SPINORIAL CLASSIFICATION OF SPIN(7) STRUCTURES

LUCÍA MARTÍN-MERCHAN

ABSTRACT. We describe the different classes of Spin(7) structures in terms of spinorial equations. We relate them to the spinorial description of $G_2$ structures in some geometrical situations. Our approach enables us to analyze invariant Spin(7) structures on quasi abelian Lie algebras.

1. Introduction

Berger’s list [2] (1955) of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds contains the so-called exceptional holonomy groups, $G_2$ and Spin(7), which occur in dimensions 7 and 8 respectively. Non-complete metrics with exceptional holonomy were given by Bryant in [3], complete metrics were obtained by Bryant and Salamon in [4], but compact examples were not constructed until 1996, when Joyce published [12], [13] and [14].

The remaining groups of Berger’s list different from $SO(n)$, called special holonomy groups, are $U(n)$, $SU(n)$, $Sp(n)$ and $Sp(n) \cdot Sp(1)$. If the holonomy of a Riemannian manifold is contained in a group $G$, the manifold admits a $G$ structure, that is, a reduction to $G$ of its frame bundle. Therefore, holonomy is homotopically obstructed by the presence of $G$ structures. Examples of manifolds endowed with $G$ structures for some of the holonomy groups in the Berger list are not only easier to obtain than manifolds with holonomy in $G$, but also relevant in $M$-theory, especially if they admit a characteristic connection [10], that is, a metric connection with totally skew-symmetric torsion whose holonomy is contained in $G$.

It is worth mentioning that Ivanov proved in [11] that each manifold with a Spin(7) structure admits a unique characteristic connection. Moreover, Friedrich proved in [9] that Spin(7) is the unique compact simple Lie group $G$ such that all the $G$ structures admit a unique characteristic connection.

The Lie group $G_2$ is compact, simple and simply connected. It consists of the endomorphisms of $\mathbb{R}^7$ which preserve the cross product from the imaginary part of the octonions [22]. Hence, a $G_2$ structure on a manifold $Q$ determines a 3-form $\Psi$, a metric and an orientation. In [7], Fernández and Gray classify $G_2$ structures into 16 different classes in terms of the $G_2$ irreducible components of $\nabla \Psi$. Related to this, the analysis of the intrinsic torsion in [5] allowed to obtain equations involving $d\Psi$ and $d(\ast \Psi)$ for each of the 16 classes, determined by the $G_2$ irreducible components of $\Lambda^2 T^* Q$ and $\Lambda^5 T^* Q$. In particular, one obtains that the holonomy of $Q$ is contained in $G_2$ if and only if $d\Psi = 0$ and $d(\ast \Psi) = 0$. The Lie group Spin(7) is also compact, simple and simply connected. It is the group of endomorphisms of $\mathbb{R}^8$ which preserve the triple cross product from the octonions [22]. Thus, a Spin(7) structure on a manifold $M$ determines 4-form $\Omega$, a metric and an orientation. In [6], Fernández classifies Spin(7) structures into 4 classes in terms of differential equations for $d\Omega$, which are determined by the Spin(7)-irreducible components of $\Lambda^2 T^* M$. Parallel structures verify $d\Omega = 0$, locally conformally parallel structures satisfy $d\Omega = \theta \wedge \Omega$ for a closed 1-form $\theta$ and balanced structures verify $\ast (d\Omega) \wedge \Omega = 0$. A generic Spin(7) structure, which does not satisfy any of the previous conditions, is called mixed.

The relationship between $G_2$ and Spin(7) structures was firstly explored by Martín-Cabrera in [17]. Each oriented hypersurface of a manifold equipped with a Spin(7) structure naturally inherits a $G_2$ structure whose type is determined by the Spin(7) structure of the ambient manifold and some extrinsic information of the submanifold, such as the Weingarten operator. Following the same viewpoint, Martín-Cabrera constructed Spin(7) structures on $S^3$-principal bundles over $G_2$ manifolds in [18]. Both approaches allowed to construct manifolds with $G_2$ and Spin(7) structures of different pure types.

It turns out that manifolds admitting $SU(3)$, $G_2$ and Spin(7) structures are spin and their spinorial bundle has a unitary section $\eta$ which determines the structure. In [1], spinorial formalism was used to deal with the distinct aspects of $SU(3)$ and $G_2$ structures, such as the classification of both types of structures, $SU(3)$ structures on hypersurfaces of $G_2$ manifolds and different types of Killing spinors. A

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clear advantage of this viewpoint is that a unique object, the spinor, encodes the whole geometry of the structure. For instance, a $G_2$ structure on a Riemannian manifold $(Q, g)$ with associated 3-form $\Psi$ is determined by a suitable spinor $\eta$ according to the formula $\Psi(X, Y, Z) = \langle X\eta, Y\eta, Z\eta \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the spinorial bundle and juxtaposition of vectors indicates the Clifford product. Any oriented hypersurface $Q'$ with normal vector field $N$ inherits an SU(3)$ structure implicitly defined by $\Psi = N^* \omega + \text{Re}(\Theta)$, where $N^*(X) = g(N, X)$ for $X \in TQ$. But both the Kahler form $\omega$ and the $(3,0)$-form $\text{Re}(\Theta)$ turn out to be determined by the same spinor $\eta$.

In this paper we follow the ideas of [1] to describe the geometry of Spin(7) structures from a spinorial viewpoint, starting from the classification of these structures, continuing to analyze the relationship between $G_2$ and Spin(7) structures and finishing with the study of invariant Spin(7) structures on quasi abelian Lie algebras.

Our first result, Theorem 4.5 in section 4, describes each type of Spin(7) structure with spinorial equations. To state it, we have to mention that if the structure is determined by a spinor $\eta$ and $R$ is a Spin(7) reduction of the frame bundle of the manifold, there is a natural isomorphism $c: R \times_{\text{Spin}(7)} \text{spin}(7)^\bot \rightarrow \langle \eta \rangle^\bot$ (see Lemma 3.1, section 3 for details).

**Theorem 1.1.** Let $D$ be the Dirac operator of the spinorial bundle and take $V \in TM$ such that $D\eta = V\eta$. The Spin(7) structure $\Omega$ defined by $\eta$ is:

1. Parallel if $\nabla\eta = 0$.
2. Locally conformally parallel if $i(V)\Omega = 28 \text{alt}(e^{-1}\nabla\eta)$.
3. Balanced if $D\eta = 0$.

Our techniques also allow us to identify the intrinsic torsion of the structure and to obtain the formula for the unique characteristic connection of each Spin(7) structure, given by Ivanov in [11, Theorem 1.1].

We also introduce the concept of $G_2$ distributions, a general setting to relate $G_2$ and Spin(7) structures.

**Definition 1.2.** Let $(M, g)$ be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. We say that $\mathcal{D}$ has a $G_2$ structure if the principal SO(7) bundle $P(\mathcal{D})$ is spin and the spinorial bundle $\Sigma(\mathcal{D})$ admits a unitary section.

This construction allows us to obtain the results which appear in [17] and [18] about $G_2$ structures on hypersurfaces of Spin(7) manifolds and $S^1$-principal bundles over $G_2$ manifolds. Related to this, we also study warped products of manifolds admitting a $G_2$ structure with $\mathbb{R}$.

The formalism of $G_2$ distributions enables us to study invariant Spin(7) structures on quasi-abelian Lie algebras, that is, Lie algebras with a codimension 1 abelian ideal. To state the result, which is Theorem 7.7, suppose that the Lie algebra is $\mathfrak{g} = \langle e_0, \ldots, e_7 \rangle$ with abelian ideal $\mathbb{R}^7 = \langle e_1, \ldots, e_7 \rangle$ and it is endowed with the canonical metric and volume form.

**Theorem 1.3.** Denote by $\mathcal{E} = ad(e_0)|_{\mathfrak{g}^7}$ and let $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ be the symmetric and skew-symmetric parts of the endomorphism. Then, $\mathfrak{g}$ admits a Spin(7) structure of type:

1. Parallel, if and only if $\mathcal{E}_{13} = 0$ and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2) i$, for some $0 \leq \lambda_1 \leq \lambda_2$.
2. Locally conformally parallel and non-parallel if and only if $\mathcal{E}_{13} = h\text{Id}$ with $h \neq 0$ and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2) i$, for some $0 \leq \lambda_1 \leq \lambda_2$.
3. Balanced if and only if $\mathfrak{g}$ is unimodular and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2) i$, for some $0 \leq \lambda_1 \leq \lambda_2$.

Moreover, if $\mathcal{E}_{24} \neq 0$ then it admits a Spin(7) structure of mixed type.

From this, it follows (Corollary 7.8) that there are no quasi abelian solvmanifolds which admit a locally conformally parallel Spin(7) structure. In addition, this result allows us to give an example of a nilmanifold admitting both an invariant balanced structure and an invariant mixed structure. A compact nilmanifold admitting a parallel structure is also obtained as a quotient of a simply connected Lie group whose Lie algebra is quasi abelian. Despite not being diffeomorphic to a torus, it is flat. Indeed, we prove that quasi abelian Lie algebras which admit an invariant Spin(7) parallel structure are flat (Corollary 7.9).

This paper is organized as follows. Section 2 contains a review of algebraic aspects of Spin(7) geometry. Section 3 identifies the intrinsic torsion of the Levi-Civita connection with a spinor, section 4 is devoted to obtain the classification of Spin(7) structures in terms of spinors and section 5 provides an alternative proof of the existence of the characteristic connection. Section 6 provides a complete analysis of $G_2$ structures on distributions and then focuses on the particular cases described above. Finally, section 7 deals with invariant structures on quasi abelian Lie algebras and provides compact examples.
In this section we introduce some aspects of Clifford algebras, 8-dimensional spin manifolds and \(\text{Spin}(7)\) representations, which can be found in \([8],[15]\) and \([22]\) as well as the notations that we will use in the sequel.

2. Preliminaries

2.1. \(\text{Spin}(7)\) Structures. Let \((M, g)\) be an oriented Riemannian 8-manifold and let \(P(M)\) be the associated \(\text{SO}(8)\) frame bundle. Provided that \(M\) is spin, that is \(w_2(M) = 0\), we can take a \(\text{Spin}(8)\) principal bundle \(\tilde{P}(M)\) over \(M\) which is a double covering \(\pi: \tilde{P}(M) \to P(M)\) equivariant under the adjoint action \(\text{Ad}: \text{Spin}(8) \to \text{SO}(8)\). We may also denote by \(\text{Ad}\) the induced action of \(\text{Spin}(8)\) on \(TM\).

The associated spinorial bundle is \(\Sigma(M) = \tilde{P}(M) \times_{\rho} \mathbb{R}^{16}\) where we have denoted by \(\rho: \text{Spin}(8) \to \text{SO}(16)\) the real spinorial representation, constructed by restricting the isomorphism \(\text{Cl}_8 \cong \text{GL}(16)\) and equipping \(\mathbb{R}^{16}\) with a metric \((\cdot, \cdot)\) which makes the Clifford product a skew-symmetric endomorphism.

The induced metric on \(\Sigma(M)\) will be denoted in the same way, and the elements of this bundle by \(\phi = [\tilde{F}, v]\), where \(\tilde{F} \in \tilde{P}(M)\) and \(v \in \mathbb{R}^{16}\).

The Clifford multiplication with a vector field is extended to an action of \(\Lambda^kT^*M\) defined as follows.

1. The product with a covector is defined by \(X^*\phi = X\phi\), where we used the canonical identification between the tangent and the cotangent bundle: \(X^* = g(X, \cdot)\).
2. If the product is defined on \(\Lambda^k T^* M\) when \(\ell \leq k\), we define

\[
(X^* \wedge \beta)\phi = X(\beta\phi) + (i(X)\beta)\phi,
\]

where \((i(X)\beta)\) denotes the contraction, \(\beta \in \Lambda^k T^* M\) and \(X \in TM\). This product is extended lineary to \(\Lambda^{k+1} T^* M\).

