Holomorphic mappings between domains with low boundary regularity

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Abstract. We study the boundary regularity of proper holomorphic mappings between strictly pseudoconvex domains with boundaries of low regularity.

Keywords: strictly pseudoconvex domain, proper holomorphic mapping, boundary regularity.

§1. Introduction

This paper considers the old problem of the precise boundary regularity of a proper holomorphic mapping between two strictly pseudoconvex domains in the case when at least one of the boundaries is of regularity exactly $C^2$. We prove the following theorem.

Theorem 1.1. Let $\Omega_1$ and $\Omega_2$ be bounded strictly pseudoconvex domains in $\mathbb{C}^n$. Suppose that the boundary of $\Omega_1$ is of class $C^{2+\varepsilon}$ with $\varepsilon > 0$ and the boundary of $\Omega_2$ is of class $C^2$. Assume that $f: \Omega_1 \to \Omega_2$ is a proper holomorphic mapping. Then $f$ extends to a mapping of class $C^\alpha(\overline{\Omega}_1)$ for each $\alpha \in [0, 1]$.

At present the boundary regularity of proper or biholomorphic mappings between strictly pseudoconvex domains is very well understood. Fefferman [1] proved that a biholomorphic mapping between strictly pseudoconvex domains with boundaries of class $C^\infty$ extends to a $C^\infty$ diffeomorphism between their closures. His proof was based on the study of asymptotic behaviour of the Bergman kernel near the boundary. Later several different approaches were explored. Some of them enable one to study the case when the boundaries of the domains are of finite smoothness. Pinchuk and Khasanov [2] proved that for every real $s > 2$, a proper holomorphic mapping between strictly pseudoconvex domains with $C^s$-boundaries extends to the boundary as a mapping of class $C^{s-1}$ if $s$ is not an integer or of class $C^{s-1-\varepsilon}$ with any $\varepsilon > 0$ when $s$ is an integer. Khurumov [3] proved that a similar result still remains true with the loss of regularity at $1/2$. The question of the precise regularity in the case when at least one domain has boundary of class $C^2$ has remained open for a long time. Khurumov announced without any details that his result also remains true in this case but, to the best of my knowledge, a detailed proof is not

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available. The only well-known result (see, for example, [4]) states that the mapping extends to the boundary as a Hölder 1/2-continuous mapping. Theorem 1.1 takes the first step towards a definitive answer.

Our main tool is the result of Chirka, Coupet and the author [5] giving a precise boundary regularity (in the Hölder scale) of a complex disc with boundary glued to a totally real manifold of class $C^1$. This enables us to improve the boundary regularity of a mapping. Note that Theorem 1.1 is new even in the case when the boundary of $\Omega_1$ is of class $C^\infty$ or real analytic (for example, when $\Omega_1$ is the unit ball). It is quite possible that the method of the present paper also enables one to deal with the case when the boundary of $\Omega_1$ is of class exactly $C^2$. In the last section we sketch suitable modifications of our argument.

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§2. Preliminaries

We briefly recall some well-known definitions and basic notation.

2.1. Classes of domains and functions. Let $\Omega$ be a domain in $\mathbb{C}^n$. For a positive integer $k$, denote by $C^k(\Omega)$ the space of $C^k$-smooth complex-valued functions on $\Omega$. Also $C^k(\overline{\Omega})$ denotes the class of functions whose partial derivatives up to order $k$ extend to continuous functions on $\overline{\Omega}$. Let $s > 0$ be a non-integral real number and let $k$ be its integer part. Then $C^s(\Omega)$ denotes the space of functions in $C^k(\overline{\Omega})$ whose partial derivatives of order $k$ are (globally) Hölder $(s-k)$-continuous on $\Omega$. These derivatives automatically satisfy the Hölder condition on $\overline{\Omega}$ so that the notation $C^s(\overline{\Omega})$ for this space of functions is also appropriate.

A closed real submanifold $E$ of a domain $\Omega \subset \mathbb{C}^n$ is of class $C^s$ (with real $s \geq 1$) if, for every point $p \in E$, one can find an open neighbourhood $U$ of $p$ and a map $\rho: U \to \mathbb{R}^d$ of maximal rank $d < 2n$ and of class $C^s$ such that $E \cap U = \rho^{-1}(0)$. Then $\rho$ is called a (vector-valued) local defining function for $E$. The positive integer $d$ is the real codimension of $E$. In the most important special case $d = 1$ we obtain the class of real hypersurfaces.

