Virasoro Symmetry in a 2h-dimensional Model and Its Implications

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Abstract

The set of two partial differential equations for the Appell hypergeometric function in two variables $F_4(\alpha, \beta, \gamma, \alpha + \beta - \gamma + 2 - h; x, y)$ is shown to arise as a null vector decoupling relation in a 2h-dimensional generalisation of the Coulomb gas model. It corresponds to a level two singular vector of an intrinsic Virasoro algebra.

Dedicated to the memory of Mitko Stoyanov
1. Introduction

The hypergeometric function is an ubiquitous object of the two-dimensional conformal field theories, providing examples of 4-point correlation functions of various models. The reason behind this is that it is the simplest example of a solution of null vector decoupling equations associated with singular vectors in Virasoro algebra Verma modules. Thus the second order hypergeometric equation appears as a differential operator realisation of a singular vector at level two [1].

Our aim in this paper is to demonstrate that a hidden Virasoro algebra plays a similar role in a higher dimensional conformal model. In particular the singular vector at level two gives rise in even 2h-dimensional space-time to a pair of second order linear partial differential equations. These are the Appell - Kampé de Fériet (AK) equations, [2], [3]

\[
(1 - x) \frac{\partial^2}{\partial x^2} - y^2 \frac{\partial^2}{\partial y^2} - 2xy \frac{\partial^2}{\partial x \partial y} + \gamma \frac{\partial}{\partial x} - (\alpha + \beta + 1)(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) - \alpha \beta \]  
\[
(1 - y) \frac{\partial^2}{\partial y^2} - x^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} + \gamma' \frac{\partial}{\partial y} - (\alpha + \beta + 1)(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) - \alpha \beta \]  

satisfied by the Appell hypergeometric functions of type

\[
F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_m (\gamma')_n}{m! n!} x^m y^n ,
\]

(1.2)

\[
(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha), \text{ with } \alpha + \beta - \gamma - \gamma' = h - 2.
\]

The two variables

\[
x = \frac{r_{13} r_{24}}{r_{14} r_{23}}, \quad y = \frac{r_{12} r_{34}}{r_{14} r_{23}}; \quad r_{ij} := \frac{x_{ij}^2}{r_{ij}},
\]

(1.4)

are the two anharmonic ratios, made of the coordinates \( x_i \in \mathbb{R}^{(2h)} \) of a 4-point conformal invariant. The model is a 2h-dimensional generalisation of the two-dimensional Coulomb gas model with a charge at infinity [4], described by a (sub)-canonical field with logarithmic propagator,

\[
\langle \phi(x_1) \phi(x_2) \rangle \sim ((-\Box)^h)^{-1} = \frac{1}{(4\pi)^h \Gamma(h)} \log x_{12}^2 ,
\]

(1.5)

and scalar fields realised by vertex operators \( V_{\alpha}(x) = e^{i\alpha \phi(x)} \); it was studied in [5]. In the two-dimensional case the system of equations (1.1) reduces, after proper change of variables, to a linear combination of two (chiral) hypergeometric equations.

The appearance of a Virasoro algebra in a four-dimensional context was pointed out many years ago by Dimitar Stoyanov [6], who was studying the infinite dimensional Lie algebras preserving the solutions of the Laplace equation; one of the two algebras he had constructed, contains a subalgebra isomorphic to the Virasoro algebra; see also [4], for a more recent development.
1.1. The 4-point function

We recall here briefly the construction in [5].

Consider the 4-point function in dimension 2 described by only one integration. It is written in terms of vertex operators (VO) as

\[
\langle \psi_4(x_4) \psi_3(x_3) \psi_2(x_2) \psi_1(x_1) \rangle = \int d^2 h \ x_5 \ \langle V_{\alpha_4} (x_5) V_{\alpha_4} (x_4) V_{\alpha_3} (x_3) V_{\alpha_2} (x_2) V_{\alpha_1} (x_1) \rangle
\]

\[
\begin{align*}
&= \prod_{1 \leq i < j \leq 4} r_{ij}^{2 \alpha_i \alpha_j} \ \int d^2 h \ x_5 \ \prod_{i=1}^4 r_{i5}^{\delta_i} \\
&= \pi^h \ \prod_i \frac{1}{\Gamma(\delta_i)} \ \prod_{1 \leq i < j \leq 4} r_{ij}^{2 \alpha_i \alpha_j} \ r_{12}^{h-\delta_1-\delta_2} r_{13}^{h-\delta_1-\delta_3} r_{23}^{h-\delta_1-h} r_{14}^{\delta_4} \ F(x,y), \ \delta_i = -2 \alpha_+ \alpha_i
\end{align*}
\]

and the conformal invariance imposes the condition

\[
\sum_{i=1}^4 \delta_i = 2h \leftrightarrow \sum_{i=1}^4 \alpha_i + \alpha_+ = 2\alpha_0 (= \alpha_+ - \frac{h}{\alpha_+}).
\]

The charges are parametrised as

\[
\alpha^J = J \sqrt{\frac{h}{t}} = -J \alpha_+, \quad 2\alpha_0 = \sqrt{h} (\sqrt{t} - \frac{1}{\sqrt{t}})
\]

and the scaling dimension is \(d = 2\triangle(\alpha) = 2\alpha(\alpha - 2\alpha_0)\), or,

\[
\triangle(\alpha^J) = h \triangle_J, \ \triangle_J = J (J + 1 - t)/t = \triangle_{t-1-J}.
\]