For instance, we have:

\[
(X^* \wedge Y^*)\phi = (XY + g(X, Y))\phi, \\
(X^* \wedge Y^* \wedge Z^*)\phi = (XYZ + g(X, Y)Z - g(X, Z)Y + g(Y, Z)X)\phi.
\]

The volume form \(\nu_8\) of \(R^8\) provides \(\mathbb{R}^{16}\) with a \(\text{Spin}(8)\) equivariant endomorphism:

\[
\nu_{8}: \mathbb{R}^{16} \to \mathbb{R}^{16}, \quad \phi \mapsto \nu_{8}\phi.
\]

Since \(\nu_{8}^2 = 1\), there is a splitting \(\mathbb{R}^{16} = \Delta^+ \oplus \Delta^-\) where \(\Delta^\pm\) is the eigenspace associated to \(\pm 1\). Therefore, \(\Sigma(M) = \Sigma(M)^+ \oplus \Sigma(M)^-\), where \(\Sigma(M)^\pm = \tilde{P}(M) \times_{\rho} \Delta^\pm\). Also note that \(X(\Sigma(M)^\pm) \subset \Sigma(M)^\mp\) if \(X \in \mathcal{X}(M)\).

At each \(p \in M\), the action \(\text{Spin}(8) \to \text{SO}(\Sigma_p(M)^+), \tilde{\varphi}[\tilde{F}, v] = [\tilde{F}, \rho(\tilde{\varphi})v]\) is a double covering, so that the existence of a unitary spinor \(\eta \in \Gamma(\Sigma^p(M)^+)\) determines an identification between \(\text{Spin}(7)\) and the stabilizer of \(\eta_p\), \(\text{Stab}(\eta_p)\). Besides, the restriction \(\text{Ad}: \text{Spin}(8) \to \text{SO}(T_pM)\) to \(\text{Stab}(\eta_p)\) is injective since \(\text{ker}(\text{Ad}) = \{1, -1\}\) and \(-1 \notin \text{Stab}(\eta_p)\).

The previous considerations allow us to define a 4-form \(\Omega\) on \(M\) such that \(\text{Ad}(\text{Stab}(\eta_p)) = \text{Stab}(\Omega_p)\).

Indeed, observe that there is a well defined map:

\[
TM \times TM \times TM \to TM, \quad (X, Y, Z) \mapsto X \times Y \times Z \text{ s.t. } (X \times Y \times Z)\eta = (X^* \wedge Y^* \wedge Z^*)\eta,
\]

which turns out to be a positive triple product, that is, it verifies \([22, \text{Definition} 6.1]\):

1. The vector \(X \times Y \times Z\) is perpendicular to \(X, Y\) and \(Z\).
2. \(\|X \times Y \times Z\| = \|X^* \wedge Y^* \wedge Z^*\|\).
3. If we take orthonormal vectors \(W, X, Y, Z\) such that \(W\) is perpendicular to \(X \times Y \times Z\), then \(X \times Y \times (X \times Z \times W) = Y \times Z \times W\).

The first property follows from (2) and the second one is obvious. To check the third one we observe that \(Y\) is perpendicular to \(X \times Z \times W\) since \(g(W, X \times Y \times Z) = (W\eta, XYZ\eta) = (Y\eta, XZW\eta) = g(Y, X \times Z \times W)\), and therefore:

\[
X \times Y \times (X \times Z \times W)\eta = XYXZW\eta = YZW\eta = (Y \times Z \times W)\eta.
\]

**Definition 2.1.** The associated 4-form to the triple cross product is:

\[
\Omega(W, X, Y, Z) = ((X \times Y \times Z)\eta, W\eta) = ((XYZ + g(X, Y)Z - g(X, Z)Y + g(Y, Z)X)\eta, W\eta) = \frac{1}{2}((-XYZ + WYZX)\eta, \eta).
\]

Since \(\tilde{\varphi}(X\phi) = Ad(\tilde{\varphi})(X)(\tilde{\varphi}\phi)\) if \(\tilde{\varphi} \in \text{Spin}(8), X \in TM\) and \(\phi \in \Sigma(M)\), it is not hard to check that \(\text{Stab}(\eta_p) = \text{Stab}(\Omega_p)\).

Some important properties of this form are the following:
1. If \((X_0, \ldots, X_T)\) is an orthonormal oriented basis and \(\sigma\) is a permutation then \(*\Omega = \Omega\) since \(X_{\sigma(0)}X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)} = (-1)^{\text{sgn}(\sigma)}X_{\sigma(4)}X_{\sigma(5)}X_{\sigma(6)}X_{\sigma(7)}\).

2. Given orthonormal vector fields \(e_0, e_1, e_2, e_4\) such that \(e_4\) is perpendicular to \(e_3 = e_0 \times e_1 \times e_2\), we can find [22, Theorem 7.12] an orthonormal frame \((e_0, \ldots, e_7)\) such that:

\[
\Omega = e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}
\]

where we have used the short-hand notation \(e^i\) for \(g(e_i, \cdot)\) and \(e^{ijkl}\) for \(e^i \wedge e^j \wedge e^k \wedge e^l\). We will also denote the Clifford product \(e_ie_j\) by \(e_{ij}\) and so on. A frame of this type will be called a Cayley frame. Since those frames verify \((e_0 \cdots e_7)\eta = \eta\), they are positively oriented.

2.2. Spin(7) representations. Let us denote the standard basis of \(\mathbb{R}^8\) by \((e_0, \ldots, e_7)\), and the standard Spin(7) structure of \(\mathbb{R}^8\) by \(\Omega_0\), given by (3).

The canonical representation of \(\text{Spin}(7) = \text{Stab}(\Omega_0) \subset \text{SO}(8)\) on \(\Lambda^k \mathbb{R}^8\) induces an orthogonal decomposition of this space into irreducible \(\text{Spin}(7)\) invariant subspaces. The expression \(\Lambda^k_\mathbb{R}^8\) denotes such an \(l\)-dimensional subspace of \(\Lambda^k \mathbb{R}^8\). Observe that Hodge star operator \(*\) gives isomorphisms between \(\Lambda^k \mathbb{R}^8\) and \(\Lambda^{8-k} \mathbb{R}^8\) determining that \(\Lambda^k_\mathbb{R}^8 = *\Lambda^{8-k}_\mathbb{R}^8\) if \(k \leq 4\). We are going to describe briefly the splitting at degrees \(k = 2\) and \(k = 3\) but a complete proof can be found in [22, Theorem 9.8]. The decomposition goes as follows:

\[
\begin{align*}
\Lambda^2 \mathbb{R}^8 &= \Lambda^2_2 \mathbb{R}^8 \oplus \Lambda^2_7 \mathbb{R}^8, \\
\Lambda^3 \mathbb{R}^8 &= \Lambda^3_3 \mathbb{R}^8 \oplus \Lambda^3_8 \mathbb{R}^8.
\end{align*}
\]

The first one comes from the orthogonal splitting \(\Lambda^2 \mathbb{R}^8 = \text{so}(8) = \text{spin}(7) \oplus m\), where \(m = \text{spin}(7) \perp\). An alternative description may be done by considering the map:

\[
\Lambda^2 \mathbb{R}^8 \to \Lambda^2 \mathbb{R}^8, \quad \beta \mapsto *(\beta \wedge \Omega_0),
\]

which is \(\text{Spin}(7)\)-equivariant, symmetric and traceless. Therefore, \(\Lambda^2 T^* M\) splits into eigenspaces which must coincide with the previous ones due to the irreducibility. It can be checked that the eigenvalues are 3 on \(\Lambda^2_2 \mathbb{R}^8\) and \(-1\) on \(\Lambda^2_7 \mathbb{R}^8\). Moreover, the set \(\{\alpha_j = e^0j - i(e_0)i(e_j)\Omega_0\}_{j=1}^{7}\) is a basis of \(\Lambda^2_7 T^* M\).

The subspaces involved in the second splitting are:

\[
\begin{align*}
\Lambda^3_3 \mathbb{R}^8 &= i(8)\Omega_0, \\
\Lambda^3_8 \mathbb{R}^8 &= \ker(\cdot \wedge \Omega_0: \Lambda^3 \mathbb{R}^8 \to \Lambda^7 \mathbb{R}^8).
\end{align*}
\]

Finally, a \(\text{Spin}(7)\) structure on the Riemannian manifold \((M, g)\) determines a canonical splitting of \(\Lambda^k T^* M\). If we take the \(\text{Spin}(7)\) reduction \(R\) of the \(\text{SO}(8)\) principal bundle given by the Cayley frames, then those are given by \(\Lambda^k_\mathbb{R}^8 T^* M = R \times_{\text{Spin}(7)} \Lambda^k_\mathbb{R}^8\).

3. The intrinsic torsion

We are going to compute the intrinsic torsion of the Levi-Civita connection, \(\Gamma \in TM \otimes \Lambda^2_7 T^* M\). Recall that the Levi-Civita connection \(\nabla\) on \(TM\) induces a connection \(\omega\) on \(P(M)\). Then a connection on the \(\text{Spin}(7)\) reduction \(R\) is defined by \(\omega' = p(\omega)|_{TR}\), where \(p\) denotes the orthogonal projection to \(\text{spin}(7)\). The connection that \(\omega'\) induces on \(TM\) is denoted by \(\nabla'\) and determines the intrinsic torsion by means of the expression:

\[
\nabla_X Y = \nabla'_X Y + \Gamma(X)Y.
\]

The skew-symmetric endomorphism \(\Gamma(X)\) can be identified with a 2-form which lies in \(R \times_{\text{Spin}(7)} m = \Lambda^2_7 T^* M\) for each \(X \in TM\). To compute it, we will first prove that the vector bundles \(\Lambda^2_7 T^* M\) and \(H = \eta^\perp\) are isomorphic:

Lemma 3.1. There is a well defined \(\text{Spin}(7)\)-equivariant map

\[
\Lambda^2 T^* M \to H, \quad \alpha \mapsto \alpha\eta,
\]

whose kernel is \(\Lambda^2_7 T^* M\). Indeed, its restriction \(c: \Lambda^2_7 T^* M \to H\) is an isomorphism whose inverse is given by \((c^{-1})\phi(X, Y) = \frac{1}{4}(\phi, (XY + g(X, Y))\eta)\).
Proof. The spinor $\beta \eta$ is perpendicular to $\eta$ if $\beta \in \Lambda^2 T^* M$. Therefore, the map is well-defined and it is $\text{Spin}(7)$-equivariant since $\text{Spin}(7) = \text{Stab}(\eta_p)$.

To prove that $c$ is an isomorphism, we first claim that if $(e_0, \ldots, e_7)$ is a Cayley frame then $\alpha_j \eta = 4e^{0j} \eta$. Observe that we only need to check this formula for $j = 1$ since $c$ is $\text{Spin}(7)$-equivariant and $G_2 = \text{Spin}(7) \cap \text{Stab}(e_0)$ acts transitively on the 6-sphere generated by $(e_1, \ldots, e_7)$. In this case, $\alpha_1 = e^{03} + e^{23} - e^{45} - e^{67}$ and if $(i, j) \in \{(2, 3), (5, 4), (7, 6)\}$ we have that $\Omega(e_0, e_i, e_j) = 1$. The previous equality means that $e_0 \eta = e_1 \eta$, so that $e^{01} \eta = e^{03} \eta$.

Moreover, since $(e^{0j} \eta)_{i=1}^7$ is an orthonormal basis of $H$ we have that

$$c^{-1}(\phi) = \frac{1}{4} \sum_{i=1}^7 (\phi, e^{0i} \eta) e_i.$$ 

If $X = e_0, Y = e_1$ are orthonormal vectors then $\alpha_j(e_0, e_1) = (e^{0j} - i(e_0) i(e_j)) e_0, e_1 = \delta_{j1}$. Hence, $c^{-1}(\phi(e_0, e_1)) = \frac{1}{4} (\phi, e_0 e_1) \eta$.

Finally, by dimensional reasons the Clifford product with $\eta$ must vanish on $\Lambda^2_3 T^* M$. $\square$

The previous result enables us to find a formula for the intrinsic torsion:

**Proposition 3.2.** The intrinsic torsion is given by $\Gamma(X) = 2c^{-1} \nabla_X \eta$.

**Proof.** We also denote by $\nabla$ and $\nabla'$ the induced connections on the spinorial bundle. According to [8, p. 60] we have that:

$$\nabla_X \phi = \nabla'_X \phi + \frac{1}{2} \Gamma(X) \phi,$$

where $\Gamma(X)$ acts on $\phi$ as a 2-form. Since the holonomy of the connection $\nabla'$ is contained in $\text{Spin}(7)$ and $\text{Stab}(\eta_p) = \text{Spin}(7)$ we have that $\nabla' \eta = 0$. Finally, if $X \in TM$ then $\nabla_X \eta \in H$ and $\Gamma(X) \in R \times \text{Spin}(7)$ $m$ thus, Lemma 3.1 shows that $\Gamma(X) = 2c^{-1} \nabla_X \eta$. $\square$

4. Spinorial classification of $\text{Spin}(7)$ structures

Spin structures are classified [6] according to the $\text{Spin}(7)$ irreducible parts of $*d\Omega$ on $\Lambda^3 T^* M$ in the following pure types:

**Definition 4.1.** A $\text{Spin}(7)$-structure given by $\Omega$ is said to be:

1. Parallel, if $*d\Omega = 0$.
2. Locally conformally parallel, if $*d\Omega \in \Lambda^3_3 T^* M$.
3. Balanced if $*d\Omega \in \Lambda^3_5 T^* M$.

