A NOTE ON THE CRITICAL GROUP OF A LINE GRAPH

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Abstract. This note answers a question posed by Levine in [3]. The main result is Theorem 1 which shows that under certain circumstances a critical group of a directed graph is the quotient of a critical group of its directed line graph.

1. Introduction

Let $G$ be a finite multidigraph with vertices $V$ and edges $E$. Loops are allowed in $G$, and we make no connectivity assumptions. Each edge $e \in E$ has a tail $e^-$ and a target $e^+$. Let $\mathbb{Z}V$ and $\mathbb{Z}E$ be the free abelian groups on $V$ and $E$, respectively. The Laplacian $\Delta_G$ of $G$ is the $\mathbb{Z}$-linear mapping $\Delta_G : \mathbb{Z}V \to \mathbb{Z}V$ determined by $\Delta_G(v) = \sum_{(v,u) \in E} (u - v)$ for $v \in V$. Given $w_* \in V$, define

$$\phi = \phi_{G,w_*} : \mathbb{Z}V \to \mathbb{Z}V$$

$$v \mapsto \begin{cases} 
\Delta_G(v) & \text{if } v \neq w_* \\
w_* & \text{if } v = w_* 
\end{cases}.$$

The critical group for $G$ with respect to $w_*$ is the cokernel of $\phi$: $K(G,w_*) := \text{cok} \phi$.

The line graph, $\mathcal{L}G$, for $G$ is the multidigraph whose vertices are the edges of $G$ and whose edges are $(e,f)$ with $e^+ = f^-$. As with $G$, we have the Laplacian $\Delta_{\mathcal{L}G}$ and the critical group $K(\mathcal{L}G,e_*) := \text{cok} \phi_{\mathcal{L}G,e_*}$ for each $e_* \in E$.

If every vertex of $G$ has a directed path to $w_*$ then $K(G,w_*)$ is called the sandpile group for $G$ with sink $w_*$. A directed spanning tree of $G$ rooted at $w_*$ is a directed subgraph containing all of the vertices of $G$, having no directed cycles, and for which $w_*$ has out-degree 0 and every other vertex has out-degree 1. Let $\kappa(G,w_*)$ denote the number of directed spanning trees rooted at $w_*$. It is a well-known consequence

1The mapping $\Lambda : \mathbb{Z}V \to \mathbb{Z}V$ defined by $\Lambda(f)(v) = \sum_{(v,u) \in E} (f(v) - f(u))$ for $v \in V$ is often called the Laplacian of $G$. It is the negative $\mathbb{Z}$-dual (i.e., the transpose) of $\Delta_G$. 

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of the matrix-tree theorem that the number of elements of the sandpile group with sink $w_*$ is equal to $\kappa(G, w_*)$. For a basic exposition of the properties of the sandpile group, the reader is referred to [2].

In his paper, [3], Levine shows that if $e_* = (w_*, v_*)$, then $\kappa(G, w_*)$ divides $\kappa(LG, e_*)$ under the hypotheses of our Theorem 1. This leads him to ask the natural question as to whether $K(G, w_*)$ is a subgroup or quotient of $K(LG, e_*)$. In this note, we answer this question affirmatively by demonstrating a surjection $K(LG, e_*) \to K(G, w_*)$. Further, in the case in which the out-degree of each vertex of $G$ is a fixed integer $k$, we show the kernel of this surjection is the $k$-torsion subgroup of $K(LG, e_*)$. These results appear as Theorem 1 and may be seen as analogous to Theorem 1.2 of [3]. In [3], partially for convenience, some assumptions are made about the connectivity of $G$ which are not made in this note.

For related work on the critical group of a line graph for an undirected graph, see [1].

2. Results

Fix $e_* = (w_*, v_*) \in E$. Define the modified target mapping

$$\tau: \mathbb{Z}E \to \mathbb{Z}V$$

$$e \mapsto \begin{cases} 0 & \text{if } e = e_*, \\ e^+ & \text{if } e \neq e_*. \end{cases}$$

Also define

$$\rho: \mathbb{Z}E \to \mathbb{Z}V$$

$$e \mapsto \begin{cases} \Delta_G(w_*) - v_* - w_* + e^+ & \text{if } e \neq e_*, \\ 0 & \text{if } e = e_. \end{cases}$$

Let $k$ be a positive integer. The graph $G$ is \textit{k-out-regular} if the out-degree of each of its vertices is $k$.

\textbf{Theorem 1.} If $\text{indeg}(v) \geq 1$ for all $v \in V$ and $\text{indeg}(v_*) \geq 2$, then $\rho: \mathbb{Z}E \to \mathbb{Z}V$ descends to a surjective homomorphism $\overline{\rho}: K(LG, e_*) \to K(G, w_*)$.

