AdS/SCFT in Superspace

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Abstract

A discussion of the AdS/CFT correspondence in IIB is given in a superspace context. The main emphasis is on the properties of SCFT correlators on the boundary which are studied using harmonic superspace techniques. These techniques provide the easiest way of implementing the superconformal Ward identities. The Ward identities, together with analyticity, can be used to give a compelling argument in support of the non-renormalisation theorems for two- and three-point functions, and to establish the triviality of extremal and next-to-extremal correlation functions. Further simplifications in other correlators are also implied. The OPE is also briefly discussed.
1 Introduction

One of the key points of the AdS/CFT conjecture \cite{1} is that the isometry group of AdS spacetime coincides with the conformal group of the boundary Minkowski space. In the supersymmetric context, these groups are extended to supergroups, and their actions as isometries and superconformal transformations are most easily presented in superspace. In this paper we shall be mainly concerned with IIB supergravity on $AdS_5 \times S^5$ and its relation to $N = 4$ super Yang-Mills theory (SYM) on the boundary. In this case the supergroup in question is $PSU(2, 2|4)$.

A brief discussion of the basic geometrical set-up and the KK supermultiplets is given in section 2 and the rest of the paper is devoted to a review of the analytic superspace approach to correlation functions of the composite operators which correspond to the KK supermultiplets in the AdS/CFT context.

2 KK supermultiplets

The basic geometrical set-up can be summarised by the following diagram

$$\begin{align*}
AdS^{5,5|32} & \rightsquigarrow M^{4|16} \times S^5 \\
\downarrow & \\
AdS^{5|32} & \rightsquigarrow M^{4|16}
\end{align*}$$

where the squiggly arrows denote passing to the boundary. Each of these superspaces is a coset space of the supergroup $PSU(2, 2|4)$, with the notation indicating the (even|odd) dimensions \cite{2, 3}. Thus $AdS^{5,5|32}$ denotes the superspace whose body is $AdS_5 \times S^5$ and which has 32 odd dimensions while $AdS^{5|32}$ is the $D = 5, N = 8$ superspace which has $AdS_5$ for its body and which has 32 odd dimensions. The boundary space on the bottom row is $D = 4, N = 4$ Minkowski superspace. The isotropy group of $AdS^{5,5|32}$ is $Spin(1,4) \times USp(4)$ while that of $AdS^{5|32}$ is $Spin(1,4) \times SU(4)$ and the former fibres over the latter with fibre $S^5$. The boundary superspaces are most easily described by viewing $PSU(2,2|4)$ as the $D = 4, N = 4$ superconformal group. The generators of the corresponding Lie superalgebras are $\{D, P, K, M, N; Q, S\}$ standing for dilations ($D$), translations ($P$), Lorentz transformations ($M$), internal $SU(4)$ transformations ($N$), and $Q$ and $S$-supersymmetry transformations. The isotropy group of $M^{4|16}$ is generated by $\{D, K, M, N; S\}$ while the isotropy group of $M^{4|16} \times S^5$ is generated by $\{D, K, M, N'; S\}$, where $N'$ denotes the generators of the $USp(4)$ subgroup of $SU(4)$.

Linearised type IIB supergravity is described by a chiral field strength superfield $A$ satisfying an additional fourth-order constraint which is schematically of the form $D^4A \sim D^2A$. This is the case for both Minkowski and $AdS^{5,5|32}$ backgrounds \cite{3}. In the latter case we can expand this superfield in harmonics on $S^5$ to obtain a family of chiral superfields on $AdS^{5|32}$ which fall into symmetric traceless representations of $SO(6)$, the isometry group of $S^5$. Passing to the boundary, $D = 4, N = 4$ super Minkowski space, we find a family of $D = 4, N = 4$ chiral field strength superfields which are now off-shell, and which also satisfy fourth-order constraints. These fields can be expressed in terms of a family of unconstrained Grassmann-analytic prepotentials on an appropriate harmonic superspace, which will be discussed below, and these prepotentials couple naturally to composite operators of the boundary SCFT which are both G-analytic and harmonic analytic \cite{3}. In fact they are operators of the form $\text{tr}(W^p)$, $p = 2, 3, \ldots$ where $W$ is the SYM field strength superfield. This identification of the KK multiplets \cite{3} and SCFT multiplets was first made in \cite{6}.  