Let $J$ be the standard complex structure on $\mathbb{C}^n$. In other words, $J$ acts on a vector $v$ by multiplication by $i$, that is, $Jv = iv$. For every $p \in E$, the holomorphic tangent space $H_pE := T_pE \cap J(T_pE)$ is the maximal complex subspace of the tangent space $T_pE$ of $E$ at $p$. Clearly, $H_pE = \{v \in \mathbb{C}^n : \partial \rho(p)v = 0\}$. The complex dimension of $H_pE$ is called the CR-dimension of $E$ at $p$. A manifold $E$ is called a CR (Cauchy–Riemann) manifold if its CR-dimension is independent of $p \in E$.

A real submanifold $E \subset \Omega$ is said to be generic (or generating) if the complex span of $T_pE$ coincides with $\mathbb{C}^n$ for all $p \in E$. Note that every generic manifold of real codimension $d$ is a CR manifold of CR-dimension $n-d$. A function $\rho = (\rho_1, \ldots, \rho_d)$ defines a generic manifold if $\partial \rho_1 \wedge \cdots \wedge \partial \rho_d \neq 0$. Of special importance are the so-called totally real manifolds, that is, submanifolds $E$ such that $H_pE = \{0\}$ for every $p \in E$. A totally real submanifold in $\mathbb{C}^n$ is generic if and only if its real dimension is equal to $n$, which is the maximal possible value for the dimension of a totally real manifold.
Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Suppose that its boundary $b\Omega$ is a (compact) real hypersurface of class $C^s$ in $\mathbb{C}^n$. Then there is a $C^s$-smooth real function $\rho$ on a neighbourhood $U$ of the closure $\overline{\Omega}$ such that $\Omega = \{ \rho < 0 \}$ and $d\rho|_{b\Omega} \neq 0$. We call such a function $\rho$ a global defining function. If $s \geq 2$, one can consider the Levi form of $\rho$:

$$L(\rho, p, v) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)v_j \overline{v}_k.$$  \([1]\)

A bounded domain $\Omega$ with $C^2$-boundary is said to be strictly pseudoconvex if $L(\rho, p, v) > 0$ for every non-zero vector $v \in H_p(b\Omega)$.

2.2. The Kobayashi–Royden pseudometric. Denote by $D = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ the unit disc in $\mathbb{C}$. We also denote by $B$ the unit ball in $\mathbb{C}^n$ (the dimension $n$ will be clear from the context). Let $\Omega$ be a domain in $\mathbb{C}^n$. We denote by $\mathcal{O}(D, \Omega)$ the class of holomorphic maps from $D$ to $\Omega$.

Let $z$ be a point in $\Omega$ and let $v$ be a tangent vector at $z$. The infinitesimal Kobayashi–Royden pseudometric $F_\Omega(z, v)$ is defined as

$$F_\Omega(z, v) = \inf \left\{ \lambda > 0 : \exists h \in \mathcal{O}(D, \Omega) \text{ with } h(0) = z, h'(0) = \frac{v}{\lambda} \right\}. \quad (2)$$

This is a non-negative upper-semicontinuous function on the tangent bundle of $\Omega$. Its integrated form coincides with the ordinary Kobayashi distance. The Kobayashi–Royden metric is decreasing under holomorphic mappings: if $f: \Omega \to \Omega'$ is a holomorphic mapping between two domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, then

$$F_{\Omega'}(f(z), df(z)v) \leq F_\Omega(z, v). \quad (3)$$

In fact, this is the largest metric in the class of infinitesimal metrics that are decreasing under holomorphic mappings. It is easy to obtain an upper bound for $F_\Omega$. Indeed, let $z + R\mathbb{B}$ with $R = \text{dist}(z, b\Omega)$ be the ball contained in $\Omega$. It follows by the holomorphic decreasing property applied to the natural inclusion $i: z + R\mathbb{B} \to \Omega$ that the Kobayashi–Royden metric of this ball is bigger than $F_\Omega$. This gives the upper bound

$$F_\Omega(z, v) \leq \frac{|v|}{\text{dist}(z, b\Omega)}. \quad (4)$$

Lower bounds require a considerably more subtle analysis. They have been obtained by several authors employing various methods. Quite general estimates can be established using plurisubharmonic functions.

