ON FINSLER TRANSNORMAL FUNCTIONS

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Abstract. In this note we discuss a few properties of transnormal Finsler functions, i.e., the natural generalization of distance functions and isoparametric Finsler functions. In particular, we prove that critical level sets of an analytic transnormal function are submanifolds, and the partition of $M$ into level sets is a Finsler partition, when the function is defined on a compact analytic manifold $M$.

1. Introduction

Let $(M, F)$ be a forward complete Finsler manifold. A function $f : M \to \mathbb{R}$ is called $F$-transnormal function if $F(\nabla f)^2 = b(f)$ for some continuous function $b$.

Transnormal functions on Riemannian manifolds have been focus of researchers in the last decades. In particular, if $b \in C^2(f(M))$ then the level sets of $f$ are leaves of the so called singular Riemannian foliation and the regular level sets are equifocal hypersurfaces; see e.g., [14], [15] and [2, Chapter 5].

In Finsler geometry, the study of transnormal functions has just begun, see [6] but there are already some interesting applications in wildfire modeling, see [11].

The most natural example of a transnormal function on a Finsler space is the distance function on a Minkowski space. More precisely consider a Randers Minkowski space $(V, Z)$ and define $f(x) := d(0, x)$. It is well known that in this example $b = 1$, see [16, Lemma 3.2.3]. And already here one can see a phenomenon that does not exist in the Riemannian case. The regular level set $f^{-1}(c_1)$ is forward parallel to $f^{-1}(c_2)$ if $c_1 < c_2$ but $f^{-1}(c_2)$ is not forward parallel to $f^{-1}(c_1)$ and hence the partition $F = \{f^{-1}(c)\}_{c \in (0, \infty)}$ is not a Finsler partition of $V \setminus 0$, recall basic definitions and examples in Sections 2 and 3 respectively.

The above observation leads us to the 3 natural questions we want to address here.

Question 1.1. If $f$ is a transnormal function on a forward complete Finsler manifold $(M, F)$, is the (regular) level set $f^{-1}(c)$ forward parallel to the (regular) level set $f^{-1}(d)$ when $c < d$? In this case, is the distance from $f^{-1}(c)$ to $f^{-1}(d)$ described by $\int_c^d \frac{1}{\sqrt{b(s)}} ds$ as it was in the Riemannian case?

A positive answer to Question 1.1 has already been given at [6]. In Section 4 we review this fact, see Proposition 4.4.

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Question 1.2. Under which conditions the level sets of a transnormal function are parallel to each other, i.e., $F = \{ f^{-1}(c) \}$ is a Finsler partition?

As we have remarked before, level sets of transnormal functions, do not need to be equidistant, and hence some hypothesis is needed to assure equidistance between the level sets, i.e, that the partition $F = \{ f^{-1}(c) \}_{c \in f(M)}$ is a Finsler partition. The next result approaches Question 1.2 and will be discussed in Section 5.

Theorem 1.3. Let $(M, F)$ be a connected, compact analytic Finsler manifold and $f : M \to \mathbb{R}$ a $F$-transnormal and analytic function with $f(M) = [a, b]$. Suppose that the level sets are connected and $a$ and $b$ are the only singular values at $[a, b]$. Then

(a) the critical level sets $f^{-1}(a)$ and $f^{-1}(b)$ are submanifolds.
(b) The level sets are equidistant to each other, i.e, $F = \{ f^{-1}(c) \}_{c \in [a, b]}$ is a Finsler partition. In particular for each regular value $c$, $f^{-1}(c)$ is a future and past cylinder over each singular level set.

Remark 1.4. It follows from the above theorem, that the regular level set $f^{-1}(c)$ are equifocal hypersurface, recall definition at [1].

Finally, inspired by [1] it is also natural to ask.

Question 1.5. Under which conditions the level sets of a transnormal function on a Finsler manifold are level sets of singular Riemannian foliation for some Riemannian metric?

In section 6 we approach this question by using the result of C. Qian, Z. Tang [12] about Morse-Bott functions.

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2. Preliminaries

In this section we fix some notations and briefly review a few facts about Finsler geometry and Finsler partitions which will be used in this note. For more details see [10], [1] and [4].

2.1. Finsler metrics. Let $V$ be a vector space and $F : V \to [0, +\infty)$ a function. We say that $F$ is a Minkowski norm and $(V, F)$ is a Minkowski space if:

(a) $F$ is smooth on $V \setminus \{ 0 \}$,
(b) $F$ is positive homogeneous of degree 1, that is $F(\lambda v) = \lambda F(v)$ for every $v \in V$ and $\lambda > 0$,
(c) for every $v \in V \setminus \{ 0 \}$, the fundamental tensor of $F$ defined as

\[
g_v(u, w) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(v + tu + sw)|_{t=s=0}
\]

for any $u, w \in V$ is a positive-definite bilinear symmetric form.

Now let us consider a manifold $M$. We say that a function $F : TM \to [0, +\infty)$ is a Finsler metric if $F$ is smooth on $TM \setminus \{ 0 \}$, and for every $p \in M$, $F_p = F|_{T_p M}$ is a Minkowski norm on $T_p M$.

