All tree-level amplitudes in $\mathcal{N} = 4$ SYM

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Abstract

We give an explicit formula for all tree amplitudes in $\mathcal{N} = 4$ SYM, derived by solving the recently presented supersymmetric tree-level recursion relations. The result is given in a compact, manifestly supersymmetric form and we show how to extract from it all possible component amplitudes for an arbitrary number of external particles and any arrangement of external particles and helicities. We focus particularly on extracting gluon amplitudes which are valid for any gauge theory. The formula for all tree-level amplitudes is given in terms of nested sums of dual superconformal invariants and it therefore manifestly respects both conventional and dual superconformal symmetry.

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1 Introduction

Gluon scattering amplitudes are known to have many remarkable properties. In a recent paper [1], it was discovered that in $\mathcal{N} = 4$ SYM, scattering amplitudes exhibit a new, dual superconformal symmetry. This new symmetry appears in addition to all previously known symmetries of the amplitudes. It was also shown that this dual superconformal symmetry can be understood through the AdS/CFT correspondence, where it appears as a symmetry of the $AdS_5 \times S^5$ string sigma model [2, 3]. In this paper we will construct a solution for all tree-level amplitudes in $\mathcal{N} = 4$ SYM and show how explicitly how it respects dual superconformal symmetry.

The firsthint at an unexpected simplicity in gluon scattering amplitudes were the MHV amplitudes conjectured by Parke and Taylor [4] (and later proved by Berends and Giele [5]). For amplitudes having generic helicity configurations, Witten argued that they have remarkable properties in twistor space [6]. This conjecture was verified for NMHV amplitudes [7, 8], however the explicit formulae for these amplitudes are rather complicated. Since tree level gluon amplitudes in $\mathcal{N} = 4$ SYM are equal to gluon amplitudes in any gauge theory, including QCD, it is no restriction to consider amplitudes in $\mathcal{N} = 4$ SYM instead. Keeping this in mind and having observed that $\mathcal{N} = 4$ SYM amplitudes have an additional symmetry, dual superconformal symmetry, it seems natural to write the amplitudes in a manifestly supersymmetric way. The appropriate on-shell $\mathcal{N} = 4$ superspace was introduced by Nair [9], who used it to write down the MHV super-amplitudes. Employing this superspace will allow us to make the additional symmetry properties of the amplitudes manifest and hopefully lead to simpler expressions than the previously available ones. Indeed, it was conjectured [1] and later proved [10] that NMHV tree level amplitudes written in this superspace have a remarkably simple form, they are just given by a sum over certain dual superconformal invariants. It seems natural to expect that one can go beyond NMHV amplitudes and that generic $\mathcal{N}^p$MHV amplitudes will have a relatively simple form when written in superspace. Since these super-amplitudes are not yet known we compute them in this paper.

The state-of-the-art method for computing tree-level scattering amplitudes in gauge theory are the BCF on-shell recursion relations [11, 12]. Recently, these recursion relations have been written for $\mathcal{N} = 4$ SYM in on-shell superspace [13, 14, 15, 16, 17]. We will use the form presented in [14, 15, 16]. This is precisely the tool we need to study tree-level super-amplitudes for arbitrary helicity configurations. The supersymmetric recursion relations have been used very recently to verify that tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM are covariant under dual conformal transformations [15].

In this paper, we use the supersymmetric recursion relations to compute tree-level amplitudes in $\mathcal{N} = 4$ SYM. As we will see, writing the recursion relations in superspace makes it significantly simpler to solve them. We use the explicit solutions for NMHV, NNMHV, and NNNMHV amplitudes as examples to study the general pattern and then we present a solution for all amplitudes in terms of nested sums. Our result on NMHV amplitudes confirms the result of [10], while our results for generic non-MHV amplitudes are new.

We then study the symmetries of our solution and show how the conventional superconformal symmetry of $\mathcal{N} = 4$ SYM is realised on the amplitudes. We also study the dual superconformal...
symmetry that the tree-level super-amplitudes should exhibit [1]. This symmetry is a generalisation of dual conformal symmetry, which first appeared as a property of loop integrals in the perturbative expansion of MHV amplitudes [18, 19, 20] and then, in the context of the AdS/CFT correspondence, as the isometry of a T-dual AdS$_5$ in [21, 22] and finally as an anomalous Ward identity for MHV amplitudes [23, 24]. This last manifestation of dual conformal symmetry is based on a conjectured duality between MHV amplitudes and Wilson loops [21, 25, 26] which has been confirmed in perturbation theory up to two loops [23, 24, 27, 28, 29].

The paper is organised as follows. In section 2 we introduce the necessary superspace definitions and briefly review the extension of the BCF recursion relations to superspace. In section 3, we show how to solve the supersymmetric recursion relations in the NMHV case, and in section 4 in the NNMHV case. Based on the previous sections, we give in section 5 the solution to the supersymmetric relations for the generic non-MHV case. In section 6 we discuss both the conventional and dual superconformal symmetry of our solutions. Section 7 serves to explain how to extract gluon scattering amplitudes from our super-amplitudes. Section 8 contains our conclusions. There are three appendices. In appendix A we discuss the behaviour of our results under the collinear limit. In appendix B we describe how our solution for all tree-level super-amplitudes could be automatised. In appendix C we give the generators of the ordinary as well as the dual superconformal algebra.

2 Amplitudes and supersymmetric recursion relations

In this paper, we will be discussing colour-ordered scattering amplitudes. The tree-level MHV gluon amplitudes mentioned in the introduction are given by [4, 5] 1

$$A(1^-, 2^+, \ldots, j^-, \ldots, n^+) = \delta^{(4)}(p) \frac{\langle 1 j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle},$$  

(1)

where $p = \sum_{i=1}^{n} \lambda_i^\alpha \lambda_i^\dot{\alpha}$ is the total momentum and $\langle ij \rangle = \lambda_i^\alpha \lambda_j^\dot{\alpha}$. In order to shed more light on gluon scattering amplitudes of arbitrary helicity configurations and make their symmetries manifest, it is useful to consider scattering amplitudes in $\mathcal{N} = 4$ SYM, which has many exceptional properties. Using Grassmann variables $\eta^A$ we can write down a super-wavefunction

$$\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \hat{\Gamma}^D(p)$$

$$+ \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p),$$  

(2)

which incorporates as its components all on-shell states of $\mathcal{N} = 4$ SYM. Since the $\mathcal{N} = 4$ supermultiplet is PCT self-conjugate, we could have equally chosen a anti-chiral representation (see [1, 10] for more explanations). Then we can define super-amplitudes as

$$\mathcal{A}_n(\lambda, \bar{\lambda}, \eta) = \mathcal{A}(\Phi_1 \ldots \Phi_n).$$  

(3)

1In this paper we omit the standard factor of $i/(2\pi)^4$ in the normalisation of the amplitudes.
In this paper we will be discussing exclusively tree-level amplitudes. The $\mathcal{N} = 4$ supersymmetric version of the MHV tree-level amplitude (1) then reads $[9]$

$$A_{\text{MHV}}^n(\lambda, \tilde{\lambda}, \eta) = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle},$$

where $q = \sum_{i=1}^n \lambda_i^a \eta_i^A$. The appearance of $\delta^{(8)}(q)$ is dictated by $\mathcal{N} = 4$ supersymmetry, and can be thought of as imposing super-momentum conservation, just as $\delta^{(4)}(p)$ ensures momentum conservation.

The full tree-level super-amplitude (3) contains not just MHV but all possible $\mathcal{N}^p$MHV super-amplitudes and has the factors $\delta^{(4)}(p)$ and $\delta^{(8)}(q)$ for the same reason. It is convenient to factor out the MHV tree-level super-amplitude (4) and write the remaining factor as $\mathcal{P}_n$:

$$A_n = A_{\text{MHV}}^n \mathcal{P}_n.$$  

(5)

The factor $\mathcal{P}_n$ has an expansion in the Grassmann parameters $\eta$,

$$\mathcal{P}_n = \mathcal{P}_{\text{MHV}}^n + \mathcal{P}_{\text{NMHV}}^n + \ldots \mathcal{P}_{\text{NMHV}}^n.$$ 

(6)

Of course $\mathcal{P}_{\text{MHV}}^n = 1$ while $\mathcal{P}_{\text{NMHV}}^n$ has Grassmann degree 4 and the remaining terms increase in Grassmann degree in units of 4 up to $\mathcal{P}_{\text{NMHV}}^n$ which is of degree $4n - 16$.

The super-amplitude $A_{\text{MHV}}^n$ contains the pure gluon amplitude (1) as a component in the expansion in the Grassmann parameters $\eta$,

$$A_{\text{MHV}}^n = (\eta_1)^4 (\eta_j)^4 A(1^-, 2^+, \ldots, j^-, \ldots, n^+) + \ldots,$$

(7)

where $(\eta)^4 = 1/4! \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta_D$. The full super-amplitude $A_n$ contains all gluon amplitudes (with arbitrary total helicity) as well as all amplitudes with fermions and scalars in $\mathcal{N} = 4$ SYM. The superspace formulation of the amplitudes has the advantage that supersymmetric Ward identities are automatically satisfied. Another advantage is that, as was conjectured in [1] and proved in [10], NMHV amplitudes have a particular simple form when written in superspace, namely

$$A_{\text{NMHV}}^n = A_{\text{MHV}}^n \mathcal{P}_{\text{NMHV}}^n = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle} \sum_{1<i<j<n} R_{n;i,j},$$

(8)

where $R_{n;i,j}$ are dual superconformal invariants [1].

