Hypercontracts

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Abstract. Contract theories have been proposed to formally support distributed and decentralized system design while ensuring safe system integration. In this paper we propose \textit{hypercontracts}, a generic model with a richer structure for its underlying model of components, subsuming simulation preorders. While this new model remains generic, it provides a much more elegant and richer algebra for its key notions of refinement, parallel composition, and quotient, and it allows inclusion of new operations. On top of these foundations, we propose \textit{conic hypercontracts}, which are still generic but come with a finite description.

1 Introduction

The need for compositional algebraic frameworks to design and analyze Cyber-Physical Systems is widely accepted. The aim is to support distributed and decentralized system design based on a proper definition of \textit{interfaces} supporting the specification of subsystems having a partially specified context of operation, and subsequently guaranteeing safe system integration. Over the last few decades, we have seen the introduction of several formalisms to do this: interface automata \cite{23,11,20,8}, process spaces \cite{23}, modal interfaces \cite{17,19,18,28,5}, assume-guarantee (AG) contracts \cite{6}, rely-guarantee reasoning \cite{15,16,10,13}, and variants of these. The interface specifications state \textit{(i)} what the component guarantees and \textit{(ii)} what it assumes from its environment in order for those guarantees to hold, i.e., all these frameworks implement a form of assume-guarantee reasoning.

These algebraic frameworks have a notion of a component, of an environment, and of a specification, also called contract to stress the give-and-take dynamics between the component and its environment. They all have notions of satisfaction of a specification by a component, and of contract composition. Among the various contract theories, assume-guarantee contracts require users to state the assumptions and guarantees of the specification explicitly, while interface theories express a specification as a game played between the specification environments and implementations. Experience tells that engineers in industry find the explicit expression of a contract’s assumptions and guarantees natural (see \cite{7} chapter 12), while interface theories are perceived as a less intuitive mechanism for writing specifications; however, interface theories in general come with the most efficient algorithms, making them excellent candidates for
internal representations of specifications. Some authors (chapter 10) have therefore proposed to translate contracts expressed as pairs (assumptions, guarantees) into some interface model, where algorithms are applied. This approach has the drawback that results cannot be traced back to the original (assumptions, guarantees) formulation.

The most basic definition of a property in the formal methods community is “a set of traces.” This notion is based on the behavioral approach to system modelling: we assume we start with a set of behaviors $B$, and properties are defined as subsets of $B$. In this approach, design elements or components are also defined as subsets of $B$. The difference between components and properties is semantics: a component collects the behaviors that can be observed from that component, while a property collects the behaviors meeting some criterion of interest. We say a component $M$ satisfies a property $P$, written $M \models P$, when $M \subseteq P$, that is, when the behaviors of $M$ meet the criterion that determines $P$. Properties of this sort are also called trace properties. Many design qualities are of this type, such as safety. But there are many system attributes that can only be determined by analyzing multiple traces such as mean response times, security attributes, and reliability. This suggests the need for a richer formalism for expressing design attributes: hyperproperties.

**Hyperproperties** are subsets of $2^B$. A component $M$ satisfies a hyperproperty $H$ if $M \in H$. Since hyperproperties allow us to define exactly what components satisfy them, we can define them using any number of behaviors of a component (as opposed to trace properties which can only predicate about single traces). One of our contributions is an assume-guarantee theory that supports the expression of arbitrary hyperproperties. As we present our theory, we will use the following running example.

**Running example:** Consider the digital system shown in Figure 1a; this system is similar to those presented in [27,22] to illustrate the non-interference property in security. Here, we have an $s$-bit secret data input $S$ and an $n$-bit public input $P$. The system has an output $O$. There is also an input $H$ that is equal to zero when the system is being accessed by a user with low-privileges, i.e., a user not allowed to use the secret data, and equal to one otherwise. We wish the overall system to satisfy the property that for all environments with $H = 0$, the implementations can only make the output $O$ depend on $P$, the public data, not on the secret input $S$. To see why a trace property cannot capture this requirement, suppose for simplicity that all variables are 1-bit-long. A trace property that refines the required non-interference property is $P = \{(H = 0, P = 1, S = 1, O = 1), (H = 0, P = 0, S = 1, O = 0), (H = 0, P = 1, S = 0, O = 1), (H = 0, P = 0, S = 0, O = 0)\}$. A valid implementation $M$ of $P$ is $M = \{(H = 0, P = 1, S = 1, O = 1), (H = 0, P = 0, S = 0, O = 0)\}$, but the component $M$ leaks the value of $S$ in its output. We conclude that non-interference does not behave as a trace property. In our development, we will use hypercontracts first to express this top-level assume-guarantee requirement, and then to find a component that added to a partial implementation of the system results in a design that meets the top-level spec.
Fig. 1: (a) A digital system with a secret input $S$ and a public input $P$. The overall system must meet the requirement that the secret input does not affect the value of the output $O$ when the signal $H$ is deasserted (this signal is asserted when a privileged user uses the system). Our agenda for this running example is the following: (b) we will start with two components satisfying hypercontracts $C_1$ and $C_2$ characterizing information-flow properties of their own; (c) their composition $C_c$ will be derived. Through the quotient $C_q$, we will discover the functionality that needs to be added in order for the design to meet the top-level information-flow spec $C$.

Non-interference, introduced by Goguen and Meseguer [12], is a common information-flow attribute, a prototypical example of a design quality which trace properties are unable to capture [9]. It can be expressed with hyperproperties, and is in fact one reason behind their introduction.

Suppose $\sigma$ is one of the behaviors that our system can display, understood as the state of memory locations through time. Some of those memory locations we call privileged, some unprivileged. Let $L_0(\sigma)$ and $L_f(\sigma)$ be the projections of the behavior $\sigma$ to the unprivileged memory locations of the system, at time zero, and at the final time (when execution is done). We say that a component $M$ meets the non-interference hyperproperty when

$$\forall \sigma, \sigma' \in M. \ L_0(\sigma) = L_0(\sigma') \Rightarrow L_f(\sigma) = L_f(\sigma'),$$

i.e., if two traces begin with the unprivileged locations in the same state, the final state of the unprivileged locations matches.

Non-interference is a downward-closed hyperproperty [27][22], and a 2-safety hyperproperty—hyperproperties called $k$-safety are those for the refutation of which one must provide at least $k$ traces. In our example, to refute the hyperproperty, it suffices to show two traces that share the same unprivileged initial state, but which differ in the unprivileged final state.

Abstract contract theories. Since many contract frameworks have been proposed, there have also been efforts to systematize this knowledge by building high-level theories of which existing contract theories are instantiations. Thus, Bauer et al. [4] describe how to build a contract theory if one has a specification theory available. Benveniste et al. [7] provide a meta-theory that builds contracts starting from an algebra of components. They provide several operations on contracts and show how this meta-theory can describe, among others, interface automata, assume-guarantee contracts, modal interfaces, and rely-guarantee reasoning. This meta-theory is, however, low-level, specifying contracts as unstruc-
tured sets of environments and implementations. As a consequence, important concepts such as parallel composition and quotient of contracts are expressed in terms that are considered too abstract—see [7], chapter 4. For example, no closed form formula is given for the quotient besides its abstract definition as adjoint of parallel composition.

**Contributions.** In this paper, we provide a theory of contracts, called *hyper-contracts*, that addresses the above deficiencies by requiring more structure in definition of components, environments, and implementations. This additional structure does not restrict the applicability of our theory, however. This theory is built in three stages. We begin with a theory of components. Then we state what are the sets of components that our theory can express; we call such objects compsets, which are equivalent to hyperproperties in behavioral formalisms [22]. From these compsets, we build hypercontracts. We provide closed-form expressions for hypercontract manipulations. Then we show how the hypercontract theory applies to two specific cases: downward-closed hypercontracts and interface hypercontracts (equivalent to interface automata). The main difference between hypercontracts and the meta-theory of contracts is that hypercontracts are more structured: the meta-theory of contracts defines a theory of components, and uses these components in order to define contracts. Hypercontracts use the theory of components to define compsets, which are the types of properties that we are interested in representing in a specific theory. Hypercontracts are built out of compsets, not out of components.

To summarize, our contributions are the following: (i) a new model of *hyper-contracts* possessing a richer algebra than the metatheory of [7] and capable of expressing any lattice of hyperproperties and (ii) a calculus of *conic hypercontracts* offering finite representations of downward-closed hypercontracts.

## 2 Preliminaries

**Preorders.** Many concepts in this paper will be inherited from preorders. We recall that a preorder \((P, \leq)\) consists of a set \(P\) and a relation \(\leq\) which is transitive (i.e., \(a \leq b\) and \(b \leq c\) implies that \(a \leq c\) for all \(a, b, c \in P\)) and reflexive (\(a \leq a\) for all \(a \in P\)). A partial order is a preorder whose relation is also antisymmetric (i.e., from \(a \leq b\) and \(b \leq a\) we conclude that \(a = b\)).

Our preorders will come equipped with a partial binary operation called composition, usually denoted \(\times\). Composition is often understood as a means of connecting elements together and is assumed to be monotonic in the preorder, i.e., we assume composing with bigger elements yields bigger results: \(\forall a, b, c \in P. a \leq b \Rightarrow a \times c \leq b \times c\). We will also be interested in taking elements apart. For a notion of composition, we can always ask the question, for \(a, b \in P\), what is the largest element \(b \in P\) such that \(a \times b \leq c\)? Such an element is called *quotient* or *residual*, usually denoted \(c/a\). Formally, the definition of the quotient \(c/a\) is

\[
\forall b \in P. a \times b \leq c \text{ if and only if } b \leq c/a,
\]
which means that the quotient is the right adjoint of composition (in the sense of category theory). A synonym of this notion is to say that composing by a fixed element $a$ (i.e., $b \mapsto a \times b$) and taking quotient by the same element (i.e., $c \mapsto c/a$) form a Galois connection. A description of the use of the quotient in many fields of engineering and computer science is given in [14].

A partial order for which every two elements have a well-defined LUB (aka join), denoted $\lor$, and GLB (aka meet), denoted $\land$, is a lattice. A lattice in which the meet has a right adjoint is called Heyting algebra. This right adjoint usually goes by the name exponential, denoted $\to$. In other words, the exponential is the notion of quotient if we take composition to be given by the meet, that is, for a Heyting algebra $H$ with elements $a, c$, the exponential is defined as

$$\forall b \in H. \ a \land b \leq c \text{ if and only if } b \leq a \to c,$$

which is the familiar notion of implication in Boolean algebras.

3 The theory of hypercontracts

Our objective is to develop a theory of assume-guarantee reasoning for any kind of attribute of Cyber-Physical Systems. We do this in three steps:

1. we consider components coming with notions of preorder (e.g., simulation) and parallel composition;
2. we discuss the notion of a compset and give it substantial algebraic structure—unlike the unstructured sets of components considered in the metatheory of [7];
3. we build hypercontracts as pairs of compsets with additional structure—capturing environments and implementations.

In this section we describe how this construction is performed, and in the next we show specialized hypercontract theories.

3.1 Components

In the theory of hypercontracts, the most primitive concept is the component. Let $(\mathcal{M}, \leq)$ be a preorder. The elements $M \in \mathcal{M}$ are called components. We say that $M$ is a subcomponent of $M'$ when $M \leq M'$. If we represented components as automata, the statement “is a subcomponent of” is equivalent to “is simulated by.”

There exists a partial binary operation, $\parallel: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, monotonic in both arguments, called composition. If $M \parallel M'$ is not defined, we say that $M$ and $M'$ are non-composable (and composable otherwise). A component $E$ is an environment for component $M$ if $E$ and $M$ are composable. We assume that composition is associative and commutative.

Similarly, we assume the existence of a second, partial binary operation that is the right adjoint of composition: the quotient $\frac{\mathcal{M}}{\mathcal{M}}$ for the component theory.
Given two components $M$ and $M'$, the quotient, denoted $M/M'$, is the largest component $M''$ satisfying $M' \parallel M'' \leq M$. In other words, it gives us the largest component whose composition with $M'$ is a subcomponent of $M$.

