A new vision for the measure of circularity using a certain representation space

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Abstract. Detecting and characterizing families of plane circles is a fundamental concern in image processing. Many methods have been developed for these purposes, often on ad hoc bases, for example in the context of metrology. This paper is based on the consideration of circles as topological points in a certain 3D representation space (RS). Instead of the natural RS with \((a, b, R)\) coordinates, \((a, b)\) for the center and \(R\) for the radius of the circle, another RS, denoted \(\Sigma\), is studied, with the third coordinate changed from \(R\) to \(c = a^2 + b^2 - R^2\). \(\Sigma\) is connected with Voronoi closest et farthest point diagrams and their “lifted” version. Moreover, \(\Sigma\) possesses a canonical measure, introduced by Stoka. This paper is focused on families of circles \(S(P, Q)\) constrained to have two fixed sets of internal \((P_i)\) and external \((Q_j)\) points, representable as polyhedra in \(\Sigma\). The Stoka measure of these polyhedra is shown to be an adequate measure of circularity expressed by a simple formula. The power of this approach is shown on two applications: circularity assessment as used in metrology, and Fillmore formula. This approach has several ramifications, for example in Hyperbolic Geometry.

1. Introduction

Two branches of Mathematics that have been elaborated in the XXth century, Topology and Measure Theory will prove their usefulness for the study of circles in the plane.

Let us begin with topology, a discipline that can be described as the science of proximity. Topology considers a family of “objects” of a same kind, as points in a certain space, that we will call here a Representation Space (RS), each one with its own "vicinity systems".

The practical usefulness of a RS can be compared to the interest of a transform, like Laplace Transform for solving problems. The initial question is translated from the initial world into a certain RS, solved there, then transferred back into its equivalent in the initial world. A fundamental assumption is that the way–and–back transfers are done faithfully, i.e. in a bijective way.

A RS may, in some cases, be given a linear space structure; i.e. any two objects can be linearly combined and the result has to be meaningful as an object of the same family. Such is the case, for example, with functions, in \(L^2\) space, where the "least squares" quantity : \(\left(\int_{-\infty}^{\infty} |f(x) - g(x)|^2 \, dx\right)^{1/2}\) informs about the vicinity of \(f\) with respect to \(g\).

In many other instances, the situation is not linear; usually, closeness is defined by a distance or a metric. Among the most common metrics, one finds "attribute metrics"; a set of numerical attributes or signatures \((s_1, s_2, \ldots)\) are computed for a same object \(o\); distance from object \(O\) to object \(O'\) is then defined, for example, as a weighted sum \(\sum_k w_k |s_k - s'_k|\) or \((\sum_k w_k |s_k - s'_k|^2)^{1/2}\).
But, for geometrical objects, like plane circles on which we are going to work, this kind of distance, will have to comply for example with translation, rotation, zoom/antizoom effect, etc. This will dramatically reduce the number of possible metrics.

A valuable alternative to the distance approach, somewhat overlooked in the applications, is the use of a measure, in association with a certain RS. This is the approach we are going to use in this paper.

What is a measure? First of all, a measure deals with certain sets \( \Omega \) of objects instead of the objects \( \omega \) themselves, and attributes them a positive (or zero) value (intuitively, this value is a "size"). The main property of a measure is its additivity for disjoint sets of objects:

\[
\forall i, j \mu(\Omega_i \cap \Omega_j) = 0 \Rightarrow \mu(\Omega_1 \cup \Omega_2 \cup \ldots) = \mu(\Omega_1) + \mu(\Omega_2) + \ldots
\] (1)

A special category of measures are probability measures, generally representable as an integral
\[
\mu(A) = \int_A f(x) \, dx
\]
where integration domain \( A \) is an event (i.e. a set of elementary events) and \( f(x) \) is a probability density function. A specificity of these measures is that they are bounded: \( \mu(A) \leq 1 \) for all \( A \).

The measure we are going to define for circles will also be represented in this way, as an integral (see (9)), but without a boundness assumption. Thus, with this measure, only some particular sets of circles will be measurable, in particular with the meaning "having a finite measure".

