A Faster Product for $\pi$ and a New Integral for $\ln \frac{\pi}{2}$

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1. INTRODUCTION. In [15] we derived an infinite product representation of $e^\gamma$, where $\gamma$ is Euler’s constant:

$$e^\gamma = \left( \frac{2}{1} \right)^{1/2} \left( \frac{2^2}{1 \cdot 3} \right)^{1/3} \left( \frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{1/4} \left( \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/5} \cdots. \quad (1)$$

Here the $n$th factor is the $(n + 1)$th root of the product

$$\prod_{k=0}^{n} \left( \frac{(-1)^{k+1/n}}{k} \right).$$

In the process we noticed a strikingly similar product representation of $\pi$:

$$\frac{\pi}{2} = \left( \frac{2}{1} \right)^{1/2} \left( \frac{2^2}{1 \cdot 3} \right)^{1/4} \left( \frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{1/8} \left( \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/16} \cdots. \quad (2)$$

In this note we give three proofs of (2). The third leads to an analog for $\ln(\pi/2)$ of integrals for $\ln(4/\pi)$ [14] and $\gamma$ [13], [14], [15]:

$$\ln \frac{\pi}{2} = - \int_{[0,1]} \frac{1-x}{(1+x) \ln x} \, dx, \quad (3)$$

$$\ln \frac{4}{\pi} = - \int_{[0,1]} \frac{1-x}{(1+xy) \ln xy} \, dx \, dy, \quad (4)$$

$$\gamma = - \int_{[0,1]} \frac{1-x}{(1-xy) \ln xy} \, dx \, dy.$$  

Using (3), we sketch a derivation of (1) and (2) from the same function (a form of the polylogarithm [7]), accounting for the resemblance between the two products. The function also leads to a product for $e$ (due to J. Guillera [5]),

$$e = \left( \frac{2}{1} \right)^{1/1} \left( \frac{2^2}{1 \cdot 3} \right)^{1/2} \left( \frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{1/3} \left( \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/4} \cdots. \quad (5)$$
surprisingly close to product (1) for $e^\gamma$.

2. THE ALTERNATING ZETA FUNCTION. The logarithm of product (1), namely,

$$\gamma = \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n}(-1)^{k+1} \binom{n}{k} \ln(k+1),$$

reminded us of the series (see [6] and [11])

$$\zeta^*(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} (k+1)^{-s} \quad (s \in \mathbb{C}),$$

which gives the analytic continuation of the alternating zeta function $\zeta^*(s)$. The latter is defined by the Dirichlet series (see [12])

$$\zeta^*(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} \quad (\Re(s) > 0).$$

(For example, using the classic formula $\zeta^*(1) = \ln 2$ for the alternating harmonic series—for a new proof see [12]—one can derive the series $\ln 2 = \sum_{n=1}^{\infty} (2^n n)^{-1}$ from (7) by considering it when $s = 1$.) Differentiating (7) termwise and substituting the value of the derivative of $\zeta^*$ at $s = 0$,

$$\zeta^*'(0) = \frac{1}{2} \ln \frac{\pi}{2},$$

(see [11]), yields the series

$$\ln \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \ln(k+1),$$

and exponentiation produces product (2).

3. WALLIS'S PRODUCT AND EULER'S TRANSFORM. The pair of infinite products (1) and (2) calls to mind another pair, Wallis's product for $\pi$ [17] and Pippenger's product for $e$ [10]:
It is interesting to note that products (2) and (12), whose factors have exponents $1/2^n$, converge rapidly to numbers $\pi/2$ and $e/2$ whose irrationality has been proved (see, for example, [9]), whereas product (1), with exponents $1/(n+1)$, converges less rapidly to a number $e^\gamma$ whose (expected) irrationality has not yet been proved.

We give a second proof of (2), using (11) and Euler's transformation of series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{k+1},$$

valid for any convergent series of complex numbers [7, sec. 33B], [11]. Applying (13) to the logarithm of Wallis's product

$$\ln \frac{\pi}{2} = \sum_{n=1}^{\infty} (-1)^{n-1} \ln \frac{n+1}{n}$$

gives

$$\ln \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln \frac{k+2}{k+1}. \quad (15)$$

If we replace $n$ by $n-1$, write the last logarithm as $\ln(k+2) - \ln(k+1)$, and the sum on $k$ as the difference of two sums in the first of which we replace $k$ by $k-1$, then the recursion $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ leads to (10), completing the second proof of (2). The first proof is basically the same, because in [11] we use Wallis's product to evaluate (9), and we take the Euler transform of (8) to get (7) for complex $s$ with $\Re(s) > 0$.

