The inner derivative and product are related by the Leibnitz rule.

Main definitions and introduce some notations in the general case, following [1, pp. 199–221].

If \( U \) is a symmetric rank \( m \) tensor field, \( f \) is a Killing tensor field of rank \( \geq 3 \). We obtain two necessary conditions on a Riemannian metric on the 2-torus for the existence of Killing tensor fields. The first condition is valid for Killing tensor fields of arbitrary rank and relates to closed geodesics. The second condition is obtained for rank 3 Killing tensor fields and pertains to isolines of the Gaussian curvature.

KILLING TENSOR FIELDS ON THE 2-TORUS

V. A. Sharafutdinov

Abstract: A symmetric tensor field on a Riemannian manifold is called a Killing field if the symmetric part of its covariant derivative equals zero. There is a one-to-one correspondence between Killing tensor fields and first integrals of the geodesic flow which depend polynomially on the velocity. Therefore Killing tensor fields relate closely to the problem of integrability of geodesic flows. In particular, the following question is still open: does there exist a Riemannian metric on the 2-torus which admits an irreducible Killing tensor field of rank \( \geq 3 \)?

The differential operator \( \sigma \nabla : C^\infty(S^m) \rightarrow C^\infty(S^{m+1}) \), where \( \nabla \) is the covariant derivative with respect to the Levi-Civita connection, is called the inner differentiation. We say that \( f \in C^\infty(S^m) \) is a Killing tensor field if

\[
df = 0. \tag{1.1}
\]

The inner derivative and product are related by the Leibnitz rule \( d(fh) = (df)h + f(dh) \), which implies the statement: If \( f \) and \( h \) are Killing tensor fields, then \( fh \) is a Killing field too. A Killing tensor field \( f \in C^\infty(S^m) \) (\( m \neq 2 \)) is said to be irreducible if \( f \) cannot be represented as a finite sum \( f = \sum_{i} u_i v_i \), where all \( u_i \) and \( v_i \) are Killing tensor fields of positive rank. In the case of \( m = 2 \), we additionally require \( f \) to be different from \( cg \) (\( c = \text{const} \)). The requirement eliminates the metric tensor from the list of irreducible Killing fields.

§ 1. Introduction

Although the main topic of this paper concerns Killing tensor fields on the 2-torus equipped by a Riemannian metric, the problem can be posed for any Riemannian manifold. Here, we present the main definitions and introduce some notations in the general case, following [1, § 3.3] whenever possible.

Given a Riemannian manifold \((M, g)\), let \( \tau_M^0 \) be the cotangent bundle and let \( S^m \tau_M^0 \) be the bundle of symmetric rank \( m \) covariant tensors. The latter notation will be mostly abbreviated to \( S^m \) on assuming the manifold to be known from the context. The space \( C^\infty(S^m) \) of smooth sections of the bundle is the \( C^\infty(M) \)-module of smooth covariant symmetric tensor fields of rank \( m \) on \( M \). The sum \( S^n = \bigoplus_{m=0}^\infty S^m \) is the bundle of graded commutative algebras with respect to the product \( fh = \sigma(f \otimes h) \), where \( \sigma \) is the symmetrization. If \((U; x^1, \ldots, x^n)\) is a local coordinate system on \( M \), then the space \( C^\infty(S^m) \) of smooth sections over \( U \) is the free commutative \( C^\infty(U) \)-algebra with generators \( dx^i \in C^\infty(\tau_M^0; U) \) (\( 1 \leq i \leq n \)), i.e., every field \( f \in C^\infty(S^m) \); \( U \) can be uniquely represented as \( f = f_{i_1 \ldots i_m} dx^{i_1} \ldots dx^{i_m} \). The coefficients \( f_{i_1 \ldots i_m} \in C^\infty(U) \), called coordinates (or components) of \( f \) (with respect to the given coordinate system), are symmetric in the indices \( (i_1, \ldots, i_m) \).

The differential operator \( d = \sigma \nabla : C^\infty(S^m) \rightarrow C^\infty(S^{m+1}) \), where \( \nabla \) is the covariant derivative with respect to the Levi-Civita connection, is called the inner differentiation. We say that \( f \in C^\infty(S^m) \) is a Killing tensor field if

\[
df = 0. \tag{1.1}
\]

The inner derivative and product are related by the Leibnitz rule \( d(fh) = (df)h + f(dh) \), which implies the statement: If \( f \) and \( h \) are Killing tensor fields, then \( fh \) is a Killing field too. A Killing tensor field \( f \in C^\infty(S^m) \) (\( m \neq 2 \)) is said to be irreducible if \( f \) cannot be represented as a finite sum \( f = \sum_{i} u_i v_i \), where all \( u_i \) and \( v_i \) are Killing tensor fields of positive rank. In the case of \( m = 2 \), we additionally require \( f \) to be different from \( cg \) (\( c = \text{const} \)). The requirement eliminates the metric tensor from the list of irreducible Killing fields.

Being written in coordinates for a rank \( m \) tensor field, (1.1) is a system of \( \binom{n+m}{m+1} \) first order linear differential equations in \( \binom{n+m-1}{m} \) coordinates of \( f \), where \( n = \dim M \). Since the system is overdetermined,
Fourier series in spherical harmonics with respect to the variable $\xi$ and metrics. In this paper, we discuss the more modest problem: Given a Riemannian manifold $(M, g)$ on a given manifold which admits irreducible Killing tensor fields of a given rank and, if possible, find all such fields. Questions are open on the metrics admitting irreducible Killing tensor fields of rank $3$. As far as we know, most of the mathematicians, starting with the classical works of G. Darboux [2] and J. Birkhoff [3], and is still topical now. We do not present the corresponding references here because of the volume limitation and refer the reader to [4] where a lengthy reference list is presented. In particular, the metrics on surfaces are classified that admit irreducible Killing tensor fields of rank 1 and 2. But as far as we know, most of the questions are open on the metrics admitting irreducible Killing tensor fields of rank $\geq 3$.

The problem is traditionally posed as follows: Determine whether there exist Riemannian metrics on a given manifold which admit irreducible Killing tensor fields of a given rank and, if possible, find all these metrics. In this paper, we discuss the more modest problem: Given a Riemannian manifold $(M, g)$, determine whether it admits irreducible Killing tensor fields of rank $m$ and, if possible, describe all these fields. As is demonstrated in the next paragraph, the problem in our setting can be efficiently solved in principle if we can solve elliptic equations on the given manifold.

Unless otherwise indicated, the term “Riemannian manifold” means a smooth (i.e., of class $C^\infty$) compact boundaryless manifold with a smooth Riemannian metric. Given a Riemannian manifold $(M, g)$, let $-\delta : C^\infty(S^{m+1}) \to C^\infty(S^m)$ be the adjoint operator to $d$ with respect to the natural $L^2$ dot product on $C^\infty(S^m)$. The operator $\delta$ is called the divergence and is expressed by the formula $(\delta f)_{i_1 \cdots i_m} = g^{pq} \nabla_p f_{q i_1 \cdots i_m}$ in local coordinates. Since $\delta d$ is an elliptic operator [1, Theorem 3.3.2], $\delta d$ has finite-dimensional kernel. The kernel coincides with the space of rank $m$ Killing tensor fields as is seen from the equality $(\delta df, f)_{L^2} = -(df, df)_{L^2}$. If we found the kernel of $\delta d$ for a given $m$ and for all less ranks, then we will be able to describe efficiently all irreducible Killing tensor fields of rank $m$ on $(M, g)$.

The dimension $\binom{n+m-1}{m}$ of the bundle $S^m$ grows fast with $m$ and $n = \dim M$. In Section 2, we will reduce the problem to a similar question for the elliptic operator $\delta pd$ ($p$ will be defined later) which acts on a bundle of the less dimension $\frac{n+2m-2}{n+m-2} \binom{n+m-2}{m}$. In particular, the latter bundle is two-dimensional in the case of $n = 2$. The reduction will be done by expanding the polynomial $F \in C^\infty(\Omega M)$ in the Fourier series in spherical harmonics with respect to the variable $\xi$ and replacing the equation $HF = 0$.
by a chain of equations relating spherical harmonics of different degrees. The main result of Section 2 is as follows: A Killing rank $m$ tensor field $f$ is determined by its highest harmonic $pf$ uniquely up to a Killing tensor field of rank $m - 2$. The method of spherical harmonics is widely used for the numerical solution of the kinetic equation and some of its relatives [5, Chapter 8], but sometimes the method works successfully in theoretical questions too. For example, some version of the method was used in [6] for proving the spectral rigidity of a negatively curved surface.

In Section 3, we consider Killing tensor fields on the 2D torus. Owing to the existence of global isothermal coordinates, the kernel of $\delta pd$ can be explicitly described and turns out to be a two-dimensional space. Theorem 3.4 gives some necessary and sufficient condition (of a nonlocal nature) on the metric for the existence of a rank $m$ irreducible Killing tensor field. Unfortunately, the verification of the condition is not much easier than the initial problem. So, the main question remains open: Does there exist a Riemannian metric on the 2-torus which admits irreducible Killing tensor fields of rank $m \geq 3$? Nevertheless, the necessary condition of Corollary 3.5 allows us to give the negative answer to the question for many specific metrics. All we need is to find two closed geodesics such that certain functions $\varphi_m$ and $\psi_m$ produce linearly independent integrals over that geodesics, while the functions $\varphi_m$ and $\psi_m$ are explicitly expressed through the metric and direction of a geodesic.

System (1.1) has the following interesting property. Each equation of the system is a first order linear differential equation in the coordinates of $f$, but the system cannot be solved with respect to all first order derivatives of the coordinates. Nevertheless, as shown in [1, Theorem 2.2.2], after $m$-times differentiation, we obtain a system that can be solved with respect to all $(m + 1)$ order derivatives of coordinates of $f$. In particular, a rank $m$ Killing tensor field on a connected manifold (not necessarily compact) is uniquely determined by the values of its derivatives of order $\leq m$ at one point. This allows us to estimate from above the dimension of the space of rank $m$ Killing fields by some quantity that is explicitly expressed through $m$ and $n$. We will use some version of this approach in Section 4 in studying rank 3 Killing tensor fields on the 2-torus. In this way, we obtain some new necessary condition that is related to the behavior of isolines of the Gaussian curvature. We observe also (although the observation is not used in this paper) that the corresponding system for conformal Killing tensor fields possesses a similar property too (see [7]).