For example, this approach leads to the following result inspired by the work of Sibony [6].

**Proposition 2.1.** Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $\rho$ be a negative $C^2$-smooth plurisubharmonic function on $\Omega$. Suppose that the partial derivatives of $\rho$ are bounded on $\Omega$ and there is a constant $C_1 > 0$ such that

$$L(\rho, z, v) \geq C_1 |v|^2 \quad (5)$$
for all \( z \) and \( v \). Then there is a constant \( C_2 > 0 \), depending only on the \( C^2 \)-norm of \( \rho \), such that

\[
F_\Omega(z, v) \geq C_2 \left( C_1^2 \frac{|\partial \rho(z), v|^2}{|\rho(z)|^2} + C_1 \frac{|v|^2}{|\rho(z)|^2} \right).
\]

(Note that \( \rho \) is not assumed to be a defining function of \( \Omega \), although this special case is particularly important in applications. The original argument of Sibony assumes that \( \Omega \) is globally bounded, but this condition can be dropped. In fact, the estimate (6) holds on an open subset of \( \Omega \) where (5) is satisfied. Therefore it can be used to localize the Kobayashi–Royden metric. Note also that \( \Omega \) is not assumed to be bounded or hyperbolic. Of course, in the present paper the bounded case is sufficient.)

The following localization principle for the Kobayashi–Royden pseudometric is established in [5]. The proof is also based on the methods of pluripotential theory.

**Proposition 2.2.** Let \( D \) be a domain in \( \mathbb{C}^n \). Suppose that \( u \) is a negative plurisubharmonic function on \( D \) such that the following conditions hold for some constants \( \varepsilon, B > 0 \):

(i) the function \( u(z) - \varepsilon |z|^2 \) is plurisubharmonic on \( D \cap 3B \);
(ii) \( |u| \leq B \) on \( 2B \).

Then there is a positive constant \( M = M(\varepsilon, B) \), independent of \( u \), such that

\[
F_D(w, \xi) \geq M|\xi| |u(w)|^{-1/2}
\]

when \( w \in D \cap 2B \).

Note that this result also gives a lower bound for the metric. This will be used in an essential way in our proof.

**2.3. Complex discs.** Consider the wedge-type domain

\[
W = \{ z \in \mathbb{C}^m : \phi_j(z) < 0, j = 1, \ldots, m \}
\]

with edge

\[
E = \{ z \in \mathbb{C}^m : \phi_j(z) = 0, j = 1, \ldots, m \}.
\]

We assume that the defining functions \( \phi_j \) are of class \( C^{1+\alpha} \) with \( \alpha > 0 \). Furthermore, as usual, we assume that \( E \) is a generic manifold, that is, \( \partial \phi_1 \wedge \cdots \wedge \partial \phi_m \neq 0 \) in a neighbourhood of \( E \).

Given \( \delta > 0 \) (which is supposed to be small enough), we also define a shrunken wedge

\[
W_\delta = \left\{ z \in \mathbb{C}^m : \phi_j - \delta \sum_{l \neq j} \phi_l < 0, j = 1, \ldots, m \right\} \subset W.
\]
Proposition 2.3. Fix $\delta > 0$. Then there is a map $H: \mathbb{D} \times \mathbb{R}^{m-1} \to W$ of class $C^{1+\alpha}(\mathbb{D} \times \mathbb{R}^{m-1})$, $H: (\zeta, t) \mapsto h_t(\zeta)$, with the following properties:

(i) for every $t \in \mathbb{R}^{m-1}$, the map $\zeta \mapsto h_t(\zeta)$ is holomorphic on $\mathbb{D}$ and $h(\mathbb{D}^+)$ is contained in $E$;

(ii) the curves $h_t(\mathbb{D}^+)$ form a foliation of $E$;

(iii) every disc $h_t(\mathbb{D})$ is transversal to $E$;

(iv) $W_\delta \subset \bigcup_t h_t(\mathbb{D})$.

