Evolution of Electron Distribution Driven by Nonlinear Resonances With Intense Field-Aligned Chorus Waves

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Abstract Resonant electron interaction with whistler mode chorus waves is recognized as one of the main drivers of radiation belt dynamics. For moderate wave intensity, this interaction is well described by quasi-linear theory. However, recent statistics of parallel propagating chorus waves have demonstrated that 5–20% of the observed waves are sufficiently intense to interact nonlinearly with electrons. Such interactions include phase trapping and phase bunching (nonlinear scattering) effects not described by quasi-linear diffusion. For sufficiently long (large) wave packets, these nonlinear effects can result in very rapid electron acceleration and scattering. In this paper we introduce a method to include trapping and nonlinear scattering into the kinetic equation describing the evolution of the electron distribution function. We use statistics of Van Allen Probes and Time History of Events and Macroscale Interactions during Substorm observations to determine the probability distribution of intense, long wave packets as a function of power and frequency. Then we develop an analytical model of individual particle resonance with an intense chorus wave packet and derive the main properties of this interaction: probability of electron trapping, energy change due to trapping and nonlinear scattering. These properties are combined in a nonlocal operator acting on the electron distribution function. When multiple waves are present, we average the obtained operator over the observed distributions of waves and examine solutions of the resultant kinetic equation. We also examine energy conservation and its implications in systems with nonlinear wave-particle interaction.

1. Introduction

The dynamics of Earth’s radiation belts is partly controlled by efficient resonant interactions of electrons with whistler mode waves (Andronov & Trakh tengerts, 1964; Kennel & Petschek, 1966). Different modes of these waves are responsible for warm electron precipitation and aurora formation (lower- and upper-band chorus waves, see, e.g., Kasahara et al., 2018; Ni et al., 2016; Thorne et al., 2010), electron acceleration and formation of relativistic electron fluxes in the heart of the outer radiation belt (lower-band chorus waves, see, e.g., Li et al., 2014; Reeves et al., 2013; Thorne et al., 2013), electron scattering and flux reduction during radial diffusion (hiss waves, see, e.g., Ma et al., 2016; Mourenas et al., 2017), electron scattering in the inner radiation belt (very low frequency waves from ground-based transmitters, see, e.g., Agapitov, Artemyev, Mourenas, Kasahara, & Krasnoselskikh, 2014; Ma, Mourenas, et al., 2017; Subbotin et al., 2011). Depending on their actual characteristics, such whistler mode waves can resonantly interact with electrons over a wide range of energies starting from 100 eV (very oblique chorus waves, see, e.g., Artemyev, Agapitov, et al., 2015; Agapitov, et al., 2015b) and up to please change to ~1-5 MeV (intense parallel propagating chorus waves, see, e.g., Omura et al., 2007). The wave frequency and obliquity define electron resonant energies (see Artemyev, Agapitov, et al., 2016; Shklyar & Matsumoto, 2009, and references therein) and the wave intensity determines the regime of resonant interaction. Low-intensity waves provide electron diffusive scattering, traditionally described by quasi-linear theory (Kennel & Engelman, 1966; Vedenov et al., 1962). In the inhomogeneous geomagnetic field of the radiation belts, the applicability of quasi-linear theory can be justified even for narrowband waves (e.g., Albert, 2010; Karpman,
Unlike low-intensity waves whose effects on particles have been modeled successfully by diffusion, the effect of intense whistler mode wave on the dynamics of the radiation belts has not yet been fully investigated. There is a common understanding that the generation of intense whistler mode waves is controlled by nonlinear wave-particle interaction (see reviews Omura et al., 2013; Shklyar & Matsumoto, 2009, and references therein), including electron trapping into resonance and a significant modification of the resonant electron distribution function (Karpman et al., 1974; Nunn, 1974). The theory of nonlinear wave generation reproduces many observed properties of intense whistler mode waves really well (e.g., Demekhov, 2011; Demekhov et al., 2017; Katoh & Omura, 2007; Omura et al., 2008; Shklyar, 2017; Tao et al., 2017). However, the exact role played by the intense whistler mode waves in electron acceleration still remains somewhat controversial. Electron trapping (Nunn, 1971) can provide a very long resonant interaction (over a fraction of the electron bounce period along magnetic field line), corresponding to a significant change of particle energy (comparable to the initial particle energy, see Agapitov, Artemyev, Mourenas, Krasnoselskikh, et al., 2014; Artemyev et al., 2012; Bortnik et al., 2008; Omura et al., 2007; Summers & Omura, 2007; Tao et al., 2013). However, the effects of trapping should be almost compensated by resonant nonlinear scattering, which provides electron drift in energy space in the opposite direction to trapping acceleration (see Shklyar, 2011, and references therein). Although the fine balance between trapping and nonlinear scattering is expected to ultimately establish a kind of diffusive regime of particle acceleration (see Shklyar, 1981; Solovev & Shkliar, 1986), the actual time required for establishing such a regime can be very long. Thus, when trapping and nonlinear scattering processes do not exactly balance each other, they can potentially provide a very rapid (compared with quasi-linear diffusion) net acceleration of electrons (e.g., Demekhov et al., 2009; Hsieh & Omur, 2017; Nunn & Omura, 2015). Models of such rapid acceleration have been able to explain some observations of fast increase of energetic electron flux (e.g., Agapitov et al., 2015a; Foster et al., 2017; Mozer et al., 2016), but the global importance of such fast acceleration processes in the overall radiation belt dynamics remains unknown.

To include the nonlinear resonant interactions into global models of wave-particle interaction in the radiation belts one needs: (i) representative statistics of intense wave characteristics based on spacecraft measurements and (ii) a universal approach for the description of the evolution of the electron distribution due to nonlinear interactions. There is a lack of information about fine wave characteristics such as the intensity and duration of individual wave packets. Because most wave statistics are collected for use in quasi-linear diffusion codes, their focus is on the average wave intensity, over long time intervals (see, e.g., Agapitov et al., 2013; Li et al., 2013; Meredith et al., 2012). Such averaging mixes two different wave populations, low-intensity waves and transient bursts of high-intensity waves, and it does not allow to consider them separately afterward. Thus, most wave statistics do not contain any information about wave packet characteristics, such as the duration of wave packets (or wave packet modulation), which are critical for modeling nonlinear wave-particle interaction (Artemyev et al., 2012; Tao et al., 2012, 2013) but not important for quasi-linear models. Recently, Zhang et al. (2018) analyzed Time History of Events and Macroscale Interactions During Substorms (THEMIS) and Van Allen Probe measurements of intense parallel chorus whistler mode waves and identified the portion of the wave population that may potentially interact with electrons in the nonlinear regime. Most of intense chorus waves (~95–99%) were found to propagate in the form of short wave packets, allowing resonant interaction with electrons over only a very short time, despite their large intensity (Mourenas et al., 2018; Zhang et al., 2018). Only a few the intense waves propagate in the form of long wave packets and can efficiently accelerate trapped electrons. In this study we use the statistics collected by Zhang et al. (2018) to quantify how these few percent of intense wave packets that are sufficiently long to produce a significant nonlinear acceleration affect the electron distribution.

Although the test particle approach provides a satisfactory description of the effects of the nonlinear resonant interactions on the electron distribution (Agapitov et al., 2016; Bortnik et al., 2008; Nunn & Omura, 2015; Yoon et al., 2013), this approach cannot be used to calculate the long-term dynamics of the radiation belts. Therefore, theoretical estimates of electron acceleration/deceleration due to nonlinear trapping and scattering (e.g., Albert, 1993; Bell, 1984; Shklyar, 1981) should be combined in some operator acting on the full electron distribution function to complement and generalize the usual quasi-linear diffusion equation. Several examples of such operators were developed numerically (e.g., Hsieh & Omura, 2017; Kubota & Omura, 2018; Omura et al., 2015) and analytically in the two opposite limiting cases of infinitely long wave packets (e.g., Artemyev, Vasiliev, Mourenas, Agapitov, Krasnoselskikh, Boscher, et al. 2014; Artemyev, Neishtadt, et al., 2016) and short wave
not a local process. Two red arrows connecting these different areas show that trapping acceleration is not a local process.

Figure 1. A schematic view of particle transport in phase space. The gray color indicates the parametric area of the expected phase space density depletion: Particles are scattered from this region to smaller energy/pitch angle, whereas particle scattered into this region come from a lower phase space density region. Black color indicates the areas of the expected parametric space density increase: Particles are scattered (to lower energies) or transported by trapping (to higher energies) into these regions. Two red arrows connecting these different areas show that trapping acceleration is not a local process.

