Equivalence of Projections as Gauge Equivalence on Noncommutative Space

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Abstract

Projections play crucial roles in the ADHM construction on noncommutative $\mathbb{R}^4$. In this article a framework for the description of equivalence relations between projections is proposed. We treat the equivalence of projections as “gauge equivalence” on noncommutative space. We find an interesting application of this framework to the study of $U(2)$ instanton on noncommutative $\mathbb{R}^4$: A zero winding number configuration with a hole at the origin is “gauge equivalent” to the noncommutative analog of the BPST instanton. Thus the “gauge transformation” in this case can be understood as a noncommutative resolution of the singular gauge transformation in ordinary $\mathbb{R}^4$. 
1 Introduction

The concept of smooth space-time manifold should be modified at the Planck scale due to the quantum fluctuations, and we except the short scale structure of space-time has noncommutative nature. When the coordinates of the space are noncommutative, we except the appearance of short scale cut off at the noncommutative scale. For example, instantons on noncommutative $\mathbb{R}^4$ constructed by the ADHM method [1] never become singular [2], due to the cut off in the size of instanton.\textsuperscript{1} Although the noncommutativity in this case is quite simple, the construction reveals deep insights in the nature of gauge theory on noncommutative space. Indeed, the precise mechanism that leads to the absence of singularity is quite nontrivial. In order to construct instantons on noncommutative $\mathbb{R}^4$, one needs to project out some states in Hilbert space, where the Hilbert space is introduced to represent the algebra of noncommutative $\mathbb{R}^4$. Since noncommutative $\mathbb{R}^4$ is defined by the whole Hilbert space and projection removes some of the states in this Hilbert space, projection can be interpreted as a change of topology of the base manifold. More precisely, projection removes some points from $\mathbb{R}^4$ and creates holes. Hence instantons on noncommutative $\mathbb{R}^4$ indicates the necessity for the unified description of gauge fields and geometry [2][3].

In this article a framework for the description of equivalence relations between projections is proposed. We treat the equivalence of projections as a kind of gauge equivalence. Hence the formalism of this framework is similar to the gauge theory. However since the projection contains information of the Hilbert space which represents noncommutative $\mathbb{R}^4$, the transformation between equivalent projections may be regarded as a noncommutative analog of coordinate transformation. Therefore this is a possible framework for the unified description of gauge fields and geometry. We find an interesting application of this framework to the study of $U(2)$ instanton on noncommutative $\mathbb{R}^4$.

2 Equivalence of Projections as Gauge Equivalence on Noncommutative Space

In this section we explain the notion of the equivalence of projections in a concrete example, the gauge theory on noncommutative $\mathbb{R}^4$. However it is obvious that following arguments can be extended to gauge theory on more general noncommutative space.

\textsuperscript{1}This is the case when the noncommutativity of the coordinates has self-dual part (and instantons are anti-self-dual)[5].
Reviews on Gauge Theory on Noncommutative $\mathbb{R}^4$

The noncommutative $\mathbb{R}^4$ we shall consider is described by an algebra generated by the noncommutative coordinates $x^\mu$ ($\mu = 1, \cdots, 4$) which satisfy the following commutation relations:

$$[x^\mu, x^\nu] = i\theta^\mu\nu,$$  

(2.1)

where $\theta^\mu\nu$ is real and constant. In this article we consider the case where the $\theta^\mu\nu$ is self-dual, and set

$$\theta^{12} = \theta^{34} = \frac{\zeta}{4}, \quad \zeta > 0 \quad \text{(others: zero)},$$  

(2.2)

for simplicity. Next we introduce the complex noncommutative coordinates by

$$z_1 = x_2 + ix_1, \quad z_2 = x_4 + ix_3.$$  

(2.3)

Their commutation relations become

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\zeta}{2} \quad \text{(others: zero)}.\quad (2.4)$$

We start with the algebra $\text{End} \mathcal{H}$ of operators acting in the Hilbert space $\mathcal{H} = \sum_{(n_1, n_2) \in \mathbb{Z}^2_{\geq 0}} \mathbb{C} |n_1, n_2\rangle$, where $z$ and $\bar{z}$ are represented as creation and annihilation operators:

$$\sqrt{\frac{2}{\zeta}} z_1 |n_1, n_2\rangle = \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_1 |n_1, n_2\rangle = \sqrt{n_1} |n_1 - 1, n_2\rangle,$$

$$\sqrt{\frac{2}{\zeta}} z_2 |n_1, n_2\rangle = \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_2 |n_1, n_2\rangle = \sqrt{n_2} |n_1, n_2 - 1\rangle. \quad (2.5)$$

The commutation relations in (2.1) have automorphisms of the form $x^\mu \mapsto x^\mu + c^\mu$, where $c^\mu$ is a commuting real number. These automorphisms are generated by unitary operator $U_c$:

$$U_c := \exp[c^\mu \hat{\partial}_\mu],$$  

(2.6)

where we have introduced derivative operator $\hat{\partial}_\mu$ by

$$\hat{\partial}_\mu := iB_{\mu\nu} x^\nu.$$  

(2.7)

Here $B_{\mu\nu}$ is a inverse matrix of $\theta^{\mu\nu}$. $\hat{\partial}_\mu$ satisfies following commutation relations:

$$[\hat{\partial}_\mu, x^\nu] = \delta^\nu_\mu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = iB_{\mu\nu}.\quad (2.8)$$
One can check the following equation:

\[ U_c x^\mu U_c^\dagger = x^\mu + c^\mu. \]  \hspace{1cm} (2.9)

We define **derivative of operators** \( \hat{O} \in \text{End} \mathcal{H} \) by

\[ \partial_\mu \hat{O} := \lim_{\delta c^\mu \to 0} \frac{1}{\delta c^\mu} \left( U_{\delta c^\mu} \hat{O} U_{\delta c^\mu}^\dagger - \hat{O} \right) = [\hat{\partial}_\mu, \hat{O}]. \]  \hspace{1cm} (2.10)

