On the existence theory for nonlinear plate equations

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Abstract. This paper is devoted to the theoretical analysis of some nonlinear plate equations in $\mathbb{R}^n \times (0, \infty)$, $n \geq 1$, with nonlinearity involving a type of polynomial behavior. We prove the existence and uniqueness of global mild solutions for small initial data in $L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$-spaces. We also prove the existence and uniqueness of local and global solutions in the framework of Bessel-potential spaces $H^s_p(\mathbb{R}^n) = (I - \Delta)^{s/2}L^p(\mathbb{R}^n)$. In order to derive the existence results, we develop new time decay estimates of the solution of the corresponding linear problem.

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1. Introduction

In recent years, the family of Cauchy problems describing deformations of elastic plates under the consideration of several physical mechanisms, including different types of interactions between sources of energy dissipation, rotational inertia and effects of nonlinearities, have caught a lot of attention by many authors [6–8,10]. With the aim of contributing to the theoretical development of this type of models, in this paper we consider the following nonlinear plate equation describing the evolution of a vertical displacement of a plate under the action of rotational inertia effects

$$u_{tt} - \mu \Delta u_{tt} + D_u \Delta^2 u - \nu^2 \Delta u_t = \delta(-\Delta)^\theta |u|^\lambda, \quad x \in \mathbb{R}^n, \quad t > 0,$$

(1.1)

where $0 \leq \theta \leq 1$, $D_u > 0$, and $\mu, \nu, \delta \geq 0$. The simplest submodel of (1.1) is given by $u_{tt} + D_u \Delta^2 u = f$, which comes from the momentum balance equation in the description of small deflection of thin plates, under the action of a distributed transverse load $f$ acting on the plate per unit area, where the coefficient $D_u$ represents the flexural rigidity of the plate. In particular, this undamped plate equation appears as a linear model describing the vibration of stiff objects where the potential energy involves curvature-like terms which lead to the bi-laplacian operator, see Denk and Schaubelt [4]. In the general model (1.1), the term $\Delta u_{tt}$ corresponds to the rotational inertia effects, and $-\nu^2 \Delta u_t$ corresponds to a dissipative term which is added to incorporate the loss of energy. Model (1.1) can be derived from the thermoelastic plate equations in $\mathbb{R}^n$, $n \geq 1$, where the heat conduction is described by the Fourier law, that is,

$$\begin{cases}
    u_{tt} - \mu \Delta u_{tt} + \Delta^2 u - \nu^2 \Delta u_t = \delta(-\Delta)^\theta |u|^\lambda, \\
    \tau_t - \Delta \tau - \nu \Delta u_t = 0.
\end{cases}$$

(1.2)

Then, neglecting the variations in time for temperature we get $\Delta \tau = -\nu \Delta u_t$, which replacing in (1.2) gives (1.1). Equation (1.1) (in the case $\delta = 0$) and related models including a complete dynamic between the displacement, the thermal moment and the heat flux, has attracted the attention of researchers, and many interesting results have been obtained (see [7–11] and references therein). Depending of the choice of the involved parameters, the resulting equation (system) represents several kinds of thermoelastic plates models. Moreover, different qualitative behaviors occur depending of the domain where the equations
are defined (bounded domains, exterior domains, the half space, the whole space $\mathbb{R}^n$, etc.). In particular, in Racke and Ueda [10], by considering in (1.1) $\delta = 0$ and $\mu = 0$ and $\Delta b(\Delta u)$ in place of $\Delta^2 u$, with $b$ a given smooth function which satisfies $b'(0) > 0$ and $b(0) = 0$, the authors obtained the existence of global solution $u$ in the class $(u_t, \Delta u_t) \in C([0, \infty); H^{s+2}(\mathbb{R}^n))$ and $u_t \in C^1([0, \infty); H^s(\mathbb{R}^n))$, $s \geq \lceil n/2 \rceil + 1$, for initial data $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$ satisfying $\|(u_0, \Delta u_0)\|_{H^{s+2}}$ being small enough. For $\mu > 0$ and $\delta = 0$, in [10] the authors proved the existence of global solution $u$ of (1.1) in the class $u_t \in C([0, \infty); H^{s+2}(\mathbb{R}^n))$ with $\Delta u \in C([0, \infty); H^{s+1}(\mathbb{R}^n))$ and initial data $(u_0, u_1)$ with $\|\Delta u_0\|_{H^{s+1}} + \|u_1\|_{H^{s+2}}$ being small enough. The results of [10] were obtained by combining a local existence result with a set of a priori estimates.

Considering in (1.1) the action of a frictional displacement $u_t$ in place of $-\nu^2 \Delta u_t$ and a polynomial nonlinearity (with $\theta = 0$), in D’Abbicco [3] the author proved the existence of global solutions $u$ in the class $u \in C([0, \infty); H^2(\mathbb{R}^n)) \cap C^1([0, \infty); H^1(\mathbb{R}^n))$ with small initial data $(u_0, u_1) \in (L^1 \cap H^2) \times (L^1 \cap H^1)$. The author also derives optimal estimates for $u(t, \cdot)$ in the $L^r$-norm, $r \geq 2$. Nonlinearities of kind $|u_t|^p$ have been considered in [5,13].

In (1.1), the presence of the inertial term $-\Delta u_{ttt}$ generates additional difficulties to derive decay estimates for the solution of the linear problem, in comparison with the corresponding model without inertial term. The decay of solutions to the associated linearized problem is crucial to obtain existence results, if we use a fixed point argument. These decays are usually proved by using explicit representative formula of the solution of the corresponding linear equation, as happens in the Schrödinger equation, or through the solution of the linear equation in terms of the Fourier transform, when we do not have the explicit formula for the inverse of the solution in Fourier variables (see [14]). In our case we do not have an explicit formula for the solution of the linear problem associated to (1.1). In fact, the solution of the linear problem associated to (1.1) is given by

$$e^{t\varphi(\xi)} \left[ \frac{\varphi(\xi) \sin(t\phi(\xi))}{\phi(\xi)} + \cos(t\phi(\xi)) \right] \hat{u}_0(\xi) + e^{t\varphi(\xi)} \frac{\sin(t\phi(\xi))}{\phi(\xi)} \hat{u}_1(\xi),$$

for which the phase is $\varphi(\xi) = -\frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)}$. We obtain time-decay rates for the solution of the corresponding linear system, which allow us to prove the existence of global solution for (1.1) and $\theta = 1$ in energy spaces $H^s(\mathbb{R}^n)$, for $s > \frac{n-2}{2}$, under suitable small and regular initial data, compensating the loss of regularity created by the inertial term. Explicitly, for $s > \frac{n-2}{2}$, we prove that the solution $\partial_t S(t)u_0(x) + S(t)\Delta u_1(x)$ of the linear problem related to (1.1) satisfies

$$\|\partial_t S(t)g\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{s}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{t}{4}} \|g\|_{H^{s+1}(\mathbb{R}^n)},$$

$$\|S(t)\Delta g\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{s}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{t}{4}} \|g\|_{H^{s+2}(\mathbb{R}^n)},$$

$$\|\partial_t^{s+1} S(t)g\|_{H^{s-k}(\mathbb{R}^n)} \leq C \|g\|_{H^{s}(\mathbb{R}^n)},$$

$$\|\partial_t^k S(t)\Delta g\|_{H^{s-k}(\mathbb{R}^n)} \leq C \|g\|_{H^{s+1}(\mathbb{R}^n)},$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$, $t > 0$, $s \in \mathbb{R}$ and $k = 0, 1$.

On the other hand, in the framework of Bessel-potential spaces $H^s_p(\mathbb{R}^n) = (I - \Delta)^{s/2} L^p(\mathbb{R}^n)$, we also derive some estimates for the solutions of the corresponding linear system which allow us to obtain the existence and uniqueness of local and global solutions. Explicitly, for $\sigma < 1 - n$, $2 \leq p \leq \infty$, we obtain that the solution of the linear problem satisfies

$$\|\partial_t^k S(t)g\|_{H^{s-k}_p(\mathbb{R}^n)} \leq C t^{-\frac{s}{2}(1-\frac{\sigma}{2})} \|g\|_{L^p(\mathbb{R}^n)},$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$, $k = 1, 2$ and $t > 0$. Using previous estimates we are able to study the nonlinear problem. Thus, the novelty of this work is summarized in the following aspects: First, we obtain time-decay estimates of the solution of the linear problem in energy spaces $H^s$ and Bessel-potential $H^s_p$ spaces, as well as a set of estimates corresponding to the operator $\Delta_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^\theta$ in $H^s$ and $H^s_p$ spaces.
spaces. Second, we prove existence and uniqueness of global solutions in the class $u \in C([0, \infty), H^s(\mathbb{R}^n)) \cap C^1([0, \infty), H^{s-1}(\mathbb{R}^n))$, for $s \geq \frac{n+2}{2}$, for small initial data $(u_0, u_1)$ in $L^1 \cap H^s \cap H^{s+2}$, which is weaker than the previous classes of initial data (cf. [10]). We also prove the existence and uniqueness of local and global solutions in the framework of Bessel-potential spaces $H^s_p(\mathbb{R}^n)$ (see Theorems 2.2 and 2.3 below).

This paper is organized as follows. In Sect. 2, we set the main results. In Sect. 3, we derive time decay estimates of the linear solution in $H^s$ and $H^s_p$-spaces, as well as a set of estimates corresponding to the operator $\Lambda(t) = S(t)(I-\Delta)^{-1}(-\Delta)^{\theta}$. In Sect. 4, we prove the existence and uniqueness of global in time solutions in the framework of $H^s$ and $H^s_p$-spaces, and finally, in Sect. 5, we prove the existence and uniqueness of local in time solutions.

### 2. Main results

Before establishing the main results, we solve the corresponding linear problem associated to (1.1) which is given by

\[
\begin{aligned}
&u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u_t = 0, & x \in \mathbb{R}^n, & t > 0, \\
&u(x, 0) = u_0(x), & u_t(x, 0) = \Delta u_1(x), & x \in \mathbb{R}^n.
\end{aligned}
\] (2.1)

Using the Fourier transform we obtain the second-order differential equation

\[(1 + |\xi|^2)\hat{u}_{tt} + |\xi|^2\hat{u}_t + |\xi|^4\hat{u} = 0.\]

The characteristic roots of the full symbol

\[(1 + |\xi|^2)t^2 + |\xi|^2r + |\xi|^4 = 0,\]

are given by

\[
\begin{aligned}
r_0 &= r_0(\xi) = -\frac{|\xi|^2}{2(1 + |\xi|^2)} + \frac{|\xi|^2\sqrt{3 + 4|\xi|^2}}{2(1 + |\xi|^2)}i = \varphi(\xi) + \phi(\xi)i, \\
r_1 &= r_1(\xi) = -\frac{|\xi|^2}{2(1 + |\xi|^2)} - \frac{|\xi|^2\sqrt{3 + 4|\xi|^2}}{2(1 + |\xi|^2)}i = \varphi(\xi) - \phi(\xi)i.
\end{aligned}
\]

After applying the Fourier transform, we can write

\[
\hat{\tilde{u}}(\xi, t) = \frac{r_0e^{rt} - r_1e^{-rt}}{r_0 - r_1}\hat{u}_0(\xi) + \frac{e^{rt} - e^{-rt}}{r_0 - r_1}\hat{u}_1(\xi) = e^{t\varphi(\xi)}\left[\frac{\varphi(\xi)\sin(t\phi(\xi))}{\phi(\xi)} + \cos(t\phi(\xi))\right]\hat{u}_0(\xi) + e^{t\varphi(\xi)}\frac{\sin(t\phi(\xi))}{\phi(\xi)}\hat{u}_1(\xi).
\]

Then, the global solution of the linear problem is given by

\[u(x, t) = \partial_t S(t)u_0(x) + S(t)\Delta u_1(x),\]

From the Duhamel principle, the solution of (1.1) with initial data $u(x, 0) = u_0(x), u_t(x, 0) = \Delta u_1(x)$, is given by

\[u(x, t) = \partial_t S(t)u_0(x) + S(t)\Delta u_1(x) - \int_0^t S(t - \tau)(I - \Delta)^{-1}(-\Delta)^{\theta}u(x, \tau)|^3d\tau,
\] (2.2)
where
\[ S(t)v(x) = \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{2(1+|\xi|^2)}{|\xi|^2 \sqrt{3|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3|\xi|^2} + \xi^T x}{2(1+|\xi|^2)} \right) \hat{v}(\xi) e^{ix \cdot \xi} d\xi, \]
\[ \partial_t S(t)v(x) = \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \left[ \cos \left( \frac{|\xi|^2 \sqrt{3|\xi|^2} + \xi^T x}{2(1+|\xi|^2)} \right) - \frac{1}{\sqrt{3|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3|\xi|^2} + \xi^T x}{2(1+|\xi|^2)} \right) \right] \hat{v}(\xi) e^{ix \cdot \xi} d\xi. \]

Henceforth we denote \( \Lambda_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^\theta \). Now we are in position to establish the main results of this paper.

