A NOTE ON CYCLES OF CURVES IN A PRODUCT OF PAIRS

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ABSTRACT. We discuss the role of subdivisions of tropical moduli spaces in logarithmic Gromov–Witten theory, and use them to study the virtual class of curves in a product of pairs. Our main result is that the cycle-valued logarithmic Gromov–Witten theory of $X \times Y$ decomposes into a product of pieces coming from $X$ and $Y$, but this decomposition must be considered in a blowup of the moduli space of curves. This blowup is specified by tropical moduli data. As an application, we show that the cycle of curves in a toric variety with fixed contact orders is a product of virtual strict transforms of double ramification cycles. The formalism we outline offers a unified viewpoint on a number of recent results in logarithmic Gromov–Witten theory, including works of Herr, Holmes–Pixton–Schmitt, and Nabijou and the author.

1. INTRODUCTION

Logarithmic Gromov–Witten theory concerns cycles on moduli spaces of stable maps from pointed curves $(C, p_1, \ldots, p_n)$ to a pair $(Z, D_Z)$, where $Z$ is a smooth variety and $D_Z$ is a simple normal crossings divisor. The logarithmic structure prescribes the tangency orders of each point $p_i$ along each divisor $D_j$. This tangency order is equal to the scheme theoretic contact order for non-degenerate maps, but also remains locally constant in families. The resulting moduli space $K_{\Gamma}(Z)$ is equipped with a virtual fundamental class and invariants are integrals of evaluation cycles from $Z$ and tautological classes against the virtual class $[1, 10, 13]$.

A basic property of the virtual structure of the spaces $K_{\Gamma}(Z)$ that has appeared in recent work in the subject is the behaviour under products $[14, 16, 21, 24]$. We revisit the question in this note from the viewpoint of tropical moduli theory. Let $(X, D_X)$ and $(Y, D_Y)$ be smooth projective varieties equipped with simple normal crossings divisors, and let $(Z, D_Z)$ be the product.

1.1. Modified products. Fix the genus, curve class, marked points, and their contact orders for maps to $Z$. This determines discrete data for maps to $X$ and $Y$ as well. We will package the discrete data in the symbol $\Gamma$, and use it flexibly to name the discrete data on moduli spaces of maps to $X, Y$, and $Z$. The moduli space of maps from smooth curves to $Z$ with given contact orders is naturally identified with the fiber product the mapping spaces for $X$ and $Y$, over $M_{g,n}$. This product decomposition breaks down over $\overline{M}_{g,n}$.

For simplicity, assume that the genus and markings are in the stable range. Given spaces $K^\dagger$ and $K$ equipped with virtual class, if there is a morphism $K^\dagger \to K$ such that pushforward identifies virtual classes, we say that $K^\dagger$ is a virtual birational model of $K$. Our main result explains that the product formula continues to hold provided the moduli spaces are replaced by virtual birational models.
Theorem A. There exists an explicit logarithmic modification of the moduli space of stable curves

\[ \overline{M}_\Gamma \rightarrow \overline{M}_{g,n} \]

and virtual birational models \( K_\Gamma(W) \rightarrow K_\Gamma(W) \) for \( W = X, Y, Z \), fitting into a diagram

\[ \begin{array}{ccc}
K_\Gamma(Z) & \xrightarrow{\vartheta} & K_\Gamma(X) \times K_\Gamma(Y) \\
\overline{M}_\Gamma & \xrightarrow{\Delta} & \overline{M}_\Gamma \times \overline{M}_\Gamma.
\end{array} \]

There is an equality of virtual classes

\[ \vartheta_* [K_\Gamma(Z)]_{\text{vir}} = \Delta^! [K_\Gamma(X) \times K_\Gamma(Y)]_{\text{vir}} \text{ in } A_*(P_\Gamma(X \times Y); \mathbb{Q}). \]

1.2. Geometry of the modification. The necessity of this modification can be easily captured in examples, and we comment further on it below. The modification \( \overline{M}_\Gamma \rightarrow \overline{M}_{g,n} \) is easily described. The morphisms \( \pi_W : K_\Gamma(W) \rightarrow \overline{M}_{g,n} \) are typically ill-behaved at the level of logarithmic structures. This manifests concretely as follows. If \( S \) is a codimension \( k \) stratum of \( \overline{M}_{g,n} \), its preimage, which is a logarithmic stratum, may not have virtual codimension \( k \) in \( K_\Gamma(W) \). This is equivalent to the statement that the morphism \( \pi_W \) is not integral and saturated as a logarithmic morphism, or that it is not flat with reduced fibers at the level of Artin fans [4].