Moreover, if $G$ is $k$-out-regular, the kernel of $\overline{\rho}$ is the $k$-torsion subgroup of $K(LG, e_*)$.

\textbf{Proof.} Let $\rho_0: \mathbb{Z}V \to \mathbb{Z}V$ be the homomorphism defined on vertices $v \in V$ by

$$\rho_0(v) := \Delta_G(w_*) - v_* - w_* + v$$
so that \( \rho = \rho_0 \circ \tau \). The mapping \( \rho_0 \) is an isomorphism, its inverse being itself:

\[
\rho_0^2(v) = \rho_0(\Delta_G(w_*) - v_* - w_* + v)
= \sum_{e^- = w_*} (\rho_0(e^+) - \rho_0(w_*)) - \rho_0(v_*) - \rho_0(w_*) + \rho_0(v)
= \Delta_G(w_*) - \rho_0(v_*) - \rho_0(w_*) + \rho_0(v)
= v.
\]

Let \( \psi : \mathbb{Z}V \to \mathbb{Z}V \) be the homomorphism defined on vertices \( v \in V \) by

\[
\psi(v) := \begin{cases} 
\Delta_G(v) & \text{if } v \neq w_*, \\
\Delta_G(w_*) - v_* & \text{if } v = w_*.
\end{cases}
\]

Let \( \phi_G \) and \( \phi_{LG} \) denote \( \phi_{G,w_*} \) and \( \phi_{LG,e_*} \), respectively. We claim the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}E & \xrightarrow{\phi_{LG}} & \mathbb{Z}E \\
\tau \downarrow & & \tau \downarrow \\
\mathbb{Z}V & \xrightarrow{\psi} & \mathbb{Z}V \\
\phi \downarrow & & \rho_0 \downarrow \\
\mathbb{Z}V & \xrightarrow{\phi_G} & \mathbb{Z}V.
\end{array}
\]

To prove commutativity of the top square of the diagram, first suppose \( e \neq e_* \). Then

\[
\tau(\phi_{LG}(e)) = \tau(\Delta_{LG}(e)) = \tau \left( \sum_{f^- = e^+} (f - e) \right).
\]

If \( e \neq e_* \) and \( e^+ \neq w_* \), then

\[
\tau \left( \sum_{f^- = e^+} (f - e) \right) = \sum_{f^- = e^+} (f^+ - e^+) = \Delta_G(e^+) = \psi(\tau(e)).
\]
On the other hand, if \( e \neq e_\ast \) and \( e^+ = w_\ast \), then

\[
\tau \left( \sum_{f^- = e^+} (f - e) \right) = \sum_{f^- = e^+, f \neq e_\ast} (f^+ - e^+) + \tau(e_\ast - e)
\]

\[
= \sum_{f^- = e^+, f \neq e_\ast} (f^+ - e^+) - w_\ast
\]

\[
= \Delta_G(w_\ast) - v_\ast = \psi(\tau(e)).
\]

Therefore, \( \tau(\phi_{\mathcal{L}G}(e)) = \psi(\tau(e)) \) holds if \( e \neq e_\ast \). Moreover, the equality still holds if \( e = e_\ast \) since \( \tau(e_\ast) = 0 \). Hence, the top square of the diagram commutes.

To prove that the bottom square of the diagram commutes, there are two cases. First, if \( v \neq w_\ast \), then

\[
\rho_0(\psi(v)) = \sum_{(v,u) \in E} (\rho_0(u) - \rho_0(v)) = \sum_{(v,u) \in E} (u - v) = \Delta_G(v) = \phi_G(v).
\]

Second, if \( v = w_\ast \), then

\[
\rho_0(\psi(v)) = \rho_0(\Delta_G(w_\ast) - v_\ast) = \Delta_G(w_\ast) - \rho_0(v_\ast) = w_\ast = \phi_G(v).
\]

From the commutativity of the diagram, the cokernel of \( \psi \) is isomorphic to \( K(G,w_\ast) \), and \( \rho = \rho_0 \circ \tau \) descends to a homomorphism \( \overline{\rho}: K(\mathcal{L}G,e_\ast) \to K(G,w_\ast) \) as claimed. The hypothesis on the in-degrees of the vertices assures that \( \tau \), hence \( \overline{\rho} \), is surjective.

