Unital versions of the higher order peak algebras

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Abstract. We construct unital extensions of the higher order peak algebras defined by Krob and the third author in [Ann. Comb. 9 (2005), 411–430], and show that they can be obtained as homomorphic images of certain subalgebras of the Mantaci-Reutenauer algebras of type $B$. This generalizes a result of Bergeron, Nyman and the first author [Trans. AMS 356 (2004), 2781–2824].

Résumé. Nous construisons des extensions unitaires des algèbres de pics d’ordre supérieur définies par Krob et le troisième auteur dans [Ann. Comb. 9 (2005), 411–430], et nous montrons qu’elles peuvent être obtenues comme images homomorphes de certaines sous-algèbres des algèbres de Mantaci-Reutenauer de type $B$. Ceci généralise un résultat dû à Bergeron, Nyman et au premier auteur [Trans. AMS 356 (2004), 2781–2824].

Keywords: Descent algebras, Noncommutative symmetric functions, Peak algebras

1 Introduction

A descent of a permutation $\sigma \in S_n$ is an index $i$ such that $\sigma(i) > \sigma(i+1)$. A descent is a peak if moreover $i > 1$ and $\sigma(i) > \sigma(i-1)$. The sums of permutations with a given descent set span a subalgebra of the group algebra, the descent algebra $\Sigma_n$. The peak algebra $\mathcal{P}_n$ of $\mathfrak{S}_n$ is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit. Descent algebras can be defined for all finite Coxeter groups [19]. In [2], it is shown that the peak algebra of $\mathfrak{S}_n$ can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group $B_n$.

The direct sum of the peak algebras turns out to be a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with $\text{Sym}$, the Hopf algebra of noncommutative symmetric functions [9]. As explained in [5], it turns out that a fair amount of results on the peak algebras can be deduced from the case $q = -1$ of a $q$-identity of [11]. Specializing $q$ to other roots of unity, Krob and the third author introduced and studied higher order peak algebras in [12]. Again, these are non-unital, and it is natural to ask whether the construction of [2] can be extended to this case.

\textsuperscript{1}Partially supported by NSF grant DMS-0600973.
\textsuperscript{2}Partially supported by ANR grant 06-BLAN-0380.
We will show that this is indeed possible. We first construct the unital versions of the higher order peak algebras by a simple manipulation of generating series. We then show that they can be obtained as homomorphic images of the Mantaci-Reutenauer algebras of type $B$. Hence no Coxeter groups other than $B_n$ and $S_n$ are involved in the process; in fact, the construction is related to the notion of superization, as defined in [16], rather than to root systems or wreath products.

2 Notations and background

2.1 Noncommutative symmetric functions

We will assume familiarity with the notations of [9] and with the main results of [12]. We recall a few definitions for the convenience of the reader.

The Hopf algebra of noncommutative symmetric functions is denoted by $\text{Sym}$, or by $\text{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet $A$. Linear bases of $\text{Sym}_n$ are labelled by compositions $I = (i_1, \ldots, i_r)$ of $n$ (we write $I \vDash n$). The noncommutative complete and elementary functions are denoted by $S_n$ and $\Lambda_n$, and $S_I = S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by $R_I$. The descent set of $I$ is $\text{Des}(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_r - 1\}$. The descent composition of a permutation $\sigma \in S_n$ is the composition $I = D(\sigma)$ of $n$ whose descent set is the descent set of $\sigma$.

Recall from [8] that for an infinite totally ordered alphabet $A$, $\text{FQSym}(A)$ is the subalgebra of $\mathbb{C}[A]$ spanned by the polynomials $G_\sigma(A) = \sum_{\text{std}(w) = \sigma} w$, (1) that is, the sum of all words in $A^n$ whose standardization is the permutation $\sigma \in S_n$. The noncommutative ribbon Schur function $R_I \in \text{Sym}$ is then $R_I = \sum_{D(\sigma) = I} G_\sigma$. (2)

This defines a Hopf embedding $\text{Sym} \rightarrow \text{FQSym}$. The Hopf algebra $\text{FQSym}$ is self-dual under the pairing $(G_\sigma, G_\tau) = \delta_{\sigma, \tau^{-1}}$ (Kronecker symbol). Let $F_\sigma := G_{\sigma^{-1}}$, so that $\{F_\sigma\}$ is the dual basis of $\{G_\sigma\}$. The internal product $\ast$ of $\text{FQSym}$ is induced by composition $\circ$ in $S_n$ in the basis $F$, that is,

