The affine Hecke category is a monoidal colimit

James Tao

jamestao@mit.edu

February 24, 2021
Overview

Main Idea

A monoidal category $\mathcal{C}$ with a stratification indexed by a Coxeter group $W$ can often be expressed as the monoidal colimit of subcategories $\mathcal{C}_J \subset \mathcal{C}$ indexed by finite type standard subgroups $W_J \subset W$.

This talk will have two parts:

1. Colimit theorems for $\mathcal{C} = \mathcal{D}(\mathcal{L}G), \mathcal{D}(I \backslash \mathcal{L}G/I)$. The category of words and bistratified descent.  
   *(Joint with Roman Travkin.)*

2. An application-in-progress: constructing $\mathcal{H}^{\text{aff}} \rightarrow \mathcal{H}^{\text{fin}}$ in type $A$. Deformed affine Hecke categories.  
   *(Joint with Kostya Tolmachov.)*
Two colimit theorems

\[ G \] – algebraic group, semisimple and simply-connected.
\[ W_f \] – affine Weyl group of \( \mathcal{L}G \).
\[ I \] – set of affine simple reflections.

For \( J \subset I \), let \( P_J \subset \mathcal{L}G \) be the standard parahoric of type \( J \).

**Theorem (monoidal colimits)**

(i) \[ \mathcal{D}(\mathcal{L}G) \simeq \colim_{J \subset I} \mathcal{D}(P_J) \]

(ii) \[ \mathcal{D}(I \backslash \mathcal{L}G / I) \simeq \colim_{J \subset I} \mathcal{D}(I \backslash P_J / I) \]

**Remarks.** To remove the ‘semisimple and simply-connected’ hypothesis, change the colimit indexing diagram to Varshavsky’s ‘category of parahorics.’

Analogues for monodromic Hecke categories, Kac–Moody groups, …
## Motivation: generators and relations (part 1)

| Monoidal object               | Presentation          | Colimit thm.                                      | Cat. level |
|-------------------------------|-----------------------|---------------------------------------------------|------------|
| Weyl group                    | Simple reflections    | $A = \operatorname{colim} A_J$                    | sets       |
| Hecke algebra                 | 1-term relations      |                                                   |            |
|                               | 2-term relations      |                                                   |            |
|                               |                       | $|J| \leq 2$                                        |            |
| Weyl group (as discrete Picard grpd) | Simple reflections | $A \simeq \operatorname{colim} A_J$              | categories |
| Hecke category                | 1-term relations      |                                                   |            |
|                               | 2-term relations      |                                                   |            |
|                               | 3-term relations      |                                                   |            |
|                               |                       | $|J| \leq 3$                                        |            |
| Weyl group (as discrete top. group) |                       |                                                   |            |
| Hecke $\infty$-category       | ??                    | $A \simeq \operatorname{colim} A_J$              | $\infty$-categories |
|                               |                       | $J \text{ f.t.}$                                  |            |

Presentation of Weyl group (as discrete Picard grpd): Thm. 1.17, *Diagrammatics for Coxeter groups*, Elias–Williamson (2017)

Presentation of Hecke category: Thm. 1.11, *Tilting modules and the p-canonical basis*, Riche–Williamson (2018)
Motivation: generators and relations (part 2)

| Relation          | Geometric origin     | Name                              |
|-------------------|----------------------|-----------------------------------|
| 1-term relations  | codimension-1 faces  | quadratic relations “wall crossing”|
| 2-term relations  | codimension-2 faces  | braid relations                   |
| 3-term relations  | codimension-3 faces  | Zamolodchikov relations           |

Image source: https://www.math.umd.edu/~jda/kac/A3.gif
Proof: The category of words (part 1)

**Word** is the following category:

- Objects are sequences \((w_1, \ldots, w_n)\) in \(W_f\). Each \(w_i\) is ‘finite type.’
- A morphism \(\varphi : (w_1, \ldots, w_{n_1}) \to (w'_1, \ldots, w'_{n_2})\) is an ordered map
  \[
  \varphi^* : \{1, \ldots, n_1\} \to \{1, \ldots, n_2\}
  \]
  satisfying that
  \[
  w'_j \geq_{\text{Bruhat}} (\text{Demazure product of } w_i \text{ for } i \in \varphi^{-1}(j)),
  \]
  for all \(j \in \{1, \ldots, n_2\}\).
Proof: The category of words (part 2)

**Key Idea**

*Word* governs ‘convolution’ products of Schubert varieties.

