Forward-backward stochastic differential equations driven by
$G$-Brownian motion under weakly coupling condition

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Abstract. In this paper, we obtain the existence and uniqueness theorem of $L^p$-solution for coupled forward-backward stochastic differential equations driven by $G$-Brownian motion ($G$-FBSDEs) with arbitrary $T$ under weakly coupling condition. Specially, the result for $p \in (1, 2)$ is completely different from the one for $p \geq 2$. Furthermore, by considering the dual linear FBSDE under a suitable reference probability, we establish the comparison theorem for $G$-FBSDEs under weakly coupling condition.

Key words. $G$-expectation; $G$-Brownian motion; Backward stochastic differential equation; Comparison theorem

AMS subject classifications. 60H10

1 Introduction

The classical fully coupled forward-backward stochastic differential equation (FBSDE) has the following form

\begin{equation}
\begin{aligned}
    dX_t &= b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dW_t, \\
    dY_t &= f(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \\
    X_0 &= x_0, \quad Y_T = \phi(X_T),
\end{aligned}
\end{equation}

where $W$ is classical standard Brownian motion. There are many literatures to study the existence and uniqueness of the solution to FBSDE (1.1). Antonelli [1] first obtained the existence and uniqueness result by fixed point approach for small $T$. Ma et al. [18] introduced the four step scheme to first obtain the existence and uniqueness theorem for arbitrary $T$. Hu, Peng [13] and Yong [31] introduced the method of continuation to study FBSDE (1.1). Pardoux and Tang [21] obtained the existence and uniqueness theorem for arbitrary $T$ by fixed point approach under weakly coupling condition. For more results on this topic, the reader may refer to [4, 19, 25] and the references therein. The applications of the theory of FBSDEs in finance can be found in Ma and Yong’s book [20]. Wu [30] studied the comparison theorem for FBSDE (1.1) by duality method (see also [9, 10]).
Motivated by volatility uncertainty in finance (see [2, 17]), Peng [22, 23] introduced a type of consistent sublinear expectation, called the $G$-expectation $\hat{E}[\cdot]$. The related $G$-Brownian motion $B$ and Itô's calculus with respect to $B$ were constructed. Moreover, the theory of stochastic differential equation driven by $G$-Brownian motion ($G$-SDE) has been established.

Hu et al. [7] studied the backward stochastic differential equation driven by $G$-Brownian motion ($G$-BSDE). The theory of quadratic $G$-BSDE has been established in [12], and the wellposedness of a type of multi-dimensional $G$-BSDE can be found in [15]. Soner et al. [27] (see also [3]) studied a new type of fully nonlinear BSDE, called 2BSDE, by different formulation and method. The theory of 2BSDE with random terminal time has been obtained in [14].

Recently, Lu and Song [16], and Zheng [32] studied the following coupled forward-backward stochastic differential equation driven by $G$-Brownian motion ($G$-FBSDE):

\[
\begin{align*}
\frac{dX_t}{dt} &= b(t, X_t, Y_t)dt + h(t, X_t, Y_t)d\langle B \rangle_t + \sigma(t, X_t, Y_t)dB_t, \\
\frac{dY_t}{dt} &= f(t, X_t, Y_t, Z_t)dt + g(t, X_t, Y_t, Z_t)d\langle B \rangle_t + Z_tdB_t + dK_t, \\
X_0 &= x_0 \in \mathbb{R}^n, \ Y_T = \phi(X_T) \in \mathbb{R}.
\end{align*}
\]

(1.2)

By fixed point approach, they obtained that $G$-FBSDE (1.2) has a unique $L^2$-solution $(X, Y, Z, K)$ for small $T$. Wang and Yuan [29] studied the minimal solution of $G$-FBSDE (1.2) with monotone coefficients under the assumption that $\sigma(\cdot)$ is independent of $Y$ and $n = 1$.

In this paper, we first study the $L^p$-solution of $G$-FBSDE (1.2) for arbitrary $T$ under weakly coupling condition. By fixed point approach, we obtain that $G$-FBSDE (1.2) has a unique $L^p$-solution $(X, Y, Z, K)$ with $p \geq 2$ for arbitrary $T$ under weakly coupling condition. But for $p \in (1, 2)$, in order to get contractive mapping for $\hat{X}$, we need the assumption that $\sigma(\cdot)$ does not depend on $Y$. The key reason is that the Doob inequality for $G$-martingale (see [20, 28]) is different from the classical case and

\[
\left( \int_0^T |\hat{Y}_t|^2 dt \right)^{p/2} \leq C \int_0^T |\hat{Y}_t|^p dt
\]

does not hold for $p \in (1, 2)$.

It is well known that the comparison theorem plays an important role in the theory of BSDEs. So, the other purpose of this paper is to establish the comparison theorem for $G$-FBSDEs under weakly coupling condition. The key point to prove the comparison theorem is to solve the linear $G$-BSDE. Since the solvability of the dual linear $G$-FBSDE is unknown, we cannot use the method in [8] to prove the comparison theorem. In order to overcome this difficulty, we must choose a suitable reference probability $P^*$ and consider the dual linear FBSDE under $P^*$. The BSDE in this dual equation is different from the one in [11] and studied in [19]. By fixed point approach under weakly coupling condition, we can still obtain the solvability of this dual linear FBSDE under $P^*$. Based on this, we can further obtain the comparison theorem.

The paper is organized as follows. In Section 2, we recall some basic results of $G$-expectations, $G$-SDEs and $G$-BSDEs. The existence and uniqueness theorem, and the related estimates of $L^p$-solution for $G$-FBSDEs have been established in Section 3. In Section 4, we obtain the comparison theorem for $G$-FBSDEs.
2 Preliminaries

We recall some basic results of $G$-expectations, $G$-SDEs and $G$-BSDEs. The readers may refer to Peng’s book [24, 1] and [3] for more details.

Let $T > 0$ be given and let $\Omega_T = C_0([0,T];\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ with $\omega_0 = 0$. The canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0,T]$. For any fixed $t \leq T$, set

$$\text{Lip}(\Omega_t) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : N \geq 1, t_1 < \cdots < t_N \leq t, \varphi \in C_bLip(\mathbb{R}^{d \times N})\},$$

where $C_bLip(\mathbb{R}^{d \times N})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d \times N}$.

Let $G : S_d \to \mathbb{R}$ be a given monotonic and sublinear function, where $S_d$ denotes the set of $d \times d$ symmetric matrices. In this paper, we only consider non-degenerate $G$, i.e., there exists a $\gamma > 0$ such that

$$G(A) - G(B) \geq \frac{\gamma}{2} \text{tr}[A - B] \text{ for } A \geq B.$$

Peng [22, 23] constructed a consistent sublinear expectation space $(\Omega_T, \text{Lip}(\Omega_T), \hat{\mathcal{E}}, (\hat{\mathcal{E}}_t)_{t \in [0,T]})$, called $G$-expectation space, such that, for $0 \leq t < s \leq T$, $\xi_i \in \text{Lip}(\Omega_t)$, $i \leq m$, $\varphi \in C_bLip(\mathbb{R}^{m+d})$,

$$\hat{\mathcal{E}}_t[\varphi(\xi_1, \ldots, \xi_m, B_s - B_t)] = \psi(\xi_1, \ldots, \xi_m),$$

where $\psi(x_1, \ldots, x_m) = u(s - t, 0)$, $u$ is the solution of the following $G$-heat equation:

$$\partial_t u - G(D^2 u) = 0, \ u(0, x) = \varphi(x_1, \ldots, x_m, x).$$

The canonical process $(B_t)_{t \in [0,T]}$ is called the $G$-Brownian motion under $\hat{\mathcal{E}}$.

For each $t \in [0,T]$, denote by $L^1_{\mathcal{G}}(\Omega_t)$ the completion of $\text{Lip}(\Omega_t)$ under the norm $\|X\|_{L^1_{\mathcal{G}}} := (\hat{\mathcal{E}}[|X|^p])^{1/p}$ for $p \geq 1$. It is clear that $\hat{\mathcal{E}}_t$ can be continuously extended to $L^1_{\mathcal{G}}(\Omega_T)$ under the norm $\|\cdot\|_{L^1_{\mathcal{G}}}$.

**Definition 2.1** A process $(M_t)_{t \leq T}$ is called a $G$-martingale if $M_T \in L^1_{\mathcal{G}}(\Omega_T)$ and $\hat{\mathcal{E}}_t[M_T] = M_t$ for $t \leq T$.

The following theorem is the representation theorem of $G$-expectation.

**Theorem 2.2** (24, 11) There exists a unique weakly compact and convex set of probability measures $\mathcal{P}$ on $(\Omega_T, B(\Omega_T))$ such that

$$\hat{\mathcal{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for all } X \in L^1_{\mathcal{G}}(\Omega_T),$$

where $B(\Omega_T) = \sigma\{B_s : s \leq T\}$.

The capacity associated to $\mathcal{P}$ is defined by

$$c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for } A \in B(\Omega_T).$$

A set $A \in B(\Omega_T)$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s.

In order to study $G$-FBSDE, we need the following spaces and norms.

