Wiener’s ‘closure of translates’ problem and Piatetski’s uniqueness phenomenon

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Abstract. Wiener characterized the cyclic vectors (with respect to translations) in $\ell^p(\mathbb{Z})$ and $L^p(\mathbb{R})$, $p = 1, 2$, in terms of the zero set of the Fourier transform. He conjectured that a similar characterization might be true for $1 < p < 2$. Our main result contradicts this conjecture.

1. Introduction

1.1. Let $G$ be a locally-compact abelian group, and $1 \leq p < \infty$. A function $f \in L^p(G)$ is called a cyclic vector if the translates of $f$ span the whole space. It is well-known that $f \in L^p(T)$ (where $T$ is the circle group) is a cyclic vector if and only if all the Fourier coefficients of $f$ are non-zero. The same is true for general compact groups (see [23]).

In the non-compact case the situation is not so easy. Wiener [24] characterized the cyclic vectors in $L^p(\mathbb{R})$ (or $\ell^p(\mathbb{Z})$) only for $p = 1$ and $2$. We formulate the result for $\ell^p(\mathbb{Z})$, the $L^p(\mathbb{R})$ case is similar.

**Theorem A** (Wiener). Let $c = \{c_n\}, \ n \in \mathbb{Z}$.

(i) $c$ is a cyclic vector in $\ell^2(\mathbb{Z})$ if and only if the Fourier transform

$$\hat{c}(t) := \sum_{n \in \mathbb{Z}} c_n e^{int}$$

is non-zero almost everywhere.

(ii) $c$ is cyclic in $\ell^1(\mathbb{Z})$ if and only if $\hat{c}(t)$ has no zeros.

Part (i) of the result is an easy consequence of the unitarity of the Fourier transform. Part (ii) is more delicate, the proof in this case is based on the fact that the space $\ell^1(\mathbb{Z})$ is a convolution algebra.

In both cases the result can be stated as follows: the vector $c$ is cyclic if and only if the set

$$Z_c := \{t \in T : \hat{c}(t) = 0\}$$

of the zeros of the Fourier transform is “small” in a certain sense. Wiener conjectured (see [24], p. 93) that a similar result should be true for $\ell^p$ spaces, at least for $1 < p < 2$. This ‘closure of translates’ problem of Wiener have been studied by Beurling [5], Pollard [20], Herz [7], Newman [18] and other authors (see, in particular, [6, 13, 21, 22]).
First of all, one should define precisely how to understand the zero set. The answer is obvious if the vector $c$ is assumed to be in $\ell^1(\mathbb{Z})$ (or $L^1(\mathbb{R})$). The above mentioned authors have studied the problem under this assumption. We shall also keep it.

A more serious question – what kind of “smallness” one should consider? It is easy to show (see [22]) that for $\ell^p$, $1 < p < 2$, the condition in part (i) of Theorem A is not sufficient, and the condition in part (ii) is not necessary for cyclicity. So one should look for an “intermediate measurement” of smallness.

Beurling [5] proved that if $Z_\mathbb{Z}$ has Hausdorff dimension less than $2(p - 1)/p$ then $c$ is a cyclic vector in $\ell^p$. This inequality is sharp, but it does not provide a characterization of the cyclic vectors (see [18]).

On the other hand, it is well-known that not only metrical but also arithmetical conditions of “smallness” may play an important role in problems of harmonic analysis. In the above cited papers, various metrical and non-metrical properties of the zero set of cyclic vectors in $L^p(\mathbb{R})$ ($\ell^p(\mathbb{Z})$) have been studied. A number of interesting results were obtained, however none of them provides a complete characterization of the cyclic vectors.

It turns out that this is impossible. Our main result is:

**Theorem 1.** Let $1 < p < 2$. Then there is a compact set $K$ on the circle $\mathbb{T}$ with the following properties:

(a) If a vector $c$ has fast decreasing coordinates, say $\sum_{n \in \mathbb{Z}} |c_n| |n|^\varepsilon < \infty$ for some $\varepsilon > 0$, and $\hat{c}$ vanishes on $K$, then $c$ is not cyclic in $\ell^p(\mathbb{Z})$.

(b) There exists $c \in \ell^1(\mathbb{Z})$, such that $\hat{c}$ vanishes on $K$, and $c$ is a cyclic vector in $\ell^p(\mathbb{Z})$.

It follows that, contrary to Wiener’s conjecture, no characterization of the cyclic vectors exists in terms of the zeros of the Fourier transform:

**Corollary 1.** Given any $1 < p < 2$ one can find two vectors in $\ell^1(\mathbb{Z})$, such that one is cyclic in $\ell^p(\mathbb{Z})$ and the other is not, but their Fourier transforms have an identical set of zeros.

The same result follows for $L^p(\mathbb{R})$.

**1.2.** Our approach to the problem is based on its relation to the uniqueness problem in Fourier analysis, or more specifically, to an aspect of it which we call the “Piatetski phenomenon”.

Recall that a set $K \subset \mathbb{T}$ is called a set of uniqueness (U-set) if whenever a trigonometric series

$$\sum_{n \in \mathbb{Z}} c_n e^{int}$$

converges to zero at every point $t \notin K$ then all the coefficients $c_n$ are zero. Otherwise $K$ is called a set of multiplicity (M-set). Classical Riemannian theory allows one to characterize the compact M-sets as the compacts which support a non-zero distribution $S$ with Fourier transform $\hat{S}(n) \to 0$ as $|n| \to \infty$ (see [10]).

It was D. E. Menshov who discovered (1916) that a set of Lebesgue measure zero can be an M-set. In fact, Menshov constructed a measure with compact
support of Lebesgue measure zero, whose Fourier transform vanishes at infinity (see [3]). It was believed within a long time that every compact M-set must support such a measure. This was disproved in 1954 by Piatetski-Shapiro [19].

This striking result can be put in a more general context. Let \( B \) be a Banach space of sequences (bigger than \( \ell^2 \)). Let us say that it satisfies the Piatetski phenomenon if there is a compact \( K \) on \( \mathbb{T} \), which supports a (non-zero) distribution \( S \) with Fourier transform \( \widehat{S} \in B \), but which does not support such a measure. Piatetski’s result says that the space \( c_0 \) satisfies the phenomenon.

As a contrast, classical results from potential theory (see [4], [10, Chapter III]) show that weighted \( \ell^2 \) spaces (under some conditions on the weight) do not satisfy the phenomenon. In particular, if \( K \) supports a distribution \( S \) such that

\[
\sum_{n \neq 0} |\widehat{S}(n)|^2 |n|^{-\alpha} < \infty \quad (0 < \alpha \leq 1),
\]

then it also supports a probability measure satisfying this condition.

1.3. Our concern, motivated by the Wiener problem, was:

Do \( \ell^q \) spaces, \( q > 2 \), satisfy Piatetski’s phenomenon?

The answer is yes:

**Theorem 2** ([15]). Given any \( q > 2 \) there is a compact set \( K \subset \mathbb{T} \) such that

(a’) \( K \) supports a non-zero distribution \( S \) such that \( \widehat{S} \in \ell^q \).

(b’) \( K \) does not support any non-zero measure \( \mu \) such that \( \widehat{\mu} \in \ell^q \).

The approach used in [15] as well as in the present paper was inspired by Piatetski-Shapiro’s original ideas, further developed by Körner [14], Kaufman [12] and Kahane [10, pp. 213–216].

The role of Theorem 2 in the cyclicity problem is clarified by the following observation (see Section 2.2 below):

The condition (a’) is equivalent to part (a) of Theorem 1, while the condition (b’) is necessary for part (b), with \( q = p/(p-1) \).

So the compatibility of (a’) and (b’) provides a necessary chance for the existence of a counter-example to Wiener’s conjecture. Such an example was indeed sketched in [16] (in a weaker form than in Theorem 1).

