LOG RAMIFICATION VIA CURVES IN RANK 1

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ABSTRACT. We prove that in the smooth rank 1 case, the Abbes-Saito log conductor is obtained by restriction to curves. Consequently, we prove an expectation of Esnault and Kerz. As an application, we translate recent results in higher class field theory of Kerz and S. Saito to the log-ramification context. We also conjecture that our results hold in arbitrary finite rank.

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1. INTRODUCTION

Let $R$ be a discrete valuation ring with residue field $\kappa$, fraction field $K = \text{Frac}(R)$, and $L$ a finite Galois extension of $K$. If $\kappa$ is perfect, there is a satisfactory ramification theory for $L/K$, see for example [Ser 68]. On the other hand, if $\kappa$ is not perfect this theory is not as well-established; for a comprehensive survey on ramification see [Xia-Zhu 13].

In the 1980’s, Brylinski and Kato defined conductors when $\kappa$ is not necessarily perfect and $L/K$ is attached to a character of rank 1, see [Bry 83] and [Kat 89]. Recently, Abbes and Saito succeeded in providing a general definition of higher ramification groups attached to $\ell$-adic sheaves of finite rank that agrees with the classical case of a perfect
residue field; see [Abb-Sai 02] and [Abb-Sai 11]. Consequently, they also define a conductor in [Abb-Sai 11]. Moreover, their conductor agrees with that of Brylinski and Kato’s in rank 1, see [Abb-Sai 09].

In another direction, in the 1970’s Deligne had initiated a program of measuring ramification of sheaves along a divisor in terms of transversal curves, see [Del 76], which was further developed by Laumon in [Lau 81] and [Lau 82]. We remark that one of the principal aims of these works was to achieve new Euler-Poincaré formulas for surfaces (over algebraically closed fields). Let us also mention work by Brylinski [Bry 83] and Zhukov [Zhu 02].

A natural question then became if one could follow Deligne’s program and express Abbes-Saito’s conductor in terms of curves. In this direction, Matsuda established results in the so-called ‘non-log’ case of Brylinski-Kato’s conductor, see [Mat 97] and [Ker-Sai 13, Coro 2.7]. There are also stronger results for the ‘non-log’ case recently obtained by T. Saito, see [Sai 13]. For a modern survey on wild ramification in the sheaf-theoretic context, see [Sai 10].

The main contribution of this article is the description of Brylinski-Kato’s ‘log’ conductor in terms of curves, see Theorem 6.1. In other words, we establish that Abbes-Saito’s ‘log’ conductor in rank 1 is given in terms of curves. Our approach is a slight generalization of Deligne’s original idea of analyzing ramification via transversal curves and instead considering all curves on a scheme with special attention to tangent curves. We conjecture that our result also holds for ℓ-adic sheaves of finite rank, see Conjecture A as well as Conjecture B below. We hope our conjectures can be used for non-abelian higher class field theory.

As applications of our main result, we affirm an expectation of Esnault and Kerz [Esn-Ker 12, §3] in the smooth rank 1 case, see Theorem 7.1 and we also obtain a reformulation of the recent ‘non-log’ Existence Theorem of higher class field theory obtained by Kerz and S. Saito, [Ker-Sai 13, Coro II] in the ‘log’ case, see [8].

The work presented here forms part of my PhD thesis.

2. Acknowledgements

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3. Conjectures A and B

Definition 3.1. For a scheme $X$, we let $Cu(X)$ denote the set of normalizations of closed integral one-dimensional subschemes of $X$.

For $\bar{C} \in Cu(X)$ let

$\bar{\phi} : \bar{C} \to X$,

denote the induced morphism to $X$.

Definition 3.2. Given a Cartier divisor $D$ and a codimension one point $y$ on a locally Noetherian scheme, we denote the multiplicity of $y$ in $D$ by $m_y(D)$.
Fix a perfect field $k$. Let $X/k$ be a normal variety and $j : U \to X$ a $k$-smooth open subscheme such that the closed complement $E := X \setminus U$ is the support of an effective Cartier divisor and let $D \in \text{Div}^+(X)$ be an effective Cartier divisor with $\text{Supp}(D) \subset E$. Let $F$ be a smooth $\ell$-adic sheaf of finite rank on $U$.

Denote by $\text{Sw}_z(-)_{\text{log}}$ the Abbes-Saito log conductor of a constructible sheaf at a codimension 1 point of a scheme, see [Abb-Sai 11, Definition 8.10 (i)]. We omit the subscript ‘log’ when the scheme has dimension one. Note that the definition of this conductor simplifies in rank 1, see Def. 4.3 and Prop. 4.5 below.

3.1. Conjecture A.

**Conjecture A.** Fix a smooth irreducible component $D_i$ of $D$ and denote the generic point of $D_i$ by $\xi$. Then

$$\sup_{\tilde{\phi}} \frac{\text{Sw}_z(\tilde{\phi}^*j_i F)}{m_z(\tilde{\phi}^*D_i)} = \text{Sw}_\xi(j_i F)_{\text{log}},$$

where the supremum ranges over $\tilde{C} \in \text{Cu}(X)$ with $\tilde{\phi}(\tilde{C}) \not\subset D_i$ and $z$ is a closed point of $\tilde{C}$ satisfying $m_z(\tilde{\phi}^* D) > 0$.

**Remark 3.3.** In [Zhu 02, Remark 2.5.3], a limit similar to that in (1) is mentioned for a smooth surface defined over an algebraically closed field.

3.2. Conjecture B. Instances of the following ideas are related to previous work by Deligne [Del 76], Matsuda [Mat 97], and Russel [Rus 10, Lemma 3.31].

**Definition 3.4.** ([Esn-Ker 12, Def. 3.6]) We say that the ramification of $F$ is bounded by $D$ if for each $C \in \text{Cu}(U)$ we have the inequality of divisors in $\text{Div}^+(\tilde{C})$:

$$\text{Sw}(\phi^* F) \leq \tilde{\phi}^* (D),$$

where $\phi : C \to U$ is the induced morphism from $C$ to $U$ and $\tilde{\phi} : \tilde{C} \to X$ is the normalization of the closure of $C$ in $X$ and

$$\text{Sw}(\phi^* F) := \sum_{z \in |\tilde{C}|} \text{Sw}_z(\tilde{\phi}^* j_i F).[z] \in \text{Div}^+(\tilde{C}).$$

In this case, we formally write

$$\text{Sw}(F) \leq D.$$

**Conjecture B.** Keep the hypothesis in Conjecture A. We conjecture that the following are equivalent (cf. [Esn-Ker 12, §3]).

1. $\text{Sw}(F) \leq D$.
2. for each open immersion $j' : U' \subset X'$ over $k$ such that $X' \setminus U$ is a simple normal crossing divisor and for a morphism $h : X' \to X$, that extends $j : U \to X$,

$$\text{Sw}_{\xi'}(h'^* j_i F)_{\text{log}} \leq m_{\xi'}(h'^* (D)),$$

for each generic point $\xi' \in D' := X' \setminus U$.

Our main result is that if $X/k$ is smooth and $D$ is a simple normal crossing divisor on $X$, then if $F$ has rank 1 Conjecture A is true (see Theorem 6.1) and implies Conjecture B (Theorem 7.1). We do not require resolution of singularities; e.g. we do not require $X/k$ to be proper in Theorem 6.1. On the other hand, if one insists that we begin with a smooth variety $U/k$ and a proper, but not-necessarily smooth, variety $X/k$ that contains
U as an open dense subscheme, then resolution of singularities seems to be required to apply our techniques.

In the case with $D = 0$ (and arbitrary rank), the equivalence in Conjecture B is shown by Kerz and Schmidt in [Ker-Sch 10] relying on work of Wiesend in [Wie 06]. Moreover, in the ‘non-log’ rank 1 setting Conjecture B follows by work of Matsuda on Kato’s Artin conductor, see [Mat 97] and [Ker-Sai 13 Coro 2.7].

4. BACKGROUND

The Brylinski-Kato filtration is introduced in §4.1 from which we derive a generalized conductor for finite extensions of discrete valuation fields with not-necessarily separable residue field extensions, see Def. 4.3. In the next section, we provide an example that gives evidence for computing the log conductor in terms of a limit of conductors of curves tangent to a given divisor, see §5.1.

4.1. Conductor for general discrete valuation fields. We mostly follow the notation in [Abb-Sai 09 §9]. Let $(R, v)$ be a discrete valuation ring of characteristic $p > 0$ with normalized valuation $v$. Let $K$ be the fraction field of $R$ and fix a separable closure $K^{\text{sep}}$ of $K$. Let $W_n(K)$ be denote the truncated ring of Witt vectors of length $n \geq 0$ over $K$. The Brylinski-Kato filtration on $W_n(K)$ is an increasing filtration and is defined as follows.

