Direct numerical simulation of anisotropic turbulent flow for incompressible fluid

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Abstract. Equation for small-scale velocity or pulsation is a starting point to build almost all models of turbulence. Transport equation for Reynolds stresses, dissipation rate, kinetic energy are derived from this equation, adding some assumptions about structure of terms, including in these equations. Equation for small-scale velocity can be simplified, if we assume that large-scale velocity and its gradients are constants instead of linear profile for large-scale velocity. We implemented the direct numerical simulation of this equation under the simple shear, leading to anisotropy. The nonlinear helicity terms were computed in spectral space, using the three-dimensional Fast Fourier transformation, then, the inverse Fast Fourier transformation was used to return in physical space. Aliasing terms were not removed. Four–order Runge-Kutta method was used for integration in time. Evolution of Reynolds stresses in time were computed.

1. Introduction
A basis for direct numerical simulation of fully developed turbulence was laid down in [1, 2]. Numerical solutions of Navier–Stokes equations for incompressible fluids with more powerful computers have been investigated in [3-5]. The pumping of energy in the isotropic flow was implemented by some random force with known statistical properties.

There is a more realistic picture, when energy from large-scale flow is transferred by an anisotropic shear term to small-scale eddies with a size less than the integral scale of turbulence. There are the results of the simplified models of anisotropic cascades [6] and the direct numerical simulation of anisotropic turbulence [7]. The prevailing assumption of the anisotopic turbulence simulation is a linear profile of large-scale or mean velocity [7]. A simplified form of the small-scale velocity transport equation, when large-scale velocity and its gradients were considered as constants, was proposed in [8]. The simplicity of this equation should lead to a better analytical analysis. Thus, the motivation of our simulation is to reveal unexplored properties of a new transport equation.

What is known about the simulation of small-scale velocity, when large-scale velocity has linear profile? The next system of differential equations was derived from Navier–Stokes equations [7]:

\[(\partial_t + x_2 \partial_1)u_1 + u_2 = H_1 + \partial_1 p = \omega \Delta u_1,\]

\[(\partial_t + x_2 \partial_1)u_2 - H_2 + \partial_2 p = \omega \Delta u_2,\]
\[(\partial_x + x_2 \partial_y)u_3 - H_3 + \partial_z p = \varepsilon \Delta u_3,\]
\[\hat{\partial}_j u_j = 0,\]

where \(H_i = -u_i \partial_j u_j\) - nonlinear terms, \(p\) – kinematics pressure, lower index corresponds to three spatial directions, the summing over repeating indexes is assumed. The equations are transformed to dimensionless form by means of the velocity tensor – gradient invariant \(S\) and the length \(L\), which is equal to integral scale. Large-scale velocity is \(U = (x_2, 0, 0)\) in dimensionless variables. Flow depends on the Rossby number, \(Ro = \frac{u}{SL}\),

where \(u\) is the small-scale velocity scale and Rossby number

\[\varepsilon = \frac{v}{SL^2},\]

where \(v\) is viscosity, \(Re = \frac{1}{\varepsilon}\) is Reynolds number.

Unfortunately, results were given only for two-dimensional flow. In paper [9], the system of ordinary differential equation for the velocity Fourier components was solved at the grid with \(512 \times 512 \times 512\) points under the action of simple shear. The integration in time was carried out up to \(St = 2\).

An agreement with Kolmogorov energy spectrum \((\propto k^{-5/3})\) and Lumley spectrum \((\propto k^{-7/3})\) for off-diagonal components \(E_{ij}\) \((i \neq j)\) was achieved. Recently, an experimental paper was published with a measurement of all components \(E_{ij}\) [10], a comparison with the dns results is not available.

This system of differential equation was numerically solved also in [11], aperiodic oscillations of energy and Reynolds’s stress were revealed. An anisotropy, caused by the gradients of mean velocity, was investigated in [12].