**Theorem 2.1.** (Global-in-time solutions in \( H^s \)) Let \( \lambda \geq 3, \theta = 1, \) and consider \( s > \frac{n-2}{2} \) with \( n(\lambda - 2) > 2 \). There exists \( \delta > 0 \) such that if
\[ C(\|u_0\|_{L^1} + \|u_0\|_{H^{s+1}} + \|u_1\|_{L^1} + \|u_1\|_{H^{s+2}}) \leq \delta/2, \]
for some \( C > 0 \), then the initial value problem \((1.1)\) has a unique global solution \( u \in C([0, \infty), H^s(\mathbb{R}^n)) \cap C^1([0, \infty), H^{s-1}(\mathbb{R}^n)) \) satisfying
\[ \sup_{0 < t < \infty} (1 + t)^{\frac{2}{s}} \|u(t)\|_{L^\infty} + \|u(t)\|_{H^s} + \|u(t)\|_{H^{s-1}} \leq \delta. \]

Let us also define the initial data space \( \mathcal{I}_0 \) as the set of pairs \([u_0, u_1] \in [S'(\mathbb{R}^n)]^2 \) such that the norm
\[ \|[u_0, u_1]\|_{\mathcal{I}_0} := \sup_{0 < t < \infty} t^\alpha \|\partial_t S(t)u_0\|_{H^s} + \|S(t)\Delta u_1\|_{H^s} + \sup_{0 < t < \infty} t^\beta (\|\partial_t^2 S(t)u_0\|_{H^{s-1}} + \|\partial_t S(t)\Delta u_1\|_{H^{s-1}}), \]
with \( \alpha = \frac{1}{n-1}(2 - \theta - \frac{n}{2}(1 - \frac{2}{p})) \) and \( \beta = \alpha + 1 - \theta, \lambda \geq 2, \theta \in (\frac{2-n}{2}, 1) \) if \( n = 1, 2, \) and \( \theta \in [0, 1] \) if \( n \geq 3 \). We also consider the norm
\[ \|u\|_{X_{\alpha, \beta}^s} := \sup_{0 < t < \infty} (t^\alpha \|u(t)\|_{H^s} + t^\beta \|u(t)\|_{H^{s-1}}). \]

**Theorem 2.2.** (Global-in-time solutions in \( H^p_\sigma \)) Let \( \lambda \geq 2, \theta \in (\frac{2-n}{2}, 1) \) if \( n = 1, 2, \) and \( \theta \in [0, 1] \) if \( n \geq 3, \) and \( \frac{1}{\lambda} > \alpha > 0 \). Assume \( 2 \leq p \leq q \leq \infty, \frac{n}{2}(1 - \frac{2}{p}) < 1 \) and consider \( \sigma, s \) such that \( s > \sigma, n\left(\frac{1}{p} - \frac{1}{q}\right) \leq \sigma < 3 - n - 2\theta \) and \( (\frac{1}{p} + \frac{1}{p} - 1)\frac{n}{\lambda - 1} + \sigma \leq s < \min\{\frac{n}{q}, \lambda - 1\} + \sigma \). There exists \( \delta > 0 \) such that if
\[ \|[u_0, u_1]\|_{\mathcal{I}_0} \leq \delta/2, \]
then the initial value problem \((1.1)\) has a unique global solution \( u \in C([0, \infty), H^p_\sigma(\mathbb{R}^n)) \cap C^1([0, \infty), H^{p-1}_\sigma(\mathbb{R}^n)) \) satisfying \( \|u\|_{X_{\alpha, \beta}^p} \leq \delta. \)

**Theorem 2.3.** (Local-in-time solutions) Let \( \lambda \geq 2, \theta \in (\frac{2-n}{2}, 1) \) if \( n = 1, 2, \) and \( \theta \in [0, 1] \) if \( n \geq 3. \) Assume \( 2 \leq p \leq q \leq \infty, \) and consider \( \sigma, s \) such that \( s > \sigma, n\left(\frac{1}{p} - \frac{1}{q}\right) \leq \sigma < 3 - n - 2\theta \) and \( (\frac{1}{p} + \frac{1}{p} - 1)\frac{n}{\lambda - 1} + \sigma \leq s < \min\{\frac{n}{q}, \lambda - 1\} + \sigma, \) and \( 1 > \frac{n}{q}(1 - \frac{2}{p})\lambda. \) Then, if \([u_0, u_1] \in H^{p+1-\sigma}_\sigma(\mathbb{R}^n) \times H^{p-\sigma} \), there exists \( 0 < T < \infty \) such that the initial value problem \((1.1)\) has a unique local in time solution \( u \in C([0, T]; H^p_\sigma(\mathbb{R}^n)) \cap C^1([0, T]; H^{p-1}_\sigma(\mathbb{R}^n)). \)

3. Time decay estimates in \( H^s \) and \( H^p_\sigma \)

The first aim of this section is to derive some decay estimates of the family of operators \( \Lambda_\theta(t), \partial_t \Lambda_\theta(t), S(t) \) and \( \partial_t S(t) \) on \( L^\infty(\mathbb{R}^n), H^s(\mathbb{R}^n) \) and \( H^p_\sigma(\mathbb{R}^n) \) spaces.
### 3.1. Estimates in $L^\infty(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$

**Lemma 3.1.** Let $a > -n$. There exists a positive constant $C_n$, that only depends on $n$, such that

$$
\int_{|\xi| \leq 1} e^{- \frac{|\xi|^2}{4}} |\xi|^a d\xi \leq C_n 2^{a+n-1} t^{- \frac{n+a}{2}}.
$$

**Proof.** Making the change of variable $\eta = \frac{\xi}{\sqrt{t}}$, we obtain

$$
\int_{|\xi| \leq 1} e^{- \frac{|\xi|^2}{4}} |\xi|^a d\xi = 2^{a+n} t^{- \frac{n+a}{2}} \int_{|\eta| \leq \frac{\sqrt{t}}{2}} e^{-|\eta|^2} |\eta|^a d\eta \leq 2^{a+n} t^{- \frac{n+a}{2}} \int_{\mathbb{R}^n} e^{-|\eta|^2} |\eta|^a d\eta. \tag{3.1}
$$

Now, using spherical coordinates in $\mathbb{R}^n$ and the change of variable $r = \sqrt{x}$, we get

$$
\int_{\mathbb{R}^n} e^{-|\eta|^2} |\eta|^a d\eta = C_n \int_0^\infty e^{-r^2} r^{a+n-1} dr = \frac{C_n}{2} \int_0^\infty e^{-x} x^{\frac{n+a}{2}-1} dx = \frac{C_n}{2} \Gamma\left(\frac{n+a}{2}\right), \tag{3.2}
$$

where $\Gamma$ is the Gamma function. From the assumption $a > -n$, it holds that $\Gamma\left(\frac{n+a}{2}\right)$ is finite because $\text{Re}\left(\frac{n+a}{2}\right) > 0$. Combining (3.1) and (3.2), we obtain the desired result. \hfill \Box

At this point, using the previous lemma, we can establish estimates in the norm $L^\infty$ for the linear operator $\Lambda_\theta(t)$ appearing in (2.2). For any positive elements $A = A(\xi,t)$ and $B = B(\xi,t)$, the notation $A \leq B$ means that there exists a positive constant $c$, which does not depend on $t$ or $\xi$, such that $A \leq cB$.

**Lemma 3.2.** Let $\theta \in (\frac{2-n}{2}, 1]$ if $n = 1, 2$, and $\theta \in [0, 1]$ if $n \geq 3$. Assume that $s > \frac{n+4\theta-6}{2}$. Then, there is a constant $C = C_n > 0$ such that

$$
\|\Lambda_\theta(t)g\|_{L^\infty(\mathbb{R}^n)} \leq C t^{\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{t}{4}} \|g\|_{H^s(\mathbb{R}^n)},
$$

for all $g \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ and $t > 0$, being $\Lambda_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^{\theta}$.

**Proof.** Since the $\sin(\cdot)$ function is bounded, $\frac{2}{\sqrt{3+4|\xi|^2}} \leq 1$ and $e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \leq e^{-\frac{t}{4}}$ for $|\xi| \geq 1$, we obtain

$$
|\Lambda_\theta(t)g(x)| \leq \left| \int_{|\xi| \leq 1} e^{- \frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{2|\xi|^{2(\theta-1)}}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix \cdot \xi} d\xi \right| + \left| \int_{|\xi| \geq 1} e^{- \frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{2|\xi|^2|\xi|^{-\theta}|\xi|^{-\frac{2(\theta-1)}{2}}}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix \cdot \xi} d\xi \right| \leq \|\tilde{g}\|_{L^\infty} \int_{|\xi| \leq 1} e^{- \frac{|\xi|^2}{2(1+|\xi|^2)}} |\xi|^{2(\theta-1)} d\xi + e^{-\frac{t}{4}} \int_{|\xi| \geq 1} \frac{2|\xi|^2|\xi|^{-\theta}|\xi|^{-\frac{2(\theta-1)}{2}}}{\sqrt{3+4|\xi|^2}} |\tilde{g}(\xi)| d\xi.
$$

Taking into account that $\|\tilde{g}\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}$ and $e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \leq e^{-\frac{t}{4}}$ for $|\xi| \leq 1$, we get

$$
|\Lambda_\theta(t)g(x)| \leq \|g\|_{L^1(\mathbb{R}^n)} \int_{|\xi| \leq 1} e^{- \frac{|\xi|^2}{4}} |\xi|^{2(\theta-1)} d\xi + e^{-\frac{t}{4}} \int_{|\xi| \geq 1} \frac{2|\xi|^2|\xi|^{-\theta}|\xi|^{-\frac{2(\theta-1)}{2}}}{\sqrt{3+4|\xi|^2}} |\tilde{g}(\xi)| d\xi.
$$
Now, from Lemma 3.1 and using the Cauchy–Schwarz inequality, we have

$$|\Lambda_\theta(t)g(x)| \leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{H^s(\mathbb{R}^n)} \left( \int_{|\xi| \geq 1} \frac{4|\xi|^{-2s}|\xi|^{4(\theta-1)}}{3+4|\xi|^2} d\xi \right)^{1/2}.$$ 

By applying spherical coordinates and since $-2s + 4\theta - 6 + n < 0$, we obtain

$$|\Lambda_\theta(t)g(x)| \leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{H^s(\mathbb{R}^n)} \left( \int_1^\infty r^{-2s+4\theta+n-7} dr \right)^{1/2}$$

$$\leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{H^s(\mathbb{R}^n)},$$

which finishes the proof of the lemma.

In order to deal with the existence of solutions in the energy spaces $H^s(\mathbb{R}^n)$, we need to obtain time decay estimates for $\Lambda_\theta(t)$ of kind $(1 + t)^{-\frac{n+2(\theta-1)}{2}}$ in place of $t^{-\frac{n+2(\theta-1)}{2}}$. Otherwise, we would need stronger restrictions on the parameters $\lambda$ and $\theta$ that make impossible to have a nonempty set of constraints for the existence of mild solutions.

**Lemma 3.3.** Let $\theta \in \left( \frac{2-n}{2}, 1 \right]$ if $n = 1, 2$, and $\theta \in [0, 1]$ if $n \geq 3$. Consider $s > \frac{n+4\theta-6}{2}$. Then, there is a constant $C = C_{n, \theta} > 0$, depending only on $n$ and $\theta$, such that

$$\|\Lambda_\theta(t)g\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{t}{4}} \|g\|_{H^s(\mathbb{R}^n)},$$

for all $g \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ and $t > 0$, being $\Lambda_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^{\theta}$.

