Singular Phenomena of Solutions for Nonlinear Diffusion Equations involving $p(x)$-Laplacian Operator

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Abstract: The authors of this paper study singular phenomena (vanishing and blowing-up in finite time) of solutions to the homogeneous Dirichlet boundary value problem of nonlinear diffusion equations involving $p(x)$-Laplacian operator and a nonlinear source. The authors discuss how the value of the variable exponent $p(x)$ and initial energy (data) affect the properties of solutions. At the same time, we obtain the critical extinction and blow-up exponents of solutions.

Keywords: $p(x)$-Laplacian Operator; Blow-up; Extinction.
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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded simply connected domain and $0 < T < \infty$. Consider the following quasilinear degenerate parabolic problem:

\[
\begin{cases}
    u_t = \text{div}(|\nabla u|^{p(x)-2}\nabla u) + u^{r-2}u, \quad (x, t) \in Q_T, \\
    u(x, t) = 0, \quad (x, t) \in \Gamma_T, \\
    u(x, 0) = u_0(x), \quad x \in \Omega,
\end{cases}
\]

(1.1)

where $Q_T = \Omega \times (0, T]$, $\Gamma_T$ denotes the lateral boundary of the cylinder $Q_T$, It will be assumed throughout the paper that the exponent $p(x)$ is continuous in $\Omega$ with logarithmic module of continuity:

\[
1 < p^- = \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ = \sup_{x \in \Omega} p(x) < \infty,
\]

(1.2)

\[
\forall x \in \Omega, \quad y \in \Omega, \quad |x - y| < 1, \quad |p(x) - p(y)| \leq \omega(|x - y|),
\]

(1.3)

where

\[
\limsup_{\tau \to 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty.
\]

Problem (1.1) occurs in mathematical models of physical processes, for example, nonlinear diffusion, filtration, elastic mechanics and electro-rheological fluids, the readers may refer to

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When \( p \) is a fixed constant, the authors in [6] discussed the extinction and non-extinction of solutions by applying a comparison theorem and energy estimate methods. Besides, in [7], the authors studied blowing-up of solutions with positive initial energy. However, we point out that the methods used in [6, 7] fail in solving our problems. The main reason is that

\[
\| \nabla u \|_{p(\cdot), \Omega} \neq \left[ \int_{\Omega} |\nabla u|^{p(x)} \, dx \right]^{\frac{1}{p(x)}},
\]

\[
\int_{\Omega} u^{m} |\nabla u|^{p(x)} \, dx \neq \int_{\Omega} \left( \frac{p(\cdot)}{m + p(\cdot)} \right)^{p(\cdot)} |\nabla u|^\frac{m + p(\cdot)}{p(\cdot)} \, dx;
\]

\[
\text{div}(|\nabla (\lambda u)|^{p(x)-2} \nabla (\lambda u)) \neq \lambda^{p(x)-1} \text{div}(|\nabla u|^{p(x)-2} \nabla u).
\]

Due to the lack of homogeneity, we have to look for new methods or techniques to study properties of solutions to the problem. Fortunately, we construct a new control function and apply suitable embedding theorems to prove that the solution blows up in finite time when the initial energy is positive, which improves the result in [10]. Subsequently, we find that the solution represents different properties when \( p(x) \) belongs to different intervals or when the initial data is sufficiently small or strictly bigger than zero. As we know, such results are seldom seen for the problem with variable exponents. By applying energy estimate method and comparison principle for ODE, we prove that the solution of Problem (1.1) develops a nonempty set \( \{ x \in \Omega, \ u(x, t) = 0 \} \), the so called dead core, after finite time, or remains positive when \( p(x) \) belongs to different intervals.

The outline of this paper is the following: In Section 2, we shall introduce the function spaces of Orlicz – Sobolev type, give the definition of the weak solution to the problem and prove that the weak solution blows up in finite time for a positive initial energy; Section 3 will be devoted to studying the critical extinction exponent.

## 2 Critical Blow-up Exponent

In this section, we will study the blowing-up of the weak solutions when the initial energy is less than a positive constant. Let us introduce the Banach spaces

\[
L^{p(x)}(\Omega) = \left\{ u(x)|u \text{ is measurable in } \Omega, \ A_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} \, dx < \infty \right\},
\]

\[
\|u\|_{p(\cdot)} = \inf\{\lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1\};
\]

\[
W^{1,p(x)}(\Omega) := \{ u : u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \};
\]

\[
\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(\cdot), \Omega} + \|\nabla u\|_{p(\cdot), \Omega};
\]

\[
V(\Omega) = \left\{ u | u \in \mathbb{L}^{2}(\Omega) \cap W^{1,1}_{0}(\Omega), u \in W^{1,p(x)}(\Omega) \right\},
\]

\[
\|u\|_{V(\Omega)} = \|u\|_{2, \Omega} + \|\nabla u\|_{p(\cdot), \Omega};
\]

\[
H(Q_T) = \left\{ u : [0, T] \mapsto V(\Omega) | u \in L^{2}(Q_T), |\nabla u| \in L^{p(x)}(Q_T), u = 0 \text{ on } \Gamma_T \right\},
\]