**Running example.** In order to reason about possible decompositions of the system shown in Figure 1a, we introduce the internal variables $O_1$ and $O_2$, as shown in Figure 1b. They have lengths $o_1$ and $o_2$, respectively. The output $O$ has length $o$. For simplicity, we will assume that the behaviors of the entire system are stateless. In that case, the set of components $\mathcal{M}$ is the union of the following sets:

- For $i \in \{1, 2\}$, components with inputs $H, S, P$, and output $O_i$, i.e., the sets $\{(H, S, P, O_1, O_2, O) \mid \exists f \in (2^1 \times 2^s \times 2^n \rightarrow 2^{o_i}). O_i = f(H, S, P)\}$.
- Components with inputs $H, S, P, O_1, O_2$, and output $O$, i.e., the set $\{(H, S, P, O_1, O_2, O) \mid \exists f \in (2^1 \times 2^s \times 2^{o_1} \times 2^{o_2} \rightarrow 2^o). O = f(H, S, P, O_1, O_2)\}$. We also consider components any subset of these components, as these correspond to restricting inputs to subsets of their domains.

In this theory of components, composition is carried out via set intersection. So for example, if for $i \in \{1, 2\}$ we have functions $f_i \in (2^1 \times 2^s \times 2^n \rightarrow 2^{o_i})$ and components $M_i = \{(H, S, P, O_1, O_2, O) \mid O_i = f_i(H, S, P)\}$, the composition of these objects is

$$M_1 \parallel M_2 = \{(H, S, P, O_1, O_2, O) \mid O_1 = f_1(H, S, P) \land O_2 = f_2(H, S, P)\}$$

which is the set intersection of the components’s behaviors.

### 3.2 Compsets

$\text{CmpSet}$ is a lattice whose objects are sets of components, called *compsets*. Thus, compsets are equivalent to hyperproperties when the underlying component theory represents components as sets of behaviors. In general, not every set of components is necessarily an object of $\text{CmpSet}$.

$\text{CmpSet}$ comes with a notion of satisfaction. Suppose $M \in \mathcal{M}$ and $H$ is a compset. We say that $M$ *satisfies* $H$ or conforms to $H$, written $M \models H$, when $M \in H$. For compsets $H, H'$, we say that $H$ *refines* $H'$, written $H \models H'$, when $M \models H \Rightarrow M \models H'$, i.e., when $H \subseteq H'$.

Since we assume $\text{CmpSet}$ is a lattice, the greatest lower bounds and least upper bounds of finite sets are defined. Observe, however, that although the partial order of $\text{CmpSet}$ is given by subsetting, the meet and join of $\text{CmpSet}$ are not necessarily intersection and union, respectively, as the union or intersection of any two elements are not necessarily elements of $\text{CmpSet}$.

**Running example.** We are particularly interested in writing non-interference specifications. Regarding the system shown in Figure 1a we require the top level component to generate the output $O$ independently from the secret input $S$. We build our theory of compsets by letting the set $2^M$ be the set of elements of $\text{CmpSet}$. This means that any set of components is a valid compset. The
components meeting the top-level non-interference property are those belonging to the compset \( \{(H, S, P, O_1, O_2, O) \mid \exists f \in (2^1 \times 2^n \rightarrow 2^n). O = f(H, P)\} \), i.e., those components for which \( H \) and \( P \) are sufficient to evaluate \( O \). This corresponds exactly to those components that are insensitive to the secret input \( S \). The join and meet of these compsets is given by set union and intersection, respectively.

**Composition and quotient** We extend the notion of composition to \( \text{CmpSet} \):

\[
H \parallel H' = \left\{ M \parallel M' \mid M \models H, M' \models H', \text{and } M \text{ and } M' \text{ are composable} \right\}. \quad (3)
\]

Composition is total and monotonic, i.e., if \( H' \leq H'' \), then \( H \parallel H' \leq H \parallel H'' \). It is also commutative and associative, by the commutativity and associativity, respectively, of component composition.

We assume the existence of a second (but partial) binary operation on the objects of \( \text{CmpSet} \). This operation is the right adjoint of composition: for compsets \( H \) and \( H' \), the residual \( H/H' \) (also called quotient), is defined by the universal property (1). From the definition of composition, we must have

\[
H/H' = \{ M \in \text{M} \mid \{M\} \parallel H' \subseteq H \}. \quad (4)
\]

**Downward-closed compsets** The set of components was introduced with a partial order. We say that a compset \( H \) is downward-closed when \( M' \leq M \) and \( M \models H \) imply \( M' \models H \), i.e., if a component satisfies a downward-closed compset, so does its subcomponent. Section 5.2 treats downward-closed compsets in detail.

### 3.3 Hypercontracts

**Hypercontracts as pairs (environments, closed-system specification).** A hypercontract is a specification for a design element that tells what is required from the design element when it operates in an environment that meets the expectations of the hypercontract. A hypercontract is thus a pair of compsets:

\[
\mathcal{C} = (\mathcal{E}, \mathcal{S}) = (\text{environments, closed-system specification}).
\]

\( \mathcal{E} \) states the environments in which the object being specified must adhere to the specification. \( \mathcal{S} \) states the requirements that the design element must fulfill when operating in an environment which meets the expectations of the hypercontract. We say that a component \( E \) is an environment of hypercontract \( \mathcal{C} \), written \( E \models^E \mathcal{C} \), if \( E \models \mathcal{E} \). We say that a component \( M \) is an implementation of \( \mathcal{C} \), written \( M \models^I \mathcal{C} \), when \( M \parallel E \models \mathcal{S} \) for all \( E \models \mathcal{E} \). We thus define the set of implementations \( \mathcal{I} \) of \( \mathcal{C} \) as the compset containing all implementations, i.e., as the quotient:

\[
\text{implementations} = \mathcal{I} = \mathcal{S}/\mathcal{E}.
\]
A hypercontract with a nonempty set of environments is called compatible; if it has a nonempty set of implementations, it is called consistent. For \( S \) and \( I \) as above, the compset \( E' = S/I \) contains all environments in which the implementations of \( C \) satisfy the specifications of the hypercontract. Thus, we say that a hypercontract is saturated if its environments compset is as large as possible in the sense that adding more environments to the hypercontract would reduce its implementations. This means that \( C \) satisfies the following fixpoint equation:

\[
E = S/I = S/(S/E).
\]

Hypercontracts as pairs \((\text{environments, implementations})\). Another way to interpret a hypercontract is by telling explicitly which environments and implementations it supports. Thus, we would write the hypercontract as \( C = (E, I) \).

The lattice \( \text{Contr} \) of hypercontracts. Just as with \( \text{CmpSet} \), we define \( \text{Contr} \) as a lattice formed by putting together two compsets in one of the above two ways. Not every pair of compsets is necessarily a valid hypercontract. We will define soon the operations that give rise to this lattice.

Preorder We define a preorder on hypercontracts as follows: we say that \( C \) refines \( C' \), written \( C \leq C' \), when every environment of \( C' \) is an environment of \( C \), and every implementation of \( C \) is an implementation of \( C' \), i.e., \( E \models^E C' \Rightarrow E \models^E C \) and \( M \models^I C \Rightarrow M \models^I C' \). We can express this as

\[
E' \leq E \quad \text{and} \quad S/E = I \leq I' = S'/E'.
\]

Any two \( C, C' \) with \( C \leq C' \) and \( C' \leq C \) are said to be equivalent since they have the same environments and the same implementations. We now obtain some operations using preorders which are defined as the LUB or GLB of \( \text{Contr} \). We point out that the expressions we obtain are unique up to the preorder, i.e., up to hypercontract equivalence.

GLB and LUB From the preorder just defined, the GLB of \( C \) and \( C' \) satisfies: \( M \models^I C \land C' \) if and only if \( M \models^I C \) and \( M \models^I C' \); and \( E \models^E C \land C' \) if and only if \( E \models^E C \) or \( E \models^E C' \).

Conversely, the least upper bound satisfies \( M \models^I C \lor C' \) if and only if \( M \models^I C \) or \( M \models^I C' \), and \( E \models^E C \lor C' \) if and only if \( E \models^E C \) and \( E \models^E C' \).

The lattice \( \text{Contr} \) has hypercontracts for objects (up to contract equivalence), and meet and join as just described.

Parallel composition The composition of hypercontracts \( C_i = (E_i, I_i) \) for \( 1 \leq i \leq n \), denoted \( ||_i C_i \), is the smallest hypercontract \( C' = (E', I') \) (up to equivalence) meeting the following requirements:
– any composition of implementations of all $C_i$ is an implementation of $C'$; and
– for any $1 \leq j \leq n$, any composition of an environment of $C'$ with implementations of all $C_i$ (for $i \neq j$) yields an environment for $C_j$.

These requirements were stated for the first time by Abadi and Lamport [1].

Using our notation, this composition principle becomes

$$C \parallel C' = \bigwedge \left\{ (\mathcal{E}', \mathcal{I}') \in \text{Contr} \bigg| \begin{array}{l}
\left[ \mathcal{I}_1 \parallel \ldots \parallel \mathcal{I}_n \leq \mathcal{I}', \right.
\left. \forall I_j \parallel \mathcal{I}_j \leq \mathcal{I}_j' \right] \quad \text{for all } 1 \leq j \leq n
\end{array} \right\},$$

(5)

where the notation $\hat{\mathcal{I}}_j$ indicates that the composition $\mathcal{I}_1 \parallel \ldots \parallel \hat{\mathcal{I}}_j \parallel \ldots \parallel \mathcal{I}_n$ includes all terms $\mathcal{I}_i$, except for $\mathcal{I}_j$.

**Running example.** Coming back to the example shown in Figure 1, we want to state a requirement for the top-level component that for all environments with $H = 0$, the implementations can only make the output $O$ depend on $P$, the public data. We will write a hypercontract for the top-level. We let $C = (\mathcal{E}, \mathcal{I})$, where

$$\mathcal{E} = \{ M \in \mathbb{M} \mid \forall (H, S, P, O_1, O_2, O) \in M. H = 0 \}$$

$$\mathcal{I} = \{ M \in \mathbb{M} \mid \exists f : (2^n \rightarrow 2^n). \forall (H, S, P, O_1, O_2, O) \in M. H = 0 \rightarrow O = f(P) \}.$$

The environments are all those components only defined for $H = 0$. The implementations are those such that the output is a function of $P$ when $H = 0$.

Let $f^* : 2^n \rightarrow 2^n$. Suppose we have two hypercontracts that require their implementations to satisfy the function $O_i = f^*(P)$, one implements it when $S = 0$, and the other when $S \neq 0$. For simplicity of syntax, let $s_1$ and $s_2$ be the propositions $S = 0$ and $S \neq 0$, respectively. Let the two hypercontracts be $C_i = (\mathcal{E}_i, \mathcal{I}_i)$ for $i \in \{1, 2\}$. We won’t place restrictions on the environments for these hypercontracts, so we obtain $\mathcal{E}_i = \mathbb{M}$ and

$$\mathcal{I}_i = \{ M \in \mathbb{M} \mid \forall (H, S, P, O_1, O_2, O) \in M. s_i \rightarrow O_i = f^*(P) \}.$$

We now evaluate the composition of these two hypercontracts: $C_c = C_1 \parallel C_2 = (\mathcal{E}_c, \mathcal{I}_c)$, yielding $\mathcal{E}_c = \mathbb{M}$ and

$$\mathcal{I}_c = \{ M \in \mathbb{M} \mid \forall (H, S, P, O_1, O_2, O) \in M. \}
(s_1 \rightarrow O_1 = f^*(P)) \land (s_2 \rightarrow O_2 = f^*(P)) \}.$$

**Mirror or reciprocal** We assume we have an additional operation on hypercontracts, called both mirror and reciprocal, which flips the environments and implementations of a hypercontract: $C^{-1} = (\mathcal{E}, \mathcal{I})^{-1} = (\mathcal{I}, \mathcal{E})$ and $C^{-1} = (\mathcal{E}, S)^{-1} = (S/\mathcal{E}, S)$. This notion gives us, so to say, the hypercontract obeyed by the environment. The introduction of this operation assumes that for every
hypercontract \( \mathcal{C} \), its reciprocal is also an element of \( \text{Contr} \). Moreover, we assume that, when the infimum of a collection of hypercontracts exists, the following identity holds:

\[
(\bigwedge_i \mathcal{C}_i)^{-1} = \bigvee_i \mathcal{C}_i^{-1}.
\]  

**Hypercontract quotient** The quotient or residual for hypercontracts \( \mathcal{C} = (\mathcal{E}, \mathcal{I}) \) and \( \mathcal{C}' = (\mathcal{E}', \mathcal{I}') \), written \( \mathcal{C}' / \mathcal{C} \), has the universal property (1), namely \( \forall \mathcal{C}' \parallel \mathcal{C} \parallel \mathcal{C}' \leq \mathcal{C} \) if and only if \( \mathcal{C} \leq \mathcal{C}' / \mathcal{C} \). We can obtain a closed-form expression using the reciprocal:

**Proposition 1.** The hypercontract quotient obeys \( \mathcal{C}' / \mathcal{C} = (\mathcal{C}'^\perp / \mathcal{C})^{-1} \).

