On the $L^\infty-$maximization of the solution of Poisson’s equation:
Brezis-Gallouet-Wainger type inequalities and applications

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Abstract

For the solution of the Poisson problem with an $L^\infty$ right hand side
\[
\begin{aligned}
-\Delta u(x) &= f(x) \quad \text{in } D, \\
\phantom{u} u &= 0 \quad \text{on } \partial D,
\end{aligned}
\]
we derive an optimal estimate of the form
\[
\|u\|_\infty \leq \|f\|_\infty \sigma_D(\|f\|_1/\|f\|_\infty),
\]
where $\sigma_D$ is a modulus of continuity defined in the interval $[0, |D|]$ and depends only on the domain $D$. In the case when $f \geq 0$ in $D$ the inequality is optimal for any domain and for any values of $\|f\|_1$ and $\|f\|_\infty$. We also show that
\[
\sigma_D(t) \leq \sigma_B(t), \quad \text{for } t \in [0, |D|],
\]
where $B$ is a ball and $|B| = |D|$. Using this optimality property of $\sigma$, we derive Brezis-Gallouet-Wainger type inequalities on the $L^\infty$ norm of $u$ in terms of the $L^1$ and $L^\infty$ norms of $f$. The estimates have explicit coefficients depending on the space dimension $n$ and turn to equality for a specific choice of $u$ when the domain $D$ is a ball. As an application we derive $L^\infty - L^1$ estimates on the $k$-th Laplace eigenfunction of the domain $D$.

1 Introduction and problem setting

The Brezis-Gallouet inequality \cite{BG} is a crucial tool in establishing the unique solvability of initial boundary value problem for the nonlinear Schrödinger equation with zero Dirichlet data on a smooth domain in $\mathbb{R}^2$ as shown by Brezis and Gallouet in \cite{BG}. The inequality asserts the following: For any smooth bounded domain $\Omega$, there exists a constant $C > 0$ depending only on $\Omega$, such that for any function $u \in H^2(\Omega)$ with $\|u\|_{H^1(\Omega)} \leq 1$, there holds:
\[
\|u\|_{L^\infty(\Omega)} \leq C(\Omega) \left(1 + \sqrt{\log(1 + \|u\|_{H^1(\Omega)})}\right).
\] (1.1)
Brezis and Wainger \cite{BW} extended (1.1) to higher dimensions proving the following: Assume $k, l, n \in \mathbb{N}$ and $q > 0$ are such that $1 \leq k < l$, $k \leq n$ and $ql > n$. Then there exists a constant $C > 0$ such that for any $u \in W^{k,q}(\mathbb{R}^n)$ with $\|u\|_{W^{k,q}(\mathbb{R}^n)} = 1$, one has the estimate
\[
\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \left(1 + \left(\log(1 + \|u\|_{W^{k,q}(\mathbb{R}^n)})\right)^{\frac{q}{n}}\right).
\] (1.2)

The Brezis-Gallouet-Wainger and similar inequalities and sharp constants in them have been extensively studied in different frameworks in \cite{BaPe, BaPe2, Pa, Pe}. The estimate has been known to be a central tool in the study of several types of partial differential equations \cite{Pe}, in particular Navier-Stokes equations and turbulence \cite{Pe, Pe2}. In the present paper we are concerned with estimating the $L^\infty$ norm of a function $u$ by the $L^1$ and $L^\infty$ norms of the Laplacian $\Delta u$. More precisely we are concerned with the following

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question: Given a bounded connected open Lipschitz set \( D \subset \mathbb{R}^n \) and a number \( 0 \leq \beta \leq |\Omega| \), let \( u_f \) be the unique weak solution of the Dirichlet boundary value problem

\[
\begin{cases}
-\Delta u_f = f & \text{in } D, \\
u_f = 0 & \text{on } \partial D,
\end{cases}
\]  

(1.3)

where the function \( f \) belongs to the class

\[
\mathcal{C} = \{ f : D \to \mathbb{R} : |f(x)| \leq 1 \text{ a.e. in } D, \text{ and } \int_D |f(x)| dx = \beta \}.
\]  

(1.4)

We consider the maximization problem

\[
\sup_{f \in \mathcal{C}} \Phi(f),
\]  

(1.5)

where

\[
\Phi(f) = \sup_{x \in D} |u_f(x)|.
\]  

(1.6)

