Signed distance Laplacian matrices for signed graphs

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\section*{ABSTRACT}
A signed graph is a graph whose edges are labelled either positive or negative. Corresponding to the two signed distance matrices defined for signed graphs, we define two signed distance Laplacian matrices. We characterize singularity and calculate the rank of these matrices and find signed distance Laplacian spectra of some classes of unbalanced signed graphs. We derive most of these results by proving them more generally for weighted signed graphs.

\section*{1. Introduction}

Throughout this article, unless otherwise mentioned, by a graph we mean a finite, simple graph. For any terms which are not mentioned here, the reader may refer to [1].

A signed graph $\Sigma = (G, \sigma)$ is an underlying graph $G = (V, E)$ with a signature function $\sigma : E \rightarrow \{1, -1\}$. Two types of signed distance matrices were introduced by Hameed et al. in [2]. In this paper, we define two corresponding signed distance Laplacian matrices for signed graphs, generalizing the distance Laplacian matrix introduced by Aouchiche and Hansen [3] for unsigned graphs, and characterize singularity and rank of these matrices in terms of balance in signed graphs. (A signed graph is balanced if in every cycle the product of its edge signs is positive.) We also study the signed distance Laplacian spectra of some classes of unbalanced signed graphs.

Our work provides a wider context for some known results on distance Laplacians of unsigned graphs. For instance, the unsigned distance Laplacian matrix, which is the same as the matrix of an all positive signed graph, is always singular [3]. We show that this is due to the fact that an all positive signed graph is balanced, by generalizing it to a rank formula for signed-graph distance Laplacians (Theorem 3.5). However, we have not studied the
second largest eigenvalue, as was done in [4] for unsigned graphs; we leave that for future work.

We recall the definition of signed distances and corresponding signed distance matrices from [2]. Given a signed graph $\Sigma = (G, \sigma)$, the sign of a path $P$ in $\Sigma$ is defined as $\sigma(P) = \prod_{e \in E(P)} \sigma(e)$. A shortest path between two given vertices $u$ and $v$ is denoted by $P_{(u,v)}$ and the collection of all shortest paths $P_{(u,v)}$ by $\mathcal{P}_{(u,v)}$; and $d(u, v)$ denotes the usual distance between $u$ and $v$. In discussing distances we assume $\Sigma$ is connected.

**Definition 1.1 (Signed distance [2]):** Auxiliary signs are defined as:

(S1) $\sigma_{\max}(u, v) = -1$ if all shortest $uv$-paths are negative, and $+1$ otherwise.
(S2) $\sigma_{\min}(u, v) = +1$ if all shortest $uv$-paths are positive, and $-1$ otherwise.

Signed distances are:

(d1) $d_{\max}(u, v) = \sigma_{\max}(u, v)d(u, v) = \max\{\sigma(P_{(u,v)}): P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u, v)$.
(d2) $d_{\min}(u, v) = \sigma_{\min}(u, v)d(u, v) = \min\{\sigma(P_{(u,v)}): P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u, v)$.

And the signed distance matrices are:

(D1) $D_{\max}(\Sigma) = (d_{\max}(u, v))_{n \times n}$.
(D2) $D_{\min}(\Sigma) = (d_{\min}(u, v))_{n \times n}$.

**Definition 1.2 ([2]):** Two vertices $u$ and $v$ in a signed graph $\Sigma$ are said to be distance-compatible (briefly, compatible) if $d_{\min}(u, v) = d_{\max}(u, v)$. And $\Sigma$ is said to be (distance-) compatible if every two vertices are compatible. Then $D_{\max}(\Sigma) = D_{\min}(\Sigma) = D_{\pm}(\Sigma)$.

The two complete signed graphs from the distance matrices $D_{\max}$ and $D_{\min}$ are defined as follows.

**Definition 1.3 ([2]):** The associated signed complete graph $K_{D_{\max}}(\Sigma)$ with respect to $D_{\max}(\Sigma)$ is obtained by joining the non-adjacent vertices of $\Sigma$ with edges having signs $\sigma(\cdot \, \cdot) = \sigma_{\max}(\cdot \, \cdot)$.