Lemma 2.1. The fundamental tensor $g_v$ fulfills the following properties:
Lemma 2.2 (Chern's connection). Given a vector field $V$ without singularities on an open set $U \subset M$ there exists a unique affine connection $\nabla^V$ on $U$ that satisfies the following properties:

1. $\nabla^X_Y - \nabla^Y_X = [X, Y]$ for every vector field $X$ and $Y$ on $U$,
2. $X \cdot g_v(Y, Z) = g_v(\nabla^X_Y Z) + g_v(Y, \nabla^X_Z) + 2C_v(\nabla^X V, Y, Z)$,

where $X, Y,$ and $Z$ are vector fields on $U$ and $C_v$ is the Cartan tensor, i.e.,

$$C_v(w_1, w_2, w_3) := \frac{1}{4} \frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} F^2(v + \sum_{i=1}^3 s_i w_i) |_{s_1 = s_2 = s_3 = 0}$$

for $v \in TM \setminus 0$ and $w_i \in T_{\pi(v)} M$, where $\pi : TM \to M$ is the canonical projection.

Among other properties Cartan tensor satisfies:

$$C_v(v, w_1, w_2) = C_v(w_1, v, w_2) = C_v(w_1, w_2, v) = 0$$

Let $\gamma : I \subset \mathbb{R} \to M$ be a piece-wise smooth curve. As usual we can induce on the pullback bundle $\gamma^*(TM)$ over $I$ a covariant derivative $\frac{\partial}{\partial t}$. A curve $\gamma$ is called geodesic if $\frac{\partial^2 \gamma}{\partial t^2}(t) = 0, \forall t \in I$. Like in Riemannian geometry, for every vector $v \in TM$, there exists a unique maximal geodesic $\gamma_v : (a, b) \to M$ such that $\gamma_v(0) = v$ and one can define the exponential map in an open subset $U \subset TM$ for those vectors $v$ such that the maximal interval of definition $(a, b)$ of $\gamma_v$ includes the value 1. Then $\exp : U \to M$ is defined as $\exp(v) = \gamma_v(1)$. This map is smooth away from the zero section and $C^1$ on the zero section. Define the Finsler distance $d(p, q)$ as the infimum of the lengths of all piecewise smooth curve joining $p$ to $q$, where the length of a curve $\alpha : [a, b] \to M$ (with $p = \alpha(a)$ and $q = \alpha(b)$) is defined as $1(\alpha) = \int_a^b F(\alpha'(t)) dt$. Note that $d(p, q)$ may not be equal to $d(q, p)$. Geodesics locally minimize 1 among piecewise smooth curves, and hence locally realize the distance. More generally, a geodesic $\gamma$ minimizes (in some interval) the distance with a submanifold $P$ if it is orthogonal to $P$, i.e., if $g_{\gamma'(t)}(\gamma'(t), u) = 0$ for all $u \in T_{\gamma(t)} P$.

2.2. Finsler partition. Let $(M, F)$ be a Finsler manifold. A partition $\mathcal{F} = \{L\}$ of $M$ into connected immersed smooth submanifolds (the leaves) is called a Finsler partition if each geodesic $\gamma : (a, b) \to M$, with $0 \in (a, b) \subset \mathbb{R}$ orthogonal to the leaf $L_{\gamma(0)}$ is horizontal, i.e., is orthogonal to each leaf it meets. In addition a partition is called a singular foliation if for each $v \in T_pL_p$ there exists a smooth vector field $X$ tangent to the leaves so that $X(p) = v$.
Given a plaque $P_q$ of a leaf $L$ (i.e., a “small” relatively compact neighborhood of $q$ in $L$) the set of all (non zero) orthogonal vectors to $P_q$ at $q$, denoted as $\nu_qP_q$ is called the orthogonal cone and, as the name suggests, it is not always a subspace (without zero) but a cone.

Recall that $U^+$ is called a (future) tubular neighborhood (of radius $\epsilon$) of the plaque $P_q$ if exp sends $\nu(P_q) \cap F^{-1}((0, \epsilon))$ diffeomorphically to $U^+ \setminus P_q$, and all the orthogonal unit speed geodesics from the plaque minimize the distance from the plaque, at least in the interval $[0, \epsilon]$. If we restrict the exponential map exp to the $\epsilon$-orthogonal cone bundle $\nu(P_q) := \nu(P_q) \cap F^{-1}(\epsilon)$, then exp sends $\nu(P_q)$ to the so-called future cylinder $C^+_r(P_q)$. Alternatively, defining $f_+ : U^+ \to [0, +\infty)$ as the (future) distance $f_+(x) := d(P_q, x)$, we can define $C^+_r(P_q) := f_+^{-1}(r)$. Similarly one can define a past (or reverse) tubular neighborhood $U^-$, and past cylinder $C^-_r(P_q)$ considering the reverse metric $F^-(v) := F(-v)$; see e.g. [13, Section 1.5].

**Definition 2.3.** We will say that a partition $\mathcal{F}$ is locally forward (resp. backward) equidistant if given a plaque $P_q$, a future tubular neighborhood $U^+$ (resp. a reverse or past tubular neighborhood $U^-$) of $P_q$ and a point $x \in U^+$ (resp. $x \in U^-$) which belongs to the future cylinder $C^+_r(P_q)$ (resp. the past cylinder $C^-_r(P_q)$), then the plaque $P_x \subset U^+$ (resp. $P_x \subset U^-$) is contained in $C^+_r(P_q)$ (resp. $C^-_r(P_q)$).

**Lemma 2.4** ([13]). A partition $\mathcal{F}$ is Finsler if and only if its leaves are locally forward and backward equidistant.

In the particular case of a foliation of codimension 1 given by pre images of a function $f : M \to \mathbb{R}$ we have the following useful definition.