Following [8], \(F(x,y)\) is given by the two fold Mellin integral

\[
F(x,y) = \frac{1}{(2\pi i)^2} \int \int \frac{d\tau}{\tau^2} \int \frac{dt}{t^2} x^{\gamma} y^{\gamma'} \Gamma(-s) \Gamma(-t)
\]

\[
\Gamma(\delta_4 + s + t) \Gamma(h - \delta_1 + s + t) \Gamma(\delta_1 + \delta_2 - h - t) \Gamma(\delta_1 + \delta_3 - h - s)
\]

with the paths of integration running parallel to the imaginary axis. Closing the contours to the right and taking into account the poles of the gamma factors produces a linear combination of four infinite sums, that can be identified with the four linearly independent solutions of the AK equations

\[
F(x,y) = A F_4(\alpha, \beta, \gamma, \gamma'; x, y)
\]

\[
+ B x^{1-\gamma} F_4(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \gamma'; x, y)
\]

\[
+ C y^{1-\gamma'} F_4(\alpha + 1 - \gamma', \beta + 1 - \gamma', 2 - \gamma', \gamma'; x, y)
\]

\[
+ D x^{1-\gamma} y^{1-\gamma'} F_4(\alpha + 2 - \gamma - \gamma', \beta + 2 - \gamma - \gamma', 2 - \gamma, 2 - \gamma'; x, y).
\]

\]

\[\text{(1.7)}\]
Here
\[ \alpha = \delta_4, \quad \beta = h - \delta_1, \quad \gamma = 1 + h - \delta_1 - \delta_3, \quad \gamma' = 1 + h - \delta_1 - \delta_2, \quad (1.12) \]

and these parameters satisfy (1.3) as a result of the constraint (1.7).

In the limit \( y \to 0, x \to 1 \) the four terms in (1.11) combine to produce two leading order terms in the powers of \( y \). In [3] it was observed that when \( \alpha_1 = \frac{1}{2 \sqrt{t}} \) these two terms correspond to the contribution of scalar fields of dimensions \( d = 2 \Delta (\alpha_2 \pm \alpha_1) \). The same is true with the other choice of screening charge, i.e., \( \alpha_+ \to 2 \alpha_0 - \alpha_+ \) and \( \alpha_1 = -\sqrt{\frac{t}{2}} \). The Symanzik method [8] was further used to analyse the small distance behaviour in the case of two integrals, i.e., two screening charges of any of the two types, and respectively the choices \( \alpha_1 = \frac{1}{\sqrt{t}}, \alpha_1 = -\sqrt{t}, \alpha_1 = \frac{1}{2}(\frac{1}{\sqrt{t}} - \sqrt{t}) \). The result is three terms in the first two cases, or four terms in the last one, again in full analogy with the two-dimensional case. This suggests that there is some hidden Virasoro symmetry in this higher dimensional problem, which can explain in algebraic terms the results in [3]. In the next sections we shall develop an operator approach, which will allow to construct and exploit this algebra.

2. Fock space quantisation of the 2h-dimensional sub-canonical field

We choose complex Euclidean coordinates \( z_a = e^{i\tau_n} n_a, n \in S^{2h-1} \subset \mathbb{R}^{2h}, \tau \in \mathbb{C} \), \( z^2 = \sum_{a=1}^{2h} z_a^2 \). For real \( \tau \) one recovers the compactified Minkowski space \( S^1 \times S^{2h-1} \). We shall mostly use the real Euclidean coordinates \( x_a = e^{t} n_a \) with \( t = i\tau \) - real; both notations \( z_a \) and \( x_a \) will appear throughout the paper.

The field \( \phi(z) \) satisfying \( \Box^h \phi(z) = (\sum_a \partial^2_{z_a})^h \phi(z) = 0 \) admits the mode expansion
\[
\phi(z) = 2q - i b_0 \log z^2 + 2i \sum_{n \neq 0} \frac{b_n(z)}{n} = 2q - i b_0 \log z^2 + 2i \sum_{n \neq 0} (z^2)^{-\frac{n}{2}} b_n(\hat{z}) \quad (2.1)
\]

with commutation relations
\[
[b_n(z_1), b_{-m}(z_2)] = n \cos n \theta_{12} \delta_{nm} (\frac{z_2^2}{z_1^2})^{\frac{n}{2}}, \quad [b_0, q] = -i, \quad (2.2)
\]

Here \( \hat{z} = z/\sqrt{z^2}, \cos \theta_{12} = \hat{z}_1 \cdot \hat{z}_2 \) and \( b_n(\rho z) = \rho^{-n} b_n(z) \). The one-dimensional projection of (2.2) with \( z_i = \sqrt{z_i^2} e, e^2 = 1, i = 1, 2 \) (so that \( \cos \theta_{12} = 1 \)) reads
\[
[b_n(e), b_{-m}(e)] = n \delta_{nm}. \quad (2.3)
\]

It is assumed that \( b_n(z) |0\rangle = 0, \langle 0| b_{-n}(z) = 0, n \geq 0 \).
2.1. Relation to the free field quantisation

For simplicity of presentation we restrict here to the four-dimensional case, \(2\hbar = 4\). The modes \(b_n(z), \ n \neq 0\) can be constructed as linear combinations of the free field modes \(a_n(z)\) described in [3]

\[
[a_n(z_1), a_{-m}(z_2)] = \frac{1}{z_1} [a_n^*(\frac{z_1}{z_2}), a_{-m}(z_2)] = \delta_{nm} \frac{1}{z_2} (\frac{z_2^2}{z_1^2})^{n/2} C_{n-1}^{(1)}(\hat{z}_1 \cdot \hat{z}_2), \ n > 0 \tag{2.4}
\]

where \(C_{n}^{(1)}(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}\). The modes \(a_n(z)\) are homogeneous \(a_n(\rho z) = \rho^{-n-1} a_n(z)\), harmonic variables \(\Box\ a_n(z) = 0\). For \(n > 0\), \(a_n(z), a_n^*(z)\) are polynomials, realising an irrep of \(SO(4)\) of dim \(n^2\) (i.e., \(a_n(z) = z_{\mu_1} ... z_{\mu_n}\), \(a_{\mu_1}...\mu_n\), where \(a_{\mu_1}...\mu_n\) are symmetric, traceless tensors), while \(a_n(z) := \frac{1}{\sqrt{z}} a_n^*(\frac{1}{z})\).