- $M^0(0,T) := \{\eta_t = \sum_{i=0}^{N-1} \xi_i I_{(t_i, t_{i+1})}(t) : N \in \mathbb{N}, 0 = t_0 < \cdots < t_N = T, \ \xi_i \in \text{Lip}(\Omega_{t_i})\}$.
For each \( \eta \in M^p_G(0, T) \) with \( p \geq 1 \), denote \( \eta = (\eta^1, \ldots, \eta^d)^T \in M^2_G(0, T; \mathbb{R}^d) \), the \( G \)-Itô integral \( \int_0^T \eta^i \, dB_i \) is well defined. Similar for \( L^p_G(\Omega; \mathbb{R}^n) \) and \( S^p_G(0, T; \mathbb{R}^n) \).

For simplicity of presentation, we suppose \( d = 1 \) throughout the paper. The results still hold for \( d > 1 \).

Under this case, the non-degenerate \( G \) is

\[
G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \bar{\sigma}^2 a^-) \quad \text{for} \quad a \in \mathbb{R},
\]

where \( 0 < \bar{\sigma} \leq \bar{\sigma} < \infty \). If \( \underline{\sigma} = \bar{\sigma} \), then \( \bar{\sigma}^{-1}B \) is a classical standard Brownian motion. So we suppose \( \underline{\sigma} < \bar{\sigma} \) in the following.

Let \( \langle B \rangle \) be the quadratic variation process of \( B \). By Corollary 3.5.5 in Peng [24], we have

\[
\underline{\sigma}^2 s \leq \langle B \rangle_{t+s} - \langle B \rangle_t \leq \bar{\sigma}^2 s \quad \text{for each} \quad t, s \geq 0.
\] (2.1)

Since \( B \) is a martingale under each \( P \in \mathcal{P} \), by Theorem 2.2 and the Burkholder-Davis-Gundy inequality, for each \( p > 0 \) and \( \| \eta \|_{M^p_G(0, T)} < \infty \), there exists a constant \( C(p) > 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \int_0^t \eta_s \, dB_s \right|^p \right] \leq C(p) \mathbb{E} \left[ \left( \int_0^T |\eta_s|^2 \, dB_s \right)^{p/2} \right] \leq \bar{\sigma}^p C(p) \mathbb{E} \left[ \left( \int_0^T |\eta_s|^2 \, ds \right)^{p/2} \right].
\] (2.2)

In the following, we consider the following \( G \)-FBSDE:

\[
\begin{align*}
\frac{dX_t}{dt} & = b(t, X_t, Y_t) dt + h(t, X_t, Y_t) d\langle B \rangle_t + \sigma(t, X_t, Y_t) dB_t, \\
\frac{dY_t}{dt} & = f(t, X_t, Y_t, Z_t) dt + g(t, X_t, Y_t, Z_t) d\langle B \rangle_t + Z_t dB_t + dK_t, \\
X_0 & = x_0 \in \mathbb{R}^n, \quad Y_T = \phi(X_T),
\end{align*}
\] (2.3)

where \( b, h, \sigma : [0, T] \times \Omega_T \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), \( f, g : [0, T] \times \Omega_T \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), \( \phi : \Omega_T \times \mathbb{R}^n \to \mathbb{R} \). We need the following assumptions:

\textbf{(H1)} There exists a \( \beta > 1 \) such that \( b(., ., y), \ h(., ., y) \in M^{1, \beta}_G(0, T; \mathbb{R}^n), \ \sigma(., ., y) \in M^{2, \beta}_G(0, T; \mathbb{R}^n), \ f(., ., y, z), \ g(., ., y, z) \in M^{1, \beta}_G(0, T) \) and \( \phi(x) \in L^p_G(\Omega_T) \) for each \( (x, y, z) \in \mathbb{R}^{n+2} \);
(H2) There exist constants $L_i > 0$, $i = 1, 2, 3$, such that, for each $t \leq T$, $\omega \in \Omega_T$, $x, x' \in \mathbb{R}^n$, $y, y', z, z' \in \mathbb{R}$,

$$|b_j(t, x, y) - b_j(t, x', y')| + |h_j(t, x, y) - h_j(t, x', y')| + |\sigma_j(t, x, y) - \sigma_j(t, x', y')|$$

$$\leq L_1|x - x'| + L_2|y - y'|$$, for $j = 1, \ldots, n$,

$$|f(t, x, y, z) - f(t, x', y', z')| + |g(t, x, y, z) - g(t, x', y', z')|$$

$$\leq L_3|x - x'| + L_4(|y - y'| + |z - z'|),$$

$$|\phi(x) - \phi(x')| \leq L_5|x - x'|,$$

where $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))^T$, $h(\cdot) = (h_1(\cdot), \ldots, h_n(\cdot))^T$, $\sigma(\cdot) = (\sigma_1(\cdot), \ldots, \sigma_n(\cdot))^T$.

Now we give the $L^p$-solution of G-FBSDE (2.3), similar for G-SDE and G-BSDE.

**Definition 2.3** For each fixed $p \in (1, \beta)$, $(X, Y, Z, K)$ is called an $L^p$-solution of G-FBSDE (2.3) if the following properties hold:

(i) $X \in S^p_G(0, T; \mathbb{R}^n)$, $Y \in S^p_G(0, T)$, $Z \in M^p_G(0, T)$, $K$ is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in L^p_G(\Omega_T)$;

(ii) $(X, Y, Z, K)$ satisfies G-FBSDE (2.3).

The following is the standard estimates of G-SDE and G-BSDE.

**Theorem 2.4** Suppose assumptions (H1) and (H2) hold. For each $p \in (1, \beta)$ and $(y^{(i)}_t)_{t \leq T} \in S^p_G(0, T)$, $i = 1, 2$. Let $(X^{(i)}_t)_{t \leq T} \in S^p_G(0, T; \mathbb{R}^n)$ be the solution of G-SDE

$$dX^{(i)}_t = b(t, X^{(i)}_t, y^{(i)}_t)dt + h(t, X^{(i)}_t, y^{(i)}_t)d(B)_t + \sigma(t, X^{(i)}_t, y^{(i)}_t)dB_t, \ X^{(i)}_0 = x_0,$$

for $i = 1, 2$. Then there exists a deterministic function $C_1(p, T, L_1, \bar{\sigma}) > 0$, which is continuous in $p$, such that

$$\mathbb{E} \left[ \sup_{t \leq T} |X^{(1)}_t - X^{(2)}_t|^p \right] \leq C_1(p, T, L_1, \bar{\sigma}) \mathbb{E} \left[ \left( \int_0^T |\dot{b}_t| + |\dot{h}_t|dt \right)^p + \left( \int_0^T |\dot{\sigma}_t|^2 dt \right)^{p/2} \right], \quad (2.4)$$

where $\dot{b}_t = b(t, X^{(2)}_t, y^{(1)}_t) - b(t, X^{(1)}_t, y^{(1)}_t)$, $\dot{h}_t = h(t, X^{(2)}_t, y^{(1)}_t) - h(t, X^{(1)}_t, y^{(1)}_t)$, $\dot{\sigma}_t = \sigma(t, X^{(2)}_t, y^{(1)}_t) - \sigma(t, X^{(1)}_t, y^{(1)}_t)$.

**Proof.** For the convenience of the reader, we sketch the proof. Set $\hat{X}_t = X^{(1)}_t - X^{(2)}_t$. For each given $t_0 \in [0, T]$ and $\delta > 0$, we have

$$\hat{X}_t = \hat{X}_{t_0} + \int_{t_0}^t \dot{\hat{b}}(s)ds + \int_{t_0}^t \dot{\hat{h}}(s)d(B)_s + \int_{t_0}^t \dot{\hat{\sigma}}(s)dB_s, \ t \in [t_0, t_0 + \delta],$$

where $|\dot{b}(s)| = |b(s, X^{(1)}_s, y^{(1)}_s) - b(s, X^{(2)}_s, y^{(2)}_s)| \leq nL_1|\hat{X}_s| + |\hat{b}_s|$, similarly, $|\dot{h}(s)| \leq nL_1|\hat{X}_s| + |\hat{h}_s|$, $|\dot{\sigma}(s)| \leq nL_1|\hat{X}_s| + |\hat{\sigma}_s|$. Then we get

$$\sup_{t \in [t_0, t_0 + \delta]} |\hat{X}_t|^p \leq 4^{p-1} \left( |\hat{X}_{t_0}|^p + \left( \int_{t_0}^{t_0 + \delta} |\dot{\hat{b}}(s)|ds \right)^p + \sigma^2 \left( \int_{t_0}^{t_0 + \delta} |\dot{\hat{h}}(s)|ds \right)^p + \sup_{t \in [t_0, t_0 + \delta]} \left( \int_{t_0}^{t} |\dot{\hat{\sigma}}(s)d(B)_s| \right)^p \right).$$
By (2.2), we can deduce
\[
\hat{E} \left[ \sup_{t \in [t_0, t_0 + \delta]} \left| \int_{t_0}^{t_0 + \delta} \tilde{\sigma}(s) dB_s \right|^p \right] \leq n^p \sigma^p C(p) \hat{E} \left[ \left( \int_{t_0}^{t_0 + \delta} |\tilde{\sigma}(s)|^2 ds \right)^{p/2} \right].
\]