The question is, what is the function \( f \) having an \( L^\infty \) norm bounded by 1 and a fixed \( L^1 \) norm, that maximizes the \( L^\infty \) norm of the solution \( u_f \)? As will be seen in the next section, we obtain a Brezis-Gallouet-Wainger type inequality on the \( L^\infty \) norm of \( u \) in terms of \( L^1 \) and \( L^\infty \) norms of the Laplacian \( \Delta u \) in all dimensions. For given \( D \subset \mathbb{R}^n \) and \( f \in \mathcal{C} \) with the additional property that \( f \geq 0 \) in \( D \), we also characterize the points \( \hat{x} \) of the function \( u \), \( n \) again with explicit coefficients. Typically working with \( L^1 \) and \( L^\infty \) norms of solutions of partial differential equations is a more delicate task than dealing with \( L^p \) norms for \( 1 < p < \infty \). It is well known for instance, that one cannot estimate the \( L^\infty \) norm of the solution \( u \) by the \( L^1 \) norm of \( f \). The trivial counter-example for the Poisson problem in the ball \( B_1 \)

\[
\begin{cases}
-\Delta u(x) = f(x) & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1,
\end{cases}
\]  

(1.7)

is given by the family of radially symmetric functions

\[
u_r(|x|) = \begin{cases} 
F(1) - F(|x|) & \text{in } B_1 \setminus B_r, \\
F(1) - F(r) + \frac{1}{2r^2}|x|^2 & \text{in } B_r,
\end{cases}
\]  

(1.8)

where \( F(|x|) \) is the fundamental solution multiplied by a dimensional constant such that \( F'(r) = \frac{1}{2r} \). Then all the functions \( f_r(x) = -\Delta u_r(x) = \frac{1}{2r^2} \chi_{B_r}(x), 0 < r < 1 \), have the same \( L^1 \) norm, while the family of solutions \( u_r \) are unbounded in \( L^\infty \).

2 Main results

In what follows all the norms will be over the domain \( D \) unless otherwise specified. Thus for the sake of convenience we will leave out the domain \( D \) from the norm notation \( \|v\|_{L^p(\Omega)} \) simply using the notation \( \|v\|_p \) when there is no ambiguity. We consider first the case when \( f \) is nonnegative. For that purpose denote

\[
\mathcal{C}^+ = \{ f : D \to \mathbb{R} : 0 \leq f(x) \leq 1 \text{ a.e. in } D, \text{ and } \int_D f(x) dx = \beta \}.
\]  

(2.1)

We have the following existence and characterization theorem.

Theorem 2.1 (Nonnegative Laplacian). The maximization problem

\[
\sup_{f \in \mathcal{C}^+} \Phi(f),
\]  

(2.2)
has a solution \( \hat{f} \in C^+ \). For a maximizer \( \hat{f} \) there exists a unique point \( \hat{x} \) such that
\[
\hat{f}(x) = \chi_{\{y : G(\hat{x}, y) > a_2\}}(x) \in C^+,
\] (2.3)
where \( G(x, y) \) is the Green’s function and \( a_2 \) is chosen to satisfy
\[
|\{y : G(\hat{x}, y) > a_2\}| = \beta.
\] (2.4)
Moreover, for the solution \(-\Delta \hat{u} = \hat{f} \), \( \hat{u} \in H^1_0(D) \), one has the maximality inequality
\[
\hat{u}(x) \leq \hat{u}(\hat{x}) \quad \text{for any} \quad x \in D,
\] (2.5)
and the point \( 
\hat{x} \) satisfies the condition
\[
0 = \nabla \hat{u}(\hat{x}) = \int_D \nabla \chi_{\{y : G(\hat{x}, y) > a_2\}} \, dy.
\] (2.6)

**Theorem 2.2** (Modulus of continuity). For any \( f \in L^\infty \), such that \( \|f\|_\infty > 0 \) we have the inequality
\[
\|u\|_\infty \leq \|f\|_\infty \sigma_D(\|f\|_1/\|f\|_\infty),
\] (2.7)
where \( \sigma_D \) is a modulus of continuity defined in the interval \([0, |D|]\) and depending only on the domain \( D \). Moreover, for any domain \( D \) and values of \( \|f\|_1 \) and \( \|f\|_\infty \) there exists a function \( f \in C^+ \), such that the inequality (2.7) turns to an equality.

Next we give an optimality condition on \( \sigma_D \).

