A conjecture concerning the $q$-Onsager algebra

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Abstract

The $q$-Onsager algebra $\mathcal{O}_q$ is defined by two generators $W_0, W_1$ and two relations called the $q$-Dolan/Grady relations. Recently Baseilhac and Kolb obtained a PBW basis for $\mathcal{O}_q$ with elements denoted

$$\{B_{n\delta + \alpha_0}\}_{n=0}^\infty, \quad \{B_{n\delta + \alpha_1}\}_{n=0}^\infty, \quad \{B_n\}_{n=1}^\infty.$$ 

In their recent study of a current algebra $\mathcal{A}_q$, Baseilhac and Belliard conjecture that there exist elements

$$\{W_{-k}\}_{k=0}^\infty, \quad \{W_{k+1}\}_{k=0}^\infty, \quad \{G_{k+1}\}_{k=0}^\infty, \quad \{\tilde{G}_{k+1}\}_{k=0}^\infty$$

in $\mathcal{O}_q$ that satisfy the defining relations for $\mathcal{A}_q$. In order to establish this conjecture, it is desirable to know how the elements on the second displayed line above are related to the elements on the first displayed line above. In the present paper, we conjecture the precise relationship and give some supporting evidence. This evidence consists of some computer checks on SageMath due to Travis Scrimshaw, and a proof of our conjecture for a homomorphic image of $\mathcal{O}_q$ called the universal Askey-Wilson algebra.

Keywords. $q$-Onsager algebra; $q$-Dolan/Grady relations; PBW basis; tridiagonal pair.

2020 Mathematics Subject Classification. Primary: 17B37. Secondary: 05E14, 81R50.

1 Introduction

We will be discussing the $q$-Onsager algebra $\mathcal{O}_q$ [3,25]. This infinite-dimensional associative algebra is defined by two generators $W_0, W_1$ and two relations called the $q$-Dolan/Grady relations; see Definition 3.1 below. One can view $\mathcal{O}_q$ as a $q$-analog of the universal enveloping algebra of the Onsager Lie algebra $\mathcal{O}$ [14,21,22].

The algebra $\mathcal{O}_q$ originated in algebraic combinatorics [25]. There is a family of algebras called tridiagonal algebras [25, Definition 3.9] that arise in the study of association schemes [24, Lemma 5.4] and tridiagonal pairs [15, Theorem 10.1], [25, Theorem 3.10]. The algebra $\mathcal{O}_q$ is the “most general” example of a tridiagonal algebra [17, Section 1.2]. A finite-dimensional irreducible $\mathcal{O}_q$-module is essentially the same thing as a tridiagonal pair of $q$-Racah type [25].
Theorem 3.10]. These tridiagonal pairs are classified up to isomorphism in [16, Theorem 3.3]. To our knowledge the \(q\)-Dolan/Grady relations first appeared in [24, Lemma 5.4].

The algebra \( \mathcal{O}_q \) has applications outside combinatorics. For instance, \( \mathcal{O}_q \) is used to study boundary integrable systems [2–6, 8–10, 12]. The algebra \( \mathcal{O}_q \) can be realized as a left or right coideal subalgebra of the quantized enveloping algebra \( U_q(\hat{s}l_2) \); see [4, 5, 18]. The algebra \( \mathcal{O}_q \) is the simplest example of a quantum symmetric pair coideal subalgebra of affine type [18, Example 7.6]. A Drinfeld type presentation of \( \mathcal{O}_q \) is obtained in [19], and this is used in [20] to realize \( \mathcal{O}_q \) as an \( \iota \) Hall algebra of the projective line. There is an injective algebra homomorphism from \( \mathcal{O}_q \) into the algebra \( \square_q \) [27, Proposition 5.6], and a noninjective algebra homomorphism from \( \mathcal{O}_q \) into the universal Askey-Wilson algebra \( \Delta_q \) [26, Sections 9, 10].

In [11, Theorem 4.5], Baseilhac and Kolb obtain a Poincaré-Birkhoff-Witt (or PBW) basis for \( \mathcal{O}_q \). They obtain this PBW basis by using a method of Damiani [13] along with two automorphisms of \( \mathcal{O}_q \) that are roughly analogous to the Lusztig automorphisms of \( U_q(\hat{s}l_2) \).

The PBW basis elements are denoted
\[
\{ B_{n\delta + \alpha_0} \}_{n=0}^\infty, \quad \{ B_{n\delta + \alpha_1} \}_{n=0}^\infty, \quad \{ B_{n\delta} \}_{n=1}^\infty.
\]
(1)

In [8] Baseilhac and Koizumi introduce a current algebra \( A_q \) for \( \mathcal{O}_q \), in order to solve boundary integrable systems with hidden symmetries. In [12, Definition 3.1] Baseilhac and Shigechi give a presentation of \( A_q \) by generators and relations. The generators are denoted
\[
\{ W_{-k} \}_{k=0}^\infty, \quad \{ W_{k+1} \}_{k=0}^\infty, \quad \{ G_{k+1} \}_{k=0}^\infty, \quad \{ \tilde{G}_{k+1} \}_{k=0}^\infty
\]
and the relations are given in (20)–(30) below. One can view \( A_q \) as a \( q \)-analog of the universal enveloping algebra of \( \mathcal{O} \) [7, Definition 4.1, Theorem 2], so it is natural to ask how \( A_q \) is related to \( \mathcal{O}_q \). Baseilhac and Belliard investigate this issue in [6]; their results are summarized as follows. In [6, line (3.7)] they show that \( W_0, W_1 \) satisfy the \( q \)-Dolan/Grady relations. In [6, Section 3] they show that \( A_q \) is generated by \( W_0, W_1 \) together with the central elements \( \{ \Delta_n \}_{n=1}^\infty \) defined in [6, Lemma 2.1]. In [6, Section 3] they consider the quotient algebra of \( A_q \) obtained by sending \( \Delta_n \) to a scalar for all \( n \geq 1 \). The construction yields an algebra homomorphism \( \Psi \) from \( \mathcal{O}_q \) onto this quotient. In [6, Conjecture 2] Baseilhac and Belliard conjecture that \( \Psi \) is an isomorphism. If the conjecture is true then there exists an algebra homomorphism \( A_q \rightarrow \mathcal{O}_q \) that sends \( W_0 \mapsto W_0 \) and \( W_1 \mapsto W_1 \). In this case there exist elements
\[
\{ W_{-k} \}_{k=0}^\infty, \quad \{ W_{k+1} \}_{k=0}^\infty, \quad \{ G_{k+1} \}_{k=0}^\infty, \quad \{ \tilde{G}_{k+1} \}_{k=0}^\infty
\]
in \( \mathcal{O}_q \) that satisfy the relations (20)–(30). In order to make progress on the above conjecture, it is desirable to know how the elements (2) are related to the elements in (1). In the present paper, we conjecture the precise relationship and give some supporting evidence. Our conjecture statement is Conjecture 6.1. Our supporting evidence consists of some computer checks on SageMath (see [23]) due to Travis Scrimshaw, and a proof of the conjecture at the level of the algebra \( \Delta_q \) mentioned above.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the algebra \( \mathcal{O}_q \), and describe the PBW basis due to Baseilhac and Kolb. In Sections
4. 5 we develop some results about generating functions that will be used in Conjecture 6.1. In Section 6 we state Conjecture 6.1 and explain its meaning. In Section 7 we present our evidence supporting Conjecture 6.1. In Section 8 we give some comments. In Appendices A, B we display in detail some equations from the main body of the paper.

2 Preliminaries

Throughout the paper, the following notational conventions are in effect. Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and integers \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \). Let \( \mathbb{F} \) denote a field. Every vector space mentioned is over \( \mathbb{F} \). Every algebra mentioned is associative, over \( \mathbb{F} \), and has a multiplicative identity.

Definition 2.1. (See [13, p. 299].) Let \( \mathcal{A} \) denote an algebra. A Poincaré-Birkhoff-Witt (or PBW) basis for \( \mathcal{A} \) consists of a subset \( \Omega \subseteq \mathcal{A} \) and a linear order \( < \) on \( \Omega \) such that the following is a basis for the vector space \( \mathcal{A} \):

\[
\prod_{i=1}^{n} a_i \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n.
\]

We interpret the empty product as the multiplicative identity in \( \mathcal{A} \).

We will be discussing generating functions. Let \( \mathcal{A} \) denote an algebra and let \( t \) denote an indeterminate. For a sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements in \( \mathcal{A} \), the corresponding generating function is

\[
a(t) = \sum_{n \in \mathbb{N}} a_n t^n.
\]

The above sum is formal; issues of convergence are not considered. We call \( a(t) \) the generating function over \( \mathcal{A} \) with coefficients \( \{a_n\}_{n \in \mathbb{N}} \). For generating functions \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) and \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) over \( \mathcal{A} \), their product \( a(t)b(t) \) is the generating function \( \sum_{n \in \mathbb{N}} c_n t^n \) such that \( c_n = \sum_{i=0}^{n} a_i b_{n-i} \) for \( n \in \mathbb{N} \). The set of generating functions over \( \mathcal{A} \) forms an algebra.