2. Intense Whistler Mode Wave Characteristics

Two wave characteristics that are the most important for the nonlinear wave-particle interaction are the wave amplitude $B_w$ (square root of wave intensity) and the duration of the wave packet, given here by the number $\beta$ of wave periods within one packet. The normalized wave amplitude is $w = R\Omega_{ce}B_w/cB_0$, where $B_0$ is the equatorial background magnetic field, $\Omega_{ce}$ the equatorial electron gyrofrequency, $R = R_LL$ the spatial scale of the background magnetic field inhomogeneity, and $c$ the speed of light. If $Bw > 2$, nonlinear interaction is possible, while for $Bw < 1 – 2$ the wave is not sufficiently intense to trap particles (for more details, see Mourenas et al., 2018; Zhang et al., 2018, and references therein). A more precise evaluation of the critical wave amplitude necessary for the nonlinear interaction depends on additional wave characteristics (e.g., frequency drift and resonant particle energy and pitch angle (see, e.g., Bell, 1986; Karpman, 1974; Le Queau & Roux, 1987), but the aforementioned simplified $Bw$ criterion is sufficiently accurate to process the observed wave statistics and provide first-order estimates of the impact on the electron distribution. We use lower-band chorus wave statistics collected by Zhang et al. (2018) for $Bw > 2$. For all selected waves, the peak amplitude exceeds 200 pT. Five years of THEMIS (Angelopoulos, 2008) and 3 years of Van Allen Probes (Mauk et al., 2013) wave measurements have been analyzed to identify intense chorus wave packets and determine their characteristics. On board THEMIS, wave fields are measured by search coil magnetometers (Le Contel et al., 2008) and the electric field instrument (Bonnell et al., 2008). We also use measurements of three components of the background magnetic field by the fluxgate magnetometer (Auster et al., 2008). The two identical Van Allen Probes measure electric and magnetic field waveforms using the Electric Fields and Waves (Wygant et al., 2013) and the Electric and Magnetic Field Instrument Suite and Integrated Science (Kletzing et al., 2013) detectors. The data is transmitted at 35,000 samples per second over 6-s intervals in burst mode.

Due to the difference between THEMIS and Van Allen Probe operational modes (the absolute majority of the THEMIS waveforms measurements are triggered by plasma injections), the occurrence rate of $Bw > 2$ waves are different for data sets collected by THEMIS and the Van Allen Probes, but on average there are $<1\%$ of intense ($Bw > 2$) parallel chorus waves propagating along magnetic field lines in the form of long wave packets (with $\beta \geq 50$). One example of a long wave packet is shown in Figure 2. The main wave parameters, $B_w$ and frequency $\omega_0$, are defined for each such packet and normalized on the background characteristics: $B_w/B_0$ and $\omega_0/\Omega_{ce}$, where $B_0$ and $\Omega_{ce}$ are evaluated using the equatorial geomagnetic field. Because most of THEMIS and Van Allen Probe measurements are not equatorial, we use $L^*$ for Van Allen Probe data to approximate the
equatorial field and $B_0$, geocentric solar magnetospheric measurements of THEMIS (at large $L$-shell) as a proxy of the equatorial field. All THEMIS measurements are restricted to the near-equatorial region with $B_z > \sqrt{B_x^2 + B_y^2}$.

The number of chorus wave packets with $Bw > 2$ and $\beta > 50$ captured by THEMIS and the Van Allen Probes per day of intense chorus wave measurements are shown in Figure 3a as a function of $L$-shell. Van Allen Probes ($L < 6$) captured much more wave packets than THEMIS ($L > 6$), but the occurrence rate of intense waves is larger at larger $L$ (where THEMIS spacecraft provide most of the statistics). This is mainly due to the shorter total time of waveform measurements by THEMIS (see details in Zhang et al., 2018). The parameter $\beta^*$ is the number of wave periods with wave magnetic field larger than half of the peak wave packet magnetic field. The distribution of the number of observed wave packets with $\beta^* > 50$ (dotted lines in Figure 3a) shows that packets with $\beta^* > 50$ represent only $\sim 15-20\%$ of all long wave packets with $\beta > 50$. The remaining $\sim 80-85\%$ of these long wave packets are actually still very localized; that is, most of the wave intensity is located near the position of peak intensity, while the remaining part (the tails) of the wave packet is much less intense. This factor can be important for realistic estimates of the efficiency of nonlinear wave-particle interaction in the radiation belts.

We separate the $L^*$-shell range into two intervals: (1) the outer radiation belt with $L^*$ between the plasmasphere (as defined by the model from O’Brien & Moldwin, 2003) and the geostationary orbit $L^* \sim 6.6$ and (2) the injection region at $L^* \in [6.6, 9]$. These two $L^*$-shell ranges roughly

![Figure 2](image1.png)

**Figure 2.** (a, b) Example of long wave packets, wave packet, $B_\perp$, and $B_\perp/B_0$, spectrum with $\omega/\Omega_{ce}$. RBSP = Radiation Belt Storm Probes.

![Figure 3](image2.png)

**Figure 3.** (a) Number of intense ($Bw > 2$ and peak amplitude $> 200\mu T$) and long ($\beta > 50$) chorus wave packets observed per day of intense ($Bw > 2$) wave measurements as a function of $L^*$ (dash-dotted lines) and the total number of observed intense wave packets with $\beta > 50$ (solid lines) or $\beta^* > 50$ (dotted lines) and (b and c) distributions of $B_w/\omega/\Omega_{ce}(B)$ for $Bw > 2$ and $\beta > 50$, for two $L^*$-shell ranges. RBSP = Radiation Belt Storm Probes; THEMIS = Time History of Events and Macroscale Interactions during Substorms.
separate Van Allen Probe measurements (outer radiation belt) and THEMIS measurements (injection region). For both \( L^* \)-shell ranges, we plot the wave packet distributions in the \((B_w/B_0, \alpha/\Omega_{ce})\) space. Figures 3b and 3c show the presence of a significant portion of intense \((B_w/B_0 \in [10^{-3}, 10^{-2}])\) waves with \( \alpha/\Omega_{ce} \in [0.2, 0.4] \). We use these wave distributions to estimate the average effect of the nonlinear wave-particle interaction on the evolution of the electron distribution.

3. Nonlinear Resonances

We consider a simple planar magnetic field model (Bell, 1984) with a single vector potential component \( A_y = -xB_0(z) \), where \( z \) is a field-aligned coordinate and \( B_0(z) \) mimics the dipolar geomagnetic field \( B_0 = \sqrt{1 + 3 \sin^2 \lambda / \cos^2 \lambda} \) where \( dz/d\lambda = R \sqrt{1 + 3 \sin^2 \lambda} \) and \( R = R_L \). The parallel propagating whistler mode wave is described by two components of the vector potential: \( A_y = (B_w/k) \sin \phi \) and \( A_y = (B_w/k) \cos \phi \), where \( B_w \) and \( k \) are the wave amplitude and wave vector \((B_w \) and \( k \) depend on \( z/R \) and \( \phi \) is the wave phase \((d\phi/dz = k, d\phi/dt = -\omega) \). The Hamiltonian of a relativistic electron (the charge being \(-e\) and the rest mass \( m_e \)) in such electromagnetic fields is

\[
H = \sqrt{m_e^2c^4 + c^2p_z^2 + (ep_x + eA_x)^2 + c^2 (A_y + A_y)^2}.
\]