**Theorem 2.3** (Estimates on \( \sigma_D \) and \( \|u\|_\infty \)). We have the optimality estimate
\[
\sigma_D(t) \leq \sigma_B(t), \quad \text{for} \quad t \in [0, |D|],
\] (2.8)
where \( B \) is a ball with the same measure as \( D \), i.e., \( |B| = |D| \). The function \( \sigma_B \) can be calculated explicitly and has the form
\[
\sigma_B(t) = \frac{(n-1)^2 + 1}{2n(n-2)\omega_n^{2/n}} t^{\frac{n+1}{2}} \left[ 1 - \frac{1}{n(n-2)\omega_n R^{n-2}} t \right], \quad \text{for} \quad n > 2,
\]
\[
\sigma_B(t) = \left( \frac{1}{2} \ln \pi + \frac{1}{2\pi} (1 + \ln R) \right) t - \frac{1}{4\pi} t \ln t, \quad \text{for} \quad n = 2,
\]
where \( R \) is the radius of \( B \) and \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). For the norm \( \|u\|_\infty \) we have the estimates
\[
\|u\|_\infty \leq \frac{(n-1)^2 + 1}{2n(n-2)\omega_n^{2/n}} \|f\|_\infty^{\frac{n+2}{2}} \|f\|_\infty^{\frac{n-2}{2}} - \frac{1}{n(n-2)\omega_n R^{n-2}} \|f\|_1, \quad \text{for} \quad n > 2
\] (2.10)
\[
\|u\|_\infty \leq \left( \frac{1}{2} \ln \pi + \frac{1}{2\pi} (1 + \ln R) \right) \|f\|_1 - \frac{1}{4\pi} \|f\|_1 \ln \frac{\|f\|_1}{\|f\|_\infty}, \quad \text{for} \quad n = 2,
\]
here, as already mentioned, \( R = \left( \frac{|D|}{\omega_n} \right)^{\frac{1}{n}} \).

For the general case we have the following theorem which follows from Theorem 2.3.

**Theorem 2.4** (Sign-changing Laplacian). The solution \( u_f \in H^1_0(D) \) of the Poisson problem
\[
-\Delta u = f \quad \text{in} \quad D,
\]
where \( f \in L^\infty(D) \), fulfills the following inequalities:
\[
\|u_f\|_\infty \leq \max \left\{ \|f^+\|_\infty \sigma_D(\|f^+\|_1/\|f^+\|_\infty), \|f^-\|_\infty \sigma_D(\|f^-\|_1/\|f^-\|_\infty) \right\},
\] (2.11)
where \( f^\pm = \max(\pm f, 0) \). Also we have the bound
\[
\|v\|_\infty \leq \|f\|_\infty \left[ \sigma_D \left( \frac{1}{2} \left[ \frac{1}{\|f\|_1} |I_f| + |D| \right] \right) - v(\hat{x}) \right],
\] (2.12)
where the function \( v \in H^1_0(D) \) is the solution to the Poisson problem \(-\Delta v = 1 \) in \( D \), the point \( \hat{x} \in D \) is the point where the maximum of \( |u| \) is achieved, and \( I_f = \int_D f(x) \, dx \). Moreover, one has the same estimates as in (2.10):
\[
\|u_f\|_\infty \leq \frac{(n-1)^2 + 1}{2n(n-2)\omega_n^{2/n}} \|f\|_1^{\frac{n+2}{2}} \|f\|_\infty^{\frac{n-2}{2}} - \frac{1}{n(n-2)\omega_n R^{n-2}} \|f\|_1, \quad \text{for} \quad n > 2
\] (2.13)
\[
\|u_f\|_\infty \leq \left( \frac{1}{2} \ln \pi + \frac{1}{2\pi} (1 + \ln R) \right) \|f\|_1 - \frac{1}{4\pi} \|f\|_1 \ln \frac{\|f\|_1}{\|f\|_\infty}, \quad \text{for} \quad n = 2.
2.1 An application to the Laplace eigenfunctions

As a consequence of the estimates (2.13), we derive an estimate on the \(L^\infty\) norm of the \(k\)-th eigenfunction of the Laplace operator in the domain \(D\). Assume as usual that \((\lambda_k, u_k)\) is the \(k\)-th eigenvalue-eigenfunction pair of the Laplace operator, i.e.,

\[
\begin{cases}
-\Delta u_k = \lambda_k u_k & \text{in } D, \\
u_k = 0 & \text{on } \partial D.
\end{cases}
\] (2.14)

We have the following theorem.

**Theorem 2.5.** The following estimates hold:

\[
\|u_k\|_{\infty} \leq \frac{2}{n^n(n-2)\omega_n} \left[ \lambda_k^{\frac{2}{n}} \left( (n-1)^2 + 1 \right)^{\frac{1}{n}} - \frac{\lambda_k n^{n-1}}{R^{n-2}} \right] \|u_k\|_1, \text{ for } n > 2
\] (2.15)

\[
\|u_k\|_{\infty} \leq \lambda_k \left( \ln \pi + \frac{1}{\pi} (1 + \ln R) + \frac{\lambda_k}{8\pi^2} \right) \|u_k\|_1, \text{ for } n = 2.
\]

3 Proof of the main results

**Proof of Theorem 2.5** The proof is divided into several steps. Of course, in the first step we will be proving the existence part of the theorem.

