EQUIVARIANT CARTAN–EILENBERG SUPERGERBES FOR THE GREEN–SCHWARZ SUPERBRANES
II. EQUIVARIANCE IN THE SUPER-MINKOWSKIAN SETTING

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Abstract. This is a continuation of a programme, initiated in Part I [arXiv:1706.05682], of geometrisation, compatible with the supersymmetry present, of the Green–Schwarz super-(p + 2)-cocycles coupling to the topological charges carried by super-p-branes on reductive homogeneous spaces of supersymmetry groups described by Green–Schwarz(-type) super-σ-models. In the present part, higher-geometric realisations of the various supersymmetries – both global and local – of these field theories are studied at length in the form of – respectively – families of gerbe (1-)isomorphisms indexed by the global-supersymmetry group and equivariant structures with respect to supersymmetry actions amenable to gauging. The discussion, employing an algebroidal analysis of the so-called small gauge anomaly, leads to a novel definition of a supersymmetric equivariant structure on the Cartan–Eilenberg super-p-gerbe of Part I with respect to actions of distinguished normal subgroups of the supersymmetry group. This is exemplified by the Ad-equivalent structure on the Green–Schwarz super-p-gerbes for \( p \in \{0, 1\} \) over the super-Minkowski space, whose existence conforms with the classical results for the Grassmann-even counterparts of the corresponding super-σ-models. The study also explores the fundamental tangential gauge supersymmetry of the Green–Schwarz super-σ-model known as \( \kappa \)-symmetry. Its geometrisation calls for a transcription of the field theory to the dual topological Hughes–Polchinski formulation. Natural conditions for the transcription are identified and illustrated on the example of the super-Minkowskian model of Part I. In the dual formulation, the notion of an extended Hughes–Polchinski p-gerbe unifying the metric and topological degrees of freedom of the Green–Schwarz super-σ-model is advanced. Its compatibility with \( \kappa \)-symmetry is ensured by the existence of a linearised equivariant structure. The results reported herein lend strong structural support to the geometrisation scheme postulated in Part I.

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1. Introduction

Symmetry is one of the central concepts in the mathematical description of physical phenomena, both classical and quantum, and compatibility with the symmetry assumed figures among the most natural and potent criteria in selecting consistent models from among a variety of generally conceivable ones with a given field content. The abstract notion has two common instantiations: There exist global (or rigid) symmetries that operate on the fibre $F$ of the configuration bundle of the field theory as its automorphisms covering the identity in its base (the spacetime $X$), and local (or gauge) ones realised by locally smooth profiles of field transformations over the spacetime with values in the group $G$ of symmetries, the latter transformations being induced, in the gauging procedure, by elements of the (Fréchet) group of global sections of the adjoint bundle $\text{Ad} P_G = P_G \times_{\text{Ad}} G \to X$ associated with some principal $G$-bundle $P_G \to X$ acting on global sections of the associated bundle $P_G \times \lambda F \to X$ determined by (and modelled on) an action $\lambda : G \times F \to F$ of the (global-)symmetry group on $F$. While global symmetries map to one another field configurations within level sets of the Dirac–Feynman amplitude and give rise, in the case of continuous symmetries, to Noether hamiltonians in involution with respect to the Poisson bracket of the field theory that furnish a field-theoretic realisation of the symmetries on the classical space of states $\mathcal{P}_F$ of the field theory, local symmetries effect a reduction of that space to its (physical) subspace given, in the regular case, by the space of leaves $\mathcal{P}_F/\ker \Omega_F$ of the characteristic foliation of the presymplectic form $\Omega_F$ of the field theory, the leaves being generated by flows of the fundamental vector fields of the gauge-group action on $\mathcal{P}_F$. From the point of view of the typical fibre $F$ of the configuration bundle, this reduction leads – via integration of the non-dynamical gauge field that enters the field theory with $G$ gauged – to a descent $F \searrow F/G$ of the field theory to the space $F/G$ of orbits of the symmetry-group action whenever the latter space is actually a manifold (which happens, e.g., when $\lambda$ is (smooth, free and proper), and more generally, one may think of the gauging procedure as a systematic construction of a field theory with physical degrees of freedom quantified by $F/G$ and dynamics inherited from the field theory from before the gauging within the larger configuration manifold $F$, cp, e.g., Ref. [GSW13, Sec. 9]. In fact, this way of thinking of the gauging procedure serves to justify the necessity of the incorporation of all isoclasses of principal $G$-bundles in the construction of the field theory with the symmetry group $G$ gauged as these correspond to the various gauge orbits of (possibly discontinuous) field configurations twisted by the action of $G$ (the so-called twisted sector) – this is, in particular, the picture that emerges from the construction of the topological gauge-symmetry defect of Ref. [Sus12, Sec. 8.3] in the two-dimensional non-linear $\sigma$-model, cp also Ref. [Sus13].

Generically, any additional differential-geometric structure on the fibre $F$ that enters the definition of the Dirac–Feynman amplitude, such as a metric tensor, a differential form or a geometrisation of a (relative) de Rham cocycle (a principal $C^\ast$-bundle, a bundle gerbe, a gerbe (bi)module etc.), becomes a source of potential obstruction against the gauging which manifests itself through the so-called gauge anomaly (the small one in the case of gauge transformations homotopic to the identity, or the large one in the remaining cases) and leads to various inconsistencies in the quantum theory. This is easy to understand in the previously invoked picture of the fibrewise descent $F \searrow F/G$ which over $X$ is realised by the sequence of redefinitions

$$F \longrightarrow X \times F \quad \begin{array}{c} \longrightarrow \ G \times F \quad \begin{array}{c} \longrightarrow \ P_G \times F \quad \begin{array}{c} \longrightarrow \ F \end{array} \end{array} \end{array}$$

\[\begin{array}{c} \text{pr}_1 \end{array} \quad \begin{array}{c} \pi_{P_G} \circ \text{pr}_1 \end{array} \quad \begin{array}{c} [\pi_{P_G} \circ \text{pr}_1] \end{array} \]

of the configuration bundle, taking us all the way from the original configuration bundle to the associated bundle, the latter playing the rôle of the configuration bundle of the field theory with $G$ gauged, and hence with the physical degrees of freedom effectively described by $F/G$. The last leg of the above sequence in which the said additional structure is augmented by a tensorial object from the product manifold $P_G \times F$ in a suitable generalisation of the standard minimal-coupling procedure, as discussed at length in Ref. [GSW13], requires $G$-invariance of the tensorial data of the original theory and the

\[\text{We assume the configuration bundle to be trivial for the sake of simplicity. This assumption can readily be dropped.}\]

\[\text{This is just the classical action functional multiplied by } \frac{1}{i} \text{ and exponentiated, which we take to be the fundamental object in the classical and, formally at least, quantum-mechanical description of the field theory.}\]

\[\text{Actually, the corresponding Noether currents may furnish a realisation of a nontrivial central extension of the Lie algebra of the symmetry Lie group.}\]
existence of equivariant extensions of the differential forms involved, whereas in the case of geometrisations of de Rham cocycles, it calls for a full-fledged G-equivariant structure, cp Refs. [GR02, GSW10, GSW13], and also Refs. [Sus12, Sus13] for an independent justification invoking gauge-symmetry defects.

The presence of additional structure on $F$ precisely of the type mentioned above is one of the (co)defining features of a class of low-dimensional field theories describing simple geometric dynamics of point-like and extended objects (material points, loops, membranes etc.) carrying topological charge, known as non-linear (super-)σ-models with the Wess–Zumino (WZ) term. These models arise naturally in (super)string theory, but occur and find numerous applications also in the theory of condensed matter as well as statistical physics. Here, the structure in question describes the coupling of a (relative) de Rham cocycle, to be understood as an external gauge field, to the charge current determined by the trajectory of the extended object in the ambient space(time), and as such is given by a Cheeger–Simons differential character generalising the holonomy functional for a principal $\mathbb{C}^\ast$-bundle. The generalisation pertains to a (higher-)geometric object associated with the de Rham $((p + 2)\times)$-cocycle that goes under the general name of a (bundle) $(p\times)$-gerbe (or to a derived structure, such as, e.g., a gerbe module). The existence of a family of gerbe isomorphisms indexed by elements of the global symmetry group of the σ-model is a prerequisite for the existence of a lift of the global symmetry to the quantum theory as cohomological data of the isomorphisms induce equivalences among classical states of the theory and transgress to automorphisms of the prequantum bundle defined by the gerbe that cover those equivalences, cp Refs. [Sus11]. An equivariant structure on the gerbe with respect to the (global) symmetry, on the other hand, has been proven necessary and sufficient for a non-anomalous gauging of that symmetry, cp Refs. [GSW11a, GSW10, GSW13, Sus12], and gauging itself is readily seen to be an important and convenient tool in the exploration of the moduli space of field theories of the type under consideration. Thus, clearly, determination of consistent realisations of field-theoretic symmetries in the higher geometry associated with such models and examining their amenability to gauging is a fundamental task in any attempt at elucidating their nature.

A distinguished place, in the context of symmetry analysis, among the field theories referred to above is occupied by σ-models with group manifolds and their homogeneous spaces as targets $F$. In the former case, the loop (and path) dynamics is captured by the Wess–Zumino–Witten (WZW) model of Ref. [Wit84], studied at great length and from a variety of angles since its inception, and in the latter case, it is modelled by a gauged variant of the same field theory along the lines of Ref. [KPSY89]. In both settings, (global) symmetries are naturally induced from left and right translations on the target Lie group $G$ and lead to a structurisation of the Hilbert space of the theory as a direct sum (finite for compact groups) of tensor products of complex-conjugate pairs of irreducible modules of a certain central extension of the loop group $L G$ of $G$, which ultimately paves the way to their complete resolution by a method due to Knizhnik and Zamolodchikov advanced in Ref. [KZ84]. Incidentally, this method has also served to establish a remarkable direct and constructive relation between these theories and the three-dimensional topological gauge field theory of Chern and Simons on a cylinder $\mathbb{R} \times \Sigma$ over the Riemann surface $\Sigma$ of the WZW model: The so-called conformal blocks whose sesquilinear combinations give correlation functions of the latter are identified with those sections of the bundle of states of the geometrically quantised Chern–Simons theory that are covariantly constant with respect to the Knizhnik–Zamolodchikov connection, cp Refs. [Wit89, GSW10, GSW13, Sus12], and also Ref. [Gaw99] for a modern review. A suitable combination of these results for a Lie group $G$ and its Lie subgroup $H$ then leads to explicit formulæ for the correlators of the σ-model on the homogeneous space $G/H$ realised as a WZW model with the subgroup $H$ of the global symmetry group $G$ gauged along the lines of Refs. [GK89b, KPSY89, GK89a]. The σ-model with the Lie-group target $G$ is defined in terms of canonical bi-invariant tensorial data of the group manifold, to wit, the Cartan–Killing metric and the canonical (Cartan) 3-cocycle on $G$. The latter geometrise in the form of a bundle (1-)gerbe reconstructed explicitly by Chatterjee in Ref. [Cha98] (for $G = SU(N), N \in \mathbb{N}^\ast$), by Gawędzki and Reis in Ref. [GR02] (independently for $G = SU(N), N \in \mathbb{N}^\ast$ and for their orbifolds with respect to subgroups of the centre), by Meinrenken in Ref. [Mei03] (for all compact simple 1-connected Lie groups) and, finally, by Gawędzki and Reis in Ref. [GR03] (for all compact simple connected but non-simply connected Lie groups). A gerbe-theoretic symmetry analysis of the WZW model aimed at elucidating the deeper nature of the gauging procedure in this model was initiated already in the papers [GR02, GR03] introducing higher geometry into the well-studied field- and string-theoretic context, and a nontrivial relation between its results and the categorial Seiberg–Witten data of the WZW model (in the so-called simple-current sector) was established in Ref. [RS09]. The theoretical programme was subsequently extended to orientifold WZW models through the introduction of the so-called Jandl structure on the bundle gerbe in Refs. [SSW07, GSW11b, GSW11a], and culminated
in the formulation of a universal gauge principle in Ref. [GSW10] (for the monophase $\sigma$-model) and Refs. [GSW13, Sus12] (for a $\sigma$-model in the presence of an arbitrary (conformal) defect). The principle emphasises the fundamental rôle of an $H$-equivariant structure in effecting the reduction $G \xrightarrow{\pi} G/H$ of the fibre of the configuration bundle of the $\sigma$-model and becomes a rich source of intuitions to be employed in the study of field theories with a less manifest symmetry content.

Another distinguished class of related field theories with a global symmetry built into their very definition consists of the Green–Schwarz-type super-$\sigma$-models describing the dynamics of the super-$p$-branes of superstring theory on homogeneous spaces $G/H$ of Lie supergroups $G$ (with respect to certain so-called vacuum isotropy groups $H \subset G$) interpreted as supersymmetry groups. These models were introduced in the pioneering works [GS84a, GSS84b, BST86, BST87, AETW87, MT98, DWW98, Cla91, AF08, FG12, DFG10]. They exhibit both global and local supersymmetry, the former being induced in an obvious fashion from left translations on the mother Lie supergroup $G$ and the latter, known as $\kappa$-symmetry and discovered by de Azcárraga and Lukierski in Refs. [AL83] and by Siegel in Refs. [Sie83, Sie84], having the peculiar nature of a purely Lie-superalgebraic (that is infinitesimal) invariance of the dynamics generated by right tangential shifts whose commutator algebra closes on extremals of the action functional, and that solely upon augmentation by generators of worldvolume (i.e., spacetime) diffeomorphisms. The physical motivation behind these models and their mathematical structure were reviewed and expanded upon at length in Refs. [Sus17, Sus18], where, moreover, an extensive bibliography on the subject can be found, and so below, we focus on their higher-geometric structure elaborated in those papers.\footnote{Cp also Ref. [SS14] for an alternative formal approach to the super-$\sigma$-models on the super-Minkowski space, based on the concept of a Lie $(p+1)$-superalgebra associated, in the spirit of Refs. [BC04, BH11, Hul11], with the Green–Schwarz (GS) super-$\sigma$-model and becomes a rich source of intuitions to be employed in the study of field theories with a less manifest symmetry content.}

The concept was revived and employed constructively in Ref. [Sus17] in which a full-fledged programme of geometrisation of the cohomological structures underlying the super-$\sigma$-models was laid out, and to which, consequently, we shall refer as Part I henceforth (the reference being inherited by section, proposition and theorem labels). It has led to a hands-on construction of novel (super)geometric correspondences between the Cartan–Eilenberg ($\mathrm{CaE}$) cohomology $\mathrm{CaE}^\bullet(G) = H^\bullet(\mathrm{ad})^G$ of the relevant Lie supergroup $G$ (the super-Minkowski supergroup in the former case and $SU(2,2|4)$ in the latter case) and the cohomology $H^\bullet(g, \mathbb{R})$ of its Lie superalgebra $g$ with values in the trivial module $\mathbb{R}$, in conjunction with the familiar interpretation of the second cohomology group $H^2(g, \mathbb{R})$ in terms of (equivalence classes of) central extensions of the Lie superalgebra. Representatives of classes $H^2(g, \mathbb{R})$ were induced from the GS super-$\sigma$-models and gave rise, upon integration of the associated central extensions of $g$ to Lie-supergroup extensions akin to those originally conceived by de Azcárraga et al. in Ref. [dAIPB00], to a hierarchy of surjective submersions with connective structure defining Murray’s geometrisation scheme. The construction has produced intrinsically supersymmetric super-$\sigma$-gerbes (for $p \in \{0,1,2\}$ as above) for the nontrivial $\mathrm{CaE}$ super-$\sigma$-models of the super-Minkowski space and a trivial super-$1$-gerbe for the trivial $\mathrm{CaE}$ super-$3$-cocycle of the Metsaev–Tseytlin super-$\sigma$-model on super-$\text{AdS}_5 \times \mathbb{S}^5$. The incompatibility of the latter with (the dual of) the Inönü–Wigner contraction that flattens super-$\text{AdS}_5 \times \mathbb{S}^5$ all the way to
the \((9 + 1)\)-dimensional super-Minkowski space and – crucially for the associated superstring theory – relates the Metsaev–Tseytlin super-\(\sigma\)-model to its GS counterpart on that flat superspace, together with the structural obstructions, determined in Ref. \([\text{Sus18a}]\), against the existence of corrected trivialisations of the Metsaev–Tseytlin super-3-cocycle compatible with the contraction, seem to indicate that an alternative definition of the super-\(\text{AdS}_5 \times S^5\) with the same body and the same flat limit should be sought (e.g., along the lines of Ref. \([\text{HK500}]\) on which a contractible super-1-gerbe (for the new super-3-cocycle) could be erected.

The study reported in the present paper arises at the confluence of the various field-theoretic, super-geometric and -algebraic ideas and constructions invoked above. It is to be regarded as a continuation of the geometrisation project advanced in Part I, and an indispensable consistency check of its hitherto results, undertaken with view to extending the project to more complex supergeometric backgrounds, with, in particular, a topologically nontrivial and curved body of the target supermanifold. Motivated by a firm understanding of the profound physical significance of the existence of higher-geometric realisations of global and local symmetries of a given field theory (with an underlying gerbe-theoretic structure), we conduct a thorough investigation of supersymmetry-equivariance \textit{sensu largo} of the model super-\(p\)-gerbes over the super-Minkowski space constructed in Sec. I.5, drawing useful insights and borrowing concrete Lie-algebroidal and symplectic tools from the well-developed gerbe theory of the WZW model along the way. More specifically, we reappraise the global supersymmetry of the GS super-\(\sigma\)-models from the vantage point of gerbe theory and identify the group-theoretic data of a consistent lift of that supersymmetry to an arbitrary CaE super-\(p\)-gerbe, including the GS super-\(p\)-gerbes of immediate interest. These data are then used to define the notion of a \textit{supersymmetric} \(H\)-equivariant structure on a CaE super-\(p\)-gerbe for any global-symmetry group \(H\) amenable to gauging, as exemplified by the Ad-equivariant structure on the GS super-\(p\)-gerbes of Sec. I.5 (for \(p \in \{0, 1\}\), for which this makes sense). Finally, we examine the all-important issue of a super-\(p\)-gerbe implementation of the peculiar tangential gauge supersymmetry of the GS super-\(\sigma\)-model by first transcribing the original field theory into an equivalent formulation of a purely topological nature (going back to the work \([\text{HP80}]\) of Hughes and Polchinski) and subsequently verifying the existence of a suitable (linearised) equivariant structure with respect to a geometrisation of the gauge supersymmetry obtained under the transcription on an extension of the GS super-\(p\)-gerbe associated with the topological Hughes–Polchinski model. The findings discussed hereunder provide solid evidence in favour of the proposal of Part I for the geometrisation of the CaE super-(\(p + 2\))-cocycles defining super-\(\sigma\)-models on homogeneous spaces of supersymmetry groups and give us sound and strong motivation for the ongoing further research into the higher-supersymmetric structures introduced \textit{ibidem}, to be reported in future publications.

The paper is organised as follows:

- In Section 2, we review (and generalise through Thm. 2.3) the various modes of description (algebraic, group-theoretic, symplectic, algebroidal, groupoidal and fibre bundle-theoretic) of symmetries of a \(\sigma\)-model with the WZ term as well as the quantitative measures of obstruction against their gauging (the small and large gauge anomalies), and recapitulate, after Ref. \([\text{GSW10} \text{GSW13}]\), their gerbe-theoretic incarnations, presenting – in particular – a full-fledged version of an equivariant structure on a \(p\)-gerbe (for \(p \in \{0, 1\}\)), restricted with hindsight) with respect to a global symmetry amenable to gauging.

- In Section 3, we recall the symmetry content of the two lowest-dimensional \(\sigma\)-models with a Lie-group target: the model of a geodesic flow on the group manifold and the WZW model, laying special emphasis on the identification of those global symmetries that admit a non-anomalous gauging.

- In Section 4, upon recalling the definitions of the generic CaE super-\(p\)-gerbes for \(p \in \{0, 1\}\) alongside their physically motivated instantiations over the super-Minkowski space, worked out in Part I, we systematically derive the notion of a supersymmetric \(H\)-equivariant structure on a CaE super-\(p\)-gerbe (for \(p\) as above) over a Lie supergroup in Sections 4.1 and 4.2 (Defs. 4.6 and 4.10) and, upon specialised of the general definitions to the setting in hand, prove the existence of supersymmetric Ad-equivariant structures on the GS super-\(p\)-gerbes over the super-Minkowski space of Part I (Thms. 4.8 and 4.12).

- In Section 5, we identify, in Thms. 5.1 and 5.2, the supergeometric (Lie supergroup-theoretic) circumstances under which the Green–Schwarz-(type) super-\(\sigma\)-model in the standard Nambu–Goto formulation with a homogeneous space of a Lie supergroup as a supertarget admits a dual Hughes–Polchinski formulation. The general results are illustrated on the example of the GS super-\(\sigma\)-model with the super-Minkowskian supertarget in Prop. 5.3.
• In Section 3 we first derive an explicit geometric implementation, in the form of the (translational) \( \kappa \)-symmetry superalgebras of Prop. 3.1 and 3.3, of the tangential gauge supersymmetry of the GS super-\( \sigma \)-model for the super-\( p \)-brane with \( p \in \{0, 1\} \) in the previously obtained dual Hughes–Polchinski formulation, and subsequently lift it, in conformity with the gauge principle of Section 3, to a full-fledged linearised equivariant structure on the extended Hughes–Polchinski \( p \)-gerbe of Defs. 6.4 and 6.5 in Thms. 6.7 and 6.8.

• In Section 4 we summarise our results and indicate directions of potential future research motivated by our findings.

• Appendices A–G contain proofs of the propositions and theorems stated in the main text of the paper.

2. Global symmetries of the \( \sigma \)-model & their gauging

The subject of our study is the monophase non-linear (super-)\( \sigma \)-model – a lagrangian field theory of smooth embeddings of a closed \((p + 1)\)-dimensional spacetime (worldvolume) \( \Omega_p \) in the fibre of the covariant configuration bundle \( \Omega_p \times M \rightarrow \Omega_p \) given by the (super)manifold (target space) \( M \), the latter being endowed with a symmetric bilinear form \( g \) on the tangent sheaf of \( M \) (a potentially degenerate metric tensor). The definition of the theory was reviewed in detail in Part I. Here, we recall merely those aspects of the definition and canonical description in the Graßmann-even setting that will prove essential in the symmetry analysis to follow.

The theory is determined by the principle of least action applied to the functional (the Dirac–Feynman amplitude)

\[
\mathcal{A}_{\text{DF}}^{(p)} : [\Omega_p, M] \rightarrow U(1) : x \mapsto e^{i S^{(p)}_{\sigma, \text{metr}}[x]} \cdot \text{Hol}_{G^{(p)}}[x]
\]

defined in terms of the metric action functional (in the Nambu–Goto formulation)

\[
S^{(p)}_{\sigma, \text{metr}}[x] = \int_{\Omega_p} \text{Vol}(\Omega_p) \sqrt{\text{det}(g^{(p+1)}(x^* g))}
\]

and the \((p + 1)\)-holonomy

\[
\text{Hol}_{G^{(p)}}[x] = \iota_p([x^* G^p])
\]
of a \( p \)-gerbe, given by the image of the isofill of its pullback \( x^* G^{(p)} \) to \( \Omega_p \) under the isomorphism

\[
\iota_p : \mathcal{W}^{p+2}(\Omega_p, 0) \xrightarrow{\sim} U(1)
\]

between the group \( \mathcal{W}^{p+2}(\Omega_p, 0) \) of isoclasses of flat \( p \)-gerbes over \( \Omega_p \) and \( U(1) \). The \( p \)-gerbe is a geometrisation, recalled at length in Part I, of a de Rham \((p + 2)\)-cocycle (the curvature of the \( p \)-gerbe)

\[
\text{curv}(G^{(p)}) \equiv \frac{H}{(p + 2)} \in Z^{p+2}_{\text{dR}}(M)
\]

with periods in

\[
\text{Per}(\frac{H}{(p + 2)}) \subset 2\pi \mathbb{Z}.
\]

It admits a sheaf-theoretic description whose data enter an explicit formula for the \((p + 1)\)-holonomy (written out, for \( p = 1 \), in Part I). The \((p + 1)\)-holonomy is an example of a real Cheeger–Simons differential character modulo \( 2\pi \mathbb{Z} \) of degree \( p + 1 \) in the sense of the definition given in Ref. [CJ85]. Indeed, writing

\[
h_{G^{(p)}}(x(\Omega_p)) \equiv \text{Hol}_{G^{(p)}}[x],
\]

we readily convince ourselves that the homomorphism

\[
h_{G^{(p)}} : \text{Hom}(Z_{p+1}(M), U(1)), \quad U(1) \equiv \mathbb{R}/2\pi \mathbb{Z}
\]
satisfies the basic property

\[
\forall \epsilon_{p+2} \in C_{p+2}(M) : h_{G^{(p)}}(\partial \epsilon_{p+2}) = \varepsilon_{G^{(p)}}(\epsilon_{p+2})
\]

for a \((p + 2)\)-cochain

\[
\varepsilon_{G^{(p)}} \in \text{Hom}(C_{p+2}(M), U(1))
\]
explicitly given by
\[
\varepsilon_{G^{(p)}} = e^{i \int \text{curv}(G^{(p)})} : C_{p+2}(M) \to \mathbb{R}/2\pi \mathbb{Z} : \epsilon_{p+2} \mapsto \exp \left( i \int_{\varepsilon_{p+2}} \text{curv}(G^{(p)}) \right).
\]

As a consequence of the above, we derive, for any (locally smooth) vector field \( V \) over \( M \) with a (local) flow
\[
\Phi_V \left( t, x(\cdot) \right) : ] - \varepsilon, \varepsilon [ \times \Omega_p \to M : (t, \sigma) \mapsto \Phi_V(t, x(\sigma)), \quad \varepsilon > 0,
\]
the useful identity
\[
\frac{1}{h_{G^{(p)}}(x(\Omega_p))} \frac{d}{dt} |_{t=0} h_{G^{(p)}} \circ \Phi_V(t, x(\Omega_p)) = i \int_{\Omega_p} x^* (V \cup \text{curv}(G^{(p)})),
\]
with the help of which we establish the Euler–Lagrange equations of the (super-)\( \sigma \)-model and identify its infinitesimal global symmetries, recalled in what follows.

The nature of the topological term in the Dirac–Feynman amplitude is also reflected in the construction, in the first-order formalism of Tulczyjew, Gawędzki, Kijowski and Szczyrba (cp. Refs. [Gaw72, Kij73, Kij74, KS76, Szc76, KT79]), of the (pre)symplectic space of states of the monophase theory and its infinitesimal global symmetries, recalled in what follows.

The definition of the relevant presymplectic form \( \Omega^{(p)}_\sigma \) on the space of states (of a single \( p \)-loop \( x \in S^p M \equiv [S^p, M] \) modelled on \( S^p_\sigma \) \( P^{(p)}_\sigma = \pi^{*}_{T^*S^p M} \), the space itself being coordinatised by Cauchy data \( \Psi^* P_{\varepsilon_p} \equiv (x^\mu, p_\mu) \) of extremals \( \Psi \) of \( \mathcal{A}_{\mathcal{A}M} \), supported on the model Cauchy section, or equitemporal slice, \( \varepsilon_p \equiv S^p \cap \Omega_p \) of the worldvolume, can be expressed as
\[
\left( P^{(p)}_\sigma, \Omega^{(p)}_\sigma \right) = \left( T^* \pi^{*}_{T^*S^p M} + \pi^*_{T^*S^p M} \int_{\varepsilon_p} \text{ev}^*_{p+2} \pi^*_{T^*S^p M} \right)
\]
in terms of the bundle projection \( \pi^{*}_{T^*S^p M} : T^* S^p M \to S^p M \), the canonical (action) 1-form \( \theta_{T^*S^p M} \) on \( T^* S^p M \) with the local presentation
\[
\theta_{T^*S^p M}[x, p] = \int_{\varepsilon_p} \text{Vol}(\varepsilon_p) p_\mu(\cdot) \delta x^\mu(\cdot),
\]
and the standard evaluation map
\[
\text{ev}_p : \varepsilon_p \times S^p M \to M : (\varphi, \gamma) \mapsto \gamma(\varphi).
\]

Above, the \( p_\mu \) is the component of the ‘kinetic’ momentum associated with the \( p \)-loop (local) position coordinate \( x^\mu \) which, in the lagrangian description, is given by the derivative of the metric term of the lagrangian density in the direction of the ‘velocity’ field \( \partial_\nu x^\mu \).

The 2-form gives rise to a Poisson bracket of hamiltonians on \( P^{(p)}_\sigma \), i.e. those smooth functions \( h \) on \( P^{(p)}_\sigma \) for which there exist smooth vector fields \( \nabla \), termed (globally) hamiltonian, satisfying the relation
\[
\nabla \cup \Omega^{(p)}_\sigma = -\partial h.
\]

Indeed, for any two such functionals \( h_A, A \in \{1, 2\} \), and the corresponding vector fields \( \nabla_A \), we may define a bracket
\[
\{ h_1, h_2 \}_{\Omega^{(p)}_\sigma}[\Psi^* P_{\varepsilon_p}] := \nabla_2 \cup \nabla_1 \cup \Omega^{(p)}_\sigma[\Psi^* P_{\varepsilon_p}],
\]
and the Jacobi identity follows automatically from the closedness of \( \Omega_\sigma \). The above construction of the (pre)symplectic structure on the space of states of the \( \sigma \)-model illuminates the rôle of the geometrisation \( G^{(p)} \) in bridging the gap between the classical theory and its quantisation. Indeed, there exists a cohomology map (the transgression map of Ref. [Gaw88])
\[
\tau_p : H^{p+1}(M, D(p+1)^*) \to H^1(S^p M, D(1)^*)
\]
between the (real) Deligne–Beilinson cohomology groups, associated with the respective Deligne complexes \( D(n)^* \), \( n \in \{1, p+1\} \) of sheaves (of locally smooth \( U(1) \)-valued maps \( U(1)_{\chi_n} \) and locally smooth \( k \)-forms \( \Omega^k(X_n), k \in \mathbb{N} \) over \( X_n \in \{X_1 = S^p M, X_{p+1} = M\}, \)
\[
\begin{align*}
D(n)^* : U(1)_{\chi_n} & \xrightarrow{\text{dlog}} \Omega^1(X_n) \xrightarrow{\text{d}} \Omega^2(X_n) \xrightarrow{\text{d}} \cdots \xrightarrow{\text{d}} \Omega^n(X_n),
\end{align*}
\]
of isomorphism classes of bundle \((n-1)\)-gerbes with connective structure over \(X_n\), that is \(p\)-gerbes with connective structure over \(M\) and principal \(C^*\)-bundles with (principal) connection over \(S^p M\), respectively, which explicitly assigns the isoclass of the so-called **transgression bundle**

\[
\mathbb{C}^* \xrightarrow{\mathcal{L}_{\mathcal{G}(p)}} \mathcal{L}_{\mathcal{G}(p)} \\
\pi_{\mathcal{L}_{\mathcal{G}(p)}}, \quad \mathcal{L}_{\mathcal{G}(p)} \in \tau_p([\mathcal{G}(p)])
\]

to the isoclass of the \(p\)-gerbe \(\mathcal{G}(p)\). The transgression bundle may subsequently be pulled back to the space of states \(\mathcal{P}_\sigma^{(p)} \equiv T^*S^p M\) of the \(\sigma\)-model along the bundle projection \(\pi_{T^*S^p M}\) and tensored with the trivial principal \(C^*\)-bundle \(C^* \xrightarrow{\mathcal{L}_{\mathcal{G}(p)}} \mathcal{P}_\sigma^{(p)}\)

\[
\mathbb{C}^* \xrightarrow{\mathcal{P}_\sigma^{(p)} \times \mathbb{C}^*} \mathcal{P}_\sigma^{(p)} \equiv \mathcal{I}_{\mathcal{G}(p)}\big[\mathcal{P}_\sigma^{(p)}\big]
\]

(2.4)

with the global principal \(C^*\)-connection 1-form

\[
\mathcal{A}[x, p, z] = \frac{dx}{dz} + \vartheta_{T^*S^p M}[x, p],
\]

whereby the **prequantum bundle** of the \(\sigma\)-model:

\[
\mathbb{C}^* \xrightarrow{\mathcal{P}_\sigma^{(p)} \times \mathbb{C}^*} \mathcal{P}_\sigma^{(p)} \equiv \mathcal{I}_{\mathcal{G}(p)}\big[\mathcal{P}_\sigma^{(p)}\big]
\]

(2.5)

is obtained, with a principal \(C^*\)-connection 1-form of curvature

\[
\text{curv}(\mathcal{L}_\sigma^{(p)}) \equiv \Omega_\sigma^{(p)}.
\]

Its suitably polarised sections are identified with wave functionals of the \(\sigma\)-model and compose the Hilbert space of the field theory,

\[
\mathcal{H}_\sigma = \Gamma_{\text{pol}}(\mathcal{L}_\sigma^{(p)}).
\]

A detailed discussion of the canonical description of the \(\sigma\)-model thus defined and its adaptations to the multi-phase setting can be found in Refs. [Sus11, Sus12].

The above considerations pave the way to a systematic analysis of symmetries of the (super-)\(\sigma\)-model. This has been well understood in the Graßmann-even setting, which we recapitulate hereunder with view to adaptation to the supergeometric setting of interest. Let us, first, deal with the infinitesimal description of continuous symmetries, both local (or gauge) and global (or rigid), in terms of the associated fundamental vector fields on the space of states. Vector fields on \(\mathcal{P}_\sigma^{(p)}\) whose flows realise the former span the kernel of \(\Omega_\sigma^{(p)}\), cp. Ref. [Gaw72]. Among those of the latter kind, we find canonical lifts \(\mathcal{K} \in \Gamma(\mathcal{T}\mathcal{P}_\sigma^{(p)})\), from \(M\) to \(T^*S^p M\), of fundamental vector fields \(\mathcal{K} \in \Gamma(\mathcal{T}M)\) associated with (left) automorphisms of the (typical) fibre \(M\) of the covariant configuration bundle \(\Omega_p \times M\) of the \(\sigma\)-model. As was demonstrated in Ref. [GSW10], they come from Killing vector fields \(\mathcal{K} \in \Gamma(\mathcal{T}M)\) of the target-space metric \(g\) (their flows preserve the metric term of the action functional) that satisfy the strong invariance condition

\[
\mathcal{K} \hookrightarrow \mathcal{H} = -d \kappa
\]

(2.6)

\(^5\text{Cp., e.g., Ref. [SIMP97], Sec. 4B].}

\(^6\text{The condition implies the weaker one: } \mathcal{L}_\mathcal{K} \mathcal{H} = 0, \text{ and the latter integrates to the invariance condition of the } p\text{-gerbe curvature with respect to the action of (the connected component of the unit of) the global-symmetry group of the } \sigma\text{-model.}\)
for some $\kappa \in \Omega^p(M)$. The above derives directly from Eq. (2.1) and ensures invariance of the topological factor in the Dirac–Feynman amplitude under the flow of $K$. Vector fields satisfying condition (2.6) will be called \textbf{generalised hamiltonian with respect to $\kappa$} (p + 2), by analogy with Eq. (2.2). They span a Lie subalgebra within the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of smooth vector fields on $M$ which we denote as

$$\mathfrak{g}_\sigma = \bigg\{ K \in \Gamma(TM) \mid \mathcal{L}_K \mathcal{g} = 0 \land \exists \kappa_{\sigma} \in \Omega^p(M) : K \lhd H = -d\kappa \bigg\} \mathbb{R}$$

in what follows. Together with the corresponding $p$-forms they compose \textbf{generalised hamiltonian sections} $(\mathcal{K}, \kappa)$ of the \textbf{generalised tangent bundle of type $(1,p)$}

$$\mathcal{E}^{(1,p)} = TM \oplus_M \bigwedge \mathcal{T}^*M \longrightarrow M : (v, \omega) \mapsto \pi_T(v)$$

over the target space. We shall denote the $\mathbb{R}$-linear subspace of generalised hamiltonian sections as

$$\mathfrak{H}^{(p)} = \bigg\{ \mathfrak{H} \equiv (\mathcal{K}, \kappa) \in \Gamma(\mathcal{E}^{(1,p)}) \mid \mathcal{L}_K \mathcal{g} = 0 \land \mathcal{K} \lhd H = -d\kappa \bigg\} \mathbb{R}.$$

It forms an algebra (over $\mathbb{R}$) with respect to the Vinogradov-type (skew) bracket, twisted (in the sense of Ševera–Weinstein, cp. Ref. [SW01]) by the $(p + 2)$-form $H \lhd (p + 2)$.

$$\mathcal{H} = \bigg[ (\mathcal{K}_1, \kappa_1), (\mathcal{K}_2, \kappa_2) \bigg] \mapsto \bigg[ [\mathcal{K}_1, \mathcal{K}_2], \mathcal{L}_{\mathcal{K}_1} \kappa_2 - \mathcal{L}_{\mathcal{K}_2} \kappa_1 \bigg]$$

(2.7)

a fact first noted in Ref. [AS03] (for $p = 1$) and subsequently generalised (to the polyphase setting) and exploited in Ref. [Sus12]. In the case $p = 1$, this structure may equivalently be understood as coming from the standard (i.e., untwisted) Courant bracket on Hitchin’s generalised tangent bundle (of type $(1,1)$) $E^{1,1}M$ twisted by the Čech–de Rham data of the gerbe geometrising $\mathcal{H}$. This interpretation of the Ševera–Weinstein twist was originally advanced in Ref. [Hit06] and elaborated in Ref. [Sus12].

The deeper field-theoretic meaning of the algebraic structure $\mathfrak{H}_\sigma$ in the (canonical) description of rigid symmetries of the $\sigma$-model and their gauging was discovered in Ref. [Sus12]. Here, the point of departure is the construction of the \textbf{Noether currents} of the theory. These are (spatial) densities of the standard \textbf{Noether hamiltonians} (or \textbf{charges}) $Q_{\mathcal{K}}$ of the symmetry,

$$Q_{\mathcal{K}}^{(p)} = \int_{\mathcal{E}^p} \text{Vol}(\mathcal{E}^p) \mathcal{J}_\mathcal{K}(\cdot) \equiv \nabla_{\mathcal{K}} \lhd \theta_{T^*S^pM} + \kappa_{\mathcal{K}}^{(p)}$$

(2.8)

defined in terms of the covariant lifts, to $P_{\mathcal{K}}^{(p)}$, of $\mathcal{K}$ and those of the generalised hamiltonian vector fields $\mathcal{K}$, denoted as $\nabla_{\mathcal{K}}$ and determined by the strong equivariance condition

$$\mathcal{L}_{\nabla_{\mathcal{K}}} \theta_{T^*S^pM} = 0,$$

which we may think of as the condition of preservation of the canonical connection 1-form $\theta_{T^*S^pM}$ on $TP^{(p)}$. The lifts take the explicit form

$$\nabla_{\kappa}^{(p)} = \int_{\mathcal{E}^p} \text{Vol}(\mathcal{E}^p) \left( \mathcal{K}^\mu x(\cdot) \right) \delta_{\mathcal{E}^p} \partial_\mu \mathcal{K}^\mu x(\cdot) + \kappa_{\mathcal{K}}^{(p)}$$

(2.9)

and satisfy the standard hamiltonian relations

$$\nabla_{\mathcal{K}} \lhd \Omega_{\mathcal{K}}^{(p)} = -\delta \mathcal{J}_{\mathcal{K}}^{(p)}.$$

The Noether current associated with the generalised hamiltonian section $\mathfrak{H}$ reads ($\mathcal{F}$ is the normalised tangent $p$-vector field on $S^p$)

$$\mathcal{J}_{\mathfrak{H}}^{(p)}(\cdot) = \mathcal{K}^\mu x(\cdot) P_\mu(\cdot) + (x_\ast \mathcal{K})(x(\cdot)) \kappa_{\mathcal{K}}^{(p)}.$$
For closed embedded worldvolumes $x(\Omega_p)$, the Noether charges furnish a faithful realisation of the symmetry algebra $\mathfrak{g}_\sigma$

$$\{Q^{(p)}_{\bar{R}_1}, Q^{(p)}_{\bar{R}_2}\}_{\Omega_p} = [\bar{K}_1, \bar{K}_2] \tag{2.10}$$

with $\omega$ as desired

$$-d(\bar{K}_2 \cup \bar{K}_1 \cup H) = [\bar{K}_1, \bar{K}_2] \cup H,$$

at least as long as we do not consider winding states that wrap around noncontractible $p$-cycles in $M$. In order to see the departure of the field-theoretic realisation of the symmetry algebra from its target-space model $\mathfrak{g}_\sigma$ directly, and in so doing understand the role of $\mathfrak{g}^{(p)}(\sigma)$, we should pass to the Poisson algebra of Noether currents. In the case $p = 1$, the Noether currents furnish an anomalous field-theoretic realisation of $\mathfrak{g}^{(p)}(\sigma)$, of the simple form $(t$ and $\phi, \phi'$ are respectively the time and space coordinate on $\Omega_1$)

$$\{J^{(1)}_{\bar{R}_1}(t, \phi), J^{(1)}_{\bar{R}_2}(t, \phi')\}_{\Omega_p} = J^{(1)}_{[\bar{R}_1, \bar{R}_2]}(t, x^{\alpha}(t, \phi, \phi')) \delta(-\phi - \phi') - 2 \{\bar{R}_1, \bar{R}_2\}(t, x^{\alpha}(t, \phi, \phi')) \delta'(-\phi - \phi'),$$

in which

$$\langle \cdot, \cdot \rangle : \Gamma(E^{1,1}M)^{\otimes 2} \to C^\infty(M, \mathbb{R}) : \left((V_1, \omega_1), (V_2, \omega_2)\right) \mapsto \frac{1}{2} \langle V_1 \cup \omega_2 + V_2 \cup \omega_1 \rangle$$

is a natural non-degenerate pairing on $\Gamma(E^{1,1}M)$.

**Remark 2.1.** It is sometimes convenient to transcribe the canonical description of the $\sigma$-model in Cartan’s Vielbein formalism in which the local coordinate 1-form fields $dx^\mu$, $\mu \in 1, \dim M$ are replaced by the Vielbein fields

$$e^a(x) = E^a_\mu(x) dx^\mu$$

carrying tangent-space indices $a \in 1, \dim M = 1, \dim M$. The action 1-form on $T^*S^pM$ now reads

$$\Phi_{\bar{R}}[x, P] = \int_{\Omega_p} \text{Vol}(\mathcal{E}_p) P_a(\cdot) e^a(\cdot),$$

with

$$P_a \equiv p_\mu E^{-1}_a^\mu(x).$$

It may happen, as it does in the supergeometric setting of immediate interest to us, that the transcription reveals a degeneracy of the metric tensor

$$g_{\langle \cdot, \cdot \rangle} = g_{a b} e^a \otimes e^b \equiv g_{a \mu \nu} dx^\mu \otimes dx^\nu,$$

that is $g_{a b} = 0$ for some $a, b \in 1, \dim M$. We then obtain a natural reduction of the (tangent-)momentum degrees of freedom as the components $P_a$ corresponding to the degeneracy directions drop out from $\Phi_{\bar{R}}|_{S^pM}$. Such a phenomenon was encountered and described in the canonical analysis of the GS super-$\sigma$-model on the super-Minkowski space in Part I, and the results of that analysis will be invoked below.

Passing, for a while, from the infinitesimal to the global level of realisation of $\sigma$-model symmetries, we encounter isometric group actions on the target space $M$ — indeed, in the case of discrete symmetries, that is all there is to it. Denote the action of the symmetry group $G_\sigma$ (for continuous symmetries, we take it to be a Lie group with the Lie algebra $\mathfrak{g}_\sigma$, acting smoothly) on the manifold $M$ as

$$\ell : G_\sigma \times M \to M.$$  

We (always) begin with global (or rigid) symmetries. These are neatly captured by families, indexed by the symmetry group $G_\sigma$, of gerbe 1-isomorphisms

$$\Phi_g : \Phi_g : f^*G^{(p)} \xrightarrow{\Phi_g} G^{(p)}, \quad g \in G_\sigma$$

that transgress, through a cohomological mechanism structurally analogous to the one described around (2.3) (and detailed in Ref. [Sus12]), to automorphisms of the prequantum bundle. The 1-isomorphisms

$^7$Note the purely classical nature of the anomaly in question.