Let \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) denote a generating function over \( \mathcal{A} \). We say that \( a(t) \) is normalized whenever \( a_0 = 1 \). If \( 0 \neq a_0 \in \mathbb{F} \) then define

\[
a(t)^\vee = a_0^{-1} a(t),
\]

and note that \( a(t)^\vee \) is normalized.

Fix a nonzero \( q \in \mathbb{F} \) that is not a root of unity. Recall the notation

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}.
\]

3 The \( q \)-Onsager algebra \( \mathcal{O}_q \)

In this section we recall the \( q \)-Onsager algebra \( \mathcal{O}_q \). For elements \( X, Y \) in any algebra, define their commutator and \( q \)-commutator by

\[
[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1} YX.
\]
Note that
\[ [X, [X, [X, Y]_q]]_{q^{-1}} = X^3Y - [3]_q X^2YY + [3]_q XYX^2 - YX^3. \]

**Definition 3.1.** (See [3, Section 2], [25, Definition 3.9].) Define the algebra \( O_q \) by generators \( W_0, W_1 \) and relations

\begin{align*}
[W_0, [W_0, W_0, W_1]_{q^{-1}}] &= (q^2 - q^{-2})^2[W_1, W_0], \tag{4} \\
[W_1, [W_1, W_0, W_0]_{q^{-1}}] &= (q^2 - q^{-2})^2[W_0, W_1]. \tag{5}
\end{align*}

We call \( O_q \) the \( q \)-Onsager algebra. The relations (4), (5) are called the \( q \)-Dolan/Grady relations.

**Remark 3.2.** In [11] Baseilhac and Kolb define the \( q \)-Onsager algebra in a slightly more general way that involves two scalar parameters \( c, q \). Our \( O_q \) is their \( q \)-Onsager algebra with \( c = q^{-1}(q^{-1})^2 \).

In [11], Baseilhac and Kolb obtain a PBW basis for \( O_q \) that involves some elements

\[ \{B_{n\delta + \alpha_0}\}_{n=0}^\infty, \quad \{B_{n\delta + \alpha_1}\}_{n=0}^\infty, \quad \{B_{n\delta}\}_{n=1}^\infty. \tag{6} \]

These elements are recursively defined as follows. Writing \( B_0 = q^2 W_1 W_0 - W_0 W_1 \) we have

\[ B_{\alpha_0} = W_0, \quad B_{\delta + \alpha_0} = W_1 + \frac{q[B_\delta, W_0]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{7} \]

\[ B_{n\delta + \alpha_0} = B_{n-2\delta + \alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta + \alpha_0}]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{8} \]

and

\[ B_{\alpha_1} = W_1, \quad B_{\delta + \alpha_1} = W_0 - \frac{q[B_\delta, W_1]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{9} \]

\[ B_{n\delta + \alpha_1} = B_{n-2\delta + \alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta + \alpha_1}]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{10} \]

Moreover for \( n \geq 2 \),

\[ B_{n\delta} = q^{-2} B_{(n-1)\delta + \alpha_1} W_0 - W_0 B_{(n-1)\delta + \alpha_1} + (q^{-2} - 1) \sum_{\ell=0}^{n-2} B_{\ell\delta + \alpha_1} B_{(n-\ell-2)\delta + \alpha_1}. \tag{11} \]

By [11] Proposition 5.12 the elements \( \{B_{n\delta}\}_{n=1}^\infty \) mutually commute.

**Lemma 3.3.** (See [11, Theorem 4.5].) Assume that \( q \) is transcendental over \( \mathbb{F} \). Then a PBW basis for \( O_q \) is obtained by the elements (6) in any linear order.

**Definition 3.4.** We define a generating function in the indeterminate \( t \):

\[ B(t) = \sum_{n \in \mathbb{N}} B_{n\delta} t^n, \quad B_{0\delta} = q^{-2} - 1. \tag{12} \]

In Section 6 we will make a conjecture about \( B(t) \). In Sections 4, 5 we motivate the conjecture with some comments about generating functions.
4 Generating functions over a commutative algebra

Throughout this section the following notational conventions are in effect. We fix a commutative algebra \( A \). Every generating function mentioned is over \( A \).

The following results are readily checked.

Lemma 4.1. A generating function \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) is invertible if and only if \( a_0 \) is invertible in \( A \). In this case \((a(t))^{-1} = \sum_{n \in \mathbb{N}} b_n t^n \) where \( b_0 = a_0^{-1} \) and for \( n \geq 1 \),

\[
b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k}.
\]

Lemma 4.2. For generating functions \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) and \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) the following are equivalent:

(i) \( a(t) = b(qt) b(q^{-1}t) \);

(ii) \( a_n = \sum_{i=0}^{n} b_i b_{n-i} q^{2i-n} \) for \( n \in \mathbb{N} \).

Lemma 4.3. For a normalized generating function \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \), there exists a unique normalized generating function \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) such that

\[
a(t) = b(qt) b(q^{-1}t).
\]

Moreover for \( n \geq 1 \),

\[
b_n = a_n - \sum_{i=1}^{n-1} b_i b_{n-i} q^{2i-n} \frac{q^n + q^{-n}}{q^n + q^{-n}}.
\]

Definition 4.4. Referring to Lemma 4.3, we call \( b(t) \) the \( q \)-square root of \( a(t) \).

Lemma 4.5. For generating functions \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) and \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) the following are equivalent:

(i) \( a(t) = b\left(\frac{a+q^{-1}}{q+q^{-1}}\right) \);

(ii) \( a_0 = b_0 \) and for \( n \geq 1 \),

\[
a_n = \sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} (-1)^\ell \binom{n-1-\ell}{\ell} [2]^n [2]_{q^{-2\ell}} b_{n-2\ell}.
\]

Proof. Note that for \( k \in \mathbb{N} \),

\[
(1-t)^{-k-1} = \sum_{\ell \in \mathbb{N}} \binom{k+\ell}{\ell} t^\ell.
\]
We have
\[ b\left(\frac{q + q^{-1}}{t + t^{-1}}\right) = \sum_{n \in \mathbb{N}} \left(\frac{q + q^{-1}}{t + t^{-1}}\right)^n b_n = b_0 + \sum_{k \in \mathbb{N}} \left(\frac{q + q^{-1}}{t + t^{-1}}\right)^{k+1} b_{k+1}. \]

We have
\[ \frac{q + q^{-1}}{t + t^{-1}} = [2]_q t(1 + t^2)^{-1}. \]

By this and (14) we find that for \( k \in \mathbb{N} \),
\[ \left(\frac{q + q^{-1}}{t + t^{-1}}\right)^{k+1} = [2]_q^{k+1} t^{k+1} \sum_{\ell \in \mathbb{N}} (-1)^\ell \binom{k + \ell}{\ell} t^{2\ell}. \]

By these comments
\[ b\left(\frac{q + q^{-1}}{t + t^{-1}}\right) = b_0 + \sum_{k, \ell \in \mathbb{N}} (-1)^\ell \binom{k + \ell}{\ell} [2]_q^{k+1} b_{k+1} t^{k+1+2\ell} \]
\[ = b_0 + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} (-1)^\ell \binom{n - 1}{\ell - \ell} [2]_q^{n-2\ell} b_{n-2\ell} t^n. \]

The result follows. \( \square \)

**Lemma 4.6.** For a generating function \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \), there exists a unique generating function \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) such that
\[ a(t) = b\left(\frac{q + q^{-1}}{t + t^{-1}}\right). \]

Moreover \( b_0 = a_0 \) and for \( n \geq 1 \),
\[ b_n = \frac{a_n - \sum_{\ell=1}^{\lfloor (n-1)/2 \rfloor} (-1)^\ell \binom{n-1-\ell}{\ell} [2]_q^{n-2\ell} b_{n-2\ell}}{[2]_q^n}. \]

**Proof.** This is a routine consequence of Lemma 4.5. \( \square \)

**Definition 4.7.** Referring to Lemma 4.6, we call \( b(t) \) the \( q \)-symmetrization of \( a(t) \).

We now combine the above constructions.