\[
\|u\|_{H(Q_T)} = \|u\|_{2, Q_T} + \|\nabla u\|_{p(\cdot), Q_T},
\]

and denote by \( H'(Q_T) \) the dual of \( H(Q_T) \) with respect to the inner product in \( L^{2}(Q_T) \). From [5], we know that Condition (1.3) can imply that \( W^{1,p(x)}_{0}(\Omega) := \{ u : u \in W^{1,p(x)}(\Omega), u = 0 \text{ on } \partial \Omega \} \) is the closure of \( C^{\infty}_{0}(\Omega) \) in \( W^{1,p(x)}(\Omega) \).
**Definition 2.1.** A function \( u(x,t) \in H(Q_T) \cap L_\infty(0,T;L^2(\Omega)) \), \( u_t \in H'(Q_T) \) is called a weak solution of Problem (1.1) if for every test-function

\[
\xi \in \mathcal{Z} \equiv \{\eta(x) : \eta \in H(Q_T) \cap L_\infty(0,T;L^2(\Omega)), \eta_t \in H'(Q_T)\},
\]

and every \( t_1, t_2 \in [0,T] \) the following identity holds:

\[
\int_{t_1}^{t_2} \int_\Omega [u\xi_t - |\nabla u|^{p(x)-2}\nabla u\nabla \xi + ru^{r-2}u\xi]dxdt = \int_\Omega u\xi dx|_{t_1}^{t_2}.
\]

(2.1)

For the existence of solutions to Problem (1.1), we have the following theorem

**Theorem 2.1.** Suppose that Conditions (1.2) – (1.3) are fulfilled. Then for every \( u_0 \in W_0^{1,p(x)}(\Omega) \cap L_\infty(\Omega) \), there exists a \( T^* > 0 \) such that Problem (1.1) has at least one weak solution \( u \in H(Q_T^*) \), \( u_t \in H'(Q_T^*) \) in the sense of Definition (2.1).

Define

\[
E(t) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{1}{r} \int_\Omega |u|^r dx.
\]

For the sake of simplicity, we give some notations used below. By Corollary 3.34 in [5], we know that \( W_0^{1,p(x)}(\Omega) \rightharpoonup L^r(\Omega)(1 < r < \frac{Np}{N-p}) \). Let \( B \) be the constant of the embedding inequality

\[
\| u \|_r \leq B \| \nabla u \|_{p(.)}, \forall u \in W_0^{1,p(x)}(\Omega).
\]

Set \( E_1 = (\frac{r-p^+}{rp^+})B_1^{\frac{r-p^+}{p^+}}, \alpha_1 = B_1^{\frac{r-p^+}{p^+}} \), where \( B_1 = \max\{B,1\} \). Our main result is

**Theorem 2.2.** Assume that \( p(x) \) satisfies (1.2) – (1.3) and the following conditions hold

\[
(H_1) \ u_0 \in L^2(\Omega) \cap W_0^{1,p(x)}(\Omega), \ E(0) < E_1, \ \min\{\|\nabla u_0\|^{p(x)}_{p(x)} \}, \|\nabla u_0\|^{p(x)}_p \} > \alpha_1;
\]

\[
(H_2) \ \max\{1,\frac{2N}{N+2}\} \leq p^- < N, \ \max\{2,p^+\} < r \leq \frac{2N+(N+2)(p^--1)}{N},
\]

then the solution of Problem (1.1) blows up in finite time.

In order to prove this theorem, we first give some lemmas.

**Lemma 2.1.** Suppose that \( u(x,t) \in H(Q_T) \cap L_\infty(0,T;L^2(\Omega)) \), \( u_t \in H'(Q_T) \) is a weak solution of Problem (1.1) and \( 2 < r \leq \frac{2N+(N+2)(p^-1)}{N} \), then the following conclusions hold

(i) \( u_t \in L^2(Q_T), |\nabla u| \in L_\infty(0,T;L^{p(x)}(\Omega)); \)

(ii) \( u \in C(0,T;L^r(\Omega)), \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \in C(0,T); \)

(iii) \( E(t) \in C[0,T] \cap C^1(0,T); \)

(iv) \( E(t) \) is non-increasing with respect to \( t \) and satisfies the following identity

\[
E'(t) = -\|u_t\|_2^2 \leq 0.
\]
Proof. A weak solution $u(x, t)$ to Problem (1.1) is a limit function of the sequence of Galerkin’s approximation

$$u^{(m)} = \sum_{k=1}^{m} c_k^{(m)} \varphi_k, \quad \varphi_k \in W_0^{1, p^+}(\Omega), \ c_k^{(m)} \in C^1(0, T).$$

