TWO WEIGHT INEQUALITY FOR BERGMAN PROJECTION

JOSÉ ÁNGEL PELÁEZ AND JOUNI RÄTYYÄ

Abstract. The motivation of this paper comes from the two weight inequality
\[ \| P_\omega (f) \|_{L^p_v} \leq C \| f \|_{L^p_v}, \]
for the Bergman projection \( P_\omega \) in the unit disc. We show that the boundedness of \( P_\omega \) on \( L^p_v \) is characterized in terms of self-improving Muckenhoupt and Bekolle-Bonami type conditions when the radial weights \( v \) and \( \omega \) admit certain smoothness. En route to the proof we describe the asymptotic behavior of the \( L^p \)-means and the \( L^p_v \)-integrability of the reproducing kernels of the weighted Bergman space \( A^2_\omega \). These estimates are further applied to characterize the positive Borel measures \( \mu \) such that the Toeplitz operator \( T_\mu \) belongs to the \( p \)-Schatten-Von Neumann ideal.

1. Introduction

If \( X \) is a Hilbert space of functions on a domain \( \Omega \) of the complex plane such that the point evaluations \( L_z \) are bounded linear functionals on \( X \), then the Riesz representation theorem ensures the existence of the unique reproducing kernels \( K_z \in X \) such that
\[ L_z f = f(z) = \langle f, K_z \rangle_X, \]
for all \( f \in X \). Reproducing kernels play a fundamental role in solutions of many important problems in operator theory in spaces of analytic functions. In the case of the Hardy space \( H^2 \) and the classical weighted Bergman space \( A^2_\alpha \) of the unit disc \( \mathbb{D} \), the kernels are given by the neat expressions \( (1 - \zeta \zeta^-1)^{-1} \) and \( (1 - \zeta \zeta^-1)^{-(2+\alpha)} \) that are easy to work with. Such useful formulas can also be obtained for the Hardy and Bergman spaces on more general domains and for the classical Fock space of entire functions. However, the general situation is much more complicated because of the lack of explicit expressions. This is the case, for example, if one considers the reproducing kernels of the weighted Bergman space \( A^2_\omega \) induced by a weight \( \omega \).

In this study we are interested in the \( L^p \)-behavior of the reproducing kernels \( B^2_\omega \) of the weighted Bergman space \( A^2_\omega \) of \( \mathbb{D} \) induced by a regular or a rapidly increasing (radial) weight \( \omega \). Postponing the exact definitions of these weights to the next section, we will denote these classes of weights by \( \mathcal{R} \) (for regular) and \( \mathcal{I} \) (for rapidly increasing). The weighted Bergman spaces induced by rapidly increasing weights form a natural setting for exploring the change of function-theoretic properties from the classical weighted Bergman space \( A^2_\omega \) to the Hardy spaces \( H^p \). In contrast to this, the regular weights induce a family of weighted Bergman spaces that is a framework for an extension of the classical theory on the standard Bergman spaces \( A^2_\alpha \). These statements become more apparent by the inclusions \( H^p \subset A^2_\omega \subset \bigcap_{\alpha \geq -1} A^2_\alpha \), valid for every \( \omega \in \mathcal{I} \), and the fact that each standard weight is regular.

Date: June 12, 2014.

Key words and phrases. Bergman space, reproducing kernel, Bergman projection, Toeplitz operator, Muckenhoupt class, Bekolle-Bonami class, regular weight, rapidly increasing weight.

This research was supported in part by the Ramón y Cajal program of MICINN (Spain); by Ministerio de Educación y Ciencia, Spain, projects MTM2011-25502 and MTM2011-26538; by La Junta de Andalucía, (FQM210) and (P09-FQM-4468); by Academy of Finland project no. 268009, by Väisälä Foundation of Finnish Academy of Science and Letters, and by Faculty of Science and Forestry of University of Eastern Finland project no. 930349.
The first of our main results describes the asymptotic behavior of the $L^p$-means of the reproducing kernel $B^{w}_{\omega}$ (or its derivatives), provided $\omega \in \mathcal{I} \cup \mathcal{R}$. The latter part of this theorem reveals a precise estimate for the $L^p_{\omega}$-integral of $B^{w}_{\omega}$ when both $\omega$ and $v$ belong to $\mathcal{I} \cup \mathcal{R}$. It is obvious that such kernel estimates have a large number of applications in the operator theory. In this study we will focus on the Toeplitz operator

$$T_{\mu}(f)(z) = \int_{\mathbb{D}} f(\zeta) B^{\omega}_{\mu}(z) \, d\mu(\zeta),$$

where $\mu$ is a positive Borel measure on $\mathbb{D}$. If $d\mu(\zeta) = \omega(\zeta) \, dA(\zeta)$ we obtain the Bergman projection $P_{\omega}$. By considering its sublinear counterpart

$$P^{\omega}_{\mu}(f)(z) = \int_{\mathbb{D}} |f(\zeta)||B^{\omega}_{\zeta}(z)|\omega(\zeta)\,dA(\zeta),$$

we will show that $P_{\omega}$ is bounded on $L^p_{\mu}$ if $\omega \in \mathcal{R}$ and $p > 1$. The situation is different for $\omega \in \mathcal{I}$ because then $P^{\omega}_{\mu}$ is not bounded on $L^p_{\mu}$. These results emphasize the general phenomena that many finer function-theoretic properties valid for $A^p_{\mu}$ just simply break down for $A^p_{\omega}$ induced by $\omega \in \mathcal{I}$.

As the main objective of this study we will characterize the pairs $(\omega, v)$ of regular weights (admitting certain smoothness) for which

$$\|P_{\omega}(f)\|_{L^p_{\nu}} \lesssim \|f\|_{L^p_{\nu}}, \quad f \in L^p_{\mu},$$

in terms of a neat condition that compares the spaces $L^p_{\mu}$ and $L^p_{\nu}$ by size. This condition is equivalent, on one hand, to a Muckenhoupt-type condition related to Hardy operators, and, on the other hand, to a generalization of the classical Bekolle-Bonami condition. Both conditions are self-improving and that plays a crucial role in the proof.

In the last part of the paper we will use our $L^p_{\mu}$-estimates for the reproducing kernels and intricate ideas due to Luecking [15] to describe those measures $\mu$ such that the Toeplitz operator $T_{\mu}$ belongs to the $p$-Schatten class $S_p(A^2_{\mu}).$

Throughout the paper, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

1.1. Background on weights. Before presenting the main results, we will shortly discuss the classes of regular and rapidly increasing weights. For further information on these classes, see [21] Chapter 1] and the references therein.

A function $\omega : \mathbb{D} \to (0, \infty)$, integrable over the unit disc $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. The distortion function of a radial weight is

$$\psi_{\omega}(r) = \frac{1}{\omega(r)} \int_{r}^{1} \omega(s) \, ds, \quad 0 \leq r < 1.$$ 

For short, we also write

$$\hat{\omega}(z) = \int_{|z|}^{1} \omega(s) \, ds, \quad z \in \mathbb{D}.$$ 

The distortion function is an efficient tool in classifying radial weights. For that purpose, we call a radial weight $\omega$ regular, denoted by $\omega \in \mathcal{R}$, if it is continuous and

$$\psi_{\omega}(r) \asymp (1 - r), \quad 0 \leq r < 1.$$ 

This asymptotic equality says that $\omega(r)$ behaves as its integral average over $(r, 1)$. It is known that if $\omega \in \mathcal{R}$, then for each $s \in [0, 1)$ there exists a constant $C = C(s, \omega) > 1$ such that

$$C^{-1}\omega(t) \leq \omega(r) \leq C\omega(t), \quad 0 \leq r \leq t \leq r + s(1 - r) < 1.$$

(1.1)
Further, a continuous radial weight $\omega$ is regular if and only if it admits the local regularity \((1.1)\) and
\[
\psi_\omega(r) \leq C(1 - r), \quad 0 \leq r < 1,
\]
for some constant $C = C(\omega) > 0$. A radial continuous weight $\omega$ is called rapidly increasing, denoted by $\omega \in \mathcal{I}$, if
\[
\lim_{r \to 1-} \frac{\psi_\omega(r)}{1 - r} = \infty.
\]
Even if standard examples of rapidly increasing weights satisfy \((1.1)\), the weights in $\mathcal{I}$ are by no means necessarily increasing functions and may actually admit a strong oscillatory behavior.

2. Main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$. Let $\mathcal{T}$ be the boundary of $\mathbb{D}$, and let $D(a, r) = \{ z : |z - a| < r \}$ denote the Euclidean disc of center $a$ and radius $r$. If $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, set
\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,
\]
\[
M_\infty(r, f) = \sup_{|z| = r} |f(z)|.
\]
For $0 < p \leq \infty$, the Hardy space $H^p$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty$. For $0 < p < \infty$ and a weight $\omega$, the weighted Bergman space $A^p_\omega$ is the space of $f \in \mathcal{H}(\mathbb{D})$ for which
\[
\|f\|_{A^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) < \infty,
\]
where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, we write $A^p_\omega$ for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1 - |z|^2)^\alpha$, where $-1 < \alpha < \infty$.

If $\omega \in \mathcal{I} \cup \mathcal{R}$, the norm convergence in $A^p_\omega$ implies the uniform convergence on compact subsets of $\mathbb{D}$, and therefore $A^p_\omega$ is a closed subspace of $L^p_\omega$. In particular, each point evaluation $L_a(f) = f(a)$ is a bounded linear functional on $A^p_\omega$, and hence there exist unique reproducing kernels $B^*_a \in A^2_\omega$ such that $\|L_a\| = \|B^*_a\|_{A^2_\omega}$ and
\[
f(a) = \langle f, B^*_a \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(z) \overline{B^*_a(z)} \omega(z) \, dA(z), \quad f \in A^2_\omega.
\]
Each radial weight $\omega$ is closely related to its associated weight
\[
\omega^*(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} \, ds, \quad z \in \mathbb{D} \setminus \{0\},
\]
by the Littlewood-Paley identity
\[
\|f\|^2_{A^2_\omega} = 4 \|f^*\|^2_{A^2_{\omega^*}} + \omega(\mathbb{D}) |f(0)|^2, \quad (2.1)
\]
which is a special case of a more general formula for $A^p_\omega$ \cite[Theorem 4.2]{20}. If $\omega \in \mathcal{I} \cup \mathcal{R}$, \cite[Lemma 1.6]{20} asserts that $\omega^*(z) \propto (1 - |z|)\overline{\omega}(z)$ for $z$ bounded away from zero.

In order to state our main results concerning the $L^p$-behavior of reproducing kernels, we need to introduce a family of Hilbert spaces. To do this, write $\omega_\beta(z) = (1 - |z|)^\beta \overline{\omega}(z)$ for all $\beta \in \mathbb{R}$ and $z \in \mathbb{D}$. For $-\infty < \alpha < 2$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, the Hilbert space $H_\alpha(\omega^*)$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that
\[
\int_{\mathbb{D}} |f'(z)|^2 \omega^*_\alpha(z) \, dA(z) < \infty.
\]
By (2.1), we deduce the identity $H_0(\omega^*) = A^2_0$. We will work with the inner product on $H_0(\omega^*)$ defined by

$$
\langle f, g \rangle_{H_0(\omega^*)} = f(0)g(0) + \int_D f'(z)\overline{g'(z)}\omega^*_{-\alpha}(z) \, dA(z). \quad (2.2)
$$

Let $K_{a,\omega}^\alpha \in H_0(\omega^*)$ be the corresponding reproducing kernels, that is,

$$
f(a) = f(0)K_{a,\omega}^\alpha(0) + \int_D f'(z)\frac{\partial K_{a,\omega}^\alpha(z)}{\partial z} \omega^*_{-\alpha}(z) \, dA(z), \quad f \in H_0(\omega^*). \quad (2.3)
$$

When a closed formula for the Bergman kernel $B_a^\omega$ exists, then the asymptotic growth of its $L^p$-means can be determined. For example, if the inducing weight is $\omega(z) = (1 - |z|^2)\alpha$, then an appropriate interpretation of the well-known $L^p$-estimate allows us to write

$$
M_p^p(r, B_a^\omega) \asymp \int_0^{|a|r} \frac{dt}{(1-t)^{2+\alpha}} \asymp \int_0^{|a|r} \frac{dt}{\overline{\omega}(1-t)^{p(1-\alpha)}}, \quad r, |a| \to 1^-,
$$

and, for $v(z) = (1 - |z|^2)^\beta$, we therefore have

$$
\|B_a^\omega\|_{L^p}^p \asymp \int_0^1 (1-r)^\beta \left( \int_0^{|a|r} \frac{dt}{(1-t)^{2+\alpha}} \right) \, dr \asymp \int_0^{|a|r} \frac{\hat{\nu}(r)}{\overline{\omega}(r)^{p(1-r)^{1+\alpha}}}, \quad \beta \to 1^-.
$$

This last one is a standard Bergman kernel estimate in the unit disc, attributed to Forelli and Rudin [13], that is usually written in a slightly different form, see [20, Lemma 3.10]. The $L^p$-behavior of the kernel $B_a^\omega$ can also be controlled in terms of off-diagonal pointwise estimates if the inducing weight tends to zero at least exponentially, as $|z| \to 1^-$, see [4] [14] [24].

Our first result shows that the discussion above regarding standard weights actually describes a general phenomenon rather than a particular case.

**Theorem 1.** Let $0 < p < \infty$, $\omega \in I \cup R$, $-\infty < \alpha < 2$ and $N \in N \cup \{0\}$. Then the following assertions hold:

(i) $M_p^p(r, (K_a^{\alpha,\omega})^N) \asymp \int_0^{|a|r} \frac{dt}{\overline{\omega}(1-t)^{p(N+1-\alpha)}}, \quad r, |a| \to 1^-.$

(ii) If $v \in I \cup R$, then

$$
\| (K_a^{\alpha,\omega})^N \|_{A_p^p} \asymp \int_0^{|a|r} \frac{\hat{\nu}(r)}{\overline{\omega}(r)^{p(1-r)^N}}, \quad |a| \to 1^-.
$$

Moreover, $K_a^{\alpha,\omega}$ can be replaced by $B_a^\omega$ in both (i) and (ii).

It is clear by the proof that the asymptotic inequality $\asymp$ in (2.4) is actually valid for any radial weight $v$, see [3,20] below. The following consequence of Theorem 1 is often more useful in praxis than the theorem itself.

**Corollary 2.** Let $0 < p < \infty$, $\omega \in I \cup R$, $-\infty < \alpha < 2$ and $N \in N \cup \{0\}$. Then the following assertions hold.

(i) $M_p^p(r, (K_a^{\alpha,\omega})^N) \asymp \frac{1}{\overline{\omega}(1-|a|r)^{p(N+1-\alpha)-1}}, \quad r, |a| \to 1^-,$

if and only if

$$
\int_0^{|a|r} \frac{dt}{\overline{\omega}(1-t)^{p(N+1-\alpha)}} \lesssim \frac{1}{\overline{\omega}(1-|a|r)^{p(N+1-\alpha)-1}}, \quad |a| \to 1^-.
$$

(ii) If $v \in I \cup R$, then

$$
\| (K_a^{\alpha,\omega})^N \|_{A_p^p} \asymp \frac{\hat{\nu}(a)}{\overline{\omega}(1-r)^{p(N+1-\alpha)-1}}, \quad |a| \to 1^-.
$$
Theorem 3. Let \( f \) consists of is more involved and relies on Theorem 1(i) and a result of Muckenhoupt on Hardy \( \omega \) clarifies the situation when there are plenty of bounded projections on \( L^p \) also Theorem 4 below. The situation is completely different for \( P \) and is smooth enough \( \[9, 10, 11, 24\] \). However, to characterize the class of weights for \( A \) of differentiation), decomposition norm estimates for \( v \) step, we will prove (ii) for \( v \) order to give a more uniform treatment we will argue differently. Indeed, as the second These estimates could then be used to establish the corresponding cases in (ii), but in \( v \) in (i) for \( A \), which was one of the first objects that emerged in the operator theory on Bergman \( v \) kernels. The most natural example of such operators is the Bergman projection \( P_\alpha(f)(z) = \frac{f(z)}{1 - \alpha^2} \), induced by analytic symbols \( f \) and \( g \).\[2\].

It is well-known that \( P_\alpha : L^p \to L^p \) is bounded for \( p > 1 \), but in contrast to this \( P_\omega \) fails to be bounded on \( L^p \) for \( p > 1 \) if \( \omega \) decreases sufficiently fast (at least exponentially) and is smooth enough \( \[9, 10, 11, 24\] \). However, to characterize the class of weights for which \( P_\omega : L^p \to L^p \) is bounded, is an open problem \( \[10, p. 116\] \). Our next result clarifies the situation when \( \omega \in I \cup R \). In the statement \( B \) denotes the Bloch space that consists of \( f \in H(\mathbb{D}) \) such that

\[ \|f\|_B = \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|^2) + |f(0)| < \infty. \]

Theorem 3. Let \( 1 < p < \infty \).

