Symplectic Geometry of Supersymmetry and Nonlinear Sigma Model

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Recently it has been argued, that Poincaré supersymmetric field theories admit an underlying loop space hamiltonian (symplectic) structure. Here shall establish this at the level of a general $N = 1$ supermultiplet. In particular, we advocate the use of a superloop space introduced in [2], and the necessity of using nonconventional auxiliary fields. As an example we consider the nonlinear $\sigma$-model. Due to the quartic fermionic term, we conclude that the use of superloop space variables is necessary for the action to have a hamiltonian loop space interpretation.

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1. Introduction. Recently, a conceptually new approach has been developed to describe Poincaré supersymmetric field theories. In this approach supersymmetric theories are interpreted in terms of loop space symplectic geometry \([1, 2]\). The idea originates from Witten, and was presented by Atiyah in \([3]\). He considered the path integral for a supersymmetric spinning particle in a gravitational background, to derive the Atiyah-Singer index theorem for a Dirac operator on a Riemannian manifold. He argued that the path integral admits an underlying loop space symplectic structure such that the relevant Hamiltonian flow is integrable. As a consequence, he could then evaluate the path integral by localization methods, using an infinite dimensional analog of the Duistermaat-Heckman integration formula \([4]\). A more detailed mathematical investigation was subsequently discussed by Bismut \([5]\), who also discussed some generalizations.

An approach to include an arbitrary gauge field background was considered in \([6]\). Subsequently in \([1, 2]\) it was argued by explicit analysis of various examples, that a loop space symplectic structure is not just a property of the particular model but is rather a characteristic feature of generic supersymmetric theories.

Here we shall consider the symplectic interpretation of generic \(N = 1\) Poincaré supersymmetric theories, at the level of a general supermultiplet. We explain the Hamiltonian loop space construction in a model independent fashion, and argue in favor of a superloop space and a new auxiliary field construction on geometrical grounds. In particular, we show how our construction applies to the supersymmetric nonlinear sigma model, with a quartic fermionic self-interaction. A conventional, bosonic loop space construction does not apply in this case, since a four-form does not admit any natural geometric interpretation in terms of symplectic geometry. However, in the superloop space the quartic fermion term turns out to be just a symplectic two-form.

2. Superloop space symplectic geometry. We shall consider a supersymmetric field theory, with a generic (bosonic or fermionic) field \(\Phi(x, t)\) vanishing in the spatial infinity and periodic in time:

\[
\begin{align*}
\Phi(x, t) &\to 0, \text{ if } x \to \infty, \\
\Phi(x, t) &= \Phi(x, t + T).
\end{align*}
\]

With these boundary conditions, we can view the fields as defined on a loop space. As we shall see, in a supersymmetric theory the fields can be naturally divided into two different categories: Half of the fields are interpreted as loop space coordinates, and the other half as the corresponding loop space one-forms. We denote these fields as \(\phi(x, t)\) and \(\xi(x, t) \sim \delta \phi(x, t)\) respectively, and we emphasize that we do not necessarily identify bosonic fields as coordinates and fermionic as one-forms: The loop space can have both bosonic and fermionic coordinates, and the corresponding one-forms are then fermionic and bosonic respectively. In particular, functionals of the original fields are now loop space differential forms. We define the loop space exterior derivative

\[
d = \int dx \int dt \xi(x, t) \frac{\delta}{\delta \phi(x, t)}.
\]
and inner multiplication by introducing a preferred vector field: On the loop space, there is a natural family of vector fields — the time-like derivatives of coordinates. However, in the following we shall find it more convenient to consider loops parametrized by light-cone coordinates, and thus we consider inner multiplication by vector fields

$$i_X = \int dx \int dt \partial_\tau \phi(x,t) \frac{\delta}{\delta \xi(x,t)},$$

where \(\tau\) denotes one of the light-cone variables \(x \pm t\). We shall prove that for a generic \(N = 1\) supersymmetric field theory the generators of Poincaré supersymmetry transformations can be expressed as

$$Q = d + i_X.$$  

with respect to the light-cone vector fields in the various light-cone directions. In refs. \[1, 2\], this was already shown to be the case in various examples.