Products (12) and (11) are linked by Stirling’s asymptotic formula $n! \sim (n/e)^n \sqrt{2\pi n}$: the formula is proved in [2] using (11) and is used in [10] to establish (12). Products (1) and (2) are linked by transformations: a hypergeometric one [15] for (1) and Euler's for (2). (To strengthen the link, we can write series (14) and (15) as integrals of hypergeometric functions—compare [15, Proof 1]—and then obtain (15) from (14) by a hypergeometric transformation equivalent to (13).) However, this link does not explain the remarkable resemblance between (1) and (2).

Euler's transformation accelerates the rate of convergence of a slowly converging series such as (14) (see [7, sec. 35B]). Thus, product (2) converges faster than product (11), as Figure 1 shows.
4. AVOIDING EULER. A third proof of (2) (due in part to S. Zlobin [18]) avoids using Euler’s transformation altogether (compare the proofs avoiding hypergeometric functions in [15]). We show that

\[ I := \int_0^1 \int_0^\infty x^n \left( \frac{1-x}{2} \right)^{n+1} \, dx \, dy = \ln \frac{\pi}{2}. \]  

This implies (2), because if we factor \((1-x)\) from the integrand and use the binomial theorem, then termwise integration (justified since the integrand is majorized by the series \(\sum 2^{-n-1}\)) yields (15) and, therefore, (2). To prove (16), we use the geometric series summations

\[ \sum_{n=0}^{\infty} \left( \frac{1-x}{2} \right)^{n+1} = \frac{1-x}{1+x} = \frac{(1-x)^2}{1-x^2} = \sum_{n=0}^{\infty} (1-x)^2 x^{2n} \]  

(17)
to write

\[ I = \int_0^\infty \int_0^1 \sum_{n=0}^{\infty} (1-x)^2 x^{y+2n} \, dx \, dy. \]

The integrand is majorized by \(\sum (n+1)^{-2}\) (because

\[ \max_{0 \leq x \leq 1} (1-x)^2 x^{2n} = \left( \frac{1}{n+1} \right)^2 \left( \frac{n}{n+1} \right)^2 < \frac{1}{(n+1)^2} \]

and \(x^y \leq 1\), so we may perform the integrations term by term, which by invoking (11) gives
\[ I = \sum_{n=0}^{\infty} \ln \frac{(2n+2)^2}{(2n+1)(2n+3)} = \ln \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots = \ln \frac{\pi}{2}. \]

This proves (16) and completes the third proof of (2).

**Proof of (3).** Equation (16) and the first equality in (17) yield

\[ \int_0^\infty \int_0^1 x^y \frac{1-x}{1+x} \, dx \, dy = \ln \frac{\pi}{2}. \]

Reversing the order of integration (permitted since the integrand is nonnegative), we integrate with respect to \( y \) and arrive at formula (3).

Alternatively, one can derive (3) from (4) by making the change of variables \( u = xy, v = 1-x \) and integrating with respect to \( v \): the result is \( \ln 2 \) minus integral (3) (with \( u \) in place of \( x \)), and equality (3) follows.

**5. RELATING THE PRODUCTS FOR \( \pi \) AND \( e^γ \).** Recall that we derived product (2) for \( \pi \) from the alternating zeta function \( \zeta^*(s) \). Omitting details, we sketch a derivation of product (1) for \( e^γ \) from a generalization of \( \zeta^*(s) \). This accounts for the resemblance between the two products. (Formulas (18) and (19) are due to J. Guillera [5].)

We generalize series (7) for \( \zeta^*(s) \) by defining the function

\[ f(t,s) = \sum_{n=0}^{\infty} t^{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s} \quad ( -1 < t < 1, \ s \in \mathbb{C}), \]  

(18)

so that \( f(1/2, s) = \zeta^*(s) \). Using integral (16) but replacing \( (1-x)/2 \) with \( t(1-x) \), we can show that the formula obtained from (3) and (9),

\[ \zeta^*(0) = -\frac{1}{2} \int_0^1 \frac{1-x}{(1+x) \ln x} \, dx, \]

extends to

\[ f'(t, 0) = -t^2 \int_0^1 \frac{1-x}{(1-t(1-x)) \ln x} \, dx, \]  

(19)

where the prime ' is shorthand for \( \partial / \partial s \).

