal{D})^*$ is a penetration sequence of length $k$ if $m_1 \leq \partial s_1$ holds, and if one has $s_i m_i \neq \Delta$, $\partial s_{i+1} \land s_i = 1$, and $m_{i+1} = \partial s_{i+1} \land s_i m_i$ for $i = 1, \ldots, k - 1$.

Notation 8. The regular languages $\mathcal{L}_M, \mathcal{L}_D$ and $\text{PSeq}_M$ are factorial (that is, closed under taking subwords). By [Sim08, Corollary 4], there exist constants $p_M, q_M, r_M \in \mathbb{N}$ and $\alpha_M, \beta_M, \gamma_M \in \{0\} \cup [1, \infty]$ such that one has

$$|\text{PSeq}^{(k)}_M| \in \Theta(k^{p_M} \alpha_M^k), \quad |\mathcal{L}^{(k)}_M| \in \Theta(k^{q_M} \beta_M^k) \quad \text{and} \quad |\mathcal{L}^{(k)}_D| \in \Theta(k^{r_M} \gamma_M^k).$$

One of the key results of [GT13] was that, if $x \in \mathcal{L}^{(k)}_M$ and $y \in \mathcal{A}_M$ are chosen with uniform probability, the expected value of the penetration distance is uniformly bounded (that is, there exists a bound that is independent of $k$) if one has $\alpha_M < \beta_M$ [GT13, Theorem 4.7].

2.3 Zappa–Szép products

Zappa–Szép products of Garside monoids were considered in [Pic01] (there called “crossed products”) and [GT14]. The notion of Zappa–Szép products generalises direct and semidirect products; the fundamental property being the existence of unique decompositions of the elements of a monoid as products of elements of two submonoids.
Definition 9 ([GT14, Definition 1]). Let $K$ be a monoid with two submonoids $G$ and $H$. We say that $K$ is the (internal) Zappa–Szép product of $G$ and $H$, written $K = G ∙ H$, if for every $k ∈ K$ there exist unique $g_1, g_2 ∈ G$ and $h_1, h_2 ∈ H$ such that $g_1 h_1 = k = h_2 g_2$.

It was shown in [GT14] that the Zappa–Szép product $K = G ∙ H$ is a Garside monoid if and only if both $G$ and $H$ are Garside monoids [GT14, Theorem 31, Theorem 34].

If Garside structures for $K$, $G$ and $H$ are chosen in a compatible way, then normal forms in $K$ can be completely described in terms of normal forms in $G$ and in $H$:

Definition 10 ([GT14, Definition 37]). A Zappa–Szép product $K = G ∙ H$ is called a Garside Zappa–Szép product if $K$ is a Garside monoid (and hence $G$ and $H$ are also Garside monoids) and the Garside elements $∆_K$, $∆_G$ and $∆_H$ of $K$, $G$ and $H$, respectively, are chosen such that $∆_K = ∆_G ∆_H$.

Theorem 11 ([GT14, Theorem 28, Theorem 38, Corollary 52]). Suppose that $K = G ∙ H$ is a Garside Zappa–Szép product.

Then the following hold.

1. The map $G × H → K$ given by $(g, h) → g ∨ h$ is a poset isomorphism $(G, ≼_G) × (H, ≼_H) → (K, ≼_K)$. Similarly, the map $G × H → K$ given by $(g, h) → g ˘ v h$ is a poset isomorphism $(G, ≻_G) × (H, ≻_H) → (K, ≻_K)$.

2. For all $g ∈ G$ and $h ∈ H$, one has

$$\inf_K (g ∨ h) = \min(\inf_G (g), \inf_H (h))$$

$$\sup_K (g ∨ h) = \max(\sup_G (g), \sup_H (h))$$

3. The map $ψ : \overline{L}_G × \overline{L}_H → \overline{L}_K$ given by

$$ψ(\begin{array}{c}g_1 | g_2 | \cdots | g_m \end{array}, \begin{array}{c}h_1 | h_2 | \cdots | h_n \end{array}) = \text{NF}(\begin{array}{c}(g_1 g_2 \cdots g_m) ∨ (h_1 h_2 \cdots h_n) \end{array})$$

is a bijection.

3 Essential simple elements

Throughout this section, let $M$ be a Garside monoid with Garside element $∆$, set of atoms $A$ and set of proper simple elements $D$, and let $L$ be the words over the alphabet $D$ that are in normal form.

By the acceptor graph $Γ$ of $L$ we shall mean the labelled directed graph with vertex set $D$ and, for $x, y ∈ D$, an edge labelled $y$ from $x$ to $y$ iff $∂x ∧ y = 1$. Paths in this graph are in 1-to-1 correspondence with words in normal form. This graph can be made into a deterministic finite state automaton (DFA) accepting $L$ by adding an initial vertex $1_Γ$ with an edge labelled $y$ from $1_Γ$ to $y$ for each $y ∈ D$.

Definition 12. Say that a simple element $x ∈ D$ is essential if for all $K ∈ \mathbb{N}$ there exists a word $x_1 | x_2 | \cdots | x_n ∈ L$ such that $x = x_i$ for some $K < i < n − K$. 

6
Let $\mathcal{E}_{\text{ss}} = \mathcal{E}_{\text{ss}M}$ denote the set of all essential simple elements and $\mathcal{L}_{\text{Ess}}$ be the restriction of $\mathcal{L}$ to essential simple elements.

$$\mathcal{L}_{\text{Ess}} := \mathcal{L} \cap \mathcal{E}_{\text{ss}}^*$$

**Example 13.** In $M = \langle a, b \mid aba = b^2 \rangle^+$ one has $\partial b \wedge x = b^2 \wedge x \neq 1$ for all proper simple elements $x \in \mathcal{D}$, whence $b$ can only occur as the final canonical factor and $b^2$ can only occur as the first canonical factor in a word in normal form. That is, the simple elements $b$ and $b^2$ are not essential.

**Proposition 14.** A proper simple element $x \in \mathcal{D}$ is essential if and only if its complement $\partial x$ is essential.

**Proof.** Suppose that $x \in \mathcal{D}$ is essential and that we are given $K \in \mathbb{N}$. As $x$ is essential there exists a word

$$w_n \cdots | w_2 | w_1 | x | y_1 | y_2 | \cdots | y_m \in \mathcal{L}$$

with $n, m > K$. As for any $s, t \in \mathcal{D}$ and any $k \in \mathbb{Z}$ one has $\partial s \wedge t = 1$ iff $\partial (\partial^{2k+1}t) \wedge \partial^{2k+3}s = 1$, the word

$$\partial^{2m+1}y_m \cdot \cdots \cdot \partial^{3}y_2 \cdot \partial^{-1}y_1 \cdot \partial x \cdot \partial^{3}w_1 \cdot \partial^{5}w_2 \cdot \cdots \cdot \partial^{2n+1}w_n$$

is also in normal form and hence lies in $\mathcal{L}$. Therefore $\partial x$ is essential.

