A STUDY OF COMPATIBLE DEFORMATIONS IN NON-ARCHIMEDEAN GEOMETRY

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Abstract. In 2010, Hrushovski–Loeser showed that the Berkovich analytification of a quasi-projective variety over a non-Archimedean valued field admits a deformation retraction onto a finite simplicial complex. In this article, we adapt the tools and methods developed by Hrushovski–Loeser to study if such deformation retractions can be obtained to be compatible with respect to a given morphism. Amongst other results, we show that compatible deformation retractions exist over a constructible partition of the base and prove the general statement in the case of a morphism of relative dimension 1 where the target is a smooth connected curve.

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1. Introduction

Over the course of the twentieth century there have been several approaches, each with its merits, towards developing a theory of geometry over non-Archimedean valued fields similar to the theory over the complex numbers. However it was only in the late 1980’s that Berkovich proposed a theory that provided for the first time non-Archimedean analytic spaces with good topological properties similar to those that one takes for granted in complex geometry.

Berkovich proved several important results that shed light on the topological nature of such analytic spaces. In [1], he showed that Berkovich analytic spaces are locally compact and locally path connected. Furthermore, there is an analytification functor that takes varieties over a non-Archimedean complete field \( K \) to Berkovich \( K \)-analytic spaces and several GAGA type results that serve to connect the topology of the analytification of the variety to scheme theoretic properties of the variety. In [2], he showed that every smooth \( K \)-analytic space is locally contractible. More generally, if \( X \) is locally isomorphic to a strictly \( K \)-analytic domain of a smooth \( K \)-analytic space then \( X \) is locally contractible. The question of local contractibility and more general questions regarding the nature of the homotopy type of a general \( K \)-analytic space remained open until 2010, when Hrushovski and Loeser used techniques from Model theory to study the homotopy types of the Berkovich analytification of quasi-projective varieties over a valued field. In [7], they showed that the homotopy types of such analytifications were determined by finite simplicial complexes embedded in the analytifications. Furthermore, they proved that these spaces are locally contractible and there can be only finitely many distinct homotopy types of the analytifications of quasi-projective varieties that vary in a family. It is important to note that they did not require any assumption of smoothness on the underlying varieties.

The work of Hrushovski and Loeser in [7] offers a theory of non-Archimedean geometry which enables one to use powerful techniques and tools from Model theory such as the notion of definability and model theoretic compactness. A brief overview of [7] can be found in the excellent survey article [3]. Inspired by the results and ideas in [7], we study the extent to which the deformations of the analytifications of quasi-projective varieties onto finite simplicial complexes is functorial. More precisely, we study the validity of Statement 1.

By a variety over a field \( F \), we mean a reduced and separated scheme of finite type over \( F \). We fix a field \( K \) that is non-Archimedean non-trivially real valued, algebraically closed and complete.

**Statement 1**: Let \( \phi : V' \to V \) be a flat surjective morphism between quasi-projective \( K \)-varieties of finite type. There exist deformation retractions \( H : I \times V'^{\text{an}} \to V^{\text{an}} \) and \( H' : I \times V'^{\text{an}} \to V^{\text{an}} \) which are compatible with the morphism \( \phi^{\text{an}} \) i.e. the following diagram commutes.

\[
\begin{array}{ccc}
I \times V'^{\text{an}} & \xrightarrow{H'} & V^{\text{an}} \\
\downarrow{\text{ad} \times \phi^{\text{an}}} & & \downarrow{\phi^{\text{an}}} \\
I \times V^{\text{an}} & \xrightarrow{H} & V^{\text{an}}
\end{array}
\]

Furthermore, if \( e \) denotes the end point of the interval \( I \) then the images of the deformations \( \Upsilon := H(e, V^{\text{an}}) \) and \( \Upsilon' := H'(e, V'^{\text{an}}) \) are homeomorphic to finite simplicial complexes.

1.1. Hrushovski-Loeser spaces. As mentioned above, in order to study the homotopy types of certain non-Archimedean analytic spaces, Hrushovski and Loeser
introduced a model theoretic analogue of the Berkovich space. The first order properties of non-Archimedean valued fields can be described within the theory ACVF of algebraically closed valued fields. Any model of ACVF is an algebraically closed valued field \( L \) whose value group we denote \( \Gamma(L) \) and residue field we denote \( k(L) \). Let \( \text{val} : L \to \Gamma(L) \) denote the valuation. Note that we write the group structure on \( \Gamma(L) \) additively.

Let \( F \) be a valued field. Analogous to the analytification functor, to any variety \( V \) over \( F \), Hrushovski–Loeser associate a functor \( \hat{V} \) that takes a model \( L \) of ACVF extending \( F \) to the set \( \hat{V}(L) \) of \( L \)-definable stably dominated types that concentrate on \( V \). The set \( \hat{V}(L) \) can be endowed with a topology whose pre-basic open sets are of the form \( \{ p \in \hat{U} | \text{val}(f)(p) \in O \} \) where \( O \) is an open subspace of the value group \( \Gamma_{\infty}(L) \). \(^1\) \( U \) is a Zariski open subset of \( V \times_F L \), \( f \) is regular on \( U \), \( \text{val}(f) : U \to \Gamma_{\infty} \) is the function \( u \mapsto \text{val}(f(u)) \) and \( \text{val}(f)_w \) which is induced by \( \text{val}(f) \) maps types on \( U \) to types on \( \Gamma_{\infty} \). A map \( f : V' \to V \) of \( F \)-varieties induces a continuous map \( \hat{f} : \hat{V'} \to \hat{V} \). Furthermore, we have an injection \( V \hookrightarrow \hat{V} \) such that the topology induced on \( V \) is the valuative topology. More generally, this formalism allows us to define \( \hat{S} \) when \( S \) is any definable subset of \( V \) and endow it with the induced topology.

The space \( \hat{V} \) is closely related to the associated Berkovich space and homotopies constructed on \( \hat{V} \) induce homotopies on \( V^{an} \) (cf. [7, §14]). Furthermore, if \( L \) is an extension of \( F \) that is maximally complete and \( \Gamma(L) = \mathbb{R} \) then \( \hat{V}(L) = V_L^{an} \). Lastly, \( \hat{V} \) is almost always not definable, it is however strictly pro-\( F \)-definable i.e. a pro-object in the category of \( F \)-definable sets.

As stated previously, one of the goals of [7] is to study the homotopy type of the space \( \hat{V} \). Theorem 11.1.1 in loc.cit. implies in particular that given a quasi-projective \( F \)-variety \( V \) there exists a deformation retraction \( H : I \times \hat{V} \to \hat{V} \) such that the image of \( H \) is an iso-definable subset \( \Upsilon \subset \hat{V} \) that admits an \( F \)-definable homeomorphism with a definable subset of \( \Gamma_{\infty}^w \) where \( w \) is a finite \( F \)-definable set. The interval \( I \) is a generalized interval which means that it is obtained by glueing copies of \([0, \infty] \) end to end.

1.2. Compatible deformations exist generically. In §5, we prove a version of Statement 1 for Hrushovski–Loeser spaces that holds generically over the base. We show that if \( \phi : V' \to V \) is a morphism between quasi-projective varieties then there exists an open dense subspace \( \hat{U} \subset \hat{V} \) such that we have deformations \( H' \) and \( H \) of \( \hat{V} \) and \( \hat{U} \) respectively which are compatible with the morphism \( \phi \). Since our result holds only generically, we do not require that the morphism \( \phi : V' \to V \) be flat. The precise statement is as follows.

**Theorem 1.1.** Let \( \phi : V' \to V \) be a morphism between quasi-projective \( K \)-varieties whose image is dense. Let \( G \) be a finite algebraic group acting on \( V' \) which restricts to a well defined action along the fibres of the morphism \( \phi \) and \( \xi : V' \to \Gamma_{\infty} \) be a finite collection of \( K \)-definable functions. There exists an open dense subset \( U \subseteq V \) such that if \( V_U' := \phi^{-1}(U) \) then there exists a generalized interval \( I \), deformation retractions \( H : I \times \hat{U} \to \hat{U} \) and \( H' : I \times V_U' \to V_U' \) which satisfy the following properties.

1. The images of \( H' \) and \( H \) are iso-definable \( \Gamma \)-internal subsets of \( V_U' \) and \( \hat{U} \) respectively. Let \( H' \) and \( H \) denote the images of \( H' \) and \( H \) respectively.

2. The homotopy \( H' \) is invariant for the action of the group \( G \), respects the levels of the functions \( \xi \), and is compatible with the homotopy \( H \).

\(^1\) \( \Gamma_{\infty}(L) := \Gamma(L) \cup \{ \infty \} \).
(3) The homotopy $H'$ is Zariski generalizing.

(4) If the fibres of the morphism $\phi$ are pure over an open dense subset of $V$ and of dimension $n$ then for every $z \in Y$, $Y'_z$ is pure of dimension $n$ where the notion of dimension for $\Gamma$-internal sets is as introduced in [7, §8.3]. If $Y \subset V$ is an irreducible component then $Y \cap Y$ is pure of dimension $\dim(Y)$.

(5) If $Z \subset V$ is a Zariski closed subset such that for every irreducible component $Y$ of $V$, $Z \cap Y$ is strictly contained in $Y$ then $\hat{Z} \cap Y = \emptyset$.

(6) If the variety $V$ is integral, projective and normal then the deformation retraction $H$ extends to a deformation retraction $I \times \hat{V} \to \hat{V}$ which preserves the closed subset $V \setminus U$. Furthermore, if $d_{\text{ord}}: V \to \Gamma_\infty$ denotes the schematic distance [7, §3.12] to $V \setminus U$ then $H$ respects $d_{\text{ord}}$. (In the event that $V$ satisfies assertions (1)-(5), we assume that (6) is vacuously true.)

Theorem 1.1 implies the following theorem concerning the analytifications of morphisms between quasi-projective varieties. It can be easily deduced using [7, Corollary 14.1.6].

**Corollary 1.2.** Let $\phi: V' \to V$ be a surjective morphism between quasi-projective $K$-varieties. There exists a finite partition $V$ of $V$ into locally closed sub-varieties such that for every $W \in V$, there exists a generalized real interval $I_W$ and a pair of deformation retractions

$$H'_W: I_W \times V'_W \to V'_W$$

and

$$H_W: I_W \times W \to W$$

which are compatible with respect to the morphism $(\phi|_{V'_W})^{\text{an}}$ and whose images are homeomorphic to finite simplicial complexes.

1.2.1. **Sketch of the proof of Theorem 1.1.** If $V = \text{Spec}(K)$ then Theorem 1.1 is essentially [7, Theorem 11.1.1]. Recall that in this case, the construction of the homotopy $H'$ is the composition of inflation and curve homotopies built using the following commutative diagram

$$
\begin{array}{ccc}
V'_1 & \xrightarrow{\varphi'} & E & \xrightarrow{p} & F \\
\downarrow{\varphi'} & & \downarrow{b} & & \\
V' & \xrightarrow{g} & \mathbb{P}^m & & \\
\downarrow{\phi} & & & & \\
\text{Spec}(K) & & & & 
\end{array}
$$

where the square is cartesian, $b$ is the blow up of $\mathbb{P}^m$ at a point, $F := \mathbb{P}^{m-1} \subset \mathbb{P}^m$ and $p$ is the projection.

To prove Theorem 1.1, we make use of the tools developed in [7]. As a first step, we use Lemma 4.2 to reduce to the case where $V$ is integral, normal and projective, $V'$ is projective and there exists a Zariski open dense subset $U \subseteq V$ such that the fibres of the morphism $\phi$ are pure and equidimensional over $U$. In this situation,
after shrinking $U$, Lemma 4.3 implies the following diagram

\[
\begin{array}{ccc}
V'_U & \xrightarrow{g'_U} & E \times U \\
\downarrow & & \downarrow h_U \\
V_U & \xrightarrow{g_U} & \mathbb{P}^m \times U \\
\downarrow & & \downarrow \\
U & & \\
\end{array}
\]

where the square is cartesian, $b_U$ is of the form $b \times \text{id}_U$ where $b: E \to \mathbb{P}^m$ is the blow up at a point and $p_U := p \times \text{id}_U$ where $p: E \to F$ is the projection map.

The deformation retraction $H'$ as required by Theorem 1.1 is the composition of homotopies $H_{\mathcal{I}_f} \circ H_{\mathcal{I}_f} \circ H'_{\text{curve}_f} \circ H'_{\text{inff}} \circ H'_{\text{primary}}$. The homotopy $H'_{\text{curve}_f}$ is constructed on the Hrushovski–Loeser space associated to a suitable constructible subset of $\mathcal{V}_U$ while making use of the fact that $p'_U := p_U \circ g'_U$ is a fibration of curves. We can extend to a homotopy on the entire space $\mathcal{V}_U$ by composing with an inflation homotopy that makes use of the finite morphism $g'_U$ and by choosing an appropriate horizontal divisor over $U$ using Lemma 4.6 that guarantees that outside of this divisor, for every $u \in U$, the morphism $\mathcal{V}_{Uu} \to \mathcal{E}_U$ is a homeomorphism locally around the simple points. We hence get a homotopy on $\mathcal{V}_{U}$ such that the fibres of the morphism $\mathcal{P}_{U}$ restricted to this image are $\Gamma$-internal. We then show as in [7, Theorem 6.4.4] that we can reduce to proving Theorem 1.1 for a certain pseudo-Galois cover of $F \times U$. Note that it is not automatic that a homotopy of $\mathcal{V}_U$ that restricts to a well defined homotopy on the exceptional divisor descends to a homotopy on $\mathcal{V}_U$. To rectify this, we begin with an inflation homotopy $H'_{\text{inff}-\text{primary}}$ on $\mathcal{V}_U$ that enables us to escape the exceptional divisor. The composition of the sequence of inflation and curve homotopies does not fix their image. We construct as in [7, §11.5] a tropical homotopy $H'_{\mathcal{I}_f}$ on an iso-definable $\Gamma$-internal subset of $\mathcal{V}_U$ such that the composition $H'$ is indeed a deformation retraction. At every stage, if necessary, we shrink the open set $U$.

An important point to note is that the homotopies $(H'_{\text{inff}-\text{primary}})$, $H'_{\text{inff}}$, $H'_{\text{curve}_f}$ and $H'_{\mathcal{I}_f}$ move along the fibres of the morphism $(\phi_U)$, $\phi_U \circ b_U$. Hence, we see that $H'$ first seeks to deform $\mathcal{V}_U$ into a subset whose fibres are $\Gamma$-internal over $U$ followed by a deformation induced by an appropriate deformation retraction of the base. This is the content of Proposition 5.2.

1.3. When the base is a curve. In §6, we study Statement 1 when the base is a smooth connected curve. While Theorem 1.1 shows the existence of deformations generically over the base, Theorem 6.1 proves that if the base is a smooth connected curve then we can get compatible homotopies over Zariski open neighbourhoods of an arbitrary point. The precise statement is Theorem 6.1. As before, [7, Corollary 14.1.6] implies the following analogous statement for Berkovich spaces.

**Corollary 1.3.** Let $S$ be a smooth connected $K$-curve and $X$ be a quasi-projective $K$-variety. Let $\phi: X \to S$ be a surjective morphism such that every irreducible component of $X$ dominates $S$. Let $s \in S(K)$. There exists a Zariski open subset $U \subset S$ containing $s$ and a pair of deformation retractions

$$H': I \times X^\text{an}_U \to X^\text{an}_U$$
and

\[ H : I \times U^{an} \to U^{an} \]

which are compatible with respect to the morphism \( \phi^{an} \) and whose images are homeomorphic to finite simplicial complexes. The interval \( I \) is a generalized real interval.

One observes from the proof of Theorem 1.1 that the existence of the curve and inflation homotopies relies on the existence of divisors which are flat over the base. The reason we can prove the stronger result when the base is of dimension 1 is a consequence of the following fact. Let \( \phi : X \to S \) be as in the statement of Theorem 6.1 and assume in addition that the fibres of \( \phi \) are pure. Let \( D \subset X \) be Zariski closed and generically of codimension 1 and pure in the fibres of \( \phi \) i.e. there exists an open set \( U \subset X \) such that if \( s \in U \) then \( D \cap X_s \) is of codimension 1 in \( X_s \). We suppose in addition that if \( \eta \in S \) is the generic point of \( S \) then \( D_\eta = D \) where \( D_\eta \) denotes the Zariski closure of \( D_\eta \) in \( X \). It then follows that \( D \) is of codimension 1 and pure in every fibre of \( \phi \). This follows from dimensionality concerns.

1.3.1. When \( \phi \) is of relative dimension 1. One of the advantages of working within the framework of Hrushovski–Loeser is the flexibility it allows us when constructing our homotopies. For instance, the construction of the inflation homotopy in \([7, \text{Lemma 10.3.2}]\) makes use of a suitable cut-off to extend a homotopy to the entire space. This is the motivation behind Lemma 6.6 which shows how we may extend homotopies that move along the fibres of a projective morphism in such a way that the extensions coincide with the original homotopy over a reasonably large subspace of the base. This result is the key ingredient in our proof of Theorem 6.9 which verifies Statement 1 when \( S \) is a smooth connected curve, the morphism \( \phi \) is flat and the fibres of \( \phi \) are of dimension 1. Since the fibres are of dimension 1, we go through the steps in the proof of Theorem 6.1 to find that for every \( s \in S \), there exists a neighbourhood \( U \) of \( s \) and a homotopy \( h_U : [0, \infty] \times X_U \to \tilde{X}_U/U \) whose image is relatively \( \Gamma \)-internal. We then extend suitable cut-offs of the homotopies \( h_U \) to the whole of \( X \) using Lemma 6.6. As before, we have the following analogous result for Berkovich spaces.

**Corollary 1.4.** Let \( S \) be a smooth connected \( K \)-curve and \( X \) be a quasi-projective \( K \)-variety. Let \( \phi : X \to S \) be a surjective morphism such that every irreducible component of \( X \) dominates \( S \). We assume in addition that the fibres of \( \phi \) are of dimension 1. There exists a pair of deformation retractions

\[ H' : I \times X^{an} \to X^{an} \]

and

\[ H : I \times S^{an} \to S^{an} \]

which are compatible with respect to the morphism \( \phi^{an} \) and whose images are homeomorphic to finite simplicial complexes. The interval \( I \) is a generalized real interval.

1.4. Locally trivial morphisms. In §7, we study the validity of Statement 1 in the context of locally trivial morphisms of relative dimension 1. A morphism \( \phi : X \to S \) is said to be locally trivial if for every point \( s \in S(K) \), there exists a Zariski open neighbourhood \( U \) of \( s \) and a \( U \)-isomorphism \( f_U : X_U \to V \times U \) where \( V \) is a quasi-projective \( K \)-variety that is independent of \( s \) and \( U \). In the event that \( \tilde{V} \) admits a homotopy whose image is iso-definable \( \Gamma \)-internal and the generalized interval over which the homotopy runs is \([0, \infty]\), we can employ Lemma 6.6 to verify the existence of compatible homotopies on \( \tilde{X} \) and \( \tilde{S} \) whose images are iso-definable and \( \Gamma \)-internal. This is the content of Corollary 7.2.

In the case when \( V = \mathbb{P}^1_K \), we refer to the morphism as a \( \mathbb{P}^1 \)-bundle. The class of \( \mathbb{P}^1 \)-bundles allows us examples of objects for which relative homotopies can be
constructed explicitly. We choose a system of coordinates on \( \mathbb{P}^1 \). Recall from [7, §7.5] that if we are given a divisor \( D \) on \( \mathbb{P}^1 \) then we have a canonical deformation retraction of \( \hat{\mathbb{P}}^1 \) onto the convex hull of \( D \). In the case of a trivial family i.e. if \( \phi: X \rightarrow S \) is isomorphic over \( S \) to \( S \times \mathbb{P}^1 \) and \( D \subset X \) is a divisor that is finite and surjective over \( S \), Hrushovski and Loeser construct deformations of \( \hat{X} \) that move along the fibres of the morphism \( \hat{\phi} \) and whose image over a simple point \( s \in S \) is the convex hull of the divisor \( D_s \) in \( \hat{\mathbb{P}}^1 \times \{ s \} \). In Theorem 7.7, we show that the existence of a suitable horizontal divisor \( D \) on \( X \) enables us to glue the standard homotopies on each trivializing chart with respect to the restriction of the divisor \( D \) to that chart.

1.5. Related work. An interesting variation of Statement 1 can be found in the tropical geometry literature. In [11], Ulirsch constructs a functorial tropicalization morphism for fine and saturated log schemes. More precisely, let \( k \) be a trivally valued field. Given a fine and saturated log scheme \( X \) which is locally of finite type over \( k \), we can define a generalized cone complex \( \Sigma_X \) and construct a continuous tropicalization morphism \( \text{trop}_{X}: X^\Sigma \rightarrow \Sigma_X \) where \( \Sigma_X \) is the canonical extension of \( \Sigma_X \). Theorem 1.1 of loc.cit. proves an analogue of Statement 1 in this context by showing that if \( f: X' \rightarrow X \) is a morphism of fine and saturated log schemes locally of finite type over \( k \) then we have an induced morphism of generalized cone complexes \( \Sigma(f): \Sigma_{X'} \rightarrow \Sigma_X \) such that \( \Sigma(f) \circ \text{trop}_{X'} = \text{trop}_X \circ f^\Sigma \).

Acknowledgements: The work presented here is inspired and motivated by the theory of non-Archimedean geometry introduced in [7]. We are greatly indebted to the tools and techniques developed in this paper. We are grateful to François Loeser for his patient explanations of several difficult parts of loc.cit. and his advice regarding this project. We must also thank Yimu Yin and Andreas Gross for the many discussions. We are grateful as well to Kavli, IPMU for the support during the writing of the article.

2. Hrushovski-Loeser spaces

Model theory seeks to understand mathematical structures, the first order sentences that hold true in these structures and those sets which are defined by first order formulae. Thus, model theory gives us tools applicable in a variety of settings alongside providing us a framework which permits us to relate different mathematical structures. The Hrushovski-Loeser space allows us to make use of these powerful tools to study the topology of objects that occur in non-Archimedean geometry. This approach which was first introduced in 2010 opens up a completely new perspective when thinking of certain Berkovich spaces. In what follows, we provide a brief introduction of the Hrushovski-Loeser space. The details can be found in [7]. We assume that the reader is familiar with the fundamental notions of model theory such as languages, structures, formulae, theories and types. Chapters 1 - 4 of [8] explain these concepts perfectly. The following two sections are more or less reproductions of parts of the paper [13].

2.1. The theory ACVF. Hrushovski-Loeser spaces can be thought of as a model theoretic analogue of Berkovich spaces, developed within the framework of ACVF - the theory of algebraically closed non-trivially valued fields. Note that working within ACVF does not prevent us from studying objects which defined over valued fields which are not algebraically closed. Let us now introduce the language \( L_{k,\Gamma} \) to describe ACVF and then explain why we extend this language to \( L_\omega \).
Definition 2.1. The multi-sorted language $\mathcal{L}_{k,\Gamma}$ is given by specifying the following set of data.

1. A set $\mathcal{S}$ of sorts - consisting of a valued field sort $VF$, a residue field sort $k$ and a value group sort $\Gamma$.
2. A set of function symbols $\mathcal{F} = \{+_{VF}, \times_{VF}, -_{VF}, +_k, \times_k, -_k, +_\Gamma, \text{val}, \text{Res}\}$ which are defined on appropriate sorts. For instance, $+_{VF}: VF \times VF \rightarrow VF$, val: $VF^* \rightarrow \Gamma$, Res: $VF^2 \rightarrow k$ and so on.
3. A set of relation symbols $\mathcal{R} = \{<\}$ defined on specific sorts. For instance, $< \subset \Gamma \times \Gamma$.
4. A set of constant symbols $\mathcal{C} = \{0_{VF}, 1_{VF}, 0_k, 1_k, 0_\Gamma\}$ which are sort specific.

ACVF is that $\mathcal{L}_{k,\Gamma}$-theory whose models are algebraically closed non-trivially valued fields. Formally, we provide the following definition.

Definition 2.2. The $\mathcal{L}_{k,\Gamma}$-theory ACVF consists of the set of sentences such that if $M$ is a model of ACVF then $VF(M)$ is an algebraically closed valued field with valuation val, value group $\Gamma(M)$ and residue field $k(M)$. It follows that the value group $\Gamma(M)$ is a non-trivial dense, linear ordered abelian group, the residue field $k(M)$ is an algebraically closed field, the map val: $VF(M)^* \rightarrow \Gamma(M)$ is a surjective homomorphism that satisfies the strong triangle inequality i.e. if $x, y \in VF(M)^*$ then $\text{val}(x + y) \geq \min\{\text{val}(x), \text{val}(y)\}$.

Lastly, the function Res maps $(x, y) \in VF^2$ to the residue in $k(M)$ of $xy^{-1}$ if $\text{val}(xy^{-1}) \geq 0$ and 0 otherwise, such that Res is surjective, $\text{Res}(_{-1})$ is a homomorphism etc.

Notation: In this section, let $K$ be a model of ACVF. We often abuse notation and use $K$ itself to denote the valued field $VF(K)$ and $k$ to denote the residue field $k(K)$.

Remark 2.3. (1) A classical result of A. Robinson states that the completions of ACVF are the theories ACVF$_{p,q}$ where the residue field has characteristic $p$, the valued field has characteristic $p$ and either $p = 0$ or $p \neq 0$ and $q = p$. Complete theories are defined in [8, Definition 2.2.1].

2. An important point to note is that ACVF admits elimination of quantifiers in the language $\mathcal{L}_{k,\Gamma}$. Quantifier elimination is defined in [8, Definition 3.1.1].

Although the language $\mathcal{L}_{k,\Gamma}$ describes ACVF, it does not eliminate imaginaries. To rectify this problem we expand the language $\mathcal{L}_{k,\Gamma}$ by adding certain sorts which we refer to as geometric sorts. We provide more details below.

2.2. Definable sets. We introduce the notion of a definable set as in [7, Section 2.1] which is of a more geometric flavour than that presented in [8]. Let $\mathcal{L}$ be a multi-sorted language and $T$ be a complete $\mathcal{L}$-theory which admits quantifier elimination. We fix a large saturated model $U$ of $T$ and assume that any model of $T$ which is of any interest to us will be contained in $U$. Furthermore, we can assume that if $A \subset U$ is of cardinality strictly less than the cardinality of $U$ then any type consisting of formulas defined over $A$ has a realization in $U$. This technical condition will be of use to us when we discuss the notion of definable types in §2.4.2. By a set of parameters, we will mean a small subset of $U$. Recall that a subset is said to be small if it is of cardinality strictly smaller than the degree of saturation of $U$ which we assume to be $\text{card}(U)$.