The present paper contains a detailed presentation and an extension of the results obtained in [15, 16]. It is organized as follows.

In Section 2 we give some preliminary background and auxiliary lemmas.

Section 3 is the key one in the paper. Our main tools are special measures on the circle, defined by Riesz-type products, and a version of Bernstein stochastic exponential estimate.

In Section 4 we construct a Helson set with the property (a’) above. In Section 5 we prove that every Helson set admits a vector with the property (b). So Theorem 1 follows.

The non-periodic version is considered in Section 6. The last Section 7 contains some additional remarks, in particular we discuss there the relation of Theorem 1 to Malliavin’s celebrated “non-synthesis” phenomenon.
2. Preliminaries. Lemmas.

2.1. Notation. In what follows $\mathbb{T}$ is the circle group $\mathbb{R}/2\pi\mathbb{Z}$.

As usually $C(\mathbb{T})$ is the space of continuous complex functions on $\mathbb{T}$, with the norm $\|f\|_\infty := \sup |f(t)|$, $t \in \mathbb{T}$. By a “measure” on $\mathbb{T}$ we always mean an element of the dual space $M(\mathbb{T})$, that is, a finite complex Borel measure.

We denote by $\{\hat{S}(n)\}$, $n \in \mathbb{Z}$, the Fourier coefficients of a Schwartz distribution $S$ on $\mathbb{T}$. It will also be convenient to keep the notation $\hat{c}$ for the Fourier transform of a vector $c \in \ell^1(\mathbb{Z})$ as defined in (1).

Let $A_p(\mathbb{T})$, $1 \leq p \leq \infty$, denote the Banach space of distributions $S$ on $\mathbb{T}$ with Fourier coefficients belonging to $\ell^p(\mathbb{Z})$, endowed with the norm $\|S\|_{A_p} := \|\hat{S}\|_{\ell^p}$.

For $p = 1$ this is the Wiener algebra $A(\mathbb{T})$ of absolutely convergent Fourier series (see [9]). Throughout we will use the following standard properties,

\[ \|f\|_\infty \leq \|f\|_{A_1}, \quad f \in A(\mathbb{T}), \]
\[ \|f \cdot g\|_{A_p} \leq \|f\|_{A_p} \|g\|_{A_p}, \quad f \in A(\mathbb{T}), \quad g \in A_p(\mathbb{T}). \]

By the “support” of a measure or a distribution we refer to the closed support.

2.2. Cyclic vectors. In this section we refer to some basic results about cyclicity. The results go back to Beurling [5], Pollard [20], Herz [7] and Newman [18]. Actually the first three authors considered Wiener’s problem in $L^p(\mathbb{R})$ rather than $\ell^p(\mathbb{Z})$, but, as the last author mentioned, “the distinction is not vital”. See also [10, pp. 111–112].

It would be convenient to reformulate the concept of cyclicity in an equivalent way, using the following

Definition. An element $f \in A_p(\mathbb{T})$, $1 \leq p \leq \infty$, is called a cyclic vector (with respect to multiplication by exponentials) if the set $\{P(t)f(t)\}$, where $P$ goes through all trigonometric polynomials, is dense in $A_p(\mathbb{T})$.

Clearly, a vector $c \in \ell^p(\mathbb{Z})$ is cyclic (with respect to translations) if and only if its Fourier transform $f := \hat{c}$ is cyclic in $A_p(\mathbb{T})$ in the sense just defined.

Below $f$ is assumed to belong to the Wiener algebra $A(\mathbb{T})$, $Z_f$ denotes the set of the zeros of $f$, and $q = p/(p - 1)$ is the exponent conjugate to $p$. Then:

(i) $f$ is cyclic in $A_p(\mathbb{T})$ if and only if there is a sequence of trigonometric polynomials $P_n$ such that
\[ \lim_{n \to \infty} \|1 - P_n \cdot f\|_{A_p} = 0. \]

(ii) If $Z_f$ is finite then $f$ is cyclic in $A_p(\mathbb{T})$ for every $p > 1$ (see [23], p. 96).

The next propositions follow by duality.

(iii) If $f$ is a non-cyclic vector in $A_p(\mathbb{T})$ then there is a non-zero distribution $S \in A_q(\mathbb{T})$, supported by $Z_f$.

(iv) If there is a non-zero measure $\mu \in A_q(\mathbb{T})$, supported by $Z_f$, then $f$ is a non-cyclic vector in $A_p(\mathbb{T})$. 

(v) If \( f \) is continuously differentiable, and there is a non-zero distribution \( S \in A_q(T) \) supported by \( Z_f \), then \( f \) is a non-cyclic vector in \( A_p(T) \).

Actually (see [7]) the smoothness condition in (v) can be reduced up to \( f \in \text{Lip}(\varepsilon), \varepsilon > 0 \), or, in terms of the Fourier coefficients of \( f \), up to

\[
\sum_{n \in \mathbb{Z}} |\hat{f}(n)| |n|^\varepsilon < \infty \quad \text{for some } \varepsilon > 0.
\]

Observe that if \( p > 2 \) then \( A_q(T) \) is a functional space (embedded in \( L^2(T) \)).

So the conditions (iii)–(iv) imply:

A function \( f \in A(T) \) is a cyclic vector in \( A_p(T) \), \( p > 2 \), if and only if its zero set \( Z_f \) does not support any non-zero function \( g \in A_q(T) \).

This condition is not very effective, but it shows that the cyclicity of a vector \( c \in \ell^1 \) in the space \( \ell^p \), \( p > 2 \), admits characterization in terms of the zero set of the Fourier transform (as Wiener thought). By the way, if \( Z_f \) has Lebesgue measure zero then this obviously implies cyclicity, but not vice versa [18] (see also [11], pp. 101–102).

The case \( p = \infty \) was also considered. Newman [18] proved that a vector \( c \in \ell^1(Z) \) is cyclic in the space \( c_0(Z) \) if and only if \( Z_c \) is nowhere dense in \( T \).

Now let \( 1 < p < 2 \) (the case where Wiener’s conjecture was most certain). Then \( A_q(T) \) is not a functional space, and (iii)–(v) only imply the following:

Let \( f \in A(T) \) be smooth \( (f \in \text{Lip}(\varepsilon), \varepsilon > 0) \). Then it is cyclic in \( A_p(T) \) if and only if the zero set \( Z_f \) does not support any non-zero distribution \( S \in A_q(T) \).

However we will see that without the smoothness condition, the zero set \( Z_f \) does not provide a characterization of the cyclic vectors.

### 2.3. Auxiliary polynomials.

We shall use trigonometric polynomials with the following properties.

**Lemma 1.** Given any \( q > 2 \) and any \( \gamma > 0 \) there is a real trigonometric polynomial \( \varphi = \varphi_{q,\gamma} \) such that

\[
\hat{\varphi}(0) = 0, \quad \|\varphi\|_\infty \leq 1, \quad \|\varphi\|_{L^2} = \frac{1}{2}, \quad \|\varphi\|_{A_q} < \gamma.
\]  

(2)

Here and below \( \| \cdot \|_{L^2} \) denotes the \( L^2 \) norm on \( T \) with respect to the normalized Lebesgue measure.