**Definition 2.5** (Forward parallel level sets). Let $f : M \to \mathbb{R}$ be a smooth function and $f^{-1}(c_1)$ and $f^{-1}(c_2)$ two regular level sets, with $c_1 < c_2$. We say that $f^{-1}(c_1)$ is is forward parallel to $f^{-1}(c_2)$ if each geodesic that starts orthogonal to $f^{-1}(c_1)$ and meets $f^{-1}(c_2)$ is orthogonal to $f^{-1}(c_2)$.

### 3. Basic remarks and examples

In this section we discuss a few basic examples of transnormal functions on Finsler manifolds stressing differences between them and transnormal functions on Riemannian manifolds.

Along this section we restrict our attention to a special but important type of Finsler metric. A Finsler metric $Z : TM \to [0, +\infty)$ is said to be a Randers metric with Zermelo Data $(h, W)$, for a Riemannian metric $h$ and smooth vector field $W$ with $h(W, W) < 1$ on $M$ (the wind), if $Z$ is the solution of

$$h\left(\frac{v}{Z(v)} - W, \frac{v}{Z(v)} - W\right) = 1, v \in TM \setminus 0.$$ (3.1)

Equivalently we can define $Z(v) = \alpha(v) + \beta(v)$ where $\alpha$ is a Riemannian norm and $\beta$ a 1-form (with $\alpha(\beta) < 1$ ) both determined by $(h, W)$; recall [13].

**Lemma 3.1.** Let $f : U \subset M \to \mathbb{R}$ be a smooth function without critical points on $U$. Let $Z$ be a Randers metric with Zermelo data $(h, W)$. Let $\nabla f$ and $\tilde{\nabla} f$ be the gradients with respect to $Z$ and $h$. Then

\begin{enumerate}
  \item $\frac{\nabla f}{Z(\nabla f)} = \tilde{\nabla} f$
  \item $Z(\nabla f) = \|	ilde{\nabla} f\| + df(W)$
\end{enumerate}
where $\|v\| = \sqrt{h(v,v)}$.

Proof. For a Randers metric $Z$ it is well known (e.g., [8, Cor. 4.17]) that:

$$g_v(v,u) = \frac{Z(v)}{\mu_0(v)}(h(v - Z(W), u))$$

where $\mu = 1 - \alpha(\beta)^2$. Eq. (3.2) and the definition of gradient imply

$$h(\nabla f, u) = df(u) = g\nabla f(\nabla f, u) = \frac{Z(\nabla f)}{\mu_0(\nabla f)}(h(\nabla f - Z(\nabla f)W, u))$$

Therefore, by setting $k := \frac{Z(\nabla f)}{\mu_0(\nabla f)} > 0$ we have:

$$\nabla f = k(\nabla f - Z(\nabla f)W)$$

By taking the norm $\|\cdot\|$ on both sides of Eq. (3.3) and replacing $v = \nabla f$ in Eq. (3.1) we infer

$$\|\nabla f\| = k\|\nabla f - Z(\nabla f)W\| = kZ(\nabla f)$$

and hence $k = \frac{\|\nabla f\|}{Z(\nabla f)}$. This together with Eq. (3.3) finish the proof of item (a).

Item (a), Eq. (3.2) and item (b) of Lemma 2.1 imply

$$Z^2(\nabla f) = g\nabla f(\nabla f, \nabla f) = kh(\nabla f - Z(\nabla f)W, \nabla f) = kh\left(\frac{\nabla f}{k} + \frac{\nabla f}{k} + Z(\nabla f)W\right) = \frac{\|\nabla f\|^2}{k} + Z(\nabla f)df(W)$$

The above equation finishes the proof of item (b) because $k = \frac{\|\nabla f\|}{Z(\nabla f)}$.

Remark 3.2. It was proved in [1, Proposition 2.12] that a partition $\mathcal{F} = \{L\}$ given by a submersion on a Randers space $(M, Z)$ with Zermelo data $(h, W)$ is Finsler, i.e., its leaves are locally forward and backward equidistant, if and only if $W$ is $\mathcal{F}$ foliated vector field (i.e., it projects to a vector field on the basis of the submersion) and $\mathcal{F}$ is Riemannian with respect to $h$. This result together with Lemma 3.1 imply the following result: Let $f : M \to \mathbb{R}$ be a Z-transnormal function on a Randers space $(M, Z)$ with Zermelo data $(h, W)$. Then the regular level sets are leaves of a Finsler partition, if and only if $W$ is $\mathcal{F}$ foliated vector field.

Example 3.3. Let $(V, Z)$ be a Randers Minkowski space with Zermelo data $(h, W)$ and define $f(x) := d(0, x)$. From [10, Lemma 3.2.3] we know that $b = 1$, i.e, $f : V \setminus \{0\} \to \mathbb{R}$ is a Z-transnormal function. As we will see in Section 4 the partition $\mathcal{F} = \{f^{-1}(c)\}_{c>0}$ is forward parallel. Remark 3.2 above implies that this partition is not a Finsler partition. This also follows from Lemma 2.4 because future spheres with center at 0 (i.e., translation of $h$-spheres in the direction of $W$) are not the same as the past spheres with the center at 0 (i.e., translation of $h$-spheres in opposite direction of $W$). As we have stressed in the introduction,
this phenomenon is different from what happens in the Riemannian case, where transnormality already implies that the level sets are equidistant.

Remark 3.4. Let \((M, Z)\) be a Randers space with Zermelo data \((h, W)\). Let \(f : M \to \mathbb{R}\) be a smooth \(h\)-transnormal function with \(b \in C^2(f(M))\). Suppose also that \(W\) is a \(F\)-foliated vector field, where \(F = \{f^{-1}(c)\}\). Using Lemma 3.1 it is possible to check that \(f\) is a \(Z\)-transnormal function with \(b \in C^0(f(M))\). As we are going to see below, there is a simple example where \(b \notin C^2(f(M))\). This indicates another phenomenon that is different from the Riemannian case, where the assumption \(b \in C^2(f(M))\) is natural.