Velocity \(\mathbf{V}\) was represented as a sun of small-scale velocity \(\mathbf{u}\) and large-scale velocity \(\mathbf{U}\):

\[
\mathbf{V} = \mathbf{u} + \mathbf{U}.
\]

We obtain the next equation from Navier-Stokes equation for incompressible fluid:

\[
\hat{\partial}_i u_a + u_i \hat{\partial}_j u_a + U_j \hat{\partial}_j u_a + u_i \hat{\partial}_j U_a = -\hat{\partial}_a p + \mathbf{v} \Delta u_a
\]

Authors of the paper [12] considered shear \(\hat{\partial}_j U_a\) as anisotropic random forces:

\[
\mathbf{f} = (0, 0, f_z(x)),
\]

where \(f_z(x) = F_1 \cos(2\pi x / L_x + \phi_1(t)) + F_2 \cos(2\pi x / L_x + \phi_2(t))\), \(F_1\) and \(F_2\) - are constants, random fazes \(\phi_1(t)\) and \(\phi_2(t)\) are mutually independent, homogeneous distributed, \(\hat{\partial}\) - correlated in time. We can transit to the coordinate system, moving with local velocity \(\mathbf{U}\), and nullify corresponding term in the transport equation. The inverse transformation in the initial coordinate system, being at rest, is given by Galileo transformation [13]:

\[
u_i(k, t) = \exp[-ik_j U_j(t - t_0)]u_i(k, t)]
\]
The results of simulation are in agreement with Kolmogorov’s and Lumley’s spectra.

2. Transport equation for polarization components of small-scale velocity

Using the approach [3], let us transform the small-scale velocity \( \mathbf{v}(x, t) = (v_1, v_2, v_3) \) in Fourier series in spatial variables:

\[
\mathbf{v}(x, t) = \sum_k \mathbf{v}_k(t) \exp(i\mathbf{k} \cdot \mathbf{x}) ,
\]

where \( i = \sqrt{-1} \), \( \mathbf{k} = (2\pi / L) \mathbf{n} \) is the wave vector, \( \mathbf{n} = (n_1, n_2, n_3) \) is a vector with the integer components, here \( L \) is the integral scale of turbulence. We should consider \( \mathbf{n} \) in numerical simulation as finite values:

\[
-\frac{N}{2} \leq n_i < \frac{N}{2}, \quad -(2\pi / L) \frac{N}{2} \leq k_i < (2\pi / L) \frac{N}{2} ,
\]

where \( i = 1,2,3 \) and \( N \) is a sufficient large number, being a positive integer power of 2 for using Fast Fourier Transform. These inequality can be rewrote as \( |\mathbf{n}| < \frac{N}{2} \equiv K \) or \( |\mathbf{k}| < k_{\text{max}} \equiv (2\pi / L)K \).

We seek the solution as:

\[
\mathbf{v}(x, t) = \sum_{|\mathbf{k}| < k_{\text{max}}} \mathbf{v}_k(t) \exp(i\mathbf{k} \cdot \mathbf{x}) .
\]

Let us define the collocation points as the following ones:

\[
x_j = \frac{L}{N}(j_1, j_2, j_3), \quad |\mathbf{j}| < N / 2 .
\]

Then, an approximate solution (3) of the small-scale transport equation in the collocation points shall be:

\[
\mathbf{v}(\mathbf{x}_j, t) = \sum_{|\mathbf{k}| < k_{\text{max}}} \mathbf{v}_k(t) \exp(i\mathbf{k} \cdot \mathbf{x}_j) .
\]

The inverse Fourier transform with finite number of terms shall be:

\[
\mathbf{v}_k(t) = \frac{1}{N^3} \sum_{|\mathbf{j}| < K} \mathbf{v}(\mathbf{x}_j, t) \exp(-i\mathbf{k} \cdot \mathbf{x}_j)
\]

Due to the wave number cutoff, the nonlinear terms of the transport equation will have the wave numbers out of the range (2), it is so-called aliasing error. Since the small-scale velocity \( \mathbf{v}(\mathbf{x}, t) \) has the real components, there is an equality:

\[
\mathbf{v}_k = \mathbf{v}^*_k
\]

