CLASSIFICATION OF COMPLEX SYSTEMS BY THEIR SAMPLE-SPACE SCALING EXPONENTS

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CLASSIFICATION OF STATISTICAL COMPLEX SYSTEMS

- Many examples of complex system are statistical (stochastic) systems
- Statistical complex systems near the thermodynamic limit $N \rightarrow \infty$
  can be characterized by asymptotics of its sample space $W(N)$
  - space of all possible configurations
- Asymptotic behavior can be described by SCALING EXPANSION
  - Coefficients of scaling expansion correspond to scaling exponents
  - Scaling exponents completely determine universality classes
- We can find corresponding extensive entropy
  - generalization of $(c,d)$-entropy *

* R.H., S.T. EPL 93 (2011) 20006
RESCALING THE SAMPLE SPACE

- How the sample space changes when we rescale its size $N \mapsto \lambda N$?
  - The ratio behaves like $\frac{W(\lambda N)}{W(N)} \sim \lambda^{c_0}$ for $N \to \infty$.
  - The exponent $c_0$ can be extracted by $\frac{d}{d\lambda}|_{\lambda=1} c_0 = \lim_{N \to \infty} \frac{NW'(N)}{W(N)}$.
  - For the leading term we have $W(N) \sim N^{c_0}$.

- Is it only possible scaling? We have $\frac{W(\lambda N)}{W(N)} \frac{N^{c_0}}{(\lambda N)^{c_0}} \sim 1$.
  - Let us use the other rescaling $N \mapsto N^\lambda$.
  - The we get that $\frac{W(N^\lambda)}{W(N)} \frac{N^{c_0}}{N^{\lambda c_0}} \sim \lambda^{c_1}$.
  - First correction is $W(N) \sim N^{c_0} (\log N)^{c_1}$.
  - It is the same scaling like for $(c, d)$-entropy.

- CAN WE GO FURTHER?
We define the set of rescalings $r^{(n)}_\lambda (x) := \exp^{(n)}(\lambda \log^{(n)}(x))$

- $f^{(n)}(x) = f(f(\ldots(f(x))\ldots))$ 
  - $n$ times
- $r^{(0)}_\lambda (x) = \lambda x$, $r^{(1)}_\lambda (x) = x^\lambda$, $r^{(2)}_\lambda (x) = e^{\log(x)^\lambda}$, ...
- They form a group: $r^{(n)}_\lambda (r^{(n)}_{\lambda'}) = r^{(n)}_{\lambda\lambda'}$, $(r^{(n)}_\lambda)^{-1} = r^{(n)}_{1/\lambda}$, $r^{(n)}_1(x) = x$

We repeat the procedure:

- We take $N \mapsto r^{(2)}_\lambda (N)$
- $\frac{W(r^{(2)}_\lambda (N))}{W(N)} \frac{N^{c_0}(\log N)^{c_1}}{r^{(2)}_\lambda (N)^{c_0}(\log r^{(2)}_\lambda (N))^{c_1}} \sim \lambda^{c_2}$
- Second correction is $W(N) \sim N^{c_0}(\log N)^{c_1}(\log \log N)^{c_2}$
Rescaling the Sample Space III

General correction is given by
\[ \frac{W(r^{(k)}(N))}{W(N)} \prod_{j=0}^{k-1} \left( \frac{\log(j) N}{\log(j) (r^{(k)}_\lambda(N))} \right)^{c_j} \sim \lambda^{c_k} \]

Possible issue: what if \( c_0 = +\infty \)? Then \( W(N) \) grows faster than any \( N^{\alpha} \)

- We replace \( W(N) \mapsto \log W(N) \)
- The leading order scaling is \( \frac{\log W(\lambda N)}{\log W(N)} \sim \lambda^{c_0} \) for \( N \to \infty \)
- So we have \( W(N) \sim \exp(N^{c_0}) \)

If this is not enough, we replace \( W(N) \mapsto \log^{(l)} W(N) \)

so that we get finite \( c_0 \)

General expansion of \( W(N) \) is
\[ W(N) \sim \exp^{(l)} (N^{c_0} (\log N)^{c_1} (\log \log N)^{c_2} \ldots) \]