$^8$By a mild abuse of notation, we use the same symbol for the functional $\Phi_{T^*S^pM}$ of the new coordinates $x, P$. 

can be regarded as geometrisations of the invariance condition (2.6). In the framework of the local field theory, we are led to demand that a global symmetry of a given field-theoretic model, which can be interpreted passively as invariance of the model under certain distinguished transformations of the reference system (in the space of internal degrees of freedom), and hence also as an equivalence between its distinguished classical configurations, be promoted to a local one, or gauged – this is, morally, the content of the universal gauge principle. The rôle of the symmetry algebra \( (\mathfrak{g}_\sigma^{(p)(H)}_{\mathcal{V}}) \) in the description of the gauging of the Lie group \( G_\sigma \) of continuous global symmetries was clarified by the author in Ref. [Sus12], cp. also Ref. [GSW13], for the case \( p = 1 \): The purely algebraic (infinitesimal) component of the obstruction against the gauging of \( G_\sigma \), termed the small gauge anomaly in Ref. [Sus12], is quantified by the departure of the \( C^\infty (M, \mathbb{R}) \)-linear span of \( \mathfrak{g}_\sigma^{(p)} \) equipped with the \( (3) \)-twisted Vinogradov(-type) bracket \( [\cdot, \cdot]^{(3)}_{\mathcal{V}} \) from a Lie algebroid. More specifically, pick up an arbitrary basis \( \{ \mathfrak{r}_A = (\mathcal{K}_A, \kappa_A) \}_{A \in \text{dim } G_\sigma}^{(1)} \) of \( \mathfrak{g}_\sigma^{(p)} \) determined by a basis \( \{ t_A \}_{A \in \text{dim } G_\sigma}^{(1)} \) of the Lie algebra \( \mathfrak{g}_\sigma \) as per

\[
\mathcal{K}_A (x) \equiv \frac{d}{dt} \big|_{t=0} \ell_{e^{-\kappa_A t}} (x), \quad x \in M,
\]

and consider

\[
\left( \mathfrak{g}_\sigma^{(p)} \right)_{C^\infty (M, \mathbb{R})} := \bigoplus_{A \in \text{dim } G_\sigma} C^\infty (M, \mathbb{R}) \mathfrak{r}_A.
\]

If we now restrict \( [\cdot, \cdot]^{(3)}_{\mathcal{V}} \) to \( \left( \mathfrak{g}_\sigma^{(p)} \right)_{C^\infty (M, \mathbb{R})} \), the bracket does not close on \( \left( \mathfrak{g}_\sigma^{(p)} \right)_{C^\infty (M, \mathbb{R})} \) and we generically find two anomalies (inherited from \( \Gamma (\mathcal{E}^{1,1} M) \)): the Jacobi anomaly quantified by the basis Jacobiators \( (A, B, C) \in 1, \text{dim } G_\sigma \)

\[
\text{Jac}(\mathfrak{r}_A, \mathfrak{r}_B, \mathfrak{r}_C) = \left( [\mathfrak{r}_A, \mathfrak{r}_B]^{(p+2)}_{\mathcal{V}}, \mathfrak{r}_C \right) + \left( [\mathfrak{r}_C, \mathfrak{r}_A]^{(p+2)}_{\mathcal{V}}, \mathfrak{r}_B \right) + \left( [\mathfrak{r}_B, \mathfrak{r}_C]^{(p+2)}_{\mathcal{V}}, \mathfrak{r}_A \right)
\]

and the Leibniz anomaly quantified by the basis expressions \( (f \in C^\infty (M, \mathbb{R})) \)

\[
\text{Leib}(\mathfrak{r}_A, \mathfrak{r}_B, f) = [\mathfrak{r}_A, f \mathfrak{r}_B]^{(p+2)}_{\mathcal{V}} - f [\mathfrak{r}_A, \mathfrak{r}_B]^{(p+2)}_{\mathcal{V}} - (\mathcal{K}_A \cdot df) \mathfrak{r}_B.
\]

These measure the departure of \( \left( \mathfrak{g}_\sigma^{(1)} \right)_{C^\infty (M, \mathbb{R})} \) with \( [\cdot, \cdot]^{(3)}_{\mathcal{V}} \) restricted to it and with the obvious anchor \( \text{pr}_1 : \bigoplus_{A \in \text{dim } G_\sigma} C^\infty (M, \mathbb{R}) \mathfrak{r}_A \rightarrow \Gamma (TM) \) from a Lie algebroid. We have the fundamental

**Theorem 2.2 (Sus12, Thms. 8.21 & 8.25).** The small gauge anomaly in the two-dimensional \( \sigma \)-model vanishes iff the \( H \)-twisted Vinogradov(-type) bracket \( [\cdot, \cdot]^{(3)}_{\mathcal{V}} \) closes on \( \left( \mathfrak{g}_\sigma^{(1)} \right)_{C^\infty (M, \mathbb{R})} \) and both anomalies: the Jacobi anomaly and the Leibniz anomaly vanish in \( \left( \mathfrak{g}_\sigma^{(1)} \right)_{C^\infty (M, \mathbb{R})} \), which happens iff the following conditions

\[
\mathcal{L}_{\mathcal{K}_A \kappa_B} = f_{AB}^{(1)} C^{\mathcal{K}_C} \quad \wedge \quad \mathcal{K}(A \cdot \kappa_B) = 0
\]

are satisfied for any \( A, B, C \in 1, \text{dim } G_\sigma \). The ensuing Lie algebroid is then canonically isomorphic with the action algebroid \( g_\sigma \mathcal{M} \), that is with the tangent Lie algebroid of the action groupoid

\[
(2.11) \quad G_\sigma \mathcal{M} : \quad G_\sigma \times M \xrightarrow{\ell} M.
\]

In fact, the algebroidal interpretation of the small gauge anomaly offered by the last theorem readily extends to the (lower and) higher dimensional \( \sigma \)-models, which we demonstrate next, with view to applying our intuition thus derived in the supergeometric setting. To this end, we now briefly review the logic behind the Universal Gauge Principle (for \( \sigma \)-models with the WZ term) first laid out in Ref. [GSW10], focusing on the topological factor in the Dirac–Feynman amplitude. The idea behind the Principle is to ‘descend’ the \( \sigma \)-model to the space \( M/G_\sigma \) of orbits of the action \( \ell \) of the symmetry group \( G_\sigma \). As elucidated in Ref. [GSW13] Sec. 9 and – from an alternative worldvolume perspective

\[\text{In the gauging procedure developed in Refs. [GSW10, 12], the metric on the target space, defining the metric term in the action functional, is assumed invariant, so that the metric term can be rendered gauge-invariant according to the standard minimal-coupling recipe.} \]
In Ref. [Sus12, Sec. 8], this can be achieved effectively by lifting the original \( \sigma \)-model to the product manifold \( P_{G_\sigma} \times M \), composed of the original target space \( M \) and of a principal \( G_\sigma \)-bundle
\[
\begin{array}{ccc}
G_\sigma & \longrightarrow & P_{G_\sigma} \\
\downarrow & & \downarrow \pi_{G_\sigma} \\
\Omega_p & \longrightarrow & P_{G_\sigma}/\Gamma 
\end{array}
\]
over the worldvolume, endowed with the defining right action
\[
r : P_{G_\sigma} \times G_\sigma \longrightarrow P_{G_\sigma}
\]
and with a principal \( G_\sigma \)-connection 1-form \( A \in \Omega^1(P_{G_\sigma} \otimes \mathfrak{g}_\sigma) \), and by extending its data (i.e., the metric \( g \) and the \( p \)-gerbe \( G^{(p)}_\sigma \) on \( M \)) ‘minimally’ with the use of \( A \) so that the extended structure descends to (that is canonically induces) a \( \sigma \)-model on the total space of the associated bundle
\[
\begin{array}{ccc}
M & \longrightarrow & \Omega_p \times M \equiv (P_{G_\sigma} \times M)/G_\sigma \\
\downarrow & & \downarrow \Omega_p 
\end{array}
\]
The latter acquires the interpretation (standard in field theory) of the covariant configuration bundle of the gauged \( \sigma \)-model – its global sections are identified with the lagrangean fields of the gauged \( \sigma \)-model, invariant – by construction – under arbitrary gauge transformations from the Lie–Fréchet group \( \Gamma(\text{Ad} P_{G_\sigma}) \) of global sections of the adjoint bundle \( \text{Ad} P_{G_\sigma} \equiv P_{G_\sigma} \times_{\text{Ad} G_\sigma} \), or the gauge group of the descended \( \sigma \)-model. The group acts on the fields in a manner modeled (fibrewise) on \( \ell \). In the present context, the word ‘minimally’ is to be understood as ‘through a tensorial correction that depends polynomially on \( A \). As the first step towards a full-fledged gauged \( \sigma \)-model, we consider the topologically trivial (or untwisted) gauging sector, which amounts to taking \( P_{G_\sigma} \) trivial,
\[
P_{G_\sigma} \equiv \Omega_p \times G_\sigma,
\]
equipped with the principal \( G_\sigma \)-connection 1-form
\[
A(\sigma, g) = (\text{id}_{\Omega^1(\ell g)} \otimes T_c \text{Ad}_{\sigma^{-1}})(A(\sigma)) + \theta_L(g)
\]
determined by the global primitive
\[
A \equiv A^A \otimes_{\mathbb{R}} t_A \in \Omega^1(\Omega_p) \otimes_{\mathbb{R}} \mathfrak{g}_\sigma
\]
of the curvature (2-form) of \( P_{G_\sigma} \). Above,
\[
(2.12) \quad \theta_L = \theta^A_L \otimes_{\mathbb{R}} t_A \in \Omega^1(G_\sigma) \otimes_{\mathbb{R}} \mathfrak{g}_\sigma
\]
is the standard \( \mathfrak{g}_\sigma \)-valued left-invariant Maurer–Cartan 1-form on \( G_\sigma \). In this simple situation, the ‘minimal’ extension alluded to above can be formulated entirely in terms of objects supported on the extended target space
\[
\tilde{M}_p := \Omega_p \times M.
\]
Indeed, the task boils down to finding \( \Omega^*(M) \)-valued tensors
\[
\left( \alpha_{A_1A_2...A_k} \right)_{A_1A_2,...,A_k \in \{\text{dim} \mathfrak{g}_\sigma\}} \in \Omega^{p+1-k}(M)^{\times k \text{dim} G_\sigma}, \quad k \in [0, p + 1]
\]
with the property that the \( p \)-holonomy, computed along extended embeddings
\[
\tilde{x} \equiv (\text{id}_{\Omega_p}, x) : \Omega_p \rightarrow \tilde{M}_p : \sigma \mapsto (\sigma, x(\sigma)),
\]
of the extension
\[
(2.13) \quad \tilde{G}^{(p)}_A := \text{pr}_2^* G^{(p)}_A \otimes \mathcal{I}_{\langle (p+1) \rangle A}
\]
\[\text{10}^{\text{In the case of the } p \text{-gerbe } G^{(p)}_{\sigma}, \text{ this means ‘by tensoring with a trivial gerbe whose global curving (the primitive of the curvature) is given by a tensor with a polynomial dependence upon } A.’}\]
of the pullback of the original gerbe \( \text{pr}_2^* G^{(p)} \) to \( \tilde{M}_p \) by the trivial \( p \)-gerbe \( \mathcal{I}_{\bar{\theta}_A} \) over \( \tilde{M}_p \) defined by the globally smooth (curving) \((p+1)\)-form

\[
\theta_{\bar{\mathcal{I}}_A} := (-1)^p \sum_{k=1}^{p+1} \frac{1}{k!} \text{pr}_2^* \alpha_{A_1 A_2 \ldots A_k} \wedge \text{pr}_1^* \left( A^{A_1} \wedge A^{A_2} \wedge \ldots \wedge A^{A_k} \right) \in \Omega^{p+1}(\tilde{M}_p)
\]

is invariant under simultaneous infinitesimal \((\varepsilon \geq 0)\) global gauge transformations of the extended embedding

\[
\widetilde{\varphi} \mapsto \left( \text{id}_{\Omega_p} \times \varepsilon \right) \circ \left( \text{id}_{\Omega_p} , \gamma_{\hat{\chi}} , x \right) =: \tilde{\gamma}_{\hat{\chi}} \varphi,
\]

with

\[
\tilde{\gamma}_{\hat{\chi}} \varphi(\sigma) \equiv (\sigma , \ell, \gamma_{\hat{\chi}}(\sigma)(x(\sigma)) , \gamma_{\hat{\chi}}),
\]

and of the gauge field

\[
A \mapsto \left( \text{id}_{\Omega^1(\Omega_p)} \otimes \mathcal{T}_c \text{Ad}_{\hat{\chi}} \right)(A) - \left( \gamma_{\hat{\chi}} \right)^* \theta_R =: \gamma_{\hat{\chi}} A,
\]
determined by an arbitrary smooth map

\[
\gamma_{\hat{\chi}} := e^{-t X^A(\cdot) t_A} : \Omega_p \to G_{\sigma}, \quad t \in [-\varepsilon, \varepsilon],
\]

and written in terms of the right-invariant counterpart

\[
\theta_R = \theta_R^\alpha \otimes_R t_A \in \Omega^1(\mathbb{G}_{\sigma}) \otimes R g_{\sigma}
\]

of \((\mathbb{G}_{\sigma})\). In analogy with the two-dimensional case, the obstruction against such invariance of (the WZ term of) the extended \((p+1)\)-dimensional \( \sigma \)-model shall be termed the small gauge anomaly. We have

**Theorem 2.3.** The small gauge anomaly in the \((p+1)\)-dimensional \( \sigma \)-model (with an invariant metric term) vanishes iff the following conditions are satisfied:

- the \( \Omega^*(M) \)-valued tensors \((\alpha_{A_1 A_2 \ldots A_k})_{A_1,A_2,\ldots,A_k \in \dim G_{\sigma}}\) are determined by the formula:

\[
\alpha_{A_1 A_2 \ldots A_k} := (-1)^{k(2p-k-1)} \frac{K_{A_1} J K_{A_2} J \cdots J K_{A_k} J K_{A_{k+1}}}{k!}, \quad k \in \mathbb{N},
\]

with the \( \kappa_A \equiv \alpha_A \) composing, together with the respective \( \kappa_A \), generalised Hamiltonian sections \( (\kappa_A, \kappa_B) \) of \( \mathcal{E}^{(1,p)} M \);

- the \( \mathcal{H} \) \((p+2)\)-twisted Vinogradov\(-\)type bracket \([[\gamma_H]]_{(p+2)} \) closes on \( \left( \Omega^{(p)}_{\sigma} \right)_{C^\infty(M,\mathbb{R})} \); and

- the Jacobi anomaly and the Leibniz anomaly vanish in \( \left( \Omega^{(p)}_{\sigma} \right)_{C^\infty(M,\mathbb{R})} \),

which happens iff the following conditions

\[
\mathcal{L}_{\mathcal{K}_A K_B} = \mathfrak{f}_{AB}^C \mathcal{K}_C \quad \wedge \quad \mathcal{K}_A \wedge \mathcal{K}_A = 0
\]

are satisfied by the basis generalised Hamiltonian sections

\[
(\mathcal{K}_A, \mathcal{K}_B)_{(p)} \in \mathbb{C}^{1, \dim G_{\sigma}}
\]

for any \( A, B, C \in \mathbb{C}^{1, \dim G_{\sigma}} \). The ensuing Lie algebroid is then canonically isomorphic with the action algebroid \( g_{\sigma} \times M \).

**Proof.** A proof is given in App. A. \( \square \)

It will be convenient to cast the conditions listed in the theorem above in the index-free notation with view to their direct transcription into the supergeometric setting. Thus, we obtain – for arbitrary \( X, Y \in g_{\sigma} \) – the equivalent conditions

\[
\mathcal{L}_{\mathcal{K}_A} \mathcal{K}_B = \mathcal{K}_{[X,Y]} \quad \wedge \quad \mathcal{K}_A \wedge \mathcal{K}_B + \mathcal{K}_Y \wedge \mathcal{K}_X = 0
\]

written for

\[
\mathcal{K}_X \wedge \mathcal{H} =: -d\mathcal{K}_X,
\]
The appearance of the action groupoid in the present context is by no means a coincidence – indeed, the groupoid of principal bundles with $G_\sigma \ltimes M$ as the structure groupoid (cp. Ref. [MM03]) was shown in Ref. [Sus12] to naturally quantify the data of the relevant gauged $\sigma$-model: the choice of the principal bundle $P_{G_\sigma} \rightarrow \Sigma$ with the structure group $G_\sigma$ and a choice of a global section of the associated bundle $P_{G_\sigma} \times_\ell M$, the latter section being identified with a lagrangean field of the gauged $\sigma$-model. Furthermore, it is over the nerve

$$d_0^{(m)}(g_m, g_{m-1}, \ldots, g_1, x) = (g_m, g_{m-1}, \ldots, g_2, \ell_{g_1}(x)),$$

$$d_0^{(m)}(g_m, g_{m-1}, \ldots, g_1, x) = (g_m, g_{m-1}, \ldots, g_1, x),$$

that the full-fledged gauging procedure was developed, for $p = 1$ but in a manner that admits straightforward and natural generalisations, in Refs. [GSW10, GSW13] (for both discrete and Lie symmetry groups) and ultimately justified, in its structural form proposed by the authors, in terms of a generalised worldsheet gauge-defect construction in Ref. [Sus12]. The procedure consists in replacing the extended $p$-gerbe \( \tilde G^{(p)}_A \) over $\tilde M_p$ with its analogon

$$\tilde G^{(p)}_A := pr_2^*G^{(p)} \otimes \mathcal{I}_{\ell_{\tilde M_p}},$$

over the product

$$P_{G_\sigma} \times M$$

of an arbitrary principal $G_\sigma$-bundle over the worldvolume, endowed with a principal $G_\sigma$-connection 1-form $A$. The latter enters the definition of the trivial $p$-gerbe $\mathcal{I}_{\ell_{\tilde M_p}}$ in exactly the same fashion as $A$ did in the previously considered case of $\mathcal{I}_{\ell_{\tilde M_p}}$. The manifold $P_{G_\sigma} \times M$ admits a free and proper action of the symmetry group $G_\sigma$,

$$\bar{\ell} : G_\sigma \times (P_{G_\sigma} \times M) \rightarrow P_{G_\sigma} \times M : (g, (p, m)) \mapsto (r_{g^{-1}}(p), \ell_g(m)).$$

and the gauged $\sigma$-model is defined as the $\sigma$-model with the target space $P_{G_\sigma} \times_\ell M$ with the metric descended from the minimally extended metric

$$g_A := pr_2^*g - pr_2^*g(K_A, \cdot) \otimes pr_1^*A^A - pr_1^*A^A \otimes pr_2^*g(K_A, \cdot) + pr_2^*g(K_A, K_B) pr_1^*(A^A \otimes A^B)$$

on $P_{G_\sigma} \times M$, readily proven $G_\sigma$-basic (with respect to $\bar{\ell}$), and a $p$-gerbe descended from $\tilde G^{(p)}_A$, provided the latter does descend to the smooth quotient. The necessary and sufficient condition for this to work is the existence of a $G_\sigma$-equivariant structure on the $p$-gerbe $G^{(p)}_A$ of the $\sigma$-model. It deserves to be emphasised that the vanishing of the small gauge anomaly is central to the existence of the $G_\sigma$-equivariant structure independently of the gauging, the latter being – after all – a field-theoretic concept. Indeed, the structure builds upon the existence of a $p$-gerbe isomorphism

$$\theta_{\bar{\ell} \cdot \theta_L} : d_0^{(1)}*G^{(p)} \rightarrow d_0^{(1)}*G^{(p)} \otimes \mathcal{I}_{\ell_{\tilde M_p}},$$

in which $\mathcal{I}_{\ell_{\tilde M_p}}$ is the trivial $p$-gerbe over $G_\sigma \times M$ with the global curving

$$\theta_{\bar{\ell} \cdot \theta_L} \equiv \sum_{k=1}^{p+1} (-1)^{p-k} pr_2^*g_A_{A_1 A_2 \ldots A_{k}} \wedge pr_1^*(\theta^{A_1}_L \wedge \theta^{A_2}_L \wedge \ldots \wedge \theta^{A_k}_L) \in \Omega^{p+1}(G_\sigma \times M),$$
with the $\alpha_{A_1A_2\ldots A_k}$ subject to the constraints listed in Thm 2.3. We shall next recall the complete definition of the structure for the two cases: $p \in \{0, 1\}$ to be translated into the supersymmetric setting in what follows.

In the case of a 0-gerbe $G^{(0)}$, or a principal $C^{\ast}$-bundle with (principal) connection, an equivariant structure is a connection-preserving isomorphism

$$\Upsilon_0 : d_1^{(1)\ast} G^{(0)} \cong d_0^{(1)\ast} G^{(0)} \otimes I_{\varrho_{-\theta_{\ell}}(1)}$$

of principal $C^{\ast}$-bundles over the arrow manifold $G_{\sigma} \times M$ of the action groupoid (the second factor in the tensor product is a trivial principal $C^{\ast}$-bundle with the global base component of the principal $C^{\ast}$-connection indicated, $cp$. Eqs. (2.3) and (2.4)), written in terms of the distinguished 1-form (trivially equal to zero in the discrete case)

$$(2.18) \quad \varrho_{-\theta_{\ell}} = -\text{pr}_2^* \kappa_A \text{pr}_1^* \theta_L^A.$$

The isomorphism is further required to satisfy the coherence condition

$$\left( d_0^{(2)\ast} \Upsilon_0 \circ \text{id}_{I_{\varrho_{-\theta_{\ell}}(2)}} \right) \circ d_2^{(2)\ast} \Upsilon_0 = d_1^{(2)\ast} \Upsilon_0$$

over $G_{\sigma}^2 \times M$.

For a 1-gerbe $G^{(1)}$, we have a 1-isomorphism

$$\Upsilon_1 : d_1^{(1)\ast} G^{(1)} \cong d_0^{(1)\ast} G^{(1)} \otimes I_{\varrho_{-\theta_{\ell}}(2)}$$

of 1-gerbes over $G_{\sigma} \times M$, written in terms of the distinguished 2-form (trivially equal to zero in the discrete case)

$$(2.19) \quad \varrho_{-\theta_{\ell}} = \text{pr}_2^* \kappa_A \wedge \text{pr}_1^* \theta_L^1 - \frac{1}{2} \text{pr}_2^* (\kappa_A \cup \kappa_B) \text{pr}_1^* (\theta_L^A \wedge \theta_L^B),$$

alongside a 2-isomorphism

$$\left( d_1^{(1)\ast} \circ d_1^{(2)\ast} \right) G^{(1)} \cong \left( d_1^{(1)\ast} \circ d_0^{(2)\ast} \right) G^{(1)} \otimes I_{\varrho_{-\theta_{\ell}}(2)}$$

between the 1-isomorphisms over $G_{\sigma}^2 \times M$, satisfying, over $G_{\sigma}^3 \times M$, the coherence condition

$$(2.20) \quad d_1^{(1)\ast} \gamma_1 \circ \text{id}_{(d_2^{(2)\ast} d_1^{(1)\ast}) \Upsilon_1} \circ d_2^{(3)\ast} \gamma_1 \in \left( d_2^{(3)\ast} \gamma_1 \otimes \text{id}_{(d_2^{(2)\ast} d_1^{(1)\ast}) \Upsilon_1} \right) \circ \text{id}_{(d_2^{(3)\ast} d_1^{(1)\ast}) \Upsilon_1} \in \text{Aut}(G_{\sigma}^3 \times M).$$

As stated earlier, the $G_{\sigma}$-equivariant structure ensures descent of the extended $p$-gerbe $\tilde{G}^{(p)}_{\sigma}$ to the fibre $P_{G_{\sigma}} \times M$ of the covariant configuration bundle of the gauged $\sigma$-model. An alternative, more directly physical interpretation of the $G_{\sigma}$-equivariant structure was given in Ref. [Gau12]. According to the reasoning detailed there, the equivariant structure provides the necessary and sufficient data for an arbitrary topological gauge-defect embedded in the worldvolume that implements the gauge symmetry. Whenever the action $\ell$ is free and proper, so that the orbit space $M / G_{\sigma}$ carries the structure of a manifold, all this implies that the $\sigma$-model descends to the orbit manifold $M / G_{\sigma}$ in that it determines – through integration of the non-dynamical gauge field $\mathcal{A}$ – a $\sigma$-model with the latter as the target space with a metric and a $p$-gerbe over it, and equivalence classes of such descended $\sigma$-models are essentially enumerated by inequivalent $G_{\sigma}$-equivariant structures on $G^{(p)}$. If $M / G_{\sigma}$ is not a manifold, on the other hand, it makes sense to regard the gauged $\sigma$-model as the definition of the induced $p$-loop mechanics on the space $M / G_{\sigma}$ – indeed, it is defined over a manifold directly related to the homotopy ($G_{\sigma}$-)quotient of $M$. 

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3. Symmetry analysis in the WZW $\sigma$-model on a compact Lie group

Geometries with a particularly rich and highly structured symmetry content are compact simple Lie groups. Let $G$ be such a group, which we, furthermore, assume to be 1-connected\(^{11}\) and let $\mathfrak{g} \equiv T_e G$ be its Lie algebra on which we fix a (negative definite) Killing form

$$\kappa_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R},$$

unique up to a normalisation constant $k \in \mathbb{R}^*$,

$$\kappa_\mathfrak{g}(X,Y) = k \text{tr}_\mathfrak{g}(\text{ad}_X \circ \text{ad}_Y), \quad X,Y \in \mathfrak{g}$$

(the normalisation of the trace over $\mathfrak{g}$ is determined by declaring the value of $\text{tr}_\mathfrak{g}(\text{ad} \circ \text{ad})$ on the long roots of $\mathfrak{g}$). In what follows, we fix a pseudo-orthonormal basis $\{t_A\}_{A \in \text{dim}_\mathfrak{g}}$ in $\mathfrak{g}$, satisfying the structure equations

$$[t_A,t_B] = f_{AB}^C t_C,$$

such that

$$f_{BC}^D f_{AD}^C = \text{tr}_\mathfrak{g}(\text{ad}_{t_A} \circ \text{ad}_{t_B}) = -\frac{1}{2} \delta_{AB}.$$ Using the Killing form, we may raise and lower Lie-algebra indices, and we find, for

$$f_{ABC} := -\frac{1}{2} \delta_{DC} f_{AB}^D,$$

the skew-symmetry property:

$$f_{ABC} \equiv f_{[ABC]}.$$

We then readily prove the implication, valid for any $X^C \in \mathbb{R}$,

$$\forall_{A,B \in \text{dim}_\mathfrak{g}} : f_{ABC} X^C = 0 \implies \forall_{C \in \text{dim}_\mathfrak{g}} : X^C = 0.$$

When endowed with the Cartan–Killing metric

$$g^{(k)}_{\text{CK}} := \kappa_\mathfrak{g} \circ (\theta_L \otimes \theta_L) = -\frac{1}{2} \delta_{AB} \theta^A_L \otimes \theta^B_L : \text{TG} \otimes_{G,R} \text{TG} \longrightarrow \mathbb{R}$$

$$\equiv \kappa_\mathfrak{g} \circ (\theta_R \otimes \theta_R) = -\frac{1}{2} \delta_{AB} \theta^A_R \otimes \theta^B_R,$$

which we write out, equivalently, in terms of the left- and right-invariant $\mathfrak{g}$-valued Maurer–Cartan 1-forms (2.12) and (2.14), the group manifold becomes a target space of a family of $\sigma$-models for embeddings

$$g \in [\Omega_p,G]$$

in which the metric term is traditionally expressed, in the so-called Polyakov formulation (and in the conformal gauge on the Lorentzian worldvolume $(\Omega_p,\eta)$, $\eta = \text{diag}(-1,1,1,\ldots,1)$ with (local) coordinates $(\sigma^a)_{a \in \partial \Omega_p}$ for which $\partial_a \equiv \frac{\partial}{\partial \sigma^a}$ are the coordinate derivations), as

$$S_{P,\text{met}}^{(p,k)} = \int_{\Omega_p} \text{Vol}(\Omega_p) L_{P,\text{met}}^{(p,k)}(g,g^{-1}\partial g)$$

in terms of the metric lagrangean (density)

$$L_{P,\text{met}}^{(p,k)}(g,g^{-1}\partial g) = \frac{k}{4} \eta^{ab} g^* \kappa_\mathfrak{g}^{(k)}((\partial_a \otimes \partial_b) g) \equiv -\frac{k}{2} \eta^{ab} \delta_{AB} \left((\partial_a \otimes g^* \theta^A_L) (\partial_b \otimes g^* \theta^B_L) \right),$$

the latter containing a $p$-dependent normalisation constant $\lambda_p \in \mathbb{R}$. With the standard definition of the (chiral) kinetic momentum,

$$P_H = \frac{\partial L_{P,\text{met}}^{(p,k)}}{\partial g^* \theta^A_L} = \frac{k}{4} \lambda_p \delta_{AB} \left((\partial_0 \otimes g^* \theta^A_L) \right), \quad \text{H} \in \{L,R\},$$

we find the action 1-form

$$\theta_{T_S^{\text{met}}}[g,P^H] = \int_{\mathcal{C}_p} \text{Vol}(\mathcal{C}_p) P_A^H(\cdot) g^* \theta^A_L(\cdot) \equiv \int_{\mathcal{C}_p} \text{Vol}(\mathcal{C}_p) P_A^R(\cdot) g^* \theta^A_R(\cdot).$$

\(^{11}\)The non-simply connected ones can be viewed as orbifolds of their simply connected counterparts with respect to the natural action of (a subgroup of) the centre $Z \subset G$. On the level of the corresponding $\sigma$-model, the passage from the simply connected target to its orbifold is effected by the gauging of the discrete global-symmetry group $Z$, cp Refs. GR02, GR03, GSW08, GSW11a, GSW10.
3.1. **The geodesic flow on the group manifold.** For $p = 0$ (and $\lambda_0 = 1$), we obtain the $\sigma$-model of the geodesic flow on the Lie group,

$$A_{DF}^{(0,k),\text{geod}} : [\Omega_0, G] \rightarrow U(1) : g \mapsto e^{iS_{\text{g-metr}}^{(0,k)}[g]} ,$$
classically driven by the Euler–Lagrange equations

$$\partial_0 (\partial_0 \lrcorner g^* \theta_L) = 0$$
and analysed at length in Ref. [Gaw99, Sec. 2]. The Hilbert space of the theory is the space $L^2(G, d\mu_H)$ of functions on $G$ square-integrable with respect to a suitably normalised Haar measure $d\mu_H$. It decomposes into a direct sum of tensor products of (representatives of isoclasses of) conjugate modules $\mathcal{V}_\lambda$ and $\mathcal{V}_{\overline{\lambda}}$ labelled by dominant highest weights (DHW) of $\mathfrak{g}$,

$$\mathcal{H}_\sigma^{(0,k),\text{geod}} \equiv L(G, d\mu_H) = \bigoplus_{\lambda \in \text{DHW}(\mathfrak{g})} \mathcal{V}_\lambda \otimes \mathcal{V}_{\overline{\lambda}} .$$

The structure reflects the non-anomalous bi-chiral (left-right) symmetry of the theory that follows from the bi-invariance of the Cartan–Killing metric: We have two commuting actions of $G$ on $\mathcal{H}_\sigma^{(0,k),\text{geod}}$ induced by left and right translations on the group.

Let us consider a topological correction to the geodesic flow that occurs when a 2-form field is turned on and the material point is endowed with a topological charge to which the field couples as described earlier. While there is no ‘canonical’ choice of a 2-form $H$ on $G$, we might take the requirement of preservation of the full bi-chiral global symmetry of the geodesic flow under the topological perturbation as guidance. The latter field should then be assumed bi-invariant (the coefficients $h_{AB} = -h_{BA}$, $A, B \in \mathfrak{g}$ are constant $G$-invariant tensors)

$$H = h_{AB} \theta^A_L \wedge \theta^B_L , \quad (T_c \text{Ad}_g)^C_A (T_c \text{Ad}_g)^D_B h_{CD} = h_{AB} , \quad g \in G ,$$

where

$$T_c \text{Ad}_g(t_A) =: (T_c \text{Ad}_g)^B_A t_B .$$

Indeed, the infinitesimal version of the chiral invariance conditions, equivalent to the existence of potentials $\kappa^{(0)}_A \in C^\infty(G, \mathbb{R})$, $H \in \{L, R\}$ for the 1-forms

$$H_A \lrcorner H = -dh^{(0)}_A ,$$

written in terms of the left- ($L_A$) and right-invariant ($R_A$) vector fields on $G$, implies

$$\mathcal{L}_{H_A} H = 0 , \quad H_A \in \{L_A, R_A\} , \quad A \in \overline{1, \dim \mathfrak{g}}$$

owing to the assumed closedness of $H$. The entire mathematical theory necessary to discuss such corrections is contained in the seminal paper [CE48] by Chevalley and Eilenberg, to which we refer the Reader for proofs of the theorems invoked hereunder.

First of all, let us note that all bi-invariant 2-forms are allowed here as all of them are closed. Furthermore, since the Cartan–Eilenberg cohomology $H^*_{\text{CE}}(G, \mathbb{R})^G \equiv \text{Ce}^*\text{e}^* (G, \mathbb{R})$ of left-invariant forms on the Lie group $G$ is isomorphic with the Chevalley–Eilenberg cohomology $\text{Ce}^*\text{e}^* (\mathfrak{g}, \mathbb{R})$ of its Lie algebra with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$,

$$\text{Ce}^*\text{e}^* (G, \mathbb{R}) \equiv \text{Ce}^*\text{e}^* (\mathfrak{g}, \mathbb{R})$$

and $\text{Ce}^2(\mathfrak{g}, \mathbb{R}) = 0$ for $G$ simple by the Second Whithead Lemma, we may write

$$H = dB^{(1)}_{(2)}$$
for some left-invariant 1-form

$$B = b_A \theta^A_L \in \Omega^1(G)^{(L)} ,$$

with constant coefficients $b_A \in \mathbb{R}$, $A \in \overline{1, \dim \mathfrak{g}}$ that satisfy the identities

$$f^{(0)}_{AB} h_{CD} = -2h_{AB} , \quad A, B \in \overline{1, \dim \mathfrak{g}} .$$

Taking into account (the infinitesimal form of) the right invariance of $H$,

$$f^{(D)}_{AB} h_{CD} = 0 , \quad A, B, C \in \overline{1, \dim \mathfrak{g}} ,$$

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we obtain the constraints
\[ f_{AB}^D f_{CD}^E b_{DE} = 0, \quad A, B, C \in 1, \text{dim } \mathfrak{g}. \]
In the light of Eq. (3.1), these imply
\[ f_{CD}^E b_{DE} = 0, \quad C, D \in 1, \text{dim } \mathfrak{g}, \]
and so also
\[ b_A = 0, \quad A \in 1, \text{dim } \mathfrak{g}, \]
whence, in particular,
\[ H_{(2)} \equiv 0. \]
Note also that a non-invariant correction to the (vanishing) primitive \( B \) of the latter 2-form would necessarily have to be closed, and therefore, in consequence of the implication
\[ G \text{ compact} \quad \Longrightarrow \quad H^*_{\text{dR}}(G, \mathbb{R}) \cong \text{CaE}^*(G), \]
of the isomorphism (3.3), and of the First Whitehead Lemma: \( \text{CE}^1(\mathfrak{g}, \mathbb{R}) = 0 \) for \( G \) simple, altogether resulting in
\[ H^1_{\text{dR}}(G, \mathbb{R}) = 0, \]
that correction would actually be exact, so that – by the end of the day – we conclude that there are no non-trivial bi-chiral topological corrections to the \( \sigma \)-model for the geodesic flow on a compact simple 1-connected Lie group \( G \).

The (pre)symplectic form of the bi-chiral \( \sigma \)-model reads
\[ \Omega^{(0)}_{\sigma} [g, \mathfrak{P}^B] = \delta \left( P^B_{\mathfrak{H}} \theta^A_{\mathfrak{H}}(g) \right), \]
and the corresponding covariant lifts of the left- and right-invariant vector fields on \( G \) to \( P^{(0)}_{\mathfrak{P}} \) read
\[ \mathcal{L}_A [g, \mathfrak{P}] = L_A(g) + f_{AB}^C P^L_{C} \frac{\delta}{\delta P^B}, \]
and
\[ \mathcal{R}_A [g, \mathfrak{P}] = R_A(g) - f_{AB}^C P^R_{C} \frac{\delta}{\delta P^B}, \]
respectively. The associated Noether currents/charges
\[ J^{(0)}_{HA} (\cdot) = P^B_{\mathfrak{H}} (\cdot) \equiv Q_{HA} [g, \mathfrak{P}] \]
furnish a non-anomalous realisation of \( \mathfrak{g}_e \), that lifts to the Hilbert space of the theory (as stated earlier) and is amenable to gauging, cp Ref. [FG94]. As a result, also the adjoint symmetry \( \text{Ad}(G) \subset G \times G \) (with elements \((g, g^{-1}) \), \( g \in G \)) can be gauged through the standard minimal-coupling construction (cp ib.).

3.2. The Wess–Zumino–Witten loop dynamics on the group manifold. For \( p = 1 \) and \( \lambda_1 = \frac{1}{24\pi} \), the choice of the manifestly bi-chiral Cartan 3-cocycle
\[ H_k \equiv \frac{1}{24\pi} \kappa_{g} \circ \left( [\cdot, \cdot] \otimes \text{id}_{\mathfrak{g}} \right) \circ \left( \theta_L \wedge \theta_L \wedge \theta_L \right) = \frac{k}{24\pi} f_{ABC} \theta^A_L \wedge \theta^B_L \wedge \theta^C_L, \]
with the Lie bracket viewed, naturally, as a mapping \( \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} \), defines the so-called Wess–Zumino–Witten \( \sigma \)-model and puts us in the much-studied context of rational conformal field theory. The 3-cocycle with \( k = 1 \) generates the third cohomology group \( H^3(G, 2\pi \mathbb{Z}) \subset H^3(G, \mathbb{R}) \) of the compact simple 1-connected Lie group \( G \), and so for any integer value of the level \( k \in \mathbb{Z} \), we obtain a geometrisation of the corresponding 3-cocycle \( H_k \) in the form of the \( k \)-th (Deligne-)tensor power of the so-called
\[ \text{basic (1-)gerbe } \mathcal{G}_{b} \equiv \mathcal{G}^{(1)}_{k=1}, \text{ i.e., we have (up to a 1-isomorphism)} \]
\[ \mathcal{G}^{(1)}_k \equiv \mathcal{G}^b_k. \]
Basic gerbes for all compact simple 1-connected Lie groups were first explicitly constructed for \( G = \text{SU}(N), \ N \in \mathbb{N}^* \) by Gawędzki and Reis in Ref. [GR03], and subsequently for arbitrary \( G \) by Meinrenken in Ref. [Mei03]. The ensuing bi-chiral \( \sigma \)-model (at level \( k \))
\[ \mathcal{A}^{(1,k)}_{DF} : [\Omega_1, G] \to U(1) : g \mapsto e^{i S_{\text{P. metr}}[g]} \cdot \text{Hol}^{(1)}_{\mathfrak{g}^k} (g) \equiv e^{i S_{\text{P. metr}}[g]} \cdot \text{Hol}^k_{\mathfrak{g}^k} (g), \]
with its Euler–Lagrange equations
\[ \partial_t (\partial_\tau g^* \theta_L) = 0, \]
or – equivalently –
\[ \partial_t (\partial_\tau g^* \theta_R) = 0, \]
expressed in terms of the partial derivatives \( \partial_\pm \equiv \frac{\partial}{\partial \tau^\pm} \) along the light-cone coordinates
\[ \sigma^\pm = \sigma^0 \pm \sigma^1, \]
has a classical space of (1-loop) states
\[ P^{(1)}_\sigma = T^* LG \triangleright (g, P^H) \]
equipped with the presymplectic structure
\[ \Omega^{(1)}_\sigma[g, P^H] = \int_{S^1} \text{Vol}(S^1) \left[ \delta\left(P^L_L (\cdot) g^* \theta^A_L (\cdot) \right) + \frac{k}{8\pi} f_{ABC} \left( \nabla g^* \theta^B_L (\cdot) \right) \right], \]

that admits a Hamiltonian realisation of left and right translations on the group manifold through Noether charges
\[ Q_{\Lambda_A}[g, P^L] = \int_{S^1} \text{Vol}(S^1) \left( P^L_A - \frac{k}{8\pi} \delta_{AB} \nabla g^* \theta^B_L (\cdot) \right), \quad Q_{\Pi_A}[g, P^R] = \int_{S^1} \text{Vol}(S^1) \left( P^R_A + \frac{k}{8\pi} \delta_{AB} \nabla g^* \theta^B_R (\cdot) \right). \]

The latter have the general structure \((2.8)\), with the covariant lifts of the left- and right-invariant vector fields on \( G \) given by
\[ \hat{\Pi}_A[g, P^H] = \int_{S^1} \text{Vol}(S^1) \left( H_A (g) - \epsilon_H \frac{f_{ABC}}{8\pi} P^H_C (\cdot) \right), \quad \epsilon_H = \begin{cases} -1 & \text{if } H = L, \\ +1 & \text{if } H = R. \end{cases} \]

and with the relevant generalised Hamiltonian sections of \( E^{1,1} G \) of the form
\[ \Sigma_A \equiv \left( H_A, \kappa_A^H \right) = \left( H_A, \frac{k}{8\pi} \epsilon_H \delta_{AB} \theta^B_H \right), \quad \Sigma \in \{ \Sigma, \Pi \}. \]

Upon recalling Eq. \( (3.2) \), we readily recognise the currents (in the standard normalisation):
\[ J_{\lambda_A}^{(1)} = -k \delta_{AB} \partial_\tau g^* \theta^B_L, \quad J_{\Pi_A}^{(1)} = -k \delta_{AB} \partial_\tau g^* \theta^B_R, \]
corresponding to the two chiral covariant lifts as the currents of the mutually commuting chiral (centrally extended) loop-group symmetries of the WZW \( \sigma \)-model
\[ LG \times [\Omega, G] \times LG \rightarrow [\Omega, G] : \left( h_+, g, h_- \right) \mapsto \left( h_+ \circ \pi_+ \cdot g \cdot h_- \circ \pi_- \right), \]
written for \( \pi_\pm (\sigma^0, \sigma^1) = \sigma^\pm \). It is customary to work with the \( \sigma \)-valued chiral currents
\[ J_{\lambda_A}^{(1)} = J_{\Pi_A}^{(1)} t_A, \quad J_{\lambda_A}^{(1)} = J_{\Pi_A}^{(1)} t_A. \]

These lift to quantum symmetries and decompose the Hilbert space of the \( \sigma \)-model into modules of the central extension \( \mathfrak{g} \) of the loop algebra \( \mathfrak{l} \) of the Lie algebra \( \mathfrak{g} \),
\[ 0 \rightarrow \mathbb{R} \rightarrow \mathfrak{g} \rightarrow \mathfrak{l} \mathfrak{g} \rightarrow 0, \]
generated by the Laurent modes of either loop-group symmetry current, and by an extra central generator \( K \) (the image of \( 1 \in \mathbb{R} \) in the above short exact sequence). Thus, the Hilbert space takes the form of the direct sum
\[ \mathcal{H}_{(1; k)}^{(1)} = \bigoplus_{\lambda \in \text{IHW}_{\lambda}(\mathfrak{g})} \hat{V}(\lambda, k) \otimes \overline{V}(\lambda, k) \]
of (Hilbert-space completion of) tensor products of the irreducible chiral modules \( \hat{V}(\lambda, k) \) of \( \mathfrak{g} \) with their complex conjugates, labelled by the so-called integrable highest weights (IHW) of the affine Kac–Moody algebra \( \mathfrak{g} \) at level \( k \). The weights of interest are those associated with the irreducible highest-weight representations of the horizontal algebra \( \mathfrak{g} \subset \mathfrak{g} \) of the highest weight \( \lambda \) subject to the integrability constraint
\[ -\kappa_\theta (\theta, \lambda) \leq k^2 \]
in which \( \theta \) is the highest root of \( \mathfrak{g} \) (that is a root \( \theta \) such that for any positive root \( \alpha \) of \( \mathfrak{g} \) the vector \( \theta + \alpha \) is not a root). The central generator \( K \) acts as \( k \text{id}_{\mathcal{H}^{(1; k)}} \) in the field-theoretic setting in hand.
Implicit in the above structure of the Hilbert space is the existence of a non-anomalous conformal symmetry realised by two chiral copies of the Virasoro algebra \(\text{Vir} \) – a central extension of the Witt algebra \(\text{Witt} \) of vector fields on the unit circle,

\[
0 \longrightarrow \mathbb{R} \longrightarrow \text{Vir} \longrightarrow \text{Witt} \longrightarrow 0,
\]

whose non-central generators can be identified with the Laurent modes of the chiral components of the energy-momentum tensor constructed from the loop-symmetry currents à la Sugawara:

\[
(3.5) \quad T_{++} = \frac{k}{2\pi} \kappa_g (J_+^{(1)}, J_+^{(1)}), \quad T_{--} = \frac{k}{2\pi} \kappa_g (J_-^{(1)}, J_-^{(1)}),
\]

and whose central generator \(C\) (the image of 1 \(\in\mathbb{R}\) in the above short exact sequence) acts as \(cI_{\mathcal{H}_g^{(1,k)}}\) on the Hilbert space of the field theory, returning the central charge of the WZW \(\sigma\)-model

\[
c = \frac{k \dim G}{k + g'(g)},
\]

where \(g'(g)\) is the dual Coxeter number of \(g\). Conformality of the theory in the quantum régime is ensured by the Ricci-flatness of the torsion-full Weitzenböck connection(s)

\[
\Gamma_{ABC}^A = \left\{ \begin{array}{c} A \\ B \\ C \end{array} \right\} \pm \frac{3}{2} \left( g_{BC}^{(k)} \right)^{-1} A D h^{k}_{DBC},
\]

written out in the Riemann normal coordinates \(\{X^A\}^{A \in \text{dim} \mathfrak{g}}\) on the group manifold in which we have Vielbeine

\[
\theta^A_L(X) = E^A_B(X) \, dX^B,
\]

so that

\[
\left( g_{BC}^{(k)} \right)^{AB}(X) = -\frac{k}{2\pi} \delta_{CD} E^C_A(X) E^D_B(X),
\]

and obtained from the Levi-Civita connection of the Cartan–Killing metric (with Christoffel symbols \(\{ \gamma_{BC} \}\)) by the addition of the torsion term induced from the components of the Cartan 3-form

\[
h_{ABC}^k(X) = \frac{k}{2\pi} f_{IJK} E^I_A(X) E^J_B(X) E^K_C(X).
\]

When looking for rigid symmetries amenable to gauging, it is natural to consider \((\mathbb{R}\)-linear combinations of the chiral generalised hamiltonian sections \(\mathcal{L}_A\) and \(\mathcal{R}_A\). While their respective \(C^\infty(G, \mathbb{R})\)-linear spans

\[
\mathfrak{g}_L = \bigoplus_{A=1}^{\text{dim} \mathfrak{g}} C^\infty(G, \mathbb{R}) \mathcal{L}_A, \quad \mathfrak{g}_R = \bigoplus_{A=1}^{\text{dim} \mathfrak{g}} C^\infty(G, \mathbb{R}) \mathcal{R}_A
\]

have vanishing Jacobi anomalies\footnote{Note that the Lie algebra of the right-invariant vector fields has structure constants \(-f_{AB}^C\). So does the Lie algebra of the vector fields \(-L_A\).} and non-vanishing Leibniz anomalies,

\[
\mathcal{L}_H \kappa_B^H + \epsilon_H f_{AB}^C \kappa_C^H = 0,
\]

\[
H_A \cup \kappa_B^H + H_B \cup \kappa_A^H = \frac{k}{2\pi} \epsilon_H \delta_{AB},
\]

on those of the combinations

\[
\mathfrak{g}_A := \mathcal{R}_A - \mathcal{L}_A \equiv \left( R_A - L_A, \frac{k}{2\pi} \delta_{AB} (\theta^R_B + \theta^L_B) \right) \equiv \left( V_A, \kappa_A^V \right)
\]

that correspond to the twisted-diagonal embedding

\[
\text{Ad}() : G \rightarrow G \times G : g \rightarrow (g, g^{-1}),
\]

both anomalies vanish, and the \(H_k\)-twisted Vinogradov bracket is readily seen to close. Indeed, for the former anomaly, we obtain

\[
\mathcal{L}_{V_A} \kappa_B^V + f_{AB}^C \kappa_C^V = -\mathcal{L}_{R_A} \kappa_B^V - \mathcal{L}_{L_A} \kappa_B^R = \frac{k}{2\pi} \delta_{BC} \left( \mathcal{L}_{R_A} \theta^C_L - \mathcal{L}_{L_A} \theta^C_R \right) = 0,
\]

and for the latter, we establish the equality

\[
V_A \cup \kappa_B^V + V_B \cup \kappa_A^V = -(L_A \cup \kappa_B^R + R_A \cup \kappa_B^L + L_B \cup \kappa_A^R + R_B \cup \kappa_A^L) = -\frac{k}{2\pi} \left( \delta_{BC} \left( L_A \cup \theta^C_R - R_A \cup \theta^C_L \right) + \delta_{AC} \left( L_B \cup \theta^C_R - R_B \cup \theta^C_L \right) \right)
\]

\[
= -\frac{k}{2\pi} \left( \left( T_e \text{Ad}_g \right)_{AB} - \left( T_e \text{Ad}_{g^{-1}} \right)_{AB} + \left( T_e \text{Ad}_g \right)_{BA} - \left( T_e \text{Ad}_{g^{-1}} \right)_{BA} \right).
\]
The $T_e\mathrm{Ad}$-invariance of the Killing form implies the identity
\[
(T_e\mathrm{Ad}_{\gamma^{-1}})_{BA} = (T_e\mathrm{Ad}_g)_{AB},
\]
whence also the vanishing of the Leibniz anomaly for $\mathrm{Ad}(G)$. Thus, the adjoint action of the group $G$ (or, indeed, of $G/\mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the centre of $G$) on itself, and so also of any Lie subgroup $H \subset G$ (or, indeed, of $H/\mathcal{Z}(H)$) is a candidate for a gauge symmetry of the WZW $\sigma$-model. What remains to be checked is the vanishing of the large gauge anomaly that quantifies obstructions against the existence of an $\mathrm{Ad}(G)$-equivariant (resp. $\mathrm{Ad}(H)$-equivariant) structure on $G_k^{(1)}$. This problem was conveniently reformulated in Ref. [GSW10, Sec. 4.2] (and subsequently extended to the WZW $\sigma$-model with maximally symmetric defects in Ref. [SW13, Sec.5]) and solutions, i.e., 1-gerbes $G_k$ with $k$ for which there exists an $\mathrm{Ad}(H)$-equivariant structure, were found, for a large class of cases, in Ref. [FGT11].

The punchline of the examination conducted hitherto is that in the bi-chirally symmetric $\sigma$-models (for $p \in \{0,1\}$) on compact simple 1-connected Lie groups the subgroup $\mathrm{Ad}(G) \subset G \times G$ of the full left-right rigid-symmetry group admits a non-anomalous gauging (although this may be the case for distinguished values of the normalisation constant $k \in \mathbb{N}^\times$ exclusively). In fact, the topological degrees of freedom decouple from the metric ones in that the existence of the $\mathrm{Ad}$-equivariant structure on the gerbe $G_k^{(p)}$ of the $\sigma$-model ensures the amenability of the adjoint symmetry $\mathrm{Ad}(G)$ to gauging. The $\mathrm{Ad}$-equivariant structure is independent of the explicit form of the metric on $\mathfrak{g}$ and is a property of the gerbe itself. We shall take this insight as a basis of our intuition regarding the super-variants of the bosonic $\sigma$-models reviewed above.

4. THE GREEN–SCHWARZ SUPER-$\sigma$-MODEL & ITS $\mathrm{Ad}$-EQUIVARIANCE

The field theories of immediate interest to us are theories of generalised embeddings
\[
\xi \equiv (\theta^\alpha, x^a) \in \left[\Omega_p, s\text{Man}_{\mathbb{R}^d,1}\right]
\]
of a closed worldvolume $\Omega_p$ (as earlier) of dimension $p + 1 \in \mathbb{N}_{\geq 0}$, written in terms of the global-coordinate mappings: the Graßmann-odd $\{\theta^\alpha\}_{\alpha \in \mathcal{T}, \mathcal{D}_d,1}$ and the Graßmann-even $\{x^a\}_{a \in \mathbb{R}^d}$, that belong to the (generalised) mapping supermanifold
\[
\left[\Omega_p, s\text{Man}_{\mathbb{R}^d,1}\right] \equiv \mathbf{Hom}_{\text{Man}}(\Omega_p, s\text{Man}_{\mathbb{R}^d,1}),
\]
given by the internal Hom
\[
\mathbf{Hom}_{\text{Man}}(\Omega_p, s\text{Man}_{\mathbb{R}^d,1}) \equiv \mathbf{Hom}_{\text{Man}}(- \times \Omega_p, s\text{Man}_{\mathbb{R}^d,1}) \in \text{Obj Fun}(\text{sMan}_{\mathbb{R}^d,1}, \text{Set})
\]
in the category $\text{sMan}$ of supermanifolds. Here, the supertarget is the supermanifold
\[
s\text{Man}_{\mathbb{R}^d,1} = (\mathbb{R}^{x_d+1}, C^\infty(\mathbb{R}, \mathbb{R}) \otimes \mathbb{R}^x D_{d,1}), \quad D_{d,1} = \text{dim } S_{d,1},
\]
where $S_{d,1}$ denotes a suitable Majorana-spinor representation of the spin group $\text{Spin}(d,1)$ of the Clifford algebra $\text{Cliff}(\mathbb{R}^{d,1})$ of the standard Minkowski (quadratic) space $\mathbb{R}^{x_d+1} \equiv (\mathbb{R}^{x_d+1}, \eta)$, $\eta = \text{diag}(-, +, +, \ldots, +)$, with generators $\{\Gamma^a\}_a \in \mathbb{R}^d$. The supertarget carries a natural Lie-supergroup structure defined by the binary operation
\[
m_1 : s\text{Man}_{\mathbb{R}^d,1} \times s\text{Man}_{\mathbb{R}^d,1} \to s\text{Man}_{\mathbb{R}^d,1}
\]
\[
\left(\left(\theta_1^\alpha, x_1^a\right), \left(\theta_2^\alpha, x_2^a\right)\right) \mapsto \left(\theta_1^\alpha + \theta_2^\alpha, x_1^a + x_2^a, \frac{1}{2} \theta_1^\alpha \left(\Gamma C \Gamma^a\right)_{\alpha \beta} \theta_2^\beta\right),
\]
in which $C$ is the charge-conjugation matrix with the properties
\[
(\Gamma a_1 a_2 \ldots a_k)^T = C \Gamma a_1 a_2 \ldots a_k, \quad k \in 0, d+1,
\]
written for
\[
C \Gamma a_1 a_2 \ldots a_k = C \Gamma^{[a_1} \Gamma^{a_2} \ldots \Gamma^{a_k]} = \Gamma^{a_1 a_2 \ldots a_k}.
\]
The latter are taken in a (Majorana-spinor) representation in which the fundamental Fierz identity
\[
\left(\Gamma_{(a_1 a_2 \ldots a_p)^\gamma} \Gamma^{a_1} \Gamma^{a_2} \ldots \Gamma^{a_p} \Gamma^{(a_1 a_2 \ldots a_p)} \right)^\gamma_\delta = 0
\]
\footnote{This is just the integrated version of its ad-invariance.}
holds true, written in terms of
\[ \Gamma_{a_1a_2...a_p} = \eta_{a_1b_1} \eta_{a_2b_2} ... \eta_{a_pb_p} \Gamma^{b_1b_2...b_p}, \]

The identity constrains the admissible values of \( d \) and \( p \) heavily, and the resulting spectrum of (classically) consistent models is known as the ‘old brane scan’, cp. Ref. [AETW87].