Vorticity is defined as

\[
\omega = \nabla \times \mathbf{v}(\mathbf{x}, t) ,
\]

kinetic energy of small-scale eddies is equal to:

\[
E = \frac{1}{2} \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) ,
\]

subgrid Reynolds tensor is defined as:

\[
R_{ij} = v_i(\mathbf{x}, t)v_j(\mathbf{x}, t)
\]
Vorticity $\omega$ has an expansion into discrete Fourier series as:

$$\omega(x_j, t) = \sum_k i\mathbf{k} \times \mathbf{v}_k \exp(i\mathbf{k} \cdot x_j)$$ (11)

Kinetic energy (9) and subgrid Reynolds tensor (10) have the similar expansion into Fourier series. Combining the results of [3,8], transport equations for two polarization Fourier components of the small-scale velocity will be for anisotropic turbulence as:

$$\partial_j u_1 + i U_j k_j u_1 = a_{1\mu} u_\mu - \frac{\epsilon^1 \cdot [k \times (k \times s_k)]}{k^2} - v_1 k^2 u_1$$ (12)

$$\partial_j u_2 + i U_j k_j u_2 = a_{2\mu} u_\mu - \frac{\epsilon^2 \cdot [k \times (k \times s_k)]}{k^2} - v_2 k^2 u_2$$ (13)

where summation over repeating indices is assumed $(j; j = 1,2,3)$, $(\mu; \mu = 1,2)$, $u_1(k,t), u_2(k,t)$ are two polarization components of the small-scale velocity:

$$\mathbf{v}(k,t) = u_e \mathbf{e}^1 + u_2 \mathbf{e}^2; \quad u_1 = \mathbf{v} \cdot \mathbf{e}^1, u_2 = \mathbf{v} \cdot \mathbf{e}^2,$$ (14)

directing along the unit polarization vectors $\mathbf{e}^1$ and $\mathbf{e}^2$, these are orthogonal to each other and to vector $k(k_1, k_2, k_3)$, its components vary from $k_{\text{max}} = 2\pi / L$ (integral scale of turbulence, separating large-scale motion and small-scale eddies) up to some $k_{\text{min}}$ (a theoretical value $k_{\text{min}} = \infty$ and $k_{\text{min}}$ in computations is limited by the computer capacity), $i = \sqrt{−1}$, helicity $\mathbf{s} = \mathbf{u} \times \omega$, $\omega = \nabla \times \mathbf{u}$ is vorticity, $\mathbf{U}$ is large-scale velocity, moving coordinate system $\mathbf{U} = 0$, four coefficients are defined as $a_{\mu\nu} = -\epsilon_{\mu\nu} \epsilon_{\eta\nu} \nabla \mathbf{e}^\mu / \mathbf{e}^\nu$ and $\mathbf{s} = \nabla \times \mathbf{u}$ - molecular viscosity. An option of unite vectors $\mathbf{e}^1$ and $\mathbf{e}^2$ is sufficiently arbitrary.

The components of strain rate tensor are defined as:

$$S_{ij} = \frac{1}{2} \left( \partial_i U_j + \partial_j U_i \right),$$

one of its invariant $S = \sqrt{2S_{ij}S_{ji}}$.

Let us define the unit vector $\mathbf{e}$, as:

$$\mathbf{e} = (\sqrt{2}, \pi, 0) \cdot \frac{1}{\sqrt{2 + \pi^2}}.$$ (15)

After that, we define the unit vectors $\mathbf{e}^1$ and $\mathbf{e}^2$:

$$\mathbf{e}^1 = \frac{\mathbf{e} \times \mathbf{k}}{|\mathbf{e} \times \mathbf{k}|}.$$ (16)

(vector $\mathbf{e}$ in (15) are chosen as nonparallel to vector $\mathbf{k}$, given in three-dimensional rectangular grid, used for numerical simulation)

$$\mathbf{e}^2 = \frac{\mathbf{k} \times \mathbf{e}^1}{|\mathbf{k} \times \mathbf{e}^1|} = \frac{\mathbf{k} \times \mathbf{e}^1}{k}.$$ (17)
It is clear that the denominator in (16) and (17) are not zeros. We have a relation, following from (11), that \( \omega(k, t) = \mathbf{i}k \times \mathbf{v}(k, t) \). Also, we obtain from formulas (14)-(17), that:

\[
\omega(k, t) = i k (u_i e^2 - u_2 e^1) \tag{18}
\]

We have the only nonzero component of velocity tensor- gradient for simple shear:

\[
\partial_i U_2 = S \neq 0
\]

It can be easily obtained for this case: \( a_{11} = -e_1^2 e_1^1 S \), \( a_{12} = -e_2^1 e_1^1 S \), \( a_{21} = -e_2^2 e_1^1 S \), \( a_{22} = -e_2^2 e_2^2 S \).

