ON A DESINGULARIZATION OF THE MODULI SPACE
OF NONCOMMUTATIVE TORI

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Abstract. It is shown that the moduli space of the noncommutative tori
admits a natural desingularization by the group Ext \((A_\theta, A_\theta)\). Namely, we
prove that the moduli space of pairs \((A_\theta, \text{Ext} \ (A_\theta, A_\theta))\) is homeomorphic to a
punctured two-dimensional sphere. The proof is based on a correspondence (a
covariant functor) between the complex and noncommutative tori.

1. Introduction

A. Let \(0 < \theta < 1\) be an irrational number, whose regular continued fraction
has the form \(\theta = [a_0, a_1, a_2, \ldots]\). Consider an AF-algebra \(A_\theta\) given by the Bratteli
diagram in Figure 1. The \(a_i\) indicate the number of edges in the upper row of

\[
\begin{array}{ccccccc}
& a_0 & a_1 & \cdots & & & \\
\vdots & & & & & & \vdots
\end{array}
\]

Figure 1. The Bratteli diagram of the AF-algebra \(A_\theta\).

the diagram. With a moderate abuse of the terminology, we shall call \(A_\theta\) a non-
commutative torus. (Note that a standard definition of the noncommutative torus
– a universal \(C^*\)-algebra generated by the unitaries \(u, v\) satisfying the commuta-
tion relation \(vu = e^{2\pi i} uv\) – is not an AF-algebra. However, the two objects are
isomorphic at the level of their dimension groups \([8], [9]\).)

B. Recall that the noncommutative tori \(A_\theta, A_\theta'\) are said to be stably isomorphic,
whenever \(A_\theta \otimes \mathcal{K} \cong A_\theta' \otimes \mathcal{K}\), where \(\mathcal{K}\) is the AF-algebra of the compact operators.

It is well known that the AF-algebras \(A_\theta, A_\theta'\) are stably isomorphic if and only
if \(\theta' \equiv \theta \mod \text{SL}(2, \mathbb{Z})\), i.e. \(\theta' = (a \theta + b) / (c \theta + d)\), where \(a, b, c, d \in \mathbb{Z}\) and
\(ad - bc = 1\) \([3]\). It is easy to see that the stable isomorphism is an equivalence
relation, which splits the set \(\{A_\theta \mid 0 < \theta < 1, \ \theta \in \mathbb{R} - \mathbb{Q}\}\) into the disjoint
equivalence classes. By \(\mathcal{M}\) we shall understand a collection of such classes, or the
“moduli space” of the noncommutative tori. An examination of $\mathcal{M}$ as a topological space (with the topology induced by $\mathbb{R}$) shows that the points of $\mathcal{M}$ have no disjoint neighborhoods, since each orbit $\{\theta' \in \mathbb{R} \mid \theta' \equiv \theta \mod SL(2, \mathbb{Z})\}$ is dense in the real line $\mathbb{R}$. A question arises as to how to “desingularize” the (non-Hausdorff) moduli space $\mathcal{M}$.

C. Let $A, B$ be a pair of the $C^*$-algebras. Recall that an extension of $A$ by $B$ is a $C^*$-algebra $E$ filling the short exact sequence $0 \to B \to E \to A \to 0$ of the $C^*$-algebras. If $A$ is a separable nuclear $C^*$-algebra, the $\text{Ext}(A, B)$ is an additive abelian group, whose group operation is a sum of the two extensions. The $\text{Ext}(A, B)$ is a homotopy invariant in both variables. The extensions $E_1, E_2$ are said to be stably equivalent if there exists an isomorphism $\psi : E_1 \otimes K \cong E_2 \otimes K$, such that $\psi \circ \alpha_1(B \otimes K) = \alpha_2(B \otimes K)$, where $\alpha_i : B \to E_i, i = 1, 2$. We shall further restrict to the case $A = B = \mathbb{K}$ and study the stable equivalence classes of the group $\text{Ext}(\mathbb{K}, \mathbb{K})$. Using the classification results of D. Handelman [5], it will develop that the group $\text{Ext}(\mathbb{K}, \mathbb{K}) \cong \text{Hom}(K_0(\mathbb{K}), \mathbb{R}) \cong \mathbb{R}$. Moreover, the $\text{Ext}(\mathbb{K}, \mathbb{K})/\text{stable equivalence} \cong \mathbb{R}/\mathbb{Z}$.