The square of \(Q\) admits a definite geometric meaning:

$$Q^2 = (d + i_X)^2 = di_X + i_X d = \mathcal{L}_X,$$

is the Weyl formula for Lie derivative in the light-cone direction. Making use of eqs. (2) and (3) one then obtains

$$Q^2 \sim \partial \phi \frac{\delta}{\delta \tau} + \partial \xi \frac{\delta}{\delta \tau} \equiv \frac{\partial}{\partial \tau}.$$  

(Here and afterwards integration over space-time is assumed without writing it explicitly.) As we will see below eq. (6) is a representation of supersymmetry algebra.

We shall find, that the action of a supersymmetric model can be naturally divided divided into a sum of a loop space scalar \(\mathcal{H}\) and a loop space two form \(\omega\):

$$S = \mathcal{H} + \omega,$$  

For explicit examples see below and refs. \[1, 2\].

Due to our boundary conditions the space-time integrals of total derivatives vanish:

$$\int dx \int dt \partial_\tau F(\phi, \xi) = 0.$$  

Hence (3) actually becomes

$$Q^2 = 0.$$  

When we investigate the consequences of supersymmetry of the action — \(Q S = 0\) — we discover the following equations by separating differential forms of different degrees:

$$d\omega = 0,$$

$$d\mathcal{H} + i_X \omega = 0.$$  

(11) implies that \(\omega\) can be interpreted as a symplectic two-form and from (11) we conclude that \(X\) is the Hamiltonian vector field corresponding to \(\mathcal{H}\). The concrete
form of $X$ is determined by the supersymmetry algebra in a model-independent way. As $X$ has a very simple structure (3), we can integrate the corresponding "Hamiltonian equations of motion" in the loop space getting constant modes as the solution. From the point of view of Duistermaat-Heckman integration formula, this means that the path integral corresponding to a supersymmetric action is localized to constant modes. (See refs. [1, 2, 3] for discussion.)

We also observe, that $\mathcal{H}$ is uniquely (up to a total derivative and a constant) determined by $\omega$. For a given $\omega$ one can locally find the corresponding symplectic potential $\vartheta$ that fulfills the following condition:

$$d \vartheta = \omega.$$  \hspace{1cm} (12)

Acting with $Q^2$ on $\vartheta$ one gets

$$d(i_X \vartheta) + i_X \omega = 0,$$  \hspace{1cm} (13)

and taking into account eq. (11) we identify:

$$i_X \vartheta = \mathcal{H}.$$  \hspace{1cm} (14)

In order to establish uniqueness of the choice of $\mathcal{H}$ let us choose another potential $\vartheta' = \vartheta + d\varphi$ for some scalar $\varphi$. We get

$$i_X \vartheta' = i_X \vartheta + i_X d\varphi = i_X \vartheta + \frac{d}{d\tau} \varphi = \mathcal{H}.$$  \hspace{1cm} (15)

Here we used (8) to put $\dot{\varphi} = 0$. On the other hand one might assume, that the true Hamiltonian $\mathcal{H}'$ differs from $i_X \vartheta$. However, it follows from the supersymmetry (11), that for $\mathcal{H} = i_X \vartheta$ and $\mathcal{H}'$ one has

$$d(\mathcal{H} - \mathcal{H}') = 0,$$  \hspace{1cm} (16)

Thus, modulo a total derivative $\mathcal{H}$ and $\mathcal{H}'$ can differ only by a constant mode, and from (13) and (14) we can locally write the action as a supersymmetry variation,

$$S = (d + i_X)\vartheta.$$  \hspace{1cm} (17)

3. Explicit constructions. In this section we shall explicitly realize the geometrical structures of the previous section using $N = 1$ super-Poincaré algebra in four dimensions (see e.g. [4]):

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu C)P_\mu.$$  \hspace{1cm} (18)

