The Schwarz-Milnor lemma for braids and area-preserving diffeomorphisms

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Abstract

We prove a number of new results on the large-scale geometry of the $L^p$-metrics on the group of area-preserving diffeomorphisms of each orientable surface. Our proofs use in a key way the Fulton-MacPherson type compactification of the configuration space of $n$ points on the surface due to Axelrod-Singer and Kontsevich. This allows us to apply the Schwarz-Milnor lemma to configuration spaces, a natural approach which we carry out successfully for the first time. As sample results, we prove that all right-angled Artin groups admit quasi-isometric embeddings into the group of area-preserving diffeomorphisms endowed with the $L^p$-metric, and that all Gambaudo-Ghys quasi-morphisms on this metric group coming from the braid group on $n$ strands are Lipschitz. This was conjectured to hold, yet proven only for low values of $n$ and the genus $g$ of the surface.

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1 Introduction and main results

1.1 Introduction

The $L^2$-length of a path of volume-preserving diffeomorphisms, which describes a time-dependent flow of an ideal incompressible fluid, corresponds to the hydrodynamic action of the flow in the same way as the length of a path in a Riemannian manifold corresponds to its energy (cf. [50]). Indeed, it is the length of this path with respect to the formal right-invariant Riemannian metric on the group $\mathcal{G}$ of volume preserving diffeomorphisms introduced by Arnol’d in [14]. The $L^1$-length of the same path has a dynamical interpretation as the average length of a trajectory of a point under the flow.

Therefore, following the principle of least action, it makes sense to consider the infimum of the lengths of paths connecting two fixed volume-preserving diffeomorphisms. This gives rise to a right-invariant
distance function (metric) on $G$. Taking the identity transformation as the initial point, Arnol’d observes that a path whose $L^2$-length is minimal (and equal to the distance) necessarily solves the Euler equation of an ideal incompressible fluid.

It follows from works of Ebin and Marsden [21] that for diffeomorphisms in $G$ that are $C^2$-close to the identity, the infimum is indeed achieved. Further, more global results on the corresponding Riemannian exponential map were obtained in [19, 51] (see also [20]). In [19, 50] Shnirel’mann showed, among a number of surprising facts about this subject, that in the case of the ball of dimension 3, the diameter of the $L^2$-metric is bounded. This result is conjectured to hold for all compact simply connected manifolds of dimension 3 or larger (see [23, 38, 2]), while its analogue in the non-simply-connected case is false [23, 7]. Furthermore, Shnirel’mann has conjectured that for compact manifolds of dimension 2, the $L^2$-diameter is infinite.

Shnirel’mann’s conjecture, and its analogues for $L^p$-metrics, with $p \geq 1$ are by now proven. It follows from results of Eliashberg and Ratiu [23] that on compact surfaces (possibly with boundary) other than $T^2$ and $S^2$, Shnirel’mann’s conjecture holds for all $p \geq 1$. Their arguments rely on the Calabi homomorphism $\text{Cal}$ from the compactly supported Hamiltonian group $\text{Ham}_c(M, \sigma)$ to the real numbers in the case of a surface $M$ with non-empty boundary ($\sigma$ is the area form), and on non-trivial first cohomology combined with trivial center of the fundamental group in the closed case. For the two-torus $T^2$ Shnirel’mann’s conjecture holds by [14] Appendix A. Finally, the case of $S^2$ was settled in [14] by means of differential forms on the configuration space related to the cross-ratio map. In [43] the second author gave a new uniform proof of Shnirel’mann’s conjecture for all compact surfaces.

The methods that were used to prove Shnirel’mann’s conjecture are two-dimensional in nature, and have to do with braiding of trajectories of time-dependent two-dimensional Hamiltonian flows (in extended phase space). Indeed, Shnirel’mann has proposed to use relative rotation numbers to bound from below the $L^2$-lengths of two-dimensional Hamiltonian paths in $M$. This direction is related to the method of [23] by a theorem of Fathi [24] and Gambaudo and Ghys [27] (see also [48, 31, 33]). This theorem shows that the Calabi homomorphism is proportional to the relative rotation number of the trajectories of two distinct points in the two-disc $\mathbb{D}$ under a Hamiltonian flow, averaged over the configuration space of ordered pairs of distinct points $(x_1, x_2)$ in the two-disc.

This line of research was notably pursued in [29], and further in [41, 18, 17, 9, 39, 14] obtaining quasi-isometric and bi-Lipschitz embeddings of various groups (right-angled Artin groups and additive groups of finite-dimensional real vector spaces) into $\text{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, into $\ker(\text{Cal}) \subset \text{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, and into $\text{Ham}(S^2, \sigma)$ endowed with their respective $L^p$-metrics (see [10] for similar embedding results on manifolds with a sufficiently complicated fundamental group). In all cases, the key technical estimate is an upper bound, via the $L^p$-length of an isotopy of volume-preserving diffeomorphisms, of the average, over all points in a configuration space of the manifold, of the word length in the fundamental group of the trace of the point under the induced isotopy (closed up to a loop by a system of short paths on the configuration space).

Such estimates were initially produced by means of analyzing relative rotation numbers of pairs of braids, or quadruples as in [14]. However, in the case of a single braid, as observed by Polterovich, a simpler estimate is possible via the Schwarz-Milnor lemma [23, 51, 41]. Indeed, according to this result, the universal cover of compact proper length space, in this case our surface, is quasi-isometric to its fundamental group. In this way one can control the word length of trajectories in terms of their lengths, which allows one to prove the required upper bound. A similar estimate in the case of braids on more than one strand is not as readily available, because the configuration space $X_n(M)$ of $n$ points on $M$ is not a compact metric space. The arguments of the second author in [43] rely crucially on a new complete metric on $X_n(M)$. However, with this metric $X_n(M)$ still can not be considered to be a compact metric space from the point of view of the Schwarz-Milnor lemma.

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1 “Schwarz” is a transliteration from Russian; alternative spellings “Švarc”, “Schwartz” sometimes appear in the literature.
In this paper we show how to successfully carry out the strategy of the Schwarz-Milnor lemma for configuration spaces. While hints of a similar approach can be discerned in \[26\] in the special case of the two-disc and of double collisions, it was not known earlier to be applicable in the general context discussed herein. Specifically, we consider a metric on \(X_n(M)\) coming from a natural compactification \(\overline{X}_n(M)\) thereof, which is a compact geodesic metric space. This compactification is equivariant under the action of the diffeomorphism group \(\text{Diff}(M)\) on \(X_n(M)\) in the sense that the action extends naturally to the compactification \[4\]. Furthermore, the map \(\pi_1(X_n(M)) \to \pi_1(\overline{X}_n(M))\) induced by the inclusion is an isomorphism. Hence we may apply the Schwarz-Milnor lemma to \(\overline{X}_n(M)\). This, and further comparison to the metric from \[13\] allows us to prove our main estimate.

The compactification we use was introduced by Axelrod-Singer in \[8\] and by Kontsevich in \[40, 41\], inspired by the Fulton-MacPherson compactification in algebraic geometry \[25\]. The compactification \(\overline{X}_n(M)\) is roughly speaking a certain positive oriented blow-up of \(M^n\) along its multi-diagonals. For instance, one part of its codimension 1 boundary stratum is identified with the disjoint union of spaces of the form \(N_1(D_{ij}(M))|D_{ij}^0(M)\) where \(N_1\) is the unit normal bundle, \(D_{ij}(M) \subset M^n\) is the submanifold of points \((x_1, \ldots, x_n)\) where \(x_i = x_j\) and \(D_{ij}^0(M) \subset D_{ij}(M)\) is the open dense subset where \(x_k \neq x_i = x_j\) for all \(k \in \{1, \ldots, n\} \setminus \{i, j\}\). Note that this is an \(S^1\)-bundle. Intuitively, from a physical perspective, this means that one resolves a double collision of points by recording the collision point and the direction in which they have collided. Other parts of the codimension 1 stratum correspond to simple \(k\)-tuple collisions, and correspond to \(S^{2k-3}\)-bundles normal to \(k\)-diagonals. Higher strata correspond to more complicated collisions modeled by suitable graphs. It will, however, be technically most convenient for us to use a model of this compactification recently constructed directly as a subspace of a Euclidean space by Sinha \[52\]. We describe this construction in Section 2 below.

