Low energy constants of $\chi$PT from the instanton vacuum model

K. Goeke,¹ M.M. Musakhhanov,² and M. Siddikov²,†

¹Institut für Theoretische Physik II, Ruhr-Universität-Bochum, D-44780 Bochum, Germany
²Theoretical Physics Dept, Uzbekistan National University, Tashkent 700174, Uzbekistan

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In the framework of the instanton vacuum model we make expansion over the current mass $m$ and number of colors $N_c$ and evaluate $O(1/N_c, m, m/N_c, m \ln m/N_c)$-corrections to the dynamical quark mass $M$, the quark condensate $\langle \bar{q}q \rangle$, the pion mass $M_\pi$ and decay constant $F_\pi$. There are several sources of these corrections: meson loops, finite size of the instanton distribution and the quark-quark “tensor” interaction terms. In contrast to the expectations, we found that numerically the $1/N_c$-corrections to dynamical mass are large and mostly come from meson loops. As a consequence, we have large $1/N_c$-corrections to all the other quantities. To provide the values of $F_\pi(m = 0), \langle \bar{q}q(m = 0) \rangle$ in agreement with $\chi$PT, we offer a new set of parameters $\rho, R$. Finally, we find the low-energy $SU(2)_f$ chiral lagrangian constants $l_3, l_4$ in a rather good correspondence with the phenomenology.

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I. INTRODUCTION

The spontaneous breaking of chiral symmetry ($S\chi SB$) is one of the most important phenomena of hadron physics. It defines the properties of all the light mesons and baryons. Using the general idea of chiral symmetry, the Chiral Perturbation Theory ($\chi$PT) was proposed in [3] for evaluation of the QCD hadronic correlators at low-energy region, where the expansion parameters are light quark current masses $m$ and pion momenta $p$. The basic tool is the phenomenological effective lagrangian, which has a form of the infinite series in these parameters $p$ and $m$. Naturally, the low-energy constants (LEC) of the series expansion are not fixed. Up to now they were extracted only from the experimental data. Recent progress in lattice calculations provide us with rough estimates of LEC. The main problem of lattice evaluations are the still-large pion masses $M_\pi$ available on the finite size lattices.

QCD instanton vacuum model, often referred as the instanton liquid model, provides a very natural nonperturbative explanation of the $S\chi SB$ [4,5]. It provides a consistent framework for description of the pions and thus may be used for evaluation of the LEC.

Quasiclassical considerations show that it is energetically favourable to have lumps of strong gluon fields (instantons) spread over 4-dimensional Euclidian space. Such fields do strongly modify the quark propagation due to the t’Hooft type quark-quark interactions in the background of the instanton vacuum field. This background is assumed as a superposition of $N_+$ instantons and $N_-$ antiinstantons

$$A_\rho(x) = \sum_{I=1}^{N_+} A^{\rho}_I(\xi_I, x) + \sum_{A=1}^{N_-} A^{\rho}_A(\xi_A, x),$$

where $\xi = (\rho, z, U)$ are the (anti)instanton collective coordinates – size, position and color orientation. The most essential for the low-energy processes are the would-be quark zero modes, which result in a very strong attraction in the channels with quantum numbers of vacuum, appearance of the quark condensate and generation of the dynamical quark mass (see the reviews [2, 3]).

The main parameters of the model are the average inter-instanton distance $R$ and the average instanton size $\rho$. The estimates of these quantities are

$$\rho \simeq 0.33 \text{ fm}, \quad R \simeq 1 \text{ fm}, \quad \text{(phenomenological)}$$

$$\rho \simeq 0.35 \text{ fm}, \quad R \simeq 0.95 \text{ fm}, \quad \text{(variational)}$$

and have $\sim 10 – 15\%$ uncertainty.

Recent computer investigations [6] of a current mass dependence of QCD observables within instanton liquid model show that the best correspondence with lattice QCD data is obtained for

$$\rho \simeq 0.32 \text{ fm}, \quad R \simeq 0.76 \text{ fm}.$$
The main purpose of this paper is evaluation of $O(1/N_c, m, m/N_c, m/N_c \ln m)$-corrections to different physical observables. There are several sources of such NLO corrections:

1. At purely gluonic sector of the instanton vacuum model the width of the instanton size distribution is $O(1/N_c)$. The account of the finite width leads to rather small corrections \([13]\). In the following we will check the accuracy of $\delta$-function type of the instanton size distribution by direct evaluation of the finite width corrections.

2. The back-reaction of the light quark determinants to the instanton vacuum properties is formally controlled by $N_f/N_c$-factor \([12]\). It does not sizably change the distribution over $N_+ + N_-$, but radically change the distribution over $N_+ - N_-$. Any $m_f = 0$ leads to $\delta$-function type of the distribution \([14]\). In the following we take $N_+ = N_-$. 

3. There are the quark-quark tensor interaction terms which are $1/N_c$-suppressed and thus are absent in the LO effective action. These terms correspond to nonplanar diagrams in old-fashioned diagrammatic technique.

4. The contribution of meson quantum fluctuations (meson loops) has to be taken into account.

In the first few sections we study the role of the meson loops which give the dominant contribution. At the end we also estimate the contributions of finite width of instanton size distribution, and above-mentioned tensor interaction term.

We consider parameters $\rho, R$ as free within their $\sim 15\%$ uncertainty and fix them from the requirement $F_{\pi, m=0} = 88 MeV, \langle \bar{q}q \rangle_{m=0} = (255 MeV)^3$ with account of NLO corrections, as it is requested by $\chi$PT \([1]\). We found the values

$$\rho = 0.350 fm, \quad R = 0.856 fm,$$

in agreement with the above-given estimates \([28]\).

Note that though the evaluation of the meson loop corrections in the instanton vacuum model is similar to the earlier meson loop evaluations \([15, 18, 17, 18]\) in the NJL model, there are a few differences which should be mentioned:

1. As it has been already mentioned, the meson loop corrections are not the only sources of $1/N_c$-corrections in the instanton model.

2. Due to nonlocal formfactors there is no need to introduce independent fermion and boson cutoffs $\Lambda_f, \Lambda_b$. The natural cutoff scale for all the loops (including meson loops) is the inverse instanton size $\rho^{-1}$.

3. The quark coupling constant is defined through the saddle-point equation in the instanton model whereas it is a fixed external parameter in NJL.

In Section \([11]\) using the above mentioned assumptions and zero-mode approximation (see below), we calculate the quark determinant in the gluon background \([11]\) and flavour vector $v_\mu$, axial-vector $a_\mu$, scalar $s$ and pseudoscalar $p$ external fields beyond chiral limit. Since the zero-mode approximation leads to a violation of the flavour local gauge invariance, we formally restore it by introducing the path-ordered exponents. The price of such restoration is the path-dependence. We show, however, that the quantities considered in this paper do not depend on the choice of the path (see Sec. \([X]\).  

After that, we average the quark determinant over instanton collective coordinates as it was described in \([19, 20, 21]\), using smallness of packing parameter \((\frac{\rho}{\Lambda})^3 \sim 0.01\). This leads to the partition function $Z_N[v, a, s, p, m]$, which is a generating functional of various quark correlators. The partition function is represented as a path integral over the fermions, which are in the present formalism constituent quarks with 't Hooft-like $2N_f$-legs nonlocal interaction term with the coupling constant $\lambda$, defined dynamically from the appropriate equation. In the following we consider only $N_f = 2$ case.

The simplest and the most important quark correlator is the quark propagator, since all the other observables are sensitive to its properties. In section \([III]\) we calculate the dynamical quark mass $M_f$. We found in the instanton vacuum model that contrary to the large-$N_c$ expectations, the meson-loop corrections to $M(m)$ are large ($\sim 50\%$) even in the chiral limit. These corrections are intimately related to the equation for the coupling $\lambda$. Such corrections are absent in NJL-type models.

All meson-loop corrections to physical observables can be splitted into two parts: ”indirect” (which are dominant and come from the dynamical mass shift) and ”direct” (which are relatively small and thus obey $1/N_c$-counting rules.)

In sections \([IV, V, VI]\) we calculate the meson-loop corrections to the quark condensate $\langle \bar{q}q \rangle$, pion mass $M_\pi$ and decay constant $F_\pi$. From $O(m, 1/N_c)$ corrections to the latter quantities in section \([VII]\) we extract the low-energy parameters $l_3, l_4$ of the phenomenological Gasser-Leutwyler $SU(2)_f$ lagrangian \([1]\). These two constants are especially important for chiral perturbation theory \([22, 23]\) since they relate the physical observables $F_\pi, M_\pi$ with their parameters. In section \([IX]\) we check that the chiral log theorems \([1, 24, 25]\) are satisfied. In section \([X]\) we demonstrate that our results are path independent. Appendix \([A]\) contains explicit expressions of the nonlocal vertices of vector and axial currents. In Appendix \([B]\) we give details of evaluation of finite width corrections, and in Appendix \([C]\) we study the structure of the correlator \(\langle J_{\mu}^{\pm}, J_{\nu}^{\pm} \rangle\).
II. LIGHT QUARKS PARTITION FUNCTION IN EXTERNAL FIELDS

The light quark partition function \(Z[m,v,a,s,p]\) with quark mass \(m\) and in external vector \(v\), axial-vector \(a\), scalar \(s\) and pseudoscalar \(p\) fields is defined as

\[
Z[v,a,s,p,m] = \int D A \, e^{-\frac{1}{4} F} \det(iD + im + \bar{\psi} + \bar{\alpha} \gamma_5 + s + p \gamma_5).
\]

The basic assumption of the instanton model is that one can evaluate the integral in quasiclassical approximation, making expansion around the classical vacuum \(1\). The first evaluation of the partition function \(5\) was performed in Ref. \([13, 14]\) in the absence of the external fields and in the chiral limit. The main purpose of the present paper is the extension of the result of \([13, 14]\) to the case of nonzero quark mass \(m\) and external \(v, a, s, p\) fields. While inclusion of the spin-0 fields \(s, \vec{p}\) is trivial, spin-1 fields \(v, a, \alpha\) deserve special attention due to the nonhomogeneous transformation under flavour rotations. Following \([13, 14]\) we split the quark determinant into the low- and high-frequency parts according to

\[
\det = \det_{\text{high}} \cdot \det_{\text{low}} \text{ and concentrating on the evaluation of } \det_{\text{low}}, \text{ which is responsible for the low-energy domain.}
\]

The high-energy part \(\det_{\text{high}}\) is responsible mainly for the perturbative coupling renormalization.

The starting point of our consideration is the zero-mode approximation formulated in \([13, 14, 20]\)

\[
S_i = \frac{1}{\bar{\rho} + A_i + im} = \frac{1}{\bar{\rho} + \frac{\Phi_{i,0}}{im}} \langle \Phi_{i,0} \rangle \langle \Phi_{i,0} \rangle
\]

Here the zero-modes \(\Phi_{i,0}\) are also functions of the instanton collective coordinates \(\xi_i\). This approximation is good for small values of \(m\) (chiral limit) and indicates that the main contribution to the quark propagator is due to the zero-modes. The extension of \(6\) beyond the chiral limit was proposed in our previous works \([21, 22, 23, 24]\) as

\[
S_i = S_0 + S_0 \bar{\rho} \left[ \frac{\Phi_{0,0}}{\langle \Phi_{0,0} \rangle} \right] \bar{\rho} S_0
\]

where

\[
c_i = -\langle \Phi_{0,0} | \bar{\rho} S_0 | \Phi_{0,0} \rangle.
\]

The advantage of this approximation is that it gives correct projection of \(S_i\) to the zero-modes:

\[
S_i | \Phi_{0,0} \rangle = \frac{1}{im} \langle \Phi_{0,0} \rangle, \quad \langle \Phi_{0,0} | S_i = \langle \Phi_{0,0} | \frac{1}{im},
\]

while the similar projection of \(S_i\) given by Eq. \(6\) is valid only at \(m \to 0\) limit.