D. An objective of the note is to show that the moduli of the pairs $(\mathbb{K}, \text{Ext}(\mathbb{K}, \mathbb{K}))$ under the stable equivalence is no longer a non-Hausdorff topological space, but a two-dimensional orbifold (a punctured sphere). To prove this result we shall use the Teichmüller space of a torus (a space of the complex structures on the torus) [6]. Namely, Hubbard and Masur established a homeomorphism between the Teichmüller space $T_g$ of a surface of genus $g \geq 1$ and the space of quadratic differentials on it. We shall use the homeomorphism to extend the action of the modular group $SL(2, \mathbb{Z})$ from the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\} \cong T_1$ to the space $(\mathbb{K}, \text{Ext}(\mathbb{K}, \mathbb{K}))$. Denote by $\tilde{\mathcal{M}}$ the set of pairs $(\mathbb{K}, \text{Ext}(\mathbb{K}, \mathbb{K}))$ modulo the stable equivalence. One obtains the following (natural) desingularization of the moduli space of the noncommutative tori.

**Theorem 1.** $\tilde{\mathcal{M}}$ is a punctured two-dimensional sphere.

2. **Proof**

We shall split the proof into two lemmas. The background material is mostly standard, and we shall recall in passing some important notation and ideas.

**Lemma 1.** $\tilde{\mathcal{M}}$ is a two-dimensional orbifold.

*Proof of Lemma 1.* We shall use a standard dictionary existing between the $AF$-algebras and their dimension groups [2]. Instead of dealing with the $AF$-algebra $\mathbb{K}$, we shall work with its dimension group $G_\theta = (G, G^+)$, where $G \cong \mathbb{Z}^2$ is the lattice and $G^+ = \{(x, y) \in \mathbb{Z}^2 \mid x + \theta y \geq 0\}$ is a positive cone of the lattice. The $G_\theta$ is the additive abelian group with an order, which defines the $AF$-algebra $\mathbb{K}$ up to a stable isomorphism.

Under the dictionary, the extension problem for the $AF$-algebra $\mathbb{K}$ translates as an extension problem for the dimension groups $G_\theta \to E \to G_\theta$ (we omit the zeros in the exact sequence). An important result of Handelman establishes the intrinsic classification of the extensions of the simple dimension group by a simple dimension group; see Theorem III.5 of [3]. Let us recall the classification as it is exposed in [4] Theorem 17.5 and Corollary 17.7. We shall adopt the same notation as in the cited work.
Let $H$ be a dense subgroup of the real line $\mathbb{R}$ and $K$ a nonzero dimension group. Let $E$ be the abelian group $H \oplus K$, and let $\tau : H \to E$ and $\pi : E \to K$ be a natural injection and projection maps. Assume that $f : K \to \mathbb{R}$ is a homomorphism of the dimension groups. Then: (i) $E$ is a dimension group with the positive cone

\[ E_+^f = \{(0,0)\} \cup \{(x,y) \in E \mid y \geq 0 \text{ and } x + f(y) > 0\}, \]

which gives an extension \( H \to (E,E_+^f) \to K \) of $H$ by $K$; (ii) if $f,f' : K \to \mathbb{R}$ are the group homomorphisms, then the extensions $E_f,E_{f'}$ are equivalent if and only if $(f-f')(K) \subseteq H$.