$F_\sigma \ast F_\tau = F_{\sigma \circ \tau}$ and $G_\sigma \ast G_\tau = G_{\tau \circ \sigma}$. (3)

Each subspace $\text{Sym}_n$ is stable under this operation, and anti-isomorphic to the descent algebra $\Sigma_n$ of $S_n$. For $f_i \in \text{FQSym}$ and $g \in \text{Sym}$, we have the splitting formula

$(f_1 \cdots f_r) \ast g = \mu_r \cdot (f_1 \otimes \cdots \otimes f_r) \ast_r \Delta^r g$, (4)

where $\mu_r$ is $r$-fold multiplication, and $\Delta^r$ the iterated coproduct with values in the $r$-th tensor power.

2.2 The Mantaci-Reutenauer algebra of level 2

We denote by $\text{MR}$ the free product $\text{Sym} \ast \text{Sym}$ of two copies of the Hopf algebra of noncommutative symmetric functions [14]. That is, $\text{MR}$ is the free associative algebra on two sequences $(S_n)$ and $(S_n)$ $(n \geq 1)$. We regard the two copies of $\text{Sym}$ as noncommutative symmetric functions on two auxiliary
alphanets: $S_n = S_n(A)$ and $S_n = S_n(\overline{A})$. We denote by $F \mapsto \overline{F}$ the involutive automorphism which exchanges $S_n$ and $S_n$. The bialgebra structure is defined by the requirement that the series

$$
\sigma_1 = \sum_{n \geq 0} S_n \text{ and } \overline{\sigma}_1 = \sum_{n \geq 0} S_{\overline{n}}
$$

are grouplike. The internal product of MR can be computed from the splitting formula and the conditions that $\sigma_1$ is neutral, $\overline{\sigma}_1$ is central, and $\overline{\sigma}_1 * \overline{\sigma}_1 = \sigma_1$.

In [15], an embedding of MR in the Hopf algebra BFQSym of free quasi-symmetric functions of type $B$ (spanned by colored permutations) is described. Under this embedding, left $*$-multiplication by $\Lambda_n = G_{n,n-1,...,2,1}$ corresponds to right multiplication by $n \cdot n - 1 \cdot 2 \cdot 1$ in the group algebra of $B_n$. This implies that left $*$-multiplication by $\lambda_1$ is an involutive anti-automorphism of BFQSym, hence of MR.

2.3 Noncommutative symmetric functions of type $B$

The hyperoctahedral analogue BSym of Sym, defined in [6], is the right Sym-module freely generated by another sequence $(\tilde{S}_n)$ ($n \geq 0$, $\tilde{S}_0 = 1$) of homogeneous elements, with $\tilde{\sigma}_1$ grouplike. This is a coalgebra, but not an algebra. It is endowed with an internal product, for which each homogeneous component BSym is anti-isomorphic to the descent algebra of $B_n$.

3 Solomon descent algebras of type $B$

3.1 Descents in $B_n$

The hyperoctahedral group $B_n$ is the group of signed permutations. A signed permutation can be denoted by $w = (\sigma, \epsilon)$ where $\sigma$ is an ordinary permutation and $\epsilon \in \{\pm 1\}^n$, such that $w(i) = \epsilon_i \sigma(i)$. If we set $w(0) = 0$, then, $i \in [0, n-1]$ is a descent of $w$ if $w(i) > w(i+1)$. Hence, the descent set of $w$ is a subset $D = \{i_0, i_0 + i_1, \ldots, i_0 + i_1 + \cdots + i_r \}$ of $[0, n-1]$. We then associate to $D$ a so-called type-B composition (a composition whose first part can be zero) $(i_0 - 0, i_1, \ldots, i_{r-1}, n - i_{r-1})$. The sum of all signed permutations whose descent set is contained in $D$ is mapped to $\tilde{S}^D := \tilde{S}_{i_0} \tilde{S}_{i_1}^\prime \cdots \tilde{S}_{i_{r-1}}^\prime$ by Chow’s anti-isomorphism [6], where $\tilde{I}' = (i_1, \ldots, i_r)$.

