Shape derivative of the energy functional for the bending of elastic plates with thin defects

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Abstract. The paper deals with an equilibrium problem for a homogeneous isotropic elastic plate with a thin rigid inclusion and interfacial crack. We provide an explicit formula for the first shape derivative of the energy functional in the direction of a given vector field by means of a volume integral. For specific examples of the vector field, we derive some representations of the formula in terms of path-independent contour integrals.

1. Setting of the problem
Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with boundary \( \partial \Omega \) of the class \( C^{1,1} \) and let \( \gamma \) be a smooth curve without self-intersections. We assume that \( \gamma \) can be extended to a closed curve \( \Sigma \) of the class \( C^{1,1} \) without self-intersections so that \( \Omega \) is divided into two subdomains \( \Omega_1 \) and \( \Omega_2 \). The boundary of \( \Omega_1 \) is \( \Sigma \), and the boundary of \( \Omega_2 \) is \( \Sigma \cup \partial \Omega \). The outward pointing unit normals to \( \Sigma \) and \( \partial \Omega \) are denoted by \( \nu = (\nu_1, \nu_2) \) and \( n = (n_1, n_2) \), respectively. Also, we put \( \Omega_\gamma = \Omega \setminus \gamma \) and assume that the domain \( \Omega \) can be divided into two subdomains \( \Omega_3 \) and \( \Omega_4 \) with Lipschitz boundaries such that \( \gamma \subset \partial \Omega_3 \cap \partial \Omega_4 \) and \( H^1(\partial \Omega_i \cap \partial \Omega) > 0 \), \( i = 3, 4 \), where \( H^1 \) denotes the one-dimensional Hausdorff measure. The last condition guarantees that Friedrichs–Poincaré’s inequality is satisfied in the non-Lipschitz domain \( \Omega_\gamma \). The geometrical setup is shown in Figure 1.

The domain \( \Omega_\gamma \) is occupied by the middle plane of a homogeneous isotropic elastic plate, and the curve \( \gamma \) corresponds to a thin rigid inclusion. To describe the vertical deflection of the thin rigid inclusion, we introduce the space of rigid displacements

\[
L(\gamma) = \{ l(x) = a_0 + a_1 x_1 + a_2 x_2 \mid a_i \in \mathbb{R}, \ i = 0, 1, 2, \ x \in \gamma \}.
\]

In conformity with positive and negative directions of the normal \( \nu \), there are positive crack face \( \gamma^+ \) and negative crack face \( \gamma^- \), and \( v^\pm \) are the traces of a function \( v \) on \( \gamma^\pm \). The inclusion \( \gamma \) is delaminated on \( \gamma^+ \) so that there is a crack on the interface between two media.

We are now in a position to formulate the equilibrium problem (see [1, 2]). For a given
external force $f \in C^1(\Omega)$, we want to find two functions $w$ and $l_0 \in L(\gamma)$ satisfying

$$D\Delta^2 w = f \quad \text{in} \quad \Omega_\gamma,$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,$$

$$m^-_\nu = 0, \quad w^- = l_0 \quad \text{on} \quad \gamma,$$

$$m^+_\nu = t^\nu_\nu = 0 \quad \text{on} \quad \gamma,$$

$$\int_{\gamma^-} t^-_\nu \bar{w} \, ds_x = 0 \quad \forall \bar{w} \in K^0.$$

Here the set $K^0$ of admissible displacements is

$$K^0 = \{ w \in H^2_{\partial \Omega}(\Omega_\gamma) \mid w|_{\gamma^-} \in L(\gamma) \},$$

where the Sobolev space $H^2_{\partial \Omega}(\Omega_\gamma)$ is defined as

$$H^2_{\partial \Omega}(\Omega_\gamma) = \{ w \in H^2(\Omega_\gamma) \mid w = \partial w/\partial n = 0 \text{ on} \partial \Omega \}.$$

The bending moment $m_\nu$ and transverse force $t_\nu$ are given by the relations

$$m_\nu = D \left( \kappa \Delta w + (1 - \kappa) \frac{\partial^2 w}{\partial \nu^2} \right),$$

$$t_\nu = D \frac{\partial}{\partial \nu} \left( \Delta w + (1 - \kappa) \frac{\partial^2 w}{\partial \tau^2} \right), \quad \tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1),$$

where $D$ is the flexural rigidity and $\kappa$ is the Poisson ratio, $0 < \kappa < 1/2$.