The action of two derivatives commutes:

\[ \partial_\mu \partial_\nu \hat{O} - \partial_\nu \partial_\mu \hat{O} = [\hat{\partial}_\mu, [\hat{\partial}_\nu, \hat{O}]] - (\mu \leftrightarrow \nu) = 0. \]  \hspace{1cm} (2.11)

Operator \( \hat{O} \) is called **bounded operator** if

\[ \forall |\phi\rangle \in \text{Dom}(\hat{O}), \quad ||\hat{O}|\phi\rangle|| \leq C|||\phi\rangle||, \]  \hspace{1cm} (2.12)

for some constant \( C > 0 \), where \( \text{Dom}(\hat{O}) \) is a domain of operator \( \hat{O} \). The norm of bounded operators are defined by

\[ ||\hat{O}|| := \sup |||\hat{O}|\phi\rangle|| \quad |\phi\rangle \neq 0, |\phi\rangle \in \text{Dom}(\hat{O}), \]  \hspace{1cm} (2.13)

where sup means the supremum. We call the operator **smooth** when the derivative of the operator is a bounded operator. We shall consider the algebra of smooth bounded operators and denote this algebra by \( \mathcal{A} \).

The \( U(n) \) gauge field on noncommutative \( \mathbb{R}^4 \) is defined as follows. First we consider \( n \)-dimensional vector space \( \mathcal{A}^n := \mathbb{C}^n \otimes \mathcal{A} \). The elements of \( \mathcal{A}^n \) can be thought of as \( n \)-dimensional vectors with their entries in \( \mathcal{A} \). Let us consider the unitary action on the element of \( \mathcal{A}^n \):

\[ \phi \to U\phi. \]  \hspace{1cm} (2.14)

Here \( U \in \mathbb{M}_n(\mathcal{A}) \) \((\mathbb{M}_n(\mathcal{A})\) denotes the algebra of \( n \times n \) matrices with their entries in \( \mathcal{A} \) and satisfying \( UU^\dagger = U^\dagger U = \text{Id}_{\mathbb{M}_n(\mathcal{A})} \), where \( \text{Id}_{\mathbb{M}_n(\mathcal{A})} \) is the identity operator in \( \mathbb{M}_n(\mathcal{A}) \). In general \( U \) depends on \( z \) and \( \bar{z} \), and hence we regard this unitary transformation as gauge transformation. We define the action of exterior derivative \( d \) by

\[ da := (\partial_\mu a) dx^\mu, \quad a \in \mathcal{A}. \]  \hspace{1cm} (2.15)

We define the covariant derivative of \( \phi \in \mathcal{A}^n \) as a derivative which transforms covariantly under the gauge transformation (2.14), i.e.

\[ D\phi \to UD\phi, \quad D = d + A. \]  \hspace{1cm} (2.16)
Here the $U(n)$ gauge field $A$ is introduced to ensure the covariance. $A$ is a matrix valued one-form: $A = A_\mu dx^\mu$ and $A_\mu \in M_n(\mathcal{A})$ is anti-Hermitian. $dx^\mu$ commute with $x^\mu$ and anti-commute among themselves, and hence $d^2a = 0$ for $a \in \mathcal{A}$. From (2.14) and (2.16), the covariant derivative transforms as

$$D \rightarrow UDU^\dagger.$$  

(2.17)

Hence the gauge field $A$ transforms as

$$A \rightarrow UAU^\dagger + UdU^\dagger.$$  

(2.18)

The field strength is defined by

$$F := D^2 = dA + A^2.$$  

(2.19)

We can construct a gauge invariant action $S$ as follows:

$$S = -\frac{1}{g^2}(2\pi)^2 \sqrt{\text{det} \theta} \text{Tr} F \wedge *F,$$

(2.20)

where $\text{Tr}$ denotes the trace over $\mathcal{H}^n := \mathbb{C}^n \otimes \mathcal{H}$ and $*$ is the Hodge star.\footnote{In this paper we only consider the case where the metric on $\mathbb{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$.} If we use the operator symbols and the star product, (2.20) can be rewritten as\footnote{For the explicit form of the map from operators to operator symbols, see for example [6][2].}

$$S = -\frac{1}{4g^2} \int d^4x \text{tr} F_{\mu\nu} * F^{\mu\nu}.$$  

(2.21)

Here $\text{tr}$ denotes the trace over the $U(n)$ gauge group. In the above, and throughout this article, we use the same letters for operators and corresponding operator symbols for notational simplicity.

Next let us consider gauge theory with projection [2].\footnote{For the roles of projections in noncommutative geometry, see for example [12][13].} A projection $p$ is an Hermitian idempotent element in $M_n(\mathcal{A})$: $p^\dagger = p$, $p^2 = p$. We consider vector space $p\mathcal{A}^n := \{ \phi_p \in \mathcal{A}^n : \phi_p = p\phi_p \}$. We can consider a unitary action on $p\mathcal{A}^n$ (which is unitary in the restricted vector space $p\mathcal{A}^n$):

$$\phi_p \rightarrow U\phi_p, \quad U^\dagger U = UU^\dagger = p.$$  

(2.22)

We can construct covariant derivative $D_p$ for $p\mathcal{A}^n$ by

$$D_p = pd + A_p, \quad A_p = pA_p.$$  

(2.23)

We require $D_p\phi_p$ to transform covariantly under the unitary transformation:

$$D_p\phi_p \rightarrow UD_p\phi_p.$$  

(2.24)
Then the covariant derivative $D_p$ must transforms as

$$D_p \rightarrow UD_pU^\dagger.$$  \hspace{1cm} (2.25)