3.2 Noncommutative supersymmetric functions

An embedding of BSym as a sub-coalgebra and sub-Sym-module of MR can be deduced from [13]. To describe it, let us define, for $F \in \text{Sym}(A)$,

$$
F^\beta = F(A|\overline{A}) = F(A - q \overline{A})|_{q = -1}
$$

(the supersymmetric version of $F$). The superization of $F \in \text{Sym}(A)$ can also be given by

$$
F^\mathfrak{s} = F * \sigma_1^\mathfrak{s}.
$$

Indeed, $\sigma_1^\mathfrak{s}$ is grouplike, and for $F = S^I$, the splitting formula gives

$$
(S_{i_1} \cdots S_{i_r}) * \sigma_1^\mathfrak{s} = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\mathfrak{s} \otimes \cdots \otimes \sigma_1^\mathfrak{s})] = S^{I'}. 
$$
We have
\[ \sigma_1^\sharp = \bar{\lambda}_1 \sigma_1 = \sum \Lambda_i S_j . \]

The element \( \bar{\sigma}_1 \) is central for the internal product, and
\[ \bar{\sigma}_1 * F = \bar{F} = F * \bar{\sigma}_1 . \]

Hence,
\[ \bar{\sigma}_1 * \sigma_1^\sharp = \bar{\lambda}_1 \bar{\sigma}_1 =: \sigma_1^\flat . \]

The basis element \( \tilde{S}^I \) of \( \text{BSym} \), where \( I = (i_0, i_1, \ldots, i_r) \) is a type \( B \)-composition, can be embedded as
\[ \tilde{S}^I = S_{i_0}(A) S_{i_1} \cdot \ldots \cdot S_{i_r}(A|\bar{A}) . \]

We will identify \( \text{BSym} \) with its image under this embedding.

### 3.3 A proof that \( \text{BSym} \) is \( * \)-stable

We are now in a position to understand why \( \text{BSym} \) is a \( * \)-subalgebra of \( \text{MR} \). The argument will be extended below to the case of unital peak algebras. Let \( F, G \in \text{Sym} \). We want to understand why \( \sigma_1 F^\sharp * \sigma_1 G^\sharp \) is in \( \text{BSym} \). Using the splitting formula, we rewrite this as
\[ \mu \left[ (\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp \right] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)}) (F^\sharp * \sigma_1 G^\sharp_{(2)}) . \]

We now only have to show that each term \( F^\sharp * \sigma_1 G^\sharp_{(2)} \) is in \( \text{Sym}^\sharp \). We may assume that \( F = S^I \), and for any \( G \in \text{Sym} \),
\[ S^I * \sigma_1 G^\sharp = \sum_{(G)} \mu \left[ (S^I_{i_1} \otimes \ldots \otimes S^I_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \ldots \otimes \sigma_1 G^\sharp_{(r)}) \right] \]
so that it is sufficient to prove the property for \( F = S_n \). Now,
\[ \sigma_1^\sharp * \sigma_1 G^\sharp = (\bar{\lambda}_1 \sigma_1) * \sigma_1 G^\sharp \]
\[ = \sum_{(G)} (\bar{\lambda}_1 * \sigma_1 G^\sharp_{(1)}) (\sigma_1 G^\sharp_{(2)}) \]
\[ = \sum_{(G)} (\bar{\sigma}_1 * \bar{\lambda}_1 * \sigma_1 G^\sharp_{(1)}) \cdot \sigma_1 G^\sharp_{(2)} \]

Now,
\[ \lambda_1 * \sigma_1 G^\sharp_{(1)} = (\lambda_1 * G^\sharp_{(1)}) (\lambda_1 * \sigma_1) = (\lambda_1 * G^\sharp_{(1)}) \lambda_1 , \]

since \( \lambda_1 \) is an anti-automorphism. We then get
\[ \sigma_1^\sharp * \sigma_1 G^\sharp = \sum_{(G)} (\bar{\sigma}_1 * ((\lambda_1 * G^\sharp_{(1)}) \lambda_1) \cdot \sigma_1 G^\sharp_{(2)}) \]
\[ = \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * G^\sharp_{(1)}) \cdot (\bar{\sigma}_1 * \lambda_1) \sigma_1 G^\sharp_{(2)} \]
\[ = \sum_{(G)} (\bar{\lambda}_1 * G^\sharp_{(1)}) \cdot \sigma_1 G^\sharp_{(2)} \]
Now, the result will follow if we can prove that \( \tilde{\lambda}_1 \ast G^\sharp \) is in \( \text{Sym}^\sharp \) for any \( G \in \text{Sym} \).