Recently the quark determinant \(\det(\bar{\rho} + A_i + im) = \exp(\Gamma_f)\) was calculated in the field of a single instanton and beyond the chiral limit \([21, 31]\). The \(\Gamma_f\) can be represented as \(\Gamma_f = -\ln m\rho - 2 \Gamma_s\), where we take the scale equal to \(1/\rho\). Here the first term is due-to the zero-mode. At small \(m\rho\) the \(\Gamma_s = 0.145873 + 0.5(m\rho)^2 \ln(m\rho) - 0.0580(m\rho)^2\). First, independent on \(m\) term was calculated at \([32]\). It is especially interesting for us the \(m\)-dependent terms in \(\Gamma_s\). In the region \(m\rho \sim 0.2\) (corresponding to the strange current quark mass) the third term is of order \(0.2\%\) of the second one and is negligible. So, the main problem is to reproduce the second term in \(\Gamma_s\). Simple calculations with using the Eq. \(7\) lead to the \(\Gamma_s = (0.5 \ln m\rho + 0.0767)(m\rho)^2\). Again, at the region \(m\rho < 0.2\) we may neglect by \((m\rho)^2\) term. So, our approximation \(7\) leads to the main \((m\rho)^2\) term in \(\Gamma_s\), which coincide with exact calculations \([30, 31]\) in a sharp contrast to the zero-mode approximation \([6]\), which leads to \(\Gamma_s = 0\).

We use the notations

\[
\tilde{S} = \frac{1}{\bar{\rho} + A + \bar{V} + im}, \quad \tilde{S}_i = \frac{1}{\bar{\rho} + \tilde{A}_i + \bar{V} + im}, \quad \tilde{S}_0 = \frac{1}{\bar{\rho} + im}
\]

for the quark propagators in the instanton background \([1]\) or individual instantons \(A_i\) and in the external flavour fields \(\bar{V} = \bar{\psi} + \bar{\alpha} \gamma_5 + s + p \gamma_5\), assumed to be weak. Our purpose is to evaluate the quark propagator \(\tilde{S}\), which is just the inverse of the argument of \(\det\) in \(\mathbf{8}\).

We can expand the quark propagator \(\tilde{S}\) with respect to a single instanton field \(A_i\):

\[
\tilde{S} = \tilde{S}_0 + \sum_i (\tilde{S}_i - \tilde{S}_0) + \sum_{i \neq j} (\tilde{S}_i - \tilde{S}_0) \tilde{S}_0^{-1} (\tilde{S}_j - \tilde{S}_0) + \ldots \tag{12}
\]

Now we are going to make expansion over \(V\) and express \(\tilde{S}_i\) via \(S_i\). Notice that the field \(V\) transforms nonhomogeneously. If we could evaluate \(S_i\) exactly, the final result would be gauge-covariant. However, due to the zero-mode approximation expansion over \(V\) will kill the gauge-covariance of the propagator \(S_i\). In order to preserve it, we introduce auxiliary field

\[
\tilde{V}'_i = L_i(\bar{\psi} + \bar{V}) L_i - \bar{\rho},\tag{13}
\]

and the gauge connection \(L_i\), which is defined as a path-ordered exponent

\[
L_i(z_i) = \text{P exp} \left( i \int_{z_i}^{x} dy \mu(y) + a(y) \gamma_5 + \psi(x) \right)
\]

\[
L_i(z_i) = \gamma_4 L_i^1(z_i, x) \gamma_4
\]

where \(z_i\) denotes an instanton position. The field \(V'_i(x, z_i)\) under flavour rotation

\[
\psi(x) \rightarrow U(x) \psi(x)
\]

transforms according to

\[
V'_i(x, z_i) \rightarrow U^{-1}(z_i) V'_i(x, z_i) U^{-1}(z_i)
\]

and thus expansion over it does not violate the gauge covariance. The propagators \(\tilde{S}_i\) and \(\tilde{S}_0\) in terms of the field \(V_i\) have a form:

\[
\tilde{S}_i = L_i S'_i L_i', \quad S'_i = \frac{1}{\bar{\rho} + \tilde{A}_i + \bar{V}'_i + im}.
\]
It is natural to introduce $S'$ with $Z$

Expanding $S'$ over $\tilde{V}'_i$ and re-summing it, we get

\[
S_i' = S_i(1 + \sum_n (\tilde{V}'_i S_i)^n) = \sum_{i,j} \langle \Phi_0 \rangle \langle \Phi_0 \rangle | \tilde{p} S_{0i}'
\]

where

\[
b_i = \langle \Phi_0 | \tilde{p} (S_{0i}' - S_0) \tilde{p} | \Phi_0 \rangle, \quad c_i - b_i = -\langle \Phi_0 | \tilde{p} S_{0i}' \tilde{p} | \Phi_0 \rangle = \langle \Phi_0 | (im + \tilde{V}_i') \Phi_0 \rangle - \langle \Phi_0 | (im + \tilde{V}_i) S_{0i}' (im + \tilde{V}_i') | \Phi_0 \rangle
\]

Rearrangement of the Eq. (12) for the total propagator yields

\[
\langle \Phi_0 | \tilde{p} L_{i,j}' \tilde{S}_0 \rangle = \langle \Phi_0 | \tilde{p} \tilde{S}_0 \tilde{p} \rangle \langle \tilde{p} L_{i,j}' \tilde{S}_0 \rangle
\]

\[
\ln (\text{Det}_{low}) = i \text{Tr} \int dm' (\tilde{S}(m') - \tilde{S}_0(m')) = \sum_{i,j} \int dm' \text{det} \tilde{B}(m') \tilde{p} \tilde{B}(m') \tilde{p} \text{det} \tilde{B}(m) = \text{det} \tilde{B}'(m)
\]

\[
Z_N = \langle \text{Det}_{low}[v,a,s,p,m] \rangle = \langle \text{det} \tilde{B} \rangle = \prod \int D\psi_f D\bar{\psi}_f \exp \left( \int d^4x \sum_{f,g} \bar{\psi}_f (\tilde{p} + i \tilde{V}) \psi_g \right) \prod_{\pm} W_{\pm f}[\psi_f, \bar{\psi}_f]
\]
where

\[ W_{\pm}[\psi^\dagger, \psi] = \int d\xi_{\pm} \prod_f V_{\pm,f}[\psi^\dagger, \psi] \]  \hspace{1cm} (27) \]

are the t'Hooft-like non-local interaction term with \( N_f \) pairs of quark legs,

\[ \tilde{V}_{\pm}[\psi^\dagger, \psi] = \int d^4x \left( \psi^\dagger(x) \hat{L}^{-1}(x, z_{\pm}) \hat{\rho} \Phi_{\pm,0}(x; \xi_{\pm}) \right) \int d^4y \left( \Phi_{\pm,0}^\dagger(y; \xi_{\pm}) (\hat{\rho} L^{-1}(y, z_{\pm}) \psi(y)) \right), \]  \hspace{1cm} (28) \]

and gauge links are defined in \([14]\).

The form-factor of the nonlocal interaction is completely defined by the quark zero-mode. The fermion fields \( \psi^\dagger, \psi \) are interpreted as constituent quarks. For the exponentiation we use Stirling-like formula

\[ W_{\pm}^{N_{\pm}} = \int d\lambda_{\pm} \exp(N_{\pm} \ln \frac{N_{\pm}}{\lambda_{\pm}} - N_{\pm} + \lambda_{\pm} W_{\pm}), \]  \hspace{1cm} (29) \]

where \( \lambda_{\pm} \) play a role of the dynamical coupling constant, defined by saddle-point condition at the integral \([23]\). Notice that the validity of the formula \([24]\) is controlled by the large number of instantons \( N_{\pm} \).

The partition function in the instanton vacuum after integration over collective coordinates and exponentiation may be reduced to the form (for the case \( N_f = 2 \)):

\[ Z_N = \int d\lambda_+ d\lambda_- D\psi D\bar{\psi} e^{-S} \]  \hspace{1cm} (30) \]

\[ S = N_+ \ln \frac{K}{\lambda_+} - N_+ + \psi^\dagger (i \hat{\partial} + \hat{V} + im) \psi + \lambda_+ Y_2^\dagger (31) \]

\[ Y_2^\dagger = \alpha^2 \det_{f} J_{\mu\nu}^{\pm} + \beta^2 \det_{f} J_{\mu\nu}^{0} \]  \hspace{1cm} (32) \]

\[ \frac{\beta^2}{\alpha^2} := \frac{1}{8N_{\pm}} \frac{2N_{c}}{N_{c} - 1} = \frac{1}{8N_{c} - 4} = \mathcal{O} \left( \frac{1}{N_{c}} \right) \]  \hspace{1cm} (33) \]

\[ J_{\mu\nu}^{\pm} = \psi^\dagger [1 + \frac{\gamma_5}{2}] \psi', \]  \hspace{1cm} (34) \]

where the determinant is taken over the (implicit) flavour indices, and \( K \) is some inessential constant introduced to make the argument of logarithm dimensionless. From \([33]\) one can clearly see that the contribution of the tensor terms is just a \( 1/N_{c} \)-correction. For the sake of simplicity we will postpone consideration of the tensor terms contribution until the Section VII and concentrate on the first term in \([32]\).

Next step is the bosonization of the interaction term \( W_{\pm}[\psi^\dagger, \psi] \). Here we are considering only the \( N_f = 2 \) case, for which bosonization is an exact procedure, and take \( N_+ = N_- \), as discussed before. So, we get the partition function

\[ Z_N[v, a, s, p, m] \]

\[ = \int d\lambda D\Phi D\bar{\psi} D\psi \exp \left[ N \ln \frac{K}{\lambda} - N + \int dx \left( -2\Phi^2 + \psi^\dagger (\hat{\rho} + \hat{V} + im + i \frac{(2\pi \rho)^2 \lambda^{0.5}}{2g} \hat{L} \hat{F} \cdot \Gamma \hat{F} L) \psi \right) \right] \]

or, integrating over fermions,

\[ = \int d\lambda D\Phi e^{-S[\lambda, \Phi, v, a, s, p, m]} \]  \hspace{1cm} (40) \]

where,

\[ -S[\lambda, \Phi, v, a, s, p, m] = N \ln \frac{K}{\lambda} - N - 2 \int dx \Phi^2 + \text{Tr} \ln \left( \frac{\hat{\rho} + \hat{V} + im + i \frac{(2\pi \rho)^2 \lambda^{0.5}}{2g} \hat{L} \hat{F} \cdot \Gamma \hat{F} L}{\hat{\rho} + \hat{V} + im} \right) \]  \hspace{1cm} (41) \]

where \( g^2 = \frac{(N_f^2 - 1)2N_{c}}{2N_{c} - 1} \) is a color factor. Here \( \Gamma = \{1, \gamma_5, i\vec{\tau}, i\vec{\tau}\gamma_5\} \) correspond to the decomposition \( \Phi = \)

\[ \bar{\psi} D\psi \hat{F} \cdot \Gamma \hat{F} L \]
\{\Phi_0, \bar{\Phi}\} = \{\sigma, \eta, \sigma, \bar{\phi}\}, and \Phi^2 = \Phi_0^2 + \bar{\Phi}^2 = \sigma^2 + \eta^2 + \sigma^2 + \bar{\phi}^2. Notice that in [10] in contrast to the NJL model the coupling constant \lambda is not a parameter of the action but it is defined by saddle-point condition in the integral [29].

Note that the partition function [41] is invariant under local flavour rotations due to the gauge links \( L \) in the interaction term \( V_{\pm, f}[\psi^\dagger, \psi] \). However, instead of the explicit violation of the gauge symmetry due to the zero-mode approximation [57], we have unphysical dependence of the effective action on the choice of the path in the gauge link \( L \). In our evaluations we used the simplest straight-line path, though there is no physical reasons why the other choices should be excluded. In Sec. X we demonstrate explicitly that for the quantities evaluated in this article the path dependence drops out.

### III. DYNAMICAL QUARK MASS

One of the main advantages of the instanton vacuum model is the natural description of the \( S \chi SB \), which is signalled by non-zero vacuum quark condensate \( \langle \bar{q}q \rangle \). The quark-quark interaction term [27] leads to the strong attraction in the channels with vacuum (and pion) quark-quark interaction. As a consequence, there appear the nonzero vacuum expectation \( \sigma \) of scalar-isoscalar component of meson fields \( \Phi \) and related with it \( \langle \bar{q}q \rangle \). For evaluation of the partition function \( Z[m] \), it is very convenient to use the formalism of the effective action [33, 34] \( \Gamma_{eff}[m, \lambda, \Phi] \), defined as:

\[
Z_N[m] = \int d\lambda Z_N[m, \lambda] = \int d\lambda \exp(-\Gamma_{eff}[m, \lambda, \Phi])
\]

where for the sake of simplicity we dropped all the external currents which are not essential in this section, and the field \( \Phi \) is the solution of the vacuum equation

\[
\frac{\partial \Gamma_{eff}[m, \lambda, \Phi]}{\partial \Phi} = 0.
\]

Notice that the solution depends on \( \lambda \), i.e. \( \Phi = \Phi(\lambda) \).