Let $C \subset U$ be a parameter set. We extend the language $\mathcal{L}$ to $\mathcal{L}_C$ by adding constant symbols corresponding to the elements of $C$. We can expand the theory $T$
to $T_C$ whose models are those models of $T$ that contain the set $C$. An $\mathcal{L}_C$-formula $\phi$ can be used to define a functor $Z_\phi$ from the category whose objects are models of $T_C$ and morphisms are elementary embeddings to the category of sets. Suppose that the formula $\phi$ involves the variables $x_1, \ldots, x_m$ where for every $i \in \{1, \ldots, m\}$, $x_i$ is specific to a sort $S_i$ respectively. Then given an $\mathcal{L}_C$-model $M$ of $T$ i.e. a model of $T$ that contains $C$, we set \[ Z_\phi(M) = \{ \bar{a} \in S_1(M) \times \cdots \times S_m(M) | M \models \phi(\bar{a}) \}. \]

Clearly, $Z_\phi$ is well defined.

**Definition 2.4.**

- A $C$-definable set $Z$ is a functor from the category whose objects are models of $T_C$ and morphisms are elementary embeddings to the category of sets such that there exists an $\mathcal{L}_C$-formula $\phi$ and $Z = Z_\phi$.
- Let $X$ and $Y$ be $C$-definable sets. A morphism $f : X \to Y$ is said to be $C$-definable if its graph is a $C$-definable set.

The definable set $Z_\phi$ is fully determined by evaluating it on $U$, i.e. by the set $Z_\phi(U)$.

**Example 2.5.** Consider the following examples of three different classes of definable sets in ACVF using the language $\mathcal{L}_k, \Gamma$. Observe that these objects appear naturally in the study of Berkovich spaces, tropical geometry and algebraic geometry.

1. Let $K$ be a model of ACVF and $A \subset K$ be a set of parameters. Let $n \in \mathbb{N}$. A semi-algebraic subset of $K^n$ is a finite boolean combination of sets of the form \[ \left\{ x \in K^n | \text{val}(f(x)) \geq \text{val}(g(x)) \right\} \]
where $f, g$ are polynomials with coefficients in $A$. One verifies that semi-algebraic sets extend naturally to define $A$-definable sets in ACVF.

2. Let $G = \Gamma(K)$ where $K$ is as above. A $G$-rational polyhedron in $G^n$ is a finite Boolean combination of subsets of the form \[ \{(a_1, \ldots, a_n) \in G^n | \sum_i z_ia_i \leq c \} \]
where $z_i \in \mathbb{Z}$ and $c \in G$. Such objects extend naturally to define a $K$-definable subset of $\Gamma^n$ where $\Gamma$ is the value group sort.

3. Any constructible subset of $k^n$ gives a definable set in a natural way.

**Remark 2.6.** Let $F$ be a non-trivially valued field and let $V$ be an $F$-variety. The variety $V$ defines in a natural way a functor $Z_V$ as follows. Let $K$ be a model of ACVF that extends $F$. We then set \[ Z_V(K) := V \times_F K(K). \]

We add appropriate sorts to ACVF so that any such functor associated to a variety is a definable set.

2.3. **The language $\mathcal{L}_\mathcal{G}$.** As mentioned earlier, the theory ACVF does not eliminate imaginaries in the language $\mathcal{L}_k, \Gamma$. In this section, we briefly introduce an extension $\mathcal{L}_\mathcal{G}$ of $\mathcal{L}_k, \Gamma$ within which ACVF eliminates imaginaries. In fact, ACVF also admits quantifier elimination in the language $\mathcal{L}_\mathcal{G}$. The Hrushovski-Loeser spaces are defined in the language $\mathcal{L}_\mathcal{G}$ and require elimination of imaginaries for some of their fundamental properties, for instance pro-definability (cf. [7, Lemma 2.5.1, Theorem 3.1.1]).
Definition 2.7. A theory $T$ is said to eliminate imaginaries if for any model $M \models T$, any collection of sorts $S_1, \ldots, S_m$ and any $\emptyset$-definable equivalence relation $E$ on $S_1(M) \times \ldots \times S_m(M)$ there exists a definable function $f$ on $S_1(M) \times \ldots \times S_m(M)$ whose codomain is a product of sorts and is such that $aEb$ if and only if $f(a) = f(b)$.

Suppose a complete theory $T$ in a language $\mathcal{L}$ does not eliminate imaginaries. We can then extend $T$ to a complete theory $T^{eq}$ over a language $\mathcal{L}^{eq}$ so that $T^{eq}$ eliminates imaginaries. Indeed, for every $\emptyset$-definable equivalence relation $E$ on a product of sorts $S_1 \times \ldots \times S_m$, we add a sort to $\mathcal{L}$ corresponding to the quotient $(S_1 \times \ldots \times S_m)/E$ and a function symbol representing the map $\overline{\pi} \mapsto \overline{\pi}/E$. This gives the language $\mathcal{L}^{eq}$. Every model $M$ of $T$ extends canonically to a model $M^{eq}$ of $T^{eq}$. The new sorts that were added to $\mathcal{L}$ are referred to as the imaginary sorts and their elements are called imaginaries.

Given a definable set $X$ in $T$, we can associate to it an element in $\mathbb{U}^{eq}$ called its code as follows. Suppose

$$X(\mathbb{U}) = \{x \in \mathbb{U}^n | \mathbb{U} \models \phi(x, a)\}$$

where $x$ and $a$ in the definition are tuples. We define an equivalence relation by setting $y_1Ey_2$ if $\forall x (\phi(x, y_1) \leftrightarrow \phi(x, y_2))$. The element $a/E$ belongs to $\mathbb{U}^{eq}$ and we refer to it as the code of $X$.

The language $\mathcal{L}^{eq}$ is obtained from $\mathcal{L}_{k, \Gamma}$ by adjoining to it the geometric sorts $S_n$ and $T_n$ for $n \geq 1$. The sort $S_n$ is the collection of codes for all free rank $R$-submodules of $K^n$ where $R$ is the valuation ring given by $\{x \in K | \text{val}(x) \geq 0\}$. Given $s \in S_n$ for some $n$, let $\Lambda(s)$ denote the corresponding free rank $n$ $R$-submodule of $K^n$. If $M$ denotes the maximal ideal of $R$ then let $T_n$ be the set of codes for the elements in $\bigcup_{s \in S_n} \Lambda(s)/\mathbb{M}\Lambda(s)$.

2.4. Stably dominated types. Let $F$ be a valued field and let $V$ be a $F$-variety. The Hrushovski-Loeser space $\hat{V}$ associated to $V$ is the space of stably dominated types that concentrate on $V$. We begin with a discussion of the notion of a type followed by that of a definable type which is central to this story. We closely follow the treatment in §2.3 of [7].

2.4.1. Types. Let $\mathcal{L}$ be a language and $T$ be a complete $\mathcal{L}$-theory. If $z$ is a set of variables, we use $F_z$ to denote the set of $\mathcal{L}$-formulae with variables in $z$ up to equivalence in the theory $T$.

Definition 2.8. An $n$-type $p = p(x_1, \ldots, x_n)$ is a subset of $F_{\{x_1, \ldots, x_n\}}$ such that $p(x_1, \ldots, x_n)$ is satisfiable in $T$ i.e. there exists a model $M$ of $T$ and $(a_1, \ldots, a_n) \in M^n$ such that for every $f \in p$, $M \models f(a_1, \ldots, a_n)$. Furthermore, we say that the type $p$ is complete if for every $\phi \in F_{\{x_1, \ldots, x_n\}}$ either $\phi \in p$ or $\neg \phi \in p$.

Remark 2.9. (1) Let $p$ be a complete $n$-type. Let $M$ be a model of $T$ such that there exists $a := (a_1, \ldots, a_n)$ that satisfies $p$. Since $p$ is complete, observe that

$$p = \{f \in F_{\{x_1, \ldots, x_n\}} | M \models f(a)\}.$$ 

In this case, we say that $a$ realizes the type $p$. In general, given an $n$-tuple $\alpha \in \mathbb{U}^n$, we write $tp(\alpha)$ to denote the type generated by $\alpha$ i.e.

$$tp(\alpha) := \{f \in F_{\{x_1, \ldots, x_n\}} | \mathbb{U} \models f(\alpha)\}.$$ 

(2) The definition above concerns types consisting of formulae without parameters. However, it is natural and necessary to work with types defined over a set of parameters. Suppose $A \subset \mathbb{U}$ is a small set of parameters then a complete $n$-type defined over $A$ will be a complete $n$-type in the language $\mathcal{L}_A$. Using this formalism and the notion of realizations, one sees that types
provide us a tool with which to probe the first order properties of elements in elementary extensions.

Example 2.10. Let \( \mathcal{L}_r := \{+, -, 0, 1\} \) denote the language of rings where \(+, -, 0, 1\) are the binary function symbols and 0, 1 are the constant symbols. Let \( q \) be a prime number or zero. Let \( \text{ACF}_q \) denote the theory of algebraically closed fields of characteristic \( q \) i.e. the \( \mathcal{L}_r \)-theory whose models are algebraically closed fields of characteristic \( q \). In this context, the complete \( n \)-types in \( \text{ACF}_q \) take on a recognizable form from algebraic geometry.

Let \( L \) be a model of \( \text{ACF}_q \). Let \( L' \subset L \) be a subfield. Let \( p = p(x_1, \ldots, x_n) \) be a type defined over \( L' \). We can associate to \( p \) an ideal \( I_p \subset L'[x_1, \ldots, x_n] \) by setting

\[
I_p := \{ f \in L'[x_1, \ldots, x_n] \mid f(x_1, \ldots, x_n) = 0 \in p \}.
\]

It can be shown that the association \( p \mapsto I_p \) defines a bijection between the set of complete \( n \)-types defined over \( F \) and the prime ideals of \( L'[x_1, \ldots, x_n] \) (cf. [8, Example 4.1.14]). Given an ideal \( I \subset L'[x_1, \ldots, x_n] \), let \( Z(I) \subset \text{Spec}(L'[x_1, \ldots, x_n]) \) denote its vanishing locus. We say that \( p \) is the \textit{generic type} of the irreducible closed subvariety \( Z(I_p) \).

2.4.2. \textbf{Definable types.} As in the previous section, we work in the complete \( \mathcal{L} \)-theory \( T \). Let \( A \subset U \) be a small set of parameters. Let \( z \) be a set of variables and \( \mathcal{F}_z^A \) denote the set of \( \mathcal{L}_A \) formulae with free variables in \( z \) upto equivalence in \( T_A \). Let \( p \) be a complete \( n \)-type defined over \( A \). Since for every \( \phi \in \mathcal{F}_z^A \), either \( \phi \notin p \) or \( \phi \in p \), one may think of \( p \) as a Boolean retraction from \( \mathcal{F}_z^A \) to the two element Boolean algebra. Equivalently, if \( U \) is an \( A \)-definable set whose formula belongs to \( p \), then one can see \( p \) as a uniform decision to include or exclude \( A \)-definable subsets \( V \) of \( U \) according to whether \( a \in V(U) \) or \( a \notin V(U) \) where \( a \) is a realization of \( p \). Note however that these decisions are restricted to only those subsets of \( U \) that are \( A \)-definable. To broaden the scope of this definition to express decisions on subsets which are not necessarily \( A \)-definable, we introduce the notion of a definable type. Before doing so, we discuss an example we hope will be illuminating.

Example 2.11. As in Example 2.10, let \( q \) be a prime number or zero. Let \( \mathcal{L}_r \) be the language of rings and \( \text{ACF}_q \) denote the theory of algebraically closed fields of characteristic \( q \). Let \( L \) be a field of characteristic \( q \). In Example 2.10, we showed that there exists a bijection between the complete \( n \)-types defined over \( L \) and the irreducible sub-varieties of \( \mathbb{A}^n_L \). Let \( p \) denote the complete 1-type defined over \( L \) that corresponds to \( \mathbb{A}^n_L \). By this we mean that \( Z(I_p) = \mathbb{A}^n_L \) where \( I_p \subset L[x] \) is the ideal associated to \( p \) and defined in Example 2.10.

Let \( L' \) be an extension of \( L \) and \( p' \) denote the complete 1-type corresponding to \( \mathbb{A}^n_{L'} \). One sees that if one were to restrict \( p' \) to \( L \) i.e. consider only those formulae in \( p' \) with parameters in \( L \) then we get \( p \). In other words,

\[
p'_{|L} = p.
\]

The geometric object \( \mathbb{A}^1_L \) can be defined over any field and a type cannot fully express this flexibility. We see that the correct notion is that of a definable type which provides us with a compatible family of types, each element of which is associated to a model of \( \text{ACF} \).

Definition 2.12. Let \( x = \{x_1, \ldots, x_m\} \). An \( \emptyset \)-\textit{definable type} is a function

\[
d_{p;x} : \mathcal{F}_{x,y_1,\ldots} \to \mathcal{F}_{y_1,\ldots}
\]

such that for any \( y = \{y_1, \ldots, y_n\} \), \( d_{p;x} \) restricts to a Boolean retraction \( \mathcal{F}_{x,y} \to \mathcal{F}_y \). If \( A \subset U \) is a set of parameters then an \( A \)-definable type is a \( \emptyset \)-definable type in the theory \( T_A \).
Remark 2.13. Let $A \subset U$ be a small subset and let $p$ be an $A$-definable type.

(1) The definable type $p$ provides us with a compatible family of types. Indeed, if $M$ is a model of $T$ that contains $A$, then $p_M$ is the type defined over $M$ consisting of those formulae $\phi(x,b_1, \ldots, b_n)$ such that $M \models d_p, \phi(b_1, \ldots, b_n).

(2) Let $X$ be a definable set. We say that $p$ concentrates on $X$ if all the realizations of the type $p|_U$ belong to $X$.

(3) Let $X$ and $Y$ be $A$-definable sets and $f: X \to Y$ be an $A$-definable map.

Let $S_{def,X}^A$ denote the set of $A$-definable types on $X$. The map $f$ induces a map $f^*: S_{def,X}^A \to S_{def,Y}^A$ such that $d_{f^*(p)}(y, z) = d_p(x, f(x, z)).$

2.4.3. Stably dominated types. The definition of a stably dominated type for a general theory $T$ is slightly involved and not necessary for our purposes. We may hence restrict our attention to working within the theory ACVF in the language $L_G$.

Definition 2.14. Let $A \subset U$ be a set of parameters. A type $p$ over $A$ is said to be almost orthogonal to $\Gamma$ to $\Gamma$ if for any realization $a$ of $p$, $\Gamma(A(a)) = \Gamma(A)$ where $A(a)$ denotes the definable closure of the set $A \cup \{a\}$ and $\Gamma(A) := \Gamma(U) \cap dcl(A)$. An $A$-definable type $p$ is said to be orthogonal to $\Gamma$ if for every structure $B$ that contains $A$, the type $p|_B$ is almost orthogonal to $\Gamma$.

A stably dominated type is a definable type which in some sense does not enlarge the value group sort. More precisely, we mean the following.

Proposition 2.15. [7, Proposition 2.9.1] In ACVF, an $A$-definable type $p$ is stably dominated if and only if it is orthogonal to $\Gamma$.

Remark 2.16. Let $A \subset U$ be a small set of parameters and let $X$ and $Y$ be $A$-definable sets. Let $g: X \to Y$ be an $A$-definable morphism. Recall that we have a map $g_*: S_{def,X}^A \to S_{def,Y}^A$. One checks that the map $g_*$ restricts to a well-defined function from the set of $A$-definable stably dominated types that concentrate on $X$ to the set of $A$-definable stably dominated types that concentrate on $Y$.

Let $f: X \to \Gamma$ be an $A$-definable function. Suppose $p$ is a stably dominated type that concentrates on $X$ and is defined over $A$. One verifies from the definition that $f_*(p)$ concentrates at a point i.e. if $M$ is a model of ACVF then $(f_*(p))|_M$ contains the formula $x = a$ for some $a \in \Gamma(A)$.

Stably dominated types in ACVF are controlled by the residue field which is the stable part of the theory. A precise formulation of this can be found in [5, §2.2]. For an introduction to stability theory see the chapter by M. Ziegler in [9].

Example 2.17. Let $a \in VF(U)$ and $\alpha \in \Gamma(U)$. Let $B(a, \alpha)$ denote the definable set such that if $M$ is a model of ACVF that contains $\{a, \alpha\}$ then

$$B(a, \alpha)(M) := \{x \in M| \text{val}(x - a) \geq \alpha\}.$$ 

In other words, $B(a, \alpha)$ is the closed ball around $a$ of radius $\alpha$.

We can associate to $B(a, \alpha)$ a stably dominated type called its generic type which we denote $p_{B(a, \alpha)}$. The definable type $p_{B(a, \alpha)}$ is determined by a definable type concentrated on a geometric object over the residue field sort in the following sense. Let us suppose $a = 0$ and $\alpha = 0$ which corresponds to the closed unit ball. Let red: $B(0, 0) \to k$ be the reduction map i.e. red$(x) := \text{Res}(x, 1)$. Let $L$ be a model of ACVF. We then have that $b$ is a realization of $(p_{B(a, \alpha)}L)$ if and only if red$(b)$ is a realization of the generic type of $A^1_{k(L)}$.

2.5. The Hrushovski-Loeser space. We define the Hrushovski-Loeser space and provide a fundamental example of such an object.
Remark 2.20. \(\hat{\text{map}}\) then the map \(Y\) is a structure that contains denote \(A\) open sets are the \(M\) well defined map \(\hat{\text{map}}\).

Example 2.21. the resulting space would be discrete. \(M\) description can be deduced from Holly’s theorem \([\text{Holly’s theorem}]\). Restrict this definition to \((1)\)

Definition 2.19. \((1)\) A pre-basic open set of \(\hat{\text{V}}(U)\) is of the form \(\{p \in \hat{\text{O}}|\text{val}(f)_*, (p) \in W\}\)

where \(O \subset V(U)\) is a Zariski open subset of \(V\) with parameters in \(U\), \(f\) a regular function on \(O\) defined over \(U\) and \(W\) an open subset of \(\Gamma_{\infty}(U)\). Here \(\text{val}(f)\) denotes the definable map given by \(O(U) \to \text{VF}(U) \to \Gamma_{\infty}(U)\). (As \(\text{val}(f)_*(p)\) is a stably dominated type on \(\Gamma_{\infty}\), it is constant.)

(2) The set \(\hat{\text{V}}(U) \times \Gamma_{\infty}'(U)\) is given the product topology and if \(X(U) \subset \hat{\text{V}}(U) \times \Gamma_{\infty}'(U)\) then we let \(X(U)\) have the subspace topology.

Remark 2.20. Let \(C \subset U\) be a small set of parameters. Let \(f: X \to Y\) be a \(C\)-definable map of \(C\)-definable sets. The map \(f_*: S_{\text{def}, X}^U \to S_{\text{def}, Y}^U\) restricts to a well defined map \(\hat{f}: \hat{X} \to \hat{Y}\). Furthermore, if \(X(U) \subset V(U)\) for some variety \(V\), \(Y(U) \subset V'(U)\) for some variety \(V'\) and \(r \in \mathbb{N}\) and \(f\) is the restriction of a regular map then the map \(\hat{f}\) is continuous.

Let \(X\) be an \(F\)-definable subset contained in \(\text{VF}^n \times \Gamma_{\infty}'\) for some \(n, l \in \mathbb{N}\). If \(M\) is a structure that contains \(F\) then we endow \(\hat{X}(M)\) with the topology whose open sets are the \(M\)-definable open sets of \(\hat{X}\). Note that it does not make sense to provide \(\hat{X}(M)\) with the subspace topology via the inclusion \(\hat{X}(M) \subset \hat{X}(U)\) since the resulting space would be discrete.

Note that if \(V\) is an algebraic variety defined over \(F\) then the space \(\hat{V}(M)\) is Hausdorff for every model \(M\) of \(\text{ACVF}\) that extends \(F\).

Example 2.21. Observe that the affine line defines an \(\emptyset\)-definable set which we denote \(\hat{k}\). (cf. Remark 2.6). As a set \(\hat{k}(U)\) consists of the generic types of balls \(pB(a, \alpha)\) (cf. Example 2.17) where \(a \in U\) and \(\alpha \in \Gamma(U)\) as well as the simple points \(b \in U\).

Let \(M\) be a model of \(\text{ACVF}\). The topology of the space \(\hat{k}(M)\) can be described as follows. Let \(m \in M\). By an open ball around \(m\) of radius \(\alpha\), we mean the set \(O(m, \alpha) := \{x \in U|\text{val}(x - m) > \alpha\}\). A fundamental system of open neighbourhoods in \(\hat{k}(M)\) of \(m\) is given by the family \(\{O(m, \alpha)(M)\}_{\alpha \in \Gamma(M)}\). Let \(B := B(a, \alpha)\) be a closed ball with \(\alpha \in \Gamma(M)\). A fundamental system of open neighbourhoods of \(pB\) is given by sets of the form \(O\) where \(O\) is a \(M\)-definable open ball from which finitely many \(M\)-definable closed balls are removed and \(O\) contains the point \(pB\). This description can be deduced from Holly’s theorem \([\text{Holly’s theorem}]\) which effectively says that a definable subset of \(\text{VF}\) admits a swiss cheese decomposition i.e. it is the disjoint union of balls from which finitely many sub-balls are removed. By a ball in \(\text{VF}\), we mean a set of the form \(\{x \in \text{VF}|\text{val}(x - a) \equiv \alpha\}\) where \(\kappa \in \{>, =\}\) and \(\alpha \in \Gamma_{\infty}\).

After fixing coordinates, we have the following equality of sets \(\hat{\mathbb{P}}(U) = \hat{k}(U) \cup \{\infty\}\).
where $\infty$ is an $\emptyset$-definable point. The space $\hat{\mathbb{A}}^1(M)$ is an open subspace of $\hat{\mathbb{P}}^1(M)$. The open neighbourhoods of $\infty$ in $\hat{\mathbb{P}}^1(M)$ consist of the complements in $\hat{\mathbb{P}}^1(M)$ of the spaces $B(0, \alpha)(M)$ where $\alpha \in \Gamma(M)$.

**Remark 2.22.** Let $C \subset U$ be a set of parameters. Let $\text{Def}_C$ be the category whose objects are those sets which are definable with parameters in $C$ and morphisms are $C$-definable maps. By a $C$-pro-definable set, we mean a pro-object in $\text{Def}_C$ indexed by a small partially ordered set. Let $X$ be a $C$-definable set. By [7, Theorem 3.1.1], there exists a $C$-pro-definable set $E$ such that for every model $M$ of $\text{ACVF}$ that contains $C$, we have a canonical identification

$$\hat{X}(M) = E(M).$$

The main theorem - [7, Theorem 11.1.1] proved by Hrushovski-Loeser in [7] implies in particular that the homotopy type of the Hrushovski-Loeser space associated to a quasi-projective variety over a valued field is determined by a relatively simple object - a definable subset of $\Gamma_n^w$ for some $n \in \mathbb{N}$. The precise statement is as in Theorem 2.24.

**Remark 2.23.** Note that when using the machinery of Hrushovski-Loeser to construct pro-definable deformation retractions $H: I \times \hat{V} \to \hat{V}$ onto a $\Gamma$-internal set, where $V$ is a quasi-projective variety of dimension greater than 1, we can no longer suppose that the interval $I$ is $[0, \infty]$. It is usually a generalized interval which consists of glueing end to end copies of $[0, \infty]$. A more in depth, explanation can be found in [7, §3.9].

**Theorem 2.24.** Let $V$ be a quasi-projective variety over a valued field $F$ and let $X$ be a definable subset of $V \times \Gamma^w_\infty$ over some base set $A \subset VF \cup \Gamma$, with $F = VF(A)$. Then there exists an $A$-definable deformation retraction

$$H: I \times \hat{X} \to \hat{X}$$

with image an iso-definable subset $\Upsilon$ which is definably homeomorphic to a definable subset of $\Gamma^w_\infty$, for some finite $A$-definable set $w$. One can furthermore require the following additional properties for $H$ to hold simultaneously.

1. Given finitely many $A$-definable functions $\xi_i: X \to \Gamma_\infty$ with canonical extension $\hat{\xi}_i: \hat{X} \to \Gamma_\infty$, one can choose $H$ to respect the $\hat{\xi}_i$, i.e. to satisfy $\hat{\xi}_i(H(t, x)) = \xi_i(x)$ for all $(t, x) \in I \times \hat{X}$. In particular, finitely many definable subsets $U$ of $X$ can be preserved, in the sense that $H$ restricts to a homotopy of $U$.
2. Assume given, in addition, a finite algebraic group $G$ acting on $V$ and leaving $X$ globally invariant. Then the retraction $H$ can be chosen to be equivariant with respect to the $G$-action.
3. Assume $I = 0$. The homotopy $H$ is Zariski generalizing, i.e. for any Zariski open subset $U$ of $V$, $\hat{U} \cap X$ is invariant under $H$.
4. The homotopy $H$ is such that for every $e \in \hat{X}$, $H(e, H(t, x)) = H(e, x)$ for every $t$ and $x$.
5. One has $H(e, X) = \Upsilon$, i.e. $\Upsilon$ is the image of the simple points.
6. Assume $I = 0$ and $X = V$. Given a finite number of closed irreducible subvarieties $W_i$ of $V$, one can demand that $\Upsilon \cap W_i$ has pure dimension $\dim(W_i)$.

Observe that the theorem not only describes precisely the homotopy type of the space $\hat{X}$ but also allows us to construct pro-definable deformation retractions with considerable flexibility. It is this fact that we will exploit in §5. Note that all
deformation retractions considered and constructed in this paper satisfy assertions
(3), (4) and (5) above.

2.5.1. Canonical Extensions. Let $C \subset U$ be a small set of parameters and let $X$ be
a $C$-definable set. Let $Y$ be a $C$-definable subset and $f: X \to \hat{Y}$ be a $C$-definable
map. In this situation, we can extend the map $f$ to a well defined map $\hat{f}: \hat{X} \to \hat{Y}$
which we call the canonical extension of $f$. We do so as follows.