Combinatorially, in the morphism of tropical moduli stacks associated to \( \pi_W \), there are cones which do not map surjectively onto their target cone. The tropical moduli stack \( \overline{M}_{\Gamma}^{\text{trop}} \) can be subdivided until the image of \( \pi_W \) is a union of cones for each \( W = X, Y, Z \), i.e. along the image of the cycle of tropical maps of type \( \Gamma \). Toroidal geometry furnishes a birational modification \( \overline{M}_\Gamma \) of \( \overline{M}_{g,n} \), and there is a compatible refinement for mapping spaces.

In the next section we describe how such modifications can be interpreted as moduli functors on logarithmic schemes, carrying natural perfect obstruction theories, whose virtual classes have the expected properties.

1.3. Toric contact cycles. We briefly recall that given a vector of integers \( A \in \mathbb{Z}^n \) with vanishing sum, the double ramification cycle \( DR_g(A) \) is cycle on the moduli space of curves whose restriction to \( \overline{M}_{g,n} \) is the locus of curves that admit a map to \( \mathbb{P}^1 \) with ramification orders over 0 and \( \infty \) given by the positive and negative entries in \( A \) along the corresponding markings. Let \( K_{g,A}(\mathbb{P}^1)_{\mathbb{G}_m} \) be the moduli space of genus \( g \) logarithmic stable maps to \( \mathbb{P}^1 \) relative to 0 and \( \infty \) with contact orders \( A \), up to \( \mathbb{G}_m \) scaling on the target. The double ramification cycle is the pushforward of the virtual class of this mapping space under

\[ K_{g,A}(\mathbb{P}^1)_{\mathbb{G}_m} \rightarrow \overline{M}_{g,n}. \]

The class is known to lie in the tautological ring and a formula for it has been calculated [12, 17].

We consider the following generalization. Fix contact order data \( A_1 \) and \( A_2 \) in \( \mathbb{Z}^n \). Let \( K_\Gamma(\mathbb{P}^1 \times \mathbb{P}^1)_T \) be the moduli space of genus \( g \) logarithmic stable maps to \( \mathbb{P}^1 \times \mathbb{P}^1 \) with contact orders along the toric boundary given by \( (A_1, A_2) \). We consider two maps equivalent if the differ by scaling under the 2-dimensional dense torus action. Define the toric contact cycle \( TC_g(A_1, A_2) \) to be the pushforward of the virtual fundamental class under the morphism

\[ K_\Gamma(\mathbb{P}^1 \times \mathbb{P}^1)_T \rightarrow \overline{M}_{g,n}. \]
Theorem B. There exists an explicit logarithmic modification of the moduli space of stable curves
\[ \overline{\mathcal{M}}_\Gamma \xrightarrow{\pi} \mathcal{M}_{g,n} \]
and explicit lifts of the cycles DR(\( A_i \)) and TC(\( A_1, A_2 \)) in the Chow groups of \( A_\ast(\overline{\mathcal{M}}_\Gamma; \mathbb{Q}) \), such that, denoting the lifts by hats, there is an equality
\[ \hat{\text{TC}}(A_1, A_2) = \hat{\text{DR}}(A_1) \cdot \hat{\text{DR}}(A_2) \] in \( A_\ast(\overline{\mathcal{M}}_\Gamma; \mathbb{Q}) \).

The analogous statements for multifold products also holds. Note that a marked point may have nonzero contact with multiple divisors, so the statement for multifold products of \( \mathbb{P}^1 \) implies the analogous result for arbitrary toric varieties, by virtual birational invariance [4].

1.4. Recent results. The double ramification cycle can also be defined using a resolution of the Abel–Jacobi section from the moduli space of curves to the universal Picard variety, and in that context, the statement of Theorem B has been proved recently in an elegant paper of Holmes–Pixton–Schmitt [16]. In particular Section 8 of loc. cit. provides a concrete example exhibiting the failure of multiplicativity of the toric contact cycles. Our second result is a proof of their result from the Gromov–Witten viewpoint. The state of the art on resolutions of the Abel–Jacobi section may be found in [5, 15, 23].

Even more recently, Leo Herr has proved the product formula in logarithmic Gromov–Witten theory as an application of an elegant and general framework of logarithmic normal cones and logarithmic perfect obstruction theories. These yield a notion of virtually logarithmically smooth morphisms [14]. If the logarithmic obstruction theories are unwound into a piece coming from an ordinary perfect obstruction theory and another coming from a logarithmic blowup, one is led to a formula of the form presented here. In contrast, the results here are presented using the traditional framework of virtually smooth morphisms. We hope that the presentation here explains the need and naturality of a logarithmic Chow theory in the work of Herr, as well as earlier work of Barrott [7, 14].