**Proof.** We assume first that $t > 1$. Since $n + 2(\theta - 1) > 0$ and

$$t^{-\frac{n+2(\theta-1)}{2}} = (2t)^{-\frac{n+2(\theta-1)}{2}} 2^{\frac{n+2(\theta-1)}{2}} = (t + 1)^{-\frac{n+2(\theta-1)}{2}} 2^{\frac{n+2(\theta-1)}{2}} \leq (1 + t)^{-\frac{n+2(\theta-1)}{2}} 2^{\frac{n+2(\theta-1)}{2}},$$

from Lemma 3.2, we arrive at

$$\|\Lambda_\theta(t)g\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{t}{4}} \|g\|_{H^s(\mathbb{R}^n)}.$$

Using the last inequality and the fact that there exists a constant $C_{n, \theta} > 0$, which only depends on $n$ and $\theta$, such that $e^{-\frac{t}{4}} \leq C_{n, \theta}(1 + t)^{-\frac{n+2(\theta-1)}{2}}$, we obtain the desired result for $t > 1$.

Now, we consider the case $0 < t \leq 1$. Since $\frac{2(1+|\xi|^2)}{4|\xi|^2 \sqrt{3+4|\xi|^2}} \left| \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \right| \leq 1$ and, $e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \leq e^{-\frac{t}{4}}$ for $|\xi| \geq 1$, we have

$$|\Lambda_\theta(t)g(x)| \leq \int_{|\xi| \leq 1} e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}}{1+|\xi|^2} \frac{2(1+|\xi|^2)}{4|\xi|^2 \sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \bar{g}(\xi) e^{ix \cdot \xi} d\xi$$

$$+ \int_{|\xi| \geq 1} e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{2|\xi|^2 |\xi|^{-1} |\xi|^{2(\theta-1)}}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \bar{g}(\xi) e^{ix \cdot \xi} d\xi$$

$$\leq t \|\bar{g}\|_{L^\infty} \int_{|\xi| \leq 1} e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}}{1+|\xi|^2} d\xi + e^{-\frac{t}{4}} \int_{|\xi| \geq 1} \frac{2|\xi|^2 |\xi|^{-1} |\xi|^{2(\theta-1)}}{\sqrt{3+4|\xi|^2}} |\bar{g}(\xi)| d\xi.$$
Using that \( e^{-\frac{\|\xi\|^2}{2(1+\|\xi\|^2)}} \leq 1 \), the Cauchy–Schwarz inequality, spherical coordinates and taking into account that \(-2s + 4\theta + n - 6 < 0\), we arrive at

\[
|A_\theta(t)g(x)| \leq t\|\tilde{g}\|_{L^\infty} \int_{|\xi| \leq 1} 1d\xi + e^{-\frac{t}{4}} \|g\|_{H^s(\mathbb{R}^n)} \left( \int_{|\xi| \geq 1} \frac{4\|\xi\|^{-2}\|\xi\|^{4(\theta-1)}}{3+4\|\xi\|^2} |d\xi| \right)^\frac{1}{2}
\]

\[
\leq t\|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{4}} \|g\|_{H^s(\mathbb{R}^n)}.
\]

Again, since there exists a constant \( C_{n,\theta} > 0 \), such that \( e^{-\frac{t}{4}} \leq C_{n,\theta}(1 + t)^{-\frac{n+2(\theta-1)}{2}} \), we obtain

\[
|A_\theta(t)g(x)| \leq t\|g\|_{L^1(\mathbb{R}^n)} + (1 + t)^{-\frac{n+2(\theta-1)}{2}} \|g\|_{H^s(\mathbb{R}^n)}.
\] (3.3)

Also, since \( t \leq 1 \), we have

\[
t\|g\|_{L^1(\mathbb{R}^n)} \leq ||g||_{L^1(\mathbb{R}^n)} = 2^\frac{n+2(\theta-1)}{2} 2^{-\frac{n+2(\theta-1)}{2}} ||g||_{L^1(\mathbb{R}^n)} = 2^\frac{n+2(\theta-1)}{2} (1 + 1)^{-\frac{n+2(\theta-1)}{2}} ||g||_{L^1(\mathbb{R}^n)}
\]

\[
\leq 2^\frac{n+2(\theta-1)}{2} (1 + t)^{-\frac{n+2(\theta-1)}{2}} ||g||_{L^1(\mathbb{R}^n)} \leq (1 + t)^{-\frac{n+2(\theta-1)}{2}} ||g||_{L^1(\mathbb{R}^n)}.
\] (3.4)

Combining (3.3) and (3.4), we obtain the desired result.

The following lemma helps to deal with the initial data of the integro-differential equation (2.2) and its derivative on the variable \( t \).

**Lemma 3.4.** Let \( s > \frac{n-2}{2} \). There exists a constant \( C = C_n > 0 \), which only depends on \( n \), such that

\[
\|\partial_t S(t)g\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{\theta}{2}} \|g\|_{L^1(\mathbb{R}^n)} + Ce^{-\frac{t}{4}} \|g\|_{H^{s+1}(\mathbb{R}^n)},
\]

\[
\|S(t)\Delta g\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{\theta}{2}} \|g\|_{L^1(\mathbb{R}^n)} + Ce^{-\frac{t}{4}} \|g\|_{H^{s+2}(\mathbb{R}^n)},
\]

\[
\|\partial_t^{k+1} S(t)g\|_{H^{s-k}(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)},
\]

\[
\|\partial_t^k S(t)\Delta g\|_{H^{s-k}(\mathbb{R}^n)} \leq C\|g\|_{H^{s+1}(\mathbb{R}^n)},
\]

for all \( g \in \mathcal{S}(\mathbb{R}^n) \), \( t > 0 \), \( s \in \mathbb{R} \) and \( k = 0, 1 \).
Proof. First we prove the estimates on $L^\infty$. From the boundedness of $\cos(\cdot)$ and $\sin(\cdot)$ functions, and since $\frac{1}{\sqrt{3+4|\xi|^2}} \leq 1$, we obtain

\[
|\partial_t S(t)g(x)| \leq \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi)e^{i\xi \cdot x} d\xi
\]

\[
+ \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} |\xi|^s+1 |\xi|^{-(s+1)} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi)e^{i\xi \cdot x} d\xi
\]

\[
+ \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{1}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi)e^{i\xi \cdot x} d\xi
\]

\[
\leq \|\tilde{g}\|_{L^\infty} \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} d\xi + \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} |\xi|^s+1 |\xi|^{-(s+1)} |\tilde{g}(\xi)| d\xi
\]

\[
+ \int e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \frac{|\xi|^s+1}{\sqrt{3+4|\xi|^2}} |\tilde{g}(\xi)| d\xi.
\]

Since $\|\tilde{g}\|_{L^\infty} \leq \|g\|_{L^1(\mathbb{R}^n)}$, $e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \leq e^{-\frac{|\xi|^2}{4}}$ for $|\xi| \geq 1$ and, $e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \leq e^{-\frac{|\xi|^2}{2}}$ for $|\xi| \leq 1$, we have

\[
|\partial_t S(t)g(x)| \leq \|g\|_{L^1(\mathbb{R}^n)} \int e^{-\frac{|\xi|^2}{4}} d\xi + e^{-\frac{|\xi|^2}{4}} \int |\xi|^s+1 |\xi|^{-(s+1)} |\tilde{g}(\xi)| d\xi + e^{-\frac{|\xi|^2}{4}} \int \frac{|\xi|^s+1}{\sqrt{3+4|\xi|^2}} |\tilde{g}(\xi)| d\xi.
\]

By Lemma 3.1, the Cauchy–Schwarz inequality and since $-2s + n - 2 < 0$, we conclude that

\[
|\partial_t S(t)g(x)| \leq t^{-\frac{\lambda}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{|\xi|^2}{4}} \|g\|_{H^{s+1}(\mathbb{R}^n)} \left( \int |\xi|^{-2(s+1)} d\xi \right)^{1/2}
\]

\[
+ e^{-\frac{|\xi|^2}{4}} \|g\|_{H^s(\mathbb{R}^n)} \left( \int \frac{|\xi|^{-2s}}{3+4|\xi|^2} d\xi \right)^{1/2}
\]

\[
\leq t^{-\frac{\lambda}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{|\xi|^2}{4}} \|g\|_{H^{s+1}(\mathbb{R}^n)}.
\]

Therefore,

\[
\|\partial_t S(t)g\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{\lambda}{2}} \|g\|_{L^1(\mathbb{R}^n)} + C e^{-\frac{|\xi|^2}{4}} \|g\|_{H^{s+1}(\mathbb{R}^n)},
\]

which proves the first inequality of the lemma.
To obtain the second inequality we use a similar argument. Indeed,

\[
|S(t)\Delta g(x)| \leq \left| \int_{|\xi| \leq 1} e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} \frac{2(1+|\xi|^2)}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix_x \cdot x} d\xi \right|
\]

\[
+ \left| \int_{|\xi| \geq 1} e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} \frac{2(1+|\xi|^2)}{\sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix_x \cdot x} d\xi \right|
\]

\[
\leq t^{-\frac{\beta}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\beta}{2}} \int_{|\xi| \geq 1} \frac{2(1+|\xi|^2) |\xi|^{-2} |\xi|^{-2} t}{\sqrt{3+4|\xi|^2}} |\tilde{g}(\xi)| d\xi
\]

\[
\leq t^{-\frac{\beta}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\beta}{2}} \|g\|_{H^{\beta+2}(\mathbb{R}^n)} \left( \int_{|\xi| \geq 1} \frac{4(1+|\xi|^2)^2 |\xi|^{-2} t}{3+4|\xi|^2} d\xi \right)^{1/2}
\]

which finishes the proof for the operator \(S(t)\Delta\) in \(L^\infty\).

The third inequality, for \(k = 0\), follows by observing that

\[
\|\partial_t S(t) g\|_{H^s} \leq \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} (1 + |\xi|^2)^{\frac{s}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) \right|^2 d\xi
\]

\[
+ \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} (1 + |\xi|^2)^{\frac{s}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) \right|^2 d\xi
\]

\[
\leq \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s-1}{2}} \tilde{g}(\xi) \right|^2 d\xi + \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s-1}{2}} \tilde{g}(\xi) \right|^2 d\xi
\]

\[
\leq \|g\|_{H^s}^2 + \|g\|_{H^{s+1}}^2 \leq \|g\|_{H^s}^2
\]

Therefore,

\[
\|\partial_t S(t) g\|_{H^s} \leq C \|g\|_{H^s}.
\]

In a similar way, we obtain the result for \(\partial_t^2 S(t)\), namely,

\[
\|\partial_t^2 S(t) g\|_{H^{s-1}} \leq \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} (1 + |\xi|^2)^{\frac{s-1}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) \right|^2 d\xi
\]

\[
+ \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|^2 t}{2(1+|\xi|^2)}} (1 + |\xi|^2)^{\frac{s-1}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) \right|^2 d\xi
\]

\[
\leq \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s-1}{2}} \tilde{g}(\xi) \right|^2 d\xi + \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s-1}{2}} \tilde{g}(\xi) \right|^2 d\xi
\]

\[
\leq \|g\|_{H^{s+1}}^2 + \|g\|_{H^s}^2 \leq \|g\|_{H^s}^2
\]

Then, we conclude the third inequality for \(k = 1\), that is,

\[
\|\partial_t^2 S(t) g\|_{H^s} \leq C \|g\|_{H^s}.
\]
For the operator $S(t)\Delta$, we have
\[
\|S(t)\Delta g\|_{H^s}^2 \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2(1+|x|^2)}} \frac{|x|^2}{\sqrt{3+4|x|^2}} \sin \left( \frac{|x|^2 \sqrt{3+4|x|^2}t}{2(1+|x|^2)} \right) \|g\|_{H^s}^2 \, dx 
\]
\[
\leq C \int_{\mathbb{R}^n} (1 + |x|^2)^{\frac{s+1}{2}} \|\hat{g}(\xi)\|^2 \, d\xi \leq C \|g\|_{H^{s+1}}^2.
\]
Therefore, we obtain the fourth inequality, with $k = 0$, namely,
\[
\|S(t)\Delta g\|_{H^s} \leq C \|g\|_{H^{s+1}}.
\]
Finally, for the case $\partial_t S(t)\Delta$, reasoning similarly as in the proof of the last inequality, we arrive at
\[
\|\partial_t S(t)\Delta g\|_{H^{s-1}}^2 \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2(1+|x|^2)}} |x|^2 (1 + |x|^2)^{-\frac{s-1}{2}} \cos \left( \frac{|x|^2 \sqrt{3+4|x|^2}t}{2(1+|x|^2)} \right) \|g\|_{H^{s-1}}^2 \, dx 
\]
\[
+ \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2(1+|x|^2)}} \frac{|x|^2 (1 + |x|^2)^{\frac{s-1}{2}}}{\sqrt{3+4|x|^2}} \sin \left( \frac{|x|^2 \sqrt{3+4|x|^2}t}{2(1+|x|^2)} \right) \|\hat{g}(\xi)\|^2 \, d\xi 
\]
\[
\leq \int_{\mathbb{R}^n} (1 + |x|^2)^{\frac{s+1}{2}} |\hat{g}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^n} (1 + |x|^2)^{\frac{s}{2}} \|\hat{g}(\xi)\|^2 \, d\xi
\]
\[
\leq \|g\|_{H^{s+1}}^2 + \|g\|_{H^s}^2 \leq \|g\|_{H^{s+1}}^2.
\]
Then, we arrived at the fourth inequality in the statement of the lemma, for $k = 1$, that is,
\[
\|\partial_t S(t)\Delta g\|_{H^{s-1}} \leq C \|g\|_{H^{s+1}},
\]
which finishes the proof of the lemma. \qed

The proof of the next corollary follows from Lemma 3.4 and arguing as in Lemma 3.3. We omit the details.

