Surfaces in $\mathbb{R}^{N^2-1}$ based on harmonic maps $S^2 \rightarrow CP^{N-1}$

Dedicated to the memory of Alexander Reznikov

W. J. Zakrzewski §
Department of Mathematical Sciences,
University of Durham,
Durham DH1 3LE,
United Kingdom

Abstract.
We show that many surfaces in $\mathbb{R}^{N^2-1}$ can be generated by harmonic maps of $S^2 \rightarrow CP^{N-1}$. These surfaces are based on the projectors in $CP^{N-1}$ which describe maps of $S^2 \rightarrow CP^{N-1}$. In the case when these maps form the Veronese sequence all the surfaces have constant curvature.

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§ email: w.j.zakrzewski@durham.ac.uk
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1. Introduction

A few years ago, Konopelchenko et al [1, 2] initiated a construction of surfaces immersed in multidimensional spaces basing their discussion on the Weierstrass procedure generalised to higher dimensional spaces. This has led to further studies [3] and to relating the surfaces to the solutions of the $CP^{N-1}$ model [4]. Recently, together with Grundland [5], we have presented a general procedure for the construction of surfaces from the harmonic $CP^{N-1}$ maps. This approach involved writing the equation for the harmonic map as a conservation law and then observing that the coordinates of the surfaces can be constructed out of the components of the special operator which appears in the conservation law.

Our procedure has then been generalised to the supersymmetric case [6]. At this stage it has become clear that, in the holomorphic case, the projector in question is proportional to the fundamental projector of the holomorphic map.

This observation has suggested to us to look at other projectors that arise in the description of harmonic maps and to use them to construct further surfaces.

In the next section we recall the general construction of $S^2 \to CP^{N-1}$ harmonic maps and then use them to construct surfaces in $\mathbb{R}^{N^2-1}$. In the following section we look, in detail, at the case of $CP^2$ and show that, in contradistinction to the $CP^1$ case, its surfaces do not have to be of constant curvature. We calculate this curvature for two classes of such harmonic maps.

Finally, we look at the Veronese sequence and show that in this case all surfaces are of constant curvature. We finish the paper with a few comments about the generality of our results.

2. Harmonic maps $S^2 \to CP^{N-1}$ [7]

The $S^2 \to CP^{N-1}$ harmonic maps involve maps into $CP^{N-1}$, i.e.

$$C \ni \zeta = \zeta_1 + i\zeta_2 \mapsto z = (z^1, ..., z^N) \in C^N,$$

where the homogeneous coordinates $z = (z^1, ..., z^N)$ have the following property

$$z \sim z' = \lambda z \quad \text{for} \quad \lambda \neq 0.$$  

Here we have chosen to parametrise the $S^2$ by $\zeta$ and $\bar{\zeta}$ the complex variables of the plane obtained from $S^2$ by its stereographic projection. Exploiting the projective invariance [2,2] we can require that

$$z^\dagger \cdot z = 1$$

holds, where $\dagger$ denotes hermitian conjugation. However, we are still left with the gauge (phase) invariance

$$z \to z' = z e^{i\phi},$$

where $\phi$ is a real-valued function.
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It is easiest to define the $S^2 \to \mathbb{C}P^{N-1}$ harmonic maps as stationary points of the Lagrangian whose density \([7]\) is given by

$$L = \frac{1}{4} (D_\mu z)^\dagger \cdot D_\mu z, \quad z^\dagger \cdot z = 1,$$

(2.5)

where the covariant derivatives $D_\mu$ act on $z : S^2 \to \mathbb{C}P^N$ according to the formula

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z)z.$$

(2.6)

Here the index $\mu = 1, 2$ denotes the components of $\zeta_\mu$. Note that the covariant derivatives $D_\mu z$ transform under the gauge transformation \([2.4]\)

$$D_\mu z \to D_\mu z' = (D_\mu z) e^{i\phi},$$

(2.7)

so that the dependence on the phase $\phi$ drops out of the Lagrangian density \([2.5]\) and so the target space is really $\mathbb{C}P^{N-1}$.

The total Lagrangian is given by

$$L = \int L d\zeta_1 d\zeta_2$$

(2.8)

and, as we want to consider $S^2 \to \mathbb{C}P^{N-1}$ maps, we require that $L$ is finite.

