Quantum Linear Gravity in de Sitter Universe
II: On Bunch-Davies vacuum state

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In de Sitter ambient space formalism, the linear gravity can be written in terms of a minimally coupled scalar field and a polarization tensor. In this formalism, the minimally coupled massless scalar field can be quantized on Bunch-Davies vacuum state, that preserves the de Sitter invariant, the analyticity and removes the infrared divergence. The de Sitter quantum linear gravity is then constructed on Bunch-Davies vacuum state, which is also covariant, analytic and free of any infrared divergence. We conclude that the unique Bunch-Davies vacuum states can be used for construction of quantum field theory in de Sitter universe.

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\section{I. INTRODUCTION}

One of the problems of quantum field theory in de Sitter space-time is absence of a unique vacuum state for all massless and massive quantum fields. The linear gravity\textsuperscript{1} and the minimally coupled massless scalar fields\textsuperscript{2} are constructed upon Gupta-Bleuler vacuum state, which successfully removes the infrared divergence and preserve de Sitter invariant. For other quantum fields, however, the Bunch-Davies vacuum appropriately presents a unique vacuum state for construction of the quantum field theory.

The ambient space formalism ($x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}$, $\alpha, \beta = 0, 1, 2, 3, 4$) allows us to construct a linear gravity utilizing a polarization tensor and a minimally coupled massless scalar field $\phi_m(x)$\textsuperscript{1}:

$$K_{\alpha\beta}(x) = D_{\alpha\beta}(x, \partial, Z_1, Z_2)\phi_m(x),$$

where $Z_1$ and $Z_2$ are two constant 5-vectors. They can be fixed in the null curvature limit. These vectors could determine the polarization states. Usual quantization of $\phi_m$, however, not only breaks the dS symmetry, its results in appearance of infrared divergence\textsuperscript{2,\textsuperscript{†}}. Krein space quantization together with the application of Gupta-Bleuler vacuum state removes these anomalies for scalar field $\phi_m$. Appearance of negative norm states\textsuperscript{2} and the non-analyticity of the two-point function, are the negative side effects of the above method. It should be noted that in Krein space quantization for scalar field, intrinsic coordinate system was used. Thus a dichotomy vividly manifest itself in this approach to linear gravity: polarization tensor $D_{\alpha\beta}$ is presented in ambient space formalism but the scalar field $\phi_m$ is reformulated in the intrinsic coordinate system\textsuperscript{1}.

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The ambient space formalism permits us to write the scalar field \( \phi_m \) in terms of the massless conformally coupled scalar field \( \phi_c \) by using the following identity [5]:

\[
\phi_m(x) = \left[ Z_3 \cdot \partial^\top + 2Z_3 \cdot x \right] \phi_c(x),
\]

where \( Z_3^5 \) is a constant five-vector and \( \partial_3^\top = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + H^2x_\alpha x \cdot \partial \). \( \theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2x_\alpha x_\beta \) is the transverse projector. The quantum field operator \( \phi_c \) is constructed on the Bunch-Davies vacuum state in the ambient space formalism [3, 7]. Then the scalar field \( \phi_m \) and linear gravity \( K_{\alpha\beta} \) can be constructed on Bunch-Davies vacuum state. The following qualities show the advantage of the present method over the previous ones: (1) Only one formalism, namely ambient space formalism, is used for the quantization of the various spin fields. (2) A unique vacuum state, i.e. Bunch-Davies vacuum is utilized for quantum fields theory. (3) The infrared divergence is non-existence in the quantization of the scalar field \( \phi_m \) and linear quantum gravity \( K_{\alpha\beta} \). (4) The two-point functions are all analytic.

In the next section the quantization of scalar field \( \phi_c \) in ambient space formalism is recalled. The scalar field \( \phi_m \) and gravitational field \( K_{\alpha\beta} \) are then constructed upon the scalar field \( \phi_c \). Finally, the conclusions and an outlook for further investigation have been presented.