**Definition 4.1.** For $m \geq 0$ define the subgroup $\text{fil}_m W_n(K)$ of $W_n(K)$ as

$$\text{fil}_m W_n(K) = \{ (x_0, \ldots, x_{n-1}) : p^{n-i-1} v(x_i) \geq -m, \ 0 \leq i \leq n - 1 \}.$$ 

We have that for $x = (x_0, \ldots, x_{n-1}) \in W_n(K)$, then $x \in \text{fil}_m W_n(K)$ if

$$m \geq \sup \{ -p^{n-1} v(x_0), \ldots, -v(x_{n-1}), 0 \}. \quad (2)$$

This filtration yields a conductor for characters. Set

$$H^1(K) := \varprojlim_{n \geq 1} H^1(K, \mathbb{Z}/p^n \mathbb{Z}) = H^1(K, \mathbb{Q}/\mathbb{Z}).$$

**Remark 4.2.** Note that since $\text{Gal}(K^{\text{sep}}/K)$ is a compact topological group (under the usual Krull topology) and $\mathbb{Q}/\mathbb{Z}$ is a discrete topological group, then an element $\chi \in H^1(K) = \text{Hom}_{\text{cont}}(\text{Gal}(K^{\text{sep}}/K), \mathbb{Q}/\mathbb{Z})$, must have finite image in $\mathbb{Q}/\mathbb{Z}$ and this image is therefore a finite cyclic group.

Put $\eta = \text{Spec}(K)$ and recall that for $n \geq 1$ there is a short exact sequence of sheaves over $\eta_{\text{et}}$:

$$0 \rightarrow \mathbb{Z}/p^n \mathbb{Z} \rightarrow W_n \xrightarrow{F-1} W_n \rightarrow 0. \quad (3)$$

Now, the short exact sequence (3) induces a surjective connecting homomorphism of groups

$$\delta_n : W_n(K) \rightarrow H^1(K, \mathbb{Z}/p^n \mathbb{Z}),$$

with $\ker(\delta_n) = (F - 1)W_n(K)$ and so we have a canonical isomorphism of groups

$$W_n(K)/(F - 1) \cong H^1(K, \mathbb{Z}/p^n). \quad (4)$$

The isomorphism (4) classifies finite cyclic extensions of $K$ of order $p^{n'}$, for $0 \leq n' \leq n$. Moreover, there is a commutative diagram.
Now write
\[ \text{fil}_m H^1(K, \mathbb{Z}/p^n) := \delta_n(\text{fil}_m W_n(K)). \tag{6} \]

The Brylinski-Kato filtration induces a filtration on all of \( H^1(K) \) via
\[ \text{fil}_m H^1(K) := H^1(K)[p'] + \text{fil}_m H^1(K)[p^\infty], \tag{7} \]
where
\[
H^1(K)[p'] = \lim_{\substack{\ell \to 1, \ell \neq 1 \ell \equiv 0 \mod\ell}} H^1(K, \mathbb{Z}/\ell\mathbb{Z}),
\]
and
\[
H^1(K)[p^\infty] = \lim_{\substack{r \to 1}} H^1(K, \mathbb{Z}/p^r\mathbb{Z}).
\]

**Definition 4.3.** Let \( \chi \in H^1(K) \). The Swan conductor of \( \chi \), denoted by \( \text{Sw}(\chi)_{\text{log}} \), is the minimal integer \( m \geq 0 \) for which \( \chi \in \text{fil}_m H^1(K) \).

**Remark 4.4.** That this definition agrees with the classical definition when the residue field of \( K \) is finite is a theorem of Brylinski [Bry 83, Corollary to Theorem 1], and for the more general case of a perfect residue field, it is due to Kato [Kat 89].

We also record the following important result of Abbes and T. Saito.

**Proposition 4.5.** The conductor defined above in Def. 4.3 is equal to the log conductor of Abbes and Saito in rank 1.

**Proof.** The Abbes-Saito log conductor is defined in [Abb-Sai 11, Def. 8.10(i)] and that this definition agrees with ours is given by [Abb-Sai 09, Coro 9.12 and Def. 10.16]. \( \square \)

Therefore, in rank 1 the conductors under our consideration will be stated entirely in terms of Def. 4.3.

**Remark 4.6.** If \( \chi' \in H^1(K)[p'] \), then \( \text{Sw}(\chi') = 0 \), by (7) above. Hence, our study of wild ramification is reduced to that of characters in \( H^1(K)[p^\infty] \).

4.2. Witt vectors and extensions.

**Lemma 4.7.** Let \( \chi \in H^1(K, \mathbb{Z}/p^n) \) and suppose \( 0 \leq n' \leq n \). Then \( \chi \) corresponds to a cyclic \( \mathbb{Z}/p^n' \)-extension of \( K \) if and only its corresponding Witt vector is in the image of the iterated Verschiebung modulo \( (F - 1) \),
\[
V^{n-n'} : W_{n'}(K)/(F - 1) \to W_n(K)/(F - 1).
\]

**Proof.** This follows by the the isomorphism (4) and the commutative diagram (5). \( \square \)
Using the isomorphism (4), we can explicate $p$-cyclic extensions of $K$ as follows. If $L/K$ is Galois and satisfies $\text{Gal}(L/K) = \mathbb{Z}/p^n$, there are Witt vectors 
\[(x_0, \ldots, x_{n-1}) \in W_n(K), \quad (\alpha_0, \ldots, \alpha_{n-1}) \in W_n(K^{\text{sep}}),\]
such that
\[L = K[\alpha_0, \ldots, \alpha_{n-1}],\] (8)
where $(\alpha_0, \ldots, \alpha_{n-1}) \in W_n(K^{\text{sep}})$ satisfies
\[(F - 1)(\alpha_0, \ldots, \alpha_{n-1}) = (x_0, \ldots, x_{n-1}).\]
The Swan conductor of $L/K$ is then determined by the image of $x = (x_0, \ldots, x_{n-1})$ in $W_n(K)/(F - 1)$.

Namely, if $x' = (x'_0, \ldots, x'_{n-1})$ is the image of $x$ in $W_n(K)/(F - 1)$, then there is a corresponding character $\chi_{x'} \in H^1(K)$ and by (2) we have
\[\text{Sw}(\chi_{x'}) = \min \{\max_y \{-p^{n-1}v(y_0), \ldots, -v(y_{n-1}), 0\}\},\] (9)
where $\max_y$ ranges over $y \in W_n(K)$ such that
\[x - y \in (F - 1)W_n(K).\]

5. CONDUCTORS FROM VARIETIES

We mostly follow the notation in [Ker-Sai 13]. Let $k$ be a perfect field and let $X/k$ be a normal variety. Let $U \subset X$ be an open subscheme that is smooth over $k$ and such that the reduced closed complement $E := X \setminus U$ is the support of an effective Cartier divisor on $X$. The discrete valuation rings of interest to us will be those coming from codimension 1 points of $X$ ‘at infinity’ and those from closed points of certain curves on $X$.

Denote by $I$ the set of generic points of $E$; then $I \subset X^{(1)}$. For $\lambda \in I$, let $K_\lambda$ be the fraction field of the henselian discrete valuation ring $\mathcal{O}_{X,\lambda}^h$. Note that the residue field of $K_\lambda$ is the function field of the divisor $\{\lambda\}$ on $X$.

Recall that in Def. 4.3 we defined the Swan conductor for characters over discrete valuation fields. Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$.

Definition 5.1. We define
\[\chi|_\lambda \in H^1(K_\lambda, \mathbb{Q}/\mathbb{Z}),\]
to be the specialization of $\chi$ to $\text{Spec}(K_\lambda)$ induced by the canonical composition
\[\text{Spec}(K_\lambda) \to \text{Spec}(\mathcal{O}_{X,\lambda}^h) \to X.\]
The log Swan conductor of $\chi|_\lambda$ is denoted by
\[\text{Sw}_\lambda(\chi)_{\log}.\]

Remark 5.2. Observe that since $k$ is perfect and $X/k$ is assumed to be of finite type, the residue field of $K_\lambda$ has transcendence degree equal to $\dim(X) - 1$ and therefore this residue field is perfect if and only if $\dim(X) = 1$.

Definition 5.3. A curve $C$ is a one-dimensional integral scheme and a curve $C$ on a scheme is a closed subscheme that is a curve.
Definition 5.4. We let $Z_1(X, E)$ denote the set of curves $\bar{C}$ on $X$ such that $\bar{C}$ is not contained in $E$, i.e. such that $\bar{C} \not\subset \text{Supp}(E)$.

Definition 5.5. For $\bar{C} \in Z_1(X, E)$ let
$$\Psi_{\bar{C}} : \bar{C}^N \to \bar{C},$$
denote the normalization of $\bar{C}$. We put
$$\bar{C}_\infty = \{ z \in |\bar{C}^N| : \Psi_{\bar{C}}(z) \in E \}.$$When we want to specify a divisor $D' \subset \text{Supp}(E)$, we write $Z_1(X, D')$, resp. $\bar{C}_\infty(D')$.