**Proposition 4.8.** Let \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) denote a normalized generating function. Then for a generating function \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) the following are equivalent:

(i) \( b(t) \) is the \( q \)-symmetrization of the \( q \)-square root of the inverse of \( a(t) \);

(ii) \( b(t) \) is normalized and
\[ a(t) b\left(\frac{q + q^{-1}}{qt + q^{-1}t^{-1}}\right) b\left(\frac{q + q^{-1}}{q^{-1}t + qt^{-1}}\right) = 1; \] (16)
(iii) \( b(t) \) is normalized and
\[
a(qt)b\left(\frac{q + q^{-1}}{q^2t + q^{-2}t^{-1}}\right) = a(q^{-1}t)b\left(\frac{q + q^{-1}}{q^{-2}t + q^2t^{-1}}\right);
\]
(17)
(iv) \( b_0 = 1 \) and for \( n \geq 1 \),
\[
0 = [n]_q a_n + \sum_{j+k+2\ell+1=n, j,k,\ell\geq 0} (-1)^\ell \left(\frac{k + \ell}{\ell}\right) [2n - j]_q [2]_q^{k+1} a_j b_{k+1}.
\]
(18)

**Proof.** (i) \( \Rightarrow \) (ii) Let \( a_1(t) \) denote the inverse of \( a(t) \), and let \( a_2(t) \) denote the \( q \)-square root of \( a_1(t) \). By assumption \( b(t) \) is the \( q \)-symmetrization of \( a_2(t) \). The generating function \( a(t) \) is normalized, so \( a_1(t) \) is normalized by Lemma 4.1. Now \( a_2(t) \) is normalized by Lemma 4.3 and Definition 4.4. Now \( b(t) \) is normalized by Lemma 4.6 and Definition 4.7. By construction
\[
a(t)a_1(t) = 1, \quad a_1(t) = a_2(qt)a_2(q^{-1}t), \quad a_2(t) = b\left(\frac{q + q^{-1}}{t + t^{-1}}\right).
\]
Combining these equations we obtain (16).
(ii) \( \Rightarrow \) (iii) In the equation (16), replace \( t \) by \( qt \) and also by \( q^{-1}t \). Compare the two resulting equations to obtain (17).
(iii) \( \Rightarrow \) (iv) Write each side of (17) as a power series in \( t \), and compare coefficients.
(iv) \( \Rightarrow \) (i) By assumption, the generating function \( b(t) \) is normalized and satisfies (18). Let \( b'(t) \) denote the \( q \)-symmetrization of the \( q \)-square root of the inverse of \( a(t) \). From our earlier comments, the generating function \( b'(t) \) is normalized and satisfies (18). The equations (18) admit a unique solution, so \( b(t) = b'(t) \).

**Definition 4.9.** Referring to Proposition 4.8, we call \( b(t) \) the \( q \)-expansion of \( a(t) \) whenever the equivalent conditions (i)–(iv) are satisfied.

**Lemma 4.10.** Let \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) denote a normalized generating function. Let \( b(t) = \sum_{n \in \mathbb{N}} b_n t^n \) denote the \( q \)-expansion of \( a(t) \). Then for \( n \geq 1 \) the following hold:

(i) \( b_n \) is a polynomial in \( a_1, a_2, \ldots, a_n \) that has coefficients in \( \mathbb{F} \) and total degree \( n \), where we view \( a_k \) as having degree \( k \) for \( 1 \leq k \leq n \). In this polynomial the coefficient of \( a_n \) is
\[
-\left[\frac{[n]_q [2n]_q^{-1} [2]_q^n}{2}\right]^{-n}.
\]
(ii) \( a_n \) is a polynomial in \( b_1, b_2, \ldots, b_n \) that has coefficients in \( \mathbb{F} \) and total degree \( n \), where we view \( b_k \) as having degree \( k \) for \( 1 \leq k \leq n \). In this polynomial the coefficient of \( b_n \) is
\[
-\left[\frac{[n]_q [2n]_q [2]_q^n}{2}\right]^{-n}.
\]

**Proof.** (i) By (18) and induction on \( n \).
(ii) By (i) above and induction on \( n \).
5 Generating functions over a noncommutative algebra

Throughout this section the following notational conventions are in effect. We fix an algebra $B$ that is not necessarily commutative. Every generating function mentioned is over $B$.

Definition 5.1. A generating function $a(t) = \sum_{n \in \mathbb{N}} a_n t^n$ is said to be commutative whenever $\{a_n\}_{n \in \mathbb{N}}$ mutually commute.

Lemma 5.2. For a commutative generating function $a(t) = \sum_{n \in \mathbb{N}} a_n t^n$ there exists a commutative subalgebra $\mathcal{A}$ of $B$ that contains $a_n$ for $n \in \mathbb{N}$.

Proof. Take $\mathcal{A}$ to be the subalgebra of $B$ generated by $\{a_n\}_{n \in \mathbb{N}}$. Referring to Lemma 5.2 we may view $a(t)$ as a generating function over $\mathcal{A}$.

Definition 5.3. Let $a(t) = \sum_{n \in \mathbb{N}} a_n t^n$ denote a generating function that is commutative and normalized. By the $q$-expansion of $a(t)$ we mean the $q$-expansion of the generating function $a(t)$ over $\mathcal{A}$, where $\mathcal{A}$ is from Lemma 5.2. By (18) and Lemma 4.10 the $q$-expansion of $a(t)$ is independent of the choice of $\mathcal{A}$.

6 Some elements in $O_q$

In the previous two sections we discussed generating functions. We now return our attention to the $q$-Onsager algebra $O_q$. Recall from Section 1 that in [6, Conjecture 2] Baseilhac and Belliard effectively conjecture that there exist elements

$$\{W_k\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_k\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_k\}_{k \in \mathbb{N}}$$

(19)

in $O_q$ that satisfy the following relations. For $k, \ell \in \mathbb{N}$,

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q + q^{-1}),$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = \rho W_{-k-1} - \rho W_{k+1},$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = \rho W_{k+2} - \rho W_{-k},$$

$$[W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1, \ell+1}] = 0,$$  

$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,$$

$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,$$

$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0,$$

$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$

$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0,$$

$$[G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$

$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$  

(20) (21) (22) (23) (24) (25) (26) (27) (28) (29) (30)
In the above equations $\rho = -(q^2 - q^{-2})^2$. For notational convenience define

$$G_0 = -(q - q^{-1})[2]_q^2, \quad \tilde{G}_0 = -(q - q^{-1})[2]_q^2.$$  \hfill (31)

It is desirable to know how the elements $\{\tilde{G} \}_{k \in \mathbb{N}}$ are related to the elements $\{W\}_{k \in \mathbb{N}}$. In this paper we conjecture the precise relationship. We will state the conjecture shortly. Before stating the conjecture, we discuss what is involved. Let us simplify things by writing the elements $\{\tilde{G}\}_{k \in \mathbb{N}}$ in terms of $W_0, W_1, \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$. To do this, we use (21), (22) to recursively obtain $W_{-k}, W_{k+1}$ for $k \geq 1$:

\[
W_{-1} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2},
\]

\[
W_3 = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - W_1, G_2]_q/(q^2 - q^{-2})^2,
\]

\[
W_{-3} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2},
\]

\[
W_5 = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2},
\]

\[
W_{-5} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_4]_q}{(q^2 - q^{-2})^2},
\]

\[
\vdots
\]

\[
W_2 = W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2},
\]

\[
W_{-2} = W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2},
\]

\[
W_4 = W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2},
\]

\[
W_{-4} = W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]_q}{(q^2 - q^{-2})^2},
\]

\[
W_6 = W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_5]_q}{(q^2 - q^{-2})^2},
\]

\[
\vdots
\]

The recursion shows that for any integer $k \geq 1$, the generators $W_{-k}, W_{k+1}$ are given as follows. For odd $k = 2r + 1$,

\[
W_{-k} = W_1 - \sum_{\ell=0}^r \frac{[\tilde{G}_{2\ell+1}, W_0]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^r \frac{[W_1, \tilde{G}_{2\ell}]_q}{(q^2 - q^{-2})^2}, \quad (32)
\]

\[
W_{k+1} = W_0 - \sum_{\ell=0}^r \frac{[W_1, \tilde{G}_{2\ell+1}]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^r \frac{[\tilde{G}_{2\ell}, W_0]_q}{(q^2 - q^{-2})^2}. \quad (33)
\]
Recall the generating function relationship using generating functions.

The generating function $B$ is commutative by Definition 5.1 and the comment above Lemma 3.3. By (1 2) the generating function $B$ is commutative, so by Lemma 5.2 there exists a commutative subalgebra $A$ of $O_q$ that contains $B_n t$ for $n \in \mathbb{N}$. So $B(t)$ is over $A$. The $q$-expansion of $B(t)^\vee$ is over $A$, and described as follows. For the moment let $\tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n$ denote any generating function over $A$ such that $\tilde{G}_0$ satisfies (31). By Proposition 4.8 and Definitions 4.9, 5.3 we find that $\tilde{G}(t)^\vee$ is the $q$-expansion of $B(t)^\vee$.

We have expressed the elements (19) in terms of $W_0$, $W_1$, $\tilde{G}_{k+1}$, $k \in \mathbb{N}$. Next, we would like to know how the elements $\tilde{G}_{k+1}$ are related to the elements (6). We will discuss this relationship using generating functions.