The above gluing discs argument is often quite useful in the study of totally real submanifolds. It was introduced in [7] and then used by many authors. We sketch the idea of the proof. Without loss of generality, we may assume that in a neighbourhood $\Omega$ of the origin a smooth totally real manifold $E$ is defined by the equation $x = r(x, y)$, where a smooth vector-valued function $r = (r_1, \ldots, r_n)$ satisfies the conditions $r_j(0) = 0$ and $dr_j(0) = 0$. Fix a positive non-integer $s$ and, for every real function $u \in C^s(\mathbb{D})$, consider the Hilbert transform $T: u \mapsto T(u)$. It is uniquely defined by requiring that $u + iT(u)$ is the trace of a function holomorphic on $\mathbb{D}$ and $T(u)$ vanishes at the origin. Explicitly, it is given by the singular integral

$$T(u)(e^{i\theta}) = \frac{1}{2\pi} \text{v. p.} \int_{-\pi}^{\pi} u(e^{it}) \cot\left(\frac{\theta - t}{2}\right) dt.$$ 

This is a classical linear singular integral operator. It is bounded on the space $C^s(\mathbb{D})$. Let $S^+ = \{e^{i\theta}: \theta \in [0, \pi]\}$ and $S^- = \{e^{i\theta}: \theta \in ]\pi, 2\pi[\}$ be the semicircles. Fix a $C^\infty$-smooth real function $\psi_j$ on $\mathbb{D}$ such that $\psi_j|S^+ = 0$ and $\psi_j|S^- < 0$, $j = 1, \ldots, n$. Set $\psi = (\psi_1, \ldots, \psi_n)$. Consider the generalized Bishop equation

$$u(\zeta) = r(u(\zeta), T(u)(\zeta) + c) + t\psi(\zeta), \quad \zeta \in \mathbb{D}, \tag{10}$$

where $c \in \mathbb{R}^m$ and $t = (t_1, \ldots, t_m)$, $t_j \geq 0$, are real parameters. It follows by the implicit function theorem that this equation admits a unique solution $u(c, t) \in C^s(\mathbb{D})$ depending smoothly on the parameters $(c, t)$. Consider the complex discs $f(c, t)(\zeta) = P(u(c, t)(\zeta) + iT(u(c, t))(\zeta))$, where $P$ denotes the Poisson operator of harmonic extension to $\mathbb{D}$:

$$P(u)(\zeta) = \int_{-\pi}^{\pi} K_P(\zeta, t) u(e^{it}) dt.$$ 

Here $K_P$ denotes the Poisson kernel

$$K_P(\zeta, t) = \frac{1}{2\pi} \frac{1 - |\zeta|^2}{|e^{it} - \zeta|^2}.$$ 

The map $(c, t) \mapsto f(c, t)(0)$ (the centres of the discs) is of class $C^s$. Every disc is attached to $E$ along the upper semicircle. It is easy to see that this family of discs fills the wedge $W_\delta(\Omega, E)$ when $\delta > 0$ and the neighbourhood $\Omega$ of the origin are chosen small enough. Indeed, this is immediate when the function $r$ vanishes identically (that is, $E = i\mathbb{R}^m$), while the general case follows by a small perturbation argument.

A detailed proof is contained in many works (see, for example, [8]) so I skip it.
§ 3. Proof of the main result

In this section we prove Theorem 1.1. For the convenience of readers, we recall the general approach to the regularity problem for boundary values of holomorphic mappings and then explain how to modify this construction in order to improve the regularity. In what follows we use the notation $C$, $C_1$, ... for positive constants. Their values may change from line to line.

3.1. General construction. The proof of Theorem 1.1 is essentially based on the estimates for the Kobayashi–Royden metric in [5]. However, they are quite different from the classical ones.

One of the first results on the boundary behaviour of holomorphic mappings is the following proposition (see [4]).

Proposition 3.1. Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping between two strictly pseudoconvex domains in $\mathbb{C}^n$ with boundaries of class $C^2$. Then $f$ extends to $\overline{\Omega}_1$ as a H"older $1/2$-continuous mapping.

The proof is based on the estimate for the Kobayashi–Royden metric given by Proposition 2.1 and Hopf’s lemma.

To improve the boundary regularity of $f$, we recall the construction of Pinchuk and Khasanov. Let $\Omega$ be a strictly pseudoconvex domain of class $C^k$, $k \geq 2$.

