MULTIPLE POSITIVE PERIODIC SOLUTIONS TO A PREDATOR-PREY MODEL WITH LESLIE-GOWER HOLLING-TYPE II FUNCTIONAL RESPONSE AND HARVESTING TERMS

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ABSTRACT. In this paper, we study a delayed predator-prey model with the Leslie-Gower Holling-type II functional response and harvesting terms. The existence of multiple positive periodic solutions for the system and the permanence of the predator-prey model are obtained by means of the generalized Mawhin coincidence degree theory.

1. Introduction. The study of predator-prey dynamics was originated by Lotka [10] and Volterra [13] independently, who showed for a one-predator-one-prey model (known as the standard Lotka-Volterra model) that the predator and prey permanently oscillate for any positive initial conditions. Predator-prey models are arguably the building blocks of the bio- and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence [7]. In the past decades, we have seen that tremendous work was undertaken for various predator-prey models with functional response [1, 3, 4, 5, 6, 9, 11, 12, 15, 16, 19]. Recently, Hsu, Hwang and Kuang [5] studied a predator-prey model with the Hassell-Varley type functional response. It was shown that the predator free equilibrium is a global attractor only when the predator death rate is greater than its growth ability, and the local stability of the positive steady state implies its global stability with respect to positive solutions. Nindjin et al [12] investigated a two-dimensional delayed dynamical system modeling a predator-prey food chain based on a modified version of Holling type-II scheme. By constructing a Lyapunov function, a sufficient condition for global stability of the positive equilibrium was found. Jing and Yang

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discussed the impact of feedback control on a predator-prey model with functional response. It shows that the position and number of positive equilibria and limit cycles, parameter domain of stability and bifurcations of such a model can be changed by some feedback control which has a linear form. He et al [3] dealt with the dynamic behaviors of an impulsive Holling II predator-prey model with mutual interference. Some sufficient conditions are obtained through the Floquet theory which ensure the prey to become extinct. They also derived some conditions for the permanence of the system by using the comparison method involving multiple Lyapunov functions. Du and Lv [1] were concerned with a Lotka-Volterra predator-prey model with mutual interference and time delay. By applying the comparison theorem of the differential equations and constructing a suitable Lyapunov functional, sufficient conditions were established to guarantee the permanence and the existence of a unique globally attractive positive periodic solution of the system.

In [15], Wang et al considered a delayed predator-prey model with nonmonotonic functional response in the form

\[
\begin{align*}
x'(t) &= x(t) [a(t) - b(t)x(t)] - \frac{a(t)y(t)}{m + x(t)}, \\
y'(t) &= y(t) \left[ \frac{\mu(t)x(t)}{m + x(t)} - d(t) \right].
\end{align*}
\]

By applying the continuation theorem of coincidence degree theory, the existence of a periodic solution was explored. Zhu and Wang [19] discussed a predator-prey model with modified Leslie-Gower Holling-type II schemes

\[
\begin{align*}
x'(t) &= x(t) [r_1(t) - b(t)x(t) - \frac{a_1(t)y(t)}{x(t) + r}], \\
y'(t) &= y(t) \left[ r_2(t) - \frac{a_2(t)y(t)}{x(t) + r} \right].
\end{align*}
\]

The existence and global attractivity of positive periodic solutions of this model were presented. Lv and Du [11] studied a Lotka-Volterra model with mutual interference and Holling III type functional response in the form

\[
\begin{align*}
\dot{x} &= x(t)(r_1(t) - b_1(t)x(t)) - \frac{c_1(t)x^2(t)}{x(t) + r} y^m(t), \\
\dot{y} &= y(t)(-r_2(t) - b_2(t)y(t)) + \frac{c_2(t)x^2(t)}{x(t) + r} y^m(t).
\end{align*}
\]

Sufficient conditions which guarantee the existence of a positive periodic solution were obtained. Li and Takeuchi [9] explored permanence conditions for two prey-predator models with the Beddington-DeAngelis functional response, and showed that density dependence gives a negative effect compared with the models without the density dependence. Hetzer, Nguyen and Shen [4] discussed the coexistence and extinction for two species Volterra-Lotka competition systems with nonlocal dispersal. Wang and Wu [16] showed some easily verifiable conditions for the existence of positive periodic solution for a delayed ratio-dependent predator-prey system with stocking system.

There are also a large amount of works concerning the multiple periodic solutions to the predator-prey model with harvesting. For example, Lan and Zhu [8] studied the dynamics of a predator-prey system with Beddington-DeAngelis functional response where the prey is harvested at a constant rate. Wei [17] investigated a predator-prey model with harvesting terms and Holling III type functional response. Xia et al [18] considered the effects of harvesting and time delay on two different types of predator-prey systems with delayed predator specific growth and...
Holling type II functional response by applying the normal form theory of retarded functional differential equations.

