Time dependent coupled harmonic oscillators

Alejandro R. Urzúa and Héctor M. Moya-Cessa

Instituto Nacional de Astrofísica, Óptica y Electrónica,
Calle Luis Enrique Erro No. 1, Santa María Tonantzintla, Puebla, 72840, Mexico

Irán Ramos-Prieto

Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México,
Apartado Postal 48-3, 62251 Cuernavaca, Morelos, Mexico

Manuel Fernández Guasti

Departamento de Física, CBI, Universidad Autónoma Metropolitana Iztapalapa,
Apartado Postal 55-534, México D.F. 09340, Mexico

(Dated: May 22, 2019)

We show that, by using the quantum orthogonal functions invariant, we are able to solve a coupled of time dependent harmonic oscillators where all the time dependent frequencies are arbitrary. We do so, by transforming the time dependent Hamiltonian of the interaction by a set of unitary operators. In passing, we show that $N$ time dependent and coupled oscillators have a generalized orthogonal functions invariant from which we can write a Ermakov-Lewis invariant.

I. INTRODUCTION

The existence of invariants in mechanical systems for time dependent Hamiltonian has attracted considerable interest over the years [1]. Such constants of motion are of central importance in the study of dynamical systems. A variety of methods to obtain invariants of systems with one degree of freedom have been developed [2]. In particular, the time dependent harmonic oscillator (TDHO) has received much attention because of its applications in several areas of physics [3]. Among the many procedures developed to obtain invariants, a derivation for the classical TDHO has been presented, that leads directly to the orthogonal functions invariant or to the Lewis invariant [4]. The study of exact invariants has led to the nonlinear superposition principle as well as the obtainment of general solutions provided that a particular solution is known.

The extension of the theory of invariants to the quantum realm has evolved in, at least, two directions. On the one hand, the one dimensional time independent Schrödinger equation is formally equivalent to the TDHO equation. The translation between equations requires the exchange of temporal and spatial variables as well as a constant shift of the potential $V(x)$ with the appropriate scaling for the initially time dependent parameter $\Omega^2(t) \to 2m(E - V(x))$. The results obtained in the classical invariant theory are thus applicable for spatially arbitrary time independent potentials in stationary one dimensional quantum theory. Using these technique it has been possible to define coherent states associated to the TDHO [5] and amplitude-phase invariants [6]. On the other hand, quantum mechanical expressions of the classical invariant operators have been used in order to obtain exact solutions to the time dependent Schrödinger equation. To this end, the classical Hamiltonian is translated into a quantum Hamiltonian by considering the canonical coordinate and momentum as time independent operators obeying the commutation relationship $[\hat{q}, \hat{p}] = i$ (we will set $\hbar = 1$ throughout the manuscript). The quantum treatment becomes then a 1+1 dimensional problem where the wave function depends on a spatial as well as the temporal variable. A potential with an arbitrary time dependence is identified with the coordinate operator of the Hamiltonian. Exact invariants have been derived to tackle a limited class of admissible potentials [7]. The most relevant cases are the linear potential [8] and the quadratic spatial dependence that leads to the quantum mechanical time dependent harmonic oscillator (QM-TDHO).

On the other hand, the simple extension to two coupled time dependent harmonic oscillators has been considered and solution to it have been presented for a very limited case of time dependent functions [14]. Ermakov-Lewis invariant have been also proposed for systems of couple harmonic oscillators [15].

The QM-TDHO has been solved under various scenarios such as time dependent mass [9, 10] and damping [11]. Several techniques have been used to solve the corresponding time dependent Schrödinger equation such as the time-space re-scaling or transformation method and the time dependent invariant method [12]. The constant of motion that has been invoked in the latter procedure is the well known Lewis invariant [13].

*Corresponding author: hmmc@inaoep.mx
The main purpose of the present contribution is to show a method to solve the Schrödinger equation for a pair of coupled time dependent harmonic oscillators when all the time dependent functions involved are arbitrary, i.e., they are not related to each other. In passing, we write the Ermakov-Lewis invariant for \( N \) coupled time dependent harmonic oscillators.

By a series of unitary transformations, some of them, time dependent, we manage to take the Hamiltonian for the two coupled harmonic oscillators to an integrable form.