Finally, we observe that lower bounds on the average word length can often be provided by quasimorphisms - functions that are additive with respect to the group multiplication - up to an error which is uniformly bounded (as a function of two variables). The quasimorphisms we consider here were introduced and studied by Gambaudo and Ghys in the beautiful paper \[28\] (see also \[8, 14, 15, 16, 17, 18\]). These quasimorphisms essentially appear from invariants of braids traced out by the action of a Hamiltonian path on an ordered \(n\)-tuple of distinct points in the surface (suitably closed up), averaged over the configuration space \(X_n(M)\) of \(n\)-tuples of distinct points on the surface \(M\). As one of our results, we prove that all homogeneous Gambaudo-Ghys quasi-morphisms are Lipschitz in the \(L^p\)-metric for all \(p \geq 1\). This subsumes all previous results in this direction and provides a maximally general result. It also contrasts a recent result of Khanevsky according to which none of these quasi-morphisms are continuous in Hofer’s metric \[37\].

We prove further stronger results on the large-scale geometry on the \(L^p\)-metric on \(\mathcal{G} = \text{Diff}_{c,0}(M, \sigma)\) for a compact surface \((M, \sigma)\) with an area form. In particular, we provide bi-Lipschitz group monomorphisms of \(\mathbb{R}^m\) endowed with the standard (say Euclidean) metric into \((\mathcal{G}, d_{L^p})\) for each positive integer \(m\) and each \(p \geq 1\). Finally, our methods combined with an argument of Kim-Koberda \[39\] (cf. Crisp-Wiest \[18\], Benaim-Gambaudo \[4\]) show the existence of quasi-isometric group monomorphisms from each right-angled Artin group to \((\mathcal{G}, d_{L^p})\) for each \(p \geq 1\). We note that this was previously known only for \(\mathbb{D}^2\) and \(S^2\) \[39, 14\].

Let \(M_1 \to M_2\) be a measure preserving embedding of surfaces. It is an open question if the induced Lipschitz monomorphism \(\text{Diff}_0(M_1, \sigma_1) \to \text{Diff}_0(M_2, \sigma_2)\) is a quasi-isometric embedding. Note that our results provide a partial positive answer to this problem, see Remark \[18, 39\].

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\footnote{It is curious to note that the same is not true for the action of \(\text{Homeo}(M)\) as was recently proven \[42\].}
the infimum runs over all \( \tilde{\phi} \) where the infimum is taken over all paths \( \{ \phi_t \}_{t \in [0,1]} \).

The identity component of the group of compactly supported diffeomorphisms of \( M \) preserving the smooth measure \( \mu \). In other words, if \( M = \partial X \), it is the identity component of the group of measure preserving diffeomorphisms of \( X \) fixing point-wise a neighbourhood of \( \partial X \).

Fix \( p \geq 1 \). For a smooth isotopy \( \{ \phi_t \}_{t \in [0,1]} \), from \( \phi_0 = 1 \) to \( \phi_1 = \phi \), we define the \( L^p \)-length by

\[
L_p(\{\phi_t\}) = \int_0^1 \left( \frac{1}{\text{vol}(M, \mu)} \cdot \int_M |X_t|^p \, d\mu \right)^{\frac{1}{p}} \, dt,
\]

where \( X_t = \frac{d}{dt}|_{t=\tau} \phi_t \circ \phi_t^{-1} \) is the time-dependent vector field generating the isotopy \( \{ \phi_t \} \), and \( |X_t| \) is its length with respect to the Riemannian structure on \( M \). As is easily seen by a displacement argument, the \( L^p \)-length functional determines a non-degenerate norm on \( \mathcal{G} \) by the formula

\[
d_p(1, \phi) = \inf L_p(\{\phi_t\}).
\]

This in turn defines a right-invariant metric on \( \mathcal{G} \) by the formula

\[
d_p(\phi_0, \phi_1) = d_p(1, \phi_1 \phi_0^{-1}).
\]

Remark 1.2.1. Consider the case \( p = 1 \). It is easy to see that the \( L^1 \)-length of an isotopy is equal to the average Riemannian length of the trajectory \( \{ \phi_t(x) \}_{t \in [0,1]} \) (over \( x \in M \), with respect to \( \mu \)). Moreover for each \( p \geq 1 \), by Jensen’s (or Hölder’s) inequality, we have

\[
L_p(\{\phi_t\}) \geq L_1(\{\phi_t\}).
\]

Denote by \( \tilde{1} \) the identity element of the universal cover \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \). Similarly one has the \( L^p \)-pseudo-norm (that induces the right-invariant \( L^p \)-pseudo-metric) on \( \tilde{\mathcal{G}} \), defined for \( \tilde{\phi} \in \tilde{\mathcal{G}} \) as

\[
d_p(\tilde{1}, \tilde{\phi}) = \inf L_p(\{\phi_t\}),
\]

where the infimum is taken over all paths \( \{ \phi_t \} \) in the class of \( \tilde{\phi} \). Clearly \( d_p(1, \phi) = \inf d_p(1, \tilde{\phi}) \), where the infimum runs over all \( \tilde{\phi} \in \tilde{\mathcal{G}} \) that map to \( \phi \) under the natural epimorphism \( \mathcal{G} \to \tilde{\mathcal{G}} \).

Up to bi-Lipschitz equivalence of metrics \( d \) and \( d' \) are equivalent if \( \frac{1}{C} d \leq d' \leq C d \) for a certain constant \( C > 0 \) the \( L^p \)-metric on \( \mathcal{G} \) (and its pseudo-metric analogue on \( \tilde{\mathcal{G}} \)) is independent of the choice Riemannian structure and of the volume form \( \mu \) on \( M \). In particular, the question of boundedness or unboundedness of the \( L^p \)-metric enjoys the same invariance property.

**Terminology:** For a positive integer \( n \), we use \( A, B, C > 0 \) as generic notation for positive constants that depend only on \( M, \mu, g \) and \( n \).
1.2.2 Quasimorphisms

Some of our results have to do with the notion of a quasimorphism. Quasimorphisms are a helpful tool for the study of non-abelian groups, especially those that admit few homomorphisms to $\mathbb{R}$. A quasimorphism $r : G \to \mathbb{R}$ on a group $G$ is a real-valued function that satisfies

$$r(xy) = r(x) + r(y) + b_r(x, y),$$

for a function $b_r : G \times G \to \mathbb{R}$ that is uniformly bounded:

$$\delta(r) := \sup_{G \times G} |b_r| < \infty.$$

A quasimorphism $r : G \to \mathbb{R}$ is called homogeneous if $r(x^k) = kr(x)$ for all $x \in G$ and $k \in \mathbb{Z}$. In this case, it is additive on each pair $x, y \in G$ of commuting elements: $r(xy) = r(x) + r(y)$ if $xy = yx$.

For each quasimorphism $r : G \to \mathbb{R}$ there exists a unique homogeneous quasimorphism $\overline{r}$ that differs from $r$ by a bounded function:

$$\sup_{G} |\overline{r} - r| < \infty.$$

It is called the homogenization of $r$ and satisfies

$$\overline{r}(x) = \lim_{n \to \infty} \frac{r(x^n)}{n}.$$

Denote by $Q(G)$ the real vector space of homogeneous quasimorphisms on $G$.