Let us now quickly introduce the necessary information on the BCF on-shell recursion relations. They express $n$-point scattering amplitudes in terms of a sum over a product of scattering amplitudes of fewer points [11, 12]. Schematically, they read

$$A = \sum_{P_i} \sum_h A_{\mathcal{P}_i}^h (z_{P_i}) A_{R_i}^h (z_{R_i}).$$

(9)

In (9), $z_P$ indicates that in the amplitudes on the r.h.s certain momenta were shifted. For example, it is convenient to shift two adjacent legs according to

$$\hat{\lambda}_n = \tilde{\lambda}_n + z_{P_i} \hat{\lambda}_1, \quad \hat{\lambda}_1 = \lambda_1 - z_{P_i} \lambda_n.$$ 

(10)
Hatted quantities denote the shifted variables. This shift, called a $|n1\rangle$ shift, is depicted in Fig. 1. Note that the amplitudes $A_L^h(z_{P_i})$, $A_R^h(z_{P_i})$ are on-shell. Indeed, the shift parameter $z_p$ must be chosen such that this is the case, which amounts to saying that the shifted intermediate momentum $\hat{P}_i = -(\hat{\lambda}_1\hat{\lambda}_1 + \sum_{j=2}^{i-1}\lambda_j\hat{\lambda}_j)$ is on-shell, i.e.

$$ (\hat{P}_i)^2 = \left(-\sum_{j=1}^{i-1}\lambda_j\hat{\lambda}_j + z_{P_i}\lambda_n\hat{\lambda}_1\right)^2 = 0. \quad (11) $$

Note also that the propagator $P_{i}^2$ is evaluated for unshifted kinematics.

We will use the supersymmetric version of the BCF recursion relations of [14, 15, 16]. This amounts to replacing the sum over intermediate states by a superspace integral, and the on-shell amplitudes by super-amplitudes, i.e.

$$ \mathcal{A} = \sum_{P_i} \int d^4\eta_{P_i} A_L(z_{P_i}) \frac{1}{P_{i}^2} A_R(z_{P_i}) \quad (12) $$

The validity of the supersymmetric equations can be justified by relating the $z \to \infty$ behaviour of the shifted super-amplitudes $\mathcal{A}(z)$ to the known behaviour of component amplitudes [12] using supersymmetry [14, 15, 16].

For the supersymmetric equations, supersymmetry requires that in addition to (10) we also have

$$ \hat{\eta}_n = \eta_n + z_{P_i}\hat{\eta}_1 \quad (13) $$

In the following sections it will be very useful to use the dual variables [18]

$$ \lambda_i\hat{\lambda}_i = x_i - x_{i+1} \quad (14) $$

As was already mentioned, these have a natural generalisation to dual superspace [1], i.e.

$$ \lambda_i\eta_i = \theta_i - \theta_{i+1} \quad (15) $$

Following [15], in the supersymmetric recursion relations only the following dual variables get shifted,

$$ \hat{x}_1 = x_1 - z_{P_i}\lambda_n\hat{\lambda}_1, \quad \hat{\theta}_1 = \theta_1 - z_{P_i}\lambda_n\eta_1 \quad (16) $$

See Fig. 1. The fact that all other dual variables remain inert under the shift will prove useful when solving the supersymmetric recursion relations.
Figure 2: The two contributions to the supersymmetric recursion relation for NMHV amplitudes. We call term $B$ inhomogeneous and $A$ homogeneous. $B$ can be easily computed since it is built from MHV amplitudes only. \( \hat{1} \) means that $\lambda_1$ is shifted, and \( \tilde{n} \) means that $\tilde{\lambda}_n$ is shifted.

3 NMHV tree amplitudes

Here we show that it is straightforward to obtain all NMHV tree amplitudes from the supersymmetric recursion relation (12) and knowing the MHV super-amplitudes.

Apart from the $n$-point MHV super-amplitude (4) we need the 3-point MHV amplitude, which can be readily obtained from (4) for $n = 3$ by a Grassmann Fourier transform,

\[
\mathcal{A}_{\text{MHV}}^3(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}(p) \frac{\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])}{[12][23][31]}.
\]

(17)

NMHV super-amplitudes have Grassmann degree 12. Looking at (12) we see that there is a Grassmann integration, which means that the Grassmann degree of the amplitudes on the r.h.s. of (12) must add up to 16. This is only possible in two ways, $4 + 12$ and $8 + 8$, which corresponds to taking $\text{MHV}_3 + \text{NMHV}$ and $\text{MHV} + \text{MHV}$ amplitudes for $\mathcal{A}_L, \mathcal{A}_R$, respectively. It is convenient to choose a shift of two neighbouring points, e.g. a $[n1]$ shift. Then the supersymmetric recursion relation for $\mathcal{A}_{n_{\text{NMHV}}}$ reads

\[
\mathcal{A}_{n_{\text{NMHV}}} = \int \frac{d^4P}{P^2} \int d^4\eta_p \mathcal{A}_{\text{MHV}}^3(z_P) \mathcal{A}_{n_{\text{MHV}}}^{n-1}(z_P) + \sum_{i=4}^{n-1} \int \frac{d^4P_i}{P_i^2} \int d^4\eta_{P_i} \mathcal{A}_{i_{\text{MHV}}}^1(z_{P_i}) \mathcal{A}_{n_{\text{MHV}}}^{n-i+2}(z_{P_i}).
\]

(18)

The two terms in (18) are depicted in Fig. 2.

Note that the shifted lines must be on opposite sides of the exchanged line. Note also that the leg $n$ with the anti-holomorphic shift cannot connect to the $\text{MHV}_3$ amplitude since this would not be allowed by the kinematics. Similarly, an MHV$_i$ amplitude containing the leg 1 with the holomorphic shift must have at least four propagators, which explains the range of $i$ in (18).

3.1 Inhomogeneous term

The inhomogeneous term in the recursion relation (18) for NMHV amplitudes (corresponding to Fig. 2 B) can be readily calculated since it is built entirely from the known MHV amplitudes, see (4).
By writing for example the Grassmann delta function coming from $A_{4}^{\text{MHV}}(z_{P})$ in the following way,

$$
\delta^{(8)} \left( \hat{\lambda}_{1}\eta_{1} + \sum_{j=2}^{i-1} \lambda_{j}\eta_{j} - \lambda_{P_{i}}\eta_{P_{i}} \right) = \langle \hat{1}P_{i} \rangle^4 \delta^{(4)} \left( \sum_{j=2}^{i-1} \langle \hat{j}P_{i} \rangle \eta_{j} - \eta_{P_{i}} \right) \delta^{(4)} \left( \eta_{1} + \sum_{j=2}^{i-1} \langle jP_{i} \rangle \eta_{j} \right),
$$

(19)

the integration over $\eta_{P_{i}}$ can be carried out straightforwardly. In this way, we obtain the following contribution to the $n$-point NMHV amplitude:

$$
B = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\prod_{j=1}^{n}(j+1)} \sum_{i=4}^{n-1} R_{r;2,i}.
$$

(20)

Here $R_{r;st}$ is a superconformal invariant introduced in [1],

$$
R_{r;st} = \frac{\langle s s - 1 \rangle \langle t t - 1 \rangle \delta^{(4)}(\Xi_{r;st})}{x_{st}^{2} \langle r | x_{rs}x_{st} | t \rangle \langle r | x_{rs}x_{st} | t - 1 \rangle \langle r | x_{rst}x_{ts} | s \rangle \langle r | x_{rst}x_{ts} | s - 1 \rangle}.
$$

(21)

The Grassmann odd quantity $\Xi_{r;st}$ is given by

$$
\Xi_{r;st} = \langle r | x_{rs}x_{st} | \theta_{tr} \rangle + \langle r | x_{rst}x_{ts} | \theta_{sr} \rangle.
$$

(22)

Here we used the dual variables $x_{i}$ and $\theta_{i}$ defined by (14) and (15).

In the following we will often deal with the quantity $\Xi_{n;st}$ for $1 < s < t < n$. It is instructive to switch from the dual $\theta$'s in (22) to $\eta$'s,

$$
\Xi_{n;st} = \langle n \rangle \left[ x_{ns}x_{st} \sum_{i=t}^{n-1} |i\rangle \eta_{i} + x_{nt}x_{ts} \sum_{i=s}^{n-1} |i\rangle \eta_{i} \right],
$$

(23)

to see that $\Xi_{n;st}$ is independent of $\eta_{n}$ and $\eta_{1}$. Alternatively, using the $\delta^{(8)}(q)$ present in all physical amplitudes to rewrite the sums we can obtain

$$
\delta^{(8)}(q) \Xi_{n;st} = -\delta^{(8)}(q) \langle n \rangle \left[ x_{ns}x_{st} \sum_{i=1}^{t-1} |i\rangle \eta_{i} + x_{nt}x_{ts} \sum_{i=1}^{s-1} |i\rangle \eta_{i} \right],
$$

(24)

such that the only dependence on $\eta_{n-1}$ and $\eta_{1}$ on the l.h.s. of (24) is contained in $\delta^{(8)}(q)$. These facts will be useful in the following sections when carrying out superspace integrations.