We expand Hamiltonian (1) over a small parameter \( eB_w/km_ec^2 \ll 1 \) and introduce new conjugate variables, the gyrophase \( \psi \) and magnetic moment \( l_x = \int p_x dx \) (see details of Hamiltonian transformation in, e.g., Artemyev, Vasiliev, et al., 2015; Artemyev et al., 2018). The new Hamiltonian takes the form:

\[
H = m_ec^2 \gamma + U_w(z, l_x) \sin (\phi + \psi), \quad \gamma = \sqrt{1 + \frac{p_z^2}{m_e^2c^4} + \frac{2l_x \Omega_{ce}}{m_e c^2}}.
\]

where \( \gamma \) is the gamma factor of the gyroaveraged system, \( \Omega_{ce} = eB_0(z)/m_ec \) and \( U_w = \sqrt{2l_x \Omega_{ce} eB_w} / \gamma m_ec k \) is the effective wave amplitude. The Hamiltonian equations describe electron motion:

\[
\begin{align*}
\dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_x}{\gamma m_e} + \frac{\partial U_w}{\partial \psi} \sin (\phi + \psi) \\
\dot{\psi} &= \frac{\partial H}{\partial l_x} = \frac{\Omega_{ce}}{\gamma} + \frac{\partial U_w}{\partial \psi} \sin (\phi + \psi) \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -l_x \frac{\Omega_{ce}}{\gamma} - kU_w \cos (\phi + \psi) + \frac{\partial U_w}{\partial \psi} \sin (\phi + \psi) \\
\dot{l}_x &= -\frac{\partial H}{\partial \psi} = -U_w \cos (\phi + \psi). \quad \phi = kz - \omega.
\end{align*}
\]

where \( \Omega_{ce}' = d\Omega_{ce}/dz \). In the absence of waves \((U_w = 0)\), electrons move along the bounce trajectory \( \dot{z} = p_x / m_e, \dot{\psi} = -l_x \Omega_{ce}' / \gamma \) with a constant energy \( \gamma \) and magnetic moment \( l_x \). Waves disturb these bounce oscillations and can scatter or trap particles (see Figure 4). For sufficiently intense waves, as considered in this study, nonlinear scattering results in energy change with a finite mean value \( \langle \Delta \gamma \rangle = \Delta \gamma_{scat} \neq 0 \) (see details in, e.g., Albert, 2002; Solovev & Shkliar, 1986). Energy changes due to trapping and nonlinear scattering significantly exceed the square root of energy variance (see relations between \( \Delta \gamma_{scat}, \Delta \gamma_{trap} \) and energy spread in Figure 4); that is, trapping and nonlinear scattering are the dominant processes, whereas energy diffusion is much weaker. The diffusion process likely becomes important on very long time intervals, but it can be omitted over relatively short intervals (still including many resonant interactions). Therefore, we focus here on the description of trapping and nonlinear scattering of electrons and on the effects of these processes on the evolution of the full electron distribution function.

Three main characteristics define the resonant interaction: (i) the probability of trapping, \( \Pi \), gives the relative number of resonant particles trapped during one resonant interaction (the relative number of scattered particles is equal to 1 - \( \Pi \)); (ii) the energy change due to trapping \( \Delta \gamma_{trap} \); and (iii) the energy change due to nonlinear scattering \( \Delta \gamma_{scat} \). These characteristics depend on the initial particle energy \( \gamma \) and on the initial value of \( l_x, l_x \) can be written through the energy and equatorial pitch angle \( \alpha_{eq} \) as \( l_x = m_ec^2(\gamma^2 - 1) \sin^2 \alpha_{eq} / 2 \Omega_{ce} \). Thus, we use the notation \( \Pi(\gamma, \alpha_{eq}), \Delta \gamma_{scat}(\gamma, \alpha_{eq}), \Delta \gamma_{trap}(\gamma, \alpha_{eq}) \). These characteristics can be combined to construct a nonlocal (integral) operator acting on the full electron distribution function to describe its evolution due to
Figure 4. Numerical integration of 10 particle trajectories described by equation (3). All particles have the same initial energy and equatorial pitch angle (initial $I_z$) and different (randomly distributed) initial phases $\psi$. Panels show energy (left panel) and equatorial pitch angle (right panel) as a function of dimensionless time (trajectories are integrated for about a quarter of the bounce period; there is only one resonance for each trajectory). System parameters are as follows: $L = 6$, $\omega/\Omega(e) = 0.35$, $B_w = 500T \cdot \tanh(\lambda/5^\circ) \exp(-\lambda/25^\circ)^2$ for $\lambda > 0$ (i.e., the latitudinal distribution used the mimics observed wave field distribution, see Agapitov et al., 2013).

The probability of trapping, $\Pi$, can be obtained numerically using Hamiltonian equations (3). We bin $(\gamma, \alpha_{eq})$ space, and for each $(\gamma, \alpha_{eq})$ we calculated trajectories of $5 \cdot 10^3$ particles with the same initial position $z = 0$ and different randomly distributed phases $\psi$ (initial $p_z$ and $I_z$ are defined by $(\gamma, \alpha_{eq})$). Each trajectory is integrated for a half of the bounce period. The final $(\gamma, \alpha_{eq})$ determines whether this particle was trapped (increase of $\Delta \gamma$ exceeding 10% of the initial $\gamma$) or scattered ($\Delta \gamma < 0$). The relative number of trapped particles is shown in Figure 5a for one particular set of system parameters (see the figure caption). Although such a calculation of $\Pi$ is relatively straightforward, it requires a lot of test particles with sufficiently small bins in $(\gamma, \alpha_{eq})$ space. Such a procedure should be repeated for all sets of the system parameters (wave amplitudes, wave frequencies, etc.). Thus, a purely numerical determination of $\Pi$ is not very effective. Alternatively, $\Pi$ can be obtained analytically (e.g., Neishtadt, 1975; Neishtadt et al., 1989; Shklyar, 1981). We use here the approach developed in Artemyev, Mourenas, et al. (2015; see details of II calculations in Appendix A, equation (A10). We plotted $\Pi(\gamma, \alpha_{eq})$ in Figures 5a and 5b. Figure 5a demonstrates that the analytical equations describe $\Pi$ well. Thus, we shall use the analytically derived distribution $\Pi(\gamma, \alpha_{eq})$ (like the one shown in Figure 5b) to describe the trapping probability for the construction of a nonlocal operator acting on the full distribution function. The coordinates of particle resonant interaction with the wave in the $(z, p_z)$ plane are defined by the resonant condition $\dot{\phi} + \psi = 0$ and the unperturbed particle trajectory ($\gamma = \text{const}, I_z = \text{const}$). There is an additional condition for trapping, which can be formulated in a simplified form as a requirement of growth of the ratio of the wave force acting on the resonant particle over the combined mirror and inertial forces (see more accurate definition in, e.g., Neishtadt, 1975; Neishtadt et al., 2011). Particles trapped at some $z_{\text{trap}}$ escape from the resonance at $z_{\text{esc}}$. The energy change is $\Delta \gamma_{\text{trap}} = \gamma_{\text{R}}(z_{\text{esc}}) - \gamma_{\text{R}}(z_{\text{trap}})$, where energy $\gamma_{\text{R}}$ along the resonant trajectory is defined as (see equation (A5) and, e.g., Artemyev et al., 2018):

$$
\gamma_R \equiv \overline{\nu} \mp \frac{N}{\sqrt{N^2 - 1}} \sqrt{1 - 2\varepsilon_0 \overline{\nu} + \overline{\nu}^2}
$$

(4)
In equation (4), \( \sigma = \Omega_c(z)/\omega, N = k(z)\zeta/\omega, \) and \( \varepsilon_0 = \gamma - \omega(\gamma^2 - 1)^{1/2} a_{eq}/2\Omega_c(0) = \text{const}. \) The sign in equation (4) is defined by the sign of the resonant velocity \( \sim (1 - \sigma/\gamma)/N. \) For a simple cyclotron resonance with \( \sigma/\gamma > 1 \) and negative resonant velocity \( \sim (1 - \sigma/\gamma)/N < 0, \) equation (4) should be used with a \(-\), while for the turning acceleration with \( \sigma/\gamma < 1 \) and \( (1 - \sigma/\gamma)/N > 0 \) (Omura et al., 2007), equation (4) should be used with a \(+\). To estimate \( \Delta \gamma_{\text{trap}}, \) we have to determine \( z_{\text{esc}}. \) The condition of particle escape from the resonance is determined by particle motion in the \( (\zeta, \dot{\zeta}) \) phase plane, where \( \zeta = \phi + \psi. \) The computation of \( z_{\text{esc}} \) and \( \Delta \gamma_{\text{trap}} \) is described in Appendix B. The obtained analytical expression for \( \Delta \gamma_{\text{trap}} \) can be tested using numerical integration of the Hamiltonian equations (3). Figure 5c compares numerical and analytical distributions \( \Delta \gamma_{\text{trap}}(\gamma, \alpha_{eq}). \) We also indicate the energy gain due to the turning acceleration, when the direction of trapped particles motion changes and the particles gain significantly more energy than through an usual cyclotron resonance (see details in Omura et al., 2007). This comparison shows that we can use the analytical distributions from Figure 5d to characterize electron trapping acceleration.
To complete the description, we need the energy change $\Delta \gamma_{\text{scat}}$. As nonlinear scattering is a local process, $\Delta \gamma_{\text{scat}}$ depends on wave properties and background magnetic field characteristics at the resonant location in the $(\zeta, \rho_0)$ plane: $\Delta \gamma_{\text{scat}} = -\omega S_{\text{res}}/2\pi$, where

$$
S_{\text{res}} = \sqrt{\frac{8r}{g}} \int \sqrt{a (\sin \zeta_0 - \sin \zeta) - (\zeta_0 - \zeta)} \, d\zeta
$$