For a finitely-generated group $G$, with finite symmetric generating set $S$, define the word norm $| \cdot |_S : G \to \mathbb{Z}_{\geq 0}$ by

$$|g|_S = \min\{k \mid g = s_1 \cdots s_k, \forall 1 \leq j \leq k, s_j \in S\}$$

for $g \in G$. This is a norm on $G$, and as such it induces a right-invariant metric $d_G(f, g) = |gf^{-1}|_S$. This metric is called the word metric. In this setting, any quasimorphism $r : G \to \mathbb{R}$ is controlled by the word norm. Indeed, for all $g \in G$,

$$|r(g)| \leq \left(\delta(r) + \max_{s \in S} |r(s)|\right) \cdot |g|_S.$$

When a specific symmetric generating set $S$ for $G$ can be fixed, we will usually denote $| \cdot |_S$ by $| \cdot |_G$.

We refer to [17] for more information about quasimorphisms.

1.2.3 Configuration spaces and braid groups

For a manifold $M$, which in this paper is usually of dimension 2, the configuration space $X_n(M) \subset M^n$ of $n$-tuples of points on $M$ is defined as

$$X_n(M) = \{(x_1, \ldots, x_n) \mid x_i \neq x_j, 1 \leq i < j \leq n\}.$$

That is

$$X_n(M) = M^n \setminus \bigcup_{1 \leq i < j \leq n} D_{ij},$$

where for $1 \leq i < j \leq n$, the partial diagonal $D_{ij} \subset M^n$ is defined as $D_{ij} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$. Note that $D_{ij}$ is a submanifold of $M^n$ of codimension $\dim M$.

Finally, if $\dim M = 2$, we define the pure braid group of $M$ as

$$P_n(M) = \pi_1(X_n(M)).$$
Noting that the symmetric group $S_n$ on $n$ elements acts freely on $X_n(M)$, we form the quotient $C_n(M) = X_n(M)/S_n$ and define the full braid group of $M$ as

$$B_n(M) = \pi_1(C_n(M)).$$

We note that $P_n(M)$ and $B_n(M)$ enter the exact sequence $1 \to P_n(M) \to B_n(M) \to S_n \to 1$. In particular $P_n(M)$ is a normal subgroup of $B_n(M)$ of finite index. We refer to [38] for further information about braid groups.

1.2.4 Short paths and the Gambaudo-Ghys construction

Let $M$ be a compact oriented surface. Given a real valued quasimorphism $r$ on $P_n(M) = \pi_1(X_n(M), q)$ for a fixed basepoint $q \in X_n(M)$ there is a natural way to construct a real valued quasimorphism on the universal cover $\hat{G}$ of the group $\hat{G} = \text{Diff}_{c,0}(M, \sigma)$ of area preserving diffeomorphisms of the surface $M$. We shall see that in the case of $M \neq T^2$ this induces a quasimorphism on $\hat{G}$ itself, because the fundamental group of $\hat{G}$ is finite. The same is true for $M = T^2$ where we consider the group $\hat{G} = \text{Ham}(M, \sigma)$ of Hamiltonian diffeomorphisms instead. This is not a restrictive condition from the viewpoint of large-scale geometry, since by a small modification of [14] Proposition A.1, the inclusion $(\text{Ham}(T^2, \sigma), d_{\text{Lip}}) \hookrightarrow (\text{Diff}_0(T^2, \sigma), d_{\text{Lip}})$ is a quasi-isometry for all $p \geq 1$.

The construction is carried out by the following steps (cf. [28 45 6]).

1. For all $x \in X_n(M) \setminus Z$, with $Z$ a closed negligible subset (e.g. a union of submanifolds of positive codimension) choose a smooth path $\gamma(x) : [0, 1] \to X_n(M)$ between the basepoint $q \in X_n(M)$ and $x$. Make this choice continuous in $X_n(M) \setminus Z$. We first choose a system of paths on $M$ itself. Then we consider the induced coordinate-wise paths in $M^n$, and pick $Z$ to ensure that these induced paths actually lie in $X_n(M)$. After choosing the system of paths $\{\gamma(x)\}_{x \in X_n(M) \setminus Z}$ we extend it measurably to all $x \in X_n(M)$ (obviously, no numerical values computed in the paper will depend on this extension). We call the resulting choice a "system of short paths".

2. Given a path $\{\phi_t\}_{t \in [0,1]}$ in $\hat{G}$ starting at $\text{Id}$, and a point $x \in X_n(M)$ consider the path $\{\phi_t \cdot x\}$, to which we then concatenate the corresponding short paths. That is consider the loop

$$\lambda(x, \{\phi_t\}) := \gamma(x) \# \{\phi_t \cdot x\} \# \gamma(y)^{-1}$$

in $X_n(M)$ based at $q$, where $^{-1}$ denotes time reversal. Hence we obtain for each $x \in X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)$ an element $[\lambda(x, \{\phi_t\})] \in \pi_1(X_n(M), q)$ (or rather for each $x \in X_n(M)$ after the measurable extension in Step 1).

3. Consequently applying the quasimorphism $r : \pi_1(X_n(M), q) \to \mathbb{R}$ we obtain a measurable function $f : X_n(M) \to \mathbb{R}$. Namely $f(x) = r([\lambda(x, \{\phi_t\}))]$. The quasimorphism $\Phi$ on $\hat{G}$ is defined by

$$\Phi([\{\phi_t\}]) = \int_{X_n(M)} f \, d\mu^\otimes n.$$  

It is immediate to see that this function is well-defined by topological reasons. The quasimorphism property follows by the quasimorphism property of $r$ combined with finiteness of volume. The fact that the function $f$ is absolutely integrable can be shown to hold a-priori by a reduction to the case of the disc. We note, however, that by Tonelli’s theorem this fact follows as a by-product of the proof of our main theorem, and therefore requires no additional proof.

4. Of course this quasimorphism can be homogenized, to obtain a homogeneous quasimorphism $\overline{\Phi}$.

Remark 1.2.2. If $M \neq S^2$ and $M \neq T^2$, then $\pi_1(\hat{G}) = 0$ and for $M = T^2$ the same is true for $\hat{G} = \text{Ham}(M, \sigma)$. In the case $M = S^2$, by the result of Smale [58] $\pi_1(\hat{G}) = \mathbb{Z}/2\mathbb{Z}$. Hence the quasimorphisms descend to quasimorphisms on $\hat{G}$, e.g. by minimizing over the two-element fibers of the projection.
\( \tilde{G} \to G \). For \( \overline{\Phi} \), the situation is easier since by homogeneity it vanishes on \( \pi_1(G) \subset Z(\tilde{G}) \), and therefore depends only on the image in \( G \) of an element in \( \tilde{G} \). We keep the same notations for the induced quasimorphisms.

### 1.3 Main results

Recall that we work with the group \( G = \text{Diff}_{c,0}(M, \sigma) \) for \((M, \sigma)\) a compact oriented surface with an area form. Our main technical result is the following.

**Theorem 1.** For an isotopy \( \overline{\phi} = \{ \phi_t \} \) in \( G \), the average word norm of a trajectory \( \lambda(x, \overline{\phi}) \) is controlled by the \( L^1 \)-length of \( \overline{\phi} \):

\[
W(\overline{\phi}) = \int_{X_n(M) \setminus \cup (\phi_1)^{-1}(Z)} |\lambda(x, \overline{\phi})| d\mu^\otimes n(x) \leq A \cdot l_1(\overline{\phi}) + B,
\]

for certain constants \( A, B > 0 \).