The idea of using tropical geometry to correct the product formula appeared also in our earlier work with Nabijou, proving the local/logarithmic conjecture [24]. The basic strategy in that paper is to establish a version of the product formula for different logarithmic structures over the same base manifold, and therefore relate the logarithmic Gromov–Witten theory with the older relative theory for smooth pairs.

Finally, this paper serves as a companion to [25], which establishes the degeneration formula for logarithmic Gromov–Witten invariants using expanded degenerations. Products of smooth pairs provide basic examples of normal crossings pairs, and the central strategy required to prove the product formula appears in loc. cit. In particular, we rely on the correcting “naive” intersections of classes by strict transform calculations attached to subdivisions to obtain both results.

1.5. Earlier work on products. When the logarithmic structures on \( X \) and \( Y \) are both trivial, the tropical part of the geometry disappears and we recover Behrend’s product formula for ordinary Gromov–Witten theory, and indeed its proof [8]. The product formula also holds for orbifolds [6]. If \( X \) alone has trivial logarithmic structure, the product formula holds without birational modifications, and was proved by Lee and Qu [21]. In genus 0, for toric targets there is a product decomposition for the moduli space of logarithmic maps, using the logarithmic torus [28].
Acknowledgements. My view on this subject has been shaped by conversations with Davesh Maulik in the context of earlier work on the degeneration formula, and I’m grateful to him for his generosity with time. I thank Dan Abramovich for the comment, “it would all be fine if the stabilization morphism was flat, but it probably fails in general”, which finally clarified the geometry. I have also benefited from conversations with Tom Graber, Andreas Gross, Navid Nabijou, Rahul Pandharipande, and Jonathan Wise. I thank Jonathan Wise for keeping me informed of the parallel work of his student Leo Herr.

2. Subdivisions of moduli problems

Given a moduli space $\mathcal{M}$ representing a fibered category over schemes, a blowup $\tilde{\mathcal{M}}$ does not come equipped with a natural modular interpretation. However, if $\mathcal{M}$ is a logarithmic scheme representing a fibered category over logarithmic schemes, a logarithmic blowup can be described as a subcategory of the fibered category $\mathcal{M}$. This fact was observed by Kato [18], and utilized in logarithmic moduli problems in [23, 25, 26, 27]. We begin by reviewing these ideas.

2.1. Subdivisions: cone complexes and logarithmic schemes. Let $\Sigma$ be a rational polyhedral cone complex. This defines a functor from the category of rational polyhedral cones to sets:

$$F_{\Sigma} : \text{Cones} \to \text{Sets}$$

$$\tau \mapsto \text{Hom}(\tau, \Sigma).$$

A key observation is to note that if $\tilde{\Sigma} \to \Sigma$ is a subdivision of cone complexes, then $F_{\tilde{\Sigma}}$ is a subfunctor of $F_{\Sigma}$.

The link to logarithmic geometry arises as follows. Assume that $\Sigma$ is a fan embedded in a vector space, and let $X = X(\Sigma)$ be the toric variety associated to $\Sigma$. The scheme $X$ inherits a natural logarithmic structure from the toric boundary divisor, and therefore defines a functor on logarithmic schemes

$$F_X : \text{LogSch} \to \text{Sets}$$

$$S \mapsto \text{Hom}(S, X),$$

where the morphisms are taken to be in the category of logarithmic schemes.

Each logarithmic scheme $S$ comes equipped with a tropicalization $S^{\text{trop}}$, which we may assume is a single cone. There is a natural tropicalization map carrying a logarithmic scheme to its underlying polyhedral functor [9, 20]. Kato observed that there is an identification

$$F_X = F_X \times_{F_{\Sigma}} F_{\tilde{\Sigma}}.$$

In other words, just as a subdivision of a cone complex is a subfunctor of the original cone complex, a toric modification of a toric variety, interpreted as a functor on logarithmic schemes, is a subfunctor of the original functor.

This formalism does not require $X$ to be toric, only that it has a logarithmic structure. A logarithmic structure on $X$ gives rise to a functor on the category of cone complexes whose $\mathbb{R}_{\geq 0}$ points is its tropicalization. Given a logarithmic scheme $X$, a logarithmic modification is by pulling back the subfunctor defined by a refinement on its tropicalization. It is a basic observation that these subfunctors are in turn obtained from schemes with logarithmic structure. That is, they are representable by schemes with logarithmic structure [18].
Remark 2.1. The interpretation of logarithmic modifications as subfunctors, or monomorphisms of logarithmic schemes, is lost once we pass to the representing ordinary scheme. This manifests combinatorially. Passing from the logarithmic scheme to its underlying scheme is analogous to replacing a fan with its partially ordered set of faces. Any subdivision of polyhedral complexes is injective, but the map of fans induces a map on partially ordered sets that is no longer an injection.