**Running example.** We use the quotient to find the specification of the component that we need to add to the system shown in Figure 1c in order to meet the top level contract \( \mathcal{C} \). To compute the quotient, we use (11). We let \( \mathcal{C} / \mathcal{C}_c = (\mathcal{E}_q, \mathcal{I}_q) \) and obtain \( \mathcal{E}_q = \mathcal{E} \land \mathcal{I}_c \) and

\[
\mathcal{I}_q = \{ M \in \mathcal{M} | \exists f \in (2^n \to 2^n) \forall (H, S, P, O_1, O_2, O) \in M. ((s_1 \to O_1 = f^*(P)) \land (s_2 \to O_2 = f^*(P)) \to O = f(P)) \}.
\]

We can refine the quotient by lifting any restrictions on the environments, and picking from the implementations the term with \( f = f^* \). Observe that \( f^* \) is a valid choice for \( f \). This yields the hypercontract \( \mathcal{C}_3 = (\mathcal{E}_3, \mathcal{I}_3) \), defined as \( \mathcal{E}_3 = \mathcal{M} \) and

\[
\mathcal{I}_3 = \{ M \in \mathcal{M} | \forall (H, S, P, O_1, O_2, O) \in M. ((s_1 \to O_1 = f^*(P)) \land (s_2 \to O_2 = f^*(P)) \to O = f^*(P)) \}.
\]

A further refinement of this hypercontract is \( \mathcal{C}_r = (\mathcal{E}_r, \mathcal{I}_r) \), where \( \mathcal{E}_r = \mathcal{M} \) and

\[
\mathcal{I}_r = \{ M \in \mathcal{M} | \forall (H, S, P, O_1, O_2, O) \in M. ((s_1 \to O_1 = O_1) \land (s_2 \to O = O_2)) \}.
\]

By the properties of the quotient, composing this hypercontract, which knows nothing about \( f^* \), with \( \mathcal{C}_r \) will yield a hypercontract which meets the non-interference hypercontract \( \mathcal{C} \). Note that this hypercontract is consistent, i.e., it has implementations (in general, refining may lead to inconsistency).

**Merging** The composition of two hypercontracts yields the specification of a system comprised of two design objects, each adhering to one of the hypercontracts being composed. Another important operation on hypercontracts is viewpoint merging, or *merging* for short. It can be the case that the same design element is assigned multiple specifications corresponding to multiple viewpoints, or design concerns [6,24] (e.g., functionality and a performance criterion). Suppose \( \mathcal{C}_1 = (\mathcal{E}_1, \mathcal{S}_1) \) and \( \mathcal{C}_2 = (\mathcal{E}_2, \mathcal{S}_2) \) are the hypercontracts we wish to merge. Two slightly different operations can be considered as candidates for formalizing viewpoint merging:
– A weak merge which is the GLB; and
– A strong merge which states that environments of the merger should be environments of both $C_1$ and $C_2$ and that the closed systems of the merger are closed systems of both $C_1$ and $C_2$. If we let $C_1 \bullet C_2 = (E, \mathcal{I})$, we have

$$E = \lor \{ E' \in \text{CmpSet} \mid E' \leq E_1 \land E_2 \text{ and } \exists C'' = (\mathcal{E}'', \mathcal{I}'') \in \text{Contr}, \mathcal{E}' = \mathcal{E}'' \}$$

$$\mathcal{I} = \lor \left\{ I' \in \text{CmpSet} \mid I' \leq (S_1 \land S_2) / E \text{ and } (E, \mathcal{I}) \in \text{Contr} \right\}.$$

The difference is that, whereas the commitment to satisfy $S_2$ survives when under the weak merge when the environment fails to satisfy $E_1$, no obligation survives under the strong merge. This distinction was proposed in [29] under the name of weak/strong assumptions.

4 Representation of compsets and hypercontracts

We have laid out the theory of hypercontracts, built in three stages. We now discuss the issue of syntactically representing these objects. Up to now, we have written compsets explicitly as sets. Doing this, however, results in a problem of portability. Consider again the example shown in Figure 1. In our running example, we found out that we could express the property of non-interference for the top-level component through the expression

$$\{ (H, S, P, O_1, O_2, O) \mid \exists f \in (2^1 \times 2^n \rightarrow 2^o). O = f(H, P) \}. $$

What would happen if we added more internal variables to the system? Suppose, for example, that we have an additional variable $O_3$. In that case, the theory of components needs to define component behaviors also over the variable $O_3$, and the compset in question becomes

$$\{ (H, S, P, O_1, O_2, O_3, O) \mid \exists f \in (2^1 \times 2^n \rightarrow 2^o). O = f(H, P) \}. $$

This makes it clear that compsets change when the theory of components modifies its variables. Yet, we would agree that the two compsets we wrote represent the same components.

In order to have a representation of compsets which is invariant to adding new variable names to the theory of components, we assume we have a logic $\Psi$ whose formulas are denoted by compsets. We require $\Psi$ to be a lattice and the denotation map

$$\text{Den : } \Psi \rightarrow \text{CmpSet}$$

to be a lattice map. This means that $\text{Den}(\psi \land \psi') = \text{Den}(\psi) \land \text{Den}(\psi')$ and $\text{Den}(\psi \lor \psi') = \text{Den}(\psi) \lor \text{Den}(\psi')$. The Den map also provides us with the means to represent hypercontracts, as these are given by a pair of compsets.

**Example.** As an example, suppose we have a theory with only one component: a voltage amplifier with an output $O$ having the same real value as its input $I$. The component is given by $M = \{ (I, O) \in \mathbb{R}^2 \mid O = I \}$. The theory of compsets has two elements: $\emptyset$ and $\{ M \}$. Suppose we have a logic $\Psi$ with symbols $i, o$ in which the formula $\psi : i = o$ is well defined and has a denotation $\text{Den}(\psi) = \{ C \in M \mid \forall (I, O) \in C. I = O \} = \{ M \}$.
Now suppose we alter the component theory so that it has an additional real variable $T$. Now the component $M$ becomes $M' = \{ (I, O, T) \in \mathbb{R}^3 \mid O = I \}$. Observe that the description of the component $M$ has changed; yet, we could say that $M'$ is completely independent of $T$. Now suppose we have a logic $\Psi'$ with symbols $i, o, t$ in which the formula $\psi := i = o$ is also well-defined. We can build a denotation map $\text{Den}' : \Psi' \to \text{CmpSet}$ such that $\text{Den}'(\psi) = \{ C \in M \mid \forall (I, O, T) \in C. I = O \} = \{ M' \}$.

We observe in this example that we were able to use the same formula $\psi$ in order to represent a compset, even when we modified the underlying symbols on which objects were defined. In other words, representations allow us to define compsets by only using “local knowledge” about the interfaces of the components described by the compset, despite the fact that components are denoted on the set of behaviors of the entire system.

5 Behavioral modeling

In the behavioral approach to system modeling, we start with a set $B$ whose elements we call behaviors. Components are defined as subsets of $B$. They contain the behaviors they can display. A component $M$ is a subcomponent of $M'$ if $M$ contains all the behaviors of $M'$, i.e., if $M \subseteq M'$. Component composition is given by set intersection: $M \times M' \overset{\text{def}}{=} M \cap M'$. If we represent the components as $M = \{ b \in B \mid \phi(b) \}$ and $M' = \{ b \in B \mid \phi'(b) \}$ for some constraints $\phi$ and $\phi'$, then composition is $M \times M' = \{ b \in B \mid \phi(b) \land \phi'(b) \}$, i.e., the behaviors that simultaneously meet the constraints of $M$ and $M'$. This notion of composition is independent of the connection topology: the topology is inferred from the behaviors of the components. The quotient is given by implication: $M/M' = M' \to M$.

We will consider three contract theories we can build with these components. The first is based on unconstrained hyperproperties; the second is based on downward-closed hyperproperties; and the third corresponds to assume-guarantee contracts.

5.1 General hypercontracts

The most expressive behavioral theory of hypercontracts is obtained when we place no restrictions on the structure of compsets and hypercontracts. In this case, the elements of $\text{CmpSet}$ are all objects $H \in 2^B$, i.e., all hyperproperties. The meet and join of compsets are set intersection and union, respectively, and their composition and quotient are given by (3) and (4), respectively. Hypercontracts are of the form $C = (E, I)$ with all extrema achieved in the binary operations, i.e., for a second hypercontract $C' = (E', I')$, the meet, join, and composition (5) are, respectively, $C \land C' = (E \cup E', I \cap I')$, $C \lor C' = (E \cap E', I \cup I')$, and $C \parallel C' = \left( \frac{E'}{E} \cap \frac{I'}{I}, I \parallel I' \right)$. From these follow the operations of quotient, and merging.
5.2 Conic (or downward-closed) hypercontracts

We assume that $\text{CmpSet}$ contains exclusively downward-closed hyperproperties. Let $H \in \text{CmpSet}$. We say that $M \models H$ is a maximal component of $H$ when $H$ contains no set bigger than $M$, i.e., if
\[
\forall M' \models H. \ M \leq M' \Rightarrow M' = M.
\]
We let $\overline{H}$ be the set of maximal components of $H$:
\[
\overline{H} = \{M \models H \mid \forall M' \models H. \ M \leq M' \Rightarrow M' = M\}.
\]
Due to the fact $H$ is downward-closed, the set of maximal components is a unique representation of $H$. We can express $H$ as
\[
H = \bigcup_{M \in \overline{H}} 2^M.
\]
We say that $H$ is $k$-conic if the cardinality of $\overline{H}$ is finite and equal to $k$, and we write this
\[
H = \langle M_1, \ldots, M_k \rangle, \text{ where } \overline{H} = \{M_1, \ldots, M_k\}.
\]

Order  The notion of order on $\text{CmpSet}$ can be expressed using this notation as follows: suppose $H' = \langle M' \rangle_{M' \in \overline{H'}}$. Then
\[
H' \leq H \text{ if and only if } \forall M' \in \overline{H'} \exists M \in \overline{H}. \ M' \leq M.
\]

Composition  Composition in $\text{CmpSet}$ becomes
\[
H \times H' = \bigcup_{M \in \overline{H}} 2^{M \cap M'} = \langle M \cap M' \rangle_{M \in \overline{H}},
\]  \hfill (7)
Therefore, if $H$ and $H'$ are, respectively, $k$- and $k'$-conic, $H \times H'$ is at most $kk'$-conic.