Up to the Section V A it will be assumed that the only nonzero vacuum field is a condensate \( \Phi = \sigma \), which is independent of coordinates, so the effective action \( \Gamma_{eff}[m, \lambda, \Phi] \) may be replaced with effective potential \( V_{eff}[m, \lambda, \sigma] \).

In the leading order, the effective action just coincides with the action (41). Shifting \( \Phi \rightarrow \sigma + \Phi' \) and integrating over the fluctuations, we get for the meson loop correction

\[
\Gamma_{Veff}[m, \lambda, \sigma] = \frac{1}{2} \text{Tr} \ln \left( 4 \delta_{ij} - \frac{1}{\sigma^2} \text{Tr} \frac{M(p)}{p + i\mu(p)} \Gamma_i \frac{M(p)}{p + i\mu(p)} \Gamma_j \right),
\]

where \( \mu(p) = m + M(p) \) and we introduced the dynamical quark mass \( M(p) = MF^2(p); M = \frac{(2\pi\rho)^2 \lambda^{\frac{1}{2}}} {\sigma} \).

It is convenient to introduce notations for the leading order meson propagators

\[
\Pi^{-1}_i(q) = 4 + \frac{1}{\sigma^2} V_i(q)
\]

where

\[
V_i \equiv Tr(Q(p)\Gamma_i Q(p + q)\Gamma_i)
\]

and

\[
Q(p) = S(p)\Gamma_i M(p) \equiv \frac{iM(p)}{p + i\mu(p)}.
\]

With these notations vacuum equation (43) turns into

\[
\frac{\partial \Gamma_{Veff}}{\partial \sigma} = 4\sigma^2 - \frac{1}{V} Tr(Q(p)) - \frac{1}{\sigma^2} \int \frac{d^4q}{(2\pi)^4} \sum_i V_i(q)\Pi_i(q) = 0,
\]

where

\[
V_i(q) = Tr(Q^2(p)\Gamma_i Q(p + q)\Gamma_i).
\]

In the leading order (LO) over \( N_c \) this equation simplifies to

\[
4\sigma^2 = \frac{1}{V} Tr(Q(p)).
\]

The equations [38, 50] completely define \( \sigma, \sigma_{LO} \) as functions of \( \lambda \). Using (50), we may cast (45) into the form

\[
\Pi^{-1}_i(q) \Rightarrow \tilde{\Pi}^{-1}_i(q) = \frac{1}{\sigma^2V} (TrQ(p) + V_i(q))
\]

This replacement guarantees self-consistency of the 1-meson-loop approximation and complies with Goldstone theorem:

\[
\tilde{\Pi}^{-1}_i(0) = \frac{8mN_c}{\sigma^2} \int \frac{d^4p}{(2\pi)^4} \frac{M(p)}{p^2 + \mu^2(p)} = \frac{2m\langle \bar{q}q \rangle (m = 0)|_{LO}} {\sigma^2} + O(m^2).
\]

Notice that the pion propagator has a correct chiral pole if and only if the field \( \sigma \) satisfies the LO equation (50). Similar property is observed in NJL model (see, e.g., [15] and references therein).
The accuracy of the solutions (54) is numerically strongly enhanced (about a factor of 30 to solve the equations (48,53) numerically in chiral limit. The saddle-point equation for \( \lambda \) has a form
\[
\frac{\partial \Gamma_{\text{eff}}}{\partial \lambda} = \frac{N}{V} - \frac{1}{2V} \text{Tr} \left( Q(p) \right) + \frac{1}{2\sigma^2} \int \frac{d^4q}{(2\pi)^4} \sum_i \left( V_2^i(q) - V_3^i(q) \right) \Pi_i(q) = 0
\]
Notice that \( V_2 \)-term in (55) requires special attention. Formally it is next to leading order correction, while numerically it is strongly enhanced (about a factor of 30 compared to the other \( 1/N_c \)-corrections), which indicates the failure of the large-\( N_c \) expansion in (63).

If we solve the equation (53), expanding it in powers of \( 1/N_c \) with the set of parameters (41), we’ll get
\[
M_0 = 0.567 - 2.362 m
\]
\[
M_1 = \frac{1}{N_c} (-0.687 - 0.808 m - 4.197 m \ln m).
\]
Here and in the following \( M \) and \( m \) are given in GeV.

We can see that the meson loop correction \( M_1 \) is of comparable size with the LO term \( M_0 \), so we can try to solve the equations (48,53) numerically in chiral limit and then evaluate the chiral corrections to it. Such procedure gives
\[
M(m) = 0.36 - 2.36 m - \frac{m}{N_c} (0.808 + 4.197 m \ln m) \quad (55)
\]
The accuracy of the solutions (54,55) is \( \mathcal{O}(m^2, 1/N_c) \). Fig 1 represents the \( M(m) \)-dependence obtained from Eqs. (48,53). For the sake of comparison we also plotted the lattice data from (32). From the first point of view our results are \( \approx 30\% \) higher than the lattice data. However, they are given in different gauges and on different scales. Since \( M(p) \) is essentially nonperturbative object, it is not very easy to rescale the data and make comparison. Rough estimates in perturbative QCD show that the discrepancy may be attributed to the scales difference. We may conclude that we have a qualitative correspondence between our model result for \( M(m) \)-dependence and unquenched lattice data (35), as it was expected.

**IV. QUARK CONDENSATE**

The presence of the quark condensate \( \langle \bar{q}q \rangle \) is one of the most important properties of the QCD vacuum. Its value characterizes the \( \Sigma \chi SB \). In the chosen framework we can extract it directly from the effective action taking derivative over the current quark mass (30)
\[
\frac{\langle \bar{q}q \rangle}{2} = \frac{1}{2} \frac{\partial \Gamma_{\text{eff}}}{\partial m} = -\frac{1}{2} \text{Tr} \left( \frac{i}{p + i\mu(p)} - \frac{i}{p + m} \right)
\]
\[
\text{FIG. 1: \( m \)-dependence of the dynamical quark mass \( M \). The solid curve – the exact numerical solution (55) of vacuum Eqs. (48,53). The dashed curve – the solution (57), obtained by the iterations \( 1/N_c \)-expansion) with the same accuracy. Data points are from (32). Notice that the scale of the lattice data is 1.64 GeV, not \( \rho^{-1} \approx 0.6 \text{ GeV} \).
\]
\[
\text{Evaluation of (56) gives}
\]
\[
- \langle \bar{q}q \rangle(m) = ((0.00497 - 0.0343 m) N_c + (0.00168 - 0.0494 m) - 0.0580 m \ln m) [\text{GeV}^3] + \mathcal{O} \left( m^2, \frac{1}{N_c} \right)
\]

The \( \langle \bar{q}q \rangle \)-dependence is depicted on the Fig. 2. We can see again that due to the chiral logarithm the \( m \)-dependence is not linear and meson loops change the \( m \)-dependence of the quark condensate drastically.

Here and below we define as leading order (LO) the result calculated with the \( 1/N_c \)-corrections dropped everywhere except in the mass \( M(m) \), which is taken from (55) without the last \( \mathcal{O}(m/N_c, m/N_c \ln m) \)-terms. As next-to-leading order (NLO) we define the contribution of the last two terms in (55) plus ”direct” contribution of meson loops.

For the sake of comparison, we also plotted the value \( \langle \bar{q}q \rangle_0 \) which one would get using LO formulae with the mass \( M_0(m) \) taken from (54). Recall that the value \( \langle \bar{q}q \rangle(m = 0) \) = \( (255 \text{ MeV})^3 \), as well as \( F_\pi(m = 0) = 88 \text{ MeV} \), was used as the input in order to fix the parameters \( (\rho, R) \) in (2). In LO we have \( \langle \bar{q}q \rangle(m = 0.1) / \langle \bar{q}q \rangle(m =
V. QUARKS IN EXTERNAL AXIAL-VECTOR FIELD AND PION PROPERTIES

The formalism of the effective action used in one of the previous sections may be successively applied in the presence of the external axial-vector isovector field \( a_\mu = a_\mu^0 r / 2 \). The general partition function (41) is reduced in this case to the form

\[
Z_N[m, \bar{a}_\mu] = \int d\lambda \exp(-\Gamma_{eff}[m, \lambda, \bar{u}, \bar{a}_\mu])
\]

(58)

The external field \( \bar{a}_\mu \) can generate nonzero vacuum average \( \langle \bar{q}q \rangle = \bar{u} \) and shift the value of the vacuum filed \( \langle \sigma \rangle \) and saddle-point value \( \lambda \).

In this paper we restrict ourselves to the case of the soft and weak external field \( a_\mu(q) \), which can be treated in perturbative fashion, and \( q \sim M_\pi \ll \rho^{-1} \). For the purpose of this paper it is sufficient to keep only \( O(\bar{a}_\mu^2, \bar{a}_\mu \bar{u}, \bar{a}_\mu \bar{\sigma}, \bar{a}_\mu \bar{\bar{u}}) \)-terms in (45).

On general grounds, one can guess that the vacuum expectation value \( \bar{u} \sim \bar{a}_\mu \), whereas shifts of the vacuum field \( \langle \sigma \rangle \) and saddle-point value \( \lambda \) are proportional to the second power of \( \bar{a}_\mu \). Using the saddle-point equation

\[
\frac{\partial \Gamma_{eff}[m, \lambda, \bar{u}, \bar{a}_\mu]}{\partial \lambda} = 0
\]

(59)

and the vacuum equations

\[
\frac{\partial \Gamma_{eff}[m, \lambda, \bar{u}, \bar{a}_\mu]}{\partial \sigma} = 0, \quad \frac{\partial \Gamma_{eff}[m, \lambda, \bar{u}, \bar{a}_\mu]}{\partial \bar{u}} = 0,
\]

(60)

we may easily get that the shifts of \( \langle \sigma \rangle, \lambda \) contribute only to \( O(a^4) \)-terms and thus may be safely omitted in this paper.

In the leading order the effective action simply coincides with the classical \( S \) defined in (41). Using vacuum equations (40), one may show that \( \langle \sigma \rangle^2 + \langle \bar{q}q \rangle^2 = const. \) This inspires us to introduce a unitary matrix \( U \) with the properties

\[
U = u_0 + i \bar{q} q, \quad U^\dagger U = U U^\dagger = 1,
\]

(61)

\[
\langle \sigma \rangle = \sigma u_0, \quad \langle \bar{q}q \rangle = \sigma \bar{u}.
\]

(62)

where \( \sigma \) is the value found in Section III. In this representation the vacuum meson field is represented as \( \Phi = \sigma U \).

In the next to leading order one has to take into account the fluctuations of the field \( \Phi \rightarrow \sigma U + \Phi' \) and integrate over \( \Phi' \).

Meson loop contribution to \( \Gamma_{eff} \) has a form

\[
\Gamma_{eff}^{mes}[m, \lambda, \bar{u}, \bar{a}_\mu] = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S[m, \lambda, \sigma, \bar{u}, \bar{a}_\mu, \Phi]}{\delta \Phi_\mu \delta \Phi'_\mu}|_{\Phi'=0}
\]

(63)

It is convenient to rewrite the second derivative \( \delta^2 S/\delta \Phi'^2 \) as

\[
\frac{\delta^2 S}{\delta \Phi_\mu \delta \Phi'_\mu} = O(a^0) + A_{1,ij} + A_{2,ij} + O(a^3)
\]

(64)

where \( A_{1,ij} \) is of the first order in external field \( a \) and induced field \( u \), and \( A_{2,ij} \) is of the second order in them.