**Corollary 3.5.** Let $s > \frac{n+2}{2}$. There exists a constant $C = C_n > 0$ such that
\[
\|\partial_t S(t)g\|_{L^\infty(\mathbb{R}^n)} \leq C (1 + t)^{-\frac{s-1}{2}} \left[ \|g\|_{L^1(\mathbb{R}^n)} + \|g\|_{H^{s+1}(\mathbb{R}^n)} \right],
\]
\[
\|S(t)\Delta g\|_{L^\infty(\mathbb{R}^n)} \leq C (1 + t)^{-\frac{s-1}{2}} \left[ \|g\|_{L^1(\mathbb{R}^n)} + \|g\|_{H^{s+2}(\mathbb{R}^n)} \right],
\]
for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$.

### 3.2. Estimates in $H^s_p(\mathbb{R}^n)$

The previous lemmas are essential to prove the existence of global solutions in $H^s(\mathbb{R}^n)$ spaces. On the other hand, to prove the existence of global and local solutions in Bessel potential spaces $H^s_p(\mathbb{R}^n)$, we need to obtain time decay estimates for the solution of the linear problem (2.1) and for the operator $\Lambda_\theta(t)$, which acts on the nonlinear part of the integro-differential equation (2.2).

**Lemma 3.6.** Let $\theta \in (\frac{2-n}{2n}, 1]$ if $n = 1, 2$, and $\theta \in [0, 1]$ if $n \geq 3$. Assume $\sigma < 3 - n - 2\theta$, $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There exists a constant $C = C_{\sigma,n} > 0$ such that
\[
\|\Lambda_\theta(t)g\|_{H^s_p(\mathbb{R}^n)} \leq C t^{1-\theta - \frac{s}{2} \left( 1 - \frac{2}{p} \right)} \|g\|_{L^{p'}(\mathbb{R}^n)},
\]
for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$. Here $\Lambda_\theta(t) = S(t)(I - \Delta)^{-\frac{1}{2}}(-\Delta)^\theta$. 


Proof. Consider $K_{\theta}(t) = \Lambda_{\theta}(t)J^\sigma$, where $J^\sigma = (I - \Delta)\tilde{z}$ is the Bessel potential operator. By the boundedness of the $\sin(\cdot)$ function and since $(1+|\xi|^2)^\frac{3+4|\xi|^2}{2(1+|\xi|^2)} \leq 1$, we conclude that

$$|K_{\theta}(t)g(x)| \leq \left| \frac{1}{|\xi|} \int e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} \frac{2(1+|\xi|^2)^{\frac{3+4|\xi|^2}{2(1+|\xi|^2)}}}{|\xi|^2} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix \cdot \xi} d\xi \right|$$

$$+ \left| \frac{1}{|\xi|} \int e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} \frac{2(1+|\xi|^2)^{\frac{3+4|\xi|^2}{2(1+|\xi|^2)}}}{|\xi|^2} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi) e^{ix \cdot \xi} d\xi \right|$$

$$\leq \|\tilde{g}\|_{L^\infty} \int e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} |\xi|^2 \sin(\theta-1) d\xi + \|\tilde{g}\|_{L^\infty} \int e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} \frac{2(1+|\xi|^2)^{\frac{3+4|\xi|^2}{2(1+|\xi|^2)}}}{|\xi|^2} d\xi.$$

Since $\|\tilde{g}\|_{L^\infty} \leq \|g\|_{L^1(\mathbb{R}^n)}, e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} \leq e^{-\frac{|\xi|^2}{4}}$ for $|\xi| \leq 1$ and, $e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} \leq e^{-\frac{|\xi|^2}{4}}$ for $|\xi| \geq 1$, we obtain

$$|K_{\theta}(t)g(x)| \leq \|g\|_{L^1(\mathbb{R}^n)} \int e^{-\frac{|\xi|^2}{2(|\xi|^2 + t)}} |\xi|^2 \sin(\theta-1) d\xi + e^{-\frac{|\xi|^2}{4}} \|g\|_{L^1(\mathbb{R}^n)} \int \frac{2(1+|\xi|^2)^{\frac{3+4|\xi|^2}{2(1+|\xi|^2)}}}{|\xi|^2} d\xi.$$

Now, from Lemma 3.1, spherical coordinates and the assumption on $\sigma$, we have

$$|K_{\theta}(t)g(x)| \leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{4}} \|g\|_{L^1(\mathbb{R}^n)} \int \frac{1}{|\xi|} d\xi,$$

$$\leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{4}} \|g\|_{L^1(\mathbb{R}^n)} \int_1^{\infty} r^{\sigma+2\theta-3} r^{n-1} dr,$$

$$\leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{4}} \|g\|_{L^1(\mathbb{R}^n)}.$$

Using the last inequality and the fact that there exists a constant $C_{n,\theta} > 0$ (depending only on $n$ and $\theta$) such that $e^{-\frac{t}{4}} \leq C_{n,\theta} t^{-\frac{n+2(\theta-1)}{2}}$, we conclude that

$$\|K_{\theta}(t)g\|_{L^\infty} \leq t^{-\frac{n+2(\theta-1)}{2}} \|g\|_{L^1(\mathbb{R}^n)}.$$

Now, we prove that $K_{\theta}(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is continuous. We consider two cases:

- **Case 1** Assume that $t > 1$. Since

$$\left| \frac{2(1+|\xi|^2)}{|\xi|^2 \sqrt{3+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \right| \leq 1 \text{ for all } t > 0 \text{ and } \xi \in \mathbb{R}^n,$$

we have

$$\|K_{\theta}(t)g\|_{L^2(\mathbb{R}^n)}^2 \leq t^2 \int_{\mathbb{R}^n} e^{-\frac{t\xi^2}{2(1+|\xi|^2)}} \frac{2(1+|\xi|^2)(1+|\xi|^2)^{\frac{3+4|\xi|^2}{2(1+|\xi|^2)}}}{|\xi|^2} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \tilde{g}(\xi)^2 d\xi$$

$$\leq t^2 \int_{\mathbb{R}^n} e^{-\frac{t\xi^2}{2(1+|\xi|^2)}} (1+|\xi|^2)^{\frac{3}{2}} |\xi|^2 \tilde{g}(\xi)^2 d\xi = I_1 + I_2,$$

where

$$I_1 = t^2 \int_{|\xi| \leq 1} e^{-\frac{t\xi^2}{2(1+|\xi|^2)}} (1+|\xi|^2)^{\frac{3}{2}} |\xi|^2 \tilde{g}(\xi)^2 d\xi$$

$$I_2 = t^2 \int_{|\xi| > 1} e^{-\frac{t\xi^2}{2(1+|\xi|^2)}} (1+|\xi|^2)^{\frac{3}{2}} |\xi|^2 \tilde{g}(\xi)^2 d\xi.$$
and

\[ I_2 = t^2 \int_{|\xi| \geq 1} e^{-\frac{\|\xi\|^2}{1+|\xi|^2}} (1 + |\xi|^2)^{\sigma-2} |\xi|^{4\theta} |\widehat{g}(\xi)|^2 d\xi. \]

Since \( e^{-\frac{\|\xi\|^2}{1+|\xi|^2}} \leq e^{-\frac{\|\xi\|^2}{2}} \) and \( (1 + |\xi|^2)^{\sigma-2} \leq \max\{2^{\sigma-2}, 1\} \) for \( |\xi| \leq 1 \), we obtain

\[ I_1 \leq C_\sigma t^2 \int_{|\xi| \leq 1} e^{-\frac{\|\xi\|^2}{2}} |\xi|^{4\theta} |\widehat{g}(\xi)|^2 d\xi. \]

Now, note that \( e^{-\frac{\|\xi\|^2}{2}} |\xi|^{4\theta} = 4^\theta t^{-2\theta} e^{-\frac{|\xi|^2}{4^\theta t^2}} \left| \frac{\xi \sqrt{t}}{\sqrt{4^\theta}} \right|^{4\theta} \leq 4^\theta t^{-2\theta} \). Then, we can conclude that

\[ I_1 \leq C_{\sigma,\theta} t^{2(1-\theta)} \|g\|_{L^2(\mathbb{R}^n)}^2. \]  \hfill (3.6)

Next, since \( e^{-\frac{\|\xi\|^2}{1+|\xi|^2}} \leq e^{-\frac{\|\xi\|^2}{2}} \) for \( |\xi| \geq 1 \), we get

\[ I_2 \leq t^2 e^{-\frac{\|\xi\|^2}{2}} \int_{|\xi| \geq 1} (1 + |\xi|^2)^{\sigma-2} |\xi|^{4\theta} |\widehat{g}(\xi)|^2 d\xi. \]  \hfill (3.7)

The restriction \( \sigma < 3 - n - 2\theta \) implies \( \sigma < 2(1 - \theta) \), and therefore \( (1 + |\xi|^2)^{\sigma-2} |\xi|^{4\theta} \leq 1 \). From (3.7) we arrive at

\[ I_2 \leq t^2 e^{-\frac{\|\xi\|^2}{2}} \|g\|_{L^2(\mathbb{R}^n)}^2. \]

Using the fact that \( t > 1 \), there exists a constant \( C_\theta > 0 \) such that \( e^{-\frac{\|\xi\|^2}{2}} \leq C_\theta t^{-2\theta} \). Therefore,

\[ I_1 \leq C_{\sigma,\theta} t^{2(1-\theta)} \|g\|_{L^2(\mathbb{R}^n)}^2. \]  \hfill (3.8)

Combining (3.6) and (3.8) and taking the square root we have

\[ \|K_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq t^{1-\theta} \|g\|_{L^2(\mathbb{R}^n)}. \]

- **Case 2** Consider \( 0 < t \leq 1 \). Since \( \frac{2(1+|\xi|^2)}{t^2 \sqrt{2+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{2+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \leq 1 \), \( (1 + |\xi|^2)^{\sigma-2} |\xi|^{4\theta} \leq 1 \) and 
  \( e^{-\frac{\|\xi\|^2}{1+|\xi|^2}} \leq 1 \) for all \( \xi \in \mathbb{R}^n \), we obtain

\[ \|K_\theta(t)g\|^2_{L^2(\mathbb{R}^n)} \leq 2 \int_{\mathbb{R}^n} e^{-\frac{\|\xi\|^2}{1+|\xi|^2}} \frac{2(1+|\xi|^2)(1+|\xi|^2) \frac{2\pi^2}{\|\xi\|^2} |\xi|^{2\theta}}{t^2 \sqrt{2+4|\xi|^2}} \sin \left( \frac{|\xi|^2 \sqrt{2+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \left| \widehat{g}(\xi) \right|^2 d\xi \leq t^2 \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 d\xi = t^{2(1-\theta)} t^{2\theta} \|g\|^2_{L^1(\mathbb{R}^n)}. \]

Since \( 0 < t \leq 1 \) and \( \theta \geq 0 \), we have \( t^{2\theta} \leq 1 \). Then, from last inequality, we arrive at

\[ \|K_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq t^{1-\theta} \|g\|_{L^2(\mathbb{R}^n)}. \]  \hfill (3.9)

In any case, we have that \( K_\theta(t) \) is a bounded operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \).