The Lee form of $\Omega$ is the unique $\theta \in \Lambda^1 T^* M$ such that the orthogonal projection to $\Lambda^3_7 T^* M$ of $d\Omega$ is $\theta \wedge \Omega$.

**Remark 4.2.** Suppose that the structure is locally conformally parallel. Let $O$ be a contractible open set, take a primitive $f$ of $-\frac{1}{2} \theta|_O$ and define the metric $g = e^{2f} g|_O$. The associated $\text{Spin}(7)$ structure is $\Omega' = e^{f} \Omega|_O$ and it verifies $d\Omega' = 0$. Therefore, $\Omega|_O$ is conformal to a parallel structure. This justifies the name.

In order to rewrite this classification by means of $\eta$, we are going to calculate $*d\Omega$. For this purpose, consider the Dirac operator $D$ at $\Sigma(M)$ and the vector field $V$ such that

$$D\eta = V\eta. \quad (4)$$

Then, the 3-form $\gamma_8(X, Y, Z) = (D\eta, (X \times Y \times Z) \eta) = (i(V)\Omega)(X, Y, Z)$ obviously lies in $\Lambda^3_7 T^* M$.

**Proposition 4.3.** Using the previous notation, we have:

$$*d\Omega = 2(\gamma_8 - 12 \text{alt}(c^{-1} \nabla \eta)),$$

where $\text{alt}(T)(X_1, \ldots, X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} T(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ if $T$ is a section of $\otimes^n TM$.

**Proof.** Since $\nabla$ is a metric connection on the spinorial bundle and acts as a derivation for the Clifford product, we get:

$$\nabla_T \Omega(W, X, Y, Z) = \frac{1}{2} \left( ((-WXYZ + WZYX) \nabla_T \eta, \eta) + ((-WXYZ + WZYX)\eta, \nabla_T \eta) \right)$$

$$= \frac{1}{2} (-WXYZ + WZYX) \eta, \nabla_T \eta).$$
Take orthonormal vectors $X, Y, Z$ and an orthonormal oriented basis $(X_0, \ldots, X_7)$ such that $X_0 = X$, $X_1 = Y$ and $X_2 = Z$. Then,

$$
\delta \Omega(X, Y, Z) = -\sum_{i=3}^{7} \nabla_{X_i} \Omega(X_i, X, Y, Z) = -2 \sum_{i=3}^{7} (XY Z \eta, X_i \nabla_{X_i} \eta)
$$

$$
= -2(D \eta, (X \times Y \times Z) \eta) + 2(XY Z \eta, X \nabla_{X_i} \eta + Y \nabla_{Y} \eta + Z \nabla_{Z} \eta)
$$

$$
= -2(D \eta, (X \times Y \times Z) \eta) - (YZ \eta, \nabla_{X} \eta) + (XZ \eta, \nabla_{Y} \eta) - (XY \eta, \nabla_{Z} \eta)
$$

$$
= -2((D \eta, (X \times Y \times Z) \eta) - 12\text{alt}(c^{-1} \nabla \eta)(X, Y, Z))
$$

Note that the coefficient 12 comes from the normalization of alt and the expression $c^{-1}(\nabla \eta)(X, Y) = \frac{1}{2}(XY + g(X, Y) \eta, \nabla \eta)$.

We are going to decompose $*d \Omega$ according to the previous splitting.

Lemma 4.4. The 3-form $\gamma_{48} = 3 \gamma_8 - 84 \text{alt}(c^{-1} \nabla \eta)$ lies in $\Lambda^3_{16} T^* M$ and

$$
* d \Omega = \frac{2}{7} \gamma_{48} + \frac{8}{7} \gamma_8
$$

Proof. Take a unitary vector $X$ and a Cayley frame $(e_0, e_1, \ldots, e_7)$ such that $X = e_0$. Then:

$$
(\gamma_8 \wedge \Omega)(e_1, \ldots, e_7) = (D \eta, (e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356}) \eta)
$$

$$
= 7(D \eta, e_0 \eta) = 7V^*(X),
$$

$$(12 \text{alt}(c^{-1} \nabla \eta) \wedge \Omega)(e_1, \ldots, e_7) = \mathcal{S}(\nabla e_1, \eta, e_{237}) = \mathcal{S}(\nabla e_1, \eta, e_{457}) - \mathcal{S}(\nabla e_1, \eta, e_{567})
$$

$$
- \mathcal{S}(\nabla e_2, \eta, e_{467}) + \mathcal{S}(\nabla e_2, \eta, e_{475}) - \mathcal{S}(\nabla e_3, \eta, e_{567})
$$

$$
- \mathcal{S}(\nabla e_3, \eta, e_{576}) = 3(D \eta, e_0 \eta) = 3V^*(X).
$$

We denote by $\mathcal{S}$ the cyclic sums in the indices involved. To arrange the last term observe that each index appears 3 times and:

$$
\mathcal{S}(\nabla e_1, \eta, e_{237}) = (e_1 \nabla e_1 \eta + e_2 \nabla e_2 \eta + e_3 \nabla e_3 \eta, e_{123} \eta)
$$

$$
- \mathcal{S}(\nabla e_2, \eta, e_{457}) = (e_1 \nabla e_1 \eta + e_4 \nabla e_4 \eta + e_5 \nabla e_5 \eta, -e_{145} \eta)
$$

and so on. Note that we have used, as in the proof of Lemma 3.1, that $e_{123} \eta = e_0 \eta = -e_{145} \eta$.

Since Cayley bases are positive oriented, we get $*(V^*) = \frac{1}{7}(\gamma_8 \wedge \Omega) = 4 \text{alt}(c^{-1} \nabla \eta)$, so that $\gamma_{48}$ as defined above lies in $\Lambda^3_{16} T^* M$. Finally, taking into account the formula for $*d \Omega$ in Proposition 4.3, we get $*d \Omega = \frac{2}{7} \gamma_{48} + \frac{8}{7} \gamma_8$.

We can now rewrite the classification of Spin(7) structures:

Theorem 4.5. The Spin(7) structure given by $\Omega$ is:

1. Parallel if $\nabla \eta = 0$.
2. Locally conformally parallel if $i(V)\Omega = 28 \text{alt}(c^{-1} \nabla \eta)$.
3. Balanced if $D \eta = 0$.

Moreover, the Lee form is given by $\theta = \frac{8}{7} V^*$, where $V$ is defined as in the equation (4).

Proof. It is an immediate consequence of Definition 4.1 and Lemma 4.4. To compute the Lee form we have used that the projection of $d \Omega$ to $\Lambda^3_{16} T^* M$ is $-\frac{8}{7} * \gamma_8$ and the formula $i(X) \Omega = * (X^* \wedge \Omega)$, which can be easily checked taking a Cayley frame and $X = e_0$.

5. The characteristic connection.

The characteristic connection of a Spin(7) structure is a connection $\nabla^c$ with totally skew-symmetric torsion, such that $\nabla^c \Omega = 0$. The computations above allow us to prove the existence and uniqueness of the characteristic connection for manifolds with a Spin(7) structure. This is a well known result which appears in [11, Theorem 1.1]. Our proof is based on the argument of Theorem 3.1 in [9].

Consider the Spin(7)-equivariant maps which are given in terms of a local Cayley frame:

$$
\Theta : \Lambda^3 T^* M \to TM \otimes \Lambda^2 T^* M, \quad \beta \mapsto \Theta(\beta) = \sum_{j=0}^{7} e_j \otimes p_7(i(e_j) \beta),
$$

$$
\Xi : TM \otimes \Lambda^2 \to \Lambda^3 T^* M, \quad \alpha \otimes \beta \mapsto \alpha \wedge \beta = 3 \text{alt}(\alpha \otimes \beta),
$$
where \( p_T : \Lambda^2 T^* M \to \Lambda^2_0 T^* M \) is the orthogonal projection.

Note that the map \( \Xi \circ \Theta \) is symmetric and Spin(7)-equivariant, so that its eigenspaces must be \( \Lambda^2_0 T^* M \) and \( \Lambda^2_3 T^* M \). Taking \( i(e_0) \Omega \in \Lambda^2_0 T^* M \) and \( e^{123} + e^{145} \in \Lambda^2_3 T^* M \) one can show that the eigenvalues are \( \frac{7}{6} \) on \( \Lambda^2_0 T^* M \) and \( \frac{1}{6} \) on \( \Lambda^2_3 T^* M \).

**Proposition 5.1.** Given a Spin(7) structure, there exists a unique characteristic connection whose torsion \( T \in \Lambda^2 T^* M \) is given by:

\[
T = -\delta \Omega - \frac{7}{6} (\theta \wedge \Omega).
\]

**Proof.** A connection with skew-symmetric torsion \( T \in \Lambda^2 T^* M \) is given by \( \nabla_X Y = \frac{1}{4} T(X, Y, \cdot)^\# \), where \( T(X, Y, \cdot)^\# \) is the vector field such that \( (T(X, Y, \cdot)^\#)^* = T(X, Y, \cdot) \). Thus, the lift to the spinorial bundle is \( \nabla_X \phi + \frac{1}{4} T(X) \phi \).

Since the condition \( \nabla^c \eta = 0 \) is equivalent to \( \nabla^c \eta = 0 \) and the kernel of the Clifford product by \( \eta \) on \( \Lambda^2 T^* M \) is \( \Lambda^2_0 T^* M \), the set of characteristic connections is isomorphic to the set of 3-forms \( T \in \Lambda^2 T^* M \) such that

\[
-4c^{-1} \nabla_X \eta = i(X) T \eta = p_T(i(X) T), \quad \forall X \in \mathfrak{X}(M).
\]

The last equality may be rewritten as \(-4 \Theta(c^{-1} \nabla \eta) = \Theta(T) \). From the definition of \( \gamma_{48} \) given in Lemma 4.4 we have: \(-4 \Xi(c^{-1} \nabla \eta) = -12 \text{alt}(c^{-1} \nabla \eta) = \frac{1}{4} (\gamma_{48} - 3 \gamma_8) \). Finally, taking into account the eigenvalues of \( \Xi \circ \Theta \), we get:

\[
T = \frac{1}{7} (2 \gamma_{48} - \frac{4}{3} \gamma_8) = *d \Omega - \frac{4}{3} \gamma_8 = -\delta \Omega - \frac{7}{6} * (\theta \wedge \Omega).
\]

To obtain the second equality we have used the formula for \( d \Omega \) from Lemma 4.4. To check the last one, note that \( \gamma_8 = i(V) \Omega = *(V^* \wedge \Omega) = \frac{7}{8} * \theta \wedge \Omega \).

\( \square \)

6. \( G_2 \) distributions

In this section we define the notion of \( G_2 \) distribution on a Spin(7) manifold in terms of spinors, and we study the torsion of the structure with respect to a suitable connection on the distribution. Then, we relate the Spin(7) structure of the ambient manifold with the \( G_2 \) structure of the distribution. This approach enables us to study \( G_2 \) structures on submanifolds of Spin(7) manifolds, \( S^7 \)-principal fibre bundles over \( G_2 \) manifolds and warped products of manifolds admitting a \( G_2 \) structure with \( R \). Our analysis is very similar to the description of \( G_2 \) structures from a spinorial viewpoint, done in [1], which we briefly recall.