II. ERMakov-LEwIS INVARIANT FOR \( N \) COUPLED TIME DEPENDENT HARMONIC OSCILLATORS

Consider the system of differential equations for \( N \) time dependent coupled classical oscillators

\[
\begin{align*}
\ddot{u}_1 + \Omega_1^2(t)u_1 &= -\eta_{12}(t)u_2 \\
\ddot{u}_2 + \Omega_2^2(t)u_2 &= -\eta_{12}(t)u_1 - \eta_{23}(t)u_3 \\
\ddot{u}_3 + \Omega_3^2(t)u_3 &= -\eta_{23}(t)u_2 - \eta_{34}(t)u_4 \\
&\vdots \\
\ddot{u}_N + \Omega_N^2(t)u_N &= -\eta_{N-1N}(t)u_{N-1},
\end{align*}
\]

(1)

with the associated quantum Hamiltonian

\[
\hat{H}_N(t) = \frac{1}{2} (\hat{p}_1^2 + \Omega_1^2(t)\hat{x}_1^2) + \frac{1}{2} (\hat{p}_2^2 + \Omega_2^2(t)\hat{x}_2^2) + \cdots + \frac{1}{2} (\hat{p}_N^2 + \Omega_N^2(t)\hat{x}_N^2) \\
+ \eta_{12}(t)\hat{x}_1\hat{x}_2 + \eta_{23}(t)\hat{x}_2\hat{x}_3 + \cdots + \eta_{N-1N}(t)\hat{x}_{N-1}\hat{x}_N.
\]

(2)

A single time dependent harmonic oscillator has quantum orthogonal functions invariant [16],

\[
\hat{G}_1 = u_1(t)\hat{p}_1 - \dot{u}_1(t)\hat{x}_1,
\]

(3)

where \( u_1(t) \) is the solution of the equation \( \ddot{u}_1 + \Omega_1^2(t)u_1 = 0 \). This invariant may be generalized to \( N \) coupled time dependent harmonic oscillators,

\[
\hat{G}_N = u_1(t)\hat{p}_1 - \dot{u}_1(t)\hat{x}_1 + u_2(t)\hat{p}_2 - \dot{u}_2(t)\hat{x}_2 + \cdots + u_N(t)\hat{p}_N - \dot{u}_N(t)\hat{x}_N,
\]

(4)

where the \( u \)'s satisfy (1), such that

\[
\frac{\partial \hat{G}_N}{\partial t} = \dot{u}_1(t)\hat{p}_1 - \dot{u}_1(t)\hat{x}_1 + \dot{u}_2(t)\hat{p}_2 - \dot{u}_2(t)\hat{x}_2 + \cdots + \dot{u}_N(t)\hat{p}_N - \dot{u}_N(t)\hat{x}_N.
\]

(5)

On the other hand, the commutator between \( \hat{G}_N \) and \( \hat{H}_N \), is

\[
i[\hat{G}_N, \hat{H}_N] = \Omega_1^2(t)u_1\dot{\hat{x}}_1 + \dot{u}_1(t)\hat{p}_1 + \eta_{12}u_1\dot{\hat{x}}_2 \\
+ \Omega_2^2(t)u_2\dot{\hat{x}}_2 + \dot{u}_2(t)\hat{p}_2 + \eta_{12}u_2\dot{\hat{x}}_1 + \eta_{23}u_2\dot{\hat{x}}_3 \\
+ \cdots \\
+ \Omega_N^2(t)u_N\dot{\hat{x}}_N + \dot{u}_N(t)\hat{p}_N + \eta_{N-1N}u_N\dot{\hat{x}}_{N-1},
\]

(6)

by subtracting the above equations we obtain

\[
\frac{d\hat{G}_N}{dt} = \frac{\partial \hat{G}_N}{\partial t} - i[\hat{G}_N, \hat{H}_N] = -[\Omega_1^2(t)u_1 + \dot{u}_1]\dot{\hat{x}}_1 - \eta_{12}u_1\dot{\hat{x}}_2 \\
- [\Omega_2^2(t)u_2 + \dot{u}_2]\dot{\hat{x}}_2 - \eta_{12}u_2\dot{\hat{x}}_1 - \eta_{23}u_2\dot{\hat{x}}_3 \\
+ \cdots \\
- [\Omega_N^2(t)u_N + \dot{u}_N]\dot{\hat{x}}_N - \eta_{N-1N}u_N\dot{\hat{x}}_{N-1}.
\]