2
3 Harmonic superspace

In extended supersymmetry flat harmonic superspaces \[ \mathcal{M}_H = M \times F \] where \( M \) is the corresponding Minkowski superspace and \( F \) is a coset space of the internal symmetry group which is chosen to be a particular type of compact, complex manifold. For example, in \( D = 4 \), the internal symmetry group is \( SU(N) \) and the possible internal manifolds of interest are the flag manifolds \( \mathbb{F}_{k_1, \ldots, k_l} = S(U(k_1) \times U(k_2 - k_1) \times \cdots \times U(N - k_l)) / SU(N) \), where \( k_1 < k_2 \ldots < k_l < N \) are positive integers. In particular the isotropy groups \( S(U(p) \times U(N - (p+q)) \times U(q)) \) define \((N, p, q)\) harmonic superspaces \( \mathcal{M} \), a family of superspaces which generalises in a natural way the harmonic superspaces for \( N = 2 \) and \( N = 3 \) supersymmetry introduced by GIKOS, which are respectively \( (2, 1, 1) \) and \((3, 1, 1)\) harmonic superspace in the above notation.\[ \]

The harmonic superspace approach of GIKOS emphasises the group theoretic aspects of fields defined on such spaces rather than the holomorphic aspects. For this reason it has become standard practice to work on the space \( \mathcal{M}_H = M \times SU(N) \). This is equivalent to working on \( M_H \) provided that the fields are restricted to be equivariant with respect to the isotropy group, \( H \), of the relevant coset space.

In this paper we shall be concerned with \( D = 4 \) \((N, p, q)\) harmonic superspaces for the cases \((4, 2, 2)\) and \((2, 1, 1)\). A group element \( u \) is written in index notation as \( u_i^J \) where the \( i \) index is acted on by \( SU(N) \) and the \( J \) index by \( H \), and the inverse \( u^{-1} \) is denoted \( u_i^J \). The index \( I \) splits under \( H \) as \( (r, r') \) where \( r = 1, \ldots, p, r' = p+1, \ldots, N \), \( p \) being either \( 2 \), for \( N = 4 \), or \( 1 \) for \( N = 2 \). With the aid of \( u \) and its inverse \( SU(N) \) indices \( i, j, \ldots \) can be converted into \( H \) indices \( I, J, \ldots \) in an obvious manner. In particular we can construct the Grassmann odd derivatives \( D_{\alpha r} = (D_{\alpha r}, D_{\alpha r'}) \) and \( \bar{D}_\alpha = (\bar{D}_\alpha^r, \bar{D}_\alpha^{r'}) \).

The derivatives \( (D_{\alpha r}, \bar{D}_\alpha) \) anticommute and so allow the introduction of generalised chiral fields, or Grassmann-analytic (G-analytic) fields, \( F \), which satisfy

\[
D_{\alpha r} F = \bar{D}_\alpha^r F = 0
\]

For any fixed values of the \( u \)'s the derivatives \( (D_{\alpha r}, \bar{D}_\alpha) \) define a CR structure on \( M \); that is, they are basis vector fields for a subspace of the complexified tangent bundle \( T_c \) such that they anticommute and their complex conjugates are linearly independent of them at any point in \( M \). The space \( F \) can thus be viewed as the space of all CR structures of this type on \( M \). In addition, the derivatives can be combined with a subset of the right-invariant vector fields \( D_I^J \) on \( SU(N) \) to define a CR structure on \( M_H \). The right-invariant vector fields decompose under \( H \) into coset derivatives \( D_{r^s} \) (essentially \( \partial \) on \( F \)) and its conjugate \( D_{r^s}^* \), and isotropy group derivatives \( D_{r^s}, D_{r^s'} \) \((SU(2) \times SU(2) \) for the case \( N = 4 \)) and \( D_o \) \((U(1) \) in both cases).

The CR structure on \( M_H \) is then determined by the derivatives \( (D_{\alpha r}, D_{\alpha r'}, D_{r^s}) \). Since the \( D_I^J \) are characterised by

\[
D_I^Ju_K^k = \delta_K^J u_I^k - \frac{1}{N} \delta_I^J u_K^k
\]

\[
D_I^J u_K^k = -\delta_K^J u_I^k + \frac{1}{N} \delta_I^J u_K^k
\]

it is straightforward to verify that this set of derivatives does indeed specify a CR structure. On \( M_H \) we therefore have two types of analyticity - Grassmann analyticity (G-analyticity) and harmonic analyticity\[ 1(3,2,1) \) superspace was first discussed in \[ \]
\[ 2\) The role of CR structures in harmonic superspace was first emphasised by \[ .\]
(H-analyticity), the latter meaning analyticity in the usual sense on $\mathbb{F}$. A field which is both G-analytic and H-analytic will be called CR-analytic, or simply analytic, for short.

$N = 2$ harmonic superspace was used by GIKOS to give an off-shell version of the hypermultiplet (which cannot be formulated in ordinary superspace with a finite number of auxiliary fields). GIKOS also reformulated $N = 2$ Yang-Mills theory in harmonic superspace \[1\]. In this case there is still a finite number of auxiliary fields (a triplet of dimension two scalars) but the gauge group is extended and it is this extension that facilitates the solution of the $N = 2$ SYM superspace constraints in terms of a dimension zero G-analytic prepotential. The $N = 3$ theory can also be given an off-shell formulation in harmonic superspace but in this case there is both an infinite number of auxiliary fields and an extended gauge group \[10\]. We note that, although the on-shell $N = 3$ and $N = 4$ SYM theories are the same, there is no known off-shell version which has manifest $N = 4$ supersymmetry.

