Comment on ‘New variables for 1 + 1 dimensional gravity’ (2010 Class. Quantum Grav. 27 025002)

Martin Bojowald\textsuperscript{1}, Suddhasattwa Brahma\textsuperscript{1,2} and Juan D Reyes\textsuperscript{1}

\textsuperscript{1} Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802, United States of America
\textsuperscript{2} Center for Field Theory and Particle Physics, Fudan University, 200433 Shanghai, People’s Republic of China
\textsuperscript{3} Facultad de Ingeniería, Universidad Autónoma de Chihuahua, Nuevo Campus Universitario, Chihuahua 31125, Mexico

E-mail: bojowald@gravity.psu.edu, suddhasattwa.brahma@gmail.com and jdrp75@gmail.com

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Abstract

The results reported in (2010 Class. Quantum Grav. 27 025002) are special cases of a general treatment of canonical variables for dilaton gravity models published in (2009 Class. Quantum Grav. 26 035018).

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Different sets of canonical variables for 1 + 1-dimensional models of gravity without local physical degrees of freedom have been discussed in \cite{1}. These variables are related to connection formulations as used in classical theories underlying loop quantum gravity. All models of this form are contained in the class of 2-dimensional dilaton models, or equivalently in the class of Poisson Sigma models \cite{2–4}. Since these classes had already been formulated in terms of connection variables \cite{5}, there should be a strict relation between the different sets of variables. In this comment we work out the relationship.

We start with standard formulations of 1 + 1-dimensional actions for a dyad $e^a$ with volume form $\epsilon$, a connection 1-form $\omega$ and a dilaton field $\phi$. For our purpose here, it suffices to consider torsion-free models, such that the condition $De^a = 0$, using the covariant derivative given by $\omega$, is implemented by Lagrange multipliers $X_a$. The dilaton gravity action with potential $V(\phi)$ is then

$$S = -\frac{1}{2G} \int_M \left( \phi d\omega + \frac{1}{2} V(\phi) \epsilon + X_a De^a \right)$$

and takes, after integrating by parts, the form
\[ S = \frac{1}{2G} \int_M \left( e^a \wedge dX_a + \omega d\phi - X_a e^b \omega \wedge e_b - \frac{1}{2} V(\phi) e \right). \] (2)

Here, \( M \) is a \( 1 + 1 \)-dimensional manifold with coordinates \((t, x)\).

In Poisson Sigma models, one organizes the variables in new sets \( X^i = (X^-, X^+, \phi) \), \( A_i = (e^+_i, e^-_i, \omega_i) \), and \( \Lambda_i = (e^+_i, e^-_i, \omega_i) \). A canonical analysis leads to Poisson brackets
\[
\{X^i(x), A_j(y)\} = 2G \delta^i_j \delta(x-y),
\] (3)
while the \( \Lambda_i \) serve as Lagrange multipliers of first-class constraints
\[
\tilde{C}^i = \frac{1}{2G} (\{X^i\}' + P^i \Lambda_i)
\] (4)
with the Poisson tensor \([2-4]\)
\[
P = \begin{pmatrix}
0 & -\frac{1}{2}V(\phi) & -X^- \\
\frac{1}{2}V(\phi) & 0 & X^+ \\
X^- & -X^+ & 0
\end{pmatrix}
\] (5)

The variables introduced in \([5]\) are obtained by using the ‘absolute values’
\[
X := \sqrt{X^+ X^-} \quad \text{and} \quad e := \sqrt{e^+_i e^-_i}
\] (6)
and boost parameters \( \alpha \) and \( \beta \) in
\[
X^\pm = X \exp(\pm \beta) \quad \text{and} \quad e^\pm = e \exp(\pm \alpha).
\] (7)

They are related to the canonical variables
\[
Q^\epsilon = 2X \cosh(\alpha - \beta) \quad \text{and} \quad Q^\alpha = 2eX \sinh(\alpha - \beta)
\] (8)
with
\[
\{Q^\epsilon(x), e(y)\} = \{Q^\alpha(x), \alpha(y)\} = \{\phi(x), \omega_i(y)\} = 2G \delta(x-y).
\] (9)

The inverse transformation is
\[
X^\pm = \frac{e Q^\epsilon \mp Q^\alpha}{2e} \exp(\pm \alpha).
\] (10)

