Two-layered electroosmotic flow through a vertical microchannel with fractional Cattaneo heat flux

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1. Introduction

The processes of microfluidic transport have gotten a lot of attention in the last decade because of the rapid development of microfluidic devices like biochemical and biomedical instruments, micro-electro-mechanical systems (MEMS), drug delivery biochips and chemical separation devices [1–4].

The pressure-driven process has traditionally been readily used at the microscale mechanical pumping for numerous engineering applications [5–7]. However, as the length scale of microfluidic devices has shrunk, some unfavourable characteristics have emerged in the microfluidic flow only driven by the pressure actuation mechanism [8], like biomedical and biochemical sample dispersion, the loss of energy due to friction. In light of these disadvantages, some feasible and efficient flow actuation methods should be supplied as the preferable source of fluid flow in microfluidic devices.

The electroosmosis actuation mechanism is an efficient mechanism to overcome the above disadvantages. It is frequently used in microfluidic systems, owing to its relative benefits like ease of design, absence of moving parts, and efficient reconfigurability with electrical circuitry. As a result, the electroosmotic force may be used as a more effective actuation mechanism than the traditional pressure-driven method [9, 10].

Many investigations on electroosmotic flows through a microchannel/microtube have been studied for both Newtonian and non-Newtonian fluids [11–21]. Several works have been interested in studying the electroosmotic flow of two or more immiscible fluids. Torres and Escándon [22] introduced a mathematical model to study the electroosmotic and pressure-driven flow with multi-layer immiscible fluids. Moreover, they got the exact solution for the velocity distribution. Escándon et al. [23] studied the transport of multilayer immiscible Phan–Thien–Tanner fluids into a slit microchannel by electroosmotic and pressure-driven effects. In a slit microchannel, the transient electroosmotic flow of multi-layer immiscible Maxwell fluids is investigated by Escándon et al. [24]. Haiwang et al. [25] studied the problem of three immiscible layers one of them non-conducting, which is delivered by the combined interfacial viscous force of two conducting fluids and pressure gradient. Huang et al. [26] introduced a mathematical model to study the electroosmotic flow of two fluid layers. One of them is conducting non-Newtonian fluid driven by electroosmotic force and the other is non-conducting Newtonian layer driven by interface shear. In a circular microchannel, the problem of the electroosmotic flow of a fractional Oldroyd-B fluid has been investigated by Jiang et al. [27]. The generation of the entropy of MHD electroosmotic flow in two immiscible fluids was discussed by Xie and Jian [28]. They showed that the viscoelastic parameter can diminish the local rate of entropy generation. Abdellateef et al. [29] discussed the EOF of fractional second-grade fluid through a vertical microchannel. They found the solutions with add of the Laplace and finite Fourier sine transforms and their numerical inverses. The study of 3D Jeffrey fluid flow over a stretching surface with the effect of the Hall current and thermal radiation has been investigated by Sinha et al. [30].

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ABSTRACT
The unsteady electroosmotic flow (EOF) of two layers of immiscible fluids through a vertical microchannel with heat transfer will be investigated. The channel is subjected to an alternating current (AC) electric field. The channel has two fluid layers: layer (I) is filled with Newtonian fluid, while layer (II) is filled with fractional Jeffreys fluid. The fractional derivatives will be replaced by the well-known Caputo–Fabrizio fractional derivatives. The Laplace and finite Fourier sine transforms and their numerical inverses have been used to get the semi-analytical solutions for the current problem. It has been found that the electroosmosis velocity and the flow rate are higher in the Newtonian layer compared with the non-Newtonian layer. Moreover, the heat transfer delays by elevating the value of the heat relaxation time in the two layers.
In nature, flows of immiscible fluids in the pipe or channel are widespread. Due to its numerous applications in modern industry, petroleum engineering, geophysics, medicine and hydrogeology, the study of simultaneous flow of two or more immiscible fluids is very important [31]. The constitution of two fluids determines their immiscibility, which is caused by high cohesive forces between their molecules. Surface tension is an experimentally established parameter that expresses the ease with which fluids can be combined. The higher this coefficient’s value, the greater the resistance to be blended. Due to the importance of this class of flows, many investigations have been discussed [32–34]. Recently, the interest in studying the motion of immiscible fluids has been increased. Abd Elmaboud [35] discussed the flow of two layers of immiscible fluids in a vertical semi-corrugated channel. After that Abd Elmaboud et al. [36] introduced a mathematical model to discuss the electromagnetic flow for two-layer of immiscible fluids. They used the homotopy analysis method to get their solutions. Recently, Chen and Jian [37] investigated the entropy generation for two layers of Newtonian immiscible fluids. They used the method of single variable optimization to minimize the entropy generation of the thermal flow system.

Partial differential equations (PDEs) have become an important tool for studying the nonlinear physical problems of numerous models that develop in these fields [38–41]. It is worth noting that differential and integral operators of integer order are used to represent many complicated systems in chemistry, mechanics, and biology. In recent years, scientists have notice that complicated systems are better to characterize by fractional differential equations. In the literature, there are many fractional differential operators used in heat/mass transfer and viscoelasticity problems such as Caputo fractional derivative [42], Caputo–Fabrizio fractional derivative [43], Riemann–Liouville fractional derivative [44] and Atagnana–Baleanu fractional derivative [45]. There are many authors used the concept of fractional derivative in their models. Rauf et al. [46] have investigated the electroosmotic flows of multi-layer immiscible fractional Maxwell fluid using the concept of Caputo derivative. Shah and Khan [47] have discussed the effect of fractional derivative approach to the thermal analysis of a fractional non-Newtonian fluid over an infinite oscillating vertical flat plate using fractional Caputo–Fabrizio derivatives. A comparison between Caputo and Caputo–Fabrizio fractional derivatives of fractional second-grade fluid have been investigated by Asjad et al. [48]. Several investigations have been formulated to discuss fractional derivatives operators as a result of the growing interest in modelling with fractional derivatives [49–51] and the references therein.