Let $p \in \hat{X}(M)$ where $M$ is a model of ACVF that contains $C$. Let $c \models p_{/M}$
and $d \models f(c)_{/M(c)}$ (cf. Remark 2.9). By [7, Proposition 2.6.5], the type $tp(cd|M)$
is stably dominated. It follows that $tp(d|M)$ is stably dominated as well. We set
$\hat{f}(p) := tp(d|M)$. The map $\hat{f}$ is well defined and can be shown to be pro-$C$-definable.

When $X \subset \mathbb{P}^n \times \Gamma_\infty^n$, it is natural to ask what hypothesis must be placed on
the definable map $f$ to ensure that the induced map $\hat{f}$ is continuous. To this end,
we introduce the $v$ and $g$-topologies on $X$. The $v$-topology is the topology induced
by the valuation on the ground field and is hence a well defined topology. The $g$-topology
on the other hand is a Grothendieck topology and if $U \subset X$ is both $v$ and $g$ open
then $\hat{U}$ is open in $\hat{X}$.

**Definition 2.25.** ($v$ and $g$ - open sets) Let $V$ be an algebraic variety defined over
a valued field $F$. A definable set $U \subset V$ is $v$-open if it is open for the valuation
topology on $V$. A definable set $G$ is $g$-open if it is a positive Boolean combination
of Zariski closed and open subsets and sets of the form $\{u : val(f)(u) < val(g)(u)\}$
where $f, g$ are regular functions on some Zariski open set. More generally, if $U$ is
a definable subset of the variety $V$ then a set $W \subset U$ is $v$-open (g-open) if it is of
the form $U \cap O$ where $O$ is $v$-open (g-open). If $X \subset V \times \Gamma_\infty^n$ is a definable set then
$X$ is $v$-open (g-open) if its pullback via $id \times val$ to $V \times \mathbb{A}^n$ is $v$-open (g-open).

**Remark 2.26.** Observe that the $v$-topology on $\Gamma$ is discrete while the neighbor-
hoods of $\infty$ are the same as those defined by the order topology. The $g$-topology
of $\Gamma_\infty$ when restricted to $\Gamma$ coincides with the order topology while the point $\infty$
is isolated. It follows that the $v + g$ topology which is the topology generated by
the class of sets which are both $v$ and $g$-open, induces the order topology on $\Gamma_\infty$.
In general the $v, g$ and $v + g$-open sets are definable. Observe that in the case
of a variety $V$ over a valued field $F$, the collection of $v$-open sets definable over $F$
generate the valuation topology on $V$. The $g$-topology however does not necessarily
generate a topology.

**Definition 2.27.** ($v$-continuity and $g$-continuity) Let $V$ be an algebraic variety
over a valued field $F$ or a definable subset of such a variety. A definable function
$h: V \to \Gamma_\infty$ is called $v$-continuous (resp. $g$-continuous) if the pullback of any $v$
-open (resp. $g$-open) set is $v$-open (resp. $g$-open). A function $h: V \to W$ with $W$
an affine $F$-variety is called $v$-continuous (resp. $g$-continuous) if, for any regular
function $f: W \to \mathbb{A}^1$, $val \circ f \circ h$ is $v$-continuous (resp. $g$-continuous).

Let $V$ be an algebraic variety and $W \subset \mathbb{P}^m \times \Gamma_\infty^n$ be a definable set. We endow
$\hat{W}$ with the subspace topology from $\hat{\mathbb{P}}^m \times \hat{\Gamma}_\infty^n$. Let $f: V \to \hat{W}$ be a well defined
pro-definable function such that for every open subset $O \subset \hat{W}$, $f^{-1}(O)$ is $v + g$
open. In this case, we say that the map $f$ is $v + g$-continuous. By [7, Lemma 3.8.2],
$f$ induces a continuous pro-definable map $\hat{f}: \hat{V} \to \hat{W}$.

2.5.2. Simple points. Let $V$ be an $A$-definable set. For $x \in V$, the definable type
$tp(x/U)$ which concentrates on the point $x$ is stably dominated. It follows that
$tp(x/U)$ is an element of $\hat{V}(U)$. We can thus view $V$ as a subset of $\hat{V}$. This subset
of points in $\hat{V}$ is called the set of simple points.
Lemma 2.28. ([7], Lemma 3.6.1) Let $X$ be a definable subset of $\text{VF}^n$.

(1) The set of simple points of $\hat{X}$ (which we identify with $X$) is an iso-definable [7, Definition 2.2.2] and relatively definable [7, §2.2] dense subset of $\hat{X}$. If $M$ is a model of $\text{ACVF}$ then $X(M)$ is dense in $\hat{X}(M)$.

(2) The induced topology on $X$ agrees with the valuation topology on $X$.

Remark 2.29. Let $V$ be an algebraic variety and $W \subset \mathbb{P}^m \times \Gamma^n_\infty$. We endow $\hat{W}$ with the subspace topology from $\mathbb{P}^m \times \Gamma^n_\infty$. Let $f : V \to \hat{W}$ be a $\nu + q$-continuous map. Since the simple points are dense in $V$, we see that there exists exactly one morphism $\hat{V} \to \hat{W}$ that extends $f$.

3. The Berkovich space $B_F(X)$

We provide a model theoretic reinterpretation of the Berkovich space, one for which a connection with the space of stably dominated types can be easily made. Our presentation follows [7, Chapter 14].

Let $F$ be a real valued field and let $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Let $F$ denote the structure defined by the pair $(F, \mathbb{R}_\infty)$. Let $V$ be a quasi-projective $F$-variety. As a set, the Berkovich space $B_F(V)$ is defined as follows.

Definition 3.1. Let $X$ be an $F$-definable subset of $V \times \Gamma^n_\infty$ for some $l \in \mathbb{N}$. Let $B_F(X)$ be the set of almost orthogonal to $\Gamma$, $F$-types which concentrate on $X$. The notion of an almost orthogonal to $\Gamma$ type was introduced in Definition 2.14.

Let $f : X \to \Gamma_\infty$ be an $F$-definable function. Observe that if $p \in B_F(X)$ is such that $a \vdash p$ then $f(a) \in \mathbb{R}_\infty$ depends only on the type $p$ i.e. if $a_1 \vdash p$ and $a_2 \vdash p$ then $f(a_1) = f(a_2)$. We set $f(p) := f(a)$. Thus we have a well defined function $f : B_F(X) \to \mathbb{R}_\infty$.

Definition 3.2. (Topology on $B_F(X)$) Let $X$ be an $F$-definable subset of the $F$-variety $V$. The set $B_F(X)$ is endowed with the topology generated by pre-basic open sets of the form $\{q \in B_F(X \cap U) \mid \text{val}(f)_*(q) \in W\}$ where $U \subset V$ is an open affine subspace, $f$ is a regular function on $U$ and $W \subset \mathbb{R}_\infty$ is an open interval.

Lemma 3.3. The spaces $V^\text{an}$ and $B_F(V)$ are canonically homeomorphic.

Proof. This is proved in [7, 14.1].

We relate the space $B_F(V)$ to the space $\hat{V}$. Let $L$ be an algebraically closed, spherically complete valued field which contains $F$ as a substructure and whose residue field is the algebraic closure of the residue field $k(F)$ of $F$ and value group $\Gamma(L)$ is $\mathbb{R}$. Such a field is unique up to isomorphism over the structure $F$. We fix one such copy and call it $F^\text{max}$.

Lemma 3.4. There exists a surjective continuous function

$$\pi_{F,V} : \hat{V}(F^{\text{max}}) \to B_F(V)$$

such that if $X$ is an $F$-definable subset of $V$ then $\pi_{F,V}^{-1}(B_F(X)) = \hat{X}(F^{\text{max}})$.

Proof. The map is constructed in [7, §14.1]. The surjectivity of $\pi_{F,V}$ and the assertion regarding $\pi_{F,V}^{-1}(B_F(X))$ is proved in [7, Lemma 14.1.1] and [7, Proposition 14.1.2]. Nonetheless, we repeat the construction here since the map $\pi_{F,V}$ figures prominently in §6.

Let $p$ be a stably dominated type defined over $F^{\text{max}}$ that concentrates on $V$. Then $p_{F^{\text{max}}}$ is an $F^{\text{max}}$-type. Let $\pi_{F,V}(p)$ denote the $F$-type defined by those formulae with parameters in $F$ that are contained in $p_{F^{\text{max}}}$. Let $a \in V$ be such that $\pi_{F,V}(p) = \text{tp}(a/F)$. We must have that $\Gamma(F(a)) \subseteq \Gamma(F^{\text{max}}(a)) = \Gamma(F^{\text{max}}) = \Gamma(F)$. 

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It follows that $\pi_{F,V}$ is a well defined function. It can be checked that it is continuous as well.

We show that $\pi_{F,V}$ is surjective. Let $p \in B_{\pi}(V)$ and $a$ be a realization of $p$. By [5, Theorem 12.18 (ii)], the type $tp(a|_{F^{\text{max}}})$ extends to an $F^{\text{max}}$-stably dominated type which concentrates on $V$, thus defining an element of $\hat{V}(F^{\text{max}})$. \[\square\]

**Remark 3.5.** The purpose of this section was to emphasize the extent to which the Berkovich space and the Hrushovski-Loeser space are closely related. In fact, when working over certain models of ACVF, the two spaces coincide. Indeed, using the notation from Lemma 3.4, we see that if $F = F^{\text{max}}$ then by [7, Lemma 14.1.1] $\pi$ is a homeomorphism.

## 4. Required tools

Our goal in this section is to develop the tools required to prove the principal results of Sections §5 and §6.

### 4.1. Preliminary simplifications

We use the following lemmas to show that in most situations under consideration, we may suppose that we have a morphism between projective varieties whose fibres are pure over some open dense subset of the base.

**Lemma 4.1.** Let $V$ be a quasi-projective $K$-variety. Let $\phi: V' \rightarrow V$ be a morphism between quasi-projective $K$-varieties whose image is dense. Let $G$ be a finite algebraic group acting on $V'$ that restricts to a well defined action along the fibres of the morphism $\phi$ and $\xi_i: V' \rightarrow \Gamma_\infty$ be a finite collection of definable functions. There exists projective $K$-varieties $\nu^{-1}, \nu'$ and a finite type surjective morphism $\overline{\nu}: \nu' \rightarrow \nu$ with the following properties.

1. The varieties $\nu^{-1}$ and $\nu'$ are pure of dimension $\dim(V')$ and $\dim(V)$ respectively.
   - In the event that $V$ is a smooth connected $K$-curve, we can suppose $\nu^{-1}$ is the unique smooth projective $K$-curve that contains $V$ as a Zariski open dense subset.
   - If $V$ is integral and normal then we can take $\nu$ to be integral and normal as well.
2. There exist embeddings $i: V \hookrightarrow \nu$ and $i': V' \hookrightarrow \nu'$ whose images are locally closed subspaces.
3. The morphism $\overline{\nu}$ extends the induced morphism $\phi: i'(V') \rightarrow i(V)$.
4. The fibres of the morphism $\overline{\nu}$ are pure and projective over some open dense subset of $\nu'$. When $V$ is a smooth connected curve and $\phi$ is flat, the morphism $\overline{\nu}$ is flat over $\nu$ and all fibres of $\overline{\nu}$ are pure.
5. The variety $\nu'$ admits an action of the group $G$ that extends the action of $G$ on $V'$. The action of the group $G$ restricts to a well defined morphism on the fibres of $\overline{\nu}$.
6. The functions $\xi_i$ extend to definable functions $\xi'_i: \nu' \rightarrow \Gamma_\infty$.

**Proof.** As the varieties $V$ and $V'$ are quasi-projective, there exists $n,m \in \mathbb{N}$ such that we can identify $V$ and $V'$ with locally closed subspaces of $\mathbb{P}^n_K$ and $\mathbb{P}^m_K$ respectively. Let $a: V \hookrightarrow \mathbb{P}^n_K$ and $a': V' \hookrightarrow \mathbb{P}^m_K$ denote the respective immersions. Let $\overline{\nu}$ denote a projective variety contained in $\mathbb{P}^n_K$ that is pure, of dimension $\dim(V)$ and contains $V$ (cf. [7, §11.2]). In the event that $V$ is a smooth connected $K$-curve, let $\overline{\nu}$ be the unique smooth projective $K$-curve that contains $V$ as a Zariski open dense subset. If $V$ is integral and normal, then we can replace $\overline{\nu}$ with its normalization. Note that this normalization is a projective variety and contains $V$ as an
open dense subset. After increasing n if necessary, we abuse notation and assume that we have a closed immersion \( V \hookrightarrow \mathbb{P}^n_k \).

Let \( A' := (\mathbb{P}^m_k)^G \). The group \( G \) acts on the projective variety \( A' \) in the following fashion. Let \( x := (x_h)_{h \in G} \in A' \) and \( g \in G \) then we set \( g(x) := (x_{hg})_{h \in G} \). Let \( b' : V' \to A' \) denote the morphism defined by \( v \mapsto (a'(hv))_{h \in G} \). Observe that \( b' \) is a \( G \)-equivariant embedding.

We have an embedding \( c : V' \to A' \times \overline{V} \) given by \( c(v) := (b'(v), a(\phi(v))) \). Let \( p_2 : A' \times \overline{V} \to \overline{V} \) denote the projection morphism onto the second coordinate. Observe that \( p_2 \) is a \( G \)-equivariant morphism where \( \overline{V} \) is endowed with the trivial action. We identify \( V' \) with its image in \( A' \times \overline{V} \) via the embedding \( c \). Likewise, we identify the morphism \( \phi \) with the restriction of \( p_2 \) to \( c(V') \).

Let \( S \subset \overline{V} \) denote the set of images of the generic points of \( V' \) for the morphism \( \phi \) and the generic points of \( V \). Given a point \( x \in \overline{V} \), let \( A'_x \) denote the fibre over \( x \) for the morphism \( p_2 \). Recall from [4, Exercise II.3.10] that \( A'_x \) is homeomorphic to the subspace \( p_2^{-1}(x) \subset A' \times \overline{V} \). For every point \( \eta \in S \), we choose a \( G \)-equivariant pure projective variety \( X(\eta) \subset A'_x \) that contains \( V' \cap A'_x \) and whose closure in \( A' \times \overline{V} \) is of dimension \( \dim(V') \) (cf. [7, §11.2]). It follows that \( \bigcup_{\eta \in S} X(\eta) \) is a \( G \)-invariant subset of \( A' \times \overline{V} \). Let \( \overline{\eta} \) denote the Zariski closure of \( \eta \) in \( A' \times \overline{V} \) and we define \( \overline{V'} := \bigcup_{\eta \in S} \overline{X(\eta)} \) endowed with the reduced induced closed subscheme structure. Since \( G \) stabilizes \( \bigcup_{\eta \in S} \overline{X(\eta)} \), we see that \( G \) must stabilize \( \overline{V} \). Hence, \( \overline{V} \) is a projective variety which is pure of \( \dim(V') \) that contains \( V' \) and extends the action of the group \( G \).

Observe that if \( V \) is a smooth connected curve and \( \phi \) is flat, then \( S \) as defined above consists of exactly the generic point of \( V \). In this situation, our construction implies in addition that every irreducible component of \( \overline{V} \) dominates \( \overline{V} \). It follows that the morphism \( \overline{V'} \to \overline{V} \) is flat. The purity of the fibres follows from [4, Corollary III.9.6].

Let us now return to the general situation. Using [4, Corollary III.9.6] and [10, Tag 052B], we deduce that the fibres of the restriction \( p_2|_{\overline{V'}} : \overline{V'} \to \overline{V} \) are pure and equidimensional over an open dense subset of \( \overline{V} \). Observe that the morphism \( p_2|_{\overline{V'}} \) is \( G \)-equivariant. We define \( \phi := p_2|_{\overline{V'}} \). The functions \( \xi : V' \to \Gamma_\infty \) extend to definable functions on \( \overline{V} \) by setting \( \xi'_i(v) := \xi_i(v) \) for \( v \in V' \) and \( \infty \) otherwise. \( \square \)

**Lemma 4.2.** Suppose that Theorem 1.1 and Proposition 5.2 are true in the following situation.

1. Let \( G \) be a finite algebraic group acting on projective \( K \)-varieties \( V' \) and \( V \) where the action on \( V \) is trivial.
2. The variety \( V \) is integral and normal.
3. We are given a \( G \)-equivariant morphism \( \phi : V' \to V \) whose fibres are pure over some dense open subset \( U \subset V \).
4. We are given a finite family \( \{ \xi_i : V' \to \Gamma_\infty \}_{i \in I} \) of \( K \)-definable functions.

Then Theorem 1.1 and Proposition 5.2 are true in general.

**Proof.** We first consider the case of Theorem 1.1. Let the data be as given in Theorem 1.1. That is to say, let \( V \) be a pure quasi-projective \( K \)-variety and let \( \phi : V' \to V \) be a morphism between quasi-projective varieties with dense image. Let \( G \) be a finite algebraic group acting on \( V' \) which restricts to a well defined action along the fibres of the morphism \( \phi \) and \( \xi_i : V' \to \Gamma_\infty \) be a finite collection of \( K \)-definable functions.

We deduce without difficulty that we can assume \( V \) is integral and normal. By Lemma 4.1, there exists projective \( K \)-varieties \( \overline{V}, \overline{V}' \) and a finite type surjective
morphism $\overline{\phi}: \overline{V'} \to \overline{V}$ satisfying conditions (1)-(6) of 4.1. Observe from our construction in the proof of Lemma 4.1 that the variety $V'$ is not necessarily dense in $\overline{V'}$. Let $V'_1$ denote the closure of $V'$ in $\overline{V'}$. In our situation, since $V'$ is locally closed, it is open in $V'_1$. To the family of definable functions $\{\xi_i: \overline{V'} \to \Gamma_\infty\}$ which extend the functions $\xi_i$, we add the valuations of the characteristic functions of the closed subvariety $V'_1 \subset \overline{V'}$ which we denote $\epsilon_j$ i.e. the family $\{\epsilon_j\}_{j}$ is such that $\bigcap_j \epsilon_j^{-1}(\infty) = V'_1$. The hypothesis of the lemma implies the existence of a Zariski dense open subset $W \subset \overline{V}$ such that if $W' := \phi^{-1}(W)$ then there exists compatible deformation pairs $(H', \Upsilon)'$ on $\overline{W'}$ and $(H, \Upsilon)$ on $\overline{W}$ satisfying assertions (1)-(6) of Theorem 1.1. This implies in particular that the deformation $H'$ restricts to a deformation of $W' \cap V'_1$. The set $W' \cap V'$ is open in $V'_1$. Theorem 1.1 asserts that the deformation $H'$ is Zariski generalizing. Every Zariski open subset of $W' \cap V'_1$ is of the form $O \cap W' \cap V'_1$ where $O$ is a Zariski open subset of $W'$. It follows that the restriction of the deformation $H'$ to $W' \cap V'_1$ is Zariski generalizing as well. This implies that $H'$ restricts to a well defined deformation of $\overline{W' \cap V'}$. Recall that the map $V' \to V$ is a well defined morphism of quasi-projective varieties whose image is dense. We can hence shrink $W$ so that $W' \cap V' \to W \cap V$ is surjective. It follows that $H$ restricts to a well defined deformation of $\overline{W} \cap V$. The restrictions of the deformations $H'$ and $H$ to $W' \cap V'$ and $W \cap V$ must be compatible since $H'$ and $H$ are compatible. The proof in the case of Proposition 5.2 is an easy adaptation of the arguments above. \hfill \Box

Lemma 4.3. Let $f: V' \to V$ be a projective morphism of $K$-varieties such that the fibres of $f$ are pure of dimension $m$. Let $G$ be a finite algebraic group acting on $V'$ such that the morphism $f$ is $G$-equivariant when $V$ is endowed with the trivial action. For every $v \in V(K)$, there exists a Zariski open neighbourhood $U \subset V$ of $v$ such that the morphism $f: f^{-1}(U) \to U$ factors through a finite surjective $G$-equivariant morphism $p: f^{-1}(U) \to \mathbb{P}^m \times U$ over $U$ where the $G$-action on $\mathbb{P}^m \times U$ is taken to be trivial.

Suppose we are given a horizontal divisor $D \subset V'_1$. There then exists a $K$-point $z \in \mathbb{P}^m$ such that after shrinking $U$ if necessary but maintaining that it contains $v$, we have the following commutative diagram

$$
\begin{array}{ccc}
V'_{1U} & \overset{p'}{\longrightarrow} & E \times U \\
\downarrow & & \downarrow b \times id \\
V'_U & \overset{p}{\longrightarrow} & \mathbb{P}^m \times U \\
\downarrow f & & \\
U & & \\
\end{array}
$$

where $b: E \to \mathbb{P}^m$ is the blow up at the point $z$, the restriction of $(q \circ p')$ to $D$ is finite surjective onto $\mathbb{P}^{m-1} \times U$ and the square in the diagram is cartesian.

Proof. This is a relative version of [7, Lemma 11.2.1]. We may replace $V'$ with $V'/G$ and assume that the action of $G$ is trivial. Let $v \in V$. Let $n \in \mathbb{N}$ be the smallest natural number greater than or equal to $m$ such that there exists a Zariski open neighbourhood $U$ of $v$ and the morphism $f$ factors through a finite morphism $i: V'_U \to \mathbb{P}^m \times U$. The fact that there exists such an $n$ is because $f$ is projective.

\footnote{We say that $D$ is a horizontal divisor if it does not contain any irreducible component of any fibre of the morphism $f$.}
Let $V'_U$ denote the fibre over $v$. If $m = n$ then we have nothing to prove. Suppose $n > m$. Let $(z,v) \in (\mathbb{P}^m \times U)(K)$ be a point that is not contained in $i(V'_U)$. Let $C := i(V'_U) \cap \{(z) \times U\} \subset \mathbb{P}^m \times U$. Observe that $C$ is a closed subset of $\mathbb{P}^m \times U$ and hence $p_2(C)$ is a closed subset of $U$ where $p_2$ is the projection $\mathbb{P}^m \times U \to U$. Furthermore, our choice of $z \in \mathbb{P}^m$ implies that $v \notin p_2(C)$. We abuse notation and call the complement of $C$ in $U$, $U$ as well. By construction, for every $u \in U$, $(z,u) \notin i(V'_U)$. Let $p: \mathbb{P}^m \to \mathbb{P}^{m-1}$ denote the projection through the point $z$. It follows that the map $p \circ \mathrm{id}: (\mathbb{P}^m \setminus \{z\}) \times U \to \mathbb{P}^{m-1} \times U$ restricts to a finite morphism $i_1: V'_U \to \mathbb{P}^{m-1} \times U$. This contradicts our assumption that $n$ was minimally chosen.

We now prove the second part of the Lemma. Let $D \subset V'_U$ be a horizontal divisor. Let $z \in \mathbb{P}^m(K) \times \{v\}$ be a point which is not contained in the image $p(D)$. We now argue as before. Let $C' := p(D) \cap \{(z) \times U\} \subset \mathbb{P}^m \times U$. We have that $C'$ is closed and hence $p_2(C') \subset U$ is a closed subspace that does not contain $v$. We shrink $U$ so that it does not intersect $p_2(C')$. Let $b: E \to \mathbb{P}^m$ be the blow up at the point $z$. Let $V'_U := V'_U \times_{\mathbb{P}^m \times U} (E \times U)$. We thus have the diagram above and the remaining assertions can be checked from the construction.

\hfill $\square$

**Remark 4.4.** Observe in Lemma 4.3 that $V''_U'$ comes equipped with a natural action by the group $G$ and the morphism $b': V''_U' \to V'_U$ is $G$-equivariant.

**Lemma 4.5.** Let $F$ be a valued field. Let $f: V' \to V$ be a finite surjective morphism between pure $K$-varieties where $V$ is assumed to be normal. Assume there exists a proper closed subset $D \subset V$ which satisfies the following property.

- If $n$ denotes the supremum of the values $\text{card}\{ f^{-1}(v) \}$ as $v$ varies along $V$ then for every $u \in U := V \setminus D$, $n = \text{card}\{ f^{-1}(u) \}$.

We then have that for every $v \in U$ and every point $v' \in f^{-1}(v)$, there exists definable sets $W \subset V$ containing $v$ and $W' \subset V'$ containing $v'$ such that $\widehat{W}$ and $\widehat{W}'$ are open in $V$ and $V'$ respectively and $\widehat{f}$ restricts to a homeomorphism from $\widehat{W}'$ onto $\widehat{W}$.

**Proof.** Let $F$ be a model of ACVF such that $v \in U(F)$. Let $U' := f^{-1}(U)$. For every $v' \in U'$ such that $v' \mapsto v$ we choose a suitably small $F$-definable $v + g$-open subset $W'_v \subset U'$ such that if $v'_1 \neq v'_2$ are distinct preimages of $v$ then $W'_v \cap W'_{v_2} = \emptyset$. Let $W$ be the $F$-definable set $\bigcap_{v' \in f^{-1}(v)} f(W'_{v'})$. By [7, Corollary 9.7.4], the morphism $\widehat{f}$ is open. Furthermore, since $f$ is proper, [7, Lemma 4.2.26] implies that $\widehat{f}$ is a definably closed map. It follows that $\widehat{W}$ is an $\mathfrak{F}$-definable open neighbourhood of $v$. For any $v' \mapsto v$, let $W'_{v'} := f^{-1}(W) \cap W'_{v'}$.

We claim that $f^{-1}(W) = \bigcup_{v' \in f^{-1}(v)} W'_{v'}$. By definition of $W'_{v'}$, we see that $\bigcup_{v' \in f^{-1}(v)} W'_{v'} \subset f^{-1}(W)$. Hence, we are left to show that if $w' \in U'$ is such that $f(w') \in W$ then for some $v' \in f^{-1}(v)$, $w' \in W'_v$. Let $w := f(w')$. Since $w \in U$, $\text{card}(f^{-1}(w)) = n$ and by construction for every $v' \in f^{-1}(v)$, there exists $w'_{v'} \in W'_{v'}$ such that $f(w'_{v'}) = w$. Since the $W'_{v'}$ are mutually disjoint, the points $w'_{v'}$ must account for all the preimages of $w$. This implies that for some $v'$, $w'_{v'} = w$. We have thus verified the claim.