For any $\phi'_p \in pA^n$, following equation holds

$$UD_pU^\dagger \phi'_p = U(pd + A_p)U^\dagger \phi'_p = Ud(U^\dagger \phi'_p) + U A_p U^\dagger \phi'_p$$
$$= UdU^\dagger \phi'_p + U(U^\dagger d\phi'_p) + UA_pU^\dagger \phi'_p \quad (U = Up = pU)$$
$$= pd\phi'_p + (UdU^\dagger + UA_p U^\dagger)\phi'_p.$$  \hspace{1cm} (2.26)

Hence the gauge field $A_p$ transforms as

$$A_p \rightarrow UA_pU^\dagger + (dU^\dagger)A_p.$$  \hspace{1cm} (2.27)

The field strength becomes

$$F := D^2_p$$
$$= p(dA_p)p + A^2_p + pdpdp.$$  \hspace{1cm} (2.28)

Indeed, for arbitrary $\phi_p \in pA^n$,

$$F \phi_p = (pd + A_p)(pd\phi_p + A_p\phi_p)$$
$$= pd(pd\phi_p) + pd(A_p\phi_p) + A_ppd\phi_p + A^2_p \phi_p$$
$$= pd(pd\phi_p) + pdA_p\phi_p + A^2_p \phi_p,$$  \hspace{1cm} (2.29)

and since $\phi_p = p\phi_p$ and $p^2 = p$, the term $pd(pd\phi_p)$ in (2.29) becomes

$$pd(pd\phi) = pd(pd(p\phi_p))$$
$$= pd(pd(p\phi_p))$$
$$= pd(pd(p\phi_p))$$
$$= pdpdp \phi_p - pdpdp \phi_p + pdpdp \phi_p$$
$$= pdpdp \phi_p.$$  \hspace{1cm} (2.30)

We can construct action $S$ which is invariant under the unitary transformation (2.22):

$$S = -\frac{1}{g^2} (2\pi)^2 \sqrt{det \theta} \text{ Tr } F \wedge *F.$$  \hspace{1cm} (2.31)

**Equivalence of Projections** \(^5\)

However, there exists more larger class of transformations under which the action (2.31) is invariant. In this subsection we will describe these transformations. We start from the

\(^5\)For detailed explanations on the equivalence of projections, see [12][14].
definition of the equivalence of projections, and then we treat the equivalence relation as
gauge equivalence.

Projections $p$ and $q$ in the algebra $\mathbb{M}_n(A)$ are said to be equivalent, or Murray-von Neumann equivalent when\(^6\)

$$\exists U \in \mathbb{M}_n(A), \quad p = U^\dagger U \quad \text{and} \quad q = UU^\dagger, \quad (2.32)$$

and denoted as $p \sim q$. These operators satisfy following equations:

$$U = Up = qU, \quad U^\dagger = pU^\dagger = U^\dagger q. \quad (2.33)$$

$$\text{Ker} \ U = \text{Id}_{\mathbb{M}_n(A)} - U^\dagger U, \quad \text{Ker} \ U^\dagger = \text{Id}_{\mathbb{M}_n(A)} - UU^\dagger. \quad (2.34)$$

$$U^\dagger \mathcal{H}^n = U^\dagger U \mathcal{H}^n = p\mathcal{H}^n, \quad U\mathcal{H}^n = UU^\dagger \mathcal{H}^n = q\mathcal{H}^n. \quad (2.35)$$

By choosing orthonormal basis of $p\mathcal{H}^n$ and $q\mathcal{H}^n$, it is easily seen that

$$p \sim q \iff \dim p\mathcal{H}^n = \dim q\mathcal{H}^n. \quad (2.36)$$

Note that $p$ can be equivalent to the identity if $p$ has infinite rank. From (2.35), $U$ can be regarded as a map from $p\mathcal{A}^n$ to $q\mathcal{A}^n$:

$$\phi_p \rightarrow \phi_q = U\phi_p, \quad \phi_p \in p\mathcal{A}^n, \quad \phi_q \in q\mathcal{A}^n. \quad (2.37)$$

We require the covariant derivative of $\phi_p$ is also mapped in the same form as $\phi_p$:

$$D_p\phi_p \rightarrow UD_p\phi_p = D_q\phi_q, \quad (2.38)$$

where $D_q = qd + A_q$ and $A_q = qA_qq$ is a transform of $A_p$. This requirement determines the transformation rule of gauge fields $A_p \rightarrow A_q$ uniquely:

$$D_q U \phi_p = (qd + A_q)U \phi_p = UU^\dagger d(U \phi_p) + UU^\dagger A_q U \phi_p = U(pd + U^\dagger(dU) + U^\dagger A_q U)\phi_p. \quad (2.39)$$

Hence

$$A_p = U^\dagger A_q U + U^\dagger(dU)p. \quad (2.40)$$

\(^6\)Here we consider $\mathbb{M}_n(A)$ as an example, but Murray-von Neumann equivalence can be considered in any $C^*$-algebra.
Then,

\[ UA_p U^\dagger = A_q + U U^\dagger (dU) p U^\dagger \]
\[ = A_q + q (dU) U^\dagger q \]
\[ = A_q + q (d(U U^\dagger) - UdU^\dagger) q \]
\[ = A_q + q (dq - UdU^\dagger) q \]
\[ = A_q - U (dU^\dagger) q. \quad (2.41) \]

Here we have used the basic identity for projections: \( q (dq) q = 0 \). Hence we obtain the reversal formula of (2.40) consistently:

\[ A_q = U A_p U^\dagger + U (dU^\dagger) q. \quad (2.42) \]

The transformation rule (2.42) is similar to the usual gauge transformation, and therefore we also call it gauge transformation, or Murray-von Neumann gauge transformation (MvN gauge transformation) if we stress the difference from the usual gauge transformation on noncommutative space. MvN gauge transformation contains the transformation proposed in [4] as a special case.\(^7\) The transformation rule for the field strength is obtained as

\[ F_p = D_p^2 \rightarrow F_q = D_q^2 \]
\[ = U D_p U^\dagger U D_p U^\dagger = U D_p D_p U^\dagger \]
\[ = U D_p^2 U^\dagger = UF_p U^\dagger. \quad (2.43) \]