We use a Majorana representation with $\gamma^0 = -\sigma^2 \otimes I$, $\gamma^1 = -i\sigma^3 \otimes \sigma^1$, $\gamma^2 = i\sigma^1 \otimes I$, $\gamma^3 = -i\sigma^3 \otimes \sigma^3$, where we have:

$$(\gamma^\mu C)P_\mu = \begin{pmatrix} i\partial_+ & \ast & \ast & \ast \\ \ast & i\partial_+ & \ast & \ast \\ \ast & \ast & i\partial_- & \ast \\ \ast & \ast & \ast & i\partial_- \end{pmatrix},$$  \hspace{1cm} (19)
with light-cone derivatives on the diagonal and * standing for terms that are not relevant in the following. Eq. (18) suggests that $Q$ — given by eqs. (2 – 4) and satisfying (6) — can be identified with any of the $Q_\alpha$-s, where light-cone coordinates $x^\pm = x_2 \pm t$ stand for the parameter $\tau$. Different representations for gamma matrices would define different preferred lightcone directions.

In order to demonstrate that the symplectic structure is present for a general case it is sufficient to prove it for the general $N = 1$ supermultiplet containing a complex scalar $M$, pseudoscalars $C, N, D$, and a vector $A_\mu$, and two Dirac spinors $\chi$ and $\lambda$. Other multiplets can be obtained by imposing some additional constraints.

From the transformation rules of the complex $N = 1$ supermultiplet $V = (C; \chi; M, N, A_\mu; \lambda; D)$:

$$
\begin{align*}
\delta C &= \bar{\zeta} \gamma_5 \chi, \\
\delta \chi &= (M + \gamma_5 N)\zeta - i \gamma^\mu (A_\mu + \gamma^5 \partial_\mu C)\zeta, \\
\delta M &= \bar{\zeta} (\lambda - i \bar{\psi} \chi), \\
\delta N &= \bar{\zeta} \gamma_5 (\lambda - i \bar{\psi} \chi), \\
\delta A_\mu &= \bar{\zeta} (i \gamma_\mu \lambda + \partial_\mu \chi), \\
\delta \lambda &= -i \sigma^{\mu \nu} \zeta \partial_\nu A_\mu - \gamma_5 \zeta D, \\
\delta D &= -i \bar{\zeta} \bar{\psi} \gamma_5 \lambda,
\end{align*}
$$

(20)

we can find the transformation generated by any of the $Q_\alpha$-s. For $Q_1$ we get:

$$
\begin{align*}
Q_1 C &= \chi_2, \\
Q_1 \chi_1 &= i A_+, \\
Q_1 \chi_2 &= i \partial_+ C, \\
Q_1 \chi_3 &= i (M + A_2 + \partial_z C), \\
Q_1 \chi_4 &= i (N + A_x - \partial_x C), \\
Q_1 M &= \lambda_1 + \partial_+ \chi_3 + \partial_2 \chi_2 - \partial_z \chi_1, \\
Q_1 N &= \lambda_2 + \partial_+ \chi_4 - \partial_2 \chi_1 + \partial_z \chi_2, \\
Q_1 A_+ &= \partial_+ \chi_1, \\
Q_1 A_- &= -2 \lambda_3 + \partial_- \chi_1, \\
Q_1 A_x &= -\lambda_2 + \partial_1 \chi_1, \\
Q_1 A_z &= -\lambda_1 + \partial_z \chi_1, \\
Q_1 \lambda_1 &= -i (\partial_+ A_2 - \partial_2 A_+) , \\
Q_1 \lambda_2 &= -i (\partial_+ A_x - \partial_x A_+), \\
Q_1 \lambda_3 &= -i \left( \frac{1}{2} (\partial_+ A_- - \partial_- A_+) \right), \\
Q_1 \lambda_4 &= -i (D + \partial_2 A_x - \partial_x A_2), \\
Q_1 D &= -\partial_+ \lambda_4 - \partial_2 \lambda_2 + \partial_x \lambda_1.
\end{align*}
$$

(21)