Example 3.5. Consider \(f : D \to \mathbb{R}\) where \(D\) is a disc of radius smaller than 1 and \(f(x, y) = x^2 + y^2\). Let \(Z\) be the Randers metric with Zermelo data \((h_0, W)\) where \(h_0\) is the Euclidean metric of \(\mathbb{R}^2\) and \(W = (x, y)\). From Lemma 3.1 we conclude that \(b(t) = (2\sqrt{t} + 2t)^2\).

4. Question 1.1

The goal of this section is to give an alternative proof to Proposition 4.4 below, that was proved at [6].

We start by recalling the next lemma, proved at [16, Lemma 3.2.1]

Lemma 4.1. Let \((M, F)\) be a Finsler space, \(U\) be an open subset of \(M\) and \(f\) be a smooth function on \(U\) without critical points on \(U\). Set \(\hat{g} := g_{\nabla f}\) and \(\hat{F} := \sqrt{\hat{g}}\). Then

\[\nabla f = \hat{\nabla} f,\]

where \(\hat{\nabla} f\) denotes the gradient of \(f\) with respect to \(\hat{F}\). Moreover

\[F(\nabla f) = \hat{F}(\hat{\nabla} f).\]

Remark 4.2. As proved at [16, Lemma 3.2.2], the gradient \(\nabla f\) of functions on Finsler space \((M, F)\) is orthogonal to each regular level set.

We also need this other known result, that follows by using a Koszul type formula associated to the Chern connection.

Lemma 4.3. Let \(X\) be a smooth vector field without singularities on an open set \(U\). Consider the Riemannian metric \(\hat{g} := g_X\) on \(U\), the associated Riemannian connection (associated to \(\hat{g}\)) \(\hat{\nabla}\) and the Chern connection \(\nabla_X\). Then \(\nabla_X X = \hat{\nabla} X\).

In particular, \(X\) is a vector field on \(U\) so that its integral curves are geodesics (with respect to \(F\)) if and only if \(X\) has the same property with respect to \(\hat{F} := \sqrt{\hat{g}}\).

Proposition 4.4. Let \((M, F)\) be a forward complete Finsler space. Let \(f : M \to \mathbb{R}\) be a \(F\)-transnormal function, \(c < d\) regular values on \(f(M)\) so that \([c, d]\) does not have singular values. Then for each \(q \in f^{-1}(d)\)

\[d(f^{-1}(c), q) = d(f^{-1}(c), f^{-1}(d)) = \int_c^d ds \frac{1}{\sqrt{b(s)}}.\]

In addition the integral curves of the vector field \(\nabla f\) (i.e., the gradient flow), when parameterized by arc length, are horizontal geodesics joining \(f^{-1}(c)\) to \(f^{-1}(d)\) and realize the distance between these two regular leaves.
Proof. Set $U = f^{-1}([c, d])$. Then from Lemma 4.1 we conclude that
\[ \hat{g}(\nabla f, \nabla f) = b \circ f \]
on $U$. In other words $f$ is also a transnormal function with respect to $\hat{g}$ (with the same $b$). We are going to use classical results about Riemannian transnormal function, recall [15] and [2, Chapter 5]. Let $\alpha$ be an integral curve of $\nabla f$ starting at some point of $p \in f^{-1}(c)$ and $\beta$ its arc-length reparametrization. Then
- $\beta$ is a horizontal unit speed geodesic (with respect to $\hat{g}$),
- $\beta|[0, r]$ joins $f^{-1}(c)$ to $f^{-1}(d)$, where $r = \int_c^d \frac{ds}{\sqrt{b(s)}}$.
- $\beta$ meets each regular level set just once.

From Lemmas 4.3 and 4.4 and Remark 4.2 we see that $\beta$ is also the arc-length reparametrization of the integral curve of $\nabla f$ and also fulfills the properties described above for the Finsler metric $F$.

Finally consider a segment of unit speed geodesic $\gamma$ joining $f^{-1}(c)$ to a point $q \in f^{-1}(d)$ realizing the distance between them. Then it is not difficult to see that $\gamma$ is contained in $U$, it meets $f^{-1}(c)$ just at one point and at this point the velocity of $\gamma$ has the same directions as $\nabla f$. From the unicity of geodesics we conclude that $\gamma$ must coincide with one of the segments $\beta$ defined above and this conclude the proof.

\[ \Box \]

Definition 4.5. As we have seen above, given a transnormal function $f : M \to \mathbb{R}$, the integral curves of the vector field $\nabla f$ (i.e., the gradient flow), when parameterized by arc-length is a geodesic. This segment of geodesics is called $f$-segment.

Remark 4.6 (Analyticity). Assume that $f : M \to \mathbb{R}$ is an analytic function on an analytic manifold $M$. Then, as usual, local properties can be extended. For example assume that $f$ is a transnormal function in a neighborhood of a point $p$ of regular leaf $f^{-1}(c)$. Set $g(s, t) = f(\exp_{\beta(s)}(t\xi)) - f(\exp_{p}(t\xi))$ where $\xi = \frac{\nabla f}{F(f)}$ and $s \to \beta(s) \in f^{-1}(c)$ is a curve such that $\beta(0) = p$. Note that $g(s, t) = 0$ for small $s$ and $t$ because $f$ is transnormal in a neighborhood of $p$. By analyticity of $f$ we conclude that the function $g$ is always zero, i.e., regular level sets are forward parallel. This and other quite similar straightforward arguments will be extensively used in the next section.