Next we use (20) to obtain the generators $\{G_{k+1}\}_{k \in \mathbb{N}}$:

$$G_{k+1} = \tilde{G}_{k+1} + (q + q^{-1})[W_1, W_{-k}] \quad (k \in \mathbb{N}).$$

For even $k = 2r$,

$$W_{-k} = W_0 - \sum_{\ell = 0}^{r-1} \frac{[W_1, \tilde{G}_{2\ell+1}]_q}{(q^2 - q^{-2})^2} - \sum_{\ell = 1}^{r} \frac{[\tilde{G}_{2\ell}, W_0]_q}{(q^2 - q^{-2})^2},$$

$$W_{k+1} = W_1 - \sum_{\ell = 0}^{r-1} \frac{[\tilde{G}_{2\ell+1}, W_0]_q}{(q^2 - q^{-2})^2} - \sum_{\ell = 1}^{r} \frac{[W_1, \tilde{G}_{2\ell}]_q}{(q^2 - q^{-2})^2}.$$ (34) (35)

Next we use (20) to obtain the generators $\{G_{k+1}\}_{k \in \mathbb{N}}$:

$$G_{k+1} = \tilde{G}_{k+1} + (q + q^{-1})[W_1, W_{-k}] \quad (k \in \mathbb{N}).$$
if and only if
\[ B(t)\tilde{G}\left(\frac{q + q^{-1}}{q^t + q^{-1}t^{-1}}\right)\tilde{G}\left(\frac{q + q^{-1}}{q^{-1}t + qt^{-1}}\right) = -q^{-1}(q - q^{-1})^3[2]^4 \tag{38} \]

if and only if
\[ B(qt)\tilde{G}\left(\frac{q + q^{-1}}{q^{2t} + q^{-2}t^{-1}}\right) = B(q^{-1}t)\tilde{G}\left(\frac{q + q^{-1}}{q^{-2}t + qt^{-2}}\right) \tag{39} \]

if and only if for \( n \geq 1, \)
\[ 0 = [n]_q B_{n\delta} \tilde{G}_0 + \sum_{j+k+\ell+1 = n, \ j,k,\ell \geq 0} (-1)^\ell \binom{k + \ell}{\ell} [2n-j]_q [2]^{k+1} B_{j\delta} \tilde{G}_{k+1}. \tag{40} \]

In Appendix A we display (40) in detail for \( 1 \leq n \leq 8. \)

7 Supporting evidence for Conjecture 6.1

In this section we give some supporting evidence for Conjecture 6.1.

Our first type of evidence is from checking via computer. The algebra \( O_q \) has been implemented in the computer package SageMath (see [23]) by Travis Scrimshaw. Using this package Scrimshaw defined the elements (37) for \( 0 \leq k \leq 5 \) using (40) along with (32)–(35) and (36). He then had SageMath verify the relations among (20)–(30) that involved these defined elements.

Our next type of evidence has to do with the universal Askey-Wilson algebra \( \Delta_q \) [26, Definition 1.2]. This algebra is defined by generators and relations. The generators are \( A, B, C. \)

The relations assert that each of the following is central in \( \Delta_q: \)
\[ A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}. \]

For the above three central elements, multiply each by \( q + q^{-1} \) to get \( \alpha, \beta, \gamma. \) Thus
\[ A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}. \tag{41-43} \]

Each of \( \alpha, \beta, \gamma \) is central in \( \Delta_q. \) By [26, Corollary 8.3] the center of \( \Delta_q \) is generated by \( \alpha, \beta, \gamma, \Omega \) where
\[ \Omega = qABC + q^2 A^2 + q^{-2}B^2 + q^2 C^2 - qA\alpha - q^{-1} B\beta - qC\gamma. \tag{44} \]
The element \( \Omega \) is called the Casimir element. By [26, Theorem 8.2] the elements \( \alpha, \beta, \gamma, \Omega \) are algebraically independent. We write \( F[\alpha, \beta, \gamma, \Omega] \) for the center of \( \Delta_q \).

Next we summarize from [26, Section 3] how the modular group \( \text{PSL}_2(\mathbb{Z}) \) acts on \( \Delta_q \) as a group of automorphisms. By [1] the group \( \text{PSL}_2(\mathbb{Z}) \) has a presentation by generators \( \varpi, \sigma \) and relations \( \varpi^3 = 1, \sigma^2 = 1 \). By [26, Theorems 3.1, 6.4] the group \( \text{PSL}_2(\mathbb{Z}) \) acts on \( \Delta_q \) as a group of automorphisms in the following way:

\[
\begin{array}{c|ccc|cccc}
  u & A & B & C & \alpha & \beta & \gamma & \Omega \\
  \varrho(u) & B & C & A & \beta & \gamma & \alpha & \Omega \\
  \sigma(u) & B & A & C + \frac{[A,B]}{q-q^{-1}} & \beta & \alpha & \gamma & \Omega \\
\end{array}
\]

For notational convenience define

\[
C' = C + \frac{[A,B]}{q-q^{-1}}. \tag{45}
\]

Applying \( \sigma \) to (41)–(43) and using the above table, we obtain

\[
B + \frac{qAC' - q^{-1}C'A}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}}, \tag{46}
\]

\[
A + \frac{qC'B - q^{-1}BC'}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}, \tag{47}
\]

\[
C' + \frac{qBA - q^{-1}AB}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}. \tag{48}
\]

Next we explain how \( \Delta_q \) is related to \( O_q \). By [26, Theorem 2.2] the algebra \( \Delta_q \) has a presentation by generators \( A, B, \gamma \) and relations

\[
A^3B - [3]_q A^2BA + [3]_q AB^2A^2 - BA^3 = (q^2 - q^{-2})^2(BA - AB), \tag{49}
\]

\[
B^3A - [3]_q B^2AB + [3]_q BAB^2A^2 - AB^3 = (q^2 - q^{-2})^2(AB - BA), \tag{50}
\]

\[
A^2B^2 - B^2A^2 + (q^2 + q^{-2})(BAB - ABAB) = (q - q^{-1})^2(BA - AB)\gamma, \tag{51}
\]

\[
\gamma A = A\gamma, \quad \gamma B = B\gamma. \tag{52}
\]

The relations (49), (50) are the \( q \)-Dolan/Grady relations. Consequently there exists an algebra homomorphism \( \natural : O_q \to \Delta_q \) that sends \( W_0 \mapsto A \) and \( W_1 \mapsto B \). This homomorphism is not injective by [26, Theorem 10.9].

For the elements (6) and (37) we retain the same notation for their images under \( \natural \). We will show that for \( \Delta_q \) the elements (37) satisfy the relations (20)–(30).

For the algebra \( \Delta_q \) define

\[
\Psi(t) = B(t) + 1 - q^{-2}, \tag{53}
\]

where \( B(t) \) is from Definition 3.4. By (12) we have \( \Psi(t) = \sum_{n=1}^{\infty} B_n \delta t^n \). By [28, Corollary 5.7] the elements \( \{B_n \delta\}_{n=1}^{\infty} \) are contained in the subalgebra of \( \Delta_q \) generated by \( F[\alpha, \beta, \gamma, \Omega] \) and
Consequently the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ commute with $C$, so $\Psi(t)$ commutes with $C$. By this and [28, Line (5.19)] we find that

$$\Psi(t)\left(qt + q^{-1}t^{-1} + C\right)(q^{-1}t + qt^{-1} + C) \quad (54)$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{(t + t^{-1})\alpha\beta}{(t - t^{-1})^2} - \frac{\alpha^2 + \beta^2}{(t - t^{-1})^2} - (t + t^{-1})\gamma + (q + q^{-1})(t + t^{-1})C + C^2. \quad (55)$$

Upon eliminating $\Psi(t)$ from (54) using (53), we find that

$$B(t)\left(qt + q^{-1}t^{-1} + C\right)(q^{-1}t + qt^{-1} + C) \quad (55)$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{(t + t^{-1})\alpha\beta}{(t - t^{-1})^2} - \frac{\alpha^2 + \beta^2}{(t - t^{-1})^2} - (t + t^{-1})\gamma - (qt + q^{-1}t^{-1})(q^{-1}t + qt^{-1}). \quad (55)$$

Define

$$N(t) = \frac{B(t)}{q^{-2} - 1} \frac{qt + q^{-1}t^{-1} + C}{qt + q^{-1}t^{-1}} \frac{q^{-1}t + qt^{-1} + C}{q^{-1}t + qt^{-1}}. \quad (56)$$

By the above comments

$$N(t) = 1 + N_1(t)\Omega + N_2(t)\alpha\beta + N_3(t)(\alpha^2 + \beta^2) + N_4(t)\gamma, \quad (57)$$

where

$$N_1(t) = \frac{-1}{(qt + q^{-1}t^{-1})(q^{-1}t + qt^{-1})}, \quad (58)$$

$$N_2(t) = \frac{t + t^{-1}}{(t - t^{-1})^2(qt + q^{-1}t^{-1})(q^{-1}t + qt^{-1})}, \quad (59)$$

$$N_3(t) = \frac{1}{(t - t^{-1})^2(qt + q^{-1}t^{-1})(q^{-1}t + qt^{-1})}, \quad (60)$$

$$N_4(t) = \frac{t + t^{-1}}{(qt + q^{-1}t^{-1})(q^{-1}t + qt^{-1})}. \quad (61)$$

Evaluating (58)–(61) using

$$\frac{1}{qt + q^{-1}t^{-1}} = \sum_{n\in\mathbb{N}} (-1)^n q^{2n+1} t^{2n+1},$$

$$\frac{1}{q^{-1}t + qt^{-1}} = \sum_{n\in\mathbb{N}} (-1)^n q^{-2n-1} t^{2n+1},$$

$$\frac{1}{(t - t^{-1})^2} = \sum_{n\in\mathbb{N}} n t^{2n}$$

we find that the functions $N_1(t), N_2(t), N_3(t), N_4(t)$ are power series in $t$ with zero constant term. By this and (57), we may view $N(t)$ as a normalized generating function over $\mathbb{F}[\alpha, \beta, \gamma, \Omega].$
Definition 7.1. Define a generating function \( Z(t) = \sum_{n \in \mathbb{N}} Z_n t^n \) over \( \mathbb{F}[\alpha, \beta, \gamma, \Omega] \) such that \( Z_0 = q^{-2} - q^2 \) and \( Z(t)^+ \) is the \( q \)-expansion of \( N(t) \).