2.2. Logarithmic and tropical moduli of curves. In our examples, the logarithmic scheme above will be replaced by a moduli space of curves or of logarithmic stable maps to a logarithmic scheme $X$. The fan side is controlled by tropical moduli theory, which we briefly recall. We follow standard conventions for tropical curves [9].

Definition 2.2. An $n$-marked tropical curve $\square$ is a finite graph $G$ with vertex set $V$ and edge set $E$, enhanced with

1. **markings** $m : \{1, \ldots, n\} \to V$,
2. a **vertex genus function** $g : V \to \mathbb{N}$,
3. an **edge length function** $\ell : E \to \mathbb{R}_+.$

The **genus** of $\square$ is equal to

$$g(\square) = h_1(G) + \sum_{v \in V} g(v).$$

The edge length function gives $\square$ the structure of a metric space, where marked points are realized by attaching copies $\mathbb{R}_{\geq 0}$ to appropriate vertices, as “legs”. There is a notion of a family of tropical curves.

Definition 2.3. Let $\sigma$ be a cone with dual cone $S_\sigma$. A family of tropical curves over $\sigma$ is a tropical curve in the sense above, but whose length function takes values in in $S_\sigma$.

A point of $\sigma$ is a monoid homomorphism $\varphi : S_\sigma \to \mathbb{R}_{\geq 0}$, and applying it to the edge length $\ell(e) \in S_\sigma$, we obtain a positive real length for each edge and thus a tropical curve.

Given a family of logarithmic curves $\mathcal{C}/S$, there is a tropicalization $\mathcal{C}/S_{\text{trop}}$, which is a family of tropical curves with the natural genus and marking data. When $S$ is a single point with logarithmic structure, the tropicalization is obtained by decorating the edges of the dual graph of $\mathcal{C}$ with the data of the deformation parameters of the nodes. The procedure is treated in detail by [9].

We may view $\overline{M}_{g,n}$ as a category fibered in groupoids over logarithmic schemes, and to emphasize this, we temporarily introduce the notation $M_{g,n}^{\log}$. There is a tropical moduli stack $M_{g,n}^{\text{trop}}$. This tropical moduli stack is a category fibered in groupoids over the category of rational polyhedral cone complexes [9]. There is a surjective morphism

$$M_{g,n}^{\log} \to M_{g,n}^{\text{trop}}$$

by taking a family of logarithmic curves to its tropicalization.

2.3. Minimality and tropical deformations. The moduli space of stable curves $\overline{M}_{g,n}$ can be recovered as a subcategory of $M_{g,n}^{\log}$. Let $S = \text{Spec}(P \to \mathbb{C})$ and let $\mathcal{C}/S$ be a logarithmic curve. The dual graph of $\mathcal{C}$ determines a cone of the tropical moduli space $M_{g,n}^{\text{trop}}$, which we denote $\sigma(\mathcal{C})$.

The logarithmic curve over a logarithmic point is minimal if the dual monoid $\text{Hom}(P, \mathbb{R}_{\geq 0})$ is isomorphic to $\sigma(\mathcal{C})$. That is, in the corresponding family of tropical curves, there are no unexpected relationships between the edge lengths. A family is said to be minimal if it is minimal at every point. F. Kato characterizes the moduli space of stable curves as follows.
Theorem 2.4 ([19]). The moduli space of stable curves $\overline{M}_{g,n}$ can be identified with the open substack of $\mathcal{M}_{g,n}^{log}$ parameterizing minimal logarithmic curves.

In other words, there are no relations between the deformation parameters of the nodes of $\mathcal{C}$.

2.4. Mapping spaces. We also require the parallel statements for the moduli spaces of stable maps to logarithmic schemes. Let $\Sigma$ be a fan and let $\overline{\mathcal{C}}$ be a tropical curve.

Definition 2.5. A tropical map from $\overline{\mathcal{C}} \to \Sigma$ is a continuous piecewise linear map such that every face of $\overline{\mathcal{C}}$ maps to a face of $\Sigma$.

Families of tropical maps are defined in the analogous fashion, and one obtains a moduli space $T_{\Gamma}(\Sigma)$ of tropical stable maps.