We can think of $\bar{C}_\infty$ as the set of places of the global field $K(\bar{C}) = K(\bar{C}^N)$ “on the boundary” $E$. Given $\bar{C} \in Z_1(X, E)$ and $z \in \bar{C}_\infty$, we write $K(\bar{C})_z$ for the henselization of $K(\bar{C})$ at the place corresponding to $z$.

Definition 5.6. Given $\bar{C} \in Z_1(X, E)$ and $z \in \bar{C}_\infty$, we define
$$\chi|_{\bar{C}, z} \in H^1(K(\bar{C})_z, \mathbb{Q}/\mathbb{Z}),$$
to be the restriction of $\chi$ via
$$\text{Spec}(K(\bar{C})_z) \to \bar{C} \hookrightarrow X.$$Since $\dim \bar{C} = 1$, we do not need to emphasize a ‘log’ conductor and we unambiguously write
$$\text{Sw}_z(\chi|_{\bar{C}}),$$
for the conductor of $\chi|_{\bar{C}, z}$, which coincides with the classical Swan conductor (see Remark 4.4). More generally, suppose $\bar{C}$ is a curve with a given finite morphism
$$\phi : \bar{C} \to X,$$
and that $\bar{z} \in |\bar{C}|$ is a closed point satisfying $\phi(\bar{z}) \in E$. We will often emphasize the morphism $\phi$ by writing
$$\text{Sw}_z(\phi^*\chi),$$
for the conductor of $\chi$ restricted to $\bar{C}$.

5.1. Example of computing by tangent curves. Here we provide some evidence of Conjecture A, (1), in rank 1 over a surface. The curves constructed in this example will serve as a prototype for the case of higher-dimensional schemes.

Example 5.7. (Artin-Schreier for a surface.) Let $k = \mathbb{F}_p$ and $U = \text{Spec}(k[x, y][1/y])$. Take $X = \text{Spec}(k[x, y])$ so that $X \setminus U = V(y)$ and put $D = V(y)$. We consider both “fierce” and “non-fierce” wild ramification (cf. [Lau 82, §2]) to give evidence that curves tangent to $D$ handle both phenomena. Let $A$ be the henselization of $k[x, y]$ at $(y)$ and $K = \text{Frac}(A)$. Fix a non-trivial homomorphism
$$\rho : \mathbb{F}_p \to \text{GL}_1(\mathbb{F}_\ell) = \mathbb{F}_\ell^\times,$$and an Artin-Schreier equation over $A$:
$$t^p - t = x^a/y^b,$$
with $a \geq 0, b \geq 1$. We may further assume that $x^a/y^b$ is not a $p$th-power in $K$, i.e. that $p$ is prime to $a$ or $b$. This equation and $\rho$ yield a character $\chi \in H^1(K, \mathbb{Z}/p)$. Let
$$L = K[t]/(t^p - t - x^a/y^b),$$
and let $B$ be the integral closure of $A$ in $L$. Let $k(A) = k(x)$ denote the residue field of $A$ and $k(B)$ the residue field of $B$. The extension $B/A$ is fierce if the extension of residue fields $k(B)/k(A)$ is inseparable, i.e. if $k(B)$ contains a $p$-th root of $x$.

We consider the curves $C_e$ on $X$ defined by the equation $y = x^e$, for $e > 0$. Geometrically, as $e$ increases the $C_e$ progressively become more tangent to the $x$-axis in the plane near the origin. The restriction of $\chi$ to $C_e$ yields a character $\chi|_{C_e} \in H^1(K(C_e), \mathbb{Z}/p)$ that corresponds to the extension of $K(C_e)$ given by

$$t^p - t = x^a/x^b = \frac{1}{x^{be-a}}.$$

For fixed $e > 0$, let $\phi : C_e \hookrightarrow X$ denote the closed immersion of $C_e$ into $X$ and fix a closed point $z \in |C_e|$ such that $\phi(z) \in D$.

(i) Non-fierce case. For simplicity, assume $a = 1$ and that $b$ is prime to $p$. Using (9) we see that

$$\text{Sw}_{(y)}(\chi)_{\log} = b,$$

which is consistent with the classical computation (see e.g. [Lau 81, Exemple 2.2]) since the extension of residue fields $k(B)/k(A)$ is separable.

The Artin-Schreier equation over $C_e$ in this case is

$$t^p - t = \frac{1}{x^{be-1}}.$$

Again by (9) we have that if $p|(be-1)$, then by $(F-1)$ equivalence in $W_1(K)$, $\text{Sw}_z(\chi|_{C_e}) < be - 1$ and if $(be - 1, p) = 1$ then $\text{Sw}_z(\chi|_{C_e}) = be - 1$. Therefore,

$$\limsup_{e > 0} \frac{\text{Sw}_z(\chi|_{C_e})}{e} = \lim_{e \to +\infty} b - \frac{1}{e} = b,$$

as expected.

(ii) Fierce case. In this case $(a, p) = 1$ and $p|b$. For simplicity, take now $a = 1$ and $b = p$ so that the extension over $A$ is given by

$$t^p - t = x/y^p.$$

Let us demonstrate that the extension of residue fields is now purely inseparable. Clearly, $x \not\in k(A)^p$; we show that $x \in k(B)^p$. Let $v_B : L^x \to \mathbb{Q}$ be a valuation on $L$ normalized by $v_B(y) = 1$. Then $v_B(t) = -p/v_B(y^p) = -1$ since $t^p - t = x/y^p$. Noting that in $L$ we also have the relation

$$(ty)^p - t.y^p = x, \quad (10)$$

then $v(t.y^p) = -1 + p > 0$ and so $t.y^p$ is an element of the maximal ideal $m_B$ of $B$. Therefore,

$$(ty)^p \equiv x \pmod{m_B},$$

which implies $x \in k(B)^p$. Now consider the class of $x/y^p$ in $W_1(K)/(F-1)$. Since $x \not\in K^p$, then $x/y^p$ is irreducible with respect to $F - 1$ equivalence in $W_1(K)$ and as $v_y(x/y^p) = -p$, we have

$$\text{Sw}_{(y)}(\chi)_{\log} = p.$$

Restricting to $C_e$ gives

$$t^p - t = x/x^{pe} = \frac{1}{x^{pe-1}},$$

and because $(pe - 1, p) = 1, \forall e > 0$, we have
\[
\limsup_{e > 0} \frac{\text{Sw}_z(\chi|_{C_e})}{e} = \lim_{e \to +\infty} \frac{p - 1}{e} = p,
\]

as desired. \hfill \square

6. Theorem 6.1

Throughout, \(k\) is a perfect field of \(\text{char}(k) = p > 0\). Recalling the definitions from \(\S\) 5, we now state the main theorem of this article.

**Theorem 6.1.** (Conjecture A in the smooth rank 1 case) Let \(X/k\) be a smooth variety and \(U \subset X\) an open variety such that \(E := X \setminus U\) is the support of an effective Cartier divisor on \(X\). Let \(D \in \text{Div}^+(X)\) be a simple normal crossing divisor on \(X\) with \(\text{Supp}(D) \subset E\). Fix a component \(D_i\) of \(D\) and let \(\xi \in D_i\) be its generic point. Let \(\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})\). Then,

\[
\sup_{\bar{C}} \frac{\text{Sw}_z(\chi|_{C})}{m_z(\phi^*D_i)} = \text{Sw}_\xi(\chi)_{\log},
\]

where the supremum ranges over all \(\bar{C} \in Z_1(X, E)\) and where \(z\) is a point in \(\bar{C}_\infty \cap D_i^{sm}\) satisfying \(m_z(\phi^*D_i) > 0\).

From now on we abbreviate simple normal crossing divisor as \(\text{sncd}\).

We first prove the theorem at a closed point in \(\S 6.1\) and proceed to the general case in \(\S 6.2\).

6.1. Local setting. We begin by formulating the problem around a closed point; the main reason we localize at a closed point is to apply Cohen’s structure theorem which enables us to explicate Artin-Schreier-Witt extensions. We transfer back to the henselian world by means of Artin Approximation, see \(\S 6.15\).