Observe that for every $v' \in f^{-1}(v)$, $f$ restricts to a bijection from $W'_{v'}$ to $W$. By [7, Lemma 4.2.6], this implies that $\widehat{f}$ restricts to a bijection $\widehat{W}'_{v'} \to \widehat{W}$. The morphism $\widehat{f}$ is clopen when restricted to $\bigcup_{v' \in f^{-1}(v)} \widehat{W}'_{v'}$. By construction, for every $v'$, $\widehat{W}'_{v'}$ is open. It follows that $\widehat{f}$ restricts to a homeomorphism from each $\widehat{W}'_{v'}$ onto $\widehat{W}$. $\square$
In the following lemma, when we write \( v \in V \) where \( V \) is a quasi-projective variety defined over a field \( F \), we mean \( v \in V(\mathbb{U}') \) where \( \mathbb{U}' \) is chosen to be a suitable universal domain for the theory ACF.

**Lemma 4.6.** Let \( F \) be a field and let \( \phi: V' \to V \) be a morphism of quasi-projective \( F \)-varieties whose fibres are pure of dimension \( m \) for some \( m \in \mathbb{N} \). We assume that the morphism \( \phi \) factors through a finite surjective morphism \( f: V' \to P \times V \) via the projection \( p: P \times V \to V \) where \( P \) is an irreducible \( F \)-variety. There then exists a Zariski open dense subset \( U \subseteq V \) and a Zariski closed subset \( T \subset P \times U \) satisfying the following properties.

1. The restriction \( p|_T: T \to U \) is flat.
2. Given \( u \in U \), let \( n(u) \) denote the supremum of the set \( \{\text{card}\{f^{-1}(x,u)\}|x \in P\} \). We then have that
   \[
   \{(x,u) \in P \times U|\text{card}\{f^{-1}(x,u)\} < n(u)\} \subseteq T.
   \]
3. When \( V \) is a smooth curve and the morphism \( \phi \) is flat, we can take \( U = V \).

**Proof.** We can assume at the outset that \( V \) is integral. We begin by defining an ACF-definable set \( E \subset P \times V \) as follows.

Let \( M \) denote the supremum of the set \( \{\text{card}\{f^{-1}(x,u)\}|(x,u) \in P \times V\} \). There exists an ACF-definable partition \( S_1, \ldots, S_M \) of \( V \) such that if \( s \in S_i \) then
\[
\sup\{\text{card}\{f^{-1}(x,s)\}|x \in P\} = i.
\]

Let \( E' = \bigcup_{1 \leq i \leq M} E'_i \) where \( E'_i \) is the ACF-definable subset of \( P \times S_i \) consisting of the set of pairs \( (x,s) \) such that
\[
\text{card}\{f^{-1}(x,s)\} = i.
\]

Observe that \( E' \) is \( F \)-definable. Let \( E \) be the complement of \( E' \) in \( P \times V \). Hence \( E \) is a constructible subset of \( P \times V \) and for every \( v \in V \), \( E_v \) is of dimension strictly smaller than \( m \). Hence \( \dim(E) < \dim(V) + m \). Let \( \overline{E} \) denote the Zariski closure of \( E \) in \( P \times V \).

We now show how to choose \( U \) as required by the lemma. Let \( Z \subset V \) be the set of points \( v \in V \) such that \( \overline{E} \) contains \( P \times \{v\} \). The set \( Z \) is ACF-definable. Since the dimension of \( E \) is strictly smaller than \( \dim(V) + m \), there exists a Zariski open subset \( U \) of \( V \) that is disjoint from \( Z \). Let \( T := \overline{E} \cap (P \times U) \). We now shrink \( U \) further if necessary so that the restriction \( p|_T: T \to U \) is flat. This verifies assertions (1) and (2) of the lemma.

Suppose that \( V \) is a smooth curve and the morphism \( \phi \) is flat. Since \( \dim(E) \leq m \), we see that \( \overline{E} \) cannot contain any fibre of the form \( P \times \{v\} \) for some \( v \in V \). Indeed, if \( \overline{E} \) contains \( P \times \{v\} \) for some \( v \in V \) then \( P \times \{v\} \) must be an irreducible component of \( \overline{E} \) and \( E \cap P \times \{v\} \) must be dense in \( P \times \{v\} \). This is not possible because \( \dim(E_v) < m \).

If \( Y \subset \overline{E} \) is an irreducible component whose image via \( p \) is a closed point \( v \in V(F) \) then we can identify \( Y \) with a Zariski closed subset of \( P_v \) and write \( T(Y) := Y \times V \). If on the other hand \( Y \) is an irreducible component of \( \overline{E} \) which surjects onto \( V \) via \( p \) then we set \( T(Y) := Y \). If \( Y_1, \ldots, Y_m \) are the irreducible components of \( \overline{E} \) then let \( T := \bigcup_i T(Y_i) \). By construction \( T \) satisfies assertions (1) and (2) when \( U = V \).

\[ \square \]

4.2. The inflation homotopy for families.

**Lemma 4.7.** Let \( V \) be a quasi-projective variety over a valued field \( F \). Let \( m: V \times V \to \Gamma_\infty \) be a definable metric as constructed in the proof of [7, Lemma 3.10.1]. Let
$D \subset V$ be a $v$-closed subset and $x \in V \setminus D$. Then the set $\{m(x,d) | d \in D\} \subset \Gamma$ has a supremum in $\Gamma$.

Proof. Observe that the definable metric as constructed in the proof of [7, Lemma 3.10.1] is such that for any $z \in V$ and $\gamma \in \Gamma$, if $B(z, \gamma) := \{y \in V | m(y,z) \geq \gamma\}$ then the family $\{B(z, \gamma)\}_\gamma$ is a fundamental system of $v$-open neighbourhoods of $y$ in $V$. Hence, if $\{m(x,d) | d \in D\}$ does not have a supremum in $\Gamma$ then we see that $x$ must be a limit point of $D$ for the $v$-topology. However, this is not possible since $D$ is $v$-closed. \hfill $\square$

**Lemma 4.8.** Let $F$ be a valued field and let $\phi : V' \to V$ be a morphism of quasi-projective $F$-varieties which satisfies the following properties.

(1) The fibres of the morphism $\phi$ are pure of dimension $m$ for some $m \in \mathbb{N}$.

(2) There exists a proper closed subset $D' \subset V'$ such that we have a map $f : V' \setminus D' \to \mathbb{A}^m \times V$. Furthermore, for every $v \in V$ and $x \in \mathbb{A}^m \times \{v\}$, if $x' \in f^{-1}(x)$ then there exists definable sets $W \subseteq \mathbb{A}^m \times \{v\}$ containing $x$ and $W' \subseteq V'$ containing $x'$ such that $\overline{W}$ and $\overline{W'}$ are open in $\mathbb{A}^m \times \{v\}$ and $\overline{V'}$ respectively and $f_*$ restricts to a homeomorphism from $\overline{W'}$ onto $\overline{W}$.

(3) Let $\{\xi_i : V' \setminus D' \to \Gamma_\infty\}_i$ be a finite family of $v$-continuous functions which are definable over $F$. Furthermore, for every $i$ and every $v \in V$, $\xi_{i}^{-1}(\infty) \cap V'$ is either empty or the union of irreducible components of $V'$.

(4) Let $G$ be a finite group acting on $V'$ such that $D'$ is $G$-invariant and the morphisms $\phi$ and $f$ are $G$-equivariant when we endow $V$ and $\mathbb{A}^m \times V$ with the trivial action.

Then there exists a $G$-equivariant homotopy $H'_{inff} : [0, \infty) \times \overline{V'} \to \overline{V'}$ with the following properties.

(a) The homotopy $H'_{inff}$ restricts to well defined homotopies along the fibres of the morphism $\phi$.

(b) Let $v \in V$ and $X \subseteq \phi^{-1}(v)$ be a Zariski closed subset of dimension strictly smaller than dim($\phi^{-1}(v)$). Then $\hat{X} \cap H'_{inff}(0, \overline{V'}) \subset \overline{D'}$.

(c) The homotopy $H'_{inff}$ can be taken to be $G$-invariant and respects the levels of the functions $\xi_i$.

Our notation $inff$ is a concatenation of the short forms $inf$ for inflation and $f$ for families.

Proof. We adapt the proof of [7, Lemma 10.3.2] to prove the lemma. Let $h_0 : [0, \infty) \times \mathbb{A}^m \to \mathbb{A}^m$ be the standard homotopy which sends $(t, x)$ to the generic type of the closed polydisk around $x$ of valuative radius $(t, \ldots, t)$. We abuse notation and use $h_0$ to denote the homotopy $[0, \infty) \times (\mathbb{A}^m \times V) \to (\mathbb{A}^m \times V)/V$ defined by $(t, (a, v)) \mapsto (h_0(t, a) \ominus v)$. The fact that this is a well defined homotopy follows from [7, Lemma 9.8.3].

By condition (3) of the lemma and [7, Lemma 7.3.4], for each $u := (x, v) \in \mathbb{A}^m \times V$, there exists a $f_0(u) \in \Gamma$ such that for any $v' \in f^{-1}(v)$, the path $t \mapsto h_0(t, u)$ for $t \in [\gamma_0(u), \infty]$ lifts uniquely to a definable path $\overline{V'}$ starting from $v'$. Furthermore, observe that for every $t \in [0, \infty]$, $\hat{p}(h_0(t, u)) = v$. It follows that if $v' \in V'$ is such that $f(v') = u$ then any lift of the path $t \mapsto h_0(t, u)$ starting from $v'$ must belong to $\overline{V'}$. Thus, for any $v' \in f^{-1}(v)$, the path $t \mapsto h_0(t, u)$ for $t \in [\gamma_0(u), \infty]$ lifts uniquely to a path in $\overline{V'}$ starting from $v'$. The remainder of the proof can be carried out using more or less the same arguments as in the proof of [7, Lemma 10.3.2]. Note that $V'$ is not a projective variety. Hence instead of using [7, Lemma 4.2.29] as in the proof of [7, Lemma 10.3.2], we use Lemma 4.7. \hfill $\square$
5. Generic deformations

5.1. The initial set-up. The goal of this section is to summarize the constructions of Lemmas 4.3 and 4.6 and introduce notation that will be used in the proofs of Proposition 5.2 and Theorem 1.1.

Let \( \phi: V' \to V \) be a projective morphism of \( K \)-varieties such that the fibres of \( \phi \) are pure of dimension \( m \). We suppose that \( V \) is integral. As before, let \( G \) be a finite algebraic group acting on \( V' \) such that the morphism \( \phi \) is \( G \)-equivariant when \( V \) is endowed with the trivial action. Let \( \{ \xi_i: V' \to \Gamma_\infty \} \) be a finite family of \( v + g \)-continuous \( K \)-definable functions.

By Lemma 4.3, there exists a Zariski open neighbourhood \( U \subset V \) such that the morphism \( \phi: \phi^{-1}(U) \to U \) factors through a finite surjective \( G \)-equivariant morphism \( g_U: \phi^{-1}(U) \to \mathbb{P}^m \times U \) over \( U \) where the \( G \)-action on \( \mathbb{P}^m \times U \) is taken to be trivial. Let \( V'_U := \phi^{-1}(U) \). We may shrink \( U \) further and assume that all generic points of \( V'_U \) belong to the fibre over the generic point of \( U \). We have the following commutative diagram.

\[
\begin{array}{ccc}
V'_U & \xrightarrow{g_U} & \mathbb{P}^m \times U \\
\downarrow{\phi_U} & & \\
U & \leftarrow & \\
\end{array}
\]

We apply the following steps to choose a horizontal divisor in \( V'_U \).

1. We apply Lemma 4.6 to the diagram above to obtain a divisor \( D_U \subset \mathbb{P}^m \times U \) which satisfies the conditions of 4.6 and \( (\mathbb{P}^m \times U) \setminus D'_U \subset \mathbb{A}^m \times U \) for some copy of \( \mathbb{A}^m \) in \( \mathbb{P}^m \). Let \( D'_U := g_U^{-1}(D_U) \).

2. For every \( i \), let \( Y_i := \xi_i^{-1}(\infty) \). Since \( \xi_i \) is \( v + g \)-continuous, \( Y_i \) is a Zariski closed subset of \( V'_U \). We shrink \( U \) further if necessary and suppose that for every \( i \), the generic points of \( Y_i \) lie on the fibre over the generic point of \( U \). Furthermore, for every \( i \), and every irreducible component \( Y_{ij} \) of \( Y_i \), we can shrink \( U \) so that the map \( Y_{ij} \to U \) is flat. It follows from [4, Corollary 9.6] that the fibres of \( \phi_{Y_{ij}} \) are pure and equi-dimensional.

   For every \( i \), let \( J_i \) denote those indices \( j \) such that the irreducible component \( Y_{ij} \) of \( Y_i \) is of dimension strictly less than \( \dim(V'_U) \). We enlarge \( D'_U \) so that for every \( i \) and \( j \in J_i \), \( Y_{ij} \subset D'_U \).

3. We shrink \( U \) further and enlarge \( D'_U \) so that \( D'_U \) remains a horizontal divisor and \( V'_U \setminus D'_U \) is the disjoint union of irreducible varieties whose fibres over \( U \) are pure and equidimensional. This is possible by first choosing a closed subset \( A \) of the generic fibre \( V'_U \) such that \( V'_U \setminus A \) is the disjoint union of irreducible \( k(\eta) \)-varieties. Note that \( \dim(A) < \dim(V'_U) \). We can shrink \( U \) so that the Zariski closure \( A' \) of \( A \) in \( V'_U \) will satisfy the required property. We then replace \( D'_U \) with the union \( D'_U \cup A' \).

4. We have \( D'_U = g_U^{-1}(g_U(D'_U)) \). Hence, \( D'_U \) is \( G \)-invariant.

By Lemma 4.3, there exists a \( K \)-point \( z \in \mathbb{P}^m \) such that after shrinking \( U \) if necessary we can extend the diagram above to get the following commutative diagram.
That the projection \( W \)

Using that the morphism \( F \)

This is possible because the above property is true over the generic point of \( P \) for every \( i \).

For every \( w \), \( A \)

where \( \varphi \)

In §5.2, we adapt the construction in [7, §11.3] of the relative curve homotopy to the fibration \( p_{\ell}': V_{11}^U \to F \times U \) where \( F := \mathbb{P}^{m-1} \) and \( p_U := p_U \circ g_U' \).

5.2. The relative curve homotopy for families. Let the notation be as in §5.1. There exists an open subset \( W \subset F \times U \) such that \( p_U^{-1}(W) \subset E \times U \) is isomorphic to \( \mathbb{P}^1 \times W \). Indeed, if \( p: E \to F \) denotes the projection map then there exists a Zariski open subset \( F_0 \subset F \) such that \( p^{-1}(F_0) = F_0 \times \mathbb{P}^1 \). By definition, \( p_U = p \times \text{id}: E \times U \to F \times U \).

Let \( A := p_U^{-1}(W) \subset V_{11}^U \) and \( B := p_U^{-1}(W) \subset E \times U \). Furthermore, we can shrink \( W \) if necessary so that the map \( g_U': A \to B \) factors through \( A \to A' \to B \) where \( A \to A' \) is radical and for every \( w \in W \), \( A'_w \to B_w = \mathbb{P}^1 \) is generically étale. This is possible because the above property is true over the generic point of \( W \).

Using that the morphism \( F \times U \to U \) is flat and hence open, we can shrink \( U \) so that the projection \( W \to U \) is surjective.

The homotopy \( H'_{\text{curve}}^U \).

We fix three points \(-1, 0, 1 \) and \( \infty \) on \( \mathbb{P}^1 \). This is to make sure that the notions of standard homotopy and closed ball are well defined. Given a divisor \( X \) on \( B \), [7, §10.2] implies that there exists a well defined definable map \( \psi_X: [0, \infty] \times B \to B/W \) which fixes \( \widehat{X} \) and if \( w \in W \) then \( \psi_X \) restricts to a well defined homotopy on \( \widehat{B}_{w} \).

Furthermore, if \( X \cap \text{B}_{w} \) is finite, then the image of the homotopy restricted to the fibre \( B_{w} \) is a \( \Gamma \)-internal subset of \( \widehat{B}_{w} \). We emphasize that this map is a priori not continuous unless for instance we add additional hypothesis on \( X \).

By definition, for \( w \in W \), the fibres of the map \( B/W \to W \) are copies of \( \widehat{B} \).

The definable map \( \psi_X \) is constructed using the standard homotopy (cf. [7, §7.5, p.105]) on \( \mathbb{P}^1 \) and then defining cut offs of this homotopy via the divisor \( X \).

**Lemma 5.1.** After shrinking the open set \( U \) if necessary, there exists a constructible set \( C \subset V_{11}^U \) with the following properties.

1. The set \( C \) maps surjectively onto \( F \times U \) via \( p_{\ell}^U \).
2. There exists an open subset \( W' \subset F \times U \) such that \( V_{11}^U \) contains \( p_U^{-1}(W') \).
3. Let \( H'_{\text{inff}} \) be the homotopy obtained from Lemma 4.8 associated to the divisor \( D^U_{11} \) and the morphism \( g_U': V_{11}^U \setminus D^U_{11} \to E \times U \). (By construction, the image of \( g_U' \) is contained in a copy of \( \mathbb{A}^m \times U \).) The image of \( H'_{\text{inff}}(0, V_{11}^U) \) is contained in \( \widehat{C} \).
(4) There exists a deformation retraction $h'_{\text{curvesf}} : [0, \infty] \times C \to \hat{C}/F \times U$. The image $T'_2 := h'_{\text{curvesf}}(0, C) \subset V'_{1U}/F \times U$ is iso-definable and relatively $\Gamma$-internal over $F \times U$. Furthermore, $h'_{\text{curvesf}}$ respects the levels of the functions $\xi_i$ for every $i$ and is $G$-equivariant.

Proof. By [7, Lemma 11.3.2], there exists a divisor $X$ on $B$ such that for any divisor $X'$ containing $X$, $\psi_X : [0, \infty] \times B \to \hat{B}/W'$ lifts uniquely to a definable map $h : [0, \infty] \times A \to \hat{A}/W$ which is fibrewise a homotopy. We enlarge $X$ so that it contains the image $g'_U(D'_{1U} \cap A)$ and the divisor $\infty \times W$. We use

$$h'_{\text{curvesf}} : [0, \infty] \times A \to \hat{A}/W$$

to denote the unique lift of $\psi_X$. By Lemma 10.2.2 in loc.cit, we can enlarge $X$ so that the lift $h'_{\text{curvesf}}$ preserves the levels of the restrictions of the functions $\xi_i$. As the lift is unique, it is $G$-invariant.

Note from the construction in [7, Lemma 11.3.2] that $\dim(X) = \dim(W)$. Observe that $W$ is an irreducible $K$-variety. It follows that the Zariski closure of the set $\{w \in W | p^{-1}_U(w) \subset X\}$ is of dimension strictly smaller than $\dim(W)$. Let $W' \subset W$ be a Zariski open subset over which $X$ is finite. We replace $A$ with $p^{-1}_U(W')$ and $B$ with $p^{-1}_U(W')$. It then follows that $\psi_X : [0, \infty] \times B \to \hat{B}/W'$ is a well defined homotopy and its lift is a well defined homotopy $h'_{\text{curvesf}} : [0, \infty] \times A \to \hat{A}/W'$ which is $G$-invariant and in addition preserves the levels of the functions $\xi_i$. Using that the morphism $F \times U \to U$ is flat and hence open, we can shrink $U$ so that the projection $W' \to U$ is surjective.

We extend $h'_{\text{curvesf}}$ to a definable map $[0, \infty] \times A \cup D'_{11U} \to A \cup D'_{11U}$ by setting $h'_{\text{curvesf}}(t, x) = x$ for every $x \in D_{11U}$. By Lemma 11.3.3 in loc.cit., the map $h'_{\text{curvesf}}$ is a well defined homotopy with canonical extension $H'_{\text{curvesf}} : [0, \infty] \times A \cup D'_{11U} \to A \cup D'_{11U}$. We set $C := A \cup D'_{11U}$.

It remains to verify the inequality $H'_{\text{inf}}(c, V'_{1U}) \subset \hat{C}$. Observe that by construction the projection $W' \to U$ is surjective and hence the complement $S$ of $W'$ in $F \times U$ cannot contain any subset of the form $F \times u$ where $u \in U$. As a result for any $u \in U$, $p^{-1}_U(S) \cap \phi^{-1}_U(U)$ is a closed subset whose dimension is strictly smaller than that of $\phi^{-1}_U(u)$. The inflation property (cf. Lemma 4.8 (2)) of the homotopy $H'_{\text{inf}}$ implies the result.

We now prove the following proposition which is related to [7, Proposition 11.7.1] in that we treat a family of quasi-projective varieties parametrized by a quasi-projective $K$-variety. While loc.cit. shows that there exists a family of deformation retractions uniform over the base, we show that there exists a global homotopy that restricts to a homotopy on each of the fibres but compromise by treating only a suitable open subset of the family.

Proposition 5.2. Let $V$ be a pure quasi-projective $K$-variety. Let $\phi : V' \to V$ be a morphism between quasi-projective varieties whose image is dense. Let $G$ be a finite algebraic group acting on $V'$ which restricts to a well defined action along the fibres of the morphism $\phi$ and $\{\xi_i : V' \to \Gamma_{\infty}\}$ be a finite collection of $K$-definable functions. There exists an open dense subset $U \subset V$ such that if $V'_U := \phi^{-1}(U)$ then there exists a generalized interval $I$ and a homotopy $h'_{\text{rel}} : I \times V'_U \to \hat{V}'_U/\hat{U}$ which satisfies the following properties.

(1) The image of $h'_{\text{rel}}$ is a relatively $\Gamma$-internal subset of $\hat{V}'_U/\hat{U}$.

(2) The homotopy $h'_{\text{rel}}$ is invariant for the action of the group $G$ and respects the levels of the functions $\xi_i$. 

(3) The homotopy $h'_{rel}$ is Zariski generalizing.

Proof. Lemma 4.2 allows us to reduce to the case of a morphism $\phi: V' \to V$ satisfying assertions (1)-(4) of 4.2. We make use of the notation and constructions introduced in §5.1. The simplifications in §5.1 show that it suffices to prove the theorem for the morphism $\phi'_U: V'^U \to U$. Indeed, any relative homotopy on $V'^U$ that restricts to a well defined homotopy on $Z'^U_U$ must descend to a homotopy on $V'^U_U$. This is because for every $v \in U$, the image of $Z'^U_U$ for the morphism $h'_U$ is $Z'^U_U$, which is the disjoint union of Zariski closed points and a relative homotopy that restricts to a well defined homotopy on each connected component of $Z'^U_U$. By [7, Lemma 3.9.4], for every $v$ the restriction of the homotopy to $V'^U_U$ descends to a homotopy on $V'^U_U$, and likewise the homotopy on $V'^U_U$ descends to a homotopy on $V'^U_U$.

We proceed to prove the proposition by induction on the dimension of the fibres of the morphism $\phi'_U$. Note that when the dimension of the generic fibre is 0, there is nothing to prove. Let $H_{inff}'$ be the homotopy obtained from Lemma 4.8 associated to the divisor $D'_{1U}$ and the morphism $\phi'_U: V'^U \to E \times U$. By §5.1, there exists homotopies $H_{inff}'_1: I_1 \times V'^U \to V'^U_U$ and $h'_{curvesf}: I_2 \times C/\Gamma \times U \to C/\Gamma \times U$ where $C$ is a constructible subset of $V'^U_U$ where $I_1 \times I_2 = [0, \infty)$. Recall from the construction of the homotopy $h'_{curvesf}$ that there exists a Zariski open subset $W' \subset E \times U$ such that the image of $h'_{curvesf}$ when restricted to those fibres over points in $W'$ is of dimension 1. Let $H'_{curvesf}: I_2 \times \hat{C} \to \hat{C}$ denote the canonical extension of the homotopy $h'_{curvesf}$. By construction, the composition $H'_{curvesf} \circ H_{inff}'_1: (I_1 \times I_2) \times V'^U \to V'^U_U$ is a well defined homotopy. Let $\tilde{Y}'_2$ denote the image $h'_{curvesf}(0, C) \subset V'^U_U/F \times U$. By construction, $\tilde{Y}'_2$ is iso-definable and relatively $\Gamma$-internal over $E \times U$. By Lemma 6.4.1 in [7], $\tilde{Y}'_2\hat{\Gamma}$ can be identified with a prov-definable subset of $V'^U_U$ such that over a point $u \in F \times U$, $\tilde{Y}'_2u = Y'_{2u}$. The image of $H'_{curvesf} \circ H_{inff}'$ is contained in $\tilde{Y}'_2$. By construction, $H'_{curvesf} \circ H_{inff}'$ is compatible with the homotopy on $\hat{U}$ that fixes every point.

**Lemma 5.3.** There exists a pseudo-Galois cover (cf. [7, §2.12]) $f: F' \to F \times U$ and a morphism $\kappa: \tilde{Y}'_2 \times (F \times U) F' \to F' \times \Gamma M$ for some $M \in \mathbb{N}$ such that the restriction $\kappa|_{\tilde{Y}'_2 F'}: \tilde{Y}'_2 F' \to F' \times \Gamma M$ is a homeomorphism onto its image where $\tilde{Y}'_2 F' := \tilde{Y}'_2 \times (F \times U) F'$.

**Proof.** Recall that $V$ is a projective variety. Let $d_v: V \to [0, \infty]$ denote the schematic distance to $V_{bord}$ where $V_{bord} := V \times U$. By construction, the morphism $\phi'_U: V'^U \to E \times U$ is finite and hence projective. It follows that the morphism $V'^U \xrightarrow{g'_U} E \times U$ factors through an embedding $j': V'^U \to \mathbb{P}_K^N \times E \times U$ for some $N \in \mathbb{N}$. Let $V_1$ be the closure in $\mathbb{P}_K^N \times E \times V$ of $V'^U_U$. Let $p': V_1 \to V'$ be the composition of the morphisms $V_1' \to E \times V$ which is the restriction of a projection and $E \times V \to V$. Let $d'_\gamma := d_v \circ p'$.

Similarly, let $\rho: V'_1 \to [0, \infty]$ denote the schematic distance from the closure of $D_{1U}$ in $V'_1$ and $\eta: V'_1 \to [0, \infty]$ denote the schematic distance to the preimage of $(F \times V) \setminus W'$ where $W' \subset E \times U$ is as in the statement of Lemma 5.1.