Applying the same argument with $\tilde{\partial} = \partial^{-1}$, we deduce that if $\partial x$ is essential then $x$ is essential. \[\Box\]

**Lemma 15.** A simple element in $M(k)$ is essential if and only if its normal form in $M$ is a length $k$ word of essential simple elements.

$$\mathcal{E}_{\text{ss}M(k)} = \mathcal{L}^{(k)}_{\text{Ess},M}$$

**Proof.** First we will show that the $M$-normal form of every $M(k)$-essential element is a length $k$ word of $M$-essential elements. So suppose that $x \in \mathcal{E}_{\text{ss}M(k)}$ is $M(k)$-essential and that $\text{NF}_M(x) = x_1 | x_2 | \cdots | x_k$ is the $M$-normal form of $x$. As $x$ is essential, for every $K \in \mathbb{N}$ there exists a word

$$w_K \cdots | w_2 | w_1 | x | y_1 | y_2 | \cdots | y_K \in \mathcal{L}_M(k).$$

By Lemma 4 we can replace each $w_i$ and each $y_i$ by their $M$-normal forms to produce a word in $M$-normal form.

$$\text{NF}_M(w_i) = w_{i,1} | w_{i,2} | \cdots | w_{i,k}$$

$$\text{NF}_M(y_i) = y_{i,1} | y_{i,2} | \cdots | y_{i,k}$$

As each $w_i$ and $y_i$ are proper $M(k)$-simple elements we have that, for $i < K$, $w_{i,j}$ and $y_{i,j}$ are proper $M$-simple elements. Hence

$$w_{K-1,1} | \cdots | w_{K-1,k} | \cdots | w_{1,1} | \cdots | w_{1,k} | x_1 | \cdots | x_k | y_{1,1} | \cdots | y_{1,k} | \cdots | y_{K-1,1} | \cdots | y_{K-1,k}$$

is a word in $\mathcal{L}_M$ and so each $x_i$ is essential.
It remains to show that every $x_1|x_2|\cdots|x_k \in \mathcal{L}_{\mathcal{E},M}^{(k)}$ defines a $M(k)$-essential simple element. Using Lemma 5 it is clear that $x_1|x_2|\cdots|x_k$ is a proper $M(k)$-simple element. As $x_1$ and $x_k$ are essential, for any $K \in \mathbb{N}$ there exist words $w_1|w_2|w_k \in \mathcal{L}^k_M$ and $y_1|y_2|\cdots|y_k \in \mathcal{L}$ such that $w_1|x_1$ and $x_k|y_1$. By Lemma 5, if we let
\[
\tilde{w}_i = w_i K w_{i+1} \cdots w_{(i-1)K+1}
\]
\[
\tilde{x} = x_1 x_2 \cdots x_k
\]
\[
\tilde{y}_i = y_{(i-1)K+1} y_{(i-1)K+2} \cdots y_{iK}
\]
then
\[
\tilde{w}_K \cdots \tilde{w}_2 \cdot \tilde{w}_1 \cdot \tilde{x} \cdot \tilde{y}_2 \cdots \tilde{y}_K
\]
is in $M(k)$-normal form. Moreover, because each $w_i$ and each $y_i$ are proper $M$-simple elements, we know that each letter of this word is a proper $M(k)$-simple element. Therefore this word is in $\mathcal{L}^k_M$ and hence $\tilde{x}$ is essential.

**Proposition 16.** If $\mathcal{E}ss = \emptyset$ then $M = \mathbb{N}$.

**Proof.** First note that $\mathcal{L}$ must be finite: Otherwise, there would be word $w \in \mathcal{L}$ whose length is longer than the number of states in the automaton accepting $\mathcal{L}$ whence, by the pumping lemma, there would exist words $x$, $y$, $z$ such that $w = x \cdot y \cdot z$, $|y| > 1$ and $x \cdot y^i \cdot z \in \mathcal{L}$ for all $i$; in particular, every letter of $y$ would be essential in contradiction to the hypothesis.

By Lemma 15, if $\mathcal{E}ssM$ is empty then $\mathcal{E}ss_{M(k)}$ is empty for all $k$. Hence, passing to $M(k)$, where $k$ is the length of the longest word contained in $\mathcal{L}$, we may assume that all the words in $\mathcal{L}$ have length 1.

For every $s \in \mathcal{D}$ and every $a \in \mathcal{A}$ we have $sa \in \mathcal{D}$, as otherwise, $s|a$ would be a word of length 2 in $\mathcal{L}$. Choosing any maximal element $s \in \mathcal{D}$, we have $sa = \Delta$ for all atoms $a \in \mathcal{A}$, hence, by cancellativity, all the atoms must be equal. In other words there is a single atom and so $M = \mathbb{N}$.

**Proposition 17.** If $\mathcal{E}ss \neq \emptyset$ then $|\mathcal{E}ss| > 1$.

**Proof.** Suppose $\mathcal{E}ss = \{s\}$.

As $s$ is essential there exist arbitrarily long words containing $s$. So there is a word $w \in \mathcal{L}$ whose length is longer than the number of states in the automaton accepting $\mathcal{L}$ hence, by the pumping lemma, there exist words $x$, $y$, $z$ such that $w = x \cdot y \cdot z$, $|y| > 1$ and $x \cdot y^i \cdot z \in \mathcal{L}$ for all $i$.

This means that every letter of $y$ is essential. Therefore $y$ is a power of $s$ and so $s|s$, that is, $\partial s \wedge s = 1$.

By Proposition 14, $\partial s$ is essential, hence $\partial s = s$. Therefore, $\partial s \wedge s = s$, which is a contradiction.

**Definition 18.** Say that the language $\mathcal{L}$ is essentially $k$-transitive if $\mathcal{E}ss \neq \emptyset$ and for all $x, y \in \mathcal{E}ss$ there exists a word of length less than or equal to $k + 1$ in $\mathcal{L}_{\mathcal{E}ss}$ which starts with $x$ and ends with $y$.

Say that $\mathcal{L}$ is essentially transitive if it is essentially $k$-transitive for some $k$.

**Remark.** The language $\mathcal{L}$ is essentially transitive if and only if the acceptor graph $\Gamma$ has exactly one non-singleton strongly connected component (that is, exactly one strongly connected component containing at least one edge).
Proposition 19. For any pair of Garside monoids $G$ and $H$, each containing at least one proper simple element, their free product amalgamated over their Garside elements, $G *_{\Delta G=\Delta H} H$, is a Garside monoid for which every proper simple element is essential and whose language of normal forms is essentially 2-transitive.

Proof. The amalgamated product $G *_{\Delta G=\Delta H} H$ is a Garside monoid [DP99, Prop. 5.3][Fro07] whose set of proper simple elements is the disjoint union of the proper simple elements of $G$ and $H$.1

Now consider $u, v \in D^*$. If $u$ and $v$ lie in different factors then $u|v$ is in normal form. If $u$ and $v$ lie in the same factor, then choosing any proper simple element $w$ from the other factor yields a word $u|w|v$ that is in normal form.

Lemma 20. If the language of normal forms of $M(k)$ is essentially transitive then the language of normal forms of $M$ is essentially transitive.

Proof. Let $x, y \in \mathcal{ESS}_M$. Since $x$ and $y$ are essential, there exist elements $x_1, \ldots, x_{k-1}$ and $y_2, \ldots, y_k$ in $D_M$ such that $\bar{x} = x_1 \cdots x_{k-1} \cdot x$ and $\bar{y} = y \cdot y_2 \cdots y_k$. By Lemma 15, $\bar{x}, \bar{y} \in \mathcal{ESS}_M(k)$. Now as the language of normal forms in $M(k)$ is essentially transitive, we can find a path connecting $\bar{x}$ to $\bar{y}$ and then, by Lemma 4, taking the $M$-normal form of each letter gives a path in $M$ from $x$ to $y$.

Remark. The converse of Lemma 20 does not hold. For example, consider the monoid $M = \langle a, b \mid a^2 = b^2 \rangle^+$ (see Example 24). $M$ is essentially transitive, but $M(2)$ is not: We have $\mathcal{ESS}_M = \{a, b\}$ and $a|b$ as well as $b|a$. However, $\mathcal{ESS}_{M(2)} = \{ab, ba\}$ and $L_M(2) = (ab)^* \cup (ba)^*$.

Lemma 21. If the language of normal forms of $M$ is essentially transitive then for all $k_0$ there exists $k \geq k_0$ such that the language of normal forms of $M(k)$ is essentially transitive.

Proof. First note that $M(k)$ is essentially transitive if and only if for each pair $x, y$ of $M$-essential elements there exists a path $x|\cdot|x_2|\cdots|x_l|y$ such that $l$ is a multiple of $k$.

