Revisiting the conformal invariance of Maxwell’s equations in curved spacetime

Jeremy Côté¹,² · Valerio Faraoni¹ · Andrea Giusti¹

Received: 12 June 2019 / Accepted: 5 September 2019 / Published online: 10 September 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
We revisit the invariance of the curved spacetime Maxwell equations under conformal transformations. Contrary to standard literature, we include the discussion of the four-current, the wave equations for the four-potential and the field, and the behaviour of gauge conditions under the conformal transformation.

Keywords Conformal transformations · Conformal invariance · Maxwell’s equations · Curved spacetimes

1 Introduction

Conformal transformations of the spacetime metric constitute a very useful tool of general relativity [1–7]. Alternative theories of gravity use conformal transformations even more heavily: for example scalar–tensor gravity admits two representations related by a conformal transformation, the so-called Jordan and Einstein conformal frames [8,9]. The conformal transformations we refer to should not be confused with the coordinate transformations of the conformal group in flat space, which also leave the Maxwell equations invariant [10–13].

A conformal transformation is a position-dependent rescaling of the spacetime metric

\[ g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab} \]

A conformal transformation is a position-dependent rescaling of the spacetime metric

\[ g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab} \]
where the conformal factor $\Omega(x^{\alpha})$ is a (dimensionless) positive smooth function of the spacetime position $x^{\alpha}$. Conformal transformations do not change the metric signature, the sign of the magnitude of four-vectors, the angles between them and, more important, they leave the light cones and the causal structure of spacetime invariant (see Appendix D of Ref. [6]).

It is standard knowledge that the Maxwell equations in four spacetime dimensions are invariant under conformal transformations [4–7]. The physical interpretation of this fact is that, due to the fact that the photon is massless, no length or mass scale is associated with the electromagnetic field. Therefore, the Maxwell equations are not affected by a (point-dependent) rescaling of the metric which changes non-zero distances between points and the lengths of non-null vectors. In the geometric optics limit of wavelengths negligible in comparison with the radius of curvature of spacetime [6,7,14], electromagnetic waves travel along null geodesics and it is well known that a conformal transformation (1.1) leaves null geodesics invariant, apart from a change of parametrization [6,7].

The standard proof of the conformal invariance of the Maxwell equations (e.g., Appendix D of Ref. [6]) is presented in the absence of sources of the electromagnetic field and it refers to the equations satisfied by the Maxwell tensor $F_{ab}$,

$$\nabla^b F_{ab} = 0, \quad (1.2)$$

$$\nabla[a F_{bc}] = 0, \quad (1.3)$$

where $\nabla_c$ denotes the covariant derivative operator of the spacetime metric $g_{ab}$, square brackets around indices denote antisymmetrization, we follow the notation of Ref. [6], and we restrict to four spacetime dimensions (this last assumption is crucial for the conformal invariance of the Maxwell equations: in higher dimension, conformal invariance can be achieved only at the price of modifying substantially the Maxwell action [15]). Nothing is usually said about the conformal invariance of the equation satisfied by the electromagnetic four-potential $A_b$ which, in the absence of sources, is

$$\Box A_b - \nabla_b \nabla^c A_c - R^d_b A_d = 0, \quad (1.4)$$

where $\Box \equiv g^{ab} \nabla_a \nabla_b$ is d’Alembert’s operator in curved spacetime. Indeed, the standard presentation of this equation is in the Lorenz gauge $\nabla^c A_c = 0$, in which the term $-\nabla_b \nabla^c A_c$ drops out. While the physical justification for the conformal invariance of the source-free Maxwell equations (1.2), (1.3) is intuitive, three questions arise. First, are the Maxwell equations in the presence of charges and currents (described by the four-vector $j^a$)

$$\nabla^a F_{ab} = -4\pi j_b, \quad (1.5)$$

$$\nabla[a F_{bc}] = 0, \quad (1.6)$$

still conformally invariant?

$^1$ By contrast, the equations for the Proca field, which contain a mass scale, are not conformally invariant.

© Springer
The answer is not trivial because, while the Maxwell field is associated with the massless photon, with the exception of displacement currents, charges and currents are associated with matter (electrons, protons, or ions). The answer is that the Maxwell equations with sources are still conformally invariant, but apparently the proof does not appear in the literature. It is presented here, together with the scaling property of the four-current $j^a$ under conformal transformations and with the explicit verification of the conformal invariance of charge conservation.

Second, is the Maxwell equation satisfied by the electromagnetic four-potential

$$\Box A_b - \nabla_b \nabla^c A_c - R^c_{\ b\ d} A_d = -4\pi j_b$$

(written here in the presence of sources) still conformally invariant?

After all, there are substantial differences between Eq. (1.7) and the Maxwell equations (1.5), (1.6): (1.7) is a wave equation while (1.5) and (1.6) are first order equations for the field $F_{ab}$. Further, $A^b$ couples explicitly to the Ricci tensor. One might wonder what happens to $A^c$, since it is gauge-dependent. Under a conformal transformation, it seems plausible that the conformal invariance is broken. However, if these terms that break the conformal invariance are pure gauge terms (which give a vanishing contribution to $F_{ab}$), then the situation might not be as bad. It turns out (but is not usually mentioned in the literature) that Eq. (1.7) is conformally invariant and, like the Maxwell field $F_{ab}$, the four-potential $A_b$ is conformally invariant.

Third, given that the equation for $A^b$ is usually presented in the Lorenz gauge $\nabla^c A_c = 0$, is this gauge (or any gauge choice) preserved by a conformal transformation? In general, the answer is negative, as will be shown in Sect. 3.1.

Before computing the answers to the questions above, we recall the action for the electromagnetic field with sources described by the four-current $j^a$ in curved spacetime [5–7,14]

$$S_{(em)} = \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{ab} F^{ab} + 4\pi A_b j^b \right)$$

$$= \int d^4x \sqrt{-g} \left( -g^{ac} g^{bd} \nabla_{[a} A_{b]} \nabla_{[c} A_{d]} + 4\pi A_b j^b \right),$$

(1.8)

where $g$ is the determinant of the metric tensor $g_{ab}$. The variation of the action (1.8) with respect to $g^{ab}$ and $A^c$ produces the field equations (1.5) and (1.6), respectively (e.g., Ref. [7], p. 164). The stress-energy tensor of the electromagnetic field is

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{(em)}}{\delta g^{ab}} = \frac{1}{4\pi} \left( F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{de} F^{de} \right),$$

(1.9)

has vanishing trace $T \equiv g^{ab} T_{ab} = 0$, and it is not covariantly conserved in the presence of sources $j^c$ interacting with the field and exchanging energy and momentum with it. By taking the covariant divergence of $T_{ab}$ and using the field equations (1.5) and (1.6), one easily obtains

$$\nabla^b T_{ab} = -F_{ab} j^b.$$  

(1.10)
We also need the transformation properties of various spacetime quantities under conformal rescalings [4,6,7], including the inverse metric
\[
\tilde{g}^{ab} = \Omega^{-2} g^{ab}, \tag{1.11}
\]
\[
\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g}, \tag{1.12}
\]
the Christoffel symbols
\[
\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \Omega^{-1} \left( \delta^a_b \nabla_c \Omega + \delta^a_c \nabla_b \Omega - g_{bc} \nabla^a \Omega \right), \tag{1.13}
\]
and the Ricci tensor
\[
\tilde{R}_{ab} = R_{ab} - 2 \nabla_a \nabla_b \ln \Omega - g_{ab} g^{ef} \nabla_e \nabla_f \ln \Omega + 2 \nabla_a \ln \Omega \nabla_b \ln \Omega \\
- 2 g_{ab} g^{ef} \nabla_e \ln \Omega \nabla_f \ln \Omega. \tag{1.14}
\]