There are several ways to get Lemma 1. In particular one may use the Shapiro-Rudin polynomials (see [9], p. 52). Namely, for an appropriate choice of signs \( \varepsilon_n = \pm 1 \) \( (n = 1, 2, \ldots) \) the trigonometric polynomial

\[
Q_k(t) = \sum_{n=1}^{2^k} \varepsilon_n \cos nt
\]

satisfies \( \|Q_k\|_\infty \leq 2^{(k+1)/2} \). It follows that if \( k = k(q, \gamma) \) is sufficiently large, then

\[
\varphi(t) := 2^{-(k+1)/2} Q_k(t)
\]

is a real trigonometric polynomial with the properties (2).
2.4. Kahane’s lemma. One of the key arguments in [19] and [12] is based on the uniqueness theorem for power series. In Kahane’s presentation of Kaufman’s result (see [10], pp. 213–216) this point was performed as follows:

Given any \( \delta > 0 \) there is a real, signed measure \( \rho \), supported by a finite subset of the interval \((1 - \delta, 1)\), such that

\[
\int d\rho = 1 \quad \text{and} \quad \left| \int s^k d\rho(s) \right| < \delta \quad (k = 1, 2, \ldots). \tag{3}
\]

This lemma was proved in [10, p. 214] based on the Hahn-Banach theorem. Here we shall need a quantitative version, with an estimate on the total variation of the measure.

**Lemma 2.** Let an interval \( I = (a, b) \), \( 0 < a < b < \frac{1}{2} \), and \( 0 < \delta < 1 \) be given. Then there is a real, signed measure \( \rho \), supported by a finite subset of \( I \), such that (3) holds, and such that

\[
\int |d\rho| < \delta^{-c(I)}, \tag{4}
\]

where \( c(I) > 0 \) is a constant which depends only on \( I \).

We proceed to the proof of Lemma 2.

2.4.1. The measure. Given \( n \) distinct points \( s_1, \ldots, s_n \in I \), consider a measure \( \rho \) supported by these points and defined uniquely by the condition

\[
\int p(s) d\rho(s) = p(0), \quad \text{for every algebraic polynomial } p \text{ of degree } \leq n - 1.
\]

In particular,

\[
\int d\rho = 1, \quad \int s^k d\rho(s) = 0 \quad (k = 1, 2, \ldots, n - 1). \tag{5}
\]

Given any function \( f(s) \) one has \( \int f(s) d\rho(s) = p(0) \), where \( p \) is the unique polynomial of degree \( \leq n - 1 \) which interpolates \( f \) at the knots \( s_1, \ldots, s_n \). It is well-known (see, for example [1], pp. 134–135) that if \( f(s) \) is real-valued and sufficiently smooth, then there is \( 0 \leq \xi < b \) such that

\[
f(0) = p(0) + \frac{f^{(n)}(\xi)}{n!} \prod_{j=1}^{n} (0 - s_j).
\]

Applying this with \( f(s) = s^k, k \geq n \), gives

\[
\int s^k d\rho(s) = (-1)^{n-1} \binom{k}{n} \xi^{k-n} \prod_{j=1}^{n} s_j,
\]

and consequently the moments of \( \rho \) satisfy the estimate

\[
\left| \int s^k d\rho(s) \right| < 2^k \cdot b^{k-n} \cdot b^n = (2b)^k \leq (2b)^n \quad (k \geq n). \tag{6}
\]
2.4.2. The total variation. Using the Lagrange polynomials
\[ l_j(s) = \prod_{i \neq j} \frac{s-s_i}{s_j-s_i} \quad (1 \leq j \leq n) \]
one can calculate the masses
\[ \rho(\{s_j\}) = \int l_j(s) \, d\rho(s) = l_j(0) = (-1)^{n-1} \prod_{i \neq j} \frac{s_i}{s_j-s_i}. \]
We choose the points \( s_1, \ldots, s_n \) as equally spaced knots, \( s_j = a + (j-1/2)h \) where \( h = (b-a)/n \). Then
\[ \|\rho(\{s_j\})\| = \frac{1}{h^{n-1} (j-1)! (n-j)!} \prod_{i \neq j} s_i, \]
and so we have
\[ \int |d\rho| \leq \left( \frac{b}{h} \right)^{n-1} \sum_{j=1}^{n} \frac{1}{(j-1)! (n-j)!} = \frac{1}{n!} \left( \frac{nb}{b-a} \right)^{n-1} \sum_{j=1}^{n} j \binom{n}{j} \]
\[ = \frac{n^n}{n!} \left( \frac{2b}{b-a} \right)^{n-1} \leq \left( \frac{2eb}{b-a} \right)^{n-1}. \] (7)
Finally, choose \( n \) to be the least integer \( \geq \frac{\log(1/\delta)}{\log(1/2b)} > 0 \). It follows from (5), (6) and (7) that \( \rho \) satisfies both (3) and (4). This proves the lemma. \( \square \)

Remark. One can prove that the condition (4) in Lemma 2 is essentially sharp, up to the precise value of the constant \( c(I) \).

2.5. Bernstein inequality. Bernstein exponential estimates for sums of independent random variables are classical. Different versions, adopted for sums of “almost” independent variables, in various senses, are also well-known.

In particular, Azuma [2] considered the so-called multiplicatively orthogonal systems and obtained Bernstein-type exponential estimates for them.

It will be convenient for us to consider a similar version, suitable for an “almost multiplicative” system of random variables, in the following sense:

Lemma 3. Let \( X_1, \ldots, X_N \) be random variables on a probability space \((\Omega, P)\) such that \( -1 \leq X_j \leq 1 \) \( (j = 1, 2, \ldots, N) \). Suppose that
\[ \mathbb{E}(X_1) = \cdots = \mathbb{E}(X_N) = \mu > 0, \] (8)
and that there is \( 0 < \varepsilon < 1 \) such that
\[ (1-\varepsilon) \mu^{|A|} \leq \mathbb{E} \left\{ \prod_{j \in A} X_j \right\} \leq (1+\varepsilon) \mu^{|A|} \] (9)
for every non-empty subset \( A \subset \{1, 2, \ldots, N\} \), where \( |A| \) denotes the number of elements in \( A \). Define
\[ X = \frac{1}{N} \sum_{j=1}^{N} X_j. \]
Then for any \( \alpha > 0 \),
\[ P\{X < \mathbb{E}(X) - \alpha\} \leq \exp \left( -\frac{1}{8} \alpha^2 N \right) + \varepsilon \exp \left( \frac{1}{4} \alpha^2 N \right). \] (10)
Proof. Fix $\lambda > 0$. By the classical Bernstein method we can estimate the probability on the left hand side of (10) as follows,

$$P\{X < \mu - \alpha\} = P\left\{\prod_{j=1}^{N} e^{\lambda(X - X_j)} > e^{\alpha N}\right\} \leq e^{-\alpha \lambda N} \sum_{j=1}^{N} e^{\lambda(X - X_j)}. \tag{11}$$

To estimate the expectation on the right hand side we adopt the approach of [2]. Since $|\mu - X_j| \leq 2$ and by the convexity of the exponential function, we have

$$e^{\lambda(X - X_j)} \leq \cosh(2\lambda) + ((\mu - X_j)/2) \sinh(2\lambda).$$

This can be rewritten in the form $e^{\lambda(X - X_j)} \leq b - aX_j$, where

$$a := (1/2) \sinh(2\lambda), \quad b := \cosh(2\lambda) + (\mu/2) \sinh(2\lambda).$$

It follows that

$$E\sum_{A \subset \{1, \ldots, N\}} (-a)^{|A|} b^{N-|A|} \prod_{j \in A} X_j. \tag{12}$$

Now we invoke the assumption (9), which (together with the fact that $a, b$ are positive numbers) implies that the right hand side of (12) is not larger than

$$\sum_{A \subset \{1, \ldots, N\}} (-a)^{|A|} b^{N-|A|} \left\{1 + (-1)^{|A|} \varepsilon\right\} \mu^{|A|} = (b - a\mu)^N + \varepsilon(b + a\mu)^N.$$

Using the inequalities

$$b - a\mu = \cosh(2\lambda) \leq \exp(2\lambda^2),$$

$$b + a\mu = \cosh(2\lambda) + \mu \sinh(2\lambda) \leq \exp(2\lambda),$$

it follows that

$$E\prod_{j=1}^{N} e^{\lambda(X - X_j)} \leq \exp(2\lambda^2 N) + \varepsilon \exp(2\lambda N). \tag{13}$$

Finally, a combination of (11) and (13), with $\lambda = \alpha/4$, gives

$$P\{X < \mu - \alpha\} \leq \exp \left\{-\frac{1}{8}\alpha^2 N\right\} + \varepsilon \exp \left\{\frac{1}{4}\alpha(2 - \alpha)N\right\}.$$
3.1. Multiplicativity. We start with the following simple property.