To obtain notational simplicity we use redefined fields denoted by primes:

$$
M' = M + A_z + \partial_z C,
$$
\[ N' = N + A_x - \partial_z C, \]
\[ \lambda'_1 = \lambda_1 - \partial_z \chi_1, \]
\[ \lambda'_2 = \lambda_2 - \partial_z \chi_1, \]
\[ \lambda'_3 = 2\lambda_3 - \partial_z \chi_1, \]
\[ D' = D + \partial_x A_z - \partial_z A_x, \]

and thus we can rewrite (21) in a more compact form:

\[ Q_1 C = \chi_2, \]
\[ Q_1 (\chi_1, \chi_2, \chi_3, \chi_4) = (iA_+, i\partial_+ C, iM', iN'), \]
\[ Q_1 M' = \partial_+ \chi_3, \]
\[ Q_1 N' = \partial_+ \chi_4, \]
\[ Q_1 (A_+, A_-, A_x, A_z) = (\partial_+ \chi_1, -\lambda_3, -\lambda_2, -\lambda_1), \]
\[ Q_1 (\lambda'_1, \lambda'_2, \lambda'_3, \lambda_4) = (-i\partial_+ A_z, -i\partial_+ A_x, -i\partial_+ A_-, -iD'), \]
\[ Q_1 D' = -\partial_+ \lambda_4. \]

Equations (22) are exactly the definitions of the nonstandard auxiliary fields, that were introduced in [1, 2]. Eqs. (23) suggest us to write:

\[ d = \chi_2 \frac{\delta}{\delta C} + iA_+ \frac{\delta}{\delta \chi_1} + iM' \frac{\delta}{\delta \chi_3} + iN' \frac{\delta}{\delta \chi_4} + \]
\[ -\lambda'_3 \frac{\delta}{\delta A_-} - \lambda'_2 \frac{\delta}{\delta A_x} - \lambda'_1 \frac{\delta}{\delta A_z} - iD' \frac{\delta}{\delta \lambda_4}, \]

and

\[ i_{X_+} = i\partial_+ C \frac{\delta}{\delta \chi_2} + \partial_+ \chi_3 \frac{\delta}{\delta M'} + \partial_+ \chi_4 \frac{\delta}{\delta N'} + \partial_+ \chi_1 \frac{\delta}{\delta A_+} + \]
\[ -\partial_+ A_z \frac{\delta}{\delta \lambda'_3} - \partial_+ A_x \frac{\delta}{\delta \lambda'_2} - \partial_+ A_z \frac{\delta}{\delta \lambda'_1} - \partial_+ A_\frac{\delta}{\delta \lambda_4}. \]

A different choice of the preferred \( Q_\alpha \) would have lead us to different redefinitions of the fields and different division of the fields into coordinates of the loop-space and their differentials. The relation

\[ Q^2_+ = di_{X_+} + i_{X_+} d = L_{X_+} = i\partial_+ \]

is the geometric form to express the superalgebra (18).

Special cases are obtained easily by imposing additional constraints. For example, to pick up supersymmetric Maxwell theory one is to impose reality and a gauge condition:

\[ (C; \chi; M, N; A_+) = 0, \]

that leaves us with an irreducible multiplet \((A_-, A_x, A_z; \lambda; D)\). Thus we obtain the following relations:

\[ d = -\lambda'_3 \frac{\delta}{\delta A_-} - \lambda'_2 \frac{\delta}{\delta A_x} - \lambda'_1 \frac{\delta}{\delta A_z} - iD' \frac{\delta}{\delta \lambda_4}, \]

(26)
\[ i_X = -\partial_+ A_x \frac{\delta}{\delta \lambda_1} - \partial_+ A_x \frac{\delta}{\delta \lambda_2} - \partial_+ A_x \frac{\delta}{\delta \lambda_3} - \partial_+ \lambda_4 \frac{\delta}{\delta D'} . \] 

(27)

All the statements on the geometric structure of the action can be now verified.

4. Nonlinear \( \sigma \)-model. As an example, we shall now proceed to discuss the two dimensional supersymmetric nonlinear sigma model. We shall find, that in the quartic fermion term half of the fermionic degrees of freedom should be interpreted as coordinates in a superloop space, while the remaining half of the fermionic degrees of freedom are differentials in this superloop space. This then identifies the quartic term as a symplectic two form.