(i) If \( \omega \in R \), then \( P_\omega^+ : L^p \to L^p \) is bounded. In particular, \( P_\omega : L^p \to A^p \) is bounded.

(ii) If \( \omega \in R \), then \( P_\omega : L^\infty(\mathbb{D}) \to B \) is bounded.

(iii) If \( \omega \in I \), then \( P_\omega^+ \) is not bounded from \( L^p \) to \( L^p \).

Parts (i) and (ii) are straightforward applications of Theorem 1(ii). The proof of (iii) is more involved and relies on Theorem 1(i) and a result of Muckenhoupt on Hardy operators \( 17 \).

The projection \( P_\omega \) is not bounded on \( L^1_\omega \) if \( \omega \) is continuous. However, for \( \omega \in R \) there are plenty of bounded projections on \( L^1_\omega \) by \( 1 \) Proposition 2.1 and \([20, Lemma 2.1]\), see also Theorem 3 below. The situation is completely different for \( \omega \in I \) by a result due
to Shields and Williams [22, Theorem 3]. For the sake of completeness, we will rewrite this result in our language to show that there are no bounded projections from $L^1_\omega$ to $A^1_\omega$ if $\omega \in \mathcal{I}$ is smooth enough.

**Theorem A.** Let $\omega \in \mathcal{I}$ and assume that there exists an increasing function $\Psi : [0, \infty) \to [0, \infty)$, convex or concave, such that

$$\Psi(x) \simeq \frac{1}{\omega \left(1 - \frac{1}{x + 1}\right)}, \quad x \in [0, 1).$$

Then there are no bounded projections from $L^1_\omega$ to $A^1_\omega$.

Obviously, this result is strongly connected with the fact that there are no bounded projections from $L^1(\mathbb{T})$ to $H^1$ [20, Theorem 9.7].

Our main objective is to consider the Bergman projection $P_\omega$ acting on an $L^p$-space that is induced by a different weight than the kernel itself. This leads us to study the two weight inequality

$$\|P_\omega(f)\|_{L^p} \lesssim \|f\|_{L^p}, \quad f \in L^p_0.$$  \hspace{1cm} (2.8)

It is known that if $\omega(z) = (1 - |z|)^\alpha$ and $v(z) = (1 - |z|)^\beta$, then (2.8) holds if and only $(\alpha + 1) < p(\beta + 1)$, see [1] and [23, Theorem 4.24]. Moreover, Bekollé and Bonami described the weights (not necessarily radial) such that $P_\omega : L^p_0 \to A^p_0$ is bounded for $p > 1$ [3]. They also showed that these weights are exactly those for which the sublinear operator

$$P^+_\alpha(f)(z) = (\alpha + 1) \int_\mathbb{D} \frac{|f(\zeta)|(1 - |\zeta|^2)^\alpha}{|1 - z\zeta|^{2\alpha + 1}} dA(\zeta)$$

is bounded on $L^p_0$. It is worth mentioning that even if the Bekollé-Bonami weights are a kind of analogue of the Muckenhoupt class, these classes have significant differences [7].

Our contribution to the study of (2.8) is contained in the following result.

**Theorem 4.** Let $1 \leq p < \infty$ and $\omega, v \in \mathcal{R}$ such that $\kappa_\omega = \lim_{r \to 1-} \frac{\omega(r)}{1 - r}$ and $\kappa_v = \lim_{r \to 1-} \frac{v(r)}{1 - r}$ exist. Then the following assertions are equivalent:

(a) $P_\omega : L^p_0 \to L^p_0$ is a bounded operator;
(b) $P^+_\omega : L^p_0 \to L^p_0$ is a bounded operator;
(c) $\kappa_\omega < p\kappa_v$.

It is worth noticing that the class $\mathcal{R}$ contains weights $\omega$ for which $\lim_{r \to 1-} \frac{\omega(r)}{1 - r}$ does not exist. See [21, p. 266] for a concrete example.

To prove Theorem 4 for $p > 1$, we will first use the boundedness of the adjoint of $P_\omega$, with the monomials as test functions, to see that

$$\sup_{0 < r < 1} \frac{\hat{v}(r)^{\frac{1}{p} - \frac{1}{\alpha}}}{\hat{\omega}(r)^{\frac{1}{p} - \frac{1}{\alpha}}} \int_r^1 \frac{\hat{\omega}(s)^{\frac{1}{\alpha} - 1}}{\hat{v}(s)^{\frac{1}{p} - 1}(1 - s)} ds < \infty,$$  \hspace{1cm} (2.9)

that is, the integrand is a regular weight. If $\omega(z) = (1 - |z|)^\alpha$, then this is the same as saying that the radial weight $\frac{v}{(1 - |z|^p)^p}$ satisfies the corresponding Bekollé-Bonami condition. Further, we will show that (2.9) is equivalent to the Muckenhoupt-type condition

$$\sup_{0 < r < 1} \left( \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^{p}(1 - s)} ds \right)^{\frac{p}{p - 1}} \left( \int_r^1 \frac{\hat{\omega}(s)^{\frac{1}{\alpha} - 1}}{\hat{v}(s)^{\frac{1}{p} - 1}(1 - s)} ds \right)^{\frac{1}{p - 1}} < \infty,$$  \hspace{1cm} (2.10)

which in turn implies

$$\sup_{0 < r < 1} \frac{\hat{v}(r)^{p}}{\hat{\omega}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^{p}(1 - s)} ds < \infty.$$  \hspace{1cm} (2.11)
The condition \((2.10)\) follows also by the boundedness of \(P^+\) together with the result by Muckenhoupt on Hardy operators. Anyway, the condition \((2.11)\), as well as all the others, is self-improving in the sense that if it is satisfied for some \(p > 1\), then it is also satisfied when \(p\) is replaced by \(p - \delta\), where \(\delta > 0\) is sufficiently small. If we now denote by \(m\) the supremum of all such admissible \(\delta > 0\), and assume that the limits \(\kappa_\omega\) and \(\kappa_v\) exist, then it turns out that \(m = p - \frac{\omega_v}{\kappa_v} \in (0, p)\) implying, in particular, that \(\kappa_\omega < p\kappa_v\). This is then used together with Theorem 1 to show that \(P^+ : L^p_v \to L^p_v\) is bounded. Therefore the proof reveals several equivalent integral conditions that characterize the boundedness of \(P_\omega\). These conditions might be more useful than the neat inequality \(\kappa_\omega < p\kappa_v\) when one can deduce that the limits \(\kappa_\omega\) and \(\kappa_v\) do exist, but their exact values are hard to determine. It is also immediate by the proof that if \(P_\omega : L^p_v \to L^p_v\) is bounded, then \(P_\omega : L^{p-\delta}_v \to L^{p-\delta}_v\) is bounded for all \(\delta < m = p - \frac{\omega_v}{\kappa_v} \in (0, p)\), but \(P_\omega : L^{p-m}_v \to L^{p-m}_v\) just fails to be bounded, see Lemma 11 below.

The boundedness of projections play an important role in many characterizations of dual spaces and therefore it is natural to expect that Theorems 3 and 4 can be used to establish such results. Here we will only discuss two cases which are probably the most natural ones in this context. Part (ii) of the next result is probably known at least to experts working on the field.

**Theorem 5.** Let \(\omega \in \mathcal{R}\).

(i) If \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{p'} = 1\), then the dual of \(A^p_\omega\), with equivalent norms, is \(A^{p'}_\omega\) under the pairing

\[
\langle f, g \rangle_{A^{p'}_\omega} = \int_{\mathbb{D}} f(z) \overline{g(z)} \omega(z) dA(z). \tag{2.12}
\]

(ii) The dual of \(A^1_\omega\), with equivalent norms, is \(B\) under the pairing \((2.12)\).

We will present one more result whose proof relies partially on Theorem 1. This concerns a class of Toeplitz operators acting on the Dirichlet-type space \(H_\alpha(\omega^*)\). For a complex Borel measure \(\mu\) on \(\mathbb{D}\), the Toeplitz operator \(T_\mu\) associated with \(H_\alpha(\omega^*)\) is defined by

\[
T_\mu(f)(z) = \int_{\mathbb{D}} f(\zeta) K^{\alpha,\omega}(z, \zeta) d\mu(\zeta), \quad f \in H_\alpha(\omega^*),
\]

where \(K^{\alpha,\omega}(z, \zeta) = K^{\alpha,\omega}_z(\zeta)\).

The Toeplitz operator \(T_\mu\), associated with the kernel of a standard weighted Bergman space and a measure \(d\mu = \varphi dA\), has been extensively studied since the seventies [8, 16, 25]. Luecking [15] was probably one of the first authors who considered \(T_\mu\) with measures as symbols. He described those \(\mu\) for which \(T_\mu\) belongs to the Schatten-Von Neumann ideal \(S_p(H)\), where \(0 < p < \infty\) and \(H\) denotes a classical weighted Dirichlet space. This result has turned out to be useful in subsequent research on concrete operator theory. We refer to [15, 26] for some of these applications. Our contribution on Toeplitz operators acting on the Dirichlet-type space \(H_\alpha(\omega^*)\) is a generalization of the result due to Luecking [15, Theorem p. 347]. Although we follow the original proof, our context leads to severe technical difficulties in the study. In particular, the original proof uses the property that each standard reproducing Bergman kernel is essentially constant in a hyperbolically bounded region. We do not know if the same remains true for the reproducing kernels of \(H_\alpha(\omega^*)\), and therefore we are forced to circumvent certain obstacles in the proof by using different techniques.

We need some notation to state our result. For \(a \in \mathbb{D}\), define \(\varphi_a(z) = (a - z)/(1 - \overline{a}z)\). The pseudohyperbolic distance from \(z\) to \(w\) is defined by \(\varrho(z, w) = |\varphi_z(w)|\), and the pseudohyperbolic disc of center \(a \in \mathbb{D}\) and radius \(r \in (0, 1)\) is denoted by \(\Delta(a, r) = \{z : \varrho(a, z) < r\}\). For a given radial weight \(\omega\), \(\alpha \in \mathbb{R}\) and a positive Borel measure \(\mu\) on \(\mathbb{D}\),
define
\[ \hat{\mu}_{\alpha,r}(z) = \frac{\mu(\Delta(z,r))}{\omega_{\alpha}^*(z)}, \quad z \in \mathbb{D}. \]

The polar rectangle associated with an arc \( I \subset \mathbb{T} \) is
\[ R(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, \ 1 - \frac{|I|}{2\pi} \leq |z| < 1 - \frac{|I|}{4\pi} \right\}, \]
and we will write \( z_I = (1 - |I|/2\pi)\xi \), where \( \xi \in \mathbb{T} \) is the midpoint of \( I \).

Let \( \mathcal{T} \) denote the family of all dyadic arcs of \( \mathbb{T} \). Every arc \( I \in \mathcal{T} \) is of the form
\[ I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi (k + 1)}{2^n} \right\}, \]
where \( k = 0, 1, 2, \ldots, 2^n - 1 \) and \( n = \mathbb{N} \cup \{0\} \). Then the family \( \{R(I) : I \in \mathcal{T}\} \) consists of pairwise disjoint rectangles whose union covers \( \mathbb{D} \). For \( I_j \in \mathcal{T} \setminus \{I_{0,0}\} \), we will write \( z_j = z_{I_j} \). For convenience, we associate the arc \( I_{0,0} \) with the point 1/2.

**Theorem 6.** Let \( 0 < p < \infty \) and \( -\infty < \alpha < 1 \) such that \( \alpha^p < 1 \). Let \( \omega \in \mathcal{U} \mathcal{R} \mathcal{L} \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the following assertions are equivalent:

1. \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \);
2. \( \sum_{s \in \mathcal{T}} \left( \frac{\mu(R_s)}{\omega^*(z_s)} \right)^p < \infty \);
3. \( \hat{\mu}_{\alpha,r} \in L^p \left( \frac{1}{1 - |z|} \right) \) for some \( 0 < r < 1 \).

It is worth noticing that a straightforward calculation based on the pseudohyperbolic distance and the properties of regular weights show that condition (b) in Theorem 6 can be replaced by \( \sum_j \left( \frac{\mu(\Delta(a_j,r))}{\omega^*(a_j)} \right)^p < \infty \), where \( \{a_j\} \) is a \( \delta \)-lattice. Recall that \( A = \{a_k\}_{k=0}^\infty \subset \mathbb{D} \) is uniformly discrete if it is separated in the hyperbolic metric, it is an \( \varepsilon \)-net if \( \mathbb{D} = \bigcup_{k=0}^\infty \Delta(a_k,\varepsilon) \), and finally, it is a \( \delta \)-lattice if it is a \( 5\delta \)-net and uniformly discrete with constant \( \gamma = \delta/5 \).

3. Integrability of reproducing kernels

In this section we will prove Theorem 1 and then deduce Corollary 2. We will need several auxiliary results that are presented first.

3.1. Preliminary results.** We begin with auxiliary results on smooth Hadamard products, and then apply Hardy-Littlewood-inequalities to obtain estimates for \( L^p \)-means of the reproducing kernels.

Throughout this section we will assume, without loss of generality, that \( \int_0^1 \omega(s) \, ds = 1 \). For each \( n \in \mathbb{N} \cup \{0\} \), let \( r_n = r_n(\omega) \in [0,1) \) be defined by
\[ \hat{\omega}(r_n) = \int_{r_n}^1 \omega(s) \, ds = \frac{1}{2^n}. \quad (3.1) \]
Clearly, \( \{r_n\}_{n=0}^\infty \) is an increasing sequence of distinct points on \([0,1)\) such that \( r_0 = 0 \) and \( r_n \to 1^{-} \), as \( n \to \infty \). For \( x \in [0,\infty) \), let \( E(x) \) denote the integer such that \( E(x) \leq x < E(x) + 1 \), and set \( M_n = E \left( \frac{1}{1 - r_n} \right) \). Write
\[ I(0) = I_\omega(0) = \{ k \in \mathbb{N} \cup \{0\} : k < M_1 \} \]
and
\[ I(n) = I_\omega(n) = \{ k \in \mathbb{N} : M_n \leq k < M_{n+1} \} \]
for all \( n \in \mathbb{N} \). If \( f(z) = \sum_{n=0}^\infty a_n z^n \) is analytic in \( \mathbb{D} \), define the polynomials \( \Delta^\omega_n f \) by
\[ \Delta^\omega_n f(z) = \sum_{k \in I_\omega(n)} a_k z^k, \quad n \in \mathbb{N} \cup \{0\}. \]
The next result on partial sums $\Delta_n^\omega f$ together with [21, Theorem 4] is one of the principal ingredients in the proof of Theorem 1.

**Lemma 7.** Let $0 < p \leq 1$ and $\omega \in \mathcal{I} \cup \mathcal{R}$. Then
\[
\|f\|_{A_p^\omega}^p \lesssim \sum_{n=0}^{\infty} 2^{-n} \|\Delta_n^\omega f\|_{H_p}^p
\]
for all $f \in H(\mathbb{D})$.

**Proof.** Since $0 < p \leq 1$, [21, (3.13)] yields
\[
M_p^p(r, f) \leq \sum_{n=0}^{\infty} M_p^p(r, \Delta_n^\omega f) \leq \sum_{n=0}^{\infty} r^{pM_n} \|\Delta_n^\omega f\|_{H_p}^p,
\]
and hence [21, Proposition 9], with $p = 1 = \alpha$, gives
\[
\|f\|_{A_p^\omega}^p \leq \int_0^1 \left( \sum_{n=0}^{\infty} r^{pM_n} \|\Delta_n^\omega f\|_{H_p}^p \right) \omega(r) \, dr \times \sum_{n=0}^{\infty} 2^{-n} \|\Delta_n^\omega f\|_{H_p}^p,
\]
which is the inequality we wanted to prove. The last asymptotic equality follows also from [31,1 and 20, Lemmas 1.1 and 1.3].

We will need background on certain smooth polynomials defined in terms of Hadamard products. If $W(z) = \sum_{k \in J} b_k z^k$ is a polynomial and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$, then the Hadamard product
\[
(W * f)(z) = \sum_{k \in J} b_k a_k z^k
\]
is well defined.

If $\Phi : \mathbb{R} \to \mathbb{C}$ is a $C^\infty$-function such that its support, $\text{supp}(\Phi)$, is a compact subset of $(0, \infty)$, we set
\[
A_{\Phi, m} = \max_{s \in \mathbb{R}} |\Phi(s)| + \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)|,
\]
and consider the polynomials
\[
W_N^\Phi(z) = \sum_{k \in \mathbb{N}} \Phi \left( \frac{k}{N} \right) z^k, \quad N \in \mathbb{N}.
\]

With this notation we can state the next result that follows by [18, p. 111–113].

**Theorem B.** Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a $C^\infty$-function such that $\text{supp}(\Phi) \subset (0, \infty)$ is compact. Then for each $p \in (0, \infty)$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that
\[
\|W_N^\Phi * f\|_{H_p} \leq CA_{\Phi, m} \|f\|_{H_p}
\]
for all $f \in H_p$ and $N \in \mathbb{N}$.

