Two-node fluid network with a heavy-tailed random input: the strong stability case

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Abstract

We consider a two-node fluid network with batch arrivals of random size having a heavy-tailed distribution. We are interested in the tail asymptotics for the stationary distribution of a two-dimensional queue-length process. The tail asymptotics have been well studied for two-dimensional reflecting processes where jumps have either a bounded or an unbounded light-tailed distribution. However, presence of heavy tails totally changes the asymptotics. Here we focus on the case of strong stability where both nodes release fluid with sufficiently high speeds to minimise their mutual influence. We show that, like in the one-dimensional case, big jumps provide the main cause for queues to become large, but now they may have multi-dimensional features. For deriving these results, we develop an analytic approach that differs from the traditional tail asymptotic studies, and obtain various weak tail equivalences. Then, in the case of one-dimensional subexponential jump-size distributions, we find the exact asymptotics based on the sample-path arguments.

1 Introduction

Tail asymptotics problems have been studied in queueing networks and related reflecting processes for long years, but new developments are still going on. A key feature of that is an influence of the multiple boundary faces in a multidimensional state space. It requires an analysis that differs from the traditional one. Recent studies of those multidimensional processes have been mainly done in the light-tail regime where no heavy tails arise (see, e.g., [5]). On the other hand, the heavy-tail asymptotics are mostly studied for processes with single boundary faces or for certain monotone characteristics (see, e.g., [5] or [1]).

Thus, it is natural to ask how presence of heavy tails changes the tail asymptotics in multidimensional reflecting processes including queueing networks. The aim of this paper is to analyse this problem for the stationary distribution of a continuous-time reflecting process in the two-dimensional nonnegative quadrant. For this, we consider a two-node fluid network with a compound input with either Poisson or renewal arrivals, which is
a simple model but still keeps the feature of a multidimensional reflecting process. It may be viewed as a continuous-time approximation of a generalised Jackson network with simultaneous arrivals of big batches of customers.

We analyse the tail asymptotics for this fluid network as follows. First we assume Poisson arrivals and develop the analytic approach based on the stationary (balance) equation. We obtain lower and upper bounds for the stationary distributions in the coordinate and arbitrary directions (Theorems 4.1 and 4.2) and deduce the weak tail equivalence of those bounds under subexponentiality assumptions. Then we turn to the sample-path approach. We assume renewal arrivals and obtain a lower bound based on the long-tailedness of the jumps distributions and, then, under subexponentiality, show that the lower bound provides the exact asymptotics.

Our results relate to the tail asymptotics in generalised Jackson networks (and in more general classes of max-plus systems) with heavy-tailed distributions of service times that have been studied in [1] (see also [2, 3]). There the exact tail asymptotics was found only for the “maximal data” (time needed to empty the system in the steady state after stopping the input process). In tandem queues, the maximal data coincides with the stationary sojourn time of a “typical” customer. But the two notions differ when routing includes feedbacks. Another novel element of the paper is in considering a multidimensional heavy-tailed input, with possible dependence between its coordinates.

The paper is organised as follows. In Section 2 we introduce a fluid network with random jumps and discuss its dynamics. Then Sections 3 and 4 deal with the analytic approach, and Section 5 with the sample-path analysis. Appendix contains an auxiliary material that includes the analysis of the corresponding fluid model and basic definitions and properties of subexponential distributions.

## 2 Fluid network with compound input

Consider a two-node fluid network where nodes \( i = 1, 2 \) receive an input process \( \Lambda(t) = (\Lambda_1(t), \Lambda_2(t)) \), which is a compound process generated by point process \( \{N(t); t \geq 0\} \) and i.i.d. jumps \( \{(J_{1,n}, J_{2,n}); n = 1, 2, \ldots\} \). In this Section, we only assume that \( \{N(s + t); t \geq 0\} \) weakly converges to a stationary point process \( \{N^*(t); t \in \mathbb{R}\} \) as \( s \downarrow -\infty \) and its intensity \( \lambda \equiv \mathbb{E}(N^*(1)) \) is finite. In the subsequent sections, we will specify \( \{N(t)\} \) to be a Poisson process for the analytic approach, and a renewal process for the sample-path approach. In what follows, we use a shorter notation \( \mathbf{J} \equiv (J_1, J_2) \) for a random vector having the same distribution with \( (J_{1,n}, J_{2,n}) \). The joint distribution of \( (J_1, J_2) \) is denoted by \( F \), with marginals \( F_i \). We let \( m_i = \mathbb{E}J_i \) and \( \alpha_i = \lambda m_i \) for \( i = 1, 2 \).

Both nodes have infinite capacity buffers and release fluid with corresponding rates \( \mu_i, i = 1, 2 \). The \( p_{ij} \) proportion of the outflow from node \( i \) goes to node \( j \) for \( i, j = 1, 2 \), while the remaining proportion \( 1 - p_{ij} \) leaves the system. We assume that

\[
0 \leq p_{12}p_{21} < 1, \quad 0 < p_{12} + p_{21}
\] (2.1)
to exclude the trivial boundary cases including parallel queues, $p_{12} = p_{21} = 0$. Without loss of generality, we may also assume that $p_{ii} = 0$, for $i = 1, 2$.

**Remark 2.1.** One may assume that, in addition to the jump input, there are continuous fluid inputs to both queues, say with rates $\beta_1$ and $\beta_2$ respectively. Given stability, such a model may be reduced to the original one, by slowing down the release rates, namely, by replacing $\mu_i$, $i = 1, 2$ with $\mu_i(1 - (\beta_i + \beta_3 - p_{3-i,i})/(1 - p_{12}p_{21}))$.

We now introduce a buffer content process $Z(t) \equiv (Z_1(t), Z_2(t))^T$, which is defined as a nonnegative solution to the following equations:

\[
Z_1(t) = Z_1(0) + \Lambda_1(t) + p_{21}(\mu_2 t - Y_2(t)) - \mu_1 t + Y_1(t), \quad (2.2)
\]

\[
Z_2(t) = Z_2(0) + \Lambda_2(t) + p_{12}(\mu_1 t - Y_1(t)) - \mu_2 t + Y_2(t), \quad (2.3)
\]

where $Y_i(t)$ is the minimal nondecreasing process that keeps $Z_i(t)$ to stay nonnegative. As usual, we assume that sample paths are right-continuous and have left-hand limits.