**Lemma 5.** Let $\nu$ be a positive integer, and suppose that $P_j$ are trigonometric polynomials, $\deg P_j < \nu \ (j = 0, 1, \ldots, N)$. Then

$$\int_{\mathbb{T}} \left\{ \prod_{j=0}^{N} P_j(\nu^j t) \right\} \frac{dt}{2\pi} = \prod_{j=0}^{N} \left\{ \int_{\mathbb{T}} P_j(t) \frac{dt}{2\pi} \right\}.$$  

**Proof.** By Fourier expansion, the left hand side is equal to

$$\sum_{k} \left\{ \prod_{j=0}^{N} \hat{P}_j(k_j) \right\} \int_{\mathbb{T}} e^{i(k_0+k_1\nu+k_2\nu^2+\cdots+k_N\nu^N)t} \frac{dt}{2\pi},$$

where the sum goes through all integer vectors $k = (k_0, k_1, \ldots, k_N)$ such that $|k_j| \leq \deg P_j$. However it is easy to check that the only solution of the equation

$$k_0 + k_1\nu + k_2\nu^2 + \cdots + k_N\nu^N = 0$$

with $k$ as above, is $k = (0, 0, \ldots, 0)$. This implies the result. \qed

3.2. Riesz-type measures. Suppose that we are given a positive integer $N$, a real trigonometric polynomial $\varphi$ with the properties

$$\hat{\varphi}(0) = 0, \quad \|\varphi\|_{\infty} \leq 1, \quad (14)$$

and also a real trigonometric polynomial $w$ such that

$$\|w\|_{\infty} \leq 1. \quad (15)$$

Choose a large integer $\nu$, satisfying the condition

$$\nu > 2 \max \{ \deg \varphi, N \deg w \}, \quad (16)$$

and define a “Riesz-type product”

$$\lambda_s(t) = \prod_{j=1}^{N} \left( 1 + s w(t) \varphi(\nu^j t) \right), \quad 0 < s < 1. \quad (17)$$

Introduce a measure $\mu_s$ on the circle $\mathbb{T}$,

$$d\mu_s(t) = \lambda_s(t) \frac{dt}{2\pi}. \quad (18)$$

Observe first that it is a probability measure on $\mathbb{T}$. Indeed, it is clear from the properties above that $\lambda_s$ is everywhere positive. Now expand the product (17) into the form

$$\lambda_s(t) = 1 + \sum_{B} \left( s w(t) \right)^{|B|} \prod_{j \in B} \varphi(\nu^j t), \quad (19)$$

where the sum goes through all non-empty subsets $B \subset \{1, \ldots, N\}$. The condition (16) allows one to use here Lemma 5 to find that

$$\int_{\mathbb{T}} \lambda_s(t) \frac{dt}{2\pi} = 1 + \sum_{B} \left\{ \int_{\mathbb{T}} w(t)^{|B|} \frac{dt}{2\pi} \right\} \left\{ s \int_{\mathbb{T}} \varphi(t) \frac{dt}{2\pi} \right\}^{|B|}.$$

However all terms in the above sum are zero, since $\hat{\varphi}(0) = 0$. So it follows that

$$\int_{\mathbb{T}} \lambda_s(t) \frac{dt}{2\pi} = 1,$$

and this proves the claim.
3.3. Random variables. Consider random variables defined by
\[ X_j(t) = w(t) \varphi(\nu^j t), \quad 1 \leq j \leq N, \] (20)
on the probability space \((T, \mu_s)\).

It is well-known that these variables are “almost independent” with respect to the Lebesgue measure on \(T\). However, we will see that (under some additional condition) they are “almost independent” also with respect to \(\mu_s\), which is going to be essentially “singular” with respect to the Lebesgue measure.

To establish such a property we first compute the “multiplicative moments”.

**Lemma 6.** Let \(A\) be a non-empty subset of \(\{1, 2, \ldots, N\}\). Then
\[
\mathbb{E} \left\{ \prod_{j \in A} X_j \right\} = s \int_T \varphi(t)^2 |A| \left\{ \int_T w(t)^2 |A| \, dt \right\}^{2/\pi}. \tag{21}
\]

**Proof.** By (18) and (20), the left hand side of (21) is equal to
\[
\int_T \left\{ w(t)^{|A|} \prod_{j \in A} \varphi(\nu^j t) \right\} \lambda_s(t) \frac{dt}{2\pi}. \tag{22}
\]
Let us again consider the expansion (19) for \(\lambda_s\), however this time we distinguish not the constant term as before, but write instead
\[
\lambda_s(t) = (s \, w(t))^{|A|} \prod_{j \in A} \varphi(\nu^j t) + \cdots
\]
where the implicit terms correspond to all subsets \(B \subset \{1, \ldots, N\}\) which are different from \(A\). Inserting this expression into (22) one can see that the integration of the explicit term gives
\[
s^{|A|} \int_T \left\{ w(t)^{2|A|} \prod_{j \in A} \varphi^2(\nu^j t) \right\} \frac{dt}{2\pi}
\]
which, by the condition (16) and Lemma 5, provides the right hand side of (21). So it remains to show that the integrals of the other terms in the sum do not contribute anything but zero.

Indeed, if \(B\) is any subset \(\neq A\) then the corresponding term is
\[
s^{|B|} \int_T \left\{ w(t)^{|A|+|B|} \prod_{j \in A} \varphi(\nu^j t) \prod_{j \in B} \varphi^2(\nu^j t) \right\} \frac{dt}{2\pi}
\]
which, again by (16) and Lemma 5, is equal to
\[
s^{|B|} \left\{ \int_T w(t)^{|A|+|B|} \frac{dt}{2\pi} \right\} \left\{ \int_T \varphi(t) \frac{dt}{2\pi} \right\}^{|A\Delta B|} \left\{ \int_T \varphi^2(t) \frac{dt}{2\pi} \right\}^{|A\cap B|}.
\]
However this is zero, because \(\hat{\varphi}(0) = 0\), so the lemma is proved. \(\Box\)

One can see from Lemma 6 that if the trigonometric polynomial \(w\) is mostly close to 1 in modulus, then the integrals of the even powers of \(w\) which appear in (21) are almost equal to 1. We will see that in this case the \(X_1, \ldots, X_N\) form an “almost multiplicative” system of random variables (in the sense of Lemma 3) with respect to the measure \(\mu_s\).
Precisely, Lemma 6 allows one to find the expectations
\[ E(X_j) = s \|\varphi\|_{L_2}^2 \|w\|_{L_2}^2 \quad (1 \leq j \leq N). \] (23)
In particular all the \( X_j \) have the same expectation, as in (8). Now suppose that
the trigonometric polynomial \( w \) satisfies, in addition to property (15), also the
condition
\[ \left\{ \int_{T} w(t)^2 \frac{dt}{2\pi} \right\}^N > \frac{1}{1 + \varepsilon} \quad \text{for some } 0 < \varepsilon < 1. \] (24)
Then, given any non-empty \( A \subset \{1, 2, \ldots, N\} \), by (21) and Jensen’s inequality
\[ \mathbb{E} \left\{ \prod_{j \in A} X_j \right\} \geq \left\{ s \int_{T} \varphi(t)^2 \frac{dt}{2\pi} \right\} |A| \left\{ \int_{T} w(t)^2 \frac{dt}{2\pi} \right\} |A| = \prod_{j \in A} \mathbb{E}(X_j). \]
On the other hand (15), (21) and (24) imply that
\[ \mathbb{E} \left\{ \prod_{j \in A} X_j \right\} \leq \left\{ s \int_{T} \varphi(t)^2 \frac{dt}{2\pi} \right\} |A| \leq (1 + \varepsilon) \prod_{j \in A} \mathbb{E}(X_j). \]
This shows that the “almost multiplicativity” condition (9) is satisfied.