If $\Omega = \{ \rho < 0 \}$, then we explicitly have

\[ H_p(b\Omega) = \left\{ v \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p) v_j = 0 \right\}. \]

Similarly, denote by $H(b\Omega)$ the holomorphic tangent bundle of $b\Omega$ with fibre $H_p(b\Omega)$ over a point $p \in b\Omega$. Every holomorphic tangent space is a complex hyperplane in $\mathbb{C}^n$ and can be viewed as a point of the complex projective space $\mathbb{CP}^{n-1}$. Therefore the holomorphic tangent bundle is a real submanifold of dimension $2n - 1$ and class $C^{k-1}$ in $\mathbb{C}^n \times \mathbb{CP}^{n-1}$. It is well known (and easily verifiable) that this manifold is totally real when $b\Omega$ is strictly pseudoconvex [2].

Assume that $(\partial \rho/\partial z_n)(p) \neq 0$. Then

\[ H_p(b\Omega) = \left\{ v \in \mathbb{C}^n : v_n = w_1 v_1 + \cdots + w_{n-1} v_{n-1}, \quad w_j = \frac{\rho_{z_j}(p)}{\rho_{z_n}(p)} \right\}. \]

Set $\phi_j(z) = \frac{\rho_{z_j}(z)}{\rho_{z_n}(z)}$, $j = 1, \ldots, n$. If $(w_1, \ldots, w_{n-1})$ are local coordinates in $\mathbb{CP}^{n-1}$ near $H_p(b\Omega)$, then the bundle $H(b\Omega)$ is given in a neighbourhood of $(p, H_p(b\Omega))$ by the equation

\[ H(b\Omega) = \{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^{n-1} : \rho(z) = 0, \quad w_j = \phi_j(z) \} \]

and is a graph over $b\Omega$. Thus, the domain

\[ W(\Omega) = \{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^{n-1} : \rho(z) < 0 \} \]

is a domain with $C^k$-boundary in $\mathbb{C}^n \times \mathbb{C}^{n-1}$, and its boundary contains the $(2n - 1)$-dimensional totally real submanifold $H(b\Omega)$ of regularity $C^{k-1}$. 
The crucial step in the approach of Pinchuk and Khasanov is the following. Let \( f : \Omega_1 \to \Omega_2 \) be a proper holomorphic mapping between strictly pseudoconvex domains. We already know that \( f \) is of class \( C^{1/2}(b\Omega_1) \); in particular, \( f \) is defined on the boundary \( b\Omega \). We can define a lift \( F \) of \( f \) to \( \mathbb{C}^n \times \mathbb{C}^{n-1} \) by the formula

\[
F(z, P) = (f(z), df(z)P).
\]

Here \( P \) is viewed as a hyperplane in \( \mathbb{C}^n \), and \( df(z)P \) is its image under the tangent map \( df(z) \) (recall that a proper holomorphic mapping between strictly pseudoconvex domains has non-vanishing gradient). The map \( F \) is defined on \( W(\Omega_1) \) and takes it to \( W(\Omega_2) \).

The following key result belongs to Pinchuk and Khasanov.

**Lemma 3.2.** The map \( F \) extends continuously to \( W(\Omega_1) \cup H(b\Omega_1) \). This extension (again denoted by \( F \)) satisfies \( F(z, H_z(b\Omega_1)) = (f(z), H_{f(z)}(b\Omega_2)) \) for any \( z \in b\Omega \). That is, \( F(H(b\Omega_1)) \subset H(b\Omega_2) \).

The proof is based on the scaling method, which is in its turn based on the estimates of the Kobayashi–Royden metric in Proposition 2.1 and an argument using normal families. The further argument of Pinchuk and Khasanov concerning the higher regularity of \( F \) requires the \( C^0 \)-regularity of \( b\Omega_1 \) and \( b\Omega_2 \) for some \( s > 2 \). This is the reason why the \( C^2 \)-case requires a different approach.

### 3.2. Improving the regularity

Consider a wedge-type domain \( W \) given by (7) with a generic edge \( E \) given by (8). Also, consider the shrunken wedge \( W_{\delta} \) given by (9).

We assume that the defining functions \( \phi_j \) are of class \( C^{1+\alpha} \), where \( \alpha > 0 \), and \( \partial \phi_1 \wedge \cdots \wedge \partial \phi_m \neq 0 \) in a neighbourhood of \( E \).

Note that there is a constant \( C > 0 \) such that for every point \( z \in W_{\delta} \) we have

\[
C^{-1} \text{dist}(z, bW) \leq \text{dist}(z, E) \leq C \text{dist}(z, bW). \tag{13}
\]

We can now prove the following theorem.