In the present context we are interested in correlation functions of gauge-invariant composite operators which are constructed as polynomials in the field strength superfields. We shall make the assumption that, provided that we restrict our attention to separated points, it should be possible to use the equations of motion for the SYM fields and thus we shall only need to have an on-shell formulation of the underlying theory. We shall consider both $N = 2$ and $N = 4$ SYM theories, partially because it is sometimes convenient to write the latter in terms of $N = 2$ superfields (in particular, this allows checks to be carried out in perturbation theory which is not possible using on-shell $N = 4$ superfields), and partially because some of the results we shall obtain are valid for more general $N = 2$ superconformal theories.

The fields for $N = 2$ Yang-Mills are the chiral field strength $W$, $\nabla^\alpha W = 0$ and the hypermultiplet $q$, which is best viewed as a field on $N = 2$ harmonic superspace, $M_H = M \times \mathbb{C}P^1$. Off-shell the latter field is G-analytic, but on-shell it becomes H-analytic as well, and thus analytic in the above nomenclature. In the free theory $q = u^1q_i$, where $u^i = (u^1,u^2)$ for $N = 2$. It is not real, but under a suitable conjugation transforms into $\tilde{q} = u^1\tilde{q}_i$. In the non-Abelian theory it is still a short multiplet of the above type but is covariantly G-analytic. However, gauge-invariant products of the $q$’s and $\tilde{q}$’s are ordinarily analytic. We shall denote operators with $p$ powers of $q$ or $\tilde{q}$ by $O_p$.

For $N = 4$ Yang-Mills theory the field strength superfield $W_{ij} = -W_{ij}$ in super Minkowski space transforms according to the six-dimensional representation of the internal symmetry group $SU(4)$, so $W^{*ij} = \frac{1}{8}\varepsilon^{ijkl}W_{kl}$, and satisfies $\nabla_\alpha W_{jk} = \nabla_{(\alpha}W_{jk)}$. The claim is that $W_{ij}$ is equivalent to a charge 1 field $W$ on $M_H$ which is covariantly G-analytic and ordinarily $\mathbb{F}$-analytic; it is also real with respect to the real structure discussed above, where covariantly G-analytic means that

$$\nabla_\alpha W = \nabla^\alpha W = 0 \quad (4)$$

with $\nabla_\alpha = u_r^i\nabla_{\alpha i}$, etc. This claim is easily verified \[3\]. The convention for $U(1)$ charges is

$$D_\alpha u_r^i = \frac{1}{2}u_r^i; \quad D_\alpha u_r^{-i} = -\frac{1}{2}u_r^{-i} \quad (5)$$

The gauge-invariant operators $O_p := \text{tr}(W^p)$ are G-analytic in the usual sense and hence analytic. These are the conformal fields of interest in the Maldacena conjecture. Since they are in short representations of $SU(2,2|4)$, the integer $p$ cannot be affected by quantum corrections and so, since this integer determines the dimension, they do not have anomalous dimensions. This family of operators was introduced in \[4\] and it has been shown that it is in one-to-one correspondence with the Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5$ \[5\]. In particular, the family of operators includes the energy-momentum tensor $T = O_2$, first presented as a harmonic superfield in \[6\]. We shall also be able to study correlators of gauge-invariant products of hypermultiplets in $N = 2$ theories; again, these operators are analytic fields.

\[3\]The field strength can also be viewed as an analytic field on $M \times U(1)^3\setminus SU(4)$ \[13\], but this description is not so convenient from the point of view of superconformal transformations if one wishes to take maximal advantage of G-analyticity.
4 Correlators in analytic superspace

Analytic superspaces are related to harmonic superspaces rather than chiral superspaces are related to Minkowski superspaces. Such spaces are intrinsically complex and are not coset spaces of the real superconformal group. From the point of view of manifest superconformal invariance it is therefore convenient to complexify both spacetime and the superconformal group (which becomes $SL(4|N)$). An analytic superfield on harmonic superspace can then be rewritten as an unconstrained, holomorphic field on analytic superspace. We emphasise that this is simply a matter of convenience; we can return to real spacetime by imposing suitable reality conditions on the coordinates. It is to be noted that the internal flag space remains unchanged in this construction. The key point is that correlation functions of analytic operators must be analytic in the internal coordinates $y$, because we can write each operator explicitly as a polynomial in $y$ with coefficients which depend on the coordinates of Minkowski superspace. On the other hand, as we shall see below, any analytic invariant has singularities in the $y$’s. This circumstance places constraints on the correlators which are strong enough in some cases to determine them completely, as conjectured in [14].