For each pair $x, y$ of $M$-essential elements choose $z \in \mathcal{ESS}_M$ together with paths $x|x_1|x_2|\cdots|x_l|z\cdot|z_1|z_2|\cdots|z_m|z$ and $z|y_1|y_2|\cdots|y_n|y$.

As the set of essential elements is finite, we can choose an integer $k \geq k_0$ such that, for each pair $x, y$ the integer $m$ is coprime to $k$. Then for each pair $x, y$ there exists $p$ such that $l + pm + n = 0 \mod k$.

Definition 22. A word $x_1|\cdots|x_k \in L^{(k)}$ is called rigid, if the word $x_k|x_1$ is in normal form. Let $L_{ess}^{(k)}$ denote the set of rigid words in $L^{(k)}$.

Theorem 23 ([Car13, Proposition 4.1]). If the language $L$ is essentially transitive then, for sufficiently large $l$, the proportion of rigid elements in the ball of radius $l$, that is in $\bigcup_{k \leq l} L^{(k)}$, is bounded below by a positive constant.

1Free products amalgamated over standard parabolic subgroups are considered for Artin groups in [DG13] and for preGarside groups in [GP12].
Theorem 23. It is not necessarily true
that the percentage of rigid elements in
for sufficiently large
large
d essentially transitive.

Definition 26. For $y \in M$, let $\Delta_y := \{x \setminus y : x \in M\}$. The monoid $M$ is
called $\Delta$-pure if $\Delta_a = \Delta_b$ holds for any $a, b \in A$.
Theorem 27 ([Pic01, Proposition 4.7][GT14, Theorem 36]). A Garside monoid $M$ is $\Delta$-pure if and only if it is $\triangleright\triangleright$-indecomposable.

Corollary 28. If $M$ is essentially transitive then it is $\Delta$-pure.

Proof. The claim follows with Proposition 25 and Theorem 27.

The following example shows that the converse to Proposition 25 and Corollary 28 does not hold, that is, there exists a Garside monoid that is $\Delta$-pure, hence $\triangleright\triangleright$-indecomposable, but not essentially transitive.

Example 29. Consider the monoid $M = \langle a, b \mid a^2 = b^2 \rangle^+$, with the Garside structure given by the Garside element $\Delta = a^4$. The lattice of simple elements and the acceptor for the regular language of words in normal form are shown in Figure 1.

We see that the acceptor graph is not connected and so $M$ cannot be essentially transitive.

Now consider $\Delta_a = \{x \setminus a : x \in M\}$. If $x$ has $a$ as a prefix then $x \setminus a = 1$, so we can restrict our attention to elements which do not have $a$ as a prefix. This means that $x = (ba)^k$ or $x = (ba)^kb$ for some $k$. In the first case $ba^k \setminus a = (ba)^ka$, so $x \setminus a = a$. In the second case $ba^kb \setminus a = (ba)^kb^2$, so $x \setminus a = b$. Therefore $\Delta_a = a \setminus b = a^2$. Similarly, $\Delta_b = a^2$. Hence $M$ is $\Delta$-pure, and thus $\triangleright\triangleright$-indecomposable.

4 Growth rates

Throughout this section, let $M$ be a Garside monoid with Garside element $\Delta$ and set of atoms $A$, such that the set $\mathcal{D} = \text{Div}(\Delta) \setminus \{1, \Delta\}$ of proper simple elements is finite. Recall that $\alpha_M$ is the exponential growth rate of the regular language $\text{PSEQ}_M$ and that $\beta_M$ is the exponential growth rate of the regular language $\mathcal{L}_M$.

The main aim of this section is to show that $\alpha_M < \beta_M$ holds if the language $\mathcal{L}_M$ is essentially transitive and every element of $\mathcal{D}$ is essential. In particular, by [GT13, Theorem 4.7], the expectation $\mathbb{E}_{x \in \mathcal{D}}[\text{pd}]$ of the penetration
distance $pd(x, a)$ is bounded independently of $k$ if $\nu_k$ is the uniform distribution on $L^{(k)}$ and $\mu_A$ is the uniform distribution on the set $A$.

Moreover, we will show that the expectation of the penetration distance $E_{\nu_k \times \mu_A}[pd]$, with $\nu_k$ and $\mu_A$ as above, diverges if $M$ is the Garside Zappa–Szép product of two Garside monoids $G$ and $H$, such that $\beta_G, \beta_H > 1$ holds, the languages $L_G$ and $L_H$ are essentially transitive, and all proper simple elements of $G$ respectively $H$ are essential.

**Theorem 30.** If every proper simple element of $M$ is essential and the language $L_M$ of normal forms is essentially transitive, then one has $\alpha_M < \beta_M$.

**Proof.** As there is only one monoid, we drop the subscript $M$.

Consider the acceptor $\Gamma$ of $L \subseteq \langle D \rangle^*$, whose adjacency matrix is given by

$$
\Gamma_{s_1, s_2} = \begin{cases} 
1 & \partial s_2 \land s_1 = 1 \\
0 & \text{otherwise}
\end{cases}
$$

for $s_1, s_2 \in D^1$. The growth rate $\beta$ of $L$ is the Perron–Frobenius eigenvalue of the non-negative matrix $(\Gamma_{s_1, s_2})_{s_1, s_2 \in D^1}$. Let $\alpha = (x_t)_{t \in D^1}$ be an eigenvector for the eigenvalue $\beta$ of $\Gamma$.

The acceptor $\Pi$ of $\text{PSeq} \subseteq \mathcal{P}^*$, where $\mathcal{P} = \{(s, m) \in D^1 \times D^*: sm \preceq \Delta\}$, has the adjacency matrix given by

$$
\Pi_{(s_1, m_1), (s_2, m_2)} = \begin{cases} 
1 & \partial s_2 \land s_1 = 1 \text{ and } s_1m_1 \neq \Delta \text{ and } m_2 = \partial s_2 \land s_1m_1 \\
0 & \text{otherwise}
\end{cases}
$$

for $(s_1, m_1), (s_2, m_2) \in \mathcal{P}$. The growth rate $\alpha$ of $\text{PSeq}$ is the Perron–Frobenius eigenvalue of the non-negative matrix $(\Pi_{(s_1, m_1), (s_2, m_2)})_{(s_1, m_1), (s_2, m_2) \in \mathcal{P}}$, whence one has

$$
\alpha = \inf_{z \in \mathbb{R}_+^\mathcal{P}} \max_{t \in \mathcal{P}} \frac{(\Pi z)_t}{z_t}
$$

by [TW89, Theorem 3.1]. In order to prove the theorem, it is thus sufficient to construct a vector $y = (y_t)_{t \in \mathcal{P}}$ such that, for any $t \in \mathcal{P}$, one has $y_t > 0$ and $(\Pi y)_t < \beta y_t$.

To do this, consider $\tilde{\mathcal{P}} = \{(s, m) \in D^1 \times (D^1 \cup \{1\}) : sm \preceq \Delta\}$, and define a directed graph $\tilde{\Pi}$ with vertex set $\tilde{\mathcal{P}}$ via its adjacency matrix given by

$$
\tilde{\Pi}_{(s_1, m_1), (s_2, m_2)} = \begin{cases} 
1 & \partial s_2 \land s_1 = 1 \text{ and } m_2 = \partial s_2 \land s_1m_1 \\
0 & \text{otherwise}
\end{cases}
$$

for $(s_1, m_1), (s_2, m_2) \in \tilde{\mathcal{P}}$. Observe that $\tilde{\Pi}$ has $\Pi$ as a subgraph and that, locally, $\tilde{\Pi}$ resembles the graph $\Gamma$: The edges ending in the vertex $(s_1, m_1) \in \tilde{\mathcal{P}}$ of $\tilde{\Pi}$ are in bijection to the edges ending in the vertex $s_1$ of $\Gamma$. More precisely, for given $s_1, s_2$ and $m_1$, there exists an $m_2$ such that there is an edge $(s_2, m_2) \to (s_1, m_1)$ in $\tilde{\Pi}$, if and only if there is an edge $s_2 \to s_1$ in $\Gamma$, and if so, $m_2$ is uniquely determined, that is, there exists exactly one such edge. (See Figure 2 and Figure 3.)

