Linear Quadratic Gaussian Synthesis for a Heated/Cooled Rod Using
Point Actuation and Point Sensing

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Abstract—We consider a rod that is heated/cooled and sensed at multiple point locations. To stabilize it to a constant temperature we set up a Linear Quadratic Regulator that we explicitly solve by the method of completing the square to find the optimal linear state feedback for the point actuators. But we don’t assume that the whole state is measureable so we construct an infinite dimensional Kalman filter to estimate the whole state from a finite number of noisy point measurements. These two components yield a Linear Quadratic Gaussian (LQG) Synthesis for the heat equation under point actuation and point sensing.

I. INTRODUCTION

Linear Quadratic Gaussian Synthesis is a (perhaps the) standard approach to constructing a compensator for a finite or an infinite dimensional linear system. It consists of two parts. One first solves a Linear Quadratic Regulator problem for a linear state feedback law for the actuators. If the state is fully measurable this linear state feedback would asymptotically stabilize the system. But the state is usually not fully measurable, instead several linear functionals of state are available and these measurements are corrupted by noise. A Kalman filter is used to process the measurements to get an estimate of the whole state. The certainty equivalence principle is invoked and the linear feedback law is applied to estimate of the state. The result is a compensator of the same dimension as the original system. It can be shown that spectrum of the combined system and its compensator is the union of the spectrum of system under linear full state feedback with the spectrum of the error dynamics of the Kalman filter. If both these spectra lie in the open left half plane then the compensator will stabilize the original system.

This LQG approach works over both finite and infinite time horizons. In this paper we will only treat infinite time horizons. To find the gain of the optimal linear state feedback one has to solve an algebraic Riccati equation and to find the gain of the Kalman filter one has to solve a dual algebraic Riccati equation. Good software exists to solve these equations in low to medium dimensions but they can be difficult to solve in high or infinite dimensions. In infinite dimensions other difficulties can arise. The actuation could be at points in the spatial domain, e.g., boundary control. The measurements could also be at points. Such systems are not "state linear systems" in the sense of Curtain and Zwart [2], [3]. State linear systems must have bounded input and bounded output linear functionals. The usual approach to dealing with a system with point actuation and/or point sensing is to approximate it by a state linear system. Boundary control actuation is replaced by intense actuation over a short interval adjacent to the boundary, in effect, the control input multiplies a shaping function that approximates a delta function at the boundary. Point sensing is replaced by integration of the state against a shaping function that approximates a delta function at the measurement point. For more details about this we refer the reader to Chapter 6 of [2] and Chapter 9 of [3] and their extensive references. For more on boundary control of systems described by PDEs, see the treatises of Lions [9], Lasiecka-Triggiani [8] and Krstic-Smyshlyaev [7]. Hulsing [4] and Burns-Hulsing [1] have addressed the computational issues associated with boundary control.

We take a different approach, in effect, we model boundary and other point actuators by delta functions and we model point sensing also by delta functions. We use a method that we call completing the square to overcome the mathematical technicalities associated with these delta functions. To keep the discussion concrete we limit our consideration to a rod heated/cooled at boundary and other points. We make noisy measurements of its temperature at some other points. Because we focus on systems modeled by the heat equation we are able to give explicit solutions to the LQR and Kalman filtering equations using the simple technique of completing the square. In particular the infinite dimensional analogs of the algebraic Riccati equations are elliptic PDEs that we call Riccati PDEs. These Riccati PDEs can be explicitly solved in terms of the eigenfunctions of the Laplacian. We restrict our attention to Neumann boundary conditions but our methods readily extend to other self adjoint boundary conditions for the Laplacian. We first used the completing the square technique on a distributed control problem [5] but we quickly realized that it works equally well for boundary control problems. In [6] we treated the LQR control of a rod heated/cool at one end and insulated at the other. The boundary conditions were Neumann at the insulated end and Robin at the controlled end.

The rest of the paper is as follows. In the next section we treat the LQR control of the rod heated/cool at the boundary and other points. Section 3 contains an example of heating/cooling at both ends and the middle of the rod. In Section 4 we derive the Kalman filter for the rod with noisy measurements at several points by converting the filtering problem into a family of LQR problems. Section 5 contains an example of the LQG synthesis of a Kalman filter with an LQR state feedback law. The conclusion is in Section 6.

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II. LQR FOR A ROD HEATED/COOLED AT MULTIPLE POINTS

Consider a rod of length one that is heated/cooled at multiple locations. We let \( x \in [0,1] \) denote position on the rod and let \( z(x,t) \) denote the temperature of the rod at position \( x \) at time \( t \). We assume the rod can be heated or cooled at \( 0 = \xi_1 < \xi_2 < \ldots < \xi_m = 1 \). We let \( u_k(t) \) be the heating/cooling flux applied to the rod at \( \xi_k \) for \( k = 1, \ldots, m \) and \( u(t) = [u_1(t), \ldots, u_m(t)]' \). We model the rod by these equations

\[
0 = \frac{\partial z}{\partial t}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t), \quad x \in (\xi_i, \xi_{i+1}) \quad (1)
\]

\[
0 = z(\xi^{-}_k, t) - z(\xi^+_k, t), \quad k = 2, \ldots, m - 1 \quad (2)
\]

\[
\beta_k u_k(t) = \frac{\partial z}{\partial x}(\xi^+_k, t) - \frac{\partial z}{\partial x}(\xi^{-}_k, t), \quad k = 1, \ldots, m \quad (3)
\]

\[
z(x,t) = z^0(x) \quad (4)
\]

where

\[
z(\xi^+_k, t) = \lim_{x \rightarrow \xi^+_k} z(x,t), \quad z(\xi^{-}_k, t) = \lim_{x \rightarrow \xi^{-}_k} z(x,t) \quad (5)
\]

\[
\frac{\partial z}{\partial x}(\xi^+_k, t) = \lim_{x \rightarrow \xi^+_k} \frac{\partial z}{\partial x}(x,t), \quad \frac{\partial z}{\partial x}(\xi^{-}_k, t) = \lim_{x \rightarrow \xi^{-}_k} \frac{\partial z}{\partial x}(x,t)
\]

Without loss of generality we assume that \( \beta_k \geq 0 \). We also assume that \( \frac{\partial z}{\partial x}(x,t) = 0 \) outside of \([0,1]\). If the rod is not heated/cooled at its endpoints we set \( \beta_1 = 0 \) and/or \( \beta_m = 0 \).