Motivated by the above papers, we consider a general predator-prey model with both Leslie-Gower Holling-type II functional response and harvesting terms

\[
\begin{align*}
\frac{dx}{dt} &= x(t)[r_1(t) - b_1(t)x(t) - \frac{a_1(t)y(t)}{x(t)+k_1}] + D_1(t)[y(t-\tau) - x(t)] - H_1(t), \\
\frac{dy}{dt} &= y(t)[r_2(t) - b_2(t)y(t) - \frac{a_2(t)y(t)}{x(t)+k_2}] + D_2(t)[x(t-\tau) - y(t)] - H_2(t),
\end{align*}
\]

(1)

where \(x\) represents the size of the prey population, \(y\) represents the size of predator population, \(r_1, b_1, a_1, D_1, H_i (i = 1, 2)\) are positive \(T\)-periodic functions, \(k_i's\) are positive constants with the ecological interpretation as follows: \(r_1\) is the growth rate of prey, \(r_2\) is the growth rate of predator, \(a_1\) is the maximum value which per capita reduction rate of prey can attain, \(a_2\) is the maximum value which per capita reduction rate of predator can attain, \(b_1\) is the prey population decays in the competition among the preys, \(b_2\) is the predator population decays in the competition among the predators, \(D_1\) is the diffusion coefficients of prey to predator, \(D_2\) is the diffusion coefficients of predator to prey, \(H_1\) is the harvesting rate of prey, and \(H_2\) is the harvesting rate of predators. Here we introduce time delay in model (1), which is a more realistic approach to the understanding of predator-prey dynamics. Time delay plays an important role in many biological dynamical systems, being particularly relevant in ecology, where time delays have been recognized to contribute critically to the stable or unstable outcome of prey densities due to predation. Therefore, it is interesting and potentially helpful to study system (1) for its richer and more plausible dynamics in population ecology.

In the present paper, our aim is to establish some criterion to guarantee the existence of four periodic solutions of systems (1). The rest of the paper is organized as follows. Section 2 is devoted to introducing several technical lemmas. In section 3, by using a systematic qualitative analysis and employing the generalized coincidence degree theorem, we show that system (1) has at least four positive \(T\)-periodic solutions, and we also discuss the permanence of system (1). In section 4, an example is given to illustrate our main results.

2. Preliminaries. In order to present our main results on the permanence and the existence of positive periodic solutions to system (1) in a straightforward manner, in this section we briefly introduce the generalized coincidence degree theorem and some related lemmas.

Let \(X\) and \(Y\) be two Banach spaces, \(L: \text{Dom} L \subset X \to Y\) be a linear map and \(N: X \times [0, 1] \to Y\) be a continuous map. If \(\text{Im} L \in Y\) is closed and \(\text{dim Ker} L = \text{codim} \text{Im} L < +\infty\), then we call the operator \(L\) a Fredholm operator of index zero. If \(L\) is a Fredholm operator with index zero and there exists continuous projections \(P: X \to X\) and \(Q: Y \to Y\) such that \(\text{Im} P = \text{Ker} L\) and \(\text{Im} L = \text{Ker} Q = \text{Im} (I - Q)\), then \(L|_{\text{Dom} L \cap \text{Ker} P} : (I - P)X \to \text{Im} L\) has an inverse function, and we set it as \(K_p\). Assume that \(\Omega \times [0, 1] \subset X\) is an open set. If \(QN(\overline{\Omega} \times [0, 1])\) is bounded and \(K_p(I - Q)N(\overline{\Omega} \times [0, 1]) \subset X\) is relatively compact, then we say that \(N(\overline{\Omega} \times [0, 1])\) is \(L\)-compact.

Let us we recall the generalized Mawhin’s coincidence theorem.

Lemma 1. [2] Let both \(X\) and \(Y\) be Banach spaces, \(L: \text{Dom} L \subset X \to Y\) be a Fredholm operator with index zero, \(\Omega \in Y\) be an open bounded set, and \(N: \overline{\Omega} \times [0, 1] \to X\) be \(L\)-compact on \(\overline{\Omega} \times [0, 1]\). If all the following conditions

\[
\begin{align*}
\text{(i)} & \quad \text{Im} \tilde{L} \cap \text{Ker} L = \{0\}; \\
\text{(ii)} & \quad \text{codim} \text{Ker} L < +\infty; \\
\text{(iii)} & \quad L|_{\text{Dom} L \cap \text{Ker} P} : (I - P)X \to \text{Im} L\text{ has an inverse } K_p; \\
\text{(iv)} & \quad N|_{\text{Dom} N \cap \text{Ker} \tilde{P}} : (I - \tilde{P})X \to \text{Im} N\text{ has an inverse } K_{\tilde{p}}; \\
\text{(v)} & \quad QN(\overline{\Omega} \times [0, 1])\text{ is bounded and } K_{\tilde{p}}(I - Q)N(\overline{\Omega} \times [0, 1]) \subset X\text{ is relatively compact,}
\end{align*}
\]

then we have \(\text{coincidence degree} ([0, 1], \text{Dom} L) = 1\).
The following conditions hold, then the equation \( \Omega = N(x, 1) \) has at least one solution on \( \Omega \cap DomL \).