**Theorem 3.3.** Let \( N \) be an \( n \)-dimensional totally real manifold of class \( C^1 \) in \( \mathbb{C}^n \) and let \( W \) be a wedge in \( \mathbb{C}^m \). Consider a holomorphic map \( f : W \to \mathbb{C}^n \) such that \( f \) is continuous on \( W \cup E \) and \( f(E) \subset N \). Then, for every \( \delta > 0 \) and every \( \alpha < 1 \), the map \( f \) extends to a Hölder \( \alpha \)-continuous map on \( W_{\delta} \) and \( E \).

In the case when \( m = 1 \), that is, \( W \) is the unit disc, a much more general result was obtained in [5]. Here we adapt the proof in [5] to our situation.

We begin the proof of the theorem with the following well-known lemma.

**Lemma 3.4.** Let \( \phi \) be a non-negative subharmonic function on \( \mathbb{D} \) with \( \phi(\zeta) \to 0 \) as \( \zeta \) tends to an open arc \( \gamma \subset b\mathbb{D} \). Then, for every compact set \( K \subset \mathbb{D} \cup \gamma \), there is a constant \( C_K > 0 \) such that \( \phi(\zeta) < C_K(1 - |\zeta|) \) for any \( \zeta \in K \cap \mathbb{D} \).

Let \( V \) be a neighbourhood of \( \gamma \cap K \) such that the domain \( V_1 = V \cap \mathbb{D} \) is simply connected and \( \phi < 1 \) on \( V_1 \). Let \( g : \mathbb{D} \to V_1 \) be a conformal mapping. Then, by the reflection principle, \( g^{-1} \) extends holomorphically across \( \gamma \). Hence, replacing \( \phi \) by \( \phi \circ g \), we reduce the question to the case of a function which is uniformly bounded.
on $\mathbb{D}$. But then the assertion follows from an obvious estimate of the Poisson kernel. Indeed, it suffices to consider the case when $\phi(e^{it})$ vanishes on the arc $|t| < \tau$ with some $\tau > 0$. When $|\arg \zeta| \leq \tau/2$ and $|t| > \tau$, the Poisson kernel admits the estimate

$$K_P(\zeta, t) \leq \frac{1}{\pi} \frac{1 - |\zeta|}{|e^{it} - |\zeta||^2} \leq \frac{1}{\pi} \frac{1 - |\zeta|}{|e^{i\tau} - e^{i\tau/2}|^2}.$$ 

But we have

$$\phi(\zeta) \leq \int_{-\pi}^{\pi} K_P(\zeta, t) \phi(e^{it}) \, dt.$$ 

This gives the desired estimate and proves the lemma.

As a corollary we have the following assertion.

**Lemma 3.5.** Let $\psi$ be a non-negative plurisubharmonic function on $W$ such that $\psi(z) \to 0$ as $z \to E$. Then, for every fixed $\delta > 0$, there is a constant $C = C_\delta > 0$ such that $\psi(z) \leq C\dist(z, E)$ for any $z \in W_\delta$.

For the proof consider the family $(h_t)$ of complex discs constructed in Lemma 2.3. Then it suffices to apply Lemma 3.4 to $\psi \circ h_t$ since all constants are uniform with respect to $t$.

By Lemma 3.5 there is a constant $C_\delta > 0$ such that, for any $z \in W_\delta$, we have

$$\rho \circ f(z) \leq C\dist(z, E). \tag{14}$$ 

We will need a result of Harvey and Wells [9] and Chirka [10]. Since $N$ is a totally real manifold of class $C^1$, there is a non-negative function $\rho$ of class $C^2$ such that $\rho$ is strictly plurisubharmonic in a neighbourhood of $N$ and $N = \rho^{-1}(0)$. Furthermore, for each $\theta \in [0, 1]$, the function $\rho^\theta$ remains plurisubharmonic in a neighbourhood of $N$.

Suppose that $a \in E$ and $f(a) = p \in N$. We can assume that $p = 0$ and there is an $\varepsilon > 0$ such that $\rho - \varepsilon|z|^2$ is plurisubharmonic on the ball $3\mathbb{B} \subset D$.

**Lemma 3.6.** There is a constant $A > 0$ with the following property. If $z \in W_\delta$ is a point such that $f(z) \in \mathbb{B}$, then

$$\|df(z)\| \leq A \dist(z, E)^{-1/2}$$

(here we are using the operator norm of the tangent map).