To find the maps it is convenient to define

$$z = \frac{f}{|f|},$$

(2.9)

where $|f| = (f^\dagger \cdot f)^{1/2}$. In terms of $f$ the Lagrangian \([2.5]\) becomes

$$L = \int \frac{(|\partial f|^2 + |\bar{\partial} f|^2)}{|f|^4} d\zeta d\bar{\zeta}$$

(2.10)

where $|\partial f|^2 = (\partial f)^\dagger \cdot (\partial f)$ and $|\bar{\partial} f|^2 = (\bar{\partial} f)^\dagger \cdot (\bar{\partial} f)$. The Euler-Lagrange equations for $f$ take the form

$$\left(1 - \frac{f \otimes f^\dagger}{|f|^2}\right) [\partial \bar{\partial} f - \partial f \left(\frac{f^\dagger \cdot \bar{\partial} f}{|f|^2} - \bar{\partial} f \left(\frac{f^\dagger \cdot f}{|f|^2}\right)\right] = 0,$$

(2.11)

where we have introduced the holomorphic and antiholomorphic derivatives

$$\partial = \frac{\partial}{\partial (\zeta_1 + i\zeta_2)}, \quad \bar{\partial} = \frac{\partial}{\partial (\zeta_1 - i\zeta_2)}$$

(2.12)

and bar denotes complex conjugation.

Then all harmonic maps of $S^2 \to \mathbb{C}P^{N-1}$ can be constructed in the following way:

First observe that $f = f(x_+)$, i.e whose components are analytical functions of $\zeta = \zeta_1 + i\zeta_2$ automatically satisfies \([2.11]\). For the total Lagrangian $L$ to be finite we require that each component of $f$ is a polynomial in $\zeta$. Then any such $f$ describes a harmonic map of $S^2 \to \mathbb{C}P^{N-1}$ and as $f$ is analytic such a map is also holomorphic.

Other maps can be constructed from the holomorphic maps. To do this we define an operator $P_+$ by its action in $\mathbb{C}^N$

$$P_+ g = \partial g - g \frac{g^\dagger \cdot \partial g}{g^\dagger \cdot g}.$$  

(2.13)
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Then we apply this operator to our holomorphic vector $f$ obtaining $P_+ f$. And repeat this procedure applying it to $P_+ f$ obtaining $P_+^2 f$ and so on. Then, as is known \[1\], all $P_+^k f$ solve (2.11) and so represent harmonic maps $S^2 \to \mathbb{C}P^{N-1}$.

The sequence of vectors $P_+^k f$ have the following properties \[7\]:

$$
(P_+^i f)^\dagger \cdot P_+^j f = 0, \quad i \neq j,
$$

$$
\bar{\partial} (P_+^k f) = -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2}, \quad \partial \left( \frac{P_+^{k-1} f}{|P_+^{k-1} f|^2} \right) = \frac{P_+^k f}{|P_+^{k-1} f|^2}.
$$

Next we construct a sequence of projectors $P_k$ of the form

$$
P(V) = \frac{V \otimes V^\dagger}{|V|^2},
$$

where for $V$ we take $V = P_+^k f$.

Due to the properties (2.14) these projectors are mutually orthogonal and we have

$$
\sum_{k=0}^{N-1} P_k = 1.
$$

Hence only $N-1$ of $P_k$ are independent (ie we can consider only those corresponding to $k = 0, 1, ..., N-2$).

3. Surfaces in $\mathbb{R}^{N^2-1}$

The Weierstrass construction of surfaces discussed in \[1\] involves looking at a different set of equations and then using their solutions to construct surfaces. However, this set of equations was shown in \[4\] to be equivalent to the harmonic maps so one could use these maps directly - to construct our surfaces. In fact in \[5\] it was shown how to construct such surfaces and, at the same time, the generalised Weierstrass system was given.

The work in \[6\] has revealed that, in the $\mathbb{C}P^1$ case, the surfaces are related to the basic projector of the corresponding map. This suggests that we look at other projectors and use them to construct further surfaces.