The most obvious RS for circles uses \((a, b, R)\) coordinates where \((a, b)\) stands for the center and \(R\) for the radius of the generic circle. This RS can be useful, for example for Circular Hough Transform [9]. But, there is a much more interesting RS, defined by \((a, b, c)\) coordinates, where \(c = a^2 + b^2 - R^2\) (equation (3)). This paper deals with this RS, denoted \(\Sigma\).

In an initial part, this paper introduces the characterization of circles in the framework of metrology, to be re–visited later on.

Then, \(\Sigma\) space is presented with its connected Voronoi diagrams (the "closest" and the "farthest" version) associated with a finite set of points \(P_i\). A fundamental result is that these diagrams can be "unfolded" resp. into a convex (resp. concave) polyhedron, the "cup" (resp. the "cap") polyhedron, whose vertical projections give back the two kinds of diagrams.

Afterwards, the concept of "separating circles" is introduced. Being given two sets of points \(P = \{P_i\}\) and \(Q = \{Q_j\}\), a separating circle between \(P\) and \(Q\) (in this order) is a circle with all points \(P_i\) inside it and all points \(Q_j\) in its outside. The set \(S(P, Q)\) of such circles, is shown to be represented in \(\Sigma\) by a certain polyhedron. The measure \(\mu(P, Q)\) of this polyhedron provides, when returning to the geometrical plane, a measure of circularity for the given sets of points, with a 2D tractable expression for \(\mu(P, Q)\) (see (11)).

Two applications are given. The first one shows how space \(\Sigma\) gives a good understanding of the metrologists’ approach to circularity. The second one shows how more general volumes can be considered as limits of certain polyhedra, with the example of the Fillmore formula [4]-[7].

2. A certain aspect of the problem: circularity as a metrology issue

In the domain of mechanical engineering, palpation is a standard way to get accurate measurements under the form of coordinates. Concerning the metrology of circles, deviation from circularity is characterized by the width \(\Delta R\) of the thinnest covering annulus [1]; this width should be below a given threshold value for the piece to be accepted as "circular". Is this width an adequate circularity measure? One of the reasons for introducing a RS for circles is to get a clearer point of view on such an issue, and will be presented later on.
3. The $\Sigma$ representation space

3.1. Definition

Consider the following two forms for the equation of a circle with center $(a, b)$ and radius $R$

\[(x - \alpha)^2 + (y - \beta)^2 = R^2 \text{ and } x^2 + y^2 - 2ax - 2by + c = 0 \quad (2)\]

By identification, we get the important relationship

\[a^2 + b^2 - c = R^2 \quad (3)\]

By definition, $\Sigma$ is the RS with coordinates $(a, b, c)$ where $c$ is given by (3).

Due to the positivity of $a^2 + b^2 - c$ in formula 3, $\Sigma$ is an "ordinary" 3D space deprived of the interior of a paraboloid, denoted $\Pi$, with equation

\[c = a^2 + b^2\]

3.2. Embedding of the geometrical plane in $\Sigma$

A circle, being represented in $\Sigma$ in an abstract manner as a point $(a, b, c)$, looks rather disconnected from the "geometric" space with its "true" points; nevertheless, the geometric space can be "embedded" in different ways in $\Sigma$ for example by identifying the geometrical plane (coordinates $(a, b)$) with the horizontal plane in $\Sigma$ (coordinates $(a, b, 0)$).

A supplementary fact gives to this embedding a more natural aspect. Let us consider, in space $\Sigma$, the tangent cone issued from point $(a, b, c)$ to $\Pi$. It is rather easy to show that the set of contact points of the cone on $\Pi$ is an ellipse; once projected onto the $(a, b, 0)$ plane, it is a circle centered in $(a, b)$ with radius $R$ connected to $c$ by equation (3). Switching between the two points of view, i.e., a same situation considered in the geometrical space and in $\Sigma$ is often rewarding. Let us see it again. Considering only the geometrical plane, and $(\alpha, \beta)$ a fixed point in this space, a circle with equation $x^2 + y^2 - 2ax - 2by + c = 0$ passes through point $(\alpha, \beta)$ if and only if:

\[c = 2a\alpha + 2b\beta - (\alpha^2 + \beta^2) \quad (4)\]

Let us now switch to a $\Sigma$ space interpretation of (4): it is easy to check that it is the equation, in coordinates $(a, b, c)$, of tangent plane $T_{\alpha, \beta}$ to $\Pi$ at point $(a, b, c) = (\alpha, \beta, \alpha^2 + \beta^2)$. 
As an immediate step further, the set of circles that exclude (resp. include) point \( P_{a,b} \) is represented in \( \Sigma \) by the upper half space (resp. the lower half space) defined by this tangent plane \( T_{a,b} \).