We shall be concerned with the analytic superspaces associated with $(4, 2, 2)$ and $(2, 1, 1)$ harmonic superspaces. On these analytic superspaces the coordinates are

$$X^{AA'} = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \lambda^{\alpha\dot{\alpha}} \\ \pi^{a\dot{a}} & y^{aa'} \end{pmatrix}$$

(6)

Here $\alpha$ and $\dot{\alpha}$ are two-component spinor indices, while $a$ and $a'$ are internal “spinor” indices; in $N = 2$ these indices take on one value each and so can be omitted, while in $N = 4$ they take on two values and behave in a similar manner to the spacetime indices.

An analytic field $\mathcal{O}_p$ of charge $p$ transforms under superconformal transformations according to the rule

$$\delta \mathcal{O}_p = V \mathcal{O}_p + p \Delta \mathcal{O}_p$$

(7)

where the vector field $V = \delta X \frac{\partial}{\partial X}$ is determined by an infinitesimal superconformal transformation $\delta X$

$$\delta X = B + AX + XD + XCX$$

(8)

and where the function $\Delta = \text{str}(A + XC)$.

The supermatrix parameters $A, B, C$ and $D$ together make up an element $\mathcal{A}$ of the Lie superalgebra $\mathfrak{sl}(4|N)$,

$$\mathcal{A} = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix}$$

(9)

In the $N = 4$ case the term in $\mathcal{A}$ proportional to the unit matrix acts trivially on $X$ so that we indeed have an action of $\mathfrak{psl}(4|4)$.

We shall be interested in correlation functions of the form

$$\langle p_1 \ldots p_n \rangle := \langle \mathcal{O}_{p_1}(X_1) \ldots \mathcal{O}_{p_n}(X_n) \rangle$$

(10)

The Ward identity for such a correlation function, which states that it is invariant under superconformal transformations, is as follows [14]:

$$\sum_{i=1}^{n} (V_i + p_i \Delta_i) \langle p_1 \ldots p_n \rangle = 0$$

(11)
where the index $i$ refers to the point $X_i$ of the operator $O_{p_i}$.

The basic building block which one uses to analyse such correlators is the two-point function for charge one operators in the free theory, that is, $< WW >$ in $N = 4$, and $< q \tilde{q} >$ in $N = 2$, where $q$ is a hypermultiplet and $\tilde{q}$ its harmonic conjugate. This two-point function, which we shall refer to as a propagator and denote by $g_{12}$, has the following form

$$g_{12} = \begin{cases} \frac{\hat{y}_{12}}{x_{12}}, & \text{for } N = 2 \\ \frac{\hat{y}_{12}^2}{x_{12}}, & \text{for } N = 4 \end{cases}$$

where

$$\hat{y}_{12} = y_{12} - \pi_{12} x_{12}^{-1} \lambda_{12}$$

with $X_{12} := X_1 - X_2$ as usual. This formula is to be understood in the sense of matrix multiplication with $x^{-1}$ being assigned subscript indices $\dot{\alpha} \alpha$.

An arbitrary correlation function of analytic fields, as long as it is non-vanishing at lowest order in an expansion in the odd coordinates, can then be expressed as a product of propagators times a function $F$ of the $n$ coordinates which is superconformally invariant,

$$< p_1 p_2 \ldots p_n > = \prod_{i<j} (g_{ij})^{p_{ij}} F$$

The powers $p_{ij}$ must be chosen such that the prefactor multiplying $F$ on the right-hand side solves the Ward identities for the given charges; in general there will be different ways of doing this.

As we remarked above, any such correlation function must be analytic in the internal coordinates $y$, because we can write each operator explicitly as a polynomial in $y$ with coefficients which depend on the coordinates of Minkowski superspace. On the other hand, any invariant has singularities in the $y$'s. This circumstance places constraints on the functional form of $F$ - the only singularities in $y$ it can have must be those which can be cancelled by the zeroes in the propagators. It was conjectured in [14] that these constraints could be strong enough to completely determine some particular types of correlation functions and this is indeed the case as we shall demonstrate below.