$g = \omega^2(N^2 - 1)/m_2^2c^2\gamma_p$, and $r = r(z)$ (see Appendix C and, e.g., Artemyev, Vasiliev, Mourenas, Agapitov & Krasnoselskikh, 2014). The integration limits, $\zeta_0$, $\zeta$, are defined in equation (A9). The resonance location depends on the initial $(\gamma, \alpha_{\text{eq}})$, and thus, we can define $\Delta \gamma_{\text{scat}}(\gamma, \alpha_{\text{eq}})$. To check the analytical $\Delta \gamma_{\text{scat}}(\gamma, \alpha_{\text{eq}})$, we used the same approach as we used for $\Pi$. We bin $(\gamma, \alpha_{\text{eq}})$ space and for each pair of $\gamma, \alpha_{\text{eq}}$ values we numerically integrated $5 \cdot 10^4$ trajectories of test particles. Then we calculated the energy change after a single resonant interaction for each particle and averaged these changes over the nontrapped (scattered) particles. Figure 5e shows $\Delta \gamma_{\text{scat}}(\gamma, \alpha_{\text{eq}})$ obtained from the numerical integration of particle trajectories and the analytically evaluated $\Delta \gamma_{\text{scat}}(\gamma, \alpha_{\text{eq}})$. This comparison confirms that we can use the analytical distributions $\Delta \gamma_{\text{scat}}(\gamma, \alpha_{\text{eq}})$ from Figure 5f to describe the dynamics of an ensemble of charged particles.

4. Kinetic Equation

4.1. Single Wave

The nonlinear trapping and scattering of resonant electrons results in evolution of the electron distribution function $\Psi(\gamma, \alpha_{\text{eq}})$. To describe this evolution, we need a generalized kinetic equation that includes effects of nonlinear scattering (i.e., drift in energy space, see, e.g., Albert, 2002) and trapping (i.e., fast transport in energy space, e.g., see Omura et al., 2007). Let us start with a system with one wave frequency, $\omega$. The particle energy in the wave reference frame is an integral of motion (e.g., Summers et al., 1998, and references therein). Using the resonant condition $\dot{\phi} + \psi = 0$ (i.e., $k_p \gamma - \gamma c + \Omega_{\text{ce}} = 0$), this integral can be written as $m_c \gamma^2 - \omega l = \text{const}$ (alternatively, $\gamma = (\omega/2\Omega_{\text{ce}}(0)) (\gamma^2 - 1) \sin^2 \alpha_{\text{eq}} = \text{const}$). Thus, for systems with one $\omega$, the energy change directly defines the pitch angle change:

$$
\Delta \alpha_{\text{eq}} = \Delta \gamma \frac{\Omega_{\text{ce}}(0) - \omega \gamma \sin^2 \alpha_{\text{eq}}}{\omega (\gamma^2 - 1) \sin \alpha_{\text{eq}} \cos \alpha_{\text{eq}}},
$$

This relation allows us to compute $\Delta \alpha_{\text{scat}}, \Delta \alpha_{\text{trap}}$ once $\Delta \gamma_{\text{scat}}, \Delta \gamma_{\text{trap}}$ are known. The relation between $\gamma$ and $\alpha_{\text{eq}}$ can be used to examine the evolution of the 1-D distribution function $\Psi(\gamma, \alpha_{\text{eq}})$, where $\gamma_{\text{eq}} = \gamma - (\omega/2\Omega_{\text{ce}}(0)) (\gamma^2 - 1) \sin^2 \alpha_{\text{eq}}$ is a constant parameter. The kinetic equation describing the evolution of a particle distribution has been derived in Artemyev et al. (2018). However, this kinetic equation does not allow a simple averaging over wave characteristics ($\omega$, $B_{\parallel}$, etc.). Therefore, we follow the approach proposed by Omura et al. (2015) and use $\Pi$, $\Delta \gamma_{\text{trap}}$, $\Delta \gamma_{\text{scat}}$ to derive the operator acting on $\Psi$. Note that in contrast to Omura et al. (2015), we use here analytical expressions for $\Pi$, $\Delta \gamma_{\text{trap}}$, and $\Delta \gamma_{\text{scat}}$.

We bin the $(\gamma, \alpha_{\text{eq}})$ space as $\gamma = 1 + \Delta \gamma_i, \alpha_{\text{eq}} = \Delta \alpha_j$, where $i, j = 0, \ldots, M$ and $\Delta \alpha = \pi/M, \Delta \gamma = (\gamma_{\text{max}} - 1)/M$. Then for each bin $(\gamma^{(i)}, \alpha^{(j)}_{\text{eq}})$, we define $\Pi_i$, $\Delta \gamma_{\text{scat}}$, $\Delta \gamma_{\text{trap}}$, and introduce two quantities: $s_{\text{mn}}^{kl}(W)$ and $p_{\text{mn}}^{kl}(W)$ denoting the probabilities for a particle to move from the state $(\gamma^{(i)}, \alpha^{(j)}_{\text{eq}})$ to the state $(\gamma^{(m)}, \alpha^{(n)}_{\text{eq}})$ due to a single nonlinear scattering ($s_{\text{mn}}^{kl}$) and trapping ($p_{\text{mn}}^{kl}$). The letter $W$ indicates all the relevant wave packet characteristics, most importantly its amplitude $B_{\parallel}$ and frequency $\omega$. For each packet (fixed $B_{\parallel}$, $\omega$), one can view all $s_{\text{mn}}^{kl}(W)$, $p_{\text{mn}}^{kl}(W)$ as the elements of a big 4-D matrix that defines the phase space transport due to scattering and trapping. Both $s_{\text{mn}}^{kl}(W)$ and $p_{\text{mn}}^{kl}(W)$ are obtained using $\Pi_i$, $\Delta \gamma_{\text{trap}}^{kl}$, and $\Delta \gamma_{\text{scat}}^{kl}$:

\[
\begin{align*}
\begin{cases}
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Pi_i, & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Pi_i, & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Pi_i, & 
\end{cases}
\end{align*}
\]
Consider a state \((\gamma', a_{eq}^{(1)}\)) of wave parameters (different energy/pitch angle space). Red cells from the sets of \((k, l)\) indexes for scattered and trapped particles.

Equation (8) can be rewritten as

\[
\frac{\partial \Psi_y}{\partial t} = -\frac{n_y}{\tau_y} \Psi_y + \sum_{kl} \frac{n_{kl}}{\tau_{kl}} \Psi_y \left( s_{kl}^y(W) \Psi_{kl} + \sum_{eq} \rho_{eq}^y(W) \Psi_{eq} \right),
\]

where \(\Psi_y = \Psi(\gamma', a_{eq}^{(1)}\)). The summation is performed over a set of indices \((k, l)\) for which \(s_{kl}^y\) or \(\rho_{eq}^y\) is nonzero—in other words, over all the states \((\gamma^{(k)}, a_{eq}^{(l)})\) from which particles can come to the state \((\gamma', a_{eq}^{(1)})\) after a single nonlinear scattering or trapping, as shown in Figure 6. The negative term proportional to \(\Psi_y\) describes the removal of particles from the state \((\gamma', a_{eq}^{(1)}\)), while the terms with the sum describe the incoming flux of particles. There is a noticeable difference between \(s_{mn}^y(W)\) and \(\rho_{eq}^y(W)\); while the nonnull elements of \(s_{mn}^y(W)\) correspond to nearby cells (around \(k = m, l = n)\), the elements of \(\rho_{eq}^y(W)\) are more remote; see Figure 6.