**II. MASSLESS CONFORMALLY COUPLED SCALAR FIELD**

The massless conformally coupled scalar field satisfies the following field equation [7, 8]:

\[
\left( Q_0^{(1)} - 2 \right) \phi_c(x) = 0,
\]

where \( Q_0^{(1)} = -H^{-2}\partial^\top \cdot \partial^\top \) is the Casimir operator of the de Sitter group. For simplicity from now on we take \( H = 1 \) unless circumstances necessitates otherwise. In ambient space formalism the two solutions of the above field equation can be written in terms of the dS plane waves \( (x \cdot \xi)^{-1} \) and \( (x \cdot \xi)^{-2} \), where \( \xi^\alpha \) lies in the positive cone \( C^+ = \{ \xi^\alpha \in \mathbb{R}^5 | \xi \cdot \xi = 0, \xi^0 > 0 \} \) [4, 7]. These solutions can not be well defined globally in de Sitter space-time. In order to obtain well defined solutions we must define them in the complex dS space-time. The complex dS space-time is defined by [4, 7]:

\[
M_H^{(c)} = \left\{ z = x + iy \in \mathbb{C}^5; \; \eta_{\alpha\beta}z^\alpha z^\beta = (z^0)^2 - \bar{z} \cdot z - (z^4)^2 = -H^{-2} \right\} = \left\{ (x, y) \in \mathbb{R}^5 \times \mathbb{R}^5; \; x^2 - y^2 = -H^{-2}, \; x \cdot y = 0 \right\}.
\]

(II.1)

Then the field operator can be well defined in the complex dS space-time by the analytic complex de Sitter plane waves [3]:

\[
\phi_c(z) = \sqrt{\alpha_0} \int_{S^3} d\mu(\xi) \left\{ a(\bar{\xi})(z \cdot \xi)^{-2} + a^\dagger(\xi)(z \cdot \xi)^{-1} \right\},
\]

(II.2)

where \( \xi^\alpha = (1, \bar{\xi}, \xi^4) \), \( \bar{\xi}^\alpha = (1, -\bar{\xi}, \xi^4) \) and the vacuum state is defined as [3]:

\[
a(\xi)|\Omega > = 0, \quad a^\dagger(\xi)|\Omega > = |\xi >, \quad < \xi|\xi >= \delta_{S^3}(\xi - \xi'), \quad \int_{S^3} d\mu(\xi)\delta_{S^3}(\xi - \xi') = 1.
\]

The notations are defined explicitly in [5]. The vacuum state \( |\Omega > \) in this case is exactly equivalent to the Bunch-Davies vacuum state [4, 7].
The analytic two-point function in terms of complex dS plane waves is \[6, 7\]:

\[ W_c(z, z') = \langle \Omega|\phi(z)\phi(z')|\Omega \rangle = c_0 \int_{S^3} d\mu(\xi)(z \cdot \xi)^{-2}(z' \cdot \xi)^{-1}, \tag{II.3} \]

and \(c_0\) is obtain by using the local Hadamard condition. One can easily calculate \(\text{II.3}\) in terms of the generalized Legendre function \([7]\):

\[ W_c(z, z') = \frac{-iH^2}{24\pi^2}P_{-1}(H^2z \cdot z') = \frac{H^2}{8\pi^2} \frac{-1}{1 - Z(z, z')} = \frac{H^2}{4\pi^2}(z - z')^{-2}, \tag{II.4} \]

where \(Z(z, z') = -H^2z \cdot z'\). The Wightman two-point function \(W_c(x, x')\) is the boundary value (in the sense of its interpretation as a distribution function, according to the theorem A.2 in \([7]\)) of the function \(W_c(z, z')\) which is analytic in the “tuboid” \(T_{12}\) of \(M_H^{(c)} \times M_H^{(c)}\) \([7]\). The tuboid above \(M_H^{(c)} \times M_H^{(c)}\) is defined by

\[ T_{12} = \{(z, z') : z \in T^+, z' \in T^-\}, \tag{II.5} \]

where \(T^\pm\) are called forward and backward tubes of the complex dS space \(X_H^{(c)}\)

\[ T^\pm = T^\pm \cap M_H^{(c)}. \tag{II.6} \]