It is easy to verify that
\[
\left( \int_{t_0}^{t_0 + \delta} |\tilde{b}(s)| ds \right)^p \leq 2^{p-1} \left( n L_1 \int_{t_0}^{t_0 + \delta} |\tilde{X}_s| ds \right)^p + \left( \int_{t_0}^{t_0 + \delta} |\tilde{b}_s| ds \right)^p
\]
\[
\leq 2^{p-1} (n L_1 \delta)^p \sup_{t \in [t_0, t_0 + \delta]} |\tilde{X}_t|^p + 2^{p-1} \left( \int_{t_0}^{t_0 + \delta} |\tilde{b}_s| ds \right)^p
\]
and
\[
\left( \int_{t_0}^{t_0 + \delta} |\tilde{\sigma}(s)|^2 ds \right)^{p/2} \leq 2^{p/2} \left[ 2^{p/2} \left( n^2 L_1^2 \int_{t_0}^{t_0 + \delta} |\tilde{X}_s|^2 ds \right)^{p/2} + \left( \int_{t_0}^{t_0 + \delta} |\tilde{\sigma}_s|^2 ds \right)^{p/2} \right]
\]
\[
\leq 2^p (n L_1)^p \delta^{p/2} \sup_{t \in [t_0, t_0 + \delta]} |\tilde{X}_t|^p + 2^p \left( \int_{t_0}^{t_0 + \delta} |\tilde{\sigma}_s|^2 ds \right)^{p/2}.
\]

Thus we obtain
\[
\hat{E} \left[ \sup_{t \in [t_0, t_0 + \delta]} |\tilde{X}_t|^p \right] \leq 4^{p-1} \hat{E} \left[ |\tilde{X}_{t_0}|^p \right] + \lambda_1(\delta) \hat{E} \left[ \sup_{t \in [t_0, t_0 + \delta]} |\tilde{X}_t|^p \right]
\]
\[
+ \lambda_2 \hat{E} \left[ \left( \int_0^T (|\tilde{b}_t| + |\tilde{h}_t|) dt \right)^p + \left( \int_0^T |\tilde{\sigma}_t|^2 dt \right)^{p/2} \right],
\]
where
\[
\lambda_1(\delta) = 8^{p-1} \left( 1 + \bar{\sigma}^{2p} (n L_1 \delta)^p + 2 C(p) (L_1 n^2 \bar{\sigma})^p \delta^{p/2} \right), \quad \lambda_2 = 8^{p-1} \left[ 1 + \bar{\sigma}^{2p} + 2 C(p) (n \bar{\sigma})^p \right].
\]
Choosing \( \delta_0 > 0 \) such that \( \lambda_1(\delta_0) = 0.75 \), then, for \( \delta \leq \delta_0 \wedge (T - t_0) \), we get
\[
\hat{E} \left[ \sup_{t \in [t_0, t_0 + \delta]} |\tilde{X}_t|^p \right] \leq 4^p \hat{E} \left[ |\tilde{X}_{t_0}|^p \right] + 4 \lambda_2 \hat{E} \left[ \left( \int_0^T (|\tilde{b}_t| + |\tilde{h}_t|) dt \right)^p + \left( \int_0^T |\tilde{\sigma}_t|^2 dt \right)^{p/2} \right].
\]

Thus we can deduce
\[
\hat{E} \left[ \sup_{T' \leq T} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right] \leq C_1(p, T, L_1, \sigma) \hat{E} \left[ \left( \int_0^T (|\tilde{b}_t| + |\tilde{h}_t|) dt \right)^p + \left( \int_0^T |\tilde{\sigma}_t|^2 dt \right)^{p/2} \right],
\]
where
\[
C_1(p, T, L_1, \sigma) = \frac{4 \lambda_2}{4^p - 1} \left( \frac{4^{p(T + 2\delta_0) / \delta_0} - 4^p}{4^p - 1} \frac{T}{\delta_0} \right).
\] (2.5)

It is easy to check that \( C_1(p, T, L_1, \sigma) \) is continuous in \( p \). □
Remark 2.5 If \( p \geq 2 \), then
\[
\left( \int_{t_0}^{t_0 + \delta} |\dot{X}_s|^2 ds \right)^{p/2} \leq \delta^{(p-2)/2} \int_{t_0}^{t_0 + \delta} |\dot{X}_s|^p ds \leq \delta^{(p-2)/2} \sup_{t \in [t_0, t_0 + \delta]} |\dot{X}_t|^p ds.
\]

Taking \( t_0 = 0 \) and \( \delta = T \) in the proof of Theorem 2.4, we obtain
\[
\mathbb{E} \left[ \sup_{t \leq T} |\dot{X}_t|^p \right] \leq \lambda_3 \int_0^T \mathbb{E} \left[ \sup_{s \leq T} |\dot{X}_s|^p \right] ds + \lambda_4 \mathbb{E} \left[ \left( \int_0^T (|\dot{b}_t| + |\dot{g}_t|) dt \right)^p + \left( \int_0^T |\dot{\sigma}_t|^2 dt \right)^{p/2} \right],
\]
where
\[
\lambda_3 = 6^{p-1} \left[ 1 + \bar{a}^2 p \right] (\mathbb{E} T)^{p-1} + 2C(p)(\mathbb{E} T)^{p-1} \left( \int_0^T (|\dot{b}_t| + |\dot{g}_t|) dt \right)^p + \left( \int_0^T |\dot{\sigma}_t|^2 dt \right)^{p/2}
\]
\[
\lambda_4 = 6^{p-1} \left[ 1 + \bar{a}^2 p \right] \mathbb{E} T.
\]

By the Gronwall inequality, we get
\[
C_1(p, T, L_1, \bar{a}) = e^{\lambda_3 T} \lambda_4. \tag{2.6}
\]

The following theorem is Propositions 3.8 and 5.1 in [7].

Theorem 2.6 Suppose assumptions (H1) and (H2) hold. For each \( p \in (1, \beta) \) and \( (x_i^{(i)})_{t \leq T} \in S_{L_p}(0, T; \mathbb{R}^n) \), \( i = 1, 2 \). Let \( (Y_i^{(i)}, Z_i^{(i)}, K_i^{(i)})_{t \leq T} \) be the \( L_p \)-solution of G-BSDE
\[
dY_i^{(i)} = f(t, x_i^{(i)}, Y_i^{(i)}, Z_i^{(i)}) dt + g(t, x_i^{(i)}, Y_i^{(i)}, Z_i^{(i)}) dB_t + dK_i^{(i)}, \quad Y_T^{(i)} = \phi(x_T^{(i)}),
\]
for \( i = 1, 2 \). Then

(i) there exists a deterministic function \( C_2(p, T, L_1, \bar{a}, \underline{a}) > 0 \), which is continuous in \( p \), such that
\[
|\bar{Y}_t| \leq C_2(p, T, L_1, \bar{a}, \underline{a}) \mathbb{E} \mathbb{E} \left[ \left( \int_0^T |\bar{b}_t| + |\bar{g}_t| dt \right)^{p/2} \right] + \lambda_3 \mathbb{E} \left[ \sup_{t \leq T} |\bar{Y}_t|^p \right] \]
where \( \bar{Y}_t = Y_t^{(1)} - Y_t^{(2)} \), \( \bar{b}_t = \phi(x_T^{(1)}) - \phi(x_T^{(2)}) \), \( \bar{f}_t = f(s, x_s^{(1)}, Y_s^{(2)}, Z_s^{(2)}) - f(s, x_s^{(2)}, Y_s^{(2)}, Z_s^{(2)}) \), \( \bar{g}_s = g(s, x_s^{(1)}, Y_s^{(2)}, Z_s^{(2)}) - g(s, x_s^{(2)}, Y_s^{(2)}, Z_s^{(2)}) \).

(ii) there exists a deterministic function \( C_3(p, T, L_1, \bar{a}, \underline{a}) > 0 \) such that
\[
\mathbb{E} \left[ \left( \int_0^T |\bar{Z}_t|^2 dt \right)^{p/2} \right] \leq C_3(p, T, L_1, \bar{a}, \underline{a}) \left\{ \mathbb{E} \left[ \sup_{t \leq T} |Y_t^{(i)}|^p \right] + (\lambda_1 + \lambda_2)^{1/2} \left( \mathbb{E} \left[ \sup_{t \leq T} |\bar{Y}_t|^p \right] \right)^{1/2} \right\},
\]
where \( \bar{Z}_t = Z_t^{(1)} - Z_t^{(2)} \),
\[
\lambda_i = \mathbb{E} \left[ \sup_{t \leq T} |Y_t^{(i)}|^p \right] + \mathbb{E} \left[ \left( \int_0^T |f(s, x_s^{(i)}, 0, 0) + g(s, x_s^{(i)}, 0, 0))| ds \right)^p \right] \quad \text{for} \quad i = 1, 2.
\]

Remark 2.7 According to the proof of Proposition 5.1 in [7], we can deduce
\[
C_2(p, T, L_1, \bar{a}, \underline{a}) = 2^{p-1} \left[ 1 + (1 + \bar{a}^2)^p L_1 (1 + \bar{a}^2) T \right] e^{\lambda_3 T}, \tag{2.7}
\]
where
\[
\lambda_3 = pL_1 (1 + \bar{a}^2) + \frac{1}{2} pL_1^2 \bar{a}^2 (1 + \bar{a}^{-2})^2 [(p - 1)^{-1} \lor 1].
\]
3 Existence and uniqueness of $L^p$-solution for $G$-FBSDEs

For simplicity, we use $C_1(p)$ and $C_2(p)$ instead of $C_1(p,T,L_1,\sigma)$ and $C_2(p,T,L_1,\sigma,\bar{\sigma})$ respectively in the following. The first main result in this section is the existence and uniqueness of $L^p$-solution for $G$-FBSDE \eqref{eq:G-FBSDE} with $p \geq 2$.