Let \(x \in |X| \cap \text{Supp}(D)\). Write the completion of the localization of \(X\) at \(x\) as

\[X_x := \text{Spec}(\widehat{O}_{X,x}).\]

Then \(\widehat{O}_{X,x}\) is a Noetherian and complete regular local ring and by Cohen’s structure theorem (see e.g. [Liu 02, 4.2.28]) there is a finite extension \(k'/k\) such that

\[\widehat{O}_{X,x} \cong k'[[t_1, \ldots, t_d]].\]

We now assume that \(D\) is irreducible and restrict this divisor to \(X_x\). We may assume that locally around \(x\), the sncd \(D\) is defined by \((t_1 = 0)\). Now let \(U_x = X_x \setminus V(t_1)\). We prove the following

**Theorem 6.2.** (Local Tangent theorem) Let \(\chi \in H^1(U_x) = H^1(U_x, \mathbb{Q}/\mathbb{Z})\). For \(\xi\) the generic point of \(D\), there are irreducible, regular curves \(\phi_e : C_e \to X_x\) with \(C_e \in Z_1(X_x, D)\) indexed by integers \(e > 0\) such that

\[
\limsup_{e \to \infty} \frac{\text{Sw}_z(\chi|_{C_e})}{m_x(\phi_e^*D)} = \text{Sw}_\xi(\chi)_{\log}.
\]

(11)

**Remark 6.3.** Clearly, the \(C_e\) depend on the choice of a generic point \(\xi\) of \(X_x \setminus U_x\). Also note that since \(X_x\) is a local scheme, then so is \(C_e\) and \(|C_e| = \{x\}\).
Theorem 6.2 is obvious if \( \dim(U_x) = 1 \), for then we simply take \( C_e = X_x \), so from now on we assume \( \dim(U_x) > 1 \). By the previous remarks 4.2 and 4.6 we are reduced to the case where \( \chi \in H^1(U_x, \mathbb{Z}/p^{n+1}) \) for some \( n \geq 0 \) (recall \( \text{char}(k) = p > 0 \)). We prove the Local Tangent theorem 6.2 first for \( n = 0 \) in 6.1.1 and then proceed to the general case in 6.1.3. We fix the following notation. Let

\[
A_x := k'[\left[t_1, \ldots, t_d\right]],
\]

and

\[
R := \Gamma(U_x, \mathcal{O}_{U_x}) = A_x[1/t_1].
\]

Set

\[
N = Sw\chi(\log).
\]

If \( N = 0 \) (i.e. if \( \chi \) is tame at \( \xi \)), then for any subscheme \( Z \subset X_x \), the restriction \( Sw\chi(\chi|Z) = 0 \). Hence we assume that \( N > 0 \) in what follows. Furthermore, by the Artin-Schreier-Witt isomorphism (4), \( \chi \) corresponds to a unique element \((f_0, \ldots, f_n) \in W_{n+1}(R)/(F - 1)\).

6.1.1. The Artin-Schreier case. In this case, \( \chi \) corresponds to a unique element \( \bar{f} \in R/(F - 1) \). We fix a lift \( f \in R \) satisfying \( f \notin R^p \) (i.e. \( f \) is not a \( p \)-power in \( R \)) and

\[
-v_{t_1}(f) = N.
\]

Defining the curves \( C_e \). Write

\[
f = \frac{B}{t_1^N} + \frac{B'}{t_1^{N-1}},
\]

where \( B' \in A_x \) and \( B \) is a non-zero element of \( A_x/(t_1) \); \( B \) is regarded as an element of \( A_x \) via a section \( t : A_x/(t_1) \to A_x \), which exists since the residue fields of \( A_x/(t_1) \) and \( A_x \) are both \( k' \). We construct curves \( C_e \) on \( X_x \) so that the image of \( B \) in \( \Gamma(C_e, \mathcal{O}_{C_e}) \) is non-zero.

We begin by defining a surjective morphism

\[
\Phi_e : k'[\left[t_1, \ldots, t_d\right]] \to k'[\left[w\right]],
\]

that sends each parameter \( t_i \) to either a power of \( w \) or zero and satisfies \( \Phi_e(B) \neq 0 \). Write \( B \) as a sum of monomials:

\[
B = \sum_{(i_2, \ldots, i_d)} u_{(i_2, \ldots, i_d)} t_2^{i_2} \cdots t_d^{i_d},
\]

where \( u_{(i_2, \ldots, i_d)} \in k' \). From this sum, choose a non-zero multi-index \((j_2, \ldots, j_d)\) that has a minimal number of \( j_i \neq 0 \). By a change of coordinates, we may assume such a term is in the \( r - 1 \) variables \( t_2, \ldots, t_r \) for some \( r \) with \( 2 \leq r \leq d \). Denote by \( \mathbb{B}_r \) the set of monomials of \( B \) of the form \( t_2^{j_2} \cdots t_r^{j_r} \) with each \( a_i > 0 \) (1 \( \leq i \leq r \)).

**Lemma 6.4.** Given \( \mathbf{n} = (n_2, \ldots, n_r) \in \mathbb{N}^{r-1} \), define

\[
\Psi^\mathbf{n} : \mathbb{B}_r \to \mathbb{N},
\]

by

\[
\Psi^\mathbf{n}(t_2^{a_2} \cdots t_r^{a_r}) = n_2a_2 + \cdots + n_ra_r.
\]
Let $g = t_2^{a_2} \cdots t_r^{a_r} \in \mathbb{B}_r$ be the monomial that is minimal in $\mathbb{B}_r$ with respect to degree in lexicographical ordering in $\mathbb{N}^{r-1}$. There exists an element $m = (m_2, \ldots, m_r) \in \mathbb{N}^{r-1}$ depending on the minimal term $g \in \mathbb{B}_r$, satisfying

$$m_r = 1 \text{ and } \Psi^m(g) < \Psi^m(h), \forall h \in \mathbb{B}_r \setminus \{g\}.$$ (15)

The method employed in the following construction of $m$ is very similar in spirit to the classical proof of Noether normalization (see e.g. [Liu 02, Prop. 2.1.9]).

Proof. Write $b = (b_2, \ldots, b_r)$. Define

$$m = (m_2, \ldots, m_r),$$

via

$$m_r = 1, \ m_j = \sum_{i=j+1}^{r} m_i b_i, \ (2 \leq j < r).$$

Now assume $\Psi^m(h) \leq \Psi^m(g)$ for some $h = t_2^{a_2} \cdots t_r^{a_r} \in \mathbb{B}_r$. We claim that $h = g$. First suppose $r = 2$, so $m_2 = 1$. Since $g$ is minimal with respect to lexicographical ordering of degree in $\mathbb{B}_r$, we have $a_2 \geq b_2$. Moreover, the hypothesis $\Psi^m(h) \leq \Psi^m(g)$ in this case means $a_2 \leq b_2$; therefore, $a_2 = b_2$ and the claim is true for $r = 2$. Now we prove the claim for $r \geq 3$. We have

$$\sum_{i=2}^{r} m_i a_i \leq \sum_{i=2}^{r} m_i b_i$$ (16)

$$a_2 + \frac{\sum_{i=3}^{r} m_i a_i}{m_2} \leq b_2 + 1$$ (17)

$$(a_2 - b_2) + \frac{\sum_{i=3}^{r} m_i a_i}{m_2} \leq 1$$ (18)

where the second inequality is obtained by dividing by $m_2$ and using the definition $m_2 = \sum_{i=3}^{r} m_i b_i$. Since $b$ is minimal in the lexicographical ordering, we have $a_2 \geq b_2$, and so the inequality (18) implies that either

(i) $a_2 = b_2 + 1$ and $\sum_{i=3}^{r} m_i a_i = 0$, or
(ii) $a_2 = b_2$ and $\sum_{i=3}^{r} m_i a_i = \sum_{i=3}^{r} m_i b_i$.

Case (i) is impossible since the choice of $b$ and induction on $r$ gives $\sum_{i=3}^{r} m_i a_i > 0$. Only case (ii) is possible. In this case induction on $r$ gives $a_i = b_i$. Therefore $h = g$. □

Definition 6.5. We call an element $m \in \mathbb{N}^{r-1}$ satisfying (15) in Lemma 6.4 a $B$-good vector. Such a vector always has last coordinate $m_r = 1$.

Proposition 6.6. There are regular curves $C_e$ on $X_e$ indexed by integers $e > 0$ such that if the morphism defining $C_e$ is $\phi_e : C_e \hookrightarrow X_e$, then $\phi_e^*(B) \neq 0$.

Proof. Let $m = (m_2, \ldots, m_r)$ be a $B$-good vector. Given $e > 0$, our curves $C_e$ are defined as

$$C_e = \text{Spec} \left( Im \Phi_e \right) \cong \text{Spec} \left( k'[w] \right),$$
where $\Phi_e$ is the $k'$-morphism $k'[t_1, \ldots, t_d] \to k'[w]$ given by

\[
\begin{align*}
\Phi_e(t_1) &= w^e \\
\Phi_e(t_2) &= w^{m_2} \\
\Phi_e(t_3) &= w^{m_3} \\
&\vdots \\
\Phi_e(t_r) &= w^{m_r} \\
\Phi_e(t_s) &= 0, \ (s > r).
\end{align*}
\]

Since $m$ is a $B$-good vector, we have $\Phi_e(B) \neq 0$. The morphism

\[\phi_e : C_e \to X_x,\]

is defined as the morphism of affine schemes that corresponds to $\Phi_e$. Note that in terms of ideals (recall $m_r = 1$), for

\[I = (t_1 - t_1^e, t_2 - t_2^{m_2}, \ldots, t_{r-1} - t_{r-1}^{m_{r-1}}, t_{r+1}, \ldots, t_d) \leq A_x,\]

we have the equality of closed subschemes

\[C_e = V(I) \subset X_x.\]

By construction of $\Phi_e$, we may assume $B$ consists only of monomials in $t_2^{i_2} \cdots t_r^{i_r}$ with $i_j > 0$ (since $\Phi_e(t_s) = 0$ for $s > r$). That is, we write (up to units in $A_x$):

\[B = \sum_{(i_2, \ldots, i_r)} t_2^{i_2} \cdots t_r^{i_r},\]

with each $i_j > 0$ for $j = 2, \ldots, r$.