5. Question 1.2 and proof of Theorem 1.3

Let us first sketch the idea of the proof of Theorem 1.3. First we are going to show that there exists a neighborhood $U_0$ of $f^{-1}(b)$ so that $\mathcal{F}$ restricted to $U_0 \setminus f^{-1}(b)$ is a Finsler foliation, see Lemma 5.1. This will be proved using the analyticity of $f$, the fact that regular level sets, future and past cylinder have codimension 1 and Lemma 2.4. Once we have assured that $\mathcal{F}$ is a Finsler foliation on $U_0 \setminus f^{-1}(b)$, we will apply index-Morse arguments from [2, Theorem 5.63] to conclude that $f^{-1}(b)$ is in fact a submanifold, see Lemma 5.2. Finally analyticity will allow us to extend the property of being a Finsler partition on $U_0$ to whole $M$.

Now let us give a few more details about the proof through the next two lemmas and a series of claims.

Lemma 5.1. There exists a neighborhood $U_0$ of the critical level set $f^{-1}(b)$ where $\mathcal{F}$ fulfills the following property: if $x \in U_0 \setminus f^{-1}(b)$ and $\gamma$ is a geodesic so that $\gamma(0) = x$
and \( \gamma'(0) \) is orthogonal to the level set that contains \( x \), then \( \gamma \) is orthogonal to all regular level sets of \( M \) it meets.

Proof.

**Claim 1.** \( C^{-}_r(f^{-1}(b)) = f^{-1}(c) \), for each \( c < b \) and \( r^-_c = d(f^{-1}(c), f^{-1}(b)) \).

In fact let \( x_0 \in f^{-1}(b) \) be a point so that \( d(f^{-1}(c), x_0) = d(f^{-1}(c), f^{-1}(b)) = r^-_c \). Let \( \gamma : [0, r^-_c] \to M \) be a unit speed geodesic so that \( \gamma(0) \in f^{-1}(c) \) and \( \gamma(r^-_c) = x_0 \). Note that \( \gamma \) is an extension of an \( f \)-segment and minimize the distance. These facts and the analyticity imply that each \( f \)-segment starting at \( f^{-1}(c) \) meets \( f^{-1}(b) \) at the first time at \( t = r^-_c \) and this implies that \( f^{-1}(c) \subset C^-_r(f^{-1}(b)) \). Now consider \( x \in C^{-}_r(f^{-1}(b)) \) and \( \tilde{c} = f(x) \). From what we have discussed before we have that \( f^{-1}(\tilde{c}) \subset C^-_r(f^{-1}(b)) \). Therefore \( r^-_c = r^- = r \). Assume by contradiction, that \( f(x) = \tilde{c} < c \). Let \( \gamma : [0, r] \to M \) be a minimal unit speed geodesic joining \( \gamma(0) \in f^{-1}(\tilde{c}) \) to \( \gamma(r) \in f^{-1}(b) \). The fact that the regular leaves have codimension one allows us to conclude that \( \gamma \) is an \( f \)-segment and cross \( f^{-1}(c) \) at time \( t < r \) what is a contradiction with the fact that \( f^{-1}(c) \subset C^-_r(f^{-1}(b)) \). A similar contradiction happens if one supposes that \( f(x) = \tilde{c} > c \). Therefore \( f(x) = \tilde{c} = c \) i.e., \( C^-_r(f^{-1}(b)) \subset f^{-1}(c) \) and this concludes the proof of Claim 1.

From Lojasiewicz’s Theorem (recall [11] Theorem 6.3.3)) we know that the level set \( f^{-1}(b) \) is stratified into submanifolds. Let \( \Sigma \) denote a (connected) stratum with local larger dimension, i.e., if \( x \in \Sigma \) then there is a neighborhood \( U \) of \( x \) so that the only components of \( f^{-1}(b) \cap U \) are components of \( \Sigma \). For each \( x_\alpha \in \Sigma \subset f^{-1}(b) \) consider a relatively compact neighborhood \( P_\alpha \subset \Sigma \) of \( x_\alpha \) so that \( \overline{P}_\alpha \) is in the interior of \( \Sigma \) and \( C^-_{r^-}(P_\alpha) = C^-_{r^-}(f^{-1}(b)) \cap U \) for some neighborhood \( U \) of \( x_\alpha \) and for \( c \) close to \( b \). By using Claim 1 we infer the next claim.

**Claim 2.** For each \( c \) close to \( b \) the past cylinder \( C^-_{r^-}(P_\alpha) \) is an open set of \( f^{-1}(c) \).

The above claim and the analyticity of \( f \) imply:

**Claim 3.** let \( \gamma_{u_1} \) be the unit speed geodesic with \( \gamma_{u_1}'(0) = u_1 \), for \( u_1, u_2 \in \nu^1(P_\alpha) \). Then

(a) \( f(\gamma_{u_1}(t)) = f(\gamma_{u_2}(t)) \), for \( t \in \mathbb{R} \)
(b) \( \gamma_{u_1} \) is orthogonal to each regular level set of \( f \).