The important point is that under the MvN gauge transformation the action (2.31) is invariant. This is because

\[ \text{Tr} \ F_p \wedge *F_p \rightarrow \text{Tr} \ F_q \wedge *F_q \]
\[ = \text{Tr} \ U^\dagger F_p \wedge *F_p U^\dagger \]
\[ = \text{Tr} \ p F_p \wedge *F_p p \]
\[ = \text{Tr} \ F_p \wedge *F_p. \quad (2.44) \]

Here we have used eq.(2.35). The noncommutative \( \mathbb{R}^4 \) is represented by operators \( \text{End} \ \mathcal{H} \). Hence one-to-one map between Hilbert space may be regarded as a noncommutative analog of coordinate transformation. The MvN gauge transformation \( U \) can be regarded as a map from \( p \mathcal{H}^n \) to \( q \mathcal{H}^n \), and thus it can be understood as a mixture of gauge transformation and coordinate transformation on noncommutative \( \mathbb{R}^4 \).

\(^7\)However we regard that the rank of the projection does not change under this transformation as opposed to [4]. For example, \( \text{Id}_{\mathcal{H}} \) and \( \text{Id}_{\mathcal{H}} - |0,0 \rangle \langle 0,0| \) can be Murray-von Neumann equivalent since both have infinite rank (see eq.(2.36)).
3 Application to Instanton on Noncommutative $R^4$

$U(2)$ One-Instanton Solution on Ordinary $R^4$

In order to illustrate the similarity and difference between commutative and noncommu-
tative case, let us first construct the $U(2)$ one-instanton solution by the ADHM method
in the case of ordinary commutative $R^4$. In this subsection, $z$ and $\bar{z}$ represent ordinary
commuting coordinates.

In order to construct instantons by the ADHM method [7], we start from the following
data:

1. A pair of complex hermitian vector spaces $V = \mathbb{C}^k$ and $W = \mathbb{C}^n$.

2. The operators $B_1, B_2 \in Hom(V, V), I \in Hom(W, V), J = Hom(V, W)$ satisfying
the equations $\mu_R = \mu_C = 0,$ where

$$\mu_R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad (3.1)$$

$$\mu_C = [B_1, B_2] + IJ. \quad (3.2)$$

Next we define Dirac-like operator $D_z : V \oplus V \oplus W \to V \oplus V$
by

$$D_z = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix},$$

$$\tau_z = (B_2 - z_2, B_1 - z_1, I),$$

$$\sigma_z^\dagger = (- (B_1^\dagger - \bar{z}_1), B_2^\dagger - \bar{z}_2, J^\dagger). \quad (3.3)$$

The equation $\mu_R = \mu_C = 0$ is equivalent to the set of equations

$$\tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z := \Box_z, \quad \tau_z \sigma_z = 0. \quad (3.4)$$

The second equation means $\text{Im} \sigma_z \in \ker \tau_z$, and therefore $\dim \ker \tau_z/\text{Im} \sigma_z = (2k + n - k) - k = n$. Hence there are $n$ zero-eigenvalue-vectors (we call them zero-modes for short)
of $D_z :$

$$D_z \Psi^{(a)} = 0, \quad a = 1, \ldots, n. \quad (3.5)$$

We can choose orthonormal basis of the space of the zero-modes:

$$\Psi^{(a)}^\dagger \Psi^{(b)} = \delta^{ab}. \quad (3.6)$$

There is a freedom in the choice of the basis:

$$\Psi \to \Psi U^\dagger, \quad \Psi = \begin{pmatrix} \Psi^{(1)} \cdots \Psi^{(n)} \end{pmatrix}, \quad (3.7)$$
where $U$ is an $n \times n$ unitary matrix. $U$ may depend on $z$ and $\bar{z}$, and this change of basis will become $U(n)$ gauge symmetry after we construct gauge fields from the zero-modes. Anti-self-dual $U(n)$ gauge field is constructed by the formula

$$A^{ab} = \Psi^{(a)\dagger}d\Psi^{(b)},$$ (3.8)

where $a$ and $b$ are indices of $U(n)$ gauge group. There is an action of $U(k)$ that does not change (3.8):

$$(B_1, B_2, I, J) \mapsto (uB_1u^{-1}, uB_2u^{-1}, uI, Ju^{-1}), \quad u \in U(k).$$ (3.9)

Therefore the moduli space of anti-self-dual $U(n)$ gauge field with instanton number $k$ is given by

$$\mathcal{M}_0(k, n) = \mu^\dagger_\mathbb{R}(0) \cap \mu^\dagger_\mathbb{C}(0)/U(k),$$ (3.10)

where the action of $U(k)$ is the one given in (3.9). The fixed points of $U(k)$ action in $\mu^\dagger_\mathbb{R}(0) \cap \mu^\dagger_\mathbb{C}(0)$ become singularities after the $U(k)$ quotients. These singularities correspond to the instantons shrinking to zero size, and often called small instanton singularities in physical literatures.