The orthogonality of the projectors allows us to take their linear combinations. Hence, in the $\mathbb{C}P^{N-1}$ case we can take as our projector

$$
P = \sum_{k=0}^{N-2} \alpha_k P_k,
$$

where $\alpha_k$ are constants. From this matrix $P$ we construct a vector $X$, with $N^2 - 1$ components, in the following way:

First we consider the off-diagonal entries of $P$ (note that $P$ is hermitian) and use them to define $N^2 - N$ components of $X$ by taking their real and imaginary parts; ie we take

$$
X_l = P_{ij} + P_{ji}, \quad \text{for} \quad l = 1, ..., \frac{N^2-N}{2}, \quad i \neq j
$$

$$
X_l = i(P_{ij} - P_{ji}), \quad \text{for} \quad l = \frac{N^2-N}{2}, ..., N^2-N, \quad i \neq j.
$$
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The remaining \( N - 1 \) components of \( X \) are taken from the diagonal entries of \( P \).

As each projector \( P_k \) has trace one, the trace of \( P \) is also constant so that the different choices of these \( N - 1 \) components of \( X \) correspond to the shifts of the vector \( X \) and so would not alter the metric nor the curvature of the surface in \( \mathbb{R}^{N^2-1} \).

How do we choose these remaining components of \( X \)? The obvious procedure is to choose them in such a way that

\[
\sum_{i=1}^{N-1} \partial_+ X_i \partial_- X_i = 2 \sum_{i=0}^{N} \partial_+ P_{ii} \partial_- P_{ii}.
\]

(3.3)

In the \( \mathbb{C}P^1 \) case this tells us that for the last component of \( X \) we should take \( P_{11} - P_{22} \).

For larger \( N \) we have more choices; thus, as was discussed in [6], for \( \mathbb{C}P^2 \) we can take

\[
X_1 = P_{11} - P_{22}, \quad X_2 = \sqrt{3}(P_{11} + P_{22}).
\]

(3.4)

or we could make another choice. In general, for \( \mathbb{C}P^2 \), we could take

\[
P_{11} = \frac{1}{3} + aX_1 + bX_2, \quad P_{22} = \frac{1}{3} + cX_1 + dX_2.
\]

(3.5)

Then we choose \( a, b, c \) and \( d \) so that

\[
\partial_+ X_1 \partial_- X_1 + \partial_+ X_2 \partial_- X_2
\]

(3.6)

gives the same expression as

\[
\partial_+ P_{11} \partial_- P_{11} + \partial_+ P_{22} \partial_- P_{22} + \partial_+ P_{33} \partial_- P_{33}
\]

(3.7)

in which we can eliminate \( P_{33} \) using \( P_{33} = 1 - P_{11} - P_{22} \).

A simple calculation shows that we have a one-parameter family of solutions

\[
a = \frac{2}{\sqrt{3}} \cos \alpha, \quad b = \frac{2}{\sqrt{3}} \sin \alpha,
\]

\[
c = \mp \sin \alpha - \frac{1}{\sqrt{3}} \cos \alpha, \quad d = -\frac{1}{\sqrt{3}} \sin \alpha \pm \cos \alpha.
\]

(3.8)

For \( N > 2 \) the solutions are even more nonunique.

4. Properties of the surfaces

Let us consider a surface defined by \( X \) (this is clearly a surface as our vector \( X \) depends on two variables \( \zeta \) and \( \bar{\zeta} \)).

The metric on the surface, induced by the map, is, due to our choice of \( X \), given by

\[
g_{++} = \text{tr}(\partial P \partial \bar{P}), \quad g_{+-} = \text{tr}(\partial P \partial \bar{P})
\]

(4.1)

and, of course, \( g_{--} = g_{++} \).

To calculate the metric we need some properties of the projectors \( P_k \).

Note that due to (3.4) we have

\[
\partial P_k = \frac{P_k^{k+1} f \otimes (P_k^{k})^\dagger}{|P_k^{k+1} f|^2} - \frac{P_k^{k} f \otimes (P_k^{k-1})^\dagger}{|P_k^{k} f|^2}
\]

(4.2)
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(for $k = 0$ we have only the first term) and $\partial P_k = \bar{\partial} T_k$.

Then, due to the orthogonality of $P^k f$ we see that

$$g_{++} = g_{--} = 0$$

and that

$$g_{+-} = \alpha_0 \frac{|P_+ f|^2}{|f|^2} + \sum_{k=1}^{N-2} \alpha_k \left( \frac{|P^{k+1}_+ f|^2}{|P^{k+1}_+ f|^2} + \frac{|P^k f|^2}{|P^{k-1}_+ f|^2} \right).$$

As only the $g_{+-}$ component of the metric is nonzero the curvature is given by

$$K = -\frac{4}{g_{+-}} \partial \bar{\partial} \ln(g_{+-}).$$

In general, it is difficult to calculate the curvature for the expression above; hence in the next section we discuss its form in some special cases.