The open loop system, \( u_k(t) = 0 \) for \( k = 1, \ldots, m \), reduces to the standard heat equation with Neumann boundary conditions, so the open loop eigenvalues are \( \lambda_n = -n^2 \pi^2 \) for \( n = 0, 1, 2, \ldots \) and the orthonormal eigenvectors are

\[
\phi^0(x) = 1, \quad \phi^n(x) = \sqrt{2} \cos n \pi x
\]

for \( n = 1, 2, \ldots \). Notice that \( \lambda_0 = 0 \) so the open loop system is only neutrally stable. The rest of the eigenvalues are rapidly going to \(-\infty\).

We wish to stabilize the rod to some uniform temperature which we conveniently take to be \( z = 0 \) by using a Linear Quadratic Regulator (LQR). We choose some \( Q > 0 \) and a \( m \times m \) positive definite matrix \( R > 0 \) and we seek to minimize

\[
\int_0^\infty \int_0^1 Q z^2(x,t) \, dx + u'(t)R u(t) \, dt \quad (6)
\]

subject to the above dynamics.

Let \( P(x_1, x_2) \) be any symmetric function, \( P(x_1, x_2) = P(x_2, x_1) \), which is continuous on the unit square \( S = [0,1]^2 \) and suppose there is a control trajectory \( u(t) \) such that \( z(x,t) \to 0 \) as \( t \to \infty \). Then by the Fundamental Theorem of Calculus

\[
0 = \int \int_S z^0(x_1)P(x_1, x_2)z^0(x_2) \, dA + \int_0^\infty \int \int_S \frac{\partial^2 z^2}{\partial x_1^2}(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \, dt + \int_0^\infty \int \int_S z(x_1, t)P(x_1, x_2)\frac{\partial^2 z}{\partial x_2^2}(x_2, t) \, dA \, dt \quad (7)
\]

where \( dA = dx_1 dx_2 \).

We assume that \( P(x_1, x_2) \) satisfies Neumann boundary conditions in each variable. Now we formally integrate by parts twice with respect to \( x_1 \) on each subinterval \([\xi_k, \xi_{k+1}]\) ignoring the fact that we only assumed \( P(x_1, x_2) \) is continuous,

\[
\int_{\xi_k}^{\xi_{k+1}} \frac{\partial^2 z}{\partial x_1^2}(x_1, t)P(x_1, x_2)z(x_2, t) \, dx_1 = \int_{\xi_k}^{\xi_{k+1}} z(x_1, t)\frac{\partial^2 P}{\partial x_1^2}(x_1, x_2)z(x_2, t) \, dx_1 + \left[ \frac{\partial z}{\partial x_1}(x_1, t)P(x_1, x_2)z(x_2, t) \right]_{x_1=\xi_k}^{x_1=\xi_{k+1}} - \left[ \int_{\xi_k}^{\xi_{k+1}} \frac{\partial P}{\partial x_1}(x_1, x_2)z(x_2, t) \right]_{x_1=\xi_k}^{x_1=\xi_{k+1}} \quad (8)
\]

Since we assumed that \( \frac{\partial z}{\partial x_1}(x_1, t) = 0 \) off of \([0,1]\) we have \( \frac{\partial^2 z}{\partial x_1^2}(x_1, t)P(x_1, x_2)z(x_2, t) = 0 \). We sum (8) over \( k = 1, \ldots, m - 1 \) and obtain

\[
\int_0^1 \int_{\xi_1}^{\xi_{m+1}} \frac{\partial^2 z}{\partial x_1^2}(x_1, t)P(x_1, x_2)z(x_2, t) \, dx_1 \quad (9)
\]

We plug these into (7) and obtain the identity

\[
0 = \int \int_S z^0(x_1)P(x_1, x_2)z^0(x_2) \, dA + \int_0^\infty \int \int_S z(x_1, t)\nabla^2 P(x_1, x_2)z(x_2, t) \, dA \, dt + \sum_{k=1}^m \beta_k u_k(t)P(\xi_k, x_2)z(x_2, t) + \sum_{k=1}^m z(x_1, t)P(\xi_1, \xi_k)\beta_k u_k(t) \, dA \, dt \quad (10)
\]

where \( \nabla^2 P(x_1, x_2) \) denotes the two dimensional Laplacian of \( P \). We add the right side of this identity (9) to the criterion (6) to be minimized to get an equivalent criterion.

To complete the square, we would like to find a function
\( K(x) \) taking values in \( IR^{m \times 1} \) such that
\[
\int_S (u(t) - K(x_1)z(x_1,t))' R (u(t) - K(x_2)z(x_2,t)) dA
\]
\[
+ u(t) Re(t)z(x_1,t) + z(t) \nabla^2 P(x_1,x_2)z(x_2,t)
\]
\[
+ \sum_{k=1}^{m} \beta_k u_k(t) P(\xi_k,x_2)z(x_2,t)
\]
\[
+ \sum_{k=1}^{m} z(x_1,t) P(x_1,\xi_k) \beta_k u_k(t) dA
\]
Clearly the terms quadratic in \( u(t) \) agree so we compare terms bilinear in \( z(x_2,t) \) and \( u_k(t) \). This yields
\[
- \sum_{j=1}^{m} R_{k,j} K_j(x_2) = \beta_k P(\xi_k,x_2)
\]
Let \( \beta \) be a diagonal matrix with diagonal entries \( \beta_1, \ldots, \beta_m \) and let \( \bar{P}(x_2) = [P(\xi_1,x_2), \ldots, P(\xi_m,x_2)]' \), then (12) yields the equation
\[
K(x) = -R^{-1} \beta \bar{P}(x)
\]
Next we compare terms bilinear in \( z(x_1,t) \) and \( z(x_2,t) \),
\[
\int_S z(x_1,t) (\nabla^2 P(x_1,x_2) + \delta(x_1-x_2)Q) z(x_2,t) dA
\]
\[
= \int_S \beta \bar{P}(x_1)z(x_1,t)' R^{-1} \beta \bar{P}(x_2)z(x_2,t) dA
\]
This will hold if \( P(x_1,x_2) \) is a solution what we call a Riccati PDE,
\[
\nabla^2 P(x_1,x_2) + \delta(x_1-x_2)Q = \bar{P}'(x_1) \beta' R^{-1} \beta \bar{P}(x_2)
\]
Recall we only assumed that \( P(x_1,x_2) \) is continuous on the unit square so the Riccati PDE must be interpreted in the weak sense.