Now we take two independent free fields, i.e., two commuting copies \(\{a_n\}, \{a'_n\}\), \([a_n, a'_m] = 0\), each set satisfying (2.4) and define

\[
b_{-n}(z) = \sqrt{\frac{n}{2}} (a_{-n-1}(z) + z^2 a'_{-n+1}(z)), \ n > 0
\]

\[
b_n(z) = \sqrt{\frac{n}{2}} (z^2 a_{n+1}(z) - a'_{n-1}(z)), \ n > 0 \tag{2.5}
\]

so that

\[
\Box \, b_n(z) = 0, \quad b_n(\rho z) = \rho^{-n} b_n(z).
\]

Indeed (2.5) is the unique decomposition of the homogeneous polynomial of degree \(n\), subject to this equation, into a sum of homogeneous harmonic polynomials. The commutation relations (2.4) then imply (2.2). The generalisation to \(h > 2\) is straightforward with the Gegenbauer polynomials \(C^{(h-1)}_{n-h+1}(\cos \theta)\) appearing in the r.h.s. of (2.4). In the two-dimensional case \(2\hbar = 2\) the free field modes \(b_n(z)\) split into a sum of chiral pieces

\[
b_{-n}(z) = \frac{1}{2} (e^{i n(\tau + \sigma)} c_{-n}^+ + e^{i n(\tau - \sigma)} c_{-n}^-). \tag{2.6}
\]

2.2. Vertex operators

Let

\[
V_\alpha(z) =: e^{i \alpha \phi(z)} := (e^{i 2 \alpha q} e^{i \alpha \phi_c(z)} ((z^2)\alpha b_0 e^{i \alpha \phi_\mu(z)}) = V_\alpha^-(z) V_\alpha^+(z) \tag{2.7}
\]

where \(\phi_\mu(z) = \pm 2i \sum_k \frac{b_{-k}(z)}{k} \). The commutation relations (2.2) imply

\[
[b_n(z_1), V_\alpha(z_2)] = 2\alpha \left(\frac{z_2^2}{z_1}\right)^{\alpha} \cos n\theta_{12} V_\alpha(z_2). \tag{2.8}
\]
Furthermore using the relation [10]

\[ \log\left(\frac{z_{12}^2}{z_1^2}\right) = \log(1 - 2\rho \cos \theta + \rho^2) = -2 \sum_{n=1}^{\infty} \rho^n \frac{\cos n\theta}{n} \]  

(2.9)

a standard formula of the two-dimensional case is generalised:

\[ V_{\alpha_1}^+(z_1) V_{\alpha_2}^-(z_2) = (z_{12}^2)^{2\alpha_1 \alpha_2} V_{\alpha_2}^-(z_2) V_{\alpha_1}^+(z_1). \]  

(2.10)

It implies the operator product expansion

\[ V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) = (z_{12}^2)^{2\alpha_1 \alpha_2} V_{\alpha_1+\alpha_2}(z_2) + \ldots \]  

(2.11)

consistent with the 2-point function

\[ \langle 2\alpha_0|V_{2\alpha_0-\alpha}(z_1) V_{\alpha}(z_2)|0\rangle = (z_{12}^2)^{-2\Delta(\alpha)}, \quad \Delta(\alpha) = \alpha(\alpha - 2\alpha_0), \]  

(2.12)

where \(2\alpha_0\) parametrises the charge at infinity, i.e., we reproduce (1.9). The (normalised) bra and ket states are determined from the vertex operators as

\[ |\alpha\rangle = V_{\alpha}(0)|0\rangle = e^{2i\alpha q}|0\rangle, \]  

\[ \langle \alpha| = \langle 0|e^{-2i\alpha q} = \lim_{x \to \infty} (x^2)^{2\Delta(\alpha)}\langle 2\alpha_0|V_{2\alpha_0-\alpha}(x). \]  

Having (2.11) one computes the matrix elements

\[ \langle 2\alpha_0 - \alpha_{p+1}|V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)|\alpha_1\rangle = \prod_{1 \leq i < j \leq p} (z_{ij}^2)^{2\alpha_i \alpha_j}, \quad \alpha_{p+1} = 2\alpha_0 - \sum_{i=1}^{p} \alpha_i. \]  

(2.14)

The charge conservation condition in (2.14) implies the identities

\[ \sum_{i=1}^{p+1} \Delta(\alpha_i) = - \sum_{1 \leq i < j \leq p+1} 2\alpha_i \alpha_j \iff \sum_{i=1}^{p} \Delta(\alpha_i) - \Delta(\alpha_{p+1}) = - \sum_{1 \leq i < j \leq p} 2\alpha_i \alpha_j. \]  

(2.15)