Moreover, it is useful to realize that terms like $\langle r | x_{rs}x_{st} | t \rangle$ in (21) and similar terms in (22) can always be written as

$$
\langle r | x_{rs}x_{st} | t \rangle = \langle r | x_{r+1}x_{st} | t \rangle,
$$

(25)

such that it is clear that they only depend explicitly on $\lambda_{r}$, but not on $\hat{\lambda}_{r}$.

### 3.2 5-point example

In [15], the supersymmetric recursion relations were examined for the example of the five-point MHV amplitude. We will also examine this example here as it is the first example of an NMHV
amplitude. For five points, \( \text{NHMV}_5 = \overline{\text{MHV}}_5 \), and therefore we could have obtained the \( \text{NHMV}_5 \) amplitude from a Grassmann Fourier transform of the \( \text{MHV}_5 \) amplitude [10].

We immediately see that only the second term in (18) contributes, because there is no four-point amplitude of Grassmann degree 12. Hence for five points, the complete amplitude is given by (20), i.e.

\[
\mathcal{A}_{5}^{\text{NHMV}} = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\prod_{j=1}^{5} \langle j \ j + 1 \rangle} R_{5;2,4}. \tag{26}
\]

We remark the invariant \( R_{5;2,4} \) can be further simplified, but this is a special feature of the \( n = 5 \) case.

Another remark is that since the super-amplitude must have cyclic symmetry. This allows us to conclude that

\[
\delta^{(8)}(q) R_{5;2,4} = \delta^{(8)}(q) R_{1;3,5} = \delta^{(8)}(q) R_{2;4,1} = \delta^{(8)}(q) R_{3;5,2} = \delta^{(8)}(q) R_{4;1,3}. \tag{27}
\]

This is just the first example of the more general identity for \( n \) points, given in [10], where

\[
\delta^{(8)}(q) \sum_{i,j} R_{r;i,j} = \delta^{(8)}(q) \sum_{i,j} R_{r';i,j}, \tag{28}
\]

where the sum goes over all allowed (nonsingular) values of \( i, j \).

### 3.3 General solution for NHMV amplitudes

It can be seen that there is a simple pattern to how the \( n \)-point solution is generated from the \((n - 1)\)-point one. Let us check that (8) indeed solves the supersymmetric recursion relation (3.3). Comparing to (26) we see that for \( n = 5 \) equation (8) is correct.

We now proceed to prove (8) by induction. Let us assume that (8) is valid for a given number of \( n - 1 \) points. Then it follows from the cyclicity of super-amplitudes that (28) is also true for \( n - 1 \) points. Now, we notice that \( \mathcal{A}_{n-1}^{\text{NHMV}}(z_P) \) only involves the quantities \( R_{n-1;i,j} \) where the first subscript is always equal to \( n \). Cyclic symmetry allows us to insert \( \mathcal{A}_{n-1}^{\text{NHMV}}(z_P) \) into (18) in our favourite orientation. It is convenient to insert it such that the legs \( \{1, 2, 3, \ldots, n - 1\} \) of \( \mathcal{A}_{n-1}^{\text{NHMV}}(z_P) \) are identified with the legs \( \{\hat{P}, 3, 4, \ldots, n\} \) in the recursion relation (see Fig. 2).

After carrying out this change of labels in \( \mathcal{A}_{n-1}^{\text{NHMV}}(z_P) \) is is clear from equations (23) and (25) that the obtained \( R_{n;i,j} \) does not depend on the shift or on \( \eta_P \), and therefore we have

\[
A = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\prod_{j=1}^{n} \langle j \ j + 1 \rangle} \sum_{2<i<j<n} R_{n;i,j}. \tag{29}
\]

We see that (20) is just the missing first term (for \( i = 2 \)) to complete (29) to the ansatz (8) for \( n \) points, i.e.

\[
A + B = \mathcal{A}_{n}^{\text{NHMV}} = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\prod_{j=1}^{n} \langle j \ j + 1 \rangle} \sum_{1<i<j<n} R_{n;i,j}. \tag{30}
\]
Figure 3: The three contributions to the supersymmetric recursion relation for NNMHV amplitudes.

This completes the inductive proof. To prepare for the notation that we use in section 5, we will rewrite the formula for NMHV amplitudes with different labels and using $P_{n}^{\text{NMHV}}$ instead of $A_{n}^{\text{NMHV}}$,

\[ P_{n}^{\text{NMHV}} = \sum_{2 \leq a_{1}, b_{1} \leq n-1} R_{n; a_{1}, b_{1}}. \]  

(31)

4 NNMHV tree amplitudes

Before we generalise to all tree-level super-amplitudes, it is useful to look first at the next case, namely NNMHV amplitudes. In examining the recursion relation in this case we will find new features which will help us find the solution for the full super-amplitude in the next section.

The recursive relation for NNMHV amplitudes reads

\[
A_{n}^{\text{NNMHV}} = \int \frac{d^{4}P}{P^{2}} \int d^{4}\eta \hat{P} A_{3}^{\text{MHV}}(z_{P}) A_{n-1}^{\text{NNMHV}}(z_{P}) + \sum_{i=4}^{n-3} \int \frac{d^{4}P_{i}}{P_{i}^{2}} \int d^{4}\eta \hat{P}_{i} A_{i}^{\text{MHV}}(z_{P_{i}}) A_{n-i+2}^{\text{NNMHV}}(z_{P_{i}}) \]
\[
+ \sum_{i=5}^{n-1} \int \frac{d^{4}P_{i}}{P_{i}^{2}} \int d^{4}\eta \hat{P}_{i} A_{i}^{\text{NMHV}}(z_{P_{i}}) A_{n-i+2}^{\text{MHV}}(z_{P_{i}}). \]

(32)

It is very similar to the recursion relation for NMHV amplitudes, and as we will show presently, it can be solved in a similarly straightforward manner.

We wish to demonstrate that the following formula is valid for all NNMHV amplitudes,

\[
A_{n}^{\text{NNMHV}} = \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\prod(j+j+1)} \sum_{2 \leq u,v \leq n-1} R_{n; u,v} \left[ \sum_{u+1 \leq s,t \leq v} R_{n; v,u,s,t} + \sum_{v \leq s,t \leq n-1} R_{n;s,t} \right]. \]

(33)

Here $R_{n; s,t}$ is the same invariant as before and $R_{n; u,v,s,t}$ is a very similar invariant\(^2\) and is given by

\[
R_{n; u,v,s,t} = \frac{\langle s | s - 1 \rangle \langle t | t - 1 \rangle \delta^{(4)}(\Xi_{r; u,v,s,t})}{x_{st}^{2} \langle r | x_{ru} x_{uv} x_{vs} x_{st} | t \rangle \langle r | x_{ru} x_{uv} x_{vs} x_{st} | t - 1 \rangle \langle r | x_{ru} x_{uv} x_{vt} x_{ts} | s \rangle \langle r | x_{ru} x_{uv} x_{vt} x_{ts} | s - 1 \rangle}. \]

(34)

\(^2\)The generalised $R$ is invariant under dual conformal transformations but not dual superconformal transformations. Nonetheless the product $R_{n; u,v} R_{n; v,u,s,t}$ is dual superconformally invariant as we will show in section 6.
The Grassmann odd quantity $\Xi_{r;u,v,s,t}$ is given by
\begin{equation}
\Xi_{r;u,v,s,t} = \langle r | x_{ru} x_{uv} x_{us} x_{st} | \theta_{uv} \rangle + \langle r | x_{ru} x_{uv} x_{ut} x_{ts} | \theta_{st} \rangle .
\end{equation}

The new quantity $R_{r;u,v,s,t}$ is a dual conformal invariant, which follows in exactly the same way as the proof that $R_{r;s,t}$ is invariant, see [1]. As we show in section 6, the product $R_{r;u,v} R_{r;u,v,s,t}$ is a dual superconformal invariant. We always use the convention that the last two labels $s$ and $t$ satisfy $s + 2 \leq t$.

For the two terms $B_1$ and $B_2$ we find
\begin{align}
B_1 &= \frac{\delta^{(4)}(q) \delta^{(8)}(q)}{\prod_{j=1}^{n} (j \ j + 1)} \sum_{4 \leq i \leq n-3} R_{n;2,i} \sum_{i \leq s,t \leq n-1} R_{n;3,s,t}, \\
B_2 &= \frac{\delta^{(4)}(q) \delta^{(8)}(q)}{\prod_{j=1}^{n} (j \ j + 1)} \sum_{5 \leq i \leq n-1} R_{n;2,i} \sum_{3 \leq s,t \leq i} R_{n;3,s,t}.
\end{align}

Let us comment on the calculation of $B_1$ and $B_2$ in order to be able to generalise such calculations to arbitrary amplitudes in section 5.