For $g(z) = \sum_{k=0}^{\infty} b_k z^k \in H(\mathbb{D})$ and $n_1, n_2 \in \mathbb{N} \cup \{0\}$, we set
\[
S_{n_1, n_2} g(z) = \sum_{k=n_1}^{n_2-1} b_k z^k, \quad n_1 < n_2,
\]
and for each radial weight $\omega$, we write
\[
\omega_x = \int_0^1 r^{2x+1} \omega(r) \, dr, \quad x > -1.
\]

The next result is known and can be proved by summing by parts and using the M. Riesz projection theorem, see [21, Lemma E].

**Lemma C.** Let $1 < p < \infty$ and $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a monotone sequence of positive numbers. Let $(\lambda g)(z) = \sum_{k=0}^{\infty} \lambda_k b_k z^k$, where $g(z) = \sum_{k=0}^{\infty} b_k z^k$. 
Further, since \( C_{\omega} \) and \( \lambda_0 \) are nondecreasing, there exists a constant \( C > 0 \) such that
\[
C^{-1}\lambda_n \| S_{n_1, n_2} g \|_{H^p} \leq \| S_{n_1, n_2} \lambda g \|_{H^p} \leq C\lambda_n \| S_{n_1, n_2} g \|_{H^p}.
\]

We will also need an extension of this result for \( 0 < p \leq 1 \) in the case when \( \lambda_k \) is either \( \omega_k \) or \( \omega_k^{-1} \).

**Lemma 8.** Let \( 0 < p < \infty \), \( \omega \) be a radial and continuous weight and \( n_1, n_2 \in \mathbb{N} \) with \( n_1 < n_2 \). Let \( g(z) = \sum_{k=0}^{\infty} c_k z^k \) be analytic in \( \mathbb{D} \), and assume that both, \( h(z) = \sum_{k=0}^{\infty} c_k \omega_k z^k \) and \( H(z) = \sum_{k=0}^{\infty} \frac{c_k}{\omega_k} z^k \), are analytic in \( \mathbb{D} \) as well. Then the following assertions hold:

(i) There exists a constant \( C = C(p) > 0 \) such that
\[
\| S_{n_1, n_2} h \|_{H^p} \leq C \omega_{n_1 - 1} \| g \|_{H^p}.
\]

(ii) If \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( n_1 < n_2 \leq K n_1 \) for some \( K > 0 \), then there exists a constant \( C = C(p, \omega, K) > 0 \) such that
\[
\| S_{n_1, n_2} H \|_{H^p} \leq C \left( \omega_{n_1 - 1} \right)^{-1} \| g \|_{H^p}.
\]

**Proof.** (i). Define
\[
\Upsilon_{n_1}(s) = \int_0^1 r^{2n_1 s + 1} \omega(r) \, dr, \quad s \geq 0.
\]
Clearly, \( \Upsilon_{n_1} \) is a \( C^\infty \) function and
\[
|\Upsilon_{n_1}(s)| \leq \int_0^1 r^{n_1} \omega(r) \, dr, \quad s \geq 1 - \frac{1}{2n_1}.
\]
Further, since \( C(m) = \sup_{0 < r < 1} \left( \log \frac{1}{r} \right)^m r^{x/2} < \infty \), we have
\[
\left| \Upsilon_{n_1}^{(m)}(s) \right| \leq \int_0^1 \left[ \left( \log \frac{1}{r^{2n_1}} \right)^m r^{n_1} \right] r^{2n_1 s + 1 - n_1} \omega(r) \, dr
\]
\[
\leq C(m) \int_0^1 r^{n_1} \omega(r) \, dr, \quad s \geq 1 - \frac{1}{2n_1}.
\]
Therefore, by using (3.2) and (3.3), we can find a function \( \Phi_{n_1} \in C^\infty \) such that \( \text{supp}(\Phi_{n_1}) \in \left( 1 - \frac{1}{2n_1}, \frac{n_2}{n_1} \right) \),
\[
\Phi_{n_1}(s) = \Upsilon_{n_1}(s), \quad s \in \left[ 1, \frac{n_2 - 1}{n_1} \right],
\]
and
\[
A_{\Phi_{n_1}, m} = \max_{s \in \mathbb{R}} |\Phi_{n_1}(s)| + \max_{s \in \mathbb{R}} |\Phi_{n_1}^{(m)}(s)| \leq C(m) \omega_{n_1 - 1}.
\]
Therefore we can write
\[
S_{n_1, n_2} h(z) = \sum_{k=n_1}^{n_2-1} c_k \omega_k z^k
\]
\[
= \sum_{k=n_1}^{n_2-1} c_k \Phi_{n_1} \left( k \right) \left( \frac{k}{n_1} \right) z^k = \left( W_{n_1}^{\Phi_{n_1}} \ast g \right)(z), \quad z \in \mathbb{T}.
\]
Hence, by fixing \( m \) sufficiently large so that \( mp > 1 \), and using Theorem [1], we obtain
\[
\| S_{n_1, n_2} h \|_{H^p} = \| W_{n_1}^{\Phi_{n_1}} \ast g \|_{H^p} \leq C_2 A_{\Phi_{n_1}, m} \| g \|_{H^p} \leq C(m) C_2 \omega_{n_1 - 1} \| g \|_{H^p},
\]
where \( C_2 = C_2(p) > 0 \) is a constant. Thus (i) is proved.
(ii). We set \( \varphi_{n_1}(s) = (\Upsilon_{n_1}(s))^{-1} \) and will prove that
\[
A_{\varphi_{n_1}, m} = \max_{1 \leq n \leq r} |\varphi_{n_1}(s)| + \max_{1 \leq n \leq s} |\varphi_{n_1}(s)|^{(m)} \\
\leq C(m, \omega, K) \left( \frac{\omega_{n_1}}{s} \right)^{-1}, \quad m \in \mathbb{N} \cup \{0\}.
\] (3.4)

Since
\[
\varphi_{n_1}(s) \leq \frac{1}{\int_{0}^{s} r^{2n_2+1} \omega(r) \, dr}, \quad 0 \leq s \leq \frac{n_2}{n_1}
\]
for all \( 0 \leq s \leq \frac{n_2}{n_1} \), where \( \beta = \beta(\omega) = (0, \infty) \). This gives (3.4) for \( m = 0 \).

If \( m = 1 \), we may use (3.3) and (3.5) to obtain
\[
|\varphi_{n_1}(s)| = \frac{\left| \Upsilon'_{n_1}(s) \right|}{\left| \Upsilon_{n_1}(s) \right|^2} = |\Upsilon'_{n_1}(s)||\varphi_{n_1}(s)| \lesssim \left( \frac{\omega_{n_1}}{s} \right)^{-1}
\]
for all \( 1 - \frac{1}{2n_1} \leq s \leq \frac{n_2}{n_1} \). The general case is now proved by induction. Assume that (3.4) holds for \( j = 1, \ldots, m - 1 \), where \( m > 1 \). Since \( 1 = \varphi_{n_1}(s) \Upsilon_{n_1}(s) \), we have
\[
0 = (\varphi_{n_1} \Upsilon_{n_1})^{(m)}(s) = \sum_{j=0}^{m} \binom{m}{j} \Upsilon_{n_1}^{(m-j)}(s) \varphi_{n_1}^{(j)}(s),
\]
which implies
\[
|\varphi_{n_1}^{(m)}(s)| \leq \frac{\sum_{j=0}^{m-1} \left( \frac{m}{j} \right) \left| \Upsilon_{n_1}^{(m-j)}(s) \varphi_{n_1}^{(j)}(s) \right|}{|\Upsilon_{n_1}(s)|}.
\]
This together with the induction hypothesis and (3.3) gives (3.4). The proof can be completed arguing as in (i). We omit the details. 

We now turn to \( L^p \)-estimates. For that purpose we will use the fact that if \( \{ \epsilon_n \} \) is an orthonormal basis of a Hilbert space \( H \), that is continuously embedded into \( \mathcal{H}(\mathbb{D}) \), then its reproducing kernel is given by
\[
K_z(\zeta) = \sum_n \epsilon_n(\zeta) \overline{\epsilon_n(z)}
\] (3.6)
for all \( z \) and \( \zeta \) in \( \mathbb{D} \), see e.g. [20] Theorem 4.19. Recall that \( \omega_\beta(z) = (1 - |z|)^\beta \omega(z) \) for all \( \beta \in \mathbb{R} \) and \( z \in \mathbb{D} \).

**Lemma 9.** Let \(-\infty < \alpha < 2\), \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( n \in \mathbb{N} \cup \{0\} \).

(i) If \( 0 < p \leq 2 \), then
\[
M_p^r(r, (K_\alpha \omega)^{(N)}(s)) \gtrsim \int_{0}^{r} \frac{dt}{\omega(t)^p (1 - t)^{p(N+1-\alpha)}}, \quad r, |a| \to 1^-.
\]

(ii) If \( 2 \leq p < \infty \), then
\[
M_p^r(r, (K_\alpha \omega)^{(N)}(s)) \lesssim \int_{0}^{r} \frac{dt}{\omega(t)^p (1 - t)^{p(N+1-\alpha)}}, \quad r, |a| \to 1^-.
\]
Proof. By using the standard orthonormal basis \( \{ (\omega^*-\alpha(D))^{-\frac{1}{2}} \} \cup \{ z^{j+1}/((j+1)^2(\omega^*-\alpha)_j) \} \), \( j \in \mathbb{N} \cup \{0\} \), of \( H_\alpha(\omega^*) \) and (3.4) we obtain

\[
K^\alpha,\omega_a(z) = (\omega^*-\alpha(D))^{-\frac{1}{2}} + \sum_{j=0}^{\infty} \frac{z^{j+1}}{2(j+1)^2(\omega^*-\alpha)_j},
\]

which implies

\[
(K^\alpha,\omega)^{(N)}(z) = \sum_{j=N-1}^{\infty} \frac{j(j-1) \cdots (j-N+2)z^{j-N+1}\omega^{j+1}}{(j+1) (\omega^*-\alpha)_j}, \quad n \in \mathbb{N}.
\]

Therefore the classical Hardy-Littlewood inequalities [12, Theorem 6.2] applied to the dilated function show that it suffices to prove

\[
\sum_{n=N}^{\infty} \frac{r^{pn}}{(n+1)^{p(1-N)+2}(\omega^*-\alpha)_n^p} \asymp \int_0^r \frac{dt}{\hat{\omega}_N(t)(1-t)^p}, \quad r \geq 1/4,
\]

(3.8)

Assume, without loss of generality, that \( r > 1 - \frac{1}{N+1} \). Choose now \( N^* \in \mathbb{N} \) such that \( 1 - \frac{1}{N^*+1} \leq r < 1 - \frac{1}{N^*+2} \). Then [20, Lemmas 1.3 and 1.7] yield

\[
\sum_{n=N}^{N^*} \frac{r^{pn}}{(n+1)^{p(1-N)+2}(\omega^*-\alpha)_n^p} \asymp \sum_{n=N}^{N^*} \frac{1}{(n+1)^{p(1-N)+2}(\omega^*-\alpha)_n^p} \asymp \sum_{n=N}^{N^*+1} \frac{ds}{s^{p(\alpha-1-N)+2\hat{\omega}(1-\frac{1}{2n+1})^p}} \asymp \int_{1-r}^{1} \frac{ds}{s^{p(\alpha-1-N)+2\hat{\omega}(1-\frac{1}{2s})^p}} = \int_{\frac{1}{1-r}}^{r} \frac{dt}{\hat{\omega}(t)^p(1-t)^{p(N+1-\alpha)}} \asymp \int_0^{r} \frac{dt}{\hat{\omega}(t)^p(1-t)^{p(N+1-\alpha)}}.
\]

Bearing in mind [20, Lemma 1.1], the same upper bound can be proved in a similar manner, so

\[
\sum_{n=N}^{N^*} \frac{r^{pn}}{(n+1)^{p(1-N)+2}(\omega^*-\alpha)_n^p} \asymp \int_0^{r} \frac{dt}{\hat{\omega}_N(t)(1-t)^p}, \quad r \geq 1/4.
\]

(3.9)
and further, by \cite[p. 10 (ii)]{20}, there exists $M > p + 1$ such that \( \hat{\omega}(r)^p \) is essentially increasing. Hence

\[
\sum_{n=N^*}^{\infty} (n+1)^p (1-N^{-2}) \omega_n^p n^p \leq \sum_{n=N^*}^{\infty} (n+1)^p (1-N^{-2}) \omega_n^p n^p \\
\leq \frac{1}{(N^* + 1)^p (1-N^{-2})} \sum_{n=N^*}^{\infty} (n+1)^p (1-N^{-2}) \omega_n^p n^p \\
\leq \frac{(N^* + 1)^p (1-N^{-2})}{\omega(1-N^{-2})^p} \\
\leq \frac{1}{(1-r)^p (1-N^{-2}) \omega(r)^p} \\
\leq \frac{1}{\int_0^r \omega(t)^p (1-t)^p (1-N^{-2}) dt} \\
\leq \frac{1}{\int_0^r \omega(t)^p (1-t)^p (1-N^{-2}) dt},
\]

and the proof is complete. \( \square \)

### 3.2. Proof of Theorem 1

We begin with proving (ii) for $v \in \mathcal{R}$. In this case we have two advantages compared to the case $v \in \mathcal{I}$. First, the main result in \cite{19} implies the Littlewood-Paley formula

\[
\|f\|_{A_p^b}^p \leq \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^n v(z) dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{3.10}
\]

for all $0 < p < \infty$, $v \in \mathcal{R}$ and $n \in \mathbb{N}$. This allows us to assume that the order $N$ of the derivative is sufficiently large, and in that way we avoid some difficulties in the proof. Note that (3.10) fails in general for $v \in \mathcal{I}$ by \cite[Proposition 4.3]{20}. Second, when $v \in \mathcal{R}$ we have precise control over the size of the blocks $\Delta_n^b$ appearing in the decomposition of the $A_p^b$-norm of $f$.

#### Part (ii). Case $v \in \mathcal{R}$.

By the Littlewood-Paley formula (3.10) we may assume that $N > \max \left\{ \frac{1}{p} + 1, \frac{1}{p} + \frac{\alpha - 1}{p} + \frac{1}{C_1} \right\}$, where $C_1 = \inf_{r \in (0,1)} \frac{\omega(r)}{1-r} > 0$. We may also assume, without loss of generality, that $\int_1^r v(r) dr = 1$. Then Lemma \cite{7} (the case $p \leq 1$) or \cite[Theorem 4]{21} (the case $p > 1$), (3.7) and Lemma \cite{6} give

\[
\int_{\mathbb{D}} |(K_{[a]}^{(\alpha, \omega)}(N))(z)|^p v(z) dA(z) = \int_{\mathbb{D}} |(K_{[a]}^{(\alpha, \omega)}(N))(z)|^p v(z) dA(z) \\
\leq \sum_{n=0}^{\infty} 2^{-n} \left\| \Delta_n^b (K_{[a]}^{(\alpha, \omega)}(N))(z) \right\|_{H^p}^p \\
= \sum_{n=0}^{\infty} 2^{-n} \left\| \sum_{j \in \mathcal{L}_n \cap \mathcal{J}_j} \frac{j(j-1) \cdots (j-N+2) z^{j-N+1} |a|^{j+1}}{2(j+1) (\omega_{-\alpha})_j} \right\|_{H^p}^p \\
\leq \sum_{n=0}^{\infty} 2^{-n} |a|^{M_n} \left\| \sum_{j \in \mathcal{L}_n \cap \mathcal{J}_j} \frac{j(j-1) \cdots (j-N+2) z^{j-N+1}}{(j+1) (\omega_{-\alpha})_j} \right\|_{H^p}^p. \tag{3.11}
\]

Now, since $v$ is regular, \cite[Lemma 6]{21} implies $\sup_{n \geq 0} \frac{M_n+1}{M_n} < \infty$, where $M_n = E \left( \frac{1}{1-r_n} \right)$ are associated to $v$ via $\hat{\omega}(r_n) = 2^{-n}$. So, using that $\omega_{-\alpha} \in \mathcal{R}$ by \cite[Lemma 1.7]{20},
Lemma 8(ii), [21, Lemma 10] and the assumption $N > \frac{1}{p} + 1$, we get

$$\left\| \sum_{j \in I_0(n), j \geq N-1} \frac{j(j-1) \cdots (j-N+2)z^{j-N+1}}{(j+1)(\omega_{*-})_j} \right\|_{H^p} \lesssim \frac{1}{(\omega_*)_{\frac{p}{2}-1}} \frac{M_n^p}{\omega_*)_{\frac{p}{2}-1}} \left( 1 - \frac{1}{M_{n+1}} \right) \left( \frac{1}{M_{n+1}} \right) \left( 1 - \frac{1}{(1-z)^{N}} \right)$$

which together with \textbf{(3.11)} gives

$$\left\| \sum_{j \in I_0(n), j \geq N-1} \frac{j(j-1) \cdots (j-N+2)z^{j-N+1}}{(j+1)(\omega_{*-})_j} \right\|_{H^p} \lesssim \frac{1}{(\omega_*)_{\frac{p}{2}-1}} \frac{M_n^p}{\omega_*)_{\frac{p}{2}-1}} \left( 1 - \frac{1}{M_{n+1}} \right) \left( \frac{1}{(1-z)^{N}} \right)$$

which together with \textbf{(3.11)} gives

$$\int_B \left| (K_n^\alpha)(z) \right|^p v(z) dA(z) \lesssim \sum_{n=0}^{\infty} \frac{2^{-n} M_n^{(N-1)p-1}}{(\omega_*)_{\frac{p}{2}-1}^n} |a|^M_n.$$  \textbf{(3.12)}

The last step in this part of the proof consists of bounding the series in \textbf{(3.12)}.

