Abstract

The feasibility of using lattice chiral fermions which are free of $O(a)$ errors for both the heavy and light quarks is examined. The fact that the effective quark propagators in these fermions have the same form as that in the continuum with the quark mass being only an additive parameter to a chirally symmetric antihermitian Dirac operator is highlighted. This implies that there is no distinction between the heavy and light quarks and no mass dependent tuning of the action or operators as long as the discretization error $O(m^2a^2)$ is negligible. Using the overlap fermion, we find that the $O(m^2a^2)$ (and $O(ma^2)$) errors in the dispersion relations of the pseudoscalar and vector mesons and the renormalization of the axial-vector current and scalar density are small. This suggests that the applicable range of $ma$ may be extended to $\sim 0.56$ with only 5% error, which is a factor of $\sim 2.4$ larger than that of the improved Wilson action. We show that the generalized Gell-Mann-Oakes-Renner relation with unequal masses can be utilized to determine the finite $ma$ errors in the renormalization of the matrix elements for the heavy-light decay constants and semileptonic decay constants of the B/D meson.
Heavy-light quarkoniums such as B and D mesons are the primary testing ground for understanding CP violation and obtaining the CKM matrix. Both experiment and theory are needed to extract relevant quantities. For example, $|V_{ub}|/|V_{cd}|$ can be determined from the semileptonic decay rate of B/D meson, $B/D \rightarrow \pi l\nu$. But it depends on the transition form factor $|f_+(E)|$ from $B/D \rightarrow \pi$. $|V_{tb}^*V_{td}|$ can be extracted from the mass difference $\Delta m_B$ of the neutral $B-B$ mesons where it also depends on the B parameter $B_B$ and the yet unmeasured leptonic decay width $f_B$. These quantities — $|f_+(E)|, B_B,$ and $f_B$ are related to hadronic matrix elements which are best determined in lattice QCD which is a non-perturbative approach to solving QCD with controllable systematic errors [1, 2, 3].

The major challenge for incorporating heavy quarks with mass $m_Q$ on the lattice is that, in the range of lattice spacing that is amenable to numerical simulation nowadays, the condition $m_Qa \ll 1$ is far from being satisfied for the $b$ quark. There are several approaches to formulating the heavy quark on the lattice. The APE [4]–UKQCD [5] approach is to simulate with the quark mass around the charm with the $O(a)$ improved Wilson action and then extrapolate to the bottom with the guide of the heavy quark effective theory (HQET). Since the functional form for the mass dependence is not certain in this mass range and the extrapolated point is very far, this results in large errors [2, 3]. Another approach is the non-relativistic QCD (NRQCD) [6]. This involves an expansion in terms of the heavy quark mass which is considered an irrelevant dynamical scale. It has the advantage that it leads to faster numerical simulation for the heavy quark and the correction to the static limit with higher dimensional operators can be incorporated in perturbation. However, there are ‘renormalon shadow’ effects which reflect ‘large perturbative uncertainties in power divergent subtractions’ [2]. As an effective theory, it does not have a continuum limit. Thus, the discretization error cannot be removed by extrapolating the lattice results to $a \to 0$. The Fermilab formulation [7] bridges the above two approaches. For small masses, it has a continuum limit. With the heavy quark effective theory (HQET) approach [8, 9], both the lattice spacing $a$ and the inverse of the large quark mass are treated as short-distances so that the heavy quark discretization effects are lumped into the Wilson coefficients. It is shown in this case that the discretization errors due to the heavy quarks can be controlled to allow systematic reduction of the discretization errors for all $ma$.

The recent relativistic approach with anisotropic lattice [10] with the ratio $\xi = a_s/a_t$ between the spatial lattice spacing $a_s$ and the temporal spacing $a_t$ chosen to be $3–5$ can alleviate the large $m_Qa_t$ problem to a degree, but it still suffers from a large $O(m_Q^2a_t^2)$ error [11, 12]. To control the systematics to a few percent level for the dispersion relation, the condition $m_Qa_t < 0.2$ [11] must be met which is very stringent.

On the other side of the approaches to heavy-light calculations, the light quark used in the heavy-light simulation so far suffers from the well known set of problems
associated with the lack of chiral symmetry. Take the Wilson action for example; this
ultra local action breaks chiral symmetry explicitly at finite lattice spacing in order to
lift the doublers to the cut off. As a consequence, it induces numerous problems. The
quark mass has an additive renormalization which is gauge configuration dependent.
The quark condensate is mixed with unity which makes it harder to calculate. There
is no unambiguous correspondence between the fermion zero modes and topology. It
has $O(a)$ error and operators in different chiral sectors mix. Although the $O(a)$ error
can be removed with the improved action and the mixing of operators can be taken
into account, the procedure nevertheless requires fine tuning and is usually quite
involved. The more serious problem is the existence of exceptional configura-
tions. Since there is no protection by chiral symmetry, there can be zero modes even
in the presence of finite and positive quark mass on certain gauge background config-
urations. This is getting more frequent when quark mass is less than $\sim 20$ MeV and
it renders the region of pion mass less than $\sim 300$ MeV inaccessible. Unfortunately,
this is the region where chiral behavior such as the chiral logs are becoming visible.
Without admission to this region of low pion mass, reliable chiral extrapolation is not
feasible.

With the advent of the recent lattice chiral fermions, such as the domain wall
fermion, the overlap fermion, and the fixed-point-action fermion, all
the above mentioned problems associated with the light quarks can be overcome in
principle. In practice, it is shown in numerical simulations of the overlap fermion
that there is indeed no additive quark mass renormalization, no exceptional configura-
tions, and the current algebra such as the Gell-Mann-Oakes-Renner relation is
satisfied to high precision. Besides fulfilling the promise of removing the difficulties of the Wilson-like fermion, the overlap fermion has turned in extra bonuses. Its
critical slowing down is quite gentle all the way to the physical pion mass; the
$O(a^2)$ and $O(m^2a^2)$ errors are apparently small, and it can incorporate
the multi-mass inversion algorithm. We will concentrate on the overlap fermion
in this paper.

The massless overlap Dirac operator is

$$D = 1 + \gamma_5 \epsilon(H),$$

where $\epsilon(H) = H/\sqrt{H^2}$ is the matrix sign function of H which we take to be the
Hermitian Wilson-Dirac operator, i.e. $H = \gamma_5(D_w(0) - 1)$. Here $D_w(0)$ is the Wilson
fermion operator with $\kappa = 1/8$. It is shown that under the global lattice chiral
flavor non-singlet transformation $\delta \psi = T\gamma_5(1 - \frac{1}{2}D)\psi, \delta \bar{\psi} = \bar{\psi}(1 - \frac{1}{2}D)\gamma_5T,$
the fermion action $\bar{\psi}D\psi$ is invariant since the operator $D$ satisfies the Ginsparg-Wilson
relation $\{\gamma_5, D\} = D\gamma_5D$. It can be shown that the flavor non-singlet scalar, pseudoscalar, vector, and axial bilinears in the form $\bar{\psi}KT(1 - \frac{1}{2}D)\psi (K$
is the kernel which includes $\gamma$ matrices) transform covariantly as in the continuum.
The $1 - \frac{1}{2}D$ factor is also understood as the lattice regulator which projects out the
unphysical real eigenmodes at $\lambda = 2$. For the massive case, the fermion action is $\bar{\psi}D\psi + ma\bar{\psi}(1 - \frac{1}{2}D)\psi$. In this case, the Dirac operator can be written as

$$D(m) = D + ma(1 - \frac{1}{2}D).$$

(2)