The calculation of $B_1$ is very similar to that of $B$ in the section 3. Firstly, the factor $\delta^{(8)}(q) R_{n;2,i}$ is obtained from carrying out the Grassmann integral using the supermomentum conservation delta’s from the subamplitudes. Secondly, the term $\sum_{i \leq s,t \leq n-1} R_{n;3,s,t}$ comes from a similar term in the $A_{n-1}^{NMHV}$ subamplitude. As in the calculation of $A$ in section 3, it can be inserted in such a way that it does not depend on the shift, and therefore, the labels of the invariants $R$ remain unchanged.

The calculation of $B_2$ reveals a new feature. When inserting the $A_{n}^{NMHV}$ subamplitude such that its labels $\{1, 2, 3, \ldots, n-1\}$ correspond to the labels $\{2, \ldots, i-1, \hat{P}, \hat{1}\}$ of the new amplitude, an invariant $R_{n;3,s,t}$ will turn into $R_{1;i,s,t}$. Let us examine the corresponding $\Xi_{1;i,s,t}$, see (21),
\begin{align}
\Xi_{1;i,s,t} &= \langle \hat{1} | x_{1s} x_{st} | \theta_{1i} \rangle + \langle \hat{1} | x_{1i} x_{ts} | \theta_{s1} \rangle \\
&= \langle \hat{1} | x_{2s} x_{st} | \theta_{12} \rangle + \langle \hat{1} | x_{2i} x_{ts} | \theta_{s2} \rangle ,
\end{align}
where in the second line we have used relation (25) to alter the label on the first $x$ and a similar relation for the labels on the $\theta$.

With the help of (24) we see that it is independent of the internal $\eta_{P_i}$, but it does depend on $\lambda_{1}$. A quick calculation shows that
\begin{equation}
\langle \hat{1} | = \langle 1 | - z_P \langle n | = \frac{1}{\langle n | P \ | 1 \rangle} \langle n | x_{1i} x_{i2} \rangle
\end{equation}
and hence
\begin{equation}
R_{1;i,s,t} = R_{n;i,2,s,t} ,
\end{equation}
in agreement with (37).

The homogeneous term works in exactly the same way as for NMHV amplitudes - one can choose the first label on every $R$ to be $n$, then all $R$ are inert under the Grassmann integral.
This suffices to show that the formula above (33) is valid for all NNMHV amplitudes, as claimed. In view of the notation we will use in the next section, it is again useful to forget the universal MHV prefactor (4) and to change labels in (33),

\[ P_n^{\text{NNMHV}} = \sum_{2 \leq a_1, b_1 \leq n-1} R_{n,a_1,b_1} \left( \sum_{a_1+1 \leq a_2, b_2 \leq b_1} R_{n,b_1,a_1,a_2,b_2} + \sum_{b_1 \leq a_2, b_2 \leq n-1} R_{n,a_2,b_2} \right). \]  

(42)

### 5 All tree amplitudes

It is simple to continue the analysis of the preceding sections to \( N^3 \text{MHV}, N^4 \text{MHV} \) amplitudes and so on. The supersymmetric recursion relation for a generic \( N^p \text{MHV} \) amplitude reads

\[ A_n^{\text{N}^p \text{MHV}} = \int \frac{d^4P}{P^2} \int d^4 \eta_P A_3^{\text{MHV}}(z_P) A_{n-1}^{\text{N}^p \text{MHV}}(z_P) + \sum_{m=0}^{p-1} \sum_i \int \frac{d^4P_i}{P_i^2} \int d^4 \eta_{P_i} A_i^{\text{N}^m \text{MHV}}(z_{P_i}) A_{n-i+2}^{\text{N}^{(p-m-1)} \text{MHV}}(z_{P_i}). \]  

(43)

At each stage one obtains the universal prefactor \( A_n^{\text{MHV}} \) while the \( R \)-invariants from the right-hand factor are left unchanged while those from the left-hand factor acquire an additional extension, just as in the case of the NNMHV amplitudes. The results are expressed in terms of the following general dual conformal invariants,

\[ R_{n,b_1,a_1; b_2,a_2; \ldots; b_r} = \langle a a - 1 \rangle \langle b b - 1 \rangle \delta^4(\langle \xi | x_{a,a} x_{a b} | \theta_{b a} \rangle + \langle \xi | x_{b,a} x_{b a} | \theta_{b a} \rangle) \]

(44)

where the chiral spinor \( \xi \) is given by

\[ \langle \xi \rangle = \langle n | x_{n b_1} x_{b_1 a_1} x_{a_1 b_2} x_{b_2 a_2} \ldots x_{b_r} a_r \rangle. \]  

(45)

For example, we find that the \( N^3 \text{MHV} \) amplitudes are given by the formula

\[ P_n^{\text{N}^3 \text{MHV}} = \sum_{2 \leq s,t \leq n-1} R_{n,s,t} \left[ \sum_{s+1 \leq u,v \leq t} R_{n,t,s; u,v} \left( \sum_{u+1 \leq w, x \leq v} R_{n,t,s; w,x} + \sum_{v \leq w , x \leq t} R_{n,t,s; w,x} \right) \\
+ \sum_{s+1 \leq u,v \leq t} R_{n,t,s; u,v} \sum_{t \leq w, x \leq n-1} R_{n,w,x} \\
+ \sum_{t \leq u,v \leq n-1} R_{n,u,v} \left( \sum_{u+1 \leq w, x \leq v} R_{n,u,v; w,x} + \sum_{v \leq w , x \leq n-1} R_{n,u,v; w,x} \right) \right]. \]  

(46)

The three lines correspond to the three different inhomogeneous terms in the recursion relation \( A_n^{\text{NNMHV}} A_R^{\text{MHV}}, A_L^{\text{NNMHV}} A_R^{\text{MHV}} \) and \( A_L^{\text{MHV}} A_R^{\text{NNMHV}} \). It is helpful to change the summation labels and to notice that the first and second lines can be combined so that we have

\[ P_n^{\text{N}^3 \text{MHV}} = \sum_{2 \leq a_1, b_1 \leq n-1} R_{n,a_1,b_1} \left[ \\
\sum_{a_1+1 \leq a_2, b_2 \leq b_1} R_{n,b_1,a_1,a_2,b_2} \left( \sum_{a_2+1 \leq a_3, b_3 \leq b_2} R_{n,a_2,a_3,b_3} + \sum_{b_2 \leq a_3, b_3 \leq b_1} R_{n,a_1,a_3,b_3} + \sum_{b_1 \leq a_3, b_3 \leq n-1} R_{n,a_3,b_3} \right) \\
+ \sum_{b_1 \leq a_2, b_2 \leq n-1} R_{n,a_2,b_2} \left( \sum_{a_2+1 \leq a_3, b_3 \leq b_2} R_{n,a_2,a_3,b_3} + \sum_{b_2 \leq a_3, b_3 \leq b_1} R_{n,a_2,a_3,b_3} \right) \right]. \]  

(47)
Comparing the above formula to the forms of the NMHV and NNMHV amplitudes we can see the general pattern. We illustrate the full \( n\)-point super-amplitude in Fig. 4. We consider a rooted tree, with the top vertex (the root) denoted by 1. The root has a single descendent vertex with labels \( a_1, b_1 \), which corresponds the invariant \( R_{n;a_1,b_1} \). The tree is completed by passing from each vertex to a number of descendent vertices, as described in Fig. 5. For an \( n\)-point super-amplitude we have \( n-3 \) rows in the tree and we enumerate the rows by 0, 1, 2, 3, \ldots, \( n-4 \) with 0 corresponding to the root. The rule for completing the tree as given in Fig. 5 can be easily seen to imply that the number of vertices in row \( p \) is the Catalan number \( C(p) = (2^p)!/(p!(p+1)!) \).

Each vertex in the tree corresponds to an \( R \)-invariant with the first label (as always) being \( n \) and the remaining labels corresponding to those written in the vertex. We consider vertical paths in the tree, starting from the root. To each path we associate the product of the \( R \)-invariants (vertices) visited by the path, with a nested summation over all labels. The last pair of labels in a given vertex correspond to the ones which are summed first (i.e. the ones of the inner-most sum) inside the nested summation and in row \( p \) they are denoted by \( a_p, b_p \). We always take the convention that \( a_p + 2 \leq b_p \), i.e. that \( a_p < b_p \) and they are separated by at least two, which is needed for the corresponding \( R \)-invariant to be well-defined. The lower and upper limits for the last pair of summation variables, \( a_p, b_p \), are noted on the line above each vertex. The formula for the full super-amplitude is given by the sum over all vertical paths of any length, starting from the root,

\[
\mathcal{P}_n = \sum \text{vertical paths}. \tag{48}
\]

Let us now see how the formula (48) works for the first few cases. Firstly there is one path of length zero, where we start at the root (row zero) and do not go anywhere. This value of path is simply 1 and it corresponds to the MHV amplitudes,

\[
\mathcal{P}_n^{\text{MHV}} = 1. \tag{49}
\]

There is one path of length one, where we start at the root and go one step to its unique descendent. This path gives us 1 from the root, multiplied by \( R_{n;a_1,b_1} \) from the descendent.
Figure 5: The rule for going from line $p-1$ to line $p$ (for $p > 1$) in Fig. 4. For every vertex in line $p-1$ of the form given at the top of the diagram, there are $r+2$ vertices in the lower line (line $p$). The labels in these vertices start with $v_1 u_1 \ldots v_r u_r b_{p-1} a_{p-1} a_p b_p$ and they get sequentially shorter, with each step to the right removing the pair of labels adjacent to the last pair $a_p, b_p$ until only the last pair is left. The summation limits between each line are also derived from the labels of the vertex above.

vertex, summed over $a_1, b_1$ with lower limit 2 and upper limit $n-1$ so we obtain for the NMHV amplitudes,

$$P_{n}^{\text{NMHV}} = \sum_{2 \leq a_1, b_1 \leq n-1} R_{n; a_1, b_1},$$

which agrees with eq. (31).