Since $v$ is a regular weight, the definition $\hat{v}(r_n) = 2^{-n}$ and [20, Lemma 1.6] imply $v^*(r_n) \asymp 2^{-n} M_n^{-1}$. Moreover, since $\omega_* \in R$ by [20, Lemma 1.7], [20, Lemmas 1.1 and 1.3] yield

$$(\omega_*)_{\frac{p}{2}-1} \sim \frac{\omega_*}{M_n} \sim \frac{\omega_*}{M_n} \left( 1 - \frac{1}{M_n} \right) \sim \omega_* \left( 1 - \frac{1}{M_n} \right) M_n^{\alpha-1}.$$  \textbf{(3.13)}

Therefore, by using [21, Lemma 6] and [20, Lemma 1.1], we deduce

$$\sum_{n=0}^{\infty} \frac{2^{-n} M_n^{(N-1)p-1}}{(\omega_*)_{\frac{p}{2}-1}^n} |a|^M_n \sim 1 + \sum_{n=1}^{\infty} \frac{v^*(r_n) M_n^{(N-1)p}}{(\omega_* \left( 1 - \frac{1}{M_n} \right) M_n^{\alpha-1})^p} |a|^M_n$$

$$\lesssim 1 + \sum_{n=1}^{\infty} \frac{v^* \left( 1 - \frac{1}{M_n} \right) M_n^{(N-1)p-1}}{(\omega_* \left( 1 - \frac{1}{M_n} \right) M_n^{\alpha-1})^p} (M_n - M_{n-1}) |a|^M_n$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{v^* \left( 1 - \frac{1}{M_n} \right) M_n^{(N-1)p-1}}{(\omega_* \left( 1 - \frac{1}{M_n} \right) M_n^{\alpha-1})^p} \sum_{j \in I_0(n-1)} |a|^j$$

$$\lesssim 1 + \sum_{n=1}^{\infty} \frac{v^* \left( 1 - \frac{1}{j+1} \right) (j+1)^{(N-1)p-1}}{(\omega_* \left( 1 - \frac{1}{j+1} \right) (j+1)^{\alpha-1})^p} |a|^j$$

Let now $|a| \geq \frac{1}{2}$. We observe that [20, Lemmas 1.1 and 1.6] imply

$$\frac{v^* (a)}{\omega (a)^p (1 - |a|)^{(N-\alpha)p}} \propto \int_{|a|-1}^{[a]} \frac{\hat{w}(s)}{\omega(s)^p (1-s)^{(N-\alpha+1)p}} ds$$

$$\lesssim \int_0^{[a]} \frac{\hat{w}(s)}{\omega(s)^p (1-s)^{(N-\alpha+1)p}} ds.$$  \textbf{(3.15)}

Next, take $N* \in \mathbb{N}$ such that $1 - \frac{1}{N*} \leq |a| < 1 - \frac{1}{N*+1}$. Then, by \textbf{(3.15)},
\[ \sum_{j=1}^{N^{\star}} \frac{v^\ast \left( 1 - \frac{1}{j+1} \right) (j+1)^{(N-\alpha)p-1}}{\omega^\ast \left( 1 - \frac{1}{j+1} \right)^p} |a|^j \lesssim \int_2^{\frac{1+|a|}{2}} \frac{v^\ast \left( 1 - \frac{1}{s} \right)}{\omega^\ast \left( 1 - \frac{1}{s} \right)^p} x^{(N-\alpha)p-1} \, dx \]

\[ \lesssim \int_0^{\frac{1+|a|}{2}} \frac{v^\ast(s)}{\omega^\ast(s)^p (1-s)^{(N-\alpha)p+1}} \, ds \]

\[ \times \int_0^{\frac{1}{2}} \frac{\hat{\nu}(s)}{\hat{\omega}(s)^p (1-s)^{(N-\alpha+1)p}} \, ds \]

(3.16)

for all |a| \geq \frac{1}{2}. On the other hand, the function \( h(r) = \hat{\omega}(r)(1-r)^{-\frac{1}{1-p}} \) is increasing on [0, 1] by [20] (ii) p. 10, and therefore [20, Lemma 1.6] shows that

\[ \left\{ (j+1)^{1+\frac{1}{N^\star}} \omega^\ast \left( 1 - \frac{1}{j+1} \right) \right\}_{j=1}^{\infty} \]

is an essentially increasing sequence. This and (3.15) together with the fact that

\[ \left\{ (j+1)v^\ast \left( 1 - \frac{1}{j+1} \right) \right\}_{j=1}^{\infty} \]

is essentially decreasing, give

\[ \sum_{j=N^\star+1}^{\infty} \frac{v^\ast \left( 1 - \frac{1}{j+1} \right) (j+1)^{(N-\alpha)p-1}}{\omega^\ast \left( 1 - \frac{1}{j+1} \right)^p} |a|^j \]

\[ \lesssim v^\ast \left( 1 - \frac{1}{N^\star+2} \right) (N^\star+2)^{\frac{1}{2}} \sum_{j=N^\star+1}^{\infty} \frac{(j+1)^{(N-\alpha)p-2}}{\omega^\ast \left( 1 - \frac{1}{j+1} \right)^p} |a|^j \]

\[ \lesssim \frac{v^\ast \left( 1 - \frac{1}{N^\star+2} \right) (N^\star+2)^{\frac{1}{2}}}{\omega^\ast \left( 1 - \frac{1}{N^\star+2} \right)^p} \sum_{j=N^\star+1}^{\infty} (j+1)^{(N-\alpha+1+\frac{1}{N^\star+1})p-2} |a|^j \]

\[ \times \int_0^{\frac{1}{2}} \frac{\hat{\nu}(s)}{\hat{\omega}(s)^p (1-s)^{(N-\alpha+1)p}} \, ds, \]

where in the last asymptotic equality we used our choice \( N > \frac{1}{p} + \alpha - 1 + \frac{1}{1-p} \).

This combined with (3.12), (3.14) and (3.16) finishes the proof of the upper bound in (2.4), when \( v \in \mathcal{R} \).

In order to establish the same lower estimate, we will consider the cases \( p > 1 \) and \( 0 < p \leq 1 \) separately. Let first \( p > 1 \). By [21, Theorem 4], (3.7), Lemma [C] and [21]...
Lemma 10] we deduce

\[
\int_{\mathbb{D}} \left| (K_{a}^{\alpha,\omega})(N)(z) \right|^p v(z) \, dA(z) = \int_{\mathbb{D}} \left| (K_{[a]}^{\alpha,\omega})(N)(z) \right|^p v(z) \, dA(z)
\]

\[
\asymp \sum_{n=0}^{\infty} 2^{-n} \left\| \Delta_{n}^{\alpha}(K_{[a]}^{\alpha,\omega})(N)(z) \right\|^p_{H^p} = \sum_{n=0}^{\infty} 2^{-n} \left\| \sum_{j \in I_{n}(n), j \geq N-1} \frac{j(j-1) \cdots (j-N+2) z^{j-N+1} |a|^{j+1}}{2(j+1)(\omega^*_\alpha)^j} \right\|^p_{H^p}
\]

\[
\asymp \sum_{n=0}^{\infty} 2^{-n} [a]\left|a\right|^{M_{n+1}} \left\| \sum_{j \in I_{n}(n), j \geq N-1} \frac{j(j-1) \cdots (j-N+2) z^{j-N+1}}{(j+1)(\omega^*_\alpha)^j} \right\|^p_{H^p}
\]

\[
\asymp \sum_{n=0}^{\infty} 2^{-n} [a]\left|a\right|^{M_{n+1}} \sum_{j \in I_{n}(n), j \geq N-1} \frac{j(j-1) \cdots (j-N+2) z^{j-N+1}}{(j+1)(\omega^*_\alpha)^j}
\]

where in the last step we have used the fact that $M_n \asymp M_{n+1}$ and $\omega^*_\alpha \in \mathbb{R}$. Next, by using (3.13), [21, Lemma 6], [20, Lemma 1.1] and arguing in a similar manner to that in (3.14), we deduce

\[
\sum_{M_n \geq N-1} 2^{-n} [a]\left|a\right|^{M_n} \asymp \sum_{j \geq N} v^* \left(1 - \frac{j+1}{j+1}ight) \left(\frac{1}{\alpha} + 1\right) \left(\frac{1}{\omega^*_\alpha}ight)^p \left|a\right|^{j+1},
\]

Without loss of generality, we may assume that $|a| > 1 - \frac{1}{N^2}$ if $N > 2$, for otherwise let $|a| \geq \frac{1}{2}$. Take $N^* \in \mathbb{N}$ such that $1 - \frac{1}{N^*} \leq |a| < 1 - \frac{1}{N^*+1}$. Then (3.15) implies

\[
\sum_{j=0}^{N^*+1} v^* \left(1 - \frac{j+1}{j+1}\right) \left(1 - \frac{1}{\omega^*_\alpha}\right)^p \left|a\right|^{j+1} \geq \int_{0}^{\left|a\right|} \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1-s)(\omega^*_\alpha)^p} \, ds,
\]

and the desired lower bound follows.

Finally, let $0 < p \leq 1$. Then, by Lemma [9] Fubini’s theorem, [20] Lemma 1.1 and (3.16), we get

\[
\int_{\mathbb{D}} \left| (K_{a}^{\alpha,\omega})(N)(z) \right|^p v(z) \, dA(z) \geq \int_{0}^{\left|a\right|} \left( \int_{0}^{s|a|} \frac{dt}{\hat{\omega}(t)^p(1-t)^p} \right) \, v(s) \, ds
\]

\[
= \int_{0}^{\left|a\right|} \frac{\hat{v}(t)}{\hat{\omega}(t)^p(1-t)^p} \, dt
\]

\[
\asymp \int_{0}^{2\left|a\right|} \frac{\hat{v}(t)}{\hat{\omega}(t)^p(1-t)^p} \left(1 - \frac{1}{\alpha}\right)^p \, dt \quad (3.17)
\]

\[
\asymp \int_{0}^{2\left|a\right|} \frac{\hat{v}(t)}{\hat{\omega}(t)^p(1-t)^p} \, dt
\]

\[
\asymp \int_{0}^{\left|a\right|} \frac{\hat{v}(t)}{\hat{\omega}(t)^p(1-t)^p} \, dt, \quad |a| \geq \frac{1}{2}.
\]
Theorem 1(ii) for $v \in \mathcal{R}$ is now proved.

Before proving (ii) for $v \in \mathcal{I}$, we will prove (i). To do this we will use the well known inclusions

$$\mathcal{D}^p_{p-1} \subseteq H^p, \quad 0 < p < 2,$$

and

$$H^p \subseteq \mathcal{D}^p_{p-1}, \quad 2 < p < \infty,$$

where $\mathcal{D}^p_{p-1}$ denotes the space of $f \in \mathcal{H}(\mathbb{D})$ such that $\int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-1} \, dA(z) < \infty$.

**Part (i). Case $0 < p \leq 2$.** Let $r \in [\frac{1}{2}, 1)$. Then, by (3.18), and Theorem 1(ii) with the regular weight $v(z) = (1 - |z|)^{p-1}$,

$$M^p_r \left( (K^\alpha, \omega)_n^{(N)}(z), r \right) = \left\| (K^\alpha, \omega)_n^{(N)}(z) \right\|_{H^p}^p \lesssim 1 + \int_{\mathbb{D}} \left| \frac{\partial^{N+1} (K^\alpha, \omega)_n(z, ra)}{\partial^{N+1} z} \right|(1 - |z|)^{p-1} \, dA(z) \asymp 1 + \int_0^{r^{|a|}} ds \frac{\omega_N - \alpha(s)^p (1 - s)^p}{\omega_N - \alpha(s)^p (1 - s)^p}.$$

The reverse implication follows by Lemma 3(i).

**Part (i). Case $2 < p < \infty$.** It can be proved similarly, bearing in mind (3.19), Theorem 1(ii) and Lemma 3(ii).

The proof of Theorem 1(i) is now complete.

**Part (ii). Case $v \in \mathcal{I}$.** If $v$ is any radial weight, then Theorem 1(i) and Fubini’s theorem yield

$$\int_\mathbb{D} \left( (K^\alpha, \omega)_n^{(N)}(z) \right)^p v(z) \, dA(z) \asymp \int_0^1 \left( \int_0^{s^{|a|}} \frac{dt}{\omega_N - \alpha(t)^p (1 - t)^p} \right) v(s) \, ds \leq \int_0^{|a|} \frac{\hat{v}(t)}{\omega_N - \alpha(t)^p (1 - t)^p} \, dt, \quad |a| \geq \frac{1}{2}.$$

The reverse inequality for $v \in \mathcal{I}$ can be proved by arguing as in (3.17). This finishes the proof of Theorem 1(ii).

To complete the proof of Theorem 1, it remains to show that $K^0, \omega_a$ can be replaced by $B^\omega_a$. To see this, recall that $B^\omega_a(z) = \sum_{n=0}^{\infty} \frac{(z \omega_n)}{2 \omega_n}$ and $4n^2 \omega_n = \omega_n$ for all $n \in \mathbb{N}$ by [20, Theorem 4.2], applied to $p = 2$ and $f(z) = z^n$. These identities give

$$K^0, \omega_a(z, a) = \frac{1}{\omega^*(\mathbb{D})} + \sum_{n=1}^{\infty} \frac{z^n \bar{\omega}_n}{2n^2 \omega_n^{n-1}} = \frac{1}{\omega^*(\mathbb{D})} + 2 \sum_{n=1}^{\infty} \frac{z^n \bar{\omega}_n}{\omega_n} = \frac{1}{\omega^*(\mathbb{D})} - \frac{4}{\omega(\mathbb{D})} + \frac{4}{\omega(\mathbb{D})} + 4B^\omega_a(z),$$

and we conclude that $K^0, \omega_a$ can be replaced by $B^\omega_a$ in the statements of Theorem 1.

**Proof of Corollary 2.** The equivalence between the asymptotic equality (i) and (2.5) follows by Theorem 1(i) and (3.9). Moreover, (2.6) is equivalent to (2.7) by Theorem 1(ii) and (3.15).
4. Projections

We begin with proving Theorems 8 and 11. The proofs of (i) and (ii) of Theorem 8 are natural applications of Theorem 1 below, meanwhile in the proof of Theorem 8 (iii) we interpret the boundedness as a Hardy-Littlewood inequality and apply a result of Muckenhoupt to deduce that the projection cannot be bounded. As pointed out in the introduction, we will deduce Theorem 11 from a result of Shields and Williams.

The proof of Theorem 8 is more involved than that of Theorem 8. What we actually do is to prove Theorem 12 below, which contains several characterizations of those pairs of weights \((\omega, v)\) for which the projection is bounded.

**Proof of Theorem 8 (i).** Let \(1 < p < \infty\) and \(\omega \in \mathcal{R}\). Let \(h = \tilde{\omega}^{-\frac{1}{p'}}\), where \(\frac{1}{p} + \frac{1}{p'} = 1\). Since \(p > 1\), [20, Lemma 1.4(iii)] shows that \(h^{p'}\omega\) is a weight with \(\psi_{h^{p'}\omega} = \frac{p}{p-1} \psi_\omega\), and thus \(h^{p'}\omega \in \mathcal{R}\). Further, by [20, (ii) p. 10], there exist \(\alpha = \alpha(\omega) > 0\) and \(\beta = \beta(\omega) \geq \alpha\) such that \(\check{\omega}(s)(1-s)^{-\alpha}\) is essentially decreasing and \(\check{\omega}(s)(1-s)^{-\beta}\) is essentially increasing on \([0, 1]\). By using this and \(h^{p'}\omega \in \mathcal{R}\) we deduce

\[
\int_0^r \frac{h^{p'}\omega(s)}{\check{\omega}(s)(1-s)} ds \asymp \int_0^r \frac{ds}{\omega(s)^\frac{1}{p}(1-s)} \asymp \frac{1}{\check{\omega}(r)^\frac{1}{p}} = h^{p'}(r), \quad r \geq \frac{1}{2} \tag{4.1}
\]

By symmetry, a similar reasoning applies when \(p'\) is replaced by \(p\), and hence we may use Theorem 1(ii) and (1.1) to deduce

\[
\int_B |B^\omega(z, \zeta)|h^{p'}(\zeta)\omega(\zeta) dA(\zeta) \asymp h^{p'}(z), \quad z \in \mathbb{D},
\]

and

\[
\int_B |B^\omega(z, \zeta)|h^{p}(\zeta)\omega(\zeta) dA(\zeta) \asymp h^{p}(\zeta), \quad \zeta \in \mathbb{D}.
\]

It follows that \(P^+(\varphi) : L^p_\omega \to L^p_\omega\) is bounded by Schur’s test [20, Theorem 3.6].