We record the following remark and lemmas that will be used in the sequel.

**Remark 6.7.** Observe that the image $\Phi_e(B)$ is independent of $e > 0$ since $m$ is independent of $e$. If $m$ is a $B$-good vector, from now on we write the image of $B$ under the morphism $\phi_e$ constructed above as

\[\Phi_m(B) := \Phi_e(B).\]

**Lemma 6.8.** Suppose that $r > 2$. Again write $g = t_2^{b_2} \cdots t_r^{b_r}$ for the minimal element in $B_r$ with respect to lexicographic ordering in degree. Suppose that $m = (m_2, \ldots, m_r = 1)$ is a $B$-good vector. Then if $Q_i \in \mathbb{N}$ ($i = 2, \ldots, r - 1$) are integers satisfying

\[Q_i \geq \sum_{j=i+1}^{r-1} b_j Q_j, \quad (19)\]

then

\[m' = (m_2 + Q_2, \ldots, m_{r-1} + Q_{r-1}, m_r = 1),\]

is a $B$-good vector.

**Proof.** Same method as in the proof of Lemma 6.4. □

Although $f = B/t_1^N + B'/t_1^{N-1}$ is assumed to not be a $p$-power in $R = A_x[1/t_1]$, it is possible that $p \mid N$. In this case, we modify $m$ as follows. Denote by $v$ the normalized discrete valuation on $k'[w]$. 

Lemma 6.9. Recall that $N := \text{Sw}_x(\chi)_{\log}$. Suppose that $p | N$. Given a $B$-good vector $m$, there is a $B$-good vector $m'$ satisfying the following condition

$$p \nmid Ne - v(\Phi_m'(B)),$$

for infinitely many $e > 0$. \hfill (20)

Proof. If $p \nmid v(\Phi_m(B))$, take $m' = m$. So now assume $p | v(\Phi_m(B))$.

Observe that the hypothesis $p | N$ implies that $B$ is not a $p$-power in $A_x$ by our choice of $f \in R$ because otherwise $-v_{t_1}(f) < N$, which contradicts our hypothesis that $f$ is a lift of $f$ with $-v_{t_1}(f) = N$. Furthermore, we may assume that each term in $B$ is not a $p$-power: for suppose that $B$ admits a term of the form $h^p$ for some (non-zero) $h \in A_x$. Write $N = p.N'$ for some $0 < N' < N$. Then in $R/(F - 1)$ we have

$$\frac{h^p}{t_1^{N'}} = \left(\frac{h}{t_1^{N'}}\right)^p = \frac{h}{t_1^{N'}},$$

and

$$\frac{B}{t_1^N} = \frac{B - h^p}{t_1^{N'}} + \frac{h}{t_1^N}.$$

Therefore, $h$ is a term of $B'$ in the expression $f = B/t_1^N + B'/t_1^{N-1}$. Hence each term in $B$ is not a $p$-power in $A_x$.

It is then clear that there are integers $Q_j \geq 0$ such that $m' = (m_2 + Q_2, \ldots, m_r - 1 + Q_r, 1)$ is a $B$-good vector and such that

$$p \nmid v(\Phi'_m(B)) = (m_2 + Q_2).b_2 + \cdots + (m_{r-1} + Q_{r-1}).b_{r-1} + b_r.$$ 

\hfill $\square$

6.1.2. The limsup in the Artin-Schreier case. We now turn to calculating the Swan conductor over the curves $\phi_e : C_e \to X_e$. Put

$$\mathbb{K} = K(C_e) \cong \text{Frac} \left( k'[[w]] \right),$$

the function field of $C_e$. Then $\mathbb{K}$ is a complete discrete valuation field and we write $v$ for the normalized discrete valuation on $\mathbb{K}$. Denote again by

$$\Phi_e : R \to \mathbb{K},$$

the morphism of rings from Lemma 6.6 above. Write

$$c := v(\Phi_e(B)) = v(\Phi_m(B)).$$

Then the image by $\Phi_e$ of $B/t_1^N \in R$ is equal to $w^{-(Ne - c)}$ in $\mathbb{K}$. For all $e \gg 0$, the term $B/t_1^N$ of $f$ determines the non-zero rational number

$$\frac{\text{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^*D)} = \frac{\text{Sw}_x(\chi|_{C_e})}{e}.$$ 

We may further assume that $e > 0$ is sufficiently large so that $Ne - c > 0$ (recall that $N > 1$ and $c \geq 0$ are independent of $e$). We now calculate $\text{Sw}_x(\chi|_{C_e})$ using the Brylinski-Kato filtration (4.1) on $W_1(\mathbb{K})/(F - 1)$. Because of $(F - 1)$ equivalence in $\mathbb{K}$, there are two cases to consider.

(i) The first case is if $p | (Ne - c)$. Then $w^{-(Ne - c)} = w^{-p^a.N'}$ for some $a > 1$ and $N' \geq 1$ with $(N', p) = 1$. Then, $w^{-(Ne - c)}$ is equal to $w^{-N'}$ in $\mathbb{K}/(F - 1)$. In this case $\text{Sw}_x(\chi|_{C_e}) = -v(w^{-N'}) = N' < Ne - c$.

(ii) The second case is if $p \nmid (Ne - c)$. Then there is no reduction modulo $(F - 1)$, and hence $\text{Sw}_x(\chi|_{C_e}) = -v(w^{-(Ne - c)}) = Ne - c$. 


Clearly, if \( p \nmid N \), there are infinitely-many \( e > 0 \) such that case (ii) above holds. If \( p|N \), we modify \( \phi_e \) by Lemma \ref{lem:6.9} to achieve case (ii) above for infinitely-many \( e > 0 \). Therefore, to verify the asserted limsup in (11) it is sufficient to take the limit over those \( e > 0 \) in case (ii) and we have

\[
\limsup_{e \to \infty} \frac{Sw_x(\chi|c_e)}{m_x(\phi_e^* D)} = \lim_{e \to \infty} \frac{Ne - c}{e} = N,
\]

as desired. The proof of Theorem \ref{thm:6.2} for \( \chi \in H^1(U_x, \mathbb{Z}/p) \) is complete.

\[ \square \]

6.1.3. The general case: Artin-Schreier-Witt. Now given \( n \geq 1 \), we prove that Theorem \ref{thm:6.2} is true for cyclic \( \mathbb{Z}/p^{n+1} \)-extensions assuming it is true for all cyclic \( \mathbb{Z}/p^{n'} \)-extensions with \( n' < n + 1 \). The proof ends in \ref{sec:6.1.4}

So fix a character \( \chi \in H^1(K, \mathbb{Z}/p^{n+1}) \) and put

\[
N := Sw_x(\chi) > 0.
\]

Suppose \( \overline{f} \in W_{n+1}(R)/(F - 1) \) corresponds to \( \chi \). Choose elements \( f_i \in R \), for \( i = 0, \ldots, n \), such that on writing

\[
f = (f_0, \ldots, f_n) \in W_{n+1}(R),
\]

we have

(i) \[
f = \overline{f} \text{ in } W_{n+1}(R)/(F - 1),
\]

and

(ii) \[
v(f_i) = \sup_{f_i' \equiv f_i \mod (F - 1)(R)} \{v(f_i')\}, \forall i \in [0, n].
\]

Condition (ii) is motivated by the observation (9) in \ref{sec:4.2} above.