Remark 2.6. Observe that continuity of the map $\overline{\mathcal{C}} \to \Sigma$ imposes nontrivial restrictions on the edge lengths of $\overline{\mathcal{C}}$. For instance, if a collection of edges $e_1, \ldots, e_k$ form a cycle in $\overline{\mathcal{C}}$, continuity forces that the total displacement around the cycle is zero, and therefore a linear relationship between the edge lengths of the $e_i$.

Given a logarithmic scheme $X$ with tropicalization $\Sigma$, there is a fibered category $K^{log}_{\Gamma}(X)$ over logarithmic schemes, whose fiber over a logarithmic scheme is the groupoid of families of logarithmic curves over $S$ together with a map to $X$. There is again a morphism

$$K^{log}_{\Gamma}(X) \to T_{\Gamma}(\Sigma).$$

Abramovich–Chen and Gross–Siebert have developed a theory of minimal (resp. basic) logarithmic maps [1, 10, 13]. There is a subcategory $K^{log}_{\Gamma}(X)$ of minimal logarithmic maps from within the category all logarithmic maps $K^{log}_{\Gamma}(X)$. A family is said to be minimal if, at each geometric point, the associated tropical family is a cone of this tropical moduli space.

Remark 2.7. The dual viewpoint to the statement above is that a logarithmic family of maps is minimal precisely when the associated tropical family of maps is maximal. It is immediate from this that if a family of logarithmic stable maps to $X$ is minimal, the underlying family of curves is typically not minimal when $X$ has nontrivial logarithmic structure. Equivalently, in the induced map from $T_{\Gamma}(\Sigma) \to M_{g,n}^{trop}$, there are cones of the source that do not surject onto their target cones.

2.5. Variation of minimality and subcategories. Subdivisions of tropical moduli spaces produce subcategories in the following manner. Given a refinement of cone stacks

$$\mathcal{M}^{trop}_{\Gamma} \to \mathcal{M}_{g,n}^{trop},$$

base change along the tropicalization map $\mathcal{M}_{g,n}^{log} \to \mathcal{M}_{g,n}^{trop}$ gives rise to a subcategory

$$\mathcal{M}^{log}_{\Gamma} \to \mathcal{M}_{g,n}^{log}.$$ 

Inside $\mathcal{M}^{log}_{\Gamma}$, we may pick out a further subcategory of $\Gamma$-minimal objects. Specifically, an object in this category is still a family of logarithmic curves $\mathcal{C} \to S$ over a logarithmic base. The tropicalization is then a family $\overline{\mathcal{C}} \to S^{trop}$, and we obtain a moduli map

$$S^{trop} \to \mathcal{M}^{trop}_{\Gamma}.$$ 

A family of curves is $\Gamma$-minimal if, at every point of $S$, the tropicalization of the base is identified with a cone of the moduli space $\mathcal{M}^{trop}_{\Gamma}$. This subcategory of $\Gamma$-minimal objects is representable by a stack, and in fact by a logarithmic modification of $\overline{M}_{g,n}$.

Once again, there are parallel statements for tropical maps, which we employ in the next section.
To begin, we examine the product formula in an entirely combinatorial setting. Fix fans $\Sigma_X$ and $\Sigma_Y$ associated to the target varieties $X$ and $Y$, and let $\Sigma_Z$ be their product. As in the introduction, fix discrete data $\Gamma$, which, in a mild abuse of notation, will be used to describe the compatible discrete data on $X, Y$, and the product.

### 3.1. Combinatorial products

We require a basic finiteness statement.

**Lemma 3.1.** Fix discrete data $\Gamma$. The cone complex $T_\Gamma(\Sigma_Z)$ associated to the space $K_\Gamma(Z)$ of logarithmic stable maps to $Z$ is a finite-type cone complex.

**Proof.** The targets have simple normal crossings logarithmic structures, so under these hypotheses, the statement is equivalent to the combinatorial finiteness proved by Gross and Siebert [13, Section 3.1].

The forgetful morphism $K_\Gamma(Z) \to \overline{M}_{g,n}$ induces a morphism of cone complexes

$$T_\Gamma(\Sigma_Z) \to \overline{M}_{g,n}^{\text{trop}}.$$  

**Lemma 3.2.** The image of the forgetful morphism

$$T_\Gamma(\Sigma_Z) \to \overline{M}_{g,n}^{\text{trop}},$$

is supported on a finite type conical subset of the cone complex $\overline{M}_{g,n}^{\text{trop}}$.

**Proof.** This follows immediately from the combinatorial finiteness in the previous lemma.

Although the image of the morphism is a conical subset, it is not a subcomplex of the tropical moduli space of curves.