For \( G = S^I \),

\[
\tilde{\lambda}_1 \ast S^I = \lambda_1 \ast \sigma_1 \ast S^I \ast \sigma^I_1 = \lambda_1 \ast S^I \ast \sigma_1 \ast \sigma^I_1 = \lambda_1 \ast S^I \ast \sigma^I_1 .
\]  

(18)

Since left \( \ast \)-multiplication by \( \lambda_1 \) in an anti-automorphism, we only need to prove that \( \lambda_1 \ast S^\flat_n \) is of the form \( G^\sharp \). And indeed,

\[
\lambda_1 \ast S^\flat_n = \sum_{i+j=n} \lambda_1 \ast (\Lambda_i S_j)
\]

\[
= \sum_{i+j=n} (\lambda_1 \ast S_j)(\lambda_1 \ast \Lambda_i)
\]

\[
= \sum_{i+j=n} \Lambda_j S_i = S^\sharp_n .
\]  

(19)

This concludes the proof that \( \text{BSym} \) is a \( \ast \)-subalgebra of \( \text{BFQSym} \).

4 Unital versions of the higher order peak algebras

As shown in \([5]\), much of the theory of the peak algebra can be deduced from a formula of \([11]\) for \( R_I((1 - q)A) \), in the special case \( q = -1 \). In \([12]\), this formula was studied in the case where \( q \) is an arbitrary root of unity, and higher order analogs of the peak algebra were obtained. In \([2]\), it was shown that the classical peak algebra can be extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of type \( B \). In this section, we construct unital extensions of the higher order peak algebras.

Let \( q \) be a primitive \( r \)-th root of unity. All objects introduced below will depend on \( q \) (and \( r \)), although this dependence will not be made explicit in the notation. We denote by \( \theta_q \) the endomorphism of \( \text{Sym} \) defined by

\[
\tilde{f} = \theta_q(f) = f((1 - q)A) = f(A) \ast \sigma_1((1 - q)A) .
\]  

(20)

We denote by \( \tilde{P} \) the image of \( \theta_q \) and by \( P \) the right \( \tilde{P} \)-module generated by the \( S_n \) for \( n \geq 0 \). Note that \( P \) is by definition a left \( \ast \)-ideal of \( \text{Sym} \).

**Theorem 4.1** \( P \) is a unital \( \ast \)-subalgebra of \( \text{Sym} \). Its Hilbert series is

\[
\sum_{n \geq 0} \dim P_n t^n = \frac{1}{1 - t - t^2 - \ldots - t^r} .
\]  

(21)

**Proof** – Since the internal product of homogeneous elements of different degrees is zero, it is enough to show that, for any \( f, g \in \text{Sym} \), \( \sigma_1 \tilde{f} \ast \sigma_1 \tilde{g} \) is in \( P \). Thanks to the splitting formula,

\[
\sigma_1 \tilde{f} \ast \sigma_1 \tilde{g} = \mu[(\sigma_1 \otimes \tilde{f}) \ast \sum_{(g)} \sigma_1 \tilde{g}_{(1)} \otimes \sigma_1 \tilde{g}_{(2)}] = \sum_{(g)} (\sigma_1 \tilde{g}_{(1)})(\tilde{f} \ast \sigma_1 \tilde{g}_{(2)}).
\]  

(22)
Thus, it is enough to check that \( \tilde{f} \ast \sigma_1 \hat{h} \) is in \( \hat{\mathcal{P}} \) for any \( f, h \in \mathrm{Sym} \). Now,

\[
\tilde{f} \ast \sigma_1 \hat{h} = f \ast \sigma_1 ((1 - q)A) \ast \sigma_1 \hat{h},
\]

and since \( \hat{\mathcal{P}} \) is a \( \mathrm{Sym} \) left \( \ast \)-ideal, we only have to show that \( \sigma_1 ((1 - q)A) \ast \sigma_1 \hat{h} \) is in \( \hat{\mathcal{P}} \). One more splitting yields

\[
\sigma_1 ((1 - q)A) \ast \sigma_1 \hat{h} = (\lambda - q \sigma_1) \ast \sigma_1 \hat{h} = \sum_{(h)} (\lambda - q \ast \sigma_1 \hat{h}(1)) (\sigma_1 \hat{h}(2))
\]

(since left \( \ast \)-multiplication by \( \lambda - q \) is an anti-automorphism, namely the composition of the antipode and \( q^{\text{degree}} \)). The first parentheses \( (\lambda - q \ast \hat{h}(1)) \) are in \( \hat{\mathcal{P}} \) since it is a left \( \ast \)-ideal. The middle term is \( \sigma_1 ((1 - q)A) \), and the last one is in \( \hat{\mathcal{P}} \) by definition.