Then

\[
\Delta \Gamma^{mes} = \frac{1}{2} \text{Tr} \Pi_i \delta_{ij} A_{2,ij} - \frac{1}{4} \text{Tr} \Pi_i \delta_{ij} A_{1,ij}^2
\]

(65)

Expanding (65) and collecting the terms \( a_\mu a_\nu, \bar{a}_\mu \bar{a}_\nu \bar{u}_i \bar{u}_j \) and \( \bar{a}_\mu \bar{a}_\nu \bar{u}_i \bar{u}_j \), after simple but very tedious evaluations it is possible to show that in agreement with chiral symmetry expectations, the structure of the effective action is

\[
\Gamma_{eff} = \alpha_0 (\bar{a}_\mu + \bar{\bar{u}}, \bar{u})^2 + m \alpha_1 \bar{u} \bar{\sigma} \bar{u}(\bar{a}_\mu + \bar{\bar{u}}) + m \alpha_2 \bar{u}^2 + \frac{1}{2} [F_{a\bar{u}}^2 \bar{a}_\mu^2 + F_{a\bar{u}} \bar{u} \bar{a}_\mu^2 + 2F_{a\bar{u}} \bar{a}_\mu \bar{u} + F_{a\bar{u}}^2 M_\pi^2 \bar{a}_\mu^2] + O(a^3, u^3, m^2),
\]

where the constants \( F_{a\bar{u}} \) differ only beyond chiral limit:

\[
F_{a\bar{u}}^2 - F_{a\bar{u}}^2 = 2 (F_{a\bar{u}}^2 - F_{a\bar{u}}^2) = -\alpha_1 m.
\]

(66)

(67)

From (66,67) one can get that the two-point axialvector currents correlator has a form:

\[
\int d^4 x e^{-i q(\mu A_\mu^0(0))} = \delta_{ij} F^2 \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + M^2} \right) + O(q^2)
\]

(68)
We can see that $M_\pi$ has a meaning of pion mass and $F_\pi$ — pion decay constant. Numerically it is much easier to calculate $F_\pi^2$ as a constant in front of $\delta_{\mu\nu}$-term in (68), taking $\bar{a}(x) = 0, a_\mu(x) = const$ (this corresponds to $a_\mu(q \approx 0)$). Also, it is possible to show that the result of such evaluation is independent of the path choice in the transporter $L$ (see Section [IX] for more details). In a similar way, we can put $u = u(q)$ and evaluate in NLO the quantities $F_{\pi,0}^2$ and pion mass $M_\pi$. Both quantities $F_\pi$ and $M_\pi$ naturally have the chiral log terms due to the pion loops contributions. The coefficients in these chiral log terms are controlled by the low-energy theorems [24] and are reproduced analytically in Section [IX].

We can see that the specific structure of the $\Gamma_{eff}$ provides a check of the numerical calculations. Moreover, the chiral log theorems provide another check of the numerical calculations.

### A. Pion decay constant $F_\pi$ from $a^2$-term

The basic diagrams which contribute to this quantity in the leading order and in the next-to-leading order are shown schematically in the Fig. 3 and Fig. 4 respectively. Notice that in integration over $\phi$ the saddle-point is shifted to $\langle \phi \rangle \sim \bar{a}_\mu$, where the ”proportionality” sign implies some nonlocal linear operator. All the vertices on these plots should be understood as a sum of the local and nonlocal parts, as it is explained in the Appendix A and in the Fig. 10.

![Fig. 3: The basic diagrams which contribute to the $a^2$ term in $\Gamma_{eff}$. The wavy line corresponds to the external field $a_\mu(x)$, the dashed line corresponds to the intermediate meson, the bulbs correspond to all the possible (local and nonlocal) couplings of the field $a$ to the constituent quarks.](image)

Finally we have

$$F_\pi^2 = N_c \left( \left( 2.85 - \frac{0.869}{N_c} \right) - \left( 3.51 + \frac{0.815}{N_c} \right) m - \frac{44.25}{N_c} m \ln m + O(m^2) \right) \cdot 10^{-3} [\text{GeV}^2] =$$

$$\left( 7.67 - 11.35 m - 44.25 m \ln m \right) \cdot 10^{-3} [\text{GeV}^2]$$

$m$ is given in GeV and the constant in front of $O(m^0)$ is given in $\text{GeV}^2$, the constant in front of $O(m)$ is given in GeV. The $F_\pi(m)$-dependence is shown in the Fig. 5. For the sake of comparison, we also plotted the value $F_{\pi,0}$ which one would get using LO formulæ with the mass $M_0(m)$ taken from [51]. Recall that the value $F_\pi(m = 0) = 88\MeV$, as well as $\langle \bar{q}q(m = 0) \rangle = (255\MeV)^3$, was used as the input in order to fix the parameters ($\rho, R$) in [2]. The comparison between the solid curve and the long-dashed one shows that the effect of the NLO-corrections grows with $m$ and is about 40% at $m = 0.1\GeV$.

![Fig. 5: $m$-dependence of the pion decay constant $F_\pi$. The long-dashed curve is the LO contribution, the short-dashed curve is the NLO contribution, the solid curve is the total LO+NLO contribution. The dot-dashed line represents the leading-order in $1/N_c$-expansion result, evaluated with the mass $M_0$ from [51] (see text).](image)

### B. $M_\pi$ from pion propagator

Effective action $\Gamma_{eff}$, Eq. (69) at $a = 0$ has a meaning of inverse $\pi$-meson propagator at small external momentum $q \sim M_\pi$ with account of meson loops. For our purpose it is sufficient to have only the $O(q^0)$ and $O(q^2)$ terms.

The LO propagators of the mesons were defined in [15]. The inverse pion propagator with account of meson loops is given by
\[ \Pi^{(ab)^{-1}}(q) = \left[ 4\delta^{ab} + \frac{1}{\sigma^2} Tr \left( Q(p)i\gamma_5\tau^a Q(p + q)i\gamma_5\tau^b \right) \right] + \]
\[
+ \frac{1}{\sigma^4} \int \frac{d^4k}{(2\pi)^4} \Pi_{ij}(k)V_{ij}(k, q) - \frac{4}{\sigma^6} \int \frac{d^4k}{(2\pi)^4} \Pi_i(k)\Pi_j(k + q)V_i^\phi(k, q)V_j^\phi(k, q) \]
\] (70)

where the function \( Q(p) \) was defined in (47), and the vertices are

\[
V_{ij}^\phi(k, q) = 2\text{Tr} \left( Q(p)i\gamma_5\tau^a Q(p + q + k)\Gamma_i Q(p + q + k)i\gamma_5\tau^b Q(p + k)\Gamma_j \right) + \]
\[
\text{Tr} \left( Q(p)i\gamma_5\tau^a Q(p + q)\Gamma_i Q(p + q + k)i\gamma_5\tau^b Q(p + k)\Gamma_j \right) \]
\[
V_i^\phi(k, q) = \text{Tr} \left( Q(p)i\gamma_5\tau^a Q(p + q)\Gamma_i Q(p + q + k)\Gamma_j \right) \]
\] (71)

FIG. 6: Diagrams corresponding to the last two meson loop terms in (70)

The value of the pion mass \( M_\pi \) is defined as a pole position in the propagator \( \Pi(0) \). Now we would like to demonstrate that \( M_\pi^2 \sim m \), i.e. that the loop corrections do not violate the Goldstone theorem, i.e. \( \Pi^{-1}(0) \sim m \).

In the leading order over \( N \), the only contribution to \( \Pi^{-1}(q) \) comes from the first term in (70), and using equation (50) may be reduced to

\[
\Pi^{-1}(0) = \frac{8N_c}{\sigma^2} \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)}{p^2 + \mu^2(p)} \]
\] (73)

\[
\frac{1}{\sigma^4} \int \frac{d^4q}{(2\pi)^4} \Pi_i(q)V_i^\eta(q) + \frac{8N_c}{\sigma^2} \frac{M_i}{M_0} \frac{\partial}{\partial M} \left( \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)}{p^2 + \mu^2(p)} \right) \]
\[
= \frac{1}{\sigma^4} \int \frac{d^4q}{(2\pi)^4} \Pi_i(q)V_i^\eta(q) + \frac{8N_c}{\sigma^2} \frac{M_i}{M_0} \left( \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)(p^2 - M^2f^4(p) + m^2)}{(p^2 + \mu^2(p))^2} \right) \]
\] (74)

and explicit expression for \( V_3^\eta(q) \) defined in (39) is

\[
V_3^\eta(q) = 8N_c c_i \int \frac{d^4p}{(2\pi)^4} \left[ M^3 F^4(p)F^2(p + q) \left( \pm 2 \frac{(p^2 + p \cdot k) \mu(p) - p^2 \mu(p + q) + \mu^2(p) \mu(p + q)}{(p^2 + \mu^2(p))^2 (p + q)^2 + \mu(p + q)^2} \right) \right] \]
\]

where here and below
\[
c_i = \begin{cases} 1, i = \sigma, \eta \\ -3, i = \sigma, \phi \end{cases} \]
\] (75)

\[ \eta, \phi \] and \( - \) for \( \sigma, \phi \).

The vertex part \( V_i^\phi(k, q = 0) \) in the second term of (70) may be rewritten explicitly as

and the sign in the numerator of the first term is + for
\[ V_i^0(k) = -8N_c \int \frac{d^4p}{(2\pi)^4} \left[ 2c_t \frac{M^4F^6(p)F^2(p+k)(\mu(p)\mu(p+k) \pm (p^2 + p \cdot k))}{(p^2 + \mu^2(p))^2((p+k)^2 + \mu^2(p+k))} + \frac{M^2F^4(p)M^2F^4(p+k)}{p^2 + \mu^2(p)(p+k)^2 + \mu^2(p+k)} \right] \] (76)

The third term in (70) may be reduced to the form

\[ \delta V(q) = -8N_c \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)MF^2(p+q)\mu(p+q)}{p^2 + \mu^2(p)(p+q)^2 + \mu^2(p+q)} \] (77)

where \( \delta V(k) \) equals

\[ \tilde{V}(k) = 8N_c \int \frac{d^4p}{(2\pi)^4} \frac{MF^4(p)MF^2(p+k)\mu(p+k)}{p^2 + \mu^2(p)(p+k)^2 + \mu^2(p+k)} \] (78)

To eliminate the terms with double propagators in the third term, we follow the strategy used in (77) for pure NJL. Using explicit expressions for propagators, we can notice that

\[ \tilde{\Pi}_\sigma^{-1}(q) - \tilde{\Pi}_\phi^{-1}(q) = \tilde{\Pi}_\gamma^{-1}(q) - \tilde{\Pi}_\phi^{-1}(q) - \tilde{\Pi}_\gamma^{-1}(q) = \frac{16N_c}{\sigma^2} \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)\mu(p)MF^2(p+q)\mu(p+q)}{p^2 + \mu^2(p)(p+q)^2 + \mu^2(p+q)} = \frac{2V(q)}{\sigma^2} \] (79)

where

\[ V(q) = 8N_c \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)\mu(p)MF^2(p+q)\mu(p+q)}{p^2 + \mu^2(p)(p+q)^2 + \mu^2(p+q)} \] (80)

Obviously

\[ \tilde{V}(q) = V(q) + m \delta V(q) \] (81)

\[ \begin{align*}
&-\frac{m8N_c}{\sigma^2} \int \frac{d^4k}{(2\pi)^4} \tilde{\Pi}_\phi(k) \int \frac{d^4p}{(2\pi)^4} \left[ 2c_t \frac{(\mu(p)\mu(p+k) \pm (p^2 + p \cdot k))M^3F^4(p)F^2(p+k)}{(p^2 + \mu^2(p))^2((p+k)^2 + \mu^2(p+k))} \right. \\
&-\frac{M F^2(p)}{p^2 + \mu^2(p)(p+k)^2 + \mu^2(p+k)} \left( (m + c_t\mu(p)) \right] + m^2(c_t + 1) \frac{1}{\sigma^4} \int \frac{d^4k}{(2\pi)^4} \tilde{\Pi}_\gamma(k) \frac{\delta V(k)}{V(k)} \\
&\left. + \frac{8N_c}{\sigma^2} M_1 \left( \int \frac{d^4p}{(2\pi)^4} \frac{MF^2(p)(p^2 - M^2f^4(p)+m^2)}{(p^2 + \mu^2(p))^2} \right) \right]
\] (85)

We can see that the pion propagator \( \Pi_\phi^{-1}(0) \sim m \), i.e. the loop corrections do not violate the Goldstone theorem. Notice that the function \( V(k) \) is positively defined, so no unexpected poles should appear due to the term \( \int \frac{d^4k}{(2\pi)^4} \tilde{\Pi}_\phi(k) \frac{\delta V(k)}{V(k)} \).

Evaluating \( F^2_{uu} \)-term in (69) with account of meson loops and taking the ratio of (85) to \( F^2_{uu} \), we obtain the pion mass

\[ M^2_\pi = m \left( \left( 3.49 + \frac{1.63}{N_c} \right) + m \left( 15.5 + \frac{18.25}{N_c} + \frac{13.5577}{N_c} \ln m \right) + \mathcal{O}(m^2) \right) = m \left( 4.04 + 21.587m + 4.52m \ln m + \mathcal{O}(m^2) \right)[\text{GeV}^2] \] (86)

in the next-to leading order. The \( M_\pi(m) \)-dependence of the pion mass is shown on the Fig. For the sake
of comparison, we also plotted the value $M_{\pi,0}$ which one would get using LO formulae with the mass $M_0(m)$ taken from [24]. Altogether, for this observables the NLO-corrections turn out to be small.