Now, there is a \( k \in [0, n] \) such that

\[
-p^{n-k}v(f_k) = N. \tag{21}
\]

If \( k < n \), we apply the induction hypothesis as follows. Denote by \( V^{k+1} : W_{n-k}(K) \to W_{n+1}(K) \) the \((k + 1)\)-iterated Verschiebung. The diagram \ref{diag:6.2} extends to a commutative diagram

\[
\begin{array}{ccc}
W_{n-k}(K) & \xrightarrow{\delta_{n-k}} & H^1(K, \mathbb{Z}/p^{n-k}) \\
\downarrow V^{k+1} & & \downarrow p^{k+1} \\
W_{n+1}(K) & \xrightarrow{\delta_{n+1}} & H^1(K, \mathbb{Z}/p^{n+1}) \\
\mod V^{k+1} & & \mod p^{n-k} \\
W_{k+1}(K) & \xrightarrow{\delta_{k+1}} & H^1(K, \mathbb{Z}/p^{k+1}) \\
\downarrow 0 & & \downarrow 0 \\
0 & & 0
\end{array}
\]
Denote by \( \Psi^{n-k} \) the morphism corresponding to \( \text{mod } p^{n-k} \) in the diagram (22) above, so
\[
\Psi^{n-k} : H^1(K, \mathbb{Z}/p^{n+1}) \to H^1(K, \mathbb{Z}/p^{k+1}).
\]
Let \( \chi' = \Psi^{n-k}(\chi) \). Note that in terms of Witt vectors, if \( \chi \) corresponds to \( (z_0, \ldots, z_n) \in W_{n+1}(K)/(F - 1) \), then \( \chi' \) corresponds to \( (z_0, \ldots, z_k) \in W_{k+1}(K)/(F - 1) \). Observing that
\[
Sw_{\xi}(\chi)_{\log} = p^{n-k}Sw_{\xi}(\chi')_{\log},
\]
then the commutativity of the diagram (22) and our induction hypothesis applied to \( \chi' \) verifies the assertion of Theorem [6.2] in the case \( k \in [0, n - 1] \). It therefore remains to analyze the case \( k = n \), i.e.
\[
N = -v(f_n) \quad \text{and} \quad -v(f_n) > -p^{n-i}v(f_i), \forall i \in [0, n - 1],
\]
which we now place ourselves in. We first construct the morphism \( \Phi_e : R \to \mathbb{K} \) and then proceed to the calculation of \( \limsup_{n \to +1} Sw_x(\chi|_{C_e})/m_x(\phi_x^e D) \), where \( \phi_e : C_e \to X_x \) is the morphism of schemes defined by \( \Phi_e \). For each \( f_i \neq 0 \), let
\[
N_i = -v(f_i),
\]
and as in (14) write
\[
f_i = \frac{B_i}{t_1^{N_i}} + \frac{B'_i}{t_1^{N_i-1}}, \quad \text{with } 0 \neq B_i \in A_x/(t_1), B'_i \in A_x. \tag{23}
\]
In order to define \( \Phi_{e'} \), consider the term \( B_n \) of \( f_n \) and fix an integer \( e > 0 \). Then we define \( \Phi_e : R \to \mathbb{K} \) with respect to \( B_n \) by setting \( B = B_n \) so that \( \Phi_e \) is defined by a \( B \)-good vector as in Prop. [6.6]. If \( p|N := Sw_{\xi}(\chi)_{\log} \), we further choose this \( B \)-good vector to satisfy (20) in Lemma [6.9]. Then,
\[
\Phi_e(B_n) \neq 0, \forall e > 0.
\]
Note that for \( i \in [0, n] \):
\[
\Phi_e(f_i) = 0 \Leftrightarrow \Phi_e(B_i) = -t_2^e \Phi(B'_i). \tag{24}
\]
Therefore, for each \( i \in [0, n] \) with \( f_i \neq 0 \), we can only have \( \Phi_e(f_i) = 0 \) for at most one \( e > 0 \) and so
\[
\text{if } f_i \neq 0, \text{ then } \Phi_e(f_i) = 0 \text{ for at most } n + 1 - \text{many } e > 0.
\]
Set
\[
\Phi_e(f) = (\Phi_e(f_0), \ldots, \Phi_e(f_n)) \in W_{n+1}(\mathbb{K}).
\]
For each \( e \gg 0 \) we claim that \( \Phi_e(f_n) \) will determine the conductor corresponding to the image of \( \Phi_e(f) \) in \( W_{n+1}(\mathbb{K})/(F - 1) \) upon dividing by the multiplicity \( m_x(\phi_x^e D) = e \).
To prove this claim, we start with the next lemma (6.10) which shows that for \( e \to \infty \), we have the desired equality in Thm. [6.2] up to \( (F - 1) \)-equivalence. After proving this lemma it remains to analyze the computation of the conductor \( Sw(\chi|_{C_e}) \) over the induced filtration on \( W_{n+1}(\mathbb{K})/(F - 1) \).

**Lemma 6.10.** Keep the above notation and hypothesis. Again write \( v \) for the normalized discrete valuation on \( \mathbb{K} \). For a fixed \( e > 0 \), let \( m(e) \geq 1 \) denote the minimal integer for which
\[
\Phi_e(f) \in \text{fil}_{m(e)}W_{n+1}(\mathbb{K}).
\]
Then \( m(e)/e \to N \) as \( e \to \infty \).
Proof. This is clear since for each \( i \in [0, n] \), we have
\[
\Phi_e(f_i) = \frac{\Phi_e(B_i)}{t_2^{N_i, e}} + \frac{\Phi_e(B_i')}{t_2^{(N_i-1), e}}.
\]

It remains to compare the valuation of each coordinate in \( \Phi_e(f) \) relative to \( (F - 1) \) equivalence in \( W_{n+1}(K) \). We claim that modulo \( (F - 1) \) in \( W_{n+1}(K) \), the image of the \( n + 1 \)-th coordinate of \( \phi(f) \in W_{n+1}(K) \), i.e. \( \Phi_e(f_n) \), determines the conductor \( Sw(\chi|_{C_n}) \). This assertion is the content of Lemma 6.11 below.

Recall that \( v \) denotes the normalized discrete valuation on \( K \). Write \( W_{n+1} := W_{n+1}(K) \) and define a map
\[
\Gamma : W_{n+1} \to \mathbb{Z} \cup \{\pm \infty\},
\]
by
\[
\bar{x} \mapsto \sup\{-p^n.v(x_0), -p^{n-1}.v(x_1), \ldots, -v(x_n)\}.
\]

Lemma 6.11. Given \( \bar{x} = (x_0, \ldots, x_n) \) and \( \bar{y} = (y_0, \ldots, y_n) \) in \( W_{n+1} \), let \( \bar{w} = \bar{x} + (1 - F)\bar{y} \). Then, we have the implication:
\[
y_n = 0, -v(x_n) > -p^{n-m}.v(x_m), \forall m \in [0, n - 1] \Rightarrow \Gamma(\bar{w}) \geq \Gamma(\bar{x}) = -v(x_n).
\]

Proof. The proof is by induction on \( n \geq 1 \). The base case is \( n = 1 \) which is verified as follows. Setting \( z_0 = y_0 - y_0^p \) (recall \( y_1 = 0 \) here) we have
\[
\bar{w} = (x_0 + z_0, x_1 + p^{-1}(x_1^p + z_0^p - (x_0 + z_0)^p)).
\]

We divide the argument into two cases: case i) \(-p.v(z_0) \leq -v(x_1)\) and case ii) \(-p.v(z_0) > -v(x_1)\).

For case (i), we show that \( v(w_1) = v(x_1) \). Observe that
\[
v(w_1 - x_1) = v\left(\sum_{k=1}^{p-1} x_0^{p-k}z_0^k\right),
\]
and this sum consists of \( p - 1 \) monomials of degree \( p \). Hence, the conditions \(-p.v(x_0) < -v(x_1)\) and \(-p.v(z_0) \leq -v(x_1)\) yield that
\[
-v(w_1 - x_1) < -v(x_1),
\]
i.e. \( v(x_1) < v(w_1 - x_1) \) and so \( v(w_1) = v(x_1) \).

Next, for case (ii) we have
\[
-p.v(x_0) < -v(x_1) < -v(z_0),
\]
hence \( v(w_0) = v(z_0) \), which gives \( \Gamma(\bar{w}) \geq -p.v(z_0) > -v(x_1) \) and the case \( n = 1 \) is complete.

Now assume the lemma is true for \( W_n \) for a fixed \( n > 1 \). Given \( \bar{y} \in W_{n+1} \), let \( z_i = y_i - y_i^p \) for \( 1 \leq i \leq n \) (recall \( y_n = 0 \) by hypothesis). Again, the argument is divided into two cases: Case i) \(-p.v(z_{i-1}) \leq -v(x_i), \forall i \) and Case ii) \(-p.v(z_{i-1}) > -v(x_i), \exists i \), where \( 1 \leq i \leq n \).

For Case (i) we show that \( v(w_n) = v(x_n) \). To compare \( v(w_n - x_n) \) and \( v(x_n) \) we make use of the following observation. Use the polynomials \( S_j \) as in [Ser 68, II.6] to express the addition law in \( W_{n+1} \) as
\[
\bar{a} + \bar{b} = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \ldots, S_n(\bar{a}, \bar{b})).
\]

1NB: we only needed \(-p.v(z_0) \geq -v(x_1)\) for case (ii).
Providing the ring \( \mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_n] \) with the grading where \( a_i \) and \( b_i \) have weight \( p^i \), then \( \mathbb{S}(\bar{a}, \bar{b}) \) is a homogeneous polynomial of degree \( p^n \).

Thus, \( u_n - x_n \) is comprised of the following sums: for each \( m \in [0, n - 1] \), there are \( p^{n-m-1} \) monomials in \( x_m, z_m \) of polynomial degree \( p^{n-m} \). Our assumptions in this case yield that \( v(x_n) \) is strictly less than each valuation \( v(\sum_{k=1}^{p^{n-m-1}} x_m^{p^{n-m-k}} z_m^k) \) for \( m \in [0, n - 1] \), from which we conclude that \( v(x_n) < v(x_n - w_n) \). Hence, \( v(w_n) = v(x_n) \).