Let’s consider the path-integral formulation of Green’s function with the $\psi$ field in the operators and interpolation fields replaced by the lattice regulated field $\hat{\psi} = (1 - \frac{1}{2}D)\psi$. After the Grassmann integration, this regulator factor will be associated with the quark propagator in the combination $(1 - \frac{1}{2}D)D(m)^{-1}$ which can be written as

$$(1 - \frac{1}{2}D)D(m)^{-1} = (D_c + ma)^{-1}.$$  

(3)

where the operator $D_c = D/(1 - \frac{1}{2}D)$ is chirally symmetric in the continuum sense, i.e. $\{\gamma_5, D_c\} = 0$; but, unlike $D$, it is non-local. This $D_c$ has been derived in the massless case [26, 27, 28] in association with the quark condensate and the solution of the Ginsparg-Wilson relation. For the massive case, it was pointed out [29, 30] that the mass term $ma$ should be added to the operator $D_c$ not $D$ in the quark propagator and Eq. (3) was derived [31] between the $N$-flavor low-energy effective Dirac operator $D^eff_N$ from the domain wall fermion and the truncated overlap operator $D_N$ with the overlap operator $D$ being the $N \rightarrow \infty$ and $a_5 \rightarrow 0$ limit of $D_N$. Here, we derive Eq. (3) from combining the lattice regulated field $\hat{\psi} = (1 - \frac{1}{2}D)\psi$ and the inverse of $D(m)$ to form the effective quark propagator. As a result, this effective quark propagator should be used together with local currents and interpolation fields without the $1 - \frac{1}{2}D$ factor. We shall highlight the fact that the effective quark propagator $(D_c + ma)^{-1}$ has the continuum form, i.e. the inverse propagator is the sum of an chirally symmetric operator and a real mass parameter. The mass in the quark propagator is the same bare mass $m$ introduced in the fermion action. It makes no distinction between a light quark and a heavy one, just as in the continuum.

It is interesting to point out that the original overlap operator $D(m)$ has eigenvalues lying on the circle due to the fact that $D$ satisfies Ginsparg-Wilson relation and is a normal matrix. This is shown in Fig. 1 where the radius is $1 - ma/2$. On the other hand, since $D_c$ is $\gamma_5$ hermitian, i.e. $D^\dagger_c = \gamma_5 D_c \gamma_5$ and anticommutes with $\gamma_5$, it is anti-hermitian. Its eigenvalues are simply the stereographic projection of the circle onto the imaginary axis as is in the massless case [30], except in the massive case, the eigenvalue of $D_c + ma$ is shifted by $ma$ to the right. There is a one-to-one correspondence between the eigenstates of $D(m)$ and $D_c + ma$, except the ‘north pole’ at the cut-off $\lambda = 2$ which is excluded by the regulator projector. This renders the propagator exactly like the continuum situation where the eigenvalues of the Euclidean Dirac operator $\bar{\mathcal{D}} + m$ are distributed on the shifted imaginary axis. Since the overlap fermion is invariant under the lattice chiral transformation, it does not mix with dimension five operators which are not chirally invariant. Therefore, there is no $O(a)$ nor $O(ma)$ error. The only question is how large the $O(m^2 a^2)$ and
Figure 1: Stereographic projection of the eigenvalues of $D(m)$ on the circle to the imaginary axis which is shifted by $ma$.

$O(ma^2)$ systematic errors are for different quark mass, be it light or heavy. We should stress that the above discussion is not limited to the overlap fermion. It also applies to other local lattice Dirac operators $D$ which satisfy normality, $\gamma_5$-hermiticity, and the Ginsparg-Wilson relation. In general, $D_c = D/(1 - \frac{1}{2}D)$ is the chirally symmetric operator for these lattice chiral fermions.

We first examine the $O(m^2a^2)$ and $O(ma^2)$ errors in the dispersion relation. It is suggested that dispersion relation is one of the places where one can discern the $ma$ error [10, 11]. We computed the pseudoscalar and vector meson masses and energies at several lattice momentum, i.e. $p_L a = \sqrt{\pi} 2\pi / La$ with $n = 0, 1, 2, 3$. The overlap quark propagators are calculated on the $16^3 \times 28$ quenched lattice with 80 configurations generated from Iwasaki gauge action with $a = 2.00$ fm as determined from $f_\pi$ [15]. Following Refs. [10, 11], we fit the energies to the dispersion relation

$$(E(p)a)^2 = c^2(pa)^2 + (E(0)a)^2$$

where $p = 2\sin(p_L a/2)$. The dispersion relation is so defined such that the $ma$ error is reflected in the deviation of $c$ (the effective speed of light) from unity.

We see in Figs. 2 and 3 that the effective speed of light $c$ is close to unity and quite flat all the way to $ma \sim 0.5$. Since there is no $O(ma)$ error, we fit it with the form quadratic in $a$, i.e. $c = c_0 + b(\Lambda_{QCD} a)ma + d m^2 a^2$ ($\Lambda_{QCD}a = 0.188$), and find that
Figure 2: The effective speed of light $c$ from the pseudoscalar meson dispersion relation as a function of $ma$.

$c_0 = 0.982(10), b = 0.580(346), \text{ and } d = -0.279(87)$ with $\chi^2/N_{dof} = 0.1$ for the case of the pseudoscalar meson and $c_0 = 1.027(26), b = -0.32(90), \text{ and } d = -0.18(22)$ with $\chi^2/N_{dof} = 0.6$ for the vector meson. Using this to gauge how large the $ma$ error is, we see that the systematic error is less than $\sim 4\% (6\%)$ for the pseudoscalar(vector) case up to $ma \sim 0.56$. This $ma$ is $\sim 2.4$ times larger than that is admitted in the study of improved Wilson action [11] where it is found that the $O(m^2 a^2)$ error from the anisotropy of the dispersion relation is less than $\sim 5\%$ when $m_Q a_t < 0.23$. Therefore with the overlap fermion, one can hope to extend the range of $ma$ to $0.5 - 0.56$ where the systematic error is still reasonably small.

Next, we address the issue of $O(ma)$ improvement. This is essential for the Wilson-type fermions which has large $O(a)$ error. The $O(a)$ improvement for the action is usually done with the addition of the clover term. The operator improvement is more involved. We shall illustrate this by considering the axial Ward identity. It has been shown [13] that in the improved mass-independent renormalization scheme, the renormalized improved axial current and pseudoscalar density have the following form from an $O(a)$ improved action

$$
A^R_\mu = Z_A(1 + b_A m_q a)\{A_\mu + c_A a \partial_\mu P\},
$$

$$
P^R = Z_P(1 + b_P m_q a)P,
$$

where $m_q = m - m_c$ is the subtracted quark mass and $c_A, b_A \text{ and } b_P$ are improvement coefficients. The renormalization constants $Z_A$ and $Z_P$ are functions of the modified coupling $g_0^2 = g_0^2(1 + b g m_q a)$. It is argued [2] that as $m_Q \rightarrow \infty$ $A^R_\mu$ goes to $-\infty$ instead of the static limit, since $c_A < 0$ and $\partial_\mu P \propto m_Q$. Even when $m_Q a \ll 1/4$, to calculate the six parameters $- m_c, b_A, c_A, b_P, Z_P$, and $Z_A$ non-perturbatively in order
Figure 3: The effective speed of light $c$ from the vector meson dispersion relation as a function of $ma$.

to satisfy the chiral Ward identity [14] is quite a task.

