DIOPHANTINE APPROXIMATION WITH ONE PRIME, TWO SQUARES OF PRIMES AND ONE $k$-TH POWER OF A PRIME

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Abstract. Let $1 < k < 14/5$, $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ be non-zero real numbers, not all of the same sign such that $\lambda_1/\lambda_2$ is irrational and let $\omega$ be a real number. We prove that the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \leq \left( \max_j p_j \right)^{-\frac{14-5k}{5k} + \varepsilon}$$

has infinitely many solutions in prime variables $p_1, p_2, p_3, p_4$ for any $\varepsilon > 0$.

1. Introduction

This paper deals with a Diophantine inequality with prime variables involving a prime, two squares of primes and one $k$-th power of a prime. In particular we prove the following theorem:

Theorem 1. Assume that $1 < k < 14/5$, $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ be non-zero real numbers, not all of the same sign, that $\lambda_1/\lambda_2$ is irrational and let $\omega$ be a real number. The inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \leq \left( \max_j p_j \right)^{-\frac{14-5k}{5k} + \varepsilon}$$

has infinitely many solutions in prime variables $p_1, p_2, p_3, p_4$ for any $\varepsilon > 0$.

Many recent such results are known with various types of assumptions and conclusions. Many of them deal with the number of exceptional real numbers $\omega$ such that the inequality

$$|\lambda_1 p_1^k + \cdots + \lambda_r p_r^k - \omega| \leq \eta$$

has no solution in prime variables $p_1, \ldots, p_r$, for small $\eta > 0$ fixed.

Brüdern, Cook and Perelli in [BCP] dealt with binary linear forms in prime arguments, Cook and Fox in [CF] dealt with a ternary form with squares of primes that was improved in term of approximation by Harman in [H2]. Cook in [C] gave a more general description of the problem, later improved by Cook and Harman in [CH].

There are some differences between the results quoted above and our purpose: in our case the value of $\eta$ does depend on the primes $p_j$ and it will be actually a negative power of the maximum of the $p_j$ while in the papers quoted above $\eta$ is a small negative power of $\omega$. In their papers the assumption that the coefficients $\lambda_j$ are all positive is not a restriction. Moreover $k_j$...
is the same positive integer for all $j$. Nevertheless the assumption that $\lambda_1/\lambda_2$ must be irrational is still the heart of the matter.

Vaughan in [V] follows another approach, that is the same we are using in our article: dealing with a ternary linear form in prime arguments and assuming some more suitable conditions on the $\lambda_j$, he proved that there are infinitely many solutions of the problem

$$|\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r} - \omega| \leq \eta$$

when $\eta$ depends on the maximum of the $p_j$; in his case $\eta = (\max_j p_j)^{-\frac{1}{6}}$. Such result was improved by Baker and Harman in [BH] with exponent $-\frac{1}{6}$, by Harman in [H] with exponent $-\frac{1}{5}$ and finally by Matomäki in [Ma] with exponent $-\frac{3}{10}$.

Languasco and Zaccagnini in [LZ] and [LZ2] dealt with a ternary problem with a $k$-th power of a prime. In this case the value of $\eta$ is a negative power of the maximum of the $p_j$ also depending on the parameter $k$: the idea in this case is to get both the widest $k$-range and the strongest bound for the approximation.

Languasco and Zaccagnini dealt also with a quaternary form in [LZ3] with a prime and 3 squares of primes obtaining $\eta = (\max_j p_j)^{-\frac{1}{5}}$; this was improved by Liu & Sun in [LS] with $\eta = (\max_j p_j)^{-\frac{1}{4}}$ using the Harman technique. Wang & Yao in [WY] improved the approximation to the exponent $-\frac{1}{3}$; in this paper we generalized the problem to a real power $k \in \left(1, \frac{14}{5}\right)$.

2. OUTLINE OF THE PROOF

We use a variant of the classical circle method that was introduced by Davenport and Heilbronn in 1946 [DH] in order to attack this kind of Diophantine problems. The integration on a circle, or equivalently on the interval $[0, 1]$, is replaced by integration on the whole real line.

Throughout this paper $p_i$ denotes a prime number, $k \geq 1$ is a real number, $\varepsilon$ is an arbitrarily small positive number whose value could vary depending on the occurrences and $\omega$ is a fixed real number. In order to prove that (11) has infinitely many solutions, it is sufficient to construct an increasing sequence $X_n$ that tends to infinity such that (11) has at least one solution with $\max p_j \in [\delta X_n, X_n]$, with $\delta > 0$ fixed, depending on the choice of $\lambda_j$. Let $q$ be a denominator of a convergent to $\lambda_1/\lambda_2$ and let $X_n = X$ (dropping the suffix $n$) run through the sequence $X = q^{7/3}$. Set

$$S_k(\alpha) = \sum_{\delta X \leq p^k \leq X} \log p \ e(p^k \alpha)$$

(2)

$$U_k(\alpha) = \sum_{\delta X \leq p^k \leq X} e(p^k \alpha)$$

(3)

$$T_k(\alpha) = \int_{(\delta X)^{1/k}}^{X^{1/k}} e(\alpha t^k) \ dt,$$

(4)

where $e(\alpha) = e^{2\pi i \alpha}$. We will approximate $S_k$ with $T_k$ and $U_k$. 

By the Prime Number Theorem and first derivative estimates for trigonometric integrals we have

\[ S_k(\alpha) \ll X^{\frac{1}{k}}, \quad T_k(\alpha) \ll_{k, \delta} X^{\frac{1}{k} - 1} \min(X, |\alpha|^{-1}). \] (5)

Moreover the Euler summation formula implies that

\[ T_k(\alpha) - U_k(\alpha) \ll 1 + |\alpha|X. \] (6)

We also need a continuous function we will use to detect the solutions of (1), so we introduce

\[ \hat{K}_\eta(\alpha) := \max\{0, \eta - |\alpha|\} \quad \text{where} \quad \eta > 0 \]

whose inverse Fourier transform is

\[ K_\eta(\alpha) = \left( \frac{\sin(\pi \alpha \eta)}{\pi \alpha} \right)^2 \]

for \( \alpha \neq 0 \) and, by continuity, \( K_\eta(0) = \eta^2 \). It vanishes at infinity like \( |\alpha|^{-2} \) and in fact it is trivial to prove that

\[ K_\eta(\alpha) \ll \min(\eta^2, |\alpha|^{-2}). \] (7)