Let us check that the field strength constructed from (3.8) is really anti-self-dual:

$$F = dA + A^2$$
$$= d(\Psi^\dagger d\Psi) + (\Psi^\dagger d\Psi)(\Psi^\dagger d\Psi)$$
$$= d\Psi^\dagger (1 - \Psi \Psi^\dagger) d\Psi.$$ (3.11)

In the above we have suppressed the $U(n)$ indices. One of the important points in the ADHM construction is that $(1 - \Psi \Psi^\dagger)$ is a projection acting on $V \oplus V \oplus W \approx \mathbb{C}^{2k+n}$ and project out the space of zero-modes ($\approx \mathbb{C}^n$). Hence it can be rewritten as

$$1 - \Psi \Psi^\dagger = D_2^\dagger \frac{1}{D_2 D_2^\dagger} D_2$$
$$= \tau_z^\dagger \frac{1}{\tau_z \tau_z^\dagger} \tau_z + \sigma_z \frac{1}{\sigma_z \sigma_z^\dagger} \sigma_z^\dagger$$
$$= \tau_z^\dagger \frac{1}{\square_z} \tau_z + \sigma_z \frac{1}{\square_z} \sigma_z^\dagger,$$ (3.12)

where we have used the notations in (3.4). Since $\tau_z \Psi = \sigma_z^\dagger \Psi = 0$ by definition (3.5), it
follows that \( \tau_z d\Psi = -d\tau_z \Psi, \sigma^1_z d\Psi = -d\sigma^1_z \Psi \). Hence

\[
F = d\Psi^\dagger (1 - \Psi \Psi^\dagger) d\Psi \\
= d\Psi^\dagger \left( \tau_z^1 \frac{1}{\Box_z} \tau_z + \sigma^1_z \frac{1}{\Box_z} \sigma^1_z \right) d\Psi \\
= \Psi^\dagger \left( d\tau_z^1 \frac{1}{\Box_z} d\tau_z + d\sigma^1_z \frac{1}{\Box_z} d\sigma^1_z \right) \Psi \\
= \Psi^\dagger \left( 
\begin{pmatrix}
  dz_1 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{\Box_z} dz_2 & -dz_1 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_2 \frac{1}{\Box_z} dz_1 & 0 \\
 0 & dz_2 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_1 \frac{1}{\Box_z} dz_1 & 0 \\
 0 & 0 & 0
\end{pmatrix}
\right) \Psi \\
:= F^-_{\text{ADHM}}. \tag{3.13}
\]

\( F^-_{\text{ADHM}} \) is anti-self-dual: \( F^-_{\text{ADHM}} + *F^-_{\text{ADHM}} = 0 \).

Now let us construct \( U(2) \) one-instanton solution by the ADHM method. A solution to the ADHM equations is given by

\[
B_1 = B_2 = 0, \quad I = (\rho, 0), \quad J^\dagger = (0, \rho). \tag{3.14}
\]

Then the Dirac-like operator \( \mathcal{D}_z \) becomes

\[
\mathcal{D}_z = \left( 
\begin{array}{ccc}
-z_2 & -z_1 & \rho & 0 \\
\bar{z}_1 & \bar{z}_2 & 0 & \rho \\
\end{array}
\right). \tag{3.15}
\]

We can find following zero-mode:

\[
\Psi_{\text{BPST}} = \left( 
\begin{array}{c}
\rho \\
0 \\
z_2 \\
-z_1 \\
\end{array}
\right) \frac{1}{\sqrt{\rho^2 + \rho^2}}. \tag{3.16}
\]

The gauge field constructed from this zero-mode is nothing but the well known BPST instanton [8]:

\[
A_{\mu \text{BPST}} = \Psi^\dagger_{\text{BPST}} \partial_\mu \Psi_{\text{BPST}} \\
= \frac{\gamma^2}{r^2 + \rho^2} g^{-1} \partial_\mu g, \tag{3.17}
\]

where \( r = \sqrt{x^\mu x_\mu} \) and

\[
g(x) = \frac{x^\mu \sigma_\mu}{r} = \frac{1}{r} \left( 
\begin{array}{cc}
  z_2 & z_1 \\
-z_1 & \bar{z}_2 \\
\end{array}
\right), \quad \sigma_\mu = (i\tau_1, i\tau_2, i\tau_3, 1). \tag{3.18}
\]
Here $\tau_i (i = 1, 2, 3)$ are Pauli matrices. The instanton number is classified by winding number $\pi^3(U(2))$:

$$\frac{1}{16\pi^2} \int d^4x \, \text{tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{16\pi^2} \int d^4x \, \partial_{\mu} K^\mu$$

$$= -\frac{1}{24\pi^2} \int_{S^3} \text{tr} \, g^{-1} dgg^{-1} dg$$

$$= -1. \quad (3.19)$$

Here $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ and

$$K^\mu = 2 \text{tr} \, \epsilon^{\mu\nu\rho\sigma} \left( A_{\nu} \partial_{\rho} A_{\sigma} + \frac{2}{3} A_{\nu} A_{\rho} A_{\sigma} \right). \quad (3.20)$$

For later purpose let us consider the following zero-mode which is not well defined at the origin $r = 0$:

$$\Psi_{\text{sing}} = \begin{pmatrix}
\rho \bar{z}_2 & -\rho z_1 \\
\rho \bar{z}_1 & \rho z_2 \\
\rho & 0 \\
0 & \rho
\end{pmatrix} \frac{1}{r \sqrt{r^2 + \rho^2}}. \quad (3.21)$$

$\Psi_{\text{sing}}$ and $\Psi_{\text{BPST}}$ are related by the “singular” gauge transformation

$$\Psi_{\text{sing}} = \Psi_{\text{BPST}} g^{-1}(x). \quad (3.22)$$

Note that this transformation is not continuous at the origin. Therefore $\Psi_{\text{sing}}$ is not an appropriate zero-mode for the ADHM construction. However in the next subsection we will observe that in the noncommutative case, we can construct a zero-mode similar to $\Psi_{\text{sing}}$, but well defined everywhere!

The gauge field constructed from $\Psi_{\text{sing}}$ is given by

$$A_{\mu, \text{sing}} = g A_{\mu, \text{BPST}} g^{-1} + g g^{-1} \partial_{\mu} g g^{-1} = \left( \frac{r^2}{r^2 + \rho^2} - 1 \right) g \partial_{\mu} g^{-1} = \frac{\rho^2}{r^2 + \rho^2} g \partial_{\mu} g^{-1}, \quad (3.23)$$

which is singular at the origin. Note that the winding of $A_{\mu, \text{BPST}}$ is resolved by the singular gauge transformation $g$.