**Lemma 2.** [1] Let \( x > 0 \), \( y > 0 \), \( z > 0 \) and \( x > 2\sqrt{yz} \). For the functions \( f(x, y, z) = \frac{z + \sqrt{x^2 - 4yz}}{2z} \) and \( g(x, y, z) = \frac{x - \sqrt{x^2 - 4yz}}{2z} \), the following assertions hold:

1. \( f(x, y, z) \) and \( g(x, y, z) \) are monotonically increasing and monotonically decreasing with respect to the variable \( x \in (0, \infty) \);
2. \( f(x, y, z) \) and \( g(x, y, z) \) are monotonically decreasing and monotonically increasing with respect to the variable \( y \in (0, \infty) \);
3. \( f(x, y, z) \) and \( g(x, y, z) \) are monotonically decreasing and monotonically increasing with respect to the variable \( z \in (0, \infty) \).

The following is a comparison theorem for the first order differential equations.

**Lemma 3.** [16] Let \( x(t) \) and \( y(t) \) be solutions of \( x' = F(t, x) \) and \( y' = G(t, y) \). Suppose that the following assumptions hold:

(a) Both \( x(t) \) and \( y(t) \) belong to a domain \( D \subset \mathbb{R}^n \) for \( t \in [0, T] \);
(b) either \( D_x F(t, x) \) is nonnegative or \( D_x G(t, x) \) is nonnegative;
(c) \( F(t, x) \leq G(t, z) \), \( (t, z) \in [0, T] \times D \).

If \( x(t_0) \leq y(t_0) \), then \( x(t_1) \leq y(t_1) \); if \( F = G \) and \( x(t_0) < y(t_0) \), then \( x(t_1) < y(t_1) \).

We denote by \( C \) the space of all bounded continuous functions \( f: \mathbb{R} \to \mathbb{R} \), and denote by \( C_+ \) the set of all functions \( f \in C \) and \( f \geq 0 \). For the convenience of statement, throughout this paper we use the following notations

\[
\tilde{f} = \frac{1}{T} \int_0^T f(t)dt, \quad f^L = \min_{t \in [0, T]} f(t), \quad f^M = \max_{t \in [0, T]} f(t).
\]

**Lemma 4.** [16] The problem \( x' = x[a(t) - b(t)x] \) \( (x \in C_+) \) has exactly one canonical solution \( U \) if both \( a(t) \) and \( b(t) \) are \( T \)-periodic functions, \( a(t) \in C, \) \( b(t) \in C_+ \) and \( a(t) > 0 \) and \( b(t) > 0 \). Moreover, the following properties also hold:

(a) \( U \) is positive \( T \)-periodic;
(b) \( U \) is constant if \( \frac{a(t)}{b(t)} \) is constant, in this case, \( U = \frac{a(t)}{b(t)} \) \( (b(t) \neq 0) \);
(c) \( u(t) - U(t) \to 0 \) as \( t \to +\infty \), for any positive solution \( u(t) \);
(d) \( \left( \frac{a(t)}{b(t)} \right)^L \leq U \leq \left( \frac{a(t)}{b(t)} \right)^M \) \( (b(t) \neq 0) \).

**Definition 1.** [14] System \((1)\) is said to be permanent if there exists a compact region \( \Omega \subset int\mathbb{R}_+^2 \) such that each solution of system \((1)\) will eventually enter and remain in the region \( \Omega \).

3. Existence of multiple periodic solutions and permanence. Let us state our result on the existence of multiple positive periodic solutions for system \((1)\).

**Theorem 1.** If the following conditions hold:

\( (H_1) \quad (r_1^L - a_1^M M_1 - D_1^M)^2 > 4b_1^M H_1^M; \)
\( (H_2) \quad (r_2^M)^2 > 4b_1^L (H_1^L - D_1^M M_1); \)
\( (H_3) \quad (D_2^L - r_2^L)^2 > 4H_2^M (a_2^M + b_2^M); \)
(H₄) \((r₄M)^2 > 4b_7^2 \ (H_2^L - D_2^M M_1)\);

hold, then system (1) has at least four positive T-periodic solutions.

**Proof of Theorem 1.** Let \(z(t) = e^{u(t)}\) and \(y(t) = e^{v(t)}\). It follows from system (1) that

\[
\begin{align*}
    &u'(t) = \left( r_1(t) - b_1(t) e^{u(t)} \right) \frac{a_1(t) e^{v(t)}}{e^{u(t)} + k_1} + D_1(t) \left( \frac{e^{v(t-\tau)}}{e^{u(t-\tau)}} - 1 \right) - \frac{H_1(t)}{e^{u(t)}}, \\
    &v'(t) = \left( r_2(t) - b_2(t) e^{v(t)} \right) \frac{a_2(t) e^{v(t)}}{e^{u(t)} + k_2} + D_2(t) \left( \frac{e^{v(t-\tau)}}{e^{u(t-\tau)}} - 1 \right) - \frac{H_2(t)}{e^{v(t)}}.
\end{align*}
\]

(2)

Let \(X = Y = \{ (u(t), v(t))^T \in C(R, R^2) : z(t + T) \equiv z(t) \}\) be equipped with the norm

\[ \|z\| = \| (u(t), v(t))^T \| = \max_{t \in [0, T]} |u(t)| + \max_{t \in [0, T]} |v(t)|, \]

then \(X\) and \(Y\) are both Banach spaces.