The original works of Davenport-Heillbronn in [DH] and later Vaughan in [V1] and [V2] approximate directly the difference \( |S_k(\alpha) - T_k(\alpha)| \), estimating it with \( O(1) \) using the Euler summation formula. The \( L^2 \)-norm estimation approach (see [BCP] and [LZ3]) improves these estimation taking the \( L^2 \)-norm of \( |S_k(\alpha) - T_k(\alpha)| \) leading to significantly better conditions and to have a wider major arc compared to the original DH approach. In fact, setting the generalized version of the Selberg integral

\[ f_k(X, h) = \int_X^{2X} \left( \theta((x + h)^\frac{1}{k}) - \theta(x^\frac{1}{k}) - ((x + h)^\frac{1}{k} - x^\frac{1}{k}) \right)^2 \ dx, \]

we have the following lemmas.

**Lemma 1 ([LZ3], Theorem 1).** Let \( k \geq 1 \) be a real number. For \( 0 < Y < \frac{1}{2} \) we have

\[ \int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \ll_k \frac{X^{\frac{k}{2} - 2} \log^2 X}{Y} + Y^2 X + Y^2 f_k \left( X, \frac{1}{2Y} \right). \]

**Lemma 2 ([LZ3], Theorem 2).** Let \( k \geq 1 \) be a real number and \( \varepsilon \) be an arbitrarily small positive constant. There exists a positive constant \( c_1(\varepsilon) \), which does not depend on \( k \), such that

\[ f_k(X, h) \ll_k h^2 X^{\frac{k}{2} - 1} \exp \left( -c_1 \left( \frac{\log X}{\log \log X} \right)^{\frac{1}{3}} \right) \]

uniformly for \( X^{\frac{1}{k} + \varepsilon} \leq h \leq X. \)
2.1. Setting the problem. Let
\[ \mathcal{P}(X) = \{(p_1, p_2, p_3, p_4) : \delta X < p_1 < X, \ \delta X < p_2^2, p_3^2 < X, \ \delta X < p_4^k < X\} \]
and let us define
\[ \mathcal{J}(\eta, \omega, \mathcal{X}) = \int_{\mathcal{X}} S_1(\lambda_1 \alpha)S_2(\lambda_2 \alpha)S_3(\lambda_3 \alpha)S_4(\lambda_4 \alpha)K_\eta(\alpha)e(-\omega \alpha) \, d\alpha \]
where \( \mathcal{X} \) is a measurable subset of \( \mathbb{R} \).

From the definitions of the \( S_j(\lambda_j \alpha) \) and performing the Fourier transform for \( K_\eta(\alpha) \), we get
\[
\mathcal{J}(\eta, \omega, \mathbb{R}) = \sum_{\mathcal{P}(X)} \log p_1 \log p_2 \log p_3 \log p_4 \cdot \left( \max(0, \eta - \left| \lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega \right|) \right) \leq \eta (\log X)^d N(X),
\]
where \( N(X) \) actually denotes the number of solutions of the inequality \((11)\) with \((p_1, p_2, p_3, p_4) \in \mathcal{P}(X)\). In other words \( \mathcal{J}(\eta, \omega, \mathbb{R}) \) provides a lower bound for the quantity we are interested in; therefore it is sufficient to prove that \( \mathcal{J}(\eta, \omega, \mathbb{R}) > 0 \).

We now decompose \( \mathbb{R} \) into subsets such that \( \mathbb{R} = \mathcal{M} \cup m \cup t \) where \( \mathcal{M} \) is the major arc, \( m \) is the minor arc (or intermediate arc) and \( t \) is the trivial arc. The decomposition is the following:
\[
\mathcal{M} = \left[ -\frac{P}{X}, \frac{P}{X} \right], \quad m = \left[ \frac{P}{X}, R \right] \cup \left[ -R, -\frac{P}{X} \right], \quad t = \mathbb{R} \setminus (\mathcal{M} \cup m),
\]
so that \( \mathcal{J}(\eta, \omega, \mathbb{R}) = \mathcal{J}(\eta, \omega, \mathcal{M}) + \mathcal{J}(\eta, \omega, m) + \mathcal{J}(\eta, \omega, t) \).

The parameters \( P = P(X) > 1 \) and \( R = R(X) > 1/\eta \) are chosen later (see \((11)\) and \((14)\)) as well as \( \eta = \eta(X) \), that, as we explained before, we would like to be a small negative power of \( \max p_j \) (and so of \( X \), see \((21)\)).

We are expecting to have on \( \mathcal{M} \) the main term with the right order of magnitude without any special hypothesis on the coefficients \( \lambda_j \). It is necessary to prove that \( \mathcal{J}(\eta, \omega, m) \) and \( \mathcal{J}(\eta, \omega, t) \) are both \( o(\mathcal{J}(\eta, \omega, \mathcal{M})) \): the contribution from the trivial is “tiny” with respect to the main term. The real problem is on the minor arc where we will need the full force of the hypothesis on the \( \lambda_j \) and the theory of continued fractions.

Remark: from now on, anytime we use the symbol \( \ll \) or \( \gg \) we drop the dependence of the approximation from the constants \( \lambda_j, \delta \) and \( k \).

2.2. Lemmas. In this paper we will also use Lemmas 3-4-10 of \([GLZ]\) that allow us to have an estimation of mean value of \( |S_k(\alpha)|^4 \):

Lemma 3 \([GLZ] \), Lemma 3. Let \( \varepsilon > 0 \) fixed, \( k > 1, \gamma > 0 \) and let \( a(X^{1/k}; k; \gamma) \) denote the number of solutions of the inequality
\[
|n_1^k + n_2^k - n_3^k - n_4^k| < \gamma, \quad X^{1/k} \leq n_1, n_2, n_3, n_4 \leq 2X^{1/k}.
\]
We write that we must estimate by means of Lemma \(\text{one} \).  