The associated signed complete graph $K_{D_{\min}}(\Sigma)$ with respect to $D_{\min}(\Sigma)$ is obtained by joining the non-adjacent vertices of $\Sigma$ with edges having signs $\sigma(\cdot \, \cdot) = \sigma_{\min}(\cdot \, \cdot)$.

The distance Laplacian matrix of a graph was introduced and studied by Aouchiche and Hansen in [3]. We introduce the generalization to signed graphs, which is two signed distance Laplacian matrices for signed graphs, as follows.

The transmission $tr(v)$ of a vertex $v$ is defined to be the sum of the distances from $v$ to all other vertices in $G$. That is, $tr(v) = \sum_{u \in V(G)} d(v, u)$. The transmission matrix $Tr(G)$ for a graph is the diagonal matrix with diagonal entries $tr(v_i)$. 
Definition 1.4: Now we define two signed distance Laplacian matrices for signed graphs as

\[
DL^\text{max}(\Sigma) = \text{Tr}(G) - D^\text{max}(\Sigma), \\
DL^\text{min}(\Sigma) = \text{Tr}(G) - D^\text{min}(\Sigma).
\]

When \(\Sigma\) is compatible, \(DL^\text{max}(\Sigma) = DL^\text{min}(\Sigma) = DL^\pm(\Sigma)\).

2. Weighted signed graphs

To study balance in signed graphs using signed distance Laplacian matrices, it is necessary to study the Laplacian matrix of weighted signed graphs. We prove three main theorems about those matrices: first, the standard type of expression for a Laplacian in terms of an incidence matrix, and then, an expression for the determinant and a characterization of singularity. We apply them to signed distance Laplacians in the following section.

We use the notation \(u \sim v\) when the vertices \(u\) and \(v\) are adjacent and similar notation for the incidence of an edge on a vertex.

Definition 2.1: A weighted signed graph \((\Sigma, w)\) consists of a signed graph \(\Sigma = (G, \sigma)\) and a positive weight function \(w\) defined on the edges of \(\Sigma\). Its adjacency matrix \(A(\Sigma, w) = (a_{ij})_n\) is defined as the square matrix of order \(n = |V(G)|\) where

\[
a_{ij} = \begin{cases} 
\sigma(v_i, v_j)w(v_i, v_j) & \text{if } v_i \sim v_j, \\
0 & \text{otherwise}.
\end{cases}
\]

For a weighted signed graph \((\Sigma, w), w(\Sigma)\) is the product of all the weights given to the edges of \(\Sigma\).

Definition 2.2: For a weighted signed graph \((\Sigma, w)\), its weighted Laplacian matrix is defined as \(L(\Sigma, w) = D(\Sigma, w) - A(\Sigma, w)\) where the diagonal matrix \(D(\Sigma, w)\) is \(\text{diag}(\sum_{e_{v_i \sim v_j}} w(e))\). The matrix \(D(\Sigma, w)\) is the weighted degree matrix of \((\Sigma, w)\).

The (oriented) incidence matrix of a weighted signed graph is based on that of a signed graph in [5]. To discuss the incidence matrix, we orient the edges (in an arbitrary but fixed way). For an oriented edge \(\vec{e}_j = \overrightarrow{v_i v_k}\), we take \(v_i\) as the tail of that edge and \(v_k\) as its head and we write \(t(\vec{e}_j) = v_i\) and \(h(\vec{e}_j) = v_k\). Since weights are positive real numbers we can take the square root \(\sqrt{w(e)}\), following the idea in [6].

Definition 2.3: Given a weighted signed graph \((\Sigma, w)\), its (oriented) weighted incidence matrix is defined as \(H(\Sigma, w) = (\eta_{v_i e_j})\) where

\[
\eta_{v_i e_j} = \begin{cases} 
\sigma(e_j)\sqrt{w(e_j)} & \text{if } t(\vec{e}_j) = v_i, \\
-\sqrt{w(e_j)} & \text{if } h(\vec{e}_j) = v_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(H^T(\Sigma, w) = (\eta'_{e_j v_i})\) be the transpose of the weighted incidence matrix \(H(\Sigma, w)\). Thus, \(\eta'_{e_j v_i} = \eta_{v_i e_j}\). We generalize the standard formula connecting the incidence and Laplacian matrices.
Theorem 2.4: For a weighted signed graph $(\Sigma, w)$, \(L(\Sigma, w) = H(\Sigma, w)H^T(\Sigma, w)\).