For Case (ii), we divide the argument into two subcases. The first is if the inequality \(-p.v(z_{n-1}) > -v(x_n)\) holds. Then, we have

\[-p.(z_{n-1}) > -v(x_n) > -p.v(x_n-1),\]

hence, \( v(z_{n-1}) < v(x_n-1) \). Induction applied to \((w_0, \ldots, w_{n-1})\) gives \( v(w_{n-1}) = v(z_{n-1}) \) and therefore

\[\Gamma(\bar{w} = (w_0, \ldots, w_n)) \geq -p.v(z_{n-1}) > -v(x_n),\]

as desired. On the other hand, suppose \(-p.v(z_{n-1}) \leq -v(x_n)\). Here, we need to compare the remaining \( z_k \) with \( x_n \). If for some \( k' \) with \( 0 \leq k' \leq n - 2 \) the inequality \(-p^{n-k'}.v(z_{k'}) > -v(x_n)\) holds, then induction gives \( v(w_{k'}) = v(z_{k'}) \) and so

\[\Gamma(\bar{w} = (w_0, \ldots, w_n)) \geq -p^{n-k'}.v(z_{k'}).\]

Finally, if \(-p^{n-k'}.v(z_{k'}) < -v(x_n)\) for all such \( k' \), then applying a similar argument as in Case (i) above gives \( v(w_n) = v(x_n) \). \( \square \)

We now follow the principle of Lemma 6.10 to compute \( \limsup_e \text{Sw}(\chi|_{C_e})/e \) as \( e \to \infty \).

**Lemma 6.12.** For \( e > 0 \), consider the composite

\[\overline{\text{f}}_e : W_{n+1}(R) \to W_{n+1}(\mathbb{K}) \to W_{n+1}(\mathbb{K})/(F - 1).\]

For an integer \( h \geq 1 \), let \( s(h) \geq 0 \) denote the minimal integer for which

\[\overline{\text{f}}_h(f) \in \phi_s(h)(W_{n+1}(\mathbb{K})/(F - 1)).\]

Then

\[\lim_{e \to \infty} \sup_{e'} s(e)/e = \sup_{e'} s(e')/e' = N,\]

where \( e' \) ranges over all \( e' > 0 \) for which \( p \nmid v(\Phi_{e'}(f_n)) \).

**Proof.** The proof is as in Lemma 6.10 with the \( \lim_{e \to \infty} \) replaced by \( \limsup_{e \to \infty} \). \( \square \)

### 6.1.4. End of proof of Theorem 6.1

By induction we’ve reduced to the case where \( \text{Sw}_\epsilon(\chi)_{\log} = -v(f_n) \). Combining Lemmas 6.11 and 6.12 enables us to use the technique in 6.1.2 to compute \( N := \text{Sw}_\epsilon(\chi)_{\log} \) in terms of the \( C_e \); recall that the curves \( \phi_e : C_e \to X_x \) on \( X_x \) are based on the term \( B_n \) of the coordinate \( f_n \) in \( \bar{f} \). Let

\[c = v(\Phi_e(B_n)).\]

We have

\[\limsup_{e \to \infty} \frac{\text{Sw}_\epsilon(\chi|_{C_e})}{m(\phi_e^*D)} = \lim_{e \to \infty} \frac{Ne - c}{e} = N.\]

This completes the proof of the Local Tangent Theorem, Thm. 6.2. \( \square \)
Corollary 6.13. If \( \phi : C_x \to X_x \) is a curve on \( X_x \), we have the inequality
\[
Sw_x(\chi|_{C_x}) \leq m_x(\phi^* D) \cdot Sw_{\xi}(\chi)_{\log},
\]
Proof. From the above proof, this assertion is clear on pulling-back Witt vectors over discrete valuation rings from \( \mathcal{O}_{X,x}^h \) to \( \mathcal{O}_{C_x,x}^h \). \( \square \)

6.2. Global setting. We proceed to the global setting, i.e. to Thm. 6.1, via the following two propositions. Recall that \( k \) is a perfect field, \( X/k \) is a smooth variety and \( U \subset X \) is an open subscheme such that \( X \setminus U \) is the support of an effective Cartier divisor on \( X \) and \( D \) is an sncd on \( X \) with Supp(\( D \)) \( \subset \) \( E \).

Proposition 6.14. Suppose that \( D \) is a smooth irreducible divisor on \( X \) with generic point \( \xi \) and let \( x \in |X| \cap \text{Supp}(D) \). Since \( D \) and \( X \) are regular, there is a unique point \( \lambda \in \text{Spec} \mathcal{O}_{X,x}^h \) lying over \( \xi \in \text{Spec} \mathcal{O}_{X,x}^h \). Consider the henselian discrete valuation rings \( A = \mathcal{O}_{X,x}^h \) and \( B = ((\mathcal{O}_{X,x})^h)^h \). Let \( \chi \in H^1(U, \mathbb{Q}/\mathbb{Z}) \). Then
\[
Sw_{\xi}(\chi)_{\log} = Sw_{\lambda}(\chi|_{\text{Spec}(B)})_{\log}.
\]
(25)
Proof. This is clear since \( B/A \) is unramified. \( \square \)

In order to ensure that we get one-dimensional subschemes on \( X \) from the local setting above, we use henselian local rings of \( X \) since they are unions finitely-generated \( k \)-algebras, as opposed to complete local rings. In particular, we transfer the curves \( C_e \sim k[[w]] \) from Prop. 6.6 to curves \( C_h \sim k[[w]]^h \) via M. Artin’s approximation theorem, which we now recall.

Theorem 6.15. (Artin approximation, [Art 69]) Let \( R \) be a field or an excellent dvr, and let \( A^h \) be the henselization of a local \( R \)-algebra that is essentially of finite type over \( R \). Let \( \hat{A} \) be the completion of \( A^h \) with respect to the maximal ideal of \( A^h \). Then given a system of polynomial equations
\[
f(Y) = 0
\]
where \( Y = (Y_1, \ldots, Y_N) \) and \( f = (f_1, \ldots, f_s) \) with coefficients in \( A^h \) and given a solution
\[
\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N) \in \hat{A}^N
\]
and an integer \( n > 0 \), there exists a solution
\[
y = (y_1, \ldots, y_N) \in (A^h)^N
\]
such that
\[
y_k \equiv \hat{y}_k \mod m^n \cdot \hat{A},
\]
for \( k = 1, \ldots, N \).

Example 6.16. If \( L \) is a field, let \( A = L[x_1, \ldots, x_n] \), and \( \hat{A} = L[[x_1, \ldots, x_n]] \). Then
\[A^h = \{ g \in \hat{A} : \exists r \geq 0, a_i \in A \text{ s.t. } a_r g^r + \cdots + a_1 g + a_0 = 0 \},
\]
i.e. \( A^h \) is the subring of \( \hat{A} \) consisting of the elements of \( \hat{A} \) that are algebraic over \( L[x_1, \ldots, x_n] \).

\( \square \)

\( ^2 \)See [Lan 02, VIII, §4] for the definition of separability of finitely-generated field extensions.
Keep the notation from Thm. 6.2 and let \( A^h_x \) be the henselization of \( X \) at \( x \) and \( R^h = A^h_x[1/t_1] \). Because the curves \( C_e \) constructed in the proof of Prop. 6.6 are defined by algebraic relations, then Thm. 6.15 gives us curves \( C_e \) on \( \text{Spec}(A^h_x) \) for the following

**Proposition 6.17.** There are regular curves \( \phi_e : C_e \to \text{Spec}(A^h_x) \) with \( C_e \in Z_1(\text{Spec}(A^h_x), D) \) indexed by \( e > 0 \) such that

\[
\limsup_{e \to \infty} \frac{\text{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^* D)} = \text{Sw}_x(\chi)_{\log}.
\] (26)

□

We are now in a position to prove Theorem 6.1.

6.3. **Proof of Theorem.**

Proof. We may assume that \( D \) is irreducible with generic point \( \xi \). Let \( x \in |X| \cap \text{Supp}(D) \). Let \( X^h_x = \text{Spec}(A^h_x) \) and write

\[
C'_e \subset X^h_x,
\]

for the curves given in Prop. 6.17 above and consider the image \( f_x(C'_e) \) where \( f_x : X^h_x \to X \) is the canonical morphism. Since \( x \) is a closed point of \( X \) and \( X^h_x \) is the henselization of \( X \) at \( x \), it follows that that the closure of \( f_x(C'_e) \) is a curve on \( X \). In fact, the claim that the Zariski closure of \( f_x(C'_e) \) has dimension 1 follows by noting that the function field of \( f_x(C'_e) \) has transcendence degree 1 over the base field \( k \). Then the \( f_x(C'_e) \) are the desired set of curves \( \bar{C} \) on \( X \). The theorem then follows immediately on combining Proposition 6.14 and the Local Tangent Theorem 6.2. □

**Corollary 6.18.** We have that for each curve \( \bar{\phi} : \bar{C} \to X \) on \( X \) and \( z \in \bar{C}_{\infty}(D_i) \),

\[
\text{Sw}_z(\bar{\phi}^* \chi) \leq m_z(\bar{\phi}^* D_i) \cdot \text{Sw}_\xi(\chi)_{\log}.
\]

Proof. Follows by combining Prop. 6.14 and Coro. 6.13 □

7. **Theorem 7.1**

Conjecture B in our setting is the following theorem. Recall that for \( D \) an effective Cartier divisor, \( Z_1(X, D) \) denotes the set of curves on \( X \) that are not contained in \( D \) and given \( C \in Z_1(X, D) \),

\[
\bar{C}_{\infty}(D) = \{ z \in |\bar{C}^N| : \Psi_C(z) \subseteq D \},
\]

see Def. 5.4.