**Proof.** The first statement comes directly from Prime Number Theorem, while the second of [R], p. 94.

**Lemma 4 ([GLZ], Lemma 4).** Let \(k > 1, \tau > 0\). We have

\[
\int_{-\tau}^{\tau} |S_k(\alpha)|^d \, d\alpha \ll \left(\tau X^{2/k} + X^{4/k-1}\right) X^e \ll \max(\tau X^{2/k+e}, X^{4/k-1+e}).
\]

**Lemma 5 ([GLZ], Lemma 10).**

\[
\int_m \left|S_k(\lambda\alpha)\right|^d K_\eta(\alpha) \, d\alpha \ll \eta X^e \cdot \max(X^{2/k}, X^{4/k-1}).
\]

Finally, we will use the following Lemma.

**Lemma 6.**

\[
\int_m \left|S_1(\alpha)\right|^2 K_\eta(\alpha) \, d\alpha \ll \eta X \log X
\]

\[
\int_m \left|S_2(\alpha)\right|^4 K_\eta(\alpha) \, d\alpha \ll \eta X \log^2 X.
\]

**Proof.** The first statement comes directly from Prime Number Theorem, while the second estimation is based on Satz 3 of [R]. p. 94.

3. **The major arc**

Let us start from the major arc and the computation of the main term. We replace all \(S_k\) defined in (2) with the corresponding \(T_k\) defined in (4). This replacement brings up some errors that we must estimate by means of Lemma [1] the Cauchy-Schwarz and the Hölder inequalities. We write

\[
\mathcal{J}(\eta, \omega, A) = \int_A S_1(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
= \int_A T_1(\lambda_1 \alpha) T_2(\lambda_2 \alpha) T_3(\lambda_3 \alpha) T_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
+ \int_A (S_1(\lambda_1 \alpha) - T_1(\lambda_1 \alpha)) T_2(\lambda_2 \alpha) T_3(\lambda_3 \alpha) T_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
+ \int_A S_1(\lambda_1 \alpha) (S_2(\lambda_2 \alpha) - T_2(\lambda_2 \alpha)) T_3(\lambda_3 \alpha) T_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
+ \int_A S_1(\lambda_1 \alpha) S_2(\lambda_2 \alpha) (S_3(\lambda_3 \alpha) - T_3(\lambda_3 \alpha)) T_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
+ \int_A S_1(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) (S_k(\lambda_4 \alpha) - T_k(\lambda_4 \alpha)) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha
\]

\[
= J_1 + J_2 + J_3 + J_4 + J_5,
\]
say. For brevity, since the computations for \( J_2 \) and \( J_3 \) are similar to, but simpler than, the corresponding ones for \( J_4 \) and \( J_5 \), we will leave them to the reader.

### 3.1. Main Term: lower bound for \( J_1 \)

As the reader might expect the main term is given by the summand \( J_1 \).

Let \( H(\alpha) = T_1(\alpha_1)T_2(\alpha_2)T_3(\alpha_3)T_k(\alpha_4)e(-\omega\alpha) \) so that

\[
J_1 = \int_{\mathbb{R}} H(\alpha) \, d\alpha + O\left( \int_{P/X}^{+\infty} |H(\alpha)| \, d\alpha \right).
\]

Using inequalities (7) and (5),

\[
\int_{P/X}^{+\infty} |H(\alpha)| \, d\alpha \ll X^{-1} X^{\frac{1}{k}-1} \eta^2 \int_{P/X}^{+\infty} \frac{d\alpha}{\alpha^\ell} \ll X^{\frac{1}{k}+1} \eta^2 P^{-3} = o\left(X^{\frac{1}{k}+1} \eta^2\right)
\]

provided that \( P \to +\infty \). Let \( D = [\delta X, X] \times [(\delta X)^{1/k}, X^{1/k}] \times [(\delta X)^{1/k}, X^{1/k}] \); we have

\[
\int_{\mathbb{R}} H(\alpha) \, d\alpha = \int \cdots \int_D \int_{\mathbb{R}} e((\alpha_1 t_1 + \alpha_2 t_2^2 + \alpha_3 t_3^2 + \alpha_4 t_4^3 - \omega)\alpha)K_\eta(\alpha) \alpha \, dt_1 \, dt_2 \, dt_3 \, dt_4
\]

\[
= \int \cdots \int_D \max(0, \eta - |\alpha_1 t_1 + \alpha_2 t_2^2 + \alpha_3 t_3^2 + \alpha_4 t_4^3 - \omega|) \, dt_1 \, dt_2 \, dt_3 \, dt_4.
\]

Apart from trivial changes of sign, there are essentially three cases as in \([LZ1]\):

1. \( \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 < 0 \)
2. \( \lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0, \lambda_4 < 0 \)
3. \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0 \).

We deal with the second case, the other ones being similar: let us perform the following change of variables: \( u_1 = t_1 - \frac{\alpha_1}{\lambda_1}, u_2 = t_2^2, u_3 = t_3^2, u_4 = t_4^3 \), so that the set \( D \) becomes essentially \([\delta X, X]^4\). Let us define \( D' = [\delta X, (1 - \delta)X]^4 \) for large \( X \), as a subset of \( D \). The Jacobian determinant of the change of variables above is \( \frac{1}{4k}u_2^{-\frac{1}{2}}u_3^{-\frac{1}{2}}u_4^{-\frac{1}{2}} \). Then

\[
J_1 \gg \int_{\mathbb{R}} H(\alpha) \, d\alpha
\]

\[
= \int \cdots \int_{D'} \max(0, \eta - |\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4|) \, du_1 \, du_2 \, du_3 \, du_4
\]

\[
\gg X^{\frac{1}{k}+2} \int \cdots \int_{D'} \max(0, \eta - |\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4|) \, du_1 \, du_2 \, du_3 \, du_4.
\]

Now, for \( j = 1, 2, 3 \) let \( a_j = \frac{4|\lambda_j|}{|\lambda_j|}, b_j = \frac{3}{2}a_j \) and \( \mathcal{D}_j = [a_j X, b_j X] \); if \( u_j \in \mathcal{D}_j \) for \( j = 1, 2, 3 \) then

\[
\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in [2|\lambda_4|\delta X, 8|\lambda_4|\delta X]
\]
so that, for every choice of \((u_1, u_2, u_3)\) the interval

\[
[a, b] = \left[ \frac{1}{|L_4|} (-\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)), \frac{1}{|L_4|} (\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)) \right]
\]

is contained in \([\delta X, (1-\delta)X]\). In other words, for \(u_4 \in [a, b]\) the values of \(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4\) cover the whole interval \([-\eta, \eta]\). Hence for any \((u_1, u_2, u_3) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3\) we have

\[
\int_{\delta X}^{(1-\delta)X} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|) du_4
\]

\[= |\lambda_4|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|) du \gg \eta^2.
\]

Finally,

\[
J_1 \gg \eta^2 X^{\frac{1}{3} - \frac{1}{2}} \iint_{\mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3} d u_1 d u_2 d u_3 \gg \eta^2 X^{\frac{1}{6} - \frac{1}{2}} X^3 = \eta^2 X^{\frac{1}{2} + 1},
\]

which is the expected lower bound.