2 Maxwell equations with sources

Let us consider the Maxwell equations (1.5), (1.6) with sources described by the four-current \(j^a\). Under the conformal transformation (1.1), the Maxwell tensor \(F_{ab}\) will scale [6] with conformal weight \(s\),
\[
\tilde{F}_{ab} = \Omega^s F_{ab}, \tag{2.1}
\]
where a tilde denotes quantities in the conformally rescaled spacetime with metric \(\tilde{g}_{ab}\) [6]. The covariant divergence of the Maxwell tensor (needed in the first Maxwell equation) in this rescaled world is
\[
\tilde{g}^{ac} \nabla_c \tilde{F}_{ab} = \Omega^{-2} g^{ac} \nabla_c \left( \Omega^s F_{ab} \right) \\
= \Omega^{s-2} g^{ac} \nabla_c F_{ab} + s \Omega^{s-3} g^{ac} F_{ab} \nabla_c \Omega \\
= \Omega^{s-2} g^{ac} \left( \partial_c F_{ab} - \tilde{\Gamma}^e_{ca} F_{eb} - \tilde{\Gamma}^e_{cb} F_{ae} \right) \\
+ s \Omega^{s-3} g^{ac} F_{ab} \nabla_c \Omega \\
= \Omega^{s-2} g^{ac} \left[ \partial_c F_{ab} - \Gamma^e_{ca} F_{eb} - \Gamma^e_{cb} F_{ae} \\
- \Omega^{-1} \left( \delta^e_c \nabla_a \Omega + \delta^e_a \nabla_c \Omega - g_{ac} \nabla^e \Omega \right) F_{eb} \\
- \Omega^{-1} \left( \delta^e_b \nabla_a \Omega + \delta^e_a \nabla_b \Omega - g_{bc} \nabla^e \Omega \right) F_{ae} \right] \\
+ s \Omega^{s-3} g^{ac} F_{ab} \nabla_c \Omega \tag{2.2}
\]
\[
= \Omega^{s-2} \nabla^a F_{ab} - s \Omega^{s-3} F_{be} \nabla^e \Omega. \tag{2.3}
\]

Using the Maxwell equation (1.5), one obtains
\[
\tilde{g}^{ac} \nabla_c \tilde{F}_{ab} = -4\pi \Omega^{s-2} j_b - s \Omega^{s-3} F_{be} \nabla^e \Omega, \tag{2.4}
\]
while the left hand side of the second Maxwell equation (1.6) in the rescaled world is

$$\tilde{\nabla}_a \left( \Omega^s F_{bc} \right) = \Omega^s \nabla_a F_{bc} + s \Omega^{s-1} \left( \nabla_a \Omega \right) F_{bc};$$

(2.5)

it is clear that the only value of the conformal weight of $F_{ab}$ that leaves both Maxwell equations conformally invariant in the rescaled spacetime is $s = 0$. Using this value, the electromagnetic tensor with two covariant indices is conformally invariant,

$$\tilde{F}_{ab} = F_{ab}, \quad \tilde{F}^{ab} = \Omega^{-4} F^{ab}, \quad \tilde{F}_a^b = \Omega^{-2} F_a^b,$$

(2.6)

and the conformally rescaled Maxwell equations read

$$\tilde{g}^{ac} \tilde{\nabla}_c \tilde{F}_{ab} = -4\pi \tilde{j}_b,$$

(2.7)

$$\tilde{\nabla}_{[a} \tilde{F}_{bc]} = 0,$$

(2.8)

provided that the four-current transforms according to

$$\tilde{j}_b = \Omega^{-2} j_b$$

(2.9)

(we are not aware of occurrences of this last equation in the literature—these transformation properties are hinted at, although not fully discussed, in Ref. [16]).

2.1 Covariant charge conservation

One can now check explicitly that the electric charge is covariantly conserved in the conformally rescaled geometry. The covariant divergence (according to the metric $\tilde{g}_{ab}$) of the rescaled four-current $\tilde{j}^b$ given by Eq. (2.9) is

$$\tilde{g}^{ac} \tilde{\nabla}_a \tilde{j}_c = \Omega^{-2} \tilde{g}^{ac} \tilde{\nabla}_a \left( \Omega^{-2} j_c \right)$$

$$= \Omega^{-2} \tilde{g}^{ac} \left[ -2\Omega^{-3} (\nabla_a \Omega) j_c + \Omega^{-2} \tilde{\nabla}_a j_c \right]$$

$$= \Omega^{-2} \tilde{g}^{ac} \left[ -2\Omega^{-3} (\nabla_a \Omega) j_c + \Omega^{-2} \left( \partial_a j_c - \Gamma_{ac}^d j_d \right) \right]$$

$$= \Omega^{-2} \tilde{g}^{ac} \left[ -2\Omega^{-3} (\nabla_a \Omega) j_c + \Omega^{-2} \left( \partial_a j_c - \Gamma_{ac}^d j_d \right) \right.$$  

$$- \Omega^{-1} \left( \delta_a^d \nabla_c \Omega + \delta_a^c \nabla_d \Omega - g_{ac} \nabla^d \Omega \right) j_d \right]$$

$$= \Omega^{-2} \tilde{g}^{ac} \left[ -2\Omega^{-3} j_c \nabla_a \Omega + \Omega^{-2} \nabla_a j_c 
$$

$$- \Omega^{-3} \left( j_a \nabla_c \Omega + j_c \nabla_a \Omega - g_{ac} j_d \nabla^d \Omega \right) \right] = 0,$$

(2.10)

where we used Eq. (1.13) and the covariant conservation of the electric charge $\nabla^c j_c = 0$ in the unrescaled spacetime.
3 Wave equation for the four-potential

The second question to address is whether the wave equation satisfied by the four-potential $A^b$ is conformally invariant. Since $A^b$ couples explicitly to the Ricci tensor and, contrary to the Maxwell tensor $F_{ab}$, is gauge-variant, this question is not trivial.

The Maxwell equation (1.6) guarantees that the Maxwell tensor can be derived from a four-potential $A^b$ according to

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a. \tag{3.1}$$

We will now go over the standard derivation of the equation satisfied by $A^b$ in curved spacetime. However, contrary to common practice, let us allow for the presence of sources. Furthermore, we won’t fix the gauge in order to keep the presentation general.

In conjunction with Eq. (3.1), the Maxwell equation (1.5) yields

$$\Box A^b - \nabla^a \nabla_b A_a = -4\pi j^b. \tag{3.2}$$

Using the rule for the commutator of covariant derivatives in curved spacetime [6,7]

$$[\nabla_a, \nabla_b] A^c = R^c_{bd} A^e \tag{3.3}$$

and the symmetries of the Riemann tensor to write

$$R^c_{bcd} = -R^c_{b dc} = -R^c_{dcb} \equiv R^c_{db}, \tag{3.4}$$

the second term on the left hand side of Eq. (3.2) becomes

$$\nabla^c \nabla_b A_c = \nabla_b \nabla^c A_c + R_{db} A^d, \tag{3.5}$$

so that (Ref. [5], p. 569)

$$\Box A^b - \nabla_b \nabla^c A_c - R^d_{db} A^d = -4\pi j^b. \tag{3.6}$$

This equation simplifies in the Lorenz gauge $\nabla^c A_c = 0$, in which it is usually presented. Before investigating the conformal invariance of Eq. (3.6), it is necessary to establish the scaling law of $A^b$. The validity of the second Maxwell equation in the conformally rescaled spacetime guarantees that the rescaled Maxwell tensor can be written as

$$\tilde{F}_{ab} = \tilde{\nabla}^a \tilde{A}_b - \tilde{\nabla}^b \tilde{A}_a = \partial^a \tilde{A}_b - \partial^b \tilde{A}_a. \tag{3.7}$$

If the conformal weight of the four-potential is $p$, i.e., $\tilde{A}_a = \Omega^p A_a$, then

$$\tilde{F}_{ab} = p \Omega^{p-1} (\partial_a \Omega A_b - \partial_b \Omega A_a) + \Omega^p F_{ab} \tag{3.8}$$
and then the result $\tilde{F}_{ab} = F_{ab}$ gives

$$\left( \Omega^p - 1 \right) F_{ab} + p \Omega^{p-1} (\partial_a \Omega A_b - \partial_b \Omega A_a) = 0. \quad (3.9)$$

This equation is only satisfied if $p = 0$, or

$$\tilde{A}_b = A_b, \quad \tilde{A}^b = \Omega^{-2} A^b. \quad (3.10)$$

Having established this result, we can now proceed to check the conformal invariance of Eq. (1.7). First, one computes

$$\tilde{\nabla}_c \tilde{A}_b = \nabla_c A_b - A_c \nabla_b \ln \Omega - A_b \nabla_c \ln \Omega + g_{bc} A_e \nabla^e \ln \Omega. \quad (3.11)$$

Before proceeding, we discuss gauge invariance.