The situation with the lattice chiral fermion is much simpler. The overlap fermion is $O(a)$ improved, and the quark mass is not additively renormalized which is verified numerically [19]. As a result, $b_A = c_A = b_P = m_c = 0$. If one uses the axial current $A_\mu = \bar{\psi}i\gamma_\mu\gamma_5\hat{\psi}$ for simplicity, the renormalization constant $Z_A$ can be obtained through the axial Ward identity

$$Z_A \partial_\mu A_\mu = 2 Z_m m Z_P P,$$  \hspace{1cm}  (6)

where $P = \bar{\psi}i\gamma_5\hat{\psi}$. Since $Z_m = Z_S^{-1}$ and $Z_S = Z_P$ due to the fact the scalar density $\bar{\psi}\hat{\psi}$ and the pseudoscalar density $P$ are in the same chiral multiplet, $Z_m$ and $Z_P$ cancel in Eq. (6) and one can directly determine $Z_A$ non-perturbatively from the axial Ward identity using the bare mass $m$ and bare operator $P$. To avoid $O(a^2)$ error introduced by the derivative in Eq. (6), one can consider the axial Ward identity for the on-shell matrix elements between the vacuum and the zero-momentum pion state. In this case,

$$Z_A = \lim_{m \to 0, t \to \infty} \frac{2m G_{PP}(\vec{p} = 0, t)}{m_\pi G_{A4P}(\vec{p} = 0, t)},$$  \hspace{1cm}  (7)

where $G_{PP}(\vec{p} = 0, t)$ and $G_{A4P}(\vec{p} = 0, t)$ are the zero-momentum pseudoscalar-pseudoscalar and axial-pseudoscalar correlators. In the mass-independent renormalization scheme [13], the renormalization factor for the axial current matrix element which takes into account the finite $ma$ errors can be defined from Eq. (7) without taking the massless limit and neglecting the small finite $ma$ difference between the
Figure 4: \( \tilde{Z}_A(\ma) \) from the axial Ward identity on a 20\(^4\) lattice with \( a = 0.148 \) fm. The fitted curve which is explained in the text is plotted as the solid line.

The same small discretization errors in \( \ma \) are reflected in other renormalization factors which are defined, similar to \( \tilde{Z}_A(\ma) \) in Eq. (9), as
\[
\tilde{Z}_\Gamma(\ma) = Z_\Gamma(1 + b_\Gamma (\Lambda_{QCD} a) \ma + c_\Gamma \ma^2 a^2),
\]
for \( \Gamma = S, P, V, \) and \( T \) [33, 34]. In the mass-independent renormalization scheme [13], the renormalization constant \( Z_\Gamma \) is a function of the gauge coupling \( g_0^2 \) and the renormalization scale \( \mu \), i.e. \( Z_\Gamma = Z_\Gamma(\g_0^2, \mu a) \). Here in Fig. 5 we show the \( \ma \) dependence

\[
\tilde{Z}_A(\ma) = \lim_{t \to \infty} \frac{2mG_{PP}(\vec{p}=0,t)}{m_\pi G_{AA}(\vec{p}=0,t)},
\] (8)
Up to \( O(\ma^2) \) and \( O(m^2 a^2) \), \( \tilde{Z}_A(\ma) \) is
\[
\tilde{Z}_A(\ma) = Z_A(1 + b_A (\Lambda_{QCD} a) \ma + c_A m^2 a^2),
\] (9)
which is the combination of the renormalization constant \( Z_A \) and the finite \( \ma \) effects. The resultant \( \tilde{Z}_A(\ma) \) on a quenched 20\(^4\) lattice with overlap fermion for \( \ma \) \( (a = 0.148 \) fm) from 0.01505 to 0.2736 were reported before [20]. Now we show the results extended to \( \ma = 0.684 \) in Fig. 4. We see that, similar to the effective speed of light \( c \), it is quite flat all the way to \( \ma \sim 0.5 \). In analogy to fitting \( c \), we fit it with the form in Eq. (9) \( (\Lambda_{QCD} a = 0.188) \), and find that \( Z_A = 1.592(5), b_A = -0.13(9), \) and \( c_A = 0.203(22) \) with \( \chi^2/N_{dof} = 0.46 \). We also have results on a 16\(^3 \times 28 \) lattice with \( a = 0.200 \) fm and found an almost identical \( \ma \) behavior for \( \tilde{Z}_A(\ma) \) [32]. Again using this to gauge how large the \( \ma \) error is, we see that the systematic error is less than \( \sim 5\% \) up to \( \ma \sim 0.56 \) which is close to the case of \( c \) for the dispersion relation for the pseudoscalar and vector mesons in Figs. 2 and 3.
in $Z_S(ma)$ at $\mu = 2$ GeV, where the renormalization constant $Z_S^{MS}(2\text{GeV})$ is obtained from the non-perturbative renormalization in the regularization independent scheme \[\text{35, 36}\] and then perturbatively matched to the $\overline{MS}$ scheme at the scale of 2 GeV. The results are from a quenched $16^3 \times 28$ lattice with $a = 0.200$ fm \[\text{34}\]. Again, we see that it is rather flat from $ma = 0$ to $ma = 0.8$. Fitting to the form in Eq. (10) ($\Lambda_{QCD} a = 0.250$) yields $Z_S = 1.718(12)$, $b_S = -0.002(194)$, $c_S = 0.073(58)$. It gives a $ma$ error of 2.6% at $ma = 0.6$. We should mention that similar studies for the $ma$ errors for $Z_A(ma)$ and $Z_V(ma)$ are done with domain-wall fermions \[\text{37, 38}\]. The $ma$ errors seem to be larger than those found here. For example, the finite $ma$ error at $m_f a = 0.10$ is already found to be 3 – 4 % which is much larger than what we obtain for the overlap fermion at the same $ma$.

For the heavy-light quarkonium, an accurate renormalization for the vector and axial current is essential for the study of $f_B$, $f_D$ and the semi-leptonic decays of the B and D mesons. Since most of the renormalization for the composite operator with heavy and light quarks are done with perturbation in one loop, its $O(\alpha_s^2)$ correction can be large. In a recent calculation of $f_B$, and $f_D$ with NRQCD for the heavy quark, the $O(\alpha_s^2)$ error is estimated to be 10% \[\text{39}\]. Similarly, it is pointed out in the study of $f_D$ \[\text{10}\] with fermilab heavy quark that the $O(\alpha_s)$ correction can be potentially as large as 30%. In the following, we show a non-perturbative method which can determine the axial-heavy-light current renormalization with finite $ma$ error at a few percent level even with $ma$ as large as $0.5 - 0.6$. This should be of help in determining $f_B$ and $f_D$ with much less systematic errors.

The finite $ma$ errors in the renormalization of the matrix elements involving $f_B$
and $f_{D}$ and semi-leptonic decays can be accurately determined with the help of current algebra relations. The axial Ward identity for the pseudoscalar meson $P$ decay matrix element of unequal masses including the finite $ma$ factor is

$$Z_{A}(m_{1}a, m_{2}a)\langle 0|\partial_{\mu}A_{\mu12}|P \rangle = (Z_{m}(m_{1}a)m_{1} + Z_{m}(m_{2}a)m_{2})Z_{P}(m_{1}a, m_{2}a)\langle 0|P_{12}|P \rangle,$$

where $A_{\mu12} = \bar{\psi}_{1}\gamma_{\mu}\gamma_{5}\psi_{2}, P_{12} = \bar{\psi}_{1}\gamma_{5}\psi_{2}$ and $\hat{Z}_{A}(m_{1}a, m_{2}a), \hat{Z}_{m}(ma),$ and $\hat{Z}_{P}(m_{1}a, m_{2}a)$ are the products of renormalization constants and their respective finite $ma$ factors. Here $\hat{Z}_{P}(m_{1}a, m_{2}a)$ does not cancel out $\hat{Z}_{m}(m_{1}a)/\hat{Z}_{m}(m_{2}a)$ except in the massless limit and, therefore, one cannot readily use Eq. (7) to obtain $\hat{Z}_{P}(m_{1}a, m_{2}a)$ to account for the finite $ma$ correction. Fortunately, one can adopt additional information from the generalized Gell-Mann-Oakes-Renner relation for the unequal mass case, which is

$$\frac{1}{V} \int d^{4}x (\pi_{12}^{a} (x)\pi_{12}^{a}(0)) = -\frac{2[\langle \bar{\psi}_{1}\psi_{1} \rangle + \langle \bar{\psi}_{2}\psi_{2} \rangle]}{m_{1} + m_{2}},$$