Equation (8) can be rewritten as

\[
\frac{\partial \Psi_y}{\partial t} = -\frac{n_y}{\tau_y} \Psi_y + \sum_{kl} \frac{n_{kl}}{\tau_{kl}} \Psi_y R_{kl}^y(W)
\]

\[
R_{kl}^y(W) = \frac{n_{kl}}{\tau_{kl}} \left( s_{kl}^y(W) + \rho_{eq}^y(W) \right).
\]

Elements of \(R_{kl}^y(W)\) depend on wave characteristics. Note that equation (9) is linear with respect to \(\Psi_y\) and all the nonlinearity is included in \(R_{kl}^y(W)\). Equation (9) fully defines the evolution of \(\Psi_y\) in the presence of a single wave (i.e., for a given \(W\)). To generalize the results obtained to an ensemble of waves, we need to average equation (9) weighted by the probability (occurrence rate) of the different wave parameters (different sets of \(W\)).

### 4.2. Multiple Waves

Consider an ensemble of waves, where \(\rho(W)\) defines the normalized statistical weight of a certain set of parameters (e.g., each of the two probability distributions in Figure 3 could result in an empirical determination of \(\rho(W)\) as function of wave amplitude \(B_w\) and frequency \(\omega\)). Here, however, the distribution \(\rho(W)\) is normalized so that \(\int \rho(W) dW = T_{tot}/T_{int}\), where \(T_{tot}\) is the total time interval of spacecraft wave measurements and \(T_{int}\) is the cumulative time interval of observations of intense \((8w > 2)\) and long chorus wave packets. Instead of equation (9), we obtain

\[
\frac{\partial \Psi_y}{\partial t} = \int_W \left( -\frac{n_y}{\tau_y} \Psi_y + \sum_{kl} R_{kl}^y(W) \Psi_{kl} \right) \rho(W) dW.
\]

As \(\Psi_y\) does not explicitly depend on the wave properties \(W\), it can be taken out of the integral. Moreover, in the first approximation (for relatively narrowband parallel chorus waves interacting with electrons through...
Both the decelerated low-energy electron population and the accelerated high-energy population (see peaks in the energy spectra and trapping acceleration to larger energies (∼30 keV above the domain of reduced phase space density) in Figure 8) are formed after a relatively short time (∼35 min) when considering the time-averaged characteristics of intense chorus waves from Figure 3b—in spite of the presence of 30–40% waves with $B_w/B_0 < 0.0035$.

5. Evolution of the Electron Distribution

Equation (11) describes the long-term evolution of the electron distribution function due to nonlinear wave-particle interaction. To demonstrate how this theoretical approach works for realistic waves, we parameterize the distribution of prolonged and intense chorus waves from Figure 3b (at $L^* < 6.6$) and substitute it as $\rho(W)$ into equation (11). We use the normalized time $\tau = \frac{t}{c/R_A}(T_{int}/T_{tot})$, and thus, the $\rho$ distribution is normalized to unity ($\int \rho dW = 1$). We take $T_{int}/T_{tot} \approx 10^{-3}$ during active periods with $AE \geq 500$ nT, in agreement with the chorus wave statistics showing that long wave packets with $\beta > 50$ represent ~2% of intense wave observations, which themselves are present during ~10% of lower-band chorus wave measurements performed during active periods between 20 and 10 MLT—the MLT range of intense chorus emissions, representing 58% of the full MLT range (see Mourenas et al., 2018; Zhang et al., 2018). Therefore, normalized time $\tau = 1$ corresponds to ~21 s in real time.

Both nonlinear scattering and trapping have a pronounced effect on $\Psi(\gamma, \alpha_{eq})$ and $\Psi(\gamma)$. Figure 8 shows $\Psi(\gamma, \alpha_{eq})$ at $\alpha_{eq} = 100$ (−35 min), $\gamma = 30$ (−10 min), $\tau = 60$ (−21 min), and $\tau = 100$ (−35 min) of the nonlinear wave-particle interaction. The main feature of the distribution displayed in Figure 8 is a heavily reduced middle domain at $\alpha_{eq} \in [30^\circ, 50^\circ]$ and energies $\in [10^2, 10^1]$ keV. Electron interaction with an ensemble of intense chorus waves leads to an efficient nonlinear scattering toward smaller energies/pitch angles ($\Psi$ increases at energies $\sim 30$ keV below the domain of reduced phase space density) and trapping acceleration to larger energies ($\Psi$ increases at energies $\sim 3$ MeV above the domain of reduced phase space density). Both the decelerated low-energy electron population and the accelerated high-energy population (see peaks in the energy spectra $\Psi(\gamma)$ increasing around 30 keV and 3 MeV in the bottom panels of Figure 8) are formed after a relatively short time (∼35 min) when considering the time-averaged characteristics of intense chorus waves from Figure 3b—in spite of the presence of 30–40% waves with $B_w/B_0 < 0.0035$, which themselves are present during ~10% of lower-band chorus wave measurements.
Figure 8. Evolution of the electron distribution function due to nonlinear interactions with intense chorus waves. (a)–(d) The solution of equation (11) for an initial distribution $\Psi_0 = (1 + (\gamma - 1)/c_0)^{-1} \sin \alpha_{eq}$ with $c_0 = (k - 3)/50$ and $k_0 = 4$. The results of numerical calculations are displayed at times $\tau = 10$ (a), $\tau = 30$ (b), $\tau = 60$ (c), and $\tau = 100$ (d). (e)–(h) The ratio of distributions from panels (a)–(d) over the initial distribution. (i)–(l) The integrated distributions $\bar{\Psi} = \int \Psi \sin \alpha_{eq} d\alpha_{eq}$ normalized to the initial integrated distribution $\bar{\Psi}_0$. Shadowed domains in panels (a)–(h) show the parameter domains of nonresonant particles. Solid black curves in panels (a)–(h) correspond to $\Psi/\Psi_0 = 1$.

that generally are not sufficiently intense to produce nonlinear effects. Moreover, the occurrence rate of long and intense chorus wave packets can become 3–5 times higher than time-averaged level used. This can occur over short time intervals (~10 min) corresponding to plasma injections during dipolarization events at $L^* \sim 5$ (Zhang et al., 2018). In principle, this could result in an even faster evolution of $\Psi$ at low electron energy < 100 keV (corresponding to an azimuthal drift period > 1 hr) but barely accelerates the evolution of $\Psi$ at 0.4–3 MeV because the azimuthal drift period of such electrons is 5–25 min, which implies that the particles encounter such rare events (localized in MLT) during only a small fraction of one azimuthal drift period. However, we caution that various other effects can modify the efficiency of nonlinear interaction and that they should be carefully taken into account to estimate the actual time scale of the evolution of the electron distribution. These will be discussed in details in section 6.

Figure 8 demonstrates the presence of both acceleration and deceleration of resonant electrons: nonlinear scattering results in particle drift to smaller energies and trapping transports particles to higher energies. These two processes should be in fine balance due to the strong relation linking the probability of trapping and the scattering drift rate (see, e.g., Artemyev et al., 2018; Solovev & Shkliar, 1986). The analytical theory predicts that the total energy change of resonant electrons should not be large (Shklyar, 2011), because any significant energy change necessitates some energy source or sink. Thus, if the total energy varies, either the assumption of roughly constant wave intensity is not satisfied or the initial electron distribution was not chosen properly. To check the prediction of small energy change, we plotted the average particle energy $m_c^2 (\gamma - 1)$ as a function of time for $\Psi$ from Figure 8. A weak but still significant (several percent) decrease of $m_c^2 (\gamma - 1)$ indicates that the initial distribution $\Psi$ may not have been chosen fully properly, Figure 9 (left panel). However, the slope of the initial energy distribution (the $k$ parameter) does not significantly influence the rate of decrease.
Figure 9. (left panel) Evolution of the total particle energy for initial distributions $\Psi_0 = (1 + (r - 1)/c_\text{eq})^{-\kappa - 1} \sin \alpha_{\text{eq}}$ with $c_\text{eq} = (x - 3/2)/50$ and three values of $\kappa$. (right panel) Evolution of total particle energy for initial distributions $\Psi_0 = (1 + (r - 1)/c_\text{eq})^{-\kappa - 1} G(\alpha_{\text{eq}})$ with four different $G(\alpha_{\text{eq}})$ functions; $c_\text{eq} = (x - 3/2)/50$ and $\kappa = 4$.