**Remark 1.3.1.** Note that \( W(\overline{\phi}) \) depends only on the class \( \tilde{\phi} = [\overline{\phi}] \in \tilde{G} \) of \( \overline{\phi} \) in the universal cover \( \tilde{G} \) of \( G \). Hence for all compact oriented surfaces other than \( S^2 \) and \( T^2 \), \( W(\overline{\phi}) \) does not depend on the chosen isotopy \( \tilde{\phi} \), but only on the diffeomorphism \( \phi_1 \). This is because \( G \) is simply connected.

Theorem 1 has a number of consequences concerning the large-scale geometry of the \( L^1 \)-metric on \( G \). Firstly, as any quasimorphism on a finitely generated group is controlled by the word norm, we immediately obtain the following statement.

**Corollary 1.** The homogenization \( \overline{\Phi} \) of each Gambaudo-Ghys quasimorphism \( \Phi \) satisfies

\[
|\overline{\Phi}(\phi)| \leq C \cdot d_1(\phi, 1).
\]

In particular, Corollary 1 implies that all Gambaudo-Ghys quasimorphisms are continuous in the \( L^1 \)-metric (and hence in the \( L^p \)-metric, see Remark 1.3.4), a fact which was known so far only for the genus zero case (see [14]) and for the higher genus case only when one considers Gambaudo-Ghys quasimorphisms coming from the fundamental group \( P_1(M) \), see [7]. Note that none of these quasimorphisms are continuous in the Hofer metric by a recent result by Khanevsky [37].

By a theorem of Ishida [34] in the genus zero case and its generalisation to any compact oriented surface [12, Theorem 2.2] (see all well Brandenbursky-Kedra-Shelukhin [11] in the genus one case, and Brandenbursky [3] in the higher genus case), the image of the map \( Q(P_n(M)) \to Q(G) \) is infinite dimensional for \( n \geq 4 \). Thus \( Q(G) \) is an infinite-dimensional vector space. Hence by Corollary 1 we obtain in particular the following.

**Corollary 2.** The \( L^1 \)-diameter of \( G \) is infinite.

Let \( Ent^k \) be the set of products of at most \( k \) entropy zero diffeomorphisms. Theorem 1 in [12] and Corollary 1 imply the following.

**Corollary 3.** For each positive integer \( k \), the complement in \( G \) of the set \( Ent^k \) contains a ball of any arbitrarily large radius in the \( L^1 \)-metric. In particular, the set of non-autonomous diffeomorphisms contains a ball of any arbitrarily large radius in the \( L^1 \)-metric.

In what follows, we apply an argument of Kim-Koberda [39] (cf. Benaim-Gambaudo [4] and Crisp-Wiest [18]), and use Theorem 1 in order to obtain the following statement, which generalizes an answer to a question of Kapovich [35] in the case of \( S^2 \) to arbitrary compact oriented surfaces.
Corollary 4. The metric group \((G, d_1)\) admits a quasi-isometric group embedding from each right-angled Artin group endowed with the word metric.

Proof. Assume first that \(M = \Sigma_g\) a closed surface of a positive genus. The case of \(S^2\) was already done in \([14]\) and the case of surfaces with boundary will be discussed in the end of the proof. Let \(n \in \mathbb{N}\). Let \(D \subset \Sigma_g\) be a smoothly embedded open disc and \(z_1, \ldots, z_n\) disjoint points in \(D\). For each \(i\), let \(D_i \subset D\) be an embedded open disc centered at \(z_i\) such that each short geodesic between two points in \(D_i\) lies in \(D_i\), and \(D_i \cap D_j = \emptyset\) for each \(i \neq j\). We denote by \(\text{Diff}(D; D_1, \ldots, D_n) < G\) the group which consists of all diffeomorphisms in \(G\) which are compactly supported in \(D\) and act by identity on each \(D_i\). This subgroup is equipped with the \(L^1\)-metric coming from \(G\).

Lemma 1. Let \(n > 4\). We identify \(P_n\) with the pure mapping class group of the disc \(D\) punctured at \(z_1, \ldots, z_n\). Then the inclusion \(P_n \rightarrow P_n(\Sigma_g)\) is a quasi-isometric embedding.

Proof. Since \(P_n\) is quasi-isometric to \(B_n\) and \(P_n(\Sigma_g)\) is quasi-isometric to \(B_n(\Sigma_g)\), it is enough to show that the inclusion \(B_n \rightarrow B_n(\Sigma_g)\) is a quasi-isometric embedding, where \(B_n\) is identified with the mapping class group of the disc \(D\) punctured at \(z_1, \ldots, z_n\). By results of Goldberg \([30]\) (c.f. Birman [5]) we have that \(B_n\) is a subgroup of \(B_n(\Sigma_g)\). In addition, the composition

\[
B_n \xrightarrow{i} B_n(\Sigma_g) \xrightarrow{F} MCG_{g,n}
\]

is injective, where \(i\) is the inclusion, the map \(F : B_n(\Sigma_g) \rightarrow MCG_{g,n}\) is the point pushing map and \(MCG_{g,n}\) is the mapping class group of \(\Sigma_g\) punctured at \(z_1, \ldots, z_n\). Moreover, by a result of Hameedastd [32] Theorem 2] the map \(F \circ i : B_n \rightarrow MCG_{g,n}\) is a quasi-isometric embedding for \(n > 4\). Since \(F\) is Lipschitz, it follows follows that \(i\) is a quasi-isometric embedding (see [10] Lemma 2.1). \(\Box\)

Let us return to the proof of the corollary. Consider the homomorphism

\[
H : \text{Diff}(D; D_1, \ldots, D_n) \rightarrow P_n < P_n(\Sigma_g),
\]

where \(H(\phi)\) is an element in \(P_n(\Sigma_g)\) represented by the path \((\phi_t(z_1), \ldots, \phi_t(z_n))\) in the configuration space, and \(\{\phi_t\}\) is any isotopy in \(\mathcal{G}\) between the identity \(\phi_0\) and \(\phi := \phi_1\).

It follows from the results of Kim-Koberda [39] (c.f. Crisp-Wiest [18] and Benaim-Gambaudo [4]) that for each RAAG \(\Gamma\) there exists \(n\), which can be assumed to satisfy \(n > 4\), and an embedding of \(\Gamma\) into \(\text{Diff}(D; D_1, \ldots, D_n)\) whose composition with \(H\) is a quasi-isometric embedding of \(\Gamma\) into \(P_n\). By Lemma [1] it gives as well a quasi-isometric embedding of \(\Gamma\) into \(P_n(\Sigma_g)\). Then in order to obtain a quasi-isometric embedding of \(\Gamma\) into \(G\) it is enough to show that the map \(H\) is large-scale Lipschitz whenever \(n > 4\).