Now define a vector $y = (y_t)_{t \in \mathcal{P}}$ by setting $y_{(s, m)} = x_s$ for $(s, m) \in \mathcal{P}$ and a vector $\tilde{y} = (\tilde{y}_t)_{t \in \tilde{\mathcal{P}}}$ by setting $\tilde{y}_{(s, m)} = x_s$ for $(s, m) \in \tilde{\mathcal{P}}$. 

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Figure 2: Lattice of simple elements and digraphs $\Gamma$, $\Pi$ and $\tilde{\Pi}$ for the monoid $M = \langle a, b \mid a^2 = b^2 \rangle^+$. Vertices and edges in red are contained in $\tilde{\Pi} \setminus \Pi$. (That is, the digraph $\Pi$ consists of the vertices $(a, a)$, and $(b, b)$ without any edges.)

Figure 3: Lattice of simple elements and digraphs $\Gamma$, $\Pi$ and $\tilde{\Pi}$ for the monoid $M = \langle a, b \mid aba = bab \rangle^+$. Vertices and edges in red are contained in $\tilde{\Pi} \setminus \Pi$. (That is, the digraph $\Pi$ consists of the vertices $(a, a)$, $(b, a)$, $(a, ba)$, $(ab, a)$, $(ba, a)$ and $(ba, b)$ without any edges.)
Theorem 30

GT13

Notation 8

the following hold:

\[
(\tilde{\Pi}y)_{(s_1,m_1)} = \sum_{(s_2,m_2) \in \tilde{P}} \tilde{y}(s_2,m_2) = \sum_{s_2 \in D} x_{s_2} = (\Gamma x)_{s_1} = \beta \cdot x_{s_1} = \beta \cdot \tilde{y}(s_1,m_1).
\]

So it remains to show that for \( t \in \mathcal{P} \subseteq \tilde{P} \) one has \( y_t > 0 \) and \((\Pi y)_t < (\tilde{\Pi} y)_t\).

Claim. For \((s,m) \in \mathcal{P} \subseteq \tilde{P}\) one has \(y(s,m) > 0\).

As every \( s \in D \) is essential and \( \mathcal{L} \) is essentially transitive, for any \( s_1, s_2 \in D \) there exists a positive integer \( \ell \) such that \((\Gamma^\ell)_{s_1,s_2} > 0\), whence \( x_{s_2} > 0 \) implies \((\Gamma^\ell x)_{s_1} > 0\), as \( \Gamma \) and \( x \) are non-negative. As \( x \) is an eigenvector of \( \Gamma \) and \( x_{s_2} > 0 \) must hold for at least one \( s_2 \in D \), we have \( x_{s_1} > 0 \) for every \( s_1 \in D \) and thus \( y(s,m) = x_{s} > 0 \) for every \((s,m) \in \mathcal{P}\), showing the claim.

Claim. For \((s,m) \in \mathcal{P} \subseteq \tilde{P}\) one has \((\Pi y)_{(s_1,m_1)} < (\tilde{\Pi} y)_{(s_1,m_1)}\).

We obtain \( \Pi \) from \( \tilde{\Pi} \) by 1.) removing all edges ending in \((s_1,m_1)\) if \( m_1 = 1 \) or \( s_1m_1 = \Delta \); and 2.) removing the edge \((s_2,m_2) \rightarrow (s_1,m_1)\) if \( m_2 = 1 \).

So it is sufficient to show that for all \((s_1,m_1) \in \mathcal{P}\) such that \( s_1m_1 \neq \Delta \), there exists \( s_2 \in D \) such that \( \partial s_2 \land s_1m_1 = 1 \) and \( y(s_2,m_2) = x_{s_2} > 0 \). The latter holds as all proper simple elements are essential, and thus \( s_1m_1 \neq \Delta \) implies the existence of an essential \( s_2 \) such that \( s_2 | s_1m_1 \), and since \( x_{s_2} > 0 \) holds.

Remark. Theorem 30 shows that the hypotheses of [GT13, Theorem 4.8] cannot be satisfied.

Corollary 31. Assume that every proper simple element of \( M \) is essential and that the language \( \mathcal{L}_M \) of normal forms is essentially transitive, let \( \nu_k \) be the uniform probability measure on \( \mathcal{L}_M \), and let \( \mu_\mathcal{A} \) be the uniform probability distribution on the set \( \mathcal{A} \) of atoms of \( M \).

The expected value \( E_{\nu_k \times \mu_\mathcal{A}}[pd] \) of the penetration distance with respect to \( \nu_k \times \mu_\mathcal{A} \) is uniformly bounded (that is, bounded independently of \( k \)).

Proof. The claim follows with Theorem 30 and [GT13, Theorem 4.7].

We now turn to the analysis of growth rates of Garside Zappa–Szép products.

Lemma 32. For \( c > 1 \) and \( m \in \mathbb{N} \) the following hold:

1. One has \( \sum_{j=0}^{k-1} j^m c^j \in \Theta(k^m c^k) \).

2. One has \( \sum_{j=0}^{k-1} j^m \in \Theta(k^{m+1}) \).

Proof. The second claim holds by Bernoulli’s formula [GKP94, p. 283]. For the first claim, observe that for any \( k \geq 2 \) one has

\[
\frac{1}{2-mc} \leq \frac{1}{k^{m+1}} (k-1)^m c^{k-1} \leq \frac{1}{k^{m+c}} \sum_{j=0}^{k-1} j^m c^j < \frac{1}{c-1}.
\]

Lemma 33. With the notation of Notation 8 the following hold:
1. One has $\beta_M = 0$ if and only if $\gamma_M = 1$ and $r_M = 0$ hold.
2. One has $\beta_M = 1$ if and only if $\gamma_M = 1$ and $r_M \geq 1$ hold. Moreover, in this case, one has $q_M = r_M - 1$.
3. One has $\beta_M > 1$ if and only if $\gamma_M > 1$ holds. Moreover, in this case, one has $\beta_M = \gamma_M$ and $q_M = r_M$.

Proof. As $\mathcal{L}^{(k)} = \bigcup_{j=0}^{k} \Delta^{k-j} \mathcal{L}^{(j)}$ holds, one has $|\mathcal{L}^{(k)}| = \sum_{j=0}^{k} |\mathcal{L}^{(j)}|$. Firstly observe that $\beta_M = 0$ holds if and only if one has $|\mathcal{L}^{(k)}| = 0$ for sufficiently large $k$. The latter happens if and only if $|\mathcal{L}^{(k)}|$ is eventually constant, which is equivalent to $\gamma_M = 1$ and $r_M = 0$. So the first claim is shown.

Using Notation 8 and Lemma 32, we obtain

$$k_r^\gamma M^k \in \Theta \left( \sum_{j=0}^{k} j^\beta M^j \beta_M^j \right) = \begin{cases} \Theta(k^{\beta M}) & \text{if } \beta_M > 1 \\ \Theta(k^{\beta M + 1}) & \text{if } \beta_M = 1 \end{cases},$$

which implies the remaining claims. \qed

Proposition 34. Assume that $M = G \triangleright H$ is a Garside Zappa–Szép product, let $\beta_M, \gamma_M$ and $q_M, r_M$ be as in Notation 8, and let $\beta_G, \gamma_G, \beta_H, \gamma_H \in \{0\} \cup [1, \infty]$ and $q_G, r_G, q_H, r_H \in \mathbb{N}$ the corresponding constants for $G$ respectively $H$.