where $\pi_{12}^{a}(x) = \bar{\psi}_{1}\gamma_{5}\gamma_{a}^{\mu} / 2\hat{\psi}_{2}$. The proof is a generalization of the equal mass case [11] and it has been proved with the staggered fermion [12]. In fact, with the effective propagator in Eq. (3), a lot of the current algebra relations can be reproduced on the lattice with finite cutoff [13, 21]. From Eq. (3), we see that $\langle \bar{\psi}_{1}\psi_{1} \rangle = -Tr(D_{c} + m_{1})^{-1}$ which can be written as

$$Tr(D_{c} + m_{1})^{-1} = Tr\{(m_{2} - D_{c})[(m_{2} - D_{c})^{-1}(m_{1} + D_{c})^{-1}]\}
= Tr\{(m_{2} - D_{c})[\gamma_{5}(m_{2} + D_{c})^{-1}\gamma_{5}(m_{1} + D_{c})^{-1}]\},$$

where we have used the property $\gamma_{5}D_{c}\gamma_{5} = -D_{c}$. Similarly, one can write

$$Tr(D_{c} + m_{2})^{-1} = Tr[\gamma_{5}(m_{2} + D_{c})^{-1}\gamma_{5}]
= Tr\{(m_{1} + D_{c})[\gamma_{5}(m_{2} + D_{c})^{-1}\gamma_{5}(m_{1} + D_{c})^{-1}]\}.$$  

Summing up Eqs. (13) and (14), we arrive at

$$Tr[\gamma_{5}(m_{2} + D_{c})^{-1}\gamma_{5}(m_{1} + D_{c})^{-1}]
= \frac{Tr[(D_{c} + m_{1})^{-1} + (D_{c} + m_{2})^{-1}]}{m_{1} + m_{2}},$$

which is just the Gell-Mann-Oakes-Renner relation for the unequal mass case in Eq. (12). We should note that this relation is satisfied for any gauge configuration, any mass, and any source for the quark propagator as is in the equal mass case [21]. With the Gell-Mann-Oakes-Renner relation as the renormalization condition, the same relation holds for the renormalized currents. Together, one obtains the renormalization factor which includes the renormalization constant and the finite $ma$ correction

$$\hat{Z}_{P}(m_{1}a, m_{2}a)^{2} = \frac{Z_{S}(m_{1}a)\langle \bar{\psi}_{1}\psi_{1} \rangle + Z_{S}(m_{2}a)\langle \bar{\psi}_{2}\psi_{2} \rangle}{\langle \bar{\psi}_{1}\psi_{1} \rangle + \langle \bar{\psi}_{2}\psi_{2} \rangle} \frac{m_{1} + m_{2}}{Z_{S}(m_{1}a)^{-1}m_{1} + Z_{S}(m_{2}a)^{-1}m_{2}},$$

(16)
Figure 6: \( Z_P \) from Eq. (16) as a function of \( m_1 a \) with \( m_2 a \) fixed at 0.8. The fitted curve which is explained in the text is plotted as the solid line.

It is seen that for the massless case, the relation \( Z_P = Z_S \) is retrieved. Also, when the \( O(m^2a^2) \) error is negligible so that \( \tilde{Z}_S(m_1 a) = \tilde{Z}_S(m_2 a) \), one finds that \( \tilde{Z}_P(m_1 a, m_2 a) = \tilde{Z}_S(m a) = \tilde{Z}_P(m a) \). To assess the error for large \( m a \), say \( m a > 0.4 \), one can first calculate the scalar renormalization \[14, 33, 34\] and the quark condensate to obtain \( \tilde{Z}_P(m_1 a, m_2 a) \) in Eq. (16) which in turn determines \( \tilde{Z}_A(m_1 a, m_2 a) \) from the axial Ward identity in Eq. (11). This will account for the non-perturbative \( m a \) error for the axial current with unequal masses. We show, in Fig. 6, the result of \( \tilde{Z}_P(m_1 a, m_2 a) \) with \( Z_P \) determined in the \( \overline{MS} \) scheme at 2 GeV as a function of \( m_1 a \) and with \( m_2 a \) fixed at 0.8. This is obtained from \( \tilde{Z}_S(m a) \) in Fig. 3 and the quark condensates from the equal-mass Gell-Mann-Oakes-Renner relation.

We see that again the \( m a \) errors in \( \tilde{Z}_P(m_1 a, m_2 a) \) are exceedingly small. Fitting it to the form \( Z_P(1 + b_P (\Lambda_{QCD} m a + c_P m^2 a^2) (\Lambda_{QCD} a = 0.250) \), \( b_P = -0.076(245) \), and \( c_P = 0.066(74) \). We see that this value of \( \tilde{Z}_P(m_1 a, m_2 a) = 1.731(15) \) for \( m_1 a = 0 \) and \( m_2 a = 0.8 \) is within 1\% of \( Z_S = 1.718(12) \) (hence \( Z_P \)) as we presented earlier. Through Eq. (11), one is expected to obtain a non-perturbatively determined \( \tilde{Z}_A(0, m_2 a) \) which has only a few percent \( O(\Lambda_{QCD} m a^2) \) and \( O(m^2a^2) \) errors, even though \( m_2 a \) is as large as 0.8. Furthermore, a statistical error at a level of 1 – 2\% is obtained with 80 gauge configurations. From this study of the renormalization of the axial current for \( f_D \) and \( f_B \), we find that even with \( m_2 a \) as large as 0.5 – 0.6 the finite \( m a \) error is as small as a few per cent. This is a good deal better than the perturbative determination from NRQCD or the Fermilab approach which estimates a 10\% - 30\% error in the heavy-light decay constants [39, 40].

Finally, we should mention that the only major drawback of the overlap formalism is its numerical cost which is about 50 times more than that of the Wilson-Dirac
operator at \( \sim 1/5 \) of the strange mass \([19]\). This numerical overhead can be offset by extending the effective range of \( ma \) of the improved Wilson fermion by a factor of \( \sim 2.4 \) (as judged on the comparison of dispersion relations and finite \( ma \) errors in the renormalization) and the fact that the inversion of the overlap operator accommodates multi-mass algorithm \([21, 19]\) in which 20–30 masses can be included with only \( \sim 10\% \) overhead to the calculation of the lowest mass. For practical calculations, one may consider an anisotropic lattice with \( \xi = 5 \) and \( a_s^{-1} = 2 \text{ GeV}^{-1} \). Limiting \( m_Qa \) to 0.56, one maybe able to cover the quark spectrum from \( u/d \) to \( b \).

To conclude, we stress that the effective quark propagator of the lattice chiral fermions closely parallels that of the continuum. The mass is only an additive parameter to the chirally symmetric Dirac operator. The problems that plagued the previous light quark formulation for lack of chiral symmetry are basically removed by the lattice chiral fermions. The additional desirable features of the overlap operator such as the gentle critical slowing down, the multi-mass inversion, and the small \( O(m^2a^2) \) and \( O(ma^2) \) errors make it suitable for the study of both light and heavy quarks without tuning of the actions or the operators. Whether the small \( O(m^2a^2) \) and \( O(ma^2) \) errors hold for other quantities than the dispersion relation and the quark bilinear current renormalization remain to be checked. The generalized Gell-Mann-Oakes-Rener relation, extended to the unequal mass case, is shown to be able to facilitate the determination of the renormalization factor \( \tilde{Z}_A(m_1a, m_2a) \) for the calculation of the heavy-light decay constants and the semileptonic decay constants. This admits the assessment of the finite \( ma \) error and helps determine to which \( ma \) one should carry out the calculation without large systematic errors.