Following the lines of the proof of Lemma 3.1 and Theorem 6.1 in [8, 10], we know that there exists a positive constant $C = C(|\Omega|, |u_0|_{L^\infty(\Omega)}, p^+, r, N)$ such that

$$\|u^{(m)}\|_{H(Q_T)} + \|u^{(m)}\|_{L^\infty(0, T; L^2(\Omega))} + \|u_t^{(m)}\|_{H'(Q_T)} \leq C;$$  \(2.2\)

$$\|u_t^{(m)}\|_{L^2(\Omega)} + \frac{1}{N} \int \frac{1}{p(x)} |\nabla u^{(m)}|^{p(x)} \, dx = \int |u^{(m)}|^{r-2} u^{(m)} u_t^{(m)} \, dx. \quad 2.3$$

Proposition 3.1 in [1] and Inequality (2.2) yield

$$\|u^{(m)}\|_{L^{\frac{p-2(N+2)}{N}}(Q_T)} \leq \gamma \int_{Q_T} \|\nabla u^{(m)}\|^{p(x)} \, dx + \left( \sup_{0 < t < T} \int_{\Omega} |u^{(m)}|^2 \, dx \right)^{\frac{p}{p-2}} \leq C. \quad 2.4$$

Furthermore, according to $1 < r \leq \frac{2N+(N+2)(p^*-1)}{N}$ and (2.4), it is easy to verify that

$$\|u^{(m)}\|_{L^2(Q_T)} \leq C. \quad 2.5$$

By (2.2), (2.3), (2.5), we have

$$\|u_t^{(m)}\|_{L^2(Q_T)} + \frac{1}{N} \int \frac{1}{p(x)} |\nabla u^{(m)}|^{p(x)} \, dx \leq C := C(p^+, |\nabla u_0(\Omega)|_{p^+}, |\Omega|),$$

which implies $u(x, t) \in L^2(Q_T), |\nabla u| \in L^\infty(0, T; L^{p^+}(\Omega))$.

Noting that $W_0^{1, p^+}(\Omega) \hookrightarrow W_0^{1, p^+}(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^2(\Omega)$ and applying Corollary 6 in [13], we get $u \in C(0, T; L^r(\Omega))$.

Similarly as the proof of Lemma 1 in [10], we have

$$\|u_t\|_{L^2(\Omega \times (t_1, t_2))} + E(t_2) = E(t_1), \quad 0 \leq t_1 < t_2 \leq T, \quad 2.6$$

which shows $E(t) \in C[0, T], \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \in C(0, T)$ from absolute continuity of Lebesgue measure.

Letting $t_1 = t, t_2 = t + h, t, t + h \in (0, T)$, multiplying (2.6) by $\frac{1}{h}$ and according to $|u_t|_{L^2(\Omega)} \in L^2(0, T)$ and Lebesgue differentiation theorem, we have

$$E'(t) = -\int |u_t|^2 \, dx \leq 0,$$

that is $E(t) \in C^1(0, T).$ \qed

**Lemma 2.2.** Suppose that $u$ is the solution of Problem (1.1). If the condition $(H_1)$ holds and $r > \max\{2, p^+\}$, then there exists a positive constant $\alpha_2 > \alpha_1$ such that for all $t \geq 0$

$$\int \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \geq \alpha_2, \quad 2.7$$

$$\int |u|^r \, dx \geq B_1^* \max\{\alpha_2^\frac{r}{p}, \alpha_2^\frac{r}{p^+}\}. \quad 2.8$$
which contradicts \( h \) with \( \alpha \).

By the definitions of \( E \) we prove (2.10)

\[
E(t) \geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{B^r}{r} \|\nabla u\|^{p(.)}_{p(.)}
\]

\[
\geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{B^r}{r} \max \left\{ \left( \int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{r}}, \left( \int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{p^+}} \right\}^{r}
\]

\[
\geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{B^r}{r} \max \left\{ \left( \int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{r}}, \left( \int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{p^+}} \right\}
\]

\[
\Delta \frac{1}{p^+} \alpha - \frac{B^r}{r} \max \{\alpha \frac{\partial}{\partial p^-}, \alpha \frac{\partial}{\partial p^+}\} = h(\alpha),
\]

with \( \alpha = \int_{\Omega} |\nabla u|^{p(x)} dx \).

Next, we will give a simple analysis about the properties of the function \( h(\alpha) \). It is easy to prove that \( h(\alpha) \) satisfies the following properties

\[
h(\alpha) \in C[0, +\infty);
\]

\[
h'(\alpha) = \begin{cases} 
\frac{1}{p^+} - \frac{B^r}{p^+} \alpha \frac{\partial}{\partial p^-} < 0, & \alpha > 1; \\
\frac{1}{p^+} - \frac{B^r}{p^+} \alpha \frac{\partial}{\partial p^+}, & \alpha < 1;
\end{cases}
\]

\[
h'(\alpha_1) = 1, \quad 0 < \alpha_1 < 1.
\]

Although the function \( h(\alpha) \) is not differentiable at \( \alpha = 1 \), a simple analysis shows that \( h(\alpha) \) is increasing for \( 0 < \alpha < \alpha_1 \) while \( h(\alpha) \) is decreasing for \( \alpha > \alpha_1 \), and \( \lim_{\alpha \to \infty} h(\alpha) = -\infty \). Due to \( E(0) < E_1 \), then there exists a positive constant \( \alpha_2 > \alpha_1 \) such that \( h(\alpha_1) = E(0) \). By \( \min \{ |\nabla u_0|^{p^-}_{p(x)}, |\nabla u_0|^{p^+}_{p(x)} \} > \alpha_1 \), we get

\[
h(\alpha_0) = E(0) = h(\alpha_2),
\]

where \( \alpha_0 = \int_{\Omega} |\nabla u_0|^{p(x)} dx \). Once again applying the monotonicity of \( h(\alpha) \), we have \( \alpha_0 \geq \alpha_2 \).