The notation \( Z(t)^+ \) is explained in (43). The \( q \)-expansion concept is explained in Proposition 4.8 and Definition 4.9. By these explanations and Definition 7.1,

\[
N(t)Z \left( \frac{q + q^{-1}}{q t + q^{-1} t^{-1}} \right) Z \left( \frac{q + q^{-1}}{q^{-1} t + qt^{-1}} \right) = (q^2 - q^{-2})^2.
\]  

(62)

Proposition 7.2. For the algebra \( \Delta_q \),

\[
\tilde{G}(t) = Z(t)(q + q^{-1} + tC).
\]

(63)

Proof. Define the generating function \( \tilde{G}(t) = Z(t)(q + q^{-1} + tC) \). We show that \( \tilde{G}(t) = \tilde{G}(t) \). Let \( \mathcal{A} \) denote the subalgebra of \( \Delta_q \) generated by \( \mathbb{F}[\alpha, \beta, \gamma, \Omega] \) and \( C \). Note that \( \mathcal{A} \) is commutative. By construction \( \tilde{G}(t) \) is over \( \mathcal{A} \). By our comments below (53), the generating function \( B(t) \) is over \( \mathcal{A} \). By the discussion around (38), it suffices to show that

\[
B(t)\tilde{G} \left( \frac{q + q^{-1}}{q t + q^{-1} t^{-1}} \right) \tilde{G} \left( \frac{q + q^{-1}}{q^{-1} t + qt^{-1}} \right) = -q^{-1}(q - q^{-1})^3[2]_q^4.
\]

(64)

Using (56) and and (62),

\[
B(t)\tilde{G} \left( \frac{q + q^{-1}}{q t + q^{-1} t^{-1}} \right) \tilde{G} \left( \frac{q + q^{-1}}{q^{-1} t + qt^{-1}} \right)
= [2]_q^2 B(t)Z \left( \frac{q + q^{-1}}{q t + q^{-1} t^{-1}} \right) \frac{qt + q^{-1} t^{-1} + C}{qt + q^{-1} t^{-1}} Z \left( \frac{q + q^{-1}}{q^{-1} t + qt^{-1}} \right) \frac{q^{-1} t + qt^{-1} + C}{q^{-1} t + qt^{-1}}
= [2]_q^2(q^{-2} - 1)N(t)Z \left( \frac{q + q^{-1}}{q t + q^{-1} t^{-1}} \right) Z \left( \frac{q + q^{-1}}{q^{-1} t + qt^{-1}} \right)
= [2]_q^2(q^{-2} - 1)(q^2 - q^{-2})^2
= -q^{-1}(q - q^{-1})^3[2]_q^4.
\]

We have shown (64), and the result follows.

Define the generating functions

\[
W^-(t) = \sum_{n \in \mathbb{N}} W_{-n} t^n, \quad W^+(t) = \sum_{n \in \mathbb{N}} W_{n+1} t^n.
\]

By (32)–(35) we obtain

\[
W^+(t) = \frac{t[\tilde{G}(t), A]_q + [B, \tilde{G}(t)]_q}{(t^2 - 1)(q^2 - q^{-2})^2}, \quad W^-(t) = \frac{[\tilde{G}(t), A]_q + t[B, \tilde{G}(t)]_q}{(t^2 - 1)(q^2 - q^{-2})^2}.
\]

(65)

(66)
Lemma 7.3. For the algebra $\Delta_q$,

\[
W^+(t) = Z(t) \frac{(q - q^{-1})(\alpha + \beta t) - (q^2 - q^{-2})(t - t^{-1})B}{(q^2 - q^{-2})^2(t - t^{-1})}, \tag{67}
\]

\[
W^-(t) = Z(t) \frac{(q - q^{-1})(\alpha t + \beta) - (q^2 - q^{-2})(t - t^{-1})A}{(q^2 - q^{-2})^2(t - t^{-1})}. \tag{68}
\]

Proof. To obtain (67), eliminate $\tilde{G}(t)$ from (65) using (63), and evaluate the result using (61), (62). Equation (68) is similarly obtained. \qed

Define the generating function

\[
G(t) = \sum_{n \in \mathbb{N}} G_n t^n.
\]

Using (36) we obtain

\[
G(t) = \tilde{G}(t) + t(q + q^{-1})[B, W^-(t)]. \tag{69}
\]

Lemma 7.4. For the algebra $\Delta_q$ we have

\[
G(t) = Z(t)(q + q^{-1} + tC'), \tag{70}
\]

where $C'$ is from (43). \qed

Proof. Eliminate $\tilde{G}(t)$ from (69) using (63). Eliminate $W^-(t)$ from (69) using (68), and evaluate the result using (45). \qed

Let $s$ denote an indeterminate that commutes with $t$.

Lemma 7.5. For the algebra $\Delta_q$ we have

\[
[A, W^+(t)] = [W^-(t), B] = t^{-1}(\tilde{G}(t) - G(t))/(q + q^{-1}),
\]

\[
[A, G(t)]_q = [\tilde{G}(t), A]_q = \rho W^-(t) - \rho t W^+(t),
\]

\[
[G(t), B]_q = [B, \tilde{G}(t)]_q = \rho W^+(t) - \rho t W^-(t),
\]

\[
[W^-(s), W^-(t)] = 0, \quad [W^+(s), W^+(t)] = 0,
\]

\[
[W^-(s), W^+(t)] + [W^+(s), W^-(t)] = 0,
\]

\[
s[W^-(s), G(t)] + t[G(s), W^-(t)] = 0,
\]

\[
s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)] = 0,
\]

\[
s[W^+(s), G(t)] + t[G(s), W^+(t)] = 0,
\]

\[
s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)] = 0,
\]

\[
[G(s), G(t)] = 0, \quad [\tilde{G}(s), \tilde{G}(t)] = 0,
\]

\[
[\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)] = 0,
\]

where $\rho = -(q^2 - q^{-2})^2$. \qed

Proof. These relations are routinely verified using Proposition 7.2 and Lemmas 7.3, 7.4 along with (61), (62), (46), (47). \qed

Theorem 7.6. In the algebra $\Delta_q$ the elements $\{37\}$ satisfy the relations (20)–(30). \qed

Proof. This is a routine consequence of Lemma 7.5. \qed
8 Comments

In the previous section we gave some supporting evidence for Conjecture 6.1. In this section we assume that Conjecture 6.1 is correct, and provide more information about how the elements (37) are related to the elements (6). We will give a variation on (32)–(35).

Using Appendix A and

$$\delta = q^{-2}W_1W_0 - W_0W_1$$

we obtain

$$\tilde{G}_1 = -qB_\delta = [W_0, W_1]_q.$$  \hfill (71)

**Lemma 8.1.** For $k \in \mathbb{N}$,

(i) $[\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_0\tilde{G}_{k+1} - q^2[B_\delta, W_{-k}],$

(ii) $[W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_1\tilde{G}_{k+1} + [B_\delta, W_{k+1}].$

**Proof.** (i) Observe that

$$[\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_0\tilde{G}_{k+1} + q[\tilde{G}_{k+1}, W_0].$$

By (26) and (71),

$$[\tilde{G}_{k+1}, W_0] = [\tilde{G}_1, W_{-k}] = -q[B_\delta, W_{-k}].$$

The result follows.

(ii) Observe that

$$[W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_1\tilde{G}_{k+1} - q^{-1}[\tilde{G}_{k+1}, W_1].$$

By (28) and (71),

$$[\tilde{G}_{k+1}, W_1] = [\tilde{G}_1, W_{k+1}] = -q[B_\delta, W_{k+1}].$$

The result follows. \hfill \Box

**Lemma 8.2.** For $n \geq 1$,

$$W_{-n} = W_n - \frac{(q - q^{-1})W_0\tilde{G}_n}{(q^2 - q^{-2})^2} + \frac{q^2[B_\delta, W_{1-n}]}{(q^2 - q^{-2})^2},$$

$$W_{n+1} = W_{1-n} - \frac{(q - q^{-1})W_1\tilde{G}_n}{(q^2 - q^{-2})^2} - \frac{[B_\delta, W_n]}{(q^2 - q^{-2})^2}.$$  \hfill (72) \hfill (73)

**Proof.** Use the equations on the right in (21), (22) along with Lemma 8.1 \hfill \Box

We recall some notation from \([11]\). For a negative integer $k$ define

$$B_{k\delta + \alpha_0} = B_{(-k-1)\delta + \alpha_1}, \quad B_{k\delta + \alpha_1} = B_{(-k-1)\delta + \alpha_0}.$$