Let us briefly discuss the so-called Birkhoff–Kolokoltsov codifferential (the term is proposed in [8]). It is considered only in the two-dimensional case. The following observation belongs to Birkhoff [3]. If the geodesic flow of a surface admits a first integral $F(x, p)$ that is a homogeneous polynomial in the impulse $p$, then some linear combination of coefficients of the polynomial is a holomorphic function in an isothermal coordinate system. This fact was systematically used by V. N. Kolokoltsov [9]. In particular, he obtained some rule for transforming the linear combination under a holomorphic change of coordinates. The latter rule allows him to introduce the Birkhoff–Kolokoltsov codifferential as a section of the one-dimensional complex bundle $\otimes^m T^{(1,0)} M$ which depends on a degree $m$ polynomial $F(x, p)$ that is a first integral of the geodesic flow on a two-dimensional Riemannian manifold $(M, g)$. We do not use the Birkhoff–Kolokoltsov codifferential in this paper. Instead of that, we use the highest harmonic $pf$ of a Killing tensor field $f$. This is a traditional object of the method of spherical harmonics. The equivalence of these two notions is demonstrated at the end of Section 2. In particular, it turns out that Kolokoltsov’s transformation law coincides with the standard transformation law for coordinates of the tensor $pf$ under a change of coordinates. The following feature of our approach is of a principle importance: The highest harmonic is defined in the multidimensional case as well. Our Theorem 2.2 is the multidimensional generalization of Birkhoff’s result. The situation becomes much easier in the case of the 2-torus due to the existence of a global isothermal coordinate system. The above-mentioned holomorphic function is constant in this case; i.e., it is determined by two real numbers. The transformation rule for the pair of real numbers is described by Definition 3.1 that introduces constant pseudovectors of weight $m$. The latter notion is much easier than the “highest harmonic” or “Birkhoff–Kolokoltsov codifferential” and is quite enough for studying Killing tensor fields on the 2-torus.

Our approach to studying equation (4.7) in Section 4 is somewhat similar to the method of [8], where
some strong but local results are obtained for rank 3 Killing tensor fields on an arbitrary surface. A rather different approach to studying Killing tensor fields on the 2-torus was proposed in [10]; but according to the authors’ statement, the approach did not give final results so far.

§ 2. The Method of Spherical Harmonics for Killing Tensor Fields

We will use the analysis of symmetric tensor fields which has been originally developed in [11]. Later this machinery was systematically presented in [1]; but some technical details are not included into the book although we need them here. The recent paper [7] contains [11] as a proper subset. Therefore we will mostly refer the reader to [7] for proofs of technical statements.

Let \((M, g)\) be a Riemannian manifold. By \(i : S^m \to S^{m+2}\), we denote the operator of symmetric multiplication by the metric tensor; i.e., \(i f = f g = \sigma(f \otimes g)\). The adjoint of \(i\) is the contraction \(j\) with the metric tensor, defined in coordinates as \((j f)_{i_1...i_m} = g^{p q} f_{p q i_1...i_m}\). We will refer to \(j f\) as the trace of \(f\). Let \(p : S^m \to S^m\) be the orthogonal projection onto the kernel of \(j\).

Let \(\text{Ker}^m j\) be the subbundle of \(S^m\) consisting of the trace free tensors, i.e., of the tensors \(f\) satisfying \(j f = 0\). This terminology was introduced in [7]. We will use the terminology, although “the bundle of harmonic tensors” is possibly a more appropriate term for \(\text{Ker}^m j\). Observe that, for \(f \in C^\infty(\text{Ker}^m j)\), the divergence \(\delta f\) is also a trace free field since the operators \(j\) and \(\delta\) commute [7, Lemma 3.2]. Therefore the pair of mutually adjoint operators is defined as

\[
C^\infty(\text{Ker}^m j) \overset{pd}{\longrightarrow} C^\infty(\text{Ker}^{m+1} j). \tag{2.1}
\]

The operator \(\delta pd\) is elliptic as shown in [7, Theorem 2.1]. The operators \(\delta pd\) and \(pd\) have the same kernel as is seen from the equalities

\[
(\delta pd f, f)_{L^2} = -(pdf, df)_{L^2} = -(p^2 df, df)_{L^2} = -(pdf, pdf)_{L^2}.
\]

Hence \(pd\) has finite-dimensional kernel.

The further formulas are slightly different in the cases of tensor fields of even and odd rank. Therefore we will first discuss the case of an even rank and then we will present the corresponding formulas in the case of an odd rank. Both cases can be united by complication of notations.

By [7, Lemma 2.3], every field \(f \in C^\infty(S^{2m})\) can be uniquely represented as

\[
f = \sum_{k=0}^{m} i^{m-k} f^{(k)}, \quad f^{(k)} \in C^\infty(\text{Ker}^{2k} j). \tag{2.2}
\]

Representation (2.2) coincides in fact with the expansion of the polynomial \(F \in C^\infty(\Omega M)\) in the Fourier series in spherical harmonics with respect to the variable \(\xi\). Therefore the field \(f^{(k)}\) will be called the harmonic of degree \(2k\) of \(f\). In particular, \(f^{(m)} = p f\) is the highest harmonic. For convenience, we assume also that \(f^{(k)} = 0\) for \(k > m\). For an odd rank field \(f \in C^\infty(S^{2m+1})\), representation (2.2) remains true but \(f^{(k)} \in C^\infty(\text{Ker}^{2k+1} j)\) now.

For tensor fields \(f \in C^\infty(S^{2m})\) and \(b \in C^\infty(S^{2m+1})\), the equation \(df = b\) is equivalent to the following chain of equations [7, Theorem 10.2] relating their harmonics:

\[
pdf^{(k)} + \frac{2k + 2}{n + 4k + 2} \delta f^{(k+1)} = b^{(k)} \quad (k = 0, \ldots, m), \tag{2.3}
\]

where \(n = \dim M\). For \(f \in C^\infty(S^{2m+1})\) and \(b \in C^\infty(S^{2m+2})\), the chain looks as follows:

\[
\delta f^{(0)} = nb^{(0)}, \quad pdf^{(k)} + \frac{2k + 3}{n + 4k + 4} \delta f^{(k+1)} = b^{(k+1)} \quad (k = 0, \ldots, m). \tag{2.4}
\]

Systems (2.3) and (2.4) are main equations of the method of spherical harmonics. These equations can be easily generalized to the case when the solution and right-hand side of the kinetic equation \(HF = B \)
are not polynomials but arbitrary smooth functions on $\Omega M$, as well as to the case of a more general equation with terms responsible for absorption and scattering.

Thus, $f \in C^\infty(S^{2m})$ is a Killing tensor field if and only if

$$pdf^{(k)} + \frac{2k + 2}{n + 4k + 2} \delta f^{(k+1)} = 0 \quad (k = 0, \ldots, m).$$

(2.5)

Similarly, $f \in C^\infty(S^{2m+1})$ is a Killing tensor field if and only if

$$\delta f^{(0)} = 0, \quad pdf^{(k)} + \frac{2k + 3}{n + 4k + 4} \delta f^{(k+1)} = 0 \quad (k = 0, \ldots, m).$$

(2.6)

Recall [1, p. 39] that $f \in C^\infty(S^m)$ is called a potential tensor field if there exists $v \in C^\infty(S^{m-1})$ such that $f = dv$.

**Lemma 2.1.** If $f$ is a Killing tensor field, then the divergence $\delta f^{(k)}$ is a potential tensor field for every summand of (2.2).

**Proof.** For definiteness, we consider a Killing field $f$ of even rank. For $\delta f^{(0)} = 0$, the claim is trivial. Since $p$ coincides with the identity operator on $S^1$, equation (2.5) for $k = 0$ can be rewritten as $df^{(0)} + \frac{2}{n+2} \delta f^{(1)} = 0$. This implies that $\delta f^{(1)}$ is a potential field. Next, we continue the proof by induction in $k$. By [7, Lemma 2.4],

$$df^{(k)} = pdf^{(k)} + \frac{2k}{n + 4k - 2} i \delta f^{(k)}$$

for every rank $2k$ tensor field $f^{(k)}$. In our case, $pf^{(k)} = f^{(k)}$ since $j f^{(k)} = 0$ and the previous formula is simplified as follows:

$$pdf^{(k)} = df^{(k)} - \frac{2k}{n + 4k - 2} i \delta f^{(k)}.$$

By the induction hypothesis, $\delta f^{(k)} = dv$ for some $v$. Inserting this expression into the previous formula and taking the permutability of $i$ and $d$ [7, Lemma 3.2] into account, we obtain

$$pdf^{(k)} = d\left(f^{(k)} - \frac{2k}{n + 4k - 2} iv\right).$$

This gives together with (2.5)

$$d\left(f^{(k)} - \frac{2k}{n + 4k - 2} iv\right) + \frac{2k + 2}{n + 4k + 2} \delta f^{(k+1)} = 0$$

and we see that $\delta f^{(k+1)}$ is a potential field.

**Theorem 2.2.** A rank $m$ Killing tensor field is determined by its highest harmonic uniquely up to a summand of the form $iv$ where $v$ is an arbitrary rank $m - 2$ Killing tensor field. A tensor field $f \in C^\infty(Ker^m j)$ is the highest harmonic of some Killing tensor field if and only if it satisfies the equation

$$pdf = 0$$

(2.7)

and has potential divergence, i.e., $\delta f = dv$ for some $v$.

**Proof.** We consider the case of an even rank. If the highest harmonic of a Killing field $f \in C^\infty(S^{2m})$ is equal to zero, then (2.2) can be written as $f = i \sum_{k=0}^{m-1} i^{m-k-1} f^{(k)}$. The last equation of (2.5) holds trivially and other equations of (2.5) mean that $v = \sum_{k=0}^{m-1} i^{m-k-1} f^{(k)}$ is a Killing field.