We prove (2.7) by arguing by contradiction. Suppose that there exists a \( t_0 > 0 \) such that \( \int_{\Omega} |\nabla u(t_0)|^{p(x)} dx < \alpha_2 \). Since \( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \in C(0, T) \), we may choose a \( t_1 > 0 \) such that

\[
\alpha_2 > \int_{\Omega} |\nabla u(t_1)|^{p(x)} dx \geq p^- \int_{\Omega} \frac{1}{p(x)} |\nabla u(t_1)|^{p(x)} dx > \alpha_1.
\]

By the definitions of \( E(t) \) and the monotonicity of \( h(\alpha) \), we have

\[
E(t_1) > h \left( \int_{\Omega} |\nabla u(t_1)|^{p(x)} dx \right) = h(\alpha_2) = E(0),
\]

which contradicts \( E(t) \leq E(0), \forall t \geq 0 \).

Noting that \( E'(t) \leq 0 \), we get

\[
\frac{1}{r} \int_{\Omega} |u|^{r} dx \geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - E(0) = \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - h(\alpha_2)
\]

\[
= \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{1}{p^+} \alpha_2 + \frac{B^r}{r} \max \{\alpha_2, \alpha_2\} \geq \frac{B^r}{r} \max \{\alpha_2, \alpha_2\}.
\]

\[\blacksquare\]
Let $H(t) = E_1 - E(t)$, then

**Lemma 2.3.** For all $t \geq 0$, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{r} \int_\Omega |u|^r \, dx. \quad (2.11)$$

**Proof.** Since $E'(t) \leq 0$, it is very easily seen that $H'(t) \geq 0$, which shows that $H(t) \geq H(0) = E_1 - E(0) > 0$. Moreover, a simple calculation shows that

$$E_1 - \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \leq \left( \frac{r - p^+}{r p^+} \right) B_1^{r-p^+} - \frac{1}{p^+} \alpha_2 \leq h(\alpha_1) - \frac{1}{p^+} \alpha_1 < 0.$$

So

$$H(t) = E_1 - \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{1}{r} \int_\Omega |u|^r \, dx \leq \frac{1}{r} \int_\Omega |u|^r \, dx.$$

**Proof of Theorem 2.2.** Letting $G(t) = \frac{1}{2} \int_\Omega |u|^2 \, dx$, we have

$$G'(t) = \int_\Omega uu_t \, dx = - \int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{r} \, dx = - \int_\Omega p(x) \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{r} \, dx$$

$$\geq -p^+ \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{r} \, dx = \int_\Omega |u|^{r} \, dx - p^+ (E(t) + \frac{1}{r} \int_\Omega |u|^{r} \, dx)$$

$$= (1 - \frac{p^+}{r}) \int_\Omega |u|^{r} \, dx - p^+ E(t) \geq (1 - \frac{p^+}{r}) \int_\Omega |u|^{r} \, dx - p^+ E_1 + p^+ H(t)$$

$$\geq (1 - \frac{p^+}{r}) \int_\Omega |u|^{r} \, dx - p^+ E_1.$$

Inequality (2.8) shows that

$$p^+ E_1 = \frac{p^+ E_1}{B_1^r \max\{\alpha_2^{p^+}, \alpha_2^{r+}\}} \left( B_1^r \max\{\alpha_2^{p^+}, \alpha_2^{r+}\} \right) \leq \frac{p^+ E_1}{B_1^r \max\{\alpha_2^{p^+}, \alpha_2^{r+}\}} \int_\Omega |u|^{r} \, dx. \quad (2.13)$$

Moreover, $r > 2$ and Hölder’s inequality imply that

$$\int_\Omega |u|^{r} \, dx \geq |\Omega|^\frac{2}{r+2} \left( \int_\Omega |u|^2 \, dx \right)^\frac{r}{2}.$$

So, using (2.12) − (2.14), we get

$$G'(t) \geq C_0 \left( \int_\Omega |u|^2 \, dx \right)^\frac{r}{2} = C_0 G^{\frac{r}{2}}(t), \quad (2.15)$$

where

$$C_0 = \frac{(r - p^+)}{B_1^r \max\{\alpha_2^{p^+}, \alpha_2^{r+}\}} - \frac{p^+}{B_1^r \max\{\alpha_2^{p^+}, \alpha_2^{r+}\}} \int_\Omega |u|^{2} \, dx > 0.$$ Integrating (2.15) with respect to $t$ over $(0, \tau)$, we have

$$G(\tau) \geq \left( G^{1-\frac{r}{2}}(0) - \frac{r}{2} - 1 \right) C_0 \tau^{\frac{2}{r}}.$$