Let \( \gamma \) be a unit speed geodesic orthogonal to \( P_\alpha \). It is not difficult to see that there exists a \( c_0 \) so that for each \( c \in [c_0, b) \) there exists \( r^-_c > 0 \) and \( r^+_c > 0 \) so that \( f(\gamma(-r^-_c)) = c = f(\gamma(r^+_c)) \). From Claim 3 one can infer that, for each other unit speed geodesic \( \gamma_\alpha \) orthogonal to \( P_\alpha \), we have that \( \gamma_\alpha(r^+_\alpha) \in f^{-1}(c) \), and hence \( C^+_{r^+_\alpha}(P_\alpha) \subset f^{-1}(c) \). This fact and the fact that \( C^+_{r^+_\alpha}(P_\alpha) \) and \( f^{-1}(c) \) have codimension 1 imply that

**Claim 4.** There exists \( c_0 \) so that \( C^+_{r^+_\alpha}(P_\alpha) \) is an open set of \( f^{-1}(c) \) for each \( c \in [c_0, b) \).

Lemma 2.4 Claims 2 and 4 imply the next claim.

**Claim 5.** There exists a neighborhood \( U_\alpha \) of \( P_\alpha \) where \( F \) fulfills the following property: if \( x \in U_\alpha \setminus P_\alpha \) and \( \gamma \) is a geodesic so that \( \gamma(0) = x \) and \( \gamma'(0) \) is orthogonal to the level set that contains \( x \), then \( \gamma \) is orthogonal to all regular level sets of \( M \) it meets.
Let $U_0$ be the saturation of $U_a$. Claim 5 and analyticity of $f$ imply that $U_0 \setminus f^{-1}(b)$ also fulfills the property of Claim 5 and in particular $\mathcal{F}$ restricted to $U_0 \setminus f^{-1}(b)$ is a Finsler foliation, as we wanted to prove.

\begin{lemma}
$f^{-1}(b)$ is an embedded submanifold.
\end{lemma}

\begin{proof}
Let $\eta_\xi : f^{-1}(c) \to M$ be the map defined as $\eta_\xi(x) = \exp_x (t\xi)$ where $\xi = \nabla f$ at $(x, f(x))$. We will call this kind of map as an end point map pointing in the direction of $\xi$.

\begin{claim}
There exists an $\epsilon > 0$ so that $\eta_\xi : f^{-1}(c) \to f^{-1}(d)$ is a diffeomorphism between regular level sets, for each $t \in (0, r_c^-) \cup (r_c^-, r_c^- + \epsilon)$ and $c$ close to $b$ (e.g., $c > c_0$).
\end{claim}

In fact from analyticity it is easy to see that $\eta_\xi : f^{-1}(c) \to f^{-1}(d)$. In order to prove that it is a diffeomorphism, it suffices to construct the smooth inverse. Let $\xi$ be the normal vector field along $f^{-1}(d)$ pointing in the opposite (resp. same) direction of $\nabla f$ if $t \in (r_c^-, r_c^- + \epsilon)$ (resp. if $t \in (0, r_c^-)$). Define $t \to \gamma(t)$ as $\gamma(t) = \exp(t\xi)$ and define the end past map $\eta^-_\xi : f^{-1}(d) \to M$ as $\eta^-_\xi(x) := \gamma(-t)$. Analyticity and Lemma 5.1 imply that the map $\eta^-_\xi : f^{-1}(d) \to f^{-1}(c)$ is the inverse of $\eta_\xi$.

\begin{claim}
The derivative of map $\eta_{r_c^- \xi} : f^{-1}(c) \to M$ has constant rank.
\end{claim}

The idea of the proof is based on [2] Theorem 5.63. Let us briefly recall it, accepting results on Jacobi field on Finsler spaces; see [7] and [10]. For $p \in f^{-1}(c)$, consider the geodesic $t \to \gamma_p(t) = \exp_p(t\xi)$. Since $f^{-1}(c)$ is a hypersurface, we can infer that the point $\gamma_p(t)$ is a $f^{-1}(c)$-focal point of multiplicity $k$ if and only if $p$ is a critical point of $\eta_\xi$ and $\dim \ker d(\eta_\xi)_p = k$. Furthermore, for the appropriate choice of $\epsilon > 0$, Claim 6 implies that if $t \in I = [0, r_c^- + \epsilon]$, then $\eta_\xi$ may only fail to be an immersion if $t = r_c^-$. These two facts together imply that for every $x \in f^{-1}(c)$,

$$m(\gamma_x) = \dim \ker d(\eta_{r_c^- \xi})_x,$$

where $m(\gamma_x)$ denotes the number of focal points on $\gamma_x$ counted with multiplicities on $\gamma_x|_I$. From Morse Index

$$m(\gamma_p) \leq m(\gamma_x),$$

for $x \in f^{-1}(c)$ near to $p$. Since

$$\dim \ker d(\eta_{r_c^- \xi})_p \geq \dim \ker d(\eta_{r_c^- \xi})_x,$$

for $x \in f^{-1}(c)$, we conclude that

$$\dim \ker d(\eta_{r_c^- \xi})_p = \dim \ker d(\eta_{r_c^- \xi})_x,$$

for $x$ near to $p$. This and the connectivity of $f^{-1}(c)$ finish the proof of Claim 7.

\begin{claim}
The map $\eta_{r_c^- \xi} : f^{-1}(c) \to f^{-1}(b)$ is surjective and $f^{-1}(b)$ is an immersed submanifold.
\end{claim}

In fact, from Claim 7 and Claim 2 we infer that $\dim d\eta_{r_c^- \xi} = \dim \Sigma$. This fact, Claim 2 connectivity argument and definition of stratification imply that $\Sigma \subset \eta_{r_c^- \xi}(f^{-1}(c))$. Note that this also holds for each other stratum of $f^{-1}(b)$ with local larger dimension. Therefore from definition of stratification we conclude that
the map \( \eta_{r,\xi} : f^{-1}(c) \to f^{-1}(b) \) is surjective. From rank theorem we deduce that 
\( f^{-1}(b) \) is an immersed submanifold (with possible intersections).