![Graph showing $M^2_{\pi}(m)$ as a function of $m$ in GeV.]

FIG. 7: $m$-dependence of the pion mass $M_\pi$. The long-dashed curve is the LO contribution, the short-dashed curve is the NLO contribution, the solid curve is the total LO+NLO contribution. The dot-dashed line represents the leading-order result, evaluated with the mass $M_0$ from [22] (see text).

VI. FINITE WIDTH CORRECTIONS

In the previous sections it was assumed that the instanton size distribution has a zero width, i.e.

$$d(\rho) = \delta(\rho - \bar{\rho})$$

and all instantons have the same size $\bar{\rho}$. As it was shown in the early works of Diakonov, Petrov [13], this approximation is justified by the small parameter $1/N_c$, i.e.

$$\frac{\langle \rho^2 \rangle - \langle \rho \rangle^2}{\langle \rho \rangle^2} \sim O \left( \frac{1}{N_c} \right).$$

For numerical evaluations we take the value

$$\delta \rho^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2 \approx \frac{0.5599 \text{GeV}^{-2}}{N_c}$$

which follows from the two-loop size distribution of [13]. Since we are interested in all $1/N_c$ corrections, we must take the finite width into account. To do this, we must return to the formula [25]. Additional integration over $\rho$ doesn’t change the exponentiation procedure, and we get the standard $2N_f$-interaction term in the effective action $S$. However, for bosonization we should slightly modify the standard procedure. Here we consider only the case $N_f = 2$ we are mainly interested in. For this case

$$e^{-\int d^4z d\rho d(\rho)J^2(z,\rho)} =$$

$$\int D\Phi(z,\rho) \exp \left( -\frac{1}{4} \int d^4z d\rho d(\rho)\Phi^2(z,\rho) + \int d^4z d\rho d(\rho)\Phi(z,\rho)J(z,\rho) \right)$$

where $\Phi(z,\rho)$ are pure mathematical objects (matrices). Thus the effective action turns into

$$S = \frac{N}{V} \ln \lambda + 2 \int d^4z d\rho d(\rho)\Phi^2(z,\rho) -$$

$$Tr \ln \left( \hat{\rho} + im + ic\lambda^{0.5} \int d^4z d\rho d(\rho)\hat{K}(z,\rho)\Phi(z,\rho) \right)$$

where $c$ is some inessential constant and $\hat{K}_{x,y}(z,\rho) \sim \hat{\phi}(x-z,\rho)\hat{\phi}(y-z,\rho)$. The LO vacuum equations which follow from (91) are

$$\frac{N}{V} = \frac{1}{2} Tr \left( \frac{ic\lambda^{0.5} \int d^4z d\rho d(\rho)\Phi(z,\rho)\hat{K}(z,\rho)}{\hat{\rho} + im + ic\lambda^{0.5} \int d^4z d\rho d(\rho)\hat{K}(z,\rho)\Phi(z,\rho)} \right)$$

$$4\Phi_0(\rho) = Tr \left( \frac{ic\lambda^{0.5} \hat{K}(z,\rho)}{\hat{\rho} + im + ic\lambda^{0.5} \int d^4z d\rho d(\rho)\hat{K}(z,\rho)\Phi(z,\rho)} \right)$$

where $\Phi_0(\rho)$ has the quantum numbers of $\sigma$. The unknown variables in these equations are parameter $\lambda$ and function $\Phi(\rho)$. In general case these equations are very complicated and may be solved only numerically. However, for the case of small width we can make expansion over $\delta \rho^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2$. In particular, we expand the functions $\Phi(\rho)$ and $\hat{K}(\rho)$ over $\langle \rho - \bar{\rho} \rangle$, get a system of equations for the Taylor coefficients $\Phi_i$ and $\lambda$. Solving these equations, we can restore the function $\Phi(\rho)$ (at least in the small vicinity of $\bar{\rho}$), and after that evaluate the finite width contributions to all the quantities. Details of these evaluations are given in Appendix [13]. The final results for the corrections are

$$\delta F^2 = \frac{N_c}{3} \times$$

$$\left( 0.0024 - 0.020m + 0.019m^2 \right) \left[ \text{GeV}^2 \right]$$

$$\delta \langle \bar{q}q \rangle = \frac{N_c}{3} \times$$

$$\left( -0.0024 + 0.058m - 0.33m^2 \right) \left[ \text{GeV}^4 \right]$$

Substituting the numerical value [26], we get

$$\delta F^2 =$$

$$\left( 0.00045 - 0.0037m + 0.0036m^2 \right) \left[ \text{GeV}^2 \right]$$

$$\delta \langle \bar{q}q \rangle =$$

$$\left( -0.00045 + 0.011m - 0.062m^2 \right) \left[ \text{GeV}^4 \right]$$

Here $m$ is given in GeV. Thus we can see that these corrections are relatively small, $\approx 5\%$ for $F^2$ and $\approx 2.6\%$ for $\langle \bar{q}q \rangle$. 
If we define the mass $M(p)$ of the constituent quark as

$$M(p) = c \lambda^{0.5} \int d^4z dp \, d(p) \tilde{K}(z, \rho) \Phi_0(p),$$  \hspace{1cm} (97)$$
$$M(p) = MF_0^2(p) + \delta \rho^2 \delta M_2(p) + \mathcal{O}(\delta \rho^4)$$  \hspace{1cm} (98)$$

we can get the $p$-dependence shown in the Fig. We can see that for $p = 0$ the increase of the dynamical quark mass is $\delta M/M = 10\%$.

![Graph showing p-dependence of M(p)](image_url)

**FIG. 8:** Change of the $p$-dependence of the constituent quark mass $M(p)$ due to the finite width corrections. The dashed curve is the contribution of the leading order result, the dot-dashed curve is the contribution of the finite width correction, the solid curve is a total result.

**VII. TENSOR TERMS CONTRIBUTIONS**

In this section we study contribution of the $(1/N_c$-suppressed) tensor terms to the axial currents correlator $\langle a_\mu(a_{\nu'}) \rangle$ introduced in [93]. Notice that due to the identity [96] the fields $J^{1,5}_{\mu\nu}$ depend on each other. Under Lorentz transformations $J^{1,5}_{\mu\nu}$ transforms like $(1,0) + (0,1)$ whereas chiral components $J^\pm_{\mu\nu}$ transform like $(1,0)$ and $(0,1)$.

In case $N_f = 2$ evaluation of the determinant becomes especially simple:

$$det J^\pm = \frac{1}{4} \sum_i (J^{(i)}_\pm)^2 = \frac{1}{16} \left( \sum_i (J_i^\pm)^2 + (J_i^5)^2 \right)$$  \hspace{1cm} (99)$$
$$det J^{\pm}_{\mu\nu} = \frac{1}{8} \left( \sum_i (J^{(i)}_{\mu\nu})^2 + \frac{i}{2} \alpha_{\mu\nu} J^{(i)}_{\mu\nu} J^{(i)}_{\alpha\beta} \right)$$  \hspace{1cm} (100)$$

\(J^{(i)} = \psi^\dagger \Gamma_5 \psi, \quad J^{5,(i)} = \psi^\dagger \Gamma_5 \psi;\)  \hspace{1cm} (101)$$
$$J^{\pm}_{\mu\nu} = \psi^\dagger \sigma_{\mu\nu} \Gamma_i \psi;\)  \hspace{1cm} (102)$$
$$\Gamma_i = \{1, i\tilde{\tau}\}$$  \hspace{1cm} (103)$$

The term $\pm \frac{1}{8} \alpha_{\mu\nu} J^{(i)}_{\mu\nu} J^{(i)}_{\alpha\beta}$ effectively averages to zero if the vacuum is symmetric, $N_+ = N_-$. For bosonization we use standard formula

$$e^{\alpha^2 x^2/4} \sim \int d\psi e^{-y^2 - \alpha xy}$$

Notice however one subtle issue which exists for the tensor fields. The field $J_{\mu\nu}$ is antisymmetric, $J_{\nu\mu} = -J_{\mu\nu}$, thus not all the components are independent and this fact must be taken into account in bosonization to avoid double counting:

$$J^{2}_{\mu\nu} = \sum_{\mu<\nu} J^{2}_{\mu\nu} = 2 \sum_{\mu<\nu} J^{2}_{\mu\nu}$$  \hspace{1cm} (104)$$
$$\exp \left( \alpha^2 J^{2}_{\mu\nu}/8 \right) = \int D\Phi_{\mu\nu} \exp \left( - \sum_{\mu<\nu} \Phi_{\mu\nu}^2 - \alpha \sum_{\mu<\nu} J_{\mu\nu} \Phi_{\mu\nu} \right) = \int D\Phi_{\mu\nu} \exp \left( - \frac{1}{2} \Phi_{\mu\nu}^2 - \frac{1}{2} \alpha J_{\mu\nu} \Phi_{\mu\nu} \right)$$  \hspace{1cm} (105)$$

The above-mentioned antisymmetry must be also taken into account in all further differentiations over $\Phi_{\mu\nu}$. From [100] we get for the partition function

$$Z_N = \int d\lambda_+ d\lambda_- D\Phi D\Psi \Phi^\pm D\Phi^\pm e^{-S}$$  \hspace{1cm} (106)$$
$$S = -N_\pm \ln \lambda_\pm + 2 \left( \Phi_i^2 + \frac{1}{2} \Phi_{i,\mu\nu}^2 \right) + \psi^\dagger \left[ i\tilde{\sigma} + \tilde{\phi} + im \right]$$

$$i\lambda^{0.5} \bar{F}(p) \left( \alpha \Phi_i \Gamma_i + \frac{1}{2} \beta \Phi_{i,\mu\nu} \sigma_{\mu\nu} \Gamma_i \right) F(p)L^{-1} \psi$$

We can integrate out fermions and get

$$Z_N[v, T] = \int d\lambda d\Lambda D\Phi D\Phi_{\mu\nu} e^{-S}$$  \hspace{1cm} (107)$$
$$S = -N_\pm \ln \lambda_\pm + 2 \left( \Phi_i^2 + \frac{1}{2} \Phi_{i,\mu\nu}^2 \right) - \frac{1}{2} T_{\mu\nu} \Phi_{\mu\nu} + im \psi^\dagger \left[ i\tilde{\sigma} + \tilde{\phi} + im \right]$$

$$i\lambda^{0.5} \bar{F}(p) \left( \alpha \Phi_i \Gamma_i + \frac{1}{2} \beta \Phi_{i,\mu\nu} \sigma_{\mu\nu} \Gamma_i \right) F(p)L^{-1} \psi$$

where we have introduced external tensor source $T_{\mu\nu}$. 

A. Contribution to the axial currents correlator

In this section we consider the contribution of the tensor mesons \( \Phi_{\mu\nu} \) to the axial currents correlator. This contribution may be represented as a Feynman diagram shown in the Fig. 9. Straightforward evaluation of the tensor-axial coupling is

\[
2i\epsilon_{\mu\nu\rho\lambda} \Phi_{\mu\nu(q)} D(q) = \frac{c_A}{N_c} \int d^4p \frac{2\mu^3(p) + p M f(p) f'(p)(p^2 - 3\mu^2(p))}{(p^2 + \mu^2(p))^2}
\]

and the total contribution of the diagram in the Fig. 9 is

\[
F \sim 8_{A} q^2 \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \times N_c
\]

Thus we can see that this diagram is just \( O(q^2) \)-correction to the axial correlator. Since we are interested only in the LO over \( q^2 \) (evaluation of \( F_x \)), we should not evaluate this diagram.