**Necessity.** The potentiality of $\delta f^{(m)}$ was proved in Lemma 2.1, and (2.7) for $f^{(m)}$ coincides with (2.5) for $k = m$. 

159
SUFFICIENCY. Assume that \( f^{(m)} \in C^\infty(\text{Ker}^m) \) satisfies the equation \( \delta f^{(m)} = 0 \) and has potential divergence, i.e.,

\[
dv = \delta f^{(m)}
\]

for some \( v \in C^\infty(S^{2m-2}) \). We expand \( v \) into the sum of spherical harmonics

\[
v = \sum_{k=0}^{m-1} i^{m-k-1} v^{(k)}, \quad jv^{(k)} = 0.
\]

The corresponding sum for \( \delta f^{(m)} \) consists of a sole summand. Therefore (2.8) is equivalent to the following chain of equations:

\[
\begin{align*}
\text{pdv}^{(k)} + \frac{2k + 2}{n + 4k + 2} \delta v^{(k+1)} &= 0 \quad (k = 0, \ldots, m - 2), \quad \text{pdv}^{(m-1)} = \delta f^{(m)}.
\end{align*}
\]

Setting \( f^{(k)} = v^{(k)} \) for \( 0 \leq k \leq m - 2 \) and \( f^{(m-1)} = -\frac{2m}{n+4m} v^{(m-1)} \), we rewrite the system as

\[
\begin{align*}
\text{pdv}^{(k)} + \frac{2k + 2}{n + 4k + 2} \delta f^{(k+1)} &= 0 \quad (k = 0, \ldots, m - 1).
\end{align*}
\]

Together with the equation \( \text{pdf}^{(m)} = 0 \), this gives (2.5), i.e., \( f = \sum_{k=0}^{m} i^{m-k} f^{(k)} \) is a Killing field.

Assume we have found the kernel of the operator \( pd \) from (2.1) and let the tensor fields \( (f_1, \ldots, f_r) \) constitute a basis of the kernel. When does the tensor field

\[
f = \alpha_1 f_1 + \cdots + \alpha_r f_r \quad (\alpha_i \in \mathbb{C})
\]

serve as the highest harmonic of a Killing field? By Theorem 2.2, the potentiality of \( \delta f \), i.e. the solvability of the equation

\[
dv = \alpha_1 \delta f_1 + \cdots + \alpha_r \delta f_r \quad (2.9)
\]

is a necessary and sufficient condition. The sequence \( \delta f_i \) (\( 1 \leq i \leq r \)) can be linearly dependent. We choose a maximal linearly independent subsystem of the sequence and, changing enumeration, denote the subsequence as \( (\delta f_1, \ldots, \delta f_s) \) with some \( s \leq r \). Then (2.9) is replaced by the equation with fewer parameters

\[
dv = \alpha_1 \delta f_1 + \cdots + \alpha_s \delta f_s \quad (2.10)
\]

and our problem is reduced to the question: For what coefficients \((\alpha_1, \ldots, \alpha_s)\) is (2.10) solvable?

In view of (2.1), the following definition is suitable: \( f \in C^\infty(\text{Ker}^m) \) is said to be a \( j \)-potential field if there exists \( v \in C^\infty(\text{Ker}^{m-1}) \) such that \( f = pdv \). The statements “\( f \) is a potential field” and “\( f \) is a \( j \)-potential field” are not related, i.e., one of them does not imply the other for an arbitrary \( f \in C^\infty(\text{Ker}^m) \). But, for a Killing field \( f \), the divergence \( \delta f^{(k)} \) of every harmonic is a potential and \( j \)-potential tensor field. The first claim is proved in Lemma 2.1 and the second claim is directly seen from (2.5).

For the sake of completeness, we also present the following easy lemma.

**Lemma 2.3.** Assume that, for some \( m \geq 2 \), a Riemannian manifold \((M, g)\) does not admit irreducible Killing tensor fields of ranks \( 1, \ldots, m - 1 \). A rank \( m \) Killing tensor field on \((M, g)\) is irreducible unless its highest harmonic is identically zero.

**Proof.** Recall that we have eliminated the metric tensor from the list of irreducible Killing fields. Each reducible Killing field can be represented as an integer coefficients polynomial of several irreducible Killing fields and of the metric tensor. This means under hypotheses of the lemma that, for an odd \( m \), every reducible rank \( m \) Killing tensor field is identically zero; and for \( m = 2k \), every reducible rank \( m \) Killing tensor field is of the form \( cg^k \) (\( c = \text{const} \)). The highest harmonic of such a field is equal to zero.
Closing the section we will discuss the two-dimensional case. In this case the bundle \( S^m \) has dimension \( m + 1 \) and \( \text{Ker}^{m}j \) is the two-dimensional bundle for \( m > 0 \).

In a neighborhood of every point of a two-dimensional Riemannian manifold \((M, g)\), we can introduce isothermal coordinates \((x, y)\) such that the metric is expressed as

\[
g = e^{2\mu(x,y)}(dx^2 + dy^2) = \lambda(z)|dz|^2 \quad (z = x + iy, \ \lambda(z) = e^{2\mu(x,y)}).
\]  

(2.11)

The Christoffel symbols of the metric are

\[
\Gamma^1_{11} = \mu_x, \ \Gamma^1_{12} = \mu_y, \ \Gamma^1_{22} = -\mu_x, \ \Gamma^2_{11} = -\mu_y, \ \Gamma^2_{12} = \mu_x, \ \Gamma^2_{22} = \mu_y.
\]  

(2.12)

Recall that \( \Omega M \) is the unit circle bundle. If \((x, y)\) are isothermal coordinates on \( M \) and \((x, y, \xi^1, \xi^2)\) are corresponding coordinates on \( TM \), then the coordinates \((x, y, \theta)\) on \( \Omega M \) are defined as \( \xi^1 = e^{-\mu} \cos \theta \) and \( \xi^2 = e^{-\mu} \sin \theta \). Substitute (2.12) for Christoffel symbols into (1.2) to obtain the expression for the operator \( H \) in isothermal coordinates

\[
H = e^{-\mu} \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} + (-\mu_x \sin \theta + \mu_y \cos \theta) \frac{\partial}{\partial \theta} \right).
\]  

(2.13)

Given \( f \in C^\infty(\text{Ker}^m j) \), let us write down the equation \( pdf = 0 \) in isothermal coordinates. The condition \( jf = 0 \) means that

\[
f_{\frac{1}{m-k}}^{\frac{2}{m-k}} + f_{\frac{1}{m-k-2}}^{\frac{2}{m-k+2}} = 0 \quad (0 \leq k \leq m - 2).
\]  

(2.14)

The function \( F \in C^\infty(\Omega M) \) corresponding to \( f \) is obtained from the polynomial \( f_{i_1...i_m}\xi^{i_1}...\xi^{i_m} \) by putting \( \xi^1 = e^{-\mu} \cos \theta \) and \( \xi^2 = e^{-\mu} \sin \theta \). This, together with (2.14), gives

\[
F(x, y, \theta) = e^{-m\mu(x,y)}(f_{1...1}(x, y) \cos m\theta + f_{1...12}(x, y) \sin m\theta).
\]  

(2.15)

Using (2.13) and (2.15), we find the function \( HF \) and then again expand \( HF \) in Fourier series in \( \theta \); to this end it suffices to recall the elementary trigonometric formulas \( 2 \cos \theta \cos m\theta = \cos(m - 1)\theta + \cos(m + 1)\theta \), \( 2 \cos \theta \sin m\theta = \ldots \). The equation \( pdf = 0 \) is equivalent to the statement: In the Fourier series for \( HF \), the coefficients at \( \cos(m + 1)\theta \) and \( \sin(m + 1)\theta \) are equal to zero. Equating these coefficients to zero, we arrive to the Cauchy–Riemann system

\[
\frac{\partial(e^{-2m\mu}f_{1...1})}{\partial x} - \frac{\partial(e^{-2m\mu}f_{1...12})}{\partial y} = 0, \quad \frac{\partial(e^{-2m\mu}f_{1...1})}{\partial y} + \frac{\partial(e^{-2m\mu}f_{1...12})}{\partial x} = 0.
\]  

(2.16)

This derivation of (2.16) is taken from [11]. As mentioned above, this result belongs in fact to Birkhoff [3] although his terminology and argumentation are very different from ours.

§ 3. Killing Tensor Fields on the Two-Dimensional Torus

Recall (see [4, § 6.5]) that there exists a global isothermal coordinate system on the two-dimensional torus \( \mathbb{T}^2 \) with a Riemannian metric \( g \). More precisely, there exists a lattice \( \Gamma \subset \mathbb{R}^2 = \mathbb{C} \) such that \( \mathbb{T}^2 = \mathbb{C}/\Gamma \) and \( g \) is expressed by (2.11), where \( \lambda(z) = e^{2\mu(x,y)} \) is a \( \Gamma \)-periodic smooth function on the plane. The global isothermal coordinates on the torus are defined uniquely up to coordinate transformations of the two kinds: either \( z = az' + b \) or \( z = az' + b \) with complex constants \( a \neq 0 \) and \( b \). The transformations of the second kind can be eliminated from consideration if we fix an orientation of the torus and consider coordinate systems agreed with the orientation. Moreover, studying the invariance of various formulas, we will restrict exposition to considering the coordinate transformations of the form \( z = az' \) since the shift by a constant vector preserves tensor formulas. The group of the latter transformations coincides with the multiplicative group \( \mathbb{C} \setminus \{0\} \). The commutativity of the group allows us to introduce the following definition (\( i \) stands for the imaginary unit in what follows).
DEFINITION 3.1. A pseudovector field $X$ of weight $m$ on a Riemannian torus $(\mathbb{T}^2, g)$ is a map sending a global isothermal coordinate system to a pair $(X^1, X^2)$ of functions on $\mathbb{T}^2$ which are transformed by the rule

$$
(X^1 + iX^2)(z) = a^m(X'^1 + iX'^2)(z'), \quad (X^1 - iX^2)(z) = \bar{a}^m(X'^1 - iX'^2)(z')
$$

(3.1)

under the coordinate change $z = az'$. If the functions $(X^1, X^2)$ are constant, we speak of a constant pseudovector $X$ of weight $m$. Similarly, a pseudocovector field (or 1-pseudocovector) $\omega$ of weight $m$ on $(\mathbb{T}^2, g)$ is a map sending a global isothermal coordinate system to a pair $(\omega_1, \omega_2)$ of functions on $\mathbb{T}^2$ which are transformed by the rule

$$
\omega_1 + i\omega_2 = \bar{a}^{-m}(\omega'_1 + i\omega'_2), \quad \omega_1 - i\omega_2 = a^{-m}(\omega'_1 - i\omega'_2).
$$

If $X$ and $\omega$ are respectively a pseudovector and a pseudocovector field of the same weight, then $X^1\omega_1 + X^2\omega_2$ is an invariant function on the torus. Let us make the following remark on (3.1): If $X^1$ and $X^2$ are real functions in one global isothermal coordinate system, then the same is true in every global isothermal coordinate system; in this case we speak of a real pseudovector field $X$ of weight $m$.