Proof: Let \(v_1, v_2, \ldots, v_n\) and \(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m\) be the vertices and edges in \(\Sigma\), respectively. The \((i,j)\)th entry of \(HH^T\) is \(\sum_{k=1}^n \eta_{v_i e_k} \eta_{e_k v_j}^\prime\).

For \(i = j\), we have \(\eta_{v_i e_k} \eta_{e_k v_j}^\prime = \eta_{v_i e_k}^2 \neq 0\) if and only if \(e_k\) is incident to \(v_i\). Then \(\eta_{v_i e_k} \eta_{e_k v_j}^\prime = (\pm \sqrt{w(e_k)})^2 = w(e_k)\). Thus, the diagonal entry in \(HH^T\) is \(\sum_{e_k \sim v} w(e)\).

For \(i \neq j\), we have \(\eta_{v_i e_k} \eta_{e_k v_j}^\prime \neq 0\) if and only if \(e_k\) is an edge joining \(v_i\) and \(v_j\). Then \(\eta_{v_i e_k} \eta_{e_k v_j}^\prime = -\sigma(e_k)w(e_k)\).

In both cases, the \((i,j)\)th entry of \(HH^T\) coincides with the \((i,j)\)th entry of \(L(\Sigma, w)\), hence the proof.

Lemma 2.5: For a weighted signed tree \((\Sigma, w)\), \(\det L(\Sigma, w) = 0\).

Proof: A tree on \(n\) vertices has \(n-1\) edges. Thus, \(H(\Sigma, w)\) is a matrix of order \(n \times (n-1)\) and hence \(L(\Sigma, w) = H(\Sigma, w)H^T(\Sigma, w)\) has rank less than \(n\). This implies \(\det L(\Sigma, w) = 0\).

Lemma 2.6: Let \((\Sigma, w)\) be a weighted signed graph where the underlying graph is a cycle \(C_n\) of order \(n\). Then \(\det L(\Sigma, w) = 2w(C_n)(1 - \sigma(C_n))\).

Proof: Let the cycle be \(C_n = v_1\tilde{e}_1v_2\tilde{e}_2v_3\tilde{e}_3 \cdots v_{n-1}\tilde{e}_{n-1}v_n\tilde{e}_n v_1\). The weighted incidence matrix \(H(\Sigma, w)\) is

\[
\begin{pmatrix}
\sigma(e_1)\sqrt{w(e_1)} & 0 & \cdots & 0 & -\sqrt{w(e_n)} \\
-\sqrt{w(e_1)} & \sigma(e_2)\sqrt{w(e_2)} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sigma(e_{n-1})\sqrt{w(e_{n-1})} & 0 \\
0 & 0 & \cdots & -\sqrt{w(e_{n-1})} & \sigma(e_n)\sqrt{w(e_n)}
\end{pmatrix}.
\]

Expanding along the first row to find the determinant we get

\[
\det H(\Sigma, w) = \sigma(e_1)\sqrt{w(e_1)}M_{1,1} + (-1)^n \sqrt{w(e_n)}M_{1,n}
\]

where

\[
M_{1,1} = \det
\begin{pmatrix}
\sigma(e_2)\sqrt{w(e_2)} & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \sigma(e_{n-1})\sqrt{w(e_{n-1})} & 0 \\
0 & \cdots & -\sqrt{w(e_{n-1})} & \sigma(e_n)\sqrt{w(e_n)}
\end{pmatrix},
\]

\[
M_{1,n} = \det
\begin{pmatrix}
-\sqrt{w(e_1)} & \sigma(e_2)\sqrt{w(e_2)} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sigma(e_{n-1})\sqrt{w(e_{n-1})} \\
0 & 0 & \cdots & -\sqrt{w(e_{n-1})}
\end{pmatrix}.
\]
Since \( M_{1,1} \) and \( M_{1,n} \) are determinants of triangular matrices,
\[
\det H(\Sigma, w) = \sigma(e_1)\sqrt{w(e_1)}\sigma(e_2)\sqrt{w(e_2)} \cdots \sigma(e_n)\sqrt{w(e_n)} \\
+ (-1)^n \sqrt{w(e_n)}(-\sqrt{w(e_1)}) \cdots (-\sqrt{w(e_{n-1})}) \\
= (\sigma(C_n) - 1)\sqrt{w(C_n)}.
\]