There are two paths of length two. The first corresponds to descending from the root by one step and then descending once more to the left in Fig. 4. For this path we obtain 1 multiplied by $R_{n; a_1, b_1}$ multiplied by $R_{n; b_1, a_1, a_2, b_2}$ with the limits for the outer sum over $a_1, b_1$ being the same as for the NMHV case above, while the inner sum, which is over $a_2, b_2$, has lower limit $a_1+1$ and upper limit $b_1$. The second path of length two corresponds to descending to the right instead of to the left. Doing so we obtain the product $1 \times R_{n; a_1, b_1} \times R_{n; a_2, b_2}$ with summation limits in the outer sum as before and in the inner sum being $b_1 \leq a_2, b_2 \leq n-1$. Adding the two paths we obtain for the NNMHV amplitudes

$$P_{n}^{\text{NNMHV}} = \sum_{2 \leq a_1, b_1 \leq n-1} R_{n; a_1, b_1} \left( \sum_{a_1+1 \leq a_2, b_2 \leq b_1} R_{n; b_1, a_1, a_2, b_2} + \sum_{b_1 \leq a_2, b_2 \leq n-1} R_{n; a_2, b_2} \right),$$

which agrees with eq. (42).

Continuing, we find five paths of length three. They correspond precisely to the five terms in the expression (47) for the $N^3$MHV amplitudes. Generically, since the number of vertices in row $p$ of the tree in Fig. 4 is the Catalan number $C(p)$, we find $C(p)$ terms in the expression for the $N^p$MHV amplitudes. Finally, by considering the sum of all vertical paths of any length, starting from the root, we obtain the sum of all amplitudes (MHV, NMHV, NNMHV, etc.). We will denote this sum by $\mathcal{N}_2^{n-1}$, where the subscript is the lower limit of the outer-most sum and the superscript is the upper limit. Thus we have introduced an alternative notation $P_n = \mathcal{N}_2^{n-1}$.

Using this notation, we can check the result we have encoded in Fig. 4 by returning to the recursion relation (12). The relevant picture, Fig. 6, is the same as Fig. 2 except that now all vertices except the three-point MHV vertex are full super-amplitudes, as described by the sum over vertical paths in Fig. 4.
Figure 6: The two contributions to the RHS of the supersymmetric recursion relation for the full super-amplitudes. We call the first term the linear term and the second term the quadratic term. As before 1 means that $\lambda_1$ is shifted, and $\bar{n}$ means that $\bar{\lambda}_n$ is shifted.

Figure 7: Graphical representation of the term $R_{n;2,i} N_3^{i} |_{\langle n \rangle \rightarrow \langle n \rangle |_{x_{n_i} x_{i2}}} N_i^{n-1}$ in equation (52). The left tree corresponds to the first two factors $R_{n;2,i} N_3^{i}$, while the right tree corresponds to the final factor $N_i^{n-1}$. As indicated in the text, after summing over $i$ the first tree is almost what is needed to complete $N_3^{n-1}$ coming from the linear term to $N_2^{n-1}$. The missing pieces come from the right factor which can be adjoined to the left by inserting it everywhere there is a line drawn in bold so that these lines then all lead to a descendent vertex with labels $c_2, d_2$. Since the $c$ and $d$ labels are all dummy variables they can then be exchanged for the suitable $a$ and $b$ labels by a change of notation.

All terms on the RHS of the recursion relation give the universal MHV tree super-amplitude prefactor (4). As for contributions to the remaining factor $P_{\alpha}$, the first term (the linear term) on the RHS of the recursion relation produces $N_3^{n-1}$, i.e. the sum over vertical paths in Fig. 4 with the lower limit on the outer-most sum being 3 instead of 2. The second term (the quadratic term) gives the following sum

$$\sum_i R_{n;2,i} N_3^{i} |_{\langle n \rangle \rightarrow \langle n \rangle |_{x_{n_i} x_{i2}}} N_i^{n-1}.$$  (52)

Here the first factor $R_{n;2,i}$ appears from doing the Grassmann integration over $\eta_{\alpha}$ in exactly the same way as we saw for the NMHV and NNMHV amplitudes. The second factor $N_3^{i} |_{\langle n \rangle \rightarrow \langle n \rangle |_{x_{n_i} x_{i2}}}$ is the full super-amplitude corresponding to the the left factor in the quadratic term. The additional subscript on $N$ indicates the fact that, just as we saw in the case of NNMHV amplitudes, the indices on all $R$-invariants get extended by one pair of labels which can be achieved by the replacement $\langle n \rangle \rightarrow \langle n \rangle |_{x_{n_i} x_{i2}}$. The third factor $N_i^{n-1}$ is the full super-amplitude corresponding to the right factor in the quadratic term. As before there are no additional labels on the $R$-invariants in this factor. Finally there is a sum over the label $i$.

If we consider the first two factors $R_{n;2,i} N_3^{i} |_{\langle n \rangle \rightarrow \langle n \rangle |_{x_{n_i} x_{i2}}}$ together we can see that they reproduce...
a sum over vertical paths in a tree very similar to the one in Fig. 4. The differences are that the root is now $R_{n;2}$ instead of 1 (so that the first term, 1, is missing), the top $R$-invariant $R_{n;2,i}$ has its lower label fixed to be 2 and its upper label to be $i$, and all descendent vertices have at least two pairs of labels due the extension $\langle n \rangle \rightarrow \langle n \rangle x_{ni} x_{i2}$. After summing over $i$ this would almost be what is needed to complete the linear term from $N_{n}^{n-1}$ to $N_{n}^{n-1}$, except that all paths which pass through a descendent vertex with only a single pair of labels are missing. These missing paths form an overall factor for each value of $i$ and this is precisely what is contained in the third factor $N_{i}^{n-1}$. Thus we arrive finally at the fact that

$$N_{3}^{n-1} + \sum_{i} R_{n;2,i} N_{3}^{n-1} |\langle n \rangle x_{ni} x_{i2} N_{i}^{n-1} = N_{2}^{n-1},$$

which shows that the sum over vertical paths in Fig. 4 is indeed a solution to the supersymmetric recursion relation. We have already verified directly that it gives the correct answer in the first few cases, so we conclude that the $n$-point tree-level super-amplitude in $\mathcal{N} = 4$ SYM is indeed given by the sum over vertical paths in the rooted tree in Fig. 4.

6 Symmetries of the amplitudes

Tree amplitudes in $\mathcal{N} = 4$ SYM are expected to have many symmetries. First of all, $\mathcal{N} = 4$ SYM is a superconformal field theory, so the amplitudes should exhibit this symmetry in their functional forms. The MHV super-amplitudes were shown to be annihilated by all generators of the the conventional superconformal algebra in [6]. The amplitudes we have constructed in this paper are manifestly invariant under all generators of the conventional superconformal algebra except for the superconformal symmetries $s, \bar{s}, k$.

In addition to the conventional superconformal symmetry, it was conjectured in [1] that the tree-level super-amplitudes should also exhibit dual superconformal symmetry. As far as tree-level super-amplitudes are concerned, the conjecture of [1] states that they should be covariant under dual conformal transformations $K$ and the chiral superconformal transformations $S$, while they are invariant under $P, Q, \bar{Q}, \bar{S}$. They also have the obvious property that the dual dilatation weight and central charge are equal to $n$, the number of particles.

The generators of the two different realisations of the superconformal algebra are not all independent. As discussed in [1] the odd generator $\bar{q}$ coincides with $\bar{S}$, while $\bar{s}$ coincides with $\bar{Q}$. The same correspondence was observed in [2, 3] after performing a fermionic T-duality in the string sigma model. The explicit form of all generators is summarised in Appendix C.

In [15] it the dual conformal covariance of the tree-level super-amplitudes was verified recursively using the supersymmetric recursion relations. We can indeed see this symmetry in the explicit form of the solution we have presented. All quantities $R_{n; a_{1}, b_{1}, \ldots, a_{m}, b_{m}; s, t}$ are dual conformal invariants, as can be quickly verified by counting the conformal weights of the numerator and denominator. Assuming further the expected conventional superconformal invariance of the

\[\text{Following the conventions of [1] we will use lower case characters to denote the conventional superconformal generators and upper case ones for the dual superconformal generators.}\]
amplitude $s \mathcal{A} = \bar{Q} \mathcal{A} = 0$, this is sufficient to derive all the expected properties under the full dual superconformal algebra. Further we remark that if all super-amplitudes obey $s \mathcal{A} = 0$ then they also obey $s \mathcal{A} = 0$, since we could alternatively have performed the entire analysis in the anti-chiral ($\bar{\eta}$) representation for the gluon supermultiplet. Thus showing $\bar{s}$-invariance is sufficient to derive invariance under $s$ and therefore $k = \{s, \bar{s}\}$.