The integral of the VO with charge \(\alpha = \alpha_+,\) or \(\alpha = 2\alpha_0 - \alpha_+,\) i.e., scaling dimension \(2\Delta(\alpha_+) = 2h,\) provides the 2h-dimensional analog of the screening charge operator. In a 4-point matrix element with one screening charge we shall use the notation \(x_5, \alpha_5 = \alpha_+\) keeping the index 4 for the last field in the 4-point function. Then the matrix element is related to the \(x_4 \to \infty\) limit of the 4-point function in (1.6) according to

\[ \int d^{2h} x_5 \langle \alpha_4|V_{\alpha_+}(x_5) V_{\alpha_2}(x_3) V_{\alpha_2}(x_2)|\alpha_1\rangle =: \int d^{2h} x_5 A \]  

\[ = (x_3^2)^{-\Delta} x^{-2\alpha_2(\alpha_3-\alpha_1)} y^{2\alpha_1(2\alpha_0-\alpha_2)} F(x, y), \quad x = \frac{x_5^2}{x_3^2}, \quad y = \frac{x_5^2}{x_2^2}, \]  

(2.16)
where in agreement with (2.15) and using that 2α₁ = −αₚ

\[ \Delta := \sum_{i=1}^{3} \Delta(\alpha_i) - \Delta(2\alpha_0 - \sum_{i=1}^{3} \alpha_i - \alpha_+ ) = 2\alpha_1(\alpha_2 + \alpha_3 - 2\alpha_0) - 2\alpha_2\alpha_3 . \]  

(2.17)

3. A Virasoro algebra

Analogously to the one-dimensional case one can construct generators

\[ \bar{L}_n(e) = \frac{1}{2} \sum_k b_{n-k}(e) b_k(e) - 2\alpha_0(n+1)b_n(e) , \ n \neq 0 , \]

\[ \bar{L}_0(e) = \sum_{n>0} b_{-n}(e) b_n(e) + \frac{1}{2} b_0^2 - 2\alpha_0 b_0 \]  

(3.1),

which close, using the commutation relations (2.3) for collinear vectors, a Virasoro algebra

\[ c = 1 - 48\alpha_0^2 = 1 + 24h - 12h(t + \frac{1}{t}) , \]

\[ \bar{L}_0|\alpha^J\rangle = 2\Delta(\alpha^J)|\alpha^J\rangle = 2h\Delta_J|\alpha^J\rangle . \]  

(3.2)

For \( h = 1/2 \) one recovers the conventional notation for the eigenvalues of these two Virasoro generators. The algebra (3.1) extends to \( \bar{L}_n(z) = (\sqrt{z^2})^{-n} \bar{L}_n(\hat{z}) \) (fixed \( z \) in the commutator).

Let us now compute the action of the generators (3.1) on the VO. Using

\[ [b_k(\hat{z_1}), \bar{L}_{-n}(\hat{z_2})] = (b_{k-n}(\hat{z_2}) + 2\alpha_0(n-1)\delta_{k,n}) k \cos k\theta_1 , \]  

(3.3)

we obtain, e.g. for \( n > 0 \), denoting \( w := \sqrt{z^2} \)

\[ [\bar{L}_{-n}(\hat{z_1}), V_\alpha(z_2)] = 
\]

\[ 2\alpha w_2^{-n}\left( \sum_{k>0} w_2^k \cos(n-k)\theta_1 \ b_{-k}(\hat{z_1}) V_\alpha(z_2) + \sum_{k\geq0} w_2^{-k} \cos(n+k)\theta_1 V_\alpha(z_2) b_k(\hat{z_1}) \right) 
\]

\[ + 2\alpha w_2^{-n}\left( -2\alpha_0(-n+1) \cos n\theta_1 - \alpha \sum_{k=1}^{n-1} \cos(n-k)\theta_1 \cos k\theta_1 V_\alpha(z_2) \right) \]  

(3.4)

\[ = 2\alpha \sum_k w_2^{k-n} \cos(n-k)\theta_1 \ b_{-k}(\hat{z_1}) V_\alpha(z_2) : -2\Delta(\alpha)(n-1) w_2^{-n} \cos n\theta_1 V_\alpha(z_2) 
\]

\[ + \alpha^2 w_2^{-n}\left( (n-1) \cos n\theta_1 - \frac{\sin(n-1)\theta_1}{\sin\theta_1} \right) V_\alpha(z_2) . \]

The meaning of the normal product is

\[ : b_k V := b_k V , \ \text{for} \ k < 0 , \ : b_k V := V b_k \ \text{for} \ k \geq 0 . \]  

(3.5)
The second of the two equalities in (3.4) extends to $L_n, n \geq 0$ as well. In particular for $n = \pm 1, 0$ the very last term in (3.4) vanishes.

On a product of VO the (negative mode) generators act as

$$[\bar{L}_{-n}(\hat{z}_1), V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)] = \sum_{i=2}^{p} 2\alpha_i \left(2\alpha_0(n - 1) w_{i}^{-n} \cos n\theta_1 V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2) + \sum_{k} w_{i}^{-n} \cos(n - k)\theta_{1i} : b_{-k}(\hat{z}_1)(V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)) : \right)$$

$$-2 \sum_{k=1}^{n-1} \left( \sum_{i=2}^{p} w_{i}^{-n+k} \alpha_i \cos(n - k)\theta_{1i} \right) \left( \sum_{j=2}^{p} w_{j}^{-k} \alpha_j \cos k\theta_{1j} \right) V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2),$$

and

$$[\bar{L}_{0}(\hat{z}_1), V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)] = 2 \left( \sum_{i} \Delta(\alpha_i) + 2 \sum_{i<j} \alpha_i \alpha_j \right) V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)$$

$$+ \sum_{i=2}^{p} 2\alpha_i \sum_{k} w_{i}^{k} \cos k\theta_{1i} : b_{-k}(\hat{z}_1)(V_{\alpha_p}(z_p) \ldots V_{\alpha_2}(z_2)) : .$$

In the two-dimensional case the expansion (2.1) splits into chiral pieces due to the splitting (2.6) of the modes. However the generators (3.1) do not reduce to a sum of chiral generators - there is a mixed term surviving in this Virasoro algebra.