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Heavy and Light Quarks with Lattice Chiral Fermions

K.F. Liu and S.J. Dong
Dept. of Physics and Astronomy, University of Kentucky, Lexington, KY 40506

Abstract

The feasibility of using lattice chiral fermions which are free of $O(a)$ errors for both the heavy and light quarks is examined. The fact that the effective quark propagators in these fermions have the same form as that in the continuum with the quark mass being only an additive parameter to a chirally symmetric antihermitian Dirac operator is highlighted. This implies that there is no distinction between the heavy and light quarks and no mass dependent tuning of the action or operators as long as the discretization error $O(m^2a^2)$ is negligible. Using the overlap fermion, we find that the $O(m^2a^2)$ (and $O(ma^2)$) errors in the dispersion relations of the pseudoscalar and vector mesons and the renormalization of the axial-vector current and scalar density are small. This suggests that the applicable range of $ma$ may be extended to $\sim 0.56$ with only 5% error, which is a factor of $\sim 2.4$ larger than that of the improved Wilson action. We show that the generalized Gell-Mann-Oakes-Renner relation with unequal masses can be utilized to determine the finite $ma$ errors in the renormalization of the matrix elements for the heavy-light decay constants and semileptonic decay constants of the B/D meson.
Heavy-light quarkonia such as B and D mesons are the primary testing ground for understanding CP violation and obtaining the CKM matrix. Both experiment and theory are needed to extract relevant quantities. For example, $|V_{ub}|/|V_{cd}|$ can be determined from the semileptonic decay rate of B/D meson, $B/D \rightarrow \pi \nu$. But it depends on the transition form factor $|f_+(E)|$ from $B/D$ to $\pi$. $|V_{ub}|/|V_{cd}|$ can be extracted from the mass difference $\Delta m_B$ of the neutral $B - \bar{B}$ mesons where it also depends on the B parameter $B_B$ and the yet unmeasured leptonic decay width $f_B$. These quantities — $|f_+(E)|$, $B_B$, and $f_B$ are related to hadronic matrix elements which are best determined in lattice QCD which is a non-perturbative approach to solving QCD with controllable systematic errors [1, 2, 3].

The major challenge for incorporating heavy quarks with mass $m_Q$ on the lattice is that, in the range of lattice spacing that is amenable to numerical simulation nowadays, the condition $m_Q a \ll 1$ is far from being satisfied for the $b$ quark. There are several approaches to formulating the heavy quark on the lattice. The APE [4]-UKQCD [5] approach is to simulate with the quark mass around the charm with the $O(a)$ improved Wilson action and then extrapolate to the bottom with the guide of the heavy quark effective theory (HQET). Since the functional form for the mass dependence is not certain in this mass range and the extrapolated point is very far, this results in large errors [2, 3]. Another approach is the non-relativistic QCD (NRQCD) [6]. This involves an expansion in terms of the heavy quark mass which is considered an irrelevant dynamical scale. It has the advantage that it leads to faster numerical simulation for the heavy quark and the correction to the static limit with higher dimensional operators can be incorporated in perturbation. However, there are ‘renormalon shadow’ effects which reflect ‘large perturbative uncertainties in power divergent subtractions’ [2]. As an effective theory, it does not have a continuum limit. Thus, the discretization error cannot be removed by extrapolating the lattice results to $a \rightarrow 0$. The Fermilab formulation [7] bridges the above two approaches. For small masses, it has a continuum limit. With the heavy quark effective theory (HQET) approach [8, 9], both the lattice spacing $a$ and the inverse of the large quark mass are treated as short-distances so that the heavy quark discretization effects are lumped into the Wilson coefficients. It is shown in this case that the discretization errors due to the heavy quarks can be controlled to allow systematic reduction of the discretization errors for all $ma$.

The recent relativistic approach with anisotropic lattice [10] with the ratio $\xi = a_s/a_t$ between the spatial lattice spacing $a_s$ and the temporal spacing $a_t$ chosen to be 3–5 can alleviate the large $m_Q a_t$ problem to a degree, but it still suffers from a large $O(m_Q^2 a_t^2)$ error [11, 12]. To control the systematics to a few percent level for the dispersion relation, the condition $m_Q a_t < 0.2$ [11] must be met which is very stringent.

On the other side of the approaches to heavy-light calculations, the light quark used in the heavy-light simulation so far suffers from the well known set of problems
associated with the lack of chiral symmetry. Take the Wilson action for example; this ultra local action breaks chiral symmetry explicitly at finite lattice spacing in order to lift the doublers to the cut off. As a consequence, it induces numerous problems. The quark mass has an additive renormalization which is gauge configuration dependent. The quark condensate is mixed with unity which makes it harder to calculate. There is no unambiguous correspondence between the fermion zero modes and topology. It has $O(a)$ error and operators in different chiral sectors mix. Although the $O(a)$ error can be removed with the improved action and the mixing of operators can be taken into account, the procedure nevertheless requires fine tuning and is usually quite involved [13, 14]. The more serious problem is the existence of exceptional configurations. Since there is no protection by chiral symmetry, there can be zero modes even in the presence of finite and positive quark mass on certain gauge background configurations. This is getting more frequent when quark mass is less than $\sim 20$ MeV and it renders the region of pion mass less than $\sim 300$ MeV inaccessible. Unfortunately, this is the region where chiral behavior such as the chiral logs are becoming visible. Without admission to this region of low pion mass, reliable chiral extrapolation is not feasible [15].

With the advent of the recent lattice chiral fermions, such as the domain wall fermion [16], the overlap fermion [17], and the fixed-point-action fermion [18], all the above mentioned problems associated with the light quarks can be overcome in principle. In practice, it is shown in numerical simulations of the overlap fermion that there is indeed no additive quark mass renormalization [19], no exceptional configurations, and the current algebra such as the Gell-Mann-Oakes-Renner relation is satisfied to high precision [19]. Besides fulfilling the promise of removing the difficulties of the Wilson-like fermion, the overlap fermion has turned in extra bonuses. Its critical slowing down is quite gentle all the way to the physical pion mass [20, 15]; the $O(a^2)$ [19] and $O(m^2 a^2)$ [20] errors are apparently small, and it can incorporate the multi-mass inversion algorithm [21]. We will concentrate on the overlap fermion in this paper.

The massless overlap Dirac operator [17] is

$$D = 1 + \gamma_5 \epsilon(H),$$

where $\epsilon(H) = H/\sqrt{H^2}$ is the matrix sign function of $H$ which we take to be the Hermitian Wilson-Dirac operator, i.e. $H = \gamma_5 (D_w(0) - 1)$. Here $D_w(0)$ is the Wilson fermion operator with $\kappa = 1/8$. It is shown [22] that under the global lattice chiral flavor non-singlet transformation $\delta \psi = T \gamma_5 (1 - \frac{1}{2} D) \psi$, $\delta \bar{\psi} = \bar{\psi} (1 - \frac{1}{2} D) \gamma_5 T$, the fermion action $\bar{\psi} D \psi$ is invariant since the operator $D$ satisfies the Ginsparg-Wilson relation $\{ \gamma_5, D \} = D \gamma_5 D$ [23]. It can be shown that the flavor non-singlet scalar, pseudoscalar [25], vector, and axial [24, 25] bilinears in the form $\bar{\psi} K T (1 - \frac{1}{2} D) \psi$ ($K$ is the kernel which includes $\gamma$ matrices) transform covariantly as in the continuum. The $1 - \frac{1}{2} D$ factor is also understood as the lattice regulator which projects out the
unphysical real eigenmodes at $\lambda = 2$. For the massive case, the fermion action is $\bar{\psi}D\psi + ma\bar{\psi}(1 - \frac{1}{2}D)\psi$. In this case, the Dirac operator can be written as