### 3.4. Concentration.
Define a trigonometric polynomial
\[ X(t) = \frac{1}{N} \sum_{j=1}^{N} X_j(t) = w(t) \cdot \frac{1}{N} \sum_{j=1}^{N} \varphi(\nu^j t). \] (25)

“Almost independence” suggests that this average is strongly concentrated
(with respect to the measure \( \mu_s \)) near its expectation, and the rate of concen-
tration is governed by the classical exponential estimates.

Indeed, assuming (14), (15) and (24) one may use Lemma 3, which implies
\[ \mu_s \{ t : X(t) < \mathbb{E}(X) - \alpha \} \leq \exp \left( -\frac{1}{8} \alpha^2 N \right) + \varepsilon \exp \left( \frac{1}{N} N \right), \quad \alpha > 0. \] (26)

We use this to prove the following \( L^2 \)-concentration estimate.

**Lemma 7.** Suppose that (14) and (15) hold, and furthermore suppose that
\[ \|\varphi\|_{L^2} \geq \frac{1}{2} \] (27)
and
\[ \left\{ \int_{T} w(t)^2 \frac{dt}{2\pi} \right\}^N > \frac{1}{1 + e^{-N}}. \] (28)
Then, for every
\[ s \in I_0 := \left( \frac{1}{4}, \frac{1}{3} \right) \] (29)
one has
\[ \int_{\{ t : X(t) < c_1 \}} \lambda_2^2(t) \frac{dt}{2\pi} < 2 e^{-c_2 N}, \]
for some absolute positive constants \( c_1, c_2 \).

**Proof.** It follows from (23), (27), (28) and (29) that
\[ \mathbb{E}(X) = s \|\varphi\|_{L_2}^2 \|w\|_{L_2}^2 > \frac{1}{100} \quad (s \in I_0). \]
So the estimate (26) with \( \varepsilon = e^{-N} \) implies that
\[ \mu_s \{ t : X(t) < c_1 \} < 2 \exp \left( -\frac{1}{8} \left( \frac{1}{100} - c_1 \right)^2 N \right), \quad 0 < c_1 < \frac{1}{100}. \] (30)
Using (17) we also obtain the estimate
\[ \lambda_s(t) \leq \exp \left( s w(t) \sum_{j=1}^{N} \varphi(\nu^j t) \right) = \exp (sN X(t)). \] (31)

A combination of (30) and (31) then gives, for every \( s \in I_0 \),
\[ \int_{\{t : X(t) < c_1\}} \lambda^2_s(t) \frac{dt}{2\pi} \leq \left( \int_{\{t : X(t) < c_1\}} \lambda_s(t) \frac{dt}{2\pi} \right)^2 \left( \sup_{\{t : X(t) < c_1\}} \lambda_s(t) \right) < 2 \exp \left( -\frac{1}{8} (\frac{1}{N - c_1})^2 N \right) \exp \left( \frac{1}{c_1} N \right) = 2 e^{-c_2 N}, \]
for appropriate absolute positive constants \( c_1, c_2 \).

Below we continue to denote by \( c_1, c_2 \) the constants from Lemma 7, and let \( c_3, c_4, \ldots \) denote other absolute positive constants.

### 3.5. Proof of Lemma 4

Let the numbers \( q > 2, \varepsilon > 0 \) and a real, non-zero trigonometric polynomial \( u \) be given. Let \( N = N(\varepsilon) \) be a sufficiently large integer, which will be chosen later. Denote by \( \varphi = \varphi_{q, \gamma} \) the trigonometric polynomial from Lemma 1. Also let \( w = w_{N, u} \) be a real trigonometric polynomial, satisfying (15) and (28), and which has the following additional property:

\[ \text{for every } t \in \mathbb{T} \text{ either } w(t) u(t) > 0 \text{ or otherwise } |w(t)| < c_3, \] (32)

where \( c_3 := c_1/2 \). Remark that it is easy to find such \( w \) by taking an approximation of the function \( \text{sign}(u) \).

Given \( 0 < \delta < 1 \) we use Lemma 2 to find a measure \( \rho \), supported by the interval \( I_0 \) defined in (29), satisfying (3) and such that
\[ \int |d\rho| < \delta^{-c_4}, \text{ where } c_4 := c(I_0). \]

Define
\[ \lambda(t) = \int \lambda_s(t) d\rho(s). \]

One can expand the product (17) using the Fourier representation of the trigonometric polynomial \( \varphi \), and this yields the expression
\[ \lambda(t) = 1 + \sum_{\mathbf{k}} \left\{ \int s^{l(\mathbf{k})} d\rho(s) \right\} \left\{ \prod_{k_j \neq 0} \hat{\varphi}(k_j) \right\} w(t)^{l(\mathbf{k})} e^{i(k_1 \nu + k_2 \nu^2 + \ldots + k_N \nu^N) t}, \]
where the sum goes through all non-zero vectors
\[ \mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N, \quad |k_j| \leq \deg \varphi, \]
and \( l(\mathbf{k}) > 0 \) denotes the number of non-zero coordinates of \( \mathbf{k} \). Note that each polynomial \( w(t)^{l(\mathbf{k})} \) has degree \( \leq N \deg w \). So the condition (16) ensures that the summands in the above sum have disjoint spectra. Taking advantage of the fact that \( \|w(t)^{l(\mathbf{k})}\|_{A_q} \leq 1 \) (which follows from (15)) we deduce that
\[ \|1 - \lambda\|^q_{A_q} < \delta^q \sum_{\mathbf{k}} \prod_{k_j \neq 0} |\hat{\varphi}(k_j)|^q < \delta^q \left( 1 + \|\varphi\|^q_{A_q} \right)^N < \delta^q \exp(N\|\varphi\|^q_{A_q}). \]

Using (2) this implies
\[ \|1 - \lambda\|_{A_q} < \delta \exp \left( \frac{1}{q} \gamma^q N \right). \] (33)
Now consider the trigonometric polynomial \( X \) defined in (25). Set
\[
E := \{ t \in \mathbb{T} : X(t) \geq c_1 \} \quad \text{and} \quad h := \lambda \cdot 1_E,
\]
then
\[
\| \lambda - h \|_{A_q} \leq \| \lambda - h \|_{L^2(\mathbb{T})} = \| \lambda \|_{L^2(\mathbb{T} \setminus E)} \leq \int \| \lambda_s \|_{L^2(\mathbb{T} \setminus E)} |d\rho(s)|.
\]
Using Lemma 7 it follows that
\[
\| \lambda - h \|_{A_q} \leq \sqrt{2} e^{-\frac{1}{2} c_2 N} \delta - c_4. \tag{34}
\]
Let \( c_5 > 0 \) be an absolute constant so small such that, setting \( \delta := e^{-c_5 N} \), the right hand side of (34) will tend to zero as \( N \to \infty \). Next, let the number \( \gamma > 0 \) be (an absolute constant) so small such that also the right hand side of (33) will tend to zero as \( N \to \infty \). Now fix \( N = N(\varepsilon) \) so large, such that the right hand sides of both (33) and (34) will be smaller than \( \varepsilon/2 \). Having fixed \( N \), the functions \( w, \lambda, X \) and \( h \) have also been fixed, and it follows that
\[
\| 1 - h \|_{A_q} \leq \| 1 - \lambda \|_{A_q} + \| \lambda - h \|_{A_q} < \varepsilon.
\]
Finally we will define the compact \( K \), the smooth function \( f \) and the trigonometric polynomial \( P \) satisfying the properties (i) and (ii) of Lemma 4. Let \( \chi \) be a non-negative, infinitely smooth function, with integral = 1. Setting \( f := h \ast \chi \) and \( \hat{K} := \text{supp}(f) \) we obtain (i). By choosing \( \chi \) supported on a sufficiently small neighborhood of zero, we may assume that \( X(t) > c_3 \) on \( K \). Let us now set
\[
P(t) := \frac{1}{c_3 N} \sum_{j=1}^{N} \varphi(\nu^j t)
\]
and check that it satisfies condition (ii). First, due to (15) we have
\[
|P(t)| \geq P(t) w(t) = \frac{1}{c_3} X(t) > 1, \quad t \in K.
\]
Secondly, since \( \| \varphi \|_{\infty} \leq 1 \), for every \( t \in K \) we have \( |w(t)| \geq X(t) > c_3 \), and (32) implies for such \( t \) that \( w(t) u(t) > 0 \). It follows that
\[
P(t) u(t) > 0 \quad \text{on} \quad K.
\]
Lastly,
\[
\| P \|_A \leq (1/c_3) \| \varphi \|_A = C(q),
\]
and our main lemma is proved.