3.2. Bound for \(J_4\). The computations on \(J_2\) and \(J_3\) are similar to and simpler than the corresponding one on \(J_4\), so we will skip it. Using the triangle inequality,

\[
J_4 = \int_{\mathcal{E}} S_1(\lambda_1 \alpha)S_2(\lambda_2 \alpha)S_2(\lambda_3 \alpha) - T_2(\lambda_3 \alpha)T_k(\lambda_4 \alpha)K_\eta(\alpha)e(-\omega \alpha) d\alpha
\]

\[\ll \eta^2 \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha) - T_2(\lambda_3 \alpha)||T_k(\lambda_4 \alpha)| d\alpha
\]

\[\leq \eta^2 \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha) - U_2(\lambda_3 \alpha)||T_k(\lambda_4 \alpha)| d\alpha
\]

\[+ \eta^2 \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||U_2(\lambda_3 \alpha) - T_2(\lambda_3 \alpha)||T_k(\lambda_4 \alpha)| d\alpha
\]

\[= \eta^2 (A_4 + B_4),
\]

say, where \(U_2(\lambda_3 \alpha)\) is given by (3).

Using the trivial inequalities \(|S_2(\alpha)| \ll X^\frac{1}{2}\), \(|S_2(\alpha)| \ll X^\frac{1}{2}\) and then the Cauchy-Schwarz inequality,

\[
A_4 \ll X^{\frac{1}{2} + \frac{1}{2}} \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)| - U_2(\lambda_3 \alpha)| d\alpha
\]

\[\ll X^{\frac{1}{2} + \frac{1}{2}} \left( \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_{\mathcal{E}} |S_2(\lambda_2 \alpha)|^2 d\alpha \right)^{\frac{1}{2}}.
\]

Using \(\int_{\mathcal{E}} |S_1(\alpha)|^2 d\alpha \ll X \log X\) and \(\int_{\mathcal{E}} |S_2(\alpha)|^2 - U_2(\alpha)|^2 d\alpha \ll (\log X)^{-A}\) for any fixed \(A\) (Lemmas 1 and 2), we have

\[
A_4 \ll X^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} (X \log X)^{\frac{1}{2}} (\log X)^{\frac{1}{2}} = X^{1 + \frac{1}{2}} (\log X)^{\frac{1}{2}} = o\left(X^{\frac{3}{2}}\right)
\]

as long as \(A > 1\). Again using (3) and (5),

\[
B_4 \ll X^{\frac{1}{2}} \int_{\mathcal{E}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||U_2(\lambda_3 \alpha) - T_2(\lambda_3 \alpha)| d\alpha
\]

\[\ll X^{\frac{1}{2}} \int_{0}^{\infty} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)| d\alpha + X^{\frac{1}{2} + 1} \int_{\lambda_3}^{P/X} \alpha |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)| d\alpha.
\]
Remembering that $|\alpha| \leq \frac{1}{X}$ on $\mathcal{M}$ and using the Hölder inequality, trivial bounds and Lemma 6 we have
\[
B_4 \ll X X^\frac{1}{P} \left( \int_{1/X}^{P/X} |S_1(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_{1/X}^{P/X} \alpha^4 d\alpha \right)^{\frac{1}{2}} \left( \int_{1/X}^{P/X} |S_2(\alpha)|^4 d\alpha \right)^{\frac{1}{4}}
\]
\[
\ll X^{\frac{1}{P} + \frac{1}{4}} + X^{\frac{1}{P} + \frac{1}{2}} (\log X)^{\frac{1}{4}} \left( \frac{P}{X} \right)^{\frac{1}{4}} X^\frac{1}{P} (\log X)^{\frac{1}{4}} = X^{\frac{1}{P} + \frac{1}{4}} P^\frac{1}{4} \log X.
\]
Since we must have $P^\frac{1}{4} = o(X^\frac{1}{P} \log X)$, it follows that
\[
P \leq X^{\frac{1}{P}-\varepsilon}
\]
is sufficient for our purpose.