More could be said about \( \Sigma \), especially about its rich algebraic and differential geometric structure, with duality playing an important role, but this would deserve much more space; see [2].

3.3. Voronoi diagrams and Lifted Voronoi diagrams

Let us now define two kinds of Voronoi tessellations associated with the same finite set \( P \) of points \( P_i \).

The Closest (resp. the Farthest) point Voronoi cell \( V_i \) attached to point \( P_i \) (resp. \( \Lambda_j \) attached to point \( P_j \)) is defined in the following way:

\[
V_i = \{ M \ | \ \forall k \neq i, d(M, P_i) < d(M, P_k) \} = \bigcap_{k \neq i} \{ M \ | \ d(M, P_i) < d(M, P_k) \} \tag{5}
\]

\[
\Lambda_j = \{ M \ | \ \forall k, d(M, P_j) > d(M, P_k) \} = \bigcup_{k \neq j} \{ M \ | \ d(M, P_j) > d(M, P_k) \} \tag{6}
\]

(where \( d(\ldots) \) stands for the Euclidean distance in the geometrical plane). These sets \( V_i \) and \( \Lambda_j \), often called “cells”, are convex, being intersection of half-planes defined by perpendicular bisectors of (certain) pairs \( (P_i, P_j) \). The set \( V \) (resp. \( \Lambda \)) of sets \( V_i \) (resp. \( \Lambda_j \)) is called the Closest (resp. The Farthest) Point Voronoi tessellation.

Let \( c = t_i(a,b) \) be the equation of the tangent plane associated with point \( P_i \).

Consider now a fixed point \((a_0,b_0)\) in the geometrical plane. Let us first recall that, in space \( \Sigma \), the set of circles centered in \((a_0, b_0)\) is clearly a vertical half line, beginning at point \((a_0, b_0, a_0^2 + b_0^2)\).

If the values \( t_i(a_0,b_0) \) are sorted in ascending order (of \( c \)), this will correspond, due to relationship (3) between \( c \) and \( R \), to a descending order of the radii. Thus \( \min_i(t_i(a_0,b_0)) \) (resp. \( \max_i(t_i(a,b)) \)) will correspond to a maximal radius (resp. a minimal radius). In the first case, maximal radius, all circles with coordinate \( c \) such that \( c \leq \min_i(t_i(a,b)) \) will include all points \( P_i \). For a completely similar reason, all circles with coordinate \( c \) with \( c \geq \max_i(t_i(a,b)) \) will exclude all points \( P_i \).
As a consequence, with these notations, the cap and cup polyhedra whose upper (resp. lower) boundaries are resp. defined by:

\[ c = \text{cap}(a, b) = \min_i(t_i(a, b)) \quad c = \text{cup}(a, b) = \max_j(t_j(a, b)) \]  

(7)

can be defined as well as the subset (considered in \( \Sigma \)) of circles containing all points \( P_i \) (resp. containing no point \( P_i \)).

These cup and cap polyhedra will be called "lifted" polyhedra.

Moreover, it is rather easy to check that the vertical projection of the edges of the cup (resp. cap) polyhedron will exactly gives back the \( V \) (resp. \( \Lambda \)) diagram of points \( P_i \). Indeed it is a consequence of the fact that an edge in the cup or the cap polyhedron is defined by a system of two plane equations of the form

\[ c = 2aa_i + 2bb_i - (a_i^2 + b_i^2) \text{ and } c = 2aa_j + 2bb_j - (a_j^2 + b_j^2) \]  

(8)

whose intersection gives, by elimination of \( c \), in the geometrical plane, the equation

\[(a - a_i)^2 + (b - b_i)^2 = (a - a_j)^2 + (b - b_j)^2\]

of the perpendicular bissector of line segment \( P_i P_j \). The maximal (resp minimal) property of height will make this part of perpendicular bissector a boundary of cells, either in \( V \) or in \( \Lambda \).