Let $\delta_1 = \mu_1 - \mu_2 p_{21}$ and $\delta_2 = \mu_2 - \mu_1 p_{12}$. Put $X_i(t) = \Lambda_i(t) - \delta_i t$ and let $X(t) = (X_1(t), X_2(t))^T$, $Y(t) = (Y_1(t), Y_2(t))^T$, and

\[
R = \begin{pmatrix}
1 & -p_{21} \\
-p_{12} & 1
\end{pmatrix}.
\]

Then (2.2) and (2.3) may be rewritten as

\[
Z(t) = Z(0) + X(t) + RY(t), \quad t \geq 0. \quad (2.4)
\]

This is the standard definition of a reflecting process (in the nonnegative quadrant $\mathbb{R}_+^2$) for a given process $X(t)$, where $Y(t)$ is a regulator such that $Y_i(t)$ increases only when $Z_i(t) = 0$. Here $R$ is called a reflection matrix (e.g., see Section 3.5 of [9]). By (2.1), the inverse $R^{-1}$ exists and is nonnegative. This guarantees the existence of $\{Z(t); t \geq 0\}$. We refer to this process as to a two-dimensional fluid network with compound inputs.

Since $R^{-1} \alpha$ is the total inflow rate vector, the fluid network is stable if and only if

\[
R^{-1} \alpha < \mu \quad (2.5)
\]

where the inequality is strict in both coordinates. A formal proof for this stability condition can be found in [7]. Let $\Delta_i = -\mathbb{E}(X_i(1))$. Then $\Delta_i = \delta_i - \alpha_i$, and stability condition (2.5) is equivalent to

\[
\Delta_1 + \Delta_2 p_{21} > 0, \quad \Delta_1 p_{12} + \Delta_2 > 0. \quad (2.6)
\]

Under condition (2.5), the stationary distribution, say $\pi$, of $Z(t)$ uniquely exists. Let $Z \equiv (Z_1, Z_2)$ be a random vector subject to $\pi$.

We are interested in the tail behaviour of $P(c_1 Z_1 + c_2 Z_2 > x)$ as $x$ goes to infinity, for a given directional vector $c \equiv (c_1, c_2) \geq 0$ satisfying $c_1 + c_2 = 1$. In this paper, we consider this asymptotic mostly under the strong stability condition, that is

\[
\Delta_1 > 0 \quad \text{and} \quad \Delta_2 > 0. \quad (2.7)
\]

Other cases will be studied in a companion paper [6]. Under (2.7), both nodes are sufficiently fast to process fluids given the input is always maximal, and the following holds.
Lemma 2.1. (a, Sample-path majorant). On any elementary event, consider an auxiliary model of two parallel queues, with node $i = 1, 2$ having a continuous input of rate $\mu_{3-i}p_{3-i,i}$, release rate $\mu_i$, and jump input process $\Lambda_i$. Let $Z_i(t)$ be the content of node $i$ at time $t$. If $\tilde{Z}_i(0) \geq Z_i(0)$, then $\tilde{Z}_i(t) \geq Z_i(t)$, for any $t$.

(b, Stable majorant). Assume that the input is a renewal process and holds. Then the processes $\tilde{Z}_i(t)$ admit a unique stationary version, and, under the natural coupling of the input processes, $\tilde{Z}_i \geq Z_i$ a.s.

PROOF. Between any two jumps, the trajectories of $Z_i(t)$ and $\tilde{Z}_i(t)$ are Lipschitz, and at any regular point $t$ with $Z_i(t) > 0$, $\tilde{Z}_i(t) > 0$, the derivative of $Z_i$ is smaller than that of $\tilde{Z}_i$. So inequality $\tilde{Z}_i(t) \geq Z_i(t)$ is preserved between any two jumps. Since the jumps are synchronous and the jump sizes are equal, the induction argument completes the proof of (a). Then statement (b) is straightforward.

3 Analytic approach: Basic tools

In this section, we assume that

(A1) $\{\Lambda(t)\}$ is a compound Poisson process with rate $\lambda$,

and derive decomposition formulae for the stationary distribution in terms of moment generating functions. We first obtain the stationary balance equation under the stability condition

Let $C^1(\mathbb{R}^2)$ be the set of all functions from $\mathbb{R}^2$ to $\mathbb{R}$ having continuous first order partial derivatives. We write $f_i'(x_1, x_2)$ instead of $\frac{\partial}{\partial x_i} f(x_1, x_2)$, for short. It follows from that, for $f \in C^1(\mathbb{R}^2)$, the increment $f(Z(1)) - f(Z(0))$ can be expressed as integrations on $[0, 1]$ with respect to $dt$, $d\Lambda(t)$ and $dY_i(t)$ (formally by Itô’s integral formula). Then, taking the expectations with $Z(0)$ subject to the stationary distribution $\pi$ (we denote this expectation by $\mathbb{E}_\pi$) and recalling the stationary version $Z \equiv (Z_1, Z_2)$, we have

$$
\sum_{i=1}^{2} (-\delta_i \mathbb{E}_\pi(f'_i(Z))) + \lambda \mathbb{E}_\pi(f(Z + J) - f(Z)) \\
+ \mathbb{E}_1(f'_1(0, Z_2) - p_{12}f'_2(0, Z_2)) + \mathbb{E}_2(-p_{21}f'_1(Z_1, 0) + f'_2(Z_1, 0)) = 0,
$$

as long as all expectations are finite, where jump size vector $J$ is independent of everything else. Here $\mathbb{E}_i$ represents the expectation with respect to the Palm measure concerning $\{Y_i(t)\}$, that is, for any bounded measurable function $g$ on $\mathbb{R}_+$,

$$
\mathbb{E}_i(g(Z_{3-i})) = \mathbb{E}_\pi \left( \int_0^1 g(Z_{3-i}) Y_i(du) \right), \quad i = 1, 2.
$$

These expectations uniquely determine finite measures $\nu_i$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, where $\mathcal{B}(\mathbb{R}_+)$ is the Borel $\sigma$-field on $\mathbb{R}_+$. They are called boundary measures. We denote a random variable with probability distribution $\frac{1}{\nu_i(\mathbb{R}_+)} \nu_{3-i}$ by $V_i$. 