Equation (1) is a equilibrium equation, the boundary conditions (2) correspond to the clamped outer edge of the plate. The nonlocal boundary conditions (5) ensure the vanishing of the resultant force and moment along $\gamma^-$. The boundary-value problem (1)–(5) admits a variational statement. Indeed, let us define the bilinear form

$$B(u, v) = D \left( u_{11}v_{1,11} + u_{22}v_{2,22} + \kappa u_{11}v_{22} + \kappa u_{22}v_{11} + 2(1 - \kappa)u_{12}v_{12} \right).$$
Subscripts following a comma represent differentiation, so that, for example, \( u_{ij} = \partial^2 u / \partial x_i \partial x_j \). Then problem (1)–(5) corresponds to minimization of the energy functional

\[
\Pi(\Omega; w) = \frac{1}{2} \int_{\Omega} B(w, w) \, dx - \int_{\Omega} f \, w \, dx
\]

over the subspace \( K^0 \) and can be written in the form of the weak Euler–Lagrange equation

\[
w \in K^0, \quad \int_{\Omega} B(w, \bar{w}) \, dx - \int_{\Omega} f \, \bar{w} \, dx = 0 \quad \forall \bar{w} \in K^0.
\]

Since the functional \( \Pi \) is coercive and weakly lower semicontinuous over the set \( K^0 \), equation (6) has a (unique) solution.

2. Shape derivative of the energy functional

We next turn our attention to the first shape derivative of the energy functional in the direction of a given vector field. We apply the technique of shape sensitivity analysis in order to demonstrate that this derivative is well defined and derive an explicit representation to it.

For a small non-negative parameter \( \delta \), let us consider a one-parameter family of perturbations of the initial domain \( \Omega \), which is governed by a transformation \( T_\delta \). To be specific, we make use of the transformation in the form

\[
T_\delta(x) = x + \delta V(x),
\]

where \( V(x) = (V_1(x), V_2(x)) \) and \( V \) has class \( W^{2,\infty}(\mathbb{R}^2)^2 \), or, in coordinate form,

\[
y_1 = x_1 + \delta V_1(x), \quad y_2 = x_2 + \delta V_2(x), \quad x = (x_1, x_2) \in \Omega.
\]

Transformation (7) defines the perturbed domain \( T_\delta(\Omega) \) and the perturbed rigid inclusion \( \gamma_\delta = T_\delta(\gamma) \) for each fixed \( \delta \). For simplicity, we assume that the vector field \( V \) has the compact support in \( \Omega \). In this case, the outward pointing unit normal \( n \) of the domain \( \Omega \) transforms to the outward unit normal \( n_\delta \) of the domain \( T_\delta(\Omega) \). We define the perturbed domain with the thin rigid inclusion as \( \Omega_\gamma = T_\delta(\Omega) \setminus \gamma_\delta \). We assume that for each \( \delta \) the domain \( \Omega_\gamma \) satisfies the last condition from the first paragraph of Section 1.

The set of admissible displacements associated with the perturbed problem is

\[
K^\delta = \left\{ w \in H^2_{\partial \Omega}(\Omega) \mid w|_{\gamma_\delta} \in L(\gamma_\delta) \right\}.
\]

As before, there exists a unique solution of the weak Euler–Lagrange equation

\[
w^\delta \in K^\delta, \quad \int_{\Omega_\gamma} B(w^\delta, \bar{w}) \, dx - \int_{\Omega_\delta} f \, \bar{w} \, dx = 0 \quad \forall \bar{w} \in K^\delta.
\]

For the solution \( w^\delta \) of equation (8), we can find the potential deformation energy

\[
\Pi(\Omega_\gamma; w^\delta) = \frac{1}{2} \int_{\Omega_\gamma} B(w^\delta, w^\delta) \, dy - \int_{\Omega_\delta} f w^\delta \, dy.
\]

With these preliminaries, we introduce the first shape derivative of the energy functional in the direction of the vector field \( V \) as

\[
D_{V} \Pi(\Omega; w) = \lim_{\delta \downarrow 0} \frac{\Pi(\Omega_\gamma; w^\delta) - \Pi(\Omega; w)}{\delta}.
\]
The question now is whether (9) is well defined and whether there exist simple formulae to calculate it. The first main result of our study is the following theorem.