### 3.1. Differential operators

For $\hat{z}_2 = \hat{z}_1$ (3.4) reduces to a differential operator action

$$[\bar{L}_{n}(\hat{z}), V_\alpha(z)] = \left( w^{n+1} \partial_w + 2\Delta(\alpha) (n + 1) w^n \right) V_\alpha(z). \quad (3.8)$$

This realisation by differential operators admits an extension to non-collinear arguments e.g., for the matrix elements

$$\langle \alpha_4|[\bar{L}_{-n}(\hat{z}_2), V_{\alpha_3}(z_3)] V_{\alpha_2}(z_2)|\alpha_1 \rangle = D_{-n}(z_2, z_3) \langle \alpha_4|V_{\alpha_3}(z_3) V_{\alpha_2}(z_2)|\alpha_1 \rangle,$$

$$D_{-n}(z_2, z_3) = w_3^{-n} \left( \frac{w_3 \sin n\theta}{w_2 \sin \theta} \cdot z_2 \cdot \partial_{z_2} - \frac{\sin(n - 1)\theta}{\sin \theta} z_3 \cdot \partial_{z_3} \right)$$

$$- (2\Delta(\alpha_3) - \alpha_3^2)(n - 1) \frac{\cos n\theta_{23}}{w_3^n} - \alpha_3^2 \frac{\sin(n - 1)\theta_{23}}{w_3^n \sin \theta_{23}}$$

These operators satisfy the Witt algebra

$$[D_{-n}, D_{-m}] = (n - m) D_{-n-m} . \quad (3.10)$$
3.2. Ward identity

The $L_0$ Ward identity for the n-point correlation function coincides with the 2h-dimensional dilatation Ward identity. Indeed, we have

$$w \partial_w V_\alpha(x) = x \cdot \partial_x V_\alpha(x) = 2\alpha \sum_k :b_{-k}(x) V_\alpha(x) :,$$

and using the relations (2.15), (3.7), one obtains for the matrix elements

$$\left( \sum_i x_i \cdot \partial x_i - \sum_{i<j} 4\alpha_i \alpha_j \right) V_{\alpha_p}(x_p) \ldots V_{\alpha_1}(x_1) = \sum_i 2\alpha_i \sum_{k\neq 0} :b_{-k}(x_i) \left( V_{\alpha_p}(x_p) \ldots V_{\alpha_1}(x_1) \right):$$

and using the relations (2.15), (3.7), one obtains for the matrix elements

$$0 = \langle \alpha_{p+1}| V_{\alpha_p}(x_p) \ldots V_{\alpha_2}(x_2) (2\Delta(\alpha_1) - \tilde{L}_0(\hat{x}_0)) |\alpha_1\rangle$$

$$= \left( \sum_{i=2}^p x_i \cdot \partial x_i + \sum_{i=1}^p 2\Delta(\alpha_i) - 2\Delta(\alpha_{p+1}) \right) \langle \alpha_{p+1}| V_{\alpha_p}(x_p) \ldots V_{\alpha_2}(x_2) |\alpha_1\rangle .$$

The integrated version of (3.12) if, say, $V_{\alpha_p}(x_p)$ is a screening charge operator, i.e. $\alpha_p = \alpha_+$, $2\Delta(\alpha_+) = 2h$, again reduces to the standard Ward identity, obtained from (3.12) by skipping the last terms in the two sums, since using that $x_p \cdot \partial x_p + 2h = \partial x_p \cdot x_p$, these two terms do not contribute to the integral over $x_p$. In particular the Ward identity for the matrix element in (2.16) reads

$$\left( \sum_{i=2}^3 x_i \cdot \partial x_i + 2\Delta \right) \int d^2x_5 \langle \alpha_4| V_{\alpha_5}(x_5) V_{\alpha_3}(x_3) V_{\alpha_2}(x_2) |\alpha_1\rangle = 0$$

which is also checked by the explicit expression in (2.16) using that $\sum_{i=2}^3 (x_i \cdot \partial x_i) f(x, y) = 0$ for any function $f(x, y)$.

3.3. Another realisation of the Virasoro algebra

There is another realisation of the generators of the Virasoro algebra with the property that for $2h = 1$ it again reproduces the known one-dimensional expressions, namely

$$L_n(e) = \frac{1}{2} \sum_{k \neq 0, n} b_{-k}(e) b_k(e) + \frac{1}{\sqrt{2h}} b_n b_0 - \sqrt{\frac{2}{h}} \alpha_0(n+1) b_n(e)$$

$$= \tilde{L}_n(e) + \left( \frac{1}{\sqrt{2h}} - 1 \right) b_n(e) (b_0 - 2\alpha_0(n+1)) , \ n \neq 0 ,$$

$$L_0(e) = \sum_{n>0} b_{-n}(e) b_n(e) + \frac{b_0}{2h} (\frac{b_0}{2} - 2\alpha_0) = \tilde{L}_0(e) + \left( \frac{1}{2h} - 1 \right) b_0 (\frac{b_0}{2} - 2\alpha_0).$$

As for the first realisation in (3.1), the duality transformation $b_n \rightarrow 4\alpha_0 \delta_{n0} - b_{-n}$ sends $L_n \rightarrow L_{-n}$, which is an automorphism of the Virasoro algebra.
More importantly, in this new realisation the eigenvalues of $L_0$ and the central charge operator do not depend on $h$ and coincide with the eigenvalues in the one-dimensional case, (cf. (1.8))

$$c = 1 - \frac{24}{h} \alpha_0^2 = 13 - 6(t + \frac{1}{t}),$$

$$L_0|\alpha^J\rangle = \frac{\Delta(\alpha^J)}{h}|\alpha^J\rangle = \Delta_J|\alpha^J\rangle.$$  