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The stationary equation (3.1) uniquely determines the stationary distribution \( \pi \) if it holds for a sufficiently large class of function \( f \). For this, we may choose a class of exponential functions \( f(x) = e^{\theta x} \) on \( \mathbb{R}^+_2 \) for each \( \theta = (\theta_1, \theta_2) \leq 0 \), where \( \langle a, b \rangle \) stands for the inner product of vectors \( a, b \in \mathbb{R}^2 \). Let \( \mathbf{\delta} = (\delta_1, \delta_2) \), and let

\[
\varphi(\theta) = \mathbb{E}_\pi(e^{\langle \theta, Z \rangle}), \quad \varphi_{3-i}(\theta_i) = \mathbb{E}_{3-i}(e^{\theta_i Z_i}), \quad i = 1, 2,
\]

\[
\widehat{F}(\theta) = \mathbb{E}(e^{\langle \theta, J \rangle}), \quad \kappa(\theta) = \langle \delta, \theta \rangle - \lambda(\widehat{F}(\theta) - 1).
\]

Here \( -\kappa(\theta) \) is the \textit{Lévy component} of \( X(t) \). Then, (3.1) becomes

\[
\kappa(\theta) \varphi(\theta) = (\theta_1 - p_{12} \theta_2) \varphi_1(\theta_1) + (\theta_2 - p_{21} \theta_1) \varphi_2(\theta_1),
\]

as long as \( \varphi(\theta), \widehat{F}(\theta) \) and \( \varphi_i(\theta_i) \) are finite. Clearly, (3.2) is always valid for \( \theta \leq 0 \).

For convenience of computations in subsequent sections, we find first \( \nu_i(\{0\}) \) and \( \nu_i(\mathbb{R}^+) \). Clearly, they are identical with \( \varphi_i(-\infty) \) and \( \varphi_i(0) \), respectively.

**Lemma 3.1.** Under the stability condition (2.4), for \( \theta \leq 0 \),

\[
\varphi_1(\theta_2) = \delta_1 \varphi(-\infty, \theta_2) + p_{21} \varphi_2(-\infty) \quad \text{and} \quad \varphi_2(\theta_1) = \delta_2 \varphi(\theta_1, -\infty) + p_{12} \varphi_1(-\infty).
\]

**Proof.** Dividing (3.2) by \( \theta_1 \), we have

\[
\left( \delta_1 + \frac{\theta_2}{\theta_1} \frac{\lambda}{\theta_1} (\widehat{F}(\theta) - 1) \right) \varphi(\theta) = \left( 1 - p_{12} \frac{\theta_2}{\theta_1} \right) \varphi_1(\theta_2) + \left( \frac{\theta_2}{\theta_1} - p_{21} \right) \varphi_2(\theta_1).
\]

Letting \( \theta_1 \to -\infty \), we get the first equality in (3.3). The symmetry gives the second. \( \square \)

Denote the traffic intensities at nodes 1 and 2, respectively, by

\[
\rho_1 = (\alpha_1 + \alpha_2 p_{21})/(\mu_1 (1 - p_{12} p_{21})), \quad \rho_2 = (\alpha_2 + \alpha_1 p_{12})/(\mu_2 (1 - p_{12} p_{21})).
\]

**Lemma 3.2.** Under the stability condition (2.4), for \( i = 1, 2 \),

\[
\varphi_i(-\infty) = \mu_i \pi(0), \quad \varphi_i(0) = \frac{\Delta_i + \Delta_{3-i} p_{(3-i)i}}{1 - p_{12} p_{21}} = \mu_i (1 - \rho_i).
\]

In particular, \( \pi(0) = \mathbb{P}(Z = 0) < \min(1 - \rho_1, 1 - \rho_2) \), and, for \( i = 1, 2 \),

\[
\mathbb{P}(V_i > x) = \frac{\delta_{3-i}}{\mu_{3-i}(1 - \rho_{3-i})} \mathbb{P}(Z_i > x, Z_{3-i} = 0), \quad x \geq 0.
\]

**Proof.** Letting \( \theta_1 \) and \( \theta_2 \) to \( -\infty \) in (3.3), we get

\[
\varphi_1(-\infty) = \delta_1 \varphi(-\infty, -\infty) + p_{21} \varphi_2(-\infty), \quad \varphi_2(-\infty) = \delta_2 \varphi(-\infty, -\infty) + p_{12} \varphi_1(-\infty).
\]

Since \( \varphi(-\infty, -\infty) = \pi(0) \), solving these equations yields \( \varphi_i(-\infty) = \mu_i \pi(0) \). On the other hand, with putting \( \theta_2 = 0 \), letting \( \theta_1 \to -\infty \) in (3.4), and using the symmetry, we get

\[
\Delta_1 + p_{21} \varphi_2(0) = \varphi_1(0), \quad \Delta_2 + p_{12} \varphi_1(0) = \varphi_2(0).
\]
Solving these equation yields the first equality of (3.5) for \( \varphi_i(0) \). The second equality is immediate from the definitions of \( \Delta_i \) and \( \rho_i \). \( \square \)

From equations (3.5), it is easy to see that \((\varphi_1(0), \varphi_2(0))^T = \mu - R^{-1}\alpha > 0\). Also, the second equality for \( \varphi_i(0) \) yields another representation for \( \Delta_i \):

\[
\Delta_1 = \mu_1(1 - \rho_1) - \mu_2 p_{21}(1 - \rho_2), \quad \Delta_2 = \mu_2(1 - \rho_2) - \mu_1 p_{12}(1 - \rho_1).
\]

(3.7)

We now turn to the tail probabilities. To this end, we single out the moment generation function \( \varphi(s c) \) of one-dimensional random variable \( c_1 Z_1 + c_2 Z_2 \) from the the stationary equation (3.2) or (3.4). Namely, we (i) derive \( \varphi(s c) \) as a linear combination of the moment generating functions of certain measures, which may include unknown boundary measures \( \nu_1 \) and \( \nu_2 \); (ii) find lower and upper bounds for the tail probability \( \mathbb{P}_\pi(c_1 Z_1 + c_2 Z_2 > x) \) from the expression obtained in (i); then (iii) obtain the asymptotics for \( \mathbb{P}_\pi(c_1 Z_1 + c_2 Z_2 > x) \) as \( x \to \infty \) using the heavy-tailedness of jump size distributions.

Clearly, the most important step is (i). Our arguments are similar to that in deriving the Pollaczek-Khinchine formula from the stationary equation of the \( M/G/1 \) queue. However, we have to be careful because here the reflecting process is two-dimensional, while the boundary is a single point in the case of the \( M/G/1 \) queue. Also our analysis depends on direction \( c \).