5. Special cases

5.1. $CP^1$

Consider first the case of $CP^1$ maps. In this case have only holomorphic harmonic maps and the surface in $\mathbb{R}^3$ is a sphere. To see this we note that we can take

$$X_1 = \frac{W + \bar{W}}{1 + |W|^2}, \quad X_2 = i \frac{W - \bar{W}}{1 + |W|^2}, \quad X_3 = \frac{2}{1 + |W|^2},$$

and this gives us

$$X_1^2 + X_2^2 + (X_3 - \frac{1}{2})^2 = \frac{1}{4}.$$}

The induced metric is

$$g_{+-} = \frac{|P_+ f|^2}{|f|^2}.$$}

Note that as $f$ has only two components we can put $f = (1, W)$ where $W$ is a ratio of polynomials in $\zeta$. Then

$$g_{+-} = \frac{|\partial W|^2}{(1 + |W|^2)^2}$$

and

$$K = -4 \frac{(1 + |W|^2)^2}{|\partial W|^2} \partial \bar{\partial} \ln\left( \frac{|\partial W|^2}{(1 + |W|^2)^2} \right)$$

$$= 4 \frac{(1 + |W|^2)^2}{|\partial W|^2} \partial \bar{\partial} \ln((1 + |W|^2)^2) = 8 \frac{(1 + |W|^2)^2}{|\partial W|^2} \frac{|\partial W|^2}{(1 + |W|^2)^2} = 8$$

as $\partial W = 0$. This is, of course, the curvature of the $CP^1$ harmonic map and is, also, the surface generated by Konopelchenko via his Weierstrass procedure.
5.2. CP\(^2\) case

Now we have more choices. We have two classes of harmonic maps; the holomorphic ones (ie those based on \(f\)) and nonholomorphic ones (based on \(Pf\)). Our vector \(X\) has 8 components and its entries are constructed from the matrix \(P\) which in this case takes the form

\[
P = \alpha_0 P_0 + \alpha_1 P_1. \tag{5.6}
\]

(of course, in addition we have some freedom of how to choose the two components of \(X\) constructed out of the diagonal entries of \(P\)). The metric is now given by

\[
g_{+-} = (\alpha_0 + \alpha_1) \frac{|P_+ f|^2}{|f|^2} + \alpha_1 \frac{|P_+ f|^2}{|P_+ f|^2}. \tag{5.7}
\]

Let us look first at the case of the holomorphic map (ie when \(\alpha_1 = 0\) and \(\alpha_0 = 1\)). Then, \(g_{+-} = \frac{|P_+ f|^2}{|f|^2}\) and the curvature is given by

\[
K = 4 \left(2 - \frac{|P_+ f|^2}{|P_+ f|^4}\right). \tag{5.8}
\]

Note that this curvature is, in general, not constant. In the cases of the embedding of \(CP^1\) into \(CP^2\) \(f\) has only 2 components and then \(P_+ f = 0\) and so the result reduces back to the \(CP^1\) case.

The curvature is also constant when the second term in (5.8) is constant and this case corresponds to the Veronese sequence. We shall discuss this case later.

Next we look at the nonholomorphic case, ie when \(P = P_1\). Then

\[
g_{+-} = \frac{|P_+ f|^2}{|f|^2} + \frac{|P_+ f|^2}{|P_+ f|^2} \tag{5.9}
\]

and the calculation of the curvature is quite tedious. In fact, lengthy calculations give

\[
K = 4 \left(2 - \frac{|P_+ f|^6}{|P_+ f|^6} + \frac{|P_+ f|^6}{|P_+ f|^6} \right) \left|\frac{\partial \ln \frac{|f|^2}{|P_+ f|^2}}{\partial \ln \frac{|f|^2}{|P_+ f|^2}}\right|^3. \tag{5.10}
\]

Of course if we take a more general case (with both \(\alpha_0\) and \(\alpha_1\) nonvanishing) we get an even more complicated expression.