**Theorem 3.1** Suppose assumptions (H1) and (H2) hold. If $\beta > 2$ and

$$
\Lambda_p := C_1(p)C_2(p)(nL_2L_3)^p(T^p + T^{p/2})(1 + T)^p < 1 
$$

for some $p \in [2, \beta)$, then $G$-FBSDE \eqref{eq:G-FBSDE} has a unique $L^p$-solution $(X,Y,Z,K)$.

**Proof.** We first prove the uniqueness. Let $(X,Y,Z,K)$ and $(X',Y',Z',K')$ be two $L^p$-solutions of $G$-FBSDE \eqref{eq:G-FBSDE}. Set

$$
\hat{X}_t = X_t - X'_t, \quad \hat{Y}_t = Y_t - Y'_t, \quad \hat{Z}_t = Z_t - Z'_t \quad \text{for } t \in [0,T].
$$

By Theorem 2.6 we obtain

$$
\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right] \leq C_1(p)\mathbb{E}\left[\left(\int_0^T (|\hat{b}_t| + |\hat{h}_t|)dt\right)^p + \left(\int_0^T |\hat{\sigma}_t|^2dt\right)^{p/2}\right].
$$

(3.2)

where $\hat{b}_t = b(t, X'_t, Y_t) - b(t, X'_t, Y'_t)$, $\hat{h}_t = h(t, X'_t, Y_t) - h(t, X'_t, Y'_t)$, $\hat{\sigma}_t = \sigma(t, X'_t, Y_t) - \sigma(t, X'_t, Y'_t)$. It follows from (H2) that

$$
|\hat{b}_t| + |\hat{h}_t| + |\hat{\sigma}_t| \leq nL_2|\hat{Y}_t|.
$$

Thus we get

$$
\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right] \leq C_1(p)(nL_2)^p(T^{p-1} + T^{(p-2)/2}) \int_0^T \mathbb{E}[|\hat{Y}_t|^p]dt.
$$

(3.3)

By (i) of Theorem 2.6 we obtain

$$
|\hat{Y}_t|^p \leq C_2(p)\mathbb{E}_t\left[\left(|\hat{\phi}_T| + \int_t^T (|\hat{f}_s| + |\hat{g}_s|)ds\right)^p\right],
$$

where $\hat{\phi}_T = \phi(X_T) - \phi(X'_T)$,

$$
\hat{f}_s = f(s, X_s, Y_s', Z'_s) - f(s, X'_s, Y'_s, Z'_s), \quad \hat{g}_s = g(s, X_s, Y_s', Z'_s) - g(s, X'_s, Y'_s, Z'_s).
$$

From (H2), we have

$$
|\hat{\phi}_T| \leq L_3|\hat{X}_T|, \quad |\hat{f}_s| + |\hat{g}_s| \leq L_3|\hat{X}_s|.
$$

Then we deduce

$$
\mathbb{E}[|\hat{Y}_t|^p] \leq C_2(p)L_3^p(1 + T)^p\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right].
$$

(3.4)

It follows from (3.1), (3.3) and (3.4) that

$$
\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right] \leq \Lambda_p\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right],
$$

$$
\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right] \leq \Lambda_p\mathbb{E}\left[\sup_{t \leq T} |\hat{X}_t|^p\right],
$$

on 8
which implies \(\mathbb{E}\left[\sup_{t\leq T} |\hat{X}_t|^p\right] = 0\). Then, by (3.4), we obtain \(\hat{Y}_t = 0\) q.s. Since \(\hat{Y}_t\) is continuous in \(t\), we can deduce

\[
\sup_{t\leq T} |\hat{Y}_t|^p = 0 \text{ q.s.,}
\]

which implies \(\mathbb{E}\left[\sup_{t\leq T} |\hat{Y}_t|^p\right] = 0\). From (ii) of Theorem 2.6 we get

\[
\mathbb{E}\left[\left(\int_0^T |\hat{Z}_t|^2 \, dt\right)^{p/2}\right] = 0,
\]

which implies \(K = K'\) by G-FBSDE (2.3). Thus the \(L^p\)-solution of G-FBSDE (2.3) is unique.

Now we prove the existence. Set \(X_t^{(0)} = x_0\) for \(t \leq T\). Define \((X^{(m)}, Y^{(m)}, Z^{(m)}, K^{(m)}), m \geq 1\), as follows:

\[
\begin{align*}
&dX_t^{(m)} = b(t, X_t^{(m)}, Y_t^{(m)})dt + h(t, X_t^{(m)}, Y_t^{(m)})d(B)_t + \sigma(t, X_t^{(m)}, Y_t^{(m)})dB_t, \\
&dY_t^{(m)} = f(t, X_t^{(m-1)}, Y_t^{(m-1)}, Z_t^{(m-1)})dt + g(t, X_t^{(m-1)}, Y_t^{(m-1)}, Z_t^{(m-1)})d(B)_t + Z_t^{(m)}dB_t + dK_t^{(m)}, \\
&X_0^{(m)} = x_0 \in \mathbb{R}^n, Y_0^{(m)} = \phi(X_0^{(m-1)}).
\end{align*}
\]

For \(m = 1\), we first slove G-BSDE in (3.5) to get \((Y^{(1)}, Z^{(1)}, K^{(1)})\). Since \(X_t^{(0)} \in S^2_G(0, T; \mathbb{R}^n)\) for each \(\alpha < \beta\), we obtain

\[
Y^{(1)} \in S^2_G(0, T), \quad Z^{(1)} \in M^{2, \alpha}_G(0, T), \quad K^{(1)} \in L^2_G(\Omega_T),
\]

for each \(\alpha < \beta\) by Theorem 4.1 in 2.4. We then slove G-SDE in (3.5) to get \(X^{(1)}\). Obviously, \(X^{(1)} \in S^2_G(0, T; \mathbb{R}^n)\) for each \(\alpha < \beta\) by Theorem 2.4. Continuing this process, we can get

\[
X^{(m)} \in S^2_G(0, T; \mathbb{R}^n), \quad Y^{(m)} \in S^2_G(0, T), \quad Z^{(m)} \in M^{2, \alpha}_G(0, T), \quad K^{(m)} \in L^2_G(\Omega_T),
\]

for each \(\alpha < \beta\) and \(m \geq 1\). Since \(\Lambda_p\) is continuous in \(p\) and \(\Lambda_p < 1\), there exists a \(p' \in (p, \beta)\) such that \(\Lambda_{p'} < 1\). Set

\[
\check{X}^{(m)} = X^{(m)} - X^{(m-1)} \text{ for } m \geq 1, \quad \check{Y}^{(m)} = Y^{(m)} - Y^{(m-1)} \text{ and } \check{Z}^{(m)} = Z^{(m)} - Z^{(m-1)} \text{ for } m \geq 2.
\]

By Theorem 2.4 we get, for \(m \geq 2\),

\[
\mathbb{E}\left[\sup_{t\leq T} |\hat{X}_t^{(m)}|^p\right] \leq C_1(p')\mathbb{E}\left[\left(\int_0^T (|\hat{h}_t^{(m)}| + |\hat{\sigma}_t^{(m)}|)\, dt\right)^{p'} + \left(\int_0^T |\hat{\sigma}_t^{(m)}|^2 \, dt\right)^{p' \slash 2}\right],
\]

where \(\hat{h}_t^{(m)} = b(t, X_t^{(m-1)}, Y_t^{(m)}) - b(t, X_t^{(m-1)}, Y_t^{(m-1)})\)

\(\hat{\sigma}_t^{(m)} = \sigma(t, X_t^{(m-1)}, Y_t^{(m-1)}) - \sigma(t, X_t^{(m-1)}, Y_t^{(m-1)})\). Similar to the proof of (3.1), we obtain

\[
\mathbb{E}\left[\sup_{t\leq T} |\hat{X}_t^{(m)}|^p\right] \leq C_1(p')(nL_2^p) (T^{p'-1} + T^{(p'-2)/2}) \int_0^T \mathbb{E}[|\hat{Y}_t^{(m)}|^{p'}] \, dt. \tag{3.6}
\]

It follows from (i) of Theorem 2.6 that, for \(m \geq 2\),

\[
|\hat{Y}_t^{(m)}|^{p'} \leq C_2(p') \mathbb{E}_t\left[\left(\int_t^T (|\hat{f}_s^{(m)}| + |\hat{g}_s^{(m)}|)\, ds\right)^{p'}\right],
\]

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where \( \hat{\phi}_{T}^{(m)} = \phi(X_{T}^{(m-1)}) - \phi(X_{T}^{(m-2)}) \),

\[
\hat{f}_{s}^{(m)} = f(s, X_{s}^{(m-1)}, Y_{s}^{(m-1)}, Z_{s}^{(m-1)}) - f(s, X_{s}^{(m-2)}, Y_{s}^{(m-1)}, Z_{s}^{(m-1)}),
\]

\[
\hat{g}_{s}^{(m)} = g(s, X_{s}^{(m-1)}, Y_{s}^{(m-1)}, Z_{s}^{(m-1)}) - g(s, X_{s}^{(m-2)}, Y_{s}^{(m-1)}, Z_{s}^{(m-1)}).
\]

Similar to the proof of (3.4), we get

\[
E \left[ \sup_{s \leq T} |X_{s}^{(m-1)}|^{p'} \right] \leq C_{2}(p')L_{3}^{p}(1 + T)^{p'} \sup_{s \leq T} |X_{s}^{(m-1)}|^{p'}.
\]

(3.7)

By (3.6) and (3.7), we deduce

\[
E \left[ \sup_{s \leq T} |X_{s}^{(m)}|^{p'} \right] \leq C_{2}(p')L_{3}^{p}(1 + T)^{p'} \sup_{s \leq T} |X_{s}^{(m-1)}|^{p'}
\]

which implies

\[
E \left[ \sup_{s \leq T} |X_{s}^{(m)}|^{p'} \right] \leq C_{2}(p')L_{3}^{p}(1 + T)^{p'} \sup_{s \leq T} |X_{s}^{(1)}|^{p'}
\]

for \( m \geq 1 \).