For $\gamma \in [0, \infty)$, let $V_{\gamma} := \{x \in \overline{V}d_v(x) \leq \gamma \}$ and let $V'_{1V_{\gamma}}$ denote its preimage in $V'_1$ i.e. the set $\{x \in \overline{V}d_v(x) \leq \gamma \}$. Observe that $V_{1\gamma}$ is definably compact since it is a closed subset of the definably compact space $\tilde{V}$. Furthermore, $V_{\gamma} \subset U$. For similar reasons, $V'_{1V_{\gamma}}$ is definably compact and also $V'_{1V_{\gamma}} \subset V'^U_U$. Observe that $\overline{\tilde{V}'}_{2\gamma}$ is $\sigma$-compact with respect to the restrictions of $\rho$ and $\eta$.
For ease of notation, let $S := F \times U$ and for $\gamma \in \Gamma$, $S_\gamma := F \times V_\gamma$. By [7, Lemma 6.4.2], there exists a pseudo-Galois cover $f : F' \to S$ and a morphism $\kappa : \Upsilon_2 \times_S F' \to F' \times \Gamma^M_\infty$ over $F'$ for some $M \in \mathbb{N}$ such that $\hat{\kappa} : \Upsilon_2 \to F' \times \Gamma^M_\infty$ is a continuous injection where $\Upsilon_2' := \Upsilon_2 \times S F'$. We fix $\gamma \in \Gamma$. We claim that $\hat{\kappa}$ restricts to a homeomorphism on the closed subspace $\Upsilon_2' \Upsilon_{2,F'}$, where $F'_\gamma$ is the preimage of $S_\gamma$ via $f$ and $\Upsilon_2' \Upsilon_{2,F'} := \Upsilon_2 \times S', F'_\gamma$. We apply [7, Lemma 6.4.3] to $\Upsilon_2 \subset V_{1,\gamma}/S_\gamma$ to verify the claim. Note that strictly speaking, loc.cit. requires that $S_\gamma$ be a quasi-projective variety. However, one checks that the steps of the proof can be carried out in our situation.

Let $V_{2,\gamma} := \{x \in V | d_\gamma(x) < \gamma\}, S_{2,\gamma} := F \times V_{2,\gamma}$ and $F'_{2,\gamma} := f^{-1}(S_{2,\gamma})$. We have shown that $\hat{\kappa}$ restricts to a homeomorphism on $\Upsilon_2' \Upsilon_{2,F'}$. To conclude that $\hat{\kappa}$ is a homeomorphism on $\Upsilon_2' \Upsilon_{2,F'}$, it suffices to verify that the image of $\Upsilon_2' \Upsilon_{2,F'}$ via $\hat{\kappa}$ is open in $\hat{\kappa}(\Upsilon_2' \Upsilon_{2,F'})$. This follows from the fact that by construction,

$$\hat{\kappa}(\Upsilon_2' \Upsilon_{2,F'}) = \hat{\kappa}(\Upsilon_2') \cap (F'_{2,\gamma} \times \Gamma^M_\infty)$$

since $\kappa$ is over $F'$ and hence also over $S$ and $U$. Note that $F'_{2,\gamma} \times \Gamma^M_\infty$ is open in $F' \times \Gamma^M_\infty$.

$\Box$

By Lemma 5.3, there exists a pseudo-Galois cover $f : F' \to F \times U$ and a morphism $\kappa : \Upsilon_2' \times (F \times U) F' \to F' \times \Gamma^M_\infty$ for some $M \in \mathbb{N}$ such that the restriction $\hat{\kappa} : \Upsilon_2' \to \Upsilon_2' \Upsilon_{2,F'}$ is a homeomorphism onto its image where $\Upsilon_2' := \Upsilon_2' \Upsilon_{2,F'}$. We shrink $U$ if necessary and assume that it is normal. Using the arguments in [7, 6.4.4], there exists a finite collection of $K$-definable functions $\mu_j : F' \to \Gamma^\infty$ such that, if $O \subset U$ is a Zariski open set, for $I$ a generalised interval, a homotopy $a'_j : I \times F \times O \to \tilde{F} \times \tilde{O}$ which lifts to a homotopy $a'_j : I \times f^{-1}(F \times O) \to f^{-1}((F \times O)$ that preserves the levels of the functions $\mu_j$ also induces a homotopy $a''_j : I \times \tilde{F} \cap \phi_{U^{-1}}^{-1}(\tilde{O}) \to \tilde{F} \cap \phi_{U^{-1}}^{-1}(\tilde{O})$ that is $G$-invariant and respects the levels of the functions $\xi_i$.

Observe that the fibres of the morphism $p_2 \circ f$ are pure over $U$. Let $G' := \text{Aut}(F'/F \times U)$. Let $p_2 : F \times U \to U$ denote the projection map onto the second coordinate. Observe that the group $G'$ acts on $F'$ along the fibres of the composition $p_2 \circ f$. We apply the induction hypothesis to the morphism $p_2 \circ f : F' \to U$ along with the definable functions $\mu_j$ and the group $G'$. Note that the morphism $p_2 \circ f$ is projective. It follows that after shrinking $U$ if necessary, we have a well defined homotopy

$$H_{b_1} : I_3 \times \tilde{\Upsilon}_2 \cap \phi_{U^{-1}}^{-1}(U) \to \tilde{\Upsilon}_2 \cap \phi_{U^{-1}}^{-1}(U).$$

Let $\Upsilon_2' \Upsilon_{2,U} := \Upsilon_2' \Upsilon_{2,U} \phi_{U^{-1}}^{-1}(U)$. Observe that by construction, $H_{b_1}$ restricts to a well defined homotopy $I \times \Upsilon_2' \Upsilon_{2,U} \to \Upsilon_2' \Upsilon_{2,U}/U$ whose image is relatively $\Gamma$-internal over $U$. Furthermore, the composition, $H_{b_1} \circ H'_{\text{curves}^1} \circ H'_{\text{inf}^1}$ restricts to a well defined homotopy $(I_3 + I_2 + I_1) \times V'_1 \Upsilon_2 \to V'_1 \Upsilon_2/U$ which fulfills the assertions of the proposition.

$\Box$

5.3. Proof of Theorem 1.1. We begin by verifying Theorem 1.1 when the dimension of the generic fibre is zero.
Lemma 5.4. Let \( \phi: V' \rightarrow V \) be a morphism satisfying assertions (1)-(4) of 4.2. We assume in addition that the dimension of the generic fibre is 0. Then, the conclusion of Theorem 1.1 is true.

Proof. There exists a Zariski open subset \( U \subseteq V \) such that \( \phi_U: V'_U \rightarrow U \) is finite. It follows that \( V'_U / U \) is relatively \( \Gamma \)-internal and iso-definable. By our assumption in 4.2, \( V \) and hence \( U \) is normal. Let \( d: V \rightarrow \Gamma_\infty \) be the schematic distance to the closed subset \( V \setminus U \). Observe that \( V'_U \) is \( \sigma \)-compact with respect to the function \( d_U \circ d_U \). By [7, Theorem 6.4.4], there exists a finite pseudo-Galois cover \( \pi: U' \rightarrow U \) and finitely many definable functions \( \{ \xi'_j \}_j \) such that any deformation retraction of \( \hat{U} \) that lifts to a deformation retraction on \( \hat{U}' \) that respects the definable functions \( \xi'_j \) will lift to a deformation retraction of \( V'_U \) that is equivariant for the action of the group \( G \) and respects the definable functions \( \xi_i \). We may hence assume \( V'_U \rightarrow U \) is a pseudo-Galois cover and \( G = \text{Gal}(K(V'_U) / K(U)) \) where \( K \) is the function field of \( U \).

We shrink \( U \) and assume \( V'_U \) is normal. We replace \( V' \) with the normalization of \( V \) in \( K(V'_U) \). We can extend the functions \( \xi'_j \) to definable functions on \( V' \). Let \( D' \) denote the preimage of \( V \setminus U \) for the morphism \( V' \rightarrow V \). Recall that in the proof of [7, Theorem 11.1.1] applied to the variety \( V' \), the group \( G \) and the functions \( \xi'_j \), we choose a divisor of \( V' \) with suitable properties and the first homotopy on \( \hat{V} \) that we construct is an inflation homotopy with respect to this divisor. We shrink \( U \) if necessary and hence enlarge \( D' \) so that we can take it to be the divisor in the proof of loc.cit. for \( V' \). We apply the schematic distance to \( D' \) to the definable functions \( \xi'_j \).

We apply loc.cit. to the variety \( V' \), the group \( G \) and functions \( \xi'_j \) to get a deformation retraction \( H': I \times \hat{V} \rightarrow \hat{V} \) on \( \hat{V} \) which is \( G \)-equivariant. We suppose in addition that the image of \( H' \) is pure of dimension \( \dim(V') \). The deformation retraction \( H' \) descends to a deformation retraction on \( \hat{V} \) and the restrictions of \( H' \) and \( H \) to \( V'_U \) and \( \hat{U} \) respectively, clearly satisfy (1) - (4) and (6) of Theorem 1.1. To verify property (5) for \( H \), it suffices to verify that if \( Z \subset V' \) is a proper Zariski closed subset then \( \hat{Z} \cap H'(e, \hat{V}_U) \) is empty where \( e \) is the end point of the generalized interval \( I \). Note by construction, \( H' \) is the composition of homotopies \( H'_U \circ H'_0 \circ H'_{\text{curves}} \circ H'_{\text{inj}} \) where \( H'_{\text{inj}} \) is an inflation homotopy with respect to \( D' \). Furthermore, the image of \( H' \) is contained in the image of \( H'_{\text{inj}} \). It follows that if \( \hat{Z} \) intersects the image of \( H' \) non-trivially then it must intersect the image of \( H'_{\text{inj}} \) non-trivially. By construction of \( H'_{\text{inj}} \), this implies that \( \hat{Z} \cap H'(e, \hat{V}) \subseteq \hat{D}' \). It follows that \( \hat{Z} \cap H'(e, \hat{V}_U) \) is empty.

□

Proof. (of Theorem 1.1) Let the data be as given in Theorem 1.1. It suffices to treat the case of a morphism \( \phi: V' \rightarrow V \) satisfying assertions (1)-(4) of 4.2. We proceed by induction on the dimension of the generic fibre. When the generic fibre has dimension 0, Theorem 1.1 is true by Lemma 5.4.

We proceed to the general case. Suppose the dimension of the generic fibre is greater than 0. We make use of the notation and construction introduced in §5.1. By Lemma 5.1, there exists a homotopy \( h'_{\text{curves}}: [0, \infty] \times C / F \times U \rightarrow C / F \times U \) where \( C \) is a certain constructible subset of \( V'_U \). Let \( H_{\text{curves}}: [0, \infty] \times \hat{C} \rightarrow \hat{C} \) denote the canonical extension of the homotopy \( h'_{\text{curves}} \). Let \( \Upsilon_2 \) denote the image \( h'_{\text{curves}}(0, C) \subset V'_U / F \times U \). By construction, \( \Upsilon_2 \) is iso-definable and relatively \( \Gamma \)-internal over \( F \times U \). By Lemma 6.4.1 in [7], \( \Upsilon_2 \) can be identified with a pro-definable
subset of $\widetilde{V}_{1U}$ such that over a point $u \in F \times U$, $\widetilde{V}_{2u} = \Upsilon_{2u}$. By construction, $H'_{\text{curves}f}$ is compatible with the homotopy that acts trivially on $\hat{U}$.

By Lemma 5.3, there exists a pseudo-Galois cover $f: F' \to F \times U$ and a morphism $\kappa: \Upsilon_2' \times (F \times U) F' \to F' \times \Gamma^M_\infty$ for some $M \in \mathbb{N}$ such that the restriction $\kappa|_{\Upsilon_2'F'}: \Upsilon_2'F' \to F' \times \Gamma^M_\infty$ is a homeomorphism onto its image where $\Upsilon_2':= \Upsilon_2 \times (F \times U) F'$.

Using the arguments in [7, 6.4.4], there exists a finite collection of $K$-definable functions $\mu_j: F' \to \Gamma_\infty$ such that, if $O \subset U$ is a Zariski open set, for $I$ a generalised interval, a homotopy $a'_I: I \times F \times O \to \hat{F} \times O$ which lifts to a homotopy $a^I: I \times f^{-1}(F \times O) \to f^{-1}(F \times O)$ that preserves the levels of the functions $\mu_j$ also induces a homotopy $a'_I: I \times \overline{\Upsilon}_2 \cap \phi'_U^{-1}(O) \to \overline{\Upsilon}_2 \cap \phi'_U^{-1}(O)$ that is $G$-invariant and respects the levels of the functions $\xi_i$.

Let $p_2: F \times U \to U$ denote the projection map onto the second coordinate. Observe that the fibres of the morphism $p_2 \circ f$ are pure over $U$. Let $G':= \text{Aut}(F'/F \times U)$. Observe that the group $G'$ acts on $F'$ along the fibres of the composition $p_2 \circ f$. We apply the induction hypothesis to the morphism $p_2 \circ f: F' \to U$ along with the definable functions $\mu_j$ and the group $G'$. It follows that we can shrink $U$ so that if $\Upsilon'_{2,U}:= \Upsilon_2 \cap \phi'_U^{-1}(U)$ then we have a well defined homotopy

$$H'_I: I \times \overline{\Upsilon}_2 \to \overline{\Upsilon}_2$$

whose image is an iso-definable $\Gamma$-internal subset of $\widetilde{V}_{1U}$. By construction, the composition $H'_I \circ H'_{\text{curves}f}$ is a well defined homotopy on $\hat{C}$.

**Remark 5.5.** Observe that in the proof of Proposition 5.2, the homotopy we constructed on $\widetilde{V}_{1U}/U$ automatically descends to a homotopy on $\widetilde{V}_{1U}/U$. As explained before, this is because for every $u \in U$, $V'_{1u} \to V'_u$ is isomorphic outside the finite set of closed points $Z'_u \subset V'_u$. Hence, as long as the induced homotopy on $\overline{\Upsilon}_1$ preserves $\overline{Z}_1$, where $\overline{Z}_1$ is the preimage of $Z'_u$ for the morphism $V'_{1u} \to V'_u$.

Note that such an argument does not hold in the case of Theorem 1.1 since we must construct a homotopy on the base $U$ as well. To resolve this issue, we construct an inflation homotopy $H'_{\text{inf}f-\text{primary}}$ on $\overline{V}_U$ whose image does not contain $\overline{Z}$ and is hence contained in $\overline{V}_U$. We then proceed to construct suitable homotopies on $\overline{V}_{1U}$.

**5.4. The relative tropical homotopy.** Let $\Upsilon_{bcf}$ denote the image of the composition of homotopies

$$H'_I \circ H'_{\text{curves}f}.$$

Our goal in this section runs parallel to [7, §11.5]. We construct a homotopy $H'_I$ on a subset of $\Upsilon_{bcf}$ and homotopies $H'_{\text{inf}f}$ and $H'_{\text{inf}f-\text{primary}}$ by applying Lemma 4.8 such that the composition

$$H'_I \circ H'_I \circ H'_{\text{curves}f} \circ H'_{\text{inf}f} \circ H'_{\text{inf}f-\text{primary}}$$

is a well defined deformation retraction i.e. it fixes its image.

An important fact to note is that we will no longer shrink the base $U$ to adapt the proofs in [7, §11.5] to our relative setting. We construct the homotopy $H'_I$ such that its image outside of $\overline{\phi}_U$ will be controlled completely by definable functions in $\Gamma$. This enables us to choose the inflation homotopies $H'_{\text{inf}f}$ and $H'_{\text{inf}f-\text{primary}}$ so that they fix the image of $H'_I$ as well. Furthermore, $H'_I$ restricts to well defined homotopies along the fibres of $\overline{\phi}_U | \Upsilon_{bcf}$. It is hence compatible with $\overline{\phi}_U | \Upsilon_{bcf}$.
5.4.1. Preliminaries. By our induction hypothesis, $H_{b_f}^r$ descends to a deformation retraction $H_b: b_f \times \hat{U} \to \hat{U}$ whose image is a $\Gamma$-internal iso-definable set which we shall denote $\Upsilon_b$. Furthermore, $H_b$ satisfies properties (5) and (6) of Theorem 1.1. For ease of notation, until the end of the proof of Lemma 5.7, we write $\Upsilon'$ in place of $\Upsilon_{b_f}$ and $\Upsilon$ in place of $\Upsilon_b$.

We now choose continuous injective maps on $\Upsilon'$ and $\Upsilon$ into the value group sort and construct a homotopy $H_{\Gamma}^{\text{top}}$ on the image of $\Upsilon'$ for this map. However, since we are making use of two inflation homotopies (cf. Remark 5.5), where $H_{\text{inf}}^{\text{f-prim}}$ acts on $V'_U$ and not $V'^{1/}_U$, we must take this into consideration when choosing coordinates in the value group sort for $\Upsilon'$.

Recall the closed subsets $Z'_U \subset V'_U$, $Z'_U \subset V'_U$ such that for every fibre over $u \in U$, $Z'_U$ is the exceptional divisor of the map $V'_U \to V'_U$ and $V'_U \setminus Z'_U$ is isomorphic to $V'_U \setminus Z'_U$. We identify $V'_U \setminus Z'_U$ with the subset $V'_U \setminus Z'_U \subset V'_U$. If $\Upsilon'_1 := \Upsilon' \setminus Z'_U$ and $\Upsilon'_2 := \Upsilon' \setminus Z'_U$ then we have a decomposition, $\Upsilon' = \Upsilon'_1 \cup \Upsilon'_2$.

Observe that $\Upsilon'_1 \subset V'_U$. By [7, Theorem 6.2.8], there exists a $K$-definable map $\alpha_0: V'_U \to \Gamma^N_{\infty}$ such that $\alpha_0$ is continuous and its restriction to $\Upsilon'_1$ is injective. Similarly, there exists a $K$-definable map $\alpha_1: V'_U \to \Gamma^N_{\infty}$ such that $\alpha_1$ is continuous and its restriction to $\Upsilon'_2$ is injective. We abuse notation and write $\alpha_0$ and $\alpha_1$ in place of $\alpha_0$ and $\alpha_1$, respectively. Let $\alpha'_0$ denote the composition $\tilde{V}'_U \to \alpha'_0 \to \Gamma^N_{\infty}$.

Once again, by [7, Theorem 6.2.8], there exists a $K$-definable map $\alpha: U \to \Gamma^M_{\infty}$ such that the induced map $\tilde{\alpha}: \hat{U} \to \Gamma^M_{\infty}$ is continuous and restricts to an injective map on $\Upsilon$. We abuse notation and use $\alpha$ itself in place of $\tilde{\alpha}$. The proof of [7, Theorem 6.2.8] shows that if $x: \Gamma^M_{\infty} \to \Gamma^N_{\infty}$ is a coordinate then $(x \circ \alpha)^{-1}(\{z \in \Gamma^M_{\infty} | x(z) = \infty\})$ is of the form $\bar{Z}$ where $Z$ is some Zariski closed subset of $U$. Since $U$ is irreducible, we can assume that for every coordinate $x$, $x \circ \alpha$ is not identically $0$ on $\hat{U}$. Hence the locus of points $u \in U$ such that $x \circ \alpha(u) < \infty$ for every coordinate $x$ on $\Gamma^M_{\infty}$ is a Zariski open dense subset of $U$. By property (5), we deduce that we have a continuous injective definable map $\alpha: \Upsilon \to \Gamma^M_{\infty}$.

As in [7, §11.5], we may assume that $G$ acts on the coordinates of $\Gamma^N_{\infty}$ and on the coordinates of $\Gamma^M_{\infty}$ such that the map $x \mapsto (\alpha'_0(x), \alpha_1(x))$ is $G$-equivariant. We simplify notation and write $f := \tilde{\alpha}_0|\Upsilon'$ and $N := N_1 + N_2$. We define $\alpha': \Upsilon' \to \Gamma^{M+1}_{\infty}$ by $\alpha'(x) := (\alpha'_0(x), \alpha_1(x), \alpha(f(x)))$. Observe that $\alpha'$ is a well defined, injective, continuous $K$-definable map such that

$$f' \circ \alpha' = \alpha \circ f$$

where $f'$ is the projection $\Gamma^{N+1-M}_{\infty} \to \Gamma^M_{\infty}$. We abuse notation and write $\xi_i$ for the functions $\xi_i \circ \alpha'^{-1}$.

Let $W'$ and $W$ denote the images of $\Upsilon'$ and $\Upsilon$ via the maps $\alpha'$ and $\alpha$ respectively. Note that the action of $G$ on $W'$ restricts to a well defined action on the fibres of $f'|_{W'}$. Let $\rho_0: V' \to [0, \infty]$ denote the schematic distance from the closure of $D'_U$ in $V'$. Let $\eta$ and $\rho$ be as in the proof of Lemma 5.3.

Let $d_{\text{ord}}: V \to \Gamma^N_{\infty}$ denote the schematic distance to the closed subset $V \setminus U$. After modifying the injection $\alpha'$, we can assume that there exists coordinates $x_h, x_{hh}, x_v$ on $\Gamma^N_{\infty}$ and $x_v$ on $\Gamma^M_{\infty}$ such that $x_{hh} \circ \alpha', x_h \circ \alpha', x_v \circ \alpha'$ and $x_v \circ \alpha$ correspond to $\rho$, $\rho_0 \circ b'_U$, $\eta$ and $d_{\text{ord}}$ respectively where $b'_U$ is the map $V'_U \to V_U$.

**Lemma 5.6.** The space $W' \cap \Gamma^{N+M}_{\infty}$ is closed in $\Gamma^{N+M}_{\infty}$.

**Proof.** Let $x \in \Gamma^{N+M}_{\infty}$ be a limit point of the set $W' \cap \Gamma^{N+M}_{\infty}$ and $\gamma := x_v(x)$. We show that $x \in W'$. This is a consequence of the fact that $W' \cap [x_v \leq \gamma]$ is $\sigma$-compact with respect to $x_{hh}$ and $x_v$. □
Lemma 5.7. Let
\[ W'_1 := (W' \cap \Gamma^{N+M}) \bigcup [x_h = \infty]. \]
There exists a $K$-definable deformation retraction
\[ H^{\text{trop}}_{\Gamma f}; [0, \infty] \times W'_1 \to W'_1 \]
which satisfies the following properties.

1. The deformation $H^{\text{trop}}_{\Gamma f}$ leaves the functions $\xi_i$ invariant, is $G$-equivariant and preserves the fibres of the morphism $f'$.
2. Let $W'_0$ denote the image of the deformation retraction $H^{\text{trop}}_{\Gamma f}$. There exists a $K$-definable open subset $W'_0$ of $W'$ that contains $W'_0 \cap [x_h = \infty]$ and $m \in \mathbb{N}$ and $c \in \Gamma(K)$ such that $x_i \leq (m + 1)x_h + c$ for every $1 \leq i \leq N$.
3. For every $x \in W$, if $W'_x$ is pure of dimension $n$ then for every $x$, $W'_0x$ is pure of dimension $n$.

Proof. We adapt the proof of [7, 11.5.1] to our setting. Before doing so, we give a rough outline of this proof. The specific details can be found in loc.cit. This is the case when $Y$ is a point and $M = 0$. In this situation, we choose a $G$-invariant $K$-definable cell decomposition $D$ of $\Gamma^N$ that respects the definable set $W' \cap \Gamma^N$ as well as all sets of the form $[x_a = x_b]$ or $[x_a = 0]$ for all coordinates $x_a, x_b$. We define the restriction of the deformation retraction $H^{\text{trop}}_{\Gamma f}$ to $\Gamma^N$ by specifying its behaviour on each cell of $D$. Let $D_0$ denote the sub collection of cells $C \in D$ such that every coordinate $x_i$ is $h$-bounded on $C$. By this we mean that there exists $m \in \mathbb{N}$, $c \in \Gamma(K)$ and $x_i(z) \leq mx(z) + c$ for every $z \in C$. Furthermore, when every coordinate is $h$-bounded on $C$, we shall say that the cell itself is $h$-bounded.

The homotopy $H^{\text{trop}}_{\Gamma f}$ will fix every point in every element of $D_0$. Let $C \not\in D_0$. We construct an element $eC \in \mathbb{Q}^N$ such that if a coordinate $x_i$ is $h$-bounded on $C$ then $x_i(eC) = 0$. For $x \in C$ and $t \in [0, \infty)$, let $H^{\text{trop}}_{\Gamma f}(t, x) := x - teC$. We verify that the definable function $H_f$ is continuous on $[0, \infty] \times C$ by induction on the dimension of the cell $C$. An important observation which makes this possible is the following. Since $x_i(teC) \geq 0$ for $t \in [0, \infty)$ and any coordinate $x_i$, we must have that $x - teC \not\in C$ for some $t$. Let $\tau(x)$ be the smallest such $t$. Observe that $H^{\text{trop}}_{\Gamma f}(\tau(x), x)$ must lie in a lower dimensional cell. It follows that the path $t \mapsto H^{\text{trop}}_{\Gamma f}(t, x)$ begins at $x$ when $t = 0$ and traverses finitely many cells in decreasing dimensions till it finally ends up in $D_0$.

The proof in the relative case follows the same steps as in the sketch above. We highlight only those points which require more than just the obvious adaptation.

Step 1. Preliminaries.

As in loc.cit., let $A$ denote the convex subgroup of $\Gamma(U)$ which is generated by $\Gamma(K)$ and define $B := \Gamma(U)/A$. Given a definable subset $X \subset \Gamma^d$ for some $t \in \mathbb{N}$, we define $\beta X$ to be the image of $X$ in $B^t$. We choose a cell decomposition $D$ of $\Gamma^{N+M}$ which respects the definable set $W' \cap \Gamma^{N+M}$, all sets of the form $[x_a = x_b]$ and $[x_a = 0]$ for all coordinates $x_a, x_b$ and is such that its push forward $f'(D)$ is a cell decomposition of $\Gamma^M$ which respects $W \cap \Gamma^M$. By definition, for every $C \in D$, its image in $\Gamma^M$ for the projection $f'$ is a cell $C$. By [12, Proposition 3.3.5], if $a \in C$ then the fibre $C'_a$ is a cell in $\Gamma^N \times \{a\}$. We can suppose that the decomposition $\mathcal{D}$ is such that for every $C \in \mathcal{D}$, $C$ is $h$-bounded iff for every $a \in f'(C)$, $C_a$ is $h$-bounded which in turn will be equivalent to saying that there exists $a \in f'(C)$ such that $C_a$ is $h$-bounded. Indeed, we modify the existing decomposition $\mathcal{D}$ so that it has the above property as follows. Let $C' \in \mathcal{D}$ and $C : = f'(C')$. The set $C_{h\text{-bd}}$ of those points $x \in C$ such that $C'_x$ is $h$-bounded is definable. This is a consequence of the
cell decomposition. We can now refine the decomposition $\mathcal{D}$ so that it respects the preimages of the definable sets $C_h$-bd and $C$-bd for every $C$.