(7)
Rearranging the above expression
\[
\frac{d\hat{G}_N}{dt} = \frac{\partial \hat{G}_N}{\partial t} - i[\hat{G}_N, \hat{H}_N] = -[\Omega_1^2(t)u_1 + \ddot{u}_1 + \eta_{12}(t)u_2]\ddot{x}_1 \\
\quad - [\Omega_2^2(t)u_2 + \ddot{u}_2 + \eta_{12}(t)u_1 + \eta_{23}(t)u_3]\ddot{x}_2 \\
\vdots \\
\quad - [\Omega_N^2(t)u_N + \ddot{u}_N + \eta_{N-1N}(t)u_{N-1}]\ddot{x}_N
\]  
(8)
that from (1) gives zero showing that \(\hat{G}_N\) is indeed an invariant.

If we write, for the single harmonic oscillator \(u_1 = \rho_1 \exp(-i \int \frac{dt}{\rho_1})\), where \(\rho_1\) obeys the Ermakov equation [16]
\[
\ddot{\rho}_1 + \Omega_1^2(t)\rho_1 = \frac{1}{\rho_1^3}
\]  
(9)
The so-called Ermakov-Lewis invariant may be obtained from \(\hat{G}_1\) as
\[
\hat{I}_1 = \hat{G}_1\hat{G}_1^\dagger = \frac{1}{2} \left( \frac{\ddot{x}_1^2}{\rho_1} + (\rho_1\ddot{\rho}_1 - \rho_1\dot{\rho}_1)^2 \right),
\]  
(10)
such that we may write the Ermakov-Lewis invariant for the \(N\) coupled time dependent harmonic oscillators as
\[
\hat{I}_N = \hat{G}_N\hat{G}_N^\dagger.
\]  
(11)

A. The classical invariant

By doing \(\dot{\rho} \rightarrow \dot{v}\) and \(\dot{x} \rightarrow v\) in (4) we find the classical invariant
\[
G_N = u_1(t)v_1(t) - \dot{u}_1(t)v_1(t) + u_2(t)v_2(t) - \dot{u}_2(t)v_2(t) + \cdots + u_N(t)v_N(t) - \dot{u}_N(t)v_N(t),
\]  
(12)
where the \(u\)’s and \(v\)’s are linearly independent solutions of (1).

III. TWO-COUPLED TIME DEPENDENT HARMONIC OSCILLATORS

We consider the time dependent Hamiltonian for the interacting oscillators as
\[
\hat{H}(t) = \frac{1}{2} \left[ \dot{\hat{p}}_x^2 + \dot{\hat{p}}_y^2 + \Omega_x^2(t)\ddot{x}^2 + \Omega_y^2(t)\ddot{y}^2 \right] + \eta(t)\ddot{x}\ddot{y}.
\]  
(13)
The classical equations of motion for the above Hamiltonian are
\[
\ddot{u}_x + \Omega_x^2(t)u_x = -\eta(t)u_y, \quad \ddot{u}_y + \Omega_y^2(t)u_y = -\eta(t)u_x,
\]  
(14)
where the quantum invariants of each coupled oscillator are [17]
\[
\hat{G}_x = (u_x\hat{p}_x - \dot{\hat{x}}u_x) - \int \eta(t)(u_x\ddot{y} - \dddot{u}_y)\, dt,
\]  
(15)
and
\[
\hat{G}_y = (u_y\hat{p}_y - \dot{\hat{y}}u_y) - \int \eta(t)(u_y\dddot{x} - \dddot{u}_x)\, dt,
\]  
(16)
since the total invariant must comply with
\[
\frac{\partial}{\partial t} (\hat{G}_x + \hat{G}_y) - i[\hat{G}_x + \hat{G}_y, \hat{H}] = 0.
\]  
(17)
We now consider the transformation \cite{16}

\[
\hat{T}_u = e^{i \frac{\mu(x)}{2} (\hat{p}_x + \hat{p}_x)} e^{-i \frac{\nu(y)}{2} (\hat{p}_y + \hat{p}_y)} e^{i \frac{\lambda(y)}{2} (\hat{p}_y + \hat{p}_y)} e^{-i \frac{\tilde{\lambda}(t)}{2} \hat{q}^2}
\]