**U(2) One-instanton Solution on Noncommutative $\mathbb{R}^4$ and MvN Gauge Transformation**

As we have seen in the previous subsection, the moduli space of instantons $\mathcal{M}_0(k, n)$ in (3.10) has small instanton singularities. The resolution of these singularities is given in
The fixed points of $U(k)$ action are removed when we add a constant to the right hand side of (3.1):

$$\mu_R = \zeta, \quad \mu_C = 0. \quad (3.24)$$

Then the quotient space

$$\mathcal{M}_\zeta(k, n) = \mu^{-1}(\zeta \text{Id}_V) \cap \mu_C^{-1}(0) / U(k). \quad (3.25)$$

is no longer singular. The modification in (3.24) modifies the key equation (3.4) if we use ordinary commutative coordinates on $\mathbb{R}^4$. However it was found in [1] that if we use noncommutative coordinates $z_i, \overline{z}_i (i = 1, 2)$ which satisfies $[z_1, \overline{z}_1] + [z_2, \overline{z}_2] = -\zeta$, $\tau_z$ and $\sigma_z$ do satisfy (3.4).

We define operator $\mathcal{D}_z : (V \oplus V \oplus W) \otimes \mathcal{A} \to (V \oplus V) \otimes \mathcal{A}$ by the same formula (3.3):

$$\mathcal{D}_z = \left( \begin{array}{c}
\tau_z \\
\sigma_z^\dagger
\end{array} \right),$$

$$\tau_z = (B_2 - z_2, B_1 - z_1, I),$$

$$\sigma_z^\dagger = (-B_1^\dagger - \overline{z}_1), B_2^\dagger - \overline{z}_2, J^\dagger). \quad (3.26)$$

The operator $\mathcal{D}_z \mathcal{D}_z^\dagger : (V \oplus V) \otimes \mathcal{A} \to (V \oplus V) \otimes \mathcal{A}$ has a block diagonal form,

$$\mathcal{D}_z \mathcal{D}_z^\dagger = \begin{pmatrix} \mathbb{D}_z & 0 \\ 0 & \mathbb{D}_z \end{pmatrix}, \quad \mathbb{D}_z \equiv \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z, \quad (3.27)$$

which is a consequence of (3.4) and important for the ADHM construction. Next we look for solutions of the equation

$$\mathcal{D}_z \Psi^{(a)} = 0 \quad (a = 1, \ldots, n), \quad (3.28)$$

where the components of $\Psi^{(a)}$ are operators: $\Psi^{(a)} : \mathcal{A} \to (V \oplus V \oplus W) \otimes \mathcal{A}$. There is an important property that $\Psi$ must satisfy (see (3.12)):

$$1 - \Psi \Psi^\dagger = \mathcal{D}_z \mathcal{D}_z^\dagger \frac{1}{\mathbb{D}_z \mathbb{D}_z^\dagger} \mathbb{D}_z. \quad (3.29)$$

This equation contains following two requirements. First, $\Psi$ must contain all the vector zero-modes [2] on the left: The vector zero-mode $|U\rangle$ is an element of $\mathcal{H}^\oplus \oplus \mathcal{H}^\oplus \oplus \mathcal{H}^\oplus \oplus \mathcal{H}^\oplus \oplus \mathcal{H}^\oplus$ which satisfies

$$\mathcal{D}_z |U\rangle = 0. \quad (3.30)$$

The operator zero-mode $\Psi$ can be constructed from vector zero-modes. In the case when the gauge group is $U(1)$, the vector zero-modes are fully classified [10][11]. Second, (3.29)
imposes normalization condition for $\Psi$. The feature peculiar to the noncommutative case is that there may be some states in $\mathcal{H}$ which are annihilated by $\Psi^{(a)}$ for some $a$ \cite{1}\cite{2}. More precisely, all the components of $\Psi^{(a)}$ annihilate those states. Then we can normalize $\Psi^{(a)}$ only in the subspace of $\mathcal{H}$ which is not annihilated by $\Psi^{(a)}$. It means that $\Psi$ is normalized as

$$\Psi^* \Psi = p,$$

(3.31)

where $p \in M_n(\mathcal{A})$ is a projection to the states which are not annihilated by $\Psi$. If the operator zero-mode satisfies (3.29) and normalized as (3.31), we can construct anti-self-dual gauge field $A_p$ by the formula

$$A_p = \Psi^* (d\Psi) \Psi^* \Psi.$$

(3.32)

$A_p$ is anti-self-dual as a gauge connection for $p\mathcal{A}^n$, i.e. if we consider the covariant derivative for $p\mathcal{A}^n$: $D_p = pd + A_p$. When the gauge group is $U(1)$, there is a natural choice for the normalization condition:

$$\Psi^* \Psi = p_I,$$

(3.33)

where $p_I$ is a projection to the ideal states described in \cite{2}. We call the zero-mode normalized minimal operator zero-mode when it is normalized as in (3.33). Then the covariant derivative for $p_I\mathcal{A}$:

$$D_{p_I} = p_I d + A_{p_I}$$

(3.34)

gives anti-self-dual field strength.