For each \( N, k \geq 1 \), we obtain

\[
\left( E \left[ \sup_{s \leq T} |X_{s}^{(N+k)} - X_{s}^{(N)}|^{p'} \right] \right)^{1/p'} \leq \sum_{m=N+1}^{\infty} \left( E \left[ \sup_{s \leq T} |X_{s}^{(m)}|^{p'} \right] \right)^{1/p'}
\]

\[
\leq \left( 1 - \Lambda_{p'}^{1/p'} - \Lambda_{p'}^{N/p'} \left( E \left[ \sup_{s \leq T} |X_{s}^{(1)}|^{p'} \right] \right)^{1/p'} \right)^{1/p'},
\]

which tends to 0 as \( N \to \infty \). Thus there exists a \( X \in S_{G}^{p}(0, T; \mathbb{R}^{m}) \) such that

\[
E \left[ \sup_{s \leq T} |X_{s}^{(m)} - X_{s}^{(1)}|^{p'} \right] \to 0 \text{ as } m \to \infty.
\]

(3.8)

For each \( N, k \geq 1 \), similar to the proof of (3.4), we can deduce

\[
E \left[ \sup_{s \leq T} \left| Y_{s}^{(N+k)} - Y_{s}^{(N)} \right|^{p'} \right] \leq C_{2}(p')L_{3}^{p}(1 + T)^{p'} E \left[ \sup_{s \leq T} \left| X_{s}^{(N+k-1)} - X_{s}^{(N-1)} \right|^{p'} \right].
\]

(3.9)

By Doob’s inequality for \( G \)-martingale (see [26, 28]), we have

\[
E \left[ \sup_{s \leq T} \left( E \left[ \sup_{s \leq T} \left| X_{s}^{(N+k-1)} - X_{s}^{(N-1)} \right|^{p'} \right] \right)^{p/p'} \right] \leq \left( E \left[ \sup_{s \leq T} \left| X_{s}^{(N+k-1)} - X_{s}^{(N-1)} \right|^{p'} \right] \right)^{p/p'}.
\]

(3.10)

It follows from (3.8), (3.9) and (3.10) that

\[
E \left[ \sup_{s \leq T} \left| Y_{s}^{(N+k)} - Y_{s}^{(N)} \right|^{p'} \right] \to 0 \text{ as } N \to \infty.
\]

Thus there exists a \( Y \in S_{G}^{p}(0, T) \) such that

\[
E \left[ \sup_{s \leq T} \left| Y_{s}^{(m)} - Y_{s}^{(1)} \right|^{p'} \right] \to 0 \text{ as } m \to \infty.
\]

(3.11)
Noting that \( \sup_{m \geq 1} \mathbb{E}\left[ \left| X_{t}^{(m)} \right| + \left| Y_{t}^{(m)} \right| \right] < \infty \), by (ii) of Theorem \( 2.6 \) we get
\[
\mathbb{E}\left[ \left( \int_{0}^{T} |Z_{t}^{(N+k)} - Z_{t}^{(N)}|^{2} dt \right)^{p/2} \right] \to 0 \text{ as } N \to \infty.
\]
Thus there exists a \( Z \in M_{G}^{2,p}(0,T) \) such that
\[
\mathbb{E}\left[ \left( \int_{0}^{T} |Z_{t}^{(m)} - Z_{t}|^{2} dt \right)^{p/2} \right] \to 0 \text{ as } m \to \infty. \tag{3.12}
\]
From \( 2.2 \), we obtain
\[
\mathbb{E}\left[ \sup_{t \leq T} \left| \int_{t}^{T} Z_{s}^{(m)} dB_{s} - \int_{t}^{T} Z_{s} dB_{s} \right|^{p} \right] \leq 2p\mathbb{E}\left[ \sup_{t \leq T} \left| \int_{0}^{t} Z_{s}^{(m)} dB_{s} - \int_{0}^{t} Z_{s} dB_{s} \right|^{p} \right]
\leq 2p\sigma^{p}C(p)\mathbb{E}\left[ \left( \int_{0}^{T} |Z_{t}^{(m)} - Z_{t}|^{2} dt \right)^{p/2} \right] \to 0 \text{ as } m \to \infty.
\]
Since
\[
\sup_{t \leq T} \left| \int_{t}^{T} f(s, X_{s}^{(m-1)}, Y_{s}^{(m)}, Z_{s}^{(m)}) ds - \int_{t}^{T} f(s, X_{s}, Y_{s}, Z_{s}) ds \right|^{p} \leq \left( \int_{0}^{T} |f(s, X_{s}^{(m-1)}, Y_{s}^{(m)}, Z_{s}^{(m)}) - f(s, X_{s}, Y_{s}, Z_{s})| ds \right)^{p} \leq 3^{p-1}L_{1}^{p}p^{p} \sup_{s \leq T} |X_{s}^{(m-1)} - X_{s}|^{p} + 3^{p-1}L_{1}^{p}T^{p} \sup_{s \leq T} \left| Y_{s}^{(m)} - Y_{s} \right|^{p} + 3^{p-1}L_{1}^{p}T^{p/2} \left( \int_{0}^{T} |Z_{s}^{(m)} - Z_{s}|^{2} ds \right)^{p/2},
\]
we get
\[
\mathbb{E}\left[ \sup_{t \leq T} \left| \int_{t}^{T} f(s, X_{s}^{(m-1)}, Y_{s}^{(m)}, Z_{s}^{(m)}) ds - \int_{t}^{T} f(s, X_{s}, Y_{s}, Z_{s}) ds \right|^{p} \right] \to 0
\]
as \( m \to \infty \) by \( 3.8 \), \( 3.11 \) and \( 3.12 \). Similarly, we can obtain
\[
\mathbb{E}\left[ \sup_{t \leq T} \left| \int_{t}^{T} g(s, X_{s}^{(m-1)}, Y_{s}^{(m)}, Z_{s}^{(m)}) dB_{s} - \int_{t}^{T} g(s, X_{s}, Y_{s}, Z_{s}) dB_{s} \right|^{p} \right] \to 0,
\]
\[
\mathbb{E}\left[ \sup_{t \leq T} \left( \int_{0}^{t} \left| b(s, X_{s}^{(m)}, Y_{s}^{(m)}) - b(s, X_{s}, Y_{s}) \right| ds + \left| \int_{0}^{t} \left( h(s, X_{s}^{(m)}, Y_{s}^{(m)}) - h(s, X_{s}, Y_{s}) \right) dB_{s} \right| \right)^{p} \right] \to 0
\]
and
\[
\mathbb{E}\left[ \sup_{t \leq T} \left| \int_{0}^{t} \left( \sigma(s, X_{s}^{(m)}, Y_{s}^{(m)}) - \sigma(s, X_{s}, Y_{s}) \right) dB_{s} \right|^{p} \right] \to 0
\]
as \( m \to \infty \). Set
\[
K_{t} = Y_{t} - Y_{0} - \int_{0}^{t} f(s, X_{s}, Y_{s}, Z_{s}) ds - \int_{0}^{t} g(s, X_{s}, Y_{s}, Z_{s}) dB_{s} - \int_{0}^{t} Z_{s} dB_{s}
\]

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for $t \in [0,T]$. It is clear that
\[ \mathbb{E} \left[ \sup_{t \leq T} \left| K_t^{(m)} - K_t^{(p)} \right|^p \right] \rightarrow 0 \text{ as } m \rightarrow \infty. \]
Thus we can easily deduce that $K$ is a non-increasing $G$-martingale with $K_0 = 0$ and $K_T \in L^p_G(\Omega_T)$. Taking $m \rightarrow \infty$ in (3.3), we obtain that $(X,Y,Z,K)$ is an $L^p$-solution of $G$-FBSDE (2.3). \(\square\)

**Remark 3.2** For each fixed $\sigma > \underline{\sigma} > 0$, $T > 0$, $L_1 > 0$ and $p \in [2,\beta)$, it is easy to deduce from (3.11) that there exists a $\delta > 0$ satisfying $\Lambda_p < 1$ for each
\[ L_2L_3 < \delta. \] (3.13)
The condition (3.13) is called weakly coupling condition for $G$-FBSDE (2.3) (see [21] for classical FBSDE).

Now we consider the $L^p$-solution for $G$-FBSDE (2.3) with $p \in (1,2)$.

**Theorem 3.3** Suppose assumptions (H1) and (H2) hold. If $\sigma(\cdot)$ does not depend on $y$ and
\[ \tilde{\Lambda}_p := C_1(p)C_2(p)(nL_2L_3)^p(1+T)^p < 1 \] (3.14)
for some $p \in (1,2 \land \beta)$, then $G$-FBSDE (2.3) has a unique $L^p$-solution $(X,Y,Z,K)$.