Quotient  Suppose $H_q$ satisfies
\[
H' \times H_q \leq H.
\]
Let $M_q \in \overline{H_q}$. We must have
\[
M_q \times M' \models H \text{ for every } M' \in \overline{H'},
\]
which means that for each $M' \in \overline{H'}$ there must exist an $M \in \overline{H}$ such that $M_q \times M' \leq M$; let us denote by $M(M')$ a choice $M' \mapsto M$ satisfying this condition. Therefore, we have
\[
M_q \leq \bigwedge_{M' \in \overline{H'}} \frac{M(M')}{{M'}}{M'},
\]  \hfill (8)
Clearly, the largest such $M_q$ is obtained by making (8) an equality. Thus, the cardinality of the quotient is bounded from above by $k^{k'}$ since we have
\[
H_q = \left(\bigwedge_{M' \in \overline{H'}} \frac{M(M')}{M'}\right)_{\forall M' \in \overline{H}}.
\]  \hfill (9)
**Contracts** Now we assume that the objects of CmpSet are pairs of downward-closed compsets. If we have two hypercontracts \(C = (E, I)\) and \(C' = (E', I')\), their composition is

\[
C \parallel C' = \left( \frac{E}{I} \land \frac{E'}{I'} \times I' \right).
\]

We can also write an expression for the quotient of two hypercontracts:

\[
C/C' = \left( \frac{E \times I'}{I} \land \frac{E'}{E} \right).
\]

### 5.3 Interval AG contracts

Now we explore AG with a must modality. We will assume that elements of CmpSet are property intervals. In other words, if \(H\) is a compset, we can find components \(L, R \in M\) such that \(H = \{M \in M \mid L \leq M \leq R\}\). We will refer to such compsets as modal or interval compsets, and we write them as \(H = [L, R]\).

The name modal is used to indicate that a component satisfying a modal compset must implement some behaviors (those contained in \(L\)) and is only allowed to implement certain behaviors (those contained in \(R\)).

Let \(H = [L, R]\) and \(H' = [L', R']\). The operations on compsets are given by

\[
\begin{align*}
H \parallel H' &= \{M \mid M' \mid M \leq M \leq R \text{ and } L' \leq M' \leq R'\} = [L \land L', R \land R'], \\
H \land H' &= \{M \mid L \leq M \leq R \text{ and } L' \leq M \leq R'\} = [L \lor L', R \lor R'], \\
H \lor H' &= \{M \mid L \leq M \leq R \text{ or } L' \leq M \leq R'\} = [L \land L', R \lor R'], \\
\frac{H}{H'} &= \lor \left\{ \left[ L'', R'' \right] \mid \frac{H''}{H'} \leq H \right\} \\
&= [L, R \lor -R'] \text{ (only defined when } L \leq L').
\end{align*}
\]

We now state the expressions for composition and quotient.

**Proposition 2.** Suppose \(C = (E, I)\) and \(C' = (E', I')\) with \(E = [L_e, R_e]\), \(I = [L_i, R_i]\), \(E' = [L'_e, R'_e]\), and \(I' = [L'_i, R'_i]\). The composition of these hypercontracts is only defined when \(L_e = L_i = L'_e = L'_i\). Set \(L = L_e\). Then the composition \(C \parallel C' = (E_c, I_c)\) is of the form

\[
E_c = [L, (R_e \land R'_e) \lor (R'_e \land -R'') \lor (R_e \land -R_i)] \quad \text{and} \\
I_c = [L, (R_i \land -R_e) \lor (R'_e \land -R'_e)].
\]

Now suppose \(C = (E, I)\) and \(C'' = (E'', I'')\) with \(E'' = [L''_e, R''_e]\), \(I'' = [L''_i, R''_i]\), \(E = [L_e, R_e]\), and \(I = [L_i, R_i]\). The residual \(C''/C = (E_r, I_r)\) is only defined when \(L''_e \leq L_i \leq L_e \leq L''_e\). Call \(L = L_i\). The components of the quotient have the form

\[
E_r = [L, R''_e \lor (R_i \land -R_e)] \quad \text{and} \\
I_r = [L, (R_e \lor R'_e) \lor -R''_e \lor (R_e \land -R_i)].
\]
6  Receptive languages and interface hypercontracts

In this section we connect the notion of a hypercontract with specifications expressed as interface automata [2]. With interface theories, we bring in the notion of input-output profiles as an extra typing for components—so far, this was not considered in our development. This effectively partitions $\mathcal{M}$ into sets containing components sharing the same profile.

Our theory of components is constructed from a new notion called receptive languages. These objects can be understood as the trace denotations of receptive I/O automata [21]. We will consider downward-closed, 1-conic compsets, see Section 5.2. And interface hypercontracts will be pairs of these with a very specific structure. At the end of the section we show how the denotation of interface automata is captured by interface hypercontracts. One novelty of our approach is that the computation of the composition of hypercontracts, which matches that of interface automata (as we will see), is inherited from our general theory by specializing the component and compset operations.

6.1 The components are receptive languages

Fix once and for all an alphabet $\Sigma$. When we operate on words of $\Sigma^*$, we will use $\circ$ for word concatenation, and we’ll let $\text{Pre}(w)$ be the set of prefixes of a word $w$. These operations are extended to languages:

\[ L \circ L' = \{ w \circ w' \mid w \in L \text{ and } w' \in L' \}, \]

and $\text{Pre}(L) = \bigcup_{w \in L} \text{Pre}(w)$. An input-output signature of $\Sigma$ (or simply an IO signature when the alphabet is understood), denoted $(I, O)$, is a partition of $\Sigma$ in sets $I$ and $O$, i.e., $I$ and $O$ are disjoint sets whose union is $\Sigma$.

**Definition 1.** Let $(I, O)$ be an IO signature. A language $L$ of $\Sigma$ is an $I$-receptive language if

- $L$ is prefix-closed; and
- if $w \in L$ and $w' \in I^*$ then $w \circ w' \in L$.

The set of all $I$-receptive languages is denoted $L_I$.

**Proposition 3.** Let $(I, O)$ be an IO signature. Then $L_I$ is closed under intersection and union.

Under the subset order, $L_I$ is a lattice with intersection as the meet and union as the join. Further, the smallest and largest elements of $L_I$ are, respectively, $0 = I^*$ and $1 = \Sigma^*$. It so happens that $L_I$ is a Heyting algebra. To prove this, it remains to be shown that it has exponentiation (i.e., that the meet has a right adjoint).

**Proposition 4.** Let $L, L' \in L_I$. The object

\[ L' \rightarrow L = \{ w \in \Sigma^* \mid \text{Pre}(w) \cap L' \subseteq L \} \]

is an element of $L_I$ and satisfies the property (2) of the exponential.
We further explore the structure of the exponential. To do this, it will be useful to define the following set: for languages $L, L'$ and a set $\Gamma \subseteq \Sigma$, we define the set of missing $\Gamma$-extensions of $L'$ with respect to $L$ as

$$\text{MissExt}(L, L', \Gamma) = ((L \cap L') \circ \Gamma) \setminus L' \circ \Sigma^*.$$  

The elements of this set are all words of the form $w \circ \sigma \circ w'$, where $w \in L \cap L'$, $\sigma \in \Gamma$, and $w' \in \Sigma^*$. These words satisfy the condition $w \circ \sigma \not\in L'$. In other words, we find the words of $L \cap L'$ which, when extended by a symbol of $\Gamma$, leave the language $L'$, and extend these words by the symbols that make them leave $L'$ and then by every possible word of $\Sigma^*$.

**Proposition 5.** Let $L, L' \in \mathcal{L}_I$. The exponential is given by $L' \rightarrow L = L \cup \text{MissExt}(L, L', O)$.

At this point, it has been established that each $\mathcal{L}_I$ is a Heyting algebra. Now we move to composition and quotient, which involve languages of different IO signatures.

### 6.2 Composition and quotient of receptive languages

To every $I \subseteq \Sigma$, we have associated the set of languages $\mathcal{L}_I$. Suppose $I' \subseteq I$. Then $L \in \mathcal{L}_I$ if it is prefix-closed, and the extension of any word of $L$ by any word of $I^*$ remains in $L$. But since $I' \subseteq I$, this means that the extension of any word of $L$ by any word of $(I')^*$ remains in $L$, so $L \in \mathcal{L}_{I'}$. We have shown that $I \subseteq I' \Rightarrow \mathcal{L}_I \leq \mathcal{L}_{I'}$. Thus, the map $I \mapsto \mathcal{L}_I$ is a contravariant functor $2^\Sigma \rightarrow 2^{2^{\Sigma^*}}$.

Since $I' \subseteq I$ implies that $\mathcal{L}_I \leq \mathcal{L}_{I'}$, we define the embedding $\iota : \mathcal{L}_I \rightarrow \mathcal{L}_{I'}$ which maps a language of $\mathcal{L}_I$ to the same language, but interpreted as an element of $\mathcal{L}_{I'}$.

Let $(I, O)$ and $(I', O')$ be IO signatures of $\Sigma$, $L \in \mathcal{L}_I$, and $L' \in \mathcal{L}_{I'}$. The composition of structures with labeled inputs and outputs traditionally requires that objects to be composed can’t share outputs. We say that IO signatures $(I, O)$ and $(I', O')$ are compatible when $O \cap O' = \emptyset$. This is equivalent to requiring that $I \cup I' = \Sigma$. Moreover, the object generated by the composition should have as outputs the union of the outputs of the objects being composed. This reasoning leads us to the definition of composition:

**Definition 2 (composition).** Let $(I, O)$ and $(I', O')$ be compatible IO signatures of $\Sigma$. Let $L \in \mathcal{L}_I$ and $L' \in \mathcal{L}_{I'}$. The operation of language composition, $\times : \mathcal{L}_I, \mathcal{L}_{I'} \rightarrow \mathcal{L}_{I \cap I'}$, is given by

$$L \times L' = \iota L \wedge \iota' L',$$

for the embeddings $\iota : \mathcal{L}_I \rightarrow \mathcal{L}_{I \cap I'}$ and $\iota' : \mathcal{L}_{I'} \rightarrow \mathcal{L}_{I \cap I'}$. 

The adjoint of this operation is the quotient. We will investigate when the

quotient is defined. Let \( I, I' \subseteq \Sigma \) with \( I \subseteq I' \), \( L \in \mathcal{L}_I \), and \( L' \in \mathcal{L}_{I'} \). Suppose there

is \( I_r \subseteq \Sigma \) such that the composition rule \( \times : \mathcal{L}_{I'}, \mathcal{L}_{I_r} \to \mathcal{L}_I \) is defined. This

means that \( I' \cup I_r = \Sigma \) and \( I' \cap I_r = I \). Solving yields \( I_r = I \cup -I' = I \cup O' \).

Observe that the smallest element of \( \mathcal{L}_{I_r} \) is \( I_r \). Thus, the existence of a

language \( L'' \in \mathcal{L}_{I_r} \) such that \( L'' \times L' \leq L \) requires that \( L' \cap I_r \subseteq L \). Clearly, not

ey every pair \( L, L' \) satisfies this property since we can take, for example, \( L = I' \) and \( L' = \Sigma' \) to obtain \( L' \cap I_r = (I \cup O')' \not\subseteq I_r \), provided \( I' \not= \Sigma \).

We proceed to obtain a closed-form expression for the quotient, but first we

define a new operator. For languages \( L, L' \) and sets \( \Gamma, \Delta \subseteq \Sigma \), the following set

of \((L', \Gamma, \Delta)\)-uncontrollable extensions of \( L \cap L' \)

\[
\text{Unc}(L, L', \Gamma, \Delta) = \left\{ w \in L \cap L' \mid \exists w' \in (\Gamma \cup \Delta)^* \wedge \sigma \in \Gamma, w \sigma w' \in L \cap L' \wedge w \sigma w' \circ \sigma \in L \setminus L' \right\} \circ \Sigma^*.
\]  

contains: (i) all words of \( L \cap L' \) which can be uncontrollably extended to a word

of \( L' \setminus L \) by appending a word of \((\Gamma \cup \Delta)^* \) and a symbol of \( \Gamma \), and (ii) all suffixes of

such words. Equivalently, \( \text{Unc}(L, L', \Gamma, \Delta) \) contains all extensions of the words

\( w \in L \cap L' \) such that there are extensions of \( w \) by words \( w' \in (\Gamma \cup \Delta)^* \) that

land in \( L' \) but not in \( L \) after appending to the extensions \( w \circ w' \) a symbol of \( \Gamma \).