Recall from [12, Prop. 3.5] that the Hilbert series of \( \hat{\mathcal{P}} \) is

\[
\sum_{n \geq 0} \dim \hat{\mathcal{P}}_n t^n = \frac{1 - t^r}{1 - t - t^2 - \ldots - t^r}.
\]

From [12] Lemma 3.13 and Eq. (3.9), it follows that \( S_n \not\in \hat{\mathcal{P}} \) if and only if \( n \equiv 0 \) mod \( r \), so that the Hilbert series of \( \mathcal{P} \) is

\[
\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \ldots - t^r}.
\]

\[\blacksquare\]

5 Back to the Mantaci-Reutenauer algebra

The above proofs are in fact special cases of a master calculation in the Mantaci-Reutenauer algebra, which we carry out in this section.

Let \( q \) be an arbitrary complex number or an indeterminate, and define, for any \( F \in \text{MR} \),

\[
F^\sharp = F \ast \sigma_1 (A - q \bar{A}) = F \ast \sigma_1^\sharp.
\]

Since \( \sigma_1^\sharp \) is grouplike, it follows from the splitting formula that

\[
F \mapsto F^\sharp
\]

is an automorphism of \( \text{MR} \) for the Hopf structure. In addition, it is clear from the definition that it is also an endomorphism of left \( \ast \)-modules. We refer to it as the \( \sharp \) transform.
We now define
\[ \hat{\mathcal{Q}} = \mathcal{MR}^\sharp, \] (29)
the image of the $\sharp$ transform. Since the latter is an endomorphism of Hopf algebras and of left $*$-modules, $\hat{\mathcal{Q}}$ is both a Hopf subalgebra of $\mathcal{MR}$ and a left $*$-ideal. When $q$ is a root of unity, its image under the specialization $\hat{\mathcal{A}} = \mathcal{A}$ is the non-unital peak algebra $\hat{\mathcal{P}}$ of Section 4 (and for generic $q$, it is $\text{Sym}$).

Let $\mathcal{Q}$ be the right $\hat{\mathcal{Q}}$-module generated by the $S_n$, for all $n \geq 0$. Clearly, the identification $\hat{\mathcal{A}} = \mathcal{A}$ maps $\mathcal{Q}$ onto $\mathcal{P}$, the unital peak algebra of Section 4.

**Theorem 5.1** $\mathcal{Q}$ is a $*$-subalgebra of $\mathcal{MR}$, containing $\hat{\mathcal{Q}}$ as a left ideal.

**Proof** – Let $F,G \in \mathcal{MR}$. As above, we want to show that $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in $\mathcal{Q}$. Using the splitting formula, we rewrite this as
\[
\mu[(\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)}) (F^\sharp * \sigma_1 G^\sharp_{(2)})
\] (30)
and we only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in $\hat{\mathcal{Q}}$. We may assume that $F = S^I$, where $I$ is now a bicolored composition, and for any $G \in \mathcal{MR}$,
\[
S^I \ast \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^I_{i_1} \otimes \cdots \otimes S^I_{i_r}) \ast (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})]
\] (31)
so that it is sufficient to prove the property for $F = S_n$ or $\overline{S}_n$. Now,
\[
\sigma_1^\sharp \ast \sigma_1 G^\sharp = (\lambda_q - q_1) \ast \sigma_1 G^\sharp
\] (32)
which is in $\hat{\mathcal{Q}}$, since it is a subalgebra and a left $*$-ideal, and similarly,
\[
\overline{\sigma}_1^\sharp \ast \sigma_1 G^\sharp = (\lambda_q \overline{\sigma}_1) \ast \sigma_1 G^\sharp
\] (33)
is also in $\hat{\mathcal{Q}}$.

The various algebras introduced in this paper and their interrelationships are summarized in the following diagram.
\[
\hat{\mathcal{Q}} \subseteq \mathcal{Q} \subseteq \mathcal{MR} \subseteq \text{BFQSym}
\]
\[
\hat{\mathcal{P}} \subseteq \mathcal{P} \subseteq \text{Sym} \subseteq \text{FQSym}
\] (34)
Note that in the special case $q = -1$, by the results of Section 3.3, $Q_n$ is the (Solomon) descent algebra of $B_n$, $Q$ is isomorphic to $\text{BSym}$, and $P$ is the unital peak algebra of [2].