(ii). Let \(\varphi \in L^\infty(\mathbb{D})\). Then

\[
(P\varphi)'(z) = \int_D \varphi(\zeta) \frac{\partial B^\omega(z, \zeta)}{\partial z} \omega(\zeta) dA(\zeta),
\]

and hence Theorem 1(ii) gives

\[
|P\varphi)'(z)| \leq ||\varphi||_{L^\infty(\mathbb{D})} \int_B \left| \frac{\partial B^\omega(z, \zeta)}{\partial z} \right| \omega(\zeta) dA(\zeta) \asymp ||\varphi||_{L^\infty(\mathbb{D})}. \tag{1.1}
\]

Since \(P\varphi(0) \leq C||\varphi||_{L^\infty(\mathbb{D})}\) for some constant \(C = C(\omega) > 0\), it follows that \(P\varphi(\varphi) \in \mathcal{B}\) and \(\|P\varphi\|_{\mathcal{B}} \leq C||\varphi||_{L^\infty(\mathbb{D})}\).

(iii). Let first \(p > 1\). We assume that \(P^+\omega : L^p_\omega \to L^p_\omega\) is bounded and aim for a contradiction. Write \(K(r) = \int_0^r \frac{dt}{\omega(t)(1-t)}\) for short, and let \(\varphi\) be a radial function. Then Theorem 1(i) together with [20, Lemma 1.1] show that

\[
P^+\omega(\varphi)(z) \asymp \int_0^1 K(|z|s)\varphi(s)\omega(s) ds \geq K(|z|^2) \int_0^1 \varphi(s)\omega(s) ds
\]

\[
\asymp K(|z|) \int_{|z|^2}^1 \varphi(s)\omega(s) ds, \quad |z| \geq \frac{1}{2}.
\]

Therefore

\[
\|P^+\omega(\varphi)\|^p_{L^p_\omega} \gtrsim \int_0^1 \left( K(r) \int_r^1 \varphi(s)\omega(s) ds \right)^p \omega(r) dr,
\]

and since we assumed that \(P^+\omega : L^p_\omega \to L^p_\omega\) is bounded, we deduce

\[
\int_0^1 \left( K(r) \int_r^1 \varphi(s)\omega(s) ds \right)^p \omega(r) dr \lesssim ||\varphi||^p_{L^p_\omega}, \quad \varphi \in L^p_\omega.
\]
This can be rewritten as
\[ \int_0^1 \left( U(r) \int_r^1 \psi(s) ds \right)^p dr \lesssim \int_0^1 \psi^p(r) V^p(r), \quad \psi \in L^p_{\nu_p}, \tag{4.2} \]
where
\[ U(r) = \begin{cases} K(r) \omega(r)^{1/p}, & 0 \leq r < 1 \\ 0, & r \geq 1 \end{cases}, \]
and
\[ V(r) = \begin{cases} \omega(r)^{-1/p}, & 0 \leq r < 1 \\ 0, & r \geq 1 \end{cases}. \]
But (4.2) is equivalent to
\[ \sup_{0 < r < 1} \left( \int_0^r K^p(s) \omega(s) ds \right) \wtilde{\omega}(r)^p < \infty \tag{4.3} \]
by [17, Theorem 2]. Since \( K(r) \gtrsim \wtilde{\omega}(r)^{-1} \), we deduce
\[ \int_0^r K^p(s) \omega(s) ds \gtrsim \int_0^r \frac{\omega(s)}{\wtilde{\omega}(s)^p} ds \to \infty, \quad r \to 1^-. \]
Two applications of the Bernoulli-l’Hôpital theorem now give
\[ \liminf_{r \to 1^-} \frac{\int_0^r K^p(s) \omega(s) ds}{\wtilde{\omega}(r)^{-p}} \geq \frac{1}{p-1} \liminf_{r \to 1^-} \frac{K^p(r)}{\wtilde{\omega}(r)^{-p}} \]
\[ = \frac{1}{p-1} \left( \liminf_{r \to 1^-} \frac{K(r)}{\wtilde{\omega}(r)^{-1}} \right)^p \]
\[ \geq \frac{1}{p-1} \liminf_{r \to 1^-} \left( \frac{\psi_\omega(r)}{1-r} \right)^p = \infty. \tag{4.4} \]
Therefore (4.3) is false and consequently, \( P^+_\omega : L^p_\omega \to L^p_\omega \) is not bounded.

Before proving Theorem A, it is worth noticing that the hypotheses are satisfied for example, if \( \omega(r) = (1 - r)^{-\alpha} \left( \log \frac{e}{1-r} \right)^{-\alpha} \), \( \alpha > 1 \), or more generally,
\[ \omega(r) = \left( (1 - r) \prod_{n=1}^N \log_n \frac{\exp_n 0}{1-r} \left( \log_{N+1} \frac{\exp_{N+1} 0}{1-r} \right)^\alpha \right)^{-1}, \]
where \( 1 < \alpha < \infty \) and \( N \in \mathbb{N} \). Here, as usual, \( \log_n x = \log(\log_{n-1} x) \), \( \log_1 x = \log x \), \( \exp_n x = \exp(\exp_{n-1} x) \) and \( \exp_1 x = e^x \).

**Proof of Theorem A.** By [20, Lemmas 1.1 and 1.3], the function \( \Psi \) satisfies
\[ \Psi(x) \asymp \frac{1}{\int_0^1 s^x \omega(s) ds}, \quad x \in [0, 1). \]
Since \( \omega \in \mathcal{I} \) by the hypothesis, for a given \( a > 0 \), the function \( h(r) = \frac{\wtilde{\omega}(r)}{(1-r)^a} \) is increasing on \([\rho, 1)\) for some \( \rho = \rho(a) \in (0, 1) \). So \( \Psi \) satisfies condition (U) in [22, p. 5]. Therefore we may apply [22, Lemma 2 and Theorem 3] with \( d\eta(r) = r \omega(r) dr \) to deduce that if there were a bounded projection from \( L^1_\omega \) to \( A^1_\omega \), then the function
\[ x \mapsto \wtilde{\omega} \left( 1 - \frac{1}{x+1} \right) \int_{1/2}^x \frac{dt}{\wtilde{\omega} \left( 1 - \frac{1}{t+1} \right) t} \]
would be bounded. But this is impossible as is seen by the change of variable \( 1 - \frac{1}{x+1} = r \) and an application of the Bernoulli-l’Hôpital theorem similar to that in the last step in (4.4). Thus there are no bounded projections from \( L^1_\omega \) to \( A^1_\omega \). \( \square \)
4.1. **Two weight inequality.** We begin with introducing some necessary definitions. For a continuous function \( \omega : [0, 1) \to (0, \infty) \), set
\[
\tilde{\psi}_\omega(r) = \frac{1}{\omega(r)} \int_0^r \omega(s) \, ds, \quad r \in [0, 1).
\]
Further, for a weight \( \omega \), set
\[
\kappa_\omega^+ = \limsup_{r \to 1^-} \frac{\psi_\omega(r)}{1 - r} \quad \text{and} \quad \kappa_\omega^- = \liminf_{r \to 1^-} \frac{\psi_\omega(r)}{1 - r}.
\]
Several useful characterizations of regular weights are gathered to the following lemma.

**Lemma 10.** Let \( \omega \) be a radial continuous weight and \( 1 < p < \infty \). Denote \( \omega_1(r) = \omega(r)^{1-p}(1-r)^{-p} \) and \( \omega_2(r) = (\omega(r)(1-r))^{-\frac{1}{p}} \). Then the following assertions are equivalent:

(i) \( \omega \in \mathcal{R} \);

(ii) \( \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \asymp 1, \quad r \to 1^- \);

(iii) \( \omega \) satisfies (1.1) and
\[
\sup_{0 < r < 1} \left( \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \right) \left( \frac{\psi_\omega(r)}{1-r} \right)^{p-1} < \infty; \quad (4.5)
\]

(iv) \( \omega_2 \in \mathcal{R} \).

**Proof.** If \( \omega \in \mathcal{R} \), then \( \tilde{\omega}(r) \asymp \omega(r)(1-r) \) by the definition, and therefore
\[
\frac{\tilde{\psi}_{\omega_1}(r)}{1-r} = (\omega(r)(1-r))^{p-1} \int_0^r \frac{ds}{\omega(s)(1-s)^p} \asymp \int_0^r \frac{\omega(s)^{1-p}(1-s)^p}{\tilde{\omega}(r)^{1-p}},
\]
where
\[
\int_0^r \frac{ds}{\omega(s)^{1-p}(1-s)^p} \asymp \int_0^r \frac{ds}{\tilde{\omega}(s)^{1-p}(1-s)^p} \to \infty, \quad r \to 1^-.
\]
Bernoulli-L'Hôpital theorem now shows that
\[
\frac{(\kappa_\omega^-)^{p}}{p-1} \lesssim \liminf_{r \to 1^-} \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \leq \limsup_{r \to 1^-} \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \leq \frac{(\kappa_\omega^+)^p}{p-1}.
\]
Conversely, if (ii) is satisfied, then
\[
\frac{\psi_\omega(r)}{1-r} \asymp \frac{\tilde{\omega}(r)}{\left( \int_0^r \frac{ds}{\omega(s)(1-s)^p} \right)^{\frac{1}{p-1}}},
\]
where
\[
\int_0^r \frac{ds}{\omega(s)^{1-p}(1-s)^p} \asymp \int_0^r \frac{ds}{\tilde{\omega}(s)(1-s)^p} \to \infty, \quad r \to 1^-.
\]
Bernoulli-L'Hôpital theorem now yields
\[
(p-1) \left( \liminf_{r \to 1^-} \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \right)^{\frac{1}{p-1}} \lesssim \kappa_\omega^- \leq \kappa_\omega^+ \lesssim (p-1) \left( \limsup_{r \to 1^-} \frac{\tilde{\psi}_{\omega_1}(r)}{1-r} \right)^{\frac{1}{p-1}}.
\]
Thus (i) and (ii) are equivalent.

The fact that \( \omega_2 \in \mathcal{R} \) if and only \( \omega \in \mathcal{R} \) follows by \cite{20} Lemma 1.4(iii) and straightforward calculations. Thus (i), (ii) and (iv) are equivalent, and any of these conditions implies (iii). Moreover, since both factors in the left hand side of (4.5) are bounded away from zero for \( r \geq \frac{1}{2} \) by the local regularity (1.1), (iii) implies (i). The proof is now complete. \( \square \)
We next analyze the condition (ii) in Lemma 10 in the case when \( \omega \) is induced by two regular weights in a very particular manner.

**Lemma 11.** Let \( 0 < p < \infty \) and \( \omega, v \in R \) such that

\[
\sup_{0 < r < 1} \frac{\hat{\omega}(r)p}{\hat{v}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds < \infty. \tag{4.6}
\]

Define

\[
m = \sup \left\{ \delta \geq 0 : \sup_{0 < r < 1} \frac{\hat{\omega}(r)p - \delta}{\hat{v}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds < \infty \right\}
\]

and

\[
M = \inf \left\{ \delta \in \mathbb{R} : \int_0^1 \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds < \infty \right\}.
\]

Then \( 0 < m \leq M < p \). Moreover, if \( \lim_{r \to 1^-} \frac{\psi_v(r)}{1 - r} \) and \( \lim_{r \to 1^-} \frac{\psi_\omega(r)}{1 - r} \) exist, then \( m = M = p - \frac{\kappa_v}{\kappa_\omega} \). In particular, \( \frac{\kappa_v}{\kappa_\omega} < p \).

**Proof.** If the integral condition in the definition of \( m \) is satisfied for some \( \delta_0 > 0 \), then it is satisfied for all \( \delta \leq \delta_0 \); and similarly, if it fails for \( \delta_0 > 0 \), then it fails for all \( \delta \geq \delta_0 \). Further, the condition obviously fails for \( \delta = p \), and also for \( \delta = \delta(v, \omega) < p \) sufficiently large because then \( \hat{v}(r)\hat{\omega}(r)^{\delta - p} \to 0 \), \( r \to 1^- \), by [20, p. 10] since \( \omega, v \in R \) by the hypothesis. This implies \( m < p \). Furthermore, an integration by parts and the hypothesis \((4.4)\) show that

\[
\int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds \lesssim \frac{\hat{\omega}(r)p - \delta}{\hat{v}(r)} + \delta \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds, \quad \delta > 0,
\]

and hence

\[
\frac{\hat{\omega}(r)p - \delta}{\hat{v}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds \lesssim 1, \quad r \to 1^-,
\]

for all \( \delta > 0 \) sufficiently small. Thus \( m \in (0, p) \).

The integral in the definition of \( M \) is bounded for \( \delta = p \) because \( v \in R \). If \( \int_0^1 \frac{\hat{v}(s)}{\hat{\omega}(s)^{p - 0}(1 - s)} \, ds \) converges, so does the same integral with \( \delta \geq \delta_0 \) in place of \( \delta_0 \). Moreover, since \( \omega, v \in R \) by the hypothesis, by [20, p. 10] (ii), \( v/\hat{\omega}^\alpha \) is a (regular) weight for \( \alpha > 0 \) sufficiently small, and thus \( M < p \). To see that \( m \leq M \), assume on the contrary that there exists \( \delta > 0 \) such that

\[
\frac{\hat{\omega}(r)p - \delta}{\hat{v}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds \lesssim 1, \quad r \to 1^-,
\]

and \( \int_0^1 \frac{\hat{v}(s)}{\hat{\omega}(s)^p(1 - s)} \, ds < \infty \).

Then we deduce \( \hat{\omega}(r)p - \delta/\hat{v}(r) \lesssim 1 \), as \( r \to 1^- \), implying that \( v/\hat{\omega}^p \) is not a weight. This is obviously a contradiction, and thus \( 0 < m \leq M < p \).

Assume now that \( \kappa_\omega = \lim_{r \to 1^-} \frac{\psi_\omega(r)}{1 - r} \) and \( \kappa_v = \lim_{r \to 1^-} \frac{\psi_v(r)}{1 - r} \) exist. Then a direct calculation shows that for a given \( \varepsilon > 0 \),

\[
(1 - r)^{\frac{1}{\kappa_\omega} + \varepsilon} \lesssim \hat{\omega}(r) \lesssim (1 - r)^{\frac{1}{\kappa_\omega} - \varepsilon} \quad \text{and} \quad (1 - r)^{\frac{1}{\kappa_v} + \varepsilon} \lesssim \hat{v}(r) \lesssim (1 - r)^{\frac{1}{\kappa_v} - \varepsilon}
\]

for all \( r \in (0, 1) \), and further, \( \hat{v}(r)(1 - r)^{-\frac{1}{\kappa_v} - \varepsilon} \) and \( \hat{\omega}(r)^{-1}(1 - r)^{\frac{1}{\kappa_\omega} - \varepsilon} \) are essentially increasing on \([0, 1)\), see [20] (ii) p. 10 for details. To prove \( m = M \), let \( K < p - \frac{\kappa_v}{\kappa_\omega} \) be
fixed. Then, for $\varepsilon > 0$ sufficiently small,

$$
\int_0^r \frac{\hat{v}(s)}{\omega(s)^{p-K}(1-s)} \, ds \sim \int_0^r \left( \frac{\hat{v}(s)}{(1-s)^{\frac{1}{\omega}+\varepsilon}} \right) \left( \frac{(1-s)^{\frac{1}{\omega}-\varepsilon}}{\omega(s)} \right)^{p-K} \, ds \sim \int_0^r \frac{ds}{(1-s)^{\frac{1}{\omega}+\varepsilon}(p-K+1)}
$$

and hence, $m \geq p - \frac{n}{\kappa_\varepsilon}$. Similarly, for a fixed $K \in (p - \frac{n}{\kappa_\varepsilon}, p)$ and $\varepsilon > 0$ sufficiently small,

$$
\int_0^1 \frac{\hat{v}(s)}{\omega(s)^{p-K}(1-s)} \, ds \sim \int_0^1 \left( \frac{\hat{v}(s)}{(1-s)^{\frac{1}{\omega}+\varepsilon}} \right) \left( \frac{(1-s)^{\frac{1}{\omega}-\varepsilon}}{\omega(s)} \right)^{p-K} \, ds \sim \int_0^1 \frac{ds}{(1-s)^{\frac{1}{\omega}+\varepsilon}(p-K+1)} \lesssim 1
$$

and consequently, $M \leq p - \frac{n}{\kappa_\varepsilon} \leq m$. It follows that $m = M = p - \frac{n}{\kappa_\varepsilon}$ as claimed. Moreover, since $m > 0$, we deduce $\frac{n}{\kappa_\varepsilon} < p$. \hfill \Box

With these preparations we are ready to characterize bounded operators $P_\omega : L^p_v \to L^p_v$. Theorem 11 for $p > 1$, is contained in the next result.