Take \(z \in X\) and define operators \(L\), \(P\) and \(Q\) as follows, respectively

\[
L : \text{Dom} L \cap X \to Y, \quad Lz = \frac{dz}{dt}, \quad P(z) = \frac{1}{T} \int_0^T z(t) dt, \quad Q(z) = \frac{1}{T} \int_0^T z(t) dt,
\]

where \(\text{Dom} L = \{ z \in X : z(t) \in C^1(R, R^2) \}\).

Define \(N : X \times [0, 1] \to Y\) by the form

\[
N(z, \lambda) = \begin{pmatrix} \Delta_1(z, t, \lambda) \\ \Delta_2(z, t, \lambda) \end{pmatrix},
\]

where

\[
\Delta_1(z, t, \lambda) = r_1(t) - b_1(t) e^{u(t)} - \lambda \left[ \frac{a_1(t) e^{v(t)}}{e^{u(t)} + k_1} + D_1(t) \left( \frac{e^{v(t-\tau)}}{e^{u(t-\tau)}} - 1 \right) - \frac{H_1(t)}{e^{u(t)}} \right],
\]

\[
\Delta_2(z, t, \lambda) = r_2(t) - b_2(t) e^{v(t)} - \lambda \left[ \frac{a_2(t) e^{v(t)}}{e^{u(t)} + k_2} + D_2(t) \left( \frac{e^{v(t-\tau)}}{e^{u(t-\tau)}} - 1 \right) - \frac{H_2(t)}{e^{v(t)}} \right].
\]

It is easy to see that \(\text{Ker} L = R^2\), \(\dim \text{Ker} L = \text{codim} \text{Im} L\), and \(\text{Im} L = \{ z \in Y : \int_0^T z(t) dt = 0 \}\) is closed in \(Y\). Both \(P\) and \(Q\) are continuous projections satisfying

\[
\text{Im} P = \text{Ker} L, \quad \text{Im} Q = \text{Ker} Q = \text{Im} (I - Q).
\]

So \(L\) is a Fredholm operator of index zero, which implies that \(L\) has a unique inverse.

We define by \(K_p : \text{Im} L \to \text{Ker} L \cap \text{Dom} L\) the inverse of \(L\). By a straightforward calculation, we deduce

\[
K_p(z) = \int_0^t z(s) ds - \frac{1}{T} \int_0^T \int_0^t z(s) ds dt.
\]

Thus, we have

\[
QN(z, \lambda) = \begin{pmatrix} \frac{1}{T} \int_0^T \Delta_1(z, t, \lambda) dt \\ \frac{1}{T} \int_0^T \Delta_2(z, t, \lambda) dt \end{pmatrix},
\]

and

\[
K_p(I - Q) N(z, \lambda) = \begin{pmatrix} \int_0^T \Delta_1(z, t, \lambda) dt - \frac{1}{T} \int_0^T \int_0^t \Delta_1(z, t, \lambda) ds dt - \left( \frac{T}{2} - \frac{1}{T} \right) \int_0^T \Delta_1(z, t, \lambda) dt \\ \int_0^T \Delta_2(z, t, \lambda) dt - \frac{1}{T} \int_0^T \int_0^t \Delta_2(z, t, \lambda) ds dt - \left( \frac{T}{2} - \frac{1}{T} \right) \int_0^T \Delta_2(z, t, \lambda) dt \end{pmatrix}.
\]
It is not difficult to check by the Lebesgue convergence theorem that both $QN$ and $K_p(I - Q)N$ are continuous. By using the Arzela-Ascoli theorem, we know that the operator $K_p(I - Q)N(\Omega \times [0, 1])$ is compact and $QN(\overline{\Omega} \times [0, 1])$ is bounded for any open set $\Omega \in X$. So $N \in \Omega$ is $L$-compact on $\overline{\Omega}$.