that produces

\[
\begin{align*}
\hat{T}_u \hat{x} \hat{T}_u^\dagger &= u_x \hat{x} \\
\hat{T}_u \hat{y} \hat{T}_u^\dagger &= u_y \hat{y} \\
\hat{T}_u \hat{p}_x \hat{T}_u^\dagger &= \hat{p}_x \hat{p}_x + \hat{u}_x \hat{x}, \\
\hat{T}_u \hat{p}_y \hat{T}_u^\dagger &= \hat{p}_y \hat{p}_y + \hat{u}_y \hat{y},
\end{align*}
\]

where \( u \) is the solution to TDHO Eq. \((14)\).

If we transform the wave function with the transformation above, i.e.,

\[
|\phi_u(t)\rangle = \hat{T}_u |\psi(t)\rangle
\]

the Schrödinger equation

\[
i \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t) |\psi(t)\rangle
\]

has to be rewritten. In order to do it, by substitution in the above equation \((21)\) leads to:

\[
i \left( \hat{T}_u^\dagger \frac{\partial |\phi_u(t)\rangle}{\partial t} + \frac{\partial \hat{T}_u^\dagger}{\partial t} |\phi_u(t)\rangle \right) = \hat{H}(t) \hat{T}_u^\dagger |\phi_u(t)\rangle.
\]

By noting that

\[
\frac{\partial \hat{T}_u^\dagger}{\partial t} = \frac{i}{2} \hat{T}_u \left[ (u_x \hat{u}_x - \hat{u}_x^2) \hat{x}^2 - \frac{\hat{u}_x}{u_x} (\hat{p}_x \hat{x} + \hat{x} \hat{p}_x) + (u_y \hat{u}_y - \hat{u}_y^2) \hat{y}^2 - \frac{\hat{u}_y}{u_y} (\hat{p}_y \hat{y} + \hat{y} \hat{p}_y) \right],
\]

or

\[
i \frac{\partial |\phi_u(t)\rangle}{\partial t} = \frac{1}{2} \left[ \frac{\hat{p}_x^2}{u_x^2} + \Omega_x^2 \hat{u}_x + \hat{u}_x \hat{x}^2 + \frac{\hat{p}_y^2}{u_y^2} + \Omega_y^2 \hat{u}_y + \hat{u}_y \hat{y}^2 + \eta(t) u_x u_y \hat{x} \hat{y} \right] |\phi_u(t)\rangle,
\]

from \((14)\) we may rewrite the Schrödinger equation as

\[
i \frac{\partial |\phi_u(t)\rangle}{\partial t} = \frac{1}{2} \left[ \frac{\hat{p}_x^2}{u_x^2} + \frac{\hat{p}_y^2}{u_y^2} - \eta(t) u_x u_y (\hat{x} - \hat{y})^2 \right] |\phi_u(t)\rangle.
\]

We now perform a second transformation, \(|\phi_\theta\rangle = \hat{R}_\theta |\phi_u\rangle\), with \(\hat{R}_\theta = \exp[i \theta (\hat{x} \hat{p}_y - \hat{y} \hat{p}_x)]\)

\[
i \frac{\partial |\phi_\theta(t)\rangle}{\partial t} = \hat{H}_\theta(t) |\phi_\theta(t)\rangle
\]

such that the different operators are transformed according to

\[
\begin{align*}
\hat{R}_\theta \hat{x} \hat{R}_\theta^\dagger &= \hat{x} \cos \theta - \hat{y} \sin \theta, \\
\hat{R}_\theta \hat{y} \hat{R}_\theta^\dagger &= \hat{y} \cos \theta + \hat{x} \sin \theta, \\
\hat{R}_\theta \hat{p}_x \hat{R}_\theta^\dagger &= \hat{p}_x \cos \theta - \hat{p}_y \sin \theta, \\
\hat{R}_\theta \hat{p}_y \hat{R}_\theta^\dagger &= \hat{p}_y \cos \theta + \hat{p}_x \sin \theta.
\end{align*}
\]