The following table gives $\beta_M, \gamma_M, q_M, r_M$ in terms of $\beta_G, \gamma_G, \beta_H, \gamma_H$ and $q_G, r_G, q_H, r_H$:

| $\beta_M$ | $\gamma_M$ | $q_M$ | $r_M$ |
|-----------|-------------|-------|-------|
| $\beta_H = 0$ | $\gamma_H = 1$, $r_H = 0$ | $q_H = 0$ | $r_H = 0$ |
| $\beta_M = 1$ | $\gamma_M = 0$ | $q_M = 1, r_M = 1$ | $q_H = 0$ |
| $\beta_H = 1$, $q_H = q_G + 1$ | $\gamma_H = 1, r_H = r_G + 1$ | $q_M = 0$ | $r_M = 0$ |
| $\gamma_H = r_H = r_G + 1$ | $\gamma_M = 0$ | $q_M = 1$ | $r_M = 1$ |
| $\beta_H = 1$, $q_H = q_G + 1$ | $\gamma_H = r_H = r_G + 1$ | $q_M = 0$ | $r_M = 0$ |

Proof. First note that, by Lemma 33, the cases in the table are correct and exhaustive.

By Theorem 11, for any $x \in \mathcal{T}_M^{(k)}$ there are unique words $g_x \in \mathcal{L}_G^{(k)}$ and $h_x \in \mathcal{L}_H^{(k)}$ such that $k = \max\{k_G, k_H\}$. Moreover, the map $x \mapsto (g_x, h_x)$ is a bijection from $\mathcal{T}_M^{(k)}$ to $\mathcal{L}_G \times \mathcal{T}_H$. Hence one has

$$|\mathcal{T}_M^{(k)}| = |\mathcal{L}_G^{(k)}| \cdot |\mathcal{T}_H^{(k)}| + \sum_{j=0}^{k-1} |\mathcal{L}_G^{(j)}| \cdot |\mathcal{T}_H^{(j)}| + \sum_{j=0}^{k-1} |\mathcal{L}_G^{(j)}| \cdot |\mathcal{T}_H^{(k)}|$$

and thus

$$\gamma_M^k r_M^k \in \Theta \left( \gamma_G^k k_r^\gamma \gamma_H^k H_r^k + \gamma_G^k k_r^\gamma \sum_{j=0}^{k-1} \gamma_H^j j_r^n \right) + \left( \sum_{j=0}^{k-1} \gamma_G^j j_r^n \right).$$

The claimed equalities for $\gamma_M$ and $r_M$ are easily verified using Lemma 32, and the claimed equalities for $\beta_M$ and $q_M$ then follow with Lemma 33. \qed
Corollary 35. Assume that $M = G \triangleright H$ is a Garside Zappa–Szép product, and let $\beta_G$ and $\beta_H$ be the exponential growth rates of the regular languages $L_G$ respectively $L_H$.

Then $|L_M^{(k)}| \in \Theta\left(\left|L_G^{(k)}\right| \cdot \left|L_H^{(k)}\right|\right)$ holds if and only if $\beta_G > 1$ and $\beta_H > 1$.

Proof. Using Notation 8, $|L_M^{(k)}| \in \Theta\left(\left|L_G^{(k)}\right| \cdot \left|L_H^{(k)}\right|\right)$ is equivalent to $\beta_M = \beta_G \beta_H$ and $q_M = q_G + q_H$. The claim then follows with Proposition 34.

Notation 36. Assume that $M = G \triangleright H$ is a Garside Zappa–Szép product. Given $g = g_1 \cdots g_k \in L_G^{(k)}$ and $h = h_1 \cdots h_k \in L_H^{(k)}$, consider the normal form $g_1 h_1' \cdots g_k h_k' \in L_M^{(k)}$ of $g_1 \cdots g_k \lor h_1 \cdots h_k$, and define

$$
\pi_{g,h} := (g_1 h_1', m_1) \cdots (g_k h_k', m_k),
$$

where $m_k = \partial_H (h_k')$ (that is, $h_k' m_k = \Delta_H$) and $m_i = \partial_M (g_i h_i') \land_M g_{i+1} h_{i+1} m_{i+1}$ for $i = 1, \ldots, k - 1$.

Lemma 37. In the situation of Notation 36, one has $\pi_{g,h} \in \text{PSeq}_M^{(k)}$.

Proof. Using [GT14, Lemma 39], we have $\Delta_H \nless g_i h_i' m_i \neq \Delta_M$ and $m_i \neq 1$ for all $i$ by induction, so $\pi_{x,y} \in \text{PSeq}_M^{(k)}$.

Proposition 38. Assume that $M = G \triangleright H$ is a Garside Zappa–Szép product, let $\alpha_M, \beta_M$ and $p_M, q_M$ be as in Notation 8, and let $\alpha_G, \beta_G, \alpha_H, \beta_H \in \{0\} \cup [1, \infty]$ and $p_G, q_G, p_H, q_H \in \mathbb{N}$ the corresponding constants for $G$ respectively $H$.

Then one has the following:

1. If one has $\beta_G, \beta_H > 0$, then $\alpha_M = \beta_M$ holds.
2. If $\beta_G, \beta_H > 1$, then $\alpha_M = \beta_M$ and $p_M = q_M$ hold.

Proof. By Lemma 37, we have a map $L_G^{(k)} \times L_H^{(k)} \to \text{PSeq}_M^{(k)}$ given by the assignment $(g, h) \mapsto \pi_{g,h}$. This map is injective by Theorem 11, so we have $|\text{PSeq}_M^{(k)}| \geq \left|L_M^{(k)}\right| = \left|L_N^{(k)}\right|$. Thus $k^{q_G + q_H} (\beta_G \beta_H)^k \in O(k^{p_M} \alpha_M^{k}) \subseteq O(k^{\mu_M \alpha_M^{k}})$, where the final inclusion holds by [GT13, Corollary 4.4].

If $\beta_G, \beta_H > 0$, then we have $\beta_M = \beta_G \beta_H$ by Proposition 34, and thus obtain $\alpha_M = \beta_M$. Similarly, if $\beta_G, \beta_H > 1$, then we have $\beta_M = \beta_G \beta_H$ and $q_M = q_G + q_H$ by Proposition 34, and thus obtain $\alpha_M = \beta_M$ and $p_M = q_M$.

Notation 39. Assume that $M$ is a Garside monoid with set of proper simple elements $\mathcal{D}$. For $s \in \mathcal{D}$ and $k \geq 1$, we define

$$
L_M^{(k)}(s) := L_M^{(k)} \cap (\mathcal{D})^*. s = \{s_1, \ldots, s_k \in L_M^{(k)} : s_k = s\}.
$$

Lemma 40 ([GT13, Lemma 4.10]). If $M$ is a Garside monoid such that all proper simple elements of $M$ are essential and the language $L_M$ is essentially transitive, then one has $|L_M^{(k)}(s)| \in \Theta\left(\left|L_M^{(k)}\right|\right)$ for all $s \in \mathcal{D}$.

Theorem 41. Assume that $M = G \triangleright H$ is a Garside Zappa–Szép product, that all proper simple elements of $H$ are essential, that the language $L_H$ is essentially transitive, and that the exponential growth rates $\beta_G$ of $L_G$ and $\beta_H$ of $L_H$ satisfy $\beta_G, \beta_H > 1$.

If $\nu_k$ is the uniform probability measure on $L_M^{(k)}$ and $\mu_A$ is the uniform probability distribution on the set $A$ of atoms of $M$, then the expected value $E_{\nu_k \times \mu_A}[pd]$ diverges, that is, one has $\lim_{k \to \infty} E_{\nu_k \times \mu_A}[pd] = \infty$. 

Proof. For any atom $a \in A \cap H$, any $g \in \mathcal{L}^{(k)}_G$ and any $h \in \mathcal{L}^{(k)}_H (\tilde{\partial} Ha)$, the sequence $\pi_{g,h}$ defined in Notation 36 is a penetration sequence establishing $pd(x_{(g,h)}, a) \geq k$ for some $x_{(g,h)} \in M$. Moreover, the map $(g, h) \mapsto x_{(g,h)}$ is injective.