**Proof.** The proof is similar to the proof of Theorem 3.1. We omit it. \(\square\)

**Remark 3.4** If $\sigma(\cdot)$ contains $y$ and $p \in (1,2 \land \beta)$, then $p/2 < 1$ and we can not get
\[ \left( \int_0^T |\dot{Y}_t|^2 dt \right)^{p/2} \leq C \int_0^T |\dot{Y}_t|^p dt \]
in (3.13), where $C > 0$ is a constant independent of $\dot{Y}$. Thus we need the assumption that $\sigma(\cdot)$ is independent of $y$ for $p < 2$.

The following proposition is the estimates for $G$-FBSDE (2.3).

**Proposition 3.5** Suppose that $b^{(i)}(\cdot)$, $h^{(i)}(\cdot)$, $\sigma^{(i)}(\cdot)$, $f_i(\cdot)$, $g_i(\cdot)$, $\phi_i(\cdot)$ satisfy assumptions (H1) and (H2) for $i = 1, 2$. For each fixed $p \in (1,\beta)$, let $(X^{(i)}, Y^{(i)}, Z^{(i)}, K^{(i)})$ be the $L^p$-solution of $G$-FBSDE
\[
\begin{align*}
  dX_t^{(i)} &= b^{(i)}(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)})dt + h^{(i)}(t, X_t^{(i)}, Y_t^{(i)})dB_t + \sigma^{(i)}(t, X_t^{(i)}, Y_t^{(i)})d\tilde{B}_t, \\
  dY_t^{(i)} &= f_i(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)})dt + g_i(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)})dB_t + Z_t^{(i)}dB_t + dK_t^{(i)}, \\
  X_0^{(i)} &= x_i \in \mathbb{R}^n, \quad Y_T^{(i)} = \phi_i(X_T^{(i)}),
\end{align*}
\]
for $i = 1, 2$. We have the following estimates.

(i) If $p \geq 2$ and $\Lambda_p$ defined in (3.14) satisfies $\Lambda_p < 1$, then there exists a constant $C_4$ depending on $p$, $T$, $L_1$, $L_2$, $L_3$, $\underline{\sigma}$ and $\sigma$ such that
\[
\mathbb{E} \left[ \sup_{t \leq T} \left| X_t^{(i)} \right|^p \right] \leq C_4 \mathbb{E} \left[ \left( |\dot{x}| + |\dot{\phi}_T| + \int_0^T (|\dot{b}_t| + |\dot{h}_t| + |\dot{f}_t| + |\dot{g}_t|) dt \right)^p + \left( \int_0^T |\sigma_t|^2 dt \right)^{p/2} \right], \quad (3.15)
\]
\[ 12 \]
where $\tilde{X}_t = X_t^{(1)} - X_t^{(2)}$, $\hat{x} = x_1 - x_2$, $\hat{\Phi}_T = \Phi(X_T^{(2)}) - \Phi(X_T^{(1)})$, $\hat{b}_t = b^{(1)}(t, X_t^{(2)}, Y_t^{(2)}) - b^{(2)}(t, X_t^{(2)}, Y_t^{(2)})$, $\hat{h}_t = h^{(1)}(t, X_t^{(2)}, Y_t^{(2)}) - h^{(2)}(t, X_t^{(2)}, Y_t^{(2)})$, $\hat{\sigma}_t = \sigma^{(1)}(t, X_t^{(2)}, Y_t^{(2)}) - \sigma^{(2)}(t, X_t^{(2)}, Y_t^{(2)})$.

(ii) If $p \in (1, 2)$, $\sigma(\cdot)$ does not depend on $y$ and $\hat{\lambda}_p$ defined in (3.14) satisfies $\hat{\lambda}_p < 1$, then there exists a constant $C_\hat{\lambda}$ depending on $p$, $T$, $L_1$, $L_2$, $L_3$, $\hat{\lambda}$ and $\varrho$ such that

$$
\hat{\mathbb{E}} \left[ \sup_{0 \leq t \leq T} |\tilde{X}_t|^p \right] \leq C_{\hat{\lambda}} \mathbb{E} \left[ \left( \int_0^T (|\hat{b}_t| + |\hat{h}_t| + |\hat{f}_t| + |\hat{g}_t|) dt \right)^p + \left( \int_0^T |\hat{\sigma}_t|^2 dt \right)^{p/2} \right],
$$

where $\hat{\sigma}_t = \sigma^{(1)}(t, X_t^{(2)}) - \sigma^{(2)}(t, X_t^{(2)})$, $\hat{X}_t$, $\hat{x}$, $\hat{\Phi}_T$, $\hat{b}_t$, $\hat{h}_t$, $\hat{f}_t$ and $\hat{g}_t$ are the same as (i).

Proof. We only prove (i). The proof of (ii) is similar. For each $a_1 > 0$ and $a_2 > 0$, by the mean value theorem, we have

$$(a_1 + a_2)^p - a_1^p = p(a_1 + \theta a_2)^{p-1}a_2 \leq p2^{p-1}(a_1^{p-1}a_2^p),$$

where $\theta \in (0, 1)$. From this, we can deduce

$$(a_1 + a_2)^p \leq (1 + \varepsilon)a_1^p + C(p, \varepsilon)a_2^p$$

for each $\varepsilon > 0$.

where

$$C(p, \varepsilon) = p2^{p-1} + p^{p-2}(p-1)p\varepsilon^{-(p-1)}.$$

Set $\tilde{X}_t^{(i)} = X_t^{(i)} - x_i$ for $i = 1, 2$, and $\tilde{X}_t = \tilde{X}_t^{(1)} - \tilde{X}_t^{(2)}$. It is easy to check that $(\tilde{X}_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}, K_t^{(i)})$ satisfies the $G$-FBSDE

$$
\begin{align*}
\frac{d\tilde{X}_t^{(i)}}{dt} &= b^{(i)}(t, \tilde{X}_t^{(i)} + x_i, Y_t^{(i)}) dt + h^{(i)}(t, \tilde{X}_t^{(i)} + x_i, Y_t^{(i)}) dB_t + \sigma^{(i)}(t, \tilde{X}_t^{(i)} + x_i, Y_t^{(i)}) d\mathbf{B}_t, \\
\frac{dY_t^{(i)}}{dt} &= f_i(t, \tilde{X}_t^{(i)} + x_i, Y_t^{(i)}, Z_t^{(i)}) dt + g_i(t, \tilde{X}_t^{(i)} + x_i, Y_t^{(i)}, Z_t^{(i)}) dB_t + Z_t^{(i)} dB_t + K_t^{(i)}, \\
\tilde{X}_0^{(i)} &= 0 \in \mathbb{R}^n, \quad Y_0^{(i)} = \phi(\tilde{X}_T^{(i)} + x_i),
\end{align*}
$$

for $i = 1, 2$. Similar to the proof of Theorem 2.4, we have

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{X}_t| \right] \leq C_1(p) \mathbb{E} \left[ \left( \int_0^T (|\tilde{b}_t| + |\tilde{h}_t|) dt \right)^p + \left( \int_0^T |\tilde{\sigma}_t|^2 dt \right)^{p/2} \right],
$$

where $\tilde{b}_t = b^{(1)}(t, X_t^{(2)} + x_1, Y_t^{(1)}) - b^{(2)}(t, X_t^{(2)} + x_2, Y_t^{(2)})$, $\tilde{h}_t = h^{(1)}(t, X_t^{(2)} + x_1, Y_t^{(1)}) - h^{(2)}(t, X_t^{(2)} + x_2, Y_t^{(2)})$, $\tilde{\sigma}_t = \sigma^{(1)}(t, X_t^{(2)} + x_1, Y_t^{(1)}) - \sigma^{(2)}(t, X_t^{(2)} + x_2, Y_t^{(2)})$. From (H2), it is easy to verify that

$$
|\tilde{b}_t| + |\tilde{h}_t| \leq nL_2 \tilde{Y}_t + nL_1 |\tilde{x}| + |\tilde{b}_t| + |\tilde{h}_t|, \quad |\tilde{x}| \leq nL_2 \tilde{Y}_t + nL_1 |\tilde{x}| + |\tilde{\sigma}_t|,
$$

where $\tilde{Y}_t = Y_t^{(1)} - Y_t^{(2)}$. Similar to (3.3), by (3.17), we obtain, for each $\varepsilon > 0$,

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{X}_t| \right] \leq (1 + \varepsilon)C_1(p)nL_2 \mathbb{E} \left[ T^{(p-1)} + T^{(p-2)/2} \right]
$$

$$
+ C_\varepsilon \mathbb{E} \left[ \left( |\tilde{x}| + \int_0^T (|\tilde{b}_t| + |\tilde{h}_t|) dt \right)^p + \left( \int_0^T |\tilde{\sigma}_t|^2 dt \right)^{p/2} \right].
$$