VIII. GASSER-LEUTWYLER COUPLINGS

According to \[\text{[1]}, \text{the low-energy constants } \bar{l}_i \text{ of the chiral lagrangian may be extracted from the } O(m) \text{-corrections to physical quantities, e.g.}
\]

\[
M^2 = m^2 \pi \left( 1 - \frac{m^2}{4N_c^2} \bar{l}_3 + O(m^4) \right)
\]

\[
F^2 = F^2 \left( 1 + \frac{m^2}{8N_c^2} \bar{l}_4 + O(m^4) \right)
\]

where \( M, F \) are the pion mass and decay constants, \( m^2 = 2mB \) and \( B, F \) are the phenomenological parameters of the chiral lagrangian. Using our results \[\text{[63}, \text{86]}, \text{we can obtain } (m \text{ is given in GeV})
\]

\[
F^2 = 0.002847777N_c - 0.000868917 + O \left( \frac{1}{N_c} \right)
\]

\[
B = 1.7467 + \frac{0.8183}{N_c} + O \left( \frac{1}{N_c^2} \right)
\]

\[
\bar{l}_3 = \frac{-1.14251 N_c \left( 1 + \frac{0.872354}{N_c} + \frac{0.874555 \ln m}{N_c} + O \left( \frac{1}{N_c^3} \right) \right)}{1 + \frac{0.946572}{N_c} + O \left( \frac{1}{N_c^3} \right)} = 0.0738267 - 1.14251 N_c - 0.999 \ln m + O \left( \frac{1}{N_c} \right)
\]

\[
\bar{l}_4 = \frac{-0.0793814 N_c \left( 1 + \frac{0.232149}{N_c} + \frac{12.5977 \ln m}{N_c} \right)}{1 + \frac{0.468486}{N_c} + O \left( \frac{1}{N_c^3} \right)} = -0.0793814 N_c + 0.0187608 - 1.000 \ln m + O \left( \frac{1}{N_c} \right)
\]

which gives

\[
F = 88 \text{ MeV, } B = 2.019 \text{ GeV, } \bar{l}_3 = 1.84, \bar{l}_4 = 4.98
\]

at \( m = 0.0055 \text{ GeV, corresponding } M_x = 0.142 \text{ GeV, } F_x = 0.0937 \text{ GeV.}
\]

The values of \( F_x, -\langle q(q(m = 0)) \rangle = -F_x^2 B \) in \[\text{[113]}, \text{were taken as input when we fixed the parameters } \rho, R \text{ in } \text{[41]. Our values of } (\bar{l}_3, \bar{l}_4) \text{ should be compared with the phenomenological estimates } (22, 23) \text{ as well as lattice predictions } (32, 39) \text{ given in Table I}\].

IX. CHIRAL LOGS

Now we would like to discuss the chiral logarithms \( m \ln m \) generated by pion loops in the effective action
The general chiral log theorem in our approximation has a form

\[
G(M_\pi) = G(M_\pi = 0) \left( 1 + \frac{M^2_{\pi, LO} \ln M^2_{\pi, LO}}{F^2_{\pi, LO}} + \ldots \right)
\]

where \( \gamma \) is some specific numerical coefficient, and we took into account that \( F^2_{\pi, LO} \sim N_c \).

Chiral logs have two sources:

1. Shift of dynamical quark mass \( \delta M \) due to meson loops ("indirect" meson loops contribution). In this region contribution to chiral logs is obtained acting with \( \frac{\delta M}{\delta M_0} \) on the analytic LO expressions for the correlators.

2. Pion small-momentum \( q \approx 0 \) region in pion loops ("direct" contribution). In this region pion propagator may be approximated as

\[
\Pi^{-1}_\pi(q) \approx F^2_{\pi, LO} (M^2_{\pi, LO} + q^2).
\]

vertices \( A_{1,ij} \) and \( A_{2,ij} \) — as constants (independent of \( q \)), so

\[
\int \frac{d^4q}{(2\pi)^4} \Pi(q) A_\pi(q) \approx A_\pi(q = 0) \frac{M^2_{\pi, LO} \ln M^2_{\pi, LO}}{16\pi^2} + \ldots
\]

So evaluation of the chiral logs is straightforward but tedious.

The first relation for the constituent quark mass \( M \)

\[
M(m) = M_0 \left( 1 - \frac{3M^2_{\pi, LO}}{32\pi^2 F^2_{\pi, LO}} \ln M^2_{\pi, LO} + O(M^4) \right)
\]

may be most easily obtained expanding the vacuum equation \([53]\) and collecting \( 1/N_c \)-terms.

\[
-\frac{1}{2} \frac{1}{M_0} \frac{\partial}{\partial M_0} \text{Tr}[Q(p)] + \ldots
\]

Using \([120]\) and the relations

\[
\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2_{\pi, LO}} = \ldots + \frac{M^2_{\pi, LO}}{16\pi^2} \ln M^2_{\pi, LO} + O(M^4)
\]

\[
\left( \frac{\partial}{\partial M_0} \right) \text{Tr}[Q(p)] = 16N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2 f^2(p) p^2}{(p^2 + \mu^2(p))^2}
\]

we can check that chiral log in \( M_1 \) satisfies \([122]\).

In completely the same way, expanding Eqs. \([69]\) and using \([122]\), we can obtain for the chiral log in quark condensate

\[
\langle \bar{q} q(m) \rangle = \langle \bar{q} q \rangle_0 \left( 1 - \frac{3M^2_{\pi, LO}}{32\pi^2 F^2_{\pi, LO}} \ln M^2_{\pi, LO} + O(M^4) \right)
\]

Similar evaluation of the chiral logs in \( F_\pi \) and \( M_\pi \) yields

\[
F^2_{\pi}(m) = F^2(0) \left( 1 - \frac{M^2_{\pi, LO}}{8\pi^2 F^2_{\pi, LO}} \ln M^2_{\pi, LO} + O(M^4) \right)
\]

\[
M^2_{\pi, LO}(m) = M^2_{\pi, LO} \left( 1 + \frac{M^2_{\pi, LO}}{32\pi^2 F^2_{\pi, LO}} \ln M^2_{\pi, LO} + O(M^4) \right)
\]

in correspondence with low-energy theorems \([24]\) and large-\( N_c \) expansion. From \([125, 126]\) we can immediately see that the chiral logs in the low-energy couplings \( l_i \) are

\[
\begin{array}{cccccc}
\chi PT & \text{MILC} & \text{Del Debbio et. al.} & \text{ETM} & \text{Our prediction} \\
[1, 23, 40] & [41] & [39] & [42] & [23] \\
\hline
l_1 & 2.9 \pm 2.4 & 0.6 \pm 1.2 & 3.0 \pm 0.5 & 3.62 \pm 0.12 & 1.84 \\
l_1 & 4.4 \pm 0.2 & 3.9 \pm 0.5 & 4.52 \pm 0.06 & 4.98 & 3.62 \pm 0.12
\end{array}
\]

TABLE I: Estimates and predictions of the low-energy constants. The first column contains phenomenological estimates, the next three columns are lattice results from different collaborations, the last column contains our results. The first four columns of the table are taken from \([23]\).
X. PATH INDEPENDENCE

In our evaluations for the sake of convenience we chose the straight-line path in the transportor \([14]\). Now we would like to demonstrate that the results of our evaluations are path independent, though on the intermediate steps we have explicit path dependence as a consequence of the zero-mode approximation \([7]\).

Let us consider first the difference of the path integrals over two contours: \(\delta(\int v_{\mu}d\xi_{\mu}) = \int v_{\mu}d\xi_{\mu}\). Due to Stock’s theorem, this integral is reducible to the surface integral

\[
\int v_{\mu}d\xi_{\mu} = \int dS_{\mu\nu}v_{\mu}(x)
\]

where for the monochromatic field \(v_{\mu}(\xi) = v_{\mu}(q)e^{iq\xi}\) we have

\[
v_{\mu\nu}(\xi) = v_{\mu\nu}(q)e^{iq\xi} = i(q_{\mu}v_{\nu}(q) - q_{\nu}v_{\mu}(q)).
\]

Since we are interested only in \(\mathcal{O}(q^{0})\)-terms, the field strength is zero and thus integral over any closed contour is zero

\[
\int v_{\mu}d\xi_{\mu} \approx v_{\mu}(q)\int dS_{\mu\nu} = v_{\mu}(q)S_{\mu\nu} = 0
\]

For evaluation of \(F_{\pi}\) from the \((j_{\mu}^{A}, j_{\nu}^{A})\)-correlator we can take \(a_{\mu} = \text{const}\), which yields the path-independent result.

Notice that the path independence exists only for small \(q\), whereas for arbitrary \(q\) (e.g., for the "dynamical" magnetic susceptibility considered in \([43]\)) the path dependence exists and there is no other argument except simplicity why one should choose the straight-line path in the transporter \([14]\).

XI. DISCUSSION

The aim of our work was the study of the pion physics beyond the chiral limit in the framework of the instanton vacuum model. We found the generating functional of the hadronic correlators with account of \(\mathcal{O}(1/N_{c}, m, m/N_{c}, m \ln m/N_{c})\)-corrections and exploited it for evaluation of the corrections to different physical observables. The corrections considered in this paper include meson loops, finite width of instanton size distribution and quark-quark tensor interactions term.

In contrast to the expectations, we found that numerically the \(1/N_{c}\)-corrections to dynamical quark mass are large and mostly come from meson loops. As a consequence, we have large \(1/N_{c}\)-corrections to all the other quantities. To provide the values of \(F_{\pi}(m = 0), (\bar{q}q(m = 0))\) in agreement with \(\chi PT\), we offer a new set of parameters \(\rho, R\) \([8]\). Remarkably, this set of parameters is still in agreement with current phenomenological and lattice estimates \([26]\).

The main result of this paper is the evaluation of the \(F_{\pi}(m)\) and \(M_{\pi}(m)\)-dependence with account of \(\mathcal{O}(1/N_{c}, m, m/N_{c}, m/N_{c} \ln m)\)-corrections. From comparison with \(\chi PT\) we extract the values of the low energy constants \(\tilde{I}_{3}, \tilde{I}_{4}\). Our results for the values of \(\tilde{I}_{3}\) and \(\tilde{I}_{4}\) are in a satisfactory agreement with phenomenological as well as lattice estimates (See Table \([1]\)). This means that the instanton vacuum is applicable for understanding of the low-energy physics, at least on the qualitative level. Evaluation of the other LEC’s is in progress.

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APPENDIX A: NONLOCAL VERTICES

One of the important features of the chiral quark model is the nonlocal interaction of the quarks with mesons in \([14]\). On the one hand, due to the formfactors all the loop integrals in the model are convergent with an effective cut-off at the scale \(p \sim \rho^{-1}\). On the other hand, such regularization explicitly violates gauge invariance. The modification of the quark-meson coupling to

\[
\tilde{L}(x, z)F(x - z)\Phi_{i}(z)\Gamma_{i}L(z, y)F(z - y)
\]

where \(L\)-factors are defined in \([14]\), formally restores the gauge covariance. However, it introduces unphysical path dependence of all correlators with vector and axial currents. In some of the simplest cases (e.g., constant field \(a_{\mu}\)) it is a trivial matter to show that the path dependence drops out, but it doesn’t in general case. In our evaluations we choose conventional straight line path. Let us consider for definiteness the fields of the plane wave form

\[
v_{\mu}(x) = v_{\mu}(q)e^{iqx}
\]

\[
a_{\mu}(x) = a_{\mu}(q)e^{iqx}
\]

The integral along the straight-line path for such fields equals

\[
\int_{z}^{x}d\xi_{\mu}v_{\mu}(\xi) = (x - z)v_{\mu}(q)e^{iqz}e^{iq(x - z)} - 1
\]

and the same for axial field \(a\). In general the vertex \([A1]\) contains an infinite series of the terms \(v, a\). However in the next sections we’ll need the vertices only up to
$O(\alpha^2, \alpha^3)$-terms and restrict ourselves with this accuracy in this section. Then the $L$-factors may be rewritten as

$$L(z, y) = 1 + V_\mu(q)i(z - y)\frac{e^{iq(z-y)}}{iq \cdot (z - y)} + \frac{e^{iq_1 \cdot (z - y)} - 1}{iq_1 \cdot (z - y)} + \frac{e^{iq_2 \cdot (z - y)} - 1}{iq_2 \cdot (z - y)} + O(V^3)$$

(A5)

$$\bar{V}_\mu(q_1) \bar{V}_\mu(q_2)i(z - y)\frac{e^{iq_1 \cdot (z - y)} - 1}{iq_1 \cdot (z - y)} + \frac{e^{iq_2 \cdot (z - y)} - 1}{iq_2 \cdot (z - y)} + O(V^3),$$