Applying Gronwall’s inequality, we know that $G(t)$ blows up in a finite time $T_* \leq \frac{G^{1-\frac{r}{2}}(0)}{(\frac{r}{2} - 1) C_0}$. For $\frac{2N+2}{2} < p^- < p^+ < 2$, $r = 2$, $E(0) < E_1$, what happens to the solution of Problem (1.1)? The following theorem gives a positive answer.
Theorem 2.3. Suppose that \( p(x) \) satisfies (1.2) – (1.3) and the following conditions hold

\begin{align*}
(H_5) \quad & u_0 \in L^2(\Omega) \cap W^{1,p(x)}_0(\Omega), \ E(0) < E_1, \ \text{min}\{|\nabla u_0|^{p^{-}}_{p(x)}, |\nabla u_0|^{p^{+}}_{p(x)}\} > \alpha_1, \\
(H_6) \quad & \frac{2N}{N+2} < p^{-} < p^{+} < r = 2,
\end{align*}

then the solution of Problem (1.1) exists globally. Furthermore, we have

\[
\lim_{t \to +\infty} \|u\|_{L^2(\Omega)} = +\infty.
\]

Proof. By (2.15), we can easily obtain that

\[
G'(t) \geq C_02^{\frac{2-p^{-}}{2}}|\Omega|^\frac{p^{-}-2}{2}G(t), \quad t \geq 0.
\]

Moreover, by applying Gronwall’s inequality, we get

\[
\lim_{t \to +\infty} \|u\|_{L^2(\Omega)} = +\infty.
\]

This completes the proof of this theorem.

For \( p^{+} < r < 2 \), we have the following theorem

Theorem 2.4. Suppose that \( p(x) \) satisfies (1.2) – (1.3) and the following conditions hold

\begin{align*}
(H_7) \quad & u_0 \in L^\infty(\Omega) \cap L^2(\Omega) \cap W^{1,p(x)}_0(\Omega), \ E(0) \leq 0; \\
(H_8) \quad & \frac{2N}{N+2} < p^{-} < p^{+} < r < 2,
\end{align*}

then the nonnegative solution of Problem (1.1) exists globally. Furthermore, we have

\[
\lim_{t \to +\infty} \|u\|_{L^\infty(\Omega)} = +\infty.
\]

Proof. We use a trick used in [8, 12]. The function \( u^{2k-1}(k \in \mathbb{N}) \) can be chosen as a test-function in (2.1). In (2.1), let \( t_2 = t + h, \ t_1 = t \), with \( t, t + h \in (0, T) \), then

\[
\frac{1}{2k} \int_t^{t+h} \frac{d}{dt}(\int_{\Omega} u^{2k} dx) dt + \int_t^{t+h} \int_{\Omega} (2k - 1)u^{2(k-1)}|\nabla u|^{p(x)} dx dt = \int_t^{t+h} \int_{\Omega} u^{2k-2+r} dx dt.
\]

(2.16)

Dividing the last equality by \( h \), letting \( h \to 0 \) and applying Lebesgue differentiation theorem, we have that \( \forall \ t \in (0, T) \)

\[
\frac{1}{2k} \frac{d}{dt} \int_{\Omega} u^{2k} dx + \int_{\Omega} (2k - 1)u^{2(k-1)}|\nabla u|^{p(x)} dx = \int_{\Omega} u^{2k-2+r} dx.
\]

(2.17)

By Hölder’s inequality, we get

\[
|\int_{\Omega} u^{2k-2+r} dx| \leq \|u(\cdot, t)\|_{L^{2k}^2(\Omega)}^{2k-2+r} \cdot |\Omega|^\frac{2-r}{2k}, \quad k = 1, 2, \cdots.
\]

(2.18)

Combing Gronwall’s inequality with inequalities (2.17) – (2.18) and dropping the nonnegative terms, we have

\[
\|u\|_{L^{2k}(\Omega)} \leq \left( \|u_0\|_{L^{2k}(\Omega)}^{2-r} + (1 - \frac{r}{2})t|\Omega|^\frac{2-r}{2k} \right)^{\frac{1}{2-r}}.
\]