3.4. Re-visiting the metrologist’s method, using \( \Sigma \)

We are going to display the smallest annulus associated with the set of points \( P \) (metrologists’ method) in \( \Sigma \) representation space. Let us consider the cup \( V_P \) and the cap \( \Lambda_P \) polyhedra associated with set of points \( P \). Their vertical shortest distance is realized for (at least) one vertical line segment, joining, say, \((a, b, c_{\min})\) to \((a, b, c_{\max})\) (see Fig. 4). Using (3):

\[ |c_{\max} - c_{\min}| = R_{\max}^2 - R_{\min}^2 = (R_{\max} + R_{\min})(R_{\max} - R_{\min}) \]

\( R_{\max} + R_{\min} \) being almost a constant, i.e. twice the theoretical value \( R \) of the radius, we get the desired approximate proportionality between \( |c_{\max} - c_{\min}| \), i.e. the length of a minimal length vertical segment (joining the \( V \) and the \( \Lambda \)) and the width \( R_{\max} - R_{\min} = \Delta R \) of the minimal width annulus. The vertical line segment in Fig. 4 materializes the set of intermediate concentric circles between the smallest and the largest circle of the annulus, as displayed on Fig. 3.

This example has displayed one of the advantages of RS \( \Sigma \): the possibility to give a concrete representation and understanding of some complex phenomena regarding circles.

4. Points differentiation; Connection with the Voronois

4.1. Definition of \( S(P, Q) \) and \( K(P, Q) \)

Instead of a single set of non differentiated points as seen above, let us now consider two separate sets of points: a set \( P \) of interior points \( P_i \) and a set \( Q \) of exterior points \( Q_j \).

Circles containing every point \( P_i \) without including any \( Q_j \) constitute a central type of sets of circles, denoted \( S(P, Q) \). The set of the centers of circles in \( S(P, Q) \) is a subset of the geometric plane ; this set will be called the kernel associated with sets \( P \) and \( Q \) and will be denoted \( K(P, Q) \) or simply \( K \) when there is no ambiguity about the set of points (see [3]).

Remark: this kernel, when non empty, is a convex polygon with boundaries obtained as line segments, supported by perpendicular bissectors of certain points belonging to \( P \cup Q \) (see Fig. 5).

4.2. Connection between \( S(P, Q) \) and the Voronois in \( \Sigma \)

The family of circles \( S(P, Q) \) has clearly a simple 3D representation as a convex polyhedron, obtainable as the intersection of the two lifted convex polyhedra (see Fig. 6): \( S(P, Q) = \text{cap} \cap \text{cup} = \Delta(P, Q) \).
4.3. The Stoka measure

It is possible to build a similitude–invariant measure in space $\Sigma$. It has been found by Stoka [4]-[5]. We will restrict its use to polyhedron $\Delta = \Delta(P,Q)$. This measure is

$$\mu(P,Q) = \int \int \int_{\Delta} \frac{dadbdc}{R^4} \quad \text{with} \quad R^4 = (a^2 + b^2 - c)^2$$

5. A new formula

Let us define 2D cells by $V'(Q_i) = V(Q_i) \cap K$ and $\Lambda'(P_j) = \Lambda(P_j) \cap K$.

where $K$ is the kernel of the set of points $(P,Q)$. In fact, $V'(Q_i)$ (resp. $\Lambda'(P_j)$) are nothing but projections of the faces of the top (resp. bottom) of polyhedron $\Delta$.

Let us now define, for a given polygon $\omega$ in the geometric plane, and for a given point $M$ outside $\omega$:

$$I_{\omega}(M) = \int \int_{(a,b) \in \omega} \frac{dadb}{(x_M - a)^2 + (y_M - b)^2}$$

A remarkable result that we have obtained [9] is

$$\mu(P,Q) = \sum_i I_{\Lambda'(P_i)} - \sum_j I_{V'(Q_j)}$$

This formula is especially important because it expresses $\mu(P,Q)$ directly in the geometrical plane, without referring to $\Sigma$. 