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where the constant $C_0 > 0$ depends on $p$, $T$, $L_1$, $\bar{\sigma}$ and $\varepsilon$. Similar to (3.4), we can get, for each $\varepsilon > 0$,
\[
\hat{\mathcal{E}}[|\hat{Y}_t|^p] \leq (1 + \varepsilon)C_2(p)\mathcal{L}_0^p(1 + T)^p\mathcal{E} \left[ \sup_{t \leq T} \left| \hat{X}_t \right|^p \right] + C_T \mathcal{E} \left[ \left| \hat{\varepsilon} \right|^{p} + \int_0^T (|\hat{f}_t| + |\hat{g}_t|)dt \right]^p,
\]
where the constant $C_T > 0$ depends on $p$, $T$, $L_1$, $L_3$, $\bar{\sigma}$, $\bar{\alpha}$ and $\varepsilon$. Thus we obtain
\[
[1 - (1 + \varepsilon)\Lambda_p \mathcal{E} \left[ \sup_{t \leq T} \left| \hat{X}_t \right|^p \right]] \leq C_8 \mathcal{E} \left[ \left| \hat{\varepsilon} \right|^p + \int_0^T (|\hat{f}_t| + |\hat{g}_t|)dt \right]^p + \left( \int_0^T |\bar{\sigma}|^2 dt \right)^{p/2},
\]
where the constant $C_8 > 0$ depends on $p$, $T$, $L_1$, $L_2$, $L_3$, $\bar{\sigma}$, $\bar{\alpha}$ and $\varepsilon$. Since $\Lambda_p < 1$, we can take $\varepsilon_0 > 0$ such that $(1 + \varepsilon_0)\Lambda_p < 1$. Note that $|\hat{X}_t|^p \leq 2^{p-1} |\hat{X}_t|^p + |\hat{\varepsilon}|^p$, then we obtain (3.15). \(\Box\)

4 Comparison theorem for G-FBSDEs

For simplicity, we only study the comparison theorem for $p = 2$. The results for $p \neq 2$ are similar. Consider the following G-FBSDEs:
\[
\begin{align*}
\begin{cases}
\ds dB_t = h(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dB_t, \\
\ds dY_t = f(t, X_t, Y_t, Z_t)dt + g(t, X_t, Y_t, Z_t)dB_t + dK_t,
\end{cases}
\end{align*}
\]
(4.1)

\textbf{Theorem 4.1} Suppose that assumptions (H1) and (H2) hold for $i = 1, 2$ with $\beta > 2$. Then there exists a $\delta > 0$ depending on $n$, $T$, $L_1$, $\bar{\sigma}$ and $\bar{\alpha}$ such that the following results hold.

(i) If $L_2L_3 < \delta$, then $G$-FBSDE (4.1) has a unique $L^2$-solution $(X^{(i)}, Y^{(i)}, Z^{(i)}, K^{(i)})$ for $i = 1, 2$.

(ii) If $L_2L_3 < \delta$ and $\phi_1(X^{(1)}_T) \geq \phi_2(X^{(2)}_T)$ (resp. $\phi_1(X^{(1)}_T) \geq \phi_2(X^{(1)}_T)$), then we have $Y^{(1)}_0 \geq Y^{(2)}_0$.

\textbf{Proof.} From the definition of $\Lambda_p$ in (3.1) for $p \geq 2$, it is easy to deduce that there exists a $\delta_1 > 0$ depending on $n$, $T$, $L_1$, $\bar{\sigma}$ and $\bar{\alpha}$ satisfying $\Lambda_2 < 1$. By Theorem 3.1, we obtain (i) under the assumption $L_2L_3 < \delta_1$.

We only prove the case $\phi_1(X^{(2)}_T) \geq \phi_2(X^{(1)}_T)$ for (ii). The proof for $\phi_1(X^{(1)}_T) \geq \phi_2(X^{(1)}_T)$ is similar. Under the assumption $L_2L_3 < \delta_1$, it is clear that $(X^{(i)}, Y^{(i)}, Z^{(i)}, K^{(i)})$ is the $L^2$-solution of G-FBSDE (4.1) for $i = 1, 2$ under each $P \in \mathcal{P}$, where $\mathcal{P}$ is defined in Theorem 2.2. Since $\mathcal{P}$ is weakly compact and $\hat{\mathcal{E}}[K^{(2)}_T] = 0$ with $K^{(2)}_T \leq 0$, there exists a $P^* \in \mathcal{P}$ such that $K^{(2)}_T = 0$ $P^*$-a.s. Noting that $K^{(2)}_T$ is a non-increasing with $K^{(2)}_0 = 0$, we obtain $K^{(2)}_T = 0$ under $P^*$. By (2.3), we know that $d(B)_t = \gamma_t dt$ q.s. with $\gamma_t \in [\sigma^2, \bar{\alpha}^2]$.

Set $X^{(i)}_t = (X^{(i)}_{1,t}, \ldots, X^{(i)}_{n,t})^T$ for $i = 1, 2$, $X_t = (X_{1,t}, \ldots, X_{n,t})^T = X^{(1)}_t - X^{(2)}_t$, $Y_t = Y^{(1)}_t - Y^{(2)}_t$, $Z_t = Z^{(1)}_t - Z^{(2)}_t$. Since $(X^{(i)}, Y^{(i)}, Z^{(i)}, K^{(i)})$ satisfies G-FBSDE (4.1) for $i = 1, 2$ under $P^*$, we obtain $P^*$-a.s.
\[
\begin{align*}
\begin{cases}
\ds d\hat{X}_t = \left[ a^{(2)}_1(t) \hat{X}_t + a^{(2)}_2(t) \hat{Y}_t \right] dt + \left[ a^{(3)}_1(t) \hat{X}_t + a^{(3)}_2(t) \hat{Y}_t \right] dB_t, \\
\ds d\hat{Y}_t = \left[ a^{(5)}_1(t) \hat{X}_t + a^{(5)}_2(t) \hat{Y}_t + a^{(7)}_1(t) \hat{Z}_t \right] dt + \hat{Z}_t dB_t + dK^{(1)}_t, \\
\hat{X}_0 = 0 \in \mathbb{R}^n, \hat{Y}_T = (a^{(8)}_1, \hat{X}_T) + \phi_1(X^{(1)}_T) - \phi_2(X^{(2)}_T),
\end{cases}
\end{align*}
\]
(4.2)
where $a^{(1)}(t) = (a_{j,k}^{(1)}(t))_{j,k=1}^{n}$ and $a^{(2)}(t) = (a_{j}^{(2)}(t), \ldots, a_{n}^{(2)}(t))^T$ with
\[
\begin{align*}
    a_{j,k}^{(1)}(t) & = b_{j}(t, k - 1) - b_{j}(t, k) + h_{j}(t, k - 1) - h_{j}(t, k) \gamma_{j} \langle \hat{X}_{k}^{\gamma} \rangle^{-1} I_{\{\hat{X}_{k}^{\gamma} \neq 0\}}, \\
    a_{j}^{(2)}(t) & = \begin{bmatrix}
        b_{j}(t, X_{1}^{(2)}, Y_{1}^{(2)}) - b_{j}(t, X_{1}^{(2)}, Y_{2}^{(2)}) + \left( h_{j}(t, X_{1}^{(2)}, Y_{1}^{(2)}) - h_{j}(t, X_{1}^{(2)}, Y_{2}^{(2)}) \right) \gamma_{j} \langle \hat{Y}_{1} \rangle^{-1} I_{\{\hat{Y}_{1} \neq 0\}}, \\
        b_{j}(t, k) = b_{j}(t, X_{1}^{(2)}, \ldots, X_{k-1}^{(2)}, X_{k+1}^{(2)}, \ldots, X_{n}^{(2)}, Y_{1}^{(2)}),
    \end{bmatrix}
\end{align*}
\]
similar for the definition of notations $b_{j}(t, k - 1), h_{j}(t, k - 1), h_{j}(t, k), a^{(3)}(t), a^{(4)}(t), a^{(5)}(t), a^{(6)}(t), a^{(7)}(t)$ and $a_{p}^{(8)}$. From the assumption (H2), it is easy to verify that
\[
\begin{align*}
    |a^{(1)}(t)| & \leq nL_{1}(1 + \sigma^{2}), |a^{(2)}(t)| \leq nL_{2}(1 + \sigma^{2}), |a^{(3)}(t)| \leq nL_{1}, |a^{(4)}(t)| \leq nL_{2}, \\
    |a^{(5)}(t)| & \leq L_{3}(1 + \sigma^{2}), |a^{(6)}(t)| + |a^{(7)}(t)| \leq L_{1}(1 + \sigma^{2}), |a_{p}^{(8)}| \leq L_{3}.
\end{align*}
\]
Consider the following FBSDE under $P^*$:
\[
\begin{align*}
    dl_{t} & = \left[ -a^{(6)}(t)l_{t} + \langle a^{(2)}(t), p_{t} \rangle + \langle \gamma_{t}a^{(4)}(t), q_{t} \rangle \right] dt - \gamma_{t}^{-1}a^{(7)}(t)l_{t}dB_{t}, \\
    dp_{t} & = \left[ l_{t}a^{(5)}(t) - a^{(1)}(t)p_{t} - \sigma_{t}a^{(3)}(t)q_{t} \right] dt + q_{t}dB_{t} + dN_{t}, \\
    l_{0} & = 1, \quad pt = l_{T}a_{T}^{(8)} \in \mathbb{R}^{n},
\end{align*}
\]
where $N$ is a $\mathbb{R}^{n}$-valued square integrable martingale with $N_{0} = 0$ such that each component of $N$ is orthogonal to $B$ under $P^*$. By Theorem 6.1 in [1], for each $(l_{t})_{t \leq T} \in S_{P^{*}}^{2}\left(0, T\right)$, the BSDE
\[
dp_{t} = \left[ l_{t}a^{(5)}(t) - a^{(1)}(t)p_{t} - \sigma_{t}a^{(3)}(t)q_{t} \right] dt + q_{t}dB_{t} + dN_{t}, \quad pt = l_{T}a_{T}^{(8)},
\]
has a unique $L^{2}$-solution $(p, q, N)$ with $p \in S_{P^{*}}^{2}\left(0, T; \mathbb{R}^{n}\right)$ and $q \in M_{P^{*}}^{2,2}\left(0, T; \mathbb{R}^{n}\right)$, where $S_{P^{*}}^{2}\left(0, T\right)$ (resp. $M_{P^{*}}^{2,2}\left(0, T\right)$) is the completion of $S_{0}^{0}\left(0, T\right)$ (resp. $M_{0}^{0}\left(0, T\right)$) under the norm
\[
||\eta||_{S_{P^{*}}^{2}\left(0, T\right)} := \left( E_{P^{*}}\left[ \sup_{t \leq T} ||\eta_{t}||^{2} \right] \right)^{1/2}, \quad ||\eta||_{M_{P^{*}}^{2,2}\left(0, T\right)} := \left( E_{P^{*}}\left[ \int_{0}^{T} ||\eta_{t}||^{2} dt \right] \right)^{1/2}.
\]
Similar to the proof of Theorem 3.1, we can deduce that there exists a $\delta_{2} > 0$ depending on $n, T, L_{1}, \sigma$ and $\sigma$ such that FBSDE [13] has a unique $L^{2}$-solution $(l, p, q, N)$ under the assumption $L_{2}L_{3} < \delta_{2}$.