A 7-dimensional Riemannian manifold \((Q, g)\) can be equipped with a \( G_2 \) structure if it is spin and its spinorial bundle \( \Sigma(Q) \) admits a unitary section \( \eta \). A cross product is then constructed from the spinor and is determined by a 3-form \( \Psi \). Denote by \( \nabla^Q \) both the Levi-Civita connection of the manifold and its lift to the spinorial bundle; an endomorphism \( S \) of \( TQ \) is defined by the condition:

\[
\nabla^Q_X \eta = S(X) \eta.
\]

The intrinsic torsion is \(-\frac{2}{3} i(S) \Psi \) [1, Proposition 4.4], so that pure types of \( G_2 \) structures are given by the \( G_2 \) irreducible components of \( \text{End}(TQ) \). It is known that \( \text{End}(\mathbb{R}^7) = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4 \), where \( \chi_i \) are irreducible \( G_2 \) representations, defined by:

\[
\chi_1 = \{ \text{Id} \}, \quad \chi_2 = g_2, \quad \chi_3 = \text{Sym}_3^2(\mathbb{R}^7), \quad \chi_4 = \{ A : \mathbb{R}^7 \to \mathbb{R}^7 : A(X) = X \times S, \quad S \in \mathbb{R}^7 \},
\]

where \( \text{Sym}_3^2(\mathbb{R}^7) \) denotes the set of symmetric and traceless endomorphisms. The dimensions of the previous spaces are 1, 14, 27 and 7 respectively.

If we denote by \( R_Q \) a \( G_2 \) reduction of the \( SO(7) \) principal bundle \( P(Q) \) and define \( \chi_i(Q) = R_Q \times_{G_2} \chi_i \), then the pure classes of \( G_2 \) structures are determined by the condition \( S \in \chi_i(Q) \). For instance, nearly parallel \( G_2 \) structures verify \( S \in \chi_1(Q) \), almost parallel or calibrated are those with \( S \in \chi_2(Q) \), and locally conformally calibrated are such that \( S \in \chi_4(Q) \). Indeed in the nearly parallel case it holds that \( S(X) = \lambda_0 X \) for some \( \lambda_0 \in \mathbb{R} \). Moreover mixed classes are also relevant, for instance co calibrated structures which correspond to \( S \in \chi_1(Q) \oplus \chi_3(Q) \).

Taking this into account, we define \( G_2 \) structures on distributions and characterise the existence of such structures.

**Definition 6.1.** Let \((M, g)\) be an oriented 8-dimensional Riemannian manifold and let \( \mathcal{D} \) be a cooriented distribution of codimension 1. We say that \( \mathcal{D} \) has a \( G_2 \) structure if the principal \( SO(7) \) bundle \( P(\mathcal{D}) \) is spin and the spinorial bundle \( \Sigma(\mathcal{D}) \) admits a unitary section.
Lemma 6.2. Consider an oriented 8-dimensional Riemannian manifold \((M, g)\) and a cooriented distribution \(\mathcal{D}\) of codimension 1. Take a unitary vector field \(N\), perpendicular to \(\mathcal{D}\) such that \(TM = \langle N \rangle \oplus \mathcal{D}\) as oriented bundles. The manifold \(M\) is spin if and only if the bundle \(P(\mathcal{D})\) is spin. In this case, the spinorial bundles are related by \(\Sigma(\mathcal{D}) = \Sigma^+(M)\) and it holds
\[
X \cdot_D \phi = NX\phi, \quad \text{if} \quad X \in \mathcal{D}, \quad \phi \in \Sigma(\mathcal{D}),
\]where we have suppressed the symbol \(\cdot_M\) to denote the Clifford product on \(M\).

Therefore, \(M\) has a \(\text{Spin}(7)\) structure if and only if \(\mathcal{D}\) has a \(G_2\) structure.

**Proof.** The bundle \(P(\mathcal{D})\) is a reduction of \(P(M)\), since we have the following inclusion:
\[
i: P(\mathcal{D}) \to P(M), \quad (X_1, \ldots, X_7) \to (N, X_1, \ldots, X_7).
\]
Suppose that \(P(\mathcal{D})\) is spin and denote the spin bundle by \(\pi_D: \tilde{P}(\mathcal{D}) \to P(\mathcal{D})\). Then, we can define the principal \(\text{Spin}(8)\) bundle \(\tilde{P}(M) = \tilde{P}(\mathcal{D}) \times_{\text{Spin}(7)} \text{Spin}(8)\) and the map:
\[
\pi_M: \tilde{P}(M) \to P(M), \quad [\tilde{F}, \tilde{\varphi}] \to \text{Ad}(\tilde{\varphi})(i(\pi_D(\tilde{F}))),
\]
which is a double covering and \(\text{Ad}\)-equivariant. Therefore, \(M\) is spin. Reciprocally, if \(M\) is spin then the pullback \(i^*\tilde{P}(M)\) is the spin bundle of \(P(\mathcal{D})\).

Moreover, the irreducible 8-dimensional representation of \(\text{Cl}_7\) which maps the volume form to the identity can be constructed from the composition
\[
\text{Cl}_7 \to \text{Cl}_8^0 \stackrel{\nu}{\to} \text{GL}(\Delta^+)\]
where the first map is induced by the embedding \(\mathbb{R}^7 \to \text{Cl}_8^0\), \(v \to e_0v\), denoting by \((e_0, \ldots, e_7)\) the canonical basis of \(\mathbb{R}^8\).

Therefore, the spinorial bundle \(\Sigma(\mathcal{D})\) coincides with \(\Sigma(M)^+\) and the Clifford products are related by the formula (5).

From now on we assume that the manifold \((M, g)\) has a \(\text{Spin}(7)\) structure \(\Omega\), constructed from a unitary section \(\eta\) of the spinorial bundle \(\Sigma(M)^+\), as in Definition 3. We equip \(M\) with a distribution as in Lemma 6.2.

**Remarks 6.3.** In this situation, we have the following:
1. If \(\beta \in \Lambda^2 TM\) and \(\phi \in \Sigma(\mathcal{D})\) then \(\beta \cdot_D \phi = \beta \phi\).
2. There is an orthogonal decomposition \(\Sigma(\mathcal{D}) = \langle \eta \rangle \oplus (\mathcal{D} \cdot_D \eta)\).
3. The section \(\eta\) defines a cross product on \(\mathcal{D}\) by means of:
\[
(X \times Y)\eta = (X^* \wedge Y^*)\eta = (XY + g(X, Y))\eta,
\]
which is determined by \(\Psi_D(X, Y, Z) = (X\eta, Y \times Z)\eta = -(\eta, XY\Lambda Z)\).
4. The cross product is determined by \(\Psi_D = i(N)\Omega\). Therefore, using that \(\ast\Omega = \Omega\) we get \(\Omega = N^* \wedge \Psi_D + \Psi_D \ast \Psi_D\).

We equip \(\mathcal{D}\) with a suitable connection which is determined by the covariant derivative of the ambient manifold.

**Definition 6.4.** The covariant derivative of \(\mathcal{D}\) induced by \(M, \nabla^\mathcal{D}\), is given by the expression:
\[
\nabla^M_X Y = \nabla^\mathcal{D}_X Y + g(T(X), Y)N, \quad X, Y \in \mathcal{D},
\]
where \(T \in \text{End}(\mathcal{D})\) is given by:
\[
2g(T(X), Y) = -N(g(X, Y) - g([X, N], Y) - g([Y, N], X) + g([X, Y], N)).
\]

We will decompose \(T\) into its symmetric and skew-symmetric parts, which we call \(W\) and \(L\) respectively. The connection \(\nabla^\mathcal{D}\) is a metric connection and the tensor \(\mathcal{L} = -\frac{1}{2}gN^*\) measures the lack of integrability of the distribution.

We will also denote by \(\nabla^\mathcal{D}\) the lift of this connection to the spinorial bundle \(\Sigma(\mathcal{D})\). This connection is metric with respect to \((\cdot, \cdot)\) and behaves as a derivation with respect to the Clifford product. Hence, \(\nabla^\mathcal{D}\eta \in \langle \eta \rangle^+\), and there is an endomorphism of \(\mathcal{D}\), that we will call \(S_D\), such that \(\nabla^\mathcal{D}_X \eta = S_D(X) \cdot_D \eta\). Therefore, if we define \(\chi(\mathcal{D}) = R_D \times \chi_i\), where \(R_D\) is the \(G_2\) reduction of \(P(\mathcal{D})\) determined by \(\Psi_D\), we have a splitting of \(\text{End}(\mathcal{D})\) and we can decompose \(S\) according to it:
\[
S_D(X) = \lambda \text{Id} + S_2 + S_3 + S_4,
\]
where \(\lambda \in C^\infty(M)\), \(S_2 \in \chi_2(\mathcal{D})\), \(S_3 \in \chi_3(\mathcal{D})\), \(S_4 \in \chi_4(\mathcal{D})\), and let \(S \in \mathcal{D}\) be such that \(S_4(X) = X \times S\).
Finally consider the $G_2$-structure defined on $M$. First of all, since $g(\nabla X N, Y) = -g(\nabla X Y, N)$ we get that the connection $\nabla^M$ at $\Sigma(M)^+$ in the direction of $D$ is given by:

$$\nabla^M_X \eta = \nabla^D_X \eta - \frac{1}{2}N^T(X)\eta = NA(X)\eta,$$

where $A = S_D - \frac{1}{2}T$. We can decompose $L$ and $W$ according to the splitting of $\text{End}(D)$ into irreducible parts and then decompose $A$:

1. $L = L_2 + L_4$, where $L_2 \in \chi_2(D)$, $L_4 \in \chi_4(D)$ and let $L \in D$ such that $L_4(X) = X \times L$.
2. $W = h(1 + W_3)$, where $h \in C^\infty(M)$, $W_3 \in \chi_3(D)$.
3. $A = \mu(1 + A_2 + A_4)$, where $\mu = \lambda - \frac{1}{2}$, $A_2 = S - \frac{1}{2}L_2$, $A_3 = S_4 - \frac{1}{2}W_3$, $A_4 = S - \frac{1}{2}L_4$.

We will also denote $A = S - \frac{1}{2}L$.

We are going to compute $*d\Omega$ in terms of the previous endomorphisms and $\nabla^D\eta$. Our first lemma is deduced from [1, Theorems 4.6.4.8].

**Lemma 6.5.** If we take an oriented orthonormal local frame of $D$, $(X_1, \ldots, X_7)$ then

$$\sum_{i=1}^7 X_i A(X_i) \eta = -7\mu \eta - 6NA \eta.$$  

**Proof.** We will split the endomorphism $A$ into its $G_2$ irreducible components and then compute each term separately. It is obvious that $\sum_{i=1}^7 X_i \mu X_i \eta = -7\mu \eta$. Moreover,

$$\sum_{i=1}^7 X_i(X_i \times A) \eta = \sum_{i=1}^7 X_i(X_i NA - g(X_i, A)N) \eta = -6NA.$$  

Finally consider the $G_2$-equivariant map, $m: D \otimes D \to \Sigma(D)$, $m(X, Y) = XY \eta$. By dimensional reasons, its kernel must be $\chi_2(D) \oplus \chi_3(D)$. Therefore, if $k \in \{2, 3\}$ we have that:

$$\sum_{i=1}^7 X_i A_k(X_i) \eta = m \left( \sum_{i=1}^7 (A_k)_{ij} X_i X_j \right) = 0,$$

where we have denoted $(A_k)_{ij}$ the entries of the matrix $A_k$ with respect to the basis $(X_1, \ldots, X_7)$. \hfill $\square$

**Remarks 6.6.**

1. Since $\nabla^M_N \eta$ is perpendicular to $\eta$ we can take $U \in D$ such that $\nabla^M_N \eta = -NU \eta$.

In order to compute $\nabla^M_N \eta$ we may take $F = (X_0, X_1, \ldots, X_7)$ a local orthonormal frame of $M$ such that $N = X_0$, a lifting $\tilde{F} \in \tilde{F}(M)$ and write $\eta(p) = [\tilde{F}, s(p)]$. With this notation we have:

$$\nabla^M_{X_0} \eta = [\tilde{F}, ds(X_0)] + \frac{1}{2} \sum_{0 \leq i < j \leq 7} g(\nabla_{X_0} X_i, X_j) X_i X_j \eta$$

$$= [\tilde{F}, ds(X_0)] + \frac{1}{2} \left( X_0 \nabla_{X_0} X_0 + \sum_{1 \leq i < j \leq 7} g(\nabla_{X_0} X_i, X_j) X_i X_j \right) \eta.$$  

Then, $U$ depends on the local information of the section and $\nabla_{X_0} X_i$.