For $w \in W_I$, let $P_w \subset L^G$ be the closure of the $w$ Bruhat cell.

- A word $w = (w_1, \ldots, w_n)$ encodes the variety
  \[
  \tilde{\mathcal{F}}_w := P_{w_1} \times \cdots \times P_{w_n}/I
  \]

- A morphism $\varphi : w \to w'$ encodes the conv. map $\tilde{\mathcal{F}}_w \to \tilde{\mathcal{F}}_{w'}$

- Example for $\tilde{A}_2$, with simple reflections $s, t, u$:

  \[
  \begin{array}{ccc}
  (s, t, s) & \downarrow & (g_1, g_2, g_3) \\
  \downarrow & \downarrow & \\
  (s, 1, sts) & P_s \times I \times P_{sts}/I & (g_1, 1, g_2g_3)
  \end{array}
  \]
Proof: Monoidal colimits are amalgamated products

\[
\text{colim}_{J \subset I} \mathcal{D}(P_J) \simeq \text{colim}_{(J_1, \ldots, J_n)} \mathcal{D}(P_{J_1}) \otimes \cdots \otimes \mathcal{D}(P_{J_n})
\]

\[
\simeq \text{colim}_{(w_1, \ldots, w_n) \in \text{Word}} \mathcal{D}(P_{w_1}) \otimes \cdots \otimes \mathcal{D}(P_{w_n})
\]

\[
\simeq \text{colim}_{(w_1, \ldots, w_n) \in \text{Word}} \mathcal{D}
(\mathbb{P}_{w_1} \times \cdots \times \mathbb{P}_{w_n})
\]

\[
\text{colim}_{J \subset I} \mathcal{D}(I \setminus P_J / I) \simeq \text{colim}_{(w_1, \ldots, w_n) \in \text{Word}} \mathcal{D}'(I \setminus \mathbb{P}_{w_1} \times \cdots \times \mathbb{P}_{w_n} / I)
\]

\(\mathcal{D}'(-) := \mathcal{D}\text{-modules constant on each (twisted) product of cells.}\)
Proof: The category of words (part 3)

Let \( \varphi : w \to w' \) be a map.

- \( \varphi \) is a **strict embedding** if \( \varphi_* \) is a bijection and \( \varphi \neq \text{(identity)} \).
- \( \varphi \) is **birational** if and only if \( \overline{\Fl}_w \to \overline{\Fl}_{w'} \) is birational.

Let \( y \in W_I \). We define full subcategories \( \text{Word}_{\preceq y}, \text{Word}_{\prec y} \).

- \( w \in \text{Word}_{\preceq y} \) if and only if \( \overline{\Fl}_w \to \Fl := \mathcal{L}G/I \) factors through \( \Fl_y \) (the \( y \) Schubert variety).
- \( w \) is **\( y \)-relevant** if this map is birational onto \( \Fl_y \).
- \( w \in \text{Word}_{\prec y} \) if and only if this map factors through \( \partial \Fl_y \).

(\( \partial \) means ‘boundary,’ i.e. complement of open cell.)
Proof: Bistratified descent

Let $F : \text{Word} \to \mathcal{E}$ be any functor. How to compute $\text{colim} F$?

**Theorem (bistratified descent)**

Assume that, for every birational map $w \to w'$, the following diagram is cocartesian:

$$
\begin{array}{ccc}
\text{colim} & F(v) & \to & F(w) \\
\downarrow & & \downarrow & \\
F(v') & \to & F(w')
\end{array}
$$

Then, for any $y$-relevant $w \in \text{Word}_{\leq y}$, the following diagram is cocartesian:

$$
\begin{array}{ccc}
\text{colim} & F(v) & \to & F(w) \\
\downarrow & & \downarrow & \\
\text{colim} F(v') & \to & \text{colim} F(w')
\end{array}
$$