**Proof.** Set $d = \dist(z, E)$. By (13) there is a constant $C > 0$ (independent of $z$) such that the ball $z + dC\mathbb{B}$ is contained in $W$. Then it follows from (14) that the image $f(z + dC\mathbb{B})$ is contained in the domain $D_d = \{w \in D : \rho(w) < 2C_1d\}$ for a suitable choice of the constant $C_1 > 0$. Note that the strictly plurisubharmonic function $u_d(w) = \rho - 2C_1d$ is negative on $D_d$. By Proposition 2.2 there is a constant $M > 0$ (independent of $d$) such that for every $w \in D \cap \mathbb{B}$ and any vector $\xi \in \mathbb{C}^n$ we have

$$F_{D_d}(w, \xi) \geq M|\xi||u_d(w)|^{-1/2}.$$

On the other hand, for the Kobayashi–Royden metric on the ball $z + dC\mathbb{B}$ we have the equality $F_{z + dC\mathbb{B}}(z, \tau) = |\tau|/dC$ for any vector $\tau \in \mathbb{C}^m$. Since the Kobayashi–Royden metric is decreasing under holomorphic mappings, one has

$$M|df(z)\tau||u_d(f(z))|^{-1/2} \leq F_{D_d}(f(z), df(z)\tau) \leq F_{z + dC\mathbb{B}}(\zeta, \tau) = \frac{|\tau|}{dC}.$$
Therefore $|df(z)\tau| \leq M^{-1}|u_d(f(z))|^{1/2}|\tau|/dC$. Since $-2C_1 d \leq u_d(f(\zeta)) < 0$, we have $|u_d(f(\zeta))|^{1/2} \leq (2C_1 d)^{1/2}$. This implies the desired estimate and proves the lemma.

It follows from Lemma 3.6 that $f$ is Hölder 1/2-continuous on $W_\delta \cup E$ by an integration argument which is a variation of the classical Hardy–Littlewood theorem (see [11]).

Let us improve the regularity. By result of Chirka [10], $\rho^\theta$ remains plurisubharmonic for every $\theta$, $1/2 < \theta < 1$. The composition $\rho^\theta \circ f$ is defined on a neighbourhood of $W \cup E$.

Applying Lemma 3.5 to this function, we see that $\rho \circ f(z) \leq C \text{dist}(z, E)^{1/\theta}$ for all $z \in W \cup E$.

We now simply repeat the argument above. Let $z \in W_\delta$ be sufficiently close to $a$. We put $d = \text{dist}(z, E)$. The image $f(z + dC\mathbb{B})$ is contained in the domain $D_d = \{w \in D: u_d(w) = \rho(w) - 2C_1 d^{1/\theta} < 0\}$. Repeating the proof of Lemma 3.6, we obtain the inequality $||df(z)|| \leq M^{-1}|u_d(f(\zeta))|^{1/2}/dC$. Since $-2C_1 d^{1/\theta} \leq u_d(f(\zeta)) < 0$, we conclude as above that $||df(\zeta)|| \leq A \text{dist}(z, E)^{1/2\theta-1}$ in a neighbourhood of $a$ in $W$. Hence, $f$ is Hölder 1/2$\theta$-continuous on $W_\delta \cup E$. This proves Theorem 3.3.

We now complete the proof of Theorem 1.1. This follows directly from Theorem 3.3. It suffices to take $E = H(b\Omega_1)$ and $N = H(b\Omega_2)$ and apply the theorem to the lift $F(z, P)$ of $f$. The hypotheses of Theorem 3.3 hold for $F$ by Lemma 3.2. Since $H(b\Omega_1)$ is a graph over $b\Omega_1$, it is easy to choose a wedge $W \subset W(\Omega_1)$ (see (12)) with edge $H(b\Omega_1)$ in such a way that, for every $\delta > 0$, the projection of the shrunken wedge $W_\delta$ to $\mathbb{C}^n$ coincides with $\Omega$. Since $F$ is of class $C^\alpha(W_\delta \cup H(b\Omega_1))$, the original map $f$ is of class $C^\alpha(\overline{\Omega}_1)$ for any $\alpha < 1$.