(2.19)
In (2.19), letting \( k \to \infty \), we have
\[
\|u\|_{L^\infty(\Omega)} \leq \left( \|u_0\|_{L^\infty(\Omega)} + (1 - \frac{r}{2})t \right)^{\frac{1}{2-r}}, \quad t \geq 0,
\]
which implies that \( \|u\|_{L^\infty(\Omega)} \) can not blow up at any finite time. We now prove that
\[
\lim_{t \to +\infty} \|u\|_{L^\infty(\Omega)} = +\infty.
\]
If not, there exists a positive constant \( M_0 \) such that
\[
\|u\|_{L^\infty(\Omega)} \leq M_0, \quad t \geq 0.
\]
Then,
\[
\int_\Omega |u|^r \, dx \leq M_0^{r-2} \int_\Omega |u|^r \, dx.
\]
Moreover, we apply Lemma 2.1 and Inequality (2.21) to obtain
\[
G'(t) = \int_\Omega uu_t \, dx = -\int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^r \, dx
\geq -p^+ \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^r \, dx = \int_\Omega |u|^r \, dx - p^+ (E(t) + \frac{1}{r} \int_\Omega |u|^r \, dx)
= (1 - \frac{p^+}{r}) \int_\Omega |u|^r \, dx - p^+ E(t) \geq C_02^{-\frac{r}{2}} |\Omega|^\frac{r-2}{2} \int_\Omega |u|^r \, dx
\geq (1 - \frac{p^+}{r})M_0^{2-r} \int_\Omega |u|^2 \, dx = C_02^{-\frac{r}{2}} |\Omega|^\frac{r-2}{2} M_0^{2-r} G(t),
\]
which shows that
\[
\lim_{t \to \infty} \|u\|_{L^2(\Omega)} = +\infty.
\]
This is a contradiction. This completes the proof of this theorem. \( \square \)

3 Critical extinction exponent

In this section, we are devoted to the discussion of the critical extinction exponent of solutions to Problem (1.1). Namely, we mainly discuss how the ranges of \( p^+, p^- \) and the value of the initial data \( u_0 \) affect the extinction property of solutions.

**Theorem 3.1.** Suppose that \( p(x) \) satisfies (1.2) – (1.3). If the following condition holds
\[
(H_9) \quad \frac{2N}{N+2} < p^- < p^+ < r \leq 2,
\]
then the nonnegative solution of Problem (1.1) vanishes in finite time for any nonnegative sufficiently, but small initial data \( u_0(x) \). More precise speaking, we have the following estimates
\[
\begin{align*}
\|u\|_2 & \leq g(t)^{\frac{1}{r-p^-}}, \quad 0 < t < T_1, \\
\|u\|_2 & = 0, \quad t \in [T_1, \infty),
\end{align*}
\]
where \( g(t), T_1 \) satisfy

\[
g(t) = \begin{cases} 
\|u_0\|_2^{2-p^+} - K_1 + K_1 e^{(p^- - 2)t}, & r = 2; \\
\|u_0\|_2^{2-p^+} + F(u_0)t, & 1 < r < 2;
\end{cases}
\]

\[
T_1 = \begin{cases} 
\frac{1}{p^- - 2} \ln(1 - \frac{\|u_0\|_2^{p^+}}{K_1}), & 1 < r < 2; \\
\frac{\|u_0\|_2^{2-p^+}}{-F(u_0)}, & F(u_0) = (2 - p^+) \left[ 2|\Omega| \|\nabla u_0\|_{p^{-, \frac{n}{p^+}}} + C_2 \min\{\|u_0\|_2^{p^- - p^+}, 1\} \right], 1 < r < 2.
\end{cases}
\]

Here \( C_1 \) is a positive constant.

**Proof.** Multiplying the first equation in (1.1) by \( u \) and integrating over \( \Omega \times (t, t + h) \), we have

\[
\frac{1}{2} \int_\Omega u^2 dx \bigg|_{t}^{t+h} + \int_t^{t+h} \int_\Omega |\nabla u|^{p(x)} dx d\tau = \int_t^{t+h} \int_\Omega u^r dx d\tau. \tag{3.1}
\]

Dividing (3.1) by \( h \) and applying Lebesgue differentiation theorem, we have

\[
G'(t) + \int_\Omega |\nabla u|^{p(x)} dx \leq 2 \int_\Omega |u|^r dx, \tag{3.2}
\]

where \( G(t) = \int_\Omega u^2 dx \).

First we consider the case when \( r = 2 \). By means of the embedding theorem \( W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega) \), we have

\[
\|u\|_2 \leq C \|\nabla u\|_{p^-} \leq C \|\nabla u\|_{p(\cdot)} \leq C \max \left( \int_\Omega |\nabla u|^{p(\cdot)} dx \right)^{\frac{1}{p^-}}, \left( \int_\Omega |\nabla u|^{p(\cdot)} dx \right)^{\frac{1}{p^-}}. \tag{3.3}
\]

By (3.2) – (3.3), we get

\[
G'(t) + 2C_1 \min\{G^{|\tau|^{-}}(t), G^{|\tau|^{+}}(t)\} \leq 2G(t). \tag{3.4}
\]

Noting that \( 2C_1 \min\{G^{|\tau|^{-}}(t), G^{|\tau|^{+}}(t)\} > 0 \), we get

\[
\|u(\cdot, t)\|_2 \leq \|u_0\|_2 e^t,
\]

which implies that

\[
\min\{G^{|\tau|^{-}}(t), G^{|\tau|^{+}}(t)\} \geq \min\{1, \|u_0\|_2^{p^- - p^+}\} \left( G(t)e^{-2t} \right)^{\frac{p^-}{2}} e^{p^- t}. \tag{3.5}
\]