6 Further developments

6.1 Inversion of the generic $\#_q$ transform

For generic $q$, the endomorphism (27) of $MR$ is invertible; therefore

$$\hat{Q} \sim MR. \quad (35)$$

The inverse endomorphism of $MR$ arises from the transformation of alphabets

$$A \mapsto (qA + A)/(1 - q^2), \quad (36)$$

which is to be understood in the following sense:

$$\sigma_1 \left( \frac{qA + A}{1 - q^2} \right) := \prod_{k \geq 0} \sigma_{q^{2k+1}}(A) \sigma_{q^{2k}}(A). \quad (37)$$

Indeed,

$$\sigma_1 \left( \frac{qA + A}{1 - q^2} \right) * \sigma_1 (A - q\bar{A}) = \prod_{k \geq 0} \sigma_{q^{2k+1}}(A - qA) \sigma_{q^{2k}}(A - q\bar{A})$$

$$= \prod_{k \geq 0} \lambda_{-q^{2k+2}}(A) \sigma_{q^{2k+1}}(A) \lambda_{-q^{2k+1}}(A) \sigma_{q^{2k}}(A) \quad (38)$$

$$= \sigma_1 (A).$$

By normalizing the term of degree $n$ in (37), we obtain $B_n$-analogs of the $q$-Klyachko elements defined in [9]:

$$K_n(q; A, \bar{A}) := \prod_{i=1}^{n} (1 - q^{2i}) S_n \left( \frac{qA + A}{1 - q^2} \right) = \sum_{I \subseteq n} q^{\max(I)} R_I(qA + A). \quad (39)$$

This expression can be completely expanded on signed ribbons. From the expression of $R_I$ in $\text{FQSym}$, we have

$$R_I(\bar{A} + A) = \sum_{C(\sigma) = I} G_{\sigma}(\bar{A} + A) \quad (40)$$

where $\bar{A} + A$ is the ordinal sum. If we order $\bar{A}$ by

$$\bar{a}_1 < \bar{a}_2 < \ldots < \bar{a}_k < \ldots \quad (41)$$

then, arguing as in [16], we have

$$G_{\sigma}(\bar{A} + A) = \sum_{\text{std}(\tau, \epsilon) = \sigma} G_{\tau, \epsilon} \quad (42)$$
so that
\[ R_1(\bar{A} + A) = \sum_{\rho(J) = I} R_J \tag{43} \]
where for a signed composition \( J = (J, \epsilon) \), the unsigned composition \( \rho(J) \) is defined as the shape of\( \text{std}(\sigma, \epsilon) \), where \( \sigma \) is any permutation of shape \( J \).

Replacing \( \bar{A} \) by \( q \bar{A} \), one obtains the expansion of the \( q \)-Klyachko elements of type \( B \):
\[ K_n(q; A, \bar{A}) = \sum_J q^{b_{\text{maj}}(J)} R_J \tag{44} \]
where
\[ b_{\text{maj}}(J) = 2 \text{ maj}(\rho(J)) + |\epsilon|, \tag{45} \]
where \(|\epsilon|\) is the number of minus signs in \( \epsilon \).

For example,
\[ K_2(q) = R_3 + q^2 R_\tau + q^2 R_{\tau 1} + q^3 R_{\tau 1} + q^4 R_{\tau \tau}. \tag{46} \]
\[ K_3(q) = R_3 + q^3 R_\tau + q^4 R_{\tau 21} + q^5 R_{\tau \tau 1} + q^2 R_{\tau 21} + q^7 R_{\tau \tau 2} + q^2 R_{\tau 21} + q^4 R_{\tau 2} \]
\[ + q R_{\tau 22} + q^5 R_{\tau \tau 1} + q^6 R_{\tau 11} + q^7 R_{\tau 11} + q^8 R_{\tau 12} + q^3 R_{\tau 11} + q^8 R_{\tau \tau \tau} + q^5 R_{\tau \tau 2} + q^4 R_{\tau \tau 11} + q^6 R_{\tau \tau \tau} \tag{47} \]
This major index of type \( B \) is the flag major index defined in \( \text{(1)} \).