Therefore, recalling that \( K_\theta(t) = \Lambda_\theta(t) J^\sigma \), from (3.5), (3.9) and applying the Riesz–Thorin interpolation theorem, we conclude the proof of the lemma. \( \square \)

The next corollary will be useful to estimate the nonlinear part of Eq. (2.2) (see Proposition 4.1).
Corollary 3.7. Let $s \in \mathbb{R}$, $\theta \in (\frac{2-n}{2}, 1]$ if $n = 1, 2$, and $\theta \in [0, 1]$ if $n \geq 3$. There exists $C = C_n > 0$ such that

$$\|\Lambda_\theta(t)g\|_{H^s(\mathbb{R}^n)} \leq C t^{1-\theta}\|g\|_{H^s(\mathbb{R}^n)},$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$.

Proof. Repeating the second part of the proof of Lemma 3.6, with $\sigma = 0$, we obtain (3.7), that is,

$$I_2 \leq t^2 e^{-\frac{\xi}{2}} \int_{|\xi| \leq 1} (1 + |\xi|^2)^{-2} |\xi|^{4\theta} \widehat{g}(\xi)^2 \, d\xi.$$

Since $\frac{2-n}{2} < \theta \leq 1$, we have $(1 + |\xi|^2)^{-2} |\xi|^{4\theta} \leq 1$. Therefore $I_2 \leq C_\theta t^2(1-\theta)\|g\|^2_{L^2(\mathbb{R}^n)}$. Arguing as in the proof of Lemma 3.6 we arrive at

$$\|\Lambda_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq t^{1-\theta}\|g\|_{L^2(\mathbb{R}^n)}.$$

From the last inequality we have

$$\|\Lambda_\theta(t)g\|_{H^s(\mathbb{R}^n)} = \|\Lambda_\theta(t)J^s g\|_{L^2(\mathbb{R}^n)} \leq t^{1-\theta}\|J^s g\|_{L^2(\mathbb{R}^n)} = C_n t^{1-\theta}\|g\|_{H^s(\mathbb{R}^n)},$$

which finishes the proof of the corollary.

 Lemma 3.8. Let $0 \leq \theta \leq 1$, $\sigma < 3 - n - 2\theta$, $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There exists $C = C_{\sigma, n} > 0$ such that

$$\|\partial_t \Lambda_\theta(t)g\|_{H^s(\mathbb{R}^n)} \leq C t^{-\frac{\sigma}{2}(1-\frac{\sigma}{2})}\|g\|_{L^{p'}(\mathbb{R}^n)},$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$. Here $\Lambda_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^{\theta}$.

Proof. Consider $M_\theta(t) = \partial_t \Lambda_\theta(t)J^{s-1}$, where $J^s = (I - \Delta)^s$ is the Bessel potential operator. Similarly to the proof of Lemma 3.6, we have

$$|M_\theta(t)g(x)| \leq \int_{|\xi| \leq 1} e^{-\frac{\xi^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}(1 + |\xi|^2)^{\frac{s-1}{2}}}{1 + |\xi|^2} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \widehat{\eta}(\xi) e^{ix \cdot \xi} \, d\xi$$

$$+ \int_{|\xi| \geq 1} e^{-\frac{\xi^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}(1 + |\xi|^2)^{\frac{s-1}{2}}}{1 + |\xi|^2} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \widehat{\eta}(\xi) e^{ix \cdot \xi} \, d\xi$$

$$+ \int_{|\xi| \leq 1} e^{-\frac{\xi^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}(1 + |\xi|^2)^{\frac{s-1}{2}}}{1 + |\xi|^2} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \widehat{\eta}(\xi) e^{ix \cdot \xi} \, d\xi$$

$$+ \int_{|\xi| \geq 1} e^{-\frac{\xi^2}{2(1+|\xi|^2)}} \frac{|\xi|^{2\theta}(1 + |\xi|^2)^{\frac{s-1}{2}}}{1 + |\xi|^2} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2} t}{2(1+|\xi|^2)} \right) \widehat{\eta}(\xi) e^{ix \cdot \xi} \, d\xi$$

$$\leq t^{-\frac{\sigma}{2}}\|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\xi}{4}}\|g\|_{L^1(\mathbb{R}^n)} \int_{|\xi| \geq 1} |\xi|^{\sigma-3+2\theta} \, d\xi + e^{\frac{\xi}{4}}\|g\|_{L^1(\mathbb{R}^n)} \int_{|\xi| \geq 1} |\xi|^{\sigma-4+2\theta} \, d\xi$$

$$\leq t^{-\frac{\sigma}{2}}\|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\xi}{4}}\|g\|_{L^1(\mathbb{R}^n)} \int_1^\infty r^{\sigma+2\theta+n-4} \, dr + e^{\frac{\xi}{4}}\|g\|_{L^1(\mathbb{R}^n)} \int_1^\infty r^{\sigma+2\theta+n-5} \, dr$$

$$\leq (t^{-\frac{\sigma}{2}} + e^{-\frac{\xi}{4}})\|g\|_{L^1(\mathbb{R}^n)} \leq t^{-\frac{\sigma}{2}}\|g\|_{L^1(\mathbb{R}^n)}.$$
Therefore,
\[
\|M_\theta(t)g\|_{L^\infty(\mathbb{R}^n)} \leq t^{-\frac{n}{2}} \|g\|_{L^1(\mathbb{R}^n)}. \tag{3.10}
\]

Now we prove that \(M_\theta(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)\) is continuous. Indeed, since \(\sigma - 1 + 2(\theta - 1) < -n \leq -1\), we obtain \(\frac{|\xi|^{2\sigma}(1+|\xi|^2)^{\frac{\sigma-1}{2}}}{(1+|\xi|^2)^{\frac{1}{2}}(3+4|\xi|^2)} \leq 1\). From the boundedness of the functions \(\cos(\cdot), \sin(\cdot), e^{-\frac{|\xi|^2}{1+t^2}}\), we conclude that
\[
\|M_\theta(t)g\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{1+t^2}} \left| \frac{|\xi|^{2\sigma}(1+|\xi|^2)^{\frac{\sigma-1}{2}}}{1+|\xi|^2} \cos \left( \frac{|\xi|^2\sqrt{3+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \right|^2 |\hat{g}(\xi)|^2 d\xi
\]
\[+ \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{1+t^2}} \left| \frac{|\xi|^{2\sigma}(1+|\xi|^2)^{\frac{\sigma-1}{2}}}{1+|\xi|^2} \sin \left( \frac{|\xi|^2\sqrt{3+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \right|^2 |\hat{g}(\xi)|^2 d\xi \]
\[\leq C \|g\|_{L^2(\mathbb{R}^n)}. \tag{3.11}
\]

Taking the square root we have
\[
\|M_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq \|g\|_{L^2(\mathbb{R}^n)}. \tag{3.12}
\]

Then, recalling that \(M_\theta(t) = \partial_\lambda A(t) J^{\sigma-1}\), from (3.10) and (3.11) applying the Riesz–Thorin interpolation theorem, we conclude the proof of the lemma. \( \Box \)

As before, the next corollary will be useful to estimate the nonlinear part of Eq. (2.2) (see Proposition 4.1).

**Corollary 3.9.** Let \(0 \leq \theta \leq 1\) and \(s \in \mathbb{R}\). There exists \(C = C_n > 0\) such that
\[
\|\partial_\lambda A_\theta(t)g\|_{H^{s-1}(\mathbb{R}^n)} \leq C \|g\|_{H^{s-1}(\mathbb{R}^n)},
\]
for all \(g \in \mathcal{S}(\mathbb{R}^n)\) and \(t > 0\).

**Proof.** Repeating the second part of the proof of Lemma 3.8, with \(\sigma = 1\), we obtain
\[
\|\partial_\lambda A_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{1+t^2}} \left| \frac{|\xi|^{2\theta}}{1+|\xi|^2} \cos \left( \frac{|\xi|^2\sqrt{3+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \right|^2 |\hat{g}(\xi)|^2 d\xi
\]
\[+ \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{1+t^2}} \left| \frac{|\xi|^{2\theta}}{1+|\xi|^2} \sin \left( \frac{|\xi|^2\sqrt{3+4|\xi|^2}t}{2(1+|\xi|^2)} \right) \right|^2 |\hat{g}(\xi)|^2 d\xi. \]

Since \(0 \leq \theta \leq 1\), we have \(\frac{|\xi|^2}{1+|\xi|^2} \leq 1\) and \(\frac{|\xi|^{2\theta}}{1+|\xi|^2} \leq 1\). Using the facts that \(e^{-\frac{|\xi|^2}{1+t^2}} \leq 1\) and since \(\cos(\cdot), \sin(\cdot)\) are bounded functions, we arrive at
\[
\|\partial_\lambda A_\theta(t)g\|_{L^2(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)}. \]

From the last inequality, we have
\[
\|\partial_\lambda A_\theta(t)g\|_{H^{s-1}(\mathbb{R}^n)} = \|\partial_\lambda A_\theta(t) J^{s-1}g\|_{L^2(\mathbb{R}^n)} \leq \|J^{s-1}g\|_{L^2(\mathbb{R}^n)} = C_n \|g\|_{H^{s-1}(\mathbb{R}^n)},
\]
which finishes the proof of the corollary. \( \Box \)

Next lemma is useful to bound the linear part in the proof of Theorem 2.3.
Lemma 3.10. Let $\sigma < 1-n$, $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There exists a constant $C = C_\sigma > 0$, such that
\begin{align}
\|\partial_t^k S(t) g\|_{H^{p-k}(\mathbb{R}^n)} &\leq Ct^{-\frac{p}{2}(1-\frac{2}{\sigma})}\|g\|_{L^p(\mathbb{R}^n)}, \quad (3.12) \\
\|\partial_t^{k-1} S(t) \Delta g\|_{H^{p-k+1}(\mathbb{R}^n)} &\leq Ct^{-\frac{p}{2}(1-\frac{2}{\sigma})}\|g\|_{H^{p}(\mathbb{R}^n)}, \quad (3.13)
\end{align}
for all $g \in \mathcal{S}(\mathbb{R}^n)$, $k = 1, 2$ and $t > 0$.

Proof. Consider $K_1(t) = \partial_t S(t) J^{\sigma-1}$. Similarly to the proof of Lemma 3.8, we obtain
\begin{align*}
|K_1(t)g(x)| &\leq \int \left| e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \right| \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
&\quad + \int \left| e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \right| \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
&\quad + \int \left| e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \right| \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
&\quad + \int \left| e^{-\frac{|\xi|^2}{2(1+|\xi|^2)}} \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3+4|\xi|^2}}{2(1+|\xi|^2)} \right) \right| \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
&\leq t^{-\frac{p}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int |\xi|^2 \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} d\xi + e^{-\frac{t}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int |\xi|^2 \left( 1 + |\xi|^2 \right)^{-\frac{\sigma-1}{2}} d\xi \\
&\leq t^{-\frac{p}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_1^{\infty} r^{\sigma-1} r^{n-1} dr + e^{-\frac{t}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_1^{\infty} r^{\sigma-2} r^{n-1} dr \\
&\leq t^{-\frac{p}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{t}{2}} \|g\|_{L^1(\mathbb{R}^n)} \leq t^{-\frac{p}{2}} \|g\|_{L^2(\mathbb{R}^n)}.
\end{align*}
Now, it is not difficult to see that
\begin{align*}
\|K_1(t)g\|_{L^2(\mathbb{R}^n)} &\leq \|g\|_{L^2(\mathbb{R}^n)}.
\end{align*}

Then, recalling that $K_1(t) = \partial_t S(t) J^{\sigma}$, and applying the Riesz–Thorin interpolation theorem, we conclude the proof of (3.12), for $k = 1$. 
On the other hand, considering $K_2(t) = \partial_t^2 S(t) J^{\sigma - 2}$, we get
\[
|K_2(t)g(x)| \leq \int_{|\xi| \leq 1} e^{-\frac{\xi^2}{2 + (1 + |\xi|^2)}} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3 + 4|\xi|^2} + t}{2(1 + |\xi|^2)} \right) \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
+ \int_{|\xi| \leq 1} e^{-\frac{\xi^2}{2 + (1 + |\xi|^2)}} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3 + 4|\xi|^2} + t}{2(1 + |\xi|^2)} \right) \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
+ \int_{|\xi| \geq 1} e^{-\frac{\xi^2}{2 + (1 + |\xi|^2)}} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} \cos \left( \frac{|\xi|^2 \sqrt{3 + 4|\xi|^2} + t}{2(1 + |\xi|^2)} \right) \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
+ \int_{|\xi| \geq 1} e^{-\frac{\xi^2}{2 + (1 + |\xi|^2)}} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} \sin \left( \frac{|\xi|^2 \sqrt{3 + 4|\xi|^2} + t}{2(1 + |\xi|^2)} \right) \hat{g}(\xi) e^{ix \cdot \xi} d\xi \\
\leq t^{-\frac{\sigma}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\xi^2}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_{|\xi| \geq 1} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} d\xi + e^{-\frac{\xi^2}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_{|\xi| \geq 1} (1 + |\xi|^2)^{\frac{\sigma - 2}{2}} d\xi \\
\leq t^{-\frac{\sigma}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\xi^2}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_0^\infty r^{\sigma - 2} r^{n-1} dr + e^{-\frac{\xi^2}{2}} \|g\|_{L^1(\mathbb{R}^n)} \int_0^\infty r^{\sigma - 1} r^{n-1} dr \\
\leq t^{-\frac{\sigma}{2}} \|g\|_{L^1(\mathbb{R}^n)} + e^{-\frac{\xi^2}{2}} \|g\|_{L^1(\mathbb{R}^n)} \leq t^{-\frac{\sigma}{2}} \|g\|_{L^1(\mathbb{R}^n)}.
\]

Also, as before
\[
\|K_2(t)g\|_{L^2(\mathbb{R}^n)} \leq \|g\|_{L^2(\mathbb{R}^n)}.
\]

Then, recalling that $K_2(t) = \partial_t^2 S(t) J^{\sigma - 2}$, and applying the Riesz–Thorin interpolation theorem, we conclude the proof of (3.12), for $k = 2$.