**Existence of a maximizer \(f\).** We start with collecting some useful bounds. First of all note that by the estimate \(0 \leq f \leq 1\) for \(f \in C^+\) and the maximum principle \[6, Theorem 8.19\], one has on one hand that the weak solution \(u_f \in H^1_0(D)\) of \(-\Delta u = f\) in \(D\) is nonnegative in \(D\), i.e.,

\[
u_f(x) \geq 0, \quad \text{for all } f \in C^+, x \in D.
\] (3.1)

On the other hand, if \(\bar{u} \in H^1_0(D)\) is the unique weak solution of \(-\Delta \bar{u} = 1\) in \(D\), then we have \(-\Delta(\bar{u} - u_f) = 1 - f \geq 0\) in \(D\) for any \(f \in C^+\), and \(\bar{u} - u_f \in H^1_0(D)\), thus again by the maximum principle we have

\[
0 \leq u_f(x) \leq \bar{u}(x), \quad \text{for all } f \in C^+, x \in D.
\] (3.2)

By the classical regularity theory we have \(\bar{u} \in C(\bar{D})\), thus \(\|\bar{u}\|_{\infty} < \infty\). Combining the obtained bounds we arrive at

\[
0 \leq u_f(x) \leq \bar{u}(x) \leq \|\bar{u}\|_{\infty} < \infty, \quad \text{for all } f \in C^+, x \in D.
\] (3.3)

We adopt the direct method in the calculus of variations, i.e., choose a maximizing sequence \(f_k \in C^+\) for the problem (2.2), such that

\[
\Phi(f_k) > \sup_{f \in C^+} \Phi(f) - \frac{1}{k} = m - \frac{1}{k},
\] (3.4)

where we clearly have due to (3.3) the bound

\[
m = \sup_{f \in C^+} \Phi(f) \leq M = \|\bar{u}\|_{\infty} < \infty.
\]

As the sequence \(\{f_k\}\) is bounded in \(L^\infty(D)\), then by weak* compactness, there exists a function \(f_0 \in L^\infty(D)\), such that

\[
f_k \rightharpoonup f_0 \quad \text{in } L^\infty(D),
\] (3.5)

for a subsequence (not relabeled). By the convergence of the averages

\[
\int_E f_k \rightarrow \int_E f_0,
\]

for any measurable subset \(E \subset D\), we have that \(0 \leq \int_E f_0 \leq |E|\), thus we get \(0 \leq f_0(x) \leq 1\) a.e. \(x \in D\). Also it follows that \(\int_D f_0 = \beta\), thus \(f_0 \in C^+\). We aim to prove that the unique \(u_0 \in H^1_0(D)\) solution of \(-\Delta u_0 = f_0\) in \(D\) satisfies the condition \(\sup_{x \in D} |u_0(x)| = m\). Let \(u_k \in H^1_0(D)\) be the unique weak solution of the equation \(-\Delta u_k = f_k\). Then by the estimate \(\|u_k\|_{H^1} \leq C\|f\|_2\), and weak \(L^2\) compactness, there exists a function \(u \in H^1_0(D)\) such that for a subsequence (not relabeled) we have

\[
u_k \rightharpoonup u \quad \text{in } L^2(D), \quad \text{and } \nabla u_k \rightharpoonup \nabla u \quad \text{weakly in } L^2(D).
\] (3.6)
For any $\varphi \in C^1(\bar{D})$ we have

$$\int_D f_0 \varphi = \lim_{k \to \infty} \int_D f_k \varphi = \lim_{k \to \infty} \int_D \nabla u_k \nabla \varphi = \int_D \nabla u \nabla \varphi,$$

thus the function $u$ solves the equation $-\Delta u = f_0$ in $D$, therefore we have $u = u_0$. Note that as $\bar{u} \in C(\bar{D})$ and $\bar{u} = 0$ on $\partial D$, then there exists $\delta > 0$ such that $|\bar{u}(x)| < \frac{\gamma}{2}$ if $x \in D$ with $\text{dist}(x, \partial D) \leq \delta$. Taking into account (3.2), we obtain the estimate

$$0 < u_k(x) < \frac{m}{2} \quad \text{if} \quad x \in D, \quad \text{dist}(x, \partial D) \leq \delta. \quad (3.7)$$

Recall next the following classical local Hölder regularity result [3, Theorem 1.14], which we formulate below for the convenience of the reader (the below formulation is the simplified version for the Laplace operator of the original version of Theorem 8.22 in [6]).