By setting \(\theta = \pi/4\) to arrive to the integrable equation

\[
\frac{\partial |\phi_\theta(t)\rangle}{\partial t} = \left[ \frac{\hat{p}_x^2 + \hat{p}_y^2}{2\mu(t)} + \frac{\hat{p}_x \hat{p}_y}{2\nu(t)} + \lambda(t) \hat{y}^2 \right] |\phi_\theta(t)\rangle,
\]
with
\[
\frac{1}{\mu(t)} = \frac{1}{2u_x^2} + \frac{1}{2u_y^2}, \quad \frac{1}{\nu(t)} = \frac{1}{u_y^2} - \frac{1}{u_y^2}, \quad \lambda(t) = \frac{\eta(t)u_xu_y}{\sqrt{2}}.
\]

Note that the Hamiltonian in equation (28) shows that the operators involved in the variable $\hat{x}$ commute as they are simple powers of $\hat{p}_x$, such that the equation (28) is readily solvable as this operator act as a $c$-number for the variable $\hat{y}$. Therefore we have been able to split the Hamiltonian into two a term that is a free particle in $\hat{x}$ (time dependent) and a TDHO in $\hat{x}$ with an extra term, damping, proportional to $\hat{p}_x$.

In order to take the equation above to a more familiar form, we transform, with the unitary operator $\hat{D} = \exp\{i\alpha(t)\hat{p}_y\} \exp\{\beta(t)\hat{y}\}$, the above equation, namely $|\phi_D\rangle = \hat{D}|\phi\rangle$ we obtain the final and solvable form of the Hamiltonian
\[
\frac{i}{\hbar} \frac{\partial |\phi_D(t)\rangle}{\partial t} = \left[ \frac{\hat{p}_y^2}{2\mu(t)} + \frac{\hat{p}_x^2}{2\nu(t)} - \frac{\beta}{2\nu(t)} \hat{p}_x + \frac{\beta^2}{2\mu(t)} \right] |\phi_D(t)\rangle,
\]
where $\alpha$ and $\beta$ are functions not only of time but of the momentum operator $\hat{p}_x$ and obey the system of differential equations
\[
\dot{\alpha} + \frac{\beta}{\mu} - \frac{\hat{p}_x}{2\nu} = 0, \quad \dot{\beta} - 2\lambda\alpha = 0.
\]

We can note that the Hamiltonian in equation (30) has been separated in two parts: one of them a time dependent harmonic oscillator that depends only on $\hat{y}$ and $\hat{p}_y$ (and powers of them) and therefore there are Ermakov-Lewis methods to solve it and the other part that depends only on $\hat{p}_x$ (and its powers) and therefore, it is integrable [16].

Finally, it is worth to mention how the different transformations act on wavefunctions. It is not difficult to show that
\[
\hat{R}_\theta \phi(x, y) = \phi \left[ \frac{\cos(\theta) x - \sin(\theta) y}{\cos^2(\theta)}, \sin(\theta) x + \cos(\theta) y \right],
\]
where $\phi(x, y)$ is an arbitrary, but well behaved, function of $x$ and $y$.

To study the action of the $\hat{R}_\theta$ operator over an arbitrary function $F(x, y)$, we make
\[
\hat{R}_\theta = \exp\{i\theta (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\} = \hat{T}_y \hat{S}_{xy} \hat{T}_x,
\]
where
\[
\hat{T}_y = \exp\{i\tan\theta \hat{x}\hat{p}_y\}, \quad \hat{S}_{xy} = \exp\{-i \ln[\cos\theta] (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\}, \quad \hat{T}_x = \exp\{-i \tan\theta \hat{y}\hat{p}_x\}.
\]
Note that the operator $\hat{S}_{xy}$ is a product of squeeze operators [18–22] in $x$ and $y$. We can prove that as
\[
\hat{T}_x F(x) = F(x - y \tan\theta), \quad \hat{T}_y G(y) = G(y + x \tan\theta),
\]
and the action of the squeeze operators
\[
\exp(\imath r \hat{p}_x \hat{x}) x = \exp(2r)x, \quad \exp(\imath r \hat{p}_y \hat{y}) y = \exp(2r)y.
\]