**Theorem 12.** Let $1 < p < \infty$ and $\omega, v \in \mathcal{R}$ such that $\lim_{r \to 1^-} \frac{\psi_\omega(r)}{r^{\frac{n}{\kappa_\varepsilon}-1}}$ and $\lim_{r \to 1^-} \frac{\psi_v(r)}{r^{\frac{n}{\kappa_\varepsilon}-1}}$ exist. Then the following conditions are equivalent:

(a) $P^*_\omega : L^p_v \to L^p_v$ is bounded;

(b) $P_\omega : L^p_v \to L^p_v$ is bounded;

(c) $\sup_{0<r<1} \frac{v(r)}{\omega(r)^{-\frac{n}{\kappa_\varepsilon}}} \frac{r^{\frac{n}{\kappa_\varepsilon}}} {\int_0^1 \frac{\omega(s)^{\frac{n}{\kappa_\varepsilon}}} {v(s)} \, ds} < \infty$;

(d) $\sup_{0<r<1} \left( \frac{\int_0^r \frac{v(s)}{\omega(s)^{p}(1-s)^p} \, ds}{\int_0^1 \frac{\omega(s)^{\frac{n}{\kappa_\varepsilon}}} {v(s)} \, ds} \right)^{p-1} < \infty$;

(e) $\sup_{0<r<1} \frac{\omega(r)^{p}(1-r)^{p-1}}{v(r)} \frac{1}{\int_0^r \frac{v(s)}{\omega(s)^{p}(1-s)^p} \, ds} < \infty$;

(f) $\frac{n}{\kappa_\varepsilon} < p$.

**Proof.** We will prove that one condition implies another in the alphabetical order and then close the chain by establishing (f) $\Rightarrow$ (a).

The implication (a) $\Rightarrow$ (b) is obvious, so assume (b). A direct calculation shows that the adjoint of $P_\omega$, with respect to $\langle \cdot, \cdot \rangle_{L^2_v}$, is given by

$$
P^*_\omega(g)(\zeta) = \frac{\omega(\zeta)}{v(\zeta)} \int_D g(z) B^\omega(\zeta, z) v(z) \, dA(z), \quad g \in L^p_v.
$$

By the hypothesis, $P^*_\omega : L^p_v \to L^p_v$ is bounded, and hence, by choosing $g_n(z) = z^n$, $n \in \mathbb{N}$, we deduce

$$
\left( \frac{\nu_n}{\omega_n} \right)^{\frac{1}{p'}} \int_D \left| \frac{\omega(\zeta)}{v(\zeta)} \right|^{\frac{1}{p'}} v(\zeta) \, dA(\zeta) = \| P^*_\omega g_n \|_{L^{p'}_v}^{p'} \lesssim \| g_n \|_{L^p_v}^{p'} = 2v_{\omega^p}. 
$$

(4.9)
This together with [20 Lemmas 1.1 and 1.3] gives
\[
\infty > \sup_n \frac{v_n^{p'}}{v_n^{p} \omega_n} \int_0^1 s^{np'+1} \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds
\]
\[
\lesssim \sup_n \frac{\hat{\omega}(1 - \frac{1}{n})^{p'-1}}{\omega_n} \int_0^1 s^{np'+1} \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds
\]
\[
\lesssim \sup_n \left( \frac{v(1 - \frac{1}{n})^{p'-1}}{\omega(1 - \frac{1}{n})^{p'}} \int_0^1 s^{np'+1} \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)
\]
\[
\gtrsim \sup_n \left( \frac{v(1 - \frac{1}{n})^{p'-1}}{\omega(1 - \frac{1}{n})^{p'}} \int_{1-\frac{1}{n}}^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right).
\]
By choosing \( r \in [0, 1) \) such that \( r \in [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \), we deduce
\[
\sup_{r \in (0,1)} \frac{v(r)^{p'-1}}{\omega(r)^p (1-r)} \int_r^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds < \infty,
\]
which is equivalent to (c).

If (c) is satisfied, then \( (\frac{\omega}{\nu})^{\frac{1}{p'}} \) is a weight, and an application of Lemma [10] gives the implications (c) ⇒ (d) ⇒ (e). Further, (e) ⇒ (f) follows by Lemma [11].

Finally, assume that (f) is satisfied, that is, \( \frac{\omega}{\nu} < p \), and let us prove (a). The reasoning in (17) yields \( m \geq p - \frac{\omega}{\nu} > 0 \) so that
\[
\frac{\hat{\omega}(r)^{p-\varepsilon}}{\hat{v}(r)} \int_0^r \frac{\hat{v}(s)}{\hat{\omega}(s)^{p-\varepsilon}(1-s)} ds \lesssim 1, \quad r \to 1^-,
\]
for all \( \varepsilon < p - \frac{\omega}{\nu} \).

Let \( h_\delta = \hat{\omega}^\delta \), where
\[
\max \left\{ 0, -\frac{\omega}{\nu} - 1 \right\} < \delta(p-1) < \min \left\{ p-1, -\frac{\omega}{\nu} \right\}.
\]

By Hölder’s inequality,
\[
\| P_\omega^+ (f) \|_{L^p}^p \leq \int_\mathbb{D} \left( \int_\mathbb{D} |f(\zeta)|^p h_\delta(\zeta)^p |B^\omega(z, \zeta)| \frac{\omega(\zeta)}{h_\delta(\zeta)} dA(\zeta) \right) \cdot \left( \int_\mathbb{D} |B^\omega(z, \zeta)| \frac{\omega(\zeta)}{h_\delta(\zeta)} dA(\zeta) \right)^{p-1} v(z) dA(z).
\]

Since \( \omega \in \mathcal{R} \) by the hypothesis and \( \delta \in (0,1) \) by our choice, also \( \omega/h_\delta \in \mathcal{R} \) by [20] Lemma 1.4(iii)]. This and the reasoning in [11] give
\[
\int_0^r \frac{(\hat{\omega}^\delta)(s)}{\hat{\omega}(s)(1-s)} ds \lesssim \int_0^r \frac{ds}{\hat{\omega}(s)^{\delta}(1-s)} \lesssim \frac{1}{\hat{\omega}(r)^{\delta}},
\]
and hence Theorem [11(ii)] and Fubini’s theorem yield
\[
\| P_\omega^+ (f) \|_{L^p}^p \lesssim \int_\mathbb{D} \left( \int_\mathbb{D} |f(\zeta)|^p h_\delta(\zeta)^{p-1} |B^\omega(z, \zeta)| \omega(\zeta) \right) h_\delta(z)^{1-p} v(z) dA(\zeta) dA(z)
\]
\[
= \int_\mathbb{D} |f(\zeta)|^p h_\delta(\zeta)^{p-1} \omega(\zeta) \left( \int_\mathbb{D} |B^\omega(z, \zeta)| h_\delta(z)^{1-p} v(z) dA(z) \right) dA(\zeta).
\]
Since \( \delta(p-1) < -\frac{\omega}{\nu} \) by [11], arguing as in (4.8) we get that \( h_\delta^{1-p} v \) is a weight, and therefore the inner integral in the last expression above is well defined. Then, an
application of Theorem 11 (iii) (for any radial weight) gives
\[
\|P_\omega^+(f)\|_{L^p_\omega} \lesssim \int_\mathbb{D} |f(\zeta)|^p h(\zeta)^{p-1} \omega(\zeta) \left( \int_0^{\frac{1}{|\zeta|}} \frac{v(s)}{\omega(s)(1-s)} \, ds \right) \, dA(\zeta).
\]
(4.12)
Therefore we may apply Fubini’s theorem, the hypothesis \( \omega \in \mathcal{R} \) and the reasoning in (4.11) to deduce
\[
\int_0^{\frac{1}{|\zeta|}} \frac{v(t)}{\omega(t)(1-t)} \, dt \lesssim \int_0^{\frac{1}{|\zeta|}} \frac{v(t)}{\omega(t)\delta(p-1)(1-t)} \, dt + \frac{1}{\omega(\zeta)} \int_0^1 \frac{\hat{v}(t)}{\omega(t)\delta(p-1)(1-t)} \, dt.
\]
(4.13)
Next, we observe that
\[
\frac{1}{\omega(\zeta)} \int_0^1 \frac{\hat{v}(t)}{\omega(t)\delta(p-1)(1-t)} \, dt \leq \frac{\hat{v}(\zeta)}{\omega(\zeta)} \int_0^1 \frac{dt}{\omega(\zeta)\delta(p-1)(1-t)} \leq \frac{1}{\omega(\zeta)\delta(p-1)+1} \leq \int_0^{\frac{1}{|\zeta|}} \frac{v(t)}{\omega(t)\delta(p-1)} \, dt, \quad |\zeta| \geq \frac{1}{2}.
\]
Hence the first term on the right side of (4.13) dominates the second one. Now, since 
\[1 + \delta(p-1) > \frac{\omega}{\kappa_0}\] by (4.11), an application of (4.10) yields
\[
\int_0^{\frac{1}{|\zeta|}} \frac{v(s)}{\omega(s)(1-s)} \, ds \lesssim \int_0^{\frac{1}{|\zeta|}} \frac{v(s)}{\omega(s)\delta(p-1)} \, ds \lesssim \frac{\hat{v}(\zeta)}{\omega(\zeta)\delta(p-1)},
\]
which together with (4.12) gives
\[
\|P_\omega^+(f)\|_{L^p_\omega} \lesssim \int_\mathbb{D} |f(\zeta)|^p h(\zeta)^{p-1} \omega(\zeta) \left( \frac{\hat{v}(\zeta)}{\omega(\zeta)^{\delta(p-1)+1}} \right) \, dA(\zeta) \approx \|f\|_{L^p_\omega}.
\]
and thus \( P_\omega^+ : L^p_\omega \to L^p_\omega \) is bounded. \( \square \)

**Theorem 13.** Let \( \omega, v \in \mathcal{R} \) such that \( \lim_{r \to 1^-} \frac{\omega(r)}{1-r} \) and \( \lim_{r \to 1^-} \frac{v(r)}{1-r} \) exist. Then the following conditions are equivalent:

(a) \( P_\omega : L^1_v \to L^1_v \) is bounded;
(b) \( P_\omega^+ : L^1_v \to L^1_v \) is bounded;
(c) \( \sup_{0 < r < 1} \frac{\omega(r)}{v(r)} \int_0^r \frac{\hat{v}(s)}{\omega(s)(1-s)} \, ds < \infty; \)
(d) \( \frac{\kappa_0}{\kappa_0} < 1. \)

**Proof.** By (4.9), \( P_\omega : L^1_v \to L^1_v \) is bounded if and only if
\[
\sup_{\zeta \in \mathbb{D}} \left| \frac{\omega(\zeta)}{v(\zeta)} \int_\mathbb{D} \tau(z)B^\omega(\zeta, z)v(z) \, dA(z) \right| \lesssim \|\tau\|_{L^\infty}, \quad \tau \in L^\infty.
\]
(4.14)
For each \( \alpha \in \mathbb{D} \), define
\[
\tau_\alpha(z) = \begin{cases} \frac{|B^\omega(a,z)|}{B^\omega(a,z)}, & \text{if } B^\omega(a,z) \neq 0 \\ 0, & \text{if } B^\omega(a,z) = 0. \end{cases}
\]
By using this family as test functions we deduce that (4.14) is equivalent to
\[
\sup_{\zeta \in \mathbb{D}} \left| \frac{\omega(\zeta)}{v(\zeta)} \int_\mathbb{D} |B^\omega(\zeta, z)| v(z) \, dA(z) < \infty, \right.
\]
\[
(4.14)
\]
which is in turn equivalent to (c) by Theorem 1. A similar argument shows that the condition (c) characterizes the bounded linear operators

\[ \tilde{P}^{+}_{\omega}(f)(z) = \int_{D} f(\zeta)|B^{\omega}(z, \zeta)|\omega(\zeta) \, dA(\zeta) \]

on \( L^1_\omega \), and since clearly \( \tilde{P}^{+}_{\omega} \) is bounded on \( L^1_\omega \) if and only if (b) is satisfied, we have shown that (a), (b) and (c) are equivalent. To complete the proof it suffices to notice that (c)\( \iff \) (d) by Lemma 11 and its proof. \( \square \)

5. Duality

**Proof of Theorem 3** (i). The proof of this part is standard and follows by using Hölder’s inequality, the Hahn-Banach theorem and Theorem 3(i).

(ii). We begin with showing that each \( g \in B \) induces a bounded linear functional on \( A^1_\omega \). By the polarization of the identity (2.1), [20, (1.29)] and (3.10), we deduce

\[ |\langle f, g \rangle_{A^1_\omega}| \lesssim |f(0)||g(0)| + |\langle f', g' \rangle|_{A^2_\omega} \]

\[ \lesssim \|f\|_{A^1_\omega}\|g\|_B + \int_D |f'(z)||g'(z)|(1 - |z|^2)\omega(z) \, dA(z) \]

\[ \lesssim \|g\|_B\|f\|_{A^1_\omega}. \]

Assume next that \( L \) is a bounded linear functional on \( A^1_\omega \). By the Hahn-Banach theorem \( L \) can be extended to a bounded linear functional \( \tilde{L} \) on \( L^1_\omega \) with \( \|L\| = \|\tilde{L}\| \). So, there exists a unique function \( h \in L^{\infty}(D) \) such that

\[ \tilde{L}f = \int_D f(z)\overline{h(z)}\omega(z) \, dA(z), \quad f \in L^1_\omega, \]

and \( \|\tilde{L}\| = \|h\|_{L^{\infty}(D)} \). By using the restriction of this identity to functions in \( A^1_\omega \), and Fubini’s theorem we deduce

\[ Lf = \tilde{L}f = \lim_{r \to 1^-} \int_D f(rz)\overline{h(z)}\omega(z) \, dA(z) \]

\[ = \lim_{r \to 1^-} \int_D \left( \int_D f(r\zeta)|B^{\omega}(z, \zeta)\omega(\zeta)| \, dA(\zeta) \right) \overline{h(z)}\omega(z) \, dA(z) \]

\[ = \lim_{r \to 1^-} \int_D f(r\zeta)\overline{P_{\omega}(h)(\zeta)}\omega(\zeta) \, dA(\zeta). \]

The first part of the proof implies that this last limit equals to \( \langle f, P_{\omega}(h) \rangle_{A^2_\omega} \), because \( P_{\omega} : L^{\infty}(D) \to B \) is bounded by Theorem 3(ii). Thus \( \|P_{\omega}(h)\|_B \lesssim \|h\|_{L^{\infty}(D)} = \|\tilde{L}\| = \|L\| \), and the assertion is proved. \( \square \)

6. Toeplitz Operators

In this section we prove Theorem 5. We begin with recalling the natural connection between the Toeplitz operator \( T_\mu : H_\alpha(\omega^*) \to H_\alpha(\omega^*) \) and the identity operator \( I_d \) acting from \( H_\alpha(\omega^*) \) to the space \( L^2(\mu) \). Namely, the definition (2.2), Fubini’s theorem and (2.3) give

\[ \langle T_\mu(g), f \rangle_{H_\alpha(\omega^*)} = \langle g, f \rangle_{L^2(\mu)} \]

for all \( g \) and \( f \) in \( H_\alpha(\omega^*) \), and therefore \( T_\mu \) is bounded (resp. compact) on \( H_\alpha(\omega^*) \) if and only if \( I_d : H_\alpha(\omega^*) \to L^2(\mu) \) is bounded (resp. compact).

The main effort on the way to Theorem 6 consists on proving the following result.
**Theorem 14.** Let $0 < p < \infty$ and $-\infty < \alpha < 1$ such that $p\alpha < 1$. Let $\omega \in I \cup R$, and let $\mu$ be a complex Borel measure on $D$. If
\[ \sum_{R_j \in \mathcal{T}} \left( \frac{|\mu(R_j)|}{\omega_{\alpha}(z_j)} \right)^p < \infty, \] (6.2)
then $T_\mu \in S_p(H_\alpha(\omega^*))$, and there exists a constant $C > 0$ such that
\[ |T_\mu|^p \leq C \sum_{R_j \in \mathcal{T}} \left( \frac{|\mu(R_j)|}{\omega_{\alpha}(z_j)} \right)^p. \]

Conversely, if $\mu$ is a positive Borel measure on $D$ and $T_\mu \in S_p(H_\alpha(\omega^*))$, then (6.2) is satisfied.

For $p \geq 1$, Theorem 14 is proved [20, Chapter 6] in order to study the integral operator $T_\mu f(z) = \int_0^z f(\zeta)g(\zeta) d\zeta$. Therefore here we only have to deal with the range $0 < p < 1$, where the proof is actually more involved. Note that in [20] the space $H_\alpha(\omega^*)$ is defined by the condition
\[ \sum_{n=0}^{\infty} (n+1)^{n+2} \omega_{\alpha}^*(a_{n+1})^2 < \infty, \quad -\infty < \alpha < 2, \]
where $\sum_{n=0}^{\infty} a_n z^n$ is the Maclaurin series of $f \in H(D)$, and this condition is equivalent to our definition of $H_\alpha(\omega^*)$ by [20, Lemma 6.9].