2. The Dirac operator of $M$ is

$$D^M \eta = U \eta + \sum_{i=1}^7 X_i NA(X_i) \eta = (U - 6A + 7\mu N) \eta.$$  

**Lemma 6.7.** If we define the forms on $D$, $\beta_2 \in \Lambda^2 D^*$ and $\beta_3 \in \Lambda^3 D^*$ by:

$$\beta_2(X, Y) = g(A_2(X), Y), \quad \beta_3(X, Y, Z) = \text{alt}(i(A_3)(\cdot)\Psi_D)(X, Y, Z),$$

then:

1. $N^* \wedge i(N)(12\text{alt}(e^{-1}\nabla \eta)) = i(U - A)N^* \wedge \Psi_D - 2N^* \wedge \beta_2,$

2. $12\text{alt}(e^{-1}\nabla \eta) \Omega = 3i(\mu N - A)\Omega + 3\beta_3.$
Proof. The first equality is a consequence of the symmetric or skew-symmetric properties of each factor:
\[
12 \text{alt}(c^{-1} \nabla \eta)(N, X, Y) = -(XY \eta, NU \eta) - (NY \eta, NA(X) \eta) + (NY \eta, NA(X) \eta)
\]
\[
= -i(U)\Psi D(X, Y) - 2(Y \eta, (A_2(X) + X \times A) \eta)
\]
\[
= (i(U - 2A)(N^* \wedge \Psi D) - 2N^* \wedge \beta_2)(N, X, Y).
\]

To check the second one, note that \(12 \text{alt}(c^{-1} \nabla \eta)|_Q = 3 \text{alt}(i(A)\eta)\Psi D\). We compute separately each term in the decomposition of \(A\).

It is evident that \(3 \text{alt}(i(\mu \text{Id})\Psi D)(X, Y, Z) = 3\mu \Psi D(X, Y, Z)\) and \(3 \text{alt}(i(A_2)\eta)\Psi D \neq 3\beta_3\). Moreover, we have that \(\text{alt}(i(A_2)\eta)\Psi D = 0\) because \(A_2 \in \chi_2(Q)\). Finally, if \(X, Y\) and \(Z\) are orthonormal vectors in \(TQ\), then:
\[
i(A_4(X))\Psi D(Y, Z) = (X \times A\eta, Y \times Z \eta) = (X A\eta, Y Z \eta) = -(A\eta, (X \times Y \times Z) \eta).
\]

Therefore, \(3 \text{alt}(i(A_4)\eta)\Psi D(Y, Z) = -3(A\eta, X \times Y \times Z \eta)\).

From lemmas 6.5 and 6.7 and the decomposition of \(*d\eta\) obtained in Proposition 4.4 we conclude:

**Proposition 6.8.** Let \(U \in \mathcal{D}\) such that \(\nabla^N X \eta = -NU \eta\) and define the forms on \(\mathcal{D}\), \(\beta_2 \in \Lambda^2 \mathcal{D}^*\) and \(\beta_3 \in \Lambda^3 \mathcal{D}^*\) by:
\[
\beta_2(X, Y) = g(A_2(X), Y), \quad \beta_3(X, Y, Z) = \text{alt}(i(A_3)\eta)\Psi D(X, Y, Z).
\]

Then, the pure components of \(*d\eta\) in terms of the G2 structure are:
\[
(*d\Omega)_{g8} = \frac{2}{7}(-4i((A + U)N)^* \wedge \Psi + 3i((A + U)\Psi D) + 4N^* \wedge \beta_2 - 6\beta_3,
\]
\[
(*d\Omega)_{s8} = \frac{8}{7}i(U - 6A + 7\mu N)(N^* \wedge \Psi D + *D\Psi D).
\]

6.1. **Hypersurfaces.** Consider an 8-dimensional Spin\(7\) manifold \((M, g)\), whose Spin\(7\) form is constructed from a unitary section \(\eta\) of the spinorial bundle \(\Sigma(M)^+\), as in Definition 3. Let \(Q\) be an oriented hypersurface and take a unitary vector field \(N\) such that \(TM = \langle N \rangle \oplus TQ\) as oriented vector bundles.

The tubular neighbourhood theorem guarantees the existence of a cooriented distribution \(\mathcal{D}\) defined on a neighbourhood \(O\) of \(Q\) such that \(\mathcal{D}|_Q = TQ\). The coorientation is determined by a unitary extension of the normal vector field that we also denote by \(N\). Both \(\mathcal{D}\) and \(Q\) have G2 structures determined by the spinor \(\eta\); we are going to relate them using Proposition 6.8 in the manifold \(O\).

Note that the Levi-Civita connection of the hypersurface \(Q\) is \(\nabla^D|_Q\). Moreover, \(\mathcal{L}|_Q = 0\) and \(\mathcal{W}|_Q\) is the Weingarten operator. Therefore, the restriction of \(\mathcal{S}_D\) at \(Q\) is the endomorphism \(\mathcal{S}\) of the submanifold \(Q\). Decompose \(\mathcal{S}|_Q\) and \(\mathcal{W}|_Q\) with respect of the G2 splitting of \(\text{End}(\mathcal{T}Q)\):

1. \(\mathcal{S} = \lambda \text{Id} + S_2 + S_3 + S_4\)
2. \(\mathcal{W}|_Q = 7H \text{Id} + W_3\),

where \(\lambda \in C^\infty(M), S_2 \in \chi_2(Q), S_3, W_3 \in \chi_3(Q), S_4 \in \chi_4(Q)\) and \(H \in C^\infty(Q)\) is the mean curvature. We will also denote by \(S\) the vector in \(TQ\) such that \(S_4(X) = X \times S\).

**Corollary 6.9.** Let \(U \in TQ\) such that \(\nabla^N \eta|_Q = -NU \eta\) and \(\Psi Q = i(N)\Omega\). Define the forms on \(Q\), \(\beta_2 \in \Lambda^2 T^*Q\) and \(\beta_3 \in \Lambda^3 T^*Q\) by:
\[
\beta_2(X, Y) = g(S_2(X), Y), \quad \beta_3(X, Y, Z) = \text{alt}(i((S_3 - \frac{1}{2}W_3)\eta)\Psi D)(X, Y, Z).
\]

Then, the pure components of \(*d\Omega\) in terms of the G2 structure are:
\[
(*d\Omega)_{g8} = \frac{2}{7}(-4i((S + U)N)^* \wedge \Psi + 3i((S + U)\Psi Q) + 4N^* \wedge i^*\beta_2 - 6\beta_3,
\]
\[
(*d\Omega)_{s8} = \frac{8}{7}i(U - 6S + 7(\lambda - \frac{7}{2}H)N)(N^* \wedge \Psi Q + *Q \Psi Q).
\]

**Remark 6.10.** Note that the condition \(\nabla_N \eta|_Q = -NU \eta\) does not depend on the extension of the vectors. Moreover, we can compute \(U\) taking into account equation (6). The terms involved are extrinsic and not encoded in \(\mathcal{S}\) and \(\mathcal{W}\).

Therefore, the Spin\(7\) type of the ambient manifold provides relations between the G2 type of the hypersurface, the vector \(U\) and the Weingarten operator. Before stating the result, we recall that a hypersurface is said to be totally geodesic if \(\mathcal{W} = 0\), totally umbilic if \(W_3 = 0\) and minimal if \(H = 0\).
Theorem 6.11. Let \((M, g)\) be a Riemannian manifold endowed with a \(\text{Spin}(7)\) structure determined by a spinor \(\eta\). Let \(Q\) be an oriented hypersurface with normal vector \(N\) and let \(U \in TQ\) be such that \(\nabla_N\eta|_Q = -NU\eta.

1. If \(M\) has a parallel \(\text{Spin}(7)\) structure, then \(Q\) has a cocalibrated \(G_2\) structure. Moreover,
   1.1 \(S = 0\) if and only if \(Q\) is totally geodesic.
   1.2 \(S \in \chi_1(Q)\) if and only if \(Q\) is totally umbilic.
   1.3 \(S \in \chi_3(Q)\) if and only if \(Q\) is a minimal hypersurface.
2. If \(M\) has a locally conformally parallel \(\text{Spin}(7)\) structure, then \(S \in \chi_1(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)\).
   Indeed,
   2.1 \(S \in \chi_1(Q)\) if and only if \(U = 0\) and \(Q\) is totally umbilic.
   2.2 \(S \in \chi_1(Q) \oplus \chi_4(Q)\) if and only if \(Q\) is totally umbilic.
3. If \(M\) has a balanced \(\text{Spin}(7)\) structure, then:
   3.1 \(S \in \chi_2(Q) \oplus \chi_3(Q)\) if and only if \(U = 0\) and \(Q\) is a minimal hypersurface.
   3.2 \(S \in \chi_1(Q) \oplus \chi_3(Q) \oplus \chi_3(Q)\) if and only if \(U = 0\).
   3.3 \(S \in \chi_2(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)\) if and only if \(Q\) is a minimal hypersurface.

Proof. The parallel case follows from the equalities \(U = S = 0, S_2 = 0, 2\lambda = 7H\) and \(2S_3 = W_3\). The locally conformally parallel case follows from the equalities \(U = -S, S_2 = 0\) and \(2S_3 = W_3\), which imply that \(S \in \chi_1(Q) \oplus \chi_2(Q) \oplus \chi_3(Q)\). Finally the balanced case follows from \(U = 6S\) and \(2\lambda = 7H\). □

6.2. Principal bundles over a \(G_2\) manifold. Let \(Q\) be a \(G_2\) manifold and let \(\pi: M \to Q\) be a \(G = \mathbb{R}\) or \(G = S^1\) principal bundle over \(Q\); identify its algebra \(\mathfrak{g}\) with \(\mathbb{R}\).

Define the vertical field \(N(p) = \frac{d}{dt}|_{t=0} (p \exp(t))\). A connection \(\omega: TM \to \mathfrak{g}\) defines a horizontal distribution \(\mathcal{H}\) and we can define a metric on \(M\) such that:

1. The map \(d\pi: \mathcal{H}_p \to T_{\pi(p)}Q\) is an isometry
2. The vector \(N(p)\) is unitary and perpendicular to \(\mathcal{H}_p\).

The projection \(d\pi\) induces a map \(\tilde{p}: P(\mathcal{H}) \to P(Q)\) so that the pullback to \(\tilde{P}(Q)\) defines a spin structure \(\tilde{P}(\mathcal{H})\) on \(P(\mathcal{H})\). The map \(\tilde{p}: \tilde{P}(\mathcal{H}) \to \tilde{P}(Q)\), which is canonically defined, has the property that \(\tilde{p}(\tilde{\varphi} \tilde{F}) = \tilde{\varphi}(\tilde{F})\) if \(\tilde{\varphi}\) is \(\text{Spin}(8)\), inducing therefore a map between the spinorial bundles, which we call \(p\). Note that this map gives isomorphisms \(\Sigma(\mathcal{H})_p \to \Sigma(Q)_{\pi(p)}\). Moreover, let \(X \in TQ\) and denote by \(X^h\) its horizontal lift, then \(\tilde{p}(X^h) = X^\pi(\phi)\). Therefore, a section \(\eta: Q \to \Sigma(Q)\) allows us to define a section \(\eta: M \to \Sigma(H)\) by means of the expression \(p^*(\eta) = \eta\). If we denote by \(\Psi_Q\) the \(G_2\) form on \(Q\), then \(\Psi_D = \pi^*\Psi_Q\).

Furthermore, one can check that \(\nabla^Q_{X^h} Y^h = (\nabla^Q_X Y)^h\). Hence, if we take \(S \in \text{End}(Q)\) such that \(\nabla^Q_X \eta = S(X)\eta\), we get that the endomorphism of the distribution \(\mathcal{S}_D\) is the lifting of \(S\), that is:

\[
\nabla^Q_{X^h} \eta = S(X)^h \eta.
\]

Therefore the distribution \(\mathcal{H}\) and the manifold \(Q\) have the same type of \(G_2\) structure. In order to classify the \(\text{Spin}(7)\) structure on \(M\), denote the curvature of the connection \(\omega\) by:

\[
\mathfrak{L}(X, Y) = [X^h, Y^h] - [X, Y]^h \in \langle N \rangle, \quad X, Y \in TQ.
\]

Since \(\mathfrak{L}(X, Y) \in \langle N \rangle\) we also denote by \(\mathfrak{L}\) the 2-form that we obtain contracting the tensor with the metric. As a skew-symmetric endomorphism, we can decompose \(\mathfrak{L} = L_2 + L_4\) where \(L_4(X) = X \times L\) for some \(L\) in \(TQ\).