3.3. **Bound for $J_5$.** In order to provide an estimation for $J_5$, we use (7),
\[
J_5 \ll \eta^2 \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha)||S_2(\lambda_4 \alpha) - T_k(\lambda_4 \alpha)| d\alpha
\]
and then the arithmetic-geometric inequality $(ab \leq a^2 + b^2)$:
\[
J_5 \ll \eta^2 \sum_{j=1}^{3} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)||S_2(\lambda_j \alpha)|^2|S_2(\lambda_4 \alpha) - T_k(\lambda_4 \alpha)| d\alpha \right)
\]
The two terms are equivalent; then we consider only one of them
\[
J_5 \ll \eta^2 \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)|^2|S_2(\lambda_4 \alpha) - T_k(\lambda_4 \alpha)| d\alpha
\]
\[
\ll \eta^2 \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)|^2|S_2(\lambda_4 \alpha) - U_k(\lambda_4 \alpha)| d\alpha
\]
\[
+ \eta^2 \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)|^2|U_k(\lambda_4 \alpha) - T_k(\lambda_4 \alpha)| d\alpha = \eta^2 (A_5 + B_5),
\]
say. Using trivial estimates,
\[
A_5 \ll X \int_{\mathbb{R}} |S_2(\lambda_2 \alpha)|^2|S_2(\lambda_4 \alpha) - U_k(\lambda_4 \alpha)| d\alpha
\]
then using the Hölder inequality, for any fixed $A > 2$ by Lemmas 1 and 2 we have
\[
A_5 \ll X \left( \int_{\mathbb{R}} |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}} |S_2(\lambda_4 \alpha) - U_k(\lambda_4 \alpha)|^2 d\alpha \right)^{\frac{1}{2}}
\]
\[
\ll X X^{\frac{1}{4}} (\log X)^{\frac{1}{4}} \frac{P}{X} \left( X, X^A \right)^{\frac{1}{4}} \ll_A X^{1+\frac{1}{4}} (\log X)^{1-\frac{1}{4}} = o \left( X^{\frac{1}{P} + 1} \right),
\]
provided that $\frac{P}{X} \geq X^{1-\frac{1}{4}+\varepsilon}$ (condition of Lemma 2), that is,
\[
P \leq X^{\frac{1}{P}-\varepsilon}. \tag{9}
\]
Now we turn to $B_5$: by (6) we have
\[
B_5 \ll \int_{1/X}^{P/X} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)|^2 d\alpha + X \int_{1/X}^{P/X} \alpha |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)|^2 d\alpha.
\]
Using trivial estimates and Lemma 6,
\[
B_5 \ll \left( \int_0^{1/X} |S_1(\lambda_1 \alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_0^{1/X} |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{2}} + X P \left( \int_{1/X}^{P/X} |S_1(\lambda_1 \alpha)|^2 d\alpha \cdot \int_{1/X}^{P/X} |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{2}} \ll (X \log X \cdot X \log^2 X)^{\frac{1}{2}} + P(X \log X \cdot X \log^2 X)^{\frac{1}{2}} = X(\log X)^{\frac{1}{2}} + PX(\log X)^{\frac{1}{2}}.
\]
Then we need
\[
P = o\left( X^{\frac{1}{2} - \varepsilon} \right).
\]
Collecting all the bounds for \( P \), that is, (8), (9), (10) we can take
\[
P \leq \min\left( X^{\frac{1}{2} - \varepsilon}, X^{\frac{1}{3} - \varepsilon} \right).
\]
In fact, if we consider (8) and (9), we should choose the most restrictive condition between the two: if \( k \leq \frac{25}{12} \), \( P = X^{\frac{1}{2} - \varepsilon} \), otherwise, if \( \frac{25}{12} < k < \frac{14}{5} \), \( P = X^{\frac{1}{3} - \varepsilon} \).

4. The trivial arc

By the arithmetic-geometric mean inequality and the trivial bound for \( S_k(\lambda_4 \alpha) \), we see that
\[
|\mathcal{F}(\eta, \omega, t)| \ll \int_R^{+\infty} |S_1(\lambda_1 \alpha)S_2(\lambda_2 \alpha)S_3(\lambda_3 \alpha)S_4(\lambda_4 \alpha)K_{\eta}(\alpha)| d\alpha
\]
\[
\ll X^{\frac{1}{2}} \sum_{j=2}^{\infty} \int_R^{+\infty} |S_1(\lambda_1 \alpha)| |S_2(\lambda_2 \alpha)|^2 K_{\eta}(\alpha) d\alpha
\]
\[
\ll X^{\frac{1}{2}} \left( \int_R^{+\infty} |S_1(\lambda_1 \alpha)|^2 K_{\eta}(\alpha) d\alpha \right)^{\frac{1}{2}} \left( \int_R^{+\infty} |S_2(\lambda_2 \alpha)|^4 K_{\eta}(\alpha) d\alpha \right)^{\frac{1}{2}}
\]
\[
\ll X^{\frac{1}{2}} \left( \int_R^{+\infty} \frac{|S_1(\lambda_1 \alpha)|^2}{\alpha^2} d\alpha \right)^{\frac{1}{2}} \left( \int_R^{+\infty} \frac{|S_2(\lambda_2 \alpha)|^4}{\alpha^2} d\alpha \right)^{\frac{1}{2}} = X^{\frac{1}{2}} C_1^{\frac{1}{2}} C_2^{\frac{1}{2}},
\]
say. Using the PNT and the periodicity of \( S_1(\alpha) \), we have
\[
C_1 = \int_R^{+\infty} \frac{|S_1(\lambda_1 \alpha)|^2}{\alpha^2} d\alpha \ll \int_{\lambda_1 |R}^{+\infty} \frac{|S_1(\alpha)|^2}{\alpha^2} d\alpha \ll \sum_{n \geq |\lambda_1| |R|} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_1(\alpha)|^2 d\alpha \ll \frac{X \log X}{|\lambda_1| |R|}.
\]
Now using Lemma 6,
\[
C_2 = \int_R^{+\infty} \frac{|S_2(\lambda_2 \alpha)|^4}{\alpha^2} d\alpha \ll \int_{\lambda_1 |R}^{+\infty} \frac{|S_2(\alpha)|^4}{\alpha^2} d\alpha \ll \sum_{n \geq |\lambda_1| |R|} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_2(\alpha)|^4 d\alpha \ll \frac{X \log^2 X}{|\lambda_1| |R|}.
\]
Collecting \((12)\) and \((13)\),

\[
|f(\eta, \omega, t)| \ll X^\frac{1}{2} \left( \frac{X \log X}{R} \right)^{\frac{1}{2}} \left( \frac{X \log^2 X}{R} \right)^{\frac{1}{2}} \ll \frac{X^{1+\frac{1}{8}}(\log X)^{\frac{1}{2}}}{R}.
\]

Hence, remembering that \(|f(\eta, \omega, t)|\) must be \(o\left(\eta^2 X^{\frac{1}{2}+\epsilon}\right)\), i.e. of the main term, the choice

\[
R = \frac{\log^2 X}{\eta^2}
\]

is admissible.

5. The minor arc

In \([\text{LZ}2]\) Lemma 3 it is proven that the measure of the set where \(|S_1(\lambda_1 \alpha)|\) and \(|S_2(\lambda_2 \alpha)|\) are both large for \(\alpha \in m\) is small, exploiting the fact that the ratio \(\lambda_1/\lambda_2\) is irrational.

Lemma 7 (Vaughan \([\text{V3}],\) Theorem 3.1). Let \(\alpha\) be a real number and \(a, q\) be positive integers satisfying \((a, q) = 1\) and \(|\alpha - \frac{a}{q}| < \frac{1}{q^2} \). Then

\[
S_1(\alpha) \ll \left( \frac{X}{\sqrt{q}} + \sqrt{Xq} + X^\frac{1}{2} \right) \log^4 X.
\]

We now state some considerations about Lemmas \([7]\).