For a real $X$ the two equations of (3.1) are equivalent. In the general case, the equations are independent.

**Theorem 3.2.** For a two-dimensional Riemannian torus, the kernel of the elliptic operator

$$
\delta p d : C^\infty(\text{Ker}^m j) \to C^\infty(\text{Ker}^m j)
$$

(3.2)

is the two-dimensional space consisting of tensor fields $f \in C^\infty(\text{Ker}^m j)$ whose coordinates with respect to a global isothermal coordinate system are of the form

$$
f_{1...1} = e^{2m\mu}c^1, \quad f_{1...12} = e^{2m\mu}c^2,
$$

(3.3)

where $c = (c^1, c^2)$ is a constant pseudovector of weight $m$. The range of operator (3.2) consists of tensor fields $f \in C^\infty(\text{Ker}^m j)$ whose all components with respect to a global isothermal coordinate system have zero mean values; i.e.,

$$
\int_{\mathbb{T}^2} f_{1...i_m} \, d\sigma = 0,
$$

(3.4)

where $d\sigma = e^{2\mu} \, dx \, dy$ is the area element.

The first statement of the theorem is also in [9].

**Proof.** First of all, we observe that all coordinates of a field $f$ are determined by (3.3) in view of (2.14).

As we mentioned before, the kernel of (3.2) coincides with the kernel of $pd$. If $f \in C^\infty(\text{Ker}^m j)$ satisfies the equation $pdf = 0$, then by (2.16) $e^{-2m\mu}(f_{1...1} + if_{1...12})$ is a holomorphic function on the torus. Hence it is a constant function, i.e., equalities (3.3) with some complex constants $c^1$ and $c^2$ hold in every global isothermal coordinate system. These constants are real in the case of a real $f$. Starting with the rule of transforming components of a tensor field under a coordinate change, we easily see that $c = (c^1, c^2)$ is a constant pseudovector of weight $m$. The claim about the kernel of $\delta pd$ is thus proved. The claim about the range follows since $\delta pd$ is a self-adjoint operator.

In the case of $m = 1$, Theorem 3.2 leads to the well-known Clairaut integral for geodesics on a surface of revolution. Indeed, let $f$ be a Killing covector field on a Riemannian torus. The field $f$ belongs to the kernel of operator (3.2) since $\text{Ker}^1 j = S^1$. By Theorem 3.2, $f_1 = e^{2\mu}c^1$ and $f_2 = e^{2\mu}c^2$ in global isothermal coordinates for some constant vector $c$. Since $f$ is a Killing field, $f_1 \dot{x} + f_2 \dot{y} = e^{2\mu}(c^1 \dot{x} + c^2 \dot{y})$ is constant on every geodesic $\gamma(t) = (x(t), y(t))$. Without lost of generality, we can assume that $\|\dot{\gamma}\|^2 = e^{2\mu}(\dot{x}^2 + \dot{y}^2) = 1$, i.e., $\dot{x} = e^{-\mu} \cos \varphi$ and $\dot{y} = e^{-\mu} \sin \varphi$, where $\varphi = \varphi(t)$ is the angle between the geodesic and the coordinate line $y = \text{const}$. Therefore,

$$
e^{\mu}(c^1 \cos \varphi + c^2 \sin \varphi) = \text{const}
$$

on every geodesic. This is just the Clairaut integral in isothermal coordinates. In particular, if the global isothermal coordinates are chosen so that $c = (1, 0)$, the Clairaut integral takes its traditional form: $e^{\mu} \cos \varphi = \text{const}.$
Applying conversely, if $\delta f$ belongs to the range of $pd : C^\infty(Ker^m j) \rightarrow C^\infty(Ker^m j)$ and $f^s \in C^\infty(Ker^m j)$ has zero divergence: $\delta f^s = 0$.

The summands of (3.5) will be called the $j$-potential and $j$-solenoidal parts of the field $f$ respectively.

**Proof.** We seek for $v \in C^\infty(Ker^{m-1} j)$ and $f^s \in C^\infty(Ker^m j)$ satisfying $pdv + f^s = f$ and $\delta f^s = 0$. Applying $\delta$ to the former equation, we obtain

$$\delta pdv = \delta f.$$  

Conversely, if $\delta f$ belongs to the range of $\delta pd$, then (3.6) is solvable and we define $f^s$ as $f^s = f - pdv$.

Thus, all we need is to check that, for $f \in C^\infty(Ker^m j)$, every component of the field $\delta f$ with respect to global isothermal coordinates has zero mean. By (2.14), it suffices to perform the check only for the components $(\delta f)_{1\ldots1} \text{ and } (\delta f)_{1\ldots12}$. We calculate the divergence using the relation $f_{1\ldots122} = -f_{1\ldots1}$:

$$\begin{align*}
(\delta f)_{1\ldots1} &= g^{pq} \nabla_p f_{1\ldots1q} = e^{-2\mu}(\nabla_1 f_{1\ldots11} + \nabla_2 f_{1\ldots12}), \\
(\delta f)_{1\ldots12} &= g^{pq} \nabla_p f_{1\ldots12q} = e^{-2\mu}(\nabla_1 f_{1\ldots121} + \nabla_2 f_{1\ldots122}) = e^{-2\mu}(-\nabla_2 f_{1\ldots11} + \nabla_1 f_{1\ldots12}).
\end{align*}$$

Using (2.12) and the definition of covariant derivative, we compute

$$\nabla_1 f_{1\ldots1} = \frac{\partial f_{1\ldots1}}{\partial x} - m\Gamma^p_{11} f_{1\ldots1p} = \frac{\partial f_{1\ldots1}}{\partial x} - m\mu_x f_{1\ldots1} + m\mu_y f_{1\ldots12}.$$  

Similarly,

$$\begin{align*}
\nabla_2 f_{1\ldots1} &= \frac{\partial f_{1\ldots1}}{\partial y} - m\mu_y f_{1\ldots1} - m\mu_x f_{1\ldots12}, \\
\nabla_1 f_{1\ldots12} &= \frac{\partial f_{1\ldots12}}{\partial x} - m\mu_x f_{1\ldots1} - m\mu_x f_{1\ldots12}, \\
\nabla_2 f_{1\ldots12} &= \frac{\partial f_{1\ldots12}}{\partial y} + m\mu_x f_{1\ldots1} - m\mu_y f_{1\ldots12}.
\end{align*}$$

Inserting these values into (3.7), we arrive to the unexpectedly simple formulas

$$\begin{align*}
(\delta f)_{1\ldots1} &= e^{-2\mu}\left(\frac{\partial f_{1\ldots1}}{\partial x} + \frac{\partial f_{1\ldots12}}{\partial y}\right), \\
(\delta f)_{1\ldots12} &= e^{-2\mu}\left(-\frac{\partial f_{1\ldots1}}{\partial y} + \frac{\partial f_{1\ldots12}}{\partial x}\right).
\end{align*}$$

Of course, this fact should have an invariant explanation independent of coordinate calculations (the author even has a guess on such an explanation but does not discuss it here). Somehow or other, these formulas imply that the expressions

$$(\delta f)_{1\ldots1} d\sigma = \left(\frac{\partial f_{1\ldots1}}{\partial x} + \frac{\partial f_{1\ldots12}}{\partial y}\right) dxdy, \quad (\delta f)_{1\ldots12} d\sigma = \left(-\frac{\partial f_{1\ldots1}}{\partial y} + \frac{\partial f_{1\ldots12}}{\partial x}\right) dxdy$$

integrate to zero over the torus.

For every integer $m \geq 0$ and for every constant pseudovector $c = (c^1, c^2)$ of weight $m + 1$, we introduce the tensor field $Z^{m,c} \in C^\infty(Ker^m j)$ on a two-dimensional Riemannian torus by setting

$$Z^{m,c}_{1\ldots1} = e^{2\mu}(c^1 \mu_x + c^2 \mu_y), \quad Z^{m,c}_{1\ldots12} = e^{2\mu}(c^2 \mu_x - c^1 \mu_y)$$

in the global isothermal coordinates. The other components of the field are determined by (2.14), where $f$ should be replaced with $Z^{m,c}$. This is a correct definition; i.e., the field components are transformed in a proper way under a change of the global isothermal coordinates, as we can easily check on using that $c$ is a pseudovector of weight $m + 1$. In the notation $Z^{m,c}$, the first index is the rank of the field and the second index reminds the dependence on a pseudovector $c$ of weight $m + 1$. The case of $m = 0$ should be specially mentioned: $Z^{0,c} = c^1 \mu_x + c^2 \mu_y = d\mu(c)$ is an invariant function on the torus.
Theorem 3.4. If a Riemannian torus \((\mathbb{T}^2, g)\) admits a real irreducible rank \(m \geq 1\) Killing tensor field, then \(Z^{m-1,c}\) is a potential tensor field for some constant real pseudovector \(c \neq 0\) of weight \(m\).

Conversely, assume that a Riemannian torus \((\mathbb{T}^2, g)\) does not admit irreducible Killing tensor fields of rank \(1, \ldots, m - 1\) for some \(m \geq 1\) (the condition is absent in the case of \(m = 1\)). If the tensor field \(Z^{m-1,c}\) is potential for some \(c \neq 0\), then there exists a rank \(m\) irreducible Killing tensor field on the torus.