(A6)

where $V_\mu(q) = v_\mu(q) + a_\mu(q)\gamma_5$, $\bar{V}_\mu(q) = v_\mu(q) - a_\mu(q)\gamma_5$. The interaction vertices may be compactly rewritten in terms of the functions

$$G_\mu(x - z; q) := i(x - z)\frac{e^{iq(x-z)} - 1}{iq \cdot (x - z)} F(x - z)$$

(A7)

$$L(x, z)F(x - z)\Phi_i(z)\Gamma_i L(z, y)F(z - y) =$$

$$[G_\mu(p, q)\Gamma_i F(p + k + q) + F(p)\Gamma_i G_\mu(p + k + q, -q)]v_\mu(q)\Phi_i(k) +$$

$$[G_\mu(p, q_1)\Gamma_i G_\mu(p + k + q_1 + q_2, -q_2) +$$

$$\frac{1}{2} (H_\mu(p, q_1, q_2)\Gamma_i F(p + k + q_1 + q_2) + F(p)\Gamma_i H_\mu(p + k + q_1 + q_2, -q_1, -q_2)) v_\mu(q_1)v_\alpha(q_2)\Phi_i(k) +$$

$$[-G_\mu(p, q)\Gamma_i F(p + k + q) + F(p)\Gamma_i G_\mu(p + k + q, -q)]a_\mu(q)\gamma_5\Phi_i(k) +$$

$$[-G_\mu(p, q_1)\Gamma_i G_\mu(p + k + q_1 + q_2, -q_2) +$$

$$\frac{1}{2} (H_\mu(p, q_1, q_2)\Gamma_i F(p + k + q_1 + q_2) + F(p)\Gamma_i H_\mu(p + k + q_1 + q_2, -q_1, -q_2)) a_\mu(q_1)a_\nu(q_2)\Phi_i(k)$$

+ $O(\alpha^v)$ – mixing terms we are not interested in now

(A13)

Due to chiral symmetry breaking field $\sigma$ has nonzero vacuum expectation value (VEV). This generates momentum dependent constituent quark mass $\mu(p)$ and also generates new nonlocal vertices we should take into account in expansions over the fields $\Phi$ and currents $v, a$:

$$V_\sigma = \sigma_{vac} [(G_\mu(p, q)F(p + q) + F(p)G_\mu(p + q, -q)]v_\mu(q) +$$

$$[G_\mu(p, q_1)G_\mu(p + q_1 + q_2, -q_2) +$$

$$\frac{1}{2} (H_\mu(p, q_1, q_2)F(p + q_1 + q_2) + F(p)H_\mu(p + q_1 + q_2, -q_1, -q_2)) v_\mu(q_1)v_\alpha(q_2) +$$

$$[-G_\mu(p, q)F(p + q) + F(p)G_\mu(p + q, -q)]a_\mu(q)\gamma_5 +$$

$$[-G_\mu(p, q_1)G_\mu(p + q_1 + q_2, -q_2) +$$

$$\frac{1}{2} (H_\mu(p, q_1, q_2)F(p + q_1 + q_2) + F(p)H_\mu(p + q_1 + q_2, -q_1, -q_2)) a_\mu(q_1)a_\nu(q_2)$$

(A14)

(A15)

In $p$-space Fourier expansions of these functions are

$$G_\mu(p; q) := \int d^4xF_\mu(x; q)e^{ip\cdot x} =$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} F_{\mu, \mu_1 \ldots, \mu_n}(p)q_{\mu_1} \ldots q_{\mu_n}$$

$$H_{\mu\nu}(p; q_1, q_2) := \int d^4x H_{\mu\nu}(x; q_1, q_2)e^{ip\cdot x} =$$

$$\sum_{n, k=0}^{\infty} \frac{1}{(n+1)!(k+1)!} F_{\mu, \nu, \mu_1 \ldots, \mu_n, \nu_1 \ldots \nu_k}(p) \times q_{\mu_1} \ldots q_{\nu_n} q_{2\nu_1} \ldots q_{2\nu_k}$$

In terms of these functions interaction terms may be rewritten as $(\Phi_i(x) = \Phi_i(k)e^{ikx}$, integration over the dummy momenta $k, q, q_1, q_2$ is implied)

The vertices which are generated are shown on the Fig[10]
the constituent quarks.

possible (local and nonlocal) couplings of the current

wavy lines correspond to the currents \( v_\mu, a_\mu \), the dashed line corresponds to the meson, the bulbs correspond to all the possible (local and nonlocal) couplings of the current \( V \) to the constituent quarks.

\[ \Phi_0(\rho) := \phi_0 + \phi_1(\rho - \bar{\rho}) + \phi_2 \frac{(\rho - \bar{\rho})^2}{2} + ... \]  
\[ \tilde{K}(p, \rho) := \tilde{K}_0(p) + \tilde{K}_1(p)(\rho - \bar{\rho}) + \tilde{K}_2(p) \frac{(\rho - \bar{\rho})^2}{2} + ... \]  
\[ \int d\rho d(p) = 1 \]  
\[ \int \rho d\rho d(p) = \bar{\rho} \]  
\[ \int (\rho - \bar{\rho})^2 d\rho d(p) = \delta^2 \]  
\[ \Rightarrow 2 \left( \phi_1^2 + \delta^2 (\phi_1^2 + \phi_0 \phi_2) \right) = \frac{N}{V}, \]  
\[ \int d\rho d(p) \tilde{K}(p, \rho) \Phi_0(\rho) = \phi_0 \tilde{K}_0(p) + \frac{\delta^2}{2} (2\phi_1 \tilde{K}_1(p) + \phi_0 \tilde{K}_2(p) + \phi_2 \tilde{K}_0(p)) \]  
\[ \Rightarrow \phi_0 = \frac{1}{4} Tr_{p, \{f,c\}} \left( \frac{i c \lambda^{0.5} \tilde{K}_0(p)}{\bar{\rho} + im + i c \lambda^{0.5} (\tilde{K}_0(p) \phi_0 + \frac{\delta^2}{2} (2\phi_1 \tilde{K}_1(p) + \phi_0 \tilde{K}_2(p) + \phi_2 \tilde{K}_0(p)))} \right) \]  
\[ \Rightarrow \phi_1 = \frac{1}{4} Tr_{p, \{f,c\}} \left( \frac{i c \lambda^{0.5} \tilde{K}_1(p)}{\bar{\rho} + im + i c \lambda^{0.5} (\tilde{K}_0(p) \phi_0 + \frac{\delta^2}{2} (2\phi_1 \tilde{K}_1(p) + \phi_0 \tilde{K}_2(p) + \phi_2 \tilde{K}_0(p)))} \right) \]  
\[ \Rightarrow \phi_2 = \frac{1}{4} Tr_{p, \{f,c\}} \left( \frac{i c \lambda^{0.5} \tilde{K}_2(p)}{\bar{\rho} + im + i c \lambda^{0.5} (\tilde{K}_0(p) \phi_0 + \frac{\delta^2}{2} (2\phi_1 \tilde{K}_1(p) + \phi_0 \tilde{K}_2(p) + \phi_2 \tilde{K}_0(p)))} \right) \]

Equations (B6,B8,B10) form a system of four equations with four independent variables \( \phi_0 - \phi_2, \lambda \). By definition,

\[ \phi_n = \frac{\Phi_0^{(n)}(\bar{\rho})}{n!} \]

Note that none of these variables is suppressed over \( \delta \). We can make the next step and expand over the variable \( \delta \):

\[ \phi_m := \sum_n \phi_{mn} \frac{\delta^n}{n!} \]  
\[ \lambda := \sum_n \lambda_n \frac{\delta^n}{n!} \]  
\[ \lambda^{0.5} = \lambda^{0.5} \left( 1 + \delta \frac{1}{2} \frac{\lambda_1}{\lambda_0} + \delta^2 \left( \frac{1}{2} \frac{\lambda_2}{\lambda_0} - \frac{\lambda_0^2}{8 \lambda_0^2} \right) + ... \right) := \sum_n l_n \frac{\delta^n}{n!} \]

From the normalization condition we get

\[ 2\phi_0^2 = \frac{N}{V} \]  
\[ \text{In agreement with } \delta = 0 \text{ case} \]  
\[ \phi_{01} = 0 \]  
\[ \phi_{00} \phi_{02} + \phi_{10}^2 + \phi_{00} \phi_{20} = 0 \]
Equations \([\text{B16-B17]}\) give

\[
\Rightarrow \phi_{00} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_0 \tilde{K}_0(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00}} \right)
\]

In agreement with \(\delta = 0\) case

\[
0 = \phi_{01} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_1 \tilde{K}_1(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right) + \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_0^2(p)(\phi_{00} l_1 + \phi_{01} l_0)}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right)
\]

\[
l_1 \left( \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic \tilde{K}_1(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right) \right) + \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_0^2(p) \phi_{00}}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right)
\]

\[
\Rightarrow l_1 = 0 \text{ since expression inside the brackets is not zero } \Rightarrow \lambda_1 = 0
\]

\[
\Rightarrow \phi_{02} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_2 \tilde{K}_0(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00}} \right) + \frac{1}{8} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_0(p)(l_2 \phi_{00} \tilde{K}_0(p) + l_0(\phi_{00} \tilde{K}_0(p) + 2\phi_{01} \tilde{K}_1(p) + \tilde{K}_2(p) \phi_{00} + \phi_{20} \tilde{K}_0(p)))}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right)
\]

\[
\Rightarrow \phi_{10} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_1 \tilde{K}_1(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right)
\]

\[
\phi_{11} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_1 \tilde{K}_1(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right) + \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_0(p) \tilde{K}_1(p)(\phi_{00} l_1 + \phi_{01} l_0)}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right) = 0
\]

since \(\lambda_1 = 0, \phi_{01} = 0\)

\[
\Rightarrow \phi_{12} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_2 \tilde{K}_1(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right)
\]

\[
\frac{1}{8} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_1(p)(l_2 \phi_{00} \tilde{K}_0(p) + l_0(\phi_{00} \tilde{K}_0(p) + 2\phi_{01} \tilde{K}_1(p) + \tilde{K}_2(p) \phi_{00} + \phi_{20} \tilde{K}_0(p)))}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right)
\]

\[
\Rightarrow \phi_{20} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_2 \tilde{K}_2(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right)
\]

\[
\phi_{21} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_1 \tilde{K}_2(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right) + \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_0(p) \tilde{K}_2(p)(\phi_{00} l_1 + \phi_{01} l_0)}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right) = 0
\]

since \(\lambda_1 = 0, \phi_{01} = 0\)

\[
\Rightarrow \phi_{22} = \frac{1}{4} \text{Tr}_{p,\{f,c\}} \left( \frac{ic l_2 \tilde{K}_2(p)}{\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_0} \right)
\]

\[
\frac{1}{8} \text{Tr}_{p,\{f,c\}} \left( \frac{c^2 l_0 \tilde{K}_2(p)(l_2 \phi_{00} \tilde{K}_0(p) + l_0(\phi_{00} \tilde{K}_0(p) + 2\phi_{01} \tilde{K}_1(p) + \tilde{K}_2(p) \phi_{00} + \phi_{20} \tilde{K}_0(p)))}{(\tilde{p} + im + ic l_0 \tilde{K}_0(p) \phi_{00})^2} \right)
\]

Thus we finish with a definite set of equations sufficient to determine all the constants \(\phi_{m,n}, \lambda_n\).