We first write the stationary equation (3.2) with \( \theta = sc \) for a directional vector \( c \) as

\[
\left( c_1 \delta_1 + c_2 \delta_2 - \frac{\lambda}{s} (F(s c) - 1) \right) \varphi(s c) = (c_1 - p_{12} c_2) \varphi_1(c_2 s) + (c_2 - p_{21} c_1) \varphi_2(c_1 s).
\]

(3.8)

To express the coefficient of \( \varphi(s c) \) in a compact form, we introduce the integrated probability distribution \( F^I_c \) by

\[
F^I_c(x) = 1 - \frac{1}{m_c} \int_x^\infty \mathbb{P}(c_1 J_1 + c_2 J_2 > y) dy, \quad x \geq 0,
\]

where \( m_c = c_1 m_1 + c_2 m_2 \), and denote its moment generating function by \( \hat{F}^I_c \). Namely, \( \hat{F}^I_c(s) = (F(s c) - 1)/(m_c s) \). For positive \( r < 1 \), let

\[
\hat{S}^I_c(r)(s) = (1 - r)(1 - r \hat{F}_c^I(s))^{-1},
\]

be the moment generating function of the geometric sum with parameter \( r \) of \( i.i.d. \) random variables having distribution \( F^I_c \).

By the strong stability assumption (2.7), \( c_2 \delta_1 + c_2 \delta_2 - (c_1 \alpha_1 + c_2 \alpha_2) = c_1 \Delta_1 + c_2 \Delta_2 > 0 \), and therefore

\[
r_c := \frac{c_1 \alpha_1 + c_2 \alpha_2}{c_1 \delta_1 + c_2 \delta_2} < 1.
\]

Since \( \lambda m_c = c_1 \alpha_1 + c_2 \alpha_2 \), (3.8) can be rewritten as

\[
\varphi(s c) = ((c_1 - p_{12} c_2) \varphi_1(c_2 s) + (c_2 - p_{21} c_1) \varphi_2(c_1 s)) \hat{S}_c^{I(r_c)}(s).
\]

(3.9)
Thus, if the coefficients of \( \varphi_1(c_1s) \) and \( \varphi_2(c_1s) \) are positive, then we have a decomposition for the distribution of \( c_1 Z_1 + c_2 Z_2 \). However, those coefficients may be negative.

We note that, if both of \( c_1 - p_{12} c_2 \) and \( c_2 - p_{21} c_1 \) are non-positive, then \( c_1 (1 - p_{12} p_{21}) \leq 0 \) and \( c_2 (1 - p_{12} p_{21}) \leq 0 \), which contradict (2.1) and \( c \neq 0 \). Hence, either one of them is positive at least, and the following three cases are only possible under (2.7).

\[(C0) \ c_1 - p_{12} c_2 \geq 0 \text{ and } c_2 - p_{21} c_1 \geq 0 \text{ (in this case, we must have } c > 0 \).
\[(C1) \ c_1 - p_{12} c_2 \geq 0 \text{ and } c_2 - p_{21} c_1 < 0, \quad (C2) \ c_1 - p_{12} c_2 < 0 \text{ and } c_2 - p_{21} c_1 \geq 0.\]

Since (C1) and (C2) are symmetric, we consider only (C0) and (C1).

Recall that \( V_1 \) and \( V_2 \) have the probability distributions normalized by \( \nu_1 \) and \( \nu_2 \), respectively. Then, from (3.9), we have the following lemma.

**Lemma 3.3.** For directional vector \( c \geq 0 \), we have, for \( B \in \mathcal{B}(\mathbb{R}_+) \),

\[
\mathbb{P}_z(c_1 Z_1 + c_2 Z_2 \in B) = \eta^{(1)}_c \mu_1(1 - \rho_1) \mathbb{P}(c_2 V_2 + S_c^{I(re)} \in B) + \eta^{(2)}_c \mu_2(1 - \rho_2) \mathbb{P}(c_1 V_1 + S_c^{I(re)} \in B), \tag{3.10}
\]

where \( V_1 \) and \( V_2 \) are independent of \( S_c^{I(re)} \), and

\[
\eta^{(1)}_c = \frac{c_1 - p_{12} c_2}{c_1 \Delta_1 + c_2 \Delta_2}, \quad \eta^{(2)}_c = \frac{c_2 - p_{21} c_1}{c_1 \Delta_1 + c_2 \Delta_2}. \tag{3.11}
\]

Since \( \eta^{(1)}_c \mu_1(1 - \rho_1) + \eta^{(2)}_c \mu_1(1 - \rho_1) = 1 \) from (3.10) with \( B = \mathbb{R}_+ \) and \( \eta^{(i)}_c \) for \( i = 1, 2 \) are positive for (C0), we get the following lower bound.

**Corollary 3.1.** For the case (C0), we have

\[
\mathbb{P}(S_c^{I(re)} > x) \leq \mathbb{P}_z(c_1 Z_1 + c_2 Z_2 > x) \quad \text{for all } x > 0. \tag{3.12}
\]

For the case (C1), (3.10) cannot be used to get a lower bound, and we use another expression. Let \( \varphi^+(\theta) = \mathbb{E}(e^{\theta Z} 1(Z > 0)) \) and \( d_0 = \frac{1}{\delta_1 \delta_2} \mu_1 \mu_2 (1 - p_{12} p_{21}) \pi(0) \). From Lemma 3.1, we have

\[
\varphi(s c) = \varphi^+(s c) + \frac{\varphi_2(c_1 s)}{\delta_2} + \frac{\varphi_1(c_2 s)}{\delta_1} - d_0 \tag{3.13}
\]

where we have used the fact that

\[
\delta_1 p_{12} \varphi_1(-\infty) + \delta_2 p_{21} \varphi_2(-\infty) + \delta_1 \delta_2 \pi(0) = \mu_1 \mu_2 (1 - p_{12} p_{21}) \pi(0) = \delta_1 \delta_2 d_0.
\]