5.3. Veronese sequence

The Veronese choice of the vector \(f\) is such that its components are monomials of \(\zeta\) multiplied by the square roots of the coefficients of the expansion of \((1+a)^N\) in powers of \(a\). Hence \(f\) is given by

\[
f = (1, \sqrt{N} \zeta, \sqrt{N(N-1)/2} \zeta^2, ... \sqrt{N} \zeta^{N-2}, \zeta^{N-1}) \tag{5.11}
\]

With this choice all \(|P^k f|^2\) are given by the powers of \((1+ |\zeta|^2)\). Moreover, \(|f|^2 = (1+ |\zeta|^2)^{N-1}\) and the successive powers decrease by 2. Thus

\[
g_{+-} = \alpha \left(\frac{1}{1+|\zeta|^2}\right)^2 \tag{5.12}
\]
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for all harmonic maps. Only the value of $\alpha$ depends on the map.

Hence the curvature is given by

$$K = \frac{8}{\alpha}.$$  

(5.13)

Looking at the concrete examples we see that the surfaces are quite complicated. For example, if we consider the $\mathbb{C}P^2$ case and look at the two projectors $P_0$ and $P_1$ we note that the corresponding surfaces are very different.

This is clear as $P_0$ is generated by $f$ which in this case is given by

$$f = (1, \sqrt{2}\zeta, \zeta^2)$$

and so

$$P_0 = \frac{1}{(1 + |\zeta|^2)^4} \begin{pmatrix} \frac{1}{\zeta^2} & \sqrt{2}\zeta & \frac{\zeta^2}{|\zeta|^2} \\ \sqrt{2}\zeta & 2|\zeta|^2 & \sqrt{2}\zeta|\zeta|^2 \\ \frac{\zeta^2}{|\zeta|^2} & \sqrt{2}\zeta|\zeta|^2 & |\zeta|^4 \end{pmatrix}. $$  

(5.14)

On the other hand

$$P_+ f = \frac{\sqrt{2}}{1 + |\zeta|^2} \begin{pmatrix} -\sqrt{2}\zeta, (1 - |\zeta|^2), \sqrt{2}\zeta \end{pmatrix}$$

(5.15)

and so

$$P_1 = \frac{1}{(1 + |\zeta|^2)^2} \begin{pmatrix} 2|\zeta|^2 & -\sqrt{2}\zeta(1 - |\zeta|^2) & -2\zeta^2 \\ -\sqrt{2}\zeta(1 - |\zeta|^2) & (1 - |\zeta|^2)^2 & \sqrt{2}\zeta(1 - |\zeta|^2) \\ -2\zeta^2 & \sqrt{2}\zeta(1 - |\zeta|^2) & 2|\zeta|^2 \end{pmatrix}. $$(5.16)

Note that as several components of $P_1$ are proportional to each other the corresponding vector $X$ has some components the same and so changing the basis in $\mathbb{R}^8$ we observe that vector $X$ lies in a five-dimensional subspace of $\mathbb{R}^8$ and so we can take it in the form

$$X_1 = \frac{2x(1 - x^2 - y^2)}{(1 + x^2 + y^2)^2}, \quad X_2 = \frac{2y(1 - x^2 - y^2)}{(1 + x^2 + y^2)^2},$$

$$X_3 = \frac{2(x^2 - y^2)}{(1 + x^2 + y^2)^2}, \quad X_4 = \frac{4xy}{(1 + x^2 + y^2)^2},$$

$$X_5 = \sqrt{3} \frac{(1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2},$$

(5.17)

where we have defined $x$ and $y$ through $\zeta = x + iy$.

Note that the curvatures for the two cases ($P_0$ and $P_1$) are constant but different; namely:

$$K(P_0) = 4, \quad K(P_1) = 2.$$  

(5.18)

6. Conclusions

We have discussed here a possible construction of surfaces in $\mathbb{R}^{N^2-1}$ based on harmonic maps $S^2 \to \mathbb{C}P^{N-1}$. Our construction, which in a way, is a generalisation of the Weierstrass construction used by Konopelchenko and collaborators has produced many surfaces whose induced metric is related to the total Lagrangian (energy of the underlying maps). The curvatures of these surfaces can be easily calculated. In the
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$\mathbb{R}^3$ case - the surface is a sphere but for larger $N$ the surfaces are more complicated. Moreover, their curvatures are related to the curvatures of the $CP^{N-1}$ spaces. When we restrict our attention to the Veronese sequence of maps all corresponding surfaces have a constant curvature.

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