Corresponding to the operator equation $Lz = \lambda N(z, \lambda)$, we consider

$$
\begin{align*}
\left\{ \begin{array}{l}
u'(t) = \lambda \left\{ r_1(t) - b_1(t)e^{u(t)} - \lambda \left[ \frac{a_1(t)e^{u(t)}}{e^{u(t)} + \xi_1} + D_1(t) \left( \frac{e^{\nu(t) - \tau}}{e^{u(t)}} - 1 \right) \right] - \frac{H_1(t)}{e^{u(t)}} \right\}, \\
v'(t) = \lambda \left\{ r_2(t) - b_2(t)e^{v(t)} - \lambda \left[ \frac{a_2(t)e^{v(t)}}{e^{v(t)} + \xi_2} + D_1(t) \left( \frac{e^{u(t) - \tau}}{e^{v(t)}} - 1 \right) \right] - \frac{H_2(t)}{e^{v(t)}} \right\}.
\end{array} \right.
\end{align*}
$$

Let

$$
\begin{align*}
u(\xi_1) &= \max_{t \in [0, T]} u(t), \quad u(\eta_1) = \min_{t \in [0, T]} u(t), \\
v(\xi_2) &= \max_{t \in [0, T]} v(t), \quad v(\eta_2) = \min_{t \in [0, T]} v(t).
\end{align*}
$$

Obviously, we have

$$
u'(\xi_1) = \nu'(\eta_1) = 0, \quad v'(\xi_2) = v'(\eta_2) = 0.
$$

From (2), we obtain

$$
\begin{align*}
r_1(\xi_1) - b_1(\xi_1)e^{u(\xi_1)} - \lambda \left[ \frac{a_1(\xi_1)e^{u(\xi_1)}}{e^{u(\xi_1)} + \xi_1} + D_1(\xi_1) \left( \frac{e^{v(\xi_1 - \tau)}}{e^{u(\xi_1)}} - 1 \right) \right] - \frac{H_1(\xi_1)}{e^{u(\xi_1)}} &= 0, \quad (3) \\
r_2(\xi_2) - b_2(\xi_2)e^{u(\xi_2)} - \lambda \left[ \frac{a_2(\xi_2)e^{u(\xi_2)}}{e^{u(\xi_2)} + \xi_2} + D_2(\xi_2) \left( \frac{e^{v(\xi_2 - \tau)}}{e^{u(\xi_2)}} - 2 \right) \right] - \frac{H_2(\xi_2)}{e^{u(\xi_2)}} &= 0, \quad (4)
\end{align*}
$$

and

$$
\begin{align*}
r_1(\eta_1) - b_1(\eta_1)e^{u(\eta_1)} - \lambda \left[ \frac{a_1(\eta_1)e^{u(\eta_1)}}{e^{u(\eta_1)} + \xi_1} + D_1(\eta_1) \left( \frac{e^{v(\eta_1 - \tau)}}{e^{u(\eta_1)}} - 1 \right) \right] - \frac{H_1(\eta_1)}{e^{u(\eta_1)}} &= 0, \quad (5) \\
r_2(\eta_2) - b_2(\eta_2)e^{u(\eta_2)} - \lambda \left[ \frac{a_2(\eta_2)e^{u(\eta_2)}}{e^{u(\eta_2)} + \xi_2} + D_2(\eta_2) \left( \frac{e^{v(\eta_2 - \tau)}}{e^{u(\eta_2)}} - 2 \right) \right] - \frac{H_2(\eta_2)}{e^{u(\eta_2)}} &= 0. \quad (6)
\end{align*}
$$

We need to consider two cases here:

**Case 1.** Assume that $u(\xi_1) \geq v(\xi_2)$, then it gives $u(\xi_1) \geq v(\xi_1 - \tau)$. From (3), we get

$$
r_1(\xi_1) - b_1(\xi_1)e^{u(\xi_1)} > 0,
$$

which implies

$$
e^{u(\xi_1)} < \frac{r_1(\xi_1)}{b_1(\xi_1)} \leq \left( \frac{r_1}{b_1} \right)^M.
$$