Because of the associativity of the operator multiplication, there is a freedom for the choice of the operator zero-mode:

$$\Psi \to \Psi U^\dagger,$$

(3.35)

where $U^\dagger U = p, \quad UU^\dagger = q$ and $q \in M_n(\mathcal{A})$ is a projection. It is apparent that $\Psi U^\dagger$ satisfies (3.29) if $\Psi$ satisfies (3.29):

$$\Psi U^\dagger (\Psi U)^\dagger = \Psi U^\dagger U \Psi^\dagger = \Psi p \Psi^\dagger = \Psi \Psi^\dagger.$$

(3.36)

It is also easily seen that this change of zero-modes corresponds to the MvN gauge transformation. Indeed,

$$A_p = \Psi^* (d\Psi)(\Psi^* \Psi) \to (U \Psi^\dagger)(d(U \Psi^\dagger))(U \Psi^\dagger \Psi U^\dagger)$$

$$= U(\Psi^\dagger d\Psi)U^\dagger (U \Psi^\dagger \Psi U^\dagger) + U \Psi^\dagger \Psi (dU^\dagger)(U \Psi^\dagger \Psi U^\dagger)$$

$$= UA_p U^\dagger + U(dU^\dagger)q,$$

(3.37)
which is nothing but the MvN gauge transformation (2.42).

Although we can choose arbitrary “MvN gauge” (or arbitrary projection), there are not so many gauge choices which are convenient or physically interesting. In the case of $U(1)$ instanton, the most natural choice may be the one that corresponds to the projection to the ideal states. However, in the case of $U(2)$ instanton, there is another choice which is physically interesting, as will be explained below.

Let us construct $U(2)$ one-instanton solution by the ADHM method. From hereafter we set $\zeta = 2$. A solution to the ADHM equation (3.24) is given by

$$B_1 = B_2 = 0, \quad I = (\sqrt{\rho^2 + 2} \ 0), \quad J = (0 \ \rho).$$ (3.38)

Then the Dirac-like operator $D_z$ becomes

$$D_z = \begin{pmatrix} -z_2 & -z_1 & \sqrt{\rho^2 + 2} & 0 \\ \bar{z}_1 & \bar{z}_2 & 0 & \rho \end{pmatrix}. $$ (3.39)

The operator zero-mode can be obtained as

$$\Psi_{\text{min}} = \begin{pmatrix} \Psi_{\text{min}}^{(1)} & \Psi_{\text{min}}^{(2)} \end{pmatrix}, \quad \Psi_{\text{min}}^{(1)} = \frac{1}{\sqrt{\hat{N}(\hat{N} + 2 + \rho^2)}} \begin{pmatrix} \sqrt{\rho^2 + 2} \bar{z}_2 \\ \sqrt{\rho^2 + 2} \bar{z}_1 \\ z_1 \bar{z}_1 + z_2 \bar{z}_2 \\ 0 \end{pmatrix},$$

$$\Psi_{\text{min}}^{(2)} = \frac{1}{\sqrt{\hat{N}(\hat{N} + 2)(\hat{N} + 2 + \rho^2)}} \begin{pmatrix} -\rho z_1 \\ \rho z_2 \\ 0 \\ z_1 \bar{z}_1 + z_2 \bar{z}_2 + 2 \end{pmatrix},$$ (3.40)

where $\frac{1}{\sqrt{\hat{N}}}$ is defined as

$$\frac{1}{\sqrt{\hat{N}}} = \sum_{(n_1, n_2) \neq (0,0)} \frac{1}{\sqrt{n_1 + n_2}} |n_1, n_2\rangle \langle n_1, n_2|,$$ (3.41)

i.e. when we consider the inverse of $\sqrt{\hat{N}}$ we omit the kernel of $\sqrt{\hat{N}}$, that is, $|0,0\rangle$, from the Hilbert space. Hence $\frac{1}{\sqrt{\hat{N}}}$ is a well defined operator. This is an essential point in the construction of instantons on noncommutative $\mathbb{R}^4$ [2]. When $\rho = 0$, the contribution of $\Psi_{\text{min}}^{(2)}$ to the field strength vanishes whereas $\Psi_{\text{min}}^{(1)}$ reduces to the normalized minimal operator zero-mode in $U(1)$ one-instanton solution [2]. The operator zero-mode $\Psi_{\text{min}}$ is normalized as

$$\Psi_{\text{min}}^\dagger \Psi_{\text{min}} = p,$$ (3.42)
where $p$ is a projection in $\mathbb{M}_2(\mathcal{A})$:

$$p = \begin{pmatrix} \text{Id}_\mathcal{H} - |0\rangle \langle 0| & 0 \\ 0 & \text{Id}_\mathcal{H} \end{pmatrix}. \quad (3.43)$$

Although in the case where the gauge group is $U(2)$ the vector zero-modes have not been classified at the moment, we can directly check that the equation (3.29) holds in this case:

$$D_z \frac{1}{D_z D_z^\dagger} D_z = 1 - \Psi_{\text{min}} \Psi_{\text{min}}^\dagger. \quad (3.44)$$

Therefore the connection

$$D_p = pd + A_p,$$

$$A_p = \Psi_{\text{min}}^\dagger (d\Psi_{\text{min}})(\Psi_{\text{min}}^\dagger \Psi_{\text{min}}) \quad (3.45)$$

gives anti-self-dual field strength.

Since the projection $p$ has infinite rank as an operator in $\mathbb{M}_2(\mathcal{A})$, it is Murray-von Neumann equivalent to the identity operator $\text{Id}_{\mathbb{M}_2(\mathcal{A})}$. So let us MvN gauge transform $p$ to $\text{Id}_{\mathbb{M}_2(\mathcal{A})}$. In order to do so one seeks for the operator $U \in \mathbb{M}_2(\mathcal{A})$ which satisfies

$$U^\dagger U = p, \quad UU^\dagger = \text{Id}_{\mathbb{M}_2(\mathcal{A})}. \quad (3.46)$$

Of course there are (infinitely) many choices for such $U$. However there is a choice which has a physically interesting interpretation. Let us consider following operator $U$ which satisfies (3.46):

$$U^\dagger = \begin{pmatrix} \frac{1}{\sqrt{N}} \bar{z}_2 & \frac{1}{\sqrt{N+2}} \bar{z}_1 \\ -\frac{1}{\sqrt{N}} \bar{z}_1 & \frac{1}{\sqrt{N+2}} \bar{z}_2 \end{pmatrix}. \quad (3.47)$$

Notice the similarity between this operator and (the inverse of) the singular gauge transformation (3.22) in the commutative case. However the operator $U$ is well defined unlike (3.18) (remember that $\frac{1}{\sqrt{N}}$ is defined as in (3.41)). Hence the MvN gauge transformation in this case can be understood as a noncommutative resolution of the singular gauge transformation (3.22)!