### 3.1 Lorenz gauge

Let us consider again the Maxwell action

$$S_{(em)} = \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{ab} F^{ab} + 4\pi g_{ab} A^a j^b \right); \quad (3.12)$$

by applying the transformation properties of the various quantities appearing in the integrand and derived in the previous sections one obtains that, under a conformal transformation $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$, the action rewrites as

$$S_{(em)} = \int d^4x \left[ -\frac{1}{4} \sqrt{-\tilde{g}} \tilde{F}_{ab} \tilde{F}^{ab} + 4\pi \left( \Omega^4 \sqrt{-\tilde{g}} \right) \left( \Omega^{-2} \tilde{g}_{ab} \right) \left( \Omega^2 \tilde{A}^a \right) \left( \Omega^{-4} j^b \right) \right] = \int d^4x \sqrt{-\tilde{g}} \left( -\frac{1}{4} \tilde{F}_{ab} \tilde{F}^{ab} + 4\pi \tilde{A}_b \tilde{j}^b \right). \quad (3.13)$$

Therefore, the action is invariant in form under conformal transformations and the “new” Maxwell equations in the tilded world coincide with the “old” Maxwell equations in the non-tilded one. However, these are the Maxwell equations prior to gauge-fixing and their conformal invariance has nothing to do (nor does preserve) particular gauge conditions, which are not contained in the action (3.12).

Our question about the conformal invariance of the Lorenz gauge can now be answered by computing

$$\tilde{g}^{ab} \tilde{\nabla}_a \tilde{A}_b = \Omega^{-2} \left( \nabla^a A_a + 2A_a \nabla^a \ln \Omega \right). \quad (3.14)$$

The Lorenz gauge is broken by the conformal transformation unless the gradient of the conformal factor is perpendicular to the four-potential,

$$A^c \nabla_c \Omega = 0, \quad (3.15)$$
a very special condition that cannot be enforced in general. Therefore, conformal transformations break the Lorenz gauge or, in general, any gauge condition (this result appears in Ref. [17] which applies it to the study of the sharp propagation of electromagnetic waves in special curved spacetimes). However, this is a “soft” breaking of conformal invariance that can always be removed by a gauge redefinition. If, in the absence of sources, one starts with the Lorenz gauge $\nabla_c A_c = 0$, one ends with $\tilde{\nabla}_c \tilde{A}_c = 2\Omega^{-3} A_b \nabla^b \Omega \neq 0$, but it is always possible to perform a gauge transformation to restore the Lorenz gauge [6]. As such, the term introduced in the equation for the four-potential by the conformal transformation can be gauged away and gives zero contributions to $\tilde{F}_{ab}$.

3.2 Light-cone gauge

To avoid the issue of having to fix the gauge every time one performs a conformal transformation to the system, it is possible to make a different choice for the gauge-fixing condition. Indeed, an adequate alternative to the Lorenz gauge, which is usually the go-to Lorentz-invariant condition used whenever one deals with the study of gauge theories in both classical and quantum field theory, is given by the so called light-cone gauge. Let $\ell^a = \ell^a (x^a)$ be the tangent vector field to a congruence of null geodesics of the spacetime $(\mathcal{M}, g_{ab})$. Then the condition

$$\ell^a A_a = 0$$

defines the light-cone gauge. Clearly, this condition is conformally invariant and therefore all conformally equivalent frames would agree on this gauge-fixing. However, this choice comes at a price, namely this condition depends on the choice of the congruence of null geodesics and its implications for the equations of motion are not as apparent as in the case of the Lorenz gauge.

3.3 Conformal invariance of Eq. (1.7)

Continuing our calculation that led to Eq. (3.11), one computes

$$\tilde{\Box} \tilde{A}_b = \Omega^{-2} g^{ac} \tilde{\nabla}_a \tilde{\nabla}_c A_b$$

where using the transformation properties (1.11)–(1.14), one obtains

$$\tilde{\nabla}_a \tilde{\nabla}_c \tilde{A}_b = \nabla_a \nabla_c A_b - \nabla_b \ln \Omega (\nabla_a A_c + \nabla_c A_a) - A_c \nabla_a \nabla_b \ln \Omega$$

$$- 2\nabla_a A_b \nabla_c \ln \Omega - A_b \nabla_a \nabla_c \ln \Omega + g_{bc} \nabla_a (A_e \nabla^e \ln \Omega)$$

$$- 2\nabla_c A_b \nabla_a \ln \Omega + 3A_c \nabla_a \ln \Omega \nabla_b \ln \Omega + 3A_b \nabla_a \ln \Omega \nabla_c \ln \Omega$$

$$- 2g_{bc} \nabla_a \ln \Omega A_e \nabla^e \ln \Omega + 2A_a \nabla_b \ln \Omega \nabla_c \ln \Omega$$

$$+ g_{ab} (\nabla^e \ln \Omega \nabla_c A_e - A_c \nabla^e \ln \Omega \nabla_e \ln \Omega - \nabla_c \ln \Omega A_e \nabla^e \ln \Omega)$$

$$+ g_{ac} (\nabla^e \ln \Omega \nabla_e A_b - A_b \nabla^e \ln \Omega \nabla_e \ln \Omega - \nabla_b \ln \Omega A_e \nabla^e \ln \Omega).$$

$$(3.18)$$
The contraction of this equation produces the d’Alembertian
\[
\tilde{\Box} \tilde{A}_b = \Omega^{-2} \left[ \Box A_b - A_b \Box \ln \Omega - 2 \nabla_b \ln \Omega \nabla^a A_a - A^a \nabla_a \nabla_b \ln \Omega \\
+ \nabla_b \left( A_a \nabla^a \ln \Omega \right) - 2 A_a \nabla^a \ln \Omega \nabla_b \ln \Omega \\
- 2 A_b \nabla^a \ln \Omega \nabla_a \ln \Omega + \nabla^a \ln \Omega \nabla_b \nabla_a A_a \right],
\]
(3.19)

while the \( \tilde{R}_d \tilde{A}_d \) term of Eq. (3.6) is
\[
\tilde{g}^{ad} \tilde{R}_{ab} \tilde{A}_d = \Omega^{-2} \left( R_b^d A_d - 2 A^a \nabla_a \nabla_b \ln \Omega - A_b \Box \ln \Omega \\
+ 2 A_a \nabla^a \ln \Omega \nabla_b \ln \Omega - 2 A_b \nabla^a \ln \Omega \nabla_a \ln \Omega \right).
\]
(3.20)

The covariant differentiation of Eq. (3.14) gives
\[
\tilde{\nabla}_b \tilde{\nabla}^a \tilde{A}_a = \nabla_b \tilde{\nabla}^a \tilde{A}_a = \Omega^{-2} \left[ \nabla_b \nabla^a A_a + 2 A_a \nabla_a \nabla_b \ln \Omega - 2 \nabla_b \ln \Omega \nabla^a A_a \\
- 4 A_a \nabla^a \ln \Omega \nabla_b \ln \Omega + 2 \nabla_b A_a \nabla^a \ln \Omega \right].
\]
(3.21)

Putting everything together and using the transformation law (2.9) of the four-current yields
\[
\tilde{\Box} \tilde{A}_b - \tilde{\nabla}_b \tilde{\nabla}^a \tilde{A}_a - \tilde{R}_d^d \tilde{A}_d = -4 \pi \tilde{j}_b,
\]
(3.22)

which demonstrates the conformal invariance of the full equation (1.7) satisfied by the four-potential prior to fixing the gauge. Alternatively, one notes that the action with sources (1.8) is conformally invariant and, therefore, the field equation (1.7) derived from its variation (prior to any gauge fixing) is also conformally invariant. The Lorentz Lorenz gauge version of this equation commonly reported in the literature is not conformally invariant because the Lorenz gauge is broken by the conformal transformation.