Claims [1] and [8] imply that \( f^{-1}(b) \) is an embedded submanifold, as we wanted to prove. \( \square \)

Now we want to extend the property of Lemma [5.1] to whole analytic manifold \( M \).

Let \( \gamma : [0, r] \to M \) be a fixed segment of geodesic joining \( f^{-1}(b) \) to \( f^{-1}(a) \) so that 
\( r = d(f^{-1}(b), f^{-1}(a)) \).

Consider a partition \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = r \), a finite covering of 
\( [0, r] \) by open intervals \( I_i \) \((i = 0 \cdots n)\) centered at \( t_i \) such that \( U_i := f^{-1}(f(\gamma(I_i))) \) is
an open neighborhood of \( f^{-1}(c_i) \) (where \( c_i = f(\gamma(t_i)) \)) contained in the future and past neighborhoods of 
\( f^{-1}(c_i) \). Finally consider \( \{s_i\}_{i=0}^{n-1} \) so that \( 0 < s_i < t_i < \)
\( s_1 < t_2 < s_2 \cdots < s_{n-1} < t_n = r \) and \( s_i \in I_i \cap I_{i+1} \). Set \( \tilde{c}_i = f(\gamma(s_i)) \). Note that 
\( f^{-1}(\tilde{c}_0) \) is contained in the neighborhood \( U_0 \cap U_1 \). Therefore Lemma 5.1 allows us to infer that the geodesics
(starting at \( f^{-1}(\tilde{c}_0) \) pointing in the opposite direction of the gradient) arise orthogonally to \( f^{-1}(c_1) \). Hence 
\( f^{-1}(c_0) \) is contained in the connected component \( C^- \) of the past cylinder \( C^-(f^{-1}(c_1)) \) of axis \( f^{-1}(c_1) \). Therefore, since 
both have the same dimension, they coincide. On the other hand, the end point map \( \eta_{r,\xi} : C^- \to f^{-1}(c_1) \) is a diffeomorphism, where \( \xi \) is the unit normal vector along \( C^- \) pointing in the opposite direction of the gradient. Similarly, end point maps induce diffeomorphisms between \( f^{-1}(c_1) \) and (connected components of) its future cylinders. These facts together imply that Lemma 5.1 also holds in a neighborhood
of \( f^{-1}(\tilde{c}_1) \). By induction we infer that Lemma 5.1 is true in a neighborhood of 
\( f^{-1}(\tilde{c}_{n-1}) \). Following the same proof of Lemmas 5.1 and 5.2 we conclude that
Lemma 6.1 holds in a neighborhood \( U_n \setminus f^{-1}(a) \) and that the level set \( f^{-1}(a) \) is an
embedded submanifold, finishing the proof of the theorem.

6. Question [1],[20]

In this section we approach Question [1],[3] and prove the next proposition.

Proposition 6.1. Let \( (M, F) \) be a compact, connected and smooth Finsler manifold
and \( f : M \to [a, b] \) be a smooth \( F \)-transnormal function with \( F^2(\nabla f) = b(f) \), where
\( b \) is a \( C^1 \) function on \( [a, b] \). Suppose that:
(a) the level sets are connected,
(b) the critical level sets \( f^{-1}(a) \) and \( f^{-1}(b) \) are submanifolds of codimension
greater than one,
(c) \( a \) and \( b \) are the only singular values of \( [a, b] \),
(d) \( b'(a) \neq 0 \neq b'(b) \).

Then there exists a Riemannian metric on \( M \) such that \( \mathcal{F} = \{f^{-1}(c)\}_{c \in [a, b]} \) is a
singular Riemannian foliation.

Remark 6.2. As discussed by Wang [15], conditions (c) and (d) above are satisfied
by a Riemannian transnormal function if \( b \in C^2[a, b] \), and these are important
conditions, e.g., there exist examples of (Riemannian) transnormal functions where
(d) is not satisfied and the level sets of \( f \) are not even leaves of a singular foliations.
The problem in the Finsler case is that the assumption \( b \in C^2[a, b] \) seems to be too
strong, recall Remark [3],[4] and Example [5],[5]. Therefore it remains for us to assume
(c) and (d) as hypotheses. Note that one can even ask if the smoothness of \( f \) and the assumption that \( b \in C^2[a, b] \) already imply some property about the Finsler metric \( F \). For example one can ask: \textit{is the Finsler metric \( F \) already Riemannian (or Riemannian in transversal directions to the singularities) or at least reversible (in transversal directions to the singularities) when the function \( b \) is \( C^2 \)?} In particular, it would be natural to try to establish an analogy between this question and the well known fact that if \( \exp_p \) is \( C^2 \) at zero then \( F \) is a Riemannian metric.

In order to prove the above proposition we will use a result about Bott-Morse functions. Let \( f : M \to \mathbb{R} \) be a smooth function. As usual we can define the Hessian of \( f \) at a critical point \( p \in M \) as the symmetric linear operator \( \text{Hess}_p : T_pM \times T_pM \to \mathbb{R} \) defined by \( \text{Hess}_p(v, w) = \bar{v}_p \bar{w}_p f \), where \( \bar{v}, \bar{w} \) are extensions of \( v \) and \( w \), resp. Let \( Cr(f) \) denote the critical level set of \( f \). Recall that \( f \) is called a Morse-Bott function if \( Cr(f) \) is union of connected submanifolds and the kernel \( \text{Hess} f \) of singular points coincides with the tangent spaces of \( Cr(f) \). In particular if \( g \) is some Riemannian metric on \( M \) and \( S \) is a submanifold normal to \( Cr(f) \) at \( p \), then \( \text{Hess}_p \), restricted to \( T_pS \), turns to be non degenerate.