4. Helson sets and distributions

4.1. Recall the main two properties of Piatetski-Shapiro’s compact \( K \):
(I) \( K \) supports a non-zero distribution \( S \) with \( \hat{S}(n) \to 0 \) as \( |n| \to \infty \).
(II) For every non-zero measure \( \mu \) supported by \( K \),
\[
\limsup_{|n| \to \infty} |\hat{\mu}(n)| > 0.
\]
In a way, the existence of such a compact reveals a “compromise” between certain “thickness” and “thinness” conditions of a set (understood not in a metrical but rather an arithmetical sense). We will see that this compromise can be achieved under stronger conditions, in both directions.
**Definition** (see, for example [9], Chapter IV). A compact set $K$ is called a *Helson set* if it satisfies any one of the following equivalent conditions:

(i) Every continuous function on $K$ admits extension to a function in $A(\mathbb{T})$.

(ii) There is $\delta_1(K) > 0$ such that, for every measure $\mu$ supported by $K$,

$$\sup_{n \in \mathbb{Z}} |\hat{\mu}(n)| \geq \delta_1(K) \int |d\mu|.$$  \hfill (35)

(iii) There is $\delta_2(K) > 0$ such that, for every measure $\mu$ supported by $K$,

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| \geq \delta_2(K) \int |d\mu|.$$  \hfill (36)

Körner [14] and Kaufman [12] generalized Piatetski-Shapiro’s result by constructing Helson sets with the property (I) above (that is, Helson M-sets).

We will prove the following stronger theorem:

**Theorem 3.** For any $q > 2$ there is a Helson set $K$ on the circle $\mathbb{T}$, which supports a non-zero distribution $S$ such that $\hat{S} \in \ell^q$.

Clearly this also implies Theorem 2.

**4.2.** For the proof of Theorem 3 we need the following

**Lemma 8.** Let $K$ be a totally disconnected compact set on $\mathbb{T}$. Suppose that there is a constant $C > 0$ such that the following is true: given any real-valued function $h \in C(\mathbb{T})$ with no zeros in $K$, one can find a real trigonometric polynomial $P(t)$ such that

$$\inf_{t \in K} |P(t)| > 1, \quad P(t)h(t) > 0 \text{ on } K, \quad \|P\|_A \leq C.$$  \hfill (37)

Then $K$ is a Helson set.

**Proof.** It would be enough to show that there is $\delta_1(K) > 0$, such that (35) is satisfied by every measure $\mu$ supported by $K$. In fact, it is enough to prove (35) only for real, signed measures $\mu$, as one can check easily by decomposing a complex measure into its real and imaginary parts.

Let therefore $\mu$ be a real, signed measure supported by $K$, and suppose that $\int |d\mu| = 1$. Since $K$ is totally disconnected, given $\varepsilon > 0$ there is a real-valued function $h \in C(\mathbb{T})$ such that $h(t) = \pm 1$ on $K$, and $\int h \, d\mu > 1 - \varepsilon$. Let $P(t)$ be a real trigonometric polynomial satisfying (37). Then

$$\int_K P \, d\mu = \int_K |P| \, |d\mu| - \int_K |P| \, h \, d\mu = \int_K |P| \, (|d\mu| - h \, d\mu) > 1 - C \varepsilon.$$  

On the other hand,

$$\int_K P \, d\mu = \int_\mathbb{T} P \, d\mu = \sum_{n \in \mathbb{Z}} \hat{P}(-n) \hat{\mu}(n) \leq C \sup_{n \in \mathbb{Z}} |\hat{\mu}(n)|.$$  

Since $\varepsilon$ was arbitrary, this shows that (35) is true with $\delta_1(K) = C^{-1}$. \qed
Remark. It is not difficult to check that the condition in Lemma 8 is also necessary for Helson sets. For comparison, we mention another necessary and sufficient condition in a similar spirit: a compact \( K \) is a Helson set if and only if it is totally disconnected, and each \( \{0,1\} \)-valued continuous function on \( K \) admits an extension to \( \mathbb{T} \) with bounded \( A(\mathbb{T}) \) norm (see [9], p. 52).

4.3. Proof of Theorem 3. Fix \( q > 2 \). Choose a sequence \( u_j \) of real, non-zero trigonometric polynomials, which is dense in the metric space of real-valued continuous function on \( \mathbb{T} \). For a sequence \( \varepsilon_j \) we use Lemma 4 with \( \varepsilon = \varepsilon_j \) and \( u = u_j \) to choose \( K_j, f_j \) and \( P_j \). We choose the \( \varepsilon_j \) by induction, such that

\[
\varepsilon_1 < 2^{-2} \quad \text{and} \quad \|f_1 \cdot f_2 \cdots \cdot f_j\|_A \varepsilon_{j+1} < 2^{-2-j} \quad (j = 1, 2, \ldots).
\]

This condition allows to define a distribution \( S \) by the \( A_q(\mathbb{T}) \)-convergent infinite product \( \prod_{j=1}^{\infty} f_j \); indeed, the partial products \( S_j = f_1 \cdots f_j \) satisfy

\[
\|S_{j+1} - S_j\|_{A_q} = \|f_1 \cdots f_j \cdot (f_{j+1} - 1)\|_{A_q} \leq \|f_1 \cdots f_j\|_A \varepsilon_{j+1} < 2^{-2-j},
\]

hence \( S_j \) converges in \( A_q(\mathbb{T}) \) to a limit \( S \). Observe that \( S \) is non-zero, since

\[
\|S - 1\|_{A_q} \leq \sum_{j=0}^{\infty} \|S_{j+1} - S_j\|_{A_q} < \sum_{j=0}^{\infty} 2^{-2-j} < 1,
\]

and that \( S \) is supported by the compact \( K := \bigcap_{j=1}^{\infty} K_j \).

On the other hand, we will show that \( K \) is a Helson set. It is enough to check that \( K \) satisfies the conditions of Lemma 8. Indeed, for each \( j \) we have

\[
\inf_{t \in K} |P_j(t)| > 1, \quad P_j(t)u_j(t) > 0 \text{ on } K, \quad \|P_j\|_A \leq C(q).
\]

In particular, none of the \( u_j \) has a zero in \( K \). Since they are dense in the metric space of real-valued continuous function on \( \mathbb{T} \), it follows that \( K \) is totally disconnected. Let now \( h \in C(\mathbb{T}) \) be any real-valued function, with no zeros in \( K \). Choose \( j \) such that \( u_j(t)h(t) > 0 \) on \( K \), then (37) is satisfied with \( P = P_j \) and \( C = C(q) \). It therefore follows from Lemma 8 that \( K \) is a Helson set.