Now we prove inequality (3.13). From Lemma 3.6 with $\theta = 1$, we have
\[
\|S(t)\Delta g\|_{H^\sigma_p(\mathbb{R}^n)} = \|\Lambda_1(t)(I - \Delta)g\|_{H^\sigma_p(\mathbb{R}^n)} \leq Ct^{-\frac{\sigma}{2}} (1 - \frac{\sigma}{2}) \|g\|_{L^2(\mathbb{R}^n)} \leq C t^{-\frac{\sigma}{2}} (1 - \frac{\sigma}{2}) \|g\|_{H^{\sigma - 1}_p(\mathbb{R}^n)},
\]
which finishes the proof of the inequality (3.13) with $k = 1$. In a similar way, by Lemma 3.8 with $\theta = 1$, we obtain
\[
\|\partial_t S(t)\Delta g\|_{H^{\sigma - 1}_p(\mathbb{R}^n)} = \|\partial_t \Lambda_1(t)(I - \Delta)g\|_{H^{\sigma - 1}_p(\mathbb{R}^n)} \leq Ct^{-\frac{\sigma}{2}} (1 - \frac{\sigma}{2}) \|g\|_{L^{\sigma - 1}_p(\mathbb{R}^n)} \leq C t^{-\frac{\sigma}{2}} (1 - \frac{\sigma}{2}) \|g\|_{H^{\sigma - 1}_p(\mathbb{R}^n)},
\]
which finishes the proof of the Lemma. 

The next lemmas are useful to deal with the nonlinear part of the integro-differential equation (2.2).

**Lemma 3.11.** [1, 12] Let $\lambda \geq 2$, $1 < p, q < \infty$ and $s > 0$ be such that $s < \min\{\frac{n}{q}, \lambda - 1\}$ and
\[
1 - \frac{1}{p} = \frac{1}{q} + \frac{\lambda - 1}{n} \left( \frac{n}{q} - s \right).
\]

There exists a universal constant $C > 0$ such that
\[
\|f^\lambda - |g|^\lambda\|_{H^\sigma_p} \leq C \|f - g\|_{H^\sigma_p} \left[ \|f\|_{H^\sigma_p}^{\lambda - 1} + \|g\|_{H^\sigma_p}^{\lambda - 1} \right]. \tag{3.14}
\]

Furthermore, if $\lambda$ is a positive odd integer then (3.14) holds without the restriction $s < \lambda - 1$. 


Lemma 3.12. [15, Lemma 3.4] Let $\lambda \geq 2$ be a positive integer. If $u, \tilde{u} \in H^s \cap L^\infty$ and $\|u\|_{L^\infty} \leq M$, $\|\tilde{u}\|_{L^\infty} \leq M$, then

$$
\||u|^\lambda - |\tilde{u}|^\lambda|_{H^s} \leq C(M)(\|u - \tilde{u}\|_{L^\infty} (\|u\|_{H^s} + \|\tilde{u}\|_{H^s})(\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty})^{\lambda - 2} + \|u - \tilde{u}\|_{H^s}(\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty})^{\lambda - 2}),
$$

$$
\||u|^\lambda - |\tilde{u}|^\lambda|_{L^1} \leq C(M)(\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}^{\lambda - 2}(\|u\|_{L^2} + \|\tilde{u}\|_{L^2})\|u - \tilde{u}\|_{L^2}),
$$

where $C(M)$ is a constant dependent on $M$.

We finish this section by recalling the following technical lemma.

Lemma 3.13. [15, Lemma 3.5] Let $a, b$ non-negative real constants such that $b \geq a \geq 0$. Then,

$$
\int_0^t (1 + t - \tau)^{-a}(1 + \tau)^{-b}d\tau \leq C(1 + t)^{-a} \int_0^t (1 + \tau)^{-b}d\tau.
$$

4. Global solutions

Before proving the existence of global solutions, we establish an estimate of the nonlinear term of integro-differential equation (2.2) in the norm

$$
\|u\|_{Y^{s, \alpha}} := \sup_{0 < t < \infty} ((1 + t)^{\alpha_1}\|u(t)\|_{L^\infty} + \|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}),
$$

as well as in the norm

$$
\|u\|_{X^{s, \beta}} := \sup_{0 < t < \infty} (t^{\alpha}\|u(t)\|_{H^s} + t^{\beta}\|u_t(t)\|_{H^{s-1}}),
$$

with $\alpha = \frac{1}{\lambda - 1}(2 - \theta - \frac{n}{2}(1 - \frac{2}{p}))$ and $\beta = \alpha + 1 - \theta$.

Proposition 4.1. Let $\lambda \geq 3$ be a positive integer such that $\alpha_1(\lambda - 2) > 1$, $\alpha_1 = \frac{n + 2(\theta - 1)}{2}$, $\theta \in (\frac{2}{n}, 1]$ if $n = 1, 2$, and $\theta \in [0, 1]$ if $n \geq 3$, and consider $s > \frac{n + 4\theta - 6}{2}$. Then, there exists $C_1 > 0$ such that

$$
\left\| \int_0^t \Lambda_\theta(t - \tau)(|u|^\lambda - |\tilde{u}|^\lambda)d\tau \right\|_{Y^{s, \alpha_1}} \leq C_1(1 + t^{1-\theta})(\|u - \tilde{u}\|_{Y^{s, \alpha_1}}(\|u\|_{Y^{s, \alpha_1}}^{-1} + \|\tilde{u}\|_{Y^{s, \alpha_1}}^{-1})),
$$

(4.1)

where $\Lambda_\theta(t) = S(t)(I - \Delta)^{-1}(-\Delta)^\theta$.

Proof. From Lemma 3.3 we get

$$
\left\| \int_0^t \Lambda_\theta(t - \tau)(|u|^\lambda - |\tilde{u}|^\lambda)d\tau \right\|_{L^\infty} \leq C \int_0^t (1 + t - \tau)^{-\frac{n + 2(\theta - 1)}{2}} ((\|u|^\lambda - |\tilde{u}|^\lambda))_{L^1} + ((\|u|^\lambda - |\tilde{u}|^\lambda))_{H^s}) d\tau
$$

$$
:= J_1 + J_2.
$$

(4.2)
Now, from Lemma 3.12, using that $\alpha_1(\lambda - 2) > 1$ and taking into account Lemma 3.13, we bound the right hand side of (4.2) as follows

\[ J_1 \leq C \int_0^t (1 + t - \tau)^{-\alpha_1} (\|u\| L_\infty + \|\tilde{u}\| L_\infty)^{\lambda - 2} (\|u\| L_2 + \|\tilde{u}\| L_2) \|u - \tilde{u}\| L_2 d\tau \]

\[ \leq C \int_0^t (1 + t - \tau)^{-\alpha_1} (1 + \tau)^{-\alpha_1(\lambda - 2)} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 2} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1}) \|u - \tilde{u}\| Y_{\alpha_1} d\tau \]

\[ \leq C (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 1} \|u - \tilde{u}\| Y_{\alpha_1} (1 + t)^{-\alpha_1}, \quad (4.3) \]

and

\[ J_2 \leq C \int_0^t (1 + t - \tau)^{-\alpha_1} (\|u\| L_\infty + \|\tilde{u}\| L_\infty)^{\lambda - 2} (\|u\| H^s + \|\tilde{u}\| H^s) \|u - \tilde{u}\| L_\infty d\tau \]

\[ + C \int_0^t (1 + t - \tau)^{-\alpha_1} (\|u\| L_\infty + \|\tilde{u}\| L_\infty)^{\lambda - 1} \|u - \tilde{u}\| H^s d\tau \]

\[ \leq C \int_0^t (1 + t - \tau)^{-\alpha_1} (1 + \tau)^{-\alpha_1(\lambda - 1)} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 1} \|u - \tilde{u}\| Y_{\alpha_1} d\tau \]

\[ \leq C (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 1} \|u - \tilde{u}\| Y_{\alpha_1} (1 + t)^{-\alpha_1}. \quad (4.4) \]

Thus, from (4.2), (4.3) and (4.4), we conclude that

\[ \sup_{0 < t < \infty} (1 + t)^{\alpha_1} \left\| \int_0^t \Lambda_0(t - \tau)(|u|^\lambda - |\tilde{u}|^\lambda) d\tau \right\|_{L_\infty} \leq C \|u - \tilde{u}\| Y_{\alpha_1} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1}). \quad (4.5) \]

On the other hand, applying Corollary 3.7, we arrive at

\[ \left\| \int_0^t \Lambda_0(t - \tau)(|u|^\lambda - |\tilde{u}|^\lambda) d\tau \right\|_{H^s} \leq C \int_0^t (t - \tau)^{-\theta_1} \|u|^\lambda - |\tilde{u}|^\lambda\|_{H^s} d\tau \leq C t^{1-\theta_1} \int_0^t \|u|^\lambda - |\tilde{u}|^\lambda\|_{H^s} d\tau \]

\[ \leq C t^{1-\theta_1} \int_0^t ((\|u\| L_\infty + \|\tilde{u}\| L_\infty)^{\lambda - 2} (\|u\| H^s + \|\tilde{u}\| H^s) \|u - \tilde{u}\| L_\infty + (\|u\| L_\infty + \|\tilde{u}\| L_\infty)^{\lambda - 1} \|u - \tilde{u}\| H^s) d\tau \]

\[ \leq C t^{1-\theta_1} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 1} \|u - \tilde{u}\| Y_{\alpha_1} \int_0^t (1 + \tau)^{-\alpha_1(\lambda - 1)} d\tau \]

\[ \leq C t^{1-\theta_1} (\|u\| Y_{\alpha_1} + \|\tilde{u}\| Y_{\alpha_1})^{\lambda - 1} \|u - \tilde{u}\| Y_{\alpha_1}. \quad (4.6) \]
Finally, from Corollary 3.9, we obtain
\[
\left\| \int_0^t \partial_t A_{\theta}(t-\tau)(|u|^{\lambda} - |\bar{u}|^\lambda) \, d\tau \right\|_{H_s^{-1}} \leq C \int_0^t \|u\|^\lambda - \|\bar{u}\|^\lambda \|_{H_s^{-1}} \, d\tau
\]
\[
\leq C \int_0^t \left( \|u\|_{L^\infty} + \|\bar{u}\|_{L^\infty} \right)^{\lambda - 2} \left( \|u\|_{H_s^{-1}} + \|\bar{u}\|_{H_s^{-1}} \right) \|u - \bar{u}\|_{L^\infty} + \left( \|u\|_{L^\infty} + \|\bar{u}\|_{L^\infty} \right)^{\lambda - 1} \|u - \bar{u}\|_{H_{s-1}} \, d\tau
\]
\[
\leq C \left( \|u\|_{Y,\alpha_0} + \|\bar{u}\|_{Y,\alpha_0} \right)^{\lambda - 1} \|u - \bar{u}\|_{Y,\alpha_0} \int_0^t (1 + \tau)^{-\alpha_1(\lambda - 1)} \, d\tau
\]
\[
\leq C \left( \|u\|_{Y,\alpha_0} + \|\bar{u}\|_{Y,\alpha_0} \right)^{\lambda - 1} \|u - \bar{u}\|_{Y,\alpha_0}.
\]  
(4.7)