From Parseval’s Identity we obtain
\[
Q \delta(x_1-x_2) = Q \sum_{n=0}^{\infty} \cos n\pi x_1 \cos n\pi x_2
\]
We assume that \( P(x_1,x_2) \) has a similar expansion
\[
P(x_1,x_2) = \sum_{n=0}^{\infty} P^n \cos n\pi x_1 \cos n\pi x_2
\]
and we define
\[
\Gamma^n = \begin{bmatrix} \cos n\pi \xi_1 \\ \vdots \\ \cos n\pi \xi_m \end{bmatrix}' \beta' R^{-1} \beta \begin{bmatrix} \cos n\pi \xi_1 \\ \vdots \\ \cos n\pi \xi_m \end{bmatrix}
\]
Then the Riccati PDE (14) reduces to a sequence of quadratic equations
\[
-2n^2 \pi^2 P^n + Q = \Gamma^n (P^n)^2
\]
with roots
\[
P^n = \frac{-n^2 \pi^2 \pm \sqrt{n^4 \pi^4 + \Gamma^n Q}}{\Gamma^n}
\]
Since we want
\[
0 < \int_S P(x_1,x_2) \phi(x_1) \phi(x_2) dA
\]
for any nonzero function \( \phi(x) \) we choose the positive root
\[
P^n = \frac{-n^2 \pi^2 + \sqrt{n^4 \pi^4 + \Gamma^n Q}}{\Gamma^n}
\]
The roots are positive since \( Q > 0 \) and \( \Gamma^n > 0 \) if at least one \( \beta_i \neq 0 \). But as we now show they are going to zero like \( \frac{1}{n^2} \). The Mean Value Theorem applied to (15) implies there exists an \( s \) between \( n^4 \pi^4 \) and \( n^4 \pi^4 + \Gamma^n Q \) such that
\[
P^n = \frac{1}{2 \sqrt{s}}
\]
The maximum of \( \frac{1}{2 \sqrt{s}} \) between \( n^4 \pi^4 \) and \( n^4 \pi^4 + \Gamma^n Q \) occurs at \( s = n^4 \pi^4 \) so we get the estimate
\[
0 < P^n \leq \frac{1}{2n^2 \pi^2 Q}
\]
for \( n > 0 \). Hence the series
\[
P(x_1,x_2) = \sum_{n=0}^{\infty} P^n \cos n\pi x_1 \cos n\pi x_2
\]
converges uniformly to a continuous function and it is straightforward to verify that it is a weak solution of the Riccati PDE (14).

The feedback control law is
\[
u(t) = -R^{-1} \beta \int_0^t \bar{P}(x)\phi(x,t) dx
\]

III. Example One

We assume that the rod can be heated/cool at both endpoints and also at the midpoint, \( m = 3 \), \( \xi_1 = 0 \), \( \xi_2 = 0.5 \), \( \xi_3 = 1 \). \( \beta_1 = 1 \), \( \beta_2 = 2 \), \( \beta_3 = 1 \), \( R = [1,0,0,0,0,0,0,0,0,1] \) and \( \Gamma^n = 6 \) if \( n \) is even and \( \Gamma^n = 2 \) if \( n \) is odd.

The \( P^n \) are given by (15). The optimal feedback gains are
\[
K_1(x) = -\sum_{n=0}^{\infty} P^n \cos n\pi x
\]
\[
K_2(x) = -2 \sum_{k=0}^{\infty} (-1)^k P^{2k} \cos 2k\pi x
\]
\[
K_3(x) = -\sum_{n=0}^{\infty} (-1)^n P^n \cos n\pi x
\]
We assume that a closed loop eigenfunction \( \psi(x) \) is composed of different sinusoids on \([0,0.5]\) and \([0.5,1]\) with a common frequency \( \nu \). Because the system is symmetric with respect to replacing \( x \) with \( 1-x \) we expect a closed eigenfunction to reflect this symmetry,
\[
\psi(x) = \begin{cases} \cos \nu x + b \sin \nu x, & 0 \leq x \leq 0.5 \\ \cos \nu (1-x) + b \sin \nu (1-x), & 0.5 \leq x \leq 1 \end{cases}
\]
Notice such a solution immediately satisfies the continuity condition (2), and
\( \frac{\partial \psi}{\partial x}(0.5^-) = -\frac{\partial \psi}{\partial x}(0.5^+) \)

The first closed loop "boundary" condition is
\[
b = \sum_{n=0}^{\infty} \int_{0}^{0.5} P^n \cos n\pi x \left( a \cos \nu x + b \sin \nu x \right) \, dx \\
+ \sum_{n=0}^{\infty} \int_{0.5}^{1} P^n \cos n\pi x \left( a \cos \nu(1-x) + b \sin(1-x) \right) \, dx
\]