**Theorem 1.** The first shape derivative of the energy functional in the direction of the vector field $V$ exists and equals

$$
\mathcal{D}_V \Pi(\Omega_\gamma; w) = \int_{\Omega_\gamma} \left( \frac{1}{2} B(w, w) \text{div} V + B(w, \widetilde{V}) - 2D \left( V_{1,1} w_{11}^2 + \kappa w_{11} w_{22} \text{div} V \right)
+ (1 - \kappa) w_{12}^2 \text{div} V + V_{2,2}^2 + (V_{1,2} + V_{2,1}) w_{12} (w_{11} + w_{22}) \right)
- D((w_{11} + \kappa w_{22}))(V_{1,1} w_{11} + V_{2,1} w_{22})
+ (\kappa w_{11} + w_{22})(V_{1,2} w_{11} + V_{2,2} w_{12}) + 2(1 - \kappa)(V_{1,12} w_{11} + V_{2,12} w_{12}) \right) dx
- \int_{\Omega_\gamma} \left( \text{div}(fV) + f\widetilde{V} \right) dx,
$$

(10)

where the vector field $\widetilde{V}$ is such that $\widetilde{V}(x) = a_0^0 V_1(x) + a_0^0 V_2(x)$, and the constants $a_0^0$ and $a_0^0$ correspond to the function $l_0$.

**Remark.** The first shape derivative of the energy functional in the direction of the vector field $V$ is independent of the particular choice of the coordinate transformation $T_3$ in the following sense. If two vector fields $V_1$ and $V_2$ are such that the corresponding coordinate transformations $T_3^1$ and $T_3^2$ map the initial domain $\Omega_\gamma$ into the same perturbed domain $\Omega_\delta$ for each $\delta$, then $\mathcal{D}_{V_1} \Pi(\Omega_\gamma; w) = \mathcal{D}_{V_2} \Pi(\Omega_\gamma; w)$. This easily follows from the definition of the shape derivative and the uniqueness of the weak solution $w_\delta \in K^\delta$.

Let us mention the main ingredients in the proof of Theorem 1. Following ideas of [3], we calculate limit (9) by transforming the integrals expressions defined in the perturbed domain $\Omega_\delta$ back to the initial domain $\Omega_\gamma$. The transformation $T_3 : \Omega_\gamma \to \Omega_\delta$ induces an isomorphism between the spaces $H^2_{\partial\Omega}(\Omega_\delta)$ and $H^2_{\partial\Omega}(\Omega_\gamma)$ in the following way:

$$
T_3 : H^2_{\partial\Omega}(\Omega_\delta) \to H^2_{\partial\Omega}(\Omega_\gamma) : w \mapsto w \circ T_3.
$$

The lack of the one-to-one correspondence between the sets of admissible displacements $K^\delta$ and $K^0$ under the mapping $T_3$ is the main obstacle in the proof of the theorem. To overcome this difficulty, we denote by $K_\delta$ the image of the admissible set $K^\delta$ under the mapping $T_3$.

Straightforward calculations yield that

$$
K_\delta = \{ w \in H^2_{\partial\Omega}(\Omega_\gamma) \mid w_\gamma^- \in L_\delta(\gamma) \},
$$

where

$$
L_\delta(\gamma) = \{ l(x) = a_0 + a_1 x_1 + a_2 x_2 + a_3 \delta V_1(x) + a_4 \delta V_2(x) \mid a_i \in \mathbb{R}, i = 0, 1, 2, \; x \in \gamma \}.
$$

Hence, $T_3$ is also an isomorphism between the sets $K^\delta$ and $K_\delta$. Denote by $w_\delta(x)$ the image of $w^\delta(y)$ under the mapping $T_3$. The next lemmata play a key role in the derivation of the explicit formula for (9).

**Lemma 1.** Let $w \in K^0$ be the solution of problem (6). Then there exist functions $w^1_\delta$ and $w^2_\delta$ satisfying the inclusions

$$
w + \delta w^1_\delta \in K_\delta, \quad w_\delta - \delta w^2_\delta \in K^0
$$

and uniform in $\delta$ estimates

$$
\|w^i_\delta\|_{H^2_{\partial\Omega}(\Omega_\gamma)} \leq c, \quad i = 1, 2.
$$

(11)
We mention that the functions \( w_\delta^1 \) and \( w_\delta^2 \) can be defined explicitly by the relations

\[
w_\delta^1 = a_1^\delta V_1 + a_2^\delta V_2, \quad w_\delta^2 = a_1^\delta V_1 + a_2^\delta V_2.
\]

Using Lemma 1, we establish that \( w_\delta \) is Hölder continuous with exponent 1/2.