**The conclusion is:** $\text{colim} F$ can be computed via a sequence of pushouts.
Proof: Applying bistratified descent

\[
\begin{align*}
\colim_{v \to w} F(v) & \to F(w) \\
\colim_{v' \to w'} F(v') & \to F(w') \\
\end{align*}
\]

The previous diagrams correspond to blow-up squares:

\[
\begin{align*}
\partial \tilde{\Fl}_w & \to \tilde{\Fl}_w \\
\partial \tilde{\Fl}_{w'} & \to \tilde{\Fl}_{w'} \\
\end{align*}
\]

Upshot. Bistratified descent can be applied when the sheaf theory satisfies descent w.r.t. blow-up squares. (E.g. $\mathcal{D}$-modules, $\ell$-adic sheaves)
How to apply the colimit theorem (part 1)

\( \mathcal{C} \) – any stable monoidal \( \infty \)-category. Assume \( I \) is irreducible.

How to construct a monoidal triangulated functor \( F : \mathcal{H}_{\text{aff}} \to \mathcal{C} \)?

1. For \( i \in I \), choose \( F_i \in \text{Fun}^{\text{mon}}(\mathcal{H}_{I \setminus \{i\}}, \mathcal{C}) \)

2. For \( i, j \in I \), choose \( F_i|_{\mathcal{H}_{I \setminus \{i,j\}}} \xrightarrow{\sim} F_j|_{\mathcal{H}_{I \setminus \{i,j\}}} \) in \( \text{Fun}^{\text{mon}}(\mathcal{H}_{I \setminus \{i,j\}}, \mathcal{C}) \)

3. For \( i, j, k \in I \), ensure commutativity in \( \text{Fun}^{\text{mon}}(\mathcal{H}_{I \setminus \{i,j,k\}}, \mathcal{C}) \):

\[
\begin{array}{ccc}
F_i|_{\mathcal{H}_{I \setminus \{i,j,k\}}} & \xrightarrow{\sigma_{ik}|_{\mathcal{H}_{I \setminus \{i,j,k\}}} } & F_k|_{\mathcal{H}_{I \setminus \{i,j,k\}}} \\
\sigma_{ij}|_{\mathcal{H}_{I \setminus \{i,j,k\}}} & \xrightarrow{\sigma_{jk}|_{\mathcal{H}_{I \setminus \{i,j,k\}}} } & \quad \\
F_j|_{\mathcal{H}_{I \setminus \{i,j,k\}}} & & \quad \\
\end{array}
\]

4. (higher associativity constraints)
Choose a $t$-structure on $\mathcal{C}$. Restrict attention to functors $F : \mathcal{H}_{\text{aff}} \to \mathcal{C}$ which send all tilting generators into $\mathcal{C}^{\heartsuit}$.

To construct these functors, one only needs the “1-categorical colimit theorem,” which follows from the Elias–Williamson presentation.

This is because of ‘truncatedness’:

$$\text{Hom}^i_{\mathcal{C}}(F(T_1), F(T_2)) = 0 \quad \text{for } i < 0.$$ 

This corresponds to vanishing of some $\pi_1, \pi_2, \ldots$, because we are using cohomological indexing.

**To construct more general functors, one needs the “$\infty$-categorical colimit theorem.”**
Problem: In a general $\infty$-category, it’s hard to check Step 4. In an ordinary category, however, Step 4 automatically follows.

Key Idea

Choose a $t$-structure on $C$. The subcategory of $\text{Fun}^{\text{mon}}(\mathcal{H}_{I\setminus\{i\}}, C)$ which sends all tilting generators into $C^\Diamond$ is an ordinary category.

This trick may work even when there is no $t$-structure on $C$ such that the desired functor $\mathcal{H}_{\text{aff}} \to C$ sends all tilting generators into $C^\Diamond$.

Indeed, we may now use a different $t$-structure for each $i \in I$. 
From now on, \( I = \tilde{A}_{n-1} \), with vertices \( \{0, \ldots, n-1\} \), and \( G := \text{GL}_n \).