Now \( \cos n\pi (1-x) = (-1)^n \cos n\pi x \) so the first closed loop "boundary" condition becomes
\[
b = 2 \sum_{k=0}^{\infty} \int_{0}^{0.5} P^{2k} \cos 2k\pi x \left( a \cos \nu x + b \sin \nu x \right) \, dx
\]

The second closed loop "boundary" condition is
\[
\frac{\partial \psi}{\partial x}(0.5^-) - \frac{\partial \psi}{\partial x}(0.5^+) = \int_{0}^{1} K_2(x)\psi(x) \, dx
\]

From (3) and (19) we obtain
\[
2\nu (-a \sin 0.5\nu x + b \cos 0.5\nu x) = -4 \sum_{k=0}^{\infty} \int_{0}^{0.5} P^{2k} \cos 2k\pi x \left( a \cos \nu x + b \sin \nu x \right) \, dx
\]

If \( 0.5\nu \) is an odd integer, i.e., \( \nu = 4j + 2 \), then the two boundary conditions are identical so the closed loop eigenvalues and eigenvectors are \( \mu_j = -(4j+2)^2\pi^2 \) and \( \psi_j(x) = \sin((4j+1)\pi x) \) for \( j = 0, 1, 2, \ldots \) and \( x \in [0, 0.5] \). In particular the least stable closed loop eigenvalue is \(-4\pi^2 = -39.4784\) so being able to heat/cool at several points has a big impact.

IV. Kalman Filtering of the Heat Equation with Point Observations

In the previous sections we constructed optimal feedbacks to control the heat equation. These feedbacks assumed that the full state \( z(x,t) \) is known at every \( x \in [0,1] \) and \( t \geq 0 \). But in practice we may only be able to measure the temperature at a finite number of points \( z(\zeta_1, s), z(\zeta_2, s), \ldots, z(\zeta_p, s) \) where \( 0 \leq \zeta_1 < \zeta_2 < \ldots < \zeta_p \leq 1 \) for \(-\infty < s \leq t \) and these measurements may be corrupted by noise.

Our model for the measured but uncontrolled rod is
\[
\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \sum_{j=1}^{k} B_j(x)v_j(t) \\
y_i(t) = C_i z(\zeta_i, t) + D_i w_i(t) \\
\frac{\partial z}{\partial x}(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = 0
\]

where \( v_j(t), w_i(t) \) are independent, standard white Gaussian noise processes. Without loss of generality we can assume \( C_i > 0 \) for \( i = 1, \ldots, p \).

We wish to construct an estimate \( \hat{z}(x,t) \) for all \( x \in [0,1] \) based on the past measurements, \( y_i(s), s \leq t \). We assume the estimates are linear functionals of the past observations of the form
\[
\hat{z}(x,t) = \int_{-\infty}^{t} \sum_{i=1}^{p} L_i(x,s) y_i(s) \, ds
\]

Since we are taking measurements for \(-\infty < s \leq t \) we expect the filter to be stationary. Therefore it suffices to solve the problem for \( t = 0 \) and we only need to consider \( y_i(s) \) for \(-\infty < s \leq 0 \).

Given a possible set of filter gains \( L_i(x,s) \) we define \( H(x,x_1,s) \) as the solution of a driven backward generalized heat equation
\[
\frac{\partial H}{\partial s} = -\frac{\partial^2 H}{\partial x_1^2} + \sum_{i=1}^{p} L_i(x,s)C_i\delta(x_1 - \zeta_i) \quad (21)
\]
\[
H(x,x_1,0) = \delta(x - x_1) \quad (22)
\]

where \( \zeta_i \) are the sensor locations and \( H(x,x_1,s) \) satisfies Neumann boundary conditions with respect to \( x \) and \( x_1 \).

Then
\[
\hat{z}(x,0) = \sum_{i=1}^{p} L_i(x,0)D_i w_i(s) \\
\int_{-\infty}^{0} \int_{0}^{1} \left( \frac{\partial H}{\partial s}(x,x_1,s) + \frac{\partial^2 H}{\partial x_1^2}(x,x_1,s) \right) z(x_1,s) \, dx_1 
\]

We integrate by parts with respect to \( s \) assuming \( H(x,x_1,s) \to 0 \) as \( s \to -\infty \) and obtain
\[
\hat{z}(x,0) - z(x,0) = -\int_{-\infty}^{0} \int_{0}^{1} H(x,x_1,s) \frac{\partial z}{\partial s}(x_1,s) \\
-\frac{\partial^2 H}{\partial x_1^2}(x,x_1,s)z(x_1,s) \, dx_1 
\]

\[
+ \int_{-\infty}^{0} \sum_{i=1}^{p} L_i(x,s)D_i w_i(s) \\
\hat{z}(x,0) - z(x,0) = -\int_{-\infty}^{0} \int_{0}^{1} H(x,x_1,s) \left( \frac{\partial^2 z}{\partial x_1^2}(x_1,s) + \sum_{j=1}^{k} B_j(x_1)v_j(s) \right) \\
-\frac{\partial^2 H}{\partial x_1^2}(x,x_1,s)z(x_1,s) \, dx_1 
\]

\[
+ \int_{-\infty}^{0} \sum_{i=1}^{p} L_i(x,s)D_i w_i(s) \\
\hat{z}(x,0) - z(x,0) = \int_{-\infty}^{0} \sum_{i=1}^{p} L_i(x,s)D_i w_i(s) \, ds \\
-\int_{-\infty}^{0} H(x,x_1,s) \sum_{j=1}^{k} B_j(x_1)(x_1)v_j(s) \, dx_1 \, ds
\]
Since \( v_j(s) \) and \( w_i(s) \) are independent standard white Gaussian noise processes, the estimation error \( \tilde{z}(x,0) = z(x,0) - \hat{z}(x,0) \) has variance

\[
\sum_{j=1}^{k} \int_{-\infty}^{0} \int_{0}^{1} H(x,x_1,s)B_j^2(x_1)H(x,x_1,s) \, dx_1 \, ds + \sum_{i=1}^{p} \int_{-\infty}^{0} L_i(x,s)D_i^2L_i(x,s) \, ds
\]

(23)

For each \( x \in [0,1] \) this is a backward in time LQR optimal control problem with infinite dimensional state \( x_1 \rightarrow H(x,x_1,s) \) and \( p \) dimensional control \( L_i(x,s) \). Our goal is to minimize for each \( x \) the error variance subject to the backward dynamics (21) and terminal condition (22). The boundary conditions with respect to \( x \) and \( x_1 \) are Neumann.