Using Lemma 40 and Corollary 35, we have

$$E_{\nu \times \mu \times \mu} [pd] \geq \sum_{a \in A \cap H} k \cdot \frac{|\mathcal{L}^{(k)}_G| \cdot |\mathcal{L}^{(k)}_H (\tilde{\partial} Ha)|}{|\mathcal{L}^{(k)}_M| \cdot |A|} \in \Theta(k),$$

proving $\lim_{k \to \infty} E_{\nu \times \mu \times \mu} [pd] = \infty$ as claimed.

Example 42. We see in particular that essential transitivity is necessary for the statement of Theorem 30:

Consider $M = G \times H$, where $G = H = A_2 = \langle a, b \mid aba = bab \rangle^\perp$. The lattice of simple elements of $G = H$ and the acceptor $\Gamma$ of $L_G$ and $L_H$ are shown in Figure 4. One sees that all proper simple elements of $G$, respectively of $H$, and thus of $M$, are essential and the languages $L_G$ and $L_H$ are essentially transitive, and it is easy to check that $\beta_G = \beta_H = 2$.

Hence, by Theorem 41, $M$ has unbounded expected penetration distance. As every proper simple element of $M$ is essential, so $M$ satisfies all the hypotheses of Theorem 30, except for essential transitivity.

5 Artin monoids

The aim of this section is to determine the essential simple elements of Artin monoids of spherical type and to determine when these monoids are essentially transitive.

In [Car13, Lemma 3.4], Caruso shows that, in our terminology, the language of normal forms of a spherical Artin monoid of type $A$ is essentially 5-transitive. In Lemmas 48 to 50 we will generalize Caruso’s construction and reproduce this result in Proposition 51. We then go on to generalize this result to all irreducible spherical Artin monoids.

Definition 43. Suppose that $M$ is a Garside monoid. For $x \in M$ the starting set of $x$ is the set of atoms which are prefixes of $x$. Similarly, the finishing set...
of $x$ is the set of atomic suffixes of $x$.

$$S(x) = \{a \in A | a \preceq x\}$$

$$F(x) = \{a \in A | x \succeq a\}$$

For Artin monoids of spherical type, we have the following result connecting the normal form condition to the starting and finishing sets.

**Theorem 44** ([Cha95, Lemma 4.2]). If $M$ is an Artin monoid of spherical type with set of atoms $A$ and $s \in D$, then $S(\partial s) = A \setminus F(s)$.

**Corollary 45.** Suppose that $M$ is an Artin monoid of spherical type. Then, for $x, y \in D$, $x \preceq y$ if and only if $F(x) \supseteq S(y)$. 

**Proof.** We have $x \preceq y$ if and only if $\partial x \land y = 1$. The latter is equivalent to $S(\partial x) \cap S(y) = \emptyset$, so the claim follows with Theorem 44.

**Proposition 46.** Suppose that $M$ is an Artin monoid of spherical type. Then every proper simple element is essential, $E_2 s = D$.

**Proof.** Given any $x \in D$ pick an atom $a \in F(x)$ and let $x_{i+1} = \widetilde{\bigvee} S(x_i)$, where $x_0 = x_1$ then we have

$$\cdots | x_2 | x_1 | a | a | \cdots$$

Note that $x_1 \neq \Delta$ as $\Delta$ is the only element whose starting set or finishing set consists of all of the atoms.

For the rest of this section we will assume that $M$ is an Artin monoid of spherical type.

**Proposition 47.** If $M$ is reducible then it is not essentially transitive.

**Proof.** If $M$ is reducible then $M = M_1 \times M_2 \times \cdots \times M_k$ where the $M_i$ are irreducible. Hence the proposition follows by Proposition 25.

**Lemma 48.** Suppose that $a$ and $b$ are two atoms which lie in the same connected component of the Coxeter graph. Then there exist simple elements $x$ and $y$ such that $S(x) = \{a\}$, $F(x) = \{b\}$, $S(y) = A \setminus \{a\}$ and $F(y) = A \setminus \{b\}$.

**Proof.** Suppose that we have an embedded path $a = a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k} = b$ in the Coxeter graph of $M$. Let $x = a_1 a_2 \cdots a_k$. There are no subwords which match any of the relations, hence $S(x) = \{a\}$ and $F(x) = \{b\}$. Furthermore, the only way $x$ can be written as a product is as $x = (a_1 \cdots a_p)(a_{p+1} \cdots a_k)$, which can never be in normal form. Hence $x$ has canonical length 1, i.e. it is a simple element.

Let $y = \partial(a_k \cdots a_2 a_1)$, then $S(y) = A \setminus F(a_k \cdots a_2 a_1) = A \setminus \{a\}$ and, similarly, $F(y) = A \setminus S(a_k \cdots a_2 a_1) = A \setminus \{b\}$.

Suppose that $1 - 2 - \cdots - k - 1$ is a subgraph of the Coxeter graph of $M$, so we have a standard parabolic subgroup of type $A_{k-1}$. In this situation we have a map from $A_{k-1}$ to the symmetric group on the set $\{1, 2, \ldots, k\}$ given by mapping each atom $i$ to the transposition $(i, i+1)$. This map is a bijection when we restrict to the set of simple elements of this submonoid.
Suppose that \( a \in \mathcal{A}_{n-1} \) and \( x \in \mathcal{D}_{n-1} \) are an atom and a simple element of this submonoid. Let \( \pi \) be the permutation induced by \( x \). Then we have that \( a \in S(x) \) if and only if \( \pi(a+1) < \pi(a) \), and \( a \in F(X) \) if and only if \( \pi^{-1}(a+1) < \pi^{-1}(a) \).

Lemma 49 ([Car13]). Suppose that \( 1 - 2 - \cdots - k - 1 \) is a subgraph of the Coxeter graph of \( M \). Let \( u = u(1, 2, \ldots, k) \) be the braid which corresponds to the permutation

\[
\pi_u = \begin{pmatrix}
1 & 2 & 3 & \cdots & |\frac{k}{2}| & |\frac{k}{2} + 1| & |\frac{k}{2} + 2| & \cdots & k
\end{pmatrix}.
\]

Then one has

\[
S(u) = \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\}, \quad F(u) = \left\{ 1, 3, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1 \right\}.
\]

Proof. It is clear that the only atom \( a \in \mathcal{A}_{n-1} \) for which \( \pi_u(a+1) < \pi_u(a) \) is \( a = \left\lfloor \frac{k}{2} \right\rfloor \), hence \( S(u) = \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \). Similarly, \( \pi_u^{-1}(a+1) < \pi_u^{-1}(a) \) if and only if \( a \) is odd, hence \( F(u) \) consists of all the odd atoms. \( \square \)

Lemma 50 ([Car13]). Suppose that \( 1 - 2 - \cdots - k - 1 \), where \( k > 2 \), is a subgraph of the Coxeter graph of \( M \), let \( u \) be the element defined in Lemma 49, and let

\[
v = v(1, 2, \ldots, k) = (\text{rev } u) \cdot D,
\]

where \( D = \bigvee \{ a \in \mathcal{A}_M \mid a \neq \left\lfloor \frac{k}{2} \right\rfloor \} \). Then \( v \) is a simple element and its finishing set contains every atom except possibly \( \left\lfloor \frac{k}{2} \right\rfloor \), that is, \( F(v) \supseteq \mathcal{A}_M \setminus \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \).

Proof. As \( S(u) = \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \) we have that \( F(\text{rev } u) = \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \) hence \( S(\partial \text{rev } u) = \mathcal{A} \setminus \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \). Therefore \( \bigvee (\mathcal{A} \setminus \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\}) \not\subseteq \partial \text{rev } u \) and so \( v \) is simple.