Following \( \text{(1)} \) and considering the signed composition (where \( \epsilon \) is encoded as boolean vector for readability)
\[ J = (2, 1, 1, 3, 1, 2, 4, 1, 2, 2) = (211324122, 00001111110000100000) \tag{48} \]
we can take the smallest permutation of shape \( (2, 1, 1, 3, 1, 2, 4, 1, 2, 2) \), which is
\[ \alpha = 1 5 4 3 2 6 9 8 7 11 10 12 13 16 15 14 18 17 19 \tag{49} \]
sign it according to \( \epsilon \), which yields
\[ 1 5 4 3 2 6 0 9 8 7 \bar{\tau} 1 0 1 2 13 16 \bar{\tau} 1 5 14 18 17 19 \tag{50} \]
whose standardized is
\[ 8 1 1 1 0 9 1 1 2 5 4 3 6 1 2 13 14 16 7 15 18 17 19 \tag{51} \]
and has shape \( \rho(J) = (2, 1, 1, 3, 1, 2, 4, 1, 2, 2) \). The major index of \( \rho(J) \) is 55, the number of minus signs in \( \epsilon \) is 7, so \( b_{\text{maj}}(J) = 2 \times 55 + 7 = 117 \).

The major index of type \( B \) can be read directly on signed compositions without reference to signed permutations as follows: one can get \( \rho(J) \) by first adding the absolute values of two consecutive parts if the left one is signed and the second one is not, then remove the signs and proceed as before.

A different solution consists in reading the composition from right to left, then associate weight 0 (resp. 1) to the rightmost part if it is positive (resp. negative) and then proceed left by adding 2 to the weight if the two parts are of the same sign and 1 if not. Finally, add up the product of the absolute values of the parts with their weight.

For example, with the same \( J \) as above we have the following weights:
\[ J = (2, 1, 1, 3, 1, 2, 4, 1, 2, 2) \]
\[ \text{weights:} 14 12 10 9 7 5 4 3 2 0 \tag{52} \]
so that we get \( 2 \cdot 14 + 1 \cdot 12 + 1 \cdot 10 + 3 \cdot 9 + 1 \cdot 7 + 2 \cdot 5 + 4 \cdot 4 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 = 117 \).

This technique generalizes immediately to colored compositions with a fixed number \( c \) of colors 0, 1, \ldots, \( c - 1 \): the weight of the rightmost cell is its color and the weight of a part is equal to the sum of the weight of the next part and the unique representative of the difference of the colors of those parts modulo \( c \) belonging to the interval \( [1, c] \).
6.2 Generators and Hilbert series

For \( n \geq 0 \), let

\[
S_{n}^{\pm} = S_{n}(A) \pm S_{n}(\bar{A}),
\]

and denote by \( \mathcal{H}_{n} \) the subalgebra of \( MR \) generated by the \( S_{k}^{\pm} \) for \( k \leq n \). For \( n \geq 0 \), we have

\[
(S_{n}^{\pm})^{\sharp} \equiv (1 \mp q^{n})S_{n}^{\pm} \mod \mathcal{H}_{n-1},
\]

so that the \((S_{n}^{\pm})^{\sharp}\) such that \( 1 \mp q^{n} \neq 0 \) form a set of free generators in \( MR^{\sharp}\).

**Conjecture 6.1** If \( r \) is odd, a basis of \( MR^{\sharp} \) will be parametrized by colored compositions such that parts of color 0 are not \( \equiv 0 \mod r \) and parts of color 1 are arbitrary. The Hilbert series is then

\[
H_{r}(t) = \frac{1 - t^{r}}{1 - 2(t + t^{2} + \cdots + t^{r})}.
\]

If \( r \) is even, there is the extra condition that parts of color 1 are not \( \equiv r/2 \mod r \). The Hilbert series is then

\[
H_{r}(t) = \frac{1 - t^{r}}{1 - 2(t + t^{2} + \cdots + t^{r}) + t^{r/2}}.
\]

For example,

\[
H_{2}(t) = 1 + t + 2t^{2} + 4t^{3} + 8t^{4} + 16t^{5} + 32t^{6} + 64t^{7} + 128t^{8} + O(t^{9})
\]

\[
H_{3}(t) = 1 + 2t + 5t^{2} + 14t^{3} + 38t^{4} + 104t^{5} + 284t^{6} + 776t^{7} + 2120t^{8} + O(t^{9})
\]

\[
H_{4}(t) = 1 + 2t + 5t^{2} + 14t^{3} + 38t^{4} + 104t^{5} + 284t^{6} + 776t^{7} + 2120t^{8} + O(t^{9})
\]

If these conjectures are correct, the Hilbert series of the right \( MR^{\sharp} \)-modules generated by the \( S_{n} \) are respectively

\[
\frac{1}{1 - 2(t + t^{2} + \cdots + t^{r})},
\]

or

\[
\frac{1}{1 - 2(t + t^{2} + \cdots + t^{r}) + t^{r/2}}.
\]

according to whether \( r \) is odd or even.