Therefore the only nontrivial property of the super-amplitude which remains to be verified is its behaviour under the $s \mathcal{A}$ or $\bar{Q} \mathcal{A}$ supersymmetry. Following [1], we can choose a fixed frame in which $\theta_s = \theta_t = 0$. In this frame, we have

$$\Xi_{r;vu;st} = x_{tu}^2 \langle x_{ru}x_{vu} | \theta_{u}^A \rangle = x_{tu}^2 \langle \xi_{u}^A \rangle, \quad (54)$$

and when acting on $R_{r;vu;st}$ the operator $\bar{Q} \hat{A}$ is reduced to $\bar{Q} \hat{A} = \theta^A_i \partial^i + \theta^A_{u} \partial^u + \theta^A_{v} \partial^v$. Therefore we need to evaluate $\left( J \bar{Q}^E \right)$ ($\hat{\lambda}_j$ is an arbitrary projection) on

$$R_{r;vu;st} \propto \frac{\epsilon_{ABCD} \langle \xi_{u}^A \rangle \langle \xi_{v}^B \rangle \langle \xi_{w}^C \rangle \langle \xi_{x}^D \rangle}{\langle \xi_{I_1} \rangle \langle \xi_{I_2} \rangle \langle \xi_{I_3} \rangle \langle \xi_{I_4} \rangle}, \quad (55)$$

where $|I_1\rangle = |x_{tu}x_{st}|t\rangle$, and similarly for $I_2, I_3, I_4$.

It can be easily seen that in the fixed frame, $[JQ^E]$ acts trivially on $I_i$ in (55), because e.g.

$$\langle \xi [JQ^E] I_1 \rangle = \langle \xi \theta^E_u \rangle [J|x_{st}|t\rangle \quad (56)$$

is annihilated by the Grassmann delta function in the numerator of (55). Thus when acting with $[JQ^E]$ on (55), only $\langle \xi \rangle$ transforms. After using the cyclic identity for spinors and rearranging some terms we find

$$[JQ^E] R_{r;vu;st} \propto \frac{\epsilon_{ABCD} \langle \xi_{u}^A \rangle \langle \xi_{v}^B \rangle \langle \xi_{w}^C \rangle \langle \xi_{x}^D \rangle}{\langle \xi_{I_1} \rangle \langle \xi_{I_2} \rangle \langle \xi_{I_3} \rangle \langle \xi_{I_4} \rangle} \times \left[ \langle r|x_{ru}x_{vu}| \theta^E_{ur} \rangle + \langle r|x_{ru}x_{uv}| \theta^E_{ur} \rangle \right]. \quad (57)$$

Therefore, $R_{r;vu;st}$ is not dual superconformally invariant. However, in (33), it always appears multiplied by the invariant $R_{r;uv}$. In this case, the Grassmann delta function in $R_{r;uv}$ makes the variation (57) vanish, and therefore $R_{r;uv}R_{r;vu;st}$ is a dual superconformal invariant.

We conclude that the NNMHV amplitudes are dual superconformally covariant. From the discussion here and in section 5 it is easy to see that this property is true for all tree level amplitudes in $\mathcal{N} = 4$ SYM. Indeed, one can repeat the argument above to 'longer' chains of invariants that appear in equation (48). Take for example $R_{n;t,s}R_{n,t,s,u,v}R_{n,t,s,y,w,x}$ from (46). After fixing a frame where $\theta_w = \theta_x = 0$, we obtain an expression like (55) with a different $\langle \xi \rangle = \langle r|x_{rt}x_{ts}x_{su}x_{vu}| \rangle$. Because of the linearity of $[JQ^E]$ the calculation of the variation of (55) is as above, except that now we obtain two contributions, one of which vanishes thanks to $R_{n;t,s}$, and the other thanks to $R_{n;t,s,u,v}$. The crucial feature is that $R$’s with many indices share all first indices of their ‘predecessors’. This is the case by construction for all terms in (48).

Therefore we have shown that all tree-level amplitudes in $\mathcal{N} = 4$ SYM are covariant under dual superconformal transformations. This extends the result of [15], where the covariance under dual conformal transformations was shown, and confirms a conjecture made in [1].
7 Gluon scattering amplitudes from super-amplitudes

Here we wish to give some explanations on how gluon amplitudes can be extracted from our solutions and how this can be implemented, for example on a computer.

Let us first stress that any component amplitudes for arbitrary particle or helicity choice can be extracted from the super-amplitudes, see e.g. [1] for more explanations. Here we focus on the particularly simple case of gluon amplitudes.

According to (2), to each negative helicity gluon at position \( j \) is associated a factor of \((\eta_j)^4 = \eta_j^1 \eta_j^2 \eta_j^3 \eta_j^4\), and to each positive helicity gluon simply a factor of 1. Going from a given super-amplitude to a gluon component amplitude therefore just amounts to extracting specific prefactors in the \( \eta \)-expansion of the super-amplitude. An elementary example is the relation (7) between the gluon MHV amplitude (1) and the super-amplitude (4).

A less trivial example is the split-helicity NMHV amplitude,

\[
A_{\text{NMHV}}^n = (\eta_{n-2})^4 (\eta_{n-1})^4 (\eta_n)^4 A(1^+, \ldots, (n-3)^+, (n-2)^-, (n-1)^-, n^-) + \ldots, \tag{58}
\]

Therefore we could expand \( A_{\text{NMHV}}^n \) in \( \eta \) and recover the desired split-helicity gluon amplitude.\(^4\) However there is a much simpler way to do this. In order to see this, let us observe that the relation between NMHV super-amplitude and the desired gluon component can be equivalently written as a Grassmann integral

\[
A(1^+, \ldots, (n-3)^+, (n-2)^-, (n-1)^-, n^-) = \int d^4 \eta_{n-2} \int d^4 \eta_{n-1} \int d^4 \eta_n \ A_{\text{NMHV}}^n. \tag{59}
\]

In this paper we have already encountered many of such Grassmann integrals and seen that they are easy to do. We can always choose two arbitrary spinor projections of \( q^A_\alpha \) to rewrite the \( \delta^{(8)}(q^A_\alpha) \) of \( A_{\text{NMHV}}^n \) as

\[
\delta^{(8)}(q^A_\alpha) = \langle n-1 \ n \rangle^4 \delta^{(4)} \left( \eta_{n-1}^A + \sum_{i=1}^{n-2} \frac{\langle in \rangle}{\langle n-1 \ n \rangle} \eta_i^A \right) \delta^{(4)} \left( \eta_n^A + \sum_{i=1}^{n-2} \frac{\langle n-1-i \rangle}{\langle n-1 \ n \rangle} \eta_i^A \right). \tag{60}
\]

This allows us to immediately carry out the \( d^4 \eta_n \) and \( d^4 \eta_{n-1} \) integrals in (59). The remaining terms in \( A_{\text{NMHV}}^n \) are unaffected by this since they can be written in the form (24) in which they are independent of \( \eta_{n-1} \) and \( \eta_n \). Hence we obtain

\[
A(1^+, \ldots, (n-3)^+, (n-2)^-, (n-1)^-, n^-) = \delta^{(4)}(p) \frac{\langle n-1 \ n \rangle^4}{\langle 12 \rangle \ldots \langle n-1 \ n \rangle} \int d^4 \eta_{n-2} \sum_{1<s,t<n} R_{n;s,t}, \tag{61}
\]

where the \( \Xi_{n;s,t} \) in \( R_{n;s,t} \) are written in the form (24). A further simplification occurs because \( R_{n;s,t} \) only depends on \( \eta_{n-2} \) if \( t = n-1 \), see (24). Carrying out the remaining Grassmann integration using the \( \delta^{(4)}(\Xi_{n;s,n-1}) \) in \( R_{n;s,n-1} \) we obtain

\[
A(1^+, \ldots, (n-3)^+, (n-2)^-, (n-1)^-, n^-) = -\frac{\delta^{(4)}(p)}{\langle 12 \rangle \ldots \langle n-3 \ n \ n \rangle} \sum_{s=2}^{n-3} x_{s,n-1}^2 x_{s,n}^2 \langle n-2 \ | x_{n-1,s} x_{n} \ | n \rangle^3 \langle s \ s \ n \rangle \langle n-1 \ | x_{n-1,s} \ | s \ n \rangle \langle n-1 \ | x_{n-1,s} \ | s \ \rangle. \tag{62}
\]

\(^4\)Note that a Grassmann delta function is simply defined as a product, \( \delta^{(4)}(\chi^A) = 1/4! \epsilon_{ABCD} \chi^A \chi^B \chi^C \chi^D \).
This is in perfect agreement with formula (4.5) given in [30].

In more complicated situations one could for example first do some $\eta$ integrations analytically (e.g. using the $\delta^{(8)}(q)$ which is present in all physical super-amplitudes because of supersymmetry), and then do the remaining integrations/expansions on a computer. This can be easily programmed, keeping track of the overall sign (because the $\eta$’s are anticommuting variables). The resulting spinor expressions can be evaluated numerically using available packages, see e.g. [31].