(3.15)

Hence in this realisation we can use all the standard expressions for the singular vectors, as e.g., the singular vector at level two

$$\left( t L_{-1}^2 - L_{-2} \right) |\alpha^J = \frac{1}{2}\rangle$$

(3.16)

and its counterpart with $t \to 1/t$ for $\alpha^J = -\frac{1}{2}$. The new terms in (3.14) modify the action (3.4) of the generators on the fields, and in particular the one-dimensional projection is no more a simple derivative with respect to $w$. E.g. we get for the zero mode instead of (3.4)

$$[L_0(\hat{z}_1), V_\alpha(z_2)] =$$

$$2\alpha \sum_{k \neq 0} w_2^k \cos k\theta_{12} : b_{-k}(\hat{z}_1) V_\alpha(z_2) : + \frac{\Delta(\alpha)}{h} V_\alpha(z_2) + 2\alpha \left( \frac{1}{2h} - 1 \right) V_\alpha(z_2) b_0.$$  

(3.17)

Although the action of the zero mode $L_0$ is modified, it does not spoil the validity of the corresponding Ward identity. Indeed, replacing in the first line of (3.12) $\bar{L}_0 \to L_0, 2\Delta(\alpha_1) \to \frac{\Delta(\alpha_1)}{h}$, we recover again the second line, but multiplied with $1/2h$.

4. The null vector decoupling identities

The true advantage of the new realisation (3.14) is confirmed by the following

**Proposition:**

Let $2\alpha_1 = \sqrt{\frac{h}{t}}$ so that $\Delta(\alpha_1) = h(\frac{3}{4t} - \frac{1}{2})$. Then the null vector decoupling identity

$$\langle \alpha_{p+1}|V_{\alpha_p}(z_p) \cdots V_{\alpha_2}(z_2) (t L_{-1}^2(\hat{z}) - L_{-2}(\hat{z}))|\alpha_1\rangle = 0.$$  

(4.1)

holds true for any $z$.

The Proposition is proved by straightforward application of the commutator formulae derived above. Applying $L_{-1}$ on the vector $\langle \alpha_{p+1}, \ldots, \alpha_2 | := \langle \alpha_{p+1}|V_{\alpha_p}(z_p) \cdots V_{\alpha_2}(z_2)$ we obtain

$$- \langle \alpha_{p+1}, \ldots, \alpha_2 | L_{-1}(\hat{z}_0) =$$

$$\langle \alpha_{p+1}, \ldots, \alpha_2 | \sum_{i=2}^p 2\alpha_i \left( \sum_{k>1} \frac{\cos(1+k)\theta_{0i}}{w_{k+1}} b_k(\hat{z}_0) + \frac{\cos 2\theta_{0i}}{w_i^2} b_1(\hat{z}_0) + \frac{1}{\sqrt{2h}} \frac{\cos \theta_{0i}}{w_i} b_0 \right).$$

(4.2)
Denoting $\mathcal{A} = \langle \alpha_{p+1}|V_{\alpha_p}(z_p)\ldots V_{\alpha_2}(z_2)|\alpha_1 \rangle$ we have

$$-\langle \alpha_{p+1}, \ldots, \alpha_2|(L_{-1}(\hat{z}_0))|\alpha_1 \rangle = \frac{1}{\sqrt{2\hbar}} \sum_{i=2}^{p} \frac{4\alpha_i\alpha_1}{w_i} \cos \theta_{i0} \mathcal{A},$$

$$\langle \alpha_{p+1}, \ldots, \alpha_2|tL_{-2}(\hat{z}_0)|\alpha_1 \rangle = \left( \frac{t}{2\hbar} \left( \sum_{i=2}^{p} 4\alpha_i\alpha_1 \frac{\cos \theta_{i0}}{w_i} \right)^2 - \frac{t}{\sqrt{2\hbar}} \sum_{i=2}^{p} 4\alpha_i\alpha_1 \frac{\cos 2\theta_{i0}}{w_i^2} \right) \mathcal{A}. \tag{4.3}$$

Similarly using (3.6) and (3.14) we compute

$$\langle \alpha_{p+1}, \ldots, \alpha_2|(L_{-2}(\hat{z}_0))|\alpha_1 \rangle = \left( \sum_{i=2}^{p} \alpha_i \frac{\cos \theta_{i0}}{w_i} \right)^2 - \frac{1}{\sqrt{2\hbar}} \sum_{i=2}^{p} 4\alpha_i(\alpha_1 + \alpha_0) \frac{\cos 2\theta_{i0}}{w_i^2} \right) \mathcal{A}. \tag{4.4}$$

It remains to insert the concrete value of $\alpha_1$ in (3.16), which implies

$$(4\alpha_1)^2 = (-2\alpha_+)^2 = \frac{4\hbar}{t}, \quad \alpha_1 + \alpha_0 = \alpha_1 t, \tag{4.5}$$

and hence (4.1) and the second equality in (4.3) coincide, thus proving (4.1). \[ \blacksquare \]

On the other hand we can partially express these matrix elements in terms of differential operators. Thus we get

$$-\langle \alpha_{p+1}, \ldots, \alpha_2|(L_{-1}(\hat{z}_0))|\alpha_1 \rangle = \frac{1}{\sqrt{2\hbar}} \hat{z}_0 \cdot (\partial_{z_2} + \ldots + \partial_{z_p}) \mathcal{A}. \tag{4.6}$$