Now, \( \det H^T(\Sigma, w) = \det H(\Sigma, w) \), hence,
\[
\det L(\Sigma, w) = (\sigma(C_n) - 1)\sqrt{w(C_n)}(\sigma(C_n) - 1)\sqrt{w(C_n)} \\
= 2w(C_n)(1 - \sigma(C_n)).
\]

**Lemma 2.7:** Let \((\Sigma, w)\) be a signed graph, where the underlying graph is a unicyclic graph of order \( n \) with unique cycle \( C \). Then
\[
\det L(\Sigma, w) = 2w(\Sigma)(1 - \sigma(C)).
\]

**Proof:** Define the orientation of edges so that for \( i < j \) the edge \( \vec{e}_{ij} \) has tail \( v_i \) and head \( v_j \). Let \( C = v_1 \vec{e}_1 v_2 \vec{e}_2 \cdots \vec{e}_p v_1 \) be the unique cycle, and label the vertices so that each edge \( \vec{e}_{ij} \) not in \( C \) has \( v_i \) nearer to \( C \) than \( v_j \); in other words, the vertex labels increase when moving away from \( C \). Then the incidence matrix \( H(\Sigma, w) \) has the following form:
\[
\begin{pmatrix}
H(C, w) & * \\
O & \end{pmatrix}
\]

which is an upper-triangular block matrix whose first diagonal block is \( H(C, w) \) and whose other diagonal elements correspond to the heads of edges not in \( C \). Hence,
\[
\det H(\Sigma, w) = \det H(C, w) \cdot \prod_{k=p+1}^{n} (-\sqrt{w(e)}) \\
= (\sigma(C) - 1)(-1)^{n-p}\sqrt{w(C)}.
\]

Thus,
\[
\det L(\Sigma, w) = \det H(\Sigma, w) \det H^T(\Sigma, w) \\
= 2w(\Sigma)(1 - \sigma(C)).
\]

A 1-tree is a unicyclic graph. A 1-forest is a graph whose components are 1-trees.

**Lemma 2.8:** Let \((\Sigma, w)\) be a signed graph whose underlying graph is a 1-forest. Then
\[
\det L(\Sigma, w) = w(\Sigma) \prod_{\psi} 2(1 - \sigma(C_{\psi})),
\]
where the product runs over all component 1-trees \( \psi \) having unique cycle \( C_{\psi} \).
Proof: By suitable reordering of vertices and edges we can make the Laplacian $L(\Sigma, w)$ into a block diagonal matrix, where the blocks correspond to the components of the 1-forest. Thus, $\det L(\Sigma, w)$ is the product of the Laplacian determinants of the components. The components are 1-trees. By using Lemma 2.7 we get the expression for the determinant. 

A signed graph is said to be \textit{contrabalanced} if it contains no positive cycles [7].

Lemma 2.9: Let $L(\Sigma, w)$ be a weighted signed graph having $n$ vertices and let $\Psi$ be a spanning subgraph of $(\Sigma, w)$ having exactly $n$ edges. Then, $\det L(\Psi, w) \neq 0$ if and only if $\Psi$ is a contrabalanced 1-forest.

Proof: A spanning subgraph $\Psi$ of $n$ edges that is not a 1-forest must have a component that has fewer edges than vertices, that is, it is a tree $T$, and $\det L(T, w) = 0$ by Lemma 2.5. Since $\det L(\Psi, w)$ is the product of the Laplacians of the components of $(\Psi, w)$, $\det L(\Psi, w) = 0$ if $\Psi$ is not a 1-forest.