We have

$$B_{r\delta + \alpha_0} = B_{s\delta + \alpha_1} \quad (r, s \in \mathbb{Z}, \quad r + s = -1).$$  \hfill (74)
Lemma 8.3. For \( n \in \mathbb{Z} \),
\[
\frac{q[B_{\delta}, B_{n\delta+\alpha_0}]}{(q - q^{-1})(q^2 - q^{-2})} = B_{(n+1)\delta+\alpha_0} - B_{(n-1)\delta+\alpha_0},
\]
(75)
\[
\frac{q[B_{\delta}, B_{n\delta+\alpha_1}]}{(q - q^{-1})(q^2 - q^{-2})} = B_{(n-1)\delta+\alpha_1} - B_{(n+1)\delta+\alpha_1}.
\]
(76)

Proof. Use (7)–(10) and (74). \( \square \)

Proposition 8.4. For \( n \in \mathbb{N} \) the following hold in \( O_q \):
\[
W_{-n} = -(q - q^{-1})^{-1} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{k}{\ell} q^{k-2\ell} [2]_q^{r-k-2} B_{(k-2)\delta+\alpha_0} \tilde{G}_{n-k},
\]
(77)
\[
W_{n+1} = -(q - q^{-1})^{-1} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{k}{\ell} q^{2\ell-k} [2]_q^{r-k-2} B_{(k-2)\delta+\alpha_1} \tilde{G}_{n-k}.
\]
(78)

Proof. We use induction on \( n \). First assume that \( n = 0 \). Then (77), (78) hold. Next assume that \( n \geq 1 \). To obtain (77), evaluate the right-hand side of (72) using induction along with (74), (75). To obtain (78), evaluate the right-hand side of (73) using induction along with (74), (76). \( \square \)

In Appendix B we display (77), (78) in detail for \( 0 \leq n \leq 7 \).

Referring to (77) and (78), if we express each term \( \tilde{G}_{n-k} \) as a polynomial in \( B_{\delta}, B_{2\delta}, \ldots, B_{(n-k)\delta} \) using (40), then we effectively write \( W_{-n} \) and \( W_{n+1} \) in the PBW basis for \( O_q \) given in Lemma 3.3. Unfortunately the resulting formula are not pleasant.

9 Acknowledgment

The author is deeply grateful to Travis Scrimshaw for performing the computer checks mentioned at the beginning of Section 7. The author thanks Pascal Baseilhac and Nicolas Crampé for giving this paper a close reading and offering valuable suggestions.

10 Appendix A

For the \( q \)-Onsager algebra \( O_q \) we use (40) to obtain \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_8 \) in terms of \( B_{\delta}, B_{2\delta}, \ldots, B_{8\delta} \).

Recall that
\[
B_{0\delta} = q^{-2} - 1, \quad \tilde{G}_0 = -(q - q^{-1})[2]_q^2.
\]

\( \tilde{G}_1 \) satisfies
\[
0 = \begin{vmatrix}
[2]_q B_{0\delta} & [1]_q B_{1\delta} \\
0 & 1 \\
[2]_q \tilde{G}_1 & 1 & 0
\end{vmatrix}
\]

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\( \tilde{G}_2 \) satisfies

| \([4]qB_{0\delta}\) | \([3]qB_{1\delta}\) | \([2]qB_{2\delta}\) |
|---|---|---|
| \(G_0\) | 0 | 0 | 1 |
| \([2]\tilde{G}_1\) | 0 | 1 | 0 |
| \([2]^2\tilde{G}_2\) | 1 | 0 | 0 |

\( \tilde{G}_3 \) satisfies

| \([6]qB_{0\delta}\) | \([5]qB_{1\delta}\) | \([4]qB_{2\delta}\) | \([3]qB_{3\delta}\) |
|---|---|---|---|
| \(G_0\) | 0 | 0 | 0 | 1 |
| \([2]\tilde{G}_1\) | 0 | -1 | 0 | 1 | 0 |
| \([2]^2\tilde{G}_2\) | 0 | 1 | 0 | 0 | 0 |
| \([2]^3\tilde{G}_3\) | 1 | 0 | 0 | 0 | 0 |

\( \tilde{G}_4 \) satisfies

| \([8]qB_{0\delta}\) | \([7]qB_{1\delta}\) | \([6]qB_{2\delta}\) | \([5]qB_{3\delta}\) | \([4]qB_{4\delta}\) |
|---|---|---|---|---|
| \(G_0\) | 0 | 0 | 0 | 0 | 1 |
| \([2]\tilde{G}_1\) | 0 | 0 | -1 | 0 | 1 | 0 |
| \([2]^2\tilde{G}_2\) | 0 | 0 | 1 | 0 | 0 | 0 |
| \([2]^3\tilde{G}_3\) | 0 | 1 | 0 | 0 | 0 | 0 |
| \([2]^4\tilde{G}_4\) | 1 | 0 | 0 | 0 | 0 | 0 |

\( \tilde{G}_5 \) satisfies

| \([10]qB_{0\delta}\) | \([9]qB_{1\delta}\) | \([8]qB_{2\delta}\) | \([7]qB_{3\delta}\) | \([6]qB_{4\delta}\) | \([5]qB_{5\delta}\) |
|---|---|---|---|---|---|
| \(G_0\) | 0 | 0 | 0 | 0 | 0 | 1 |
| \([2]\tilde{G}_1\) | 1 | 0 | 0 | 0 | 0 | 0 |
| \([2]^2\tilde{G}_2\) | 0 | -2 | 0 | 1 | 0 | 0 |
| \([2]^3\tilde{G}_3\) | 0 | 0 | 0 | 0 | 0 | 0 |
| \([2]^4\tilde{G}_4\) | 1 | 0 | 0 | 0 | 0 | 0 |

\( \tilde{G}_6 \) satisfies

| \([12]qB_{0\delta}\) | \([11]qB_{1\delta}\) | \([10]qB_{2\delta}\) | \([9]qB_{3\delta}\) | \([8]qB_{4\delta}\) | \([7]qB_{5\delta}\) | \([6]qB_{6\delta}\) |
|---|---|---|---|---|---|---|
| \(G_0\) | 0 | 0 | 0 | 0 | 0 | 1 |
| \([2]\tilde{G}_1\) | 0 | 0 | -1 | 0 | 0 | 0 |
| \([2]^2\tilde{G}_2\) | 3 | 0 | -2 | 0 | 1 | 0 |
| \([2]^3\tilde{G}_3\) | 0 | -3 | 0 | 1 | 0 | 0 |
| \([2]^4\tilde{G}_4\) | 0 | 0 | 0 | 0 | 0 | 0 |
| \([2]^5\tilde{G}_5\) | 1 | 0 | 0 | 0 | 0 | 0 |
| \([2]^6\tilde{G}_6\) | 1 | 0 | 0 | 0 | 0 | 0 |
\( \tilde{G}_7 \) satisfies

\[
0 = \begin{bmatrix}
14_q B_{0\delta} & 13_q B_{1\delta} & 12_q B_{2\delta} & 11_q B_{3\delta} & 10_q B_{4\delta} & 9_q B_{5\delta} & 8_q B_{6\delta} & 7_q B_{7\delta} \\
G_0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
[2]q \tilde{G}_1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
[2]^2_q \tilde{G}_2 & 0 & 3 & 0 & -2 & 0 & 1 & 0 & 0 \\
[2]^3_q \tilde{G}_3 & 6 & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\
[2]^4_q \tilde{G}_4 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\
[2]^5_q \tilde{G}_5 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
[2]^6_q \tilde{G}_6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
[2]^7_q \tilde{G}_7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\( \tilde{G}_8 \) satisfies

\[
0 = \begin{bmatrix}
16_q B_{0\delta} & 15_q B_{1\delta} & 14_q B_{2\delta} & 13_q B_{3\delta} & 12_q B_{4\delta} & 11_q B_{5\delta} & 10_q B_{6\delta} & 9_q B_{7\delta} & 8_q B_{8\delta} \\
G_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
[2]q \tilde{G}_1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
[2]^2_q \tilde{G}_2 & -4 & 0 & 3 & 0 & -2 & 0 & 1 & 0 & 0 \\
[2]^3_q \tilde{G}_3 & 0 & 6 & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\
[2]^4_q \tilde{G}_4 & 10 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\
[2]^5_q \tilde{G}_5 & 0 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
[2]^6_q \tilde{G}_6 & -6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
[2]^7_q \tilde{G}_7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
[2]^8_q \tilde{G}_8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

11 Appendix B

For the \( q \)-Onsager algebra \( \mathcal{O}_q \) we use (77), (78) to obtain \( \{ W_n \}_{n=0}^7 \) and \( \{ W_{n+1} \}_{n=0}^7 \) in terms of \( \{ B_{n\delta+\alpha_0} \}_{n=0}^7 \), \( \{ B_{n\delta+\alpha_1} \}_{n=0}^7 \), \( \{ \tilde{G}_n \}_{n=0}^7 \). Recall that \( \tilde{G}_0 = -(q - q^{-1})[2]_q^2 \).