## 5 Invariants

In [13] a method for constructing non-nilpotent invariants in analytic superspace was given. However, it was pointed out in [16] that the $N = 4$ invariants given there are in fact invariant under an additional symmetry, $U(1)_Y$, and therefore under the group $PGL(4|4)$ and not just $PSL(4|4)$. In [17] it was shown that there are non-nilpotent invariants in $N = 4$ which are not invariant under $U_Y(1)$. Here we give a brief account of a systematic method which can be used to construct all the invariants. It is, in fact, a valid method for all Grassmannians of the form $G_n(2n)$ where $n$ can be either an integer (bosonic case) or a pair of integers (super case). It is based on the approach of [18]

Consider the transformation

$$\delta X_i = B + AX_i + X_i D + X_i C X_i, \ i = 1, \ldots n, \ \text{no sum on} \ i$$

where $X_i$ is the supercoordinate $X_i^{\dot{\alpha} A'}$ of the $i$th point. We are looking for functions $F(X_1, \ldots X_n)$ which are invariant under the above. We first solve for translations $B$. If we change coordinates to
\((X_1, X_{1i}), i = 2 \ldots n\) we find that \(F\) is independent of \(X_1\). Now consider the transformation of \(X_{1i}\) under \(C\),

\[
\delta_C X_{1i} = X_1 CX_i - X_i CX_1 - X_{1i} CX_{1i} + X_{1i} CX_1
\]

For the inverse we therefore have

\[
\delta X_{1i}^{-1} = C - (CX_1)_{X_{1i}}^{-1} - X_{1i}^{-1}(X_1 C)
\]

At this stage we can regard \(F\) as being a function of the \(n-1\) inverses \(X_{1i}^{-1}\) and change variables to \(X_{12}^{-1}\) and \(n-2\) variables \(Y_i\) defined by

\[
Y_i := X_{12}^{-1} - X_{1i}^{-1}, \quad i = 1, \ldots n-2
\]

We have

\[
\delta_C Y_i = -(CX_1)Y_i - Y_i (CX_1)
\]

and so the invariance of \(F\) under \(C\) implies

\[
\delta_C F = (C - (CX_1)X_{12}^{-1} - X_{12}^{-1}(X_1 C)) \frac{\partial F}{\partial X_{12}} + \sum_{i=3}^{n} (- (CX_1)Y_i - Y_i (CX_1)) \frac{\partial F}{\partial Y_i} = 0
\]

Now \(F\) is independent of \(X_1\) so the above should be valid for arbitrary values of this coordinate. Taking \(X_1 = 0\) we see that \(F\) does not depend on \(X_{12}^{-1}\). Thus \(F\) depends only on the \((n-2)\) \(Y_i\)'s and the residual transformation reduces to linear transformations of type \(A\) and \(D\). Hence, if \(F\), as a function of the \(Y_i\), is invariant under the linear \(A\) and \(D\) symmetries it will automatically be completely invariant. Now each of the parameter matrices \(A\) and \(D\) contain as many odd entries as there are odd coordinates. Solving the constraints on \(F\) due to the odd linear symmetries allows us to eliminate a further two sets of odd variables and thereby arrive at a set of \((n-2)\) sets of even coordinates \((x's\ and y's)\) and \((n-4)\) sets of odd coordinates \((\lambda's\ and \pi's)\). The final step is to construct functions from these which are invariant under standard bosonic linear symmetries which simply requires the indices to be hooked up in the right way and the weights to cancel.

From this analysis we can extract a few simple but important results \[17\]. There are no invariants for \(N = 2\) and \(N = 4\) analytic superspaces at two or three points, and the four-point invariants are non-nilpotent. The non-nilpotent invariants for any number of points, (or rather the elements of the quotient ring of invariants modulo the nilpotent ones), were constructed in \[15\]. They have the property of being power series in \(\lambda \pi\). The nilpotent invariants start at five points and in \(N = 2\) they again have to be power series in \(\lambda \pi\) due to the \(U(2)\) R-symmetry group. In \(N = 4\), however, this is not the case, because the R-symmetry group is \(SU(4)\) not \(U(4)\); the powers of \(\lambda\) and \(\pi\) in any term in any invariant need only be the same modulo 4, the latter restriction being due to invariance under the \(Z_4\) centre of \(SU(4)\). The existence of an \(N = 4\) five-point nilpotent invariant with leading term of the form \(\lambda^5\) was demonstrated in \[17\].

In \(N = 4\) these results imply that the 4-point invariants, and indeed all of the non-nilpotent invariants, have an additional symmetry, \(U_Y(1)\), which acts on the odd coordinates by \(\delta \lambda = \lambda, \delta \pi = -\pi\) with the
even coordinates being unchanged. Since this is also a symmetry of the basic propagators, it follows that $N = 4$ $n$-point functions for $n < 5$, are actually invariant under $PGL(4|4)$ and not just $PSL(4|4)$, as conjectured in [16] and confirmed in [17].

The non-nilpotent invariants are much easier to construct explicitly, as they can be expressed in terms of superdeterminants and supertraces of coordinate difference matrices. For example, there are three independent invariants at four points in $N = 2$ and they can be chosen to be

$$S = \frac{s\text{det}X_{14}s\text{det}X_{23}}{s\text{det}X_{12}s\text{det}X_{34}}, \quad T = \frac{s\text{det}X_{13}s\text{det}X_{24}}{s\text{det}X_{12}s\text{det}X_{34}} \quad (21)$$

and

$$U = \text{str}(X_{12}^{-1}X_{23}X_{34}^{-1}X_{41}) \quad (22)$$

They are all singular in the $y$ variables; for example, the leading terms in $S$ and $T$ are spacetime cross-ratios divided by internal cross-ratios.