**Theorem 3.1.** Assume $g \in L^q(D)$ for some $q > n$. Then if $v \in H^1(D)$ solves $-\Delta v = g$ in $D$, it follows that $v$ is locally Hölder continuous in $\Omega$, and for any $x \in D$, any $0 < R \leq R_0$ such that $B_{R_0} \subset D$, one has the estimate

$$\text{osc}_B v \leq CR^n(R_0^{-\gamma} \sup_{B_{R_0}(y)} |v| + \|g\|_{L^q(B_{R_0})}), \quad (3.8)$$

where the constants $C, \gamma > 0$ depend only on $n, q$ and $R_0$ and $\text{osc}_E v = \sup_E v - \inf_E v$ for any $E \subset D$.

Owing to (3.4), we choose a point $x_k \in D$ with the property that $|u_k(x_k)| > m - \frac{1}{4}$. Then (3.7) implies that for big enough $k$ one has $x_k \in D_\delta$, where $D_\delta = \{x \in D : \text{dist}(x, \partial D) \geq \delta\}$. We apply Theorem 3.1 to the function $v = u_k - u_0$ (where we take $q = 2n$) in the ball $B_\delta \subset D$, to get

$$\text{osc}_{B_{2\delta}(x_k)} (u_k - u_0) \leq C_\delta R^n(\delta^{-\gamma} \sup_{B_\delta(x_k)} |u_k - u_0| + \|f_k - f_0\|_{L^2(B_\delta)}), \quad (3.9)$$

for any $0 < R \leq \delta$. We have that $\|f_k - f_0\|_{L^{2n}(B_\delta)} \leq \|f_k - f_0\|_{L^{2n}(D)} \leq 2|D|^{\frac{1}{n}}$ and also by (3.3) that $\sup |u_k - u_0| \leq 2M$, thus we get from (3.9) the bound

$$\text{osc}_{B_{2\delta}(x_k)} (u_k - u_0) \leq CR^n, \quad (3.10)$$

for some constant $C$ depending only on $n, \delta, |D|$ and $M$. Denote next $\epsilon_k = |u_k(x_k) - u_0(x_k)|$. The goal is now to prove that $\epsilon_k \to 0$ as $k \to \infty$. Assume by contradiction that for some subsequence (not relabeled) one has $\epsilon_k \geq \epsilon > 0$. Then by (3.10) there exists a constant radius $0 < r < \delta$ such that $|u_k(x) - u_0(x)| \geq \frac{\epsilon}{2}$ for $x \in B_r(x_k)$, which implies the bound

$$\|u_k - u_0\|_{L^2(D)} \geq \int_{B_r(x_k)} |u_k(x) - u_0(x)|^2 dx \geq c_n r^n \epsilon^2,
$$

where $c_n > 0$ is a constant depending only on $n$. The last lower bound contradicts the $L^2$ convergence $u_k \to u_0$ in (3.6). The convergence $\epsilon_k \to 0$ and the inequality $|u_k(x_k)| > m - \frac{1}{4}$ give

$$\sup_{x \in D} u_0(x) \geq \liminf_{k \to \infty} u_0(x_k) \geq m,$$

which completes the proof of the existence part.

**Proof of (2.3)–(2.5).** We will utilize the following well-known argument called the bathtub principle [3, Theorem 1.14], which is formulated below.

**Theorem 3.2.** Assume $D \subset \mathbb{R}^n$ is open bounded connected and Lipschitz and assume $0 \leq \beta \leq |D|$. Let $G \in L^1(D)$ be a non-negative function without flat parts, i.e.,

$$\{|x : G(x) = l| = 0 \quad \text{for all} \quad l > 0\}$$

Then the maximization problem

$$I = \sup_{g \in C^+} \int_D G(x) g(x) dx$$

is solved by

$$\hat{g}(x) = \chi_{\{x : G(x) > \alpha\}},$$

where $\alpha$ is chosen such that

$$\{|x : G(x) > \alpha| = \beta\}.$$
Denote now $f_0 = \hat{f}$ and $u_0 = \hat{u}$. By the local Hölder continuity of the function $\hat{u}$ (Theorem 3.1) and
the property that the maximal value is taken a positive distance away from the boundary \((3.7)\), we have
that $\sup_{D} |\hat{u}(x)|$ is attained at a point $\hat{x} \in D$. Using Poisson integral formula we have that
$$\hat{u}(\hat{x}) = \int_{D} G(\hat{x}, y)\hat{f}(y)dy \geq \int_{D} G(\hat{x}, y)f(y)dy = u_f(\hat{x})$$
for any $f \in \mathcal{C}^+$. Consequently, the bathtub principle give the result
$$\hat{f}(x) = \chi_{\{y : G(\hat{x}, y) > \alpha_x\}}(x) \in \mathcal{C}^+.$$
(3.11)