The element \( D \) is a Garside element of a standard parabolic subgroup of \( M \), thus balanced, whence one has \( \mathcal{A}_M \setminus \left\{ \left\lfloor \frac{k}{2} \right\rfloor \right\} \subseteq D(\mathcal{D}) \subseteq F(v) \).

\( \square \)

Proposition 51 ([Car13]). Suppose that \( M \) is the Artin monoid of type \( A_{n-1} \), where \( n > 2 \).

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & \cdots & n - 1
\end{array}
\]

Then \( M \) is essentially 5-transitive.

Proof. By Proposition 46, one has \( \mathcal{E}_\mathcal{S} = \mathcal{D} \not\subseteq \emptyset \). Suppose \( x, y \in \mathcal{D} \). We will construct elements \( x_1, x_2, x_3, x_4 \in \mathcal{D} \) that satisfy \( x|x_1|x_2|x_3|x_4|y \).

Suppose that \( a \) is an atom in the finishing set of \( x \). By Lemma 48, there exists \( x_1 \) such that \( S(x_1) = \{ a \} \) and \( F(x_1) = \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\} \). Let \( x_2 = u(1, 2, \ldots, n-1) \), so by Lemma 49 we have \( S(x_2) = \left\lfloor \frac{n}{2} \right\rfloor \). So by Corollary 45 we have \( x|x_1|x_2 \).

Similarly, suppose that \( b \) is an atom not in the starting set of \( y \). By Lemma 48, there exists \( x_4 \) such that \( F(x_4) = \mathcal{A} \setminus \{ b \} \) and \( S(x_4) = \mathcal{A} \setminus \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\} \). Let \( x_3 = v(1, 2, \ldots, n-1) \), so by Lemma 50 we have \( F(x_3) \supseteq \mathcal{A} \setminus \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\} \), whence we have \( x_3|x_4|y \) by Corollary 45.

It remains to show that \( x_2|x_3 \), or equivalently that \( F(x_2) \supseteq S(x_3) \). We have \( F(x_2) = \{ 1, 3, \ldots, 2 \left\lfloor \frac{k}{2} \right\rfloor - 1 \} \) by Lemma 49.

Consider the permutation induced by \( x_3 = v \). The permutation induced by \( \text{rev } u \) takes the set of even numbered strings to \( \{ 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \) and the set of odd
numbered strings to \( \{ \left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n \} \), and \( \bigvee \{ a \in A_M | a \neq \left\lfloor \frac{n}{2} \right\rfloor \} \) consist of a half-twist on both of these subsets. Hence, for all \( i \) we have

\[
(2i \pm 1) \cdot x_3 < (2i) \cdot x_3.
\]

Therefore \( S(x_3) = \{ 1, 3, \ldots, 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \} = F(x_2) \), i.e. we have \( x_2 \mid x_3 \). \( \square \)

**Proposition 52.** Suppose that \( M \) is the Artin monoid of type \( B_n \), where \( n > 2 \).

Then \( M \) is essentially 5-transitive.

**Proof.** By Proposition 46, one has \( \mathcal{E}_M \neq \emptyset \). As in the proof of the previous propositions, it suffices to construct elements \( x_2, x_3 \in \mathcal{D} \) such that \( S(x_2) = \{ \left\lfloor \frac{n}{2} \right\rfloor \} \), \( x_2 \mid x_3 \) and \( F(x_3) \supseteq A \setminus \{ \left\lfloor \frac{n}{2} \right\rfloor \} \).

Let \( x_2 = u(1, 2, \ldots, n - 1) \) and \( x_3 = v(1, 2, \ldots, n - 1) \).

Simple elements in \( B_n \) correspond to signed permutations of \( \{1, \ldots, n\} \). The atoms \( i = 1, 2, \ldots, n - 1 \) give the transposition \( (i, i + 1)(-i, -i - 1) \) and 0 is the transposition \( (1, -1) \) which changes the sign of 1.

The starting set can be computed from the induced signed permutation [BB05, Proposition 8.1.2].

\[
S(x_3) = \{ i \in A \mid i \cdot x_3 > (i + 1) \cdot x_3 \}
\]

Where for 0 we use \( 0 \cdot x_3 = 0 \).

The signed permutation induced by \( \text{rev } u \) takes every odd number to a number greater than \( \frac{n}{2} \), and every even number to a number less than or equal to \( \frac{n}{2} \). The action of \( \Delta_A \setminus \{ \left\lfloor \frac{n}{2} \right\rfloor \} \) performs a half twist of the numbers greater than \( \frac{n}{2} \) and changes the sign of the numbers less than or equal to \( \frac{n}{2} \). Hence, for all \( i \) we have

\[
(2i \pm 1) \cdot x_3 > (2i) \cdot x_3.
\]

Therefore \( S(x_3) = \{ 1, 3, \ldots, 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \} = F(x_2) \), i.e. we have \( x_2 \mid x_3 \). \( \square \)

**Proposition 53.** Suppose that \( M \) is the Artin monoid of type \( D_n \), where \( n > 2 \).

Then \( M \) is essentially 5-transitive.

**Proof.** By Proposition 46, one has \( \mathcal{E}_M = \mathcal{D} \neq \emptyset \). As in the proof of the previous propositions, it suffices to construct elements \( x_2, x_3 \in \mathcal{D} \) such that \( S(x_2) = \{ \left\lfloor \frac{n}{2} \right\rfloor \} \), \( x_2 \mid x_3 \) and \( F(x_3) \supseteq A \setminus \{ \left\lfloor \frac{n}{2} \right\rfloor \} \).

Let \( x_2 = u(1, 2, \ldots, n - 1) \) and \( x_3 = v(1, 2, \ldots, n - 1) \).

Simple elements in \( B_n \) correspond to signed permutations of \( \{1, \ldots, n\} \). The atoms \( i = 1, 2, \ldots, n - 1 \) give the transposition \( (i, i + 1)(-i, -i - 1) \) and 0 is the transposition \( (1, -1) \) which changes the sign of 1.
Simple elements in $D_n$ correspond to the subset of signed permutations of \{1, \ldots, n\} which consists of all elements that change the sign of an even number of numbers. As in the type $B_n$ case, the atoms $i = 1, 2, \ldots, n - 1$ give the transposition $(i, i + 1)(-i, -i - 1)$, but now 0 is the transposition $(1, -2)(2, -1)$.

The starting set can be computed from the induced signed permutation [BB05, Proposition 8.2.2].

$$S(x_3) = \{i \in A \mid i \cdot x_3 > (i + 1) \cdot x_3\}$$

Where for 0 we use $0 \cdot x_3 = -2 \cdot x_3$.

The signed permutation induced by rev $u$ takes every odd number to a number greater than $\frac{n}{2}$, and every even number to a number less than or equal to $\frac{n}{2}$. The action of $\Delta_{A \setminus \{\frac{n}{2}\}}$ performs a half twist of the numbers greater than $\frac{n}{2}$ and changes the sign of the numbers less than or equal to $\frac{n}{2}$. Hence, for all $i$ we have

$$(2i \pm 1) \cdot x_3 > (2i) \cdot x_3.$$  

Therefore $S(x_3) = \{1, 3, \ldots, 2 \left\lceil \frac{n}{2} \right\rceil - 1\} = F(x_2)$, i.e. we have $x_2 | x_3$.

**Proposition 54.** Suppose that $M$ is the Artin monoid of type $E_6$.

Then $M$ is essentially 5-transitive.

**Proof.** By Proposition 46, one has $E_6 = D^\perp \neq \emptyset$. Following the same pattern as previous proofs, it suffices to construct elements $x_2, x_3 \in D^\perp$ such that $S(x_2) = \{3\}$, $x_2 | x_3$ and $F(x_3) \supseteq A \setminus \{3\}$.

