Analyticity of off-shell Green's functions in superstring field theory

Ratul Mahanta
Harish-Chandra Research Institute, Allahabad, India

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Collaborator: Ritabrata Bhattacharyya (CMI)

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Outline

1. Green’s functions in Local QFTs some old results

2. de Lacroix-Erbin-Sen (LES) result in closed SFT

3. Generalization of the LES result

4. Discussions/remaining questions
Green's functions in Local QFTs

1. Commutator of local operators vanishes outside lightcone,
2. Existence of a complete system of (physical) states with +ve energy

Constraints on position space correlators

Primitive analyticity of off-shell momentum space Green’s functions

(Off-shell momentum space Green’s functions are defined as the Fourier Transforms of position space correlators)
Analyticity of $G(p_1, \ldots, p_n)$ on primitive domain in complex external momenta variables

Define $P_I = \sum_{a \in I} p_a$, $I \subset \{1, \ldots, n\} \setminus \emptyset$

A collection of points $(p_1, \ldots, p_n)$ satisfying

C1. $p_1 + \cdots + p_n = 0$

C2. If $\text{Im } P_I \neq 0$ then $\text{Im } P_I$ must be timelike,
    If $\text{Im } P_I = 0$ then $-P_I^2$ must be below the threshold mass for producing multi-particle states

— Primitive domain, PD $\subset \mathbb{C}^{(n-1)D}$
For S-matrix, need to evaluate \( G(p_1, \ldots, p_n) \mid -p_a^2 = m_a^2 \) connected, amputated

C2. implies that within PD one cannot satisfy \( -p_a^2 = m_a^2 \)

PD \( \not\Rightarrow \) on-shell external momenta. We are interested in scattering amplitudes. Rescue?

\( G(p_1, \ldots, p_n) \) is a function of several complex variables
In several complex variables, sometimes the "shape" of a domain forces holomorphic extension of it to a larger domain — property of the domain, irrespective of the functions that are analytic on it

Holomorphic extension of Primitive domain includes on-shell external momenta \(\implies\) We can compute amplitudes

Properties of a subregion of PD \(\implies\) crossing symmetry of amplitudes for 2 \(\to\) 2 scattering

Bros, Epstein, Glaser (1965)
LES result in closed SFT

- Action directly written in momentum space. Has non-local vertices. No position space construction known.

- Work with perturbative expansion of $G(p_1, \ldots, p_n)$.
  Consider each Feynman diagram $F(p_1, \ldots, p_n)$.

- $F(p)$ is analytic on the domain defined by

  $\text{C1.}, \text{ C2.}, \text{ C3.}: \text{ Each } \text{Im } p_a \text{ lies on the two plane } p^0 - p^1$, if \( \text{Im } p_a \neq 0 \).

  \cite{de Lacroix, Erbin, Sen (2018)}

- Primitive domain $\supset$ LES domain $\longrightarrow$ crossing symmetry
Complex matrix \( \tilde{\Lambda} \), s.t. \( \tilde{\Lambda}^T \eta \tilde{\Lambda} = \eta = \) Minkowski metric in \( \mathbb{R}^D \)

Define its action on \( p \) as \( \tilde{\Lambda} p \equiv (\tilde{\Lambda} p_1, \ldots, \tilde{\Lambda} p_n) \)

Two corollaries from the LES paper

**Cor1.** \( F(p) \) remains analytic at \( \tilde{\Lambda} p \), if \( p \in \) LES domain

**Cor2.** At each \( p \in \) LES domain, we can allow a small open neighbourhood where \( F(p) \) remains analytic
Generalization of the LES result

Are \( F(p) \) analytic on the full primitive domain?

to answer this our starting point will be:
\( F(p) \) is analytic on extended LES domain (LESD+Cor1.+Cor2.)

We have not used any further input from closed SFT

Use fact — \( F(p) \) is a function of several complex variables

Show — the "shape" of the (LESD+Cor1.+Cor2.) forces a holomorphic extension to a larger subset of primitive domain
This forced holomorphic extension yields the full PD for 2-, 3- and 4-point functions.

Higher-point functions: How much of PD is recovered by this?

In general difficult to determine analytically.

5-point function: we analytically show at least a recovery of a large part of PD is guaranteed by our extension.
Primitive domain $\bigcup$ union of a family of mutually disjoint tube domains $\mathcal{T}_\lambda$

Indexed family some index

Primitive tube, $\mathcal{T}_\lambda = \mathbb{R}^{(n-1)D} + iC_\lambda$ ; $C_\lambda :$ cone in $\mathbb{R}^{(n-1)D}$

typically specifying various $\text{Im } P_I$ in specific lightcone (forward/backward) determines a $C_\lambda$

For 3-point function, # primitive tubes = 6
For 4-point function, # primitive tubes = 32
For 5-point function, # primitive tubes = 370 etc.

Araki, Burgoyne ‘60; Bros, Epstein, Glaser ‘64; Lassalle ‘74; Bros, Lassalle ‘75
Identify — an open tube from (LESD+Cor1.+Cor2.) inside each primitive tube

Prove — this LES tube is (path-)connected

— also non-convex for $n \geq 3$, convex for $n = 2$

Apply — Bochner’s tube theorem on the LES tube:

Any open connected tube $\mathbb{R}^m + iA$ has a holomorphic extension to the domain $\mathbb{R}^m + i\text{Ch}(A)$

$\text{Ch}(A)$: smallest convex set containing $A$, called the convex hull of $A$
Certainly non-trivial extension for \( n \geq 3 \)

For 3-point function, \# primitive tubes = 6
For 4-point function, \# primitive tubes = 32
For 5-point function, \# primitive tubes = 370

For remaining 20 LES tubes, analytically determining their convex hulls appears to be difficult

Similar feature occurs for higher-point functions as well
Certain known sign-valued map $\lambda(I)$, $I \subsetneq \{1, \ldots, n\} \setminus \emptyset$

$\mathcal{T}_\lambda$ determined by $\lambda(I) \cdot \text{Im } P_I \in$ Forward lightcone

Given any $p \in \mathcal{T}_\lambda$ can be written as a convex combination of $m$ points taken from the LES tube that resides in $\mathcal{T}_\lambda$?

$m \leq (n - 1)D + 1$ (due to a theorem by Carathéodory)

For the cases mentioned in previous slide, we explicitly give points whose convex combination produce $p$. 
Certain known sign-valued map $\lambda(I)$, $I \subseteq \{1, \ldots, n\} \setminus \emptyset$

$\mathcal{T}_\lambda$ determined by $\lambda(I) \cdot \text{Im } P_I \in \text{Forward lightcone}$

Given any $p \in \mathcal{T}_\lambda$ can be written as a convex combination of $m$ points taken from the LES tube that resides in $\mathcal{T}_\lambda$?

$m \leq (n - 1)D + 1$ (due to a theorem by Carathéodory)

In case of the 5-point function, for the remaining 20 tubes can we solve for points numerically? Here $n = 5$
Discussions/remaining questions

LES results are valid for ANY Feynman diagram that has NO mass-less internal propagators. Hence ours too.

Our analysis is perturbative in contrast to available analysis in Local QFT. Potential singularities from non-perturbative effect?

Green’s functions in Local QFTs and in Closed SFT are analytic on PD. Their large momentum behaviours are likely to differ.

Are higher-point functions in closed SFT analytic on full PD?

Thank you