Figure 5. Different Voronoi cells associated with two different set of inside $P = P'$ and outside $Q = P''$. The set of centers of circles of $S(P,Q)$ is the kernel polygon (ABCDEF, dashed thick line).

Figure 6. “Lifting” in space $\Sigma$ of Fig. 5. The interior of polyhedron $\Delta = ABCDEFGH$ is made of all circles including points $P_i$ and excluding points $Q_j$ with associated measure $\mu(P,Q)$. The stitching line (polygon ABCDEF) between top and bottom faces of polyhedron $\Delta$ has the kernel for its vertical projection.
5.1. An application

Let us fix an integer \( n > 2 \). Consider (see Fig. 7 for the case \( n = 8 \)) the following two sets of points: \( P \) with \( P_k(\cos(k\pi/n), \sin(k\pi/n)) \) \( k = 1, \ldots, n \) and \( Q \) the dilated set of \( P \) with a ratio 2 with \( O\overrightarrow{Q_j} = 2 \overrightarrow{OP}_j \). The set of circles that contain all the \( P_i \) and none of the \( Q_j \) is a somewhat discrete version of the set of circles of Fig. 2 among which can be the reference (theoretical) circle.

Let us compute the measure of the set \( S(P, Q) \). Due to the rotational invariance of the figure, it suffices to consider the triangle \( \omega \) with stripes on figure 7 and use it 8 times; formula (11) is thus reduced to

\[
\mu(S(P, Q)) = \mu(S(P, Q)) = 8(\mu(P_1, \omega) - \mu(Q_5, \omega)) = 0.174825\ldots
\]

If number \( n \) is taken larger, for example, for \( n = 16 \) (resp. \( n = 32 \)), one obtains \( \mu = 0.168404\ldots \) (resp. \( \mu = 0.166882\ldots \)), etc. This attests of a convergence. In fact, the measure of all circles like those drawn in Fig. 2 (including the smallest circle and included into the largest circle of the annulus) is obtainable by the following formula [4]:

\[
\mu = 2\pi \left( \ln(\rho) - 2\frac{\rho - 1}{\rho + 1} \right) \tag{12}
\]

where \( \rho = \frac{r_{\text{max}}}{r_{\text{min}}} \) with \( r_{\text{min}} \) and \( r_{\text{max}} \) the respective radii of the smallest and largest circle of the annulus.

Note that this formula is invariant with respect to a zoom effect.

Fillmore has given a sketchy proof of (12) in [4]. We have proven it in a more explicit way in our publication [7], and now, we have another way to prove it as the measure of a limit. We will not give an explicit proof by lack of place.

In the case considered in Fig. 7 and 8, considering two cones with a common basis instead of two regular pyramids gives the value \( \mu = 2\pi \left( \ln(2) - \frac{\pi}{2} \right) = 0.166382\ldots \)

![Figure 7. Some circles belonging to \( S(P, Q) \) have been drawn. Their centers are vertices points of the kernel, the smallest octogon. Notations \( P_i = P_i' \) and \( Q_j = P_j' \).](image)

![Figure 8. Fig. 7 considered in 3D: all the sides of the polyhedron are tangent to \( \Pi \) (not drawn) ; the vertical projection of the polyhedron is the kernel. This polyhedron, as above, is obtained as the intersection of the cap and the cup polyhedra. For values of \( n \) tending to \( \infty \), the limit figure is two cones whose Stoka measure is given by Fillmore formula (12).](image)
6. Conclusion

We have presented a partly new theory concerning plane circles. Starting with concrete circularity issues, we have shown the benefit of a treatment using the 3D representation space $\Sigma$. The innovative part of this work is in the connection that has been made between the Stoka measure $\mu$ and the projection of the faces of these polyhedra through a new and tractable formula (11).

The ability of this theory to provide alternative representations has been exemplified by a better understanding of two already developed formulas, the metrologist’s formula and Fillmore’s formula.

A whole part of the theory we are developing, and that has not been referred to here, is its connection with hyperbolic geometry. A brief account of this fact is given in article [9] and [10].

Our future work will mainly be on algorithmic study concerning formula (11).

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