That is,

$$
v(\xi_2) \leq u(\xi_1) < \ln \left( \frac{r_1}{b_1} \right)^M.
$$

**Case 2.** Assume that $u(\xi_1) < v(\xi_2)$, then it gives $u(\xi_2 - \tau) < v(\xi_2)$. From (4), we have

$$
r_2(\xi_2) - b_2(\xi_2)e^{u(\xi_2)} > 0,
$$

and

$$
u(\xi_1) < v(\xi_2) < \ln \left( \frac{r_2}{b_2} \right)^M.
$$

Set

$$M_1 = \max \left\{ \left( \frac{r_1}{b_1} \right)^M, \left( \frac{r_2}{b_2} \right)^M \right\}.$$
then we see that
\[ \max\{u(\xi_1), v(\xi_2)\} < \ln M_1. \]
In view of (3), we have
\[
\frac{b_1(\xi_1)e^{u(\xi_1)} + \frac{H_1(\xi_1)}{e^{u(\xi_1)}}}{e^{u(\xi_1)}} = r_1(\xi_1) - \lambda \frac{a_1(\xi_1)e^{v(\xi_1)}}{e^{u(\xi_1)} + k_1} + \lambda D_1(\xi_1) \left( \frac{e^{v(\xi_1)} - \tau}{e^{u(\xi_1)}} - 1 \right) \\
> r_1(\xi_1) - \frac{a_1(\xi_1)e^{v(\xi_1)}}{e^{u(\xi_1)} + k_1} - D_1(\xi_1) \\
> r_1(\xi_1) - a_1(\xi_1)M_1 - D_1(\xi_1).
\]
A direct calculation gives
\[ b_1^M e^{2u(\xi_1)} - \left( r_1^T - a_1^M M_1 - D_1^M \right) e^{u(\xi_1)} + H_1^M > 0, \]
which implies that
\[
e^{u(\xi_1)} < \frac{(r_1^T - a_1^M M_1 - D_1^M) - \sqrt{(r_1^T - a_1^M M_1 - D_1^M)^2 - 4b_1^M H_1^M}}{2b_1^M} = u_-, \]
or
\[ e^{u(\xi_1)} > \frac{(r_1^T - a_1^M M_1 - D_1^M) + \sqrt{(r_1^T - a_1^M M_1 - D_1^M)^2 - 4b_1^M H_1^M}}{2b_1^M} = u_+. \]
According to (5), we get
\[
b_1(\eta_1)e^{u(\eta_1)} + \frac{H_1(\eta_1)}{e^{u(\eta_1)}} = r_1(\eta_1) - \lambda \frac{a_1(\eta_1)e^{v(\eta_1)}}{e^{u(\eta_1)} + k_1} + \lambda D_1(\eta_1) \left( \frac{e^{v(\eta_1)} - \tau}{e^{u(\eta_1)}} - 1 \right) \\
> r_1(\eta_1) - \frac{a_1(\eta_1)e^{v(\eta_1)}}{e^{u(\eta_1)} + k_1} - D_1(\eta_1) \\
> r_1(\eta_1) - a_1(\eta_1)M_1 - D_1(\eta_1),
\]
and
\[ b_1^M e^{2u(\eta_1)} - \left( r_1^T - a_1^M M_1 - D_1^M \right) e^{u(\eta_1)} + H_1^M > 0. \]
This implies that
\[ e^{u(\eta_1)} < u_- \quad \text{or} \quad u_+ < e^{u(\eta_1)}. \]
Again from (3), one can find that
\[ b_1^T e^{2u(\xi_1)} - r_1^M e^{u(\xi_1)} + H_1^T - D_1^M M_1 < 0. \]
A straightforward calculations gives
\[ l_- = \frac{r_1^M - \sqrt{(r_1^M)^2 - 4b_1^T(H_1^T - D_1^M M_1)}}{2b_1^T} < e^{u(\xi_1)} \]
\[ < \frac{r_1^M + \sqrt{(r_1^M)^2 - 4b_1^T(H_1^T - D_1^M M_1)}}{2b_1^T} := l_. \]
Using an analogous argument, we can also derive
\[ l_- < e^{u(\eta_1)} < l_. \]
In view of Lemma 2, we can find that \( l_- < u_- < u_+ < l_+ \), and
\[
\begin{cases} 
  u(\xi_1) < \ln u_- & \text{or} & u(\xi_1) > \ln u_+ \\
  \ln l_- < u(\xi_1) < \ln l_+.
\end{cases}
\]
Similarly, we get
\[
\begin{cases}
  u(\eta_1) < \ln u_- \text{ or } u(\eta_1) > \ln u_+,
  \\
  \ln l_- < u(\eta_1) < \ln l_+.
\end{cases}
\]
Thus, we have
\[
u(\xi_1) \in (\ln l_-, \ln u_-) \cup (\ln u_+, \ln l_+), \quad u(\eta_1) \in (\ln l_-, \ln u_-) \cup (\ln u_+, \ln l_+).
\]
It follows from (4) that
\[
b_2(\xi_2)e^{\nu(\xi_2)} + \frac{H_2(\xi_2)}{e^{\nu(\xi_2)}} = r_2(\xi_2) - \lambda \frac{a_2(\xi_2)e^{\nu(\xi_2)}}{e^{\nu(\xi_2)} + \kappa_2} + \lambda D_2(\xi_2) \left( \frac{e^{u(\xi_2 - \tau)}}{e^{\nu(\xi_2)}} - 1 \right)
\]
Similarly, we get
\[
\left\{ \begin{array}{l}
  \left\{ u(\eta_1) < \ln u_- \text{ or } u(\eta_1) > \ln u_+,
  \\
  \ln l_- < u(\eta_1) < \ln l_+.
\end{array} \right. \]
A direct calculation yields
\[
e^{\nu(\xi_2)} < \frac{r_2^L - D_2^L - \sqrt{(r_2^L - D_2^L)^2 - 4H_2^M(a_2^M + b_2^M)}}{2(a_2^M + b_2^M)} := v_-,
\]
or
\[
e^{\nu(\xi_2)} > \frac{r_2^L - D_2^L + \sqrt{(r_2^L - D_2^L)^2 - 4H_2^M(a_2^M + b_2^M)}}{2(a_2^M + b_2^M)} := v_+.
\]
Discussing in a similar way, by (6) we can also deduce that
\[
e^{\nu(\eta_2)} < v_- \text{ or } e^{\nu(\eta_2)} > v_+.
\]
Again from (4), we find
\[
b_2(\xi_2)e^{\nu(\xi_2)} + \frac{H_2(\xi_2)}{e^{\nu(\xi_2)}} < r_2^M + D_2^M \frac{M_1}{e^{\nu(\xi_2)}}.
\]
After calculations, we derive
\[
m_- := \frac{r_2^M - \sqrt{(r_2^M)^2 - 4b_2^L(H_2^L - D_2^LM_1)}}{2b_2^L} < e^{\nu(\xi_2)}
\]
Similarly, we have
\[
m_- < e^{\nu(\eta_2)} < m_+.
\]
Hence, we have
\[
v(\xi_2) \in (\ln m_-, \ln v_-) \cup (\ln v_+, \ln m_+), \quad v(\eta_2) \in (\ln m_-, \ln v_-) \cup (\ln v_+, \ln m_+).
\]
It is easy to see that \(u_{\pm}, v_{\pm}, l_{\pm}, m_{\pm}\) are independent of \(\lambda\). Consider the following four sets
\[
\Omega_i = \{ z = (u, v)^T \in X | \ln l_- < u < \ln u_-, \ln m_- < v < \ln v_- \},
\]
\[
\Omega_2 = \{ z = (u, v)^T \in X | \ln l_- < u < \ln u_-, \ln v_+ < v < \ln m_+ \},
\]
\[
\Omega_3 = \{ z = (u, v)^T \in X | \ln u_+ < u < \ln l_+, \ln m_- < v < \ln v_- \},
\]
\[
\Omega_4 = \{ z = (u, v)^T \in X | \ln u_+ < u < \ln l_+, \ln v_+ < v < \ln m_+ \}.
\]
We know that \(\Omega_i \subset X\) and \(\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset\) (when \(i \neq j\)). So \(\Omega_i\)'s \((i = 1, ..., 4)\) satisfy the condition \((C_1)\) in Lemma 1.
Next we show that $QN(z, 0) \neq 0$, for $\forall z \in \partial \Omega_i \cap Ker L = \partial \Omega_i \cap R^2$ ($i = 1, ..., 4$). If it is not true, then there exists $(u, v)^T \in \partial \Omega_i$ satisfying

