Inductive characterizations of hyperquadrics

Baohua Fu

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Abstract We give two characterizations of hyperquadrics: one as non-degenerate smooth projective varieties swept out by large dimensional quadric subvarieties passing through a point; the other as LQEL-manifolds with large secant defects.

1 Introduction

We work over an algebraically closed field of characteristic zero. In [1], Ein proved that if $X$ is an $n$-dimensional smooth projective variety containing an $m$-plane $\Pi_0$ whose normal bundle is trivial, with $m \geq n/2 + 1$, then there exists a smooth projective variety $Y$ and a vector bundle $E$ over $Y$ such that $X \cong \mathbb{P}(E)$ and $\Pi_0$ is a fiber of $X \to Y$. The bound on $m$ was improved to $m \geq n/2$ by Wiśniewski in [11]. Later on, Sato [10] studied projective smooth $n$-folds $X$ swept out by $m$-dimensional linear subspaces, i.e. through every point of $X$, there passes through an $m$-dimensional linear subspace. If $m \geq n/2$, he proved that either $X$ is a projective bundle as above or $m = n/2$. In the latter case, $X$ is either a smooth hyperquadric or the Grassmanian variety parametrizing lines in $\mathbb{P}^{m+1}$.

A natural problem is to extend these results to the case where linear subspaces are replaced by quadric hypersurfaces. In this paper, we will consider a smooth projective non-degenerate variety $X \subseteq \mathbb{P}^N$ of dimension $n$, which is swept out by $m$-dimensional irreducible hyperquadrics passing through a point (for the precise definition see Sect. 3). Examples of such varieties include Severi varieties (see [12]), or more generally LQEL-manifolds of positive secant defect(see Sect. 2 below). As it
turns out, the number \( m \) is closely related to the secant defect of \( X \), which makes it hard to construct examples with big \( m \).

Our main theorem is to show (cf. Theorem 2) that if \( m > \lfloor n/2 \rfloor + 1 \), then \( N = n + 1 \) and \( X \) is itself a hyperquadric. This gives a substantial improvement to the Main Theorem 0.2 of [7], where the same claim is proved under the assumption that a general hyperquadric in the family is smooth and that \( m \geq 3n/5 + 1 \). Our proof here, based on ideas contained in [5] and [9], is much simpler and is completely different from that in [7]. However, we should point out that a more general result, without assuming the quadric subspaces pass all through a fixed point, is proven in [7].

The same idea of proof, combined with the Divisibility Theorem of [9], allows us to prove (cf. Corollary 3) that for an \( n \)-dimensional LQEL-manifold, either it is a hyperquadric or its secant defect is no bigger than \( n + 8/3 \). This improves Corollaries 0.11, 0.14 of [7]. It also gives positive support to the general believing that hyperquadrics are the only LQEL-manifolds with large secant defects.

2 Preliminaries

Let \( \delta = \delta(X) = 2n + 1 - \dim(SX) \) be the secant defect of a non-degenerate \( n \)-dimensional variety \( X \subset \mathbb{P}^N \), where

\[
SX = \bigcup_{x \neq y, x, y \in X} \langle x, y \rangle \subseteq \mathbb{P}^N
\]

is the secant variety of \( X \subset \mathbb{P}^N \).

Recall [4,7] that a smooth irreducible non-degenerate projective variety \( Z \subset \mathbb{P}^N \) is said to be conically connected (CC for short) if through two general points there passes an irreducible conic contained in \( Z \). Such varieties have been studied and classified in [4] and [5].

We begin with a simple but very useful remark, which is probably well known but we were not able to find a reference.

**Lemma 1** Let \( X \subset \mathbb{P}^N \) be a smooth projective variety and let \( z \in X \) be a point. If there exists a family of smooth rational curves of degree \( d \) on \( X \) passing through \( z \) and covering \( X \), then through two general points \( x, y \in X \) there passes such a curve.

In particular, if \( d = 1 \), then \( X \subset \mathbb{P}^N \) is a linearly embedded \( \mathbb{P}^n \). If \( d = 2 \) and if \( X \subset \mathbb{P}^N \) is non-degenerate, then \( X \subset \mathbb{P}^N \) is conically connected.

**Proof** By Theorem II.3.11 [6], there exists finitely many closed subvarieties (depending on \( z \)) \( V_i \subsetneq X \), \( i = 1, \ldots, l \), such that for any nonconstant morphism \( f : \mathbb{P}^1 \to X \) with \( f(0) = z \), \( \deg(f_*((\mathbb{P}^1))) = d \) and with \( f((\mathbb{P}^1)) \nsubseteq \bigcup_{i=1}^l V_i \), we have \( f^*T_X \) is ample. Now take a general point \( x \in X \setminus \bigcup_{i=1}^l V_i \) and a smooth rational curve \( C \subset X \) of degree \( d \) passing through \( x \) and \( z \). The above result implies that \( f^*T_X = T_X|_C \) is ample and hence that \( N_{C|X} \) is ample. Thus there exists a unique irreducible component \( W_x \) of the Hilbert schemes of rational curves of degree \( d \) contained in \( X \) and passing through \( x \) containing \( [C] \). Since \( N_{C|X} \) is ample, it is well known that deformations of \( C \) parametrized by \( W_x \) cover \( X \). Therefore given a general point \( y \in X \), we can find a smooth