Note that the simulations shown in Figure 8 do not include the loss of small pitch angle electrons when they reach their loss cone. This feature can result in an enhanced $\Psi$ at small pitch angles. Such losses can be incorporated into our model through an additional $-\Psi/\tau$ term with a large $\tau$ of the order of a quarter of the bounce period of electrons along geomagnetic field lines for pitch angles within the loss cone.

6. Discussion

In this study, we focused on the electron nonlinear resonant interactions with long and intense chorus wave packets, which can support a rapid evolution of the electron distribution via trapping and nonlinear scattering. When combined with the previous investigations of electron diffusion by low-intensity waves (see Lyons & Williams, 1984; Schulz & Lanzerotti, 1974, and references therein) and intense short wave packets (see Mourenas et al., 2018), the present investigation complements the set of theoretical tools needed to fully describe electron dynamics driven by realistic whistler mode waves in the radiation belts. Here we discuss several important questions arising from a careful analysis of solutions of equation (11).

6.1. Time Scale of Electron Distribution Evolution

Figure 8 shows that the typical time scale of the evolution of the electron distribution via trapping and scattering by long and intense chorus wave packets can be about half an hour, that is, rather short compared with $\sim 4–10$ hr typically for quasi-linear diffusion by the bulk of lower-intensity chorus waves (e.g., Li et al., 2016; Ma, Li, et al., 2017; Mourenas et al., 2014; Yang et al., 2018). Although such fast electron flux variations have been occasionally observed (e.g., Agapitov et al., 2015a; Foster et al., 2017), they are rather unusual. Therefore, we discuss below the possible effects that can slow down the nonlinear evolution of the electron distribution.

First, during low geomagnetic activity the occurrence rate of intense chorus waves is reduced (see the dependence of intense wave occurrence rate on the $AE$ index in Zhang et al., 2018). Accordingly, the characteristic time scale of electron acceleration would be considerably increased as compared with $AE \geq 500$ nT. Moreover, even during geomagnetically active periods, intense long wave packets are observed only sporadically. The time intervals when long wave packets are present mainly correspond to intervals ($\sim 10$ min) of significant plasma injections, which are a source of free energy for strong wave generation (e.g., Demekhov et al., 2017; Tao et al., 2011; Zhang et al., 2018). Such realistic chorus wave packets are limited in time and space (the typical spatial scale of the source region of chorus waves is about $\sim 500$–1,000 km, see Agapitov et al., 2017). Thus, they have smaller (and slower) effects on the electron distribution than those in Figure 8, where we considered very long wave packets present at all latitudes all the time, with their measured time-averaged occurrence rate. But if the actual wave packets are not present above some latitudinal range (if they get damped, for instance), the efficiency of nonlinear interaction can be reduced. Moreover, $\sim 1$-MeV electrons have an azimuthal drift velocity in the nearly dipolar geomagnetic field of about $\sim 200$ km/s at $L \sim 6$. Thus, the time needed for electrons to cross a 1,000-km-wide chorus wave source region corresponds to $\sim 10$ bounce
periods, during which these electrons can interact resonantly with the wave 10 times. Ten resonant interactions can be sufficient to accelerate/scatter electrons non-linearly, but for the long-term evolution of the electron distribution function described in our study, electrons would need to meet several such wave source regions during their azimuthal drift.

Quasi-linear diffusion coefficients can be directly calculated by time averaging the wave intensity. To obtain the nonlinear resonant operator in equation (11), we can average operator over a distribution of long intense wave packet characteristics and multiple it by the percentage of time when such wave packets are observed. However, nonlinear effects described by this operator may be significantly smoothed and slowed down (compared with results of Figure 8), if short periods of really intense wave packet observations are mixed with long periods of moderately intense waves. This may partly explain why the effects of the nonlinear wave-particle interaction (rapid electron energization/scattering) can be observed mostly within short periods of intense waves. These effects are barely noticeable in the long-term dynamics of the outer radiation belt.

Second, there is an important additional factor that can reduce the efficiency of the nonlinear wave-particle interaction, slowing down the global evolution of the electron distribution. Even long wave packets may not always sustain a prolonged nonlinear interaction, because of a possible destruction of the nonlinear resonance by adverse effects (caused by additional resonant sidebands of lower amplitude or nonresonant weaker waves, see Artemyev, Mourenas, et al., 2015; Nunn, 1986; Shklyar & Zimbardo, 2014). The length (and/or amplitude) of long wave packets should also probably decrease as they propagate toward higher latitudes (Tsurutani et al., 2011). This effect reduces the rate of trapping acceleration (e.g., Artemyev et al., 2012). Intense chorus wave statistics presented in Figure 3 were obtained at low latitudes, inside or near the wave source region. Moreover, Figure 3a indicates that important modulations of the wave intensity are generally present inside long wave packets, which can result in a further reduction of the efficiency of nonlinear wave-particle interaction (Artemyev, Mourenas, et al., 2015; Tao et al., 2013). Only ~15–20% of intense long wave packets (with $β > 50$) contain at least ~50 wave periods with a wave amplitude larger than the half of the peak wave packet amplitude; that is, most of the long wave packets are rather localized, with a narrow peak of wave intensity and much less intense tails. Based on this last effect alone, the typical time scale of nonlinear evolution of the electron distribution can be increased by a factor ~5–6 as compared with Figure 8, reaching already 3–4 hr, similar to quasi-linear time scales.

6.2. Nonlinear Wave-Particle Interaction Versus Quasi-Linear Models

Equation (11) describes changes of the electron distribution $Ψ$ due to the nonlinear wave-particle interaction. This equation has the form of a classical evolution equation (e.g., Kampen-Van, 2003): The time derivative $\partial Ψ/\partial t$ is equal to an operator $\hat{L}$ acting on $Ψ$: $\partial Ψ/\partial t = \hat{L}Ψ$. The same type of equation describes quasi-linear particle diffusion by an ensemble of low-intensity waves, $\partial Ψ/\partial t = \hat{D}Ψ$ (Schulz & Lanzerotti, 1974), where the diffusion operator $D \sim ∂^2/∂\gamma^2$, $∂^2/∂\alpha^2_{eq}$ is composed of quasi-linear diffusion rates that depend on the wave intensity and dispersion (see examples in Albert, 2008; Glauber & Horne, 2005; Shprits & Ni, 2009). Therefore, equation (11) can be readily incorporated into existing numerical codes solving diffusion equations. The resultant combined equation would include three operators $\partial Ψ/\partial t = T_D\hat{D}Ψ + T_L\hat{L}Ψ + T_K\hat{K}Ψ$, where coefficients $T_D$, $T_L$, and $T_K$ define the relative time intervals of spacecraft observations of low-intensity ($βw < 2$, quasi-linear regime), high-intensity long wave packets ($βw > 2, β^* > 50$, nonlinear regime), and high-intensity short wave packets ($βw > 2, β^* < 50$, nonregime) waves. The operator $\hat{K}$ describing the combined action of nonlinear trapping and scattering by short wave packets has been derived by Mourenas et al. (2018) and amounts to a simple drift term $\hat{K} = -W_e\partial Ψ/\partial γ - W_e\partial Ψ/\partial α_{eq}$. For existing Fokker-Planck numerical codes of the radiation belt dynamics, the operator $\hat{L}$ can be considered as a source/loss operator (similar to the $\tau$ operator describing particle losses $\partial Ψ/\partial t \sim -Ψ/τ$—e.g., see Balkin et al., 2012; Horne et al., 2005; Mourenas et al., 2014). However, to complete the description of nonlinear resonant effects in global codes of the radiation belt dynamics, one needs a more detailed statistics of intense wave characteristics (e.g., $W$ should include a detailed dependence on $L$-shell and geomagnetic activity level).