We now derive product (1) by evaluating the integral \( \int_0^1 t^{-1} f'(t, 0) \, dt \) in two different ways. On the one hand, a glance at (18) reveals that this integral equals the right side of (6). On the other hand, substituting (19) into the integral and reversing the order of integration gives

\[ \int_0^1 \frac{f'(t, 0)}{t} \, dt = -\int_0^1 \int_0^1 t \frac{(1-x)}{(1-t(1-x)) \ln x} \, dt \, dx = \int_0^1 \left( \frac{1}{\ln x} + \frac{1}{1-x} \right) \, dx. \]
The last is a classical integral for Euler's constant [1, sec. 10.3], [15], and (6) follows, implying (1).

Other products can be derived in the same way. For example, exponentiating the integral \( \int_0^1 t^{-2} f'(t,0) dt = 1 \) gives product (5) for \( e \), which converges more slowly than Pippenger's product for \( e \), because of the exponents \( 1/n \) in (5), versus \( 1/2^n \) in (12).

In order to identify the function \( f(t, s) \), we reverse the order of summation in (18) and sum the resulting series on \( n \). We then replace \( k \) with \( k - 1 \), obtaining

\[
f(t, s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)^s} \sum_{n=k}^{\infty} \binom{n}{k} t^n = -\sum_{k=1}^{\infty} \frac{(t/(t-1))^k}{k^s}
\]

for \( t \) satisfying \(-1 < t \leq 1/2\) and for suitable \( s \). Therefore, \( f(t, s) \) and \( \zeta^*(s) \) are related to the function

\[
F(t, s) = \sum_{k=1}^{\infty} \frac{t^k}{k^s} \quad (-1 \leq t < 1, \Re(s) > 0)
\]

by the formulas

\[
f(t, s) = -F(t/(t-1), s),
\]

\[
\zeta^*(s) = -F(-1, s),
\]

for appropriate \( t \) and \( s \). (With \( t = 1/2 \), equations (18) and (20) verify that formulas (7) and (8) for \( \zeta^*(s) \) agree.) The function \( F(t, s) \), a special case of the Lerch zeta function \( \Phi(z, s, v) \) (see [3, sec. 1.11], [16, Sec. 64]), is the polylogarithm \( \text{Li}_s(t) \) when \( s \) is an integer [7, p. 189], [16, secs. 25, 64]. Relations (18) and (21) lead to an analytic continuation of \( F(t, s) \), and thus of the polylogarithm.
REFERENCES

1. G. Boros and V. Moll, *Irresistible Integrals: Symbolics, Analysis, and Experiments in the Evaluation of Integrals*, Cambridge University Press, Cambridge, 2004.

2. A. J. Coleman, A simple proof of Stirling's formula, *Amer. Math. Monthly* 58 (1951) 334-336.

3. A. Erdélyi et al., *Higher Transcendental Functions*, The Bateman Manuscript Project, vol. 1, McGraw-Hill, New York, 1953.

4. R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Boston, 1994.

5. J. Guillera, personal communication, 25 July 2003.

6. H. Hasse, Ein Summierungsverfahren für die Riemannsche ζ-Reihe, *Math. Z.* 32 (1930) 458-464.

7. K. Knopp, *Theory and Application of Infinite Series*, Dover, New York, 1990.

8. L. Lewin, *Polylogarithms and Associated Functions*, Elsevier North-Holland, New York, 1981.

9. A. E. Parks, π, e, and other irrational numbers, *Amer. Math. Monthly* 93 (1986) 722-723.

10. N. Pippenger, An infinite product for e, *Amer. Math. Monthly* 87 (1980) 391.

11. J. Sondow, Analytic continuation of Riemann's zeta function and values at negative integers via Euler's transformation of series, *Proc. Amer. Math. Soc.* 120 (1994) 421-424.

12. __________, Zeros of the alternating zeta function on the line ℜ(s) = 1, *Amer. Math. Monthly* 110 (2003) 435-437.

13. __________, Criteria for irrationality of Euler's constant, *Proc. Amer. Math. Soc.* 131 (2003) 3335-3344.

14. __________, Double integrals for Euler's constant and ln(4/π) (preprint); available at http://arXiv.org/abs/math.CA/0211148.

15. __________, An infinite product for e^γ via hypergeometric formulas for Euler's constant γ (preprint); available at http://arXiv.org/abs/math.CA/0306008.

16. J. Spanier and K. B. Oldham, *An Atlas of Functions*, Hemisphere, New York, 1987.

17. J. Wallis, Computation of π by successive interpolations, in *A Source Book in Mathematics, 1200-1800*, D. J. Struik, ed., Princeton University Press, Princeton, 1986, pp. 244-253; reprinted in *Pi: A Source Book*, 2nd ed., L. Berggren, J. Borwein, and P. Borwein, eds., Springer-Verlag, New York, 2000, pp. 68-77.

18. S. Zlobin, personal communication, 26 May 2003.

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