IV. CONCLUSIONS

We have shown that the quantum invariant for $N$-coupled time dependent harmonic oscillators is indeed constant for arbitrary restitutive oscillator time dependent functions as well as arbitrary time dependent coupling between them. We have translated this result to its classical version. In the case of two oscillators, we have shown how to solve the Hamiltonian for arbitrary functions of time, as formerly it had been solved only when the functions were related in specific ways [14]. We did it by using the orthogonal functions invariant introduced in reference [16] that allowed us to split the Hamiltonian in such a way that it was left to solve a single time dependent harmonic oscillator, which is a well-known problem [13, 14].
The Ermakov-Lewis invariant for a single oscillator does not involve the time dependent parameters explicitly. This well known fact is not fulfilled when each coupled oscillator is considered separately. The quantum invariants $\hat{G}_x$ and $\hat{G}_y$ given by (15) and (16), involve the coupling variable $\eta(t)$. However, the Ermakov-Lewis invariant of the whole system, in this case, the two coupled oscillators, i.e. $\hat{G}_x + \hat{G}_y$ no longer involves $\eta(t)$. Therefore, the invariant of the complete system is again explicitly independent of the time varying parameters. This remark is also evinced for the $N$-coupled system. The invariant for $N$-coupled oscillators (4), is the Ermakov invariant of the complete system, it is again explicitly independent of any of the time varying parameters.

[1] S. Bouquet and H.R. Lewis, J. Math. Phys. 37 (11), 5509, (1996).
[2] J.R. Ray and J.L. Reid, Phys. Rev. A 26 (2) 1042 (1982).
[3] R.K. Colegrave and M.A. Mannan, J. Math. Phys. 29 (7), 1580, (1988).
[4] M. Fernández Guasti and A. Gil-Villegas, Phys. Lett. A 292,
[5] H Moya-Cessa and MF Guasti, Coherent states for the time dependent harmonic oscillator: the step function. Physics Letters A 311, 1-5 (2003).
[6] M Fernández Guasti and H Moya-Cessa, Amplitude and phase representation of quantum invariants for the time-dependent harmonic oscillator. Physical Review A 67, 063803 (2003).
[7] H.R. Lewis and P.G.L. Leach, J. Math. Phys. 23 (1) 165 (1982).
[8] I. Guedes, Phys. Rev. A 63 034102 (2001).
[9] H. Moya-Cessa, M. Fernández Guasti, Time dependent quantum harmonic oscillator subject to a sudden change of mass: continuous solution. Revista Mexicana de Fisica 53, 42-46 (2007).
[10] I. Ramos-Prieto, A. Espinosa-Zuniga, M. Fernández-Guasti and H. M. Moya-Cessa, Phys. Lett. B 32, 1850235 (2018).
[11] Kyu-Hwang Yeon, Duk-Hyeon Kim, Chung-In Um, Thomas F. George, and Lakshmi N. Pandey, Phys. Rev. A 55, 4023 (1997). Relations of canonical and unitary transformations for a general time-dependent quadratic Hamiltonian system
[12] J.R. Ray, Phys. Rev. A 22 (2) 729 (1982).
[13] H.R. Lewis, Phys. Rev. Lett. 18 510 (1967).
[14] D. X. Macedo and I. Guedes, Time-dependent coupled harmonic oscillators. J. Math. Phys. 53, 052101 (2012).
[15] K.-E. Thylwe and H. J. Korsch, The ‘Ermakov-Lewis’ invariants for coupled linear oscillators. J. Phys. A: Math. Gen. 31 (1998) L279-L285 (1998).
[16] M Fernández Guasti and H Moya-Cessa, Solution of the Schrödinger equation for time-dependent 1D harmonic oscillators using the orthogonal functions invariant. Journal of Physics A 36, 2069 (2003).
[17] M. Fernandez-Guasti, Energy content in linear mechanical systems with arbitrary time dependence, Phys. Lett. A 382(2018)3231-3237.
[18] Loudon, R. and Knight, P. L., “Squeezed light,”, J. Mod. Opt. 34, 709-759 (1987).
[19] Yuen, H. P., “Two-photon coherent states of the radiation field,” Phys. Rev. A 13, 2226-2243 (1976).
[20] C.M. Caves, C. M., “Quantum-mechanical noise in an interferometer,” Phys. Rev. D 23, 1693-1708 (1981).
[21] Moya-Cessa, H. and Vidiella-Barranco, A. Interaction of squeezed states of light with two-level atoms. J. of Mod. Optics 39 2481 (1992).
[22] Barnett, S.M., Beige, A., Ekert, A., Garraway, B.M., et al., "Journeys from quantum optics to quantum technology," Progress in Quantum Electronics 54, 19-45 (2017).