§ 4. Further results

The natural question arises as to whether the conclusion of Theorem 1.1 still remains true when the boundary of $\Omega_1$ is only of class $C^2$. The answer is affirmative but the proof is much more technical. We sketch two possible approaches.

First of all, we note that the assumption of $C^{2+\varepsilon}$-regularity of $b\Omega_1$ enables us to use Theorem 3.3 since it guarantees that $H(b\Omega_1)$ is of regularity $> 1$. In its turn, the hypothesis of Theorem 3.3 that $E$ is of regularity $> 1$ is used in the proof only to establish Lemma 3.5. More precisely, it is used to apply Lemma 2.3 on the gluing of complex discs. Hence it suffices to establish an analogue of Lemma 2.3 in the case when the edge $E$ is precisely of class $C^1$. This is possible but requires a much more careful analysis of the Bishop equation (10). Denote by $W^{k,p}(\mathbb{D})$ the Sobolev classes of functions admitting the Sobolev partial derivatives up to order $k$ in the class $L^p(\mathbb{D})$. One can show that the equation (10) has a unique solution in $W^{1,p}(\mathbb{D})$ for every $p > 2$. Furthermore, this solution is of class $C^1(\mathbb{D})$ and depends $C^1$-smoothly on the parameters in the interior of $\mathbb{D}$. This is sufficient in order to prove an analogue of Theorem 3.3. A detailed argument requires some additional results from geometric measure theory.

The second approach uses the fact that in order to prove Theorem 1.1, it suffices to establish Theorem 3.3 in the particular case when the wedge $W$ coincides
with \( W(\Omega_1) \) and \( E = H(b\Omega_1) \). Technically, it is more appropriate to consider holomorphic discs glued to the holomorphic tangent bundle \( H(b\Omega) \) (more precisely, to its projectivization) along the whole boundary (that is, the whole unit circle \( b\Omega \)). This class of complex discs is well known. It is exactly the stationary discs of Lempert [12] studied by several authors. Suppose that \( H(b\Omega_1) \) is defined by (11). A holomorphic disc \( z : \mathbb{D} \to \Omega_1, z : \zeta \mapsto z(\zeta) \), is said to be stationary if it admits the holomorphic lift \( (z, w) : \mathbb{D} \to \Omega_1 \times \mathbb{C}P^{n-1}, (z, w) : \zeta \mapsto (z(\zeta), w(\zeta)) \), whose boundary is glued to \( H(b\Omega_1) \), that is, \( (z, w)(b\mathbb{D}) \subset H(b\Omega_1) \). In coordinates, this is equivalent to the Bishop-type equation

\[
\rho(z(e^{i\theta})) = 0, \quad w_j = \phi_j(z(e^{i\theta})), \quad j = 1, \ldots, n-1.
\]

This is a non-linear boundary-value problem of Riemann–Hilbert type. It is well known that the linearized problem is represented by a Fredholm operator which has positive partial indices and is therefore surjective. This enables one to solve the problem by the implicit function theorem (see, for example, [13], [14]) in suitable function spaces such as \( C^\alpha(\mathbb{D}) \) or \( W^{1,p}(\mathbb{D}) \) with \( p > 2 \). In the case when \( \Omega_1 = \mathbb{B}^n \) (which corresponds to the linearized Riemann–Hilbert problem), the stationary discs belong to complex lines. Consider a point \( a \in b\Omega_1 \). One can assume that \( \mathbb{B}^n \) is tangent of order 2 to \( \Omega_1 \) at \( a \). Let \( b \in \Omega_1 \) be a point on the real inward normal to \( b\Omega_1 \) at \( a \). Consider a vector \( v \in \mathbb{C}^n \) parallel to \( H_a(b\Omega_1) \). Then the unique stationary disc through \( b \) in the direction \( v \) is a small perturbation of the complex line \( L \) through \( b \) in the direction \( v \). Note that a lift glued to \( H(b\Omega_1) \) is a large complex disc (the totally real manifold \( H(b\Omega_1) \) admits no attached small complex discs). These discs form a foliation which is a small deformation of the foliation of \( \Omega_1 \) near the boundary point \( p \in L \cap b\Omega_1 \) by the family of complex lines parallel to the holomorphic tangent space of \( b\Omega_1 \). Every disc is of class \( C^\alpha \) with \( \alpha < 1 \) and then the proof of Theorem 3.3 goes through.

A detailed presentation of these approaches will appear in a forthcoming paper.

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