6.3. Energy Conservation

The evolution of the electron distribution function due to the nonlinear wave-particle interaction includes a rapid change of the total electron energy (see Figure 9, left panel). However, the energy of a whistler mode wave is usually not sufficiently large to provide such a deceleration/acceleration (see discussion and estimates in Shklyar, 2011) and thus scattered (decelerated) electrons are the only available energy source for trapped electron acceleration. This balance imposes certain constraints on the initial electron distribution
function for models describing nonlinear resonances. This initial distribution should be consistent with conservation of the total energy for solutions of equation (11). Figure 9 (left panel) shows that one cannot easily reduce the rate of the total energy decrease simply by choosing a different electron energy distribution. We further checked the total particle energy variation for different initial pitch angle distributions. Figure 9 (right panel) shows that an initial distribution with transverse anisotropy ($\Psi \sim \sin \alpha_{eq}$) immediately loses energy, whereas a field-aligned anisotropic distribution ($\Psi \sim \cos \alpha_{eq}$) first gains energy before losing it. We also checked more specific distributions (butterfly $\Psi \sim \sin 2\alpha_{eq}$ sometimes observed in the radiation belts; see Shklyar et al., 2005; and a distribution with field-aligned beams $\Psi \sim \sin \alpha_{eq} \cos^2 \alpha_{eq}$ observed during particle injections; e.g., Mozer et al., 2016). Both show energy increase during the first stage of the evolution. This suggests that for a $\Psi$ to result in total energy conservation it should be more isotropic (i.e., something between transversely anisotropic $\sim \sin \alpha_{eq}$ and field-aligned anisotropy $\sim \cos \alpha_{eq}$). Moreover, under realistic conditions, particle anisotropy is energy dependent and can be different for <100-keV particles and ~1-MeV particles. Therefore, further investigations of particle distributions supporting the assumption of total energy conservation are needed for accurately modeling the nonlinear wave-particle interaction (see also Shklyar, 2017, for details on the relation between trapped and scattered electron populations in a system with small variation of wave intensity).

7. Conclusions

We investigated the nonlinear resonant electron interaction with intense whistler mode waves in the outer radiation belt. Combining THEMIS and Van Allen Probe observations of long and intense chorus wave packets with the analytical theory of the resonant wave-particle interaction, we constructed a generalized kinetic equation that describes the evolution of the electron distribution function. The main conclusions of this study are the following:

• The observed occurrence rate and characteristics (amplitude and frequency) of intense parallel chorus waves may provide a very rapid (over tens of minutes) electron acceleration in the outer radiation belt. But such a rapid acceleration should be observed only sporadically and locally, during time intervals of plasma injection and strong wave generation. Otherwise, the combination of very small time-averaged occurrence rate of long and intense chorus wave packets and various other effects moderating their efficacy leads to acceleration, occurring over typical times scales >3–4 hr similar to, or possibly larger than, quasi-linear diffusion time scales.

• Nonlinear trapping and scattering cause an extremely intense energy exchange between different particle populations. In the absence of a fine balance between these different particle populations, particle acceleration (deceleration) would immediately result in intense wave damping (growth). Thus, the observations of intense chorus wave emissions during many periods of resonant interaction suggest that these waves may be accompanied by particular (balanced) distribution of electrons, evolving with almost no variation of their total energy.

• Nonlinear effects of the wave-particle interaction can be included into modern codes of radiation belt dynamics, but to exclude unrealistically large particle acceleration/deceleration, such effects should be considered only with properly chosen initial particle distributions.

Appendix A : Probability of Trapping

In this appendix we derive the probability of electron trapping into resonance (see details in, e.g., Artemyev, Vasiliev, et al., 2015; Artemyev et al., 2018). We start with Hamiltonian (2) and follow the proposed in Neishtadt (1999). First, we introduce the momentum $I$ conjugate to phase $\zeta = \phi + \psi$. We use generating function $S_i = P_{z,i} + \tilde{k}(\phi + \psi) + \tilde{I}_w$, where $P_{z,i} = P_{z,i} - kl$ and $\tilde{I}_w = I_w - I$ are new momenta conjugate to unchanged coordinates $z$ and $\psi$. The new Hamiltonian $F = H + \partial S_i / \partial t$ is

$$F = -\gamma c^2 \gamma + U_w(z, \tilde{I}_w + I) \sin \zeta$$

$$\gamma = \sqrt{1 + \left(\frac{P_{z,i} + kl}{m_e c^2} \right)^2 + \frac{2\Omega_{eq}}{m_e c^2} (\tilde{I}_w + I)}.$$  \hspace{1cm} (A1)

Hamiltonian (A1) does not depend on $\psi$, and thus, $\tilde{I}_w$ is constant (we chose $\tilde{I}_w = 0$, i.e., $I = I_w$ at the initial time). Moreover, Hamiltonian (A1) does not explicitly depend on time either ($\zeta$ is a new variable), and thus, $F = \text{const.}$
This constant can be written as $F/m_e c^2 = \varepsilon_0 = \gamma - \omega (\gamma^2 - 1) \sin^2 \alpha_{eq}/2 \Omega_{ce}(0)$, where we took into account that $l_y = (1/2)(\gamma^2 - 1) \sin^2 \alpha_{eq}/\Omega_{ce}(0)$ and $I = l_y$. The resonance condition ($\zeta = 0$) for Hamiltonian (A1) is

$$\frac{\partial F}{\partial l} = -\omega + \frac{k^2 (P_z + k)}{m_e \gamma} + \frac{\Omega_{ce}}{\gamma} = 0 \quad (A2)$$

Combination of equations (A1) and (A2) gives for $l = l_y (z, P_z)$ at the resonance:

$$\frac{\partial l_y}{m_e c^2} = -\frac{P_z}{N^2} - \frac{P_x}{N m_e c} + \frac{1}{N \sqrt{N^2 - 1}} \left( 1 - \frac{\nu^2}{N^2} - 2 \frac{\nu}{N^2} \frac{P_z}{m_e c} \right)$$

where $N = k(z) c/\omega$, $\nu = \Omega_{ce} (z) / \omega$. Substituting equation (A3) into $\gamma$ from equation (A1), we obtain the resonant energy:

$$\gamma_R = \frac{N}{\sqrt{N^2 - 1}} \left( 1 - \frac{\nu^2}{N^2} - 2 \frac{\nu}{N^2} \frac{P_z}{m_e c} \right)$$

Taking into account that $\varepsilon_0 = -\omega l_y / m_e c^2 + \gamma_R = \text{const}$, we exclude $P_z$ from the equation for energy $\gamma_R$:

$$\gamma_R = \left| \frac{N}{\sqrt{N^2 - 1}} \left( 1 - 2 \nu \omega + \nu^2 \right) \right|$$

Then, we expand Hamiltonian (A1) around the resonant momentum:

$$F = \Lambda + \frac{1}{2} g (l - l_y)^2 + U_w (z, l_y) \sin \zeta,$$

$$\Lambda = -\omega l_y + m_e c^2 \gamma_R.$$  

Hamiltonian $\Lambda (z, P_z)$ describes resonant particle motion in the $(z, P_z)$ plane. We use generating function $S_z = (l - l_y) \zeta + p z$ to introduce new momenta $P_z = l - l_y$, $p = p_z - (\partial l_y / \partial \zeta) \zeta$ and new coordinate $z = z + (\partial l_y / \partial P_z) \zeta$. Near the resonance $| (\partial l_y / \partial \zeta) | \ll 1$, $| (\partial l_y / \partial P_z) | \ll 1$ and we can expand the new Hamiltonian as (see details in, e.g., Neishtadt, 1999; Neishtadt et al., 2011):

$$F = \Lambda (p_z, z) + \frac{1}{2} g \zeta^2 + U_w (z, r) \sin \zeta = \Lambda (p, s) + H_z$$

$$H_z = \frac{1}{2} g \zeta^2 - r \zeta + U_w (z, r) \sin \zeta,$$

$$U_{w} = \sqrt{\frac{2}{\gamma_R N}} \frac{e B_w}{m_e c \omega}.$$  

Hamiltonian $\Lambda (z, P_z)$ describes particle motion in the resonant phase plane $(P_z, \zeta)$ with coefficients depending on $(p, s)$ as on parameters. Dynamics of $(p, s)$ is described by Hamiltonian $\Lambda (s, p)$. The coefficient $r$ is (see details in, e.g., Artemyev et al., 2018)

$$r = \left( \frac{\partial \Lambda}{\partial z} \frac{\partial l_y}{\partial p_z} - \frac{\partial \Lambda}{\partial P_z} \frac{\partial l_y}{\partial z} \right)_{P_z = 0} = m_e c^2 \left( \frac{\partial \Lambda}{\partial z} \frac{\partial l_y}{\partial p_z} - \frac{\partial \Lambda}{\partial P_z} \frac{\partial l_y}{\partial z} \right),$$

$$= \frac{m_e c^2}{2 \gamma_R N^2} \left( 1 - \frac{\nu^2}{N^2} + \frac{\nu^2}{N^2} \right) \left( 1 - \frac{\nu^2}{N^2} + \frac{\nu^2}{N^2} \right) + \frac{2 \gamma_R (\nu - \nu^2) \ln N}{N^2}.$$