Proof. Necessity: Let \(f\) be a real irreducible rank \(m\) Killing tensor field and let \(pf\) be the highest harmonic of \(f\). The field \(pf\) is not identically zero; otherwise, \(f\) would be reducible. By Theorem 2.2, \(pf\) belongs to the kernel of \(\delta pd\) and has potential divergence. Applying Theorem 3.2, we obtain in global isothermal coordinates

\[
(pf)_{1...1} = e^{2\mu} c^1, \quad (pf)_{1...12} = e^{2\mu} c^2
\]

(3.10)

for some constant real pseudovector \(c \neq 0\) of weight \(m\). We already calculated the divergence of an arbitrary field \(pf \in C^\infty(Ker\, j)\) in the proof of Corollary 3.3, namely,

\[
(\delta pf)_{1...1} = e^{-2\mu} \left( \frac{\partial (pf)_{1...1}}{\partial x} + \frac{\partial (pf)_{1...12}}{\partial y} \right),
\]

\[
(\delta pf)_{1...12} = e^{-2\mu} \left( - \frac{\partial (pf)_{1...12}}{\partial y} + \frac{\partial (pf)_{1...1}}{\partial x} \right).
\]

Substituting (3.10) for the coordinates of \(pf\), we obtain

\[
(\delta pf)_{1...1} = 2me^{2(m-1)\mu}(c^1 \mu_x + c^2 \mu_y), \quad (\delta pf)_{1...12} = 2me^{2(m-1)\mu}(c^2 \mu_x - c^1 \mu_y).
\]

(3.11)

Comparing (3.11) with the definition (3.9) of \(Z^{m,c}\), we see that \(Z^{m-1,c} = \frac{1}{2m} \delta (pf)\) is a potential tensor field.

Sufficiency: Let \(Z^{m-1,c}\) be a potential field for some \(c \neq 0\). Define the new tensor field \(h \in C^\infty(Ker\, j)\) by setting \(h_{1...1} = e^{2\mu} c^1\) and \(h_{1...12} = e^{2\mu} c^2\) in global isothermal coordinates. By Theorem 3.2, \(h\) belongs to the kernel of the operator \(\delta pd\). Moreover, the field has the potential divergence since \(\delta h = 2mZ^{m-1,c}\). Applying Theorem 2.2, we find a Killing tensor field \(f\) whose highest harmonic is \(h\). By Lemma 2.3, \(f\) is an irreducible Killing field.

Theorem 3.4 actually reduces the problem of finding rank \(m\) Killing tensor fields on a two-dimensional Riemannian torus to the following question: For which constant pseudovectors \(c\) of weight \(m - 1\) is the equation

\[
dv = Z^{m-1,c}
\]

(3.12)
solvable? At the first sight, this equation is not easier than the initial equation (1.1). However, let us observe that the order of equation (3.12) is less than the order of (1.1). Indeed, being written in coordinates, (1.1) is a system of \(m + 2\) first order linear differential equations in the coordinates of \(f\). But the corresponding system for (3.12) consists of \(m\) equations, although the latter are inhomogeneous.

In the case of \(m = 1\), the potentiality of \(Z^{0,c}\) means that \(Z^{0,c} = c^1 \mu_x + c^2 \mu_y = 0\) (a rank 0 potential tensor field is the identically zero function). By an appropriate change of global isothermal coordinates, we can achieve \(c = (1, 0)\) and the previous equation becomes: \(\mu_x = 0\). Thus, in the case of \(m = 1\), Theorem 3.4 is equivalent to the classical result: If a two-dimensional Riemannian torus admits a nontrivial Killing vector field, then \(\mu = \mu(y)\) in some global isothermal coordinate system.

In the case of \(m = 2\), the potentiality of \(Z^{1,c}\) means the existence of \(v \in C^\infty(\mathbb{T}^2)\) such that

\[
v_x = e^{2\mu}(c^1 \mu_x + c^2 \mu_y), \quad v_y = e^{2\mu}(c^2 \mu_x - c^1 \mu_y).
\]

We again change the isothermal coordinates so that \(c = (1, 0)\) and obtain

\[
v_x = e^{2\mu} \mu_x, \quad v_y = -e^{2\mu} \mu_y.
\]

Eliminating \(v\) from the system, we arrive to the equation \(\partial^2 e^{2\mu}/\partial x \partial y = 0\). Thus, in the case of \(m = 2\), Theorem 3.4 is equivalent to the classical result: If a Riemannian torus \((\mathbb{T}^2, g)\) admits a rank 2 irreducible Killing tensor field, then \(g = (a(x) + b(y))(dx^2 + dy^2)\) in an appropriate global isothermal coordinate system.
Corollary 3.5. If a Riemannian torus \( (\mathbb{T}^2, g) \) admits a real irreducible rank \( m+1 \geq 1 \) Killing tensor field; then, for some real constant pseudovector \( c \neq 0 \) of weight \( m+1 \), we have

\[
\int \gamma e^{m\mu}((c^1 \mu_x + c^2 \mu_y) \cos m\varphi + (c^2 \mu_x - c^1 \mu_y) \sin m\varphi) \, dt = 0
\]

(3.13)

for every closed geodesic \( \gamma \), where \( \varphi = \varphi(t) \) is the angle between the geodesic and the coordinate line \( y = \text{const} \) of a global isothermal coordinate system.

\textbf{Proof.} As known (see [1, Lemma 4.3.1]), every potential tensor field belongs to the kernel of the ray transform; i.e., it integrates to zero over every closed geodesic. In our case, \( Z^{m,c} \) is a potential field by Theorem 3.4 and so

\[
\int \gamma Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} \, dt = 0
\]

(3.14)

for every closed geodesic \( \gamma \). If \( \gamma(t) = (x(t), y(t)) \), then

\[
Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} = \sum_{k \geq 0} \binom{m}{k} Z^{m,c}_{1...1 2...2} \dot{x}^{m-k} \dot{y}^k.
\]

We assume the binomial coefficient \( \binom{m}{k} = \frac{m!}{k!(m-k)!} \) to be equal to zero for \( k > m \) which allows us not to designate the upper summation limit on the right-hand side. Separating the summands corresponding to even and odd \( k \), we get

\[
Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} = \sum_{k \geq 0} \binom{m}{2k} Z^{m,c}_{1...1 2...2} \dot{x}^{m-2k} \dot{y}^{2k} + \sum_{k \geq 0} \binom{m}{2k+1} Z^{m,c}_{1...1 2...2} \dot{x}^{m-2k-1} \dot{y}^{2k+1}.
\]

Since \( jZ^{m,c} = 0 \), we have

\[
Z^{m,c}_{1...1 2...2} = (-1)^k Z^{m,c}_{1...1}, \quad Z^{m,c}_{1...1 2...2} = (-1)^k Z^{m,c}_{1...1 12}.
\]

We use these relations, transforming the previous formula to the form

\[
Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} = Z^{m,c}_{1...1} \sum_{k \geq 0} (-1)^k \binom{m}{2k} \dot{x}^{m-2k} \dot{y}^{2k} + Z^{m,c}_{1...1 12} \sum_{k \geq 0} (-1)^k \binom{m}{2k+1} \dot{x}^{m-2k-1} \dot{y}^{2k+1}.
\]

Without lost of generality, we can assume that \( ||\dot{\gamma}||^2 = e^{2\mu}(\dot{x}^2 + \dot{y}^2) = 1 \); i.e., \( \dot{x} = e^{-\mu} \cos \varphi \) and \( \dot{y} = e^{-\mu} \sin \varphi \). Then the last formula becomes

\[
Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} = e^{-\mu}(Z^{m,c}_{1...1} \cos m\varphi + Z^{m,c}_{1...1 12} \sin m\varphi).
\]

Inserting the values (3.9) of the components of \( Z^{m,c} \), we obtain

\[
Z^{m,c}_{i_1...i_m} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_m} = e^{\mu}((c^1 \mu_x + c^2 \mu_y) \cos m\varphi + (c^2 \mu_x - c^1 \mu_y) \sin m\varphi).
\]

Finally, inserting this expression into (3.14), we arrive at (3.13).

After rewriting (3.13) as

\[
ee^\mu \int \gamma (\mu_x \cos m\varphi - \mu_y \sin m\varphi) \, dt + e^\mu \int \gamma (\mu_y \cos m\varphi + \mu_x \sin m\varphi) \, dt = 0,
\]

we see that

\[
\frac{\int \gamma e^{\mu}(\mu_x \cos m\varphi - \mu_y \sin m\varphi) \, dt}{\int \gamma e^{\mu}(\mu_y \cos m\varphi + \mu_x \sin m\varphi) \, dt}
\]

(3.15)

is independent of \( \gamma \). Thus, if we succeeded in finding two closed geodesics such that the ratio (3.15) took different values for them, then the Riemannian torus would not admit a rank \( m+1 \) irreducible Killing tensor field.
§ 4. A Rank 3 Killing Tensor Field on the Two-Dimensional Torus

Let $(\mathbb{T}^2, g) = (\mathbb{R}^2/G, e^{2\mu}(dx^2 + dy^2))$ be a Riemannian torus. Given a constant pseudovector $c = (c^1, c^2)$ of weight 3, we introduce the tensor field $T^c \in C^\infty(\text{Ker}^2\gamma)$ by setting in global isothermal coordinates

$$T^c_{11} = -T^c_{22} = e^{4\mu}(-c^2\mu_x + c^1\mu_y), \quad T^c_{12} = e^{4\mu}(c^1\mu_x + c^2\mu_y).$$

(4.1)

Up to notations, (4.1) coincides with (3.9) in the case of $m = 2$. Indeed,

$$T^c = Z^{2,c}, \quad \text{where} \quad c^\perp = (-c^2, c^1).$$

If $c$ is a constant pseudovector of weight 3, then $c^\perp$ is a constant pseudovector of weight 3 too.