\(T^\pm = \mathbb{R}^5 + iV^\pm\) is the forward and backward tubes in \(\mathbb{C}^5\). The domain \(V^\pm\) stem from the causal structure on \(M_H:\)

\[ V^\pm = \left\{ x \in \mathbb{R}^5; \ x^0 \gtrless \sqrt{\parallel \vec{x} \parallel^2 + (x^4)^2} \right\}. \tag{II.7} \]

For more details, see \([7]\). The boundary value is defined for \(z = x + iy \in T^-\) and \(z' = x' + iy' \in T^+\) by

\[ Z(z, z') = Z(x, x') - i\tau \epsilon(x^0, x'^0), \]

where \(y = (-\tau, 0, 0, 0, 0) \in V^-\), \(y' = (\tau, 0, 0, 0, 0) \in V^+\) and \(\tau \to 0\). Then, one obtains \([7, 9]\):

\[ W_c(x, x') = \frac{-H^2}{8\pi^2} \lim_{\tau \to 0} \frac{1}{1 - Z(x, x') + i\tau \epsilon(x^0, x'^0)} \]

\[ = \frac{-H^2}{8\pi^2} \left[ P \frac{1}{1 - Z(x, x')} - i\pi \epsilon(x^0, x'^0)\delta(1 - Z(x, x')) \right], \tag{II.8} \]

where the symbol \(P\) is the principal part. \(Z(x, x')\) is the geodesic distance between two points \(x\) and \(x'\) on the dS hyperboloid:

\[ Z(x, x') = -H^2x \cdot x' = 1 + \frac{H^2}{2}(x - x')^2, \]

and

\[ \epsilon(x^0 - x'^0) = \begin{cases} 1 & x^0 > x'^0 \\ 0 & x^0 = x'^0 \\ -1 & x^0 < x'^0 \end{cases}. \tag{II.9} \]
III. MASSLESS MINIMALLY COUPLED SCALAR FIELD

The massless minimally coupled scalar field equation is:

\[ Q_0^{(1)} \phi_m(x) = 0. \]

This field equation is invariant under the transformation

\[ \phi'_m(x) = \phi_m(x) + \text{const}. \]

The solutions of the field equation in terms of de Sitter plane waves are \((x \cdot \xi)^{-3}\) and \((x \cdot \xi)^0\). The constant solution \(((x \cdot \xi)^0 = \text{constant})\), poses the zero mode problem [2, 4]. With just one solution \(((x \cdot \xi)^{-3})\), it is not possible to establish a proper covariant quantum field operator on the Hilbert space constructed on an unitary irreducible representation of the dS group [2–4]. Nevertheless, one can associate a massless minimally coupled scalar field with an indecomposable representation of the dS group [2]. We represent the field operator as follows. Using the followings identities

\[ Q_0^{(1)} \partial_{\alpha}^{\top} \phi(x) - \partial_{\alpha}^{\top} Q_0^{(1)} \phi(x) = 2 \partial_{\alpha}^{\top} \phi(x) + 2x_{\alpha} Q_0^{(1)} \phi(x), \]
\[ Q_0^{(1)} x_{\alpha} \phi(x) - x_{\alpha} Q_0^{(1)} \phi(x) = -2 \partial_{\alpha}^{\top} \phi(x) - 4x_{\alpha} \phi(x), \]

with \(\phi\) as an arbitrary scalar field, one can prove the existence of a magic relation between the minimally coupled and the conformally coupled scalar fields in the dS ambient space formalism [3]

\[ \phi_m(x) = \left[ Z \cdot \partial^{\top} + 2Z \cdot x \right] \phi_c(x). \tag{III.10} \]

\(Z^\alpha\) is a constant five-vector, that fixes the indecomposable representation of the dS group. The quantum field operator is defined by:

\[ U(g) \Phi_m(z, Z) U(g)^{-1} = \Phi_m(\Lambda z, \Lambda Z), \]

where \(U(g)\) is an indecomposable representation of the dS group. Such indecomposable representation can be constructed as the product of two representations of the dS group: (1) the scalar complementary series representation related to the conformally coupled scalar field [2], and (2) a five-dimensional trivial representation with respect to \(Z^{(l)}_\alpha\) [10]. For a thorough investigation regarding the five existing polarization states \(l = 0, 1, 2, 3, 4\), the reader may refer to [10]. This subject will not be pursued here since the quantum field operator can be constructed from the conformally coupled scalar field and the identity (III.10).