The Lie-supergroup structure induces distinguished global sections of the tangent sheaf \( \mathcal{T}_{s\text{Mink}^{d,1}|D_{d,1}} \) of the super-target, namely the fundamental vector fields of the left and right regular actions of \( s\text{Mink}^{d,1}|D_{d,1} \) on itself. The former are termed right-invariant vector fields on \( s\text{Mink}^{d,1}|D_{d,1} \) and are spanned (\( \mathbb{R} \)-linearly) on the generators
\[ Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} \Gamma^a_{\alpha\beta} \theta^\beta \frac{\partial}{\partial \theta^a}, \quad \alpha \in 1, D_{d,1}, \]

satisfying the Lie superalgebra
\[ \{ Q_\alpha, Q_\beta \} = -\Gamma^a_{\alpha\beta} P_a, \quad [ Q_\alpha, P_a ] = 0, \quad [ P_a, P_b ] = 0. \]

The latter go under the name of left-invariant vector fields on \( s\text{Mink}^{d,1}|D_{d,1} \) and are freely generated (over \( \mathbb{R} \)) by
\[ P_a = \frac{\partial}{\partial \theta^a}, \quad a \in 0, d, \]

forming the Lie superalgebra
\[ s\text{Mink}^{d,1}|D_{d,1} = \bigoplus_{\alpha=1}^{D_{d,1}} (Q_\alpha) \oplus d (P_a), \]

with the structure equations
\[ \{ Q_\alpha, Q_\beta \} = \Gamma^a_{\alpha\beta} P_a, \quad [ Q_\alpha, P_a ] = 0, \quad [ P_a, P_b ] = 0. \]

This Lie superalgebra is the central piece of data of the equivalent definition of the Lie supergroup \( s\text{Mink}^{d,1}|D_{d,1} \), in line with Kostant’s idea advanced in Ref. [Kos77], as the super Harish-Chandra pair
\[ s\text{Mink}^{d,1}|D_{d,1} = (Mink^{d,1}, s\text{Mink}^{d,1}|D_{d,1}), \]

with the body Lie group \( \text{Mink}^{d,1} \equiv \mathbb{R}^{d+1} \) (the Minkowski group of translations) realised trivially on the Graßmann-odd component of \( s\text{Mink}^{d,1}|D_{d,1} \) spanned on the supercharges \( Q_\alpha \).

The mapping supermanifolds are to be evaluated on the reference supermanifolds \( \mathbb{R}^{0|N}, N \in \mathbb{N}^* \) to give a proper meaning, along the lines of Ref. [Fre99], to the Dirac–Feynman amplitudes
\[ A_{\text{DF},GS,p}[\xi] = \exp \left( i S_{\text{metr},GS,p}[\xi] \right) \exp \left( i \int_{\Omega_p} (d^{-1} H)^{p+2} \right) \]

that determine the theory of embeddings in question, known as the Green–Schwarz super-\( \sigma \)-model of the super-\( p \)-brane, through the Principle of Least Action. The amplitudes are expressed in terms of the metric term, which we write in the Nambu–Goto formulation as
\[ S^{(NG)}_{\text{metr},GS,p}[\xi] = \int_{\Omega_p} \text{Vol}(\Omega_p) \sqrt{\det_{(p+1)} \left( \eta_{ab} \left( \partial_i \xi^a \partial_j \xi^b \right) \right)}, \]

where \( e^a(\theta, x) = dx^a + \frac{1}{2} \theta^a \left( CT^a \right)_{\alpha\beta} d\theta^\beta \equiv dx^a + \frac{1}{2} \theta \Gamma^a \ d\theta, \quad a \in 0, d \)

are the left-invariant super-1-forms on \( s\text{Mink}^{d,1}|D_{d,1} \) (of the total degree \( \text{Deg}(e^a) = (0, 1) \), cp App. A of Part I) dual to the \( P_a \), and of the Green–Schwarz super-\( (p + 2) \)-cocycle \( H_{(p+2)} \in Z_{\text{dR}}^{p+2}(s\text{Mink}^{d,1}|D_{d,1}) \)

defining a nontrivial class
\[ 0 \neq \left[ H_{(p+2)} \right] \in \text{CaE}^{p+2}(s\text{Mink}^{d,1}|D_{d,1}) \]

in the Cartan–Eilenberg cohomology \( \text{CaE}^* (s\text{Mink}^{d,1}|D_{d,1}) \) of the target Lie supergroup. In order to write out the super-\( (p + 2) \)-cocycle, we need to complete the basis of left-invariant super-1-forms by adjoining the remaining
\[ \text{pr}^* \sigma^\alpha(\theta, x) = \sigma^\alpha(\theta) = d\theta^\alpha, \quad \alpha \in 1, D_{d,1} \]

\[ ^{14} \text{There is a possibility to enter yet another parameter into the game, to wit, the number } N \in \mathbb{N}^* \text{ of spinor generations, but we abstain from doing it for the sake of simplicity of the presentation.} \]
classes in the cohomology group $H$ they compose the $\text{smink}^{d,1|D_{d,1}}$-valued left-invariant Maurer–Cartan super-1-form
\[
\tilde{\partial}_L = \text{pr}_1^\ast \sigma \otimes Q_\alpha + e^a \otimes P_a.
\]
With these in hand, we may now explicitly write out – for $p = 0$ and for $d = 9$ –
\[
H (2) = \text{pr}_1^\ast \sigma \wedge \Gamma_{11} \sigma,
\]
with
\[
\Gamma_{11} := i \Gamma^0 \cdot \Gamma^1 \cdots \Gamma^9,
\]
and – for $p \geq 1$ –
\[
H (p + 2) = \text{pr}_1^\ast (\sigma \wedge \Gamma_{a_1 a_2 \ldots a_p} \sigma) \wedge e^{a_1 a_2 \ldots a_p},
\]
where
\[
e^{a_1 a_2 \ldots a_p} \equiv e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p}.
\]
These admit global non-invariant primitives, as stated in

**Proposition 4.1.** [Sus17, Prop. 4.2] For any $p \in 1, 9$, the GS super-$(p + 2)$-cocycle $H (p + 2)$ of Eq. (4.4) admits a manifestly ISO$(d,1)$-invariant primitive
\[
B (\theta, x) = \frac{1}{p + 1} \sum_{k=0}^p \theta \Gamma_{a_1 a_2 \ldots a_p} \sigma (\theta) \wedge dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_k} \wedge e^{a_{k+1} a_{k+2} \ldots a_p} (\theta, x).
\]
A primitive of the super-2-form $H (2)$ of Eq. (4.3) can be chosen in the form
\[
B (\theta, x) = \theta \Gamma_{11} \sigma (\theta).
\]

In order to resolve the super-$(p + 2)$-cocycles in the Chevalley–Eilenberg cohomology instead, an idea motivated amply – after Rabin and Crane (cp Refs. [RC85, Rab87]) – in Part I from the topological perspective, we have to extend the underlying Lie superalgebras $\text{smink}^{d,1|D_{d,1}}$ in a stepwise procedure, devised by de Azcárraga et al. in Ref. [CdAIPB00] and based on the one-to-one correspondence between classes in the cohomology group $H^2(\text{smink}^{d,1|D_{d,1}}, \mathbb{R})$ of the Lie superalgebra $\text{smink}^{d,1|D_{d,1}}$ with values in its trivial module $\mathbb{R}$ and central extensions of that Lie superalgebra. The Lie supergroup that integrates the full extension of $\text{smink}^{d,1|D_{d,1}}$ on which (the pullback of) the relevant super-$(p + 2)$-cocycle $H (p + 2)$ trivialises is then taken as the surjective submersion of the super-$p$-gerbe $G^{(p)}_{\text{GS}}$ associated to $H (p + 2)$ as its (super)geometrisation, the latter developing along the same lines as the geometrisation of standard de Rham cocycles through $p$-gerbes, laid out – for $p = 1$ – by Murray and Stevenson in [Mur96, MS00]. The (super)geometrisation consists in a sequence of nested sub-(super)geometrisations (of Chevalley–Eilenberg cocycles of a decreasing rank) over powers of the basic superjective submersion fibred over its base $\text{sMink}^{d,1|D_{d,1}}$, each using the aforementioned fundamental correspondence between non-trivial 2-cocycles on the Lie superalgebra and its extensions, and the ensuing (super)geometric structure can be regarded as a usual $p$-gerbe over, however, a quotient of $\text{sMink}^{d,1|D_{d,1}}$ by the Kostelecký–Rabin discrete supersymmetry group of Ref. [KR84]. The topology of the quotient encodes the full information on the Cartan–Eilenberg cohomology of $\text{sMink}^{d,1|D_{d,1}}$ (and no more), cp Ref. [RC85], and the original super-$\sigma$-model is to be understood as – implicitly – a field theory with that quotient as the (super)target space. We shall now briefly recall the definitions of the super-$p$-gerbes thus constructed in Part I for $p \in \{0, 1\}$ as these are the structures we shall work with extensively in the remainder of the paper.

The hierarchy of the Green–Schwarz super-$p$-gerbes $G^{(p)}_{\text{GS}}$ begins with the super-0-gerbe of curvature $H (2)$. It is defined as the triple
\[
G^{(0)}_{\text{GS}} := \left( \mathcal{L}^{(0)}, \varpi_{\mathcal{L}^{(0)}}, \beta^{(2)} \right)
\]
consisting of the (trivial) principal $\mathbb{C}^\times$-bundle
\[
\mathbb{C}^\times \rightarrow \mathcal{L}^{(0)} := \text{sMink}^{d,1|D_{d,1}} \times \mathbb{C}^\times
\]

\[
\pi_{\mathcal{L}(0)}
\]
\[
\text{sMink}^{d,1|D_{d,1}}.
\]
equipped with the projection to the base
\[
\pi_{\mathcal{L}(0)} \equiv \text{pr}_1 : \text{sMink}^{d,1|D_{d,1}} \times \mathbb{C}^\times \rightarrow \text{sMink}^{d,1|D_{d,1}} : (\theta^a, x^a, z) \mapsto (\theta^a, x^a)
\]
and the principal $\mathbb{C}^\times$-connection (super-)1-form
\[
(4.6) \quad \beta^{(2)}_1(\theta, x, z) = \frac{dz}{\theta} + B_1(\theta, x), \quad B_1(\theta, x) = \theta \Sigma_{11} \sigma(\theta),
\]
satisfying the identity
\[
\frac{d}{\theta} \beta^{(2)}_1 = \pi_{\mathcal{L}(0)}^* H_1.
\]
The total space of the bundle carries the structure of a Lie-supergroup extension of $\text{sMink}^{d,1|D_{d,1}}$ with
the binary operation
\[
m_0^{(2)} : \mathcal{L}^{(0)} \times \mathcal{L}^{(0)} \rightarrow \mathcal{L}^{(0)}
\]
\[
= ((\theta_1^a, x_1^a, z_1), (\theta_2^b, x_2^b, z_2)) \mapsto (m_1((\theta_1^a, x_1^a), (\theta_2^b, x_2^b)), e^{i(\lambda^{(0)}(\theta_1^a, x_1^a), (\theta_2^b, x_2^b))) \cdot z_1 \cdot z_2}),
\]
determined by the 2-cocycle
\[
(4.7) \quad \lambda^{(0)} : \text{sMink}^{d,1|D_{d,1}} \times \text{sMink}^{d,1|D_{d,1}} \rightarrow \mathbb{R} : ((\theta_1, x_1), (\theta_2, x_2)) \mapsto \theta_1 \Sigma_{11} \theta_2
\]
that derives from $H_1$ and $\pi_{\mathcal{L}(0)}$ is promoted to the rank of a Lie-supergroup homomorphism. The
principal $\mathbb{C}^\times$-connection (super-)1-form is invariant with respect to the left regular action of that Lie
supergroup on itself.

The above is an example of

**Definition 4.2** (Def. I.5.4). Let $G$ be a Lie supergroup and let $H$ be a super-2-cocycle on $G$ representing
a class in its (left) Cartan–Eilenberg cohomology. A **Cartan–Eilenberg super-0-gerbe** of
curvature $H$ over $G$ is a triple
\[
\mathcal{G}^{(0)}_{\text{CaE}} = (YG, \pi_YG, A^{(1)})
\]
composed of
- a principal $\mathbb{C}^\times$-bundle
\[
\mathbb{C}^\times \rightarrow YG \xrightarrow{\pi_YG} G
\]
with the structure of a Lie supergroup on its total space $YG$ that fits into the short exact
sequence of Lie supergroups

\[
1 \rightarrow \mathbb{C}^\times \rightarrow YG \xrightarrow{\pi_YG} G \rightarrow 1;
\]
- a principal $\mathbb{C}^\times$-connection $A \in \Omega^1(YG)$ on $YG$ invariant with respect to the left regular action
of the latter Lie supergroup on itself.

Accordingly, an isomorphism between two Cartan–Eilenberg super-0-gerbes $(YG, \pi_YG, A^{(1)})$, $A \in \{1, 2\}$
over a common base $G$ is a connection-preserving isomorphism of principal $\mathbb{C}^\times$-bundles
\[
\varphi : Y_1G \xrightarrow{\sim} Y_2G
\]

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which is simultaneously a Lie-supergroup isomorphism, and so — altogether — an equivalence of the two extensions that fits into the commutative diagram

\[
\begin{array}{ccc}
Y_1 G & \xrightarrow{\pi_{Y_1 G}} & \mathbb{C}^x \\
\downarrow & & \downarrow \varphi \\
Y_2 G & \xrightarrow{\pi_{Y_2 G}} & G \xrightarrow{1} 1
\end{array}
\]

At the next level, we find the super-1-gerbe of curvature \( H^{(3)} \), given by the septuple

\[
\mathcal{G}^{(1)}_{GS} := \left( Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}, \pi_{Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}}, \beta^{(2)}_L, \pi_L^{(1)}, \pi_Z^{(1)}, \mathcal{A}_Z^{(1)}, \mu_Z^{(1)} \right)
\]

composed of the surjective submersion

\[
\pi_{Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}} \equiv \text{pr}_1 : Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}} \cong \text{Mink}^{d,1}|_{D_{d,1}} \times \mathbb{R}^0|_{D_{d,1}} \to \text{Mink}^{d,1}|_{D_{d,1}}
\]

and, on it, of the global primitive (curving)

\[
\beta^{(2)}_L(\theta, x, \xi) = \sigma^\alpha(\theta) \wedge e^{(2)}_\alpha(\theta, x, \xi), \quad e^{(2)}_\alpha(\theta, x, \xi) = dx^\alpha - (\Gamma_a)_{\alpha\beta} \theta^\beta (dx^a + \frac{1}{6} \theta^a \Theta^a \sigma(\theta))
\]

(4.9)

of the pullback of \( H^{(3)} \),

\[
d\beta^{(2)} = \pi^*_L Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}} H^{(3)},
\]

as well as of the (trivial) principal \( \mathbb{C}^x \)-bundle

\[
\begin{array}{ccc}
\mathbb{C}^x & \xrightarrow{\pi_Z^{(1)}} & \mathcal{L}^{(1)} := Y_1^{[2]} \text{Mink}^{d,1}|_{D_{d,1}} \times \mathbb{C}^x \\
\downarrow & & \downarrow \pi_L^{(1)} \\
Y_1^{[2]} \text{Mink}^{d,1}|_{D_{d,1}} & \to & Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}
\end{array}
\]

over the fibred square of \( Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}} \), the latter being determined by the commutative diagram (in which the \( \text{pr}_i, i \in \{1, 2\} \) are the canonical projections)

\[
\begin{array}{ccc}
Y_1^{[2]} \text{Mink}^{d,1}|_{D_{d,1}} & \xrightarrow{\text{pr}_1} & Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}} \\
\downarrow & & \downarrow \pi_{Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}} \\
Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}} & \xrightarrow{\text{pr}_2} & Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}
\end{array}
\]

and equipped with the projection to the base

\[
\pi_{Y_{1s\text{Mink}}^{d,1}|_{D_{d,1}}} \equiv \text{pr}_1 : Y_1^{[2]} \text{Mink}^{d,1}|_{D_{d,1}} \times \mathbb{C}^x \to Y_1^{[2]} \text{Mink}^{d,1}|_{D_{d,1}}
\]

\[
: \left( (\theta, x, \xi_1), (\theta, x, \xi_2), z \right) \mapsto \left( (\theta, x, \xi_1), (\theta, x, \xi_2) \right),
\]

with the principal \( \mathbb{C}^x \)-connection 1-form

\[
\mathcal{A}_Z^{(1)}(\left( (\theta, x, \xi_1), (\theta, x, \xi_2), z \right) = i \frac{dz}{z} + \mathcal{A}_Z^{(1)}(\left( (\theta, x, \xi_1), (\theta, x, \xi_2) \right),
\]
Above, $Y_{1s\text{Mink}^{d,1}|D_{d,1}}$ is a Lie supergroup that extends $s\text{Mink}^{d,1}|D_{d,1}$ centrally through the binary operation

$$m_1^{(2)} : Y_{1s\text{Mink}^{d,1}|D_{d,1}} \times Y_{1s\text{Mink}^{d,1}|D_{d,1}} \rightarrow Y_{1s\text{Mink}^{d,1}|D_{d,1}}$$

$$: \left(\left(\theta_1^\alpha, x_1^a, \xi_{1\beta}\right), \left(\theta_2^\gamma, x_2^b, \xi_{2\delta}\right)\right) \mapsto \left(m_1\left(\left(\theta_1^\alpha, x_1^a\right), \left(\theta_2^\gamma, x_2^b\right)\right), \xi_{1\alpha} + \xi_{2\alpha} + \left(\check{\Gamma}_a\right)_{\alpha\beta} \theta_1^\beta x_2^a\right)$$

and this structure induces the product Lie-supergroup structure on the fibred powers of $Y_{1s\text{Mink}^{d,1}|D_{d,1}}$. The surjective submersion $\pi_{Y_{1s\text{Mink}^{d,1}|D_{d,1}}}$ now becomes a Lie-supergroup homomorphism and the curvature is invariant with respect to the left regular action of $Y_{1s\text{Mink}^{d,1}|D_{d,1}}$ on itself. Likewise, $\mathcal{L}^{(1)}$ is a central extension of $Y_{1s\text{Mink}^{d,1}|D_{d,1}}$ (the latter being endowed with the aforementioned product Lie-supergroup structure) with the binary operation (written for $m_2^A = \left(\theta_1^\alpha, x_1^a, \xi_{1\beta}\right)$, $A \in \{1, 2\}$ and $n_2^A = \left(\theta_2^\gamma, x_2^b, \xi_{2\delta}\right)$, $A \in \{1, 2\}$)

$$m_1^{(3)} : \mathcal{L}^{(1)} \times \mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(1)} : \left(\left(m_1^{(2)}\left(m_2^1, n_2^1\right), \left(m_2^2, n_2^2\right)\right), z_1, z_2\right) \mapsto \left(\left(m_1^{(2)}\left(m_2^1, n_2^1\right), m_1^{(2)}\left(m_2^2, n_2^2\right)\right) \cdot \left(m_1^{(2)}\left(m_2^1, n_2^1\right), m_1^{(2)}\left(m_2^2, n_2^2\right)\right) \cdot z_1 \cdot z_2\right)$$

determined by the super-2-cocycle

$$d^{(1)}\left(\left(m_2^1, n_2^1\right), \left(m_2^2, n_2^2\right)\right) = e^{\theta_1^\alpha x_2^a\xi_{1\alpha}}$$

for which the bundle projection $\pi_{\mathcal{L}^{(1)}}$ is a Lie-supergroup homomorphism and the principal $\mathbb{C}^*$-connection (super-)1-form $A_{\mathcal{L}^{(1)}}$ is left-invariant. Finally, the groupoid structure is readily proven
to be a Lie-supergroup homomorphism (with respect to the natural Lie-supergroup structures on the (product) pullback bundles that it identifies). The above exemplifies an object described in

**Definition 4.3** (Def. I.5.11). Adopt the notation of Def. I.3. Let \( H \) be a super-3-cocycle on \( G \) representing a class in its (left) Cartan–Eilenberg cohomology. A **Cartan–Eilenberg super-1-gerbe** over \( G \) of curvature \( H \) is a septuple

\[
G^{(1)}_{\text{CaE}} := (YG, \pi_{YG}, B, L, \pi_L, A_L, \mu_L)
\]

composed of

- a surjective submersion
  
  \[ \pi_{YG} : YG \to G \]

with a structure of a Lie supergroup on its total space mapped onto that on \( G \) by the Lie-supergroup epimorphism \( \pi_{YG} \);

- a global primitive \( B^{(2)} \) of the pullback of \( H^{(3)} \) to it,
  
  \[ \pi_{YG}^* H = dB^{(2)} \]

which is left-invariant with respect to the left regular action of \( YG \) on itself,

\[ Yf_{g(2)}^* B = B^{(2)}, \quad y \in YG; \]

- a CaE super-0-gerbe
  
  \[
  \left( L, \pi_L, A_L \right)^{(1)}
  \]

over the fibred square \( Y[2]G \equiv YG \times_G YG \) (endowed with the natural Lie-supergroup structure induced from the product structure on \( YG^{\times 2} \) through restriction), with a principal \( C^* \)-connection 1-form \( A_L \) of curvature \( \left( \pi_L^* - \pi_Y^* \right) B^{(2)} \)

\[ \pi_L^* \left( \pi_Y^* - \pi_Y^* \right) B^{(2)} = dA_L^{(1)}; \]

- an isomorphism of CaE super-0-gerbes\(^\text{15}\)
  
  \[ \mu_L : \left( \pi_L^* \right)^{1,2} L \otimes \left( \pi_L^* \right)^{2,3} L \xrightarrow{\cong} \left( \pi_Y^* \right)^{1,3} L \]

over the fibred cube \( Y[3]G \equiv YG \times_G YG \times_G YG \times_G YG \) that satisfies the coherence (associativity) condition

\[ \left( \pi_L^* \right)^{1,2,4} \mu_{L} \circ \left( \text{id}_{\left( \pi_L^* \right)^{1,2} L} \otimes \left( \pi_L^* \right)^{2,3,4} \mu_{L} \right) = \left( \pi_L^* \right)^{1,3,4} \mu_{L} \circ \left( \left( \pi_Y^* \right)^{1,2,3} \mu_{L} \otimes \text{id}_{\left( \pi_Y^* \right)^{3,4} L} \right) \]

over the quadruple fibred product \( Y[4]G \equiv YG \times_G YG \times_G YG \times_G YG \).

Given CaE super-1-gerbes \( G^{(1)}_{\text{CaE}} = (Y_A G, \pi_{Y_A G}, B, L, A_L, \mu_L, E) \), \( A \in \{1, 2\} \) over a common base \( G \), a 1-isomorphism between them is a quintuple

\[ \Phi^{(1)}_{\text{CaE}} := (\mathcal{Y} Y_{1,2} G, \pi_{\mathcal{Y} Y_{1,2} G}, E, A_E, \alpha_E) : G^{(1)}_{\text{CaE}} \xrightarrow{\cong} G^{(1)}_{\text{CaE}} \]

composed of

- a surjective submersion
  
  \[ \pi_{\mathcal{Y} Y_{1,2} G} : \mathcal{Y} Y_{1,2} G \to Y_{1,2}G \times_G Y_{1,2}G \equiv Y_{1,2}G \]

with a structure of a Lie supergroup on its total space that lifts the product Lie-supergroup structure on the fibred product \( Y_{1,2} G \) along the Lie-supergroup epimorphism \( \pi_{\mathcal{Y} Y_{1,2} G} \),

\(^{15}\)Note that pullback along a canonical projection is consistent with the definition of a super-0-gerbe due to its equivariance.
• a CaE super-0-gerbe

\[(E, \pi_E, A_E)\]

over the total space \(YY_{1,2}G\), with a principal \(\mathbb{C}^\times\)-connection 1-form \(A_E\) of curvature \(\pi^{*}_{YY_{1,2}G}(pr^{*}_{2}B_{2})\)

\[\pi^{*}_{E}\pi^{*}_{YY_{1,2}G}(pr^{*}_{2}B_{2} - pr^{*}_{1}B_{1}) = dA_E;\]

• an isomorphism of super-0-gerbes

\[\alpha_E : (\pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G})^{*}pr^{*}_{1,3}L_{1} \otimes pr^{*}_{2}E \rightarrow pr^{*}_{1}E \otimes (\pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G})^{*}pr^{*}_{2,4}L_{2}\]

over the fibred product \(\mathcal{Y}^{[3]}YY_{1,2}G = YY_{1,2}G \times G YY_{1,2}G\), subject to the coherence constraint expressed by the commutative diagram

\[
\begin{array}{ccc}
\pi_{1,2} \circ pr_{1,3}^{*}L_{1} & \otimes & \pi_{2,3} \circ pr_{1,3}^{*}L_{1} \otimes pr_{2}^{*}E \\
pr_{1}^{*}E \otimes \pi_{1,2} \circ pr_{2,4}^{*}L_{2} & \otimes & \pi_{2,3} \circ pr_{2,4}^{*}L_{2} \\
pr_{1}^{*}E \otimes \pi_{1,2} \circ pr_{2,4}^{*}L_{2} & \otimes & \pi_{2,3} \circ pr_{2,4}^{*}L_{2}
\end{array}
\]

of isomorphisms of CaE super-0-gerbes over the fibred product \(\mathcal{Y}^{[3]}YY_{1,2}G \equiv YY_{1,2}G \times G YY_{1,2}G\), written in terms of the maps

\[\pi_{i,j} = (\pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G}) \circ pr_{i,j}, \quad (i, j) \in \{(1, 2), (2, 3), (1, 3)\},\]

\[\pi_{1,2,3} = \pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G}.\]

Given a pair of 1-isomorphisms \(\Phi^{(1)}_{CaE} = (Y^{B}Y_{1,2}G, \pi_{YY_{1,2}G}, B, \alpha_{E_{B}, \alpha_{E_{B}}}), \quad B \in \{1, 2\}\) between (1)

CaE super-1-gerbes \(\Phi^{(1)}_{CaE} = (Y^{A}G, \pi_{YY_{1,2}G}, B_{A}, L_{A}, A_{L_{A}}, \mu_{A_{L_{A}}})\), \(A \in \{1, 2\}\), a 2-isomorphism is represented by a triple

\[\varphi^{(1)}_{CaE} = (\mathcal{Y}^{1,2}YY_{1,2}G, \pi_{YY_{1,2}G}, \beta) : \Phi^{(1)}_{CaE} \Rightarrow \Phi^{(1)}_{CaE}\]

composed of

• a surjective submersion

\[\pi_{YY^{1,2}YY_{1,2}G} : \mathcal{Y}^{1,2}YY_{1,2}G \rightarrowYY_{1,2}G \times YY_{1,2}G \rightarrow YY_{1,2}G \equiv YY_{1,2}G\]

with a structure of a Lie supergroup on its total space that lifts the product Lie-supergroup structure on the fibred product \(YY_{1,2}G \times YY_{1,2}G\) along the Lie-supergroup epimorphism \(\pi_{YY_{1,2}G}\),

• an isomorphism of CaE super-0-gerbes

\[\beta : (pr_{1} \circ \pi_{YY^{1,2}YY_{1,2}G})^{*}E_{1} \rightarrow (pr_{2} \circ \pi_{YY^{1,2}YY_{1,2}G})^{*}E_{2}\]

subject to the coherence constraint expressed by the commutative diagram

\[
\begin{array}{ccc}
p_{1,1}^{*}L_{1} \otimes \pi_{1,2}^{*}E_{1} & \rightarrow & \pi_{1,1}^{*}L_{1} \otimes p_{2}^{*}L_{2} \\
pr_{1}^{*}E_{1} \otimes \pi_{1,2}^{*}E_{2} \equiv p_{1,2}^{*}L_{1} \otimes \pi_{2,2}^{*}E_{2} & \rightarrow & \pi_{2,1}^{*}E_{2} \otimes p_{2,2}^{*}L_{2} \equiv \pi_{2,1}^{*}E_{2} \otimes p_{2,2}^{*}L_{2}
\end{array}
\]
of isomorphisms of CaE super-0-gerbes over $Y[2] Y^{1,2} \mathbb{Y}_{1,2} \mathbb{G}$, with
\[
\pi_i = \text{pr}_i \circ \pi_Y Y^{1,2} \mathbb{Y}_{1,2} \mathbb{G}, \quad \pi_{j,k} = \pi_j \circ \text{pr}_k, \quad i, j, k \in \{1, 2\},
\]
\[
p_{l,m} = \text{pr}_l \circ \pi_Y Y^{1,2} \mathbb{Y}_{1,2} \mathbb{G} \circ \pi_m \times \text{pr}_l \circ \pi_Y Y^{1,2} \mathbb{Y}_{1,2} \mathbb{G} \circ \pi_m, \quad l, m \in \{1, 2\}.
\]

Having explicited the supergeometrisations of immediate relevance to the rest of our discourse, we may return to the study of symmetries of the super-Minkowskian super-$\sigma$-models in the vein of Refs. [GSW10, Sus12, GSW13].

The manifest left-invariance of the GS super-$(p + 2)$-cocycle $H_{(p + 2)}$, in conjunction with the triviality of the de Rham cohomology of $s\text{Mink}^{d,1|D_{d,1}}$, implies the existence of an extension, to the generalised tangent sheaf
\[
\mathcal{E}^{(1,p)} s\text{Mink}^{d,1|D_{d,1}} = T s\text{Mink}^{d,1|D_{d,1}} \oplus \bigwedge p^* s\text{Mink}^{d,1|D_{d,1}},
\]
of the algebra of the (left) supersymmetry generators
\[
(4.12) \quad R_{(\varepsilon, y)}(\theta, x) := \varepsilon^{\alpha} \mathcal{D}_\alpha(\theta, x) + y^I \mathcal{F}_I(\theta, x), \quad (\varepsilon, y) \in s\text{Mink}^{d,1|D_{d,1}},
\]
with the Lie bracket
\[
(4.13) \quad [R_{(\varepsilon_1, y_1)}, R_{(\varepsilon_2, y_2)}] = \varepsilon_1^{\alpha} \varepsilon_2^{\beta} \mathcal{D}_\beta - \varepsilon_2^{\alpha} \varepsilon_1^{\beta} \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = R_{(0, \pi_1 \gamma_1 \varepsilon_2)}.
\]
Indeed, we have

**Proposition 4.4.** For any $p \in (0, Y)$, the fundamental vector field $R_{(\varepsilon, y)}$ of Eq. (4.12) (defined as above for arbitrary $(\varepsilon, y) \in s\text{Mink}^{d,1|D_{d,1}}$) is generalised hamiltonian with respect to the super-$(p + 2)$-cocycle $H_{(p + 2)}$, that is, there exists a globally smooth super-$p$-form $\kappa^R_{(p)}(\varepsilon, y) \in \Omega^p(s\text{Mink}^{d,1|D_{d,1}})$ with the property
\[
R_{(\varepsilon, y)} H_{(p + 2)} = -d \kappa^R_{(p)}(\varepsilon, y).
\]
The latter can be chosen in the manifestly $\text{ISO}(d,1)$-invariant form
\[
\kappa^R_{(p)}(\varepsilon, y)(\theta, x) = -2 \varepsilon^{(1)} \Gamma_1 \theta
\]
for $p = 0$, and 
for $p > 0$
\[
\kappa^R_{(p)}(\varepsilon, y)(\theta, x) = -p y^a \beta_a(\theta, x) - 2(\varepsilon^{(a)} a_{1a2...ap}) \varepsilon^{a_{1a2...ap}}(\theta, x)
\]
\[
+ \frac{p!}{(2p+1)!} \sum_{k=1}^{\frac{p}{2}} (2p+1-2k)! \eta_{(1)}^{(a_{1a2...ap})}(\theta, x) \wedge d x^{a_2} \wedge d x^{a_3} \wedge ... \wedge d x^{a_p} \wedge \varepsilon^{a_{1a3...ap}}(\theta, x),
\]
written in terms of the super-p-forms $\beta_a$ from Eq. (B.1) and of the super-1-forms $\eta_{(1)}^{(a_{1a2...ap})}$ from Eq. (B.3).

**Proof.** A proof is given in App. B. \qed

The extension, defined in terms of the Vinogradov-type bracket of Eq. (2.7), closes on pairs of the distinguished fundamental sections
\[
(4.14) \quad \mathcal{R}_{(\varepsilon, y)} = (R_{(\varepsilon, y)}, \kappa^R_{(p)}(\varepsilon, y)) \in \Gamma(\mathcal{E}^{(1,p)} s\text{Mink}^{d,1|D_{d,1}})
\]
of the generalised tangent bundle over $s\text{Mink}^{d,1|D_{d,1}}$. We readily convince ourselves that the restriction of the $H_{(p + 2)}$-twisted Vinogradov bracket to the linear span
\[
\Phi^R_{(p)} = \left\{ \mathcal{R}_{(\varepsilon, y)} \mid (\varepsilon, y) \in s\text{Mink}^{d,1|D_{d,1}} \right\}
\]
is anomalous\footnote{Note that we (intentionally) consider Grassmann-even supervector fields here.} for $p > 0$. Indeed, we have

\footnote{As we have judiciously decided to work with linear combinations of the generators of $s\text{mink}^{d,1|D_{d,1}}$ with coefficients of the same parity as the corresponding vector fields, we may transcribe the conditions (2.16) verbatim into the current supergeometric context.}
Proposition 4.5. For any \( p \in \mathbb{T}_0 \), the \( H^{(p+2)} \)-twisted Vinogradov bracket (defined as before) has a non-vanishing Leibniz anomaly on \( \Theta_R^{(p)} \). In particular, its projection to the body reads

\[
P_{a_{p-1}} \cup P_{a_{p-2}} \cup \ldots \cup P_{a_1} \rightarrow \left( R_{(\varepsilon_1,0)} \cup R_{(\varepsilon_2,0)} \right) (\theta,x) = \frac{2\varepsilon}{\sqrt{3}} \left( \left( \varepsilon_1 \mathbf{T}^\alpha \theta \right) \left( \varepsilon_2 \mathbf{T}_{a_1 a_2 \ldots a_{p-1}} \right) + \left( \varepsilon_2 \mathbf{T}^\alpha \theta \right) \left( \varepsilon_1 \mathbf{T}_{a_1 a_2} \ldots a_{p-1} \right) \right).
\]

Proof. A proof is given in App. C. \( \square \)

Instead of the standard \( H^{(p+2)} \)-twisted Vinogradov bracket on the Graßmann-even sections of \( \mathcal{E}^{(1,p)}_{s \text{Mink}} \), we may also consider its \( \mathbb{Z}/2\mathbb{Z} \)-graded (or super-)counterpart (written in a homogeneous basis)

\[
\left[ R_A, R_B \right]_V^{H^{(p+2)}} = \left( \left[ \mathcal{K}_A, \mathcal{K}_B \right] - (-1)^{|A||B|} \mathcal{L}_{\mathcal{K}_A} \mathcal{K}_C - \frac{1}{2} d \left( \left( \mathcal{K}_A \cup \mathcal{K}_C \right) - (-1)^{|A||B|} \mathcal{K}_B \cup \mathcal{K}_A \right) + \mathcal{K}_A \cup \mathcal{K}_B \right) \left( (p+2) \right),
\]

with the corresponding small gauge anomaly quantified by the obstruction against the equivariance and \( \mathbb{Z}/2\mathbb{Z} \)-graded symmetry conditions

\[
\mathcal{L}_{\mathcal{K}_A} \mathcal{K}_B = f_{AB}^C \mathcal{K}_C , \quad \mathcal{K}_A \cup \mathcal{K}_B + (-1)^{|A||B|} \mathcal{K}_B \cup \mathcal{K}_A = 0.
\]

In the special case \( p = 0 \), the \( H \)-twisted Vinogradov superbracket does not close on the span of the basis sections

\[
\mathcal{R}_a = \left( \mathcal{L}_a, -2(\mathbf{T}_{11})_{\alpha \beta} \theta^\beta \right), \quad \mathcal{R}_a = \left( \mathcal{L}_a, 0 \right).
\]

Indeed, we find

\[
\left[ \mathcal{R}_a, \mathcal{R}_b \right]_V^{H^{(p+2)}} = -2(\mathbf{T}_{11})_{\alpha \beta} \mathcal{R}_a + 0, -2(\mathbf{T}_{11})_{\alpha \beta} \mathcal{R}_a \right),
\]

and so there is an irremovable correction \( 0, -2(\mathbf{T}_{11})_{\alpha \beta} \) to the Lie-algebroidal structure. The pathology of the structure can also be demonstrated by computing the Lie derivative

\[
\mathcal{L}_{\mathcal{R}_{(\varepsilon_1,0)}} \mathcal{R}_{(\varepsilon_2,0)} = -2\varepsilon_1 \mathcal{L}_a \cup \varepsilon_2 \mathbf{T}_{11} \sigma (\theta) = 2\varepsilon_1 \mathbf{T}_{11} \varepsilon_2,
\]

only to find disagreement with

\[
\mathcal{R}_{(\varepsilon_1,0)} \cup \mathcal{R}_{(\varepsilon_2,0)} = 0.
\]

The last two results rule out the possibility of constructing a (standard) \( s \text{Mink}^{d,1}_{D_{d,1}} \)-equivariant structure for the left regular action of the Lie supergroup on itself on the super-\( p \)-gerbes geometrising the super-(\( p+2 \))-cocycles \( H^{(p+2)} \).

Let us, next, discuss the right regular action of \( s \text{Mink}^{d,1}_{D_{d,1}} \) on itself. The crucial point to note is that while the \( \sigma^\alpha \) are left- and right-invariant, the \( e^a \) are only left-invariant and transform, under a right translation by a constant vector \( (\varepsilon, y) \), as

\[
e^{a} \left( \mathbf{m}((\theta,x),(\varepsilon,y)) \right) - e^{a}(\theta,x) = d(\varepsilon \mathbf{T}^a \theta).
\]

Consequently, the metric term of the action functional is not bi-chirally invariant, and so there is no hope for the full bi-chiral invariance of the GS super-\( \sigma \)-model\footnote{There exists, however, a rather peculiar infinitesimal right gauge invariance which we shall discuss later in the present work.}. Still, we may enquire as to the bi-invariance of the GS super-(\( p+2 \))-cocycles, the only property to be checked being their invariance under right translations. In order to provide an answer to the above question, we shall pass, once more, to the infinitesimal picture in which right translations are generated (as flows) by the left-invariant fundamental vector fields listed before. We may now consider Lie derivatives of the various GS super-(\( p+2 \))-cocycles along the generators of \( s \text{Mink}^{d,1}_{D_{d,1}} \).

For \( p = 0 \), we obtain the identities

\[
\mathcal{L}_{Q_{a}} H^{(2)} = d(Q_{a} \cup pr_1^*(\sigma \wedge \mathbf{T}_{11} \sigma)) = 2(\mathbf{T}_{11})_{\alpha \beta} pr_1^* d\sigma^\beta = 0,
\]

\[
\mathcal{L}_{P_{a}} H^{(2)} = d(P_{a} \cup pr_1^*(\sigma \wedge \mathbf{T}_{11} \sigma)) = 0.
\]

\[\]
Similarly, for \( p = 1 \), we find
\[
\mathcal{L}_{Q_0} H^{(3)} \equiv d(Q_a \cup \rho_1^a (\sigma \wedge \Gamma_a \sigma) \wedge \varepsilon^a) = 2 (\Gamma_a)_{\alpha \beta} d (\rho_1^a \sigma^\beta \wedge \varepsilon^a) = - (\Gamma_a)_{\alpha \beta} \Gamma_\gamma^a \rho_1^a (\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \varepsilon^\delta) \\
= - (\Gamma_a)_{\alpha \beta} \Gamma_\gamma^a \rho_1^a (\sigma^\beta \wedge \varepsilon^\gamma \wedge \varepsilon^\delta) = 0,
\]
where we have used the Fierz identities (4.2) and the symmetricity of the \( \Gamma^a \). Thus, the super-\((p+2)\)-cocycles for the super-0-brane \((p=0)\) and for the superstring \((p=1)\) are manifestly bi-invariant. This is not so for \( p > 1 \) as shown by the explicit computation below. Indeed, while
\[
\mathcal{L}_{P_a} H^{(p+2)} \equiv d (P_a \cup \rho_1^a (\sigma \wedge \Gamma_a \sigma) \wedge \varepsilon^a) = \rho_1^a d (\sigma \wedge \Gamma_a \sigma) = 0,
\]
where \(- \) once more \(-\) the Fierz identities (4.2) have been invoked, we also have
\[
\mathcal{L}_{Q_0} H^{(p+2)} \equiv d (Q_a \cup \rho_1^a (\sigma \wedge \Gamma_a \sigma) \wedge \varepsilon^a) = \rho_1^a d (\sigma \wedge \Gamma_a \sigma) \equiv \rho_1^a (\sigma \wedge \varepsilon^a) = \rho_1^a (\sigma \wedge \varepsilon^a) = 0.
\]
We conclude that there cannot exist, on the super-\(p\)-gerbes with \( p > 1 \), a \( G\)-equivariant structure with the embedding \( G \subset G \times G \) generated by linear combinations of both right- and left-invariant vector fields on \( G \) which are either non-chiral or left-chiral. In the distinguished cases \( p \in \{0, 1\} \), on the other hand, the possibility does exist and should be inspected carefully, which is what we turn to next.

In order to boost our intuition as to the possible equivariance scenarios in a natural direction, it is worth noting that the super-\(\sigma\)-model with \( p = 1 \) may, in fact, be regarded as a super-variant of the \( WZW \) \( \sigma\)-model on a Lie group. Indeed, with the obvious choice of the degenerate metric
\[
\bar{\eta} := \eta_{ab} \varepsilon^a \otimes \varepsilon^b : \mathcal{T} \sMink^{d_1|D_{d_1}} \otimes \sMink^{d_1|D_{d_1}} \mathcal{T} \sMink^{d_1|D_{d_1}} \mathcal{T} \sMink^{d_1|D_{d_1}} \to \mathbb{R}
\]
on the group manifold \( \sMink^{d_1|D_{d_1}} \), we readily find
\[
- \bar{\eta} \circ (\{ \cdot, \cdot \})_{\text{sm}\text{int}_{d_1|D_{d_1}}} \otimes \text{id}_{\sMink^{d_1|D_{d_1}}} \circ (\hat{\theta}_L \wedge \hat{\theta}_L) = \bar{\eta} \circ (\{ Q_a, Q_\beta \} \otimes P_a) \circ (\rho_1^a (\sigma^a \wedge \varepsilon^a)) = \eta_{ab} \Gamma_{(3)}^a \rho_1^a (\sigma^a \wedge \varepsilon^a) \subseteq H.
\]
Thus, being in mind the standard correspondence between the Nambu–Goto and Polyakov formulations of (the metric term of) the \( \sigma\)-model (the former being defined in terms of the same degenerate metric \( \bar{\eta} \) in the super-Minkowskian setting), we arrive at a super-\(\sigma\)-model structurally fully analogous with the two-dimensional \( \sigma\)-model (3.3). Our discussion of the symmetry content of the latter and of the amenability of its various rigid symmetries to gauging immediately suggest that we should look for an \( \text{Ad}(\sMink^{d_1|D_{d_1}})\)-equivariant structure on the Green–Schwarz super-1-gerbe \( \mathcal{G}^{(1)}_{\text{GS}} \) that was constructed in Part I as a supersymmetric geometrisation of \( H \). In fact, we freely extend this intuition to both super-\(p\)-gerbes \( \mathcal{G}^{(p)} \) with \( p \in \{0, 1\} \) and provide a rigorous confirmation thereof below.

Prior to launching a detailed study of the two candidates for \( \text{Ad}(\sMink^{d_1|D_{d_1}})\)-equivariant super-\(p\)-gerbes, we need to adapt the concept of supersymmetry, or invariance under the action of the supersymmetry group \( \sMink^{d_1|D_{d_1}} \), to the present context which will be developed along the lines of the construction of the Grassmannian structure reviewed at the end of Sec. 2. Since the construction unfolds over the nerve \( \mathcal{N} \\text{Ad}(\sMink^{d_1|D_{d_1}})\rightarrow \text{sm\text{ink}^{d_1|D_{d_1}}} \equiv \text{Ad}(\sMink^{d_1|D_{d_1}})\text{sm\text{ink}^{d_1|D_{d_1}}} \) of the relevant action groupoid \( \text{Ad}(\sMink^{d_1|D_{d_1}})\text{sm\text{ink}^{d_1|D_{d_1}}} \), as defined in and around Eq. (2.17), we should first look for an action of the supersymmetry group \( \sMink^{d_1|D_{d_1}} \) on each member \( \text{Ad}(\sMink^{d_1|D_{d_1}})\text{sm\text{ink}^{d_1|D_{d_1}}} \otimes \mathbb{N}^{\mathbb{N}} \text{sm\text{ink}^{d_1|D_{d_1}}} \), \( m \in \mathbb{N} \) of that object, and, in so doing, it is only natural to demand compatibility of the respective actions with the simplicial structure present. The most natural
notation of compatibility in this setting is equivariance of the face maps $d^{(m)}_i$, $i \in \{0, m\}$ with respect to the relevant actions. Actually, this choice fixes the actions uniquely once the action is defined at the lowest level of the ladder, that is on $\text{Ad}(\text{sMink}^{d,1}|D_{d,1})^{\alpha_0} \equiv \text{sMink}^{d,1}|D_{d,1}$. Let us state the result in abstraction from the particular supergeometric setting considered in the present paper. Thus, we consider a (super)manifold $\mathcal{M}$ endowed with a (left) (super)group action

$$\ell^{(0)} = \ell : G \times \mathcal{M} \rightarrow \mathcal{M},$$

fix a normal sub(-super)-group $H \subseteq G,$

$$\forall_{g \in G} : \text{Ad}_g(H) \subset H,$$

and subsequently look for a family of actions

$$\ell^{(m)} : G \times N^m(H \mathcal{M}) \rightarrow N^m(H \mathcal{M}), \quad m \in \mathbb{N}$$

of $G$ on members of the nerve $N^\bullet(H \mathcal{M}) \cong H^\bullet \times \mathcal{M}$ of the action groupoid

$$H \mathcal{M} : \xymatrix{G \times \mathcal{M} \ar[r]^-{\ell} & \mathcal{M}.}$$

determined by $\lambda$, for which all the face maps of the nerve are equivariant,

$$d^{(m)}_i \circ \ell^{(m)} = \ell^{(m-1)} \circ (\text{id}_G \times d^{(m)}_i), \quad i \in \{0, m\}, \quad m \in \mathbb{N}^\ast.$$  

This condition ensures that objects which are left-invariant with respect to the original action of $G$ pull back to objects with the same property with respect to the new action. The sought-after action on $N^\bullet(G \mathcal{M})$ reads

$$\ell^{(m)} : G \times N^m(H \mathcal{M}) \rightarrow N^m(H \mathcal{M})$$

(4.16)

$$\quad (g, (h_1, h_2, \ldots, h_n, x)) \mapsto (\text{Ad}_g(h_1), \text{Ad}_g(h_2), \ldots, \text{Ad}_g(h_n), \ell_g(m)), $$

and we have to impose the condition

$$\forall_{(g, h) \in G \times H} : \ell_g \circ \lambda_h \circ \ell_{g^{-1}} = \lambda_{\text{Ad}_g(h)}.$$  

This simple construction suggests an obvious adaptation of the notion of supersymmetry-invariance, previously formulated for tensors and geometric structures over $\mathcal{M}$, to wit,

**The Invariance Postulate:** We demand invariance of the geometric objects (tensors and their geometrisations) over components $N^m(H \mathcal{M})$ of the nerve, and equivariance of (iso)morphisms between them under the respective extensions $\ell^{(m)}$ of $\ell$.

The compatibility condition (4.17) constrains the admissible choices of $\ell$ once the action $\lambda$ is picked up. In particular, for $\lambda = \text{Ad}$ on $\mathcal{M} = G$, we are led to take $\ell = \text{Ad}$. This is the choice that we wind up studying below, and so we explicit here the adjoint action of the Lie supergroup $\text{sMink}^{d,1}|D_{d,1}$ on itself. This is given by

$$\text{Ad} : \text{sMink}^{d,1}|D_{d,1} \times \text{sMink}^{d,1}|D_{d,1} \rightarrow \text{sMink}^{d,1}|D_{d,1}$$

(4.18)

$$\quad (\varepsilon^a, y^a, (\theta^a, x^b)) \mapsto (\theta^a, x^a - \varepsilon^a \Gamma^a \theta).$$

We are now ready to perform a detailed analysis of the equivariance properties of the various objects defined over $N^\bullet(\text{Ad}(\text{sMink}^{d,1}|D_{d,1}) \equiv \text{sMink}^{d,1}|D_{d,1})$.