For a cell $C' \in \mathcal{D}$, we now define the point $e_{C'}$. Let $\mathfrak{M}$ denote the last $M$ coordinates of $\Gamma^{N+M}$ and $\mathfrak{M}$ denote the first $N$ coordinates. Let
\[ \beta' C' := \beta C' \cap [x_h = 0] \cap [x_i = 0]_{i \in \mathfrak{M}} \]
and let $e_{C'}$ denote the barycentre of $\beta' C' \cap [\sum x_i = 1]$. By construction, if $x_i \in \mathfrak{M}$ or if $x_i$ is $h$-bounded on $C'$ then $x_i(e_{C'}) = 0$. It follows that if $C' \in \mathcal{T}_0$ then $e_{C'} = 0$.

Also, if $x_i$ is not $h$-bounded on $C'$ then $x_i(e_{C'}) > 0$.

For $x \in C'$, we define $H_{tf}^{\text{trop}}(x) := x - te_{C'}$. Clearly, $H_{tf}^{\text{trop}}$ is well defined. Furthermore, if $C = f'(C')$ then by construction, we see that for every $x \in C$, the homotopy $H_{tf}^{\text{trop}}$ restricts to a well defined homotopy along the fibre over $x$.

**Step 2. Continuity and end of the proof.**

The continuity of $H_{tf}^{\text{trop}}$ and assertion (2) of the Lemma can be shown by following the arguments in [7, Lemma 11.5.1] with little change. Note that the homotopy $H_{tf}^{\text{trop}}$ preserves the closures of the cells. By Lemma 5.6, $H_{tf}^{\text{trop}}$ preserves $W' \cap \Gamma^{N+M}$. Assertion (3) can be verified using the argument in loc.cit., and considering the hyperplane $L$: $\sum_{i \in \mathbb{N}} x_i = Mx_h + K$ where $M = N(m+1)$ and $K = Nc$.

It remains to treat the case when for some $x \in W$, $W'_x \cap (\Gamma^N \times \{x\})$ is empty. We attempt a relative version of [7, Lemma 11.5.2].

Let $\eta$ denote the generic point of $U$ and $V_{\eta}'$ be the generic fibre of the morphism $\phi'_{|U}$. Recall that we assumed that every generic point of every irreducible component of $V'_{U}$ is contained in the fibre over $\eta$. Let $V'_{10\eta}, \ldots, V'_{r\eta}$ denote the irreducible components of $V'_{\eta}$ and for every $j$, we set $V_{ij}'$ to be the Zariski closure of $V_{ij}'$ in $V'_{iU}$. By construction, we have that the $V_{ij}'$ are equidimensional and there exists a Zariski open subset $U_0 \subseteq U$ such that the maps $V_{ij}' \to U_0$ are flat. By [4, Corollary 9.6], we get that the fibres of $V_{ij}'$ are pure of dimension $\dim(V_{ij}' \cap U_0)$. Recall by construction that $D_{ij}'$ contains the intersections $V_{ij}' \cap V_{ij}'$. For every $j$, $j'$ such that $j \neq j'$. For every $j$, let $W_j'$ be the image of $T_{\alpha,j} \cap [V_{ij}' \setminus \bigcup_{j' \neq j} V_{ij}']$ for the map $\alpha'$. Observe that $W_j'$ is open and
\[ W' = \bigcup_j W_j' \bigcup \{[x_h = \infty] \cap W'\}. \]

For every $x \in W$, $W_{ix}'$ is pure of dimension $\dim(V_{ij}' \cap W_{ix})$.

Let $x_1, \ldots, x_N$ denote the first $N$ coordinates of $\Gamma^{N+M}$. Recall that the preimage of the locus $[x_i = \infty]$ via the function $x_i \circ \alpha'$ is of the form $\hat{Z}_i$ where $Z_i$ is a Zariski closed subset of $V'_{iU}$.

Let $j \in \{1, \ldots, r\}$. Let $\mathfrak{P}_j \subset \{1, \ldots, N\}$ be the subset of indices such that for $p \in \mathfrak{P}_j$, $Z_p \cap V_{ij}'$ is of dimension strictly less than $\dim(V_{ij}' \cap U_0)$. We shrink $U_0$ further to get that for every such $p \in \{1, \ldots, N\}$, $Z_p \cap V_{ij}'$ is flat over $U_0$ or empty. It follows that for every $x \in U_0$ and $p \in \mathfrak{P}_j$, $Z_p \cap V_{ij}'$ is of dimension strictly less than $\dim(V_{ij}' \cap U_0)$. Since for every $x \in W$, $W_{ix}'$ is pure of dimension $\dim(V_{ij}' \cap W_{ix})$, we deduce that for every $p \in \mathfrak{P}_j$, $x_p$ is not identically $\infty$ on $W_{ix}'$. Indeed, our choice of $p$ implies that $(V_{ij}' \setminus \bigcup_{j' \neq j} V_{ij}' \cup Z_p)) \cap V'_{iU_0}$ is a non-empty Zariski open subset of $V'_{iU_0}$. By construction, if $A := (V_{ij}' \setminus \bigcup_{j' \neq j} V_{ij}' \cup Z_p) \cap V'_{iU_0}$ then $\phi'_{|U}(A) = U_0$ and hence
\(\overline{\phi}_t(\overline{A}) = \overline{U}_0.\) Note that if \(x' \in \mathcal{Y}_b\) is such that \(x' \to x\) then the homotopies \(H'_{bf}\) and \(H'_{\text{curve}}\) restrict to well defined homotopies on \(\overline{\phi}_t^{-1}(x')\). As \(H_b\) satisfies property \((5)\) of Theorem \(1.1\), we see that \(x' \in \overline{U}_0\). Since \(H'_{bf}\) and \(H'_{\text{curve}}\) are also Zariski generalizing, we must have that \(\mathcal{Y}_{bcf} \cap (\overline{\phi}_t^{-1}(x'))\) intersects \(V_{b1}^I \setminus (\bigcup_{j \neq J} V_{f1}^I \cup Z_b)\) non-trivially. It follows that \(x_p\) is not identically \(\infty\) on \(W_j^x\).

Let \(n_j\) be the cardinality of the set \(\mathfrak{P}_j\). Consider the embedding \(\alpha'_j: W'_j \to \Gamma_{\infty}^n\) given by \(z \mapsto ((x_i(z)))_{i \in \mathfrak{P}_j}, (x_i(z))_{i \in \mathbb{R}}\). Note that \(\alpha'_j\) is a homeomorphism onto its image. Let \(W_{j0}\) be the preimage of the open set \(\alpha'_j(W'_j) \cap \Gamma_{\infty}^n\). Observe that \(W_{j0}\) is \(z\)-open and \(z\)-dense in \(W_j^x\). Let \(H'_{trop}\) be the deformation retraction induced on \(W_{j0}\) from Lemma \(5.7\). Let \(W^0 := \bigcup_j W_{j0}\) and \(W^m := (W^0 \setminus [x_v = \infty]) \cup [x_h = \infty]\). Let

\[
H'_{trop}: [0, \infty] \times W^m \to W^m
\]
denote the deformation retraction whose restriction to \(W_{j0}\) for every \(j\) is induced by \(H'_{trop}\). Let \(W'_0\) denote the image of \(H'_{trop}\). Observe that \(H'_{trop}\) satisfies the following properties.

1. The deformation \(H'_{trop}\) leaves the \(\xi_i\) invariant, is \(G\)-equivariant and preserves the fibres of the morphism \(f'\).

2. There exists a \(K\)-definable open subset \(W'_0\) of \(W'\) that contains \(W^0_0 \setminus [x_h = \infty]\), \(m \in \mathbb{N}\) and \(c \in \Gamma(K)\) such that on \(W'_0 \cap W'_j\), \(x_i \leq (m + 1)x_h + c\) for every \(i \in \mathfrak{P}_j\).

5.5. Completing the proof of Theorem \(1.1\). In order to complete the proof of Theorem \(1.1\), we choose inflation homotopies \(H'_{inff}\) and \(H'_{inff-\text{primary}}\) using Lemma \(4.8\) such that the image of the composition \(H'_{bf} \circ H'_{\text{curve}} \circ H'_{inff} \circ H'_{inff-\text{primary}}\) is contained in a subspace \(P\) of \(\mathcal{Y}_{bcf}\) such that \(\alpha'_p\) is a homeomorphism. We can then define a homotopy \(H'_{t}\) on \(P\) via the tropical homotopy \(H'_{trop}\) so that the composition

\[
H'_{t} \circ H'_{bf} \circ H'_{\text{curve}} \circ H'_{inff} \circ H'_{inff-\text{primary}}\]

is well defined and fixes its image. Here we abuse notation and write \(H'_{t}\) for the homotopy on \(P\).

For every \(i \in \mathfrak{P}_t\), let \(y_i := \min\{x_i, (m + 1)x_h + c\}\). Observe that for every \(i\), \((x_i, (m + 1)x_h + c)\) is contained in \(\overline{D'}_{1U}\) and hence also in \(\overline{D'_{mU}}\). When there is no ambiguity, we simplify notation and write \(y_i\) in place of the composition \(y_i \circ \alpha'_i\).

We now construct the homotopy \(H'_{inff}\) using Lemma \(4.8\). We deduce from \((1)\) of \(\S 5.1\) that for some copy of \(\mathbb{A}^m \subset \overline{E}\), the morphism \(g_{\overline{U}}\) restricts to a map \(V_{l1U}^I \setminus D'_{1U} \to \mathbb{A}^m \times U\). By our choice of \(D'_{1U}\) in \(\S 5.1\) and Lemma \(4.5\), this map satisfies requirement \((2)\) of Lemma \(4.8\).

Let \(H'_{inff}: [0, \infty] \times \overline{V_{l1U}^I} \to \overline{V_{l1U}^I}\) be the deformation retraction constructed in Lemma \(4.8\) with respect to the morphisms \(\phi_{\overline{U}}\) and \(g_{\overline{U}|V_{l1U}^I \setminus D'_{1U}}\), the divisor \(D'_{1U}\), the family \(\{\xi_i\} \cup \{y_j \circ \alpha'_j\}\) and the group \(G\). Note that \(H'_{inff}\) is compatible with the morphism \(\phi_{\overline{U}}: V_{l1U}^I \to U\). Furthermore, \(H'_{bf} \circ H'_{\text{curve}} \circ H'_{inff}\) is a well defined homotopy which respects the levels of the functions \(\xi_i\) and is equivariant for the action of the group \(G\). Note that at this stage we cannot say that the image of \(H'_{bf} \circ H'_{\text{curve}} \circ H'_{inff}\) will be contained in \(W^m\) - the domain of the homotopy \(H'_{t}\).

Let \(Z^t_{1U} \subset V_{l1U}^I\) be as introduced in \(\S 5.1\). Recall that by definition, \(\rho_0: V_{l1U}^I \to \Gamma_{\infty}\) was defined to be the schematic distance to \(D'_{lU}\) and \(x_h \circ \alpha' = \rho_0 \circ b_U\). By construction
Let $v_i: V_U' \to \Gamma_\infty$ be the coordinates of the map $\alpha_0: \tilde{V}_U \to \Gamma_\infty^i$, i.e. $v_i$ is the composition of $\alpha_0$ and the $i$-th projection $\Gamma_\infty^i \to \Gamma_\infty$. For every $i$, let $w_i := \min\{v_i, (m+1)p_0+c\}$. Observe that $\rho_0^{-1}(\infty) = D_U$. Let $H_{inff-\text{primary}}'$ be the homotopy as provided by Lemma 4.8 for the morphism $V_U' \to U$, the morphism $gv: V_U' \to \mathbb{P}^m \times U$, the divisor $D_U$, the functions $\{w_i\}_j \cup \{\xi_i\}_j$ and the group $G$.

Observe that the composition $H_{bcf}'' \circ H'_{\text{curvesf}} \circ H_{inff}'' \circ H_{inff-\text{primary}}'$ is well defined. Let $\beta: \tilde{V}_U \to \Upsilon'$ be the retraction associated to this composition and $B := \beta(\tilde{V}_U)$.

**Lemma 5.8.** The morphism $\alpha': \Upsilon_{bcf}' \to W'$ restricts to a homeomorphism from $\beta(\tilde{V}_U)$ onto its image.

**Proof.** The morphism $\alpha'$ is injective and continuous. It suffices to show that the restriction $\alpha'|_B$ is a closed map from $B$ to $\alpha'(B)$. Note that $\alpha'(B)$ is an iso-definable subset of $\Upsilon_{bcf}'$. This is a consequence of the fact that $\beta(\tilde{V}_U') \subset \Upsilon_{bcf}'$ is iso-definable and $\Gamma$-internal and hence $\beta(\tilde{V}_U') = \beta(\tilde{V}_U)$. Let $Z \subset B$ be an iso-definable closed subset. We claim that $\alpha'(Z)$ is closed in $\alpha'(B)$. Let $a$ belong to the closure of $\alpha'(Z)$ in $\alpha'(B)$. By [7, Proposition 4.2.13], there exists a definable type $q$ that concentrates on $\alpha'(Z)$ with limit point $a$. Let $a' \in B$ be the preimage of $a$ and $q'$ be the preimage of the definable type $q$. We show that $a'$ is a limit point of $q'$. Recall the coordinate $x_\ell$ on $\Gamma_\infty^M$ which corresponds to the schematic distance $d_{\text{bord}}: V \to \Gamma_\infty \times U$.

Let $\lambda \in \Gamma$ be such that $x_\ell \circ f'(a) < \lambda$ where $f'$ is the projection $\Gamma_\infty^{M+1} \to \Gamma_\infty^M$. Note that if $U_\lambda := \{x \in V|d_{\text{bord}}(x) \leq \lambda\}$ and $V_{U_\lambda}' := (\phi_U')^{-1}(U_\lambda)$ then $\tilde{V}_{U_\lambda}'$ is definably compact. It follows that $\beta(\tilde{V}_{U_\lambda}') \subset B$ is definably compact and hence $\alpha'$ restricts to a homeomorphism from $\beta(\tilde{V}_{U_\lambda}')$ onto its image. Since $H_b$ preserves $d_{\text{bord}}$, we deduce that $B_\lambda := B \cap \tilde{V}_{U_\lambda}' = \beta(\tilde{V}_{U_\lambda}')$. By construction, we have that $f' \circ \alpha' = \alpha \circ f$ where $f = \phi_{U_\lambda}'$. Hence, $\alpha'$ restricts to a homeomorphism from $B_\lambda$ onto $[x_\ell \leq \lambda] \cap \alpha'(B)$. Note that $B_\lambda$ and $[x_\ell \leq \lambda] \cap \alpha'(B)$ are closed neighbourhoods of $a'$ and $a$ respectively. It follows that $a'$ is a limit point of $q'$. Since $Z$ is closed, $a' \in Z$. It follows that $a \in \alpha'(Z)$. This concludes the proof.

We define $H_{inff}'$ as follows. Observe firstly that $\alpha'(B) \subset W''$. This is a consequence of the inflation properties of $H_{inff}'$ and $H_{inff-\text{primary}}'$. Firstly, note that the image of $H_{inff-\text{primary}}'$ is contained in $\tilde{V}_{U_\lambda}' \times Z_{U_\lambda}'$. For any $u \in U$, if $Z \subset V_u'$ is a closed sub variety of dimension strictly smaller than $\dim(V_u)$ then the intersection of $\tilde{Z}$ with the image of $H_{inff}' \circ H_{inff-\text{primary}}'$ will be contained in $\tilde{D}_{U_\lambda}'$. Let $x \in \beta(\tilde{V}_U)$ and $t \in [0, \infty)$. We set $H_{inff}'(t, x) := \alpha'^{-1}(H_{\text{trop}}(t, \alpha'(x)))$. By Lemma 5.8, $H_{inff}'$ is a well defined homotopy. We have thus shown that $H_{inff}' \circ H_{inff} \circ H_{\text{curvesf}} \circ H_{inff} \circ H_{inff-\text{primary}}'$ is a well defined homotopy.

We now show that $H_{inff}' \circ H_{inff} \circ H_{\text{curvesf}} \circ H_{inff} \circ H_{inff-\text{primary}}'$ fixes its image. From the construction, we see that it suffices to verify that $H_{inff}$ and $H_{inff-\text{primary}}'$ fix this image. Recall that we used $W_0'$ to denote the image of $H_{inff}'$. Let $\Upsilon_0'$ denote the image of $H_{inff}'$. Let $w \in \Upsilon_0' \subset U$. We check that $H_{inff}'$ and $H_{inff-\text{primary}}'$ both fix $W_0'$. To do so we borrow the notation from the previous section. Recall that
we have a decomposition

\[ W' = \bigsqcup_{j} W'_j \sqcup ([x_h = \infty] \cap W'). \]

On \( W'_j \cap W'_k \), for every \( i \in \mathfrak{P} \), we must have that \( y_i = x_i \). Furthermore, if \( i \notin \mathfrak{P} \), then \( x_i \) is identically \( \infty \) on \( V'_j \). Since \( H'_{\text{aff}} \) respects the levels of the functions \( y_i \) for all \( i \), by [7, Lemma 8.3.1(2)], it fixes \( W'_j \cap W'_k \).

We now show that \( H'_{\text{inff-prim}} \) fixes the image of the composition \( H'_{\text{ff}} \circ H'_{\text{bf}} \circ H'_{\text{curves}} \circ H'_{\text{aff}} \circ H'_{\text{inff-prim}} \). The argument is identical to the one made above but makes use of the fact that the image of the composition must lie in the complement of \( Z^1_U \) and is hence controlled completely by the coordinates \( v_i \).

One observes that each of the homotopies in the composition respects the levels of the functions \( x_i \) and the action of the group \( G \). Furthermore, the induction hypothesis that gives rise to \( H'_{\text{ff}} \) and (3) of Lemma 5.7 guarantee that for every \( z \in T_b \), the fibre over \( z \) is pure of dimension \( n \) where \( n \) is the dimension of the generic fibre for the map \( V'_U \to U \). We now show how to ensure that the composition is Zariski generalizing. The argument is similar to the one that appears in [7, §11.6]. By Lemma 5.8, \( \alpha' \) restricts to a homeomorphism from \( \beta(V'_U) \) onto its image. Hence, by [7, Corollary 10.4.6], it suffices to verify that \( H'^{\infty_{\text{top}}} \) is Zariski generalizing. This can be done as in §11.6 of loc.cit. Lastly, our construction of \( H_b \) ensures that it satisfies properties (5) and (6) of Theorem 1.1. This concludes the proof.

\[ \square \]

6. When the base is a curve

Theorem 1.1 asserts the existence of compatible deformations generically over the base. In the proof of this theorem, at several stages, we shrunk the base so as to obtain that the family behaved in a tame manner. When the base is a curve, we do not need to shrink the base constantly to obtain tame properties of the family. This allows us to prove the following theorem.

**Theorem 6.1.** Let \( S \) be a smooth connected \( K \)-curve and \( X \) be a quasi-projective \( K \)-variety. Let \( \phi: X \to S \) be a surjective morphism such that every irreducible component of \( X \) dominates \( S \). Let \( \{x_i: X \to \Gamma_\infty\} \) be a finite collection of \( K \)-definable functions. Recall that the functions \( x_i \) extend to functions \( \xi_i: \hat{X} \to \Gamma_\infty \). Let \( G \) be a finite algebraic group acting on \( X \) such that the action of \( G \) respects the fibres of the morphism \( \phi \). Let \( s \in S(K) \). There exists a Zariski open subset \( U \subset S \) containing \( s \) and compatible homotopies \( (H, \Upsilon) \) of \( \hat{U} \) and \( (H', \Upsilon') \) of \( \hat{X}_U \) such that the following hold.

1. The homotopy \( H \) is in fact a deformation retraction.
2. The images \( \Upsilon \subset \hat{U} \) and \( \Upsilon' \subset \hat{X}_U \) are \( \Gamma \)-internal.
3. The homotopy \( H' \) respects the functions \( \xi_i \) i.e. \( \xi_i(H'(t, p)) = \xi_i(p) \) for every \( p \in \hat{X} \) and \( t \in \mathbb{I} \).
4. The action of the group \( G \) on \( X \) extends to an action on \( \hat{X} \). The deformation \( H' \) can be taken to be \( G \)-invariant i.e. for every \( g \in G' \), \( H'(t, g(p)) = g(H'(t, p)) \).
5. The homotopy \( H' \) is Zariski generalizing i.e. if \( W \subset X_U \) is a Zariski open subset then \( H' \) restricts to a well defined homotopy on \( \hat{W} \).

\[ ^3_XU := X \times_S U \]
6.1. Initial reductions.

Remark 6.2. 

(1) By arguments similar to those appearing in the proof of Lemma 4.2 and using Lemma 4.1, one shows that if Theorem 6.1 is true for \( G \)-equivariant morphisms \( \phi: X \to S \) between projective \( K \)-varieties whose fibres are pure and where \( S \) is a smooth projective connected \( K \)-curve then the theorem is true in general. Henceforth, unless otherwise stated, we will assume \( \phi: X \to S \) is a flat morphism between projective \( K \)-varieties such that the fibres of \( \phi \) are pure and \( S \) is a smooth, connected curve.

(2) The functions \( \xi_i: V' \to \Gamma_\infty \) can be taken to be \( v + g \)-continuous [7, §11.2]. It follows by Lemma 10.4.3 in loc.cit. that for every \( i \), \( \xi_i^{-1}(\infty) \) is a subvariety of \( V' \). For every \( i \), let \( Z_i := \xi_i^{-1}(\infty) \). Since \( S \) is a curve, we can realize \( Z_i \) as the union of a horizontal divisor \( Z_{ih} \) and a vertical divisor \( Z_{iv} \). Indeed, \( Z_{ih} \) is the closure of those irreducible components of \( Z_i \) that dominate \( S \) while \( Z_{iv} \) is the union of those irreducible components that map to a \( K \)-point in \( S \).

6.2. The inflation homotopy.

Remark 6.3. Let \( s \in S(K) \) be as in the statement of Theorem 6.1. We apply the first part of Lemma 4.3 to obtain a Zariski open affine neighbourhood \( U \) of \( s \) such that the morphism \( \phi \) factors through a finite \( G \)-equivariant morphism \( f: X_U \to \mathbb{P}^m \times U \) where \( G \) acts trivially on \( \mathbb{P}^m \times U \) and \( m = \dim(X_s) \). Let \( T \subset \mathbb{P}^m \times U \) be as given by Lemma 4.6. Let \( H := \mathbb{P}^m \setminus \mathbb{A}^m \). We enlarge \( T \) so that it contains \( H \times U \). Let \( D := f^{-1}(T) \). We enlarge \( T \) so that \( D \) contains the closed sub-varieties \( Z_{ih} \) introduced above and remains \( G \)-invariant.

We apply the second part of Lemma 4.3 to obtain a variety \( X'_U \) that fits into a commutative diagram

\[
\begin{array}{ccc}
X'_U & \xrightarrow{f'} & E \times U \\
\downarrow & & \downarrow \text{id} \\
X_U & \xrightarrow{f} & \mathbb{P}^m \times U \\
\downarrow & & \downarrow \phi \\
U & & \mathbb{P}^m \times U \\
\end{array}
\]

such that \( b: E \to \mathbb{P}^m \) is the blow up at a \( K \)-point \( z \), the restriction of \( (p \circ f') \) to \( b'^{-1}(D) \) is finite surjective onto \( \mathbb{P}^{m-1} \times U \) and the square in the diagram is cartesian. We define \( D' \subset X'_U \) to be the union of \( b'^{-1}(D) \) and the preimage via \( (b \times \text{id}) \circ f' \) of \( \{ z \} \times \mathbb{P}^m \). Let \( \phi' := \phi \circ b' \). Lastly, for every \( i \), we set \( \xi'_i := \xi_i \circ b' \).

6.3. Relative curves homotopy. Our goal in this section is to construct a homotopy on a family of curves similar to the construction in §5.2. The difference between the results presented here and those before is that we no longer have the freedom to shrink the base \( U \) arbitrarily. As a result, we proceed a little differently and make use crucially of the fact that the base is of dimension 1.

We adapt the construction in [7, §11.3] of the relative curve homotopy to the fibration \( p' := p \circ f' : X'_U \to F \times U \) where \( F = \mathbb{P}^{m-1} \). As in §5.2, there exists an affine open subset \( F_0 \subset F \) such that \( p^{-1}(F_0 \times U) \subset E \times U \) is isomorphic.

\[\text{Remark 6.6.} \quad \text{The action of } G \text{ on } S \text{ is assumed to be trivial.}\]
to \( \mathbb{P}^1 \times (F_0 \times U) \). Let \( W := F_0 \times U \), \( A := p'^{-1}(W) \subset X'_U \) and \( B := p^{-1}(W) \subset E \times U \).

The homotopy \( H'_{\text{curves}} \).

We fix three points \(- 0, 1, \infty \) on \( \mathbb{P}^1 \). Recall that given a divisor \( P \) on \( B \), \([7, \S 10.2]\) implies that there exists a well defined definable map \( \psi_P : [0, \infty) \times B \to \overline{B}/W \) which fixes \( \bar{P} \). These homotopies played a crucial role in \([5, 2]\).

**Lemma 6.4.** There exists a divisor \( P \subset B \) which satisfies the following properties.

1. For any divisor \( P' \subset B \) that contains \( P \), the definable map \( \psi_{P'} : [0, \infty] \times B \to \overline{B}/W \) lifts uniquely to a definable map \( h : [0, \infty] \times A \to A/W \) such that:
   
   (a) For every \( i \), the function \( h \) preserves the levels of the functions \( \xi_i \).
   
   (b) The function \( h \) is \( G \)-invariant.

2. Recall the divisor \( D' \subset X'_U \) from Remark 6.3 which is finite over \( F \times U \). We have that \( f'(D' \cap A) \subset P \).

3. Let \( W' \subset W \) be the open subspace over which the divisor \( P \) is finite. The restriction \( W' \to U \) is surjective.