Corollary 1 (Liu-Sun \([\text{LSu}],\) Corollary 2.7). Suppose that \(X \geq Z \geq X^{1-\frac{1}{4}+\epsilon}\) and \(|S_1(\lambda_1 \alpha)| > Z\). Then there are coprime integers \((a, q) = 1\) satisfying

\[
1 \leq q \leq \left( \frac{X^{1+\epsilon}}{Z} \right)^2, \quad |q \lambda_1 \alpha - a| \ll \left( \frac{X^{1+\epsilon}}{Z} \right)^2.
\]

Lemma 8 (Wang-Yao \([\text{WY}],\) Lemma 1). Suppose that \(X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{10}+\epsilon}\) and \(|S_2(\lambda_2 \alpha)| > Z\). Then there are coprime integers \((a, q) = 1\) satisfying

\[
1 \leq q \leq \left( \frac{X^{\frac{1}{2}+\epsilon}}{Z} \right)^4, \quad |q \lambda_2 \alpha - a| \ll X^{-1} \left( \frac{X^{\frac{1}{2}+\epsilon}}{Z} \right)^4
\]

Let us now split \(m\) into two subsets \(\tilde{m}\) and \(m^* = m \setminus \tilde{m}\). In turn \(\tilde{m} = m_1 \cup m_2\), where

\[
\begin{align*}
m_1 &= \{ \alpha \in m : |S_1(\lambda_1 \alpha)| \leq X^{1-\frac{1}{4}+\epsilon} \}, \\
m_2 &= \{ \alpha \in m : |S_2(\lambda_2 \alpha)| \leq X^{\frac{1}{2}-\frac{1}{10}+\epsilon} \}.
\end{align*}
\]

Using the Hölder inequality, Lemma \([5]\) and the definition of \(m_1\) we obtain

\[
|f(\eta, \omega, m)| \ll \int_{m_1} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha)||S_2(\lambda_4 \alpha)||S_k(\lambda_k \alpha)||K_\eta(\alpha) d\alpha
\]

\[
\ll \left( \max_{\alpha \in m_1} |S_1(\lambda_1 \alpha)| \right)^{\frac{1}{2}} \left( \int_{m_1} |S_1(\lambda_1 \alpha)|^2 K_\eta(\alpha) d\alpha \right)^{1/4}
\]

\[
\prod_{i=2}^{3} \left( \int_{m_1}^{3} |S^2(\lambda_i\alpha)|^4 K_\eta(\alpha) d\alpha \right)^{1/4} \left( \int_{m_1}^{3} |S^3(\lambda_4\alpha)|^4 K_\eta(\alpha) d\alpha \right)^{1/4} < X^{1/2+\varepsilon} (\eta X \log X)^{1/2}(\eta X \log^2 X)^{1/2} \left( \eta X^\varepsilon \max(X^{2/3}, X^{1/3}) \right)^{1/2}
= \eta X^{33/32+2\varepsilon} \max(X^{1/3}, X^{1/4-1/3}).
\]

Using the Hölder inequality, Lemma 15 and the definition of \(m_2\) we obtain
\[
|\mathcal{F}(\eta, \omega, m_2)| \ll \int_{m_2} \left| S_1(\lambda_1\alpha) \right| S_2(\lambda_2\alpha) |S_2(\lambda_3\alpha)| S_3(\lambda_4\alpha) K_\eta(\alpha) d\alpha
\ll \max_{\alpha \in m_2} S_2(\lambda_2\alpha) \left( \int_{m_2} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha) d\alpha \right)^{1/2}
\ll \max_{\alpha \in m_2} S_2(\lambda_2\alpha) \left( \int_{m_2} S_2(\lambda_3\alpha) |S_3(\lambda_4\alpha)| K_\eta(\alpha) d\alpha \right)^{1/4}
\ll \eta X^{33/32+2\varepsilon} \max(X^{1/3}, X^{1/4-1/3}).
\]

Both (15) and (16) must be \(o\left(\eta^2 X^{1+1/3}\right)\), consequently it is clear that for \(1 < k < 2\), \(\eta\) is a negative power of \(X\) independently from the value of \(k\). The we have the following most restrictive condition for \(k \geq 2\):
\[
\eta = \infty \left( X^{-\frac{15+5k}{28k}+\varepsilon} \right).
\]

It remains to discuss the set \(m^*\) in which the following bounds hold simultaneously
\[
|S_1(\lambda_1\alpha)| > X^{1/2+\varepsilon}, \ |S_2(\lambda_2\alpha)| > X^{1/2+\varepsilon}, \ \frac{p}{X} = X^{-\frac{1}{2}} < |\alpha| \leq \frac{\log^2 X}{\eta^2} = R.
\]

Following the dyadic dissection argument as in [H2] we divide \(m^*\) into disjoint sets \(E(Z_1, Z_2, y)\) in which, for \(\alpha \in E(Z_1, Z_2, y)\), we have
\[
Z_1 < |S_1(\lambda_1\alpha)| \leq 2Z_1, \ \ \ Z_2 < |S_2(\lambda_2\alpha)| \leq 2Z_2, \ \ \ y < |\alpha| \leq 2y
\]
where \(Z_1 = 2^{k_1} X^{1/2+\varepsilon}, \ Z_2 = 2^{k_2} X^{1/2+\varepsilon} \) and \(y = 2^{k_3} X^{1/2+\varepsilon}\) for some non-negative integers \(k_1, k_2, k_3\).

It follows that the disjoint sets are, at the most, \(\ll \log^3 X\). Let us define \(\mathfrak{A}\) as a shorthand for the set \(E(Z_1, Z_2, y)\); we have the following result about the Lebesgue measure of \(\mathfrak{A}\) following the same lines of Lemma 6 in [Mu]:

**Lemma 9.** We have \(\mu(\mathfrak{A}) \ll YX^{1/2+6\varepsilon} Z_1^{-2} Z_2^{-4}\), where \(\mu(\cdot)\) denotes the Lebesgue measure.