We have to specialize the above theorem to the case $H = K = G_{\theta}$. It is immediate from (i) that $E \cong \mathbb{Z}^4$. Note that the group homomorphisms $f : G_{\theta} \to \mathbb{R}$ are bijective with the reals $\mathbb{R}$. Indeed, we have to find all the linear maps $f : \mathbb{R}^2 \to \mathbb{R}$, such that $Ker f = x + \theta y$. (The last equation follows from the condition $f(G_{\theta}^+) > 0$.) Such maps have the form $f(p) = (p,t)$, $p,t \in \mathbb{R}^2$, where $(p,t)$ is the dot product of the two vectors. Let $t = (t_1,t_2)$. Then $f_i(-\theta y,y) = t_1(-\theta y) + t_2 y = y(t_2 - t_1 \theta) = 0$ for all $y \in \mathbb{R}$. Therefore, $t_2 = \theta t_1$ and $f_i(x,y) = t_1 x + \theta t_1 y = t_1(x + \theta y)$, $t_1 \in \mathbb{R}$. Thus, all linear maps $f : \mathbb{R}^2 \to \mathbb{R}$ with $Ker f = x + \theta y$ are bijective with the reals $t_1 \in \mathbb{R}$. In other words, $Ext (\mathbb{Z},\mathbb{Z})$ and $\mathbb{R}$ are isomorphic as additive abelian groups.

Let us find when the two extensions $E,E'$ are equivalent. Since $H = G_{\theta}$ is a subgroup of $\mathbb{R}$, one can write $H = \mathbb{Z} + \theta \mathbb{Z}$. Let $t,t'$ be the real numbers corresponding to the homomorphisms $f,f'$. Then $f(G_{\theta}) = t(\mathbb{Z}+\theta \mathbb{Z})$ and $f'(G_{\theta}) = t'(\mathbb{Z}+\theta \mathbb{Z})$. The condition $(f-f')(K) \subseteq H$ of the item (ii) will take the form $(t-t')(\mathbb{Z}+\theta \mathbb{Z}) \subseteq \mathbb{Z}+\theta \mathbb{Z}$. One gets immediately that $t = t' + n, n \in \mathbb{Z}$ as a necessary and sufficient condition for the last inclusion. In other words, the extensions $E,E'$ are equivalent if and only if $t' = t \mod \mathbb{Z}$. Thus, the equivalence classes of $Ext (\mathbb{Z},\mathbb{Z})$ are bijective with the factor space $\mathbb{R}/\mathbb{Z}$ (a unit interval).

To finish the proof of Lemma 1 let us extend the domain of definition of $\theta$ from the interval $(0,1)$ to the real line $\mathbb{R}$ by allowing $a_0$ to take on any integer value. In this way, one can identify the pairs $(\mathbb{Z},Ext (\mathbb{Z},\mathbb{Z}))$ with the points of $\mathbb{R}^2$ equipped with the usual Euclidean topology. We have seen that the points $(\theta,t) \sim (\theta',t') \in \mathbb{R}^2$ are equivalent if and only if $\theta' \equiv \theta \mod SL(2,\mathbb{Z})$ and $t' \equiv t \mod \mathbb{Z}$. Note that the action of the modular group on the second coordinate is always free. Therefore, the points $x,y$ of the space $\mathcal{M} \cong \mathbb{R}^2/\sim$ admit the disjoint neighborhoods defined, e.g., by the open balls of radius $1/3$ centered in $x$ and $y$, respectively. The balls are locally homeomorphic to the Euclidean plane, and therefore $\mathcal{M}$ is a two-dimensional orbifold.

Lemma 1 gives a (partial) desingularization of the space $\mathcal{M}$. Indeed, we have seen that the group $SL(2,\mathbb{Z}) \times \mathbb{Z}$ acts in the plane $(\mathbb{Z},Ext (\mathbb{Z},\mathbb{Z}))$ by the formula $(\theta,t) \to (\frac{a \theta + b}{c \theta + d},t + n)$, where $ad - bc = 1$ and $a,b,c,d,n \in \mathbb{Z}$. However, the last formula does not specify the action on the parameter plane $(\theta,t)$ of the modular group $SL(2,\mathbb{Z})$ alone, since the function $n = n(a,b,c,d)$ is unknown. To find how the integer $n$ depends on the integers $a,b,c,d$, we would need a special construction which involves a correspondence (a covariant functor) between the complex and noncommutative tori. Such a construction will be given in the next paragraph and is encapsulated in the following lemma.

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1That is, $f$ preserves the positive cone of $K$ and $\mathbb{R}$: $f(K^+) > 0$. 

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Lemma 2. There exists a homeomorphism \( h : \tilde{M} \to \mathbb{H} / SL(2, \mathbb{Z}) \), where \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) is the Lobachevsky plane endowed with a hyperbolic metric.