4 Wave equation for the Maxwell field

The Maxwell field \( F_{ab} \) also satisfies a wave equation. Let us consider the condition \( \nabla_{[a} F_{bc]} = 0 \); if we unpack it and differentiate both sides, we obtain
\[
\Box F_{ab} + \nabla^c \nabla_b F_{ca} + \nabla^c \nabla_a F_{bc} = 0,
\]
(4.1)
or
\[
\Box F_{ab} + g^{cd} (\nabla_d \nabla_b F_{ca} + \nabla_a \nabla_d F_{bc}) = 0.
\]
(4.2)

Using the identity
\[
[\nabla_a, \nabla_b] F_{cd} = R_{abce} F^e_d + R_{abde} F^e_c,
\]
(4.3)
one finds

\[
\nabla_d \nabla_b F_{ca} + \nabla_d \nabla_a F_{bc} = \nabla_b \nabla_d F_{ca} + \nabla_a \nabla_d F_{bc} + R_{dbec} F^e_a \\
+ R_{dbae} F^e_c + R_{dabe} F^e_c + R_{dace} F^e_b
\]

(4.4)

that, contracting with \( g^{cd} \), gives

\[
g^{cd}(\nabla_d \nabla_b F_{ca} + \nabla_d \nabla_a F_{bc}) = \nabla_b \nabla^c F_{ca} + \nabla_a \nabla^c F_{bc} + (R_{cabe} - R_{cbae}) F^{ec} \\
+ R^{e}_{bce} F^e_a + R^{e}_{ace} F^e_b.
\]

(4.5)

Now, recalling (1.5) and the properties \( R_{abc} b \equiv R_{ac}, R_{a[bc]} = 0 \), together with the symmetries of the Riemann tensor, one obtains [18,19]

\[
\Box F_{ab} + R_{abcd} F^{cd} - R_{ac} F^c_b + R_{bc} F^c_a = 4\pi (\nabla_b j_a - \nabla_a j_b).
\]

(4.6)

Since this equation is derived using only the identity (4.3) and the Maxwell equations, which have already been shown to be conformally invariant, together with the symmetries of the Riemann tensor (also invariant), we only need to show the conformal invariance of Eq. (4.3) to establish the conformal invariance of the wave equation (4.6). But Eq. (4.3) is clearly conformally invariant because it derives from \( \nabla_{[a} F_{bc]} = 0 \), which is equivalent to \( \partial_{[a} F_{bc]} = 0 \) and is conformally invariant. An explicit check is reported in the “Appendix”.

5 Conclusions

Conformal transformations of the spacetime metric are by now, a crucial tool in alternative theories of gravity. Scalar–tensor gravity and cosmology are routinely presented in both the Jordan and the Einstein conformal frames [8,9], and it is now common to switch between them. Conformal transformation constitute a very useful and widespread tool also in general relativity [1–7]. Therefore, it is appropriate to understand fully how the Maxwell field, the four-potential, four-current, and the equations they satisfy, together with relevant gauge conditions, transform under conformal transformations. This is what we have done in the previous sections. There remains to check the consistency of the various scaling laws derived above with the conformal invariance of the action (1.8), but this is easy to do. Using the transformation properties (1.1), (1.11), (1.12), (2.6), (3.10), and (2.9), one obtains

\[
\sqrt{-\tilde{g}} \ g^{ac} g^{bd} F_{ab} F_{cd} = \Omega^{-4} \sqrt{-\tilde{g}} \ \Omega^2 \tilde{g}^{ac} \Omega^2 \tilde{g}^{bd} \tilde{F}_{ab} \tilde{F}_{cd} \\
= \sqrt{-\tilde{g}} \sqrt{-\tilde{g}} \ g^{ac} g^{bd} \tilde{F}_{ab} \tilde{F}_{cd} \equiv \sqrt{-\tilde{g}} \ \tilde{F}_{ab} \tilde{F}^{ab},
\]

(5.1)

\[
\sqrt{-\tilde{g}} \ g^{ab} A_a j_b = \Omega^{-4} \sqrt{-\tilde{g}} \ \Omega^2 \tilde{g}^{ab} \tilde{A}_a \tilde{\Omega}^2 \tilde{J}_b = \sqrt{-\tilde{g}} \ \tilde{g}^{ab} \tilde{A}_a \tilde{J}_b \\
\equiv \sqrt{-\tilde{g}} \ \tilde{A}^c \tilde{J}_c.
\]

(5.2)
Putting everything together, the Maxwell action (1.8) becomes

\[
S_{(em)} = \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{ab} F^{ab} + 4\pi A_b j^b \right)
\]

\[
= \int d^4x \sqrt{-\tilde{g}} \left( -\frac{1}{4} \tilde{F}_{ab} \tilde{F}^{ab} + 4\pi \tilde{A}_b \tilde{j}^b \right),
\]

(5.3)
i.e., it is invariant in form under the conformal transformation (1.1). Hence, it produces conformally invariant field equations provided that \( F_{ab}, A^c, \) and \( j^c \) transform according to the rules discussed in the previous sections.

Using the conformal scaling laws discussed, we derive immediately the transformation property of the Maxwell stress-energy tensor (1.9)

\[
\tilde{T}_{ab} = \Omega^{-2} T_{ab}, \quad \tilde{T}^{ab} = \Omega^{-6} T^{ab}, \quad \tilde{T}_a^b = \Omega^{-4} T_a^b
\]

(5.4)
and, of course, \( \tilde{T} = \Omega^{-4} T = 0 \) [20]. This completes the analysis of the conformal invariance of electromagnetism in curved spacetime. In the geometric optics approximation, electromagnetic waves satisfying the Maxwell equations follow null rays, which obey the null geodesic equation

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0,
\]

(5.5)
where \( \lambda \) is an affine parameter along the geodesic and \( k^\alpha = dx^\alpha / d\lambda \) is the four-tangent to the null geodesic. As a consequence of the conformal invariance of the Maxwell equations, conformal transformations leave null geodesics invariant (apart from changing the parametrization to a non-affine parameter), a result that could also be established directly without knowledge of the conformal invariance of Maxwell’s theory [6,7].

Conformally invariant systems are rare in nature and the electromagnetic interaction realizing this invariance is studied intensely because it is one of only four fundamental forces and the simplest example of gauge theory.

Acknowledgements This work is supported, in part, by the Natural Science and Engineering Research Council of Canada (Grant No. 2016-03803 to V.F.) and by Bishop’s University. This research was supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

Appendix

Here we derive the wave equation satisfied by the Maxwell tensor and we check explicitly its conformal invariance.
Derivation of the wave equation for $F_{ab}$

We begin with the Maxwell equation

$$\nabla_c F_{ab} + \nabla_b F_{ca} + \nabla_a F_{bc} = 0 \quad (A.1)$$

and we take the covariant derivative $\nabla^c$ on both sides to obtain

$$0 = \nabla^c \nabla_c F_{ab} + \nabla^c \nabla_b F_{ca} + \nabla^c \nabla_a F_{bc}
= \square F_{ab} + \nabla^c \nabla_b F_{ca} + \nabla^c \nabla_a F_{bc}
= \square F_{ab} + g^{cd} (\nabla_d \nabla_a F_{bc} + \nabla_d \nabla_b F_{ca}). \quad (A.2)$$

We can then rewrite $\nabla_d \nabla_a F_{bc} + \nabla_d \nabla_b F_{ca}$ in terms of commutators as

$$\nabla_d \nabla_a F_{bc} - \nabla_a \nabla_d F_{bc} + \nabla_a \nabla_d F_{bc} + \nabla_d \nabla_a F_{bc} - \nabla_b \nabla_d F_{ca} + \nabla_b \nabla_d F_{ca}
= [\nabla_d, \nabla_a] F_{bc} + [\nabla_d, \nabla_b] F_{ca} + \nabla_a \nabla_d F_{bc} + \nabla_b \nabla_d F_{ca}. \quad (A.3)$$

The identity

$$[\nabla_a, \nabla_b] F_{cd} = R_{abc}^e F_{de} + R_{abde} F_{ce} \quad (A.4)$$

yields

$$[\nabla_d, \nabla_a] F_{bc} + [\nabla_d, \nabla_b] F_{ca} + \nabla_a \nabla_d F_{bc} + \nabla_b \nabla_d F_{ca}
= R_{dabe} F_{ce} + R_{dace} F_{be} + R_{dbce} F_{ea} + R_{dbae} F_{ce} + \nabla_a \nabla_d F_{bc} + \nabla_b \nabla_d F_{ca}. \quad (A.5)$$