Taking $\delta = \delta_{1} \wedge \delta_{2}$, we assume $L_{2}L_{3} < \delta$ in the following. Applying Itô’s formula to $\langle p_{t}, \hat{X}_{t} \rangle - l_{t}\hat{Y}_{t}$ under $P^*$, we obtain
\[
\hat{Y}_{0} = E_{P^{*}}\left[ l_{T} \left( \phi_{1}(X_{T}^{(2)}) - \phi_{2}(X_{T}^{(2)}) \right) - \int_{0}^{T} l_{t}dK_{t}^{(1)} \right].
\]
Since $\phi_{1}(X_{T}^{(2)}) \geq \phi_{2}(X_{T}^{(2)})$ and $dK_{t}^{(1)} \leq 0$, we only need to prove $l_{t} \geq 0$ $P^{*}$-a.s. for $t \in [0, T]$. Define the stopping time
\[
\tau = \inf\{t \geq 0 : l_{t} = 0\} \wedge T.
\]
It is clear that $l_{\tau} = 0$ on $\{\tau < T\}$ and $l_{T} \geq 0$ on $\{\tau = T\}$. Consider the following FBSDE on $[\tau, T]$ under $P^*$:
\[
\begin{align*}
    dl'_{t} & = \left[ -a^{(6)}(t)l'_{t} + \langle a^{(2)}(t), p'_{t} \rangle + \langle \gamma_{t}a^{(4)}(t), q'_{t} \rangle \right] dt - \gamma_{t}^{-1}a^{(7)}(t)l'_{t}dB_{t}, \\
    dp'_{t} & = \left[ l'_{t}a^{(5)}(t) - a^{(1)}(t)p'_{t} - \sigma_{t}a^{(3)}(t)q'_{t} \right] dt + q'_{t}dB_{t} + dN'_{t}, \\
    l'_{0} = l_{\tau}, \quad p'_{T} = l'_{T}a_{T}^{(8)} \in \mathbb{R}^{n}, \quad t \in [\tau, T].
\end{align*}
\]
It is easy to verify that

\[
(l_t^0, p_t^0, q_t^0, N_t^0)_{t \in [\tau, T]} = \left( l_T I_{\{\tau = T\}}, l_T a_T^{(8)} I_{\{\tau = T\}}, 0, 0 \right)_{t \in [\tau, T]}
\]
satisfies FBSDE (4.5). Obviously, \((l_t^0, p_t^0, q_t^0, N_t^0)_{t \in [\tau, T]} = (l_t, p_t, q_t, N_t - N_\tau)_{t \in [\tau, T]}\) satisfies FBSDE (4.5). Since the \(L^2\)-solution to FBSDE (4.5) is unique, we obtain \(l_t = l_T I_{\{\tau = T\}}\) for \(t \in [\tau, T]\). Thus \(l_t \geq 0\) \(P^*\)-a.s. for \(t \in [0, T]\). By (4.4), we get \(\hat{Y}_0 \geq 0\), which implies (ii). \(\square\)

Suppose \(n = 1\) in the following and consider the following G-FBSDEs:

\[
\begin{align*}
    dX_t^{(i)} &= b(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}) dt + h(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}) d\langle B \rangle_t + \sigma(t, X_t^{(i)}, Y_t^{(i)}) dB_t, \\
    dY_t^{(i)} &= f(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}) dt + g(t, X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}) d\langle B \rangle_t + Z_t^{(i)} dB_t + K_t^{(i)}, \\
    X_0^{(i)} &= x_i \in \mathbb{R}, \ Y_0^{(i)} = \phi(X_T^{(i)}), \ i = 1, 2.
\end{align*}
\]

**Theorem 4.2** Suppose that assumptions (H1) and (H2) hold with \(n = 1\) and \(\beta > 2\). Then there exists a \(\delta > 0\) depending on \(T, L_1, \sigma\) and \(\sigma\) such that the following results hold.

(i) If \(L_2 L_3 < \delta\), then G-FBSDE (4.6) has a unique \(L^2\)-solution \((X^{(i)}, Y^{(i)}, Z^{(i)}, K^{(i)})\) for \(i = 1, 2\).

(ii) If \(L_2 L_3 < \delta\), \(x_1 \geq x_2\), \(\phi(\cdot)\) is non-decreasing, \(f(\cdot)\) and \(g(\cdot)\) are non-increasing in \(x\), then we have \(Y_0^{(1)} \geq Y_0^{(2)}\).

**Proof.** The proof is similar to the proof of Theorem 4.1. For the convenience of the reader, we sketch the proof. (i) is obvious. For (ii), we can similarly find a \(P^* \in \mathcal{P}\) such that \(K_T^{(2)} = 0\) \(P^*\)-a.s. The equation (4.2) is rewritten as the following equation: \(P^*\)-a.s.

\[
\begin{align*}
    d\hat{X}_t &= \left[ a^{(1)}(t) \hat{X}_t + a^{(2)}(t) \hat{Y}_t \right] dt + \left[ a^{(3)}(t) \hat{X}_t + a^{(4)}(t) \hat{Y}_t \right] d\langle B \rangle_t, \\
    d\hat{Y}_t &= \left[ a^{(5)}(t) \hat{X}_t + a^{(6)}(t) \hat{Y}_t + a^{(7)}(t) \hat{Z}_t \right] dt + \hat{Z}_t dB_t + K_t^{(1)}, \\
    \hat{X}_0 &= x_1 - x_2, \ \hat{Y}_T = a_T^{(8)} \hat{X}_T,
\end{align*}
\]

where the notations \(a^{(1)}(t), a^{(2)}(t), a^{(3)}(t), a^{(4)}(t), a^{(5)}(t), a^{(6)}(t)\) and \(a^{(7)}(t)\) are the same as the notations in the proof of Theorem 4.1 under \(n = 1\),

\[
a_T^{(8)} = \left[ \phi(X_T^{(1)}) - \phi(X_T^{(1)}) \right] \left( \hat{X}_T \right)^{-1} I_{\{\hat{X}_T \neq 0\}}.
\]

Since \(\phi(\cdot)\) is non-decreasing, \(f(\cdot)\) and \(g(\cdot)\) are non-increasing in \(x\), it is easy to verify that

\[
a_T^{(8)} \geq 0 \text{ and } a_T^{(5)}(t) \leq 0 \text{ for } t \in [0, T]. \tag{4.8}
\]

Applying Itô’s formula to \(p_t \hat{X}_t - l_t \hat{Y}_t\) under \(P^*\), where \((l, p, q, N)\) is the \(L^2\)-solution of FBSDE (4.3) under \(n = 1\), we obtain

\[
\hat{Y}_0 = p_0(x_1 - x_2) + E_{P^*} \left[ - \int_0^T l_t dK_t^{(1)} \right].
\]
We have obtained \( l_t \geq 0 \) \( P^*\)-a.s. for \( t \in [0,T] \) in the proof of Theorem \ref{thm:main}. Thus we get
\[
\hat{Y}_0 \geq p_0(x_1 - x_2).
\] (4.9)

By (4.8), we have
\[
l_T a_T^{(8)} \geq 0 \quad \text{and} \quad l_t a_t^{(5)}(t) \leq 0 \quad \text{for} \quad t \in [0,T].
\]

By comparison theorem for BSDEs
\[
d\hat{p}_t = \left[ l_t a_t^{(5)}(t) - a_t^{(4)}(t)p_t - \gamma_t a_t^{(3)}(t)q_t \right] dt + q_t dB_t + dN_t, \quad \hat{p}_T = l_T a_T^{(8)},
\]
and
\[
d\tilde{p}_t = \left[ -a_t^{(1)}(t)\tilde{p}_t - \gamma_t a_t^{(3)}(t)\tilde{q}_t \right] dt + \tilde{q}_t dB_t + d\tilde{N}_t, \quad \tilde{p}_T = 0,
\]
we get \( p_0 \geq \tilde{p}_0 = 0 \). Thus, from (4.8), we deduce \( \hat{Y}_0 \geq 0 \), which implies (ii). \( \square \)

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