8 Conclusions

The main result of our paper is formula (48) for all tree-level amplitudes in $\mathcal{N} = 4$ SYM. The formula contains all amplitudes with arbitrary total helicity (MHV, NMHV, ..., $\overline{\text{MHV}}$). It is given in terms of vertical paths of a particular rooted tree, shown in Fig. 4. This extends previous solutions of the BCF recursion relations which applied only to the closed subset of split-helicity gluon amplitudes [30]. Our solution is written in on-shell $\mathcal{N} = 4$ superspace. It is built from dual superconformal invariants and so it manifestly exhibits both conventional and dual superconformal symmetries.

Our contains as components all amplitudes for arbitrary external states and helicities. We explained in section 7 that gluon components are particularly simple to extract, since they can be obtained from the super-amplitudes by carrying out Grassmann integrations. A crucial simplifying feature is that (71) is built from sums over products of Grassmann delta functions, which can be used to perform the aforementioned integrations. We expect that it will be possible to obtain compact expressions for previously unknown gluon components following the example in section 7.

We expect our results to be relevant for $\mathcal{N} = 8$ supergravity as well, since tree-level amplitudes in the latter theory can be obtained from those in $\mathcal{N} = 4$ SYM through the KLT relations [32]. Furthermore the methods employed here could also be directly applied to solving recursion relations directly for supergravity tree-level amplitudes [33]. It would also be interesting to see if our formula could shed light on the relation among tree-level amplitudes described in [35].

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Appendices

A Collinear limits of the super-amplitudes

Here we discuss the consistency of our results. We do not need to check whether the amplitudes have the correct multi-particle poles, since this is automatically the case when constructing amplitudes from the BCF recursion relations.

Here we check that our amplitudes have the correct collinear limit as two particles become almost collinear [34]. Consider two neighbouring particles at points $a$ and $b = a + 1$ that become collinear such that

$$p_a = z P, \quad p_b = (1 - z) P,$$

then an $n$-gluon tree amplitude is expected to behave as

$$A_n \frac{a||b}{\lambda=\pm} \sum \text{Split}_{\pm}^{\text{tree}} (a^\lambda, b^\lambda) A_n(\ldots, (a + b)^\lambda, \ldots),$$

where $\text{Split}_{\pm}^{\text{tree}}$ are certain helicity-dependent splitting functions, see [34]. The non-vanishing splitting functions diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab} = (p_a + p_b)^2 \to 0$. In the collinear limit, the spinors corresponding to the momenta $p_a$ and $p_b$ become

$$\lambda_a \to \sqrt{z} \lambda_P, \quad \tilde{\lambda}_a \to \sqrt{z} \tilde{\lambda}_P, \quad \lambda_b \to \sqrt{1 - z} \lambda_P, \quad \tilde{\lambda}_b \to \sqrt{1 - z} \tilde{\lambda}_P.$$

In the supersymmetric case, the define consistently to (65)

$$\eta_a \to \sqrt{z} \eta_P, \quad \eta_b \to \sqrt{1 - z} \eta_P.$$

By inspecting the collinear limit for the MHV super-amplitudes (4), we expect the following collinear limit for super-amplitudes at tree level,

$$A_n(\ldots, a, b, \ldots) \frac{a||b}{\lambda=\pm} \sum \text{Split}_{\pm}^{\text{tree}} (a^\lambda, b^\lambda) A_n(\ldots, (a + b)^\lambda, \ldots) \to \frac{1}{\sqrt{z(1 - z)} \langle ab \rangle} A_{n-1}(\ldots, P, \ldots).$$

(67)

Let us see if relation (67) holds for the NMHV amplitudes (8) as well. We need to analyse the behaviour of the invariants $R_{n;s,t}$ in the limit. Because of cyclic symmetry of the superamplitude, we can consider the $a = n - 1, b = n$ without loss of generality. This is advantageous because then the invariants $R_{n;s,t}$ are affected by the collinear limit only through $\lambda_n = \sqrt{1 - z} \lambda_P$. Looking at (21) we see that

$$R_{n;s,t} \xrightarrow{n-1||n} R_{P;s,t}. \quad (68)$$

We also observe that

$$R_{n;s,n-1} \xrightarrow{n-1||n} R_{P;s,n-1} \propto (n - 1 n)^2 \to 0. \quad (69)$$

Using (68) and (69) on (8) we see that indeed

$$A_n^{\text{NMHV}} (1, \ldots, n - 1, n) \xrightarrow{n-1||n} \frac{1}{\sqrt{z(1 - z)} \langle n - 1 n \rangle} A_{n-1}^{\text{NMHV}} (1, \ldots, n - 2, P).$$

(70)
Going to NNHMV amplitudes, see equation (33), we see that the behaviour of the 'longer' invariants like \( R_{n,u,v}R_{n,v,u}R_{s,t} \) under the collinear limit where particles \( n-1 \) and \( n \) become collinear is completely analogous to the NMHV case, they turn into \( R_{P(u,v)R_{P(v,u)}} \). It is then obvious that (33) obeys the collinear limit (67). This observation can be immediately generalised to arbitrary non-MHV amplitudes, see equation (71). The crucial feature is that all invariants share the same first label \( n \), which is simply replaced by \( P \) in the collinear limit.

**B Formula for all tree amplitudes**

For a computerised implementation of the full tree-level formula we have presented, it is helpful to reinterpret the iterative construction of the rooted tree in Fig. 4 in terms of explicit sums. Doing so we arrive at the following formula for \( N_{\text{PMHV}} \) amplitudes.

\[
A_{n}^{\text{PMHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\prod_{j=1}^{n}(j+j+1)} \sum_{\{a\},\{b\}\{r\}} \prod_{s=1}^{p} R_{n,b_{1}^{(s)}a_{1}^{(s)}}...b_{s}^{(s)}a_{s}^{(s)}b_{s}^{(s)} \tag{71}
\]

The \( a_{i}^{(s)}, b_{i}^{(s)} \) are related to the summation variables \( a_{j}, b_{j} \) in the following way. There are two cases. If \( r_{s} = s - 1 \) then

\[
a_{i}^{(s)} = a_{i}^{(s-1)} \quad \text{for} \quad i < s - 1, \quad a_{s-1}^{(s)} = b_{s-1}, \quad \text{and} \quad a_{s}^{(s)} = a_{s},
\]

\[
b_{i}^{(s)} = b_{i}^{(s-1)} \quad \text{for} \quad i < s - 1, \quad b_{s-1}^{(s)} = a_{s-1}, \quad \text{and} \quad b_{s}^{(s)} = b_{s},
\]

and in all other cases, i.e. for \( r_{s} < s - 1 \),

\[
a_{i}^{(s)} = a_{i}^{(s-1)} \quad \text{for} \quad i < s \quad \text{and} \quad a_{s}^{(s)} = a_{s},
\]

\[
b_{i}^{(s)} = b_{i}^{(s-1)} \quad \text{for} \quad i < s \quad \text{and} \quad b_{s}^{(s)} = b_{s}.
\]

Further, the summations over the different \( a_{i}, b_{i} \) and \( r_{i} \) are nested in the following way

\[
\sum_{\{a\},\{b\}\{r\}} = \sum_{r_{1}=0}^{r_{1+1}} \sum_{r_{2}=0}^{r_{2+1}} ... \sum_{r_{p}=0}^{r_{p+1}} \sum_{l_{r_{1}}^{(i)}=a_{1},b_{1} \leq u_{r_{1}}^{(i)}} l_{r_{2}}^{(i)} \leq a_{2},b_{2} \leq u_{r_{2}}^{(i)} ... l_{r_{p}}^{(i)} \leq a_{p},b_{p} \leq u_{r_{p}}^{(i)}
\]

where the lower limits \( l_{r_{i}}^{(i)} \) and upper limits \( u_{r_{i}}^{(i)} \) on the sums over the \( a_{i} \) and \( b_{i} \) are defined by

\[
l_{r_{i}}^{(i+1)} = a_{i} + 1, \quad l_{r_{i}}^{(s)} = b_{s-i-1} \quad \text{for} \quad s > i + 1, \quad u_{r_{i}}^{(i)} = b_{r}.
\]

By convention we take \( a_{0} = 1 \) and \( b_{0} = n - 1 \) such that \( l_{0}^{(1)} = 2 \) and \( u_{0}^{(i)} = n - 1 \).