$$\int_0^T \left( r_1(t) - b_1(t)e^{u(t)} - \frac{H_1(t)}{e^{u(t)}} \right) dt = 0,$$

$$\int_0^T \left( r_2(t) - b_2(t)e^{v(t)} - \frac{H_2(t)}{e^{v(t)}} \right) dt = 0.$$

By virtue of the mean value theorem, there exists two points $t_j \in [0, T]$ ($j = 1, 2$) such that

$$r_1(t_1) - b_1(t_1)e^{u(t_1)} - \frac{H_1(t_1)}{e^{u(t_1)}} = 0,$$

$$r_2(t_2) - b_2(t_2)e^{v(t_2)} - \frac{H_2(t_2)}{e^{v(t_2)}} = 0.$$ 

So we have

$$\ln l_- < u < \ln u_-, \quad \text{or} \quad \ln u_+ < u < \ln l_+,$$

$$\ln m_- < v < \ln v_-, \quad \text{or} \quad \ln v_+ < v < \ln m_+.$$ 

This implies $(u, v) \in \Omega_i \cap R^2$, which contradicts the fact that $(u, v) \in \partial \Omega_i \cap R^2$. So the condition $(C_2)$ in Lemma 1 holds.

Finally, we check the condition $(C_3)$ in Lemma 1. Consider the following algebraic equations

$$\begin{cases}
    r_1(t_1) - b_1(t_1)e^x - \frac{H_1(t_1)}{e^x} = 0, \\
    r_2(t_2) - b_2(t_2)e^y - \frac{H_2(t_2)}{e^y} = 0.
\end{cases} \tag{7}$$

Solving system (7), we find that it has four distinct solutions:

$$z_1^* = (x_1^*, y_1^*) = (\ln x_-, \ln y_-),$$

$$z_2^* = (x_2^*, y_2^*) = (\ln x_-, \ln y_+),$$

$$z_3^* = (x_3^*, y_3^*) = (\ln x_+, \ln y_-),$$

$$z_4^* = (x_4^*, y_4^*) = (\ln x_+, \ln y_+),$$

where

$$x_\pm = \frac{r_1 \pm \sqrt{r_1^2 - 4b_1 H_1}}{2b_1}, \quad y_\pm = \frac{r_2 \pm \sqrt{r_2^2 - 4b_2 H_2}}{2b_2}.$$ 

By Lemma 2, it is easy to verify that

$$\ln l_- < \ln x_- < \ln u_- < \ln u_+ < \ln x_+ < \ln l_+,$$

and

$$\ln m_- < \ln y_- < \ln v_- < \ln v_+ < \ln y_+ < \ln m_+.$$ 

Thus, we have $z_i^* \in \Omega_i$ ($i = 1, ..., 4$).