**Proof.** We use arguments as in the proof of Lemma 4.6 to show that there exists a constructible set \( P_1 \subset B \) such that for every \( w \in W \), \( P_{1w} := P_1 \cap B_w \) is a finite set of those points \( c \in B_w \) such that \( \text{card} \{ f'^{-1}(c) \} < \sup_{b \in B_w} \text{card} \{ f^{-1}(b) \} \). We can further enlarge \( P_1 \) so that for every \( w \in W \), if \( F B_w \subset B_w \) denotes the forward branching points \([7, \text{Definition 7.4.2}] \) of the morphism \( f'_w := f'|_{A_w} \) then the convex hull of \( P_{1w} \) contains \( F B_w \). One sees from the proof of \([7, \text{Lemma 11.3.1}] \) that \( P_1 \) can be chosen so that it continues to be constructible and finite over \( W \) i.e. for every \( w \in W \), \( P_{1w} \) is finite.

Let \( P_1 \subset B \) denote the Zariski closure of \( P_1 \) in \( B \). We claim that \( \psi_{\overline{P}_1} \) lifts uniquely to a definable map \( h_{\overline{P}_1} : [0, \infty] \times A \to A/W \). It suffices to show that if given \( w \in W \), then the restriction \( \psi_{\overline{P}_1}|_{[0, \infty] \times B_w} \) lifts to a definable map \( h_{\overline{P}_1}|_{[0, \infty] \times A_w} : [0, \infty] \times A_w \to \overline{A}_w \). This can be accomplished by the arguments in \([7, \text{Proposition 7.4.6}] \) and Lemma 4.5. Note that in \([7, \text{Proposition 7.4.6}] \), one constructs a path in \( \overline{A}_w \) (which is a lift of a path in \( B_w \)) starting from a Zariski closed point which is distinct from a point of ramification up to a forward branching point. However, the only reason to have the hypothesis that the Zariski closed point be distinct from the ramification locus is to use Lemma 7.3.1 in loc.cit. which we accomplish by Lemma 4.5. Let \( W_1 \subset W \) be the open subspace over which \( \overline{P}_1 \) is finite. We claim that the projection \( W_1 \to U \) is surjective. If \( W_1 \) was not surjective then we deduce that \( \overline{P}_1 \) must contain a subspace of the form \( E \times \{ u \} \) for some \( u \in U \) using the notation from Remark 6.3. Observe that \( \dim(\overline{P}_1) = \dim(E) = m \). Hence we must have that \( E \times \{ u \} \) is an irreducible component of \( \overline{P}_1 \) which implies that \( P_1 \cap (E \times \{ u \}) \) is dense in \( E \times \{ u \} \). By construction of \( P_1 \), this is not possible. We have thus verified the claim.

We now enlarge \( \overline{P}_1 \) to a divisor \( P \) so that the lift \( h \) of \( \psi_P \) respects the levels of the functions \( \xi_i \). We proceed as follows. We enlarge \( \overline{P}_1 \) so that it contains the divisor \( \{ \infty \} \times W \subset B \). Observe that \( B_0 := B \setminus (\{ \infty \} \times W) = A^1 \times W \) is affine. It follows that \( A_0 := f'^{-1}(B_0) \) is affine as well. By the proof of \([7, \text{Lemma 10.2.3}] \), we see that for every \( i \), there exists a finite family \( \{ \epsilon_j : B \to \Gamma_{\infty} \} \) of definable functions such that if a definable function \( [0, \infty] \times B \to \overline{B}/W \) preserves \( \epsilon_j \) for every \( j \) then any lift \( [0, \infty] \times A \to A/W \) must preserve \( \xi_i \).
Since $B_0$ is affine, for every $i, j$, the restriction $(\epsilon_{ij})|_{B_0}$ factorizes through functions of the form $\text{val}(g)$ where $g$ is a regular function on $B_0$. Hence there exists finitely many regular functions $g_1, \ldots, g_r$ on $B_0$ such that if $Z_1, \ldots, Z_r$ denotes the zeroes of the functions and $P_2 = \bigcup Z_i$ then the homotopy $\psi_{P_2}$ respects the levels of the functions $\epsilon_{ij}$ for every $i, j$. We modify $P_2$ slightly as follows. For every $t$, let $Z_{ht}$ be the union of those components of $Z_t$ which are generically finite over $W$. If $W_2 \subset W$ denotes the locus over which the Zariski closure $Z_{ht}$ in $B$ is finite then we claim that $W_2 \to U$ is surjective. Indeed, the Zariski closure $Z_{ht}$ is such that it cannot contain a subset of the form $E \times \{t\}$ for dimension reasons. Let $P_3 := \bigcup Z_{ht}$.

Let $P = P_1 \cup P_3 \cup f'(D' \cap A)$. By construction, $\psi_P$ lifts to a homotopy on $A$ and respects the levels of the functions $\xi_i$. To conclude a proof of the lemma, we must show that the lift $h$ is $G$-invariant. This follows from the uniqueness of the lift $h$.

□

Using Lemma 4.5, we deduce that our choice of $D' \subset X'_U$ implies that we can apply Lemma 4.8 to the morphism $X'_U \to U$, the map $X'_U \times D' \to \hat{\mathbb{A}}^m \times U$, the functions $\xi_i: X'_U \times D' \to \Gamma^\infty$ and the group $G$.

**Lemma 6.5.** There exists a constructible set $C \subset X'_U$ with the following properties.

1. The set $C$ maps surjectively onto $\mathbb{P}^{m-1} \times U$ via $p'$
2. Let $H'\text{inff}$ be the homotopy obtained by applying Lemma 4.8 to the morphism $X'_U \to U$, the map $X'_U \times D' \to \hat{\mathbb{A}}^m \times U$, the functions $\xi_i: X'_U \times D' \to \Gamma^\infty$ and the group $G$. The image $H'\text{inff}(e, X'_U)$ is contained in $\hat{\mathbb{C}}$.
3. Let $I_2 := [0, \infty]$. There exists a $v+g$-continuous homotopy $h'_\text{curvesf}: I_2 \times C \to C/\mathbb{P}^m \times U$. The image $\Upsilon'_2 := h'_\text{curvesf}(0, C) \subset X'_U/\mathbb{P}^{m-1} \times U$ is iso-definable and relatively $\Gamma$-internal over $\mathbb{P}^{m-1} \times U$.

**Proof.** We verify that $C := A \cup D'$ satisfies the assertions of the lemma by identical constructions and arguments as in Lemma 5.1 using Lemma 6.4 in place of [7, Lemma 11.3.2] □

6.4. **Theorem 6.1 and consequences.**

**Proof.** (Theorem 6.1) The proof is identical to that given in §5.3 of Theorem 1.1. Note that we do not claim that the homotopy $H'$ on $\hat{X}_U$ fixes its image i.e. that it is a deformation retraction. Hence, we do not require the relative tropical homotopy from §5.4. □

In the case that the morphism $\phi: X \to S$ in Remark 6.2 is of relative dimension 1 i.e. for every $s \in S(K)$, $X_s$ is a $K$-curve, we can verify that there exists deformation retractions of $\hat{X}$ and $\hat{S}$ which are compatible with $\hat{\phi}$ and whose images are $\Gamma$-internal.

The method of proof is to first show the result locally around an arbitrary point. The following lemma then allows us to glue the various relative homotopies to obtain a relative homotopy of the total family whose image is relatively $\Gamma$-internal.

We employ the following notation in the lemma below. Given a projective variety $V$, recall from [7, §3.10] the notion of a definable metric $m: V \times V \to \Gamma^\infty$. Let $D \subset V$ be a closed sub-variety of $V$. As in the proof of Lemma 10.3.2 in loc.cit., we define $\rho_D: V \to \Gamma^\infty$ as follows. For $x \in V$, set $\rho_D(x) := \sup \{m(x, d) \mid d \in D\}$. When there is no ambiguity about the divisor $D$ chosen, we simplify notation and write $\rho_D$ in place of $\rho$. 
Lemma 6.6. Let $f : V' \to V$ be a morphism of projective $K$-varieties. Let $D \subset V$ be a closed sub-variety and $D' := f^{-1}(D)$. Let $U := V \setminus D$ and $U' := V' \setminus D'$. Let $h : [0, \infty) \times U' \to \overline{U'}$ be a homotopy. Let $\epsilon \in [0, \infty)$ be $K$-definable. There exists a $K$-definable homotopy $g_{h, \epsilon} : [0, \infty] \times V' \to \overline{V'}$ such that if $x \in V$ and $\rho_D(x) \leq \epsilon$ then $g_{h, \epsilon}(0, V'_x) = h(0, V'_x)$.

Proof. We begin by verifying a certain technical condition that is necessary for the proof. Let $m'$ be a definable metric on $V'$. Let $m$ be a definable metric on $V$. Let $\gamma : U \to [0, \infty]$ be defined as $\gamma(u) := \sup \{ \gamma(u') | u' \in f^{-1}(u) \}$ where $\gamma(u')$ is the smallest element in $[0, \infty]$ such that $h(\gamma(u'), u')$ belongs to the ball $B(u', m', \rho_D(f(u'))) := \{ x \in U' | m'(u', x) \geq \rho_D(f(u')) \}$.

We claim that $\gamma$ is locally bounded on $U$ in the sense of [7, §10.1].

Let $u' \in U'$ and $t_0 \in \Gamma$. By continuity, $h^{-1}(B(u', m', \rho_D(f(u'))))$ is a $v + g$-closed neighbourhood of the point $(\infty, u') \in [0, \infty] \times U'$. The $v + g$-topology of $[0, \infty \times U'$ is the product of the $v + g$-topology on $[0, \infty)$ and the $v + g$-topology on $U'$. Hence we can assume that the closed neighbourhood of $(\infty, u')$ above contains an open neighbourhood $O_1 \times O_2$ where $O_1$ is of the form $[s, \infty)$ and $O_2$ is a $v$-open \(^5\) neighbourhood of $u'$. We can further shrink $O_2$ so that for every $o \in O_2$, $\rho_D(f(o)) = \rho_D(f(u'))$. This is a consequence of the fact that $\rho_D$ is a $v + g$-continuous function and hence $\rho_D^{-1}(\rho_D(f(u'))) = \rho_D^{-1}(\rho_D(f(u')))$ is a $v + g$-open neighbourhood of $f(u')$.

We can hence replace $O_2$ with $O_2 \cap f^{-1}(\rho_D^{-1}(\rho_D(f(u'))))$. It follows that for every $o \in O_2$, $\gamma(o) \leq s$. We have thus shown that $\gamma : U' \to \Gamma$ is locally bounded. We now deduce that as a consequence $\gamma$ is locally bounded. Indeed, let $u \in U$. There exists $\beta \in \Gamma$ large enough so that $B(u, m, \beta) \subset U$. Since $f$ is a projective morphism, we see that $f^{-1}(B(u, m, \beta))$ is bounded and definable. By [7, Lemma 10.1.7], we get that $\gamma(f^{-1}(B(u, m, \beta)))$ is bounded.

For any $\delta$, let $U_\delta := \{ u \in U | \rho_D(u) = \delta \}$ and $\gamma_1(\delta) := \sup \{ \gamma(u) | u \in U_\delta \}$. Since $U_\delta$ is bounded, by [7, Lemma 10.1.7] and the locally boundedness on the function $\gamma$, $\gamma_1(\delta) \in \Gamma$. Observe that since $\gamma_1$ is piece-wise affine, there exists $m \in N$ and $c_0 \in \Gamma$ such that if $\delta \geq 0$ then $\gamma_1(\delta) \leq m\delta + c_0$. Let $\gamma_{10}(\delta) := m\delta + c_0$. Let $\epsilon' \in \Gamma(K)$ be such that $\epsilon' > \epsilon$. Let $\gamma_2$ be a continuous function $\Gamma \to \Gamma$ which is defined as follows. For every $x \leq \epsilon$, $\gamma_2(x) = 0$. For $x \in [\epsilon, \epsilon']$, $\gamma_2(x) := (\gamma_{10}(\epsilon') \gamma_{10}(\epsilon')) x - (\gamma_{10}(\epsilon') \gamma_{10}(\epsilon')) \epsilon$ and for $x \geq \epsilon'$, $\gamma_2(x) := \gamma_{10}(x)$. By construction, $\gamma_2$ is continuous. Let $\gamma_3 : U' \to \Gamma$ be a continuous function $x \mapsto \gamma_2(\rho_D(f(x)))$.

Let $g_{h, \epsilon} : [0, \infty] \times V' \to \overline{V'}$ be defined as follows. We set $[g_{h, \epsilon}]([0, \infty] \times U') := h'[\gamma_3]$ and for every $t \in [0, \infty], d \in D'$, $g_{h, \epsilon}(t, d) := d$.

We now show that the canonical extension $G_{h, \epsilon}$ is continuous. We proceed as in [7, Lemma 10.3.2]. It suffices to verify that $G_{h, \epsilon}$ is continuous at a point $(t, d)$ with $t \in [0, \infty]$ and $d \in D'$. By definition, $G_{h, \epsilon}(t, d) = d$. Let $O \subset \overline{V'}$ be an open neighbourhood of $d$. We must show that there exists an open neighbourhood $W$ of $(t, d)$ in $[0, \infty] \times \overline{V'}$ such that $G_{h, \epsilon}$ maps the simple points of $W$ to $O$.

Let $M$ be a model of ACVF such that $O$ is pre-definable over $M$ and $d$ is an $M$-definable type. Let $z$ be a realization of $d_M$. Let $\alpha \in [0, \infty]$ be the smallest value such that $B(z, m', \alpha)^- := \{ y \in V' | m'(y) > \alpha \} \subset O$. Observe that $\alpha \in \Gamma(M(z))$.

\(^5\)It is $g$-open as well but we will use only that it is $v$-open.
Since $d$ is stably dominated, we get that $\alpha \in \Gamma(M)$. Let $W_0$ be the set of all $x \in V'$ such that $B(x, m', \alpha)^{-}$ is contained in $O$. The set $W_0$ is $v + g$-open and definable with parameters in $M$. Since $x \in W_0$, we see that $d \in W_0$.

Suppose the open set $W$ does not exist. As the simple points are dense, there exists sequences $t_i \rightarrow t$, $v_i \rightarrow d$ but $G_{h,c}(t_i, v_i) \notin O$. Since $v_i \rightarrow d$ and $W_0$ is an open neighbourhood of $d$, we get that there exists $i_0$ such that if $i \geq i_0$ then $v_i \in W_0$. It follows that $B(v_i, m', \alpha)^{-} \subset O$. The assumption that $v_i \rightarrow d$ implies that $f(v_i) \rightarrow f(d)$ which in turn implies that $\rho_d(f(v_i)) \rightarrow \infty$. Hence after increasing $i_0$ suitably, we get that if $i \geq i_0$ then $G_{h,c}(t_i, v_i) \subset B(v_i, m', \alpha)^{-}$. This implies that $G_{h,c}(t_i, v_i) \in O$ which gives a contradiction. Clearly, from the construction, $g_{h,c}(0, V'_2) = h(0, V'_2)$ for every $x \in U$. \qed

6.4.1. Deformation retractions for curves. As stated above, our goal is to show that when $\phi: X \rightarrow S$ is of relative dimension 1, there exist deformation retractions of $\hat{X}$ and $\hat{S}$ which are compatible with the morphism $\hat{\phi}$. We require the following lemmas to ensure that the homotopies we construct do indeed fix their image.

We introduce the following notation to simplify the statement and proof of Lemma 6.7. Let $F$ be a valued field and $C$ be an $F$-curve. Let $f: C \rightarrow \mathbb{P}^1_F$ be a finite morphism. Let $D \subseteq \mathbb{P}^1$ be a divisor. We say that the pair $(f, D)$ is homotopy liftable if the standard homotopy with stopping divisor $\psi_D: [0, \infty] \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ lifts to a homotopy $h: [0, \infty] \times C \rightarrow C$ via the morphism $\hat{f}$. Given a $v + g$-continuous function $\gamma: \mathbb{P}^1 \rightarrow [0, \infty]$, recall the cut-off homotopy $h_\gamma \circ f$ from [7, Lemma 10.4.6].

**Lemma 6.7.** Let $F$ be an algebraically closed valued field and let $C$ be an $F$-curve. Let $f_1: C \rightarrow \mathbb{P}^1_F$ and $f_2: C \rightarrow \mathbb{P}^1_F$ be finite morphisms. Let $D_1, D_2, D_{11}, D_{22} \subseteq \mathbb{P}^1$ be finite $F$-definable sets with the following properties.

1. The pairs $(f_1, D_1)$, $(f_1, D_{11})$, $(f_2, D_2)$ and $(f_2, D_{22})$ are homotopy liftable.
2. We have the inclusions $D_{11} \subseteq D_1$ and $D_{22} \subseteq D_2$.
3. Let $D'_1 := f_1^{-1}(D_1)$, $D'_{11} := f_1^{-1}(D_{11})$, $D'_2 := f_2^{-1}(D_2)$ and $D'_{22} := f_2^{-1}(D_{22})$.

Assume there exists a finite $F$-definable set $D' \subseteq D'_1 \cap D'_2$ containing $D'_{11}$ and $D'_{22}$ such that $f_1(D') \cap f_2(D')$ contains $\{0, \infty\}$. Let $h_1$ and $h_{11}$ ($h_2$ and $h_{22}$) lift the standard homotopies $\psi_{D_1}$ and $\psi_{D_{11}}$ ($\psi_{D_2}$ and $\psi_{D_{22}}$) respectively via $f_1$ ($f_2$). Let $\gamma_1, \gamma_2: \mathbb{P}^1 \rightarrow [0, \infty]$ be $v + g$-continuous functions. We then have that the image of the composition $h_2[\gamma_2 \circ f_2 \circ h_1[\gamma_1 \circ f_1]]$ is the intersection of the images of the homotopies $h_2[\gamma_2 \circ f_2]$ and $h_1[\gamma_1 \circ f_1]$.

**Proof.** We simplify notation and write $h_1[\gamma_1]$ in place of $h_1[\gamma_1 \circ f_1]$. Let $\mathcal{T}'_1$, $\mathcal{T}'_2$, $\mathcal{T}'_{11}$ and $\mathcal{T}'_{22}$ denote the images of the homotopies $h_1, h_2, h_{11}$, and $h_{22}$ respectively. Let $\mathcal{T}'$ denote the convex hull in $\hat{C}$ of the union of the finite set $D'$ and $\mathcal{T}'_{11} \cup \mathcal{T}'_{22}$. Furthermore, let $\mathcal{T}'_1[\gamma_1]$ and $\mathcal{T}'_2[\gamma_2]$ denote the images of the homotopies $h_1[\gamma_1]$ and $h_2[\gamma_2]$. Since $h_1$ and $h_2$ are deformation retractions, we see that it suffices to check that if $x \in \hat{C}$ lies in the image of the composition $h_2[\gamma_2 \circ h_1[\gamma_1]]$ then $x$ belongs to $\mathcal{T}'_{11}[\gamma_1] \cap \mathcal{T}'_{22}[\gamma_2]$. By construction of the standard homotopies and their lifts via finite morphisms in [7, §7.5], we see that it suffices to show that if $x \in C$ then $h_2[\gamma_2](0, h_1[\gamma_1](0, x))$ belongs to $\mathcal{T}'_{11}[\gamma_1] \cap \mathcal{T}'_{22}[\gamma_2]$. We prove the following lemma.

**Lemma 6.8.** Observe that $h_{11}(0, x) \in \mathcal{T}'_{11} \subset \mathcal{T}'$. Let $t$ be the largest element in $[0, \infty]$ such that $h_{11}(t, x) \in \mathcal{T}'$ and $p_{11} := h_{11}(t, x)$. Likewise, let $t'$ be the largest element in $[0, \infty]$ such that $h_{22}(t', x) \in \mathcal{T}'$ and $p_{22} := h_{22}(t', x)$. Then $p_{22} = p_{11}$.

**Proof.** By construction, $\mathcal{T}'$ is such that for every $a, b \in \mathcal{T}'$ there exists a path in $\hat{C}$ from $a$ to $b$ that does not intersect $\mathcal{T}'$ outside of the points $a$ and $b$. If $p_{11} \neq p_{22}$
then we deduce that there exists a path from $p_{11}$ to $p_{22}$ which does not lie in $\Gamma'$ outside of $\{p_{11}, p_{22}\}$. This is not possible. Hence $p_{11} = p_{22}$. \hfill $\Box$

Let $p := p_{11} = p_{22}$. Let $P_1$ be the path from $x$ to $p$ given by $r \mapsto h_{11}(r,x)$ for $r \in [t, \infty)$. Let $P_2$ denote the path $r \mapsto h_{22}(r,x)$ from $x$ to $p$ for $r \in [t', \infty)$. Observe that the morphism $f_1$ maps the paths $P'_1$ and $P'_2$ to paths from $f_1(x)$ to $f_1(p)$. Since there is exactly one injective path up to re-parametrization from $f_1(x)$ to $f_1(p)$ in $\mathbb{P}^1$ and this path lifts uniquely to a path in $\hat{C}$, we deduce that the images of $P'_1$ and $P'_2$ must coincide.

We identify the path $P_1$ with the closed interval $[t, \infty)$. Our discussion above implies that the path $r \mapsto h_{22}'(r,x)$ moves along $P_1$ for $r \in [t', \infty)$. Let $t'_1$ be the largest point on $[t, \infty)$ that belongs to $\Upsilon'_1[\gamma_1]$ and similarly, let $t'_2$ be the largest element that belongs to $\Upsilon'_2[\gamma_2]$. Recall that $h'_1$ and $h'_2$ are cut-offs of the homotopies $h_{11}'$ and $h_{22}'$ respectively. Hence, we see that $h'_1[\gamma_1]$ defines an injective path from $\infty$ to $t'_1$ and fixes $t'_1$. Likewise, $h'_2[\gamma_2]$ defines an injective path from $\infty$ to $t'_2$ and fixes $t'_2$.

Suppose $t'_1 \leq t'_2$. In this case, $t'_1 \in \Upsilon'_2[\gamma_2]$ and
\[ h'_2[\gamma_2] \circ h'_1[\gamma_1](0, x) = t'_1 \]
which belongs to the intersection $\Upsilon'_1[\gamma_1] \cap \Upsilon'_2[\gamma_2]$. Now, suppose $t'_1 > t'_2$. This implies that $t'_2 \in \Upsilon'_1[\gamma_1]$ and we see that the image of the composition is $t'_2$ which lies in the intersection $\Upsilon'_1[\gamma_1] \cap \Upsilon'_2[\gamma_2]$. This completes the proof. \hfill $\Box$

**Corollary 6.9.** Let $S$ be a smooth connected $K$-curve and $X$ be a quasi-projective $K$-variety. Let $\phi : X \to S$ be a surjective morphism such that every irreducible component of $X$ contains $S$. We assume in addition that the fibres of $\phi$ are of dimension 1. Let $\{\xi_i : X \to \Gamma_\infty\}$ be a finite collection of $K$-definable functions. Recall that the functions $\xi_i$ extend to functions $\xi_i : \hat{X} \to \Gamma_\infty$. Let $G$ be a finite algebraic group acting on $X$ such that the action of $G$ respects the fibres of the morphism $\phi$. There exists compatible deformation retractions $(H, T)$ of $\hat{S}$ and $(H', T')$ of $\hat{X}$ such that

1. The images $\Upsilon \subset \hat{S}$ and $\Upsilon' \subset \hat{X}$ are $\Gamma$-internal.
2. The homotopy $H'$ respects the functions $\xi_i$ i.e. $\xi_i(H'(t, p)) = \xi_i(p)$ for every $p \in \hat{X}$ and $t \in I$.
3. The action of the group $G$ on $X$ extends to an action on $\hat{X}$. The homotopy $H'$ can be taken to be $G$-equivariant i.e. for every $g \in G$ and $p \in \hat{X}$, $H'(t, g(p)) = g(H'(t, p))$.
4. The homotopy $H'$ is Zariski generalizing i.e. if $U \subset X$ is a Zariski open subset then $H'$ restricts to a well defined homotopy on $\hat{U}$.

**Proof.** Similar to Remark 6.2, we can assume at the outset that $X$ and $S$ are projective, $S$ is a smooth connected $K$-curve, and the fibres of the morphism $\phi : X \to S$ are pure.

Let $s \in S(K)$. We show that there exists a Zariski open neighbourhood $U$ of $s$ and a homotopy $H' : [0, \infty] \times X_U \to \hat{X}_U/U$ whose image $H(0, X_U)$ is relatively $\Gamma$-internal over $U$. Note that this is slightly different from the assertion in Theorem 6.1 since we are asking for a single homotopy and not a chain of homotopies as would appear if we went through the construction in the proof of loc.cit.

By the first part of Lemma 4.3, there exists a Zariski open set $U_0$ that contains $s$ and is such that the morphism $\phi : X \to S$ factors through a finite morphism $f_0 : X_{U_0} \to \mathbb{P}^1 \times U_0$. We have the following commutative diagram.
We fix coordinates on $\mathbb{P}^1$. Let $B_0 := \mathbb{P}^1 \times U_0$. Observe that any closed subset of $B_0$ that is generically finite over $U_0$ will be finite over $U_0$. We apply Lemma 6.4 to obtain a divisor $P_{00} \subset B_0$ that satisfies the following properties

1. $P_{00}$ is finite over $U_0$ and contains the closed subset $\{0, \infty\} \times U_0$.
2. For any divisor $T \subset B_0$ that contains $P_{00}$ and is finite over $U_0$, the homotopy
   \begin{align*}
   \psi_T \colon [0, \infty] \times B_0 &\to \hat{B}_0/U_0 \quad (\text{cf. \cite{7}, \S 10.2})
   \end{align*}
   lifts uniquely to a homotopy $h_T \colon [0, \infty] \times A \to A/U$. Furthermore, for every $j$, $h_T$ preserves the levels of the function $\xi_j$ and is $G$-invariant.

The fact that $\psi_T$ is a homotopy follows from the fact that $T$ is finite over $U_0$ which then implies that the lift $h_T$ is a homotopy by \cite[Lemma 10.1.1]{}. Observe that the image of $h_T$ is relatively $\Gamma$-internal.

Let $D := \{x_1, \ldots, x_n\}$ be the complement of the set $U_0 \subset S$. For every $1 \leq i \leq n$, there exists a Zariski open neighbourhood $U_i$ of $x_i$ and a divisor $P_i \subset \mathbb{P}^1 \times U_i$ that satisfies the analogous properties as fulfilled by $P_{00}$ which are listed above.