**Proof.** If \(\alpha \in \mathfrak{A}\), by Corollaries 11 and Lemma 8 there are coprime integers \((a_1, q_1)\) and \((a_2, q_2)\) such that
\[
1 \leq q_1 \ll \left( \frac{X^{1+\varepsilon}}{Z_1} \right)^2, \ \ |q_1 \lambda_1 \alpha - a_1| \ll \left( \frac{X^{1+\varepsilon}}{Z_1} \right)^2
\]
\[ 1 \leq q_2 \ll \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4. \] 

(17)

We remark that \(a_1a_2 \neq 0\) otherwise we would have \(\alpha \in \mathcal{M}\). Recalling the definitions of \(Z_1\) and \(Z_2\):

\[
q_1^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2 \ll \alpha \ll q_1^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2.
\]

\[
q_2^{-1} X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4 \ll \alpha \ll q_2^{-1} X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4.
\]

(18)

then, if \(q_1 = q_2 = 1\) we get:

\[ \alpha \gg \frac{X}{X^{\frac{1}{2}} + 2\epsilon} = X^{-\frac{1}{2} + 2\epsilon}. \]

Note that, by the conditions on \(P\) (see (11)), there is no gap between the major arc and the minor arc. Now, we can further split \(m^*\) into sets \(I(Z_1, Z_2, y, Q_1, Q_2)\) where, on each set, \(Q_j \leq q_j \leq 2Q_j\). If we explicit \(\alpha\) as in (18), we obtain the following inequalities:

\[
Q_1^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2 \ll \alpha \ll Q_1^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2.
\]

\[
Q_2^{-1} X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4 \ll \alpha \ll Q_2^{-1} X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4.
\]

As the inequalities (17) hold simultaneously, the measure of \(I\) can be bounded with the minimum of the two:

\[ \mu(I) \ll \min \left( Q_1^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2, Q_2^{-1} X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4 \right). \]

Taking the geometric mean (min(a, b) \(\leq \sqrt{ab}\)) we can write

\[
\mu(I) \ll Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} X^{-\frac{1}{2}} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2 \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^2 \ll \frac{X^{1+3\epsilon}}{Q_1^\frac{1}{2} Q_2^\frac{1}{2} Z_1 Z_2^2}. \]

(19)

Now we need a lower bound for \(Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}}\): by (17)

\[
|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2| = \left| \frac{a_2}{\lambda_2} (q_1 \lambda_1 \alpha - a_1) - \frac{a_1}{\lambda_2} (q_2 \lambda_2 \alpha - a_2) \right|
\ll q_2 |q_1 \lambda_1 \alpha - a_1| + q_1 |q_2 \lambda_2 \alpha - a_2| \ll
\]

\[
Q_2 \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2 + Q_1 X^{-1} \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4.
\]

Remembering that \(Q_1 \ll \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_1} \right)^2, Q_2 \ll \left( \frac{X^{\frac{1}{2}} + \epsilon}{Z_2} \right)^4, Z_1 \gg X^{\frac{1}{2}} + \epsilon, Z_2 \gg X^{\frac{1}{2}} + \epsilon\),
\[ |a_2q_1 \frac{a_1}{a_2} - a_1q_2| \ll \left( \frac{X^{1+\varepsilon}}{X^{2+\varepsilon}} \right)^4 \left( \frac{X^{1+\varepsilon}}{X^{2+\varepsilon}} \right)^2 \left( \frac{X^{1+\varepsilon}}{X^{2+\varepsilon}} \right)^2 X^{-1} \left( \frac{X^{1+\varepsilon}}{X^{2+\varepsilon}} \right)^4 \]
\[ \ll \frac{X^{2+4\varepsilon}X^{1+2\varepsilon}}{X^{4+4\varepsilon}X^{4+2\varepsilon}} \ll \frac{1}{4q} \] (20)

since \( X = q^{7/3} \). We would like that \( |a_2q_1| \geq q \) so that, recalling that \( q \) is the denominator of a convergent of \( \lambda_1/\lambda_2 \), we could apply the Legendre’s law of best approximation for continued fractions: in our case it must be
\[ X^{-\frac{1}{2}-6\varepsilon} < \frac{1}{4q}, \]

It turns out that for any pair \( \alpha, \alpha' \) having distinct associated products \( a_2q_1 \) (see Watson [W]),
\[ |a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q; \]
thus, by the pigeon-hole principle, there is at most one value of \( a_2q_1 \) in the interval \([rq, (r + 1)q]\) for any positive integer \( r \). \( a_2q_1 \) determines \( a_2 \) and \( q_1 \) to within \( X^\varepsilon \) possibilities (from the bound for the divisor function) and consequently also \( a_2q_1 \) determines \( a_1 \) and \( q_2 \) to within \( X^\varepsilon \) possibilities from (20).

Hence we got a lower bound for \( q_1q_2 \), remembering that in our shorthand \( Q_j \leq q_j \leq 2Q_j \):
\[ q_1q_2 = a_2q_1 \frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1} \]
and finally by (19)
\[ \mu(I) \ll X^{1+3\varepsilon}Z_1^{-1}Z_2^{-2}r^{-\frac{1}{4}}q^{-\frac{1}{4}}y^{\frac{1}{4}}. \]

Inside the interval \([rq, (r + 1)q]\), \( rq \leq |a_2q_1| \) and, in turn from (17), \( a_2 \ll q_2|\alpha| \), then
\[ rq \ll q_1q_2|\alpha| \ll \left( \frac{X^{1+\varepsilon}}{Z_1} \right)^2 \left( \frac{X^{1+\varepsilon}}{Z_2} \right)^4 y \ll yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4} \]
\[ \Rightarrow r \ll q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4}. \]

Now, we sum on every interval to get an upper bound for the measure of \( \mathcal{A} \):
\[ \mu(\mathcal{A}) \ll X^{1+3\varepsilon}Z_1^{-1}Z_2^{-2}q^{-\frac{1}{2}}y^{\frac{1}{2}} \sum_{1 \leq r \ll q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}}. \]