Now, assuming $(\Psi, w)$ is a 1-forest, by Lemma 2.8, $\det L(\Psi, w) \neq 0$ if and only if $\Psi$ contains no positive cycles, which means that $\Psi$ is a contrabalanced 1-forest.

The determinant formula for the Laplacian of a signed graph [5] generalizes to weighted signed graphs. Let $c(\Psi)$ denote the number of components of a signed graph $\Psi$.

Theorem 2.10: Let $(\Sigma, w)$ be a weighted signed graph; then

$$\det L(\Sigma, w) = \sum_{\Psi} 4^{c(\Psi)} w(\Psi),$$

where the summation runs over all contrabalanced spanning 1-forests $\Psi$ of $G$.

Proof: Since $L(\Sigma, w) = H(\Sigma, w)H^T(\Sigma, w)$, by the Binet–Cauchy theorem [8] we get

$$\det L(\Sigma, w) = \sum_J \det H(J) \det H^T(J) = \sum_J \det L(J, w),$$

where $J$ is a spanning subgraph of $G$ with exactly $n$ edges.

Thus by Lemma 2.9, $\det L(\Sigma, w) = \sum_{\Psi} \det L(\Psi)$ where the summation runs over all contrabalanced spanning 1-forests $\Psi$ of $G$. Since every cycle $C_{\Psi}$ in $\Psi$ is negative, the factor $1 - \sigma(C_{\Psi}) = 2$. That gives the formula of the theorem.

We characterize singularity of a weighted Laplacian if the signed graph is connected.

Theorem 2.11: Let $(\Sigma, w)$ be a weighted, connected signed graph. The determinant of its Laplacian matrix is equal to 0 if and only if $\Sigma$ is balanced.

Proof: In the formula of Theorem 2.10 every term is positive, so the determinant is 0 if and only if there is no contrabalanced 1-forest in $\Sigma$. Since $\Sigma$ is connected, that implies it has no negative circle, that is, it is balanced.
We strengthen this to state the rank of the weighted Laplacian. We define $b(\Sigma)$ to be the number of balanced components in $\Sigma$.

**Theorem 2.12:** Let $(\Sigma, w)$ be a weighted signed graph with $n$ vertices. The rank of its Laplacian matrix is equal to $n - b(\Sigma)$.

**Proof:** Assume $\Sigma$ is connected and balanced. Let $K$ be the result of deleting the first row and column from $L(\Sigma, w)$. Then by the weighted matrix-tree theorem (see, e.g. [9])

$$\det K = \sum_T w(T),$$

summed over all spanning trees of $\Sigma$. Since $\Sigma$ is connected, the sum is not empty, hence $\det K > 0$ so $L(\Sigma, w)$ has rank at least $n - 1$. By Theorem 2.11, $L(\Sigma, w)$ has rank at most $n - 1$. Thus, the rank is $n - 1 = n - b(\Sigma)$.

For a weighted signed graph that is not connected, the rank of its Laplacian matrix is the sum of the ranks of those of its components. We conclude that $L(\Sigma, w)$ has rank $n - b(\Sigma)$.

### 3. Properties of the signed distance laplacians

Our first result expresses the signed distance Laplacian of a connected signed graph $\Sigma$ in terms of a signed distance incidence matrix based on Definition 2.3. The underlying graph is the complete graph $K_n$, so there is a column in the incidence matrix for each edge $v_i v_k \in E(K_n)$, and the edge weights are the distances: $w(v_i v_j) = d(v_i, v_j)$. These edges are oriented arbitrarily.

**Definition 3.1:** Given a signed graph $\Sigma$, its (oriented) max-distance incidence matrix is defined as $DH_{\max}(\Sigma) = (\eta_{v_i v_j})$ where

$$\eta_{v_i v_j} = \begin{cases} 
\sigma_{\max}(v_i v_j) \sqrt{d(v_i, v_j)} & \text{if } t(v_i v_j) = v_i, \\
-\sqrt{d(v_i, v_j)} & \text{if } h(v_i v_j) = v_i, \\
0 & \text{otherwise}.
\end{cases}$$

The definition of the min-distance incidence matrix $DH_{\min}(\Sigma)$ is similar. If $\Sigma$ is compatible the two incidence matrices are equal and we write $DH^\pm(\Sigma) = DH_{\max}(\Sigma) = DH_{\min}(\Sigma)$.