5. Helson sets and cyclic vectors

5.1. The role of Helson sets in our problem is clarified by the following

**Lemma 9.** Let \( K \) be a Helson set on \( \mathbb{T} \). Then there is a function \( g \in A(\mathbb{T}) \), vanishing on \( K \), which is a cyclic vector in \( A_p(\mathbb{T}) \) for every \( p > 1 \).

For the proof we shall need the following property of Helson sets. Denote by \( C(K) \) the space of continuous functions on \( K \) with the norm

\[
\|h\|_{C(K)} = \sup_{t \in K} |h(t)|.
\]

Recall that one of the equivalent definitions of a Helson set is that every element of \( C(K) \) admits an extension to a function in \( A(\mathbb{T}) \). In the next lemma it is shown that one can actually find such extensions with arbitrarily small \( A_p \) norms.

**Lemma 10.** Let \( K \) be a Helson set, and suppose that \( \varepsilon > 0 \), \( p > 1 \) and \( h \in C(K) \) are given. Then one can find \( f \in A(\mathbb{T}) \) such that

\[
f|_K = h, \quad \|f\|_A \leq (1/\delta) \|h\|_{C(K)}, \quad \|f\|_{A_p} < \varepsilon,
\]

where \( \delta = \delta_2(K) > 0 \) is the constant from (36).
Proof. Fix $p > 1$ and $\varepsilon > 0$. Introduce a Banach space $B = B(p, \varepsilon)$ of functions $f$ on the circle $\mathbb{T}$ such that

$$\|f\|_B := \|f\|_A + (1/\varepsilon) \|f\|_{A_p} < \infty.$$ 

In other words, the space $B$ coincides with the space $A(\mathbb{T})$ but is equipped with a different (equivalent) norm. Let also $T : B \to C(K)$ be the restriction operator $f \to f|_K$, and denote by $T^*$ its dual operator.

Given a measure $\mu$ supported by $K$, by (36) we have

$$L(\mu) := \limsup_{|n| \to \infty} |\hat{\mu}(n)| \geq \delta \int |d\mu|.$$ 

Take a sequence of integers $n_j$, $|n_1| < |n_2| < \cdots$, and real numbers $\theta_j$ such that

$$\lim_{j \to \infty} \hat{\mu}(n_j) e^{-i\theta_j} = L(\mu),$$

and define

$$f_N(t) = \frac{1}{N} \sum_{j=1}^N e^{-i(n_jt + \theta_j)}.$$ 

Then $\|f_N\|_B = 1 + (1/\varepsilon) N^{(1/p)-1}$, and

$$\langle f_N, T^* \mu \rangle = \langle Tf_N, \mu \rangle = \int_K f_N(t) d\mu(t) = \frac{1}{N} \sum_{j=1}^N \hat{\mu}(n_j) e^{-i\theta_j}.$$ 

It follows that

$$\|T^* \mu\|_{B^*} \geq \lim_{N \to \infty} \frac{|\langle f_N, T^* \mu \rangle|}{\|f_N\|_B} = L(\mu) \geq \delta \int |d\mu|,$$ 

for every measure $\mu$ supported by $K$.

By a classical theorem of Banach (see [10], p. 141) this implies that for every $h \in C(K)$, the equation $Tf = h$ admits a solution $f \in B$ such that $\|f\|_B \leq (1/\delta) \|h\|_{C(K)}$. This proves the lemma.

5.2. Using Lemma 10 we can prove Lemma 9 above.

Proof of Lemma 9. It will be convenient to use Baire categories in the proof. Let $I(K)$ denote the set of functions $g \in A(\mathbb{T})$ which vanish on $K$. This is a complete metric space, under the metric inherited from $A(\mathbb{T})$. We will prove that the set of functions $g \in I(K)$ which are cyclic in $A_p(\mathbb{T})$ for every $p > 1$, is a countable intersection of open, dense sets in the space $I(K)$. By Baire’s theorem, this set is therefore non-empty (and in fact is dense in the space).

For $\varepsilon > 0$ and $p > 1$, denote by $G(\varepsilon, p)$ the set of $g \in I(K)$ for which there exists a trigonometric polynomial $P$ such that $\|1 - P \cdot g\|_{A_p} < \varepsilon$. Choose a sequence $\varepsilon_n \to 0$ and a sequence $p_n \to 1$ ($n \to \infty$), and consider the intersection

$$\bigcap_{n=1}^\infty G(\varepsilon_n, p_n).$$

According to Remark (i) from Section 2.2, a function $g \in I(K)$ belongs to this intersection if and only if it is cyclic in $A_p(\mathbb{T})$ for every $p > 1$. So to conclude the proof it remains to show that each $G(\varepsilon, p)$ is an open, dense set in $I(K)$.
Let $g_0 \in G(\varepsilon, p)$ be given. Then $\|1 - P \cdot g_0\|_{A_p} < \varepsilon$ for some trigonometric polynomial $P$. Given $\eta > 0$, suppose that $g \in I(K)$ and $\|g - g_0\|_A < \eta$. Then
\[\|1 - P \cdot g\|_{A_p} \leq \|1 - P \cdot g_0\|_{A_p} + \eta \|P\|_{A_p}.\]
It $\eta$ is chosen sufficiently small then the right hand side is smaller than $\varepsilon$. Hence $G(\varepsilon, p)$ contains the open ball $B(g_0, \eta)$ of radius $\eta$ centered at $g_0$, and this shows that $G(\varepsilon, p)$ is open.

Finally we show that $G(\varepsilon, p)$ is dense. Let a ball $B(g_0, \eta)$ in $I(K)$ be given. Choose a non-zero trigonometric polynomial $h$ such that
\[\|h - g_0\|_A < \frac{\delta}{1 + \delta} \cdot \eta,\]
where $\delta = \delta_2(K) > 0$ is the constant from (36). In particular this implies that
\[\sup_{t \in K} |h(t)| < \frac{\delta}{1 + \delta} \cdot \eta.\]
Since $h$ is non-zero, it has finitely many zeros, so by the Remarks (i) and (ii) from Section 2.2 there is a trigonometric polynomial $P$ such that $\|1 - P \cdot h\|_{A_p} < \varepsilon/2$. Now use Lemma 10 to find $f \in A(\mathbb{T})$ such that
\[f|_K = h|_K, \quad \|f\|_A < \eta/(1 + \delta), \quad \|f\|_{A_p} < \frac{\varepsilon}{2\|P\|_A},\]
and set $g := h - f$. Then clearly $g \in I(K)$. Moreover
\[\|g - g_0\|_A \leq \|h - g_0\|_A + \|f\|_A < \eta,\]
that is, $g \in B(g_0, \eta)$. Also,
\[\|1 - P \cdot g\|_{A_p} \leq \|1 - P \cdot h\|_{A_p} + \|P\|_A \|f\|_{A_p} < \varepsilon,\]
and therefore $g \in G(\varepsilon, p)$. This shows that $G(\varepsilon, p)$ is dense.

5.3. Our main result now follows:

Proof of Theorem 1. By Theorem 3 there is a Helson set $K$ satisfying the condition $(a')$ in Theorem 2. This condition is equivalent to condition $(a)$ in Theorem 1 (see Section 2.2 above). On the other hand, Lemma 9 implies that $K$ satisfies also condition $(b)$. So Theorem 1 is proved.