Apart from the polarization five-vector \(Z^{(l)}_\alpha\), the quantum field operator in complex de Sitter space-time can be defined properly from the quantum field operator of conformally coupled scalar field:

\[ \phi_m(z) = \sqrt{c_0} \int_{S^3} d\mu(\xi) \sum_{l=0}^{4} \left[ Z^{(l)} \cdot \partial^\top + 2Z^{(l)} \cdot z \right] \left\{ a(\xi)(z \cdot \xi)^{-2} + a^\dagger(\xi)(z \cdot \xi)^{-1} \right\} \]
\[ = \sqrt{c_0} \sum_{l=0}^{4} \int_{S^3} d\mu(\xi) \left\{ a(\xi) \left[ -2(Z^{(l)} \cdot \xi^\top)(z \cdot \xi)^{-3} + 2(Z^{(l)} \cdot z)(z \cdot \xi)^{-2} \right] \right\} \]
which results in the following analytic two-point function:

\[ \Phi_m (z, Z) = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \left[ (Z^{(l)} \cdot \partial \tau + 2Z^{(l)} \cdot z) \right] W_{c} (z, z'). \]

The explicit form of this function depends on the representation \( U(g) \). As a simple case, one can choose \[ (l) \]

\[ \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} Z_{\alpha}^{(l)} Z_{\beta}^{(l')} = \delta_{\alpha\beta}, \quad Z^{(l)} \cdot Z^{(l')} = \delta^{l', l}, \]

which results in the following analytic two-point function:

\[ W_{m} (z, z') = \left[ (1 - Z) \right] W_{c} (z, z'), \]

with \( W_{c} \) being the analytic two-point function of conformally coupled scalar field \( (1.4) \). By using the following relations

\[ Z = -H^2 z \cdot z', \quad \frac{\partial}{\partial z_{\alpha}} = -H^2 z_{\alpha}' \frac{d}{dZ}, \quad \partial_{\alpha}^{\tau} = (z_{\alpha} Z - z_{\alpha}') \frac{d}{dZ}, \]

\[ \partial^{\tau} \cdot \partial^{\tau} = \left( 1 - Z^2 \right) \left[ \frac{d}{dZ} + Z \frac{d^2}{dZ^2} \right], \quad z' \cdot \partial^{\tau} = \left( 1 - Z^2 \right) \frac{d}{dZ}, \]

one can show that this two-point function also satisfies the minimally coupled scalar field equation for the variables \( z \) and \( z' \). In conclusion, the analytic two-point function \( (3.13) \) is free of any infrared divergences. The two-point function in the real dS space is the boundary value of the analytic two-point function \( W_{m} (z, z') \) \( (3.13) \):

\[ \mathcal{W}_{m} (x, x') = \frac{-H^2}{8\pi^2} \left[ \partial^{\tau} \cdot \partial^{\tau} + 2x \cdot \partial^{\tau} + 2\partial^{\tau} \cdot x' - 4Z (x, x') \right] \]

\[ \times \left[ \frac{1}{1 - Z (x, x')} - i\pi \epsilon (x^0 - x'^0) \delta (1 - Z (x, x')) \right]. \]