We have

\[
W_0 = B_{\alpha_0} = -(q - q^{-1})^{-1}[2]_q^{-2} B_{\alpha_0} \tilde{G}_0.
\]

\( W_{-1} \) is equal to \( -(q - q^{-1})^{-1}[2]_q^{-3} \) times

\[
\begin{array}{c|cc}
q^{-1}B_{\alpha_1} & \tilde{G}_0 & [2]_q \tilde{G}_1 \\
B_{\alpha_0} & 0 & 1 \\
qB_{\delta+\alpha_0} & 1 & 0
\end{array}
\]

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$W_{-2}$ is equal to $-(q - q^{-1})^{-1}[2]_{q}^{-4}$ times

|                           | $\tilde{G}_0$ | $[2]_{q}\tilde{G}_1$ | $[2]_{q}^{2}\tilde{G}_2$ | $[2]_{q}^{3}\tilde{G}_3$ | $[2]_{q}^{4}\tilde{G}_4$ |
|---------------------------|---------------|----------------------|--------------------------|--------------------------|--------------------------|
| $q^{-2}B_{\delta + \alpha_1}$ | 1             | 0                    | 0                        |                          |                          |
| $q^{-1}B_{\alpha_1}$       | 0             | 1                    | 0                        |                          |                          |
| $B_{\alpha_0}$             | 2             | 0                    | 1                        |                          |                          |
| $qB_{\delta + \alpha_0}$   | 0             | 1                    | 0                        |                          |                          |
| $q^2B_{2\delta + \alpha_0}$ | 1             | 0                    | 0                        |                          |                          |

$W_{-3}$ is equal to $-(q - q^{-1})^{-1}[2]_{q}^{-5}$ times

|                           | $\tilde{G}_0$ | $[2]_{q}\tilde{G}_1$ | $[2]_{q}^{2}\tilde{G}_2$ | $[2]_{q}^{3}\tilde{G}_3$ | $[2]_{q}^{4}\tilde{G}_4$ |
|---------------------------|---------------|----------------------|--------------------------|--------------------------|--------------------------|
| $q^{-3}B_{2\delta + \alpha_1}$ | 1             | 0                    | 0                        |                          |                          |
| $q^{-2}B_{\delta + \alpha_1}$ | 0             | 1                    | 0                        |                          |                          |
| $q^{-1}B_{\alpha_1}$       | 3             | 0                    | 1                        |                          |                          |
| $B_{\alpha_0}$             | 0             | 2                    | 0                        | 1                        |                          |
| $qB_{\delta + \alpha_0}$   | 3             | 0                    | 1                        | 0                        |                          |
| $q^2B_{2\delta + \alpha_0}$ | 0             | 1                    | 0                        | 0                        |                          |
| $q^3B_{3\delta + \alpha_0}$ | 1             | 0                    | 0                        | 0                        |                          |

$W_{-4}$ is equal to $-(q - q^{-1})^{-1}[2]_{q}^{-6}$ times

|                           | $\tilde{G}_0$ | $[2]_{q}\tilde{G}_1$ | $[2]_{q}^{2}\tilde{G}_2$ | $[2]_{q}^{3}\tilde{G}_3$ | $[2]_{q}^{4}\tilde{G}_4$ | $[2]_{q}^{5}\tilde{G}_5$ |
|---------------------------|---------------|----------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $q^{-3}B_{3\delta + \alpha_1}$ | 1             | 0                    | 0                        |                          |                          |                          |
| $q^{-2}B_{2\delta + \alpha_1}$ | 0             | 1                    | 0                        |                          |                          |                          |
| $q^{-2}B_{\delta + \alpha_1}$ | 4             | 0                    | 1                        | 0                        |                          |                          |
| $q^{-1}B_{\alpha_1}$       | 0             | 3                    | 0                        | 1                        |                          |                          |
| $B_{\alpha_0}$             | 6             | 0                    | 2                        | 0                        | 1                        |                          |
| $qB_{\delta + \alpha_0}$   | 0             | 3                    | 0                        | 1                        | 0                        |                          |
| $q^2B_{2\delta + \alpha_0}$ | 4             | 0                    | 1                        | 0                        | 0                        |                          |
| $q^3B_{3\delta + \alpha_0}$ | 0             | 1                    | 0                        | 0                        | 0                        |                          |
| $q^4B_{4\delta + \alpha_0}$ | 1             | 0                    | 0                        | 0                        | 0                        |                          |

$W_{-5}$ is equal to $-(q - q^{-1})^{-1}[2]_{q}^{-7}$ times

|                           | $\tilde{G}_0$ | $[2]_{q}\tilde{G}_1$ | $[2]_{q}^{2}\tilde{G}_2$ | $[2]_{q}^{3}\tilde{G}_3$ | $[2]_{q}^{4}\tilde{G}_4$ | $[2]_{q}^{5}\tilde{G}_5$ | $[2]_{q}^{6}\tilde{G}_6$ | $[2]_{q}^{7}\tilde{G}_7$ |
|---------------------------|---------------|----------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $q^{-3}B_{4\delta + \alpha_1}$ | 1             | 0                    | 0                        |                          |                          |                          |                          |                          |
| $q^{-4}B_{3\delta + \alpha_1}$ | 0             | 1                    | 0                        |                          |                          |                          |                          |                          |
| $q^{-3}B_{2\delta + \alpha_1}$ | 5             | 0                    | 1                        | 0                        |                          |                          |                          |                          |
| $q^{-2}B_{\delta + \alpha_1}$ | 0             | 4                    | 0                        | 1                        | 0                        |                          |                          |                          |
| $q^{-1}B_{\alpha_1}$       | 10            | 0                    | 3                        | 0                        | 1                        | 0                        |                          |                          |
| $B_{\alpha_0}$             | 0             | 6                    | 0                        | 2                        | 0                        | 1                        |                          |                          |
| $qB_{\delta + \alpha_0}$   | 10            | 0                    | 3                        | 0                        | 1                        | 0                        |                          |                          |
| $q^2B_{2\delta + \alpha_0}$ | 0             | 4                    | 0                        | 1                        | 0                        | 0                        |                          |                          |
| $q^3B_{3\delta + \alpha_0}$ | 5             | 0                    | 1                        | 0                        | 0                        | 0                        |                          |                          |
| $q^4B_{4\delta + \alpha_0}$ | 0             | 1                    | 0                        | 0                        | 0                        | 0                        |                          |                          |
| $q^5B_{5\delta + \alpha_0}$ | 1             | 0                    | 0                        | 0                        | 0                        | 0                        |                          |                          |
$W_{-6}$ is equal to $-(q - q^{-1})^{-1}[2]_q^{-8}$ times

| $q^{-6}B_{5\delta + \alpha_1}$ | $\tilde{G}_0$ | $[2]_q\tilde{G}_1$ | $[2]_q^2\tilde{G}_2$ | $[2]_q^3\tilde{G}_3$ | $[2]_q^4\tilde{G}_4$ | $[2]_q^5\tilde{G}_5$ | $[2]_q^6\tilde{G}_6$ |
|-----------------------------|-------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1                          | 0           | 0                | 0                | 0                | 0                | 0                | 0                |
| $q^{-5}B_{4\delta + \alpha_1}$ | 0           | 1                | 0                | 0                | 0                | 0                | 0                |
| $q^{-4}B_{3\delta + \alpha_1}$ | 6           | 0                | 1                | 0                | 0                | 0                | 0                |
| $q^{-3}B_{2\delta + \alpha_1}$ | 0           | 5                | 0                | 1                | 0                | 0                | 0                |
| $q^{-2}B_{\delta + \alpha_1}$ | 15          | 0                | 4                | 0                | 1                | 0                | 0                |
| $q^{-1}B_{\alpha_1}$        | 0           | 10               | 0                | 3                | 0                | 1                | 0                |
| $B_{\alpha_0}$              | 20          | 0                | 6                | 0                | 2                | 0                | 1                |
| $qB_{\delta + \alpha_0}$    | 0           | 15               | 0                | 4                | 0                | 1                | 0                |
| $q^2B_{2\delta + \alpha_0}$ | 0           | 15               | 0                | 4                | 0                | 1                | 0                |
| $q^3B_{3\delta + \alpha_0}$ | 21          | 0                | 5                | 0                | 1                | 0                | 0                |
| $q^4B_{4\delta + \alpha_0}$ | 0           | 6                | 0                | 1                | 0                | 0                | 0                |
| $q^5B_{5\delta + \alpha_0}$ | 0           | 1                | 0                | 0                | 0                | 0                | 0                |
| $q^6B_{6\delta + \alpha_0}$ | 1           | 0                | 0                | 0                | 0                | 0                | 0                |