Proof of Corollary 1. Let $K$ be the compact set of Theorem 1. By the property $(b)$ there is $g \in A(\mathbb{T})$ vanishing on $K$, which is cyclic in $A_p(\mathbb{T})$. Choose a smooth (say, twice continuously differentiable) function $f$ on $\mathbb{T}$, such that $Z_f = Z_g$. In particular, $f$ vanishes on $K$. Since the Fourier coefficients of $f$ decrease sufficiently fast, the property $(a)$ implies that $f$ is a non-cyclic vector in $A_p(\mathbb{T})$. Thus our corollary is proved.

6. Non-periodic version

Here we extend the results to $L^p(\mathbb{R})$ spaces, $1 < p < 2$. This can be deduced easily from the previous results, so we may be brief. In particular we skip the formulation of the corresponding version of Theorem 1, and restrict ourselves to:
COROLLARY 2. Given any $1 < p < 2$ one can find two functions in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, such that one is cyclic in $L^p(\mathbb{R})$ and the other is not, but their Fourier transforms have the same (compact) set of zeros.

Here $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ vanishing at infinity.

It will be convenient to denote by $\hat{\mathbb{R}}$ another copy of the real line. We consider distributions on the Schwartz space $S(\hat{\mathbb{R}})$. We denote by $A_p(\hat{\mathbb{R}})$, $1 \leq p < \infty$, the space of Fourier transforms of functions in $L^p(\mathbb{R})$, with the corresponding norm. In particular, for $p = 1$ this is the Wiener algebra $A(\hat{\mathbb{R}})$ of functions with an absolutely convergent Fourier integral.

Recall that, by definition, a function $F(x) \in L^p(\mathbb{R})$ is a cyclic vector if the translates $\{F(x-y)\}, y \in \mathbb{R}$, span the whole space. Equivalently, $F$ is cyclic if the set $\{f(t)\phi(t)\}$, where $f = \hat{F}$ and $\phi$ runs over $S(\mathbb{R})$, is dense in $A_p(\hat{\mathbb{R}})$.

PROOF OF COROLLARY 2. Fix $1 < p < 2$, and take the compact $K$ of Theorem 1. By the property (b) there is $h \in A(\mathbb{T})$, vanishing on $K$, which is cyclic in $A_p(\mathbb{T})$ (with respect to multiplication by trigonometric polynomials).

We may assume that $h(t)$ is positive at some point $t$, and by rotation, that $h(\pi) > 0$. It follows that there is an interval

$$I := (-\pi + \delta, \pi - \delta), \quad \delta > 0,$$

such that $K \subset \{t : \Re h(t) \leq 0\} \subset I$. Choose a function $\chi \in S(\hat{\mathbb{R}})$, $0 \leq \chi \leq 1$, compactly supported by $(-\pi, \pi)$, and such that $\chi(t) = 1$ on $I$. Define

$$g(t) := \chi(t) h(t) + (1 - \chi(t)) e^{-t^2}, \quad t \in \hat{\mathbb{R}}.$$ 

It is easy to see that:

(i) The zero set $Z_g$ is compact, $K \subset Z_g \subset I$.

(ii) $g = \hat{G}$ for some $G \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, which implies $g \in A_p(\hat{\mathbb{R}})$.

Claim 1. The set $\{g(t)\phi(t)\}, \phi \in S(\hat{\mathbb{R}})$, is dense in $A_p(\hat{\mathbb{R}})$.

If not then, by duality, there is a (non-zero) distribution $S \in A_q(\hat{\mathbb{R}})$, $q = p/(p-1)$, such that $\langle S, g \cdot \phi \rangle = 0$ for every $\phi \in S(\hat{\mathbb{R}})$. It follows that

$$\text{supp}(S) \subset Z_g \subset I, \quad |I| < 2\pi. \quad (38)$$

We have $g(t) = h(t)$ on $I$, since $\chi(t) = 1$ on $I$. Hence

$$\langle S, h \cdot \phi \rangle = 0 \quad \text{for every } \phi \in S(\hat{\mathbb{R}}). \quad (39)$$

The condition (38) allows us to regard $S$ also as a distribution on $\mathbb{T}$. It is well-known that under this condition the following equivalence holds:

$$S \in A_q(\mathbb{T}) \iff S \in A_q(\hat{\mathbb{R}}).$$

But $h$ is cyclic in $A_p(\mathbb{T})$, so (39) implies that $S = 0$, which proves the claim.

Now take an arbitrary function $f \in S(\hat{\mathbb{R}})$ with $Z_f = Z_g$. 
Claim 2. The set \( \{ f(t) \phi(t) \}, \) \( \phi \in S(\hat{\mathbb{R}}) \), is not dense in \( A_p(\hat{\mathbb{R}}) \).

Indeed, the property (a) from Theorem 1 implies that \( K \) supports a (non-zero) distribution \( S \in A_q(\mathbb{T}) \). As above we can regard it as a distribution on \( \hat{\mathbb{R}} \), belonging to \( A_q(\mathbb{R}) \). But \( f \) is a smooth function in \( A_p(\mathbb{R}) \) and \( f|_{K} = 0 \). So

\[ \langle S, f \cdot \phi \rangle = 0 \quad \text{for every } \phi \in S(\hat{\mathbb{R}}). \]

This means that the inverse Fourier transform of \( f \) is a function \( F \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}) \) which is non-cyclic in \( L^p(\mathbb{R}) \). So our corollary is proved. \( \square \)

7. Remarks

7.1. Theorem 1 may be put into the context of the theory of translation-invariant subspaces. A linear subspace \( M \subset L^p(G) \) is called translation-invariant if whenever \( f \) belongs to \( M \) then so do all of its translates. Observe that \( f \in L^p(G) \) is a cyclic vector if and only if it does not belong to any proper closed translation-invariant subspace of \( L^p(G) \).

It is well-known that any closed translation-invariant subspace in \( \ell^2(\mathbb{Z}) \) can be uniquely recovered from the set of the common zeros of the Fourier transforms of its elements.

This is not the case in \( \ell^1(\mathbb{Z}) \). Malliavin’s “non-synthesis” example [17] means that different closed translation-invariant subspaces in this space may have the same set of common zeros. More precisely, given a compact set \( K \subset \mathbb{T} \) consider the invariant subspaces

\[ I(K) = \{ c \in \ell^1(\mathbb{Z}) : \hat{c} \text{ vanishes on } K \}, \]

\[ J(K) = \{ c \in \ell^1(\mathbb{Z}) : \hat{c} \text{ vanishes on some open set containing } K \}. \]

Then for some compacts \( K \) the closure of \( J(K) \) is strictly smaller than \( I(K) \).

Kahane [8] (see also [10], p. 121) showed that such a result still holds if one takes the closures of \( J(K) \) and \( I(K) \) in \( \ell^p(\mathbb{Z}) \), \( 1 < p < 2 \).

Theorem 1 reveals a sharper phenomenon in these spaces, which is not possible in \( \ell^1(\mathbb{Z}) \). Namely, there is a compact \( K \) such that the \( \ell^p \)-closures satisfy

\[ \text{Clos} \ J(K) \subset \not\subseteq \text{Clos} \ I(K) = \ell^p(\mathbb{Z}). \]

7.2. Strictly speaking, it is not necessary to require that \( c \in \ell^1 \) in order to have the zero set \( Z_{\hat{c}} \) well-defined. The continuity of \( \hat{c} \) is sufficient for that, as appeared in the weaker version of Theorem 1 proved in [16]. The advantage of the present version, however, seems to be substantial, since very little is known about the relation between the cyclicity in \( \ell^p \) (\( 1 < p < 2 \)) and the zero set \( Z_{\hat{c}} \), unless \( c \in \ell^1 \).

7.3. Perhaps the most interesting problem left open is: could one characterize in reasonable terms the cyclic vectors \( c \) in \( \ell^p \), \( 1 < p < 2 \), under the standard assumption \( c \in \ell^1 \) with no extra restrictions?
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