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19In the case of geometrisations ($n$-gerbes), it is natural to require the existence of isomorphisms between them and their pullbacks along the $\ell^{(m)}_g$, $g \in G$.
4.1. The Ad-equivariant Green–Schwarz super-0-gerbe. The small gauge anomaly for the adjoint action of the supersymmetry group $s\text{Mink}^{9,1|32}$ in the super-0-brane model can be read off from the structure of the basis sections of $E^{1,0}s\text{Mink}^{9,1|32}$ corresponding to this action. These are

$$\mathcal{V}_\alpha = \mathcal{R}_\alpha - \mathcal{L}_\alpha, \quad \alpha \in 1,32$$

$$\mathcal{V}_a = \mathcal{R}_a - \mathcal{L}_a, \quad a \in \overline{0,9},$$

where

$$\mathcal{R}_\alpha(\theta,x) = (\mathcal{Q}_\alpha(\theta,x), -2(\mathcal{T}_{11})_{\alpha\beta}\theta^\beta), \quad \mathcal{L}_\alpha(\theta,x) = (Q_\alpha(\theta,x), -2(\mathcal{T}_{11})_{\alpha\beta}\theta^\beta),$$

$$\mathcal{R}_a(\theta,x) = (\mathcal{P}_a(\theta,x), 0), \quad \mathcal{L}_a(\theta,x) = (P_a(\theta,x), 0),$$

and so

$$\mathcal{V}_\alpha(\theta,x) = (-\mathcal{T}_{11}^{\alpha\beta}\theta^\beta \partial_a, 0), \quad \mathcal{V}_a(\theta,x) = (0,0).$$

The superbracket of the basis sections is identically zero, which makes it anomaly free (here, the $f_{AB}^C$ are the structure constants of $s\text{Mink}^{9,1|32}$)

$$[\mathcal{V}_\alpha, \mathcal{V}_\beta]_V^{(2)} = 0 \equiv -f_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\mathcal{V}_a, \mathcal{V}_b]_V^{(2)} = 0 \equiv -f_{ab}^\gamma \mathcal{V}_\gamma.$$

This can be seen independently by verifying the relevant equivariance and symmetry identities [4.15] which are satisfied trivially. The conclusion of our (super)algebroidal analysis is that, as expected, the small gauge anomaly vanishes for the adjoint action. From the analysis, we also extract the relevant super-1-form

$$\theta - \tilde{\theta}_{(1)} \equiv -pr_2^* A \tilde{\theta}_L = 0.$$ 

(4.19)

Eq (2.18). Thus equipped, we may next examine the large gauge anomaly.

In order to be able to develop some intuition as to the right questions to be asked in the super-Minkowskian setting, let us abstract from its peculiarities for a while and consider an arbitrary principal $C^\times$-bundle

$$\begin{array}{ccc}
\mathbb{C}^\times & \longrightarrow & \mathfrak{Y}M \\
\downarrow \pi_{\mathfrak{Y}M} & & \downarrow \pi_{\mathfrak{Y}M} \\
\mathcal{M} & \longrightarrow & \mathcal{M}
\end{array}$$

over a supermanifold $\mathcal{M}$ with a principal $C^\times$-connection $A$ of curvature $H$. Assume given, as before, an action $\ell : G \times \mathcal{M} \longrightarrow \mathcal{M}$ of the supersymmetry Lie supergroup $G$ on the base of the bundle that lifts to the total space of the bundle as an action (possibly projective)

$$\begin{array}{ccc}
G \times \mathfrak{Y}M & \overset{\mathcal{Y}_\ell}{\longrightarrow} & \mathfrak{Y}M \\
\downarrow \text{id}_G \times \pi_{\mathfrak{Y}M} & & \downarrow \pi_{\mathfrak{Y}M} \\
G \times \mathcal{M} & \overset{\ell}{\longrightarrow} & \mathcal{M}
\end{array}$$

(4.20)

commuting with the defining action

$$r^\mathfrak{Y}M : \mathfrak{Y}M \times C^\times \longrightarrow \mathfrak{Y}M$$

of the structure group $C^\times$ on $\mathfrak{Y}M$,

$$\forall_{(g,z)\in G \times C^\times} : \mathcal{Y}_{\ell g} \circ r^\mathfrak{Y}M_2 = r^\mathfrak{Y}M_2 \circ \mathcal{Y}_{\ell g},$$

and preserving the principal $C^\times$-connection super-1-form,

$$\forall_{g\in G} : \mathcal{Y}_{\ell g}^* A = A.$$
The above lift will be used to verify the invariance of geometric objects and the equivariance of maps referred to in the Invariance Postulate. In particular, it is readily seen to induce a family \( \phi \equiv \{ \phi_g \}_{g \in G} \) of (connection-preserving) principal \( C^\ast \)-bundle isomorphisms

\[
\ell_g^*YM \xrightarrow{\phi_g} YM , \quad g \in G ,
\]

written for the pullback principal \( C^\ast \)-bundle with the total space \( \ell_g^*YM \) which we may take in the form

\[
\ell_g^*YM \equiv YM
\]

so that the corresponding pullback connection super-1-form reads

\[
\tau_g^*A \equiv \gamma\ell_g^*A = A .
\]

With the pullback data thus chosen, we obtain the identities

\[
\ell_g^*YM \equiv YM ,
\]

whence also the isomorphisms sought after

\[
\phi_g \equiv \text{id}_{YM} .
\]

Their existence permits us to think of the pair \((\ell, \gamma\ell)\) as an effective realisation of supersymmetry in the present context in which we seek a supersymmetric \( H \)-equivariant structure on \( YM \).

The concept of equivariance is based on the assumption of existence of an action \( \lambda : H \times M \rightarrow M \) of a normal Lie sub-supergroup \( H \subset G \) on \( M \), with property (4.17). The latter map enters the definition of a (connection-preserving) principal \( C^\ast \)-bundle isomorphism

\[
\lambda^*YM \xrightarrow{\tau_g} pr_2^*YM \otimes \mathcal{I}_g \quad (1)
\]

over \( H \times M \) in which \( \lambda^*YM \) and \( pr_2^*YM \) are the pullback principal \( C^\ast \)-bundles described by the respective commutative diagrams

\[
(4.21)
\]

and \( \mathcal{I}_g \) is the trivial principal \( C^\ast \)-bundle with the global connection 1-form \( \varrho \in \Omega^1(H \times M) \) on its base \( H \times M \), of curvature

\[
d \varrho_{(1)} = (\lambda^* - pr_2^*) H_{(2)} .
\]
The connection 1-form has to obey the identity (the $d_i^{(2)}$ are the face maps of $\mathbb{N}^*(\mathbb{H} \ltimes \mathcal{M})$)

\[(d_0^{(2)} + d_2^{(2)} - d_1^{(2)}) \varrho = 0\]

in order that the coherence constraint

\[(d_0^{(2)} \ast \varrho_0 \otimes \text{id}_{d_2^{(2)} \ast \varrho}) \circ d_2^{(2)} \ast \varrho_0 = d_1^{(2)} \ast \varrho_0\]

may be imposed upon $\Upsilon_0$.

It is straightforward to extract from the above definitions structural properties of the functional realisation of $\Upsilon_0$. Indeed, we have

$$\Upsilon_0 : (H \times \mathcal{M}) \times_\lambda \mathcal{M} \to (H \times \mathcal{M}) \times \text{pr}_2 \mathcal{M} : ((h,m),y) \mapsto ((h,m),\varphi_h(y))$$

for some smooth maps

$$\varphi_\cdot : H \times \mathcal{M} \to \mathcal{M}$$

that satisfy

$$\pi_{\mathcal{M}} \circ \varphi_h(y) = \lambda_{h^{-1}} \ast \pi_{\mathcal{M}}(y)$$

for

$$\pi_{\mathcal{M}}(y) \equiv \lambda_h(m),$$

and, in virtue of Eq. (4.23),

$$\forall h_1, h_2 \in H : \varphi_{h_1 \circ h_2} = \varphi_{h_2} \circ \varphi_{h_1}.$$  

Preservation of the connection by $\Upsilon_0$ is reflected in the condition (valid for any $h$ and $y$ as above)

\[\left(\left(\varphi \circ (\text{pr}_1 \times \text{id}_{\mathcal{M}})\right)^{\ast} - \text{pr}_2^{\ast}\right) \mathcal{A}(\varphi_{h_1})(h,m) = \varrho(h,m).\]

We may, next, discuss the supersymmetry of the $H$-equivariant structure thus defined. The first step towards it is the definition of a lift of the realisation $\ell^{(1)}$ of supersymmetry on the common base $H \times \mathcal{M}$ of the bundles $\mathcal{M}$ and $\mathcal{L}^\ast \mathcal{M}$ to the respective total spaces. There is an obvious candidate:

$$\Upsilon \ell^{(1)} : G \times ((H \times \mathcal{M}) \times \mathcal{M}) \to (H \times \mathcal{M}) \times \mathcal{M} : (g,((h,m),y)) \mapsto ((\text{Ad}_g(h),\ell_g(m)),\Upsilon_0 \ell_g(y))$$

for the precursor of both realisations, and we merely have to check that it restricts to either sub(supersymmetry) manifold. This is trivial in the case of $\text{pr}_2^* \mathcal{M}$, whereas for $\mathcal{L}^* \mathcal{M}$, we have to use identity (4.17) to obtain, for any $y \in \mathcal{M}$ such that $\pi_{\mathcal{M}}(y) = \lambda_h(m)$,

$$\pi_{\mathcal{M}}(\Upsilon_0 \ell_g(y)) = \ell_g \circ \pi_{\mathcal{M}}(y) = \ell_g \circ \lambda_h(m) = \lambda_{\text{Ad}_g(h)}(\ell_g(m)),$$

as desired. With the action well-defined, we require that the 1-isomorphism of the $H$-equivariant structure be $G$-equivariant,

\[(4.25) \quad \forall g \in G : \Upsilon_0 \circ \Upsilon_\ell^{(1)} \mid_{\mathcal{L}^* \mathcal{M}} = \Upsilon_\ell^{(1)} \circ \Upsilon_0,\]

and that the corresponding connection 1-form $\varrho$ be $G$-invariant,

\[(4.25) \quad \forall g \in G : \ell_g^{(1)} \ast \varrho = \varrho \circ \ell_g^{(1)} \ast \varrho = \varrho .\]

Condition (4.25) is readily seen to transcribe into the statement of equivariance:

$$\forall (g,h) \in G \times H : \varphi_{\text{Ad}_g(h)} = \Upsilon_\ell \circ \varphi_h.$$  

We summarise a specialisation of our findings to the setting of immediate interest in

**Definition 4.6.** Let $\varrho^{(0)}_{\text{CaE}} = (\Upsilon G, \pi_{\mathcal{G}}, \mathcal{A})^{(1)}$ be a Cartan–Eilenberg super-0-gerbe of curvatures $\mathcal{H}$, as in Def. 4.2, endowed with a lift

$$\Upsilon \text{Ad} : G \times \mathcal{G} \to \mathcal{G}.$$
of the adjoint action $\text{Ad}$ to the total space of the bundle $YG$, described by the commutative diagram

$$
\begin{array}{c}
G \times YG \\
\downarrow \text{id}_G \times \piYG \\
G \times G
\end{array}
\xrightarrow{\text{id}_G \times \piYG}
\begin{array}{c}
YAd \\
\downarrow \piYG
\end{array}
\begin{array}{c}
YG \\
\downarrow \piYG
\end{array}
\xrightarrow{YAd}
(4.26)
$$

and required to commute with the defining action $r^{YG_\cdot}$, $YG \times \mathbb{C}^\times \to YG$ of the structure group $\mathbb{C}^\times$ on $YG$,

$$
\forall_{(g,z) \in G \times \mathbb{C}^\times} : YAd_g \circ r^{YG_\cdot}_z = r^{YG_\cdot}_z \circ YAd_g
$$

and to preserve the principal connection super-1-form

$$
\forall g \in G : YAd^*_g A^{(1)} = A^{(1)}
$$

(4.27)

in which the $f^{C}_{AB}$ are the structure constants of the Lie superalgebra of $G$,

$$
[L_A, L_B] = f^{C}_{AB} L_C.
$$

Define a super-1-form on $G \times G$ by the formula

$$
\theta^{(0)} \circ \tilde{\theta}_L = -pr_2^* \kappa^V_A pr_1^* \tilde{\theta}_L^{(0)},
$$

expressed in terms of components $\tilde{\theta}_L^{(0)}$ of the left-invariant Maurer–Cartan super-1-form on $G$, and denote by $\mathcal{I}_{\theta^{(0)}}$ the trivial principal $\mathbb{C}^\times$-bundle over $G \times G$ with the global connection 1-form [4.28]. A

**supersymmetric Ad-equivariant structure on $G^{(0)}_{\text{CaE}}$ relative to $\theta^{(0)}_L$** is a connection-preserving isomorphism

$$
\begin{array}{c}
\text{Ad}^*YG \\
\downarrow \text{pr}_2^* YG \oplus \mathcal{I}_{\theta^{(0)}_L}
\end{array}
\xrightarrow{Y_0}
\begin{array}{c}
\text{pr}_2^* YG \\
\downarrow \text{pr}_1
\end{array}
\begin{array}{c}
G \times G \\
\downarrow \text{id}_{G \times G}
\end{array}
\xrightarrow{\text{id}_{G \times G}}
\begin{array}{c}
G \times G
\end{array}
$$

of principal $\mathbb{C}^\times$-bundles over $G \times G$ subject to the coherence constraint

$$
(d^{(2)}_0 \cdot Y_0 \oplus \text{id}_G \circ \mathcal{I}_{\theta^{(0)}_L} \circ d^{(2)}_2) \circ d^{(2)}_1 \cdot Y_0 = d^{(2)}_1 \cdot Y_0
$$

over $G^{*2} \times G$, written in terms of the face maps $d^{(2)}_i$, $i \in \{0, 1, 2\}$ of the nerve $N^*(G \bowtie G) \equiv G^{*2} \times G$ of the action groupoid

$$
(4.29)
$$

$$
\text{GoG} : G \times G \xrightarrow{\text{Ad} \circ \text{pr}_2} G
$$

associated with the adjoint action of $G$ on itself, and such that the identities

$$
\forall g \in G : Y_0 \circ \text{Ad}_g^{(1)} \circ \text{Ad}^*YG = \text{Ad}_g^{(1)} \circ Y_0,
$$

$$
\forall g \in G :
$$
satisfying the identities

\[ \pi_{YG} \circ \varphi_h(y) = \text{Ad}_{h^{-1}} \circ \pi_{YG}(y), \quad ((\varphi \circ (pr_1 \times \text{id}_{YG}))^* - pr_2^* A)((h, g), y) = \varphi_{-\bar{\eta}_L}(h, g) \]

for all \( h \in G \) and

\[ \pi_{YG}(y) = \text{Ad}_h(g), \]

as well as

\[ \forall_{g_1, g_2 \in G} : \varphi_{g_1 \cdot g_2} = \varphi_{g_2} \circ \varphi_{g_1} \]

and

\[ \forall_{(g, h) \in G \times G} : \varphi_{\text{Ad}_g(h)} = Y\text{Ad}_g \circ \varphi_h. \]

**Remark 4.7.** Note that the other identities:

\[ \forall_{g \in G} : \text{Ad}_{g}^{(1) \cdot \ast} \varphi_{-\bar{\eta}_L} = \varphi_{-\bar{\eta}_L} \]

and \((4.22)\), to be imposed for

\[ \text{Ad}^{(1)} : G \times (G \times G) \rightarrow G \times G : (g, (h, k)) \mapsto (\text{Ad}_g(h), \text{Ad}_g(k)) \]

and the face maps \( \delta_{1}^{(2)}, \ i \in \{0, 1, 2\} \) of \( N^*(G \circ G) \), are satisfied automatically for \( \varphi_{-\bar{\eta}_L} \) of Eq. \((4.28)\).

Indeed, the first of conditions \((4.42)\) integrates to

\[ \forall_{g \in G} : \text{Ad}_{g}^{(1) \cdot \ast} \kappa^V_{A} = (T_{e\text{Ad}_g^{-1}})^{(B)} \kappa^V_{B}, \]

and so we obtain

\[ \text{Ad}_{g}^{(1) \cdot \ast} \varphi_{-\bar{\eta}_L} = -\text{pr}^{2 \ast}_{g} \text{Ad}_{g}^{(1) \cdot \ast} \varphi_{-\bar{\eta}_L} = -\text{pr}_{g}^{2 \ast} \varphi_{-\bar{\eta}_L} (T_{e\text{Ad}_g^{-1}})^{(B)} \kappa^{V}_{A} = -\text{pr}_{g}^{2 \ast} \kappa^{V}_{B} \varphi_{-\bar{\eta}_L} = \varphi_{-\bar{\eta}_L}. \]

A similar argument shows that identity \((4.22)\) holds true in the present setting.

Last, we shall specialise the general construction of a supersymmetric H-equivariant structure to the case of a trivial principal \( C^\times \)-bundle \( \mathcal{Y}M \equiv \mathcal{M} \times \mathbb{C}^\times \) equipped with a global connection 1-form,

\[ A((m, z)) = \frac{idz}{z} + A((m)). \]

In this case, the lift of the realisation \( \ell_g, g \in G \) of supersymmetry on \( \mathcal{M} \) to \( \mathcal{Y}M \equiv \mathcal{M} \times \mathbb{C}^\times \) can be written as

\[ \forall \ell_g : \mathcal{M} \times \mathbb{C}^\times \ni (m, z) \mapsto (\ell_g(m), e^{ip_{\mathcal{M}}(m)} \cdot z) \]

in terms of smooth maps

\[ \mu_g : \mathcal{M} \rightarrow \mathbb{R}, \quad g \in G \]

subject to the constraints

\[ (\ell_g^* - id_{\mathcal{M}})^{(1)} = d\mu_g \]

resulting from the imposition of the requirement of invariance of the connection.

The isomorphism \( \Upsilon_0 \) takes the form

\[ \Upsilon_0 : (H \times \mathcal{M}) \times_{\lambda} (\mathcal{M} \times \mathbb{C}^\times) \rightarrow (H \times \mathcal{M}) \times_{pr_2} (\mathcal{M} \times \mathbb{C}^\times) \]

\[ : ((h, m), (\lambda_h(m), z)) \mapsto ((h, m), (m, e^{-i \chi_h(m)} \cdot z)) \]
for smooth maps
\[ \chi_\cdot : H \times \mathcal{M} \rightarrow \mathbb{R}, \]

and the condition of preservation of the connection reads
\[ (\lambda^* - pr_2^* \lambda)_{(1)} \equiv \theta_{(1)} = d\chi_\cdot. \]

The coherence condition (4.24) now boils down to
\[ (\forall h_1, h_2 \in H : \chi_{h_1, h_2} = \lambda_{h_2}^* \chi_{h_1} + \chi_{h_2}). \]

Finally, we readily derive a transcription of the condition of supersymmetry-equivariance of \( \Upsilon_0 \):
\[ (\forall h \in H : \lambda^*_h \mu_g = \ell^*_h \chi_{Ad_g(h)} - \chi_h). \]

Our hitherto considerations enable us to phrase the anticipated

**Theorem 4.8.** The Green–Schwarz super-0-gerbe \( G_{\text{GS}}^{(0)} \) of Def. I.5.2, recalled on p. 24, carries a canonical supersymmetric \( \text{Ad.}-\text{equivariant} \) structure \( \Upsilon_0 \) with respect to the adjoint action of the Lie supergroup \( \text{sMink}^{9,1|32} \) on itself relative to the super-1-form \( \rho_{\text{Mink}} = 0 \), as described in Def. I.4.

**Proof.** A proof is given in App. D. \( \square \)

**Remark 4.9.** When \( \mathcal{M} \equiv G \) is the supersymmetry group itself, with \( \ell \equiv \text{Ad} \), and the principal \( \mathbb{C}^* \)-bundle \( \Upsilon G \) is a Cartan–Eilenberg super-0-gerbe \( G_{\text{CaE}}^{(0)} = (\Upsilon G, \pi G, \mathcal{A}) \) of Def. I.2, we could additionally demand, in Def. I.4, that each \( \Upsilon \ell_g \) be a Lie-supergroup homomorphism. For \( \Upsilon G \) trivial and the binary operation on the central extension \( \Upsilon G \equiv G \times \mathbb{C}^* \) given by the formula \( (\Lambda \equiv \mathbb{R} \text{-valued} \text{ 2-cocycle on } G) \)
\[ \Upsilon m : (G \times \mathbb{C}^*)^2 \rightarrow G \times \mathbb{C}^* : ((g_1, z_1), (g_2, z_2)) \mapsto (m(g_1, g_2), e^{i \Lambda(g_1, g_2)} \cdot z_1 \cdot z_2), \]

written in terms of the group operation \( m : G^2 \rightarrow G \) on \( G \), we should then obtain, furthermore, the extra constraints
\[ (m^* - pr_1^* - pr_2^*) \mu_g = (\text{Ad}_g^{x^2} \cdot \text{id}^*_{\text{G} \times G}) \Lambda. \]

These are trivially satisfied in the super-Minkowskian setting as for \( \Lambda = \lambda^{(0)} \) of Eq. (I.7) we find
\[
(\text{Ad}_{\text{G} \times G}^{x^2} \cdot \text{id}^*_{\text{G} \times G}^{(9,1|32)}) \lambda^{(0)}((\theta_1, x_1), (\theta_2, x_2)) \\
= \lambda^{(0)}((\theta_1, x_1 - \varepsilon \Gamma \theta_1), (\theta_2, x_2 - \varepsilon \Gamma \theta_2)) - \lambda^{(0)}((\theta_1, x_1), (\theta_2, x_2)) \equiv 0,
\]
consistently with the result \( \mu^{(\varepsilon, y)} \equiv 0 \) derived in App. D. Nevertheless, we consider this structural condition unnecessarily restrictive in general. Indeed, the sole rationale behind it is the ability to internalise the ensuing isomorphisms \( \phi_g, g \in G \) in the Lie-supergroup category. However, thinking of these isomorphisms as symmetries of the corresponding super-\( \sigma \)-model merely requires invariance of the Dirac–Feynman amplitudes under \( \ell \equiv \text{Ad} \), and this calls for an arbitrary (connection-preserving) principal \( \mathbb{C}^* \)-bundle isomorphism. In fact, as we shall see in the next example, there are situations in which the more restrictive definition of a supersymmetric \( H \)-equivariant structure on a super-\( p \)-gerbe over the supersymmetry group \( G \) actually fails.

### 4.2. The \text{Ad.-equivariant Green–Schwarz super-1-gerbe}

As before, we begin with the derivation of the small gauge anomaly for the adjoint action of the supersymmetry group \( \text{sMink}^{(d,1)\overrightarrow{D}_{d,1}} \). The relevant basis sections of \( \mathcal{E}^{1,1}_{\text{SMink}^{(d,1)\overrightarrow{D}_{d,1}}} \) are
\[ \mathfrak{R}_a = \mathfrak{R}_a - \mathcal{L}_a, \quad a \in \Gamma_{\overrightarrow{D}_{d,1}} \]
\[ \mathfrak{U}_a = \mathfrak{R}_a - \mathcal{L}_a, \quad a \in \overrightarrow{D}_{d,1}, \]

where
\[ \mathfrak{R}_a(\theta, x) = (\mathfrak{P}_a(\theta, x), -2(\Gamma a)_{\alpha \beta} \theta^\beta (dx^\alpha - \frac{1}{6} \theta \Gamma^a \sigma(\theta))) , \]
\[ \mathfrak{L}_a(\theta, x) = (Q_a(\theta, x), -2(\Gamma a)_{\alpha \beta} \theta^\beta (dx^\alpha + \frac{1}{6} \theta \Gamma^a \sigma(\theta))) , \]
\[ \mathfrak{R}_a(\theta, x) = (\mathfrak{P}_a(\theta, x), -\theta \Gamma_a \sigma(\theta)) , \]
\[ \mathfrak{L}_a(\theta, x) = (Q_a(\theta, x), -\theta \Gamma_a \sigma(\theta)) . \]
and so
\[ \mathfrak{W}_a(\theta, x) = \left( -\mathcal{T}_{a, \beta} \theta \partial_a, \frac{2}{3} (\mathcal{T}_a)_{a, \beta} \theta \mathcal{T}^\beta \sigma(\theta) \right), \quad \mathfrak{W}_a(\theta, x) = (0, 0). \]
The superbracket of the basis sections is – once more – identically zero, and hence anomaly free
\[ \left[ \mathfrak{W}_a, \mathfrak{W}_b \right]_V^{(2)} = 0 \equiv -\mathcal{T}_{a, \beta} \mathfrak{W}_b \equiv -f_{ab} A \mathfrak{W}_b, \quad \left[ \mathfrak{W}_a, \mathfrak{W}_b \right]_V^{(2)} = 0 \equiv -f_{ab} A \mathfrak{W}_a. \]

Equivalently, we check the equivariance and symmetry identities (4.15). Altogether, this means that, once again, we first perform our analysis in abstraction from the concrete supergeometric context in hand so as to avoid the situation in which peculiarities of the latter obscure or deform the general concept whose construction, significantly more complex than its lower-dimensional counterpart discussed in Sec. 4.1, is guided by the postulate of naturality in the categories of bundle gerbes and general concept whose construction, significantly more complex than its lower-dimensional counterpart discussed in Sec. 4.1, is guided by the postulate of naturality in the categories of bundle gerbes and super-1-forms, and then further to the super-Minkowskian setting. Thus, consider the super-1-gerbe \( G^{(1)} = (\mathcal{M}, \pi_{\mathcal{M}}, L, \mathcal{A}_L, \mu_{L}) \) of curvature \( H \in \Omega^3(\mathcal{M}) \), described by the diagram
\[
\begin{array}{ccc}
\mu_{L} : pr_{1, 2}^* L \otimes pr_{2, 3}^* L & \overset{\mu_{L}}{\longrightarrow} & pr_{1, 3}^* L \\
Y[3] & \overset{pr_{1, 2}}{\longrightarrow} & Y[2] \\
\partial L, \mathcal{A}_L & \overset{pr_{1, 3}}{\longrightarrow} & \mathcal{M}, \mathcal{M} \\
\end{array}
\]

and composed of a surjective submersion \( \pi_{\mathcal{M}} \) with a global primitive (curving) \( B \in \Omega^2(\mathcal{M}) \) of the pullback of \( H \) to its total space \( \mathcal{M} \),
\[ \pi_{\mathcal{M}}^* H = d B. \]
and of a principal \( C^\ast \)-bundle \( L \) over the fibred square
\[ \mathcal{Y}[2] = \{ (y_1, y_2) \in \mathcal{M}^2 \mid \pi_{\mathcal{M}}(y_1) = \pi_{\mathcal{M}}(y_2) \}, \]
endowed with a principal \( C^\ast \)-connection super-1-form \( \mathcal{A}_L \in \Omega^1(L) \) of curvature \( (pr_2^* - pr_1^*) B \) and a (connection-preserving) principal \( C^\ast \)-bundle isomorphism \( \mu_{L} \) over the fibred cube
\[ \mathcal{Y}[3] = \{ (y_1, y_2, y_3) \in \mathcal{M}^3 \mid \pi_{\mathcal{M}}(y_1) = \pi_{\mathcal{M}}(y_2) = \pi_{\mathcal{M}}(y_3) \}, \]
that induces a groupoid structure on its fibres, being subject to the associativity constraint
\[
\text{pr}_{1, 2, 4} \mu_{L} \circ (\text{id}_{\text{pr}_{1, 2}^* L} \otimes \text{pr}_{2, 3, 4}^* \mu_{L}) = \text{pr}_{1, 3, 4} \mu_{L} \circ (\text{pr}_{1, 2, 3} \mu_{L} \otimes \text{id}_{\text{pr}_{3, 4}^* L})
\]
over
\[ \mathcal{Y}[4] = \{ (y_1, y_2, y_3, y_4) \in \mathcal{M}^4 \mid \pi_{\mathcal{M}}(y_1) = \pi_{\mathcal{M}}(y_2) = \pi_{\mathcal{M}}(y_3) = \pi_{\mathcal{M}}(y_4) \}, \]

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As before, we assume $\mathcal{M}$ to be equipped with an action $\ell : G \times \mathcal{M} \to \mathcal{M}$ of the supersymmetry Lie supergroup $G$ that lifts to the total space $\mathcal{Y}\mathcal{M}$ of the surjective submersion as in Diag. 4.20, and so also to the fibre products

$$\mathcal{Y}^{[n]}\mathcal{M} = \left\{ (y_1, y_2, \ldots, y_n) \in \mathcal{Y}\mathcal{M}^\times^n \mid \pi_{\mathcal{Y}\mathcal{M}}(y_1) = \pi_{\mathcal{Y}\mathcal{M}}(y_2) = \ldots = \pi_{\mathcal{Y}\mathcal{M}}(y_n) \right\}$$

as per

$$\mathcal{Y}^{[n]}\ell : G \times \mathcal{Y}^{[n]}\mathcal{M} \to \mathcal{Y}^{[n]}\mathcal{M} : (g, (y_1, y_2, \ldots, y_n)) \mapsto \left( Y \ell_g(y_1), Y \ell_g(y_2), \ldots, Y \ell_g(y_n) \right),$$

in such a manner that the curving $B$ is preserved,

$$\forall_{g \in G} : \mathcal{Y}\ell^*_g B = B,$$

as well as to the total space $L$ of the principal $\mathbb{C}^\times$-bundle as an action (possibly projective)

$$G \times L \xrightarrow{L\ell} L$$

in such a manner that it commutes with the defining action

$$r_L^L : L \times \mathbb{C}^\times \to L$$

of the structure group $\mathbb{C}^\times$ on $L$,

$$\forall_{(g, z) \in G \times \mathbb{C}^\times} : L\ell_g \circ r_L^L = r_L^L \circ L\ell_g,$$

and preserves the principal $\mathbb{C}^\times$-connection super-1-form,

$$\forall_{g \in G} : L\ell^*_g A_L \equiv A_L \in \mathcal{A}_L.$$

The above lifts then induce actions on the pullback bundles

$$\mathcal{Y}^{[3]}\mathcal{M} \xrightarrow{pr_{i,j}} \mathcal{Y}^{[3]}\mathcal{M} \times_{pr_{i,j}} L \xrightarrow{pr_2} L$$

over $\mathcal{Y}^{[3]}\mathcal{M}$ given by

$$L_{i,j} : G \times pr_{i,j}^* L \to pr_{i,j}^* L : \left( g, ((y_1, y_2, y_3), p) \right) \mapsto \left( Y \ell_g(y_1), Y \ell_g(y_2), Y \ell_g(y_3), L\ell_g(p) \right),$$

and hence also on the tensor-product bundle

$$L_{1,2;3} : G \times (pr_{1,2}^* L \otimes pr_{2,3}^* L) \to pr_{1,2}^* L \otimes pr_{2,3}^* L$$

written in the notation

$$\left( (y_1, y_2, y_3), p_1 \right) \otimes \left( (y_1, y_2, y_3), p_2 \right) = \left\{ \left( (y_1, y_2, y_3), r_{L}^L(p_1), (y_1, y_2, y_3), r_{L}^L(p_2) \right) \mid z \in \mathbb{C}^\times \right\},$$

and we assume equivariance of the groupoid structure with respect to these induced actions,

$$\forall_{g \in G} : \mu_L \circ L_{1,2;3} \ell_g = L_{1,2;3} \ell_g \circ \mu_L.$$

The triple $(\ell, \mathcal{Y}\ell, \mathcal{L}\ell)$ of actions serves to distinguish invariant geometric objects and equivariant maps between them referred to in the Invariance Postulate. By way of a sanity check, we verify that they give rise to isomorphisms of (super-)1-gerbes

$$\Phi_g : \ell^* g \mathcal{G}^{(1)} \xrightarrow{\sim} \mathcal{G}^{(1)}, \quad g \in G,$$
understood as in Sec. 1.2.1. To this end, we make a convenient choice of the surjective submersion of the pullback 1-gerbe,

\[ \ell_g^* \YM \equiv \YM \xrightarrow{\ell_g \equiv \ell_g} \YM \]

\[ \pi^{\ell_g}_{\YM} = \pi_{\YM}, \]

\[ \mathcal{M} \xrightarrow{\ell_g} \YM \]

\[ \mathcal{Y}^{[2]} \mathcal{M} \]

for which we compute the corresponding curving

\[ \ell_g^* \mathcal{B} \equiv \YM_{\ell_g} = \mathcal{B}. \]

Next, we erect the pullback principal \( \mathbb{C}^* \)-bundle \( \widehat{\ell}_g^* L \) over \( \ell_g^* \YM \times \mathcal{M} \ell_g^* \YM \equiv \mathcal{Y}^{[2]} \mathcal{M} \) by, once more, choosing the pullback judiciously in the form

\[ \widehat{\ell}_g^* L \equiv \mathcal{Y}^{[2]} \ell_g^* L \]

\[ \pi_{\widehat{\ell}_g^* L} = \pi_L, \]

\[ \mathcal{Y}^{[2]} \mathcal{M} \xrightarrow{\ell_g} \YM \]

\[ \mathcal{Y}^{[2]} \mathcal{M} \]

so that – in particular – we obtain the pullback connection super-1-form

\[ \overline{\mathcal{Y}^{[2]} \ell_g^* A_L} \equiv \mathcal{L} \ell_g^* A_L = \mathcal{A}_L. \]

It is now completely straightforward to see that the pullback groupoid structure reads

\[ \overline{\ell}_g^* \mu_L \equiv \mathcal{L}^1 \ell_g \mu_L \circ \mathcal{L} \ell_{1,2,3} \ell_g = \mu_L, \]

and so, by the end of the day, we obtain the identity

\[ \ell_g^* \mathcal{G}^{(1)} \equiv \mathcal{G}^{(1)}, \]

or

\[ \Phi_g \equiv \mathrm{id}_{\mathcal{G}^{(1)}}. \]

Thus, once more, we are led to think of the triple \( (\ell_g, \YM_{\ell_g}, \mathcal{L}) \) as an effective higher-geometric realisation of supersymmetry, to be employed in the definition of a supersymmetric H-equivariant structure on \( \mathcal{G}^{(1)} \).

As in the case of a supersymmetric H-equivariant super-0-gerbe, the point of departure is an action \( \lambda : H \times M \rightarrow \mathcal{M} \) of a normal Lie sub-supergroup \( H \subset G \) on \( \mathcal{M} \) that satisfies Eq. (1.17). With this, we associate an isomorphism

\[ \Upsilon_1 : \lambda^* \mathcal{G}^{(1)} \xrightarrow{\cong} \mathcal{P}_{\ell_g} \mathcal{G}^{(1)} \otimes \mathcal{I}_{\ell_g} \]

of 1-gerbes over \( H \times \mathcal{M} \), written in terms of a trivial 1-gerbe \( \mathcal{I}_{\ell_g} \) with a global curving \( \varrho \in \Omega^2(H \times \mathcal{M}) \) of curvature

\[ d \varrho = (\lambda^* \circ \mathcal{P}_{\ell_g} \mathcal{G}^{(1)}) \]

that satisfies the identity (the \( d^{(2)}_i \) are the face maps of \( \mathbb{N}^*(H \times \mathcal{M}) \))

\[ (d^{(2)*}_0 + d^{(2)*}_2 - d^{(2)*}_1) \varrho = 0. \]

The above is a necessary condition for the existence of a 2-isomorphism

\[ (4.33) \quad \gamma_1 : \left( d^{(2)*}_0 \Upsilon_1 \otimes \mathrm{id}_{\mathcal{P}_{\ell_g} \mathcal{G}^{(1)}} \right) \circ d^{(2)*}_2 \Upsilon_1 \xrightarrow{\cong} d^{(2)*}_1 \Upsilon_1 \]
of 1-isomorphisms over $H^\times 2 \times M$, the latter being subject to the coherence constraint

$$d_1^{(3)} \ast \gamma_1 \circ (\text{id}_{d_1^{(2)}} \ast \rho_1) \ast d_1^{(3)} \ast \gamma_1 = d_2^{(3)} \ast \gamma_1 \circ (\text{id}_{d_2^{(2)}} \ast \rho_2) \ast d_2^{(3)} \ast \gamma_1 \circ \text{id}_{d_2^{(2)}} \ast \rho_2 \ast d_2^{(3)} \ast \gamma_1 \circ \text{id}_{d_2^{(2)}} \ast \rho_2 \ast d_2^{(3)} \ast \gamma_1$$

over $H^\times 3 \times M$. In what follows, we shall need a more explicit description of the various components of the $H$-equivariant structure on which an action of the supersymmetry group $G$ will be either assumed or induced. We begin with the data of the 1-isomorphism $\Upsilon_1$. These consist of a principal $\mathbb{C}^*$-bundle

$$\mathbb{C}^* \to E \quad \pi_E$$

$$\lambda^*YM \times_{H \times M} \text{pr}_2^*YM$$

with a principal $\mathbb{C}^*$-connection $A_E \in \Omega^1(E)$ over the fibred product of the two pullbacks of the surjective submersion $YM$ of the 1-gerbe $G^{(1)}$ that we choose in the form determined by the commuting diagrams (4.21). The said fibred product fits into the commutative diagram

$$Y_{\lambda^2}M \equiv \lambda^*YM \times_{H \times M} \text{pr}_2^*YM \xrightarrow{\text{pr}_2} \text{pr}_2^*YM$$

$$\lambda^*YM \xrightarrow{\text{pr}_1} H \times M$$

Over its fibred square

$$Y_{\lambda^2\lambda^2}M \equiv \lambda^*YM \times_{H \times M} \text{pr}_2^*YM \times_{H \times M} \lambda^*YM \times_{H \times M} \text{pr}_2^*YM,$$

we find the last piece of data associated with $\Upsilon_1$, to wit, a (connection-preserving) isomorphism

$$\alpha_E : \text{pr}_1^* \lambda^{x^2} L \otimes \text{pr}_2^* E \xrightarrow{\sim} \text{pr}_1^* \lambda^{x^2} L \otimes \text{pr}_2^* \text{pr}_2^* \lambda^{x^2} L$$

of principal $\mathbb{C}^*$-bundles determined by the diagrams

$$\text{pr}_1^* \lambda^{x^2} L \equiv Y_{\lambda^2\lambda^2}M \times_{\lambda^2\lambda^2} \lambda^{x^2} L \xrightarrow{\text{pr}_2} \lambda^{x^2} L \equiv Y_{\lambda^2}M \times_{\lambda^2} L \xrightarrow{\text{pr}_2} L$$

$$\text{pr}_1 \lambda^{x^2} L \equiv Y_{\lambda^2\lambda}M \equiv \lambda^*YM \times_{H \times M} \lambda^*YM \lambda^{x^2} \equiv \lambda^{x^2} \equiv \text{pr}_2^* Y_{[2]}M$$

$$\text{pr}_2^* \lambda^{x^2} L \equiv Y_{\lambda^2\lambda^2}M \times_{\lambda^2\lambda^2} \lambda^{x^2} L \xrightarrow{\text{pr}_2} \lambda^{x^2} L \equiv Y_{\lambda^2}M \times_{\lambda^2} L \xrightarrow{\text{pr}_2} L$$

and

$$\text{pr}_1 E \equiv Y_{\lambda^2\lambda^2}M \times_{\lambda^2\lambda^2} E \xrightarrow{\text{pr}_2} E$$

$$\text{pr}_1 \lambda^{x^2} L \equiv Y_{\lambda^2\lambda}M \equiv \lambda^*YM \times_{H \times M} \lambda^*YM \lambda^{x^2} \equiv \lambda^{x^2} \equiv \text{pr}_2^* Y_{[2]}M$$