Before proceeding to the auxiliary results needed, we show how Theorem 6 follows once Theorem 14 is proved.

**Proof of Theorem 6.** The equivalence of (a) and (b) follows by Theorem 14. Let now $r \in (0,1)$ be fixed. Divide each polar rectangle $R_j$ into $K^2$ disjoint subrectangles $R_j^k$, $k = 1, \ldots, K^2$, of approximately equal size in the hyperbolic sense such that $R_j^k \subset D(z,r)$ for all $z \in R_j^k$, $k = 1, \ldots, K^2$ and $R_j \in \mathcal{T}$. Then, by [20, Lemma 1.7] and (1.1),
\[ \sum_{R_j \in \mathcal{T}} \left( \frac{\mu(R_j)}{\omega_{\alpha}(z_j)} \right)^p \leq \sum_{R_j \in \mathcal{T}} \sum_{k=1}^{K^2} \left( \frac{\mu(R_j^k)}{\omega_{\alpha}(z_j)} \right)^p \]
\[ \approx \sum_{R_j \in \mathcal{T}} \sum_{k=1}^{K^2} \int_{R_j^k} \left( \frac{\mu(R_j^k)}{\omega_{\alpha}(z_j)} \right)^p dA(z) \]
\[ \leq \sum_{R_j \in \mathcal{T}} \sum_{k=1}^{K^2} \int_{R_j^k} \left( \frac{\mu(D(z,r))}{\omega_{\alpha}(z_j)} \right)^p dA(z) \]
\[ \approx \int_{D} \left( \hat{\mu}_{\alpha,r}(z) \right)^p dA(z) \]
and hence (c) implies (b) is proved. The fact that (b) implies (c) can be proved by a reasoning similar to that above bearing in mind that each disc $\Delta(z,r)$ intersects at most $N = N(r) \in \mathbb{N}$ squares of the lattice $\mathcal{T}$. This finishes the proof of the theorem. \hfill \Box

### 6.1. Preliminary results
Auxiliary results needed in the proof of Theorem 14 are gathered to this section. The first lemma shows that the derivatives of the reproducing kernel $K_{\alpha}\omega$ are essentially constant in sufficiently small hyperbolic discs centered at the point $a$.

**Lemma 15.** Let $\omega \in I \cup R$, $-\infty < \alpha < 1$ and $N \in \mathbb{N} \cup \{0\}$. Then there exists $r_0 = r_0(\omega,\alpha,N) \in (0,1)$ such that $\left| \frac{\partial^N K_{\alpha,\omega}(a,z)}{\partial^N a} \right| \leq \left| \left( \frac{\partial^N K_{\alpha,\omega}(a,z)}{\partial^N a} \right) \right|_{z=a}$ for all $a \in D$, $z \in \Delta(a,r)$ and $0 < r \leq r_0$. 

Proof. The proof is similar to that of [20, Lemma 6.4]. By using (3.7) we obtain

\[
\left| \left( \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right) \right|_{z=a} = \frac{1}{|a|^N} \sum_{j \geq N} \frac{j(j-1) \cdots (j-N+1)|a|^{2j}}{2^j(\omega_{a}^{*})_{j-1}} \times \sum_{j \geq N} \frac{|a|^{2j}}{j^{2-N}(\omega_{a}^{*})_{j-1}}, \quad N \in \mathbb{N}.
\]

On the other hand, [20, Lemmas 6.3 and 6.10] yield

\[
\frac{1}{(1-|a|)^N \omega_{a}^{*}(a)} = \frac{1}{\omega_{N-a}(a)} \times \| K^{\alpha,-N,\omega}_{a} \|^2_{H_{N-a}(\omega^*)} \times \sum_{j \geq 1} \frac{|a|^{2j}}{2^j(\omega_{N-a}^{*})_{j-1}} \times \sum_{j \geq 1} \frac{|a|^{2j}}{j^{2-N}(\omega_{a}^{*})_{j-1}},
\]

and it follows that

\[
\left| \left( \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right) \right|_{z=a} \times \frac{1}{(1-|a|)^N \omega_{a}^{*}(a)}, \quad |a| \to 1^{-}.
\]

Next, by [20, Lemma 1.6],

\[
\left| \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right| \leq \left( \sum_{j \geq N} \frac{j(j-1) \cdots (j-N+1)|a|^{2j}}{2^j(\omega_{a}^{*})_{j-1}} \right)^{\frac{1}{2}} \times \left( \sum_{j \geq N} \frac{j(j-1) \cdots (j-N+1)|z|^{2j}}{2^j(\omega_{a}^{*})_{j-1}} \right)^{\frac{1}{2}}.
\]

(6.3)

\[
\times \left( \frac{1}{(1-|a|)^N \omega_{a}^{*}(a)(1-|z|)^N \omega_{a}^{*}(z)} \right)^{\frac{1}{2}} \times \left( \frac{1}{(1-|a|)^N \omega_{a}^{*}(a)} \times \left| \left( \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right) \right|_{z=a} \right).
\]

for all \( z \in \Delta(a, r) \). To obtain the same lower bound, let \( z \in \Delta(a, r_0) \), where \( r_0 \in (0, 1) \) is to be fixed later, and note first that

\[
\left| \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right| \geq \left| \left( \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right) \right|_{z=a} - \max_{\zeta \in [a, z]} \left| \frac{\partial^{N+1} K^{\alpha,\omega}(a, z)}{\partial^N a \partial \zeta} \right|_{z=a} |z-a|
\]

\[
\geq \left| \left( \frac{\partial^N K^{\alpha,\omega}(a, z)}{\partial^N a} \right) \right|_{z=a} - \max_{\zeta \in [a, z]} \left| \frac{\partial^{N+1} K^{\alpha,\omega}(a, z)}{\partial^N a \partial \zeta} \right| \cdot Cr_0(1-|a|),
\]

where \( C = C(r_0) > 0 \) is a constant. Now the Cauchy formula and a reasoning similar to that in (6.3) yields

\[
\max_{\zeta \in [a, z]} \left| \frac{\partial^{N+1} K^{\alpha,\omega}(a, z)}{\partial^N a \partial \zeta} \right| \lesssim \frac{1}{(1-|a|)^{N+1} \omega_{a}^{*}(a)}
\]

and the desired lower bound follows by choosing \( r_0 \) sufficiently small. \( \square \)

In the next lemma we define an auxiliary linear operator induced by a sequence \( \{b_j\} \subset \mathbb{D} \) and a derivative of the reproducing kernel \( K^{\alpha,\omega}_{a} \). It turns out that this operator is bounded from any separable Hilbert space to \( H_a(\omega^*) \) whenever \( \{b_j\} \) is uniformly discrete, and even onto if \( \{b_j\} \) is a \( \delta \)-lattice.

**Lemma 16.** Let \( \omega \in \mathcal{I} \cup \mathcal{R} \), \(-\infty < \alpha < 1 \) and \( N \in \mathbb{N} \). For \( a \in \mathbb{D} \setminus \{0\} \), define

\[
h^{N,\omega_{a}^{*}}(z) = (1-|a|)^N (\omega_{a}^{*}(a))^{\frac{1}{2}} \frac{\partial^N K^{\alpha,\omega}(z, a)}{\partial^N a}, \quad z \in \mathbb{D},
\]
and set $h_0^{N,\omega_{-\alpha}} \equiv 0$. Let $\{b_j\}_{j=0}^{\infty}$ be a uniformly discrete sequence ordered by increasing moduli, and let $\{c_j\}_{j=0}^{\infty}$ be an orthonormal basis of a Hilbert space $H$. Let $J$ be the linear operator such that $J(e_0) = \frac{1}{(\omega_{-\alpha}(D))^{1/2}}$, $J(e_j) = \frac{x^j}{f^{2}(\omega_{-\alpha})_{j-1}}$ for all $j = 1, \ldots, N - 1$, and $J(e_j) = h_j^{N,\omega_{-\alpha}}$ if $j \geq N$. Then $J : H \to H_\alpha(\omega^*)$ is bounded. Moreover, if $\{b_j\}$ is a $\delta$-lattice for some $\delta \in (0,1)$, then $J$ is onto.

**Proof.** We first observe that $J : H \to H_\alpha(\omega^*)$ is bounded if

$$
\left\| J \left( \sum_{j} c_j e_j \right) \right\|_{H_\alpha(\omega^*)} \lesssim \left( \sum_{j} |c_j|^2 \right)^{1/2}
$$

for all sequences $\{c_j\}$ in $\ell^2$. Next, $N$ differentiations of the reproducing formula (2.3)

give

$$
f^{(N)}(a) = \int_D f'(z) \frac{\partial^{N+1} K^{\alpha,\omega}(a,z)}{\partial a \partial^{N} z} \omega_{-\alpha}(z) \, dA(z), \quad a \in \mathbb{D}, \quad f \in H_\alpha(\omega^*). \tag{6.4}
$$

By (3.7), $h_j^{N,\omega_{-\alpha}}$ vanishes at the origin for all $a \in \mathbb{D}$. Therefore, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_\alpha(\omega^*)$, then (6.4) and the Cauchy-Schwarz inequality yield

$$
\left| \left\langle J \left( \sum_{j=0}^{\infty} c_j e_j \right), f \right\rangle \right|_{H_\alpha(\omega^*)} \lesssim \left( \sum_{j} |c_j|^2 \right)^{1/2} \left( \sum_{j} |c_j|^2 \right)^{1/2} \left( \sum_{j} |c_j|^2 \right)^{1/2}.
$$

Now, since $|f^{(N)}|^2$ is subharmonic, (1.1) for the regular weight $\omega_{-\alpha+2N-2}$ and the assumption that $\{b_j\}$ is uniformly discrete, give

$$
\sum_{j=N}^{\infty} (1 - |b_j|) \omega_{-\alpha}(b_j) |f^{(N)}(b_j)|^2 \lesssim \int_D |f^{(N)}(z)|^2 \omega_{-\alpha+2N-2}(z) \, dA(z).
$$

Consequently, $N - 1$ applications of (2.1) show that $J : H \to H_\alpha(\omega^*)$ is bounded.

Let now $\{b_j\}$ be a $\delta$-lattice for some $\delta \in (0,1)$. To see that in this case the operator $J : H \to H_\alpha(\omega^*)$ is onto, note first that the adjoint of $J$ is

$$
J^*(f) = a_0 (\omega_{-\alpha}(D))^{1/2} c_0 + 2 \sum_{j=1}^{N-1} a_j (\omega_{-\alpha})_{j-1}^{1/2} c_j + \sum_{j=N}^{\infty} (1 - |b_j|) \omega_{-\alpha}(b_j) |f^{(N)}(b_j)|^2.
$$

Since the weight $\omega_{-\alpha+2N-2}$ is regular, it satisfies the $C_p$ property defined at (1.4) p. 321] by (1.1). Therefore [14, Theorem 3.14] and $N - 1$ applications of (2.1) imply that $J^*$
is bounded below, and hence injective. In particular, \( \text{Ker}(J^*) = \{0\} \), and thus \( J \) is onto.

The third auxiliary result needed is a consequence of Theorem 11 (or more precisely of Corollary 2).

**Corollary 17.** Let \( 0 < p < \infty \), \( \omega \in \mathcal{I} \cup \mathcal{R} \), \( -\infty < \alpha < 2 \) and \( N \in \mathbb{N} \) such that \( (1 - |z|)^{Np-2}(\omega^*_\alpha(z))^{\frac{p}{p-1}} \) is a regular weight. Then

\[
\int_{\mathbb{D}} |(K^\alpha_\omega)^{(N)}(z)|^p (1 - |z|)^{Np-2}(\omega^*_\alpha(z))^{\frac{p}{p-1}} dA(z) \propto (\omega^*_\alpha(a))^{\frac{p}{p-1}}, \quad |a| > \frac{1}{2}
\]

**Proof.** Since \( W(z) = (1 - |z|)^{Np-2}(\omega^*_\alpha(z))^{\frac{p}{p-1}} \) is regular by the hypothesis,

\[
\hat{W}(z) \propto (1 - |z|)^{Np-2}(\omega^*_\alpha(z))^{\frac{p}{p-1}} \propto (1 - |z|)^{p(N + \frac{1}{2})-1}\hat{\omega}(z)^{\frac{p}{p-1}},
\]

and so \cite{20} Lemma 1.6] gives

\[
\int_0^{[a]} \frac{|\hat{W}(s)|}{\omega(s)^p(1 - s)^{p(N+1-\alpha)}} ds \propto \int_0^{[a]} \frac{1}{\omega(s)^{\frac{p}{p-1}}(1 - s)^{p\frac{1}{2}+1}} ds
\]

\[
\leq \frac{1}{\omega(a)^{\frac{p}{p-1}}} \int_0^{[a]} \frac{ds}{(1 - s)^{p\frac{1}{2}+1}} \propto \frac{1}{\omega(a)^{\frac{p}{p-1}}(1 - |a|)^{p\frac{1}{2}}} \propto (\omega^*_\alpha(a))^{-\frac{p}{p-1}}.
\]

Consequently, (6.5) follows from Corollary 2. \( \square \)

**Proof of Theorem 14.** The reasoning employed here is based on ideas from \cite{15}. By \cite{20} Theorem 6.11 it is enough to deal with the case \( 0 < p < 1 \). Further, by \cite{20} p. 10 (ii)] we can fix \( N = N(p, \alpha, \omega) \) large enough such that \( (1 - |z|)^{Np-2}(\omega^*_\alpha(z))^{\frac{p}{p-1}} \) is a regular weight. This simple observation is one of the key steps in the proof.

Let \( \mu \) be a complex Borel measure on \( \mathbb{D} \) such that (6.2) is satisfied. By \cite{20} Theorem 6.11, \( T_\mu \in \mathcal{S}_1(H_\alpha(\omega^*)) \) and, in particular, \( T_\mu \) is compact on \( H_\alpha(\omega^*) \). We will show that \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \). To do this, Corollary 17 will be used.

Let \( \{h_j\}_{j=0}^\infty \) be a \( \delta \)-lattice. Then, for a fixed basis \( \{e_j\}_{j=0}^\infty \) of \( H_\alpha(\omega^*) \), the operator \( J : H_\alpha(\omega^*) \to H_\alpha(\omega^*) \), defined on Lemma 16 is bounded and onto. Therefore, by \cite{26} Proposition 1.30], \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \) if and only if \( J^*T_\mu J \in \mathcal{S}_p(H_\alpha(\omega^*)) \). This together with \cite{15} Lemma 5] (see also \cite{26} Proposition 1.29]) shows that it suffices to prove

\[
\sum_{j=0}^\infty \sum_{k=0}^\infty |\langle T_\mu(h_j), h_k \rangle|^p \lesssim \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu(R_j)}{\omega^*_\alpha(z_j)} \right)^p,
\]

where \( J(e_j) = h_j \). To see this, we first observe that (6.1) yields

\[
|\langle T_\mu(h_j), h_k \rangle| \leq \int_{\mathbb{D}} |h_j(z)||h_k(z)| d|\mu|(z) = \sum_{R_n \in \mathcal{Y}} \int_{R_n} |h_j(z)||h_k(z)| d|\mu|(z)
\]