Corollary 6.12. Suppose that \(\nabla^Q_X \eta = \mathfrak{L}(X) \odot \eta\) with \(\mathfrak{S}(X) = \lambda d + S_2 + S_3 + S_4\) where \(\lambda \in C^\infty(Q), S_2 \in \chi_2(Q), S_3 \in \chi_3(Q), S_4 \in \chi_4(Q)\) and let \(S \in TQ\) be such that \(S_4(X) = X \times S\). Define \(\beta_2 \in \Lambda^2 T^*Q\) and \(\beta_3 \in \Lambda^3 T^*Q\) by:

\[
\beta_2(X, Y) = g \left( S_2(X) - \frac{1}{4} L_2(X), Y \right), \quad \beta_3(X, Y, Z) = \text{alt}(i(S_3(\cdot))) \Psi_Q)(X, Y, Z).
\]

The pure components of \(*d\Omega\) in terms of the \(G_2\) structure are:

\[
(*d\Omega)_4 = \frac{2}{7} \left( -4i(S^h + \frac{1}{2} L^h) N^\ast \pi^* \Psi_Q + 3i(S^h + \frac{1}{2} L^h) \pi^*(*Q \Psi_Q) \right) - 4 N^\ast \pi^* \beta_2 + 6 \pi^* \beta_3,
\]

\[
(*d\Omega)_8 = \frac{8}{7} i \left( \frac{15}{4} L^h - 6 S^h + 7 \lambda N \right) (N^\ast \pi^* \Psi_Q + \pi^*(*Q \Psi_Q)).
\]
Suppose that $\omega = 0$. Moreover, if $\omega$ according to Koszul formulas we have:

$$L(\bar{M}) = \frac{1}{2} \pi^* \mathcal{L}(X, Y), \quad U = \frac{1}{2} L^h.$$

First of all, since the connection $\omega$ is left-invariant we have that $[X^h, N] = 0$ if $X \in TQ$. Thus, $\mathcal{W} = 0$. Moreover, $\mathcal{L}(X^h)(Y^h) = \frac{1}{2} \mathcal{L}(X, Y)$. Furthermore, let $F = (X_1, \ldots, X_7)$ be a local frame of $\mathcal{H}$ which lifts some local frame of $TQ$. Take a lift $\tilde{F} \in \mathcal{P}(\mathcal{H})$ and write $q(p) = [\tilde{F}, s(p)]$. We denote $X_0 = N$ and compute $U$ using the formula (6).

By definition, if $\bar{q}(\pi(p)) = [\bar{p}(F(p)), \bar{s}(\pi(p))]$ then $s(p) = \bar{s}(\pi(p))$ so that $ds_p(N) = 0$. Besides, according to Koszul formulas we have:

$$\nabla_N N = 0,$$

$$g(\nabla_N X_i, X_j) = -\frac{1}{2} g([X_i, X_j], N) = -\frac{1}{2} g(N, \mathcal{L}(d\pi(X_i), d\pi(X_j))).$$

Therefore, if we define $\gamma_i(X, Y) = g(\bar{L}_i(X), Y)$, for $i \in \{2, 4\}$, then:

$$\nabla_N \eta = \frac{1}{4} \pi^* \mathcal{L} \eta = -\frac{1}{4} \pi^* \gamma_4 \eta = -\frac{3}{4} N L^h \eta,$$

where we have used that $\pi^* \gamma_2 \eta = 0$ because $\mathfrak{g}_2 \subset \mathfrak{sp}(7) = \Lambda^2 \mathbb{R}^8$ and $\pi^* \gamma_4 = -i(N)(L^h)\Omega$ so that $\pi^* \gamma_4 \eta = 3N L^h \eta$, as we noted in the proof of Lemma 3.1.

### 6.3. Warped products

We analyze Spin(7) structures on warped products of a $G_2$ manifold with $\mathbb{R}$. Recall that a warped product of two Riemannian manifolds $(X_1, g_1)$ and $(X_2, g_2)$ is $(X_1 \times X_2, g_1 + f_1 g_2)$ where $f_1 : X_1 \to \mathbb{R}$ is a smooth function. Therefore, we have to distinguish two cases.

#### 6.3.1. Consider a $G_2$ manifold $(Q, g)$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$. Define the Riemannian manifold $(M = Q \times \mathbb{R}, e^{2f} g + dt^2)$. This is the so-called spin cone.

The distribution $\mathcal{D} = TQ$ obviously admits a $G_2$ structure. The spinorial bundle is given by $\Sigma(M) = \Sigma(TQ \times \mathbb{R}) = \Sigma(Q) \times \mathbb{R}$ and Clifford products are related by $(X, \phi, t) = e^{-f} X \cdot \mathcal{D} (\phi, t) = e^{-f} \frac{\partial}{\partial t} X(\phi, t)$ if $X \in TQ$. In the last expression, we have suppressed the symbol $\cdot$ to denote the Clifford product on $M$.

A unitary section $\eta$ is constructed from a section $\bar{\eta} : Q \to \Sigma(Q)$ by defining $\eta : M \to \Sigma(\mathcal{D})$, $\eta(x, t) = (\bar{\eta}(x), t)$. If we denote by $\Psi_Q$ the $G_2$ form on $Q$, then $\Psi_\mathcal{D} = e^{3f} \pi^* \Psi_Q$ and $*_{\mathcal{D}} (\Psi_\mathcal{D}) = e^{4f} *_{Q}(\Psi_Q)$. In addition, since $\nabla^Q_X Y = \nabla^Q_X Y$ when $X, Y \in TQ$, we have that $\nabla^Q_X \eta = e^{-f} S(X) \cdot \mathcal{D} \eta$, if $X \in TQ$ and $\nabla^Q_X \bar{\eta} = S(X) \bar{\eta}$. That is, $\mathcal{S} = e^{-f} S$.

**Corollary 6.13.** Suppose that $\nabla^Q_X \eta = S(X) \cdot \mathcal{Q} \subset \mathbb{Q}$ with $S(X) = \lambda \text{Id} + S_2 + S_3 + S_4$ where $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$. Let $S \in TQ$ be such that $S_4(X) = X \times S$. Denote by $\Psi_Q$ the $G_2$-form on $Q$ and define $\beta_2 \in \Lambda^2 T^* Q$ and $\beta_3 \in \Lambda^3 T^* Q$ by:

$$\beta_2(X, Y) = g(S_2(X), Y), \quad \beta_3(X, Y, Z) = \text{alt}(i(S_3(\cdot))\Psi_Q)(X, Y, Z).$$

The pure components of $*d\Omega$ in terms of the $G_2$ structure are:

$$(*d\Omega)_4 = \frac{2}{7} \left(-4e^{2f} i(S) dt \wedge \pi^* \Psi_Q + 3e^{3f} i(S) \pi^* (*_{Q} \Psi_Q) + 4e^f dt^* \wedge \pi^* \beta_2 - 6e^{2f} \pi^* \beta_1, \right.$$  

$$(*d\Omega)_8 = \frac{8}{7} \left(-6e^{-f} S + 7(\lambda e^{-f} + \frac{1}{2} f')(\frac{\partial}{\partial t}) \right)(e^{3f} dt \wedge \pi^* \Psi_Q + e^{4f} \pi^* (*_{Q} \Psi_Q)).$$

**Proof.** The result follows immediately from Proposition 6.8 once we check that $\mathcal{W} = -f' \text{Id}$, $\mathcal{L} = 0$ and $U = 0$.

Since the distribution $\mathcal{D}$ is integrable, we have that $\mathcal{L} = 0$. Take an orthonormal frame of $TQ$, $(X_1, \ldots, X_7)$ and note that $W(X_i, X_j) = -f e^{2f} \delta_{ij}$ so that $W = -f'$. Moreover, using the Koszul formulas we get:

$$\nabla_X \frac{\partial}{\partial t} = 0 = \nabla_{\frac{\partial}{\partial t}} (e^{-f} X_i).$$

Therefore, using formula (6) we conclude that $\nabla_{\frac{\partial}{\partial t}} \eta = 0$. 

\[\mathbb{Q} \]
6.3.2. Consider a $G_2$ manifold $(Q, g)$ and a smooth function $f : Q \to \mathbb{R}$. Define the Riemannian manifold $(M = Q \times \mathbb{R}, g + e^{2f}dt^2)$.

The distribution $\mathcal{D} = TQ$ obviously admits a $G_2$ structure. The spinorial bundle is given by $\Sigma(M)^+ = \Sigma(TQ \times \mathbb{R}) = \Sigma(Q) \times \mathbb{R}$ and the Clifford products are related by $(X \cdot_Q \phi, t) = X \cdot \mathcal{D} \phi, t) = e^{-f} \frac{\partial}{\partial t} X(\phi, t)$ if $X \in TQ$. We have suppressed again the symbol $\cdot$ to denote the Clifford product on $M$.

A unitary section $\eta$ is constructed from a section $\eta : Q \to \Sigma(Q)$ by defining $\eta : M \to \Sigma(\mathcal{D})$, $\eta(x, t) = (\eta(x), t)$. If we denote by $\Psi_Q$ the $G_2$ form on $Q$, then $\Psi_\mathcal{D} = \pi^* \Psi_Q$ and $\ast_\mathcal{D}(\Psi_\mathcal{D}) = \ast_Q(\Psi_Q)$. In addition, since $\nabla_X^Q Y = \nabla_X^Q Y$ when $X, Y \in TQ$, if we take $S \in \text{End}(TQ)$ with $\nabla_X^S \eta = S(X)\eta$, then $S_\mathcal{D} = S$.

**Corollary 6.14.** Suppose that $\nabla_X^Q \eta = S(X) \cdot_Q \eta$ with $S(X) = \lambda d + S_2 + S_3 + S_4$ where $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$. Let $S \in TQ$ be such that $S_4(X) = X \times S$. Denote by $\Psi_Q$ the $G_2$-form on $Q$ and define $\beta_2 \in \Lambda^2 T^*Q$ and $\beta_3 \in \Lambda^3 T^*Q$ by:

$$
\beta_2(X, Y) = g(S_2(X, Y)), \quad \beta_3(X, Y, Z) = \text{alt}(i(S_3(\cdot))\Psi_Q)(X, Y, Z).
$$

The pure components of $\ast \mathrm{d}\Omega$ in terms of the $G_2$ structure are:

$$
(*\mathrm{d}\Omega)_\mathcal{D} = \frac{2}{7} \left(-4i \left(S + \frac{1}{2} \text{grad}(f)\right) e^f dt \wedge \pi^* \Psi_Q + 3i \left(S + \text{grad}(f)\right) \pi^*(\ast_Q \Psi_Q)\right) + 4e^f dt \wedge \pi^* \beta_2 - 6\pi^* \beta_3,
$$

$$
(*\mathrm{d}\Omega)_\mathcal{D} = \frac{8i}{7} \left(\frac{1}{2} \text{grad}(f) - 6S + 7\lambda e^{-f} \frac{\partial}{\partial t}\right) (e^f dt \wedge \pi^* \Psi_Q + \pi^*(\ast_Q \Psi_Q)).
$$

**Proof.** The result follows immediately from Proposition 6.8 once we check that $W = 0$, $\mathcal{L} = 0$ and $U = \frac{1}{2} \text{grad}(f)$.

Since the distribution $\mathcal{D}$ is integrable, we have that $\mathcal{L} = 0$. Take an orthonormal frame of $TQ$, $(X_1, \ldots, X_7)$ and note that $W(X_i, X_j) = 0$. Moreover, using the Koszul formulas we get:

$$
g(\nabla_{e^{-f} \frac{\partial}{\partial t}} X_i, X_j) = 0,$$

$$
g \left(\nabla_{e^{-f} \frac{\partial}{\partial t}} e^{-f} \frac{\partial}{\partial t}, X_i\right) = -X_i(f).
$$

Therefore, using formula (6) we conclude that $\nabla_N \eta = -\frac{1}{2}e^{-f} \left(\frac{\partial}{\partial t}\right) \text{grad}(f)\eta$. 

7. Spin(7) structures on quasi abelian Lie algebras

As an application of the previous section, we are going to study Spin(7) structures on quasi abelian Lie algebras. The geometric setting will be that of a simply connected Lie group with an invariant Spin(7) structure, endowed with an integrable distribution which inherits a $G_2$ structure. The integral submanifolds of the distribution are actually flat, so that the $G_2$ distribution will be parallel, while we will have non-trivial Weingarten operators. In some cases, finding a lattice in the Lie group will allow us to give compact examples.