We plug in the dynamics of \( H \) to get

\[
0 = \int\int_S P(x_1,x_2)H(x,x_1,0)H(x_2,0) \, dA \\
- \int_{-\infty}^{0} \int_S P(x_1,x_2) \frac{\partial H}{\partial x_1}(x_1,s)H(x_2,s) \, dA \, ds \\
+ \int_{-\infty}^{0} \int_S P(x_1,x_2)H(x_1,s) \frac{\partial^2 H}{\partial x_1^2}(x_2,s) \, dA \, ds \\
+ \sum_{i=1}^{p} L_i(x,s)C_i\delta(x_1 - \zeta_i) \, dA \, ds \\
- \int_{-\infty}^{0} \int_S P(x_1,x_2)H(x_1,s) \times \frac{\partial^2 H}{\partial x_2^2}(x_2,t) + \sum_{i=1}^{p} L_i(x,s)C_i\delta(x_2 - \zeta_i) \, dA \, ds
\]

Then we integrate by parts twice to get

\[
0 = \int\int_S P(x_1,x_2)H(x,x_1,s)H(x_2,s) \, dA \\
+ \int_{-\infty}^{0} \int_S \nabla^2 P(x_1,x_2)H(x_1,s)H(x_2,s) \, dA \, ds \\
- \int_{-\infty}^{0} \int_0^1 H(x_2,s) \sum_{i=1}^{p} P(x_1,\zeta_2)L_i(x,s)C_i \, dx_2 \, ds \\
- \int_{-\infty}^{0} \int_0^1 H(x_1,s) \sum_{i=1}^{p} P(x_1,\zeta_1)L_i(x,s)C_i \, dx_1 \, ds
\]

where \( \nabla^2 \) is the two dimensional Lagrangian with respect to \( x_1 \) and \( x_2 \).

We add the right side of this identity to the estimation error variance (23) to get an equivalent quantity to be minimized

\[
\sum_{j=1}^{k} \int_{-\infty}^{0} \int_S H(x,x_1,s)B_j^2(x_1)\delta(x_1 - x_2) \times H(x_2,s) \, dA \, dt \\
+ \int_{-\infty}^{0} \int_S P(x_1,x_2)H(x_1,s)H(x_2,s) \, dA \\
+ \int_{-\infty}^{0} \int_S \nabla^2 P(x_1,x_2)H(x_1,s)H(x_2,s) \, dA \, ds \\
- \int_{-\infty}^{0} \int_0^1 H(x_2,s) \sum_{i=1}^{p} P(\zeta_2)L_i(x,s)C_i \, dx_2 \, ds \\
- \int_{-\infty}^{0} \int_0^1 H(x_1,s) \sum_{i=1}^{p} P(x_1,\zeta_1)L_i(x,s)C_i \, dx_1 \, ds
\]

For each \( i = 1, \ldots, p \) and each \( x \in [0,1] \) we would to choose \( K_i(x_1) \) so the time integrand of the quantity to be minimized is a perfect square of the form

\[
\int\int_S \sum_{i=1}^{p} (L_i(x,s) - K_i(x_1)H(x_1,s)) \times D_i^2(L_i(x,s) - K_i(x_2)H(x_2,s)) \, dA
\]

Clearly the terms quadratic in \( L_i(x,s) \) match up so we compare terms bilinear in \( L_i(x,s) \) and \( H(x_1,s) \)

\[
\int_{0}^{1} \sum_{i=1}^{p} L_i(x,s)D_i^2K_i(x_2)H(x_2,s) \, dx_2 \\
= \int_{0}^{1} \sum_{i=1}^{p} H(x_2,s)P(\zeta_2)L_i(x,s)C_i \, dx_2
\]

This will hold if

\[
D_i^2K_i(x_2) = P(\zeta_2)C_i
\]

for \( i = 1, \ldots, p \) so we define

\[
K_i(x_2) = D_i^{-2}P(\zeta_2)C_i
\]

Then we compare terms bilinear in \( H(x_1,s) \) and \( H(x_2,s) \) and obtain

\[
\sum_{j=1}^{k} \int_{-\infty}^{0} \int_S H(x,x_1,s)B_j^2(x_1)\delta(x_1 - x_2)H(x_2,s) \, dA \\
+ \int_{-\infty}^{0} \int_S \nabla^2 P(x_1,x_2)H(x_1,s)H(x_2,s) \, dA \\
= \int_{-\infty}^{0} \int_S \sum_{i=1}^{p} K_i(x_1)H(x_1,s)D_i^2K_i(x_2)H(x_2,s) \, dA \\
= \int_{-\infty}^{0} \int_S \sum_{i=1}^{p} H(x_1,s)P(\zeta_1,x_1)C_i^2D_i^{-2}P(\zeta_2,x_2) \times H(x_2,s) \, dA
\]
So we are looking for a weak solution to what we call the filter Riccati PDE,
\[ 0 = \nabla^2 P(x_1, x_2) + \sum_{j=1}^{k} B_j^2(x_1) \delta(x_1 - x_2) \]
\[ - \sum_{i=1}^{p} P(x_1, \zeta_i) C_i^2 D^{-2} P(\zeta_i, x_2) \]

Because it satisfies Neumann boundary conditions we guess that \( P(x_1, x_2) \) has an expansion
\[ P(x_1, x_2) = \sum_{n=0}^{\infty} P^n \phi_n(x_1) \phi_n(x_2) \] (24)

We plug this into the weak form of the filter Riccati PDE and we obtain for \( n = 0, 1, 2, \ldots \) the equations
\[ 0 = c_n (P^n)^2 + (2n^2 \pi^2) P^n - b_n \]
where \( c_n = \sum_{i=1}^{p} C_i^2 \) and \( b_n = \sum_{j=1}^{k} \int_0^1 B_j^2(x_1) \phi_n^2(\zeta_j) \, dx_1 \)

The solutions to these equations are
\[ P^n = \frac{-n^2 \pi^2 \pm \sqrt{n^4 \pi^4 + c_n b_n}}{c_n} \] (25)

Since we expect the error covariance to be at least nonnegative we take the positive sign in (25). If we assume \( c_n > 0 \) and \( b_n > 0 \) then \( P^n \) will be positive.