Proof of Lemma 2. Let \( X \) be a topological surface of genus \( g \geq 0 \). The Teichmüller space \( T_g \) of \( X \) consists of the equivalence classes of the complex structures on \( X \). The space \( T_g \) is an open ball of the (real) dimension \( 6g - 6 \) if \( g \geq 2 \) and \( 2g \) if \( g = 0, 1 \). By \( \text{Mod} \, X \) we designate a group of the orientation-preserving diffeomorphisms of \( X \) modulo the trivial ones. The points \( S, S' \in T_g \) are equivalent if there exists a conformal map \( f \in \text{Mod} \, X \) such that \( S' = f(S) \). The moduli of conformal equivalence is denoted by \( \mathcal{M}_g = T_g / \text{Mod} \, X \). The space \( \mathcal{M}_g \) is a (classical) moduli space, whose definition dates back to Riemann.

Let \( S \in T_g \) be a Riemann surface thought of as a point in the Teichmüller space, and let \( H^0(S, \Omega^{g2}) \) be the space of the holomorphic quadratic forms on \( S \). The fundamental theorem of Hubbard and Masur says that there exists a homeomorphism \( h_S : H^0(S, \Omega^{g2}) \to T_g \) [6, p. 224]. The space \( H^0(S, \Omega^{g2}) \) is a real vector space of dimension \( 6g - 6 \), where \( g \geq 2 \). It has been shown in the above cited work that \( H^0(S, \Omega^{g2}) \cong \text{Hom} \, (H_1(\tilde{X}, \tilde{\Gamma})^-; \mathbb{R}) \) defined by the formula

\[
(2) \quad \omega \longmapsto \left( \gamma \mapsto \text{Im} \int_\gamma \omega \right),
\]

where \( H_1(\tilde{X}, \tilde{\Gamma})^- \) is the odd part in the homology of a double cover \( \tilde{X} \) of \( X \) ramified at the zeroes \( \tilde{\Gamma} \) of the odd multiplicity of the quadratic form \( \tilde{\omega} \) [6, p. 232]. (The symbols and formulas will simplify as we come to the complex torus — our principal case.) It has been proved that \( H_1(\tilde{X}, \tilde{\Gamma})^- \cong \mathbb{Z}^{6g - 6} \).

Let \( X = T^2 \), i.e. \( g = 1 \). In this case each quadratic differential form is the square of a holomorphic abelian form (a one-form), i.e. \( H^0(S, \Omega^{g2}) = H^0(S, \Omega) \). Therefore \( \tilde{X} = X = T^2 \), \( \tilde{\Gamma} = \emptyset \) and \( H_1(\tilde{X}, \tilde{\Gamma})^- = H_1(T^2) \cong \mathbb{Z}^2 \). In other words, one gets a homeomorphism \( h_S : \text{Hom} \, (\mathbb{Z}^2, \mathbb{R}) \to T_1 \). As we have seen earlier, \( \text{Hom} \, (\mathbb{Z}^2, \mathbb{R}) = \{ t_1 \mathbb{Z} + t_2 \mathbb{Z} \mid t_1, t_2 \in \mathbb{R} \} = \{ t(Z + \theta Z) \mid \theta, t \in \mathbb{R} \} \), where \( t = t_1, \theta = t_2/t_1 \). On the other hand, the Teichmüller space \( T_1 \cong \mathbb{H} \), where \( \mathbb{H} = \{ \tau = x + iy \in \mathbb{C} \mid y > 0 \} \) is a (Lobachevsky) upper half-plane and \( \tau \) is a modulus of the complex torus \( C/(Z + \tau Z) \) [10, pp. 6-14]. Thus, we have a homeomorphism \( h_S : (\mathbb{A}_\theta, \text{Ext}_t(\mathbb{A}_\theta, \mathbb{A}_\theta)) \to \mathbb{H} \).