J.K., R.H., S.T. New J. Phys. 20 (2018) 093007
SCALING EXPANSION

- Previous formula can be expressed in terms of Poincaré asymptotic expansion

\[ W^{(l)}(N) \equiv \log^{(l+1)}(W(x)) = \sum_{j=0}^{n} c_j^{(l)} \log^{(j+1)}(N) + \mathcal{O}(\phi_n(N)) \]

- Coefficients of the expansion are *scaling exponents* and can be calculated from:

\[ c_k^{(l)} = \lim_{N \to \infty} \log^{(k)}(N) \left( \log^{(k-1)} \left( \ldots \left( \log N \left( \frac{NW'(N)}{\prod_{i=0}^{l} \log^{(i)}(W(N))} - c_0^{(l)} \right) - c_1^{(l)} \right) \ldots \right) - c_k^{(l)} \right) \]
EXTENSIVE ENTROPY

- We can do the same procedure with entropy $S(W)$
- **Leading order scaling:** $\frac{S(\lambda W)}{S(W)} \sim \lambda^{d_0}$
- **First correction** $\frac{S(W^\lambda)}{S(W)} \frac{W^{d_0}}{W^{\lambda d_0}} \sim \lambda^{d_1}$
  - First two scalings correspond to $(c, d)$-entropy
    
    for $c = 1 - d_0$ and $d = d_1$

- **Scaling expansion** of entropy
  
  $S(W) \sim W^{d_0} (\log W)^{d_1} (\log \log W)^{d_2} \ldots$

- Requirement of **extensivity** $S(W(N)) \sim N$ determines the relation between $c$ and $d$:
  
  - $d_l = 1/c_0$,  
  - $d_{l+k} = -c_k/c_0$ for $k = 1, 2, \ldots$
EXAMPLES
| Process                        | $d_0$ | $d_1$ | $d_2$ | $S(W)$                      |
|-------------------------------|-------|-------|-------|----------------------------|
| Random walk                   | 0     | 1     | 0     | $\log W$                   |
| $W(N) = 2^N$                  |       |       |       |                             |
| Aging random walk             | 0     | 2     | 0     | $(\log W)^2$               |
| $W(N) \approx 2^{\sqrt{N}/2} \sim 2^{N^{1/2}}$ |       |       |       |                             |
| Magnetic coins *              | 0     | 1     | -1    | $\log W / \log \log W$    |
| $W(N) \approx N^{N/2} e^{2\sqrt{N}} \sim e^N \log N$ |       |       |       |                             |
| Random network                | 0     | 1/2   | 0     | $(\log W)^{1/2}$           |
| $W(N) = 2^{\binom{N}{2}} \sim 2^{N^2}$ |       |       |       |                             |
| Random walk cascade           | 0     | 0     | 1     | $\log \log W$             |
| $W(N) = 2^{2^N} - 1 \sim 2^{2^N}$ |       |       |       |                             |

* H. Jensen et al. J. Phys. A: Math. Theor. 51 375002
PARAMETER SPACE OF \((C, D)\) - Entropy

HOW DOES IT CHANGE FOR ONE MORE SCALING EXPONENT?
PARAMETER SPACE OF \((D_0, D_1, D_2)\)-ENTROPY

To fulfill SK axiom 2 (maximality): \(d_l > 0\), to fulfill SK axiom 3 (expandability): \(d_0 < 1\)
Fields of possible applications of scaling expansions:

- Non-equilibrium thermodynamics
- Information geometry
- Critical phase transitions
- Information theory (Shannon-Khinchin axioms)
- Statistical inference (Shore-Johnson axioms)
- Super-exponential processes
- Processes with structures

* J.K., R.H., S.T. Entropy 21(2) (2019) 112
† P. Tempesta, Proc. R. Soc. A 472 (2016) 2195
‡ P.J., J.K. Phys. Rev. Lett. 122 (2019), 120601
I am excited to discuss any possible application of scaling expansions during the welcome reception or later.

ENJOY THE WELCOME RECEPTION!