Applying the Mean Value Theorem to (25) we get the estimate
\[ 0 < P^n \leq \frac{1}{2n^2 \pi^2} b_n \]
so the series (24) converges to a continuous function \( P(x_1, x_2) \).

Having found \( P^n \) we have
\[ K_i(x_1) = D_i^{-2} \sum_{n=0}^{\infty} P^n \phi_n(\zeta_i) \phi_n(x_1) \] (26)
\[ L_i(x, s) = \int_0^1 K_i(x_1) H(x_1, x, s) \, dx_1 \] (27)

Since \( H(x, x_1, s) \) satisfies Neumann boundary conditions with respect to \( x \) and \( x_1 \) we assume that the solution to (21) takes the form
\[ H(x, x_1, s) = \sum_{m,n=0}^{\infty} \gamma_{m,n}(s) \phi_m(x) \phi_n(x_1) \] (28)

where \( \phi_n(x) \) are the orthonormal open loop eigenfunctions \( (5) \).

We plug this into the dynamics of \( H \) and we get a sequence of ODEs,
\[ \frac{d}{ds} \gamma_{m,n}(s) = n^2 \pi^2 \gamma_{m,n}(s) \]
\[ + \sum_{i=1}^{p} \sum_{l=1}^{\infty} P^l C_i \phi_l(\zeta_i) \phi_n(\zeta_i) \gamma_{m,l}(s) \] (29)

The terminal condition \( H(x, x_1, 0) = \delta(x - x_1) \) implies
\[ \gamma_{m,n}(0) = \delta_{m,n} \] (30)

Unfortunately in general these equations (29) are an infinite family of coupled linear ODEs but there is at least one case where they decouple.

**Assumption:** Henceforth we shall make the standing assumption that \( B_j(x) = B_j \), a constant. Then WLOG we can then assume that there is only one driving noise, \( k = 1 \).

**Theorem:** If the Assumption holds then
1) If \( n > 0 \) then \( P^n = 0 \).
2) \( L_n(x) = L_i, \) a constant, for \( i = 1, \ldots, p \).
3) In the expansion (28), \( \gamma_{m,n}(s) = 0 \) if \( m \neq n \) and
\[ \gamma_{0,0}(s) = \exp(\sum_{i=1}^{p} P^0 C_i s) \] (31)
\[ \gamma_{m,n}(s) = \exp(n^2 \pi^2 s) \] (32)
for \( n > 0 \).
4) \[ L_i(x, s) = L_i(s) = K_i \gamma_{0,0}(s) \]
\[ \frac{d}{ds} L_i(s) = L_i(s) \left( \sum_{j=1}^{p} C_j \right) \]
5) \( H(x, x_1, s) \) is symmetric in \( x \) and \( x_1 \), \( H(x, x_1, s) = H(x_1, x, s) \).

**Proof:** If \( B(x) = B \), a constant then for \( n > 0 \), \( b_n = 0 \) so \( P^n = 0 \) so (1) holds.
Then (26) implies \( L_i(x_1) = D_i^{-2} P^0 \) so (2) holds.
Under the Assumption the ODEs (29) reduce to
\[ \frac{d}{ds} \gamma_{m,n}(s) = n^2 \pi^2 \gamma_{m,n}(s) + \sum_{i=1}^{p} P^0 C_i \phi_n(\zeta_i) \gamma_{m,0}(s) \]

In particular if \( m = n = 0 \) then
\[ \frac{d}{ds} \gamma_{0,0}(s) = \sum_{i=1}^{p} P^0 C_i \gamma_{0,0}(s) \] (33)
and it satisfies the terminal condition \( \gamma_{0,0}(0) = 0 \) so (31) holds.
If \( m > 0 \) then
\[ \frac{d}{ds} \gamma_{m,0}(s) = \sum_{i=1}^{p} P^0 C_i \phi_n(\zeta_i) \gamma_{m,0}(s) \] (34)
and it satisfies the terminal condition \( \gamma_{m,0}(0) = 0 \) so \( \gamma_{m,0}(s) = 0 \) for \( s \leq 0 \).
If \( m > 0 \) and \( n \neq m \) then
\[ \frac{d}{ds} \gamma_{m,n}(s) = n^2 \pi^2 \gamma_{m,n}(s) \] (35)
and it satisfies the terminal condition \( \gamma_{m,0}(0) = 0 \) so \( \gamma_{m,n}(s) = 0 \) for \( s \leq 0 \).

Then (27) implies
\[ L_i(x, s) = L_i(s) = D_i^{-2} P^0 \gamma_{0,0}(s) \] (36)
\[ \frac{d}{ds} L_i(s) = L_i(s) \left( \sum_{j=1}^{p} P^0 C_j \right) \] (37)

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If \( m = n > 0 \) then
\[
d
d\gamma_{n,n}(s) = n^2 \pi^2 \gamma_{n,n}(s) \tag{38}
\]
and it satisfies the terminal condition \( \gamma_{n,n}(0) = 1 \) so (32) holds. So (4) holds and (5) follows immediately from (4).

QED.

Notice that \( H(x, x_1, s) \to 0 \) as \( s \to -\infty \) and \( H(x, x_1, s) \) satisfies Neumann boundary conditions with respect to \( x \) and \( x_1 \).