$$D(m) = D + ma(1 - \frac{1}{2}D). \tag{2}$$

Let's consider the path-integral formulation of Green's function with the $\psi$ field in the operators and interpolation fields replaced by the lattice regulated field $\bar{\psi} = (1 - \frac{1}{2}D)\psi$. After the Grassmann integration, this regulator factor will be associated with the quark propagator in the combination $(1 - \frac{1}{2}D)D(m)^{-1}$ which can be written as

$$(1 - \frac{1}{2}D)D(m)^{-1} = (D_c + ma)^{-1}. \tag{3}$$

where the operator $D_c = D/(1 - \frac{1}{2}D)$ is chirally symmetric in the continuum sense, i.e. $\{\gamma_5, D_c\} = 0$; but, unlike $D$, it is non-local. This $D_c$ has been derived in the massless case [26, 27, 28] in association with the quark condensate and the solution of the Ginsparg-Wilson relation. For the massive case, it was pointed out [29, 30] that the mass term $ma$ should be added to the operator $D_c$ not $D$ in the quark propagator and Eq. (3) was derived [31] between the $N$-flavor low-energy effective Dirac operator $D_N^{eff}$ from the domain wall fermion and the truncated overlap operator $D_N$ with the overlap operator $D$ being the $N \to \infty$ and $a_5 \to 0$ limit of $D_N$. Here, we derive Eq. (3) from combining the lattice regulated field $\bar{\psi} = (1 - \frac{1}{2}D)\psi$ and the inverse of $D(m)$ to form the effective quark propagator. As a result, this effective quark propagator should be used together with local currents and interpolation fields without the $1 - \frac{1}{2}D$ factor. We shall highlight the fact that the effective quark propagator $(D_c + ma)^{-1}$ has the continuum form, i.e. the inverse propagator is the sum of an chirally symmetric operator and a real mass parameter. The mass in the quark propagator is the same bare mass $m$ introduced in the fermion action. It makes no distinction between a light quark and a heavy one, just as in the continuum.

It is interesting to point out that the original overlap operator $D(m)$ has eigenvalues lying on the circle due to the fact that $D$ satisfies Ginsparg-Wilson relation and is a normal matrix. This is shown in Fig. 1 where the radius is $1 - ma/2$. On the other hand, since $D_c$ is $\gamma_5$ hermitian, i.e. $D_c^\dagger = \gamma_5 D_c \gamma_5$ and anticommutes with $\gamma_5$, it is anti-hermitian. Its eigenvalues are simply the stereographic projection of the circle onto the imaginary axis as is in the massless case [30], except in the massive case, the eigenvalue of $D_c + ma$ is shifted by $ma$ to the right. There is a one-to-one correspondence between the eigenstates of $D(m)$ and $D_c + ma$, except the 'north pole' at the cut-off $\lambda = 2$ which is excluded by the regulator projector. This renders the propagator exactly like the continuum situation where the eigenvalues of the Euclidean Dirac operator $\slashed{D} + m$ are distributed on the shifted imaginary axis. Since the overlap fermion is invariant under the lattice chiral transformation, it does not mix with dimension five operators which are not chirally invariant. Therefore, there is no $O(a)$ nor $O(ma)$ error. The only question is how large the $O(m^2a^2)$ and
$O(ma^2)$ systematic errors are for different quark mass, be it light or heavy. We should stress that the above discussion is not limited to the overlap fermion. It also applies to other local lattice Dirac operators $D$ which satisfy normality, $\gamma_5$-hermiticity, and the Ginsparg-Wilson relation. In general, $D_c = D/(1 - \frac{1}{2}D)$ is the chirally symmetric operator for these lattice chiral fermions.

We first examine the $O(m^2a^2)$ and $O(ma^2)$ errors in the dispersion relation. It is suggested that dispersion relation is one of the places where one can discern the $ma$ error [10, 11]. We computed the pseudoscalar and vector meson masses and energies at several lattice momentum, i.e. $p_L a = \sqrt{n} 2\pi/La$ with $n = 0, 1, 2, 3$. The overlap quark propagators are calculated on the $16^3 \times 28$ quenched lattice with 80 configurations generated from Iwasaki guage action with $a = 2.00$ fm as determined from $f_\pi$ [15]. Following Refs. [10, 11], we fit the energies to the dispersion relation

$$\begin{align*}
(E(p)a)^2 &= c^2(pa)^2 + (E(0)a)^2
\end{align*}$$

where $p = 2s in(p_L a/2)$. The dispersion relation is so defined such that the $ma$ error is reflected in the deviation of $c$ (the effective speed of light) from unity.

We see in Figs. 2 and 3 that the effective speed of light $c$ is close to unity and quite flat all the way to $ma \sim 0.5$. Since there is no $O(ma)$ error, we fit it with the form quadratic in $a$, i.e. $c = c_0 + b(\Lambda_{QCD} a) ma + d m^2 a^2 (\Lambda_{QCD} a = 0.188)$, and find that...
\[ E_p^2 = c^2 \cdot p^2 + m_p^2 \]

Figure 2: The effective speed of light \( c \) from the pseudoscalar meson dispersion relation as a function of \( ma \).

\( c_0 = 0.982(10), b = 0.580(346), \) and \( d = -0.279(87) \) with \( \chi^2 / N_{\text{dof}} = 0.1 \) for the case of the pseudoscalar meson and \( c_0 = 1.027(26), b = -0.32(90), \) and \( d = -0.18(22) \) with \( \chi^2 / N_{\text{dof}} = 0.6 \) for the vector meson. Using this to gauge how large the \( ma \) error is, we see that the systematic error is less than \( \sim 4\%(6\%) \) for the pseudoscalar(vector) case up to \( ma \sim 0.56 \). This \( ma \) is \( \sim 2.4 \) times larger than that is admitted in the study of improved Wilson action [11] where it is found that the \( O(m^2a^2) \) error from the anisotropy of the dispersion relation is less than \( \sim 5\% \) when \( m_Qa_i < 0.23 \). Therefore with the overlap fermion, one can hope to extend the range of \( ma \) to \( 0.5 - 0.56 \) where the systematic error is still reasonably small.

Next, we address the issue of \( O(ma) \) improvement. This is essential for the Wilson-type fermions which has large \( O(a) \) error. The \( O(a) \) improvement for the action is usually done with the addition of the clover term. The operator improvement is more involved. We shall illustrate this by considering the axial Ward identity. It has been shown [13] that in the improved mass-independent renormalization scheme, the renormalized improved axial current and pseudoscalar density have the following form from an \( O(a) \) improved action

\[
\begin{align*}
A_\mu^R &= Z_A(1 + b_A m_q a) \{ A_\mu + c_A a \bar{\partial}_\mu P \}, \\
P^R &= Z_P(1 + b_P m_q a) P, 
\end{align*}
\]

(5)

where \( m_q = m - m_c \) is the subtracted quark mass and \( c_A, b_A \) and \( b_P \) are improvement coefficients. The renormalization constants \( Z_A \) and \( Z_P \) are functions of the modified coupling \( \tilde{g}_0^2 = g_0^2(1 + b_q m_q a) \). It is argued [2] that as \( m_Q \to \infty \) \( A_\mu^R \) goes to \( -\infty \) instead of the static limit, since \( c_A < 0 \) and \( \partial_\mu P \propto m_Q \). Even when \( m_Q a \ll 1/4 \), to calculate the six parameters \(- m_c, b_A, c_A, b_P, Z_P, \) and \( Z_A \) non-perturbatively in order
Figure 3: The effective speed of light $c$ from the vector meson dispersion relation as a function of $ma$. 

to satisfy the chiral Ward identity [14] is quite a task.