**Lemma 3.4.** For the case (C1), let \( r'_c = \frac{c_1 \alpha_1 + c_2 \alpha_2}{c_1 (\delta_1 + \delta_2 p_{21})}, \) then \( 0 < r'_c < 1 \) and

\[
\varphi(s c) = \left( \frac{d^{(1)}_c}{\delta_1} \varphi_1(c_2 s) + d^{(2)}_c (\varphi^+(s c) - d_0) \right) S^{I(r'_c)}_c(s), \tag{3.14}
\]
Theorem 4.1. Therefore
\[ d_c^{(1)} = \frac{\delta_1(c_1 - p_{12}c_2) + \delta_2(p_{21}c_1 - c_2)}{c_1(\delta_1 + \delta_2p_{21})(1 - r_c')}, \quad d_c^{(2)} = \frac{\delta_2(p_{21}c_1 - c_2)}{c_1(\delta_1 + \delta_2p_{21})(1 - r_c')} \]

This yields (3.14) because \( \alpha_1/\mu_1 < r_1' = \alpha_1/(\delta_1 + \delta_2p_{21}) < 1 \). Since \( d_c^{(1)} > 0 \) and \( d_c^{(2)} > 0 \), the right-hand side of (3.14) represents the convolutions of two distributions on \([0, \infty)\), and this leads to (3.15).

\[ \mathbb{P}(S_c^{(r_c')} > x) \leq \mathbb{P}_\pi(c_1Z_1 + c_2Z_2 > x), \quad x \geq 0. \] (3.15)

**Proof.** Multiplying \((p_{21}c_1 - c_2)\delta_2\) with (3.13) and adding to (3.9), we have
\[ \left( c_1(\delta_1 + \delta_2p_{21}) - (c_1\alpha_1 + c_2\alpha_2)\hat{F}_c(s) \right) \varphi(sc) = \frac{1}{\delta_1} (c_1(p_{12}c_2) + \delta_2(p_{21}c_1 - c_2))\varphi_1(c_2s) + \delta_2(p_{21}c_1 - c_2) \left( \varphi^+(sc) - d_0 \right). \]

This yields (3.14) because \( c_1(\delta_1 + \delta_2p_{21}) - (c_1\alpha_1 + c_2\alpha_2) = c_1(\Delta_1 + \Delta_2p_{21}) + \alpha_2(c_{1p_{21}} - c_2) > 0 \). Since \( d_c^{(1)} > 0 \) and \( d_c^{(2)} > 0 \), the right-hand side of (3.14) represents the convolutions of two distributions on \([0, \infty)\), and this leads to (3.15).

4 Analytic approach: Bounds and tail asymptotics

We continue to assume (A1), and consider the tail probability \( \mathbb{P}_\pi(c_1Z_1 + c_2Z_2 > x) \) for directional vector \( c \geq 0 \). We start with \( c = (1, 0)^T \) where results are obtained under weaker assumptions.

**Theorem 4.1.** Assume (A1) and that the system is stable and that \( \Delta_1 > 0 \). We have
\[ \mathbb{P}(S_c^{(r_1')} > x) \leq \mathbb{P}(Z_1 > x) \leq \mathbb{P}(S_c^{(r_1)}) > x, \quad x > 0, \] (4.1)

where \( r_1 = \alpha_1/\delta_1 \) and \( r_1' = \alpha_1/(\delta_1 + \delta_2p_{21}) \).

**Proof.** The upper bound in (4.1) is immediate from Lemma 2.1 because \( \tilde{Z}_1 \) is subject to the stationary workload distribution of the \( M/G/1 \) queue. It also can be analytically obtained from (3.9). The lower bound is already obtained in Lemma 3.4.

**Remark 4.1.** Clearly, \( \Delta_1 > 0 \) and \( \mu_2 - \mu_1p_{12} = \delta_2 > 0 \) imply that
\[ \frac{\alpha_1}{\mu_1} < r_1' = \frac{\alpha_1}{\mu_1(1 - p_{12}p_{21})} < r_1 = \frac{\alpha_1}{\mu_1 - \mu_2p_{21}} < 1. \]

By arguments similar to that in Lemma 2.1, \( \mathbb{P}(S_c^{(p_1)} > x) \) with \( p_1 = \alpha_1/\mu_1 \) also gives a lower bound, but the lower bound in (4.1) is tighter than that because \( p_1 < r_1' \).

To compare tails of distributions, we recall the following notion. Distribution functions \( F \) and \( G \) are weakly tail-equivalent if, for \( \bar{F}(x) = 1 - F(x) \) and \( \bar{G}(x) = 1 - G(x) \),
\[ 0 < \liminf_{x \to \infty} \frac{\bar{F}(x)}{\bar{G}(x)} \leq \limsup_{x \to \infty} \frac{\bar{F}(x)}{\bar{G}(x)} < \infty, \]
These observations and Theorem 4.2 lead to the following result.

Hence, Theorem 4.1 yields

**Corollary 4.1.** Under the assumptions of Theorem 4.1, if $F^I_1$ is subexponential, then the distribution of $Z_1$ is weakly tail-equivalent to $F^I_1$.

We next consider the case of directional vector $c > 0$ for options (C0) and (C1).

**Theorem 4.2.** Assume (A1) and the strong stability (2.7) to hold, and recall the definition (3.11) of $\eta^{(i)}$. Let $x \geq 0$ and $c \geq 0$ be a directional vector. For the case (C0),

$$
\mathbb{P}(S^{I(re)}_c > x) \leq \mathbb{P}_\pi(c_1Z_1 + c_2Z_2 > x) \leq \delta_1\eta^{(1)}_c \mathbb{P}(c_2S^I_2 + S^{I(re)}_c > x) + \delta_2\eta^{(1)}_c \mathbb{P}(c_1S^I_1 + S^{I(re)}_c > x). \tag{4.3}
$$

For the case (C1),

$$
\mathbb{P}(S^{I(re)}_c > x) \leq \mathbb{P}_\pi(c_1Z_1 + c_2Z_2 > x) \leq \delta_1\eta^{(1)}_c \mathbb{P}(c_2S^I_2 + S^{I(re)}_c > x). \tag{4.4}
$$

Here random variables $S^{I(r_1)}_1$ and $S^{I(r_2)}_2$ are assumed to be independent of $S^{I(re)}_c$.