**Proposition 6.** Let \((I, O)\) and \((I', O')\) be IO signatures of \( \Sigma \) such that \( I \subseteq I' \).

Let \( L \in \mathcal{L}_I \) and \( L' \in \mathcal{L}_{I'} \). Let \( I_r = I \cup O' \), and assume that \( L' \cap I_r \subseteq L \). Then

the largest \( L'' \in \mathcal{L}_{I_r} \) such that \( L'' \parallel L' \leq L \) is denoted \( L/L' \) and is given by

\[
L/L' = (L \cap L' \cup \text{MissExt}(L, L', O')) \setminus \text{Unc}(L, L', O', I).
\]

We have defined receptive languages together with a preorder and a com-

position operation with its adjoint. These objects will constitute our theory of

components, i.e., \( \mathcal{M} = \oplus_{I \in 2^\Sigma} \mathcal{L}_I \).

### 6.3 Compsets and interface hypercontracts

Using the set of components just defined, we proceed to build compsets and

hypercontracts. The compsets contain components adhering to the same IO

signature. Thus, again the notion of an IO signature will partition the set of

compsets (and the same will happen with hypercontracts). This means that for

every compset \( H \), there will always be an \( I \subseteq \Sigma \) such that \( H \subseteq \mathcal{L}_I \).

For \( I \subseteq \Sigma \), and \( L \in \mathcal{L}_I \), we will consider compsets of the form

\[
\{ M \in 2^L \mid I^* \subseteq M \}, \text{ denoted by } [0, L],
\]

where 0 is \( I^* \), the smallest element of \( \mathcal{L}_I \), i.e., the compsets are all \( I \)-receptive lan-

guages smaller than \( L \). We will focus on hypercontracts whose implementations
have signature \((I, O)\) and whose environments have \((O, I)\). Thus, hypercontracts will consist of pairs \(C = (E, S)\) of \(O\)- and \(\emptyset\)-receptive compsets, respectively. We will let
\[
S = [0, S] = \{ M \in \mathcal{L}_\emptyset \mid M \subseteq S \}
\]
for some \(S \in \mathcal{L}_\emptyset\). We will restrict the environments \(E \in \mathcal{E}\) to those that never extend a word of \(S\) by an input symbol that \(S\) does not accept. The largest such environment is given by
\[
E_S = S \cup \text{MissExt}(S, S, O).
\] (13)
Since \(S\) is prefix-closed, so is \(E_S\). Moreover, observe that \(E_S\) adds to \(S\) all those strings that are obtained by continuations of words of \(S\) by an output symbol that \(S\) does not produce. This makes \(E_S\) \(O\)-receptive. The set of environments is thus \(E = [O^*, E_S]\).

Having obtained the largest environment, we can find the implementations. These are given by \(I = [I^*, M_S]\) for \(M_S = S/E_S\). Plugging the definition, we have
\[
M_S = (S \cap E_S \cup \text{MissExt}(S, E_S, I)) \setminus \text{Unc}(S, E_S, I, \emptyset).
\]
There is no word of \(I^*\) which can extend a word of \(S\) into \(E_S \setminus S\). Thus,
\[
M_S = S \cup \text{MissExt}(S, S, I).
\]
Observe that \(S\) and \(\text{MissExt}(S, S, I)\) are disjoint (same for \(\text{MissExt}(S, S, O)\)). Thus, \(E_S \times M_S = S\). In summary, we observe that our hypercontracts are highly structured. They are in 1-1 correspondence with a language \(S \in \mathcal{L}_\emptyset\) and an input alphabet \(I \subseteq \Sigma\), i.e., there is a set isomorphism
\[
\mathcal{L}_\emptyset, 2^\Sigma \rightleftharpoons \text{Contr}.
\] (14)
Indeed, given \(S\) and \(I\), we build \(E_S\) by extending \(S\) by \(\Sigma \setminus I = O\), and \(M_S\) by extending \(S\) by \(I\). After this, the hypercontract has environments, closed systems, and implementations \([O^*, E_S]\), \([\emptyset, S]\), and \([I^*, M_S]\), respectively.

### 6.4 Hypercontract composition

Let \(S, S' \in \mathcal{L}_\emptyset\). We consider the composition of the interface hypercontracts \(C_R = C_S || C_{S'}\), where \(C_S = ([0, E_S], [0, S])\), \(C_{S'} = ([0, E_{S'}], [0, S'])\) and \(E_S\) and \(E_{S'}\) have signatures \((O, I)\) and \((O', I')\), respectively. From the structure of interface hypercontracts, we have the relations
\[
E_S = S \cup \text{MissExt}(S, S, O) \quad \text{and} \quad E_{S'} = S' \cup \text{MissExt}(S', S', O').
\]
Moreover, the implementations of \(C, C'\) are, respectively, \(I = [I^*, M_S]\) and \(I' = [I'^*, M_{S'}]\), where
\[
M_S = S \cup \text{MissExt}(S, S, I) \quad \text{and} \quad M_{S'} = S' \cup \text{MissExt}(S', S', I').
\]
The composition of these hypercontracts is defined if \((I, O)\) and \((I', O')\) have compatible signatures. Suppose \(C_R = C_S \parallel C_{S'} = ([0, E_R], [0, R])\) for some \(R \in L_0\). Then the environments must have signature \((O \cup O', I \cap I')\), and the implementations \((I \cap I', O \cup O')\).

Finally, as usual, \(E_R = R \cup \text{MissExt}(R, R, O \cup O')\) and \(M_R = R \cup \text{MissExt}(R, R, I \cap I')\) are the maximal environment and implementation. \(R\) is determined as follows:

**Proposition 7.** Let \(C_S\) and \(C_{S'}\) be interface hypercontracts and let \(C_R \overset{\text{def}}{=} C_S \parallel C_{S'}\). Then \(R\) is given by the expression

\[
R = (S \cap S') \setminus [\text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O)].
\]

The quotient for interface hypercontracts follows from Proposition 1.

### 6.5 Connection with interface automata

Now we explore the relation of interface hypercontracts with interface automata. Let \((I, O)\) be an IO signature. An \(I\)-interface automaton \([2]\) is a tuple \(A = (Q, q_0, \rightarrow)\), where \(Q\) is a finite set whose elements we call states, \(q_0 \in Q\) is the initial state, and \(\rightarrow \subseteq Q \times \Sigma \times Q\) is a deterministic transition relation (there is at most one next state for every symbol of \(\Sigma\)). We let \(A_I\) be the class of \(I\)-interface automata, and \(A = \bigoplus_{I \in 2^\Sigma} A_I\). In the language of interface automata, input and output symbols are referred to as actions.

Given two interface automata \((IA)\) \(A_i = (Q_i, q_{i,0}, \rightarrow_i) \in A_I\) for \(i \in \{1, 2\}\), we say that the state \(q_1 \in Q_1\) refines \(q_2 \in Q_2\), written \(q_1 \leq q_2\), if

- for all \(i \in O, q_1' \in Q_1, q_1 \rightarrow_i q_1' \Rightarrow \exists q_2' \in Q_2, q_2 \rightarrow_i q_2'\) and \(q_1' \leq q_2'\)

We say that \(A_1\) refines \(A_2\), written \(A_1 \leq A_2\), if \(q_{1,0} \leq q_{2,0}\). This defines a preorder in \(A_I\).

**Mapping to interface hypercontracts** Suppose \(A = (Q, q_0, \rightarrow) \in A_I\). We define the language of \(A\), denoted \(\ell (A)\), as the set of words obtained by “playing out” the transition relation, i.e.,

\[
\ell (A) = \left\{ \sigma_0 \sigma_1 \ldots \sigma_n \mid \exists q_1, \ldots, q_{n-1}, \quad q_i \rightarrow_i q_{i+1} \text{ for } 0 \leq i < n \right\}.
\]

Since \(\ell (A)\) is prefix-closed, it is an element of \(L_\emptyset\).

From Section 6.3 we know that interface hypercontracts are isomorphic to a language \(S\) of \(L_\emptyset\) and an IO signature \(I\). The operation \(A \mapsto \ell (A)\) maps an \(I\)-receivable interface automaton \(A\) to a language of \(L_\emptyset\). Composing this map with the map \([14]\) discussed in Section 6.3 we have maps \(A \to L_\emptyset, 2^L \rightarrow \text{Constr}\).

Thus, the interface hypercontract associated to \(A \in A_I\) is \(C_A = ([0, E_{\ell (A)}], [0, \ell (A)])\), where \(E_{\ell (A)} \in L_0\) is given by \([13]\). The following result tells us that refinement of interface automata is equivalent to refinement of their associated hypercontracts.

**Proposition 8.** Let \(A_1, A_2 \in A_I\). Then \(A_1 \leq A_2\) if and only if \(C_{A_1} \leq C_{A_2}\).
Composition

Let $A_1 = (Q_1, q_{1,0}, \rightarrow_1) \in A_{I_1}$ and $A_2 = (Q_2, q_{2,0}, \rightarrow_2) \in A_{I_2}$. The composition of the two IA is defined if $I_1 \cup I_2 = \Sigma$. In that case, the resulting IA $A_1 \parallel A_2$, has IO signature $(I_1 \cap I_2, O_1 \cup O_2)$. The elements of the composite IA are $(Q, (q_1, q_2), 0, \rightarrow_c)$, where the set of states and the transition relation are obtained through the following algorithm:

- Initialize $Q := Q_1 \times Q_2$. For every $\sigma \in \Sigma$, $(q_1, q_2) \rightarrow_2 (q'_1, q'_2)$ if $q_1 \rightarrow_1 q'_1$ and $q_2 \rightarrow_2 q'_2$.
- Initialize the set of invalid states to those states where one interface automaton can generate an output action which the other interface automaton does not accept:

$$N := \{(q_1, q_2) \in Q_1 \times Q_2 \mid \exists q'_2 \in Q_2, \sigma \in O_2 \forall q'_1 \in Q_1, q_2 \rightarrow_2 q'_2 \wedge \neg (q_1 \rightarrow_1 q'_1) \text{ or } \exists q'_1 \in Q_1, \sigma \in O_1 \forall q'_2 \in Q_2, q_1 \rightarrow_1 q'_1 \wedge \neg (q_2 \rightarrow_2 q'_2)\}.$$

- Also deem invalid a state such that an output action of one of the interface automata makes a transition to an invalid state, i.e., iterate the following rule until convergence:

$$N := N \cup \{(q_1, q_2) \in Q_1 \times Q_2 \mid \exists q'_1, q'_2 \in N, \sigma \in O_1 \cup O_2, (q_1, q_2) \rightarrow_2 (q'_1, q'_2)\}.$$

- Now remove the invalid states from the IA:

$$Q := Q \setminus N \text{ and } \rightarrow_c := \rightarrow_2 \setminus \{(q, \sigma, q') \in \rightarrow_2 \mid q \in N \text{ or } q' \in N\}.$$

It turns out that composing IA is equivalent to composing their associated hypercontracts:

**Proposition 9.** Let $A_1, A_2 \in A_{I_1}$. Then $C_{A_1 \parallel A_2} = C_{A_1 \parallel A_2}$.

Propositions 8 and 9 express that our model of interface hypercontracts is equivalent to Interface Automata. We observe that the definition for the parallel composition of interface hypercontracts is straightforward, unlike for the Interface Automata (the latter involves the iterative pruning of invalid states). In fact, in our case this pruning is hidden behind the formula (12) defining the set Unc().