The coherence condition of the general type (1.2.4) obeyed by the isomorphism $\alpha_E$ over the fibred product $Y_{\lambda^2\lambda^2}M \times_{H \times M} Y_{\lambda^2}M$, while important for the construction of the $H$-equivariant structure,
does not enter the discussion of the supersymmetry of the latter, therefore we leave the somewhat
tedious but otherwise completely straightforward derivation of its detailed description to the Reader
and pass to the deciphering of the last datum of the H-equivariant structure, that is the 2-isomorphism
\( \gamma_1 \) over \( H^{x^2} \times M \). The relevant pullback 1-isomorphisms
\[
d_i^{(2)} \circ \gamma_1 : d_i^{(2)} \ast \lambda^* G^{(1)} \rightarrow d_i^{(2)} \ast pr_2^* G^{(1)} \otimes T_{d_i^{(2)} \ast \varphi}, \quad i \in \{0, 1, 2\}
\]
geometrise as principal \( \mathbb{C}^* \)-bundles over the fibred products
\[
Y_{\lambda^2, M}^2 \equiv \left( (H^{x^2} \times M) \times d_i^{(2)} \lambda^* YM \right) \times_{H^{x^2} \times M} \left( (H^{x^2} \times M) \times d_i^{(2)} pr_2^* YM \right) \xrightarrow{pr_2} \left( (H^{x^2} \times M) \times d_i^{(2)} pr_2^* YM \right)
\]
with the component factors determined by the respective commutative diagrams
\[
d_i^{(2)} \ast \lambda^* YM \equiv \left( H^{x^2} \times M \right) \times d_i^{(2)} \lambda^* YM \xrightarrow{pr_2} \lambda^* YM
\]
and
\[
d_i^{(2)} \ast pr_2^* YM \equiv \left( H^{x^2} \times M \right) \times d_i^{(2)} pr_2^* YM \xrightarrow{pr_2} pr_2^* YM
\]
The bundles in question are the pullbacks
\[
Y_{\lambda^2, M}^2 \times _{pr_2^*} E \xrightarrow{pr_2} E
\]
and so, upon identifying
\[
d_i^{(2)} \ast pr_2^* YM \xrightarrow{\equiv} d_i^{(2)} \ast \lambda^* YM : \left \{ (h_1, h_2, m), (h_1, \lambda_{h_2}(m), y) \right \} \rightarrow \left \{ (h_1, h_2, m), (h_2, m), y \right \},
\]
\[
d_i^{(2)} \ast \lambda^* YM \xrightarrow{\equiv} d_i^{(2)} \ast \lambda^* YM : \left \{ (h_1, h_2, m), (h_1, \lambda_{h_2}(m), y) \right \} \rightarrow \left \{ (h_1, h_2, m), (h_1 \cdot h_2, m), y \right \},
\]
and
\[
d_i^{(2)} \ast pr_2^* YM \xrightarrow{\equiv} d_i^{(2)} \ast pr_2^* YM : \left \{ (h_1, h_2, m), (h_2, m), y \right \} \rightarrow \left \{ (h_1, h_2, m), (h_1 \cdot h_2, m), y \right \},
\]
and defining
\[
Y_{\lambda^2, M}^2 \equiv d_2^{(2)} \ast \lambda^* YM \times_{H^{x^2} \times M} d_2^{(2)} \ast pr_2^* YM \times_{H^{x^2} \times M} d_0^{(2)} \ast pr_2^* YM \tag{4.35}
\]

```
we arrive at the pullback principal $\mathbb{C}^*$-bundles

\[
\begin{array}{cccc}
\text{pr}^*_1(Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \equiv & Y_{\lambda;2}^2 M \times_{\text{pr}^*_1} (Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \rightarrow \ Y_{\lambda;2}^2 M \times_{pr^*_2} E \\
\text{pr}^*_1 Y_{\lambda;2}^2 M & \rightarrow & Y_{\lambda;2}^2 M \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{pr}^*_2(Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \equiv & Y_{\lambda;2}^2 M \times_{\text{pr}^*_2} (Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \rightarrow \ Y_{\lambda;2}^2 M \times_{pr^*_2} E \\
\text{pr}^*_1 Y_{\lambda;2}^2 M & \rightarrow & Y_{\lambda;2}^2 M \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{pr}^*_1(Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \equiv & Y_{\lambda;2}^2 M \times_{\text{pr}^*_1} (Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \rightarrow \ Y_{\lambda;2}^2 M \times_{pr^*_2} E \\
\text{pr}^*_1 Y_{\lambda;2}^2 M & \rightarrow & Y_{\lambda;2}^2 M \\
\end{array}
\]

and

\[
\begin{array}{cccc}
\text{pr}^*_2(Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \equiv & Y_{\lambda;2}^2 M \times_{\text{pr}^*_2} (Y_{\lambda;2}^2 M \times_{pr^*_2} E) & \rightarrow \ Y_{\lambda;2}^2 M \times_{pr^*_2} E \\
\text{pr}^*_2 Y_{\lambda;2}^2 M & \rightarrow & Y_{\lambda;2}^2 M \\
\end{array}
\]

The tensor product of the first two,

\[
\mathbb{C}^* \rightarrow \text{pr}^*_1(Y_{\lambda;2}^2 M \times_{pr^*_2} E) \otimes \text{pr}^*_2(Y_{\lambda;2}^2 M \times_{pr^*_2} E) \rightarrow \ Y_{\lambda;2}^2 M
\]

is the principal $\mathbb{C}^*$-bundle of the product 1-isomorphism \( (d_0^{(2)} \ast \text{id}_2 \ast \text{id}_2, \cdot) \circ q_2 \) \( = \lambda \). Thus prepared, we may, at long last, formulate the conditions of supersymmetry of the H-equivariant structure.

The first of the conditions to be imposed involves the formerly introduced extension \( \ell^{(1)} : G \times (H \times M) \rightarrow H \times M \) of \( \ell \), \( cp \ Eq. (1.16) \), and reads

\[
\forall \gamma \in G : \ell^{(1)} \ast g \ (2) = \gamma (2).
\]

Next, we note the invariance of the curvature of (the principal $\mathbb{C}^*$-connection on) \( E \) under the induced action

\[
Y_{\lambda;2} \ell^{(1)} : G \times Y_{\lambda;2} M \rightarrow Y_{\lambda;2} M
\]

\[
(\gamma, ((h, m), y_1), ((h, m), y_2)) \rightarrow (((Ad_\gamma(h), \ell_\gamma(m)), Y\ell_\gamma(y_1)), ((Ad_\gamma(h), \ell_\gamma(m)), Y\ell_\gamma(y_2)))
\]

(4.36)

of the supersymmetry group on the base of the bundle,

\[
Y_{\lambda;2} \ell^{(1)} \ast (pr^*_2(pr^*_2 B + pr^*_1 \gamma) - pr^*_2 pr^*_2 B) = pr^*_2(pr^*_2 Y\ell^*_g B + pr^*_1 \ell^{(1)} \ast \gamma - pr^*_2 pr^*_2 Y\ell^*_g B)
\]

\[
= pr^*_2(pr^*_2 B + pr^*_1 \gamma) - pr^*_2 pr^*_2 B
\]
and demand the existence of a lift

\[
\begin{array}{ccc}
G \times E & \xrightarrow{E_{\ell}} & E \\
\downarrow{\text{id}_G \times \pi_E} & & \downarrow{\pi_E} \\
G \times Y_{\lambda_2}M & \xrightarrow{\gamma_{\lambda_2}^{(1)}} & Y_{\lambda_2}M
\end{array}
\]

of the induced action to the total space of the bundle which commutes with the defining action \( r^E : E \times \mathbb{C}^* \to E \) of the structure group \( \mathbb{C}^* \) on \( E \),

\[
\forall_{(g, z) \in \mathbb{C}^* \times E} : E_{\ell g} \circ r^E_z = r^E_z \circ E_{\ell g},
\]

and for which the identity

\[
\forall g \in G : E_{\ell g}^* A_E = A_E^{(1)}
\]

obtains. Commutativity of \( E_{\ell} \) with \( r^E \) in conjunction with that of \( L_{\ell} \) with \( r^L \) (assumed previously) enables us to induce actions of \( G \) on the total spaces of the tensor-product bundles \( \text{pr}_{1,3,3}^* \mathcal{X}^{x^*} \times \mathcal{Y} \text{pr}_{3,4}^* E \) and \( \text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E \) through (here, we use the shorthand notation \( \tilde{y}_\alpha = ((h, m), y_\alpha), \alpha \in \{1, 2, 3, 4\} \))

\[
[L_{\lambda} E_{\ell}] : G \times (\text{pr}_{1,3,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E) \to \text{pr}_{1,3,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E
\]

\[
\leadsto (g, ((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)), ((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4), e)) \mapsto \to (\text{pr}_{1,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E)
\]

\[
(4.37) \quad \circ ((\text{pr}_{1,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E), (\text{pr}_{1,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E), (\text{pr}_{1,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E), (\text{pr}_{1,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* E))
\]

and

\[
[EL_{2\ell}] : G \times (\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E) \to \text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E
\]

\[
\leadsto (g, ((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4), e)) \mapsto ((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4), ((\tilde{y}_2, \tilde{y}_4), l)) \mapsto \to (\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E)
\]

\[
(4.38) \quad \circ ((\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E), (\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E), (\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E), (\text{pr}_{1,2,3}^* \mathcal{X}^{x^*} \times \text{pr}_{3,4}^* \mathcal{Y} \text{pr}_{2,4}^* E))
\]

respectively, and we further require equivariance of the isomorphism \( \alpha_E \) with respect to these,

\[
\forall g \in G : \alpha_E \circ [L_{\lambda} E_{\ell}]_g = [E_{\ell} L_{2\ell}]_g \circ \alpha_E.
\]

Finally, we impose the requirement of equivariance upon the isomorphism of principal \( \mathbb{C}^* \)-bundles contained in the definition of the 2-isomorphism \( \gamma_1 \). To this end, we use the structure obtained hitherto to induce actions of the supersymmetry group on the total spaces \( \text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E) \otimes \text{pr}_{2,4}^* (Y_{\lambda_2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E) \) of the principal \( \mathbb{C}^* \)-bundles of \( (d_0^{(2)})^* Y_1 \otimes \text{id}_{z^*(2,1)^*} ) ^{(d_1^{(2)})^*} Y_1 \), and demand equivariance of \( \gamma_1 \) with respect to these actions. We have for \( \tilde{m}_{1,2} \equiv (h, h, m), \tilde{y}_n \equiv (d_1^{(2)}(\tilde{m}_{1,2}), y_n), \alpha \in \{1, 2, 3\}, \ i \in \{0, 1, 2\} \)

\[
[Y_{\lambda_2,2,0}^* E_{\ell}] : G \times (\text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E)) \otimes \text{pr}_{2,3}^* (Y_{\lambda_2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E)
\]

\[
\leadsto \text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E) \otimes \text{pr}_{2,3}^* (Y_{\lambda_2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E)
\]

\[
\leadsto (g, (((\tilde{m}_{1,2}, \tilde{y}_1^n), (\tilde{m}_{1,2}, \tilde{y}_2^n), (\tilde{m}_{1,2}, \tilde{y}_3^n)), ((\tilde{m}_{1,2}, \tilde{y}_1^n), (\tilde{m}_{1,2}, \tilde{y}_2^n), e_1))) \otimes
\]

\[
(4.39) \quad \circ (((\tilde{m}_{1,2}, \tilde{y}_1^n), (\tilde{m}_{1,2}, \tilde{y}_2^n), (\tilde{m}_{1,2}, \tilde{y}_3^n)), ((\tilde{m}_{1,2}, \tilde{y}_1^n), (\tilde{m}_{1,2}, \tilde{y}_2^n), e_2))) \mapsto \to (\text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E), (\text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E), (\text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E), (\text{pr}_{1,2}^* (Y_{\lambda_2,2,3}^* \mathcal{X} \times \text{pr}_{2,3}^* E))
\]

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\[\oplus(((\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{11}, \gamma^2_{12}), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12})), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12}))),\]

\[(4.39)\]

\[(((\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{11}, \gamma^2_{12}), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12})), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12})), E\ell_g(e_2))\]

and

\[Y^2_{\lambda;2;1}E\ell_. : G \times pr^*_1,3(E\gamma^2_{\lambda;2;1}M \times pr^*_2,2E) \rightarrow Y^2_{\lambda;2;1}M \times pr^*_2,2E\]

\[:(g, (((m_{1,2}, \gamma^1_{11}), (m_{1,2}, \gamma^2_{11})), (m_{1,2}, \gamma^2_{11}, \gamma^2_{12})), ((m_{1,2}, \gamma^1_{11}, \gamma^2_{11})), (m_{1,2}, \gamma^1_{11}, \gamma^2_{12})), \epsilon)) \rightarrow\]

\[\rightarrow (((\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{11}, \gamma^2_{12}), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12})), (\ell^2_g(m_{1,2}), \gamma^1_{11}, \gamma^2_{12})), E\ell_g(e_2))\]

\[(4.40)\]

The equivariance condition now takes the form

\[\forall_{g \in G} : Y^2_{\lambda;2;2,0}E^2 \epsilon_g = Y^2_{\lambda;2;1}E\ell_g \circ \gamma_1.\]

Upon specialisation, our general considerations yield

**Definition 4.10.** Let \(G_{C^{(1)}E}^{(1)} = (YG, \pi_{YG}, B, L, \pi_L, A_L, \mu_L)\) be a Cartan–Eilenberg super-1-gerbe of curvature \(H^{(1)}\), and endowed with a lift \(YAd_. : G \times YG \rightarrow YG\) of the adjoint action \(Ad_.\) to the total space \(YG\) of the surjective submersion \(\pi_{YG}\) described by \(Diag.(4.26)\) and required to preserve the curving

\[(4.41)\]

\[\forall_{g \in G} : YAd^*_g B = B^{(2)}_.\]

Assume that the induced action \(Y^{[2]} Ad_. = (YAd_. \circ pr_{1,2}, YAd_. \circ pr_{1,3}) : G \times Y^{[2]}G \rightarrow Y^{[2]}G\) lifts further to an action

\[LAd_. : G \times L \rightarrow L\]

of \(G\) on the total space of the bundle \(L\) described by the commutative diagram

\[G \times L \xrightarrow{LAd_.} L\]

\[\xrightarrow{id_G \times \pi_L} G \times Y^{[2]}G \xrightarrow{Y^{[2]}Ad_.} Y^{[2]}G\]

and required to commute with the defining action \(r^L : L \times \mathbb{C}^x \rightarrow L\) of the structure group \(\mathbb{C}^x\) on \(L\),

\[\forall_{(g, z) \in G \times \mathbb{C}^x} : LAd_g \circ r^L = r^L \circ LAd_g,\]

and to preserve the principal connection super-1-form

\[\forall_{g \in G} : LAd^*_g A_L = A_L^{(1)}_.\]

Suppose also that the groupoid structure is equivariant with respect to the natural actions of \(G\) on its domain,

\[L_{1,2,2,3}Ad_. : G \times (pr_{1,2}^* L \otimes pr_{2,3}^* L) \rightarrow pr_{1,2}^* L \otimes pr_{2,3}^* L\]

\[:(g, ((y_1, y_2, y_3), p_1) \otimes ((y_1, y_2, y_3), p_2)) \rightarrow\]

\[\rightarrow ((YAd_g(y_1), YAd_g(y_2), YAd_g(y_3)), LAd_g(p_1)) \otimes\]

\[\otimes ((YAd_g(y_1), YAd_g(y_2), YAd_g(y_3)), LAd_g(p_2))\]

and codomain,

\[L_{1,3}Ad_. : G \times pr_{1,3}^* L \rightarrow pr_{1,3}^* L\]

\[:(g, ((y_1, y_2, y_3), p)) \rightarrow ((YAd_g(y_1), YAd_g(y_2), YAd_g(y_3)), LAd_g(p))\].
induced from $\mathcal Y\text{Ad}.$ and $L\text{Ad}.$ with the above properties, so that the identities
\[
\forall g \in G : \mu_L \circ L_{1,2,3} \text{Ad}_g = L_{1,3} \text{Ad}_g \circ \mu_L
\]
hold true.

Assume, further, the existence of super-1-forms $(\kappa^V_A)_{A \in \dim G}$ given by $(L_A$ and $R_A$ are the left- and right-invariant vector fields on $G$, respectively)
\[
d\kappa^V_A = -V_A \wedge H, \quad V_A = R_A - L_A
\]
and satisfying the identities
\[
(4.42) \quad \mathcal L_{V_A} \kappa^V_B = -f_{AB} \mathcal L_C \kappa^V_C, \quad V_A \wedge \kappa^V_B + (-1)^{|A||B|} V_B \wedge \kappa^V_A = 0
\]
in which the $f_{AB}^C$ are the structure constants of the Lie superalgebra of $G$ (in a homogeneous basis),
\[
[L_A, L_B] = f_{AB}^C L_C.
\]

Define a super-2-form on $G \times G$ by the formula
\[
\theta_{\pi, \mathcal Y} = \text{pr}^2_2 \kappa^V \wedge \text{pr}^1_1 \mathcal L^V, \quad \text{pr}^1_1 \mathcal L^V = \frac{1}{2} \text{pr}^1_1 \left(\mathcal L_{V_A} \kappa^V_B - \frac{1}{2} \mathcal L_{\mathcal L_{V_A} \kappa^V_B} \right),
\]
expressed in terms of components $\mathcal L^V$ of the left-invariant Maurer–Cartan super-1-form on $G$, and denote by $\mathcal T_{\pi, \mathcal Y}$ the trivial super-1-gerbe over $G \times G$ with the global connection 2-form $(4.43)$. A

**supersymmetric Ad.-equivariant structure on** $\mathcal G^{(1)}_{\mathcal C, \mathcal A, \mathcal B}$ **relative to** $\theta_{\pi, \mathcal Y}$ **is a pair** $(\mathcal Y_1, \gamma_1)$ composed of a super-1-gerbe 1-isomorphism
\[
\mathcal Y_1 : \text{Ad}^* \mathcal G^{(1)} \overset{\pi}{\longrightarrow} \text{pr}^2_2 \mathcal G^{(1)} \otimes \mathcal T_{\pi, \mathcal Y}
\]
and of a super-1-gerbe 2-isomorphism
\[
\gamma_1 : (d^{(2)}_{1} \circ \mathcal Y_1 \otimes \text{id}_{d^{(2)}_{1}, \pi_{\mathcal Y}}) \circ d^{(2)}_{1} \cdot \mathcal Y_1 \overset{\pi}{\longrightarrow} d^{(2)}_{1} \cdot \mathcal Y_1,
\]
written in terms of the face maps $d^{(2)}_{i}$, $i \in \{0, 1, 2\}$ of the nerve $N^\bullet(\mathcal G \mathcal C \mathcal A) \equiv G^\bullet \times G$ of the action groupoid (4.29) and subject to the coherence constraint
\[
d^{(3)}_{1} \circ \gamma_1 \bullet (\text{id}_{d^{(2)}_{1}, \mathcal C} \circ d^{(3)}_{1} \circ \gamma_1) = d^{(3)}_{1} \circ \gamma_1 \bullet \left(\left(\left(d^{(3)}_{1} \circ \gamma_1 \otimes \text{id}_{d^{(2)}_{1}, \mathcal C} \circ d^{(3)}_{1} \circ \gamma_1} \right) \circ \text{id}_{d^{(2)}_{1}, \mathcal C} \circ d^{(3)}_{1} \circ \gamma_1} \right)
\]
over $N^\bullet(\mathcal G \mathcal C \mathcal A)$, and such that the following conditions are satisfied
(i) there exists a lift
\[
\begin{array}{ccc}
G \times E & \overset{E \text{Ad}}{\longrightarrow} & E \\
\downarrow \text{id}_G \times \pi_E & & \downarrow \pi_E \\
G \times Y_{\mathcal A2G} & \overset{Y_{\text{Ad}2 \mathcal A^{(1)}}}{\longrightarrow} & Y_{\text{Ad}2G}
\end{array}
\]
of the action $Y_{\text{Ad}2 \mathcal A^{(1)}}$ induced, as in Eq. (4.33), on the base of the principal $\mathbb C^*$-bundle
\[
\mathbb C^* \overset{\pi_E}{\longrightarrow} E
\]

\[
(4.44) \quad \text{Ad}^* YG \times_{G \times G} \text{pr}^2_2 YG = Y_{\text{Ad}2G}
\]
of $\mathcal Y_1$ to its total space which commutes with the defining action $r^E : E \times \mathbb C^* \longrightarrow E$ of the structure group $\mathbb C^*$ on $E$,
\[
\forall_{(g, z) \in G \times \mathbb C^*} : E \text{Ad}_g \circ r^E_z = r^E_{zg} \circ E \text{Ad}_g
\]
and which preserves the principal $C^*$-connection super-1-form $A_E$ of $E$, \[ \forall_{g \in G} : E \text{Ad}^*_g A_E = A_E; \]

(ii) the principal $C^*$-bundle isomorphism

\[ (4.45) \]

\[ \alpha_E : \text{pr}^*_1,3 \Ad^{x^2} L \odot \text{pr}^*_4,4 E \xrightarrow{z} \text{pr}^*_1,4 E \odot \text{pr}^*_2,4 \tilde{p}^{x^2} L \]

of $\Upsilon_1$, defined as in Eq. (4.34), is equivariant with respect to the actions

\[ [L_{\text{Ad}} E \text{Ad}] : G \times (\text{pr}^*_1,3 \Ad^{x^2} L \odot \text{pr}^*_4,4 E) \rightarrow \text{pr}^*_1,3 \Ad^{x^2} L \odot \text{pr}^*_4,4 E \]

and

\[ [E L_{\text{Ad}2} \text{Ad}] : G \times (\text{pr}^*_1,4 E \odot \text{pr}^*_2,4 \tilde{p}^{x^2} L) \rightarrow \text{pr}^*_1,4 E \odot \text{pr}^*_2,4 \tilde{p}^{x^2} L \]

induced, as in Eqs. (4.37) and (4.38), respectively, on the total spaces of its domain and codomain, that is $\alpha_E$ satisfies the identities

\[ \forall_{g \in G} : \alpha_E \circ [L_{\text{Ad}} E \text{Ad}]_g = [E L_{\text{Ad}2} \text{Ad}]_g \circ \alpha_E; \]

(iii) the isomorphism $\gamma_1$ is equivariant with respect to the actions

\[ [\Upsilon^4_{\text{Ad}2;2,0} E^2 \text{Ad}] : G \times (\text{pr}^*_1,2 (\Upsilon^2_{\text{Ad}2;2} M \times p^{x^2} E) \odot \text{pr}^*_2,3 (\Upsilon^2_{\text{Ad}2;0} M \times p^{x^2} E)) \rightarrow \text{pr}^*_1,2 (\Upsilon^2_{\text{Ad}2;2} M \times p^{x^2} E) \odot \text{pr}^*_2,3 (\Upsilon^2_{\text{Ad}2;0} M \times p^{x^2} E) \]

and

\[ \Upsilon^4_{\text{Ad}2;1} E \text{Ad} : G \times \text{pr}^*_1,3 (\Upsilon^2_{\text{Ad}2;1} M \times p^{x^2} E) \rightarrow \text{pr}^*_1,3 (\Upsilon^2_{\text{Ad}2;1} M \times p^{x^2} E) \]

induced, as in Eqs. (4.39) and (4.40), respectively, on the total spaces of its domain and codomain, that is $\gamma_1$ satisfies the identities

\[ \forall_{g \in G} : \gamma_1 \circ [\Upsilon^4_{\text{Ad}2;2,0} E^2 \text{Ad}]_g = \Upsilon^4_{\text{Ad}2;1} E \text{Ad}_g \circ \gamma_1. \]

Remark 4.11. The additional invariance constraints [(1.30) and (1.24)] to be imposed upon $\vartheta_{\tilde{\vartheta}_i}$ are satisfied automatically in the present setting for exactly the same reason as in the case of the super-0-gerbe, cp Rem. [1.7]

Our general discussion culminates in

Theorem 4.12. The Green–Schwarz super-1-gerbe $\mathcal{C}_{\text{GS}}^{(1)}$ of Def. I.5.9, recalled on p. 25, carries a canonical supersymmetric Ad.-equivariant structure $(\Upsilon_1, \gamma_1)$ with respect to the adjoint action of the Lie supergroup $s\text{Mink}^{d,1|D_{d,1}}$ on itself relative to the super-1-form

\[ \vartheta_{\tilde{\vartheta}_i} \left( (\theta_1, x_1), (\theta_2, x_2) \right) = -\frac{2}{3} (\theta_1 \Gamma^a d\theta_1) \wedge (\theta_2 \Gamma^a d\theta_2), \quad (\theta_i, x_i) \in s\text{Mink}^{d,1|D_{d,1}}, \quad i \in \{1, 2\}, \]

as described in Def. [1.10].

Proof. A proof is given in App. E. \[ \square \]

Remark 4.13. As in the case of the Cartan–Eilenberg super-0-gerbe, we could, in principle, insist on homomorphicity of the lifts $Y\text{Ad}$ and $L\text{Ad}$ as an additional constraint to be imposed whenever $M \equiv G$ with $\ell. \equiv \text{Ad}$. And just as in that case, the symmetry argument permits us to view this constraint as unjustifiably restrictive. Moreover, it is not hard to convince oneself that homomorphicity actually fails for the lifts derived in the constructive proof of Thm. 4.12.

Altogether, the above in-depth analysis provides an unequivocal confirmation of our original expectation regarding compatibility of the Ad.-equivariant structure on the GS super-$p$-gerbes with $p \in \{0, 1\}$ with supersymmetry (realised in the adjoint), based upon the intuitions derived from the study of the bosonic $\sigma$-models of Sec. 3.1 and Sec. 3.2.
5. The Two Faces of the $p$-brane Dynamics on a Homogeneous Space

One of the fundamental features of the Green–Schwarz(-type) super-$\sigma$-model with the super-Minkowskian target is the presence of a rather peculiar gauge supersymmetry discovered by de Azcárraga and Lukierski in Ref. [IAL83] and by Siegel in Refs. [Sie83, Sie84], and regarded as a consistency constraint in subsequent attempts at constructing super-$\sigma$-models on curved supermanifolds. In its original rendering in the Nambu–Goto (resp. Polyakov) formulation of the field theory under consideration, the supersymmetry, whose prime rôle is to restore balance between the bosonic and the fermionic degrees of freedom in the effective field theory of (the excitations of) the extended object described by the super-$\sigma$-model, perturbs both components of the action functional — the (induced-)metric one and the topological one — and it is solely a suitably relatively normalised combination of the two that is left unchanged. Thus, the symmetry is a mechanism that fixes the action functional of the field theory. As there is currently no geometric structure known to unify the metric and gerbe-theoretic components of the (super-)$\sigma$-model background\footnote{See, however, Ref. [Sus12], where some ideas in this spirit were articulated.} and the symmetry is of an inherently local nature, there seems to be no hope \textit{a priori} for a meaningful geometrisation of the latter. The first clearcut hint that the situation might, after all, not be so hopeless as it looks came out of the studies reported in Refs. [McA00, GKW06a] in which the supersymmetry was identified as one generated by tangential (or infinitesimal) right translations in certain distinguished Graßmann-odd directions in the target Lie supergroup. A full-fledged geometrisation of the symmetry became attainable, and was realised on the level of the relevant action functional with the super-Minkowskian target in Ref. [GKW06a], only upon reformulation of the original super-$\sigma$-model along the lines of Ref. [HP86]. The reformulation, to be recapitulated and elaborated hereunder, calls for a change of perspective: We should abandon our study of spacetime symmetries by Salam, Strathdee and Isham in Refs. [SS69b, ISS71]. The scheme of realisation of the (super)group $G$ on $G/H$ and its field-theoretic ramifications, originally derived by Schwinger and Weinberg in Refs. [Sch67, Wei68] in the context of effective field theories with chiral symmetries, and subsequently elaborated in Refs. [CCWZ69, SS69a], and adapted to the study of spacetime symmetries by Salam, Strathdee and Isham in Refs. [SS69b, SS71]. The scheme was successfully employed in the setting of a supersymmetric field theory by Akulov and Volkov et al. in Refs. [VA72, VA73, LR78, UZ82, IK82, SW83, FMW83, BW84], and this is the variant that we encounter below.

Thus, let $G$ be a Lie supergroup, to be referred to as the \textbf{supersymmetry group} in what follows, and let $H$ be a closed Lie subgroup of $G$, to be termed the \textbf{isotropy group}, with a distinguished closed Lie subgroup

\begin{equation}
\mathfrak{h}_{\text{vac}} \subseteq \mathfrak{h},
\end{equation}

to be termed the \textbf{vacuum isotropy group}. Let the corresponding Lie (super)algebras be: $\mathfrak{g}$ for $G$, to be called the \textbf{supersymmetry algebra}, and $\mathfrak{h} \supseteq \mathfrak{h} \oplus \mathfrak{t}$ (resp. $\mathfrak{h}_{\text{vac}} \supseteq \mathfrak{h}_{\text{vac}} \oplus \mathfrak{t}$) for $H$ (resp. $\mathfrak{h}_{\text{vac}}$), to be called the \textbf{isotropy algebra} (resp. the \textbf{vacuum isotropy algebra}). We shall denote the direct-sum complement of $\mathfrak{h}$ within $\mathfrak{g}$ as $\mathfrak{t}$,

\begin{equation}
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h},
\end{equation}

further assuming it to be an ad-module of the isotropy algebra,

\begin{equation}
[\mathfrak{h}, \mathfrak{t}] \subseteq \mathfrak{t},
\end{equation}

with a distinguished Lie (super)algebra $\mathfrak{g}$. Let the corresponding Lie (super)algebras be: $\mathfrak{g}$ for $G$, to be called the \textbf{supersymmetry algebra}, and $\mathfrak{h} \supseteq \mathfrak{h} \oplus \mathfrak{t}$ (resp. $\mathfrak{h}_{\text{vac}} \supseteq \mathfrak{h}_{\text{vac}} \oplus \mathfrak{t}$) for $H$ (resp. $\mathfrak{h}_{\text{vac}}$), to be called the \textbf{isotropy algebra} (resp. the \textbf{vacuum isotropy algebra}). We shall denote the direct-sum complement of $\mathfrak{h}$ within $\mathfrak{g}$ as $\mathfrak{t}$,
which qualifies decomposition (5.2) as \textbf{reductive}. The Lie superalgebra $\mathfrak{g}$ admits a super-grading

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$$

in which $\mathfrak{g}^{(0)}$ is the Graßmann-even Lie subalgebra of $\mathfrak{g}$,

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(0)},$$

containing $\mathfrak{h}$,

$$\mathfrak{h} \subset \mathfrak{g}^{(0)},$$

and $\mathfrak{g}^{(1)}$ is the Graßmann-odd ad-module thereof,

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}.$$

The super-grading is inherited by the subspace $\mathfrak{t}$,

$$\mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)}.$$

The direct-sum complement of $\mathfrak{h}_\text{vac}$ within $\mathfrak{h}$ shall be denoted as $\mathfrak{d}$,

$$\mathfrak{h} = \mathfrak{d} \oplus \mathfrak{h}_\text{vac}.$$

Finally, we distinguish a subspace

$$\mathfrak{t}^{(0)}_{\text{vac}} \subset \mathfrak{t}^{(0)}$$

closed under the ad-action of the vacuum isotropy algebra,

$$[\mathfrak{h}_\text{vac}, \mathfrak{t}^{(0)}_{\text{vac}}] \subset \mathfrak{t}^{(0)}_{\text{vac}}.$$

Its direct-sum complement within $\mathfrak{t}^{(0)}$ shall be denoted as $\mathfrak{c}$,

$$\mathfrak{t}^{(0)} = \mathfrak{t}^{(0)}_{\text{vac}} \oplus \mathfrak{c}.$$

We assume the decomposition

$$\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{h}_\text{vac}, \quad \mathfrak{f} = \mathfrak{t} \oplus \mathfrak{d}$$

to be reductive as well,

$$[\mathfrak{h}_\text{vac}, \mathfrak{f}] \subset \mathfrak{f}.$$

The adjoint action of $\mathfrak{h}_\text{vac}$ on $\mathfrak{t}^{(0)}_{\text{vac}}$ is taken to integrate to a \textbf{unimodular} (adjoint) action of the Lie group $H_\text{vac}$ on the same space, \textit{i.e.},

$$\forall \mathfrak{h}_{\text{vac}} : \det \left( T_c A \text{d}_{\mathfrak{h}_\text{vac}} \right) = 1.$$

We set

$$(D, \delta, \hat{\delta}, d, p) := (\dim \mathfrak{g} - 1, \dim \mathfrak{t} - 1, \dim \mathfrak{f} - 1, \dim \mathfrak{t}^{(0)} - 1, \dim \mathfrak{t}^{(0)}_{\text{vac}} - 1)$$

and denote the respective homogeneous basis vectors (generators) of the various (complexified) subalgebras and subspaces as

$$\mathfrak{g} = \bigoplus_{A=0}^{D} (t_A)^C, \quad \mathfrak{t} = \bigoplus_{A=0}^{d} \left( P_A \right)^C, \quad \mathfrak{h} = \bigoplus_{S=1}^{D-\delta} (J_S)^C,$$

$$\mathfrak{t}^{(0)} = \bigoplus_{\mu=0}^{d} (P_\mu)^C, \quad \mathfrak{t}^{(1)} = \bigoplus_{\alpha=1}^{\delta-\hat{\delta}} (Q_\alpha)^C, \quad \mathfrak{t}^{(0)}_{\text{vac}} = \bigoplus_{\alpha=0}^{p} \left( P_{\alpha} \right)^C,$$

$$\mathfrak{c} = \bigoplus_{\kappa=1}^{\hat{\delta}-\delta} \left( P_\kappa \right)^C, \quad \mathfrak{h}_\text{vac} = \bigoplus_{S=1}^{D-\hat{\delta}} (J_S)^C, \quad \mathfrak{d} = \bigoplus_{S=D-\hat{\delta}+1}^{D-\delta} (J_S)^C.$$

These satisfy structure relations

$$[t_A, t_B] = f^{C}_{AB} t_C$$

in which the $f^{C}_{AB}$ are structure constants with symmetry properties, expressed in terms of the Graßmann parities $|A| \equiv |t_A|$ and $|B| \equiv |t_B|$ of the respective generators $t_A$ and $t_B$,

$$f^{C}_{AB} = (-1)^{|A||B|+1} f^{C}_{BA} \in \mathbb{C}.$$

In the specific examples listed above, $\mathfrak{t}$ is the linear span of supertranslations, and so – in particular – it is promoted to the rank of a Lie sub-superalgebra in the super-Minkowskian setting. We shall call the superspace

$$\mathfrak{h}_\text{vac}^{(0)} := \mathfrak{t}^{(0)} \oplus \mathfrak{h}_\text{vac}^{(0)}$$
the even tangential vacuum-symmetry space, and the Lie supergroup
\[ S_{\text{vac}}^{(0)} := \mathfrak{g}_{\text{vac}}^{(0)} \times H_{\text{vac}}, \]
abelian in its first factor and with the semidirect product determined by the adjoint action of the vacuum isotropy group \( H_{\text{vac}} \), the even tangential vacuum-symmetry group.

The homogeneous space \( G/K, \ K \in \{ H, H_{\text{vac}} \} \) can be realised locally as a section of the principal bundle
\[ K \longrightarrow G \]
\[ \pi_{G/K} \]
\[ G/K \]
with the structure group \( K \). Thus, we shall work with a family of sub-supermanifolds embedded in \( G \) by the respective (local) sections
\[ \sigma_i^K : \mathcal{O}_i^K \longrightarrow G : gK \longmapsto g \cdot h_i^K(g), \quad i \in I^K, \]
of the submersive projection on the base \( \pi_{G/K} \), associated with a trivialising cover \( \mathcal{O}^K = \{ \mathcal{O}_i^K \}_{i \in I^K} \) of the latter,
\[ G/K = \bigcup_{i \in I^K} \mathcal{O}_i^K. \]
The redundancy of such a realisation over any non-empty intersection, \( \mathcal{O}_j^K \equiv \mathcal{O}_i^K \cap \mathcal{O}_j^K \neq \emptyset \), is accounted for by a collection of locally smooth (transition) maps
\[ h_{ij}^K : \mathcal{O}_j^K \longrightarrow K \subset G \]
fixed by the condition
\[ \forall x \in \mathcal{O}_j^K : \sigma_j^K(x) = \sigma_i^K(x) \cdot h_{ij}^K(x). \]

The homogeneous space admits a natural action of the supersymmetry group induced by the left regular action
\[ \ell : G \times G \longrightarrow G : (g', g) \longmapsto g' \cdot g \equiv \ell g'(g), \]
namely,
(5.5) \[ [\ell^K] : G \times G/K \longrightarrow G/K : (g', gK) \longmapsto (g' \cdot g)K. \]
The latter is transcribed, through the \( \sigma_i^K \), into a geometric realisation of \( G \) on the image of \( G/K \) within \( G \), with the same obvious redundancy. Indeed, consider a point \( x \in \mathcal{O}_i^K \) and an element \( g \in G \). Upon choosing an arbitrary index \( j \in I^K \) with the property
\[ \pi(x; g') := \pi_{G/K}(g' \cdot \sigma_i^K(x)) \in \mathcal{O}_j^K, \]
we find a unique \( \mathbf{L}_{ij}^K(x; g') \in K \) defined (on some open neighbourhood of \( (x, g') \)) by the condition
\[ g' \cdot \sigma_j^K(x) = \sigma_j^K(\pi(x; g')) \cdot \mathbf{L}_{ij}^K(x; g')^{-1}. \]
Note that for \( \pi(x; g') \in \mathcal{O}_j^K \) we have
\[ \mathbf{L}_{ij}^K(x; g') = \mathbf{L}_{ij}^K(x; g') \cdot h_{ij}^K(\pi(x; g')), \]
so that the two realisations of the action are related by a compensating transformation from the structure group \( K \).

While there is no natural action of \( G \) on \( G/K \) induced from right translations on the (super)group, once the realisation of the homogeneous space within \( G \) is fixed in the form given above, we may contemplate infinitesimal perturbations of the sections \( \sigma_i^K \), \( i \in I^K \) along the flows of left-invariant vector fields on \( G \). For \( K = H_{\text{vac}} \) and a particular choice of the \( \sigma_i^{H_{\text{vac}}} \), to be described below, there exists a subspace (even, sometimes, a Lie sub-superalgebra) \( \mathfrak{g}_{\text{vac}} \subset \mathfrak{g} \) containing \( \mathfrak{g}_{\text{vac}}^{(0)} \) such that the left-invariant vector fields \( L_{X(\ell)} \) associated with arbitrary (locally) smooth maps \( X \in \{ \Omega_p, \mathfrak{g}_{\text{vac}} \} \) preserve the Green-Schwarz action functional in the dual Hughes-Polchinski formulation given below. Since

\[ ^{21} \text{We transplant freely the standard constructions from the theory of Lie groups and manifolds with smooth Lie-group actions, and in particular – their homogeneous spaces, into the supergeometric setting. That this makes perfect sense follows from Kostant’s seminal study [Kos77], cp also Ref. [Kos82].} \]
further elucidation of the concept requires a definition of the relevant field theory and the $\sigma_i^K$, we postpone the discussion of the details until Sec. 5.2.

The first step towards the advocated systematic construction of supersymmetric lagrangian field theories with the fibre of the configuration bundle given by $G/K$, realised within $G$ as above, consists in modelling the differential geometry of the homogeneous space in terms of the Cartan differential calculus on $G$. In other words, we seek, in the sheaf $T^*(G/H)$ of (super)differential forms on $G/K$, dual to its tangent sheaf $\mathcal{T}(G/H)$, global sections descended from the Lie supergroup $G$. Denote the relevant direct-sum decomposition of the supersymmetry algebra as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{t}, \quad \{(l, t) \in \{(t, h), (f, h_{\text{vac}})\}\}$$

and the corresponding generators as

$$\mathfrak{l} = \bigoplus_{\ell=0}^{\dim \mathfrak{l} - 1} \langle T_\ell \rangle_{\mathcal{C}}, \quad \mathfrak{t} = \bigoplus_{Z=1}^{\dim \mathfrak{t}} \langle J_Z \rangle.$$

Among the global sections of $T^*(G/H)$, there are superdifferential forms whose pullbacks along $\pi_{G/K}$ are linear combinations of wedge products of the components of the left-invariant $\mathfrak{g}$-valued Maurer–Cartan super-1-form $\theta_L = \theta^A_L \otimes t_A$ on $G$ along $\mathfrak{l}$, with arbitrary $K$-invariant tensors as coefficients. Indeed, the said components transform tensorially under right $K$-translations on $G$, and so the combinations are manifestly $K$-basic. Consequently, pullbacks, along the local sections $\sigma_i^K$ over $\mathcal{O}_i^K$, of super-$k$-forms

$$\omega = \omega_{\zeta_1 \zeta_2 \ldots \zeta_k} \theta^\zeta_{\zeta_1} \wedge \theta^\zeta_{\zeta_2} \wedge \ldots \wedge \theta^\zeta_{\zeta_k}, \quad \zeta_1, \zeta_2, \ldots, \zeta_k \in \overline{0, \dim l - 1},$$

with – for any left-invariant vector field $L_{T_\zeta}$ associated with $T_\zeta \in \mathfrak{l}$ in the standard manner –

$$L_{T_\zeta} \theta^\zeta_{\zeta_i} = \delta^\zeta_{\zeta_i},$$

and – for any $h \in K$ –

$$\omega_{\zeta_1 \zeta_2 \ldots \zeta_k} \left(T_e Ad_h\right)^{\zeta_1}_{\zeta_1} \left(T_e Ad_h\right)^{\zeta_2}_{\zeta_2} \ldots \left(T_e Ad_h\right)^{\zeta_k}_{\zeta_k} = \omega_{\zeta_1 \zeta_2 \ldots \zeta_k}$$

do not depend on the choice of the local section and hence glue smoothly over non-empty intersections $\mathcal{O}_i^K$ to global superdifferential forms on $G/K$, mentioned earlier.

Passing to the two classes of supersymmetric field theories of particular interest to us, and intimately related to one another, to wit, the Nambu–Goto super-$\sigma$-model of smooth embeddings of a $(p+1)$-dimensional worldvolume $\Omega_p$ of a super-$p$-brane in $G/H$ and the Hughes–Polchinski model of smooth embeddings of $\Omega_p$ in $G/H_{\text{vac}}$, but forward in Ref. [HP81] and elaborated in Ref. [G/F90], we note that the main supergeometric datum that enters the definition of both models (in correspondence) is a distinguished Cartan–Eilenberg super-$(p+2)$-cocycle $\chi_{(p+2)} \in Z_{(p+2)}^{dR}(G)^G$ on $G$ given by a linear combination

$$\chi = \frac{1}{(p+2)!} \chi_{\underline{A_{1,2,2}} \ldots \underline{A_{2,1,2}}} \delta_{L_{\underline{A_{1,2,2}}}} \wedge \delta_{L_{\underline{A_{2,1,2}}}} \wedge \ldots$$

of $(p+2)$-fold wedge products of the components $\delta_{L_{\underline{A}}} \ A \in \overline{0, \delta}$ of the Maurer–Cartan super-1-form $\theta_L$ along $t \in \mathfrak{l}$ with $H$-invariant (and so also $H_{\text{vac}}$-invariant) tensors $\chi_{\underline{A_{1,2,2}} \ldots \underline{A_{2,1,2}}}$ as coefficients. Clearly, the super-$(p+2)$-cocycle descends to $G/H$ (and so also to $G/H_{\text{vac}}$), that is, there exists a (unique) Green–Schwarz super-$(p+2)$-cocycle $H^K_{(p+2)} \in Z_{(p+2)}^{dR}(G/K)^G$ with the property

$$\chi = \pi_{G/K}^* H^K_{(p+2)},$$

or, equivalently,

$$H^K_{(p+2)} \equiv H^K_{(p+2)} |_{\mathcal{O}_K} = \sigma_i^K |_{\mathcal{O}_K} = \chi_{(p+2)} |_{\mathcal{O}_K}.$$
forming, under the induced action $[t^K]_j^\ell B_j^K (x) = B_j^K (x) + d\Delta_j^K g(x)$
valid for all $(g, x) \in G \times O^K$, for $j \in I^K$ such that $[t^K]_j g(x) \in O_j^K$, and for some $\Delta_j^K \in \Omega^\ell (O_j^K)$
such that the WZ term in the relevant DF amplitude is invariant under G-translations (i.e., under supersymmetry transformations). In fact, in the most studied examples, $\chi$ is de Rham-exact, and so it is the behaviour of its globally smooth primitive under left G-translations and right K-translations that determines its cohomological status on $G/K$, and – through the latter – the well-definedness of the corresponding field theory.

With the above general observations in hand, we are, at long last, ready to give the definitions of the two classes of field theories that we shall study (upon specialisation) in the remainder of the present paper.

5.1. The standard Nambu–Goto formulation of the super-$\sigma$-model. The Nambu–Goto super-$\sigma$-model requires yet another datum: a metric tensor $g$ (possibly degenerate in the Graßmann-odd directions) on $G/H$ descended from a left-G-invariant and right-H-basic symmetric bilinear tensor $g$ on $G$ as

$$\pi^*_G H = g_{AB} \theta_A^B \theta_B^A \equiv g,$$

where the $g_{AB}$ are components of an H-invariant tensor,

$$g_{AB} (T_e \text{Ad}_h)_{A}^C (T_e \text{Ad}_h)_{B}^D = g_{C D}, \quad h \in H.$$

Given the pair $(\chi, g)$, we define the super-$\sigma$-model as the theory of smooth embeddings

$$\xi \in [\Omega, G/H]$$

of the $(p + 1)$-dimensional worldvolume $\Omega(p)$ in the homogeneous space $G/H$ of the (super)symmetry group $G$ determined by (the principle of least action for) the Dirac–Feynman amplitude for an action functional constructed in the following fashion. Let $(\theta^\alpha_i, X^\mu_i)$ be local coordinates on $O_i^H$, centred on a reference point $g_1 \in O_1^H$ (with $(\theta^\alpha_i, X^\mu_i)(g_1 H) = (0, 0)$), and consider the corresponding local sections of the principal H-bundle $G \rightarrow G/H$ of the form

$$\sigma_i^H : O_i^H \rightarrow G : Z_i \equiv (\theta^\alpha_i, X^\mu_i) \mapsto g_1 \cdot g_i(X_i) \cdot e^{\Theta_i(Z_i)}, \quad i \in I^H,$$

with

$$g_i(X_i) = e^{X^\mu_i P_\mu},$$

and

$$\Theta_i^H(Z_i) = \theta^\alpha_i f_{i \beta} \Theta_i^H(X_i),$$

the latter depending, in general, upon the Graßmann-even coordinate (through some functions $f_{i \beta}^\alpha$). Next, take an arbitrary tesselation $\Delta(\Omega(p))$ of $\Omega(p)$ subordinate, for a given map $\xi$, to the open cover $\{O_i^H\}_{i \in I^H}$, as reflected by the existence of a map $i : \Delta(\Omega(p)) \rightarrow I^H$ with the property

$$\forall \xi \in \Delta(\Omega(p)) : \xi(\zeta) \in O_{i(\zeta)}^H.$$

Let $\mathcal{C} \subset \Delta(\Omega(p))$ be the set of $(p + 1)$-cells of the tesselation,

$$\Omega(p) = \bigcup_{\tau \in \mathcal{C}} \tau.$$

The Nambu–Goto action functional is now given by the sum

$$S_{GS,p}^{(NG)}[\xi] = S_{GS,metr,p}^{(NG)}[\xi] + S_{GS,loop,p}^{(NG)}[\xi]$$

of the metric term

$$S_{GS,metr,p}^{(NG)}[\xi] := \sum_{\tau \in \mathcal{C}} S_{GS,metr,p}^{(\tau)}[\xi_\tau], \quad \xi_\tau := \xi |_{\tau},$$

and

$$S_{GS,loop,p}^{(\tau)}[\xi_\tau] = -\frac{1}{2} \int_{\tau} \text{Vol}(\Omega) \sqrt{\text{det}_{(p+1)}} \left( g_{AB} \left( \partial_0 \circ (\sigma_i^H \circ \xi_\tau)^* \theta_A^B \right) (\partial_0 \circ (\sigma_i^H \circ \xi_\tau)^* \theta_B^A) \right),$$

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expressed in terms of the (local) coordinate vector fields \( \partial_a = \frac{\partial}{\partial \sigma^a} \), \( a \in \overline{0,p} \) on \( \tau \subset \Omega_p \), and of the Wess–Zumino term which may be formally written as the integral

\[
S^{(NG)}_{\text{GS,top,p}}[\xi] = \int_{\Omega_p} d^{p+1}x \frac{1}{(p+2)} H
\]

of a primitive of (the pullback of) the (GS super-\((p+2)\)-cocycle). Generically, the latter is not a globally smooth \( G \)-invariant \((p+1)\)-form, and so either we restrict to a class of embeddings \( \xi \) with \( \xi(\Omega_p) \subset O_i^H \) for some \( i \in I^H \) \((p+1)\)-symmetry transformations from a vicinity of the identity (resp. cut the worldvolume open, in which case we may sometimes define the action functional as above but lose the possibility to compare values taken by the functional on maps with cobordant images in \( G/H \), or we write it out in terms of worldvolume \((de \text{Rham})\) currents associated with the tesselation \( \Delta(\Omega_p) \) \((or \text{suitable refinement thereof})\), whereupon it sums up to \((or \text{local presentation of})\) the volume holonomy, along \( \xi(\Omega_p) \), of the geometrisation of the (GS super-\((p+2)\)-cocycle. As argued before, the geometrisation allows for a rigorous definition of the topological WZ term of the (super-)\( \sigma \)-model in arbitrary topological circumstances.