**Question**

Is there a (monoidal) functor \( \mathcal{H}_{\text{aff}} \rightarrow \mathcal{H}_{I \setminus \{0\}} \) which is compatible with the following map of braid groups \( \mathbb{B}_{\text{aff}} \rightarrow \mathbb{B}_{\text{fin}} \)?

Bezrukavnikov’s equivalence states that \( \mathcal{H}_{\text{aff}} \simeq D^b(\text{Coh}^G(\text{St})) \). Tolmachov’s thesis constructs a functor \( \text{Perf}^G(\text{St}) \rightarrow \mathcal{H}_{I \setminus \{0\}} \).

Image source: Tolmachov’s thesis
Tolmachov’s thesis: $\text{Perf}^G(\text{St}) \rightarrow \mathcal{H}_{\text{fin}}$ in type A (part 2)

**Key Idea for Bezrukavnikov’s equivalence**

$\text{Perf}^G(\text{St})$ is generated by the vector bundles $\mathcal{O}(\lambda, \mu) \otimes \mathbb{C} \ V$ and maps (highest weight arrows, monodromy endomorphisms) subject to some relations. The object $\mathcal{O}(\lambda, \mu) \otimes \mathcal{V}$ corresponds to $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu \in \mathcal{H}_{\text{aff}}$.

**Question 1:** Where should $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu$ map to?

Weight decomposition $V = \bigoplus \lambda V_\lambda$.

Recall that $\mathcal{Z}_V$ is an iterated extension, in which $J_\lambda$ occurs $\dim V_\lambda$ times.

$\mathcal{B}_{\text{aff}} \rightarrow \mathcal{B}_{\text{fin}}$ forces $J_\lambda \mapsto \mathbb{L}_\lambda$ (Jucys–Murphy sheaves).

Anything in $\mathcal{B}_{\text{fin}}$ convolved with $\Xi$ yields $\Xi$.

**Answer:** $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu \mapsto \mathbb{L}_\lambda \star (V \otimes \mathbb{C} \Xi) \star \mathbb{L}_\mu$. 
Tolmachov’s thesis: $\text{Perf}^G(\text{St}) \to \mathcal{H}_{\text{fin}}$ in type $A$ (part 3)

To get maps and relations, need to ‘take apart’ $V \otimes_{\mathbb{C}} \Xi$. Thus, we ask:

**Question 2:** Under $\mathcal{H}_{\text{aff}} \to \mathcal{H}_{\text{fin}}$, where should $\mathcal{Z}_V$ map to?

**Answer:** For $V_{\text{std}}$, it’s an “averaged” parabolic Springer sheaf.

Let $P$ be the parabolic which fixes a line. $G\backslash (G \times U_P) \simeq P \backslash U_P$

Parabolic Springer sheaf: $\text{Spr}_P := \pi_*\mathbb{C}u_P^{\mathbb{P}}[2 \dim U_P]$

Pull-push $\text{Spr}_P$, then force it to be $T$-monodromic.
Tolmachov’s thesis: $\text{Perf}^G(St) \to \mathcal{H}_{\text{fin}}$ in type $A$ (part 4)

What about $\wedge^k V_{\text{std}}$?

\[ \lambda_k \quad \text{– partition of } n \text{ given by the ‘hook’ } (k, 1, \ldots, 1). \]

\[ \text{IC}_{\lambda_k} \quad \text{– IC-complex of the unipotent orbit in } G = \text{GL}_n \text{ given by } \lambda_k. \]

Main Theorem of Tolmachov’s thesis

(a) $\wedge^k \text{Spr}_P \simeq \text{IC}_{\lambda_k} \oplus \text{IC}_{\lambda_{k+1}}$ for $1 \leq k \leq n - 1$.

(b) $\wedge^n \text{Spr}_P \simeq \text{IC}_{\lambda_n}$.

(c) $\text{IC}_{\lambda_n}$ becomes invertible after some averaging.

(d) $\wedge^{n+1} \text{Spr}_P = 0$.

Finally, bootstrap from $\wedge^k V_{\text{std}}$ to all $\text{GL}_n$-reps via a Tannakian argument.
To apply the colimit theorem, we need to do Steps 1, 2, 3 from before.