Here $D = (\varepsilon / \omega N) \partial \ln \Omega_{ce} / \partial s$ is the ratio of the spatial scales. To define the probability of trapping, we introduce the area in the $(P_z, \zeta)$ plane filled by trapped particle trajectories, $S_{\text{res}} = \oint P_z d\zeta$:

$$S_{\text{res}} = \sqrt{\frac{B_r}{g}} \int_{-\infty}^{\infty} \sqrt{a (\sin \zeta - \sin \zeta_0) - (\zeta_+ - \zeta - \zeta_0) d\zeta},$$

$$= \frac{\sqrt{B_r}}{g} \int_{-\infty}^{\infty} \sqrt{a (\sin \zeta - \sin \zeta_0) - (\zeta_+ - \zeta - \zeta_0) d\zeta}.$$  

$$= \frac{\sqrt{B_r}}{g} \int_{-\infty}^{\infty} \sqrt{a (\sin \zeta - \sin \zeta_0) - (\zeta_+ - \zeta - \zeta_0) d\zeta}.$$
where $a = |U_w|/r$, and integration limits, $\zeta_{\pm} = \zeta_{\pm}(s)$, are solutions of equations $a \cos \zeta_+ = 1$ and $a(\sin \zeta_+ - \sin \zeta_-) = (\zeta_+ - \zeta_-) = 0$. For $|D| \ll 1$, the probability of trapping is (see Neishtadt, 1975, 1999):

$$\Pi = \frac{1}{2|\pi r|} \left( \frac{\partial \Lambda}{\partial s} \frac{\partial S_{\text{res}}}{\partial p} - \frac{\partial \Lambda}{\partial p} \frac{\partial S_{\text{res}}}{\partial s} \right) = \frac{\{\Lambda, S_{\text{res}}\}_{s,p}}{2|\pi r|}.$$  \hspace{1cm} (A10)

This probability is small: $\Pi \sim \sqrt{|D|}$. Numerical verifications of equation (A10) can be found in Figure 4, Artemyev et al. (2013), and Artemyev, Vasiliev, et al. (2015).

**Appendix B: Trapped Particle Acceleration**

In this appendix we estimate the acceleration rate of trapped particles. We would like to determine the energy change $\Delta \gamma_{\text{trap}}$ due to trapping as a function of the initial particle energy and pitch angle, $(\gamma, \alpha_{\text{eq}})$ the corresponding pitch angle change is defined by equation (6). To be trapped, a particle should resonate with the wave at a location where the probability of trapping $\Pi > 0$. Equation (A10) shows that $\Pi$ is proportional to the change of $S_{\text{res}}$ along the resonant trajectory: $dS_{\text{res}}/dt = \partial S_{\text{res}}/\partial t + \{\Lambda, S_{\text{res}}\}_{s,p}$. Therefore, particles become trapped at some $S_{\text{res,trap}}$ with $dS_{\text{res,trap}}/dt > 0$.

Dynamics of trapped particles are defined by the Hamiltonian $H_\zeta$ in equation (A7). This is a classical pendulum Hamiltonian with coefficients depending on slowly changing parameters $(s, p)$ determined by Hamiltonian $\Lambda$ in equation (A7) (note that the time scale of $(s, p)$ variation is much larger than the time scale of $(\zeta, P_\zeta)$ variations (see details in, e.g., Artemyev, Vasiliev, et al., 2015; Neishtadt, 1999). The phase portrait of Hamiltonian $H_\zeta$ is shown in Figure B1a. There are closed trajectories oscillating around the resonance $P_\zeta = 0$ (trapped trajectories) confined by the separatrix $S$. The area surrounded by the separatrix, $S_{\text{res}}$, changes with time (with $(s, p)$) and when this area becomes larger than the area surrounded by the particle trajectory, $2\pi k_\zeta$, particles becomes trapped (this determines the condition of $dS_{\text{res}}/dt > 0$). The area $2\pi k_\zeta = \oint P_\zeta d\zeta$ (integrated along a particle trajectory) is the adiabatic invariant of the system described by Hamiltonian $H_\zeta$ (e.g., Landau &

![Figure B1.](image-url)
Lifshitz, 1988), and thus, $I_c$ is conserved during trapped motion. The equation $I_c = \text{const}$ defines the condition for particle escape from resonance: $2\pi l_c = S_{\text{res}}$ and $dS_{\text{res}}/dt < 0$. Thus, the position of particle escape from the resonance, $z_{\text{esc}}$, is determined by equation $S_{\text{res,esc}} = S_{\text{res}}$ and $dS_{\text{res,esc}}/dt < 0$ (see scheme in Figure B1b). The energy gain of trapped particles is $\Delta E_{\text{trap}} = f_k(z_{\text{esc}}) - f_k(z_{\text{trap}})$ with $f_k$ given by equation (A5). To construct $\Delta E_{\text{trap}}(\gamma, \nu_{\text{eq}})$, we determine the $S_{\text{res}}$ profile along the resonant trajectory (equation (A9)) for each $(\gamma, \nu_{\text{eq}})$, define positions of particle trapping and escape and calculate the difference of the resonant energy $\gamma_k$ between these positions.

Figures B1c and B1d compare analytical predictions of trapped particle motion with results of the numerical integration of Hamiltonian equations. Figure B1c demonstrates that trapped particle energy (black curve) coincides with equation (A5) (red curve). Particle becomes trapped and escape from the resonance with the same area $S_{\text{res}}$ (gray curve). During the trapped motion, the particle oscillates in the $(\zeta, P)$ plane (panel d) and the area surrounded by the particle trajectory is conserved (blue crosses in panel c). This area, $2\pi l_c$, is equal to $S_{\text{res,trap}}$, and thus, equation $2\pi l_c = S_{\text{res}}$ defines the position of particle escape from resonance.

Appendix C: Scattering Amplitude

In this appendix we estimate the particle energy change due to nonlinear scattering, $\Delta E_{\text{scat}}$. The conservation of $e_\| = \gamma - \alpha l/m_c c^2$ shows that $\Delta E = \alpha \Delta l/m_c c^2$. To evaluate $\Delta l$, we consider Hamiltonian (A1):

$$\Delta l = -2 \int_{-\infty}^{t_g} \frac{dI}{d\zeta} dt = -2 \int_{-\infty}^{c^2} \frac{U_{w,R} \cos \zeta}{gP_b} \frac{dc}{{\gamma} \sqrt{a \cos \zeta} \sqrt{\sin \zeta - \sin \zeta}} \frac{d\zeta}{\zeta - \zeta},$$

where $t_g$ is the time of the resonant interaction and we use $d \zeta = dc/gP_c$ (see Hamiltonian $H_c$ in equation (A7)). The change $\Delta l$ depends on the phase $\zeta$, or alternatively on $\xi$, where $2\pi x = a \sin \zeta - \sin \zeta$. It can be shown that $\Delta l$ is a periodic function of $\zeta$ (see, e.g., Artemyev, Vasiliev, Mourenas, Agapitov & Krasnoselskikh, 2014; Karpman et al., 1975; Neishstadt, 1999) and $\xi$ is distributed uniformly for any reasonable initial set of wave phases $\zeta$ (e.g., Itin et al., 2000). Therefore, to derive the average value of the energy change $\Delta E_{\text{scat}} = \alpha E(\Delta l)/m_c c^2$, we need to average $\Delta l$ over $\xi \in [0, 1]$. An important property of the function $\Delta l(\xi)$ is $\langle \Delta l \rangle = -S_{\text{res}}/2\pi$ (see Dolgopyat, 2012; Karpman et al., 1975; Neishstadt, 1999; Neishstadt & Vasiliev, 2006). Thus, we can express $E_{\text{scat}}$ as follows:

$$\Delta E_{\text{scat}} = \frac{\omega \sqrt{m_c c^2}}{E_{\text{scat}}} \langle \Delta l \rangle = -\frac{\omega \sqrt{m_c c^2}}{E_{\text{scat}}} \frac{S_{\text{res}}}{2\pi},$$

Equation (C2) shows that $\Delta E_{\text{scat}}$ is about $\sqrt{|D|}$; that is, $\Delta E_{\text{scat}}$ is of the same order as the probability of trapping $\Pi \sim \sqrt{|D|}$ (see a discussion in Shklyar, 2011; Shklyar & Matsumoto, 2009).

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