Let $f \in C^\infty(S^3)$ be a rank 3 real irreducible tensor field on the torus $(\mathbb{T}^2, g)$. Equations (2.6) look in this case as follows:

$$\delta f^{(0)} = 0, \quad pdf^{(0)} + \frac{1}{2}\delta f^{(1)} = 0, \quad pdf^{(1)} = 0.$$  

(4.2)

By Theorem 3.2, the last equation of the system means that in global isothermal coordinates

$$f^{(1)}_{111} = -f^{(1)}_{122} = \frac{1}{3}c^1e^{6\mu}, \quad f^{(1)}_{112} = -f^{(1)}_{222} = \frac{1}{3}c^2e^{6\mu}$$

(4.3)

for some real constant pseudovector $0 \neq c = (c^1, c^2)$ of weight 3. The coefficient $1/3$ is included here to simplify further formulas. We already calculated the divergence of this tensor field (formulas (3.11)): $\delta f^1 = 2Z^{2,c}$. Hence (4.2) is reduced to the following:

$$\delta f^{(0)} = 0, \quad pdf^{(0)} = -Z^{2,c}.$$  

(4.4)

We calculate the divergence of the covector field $f^{(0)}$ by the standard rules

$$\delta f^{(0)} = g^{pq}\nabla_pf_q^{(0)} = e^{-2\mu}\left(\frac{\partial f^{(0)}_1}{\partial x} + \frac{\partial f^{(0)}_2}{\partial y}\right).$$

Therefore the first equation of (4.4) looks in global isothermal coordinates as follows:

$$\frac{\partial f^{(0)}_1}{\partial x} + \frac{\partial f^{(0)}_2}{\partial y} = 0.$$  

We satisfy this equation by setting

$$f^{(0)}_1 = -\nabla_2u = -u_y, \quad f^{(0)}_2 = \nabla_1u = u_x,$$

(4.5)

where $u(x, y)$ is a smooth real function on the plane whose partial derivatives $u_x$ and $u_y$ are $\Gamma$-periodic. The function $u$ is determined by the field $f^{(0)}$ uniquely up to an additive constant.

The definition of $d$ and (4.5) yield

$$(df^{(0)})_{11} = -\nabla_1\nabla_2u, \quad (df^{(0)})_{12} = \frac{1}{2}(\nabla_1\nabla_1u - \nabla_2\nabla_2u), \quad (df^{(0)})_{22} = \nabla_1\nabla_2u.$$  

Observe that $df^{(0)}$ turns out to be a trace free field: $j(df^{(0)}) = e^{-2\mu}((df^{(0)})_{11} + (df^{(0)})_{22}) = 0$. Therefore $pdf^{(0)} = df^{(0)}$. Thus, the second of the equations (4.4) is written in isothermal coordinates as the system

$$\nabla_1\nabla_2u = Z^{2,c}_{11}, \quad \frac{1}{2}(\nabla_1\nabla_1u - \nabla_2\nabla_2u) = -Z^{2,c}_{12}.$$  

Comparing (3.9) and (4.1), we see that $Z^{2,c}_{11} = T^c_{11}$ and $Z^{2,c}_{12} = -T^c_{11}$. Therefore the previous system can be rewritten as

$$\frac{1}{2}(\nabla_1\nabla_1u - \nabla_2\nabla_2u) = T^c_{11}, \quad \nabla_1\nabla_2u = T^c_{12}.$$  

(4.6)
Let us rewrite system (4.6) in invariant form. To this end, we observe first of all that the Hessian $\nabla\nabla u = (\nabla_i \nabla_j u)$ of $u$ is a well-defined symmetric tensor field on the torus since the partial derivatives of $u$ are $\Gamma$-periodic. The Riemannian Laplacian $\Delta u = \text{tr}(\nabla\nabla u) = g^{ij}\nabla_i \nabla_j u$ is a well-defined function on the torus. Let us now consider the trace free part of the Hessian

$$\nabla\nabla u - \frac{1}{2}(\Delta u)g,$$

where $g$ is the metric tensor. In isothermal coordinates

$$\left(\nabla\nabla u - \frac{1}{2}(\Delta u)g\right)_{11} = -\left(\nabla\nabla u - \frac{1}{2}(\Delta u)g\right)_{22} = \frac{1}{2}(\nabla_1 \nabla_1 u - \nabla_2 \nabla_2 u),$$

$$\left(\nabla\nabla u - \frac{1}{2}(\Delta u)g\right)_{12} = \nabla_1 \nabla_2 u.$$

Comparing these equalities with (4.6), we see that (4.6) is equivalent to the equation

$$\nabla\nabla u - \frac{1}{2}(\Delta u)g = T^c. \quad (4.7)$$

The latter arguments can be reversed: If $u \in C^\infty(\mathbb{R}^3)$ is a solution to (4.7) with $\Gamma$-periodic partial derivatives, then (4.3) and (4.5) define a rank 3 Killing tensor field that is irreducible in the case of $c \neq 0$. We have thus proved

**Lemma 4.1.** A Riemannian torus $(\mathbb{R}^2/\Gamma, e^{2\mu}(dx^2 + dy^2))$ admits a rank 3 irreducible Killing tensor field if and only if, for some constant pseudovector $c \neq 0$ of weight 3, equation (4.7) has a solution $u \in C^\infty(\mathbb{R}^2)$ with $\Gamma$-periodic derivatives $u_x$ and $u_y$.

In this section, we will obtain two necessary solvability conditions for equation (4.7) which are of some interest.

Let us demonstrate that (4.7) can be solved with respect to all third order derivatives of the function $u$. Now, we perform our calculations in arbitrary coordinates. We introduce the temporary notation $v = \frac{1}{2} \Delta u$. Differentiate (4.7) to obtain

$$\nabla_i \nabla_j \nabla_k u = g_{jk} \nabla_i v + \nabla_i T^c_{jk}. \quad (4.8)$$

By the commutator formula for covariant derivatives,

$$\nabla_1 \nabla_2 u - \nabla_2 \nabla_1 u = -R^{1}_{112} \nabla_1 u - R^{2}_{112} \nabla_2 u,$$

$$\nabla_1 \nabla_2 u - \nabla_2 \nabla_1 u = -R^{1}_{212} \nabla_1 u - R^{2}_{212} \nabla_2 u,$$

where $R = (R^i_{jkl})$ is the curvature tensor. Substituting values (4.8) for third order derivatives into left-hand sides of these equalities, we arrive to the system

$$g_{12} \nabla_1 v - g_{11} \nabla_2 v = -R^{1}_{112} \nabla_1 u - R^{2}_{112} \nabla_2 u + \nabla_2 T^c_{11} - \nabla_1 T^c_{12},$$

$$g_{22} \nabla_1 v - g_{12} \nabla_2 v = -R^{1}_{212} \nabla_1 u - R^{2}_{212} \nabla_2 u + \nabla_2 T^c_{12} - \nabla_1 T^c_{22}.$$
and the two previous formulas can be written uniformly:
\[
\nabla_i v = -R^i_{12} \nabla_i u + \nabla^p T^c_{ip},
\]
where \( \nabla^p = g^{pq} \nabla_q \). Recall that \( R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}) \) in the two-dimensional case where \( K \) is the Gaussian curvature. Hence \( R^i_{12} = K \) and the previous formula takes the form
\[
\nabla_i v = -K \nabla_i u + \nabla^p T^c_{ip}.
\]
Inserting this value into (4.8), we obtain the final formula
\[
\nabla_i \nabla_j \nabla_k u = g_{jk}( -K \nabla_i u + \nabla^p T^c_{jp} ) + \nabla_i T^c_{jk}.
\] (4.9)

Now, we will derive some solvability condition for (4.9). In the case when the third order partial derivatives appear on the left-hand sides, the standard approach for deriving such solvability conditions consists of differentiating equations and using the symmetry of fourth order partial derivatives. In our case, the covariant derivatives stand on the left-hand sides. Therefore we need to use the corresponding commutator formulas.

Differentiate (4.9) to obtain
\[
\nabla_i \nabla_j \nabla_k \nabla \ell u = g_{k\ell}( -K \nabla_i \nabla_j u - \nabla_i K \cdot \nabla_j u + \nabla_i \nabla^p T^c_{jp} + \nabla_i \nabla^\ell T^c_{ip} ) + \nabla_i \nabla_j T^c_{k\ell}.
\]
Then we alternate this equality in the indices \((i, j)\) and write the result as follows:
\[
g_{k\ell}( -\nabla_j K \cdot \nabla_i u + \nabla_i K \cdot \nabla_j u - \nabla_i \nabla^p T^c_{jp} + \nabla_j \nabla^\ell T^c_{ip} )
= ( \nabla_i \nabla_j T^c_{k\ell} - \nabla_j \nabla_i T^c_{k\ell} ) - ( \nabla_i \nabla_j \nabla_k \nabla \ell u - \nabla_j \nabla_i \nabla_k \nabla \ell u ).
\] (4.10)
Let us demonstrate that the right-hand side of this formula is identically zero. Indeed, by the commutator formula for covariant derivatives,
\[
\nabla_i \nabla_j T^c_{k\ell} - \nabla_j \nabla_i T^c_{k\ell} = -R^p_{kij} T^c_{p\ell} - R^p_{ijp} T^c_{kp},
\]
\[
\nabla_i \nabla_j \nabla_k \nabla \ell u - \nabla_j \nabla_i \nabla_k \nabla \ell u = -R^p_{kij} \nabla_p \nabla \ell u - R^p_{ijp} \nabla_k \nabla \ell u.
\] (4.11)
Substituting (4.7) for the second order derivatives of the function \( u \) into the right-hand side of the last formula, we obtain
\[
\nabla_i \nabla_j \nabla_k \nabla \ell u - \nabla_j \nabla_i \nabla_k \nabla \ell u = -R^p_{kij} T^c_{p\ell} - R^p_{ijp} T^c_{kp} - \frac{1}{2} (\Delta u) R_{kij} - \frac{1}{2} (\Delta u) R_{kij}.
\]
The sum of the two last terms on the right-hand side is equal to zero in view of the symmetries of the curvature tensor and the formula is simplified as follows:
\[
\nabla_i \nabla_j \nabla_k \nabla \ell u - \nabla_j \nabla_i \nabla_k \nabla \ell u = -R^p_{kij} T^c_{p\ell} - R^p_{ijp} T^c_{kp}.
\]
From this and (4.11), we see that the right-hand side of (4.10) is indeed equal to zero. Now, (4.10) takes the form
\[
-\nabla_j K \cdot \nabla_i u + \nabla_i K \cdot \nabla_j u = \nabla_i \nabla^p T^c_{jp} - \nabla_j \nabla^\ell T^c_{ip}.
\]
Setting \((i, j) = (1, 2)\) here, we arrive to the equality
\[
-\nabla_2 K \cdot \nabla_1 u + \nabla_1 K \cdot \nabla_2 u = \nabla_1 \nabla^p T^c_{2p} - \nabla_2 \nabla^p T^c_{1p}
\]
that can be written as
\[
\left( -\nabla_2 K \frac{\partial}{\partial x^1} + \nabla_1 K \frac{\partial}{\partial x^2} \right) u = \nabla_1 \nabla^p T^c_{2p} - \nabla_2 \nabla^p T^c_{1p}.
\] (4.12)
Let $\nabla^\perp K$ be the vector field obtained from the gradient $\nabla K$ by rotating through the right angle in the positive direction. We assume the torus to be oriented and the coordinate system to be agreed with the orientation so that the shortest rotation from $\partial/\partial x^1$ to $\partial/\partial x^2$ goes in the positive direction. Then