The situation with the lattice chiral fermion is much simpler. The overlap fermion is $O(a)$ improved, and the quark mass is not additively renormalized which is verified numerically [19]. As a result, $b_A = c_A = b_P = m_c = 0$. If one uses the axial current $A_\mu = \bar{\psi} i\gamma_\mu \gamma_5 \psi$ for simplicity, the renormalization constant $Z_A$ can be obtained through the axial Ward identity

$$Z_A \partial_\mu A_\mu = 2Z_m m Z_P P, \quad (6)$$

where $P = \bar{\psi} \gamma_5 \psi$. Since $Z_m = Z_S^{-1}$ and $Z_S = Z_P$ due to the fact the scalar density $\bar{\psi} \psi$ and the pseudoscalar density $\bar{P}$ are in the same chiral multiplet, $Z_m$ and $Z_P$ cancel in Eq. (6) and one can directly determine $Z_A$ non-perturbatively from the axial Ward identity using the bare mass $m$ and bare operator $P$. To avoid $O(a^2)$ error introduced by the derivative in Eq. (6), one can consider the axial Ward identity for the on-shell matrix elements between the vacuum and the zero-momentum pion state. In this case,

$$Z_A = \lim_{m \to 0, t \to \infty} \frac{2m G_{PP}(\vec{p} = 0, t)}{m_\pi G_{A_A P}(\vec{p} = 0, t)}, \quad (7)$$

where $G_{PP}(\vec{p} = 0, t)$ and $G_{A_A P}(\vec{p} = 0, t)$ are the zero-momentum pseudoscalar-pseudoscalar and axial-pseudoscalar correlators. In the mass-independent renormalization scheme [13], the renormalization factor for the axial current matrix element which takes into account the finite $ma$ errors can be defined from Eq. (7) without taking the massless limit and neglecting the small finite $ma$ difference between the
Figure 4: $\hat{Z}_A(ma)$ from the axial Ward identity on a $20^4$ lattice with $a = 0.148$ fm. The fitted curve which is explained in the text is plotted as the solid line.

The renormalization of the pseudoscalar and scalar densities \[33, 34\],

$$
\hat{Z}_A(ma) = \lim_{t \to \infty} \frac{2mG_{pp}(\vec{p} = 0, t)}{m\pi G_{A\pi}(\vec{p} = 0, t)}.
$$

(8)

Up to $O(ma^2)$ and $O(m^2a^2)$, $\hat{Z}_A(ma)$ is

$$
\hat{Z}_A(ma) = Z_A(1 + b_A (\Lambda_{QCD} a) ma + c_A m^2a^2),
$$

(9)

which is the combination of the renormalization constant $Z_A$ and the finite $ma$ effects. The resultant $\hat{Z}_A(ma)$ on a quenched $20^4$ lattice with overlap fermion for $ma$ ($a = 0.148$ fm) from 0.01505 to 0.2736 were reported before [20]. Now we show the results extended to $ma = 0.684$ in Fig. 4. We see that, similar to the effective speed of light $c$, it is quite flat all the way to $ma \sim 0.5$. In analogy to fitting $c$, we fit it with the form in Eq. (9) ($\Lambda_{QCD} a = 0.188$), and find that $Z_A = 1.592(5)$, $b_A = -0.13(9)$, and $c_A = 0.203(22)$ with $\chi^2/N_{dof} = 0.46$. We also have results on a $16^3 \times 28$ lattice with $a = 0.200$ fm and found an almost identical $ma$ behavior for $\hat{Z}_A(ma)$ [32]. Again using this to gauge how large the $ma$ error is, we see that the systematic error is less than $\sim 5\%$ up to $ma \sim 0.56$ which is close to the case of $c$ for the dispersion relation for the pseudoscalar and vector mesons in Figs. 2 and 3.

The same small discretization errors in $ma$ are reflected in other renormalization factors which are defined, similar to $\hat{Z}_A(ma)$ in Eq. (9), as

$$
\hat{Z}_\Gamma(ma) = Z_\Gamma(1 + b_\Gamma (\Lambda_{QCD} a) ma + c_\Gamma m^2a^2),
$$

(10)

for $\Gamma = S, P, V$, and $T$ \[33, 34\]. In the mass-independent renormalization scheme \[13\], the renormalization constant $Z_\Gamma$ is a function of the gauge coupling $g_0^2$ and the renormalization scale $\mu$, i.e. $Z_\Gamma = Z_\Gamma(g_0^2, \mu a)$. Here in Fig. 5, we show the $ma$ dependence...
Figure 5: $\tilde{Z}_S(\ma)$ with $Z^{\overline{MS}}_S(2\text{ GeV})$ from the non-perturbative renormalization on a $16^3 \times 28$ lattice with $a = 0.200$ fm. The fitted curve which is explained in the text is plotted as the solid line.

in $\tilde{Z}_S(\ma)$ at $\mu = 2$ GeV, where the renormalization constant $Z^{\overline{MS}}_S(2\text{GeV})$ is obtained from the non-perturbative renormalization in the regularization independent scheme [35, 36] and then perturbatively matched to the $\overline{MS}$ scheme at the scale of 2 GeV. The results are from a quenched $16^3 \times 28$ lattice with $a = 0.200$ fm [34]. Again, we see that it is rather flat from $\ma = 0$ to $\ma = 0.8$. Fitting to the form in Eq. (10) $(\Lambda_{QCD} a = 0.250)$ yields $Z_S = 1.718(12)$, $b_S = -0.002(194)$, $c_S = 0.073(58)$. It gives a $\ma$ error of 2.6\% at $\ma = 0.6$. We should mention that similar studies for the $\ma$ errors for $\tilde{Z}_A(\ma)$ and $\tilde{Z}_V(\ma)$ are done with domain-wall fermions [37, 38]. The $\ma$ errors seem to be larger than those found here. For example, the finite $\ma$ error at $m_f a = 0.10$ is already found to be 3 - 4\% which is much larger than what we obtain for the overlap fermion at the same $ma$.

For the heavy-light quarkonium, an accurate renormalization for the vector and axial current is essential for the study of $f_B$, $f_D$ and the semi-leptonic decays of the B and D mesons. Since most of the renormalization for the composite operator with heavy and light quarks are done with perturbation in one loop, its $O(\alpha_s^2)$ correction can be large. In a recent calculation of $f_{B_s}$ and $f_{D_s}$ with NRQCD for the heavy quark, the $O(\alpha_s^2)$ error is estimated to be 10\% [39]. Similarly, it is pointed out in the study of $f_{D_s}$ [40] with fermilab heavy quark that the $O(\alpha_s)$ correction can be potentially as large as 30\%. In the following, we show a non-perturbative method which can determine the axial heavy-light current renormalization with finite $\ma$ error at a few percent level even with $\ma$ as large as 0.5 - 0.6. This should be of help in determining $f_B$ and $f_D$ with much less systematic errors.

The finite $\ma$ errors in the renormalization of the matrix elements involving $f_B$
and $f_D$ and semi-leptonic decays can be accurately determined with the help of current algebra relations. The axial Ward identity for the pseudoscalar meson $P$ decay matrix element of unequal masses including the finite $m a$ factor is

$$\bar{Z}_A(m_1 a, m_2 a) \langle 0 | \partial_{\mu} A_{\mu 12} | P \rangle = (\bar{Z}_m(m_1 a) m_1 + \bar{Z}_m(m_2 a) m_2) \bar{Z}_P(m_1 a, m_2 a) \langle 0 | P_{12} | P \rangle,$$  

(11)

where $A_{\mu 12} = \bar{\psi}_1 i \gamma_\mu \gamma_5 \psi_2$, $P_{12} = \bar{\psi}_1 i \gamma_5 \psi_2$ and $\bar{Z}_A(m_1 a, m_2 a)$, $\bar{Z}_m(m a)$, and $\bar{Z}_P(m_1 a, m_2 a)$ are the products of renormalization constants and their respective finite $ma$ factors. Here $\bar{Z}_P(m_1 a, m_2 a)$ does not cancel out $\bar{Z}_m(m_1 a)/\bar{Z}_m(m_2 a)$ except in the massless limit and, therefore, one cannot readily use Eq. (7) to obtain $\bar{Z}_A(m_1 a, m_2 a)$ to account for the finite $ma$ correction. Fortunately, one can adopt additional information from the generalized Gell-Mann-Oakes-Renner relation for the unequal mass case, which is

$$\frac{1}{V} \int d^4x \langle \pi_{12}^a \rangle(x) \pi_{12}^a(0) = \frac{-2[\langle \bar{\psi}_1 \psi_1 \rangle + \langle \bar{\psi}_2 \psi_2 \rangle]}{m_1 + m_2},$$  