Theorem 2.4 yields formulas for the signed distance Laplacians.

**Theorem 3.2:** For a connected signed graph $\Sigma$,

$$DL_{\max}(\Sigma) = DH_{\max}(\Sigma)DH_{\max}(\Sigma)^T,$$

$$DL_{\min}(\Sigma) = DH_{\min}(\Sigma)DH_{\min}(\Sigma)^T,$$

$$DL^\pm(\Sigma) = DH^\pm(\Sigma)DH^\pm(\Sigma)^T.$$

Next, we give formulas for determinants based on Theorem 2.10. We define for a subgraph $\Psi$ of $K^{D_{\max}}(\Sigma)$ the value $d^{\ast_{\max}}(\Psi) = \prod_{v_i v_j \in E(\Psi)} d_{\max}(v_i, v_j)$ and similarly $d^{\ast_{\min}}(\Psi)$ and $d^{\ast_{\pm}}(\Psi)$. 

Theorem 3.3: Let $\Sigma$ be a connected signed graph; then

$$\det DL^{\text{max}}(\Sigma) = \sum_{\Psi} 4^c(\Psi) d^{\ast \text{max}}(\Psi)$$

with similar formulas for $\det DL^{\text{min}}(\Sigma)$ and $\det DL^{\pm}(\Sigma)$, where the summation runs over all contrabalanced spanning 1-forests $\Psi$ of $K_{D^{\text{max}}}(\Sigma)$, $K_{D^{\text{min}}}(\Sigma)$, or $K_{D^{\pm}}(\Sigma)$, respectively.

Finally, we characterize when the signed distance Laplacian matrices are singular. We recall the characterization theorem for balance in signed graphs using signed distances proved by Hameed et al. in [2].

Lemma 3.4 ([2]): For a connected signed graph $\Sigma$ the following statements are equivalent:

(i) $\Sigma$ is balanced.
(ii) The associated signed complete graph $K_{D^{\text{max}}}(\Sigma)$ is balanced.
(iii) The associated signed complete graph $K_{D^{\text{min}}}(\Sigma)$ is balanced.
(iv) $D^{\text{max}}(\Sigma) = D^{\text{min}}(\Sigma)$ and the associated signed complete graph $K_{D^{\pm}}(\Sigma)$ is balanced.

Now we are ready to characterize singularity.

Theorem 3.5: The following properties of a connected signed graph $\Sigma$ are equivalent.

(i) The max-signed distance Laplacian determinant $\det DL^{\text{max}}(\Sigma) = 0$.
(ii) The min-signed distance Laplacian determinant $\det DL^{\text{min}}(\Sigma) = 0$.
(iii) $DL^{\text{max}}(\Sigma) = DL^{\text{min}}(\Sigma)$ and $\det DL^{\pm}(\Sigma) = 0$.
(iv) $\Sigma$ is balanced.

More precisely, the rank of the distance Laplacian matrices of $\Sigma$ is $n-1$ if $\Sigma$ is balanced and $n$ if it is unbalanced.

Proof: Corresponding to the associated signed complete graph $K_{D^{\text{max}}}(\Sigma)$, we define a weighted signed complete graph $(K_{D^{\text{max}}}(\Sigma), w)$ and $(K_{D^{\text{min}}}(\Sigma), w)$, where $w(e) = d(u, v)$ for an edge $e = uv$. Then

$$L(K_{D^{\text{max}}}(\Sigma), w) = DL^{\text{max}}(\Sigma)$$

and

$$L(K_{D^{\text{min}}}(\Sigma), w) = DL^{\text{min}}(\Sigma).$$

Thus, by Theorem 2.11, $\det DL^{\text{max}}(\Sigma) = 0$ if and only if $K_{D^{\text{max}}}(\Sigma)$ is balanced, and $\det DL^{\text{min}}(\Sigma) = 0$ if and only if $K_{D^{\text{min}}}(\Sigma)$ is balanced. Hence, by Theorem 3.4 we get the required characterization.

The exact rank of the distance Laplacian matrices is obtained from Theorem 2.12.