$$\nabla^\perp K = (g_{11}g_{22} - g_{12}^2)^{-1/2} \left( - \nabla_2 K \frac{\partial}{\partial x^1} + \nabla_1 K \frac{\partial}{\partial x^2} \right);$$

and (4.12) takes the final form

$$(\nabla^\perp K) u = \Phi^c,$$  \hspace{1cm} (4.13)

where

$$\Phi^c = (g_{11}g_{22} - g_{12}^2)^{-1/2} (\nabla_1 \nabla^p T_2^p - \nabla_2 \nabla^p T_1^p).$$  \hspace{1cm} (4.14)

In particular, (4.13) implies that $\Phi^c$ is a well-defined smooth function on the torus. The latter fact can be proved directly by checking that the right-hand side of (4.14) is independent of the choice of coordinates.

Substituting values (4.1) for the components of the tensor $T^c$ into (4.14) and performing some easy calculations, we obtain the following expression for $\Phi^c$ in global isothermal coordinates:

$$\Phi^c = c^1 \Lambda_1 + c^2 \Lambda_2,$$  \hspace{1cm} (4.15)

where

$$\Lambda_1 = \mu_{xxx} - 3\mu_{xxy} + 10\mu_{xxx} - 20\mu_{yyx} - 10\mu_{xxy} + 8\mu_x^3 - 24\mu_x \mu_y^2,$$

$$\Lambda_2 = 3\mu_{xxx} - 3\mu_{yyx} + 20\mu_{xxx} - 10\mu_{yyx} + 24\mu_x \mu_y - 8\mu_y^3.$$  \hspace{1cm} (4.16)

As we know, $c = (c^1, c^2)$ is a pseudovector of weight 3. Equality (4.15) gives us an impetus to the suggestion: $\Lambda = (\Lambda_1, \Lambda_2)$ must be a 1-pseudoform of weight 3. This fact is not obvious from (4.16). To clarify the situation, let us find the complex version of (4.16). Using the equalities

$$e^{2i\mu} = \lambda, \quad \partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = i(\partial_z - \partial_{\bar{z}}),$$

and performing some easy calculations, we transform (4.16) to obtain

$$\Lambda_1 = \frac{2}{\lambda} \left( \frac{\lambda_{zzz} + \lambda_{zz\bar{z}}}{\lambda} + 4 \frac{\lambda_z \lambda_{zz} + \lambda_{z\bar{z}} \lambda_{\bar{z}z}}{\lambda^2} - 2 \frac{\lambda_{zz}^2 + \lambda_{z\bar{z}}^2}{\lambda^3} \right),$$

$$\Lambda_2 = i \left( 2 \frac{\lambda_{zz} - \lambda_{z\bar{z}}}{\lambda} + 4 \frac{\lambda_z \lambda_{zz} - \lambda_{z\bar{z}} \lambda_{\bar{z}z}}{\lambda^2} - 2 \frac{\lambda_{zz}^2 - \lambda_{z\bar{z}}^2}{\lambda^3} \right).$$  \hspace{1cm} (4.17)

The invariant nature of these formulas is now obvious, which is expressed in our language as follows: $\Lambda = (\Lambda_1, \Lambda_2)$ is a 1-pseudoform of weight 3. The summands on the right-hand sides of (4.17) give us the three examples of 1-pseudoforms of weight 3:

$$\Lambda^{(1)} = (\Lambda_1^{(1)}, \Lambda_2^{(1)}) = \frac{1}{\lambda} (\lambda_{zzz} + \lambda_{zz\bar{z}}, i(\lambda_{zzz} - \lambda_{zz\bar{z}})),$$

$$\Lambda^{(2)} = (\Lambda_1^{(2)}, \Lambda_2^{(2)}) = \frac{1}{\lambda^2} (\lambda_z \lambda_{zz} + \lambda_z \lambda_{z\bar{z}}, i(\lambda_z \lambda_{zz} - \lambda_z \lambda_{z\bar{z}})),$$

$$\Lambda^{(3)} = (\Lambda_1^{(3)}, \Lambda_2^{(3)}) = \frac{1}{\lambda^3} (\lambda_z^2 + \lambda_{z\bar{z}}^2, i(\lambda_z^3 - \lambda_{z\bar{z}}^3)).$$  \hspace{1cm} (4.18)

We emphasize that there is no ambiguity in the definition; i.e., these 1-pseudoforms are completely determined by the metric $g$ as well as the 1-pseudoform of weight 3 participating in (4.15)

$$\Lambda = 2\Lambda^{(1)} + 4\Lambda^{(2)} - 2\Lambda^{(3)}. $$  \hspace{1cm} (4.19)

The expression on the left-hand side of (4.13) is the derivative of $u$ along the isoline $\gamma$ of the function $K$ which is parametrized so that $\|\gamma\| = \|\nabla K\|$. Integrating (4.13) over $\gamma$, we arrive to the following theorem.
**Theorem 4.2.** Let \((\mathbb{T}^2, g) = (\mathbb{R}^2/\Gamma, \lambda)\) be a two-dimensional Riemannian torus. If the torus admits a real irreducible rank 3 Killing tensor field; then the 1-pseudoform \(\Lambda\) of weight 3, which is defined by (4.18) and (4.19), is such that there are a real constant pseudovector \(c \neq 0\) of weight 3 and real function \(u \in C^\infty(\mathbb{R}^2)\) with \(\Gamma\)-periodic partial derivatives satisfying the following statement:

Let a curve \(\gamma : [a, b] \to \mathbb{T}^2\) be a part of an isoline \(\{K = K_0\}\) of the Gaussian curvature \(K\). Assume that \(\gamma\) does not contain critical points of \(K\) and is parametrized so that \(\|\dot{\gamma}\| = \|\nabla K\|\). Let \(\tilde{\gamma} : [a, b] \to \mathbb{R}^2\) be the lift of \(\gamma\) with respect to the covering \(\mathbb{R}^2 \to \mathbb{R}^2/\Gamma = \mathbb{T}^2\). Then

\[
\int_a^b (c^1 \Lambda_1(\gamma(t)) + c^2 \Lambda_2(\gamma(t)) \, dt = u(\tilde{\gamma}(b)) - u(\tilde{\gamma}(a)).
\]  

(4.20)

Since \(u \in C^\infty(\mathbb{R}^2)\) has \(\Gamma\)-periodic derivatives \(u_x\) and \(u_y\), the function \(u\) can be uniquely represented as

\[
u(x, y) = w(x, y) + \alpha_1 x + \alpha_2 y,
\]

(4.21)

where \(\alpha_1, \alpha_2 \in \mathbb{R}\) and \(w\) is a \(\Gamma\)-periodic function. The expression \(\alpha = \alpha_1 \, dx + \alpha_2 \, dy\) is a well-defined closed 1-form on the torus independent of the choice of global isothermal coordinates. Let \(\sigma = [\alpha] \in H^1(\mathbb{T}^2, \mathbb{R})\) be the one-dimensional homology class defined by the form \(\alpha\).

Now, we consider the case of a closed curve \(\gamma : [a, b] \to \mathbb{T}^2\) participating in Theorem 4.2. If \(\tilde{\gamma}(t) = (\tilde{\gamma}^1(t), \tilde{\gamma}^2(t))\) is the lift of \(\gamma\), then the vector \(\tilde{\gamma}(b) - \tilde{\gamma}(a)\) belongs to the lattice \(\Gamma\). Using (4.21), we write the right-hand side of (4.20) as

\[
u(\tilde{\gamma}(b)) - \nu(\tilde{\gamma}(a)) = [w(\tilde{\gamma}(b)) - w(\tilde{\gamma}(a))] + [\alpha_1(\tilde{\gamma}^1(b) - \tilde{\gamma}^1(a)) + \alpha_2(\tilde{\gamma}^2(b) - \tilde{\gamma}^2(a))].
\]

The difference in the first brackets is equal to zero since \(w\) is a \(\Gamma\)-periodic function. The expression in the second brackets is obviously equal to \(\langle \sigma, [\gamma] \rangle\), where \([\gamma] \in H_1(\mathbb{T}^2, \mathbb{R})\) is the one-dimensional homology class determined by the cycle \(\gamma\) and \(\langle \cdot, \cdot \rangle : H^1(\mathbb{T}^2, \mathbb{R}) \times H_1(\mathbb{T}^2, \mathbb{R}) \to \mathbb{R}\) is the canonical paring of one-dimensional de Rham cohomologies and homologies. In this way we arrive to the following theorem.

**Theorem 4.3.** Let \((\mathbb{T}^2, g)\) be a two-dimensional Riemannian torus. If the torus admits a real irreducible rank 3 Killing tensor field, then the 1-pseudoform \(\Lambda\) of weight 3 which is defined by (4.18) and (4.19) is such that there exist a real constant pseudovector \(c \neq 0\) of weight 3 and cohomology class \(\sigma \in H^1(\mathbb{T}, \mathbb{R})\) such that

\[
\int_{\gamma} (c^1 \Lambda_1 + c^2 \Lambda_2) \, dt = \langle \sigma, [\gamma] \rangle
\]

(4.22)

for every closed curve \(\gamma : [a, b] \to \mathbb{T}^2\) which is a part of an isoline \(\{K = K_0\}\), does not contain critical points of \(K\), and is parametrized so that \(\|\dot{\gamma}\| = \|\nabla K\|\).