**C Conventional and dual superconformal generators**

In this appendix we give the conventional and dual representations of the superconformal algebra. We begin by listing the commutation relations of the algebra \( u(2,2|4) \). The Lorentz generators
\(M_{\alpha \beta}, \overline{M}_{\dot{\alpha} \dot{\beta}}\) and the \(su(4)\) generators \(R^A_B\) act canonically on the remaining generators carrying Lorentz or \(su(4)\) indices. The dilatation \(\mathbb{D}\) and hypercharge \(\mathbb{B}\) act via

\[
[\mathbb{D}, J] = \text{dim}(J), \quad [\mathbb{B}, J] = \text{hyp}(J).
\]

The non-zero dimensions and hypercharges of the various generators are

\[
\begin{align*}
\text{dim}(\mathbb{P}) &= 1, & \text{dim}(Q) &= \text{dim}(\overline{Q}) = \frac{1}{2}, & \text{dim}(S) &= \text{dim}(\overline{S}) = -\frac{1}{2}, \\
\text{dim}(\mathbb{K}) &= -1, & \text{hyp}(Q) &= \text{hyp}(\overline{S}) = \frac{1}{2}, & \text{hyp}(\overline{Q}) &= \text{hyp}(S) = -\frac{1}{2}.
\end{align*}
\]

The remaining non-trivial commutation relations are,

\[
\begin{align*}
\{Q_{\alpha A}, \overline{Q}^B_{\dot{\alpha}}\} &= \delta^B_A \mathbb{P}_{\alpha \dot{\alpha}}, & \{S^A_{\dot{\alpha}}, \overline{S}^B_{\alpha}B\} &= \delta^A_B \mathbb{K}_{\alpha \dot{\alpha}}, \\
[\mathbb{P}_{\alpha \dot{\alpha}}, S^A_B] &= \delta^B_A \mathbb{Q}_{\alpha \dot{\alpha}}, & [\mathbb{K}_{\alpha \dot{\alpha}}, Q^B_{\alpha}] &= \delta^B_A \mathbb{S}_{\alpha \dot{\alpha}}, \\
[\mathbb{P}_{\alpha \dot{\alpha}}, \overline{S}^B_{\dot{\alpha}}] &= \delta^B_A \mathbb{Q}_{\alpha \dot{\alpha}}, & [\mathbb{K}_{\alpha \dot{\alpha}}, \overline{Q}^B_{\dot{\alpha}}] &= \delta^B_A \mathbb{S}_{\alpha \dot{\alpha}}, \\
[\mathbb{K}_{\alpha \dot{\alpha}}, \mathbb{P}^B_{\beta}] &= \delta^B_\beta \mathbb{D}_\alpha + M_{\alpha}^B \delta^B_\beta + \overline{M}^{\beta}_{\alpha} \delta^B_\beta, \\
\{Q^A_{\alpha}, S^B_\beta\} &= \mathbb{M}^A_{\beta} \delta^B_\beta + \delta^A_\beta \mathbb{R}^B_A + \frac{1}{2} \delta^A_\beta \delta^B_A (\mathbb{D} + \mathbb{C}), \\
\{\overline{Q}^{\dot{A}}_{\dot{\alpha}}, \overline{S}_\beta^{\dot{B}}\} &= \mathbb{M}^{\dot{A}}_{\dot{B}} \delta^\beta_\dot{\alpha} - \delta^\beta_\dot{\beta} \mathbb{R}^A_B + \frac{1}{2} \delta^\beta_\dot{\beta} \delta^A_B (\mathbb{D} - \mathbb{C}).
\end{align*}
\]

Note that in writing the algebra relations we are obliged to choose the \(su(4)\) chirality of the odd generators. The relations above are valid directly for the dual superconformal generators. For the conventional realisation of the algebra, one should simply swap all \(su(4)\) chiralities appearing in the commutation relations. We now give the generators in both the conventional and dual representations of the superconformal algebra. We will use the following shorthand notation:

\[
\begin{align*}
\partial_{i\alpha \dot{\alpha}} &= \frac{\partial}{\partial \lambda_i^{\alpha \dot{\alpha}}}, & \partial_{iA} &= \frac{\partial}{\partial \theta_i^{A}}, & \partial_{i\alpha} &= \frac{\partial}{\partial \lambda_i^{\alpha}}, & \partial_{i\dot{\alpha}} &= \frac{\partial}{\partial \lambda_i^{\dot{\alpha}}}, & \partial_{i\dot{\alpha}} &= \frac{\partial}{\partial \eta_i^{\dot{\alpha}}}, \quad (81)
\end{align*}
\]

We first give the generators of the conventional superconformal symmetry, using lower case characters to distinguish these generators from the dual superconformal generators which follow afterwards.

\[
\begin{align*}
p^{\alpha \dot{\alpha}} &= \sum_i \lambda_i^{\alpha \dot{\alpha}}, & k_{\alpha \dot{\alpha}} &= \sum_i \partial_{i\alpha} \partial_{i\dot{\alpha}}, \\
\overline{m}_{\dot{\alpha} \beta} &= \sum_i \lambda_i(\partial_{i\dot{\alpha}} \partial_{i\beta}), & m_{\alpha \beta} &= \sum_i \lambda_i(\partial_{i\alpha} \partial_{i\beta}), \\
d &= \sum_i \left[ \frac{1}{2} \lambda_i^{\alpha \dot{\alpha}} \partial_{i\alpha} + \frac{1}{2} \lambda_i^{\dot{\alpha} \dot{\alpha}} \partial_{i\dot{\alpha}} + 1 \right], & r^A_B &= \sum_i \left[ -\eta_i^A \partial_{iB} + \frac{1}{4} \eta_i^C \partial_{iC} \right], \\
q_{\alpha A} &= \sum_i \lambda_i^{\alpha} \eta_i^A, & q^\alpha_\dot{\alpha} &= \sum_i \lambda_i^\dot{\alpha} \partial_{i\dot{\alpha}}, \\
s_{\alpha A} &= \sum_i \partial_{i\alpha} \partial_{iA}, & s^A_{\dot{\alpha}} &= \sum_i \eta_i^A \partial_{i\dot{\alpha}}. \\
c &= \sum_i \left[ 1 + \frac{1}{2} \lambda_i^{\alpha \dot{\alpha}} \partial_{i\alpha} - \frac{1}{2} \lambda_i^{\dot{\alpha} \dot{\alpha}} \partial_{i\dot{\alpha}} - \frac{1}{2} \eta_i^A \partial_{iA} \right] \quad (82)
\end{align*}
\]
We can construct the generators of dual superconformal transformations by starting with the standard chiral representation and extending the generators so that they commute with the constraints,
\[ (x_i - x_{i+1})_{a\dot{a}} - \lambda_i \dot{\lambda}_{i\dot{a}} = 0, \quad (\theta_i - \theta_{i+1})^A_{a\alpha} - \lambda_{ia} \eta_i^A = 0. \] (83)

By construction they preserve the surface defined by these constraints, which is where the amplitude has support. The generators are
\[
P_{a\dot{a}} = \sum_i \partial_{i a\dot{a}},
\]
(84)
\[
Q_{aA} = \sum_i \partial_{i aA},
\]
(85)
\[
\overline{Q}_{\dot{a}}^A = \sum_i \left[ \theta_i^a A \partial_{i a\dot{a}} + \eta_i A \partial_{i a\dot{a}} \right],
\]
(86)
\[
M_{a\beta} = \sum_i \left[ x_i (\alpha \partial_{i \beta\dot{a}} + \theta_i^A \partial_{i \beta A} + \lambda_i \partial_{i \beta}) \right],
\]
(87)
\[
\overline{M}_{\dot{a}\beta} = \sum_i \left[ x_i (\dot{a} \partial_{i a\dot{a}} + \lambda_i \partial_{i \dot{a}\beta}) \right],
\]
(88)
\[
R^A_B = \sum_i \left[ \theta_i^a A \partial_{i aB} + \eta_i^A \partial_{i aB} - \frac{1}{4} \delta^A_B \theta_i^C \partial_{i aC} - \frac{1}{4} \eta_i^C \partial_{i aC} \right],
\]
(89)
\[
D = \sum_i \left[ -x_i \partial_{i a\dot{a}} - \frac{1}{2} \theta_i^a A \partial_{i a\dot{a}} - \frac{1}{2} \lambda_i \partial_{i a} - \frac{1}{2} \lambda_i \partial_{i a\dot{a}} \right],
\]
(90)
\[
C = \sum_i \left[ -\frac{1}{2} \lambda_i \partial_{i a} + \frac{1}{2} \lambda_i \partial_{i a\dot{a}} + \frac{1}{2} \eta_i A \partial_{i a} \right],
\]
(91)
\[
S^A_{\dot{aA}} = \sum_i \left[ -\theta_i^\dot{a} B \partial_{i aB} + x_i \beta A \partial_{i \dot{a} B} + \lambda_{ia} \theta_i^A \partial_{i \dot{a} B} + x_i \beta A \partial_{i aB} + \lambda_{i a} \partial_{i \dot{a} B} + x_i \beta A \partial_{i aB} + \lambda_{i a} \partial_{i \dot{a} B} \right],
\]
(92)
\[
\overline{S}_{\dot{a} A} = \sum_i \left[ x_i \beta \partial_{i \dot{a} A} + \dot{\lambda}_{i a} \partial_{i A} \right],
\]
(93)
\[
K_{a\dot{a}} = \sum_i \left[ x_i \beta A \partial_{i \dot{a} B} + x_i \beta A \partial_{i aB} + x_i \beta A \partial_{i \dot{a} B} + x_i \beta A \partial_{i aB} + \dot{\lambda}_{i a} \partial_{i \dot{a} B} + \dot{\lambda}_{i a} \partial_{i aB} \right].
\]
(94)

We also have the hypercharge \( B \),
\[
B = \sum_i \left[ -\frac{1}{2} \theta_i^a A \partial_{i aA} - \frac{1}{2} \lambda_i \partial_{i a} + \frac{1}{2} \lambda_i \partial_{i a\dot{a}} \right]
\]
(95)

Note that if we restrict the dual generators \( \overline{Q}, \overline{S} \) to the on-shell superspace they become identical to the conventional generators \( \overline{s}, \overline{q} \).

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