4. Signed distance Laplacian spectrum

Now, we move to the spectra of signed distance Laplacians, especially the spectrum of $DL^{\pm}$ for compatible graphs. First, we give a spectral characterization of balance using the signed distance Laplacian.
Switching a signed graph $\Sigma$ by a switching function $\zeta : V \to \{+1, -1\}$ means multiplying each edge sign $\sigma(uv)$ to get a new sign $\sigma^\zeta(uv) = \zeta(u)\sigma(uv)\zeta(v)$. The new signed graph is denoted by $\Sigma^\zeta$. Switching changes distance matrices through conjugation by $S$, the $n \times n$ diagonal matrix with $\zeta$ on the diagonal; thus, for example, $D_{\text{min}}^\Sigma(\Sigma^\zeta) = SD_{\text{min}}^\Sigma(\Sigma)S^{-1}$ [2]. It follows that switching does not change the spectra of the distance matrices [2] and also not of the distance Laplacians, as we now prove.

**Lemma 4.1:** Let $\Sigma$ be a signed graph and $\zeta$ a switching function. The distance Laplacian matrices of $\Sigma^\zeta$ are similar to those of $\Sigma$ and therefore have the same spectra.

**Proof:** We use $D_{\text{min}}^\Sigma$ as an example, the other proofs being similar. Since $D_{\text{min}}(\Sigma^\zeta) = SD_{\text{min}}^\Sigma(\Sigma)S^{-1}$, we have

$$S. D_{\text{min}}(\Sigma)S^{-1} = S(\text{Tr}(G) - D_{\text{min}}(\Sigma))S^{-1} = \text{Tr}(G) - D_{\text{min}}(\Sigma^\zeta)$$

$$= D_{\text{min}}(\Sigma^\zeta).$$

Thus, $D_{\text{min}}(\Sigma^\zeta)$ is similar to $D_{\text{min}}(\Sigma)$. $\blacksquare$

**Theorem 4.2:** A signed graph $\Sigma$ is balanced if and only if $D_{\text{max}}^\Sigma(\Sigma) = D_{\text{min}}^\Sigma(\Sigma) = D^\pm_{\Sigma} = D^\pm(\Sigma)$ and $D^\pm(\Sigma)$ is cospectral with $D(L(G))$, where $D(L(G))$ denotes the distance Laplacian matrix of the underlying graph $G$.

**Proof:** Suppose $\Sigma = (G, \sigma)$ is balanced. Then $\Sigma$ can be switched to an all positive signed graph $\Sigma^\zeta = (G, +)$. Now, $D^\pm(\Sigma)$ exists [2] and, by Lemma 4.1, $D^\pm(\Sigma)$ is similar to $D^\pm(G, +) = D(L(G))$, which implies that $D^\pm(\Sigma)$ is cospectral with $D(L(G))$.

Conversely, suppose $D_{\text{max}}^\Sigma(\Sigma) = D_{\text{min}}^\Sigma(\Sigma) = D^\pm(\Sigma)$ and $D^\pm$ is cospectral with $D(L(G))$. Thus, det $D^\pm(\Sigma) = \text{det} D(L(G)) = 0$ and hence, by Theorem 3.5, $\Sigma$ is balanced. $\blacksquare$

Our next result is that, under a regularity condition, the signed distance spectrum gives the signed distance Laplacian spectrum. A signed graph $\Sigma$ is $t$-transmission regular if $n(v) = \sum_{u \in V(G)} d(v, u) = t$ for all $v \in V(G)$. Odd cycles $C_{2k+1}$ are $k(k+1)$-transmission regular and even cycles $C_{2k}$ are $k^2$-transmission regular.

**Theorem 4.3:** If the signed graph $\Sigma$ is $t$-transmission regular, then the signed distance Laplacian eigenvalues of $D_{\text{max}}^\Sigma(\Sigma)$(or $D_{\text{min}}^\Sigma(\Sigma)$) are $t - \lambda$, where $\lambda$ is an eigenvalue of $D_{\text{max}}(\Sigma)$(or $D_{\text{min}}(\Sigma)$).