This fact follows from our main result, Theorem [4] Let us prove it. Set

\[
X(D_1, \ldots, D_n) := \{(x_1, \ldots, x_n) \in X_n(\Sigma_g)| x_i \in D_i\}.
\]

Let \(\{\phi_t\}\) be any isotopy in \(\mathcal{G}\) between the identity \(\phi_0 = 1\) and \(\phi := \phi_1\). It follows from Theorem [4] that there exist positive constants \(A\) and \(B\) which depend only on \(n\) such that

\[
\int_{X(D_1, \ldots, D_n)} |\lambda(x, \phi)||P_n(\Sigma_g)| d\mu^{\otimes n}(x) \leq A \cdot l_1(\phi) + B.
\]

Note that for each \(x \in X(D_1, \ldots, D_n)\) we have \(|\lambda(x, \phi)| = H(\phi)\). Thus

\[
\int_{X(D_1, \ldots, D_n)} |\lambda(x, \phi)||P_n(\Sigma_g)| d\mu^{\otimes n}(x) = \text{vol}(X(D_1, \ldots, D_n)) \cdot |H(\phi)|_{P_n(\Sigma_g)},
\]

\[
\frac{\text{vol}(X(D_1, \ldots, D_n))}{|H(\phi)|_{P_n(\Sigma_g)}} \leq A \cdot l_1(\phi) + B.
\]
and we obtain the result, i.e., $H$ is large-scale Lipschitz:

$$|H(\phi)|_{\mathcal{P}_n(\Sigma_n)} \leq \frac{A}{\text{vol}(X(D_1, \ldots, D_n))} \cdot l_1(\phi) + \frac{B}{\text{vol}(X(D_1, \ldots, D_n))}.$$  

Assume now that $M_1$ is a surface with boundary. We can embed $M_1$ in a closed surface $M_2 = \Sigma_g$ such that the embedding is area preserving. This induces a monomorphism $\iota : \text{Diff}_{c,0}(M_1, \sigma_1) \to \text{Diff}_{c,0}(M_2, \sigma_2)$ (note that elements of $\text{Diff}_{c,0}(M_1, \sigma_1)$ fix point-wise the neighbourhood of the boundary, so they can be extended by the identity). Since $\iota$ is Lipschitz, and the embedding of $\Gamma$ to $\text{Diff}_{c,0}(M_2, \sigma_2)$ can be arranged to factor through $\text{Diff}_{c,0}(M_1, \sigma_1)$, we get a quasi-isometric embedding to $\text{Diff}_{c,0}(M_1, \sigma_1)$ (see [10] Lemma 2.1).

**Remark 1.3.2.** We note that Corollary 4 implies Corollary 2 providing the latter with a proof that does not use quasimorphisms.

Furthermore, note that Corollary 4 implies that for each $k \in \mathbb{N}$ there is a quasi-isometric embedding $i_k : \mathbb{Z}^k \to (G, d_1)$. Moreover, there exists a $k$-tuple of autonomous Hamiltonian flows (one-parameter subgroups) $\{\phi_t \}_{t \in \mathbb{R}}$ which have disjoint supports, and $i_k(e_j) = \phi_j^1$ for each $1 \leq j \leq k$, where $e_j = (0, \ldots, 1, \ldots, 0)$ and $1$ lies in the $j$-th entry. The above $k$-tuple of flows defines a homomorphism $j_k : \mathbb{R}^k \to (G, d_1)$ such that $i_k = j_k|_{\mathbb{Z}^k}$. Since $\mathbb{R}^k$ is quasi-isometric to $\mathbb{Z}^k$, we obtain the following statement:

**Corollary 5.** The metric group $(G, d_1)$ admits a quasi-isometric embedding from $(\mathbb{R}^k, d)$ where $d$ is any metric on $\mathbb{R}^k$ induced by a vector-space norm.

The following lemma allows us to prove that the quasi-isometric embeddings of $\mathbb{Z}^k$, $\mathbb{R}^k$ from Corollary 5 are in fact bi-Lipschitz embeddings.

**Lemma 2.** Let $(G, d)$ be a metric group and let $A \subset V$ be a subgroup of a normed linear space. Then each Lipschitz quasi-isometric embedding $j : A \to G$ of metric groups is a bi-Lipschitz embedding. If $A$ is discrete and finitely generated then each homomorphism $j : A \to G$ is Lipschitz.

**Proof.** Let $j : A \to G$ be a homomorphism. If $A$ is discrete and finitely generated, then $A \cong \mathbb{Z}^l$ for $l \in \mathbb{Z}_{>0}$, the norm being equivalent to the standard norm on $\mathbb{Z}^l$. The upper bound $d(j(x), 1) \leq C|x|$ for all $x \in A$ is now immediate. Hence $j$ is Lipschitz.

Suppose now that $A \subset V$ is a subgroup and $j : A \to G$ is a Lipschitz quasi-isometric embedding. Let $C \geq 1$ be such that

$$\frac{1}{C}|x| - B \leq d(j(x), 1) \leq C|x| \quad \text{(1)}$$

for a constant $B \geq 0$ and all $x \in A$. We claim that the inequality (1) holds with $B = 0$. Indeed, consider (1) for $x^m$ where $m \in \mathbb{Z}_{>0}$. We have

$$\frac{1}{C}|m|x| - B \leq d(j(x^m), 1) = d(j(x)^m, 1) \leq m \cdot d(j(x), 1),$$

where the last inequality is due to the right-invariance of $d$. Dividing by $m$ yields

$$\frac{1}{C}|x| - B/m \leq d(j(x), 1),$$

which finishes the proof by taking limits as $m \to \infty$.

This immediately implies the following strengthening of Corollary 5 since $j_k$ is evidently Lipschitz.
**Corollary 6.** The metric group \((G, d_1)\) admits a bi-Lipschitz embedding from \((\mathbb{R}^k, d)\) where \(d\) is any metric on \(\mathbb{R}^k\) induced by a vector-space norm.

**Remark 1.3.3.** It should be possible to prove Corollary 6 by using quasimorphisms, as in [14].

**Remark 1.3.4.** Let \(p \geq 1\). Note that since, by Jensen's (or Hölder's) inequality,

\[
d_1 \leq d_p,
\]

all the above results for \(d_1\) continue to hold for \(d_p\).

Recall that if \(M\) has boundary then we assume that elements of \(\text{Diff}_{c,0}(M, \sigma_1)\) fix point-wise an open neighbourhood of \(\partial M\). Thus if \((M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)\) is a measure preserving embedding of manifolds, then extending a diffeomorphism on \(M_1\) by the identity to a diffeomorphism of \(M_2\) gives a well defined monomorphism \(\iota: \text{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \text{Diff}_{c,0}(M_2, \sigma_2)\).

We finish this section with a question.

**Question 1.** Let \((M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)\) be a measure preserving embedding of surfaces. Is the monomorphism \(\iota: \text{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \text{Diff}_{c,0}(M_2, \sigma_2)\) a quasi-isometric embedding?

**Remark 1.3.5.** Note that this question is motivated by our results since the statement holds for the compositions of \(\iota\) with the embeddings of right-angled Artin groups and \((\mathbb{R}^k, d)\) from the proofs of Corollary 4 and Corollary 6.

### 1.4 Outline of the proof

In order to show Theorem 1, we define a Riemannian metric \(g\) on \(X_n(M)\) such that \(d_g\), the geodesic metric on \(X_n(M)\) induced by \(g\), extends to the geodesic metric on the compactification \(\overline{X}_n(M)\). This allows us to use the Schwarz-Milnor lemma.

More precisely, Theorem 1 is a consequence of the following two propositions (we defer the proofs to Section 2).

**Proposition 1.1.** Let \(\lambda\) be a piecewise \(C^1\) loop in \(X_n(M)\) based at \(q\). Let \(S\) be a finite generating set of \(P_n(M)\). The word norm of the class \([\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)\) with respect to \(S\) satisfies

\[
||[\lambda]|_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,
\]

for constants \(A_0, B_0 > 0\) depending only on \(S\) and \(n\).

The next proposition says that the average length of loops \(\lambda(x, \phi)\) is controlled by \(l_1(\phi)\).

**Proposition 1.2.** Let \(\phi = \{\phi_t\}\) be an isotopy in \(G\) such that \(\phi_0 = 1\). There exist constants \(A_1, B_1 > 0\) depending only on \(n\), such that

\[
\int_{X_n(M) \setminus \bigcup \phi_t^{-1}(Z)} l_g(\lambda(x, \phi_t)) d\mu^{\otimes n}(x) \leq A_1 \cdot l_1(\phi) + B_1.
\]

To show this proposition, we first compare a metric \(g\) to an auxiliary metric \(g_0\). The metric \(g_0\) does not extend to a metric on the compactification \(\overline{X}_n(M)\), but it is relatively easy to show [53, Lemma 5.2], that the inequality from Proposition 1.2 holds for \(g_0\).
2 Proofs.