5.2. The alternative Hughes–Polchinski formulation of the (super-)\( \sigma \)-model. There is an alternative to the standard (Nambu–Goto) formulation of the (super-)\( \sigma \)-model with a homogenous space of a (super)group as a (super)target that was originally conceived in Ref.\[HPS8 \]and elaborated significantly in Ref.\[GT9 \]. Here, we use its full-fledged version and draw on an in-depth geometric understanding thereof worked out, in the context of immediate interest, in Ref.\[McA00,McA10 \]and Refs.\[Wes00,GKW06b,GKW06a \]. The formulation introduces into the lagrangean density, among other fields, Goldstone fields for the global spacetime symmetries of the (super-)\( \sigma \)-model broken by the ‘vacuum’ of the theory, \( i.e. \), by the embedding of the membrane in the ambient Lie (super)group \( G \) described by a classical field configuration, and subjects them to the inverse Higgs mechanism of Ref.\[IO7 \]to remove some of them in a manner consistent with the surviving ‘vacuum’ symmetries. In this procedure, the Cartan geometry of the homogeneous space \( G/H_{\text{vac}} \) employed in the construction of the action functional proves instrumental. Indeed, the mechanism boils down to the imposition of geometric constraints that restrict the tangents of classical field configurations embedded in a \( \text{(local)} \) model of \( G/H_{\text{vac}} \) in \( G \) \((cp \text{above})\) to a submanifold giving a model of \( G/H \) \( (\text{note the extension of the isotropy group}) \) in the same mother Lie (super)group \( G \) – these constraints can be expressed as the conditions of the vanishing, on classical field configurations, of the pullbacks along the local coset section \( \sigma_i^{H_{\text{vac}}} \) \((\text{of those components of the Maurer–Cartan super-1-form on } G, \text{restricted to the section, which correspond to the broken (infinitesimal) symmetries in } g) \). Technically, this means that the Goldstone fields eliminated in the procedure are expressed in terms of the remaining fields of the theory, and in particular – in terms of the derivatives of the surviving Goldstone fields, whence also the name of the mechanism.

The Hughes–Polchinski super-\( \sigma \)-model is a theory of smooth embeddings

\[
\tilde{\xi} \in [\Omega_p, G/H_{\text{vac}}].
\]

Its basic building blocks are, as previously, the components \( \theta_i^\Delta, \Delta \in \overline{0,\delta} \) of the Maurer–Cartan super-1-form. However, this time, we introduce the additional Goldstone fields \( \phi_i^S, \hat{S} \in D-\delta+1, D-\delta \) by pulling back the \( \theta_i^\Delta \) along the distinguished local sections

\[
\sigma_i^{H_{\text{vac}}} : O_i^{H_{\text{vac}}} \to G : \tilde{Z}_i \equiv (\theta_i^\alpha, X_i^\mu, \phi_i^S) \to \tilde{g}_i(\xi_i) \cdot g_i(X_i) \cdot e^{\theta_i(\theta_i, X_i)} \cdot e^{\phi_i^S} J_S, \quad i \in I^{H_{\text{vac}}},
\]

expressed in terms of local coordinates \((\theta_i^\alpha, X_i^\mu, \phi_i^S) \equiv (\theta_i^\Delta, \phi_i^S) \) on \( O_i^{H_{\text{vac}}} \), centred on a reference point \( \tilde{g}_i \). Consequently, in the previously introduced notation

\[
ad f_S(\mu) = [J_S, \mu] = f_S^\alpha \nu \nu P_\nu \quad \text{and} \quad \ad f_S(Q_\alpha) = [J_S, Q_\alpha] = f_S^\beta \alpha Q_\beta,
\]

consistent with the assumed structure of \( g \), we obtain coordinate expressions for the component 1-forms

\[
\sigma_i^{H_{\text{vac}}} \cdot \theta_i^\mu(\xi_i, \phi_i) \otimes P_\mu = e^\mu_\Delta(\xi_i) \left( e^{\phi_i^S} \right)_\mu \left( e^{-\lambda(\phi_i)} \right)_\mu \otimes P_\nu,
\]

\[
\sigma_i^{H_{\text{vac}}} \cdot \theta_i^\alpha(\xi_i, \phi_i) \otimes R_\alpha = e^\alpha_\Delta(\xi_i) \left( e^{\phi_i^S} \right)_\alpha \left( e^{-\lambda(\phi_i)} \right)_\alpha \otimes Q_\alpha,
\]

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where we have used the shorthand notation
\[ \Lambda(\phi_i)_\mu^\lambda = \phi_i^B f^{\lambda B}_{\mu} \], \quad \Lambda(\phi_i)_\alpha^\beta = \phi_i^B f^B_{\alpha} \beta .

and the (G/H)-reduced Vielbeine
\[ (5.11) \quad \sigma^\text{H}_{\text{vac}} \ast \theta^A_L(\xi_i, 0) = : e^{\frac{\Lambda}{2}}(\xi_i) d\xi^B_i . \]

Define the auxiliary matrix
\[ \Lambda(\phi_i)_\equiv (\Lambda(\phi_i)_\equiv^B_B = \phi_i^B f^{\equiv B}_{\equiv} B) \Lambda_B \equiv_s, \]

with the obvious decomposition
\[ \Lambda(\phi_i) = (\Lambda(\phi_i), \Lambda(\phi_i)) \in \text{End}_G(t(0)) \oplus \text{End}_G(t(1)) \in \text{End}_G(t) . \]

In view of our assumptions regarding the structure of the Green–Schwarz (p + 2)-form
\[ \chi \equiv \chi(\xi, \phi_i) = \chi(\xi_i, 0) . \]

At this stage, it suffices to demand, as we have in Eq. (5.4), that the Lie-algebra action (5.3) integrate
\[ (5.12) \quad \beta^{\text{(HP)}}_{(p + 1)} = \frac{1}{(p+1)!} \epsilon_{\overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_p}} \rho_0^B \wedge \rho_2^B \wedge \ldots \wedge \rho_{p+1}^B , \]

written in terms of the standard totally antisymmetric symbol
\[ \epsilon_{\overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_p}} = \begin{cases} \text{sign} \left( \begin{array}{cccc} 0 & 1 & \ldots & p \\ \overline{\alpha_0} & \overline{\alpha_1} & \ldots & \overline{\alpha_p} \end{array} \right) & \text{if } \{ \overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_p} \} = 0, p \\ 0 & \text{otherwise} \end{cases} \]

and corresponding, under the cochain map \( \gamma \) of Thm. I.C.7, to the volume form on \( t(0) \). The latter subspace has a clearcut physical interpretation, to wit, it models the tangent of the body of the embedded super-p-brane worldvolume within G. Technically, this means that for a given embedding \( \overline{\xi} \) and for a tesselation \( \Delta(\Omega_p) \) of the worldvolume subordinate – with respect to \( \overline{\xi} \) – to the trivialising open cover \( \text{G}^\text{H}_{\text{vac}} \) of \( G/H_{\text{vac}} \), the subspace \( T_{\overline{\xi}_{\text{H}_{\text{vac}}}} \text{G}^{\text{H}_{\text{vac}}}(\sigma_{\overline{\sigma}}^{\text{H}_{\text{vac}}} \circ \overline{\xi}(\sigma)) \) \( (p + 1) \)-cells of \( \Delta(\Omega_p) \), spanned on the left-invariant vector fields \( L_{\overline{P}_a} \), \( a \in 0, p \), coincides with the Graßmann-even component of the tangent \( T_{\overline{\xi}_{\text{H}_{\text{vac}}}} \text{G}^{\text{H}_{\text{vac}}}(\sigma_{\overline{\sigma}}^{\text{H}_{\text{vac}}} \circ \overline{\xi}(\sigma)) \) of the embedded worldvolume in the so-called static gauge for \( \overline{\xi} \). This choice is accompanied by the identification of the Lie subalgebra \( h_{\text{vac}} \subset \mathfrak{h} \) of infinitesimal symmetries of the (statically) embedded worldvolume.

Given all this, we may, finally, write out the Hughes–Polchinski action functional as the sum
\[ (5.14) \quad S_{\text{GS}, p}^{\text{(HP)}}[\overline{\xi}] = S_{\text{GS}, p, \text{metr}, p}^{\text{(HP)}}[\overline{\xi}] + S_{\text{GS}, \text{top}, p}^{\text{(HP)}}[\overline{\xi}] \]

of the topological WZ term
\[ (5.15) \quad S_{\text{WZ}, \text{top}, p}^{\text{(HP)}}[\overline{\xi}] = \int_{\Omega_p} d^{p+1}\overline{\xi}^* \overline{H}_{(p+2)}^+, \quad \overline{H}_{(p+2)}^+ |_{\text{G}^{\text{H}_{\text{vac}}}} \equiv \sigma^\text{H}_{\text{vac}} \ast \chi \ast_{(p+1)} \]

to be understood as in the NG model, and of the complementary ‘metric’ term
\[ S_{\text{GS, metr}, p}^{\text{(HP)}}[\overline{\xi}] : = \sum_{\overline{\xi}_{\overline{\tau}}} S_{\text{GS, metr}, p}^{(\overline{\tau})}[\overline{\xi}_{\overline{\tau}}], \quad \xi_{\overline{\tau}} := \xi |_{\overline{\tau}} \]
\[ S_{\text{GS, metr}, p}^{(\overline{\tau})}[\overline{\xi}] = \int_{\overline{\tau}} \left( \sigma^\text{H}_{\text{vac}} \circ \overline{\xi}_{\overline{\tau}} \right)^* \beta^{\text{(HP)}}_{(p + 1)} . \]
There is a class of supertargets for which we may establish a direct relation between the two formulations of the Green–Schwarz super-σ-model, which we phrase as

**Theorem 5.1.** Let $G$ be a Lie supergroup with the Lie superalgebra decomposing reductively

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h} \equiv \mathfrak{f} \oplus \mathfrak{h}_{\text{vac}}$$

as described at the beginning of Sec. 4, and let $H_{\text{vac}} \subseteq H \subset G$ be its Lie subgroups with the Lie algebras $\mathfrak{h}$ and $\mathfrak{h}_{\text{vac}}$, respectively, the two algebras satisfying the structural relations given ibidem. The Green–Schwarz super-σ-model on the homogeneous space $G/H_{\text{vac}}$ in the Hughes–Polchinski formulation determined by the action functional $S_{\text{GS},p}^{(\text{HP})}$ of Eq. (5.13), with the metric term $\beta$ and the topological term $\chi$, is equivalent to the Green–Schwarz super-σ-model on the homogeneous space $G/H$ in the Nambu–Goto formulation defined by the action functional $\sigma$ with the metric term $\delta$ for the metric $g = \kappa(0)\mathfrak{t}_{\text{vac}}^{(0)}\otimes e$ given by the restriction, to $\mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e}$, of the Cartan–Killing metric $\kappa(0)$ on the Lie algebra $\mathfrak{t}^{(0)} \oplus \mathfrak{h} \equiv \mathfrak{g}^{(0)}$ and with the topological term $\tau$, if the following conditions are satisfied:

- **(E1)** $\kappa(0)$ defines an orthogonal decomposition
  $$\mathfrak{g}^{(0)} = \mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e} \oplus \mathfrak{h}$$
  such that $\kappa(0)\mathfrak{t}_{\text{vac}}^{(0)}\otimes e$ is non-degenerate;

- **(E2)** $S_{\text{GS},p}^{(\text{HP})}$ is restricted to field configurations satisfying the inverse Higgs constraint

  $$\forall (\sigma, \tau) \in \mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e} : \left(\sigma^{\text{H}_{\text{vac}} \circ \xi^{-1}}\right)^* \sigma_L = 0$$

  whose solvability is ensured by the invertibility – in an arbitrary (local) coordinate system $\{\sigma^A\}^{\text{H}_{\text{vac}}} \oplus \{\tau_b\}$ on $\mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e}$ – of the (tangent-transport) operator

  $$\mathfrak{e} \partial_{\Delta} (\xi_{\tau} (\sigma)) \partial_{\Delta} \sigma = 0,$$

  written in terms of the reduced Vielbein field $\mathfrak{e}^\mu_\Delta$ of Eq. (5.11).

The latter constraint is equivalent to the Euler–Lagrange equations of $S_{\text{GS},p}^{(\text{HP})}$ obtained by varying the functional in the direction of the Goldstone fields $\phi^S$, $\bar{S} \in D - \delta + 1, D - \delta$.

**Proof.** A proof is given in App. F. □

The assumptions of the last proposition exclude important – both mathematically and physically – examples of supertargets such as the super-Minkowski space for which the Killing metric degenerates in the Graßmann-odd translational directions. At the same time, they suggest very clearly a generalisation that does not – a priori – constrain the structure of the underlying Lie algebra $\mathfrak{g}^{(0)}$. Thus, we formulate

**Theorem 5.2.** Let $G$ be a Lie supergroup with the Lie superalgebra decomposing reductively

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h} \equiv \mathfrak{f} \oplus \mathfrak{h}_{\text{vac}}$$

as described at the beginning of Sec. 4, and let $H_{\text{vac}} \subseteq H \subset G$ be its Lie subgroups with the Lie algebras $\mathfrak{h}$ and $\mathfrak{h}_{\text{vac}}$, respectively, the two algebras satisfying the structural relations given ibidem. If condition (E2) of Thm. 5.1 is satisfied in conjunction with the condition

- **(E1')** there exist non-degenerate bilinear symmetric forms: $\gamma$ on $\mathfrak{t}_{\text{vac}}^{(0)}$ and $\tilde{\gamma}$ on $\mathfrak{e}$ with respective presentations

  $$\gamma = \gamma_{\Delta} \tau_A \otimes \tau^A, \quad \tilde{\gamma} = \tilde{\gamma}_{\Delta} \tau_A \otimes \tilde{\tau}^A$$

  in the basis $\{\tau_A\}^{\text{Ad(D)}}$ of $\mathfrak{g}$ dual to $\{t_A\}^{\text{Ad(D)}}$,

  $$\tau_A (t_B) = \delta^A_B, \quad A, B \in \overline{0, D},$$

  for which the following identities hold true

  $$\gamma^{-1} \mathfrak{e} = \mathfrak{f} \tilde{\gamma}_{\Delta} = f \tilde{\gamma}_{\Delta} \mathfrak{e},$$

  where

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the Green–Schwarz super-σ-model on the homogeneous space $G/H_{\text{vac}}$ in the Hughes–Polchinski formulation determined by the action functional $S_{\text{GS},p}^{(\text{HP})}$ of Eq. (5.14), with the metric term (5.13) and the topological term (5.13), is equivalent to the Green–Schwarz super-σ-model on the homogeneous space $G/H$ in the Nambu–Goto formulation defined by the action functional (5.6), with the metric term (5.7) for the metric

$$g = \gamma \otimes \gamma$$

and with the topological term (5.8).

The inverse Higgs constraint is equivalent to the Euler–Lagrange equations of $S_{\text{GS},p}^{(\text{HP})}$ obtained by varying the functional in the direction of the Goldstone fields $\phi^S$, $\overline{S} \in D - \delta + 1, D - \delta$.

**Proof.** The proof is entirely analogous to that of Thm. 5.1, with identity (5.17) playing the structural rôle of identity (5.1), the latter being satisfied automatically under the assumptions of Thm. 5.1. □

While we are not going to make essential use of that in what follows, it is to be noted that the canonical description of the Hughes–Polchinski model is highly singular in that the corresponding presymplectic form does not depend on the kinetic momentum. In the light of the above proposition, the latter is reintroduced into the canonical description only through the imposition of the inverse Higgs constraint.

A specialisation of the scenario referred to in the above theorems which is particularly interesting from the physical point of view is one in which we have

$$[\mathfrak{t}_{\text{vac}}^{(0)}, \mathfrak{t}_{\text{vac}}^{(0)}] \subset \mathfrak{g}_{\text{vac}}^{(0)}$$

and there exists a subspace

$$\mathfrak{t}_{\text{vac}}^{(1)} \subset \mathfrak{t}^{(1)}$$

stable under the adjoint action of the vacuum isotropy algebra,

$$[\mathfrak{h}_{\text{vac}}^{(1)}, \mathfrak{t}_{\text{vac}}^{(1)}] \subset \mathfrak{t}_{\text{vac}}^{(1)}$$

and such that

$$\{\mathfrak{t}_{\text{vac}}^{(1)}, \mathfrak{t}_{\text{vac}}^{(1)}\} \subset \mathfrak{g}_{\text{vac}}^{(0)}, \quad [\mathfrak{t}_{\text{vac}}^{(0)}, \mathfrak{t}_{\text{vac}}^{(1)}] \subset \mathfrak{t}_{\text{vac}}^{(1)}.$$ 

We shall dub the Lie superalgebra

$$\mathfrak{g}_{\text{vac}} := \mathfrak{g}_{\text{vac}}^{(0)} \oplus \mathfrak{t}_{\text{vac}}^{(1)}$$

the **tangential vacuum-symmetry superalgebra**, and the Lie supergroup

$$S_{\text{vac}} := \mathfrak{g}_{\text{vac}} \rtimes H_{\text{vac}}$$

(defined in analogy with its even counterpart $S_{\text{vac}}^{(0)}$) shall be called the **tangential vacuum-symmetry supergroup**. The name is justified by physical considerations that feature such structures with the additional property, to be termed the $\kappa$-**symmetry condition**, for all $X \in \Omega_p, g_{\text{vac}}$

$$\forall X \in \Omega_p, g_{\text{vac}} : \frac{d}{dt} |_{t=0} S_{\text{GS},p}^{(\text{HP})}[\Phi_{L,X}(\xi(\cdot), t)],$$

in whose definition $\Phi_{L,X}(\sigma), \sigma \in \Omega_p$ is the flow of the left-invariant vector field associated with the Lie-superalgebra vector $X(\sigma) \in \mathfrak{g}_{\text{vac}}$.

We conclude the present section with an explicit identification of particular circumstances in which the transcription from the Nambu–Goto picture to the Hughes–Polchinski one can be realised.

**Proposition 5.3.** Let $d \in \mathbb{N}$ and $p \in 0, d$ be such that there exists one of the Majorana-spinor representations of the Clifford algebra $\text{Cliff}(\mathbb{R}^{d,1})$ described in Sec. I.4.2 and consider the corresponding Minkowski spacetime $(\mathbb{R}^{d+1}, \eta) \equiv \mathbb{R}^{d,1}$, regarded as a Lie group of translations. Take the decomposition of its Lie algebra

$$\bigoplus_{\mu=0}^{d} (P_\mu) \equiv \mathfrak{t}^{(0)}$$

into subspaces

$$\mathfrak{t}^{(0)} = \mathfrak{t}_{\text{vac}}^{(0)} \ominus \epsilon,$$

(5.18)

$$\mathfrak{t}_{\text{vac}}^{(0)} = \bigoplus_{\mu=0}^{p} (P_\mu) \subset \mathfrak{t}_{\text{vac}}^{(0)}$$

and $\epsilon = \bigoplus_{\rho=p+1}^{d} (P_\rho)$. 



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orthogonal with respect to $\eta$. Next, extend the above Lie algebra to the full Poincaré algebra $\mathfrak{iso}(d,1)$ of the Poincaré group $\text{ISO}(d,1) = \mathbb{R}^{d,1} \rtimes \text{SO}(d,1)$ by adjoining the generators of the Lorentz algebra

$$\mathfrak{so}(d,1) = \bigoplus_{\mu, \nu = 0}^{d-1} \{J_{\mu\nu}\} \equiv \mathfrak{h},$$

further decomposed, relative to the splitting (5.18), into the Lie subalgebra

$$\mathfrak{j}_\mathfrak{d} = \bigoplus_{\mu, \nu = 0}^{d-1} \{J_{\mu\nu}\} \equiv \mathfrak{h}_{\text{vac}}$$

of Lorentz transformations preserving (5.18) and integrating to the vacuum isotropy group

$$\mathfrak{H}_{\text{vac}} = \text{SO}(p,1) \times \text{SO}(d-p),$$

and its direct-sum completion

$$\bigoplus_{(\alpha, \beta) \in p+1, d} \{J_{\alpha\beta}\} \equiv \mathfrak{d}.$$

Finally, embed the Poincaré algebra, with its reductive decomposition

$$\mathfrak{iso}(d,1) = \mathfrak{t}^{(0)} \oplus \mathfrak{h},$$

as a Lie subalgebra in the super-Poincaré superalgebra $\mathfrak{siso}(d,1|D_{d,1})$ of the super-Poincaré supergroup $\mathfrak{sISO}(d,1|D_{d,1})$, as described in Sec. I.4.1, by adjoining the Majorana-spinor supercharges $\{Q_\alpha\}_{\alpha = 1}^{D_{d,1}}$ spanning a Majorana-spinor module $S_{d,1}$ of dimension $D_{d,1}$ (as above) with respect to Spin$(d,1)$ and forming a reductive decomposition

$$\mathfrak{g} \equiv \mathfrak{sISO}(d,1|D_{d,1}) = \mathfrak{t} \oplus \mathfrak{h}, \quad \mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)}, \quad \mathfrak{t}^{(1)} = \bigoplus_{\alpha = 1}^{D_{d,1}} \{Q_\alpha\} \equiv \mathfrak{sISO}(d,1|D_{d,1}).$$

An arbitrary projector

$$\mathcal{P} \in \text{End}_\mathbb{C} S_{d,1}$$

satisfying the identity

$$\mathcal{P}^T = C (1_{D_{d,1}} - \mathcal{P}) C^{-1}$$

and correlated with decomposition (5.18) through the relations

$$\{\mathcal{P}, \Gamma^\alpha\} = \Gamma^\alpha, \quad \mathcal{P} \in 0, p \quad \mathfrak{P}[\mathcal{P}, \Gamma^\alpha] = 0 \quad \mathcal{P} \in p+1, d$$

determines a tangential vacuum-symmetry superalgebra

$$\mathfrak{g}_{\text{vac}} := \{\mathfrak{t}^{(0)}_{\text{vac}} \oplus \mathfrak{h}_{\text{vac}}\} \oplus \{\mathfrak{t}^{(1)}_{\text{vac}}\}, \quad \mathfrak{t}^{(1)}_{\text{vac}} := \text{im} \mathcal{P}.$$

The data listed above satisfy the assumptions of Thm. 5.3, and so the corresponding Green–Schwarz super-$\sigma$-model on $\mathfrak{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))$ in the Hughes–Polchinski formulation, with the even tangential vacuum-symmetry group $(\mathfrak{t}^{(0)}_{\text{vac}} \oplus \mathfrak{h}_{\text{vac}}) \times \mathfrak{H}_{\text{vac}}$, is equivalent to the Green–Schwarz super-$\sigma$-model on $\mathfrak{sMink}^{d,1|D_{d,1}} = \mathfrak{sISO}(d,1|D_{d,1})/\text{SO}(d,1)$ in the Nambu–Goto formulation.

Proof. A proof is given in App. E. 

The above proposition invokes the field theory, determined unequivocally by the previous considerations, that will be studied at some length in the last part of the present work and hence merits a separate

**Definition 5.4.** The Green–Schwarz super-$\sigma$-model on $\mathfrak{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))$ in the Hughes–Polchinski formulation is the theory of smooth embeddings $\tilde{\xi} \in \mathfrak{P}[\Omega_p, \mathfrak{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))]$, realised within $\mathfrak{sISO}(d,1|D_{d,1})$ by the distinguished local sections of the form (5.9), of a closed $(p+1)$-dimensional worldvolume $\Omega_p$ in the homogeneous space $\mathfrak{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))$ determined by the principle of least action applied to the Dirac–Feynman functional

$$\mathcal{A}_{\text{DF, p}}^{(\text{HP})} : \left[\Omega_p, \mathfrak{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))\right] \to U(1) : \tilde{\xi} \mapsto e^{iS_{\text{GS, p}}^{(\text{HP})}\tilde{\xi}}.$$

The latter is written in terms of the action functional (5.14) in which $\chi$ is the (unique) Cartan–Eilenberg super-$(p+2)$-cocycle on $\mathfrak{sISO}(d,1|D_{d,1})$ that descends to the corresponding supersymmetric de Rham super-$(p+2)$-cocycle $H^{(p+2)}$ given in Eqs. (4.3) and (4.4).
In the situation captured by the $\kappa$-symmetry condition, it is tempting to think of $S_{\text{vac}}$ as a (tangential) gauge group of the field theory $S_{\text{GS,p}}^{(\text{HP})}$, with $\mathfrak{g}_{\text{vac}}$ realised as a supervector space (not as a Lie superalgebra). In fact, this point of view is implicitly built into the definition of the tangential vacuum-symmetry superalgebra. Indeed, the demand that the Lie superbracket of $\mathfrak{g}$ close on its component $\mathfrak{g}_{\text{vac}}$ is motivated by classical (gauge) field theory in which we want the vector fields that generate gauge transformations to determine a regular foliation in the space of states of the field theory (the characteristic foliation of the presymplectic form), and hence – in keeping with the Frobenius theorem – require integrability of their span. We shall devote the remainder of the present paper to a careful study of the higher-geometric aspect of the Hughes–Polchinski formulation of the lowest-dimensional Green–Schwarz super-$\sigma$-models in the super-Minkowskian setting in which this structure arises naturally.

6. The pure-supergerbe description of the super-$\sigma$-model & its $\kappa$-symmetry

Our interest in the field-theoretic correspondence stated in Thms. 5.1 and 5.2 has been fuelled by the amalgamation of the metric and gerbe-theoretic structures on the superspace of the GS super-$\sigma$-model in the Nambu–Goto formulation into a purely gerbe-theoretic structure on the larger superspace of the GS super-$\sigma$-model in the Hughes–Polchinski formulation that should enable us to circumnavigate the obstruction, mentioned earlier, against geometrisation of the peculiar tangential gauge supersymmetry of Refs. [35, 36] along the lines of Sec. 4. In the present section, we provide a corroborating of that expectation by first identifying the simple and natural implementation of the supersymmetry on the superspace of the HP super-$\sigma$-model, and then by lifting it consistently to the higher-geometric object associated with the topological HP action functional that we construct hereunder and dub the extended Hughes–Polchinski $p$-gerbe. The local supersymmetry under consideration has its peculiarities, to be detailed below, that preclude the construction of a completely standard equivariant structure on the extended $p$-gerbe. As a result of this, the analysis to follow provides us with a novel geometric instantiation, laid out in and around Thms. 6.7 and 6.8 of a local field-theoretic $\kappa$-symmetry in the presence of a topological charge.

6.1. The Cartan supergeometry of Siegel’s linearised gauge supersymmetry. The point of departure of our analysis is the HP formulation of the GS super-$\sigma$-model of embeddings of the $(p+1)$-dimensional worldvolume $\Omega_p$ in the homogeneous space $\text{sISO}(d,1|D_{d,1})/(\text{SO}(p,1) \times \text{SO}(d-p))$, which we scrutinise below for $p \in \{0,1\}$ by way of an illustration of the generic phenomenon. With hindsight, we focus on the right regular action (we are using the notation of Sec. I.4.1)

\[ \varphi : \text{sISO}(d,1|D_{d,1}) \times \text{sISO}(d,1|D_{d,1}) \longrightarrow \text{sISO}(d,1|D_{d,1}) \]

\[ (\theta^\alpha, x^\mu, \phi^{\alpha\lambda}, (e^\beta, y^\rho, \psi^{\sigma\tau})) \longmapsto (\theta^\alpha + S(\phi)_{\alpha\beta}^\gamma e^\beta, x^\mu + L(\phi)^{\mu\nu} y^\nu - \frac{1}{2} \theta^{\nu\mu} S(\phi) \varepsilon, \tilde{\phi}^{\lambda\rho}(\phi, \psi)) \]

of the Lie supergroup $\text{sISO}(d,1|D_{d,1})$, in which the homogeneous space is realised by the distinguished sections \([35, 36]\), on itself. The action engenders left-invariant vector fields on $\text{sISO}(d,1|D_{d,1})$, and it is the distinguished components along $t \equiv \bigoplus_{\mu=0}^{d-1} \langle F_{\mu} \rangle_{\mathcal{C}} \oplus \bigoplus_{\alpha=1}^{D_{d,1}} \langle Q_{\alpha} \rangle_{\mathcal{C}}$ of the dual Maurer–Cartan super-1-form that we shall use as the building blocks of (the ‘metric’ term of) the relevant action functional. These come in two families (written in the (local) coordinates $(\theta, x, \phi) \in \text{sMink}^{d,1|D_{d,1}} \times \text{SO}(d,1))$

\[ \Sigma^\alpha_L(\theta, x, \phi) = S(-\phi)^{\alpha\beta} \sigma^\beta (\theta) \equiv S(-\phi)^{\alpha\beta} d\theta^\beta, \quad a \in 1, D_{d,1}, \]

\[ \theta^{\mu\nu}_L(\theta, x, \phi) = L(-\phi)^{\mu\nu} e^\nu(\theta, x) \equiv L(-\phi)^{\mu\nu} (dx^\nu + \frac{1}{2} \theta^{\nu\mu} d\theta), \quad \mu \in 0, d, \]

cp Sec. I.4.1. Following Def. 5.3, we may now write the action functional in the form

\[ S_{\text{GS,0}}^{(\text{HP})}[\theta, x, \phi_{\text{HP}}] = \int_{\Omega_p} (\lambda_0 \theta^{\mu\nu}_L(\theta, x, \phi_{\text{HP}})(\cdot) + \theta^{\Gamma_{11}} \sigma(\theta)(\cdot)) \]

(6.3)

with $\phi_{\text{HP}} \in \text{SO}(9,1)$ in the restricted form $(\overline{a} \in 1,9)$

\[ \phi^{\alpha\beta}_{\text{HP}} = (\delta^{\alpha\beta}_{\nu} - \delta^{\nu\sigma} \delta^{\alpha\beta}_{\sigma}) \phi^{\nu\sigma} \]

for $p = 0$, or for $p > 0$

\[ S_{\text{GS,p}}^{(\text{HP})}[\theta, x, \phi_{\text{HP}}] = \int_{\Omega_p} (\lambda_0^{p+1} \epsilon^{\overline{a}_1 \cdots \overline{a}_p}_L (\phi_{\text{HP}}^\alpha \wedge \phi_{\text{HP}}^\beta \wedge \cdots \wedge \phi_{\text{HP}}^\alpha) (\theta, x, \phi_{\text{HP}})(\cdot)) \]

(6.4)
\[ r_{\kappa}^\alpha(\kappa, y) := \text{sISO}(d, 1 | D_{d, 1}) \bigcap \{ (\theta^\alpha, x^\mu, \phi^{a\beta}) \mapsto (\theta^\alpha + \tilde{\kappa}^{\alpha}(\phi), x^\mu + \tilde{\gamma}^\mu(\phi) - \frac{1}{2} \theta^\alpha \Gamma^\mu \tilde{\kappa}(\phi), \phi^{a\beta}) \}, \]

with, this time, \( \phi_{\text{HP}} \in \text{SO}(d, 1) \) given by \( \bar{a} \in \mathbb{R}^d \)

\[ \phi_{\text{HP}}^\alpha = (\delta^\alpha_0 - \delta^\alpha_1 - \delta^\alpha_2) \phi_0 + (\delta^\alpha_1 \delta^\alpha_2 - \delta^\alpha_2 \delta^\alpha_1) \phi_1. \]

Above, we have reinstated a parameter \( \lambda_0 \in \mathbb{R}^n \) that quantifies the relative parametrisation of the two terms in the action functional. This parameter passes to the NG formulation upon integrating out the (unphysical) Goldstone degrees of freedom in the HP action functional. There, as in the HP action functional itself, its value does not affect the global symmetries of the super-\( \sigma \)-model in a qualitative manner, and so it remains arbitrary as long as we consider those symmetries only. Its status changes dramatically in the context of local symmetries for which we look, as heralded several times already, among infinitesimal (or tangential) shifts of the coordinates \( \theta^\alpha, x^\mu \) and \( \phi^{a\beta} \) induced by the right translations of (6.1) that is

\[ r_{\kappa}^\alpha(\kappa, y) : \text{sISO}(d, 1 | D_{d, 1}) \bigcap \{ (\theta^\alpha, x^\mu, \phi^{a\beta}) \mapsto (\theta^\alpha + \tilde{\kappa}^{\alpha}(\phi), x^\mu + \tilde{\gamma}^\mu(\phi) - \frac{1}{2} \theta^\alpha \Gamma^\mu \tilde{\kappa}(\phi), \phi^{a\beta}) \}, \]

where

\[ \tilde{\kappa}^{\alpha}(\phi) := S(\phi)_{\beta}^{\alpha} \kappa^\beta, \quad \tilde{\gamma}^\mu(\phi) := L(\phi)_{\mu}^{\rho} y^\rho \]

are the \( \text{SO}(d, 1) \)-rotated counterparts of the \( (\kappa^\alpha, y^\mu) \). These shifts satisfy the algebra

\[ [r_{(\kappa, y_1)}^{\kappa}, r_{(\kappa, y_2)}^{\kappa}] = r_{(0, \kappa, \Gamma)}^{\kappa}. \]

Note, in particular, that purely Grassmann-odd translations do not form a subalgebra in \( \text{sISO}(d, 1 | D_{d, 1}) \) which implies that closing the algebra of infinitesimal symmetries will require extending the space of such translations by (some) purely Grassmann-even ones.

For the field-theoretic analysis to follow, we shall also need the linearised transformations of the left-invariant super-1-forms of (6.2) under the above right shifts. These read

\[ (\Sigma^\alpha_L, \theta^\alpha_L)(\theta, x, \phi) \mapsto (\Sigma^\alpha_L, \theta^\alpha_L)(\theta, x, \phi) + \left( S(-\phi)_{\beta}^{\alpha} d\kappa^\beta(\phi), \tilde{L}(\phi)_{\mu}^{\rho} d\tilde{\gamma}^\mu(\phi) + \kappa^\alpha \Sigma_L(\theta, x, \phi) \right) + o(\kappa^2) \]

determine the (linearised) variation of the action functional of the HP super-\( \sigma \)-model. Below, we write out and examine the variations in the particular cases \( p \in \{0, 1\} \) in which we shall subsequently look for a suitable lift of the symmetry to the corresponding extended p-gerbes. The symmetry is discussed extensively, in the geometric and the (classical) field-theoretic contexts, in Refs. [McA06, GKW06k, cp also Ref. [dAIME05]].

6.1.1. The k-symmetry of the super-0-brane. The integrand of the HP action functional for the super-0-brane in the homogeneous space \( \text{sISO}(9,1|32)/\text{SO}(9) \) is given by the restriction of the super-1-form

\[ \tilde{\beta}^{(\lambda_0)}(\lambda_0) = \lambda_0 \theta_0^{(1)} + \pi_9^{(1)} B^{(1)}, \]

with \( B^{(1)} \) of Prop. [L] pulled back along the canonical projection

\[ \pi_9 : \text{sISO}(9,1|32) \longrightarrow \text{sISO}(9,1|32)/\text{SO}(9,1) \equiv \text{sMink}^{9,1|32}, \]

to the HP section with \( \phi \equiv \phi_{\text{HP}} \) as in Eq. (6.4). Its variation under the Grassmann-odd shift of the lagrangean field reads

\[ r_{\kappa^\alpha}(\lambda_0) = \lambda_0 \Sigma_L(\theta, x, \phi) \tilde{\gamma}^\mu(\phi) + 2\sigma(\theta) \tilde{\kappa}(\phi) + d(\theta^\alpha \Gamma^\mu \tilde{\kappa}(\phi)) \]

\[ = -4\Sigma_L(\theta, x, \phi) \tilde{\gamma}^\mu(\phi) + d(\theta^\alpha \Gamma^\mu \tilde{\kappa}(\phi)) \]

where

\[ \tilde{F}(\theta, x, \phi, (\kappa, 0)) = \theta^\alpha \Gamma^\mu \tilde{\kappa}(\phi). \]

The operator

\[ \mu_0^{(1)} : \varepsilon_0 \lambda_0 \mu_{\text{HP}}^{(1)} \in \text{End}_{\mathbb{C}}(S_{9,1}), \quad \varepsilon_0 \in \{ -1, +1 \}. \]
appearing in the first term of the variation is a projector iff

$$(\varepsilon_0, \lambda_0) \in \{(-1, -2), (1, 2)\},$$

and it then suffices to take

(6.11) $\kappa \in \ker P_{(\varepsilon_0, \lambda_0)}^{(0)}$

to obtain a tangential symmetry of the action functional. The difference between the two choices is immaterial, hence we set, say,

$$(\varepsilon_0, \lambda_0) = (1, 2)$$

and proceed with the symmetry analysis of the action functional associated with the super-1-form

(6.12) $\overline{\beta}_{(2)}^{(2)}(\theta, x, \phi) = 2\theta_L^0(\theta, x, \phi) + \theta T_{11}^0(\theta, x, \phi).$

Note that the complementary projector

$$P^{(0)} := 1_{32} - P^{(1,2)}_{(1,2)} = \frac{1_{32} - \pi^0 \Gamma_{11}}{2}$$

is precisely of the type described in Prop. 5.3 as

$$P^{(0)} T = \frac{1}{2} (1_{32} - \Gamma_{11}^T \cdot \Gamma_{0}^0 \Gamma_{11}^T) = \frac{1}{2} (1_{32} - C \cdot \Gamma_{11} \cdot C^{-1}) = C \cdot \frac{1}{2} (1_{32} - \Gamma_{11} \cdot \Gamma_{0}) \cdot C^{-1} = C \cdot \frac{1}{2} (1_{32} - P^{(0)}) \cdot C^{-1}$$

and, for any $\overline{\alpha} \in \Gamma_{0},$

$$P^{(0)} \cdot \Gamma_{0} = \frac{1}{2} (\Gamma_{0}^0 - \Gamma_{0} \cdot \Gamma_{11} \cdot \Gamma_{0}^0) = \frac{1}{2} (\Gamma_{0}^0 + \Gamma_{0} \cdot \Gamma_{11} \cdot \Gamma_{0}^0) \equiv \Gamma_{0} \cdot (1_{32} - P^{(0)}),$$

$$P^{(0)} \cdot \Gamma_{0} \equiv \frac{1}{2} (\Gamma_{0}^0 - \Gamma_{0} \cdot \Gamma_{11} \cdot \Gamma_{0}^0) = \frac{1}{2} (\Gamma_{0}^0 + \Gamma_{0} \cdot \Gamma_{11} \cdot \Gamma_{0}^0) \equiv \Gamma_{0} \cdot (1_{32} - P^{(0)}),$$

(6.12) $\overline{\beta}_{(2)}^{(2)}(\theta, x, \phi, (0, y)) - \overline{\beta}_{(1)}^{(2)}(\theta, x, \phi) = 2L (-\phi)^0 \mu d\theta^{\mu} = 2d\theta^0 y^\mu \eta_{\mu\nu} \theta_L^0(\theta, x, \phi) + 2d\theta^0 y^\mu \delta_{\mu\nu} \theta_L^0(\theta, x, \phi).$

The variation descends unscathed unto the locus of the inverse Higgs constraint\footnote{22} and so we are bound to set

$$y^\mu = y \delta_0^\mu, \quad \mu \in \mathbb{R}_{0,9}$$

for $y \in \mathbb{R}$ arbitrary, whereupon we arrive at

$$\overline{r}_{(2)}(\theta, x, \phi, (0, y \delta_0)) = \overline{\beta}_{(1)}^{(2)}(\theta, x, \phi) = d\overline{F}(\theta, x, \phi).$$

where

$$\overline{F}(\theta, x, \phi, (0, y \delta_0)) = 2y.$$

The distinguished shifts in the time direction may be seen to engender diffeomorphisms of the super-0-brane worldline in the so-called static gauge, cp. Ref. (6.12) and so insisting on their presence among gauge-symmetry generators is physically perfectly justified. Putting the pieces together, we establish a general gauge variation

$$\overline{r}_{(2)}(\theta, x, \phi, (P^{(0)} \kappa, y \delta_0)) - \overline{\beta}_{(1)}^{(2)}(\theta, x, \phi) = d\overline{F}(\theta, x, \phi, (P^{(0)} \kappa, y \delta_0)),$$

with

(6.13) $\overline{F}(\theta, x, \phi, (P^{(0)} \kappa, y \delta_0)) = 2y + \theta T_{11}^0(\theta, x, \phi).$

We complete the analysis of the tangential vacuum-symmetry superalgebra with the examination of the commutator (6.9) of two gauge shifts. This is dictated by the algebra

$$\kappa_1 \Gamma_{0} \kappa_2 \equiv \kappa_1 P^{(0)} T \cdot \Gamma_{0} \cdot P^{(0)} \kappa_2 = \kappa_1 C \cdot (1_{32} - P^{(0)})^2 \cdot \Gamma_{0} \kappa_2 = \kappa_1 C \cdot (1_{32} - P^{(0)}) \cdot \Gamma_{0} \kappa_2$$
The tangential vacuum-symmetry superalgebra of the Green–Schwarz super-
the set of constraints is complete.

The spinor rotations). The last conclusion applies (trivially) also to the other set of constraints, and so
(we use (G.1) (in conjunction with the Cartan–Dieudonné theorem) to commute the projectors through
their variation under a general symmetry transformation,

\[ r^{\kappa} \theta^2_{\kappa} (\theta, x, \phi) - \theta^2_{\kappa} (\theta, x, \phi) = y \theta^2_{\kappa} (\theta, x, \phi) + \kappa \Gamma^\alpha \mathbf{P}(\alpha, 2) \Sigma_{\alpha} (\theta, x, \phi). \]

The demand that the above vanish yields secondary constraints (recall that the \( \Gamma^\alpha \) are invertible):

\[ \theta^2_{\kappa} (\theta, x, \phi) \mid_\Gamma = 0, \quad \tilde{a} \in \Gamma, \quad \mathbf{P}(\alpha, 2) \Sigma_{\alpha} (\theta, x, \phi) \mid_\Gamma = 0 \]

of which the former were identified in Ref. [GKW06a] as field equations of the super-\( \sigma \)-model under study. Constraints analogous to the latter one were encountered in the study of gauge supersymmetries of the GS super-\( \sigma \)-model in Ref. [McA00]. Clearly, the constraint enforces the removal of the pure-gauge Graßmann-odd degrees of freedom and as such possesses a status fundamentally different from the other ones. In the light of the rôle it plays, it makes sense to impose in conjunction with the inverse Higgs constraint and call it the \( \kappa \)-gauge constraint. In the case in hand, it does not affect our earlier conclusions. Neither does it lead to any secondary constraints upon variation in a direction \((\mathbf{P}(\alpha, 2) \kappa, y \delta^\alpha)\) as

\[ r^{\kappa} \mathbf{P}(\alpha, 2) \Sigma_{\alpha} (\theta, x, \phi) = \mathbf{P}(\alpha, 2) \Sigma_{\alpha} (\theta, x, \phi) + \mathbf{P}(\alpha, 2) \cdot S(-\phi) \theta^2_{\kappa} (\theta, x, \phi) = \mathbf{P}(\alpha, 2) \Sigma_{\alpha} (\theta, x, \phi) \]

(we use (3.3) in conjunction with the Cartan–Dieudonné theorem) to commute the projectors through the
spinor rotations). The last conclusion applies (trivially) also to the other set of constraints, and so the
set of constraints is complete.

Our findings are summarised in

**Proposition 6.1.** The tangential vacuum-symmetry superalgebra of the Green–Schwarz super-\( \sigma \)-model
for the super-\( 0 \)-brane is the Lie (sub-)superalgebra

\[ \mathcal{G}^{(G0, 0)} \equiv \left\{ Q^\alpha := \mathbf{P}(\alpha, 2) Q^\beta | \alpha \in \Gamma, 2 \right\} \right\}_C \oplus \left\{ P_0 \right\}_C \oplus \mathfrak{sO}(9, 1) \subset \mathfrak{s} \text{mink}^{9, 1|32}, \]

written for

\[ \mathbf{P}(\alpha, 2) = \frac{1}{2 \alpha \Gamma_1} \Gamma_{1\alpha}. \]

Its supertranslation sub-superalgebra

\[ t^{(G0, 0)} \equiv \left\{ Q^\alpha := \mathbf{P}(\alpha, 2) Q^\beta | \alpha \in \Gamma, 2 \right\} \right\}_C \oplus \left\{ P_0 \right\}_C \]

has the basis Lie superbrackets \((\alpha, \beta \in \Gamma, 2)\)

\[ \{Q^\alpha, Q^\beta\} = \left( \mathbf{P}(\alpha, 2) \cdot \Gamma^{(0)} \right)_{\alpha \beta} P_0, \quad [P_0, Q^\alpha] = 0. \]

All elements of the latter satisfy the \( \kappa \)-symmetry condition, and so define gauge symmetries of
the Hughes–Polchinski super-\( \sigma \)-model of Def. [5.3] (with \( p = 0 \)), realised linearly \(^2\) on its fields – accordingly, we shall call \( t^{(G0, 0)} \) the \( \text{(translational) } \kappa \)-symmetry superalgebra of the super-\( 0 \)-brane. The symmetries preserve the space of its restricted (classical) field configurations \( \mathcal{D}_0 \subset \mathfrak{s} \text{ISO}(9, 1|32) \) defined by the family of constraints: the inverse Higgs constraint

\[ \theta^2_{\kappa} \mid_\Gamma \mathcal{D}_0 = 0, \quad \tilde{a} \in \Gamma, \]

\[ ^2\text{This means, just to reiterate, that we consider a realisation of the supervector space } t^{(G0, 0)}, \text{ not of the Lie superalgebra.} \]
and the $\kappa$-gauge constraint

\begin{equation}
(1_{32} - p^{(0)}) \Sigma L \upharpoonright_\mathcal{T} \mathcal{G}_0 = 0,
\end{equation}

together with the dynamical constraints

\begin{equation}
\theta_1^{\mathcal{G}_0} \upharpoonright_\mathcal{T} \mathcal{G}_0 = 0, \quad \tilde{a} \in \mathcal{T} \mathcal{G}.
\end{equation}

6.1.2. The $\kappa$-symmetry of the Green–Schwarz superstring. In the case of the superstring super-$\sigma$-model, 
the integrand of the HP action functional is the restriction of the super-2-form

\[ \bar{\beta}^{(\lambda_1)}_{(2)} = \lambda_1 \theta_1^0 \wedge \theta_1^1 + \pi_2^b B, \]

with $B$ of Prop. 4.1 pulled back along the canonical projection

\begin{equation}
\pi_d : sISO(d,1|D_{d,1}) \longrightarrow sISO(d,1|D_{d,1})/SO(d,1) \equiv sMink^{d,1|D_{d,1}}
\end{equation}
to the HP section with $\phi \equiv \phi_{\text{HP}}$ as in Eq. (5.7), and we readily compute, invoking Eqs. (I.4.3), (I.4.4) and (I.4.5) along the way,

\[ \rho^\kappa \ast \bar{\beta}^{(\lambda_1)}_{(2)}((\theta, x, \phi), (\kappa, 0)) = - \bar{\beta}^{(\lambda_1)}_{(2)}(\theta, x, \phi) \]

where

\begin{equation}
\tilde{\mathbb{E}}((\theta, x, \phi), (\kappa, 0)) = \theta \Gamma_{\mu} \mathcal{K}(\phi) e^\mu(\theta, x).
\end{equation}

The operator

\[ P_{(\epsilon_1, \lambda_1)}^{(1)} = \epsilon_1 \frac{1}{2} \frac{1}{\lambda_1 \theta_1^0 \wedge \theta_1^1} \in \text{End}_\mathbb{C}(S_{d,1}), \quad \epsilon_1 \in \{-1, +1\} \]
an appearing in the above expression is a projector iff

\[ (\epsilon_1, \lambda_1) \in \{-1, -2, (1, 2)\}, \]

and so for

\[ \kappa \in \text{ker} P_{(\epsilon_1, \lambda_1)}^{(1)} \]

we obtain a gauge symmetry of the space of field configurations subject to the inverse Higgs constraint, i.e., of the corresponding Nambu–Goto dynamics. Once again, we set, without any loss of generality,

\[ (\epsilon_1, \lambda_1) = (1, 2) \]

and continue our analysis for the super-2-form

\begin{equation}
\bar{\beta}^{(2)}_{(2)}(\theta, x, \phi) = 2(\theta_1^0 \wedge \theta_1^1)(\theta, x, \phi) + \theta \Gamma_{\mu} \sigma(\theta) \wedge dx^\mu.
\end{equation}

Inspection of the complementary projector

\[ P_{(1,2)}^{(1)} = 1_{D_{d,1}} - P_{(1,2)}^{(1)} \]

reveals its expected properties

\begin{align*}
P_{(1,2)}^{(1)} & = \frac{1}{2} \left( 1_{D_{d,1}} + \left( \Gamma^0 \right)^T \cdot \left( \Gamma^0 \right)^T \right) = \frac{1}{2} \left( 1_{D_{d,1}} + C \cdot \Gamma^1 \cdot C^{-1} \cdot C \cdot \Gamma^0 \cdot C^{-1} \right) = C \cdot \frac{1}{2} \left( 1_{D_{d,1}} + \Gamma^1 \cdot \Gamma^0 \right) \cdot C^{-1} \\
& = C \cdot \frac{1}{2} \left( 1_{D_{d,1}} - \Gamma^0 \cdot \Gamma^1 \right) \cdot C^{-1} \equiv C \cdot \left( 1_{D_{d,1}} - P_{(1,2)} \right) \cdot C^{-1}
\end{align*}

and, for any $\tilde{a} \in \mathcal{T} \mathcal{G}$,

\begin{align*}
P_{(1,2)}^{(1)} \Gamma^0 & \equiv \frac{1}{2} \left( \Gamma^0 + \Gamma^0 \cdot \Gamma^1 \cdot \Gamma^0 \right) = \frac{1}{2} \left( \Gamma^0 - \Gamma^0 \cdot \Gamma^1 \cdot \Gamma^0 \right) = \Gamma^0 \cdot \left( 1_{D_{d,1}} - P_{(1,2)} \right), \\
P_{(1,2)}^{(1)} \Gamma^1 & \equiv \frac{1}{2} \left( \Gamma^1 + \Gamma^0 \cdot \Gamma^1 \cdot \Gamma^1 \right) = \frac{1}{2} \left( \Gamma^1 - \Gamma^0 \cdot \Gamma^1 \cdot \Gamma^1 \right) = \Gamma^1 \cdot \left( 1_{D_{d,1}} - P_{(1,2)} \right), \\
\end{align*}

63
\[ p^{(1)} \cdot \Gamma^\alpha = \frac{1}{2} (\Gamma^\alpha + \Gamma_0 \cdot \Gamma_1 \cdot \Gamma^\alpha) = \frac{1}{2} (\Gamma^\alpha - \Gamma_0 \cdot \Gamma_1 \cdot \Gamma^\alpha) = \frac{1}{2} (\Gamma^\alpha + \Gamma^\alpha \cdot \Gamma_0 \cdot \Gamma_1) \equiv \Gamma^\alpha \cdot p^{(1)}. \]

Taking into account the source of the \( \kappa \)-gauge constraint in the super-0-brane model, we interrupt the derivation of the algebra of gauge symmetries to inspect implications of the requirement of invariance of the inverse Higgs constraints under the purely Grassmann-odd shifts from \( \ker P^{(1)}_{(1,2)} \). These we read off from the direct calculation

\[ r^{\kappa \ast} \beta_L^0((\theta, x, \phi), (\kappa, 0)) = \theta_L^0(\theta, x, \phi) + \kappa \Gamma^\alpha \cdot P^{(1)}_{(1,2)} \Sigma_L(\theta, x, \phi) \]

which – as before – yields the \( \kappa \)-gauge constraint

\[ P^{(1)}_{(1,2)} \Sigma_L \neq 0, \]

to be imposed jointly with the inverse Higgs constraint in what follows.

Returning to the reconstruction of the tangential vacuum-symmetry superalgebra, we compute the variation of the lagrangian density of the superstring super-\( \sigma \)-model under Grassmann-even shifts,

\[
\begin{align*}