Furthermore we use

$$\langle \alpha_{p+1}, \ldots, \alpha_2|\left( \frac{\cos 2\theta_{0i}}{w_i^2} b_1(\hat{z}_0) + \frac{1}{\sqrt{2\hbar}} (\hat{z}_0 \cdot \partial_{z_i} b_1(z_i)) \right) L_{-1}(\hat{z}_0)|\alpha_1 \rangle = \frac{1}{\sqrt{2\hbar}} \frac{2\alpha_1 \cos 2\theta_{0i}}{w_i^2} (1 - \frac{1}{\sqrt{2\hbar}}) \mathcal{A}. \tag{4.7}$$

This equality is true also when $z_0$ coincides with some $z_i$ since then we can use the scaling property of the modes $(z \cdot \partial_z) b_n(z) = -nb_n(z)$. Hence

$$\langle \alpha_{p+1}, \ldots, \alpha_2|L_{-1}^2(\hat{z}_0)|\alpha_1 \rangle = \frac{1}{2\hbar} (\hat{z}_0 \cdot (\partial_{z_2} + \ldots + \partial_{z_p}))^2 \mathcal{A} - \sum_{i=2}^{p} 4\alpha_i \alpha_1 \frac{\cos 2\theta_{0i}}{w_i^2} \left( \frac{1}{\sqrt{2\hbar}} - \frac{1}{2\hbar} \right) \mathcal{A}. \tag{4.8}$$
Combining with (4.4) we can write
\[
\frac{2h}{t} \langle \alpha_{p+1}, \ldots, \alpha_2 | (tL^2_{-1} (z_0) - L_{-2} (z_0)) | \alpha_1 \rangle = \left( (z_0 \cdot (\partial_{z_2} + \ldots + \partial_{z_p})) \right)^2
\]
\[- \sum_{i=2}^{p} \delta_{0i} z_0 \cdot (\partial_{z_2} + \ldots + \partial_{z_p}) + \left( \sum_{i=2}^{p} 4 \alpha_i \alpha_1 \frac{w_0}{w_i} \cos \theta_{0i} \right)^2 + \sum_{i=2}^{p} 4 \alpha_i \alpha_1 \frac{w_0^2}{w_i^2} \cos 2 \theta_{0i} \right) A.
\]

4.1. The 3-point null vector decoupling condition

We consider first the 3-point null vector decoupling condition, which determines the possible "fusions" with the fundamental field with \( \triangle (\alpha^J = \frac{1}{2}) \). The 3-point matrix element (with one screening charge) is determined by the \( L_0 \) Ward identity as
\[
\int A = \int d^{2h} x_5 \langle \alpha_3 | V_{\alpha_+} (x_5) V_{\alpha_2} (x_2) | \alpha_1 \rangle = w_2^{2a},
\]
\[- a = \triangle (\alpha_1) + \triangle (\alpha_2) - \triangle (\alpha_3). \quad (4.10)
\]
Choosing the argument of the Virasoro generators as \( z_0 = x_2 \) we can represent (4.9) fully in terms of derivatives
\[
\langle \alpha_3 | V_{\alpha_+} (x_5) V_{\alpha_2} (x_2) (tL^2_{-1} (x_2) - L_{-2} (x_2)) | \alpha_1 \rangle \equiv \left( (x_2 \cdot (\partial_{x_2} + \partial_{x_5})) \right)^2 + \frac{2h}{t} (1-t) x_2 \cdot (\partial_{x_2} + \partial_{x_5}) - \frac{4h}{t} \triangle (\alpha_2) - \frac{2h}{t} \partial_{x_5} \cdot x_{52} \frac{x_2 \cdot x_5}{r_{5}} A.
\]

Dropping the derivative terms with respect to \( x_5 \) gives for the null vector equation
\[
\left( (w_2 \partial_{w_2})^2 + \frac{2h}{t} (1-t) w_2 \partial_{w_2} - \frac{4h}{t} \triangle (\alpha_2) \right) \int A = 0, \quad (4.12)
\]
\[\Rightarrow 2a \left( \frac{2h}{t} (1-t) + 2a \right) - \frac{4h}{t} \triangle (\alpha_2) = 0. \]

This equation for \( a \) does not depend on the charge \( \alpha_2 \) itself but rather on the scaling dimension \( 2 \triangle (\alpha_2) \) and is the same as the one for the 3-point matrix element without a screening charge. Accordingly we obtain as in the one-dimensional case \( 2h = 1 \) two solutions for \( \triangle (\alpha_3) = a + \triangle (\alpha_1) + \triangle (\alpha_2) \),
\[
\triangle (\alpha_3) = \triangle (\alpha_2 + \alpha_1), \quad \triangle (\alpha_3) = \triangle (\alpha_2 - \alpha_1) = \triangle (\alpha_2 + \alpha_1 + \alpha_5); \quad (4.13)
\]
the first corresponds to the screeningless case, the second to the matrix element (4.10).
4.2. The 4-point null vector decoupling condition

We shall now apply the relations (4.9) for the 4-point matrix element with one screening charge. In this case we can specialise the argument of the generators $L_{-n}(z_0)$ in (4.9) to the coordinate of each of the two middle vertex operators $z_0 = x_2$, or $z_0 = x_3$ and thus obtain two identities

\[ 0 = \frac{2h}{t} \int d^2 x_5 \langle \alpha_4 | V_{\alpha_4}(x_5) V_{\alpha_3}(x_3) V_{\alpha_2}(x_2) (tL_{-1}^2(x_i) - L_{-2}(x_i)) | \alpha_1 \rangle \]

\[ \equiv D_i \int d^2 x_5 \mathcal{A} + \int d^2 x_5 I_i, \quad i = 2, 3. \]  \hspace{1cm} (4.14)