2.1 Compactification of the configuration space

In this section $M$ is a compact $m$-dimensional manifold. Below we describe the compactification of $X_n(M)$ using an embedding of the configuration space into a high-dimensional Euclidean space. We follow closely the construction given in [52].

Let us start with describing a family of maps on $X_n(\mathbb{R}^d)$. Let $\pi_{ij}: X_n(\mathbb{R}^d) \to S^{d-1}$ be defined by the formula

$$\pi_{ij}(x) = \frac{x_i - x_j}{|x_i - x_j|},$$

where $0 \leq i < j \leq n$ and $x = (x_1, \ldots, x_n) \in X_n(\mathbb{R}^d)$. Let $s_{ijk}: X_n(\mathbb{R}^d) \to [0, \infty]$ be defined by the formula

$$s_{ijk}(x) = \frac{|x_i - x_j|}{|x_i - x_k|},$$

where $0 \leq i < j < k \leq n$ and $[0, \infty]$ is the one point compactification of $[0, \infty)$. Let $i: X_n(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ be the standard inclusion and let $A_n[\mathbb{R}^d] = (\mathbb{R}^d)^n \times (S^{n-1})^n \times [0, \infty]^n$. We consider the embedding of $X_n(\mathbb{R}^d)$ to the ambient space $A_n[\mathbb{R}^d]$ given by the product map

$$\alpha_n = i \times (\pi_{ij}) \times (s_{ijk}): X_n(\mathbb{R}^d) \to A_n[\mathbb{R}^d].$$

Suppose $M$ is a submanifold of $\mathbb{R}^d$. Let $\overline{X}_n(M)$ be the closure of the image $\alpha_n(X_n(M))$ in $A_n[\mathbb{R}^d]$. Then $\overline{X}_n(M)$ is a manifold with boundary, and the inclusion $X_n(M) \hookrightarrow \overline{X}_n(M)$ is a homotopy equivalence [52, Theorem 4.4 and Corollary 4.5].

2.2 Two metrics on $X_n(M)$ and proof of Proposition 1.1

On $[0, \infty]$ we introduce the structure of a smooth manifold by a 1-map atlas $e^{-x}: [0, \infty] \to [0, 1]$. Let $g_{\text{exp}}$ be the Riemannian metric on $[0, \infty]$ given by the pull-back of the Euclidean metric from $[0, 1]$. In particular, we have that $\langle \partial_x | g_{\text{exp}} \rangle = e^{-x}$, where $\partial_x$ is a standard 'unit' vector field on $[0, \infty)$.

Let $\text{euc}$ be the metric on $A_n[\mathbb{R}^d]$ given by the product of standard metrics on $\mathbb{R}^d$, $S^{d-1}$ and $([0, \infty], g_{\text{exp}})$. The first metric on $X_n(\mathbb{R}^d)$ we want to consider is defined to be the pull-back of $\text{euc}$ to $X_n(\mathbb{R}^d)$ by the map $\alpha_n$:

$$g = \alpha_n^*(\text{euc}).$$

Now, since we regard $M$ as a submanifold of $\mathbb{R}^d$, $g$ induces a Riemannian metric on $X_n(M)$.

The second metric is defined as follows. Let $x = (x_1, \ldots, x_n) \in X_n(\mathbb{R}^d)$. Let $d(x)$ denote the minimal distance between the points in $x$, that is:

$$d(x) = \min\{|x_i - x_j| : i \neq j\},$$

where $|x - y|$ denotes the standard Euclidean distance between vectors $x, y \in \mathbb{R}^d$.

Note that on $X_n(\mathbb{R}^d)$ we have the Euclidean metric restricted from $(\mathbb{R}^d)^n$. We rescale this metric by the factor $\frac{1}{n}$, i.e. we define a metric $g_0$ on $X_n(\mathbb{R}^d)$ by

$$|v|_{g_0} = \frac{|v|}{d(x)},$$

where $v \in T_x(X_n(\mathbb{R}^d)) = (\mathbb{R}^d)^n$ and $|v|$ is the Euclidean length of $v \in (\mathbb{R}^d)^n$. One should note that $d(x)$, and consequently $g_0$, is continuous, but not differentiable. A manifold with such a metric is
called a $C^0$-Riemannian manifold. On a $C^0$-Riemannian manifold one defines the lengths of paths and a geodesic metric in the same way as in the smooth case. Again, $g_0$ restricts to a $C^0$-Riemannian metric on $X_n(M)$.

**Proof of Proposition 1.1.** We need to show that

$$||\lambda||_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,$$

where $\lambda$ be a piecewise $C^1$ loop in $X_n(M)$ based at $q$, $[\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)$ and $||\lambda||_{P_n(M)}$ is the word norm of $[\lambda]$.

By [22], Theorem 4.4, $X_n(M)$ is a manifold with corners and $X_n(M)$ is its interior. Moreover, the coordinate charts on $X_n(M)$ are defined in such a way that the embedding of $X_n(M)$ into $A_n[\mathbb{R}^d]$ is smooth. Thus one can restrict the metric $euc$ to $X_n(M)$. In other words, $g = \alpha_*^n(euc)$ extends to the compactification $\overline{X}_n(M)$. Now the proposition follows directly from the Schwarz-Milnor lemma [22] [34] [44]. More precisely, consider the following formulation of this result [15], Proposition 8.19.

**Lemma 3.** Let a group $\Gamma$ act properly and discontinuously by isometries on a proper length space $X$. If the action is cocompact, then $\Gamma$ is finitely generated and for any choice of base-point $x_0 \in X$, the map $\Gamma \to X$, $h \mapsto h \cdot x_0$, is a quasi-isometry.

Denote by $g_c$ the metric on $X_n(M)$ obtained from $euc$. We apply Lemma 3 to $X = \overline{X}_n(M)$, $\Gamma = \pi_1(X_n(M), \pi(x_0)) \cong \pi_1(X_n(M), \pi(x_0))$, and the length space structure on $X$ being induced by the lift $\tilde{g}_c = \pi^* g_c$ to $X$ of $g_c$ on $K = X/\Gamma = \overline{X}_n(M)$ by the natural projection map $\pi : X \to K$. Note that $d_{\tilde{g}_c}(x_0, h \cdot x_0) \leq l_{g_c}(\lambda)$ where $\lambda$ is any $C^1$ loop in $X_n(M)$ based at $\pi(x_0)$ representing the element $h$. Moreover, since $g_c$ is an extension of $g$, we have $l_{\tilde{g}_c}(\lambda) = l_g(\lambda)$.

### 2.3 Proof of Proposition 1.2

We begin with the proof of the following technical result.

**Lemma 4.** There exists $C > 0$ depending only on $n$, such that $|v|_g \leq C \cdot |v|_{g_0}$ for every $v \in T(X_n(M))$.