7 Conclusions

We proposed hypercontracts, a generic model of contracts providing a richer algebra than the metatheory of [7]. We started from a generic model of components equipped with a simulation preorder and parallel composition. On top of them,
we considered compsets (or hyperproperties, for behavioral formalisms), which are lattices of sets of components equipped with parallel composition and quotient; compsets are our generic model formalizing “properties.” Hypercontracts are then defined as pairs of compsets specifying the allowed environments and either the obligations of the closed system or the set of allowed implementations—both forms are useful.

We specialized hypercontracts by restricting them to pairs of downward closed compsets (where downward closed refers to the component preorder), and then to conic hypercontracts, whose environments and closed systems are described by a finite number of components. Conic hypercontracts include Assume/Guarantee contracts as a specialization. We illustrated the versatility of our model on the definition of contracts for information flow in security.

The flexibility and power of our model suggests that a number of directions that were opened in [7], but not explored to their end, can now be re-investigated with more powerful tools: contracts and testing, subcontract synthesis (for requirement engineering), contracts and abstract interpretation, contracts in physical system modeling. Furthermore, contracts were also developed in the neighbor community of control, which motivates us to establish further links. In particular, Phan-Minh and Murray [26, 25] introduced the notion of reactive contracts. Saoud et al. [30, 31] proposed a framework of Assume/Guarantee contracts for input/output discrete or continuous time systems. Assumptions vs. Guarantees are properties stated on inputs vs. outputs; with this restriction, reactive contracts are considered and an elegant formula is proposed for the parallel composition of contracts.

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A Proofs

A.1 Proofs: Hypercontracts

Proof (Proposition 1).

Let \( C'' \subseteq C \) be \( C' \mid C = C' \leq C'' \) = \( \bigvee \left\{ (C', I') \mid \left[ I \parallel I' \leq I'', \quad \mathcal{E}'' \parallel I \leq \mathcal{E}', \quad \mathcal{E}'' \parallel I' \leq \mathcal{E} \right] \right\} \)

= \left( \left( \bigvee \left\{ (C', I') \mid \left[ I \parallel I' \leq I'', \quad \mathcal{E}'' \parallel I \leq \mathcal{E}', \quad \mathcal{E}'' \parallel I' \leq \mathcal{E} \right] \right\} \right)^{-1} \right)^{-1}.

Proof (Proposition 2). We consider contract composition. Let \( C = (\mathcal{E}, I) \) and \( C' = (\mathcal{E}', I') \) with \( \mathcal{E} = [L_c, R_c], I = [L_i, R_i], \mathcal{E}' = [L_c', R_c'], \) and \( I' = [L_i', R_i'] \). Their composition of these two contracts \( C \parallel C' = (\mathcal{E}_c, I_c) \) requires us to compute

\[ \mathcal{E}_c = (\mathcal{E}' / (I / \mathcal{E})) \land (\mathcal{E} / (I' / \mathcal{E}')) \quad \text{and} \quad I_c = ((I / \mathcal{E}) \parallel (I' / \mathcal{E}')) / \mathcal{E}_c. \]

Since we have to compute \( I / \mathcal{E} \), we must have \( L_i \leq L_c \). Similarly, to compute \( I' / \mathcal{E}' \), we need \( L_i' \leq L_c' \). Now, to compute \( \mathcal{E}' / (I / \mathcal{E}) \) and \( \mathcal{E} / (I' / \mathcal{E}') \), we must have \( L_c' \leq L_i \) and \( L_c \leq L_i' \). Then

\[ L_c' \leq L_i \leq L_c \quad \text{and} \quad L_c \leq L_i' \leq L_c', \]

so we must have \( L_c = L_i = L_c' = L_i' \) for contract composition to be well defined. To simplify notation, let \( L = L_c = L_i = L_i' = L_c' \). We obtain

\[ \mathcal{E}_c = (\mathcal{E}' / (I / \mathcal{E})) \land (\mathcal{E} / (I' / \mathcal{E}')) = (\mathcal{E}' / [L, R_i \lor \neg R_e]) \land (\mathcal{E} / [L, R_i' \lor \neg R_e']) \]

= \([L, R_c' \lor \neg (R_i \lor \neg R_e)] \lor [L, R_c \lor \neg (R_i' \lor \neg R_e')]

= \([L, (R_c \lor R_c') \lor (R_i \lor \neg R_i') \lor (R_e \lor \neg R_e)]\]

and

\[ I_c = ((I / \mathcal{E}) \parallel (I' / \mathcal{E}')) / \mathcal{E}_c = ([L, R_i \lor \neg R_e] \parallel [L, R_i' \lor \neg R_e']) / \mathcal{E}_c \]

= \([L, (R_i \lor \neg R_e) \lor (R_i' \lor \neg R_e')] / \mathcal{E}_c\]

= \([L, (R_i \lor \neg R_e) \lor (R_i' \lor \neg R_e')] / \mathcal{E}_c\).
Finally, we seek expressions for the residual. Let $C = (E, I)$ and $C'' = (E'', I'')$ with $E'' = [L''_c, R''_c]$, $I'' = [L''_i, R''_i]$, $E = [L_c, R_c]$, and $I = [L_i, R_i]$. The residual $C'' / C = (E, I)$ is given by

$$E_r = E'' \parallel (I / E)$$ and $$I_r = ((E / E'') \land ((I'' / E'') / (I / E))) / E_r.$$

To compute $I / E$, $E / E''$, and $I'' / E''$, we must have

$$L_i \leq L_c \leq L'_c$$ and $$L_i \leq L_i' \leq L_c'.$$

We compute

$$E_r = E'' \parallel [L_i, R_i \cup \lnot R_c] = [L_i, R''_c \cap (R_i \cup \lnot R_c)]$$ and

$$I_r = ((E / E'') \land ((I'' / E'') / (I / E))) / E_r = (([L_c, R_c \cup \lnot R''_c] \land ([L''_c, R''_c \cup \lnot R''_c] / [L_c, R_i \cup \lnot R_c])) / E_r.$$

We have the further constraint $L''_i \leq L_i$. Thus, $L''_i \leq L_i \leq L_c \leq L'_c$ and

$$I_r = ([L_c, R_c \cup \lnot R''_c] \land [L''_c, (R''_c \cup \lnot R''_c) \cup (R_c \cap \lnot R_i)]) / E_r = [L_c, (R_c \cap R''_c) \cup \lnot R''_c \cup (R_c \cap \lnot R_i)] / [L_c, (R_c \cap R''_c) \cup \lnot R''_c \cup (R_c \cap \lnot R_i)].$$

We have the additional constraint $L_c \leq L_i$. Thus, we have $L_c = L_i = L$, and we have $L''_i \leq L \leq L'_c$ and

$$I_r = [L, (R_c \cap R''_c) \cup \lnot R''_c \cup (R_c \cap \lnot R_i)].$$

### A.2 Proofs: Receptive languages and hypercontracts

**Proof (Proof of Proposition 3).** Suppose $L,L' \in L_I$. If $w$ is contained in $L \cap L'$, and $w_p$ is a prefix of $w$, then $w$ is contained in both $L$ and $L'$, and so is $w_p$, which means intersection is prefix-closed. Moreover, for any $w' \in I^*$, we have $w \circ w' \in L$ and $w \circ w' \in L'$, so $w \circ w' \in L \cap L'$. We conclude that $L \cap L' \in L_I$.

Similarly, if $w$ is contained in $L \cup L'$, then we may assume that $w \in L$. Any prefix $w_p$ of $w$ is also contained in $L$, so $w_p \in L \cup L'$, meaning that union is prefix-closed. In addition, for every $w' \in I^*$, we have $w \odot w' \in L$, so $w \odot w' \in L \cup L'$. This means that $L \cup L' \in L_I$.

**Proof (Proof of Proposition 4).** First we show that $L' \to L \in L_I$. Let $w \in L' \to L$. If $w_p$ is a prefix of $w$ then $\text{Pre}(w_p) \cap L' \subseteq \text{Pre}(w) \cap L \subseteq L$, so $L' \to L$ is prefix-closed.

Now suppose $w \in L' \to L$ and $w \in L$. Then for $w_1 \in I^*$, $w \circ w_1 \in L$, so $\text{Pre}(w \circ w_1) \subseteq L$. Suppose $w \in L' \to L$ and $w \notin L$. Let $n$ be the length of $w$. Since $w \notin L$, $n > 0$ (the empty string is in $L$). Write $w = \sigma_1 \ldots \sigma_n$ for $\sigma_i \in \Sigma$. Let $k \leq n$ be the largest natural number such that $\sigma_1 \ldots \sigma_k \in L'$ (note that $k$ can be zero). If $k = n$, then $w \in L' \cap \text{Pre}(w) \subseteq L$, which is forbidden by our assumption that $w \notin L$. Thus, $k < n$. Define $w_p = \sigma_1 \ldots \sigma_{k+1}$.
Clearly, \( w_p \notin L' \). For any \( w_\Sigma \in \Sigma^* \), since \( L' \) is prefix-closed, we must have \( \text{Pre}(w \circ w_\Sigma) \cap L' = \text{Pre}(w_p) \cap L' = \text{Pre}(w) \cap L' \subseteq L \). We showed that any word of \( L' \rightarrow L \) extended by a word of \( I^* \) remains in \( L' \rightarrow L \). We conclude that \( L' \rightarrow L \in L_I \).

Now we show that \( L' \rightarrow L \) has the properties of the exponential. Suppose \( L'' \in L_I \) is such that \( L' \cap L'' \subseteq L \). Let \( w \in L'' \). Then \( \text{Pre}(w) \cap L' \subseteq L \), which means that \( L'' \leq L' \rightarrow L \). On the other hand,

\[
L' \cap (L' \rightarrow L) = L' \cap \{ w \in \Sigma^* \mid \text{Pre}(w) \cap L' \subseteq L \} \subseteq L.
\]

Thus, any \( L'' \leq L' \rightarrow L \) satisfies \( L'' \cap L' \leq L \). This concludes the proof.

**Proof (Proof of Proposition 5).** Suppose \( w \in L' \cap L \). Since \( L \) and \( L' \) are I-receptive, \( w \circ \sigma \in L \cap L' \) for \( \sigma \in I \). Assume \( \sigma \in O \). If \( w \circ \sigma \notin L' \), then we can extend \( w \circ \sigma \) by any word \( w' \in \Sigma^* \), and this will satisfy \( \text{Pre}(w \circ \sigma \circ w') \cap L' = \text{Pre}(w) \cap L' \subseteq L \) due to the fact \( L' \) is prefix-closed. If \( w \circ \sigma \in L' \setminus L \), then \( w \notin L' \rightarrow L \). Thus, we can express the exponential using the closed-form expression of the proposition.

**Proof (Proof of Proposition 6).** Suppose \( w \in L/L' \) and \( w \in L \cap L' \). We have not lost generality because \( \epsilon \in L \cap L' \). We consider extensions of \( w \) by a symbol \( \sigma \):

a. If \( \sigma \in I \), \( \sigma \) is an input symbol for both \( L' \) and the quotient.
   i. \( L \) is receptive to \( I \), so \( w \circ \sigma \in L \);
   ii. \( L' \) is receptive to \( I \subseteq I' \), so \( w \circ \sigma \in L' \); and
   iii. \( L/L' \) must contain \( w \circ \sigma \) because the quotient is \( L \)-receptive.

b. If \( \sigma \in O \cap I' \), then \( \sigma \) is an output of the quotient, and an input of \( L' \).
   i. \( L' \) is \( I' \)-receptive, so \( w \circ \sigma \in L' \);
   ii. \( \sigma \) is an output symbol for both \( L \) and \( L/L' \), so none of them is required to contain \( w \circ \sigma \); and
   iii. if \( w \circ \sigma \in L' \setminus L \), the extension \( w \circ \sigma \) cannot be in the quotient. Otherwise, it can.

c. If \( \sigma \in O' \), \( \sigma \) is an output for \( L' \) and an input for the quotient.
   i. Neither \( L \) nor \( L' \) are \( O' \)-receptive;
   ii. \( L/L' \) is \( O' \)-receptive, so we must have \( w \circ \sigma \in L/L' \); and
   iii. if \( w \circ \sigma \in L' \setminus L \), we cannot have \( w \circ \sigma \in L/L' \).