**Remark 4.2.** For the case (C0), $c > 0$, so (4.3) does not contradict (4.1).

**Proof.** In the case (C0), by Lemma 3.2, Theorem 4.1 and its symmetric version,

$$
\varphi_3-i(0)\mathbb{P}(V_i > x) \leq \delta_3-i\mathbb{P}(Z_i > x) \leq \delta_3-i\mathbb{P}(S^{I(r_i)}_i > x), \quad i = 1, 2.
$$

Hence, Lemma 3.3 yields the upper bound of (4.3), and its lower bound is obtained by Corollary 3.1. In the case (C1), the upper bound of (4.4) is immediate from Lemma 3.3 while the lower bound is already obtained in Lemma 3.4.

Assume now that the three distributions, $F^I_1$, $F^I_2$ and $F^I_c$ are all subexponential. Then

$$
\mathbb{P}(c_iS^{I(r_i)}_i > x) \sim \frac{r_1}{1 - r_1} F^I_i(x/c_i), \quad i = 1, 2, \quad \mathbb{P}(S^{I(re)}_c > x) \sim \frac{r_c}{1 - r_c} F^I_c(x).
$$

Similar asymptotic equivalences hold when one replaces $r_c$ by either $r_c'$ or $r_c''$. Clearly,

$$
F^I_c(x) = \frac{1}{m_c} \int_{x}^{\infty} \mathbb{P}(c_1J_1 + c_2J_2 > y)dy \geq \frac{c_1}{m_c} \int_{x/c_1}^{\infty} \mathbb{P}(J_1 > y)dy = \frac{c_1m_1}{m_c} F^I_1(x/c_1),
$$

therefore,

$$
\mathbb{P}(c_1S^{I(r_1)}_1 > x) \leq (1 + o(1)) \frac{m_c r_1}{(1 - r_1)c_1m_1} F^I_c(x) \leq (1 + o(1)) \frac{(1 - r_c)m_c r_1}{r_c(1 - r_1)c_1m_1} \mathbb{P}(S^{I(re)}_c > x)
$$

and, finally,

$$
\mathbb{P}(c_1S^{I(r_1)}_1 + S^{I(re)}_c > x) \leq (1 + o(1)) K F^I_c(x).
$$

Here constant $K := ((1 - r_c)m_c r_1)/(r_c(1 - r_1)c_1m_1) + (r_c)/(1 - r_c)$ is positive and finite. These observations and Theorem 4.2 lead to the following result.
Corollary 4.2. In the both cases of Theorem 4.2, if the three distributions $F_1^I$, $F_2^I$, and $F_c^I$ are subexponential, then the distribution of $c_1Z_1 + c_2Z_2$ is weakly tail-equivalent to $F_c^I$.

Remark 4.3. Subexponentiality of both $F_1^I$ and $F_2^I$ may not imply that of $F_c^I$, even if jump sizes $J_1$ and $J_2$ are independent. It holds if the distributions are regularly varying. In general, necessary and sufficient conditions for subexponentiality of the convolution of subexponential distributions is given in Embrechts and Goldie [3] (see also Theorem 3.33 in [3] for complete characterization).

## 5 Sample-path approach

Throughout the section, we assume that

(A2) The point process $\{N(t)\}$ is a renewal process with i.i.d. inter-arrival times having a general distribution with finite mean $a \equiv 1/\lambda$, and jumps are one-dimensional, that is, for $p_1, p_2 > 0$ satisfying $p_1 + p_2 = 1$,

$$F(x, y) = p_1F_1(x) + p_2F_2(y), \quad x, y \geq 0.$$  \hspace{1cm} (5.1)

Based on fluid dynamics considered in Appendix A, we provide a lower bound for the tail probabilities assuming only that the integrated tail distributions $F_1^I$ and $F_2^I$ are long-tailed (see Appendix B for definitions). Then we get the exact asymptotics in the case of subexponential distribution.

**Lower bound.** In what follows, we use notation $LB(x)$ for the lower bound for the probability $\mathbb{P}(c_1Z_1 + c_2Z_2 > x)$. By Corollary A.1 for $c_1, c_2 \geq 0$ with $c_1 + c_2 = 1$,

$$LB(x) = (1 + o(1)) \sum_{n=1}^{\infty} \left( p_1 \mathbb{P}(J_{1,-n} > x/c_1 + na(\Delta_1 + p_{21}\Delta_2)) + p_2 \mathbb{P}(J_{2,-n} > x/c_2 + na(\Delta_2 + p_{12}\Delta_1)) \right), \quad x > 0.$$  \hspace{1cm} (5.2)

Here, we have used the following heuristics: the lower bound asymptotics in our model is equivalent to that in an auxiliary fluid model where (i) possible time instants of big jumps are $-na$, $n = 1, 2, \ldots$, and (ii) all further input jumps (after the big one) are replaced by continuous fluid inputs with constant rates $\alpha_1$ and $\alpha_2$, respectively. So the lower bound constitutes a probability of a union of infinite number of trajectories, with a single big random jump followed by a continuous path. Since the probability of simultaneous occurrence of two or more big jumps is negligibly small, the probability of a union of events may be replaced by the sum of probabilities. These heuristic arguments can be justified by exact and complete mathematical calculations and statements along the lines of, say, [1] or/and [4].

Assume both $F_1^I$ and $F_2^I$ to be long-tailed, then (5.2) yields

$$LB(x) = (1 + o(1)) \frac{\alpha_1}{\Delta_1 + p_{21}\Delta_2} F_1^I(x/c_1) + (1 + o(1)) \frac{\alpha_2}{\Delta_2 + p_{12}\Delta_1} F_2^I(x/c_2).$$  \hspace{1cm} (5.3)
where, by convention, \( x/0 = \infty \) and \( F_i'(\infty) = 0 \). In particular, for \( c_1 = 1, c_2 = 0, \)
\[
\frac{r'_1}{1 - r'_1} = \frac{\alpha_1}{\mu_1(1 - p_{12}p_{21})} - \alpha_1 \lesssim \frac{\alpha_1}{\Delta_1 + p_{21}\Delta_2} < \frac{\alpha_1}{\Delta_1}.
\]
Hence, (5.3) gives a better (tighter) bound than the one in (4.1) (see (4.2)).

**Tail equivalence.** We refer to Lemma 2.1. It is known that, for a single-node queue \( \tilde{Z}_i \) with subexponential batch distributions, the stationary content is large due to a single large value of one of batches, and the tail distribution of the stationary content is equivalent to the integrated tail \( \tilde{F}_i'(x) \). Then, by Lemma 2.1 and by lower bound (5.3) with \( c_1 = 1 \), the tail asymptotics for the stationary \( Z_i \) is weakly tail-equivalent to \( \tilde{F}_i'(x) \).