In particular, the second quantity in Eq. (A.2) becomes

$$g^{cd} (\nabla_d \nabla_a F_{bc} + \nabla_d \nabla_b F_{ca})
= g^{cd} R_{dabe} F_{ce} + g^{cd} R_{dace} F_{be} + g^{cd} R_{dbce} F_{ea}
+ g^{cd} R_{dbae} F_{ce} + g^{cd} \nabla_a \nabla_d F_{bc} + g^{cd} \nabla_b \nabla_d F_{ca}
= R_{dabe} F^{ed} + g^{cd} R_{dace} F_{be} + g^{cd} R_{dbce} F_{ea}
+ R_{dbae} F^{de} + \nabla_a \nabla^c F_{bc} + \nabla_b \nabla^c F_{ca}
= F^{ed} R_{dabe} - R_{dabe} + g^{cd} R_{dace} F_{be}
+ g^{cd} R_{dbce} F_{ea} + \nabla_a \nabla^c F_{bc} + \nabla_b \nabla^c F_{ca}. \quad (A.6)$$

The properties of the Riemann tensor $R_{abc} \equiv R_{abcd}$, $R_{abc}^d = -R_{bac}^d$, $R_{[abc]^d} = 0$, $R_{abcd} = R_{cdab}$, and $R_{abcd} = -R_{abdc}$ imply that

$$R_{dabe} - R_{dbae} = R_{dabe} + R_{bdae} = -R_{abde}, \quad (A.7)$$

where the last equality uses the Bianchi identity $R_{[abc]^d} = 0$. Also note that

$$g^{cd} R_{dace} = -g^{cd} R_{adce} = g^{cd} R_{deac} = R_{ade} = R_{ae} \quad (A.8)$$
and
\[ g^{cd} R_{dbce} = -g^{cd} R_{bdce} = g^{cd} R_{bdec} = R_{bde}^d = R_{be}. \] (A.9)

Therefore, Eq. (A.6) becomes
\[ F^{cd} (-R_{abde}) + R_{ae} F_{b}^e + R_{be} F_{a}^e + \nabla_a \nabla^c F_{bc} + \nabla_b \nabla^c F_{ca} \]
\[ = R_{abed} F^{cd} + R_{ae} F_{b}^e + R_{be} F_{a}^e + \nabla_a \nabla^c F_{bc} + \nabla_b \nabla^c F_{ca}. \] (A.10)

Finally, Eq. (A.2) becomes
\[ 0 = \Box F_{ab} + R_{abed} F^{cd} + R_{ae} F_{b}^e + R_{be} F_{a}^e = \Box F_{ab} + R_{abed} F^{cd} + R_{ae} F_{b}^e + R_{be} F_{a}^e + 4\pi (\nabla_a j_b - \nabla_b j_a). \] (A.11)

where the last line is due to \( \nabla^c F_{ca} = -4\pi j_a \). This gives us the wave equation for the Maxwell tensor
\[ \Box F_{ab} + R_{abed} F^{cd} + R_{ae} F_{b}^e + R_{be} F_{a}^e = 4\pi (\nabla_b j_a - \nabla_a j_b). \] (A.12)

**Conformal invariance of the wave equation for \( F_{ab} \)**

To check explicitly the conformal invariance of Eq. (A.12), we look at each term separately.

**The d’Alembertian**

First, we examine the \( \Box F_{ab} \) term, recalling the transformation properties (2.6) of the Maxwell tensor and (1.13) of the Christoffel symbols, with
\[ \tilde{\Box} \tilde{F}_{ab} = \tilde{\nabla}^c \tilde{\nabla}_c \tilde{F}_{ab} = \tilde{g}^{cd} \tilde{\nabla}_d \tilde{\nabla}_c \tilde{F}_{ab}. \] (A.13)

Let us begin by calculating \( \tilde{\nabla}_c \tilde{F}_{ab} \).
\[
\begin{align*}
\tilde{\nabla}_c \tilde{F}_{ab} &= \tilde{\nabla}_c F_{ab} \\
&= \partial_c F_{ab} - \Gamma^e_{ca} F_{eb} - \Gamma^e_{cb} F_{ae} \\
&= \partial_c F_{ab} - \Gamma^e_{ca} F_{eb} - \Gamma^e_{cb} F_{ae} - \delta_c^e \nabla_a \left( \ln \Omega \right) F_{eb} \\
&\quad - \delta_c^e \nabla_a \left( \ln \Omega \right) F_{eb} + g_{ca} \nabla^e \left( \ln \Omega \right) F_{eb} - \delta_c^e \nabla_b \left( \ln \Omega \right) F_{ae} \\
&\quad - \delta_c^e \nabla_b \left( \ln \Omega \right) F_{ae} + g_{cb} \nabla^e \left( \ln \Omega \right) F_{ae} \\
&= \nabla_c F_{ab} - \nabla_a \left( \ln \Omega \right) F_{eb} - \nabla_c \left( \ln \Omega \right) F_{ab} + g_{ca} \nabla^e \left( \ln \Omega \right) F_{eb} \\
&\quad - \nabla_b \left( \ln \Omega \right) F_{ae} - \nabla_c \left( \ln \Omega \right) F_{ab} + g_{cb} \nabla^e \left( \ln \Omega \right) F_{ae} \\
&= \nabla_c F_{ab} - \nabla_a \left( \ln \Omega \right) F_{eb} - \nabla_b \left( \ln \Omega \right) F_{ac} + 2\nabla_c \left( \ln \Omega \right) F_{ab} \\
&\quad + g_{ca} \nabla^e \left( \ln \Omega \right) F_{eb} + g_{cb} \nabla^e \left( \ln \Omega \right) F_{ae}. \end{align*}
\] (A.14)
Then, we take another covariant derivative of Eq. (A.14):

\[ \tilde{\nabla}^d \left( \tilde{\nabla}_c \tilde{F}_{ab} \right) = \partial_d \left( \tilde{\nabla}_c F_{ab} \right) - \tilde{\Gamma}^e_{dc} \left( \tilde{\nabla}_e F_{ab} \right) - \tilde{\Gamma}^e_{da} \left( \tilde{\nabla}_c F_{eb} \right) - \tilde{\Gamma}^e_{db} \left( \tilde{\nabla}_c F_{ae} \right). \]  

(A.15)

To analyze Eq. (A.15), we look at each component separately. The transformation property (1.13) of the Christoffel symbols yields

\[ \tilde{\Gamma}^e_{dc} (\tilde{\nabla}_e F_{ab}) = \Gamma^e_{dc} (\tilde{\nabla}_e F_{ab}) + \delta^e_d \nabla_c (\ln \Omega) \tilde{\nabla}_e F_{ab} + \delta^e_d \nabla_c (\ln \Omega) \tilde{\nabla}_e F_{ab} - g_{dc} \nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ab} = \Gamma^e_{dc} (\tilde{\nabla}_e F_{ab}) + \nabla_c (\ln \Omega) \tilde{\nabla}_d F_{ab} + \nabla_d (\ln \Omega) \tilde{\nabla}_c F_{ab} - g_{dc} \nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ab}. \]  

(A.16)

\[ \tilde{\Gamma}^e_{da} (\tilde{\nabla}_c F_{ab}) = \Gamma^e_{da} (\tilde{\nabla}_e F_{ab}) + \nabla_a (\ln \Omega) \tilde{\nabla}_c F_{ab} + \nabla_d (\ln \Omega) \tilde{\nabla}_e F_{ab} - g_{da} \nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ab}. \]  

(A.17)

\[ \tilde{\Gamma}^e_{db} (\tilde{\nabla}_c F_{ae}) = \Gamma^e_{db} (\tilde{\nabla}_e F_{ae}) + \nabla_b (\ln \Omega) \tilde{\nabla}_c F_{ad} - g_{db} \nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ae}. \]  

(A.18)

Putting the above terms together, one finds

\[ \tilde{\nabla}^d (\tilde{\nabla}_c \tilde{F}_{ab}) = \left[ \partial_d \left( \tilde{\nabla}_c F_{ab} \right) - \Gamma^e_{dc} \left( \tilde{\nabla}_e F_{ab} \right) - \Gamma^e_{da} \left( \tilde{\nabla}_c F_{eb} \right) - \Gamma^e_{db} \left( \tilde{\nabla}_c F_{ae} \right) \right] - \nabla_c (\ln \Omega) \tilde{\nabla}_d \tilde{F}_{ab} - \nabla_d (\ln \Omega) \tilde{\nabla}_c \tilde{F}_{ab} + g_{dc} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ab} - \nabla_d (\ln \Omega) \tilde{\nabla}_c \tilde{F}_{db} + g_{da} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ab} - g_{db} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ae} = \nabla_d \tilde{\nabla}_c \tilde{F}_{ab} - \nabla_c (\ln \Omega) \tilde{\nabla}_d \tilde{F}_{ab} - \nabla_a (\ln \Omega) \tilde{\nabla}_c \tilde{F}_{db} - \nabla_b (\ln \Omega) \tilde{\nabla}_c \tilde{F}_{ad} - 3 \nabla_d (\ln \Omega) \tilde{\nabla}_c \tilde{F}_{ab} + g_{dc} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ab} + g_{da} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ab} + g_{db} \nabla^e (\ln \Omega) \tilde{\nabla}_e \tilde{F}_{ae}. \]  