\[
\leq \sum_{R_n \in \mathcal{Y}} |h_j(\tilde{z}_{j,n})||h_k(\tilde{z}_{k,n})||\mu|(R_n),
\]
where \( |h_j(\tilde{z}_{j,n})| = \max_{z \in \mathcal{R}_n} |h_j(z)| \) for each \( j \in \mathbb{N} \cup \{0\} \). Since \( 0 < p < 1 \) by the assumption, we deduce
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\langle T_{\mu}(h_j), h_k \rangle|^p \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |h_j(\tilde{z}_{j,n})|^p |h_k(\tilde{z}_{k,n})|^p |\mu|(R_n)^p
\]
\[
= \sum_{R_n \in \mathcal{Y}} |\mu|(R_n)^p \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |h_j(\tilde{z}_{j,n})|^p |h_k(\tilde{z}_{k,n})|^p \right).
\]
Consequently, the proof will be finished once we prove
\[
\sum_{j=0}^{\infty} |h_j(\tilde{z}_{j,n})|^p \leq C \left( \omega_{\alpha}(z_n) \right)^{-\frac{p}{2}}, \tag{6.6}
\]
where the constant \( C > 0 \) is independent of \( n \). Clearly,
\[
\sum_{j=0}^{\infty} |h_j(\tilde{z}_{j,n})|^p \lesssim N + \sum_{j=N}^{\infty} (1 - |b_j|)^{Np} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \left| \frac{\partial^{N} K^{\alpha,\omega}(\tilde{z}_{j,n}, b_j)}{\partial^{N} b_j} \right|^p
\]
\[
= N + \sum_{j=N}^{\infty} (1 - |b_j|)^{Np} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \left| \frac{\partial^{N} K^{\alpha,\omega}(b_j, \tilde{z}_{j,n})}{\partial^{N} b_j} \right|^p.
\]
Since \( \{\tilde{z}_{j,n}\} \subset \mathcal{R}_n \), by applying the subharmonicity to the second variable of \( \frac{\partial^{N} K^{\alpha,\omega}(z,\omega)}{\partial^{N} z} \), and then to the first one, and using (1.1) for the regular weight \( \omega_{-\alpha} \), we get
\[
\sum_{j=N}^{\infty} (1 - |b_j|)^{Np} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \left| \frac{\partial^{N} K^{\alpha,\omega}(b_j, \tilde{z}_{j,n})}{\partial^{N} b_j} \right|^p
\]
\[
\lesssim \frac{1}{(1 - |z_n|)^2} \int_{\Delta(z_n,r)} \left( \sum_{j=N}^{\infty} (1 - |b_j|)^{Np} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \left| \frac{\partial^{N} K^{\alpha,\omega}(b_j, \zeta)}{\partial^{N} b_j} \right|^p \right) dA(\zeta)
\]
\[
\lesssim \frac{1}{(1 - |z_n|)^2} \int_{\Delta(z_n,r)} \left( \sum_{j=N}^{\infty} (1 - |b_j|)^{Np-2} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \frac{\partial^{N} K^{\alpha,\omega}(z, \zeta)}{\partial^{N} z} \right) dA(z) \right) dA(\zeta)
\]
\[
\lesssim \frac{1}{(1 - |z_n|)^2} \int_{\Delta(z_n,r)} \left( \sum_{j=N}^{\infty} \int_{\Delta(b_j,r)} \left| \frac{\partial^{N} K^{\alpha,\omega}(z, \zeta)}{\partial^{N} z} \right|^p (1 - |z|)^{Np-2} \left( \omega_{-\alpha}(z) \right)^{\frac{p}{2}} dA(z) \right) dA(\zeta)
\]
for a suitably chosen \( r \in (0,1) \). Finally, by using that \( \{b_j\} \) is uniformly discrete and applying Corollary 17, we deduce
\[
\sum_{j=N}^{\infty} (1 - |b_j|)^{Np} \left( \omega_{-\alpha}(b_j) \right)^{\frac{p}{2}} \left| \frac{\partial^{N} K^{\alpha,\omega}(b_j, \tilde{z}_{j,n})}{\partial^{N} b_j} \right|^p
\]
\[
\lesssim \frac{1}{(1 - |z_n|)^2} \int_{\Delta(z_n,r)} \left( \int_{\mathbb{D}} \left| \frac{\partial^{N} K^{\alpha,\omega}(z, \zeta)}{\partial^{N} z} \right|^p (1 - |z|)^{Np-2} \left( \omega_{-\alpha}(z) \right)^{\frac{p}{2}} dA(z) \right) dA(\zeta)
\]
\[
\lesssim \frac{1}{(1 - |z_n|)^2} \int_{\Delta(z_n,r)} \frac{dA(\zeta)}{(\omega_{-\alpha}(\zeta))^{\frac{p}{2}}} \lesssim \frac{1}{(\omega_{-\alpha}(z_n))^{\frac{p}{2}}}
\]
The inequality (6.6) follows and thus the first assertion is proved.

Before proving the second part of the assertion, we introduce the necessary notation. Each \( a \in \mathbb{D} \setminus \{0\} \) induces the polar rectangle
\[
R(a) = \left\{ z \in \mathbb{D} : |\arg az| < \frac{1 - |a|}{2}, \quad |a| \leq |z| < \frac{1 + |a|}{2} \right\}.
\]
For convenience, we also denote by $R_{-1}(a)$ the rectangle induced by the point $(2|a| - 1)e^{i(\arg a - (1-|a|))}$, and if $|a| < \frac{1}{2}$, we set $R_{-1}(a) = D(0,1/2)$. Further, we denote

$$Q(a) = R(a) \cup R_{-1}(a), \quad a \in \mathbb{D} \setminus \{0\}.$$  

Let now $\mu$ be a positive Borel measure on $\mathbb{D}$, and assume that $T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*))$, where $0 < p < 1$. For each $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that $\{z_k\}$ (the sequence which induces the partition $\{R(I) : I \in \mathcal{Y}\}$) can be divided into $M$ subsequences $\{z_k^{(j)}\}$, $j = 1, \ldots, M$, such that $\rho(z_k^{(j)}, z_l^{(j)}) > 1 - \varepsilon$ for all $n \neq k$ and $j = 1, \ldots, M$. Therefore, if $R_k^{(j)} \in \mathcal{T}$ denotes the element containing $z_k^{(j)}$, then $\rho(Q_k^{(j)}, Q_k^{(j)}) > 1 - \delta(\varepsilon)$, for all $n \neq k$ and $j = 1, \ldots, M$, and $\delta = \delta(\varepsilon) \to 0$, as $\varepsilon \to 0$. The choice of $\varepsilon$ will be made later.

Next, we need to do a new partition in order to ensure that the kernels involved in the proof are essentially constant on certain subrectangles (small regions in the hyperbolic sense). To do this, divide each $R_k^{(j)} \in \mathcal{T}$, $k \in \mathbb{N}$, $j = 1, \ldots, M$, into $P^2$ disjoint rectangles $\{R_k^{(j,l)}\}_{l=1}^{P^2}$ (of approximately equal size in the hyperbolic sense) where $P \in \mathbb{N}$ is sufficiently large so that we may use Lemma 15 to deduce

$$\sup_{z \in R_k^{(j,l)}} \left| \frac{\partial^N K^{\alpha,\omega}(z_k^{(j,l)}, z)}{\partial^N z_k^{(j,l)}} \right| \lesssim \frac{1}{(1 - |z_k^{(j,l)}|)^N \omega_{-\alpha}(z_k^{(j,l)})}, \quad z \in R_k^{(j,l)},$$

(6.7)

where the constants of comparison do not depend on $k$, $j$, and $l$. Moreover, we obviously have

$$\rho(Q(z_k^{(j,l)}), Q(z_k^{(j,l)})) > 1 - \delta, \quad n \neq k, \quad j = 1, \ldots, M, \quad l = 1, \ldots, P^2.$$  

Let $\mu_{j,l} = \left(\sum_k \chi_{R_k^{(j,l)}}\right)\mu$. We note that $T_\mu$ and $T_{\mu_{j,l}}$ are both diagonalizable as positive operators and $|T_{\mu_{j,l}}| \leq |T_\mu|$, see [15] p. 359 for details. Fix now indexes $j$ and $l$, and write $\nu = \mu_{j,l}$ and $b_k = z_k^{(j,l)}$ for short. Write also $h_k = h_{b_k}^{\omega_{-\alpha}}$, and let $J$ be the operator such that $J(e_j) = 0$ if $j < N$ and $J(e_j) = b_j$ if $j \geq N$. Then, by Lemma 16 $J^*T_\nu J \in \mathcal{S}_p(H_\alpha(\omega^*))$, whenever $T_\nu \in \mathcal{S}_p(H_\alpha(\omega^*))$. Further, $J^*T_\nu J = D + E$, where $D$ is the diagonal operator

$$D(f) = \sum_k \langle T_\nu(h_k), h_k \rangle_{H_\alpha(\omega^*)} \langle f, e_k \rangle_{H_\alpha(\omega^*)} e_k,$$

and $E$ is the remainder

$$E(f) = \sum_{n \neq k} \sum_k \langle T_\nu(h_k), h_n \rangle_{H_\alpha(\omega^*)} \langle f, e_k \rangle_{H_\alpha(\omega^*)} e_n.$$  

Now, bearing in mind (6.7), we deduce

$$|D|^p = \sum_k \langle T_\nu(h_k), h_k \rangle^p_{H_\alpha(\omega^*)} = \sum_k \left( \int_{\mathbb{D}} |h_k(z)|^2 \, d\nu(z) \right)^p$$

$$\geq \sum_k \left( \int_{R_b(h_k)} |h_k(z)|^2 \, d\nu(z) \right)^p$$

$$\geq c_1 \sum_k \left( (1 - |b_k|)^{2N} \omega_{-\alpha}(b_k) \frac{\nu(R_b(h_k))}{(1 - |b_k|)^{2N} \omega_{-\alpha}(b_k)} \right)^p$$

$$= c_1 \sum_k \left( \frac{\nu(R_b(h_k))}{\omega_{-\alpha}(b_k)} \right)^p$$
for some constant $c_1 > 0$ depending only on $\alpha$, $\omega^*$, $N$ and $p$, and where $R(b_k)$ is the square induced by $b_k$.

To deal with $E$, we may argue as in the first part of the proof to obtain

$$|E|^p \leq \sum_{n} \sum_{k \neq n} |\langle T_v(h_k), h_n \rangle|_{H_{\alpha}(\omega^*)}^p \leq \sum_{n} \sum_{k \neq n} \left( \sum_{j \in R(b_j)} |h_k(z)| |h_n(z)| \, dv(z) \right)^p$$

$$= \sum_{n} \sum_{k \neq n} \left( \sum_{j} (1 - |b_k|)^N (\omega_{\alpha}(b_k))^\frac{p}{2} (1 - |b_n|)^N (\omega_{\alpha}(b_n))^\frac{p}{2} \right.$$

$$\cdot \int_{R(b_j)} \left| \frac{\partial^N K_{\alpha,\omega}(z, b_k)}{\partial^N b_k} \right| \left| \frac{\partial^N K_{\alpha,\omega}(z, b_n)}{\partial^N b_n} \right| \, dv(z) \right)^p$$

$$\cdot \nu(R(b_j))^p \left( \sum_{n} \sum_{k \neq n} (1 - |b_k|)^p (\omega_{\alpha}(b_k))^\frac{p}{2} (1 - |b_n|)^p (\omega_{\alpha}(b_n))^\frac{p}{2} \right.$$

$$\cdot \left| \frac{\partial^N K_{\alpha,\omega}(z_k, b_k)}{\partial^N b_k} \right| \left| \frac{\partial^N K_{\alpha,\omega}(z_n, b_n)}{\partial^N b_n} \right| \right)^p,$$

where $\left| \frac{\partial^N K_{\alpha,\omega}(z_k, b_k)}{\partial^N b_k} \right| = \max_{z \in R(b_j)} \left| \frac{\partial^N K_{\alpha,\omega}(z, b_k)}{\partial^N b_k} \right|$ for each $j \in \mathbb{N} \cup \{0\}$.

Next, bearing in mind that $\omega_{\alpha}$ is regular, by using subharmonicity, we deduce

$$\sum_{k} (1 - |b_k|)^p (\omega_{\alpha}(b_k))^\frac{p}{2} \left| \frac{\partial^N K_{\alpha,\omega}(z_k, b_k)}{\partial^N b_k} \right|^p$$

$$\lesssim \sum_{k} \int_{\Delta(b_k,r)} (1 - |u|)^{Np-2} (\omega_{\alpha}(z))^\frac{p}{2} \left| \frac{\partial^N K_{\alpha,\omega}(z_k, u)}{\partial^N b_k} \right|^p \, dA(u).$$

Choose now constants $0 < s_1 < s$ such that $\Delta(z, s_1) \subset \Delta(b_j, s)$ for any $z \in R(b_j)$, $j \in \mathbb{N}$. Then, by using subharmonicity again, we get

$$\sum_{k} (1 - |b_k|)^p (\omega_{\alpha}(b_k))^\frac{p}{2} \left| \frac{\partial^N K_{\alpha,\omega}(z_k, b_k)}{\partial^N b_k} \right|^p$$

$$\lesssim \sum_{k} \int_{\Delta(b_k,r)} (\omega_{2N - \frac{4}{p} - \alpha}(u))^\frac{p}{2} \left( \frac{1}{(1 - |b_j|)^2} \int_{\Delta(b_j,s)} \left| \frac{\partial^N K_{\alpha,\omega}(z, u)}{\partial^N b_k} \right|^p \, dA(z) \right) \, dA(u).$$

Next, we choose $r \in (0, 1)$ such that $\Delta(z, r) \subset Q(z)$, $z \in \mathbb{D} \setminus \{0\}$. Since the sets $\{Q(b_k)\}$ are disjoint, we get

$$\sum_{k} (1 - |b_k|)^p (\omega_{\alpha}(b_k))^\frac{p}{2} \left| \frac{\partial^N K_{\alpha,\omega}(z_k, b_k)}{\partial^N b_k} \right|^p$$

$$\lesssim \int_{\cup_k Q(b_k)} (\omega_{2N - \frac{4}{p} - \alpha}(u))^\frac{p}{2} \left( \frac{1}{(1 - |b_j|)^2} \int_{\Delta(b_j,s)} \left| \frac{\partial^N K_{\alpha,\omega}(z, u)}{\partial^N b_k} \right|^p \, dA(z) \right) \, dA(u),$$
It follows that for each \( \eta > \omega \) sufficiently large such that \( (\eta - \omega)^2 \approx \omega - \alpha \), and hence

\[
\int_{U \cup U \neq k} Q(b_k) \times Q(b_n) \frac{(\omega^* - \alpha)}{p}(u) \frac{1}{p} (\omega^* - \alpha + 2N - \frac{4}{p})(v) \frac{1}{p}
\]

\[
\cdot \left( \frac{1}{(1 - |b_j|)^4} \int_{\Delta(b_j) \Delta(b_j)} \left| \frac{\partial N K^{\alpha,\omega}(u, v)}{\partial N_N} \right|^p \left| \frac{\partial N K^{\alpha,\omega}(z, v)}{\partial N_N} \right|^p \ dA(u) \ dA(v) \right) \ dA(u) \ dA(v)
\]

\[
\leq \int_G (\omega^* - \alpha + 2N - \frac{4}{p})(u) \frac{1}{p} (\omega^* - \alpha + 2N - \frac{4}{p})(v) \frac{1}{p}
\]

\[
\cdot \left( \frac{1}{(1 - |b_j|)^4} \int_{\Delta(b_j) \Delta(b_j)} \left| \frac{\partial N K^{\alpha,\omega}(u, v)}{\partial N_N} \right|^p \left| \frac{\partial N K^{\alpha,\omega}(z, v)}{\partial N_N} \right|^p \ dA(u) \ dA(v) \right) \ dA(u) \ dA(v)
\]

\[
\leq \frac{1}{(1 - |b_j|)^4} \int_{\Delta(b_j) \Delta(b_j)} \left[ \left( \frac{\partial N K^{\alpha,\omega}(u, v)}{\partial N_N} \right|^p \left| \frac{\partial N K^{\alpha,\omega}(z, v)}{\partial N_N} \right|^p \ dA(u) \ dA(v) \right] \ dA(u) \ dA(v)
\]

where \( G = \{(u, v) : \rho(u, v) > 1 - \delta\} \supset \cup_{n \neq k} Q(b_k) \times Q(b_n) \). Since we have chosen \( N \) sufficiently large such that \( (\omega^* - \alpha + 2N - \frac{4}{p}) \) is a regular weight, Corollary 17 shows that the double inner integral is uniformly bounded by a positive constant times

\[
(\omega^* - \alpha)(\omega^* - \alpha)^{-\frac{p}{2}} (\omega^* - \alpha)^{-\frac{p}{2}} \times (\omega^* - \alpha)^{-\frac{p}{2}} - p,
\]

and hence

\[
\sum_n \sum_{k \neq n} \int \leq o((\omega^* - \alpha)(\omega^* - \alpha)^{-p}), \quad \delta \to 0^+.
\]

It follows that for each \( \eta > 0 \) there exists \( M = M(\eta) \in \mathbb{N} \) such that

\[
|E|^p \leq \eta \sum_k \left( \frac{\mu(R(z_k))}{\omega^* - \alpha(z_k)} \right)^p, \quad j = 1, \ldots, M.
\]

By choosing now \( M \) big enough, we get \( \eta \) small enough for which

\[
\sum_k \left( \frac{\mu(R(z_k))}{\omega^* - \alpha(z_k)} \right)^p \leq |J^* T_{\mu_{j,l}} J|^p \leq |T_{\mu_{j,l}}|^p \leq |T_{\mu}|^p
\]

for each \( j = 1, \ldots, M \) and \( l = 1, \ldots, P^2 \). If now \( \mu_j = \left( \sum_k \chi_{R_{k}^{(j)}(\mu)} \right) \), then

\[
\sum_k \left( \frac{\mu_j(R_{k})}{\omega^* - \alpha(z_k)} \right)^p = \sum_k \left( \frac{\sum_{l=1}^{P^2} \mu(R_{k}^{(j,l)})}{\omega^* - \alpha(z_k)} \right)^p \leq \sum_k \sum_{l=1}^{P^2} \left( \frac{\mu(R_{k}^{(j,l)})}{\omega^* - \alpha(z_k)} \right)^p \leq P^2 |T_{\mu}|^p \quad j = 1, \ldots, M.
\]

This being true for each \( j \), we get the assertion for compactly supported \( \mu \). If \( \mu \) has not a compact support, then we may apply this to \( \mu_r = \chi_{D(0,r)} \mu \), and then use standard arguments to deduce \( |T_{\mu_r}|^p \leq |T_{\mu}|^p \) and finally let \( r \to 1^- \) to complete the proof. \( \square \)

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