First of all, let us recall the following definition:

**Definition 7.1.** A Lie algebra $\mathfrak{g}$ is called quasi abelian if it contains a codimension 1 abelian ideal $\mathfrak{h}$.

The information of $\mathfrak{g}$ is then encoded in $ad(x)$ for any vector $x$ transversal to $\mathfrak{h}$. The following result shows that $\mathfrak{h}$ is unique in $\mathfrak{g}$ with exception of the Lie algebras $\mathbb{R}^n$ and $L_3 \oplus \mathbb{R}^{n-3}$, where $L_3$ is the Lie algebra of the 3-dimensional Heisenberg group, which is generated by $x, y, z$ with relations $[x, y] = z$ and $[x, z] = [y, z] = 0$.

**Lemma 7.2.** Let $\mathfrak{g}$ be a $n$-dimensional quasi abelian Lie algebra with $n \geq 3$. If $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ or $L_3 \oplus \mathbb{R}^{n-3}$, then it has a unique codimension 1 abelian ideal. Moreover, codimension 1 abelian ideals on $L_3 \oplus \mathbb{R}^{n-3}$ are parametrized by $\mathbb{RP}^1$.

**Proof.** Suppose that $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ and let $\mathfrak{h}$ be a codimension 1 abelian ideal with a transversal vector $x$. Let $\mathfrak{h}'$ be a codimension 1 abelian ideal different from $\mathfrak{h}$. If $u \in \mathfrak{h}$ is such that $x + u \in \mathfrak{h}'$ and $v \in \mathfrak{h} \cap \mathfrak{h}'$, then $0 = [x + u, v] = ad(x)(v)$. Since $\mathfrak{h} \cap \mathfrak{h}'$ is $(n - 2)$-dimensional and $\mathfrak{g}$ is not abelian we conclude that $\mathfrak{h} \cap \mathfrak{h}' = \ker(ad(x)|_{\mathfrak{h}'})$ and $ad(x)(\mathfrak{h}) = \langle z \rangle$ for some $z \in \mathfrak{h}$. Take $y \in \mathfrak{h}$ with $[x, y] = z$ and observe that $z \in \langle [\mathfrak{g}, \mathfrak{g}] \rangle \subseteq \mathfrak{h}'$, that is, $z \in \mathfrak{h} \cap \mathfrak{h}'$ and $[x, z] = 0$. Therefore, $\mathfrak{g}$ is isomorphic to $L_3 \oplus \mathbb{R}^{n-3}$.

Moreover, from the discussion above we get that $\mathfrak{h}' = \langle v, z \rangle \oplus \mathbb{R}^5$ for some $v \in \langle x, y \rangle$. Conversely, all the subspaces of the previous form are actually codimension 1 abelian ideals. Therefore, they are parametrized by $\mathbb{RP}^1$.

$\square$
An invariant Spin(7) structure on a Lie group is determined by the choice of a Spin(7) form $\Omega$, which is in turn determined by a direction of the spinnorial space $\Delta^+$. Define the set $\mathcal{QA}$ with elements $(g, h, g, \nu, \Omega)$ where $g$ is a non-trivial quasi abelian Lie algebra with a marked codimension 1 abelian ideal $h$, $g$ is a metric on $g$, $\nu$ is a volume form on $g$ and $\Omega$ is a Spin(7) structure on $(g, \nu)$. We will say that $\varphi': (g, h, g, \nu, \Omega) \to (g', h', g', \nu', \Omega')$ is an isomorphism if $\varphi$ is an isomorphism of Lie algebras such that $\varphi'(h) = h'$, $(\varphi')^*g' = g$, $\varphi^*\nu' = \nu$ and $\varphi^*\Omega = \Omega$.

**Lemma 7.3.** The set $\overline{\mathcal{QA}}$ of isomorphisms classes of $\mathcal{QA}$ is given by:
$$\overline{\mathcal{QA}} = \left( \left( \text{End}(\mathbb{R}^7) - \{0\} \right) \times \mathbb{P}(\Delta^+) \right) / \text{O}(7),$$
where $\text{O}(7)$ acts via
$$\varphi \cdot (E, [\eta]) = (\det(\varphi)\varphi \circ E \circ \varphi^{-1}, [\rho(\varphi)\eta]),$$
where $\rho$ is a lifting to Spin(8) of the unique $\varphi' \in \text{SO}(8)$ such that $\varphi'|_{\mathbb{R}^7} = \varphi$.

*Proof.* A map $\left( \text{End}(\mathbb{R}^7) - \{0\} \right) \times \mathbb{P}(\Delta^+) \to \mathcal{QA}$ can be defined as follows. Take $(E, \eta)$ and define the Lie structure on $\mathbb{R}^8$ with oriented basis $(\vec{e}_0, \ldots, \vec{e}_7)$ such that $\mathbb{R}^7 = \langle \vec{e}_1, \ldots, \vec{e}_7 \rangle$ is a maximal abelian ideal and $E = ad(e_0)|_{\mathbb{R}^7}$. We will endow this algebra with the canonical metric, the standard volume form and the spin structure determined by $\eta$.

It is obvious that a representative of each element of $\overline{\mathcal{QA}}$ can be chosen to live in the image of our map. Moreover, if two structures given by $(E, \eta)$ and $(E', \eta')$ are isomorphic via $\varphi'$, we have the following:
1. $\varphi'(e_0) = \pm e_0$ and $\varphi'|_{\mathbb{R}^7} \in \text{O}(7)$, since $\varphi'$ preserves the metric and the orientation.
2. Denote by $\tilde{\varphi}$ any lifting of $\varphi'$ to Spin(8). Since $(\varphi')^*\Omega' = \Omega$, we have that $\text{Stab}(\Omega) = (\varphi')^{-1} \circ \text{Stab}(\Omega') \circ \varphi'$. Thus $\text{Stab}(\eta) = \tilde{\varphi}^{-1} \text{Stab}(\eta') \tilde{\varphi}$, but $\text{Stab}(\rho(\tilde{\varphi})^{-1}\eta') = \tilde{\varphi}^{-1} \text{Stab}(\eta') \tilde{\varphi}$, so that $\eta = \pm \rho(\tilde{\varphi})^{-1}\eta'$. $\varphi \circ E = \det(\varphi)E' \circ \varphi$, since $\varphi'$ is an isomorphism of Lie algebras.

From now on we denote by $(\mathbb{R}^8, [E, [\eta]])$ to $(g, h, g, \nu, \Omega) \in \mathcal{QA}$ where $g$ is the Lie algebra $\mathbb{R}^8$ with maximal abelian ideal $h = \mathbb{R}^7$, $ad(e_0) = E$, $g$ is the canonical metric, $\nu$ is the canonical volume form and the Spin(7) form $\Omega$ by $[\eta]$.

**Remark 7.4.** To obtain an analogue of Lemma 7.3, suppressing the condition $\varphi'(h) = h'$ in the definition of isomorphism, we have to treat separately the case of the Lie algebra $L_3 \oplus \mathbb{R}^5$. For this purpose, define $E(x) = e_1^*(x)e_2$ and observe that lemmas 7.2 and 7.3 allow us to suppose that any isomorphism of algebras with underlying Lie algebra $L_3 \oplus \mathbb{R}^5$ is represented by $\varphi': (\mathbb{R}^8, \lambda E, [\eta]) \to (\mathbb{R}^8, \lambda E, [\eta'])$, for some $\lambda, \lambda' \neq 0$.

The set $\varphi'(\mathbb{R}^7)$ is a codimension 1 abelian ideal, hence Lemma 7.2 guarantees that $\varphi'(e_0) = \cos(\theta)e_0 + \sin(\theta)e_1$. Denote $\mathbb{R}^6 = \langle e_2, \ldots, e_7 \rangle$ and let $v, v' \in \mathbb{R}^6$ be such that $\varphi'(v) = -\mu \sin(\theta)e_0 + \mu \cos(\theta)e_1 + v'$. Then, $\varphi'(v, u, v') = [\cos(\theta)e_0 + \sin(\theta)e_1, -\mu \sin(\theta)e_0 + \mu \cos(\theta)e_1 + v'] = \mu \lambda' e_2$. Therefore $\mu = 0$, $\mathbb{R}^6$ is $\varphi'$-invariant and $\varphi'(e_1) = \mp \sin(\theta)e_0 \pm \cos(\theta)e_1$.

Denote by $\varphi_1$ the restriction of $\varphi'$ to $(e_0, e_1)$ and note that $\lambda \varphi'(e_2) = \varphi'(e_0, e_1) = [\varphi'(e_0), \varphi'(e_1)] = \det(\varphi_1)\lambda e_2$, hence $\varphi'(e_2) = \det(\varphi_1)\lambda^2 e_2$ and $|\lambda| = |\lambda'|$. Then, $\varphi'$ is determined by $\varphi_1$ and $\varphi_2 = \varphi'|_{\mathbb{R}^5}$, where $\mathbb{R}^5 = \langle e_3, \ldots, e_7 \rangle$, under the conditions $\det(\varphi_2) = 1$ and $\varphi'(e_2) = \det(\varphi_1)\lambda^2 e_2$.

The condition over the spinor is obviously $\eta' = \pm \rho(\tilde{\varphi})\eta$, where $\tilde{\varphi}$ is any lifting of $\varphi'$ to Spin(8).

In the following result we describe the action which appears in Lemma 7.3.

**Lemma 7.5.** Under the action of $\text{O}(7)$ on $\text{End}(\mathbb{R}^7)$,
$$\varphi \cdot E = \det(\varphi)E \circ \varphi^{-1},$$
the sets $\{\text{Id}\}$, $\text{Sym}^2_+(\mathbb{R}^7)$ and $\Lambda^3\mathbb{R}^7$ are parametrized respectively by:

1. $\{0, \infty\}$,
2. $\{\lambda_1, \ldots, \lambda_\ell : \lambda_1 \leq \lambda_{j+1}, \sum_{j=1}^\ell \lambda_j = 0\}$, where $\lambda_1, \ldots, \lambda_\ell \sim (-\lambda_7, \ldots, -\lambda_1)$,
3. $\{\lambda_1, \lambda_2, \lambda_3 : 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3\}$.

*Proof.* The first claim is obvious and the second follows from the fact that each symmetric matrix has an oriented orthonormal basis of ordered eigenvectors. Note also that $-\text{Id} \cdot \text{diag}(\lambda_1, \ldots, \lambda_7) = \text{diag}(-\lambda_7, \ldots, -\lambda_1)$, hence $\lambda_1, \ldots, \lambda_7 \sim (-\lambda_7, \ldots, -\lambda_1)$ is related to $(-\lambda_7, \ldots, -\lambda_1)$.

If $E$ is a skew-symmetric endomorphism of $\mathbb{R}^7$ we can find a hermitian basis in $\mathbb{C}^7$ of eigenvectors and the eigenvalues are of the form $(-\lambda_3 i, -\lambda_2 i, \lambda_1 i, 0, \lambda_1 i, \lambda_2 i, \lambda_3 i)$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$. Moreover, the
real parts of the eigenspaces associated to $-\lambda_i i$ and $\lambda_i i$ coincide. Thus, we can find a positive oriented orthonormal basis $(v_1, w_1, \ldots, v_k, w_k, u_1, \ldots, u_{7-2k})$ of $\mathbb{R}^7$, such that $E(v_j) = \lambda_j v_j$ and $E(u_j) = 0$. Finally note that $(\lambda_1, \lambda_2, \lambda_3)$ are invariantly defined in the orbit.

In Lemma 7.3, the second factor of the product of $QA$ depends on $\text{Stab}(E)$ under the action defined by (8) and it is determined by the number of equal eigenvalues. Now we compute the invariants that we defined for $G_2$ distributions on $\mathbb{R}^7$.