Let us now observe that the set $\{x : \hat{u}(x) = \hat{u}_{\text{max}}\}$ consists of one element. Assume in contradiction
$\hat{x}, \hat{\tilde{x}} \in \{x : \hat{u}(x) = \hat{u}_{\text{max}}\}$ for some $\hat{x} \neq \hat{\tilde{x}}$. Then by \((3.11)\) we have
$$\hat{f}(x) = \chi_{\{y : G(\hat{x}, y) > \alpha_x\}}(x) = \chi_{\{y : G(\hat{\tilde{x}}, y) > \alpha_x\}}(x),$$
thus
$$G(\hat{x}, y) - \frac{\alpha_x}{\alpha_{\tilde{x}}}G(\hat{\tilde{x}}, y) \equiv 0,$$
for $y \in D \setminus \{y : G(\hat{x}, y) > \alpha_{\tilde{x}}\}$. This implies $\hat{x} = \hat{\tilde{x}}$ and $\alpha_{\tilde{x}} = \alpha_{\hat{x}}$, which is a contradiction.

**Proof of \(2.6\).** The optimality condition
$$0 = \nabla u(\hat{x}) = \int_{D} \nabla_x G(\hat{x}, y)\chi_{\{G(\hat{x}, y) > \alpha_x\}}dy.$$
follows from $\hat{x} \in \{x : \hat{u}(x) = \hat{u}_{\text{max}}\}$. The theorem is now proven.

**Proof of Theorem 2.2.** Without loss of generality we can assume that $\|f\|_\infty = 1$, the general case following from linearity. If $f \in \mathcal{C}^+$, then the proof follows from Theorem 2.1 with
$$\sigma(\|f\|_1) = u(\hat{x}) = \int_{D} G(\hat{x}, y)\chi_{\{G(\hat{x}, y) > \alpha_x\}}dy.$$
(3.12)

In the case when $f \in \mathcal{C}$ is not necessarily nonnegative, since $f$ and $|f|$ have same the $L^1$ and $L^\infty$ norms and
$$u_{-|f|} \leq u_f \leq u_{|f|},$$
then Theorem 2.2 obviously holds.

### 3.1 Estimates on $\sigma_D$ for nonnegative $f$

**Proof of Theorem 2.6.** The main tool in the proof is the following result of Talenti [15].

**Lemma 3.1** (Talenti’s Lemma). Let $f$ be a smooth nonnegative function in $D \subset \mathbb{R}^n$, and let $u$ solve the
Dirichlet boundary value problem
$$\begin{cases}
-\Delta u(x) = f(x) & \text{in } D, \\
u = 0 & \text{on } \partial D.
\end{cases}$$
(3.13)

Assume $f^*$ and $u^*$ are the symmetric decreasing rearrangements of respectively $f$ and $u$ defined in a ball $B$, such that $|B| = |D|$. Then if $v$ solves the problem
$$\begin{cases}
-\Delta v(x) = f^*(x) & \text{in } B, \\
v = 0 & \text{on } \partial B,
\end{cases}$$
(3.14)
then
$$u^*(x) \leq v(x), \quad \text{for all } x \in B.$$
Moreover, the equality holds if and only if $D = B$ and $f = f^*$.

We get the following simple corollary of Talenti’s lemma.

**Lemma 3.2.** Let $G^*_x, B(y)$ be the symmetric decreasing rearrangement of $G_D(\hat{x}, y)$ with respect to the $y$
variable. Then
$$G^*_x, B(y) \leq G_B(0, y) \quad \text{for } y \in B \setminus \{0\},$$
where $G_B$ is the Green’s function in the ball $B$ with $|B| = |D|$.
3 PROOF OF THE MAIN RESULTS

Proof. Observe that
\[
\begin{cases}
-\Delta_x G_D(x, y) = \delta_y(x) & \text{in } D, \\
G_D(x, y) = 0 & \text{for } x \in \partial D,
\end{cases}
\]
and
\[
\begin{cases}
-\Delta G_B(0, x) = \delta_y(x) & \text{in } B, \\
G(0, x) = 0 & \text{on } \partial B,
\end{cases}
\]
where \(\delta_y(x)\) is the Dirac measure in the point \(y\). The proof would follow from Lemma 3.1 if we could take \(f\) a Dirac measure. This is a simple generalization which trivially follows from approximation \(\frac{1}{\omega_n} \delta_{B_n(y)}(x) \to \delta_y(x)\), we omit the details here.