Here $\mathcal{A}$ is the matrix element in (2.16) and $D_i$ are differential operators

\[
D_2 = D(x_2, x_3; \alpha_3) \\
= (x_2 \cdot D)^2 - x_2 \cdot D - (x_2 \cdot \partial x_2)^2 - \rho^2 (x_3 \cdot \partial x_3)^2 - 2\rho \cos \theta x_2 \cdot \partial x_2 x_3 \cdot \partial x_3 \\
+ (2h - 4)(x_2 \cdot D - x_2 \cdot \partial x_2 - \rho \cos \theta x_3 \cdot \partial x_3) + x_2 \cdot \partial x_2 + \rho^2 x_3 \cdot \partial x_3 \\
+ (2 + \frac{2h}{t}(1 - t))(1 + \rho \cos \theta)(x_2 \cdot D - x_2 \cdot \partial x_2) - (\rho^2 + \rho \cos \theta)x_3 \cdot \partial x_3 \\
+ 4\tilde{\alpha}\tilde{\beta} x \rho^2 \sin^2 \theta + \frac{4h}{t} \Delta(\alpha_3)\rho^2 \sin^2 \theta, \\
D_3 = D(x_3, x_2; \alpha_2),
\]

with

\[
D = \partial x_2 + \partial x_3, \quad \rho^2 = \frac{x_2^2}{x_3} = y \frac{x_2}{x_3}, \quad 2\rho \cos \theta = 2\frac{x_2 x_3}{x_3^2} = \frac{x + y - 1}{x}.
\]

Furthermore these operators are expressed as

\[
\frac{x_2^2}{x_2^3} D_i = (x_3^2)^{-\Delta} \left( (x + y - 1)^2 - 4xy \right) D_i(x, y) (x_3^2)^\Delta, \quad i = 2, 3. \]  \hspace{1cm} (4.16)

The operators $D_i(x, y)$ here are given by formulae analogous to the two differential operators in (1.1), with the parameters $\alpha, \beta, \gamma, \gamma'$ in (1.12) replaced by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\gamma}'$

\[
\tilde{\alpha} = -2\alpha_2 \alpha_3 + h + \frac{2h}{t}(1 - t) - 2\Delta, \quad \tilde{\beta} = 2\alpha_2 \alpha_3, \\
\tilde{\gamma} = 1 + \frac{h}{t}(1 - t) - 2\Delta, \quad \tilde{\gamma}' = 1 + \frac{h}{t'}(1 - t), \]  \hspace{1cm} (4.17)

plus the additional terms

\[
-\frac{h}{t} \frac{d(\alpha_3)}{x} := \frac{\Delta(\Delta - \frac{h}{t}(1 - t)) - \frac{h}{t}\Delta(\alpha_3)}{x}, \quad -\frac{h}{t} \frac{\Delta(\alpha_2)}{y}
\]  \hspace{1cm} (4.18)
respectively. These operators are precisely the AK operators (1.1) when the latter are rewritten on the matrix element \((x_3^2)^\Delta \int A\), which according to (2.16) differs by a prefactor from \(F(x,y)\). The asymmetry in (4.18) is due to the choice of \(x_3^2\) as a third independent variable in both equations (4.18).

The integrands \(I_i\) of the remaining integrals in (4.14) are expected to be expressible as full derivatives in the integration variable so that these integrals vanish identically. Indeed we have checked this for the linear combination

\[
\int d^2x_5 \left( I_1 - \frac{r_2}{r_3} I_2 \right) = \frac{2h}{t} \int d^2x_5 \left( (\partial_{x_5} \cdot x_2 \frac{x_3 \cdot x_5}{r_5} - \partial_{x_5} \cdot x_3 \frac{x_2 \cdot x_5}{r_5}) \frac{x_52 \cdot x_53}{r_3} \right. \\
\left. + \partial_{x_5} \cdot x_52 \left( \frac{x_3 \cdot x_5}{r_5} \frac{x_2 \cdot x_3}{r_3} - \frac{x_2 \cdot x_5}{r_5} \right) - \partial_{x_5} \cdot x_53 \left( \frac{x_2 \cdot x_5}{r_5} \frac{x_2 \cdot x_3}{r_3} - \frac{x_3 \cdot x_5}{r_5} \frac{r_2}{r_3} \right) \right) A = 0. 
\]

In the screeningless case one recovers the same operators (4.15) but with different value \(\Delta \rightarrow \Delta' = -2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3\) to be inserted in (4.18). Changing back variables, this correlation function corresponds to a constant factor \(F(x,y)\) with parameters \(\alpha' \beta' = 0\), so that it trivially satisfies the initial equations (1.1).

### 5. Conclusions

We have revealed a hidden Virasoro symmetry in a 2h-dimensional model and have demonstrated that it leads to differential equations for the 4-point correlation functions in a way analogous to the 2-dimensional case. This symmetry allows to determine the leading short distance behaviour purely algebraically, without having to perform the complicated multiple Mellin integral computation of the Symanzik method.

The Virasoro algebra is related to a dimension two operator. One can extend it by fields of higher integer dimension, like the energy-momentum tensor, which also depends on the parameter \(\alpha_0\) - this problem is left for future investigation; see also [11] for a different approach to an analogous problem.

The four-dimensional model considered here is still an unrealistic, nonunitary toy model, see also the discussion in [5]. It remains to be seen whether it could be used as a building block in more realistic applications.

### Acknowledgments

We thank N. Nikolov, G. Sotkov and I. Todorov for the interest in this work and for a useful discussion. P.F. acknowledges the support of the Italian Ministry of Education, University and Research (MIUR). V.B.P. acknowledges the hospitality of INFN, Trieste, and the University of Northumbria, Newcastle, UK. This research is supported in part by the TMR Network EUCLID, contract HPRN-CT-2002-00325, and by the Bulgarian National Council for Scientific Research, grant F-1205/02.
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