**Lemma 3.3.** There exists a smooth (unimodular) subdivision of conical subcomplexes

$$\overline{M}_{g,n}^{\text{trop}} \to \overline{M}_{g,n}^{\text{trop}}$$

such that the images of the morphisms

$$T_\Gamma(\Sigma_Z) \to \overline{M}_{g,n}^{\text{trop}}, \quad T_\Gamma(\Sigma_X) \to \overline{M}_{g,n}^{\text{trop}} \quad \text{and} \quad T_\Gamma(\Sigma_Y) \to \overline{M}_{g,n}^{\text{trop}}$$

are each unions of cones in the subdivision $\overline{M}_{g,n}^{\text{trop}}$.

**Proof.** The lemma is a consequence of [3, Lemma 4.3], which asserts that for a morphism of polyhedral complexes, there is always a projective subdivision of the target that ensures that the image of each cone of the source is a union of cones. Toric resolution of singularities guarantees the existence of a smooth subdivision.

From this point forward, we fix a choice of subdivision $\overline{M}_{g,n}^{\text{trop}}$ of the tropical moduli space of curves realizing the lemma above.

As constructed, the morphism

$$T_\Gamma(\Sigma_X) \to \overline{M}_{g,n}^{\text{trop}}$$

may not be a morphism of cone complexes, as the image of a cone may be a union of cones, and therefore not contained in a single cone. However, we choose subdivisions $T_\Gamma(X)^\dagger, T_\Gamma(Y)^\dagger,$ and
Lemma 3.4. Each of the morphisms
\[ T_\Gamma(\Sigma Z)^\dagger \rightarrow M_\Gamma^{\text{trop}}, \quad T_\Gamma(\Sigma X)^\dagger \rightarrow M_\Gamma^{\text{trop}}, \quad \text{and} \quad T_\Gamma(\Sigma Y)^\dagger \rightarrow M_\Gamma^{\text{trop}} \]
have the property that they map cones surjectively onto cones.

3.2. Virtual strict transforms. The formalism in Section 2, together with the subdivision \( M_\Gamma^{\text{trop}} \) produced in the previous section, give rise to a birational models of the moduli spaces of curves and maps. These will be the spaces claimed to exist in the introductory statements.

In order to lift our combinatorial statements to virtual classes, we use a formalism developed by Abramovich and Wise [4]. If \( W \) is a toric variety with dense torus \( T \), the Artin fan of \( W \) is the Artin stack \( \mathcal{A}_W = [W/T] \). Both \( W \) and \( \mathcal{A}_W \) are equipped with divisorial logarithmic structure, and the morphism
\[ W \rightarrow \mathcal{A}_W \]
is strict. The Artin fan is logarithmically étale over a point. For an arbitrary logarithmic scheme \( W \), there is a replacement of the global quotient construction above, producing an Artin fan, which continues to be an Artin stack, logarithmically étale over a point; \( W \) has a strict morphism
\[ W \rightarrow \mathcal{A}_W. \]

The stack \( \mathcal{A}_W \) is essentially combinatorial, constructed from the local toric models that give charts for the logarithmic structure for \( W \). Details may be found in [2].

Manolache has defined a functorial virtual pullback in Chow homology for morphisms equipped with relative perfect obstruction theories [22].

Proposition 3.5. Let \( W \) be logarithmically smooth. The moduli stack of logarithmic maps \( \mathcal{K}_\Gamma(\mathcal{A}_W) \) from curves to \( \mathcal{A}_W \) is a logarithmically smooth stack that is locally of finite type. The natural morphism
\[ \mathcal{K}_\Gamma(W) \rightarrow \mathcal{K}_\Gamma(\mathcal{A}_W) \]
has a natural relative perfect obstruction theory. The virtual pullback of the fundamental class is equal to the virtual fundamental class of \( \mathcal{K}_\Gamma(W) \).

The virtual class defined above has natural lifts to any logarithmic modification. Any subdivision
\[ T_\Gamma(\Sigma W)^\dagger \rightarrow T_\Gamma(\Sigma W) \]
induces a logarithmic modifications
\[ \mathcal{K}_\Gamma(\mathcal{A}_W)^\dagger \rightarrow \mathcal{K}_\Gamma(\mathcal{A}_W) \quad \text{and} \quad \mathcal{K}_\Gamma(W)^\dagger \rightarrow \mathcal{K}_\Gamma(W) \]
by pulling back [4, 20]. The moduli space \( \mathcal{K}_\Gamma(\mathcal{A}_W)^\dagger \) is certainly logarithmically smooth, since it is a logarithmically étale modification of a logarithmically smooth stack. We summarize the observations concerning the virtual structure of this stack from [25, Section 3.5].