Let $x_2 = u(1, 2, 3, 4, 5) = 324135$ and $x_3 = v(1, 2, 3, 4, 5) = 1352431214546$. By direct computation we have the following starting and finishing sets.

$S(x_2) = \{3\}$ \hspace{1cm} $F(x_2) = \{1, 3, 5\}$

$S(x_3) = \{1, 3, 5\}$ \hspace{1cm} $F(x_3) = A \setminus \{3\}$

**Proposition 55.** Suppose that $M$ is the Artin monoid of type $E_7$.

Then $M$ is essentially 5-transitive.

**Proof.** By Proposition 46, one has $E_7 = D^\perp \neq \emptyset$. Again, it suffices to construct elements $x_2, x_3 \in D^\perp$ such that one has $S(x_2) = \{3\}$, $x_2 | x_3$ and $F(x_3) \supseteq A \setminus \{3\}$.

We define $x_2 = u(1, 2, 3, 4, 5, 6) = 324135$ and $x_3 = v(1, 2, 3, 4, 5, 6) = 135243121456457$. By direct computation we have the following starting and finishing sets.

$S(x_2) = \{3\}$ \hspace{1cm} $F(x_2) = \{1, 3, 5\}$

$S(x_3) = \{1, 3, 5\}$ \hspace{1cm} $F(x_3) = A \setminus \{3\}$
Proposition 56. Suppose that $M$ is the Artin monoid of type $E_8$.  

Then $M$ is essentially 5-transitive.

Proof. By Proposition 46, one has $\mathcal{E}_S = D^\ell \neq \emptyset$. Again, it suffices to construct elements $x_2, x_3 \in \mathcal{D}$ such that $S(x_2) = \{4\}$, $x_2|x_3$ and $F(x_3) \supseteq A \setminus \{4\}$.

Let $x_2 = u(1, 2, 3, 4, 5, 6, 7) = 4352461357$ and $x_3 = v(1, 2, 3, 4, 5, 6, 7) = 1357246354123121567565$. By direct computation we have the following starting and finishing sets.

\[
\begin{align*}
S(x_2) &= \{4\} & F(x_2) &= \{1, 3, 5, 7\} \\
S(x_3) &= \{1, 3, 5, 7\} & F(x_3) &= A \setminus \{4\}
\end{align*}
\]

Proposition 57. Suppose that $M$ is the Artin monoid of type $F_4$.  

Then $M$ is essentially 4-transitive.

Proof. By Proposition 46, one has $\mathcal{E}_S = D^\ell \neq \emptyset$. By Lemma 48, it suffices to construct an element $x_2 \in \mathcal{D}$ such that $S(x_2) = \{3\}$ and $F(x_2) = A \setminus \{2\}$.

Let $x_2 = 321343$. By direct computation we have the required starting and finishing sets.

Proposition 58. Suppose that $M$ is the Artin monoid of type $H_3$.  

Then $M$ is essentially 4-transitive.

Proof. By Proposition 46, one has $\mathcal{E}_S = D^\ell \neq \emptyset$. By Lemma 48, it suffices to construct an element $x_2 \in \mathcal{D}$ such that $S(x_2) = \{2\}$ and $F(x_2) = A \setminus \{2\}$.

Let $x_2 = 213$. By direct computation we have the required starting and finishing sets.

Proposition 59. Suppose that $M$ is the Artin monoid of type $H_4$.  

Then $M$ is essentially 5-transitive.

Proof. By Lemma 48, it suffices to construct elements $x_2, x_3 \in \mathcal{D}$ such that $S(x_2) = \{3\}$, $x_2|x_3$ and $F(x_3) = A \setminus \{3\}$.

Let $x_2 = 324$ and $x_3 = 243121214$. By direct computation we have the following starting and finishing sets.

\[
\begin{align*}
S(x_2) &= \{3\} & F(x_2) &= \{2, 4\} \\
S(x_3) &= \{2, 4\} & F(x_3) &= A \setminus \{3\}
\end{align*}
\]
Proposition 60. Suppose that $M$ is the Artin monoid of type $I_2(p)$, where $p \geq 3$.

\[
\begin{array}{ccc}
a & \overset{p}{\longrightarrow} & b
\end{array}
\]

Then $M$ is essentially 2-transitive.

Proof. We have $p \geq 3$ by assumption, so $ab$ and $ba$ are distinct proper simple elements. The proper simple elements fall into one of four types:

1. $b(ab)^* = (ba)^*b$
2. $a(ba)^*b$
3. $(ab)^*a$
4. $(ab)^* = (ab)^*a$

Writing $(i)$ and $(i')$ for arbitrary simple elements of type $i$, we have

1. $(1)(1')$ \quad $(1)(ba(2'))$ \quad $(1)(3')$ \quad $(1)(ba(4'))$
2. $(2)(1')$ \quad $(2)(ba(2'))$ \quad $(2)(3')$ \quad $(2)(ba(4'))$
3. $(3)(ab)(1')$ \quad $(3)(2')$ \quad $(3)(ab)(3')$ \quad $(3)(4')$
4. $(4)(ab)(1')$ \quad $(4)(2')$ \quad $(4)(ab)(3')$ \quad $(4)(4')$

and thus 2-transitivity.

Combining the classification of Artin monoids of spherical type with the above propositions we have the following theorem.

Theorem 61. Let $M$ be an Artin monoid of spherical type with more than one atom.

The language of normal forms in $M$ is essentially transitive if and only if $M$ is irreducible. Moreover, if the language is essentially transitive then it is essentially 5-transitive.

Corollary 62. Let $M$ be an irreducible Artin monoid of spherical type, let $\nu_k$ be the uniform probability measure on $L_M^{(k)}$, and let $\mu_A$ be the uniform probability distribution on the set $A$ of atoms of $M$.

The expected value $E_{\nu_k \times \mu_A}[pd]$ of the penetration distance with respect to $\nu_k \times \mu_A$ is uniformly bounded (that is, bounded independently of $k$).

Proof. Corollary 31, Proposition 46 and Theorem 61 imply the claim.

Recall that $\beta_M$ is the exponential growth rate of the regular language $L_M^{(k)}$.

Lemma 63. If $M$ is an irreducible Artin monoid of spherical type with more than one atom, then one has $\beta_M > 1$.

Proof. As there is only one monoid, we drop the subscript $M$.

Consider two atoms $a \neq b$ of $M$. As $L_M$ is essentially transitive, there exist $s_1, \ldots, s_k, t_1, \ldots, t_l \in \mathcal{D}$ such that $a|s_1| \cdots |s_k|b|t_1| \cdots |t_l|a$. Moreover, we have $a|a$ by Corollary 45. Thus, one has $L^{(N(k+l+2))} \geq 2^N$, showing the claim.

Corollary 64. Let $M = G \triangleright H$, where $G$ and $H$ are irreducible Artin monoids of spherical type with more than one atom, let $\nu_k$ be the uniform probability measure on $L_M^{(k)}$, and let $\mu_A$ be the uniform probability distribution on the set $A$ of atoms of $M$.

The expected value $E_{\nu_k \times \mu_A}[pd]$ diverges, that is, $\lim_{k \to \infty} E_{\nu_k \times \mu_A}[pd] = \infty$.

Proof. Theorem 41, Proposition 46, Theorem 61 and Lemma 63 imply the claim.
Using the terminology of this paper, Dehornoy asked in [Deh07, Question 3.13] whether for the braid monoid, that is the Artin monoid of type $A_n$, one has $|L(s)^{(k)}| \in \Theta(L^{(k)})$ for all $s \in \mathcal{D}$. The answer is affirmative for all irreducible Artin monoids of spherical type:

**Corollary 65.** If $M$ is an irreducible Artin monoid of spherical type and $s \in \mathcal{D}$, then one has $|L(s)^{(k)}| \in \Theta(L^{(k)})$.

**Proof.** The claim follows from Lemma 40, Proposition 46 and Theorem 61.

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