(A.19)

To calculate \( \square \tilde{F}_{ab} \), we need \( \tilde{\nabla}^c \tilde{\nabla}_d \left( \tilde{\nabla}_c \tilde{F}_{ab} \right) = \Omega^{-2} g^{cd} \tilde{\nabla}_d \left( \tilde{\nabla}_c \tilde{F}_{ab} \right) \). From Eq. (A.19) (and dropping the \( \Omega^{-2} \) factor for now), we see that

\[ g^{cd} \tilde{\nabla}_d \left( \tilde{\nabla}_c \tilde{F}_{ab} \right) = \nabla^c \left( \tilde{\nabla}_c F_{ab} \right) - \nabla_d (\ln \Omega) \left( \tilde{\nabla}_d F_{ab} \right). \]
\[-\nabla_a (\ln \Omega) \ g^{cd} \left( \tilde{\nabla}_c F_{db} \right) - \nabla_b (\ln \Omega) \ g^{cd} \left( \tilde{\nabla}_c F_{ad} \right) \]
\[-3 \nabla^c (\ln \Omega) \left( \tilde{\nabla}_c F_{ab} \right) + 4 \nabla^e (\ln \Omega) \left( \tilde{\nabla}_e F_{ab} \right) \]
\[+ \nabla^e (\ln \Omega) \left( \tilde{\nabla}_a F_{eb} \right) + \nabla^e (\ln \Omega) \left( \tilde{\nabla}_b F_{ae} \right) \]
\[= \nabla^c \left( \tilde{\nabla}_c F_{ab} \right) - g^{cd} \left( \nabla_a (\ln \Omega) \tilde{\nabla}_c F_{db} + \nabla_b (\ln \Omega) \tilde{\nabla}_c F_{ad} \right) \]
\[+ \nabla^e (\ln \Omega) \left( \tilde{\nabla}_a F_{eb} + \tilde{\nabla}_b F_{ae} \right) \]
\[= \nabla^c \left( \tilde{\nabla}_c F_{ab} \right) + g^{cd} \left( \nabla_b (\ln \Omega) \tilde{\nabla}_c F_{da} - \nabla_a (\ln \Omega) \tilde{\nabla}_c F_{db} \right) \]
\[+ \nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ab} \]  \hspace{1cm} (A.20)

The changes in the last line are due to the antisymmetry of $F_{ab}$ and to the Maxwell equation $\tilde{\nabla}_a F_{eb} = 0$. We now look at each of the above terms separately. Using Eq. (A.14), the first term is

\[
\nabla^c \left( \tilde{\nabla}_c F_{ab} \right) = \nabla^c \nabla_a F_{ab} - \nabla^c (\nabla_a (\ln \Omega) F_{cb}) \\
- \nabla^c (\nabla_b (\ln \Omega) F_{ac}) - 2 \nabla^c (\nabla_c (\ln \Omega) F_{ab}) \\
+ g_{ca} \nabla^c (\nabla^e (\ln \Omega) F_{eb}) + g_{cb} \nabla^c (\nabla^e (\ln \Omega) F_{ae}) \\
= \Box F_{ab} - \nabla^c \nabla_a (\ln \Omega) F_{cb} - \nabla_a (\ln \Omega) \nabla^c F_{cb} \\
- \nabla^c \nabla_b (\ln \Omega) F_{ac} - \nabla_b (\ln \Omega) \nabla^c F_{ac} - 2 \Box (\ln \Omega) F_{ab} \\
- 2 \nabla^c (\ln \Omega) \nabla^c F_{ab} + \nabla_a \nabla^e (\ln \Omega) F_{eb} + \nabla^e (\ln \Omega) \nabla_a F_{eb} \\
+ \nabla_b \nabla^e (\ln \Omega) F_{ae} + \nabla^e (\ln \Omega) \nabla_b F_{ae} \\
= \Box F_{ab} - \nabla_a (\ln \Omega) \nabla^c F_{cb} - \nabla_b (\ln \Omega) \nabla^c F_{ac} \\
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^c (\ln \Omega) \nabla^c F_{ab} \\
+ \nabla^e (\ln \Omega) \nabla_a F_{eb} + \nabla^e (\ln \Omega) \nabla_b F_{ae} \]  \hspace{1cm} (A.21)

while the second term is

\[
g^{cd} \left( \nabla_b (\ln \Omega) \tilde{\nabla}_c F_{da} - \nabla_a (\ln \Omega) \tilde{\nabla}_c F_{db} \right) \]
\[= g^{cd} \left( \nabla_b (\ln \Omega) \nabla_c F_{da} - \nabla_b (\ln \Omega) \nabla_d (\ln \Omega) F_{ca} \right) \\
- \nabla_b (\ln \Omega) \nabla_a (\ln \Omega) F_{dc} - 2 \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F_{da} \\
+ \nabla_b (\ln \Omega) g_{cd} \nabla^e (\ln \Omega) F_{ea} + \nabla_b (\ln \Omega) g_{ca} \nabla^e (\ln \Omega) F_{de} \\
- \nabla_a (\ln \Omega) \nabla_c F_{db} + \nabla_a (\ln \Omega) \nabla_d (\ln \Omega) F_{cb} \\
+ \nabla_a (\ln \Omega) \nabla_b (\ln \Omega) F_{dc} + 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F_{db} \\
- \nabla_a (\ln \Omega) g_{cd} \nabla^e (\ln \Omega) F_{eb} - \nabla_a (\ln \Omega) g_{cb} \nabla^e (\ln \Omega) F_{de} \}
\[= \nabla_b (\ln \Omega) \nabla^d F_{da} - \nabla_b (\ln \Omega) \nabla^c (\ln \Omega) F_{ca} \\
- 2 \nabla_b (\ln \Omega) \nabla^d (\ln \Omega) F_{da} + 4 \nabla_b (\ln \Omega) \nabla^e (\ln \Omega) F_{ea} \\
+ \nabla_b (\ln \Omega) \nabla^e (\ln \Omega) F_{ae} - \nabla_a (\ln \Omega) \nabla^d F_{db} \]
\[-\nabla_a (\ln \Omega) \nabla^c (\ln \Omega) F_{cb} + 2 \nabla_a (\ln \Omega) \nabla^d (\ln \Omega) F_{db} - 4 \nabla_a (\ln \Omega) \nabla^e (\ln \Omega) F_{eb}, \tag{A.22}\]

so that

\[
g^{cd} \left( \nabla_b (\ln \Omega) \tilde{\nabla}_c F_{da} - \nabla_a (\ln \Omega) \tilde{\nabla}_c F_{db} \right)
= \nabla_b (\ln \Omega) \nabla^d F_{da} - \nabla_a (\ln \Omega) \nabla^d F_{db}. \tag{A.23}\]

The third term is

\[
\nabla^e (\ln \Omega) \tilde{\nabla}_e F_{ab} = \nabla^e (\ln \Omega) \nabla_e F_{ab} - \nabla^e (\ln \Omega) \nabla_a (\ln \Omega) F_{eb}
- \nabla^e (\ln \Omega) \nabla_b (\ln \Omega) F_{ae} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab}
+ \nabla^e (\ln \Omega) g_{ea} \nabla^k (\ln \Omega) F_{kb} + \nabla^e (\ln \Omega) g_{eb} \nabla^k (\ln \Omega) F_{ak}
= \nabla^e (\ln \Omega) \nabla_e F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e F_{ab}. \tag{A.24}\]