**Lemma 2.** There exists a positive constant \( c \) such that

\[
\|w_\delta - w\|_{H^2_0(\Omega_\gamma)} \leq c\sqrt{\delta}.
\]

**Corollary.** \( w_\delta^2 \rightarrow \tilde{V} \) strongly in \( H^2_{\partial \Omega}(\Omega_\gamma) \) as \( \delta \downarrow 0 \).

We next apply transformation (7) to the integrals involved in \( \Pi(\Omega_\delta; \tilde{w}) \) to obtain a functional \( \Pi(\Omega_\delta; \tilde{w}_\delta) \) that defines over the space \( H^2_{\partial \Omega}(\Omega_\gamma) \) and admits the following asymptotic expansion as \( \delta \downarrow 0 \):

\[
\Pi(\Omega_\gamma; \tilde{w}_\delta) = \Pi(\Omega_\gamma; \tilde{w}) + \delta \mathcal{D}_V \Pi(\Omega_\gamma; \tilde{w}_\delta) + o(\delta),
\]

with the form \( \mathcal{D}_V \Pi(\Omega_\gamma; \tilde{w}_\delta) \) from (10).

Since the mapping \( T_\delta \) is an isomorphism between the sets \( K^\delta \) and \( K_\delta \), it follows that

\[
\frac{\Pi(\Omega_\gamma; \tilde{w}) - \Pi(\Omega_\gamma; w)}{\delta} = \frac{\Pi(\Omega_\gamma; \tilde{w}_\delta) - \Pi(\Omega_\gamma; w)}{\delta}.
\]

Taking into account that \( w \) is a minimizer of \( \Pi(\Omega_\delta, \cdot) \) over the set \( K^\delta \) and \( w_\delta \) is a minimizer of \( \Pi(\Omega_\delta, \cdot) \) over the set \( K_\delta \), we invoke Lemma 1, for each \( \delta > 0 \), to deduce the following chain of inequalities:

\[
\frac{\Pi(\Omega_\gamma; \tilde{w}_\delta) - \Pi(\Omega_\gamma; \tilde{w})}{\delta} = \frac{\Pi(\Omega_\gamma; \tilde{w} - \delta w_\delta^2) - \Pi(\Omega_\gamma; \tilde{w})}{\delta} \leq \frac{\Pi(\Omega_\gamma; \tilde{w} + \delta w_\delta^1) - \Pi(\Omega_\gamma; \tilde{w})}{\delta}.
\]

Applying Lemma 2 and its Corollary, we have no difficulty in calculating the limits as \( \delta \downarrow 0 \) of the right-hand and left-hand sides in (12) and checking that they are finite, coincide with each other, and equal to \( \mathcal{D}_V \Pi(\Omega_\gamma; \tilde{w}_\delta) \) from (10). This completes the sketch of the proof of Theorem 1.

We conclude this section by mentioning the recent papers [4, 5, 6, 7], where a similar scheme for deriving of explicit formulae to the first shape derivative of the energy functional in the direction of a given vector field can be found for elastic models (2D elasticity and Timoshenko plates) with embedded rigid and semirigid inclusions and interfacial cracks.

### 3. Path-independent integrals

Our intention next is to consider specific examples of the vector field \( V \) that lead us to path-independent energy integrals via transformations of formula (10). To do this, it is necessary to have more information on the regularity of the transverse deflection \( w \) in comparison with the variational one. We next assume that the thin rigid inclusion \( \gamma \) is rectilinear, i.e., it lies in the line \( x\nu = a, a = \text{const} \), with the tips \( C_1 \) and \( C_2 \). In this case, the standard regularity theory for uniformly elliptic differential operators shows that the solution \( w \) of problem (6) possess \( H^4 \)-regularity up to the faces \( \gamma^+ \) and \( \gamma^- \) except the thin inclusion tips \( C_1 \) and \( C_2 \).

In all cases below, we choose neighborhoods \( S \) and \( S_1 \) with Lipschitz boundaries \( \partial S \) and \( \partial S_1 \). We also assume that the boundaries of the domains \( (S_1 \setminus S) \cap \Omega_\gamma \) also satisfy the Lipschitz condition and \( f \equiv 0 \) in \( S \cap \Omega_\gamma \). Finally, denote by \( q = (q_1, q_2) \) the outward normal vector to \( \partial S \).