1. Evaluate the "ordinary" LO vacuum equation \([\text{B14-B17]}\). Obtain the values \(l_0, \phi_{00}\)
2. Substitute these values into \([\text{B20-B23]}\) and evaluate \(\phi_{10}, \phi_{20}\)
3. From \([\text{B16]}\) evaluate \(\phi_{02}\)
4. Consider \([\text{B19]}\) as a (linear !) equation w.r.t. \(l_2\) and get the value \(l_2\)
5. Evaluate directly the remaining values \(\phi_{12}, \phi_{22}\)

Notice that so far we haven’t used any particular form of the size distribution \(\rho\). The only assumption we did is that the width is small, \(\delta \rho^2 \ll \tilde{\rho}^2\)

---

**APPENDIX C: STRUCTURE OF THE \((\mathbb{J}^{\delta a}_{\mu}, \mathbb{J}^{\delta b}_{\nu})\)-CORRELATOR**

The easiest way is to start from \([\text{H1]}\), split the field \(\Phi\)

In this section we are going to prove that due to the chiral symmetry correlator \((\mathbb{J}^{\delta a}_{\mu}, \mathbb{J}^{\delta b}_{\nu})\) has a form

\[
\mathbb{J}^{\delta a}_{\mu}(x, y) = K^{\delta ab}[\delta_{\mu\nu} - \eta_{\mu\nu}] + O(\rho^2)
\]
as $\Phi = \sigma U + \Phi'$ and make the chiral rotation of the quark fields

$$\psi(x) \rightarrow U^{\gamma_5,1/2} (x) \psi(x), \quad \psi^\dagger(x) \rightarrow U^{\gamma_5,1/2} (x) \psi^\dagger(x)$$

where the matrix $U$ was defined in (62) and will be parameterized as

$$U(x) = \exp (i \vec{u} \cdot \vec{\tau}) = 1 + \frac{i \vec{u} \cdot \vec{\tau}}{2} + \ldots$$

(C3)

The transformation (C2) has Jacobian equal to one, so we have just to evaluate the rotation of the Dirac operator in $S$. This is convenient to introduce a special notation

$$i^n x_{\mu_1} \ldots x_{\mu_n} F(x) = F_{\mu_1, \ldots, \mu_n}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot (x-z)} F_{\mu_1, \ldots, \mu_n}(p),$$

(C4)

i.e. for formfactors the lower index corresponds to differentiation in momentum space (not in coordinate!) whereas for all the other quantities we will use the lower index for differentiation in coordinate space, e.g.

$$u_{\mu}(x) \equiv \frac{\partial u(x)}{\partial x^\mu} = \frac{\partial u(x)}{\partial x_{\mu}}$$

(C6)

One more agreement, for the sake of brevity sometimes we will not write out explicitly the flavour dependence for the quantities $\vec{u}$ and $\vec{a}$, implying

$$u = \frac{\vec{u} \vec{\tau}}{2}, \quad a_{\mu} = \frac{\vec{a} \gamma_{\mu} \vec{\tau}}{2},$$

(C7)

Now we are going to evaluate the result of rotation. Note that actually we are making a double expansion: up to the second order in $\vec{u} \sim \vec{a}$ and up to the second order in derivatives of the field $u$ (assuming $a_{\mu}$ to be the first order). Rotation of the local part is quite trivial, the result is

$$U^{\gamma_5,1/2} (\hat{p} + \hat{a} \gamma_5 + i m) U^{\gamma_5,1/2} = \ldots = \hat{p} + \hat{a} \gamma_5 + u_{\mu} \gamma^\mu \gamma_5 + \frac{\vec{\tau}}{2} \cdot (\hat{a}_{\mu} + \hat{u}_{\mu}) \times \vec{u} + i m \left(1 - i u \gamma_5 - \frac{u^2}{2}\right)$$

(C8)

Notice that the term containing cross-product (in isospace) doesn’t contribute since it is already a highest order in $\vec{a}, \vec{u}$ which we keep and it should be combined with another vector-isovector combined of $\vec{u}, \vec{a}$ and the derivatives. For the nonlocal part rotation is a bit more tricky since we have matrices $U$ at three different points:

\[ i M \tilde{L}(x, z) F(x - z) U^{\gamma_5}(z) L(z, y) F(z - y) + v \rightarrow i M U^{\gamma_5,1/2} (x) \tilde{L}(x, z) F(x - z) U^{\gamma_5}(z) L(z, y) F(z - y) U^{\gamma_5,1/2} (y) + v', \]

\[ v' = \frac{i M}{\sigma} U^{\gamma_5,1/2} \tilde{L} F(p) U^{\gamma_5,1/2} \Gamma \cdot \Phi'' U^{\gamma_5,1/2} F(p) U^{\gamma_5,1/2} \]

where meson fluctuations $\Phi_{\mu}$ and $\Phi'$ are related by the chiral rotation,

$$\Phi'' = U^{\gamma_5,1/2} \Phi' U^{\gamma_5,1/2}.$$

(C9)

for a moment we will consider only the LO result and ignore the fluctuations term $v$.

First, we are going to make the Taylor expansion at the point $z$,

\[ U^{\gamma_5,1/2} (x) = \left(1 - i u(z) \gamma_5 - \frac{u^2(z)}{8}\right) + \left(-i u_{\mu}(z) \gamma_5 - \frac{\vec{u}(z) \vec{u}_{\mu}(z)}{4}\right) (x - z)_{\mu} + \frac{i}{2} u_{\mu, \nu}(z) \gamma_5 - \frac{\vec{u}(z) \vec{u}_{\mu, \nu}(z)}{4} (x - z)_{\mu, \nu} + \mathcal{O}(a^2, a^2, a u, \partial^3) \]

(C10)

\[ U^{\gamma_5,1/2} (y) = \left(1 - i u(z) \gamma_5 - \frac{u^2(z)}{8}\right) + \left(i u_{\mu}(z) \gamma_5 + \frac{\vec{u}(z) \vec{u}_{\mu}(z)}{4}\right) (y - z)_{\mu} + \frac{i}{2} u_{\mu, \nu}(z) \gamma_5 - \frac{\vec{u}(z) \vec{u}_{\mu, \nu}(z)}{4} (y - z)_{\mu, \nu} + \mathcal{O}(a^2, a^2, a u, \partial^3) \]

(C11)

Using the general expressions for the nonlocal vertices from the previous Section one can show that

\[ LF(x - z) = F(x - z) - a_{\mu}(q) F_{\mu}(x - z) \gamma_5 + \frac{1}{2} a_{\mu}(q) a_\nu(q) F_{\mu, \nu}(x - z) + \mathcal{O}(a^3) \]

(C12)

\[ LF(z - y) = F(z - y) + a_{\mu}(q) F_{\mu}(z - y) \gamma_5 + \frac{1}{2} a_{\mu}(q) a_\nu(q) F_{\mu, \nu}(z - y) + \mathcal{O}(a^3) \]

(C13)
Notice that we drop the terms \( \partial_\mu (\tilde{u} \partial_\mu \tilde{u}) \) which give just a total derivative

\[
U^{\gamma, 1/2, l}(z) \bar{L} F(x - z) U^{\gamma, 1/2}(z) = F(z - y) + \left( a_\mu(q) \gamma_5 + u_\mu(z) \gamma_5 - \frac{i}{2} \bar{u}(z) \times (\tilde{a}_\mu(q) + \tilde{u}_\mu(z)) \right) F_\mu(z - y) + \\
\frac{1}{8} (\tilde{a}_\mu(q) + \tilde{u}_\mu(z)) (\tilde{a}_\nu(q) + \tilde{u}_\nu(z)) F_{\mu\nu}(z - y) + \\
\frac{i}{2} u_{\mu\nu} \gamma_5 F(x - z) + \mathcal{O}(a^3, w^3, a^2 w, a w^2, \partial_\mu (\tilde{u} \partial_\mu \tilde{u}))
\]

(C14)

\[
\Rightarrow U^{\gamma, 1/2, l}(z) \bar{L} F(x - z) U^{\gamma, 1/2}(z) L(z, y) F(z - y) U^{\gamma, 1/2, l}(y) = \ldots
\]

\[
F(x - z) F(z - y) + \left( a_\mu(q) \gamma_5 + u_\mu(z) \gamma_5 - \frac{i}{2} \bar{u}(z) \times (\tilde{a}_\mu(q) + \tilde{u}_\mu(z)) \right) (F_\mu(x - z) F(z - y) - F(x - z) F_\mu(z - y)) + \\
\frac{1}{8} (\tilde{a}_\mu(q) + \tilde{u}_\mu(z)) (\tilde{a}_\nu(q) + \tilde{u}_\nu(z)) (F_{\mu\nu}(x - z) F(z - y) + F(x - z) F_{\mu\nu}(z - y) - 2 F_\mu(x - z) F_\nu(z - y)) + \\
\frac{i}{2} u_{\mu\nu}(z) \gamma_5 (F_{\mu\nu}(x - z) F(z - y) + F(x - z) F_{\mu\nu}(z - y)) + \mathcal{O}(a^3, w^3, \ldots)
\]

(C15)

(C16)

(C17)

Note that

1. The term \( \sim \tilde{a}_\mu \times \tilde{u} \) doesn’t contribute due to its isovector structure

2. The term \( \sim u_{\mu\nu} \) may only contribute beyond the chiral limit: it contains the second derivative, and the “free” \( \tilde{u} \) (without derivatives) comes only via \( \mathcal{O}(mu) \)-term.

3. All the remaining structures have a form \((\tilde{a}_\mu(q) + \tilde{u}_\mu(z))\)

4. Beyond the chiral limit expansion of \( Tr \ln \) yields

\[
\Gamma_{eff} = a_0(\tilde{a}_\mu + \tilde{u}_\mu)^2 + m \alpha_1 \tilde{u}_\mu(\tilde{a}_\mu + \tilde{u}_\mu) + m \alpha_2 \tilde{u}_\mu^2 = \\
= \frac{1}{2} \left[ F_{aa} \tilde{a}_\mu^2 + F_{uu} \tilde{u}_\mu^2 + 2 F_{au} \tilde{a}_\mu \tilde{u}_\mu + F_{aa} M_x^2 \tilde{a}_\mu^2 \right] + \mathcal{O}(a^3, w^3, m^2),
\]

where the constants \( F_{ij} \) differ only beyond chiral limit:

\[
F_{aa} - F_{uu} = 2 (F_{aa} - F_{uu}) = - \alpha_1 m
\]

(C18)

For evaluation of the meson loop contribution (term \( v' \)) one should notice that only the second order, \( v'^2 \), is essential. Using expansions \( [C14, C15] \), one can show that in the chiral limit the result depends only upon the structure \( \tilde{a}_\mu + \tilde{u}_\mu \). Beyond the chiral limit \( \mathcal{O}(m) \)-correction has a form \( \sim m \tilde{u}_\mu(\tilde{a}_\mu + \tilde{u}_\mu) \). This completes the proof that the correlator of two axial currents has a form \( [C1] \).
[20] E. D. Salvo and M. M. Musakhanov, Eur. Phys. J. C 5, 501 (1998) [arXiv:hep-ph/9706537].
[21] M. Musakhanov, Eur. Phys. J. C 9 (1999) 235 [arXiv:hep-ph/9810295].

[22] H. Leutwyler, [arXiv:hep-ph/0612112]

[23] H. Leutwyler, [arXiv:0706.3138] [hep-ph].

[24] P. Langacker, H. Pagels, Phys. Rev. D 8 (1973) 4595.
[25] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 191 (1981) 301.
[26] C. K. Lee and W. A. Bardeen, Nucl. Phys. B 153 (1979) 210.

[27] M. Musakhanov, Nucl. Phys. A 699 (2002) 340.
[28] M. M. Musakhanov and H. C. Kim, Phys. Lett. B 572 (2003) 181 [arXiv:hep-ph/0206233].
[29] H. C. Kim, M. Musakhanov and M. Siddikov, Phys. Lett. B 608 (2005) 95 [arXiv:hep-ph/0411181].
[30] R. D. Carlitz and D. B. Creamer, Annals Phys. 118 (1979) 429.
[31] G. V. Dunne, J. Hur, C. Lee and H. Min, Phys. Rev. D 71 (2005) 085019 [arXiv:hep-th/0502087].
[32] G. ’t Hooft, Phys. Rev. D 14 (1976) 3432 [Erratum-ibid. D 18 (1978) 2199].
[33] S. R. Coleman and E. Weinberg, Phys. Rev. D 7(1973) 1888.
[34] R. Jackiw, Phys. Rev. D 9(1974) 1686.
[35] P. O. Bowman, private communication \((M(m))\)-dependence. See also P. O. Bowman, U. M. Heller, M. B. Leinweber, M. B. Parappilly, A. G. Williams and J. b. Zhang, Phys. Rev. D 71 (2005) 054507 [arXiv:hep-lat/0501019].
[36] H. C. Kim, M. M. Musakhanov and M. Siddikov, Phys. Lett. B 633 (2006) 701 [arXiv:hep-ph/0508211].
[37] V. Dmitrasinovic, H. J. Schulze, R. Tegen and R. H. Lemmer, Annals Phys. 238 (1995) 332.
[38] C. Aubin et al. [MILC Collaboration], Phys. Rev. D 70 (2004) 114501 [arXiv:hep-lat/0407028].
[39] L. Del Debbio, L. Giusti, M. Luscher, R. Petronzio and N. Tantalo, JHEP 0702 (2007) 056 [arXiv:hep-lat/0610059].
[40] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B 603 (2001) 125 [arXiv:hep-ph/0103088].
[41] C. Bernard et al., [arXiv:hep-lat/0611024].
[42] Ph. Boucaud et al. [ETM Collaboration], [arXiv:hep-lat/0701012].
[43] A. E. Dorokhov, Eur. Phys. J. C 42 (2005) 309 [arXiv:hep-ph/0505007].