The action of the \( \sigma \)-model is following:

\[
S = \frac{1}{2} \int d^2x \{ g_{ij}(\varphi)(\partial_+ \varphi^i \partial_- \varphi^j + \bar{\psi}^i D \psi^j + \tilde{F}^i \tilde{F}^j) + \frac{1}{6} R_{ijkl}(\varphi) \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l \},
\]

where

\[
\tilde{F}^i = F^i - \frac{1}{2} \Gamma^i_{jk} \bar{\psi}^j \psi^k .
\]

It contains a set of real scalar fields \( \varphi^i \), auxiliary scalars \( F^i \) and Majorana spinors \( \psi^i = (\psi_{1i}, \psi_{2i}) \).

As previously, the detailed structure of supersymmetry transformations suggest a suitable choice for the exterior derivative and inner multiplication:

\[
d = \psi_{1i} \frac{\delta}{\delta \varphi^i} - i F^i \frac{\delta}{\delta \psi_{2i}} ,
\]

(29)

\[
i_X = i \partial_+ \varphi^i \frac{\delta}{\delta \psi_{1i}} - \partial_- \psi_{2i} \frac{\delta}{\delta F^i} ,
\]

(30)

with the peculiarity of two dimensions, that one has to make no redefinitions of the fields. Another choice permitted by the superalgebra would be:

\[
d' = \psi_{2i} \frac{\delta}{\delta \varphi^i} + i F^i \frac{\delta}{\delta \psi_{1i}} ,
\]

(31)

\[
i_{X'} = i \partial_+ \varphi^i \frac{\delta}{\delta \psi_{2i}} - \partial_+ \psi_{1i} \frac{\delta}{\delta F^i} .
\]

(32)

The operators \( Q = d + i_X \) and \( Q' = d' + i_{X'} \) reproduce the standard supersymmetry transformations and so the anticommutation relations of the operators \( Q \) and \( Q' \) obey the relations of the supersymmetry algebra (see [3]):

\[
QQ' + Q'Q = 0 ,
\]

\[
QQ + QQ = 2L_X = 2i \partial_- ,
\]

(33)

\[
Q'Q' + Q'Q = 2L_{X'} = 2i \partial_+ .
\]
Following the general scheme of section 2 we can find the symplectic potential corresponding to the action (28) (if we make the choice of (29) and (30)):

\[ \vartheta = -\frac{i}{2} g_{ij} \psi^1_i \partial_- \phi^j + \frac{i}{2} g_{ij} \psi^2_i \partial_- F^j + \frac{1}{2} \Gamma_{i,j,k} \psi^j_1 \psi^k_2 \psi^1_2, \]  

(34)

The action is related to it by

\[ S = (d + iX) \vartheta = \mathcal{H} + \omega. \]

One might think that the quartic terms lead to differential forms of higher degrees than two. In fact, if we write out the relevant terms in components we obtain the following expression

\[ -\frac{1}{2} g_{ik,jl} \psi^i_1 \psi^j_1 \psi^k_2 \psi^l_2, \]  

(35)

that clearly contains the fermionic degrees of freedom \( \psi^i_1 \), that we have identified with differentials, bilinearly. Hence (35) is a closed two form in the superloop space, as expected.

5. Conclusions. We have shown, that the symplectic interpretation of supersymmetric theories can be based on the properties of Poincaré superalgebra and supersymmetry transformation laws of a general \( N = 1 \) supermultiplet. Our approach generalizes the results of [1, 2], where the geometric structures were discussed for a number concrete models. In particular, we have given a model independent definitions of the exterior derivative in superloop space (24), a contraction operator \( i_X \) with a preferred vector field \( X \) (25), and we have represented a superrotation as a sum of these (4). The action of supersymmetric models was observed to split into the sum of a scalar functional (the Hamiltonian) and a two-form (the symplectic structure), and due to supersymmetry (11) the vector field \( X \) appears to be Hamiltonian. In Section 4 we discussed how the super-loop space formalism applies to the nonlinear \( \sigma \)-model. In this case, due to the four-fermion term, in order to have a geometric interpretation it is necessary to use a superloop space.

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