Starting with a word \( w \) in the quotient, statements a and b allow or disallow extensions of that word to be in the quotient. However, statements c.ii and c.iii impose a requirement on the word \( w \) itself, i.e., if c.iii is violated, c.ii implies that \( w \) is not in the quotient. Statements a.iii and c.ii impose an obligation on the quotient to accept extensions by symbols of \( I \) and \( O' \); and those extensions may lead to a violation of c.iii. Thus, we remove from the quotient all words such that extensions of those words by elements of \( I \cup O' \) end up in \( L' \setminus L \). The expression of the proposition follows from these considerations.
Proof (Proof of Proposition 7). From the principle of hypercontract composition, we must have

\[ E_R \leq U \overset{\text{def}}{=} (E_{S'}/M_S) \land (E_S/M_{S'}) \quad \text{and} \tag{15} \]
\[ L \overset{\text{def}}{=} M_{S'} \times M_S \leq M_R. \tag{16} \]

Observe that the quotients \( E_{S'}/M_S \) and \( E_S/M_{S'} \) both have IO signature \( \mathcal{O} \cup \mathcal{O}' \), so the conjunction in (15) is well-defined as an operation of the Heyting algebra \( \mathcal{L}_{\mathcal{O} \cup \mathcal{O}'} \). We study the first element:

\[ E_{S'}/M_S = (E_{S'} \cap M_S \cup \text{MissExt}(E_{S'}, M_S, O)) \setminus \text{Unc}(E_{S'}, M_S, O, O'). \]

We attempt to simplify the terms. Suppose \( w \in E_{S'} \cap (M_S \setminus S) \). Then all extensions of \( w \) lie in \( M_S \setminus S \). This means that \( \text{MissExt}(E_{S'}, M_S, O) = \text{MissExt}(E_{S'}, S, O) \). Moreover, if a word is an element of \( E_{S'} \setminus S' \), all its extensions are in this set, as well (i.e., it is impossible to escape this set by extending words). Thus, \( \text{Unc}(E_{S'}, M_S, O, O') = \text{Unc}(S', M_S, O, O') \). We have

\[ E_{S'}/M_S = (E_{S'} \cap M_S \cup \text{MissExt}(E_{S'}, S, O)) \setminus \text{Unc}(S', M_S, O, O'). \]

Now we can write

\[ U = \left[ \begin{array}{c}
(E_{S'} \cap M_S \cup \text{MissExt}(E_{S'}, S, O)) \setminus \\
(E_{S} \cap M_{S'} \cup \text{MissExt}(E_{S'}, S', O'))
\end{array} \right] \setminus \\
[\text{Unc}(S', M_S, O, O') \cup \text{Unc}(S, M_{S'}, O', O)]. \]

Observe that

\[ E_{S'} \cap M_S \cap \text{MissExt}(E_{S'}, S', O') = (S' \cup \text{MissExt}(S', S', O')) \cap M_S \cap \text{MissExt}(E_{S'}, S', O') = M_S \cap \text{MissExt}(E_{S'}, S', O') = M_S \cap \text{MissExt}(S, S', O'). \]

The last equality comes from the following fact: if a word of \( \text{MissExt}(E_{S'}, S', O') \) is obtained by extending a word of \( (E_S \setminus S) \cap S' \) by \( O' \), the resulting word is still an element of \( E_S \), which means it cannot be an element of \( M_S \) because \( M_S \) and \( E_{S'} \) are disjoint outside of \( S \). Therefore,

\[ U = \left[ \begin{array}{c}
(S \cup S') \cup \\
(M_S \cap \text{MissExt}(S, S', O')) \cup \\
(M_{S'} \cap \text{MissExt}(S', S, O)) \cup \\
\text{MissExt}(E_{S'}, S, O) \cap \text{MissExt}(E_{S'}, S', O')
\end{array} \right] \setminus \\
[\text{Unc}(S', M_S, O, O') \cup \text{Unc}(S, M_{S'}, O', O)]. \tag{17} \]
We can write

\[
\begin{align*}
\text{MissExt}(E_{S'}, S, O) \cap \text{MissExt}(E_{S}, S', O') &= \\
\text{MissExt}(E_{S'}, S, O) \cap \text{MissExt}(S, S', O') \cup \\
\text{MissExt}(S', S, O) \cap \text{MissExt}(E_{S}, S', O') \cup \\
\left( \text{MissExt}(\text{MissExt}(S', S', O'), S, O) \cap \\ 
\text{MissExt}((\text{MissExt}(S, S, O), S', O')) \right).
\end{align*}
\]

Note that MissExt\((E_{S'}, S, O) \cap \text{MissExt}(S, S', O') = \text{MissExt}(S, S, O) \cap \text{MissExt}(S, S', O')\). Hence

\[
\begin{align*}
(M_S \cap \text{MissExt}(S, S', O')) &\cup \\
(\text{MissExt}(E_{S'}, S, O) \cap \text{MissExt}(S, S', O')) \\
= \text{MissExt}(S, S', O') \cap (M_S \cup \text{MissExt}(E_{S'}, S, O)) \\
= \text{MissExt}(S, S', O') \cap (M_S \cup \text{MissExt}(S, S, O)) \\
= \text{MissExt}(S, S', O').
\end{align*}
\]

Finally, we observe that the set MissExt\((\text{MissExt}(S', S', O'), S, O) \cap \text{MissExt}(M_S, S, O)\) must be empty since the words of the first term have prefixes in \(S \setminus S'\), and the second in \(S' \setminus S\). These considerations allow us to conclude that

\[
U \leq \left[ (S \cap S') \cup \text{MissExt}(S, S', O') \cup \text{MissExt}(S', S, O) \right] \setminus [\text{Unc}(S', M_S, O, O') \cup \text{Unc}(S, M_{S'}, O', O)].
\]

To simplify the expression a step further, suppose \(w \in \text{Unc}(S', M_S, O, O')\) and has a prefix in \(S' \cap (M_S \setminus S)\). Then \(w \not\in S \cap S'\). The words of MissExt\((S, S', O')\) do not have prefixes in \(S' \setminus S\), so \(w \notin \text{MissExt}(S, S', O')\). The words of MissExt\((S', S, O)\) belong to \(E_S\), which is disjoint from \(M_S\) outside of \(S\). Thus, \(w \notin \text{MissExt}(S', S, O)\).

We just learned that the words of Unc\((S', M_S, O, O')\) having a prefix in \(S' \cap (M_S \setminus S)\) are irrelevant for the inequality above. Now consider a word \(w\) of Unc\((S', M_S, O, O')\) with no prefix in \(S' \cap (M_S \setminus S)\). Let \(w_p\) be the longest prefix of \(w\) which is in \(S \cap S'\). There is a word \(w' \in (O \cup O')^*\) and a symbol \(\sigma \in O\) such that \(w_p \circ w' \in S' \cap M_S\) and \(w_p \circ w' \circ \sigma \in M_S \setminus S'\). Suppose \(w'\) is not the empty string. Then we can let \(\sigma'\) be the first symbol of \(w'\). Then \(w_p \circ \sigma' \in M_S \setminus S\), so \(\sigma' \in O'\). But this means that \(w \in \text{Unc}(S, S', O', O)\). If \(w'\) is empty, \(w_p \in S \cap S'\) and \(w_p \circ \sigma' \in M_S \setminus S'\). Since \(\sigma \in O\), \(w_p \circ \sigma \in \cap M_S\) if and only if it belongs to \(S\). Thus, \(w_p \circ \sigma' \in S \setminus S'\), which means that \(w \in \text{Unc}(S, S, O, O')\). We can thus
simplify the upper bound on $E_R$ to

$$U = \left[ (S \cap S') \cup \text{MissExt}(S, S', O') \cup \text{MissExt}(S', S, O) \right] \setminus \left[ \text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O) \right].$$  

(18)

Define $\hat{R} \overset{\text{def}}{=} (S \cap S') \setminus [\text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O)]$. We want to show that $U = \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, O \cup O')$. Note that we only have to prove that

$$\text{MissExt}(\hat{R}, \hat{R}, O \cup O') = \left[ \text{MissExt}(S, S', O') \cup \text{MissExt}(S', S, O) \right] \setminus \left[ \text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O) \right].$$  

(19)

Proof. Suppose $w \in \text{MissExt}(S, S', O') \setminus \text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O)]$. Write $w = w_p \circ \sigma \circ w'$, where $w_p$ is the longest prefix of $w$ which lies in $S \cap S'$, $\sigma \in O'$, and $w' \in \Sigma^*$. $w_p \notin \text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O]$ because all its extensions would be in this set if $w_p$ were in this set, and we know that $w$ is not in this set. It follows that $w_p \in \hat{R}$ and since $w_p \circ \sigma \notin \hat{R}$, $w_p \circ \sigma$ and all its extensions are in $\hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, O \cup O')$. Thus, $w \in \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, O \cup O')$

The same argument applies when $w \in \text{MissExt}(S', S, O) \setminus \text{Unc}(S', S, O, O') \cup \text{Unc}(S, S', O', O)]$. We conclude that the right hand side of (19) is a subset of the left hand side.

Now suppose that $w \in \text{MissExt}(\hat{R}, \hat{R}, O \cup O')$ and write $w = w_p \circ \sigma \circ w'$, where $w_p$ is the longest prefix of $w$ contained in $\hat{R}$, $\sigma \in O \cup O'$, and $w' \in \Sigma^*$. From the definition of $\hat{R}$, $w_p \circ \sigma \in S \cap S'$. Suppose $w_p \circ \sigma \in S \cap S'$. Then $w_p \circ \sigma \in \text{Unc}(S', S, O, O') \cup 
\text{Unc}(S, S', O', O)]$, which means that $w_p$ also belongs to this set (because $\sigma \in O \cup O'$). This contradicts the fact that $w_p \in \hat{R}$, so our assumption that $w_p \circ \sigma \in S \cap S'$ is wrong. Then $w_p$ is also the longest prefix of $w$ contained in $S \cap S'$.

Without loss of generality, assume $\sigma \in O$. Suppose $w_p \circ \sigma \notin S$. Then $w \in \text{MissExt}(S', S, O)$. Moreover, since $w_p \in \hat{R}$, $w_p \circ \sigma \notin [\text{Unc}(S', S, O, O') \cup 
\text{Unc}(S, S', O', O)].$ Since $w_p \circ \sigma \notin S \cap S'$, we have $w \notin [\text{Unc}(S', S, O, O') \cup 
\text{Unc}(S, S', O', O)].$ Thus, $w$ is in the right hand set of (19).

Now suppose $w_p \circ \sigma \notin S'$ and $w_p \circ \sigma \in S$. If $w_p \circ \sigma \in S$, then $w_p \in \text{Unc}(S', S, O, O')$, which contradicts the fact that $w_p \in \hat{R}$. We must have $w_p \circ \sigma \notin S$, which we already showed implies that $w$ is in the right hand set of (19).

An analogous reasoning applies to $\sigma \in O'$. We conclude that the right hand side of (19) is a subset of the left hand side, and this finishes the proof of their equality.
This result and \[15\] tell us that \(E_R \leq \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, O \cup O')\). Now we study the constraint \[16\]. We want to show that \(\hat{R}\) yields the tightest bound \(L \leq \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\) which also respects the bound \[15\].