&\quad r^{\kappa \ast} \beta_{(2)}((\theta, x, \phi), (0, y)) - \beta_{(2)}((\theta, x, \phi)) \\
&= 2L(-\phi) \mu \delta \theta L(\theta, x, \phi) + 2\eta \phi L(\theta, x, \phi) \wedge L(-\phi) \mu \delta \theta \mu \sigma(\theta) \wedge \delta \theta \mu \\
&= 2y^\theta \eta \mu \eta \phi \eta \phi L(\theta, x, \phi) + 2y^\theta \eta \mu \theta L(\theta, x, \phi) - 2y^\theta \eta \mu \theta L(\theta, x, \phi) + \delta \theta \mu \sigma(\theta)
\end{align*}
\]

for the Maurer–Cartan equations of Sec. I.4.1) whose projection to the common locus of the inverse Higgs and \( \kappa \)-gauge constraints takes the form

\[
\begin{align*}
&\quad (r^{\kappa \ast} \beta_{(2)}((\theta, x, \phi), (0, y)) - \beta_{(2)}((\theta, x, \phi)))|_{\theta_L = 0, \varphi \in \mathbb{Z}} \\
&= y^\theta \Sigma_L \wedge \Gamma^\alpha \Sigma_L + 2y^\theta \delta \theta \Sigma_L \wedge \theta L(\theta, x, \phi) + \delta \theta \Sigma_L \wedge \theta L(\theta, x, \phi)
\end{align*}
\]

where

\[ \bar{E}((\theta, x, \phi), (0, y)) = 2y^\theta \theta_L(\theta, x, \phi) - 2y^\theta \theta_L(\theta, x, \phi) - \bar{g} \mu(\theta, x, \phi) \wedge \bar{g} \mu(\theta, x, \phi) \]

From the above, we infer the condition

\[ y^\mu = y^0 \delta^\mu_0 + y^1 \delta^\mu_1, \quad \mu \in \mathbb{Z}, \]

to be obeyed by admissible gauge transformations. For these, we obtain coherence constraints

\[ r^{\kappa \ast} \beta_L^0((\theta, x, \phi), (0, y^0 \delta_0 + y^1 \delta_1)) = \theta_L^0(\theta, x, \phi) + y^0 \theta_L^0(\theta, x, \phi) - \theta_L^0(\theta, x, \phi), \]

and so additional conditions

\[ \theta_L^0 = 0, \quad (\mu, \nu) \in \{0, 1\} \]

have to be imposed for consistency. Altogether, we find, for the admissible gauge (super)translations,

\[
\begin{align*}
&\quad r^{\kappa \ast} \beta_{(2)}((\theta, x, \phi), \{(p^{(1)} \kappa, y^0 \delta_0 + y^1 \delta_1)\}) - \beta_{(2)}((\theta, x, \phi)) \\
&= d\bar{E}((\theta, x, \phi), \{(p^{(1)} \kappa, y^0 \delta_0 + y^1 \delta_1)\})
\end{align*}
\]

with

\[ \bar{E}((\theta, x, \phi), \{(p^{(1)} \kappa, y^0 \delta_0 + y^1 \delta_1)\}) = 2y^0 \theta_{(1)}^0(\theta, x, \phi) - 2y^1 \theta_{(1)}^0(\theta, x, \phi) - \bar{g} \mu(\theta, x, \phi) \wedge \bar{g} \mu(\theta, x, \phi)
\]

Note that here, as in the previous case, the constraints found are preserved by the gauge transformations, and so can be imposed self-consistently.
We conclude our discussion of the gauge symmetries of the superstring by verifying their closure under the Lie superbracket of \( \text{smint}^{d,1}|D_{d,1} \). Taking into account the identities, valid for \( \kappa_1, \kappa_2 \in \ker P^{(1)}_{(1,2)} \),

\[
\kappa_1 \Gamma^0 \kappa_2 = 0, \quad \bar{a} \in 2, d
\]

and

\[
\kappa_2 \in \ker P^{(1)}_{(1,2)} \iff \Gamma^1 \kappa_2 = -\Gamma^0 \kappa_2,
\]

we reduce the commutator of two gauge transformations as

\[
(6.20) \quad \left[ r^K_{(\beta)}(\kappa_1,0), r^K_{(\beta)}(\kappa_2,0) \right] = r^K_{(0,\kappa_1 \Gamma^0 \kappa_2 \delta_0 + \kappa_1 \Gamma^0 \delta_1)} = r^K_{(0,\kappa_1 \Gamma^0 \kappa_2 (\delta_0 - \delta_1))}.
\]

**Remark 6.2.** It is amusing to note that the latter result is in keeping with the symmetry analysis of the WZW \( \sigma \)-model of the bosonic string on a (compact) Lie group. Indeed, one-sided regular translations on the group manifold induce chiral gauge symmetries of the latter field theory. These extend, through the Sugawara construction \( 6,3 \), the algebra of (conformal) worldsheet diffeomorphisms. When read in the static gauge underlying the interpretation of the supertarget in the directions \( x^\alpha, \bar{a} \in \{0, 1\} \) in terms of diffeomorphisms of the embedded worldsheet, the formula for the commutator of a pair of Grassmann-odd translations above seems to be a manifestation of the fact that the right gauge symmetry on the Lie (super)group depends solely on the light-cone coordinate \( \sigma^0 - \sigma^1(\equiv x^0 - x^1) \) and generates, à la Sugawara, the chiral left-moving copy of the Virasoro algebra.

We recapitulate our discussion in

**Proposition 6.3.** The tangential vacuum-symmetry superalgebra of the Green–Schwarz super-\( \sigma \)-model for the superstring in the Lie (sub-)superalgebra

\[
\mathfrak{g}^{(\text{GS,1})}_{\text{vac}} \equiv \left\{ Q^\alpha_\beta := P^{(1)} \alpha \beta Q_\beta \mid \alpha \in \frac{1}{4} D_{d,1}, \Gamma^1 \right\} \subset \langle P_0, P_1 \rangle_\mathbb{C} \oplus \langle J_{01} \rangle_\mathbb{C} \oplus \mathfrak{so}(d - 1) \subset \text{smint}^{d,1}|D_{d,1},
\]

written for

\[
P^{(1)} = \frac{1}{2} \rho_{d,1} + \Gamma^0 \Gamma^1.
\]

Its supertranslation sub-superalgebra

\[
\mathfrak{t}^{(\text{GS,1})}_{\text{vac}} \equiv \left\{ Q^\alpha_\beta := P^{(1)} \alpha \beta Q_\beta \mid \alpha \in \frac{1}{4} D_{d,1}, \Gamma^1 \right\} \subset \langle P_0, P_1 \rangle_\mathbb{C}
\]

has the basis Lie superbrackets (\( \alpha, \beta \in \frac{1}{4} D_{d,1}, \bar{a} \in \{0, 1\} \))

\[
\{ Q^\alpha_\beta, Q^\gamma_\delta \} = (P^{(1)} \Gamma^0 \alpha \beta \delta_0 + P^{(1)} \Gamma^0 \alpha \beta \delta_1) P_0 + (P^{(1)} \Gamma^0 \alpha \beta \delta_0 + P^{(1)} \Gamma^0 \alpha \beta \delta_1) P_1,
\]

\[
\Gamma^1 \kappa_2 = -\Gamma^0 \kappa_2,
\]

\[
\kappa_2 \in \ker P^{(1)}_{(1,2)} \iff \Gamma^1 \kappa_2 = -\Gamma^0 \kappa_2,
\]

\[
(6.22) \quad (1_{D_{d,1}} - P^{(1)}) \Sigma_{\mathbb{C}} |_{\mathcal{T} \mathcal{F}_1} = 0,
\]

All elements of the latter satisfy the \( \kappa \)-symmetry condition, and so define gauge symmetries of the Hughes–Polchinski super-\( \sigma \)-model of Def. \( 5.2 \) (with \( p = 1 \)), realised linearly \( 3 \) on its fields – accordingly, we shall call \( \mathfrak{t}^{(\text{GS,1})}_{\text{vac}} \) the (translational) \( \kappa \)-symmetry superalgebra of the superstring. The symmetries preserve the space of its restricted (classical) field configurations \( \mathcal{F}_1 \subset \text{slSO}(d,1|D_{d,1}) \) defined by the family of constraints: the inverse Higgs constraint

\[
(6.21) \quad \theta^\alpha_{\bar{a}} |_{\mathcal{T} \mathcal{F}_1} = 0, \quad \bar{a} \in 2, d
\]

and the \( \kappa \)-gauge constraint

\[
(6.22) \quad (1_{D_{d,1}} - P^{(1)}) \Sigma_{\mathbb{C}} |_{\mathcal{T} \mathcal{F}_1} = 0,
\]

together with the dynamical constraints

\[
(6.23) \quad \theta^a_\alpha |_{\mathcal{T} \mathcal{F}_1} = 0, \quad (a, \bar{a}) \in \{0, 1\} \times 2, d.
\]

\( 24 \) \( \bar{C}p \) the footnote on p. 68.
6.2. The extended Hughes–Polchinski gerbes. Having understood the (super)group-theoretic origin of $\kappa$-symmetry in the framework of Cartan geometry of the extended supersymmetry group $\text{sISO}(d,1|D_{d,1})$, we may next – in the spirit of Sec.I.5.1 – look for a geometrisation of the super-$\sigma$-model and the corresponding gerbe-theoretic extension of its gauge-symmetry analysis. This is more than well justified as the relevant action functional $S^{(\text{HP})}_{\text{metr.GS.p}}$ has the structure of a (super-)p-gerbe holonomy, with the “metric” term $S^{(\text{HP})}_{\text{metr.GS.p}}$ of (5.14) determined by a manifestly supersymmetric super-$(p+1)$-form and hence defining a trivial super-p-gerbe on the extended super-target. The analysis of the preceding section suggests that the ensuing simple picture of a (Deligne) tensor product of the latter trivial super-p-gerbe with the pullback of the super-p-gerbe from $\text{sMink}^{d,1|D_{d,1}}$ to the super-Poincaré supergroup $\text{sISO}(d,1|D_{d,1})$ along the canonical projection be refined though incorporation of the tangential constraints deduced from the $\kappa$-symmetry analysis, so that ultimately we wind up with a restriction of the product gerbe to the Hughes–Polchinski section $\mathcal{P}_p$ defined as in Props.6.1 and 6.3. Thus, we arrive at

**Definition 6.4.** Let $\mathcal{G}^{(0)}_{\text{GS}}$ be the Green–Schwarz super-0-gerbe over $\text{sMink}^{9,1|32}$ of Def.I.5.2, recalled in Sec.6. The **extended Hughes–Polchinski 0-gerbe** over $\text{sISO}(9,1|D_{9,1})$ is the tensor product

$$
\tilde{\mathcal{G}}^{(0)}_{\text{HP}} := \pi_9 \mathcal{G}^{(0)}_{\text{GS}} \otimes \mathcal{I}_2 \beta^{(\text{HP})}_{(1)}
$$

of the trivial (super-)0-gerbe $\mathcal{I}_2 \beta^{(\text{HP})}_{(1)}$ equipped with the principal $\mathbb{C}^\times$-connection with the global base component

$$2 \beta^{(\text{HP})}_{(1)} = 2 \theta^0_L$$

with the pullback of $\mathcal{G}^{(0)}_{\text{GS}}$ along the canonical projection (6.10).

and the analogous

**Definition 6.5.** Let $\mathcal{G}^{(1)}_{\text{GS}}$ be the Green–Schwarz super-1-gerbe over $\text{sMink}^{d,1|D_{d,1}}$ of Def.I.5.9, recalled in Sec.6. The **extended Hughes–Polchinski 1-gerbe** over $\text{sISO}(d,1|D_{d,1})$ is the tensor product

$$
\tilde{\mathcal{G}}^{(1)}_{\text{HP}} := \pi_9 \mathcal{G}^{(1)}_{\text{GS}} \otimes \mathcal{I}_2 \beta^{(\text{HP})}_{(2)}
$$

of the trivial (super-)1-gerbe $\mathcal{I}_2 \beta^{(\text{HP})}_{(2)}$ equipped with the curving with the global base component

$$2 \beta^{(\text{HP})}_{(2)} = 2 \theta^0_L \wedge \theta_L^1$$

with the pullback of $\mathcal{G}^{(1)}_{\text{GS}}$ along the canonical projection (6.17).

Let us unwrap the above definitions with view to our subsequent considerations. Thus, the extended HP 0-gerbe is the triple

$$
\tilde{\mathcal{G}}^{(0)}_{\text{HP}} = \left( \tilde{\mathcal{Z}}^{(0)}, \pi_{\tilde{\mathcal{Z}}^{(0)}}, \tilde{\beta}^{(0)}_{(1)} \right)
$$

that consists of the principal $\mathbb{C}^\times$-bundle

$$
\begin{array}{ccc}
\mathbb{C}^\times & \xrightarrow{\pi_{\tilde{\mathcal{Z}}^{(0)}}} & \tilde{\mathcal{Z}}^{(0)} := \text{sISO}(9,1|32) \times_{\pi_9} (\text{sMink}^{9,1|32} \times \mathbb{C}^\times) \\
\downarrow & & \uparrow_{\pi_{\tilde{\mathcal{Z}}^{(0)}} := \text{pr}_1} \\
& \text{sISO}(9,1|32) & 
\end{array}
$$
defined in terms of the pullback
\[
\text{sISO}(9,1|32) \times_{\pi_9} \left( \text{sMink}^9,1|32 \times C^\ast \right) \xrightarrow{\pi_9 \circ \text{pr}_2} \text{sMink}^9,1|32 \times C^\ast
\]
\[
\text{sISO}(9,1|32) \xrightarrow{\pi_9} \text{sMink}^9,1|32
\]
and equipped with the principal \( C^\ast \)-connection super-1-form
\[
\bar{\beta} := \pi_9^* \beta \biggl( \frac{2}{1} \biggr) + 2\pi_9^* \beta \biggl( \frac{\text{HP}}{0} \biggr)
\]
The extended HP 1-gerbe is the septuple
\[
\bar{\mathcal{G}}^{(1)}_{\text{HP}} = \left( \bar{\mathcal{Y}}_{\text{sISO}(d,1|D_{d,1})}, \pi_{\text{sISO}(d,1|D_{d,1})}, \bar{\beta}, \mathcal{F}^{(1)}(1), \pi_{\mathcal{F}(1)}, A_{\mathcal{F}(1)}, \mu_{\mathcal{F}(1)} \right)
\]
composed of the pullback surjective submersion
\[
\bar{\mathcal{Y}}_{\text{sISO}(d,1|D_{d,1})} := \text{sISO}(d,1|D_{d,1}) \times_{\pi_d} \left( \text{sMink}^{d,1|D_{d,1}} \times \mathbb{R}^0[D_{d,1}] \right) \xrightarrow{\pi_d \circ \text{pr}_2} \text{sMink}^{d,1|D_{d,1}} \times \mathbb{R}^0[D_{d,1}] \equiv Y_1 \text{sMink}^{d,1|D_{d,1}},
\]
with the curving super-2-form
\[
\bar{\beta} := \pi_d^* \beta \biggl( \frac{2}{1} \biggr) + 2\pi_d^* \beta \biggl( \frac{\text{HP}}{0} \biggr)
\]
on its total space and of the pullback principal \( C^\ast \)-bundle
\[
C^\ast \xrightarrow{\mathcal{F}^{(1)}(1)} := \pi_d^{x^2} \ast \mathcal{L}^{(1)} \]
\[
\bar{\mathcal{Y}}^{[2]}_{\text{sISO}(d,1|D_{d,1})} \equiv \bar{\mathcal{Y}}_{\text{sISO}(d,1|D_{d,1})} \times_{\text{sISO}(9,1|32)} \bar{\mathcal{Y}}_{\text{sISO}(d,1|D_{d,1})}
\]
with the total space
\[
\pi_{\mathcal{F}(1)} := \pi_d^{x^2} \ast \mathcal{L}^{(1)} \xrightarrow{\pi_{\mathcal{F}(1)}} \mathcal{L}^{(1)}
\]
\[
\bar{\mathcal{Y}}^{[2]}_{\text{sISO}(d,1|D_{d,1})} \xrightarrow{\pi_d^{x^2} \circ \text{pr}_2} \mathcal{Y}_1^{[2]} \text{sMink}^{d,1|D_{d,1}}
\]
equipped with the principal \( C^\ast \)-connection super-1-form
\[
A_{\mathcal{F}(1)} := \pi_d^{x^3} \ast A_{\mathcal{F}(1)}(1) \equiv \text{pr}_2^* A_{\mathcal{F}(1)}(1)
\]
and with the pullback groupoid structure
\[
\mu_{\mathcal{F}(1)} = \pi_d^{x^3} \ast \mu_{\mathcal{F}(1)}(1) \equiv \text{pr}_2^* \mu_{\mathcal{F}(1)}(1)
\]
on its fibres.

6.3. A linearised \( \kappa \)-equivariant structure of the extended Hughes–Polchinski gerbe. The purely gerbe-theoretic nature of the HP action functional of the GS super-\( \sigma \)-model in conjunction with the presence of a gauge supersymmetry rederived at the beginning of the present section, give rise to the hypothesis, based on former studies reported in Refs. [GSW10, GSW13, Sus13], that the extended HP \( p \)-gerbe should be endowed with an equivariant structure of some sort with respect to the action of the corresponding (translational) \( \kappa \)-symmetry superalgebra. \textit{A priori}, such an informed guess is confronted with several more or less obvious obstacles: First of all, the very existence of the symmetry necessitates imposition of constraints on the admissible field configurations – in particular, the symmetry algebra does not seem to close on non-classical field configurations (\textit{cp.}, \textit{e.g.}, Ref. [McA00, GKW06a]). Luckily,
the constraints, enumerated in Props. 6.1 and 6.3, including the dynamical ones, are (super)geometric in nature, i.e., they can be treated as linear conditions to be imposed on sections of the tangent sheaf of the super target, distinguishing the HP section $\mathcal{D}_p$, $p \in \{0,1\}$ within it. Secondly, the symmetry in question is tangential (or infinitesimal) and it is the supervector space $t^{(\text{GS},\rho)}_{\text{vac}}$ and not the associated Lie superalgebra that is being represented, whence we may anticipate that a linearisation of sorts will have to be implemented when lifting it to the relevant $p$-gerbe. Such a linearisation is bound to result in a non-standard – if any at all – notion of a $t^{(\text{GS},\rho)}_{\text{vac}}$-equivariant structure on $\mathcal{E}^{(p)}_{\text{HP}}$. Lastly, in keeping with the reasoning advanced in Sec. 6.1, we ought to demand and verify compatibility of any such structure with the global supersymmetry present, the latter being quantified, just as in the Nambu–Goto picture, by the super-Poincaré supergroup $\text{sISO}(d,1|D_{d,1})$. Now, the HP section $\mathcal{D}_p$ is defined in terms of manifestly supersymmetric super-1-forms, therefore it is preserved by supersymmetry transformations. So is the trivial correction $\mathcal{I}_{\theta}^{\beta}_{\text{HP}}$, with its manifestly supersymmetric (global) base curving $2\beta^{(\text{HP})}$.

Remark 6.6. In order to be able to exploit directly the results of our detailed discussion of $\rho$-equivariant structures on (super-)gerbes (for $p \in \{0,1\}$), presented in Secs. 6.1 and 6.2, respectively, we replace the right action $r_\rho$ of the supervector space $t^{(\text{GS},\rho)}_{\text{vac}}$ on $\text{sISO}(d,1|D_{d,1})$, defined in Eq. (1.8), with its *left* counterpart

$$\lambda_\rho : t^{(\text{GS},\rho)}_{\text{vac}} \times \text{sISO}(d,1|D_{d,1}) \rightarrow \text{sISO}(d,1|D_{d,1}) : (X,(\theta,x,\phi)) \mapsto r^\rho_{-\lambda}(\theta,x,\phi).$$

In what follows, we shall use the shorthand notation

$$t_{\text{Mink}}^{(\text{GS},p)} \equiv t^{(\text{GS},p)}_{\text{vac}} \times \text{sISO}(d_p,1|D_{d_p,1})$$

(with $d_0 = 9$ and $d_p = d$ as predicted by the old brane scan) for the sake of transparency.

6.3.1. The $\kappa$-equivariant extended HP $\theta$-gerbe. Reasoning as in Sec. 4.1, we seek to construct a connection-preserving isomorphism

$$\tilde{\mathcal{F}}^\kappa_{\theta} : \lambda^\kappa \cdot \tilde{\mathcal{F}}^{(0)}_{\text{HP}} \xrightarrow{\cong} \text{pr}_2^\kappa \mathcal{E}^{(0)}_{\text{GS}} \otimes \mathcal{I}_{\mathcal{F}^{(0)}_{\theta}}$$

between the principal $\mathbb{C}^\times$-bundle

$$\lambda^\kappa \cdot \mathcal{F}^{(0)}_{\theta} = t_{\text{Mink}}^{(\text{GS},0)} \times \lambda^\kappa \cdot \mathcal{F}^{(0)}_{\theta} \xrightarrow{\cong} \text{pr}_2^\kappa \mathcal{F}^{(0)}_{\theta} \xrightarrow{\pi_{\mathcal{F}^{(0)}}} \mathcal{E}^{(0)}_{\text{GS}} \xrightarrow{\lambda^\kappa} \text{sISO}(9,1|32)$$

25 We focused on the translational component of the supersymmetry group before but the extension of our analysis to the semidirect product of the super-Minkowski supergroup with the Lorentz group $\text{SO}(d,1)$ proceeds without any obstruction.
with the principal $C^*$-connection super-1-form
\[ \hat{\lambda}^\ast \beta \equiv \text{pr}_{2}^\ast \left( \bar{\rho}_{0}^\ast \beta^{(2)}_{\ast} + 2\pi_{\mathcal{F}(\mathcal{G})}^\ast \beta^{(\text{HP})}_{\ast} \right), \]
and the principal $C^*$-bundle
\[
\begin{array}{l}
\text{pr}_{2}^\ast \mathcal{F}(0) \otimes \mathcal{F}_{\mathcal{P}} = tM\mathcal{M}_{\text{vac}}^{(\mathcal{G},\mathcal{S})} \times_{\text{pr}_{2}} \mathcal{F}(0) \\
\end{array}
\]
the latter having its principal $C^*$-connection super-1-form
\[ \hat{\rho}_{0}^\ast \beta \equiv \text{pr}_{2}^\ast \left( \bar{\rho}_{0}^\ast \beta^{(2)}_{\ast} + 2\pi_{\mathcal{F}(\mathcal{G})}^\ast \beta^{(\text{HP})}_{\ast} \right) \]
corrected by the pullback of some super-1-form $\hat{\rho}^\ast \beta \in \Omega^1(tM\mathcal{M}_{\text{vac}}^{(\mathcal{G},\mathcal{S})})$ to be derived. As mentioned earlier, the gauge symmetry of the super-$\sigma$-model that we are reconstructing has a very concrete effect, to wit, it restores balance between the bosonic and the fermionic degrees of freedom in the (effective) field theory. Hence, we anticipate the super-$\sigma$-model to actually descend to the space of gauge orbits, or to geometrical speaking to the space of orbits of $\lambda^\ast$ (cp Ref. [GSW13, Sec. 9]). In the light of the arguments given in Refs. [GSW10, GSW13], this happens – at least, in the case of free and proper actions – iff $\hat{\rho}^\ast \beta \equiv 0$, and so it is only natural to expect the latter super-1-form to vanish. This physical expectation receives a direct confirmation from the comparison of the base components of the principal $C^*$-connection super-1-forms involved that yields the familiar result (here, $\kappa = P^{(0)}(0)$) 

\[ \left( \lambda^\ast_{\ast} \cdot \text{pr}_{2}^\ast \right) \beta^{(2)}_{\ast}((\kappa, y_{\delta_{0}}), (\theta, x, \phi)) = d\hat{F}((\theta, x, \phi), (-\kappa, -y \delta_{0})) \]

with $\hat{F}$ given in Eq. (6.13). From the above, we immediately read off the data of the isomorphism $\hat{\Upsilon}_{0}^\ast$ sought after in the form

\[ \begin{array}{l}
\hat{\Upsilon}_{0}^\ast : tM\mathcal{M}_{\text{vac}}^{(\mathcal{G},\mathcal{S})} \times_{\lambda^\ast} \mathcal{F}(0) \to tM\mathcal{M}_{\text{vac}}^{(\mathcal{G},\mathcal{S})} \times_{\text{pr}_{2}} \mathcal{F}(0) \\
\end{array} \]

\[ \begin{array}{l}
\text{with } \chi(\kappa, y_{\delta_{0}})(\theta, x, \phi) = \hat{F}((\theta, x, \phi), (-\kappa, -y \delta_{0})) = -2y - \theta \Gamma_{11} \kappa(\phi). \\
\end{array} \]

We have cast the data of the isomorphism in the above form so as to be able to refer directly to the results of the analysis conducted in Sec. 4.3 when answering the question about the coherence of the isomorphism found, as quantified by Eq. (4.31). We obtain, for any two vectors $t_{\alpha} = (\kappa_{\alpha}, y_{\delta_{0}}) \in (\mathcal{G},\mathcal{S}), \alpha \in \{1, 2\},$

\[ \chi_{t_{1} + t_{2}} - \chi_{t_{1}} \chi_{t_{2}})(\theta, x, \phi) = -2(y_{1} + y_{2}) - \theta \Gamma_{11} \kappa_{1} + \chi_{t_{2}}(\phi) - \kappa_{2}(\phi) - 2y_{1} - (\theta - \kappa_{2}(\phi)) \Gamma_{11} \kappa_{1}(\phi) - 2y_{2} - \theta \Gamma_{11} \kappa_{2}(\phi) \]

As we have consistently neglected terms of order 2 in our analysis of the tangential $\kappa$-symmetry, it makes perfect sense to write the above as

\[ \left( \chi_{t_{1} + t_{2}} - \chi_{t_{1}} \chi_{t_{2}} \right)(\theta, x, \phi) = 0 + \mathcal{O} \left( \kappa_{1} \kappa_{2} y_{1} y_{2}^{2-p-q-r} \right) \]

and summarise our examination in

**Theorem 6.7.** The extended Hughes–Polchinski 0-gerbe $\hat{\Phi}_{\text{III}}^{(0)}$ of Def. 6.4 is endowed with a **linearised** $t_{\text{vac}}^{(\mathcal{G},\mathcal{S})}$-equivariant structure relative to $\hat{\rho}^\ast \beta \equiv 0$, as explicit above.
6.3.2. The $\kappa$-equivariant extended HP 1-gerbe. This time, we follow the logic of Sec. 4.2. Thus, we begin our study by looking for a 1-isomorphism
\[ \mathfrak{T}_1^\kappa : \lambda^* s\text{ISO}(d, 1 | D_d) \cong \text{pr}_2^* s\text{ISO}(1) \otimes \mathcal{I}_{\mathcal{F}} \]
between the 1-gerbes over $tM_{\text{vac}}^{(\text{GS}, 1)}$: on the one hand, the pullback 1-gerbe
\[ \lambda^* s\text{ISO}(d, 1 | D_d) = (\lambda^* s\text{ISO}(d, 1 | D_d), \pi_{\lambda^* s\text{ISO}(d, 1 | D_d)}, \lambda^* \beta, \lambda^* \mathcal{F}(1), \lambda^* \mu_{\mathcal{G}(1)}) \]
with the surjective submersion
\[ \lambda^* \mathcal{Y}\text{ISO}(d, 1 | D_d) \rightarrow \mathcal{Y}\text{ISO}(d, 1 | D_d) \]
and, on its total space, the curving
\[ \lambda^* \beta \equiv \text{pr}_2^* \left( \pi_{\lambda^* s\text{ISO}(d, 1 | D_d)} \beta^{(2)} + 2 \pi_{s\text{ISO}(d, 1 | D_d)} \beta^{(2)} \right), \]
as well as the principal $\mathbb{C}^*$-bundle
\[ \lambda^* \mathcal{F}(1) = (\lambda^* \mathcal{Y}\text{ISO}(d, 1 | D_d) \times_{tM_{\text{vac}}^{(\text{GS}, 1)}} \lambda^* \mathcal{Y}\text{ISO}(d, 1 | D_d)) \times \lambda^* \mathcal{F}(1) \]
with the total space given by the fibred product
\[ (\lambda^* \mathcal{Y}\text{ISO}(d, 1 | D_d) \times_{tM_{\text{vac}}^{(\text{GS}, 1)}} \lambda^* \mathcal{Y}\text{ISO}(d, 1 | D_d)) \times \lambda^* \mathcal{F}(1) \]
and, on it, with the principal $\mathbb{C}^*$-connection
\[ \lambda^* \mu_{\mathcal{G}(1)} = \text{pr}_2^* \text{pr}_2^* A_{\mathcal{G}(1)}, \]
alongside the fibrewise groupoid structure $\lambda^* \mu_{\mathcal{G}(1)} \equiv \text{pr}_2^* \text{pr}_2^* A_{\mathcal{G}(1)}$, and, on the other hand, the product 1-gerbe
\[ \text{pr}_2^* \mathcal{G}(1) \otimes \mathcal{I}_{\mathcal{F}} = \left( \text{pr}_2^* s\text{ISO}(d, 1 | D_d), \pi_{\text{pr}_2^* s\text{ISO}(d, 1 | D_d)}, \text{pr}_2^* \beta, \pi_{\text{pr}_2^* s\text{ISO}(d, 1 | D_d)} \beta^{(2)} \right), \]
with the surjective submersion
\[ \text{pr}_2^* s\text{ISO}(d, 1 | D_d) = tM_{\text{vac}}^{(\text{GS}, 1)} \times_{\text{pr}_2^* s\text{ISO}(d, 1 | D_d)} s\text{ISO}(d, 1 | D_d) \]
and the curving
\[ \text{pr}_2^* \beta \equiv \text{pr}_2^* \left( \pi_{\text{pr}_2^* s\text{ISO}(d, 1 | D_d)} \beta^{(2)} + 2 \pi_{s\text{ISO}(d, 1 | D_d)} \beta^{(2)} \right) + \text{pr}_1^* \beta \]
corrected by that of the trivial gerbe $\bar{\beta} \in \Omega^2(tM_{\text{vac}}^{(\text{GS}, 1)})$, to be fixed in a direct computation, and with the principal $\mathbb{C}^*$-bundle
\[ \text{pr}_2^* \mathcal{F}(1) = \left( \text{pr}_2^* s\text{ISO}(d, 1 | D_d) \times_{tM_{\text{vac}}^{(\text{GS}, 1)}} \text{pr}_2^* s\text{ISO}(d, 1 | D_d) \right) \times \text{pr}_2^* \mathcal{F}(1) \]
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with the total space given by the fibred product
$$\left( \text{pr}_2^2 \tilde{\mathcal{Y}} \text{ISO}(d,1|D_{d,1}) \times_{\lambda^* \text{pr}_2} \text{pr}_2^2 \tilde{\mathcal{Y}} \text{ISO}(d,1|D_{d,1}) \right) \times_{\pi^* \text{pr}_2} \mathcal{F}(1)^{(1)}$$
and, on it, with the principal $\mathbb{C}^*$-connection
$$\tilde{\mathcal{F}}^{[2]}_{\eta^*} = \text{pr}_{2}^* \mathcal{A}(1)^{(1)}$$
alongside the fibrewise groupoid structure $\tilde{\mathcal{F}}^{[3]}_{\eta^*} \mu \mathcal{F}(1) \equiv \tilde{\mathcal{F}}^{[3]}_{\eta^*} \mu \mathcal{F}(1)$. Data of the principal $\mathbb{C}^*$-bundle of $\tilde{\mathcal{Y}}_1$ can be extracted from comparison of the relevant pullbacks of the curvings (6.27) and the term $\tilde{\mathcal{F}}^{[2]}_{\eta^*}$ in (8.24) to its base (written in the notation of Sec. 6.2).

$$\nabla_{\lambda^*}^2 \text{ISO}(d,1|D_{d,1}) \equiv \lambda^* \text{pr}_2^2 \tilde{\mathcal{Y}} \text{ISO}(d,1|D_{d,1}) \times_{\lambda^* \text{pr}_2} \text{pr}_2^2 \tilde{\mathcal{Y}} \text{ISO}(d,1|D_{d,1})$$

$$\exists \left( (\kappa = \mathcal{P}^*(\kappa), y = 0^\delta \delta_0 + y^\phi \delta_1), (\theta, x, \phi), (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1) \right) \right)$$

$$\left( (\kappa, \theta, x, \phi), (\theta, x, \phi, \xi^2) \right) \equiv \left( (h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2)) \right).$$

We find

$$\left( \text{pr}_2^2 \tilde{\mathcal{F}}^{[2]}_{\eta^*} = \text{pr}_{1}^* \tilde{\mathcal{F}}^{[2]}_{\eta^*} \right) \left( ((h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2))) \right)$$

$$\equiv \left( \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^2 + \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^1 \right)$$

$$\equiv \left( \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^2 \right) + \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^1$$

$$\equiv \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^2 + \mathcal{F}^{[2]}_{\eta^*} (\theta, x, \phi), \xi^1$$

where $\tilde{\mathcal{F}}^{[2]}_{\eta^*}$ is the super-1-form given in Eq. (6.27). Accordingly, we set

$$\tilde{\mathcal{F}}^{[2]}_{\eta^*} = 0$$

and postulate the principal $\mathbb{C}^*$-bundle of $\tilde{\mathcal{Y}}_1$ to be the trivial one

$$\mathbb{C}^* \to \mathcal{E}^\times := \nabla_{\lambda^*}^2 \text{ISO}(d,1|D_{d,1}) \times \mathbb{C}^*$$

$$\nabla_{\lambda^*}^2 \text{ISO}(d,1|D_{d,1})$$

equipped with the principal $\mathbb{C}^*$-connection super-1-form $(z \in \mathbb{C}^*$ is a point in the fibre)

$$\mathcal{A}^{\times} : \left( ((h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2))), z \right)$$

$$= \frac{d\zeta}{dz} + \tilde{\mathcal{F}}^{[2]}_{\eta^*} ((h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2))).$$

Next, we pass to the fibred product

$$\nabla_{\lambda^*}^2 \text{ISO}(d,1|D_{d,1}) \equiv \nabla_{\lambda^*}^2 \text{ISO}(d,1|D_{d,1}) \times_{\lambda^* \text{pr}_2} \text{pr}_2^2 \tilde{\mathcal{Y}} \text{ISO}(d,1|D_{d,1})$$

$$\exists \left( (h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2))) \right)$$

$$\left( (h, (\theta = \tilde{\kappa}(\phi), x - y + \frac{1}{2} \theta \Gamma \tilde{\kappa}(\phi), \phi, \xi^1)), (h, (\theta, x, \phi, \xi^2))) \equiv (\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3, \tilde{\xi}^4)$$
and compute, over it, the relevant base components of the various pullback principal $C^*$-bundles

\[
(pr_{1,3}, \mathcal{L}^{x\times2}_A \mathcal{G}(\cdot)^{(1)}) + pr_{3,4}E - pr_{1,2}E - pr_{2,4}^m \mathcal{G}(\cdot)^{(1)}(C^i, C^2, \xi^2, \xi^2) = \mathcal{A}(\cdot)^{(1)}((\theta - \kappa(\phi), x - y + \frac{1}{2} \theta \nabla \xi^2, \xi^3, \theta, \xi^4)) \\
+ \mathcal{E}(h, (\theta - \kappa(\phi), x - y + \frac{1}{2} \theta \nabla \xi^2, \xi^3)) = \mathcal{E}(h, (\theta, x, \phi, \xi^4)) \\
- \mathcal{A}(\cdot)^{(1)}((\theta, x, \phi, \xi^4)) = \theta - \kappa(\phi)^{c31} - \mathcal{E}(\theta, x, \phi, -(\kappa, -y)) + \theta^{c33} - \kappa(\phi)^{c33} - \theta^{c32} = 0.
\]

cp Eq. (4.34). From the above calculation, we readily infer triviality of the principal $C^*$-bundle isomorphism of $T_1^{(1)}$

\[
\alpha_E : pr_{1,3}^\mathcal{L}^{x\times2}_A \mathcal{G}(\cdot)^{(1)} \times pr_{3,4}E^\mathcal{G}(\cdot)^{(1)} \times pr_{1,2}E^\mathcal{G}(\cdot)^{(1)} \times pr_{2,4}^m \mathcal{G}(\cdot)^{(1)}
\]

written in terms of the face maps $d^{(2)}_{i}, i \in \{0, 1, 2\}$ of the nerve $N^\mathcal{N}(t^{(1)}_{\text{vac}}^{(1)} \text{stISO}(d, 1, D_{d, d}))$ of the (linearised supervector space-)action groupoid $t^{(1)}_{\text{vac}}^{(1)} \text{stISO}(d, 1, D_{d, d})$ for which we introduce the shorthand notation

\[
t^{(1)}_{\text{vac}}^{(1)}[n] \equiv t^{(1)}_{\text{vac}}^{(1)} \times n \text{stISO}(d, 1, D_{d, d}) \equiv N^\mathcal{N}(t^{(1)}_{\text{vac}}^{(1)} \text{stISO}(d, 1, D_{d, d})), \quad n \in \mathbb{N}.
\]

To this end, we pull back the base component $\mathcal{E}$ of the principal $C^*$-connection $A_E^{(1)}$ of $E^\mathcal{G}(\cdot)^{(1)}$ to the common base (subject to the various identifications indicated in Eq. (4.33))

\[
\mathcal{Y}^{(1)}_{\lambda^{x\times2} \text{stISO}(d, 1, D_{d, d})} = d^{(2)}_{i} \lambda^{x\times2} \mathcal{Y} \text{stISO}(d, 1, D_{d, d}) \times \mathcal{Y}^{(1)}_{\text{vac}^{(1)} \text{stISO}(d, 1, D_{d, d})} \times d^{(2)}_{i} \lambda^{x\times2} \mathcal{Y} \text{stISO}(d, 1, D_{d, d}) \times \mathcal{Y}^{(1)}_{\text{vac}^{(1)} \text{stISO}(d, 1, D_{d, d})}
\]

of the pullback bundles identified by the connection-preserving principal $C^*$-bundle isomorphism of $\pi_1^{(1)}$, and subsequently compute their relevant sign-weighted sum. Denote, similarly as in the proof of Thm. 4.12 given in App. [8]

\[
m_{1,2,3} \equiv ((\kappa_1, y_1), (\kappa_2, y_2), (\theta, x, \phi)), \quad m_{2,3} \equiv ((\kappa_2, y_2), (\theta, x, \phi)),
\]

\[
m_{1,2,3}^{(1)} = ((\kappa_1, y_1), (\theta - \kappa_2)(\phi), x - \nabla \xi^1(\phi)), \quad m_{2,3}^{(1)} \equiv ((\kappa_1 + \kappa_2, y_1 + y_2), (\theta, x, \phi)),
\]

\[
\xi^1 \equiv (\theta - \kappa_1 + \kappa_2(\phi), x - y_1 + y_2(\phi) + \frac{1}{2} \theta \nabla \kappa_1(\phi, \phi, \xi^1)),
\]

\[
\xi^2 \equiv (\theta - \kappa_2(\phi), x - \nabla \xi^2(\phi), \phi, \xi^2), \quad \xi^3 \equiv (\theta, x, \phi, \xi^3),
\]

\[\text{Note the absence of the second tensor component in the left-most 1-isomorphism (cp Eq. (4.33)) that follows from our earlier identification (4.27).}\]
to write, (relatively) compactly,

$$\widetilde{Y}_{\lambda^2}^{\alpha,22}\text{ISO}(d,1\mid D_{d,1}) \ni (\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{2,3},\bar{\xi}^3)\rangle)$$

$$=\bar{\gamma}_{1}^{-1}(\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{2,3},\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{12,3},\bar{\xi}^3)\rangle).$$

We may now calculate, going along the lines of the derivation of identity \((1.24)\),

$$\text{pr}_{1,2}^*\text{pr}_{2}^* \widetilde{E}\langle \langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{2,3},\bar{\xi}^2)\rangle\rangle$$

$$+\text{pr}_{2,3}^*\text{pr}_{2}^* \widetilde{E}\langle \langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{2,3},\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{12,3},\bar{\xi}^3)\rangle\rangle$$

$$-\text{pr}_{1,3}^*\text{pr}_{2}^* \widetilde{E}\langle \langle \bar{m}_{1,2,3},(\bar{m}_{12,3},\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{2,3},\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{12,3},\bar{\xi}^3)\rangle\rangle$$

$$= \widetilde{E}(\langle \bar{m}_{1,\lambda^2,23},\bar{\xi}^1\rangle,\langle \bar{m}_{1,\lambda^2,23},\bar{\xi}^2\rangle) + \widetilde{E}(\langle \bar{m}_{2,3},\bar{\xi}^2\rangle,\langle \bar{m}_{12,3},\bar{\xi}^3\rangle) - \widetilde{E}(\langle \bar{m}_{12,3},\bar{\xi}^1\rangle,\langle \bar{m}_{12,3},\bar{\xi}^3\rangle)$$

$$= -\widetilde{E}(\langle \theta - \kappa_2(\phi),x - y_2(\phi) + \frac{1}{\theta} \theta \Gamma \kappa_2(\phi),(-\kappa_1,-y_1)\rangle + \langle \theta - \kappa_2(\phi)\rangle^a d\xi^{21}_a + \kappa_1(\phi)^a d\xi^1_a$$

$$-\widetilde{E}(\langle \theta,x,\phi,(-\kappa_2,-y_2)\rangle + \theta^a d\xi^{32}_a + \kappa_2(\phi)^a d\xi^2_a$$

$$+\widetilde{E}(\langle \theta,x,\phi,(-\kappa_1,-\kappa_2,-y_1-y_2)\rangle - \theta^a d\xi^{31}_a - (\kappa_1 + \kappa_2)(\phi)^a d\xi^3_a)$$

$$= -\widetilde{E}(\langle \theta,x,\phi,(-\kappa_1,-y_1)\rangle - \widetilde{E}(\langle \theta,x,\phi,(-\kappa_2,-y_2)\rangle + \widetilde{E}(\langle \theta,x,\phi,(-\kappa_1 - \kappa_2,-y_1-y_2)\rangle$$

$$+\Theta(\kappa_1^m \kappa_2^n y_1, y_2) = 0 + \Theta(\kappa_1^m \kappa_2^n y_1, y_2)$$

whereupon we conclude that \(\bar{\gamma}_{1}^{\kappa}\) can be – up to corrections of order 2 in the \(\kappa\)-translations – taken in the trivial form

$$\bar{\gamma}_{1}^{\kappa} : \text{pr}_{1,2}^* \langle \widetilde{Y}_{\lambda^2}^{22}\text{ISO}(d,1\mid D_{d,1}) \times_{\text{pr}_{2}^*} E^c \rangle \otimes \text{pr}_{2,3}^* \langle \widetilde{Y}_{\lambda^2}^{22}\text{ISO}(d,1\mid D_{d,1}) \times_{\text{pr}_{2}^*} E^c \rangle \longrightarrow$$

$$\text{pr}_{1,3}^* \langle \widetilde{Y}_{\lambda^2}^{22}\text{ISO}(d,1\mid D_{d,1}) \times_{\text{pr}_{2}^*} E^c \rangle$$

$$= (\langle \bar{m}_{1,2,3},(\langle \bar{m}_{1,\lambda^2,23},(\bar{\xi}^1)\rangle,\langle \bar{m}_{1,\lambda^2,23},(\bar{\xi}^2)\rangle,\langle \bar{m}_{1,\lambda^2,23},(\bar{\xi}^3)\rangle\rangle,\bar{z}_1) \otimes (\langle \bar{m}_{1,2,3},(\langle \bar{m}_{2,3},(\bar{\xi}^2)\rangle,\langle \bar{m}_{12,3},(\bar{\xi}^3)\rangle)\rangle,\bar{z}_1 \cdot \bar{z}_2),$$

written for

$$\bar{m}_{1,2,3} \equiv (\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},(\bar{\xi}^1)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{1,\lambda^2,23},(\bar{\xi}^2)\rangle,\langle \bar{m}_{1,2,3},(\bar{m}_{12,3},(\bar{\xi}^3)\rangle)\rangle).$$

The latter is manifestly coherent. Our findings are neatly encapsulated in

**Theorem 6.8.** The extended Hughes–Polchinski 1-gerbe \(\widetilde{G}_{\text{HP}}^{(1)}\) of Def. 2.3, is endowed with a **linearised** \((\text{GS},\ell)\)-equivariant structure relative to \(\bar{\gamma}^{\kappa} = 0\), as explicit above.

Of course, a full understanding of the structural observations reported in this closing section of the paper, which are hoped to have shed some light on the gerbe-theoretic aspect of the \(\kappa\)-symmetry of the Green–Schwarz super-\(\sigma\)-model, would require a thorough examination of the global supersymmetry in the presence of differential constraints imposed with view to verifying compatibility of the linearised \((\text{GS},\ell)\)-equivariant structure with it. This we leave to a future work.
7. Conclusions & Outlook

In the present paper, we have studied at considerable length the issue of (super)symmetry in the context of the geometrisation scheme, based on the notion of the Cartan–Eilenberg super-\(p\)-gerbe and exemplified amply in Part I, of the supersymmetry-invariant refinement of the de Rham cohomology of the Lie supergroup in which that issue takes on the form of a consistent lift of the geometric action of the (super)symmetry group from the base of the geometrisation to its total space endowed with extra connective structure (curvings, connections, isomorphisms). Our discussion, conducted from the vantage point of higher (super)geometry and employing the formal tools as well as insights developed in former gerbe-theoretic treatments of symmetry, in its various guises, in the purely Graßmann-even setting in, i.a., Refs. [GSW10, Sus11, GSW13, Sus12], is founded upon the concept of equivariance with respect to an action \(\lambda\) of a Lie sub-supergroup \(H \subset G\) of the supersymmetry group \(G\) that has been transplanted into the supergeometric setting in a manner compatible with the (global) supersymmetry present in it in the form of a family of super-\(p\)-gerbe isomorphisms indexed by \(G\). As a result, we have come up with the novel notion of a supersymmetric \(H\)-equivariant structure on the super-\(p\)-gerbe (explicit for \(p \in \{0, 1\}\), but amenable to obvious generalisations) elaborated in Secs. 4.1 and 4.2 and encapsulated in Defs. 4.6 and 4.10 for the important special case of \((H, \lambda) = (G, \text{Ad})\). The abstract notion has been illustrated and, through that, justified \textit{ex post} by the examples of supersymmetric Ad-\(p\)-equivariant structures, identified in Thms. 4.8 and 4.12 on the Green–Schwarz super-\(p\)-gerbes on the super-Minkowski space \(s\text{Mink}^{\text{p,1}}|_{\text{D}_{\text{f,1}}}\) constructed in Part I. Their existence has been shown to conform with the intuition that follows from the symmetry analysis of the bosonic counterparts of the associated WZW-(type) super-\(\sigma\)-models reviewed in Secs. 3.1 and 3.2. Another incarnation of supersymmetry examined in the present paper as giving rise to an equivariant structure on the geometrisation of a de Rham (super-)cocycle is the \textit{gauge} (i.e., local) right tangential supersymmetry of the Green–Schwarz super-\(\sigma\)-model discovered in Refs. [IALS83, Sie83, Sie84] and known under the name of \(\kappa\)-symmetry. Its purely geometric analysis, inspired and organised largely by the approach of Refs. [McA00, Wes00, GKW06a, GKW06b, McA10], assumes its point of departure the correspondence between the Nambu–Goto and the Hughes–Polchinski formulations of the Green–Schwarz super-\(\sigma\)-model on a homogeneous space of a Lie supergroup, originally proposed in Ref. [HP86]. Here, we have rigorously pinned down the circumstances, stated in Thms. 7.1 and 7.2, under which the correspondence occurs in a large class of supergeometries. These have been exemplified, through Prop. 7.3, by the super-Minkowskian background of immediate interest. Upon putting the correspondence thus elucidated in the context of (super-)gerbe theory, and in conjunction with the concrete constructions of Part I as well as with the discussion of equivariant structures on (super-)gerbes from previous sections of the present paper, we have been led to the construction, laid out in Defs. 6.4 and 1.3 of novel supergeometric objects dubbed extended Hughes–Polchinski \(p\)-gerbes. These are readily seen to unify, in a natural and tractable fashion, the formal description of the metric and topological degrees of freedom of the (super-)\(\sigma\)-model, and, in a direct consequence thereof, afford a particularly neat identification of a Lie-superalgebraic structure of the gauge symmetry under consideration given by the (translational) and, in a direct consequence thereof, afford a particularly neat identification of a Lie-superalgebraic fashion, the formal description of the metric and topological degrees of freedom of the (super-)model.

Of the constructions of Part I. Another natural idea is the corroboration of further bosonic intuitions regarding the Green–Schwarz super-\(p\)-gerbes of Part I, such as, \textit{e.g.}, the existence and concrete realisation of a multiplicative structure, suggested by the findings of Refs. [JM+05, Val10, GW09]. On the more field-theoretic note, we remark that our construction of the supersymmetry-(\(\text{Ad}\text{-})equivariant...
structure on the super-$p$-gerbe begs for a logical conclusion in the form of a hands-on construction of Green–Schwarz super-$\sigma$-models with the supersymmetry group $\mathfrak{sMink}^{d,1|D_{d,1}}$ in its adjoint realisation gauged along the lines of Refs. [GSWI13, Sus12]. With such a maximal choice of the global-symmetry group to be gauged, one should expect, on the basis of the bosonic experience gathered in Refs. [Gaw99, GTTNB04, Gaw02], the emergence of a topological field theory of the (super-)Chern–Simons type, an interesting object of prospective study in its own right, and of a well-established gerbe-theoretic nature.

Unification of the metric and topological degrees of freedom and the ensuing simple higher-geometric picture of $\kappa$-symmetry obtained in our analysis with the help of the correspondence between the two formulations of the Green–Schwarz super-$\sigma$-model opens a number of separate avenues of further study. Thus, we are confronted with the question of compatibility of the extended gerbes and their $t^{(\mathbb{G}_S,p)}_{vac}$-equivariant structure with the (global) supersymmetry quantified in the setting in hand by the Lie supergroup $\mathfrak{sisO}(d,1|D_{d,1})$ and broken spontaneously by the vacuum of the Hughes–Polchinski super-$\sigma$-model. It would also be desirable to establish a relation of the Lie-superalgebraic description of $\kappa$-symmetry derived in the present paper to the alternative approach to geometrisation of Green–Schwarz super-$(p+2)$-cocycles through Lie $(p−1)$-superalgebras (and $L_\infty$-superalgebras) and the corresponding Lie $(p−1)$-supergroups, rooted in the works [BC04, BHI1, Hue11] of Baez et al. and advocated by Schreiber et al. in Ref. [FSS14]. Furthermore, it is natural to enquire as to the applicability of the correspondence between the two formulations of the Green–Schwarz super-$\sigma$-model and its ramifications in other physically motivated supergeometric setting, such as, e.g., that of the super-$\sigma$-models on supertargets with the body of the general type $\text{AdS}_{p+2} \times S^{d−p−2}$ – partial results in this direction have been reported in Ref. [Sus18b]. Finally, and independently (also from the supergeometric context), one is tempted to exploit the correspondence in a study and, in particular, a potential gerbe-theoretic geometrisation of non-geometric dualities of non-linear $\sigma$-models, such as, e.g., the essentially field-theoretic $T$-duality of the loop dynamics determined by the two-dimensional $\sigma$-model with a toroidally fibred target space. We shall certainly return to these ideas in a future work.
APPENDIX A. A PROOF OF THEOREM 2.3

First, we examine the requirement of gauge invariance imposed upon the extended $p$-holonomy

$$\text{Hol}_{\varpi(0)}[\overline{x}] = \text{Hol}_{\varpi(0)}[x] \cdot \exp\left( i \int_{\Omega_p} \overline{x}^* \left( \frac{\partial}{\partial x^i} \right) \right)$$

and in this way fix the explicit form of the $\Omega^*(M)$-valued tensors $(\alpha_{A_1:A_2:...:A_k})_{A_1,A_2,...,A_k \in T_1 \dim G_x}$. Thus, we impose the condition

$$\frac{d}{dt} I = 0 \text{,}$$

from which we obtain, with the help of Eq. (2.1) (and for $\partial \Omega_p = \emptyset$), the identity

$$0 = \int_{\Omega_p} [X_A^*] \overline{x}^* [\mathcal{K}_A] = \left( \text{pr}_2^* \mathcal{H}_{(p+1)} + (-1)^p \frac{1}{p!} \mathcal{K}_A \right) \mathcal{H}_{(p+1)} + \mathcal{H}_{(p+1)} \mathcal{H}_{(p+1)}$$

$$= \int_{\Omega_p} [X_A^*] \overline{x}^* [\mathcal{K}_A] = \left( \text{pr}_2^* \mathcal{H}_{(p+1)} + (-1)^p \frac{1}{p!} \mathcal{K}_A \right) \mathcal{H}_{(p+1)} + \mathcal{H}_{(p+1)} \mathcal{H}_{(p+1)}$$