Now suppose that \( G \), hence \( \mathcal{L}G \), is \( k \)-out-regular. This part of our proof is an adaptation of that given for Theorem 1.2 in [3]. Since \( \rho_0 \) is an isomorphism, it suffices to show that the kernel of the induced map, \( \overline{\tau}: K(\mathcal{L}G,e_\ast) \to \operatorname{cok} \psi \) has kernel equal to the \( k \)-torsion of \( K(\mathcal{L}G,e_\ast) \).

To this end, define the homomorphism \( \sigma: \mathbb{Z}V \to \mathbb{Z}E \), given on vertices \( v \in V \) by

\[
\sigma(v) := \sum_{e^- = v} e.
\]

We claim that the image of \( \sigma \circ \psi \) lies in the image of \( \phi_{\mathcal{L}G} \), so that \( \sigma \) induces a map, \( \overline{\sigma} \), between \( \operatorname{cok} \psi \) and \( K(\mathcal{L}G,e_\ast) \). To see this, first note
that for \( v \in V \),
\[
\sigma(\Delta G(v)) = \sigma\left( \sum_{e^- = v} e^+ - kv \right) \\
= \sum_{e^- = v} \sum_{f^- = e^+} f - k \sum_{e^- = v} e \\
= \sum_{e^- = v} \Delta_{LG}(e)
\]
Therefore, for \( v \neq w_* \), it follows that \( \sigma(\psi(v)) \) is in the image of \( \phi_{LG} \).
On the other hand, using the calculation just made,
\[
\sigma(\Delta G(w_*) - v_*) = \sum_{e^- = w_*} \Delta_{LG}(e) - \sum_{f^- = v^*} f \\
= \sum_{e^- = w_*} \Delta_{LG}(e) - \left( \sum_{f^- = v^*} f - k e_* + k e_* \right) \\
= \sum_{e^- = w_*} \Delta_{LG}(e) - \Delta_{LG}(e_*) - k e_* \\
= \sum_{e^- = w_*, e \neq e_*} \Delta_{LG}(e) - k e_* ,
\]
which is also in the image of \( \phi_{LG} \).
Now note that for \( e \neq e_* \),
\[
\overline{\sigma}(\overline{\tau}(e)) = \sum_{f^- = e^+} f = \Delta_{LG}(e) + k e = k e \in K(LG, e_*).
\]
Thus, the kernel of \( \overline{\tau} \) is contained in the \( k \)-torsion of \( K(LG, e_*) \), and to show equality it suffices to show that \( \overline{\sigma} \) is injective.

The case where \( k = 1 \) is trivial since there are no \( G \) satisfying the hypotheses: if \( G \) is 1-out-regular and \( \text{indeg}(v) \geq 1 \) for all \( v \in V \), then \( \text{indeg}(v) = 1 \) for all \( v \in V \), including \( v_* \). So suppose that \( k > 1 \) and that \( \eta = \sum_{v \in V} a_v v \) is in the kernel of \( \overline{\sigma} \). We then have
\[
(1) \quad \sigma(\eta) = \sum_{v \in V} \sum_{e^- = v} a_v e = \sum_{e \neq e_*} b_e \Delta_{LG}(e) + c e_*
\]
for some integers \( b_e \) and \( c \). Comparing coefficients in (1) gives
\[
(2) \quad a_{e^-} = \sum_{f^+ = e^-, f \neq e_*} b_f - k b_e \quad \text{for } e \neq e_* .
\]
Define

\[ F(v) = \frac{1}{k} \left( \sum_{f^+=v, f \neq e_\ast} b_f - a_v \right). \]

From (2),

\[ F(e^-) = b_e \quad \text{for} \quad e \neq e_\ast. \] (3)

Since \( k > 1 \), for each vertex \( v \), we can choose an edge \( e_v \neq e_\ast \) with \( e_v^- = v \). By (2) and (3), for all \( v \in V \),

\[ a_v = \sum_{f^+ = v, f \neq e_\ast} b_f - k e_v = \sum_{f^+ = v, f \neq e_\ast} F(f^-) - k F(v). \]

Therefore, as an element of \( \text{cok} \psi \),

\[
\eta = \sum a_v v = \sum_{e \neq e_\ast} F(e^-) e^+ - \sum_{v \in V} k F(v) v \\
= \sum_{v \in V, v \neq w_\ast} F(v) \left( \sum_{e^- = v} e^+ - k v \right) + F(w_\ast) \left( \sum_{e^- = w_\ast, e \neq e_\ast} e^+ - k w_\ast \right) \\
= \sum_{v \in V, v \neq w_\ast} F(v) \Delta_G(v) + F(w_\ast) (\Delta_G(w_\ast) - v_\ast) \\
= 0,
\]

which shows that \( \sigma \) is injective.

References

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