6 2 and 3-point functions, anomalies and non-renormalisation

The forms of the two- and three-point functions are determined completely by superconformal invariance as one might expect. One has [14]

$$< p_1 p_2 > \sim \delta_{p_1 p_2} (g_{12})^{p_1} \quad (23)$$

and

$$< p_1 p_2 p_3 > \sim (g_{12})^{p_{12}} (g_{23})^{p_{23}} (g_{31})^{p_{31}} \quad (24)$$

where

$$p_{ij} = p_i + p_j - p_k \quad (25)$$

and where $k \neq i, j$.

Let us now consider the field $O_2 := T$ in $N = 4$. This field is the $N = 4$ supercurrent; it contains among its components the traceless conserved energy-momentum tensor and the conserved $SU(4)$ currents. These conservation laws and tracelessness conditions arise because the superfield is constrained. On the other hand, we know that even in a superconformal field theory some at least of these conditions will be spoiled in the quantum theory by anomalies, so that the three-point function cannot be analytic. The resolution to this is that it is only formally analytic - closer inspection reveals that it is necessary to regulate the $x$ singularities, and this is where the anomaly creeps in. For example, we can extract from the above formula for the three-point function the correlator for three $SU(4)$ currents. It agrees with the expression obtained in components in [19]. Directly testing the conservation of any one of these currents then produces a non-zero answer due to the well-known triangle graph [19].

In fact, the one-loop result for two and three-point functions of the supercurrent is exact as was shown in [19] using a result proven in [20]. Another way of thinking about this is to couple the theory to $N = 4$ conformal supergravity [21, 22]. Superspace non-renormalisation theorems indicate that the only divergences in this theory occur at one loop [23, 24]. Since these divergences are directly related to the
conformal anomaly [25], one arrives at the same conclusion [22]. This means that the coefficient of the two- and three-point functions of $T$ cannot depend on the coupling - in other words, the only contribution is at lowest order and all other contributions vanish.

7 The reduction formula and non-renormalisation for 2 and 3-point functions with arbitrary charges

Although the anomaly argument given above shows that the non-renormalisation theorem holds for the supercurrent, it has long been suspected that this must be the case for arbitrary analytic operators [26]. One way of seeing this in the $N = 4$ context is to use the reduction formula which relates the derivative of an $n$-point function to an $(n + 1)$-point function with an insertion of the integrated on-shell action. This formula was first proposed in the current context in [16]. Now the integrated on-shell action can be written as a superaction in $N = 4$ super Minkowski space [24], and this translates into a certain integral in analytic superspace, so that the reduction formula is

$$\frac{\partial}{\partial \tau} < p_1 \ldots p_n > \sim \int d\mu_0 < T(X_0)p_1 \ldots p_n >$$

(26)

The measure $d\mu$ is

$$d\mu := d^4 x d^4 y d^4 \lambda$$

(27)

or, in harmonic superspace notation, $d\mu = d^4 x d\mu(D_{\alpha r'})^4$.

We have seen previously that two-, three- and four-point functions do not involve nilpotent invariants and therefore only involve the odd coordinates in the combination $\lambda \pi$ (since the same is true for the propagators). For $n \leq 3$ any correlator on the left of (26) has a non-vanishing leading term in a power series expansion in the odd variables and so in order for there to be a non-vanishing contribution to its derivative with respect to the coupling there would have to be a contribution to the integrand on the right of (26) of the form $\lambda^4$ times some function of the even coordinates. Since there are no such terms we conclude that this derivative must vanish and hence that all two- and three-point analytic correlators are non-renormalised.

There is a slight loophole in the above argument and that is that the integral on the RHS of (26) means that contact terms could in principle contribute [21]. Such local terms could either be associated with the anomalies or they could be superconformally invariant contact terms. In [28] a search for invariant contact terms was made and none were found with the right fermion structure to contribute to the integral. We thus conclude that invariant contact terms cannot spoil the argument. On the other hand, the anomalous terms, at least in the case of the energy-momentum tensor, are already present in the three-point functions as we have seen. Although one has to be careful when dealing with the contact points arising from such expressions, it is nevertheless the case that their fermion structure is such that they cannot contribute to this sort of integral. It seems likely that this situation will also hold true for the higher-charge operators: any local terms contributing on the RHS would have to have a different fermion structure to that of the $U(1)_Y$-invariant expressions which are valid for three or four separated points. There is no evidence that invariant contact terms like this exist, and, on the other hand, it is to be expected that the anomalies should arise from the formally analytic expressions for separated points.