**Proof.** Observe that \(L = (S' \cup \text{MissExt}(S', S', I')) \cap (S \cup \text{MissExt}(S, S, I))\). First we will show that \(L \subseteq \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\). Suppose \(w \in L\). Then \(w\) belongs to at least one of the sets (1) \(S \cap S'\), (2) \(S \cap \text{MissExt}(S', S', I')\), (3) \(S' \cap \text{MissExt}(S, S, I)\), or (4) \(\text{MissExt}(S, S, I) \cap \text{MissExt}(S', S', I')\). We analyze each case:

1. Suppose \(w \in S \cap S'\). If \(w \in \hat{R}\), then clearly \(w \in \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\).
   Suppose \(w \notin \hat{R}\). Then there is word \(w' \in (O \cup O')^*\) and either a symbol \(\sigma \in O\) such that \(w \circ w' \circ \sigma \in S' \setminus S\) or a symbol \(\sigma \in O'\) such that \(w \circ w' \circ \sigma \in S' \setminus S\).
   Write \(w = w_p \circ w''\) such that \(w''\) is the longest suffix of \(w\) which belongs to \(O \cup O'\). It follows that the last symbol of \(w_p\) is an element of \(I \cap I'\). Since \(w \notin \hat{R}\), neither does \(w_p\). This shows that for every word \(w_r \circ \sigma_r \in S \cap S'\) such that \(w_r \in \hat{R}\) but \(w_r \circ \sigma_r \notin \hat{R}\), we must have \(\sigma_r \in I \cap I'\).
   Let \(w_p'\) be the longest prefix of \(w_p\) which lies in \(\hat{R}\). By assumption, \(\hat{R}\) is not empty. If we write \(w = w_p' \circ w''\), the first symbol of \(w''\) is in \(I \cap I'\). Thus, \(w \in \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\).

2. Observe that \(\text{MissExt}(S', S', I') = \text{MissExt}(S' \cap \hat{R}, S', I') \cup \text{MissExt}(S', S', I' \cap S)\). Moreover, \(S \cap \text{MissExt}(S', S', I') \subseteq \text{MissExt}(S \cap S', S', I' \cap S) \subseteq \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\).
   Suppose \(w \in S \cap \text{MissExt}(S', S', I' \cap S)\). Then \(w \in \text{Unc}(S', S, O, O')\), so \(w \notin \hat{R}\). Let \(w_i\) be the longest prefix of \(w\) which lies in \(S \cap S'\). Then \(w_i \notin \hat{R}\), either.
   Let \(w_p\) be the longest prefix of \(w_i\) which is in \(\hat{R}\). Then \(w_i \in \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\), and therefore, so does \(w\).

3. If \(w \in S' \cap \text{MissExt}(S, S, I)\), an analogous reasoning applies.

4. Suppose \(w \in \text{MissExt}(S, S, I) \cap \text{MissExt}(S', S', I')\). If \(w\) has a prefix in \(S' \cap \text{MissExt}(S, S, I)\) or \(S \cap \text{MissExt}(S', S', I')\), then the reasoning of the last two points applies, and we have \(w \in \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\).
   Suppose \(w\) has no such a prefix, and write \(w = w_p \circ w'\), where \(w_p\) is the longest prefix of \(w\) which lies in \(S \cap S'\). Let \(\sigma\) be the first symbol of \(w'\). Then \(w_p \circ \sigma \in \text{MissExt}(S, S, I) \cap \text{MissExt}(S', S', I')\), which means that \(\sigma \in I \cap I'\). Thus, \(w \in \text{MissExt}(S \cap S', S \cap S', I \cap I') \subseteq \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\).

We have shown that \(L \subseteq \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\). Now suppose \(w \in \hat{R} \cup \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\). If \(w \in \hat{R}\) then clearly \(w \in S \cap S' \subseteq L\). Suppose \(w \in \text{MissExt}(\hat{R}, \hat{R}, I \cap I')\) and let \(w_r\) be the longest prefix of \(w\) contained in \(\hat{R}\) and \(\sigma_r\) the symbol that comes immediately after \(w_r\) in \(w\). Clearly \(\sigma_r \in I \cap I'\).

If \(w_r \circ \sigma_r \in S \setminus S'\), then \(w_r \circ \sigma_r\) cannot be an element of \(\hat{R}\). If it were, we would have \(E_R \times S \not\subseteq E_S\), violating the bound \[15\]. The same applies when \(w_r \circ \sigma_r \in S' \setminus S\).

If \(w_r \circ \sigma_r \notin S \cup S'\), then \(w_r \circ \sigma_r \in \text{MissExt}(S', S', I') \cap \text{MissExt}(S, S, I) \subseteq L\).

If \(w_r \circ \sigma_r \in S \cap S'\), then \(w_r \circ \sigma_r \in \text{Unc}(S', S, O, O') \cup \text{Unc}(S', S, O', O)\), which means that \(w_r \circ \sigma_r\) is not allowed to be an element of \(\hat{R}\); otherwise, there would be a contradiction of \(15\).
We conclude that $R = \hat{R}$.

Proof (Proof of Proposition 8). Suppose that $A_1 \subseteq A_2$. We want to show that $M_{\ell(A_1)} \subseteq M_{\ell(A_2)}$ and $E_{\ell(A_2)} \subseteq E_{\ell(A_1)}$. We proceed by induction in the length $n$ of words, i.e., we will show that this relations hold for words of arbitrary length.

Consider the case $n = 1$. Suppose $\sigma \in M_{\ell(A_1)} \cap \Sigma$. If $\sigma \in I$, then $\sigma \in M_{\ell(A_2)}$ because of I-receptivity. If $\sigma \in O$, then $\sigma \in \ell(A)$, so there exists $q_1 \in Q_1$ such that $q_1,0 \xrightarrow{\sigma} q_1$, which means that there exists $q_2 \in Q_2$ such that $q_2,0 \xrightarrow{\sigma} q_2$. Thus, $\sigma \in \ell(A_2) \subseteq M_{\ell(A_2)}$. We have shown that $M_{\ell(A_1)} \subseteq M_{\ell(A_2)}$ for $n = 1$. An analogous reasoning shows that $E_{\ell(A_2)} \subseteq E_{\ell(A_1)}$.

Suppose the statement is true for words of length $l$. Let $w \circ \sigma \in M_{\ell(A_1)}$, where $w \in \Sigma^*$ is a word of length $n$, and $\sigma \in \Sigma$. By the inductive assumption, $w \in M_{\ell(A_2)}$.

– If $\sigma \in I$, then $w \circ \sigma \in M_{\ell(A_2)}$ due to I-receptiveness.

– Let $\sigma \in O$ and $w \notin \ell(A_1)$. Then we can write $w = w_p \circ w'$, where $w_p$ is the longest prefix of $w$ which lies in $\ell(A_1)$ (suppose it has length $l$). Let $\sigma'$ be the first symbol of $w'$; clearly $\sigma' \in I$. Since $w \in \ell(A_2)$ and this set is prefix-closed, $w_p \in \ell(A_2)$. Since $w_p \in \ell(A_1) \cap \ell(A_2)$, there exist $\{q_{j,i} \in Q_j\}_{j=1}^k$ such that $q_{j,i-1} \xrightarrow{w_i} q_{j,i}$ for $0 < i \leq k$, where $w_i$ is the $i$-th symbol of $w_p$. Since the IA are deterministic, we must have $q_{1,i} \leq q_{2,i}$. Suppose there were a $q_2 \in Q_2$ such that $q_{2,k} \xrightarrow{\sigma} q_2$; since $q_{1,k} \leq q_{2,k}$ and $\sigma' \in I$, this would mean that there exists $q_1 \in Q_1$ such that $q_{1,k} \xrightarrow{\sigma'} q_1$, which would mean that $w_p \circ \sigma' \in \ell(A_1)$, a contradiction. We conclude that such $q_2$ does not exist, which means that $w_p \circ \sigma' \notin \ell(A_2)$, which means that $w \circ \sigma \in M_{\ell(A_2)}$ because of I-receptiveness.

– Finally, if $\sigma \in O$ and $w \in \ell(A_1)$, then there exist $\{q_{i,j} \in Q_j\}_{i=1}^n$ such that $q_{1,i-1} \xrightarrow{w_i} q_{1,i}$ for $0 < i \leq n$, where $w_i$ is the $i$-th symbol of $w$. Since $w \circ \sigma \in M_{\ell(A_1)}$, $w \in \ell(A_1)$, and $\sigma \in O$, we must have $w \circ \sigma \in \ell(A_1)$. This means that there must exist $q_{1,n+1} \in Q_1$ such that $q_{1,n} \xrightarrow{\sigma} q_{1,n+1}$. We know that $w \in M_{\ell(A_2)}$ by the induction assumption. If $w \notin \ell(A_2)$, then clearly $w \circ \sigma \in M_{\ell(A_2)}$. If $w \in \ell(A_2)$, there are states $\{q_{2,i} \in Q_2\}_{i=1}^n$ such that $q_{2,i-1} \xrightarrow{w_i} q_{2,i}$ for $0 < i \leq n$. Moreover, there exists $q_{n+1} \in Q_1$ such that $q_n \xrightarrow{\sigma} q_{n+1}$ and $q_{1,n} \leq q_{2,n}$, there must be a $q_{2,n+1} \in Q_2$ such that $q_{2,n} \xrightarrow{\sigma} q_{2,n+1}$, which means that $w \circ \sigma \in M_{\ell(A_2)}$.

We have shown that $M_{\ell(A_1)} \subseteq M_{\ell(A_2)}$. An analogous argument proves that $E_{\ell(A_2)} \subseteq E_{\ell(A_1)}$.

Now suppose that $M_{\ell(A_1)} \subseteq M_{\ell(A_2)}$ and $E_{\ell(A_2)} \subseteq E_{\ell(A_1)}$. We want to show that $q_{1,0} \leq q_{2,0}$. We proceed by coinduction.

Let $n$ be a natural number. Suppose there exist sets $\{q_{j,i} \in Q_j\}_{i=1}^n$ with $j \in \{1,2\}$ such that $q_{j,i} \leq q_{2,i}$ for all $i$ and a word $w$ of length $n$ such that $q_{j,i-1} \xrightarrow{w_i} q_{j,i}$ for $0 < i \leq n$. Suppose there exists $q_{1,n+1} \in Q_1$ and $\sigma \in O$ such that $q_{1,n} \xrightarrow{\sigma} q_{1,n+1}$. Then $w \circ \sigma \in M_{\ell(A_1)} \subseteq M_{\ell(A_2)}$. Observe that $w \in \ell(A_2)$, so we must have $w \circ \sigma \in \ell(A_2)$, which means that there must be a $q_{2,n+1} \in Q_2$ such that $q_{2,n} \xrightarrow{\sigma} q_{2,n+1}$. We assume that $q_{1,n+1} \leq q_{2,n+1}$. Similarly, suppose there exists $q'_{2,n+1} \in Q_2$ and $\sigma \in I$ such that $q_{2,n} \xrightarrow{\sigma} q'_{2,n+1}$. Then $w \circ \sigma \in E_{\ell(A_2)} \subseteq E_{\ell(A_1)}$. Hypercontracts 31
Since $w \in \ell (A_1)$, we must have $w \circ \sigma \in \ell (A_1)$. Thus, there must exist $q'_{1,n+1} \in Q_1$ such that $q_{1,n} \xrightarrow{\sigma} q'_{1,n+1}$. We assume that $q_{1,n+1} \leq q_{2,n+1}$. This finished the coinductive proof.

Proof (Proof of Proposition 9). Let $A_i$ have IO signatures $(I_i, O_i)$ for $i \in \{1, 2\}$. For composition to be defined, we need $I_1 \cup I_2 = \Sigma$. Let $C_{A_i}$ be the interface contract associated with $A_i$. From Proposition 7 and Section 6.3, the composition $C_{A_1} \parallel C_{A_2}$ is isomorphic to $I_1 \cap I_2$ and the $L_\emptyset$ language $R = (\ell (A_1) \cap \ell (A_2)) \setminus \{\text{Unc}(\ell (A_1), \ell (A_2), O_2, O_1) \cup \text{Unc}(\ell (A_2), \ell (A_1), O_1, O_2)\}$. From Section 6.5, we deduce that $\ell (A_1 \parallel A_2) = R$. The proposition follows.