Similar result holds for the linear sum \( c_1Z_1 + c_2Z_2 \) if we assume the tails \( \tilde{F}_1(x) \) and \( \tilde{F}_2 \) to be equivalent. But here we can get more.

**Exact asymptotics.** We first consider exact tail asymptotics for \( c_1 = 1, c_2 = 0 \), with applying the “squeeze principle” (see Theorem 8 in [1]). Thus, we focus on the tail asymptotic for \( Z_1 \). This is done by the following two steps.

(1st Step) Assume that the distributions \( F_i' \) are subexponential. Assume that both the systems run in the stationary regime and let \( Z_1 = Z_1(0) \) and \( \tilde{Z}_1 = \tilde{Z}_1(0) \). Following the lines of Theorem 5.4* in [5], one can show that, for \( i = 1, 2, \)
\[
\mathbb{P}(\tilde{Z}_1 > x) = (1 + o(1)) \sum_{n=0}^{\infty} p_1 \mathbb{P}(\tilde{Z}_1 > x, J_{i,(-n)} > x + an\Delta_i), \quad x \to \infty,
\]
Since \( Z_1 \leq \tilde{Z}_1 \) a.s., this and the fact that \( \lim \inf_{x \to \infty} \mathbb{P}(Z_1 > x) / \mathbb{P}(\tilde{Z}_1 > x) > 0 \) yields
\[
\mathbb{P}(Z_1 > x) = \mathbb{P}(Z_1 > x, \tilde{Z}_1 > x) = \sum_{n=0}^{\infty} p_1 \mathbb{P}(Z_1 > x, J_{1,(-n)} > x + an\Delta_1) + o(\mathbb{P}(\tilde{Z}_1 > x))
\]
\[
= (1 + o(1)) \sum_{n=0}^{\infty} p_1 \mathbb{P}(Z_1 > x, J_{i,(-n)} > x + an\Delta_i), \quad x \to \infty,
\]
(2nd Step) If there is only one big jump before time 0 and the other jumps are uniformly approximated by a fluid limit, then it follows from Lemma A.1 that \( Z_1(0) > x \) occurs only if there is a \((-n)\)-th arrival, whose arrival time is denoted by \( T_{-n} < 0 \), such that \( J_{i,(-n)} > x + (-T_{-n})(\Delta_1 + p_{21}\Delta_2) \). Applying similar arguments to those in the proof of Theorem 8 on [1] with help of (5.4), we can get that
\[
\mathbb{P}(Z_1 > x) = (1 + o(1)) \sum_{n=0}^{\infty} p_1 \mathbb{P}(J_{1,(-n)} > x + an(\Delta_1 + p_{21}\Delta_2)), \quad x \to \infty.
\]
Hence, we have the upper bound for \( \mathbb{P}(Z_1 > x) \) which coincides with the lower bound.

By symmetry, we have a similar result for \( c_1 = 0, c_2 = 1 \). We finally assume that \( c_1 > 0, c_2 > 0 \). If in addition \( F_i'(x/c_1) \) and \( F_i'(x/c_2) \) are weakly tail-equivalent, then, due to Theorem 3.33 from [5], the distribution of \( c_1\tilde{Z}_1 + c_2\tilde{Z}_2 \) is also subexponential and
\[
\mathbb{P}(c_1\tilde{Z}_1 + c_2\tilde{Z}_2 > x) = (1 + o(1)) \left( \mathbb{P}(\tilde{Z}_1 > x/c_1) + \mathbb{P}(\tilde{Z}_2 > x/c_2) \right)
\]
where we again use convention $x/c_i = \infty$ if $c_i = 0$. Then, similarly,

$$
\mathbb{P}(c_1Z_1 + c_2Z_2 > x) = (1 + o(1)) \sum_{n=0}^{\infty} p_1 \mathbb{P}(J_{1,(-n)} > x/c_1 + an(\Delta_1 + p_{12}\Delta_2)) + (1 + o(1)) \sum_{n=0}^{\infty} p_2 \mathbb{P}(J_{2,(-n)} > x/c_2 + an(\Delta_2 + p_{12}\Delta_1)).
$$

Thus, we again have the upper bound which coincides with the lower bound (5.2), and there we have the following theorem.

**Theorem 5.1.** Assume that (A2) holds, both $\Delta_1$ and $\Delta_2$ are positive, distributions $F_1^I(x/c_1)$ and $F_2^I(x/c_2)$ are both subexponential and weakly tail-equivalent. Then (5.3) provides the exact asymptotics for $\mathbb{P}(c_1Z_1 + c_2Z_2 > x)$. If, say, $c_1 = 1$, then the first term in (5.3) gives the exact asymptotics for $\mathbb{P}(Z_1 > x)$.

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**Appendix**

A Analysis of a pure fluid model

Assume again $\Delta_1 > 0$ and $\Delta_2 > 0$. Consider an auxiliary pure fluid model with continuous fluid input rates $\alpha_1, \alpha_2$, service rates $\mu_1, \mu_2$ and transition fractions $p_{12}, p_{21}$ as described in Section 2. We use the same notation $Z_1(t), Z_2(t)$ as before, but for the deterministic buffer quantities.

Let $t > 0$ be fixed. We assume that the fluid model starts at negative time $-t$ from levels $y_1, y_2$ (this means $Z_1(-t) = y_1, Z_2(-t) = y_2$) and want to identify conditions on $y_1, y_2$ for

$$c_1Z_1 + c_2Z_2 \geq x \quad \text{(A.1)}$$

to hold where again $c_1, c_2 \geq 0$ and $c_1 + c_2 = 1$ are given constants, and where $Z_i = Z_i(0)$.

**Case 1:** $c_1 = 1, c_2 = 0$. Thus, we have to find conditions for $Z_1 \geq x$.

Due to monotonicity properties of fluid limits (see, e.g., Lelarge [8]), under the stability conditions, if the fluid model starts from a non-zero initial value at time $-t$ and if some coordinate, say $i$, becomes zero, $Z_i(u) = 0$ at time $u > -t$, then it stays at zero, $Z_i(v) = 0$ for all $u \leq v \leq 0$.