Let $W := \bigcap_{0 < \xi \leq n} U_i$. For every $i$, let $P_i' := f_i^{-1}(P_i)$. Let $P'$ denote the Zariski closure in $X$ of the closed subset $\bigcup P_i'$. For every $i$, define $P_i := f_i(P' \cap X_{U_i})$ and $P_i' := f_i^{-1}(f_i(P' \cap X_{U_i}))$. Observe that $P_i$ also satisfies the analogous versions of the points (1) - (2) as fulfilled by $P_{00}$ above. Hence, we see that for every $i$, the homotopies $\psi_i$ lift to homotopies $h_i$ which respect the definable function $\xi_i$ and the action of the group $G$.

Let $D_i := S \setminus U_i$. Let $m$ be a metric on $S$. Recall the function $p_D \colon S \to [0, \infty]$ defined as $x \mapsto \sup\{m(x, d) | d \in D\}$. Let $\epsilon \in [0, \infty](K)$ and $\delta \in [0, \infty](K)$ be such that the following is satisfied. For every $i$, $B(x_i, m, \epsilon) \subset \{x \in S | p_{D_i}(x) \leq \delta\}$ where $B(x_i, m, \epsilon) := \{y \in S | m(x_i, y) \geq \epsilon\}$.

For every $i$, we extend the homotopy $h_i$ to the whole of $X$ such that the image of the composition of the extended $h_i$ will be relatively $\Gamma$-internal over $S$. We proceed as follows.

Note that
\[ \{x \in S | p_{D_i}(x) \geq \epsilon\} = \bigcup_{x \in D_i} B(x, m, \epsilon). \]

By Lemma 6.6, we can extend $h_0$ to a homotopy $h_f_0 \colon [0, \infty] \times X \to \hat{X}/S$ such that for every $x \in S$ with $p_{D_0}(x) \leq \epsilon$, $h_f_0(0, X'_0) = h_0(0, X'_0)$. This implies in particular that the image of $h$ is relatively $\Gamma$-internal over $\{x \in S | p_{D_0}(x) \leq \epsilon\}$. Likewise, for every $i \geq 1$, we apply Lemma 6.6 to extend $h_i$ to a homotopy $h_i : [0, \infty] \times X \to \hat{X}/S$ such that the image of $h_i$ is relatively $\Gamma$-internal over $\{x \in S | p_{D_i}(x) \leq \delta\}$. Hence the image of $h_i$ is relatively $\Gamma$-internal over $B(x_i, m, \epsilon)$ since $B(x_i, m, \epsilon) \subset \{x \in S | p_{D_i}(x) \leq \delta\}$.

Let $h := h_{f_0} \circ \ldots \circ h_{f_0} \colon [0, \infty] \cup \ldots \cup [0, \infty] \times X \to \hat{X}/S$ be the composition of the homotopies described above. Observe from the construction that the image of $h$ is relatively $\Gamma$-internal over $S$. Let $\Upsilon \subset \hat{X}/S$ denote the image of the homotopy $h$. By Lemma 6.7, the homotopy $h$ fixes its image.

By \cite[Theorem 6.4.4]{7}, there exists a pseudo-Galois cover $\alpha : S' \to S$ and a finite collection of $K$-definable functions $\mu_j : S' \to \Gamma_\infty$ such that, for $I$ a generalised interval, a homotopy $\beta : I \times S \to \hat{S}$ which lifts to a homotopy $a' : I \times S' \to \hat{S}'$ that
preserves the levels of the functions $\mu_j$ also induces a homotopy $a: I \times \hat{\mathbb{T}} \rightarrow \hat{\mathbb{T}}$ that is $G$-invariant and respects the levels of the functions $\xi_i$. The homotopy $a$ is compatible with the homotopy $\beta$ for the morphism $\delta$. We apply [7, Remark 11.1.3 (2)] to complete the proof while noting that since each of the deformations in the composition are Zariski generalizing, the composition is Zariski generalizing as well.

7. Locally trivial morphisms

In this section, we show that compatible homotopies can be constructed for locally trivially morphisms. Unfortunately, our method doesn’t guarantee that the homotopy on the source fixes its image and hence it might not be a deformation retraction. It is possible that in this situation an additional hypothesis is required. We demonstrate this explicitly for $\mathbb{P}^1$-bundles which satisfy an additional finiteness requirement. The method presented below adapts the proof of the relative curve homotopy in [7].

7.1. Locally trivial morphisms. Let $m$ be a definable metric on a $K$-variety $S$. Suppose $D \subset S$ is a Zariski closed subset. Recall the $v+g$-continuous function, $\rho_D: S \rightarrow \Gamma_\infty$ given by $x \mapsto \sup\{m(x,d)|d \in D\}$.

**Proposition 7.1.** Let $S$ be a projective $K$-variety. Let $m$ be a definable metric on $S$. Suppose that for every $x \in S(K)$, we are given a Zariski open neighbourhood $U_x \subset S$ of $x$ defined over $K$. Let $D_x := S \setminus D_x$. We have that there exists finitely many points $\{x_1,\ldots,x_n\} \subset S(K)$ and $\{\epsilon_1,\ldots,\epsilon_n\} \subset \Gamma(K)$ such that

$$S = \bigcup_{1 \leq i \leq n} \{x \in U_{x_i} | \rho_{D_{x_i}}(x) \leq \epsilon_i\}.$$

**Proof.** Let $\langle a_i \rangle$ be a sequence in $\Gamma(K)$. We define a sequence $\langle x_i \rangle \subset S(K)$ as follows. Let $x_1 \in S(K)$. Given $x_i$, let $x_{i+1}$ be a point in $F_i := S \setminus (\bigcup_{1 \leq j \leq i} U_{x_j})$ if $F_i$ is not empty. Otherwise, the sequence must terminate. Let us assume that it terminates at $n \in \mathbb{N}$ i.e. $F_n = \emptyset$.

We choose an $n-1$-tuple $(\delta_1,\ldots,\delta_{n-1}) \in \prod_{1 \leq i \leq n-1} [a_i,\infty)(K)$ which satisfies the following properties. Firstly, $\delta_{i-1}$ is such that $\{y \in S(\rho_{F_{i-1}}(y) \geq \delta_{i-1}) \cap D_{x_{i-1}} = \emptyset$. This is possible because $F_{n-1}$ is disjoint from $D_{x_{n-1}}$ and $\rho_{F_{n-1}}$ is $v+g$-continuous and hence bounded in $\Gamma$ on $D_{x_{n-1}}$.

Suppose $1 \leq i \leq n-1$. Having chosen $\delta_i$, we choose $\delta_{i-1}$ such that the following is true.

1. $\delta_{i-1} \in [a_{i-1},\infty)$ and $\delta_{i-1} > \delta_i$.
2. If $F_{i-1,\delta_i} := \{x \in F_{i-1} | \rho_{F_{i-1}}(x) \leq \delta_i\}$

and $D_{x_i,\delta_i} := \{x \in D_{x_i} | \rho_{F_{i-1}}(x) \leq \delta_i\}$ then $\{y \in S(\rho_{F_{i-1}}(y) \geq \delta_{i-1}) \cap D_{x_i,\delta_i} = \emptyset$.

The existence of $\delta_{i-1}$ can be deduced as follows. Observe that $F_i = D_{x_i} \cap F_{i-1}$. Hence, we see that $D_{x_i,\delta_i}$ is disjoint from $F_{i-1}$. We now use the fact that $\rho_{F_{i-1}}$ is $v+g$-continuous and hence bounded on $D_{x_i,\delta_i}$.

We choose an $n-1$-tuple $(\epsilon_1,\ldots,\epsilon_{n-1}) \in [0,\infty)(K)$ such that the following holds. For $1 \leq i \leq n-1$, if $A_i := S \setminus (\bigcup_{1 \leq j \leq i} \{x \in U_{x_j} | \rho_{D_{x_j}}(x) \leq \epsilon_j\})$
then \( A_i \subseteq \{ x \in S | \rho_{F_i}(x) \geq \delta_i \} \). We construct \((\epsilon_1, \ldots, \epsilon_{n-1})\) as follows. We set \( \epsilon_1 = \delta_1 \). Suppose, we have chosen \((\epsilon_1, \ldots, \epsilon_i)\) with \( i \leq n - 2 \) appropriately. Let \( \epsilon_{i+1} \in \Gamma(K) \) be such that \( \epsilon_{i+1} > \delta_i \). We claim that \( A_{i+1} \subseteq \{ x \in S | \rho_{F_{i+1}}(x) \geq \delta_{i+1} \} \).

Let \( z \in A_{i+1} \) and suppose that \( z \notin \{ x \in S | \rho_{F_{i+1}}(x) \geq \delta_{i+1} \} \) i.e. \( \rho_{F_{i+1}}(z) < \delta_{i+1} \). Since \( z \in A_{i+1} \), we have that \( z \in A_i \) and hence \( z \in \{ x \in S | \rho_{F_i}(x) \geq \delta_i \} \). Let \( y \in F_i \) be such that \( \rho_{F_i}(z) = m(z,y) \). We must have that \( y \in F_{i, \delta_{i+1}} \). Indeed, suppose that for some \( y' \in F_{i+1}, m(y',y) \geq \delta_{i+1} \). It follows that

\[
m(y',z) \geq \inf\{m(y',y),m(y,z)\} \\
\geq \inf\{\delta_{i+1},\delta_i\} \\
\geq \delta_i \text{ (by our choice of } \delta_i)\]

This contradicts our initial assumption that \( \rho_{F_{i+1}}(z) < \delta_{i+1} \).

By assumption, \( \rho_{D_{i+1}}(z) > \epsilon_{i+1} \). Using arguments as above and the fact that \( \epsilon_{i+1} > \delta_i > \delta_{i+1} \), we get that if \( x \in D_{\epsilon_{i+1}} \) is such that \( \rho_{D_{\epsilon_{i+1}}}(z) = m(x,z) \) then \( x \in D_{\delta_{i+1}} \).

Observe that

\[
m(x,y) \geq \inf\{m(x,z),m(y,z)\} \\
\geq \inf\{\epsilon_{i+1},\delta_i\} \\
\geq \delta_i \]

However this is not possible by our choice of \( \delta_i \).

Finally, we choose \( \epsilon_n \in \Gamma(K) \) such that the following holds. By construction, \( F_n := D_{\epsilon_n} \cap F_n - 1 \) is empty. Recall that we chose \( \delta_{n-1} \) such that \( \{ y \in S | \rho_{F_{n-1}}(y) \geq \delta_{n-1} \} \cap D_{\epsilon_{n}} = \emptyset \). It follows that \( \rho_{D_{\epsilon_{n}}} \) is bounded in \( \Gamma \) when restricted to \( \{ y \in S | \rho_{F_{n-1}}(y) \geq \delta_{n-1} \} \). Let \( \epsilon_n \) be such that \( \{ a \in S | \rho_{D_{\epsilon_{n}}}(a) \geq \epsilon_n \} \cap \{ y \in S | \rho_{F_{n-1}}(y) \geq \delta_{n-1} \} = \infty \). One checks easily from the construction that \((\epsilon_1, \ldots, \epsilon_n)\) satisfies the required property.

\[\square\]

Proposition 7.1 used in conjunction with Lemma 6.6 allows us to construct compatible deformations for certain locally trivial morphisms.

**Corollary 7.2.** Let \( V \) be a projective \( K \)-variety and \( \psi: [0, \infty] \times V \to \hat{V} \) be a homotopy whose image is \( \Gamma \)-internal. Let \( \phi: X \to S \) be a morphism of projective varieties such that for every \( s \in S(K) \), there exists a Zariski open neighbourhood \( U_s \) of \( s \) and a \( U_s \)-isomorphism \( f_U: X_{U_s} \to U_s \times V \). Let \( \xi_i: S \to \Gamma_{\infty} \) be finitely many \( K \)-definable functions. We then have that there exists a homotopy \( H': I \times \hat{X} \to \hat{X} \) and a deformation retraction \( H: I \times \hat{S} \to \hat{S} \) which are compatible for the morphism \( \hat{\phi} \). Furthermore, the deformation \( H \) respects the levels of the functions \( \xi_i \) for every \( i \).

**Proof.** Recall from [7, Remark 9.8.4] that if \( V_1 \) and \( V_2 \) are varieties over a valued field and \( \psi: I \times V_1 \to \hat{V}_1 \) is \( v + g \)-continuous then the map \( (t,u,v) \mapsto \psi(t,u) \otimes v \) defines a \( v + g \)-continuous map \( \phi: I \times (V_1 \times V_2) \to (V_1 \times V_2)/V_2 \). It follows from this and our assumptions on \( \phi \) and \( V \) that for every \( s \in S(K) \) there exists a Zariski open neighbourhood \( U_s \subseteq S \) of \( s \) and a homotopy \( h'_{U_s}: [0, \infty] \times X_{U_s} \to \hat{X}_{U_s} \) such that the image is relatively \( \Gamma \)-internal.

Let \( m \) be a definable metric on \( V \). We apply Proposition 7.1 to the given data and obtain finite sets \( \{s_1, \ldots, s_m\} \subseteq S(K) \) and \( \{\epsilon_1, \ldots, \epsilon_m\} \subseteq \Gamma(K) \) such that \( S = \bigcup_{1 \leq i \leq m} \{ x \in S | \rho_{D_i}(x) \leq \epsilon_i \} \) where \( D_i := S - U_{s_i} \). For every \( 1 \leq i \leq m \), we simplify notation and write \( U_i := U_{s_i}, D_i := D_{s_i} \) and \( h'_{U_{s_i}} := h'_{U_i} \).
By Lemma 6.6, we can extend the homotopy $h'_i : [0, \infty] \times X_i \to \tilde{X}/\tilde{U}_i$ to a homotopy $h'_i : [0, \infty] \times X \to \tilde{X}/\tilde{S}$ such that for every $s$ with $\rho_{D_i}(s) \leq \epsilon_i$, $h'_i(0, X_{ia}) = h'_i(0, X_{ia})$. In particular the image of $h'_i$ is relatively $\Gamma$-internal over $\{s \in S \mid \rho_{D_i}(s) \leq \epsilon_i\}$. Let $h'_i := \Sigma_{i \leq s \leq m} h'_i : [0, \infty] \cup \ldots \cup [0, \infty] \times X \to \tilde{X}/\tilde{S}$ be the composition of the homotopies $h'_i$. Let $I_1$ be the gluing of the $m$-copies of the intervals $[0, \infty]$. By construction, $h'_i : I_1 \times X \to \tilde{X}/\tilde{S}$ and its image $\Gamma'$ is relatively $\Gamma$-internal.

By [7, Theorem 6.4.4] there exists a finite pseudo-Galois cover $f : S' \to S$ and a finite number of $K$-definable functions $\{\mu_j : S \to \Gamma_\infty\}$ such that a homotopy $H : I_2 \times S \to \tilde{S}$ that lifts to a homotopy $I_2 \times S' \to \tilde{S}'$ and respects the level of the functions $\{\mu_j\}$ must also give a homotopy $H'_j : I_2 \times Y \to \tilde{\tilde{Y}}$ that is compatible with the map $Y \to S$. By [7, Theorem 11.1.1], there exists a deformation $H$ that satisfies these properties, has a $\Gamma$-internal image and also respects the definable functions $\xi_i$. By construction, the composition $H' := H'_1 \circ H'_j : I_2 + I_1 \times X \to \tilde{X}$ is a homotopy compatible with $H$ and has $\Gamma$-internal image.

Corollary 7.3. Let $C$ be a projective $K$-curve and $S$ be a projective $K$-variety. Let $\phi : X \to S$ be a projective morphism such that for every $s \in S$, there exists a Zariski open neighborhood $U$ of $s$ and a $U$-isomorphism $f_U : X_U \to C \times U$. Let $\{\xi_i : S \to \Gamma_\infty\}$ be a finite family of $K$-definable functions.

We then have that there exists a homotopy $H' : I_1 \times \tilde{X} \to \tilde{X}$ and a deformation $H : I_1 \times \tilde{S} \to \tilde{S}$ which are compatible for the morphism $\phi$. Furthermore, the deformation $H$ respects the levels of the functions $\xi_i$ for every $i$.

Proof. Let $C$ be as in the statement of the corollary. [7, Theorem 7.5.1] implies the existence of a deformation $\psi : [0, \infty] \times C \to \tilde{C}$ whose image is $\Gamma$-internal. Applying Corollary 7.2 proves the result. □

7.1.1. Compatible deformations for $\mathbb{P}^1$-bundles. Note that Corollary 7.3 doesn’t guarantee that the existence of compatible deformation retractions. It is possible that this is true only after an additional hypothesis. We present such an instance of adding a hypothesis to guarantee the existence of compatible deformations in the case of $\mathbb{P}^1$-bundles.

Let $S$ be a quasi-projective $K$-variety.

Definition 7.4. A morphism $\phi : X \to S$ is a $\mathbb{P}^1$-bundle if there exists a finite Zariski open covering $\{U_i\}_{1 \leq i \leq m}$ of $S$ such that for every $i$, there exists an isomorphism $g_i : X_i := \phi^{-1}(U_i) \to U_i \times \mathbb{P}^1_k$ over $U_i$. We encode the data of the $\mathbb{P}^1$-bundle using the tuple $(\phi : X \to S, \{U_i\}, \{g_i\})$.

Remark 7.5. For $i, j$, the transition maps $g_{ij} := [g_j \circ g_i^{-1}]_{U_i \cap U_j} : (U_i \cap U_j) \times \mathbb{P}^1_k \to (U_i \cap U_j) \times \mathbb{P}^1_k$ are isomorphisms over $U_i \cap U_j$ i.e., for every $u \in U_i \cap U_j$, $g_{ij}^{-1}(u) : g_{ij}^{-1}(u) : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a well defined automorphism. Let $X_{ij} := X_i \cap X_j$.

The projective bundles we consider will satisfy the following finiteness hypothesis. For the remainder of this section, we fix a system of coordinates on $\mathbb{P}^1_k$.

(F) There exists a divisor $D \subset X$ such that

1. The morphism $\phi$ restricts to a finite map from $D$ onto $S$.
2. For every $1 \leq i \leq m$, $g_i(D \cap X_i)$ contains $U_i \times \{0, \infty\}$.
Remark 7.6. Let \( \phi : X \to S, \{U_i\}, \{g_i\} \) be a rank-1 projective bundle. Let \( X_i = \phi^{-1}(U_i) \). To construct a homotopy \( H^i : I \times X \to \tilde{X}/S \) it suffices to construct a family of homotopies \( H^i_t : I \times Y_i \to \tilde{Y}_i/\tilde{U}_i \) which satisfy the obvious gluing conditions. Precisely, if \( u \in U_i \cap U_j \) then \( H^i_t \) restricts to a homotopy on the fibre \( \tilde{Y}^i_{tu} = \tilde{P}^1_K \) such that for every \( t \in I \) and \( x \in \tilde{Y}^i_{tu} \), \( H^i_t(t,x) = \overline{g}_{ji}(H^j_t(t, \overline{g}_{jj}(x))) \).

Theorem 7.7. Let \( \{ \phi : X \to S, \{U_i\}, \{g_i\} \} \) be a \( \mathbb{P}^1 \)-bundle which satisfies the hypothesis (F). There exists a pair of deformation retractions (\( H^i_t : I \times \tilde{X} \to \tilde{X}, \tilde{Y} \)) and (\( H : I \times \tilde{S} \to \tilde{S}, \tilde{Y} \)) which are compatible for the morphism \( \overline{\phi} \) and whose images \( \tilde{Y}' \subset \tilde{X}, \tilde{Y} \subset \tilde{S} \) are \( \Gamma \)-internal subsets.

Proof. For every \( i \), let \( Y^i := U_i \times \mathbb{P}^1_K \). By assumption, there exists a closed subset \( D \subset X \) such that \( \phi_D : D \to S \) is finite and for every \( i \in \{1, \ldots, n\} \), \( g_i(D \cap X_i) \subset Y_i \) contains \( U_i \times \{0, \infty\} \). Let \( E_i := g_i(D \cap X_i) \). Let \( p_i : Y_i \to U_i \) be the projection map. We have that \( p_{ij|E_i} : E_i \to U_i \) is finite. Recall from the paragraph above [7, Lemma 9.5.3], the construction of the homotopy \( \psi_{E_i} \). The homotopy \( \psi_{E_i} \) is such that for every \( u \in U_i \), \( \psi_{E_i} \) restricts to the homotopy on the fibre \( Y^i_{tu} = \tilde{P}^1_K \) defined by the standard homotopy with cut-off determined by \( E_i \cap \tilde{Y}_t \). By [7, Lemma 10.2.1], the homotopy \( \psi_{E_i} : [0, \infty] \times Y_i \to \tilde{Y}_i/\tilde{U}_i \) is well defined and continuous.

By pulling back via the isomorphism \( g_i \), we have for every \( i \), a homotopy

\[
\psi^i_t : [0, \infty] \times X_i \to \tilde{X}_i/\tilde{U}_i
\]

whose image is relatively \( \Gamma \)-internal. We claim that these homotopies glue to give a homotopy on \( \tilde{X}/S \). We verify the claim as follows.

Let \( i, j \in \{1, \ldots, n\} \). Let \( u \in U_i \cap U_j \). The homotopies \( \psi^i_t \) and \( \psi^j_t \) restrict to define homotopies \( \psi^i_{t,u}, \psi^j_{t,u} : [0, \infty] \times X_u \to \tilde{X}_u \) where \( X_u := \phi^{-1}(u) \). We show that these homotopies coincide. Since, \( i \) and \( j \) were chosen arbitrarily, this will imply that the homotopies \( \psi^i_t \) glue together. Since \( \psi^i_t \) and \( \psi^j_t \) are definable maps, it suffices to consider the case when \( u \) is defined over \( K \).

The isomorphisms \( g_i \) and \( g_j \) imply a definable automorphism \( g_{i,j,u} := g_{j,u} \circ g_{i,u}^{-1} : \mathbb{P}^1_{\mathbb{K}(u)} \to \mathbb{P}^1_{\mathbb{K}(u)} \) where \( g_{i,u} := g_i|X_u \). Let \( \psi_{E_{i,u}}, \psi_{E_{j,u}} : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the homotopies induced by \( \psi_{E_i} \) and \( \psi_{E_j} \). As in Remark 7.6, it suffices to verify that for every \( x \in \mathbb{P}^1_{\mathbb{K}(u)} \) and \( t \in [0, \infty] \), \( \overline{g}_{i,j,u}(\psi_{E_{i,u}}(t,x)) = \psi_{E_{j,u}}(t, g_{i,j,u}(x)) \). Since, \( \mathbb{P}^1 \) is definable, we reduce to showing the above equality when \( x \in \mathbb{P}^1_{\mathbb{K}(u)}(K^{max}) \) and \( t \in [0, \infty]\cap(K^{max}) = [0, \infty]\cap(\mathbb{R}) \). In this situation, we make use of the fact that \( \mathbb{P}^1_{\mathbb{K}(u)}(K^{max}) \) is homeomorphic to the Berkovich space \( B_{K^{max}}(\mathbb{P}^1) \).

If \( x \in E_i \) then for every \( t \), \( \overline{g}_{i,j,u}(\psi_{E_{i,u}}(t,x)) = \psi_{E_{j,u}}(t, g_{i,j,u}(x)) \). Let \( Y_i \subset \mathbb{P}^1 \) denote the convex hull of \( E_{i,u} \) and likewise, \( Y_j \subset \mathbb{P}^1 \) denote the convex hull of \( E_{j,u} \). Note that \( \overline{g}_{i,j,u}^{-1}(Y_j) = Y_i \). Let \( x \in \mathbb{P}^1(K^{max}) \setminus Y_i \) and \( O \) be the connected component of \( \mathbb{P}^1(K^{max}) \setminus Y_i \) that contains \( x \). As \( \overline{g}_{i,j,u} \) is a homeomorphism and \( \overline{g}_{i,j,u}(Y_i) = Y_j \), there exists a connected component \( O' \) of \( \mathbb{P}^1 \setminus Y_j \) such that \( g_{i,j,u}(O) = O' \). Note that these connected components are Berkovich open balls. Since the morphism \( g_{i,j,u} \) is algebraic, the restriction \( g_{i,j,u|O} \) at the level of algebras must be of the form \( x \mapsto ax + b \) where \( a \in K^* \) and \( b \in K \). It can then be checked by explicit calculation that \( \overline{g}_{i,j,u}(\psi_{E_{i,u}}(t,x)) = \psi_{E_{j,u}}(t, g_{i,j,u}(x)) \).

We have thus shown that there exists a definable homotopy \( \psi : [0, \infty] \times X \to \tilde{X}/S \) whose image is relatively \( \Gamma \)-internal. Let \( \tilde{Y} \) denote the image of \( \psi \). Let \( \tilde{S} \) be a projective \( K \)-variety that contains \( S \) as an open dense subvariety. Let \( d_{\text{bord}} : \tilde{S} \to \Gamma_{\infty} \) be the schematic distance to \( S_{\text{bord}} := \tilde{S} \setminus S \). Observe that \( \tilde{Y} \) is \( \sigma \)-compact.
with respect to the function \( d_{\operatorname{ord}} \circ \hat{\phi} \). By [7, Theorem 6.4.4], there exists a pseudo-Galois cover \( f: S' \to S \) and a morphism \( \kappa: \Upsilon' := \Upsilon \times_S S' \to S' \times \Gamma_M^\infty \) for some \( M \in \mathbb{N} \) such that the restriction \( \hat{\kappa}|_{\hat{\Upsilon}'}: \hat{\Upsilon}' \to \hat{S}' \times \Gamma_M^\infty \) is a homeomorphism onto its image. By loc.cit., we see that there exists a finite number of \( K \)-definable functions \( \{ \mu_j: S \to \Gamma^\infty \} \) such that a homotopy \( H: I \times S \to \hat{S} \) that lifts to a homotopy \( I \times S' \to \hat{S}' \) respects the level of the functions \( \{ \mu_j \} \) must also give a homotopy \( H_1': I \times \Upsilon \to \hat{\Upsilon} \) that is compatible with the map \( \Upsilon \to S' \). By [7, Theorem 11.1.1], the homotopy \( H \) can be chosen so that it satisfies these properties and in addition has a \( \Gamma \)-internal image. By construction, the composition \( H_1' \circ \psi: I + [0, \infty] \times X \to \hat{X} \) is compatible with \( H \) and has \( \Gamma \)-internal image. Clearly, \( H_1' \circ \psi \) is a deformation retraction. \( \square \)

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