**IV. LINEAR QUANTUM GRAVITY**

In the previous paper, we show that the linear gravity in dS ambient space formalism can be written in terms of the massless minimally coupled scalar field \( \phi_{m} \) (for gauge fixing parameter \( c = \frac{2}{3} \) \[ 1 \])

\[ \mathcal{K}_{\alpha\beta} (x) = \mathcal{D}_{\alpha\beta} (x, \partial, Z_1, Z_2) \phi_{m} (x, Z_3), \]

where

\[ \mathcal{D} (x, \partial, Z_1, Z_2) = \left( -\frac{2}{3} \theta Z_1 \cdot + S Z_1^\top + \frac{1}{9} D_2 \right) \left[ H^2 x Z_1 \cdot Z_1 \cdot \partial^{\tau} + \frac{2}{3} H^2 D_1 Z_1 \cdot \right] \]
\[
(Z_2^\top - \frac{1}{2}D_1 [Z_2 \cdot \partial^\top + 2H^2 x \cdot Z_2]) \].
\]

\text{(IV.2)}

The operator \( D_1 \) is \( D_1 = H^{-2} \partial^\top \) and the operator \( D_2 \) is the generalized gradient
\[
D_2 K = H^{-2} S(\partial - H^2 x)K,
\]
\text{(IV.3)}

where \( S \) is the symmetrizer operator and \( K \) is a vector field. \( Z_1, Z_2 \) and \( Z_3 \) are the constant five-vectors. They determine (or in other words "fix") the specific indecomposable representation of de Sitter group. The tensor field \( K_{\alpha\beta} \) transforms by an indecomposable representation of de Sitter group.

The linear gravitational field operator in complex de Sitter space-time is defined by:
\[
K_{\alpha\beta}(z) = \sqrt{c_0} \int_{S^3} d\mu(\xi) \sum_{l_i=0}^4 D_{\alpha\beta}(z, \partial, Z_1^{(l_1)}, Z_2^{(l_2)}) [Z_3^{(l_3)} \cdot \partial^\top + 2Z_3^{(l_3)} \cdot z] \]
\[
\times \left\{ a(\xi)(z \cdot \xi)^{-2} + a^\dagger(\xi)(z \cdot \xi)^{-1} \right\}. \]

For simplicity the conditions (III.12) are imposed on the constant five-vectors \( Z_1, Z_2 \) and \( Z_3 \).

Using the conditions (III.12), the bi-tensor two-point function can be written in the following form \((c = \frac{2}{5})\) [1]
\[
W_{\alpha\beta\alpha'\beta'}(x, x') = \Delta_{\alpha\beta\alpha'\beta'}(x, x') W_m(x, x'), \]
\text{(IV.4)}

where \( W_m \) is the two-point function for the massless minimally coupled scalar field (III.14) and

\[
\Delta(x, x') = -\frac{2}{3} S' \theta \theta' \cdot \left( \theta \cdot \theta' - \frac{1}{2} D_1 \left[ 2H^2 x \cdot \theta' + \theta' \cdot \partial^\top \right] \right)
\]
\[
+ SS' \theta \cdot \theta' \left( \theta \cdot \theta' - \frac{1}{2} D_1 \left[ 2H^2 x \cdot \theta' + \theta' \cdot \partial^\top \right] \right)
\]
\[
+ \frac{H^2}{9} S' D_2 \left( \frac{2}{3} D_1 \theta' + x \theta' \cdot -H^{-2} \theta' \cdot \partial^\top \right) \left( \theta \cdot \theta' - \frac{1}{2} D_1 \left[ 2H^2 x \cdot \theta' + \theta' \cdot \partial^\top \right] \right). \]
\text{(IV.5)}

This two-point function, defined completely on de Sitter ambient space formalism, is analytic, de Sitter covariant and free of any infrared divergence. The two-point function is also constructed on Bunch-Davies vacuum state.

V. CONCLUSION AND OUTLOOK

In a series of papers, we constructed the massless and massive fields with spin \(= 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \) in de Sitter ambient space formalism (for review see [5]). It is shown that a unique Bunch-Davies vacuum state does exist for quantum fields in this space, which includes the massless minimally coupled scalar field and the linear quantum gravity. The infrared divergence is non-existent in either quantization of the scalar field \( \phi_m \) or the linear quantum gravity \( K_{\alpha\beta} \). The two-point functions are all analytic in this construction. Since the quantum field theory in our formalism is completely unitary and analytic, a unitary supergravity in de Sitter ambient space formalism seems quite plausible, which will be studied in a forthcoming paper.
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