**Proof.** Since $\alpha_n$ is a product map, in order to get a bound on the norm $|v|_g$, we need to bound the norms of the vectors $D\pi_{ij}(v)$ and $D\delta_{ijk}(v)$. Let $x = (x_1, \ldots, x_n) \in X_n(M)$ and let $v = (v_1, \ldots, v_n) \in T_x(X_n(M))$. There exists $\epsilon > 0$ such that $x + tv \in X_n(\mathbb{R}^d)$ for every $t < \epsilon$. Note that even though the path $(x + tv)_{t \in [0, \epsilon)}$ is not contained in $X_n(M)$, it still represents the tangent vector $v \in T_x(X_n(M)) < T_x(X_n(\mathbb{R}^d))$. We have

$$D\pi_{ij}(v) = \frac{d}{dt}_{|t=0} \pi_{ij}(x + tv) = \frac{d}{dt}_{|t=0} \frac{x_i - x_j + t(v_i - v_j)}{|x_i - x_j + t(v_i - v_j)|} = \frac{v_i - v_j}{|x_i - x_j|} \cdot \frac{(x_i - x_j)(v_i - v_j, x_i - x_j)}{|x_i - x_j|^3}.$$

By definition $D\pi_{ij}(v)$ is a vector in $\mathbb{R}^d$ tangent to a $(d - 1)$-dimensional sphere at point $\pi_{ij}(x)$. By $|D\pi_{ij}(v)|$ we denote its Euclidean length.
Applying the inequalities \( \langle v_i - v_j, x_i - x_j \rangle \leq |v_i - v_j||x_i - x_j| \) and \(|v_i - v_j| \leq |v_i| + |v_j|\), we obtain:

\[
|D\pi_{ij}(v)| \leq 2 \frac{1}{|x_i - x_j|}(|v_i| + |v_j|) \\
\leq 2 \frac{1}{d(x)}(|v_i| + |v_j|) \\
\leq 2 \frac{1}{d(x)}(\sum_i |v_i|).
\]

Similarly, for \( s_{ijk} \) we compute (already in the map):

\[
De^{-s_{ijk}}(v) = \frac{d}{dt}|_{t=0} e^{-s_{ijk}(x+tv)} \\
= \frac{d}{dt}|_{t=0} e^{-\frac{|x_i-x_j+(v_i-v_j)|}{|x_i-x_k+(v_i-v_k)|}} \\
= \left[ \frac{\langle v_i - v_j, x_i - x_j \rangle}{|x_i - x_j|} - \frac{|x_i - x_j||x_i - x_k, v_i - v_k|}{|x_i - x_k|^3} \right] e^{-\frac{|x_i-x_j|}{|x_i-x_k|}}.
\]

By definition \( |Ds_{ijk}(v)|_{g_{exp}} = |De^{-s_{ijk}}(v)| \). Using similar inequalities as before we get:

\[
|Ds_{ijk}(v)|_{g_{exp}} \leq \left[ \frac{|v_i| + |v_j| + (|v_i| + |v_k|)|x_i - x_j|}{|x_i - x_k|} \right] \frac{|x_i - x_k|}{|x_i - x_j|} \\
\leq \frac{1}{d(x)}(2|v_i| + |v_j| + |v_k|) \\
\leq 2 \frac{1}{d(x)}(\sum_i |v_i|).
\]

Finally, since \( \sum_i |v_i| \leq \sqrt{n}|v| \), we get:

\[
|D\pi_{ij}(v)| \leq \frac{\sqrt{n}}{d(x)}|v|,
\]

\[
|Ds_{ijk}(v)|_{g_{exp}} \leq 2 \frac{\sqrt{n}}{d(x)}|v|.
\]

We assume that \( M \) is compact, therefore there exists \( A > 0 \) such that \( d(x) \leq A \) for every \( x \in X_u(M) \).
Now we can bound the pulled-back metric \( |\cdot|_{g'} \):

\[
|v|^2_g = |Dv|^2 + \sum_{i,j} |D\pi_{ij}(v)|^2 + \sum_{i,j,k} |Ds_{ijk}(v)|_{g_{exp}}^2 \\
\leq 2 \frac{A^2}{d(x)^2}|v|^2 + \sum_{i,j} \frac{n}{d(x)^2} |v|^2 + \sum_{i,j,k} \frac{4n}{d(x)^2} |v|^2 \\
= C' \frac{|v|^2}{d(x)^2} = C' \frac{|v|^2}{g_0},
\]

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for \( C = A^2 + n\binom{n}{2} + 4n\binom{n}{3} \).

\[ \square \]

Remark 2.3.1. In the proof of Lemma 4, the inequality \( e^{-x} < \frac{1}{x} \) is used to show the bound on the length of \( Ds_{ijk} \). Indeed, the choice of the function \( e^{-x} : [0, \infty) \to [0, 1] \) to define a smooth structure and the metric on \([0, \infty]\) is not completely arbitrary. For a different identification, e.g. \( \ln(\frac{x+y}{x+y+1}) : [0, \infty) \to [0, 1] \), we get a different smooth structure on \([0, \infty]\) (in the sense, that the identity map is not smooth) and a metric which is not equivalent to \( \alpha_{exp} \). Then after pull-back by \( \alpha_{n} \) we get a metric on \( X_{n}(M) \) which is not equivalent to \( g \) and for this metric Lemma 4 might not hold.

**Proof of Proposition 1.2.** We need to show that

\[
\int_{X_{n}(M) \cup \{ \phi_1 \}^{-1}(Z)} l_{g}(\lambda(x, \overline{\phi})) \mu_{\otimes^n}(x) \leq A_{1} \cdot l_{1}(\overline{\phi}) + B_{1},
\]

where \( \overline{\phi} = \{ \phi_{t} \} \) is an isotopy in \( G \) such that \( \phi_{0} = 1 \).

A similar inequality was proven in [K] for a metric which is equivalent to \( g_{0} \). Let us first describe this metric. By \( d_{M} \) denote the geodesic metric on \( M \) induced by the restriction of the standard Riemannian metric on \( \mathbb{R}^{d} \). Let \( x \in X_{n}(M) \) and \( v \in T_{x}(X_{n}(M)) \). We define

\[
|v|_{g_{0}} = \frac{|v|}{d_{M}(x)},
\]

where \( d_{M}(x) = \min\{ d_{M}(x_{i}, x_{j}) : i \neq j \} \) and \( |v| \) is the Euclidean length of \( v \) seen as a vector in \( (\mathbb{R}^{d})^{n} \).

The difference between the function \( d \) used to define \( g_{0} \) and \( d_{M} \) is that in \( d \) we measure the distance between points \( x_{i} \) and \( x_{j} \) in the ambient space \( \mathbb{R}^{d} \) and in \( d_{M} \) inside \( M \). Since \( M \) is compact, clearly \( d \) and \( d_{M} \) are equivalent and consequently \( g_{b} \) and \( g_{0} \) are equivalent.

It follows from [K] Lemma 5.2] that

\[
\int_{X_{n}(M) \cup \{ \phi_1 \}^{-1}(Z)} l_{g_{0}}(\{ \phi_{t} \cdot x \}) \mu_{\otimes^n}(x) \leq C' \cdot l_{1}(\overline{\phi}).
\]

Recall that \( \{ \phi_{t} \} \) is a path in \( X_{n}(M) \) (\( \phi_{t} \) acts on \( x \in X_{n}(M) \) component-wise). Since \( g_{0} \) and \( g_{b} \) are equivalent, this inequality holds as well for \( g_{0} \) (with a possibly different constant).

Let us now focus on the metric \( g \). Since the geodesic metric \( d_{g} \) defined by \( g \) extends to the compactification of \( X_{n}(M) \), the diameter of \( (X_{n}(M), d_{g}) \) is finite. We can choose the system of short paths \( \gamma(x) \) such that \( l_{g}(\gamma(x)) \leq D \) for some \( D > 0 \) and every \( x \in X_{n}(M) \setminus Z \).

Thus for every \( x \in X_{n}(M) \setminus Z \cup \{ \phi_1 \}^{-1}(Z) \) we have

\[
l_{g}(\lambda(x, \overline{\phi})) \leq l_{g}(\gamma(x)) + l_{g}(\{ \phi_{t} \cdot x \}) + l_{g}(\gamma(\phi_{1}(x))) \leq 2D + l_{g}(\{ \phi_{t} \cdot x \}).
\]

Finally, due to Lemma 4 we have \( l_{g}(\{ \phi_{t} \cdot x \}) \leq C' \cdot l_{g_{0}}(\{ \phi_{t} \cdot x \}) \) and the proposition follows. \[ \square \]

**References**

[1] Vladimir I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits*, Ann. Inst. Fourier 16 (1966), 319–361.