8 Extremal correlators

For four points or more the restrictions imposed on correlators by analyticity are most evident when the charges of the operators are chosen in an appropriate fashion. It was observed in AdS supergravity that
certain correlators seemed to be particularly simple in such cases \[30, 31, 32\]. A test of the Maldacena conjecture, therefore, is to check whether this is the case on the field theory side. These so-called extremal correlators are those for which one of the charges \(p_1\), say, is equal to the sum of all of the others,

\[ p_1 = \sum_{i=2}^{n} p_i \quad (28) \]

In the SCFT the prefactor \(P\) in \(< p_1 \ldots p_n > = PF\) can be written uniquely as

\[ P = \prod_{i=2}^{n} (g_{1i})^{p_i} \quad (29) \]

This in turn means that the function of invariants which multiplies the prefactor is only allowed to have singularities of a rather special type. Typically, the singularities that arise in the invariants are of the form \((y_{12}y_{34})^{-1}\) whereas the zeroes in \(P\) involve products of \(y\)'s where each factor has a subscript 1. This rather rough argument indicates how it can be that the only functions of invariants allowed for in extremal correlators are in fact constant. It is confirmed in more detail in \[33\] for four points and extended to \(n\) points in \[34\] using \(N = 2\) superfields and the reduction formula which also allows one to show that the constant does not depend on the coupling, in a similar fashion to the non-renormalisation theorems for two- and three-point functions. Moreover, it is also possible to show that next-to-extremal correlators, for which \(p_1 = (\sum_{i=2}^{n} p_i) - 2\), are also trivial and non-renormalised, and this has been confirmed on the supergravity side \[35\]. Similar results have been shown in M-theory on \(AdS_{7(4)} \times S^{4(7)}\) \[37\] and seem also to be true on the field theory side in \(D = 6(2, 0)\) SCFT \[36\].

9 4-point functions

Initially the analytic superspace programme focused on the analysis of correlators of four charge two hypermultiplet composites in \(N = 2\), and it was conjectured that for such low charges it might be possible to determine such functions completely by symmetry and analyticity. However, this has turned out not to be the case. We shall not give the details here but outline the main points.

The correlator to be studied is \(< 2222 >\), although the operators need not be identical (there are three such charge two operators in \(N = 4\), for example). It can be written in the form

\[ < 2222 > = (g_{12})^2(g_{34})^2F \quad (30) \]

where \(F\) is a function of invariants. At four points in \(N = 2\) there are three independent super-invariants which we discussed above and which correspond to the three purely bosonic invariants which arise at lowest order. These are two \(x\) cross-ratios \(s, t\) and one \(y\) cross-ratio \(v\), where

\[ s = \frac{x_{14}^2x_{23}^2}{x_{12}^2x_{34}^2}, \quad t = \frac{x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2} \quad (31) \]

and

\[ v = \frac{y_{14}y_{23}}{y_{12}y_{34}} \quad (32) \]

Due to the fact that the propagators behave as \((y_{12})^2(y_{34})^2\) it follows that \[38\]

\[ F = a_1(S', T') + a_2(S', T')V + a_3(S', T')V^2 \quad (33) \]
where \( S', T' \) are \( V \) are the three functions which (uniquely) extend \( s, t \) and \( v \) to super-invariants. Explicitly,

\[
S' = SV, \quad T' = T(1 + V), \quad V = \frac{T + U - 1}{1 + S - T}
\]

where the simpler invariants \( S, T \) and \( U \) are given in equations (24) and (25).

The idea is to expand \( F \) in a power series in odd variables and study each order to see what restrictions, if any, are placed on the functions \( a_i \). This can be done using a set of partially supersymmetric variables \( \Lambda, \Pi \) which are also analytic in \( y \). In terms of these variables the expansion goes up to fourth order in \( \Lambda \Pi \), but the only constraints that arise occur at first order. Alternatively, one can use harmonic variables. One finds that two combinations of the three functions \( a_i \) satisfy two coupled partial differential equations while \( a_3 \) is unconstrained \([38]\).

Although this result is not as strong as originally had been hoped, nevertheless analyticity does yield results at first order which go beyond what might expect from naive symmetry considerations. Recently this result has been slightly strengthened, in the \( N = 4 \) case, using symmetry considerations (under interchange of points) and the reduction formula \([39]\). The result is that the entire amplitude of four supercurrents is determined in terms of one function of \( s \) and \( t \).