The system of equations (12), (13) turns into the next system:

\[
\partial_i u_1 = -e_1^2 e_1^1 S u_1 - e_2^1 e_1^2 S u_2 - \frac{e^1 \cdot [\mathbf{k} \times (\mathbf{k} \times S_k)]}{k^2} - \nu k^2 u_1 \tag{19}
\]

\[
\partial_i u_2 = -e_1^2 e_2^1 S u_1 - e_2^2 e_1^2 S u_2 - \frac{e^2 \cdot [\mathbf{k} \times (\mathbf{k} \times S_k)]}{k^2} - \nu k^2 u_2 \tag{20}
\]

Using well-known formula for double vector product: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \), where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) - arbitrary vectors.

We deduce: \( \mathbf{k} \times (\mathbf{k} \times S_k) = \mathbf{k}(\mathbf{k} \cdot S_k) - S_k k^2 \).

In result, we obtain the next system of equations from the system (19) and (20):

\[
\partial_i u_1 = -e_1^2 e_1^1 S u_1 - e_2^1 e_1^2 S u_2 - \frac{(e^1 \cdot \mathbf{k})(\mathbf{k} \cdot S_k) - k^2 (e^1 \cdot S_k)}{k^2} - \nu k^2 u_1 \tag{21}
\]

\[
\partial_i u_2 = -e_1^2 e_2^1 S u_1 - e_2^2 e_1^2 S u_2 - \frac{(e^2 \cdot \mathbf{k})(\mathbf{k} \cdot S_k) - k^2 (e^2 \cdot S_k)}{k^2} - \nu k^2 u_2 \tag{22}
\]

Using expressions (3)-(5), we can write down the explicit expressions for the unit vectors in Cartesian coordinate system:

\[
e^1 = \frac{e_2 k_3 - e_3 k_2}{\sqrt{(e_2 k_3 - e_3 k_2)^2 + (e_1 k_3 - e_3 k_1)^2 + (e_1 k_2 - e_2 k_1)^2}} \tag{23}
\]

and, taking into account (15), we have:

\[
e^1 = \frac{e_2 k_3 - e_3 k_2}{\sqrt{k_3^2 + (e_1 k_2 - e_2 k_1)^2}} \tag{24}
\]

for unit vector \( e^2 \), taking into account the mutual orthogonality vector \( e^1 \) and vector \( \mathbf{k} \), we obtain the relation:

\[
e^2 = \frac{(k e_3^1 - k e_2^1) i_1 - (k e_1^1 - k e_2^1) i_2 + (k e_1^1 - k e_2^1) i_3}{k} \tag{25}
\]

Also, taking into account orthogonality of \( e_1 \) and \( e_2 \) to vector \( \mathbf{k} : e_1 \cdot \mathbf{k} = e_2 \cdot \mathbf{k} = 0 \), system of equation (21), (22) simplifies:

\[
\partial_i u_1 = -e_1^2 e_1^1 S u_1 - e_2^1 e_1^2 S u_2 + e^1 \cdot S_k - \nu k^2 u_1 \tag{26}
\]
\[ \partial_t u_2 = -e_2^2 e_1^1 S u_1 - e_2^2 e_1^2 S u_2 + e^2 \cdot S_k - \nu k^2 u_2 \]  
(27)

Thus, for given vector \( \mathbf{k}(k_1, k_2, k_3) \) we calculate vector \( e^1 \), using (16), and vector \( e^2 \), using (17).