We apply this to get the spectrum of $D^\pm$ for an odd unbalanced cycle. We choose this example because if a cycle is positive, the spectrum is that of the unsigned cycle, and if it is negative but even, the two signed distance Laplacians $D_{\text{max}}^\Sigma$ and $D_{\text{min}}^\Sigma$ are unequal so $D^\pm$ is not defined.

For the odd unbalanced cycle $C_{n}$, where $n = 2k + 1$, there is a unique shortest path between any two vertices. Thus, $D_{\text{max}}^\Sigma(C_{n}) = D_{\text{min}}^\Sigma(C_{n}) = D^\pm(C_{n})$. The signed distance spectrum of an odd unbalanced cycle is given in [2]. Thus, we get, the signed distance Laplacian spectrum of $C_{n}$ as an immediate corollary.
**Example 4.4:** For an odd unbalanced cycle $C_n^-$, where $n = 2k + 1$, $DL^\pm$ has the eigenvalues

$$k(k + 1) - k(-1)^k - \frac{1 - (-1)^k}{2}$$

with multiplicity 1 and

$$k(k + 1) - \frac{k(-1)^j}{\sin \left(\frac{(2j+1)\pi}{2n}\right)} - \frac{\sin^2 \left(\frac{(2j+1)k\pi}{2n}\right)}{\sin^2 \left(\frac{(2j+1)\pi}{2n}\right)}$$

with multiplicity 2 for each $j = 0, 1, 2, \ldots, k - 1$.

Like the odd cycle, the complete graph is geodetic, hence distance compatible, so $DL^\pm$ exists.

**Example 4.5:** For an all negative complete graph $-K_n$, $DL^\pm$ has the eigenvalues $n$ with multiplicity $n-1$ and 0 with multiplicity 1. The distance Laplacian matrix is $DL^\pm(-K_n) = nI_n - J_n$, where $I_n$ is the identity matrix and $J_n$ is the all-ones matrix of order $n$. The eigenvalues of $J_n$ are 0 with multiplicity $n-1$ and $n$ with multiplicity 1. Hence, $DL^\pm(-K_n)$ has the eigenvalues $n$ with multiplicity $n-1$ and 0 with multiplicity 1.

The eigenvalues are shifted from the negatives of those of $DL(K_n)$, because $DL^\pm(-K_n) = 2(n-1)I_n - DL(K_n)$. Such a simple relation between a graph and its all negative signed graph is truly exceptional, because it depends on all distances being equal. Thus, we expect the spectrum of $DL^\pm(-G)$ to be difficult, even for geodetic graphs.

Our next example shows that, for an even cycle, the distance Laplacian spectrum depends on the sign function. We denote the distance Laplacian characteristic polynomials of $\Sigma$ by $f(DL^\text{max}(\Sigma), \lambda)$ and $f(DL^\text{min}(\Sigma), \lambda)$.

**Example 4.6:** Consider the two unbalanced even cycles $C_6$ with different sign functions, given in Figures 1 and 2. Although both signed cycles are distance incompatible, both have

![Figure 1. C_6^-](image)
equal minimum and maximum characteristic polynomials: \( f(DL_{\text{max}}) = f(DL_{\text{min}}) \). Using computational software for finding spectra, we get

\[
\begin{align*}
  f(DL_{\text{max}}(C^{-}_6), \lambda) &= f(DL_{\text{min}}(C^{-}_6), \lambda) \\
  &= \lambda^6 - 54\lambda^5 + 1158\lambda^4 - 12,536\lambda^3 + 71,721\lambda^2 - 203,802\lambda + 221,128, \\
  f(DL_{\text{max}}(C^{-}_{6a}), \lambda) &= f(DL_{\text{min}}(C^{-}_{6a}), \lambda) \\
  &= \lambda^6 - 54\lambda^5 + 1158\lambda^4 - 12,536\lambda^3 + 71,505\lambda^2 - 201,354\lambda + 219,040.
\end{align*}
\]

Thus, \( C^{-}_6 \) and \( C^{-}_{6a} \) are neither \( DL_{\text{max}} \) cospectral nor \( DL_{\text{min}} \) cospectral. We do not know a reason that the three highest coefficients for both graphs agree.

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