In canonical variables, the constraints are
\[
\tilde{C}^\pm = \frac{1}{2G} \left( \left( \frac{e Q^\epsilon \mp Q^\alpha}{2e} \right) \pm \frac{e Q^\epsilon \pm Q^\alpha}{2e} (\omega_i + \alpha') \pm \frac{1}{2} V(\phi) e \right) \exp(\pm \alpha)
\] (11)
and
\[
\tilde{C}^3 = \frac{1}{2G} (\phi' + Q^\alpha).
\] (12)

For spherically symmetric gravity, corresponding to a specific dilaton potential \( V(\phi) = -2/\sqrt{\phi} \) \([4]\), one can compare the new canonical variables to those used in real connection formulations such as \([6]\), denoted as \((K_x, E^x; K_s, E^s; \eta, P^\psi)\) and with Poisson brackets
\[
\{K_x(x), E^s(y)\} = 2G \delta(x-y), \quad \{K_x(x), E^s(y)\} = G \delta(x-y).
\] (13)
Comment

(Note that there is no factor of two in the second equation because the pair \((K_\phi, E_\phi)\) represents two angular directions which were independent before symmetry reduction.) They are related to \((Q^e, e; Q^\alpha, \alpha; \phi, \omega_x)\) by the canonical transformation

\[
Q^e = 2\sqrt{2}(E^e)^{1/4}K_\phi, \quad e = \frac{E^e}{\sqrt{2}(E^e)^{1/4}} \quad (14)
\]

\[
Q^\alpha = P^\alpha, \quad \alpha = -\eta \quad (15)
\]

\[
\omega_x = -\left(K_x + \frac{E^e}{2E_x}K_\phi - \eta'\right), \quad \phi = E^e. \quad (16)
\]

(Unlike the pair \((K_\phi, E_\phi)\), the pair \((e, Q^e)\) is defined with a factor of two in the Poisson bracket (9).) These relations have been derived in [5]; see equation (42) in this paper. Also in [5], the spherically symmetric Hamiltonian constraint has been obtained as

\[
H[N] = C^+ [2^{-1/2}N\phi^{1/4}\exp(-\alpha)] - C^- [2^{-1/2}N\phi^{1/4}\exp(\alpha)]
\]

\[
= \frac{1}{2G} \int d\chi N \left(\frac{K_\phi^2 E^e}{\sqrt{E^e}} + 2\sqrt{E^e}K_x K_\phi - \frac{1}{2} E^\phi V(E^\phi) \right.
\]

\[
- \left(\frac{(E^\phi)'^2}{4E^\phi \sqrt{E^e}} + \sqrt{E^e}(E^\phi)'(E^\phi)^' \right) \frac{(E^\phi)^''}{(E^\phi)^2} - \frac{\sqrt{E^e}(E^\phi)^''}{E^\phi}. \quad (17)
\]

For an arbitrary dilaton potential, this formulation generalizes the connection formulation of spherically symmetric canonical gravity to arbitrary \(1+1\)-dimensional dilaton models. Deriving just this kind of result was the aim of [1]. In fact, the definitions of [1] are nothing but the results of [5] with different names chosen for the new variables. The expression for \(e\) in (14) is the same as (51) in [1], \(Q^e\) in (14) is (57), and \(\omega_x\) in (16) is (60).

There is a different formulation in [1] for the specific case of the CGHS model (constant dilaton potential). This model, as presented in [1] has a Hamiltonian constraint with only \(K_xK_\phi\) in its kinetic part but no contribution of \(K_\phi^2\), unlike what we have in (17). However, this formulation is not new either, even though it does not correspond to a connection formulation as defined in [5]. It rather amounts to using the original canonical variables (8) of dilaton models, along with \(\omega_x\), and just renaming them as \(\omega_x \equiv K_x, Q^e \equiv K_\phi\) and \(e \equiv E^e\). Except for the new names, these variables, in particular \(Q^e\), have been introduced in [5]; see equation (20) in this paper. Except for using the SU(1,1)-invariant \(Q^e\), they correspond to a first-order formulation of Poisson Sigma models. The constraints (11) in these variables depend on \(\omega_x\) in a linear fashion, and so does the Hamiltonian constraint obtained by a linear combination. In these variables, therefore, the Hamiltonian constraint has only one term \(Q^e \omega_x = K_x K_\phi\) but no contribution from \(K_\phi^2\) (which would amount to \((Q^e)^2\), but there is no such term in (11)). The quadratic term is introduced in (17) by applying the canonical transformation (14) and (16) via the product \(Q^e \omega_x\) in (11).

Even though [1] cites [5], it misses the close relationship between the results.

Acknowledgments

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