After the MvN gauge transformation the covariant derivative takes the familiar form, i.e. without projection operator on the left side of the derivative:

$$D = d + A. \quad (3.48)$$

Here the gauge field $A$ is constructed from the gauge transformed zero-mode $\Psi_{\text{BPST}}^* = \Psi_{\text{min}} U^\dagger$:

$$A = A_{\text{BPST}}^* = \Psi_{\text{BPST}}^* d\Psi_{\text{BPST}}^*. \quad (3.49)$$
If we express the gauge fields using operator symbols, the long \( r \) behavior of \( A_{\text{BPST}}^* \) is the same as that of the BPST instanton \( A_{\text{BPST}} \) in commutative case, and the instanton number \( \frac{1}{8\pi^2} \int d^4x \, \text{tr} \, F \wedge F \) is classified by \( \pi^3(U(2)) \), as in (3.19). On the other hand the large \( r \) behavior of \( A_p \) which is constructed from \( \Psi_{\min} \) is the same as the one in singular configuration \( A_{\text{sing}} \) in commutative case. Therefore the instanton number is not classified by \( \pi^3(U(2)) \) in this gauge. However the instanton number itself does not change under the MvN gauge transformation, and in this case the instanton number count the dimension of the projection \( (1 - p) \), as described below. We define new gauge field \( A'_\mu \) for a notational convenience:

\[
p(\hat{\partial}_\mu + A_{\mu p})p = \hat{\partial}_\mu - (1 - p)\hat{\partial}_\mu - (1 - p)\hat{\partial}_\mu (1 - p) + A_{\mu p} = \hat{\partial}_\mu + A'_{\mu}.
\]

(3.50)

Here \( \hat{\partial}_\mu \) is the derivative operator (2.7). \( A' \) is not an MvN gauge transform of \( A_p \). The field strength of \( A'_\mu \) is given as

\[
F'_{\mu\nu} = [\hat{\partial}_\mu + A'_{\mu}, \hat{\partial}_\nu + A'_{\nu}] - [\hat{\partial}_\mu, \hat{\partial}_\nu] = p(iB_{\mu\nu} + F^{-}_{\mu\nu\min})p - iB_{\mu\nu} = (1 - p)iB_{\mu\nu}(1 - p) + F_{\mu\nu\min}.
\]

(3.51)

where \( F^{-}_{\min} = D^2_p \). It can be shown that the operator symbol of \( A_p \) decays like \( O(r^{-3}) \) for large \( r \) (recall the resemblance between the minimal zero-mode (3.40) and the singular zero-mode (3.21)). The operator symbol of \( (1 - p) \) decays like \( \sim e^{-\frac{3}{2}r^2} \). \( \theta^{\mu\nu} \) in the star product appear as a multiplication of the combination \( \theta/r^2 \) for large \( r \) and hence does not contribute to the surface integral at large \( r \). Taking all these accounts, the instanton number of \( A' \) vanishes:

\[
\frac{1}{16\pi^2} (2\pi)^2 \sqrt{\det \theta} \text{Tr} F'_{\mu\nu} \tilde{F}'_{\mu\nu} = \frac{1}{16\pi^2} \int d^4x \, \text{tr} \, F'_{\mu\nu} \star \tilde{F}'_{\mu\nu} = \frac{1}{16\pi^2} \int d^4x \, \partial_\mu K^\mu = 0.
\]

(3.52)

Here

\[
K^\mu = 2 \text{tr} \, \epsilon^{\mu\nu\rho\sigma} \left( A'_{\nu} \star \partial_\rho A'_{\sigma} + \frac{2}{3} A'_{\nu} \star A'_{\rho} \star A'_{\sigma} \right).
\]

(3.53)

On the other hand,

\[
\frac{1}{16\pi^2} (2\pi)^2 \sqrt{\det \theta} \text{Tr} F'_{\mu\nu} \tilde{F}'_{\mu\nu} = \frac{1}{16\pi^2} (2\pi)^2 \sqrt{\det \theta} \text{Tr} \left[ (1 - p)B_{\mu\nu} \tilde{B}^{\mu\nu}(1 - p) - F^{-}_{\mu\nu\min} \tilde{F}^{-}_{\mu\nu\min} \right],
\]

(3.54)
($B_{\mu \nu}$ is self-dual as we have set $\theta^{\mu \nu}$ self-dual). Thus the instanton number counts the dimension of the projection $(1 - p)$:

$$
\frac{1}{16\pi^2} (2\pi)^2 \sqrt{det\theta} \text{Tr} F^{-\mu \nu \min}_\mu F^\mu \nu^{-\min}_\mu \\
= \frac{1}{16\pi^2} (2\pi)^2 \sqrt{det\theta} \text{Tr} (1 - p) B_{\mu \nu} \tilde{B}^{\mu \nu} (1 - p) \\
= \text{dim} (1 - p).
$$

(3.55)

4 Conclusion

In this article the formalism that describes the equivalence of projections as a kind of gauge equivalence on noncommutative space is given. We apply this formalism to the $U(2)$ one-instanton solution on noncommutative $\mathbb{R}^4$. The gauge equivalence between BPST type configuration with winding number one and the configuration without winding but with projection is shown. In this case the gauge transformation can be understood as a non-commutative resolution of the singular gauge transformation in ordinary $\mathbb{R}^4$. Recall that the projection describes holes on noncommutative $\mathbb{R}^4$ [2]. Hence this formalism gives a unified description to the intriguing mixing of gauge fields and geometry in noncommutative space [2][3].

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