**Theorem 7.1.** (Conjecture B in the smooth rank 1 case) Let \( X/k \) be a smooth variety and \( U \subset X \) open variety such that \( E := X \setminus U \) is the support of an effective Cartier divisor on \( X \). Let \( \chi \in H^1(U, \mathbb{Q}/\mathbb{Z}) \). Let \( D \in \text{Div}^{\geq} \) be a regular sncd with \( \text{Supp}(D) \subset E \). The following are equivalent.

(i) For each curve \( \bar{\phi} : \bar{C} \to X \) and each \( z \in \bar{C}_{\infty}(D) \),

\[
\text{Sw}_z(\bar{\phi}^* \chi) \leq m_z(\bar{\phi}^* D).
\]

(ii) For each generic point \( \xi \in D \),

\[
\text{Sw}_\xi(\chi)_{\log} \leq m_\xi(D).
\]
7.1. **Proof of Theorem 7.1**

**Proof.** Since the divisor in question is \( D \), we simply write \( \bar{C}_\infty = \bar{C}_\infty(D) \).

(i)⇒(ii): Since \( 1 \leq m_\xi(D) \), it suffices to prove \( \text{Sw}_\xi(\chi)_{\log} \leq 1 \).

By hypothesis, for each point \( z \in \bar{C}_\infty \)

\[
\frac{\text{Sw}_z(\bar{\phi}^*\chi)}{m_z(\bar{\phi}^*D)} \leq 1.
\]

By the Local Tangent theorem, Thm. 6.2, we know that

\[
\text{Sw}_\xi(\chi)_{\log} = \sup_{\bar{C}} \frac{\text{Sw}_z(\bar{\phi}^*\chi)}{m_z(\bar{\phi}^*D)},
\]

and therefore

\[
\text{Sw}_\xi(\chi)_{\log} \leq 1.
\]

(ii)⇒(i): We may assume \( \chi \in H^1(U, \mathbb{Z}/p^n) \) for some \( n \geq 1 \).

First suppose that \( D \) is irreducible with generic point \( \lambda \).

By Coro. 6.18, we may assume that the multiplicity \( m_z(\bar{\phi}^*D) \) is non-zero, i.e. that \( m_z(\bar{\phi}^*D) > 0 \). There is an integer \( M \geq 1 \) such that

\[
m_z(\bar{\phi}^*D) = M \cdot m_\lambda(D).
\]

The proof of Thm. 6.2 and Coro. 6.18 yields that

\[
\text{Sw}_z(\bar{\phi}^*\chi) \leq M \cdot \text{Sw}_\lambda(\chi)_{\log}.
\]

By hypothesis, \( \text{Sw}_\lambda(\chi)_{\log} \leq m_\lambda(D) \) and so

\[
\text{Sw}_z(\bar{\phi}^*\chi) \leq M \cdot m_\lambda(D) = m_z(\bar{\phi}^*D),
\]

as desired.

These arguments and purity of the branch locus complete the proof for the case where \( D \) has more than one component. \( \square \)

8. **CLASS GROUPS WITH MODULUS AND EXISTENCE THEOREM**

Recently, Kerz and S. Saito proved an existence theorem for varieties using, in particular, bounded extensions and conductors. One of their results is a generalization of the existence theorem for ray class groups and ray class fields in the classical case. They employ a so-called ‘non-log’ filtration that is similar to the ‘log’ filtration in rank 1, the latter being the Brylinski-Kato filtration, see Def. 4.1. Since the log and non-log filtrations are nested in rank 1, see Lemma 8.1 below, we are able to use our result Thm. 7.1 to reformulate one of their main results [Ker-Sai 13, Coro II], to the log case.

We recall the non-log filtration used in [Ker-Sai 13], cf. [Mat 97]. Let \( K \) be a henselian discrete valuation field of characteristic \( p > 0 \). Recall the Brylinski-Kato filtration \( \text{fil}_m \) on \( W_n(K) \) in Def. 4.1 and set

\[
\text{fil}_{m_\text{nonlog}} W_n(K) := \text{fil}_m W_n(K) + V^{n-n'} \text{fil}_m W_{n'}(K), \tag{27}
\]

where \( n' = \min\{ n, \text{ord}_p(m+1) \} \).

As in Def. (7) from 4.1, definition (27) yields a so-called non-log filtration on all of \( H^1(K) \), which we write as \( \text{fil}_{m_\text{nonlog}} \). This filtration satisfies the following properties.

**Lemma 8.1.** (see [Mat 97] and [Kat 89])

The following properties are satisfied by the non-log filtration.
(i) $\fil_{m}^{\nonlog}H^1(K)$ is the subgroup of tamely ramified characters
(ii) $\fil_{m}^{\nonlog}H^1(K_{\delta}) \subset \fil_{m}^{\nonlog}H^1(K) \subset \fil_{m-1}^{\nonlog}H^1(K)$.
(iii) $\fil_{m}^{\nonlog}H^1(K) = \fil_{m-1}^{\nonlog}H^1(K)$ if $(m, p) = 1$.

For simplicity, let $X/k$ a normal variety and $U/k$ an open subscheme such that the reduced closed complement $X \setminus U$ is a (nontrivial) sncd on $X$.

Consider the following property on a character $\chi \in H^1(U) = \text{Hom}_{\cont}(\pi_{ab}^1(U), \mathbb{Q}/\mathbb{Z})$:
for each generic point $\xi \in D$,
$$\text{Sw}_{\xi}(\chi)_{\log} \leq m_{\xi}(D),$$
(28)
(recall Def. 5.1).

**Definition 8.2.** For $D$ as above, define $\fil_{D}H^1(U)$ to be the subgroup of $H^1(U)$ that consists of those $\chi$ satisfying (28) above. Also define
$$\pi_{1}^{ab}(X, D)_{\log} = \text{Hom}_{\cont}(\fil_{D}H^1(U), \mathbb{Q}/\mathbb{Z}),$$
with the pro-finite topology of the dual.

The topological group $\pi_{1}^{ab}(X, D)_{\log}$ is a quotient of $\pi_{1}^{ab}(U)$ which should be thought of as classifying extensions of $U$ with “ramification bounded by $D$”.

Following [Rus 10, §3.4], let $Z_{0}(X, D)$ be the subgroup of 0-cycles on $X \setminus D$ and set
$$\mathcal{R}(X, D) = \{(C, f) : C \in Z_{1}(X, E), f \in K(C)^{\times}, \tilde{f} \equiv 1 \mod D^{\#}\},$$
where $\tilde{f}$ is the image of $f$ in $K(C^{N})$ and $D^{\#} = (D - D_{\text{red}}) \cdot C^{N} + (D \cdot C^{N})_{\text{red}}$.

Let $R_{0}(U, D)$ be the subgroup of $Z_{0}(X \setminus D)$ generated by $\text{div}(f)_{C}$ for $(C, f) \in \mathcal{R}(X, D)$. We set
$$C(X, D)_{\log} = Z_{0}(X \setminus D)/R_{0}(X, D).$$

Now consider the degree-zero parts
$$C(X, D)_{\log}^{0} = \ker[C(X, D)_{\log} \xrightarrow{\text{deg}} \mathbb{Z}],$$
and
$$\pi_{1}^{ab}(X, D)_{\log}^{0} = \ker[\pi_{1}^{ab}(X, D)_{\log} \rightarrow \pi_{1}^{ab}(\text{Spec}(k))].$$

Using our Theorem 7.1 and Lemma 8.1 combined with [Ker-Sai 13, Coro II], we have the following log-version of their Existence theorem:

**Theorem 8.3.** (Existence theorem, with log filtration) Suppose $k$ is a finite field with $\text{char}(k) \neq 2$. Let $X/k$ be a proper and smooth $k$-variety and $U \subset X$ an open $k$-subscheme such that the reduced closed complement $D = X \setminus U$ is an sncd on $X$. Then
$$C(X, D)_{\log}^{0} \rightarrow \pi_{1}^{ab}(X, D)_{\log}^{0},$$
is an isomorphism of finite abelian groups.
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