Proposition 3.6. The morphism \( \pi : \mathcal{K}_\Gamma(W)^\dagger \rightarrow \mathcal{K}_\Gamma(\mathcal{A}_W)^\dagger \) has a relative perfect obstruction theory. The virtual pullback of the fundamental class defines a virtual fundamental class
\[ [\mathcal{K}_\Gamma(W)^\dagger]^{\text{vir}} := \pi^![\mathcal{K}_\Gamma(\mathcal{A}_W)^\dagger] \]
for $K_F(W)\dagger$. Moreover, pushforward along the modification

$$K_F(W)\dagger \to K_F(W)$$

identifies virtual classes.

The virtual class $[K_F(W)\dagger]_{vir}$ functions as a virtual strict transform of the virtual class on $K_F(W)$.

3.3. Proof of Theorem A. Let $\mathcal{M}_F$ be the logarithmic modification of $\mathcal{M}_{g,n}$ defined by pulling back the tropical subdivision $\mathcal{M}_{trop}^F \to \mathcal{M}_{trop}^{g,n}$. Consider the following commutative diagram.

$$
\begin{array}{ccc}
K_F(Z)\dagger & \xrightarrow{\vartheta} & P(X \times Y) \\
\downarrow{\varphi} & & \downarrow{\phi} \\
K_F(\mathfrak{A}_Z)\dagger & \xrightarrow{\nu} & P(\mathfrak{A}_X \times \mathfrak{A}_Y) \\
\downarrow{\psi} & & \downarrow{\psi} \\
\mathcal{M}_G & \xrightarrow{\Delta} & \mathcal{M}_G \times \mathcal{M}_G.
\end{array}
$$

We are now led to the main result, which asserts that after these logarithmic modifications, the virtual pullbacks in the above diagram are compatible.

**Theorem 3.7.** There is equality of Chow homology classes on the space $P(X \times Y)$:

$$\vartheta_*[K_F(Z)\dagger]_{vir} = \Delta^![K_F(X)\dagger \times K_F(Y)\dagger]_{vir}.$$

**Proof.** We first inspect the vertical map $\psi$, which, on each factor forgets the data of the map and stabilizes if necessary. We claim that this map is flat with reduced fibers. In order to see this, we first note that the morphism is a toroidal morphism of toroidal embeddings. This is established in [4, Proposition 3.2]. We can therefore use the polyhedral criteria for equidimensionality [3, Lemma 4.1] and reducedness [3, Lemma 5.2]. These are satisfied by construction.

The flatness of $\psi$ leads to the main conclusions necessary for the proof. The flatness and reducedness implies that the two squares on the right are Cartesian in the categories of fine and saturated logarithmic stacks, as well as in ordinary stacks. The morphisms $\varphi$ and $\phi$ each give relative obstruction theories and these are compatible, and the proof of compatibility is identical to [8, Proposition 6]. As noted above, the morphism $\psi$ is a flat morphism by construction, and therefore $g$ is the flat base change of $\Delta$. In turn, $\Delta$ is the diagonal morphism on a smooth Deligne–Mumford stack, and is therefore a local complete intersection morphism. Since the property of being local complete intersection is stable under flat base change, we see $g$ is also a local complete intersection morphism. The obstruction theories given by $\Delta$ and $g$ are therefore compatible [29, Tag 069I].

Finally, we consider the morphism $\nu$. Since the stack $P(\mathfrak{A}_X \times \mathfrak{A}_Y)$ is a fiber product in ordinary schemes, it parameterizes data

$$(C, \tilde{C}_1, \tilde{C}_2, \tilde{C}_1 \to \mathfrak{A}_X, \tilde{C}_2 \to \mathfrak{A}_Y).$$

That is families of curves, together with two destabilizations of the curve, respectively equipped with logarithmic morphisms to $\mathfrak{A}_X$ and to $\mathfrak{A}_Y$. We identify the tropicalization of this moduli problem analogously, as parameterizing data of piecewise linear families

$$(\zeta, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_1 \to \Sigma_X, \tilde{\zeta}_2 \to \Sigma_Y).$$
Let \( \sigma \) be the base of such a family. Note that fiberwise over \( \sigma \), \( \tilde{e}_1 \) and \( \tilde{e}_2 \) are each simply subdivisions of \( \tilde{e} \). In particular, after a further subdivision of \( \sigma \), there exists a common refinement \( \tilde{e} \) of these two curves, admitting maps to \( \Sigma_X \) and \( \Sigma_Y \), and therefore to \( \Sigma_Z \). Observe that the morphism 
\[ K(\mathcal{O}_Z) \to P(\mathcal{O}_X \times \mathcal{O}_Y) \]
is pulled back from the morphism of their tropicalizations. The argument above shows that the analogous morphism of tropicalizations is a subdivision. The pullback is therefore a logarithmic modification, and we conclude that \( \nu \) has pure degree 1.