By (3.4) – (3.5), we get

\[
\frac{d(Ge^{-2t})}{dt} \leq -2C_1 \min\{1, \|u_0\|_2^{p^- - p^+}\} \left( G(t)e^{-2t} \right)^{\frac{p^-}{2}} e^{(p^- - 2)t}, \quad G(0) = \|u_0\|_2^2 > 0. \tag{3.6}
\]

Gronwall’s inequality implies that the solution of Inequality (3.6) satisfies the following estimate

\[
G(t) \leq \left[ \left( \|u_0\|_2^{2-p^+} - K_1 + K_1 e^{(p^- - 2)t} \right) \right]^{\frac{2}{2-p^+}}.
\]
Suppose that $0 < \varepsilon < \epsilon$. From [11], we know $\Phi \in \Omega$. By (3.1), a simple analysis shows that $F$ is positively bounded from below. Due to $2 > r > p$, we may choose sufficiently small $\|u_0\|_2$ such that $F(u_0) < 0$. Furthermore, a simple analysis shows that $F(u(t))$ is decreasing with respect to $t$. Hence, we obtain that
\[
F(u(t)) \leq F(u_0) < 0, \quad \forall \ t \geq 0.
\]
By (3.8) – (3.9), we arrive at the following relations
\[
\begin{align*}
0 < y(t) & \leq y(0); \\
y(t) & \leq y(0) + F(u_0)t, \quad 0 < t < T_1 = \frac{y(0)}{F(u_0)}; \\
y(t) & = 0, \quad t \geq T_1.
\end{align*}
\]
It is easy to verify that $y^{2-p^+}(t)$ is an upper-solution of (3.7), then according to comparison principle for ODE in [13], we get
\[
\|u\|_2^2 \leq y^{2-p^+}(t), \quad 0 < t < T_1.
\]

When $r < p^+ < 2$, we have

**Theorem 3.2.** Suppose that $p(x)$ satisfies (1.2) – (1.3). If the following condition holds
\[
(H_{10}) \quad \frac{2N}{N+2} < r < p^- < p^+ \leq 2,
\]
then the nonnegative solution of Problem (1.1) does not vanish in finite time for any initial data positively bounded from below.

**Proof.** Let $\lambda_1 > 0$ and $\Phi > 0$ be the first eigenvalue and eigenfunction of the following problem
\[
\begin{align*}
-\text{div}(|\nabla \Phi|^{p(x)-2}\nabla \Phi) & = \lambda_1 |\Phi|^{p(x)-2}\Phi, \quad x \in \Omega; \\
\Phi & = 0, \quad x \in \partial \Omega.
\end{align*}
\]
From [11], we know $\Phi \in W_0^{1,p(x)}(\Omega)$ satisfies that the following facts
\[
\Phi > 0, \quad x \in \Omega, \quad M = \sup_{x \in \Omega} |\Phi| < \infty.
\]
For $0 < \varepsilon < \min\{1, \frac{(1+\lambda_1)r^-}{cM}, \frac{\min u_0}{cM}\}$, we consider the auxiliary problem
\[
\begin{align*}
v_t - \text{div}(|\nabla v|^{p(x)-2}\nabla v) & = \frac{\lambda_1 \varepsilon^r}{\varepsilon + \lambda_1 \varepsilon^r}, \quad (x, t) \in Q_T, \\
v(x, t) & = 0, \quad (x, t) \in \Gamma_T, \\
v(x, 0) & = u_0 > 0, \quad x \in \Omega.
\end{align*}
\]

Applying Hölder’s inequality and Inequality (3.2) – (3.3), we obtain
\[
G'(t) + 2C_1 \min\{G^{\frac{r^+}{r^-}}, G^{\frac{p^-}{p^+}}\} \leq 2|\Omega|^{\frac{2-r}{2}}G^\frac{r}{2}(t).
\]
Now, we choose $A = C_1 \min\{\|u_0\|^{\frac{r^-}{r^+}}, 1\}$, $B = 4|\Omega|^{\frac{2-r}{2}}$. Let us consider the following problem
\[
\begin{align*}
y'(t) & = \frac{2-p^+}{2}By^{\frac{2-p^+}{r^+}} - \frac{2-p^+}{2}A := F(u(t)), \\
y(0) & = \|u_0\|^{\frac{2-p^+}{2}} > 0.
\end{align*}
\]
Due to $2 > r > p^+$, we may choose sufficiently small $\|u_0\|_2$ such that $F(u_0) < 0$. Furthermore, a simple analysis shows that $F(u(t))$ is decreasing with respect to $t$. Hence, we obtain that
\[
F(u(t)) \leq F(u_0) < 0, \quad \forall \ t \geq 0.
\]
It is easy to prove that the solution $u$ of Problem (1.1) is an upper-solution to Problem (3.10). Using the comparison principle in [12], we get $v(x, t) \leq u(x, t)$, $(x, t) \in Q_T$.