Putting these three pieces together, we obtain

\[
g^{cd} \tilde{\nabla}_d \left( \tilde{\nabla}_c \tilde{F}_{ab} \right) = \Box F_{ab} - \nabla_a (\ln \Omega) \nabla^c F_{cb} - \nabla_b (\ln \Omega) \nabla^c F_{ac}
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla_c (\ln \Omega) \nabla^e F_{ab} + \nabla^e (\ln \Omega) \nabla_a F_{eb}
+ \nabla^e (\ln \Omega) \nabla_b F_{ae} + \nabla_b (\ln \Omega) \nabla^d F_{da} - \nabla_a (\ln \Omega) \nabla^d F_{db}
+ \nabla^e (\ln \Omega) \nabla_e F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab}
= \Box F_{ab} - 2 (\nabla_a (\ln \Omega) \nabla^e F_{eb} + \nabla_b (\ln \Omega) \nabla^e F_{ae})
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab}
= \Box F_{ab} - 8 \pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b)
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab}, \tag{A.25}\]

where the final equality is due to the Maxwell equation $\nabla^b F_{ba} = -4 \pi j_a$. Finally, this gives us the d’Alembertian

\[
\tilde{\Box} \tilde{F}_{ab} = \Omega^{-2} \left[ \Box F_{ab} - 8 \pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b)
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right]. \tag{A.26}\]

The Riemann term

The next term is $R_{abcd} F^{cd}$. The transformation property of the Riemann tensor is usually reported for $R_{abc}^\ d [4,6,7]$ and the Riemann tensor with all indices covariant picks up a $\Omega^2$ factor due to the metric $\tilde{g}_{de}$:

\[
\frac{\tilde{R}_{abcd}}{\Omega^2} = R_{abcd} + g_{ad} \nabla_b \nabla_c (\ln \Omega) - g_{bd} \nabla_a \nabla_c (\ln \Omega) - g_{ca} \nabla_b \nabla_d (\ln \Omega)
\]

\[\tilde{\Box} \tilde{F}_{ab} = \Omega^{-2} \left[ \Box F_{ab} - 8 \pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b)
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right]. \tag{A.26}\]

\[\tilde{\Box} \tilde{F}_{ab} = \Omega^{-2} \left[ \Box F_{ab} - 8 \pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b)
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right]. \tag{A.26}\]
\[+g_{cb} \nabla_a \nabla_d (\ln \Omega) + g_{bd} \nabla_a (\ln \Omega) \nabla_c (\ln \Omega)\]
\[-g_{ad} \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) - g_{bc} \nabla_a (\ln \Omega) \nabla_d (\ln \Omega)\]
\[+g_{ac} \nabla_b (\ln \Omega) \nabla_d (\ln \Omega) - g_{bd} g_{ca} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega)\]
\[+g_{ad} g_{cb} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega).\] (A.27)

Projecting this onto the electromagnetic tensor \( \tilde{F}^{cd} \) gives (note that there is a \( \Omega^{-4} \) factor from \( \tilde{F}^{cd} \))

\[
\Omega^2 \tilde{R}_{abcd} \tilde{F}^{cd} = R_{abcd} F^{cd} + g_{ad} \nabla_b \nabla_c (\ln \Omega) F^{cd} - g_{bd} \nabla_a \nabla_c (\ln \Omega) F^{cd}
\]
\[-g_{ca} \nabla_b \nabla_d (\ln \Omega) F^{cd} + g_{cb} \nabla_a \nabla_d (\ln \Omega) F^{cd}\]
\[+g_{bd} \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^{cd} - g_{ad} \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^{cd}\]
\[-g_{bc} \nabla_a (\ln \Omega) \nabla_d (\ln \Omega) F^{cd} + g_{ac} \nabla_b (\ln \Omega) \nabla_d (\ln \Omega) F^{cd}\]
\[-g_{ad} g_{cb} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F^{cd} + g_{ad} g_{cb} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F^{cd}\]
\[= R_{abcd} F^{cd} + \nabla_b \nabla_c (\ln \Omega) F^{cd} - \nabla_a \nabla_c (\ln \Omega) F^{cd}\]
\[-\nabla_b \nabla_d (\ln \Omega) F^{cd} + \nabla_a \nabla_d (\ln \Omega) F^{cd}\]
\[+\nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^{cd} - \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^{cd}\]
\[-\nabla_c (\ln \Omega) \nabla_d (\ln \Omega) F^{cd} + \nabla_b (\ln \Omega) \nabla_d (\ln \Omega) F^{cd}\]
\[-2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F^{cd}\]
\[= R_{abcd} F^{cd} + \nabla_b \nabla_c (\ln \Omega) F^{cd} - \nabla_a \nabla_c (\ln \Omega) F^{cd}\]
\[+\nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^{cd} - \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^{cd}\]
\[-\nabla_c (\ln \Omega) \nabla_d (\ln \Omega) F^{cd} + \nabla_b (\ln \Omega) \nabla_d (\ln \Omega) F^{cd}\]
\[-2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F^{cd}\]. (A.28)

**The Ricci terms**

We now look at the two Ricci terms. The Ricci tensor transforms according to Eq. (1.14), then the two Ricci terms in Eq. (A.2) become

\[
\tilde{R}_{bc} \tilde{F}^c_a = \Omega^{-2} \left( R_{bc} F^c_a - 2 \nabla_b \nabla_c (\ln \Omega) F^c_a \right.
\]
\[-g_{bc} F^c_a \Box (\ln \Omega) + 2 \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^c_a
\]
\[-2 g_{bc} F^c_a \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) \right)
\]
\[
\tilde{\mathcal{R}}_{abc} \tilde{\mathcal{F}}^e_{b} = \Omega^{-2} \left( \tilde{R}_{bc} F^c_a - \nabla_b \nabla_c (\ln \Omega) F^e_a - \nabla_b \square (\ln \Omega) \right) + 2 \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^c_a - \nabla_b \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) \quad (A.29)
\]

\[
\tilde{\mathcal{R}}_{ac} \tilde{\mathcal{F}}^c_{b} = \Omega^{-2} \left( \tilde{R}_{ac} F^e_c - 2 \nabla_a \nabla_c (\ln \Omega) F^e_b - \nabla_a \square (\ln \Omega) \right) + 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^e_b - 2 F_{ab} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) \quad (A.30)
\]

Therefore, the difference between the two terms becomes
\[
\tilde{\mathcal{R}}_{bc} \tilde{F}^e_a - \tilde{\mathcal{R}}_{ac} \tilde{F}^c_b = \Omega^{-2} \left( \tilde{R}_{bc} F^c_a - \nabla_b \nabla_c (\ln \Omega) F^e_a - \nabla_b \square (\ln \Omega) \right) + 2 \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^c_a - \nabla_b \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) - R_{ac} F^e_c + 2 \nabla_a \nabla_c (\ln \Omega) F^e_b + F_{ab} \square (\ln \Omega) - 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^e_b + 2 F_{ab} \nabla^e (\ln \Omega) \nabla_e (\ln \Omega)
\]

\[
= \Omega^{-2} \left( \tilde{R}_{bc} F^c_a - R_{ac} F^e_c + \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F^e_a - 2 \nabla_b \nabla_c (\ln \Omega) F^c_a - \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F^e_b + 2 \nabla_a \nabla_c (\ln \Omega) F^e_b + 2 \square (\ln \Omega) F_{ab} + 4 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right) \quad (A.31)
\]

### The source terms

Finally, we can check the terms such as \( \nabla_a j_b \) under a conformal transformation (recall that \( \tilde{j}_b = \Omega^{-2} j_b \)). One obtains
\[
\tilde{\mathcal{V}}_{a} \tilde{j}_b = \tilde{\mathcal{V}}_{a} \left( \Omega^{-2} j_b \right)
\]

\[
= -2 \Omega^{-3} \tilde{\mathcal{V}}_{a} \Omega j_b + \Omega^{-2} \tilde{\mathcal{V}}_{a} j_b
\]

\[
= \Omega^{-2} \left( \tilde{\mathcal{V}}_{a} j_b - 2 \nabla_a (\ln \Omega) j_b \right)
\]

\[
= \Omega^{-2} \left( \partial_a j_b - \tilde{\mathcal{V}}_{a} j_b - 2 \nabla_a (\ln \Omega) j_b \right)
\]

\[
= \Omega^{-2} \left( \partial_a j_b - \tilde{\mathcal{V}}_{a} j_b - 2 \nabla_a (\ln \Omega) j_b \right)
\]

\[
= \Omega^{-2} \left( \nabla_a j_b - \nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b + g_{ab} \nabla^d (\ln \Omega) j_d \right)
\]

\[
= \Omega^{-2} \left( \nabla_a j_b - \nabla_b (\ln \Omega) j_a - 3 \nabla_a (\ln \Omega) j_b + g_{ab} \nabla^d (\ln \Omega) j_d \right) \quad (A.32)
\]