From (4.5), (4.6) and (4.7), we obtain the desired result.

\[ \square \]

**Proposition 4.2.** Let \( \lambda \geq 2, \theta \in (\frac{2-n}{2}, 1) \) if \( n = 1, 2 \), and \( \theta \in [0, 1] \) if \( n \geq 3 \), and \( \frac{1}{\lambda} > \alpha > 0 \). Assume \( 2 \leq p \leq q \leq \infty, \frac{p}{2}(1 - \frac{2}{p}) < 1 \) and consider \( \sigma, s \) such that \( s > \sigma, n(\frac{1}{p} - \frac{1}{q}) \leq \sigma < 3 - n - 2\theta \) and \( (\frac{\lambda}{q} + \frac{\sigma}{p} - 1) \frac{\alpha}{\lambda} + \sigma \leq s < \min\{\frac{\sigma}{q}, \lambda - 1\} + \sigma \). Then, there exists a constant \( C_2 > 0 \) such that
\[
\left\| \int_0^t A_{\theta}(t-\tau)(|u|^{\lambda} - |\bar{u}|^\lambda) \, d\tau \right\|_{X_{\alpha,\beta}^s} \leq C_2 \|u - \bar{u}\|_{X_{\alpha,\beta}^s} \left( \|u\|_{X_{\alpha,\beta}^s}^{\lambda - 1} + \|\bar{u}\|_{X_{\alpha,\beta}^s}^{\lambda - 1} \right).
\]  
(4.8)

**Proof.** From Lemma 3.6, the embedding \( H_p^s \subset H_q^{s-\sigma} \), Lemma 3.11, and recalling the integrability of the Beta function (notice that \( 1 > \alpha \lambda \), and \( \alpha > 0 \) implies that \( 2 > \theta + \frac{n}{2}(1 - \frac{2}{p}) \)), we arrive at
\[
\left\| \int_0^t A_{\theta}(t-\tau)(|u|^{\lambda} - |\bar{u}|^\lambda) \, d\tau \right\|_{H_p^s} \leq C \int_0^t (t - \tau)^{1 - \theta - \frac{\lambda}{q}(1 - \frac{2}{p})} \left( \|u\|^\lambda - \|\bar{u}\|^\lambda \right) \|H_q^{-s}\, d\tau
\]
\[
\leq C \int_0^t (t - \tau)^{1 - \theta - \frac{\lambda}{q}(1 - \frac{2}{p})} \|u - \bar{u}\|_{H_q^{-s}} \left( \|u\|_{H_q^{-s}}^{\lambda - 1} + \|\bar{u}\|_{H_q^{-s}}^{\lambda - 1} \right) \|H_q^{-s}\, d\tau
\]
\[
\leq C \int_0^t (t - \tau)^{1 - \theta - \frac{\lambda}{q}(1 - \frac{2}{p})} \|u - \bar{u}\|_{H_p^s} \left( \|u\|_{H_p^s}^{\lambda - 1} + \|\bar{u}\|_{H_p^s}^{\lambda - 1} \right) \|H_p^s\, d\tau
\]
\[
\leq C \|u - \bar{u}\|_{X_{\alpha,\beta}^s} \left[ \|u\|_{X_{\alpha,\beta}^s}^{\lambda - 1} + \|\bar{u}\|_{X_{\alpha,\beta}^s}^{\lambda - 1} \right] \int_0^t (t - \tau)^{1 - \theta - \frac{\lambda}{q}(1 - \frac{2}{p})} \tau^{-\alpha \lambda} \, d\tau
\]
\[
\leq C t^{2 - \theta - \frac{\lambda}{q}(1 - \frac{2}{p}) - \alpha \lambda} \|u - \bar{u}\|_{X_{\alpha,\beta}^s} \left[ \|u\|_{X_{\alpha,\beta}^s}^{\lambda - 1} + \|\bar{u}\|_{X_{\alpha,\beta}^s}^{\lambda - 1} \right]
\]
\[
= C t^{-\alpha} \|u - \bar{u}\|_{X_{\alpha,\beta}^s} \left[ \|u\|_{X_{\alpha,\beta}^s}^{\lambda - 1} + \|\bar{u}\|_{X_{\alpha,\beta}^s}^{\lambda - 1} \right].
\]

Multiplying the last inequality by \( t^\alpha \) we have
\[
t^\alpha \left\| \int_0^t A_{\theta}(t-\tau)(|u|^{\lambda} - |\bar{u}|^\lambda) \, d\tau \right\|_{H_p^s} \leq C \|u - \bar{u}\|_{X_{\alpha,\beta}^s} \left[ \|u\|_{X_{\alpha,\beta}^s}^{\lambda - 1} + \|\bar{u}\|_{X_{\alpha,\beta}^s}^{\lambda - 1} \right].
\]  
(4.9)
On the other hand, from Lemma 3.8, the embedding \( H^s_p \subset H^{s-\sigma}_q \), Lemma 3.11, and recalling the integrability of the Beta function, we get

\[
\left\| \int_0^t \partial_t \Lambda_\theta(t - \tau) (|u|^{\lambda} - |\tilde{u}|^{\lambda}) d\tau \right\|_{H^{s-\sigma}_p} \leq C \int_0^t (t - \tau)^{-\frac{s}{2}(1 - \frac{2}{p})} \left( \|u\|_{H^{s-\sigma}_p}^{\lambda} + \|\tilde{u}\|_{H^{s-\sigma}_p}^{\lambda} \right) d\tau
\]

\[
\leq C \int_0^t (t - \tau)^{-\frac{s}{2}(1 - \frac{2}{p})} \|u - \tilde{u}\|_{H^{s-\sigma}_p} (\|u\|_{H^{s-\sigma}_p}^{\lambda-1} + \|\tilde{u}\|_{H^{s-\sigma}_p}^{\lambda-1}) d\tau
\]

\[
\leq C \|u - \tilde{u}\|_{X^{s,\beta}_{\alpha,\beta}} (\|u\|_{X^{s,\beta}_{\alpha,\beta}}^{\lambda-1} + \|\tilde{u}\|_{X^{s,\beta}_{\alpha,\beta}}^{\lambda-1}) \int_0^t (t - \tau)^{-\frac{s}{2}(1 - \frac{2}{p})} \tau^{-\alpha\lambda} d\tau
\]

Multiplying by \( t^\beta \), we obtain

\[
t^\beta \left\| \int_0^t \partial_t \Lambda_\theta(t - \tau) (|u|^{\lambda} - |\tilde{u}|^{\lambda}) d\tau \right\|_{H^{s-\sigma}_p} \leq C \|u - \tilde{u}\|_{X^{s,\beta}_{\alpha,\beta}} (\|u\|_{X^{s,\beta}_{\alpha,\beta}}^{\lambda-1} + \|\tilde{u}\|_{X^{s,\beta}_{\alpha,\beta}}^{\lambda-1}). \tag{4.10}
\]

From (4.9) and (4.10), taking the supremum for \( t > 0 \), we obtain the desired result.

\[ \square \]

**4.1. Proof of Theorem 2.1**

**Proof.** We consider the closed ball

\[ B_{\delta_1} = \{ u \in C([0, \infty), H^s(\mathbb{R}^n)) \cap C((0, \infty); L^\infty(\mathbb{R}^n)) \cap C^1([0, \infty), H^{s-1}(\mathbb{R}^n)) : \|u\|_{Y^{s,\alpha_1}} \leq \delta_1 \}, \quad \delta_1 > 0, \]

endowed with the complete metric \( d(\cdot, \cdot) \), defined by \( d(u, \tilde{u}) = \|u - \tilde{u}\|_{Y^{s,\alpha_1}} \). Then, we prove that the map \( \Phi \) defined by

\[
\Phi(u) = \partial_t S(t) u_0(x) + S(t) \Delta u_1(x) - \int_0^t \Lambda(t - \tau) \|u(\tau)|^{\lambda} d\tau,
\]

is a contraction on \( (B_{\delta_1}, d) \). Let \( \delta_1 > 0 \) such that \( 2C_1\delta_1^{\lambda-1} < 1 \), being \( C_1 \) the constant in (4.1). From the assumption on the initial data, Lemma 3.4, Corollary 3.5, and Proposition 4.1 with \( \tilde{u} = 0 \), and \( \theta = 1 \), we
obtain (for all \( u \in B_{\delta} \)) that
\[
\|\Phi(u)\|_{Y^{\alpha,1}} \leq \|\partial_t S(t)u_0(x) + S(t)\Delta u_1(x)\|_{Y^{\alpha,1}} + \left\| \int_0^t \Lambda(t-\tau)|u(x,\tau)|^{\lambda}d\tau \right\|_{Y^{\alpha,1}}
\]
\[
\leq \sup_{0 < t < \infty} ((1 + t)^{\frac{\theta}{2}}(\|\partial_t S(t)u_0\|_{L^\infty} + \|\partial_t S(t)u_0\|_{H^s} + \|\partial_t^2 S(t)u_0\|_{H^{s-1}}) + \sup_{0 < t < \infty} ((1 + t)^{\frac{\theta}{2}}\|S(t)\Delta u_1\|_{L^\infty} + \|S(t)\Delta u_1\|_{H^s} + \|\partial_t S(t)\Delta u_1\|_{H^{s-1}}) + C_1\|u\|_{Y^{\alpha,1}}^\lambda
\]
\[
\leq \sup_{0 < t < \infty} ((\|u_0\|_{L^1} + \|u_0\|_{H^{s+1}} + \|\partial_t S(t)u_0\|_{H^s} + \|\partial_t^2 S(t)u_0\|_{H^{s-1}}) + \sup_{0 < t < \infty} ((\|u_1\|_{L^1} + \|u_1\|_{H^{s+2}} + \|S(t)\Delta u_1\|_{H^s} + \|\partial_t S(t)\Delta u_1\|_{H^{s-1}}) + C_1\|u\|_{Y^{\alpha,1}}^\lambda
\]
\[
\leq C((\|u_0\|_{L^1} + \|u_0\|_{H^{s+1}} + \|u_1\|_{L^1} + \|u_1\|_{H^{s+2}}) + C_1\delta_1^\lambda \leq \delta_1.
\]
Thus, \( \Phi(B_{\delta_1}) \subset B_{\delta_1} \). Now, taking \( u, \tilde{u} \in B_{\delta_1} \), from Proposition 4.1 we get
\[
\|\Phi(u) - \Phi(\tilde{u})\|_{Y^{\alpha,1}} \leq C_1\|u - \tilde{u}\|_{Y^{\alpha,1}}(\|u\|_{Y^{\alpha,1}}^{\lambda - 1} + \|\tilde{u}\|_{Y^{\alpha,1}}^{\lambda - 1})
\]
\[
\leq 2C_1\delta_1^{\lambda - 1}\|u - \tilde{u}\|_{Y^{\alpha,1}}.
\](4.11)

Since \( 2C_1\delta_1^{\lambda - 1} < 1 \), the map \( \Phi \) is a contraction on \( B_{\delta_1} \). Consequently, we have a unique fixed point in \( B_{\delta_1} \), which is the unique solution \( u \), satisfying \( \|u\|_{Y^{\alpha,1}} \leq \delta_1 \) of the integral equation (2.2). Finally, the time continuity of the solution can be obtained in the standard way; therefore we omit it (see for instance Banquet and Villamizar-Roa [2]).