(12)

where $\pi_{12}^a(x) = \bar{\psi}_1 \gamma_5 \tau^a / 2 \gamma_2$. The proof is a generalization of the equal mass case [41] and it has been proved with the staggered fermion [42]. In fact, with the effective propagator in Eq. (3), a lot of the current algebra relations can be reproduced on the lattice with finite cutoff [43, 21]. From Eq. (3), we see that $\langle \bar{\psi}_1 \psi_1 \rangle = -Tr(D_c + m_1)^{-1}$ which can be written as

$$Tr\ (D_c + m_1)^{-1} = Tr\{(m_2 - D_c)(m_2 - D_c)^{-1}(m_1 + D_c)^{-1}\}$$

$$= Tr\{((m_2 - D_c)\gamma_5 m_2 + D_c)^{-1}\gamma_5 (m_1 + D_c)^{-1}\},$$  

(13)

where we have used the property $\gamma_5 D_c \gamma_5 = -D_c$. Similarly, one can write

$$Tr\ (D_c + m_2)^{-1} = Tr[\gamma_5 (m_2 + D_c)^{-1}\gamma_5]$$

$$= Tr\{(m_1 + D_c)\gamma_5 (m_2 + D_c)^{-1}\gamma_5 (m_1 + D_c)^{-1}\}. $$  

(14)

Summing up Eqs. (13) and (14), we arrive at

$$Tr[\gamma_5 (m_2 + D_c)^{-1}\gamma_5 (m_1 + D_c)^{-1}]$$

$$= \frac{Tr[(D_c + m_1)^{-1} + (D_c + m_2)^{-1}]}{m_1 + m_2},$$  

(15)

which is just the Gell-Mann-Oakes-Renner relation for the unequal mass case in Eq. (12). We should note that this relation is satisfied for any gauge configuration, any mass, and any source for the quark propagator as is in the equal mass case [21]. With the Gell-Mann-Oakes-Renner relation as the renormalization condition, the same relation holds for the renormalized currents. Together, one obtains the renormalization factor which includes the renormalization constant and the finite $ma$ correction

$$\bar{Z}_P(m_1 a, m_2 a)^2 = \frac{\bar{Z}_S(m_1 a) \langle \bar{\psi}_1 \psi_1 \rangle + \bar{Z}_S(m_2 a) \langle \bar{\psi}_2 \psi_2 \rangle}{\langle \bar{\psi}_1 \psi_1 \rangle + \langle \bar{\psi}_2 \psi_2 \rangle} = \frac{m_1 + m_2}{\bar{Z}_S(m_1 a)^{-1} m_1 + \bar{Z}_S(m_2 a)^{-1} m_2}.$$

(16)
Figure 6: $\tilde{Z}_P$ from Eq. (16) as a function of $m_1 a$ with $m_2 a$ fixed at 0.8. The fitted curve which is explained in the text is plotted as the solid line.

It is seen that for the massless case, the relation $Z_P = Z_S$ is retrieved. Also, when the $O(m^2 a^2)$ error is negligible so that $\tilde{Z}_S(m_1 a) = \tilde{Z}_S(m_2 a)$, one finds that $\tilde{Z}_P(m_1 a, m_2 a) = \tilde{Z}_S(m a) = \tilde{Z}_P(m a)$. To assess the error for large $m a$, say $m a > 0.4$, one can first calculate the scalar renormalization [44, 33, 34] and the quark condensate to obtain $\tilde{Z}_P(m_1 a, m_2 a)$ in Eq. (16) which in turn determines $\tilde{Z}_A(m_1 a, m_2 a)$ from the axial Ward identity in Eq. (11). This will account for the non-perturbative $m a$ error for the axial current with unequal masses. We show, in Fig. 6, the result of $\tilde{Z}_P(m_1 a, m_2 a)$ with $Z_P$ determined in the $\overline{MS}$ scheme at 2 GeV as a function of $m_1 a$ and with $m_2 a$ fixed at 0.8. This is obtained from $\tilde{Z}_S(m a)$ in Fig. 3 and the quark condensates from the equal-mass Gell-Mann-Oakes-Renner relation.

We see that again the $m a$ errors in $\tilde{Z}_P(m_1 a, m_2 a)$ are exceedingly small. Fitting it to the form $Z_P(1 + b_P (\Lambda QCD a) m a + c_P m^2 a^2) (\Lambda QCD a = 0.250)$ gives $Z_P = 1.731(15)$, $b_P = -0.076(245)$, and $c_P = 0.066(74)$. We see that this value of $\tilde{Z}_P(m_1 a, m_2 a) = 1.731(15)$ for $m_1 a = 0$ and $m_2 a = 0.8$ is within 1% of $Z_S = 1.718(12)$ (hence $Z_P$) as we presented earlier. Through Eq. (11), one is expected to obtain a non-perturbatively determined $\tilde{Z}_A(0, m_2 a)$ which has only a few percent $O(\Lambda QCD a^2)$ and $O(m^2 a^2)$ errors, even though $m_2 a$ is as large as 0.8. Furthermore, a statistical error at a level of 1 – 2% is obtained with 80 gauge configurations. From this study of the renormalization of the axial current for $f_D$ and $f_B$, we find that even with $m_2 a$ as large as 0.5 – 0.6 the finite $m a$ error is as small as a few per cent. This is a good deal better than the perturbative determination from NRQCD or the Fermilab approach which estimates a 10% – 30% error in the heavy-light decay constants [39, 40].

Finally, we should mention that the only major drawback of the overlap formalism is its numerical cost which is about 50 times more than that of the Wilson-Dirac
operator at $\sim 1/5$ of the strange mass [19]. This numerical overhead can be offset by extending the effective range of $ma$ of the improved Wilson fermion by a factor of $\sim 2.4$ (as judged on the comparison of dispersion relations and finite $ma$ errors in the renormalization) and the fact that the inversion of the overlap operator accommodates multi-mass algorithm [21, 19] in which 20–30 masses can be included with only $\sim 10\%$ overhead to the calculation of the lowest mass. For practical calculations, one may consider an anisotropic lattice with $\xi = 5$ and $a_s^{-1} = 2 \text{GeV}^{-1}$. Limiting $m_{QA_i}$ to 0.56, one maybe able to cover the quark spectrum from $u/d$ to $b$.

To conclude, we stress that the effective quark propagator of the lattice chiral fermions closely parallels that of the continuum. The mass is only an additive parameter to the chirally symmetric Dirac operator. The problems that plagued the previous light quark formulation for lack of chiral symmetry are basically removed by the lattice chiral fermions. The additional desirable features of the overlap operator such as the gentle critical slowing down, the multi-mass inversion, and the small $O(m^2a^2)$ and $O(ma^2)$ errors make it suitable for the study of both light and heavy quarks without tuning of the actions or the operators. Whether the small $O(m^2a^2)$ and $O(ma^2)$ errors hold for other quantities than the dispersion relation and the quark bilinear current renormalization remain to be checked. The generalized Gell-Mann-Oakes-Renner relation, extended to the unequal mass case, is shown to be able to facilitate the determination of the renormalization factor $Z_A(m_1a, m_2a)$ for the calculation of the heavy-light decay constants and the semileptonic decay constants. This admits the assessment of the finite $ma$ error and helps determine to which $ma$ one should carry out the calculation without large systematic errors.

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