1. $F_i : \mathcal{H}_{I \setminus \{i\}} \to \mathcal{H}_{I \setminus \{0\}}$ is conjugation by a specific element $b_i \in \mathbb{B}_{aff}$.

2. $\sigma_{ij} : F_i|_{\mathcal{H}_{I \setminus \{i,j\}}} \sim F_j|_{\mathcal{H}_{I \setminus \{i,j\}}}$ expresses that $b_j^{-1}b_i$ centralizes $\mathcal{H}_{I \setminus \{i,j\}}$.
   (Use centrality of $\Delta^2_{w_0}$, Prop. 5.4, Monodromic model for Khovanov–Rozansky homology, Bezrukavnikov–Tolmachov)
3. Want to check \( \sigma_{jk} \circ \sigma_{ik} \simeq \sigma_{ik} \) in \( \text{Fun}^{\text{mon}}(\mathcal{H}_{I \setminus \{i,j,k\}}, \mathbb{C}) \).

Right now, we do not know if this is true.

What if Step 3 fails?

That is, what if \( \sigma_{ik}^{-1} \circ \sigma_{jk} \circ \sigma_{ik} \) is not the id. natural transformation?

**Key Idea (deformed affine Hecke category)**

One may define a new category \( \mathcal{H}_{aff}^{(\alpha)} \) by deforming \( \mathcal{H}_{aff} \) using the ‘cocycle’ \( \sigma_{ik}^{-1} \circ \sigma_{jk} \circ \sigma_{ik} \). By construction, there will be a monoidal functor

\[
\mathcal{H}_{aff}^{(\alpha)} \rightarrow \mathcal{H}_{I \setminus \{0\}}.
\]
Interpret the colimit diagram in $\mathcal{H}_{\text{aff}} \simeq \operatorname{colim}_{J \subsetneq I} \mathcal{H}_J$ as follows:

1. For $i \in I$, write down the category $\mathcal{H}_{I \setminus \{i\}}$.

2. For $i, j \in I$, write down the identity functor
   
   $$
   \begin{array}{ccc}
   \mathcal{H}_{I \setminus \{i,j\}} & \longrightarrow & \mathcal{H}_{I \setminus \{i\}} \\
   \downarrow \text{Id} & & \downarrow \text{Id} \\
   \mathcal{H}_{I \setminus \{i,j\}} & \longrightarrow & \mathcal{H}_{I \setminus \{j\}}
   \end{array}
   $$

3. For $i, j, k \in I$, write down the trivial commutativity natural iso
   
   $$
   \begin{array}{ccc}
   \mathcal{H}_{I \setminus \{i,j,k\}} & \longrightarrow & \mathcal{H}_{I \setminus \{i\}} \\
   \downarrow \text{Id} & & \downarrow \text{Id} \\
   \mathcal{H}_{I \setminus \{i,j,k\}} & \longrightarrow & \mathcal{H}_{I \setminus \{j\}} \\
   \downarrow \text{Id} & & \downarrow \text{Id} \\
   \mathcal{H}_{I \setminus \{i,j,k\}} & \longrightarrow & \mathcal{H}_{I \setminus \{k\}}
   \end{array}
   $$
Define $\mathcal{H}^{(\alpha)}_{\text{aff}} := \lim_{J \subseteq I} \mathcal{H}_J$ using a modified colimit diagram:

Steps 1 and 2 are the same as before.

3'. For $i, j, k \in I$, write down a nontrivial commutativity natural iso

\[
\begin{array}{ccc}
\mathcal{H}_{I \setminus \{i, j, k\}} & \xrightarrow{\text{Id}} & \mathcal{H}_{I \setminus \{i, j, k\}} \\
\downarrow & & \downarrow \\
\mathcal{H}_{I \setminus \{i, j, k\}} & \xrightarrow{\text{Id}} & \mathcal{H}_{I \setminus \{i, j, k\}}
\end{array}
\]

4. For $i, j, k, \ell \in I$, the following tetrahedron must commute:

\[
\begin{array}{ccc}
\mathcal{H}_{I \setminus \{i, j, k, \ell\}} & \xrightarrow{\text{Id}} & \mathcal{H}_{I \setminus \{i, j, k, \ell\}} \\
\downarrow & & \downarrow \\
\mathcal{H}_{I \setminus \{i, j, k, \ell\}} & \xrightarrow{\text{Id}} & \mathcal{H}_{I \setminus \{i, j, k, \ell\}}
\end{array}
\]