The cases \( r = 1 \) and \( r = 2 \) are easily proved as follows. Assume first that \( q = 1 \). Set

\[
f = 1 + (\sigma_{1}^{+})^{2} = (\sigma_{1} + \lambda(-))(A - \bar{A}),
\]

\[
g = (\sigma_{1}^{-})^{2} - 1 = (\sigma_{1} - \lambda(-))(A - \bar{A}).
\]

Then, \( f^{2} = g^{2} + 4 \), so that

\[
f = 2 \left( 1 + \frac{1}{q^{2}} \right)^{1/2}
\]

which proves that the \((S_{n}^{+})^{2}\) can be expressed in terms of the \((S_{n}^{-})^{2}\).

Similarly, for \( q = -1 \), one can express

\[
f = \sum_{n \geq 1} (S_{2n}^{+})^{2} + \sum_{n \geq 0} (S_{2n+1}^{-})^{2}
\]
in terms of
\[ g = \sum_{n \geq 1} (S_{2n}^-)^2 + \sum_{n \geq 0} (S_{2n+1}^+)^2 \]  
(66)

since, as is easily verified,
\[ (f + 1)^2 = g^2 + 4, \text{ i.e., } f = -2 + 2 \left( 1 + \frac{1}{4}g^2 \right) \frac{1}{2}. \]  
(67)

Apparently, this approach does not work anymore for higher roots of unity.

7 Appendix: monomial expansion of the \((1 - q)\)-kernel

The results of [16, 7] allow us to write down a new expansion of \(S_n((1 - q)A)\), in terms of the monomial basis of [3]. The special case \(q = 1\) gives back a curious expression of Dynkin’s idempotent, first obtained in [3].

Let \(\sigma\) be a permutation. We then define its left-right minima set \(LR(\sigma)\) as the values of \(\sigma\) that have no smaller value to their left. We will denote by \(lr(\sigma)\) the cardinality of \(LR(\sigma)\). For example, with \(\sigma = 46735182\), we have \(LR(\sigma) = \{4, 3, 1\}\), and \(lr(\sigma) = 3\).

Let us now decompose \(S_n((1 - q)A)\) on the monomial basis \(M_\sigma\) (see [4]) of \(\mathbb{F}Q\text{Sym}\). Thanks to the Cauchy formula of \(\mathbb{F}Q\text{Sym}\) [7], we have
\[ S_n((1 - q)A) = \sum_\sigma S_\sigma(1 - q)M_\sigma(A), \]  
(68)

where \(S\) is the dual basis of \(M\). Given the transition matrix between \(M\) and \(G\), we see that
\[ S_\sigma = \sum_{\tau \leq \sigma^{-1}} F_\tau, \]  
(69)

where \(\leq\) is the right weak order, e.g., \(S_{123}^{312} = F_{123} + F_{213} + F_{231}\). Thanks to [16], we know that \(F_\sigma(1 - q)\) is either \((-q)^k\) if \(\text{Des}(\sigma) = \{1, \ldots, k\}\) or \(0\) otherwise. Let us define hook permutations of hook \(k\) the permutations \(\sigma\) such that \(\text{Des}(\sigma) = \{1, \ldots, k\}\). Now, \(S_\sigma(1 - q)\) amounts to compute the list of hook permutations smaller than \(\sigma\). Note that hook permutations are completely characterized by their left-right minima. Moreover, if \(\tau\) is smaller than \(\sigma\) in the right weak order, then \(LR(\tau) \subset LR(\sigma)\).

Hence all hook permutations smaller than a given permutation \(\sigma\) belong to the set of hook permutations with left-right minima in \(LR(\sigma)\). Since by elementary transpositions decreasing the length, one can get from \(\sigma\) to the hook permutation with the same left-right minima and then from this permutation to all the others, we have:

**Theorem 7.1** Let \(n\) be an integer. Then
\[ S_n((1 - q)A) = \sum_{\sigma \in S_n} (1 - q)^{lr(\sigma)}M_\sigma. \]  
(70)

\[ \Psi_n = \sum_{\sigma \in S_n, \sigma(1) = 1} M_\sigma, \]  
(71)

where \(\Psi_n\) is the noncommutative power sum associated with Dynkin’s idempotent [11 Prop. 5.2].
Acknowledgements

This work started during a stay of the first author at the University Paris-Est Marne-la-Vallée. The authors would also like to thank the contributors of the MuPAD project, and especially those of the combinat package, for providing the development environment for their research (see [10] for an introduction to MuPAD-Combinat).

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