Let $L_2 = y_2/\Delta_2$. Assume first that $L_2 \geq t$. Then, at any time instant $u \in (-t, 0)$,

- (i) the input rate to Queue 1 is $\alpha_1 + \mu_2 p_{21}$;
- (ii) the output rate from Queue 1 is $\mu_1$;
- (iii) the input rate to Queue 2 is $\alpha_2 + \mu_1 p_{12}$;
- (iv) the output rate from Queue 2 is $\mu_2$.
Then $Z_1 \geq x$ means that

$$y_1 \geq x + t\Delta_1.$$  \hspace{1cm} (A.2)

Assume now that $L_2 < t$. Then, for any $u \in (-t, -t + L_2)$,
(i) the input rate to Queue 1 is $\alpha_1 + \mu_2 p_{21}$; (ii) the output rate from Queue 1 equals $\mu_1$;
(iii) the input rate to Queue 2 is $\alpha_2 + \mu_1 p_{12}$; (iv) the output rate from Queue 2 is $\mu_2$;
and, for any $u \in (-t + L_2, 0)$,
(a) the input and output rates to/from Queue 1 and the input rate to Queue 2 are as before, but (b) the output rate from Queue 2 equals to the input rate, i.e. is $\alpha_2 + \mu_1 p_{12}$.

Then condition $Z_1 \geq x$ is equivalent to

$$y_1 - x \geq L_2 \Delta_1 + (t - L_2)(\mu_1 - (\alpha_1 + p_{21}(\alpha_2 + \mu_1 p_{12}))) = t\Delta_1 + t p_{21}(\Delta_2 - y_2 p_{21}).$$  \hspace{1cm} (A.3)

Combining (A.2) and (A.3) together, we have

$$y_1 \geq x + t\Delta_1 + p_{21}(t\Delta_2 - y_2)^+.  \hspace{1cm} (A.4)$$

**Case 2:** $c_1 = 0, c_2 = 1$. This case is symmetric to the previous one.

**Case 3:** $c_1 > 0, c_2 > 0$. Following the same logics as before, we get:
if $L_2 \leq t$, then $Z_2 = 0$, and the condition on $y_1$ coincides with (A.3) if one replaces $x$ by $x/c_1$. More precisely, we get inequality:

$$y_1 \geq x/c_1 + t\Delta_1 + p_{21}(\Delta_2 - y_2).$$  \hspace{1cm} (A.5)

Similarly, if $L_1 \leq t$, then $Z_1 = 0$, and we get

$$y_2 \geq x/c_2 + t\Delta_2 + p_{12}(t\Delta_1 - y_1).$$  \hspace{1cm} (A.6)

Otherwise, if both $L_1 > t$ and $L_2 > t$, then $y_1 = Z_1 + t\Delta_1$, $y_2 = Z_2 + t\Delta_2$, and we have

$$c_1 Z_1 + c_2 Z_2 = c_1(y_1 - t\Delta_1) + c_2(y_2 - t\Delta_2) \geq x.$$  \hspace{1cm} (A.7)

Combining all three sub-cases, we arrive at the following result:

**Lemma A.1.** Consider a purely fluid model with input rates $\alpha_1, \alpha_2$, service rates $\mu_1, \mu_2$ and transition fractions $p_{12} > 0, p_{21} > 0$. Let $c_1, c_2$ be non-negative constants with $c_1 + c_2 = 1$. Let $t > 0$ and let the system start at time $-t$ from $Z_1(-t) = y_1 \geq 0$ and $Z_2(-t) = y_2 \geq 0$. If $\Delta_1 > 0$ and $\Delta_2 > 0$, then inequality $c_1 Z_1 + c_2 Z_2 \geq x$ holds if and only if

$$c_1(y_1 - t\Delta_1 - p_{21}(t\Delta_2 - y_2)^+) + c_2(y_2 - t\Delta_2 - p_{12}(t\Delta_1 - y_1)^+) \geq x.$$  \hspace{1cm} (A.8)

**Corollary A.1.** Consider a particular case where only one of two options is possible, either $y_1 > 0$ and $y_2 = 0$ or $y_1 = 0$ and $y_2 > 0$. Then (A.8) is equivalent to

$$\max(c_1(y_1 - t\Delta_1 - p_{21}t\Delta_2), c_2(y_2 - t\Delta_2 - p_{12}t\Delta_1)) > x.$$  

In turn, the latter inequality is equivalent to a union of two events,

$$\{y_1 > x/c_1 + t\Delta_1 + p_{21}t\Delta_2\} \cup \{y_2 > x/c_2 + t\Delta_2 + p_{12}t\Delta_1\},$$  \hspace{1cm} (A.9)

where one of these events is empty if the corresponding $c_i$ equals 0.
B Heavy-tailed distributions

Definitions: Distribution $F$ of a positive random variable $X$ is (1) heavy-tailed if $\mathbb{E}e^{cX} \equiv \int_0^\infty e^{cx}dF(x) = \infty$ for all $c > 0$; and light-tailed otherwise; (2) long-tailed if $F(x) > 0$ for all $x > 0$ and $F(x+1)/F(x) \to 1$ as $x \to \infty$; (3) subexponential if $F*F(x) \sim 2F(x)$ or, equivalently, $\mathbb{P}(X_1 + X_2 > x) \sim 2\mathbb{P}(X > x)$ as $x \to \infty$ (here $X_1$ and $X_2$ are two independent copies of $X$). (4) Distribution $F$ of a real-valued r.v. $X$ is subexponential if the distribution of $\max(X,0)$ is subexponential. (5) Distribution $F$ is regularly varying if $F(s) = l(x)x^{-k}$, where $k \geq 0$ and positive function $l(x)$ is slowly varying at infinity. Regularly varying distributions are subexponential.

Key Properties: (1) Any subexponential distribution is long-tailed, and any long-tailed distribution is heavy-tailed. (2) If distribution $F$ is long-tailed, then there exists a function $h(x) \to \infty$ as $x \to \infty$ such that $F(x+h(x))/F(x) \to 1$ as $x \to \infty$. (3) If $F$ is subexponential and $G(x) \sim F(x)$, then $G$ is subexponential. (4) If $X_1, X_2, \ldots$ are i.i.d. with common subexponential distribution $F$ and if $\tau$ is a light-tailed counting r.v., then $\sum_1^\tau X_i$ also has a subexponential distribution and $\mathbb{P}(\sum_1^\tau X_i > x) \sim \mathbb{E}\tau F(x)$.

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