Next we obtain from (3.12),
\[
\sigma_D(||f||_1) = u(\hat{x})
\]
\[
= \int_D G_D(\hat{x}, y) \chi_{\{G(\hat{x}, y) > \alpha \hat{x}\}} \, dy
\]
\[
= \int_B G_D^*(\hat{x}, y) \chi_{\{G_D^*(\hat{x}, y) > \alpha \hat{x}\}} \, dy
\]
\[
\leq \int_B G(0, y) \chi_{\{G(0, y) > \alpha_0\}} \, dy,
\]
where the last inequality follows from Lemma 3.2 and the fact that
\[
\{G_D^*(\hat{x}, y) > \alpha \hat{x}\} = \{G_B(0, y) > \alpha_0\}.
\]
To calculate the value of \(\sigma_B\) we utilize the example (3.8) in the ball \(B_R\):
\[
u_r(|x|) = \begin{cases}
F(R) - F(|x|) & \text{in } B_R \setminus B_r, \\
F(R) - F(r) + \frac{1}{2|\pi|} - \frac{1}{2} |x|^2 & \text{in } B_r,
\end{cases}
\]
where \(F(|x|)\) is the fundamental solution multiplied by a dimensional constant such that \(F'(r) = \frac{1}{r} \frac{1}{2}\). Computing the norms \(|f_r|_{L^1(B)}\) and \(|f_r|_{L^\infty(B)}\) of \(f_r = -\Delta u_r\) we get
\[
\sigma_B(\omega_n r^n) = \frac{1}{n} r^n (F(R) - F(r)) + \frac{1}{2} r^2,
\]
thus we obtain for \(\sigma_B\) the formula
\[
\sigma_B(t) = \frac{t}{n\omega_n}(F(R) - F((t/\omega_n)^{1/n})) + \frac{1}{2} (t/\omega_n)^{2/n}.
\]
In particular for \(n \geq 3\) we have \(F(\tau) = -\frac{1}{(n-2)|\pi|^{n-2}}\) and
\[
\sigma_B(t) = \frac{t}{n(n-2)\omega_n} \left( \frac{1}{(t/\omega_n)^{(n-2)/n}} - \frac{1}{R^{n-2}} \right) + \frac{1}{2} (t/\omega_n)^{2/n}
\]
\[
= C_1(n)t^{\frac{n}{n-2}} - C_2(n) \frac{t}{R^{n-2}},
\]
with
\[
C_1(n) = \frac{(n-1)^2 + 1}{2n(n-2)\omega_n^{2/n}} \quad \text{and} \quad C_2(n) = \frac{1}{n(n-2)\omega_n}.
\]
For \(n = 2\) we need to take \(F(\tau) = \ln \tau, \omega_2 = \pi\), which yields
\[
\sigma_B(t) = \frac{t}{2\pi} \left( \ln R - \ln((t/\pi)^{1/2}) \right) + \frac{1}{2} (t/\pi)
\]
\[
= \left( \frac{1}{2} \ln \pi + \frac{1}{2\pi} + \frac{1}{2\pi} \ln R \right) t - \frac{1}{4\pi} t \ln t.
\]
The proof is finished now.
\[
\Box
\]
3.2 Signchanging Laplacian: The first approach.

In the next two subsections we prove Theorem 2.4. Note that (2.13) is a finer estimate, which takes into account the cancelation phenomena. The first approach is to add the norm \( \|f\|_\infty \) to the right hand side of the equation \(-\Delta u = f\), while the second one to consider the positive and negative parts of \(f\), i.e., the functions \(f^+\) and \(f^-\). From the linearity we know that

\[
u_f = u_g - \|f\|_\infty v
\]

where \(g = f + \|f\|_\infty\) and \(v = u_1\) is the solution of the Poisson problem with \(f \equiv 1\). Observe that \(g \geq 0\), \(\|g\|_\infty \leq 2\|f\|_\infty\) and \(\|g\|_L^1(D) = I_f + \|f\|_\infty|D|\), where \(I_f = \int_D f\,dx\). Thus

\[
\|u_g\|_\infty \leq 2\|f\|_\infty \sigma_D \left( \frac{I_f + \|f\|_\infty|D|}{2\|f\|_\infty} \right),
\]

and

\[
\max_{x \in D} u(x) \leq \|f\|_\infty \left[ \sigma_D \left( \frac{1}{2} \left[ \frac{1}{\|f\|_\infty} I_f + |D| \right] \right) - v(\bar{x}) \right],
\]

where \(\bar{x}\) is the point where the maximum of \(u\) is achieved. Similarly we can consider the function

\[
g = \|f\|_\infty - f\]

and obtain

\[
\max_{x \in D} (-u(x)) \leq \|f\|_\infty \left[ \sigma_D \left( \frac{1}{2} \left[ -\frac{1}{\|f\|_\infty} I_f + |D| \right] \right) - v(\bar{x}) \right],
\]

where \(\bar{x}\) is the point where the minimum of \(u\) is achieved. The last two estimates yield the bound (2.12).