The estimate of the state at time \( t \) is
\[
\hat{z}(x, t) = \int_{-\infty}^{t} \sum_{i=1}^{p} L_i(s - t)y_i(s) \, ds
\]
and
\[
\frac{\partial \hat{z}}{\partial t}(x, t) = \sum_{i=1}^{p} L_i(0)y_i(t)
+ \int_{-\infty}^{t} \sum_{i=1}^{p} \frac{\partial L_i}{\partial t}(s - t)y_i(s) \, ds
\]
\[
= \sum_{i=1}^{p} L_i(0)y_i(t) + \int_{-\infty}^{t} \sum_{i=1}^{p} K_i(s - t) \left( \sum_{j=1}^{p} P^0 C_j \right) y_i(s) \, ds
\]
\[
= \sum_{i=1}^{p} L_i(0)y_i(t) + \left( \sum_{j=1}^{p} P^0 C_j \right) \hat{z}(t) \, ds
\]

Notice \( L_i(0) = K_i \).

Because \( H(x, x_1, s) \) is symmetric in \( x \) and \( x_1 \) we see that
\[
\frac{\partial^2 H}{\partial x^2}(x, x_1, s - t) = \frac{\partial^2 H}{\partial x^2}(x, x_1, s - t)
\]
so
\[
\frac{\partial \hat{z}}{\partial t}(x, t) = \sum_{i=1}^{p} K_i y_i(t)
+ \int_{-\infty}^{t} \int_{0}^{1} \sum_{i=1}^{p} K_i \frac{\partial^2 H}{\partial x^2}(x, x_1, s - t)y_i(s) \, dx_1 \, ds
\]
\[
- \sum_{i=1}^{p} K_i \int_{-\infty}^{t} \sum_{j=1}^{p} C_j H(x, \zeta_j, s)y_i(s) \, ds
\]
\[
= \sum_{i=1}^{p} K_i y_i(t) + \frac{\partial \hat{z}}{\partial x^2}(x, t)
- \sum_{j=1}^{p} K_j \int_{-\infty}^{t} \sum_{i=1}^{p} C_j H(x, \zeta_j, s)y_i(s) \, ds
\]
\[
= \sum_{i=1}^{p} K_i y_i(t) + \frac{\partial \hat{z}}{\partial x^2}(x, t)
- \sum_{j=1}^{p} K_j \int_{-\infty}^{t} \sum_{i=1}^{p} C_j H(x, \zeta_j, s)y_i(s) \, ds
\]
\[
= \sum_{i=1}^{p} K_i y_i(t) + \frac{\partial \hat{z}}{\partial x^2}(x, t) - \sum_{j=1}^{p} K_j \hat{z}(t, \zeta_j)
\]
Now we define
\[
\dot{y}_i(t) = \dot{\hat{z}}(t, \zeta_i)
\]
\[
\ddot{y}_i(t) = y_i(t) - \ddot{y}_i(t)
\]
The quantities \( \ddot{y}_i(t) \) are called the innovations. So the dynamics of optimal estimator is a copy of original dynamics driven by the innovations
\[
\frac{\partial \ddot{z}}{\partial t}(x, t) = \frac{\partial^2 \ddot{z}}{\partial x^2}(x, t) + \sum_{i=1}^{p} K_i \ddot{z}(t, \zeta_i)
\]
The error dynamics is
\[
\frac{\partial \ddot{z}}{\partial t}(x, t) = \frac{\partial^2 \ddot{z}}{\partial x^2}(x, t) - \sum_{i=1}^{p} K_i \ddot{z}(t, \zeta_i) + Bv(t) \tag{39}
\]
Because both \( z(x, t) \) and \( \hat{z}(x, t) \) satisfy Neumann boundary conditions so does \( \ddot{z}(x, t) \).

From the form of the error dynamics we see that the closed loop eigenvalues \( \eta_n \) and eigenfunctions \( \theta_n(x) \) are weak solutions of the equations
\[
\frac{\partial^2 \theta_n}{\partial x^2}(x, x_1, s - t) = \theta_n \theta_n(x)
\]
subject to Neumann boundary conditions. It is easy to see that that \( \theta_0(x) = 1 \) and \( \eta_0 = -\sum K_i < 0 \) since \( K_i > 0 \).

For \( n > 0 \) we expect that on each subinterval \( \zeta_i < x < \zeta_{i+1} \) the eigenfunctions are sinusoidal of the same frequency \( \tau_n \). The corresponding eigenvalue is \( \eta_n = -\tau_n^2 \).

The eigenfunctions \( \theta_n(x) \) are continuous at the measurement points \( \zeta_i \) but their derivatives jump
\[
\frac{\partial \theta_n}{\partial x}(x^+, \zeta_i) - \frac{\partial \theta_n}{\partial x}(x^-, \zeta_i) = K_i \theta_n(\zeta_i)
\]
Next we consider the point controlled and point measured system
\[
\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + Bv(t)
\]
\[
z(\xi_k^-, t) = z(\xi_k^+, t) \text{ for } k = 2, \ldots, m - 1
\]
\[
\beta_k u_k(t) = \frac{\partial z}{\partial x}(\xi_k^-, t) - \frac{\partial z}{\partial x}(\xi_k^+, t) \text{ for } k = 1, \ldots, m
\]
\[
y_i(t) = C_i z(\zeta_i, t) + D_i w_i(t)
\]
\[
z(x, t) = z^0(x)
\]
We use the linear feedback on the state estimate to get the control inputs
\[
\dot{u}(t) = \int_{0}^{1} K(x) \hat{z}(x, t) \, dx
\]
where \( \hat{z}(x, t) \) satisfies the Kalman filtering equation modified by the linear feedback on the state estimate,
\[
\frac{\partial \hat{z}}{\partial t}(x, t) = \frac{\partial^2 \hat{z}}{\partial x^2}(x, t) + \sum_{i=1}^{p} K_i \hat{z}(t, \zeta_i)
\]
\[
\hat{z}(\xi_k^+, t) = \hat{z}(\xi_k^+, t) \text{ for } k = 2, \ldots, m - 1
\]
\[
\beta_k \dot{u}_k(t) = \frac{\partial \hat{z}}{\partial x}(\xi_k^-, t) - \frac{\partial \hat{z}}{\partial x}(\xi_k^+, t) \text{ for } k = 1, \ldots, m
\]
\[
\hat{z}(x, t) = \hat{z}^0(x)
\]