in which we have used the shorthand notation

$$\mathcal{A}^{A_1:A_2:...:A_k} := \mathcal{A}^{A_1} \wedge \mathcal{A}^{A_2} \wedge \cdots \wedge \mathcal{A}^{A_k}.$$

The arbitrariness of the map $x$ and that of the gauge field $A$ infers that we should independently impose the constraints

$$\mathcal{K}_A \wedge \mathcal{H}_{(p+1)} = - \mathcal{K}_A,$$

and demand that the expressions in the round brackets in the second and third lines of the above formula (multiplying the $\mathcal{A}^{A_1:A_2:...:A_k}$) are nullified for each $k \in T_1 \cdot \mathcal{P}$. The first of the constraints, (A.1),
implies that the pairs \((\mathcal{K}_A, \kappa_A)\) are the basis generalised hamiltonian sections of \(\mathcal{E}^{(1,p)}\) of Eq. (2.13). The other one(s), (A.2), admit the unique solution
\[
\alpha_{A1A2…Ak+1} = (-1)^{(k-1)(2p-k)}\mathcal{K}_{A1} \cup \mathcal{K}_{A2} \cup \cdots \cup \mathcal{K}_{Ak} \cup \mathcal{K}_{Ak+1}, \quad k \in \mathbb{I}, p,
\]
subject to the symmetry constraints
\[
\mathcal{K}_A \cup \kappa_B + \mathcal{K}_B \cup \kappa_A = 2(-1)^{p-1} \alpha_{(AB)} \equiv 0.
\]

Upon substituting Eq. (A.2) into the remaining equations, we obtain, after antisymmetrisation (of the last term), the equivariance constraints
\[
0 = \mathcal{L}_{\mathcal{K}_A} \alpha_{A1A2…Ak} + (-1)^{p-k} \mathcal{L}_{\mathcal{K}_A} \alpha_{A1A2…Ak} - \sum_{l=1}^{k} (-1)^{l-1} f_{AA1}^B \alpha_{BA1A2…Ak}\
\]
\[
\mathcal{L}_{\mathcal{K}_A} \alpha_{A1A2…Ak} - \sum_{l=1}^{k} (-1)^{l-1} f_{AA1}^B \alpha_{BA1A2…Ak} = 0,
\]
to be imposed for any \(k \in \mathbb{I}, p+1\). We readily conclude that in consequence of Eq. (A.3) only the first of these, with \(k = 1\), gives a new condition, to wit,
\[
\mathcal{L}_{\mathcal{K}_A} \kappa_B - f_{AB}^C \kappa_C = 0,
\]
whereas the remaining ones immediately follow from this condition once we take into account the explicit form (A.3) of the (higher) \(S^*(M)\)-valued coefficients and the symmetry constraints (A.4).

Thus, altogether, we are left with the independent constraints (A.1), (A.3), (A.4) and (A.3).

Next, we pass to examine the conditions under which the \(C^\infty(M)\)-linear span of the basis sections
\[
\mathfrak{R}_A = (\mathcal{K}_A, \kappa_A) \quad \text{forms a Lie algebroid with the bracket} \quad [\cdot, \cdot]_{V}^{(p+2)}.
\]
We readily obtain the identities
\[
[\mathfrak{R}_A, \mathfrak{R}_B]_{V}^{H} = f_{AB}^C \mathfrak{R}_C + (0, \mathcal{L}_{\mathcal{K}_A} \kappa_B - f_{AB}^C \kappa_C - \mathcal{L}_{\mathcal{K}_A} \kappa_B)_{V}
\]
and – for any \(f \in C^\infty(M)\) –
\[
[\mathfrak{R}_A, f \mathfrak{R}_B]_{V}^{H} - f [\mathfrak{R}_A, \mathfrak{R}_B]_{V}^{H} - (\mathcal{K}_A \cup df) \mathfrak{R}_B = -df \wedge \mathcal{K}_A \cup \kappa_B,
\]
and so the requirements of the closure of \([\cdot, \cdot]_{V}^{(p+2)}\) on \(\mathcal{S}_\sigma^{(p)}\) and of the vanishing of the Leibniz anomaly boil down to (A.4) and (A.3). Upon imposition of these constraints, the bracket of the basis sections reads
\[
[\mathfrak{R}_A, \mathfrak{R}_B]_{V}^{H} = f_{AB}^C \mathfrak{R}_C,
\]

whence also the triviality of the Jacobi anomaly. The last statement of the thesis of the theorem now follows from Ref. [38,Prop. 8.24].

\section*{Appendix B. A proof of Proposition \ref{prop:generators}}

Using the elementary identities
\[ \mathcal{R}(\varepsilon,y) \ni p r_i^* \sigma^\alpha(\theta,x) = \varepsilon^\alpha, \quad \mathcal{R}(\varepsilon,y) \ni e^a(\theta,x) = y^a - \varepsilon \Gamma^a \theta \]
and invoking the proof of Prop. 4.2 from Part I (given ib.), we compute
\[ \mathcal{R}(\varepsilon,y) \ni H(\theta,x) = \frac{p}{(p+2)} \left( y^{a_1} - \varepsilon \Gamma^{a_1} \theta \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \wedge e^{a_2a_3...a_p}(\theta,x) \]
\[ \quad + 2 \varepsilon \Gamma_{a_1a_2...a_p} \sigma(\theta) \wedge e^{a_1a_2...a_p}(\theta,x) \]
\[ \equiv \frac{p}{(p+2)} \left( \varepsilon \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \wedge e^{a_2a_3...a_p}(\theta,x), \]

\begin{equation}
\beta_{a_1}(\theta,x) = \frac{1}{p} \sum_{k=1}^{p} \theta \Gamma_{a_1a_2...a_p} \sigma(\theta) \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_k} \wedge e^{a_{k+1}a_2...a_p}(\theta,x),
\end{equation}

and we may subsequently use the symmetry properties of the objects involved in conjunction with the Fierz identity \ref{fierz} to rewrite the sum in the square brackets in the last term as
\[ \mathcal{C}_n(\varepsilon)(a_2a_3...a_p)(\theta) := \left( \varepsilon \Gamma_{a_1a_2...a_p} \theta \right) \left( \sigma \wedge \Gamma^{a_1} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \]
\[ \equiv \left( \Gamma_{a_1a_2...a_p} \alpha_{\beta} \Gamma^{a_{\beta}} + \Gamma_{a_2a_3...a_p} \gamma_{\beta} \Gamma^{a_{\beta}} \right) \varepsilon^\alpha \theta^\beta \left( \sigma \wedge \sigma(\theta) \right) \]
\[ = \left( \Gamma_{a_1a_2...a_p} \alpha_{\beta} \Gamma^{a_{\beta}} \right) \left( \sigma \wedge \Gamma^{a_1} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \]
\[ = 2 \left( \varepsilon \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \left( \sigma \wedge \Gamma^{a_1} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \]
\[ = 2 \left( \varepsilon \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \left( \sigma \wedge \Gamma^{a_1} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right) \]
so that
\[ \mathcal{C}_n(\varepsilon)(a_2a_3...a_p)(\theta,x) = \frac{2}{3} \mathcal{D} \mathcal{C}_n(\varepsilon)(a_2a_3...a_p)(\theta,x), \]

Write
\begin{equation}
\eta_{a_2a_3...a_p}(\theta,x) := \left( \varepsilon \Gamma_{a_1a_2...a_p} \theta \right) \left( \sigma \wedge \Gamma^{a_1} \sigma(\theta) \right) \left( \sigma \wedge \Gamma_{a_1a_2...a_p} \sigma(\theta) \right)
\end{equation}

and note the identity
\[ \left( \sigma \wedge \Gamma^{a_2} \sigma \right)(\theta) \wedge \eta_{a_2a_3...a_p}(\theta,x) = \left( \sigma \wedge \Gamma^{a_2} \sigma \right)(\theta) \wedge \mathcal{C}_n(\varepsilon)(a_2a_3...a_p)(\theta,x), \]

following directly from Eq. \ref{fierz} rewritten in the useful form
\[ \left( \Gamma_{a_1a_2...a_p} \alpha_{\beta} \Gamma^{a_{\beta}} \right) \gamma_{\delta} = - \Gamma_{a_1a_2...a_p} \gamma_{\delta}. \]

We now obtain
\[ \mathcal{C}_n(\varepsilon)(a_2a_3...a_p) \wedge e^{a_2a_3...a_p}(\theta,x) = \frac{2}{3} \mathcal{D} \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \]
\[ = \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) + \frac{2}{3} \left( \sigma \wedge \Gamma^{a_2} \sigma \right)(\theta) \wedge \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) \]
\[ = \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) - \frac{2}{3} \left( \sigma \wedge \Gamma^{a_2} \sigma \right)(\theta) \wedge \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) \]
\[ \equiv \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) + \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right) \]
\[ = \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right), \]
and therefore
\[ \mathcal{C}_n(\varepsilon)(a_2a_3...a_p)(\theta,x) = \frac{2}{3} \mathcal{D} \left( \eta_{a_2a_3...a_p} \wedge e^{a_2a_3...a_p}(\theta,x) \right). \]
Continuing the reduction as in the previous section, we establish
\[ dx^a_2 \wedge \omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x) = \frac{2}{p} dx^a_2 \wedge \frac{2}{p-1} dx^a_2 \wedge (\omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x)) \]
whence also
\[ dx^a_2 \wedge \omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x) = -\frac{2}{p-1} dx^a_2 \wedge (\omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x)) \]
which yields
\[ \mathcal{R}(\varepsilon, y) \cup H_{(p+2)}(\theta, x) = d[p_\theta^a \beta_a(\theta, x) + 2(\varepsilon \Gamma_{a_1 a_2 \cdots a_p} \theta) e^{a_1 a_2 \cdots a_p} (\theta, x)] \]
\[ -\frac{2p}{2p+1} \left( \omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x) \right) - \frac{2p}{2p-1} \left( \omega^\varepsilon_{a_2 a_3 \cdots a_p} \wedge dx^{a_2} \wedge e^{a_3 a_4 \cdots a_p} (\theta, x) \right) \]
and so, after \( p \) steps, we arrive at the equality
\[ \mathcal{R}(\varepsilon, y) \cup H_{(p+2)} = -\delta(k^R)_{(p)}(\varepsilon, y) \]
with
\[ k^R(\varepsilon, y)(\theta, x) = -p_\theta^a \beta_a(\theta, x) - 2(\varepsilon \Gamma_{a_1 a_2 \cdots a_p} \theta) e^{a_1 a_2 \cdots a_p} (\theta, x) \]
\[ + \frac{p^2}{2p+1} \sum_{k=1}^{p} \frac{2^k(2p+1-k)!}{(p-k)!} \omega^\varepsilon_{a_2 a_3 \cdots a_p} (\theta, x) \wedge dx^{a_2} \wedge dx^{a_3} \wedge \cdots \wedge dx^{a_k} \wedge e^{a_{k+1} a_{k+2} \cdots a_p} (\theta, x) \]

**Appendix C. A proof of Proposition 4.3**

We readily compute, upon invoking the explicit form of the super \( p \)-forms \( k^R(\varepsilon, y) \) derived in App. 3.
\[ \mathcal{R}(\varepsilon_1, 0) \cup k^R(\varepsilon_2, 0)(\theta, x) = 2p(\varepsilon_2 \Gamma_{a_1 a_2 \cdots a_p} \theta) (\varepsilon_1 \Gamma_{a_1} \theta) e^{a_2 a_3 \cdots a_p} \]
\[ -\frac{p^2}{2p+1} \sum_{k=1}^{p} \frac{2^k(2p+1-k)!}{(p-k)!} \left( (\varepsilon_2 \Gamma_{a_1 a_2 \cdots a_p} \theta) + (\varepsilon_1 \Gamma_{a_1} \theta) \right) \Gamma_{a_2 a_3 \cdots a_k} \]
\[ -k(\varepsilon_1 \Gamma_{a_2} \theta) \omega^\varepsilon_{a_2 a_3 \cdots a_p} (\theta, x) \wedge dx^{a_2} \wedge e^{a_3 a_4 \cdots a_k} \]
\[ +(-1)^k(\varepsilon_1 \Gamma_{a_2} \theta) \omega^\varepsilon_{a_2 a_3 \cdots a_p} (\theta, x) \wedge dx^{a_2 a_3} \wedge e^{a_4 a_5 \cdots a_k} \]
where we have used the shorthand notation
\[ dx^{a_2 a_3 \cdots a_k} \equiv dx^{a_2} \wedge dx^{a_3} \wedge \cdots \wedge dx^{a_k} \]
Expressing the 1-forms \( dx^a \) as functional combinations of the left-invariant ones,
\[ dx^a = e^a(\theta, x) - \frac{1}{2} \theta \Gamma^a \sigma(\theta) \]
we thus arrive at the formula
\[ P_{a_{p-1}} \cup P_{a_{p-2}} \cup \cdots \cup P_{a_1} \cup \left( \mathcal{R}(\varepsilon_1, 0) \cup k^R(\varepsilon_2, 0) \right) = (p-1)! \left[ 2p(\varepsilon_2 \Gamma_{a_1 a_2 \cdots a_p} \theta) (\varepsilon_1 \Gamma_{a_1} \theta) \right] \]
\[-\frac{pl}{(2p+1)!} \sum_{k=1}^{p} \frac{2^k (2p+1-2k)!}{(p-k)!} \left( (\varepsilon_1 \Gamma_{a_1} \theta) (\varepsilon_2 \Gamma_{a_1 a_2 \ldots a_p} \theta) + (\varepsilon_2 \Gamma_{a_1} \theta) (\varepsilon_1 \Gamma_{a_1 a_2 \ldots a_p} \theta) \right) \]

whose symmetrisation in the pair \((\varepsilon_1, \varepsilon_2)\) yields

\[
P_{a_{p-1}} P_{a_{p-2}} \ldots P_{a_1} \left( (\mathcal{R}(\varepsilon_1,0) \mathcal{R}_{(p)}^R(\varepsilon_2,0) + \mathcal{R}(\varepsilon_1,0) \mathcal{R}_{(p)}^R(\varepsilon_2,0)) \right) = 2(p-1)! (p-C_p) \left( (\varepsilon_1 \Gamma^{a_1} \theta) (\varepsilon_2 \Gamma_{a_1 a_2 \ldots a_p} \theta) + (\varepsilon_2 \Gamma^{a_1} \theta) (\varepsilon_1 \Gamma_{a_1 a_2 \ldots a_p} \theta) \right),
\]

where

\[
C_p = -\frac{pl}{(2p+1)!} \sum_{k=1}^{p} \frac{2^k (2p+1-2k)!}{(p-k)!} = \frac{2p}{3},
\]

whence the desired result. \(\square\)

**APPENDIX D. A PROOF OF THEOREM 4.8**

First of all, we should verify the existence of a lift

\[
\mathrm{YAd}_{(\varepsilon, y)} : \text{sMink}^{9,1|32} \times \mathbb{C}^\ast \circlearrowleft \ : (\theta, x) \mapsto (\text{Ad}_{(\varepsilon, y)}(\theta, x), e^{i\mu_{(\varepsilon, y)}(\theta, x)} \cdot z)
\]

of the adjoint action of \text{sMink}^{9,1|32} on itself to the extension \text{sMink}^{9,1|32} \times \mathbb{C}^\ast that satisfies the identities

\[
\left( \text{Ad}^{*}_{(\varepsilon, y)} - \text{id}^{*}_{\text{sMink}^{9,1|32}} \right) B = d\mu_{(\varepsilon, y)}
\]

for \(B\) as in Eq. (1.6). Using Eq. (1.18), we obtain

\[
d\mu_{(\varepsilon, y)}(\theta, x) = 0,
\]

and so we may take

\[
\mu_{(\varepsilon, y)}(\theta, x) \equiv 0.
\]

Clearly,

\[
\forall (\varepsilon_1, y_1), (\varepsilon_2, y_2) \in \text{sMink}^{9,1|32} : \mathrm{YAd}_{(\varepsilon_1, y_1)} \circ \mathrm{YAd}_{(\varepsilon_2, y_2)} = \mathrm{YAd}_{(\varepsilon_1, y_1)}(\varepsilon_2, y_2).
\]

We conclude that there exists an adjoint realisation of the supersymmetry group \text{sMink}^{9,1|32} on the Green–Schwarz super-0-gerbe \(\mathcal{G}_{\text{GS}}^{(0)}\).

Next, we confirm the identification

\[
\theta_{(\varepsilon, y)} \equiv 0
\]

derived previously in Eq. (4.18) through a direct computation

\[
\left( \text{Ad}^{*} - \text{pr}_2^{*} \right) H \left( \theta_1, x_1 \right) \left( \theta_2, x_2 \right) = H \left( \theta_2, x_2 - \theta_1 \Gamma \theta_2 \right) - H \left( \theta_2, x_2 \right) = 0,
\]

and, accordingly, look for a principal \(\text{C}^\ast\)-bundle isomorphism

\[
\Upsilon_{0} : \left( \text{sMink}^{9,1|32} \times_{\text{Ad}} \left( \text{sMink}^{9,1|32} \times \mathbb{C}^\ast \right) \right) \left( \text{sMink}^{9,1|32} \times_{\text{pr}_2} \left( \text{sMink}^{9,1|32} \times \mathbb{C}^\ast \right) \right)
\]

subject to the constraint

\[
d\chi \left( \left( \theta_1, x_1 \right), \left( \theta_2, x_2 \right) \right) = \left( \text{Ad}^{*} - \text{pr}_2^{*} \right) B \left( \left( \theta_1, x_1 \right), \left( \theta_2, x_2 \right) \right) = B \left( \theta_2, x_2 - \theta_1 \Gamma \theta_2 \right) - B \left( \theta_2, x_2 \right) = 0,
\]

from which we read off the admissible choice

\[
\chi \left( \left( \theta_1, x_1 \right), \left( \theta_2, x_2 \right) \right) \equiv 0.
\]

The latter clearly satisfies the coherence condition

\[
\chi m_1 \left( \left( \theta_1, x_1 \right), \left( \theta_2, x_2 \right) \right) = \left( \text{Ad}^{*}_{(\theta_2, x_2)} \right) \chi \left( \theta_1, x_1 \right) + \chi \left( \theta_2, x_2 \right),
\]

and so the very last property of the isomorphism \(\Upsilon_{0}\) proposed above is its supersymmetry-equivariance. In view of the triviality of both: the \(\mu_{(\varepsilon, y)}\) and the \(\chi(\theta, x)\), the relevant identity (4.32) is satisfied automatically, and so the proof is complete.
Appendix E. A proof of Theorem 4.12

We begin with a derivation of a lift
\[ YAd : s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times (s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1}) \longrightarrow s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1} \]

of the adjoint action of \( s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \) on itself to the total space of the surjective submersion \( s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1} \equiv \mathcal{Y}_{1}s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \) from the invariance condition \( (4.11) \) which for \( \mathcal{B} \equiv \beta(2) \) as in Eq. (4.3) can be rewritten as

\[ 0 = d\theta^\alpha \wedge \left( d(\tilde{\xi}_\alpha - \xi_\alpha) + (\nabla_\alpha \theta^\beta)^{\Gamma} d\theta^\beta \right) = d(\tilde{\xi}_\alpha - \xi_\alpha - \frac{1}{3} (\varepsilon \nabla_\alpha \theta) \nabla_\alpha \theta^\beta \wedge d\theta^\alpha \]

upon invoking the relevant Fierz identity \( (4.2) \) (with \( p = 1 \)). Accordingly, we postulate

\[ YAd : s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times (s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1}) \longrightarrow s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1} \]

\[ \left( \varepsilon, (\xi, \alpha, \beta) \right) \rightarrow \left( (\alpha, \beta), \xi \right) \]

and readily check that it is an action,

\[ \mathcal{Y}_{(\varepsilon, y_1), (\varepsilon, y_2)} : YAd_{(\varepsilon, y_1)} \circ YAd_{(\varepsilon, y_2)} = YAd_{(\varepsilon, y_1)(\varepsilon, y_2)} \] .

Passing next, to the base of the principal \( C^* \)-bundle \( L \equiv \mathcal{L}^{(1)} \) of the super-1-gerbe,

\[ Y[2]s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \equiv (s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1}) \times s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \]

\[ \equiv \left( (\theta, x, \xi_1), (\theta, x, \xi_2) \right) \]

and considering the base component \( A_L \equiv A_{\mathcal{L}^{(1)}} \) of the principal \( C^* \)-connection on \( \mathcal{L}^{(1)} \equiv Y[2]s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times C^* \), given in Eq. (4.11), we obtain the identity

\[ \left( Y[2]Ad_{(\varepsilon, y)}, YAd_{(\varepsilon, y)} \right) A_L \left( (\theta, x, \xi_1), (\theta, x, \xi_2) \right) = 0 \]

that justifies setting

\[ LAd : s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathcal{L}^{(1)} \longrightarrow \mathcal{L}^{(1)} \]

\[ \left( \varepsilon, (\theta, x, \xi_1), (\theta, x, \xi_2), z \right) \rightarrow \left( YAd_{(\varepsilon, y)}(\theta, x, \xi_1), YAd_{(\varepsilon, y)}(\theta, x, \xi_2), z \right) . \]

Once more, we check homomorphicity of the lift in the first argument as well as commutativity with the defining action of the structure group \( C^* \) on \( \mathcal{L}^{(1)} \). The groupoid structure \( \mu_{\mathcal{L}^{(1)}} \) of Eq. (4.11) trivially intertwines the actions induced from \( YAd \) on its domain \( (L_{1,2,3}Ad) \) and codomain \( (L_{1,3}Ad) \), respectively. We conclude that we have a lift of the adjoint action \( Ad \) of the supersymmetry group to the total space \( Ys\text{Mink}^{d,1}|\mathcal{D}_{d,1} \equiv s\text{Mink}^{d,1}|\mathcal{D}_{d,1} \times \mathbb{R}^0|\mathcal{D}_{d,1} \) of the surjective submersion of the Green–Schwarz super-1-gerbe \( G_{\text{GS}}^{(1)} \).

In the next step, we (re)derive the expression for the super-2-form \( \theta \cdot \bar{\theta} \) stated in the thesis of the theorem, and predicted by the analysis from the beginning of Sec. 4.2 in a direct calculation. Thus, we have

\[ \left( Ad^* - pr_2^* \right) H \left( (\theta_1, x_1), (\theta_2, x_2) \right) = -d\theta_2 \wedge \nabla_\alpha d\theta_2 \wedge d(\theta_1 \nabla^\alpha \theta_2) = d(-\theta_1 \nabla^\alpha \theta_2) d\theta_2 \wedge \nabla_\alpha d\theta_2 \]

but – upon employing \( (4.2) \) –

\[ -\left( \theta_1 \nabla^\alpha \theta_2 \right) d\theta_2 \wedge \nabla_\alpha d\theta_2 = -2 \theta_1 \nabla_\alpha d\theta_2 \wedge \theta_2 \nabla^\alpha d\theta_2 \]

\[ = d(-2 \theta_1 \nabla_\alpha \theta_2 \theta_2 \nabla^\alpha d\theta_2 - 2 \theta_2 \nabla_\alpha d\theta_1 \wedge \theta_2 \nabla^\alpha d\theta_2 + 2 \theta_1 \nabla_\alpha \theta_2 d\theta_2 \nabla^\alpha d\theta_2) , \]

whence also

\[ \left( \theta_1 \nabla^\alpha \theta_2 \right) d\theta_2 \wedge \nabla_\alpha d\theta_2 = -\frac{2}{3} \theta_2 \nabla_\alpha d\theta_1 \wedge \theta_2 \nabla^\alpha d\theta_2 + d\left( \frac{2}{3} \theta_1 \nabla^\alpha \theta_2 \theta_2 \nabla_\alpha d\theta_2 \right) , \]

and so

\[ \left( Ad^* - pr_2^* \right) H \left( (\theta_1, x_1), (\theta_2, x_2) \right) = d(-\frac{2}{3} \theta_2 \nabla_\alpha d\theta_1 \wedge \theta_2 \nabla^\alpha d\theta_2) . \]
as claimed. We may now proceed with the construction of the Ad-equivariant structure.

From comparison of pullbacks of the curvings \( \tilde{\text{Ad}}^\ast \beta \) and \( \tilde{\text{pr}}_2^\ast \beta \) with \( \tilde{\text{pr}}_2^\ast \beta \), we have the isomorphism in the form

\[
\begin{align*}
Y_{\text{AdS}^3 \text{Mink}^4,1|\mathcal{D}_{4,1}} \cong (\{(0,1), (0,2), (\theta_2, x_2 - \theta_1 \Gamma \theta_2, \xi^1)\}, \{(0,1), (0,2), (\theta_2, x_2, \xi^2)\})
\end{align*}
\]

\[
\equiv m_{\text{Ad}^2},
\]

which yields, upon taking into account Eq. (E4),

\[
\begin{align*}
\left(\text{pr}_2^\ast (\tilde{\text{pr}}_2^\ast \beta + \text{pr}_2^\ast \tilde{\theta}_{\tilde{\mathcal{N}}}) - \text{pr}_1^\ast \text{Ad}^\ast \beta\right)(m_{\text{Ad}^2})
\end{align*}
\]

\[
\equiv \text{pr}_2^\ast \beta\left(\{(0,1), (0,2), (\theta_2, x_2, \xi^2)\}\right) + \tilde{\theta}_{\tilde{\mathcal{N}}}\left(\{(0,1), (0,2), (\theta_2, x_2)\}\right)
\]

\[
= \beta\left(\theta_2, x_2, \xi^2\right) + \tilde{\theta}_{\tilde{\mathcal{N}}}\left(\{(0,1), (0,2), (\theta_2, x_2)\}\right) - \beta\left(\theta_2, x_2 - \theta_1 \Gamma \theta_2, \xi^1\right)
\]

\[
= d\theta_2^\ast \left(\xi^2 - \xi_1^1\right) - \Gamma^\alpha_{\beta\gamma} \theta_{\beta}^\ast d\left(\theta_1 \Gamma^\alpha \theta_2\right) - \frac{2}{3} \theta_2^\ast \Gamma_a \theta_2 \theta_2 \Gamma^a \theta_2 d\theta_2
\]

\[
= d\theta_2^\ast \left(\xi^2 - \xi_1^1\right) - \frac{1}{2} \left(\theta_2^\ast \Gamma_a \theta_2 \right) \theta_2 \Gamma^a \theta_2 d\theta_2
\]

we extract the base component

\[
A_E(m_{\text{Ad}^2}, z) = \text{pr}_{2\ast}^\ast d\left(\xi^2 - \xi_1^1\right) - \frac{1}{2} \left(\theta_2^\ast \Gamma_a \theta_2 \right) \theta_2 \Gamma^a \theta_2 d\theta_2
\]

of the principal \( \mathbb{C}^\ast \)-connection super-1-form

\[
A_E(m_{\text{Ad}^2}, z) = \frac{1 + \text{pr}_{2\ast}^\ast d\left(\xi^2 - \xi_1^1\right)}{z} + A_E(m_{\text{Ad}^2})
\]

on the trivial principal \( \mathbb{C}^\ast \)-bundle

\[
E = Y_{\text{AdS}^3 \text{Mink}^4,1|\mathcal{D}_{4,1} \times \mathbb{C}^\ast \equiv \left(m_{\text{Ad}^2}, z\right)}
\]

of \( Y_1 \) described by Diag. (I.44). Given these, it is now straightforward to decipher the explicit form of the principal \( \mathbb{C}^\ast \)-bundle isomorphism \( \alpha_E \) of Eq. (I.43). Write

\[
m_{\text{Ad}^2}^A \equiv \{(0,1), (0,2), (\theta_2, x_2 - \theta_1 \Gamma \theta_2, \xi^4)\}, \quad A \in \{1, 3\},
\]

\[
m_{\text{Ad}^2}^B \equiv \{(0,1), (0,2), (\theta_2, x_2, \xi^B)\}, \quad B \in \{2, 4\}
\]

to obtain

\[
\begin{align*}
\left(\text{pr}_1^\ast \text{Ad}^\ast x^2 \ast A_{\mathcal{F}(1)} + \text{pr}_3^\ast A_E - \text{pr}_1^\ast \text{Ad}^\ast x^2 \ast A_{\mathcal{F}(1)}\right)(m_{\text{Ad}^2, m_2, m_3, m_4})
\end{align*}
\]

\[
= \text{pr}_2^\ast x^2 \ast A_{\mathcal{F}(1)}(m_{\text{Ad}^2, m_2, m_3, m_4}) + A_E(m_{\text{Ad}^2, m_2, m_3, m_4}) - \text{pr}_2^\ast x^2 \ast A_{\mathcal{F}(1)}(m_{\text{Ad}^2, m_2, m_3, m_4})
\]

\[
= A_{\mathcal{F}(1)}(\{(0,1, x_2 - \theta_2 \Gamma \theta_2, \xi^1), (\theta_2, x_2 - \theta_1 \Gamma \theta_2, \xi^3)\}) + A_E(m_{\text{Ad}^2, m_2, m_3, m_4}) - \text{pr}_2^\ast x^2 \ast A_{\mathcal{F}(1)}(m_{\text{Ad}^2, m_2, m_3, m_4})
\]

\[
- A_{\mathcal{F}(1)}(\{(0,1, x_2, \xi^2), (\theta_2, x_2, \xi^4)\}) = \theta_2^\ast d(\xi^3 - \xi^1)\alpha + \theta_2^\ast d(\xi^4 - \xi^3)\alpha - \theta_2^\ast d(\xi^2 - \xi^1)\alpha - \theta_2^\ast d(\xi^4 - \xi^2)\alpha = 0
\]

and, accordingly, postulate the isomorphism in the form

\[
\begin{align*}
\alpha_E : \text{pr}_1^\ast \text{Ad}^\ast x^2 \ast A_{\mathcal{F}(1)}(m_{\text{Ad}^2, m_2, m_3, m_4}) \cong \text{pr}_{1, 2, 3, 4}^\ast E \cong \text{pr}_{1, 2, 3, 4}^\ast \Gamma \theta_2, \xi^1\}
\end{align*}
\]

\[
\cong \{(0,1, x_2 - \theta_1 \Gamma \theta_2, \xi^1), (\theta_2, x_2 - \theta_1 \Gamma \theta_2, \xi^3), z_1\} \otimes (m_{\text{Ad}^2, m_2, m_3, m_4, z_2)}
\]

\[
\cong \{(0,1, x_2, \xi^2), (\theta_2, x_2, \xi^4), z_1 \cdot z_2\}.
\]

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manifestly coherent with the groupoid structure $\mu_L \equiv \mu_{\mathcal{L}^{(1)}}$ on $\mathcal{L}^{(1)}$. Finally, we compute, in the shorthand notation

\[ m_{1,2,3} \equiv ((\theta_1, x_1), (\theta_2, x_2), (\theta_3, x_3)), \quad m_{2,3} \equiv ((\theta_2, x_2), (\theta_3, x_3)), \]

\[ m_{1,\text{Ad}23} = ((\theta_1, x_1), (\theta_3, x_3 - \frac{1}{2} \theta_1 \Gamma \theta_2), (\theta_3, x_3)), \]

\[ \tilde{c}_1 = (\theta_3, x_3 - (\theta_1 + \theta_2) \Gamma \theta_3, \xi^1), \quad \tilde{c}_2 = (\theta_3, x_3 - \theta_2 \Gamma \theta_3, \xi^2), \quad \tilde{c}_3 = (\theta_3, x_3, \xi^3), \]

an appropriate combination of the base components of the pullback connections of the various factors in the domain $\text{pr}_{1,3}^*(\mathcal{Y}_A^{d,1} \times_{\text{pr}_{2,3}} \mathcal{M}) \otimes \text{pr}_{2,3}^*(\mathcal{Y}_A^{2d,0} \times_{\text{pr}_{x,y}} \mathcal{E})$ and in the codomain $\text{pr}_{1,3}^*(\mathcal{Y}_A^{d,1} \times_{\text{pr}_{2,3}} \mathcal{E})$ of the principal $\mathbb{C}^*$-bundles over

\[ \mathcal{Y}_A^{2d,0} \mathbb{M}_{\text{SMin}}^{d,1/d,1} \ni \left( (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)) \right) \]

\[ = \gamma_1^{-1}(\left( (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)) \right)) \]

related by the 2-isomorphism $\gamma_1$ sought after,

\[ \text{pr}_{1,2}^* \text{pr}_{2,3}^* A_E((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) \]

\[ + \text{pr}_{2,3}^* \text{pr}_{2,3}^* A_E((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) \]

\[ - \text{pr}_{1,2}^* \text{pr}_{2,3}^* A_E((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) \]

\[ = A_E((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) + A_E((m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) - A_E((m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) \]

\[ = \theta_3^2 d(\xi_3^1 - \xi_3^0) - \frac{1}{3} (\theta_1 \Gamma_a \theta_2 a_3) \Gamma^a d\theta_3 + \theta_3^2 d(\xi_3^1 - \xi_3^0) - \frac{1}{3} (\theta_1 \Gamma_a \theta_2 a_3) \Gamma^a d\theta_3 \]

\[ - \theta_3^2 d(\xi_3^1 - \xi_3^0) + \frac{1}{3} (\theta_1 + \theta_2) \Gamma_a \theta_3 a_3) \Gamma^a d\theta_3 = 0, \]

whereupon we conclude that $\gamma_1$ can be taken in the trivial form

\[ \gamma_1 : \text{pr}_{1,2}^* \mathcal{Y}_A^{d,1} \times_{\text{pr}_{2,3}} \mathcal{M} \otimes \mathcal{Y}_A^{2d,0} \times_{\text{pr}_{x,y}} \mathcal{E} \longrightarrow \text{pr}_{1,3}^* \mathcal{Y}_A^{d,1} \times_{\text{pr}_{2,3}} \mathcal{E} \]

\[ : (m_{1,2,3}, ((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)), (m_{2,3}, \xi^1), \xi^2, \xi^3) \]

\[ \longrightarrow (m_{1,2,3}, ((m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3))) \]

\[ = \gamma_1^{-1}(\left( (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)) \right)) \]

The latter is manifestly coherent.

We finish the proof with an inspection of the behaviour of the Ad.-equivariant structure reconstructed above under supersymmetry. There is, at this stage, one last piece of the structure that determines the relation between the pair $(\Gamma_1, \gamma_1)$ and the realisation $(\text{Ad}_1, \mathcal{Y}_A, \text{LAd}_1)$ of supersymmetry on $\mathcal{E}_{GS}^{d,1}$ that we should establish prior to verifying its expected properties. That piece of data is a lift $E\text{Ad}_1 : \mathbb{G}^1 \times \mathcal{E} \longrightarrow \mathcal{E}$ of the induced action $\mathcal{Y}_A^{d,1}$ from the base $\mathcal{Y}_A^{d,1} \mathbb{M}_{\text{SMin}}^{d,1} \times_{\text{pr}_{x,y}} \mathcal{E}$ to its total space. We fix it by demanding that it preserve the principal $\mathbb{C}^*$-connection super-1-form $A_E$. We obtain the identity (written in the previously adopted shorthand notation)

\[ \mathcal{Y}_A \mathcal{E} \mathcal{A}^{d,1}_{(c,y)} A_E(m_{Ad2}) \]

\[ = A_E(((\theta_1, x_1 - \varepsilon \Gamma \theta_1), (\theta_2, x_2 - \varepsilon \Gamma \theta_2)), (\theta_2, x_2 - (\theta_1 + \varepsilon \Gamma \theta_2), \xi^1 + \frac{1}{3} (\varepsilon \Gamma_a \theta_2 \theta_3 \Gamma^a \theta_3))) \]

\[ = (\theta_1, x_1 - \varepsilon \Gamma \theta_1), (\theta_2, x_2 - \varepsilon \Gamma \theta_2)), (\theta_2, x_2 - \varepsilon \Gamma \theta_2, \xi^2 + \frac{1}{3} (\varepsilon \Gamma_a \theta_2 \theta_3 \Gamma^a \theta_3))) \]

\[ = \gamma_1^{-1}(\left( (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_1)), (m_{1,2,3}, (m_{1,\text{Ad}23}, \tilde{c}_2)), (m_{1,2,3}, (m_{2,3}, \tilde{c}_3)) \right)) \]

and so we may set

\[ E\text{Ad}_1 : \mathcal{Y}_A \mathcal{E} \mathcal{A}^{d,1}_{(c,y)} \times \mathcal{E} \longrightarrow \mathcal{E} : ((\varepsilon, y), (m_{Ad2}, z)) \longrightarrow (\mathcal{Y}_A \mathcal{E} \mathcal{A}^{d,1}_{(c,y)} (m_{Ad2}), z). \]
It is now completely straightforward to check equivariance of \(\alpha_E\) and \(\gamma_1\).

**APPENDIX F. A PROOF OF THEOREM 5.1**

In view of the previous remarks, we first have to demonstrate that \(S_{\text{metr},G_S,p}^{(H)}\) reduces to \(S_{\text{metr},G_S,p}^{(NG)}\) upon imposing \(\text{(5.16)}\) whenever conditions \((E1)\) and \((E2)\) are satisfied. To this end, we work out explicit formulae for the relevant components of the Maurer–Cartan super-1-form. We have
\[
e^{-\Phi_{\text{ad}}^S f(P^h_2)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \phi^S_0 \phi^S_2 \cdots \phi^S_{2n} f_{S^0_{2n}} ^\hbar f_{S^0_{2n-1}} ^\hbar f_{S^0_{2n-2}} ^\hbar \cdots f_{S^0_{2}} ^\hbar f_{S^0_{1}} ^\hbar f_{S^0_{0}} ^\hbar \triangleright P^h_2,
\]
and
\[
e^{-\Phi_{\text{ad}}^S f(P^h_b)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \phi^S_0 \phi^S_2 \cdots \phi^S_{2n} f_{S^0_{2n}} ^\hbar f_{S^0_{2n-1}} ^\hbar f_{S^0_{2n-2}} ^\hbar \cdots f_{S^0_{2}} ^\hbar f_{S^0_{1}} ^\hbar f_{S^0_{0}} ^\hbar \triangleright P^h_b,
\]

Denote
\[
F(\phi) = \phi^S f_{S^0} ^\hbar, \quad \bar{F}(\phi) = \phi^S f_{S^0} ^\hbar.
\]

Furthermore, for the sake of brevity, use the symbolic notation
\[
L(\phi)^2 := Q(\phi), \quad \bar{L}(\phi)^2 := \bar{Q}(\phi)
\]
in (even) functions of \(\phi\) whose dependence on the argument factors through \(Q(\phi)\) or \(\bar{Q}(\phi)\), e.g.,
\[
e^{-\Phi_{\text{ad}}^S f(P^h_2)} = (\text{ch} L(\phi))^b \triangleright P^h_2 - \phi^S f_{S^0} ^\hbar \left(\text{sh} L(\phi) \frac{L(\phi)^2}{L(\phi)}\right)^b \triangleright P^h_c,
\]
\[
e^{-\Phi_{\text{ad}}^S f(P^h_b)} = (\text{ch} \bar{L}(\phi))^\bar{b} \triangleright P^h_c - \phi^S f_{S^0} ^\hbar \left(\text{sh} \bar{L}(\phi) \frac{L(\phi)^2}{L(\phi)}\right)^\bar{b} \triangleright P^h_b.
\]

The above then rewrites as
\[
e^{-\Lambda(\phi)} = \begin{pmatrix}
\text{ch} L(\phi) & -F(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi)} \\
-F(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi)} & \text{ch} \bar{L}(\phi)
\end{pmatrix},
\]

where the blocks correspond (in an obvious manner) to the direct summands in the decomposition \(t^{(0)}_\text{ad} = t^{(0)}_\text{vec} \oplus \mathfrak{e}\). This can be further decomposed as
\[
e^{-\Lambda(\phi)} = \begin{pmatrix}
1_p & -\varphi(\phi) \\
-\varphi(\phi) & 1_d^p
\end{pmatrix} \begin{pmatrix}
\text{ch} L(\phi) & 0 \\
0 & \text{ch} \bar{L}(\phi)
\end{pmatrix},
\]

with
\[
\varphi(\phi) = F(\phi) \cdot \frac{\text{sh} \bar{L}(\phi)}{L(\phi) \text{ch} L(\phi)}, \quad \bar{\varphi}(\phi) = \bar{F}(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)}.
\]

In view of the obvious identities
\[
F(\phi) \cdot \bar{Q}(\phi) = Q(\phi) \cdot F(\phi), \quad \bar{F}(\phi) \cdot Q(\phi) = \bar{Q}(\phi) \cdot \bar{F}(\phi),
\]
we may rewrite the last definitions in the equivalent form
\[
\varphi(\phi) = \frac{\text{sh} \bar{L}(\phi)}{L(\phi) \text{ch} L(\phi)} \cdot F(\phi), \quad \bar{\varphi}(\phi) = \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} \cdot \bar{F}(\phi).
\]

Furthermore, as
\[
F(\phi) \cdot \bar{F}(\phi) = Q(\phi), \quad \bar{F}(\phi) \cdot F(\phi) = \bar{Q}(\phi),
\]

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we obtain the relation
\[ \varphi \cdot \overline{\varphi} = \frac{\text{sh} L(\phi)}{L(\sigma) \chi(\sigma)} \cdot F(\phi) \cdot \overline{F}(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\sigma) \chi(\sigma)} = \left( \frac{\text{sh} L(\phi)}{\chi(\sigma) L(\sigma)} \right)^2 = 1_{p+1} - \frac{1}{\chi^2 L(\phi)}, \]
and similarly – the relation
\[ \overline{\varphi} \cdot \varphi = 1_{d-p} - \frac{1}{\chi^2 L(\phi)}, \]
so that we may ultimately express \( e^{-\Lambda(\phi)} \) entirely in terms of \( \varphi \) and \( \overline{\varphi} \) as
\[ e^{-\Lambda(\phi)} = \left( \begin{array}{cc} 1_{p+1} - \varphi & -\overline{\varphi} \\ \overline{\varphi} & 1_{d-p} \end{array} \right) \left( \begin{array}{cc} (1_{p+1} - \varphi \cdot \overline{\varphi})^{-\frac{1}{2}} & 0 \\ 0 & (1_{d-p} - \overline{\varphi} \cdot \varphi)^{-\frac{1}{2}} \end{array} \right) \left( \begin{array}{c} \varphi \\ \overline{\varphi} \end{array} \right). \]

We next use assumption (E1) to relate \( \overline{\varphi} \) to \( \varphi \). To this end, we compute, using the ad-invariance of the Killing form,
\[ \kappa(0)^{-1} \text{sh} f_{\overline{\sigma} \overline{d}} \kappa(0)^{-1} \overline{\sigma} = \kappa(0)^{-1} \text{sh} f_{\overline{\sigma} \overline{d}} \kappa(0)^{-1} \left[ J_\sigma, P_\overline{a} \right] P_\overline{a} = -\kappa(0)^{-1} \text{sh} \kappa(0) \left[ J_\sigma, P_\overline{a} \right] P_\overline{a} = -\kappa(0)^{-1} \text{sh} f_{\overline{\sigma} \overline{d}} \kappa(0)^{-1} \overline{\sigma}, \]
whence also
\[ \kappa(0)^{-1} f_{\overline{\sigma} \overline{d}} = -\kappa(0)^{-1} \text{sh} f_{\overline{\sigma} \overline{d}} \kappa(0)^{-1} \overline{\sigma}. \]

Taking into account that \( \overline{\varphi} \) is an odd function of the \( \varphi^a \), we then readily establish the fundamental identity
\[ \overline{\varphi}^a = -\kappa(0)^{-1} \text{sh} f_{\overline{\sigma} \overline{d}} \kappa(0)^{-1} \overline{\sigma}. \]

At this stage, we may express (the pullbacks of) the relevant components of the left-invariant Maurer–Cartan super-1-form as functions of the (local) coordinates
\[ (\xi^a)^{0,\sigma} = (\theta^a, X^a)^{0,\sigma} \]
and of the \( \varphi^a \), whereupon the imposition of the inverse Higgs constraint becomes straightforward. Thus, taking into account (6.10) as well as the hitherto results, we find the expressions
\[ \sigma^{H_{\text{vac}}} \theta^a \Sigma (\xi, \phi) = \left( e^b_{\overline{a}} (\xi) - e^a_{\overline{b}} (\phi) \right) \left( \sqrt{1_{p+1} - \varphi \cdot \overline{\varphi}} \right)^{-1} (\phi) \, d\xi^b, \]
\[ \sigma^{H_{\text{vac}}} \theta^a \Sigma (\xi, \phi) = \left( e^b_{\overline{a}} (\xi) - e^a_{\overline{b}} (\phi) \right) \left( \sqrt{1_{d-p} - \overline{\varphi} \cdot \varphi} \right)^{-1} (\phi) \, d\xi^b, \]
written in terms of the reduced Vielbeine \( e^a_{\overline{A}} \) of Eq. (6.11). Denote, for any \( \mu \in 0, d \) and for (local) coordinates \( \{ \sigma^0 \}^a \) on \( \Omega_p \),
\[ i e^a_{,\mu}(\sigma) := e^a_{\overline{A}} (\xi(\sigma)) \left( \frac{\partial \xi^b}{\partial \sigma^0} (\sigma) \right) \]
and further write
\[ i e^a_{,b} := i e^a_{,b}, \quad i e^a_{,b} := i e^a_{,b} \]
for the sake of clarity of the formulæ that follow. The solution to the inverse Higgs constraint now reads
\[ \varphi^a (\phi_1 (\sigma)) = \xi^{-1} c \left( \sigma \right) \varphi^a (\sigma), \]
or – in an obvious shorthand notation –
\[ \varphi \circ \phi_i = \xi^{-1} T \cdot i e^T. \]

Substituting this into the first of formulæ (F.3) and using Eq. (F.2) along the way, we arrive at the expression
\[ i e^a_{,b} \equiv i e^a_{,b} = \sigma^{H_{\text{vac}}} \theta^a \Sigma (\xi, \phi_i (\xi)) \]
\[ = \left( e^b_{\overline{a}} + i e^b_{\overline{a}} \kappa(0)^{-1} \text{sh} i e^b_{\overline{a}} \kappa(0)^{-1} \text{sh} \right)\left( 1_{p+1} - \frac{1}{\chi^2 L(\sigma)} \right) \left( \frac{1}{\chi^2 L(\sigma)} \right) \right)^{\frac{a}{b}}. \]
In order to simplify the above expression and prepare it for subsequent use in the reconstruction of the inverse Higgs-reduced Hughes–Polchinski action functional, let us call
\[ i \overline{\omega}_{ab} := \kappa_{ab}^{(0)}, \quad \overline{\eta}_{cd} := \kappa_{cd}^{(0)} \]
and
\[ i \overline{\eta}_{ab} := \overline{\eta}_{cd} i^{\varepsilon_a}_c i^{\varepsilon_b}_b, \quad i \overline{\eta}_{ab} := \overline{\eta}_{cd} i^{\varepsilon_a}_c i^{\varepsilon_b}_b, \]
as well as
\[ i \overline{\varepsilon}_{ab} := i \delta_{ab} + i \overline{g}_{ab} \equiv \kappa_{\mu \nu} i^{\varepsilon}_a i^{\varepsilon}_b. \]
We then obtain
\[
\kappa_{\mu \nu} \equiv \frac{i a \ \overline{\varepsilon}_{ab} \ \overline{b}}{\sqrt{1 + i \overline{g}_{cd} i^{\varepsilon}_a i^{\varepsilon}_b}} \left( \frac{1}{\sqrt{1 + i \overline{g}_{mn} i^{\varepsilon}_a i^{\varepsilon}_b}} \right) \frac{a}{d} \frac{b}{d}.
\]
At long last, we may now write out the contribution to the sought-after metric term of the reduced Hughes–Polchinski action functional from the \( (p + 1) \)-cell of the subordinate tessellation \( \Delta(\Omega_p) \) of the worldvolume \( \Omega_p \) (in an obvious shorthand notation),
\[
S_{\text{metr,GS,p}}^{(\overline{\eta})}[(\xi, \phi(\overline{\xi}))] = \int \overline{\xi} \ \text{Vol}(\Omega_p) \ \varepsilon^{a_0 a_1 \ldots a_p \nu_0 \nu_1 \ldots \nu_p} \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}) \equiv \int \overline{\xi} \ \text{Vol}(\Omega_p) \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}) \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}),
\]
whence also we finally retrieve the anticipated result
\[
S_{\text{metr,GS,p}}^{(\overline{\eta})}[(\xi, \phi(\overline{\xi}))] = \lambda_p \int \overline{\xi} \ \text{Vol}(\Omega_p) \ \sqrt{\left( \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}) \right)^2 \cdot \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}) \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p})},
\]
up to an overall constant \( \lambda_p \) (which we can always set to one by a suitable rescaling of the metric term).

Passing to the closing statement of the theorem, we shall first write out the aforementioned contribution to the metric term of the Hughes–Polchinski action functional in a form amenable to further treatment. Taking into account Eq. (F.1), we obtain – in the previously introduced notation –
\[
S_{\text{metr,GS,p}}^{(\overline{\eta})}[(\xi, \phi(\overline{\xi}))] = \int \overline{\xi} \ \text{Vol}(\Omega_p) \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}) \ \text{det}_{(p+1)}(i \overline{\varepsilon}_{a_0} i^{\varepsilon}_{a_1} \ldots i^{\varepsilon}_{a_p}),
\]
where \( M(\xi_\nu) \) is a matrix that does not depend on the \( \phi^S_\mu \), and hence does not contribute to the Euler–Lagrange equations for these fields, and where
\[
A(\xi_\nu, \phi_\nu) = \kappa + \varphi(\phi_\nu) \cdot \overline{\kappa} \cdot i \overline{\varepsilon} \cdot (\xi_\nu) \cdot i \overline{\varepsilon}^{-1} (\xi_\nu),
\]
\[
B(\xi_\nu, \phi_\nu) = \kappa + \varphi(\phi_\nu) \cdot \overline{\kappa} \cdot \varphi(\phi_\nu),
\]
The said Euler–Lagrange equations read
\[
\frac{\delta \overline{\eta}_{ab}}{\delta \phi^a} \text{tr}_{(p+1)}(A(\xi_\nu, \phi_\nu)) \cdot \frac{\delta}{\delta \phi^a} A(\xi_\nu, \phi_\nu) - \frac{1}{2} B(\xi_\nu, \phi_\nu) \cdot \frac{\delta}{\delta \phi^a} B(\xi_\nu, \phi_\nu) = 0,
\]
and so using the symmetricity of \( B(\xi_\nu, \phi_\nu) \) and taking into account arbitrariness of \( \delta \phi^a_\mu \equiv \delta \phi^a \overline{\epsilon}^a_\mu \),
they can be cast in the simple matrix form
\[
(\overline{\kappa} \cdot i \overline{\varepsilon} \cdot (\xi_\nu) \cdot i \overline{\varepsilon}^{-1} (\xi_\nu) \cdot A(\xi_\nu, \phi_\nu)) = B(\xi_\nu, \phi_\nu) \cdot \varphi(\phi_\nu).
\]
Upon multiplying both sides of the above equation by $\varphi(\phi_{iv})^T$ and invoking Eq. (5,3), we deduce from the above the identity

$$A(\xi_{iv}, \phi_{iv})^{-1} = B(\xi_{iv}, \phi_{iv})^{-1},$$

which – when used in the original equation alongside $\varphi \neq 0$ – yields the anticipated solution (4.4).

**Appendix G. A proof of Proposition 5.3.**

First of all, we check that the superbracket of $\mathfrak{so}(d,1|D_{d,1})$ does close on the distinguished subspace $g_{\text{vac}}$. To this end, we compute, for arbitrary $\alpha, \beta \in \mathfrak{g}, \omega, \nu, \gamma, \mu \in \mathfrak{g}$, and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\nu}, \tilde{\mu} \in \mathfrak{g} + 1, d - p$, the anticommutator of projected supercharges

$$\{ P^\alpha, Q_\beta, P_\mu \} = (P^T \cdot \Gamma^\mu \cdot P)_{\alpha\beta} P_\mu = \left( C \cdot \left( (1_{D_{d,1}} - P) \cdot \Gamma^\mu \cdot P \right)_{\alpha\beta} P_\mu \right.$$

$$= \left( C \cdot (1_{D_{d,1}} - P)^2 \cdot \Gamma^\alpha \right)_{\alpha\beta} P_\mu + \left( C \cdot (1_{D_{d,1}} - P) \cdot P \cdot \Gamma^\alpha \right)_{\alpha\beta} P_\mu$$

and the commutators

$$[P^\alpha, Q_\beta] = 0 \in \mathfrak{g}_{\text{vac}}(0),$$

as well as the module commutators

$$[J^\alpha_{ab}, J^\beta_{cd}] = \eta^a_{ac} J^\beta_{bd} - \eta^a_{bd} J^\beta_{ac} + \eta^b_{bc} J^\alpha_{ad} - \eta^b_{ad} J^\alpha_{bc} \in \mathfrak{g}_{\text{vac}},$$

$$[J^\alpha_{ab}, J^\beta_{cd}] = \delta^a_{ad} J^\beta_{bc} - \delta^a_{ad} J^\beta_{bc} - \delta^b_{bd} J^\beta_{ac} + \delta^b_{bd} J^\beta_{ac} \in \mathfrak{g}_{\text{vac}},$$

$$[J^\alpha_{ab}, P^\beta_{cd}] = 0 \in \mathfrak{g}_{\text{vac}}(0),$$

$$[J^\alpha_{ab}, P^\beta_{cd}] = 0 \in \mathfrak{g}_{\text{vac}}(0),$$

$$[J^\alpha_{ab}, P^\beta_{cd}] = P^\beta_{cd} [J^\alpha_{ab}, Q_\mu] = \frac{1}{2} \left( (\Gamma^\alpha_{ab} \cdot P)_{\beta\mu} \right) Q_\mu = \frac{1}{2} \left( (\Gamma^\alpha_{ab} \cdot P)_{\beta\mu} \right) Q_\mu \in \mathfrak{g}_{\text{vac}}(0),$$

the last two following from the algebra

$$\Gamma^\alpha_{ab} \cdot (1_{D_{d,1}} - P) \cdot J^\alpha_{cd} = \left( (1_{D_{d,1}} - (1_{D_{d,1}} - P)) \cdot \Gamma^\alpha_{ab} \cdot J^\alpha_{cd} \right) \equiv 0 \in \mathfrak{g}_{\text{vac}}(0).$$

Passing to the field-theoretic part of the thesis, we conclude that the adjoint action of $H_{\text{vac}} \equiv SO(p,1) \times SO(d-p)$ is tautologically unimodular (in particular, in restriction to $\mathfrak{g}_{\text{vac}}(0)$), and so it remains to verify the conditions involving the Minkowskian metric $\eta$. First, then, we note that the restrictions of that metric to the directions $\partial a$, $\tilde{a} \in \mathfrak{g} + 0, \tilde{a} \in \mathfrak{g} + 1, d - p$ in the tangent space define, respectively, the non-degenerate bilinear symmetric forms $\gamma$ and $\tilde{\gamma}$ requested by Thm. 5.2. Finally, we readily check that identity (5,17) is trivially satisfied

$$\eta_{\tilde{a} \tilde{b}} f^\tilde{a} \tilde{b} e = \eta_{\tilde{a} \tilde{b}} f^\tilde{a} \tilde{b} e \equiv -\delta_{\tilde{a} \tilde{b}} -\delta_{\tilde{a} \tilde{b}} \equiv -\delta_{\tilde{a} \tilde{b}} \tilde{e} e \equiv -\delta_{\tilde{a} \tilde{b}} \tilde{e} e.$$

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