Since $Ker L = Im Q$, we take $Jz = z$, and find

$$\deg \{ JQN(u, v, 0)^T, \Omega_i \cap Ker L, (0, 0)^T \} = sgn \left[ \begin{array}{cc}
    -b_1(t_1)e^x + \frac{H_1(t_1)}{e^x} & 0 \\
    0 & -b_2(t_2)e^y + \frac{H_2(t_2)}{e^y}
\end{array} \right] z_i^*$$

$$= sgn \left[ (r_1(t_1) - 2b_1(t_1)e^x)(r_2(t_2) - 2b_2(t_2)e^y) \right] z_i^*$$

$$= \pm 1.$$
This implies that the condition \((C_3)\) in Lemma 1 holds too. Note that \(\Omega_i (i = 1, \ldots, 4)\) satisfies all the associated conditions of Lemma 1, Consequently, system (1) has at least four \(T\)-periodic solutions.

Next, we consider the permanence of system (1) under the conditions \((H_1)-(H_6)\). We also assume that the amount of diffusion and harvesting is less than the total amount of each species. This means that the following two inequalities hold:

\[
\begin{align*}
(H_5) \quad & D_1(t)(y(t-\tau) - x(t)) + H_1(t) < x(t); \\
(H_6) \quad & D_2(t)(x(t-\tau) - y(t)) + H_2(t) < y(t).
\end{align*}
\]

It is notable that conditions \((H_5)\) and \((H_6)\) conform to the biological significance.

**Theorem 2.** Assume that \((H_1) - (H_6)\) hold. Then system (1) is permanent.

**Proof of Theorem 2.** Notice that \(R^2_+ = \{(x, y) | x \geq 0, y \geq 0\}\) is a positively invariant region of system (1). Given any positive solution \((x(t), y(t))\) of system (1), from the first equation of system (1), we have

\[
x'(t) \leq x(t)[(r_1(t) + 1) - b_1(t)x(t)].
\]

Consider the corresponding auxiliary equation

\[
z'(t) = z(t)[(r_1(t) + 1) - b_1(t)z(t)].
\]

From Lemmas 3 and 4, equation (8) has a unique positive \(T\)-periodic solution \(z_1(t)\) satisfying

\[
z_1(t) \leq \left(\frac{r_1(t) + 1}{b_1(t)}\right)^M := A^*_1.
\]

Choosing \(A_1 \geq A^*_1\), then we have

\[
\lim_{t \to +\infty} \sup x(t) \leq A_1.
\]

Similarly, we can deduce

\[
\lim_{t \to +\infty} \inf y(t) \geq A_2,
\]

where \(A_2\) is an arbitrary number satisfying

\[
0 < A_2 \leq \left[\frac{r_1(t) + 1 - a_1(t) M_1}{b_1(t)}\right]^L.
\]

From the second equation of system (1), using the same argument, we obtain

\[
\lim_{t \to +\infty} \sup y(t) \leq B_1, \quad \lim_{t \to +\infty} \inf y(t) \geq B_2,
\]

where \(B_1\) and \(B_2\) are constants satisfying

\[
B_1 \geq \left(\frac{r_1(t) + 1}{b_1(t)}\right)^M, \quad 0 < B_2 \leq \left(\frac{r_2(t) + 1 - a_2(t) M_1}{b_2(t)}\right)^L.
\]

Consequently, system (1) is permanent.
4. Example. In this section, we give an example to illustrate our main result. Consider the following model
\[
\begin{align*}
    x'(t) &= x(t) \left[ 3 + \sin t - \frac{4 + \cos t}{10} x(t) - \frac{\cos t(y(t))}{x(t) + \delta} \right] + (\cos t + 1) |y(t) - \tau| - \frac{1 + \cos t}{10}, \\
    y'(t) &= y(t) \left[ 3 + \cos t - \frac{4 + \cos t}{10} y(t) - \frac{\sin t(y(t))}{x(t) + \delta} \right] + (\sin t + 1) |x(t) - \tau| - \frac{1 + \sin t}{10},
\end{align*}
\]
where \( r_1 = 3 + \sin t, b_1 = \frac{4 + \cos t}{10}, a_1 = \cos t, D_1 = \cos t + 1, H_1 = \frac{1 + \cos t}{10}; \ r_2 = 3 + \cos t, b_2 = \frac{6 + \cos t}{10}, a_2 = \sin t, D_2 = \sin t + 1 \) and \( H_2 = \frac{1 + \sin t}{10} \).

It is easy to calculate that all the conditions of Theorem 1 hold, then system (9) has at least four \( 2\pi \)-periodic solutions.

5. Conclusion. In this work, we were concerned with a predator-prey model with the Leslie-Gower Holling-type II functional response and harvesting terms. The existence of multiple positive periodic solutions for the system were obtained by means of the generalized Mawhin coincidence degree theory. Furthermore, the permanence of the predator-prey model was also presented.

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