By partial summation on the generalized harmonic series,
\[ \sum_{1 \leq r \ll q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}} \ll (q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4})^{\frac{1}{2}} \]
then
\[ \mu(\mathcal{A}) \ll yX^{3+6\varepsilon}Z_1^{-2}Z_2^{-4}q^{-1} \ll yX^{3+6\varepsilon}Z_1^{-2}Z_2^{-4}X^{-\frac{1}{2}} \ll yX^{15+6\varepsilon}Z_1^{-2}Z_2^{-4}. \]
\[ \square \]
Using Lemma 7, we finally are able to get a bound for $\mathcal{J}(\eta, \omega, \delta)$:

\[
|\mathcal{J}(\eta, \omega, \delta)| \ll \int_{x} |S_1(\lambda_1 \alpha)| |S_2(\lambda_2 \alpha)| |S_2(\lambda_3 \alpha)| |S_k(\lambda_4 \alpha)| K_{\eta}(\alpha) d\alpha
\]

\[
\ll \left( \int_{x} |S_1(\lambda_1 \alpha)S_2(\lambda_2 \alpha)|^2 K_{\eta}(\alpha) d\alpha \right)^{\frac{1}{2}} \left( \int_{x} |S_2(\lambda_3 \alpha)|^4 K_{\eta}(\alpha) d\alpha \right)^{\frac{1}{4}}
\]

\[
\ll \left( \int_{x} |S_k(\lambda_4 \alpha)|^d K_{\eta}(\alpha) d\alpha \right)^{\frac{1}{d}}
\]

\[
\ll \left( \min \left( \eta^2, \frac{1}{y^2} \right) \right)^{\frac{1}{2}} \left( (Z_1Z_2)^2 \mu(d) \right)^{\frac{1}{2}} \left( \eta X \log^2 X \right)^{\frac{1}{2}} \left( \eta X^e \max(X^{\frac{1}{2}}, X^{\frac{1}{2}-1}) \right)^{\frac{1}{2}}
\]

\[
\ll \eta Z_2^{-1} X^{\frac{\omega}{2}+2\epsilon} X^{\frac{1}{2}+2\epsilon} \max(X^{\frac{1}{2}}, X^{\frac{1}{2}-\frac{1}{2}}) \ll \eta X^{\frac{\omega}{2}+2\epsilon} X^{\frac{1}{2}+2\epsilon} \max(X^{\frac{1}{2}}, X^{\frac{1}{2}-\frac{1}{2}})
\]

so $\eta = \infty \left( X^{-\frac{14-6k}{28}+\epsilon} \right)$ is the optimal choice.

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References

[BCP] Brüdern, J., Cook, R., Perelli, A., 1997. The values of binary linear forms at prime arguments. Proc of Sieve Methods, Exponential Sums and Their Application in Number Theory, Cambridge: Cambridge Univ Press, 87–100.

[BH] Baker, R., Harman, G., 1982. Diophantine approximation by prime numbers. Journal of the London Mathematical Society 2 (2), 201–215.

[C] Cook, R., 2001. The value of additive forms at prime arguments. Journal de Théorie des Nombres de Bordeaux 13 (1), 77–91.

[CF] Cook, R., Fox, A., 2001. The values of ternary quadratic forms at prime arguments. Mathematika 48 (1-2), 137–149.

[CH] Cook, R., Harman, G., 2006. The values of additive forms at prime arguments. Journal of Mathematics 36 (4).

[DH] Davenport, H., Heilbronn, H., 1946. On indefinite quadratic forms in five variables. Journal of the London Mathematical Society 1 (3), 185–193.

[GLZ] Gambini, A., Languasco, A., Zaccagnini, A., 2017. A diophantine approximation problem with two primes and one $k$-power of a prime. arXiv preprint arXiv:1706.00343

[GL] Ge, W., Li, W., 2016. One Diophantine inequality with unlike powers of prime variables. Journal of Inequalities and Applications (1), 1–8.

[G] Ghosh, A., 1981. The distribution of $ap^2$ modulo 1. Proceedings of the London Mathematical Society 3 (2), 252–269.
[H1] Harman, G., 1991. Diophantine approximation by prime numbers. Journal of the London Mathematical Society 2 (2), 218–226.

[H2] Harman, G., 2004. The values of ternary quadratic forms at prime arguments. Mathematika 51 (1-2), 83–96.

[HK] Harman, G., Kumchev, A.V., 2006. On sums of squares of primes. Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge Univ Press 140, 1–13.

[LS] Languasco, A., Settimi, V., 2012. On a Diophantine problem with one prime, two squares of primes and s powers of two. Acta Arithmetica 154 (4), 385–412.

[LZ1] Languasco, A., Zaccagnini, A., 2012. A Diophantine problem with a prime and three squares of primes. Journal of Number Theory 132 (12), 3016–3028.

[LZ2] Languasco, A., Zaccagnini, A., 2013. On a ternary Diophantine problem with mixed powers of primes. Acta Arithmetica 159 (4), 345–362.

[LZ3] Languasco, A., Zaccagnini, A., 2016. A Diophantine problem with prime variables. V. Kumar Murty, D. S. Ramana, and R. Thangadurai, editors, Highly Composite: Papers in Number Theory, Proceedings of the International Meeting on Number Theory, celebrating the 60th Birthday of Professor R. Balasubramanian (Allahabad, 2011), volume 23, pages 157–168. RMS-Lecture Notes Series.

[LSu] Liu, Z., Sun, H., 2013. Diophantine approximation with one prime and three squares of primes. The Ramanujan Journal 30 (3), 327–340.

[Ma] Matomäki, K., 2010. Diophantine approximation by primes. Glasgow Mathematical Journal 52 (01), 87–106.

[Mu] Mu, Q., 2016. Diophantine approximation with four squares and one k-th power of primes. The Ramanujan Journal 39 (3), 481–496.

[R] Rieger, G., 1968. Über die Summe aus einem Quadrat und einem Primzahlquadrat. J. reine angew. Math. 231, 89–100

[RS] Robert, O., Sargos, P., 2006. Three-dimensional exponential sums with monomials. Journal für die reine und angewandte Mathematik (Crelles Journal) 591, 1–20.

[V1] Vaughan, R., 1974a. Diophantine approximation by prime numbers, I. Proceedings of the London Mathematical Society 3 (2), 373–384.

[V2] Vaughan, R., 1974b. Diophantine approximation by prime numbers, II. Proceedings of the London Mathematical Society 3 (3), 385–401.

[V3] Vaughan, R., 1997. The Hardy-Littlewood method, 2nd ed. Vol. 125. Cambridge University Press.

[WY] Wang Y., Yao W., 2017. Diophantine approximation with one prime and three squares of primes. Journal of Number Theory.

[W] Watson, G., 1953. On indefinite quadratic forms in five variables. Proceedings of the London Mathematical Society 3 (1), 170–181.