$W_{-7}$ is equal to $-(q - q^{-1})^{-1}[2]_q^{-9}$ times

| $q^{-7}B_{6\delta + \alpha_1}$ | $\tilde{G}_0$ | $[2]_q\tilde{G}_1$ | $[2]_q^2\tilde{G}_2$ | $[2]_q^3\tilde{G}_3$ | $[2]_q^4\tilde{G}_4$ | $[2]_q^5\tilde{G}_5$ | $[2]_q^6\tilde{G}_6$ | $[2]_q^7\tilde{G}_7$ |
|-----------------------------|-------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1                          | 0           | 0                | 0                | 0                | 0                | 0                | 0                | 0                |
| $q^{-6}B_{5\delta + \alpha_1}$ | 0           | 1                | 0                | 0                | 0                | 0                | 0                | 0                |
| $q^{-5}B_{4\delta + \alpha_1}$ | 7           | 0                | 1                | 0                | 0                | 0                | 0                | 0                |
| $q^{-4}B_{3\delta + \alpha_1}$ | 0           | 6                | 0                | 1                | 0                | 0                | 0                | 0                |
| $q^{-3}B_{2\delta + \alpha_1}$ | 21          | 0                | 5                | 0                | 1                | 0                | 0                | 0                |
| $q^{-2}B_{\delta + \alpha_1}$ | 0           | 15               | 0                | 4                | 0                | 1                | 0                | 0                |
| $q^{-1}B_{\alpha_1}$        | 35          | 0                | 10               | 0                | 3                | 0                | 1                | 0                |
| $B_{\alpha_0}$              | 0           | 20               | 0                | 6                | 0                | 2                | 0                | 1                |
| $qB_{\delta + \alpha_0}$    | 35          | 0                | 10               | 0                | 3                | 0                | 1                | 0                |
| $q^2B_{2\delta + \alpha_0}$ | 0           | 15               | 0                | 4                | 0                | 1                | 0                | 0                |
| $q^3B_{3\delta + \alpha_0}$ | 21          | 0                | 5                | 0                | 1                | 0                | 0                | 0                |
| $q^4B_{4\delta + \alpha_0}$ | 0           | 6                | 0                | 1                | 0                | 0                | 0                | 0                |
| $q^5B_{5\delta + \alpha_0}$ | 7           | 0                | 1                | 0                | 0                | 0                | 0                | 0                |
| $q^6B_{6\delta + \alpha_0}$ | 0           | 1                | 0                | 0                | 0                | 0                | 0                | 0                |
| $q^7B_{7\delta + \alpha_0}$ | 1           | 0                | 0                | 0                | 0                | 0                | 0                | 0                |

$W_1 = B_{\alpha_1} = -(q - q^{-1})^{-1}[2]_q^{-2}B_{\alpha_1}\tilde{G}_0$.

$W_2$ is equal to $-(q - q^{-1})^{-1}[2]_q^{-3}$ times

| $q^{-1}B_{\delta + \alpha_1}$ | $\tilde{G}_0$ | $[2]_q\tilde{G}_1$ |
|-----------------------------|-------------|------------------|
| 1                          | 0           | 0                |
| $B_{\alpha_1}$              | 0           | 1                |
| $qB_{\alpha_0}$             | 1           | 0                |
\( W_3 \) is equal to \(- (q - q^{-1})^{-1} [2]_q^{-4} \) times

\[
\begin{array}{|c|c|c|c|}
\hline
q^{-2}B_{2\delta + \alpha_1} & 1 & 0 & 0 \\
q^{-1}B_{\delta + \alpha_1} & 0 & 1 & 0 \\
B_{\alpha_1} & 2 & 0 & 1 \\
qB_{\alpha_0} & 0 & 1 & 0 \\
q^2B_{\delta + \alpha_0} & 1 & 0 & 0 \\
\hline
\end{array}
\]

\( W_4 \) is equal to \(- (q - q^{-1})^{-1} [2]_q^{-5} \) times

\[
\begin{array}{|c|c|c|c|c|}
\hline
q^{-3}B_{3\delta + \alpha_1} & 1 & 0 & 0 & 0 \\
q^{-2}B_{2\delta + \alpha_1} & 0 & 1 & 0 & 0 \\
q^{-1}B_{\delta + \alpha_1} & 3 & 0 & 1 & 0 \\
B_{\alpha_1} & 0 & 2 & 0 & 1 \\
qB_{\alpha_0} & 3 & 0 & 1 & 0 \\
q^2B_{\delta + \alpha_0} & 0 & 1 & 0 & 0 \\
q^3B_{2\delta + \alpha_0} & 1 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\( W_5 \) is equal to \(- (q - q^{-1})^{-1} [2]_q^{-6} \) times

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
q^{-4}B_{4\delta + \alpha_1} & 1 & 0 & 0 & 0 & 0 \\
q^{-3}B_{3\delta + \alpha_1} & 0 & 1 & 0 & 0 & 0 \\
q^{-2}B_{2\delta + \alpha_1} & 4 & 0 & 1 & 0 & 0 \\
q^{-1}B_{\delta + \alpha_1} & 0 & 3 & 0 & 1 & 0 \\
B_{\alpha_1} & 6 & 0 & 2 & 0 & 1 \\
qB_{\alpha_0} & 0 & 3 & 0 & 1 & 0 \\
q^2B_{\delta + \alpha_0} & 4 & 0 & 1 & 0 & 0 \\
q^3B_{2\delta + \alpha_0} & 0 & 1 & 0 & 0 & 0 \\
q^4B_{3\delta + \alpha_0} & 1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\( W_6 \) is equal to \(- (q - q^{-1})^{-1} [2]_q^{-7} \) times

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
q^{-5}B_{5\delta + \alpha_1} & 1 & 0 & 0 & 0 & 0 & 0 \\
q^{-4}B_{4\delta + \alpha_1} & 0 & 1 & 0 & 0 & 0 & 0 \\
q^{-3}B_{3\delta + \alpha_1} & 5 & 0 & 1 & 0 & 0 & 0 \\
q^{-2}B_{2\delta + \alpha_1} & 0 & 4 & 0 & 1 & 0 & 0 \\
q^{-1}B_{\delta + \alpha_1} & 10 & 0 & 3 & 0 & 1 & 0 \\
B_{\alpha_1} & 0 & 6 & 0 & 2 & 0 & 1 \\
qB_{\alpha_0} & 10 & 0 & 3 & 0 & 1 & 0 \\
q^2B_{\delta + \alpha_0} & 0 & 4 & 0 & 1 & 0 & 0 \\
q^3B_{2\delta + \alpha_0} & 5 & 0 & 1 & 0 & 0 & 0 \\
q^4B_{3\delta + \alpha_0} & 0 & 1 & 0 & 0 & 0 & 0 \\
q^5B_{4\delta + \alpha_0} & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]
\( W_7 \) is equal to \(-(q - q^{-1})^{-1} [2]_q^{-8} \) times

\[
\begin{array}{c|cccccccc}
q^{-6}B_{6\delta+\alpha_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^{-5}B_{5\delta+\alpha_1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ \\
q^{-4}B_{4\delta+\alpha_1} & 6 & 0 & 1 & 0 & 0 & 0 & 0 \ \\
q^{-3}B_{3\delta+\alpha_1} & 0 & 5 & 0 & 1 & 0 & 0 & 0 \ \\
q^{-2}B_{2\delta+\alpha_1} & 15 & 0 & 4 & 0 & 1 & 0 & 0 \ \\
q^{-1}B_{\delta+\alpha_1} & 0 & 10 & 0 & 3 & 0 & 1 & 0 \ \\
B_{\alpha_1} & 20 & 0 & 6 & 0 & 2 & 0 & 1 \ \\
qB_{\alpha_0} & 0 & 10 & 0 & 3 & 0 & 1 & 0 \ \\
q^2B_{\delta+\alpha_0} & 15 & 0 & 4 & 0 & 1 & 0 & 0 \ \\
q^3B_{2\delta+\alpha_0} & 0 & 5 & 0 & 1 & 0 & 0 & 0 \ \\
q^4B_{3\delta+\alpha_0} & 6 & 0 & 1 & 0 & 0 & 0 & 0 \ \\
q^5B_{4\delta+\alpha_0} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ \\
q^6B_{5\delta+\alpha_0} & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\( W_8 \) is equal to \(-(q - q^{-1})^{-1} [2]_q^{-9} \) times

\[
\begin{array}{c|cccccccc}
q^{-7}B_{7\delta+\alpha_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^{-6}B_{6\delta+\alpha_1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ \\
q^{-5}B_{5\delta+\alpha_1} & 7 & 0 & 1 & 0 & 0 & 0 & 0 \ \\
q^{-4}B_{4\delta+\alpha_1} & 0 & 6 & 0 & 1 & 0 & 0 & 0 \ \\
q^{-3}B_{3\delta+\alpha_1} & 21 & 0 & 5 & 0 & 1 & 0 & 0 \ \\
q^{-2}B_{2\delta+\alpha_1} & 0 & 15 & 0 & 4 & 0 & 1 & 0 \ \\
q^{-1}B_{\delta+\alpha_1} & 35 & 0 & 10 & 0 & 3 & 0 & 1 & 0 \ \\
B_{\alpha_1} & 0 & 20 & 0 & 6 & 0 & 2 & 0 & 1 \ \\
qB_{\alpha_0} & 35 & 0 & 10 & 0 & 3 & 0 & 1 & 0 \ \\
q^2B_{\delta+\alpha_0} & 0 & 15 & 0 & 4 & 0 & 1 & 0 & 0 \ \\
q^3B_{2\delta+\alpha_0} & 21 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \ \\
q^4B_{3\delta+\alpha_0} & 0 & 6 & 0 & 1 & 0 & 0 & 0 & 0 \ \\
q^5B_{4\delta+\alpha_0} & 7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ \\
q^6B_{5\delta+\alpha_0} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ \\
q^7B_{6\delta+\alpha_0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

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