Remark 3.1. The inequality (2.13) is interesting because of two reasons. First for a sign changing function \(f\) it contains the integral \(I_f\) of \(f\) and not the norm \(\|f\|_1\). Second, the function \(v\) is a known positive function depending only on the domain. If the location of the point where the maximum of \(|u|\) is achieved can be estimated, then the term \(-v(\bar{x})\) would provide significant improvement of the inequality.

3.3 The second approach: Proof of Theorem 2.4

Now we can write \(f = f^+ + f^-\), and thus by linearity

\[
u_f = u_{f^+} + u_{f^-}.
\]

Since the functions on the right hand side have different signs, we get

\[
|u| \leq \max(u_{f^+}, u_{f^-})
\]

and thus

\[
\|u\|_\infty \leq \max \{ \|f^+\|_\infty \sigma_D(\|f^+\|_1/\|f^+\|_\infty), \|f^-\|_\infty \sigma_D(\|f^-\|_1/\|f^-\|_\infty) \}. \tag{3.19}
\]

Remark 3.2. The inequality (3.19) is optimal on non-connected domains. We only need to let the function \(f\) have opposite signs on those domains. By connecting the disconnected domains by narrow tubes we will not change much the inequality.

The estimates (3.13) follow from Theorems 2.2, 2.3.

4 An application to the Laplace eigenfunctions

In this section we derive an estimate on the \(L^\infty\) norm of the \(k\)–th eigenfunction of the Laplace operator in the domain \(D\). Assume as usual that \((\lambda_k, u_k)\) is the \(k\)–th eigenvalue-eigenfunction pair of the Laplace operator, i.e.,

\[
\begin{cases}
  -\Delta u_k = \lambda_k u_k & \text{in } D, \\
  u_k = 0 & \text{on } \partial D.
\end{cases}
\tag{4.1}
\]

We apply (2.10) to the pair \((u_k, \lambda_k u_k)\) to derive an estimate on the norm \(\|u_k\|_\infty\) in terms of the parameters \(n, \lambda_k\) and \(|D|\). Assume first \(n > 2\), then we have by (2.13) that

\[
\frac{1}{\lambda_k} \|u_k\|_\infty \leq \frac{(n-1)^2 + 1}{2(n-2)\omega_n^{2/n}} \|u_k\|_1 \|u_k\|_\infty^{2/n} - \frac{1}{n(n-2)\omega_n} \|u_k\|_1.
\tag{4.2}
\]
Let $\alpha, \beta > 0$ be parameters yet to be chosen such that $\alpha^2 \beta^{n-2} = 1$. Then we have by Young’s inequality
\[
\|u_k\|_2^2 \|u_k\|_{\infty}^{-2} = (\alpha \|u_k\|_1) \frac{2}{n} (\beta \|u_k\|_{\infty}) \frac{2}{n} \leq \frac{2\alpha}{n} \|u_k\|_1 + \frac{\beta(n-2)}{n} \|u_k\|_{\infty},
\]
thus we get the estimate
\[
\frac{(n-1)^2 + 1}{2n(n-2)\omega_n^{2/n}} \|u_k\|_1^2 \|u_k\|_{\infty}^{-2} \leq \frac{\alpha((n-1)^2 + 1)}{n^2(n-2)\omega_n^{2/n}} \|u_k\|_1 + \frac{\beta((n-1)^2 + 1)}{2n^2\omega_n^{2/n}} \|u_k\|_{\infty}.
\]
(4.3)

We now choose $\beta$ such that $\frac{\beta((n-1)^2 + 1)}{2n^2\omega_n^{2/n}} = \frac{1}{2\lambda_k}$, which gives
\[
\beta = \frac{n^2\omega_n^{2/n}}{\lambda_k ((n-1)^2 + 1)} , \quad \alpha = \frac{\lambda_k ((n-1)^2 + 1)}{n^{n-2}\omega_n^2}.
\]
(4.4)

A combination of (4.2), (4.3) and (4.4) gives (2.15) for $n > 2$. For the proof for $n = 2$ observe that if $\|u_k\|_{\infty} \leq \|u_k\|_1$, then (2.15) follows from (2.13). If else $\|u_k\|_{\infty} > \|u_k\|_1$, then one should utilize the inequality $\|u_k\|_1 \ln\left(\frac{\|u_k\|_{\infty}}{\|u_k\|_1}\right) \leq \sqrt{2}\|u_k\|_1 \|u_k\|_{\infty}$ and then a suitable Young’s inequality. We omit the details here.

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