The next strong result stresses the relation between Bott-Morse functions and (Riemannian) transnormal functions.

\textbf{Theorem 6.3 \textup{(12)}.} Let \( M \) be a compact smooth manifold, and \( f : M \to \mathbb{R} \) a Morse-Bott function with \( Cr(f) = M_+ \cup M_- \), where \( M_+ \) and \( M_- \) are both closed connected submanifolds of codimensions bigger than 1. Then there exists a Riemannian metric on \( M \) so that \( f \) is transnormal. In fact, the metric can be chosen so that \( M_+ \) and \( M_- \) are both totally geodesics.

Our goal is to check that the \( F \)-transnormal function that satisfies the hypothesis of Proposition 6.1 is a Bott-Morse function. Once we have proved this, our result will follow directly from Theorem 6.3 and Wang \textsuperscript{15}.

We start by recalling the definition of Finslerian \( \text{Hess}^F \) on a Finsler manifold \((M, F)\) on non singular values of \( f \).

\textbf{Definition 6.4.} Let \( f : (M, F) \to \mathbb{R} \) be a smooth function on a Finsler manifold \( M \) and \( U = \{ x \in M, \; df_x \neq 0 \} \). We define \( \text{Hess}^F f \) on \( U \) as
\[
\text{Hess}^F f : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathbb{R}
\]
\[
(Y, X) \mapsto g_{\nabla f}(\nabla_Y^{\nabla f} \nabla f, X)
\]

\textbf{Lemma 6.5.} Let \( f : (M, F) \to \mathbb{R} \) be a smooth function on a Finsler manifold \( M \) and \( U = \{ x \in M, \; df_x \neq 0 \} \). Then \( \text{Hess}^F_x(Y, X) = Y(X(f))_x - df_x(\nabla_Y^{\nabla f} X) \).

\textbf{Proof.} By the almost \( g \)-compatibility of the connection and the definition of the gradient we have
\[
Y(X(f)) = Y(dfX) = Y(g_{\nabla f}(\nabla f, X))
\]
\[
= g_{\nabla f}(\nabla_Y^{\nabla f} \nabla f, X) + g_{\nabla f}(\nabla_Y^{\nabla f} X, \nabla f)
\]
\[
+ 2C_{\nabla f}(\nabla_Y^{\nabla f} \nabla f, \nabla f, X).
\]
From Eq. 2.2 we conclude that \( C_{\nabla f}(\nabla_Y^{\nabla f} \nabla f, \nabla f, X) = 0 \). This fact and the definition of gradient imply the lemma. \( \square \)
Lemma 6.6. Let \( f : (M,F) \to \mathbb{R} \) be a smooth function which is \( F \)-transnormal with \( F^2(\nabla f) = b \circ f \) and \( U = \{ x \in M, df_x \neq 0 \} \). Then on \( U \) we have

\[
\text{Hess}^F f(\nabla f, \nabla f) = \frac{1}{2} b'(f) b(f).
\]

In particular, one can write \( \text{Hess}^F f(\nabla f_{F(\nabla f)}, \nabla f_{F(\nabla f)}) = b'(f) \).

Proof. The definition of transnormal function, and Eq. (2.2) imply

\[
\text{Hess}^F f(\nabla f, \nabla f) = \frac{1}{2} \nabla f (g_{\nabla f}(\nabla f, \nabla f))
\]

\[
= \frac{1}{2} \nabla f (F^2(\nabla f)) = \frac{1}{2} \nabla f (b \circ f)
\]

\[
= \frac{1}{2} b'(f) df(\nabla f) = \frac{1}{2} b'(f) g_{\nabla f}(\nabla f, \nabla f)
\]

\[
= \frac{1}{2} b'(f) F^2(\nabla f) = \frac{1}{2} b'(f) b(f).
\]

\[
\square
\]

Consider an arbitrary metric \( g \) and a slice \( S \) orthogonal to \( f^{-1}(b) \) at \( p \). Let \( X \) be a vector of \( T_p S \) and consider the (only) vector \( V \) at \( \nu_q(f^{-1}(b)) \), that projects to \( X \), i.e., so that \( V = X + X^v \) where \( X^v \) is tangent to \( f^{-1}(b) \), see [1, Lemma 2.9]. Let \( \gamma \) be the unit speed geodesic that contains the integral lines of \( \nabla f \) so that \( p = \lambda \gamma'(0) = V \), for the appropriate \( \lambda \neq 0 \). Lemmas 6.5 and 6.6 imply

\[
\text{(6.1)} \quad \text{Hess}_f (\gamma'(0), \gamma'(0)) = \frac{1}{2} b'(b).
\]

Eq. (6.1) and the fact that \( T_p f^{-1}(b) = \ker \text{Hess}_p f \) imply that

\[
\text{Hess}_p f(X, X) = \frac{\lambda^2}{2} b'(b) \neq 0.
\]

The last equation and the arbitrary choice of \( X \in T_p S \) imply \( \text{Hess}_p f \) is non degenerate at \( T_p S \). A similar proof is valid for \( f^{-1}(a) \) and hence \( f \) is a Morse Bott-function, as we wanted to prove.

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