Unlike Theorem 4.2, the function \(u\) is not mentioned here. Therefore Theorem 4.3 can be considered as a necessary condition for the solvability of (4.7).

**Corollary 4.4.** Under the hypotheses of Theorem 4.3, let \(\gamma\) be a contractible closed curve that is a part of an isoline \(\{K = K_0\}\) and let \(D\) be the closed domain on the torus which is homeomorphic to a disk and bounded by \(\gamma\). Assume that there is exactly one critical point of \(K\) in \(D\) and, moreover the point belongs to the interior of \(D\) and is a nondegenerate critical point either of index 0 or of index 2 (i.e., it is a point either of a local maximum or of a local minimum). Then

\[
\int_D (c^1 \Lambda_1 + c^2 \Lambda_2) \, d\sigma = 0,
\]

(4.23)

where \(d\sigma\) is the area element.
Corollary 4.5. Under the hypotheses of Theorem 4.3, let $D$ be an annulus domain on the torus which is homeomorphic to the product of a segment and circle. Assume that $D$ does not contain critical points of $K$ and both boundary circles are parts of the isolines $\{K = K_0\}$ and $\{K = K_1\}$ respectively. Then

$$\int_D (c_1^2 \Lambda_1 + c_2^2 \Lambda_2) d\sigma = \pm \langle \sigma, [\gamma] \rangle (K_1 - K_0), \quad (4.24)$$

where $\gamma$ is one of the boundary circles of $D$.

To prove Corollaries 4.4 and 4.5, it suffices to observe that, if the parametrization $\gamma(t)$ of an isoline is chosen as in Theorem 4.3, then $dt \wedge dK = \pm d\sigma$.

In closing, let us return to (4.7) and derive some new fourth order equation for $u$. Obviously, $d^2 u$ is the Hessian of $u$ and $pd^2 u$ is the trace-free part of the Hessian. Hence, (4.7) can be written as

$$pd^2 u = T^c. \quad (4.25)$$

The operator $\delta^2$ is adjoint to $pd^2$. Apply $\delta^2$ to both sides of (4.25) to obtain

$$\delta^2 pd^2 u = \delta^2 T^c. \quad (4.26)$$

Observe that the passage from (4.25) to (4.26) is not reversible. Therefore, the further conclusions should be considered as necessary conditions for the solvability of (4.7).

The fourth order operator $\delta^2 pd^2$ is quite similar to the Laplacian squared. To clarify the similarity, we perform some calculations in coordinates. First of all,

$$(pd^2 u)_{ij} = \nabla_i \nabla_j u - \frac{1}{2} g_{ij} \Delta u.$$ 

Hence,

$$\delta^2 pd^2 u = \nabla^i \nabla^j \left( \nabla_i \nabla_j u - \frac{1}{2} g_{ij} \Delta u \right) = \nabla^i \nabla^j \nabla_i \nabla_j u - \frac{1}{2} \Delta^2 u.$$ 

Permuting the derivatives $\nabla^j$ and $\nabla_i$ in the first term on the right-hand side with the help of the corresponding commutator formula, we obtain

$$\delta^2 pd^2 u = \frac{1}{2} \Delta^2 u - \nabla^i \left( R_i^j \nabla_j u \right),$$

where $R_i^j$ is the Ricci tensor. In the two-dimensional case, $R_i^j = -K \delta_i^j$, where $K$ is the Gaussian curvature, and the last formula takes the form

$$\delta^2 pd^2 u = \frac{1}{2} \Delta^2 u + \nabla^i (K \nabla_i u).$$

Equation (4.26) is thus equivalent to

$$\frac{1}{2} \Delta^2 u + \delta(K du) = \delta^2 T^c. \quad (4.27)$$

Now, we evaluate the right-hand side of (4.27). We already calculated the divergence of an arbitrary trace-free tensor field in (3.8). Applying this formula to $T^c$ and using (4.1), we find in global isothermal coordinates

$$(\delta T^c)_1 = e^{2\mu} (-c_2^2 \mu_{xx} + 2c_2^2 \mu_{xy} + 2c_2^2 \mu_{yy} - 4c_2^2 \mu_x^2 + 8c_1^1 \mu_x \mu_y + 4c_2^2 \mu_y^2), \quad (\delta T^c)_2 = e^{2\mu} (c_1^1 \mu_{xx} + 2c_2^2 \mu_{xy} - c_1^1 \mu_{yy} + 4c_1^1 \mu_x^2 + 8c_2^2 \mu_x \mu_y - 4c_1^1 \mu_y^2).$$
Inserting these values into the formula
\[ \delta^2 T^c = e^{-2\mu} \left( \frac{\partial (\delta T^c)_1}{\partial x} + \frac{\partial (\delta T^c)_2}{\partial y} \right), \]
we obtain
\[ \delta^2 T^c = -c^2 \Lambda_1 + c^1 \Lambda_2, \]
where \( \Lambda = (\Lambda_1, \Lambda_2) \) is defined by (4.16). Thus, (4.27) takes the form
\[ \frac{1}{2} \Delta^2 u + \delta (K du) = -c^2 \Lambda_1 + c^1 \Lambda_2 \]
or, in more traditional notations,
\[ \frac{1}{2} \Delta^2 u + \text{div}(K \nabla u) = -c^2 \Lambda_1 + c^1 \Lambda_2. \quad (4.28) \]
Combining (4.13) and (4.28), we arrive to the following theorem.

**Theorem 4.6.** If a Riemannian torus \( (\mathbb{R}^2/\Gamma, g) \) admits a rank 3 irreducible Killing tensor field, then there is a constant real pseudovector \( 0 \neq c = (c^1, c^2) \) such that the system of equations
\[ (\nabla^1 K) u = c^1 \Lambda_1 + c^2 \Lambda_2, \quad \frac{1}{2} \Delta^2 u + \text{div}(K \nabla u) = -c^2 \Lambda_1 + c^1 \Lambda_2 \quad (4.29) \]
has a solution \( u \in C^\infty(\mathbb{R}^2) \) with \( \Gamma \)-periodic derivatives \( u_x \) and \( u_y \). Here \( K \) is the Gaussian curvature and \( \Lambda = (\Lambda_1, \Lambda_2) \) is the 1-pseudoform of weight 3 which is defined by (4.18) and (4.19) in global isothermal coordinates.

Integrating the second of the equalities in (4.29) over the torus, we obtain
\[ -c^2 \int_{T^2} \Lambda_1 d\sigma + c^1 \int_{T^2} \Lambda_2 d\sigma = 0, \]
where \( d\sigma = e^{2\mu} \, dx dy \) is the area element. At the first sight we can imagine that the equality allows us to determine the ratio \( c^1/c^2 \). But this is wrong since the equalities
\[ \int_{T^2} \Lambda_1 d\sigma = 0, \quad \int_{T^2} \Lambda_2 d\sigma = 0 \quad (4.30) \]
always hold irrespectively of the existence of Killing fields. Indeed, for every constant pseudovector \( c = (c^1, c^2) \),
\[ c^1 \Lambda_1 + c^2 \Lambda_2 = e^{-2\mu} (\nabla_1 (\delta T^c)_2 - \nabla_2 (\delta T^c)_1) = \nabla^1 (\delta T^c)_2 - \nabla^2 (\delta T^c)_1 \]
in global isothermal coordinates. Introduce the covector field \( v = (\delta T^c)^\perp = -(\delta T^c)_2 \, dx + (\delta T^c)_1 \, dy \). The previous formula can be rewritten in terms of \( v \) as follows:
\[ c^1 \Lambda_1 + c^2 \Lambda_2 = -(\nabla^1 v_1 + \nabla^2 v_2) = -\delta v. \]
Hence
\[ c^1 \int_{T^2} \Lambda_1 d\sigma + c^2 \int_{T^2} \Lambda_2 d\sigma = 0. \]
This is equivalent to (4.30) since \( c \) is arbitrary.
References

1. Sharafutdinov V., Integral Geometry of Tensor Fields, VSP, Utrecht, The Netherlands (1994).
2. Darboux G., Lecons sur la théorie generale des surfaces et les applications géométriques du calcul infinitesimal, Gau-
thier-Villars, Paris (1891).
3. Birkhoff G. D., Dynamical Systems, Amer. Math. Soc., Providence (1927) (Amer. Math. Soc. Colloq. Publ.; V. 9).
4. Bolsinov A. V. and Fomenko A. T., Integrable Geodesic Flows on Two-dimensional Surfaces, Plenum Academic Publishers, New York (2000).
5. Case K. M. and Zweifel P. F., Linear Transport Theory, Addison-Wesley Publishing Company, Reading, Mass.; Palo Alto; London; Don Mills, Ont. (1967).
6. Guillemin V. and Kazhdan D., “Some inverse spectral results for negatively curved 2-manifolds,” Topology, 19, No. 3, 301–302 (1980).
7. Dairbekov N. S. and Sharafutdinov V. A., “On conformal Killing symmetric tensor fields on Riemannian manifolds,” Sib. Adv. Math., 21, No. 1, 1–41 (2011).
8. Matveev V. and Shevchishin V., “Differential invariants for cubic integrals of geodesic flows on surfaces,” J. Geom. Phys., 60, 833–856 (2010).
9. Kolokoltsov V. N., “Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial in the velocities,” Math. USSR-Izv., 21, No. 2, 291–306 (1983).
10. Bialy M. and Mironov A., “Cubic and quartic integrals for geodesic flow on 2-torus via a system of the hydrodynamic type,” Nonlinearity, 24, 3541–3557 (2011).
11. Sharafutdinov V. A., On Symmetric Tensor Fields on a Riemannian Manifold [in Russian] [Preprint, No. 539], Computer Center of the Siberian Division of Soviet Academy of Sciences, Novosibirsk (1984).

V. A. SHARAFUTDINOV
SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, RUSSIA
NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK, RUSSIA
E-mail address: sharaf@math.nsc.ru