We first investigate the case when an integration path surrounds the whole rigid inclusion \( \gamma \). Let the support of a smooth function \( \eta \) lies in a small neighborhood \( S_1 \) of \( \gamma \) and \( \eta = 1 \)
in a neighborhood \( S \) of \( \gamma \), \( S \subset S_1 \). Given a vector \( p = (p_1, p_2) \), we consider the coordinate transformation (7) in the form

\[
y_1 = x_1 + \delta p_1 \eta(x_1, x_2), \quad y_2 = x_2 + \delta p_2 \eta(x_1, x_2),
\]

(13)

where \( (x_1, x_2) \in \Omega_\gamma \) and \( (y_1, y_2) \in \Omega_{\gamma_b} \). Transformation (13) corresponds to a translation of the thin rigid inclusion along the vector \( p \), and the vector field \( V \) is determined by the formula \( V = pq \).

Next we substitute the vector field \( V \) into (10) and integrate by parts there. Taking into account (1)–(5), we obtain the contour integral

\[
I_{pq} = D \int_{\partial S} \left( \frac{1}{2} (p_2q_2 - p_1q_1) (w_{11}^2 - w_{22}^2) - (1 - \kappa) w_{12}^2 pq - w_{12} (p_2q_1 M_1(w) + p_1q_2 M_2(w)) \right)
\]

\[
- (M_{1,1}(w) q_1 + M_{2,2}(w) q_2) \left( \frac{\partial w}{\partial p} + a_1^0 p_1 + a_2^0 p_2 \right)
\]

\[
+ 2 (1 - \kappa) \left( w_{112} \left( p_1 w_{1,1} - \frac{a_1^0 p_1 + a_2^0 p_2}{2} \right) q_2 + w_{122} \left( p_2 w_{1,2} - \frac{a_1^0 p_1 + a_2^0 p_2}{2} \right) q_1 \right) \right) \, dx,
\]

where

\[
M_1(w) = w_{11} + \kappa w_{22}, \quad M_2(w) = \kappa w_{11} + w_{22}.
\]

It is immediate from the construction that \( I_{pq} \) is a path-independent integral, i.e., it has the same value for all closed smooth curve surrounding the thin rigid inclusion \( \gamma \).

The next step is to discuss a situation when the integration path surrounds only the thin inclusion tip. Let the support of a smooth function \( \theta \) lies in a small neighborhood \( S_1 \) of thin inclusion tip \( C_1 \) and \( \theta = 1 \) in a neighborhood \( S \) of \( C_1, S \subset S_1 \). We consider the coordinate transformation (7) in the form

\[
y_1 = x_1 + \delta_1 \tau \theta(x_1, x_2), \quad y_2 = x_2 + \delta_2 \tau \theta(x_1, x_2),
\]

(14)

where \( (x_1, x_2) \in \Omega_\gamma \) and \( (y_1, y_2) \in \Omega_{\gamma_b} \). Transformation (14) corresponds to a local translation of the thin rigid inclusion along the vector \( \tau \), which is tangential to \( \gamma \), and the vector field \( V \) is determined by the formula \( V = \tau \theta \). Substituting the vector field \( V \) into (10) and integrating by parts there, we obtain the path-independent integral

\[
I_{\tau \theta} = D \int_{\partial S} \left( \frac{1}{2} (\tau_2 q_2 - \tau_1 q_1) (w_{11}^2 - w_{22}^2) - (1 - \kappa) w_{12}^2 \tau q - w_{12} (\tau_2 q_1 M_1(w) + \tau_1 q_2 M_2(w)) \right)
\]

\[
- (M_{1,1}(w) q_1 + M_{2,2}(w) q_2) \left( \frac{\partial w}{\partial \tau} + a_1^0 \tau_1 + a_2^0 \tau_2 \right)
\]

\[
+ 2 (1 - \kappa) \left( w_{112} \left( \tau_1 w_{1,1} - \frac{a_1^0 \tau_1 + a_2^0 \tau_2}{2} \right) q_2 + w_{122} \left( \tau_2 w_{1,2} - \frac{a_1^0 \tau_1 + a_2^0 \tau_2}{2} \right) q_1 \right) \right) \, dx.
\]

(15)

The path-independent integral (15) is an analogue of the well-known Eshelby–Cherepanov–Rice \( J \)-integral from fracture mechanics.

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