Next, we construct a lower-solution to Problem (3.10). For any given $T > 0$, let $w(x, t) = \varepsilon e^{(1-\Phi)}$, then we have

$$w'(t) \leq 0, \quad \lambda_1 w^{p-1} - \frac{\lambda_1 w^r}{\varepsilon \Phi + \lambda_1 w} \leq 0, \quad (x, t) \in Q_T.$$ 

So, for any nonnegative test-function $\varphi$, we have

$$\int_{Q_T} [w u - \varphi + |\nabla w|^{p(x)-2}\nabla w \nabla \varphi - \frac{\lambda_1 w^r}{\varepsilon \Phi + \lambda_1 w} \varphi] dx dt = \int_{Q_T} \left[ \lambda_1 w^{p-1} - \frac{\lambda_1 w^r}{\varepsilon \Phi + \lambda_1 w} \right] \varphi dx dt \leq 0.$$ 

Again applying the comparison principle, we get

$$0 < w(x, t) \leq v(x, t) \leq u(x, t), \quad (x, t) \in Q_T.$$ 

That is $u$ does not vanish in finite time. \qed

**Remark 3.1.** For $\frac{2N}{N+2} < p^- < r < p^+ < 2$, what happens to the solution of Problem (1.1)? Due to technical reasons, up to now we can not prove or not whether the solution vanishes and remain positive.

**Theorem 3.3.** Suppose that $p(x)$ satisfies (1.2) – (1.3). If the following condition holds

$$(H_{11}) \quad 1 < p^- < \frac{2N}{N+2}, \quad 1 < p^+ < \frac{Np^-}{N-p^-}, \quad r \geq 2,$$

then the bounded nonnegative solution of Problem (1.1) vanishes in finite time if the initial data is sufficiently small.

**Proof.** Multiplying (1.1) by $u^s (s = \frac{2N-(N+1)p^-}{p^-})$ and integrating over $\Omega$, we get

$$\frac{1}{s+1} \int_0^{t+h} \int_{\Omega} u^{s+1} dx dt + C_1 \int_t^{t+h} \int_{\Omega} |\nabla u^\beta|^{p(x)} dx dt \leq C_2 \int_t^{t+h} \int_{\Omega} u^{s+1} dx dt, \quad (3.11)$$

with $\beta = \frac{(2-p^-)(N-p^-)}{p^-}$.

By means of the above inequality and the embedding theorem $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^{\frac{Np^-}{N-p^-}}(\Omega)$, we have

$$\|u^\beta\|_{L^{p^-}} \leq C\|\nabla u^\beta\|_{p^-} \leq C\|\nabla u^\beta\|_{p(\cdot)} \leq C \max \left[ \left( \int_{\Omega} |\nabla u^{\beta[p(\cdot)]} dx \right)^{\frac{1}{p^-}}, \left( \int_{\Omega} |\nabla u^{\beta[p(\cdot)]} dx \right)^{\frac{1}{p^+}} \right] \leq C \max[C_1^{p^-} (\|u_0\|_{L^2(\Omega)}), 1] \left( \int_{\Omega} |\nabla u^\beta|^{p(x)} dx \right)^{\frac{1}{p^-}} \leq C \left( \int_{\Omega} |\nabla u^\beta|^{p(x)} dx \right)^{\frac{1}{p^+}}. \quad (3.12)$$

Dividing (3.11) by $h$ and applying Lebesgue differentiation theorem and Inequality (3.12), we have

$$\frac{1}{s+1} G'(t) + C_2 G^{\beta[p^-]}(t) \leq C_3 G(t),$$
with $G(t) = \int_\Omega u^{s+1}dx$.

Recalling Gronwall’s inequality, there exists a $T_3 > 0$ such that

$$G(t) \leq \left[ G \frac{Np^- - p^+(N-p^-)}{2Np^-} (0) - \frac{C_2}{C_3} + \frac{C_2}{C_3} e^{\frac{C_3(2-p^-)(p^+(N-p^-)-Np^-)}{p^-p^+}} \right]^{\frac{Np^- - p^+(N-p^-)}{2Np^-}} 0 < t < T_3;$$

$$G(t) = 0, \quad t \in [T_3, \infty),$$

where

$$T_3 = \frac{\frac{p^-p^+}{C_3(2-p^-)(Np^- - p^+(N-p^-))} \ln \left[ 1 + \frac{G \frac{Np^- - p^+(N-p^-)}{2Np^-} (0)}{\frac{C_2}{C_3} - G \frac{2p^- - p^+(N-p^-)}{2Np^-} (0)} \right]}{C_2 - G \frac{2p^- - p^+(N-p^-)}{2Np^-} (0)}.$$

**Remark 3.2.** When $1 < p^- < \frac{2N}{N+2}, \frac{Np^-}{N-p^-} < p^+ < 2 \leq r$, what happens to the solution of Problem (1.1)? Due to technical reasons, up to now we can’t prove whether the solution vanishes. But, we guess that the solution may vanish for sufficiently small initial data and may not vanish for sufficiently large initial data. That is, the value of the initial data plays a role in studying the properties of solutions.

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