Let us make a transition to dimensionless variables, using time scale \((1/S) \) and length scale \((L)\):

\[ tS \rightarrow t, \quad kL \rightarrow k, \quad u_{j\mu} / (LS) \rightarrow u_{j\mu}, \quad S_k / (L S^2) \rightarrow S_k, \quad \nu \rightarrow 1/ \text{Re}_s. \]

where \( \text{Re}_s = \frac{S L^2}{\nu} \) is subgrid Reynolds number, the only parameter in the system.

The system equation (26) and (27) in dimensionless variables becomes as the following:

\[ \partial_t u_1 = -e_2^1 e_1^1 u_1 - e_2^1 e_1^2 u_2 + e^1 \cdot S_k - \frac{1}{\text{Re}_s} k^2 u_1 \]  
(28)

\[ \partial_t u_2 = -e_2^2 e_1^1 u_1 - e_2^2 e_1^2 u_2 + e^2 \cdot S_k - \frac{1}{\text{Re}_s} k^2 u_2 \]  
(29)

It seems that we obtained the simplest description of anisotropic turbulence here. All nonlinearity is in Fourier components of velocity: \( \mathbf{S} = \mathbf{v} \times \omega \).

3. Choice of equation’s parameters

Parameter \( L \) is a spatial scale, dividing large and small fluid motion, \( k_0 = 2\pi / L ; L = 1 \) – expressed in dimensionless variables. The dissipation rate \( \varepsilon \) is part of Kolmogorov’s energy spectrum \( E(k) \):

\[ E(k) = C_k \varepsilon^{2/3} k^{-5/3}, \]  
(30)

where \( C_k \approx 1.5 \) [14].

The formula (30) is used for setting up the initial conditions for small – scale polarization components of velocity \( \varepsilon = 1 \) as dimensionless variable:

\[ <|u_1|^2 + |u_2|^2> = E(k) / (4\pi k^2) \]

A computational grid in space of the wave numbers \( \mathbf{k} \) is defined as:

\( (k_{\text{max}}/k_0) \times (k_{\text{max}}/k_0) \times (k_{\text{max}}/k_0) \), where the choice of \( k_{\text{max}} \) is limited of our computer capacity.

The fast Fourier transform is usually used for \( (k_{\text{max}}/k_0) = 2^n \).

The setting up of initial condition can be defined, following to [15] as:

\[ u_a(\mathbf{k},0) = (\delta_{a\gamma} - k_a k_\gamma / k^2) r_\gamma(\mathbf{k},0), \]

or \( u_1 = \pm u_2, u_2 = \pm \frac{1}{2} E(k) / (4\pi k^2), \)

where \( r_\gamma(\mathbf{k},0) \) - statistically independent random variables with Gaussian distribution with zero mean and dispersion, being proportional to energy spectrum (30).

Following to [3], to compute \( S_k \) and \( E_k \) (which are Fourier Images of \( u \times \omega(x_j) \) and \( \frac{1}{2} u \cdot u(x_j) \) respectively) we compute Fourier vorticity: \( \alpha_k = i k \times u_k \), which transforms in physical space to obtain \( u(x_j) \) and \( \omega(x_j) \). Then, we compute variables \( S(x_j) = u(x_j) \times \omega(x_j) \) and \( E(x_j) = \frac{1}{2} u(x_j) \cdot u(x_j) \)
At last, we come back in wave-numbers space with help of Inverse Fourier transform. The Fast three dimensional Fourier Transform is used. The integration in time is implemented with help of the explicit 4-th order Runge-Kutta method.

4. The computational results and discussion
Here, we give preliminary results of dns for anisotropic incompressible turbulence, using a new simplified transport equation for the small-scale velocity [8]. We used 8x8x8 grid of the wave numbers. Subgrid Reynolds number was $Re_s = 10$. The integration in time was stopped at $St = 3$. The averaging over the random initial conditions is absent. We give a typical evolution of the polarization Fourier velocity components.

![Figure 1. Evolution of the polarization Fourier velocity components in time; $+ U_1 = \sum u_1^2$; $o U_2 = \sum u_2^2$; $St$ - dimensionless time.](image)

We see from figure 1 a tendency to isotropy. Also, we can describe the process as self-sustained one.

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