### 10 The OPE and analytic superspace

In \([40]\) the OPE of \( N = 4 \) analytic operators was investigated and it was claimed that this should close in the sense that only such operators and their derivatives appear on the right-hand side. As we shall see, this is indeed the case, provided that we consider more general analytic tensor superfields, although at the time it was not clear how operators such as the Konishi operator could be accommodated in this formalism. In \([41]\) it was shown that this operator occurs in the OPE of two supercurrents of any \( N = 1 \) SCFT. Now the \( N = 4 \) Konishi multiplet is \( \text{tr}(W_{ij} W^{*ij}) \) and does not appear to be analytic; in fact, it is a long multiplet. However, it is very easy to see using harmonic superspace notation that it does occur in the OPE of two \( N = 4 \) supercurrents even in the free theory. Consider then the free theory where \( T = W^2 \) and \( W = u^{ij} W_{ij} \), with \( u^{ij} := e^{rs} u_{ri} u_{sj} \). \( K \) is absent in the product \( W^2 \) due to the fact that it can be written as \( \frac{1}{2} \epsilon^{ijkl} W_{ij} W_{kl} \), and so drops out because of the \( u \)’s. Now consider the OPE of two \( T \)’s; it will include a contribution which involves a single contraction, and this for the free theory is simply the propagator \( g_{12} \). Hence we find

\[
T(1)T(2) \sim g_{12} W(1)W(2) + \ldots
\]

and

\[
W(1)W(2) = u^{(1)}ij u^{(2)}kl W_{ij}(1)W_{kl}(2) = u^{(1)}ij u^{(2)}kl (W_{ij}(1)W_{kl}(1) + \ldots) = u^{(1)}ij u^{(2)}kl (T_{ij,kl}(1) + \epsilon_{ijkl} K(1) + \ldots)
\]

But, since the \( u \)’s are at different points we can no longer conclude that they are annihilated when contracted with \( \epsilon_{ijkl} \). Thus the Konishi operator is certainly present in the \( N = 4 \) OPE. At first sight, this seems to be a disaster for the analytic superspace method because the OPE seems to produce operators which are no longer analytic. However, this is not the case. It turns out that the Konishi operator can be written as an operator on analytic superspace provided that one is prepared to include more general superfields. It is simply an example of a superfield which has superindices, i.e. an analytic tensor superfield.
To see this explicitly consider the OPE for two T’s in analytic superspace. Looking again at single contractions we shall have a term

\[ T(1)T(2) \sim g_{12} W(1)W(2) \] (36)

but now we regard the W’s as functions of the analytic coordinates. We can now make a Taylor expansion about point 2 in all the coordinates. Up to second order this gives

\[ W(1)W(2) = T + X_{12}^{AA'}(\partial_{AA'}W)W + \frac{1}{2} X_{12}^{BB'} X_{12}^{AA'}(\partial_{AA'}\partial_{BB'}W)W + \ldots \] (37)

The second term can clearly be arranged to give \(1/2X_{12} \cdot \partial T\), but there are obviously going to be new operators at second and higher orders which have indices. By taking linear combinations of operators with the same number of derivatives one can combine all the operators with a given number of derivatives into derivatives of known operators together with operators which are quasi-primary, i.e. that transform in a tensorial fashion under the superconformal group.

One of the operators that arises in this way at second order is

\[ O_{AB,AA'B'} = \partial_{(AA'}W\partial_{B)B'}W - \frac{1}{6} \partial_{(AA'}\partial_{B)B'}(W^2) \] (38)

where the round brackets denote generalised symmetry on the primed and unprimed indices. This means that it is symmetric on the spacetime spinor indices and antisymmetric on the internal indices. The lowest dimensional component of this operator is proportional to

\[ \epsilon^{ab}\epsilon^{a'b'} \left( \partial_{aa'}W\partial_{bb'}W - \frac{1}{6} \partial_{aa'}\partial_{bb'}(W^2) \right) \sim K \] (39)

So the operator (38) is simply the Konishi operator written as an analytic tensor field. Indeed, it can be shown that all superconformal representations (at least of the type that are of interest in field theory) can be represented on analytic superfields [42].

The conclusion, therefore, is that the analytic OPE does contain more terms than was suggested in [40] and does indeed contain multiplets such as the Konishi multiplet on the right-hand side. On the other hand, it is not incompatible with analyticity, at least in the free theory. The situation is more complicated in the interacting case but this simple example gives us hope, even in this case, that the OPE and analyticity will remain compatible. It would be interesting to pursue this topic further, particularly in the light of recent results on the \(N = 4\) OPE and the Konishi multiplet [43, 44, 45].

11 Conclusions

To summarise we have seen that superspace methods give a very explicit and clear way of implementing the group-theoretical aspects of the Maldacena conjecture for \(AdS_5 \times S^5\). On the field theory side, harmonic superspace methods provide the easiest way of implementing the superconformal Ward identities. These identities, together with analyticity, enable us to prove, almost rigorously, the non-renormalisation theorems for two- and three-point functions, and to establish the triviality of extremal and next-to-extremal correlations functions. Further simplifications in other correlators are also implied. Finally, the OPE seems to be consistent with analyticity, since apparently non-analytic operators such as the Konishi operator can be interpreted as more complicated analytic superfields.
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