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By linearity the error dynamics is still
\[ \frac{\partial \tilde{z}}{\partial t}(x,t) = \frac{\partial^2 \tilde{z}}{\partial x^2}(x,t) - \sum_{i=1}^{p} K_i \tilde{z}(t,\zeta_i) \]
\[ \tilde{z}(x,0) = \tilde{z}^0(x) = \tilde{z}^0(x) - \tilde{z}^0(x) \]

We can consider the combined system in coordinates \( z(x,t) \) and \( \tilde{z}(x,t) \) but it is more useful to consider it in coordinates \( z(x,t) \) and \( \tilde{z}(x,t) \) because in the latter coordinates the combined dynamics is block upper triangular, the error dynamics does not depend on \( z(x,t) \). In these coordinates the control is given by
\[ \hat{u}(t) = \int_0^1 K(x)(z(x,t) - \tilde{z}(x,t)) \, dx \]
so when \( \tilde{z}(x,t) = 0 \) the dynamics of \( z(x,t) \) takes the form of the original system under full state feedback. This shows the spectrum of the LQG synthesis is the union of the spectrum of the original system under LQR full state feedback and spectrum of the error dynamics of the Kalman filter. So if these spectra are in the open left half plane then the LQG synthesis is asymptotically stable, the rod goes to the desired temperature \( z(x,t) \rightarrow 0 \) and the estimation error goes to zero \( \tilde{z}(x,t) \rightarrow 0 \) in the absence of driving and observation noises, \( v_j(t) = 0 \), \( w_i(t) = 0 \).

V. Example Two

We consider a Linear Quadratic Gaussian synthesis for the heat equation. As in Example One we assume that there are three actuators, \( m = 3 \), at \( \zeta_1 = 0 \), \( \zeta_2 = 0.5 \), \( \zeta_3 = 1 \) with coefficients \( \beta_1 = 1 \), \( \beta_2 = 2 \), \( \beta_3 = 3 \) and the rest of the constants are as in Example One. We further assume that there are two sensors at \( \zeta_1 = 0.25 \), \( \zeta_2 = 0.75 \) with \( C_1 = C_2 = 1 \). The coefficient of the driving noise is \( B = 1 \) and the coefficients of measurement noise are \( D_1 = D_2 = 1 \). Then \( P^0 = \frac{\pi^2}{4} \) and \( K_1 = K_2 = \frac{\pi^2}{2} \).

The zeroth order eigenfunction is \( \theta_0(x) = 1 \) and the corresponding eigenvalue is \( \eta_0 = -\sqrt{2} \). For \( n > 0 \) the eigenfunctions are of the form
\[
\theta_n(x) = \begin{cases} 
  a_1 \cos \tau_n x + b_1 \sin \tau_n x & 0 \leq x \leq 0.25 \\
  a_2 \cos \tau_n x + b_2 \sin \tau_n x & 0.25 \leq x \leq 0.75 \\
  a_3 \cos \tau_n x + b_3 \sin \tau_n x & 0.75 \leq x \leq 1 
\end{cases}
\]

By symmetry \( \theta_n(x) = \theta_n(1 - x) \) so only need to find \( a_1, b_1, a_2, b_2 \). Since \( \theta_n(x) \) satisfies Neumann boundary conditions at \( x = 0 \), \( b_1 = 0 \) and we can take \( a_1 = 1 \).

On the interval \([0.25, 0.75]\) the solution must be symmetric around 0.5 so we make the change of coordinates \( \bar{x} = 0.5 - x \) then it must be of the form \( \theta_n(x) = \tilde{a} \cos \tau_n \bar{x} = \tilde{a} \cos \tau_n (0.5 - x) \) for \(-0.25 \leq \bar{x} \leq 0.25 \). Since \( \theta_n(x) \) is continuous at \( x = \zeta_1 = 0.25 \) which corresponds \( \bar{x} = 0.25 \) we get the condition \( \cos \tau_n \frac{\pi}{4} = \tilde{a} \cos \tau_n \frac{\pi}{4} \) so we conclude that \( \tilde{a} = 1 \).

The derivative jumps at \( x = \zeta_1 = 0.25 \) so we have
\[ 2\tau_n \sin \frac{\tau_n \pi}{4} = \sqrt{2} \cos \frac{\tau_n \pi}{4} \]
This leads to the equation
\[ \frac{\tau_n}{4} = \frac{\sqrt{2}}{16} \cot \frac{\tau_n}{4} \]
There is exactly one root \( \tau_n \) of this equation between \((n-1)\pi \) and \((n-1/2)\pi \) for each \( n = 1, 2, 3, \ldots \) so
\[ 4(n-1) < \tau_n < 4(n-1/2) \]
and so the eigenvalues \( \eta_n \) of the error dynamics satisfy
\[ -16(n-1)^2 > \eta_n > -16(n-1/2)^2 \]

We showed in Example 2 the closed loop eigenvalues under full state feedback are \( \mu_j = -(4j + 2)^2 \pi^2 \) so this Linear Quadratic Gaussian synthesis is asymptotically stable in the absence of driving and observation noises.

VI. Conclusion

We have explicitly derived an LQG synthesis for the heated/ cooled rod under point actuation and point sensing. The key tool that we used is the completing the square technique. We believe that this technique can be used to solve the LQG synthesis problems for the wave and beam equations under point actuation and point sensing.

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