All six maps are $\text{Id}$, all four triangles come from step 3'.
Key Idea (deformed affine Hecke category)

Use the natural iso’s $\sigma^{-1}_{ik} \circ \sigma_{jk} \circ \sigma_{ik}$ to define natural iso’s for Step 3’. The resulting category $\mathcal{H}^{(\alpha)}_{\text{aff}}$ admits a monoidal functor to $\mathcal{H}_{I\setminus\{0\}}$.

We expect that $\mathcal{H}^{(\alpha)}_{\text{aff}}$ can be (noncanonically) obtained from $\mathcal{H}_{\text{aff}}$ by altering the 3-term (and higher) associativity constraints for the monoidal structure.

Hence, the centers of $\mathcal{H}^{(\alpha)}_{\text{aff}}$ and $\mathcal{H}_{\text{aff}}$ are equivalent (as categories), but their (braided) monoidal structures are different.

Next steps. Describe the categories $\mathcal{H}^{(\alpha)}_{\text{aff}}$ more explicitly.
Investigate the center of $\mathcal{H}^{(\alpha)}_{\text{aff}}$ and compare with the center of $\mathcal{H}_{I\setminus\{0\}}$. 
References

B. Elias and G. Williamson, *Soergel calculus*, Represent. Theory **20** (2016), p. 295-374.

B. Elias and G. Williamson, *Diagrammatics for Coxeter groups and their braid groups*, Quantum Topology **8**(3) (2017), pp. 413–457.

S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, Asterisque, Société Mathématique de France, 2018, 397. hal-01249796v3

J. Tao and R. Travkin, *The affine Hecke category is a monoidal colimit*, preprint arXiv:2009.10998 (New version will be available soon.)

K. Tolmachov, *Towards a functor between affine and finite Hecke categories in type A*, Ph.D. thesis (2018), available at http://tolmak.khtos.com/thesis_tolmachov.pdf

R. Bezrukavnikov and K. Tolmachov, *Monodromic model for Khovanov–Rozansky homology*, preprint arXiv:2008.11379
Appendix: How to construct $\mathbb{B}_{\text{aff}} \to \mathcal{H}_{\text{aff}}$ (part 1)

"Reduced lift" presentation of braid monoid $\mathbb{B}_{\text{aff}}^+$:

$$
\mathbb{B}_{\text{aff}}^+ \simeq \left\langle t_w \text{ for } w \in W_{\text{aff}} \middle| \begin{array}{l}
t_{w_1}t_{w_2} = t_{w_1w_2} \text{ whenever } \\
\ell(w_1) + \ell(w_2) = \ell(w_1w_2)
\end{array} \right\rangle
$$

Valid even when $\mathbb{B}_{\text{aff}}^+$ is viewed as a discrete topological monoid!

Finite type: Thm. 1.7, *Action du groupe des tresses sur une catégorie*, Deligne (1997)
Arbitrary type: Generalize Deligne's proof, or apply Thm. 5.1, *Configuration spaces of labeled particles*, Dobrinskaya (2006)

Define the monoidal functor $\mathbb{B}_{\text{aff}}^+ \to \mathcal{H}_{\text{aff}}$ via $t_w \mapsto (j_w)!C = \Delta_w$. 

$\mathcal{H}_{\text{aff}}$ is a monoidal colimit
Thm. 5.2 of Dobrinskaya (2006)

\[ I \rightarrow \text{any Coxeter–Dynkin diagram} \]

The homotopy groupification of \( B_I^+ \) is the (discrete) braid group \( B_I \) if and only if \( K(\pi, 1) \) conjecture holds for \( B_I \).

Paolini and Salvetti (2020) proved the \( K(\pi, 1) \) conjecture for affine \( I \).

Universal property of homotopy groupification:

\[ (j_w)! C = \Delta_w \text{ invertible} \implies \text{Get a monoidal functor } B_{\text{aff}} \rightarrow H_{\text{aff}}. \]