**Proposition 7.6.** Consider $(\mathbb{R}^8, E, [\eta]) \in QA$ and decompose $E$ according to the $G_2$ structure induced by $\eta$, that is $E = \Id + E_2 + E_3 + E_4$, where $h \in \mathbb{R}$, $E_2 \in \mathfrak{X}_2$, $E_3 \in \mathfrak{X}_3$, $E_4 \in \mathfrak{X}_4$ and $E_4(X) = X \times E$ for some $E \in \mathbb{R}^7$. Define $\Psi_1, \beta_3 \in \Lambda^3 T^* \mathbb{R}^7$ by $\Psi = \Omega_{[G^7}$ and $\beta_3(X, Y, Z) = \text{alt}(i(E_4(\eta)\Psi))$. We have:

$$(*)d\Omega_{48} = \frac{2}{7} \left(6i(E)e^0 \wedge \Psi - \frac{9}{4} i(E) *_{\mathbb{R}^7} \Psi \right) + 6\beta_3,$$

$$(*)d\Omega_{8} = -\left(\frac{12}{7}E + 4he_0\right)(e^0 \wedge \Psi *_{\mathbb{R}^7} \Psi).$$

**Proof.** The result follows immediately from Proposition 6.8 once we check that: $\mu = -\frac{1}{2}h$, $A_2 = 0$, $A_3 = -\frac{1}{2}E_3$, $A = 0$ and $U = -\frac{1}{2}E$.

To obtain this, first observe that $\nabla^h_\eta = 0$ and $\mathcal{L} = 0$ because $h$ is an abelian ideal. From the formula of the Weingarten operator we get: $W = h\Id + E_3$. To compute $U$ we use again equation (6), obtaining that:

$$\nabla_{c_0} = \frac{3}{2}c_0 E \eta,$$

since $\nabla_{c_0} = 0$ because $h$ is an ideal and $\nabla_{c_0} = (E_2 + E_3)(e_j)$ if $j > 0$.

In the next result we characterise in terms of Lemma 7.5 the type of $\text{Spin}(7)$ structure on quasi abelian Lie algebras. For this purpose, recall that a Lie algebra is called unimodular if the volume form is not exact. In the case of the Lie algebra $(\mathbb{R}^8, E)$, it is equivalent to say that $E$ is traceless.

**Theorem 7.7.** Consider the Lie algebra $(\mathbb{R}^8, E)$ endowed with the standard metric and volume form. Denote by $E_{13}$ and $E_{24}$ the symmetric and skew-symmetric parts of the endomorphism $E \neq 0$. Then, the Lie algebra admits a Spin(7) structure of type:

1. parallel, if and only if $E_{13} = 0$ and $E_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$, $\lambda_2 > 0$ as in Lemma 7.5.
2. locally conformally parallel and non-parallel if and only if $E_{13} = h\Id$ with $h \neq 0$ and $E_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 < \lambda_1 \leq \lambda_2$, as in Lemma 7.5.
3. balanced if and only if it is unimodular and $E_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$, as in Lemma 7.5.

Moreover, if $E_{24} \neq 0$ then it admits a Spin(7) structure of mixed type.

**Proof.** We identify $E_{24}$ with a 2-form $\gamma$ which can be written with respect to a positive oriented orthonormal basis $(X_1, \ldots, X_7)$ of $\mathbb{R}^7$ as $\gamma = \lambda_1 X^{23} + \lambda_2 X^{45} + \lambda_3 X^{67}$, where $0 \leq \lambda_j \leq \lambda_{j+1}$ and $X^{ij} = X^i \wedge X^j$.

Due to Proposition 7.6, to prove the first part we have to check that under the assumption $E_{24} \neq 0$, the existence of a spinor $\eta$ such that $\gamma = 0$ is equivalent to the fact that $E_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$. This spinor exists if and only if $\rho(E_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is non-invertible for some 8-dimensional real irreducible representation $\rho : Cl_7 \to \text{End}(\mathbb{R}^8)$ which maps the volume form $\nu_7$ to the identity, since they are all equivalent [15, Proposition 5.9].

It is known that the two distinct irreducible representations of $Cl_7$ can be constructed from the octonions $\mathbb{O}$ [15, p. 51]. Specifically, those are the extension to $Cl_7$ of the maps $\rho_0 : \mathbb{R}^7 \to \text{End}(\mathbb{R}^8)$, $\rho_0(\theta)(x) = \theta x x$, where $\theta = \pm 1$ and $\mathbb{R}^7$ is viewed as the imaginary part of the octonions. Define the isometry $\varphi$ of $\mathbb{R}^7$ which maps $X_i$ to $e_i$ and note that the volume form is fixed by the extension of $\varphi$ to the Clifford algebra. The extensions of $\rho_0$ and $\varphi$ to $Cl_7$ are denoted in the same way. We check the previous condition using the representation $\rho(E) = \rho_0(\varphi) : Cl_7 \to \text{End}(\mathbb{R}^8)$, taking $\theta$ such that $\rho_0(\nu_7) = \Id$. The determinant of $\rho(E_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is given by:

$$(\lambda_1 + \lambda_2 + \lambda_3)^2(\lambda_1 + \lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_2 + \lambda_3)^2.$$

Since $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the endomorphism is non-invertible if and only if $\lambda_3 = \lambda_2 + \lambda_1$.

Finally, if $E_{24} \neq 0$ then $\rho(E_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7) \neq 0$ so that there is a spinor inducing a Spin(7) structure of mixed type. $\square$
Recall that solvmanifolds are compact quotients $G/\Gamma$, where $G$ is a simply connected solvable Lie group and $\Gamma$ is a discrete lattice. This forces the Lie algebra $\mathfrak{g}$ of $G$ to be unimodular [19, Lemma 6.2]. Therefore, using Proposition 7.6, we conclude the following:

**Corollary 7.8.** There exists no quasi abelian solvmanifold with an invariant locally conformally parallel and non-parallel Spin(7) structure.

Of course, a torus is solvmanifold which admits a parallel Spin(7) structure.

**Corollary 7.9.** If $(\mathbb{R}^8, \mathcal{E})$ is a quasi abelian Lie algebra such that $\mathcal{E}$ is skew-symmetric, then it is flat. In particular, quasi abelian Lie algebras which admit an invariant parallel Spin(7) structure are flat.

**Proof.** Let $(\mathbb{R}^8, \mathcal{E})$ be a quasi abelian Lie algebra and denote by $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ the symmetric and skew-symmetric parts of $\mathcal{E}$. It is straightforward to check that if $i, j > 0$ then:

$$\nabla_{e_i}e_0 = 0, \quad \nabla_{e_i}e_j = \mathcal{E}_{24}(e_j), \quad \nabla_{e_i}e_0 = -\mathcal{E}_{13}(e_i), \quad \nabla_{e_i}e_j = g(\mathcal{E}_{13}(e_i), e_j)e_0.$$  

From this, one can deduce that if $i, j, k > 0$, then the curvature tensor is given by:

$$R(\mathcal{E}_{13}^i, \mathcal{E}_{13}^k)(\mathcal{E}_{13}^j) = g(\mathcal{E}_{13}(e_i), (\mathcal{E} + \mathcal{E}_{24})(e_j))e_0.$$  

Therefore, if $\mathcal{E}$ is skew-symmetric then the Lie group is flat.

**Examples.** Let $\mathfrak{g}$ be a quasi abelian Lie algebra determined by an endomorphism $\mathcal{E}$. Consider the unique simply connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$. The split exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ lifts to a split exact sequence of Lie groups $0 \to (\mathbb{R}^7, +) \to G \to (\mathbb{R}/\mathbb{R}^7, +) \to 0$. This splitting and the conjugation $\epsilon$ on $G$ by the elements of $(\mathbb{R}, +)$, provide an isomorphism $(\mathbb{R}, +) \cong (\mathbb{R}^7, +)$. Therefore $d/dt_{|t=0} d(\epsilon(t)) = s\mathcal{E}$, so that $d(\epsilon(t)) = \exp(t\mathcal{E}) = \epsilon(t)$, using that the exponential of $\mathbb{R}^7$ is the identity.

**A nilmanifold with a balanced and a mixed Spin(7) structure.** Define the endomorphism of $\mathbb{R}^7$

$$\mathcal{E} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and consider the quasi abelian Lie algebra $(\mathbb{R}^8, \mathcal{E})$. Note that this is a nilpotent Lie algebra with structure equations $(0, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0)$, using Salamon notation [21].

The symmetric part of $\mathcal{E}$ is traceless and the eigenvalues of its skew-symmetric part are of the form $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$. Therefore, Theorem 7.7 guarantees the existence of an invariant Spin(7) structure of type balanced and other invariant Spin(7) structure which is mixed. To avoid computing the eigenvalues, one can observe that if we take the standard form $\Omega_8$ in $\mathbb{R}^8$, determined by a spinor $\eta$, it holds that $e_2e_3\eta = -e_4e_5\eta = -e_6e_7\eta$ and $e_1e_2\eta = -e_5e_6\eta$. Therefore, if we identify the skew-symmetric part of $\mathcal{E}$ with a 2-form, $\gamma$, we get that $\gamma\eta = 0$.

On some nilpotent Lie algebras, the existence of a lattice is guaranteed by general theorems [16]. This case is really simple and we can compute it explicitly. The matrix of the endomorphism $\exp(t\mathcal{E})$ is:

$$\begin{pmatrix} 1 & -t & t^2 \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & -2t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & t^2 \frac{1}{2} & -t^3 \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & -t & t^2 \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
If we define $\Gamma = 6\mathbb{Z}e_0 \times e(\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$, then $G/\Gamma$ is a compact manifold with $\pi_1(G/\Gamma) = \Gamma$ which inherits both a balanced and a mixed $\text{Spin}(7)$ invariant structure.

Moreover, we claim that $G/\Gamma$ is not diffeomorphic to $Q \times S^1$ for any 7-dimensional submanifold $Q$. Since $b_1(G/\Gamma) = 2$, it is sufficient to prove that if a nilmanifold $G'/\Gamma'$ is diffeomorphic to $Q \times S^1$ then, $b_1(Q \times S^1) \geq 3$, or equivalently, $b_1(Q) \geq 2$. This assertion turns out to be true because we can check that $Q$ is homotopically equivalent to a nilmanifold. On the other hand, $Q$ is an Eilenberg-MacLane space $K(1,\pi_1(Q))$, because $G'$ is contractible. On the other hand a group is isomorphic to a lattice of a nilpotent Lie group if and only if it is nilpotent, torsion-free and finitely generated [20, Theorem 2.18]. Since $\Gamma' = \pi_1(G'/\Gamma') = \pi_1(Q) \times \mathbb{Z}$, both $\pi_1(Q)$ and $\Gamma'$ verify the conditions listed above. Thus, there is a nilmanifold $Q'$ such that $\pi_1(Q') = \pi_1(Q)$, which is an Eilenberg-MacLane space $K(1,\pi_1(Q))$. Therefore, $Q'$ and $Q$ have the same homotopy type and $b_1(Q) = b_1(Q') \geq 2$, because $Q'$ is a nilmanifold.

**A compact manifold with a parallel and a mixed $\text{Spin}(7)$ structure.** Take the same spinor and basis of $\mathbb{R}^7$ as the previous example. Consider the skew-symmetric endomorphism such that $\mathcal{E}(e_2) = e_3$, $\mathcal{E}(e_4) = e_5$ and $\mathcal{E}(X) = 0$ on $\langle e_2, e_3, e_4, e_5 \rangle$. The rank of this matrix is two and it is associated to $(0,1,1)$. Therefore, Theorem 7.7 guarantees the existence of a parallel invariant $\text{Spin}(7)$ structure and other invariant $\text{Spin}(7)$ structure which is mixed. The matrix of the endomorphism $\exp(t\mathcal{E}_2)$ in the previous basis is:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos(t) & \sin(t) & 0 & 0 & 0 & 0 \\
0 & -\sin(t) & \cos(t) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(t) & -\sin(t) & 0 & 0 \\
0 & 0 & 0 & \sin(t) & \cos(t) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

If $t \in \pi\mathbb{Z}$, the previous matrix has integers coefficients so that $\gamma = \pi\mathbb{Z}e_0 \times e(\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$ is a subgroup. Moreover, $G/\Gamma$ is a compact manifold with $\pi_1(G/\Gamma) = \Gamma$ and inherits from $G$ both a parallel invariant $\text{Spin}(7)$ structure and a mixed invariant one.

According to Remark 7.9, this manifold is flat. It is the mapping torus of $\exp(\pi\mathcal{E})$: $X \to X$, where $X$ is a 7 torus. Indeed, since $\exp(\pi\mathcal{E})^2 = \text{Id}$, the 8-torus is a 2-fold connected covering of $G/\Gamma$.

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Departamento de Álgebra, Geometría y Topología, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

E-mail address: lmerchan@ucm.es