4.2. Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 is based on a fixed point argument. We prove that the map \( \Phi_{\theta} \) defined by
\[
\Phi_{\theta}(u) = \partial_t S(t)u_0(x) + S(t)\Delta u_1(x) - \int_0^t S(t - \tau)(I - \Delta)^{-1}(-\Delta)^{\theta}|u(x,\tau)|^{\lambda}d\tau,
\]
is a contraction on the closed ball \( B_{\delta_2} \) given by
\[
B_{\delta_2} = \{ u \in C([0,\infty) : H^s_p(R^n)) \cap C^1([0,\infty) : H^{s-1}_p(R^n)) : \|u\|_{X^{\alpha,\beta}} \leq \delta_2 \}, \delta_2 > 0,
\]
endowed with the complete metric \( d(\cdot,\cdot) \) defined by \( d(u, \tilde{u}) = \|u - \tilde{u}\|_{X^{\alpha,\beta}} \). Let \( \delta_2 > 0 \) such that \( 2C_2\delta_2^{\lambda - 1} < 1 \), where \( C_2 > 0 \) is the constant in (4.8). From the assumption on the initial data and Proposition 4.2 with \( \tilde{u} = 0 \), we get (for all \( u \in B_{\delta_2} \))
\[
\|\Phi_{\theta}(u)\|_{X^{\alpha,\beta}} \leq \|\partial_t S(t)u_0\|_{H^s_p} + \|S(t)\Delta u_1\|_{H^{s-1}_p} + \left\| \int_0^t S(t - \tau)(I - \Delta)^{-1}(-\Delta)^{\theta}|u(\cdot,\tau)|^{\lambda}d\tau \right\|_{X^{\alpha,\beta}}
\]
\[
\leq \sup_{0 < t < \infty} (\|\partial_t S(t)u_0\|_{H^s_p} + \|S(t)\Delta u_1\|_{H^{s-1}_p}) + \sup_{0 < t < \infty} (\|\partial_t^2 S(t)u_0\|_{H^{s-1}_p} + \|\partial_t S(t)\Delta u_1\|_{H^{s-2}_p}) + C_2\|u\|_{X^{\alpha,\beta}}^\lambda
\]
\[
\leq \frac{\delta_2}{2} + C_2\delta_2^{\lambda} \leq \delta_2.
\]
Thus, $\Phi_\delta (B_{\delta_2}) \subset B_{\delta_2}$. Now, taking $u, \tilde{u} \in B_{\delta_2}$, from Proposition 4.2 we get
\[
\| \Phi_\delta (u) - \Phi_\delta (\tilde{u}) \|_{X^p_{\alpha,\beta}} \leq C_2 \| u - \tilde{u} \|_{X^p_{\alpha,\beta}} (\| u \|_{X^p_{\alpha,\beta}}^{\lambda-1} + \| \tilde{u} \|_{X^p_{\alpha,\beta}}^{\lambda-1}) \\
\leq 2C_2 \delta_2^{\lambda-1} \| u - \tilde{u} \|_{X^p_{\alpha,\beta}}.
\]
Since $2C_2 \delta_2^{\lambda-1} < 1$, the map $\Phi$ is a contraction on $B_{\delta_2}$. Consequently, we have a unique fixed point in $B_{\delta_2}$, which is the unique solution $u$, satisfying $\| u \|_{X^p_{\alpha,\beta}} \leq \delta_2$ of the integral equation (2.2). Finally, the time continuity of the solution can be obtained in the standard way; therefore we omit it (see for instance Banquet and Villamizar-Roa [2]).

\[\square\]

5. Local solutions

Before proving the existence of local solutions, we establish some estimates of the nonlinear term of integro-differential equation (2.2) in the norm
\[
\| u \|_{Z^p_{T}} := \sup_{0 < t < T} \int_0^t \| u \|_{H^p} \, \| u(t) \|_{H^1}.
\]

Proposition 5.1. Let $\lambda \geq 2$, $0 < T < 1$, $\theta \in (\frac{2-n}{2},1]$ if $n = 1,2$, and $\theta \in [0,1]$ if $n \geq 3$. Assume $2 \leq p \leq q \leq \infty$, and consider $\sigma, s$ such that $s > \sigma, n(\frac{1}{p} - \frac{1}{q}) \leq \sigma < 3 - n - 2\theta, (\frac{2}{q} + \frac{1}{p} - 1)\frac{n}{\lambda-1} + \sigma \leq \lambda < \min\{ \frac{n}{2}, \lambda - 1 \}$ and $\sigma$ and, $\frac{n}{2} (1 - \frac{2}{p}) \lambda < 1$. Then, there exists $C_3 > 0$ such that
\[
\left\| \int_0^t \Lambda_\delta (t - \tau) (| u \|^{ \lambda} - | \tilde{u} \|^{ \lambda}) \, d\tau \right\|_{Z^p_{T}} \leq C_3 T^{2 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda} \left\| u - \tilde{u} \|_{Z^p_{T}} \right\| (\| u \|^{ \lambda-1}_{Z^p_{T}} + \| \tilde{u} \|^{ \lambda-1}_{Z^p_{T}}).
\] (5.1)

Proof. From Lemma 3.6, the embedding $H^s_p \subset H^{s-\sigma}_q$, Lemma 3.11, and recalling the integrability of the Beta function (notice that $1 > \frac{n}{2} (1 - \frac{2}{p}) \lambda$ and $\lambda \geq 2$, imply that $2 > \theta + \frac{n}{2} (1 - \frac{2}{p})$), we arrive at
\[
\left\| \int_0^t \Lambda_\delta (t - \tau) (| u \|^{ \lambda} - | \tilde{u} \|^{ \lambda}) \, d\tau \right\|_{H^p} \leq C \int_0^t (t - \tau)^{1 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda} \left\| u \|^{ \lambda-1}_{H^p} + \| \tilde{u} \|^{ \lambda-1}_{H^p} \right\| \, d\tau
\]
\[
\leq C \int_0^t (t - \tau)^{1 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda} \left\| u \|^{ \lambda-1}_{H^p} + \| \tilde{u} \|^{ \lambda-1}_{H^p} \right\| \, d\tau
\]
\[
\leq C \left\| u - \tilde{u} \|_{Z^p_{T}} \right\| (\| u \|^{ \lambda-1}_{Z^p_{T}} + \| \tilde{u} \|^{ \lambda-1}_{Z^p_{T}}) \int_0^t (t - \tau)^{1 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda} \, d\tau
\]
\[
\leq C \left[ \| u - \tilde{u} \|_{Z^p_{T}} \right\| (\| u \|^{ \lambda-1}_{Z^p_{T}} + \| \tilde{u} \|^{ \lambda-1}_{Z^p_{T}}) \int_0^t \frac{(t - \tau)^{1 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda}}{\lambda} \, d\tau
\]
\[
\leq C t^{2 - \theta - \frac{n}{2} (1 - \frac{2}{p}) (1 + \lambda)} \left\| u - \tilde{u} \|_{Z^p_{T}} \right\| (\| u \|^{ \lambda-1}_{Z^p_{T}} + \| \tilde{u} \|^{ \lambda-1}_{Z^p_{T}}).
\] (5.2)

Therefore,
\[
\sup_{0 < t < T} t^{\frac{n}{2} (1 - \frac{2}{p}) \lambda} \left\| \int_0^t \Lambda_\delta (t - \tau) (| u \|^{ \lambda} - | \tilde{u} \|^{ \lambda}) \, d\tau \right\|_{H^p} \leq C T^{2 - \theta - \frac{n}{2} (1 - \frac{2}{p}) \lambda} \left\| u - \tilde{u} \|_{Z^p_{T}} \right\| (\| u \|^{ \lambda-1}_{Z^p_{T}} + \| \tilde{u} \|^{ \lambda-1}_{Z^p_{T}}).
\] (5.2)
On the other hand, from Lemma 3.8, the embedding \( H_p^s \subset H_Q^{s-\sigma} \) and Lemma 3.11, we get

\[
\left\| \int_0^t \partial_t \Lambda_{\theta}(t-\tau)(|u|^\lambda - |\tilde{u}|^\lambda) d\tau \right\|_{H_p^{s-1}} \leq C \int_0^t (t-\tau)^{-\frac{\theta}{2}(1-\frac{2}{p})} \|(|u|^\lambda - |\tilde{u}|^\lambda)\|_{H_p^{s-\sigma}} d\tau
\]

\[
\leq C \int_0^t (t-\tau)^{-\frac{\theta}{2}(1-\frac{2}{p})} \|u - \tilde{u}\|_{H_Q^{s-\sigma}} (\|u\|_{H_Q^{s-\sigma}}^{\lambda-1} + \|\tilde{u}\|_{H_Q^{s-\sigma}}^{\lambda-1})
\]

\[
\leq C \int_0^t (t-\tau)^{-\frac{\theta}{2}(1-\frac{2}{p})} \|u - \tilde{u}\|_{H_p^{s-\sigma}} (\|u\|_{H_p^{s-\sigma}}^{\lambda-1} + \|\tilde{u}\|_{H_p^{s-\sigma}}^{\lambda-1})
\]

\[
\leq C\|u - \tilde{u}\|_{Z_T^{s,p}} \int_0^t (t-\tau)^{-\frac{\theta}{2}(1-\frac{2}{p})} \|u - \tilde{u}\|_{Z_T^{s,p}} (\|u\|_{Z_T^{s,p}}^{\lambda-1} + \|\tilde{u}\|_{Z_T^{s,p}}^{\lambda-1})
\]

Therefore,

\[
\sup_{0<t<T} t^\frac{\theta}{2}(1-\frac{2}{p}) \left\| \int_0^t \Lambda_{\theta}(t-\tau)(|u|^\lambda - |\tilde{u}|^\lambda) d\tau \right\|_{H_p^{s-1}} \leq CT^1 \frac{\theta}{2}(1-\frac{2}{p}) \lambda \|u - \tilde{u}\|_{Z_T^{s,p}} (\|u\|_{Z_T^{s,p}}^{\lambda-1} + \|\tilde{u}\|_{Z_T^{s,p}}^{\lambda-1}).
\]

From (5.2) and (5.3) we obtain the desired result. \( \square \)

### 5.1. Proof of Theorem 2.3

Proof. The proof of Theorem 2.3 is based on a fixed point argument. We will prove that the map \( \Phi_{\theta} \) defined by

\[
\Phi_{\theta}(u) = \partial_t S(t)u_0(x) + S(t)u_1(x) - \int_0^t S(t-\tau)(I - \Delta)^{-1}(-\Delta)^{\theta}u(x, \tau) |^\lambda d\tau,
\]

is a contraction on the closed ball \( B_R \), given by

\[
B_R = \{ u \in C([0, T], H_p^s(\mathbb{R}^n)) \cap C^1([0, T], H_p^{s-1}(\mathbb{R}^n)) : \|u\|_{Z_T^{s,p}} \leq R \}, \quad R > 0.
\]

From Lemma 3.10 and Proposition 5.1 with \( \tilde{u} = 0 \), we get

\[
\|\Phi_{\theta}(u)\|_{Z_T^{s,p}} \leq \|\partial_t S(t)u_0(x) + S(t)u_1(x)\|_{Z_T^{s,p}} + \int_0^t S(t-\tau)(I - \Delta)^{-1}(-\Delta)^{\theta}u(x, \tau) |^\lambda d\tau
\]

\[
\leq \sup_{0<t<T} t^\frac{\theta}{2}(1-\frac{2}{p}) (\|\partial_t S(t)u_0\|_{H_p^s} + \|\partial_t^2 S(t)u_0\|_{H_p^{s-1}})
\]

\[
+ \sup_{0<t<T} t^\frac{\theta}{2}(1-\frac{2}{p}) (\|S(t)\Delta u_1\|_{H_p^s} + \|\partial_t S(t)\Delta u_1\|_{H_p^{s-1}})
\]

\[+ C_3 T^{2-\theta-\frac{\theta}{2}(1-\frac{2}{p})} R^\lambda
\]

\[\leq C (\|u_0\|_{H_p^{s+1-\sigma}} + \|u_1\|_{H_p^{s+2-\sigma}}) + C_3 T^{2-\theta-\frac{\theta}{2}(1-\frac{2}{p})} R^\lambda.
\]
Now we take $R = 2C(\|u_0\|_{H^{s+1}} + \|u_1\|_{H^{s+2}}) > 0$ y $T > 0$ such that

$$C_3 T^{2-\theta - \frac{2}{p}(1-\frac{2}{p})\lambda} R^{\lambda - 1} < \frac{1}{2}.$$ 

Thus, from (5.4) we get $\|\Phi_\theta(u)\|_{Z^p_T} \leq \frac{R}{2} + \frac{R}{2} = R$, for $u \in B_R$, that is, $\Phi_\theta(B_R) \subseteq B_R$. On the other hand, using again Proposition 5.1, we also have

$$\|\Phi_\theta(u) - \Phi_\theta(\tilde{u})\|_{Z^p_T} \leq C_3 T^{2-\theta - \frac{2}{p}(1-\frac{2}{p})\lambda} 2R^{\lambda - 1} \|u - \tilde{u}\|_{Z^p_T},$$

for all $u, \tilde{u} \in B_R$. Consequently, $\Phi_\theta$ is a contraction in $B_R$ and then the Banach fixed point theorem ensures the existence of a unique solution $u \in Z^p_T$ of (1.1). Finally, the time continuity of the solution can be obtained in the standard way; therefore we omit it (see for instance Banquet and Villamizar-Roa [2]).

\[\Box\]

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