[2] Vladimir I. Arnold and Boris A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer-Verlag, 1998.

[3] Scott Axelrod and I. M. Singer, *Chern-Simons perturbation theory. II*, J. Differential Geom. 39 (1994), no. 1, 173–213.
[4] Michel Benaim and Jean-Marc Gambaudo, *Metric properties of the group of area preserving diffeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), no. 11, 4661–4672.

[5] Joan S. Birman, *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. **22** (1969), 213–238. MR 0243519 (39 #4840)

[6] Michael Brandenbursky, *On quasi-morphisms from knot and braid invariants*, J. Knot Theory Ramifications **20** (2011), no. 10, 1397–1417.

[7] ______, *Quasi-morphisms and $L^p$-metrics on groups of volume-preserving diffeomorphisms*, J. Topol. Anal. **4** (2012), no. 2, 255–270.

[8] Michael Brandenbursky, *Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces*, Internat. J. Math. **26** (2015), no. 9, 1550066, 29 pages. MR 3391653

[9] Michael Brandenbursky and Jarek Kędra, *On the autonomous norm on the group of area-preserving diffeomorphisms of the 2-disc*, Algebr. Geom. Topol. **13** (2013), 795–816.

[10] ______, *Quasi-isometric embeddings into diffeomorphism groups*, Groups, Geometry and Dynamics **7** (2013), no. 3, 523–534.

[11] Michael Brandenbursky, Jarek Kędra, and Egor Shelukhin, *On the autonomous norm on the group of Hamiltonian diffeomorphisms of the torus*, Commun. Contemp. Math. **20** (2018), no. 2, 1750042, 27. MR 3730755

[12] Michael Brandenbursky and Michał Marcinkowski, *Entropy and quasimorphisms*, J. Mod. Dyn. **15** (2019), 143–163. MR 3983639

[13] Michael Brandenbursky and Egor Shelukhin, *On the $L^p$-geometry of autonomous Hamiltonian diffeomorphisms of surfaces*, Math. Res. Lett. **22** (2015), no. 5, 1275–1294.

[14] ______, *The $L^p$-diameter of the group of area-preserving diffeomorphisms of $S^2$*, Geom. Topol. **21** (2017), no. 6, 3785–3810.

[15] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[16] Eugenio Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, 1970, pp. 1–26.

[17] Danny Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009.

[18] John Crisp and Bert Wiest, *Quasi-isometrically embedded subgroups of braid and diffeomorphism groups*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5485–5503.

[19] D. G. Ebin, G. Misiolek, and S. C. Preston, *Singularities of the exponential map on the volume-preserving diffeomorphism group*, Geom. Funct. Anal. **16** (2006), no. 4, 850–868.

[20] David G. Ebin, *Geodesics on the symplectomorphism group*, Geom. Funct. Anal. **22** (2012), no. 1, 202–212.

[21] David G. Ebin and Jerrold Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102–163.

[22] V. Efremović, *The proximity geometry of Riemannian manifolds*, Uspehi Mat. Nauk **8** (1953), no. 189.

[23] Yakov Eliashberg and Tudor Ratiu, *The diameter of the symplectomorphism group is infinite*, Invent. Math. **103** (1991), no. 2, 327–340.
[24] Albert Fathi, *Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux*, Ph.D. thesis, Orsay, 1980.

[25] William Fulton and Robert MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) 139 (1994), no. 1, 183–225.

[26] J.-M. Gambaudo and E. E. Pécou, *Dynamical cocycles with values in the Artin braid group*, Ergodic Theory Dynam. Systems 19 (1999), no. 3, 627–641.

[27] Jean-Marc Gambaudo and Étienne Ghys, *Enlacements asymptotiques*, Topology 36 (1997), no. 6, 1355–1379.

[28] ———, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1591–1617.

[29] Jean-Marc Gambaudo and Maxime Lagrange, *Topological lower bounds on the distance between area preserving diffeomorphisms*, Bol. Soc. Brasil. Mat. (N.S.) 31 (2000), no. 1, 9–27.

[30] Charles H. Goldberg, *An exact sequence of braid groups*, Math. Scand. 33 (1973), 69–82. MR 334182

[31] Peter Haïssinsky, *L’invariant de Calabi pour les homéomorphismes quasiconformes du disque*, C. R. Math. Acad. Sci. Paris 334 (2002), no. 8, 635–638.

[32] Ursula Hamenstadt, *Geometry of the mapping class group II: A biautomatic structure*, Preprint arXiv:0912.0137, 2009.

[33] Vincent Humilière, *The Calabi invariant for some groups of homeomorphisms*, J. Symplectic Geom. 9 (2011), no. 1, 107–117.

[34] Tomohiko Ishida, *Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk via braid groups*, Proc. Amer. Math. Soc. Ser. B 1 (2014), 43–51.

[35] Michael Kapovich, *RAAGs in Ham*, Geom. Funct. Anal. 22 (2012), no. 3, 733–755.

[36] Christian Kassel and Vladimir Turaev, *Braid groups*, Graduate Texts in Mathematics, vol. 247, Springer, New York.

[37] Michael Khanevsky, *Quasimorphisms on surfaces and continuity in the Hofer norm*, Preprint, arXiv:1906.08429, 2019.

[38] Boris Khesin and Robert Wendt, *The geometry of infinite-dimensional groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 51, Springer-Verlag, Berlin, 2009.

[39] Sang-hyun Kim and Thomas Koberda, *Anti-trees and right-angled Artin subgroups of braid groups*, Geom. Topol. 19 (2015), no. 6, 3289–3306.

[40] Maxim Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97–121.

[41] ———, *Operads and motives in deformation quantization*, vol. 48, 1999, Moshé Flato (1937–1998), pp. 35–72.

[42] Alexander Kupers, *There is no topological Fulton-MacPherson compactification*, Preprint, arXiv:2011.14855, 2020.

[43] Michal Marcinkowski, *A short proof that the L^p-diameter of Diff_0(S, area) is infinite*, Preprint arXiv:2010.03000, 2020.

[44] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry 2 (1968), 1–7.
[45] Leonid Polterovich, *Floer homology, dynamics and groups*, Morse theoretic methods in nonlinear analysis and in symplectic topology (Dordrecht), NATO Sci. Ser. II Math. Phys. Chem., vol. 217, Springer, 2006, pp. 417—-438.

[46] Pierre Py, *Quasi-morphisms de Calabi et graphe de Reeb sur le tore.*, C. R. Math. Acad. Sci. Paris 343 (2006), no. 5, 323—-328.

[47] , *Quasi-morphismes et invariant de Calabi*, Ann. Sci. E’cole Norm. Sup. (4) 39 (2006), no. 1, 177–195.

[48] Egor Shelukhin, “Enlacements asymptotiques” revisited, Ann. Math. Qué. 39 (2015), no. 2, 205–208.

[49] Alexander Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Mat. Sb. (N.S.) 128(170) (1985), no. 1, 82–109, 144.

[50] , *Generalized fluid flows, their approximation and applications*, Geom. Funct. Anal. 4 (1994), no. 5, 586–620.

[51] , *Microglobal analysis of the Euler equations*, J. Math. Fluid Mech. 7 (2005), no. suppl. 3, S387–S396.

[52] Dev P. Sinha, *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. (N.S.) 10 (2004), no. 3, 391–428.

[53] Steven Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. (1959), no. 10, 621–626.

[54] A. S. Švarc, *A volume invariant of coverings*, Doklady Akademii Nauk SSSR 105 (1955), 32–34.

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