The product formula now follows from a standard diagram chase, using the established compatibilities. The fact that \( \nu \) is pure degree 1 shows that 
\[ \vartheta_\ast [K(\mathcal{O}_Z)]^{\text{vir}} = \phi_\ast [P(\mathcal{O}_X \times \mathcal{O}_Y)] \]
by an application of Costello’s pushforward theorem [11, Theorem 5.0.1]. We also have the equality 
\[ \phi_\ast [P(\mathcal{O}_X \times \mathcal{O}_Y)] = g_\ast [K(\mathcal{O}_X) \times K(\mathcal{O}_Y)] \]
by functoriality of virtual pullbacks [22, Theorem 4.1]. The result follows.

3.4. Toric contact cycles. To deduce Theorem B, we apply the argument with a minor change. Specifically, for the rubber geometry \( K(\mathbb{P}^1)_{G_m} \), we need a replacement for \( K(\mathcal{O}_X) \). The latter can be identified as fibered category over logarithmic schemes whose fibers are \( (C, \tilde{e} \to \mathbb{R}) \), where \( C \) is a logarithmic curve, \( \tilde{e} \) is its tropicalization, and the map \( \tilde{e} \to \mathbb{R} \) is piecewise linear, balanced, with asymptotic slopes given by \( A \).

We replace \( K(\mathcal{O}_X) \) with the category \( K(\mathcal{O}_X)_{G_trop} \), where (1) the data \( \tilde{e} \to \mathbb{R} \) is only required to be determined up to additive translation on the target, and (2) \( C \) is required to be stable. The corresponding category of minimal objects of schemes is an open subset of a blowup of \( \overline{M}_{g,n} \). The moduli space is constructed in detail in [23, Section 5]. With this replacement, an identical diagram chase as above yields the claimed result.

Remark 3.8. The cycle of curves in any toric variety, not necessarily one of product type, can be decomposed into products of strict transforms of double ramification cycles. Indeed, for \( X \) toric of dimension \( r \), equipped with its toric boundary logarithmic structure, the moduli space \( K(\mathcal{O}_X) \) of logarithmic maps is virtually birational to \( K(\mathbb{P}^1)^r \) by the invariance statements of [4]. The result above may then be applied to each factor of the product.

3.5. A sample subdivision. The subdivisions constructed in Section 3.1 can be described explicitly for the double ramification and toric contact cycles. We include a figure below, which describes a piece of the subdivision for genus 2 curves mapping to rubber \( \mathbb{P}^1 \) with degree 3, totally ramified over 0 and \( \infty \).

The cone associated to the cover in Figure 1 is a ray, since by continuity of the tropical map, the three bounded edges must have the same edge length. However, this ray maps into a 3-dimensional cone in \( M^{trop}_{2,2} \). This 3-dimensional cone is dual to a codimension 3 stratum, consisting of curves with two genus 0 components and three mutual nodes. Perform a stellar subdivision of this cone. After this subdivision, the ray of the tropical rubber mapping space now maps onto a subcomplex of this cone. Similar subdivisions can be made in other cones of the moduli space.
Figure 1. A dual graph of a relative stable map to rubber $\mathbb{P}^1$ of degree 3 with total ramification. The slopes along the outer edges are each 3 while the slopes along the internal edges are all 1.

After subdividing $\mathcal{M}_{2,2}^{\text{trop}}$, we obtain a birational model $\overline{\mathcal{M}}_\Gamma$ of the space of stable curves. The refinement may be pulled back to the tropical rubber space. The resulting virtual birational model of the rubber mapping space has “virtually proper intersections” with strata. That is, the preimage of a stratum under the map

$$K_\Gamma(\mathbb{P}^1)_{G_m} \to \overline{\mathcal{M}}_\Gamma,$$

if nonempty, has the expected virtual dimension. The pushforward of the virtual class is a transverse double ramification cycle, which we view as a strict transform of the usual double ramification cycle under the blowup. Our main theorem asserts that products of pullbacks (under point-forgetting) of such transverse cycles are toric contact cycles. We note that the combinatorics that arises in identifying the locus in the tropical moduli space of curves that admit a piecewise linear map to $\mathbb{R}$ with the given slopes is essentially a manifestation of the admissible weightings that appear in Pixton’s formula for the double ramification cycle [17].

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