Let us show that \( h_S \) is equivariant in the first coordinate with respect to the action of \( \text{Mod} \, (T^2) \cong SL(2, \mathbb{Z}) \); i.e., \( \tau' \equiv \tau \mod SL(2, \mathbb{Z}) \) if and only if \( \theta' \equiv \theta \mod SL(2, \mathbb{Z}) \). Indeed, since \( \text{Hom} \, (H_1(T^2); \mathbb{R}) \cong \mathbb{H} \) the modular group \( SL(2, \mathbb{Z}) \) acts on the right-hand side by the formula \( \tau \mapsto (a\tau + b)/(c\tau + d) \) and on the left-hand side by a linear transformation \( p_1 \mapsto ap_1 + bp_2, \ p_2 \mapsto cp_1 + dp_2 \), where \( p = (p_1, p_2) \in H_1(T^2) \) and \( ad - bc = 1 \). The \( f_p(t) = p_1t_1 + p_2t_2 \) will become \( f_p(t') = t_1(ap_1 + bp_2) + t_2(cp_1 + dp_2) = p_1t'_1 + p_2t'_2 \), where \( t'_1 = at_1 + ct_2 \) and \( t'_2 = bt_1 + dt_2 \). Therefore \( \theta = t_2/t_1 \) goes to \( \theta' = t'_2/t'_1 = (b + dt_2)/(a + ct_1) \) and \( \theta' \equiv \theta \mod SL(2, \mathbb{Z}) \). (The ‘only if’ part of the statement is obtained likewise by an inversion of the formulas.)

Recall that the Lobachevsky plane \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) carries a hyperbolic metric \( ds = |dz/y | \) such that \( SL(2, \mathbb{Z}) \) acts on it by the isometries (linear-fractional transformations). The tessellation of \( \mathbb{H} \) by the fundamental regions is shown in Figure 2. Let \( \tau' = \frac{ax + by + c}{cx + dy} = T(\tau), \ \tau \in \mathbb{H} \). The number \( n = n(a, b, c, d) \in \mathbb{Z} \) we shall call a height of the transformation \( T \in SL(2, \mathbb{Z}) \) if \( n \) is equal to the number of intersections of the vertical segment \( \text{Im} \ (\tau' - \tau) \) issued from \( \tau \) with the lines of tiling \( \mathbb{H}/SL(2, \mathbb{Z}) \). (In other words, \( n \) shows how many fundamental regions apart
from $\tau$ and $\tau'$ are in the vertical direction.) We leave it to the reader to verify that
the height $n$ does not depend on a particular choice of $\tau$ in the fundamental region
or the fundamental region itself being a function of the transformation $T$ only.

Let us now define an action of the modular group on $(\mathbb{A}_\theta, \text{Ext}((\mathbb{A}_\theta, \mathbb{A}_\theta)))$. The
action is given by the formula $(\theta, t) \mapsto (\frac{a\theta + b}{c\theta + d}, t + n)$, where $n = n(a, b, c, d)$ is the
height of the transformation $T = T(a, b, c, d)$. Under the homeomorphism $h_S$, the
tessellation of $\mathbb{H}$ maps into a tessellation of the plane $(\theta, t)$. As we have shown
earlier, the action of the modular group $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is equivariant with the
action on $(\theta, t)$. On the other hand, it is known that $\mathbb{H}/SL(2,\mathbb{Z})$ is a punctured
two-dimensional sphere [10, p. 15]. Lemma 2 and Theorem 1 follow.

3. Remarks

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be a complex torus and $h_S(E_\tau) = (\mathbb{A}_\theta, \text{Ext}((\mathbb{A}_\theta, \mathbb{A}_\theta)))$ its
image under the homeomorphism $h_S$. Let us call the respective reals $\theta = \theta(\tau)$ and
t $t = t(\tau)$ a projective curvature and an area of the complex torus $E_\tau$. The projective
curvature of the complex tori with a nontrivial group of endomorphisms (complex
multiplication) is a quadratic irrationality. In the latter case, the noncommutative
tori is said to have a real multiplication. The noncommutative tori with real
multiplication can be used to construct the abelian extensions of the real quadratic
number fields, as was suggested by Yu. Manin [7]. It seems challenging at this point
to write a formula for the projective curvature and the area as a function of the
complex modulus $\tau$. It is likely that the functions will be of the class $C^0$.

Problem 1. Find a formula (if any) for the functions $\theta(\tau)$ and $t(\tau)$.

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