Similarly,
\[
\tilde{\mathcal{V}}_{b} \tilde{j}_a = \Omega^{-2} \left[ \nabla_b j_a - \nabla_a (\ln \Omega) j_b - 3 \nabla_b (\ln \Omega) j_a + g_{ba} \nabla^d (\ln \Omega) j_d \right] \quad (A.33)
\]
Therefore, the difference between the two terms is
\[
\tilde{\nabla}_a \tilde{j}_b - \tilde{\nabla}_b \tilde{j}_a = \Omega^{-2} \left[ \nabla_a j_b - \nabla_b (\ln \Omega) j_a - 3 \nabla_a (\ln \Omega) j_b + g_{ab} \nabla^d (\ln \Omega) j_d 
- \nabla_b j_a + \nabla_a (\ln \Omega) j_b + 3 \nabla_b (\ln \Omega) j_a - g_{ba} \nabla^d (\ln \Omega) j_d \right] 
= \Omega^{-2} [\nabla_a j_b - \nabla_b j_a + 2 \nabla_b (\ln \Omega) j_a - 2 \nabla_a (\ln \Omega) j_b] . \tag{A.34}
\]

**Putting everything together**

We are now ready to check the conformal invariance of the wave equation. As a reminder, we have
\[
\Box \tilde{F}_{ab} = \Omega^{-2} [\Box F_{ab} - 8\pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b) 
- 2 \Box (\ln \Omega) F_{ab} - 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab}] , \tag{A.35}
\]
\[
\tilde{R}_{abcd} \tilde{F}^{cd} = \Omega^{-2} \left( R_{abcd} F^{cd} + 2 \nabla_b \nabla_c (\ln \Omega) F_c^a 
+ 2 \nabla_a \nabla_c (\ln \Omega) F_c^b + 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F_c^b 
+ 4 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right) , \tag{A.36}
\]
\[
\tilde{R}_{bc} \tilde{F}_a^c - \tilde{R}_{ac} \tilde{F}_b^c = \Omega^{-2} \left( R_{bc} F_a^c - R_{ac} F_b^c + \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F_c^a 
- 2 \nabla_b \nabla_c (\ln \Omega) F_c^a - 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F_c^b 
+ 2 \nabla_a \nabla_c (\ln \Omega) F_c^b + 2 \Box (\ln \Omega) F_{ab} 
+ 4 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} \right) . \tag{A.37}
\]
\[
\tilde{\nabla}_a \tilde{j}_b - \tilde{\nabla}_b \tilde{j}_a = \Omega^{-2} (\nabla_a j_b - \nabla_b j_a + 2 \nabla_b (\ln \Omega) j_a - 2 \nabla_a (\ln \Omega) j_b) . \tag{A.38}
\]

The wave equation is
\[
\Box F_{ab} + R_{abcd} F^{cd} + R_{ae} F_b^e + R_{be} F_a^e + 4\pi (\nabla_a j_b - \nabla_b j_a) = 0 . \tag{A.39}
\]

Substituting the transformed quantities into the wave equation, one finds
\[
\Box \tilde{F}_{ab} + \tilde{R}_{abcd} \tilde{F}^{cd} + \tilde{R}_{ae} \tilde{F}_b^e + \tilde{R}_{be} \tilde{F}_a^e + 4\pi (\tilde{\nabla}_a \tilde{j}_b - \tilde{\nabla}_b \tilde{j}_a) 
= \Omega^{-2} (\Box F_{ab} - 8\pi (\nabla_b (\ln \Omega) j_a - \nabla_a (\ln \Omega) j_b) - 2 \Box (\ln \Omega) F_{ab} 
- 2 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} + \tilde{R}_{abcd} \tilde{F}^{cd} + 2 \nabla_b \nabla_c (\ln \Omega) F_c^a 
+ 2 \nabla_a \nabla_c (\ln \Omega) F_c^b + 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F_c^b 
+ R_{bc} F_a^c - R_{ac} F_b^c + \nabla_b (\ln \Omega) \nabla_c (\ln \Omega) F_c^a 
- 2 \nabla_b \nabla_c (\ln \Omega) F_c^a - 2 \nabla_a (\ln \Omega) \nabla_c (\ln \Omega) F_c^b 
+ 2 \nabla_a \nabla_c (\ln \Omega) F_c^b + 2 \Box (\ln \Omega) F_{ab} 
+ 4 \nabla^e (\ln \Omega) \nabla_e (\ln \Omega) F_{ab} ) .
\]
\[ +4\pi (\nabla_a j_b - \nabla_b j_a + 2\nabla_b (\ln \Omega) j_a - 2\nabla_a (\ln \Omega) j_b) \,. \tag{A.40} \]

After many cancellations, we are left with

\[ \Omega^{-2} \left[ \Box F_{ab} + R_{abed} F^{ed} + R_{ae} F_b^e + R_{be} F_a^e + 4\pi (\nabla_a j_b - \nabla_b j_a) \right] = 0, \tag{A.41} \]

where the final equality is due to the wave equation for the original (non-tilded) quantities. The wave equation (A.2) is indeed invariant under conformal transformations, as expected.

References

1. Penrose, R.: Asymptotic properties of fields and space-times. Phys. Rev. Lett. 10, 66 (1963)
2. Penrose, R.: Zero rest mass fields including gravitation: asymptotic behavior. Proc. R. Soc. Lond. A 284, 159 (1965)
3. Penrose, R.: Structure of space-time. In: De Witt, C.M., Wheeler, J.A. (eds.) Battelle Rencontres, 1967. Benjamin, New York (1968)
4. Synge, J.L.: Relativity: The General Theory. North Holland, Amsterdam (1960)
5. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
6. Wald, R.M.: General Relativity. Chicago University Press, Chicago (1984)
7. Carroll, S.M.: Spacetime and Geometry. Addison-Wesley, San Francisco (2004)
8. Dicke, R.H.: Mach’s principle and invariance under transformation of units. Phys. Rev. 125, 2163 (1962)
9. Harrison, E.R.: Scalar–tensor theory and general relativity. Phys. Rev. D 6, 2077 (1972)
10. Cunningham, E.: The principle of relativity in electrodynamics and an extension thereof. Proc. Lond. Math. Soc. 8, 77–98 (1909)
11. Weyl, H.: Gravitation und Elektrizität. Sitzungsber. der K. Preuss. Akad der Wiss. zu Berlin, pp. 465–480 (1918)
12. Fulton, T., Rohrlich, F., Witten, L.: Conformal invariance in physics. Rev. Mod. Phys. 34, 442 (1962)
13. Stephani, H.: General Relativity, 3rd edn. Cambridge University Press, Cambridge (2004)
14. Deser, S., Erli, S., Grumiller, D.: Canonical bifurcation in higher derivative, higher spin, theories. J. Phys. A 46, 214018 (2013). [arXiv:1208.0339 [hep-th]]
15. Doménech, G., Sasaki, M.: Conformal frames in cosmology. In: Proceedings, 2nd LeCosPA Symposium: Everything About Gravity, Celebrating the Centenary of Einstein’s General Relativity (LeCosPA2015)—Selected Papers (Taipei, Taiwan, December 14–18, 2015). Int. J. Mod. Phys. D 25, 1645006 (2016)
16. Sonogu, S., Faraoni, V.: Huygens’ principle and characteristic propagation property for waves in curved space–times. J. Math. Phys. 33, 625 (1992)
17. Tsagas, C.G.: Electromagnetic fields in curved spacetimes. Class. Quantum Gravit. 22, 393 (2005). [gr-qc/0407080]
18. Starko, D., Craig, W.: The wave equation in Friedmann–Robertson–Walker space-times and asymptotics of the intensity and distance relationship of a localised source. J. Math. Phys. 59, 042502 (2018)
19. Bekenstein, J.D.: Exact solutions of Einstein-conformal scalar equations. Ann. Phys. (NY) 82, 535 (1974)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.