In the current study, the unsteady EOF of two layers of immiscible fluids through a vertical microchannel with heat transfer will be investigated. The channel is subjected to an alternating current (AC) electric field. The channel has two fluid layers: layer (I) is filled with Newtonian fluid, while layer (II) is filled with fractional Jeffreys fluid (or fractional Oldroyd-B fluid). The fractional derivatives will be replaced by the well-known Caputo–Fabrizio fractional derivatives. The approximation methods will be used to acquire the solutions for the current problem owing to the difficulty of the governing equations. The Laplace and finite Fourier sine transforms and their numerical inverses will be used to get the semi-analytical solutions for the current problem. Finally, we shall see the influence for various values of the problem parameters on the velocity and temperature fields.

2. Statement of the problem

Consider the unsteady EOF of two layers of immiscible fluids through a vertical microchannel with width $H$ and the microchannel is exposed to an alternating current (AC). layer (I) (Region (I) $[0, H]$, $H$ is the width of Region (II)) is filled with Newtonian fluid, while layer (II) (Region (II) $[0, \delta']$) is filled with fractional Jeffreys fluid. The fluids move as a result of the combination of pressure gradient in the $x$-direction, AC electric field $E_x(t')$ and the natural convection force. The channel walls $y' = 0$ and $y' = H'$ are kept at the constant temperatures $T_0$ and $T_1$, respectively. Figure 1 depicts the flow geometry and Cartesian coordinate system. The fluid velocity is considered to be in the form $\mathbf{q}' = (w'(y', t'), 0, 0)$ hence the continuity equation is identically satisfied. The governing equations for the two regions are as follows:

$$\rho_1 \frac{\partial w^{(I)}(t)}{\partial t} = -\frac{\partial p'}{\partial x'} + \frac{\partial \tau^{(I)}_{yx}}{\partial y'} + \phi' E_x(t') + \rho_1 g \beta_1 (T^{(I)} - T_0),$$  \hspace{1cm} (1)

where (I) stands for regions (I) and (II), respectively. The nonzero shear stress can be described by Newtonian fluid for region (I):

$$\tau^{(I)}_{yx} = \mu_1 \frac{\partial w^{(I)}(t)}{\partial y'},$$ \hspace{1cm} (2)

and by fractional Jeffreys fluid for region (II) [27]:

$$\left[1 + \lambda_1^{(II)} \frac{\partial^\alpha}{\partial t^\alpha} \right] \tau^{(II)}_{yx} = \mu_2 \left[1 + \lambda_2^{(II)} \frac{\partial^\beta}{\partial t^\beta} \right] \frac{\partial w^{(II)}(t)}{\partial y'},$$ \hspace{1cm} (3)

where $\lambda_1$ is the relaxation time, $\lambda_2$ is the retardation time, $\alpha$ & $\beta$ are the fractional-orders that have a range $0 \leq \alpha \leq \beta \leq 1$, $\mu$ is the viscosity. Substituting from Equations (2) and (3) in Equation (1) then the governing equations can be expressed as [26, 28, 29]:

Region (I)

$$\rho_1 \frac{\partial w^{(I)}(t)}{\partial t} = -\frac{\partial p'}{\partial x'} + \mu_1 \frac{\partial^2 w^{(I)}(t)}{\partial y'^2} + \phi' E_x(t')$$

$$+ \rho_1 g \beta_1 (T^{(I)} - T_0),$$ \hspace{1cm} (4)
where the energy equation is modified by the fractional source,

\[
\rho_2 \left[ 1 + \lambda_2^{\alpha} \frac{\partial \psi}{\partial t} \right] \frac{\partial T^{(II)}}{\partial t} = - \frac{\partial \psi'}{\partial t'} + \mu_2 \left[ 1 + \lambda_2^{\beta} \frac{\partial \theta}{\partial t} \right] \frac{\partial^2 T^{(II)}}{\partial y'^2} + \varepsilon_0^{(II)} \left[ 1 + \lambda_1^{\alpha} \frac{\partial \psi}{\partial t} \right] E(t') + \rho_2 \gamma_2 \left[ 1 + \lambda_2^{\alpha} \frac{\partial \psi}{\partial t} \right] (T^{(II)} - T_0), \tag{5}
\]

The energy equation

\[
\rho C_p \left[ 1 + \lambda_3^{\gamma} \frac{\partial \theta}{\partial t} \right] \frac{\partial T'(t)}{\partial t} = k_1 \frac{\partial^2 T'(t)}{\partial y'^2} + Q_0 \left[ 1 + \lambda_3^{\gamma} \frac{\partial \theta}{\partial t} \right] (T'(t) - T_0), \tag{6}
\]

where the energy equation is modified by the fractional Cattaneo model [29, 52] and \((t)\) stands for (I) and (II) layers, \(\rho\) is the fluid density, \(\beta_1\) and \(\beta_2\) is the coefficient of thermal expansion for (I) and (II) layers, respectively, \(T'\) is the fluid temperature, \(C_p\) specific heat, \(k\) is the thermal conductivity, \(\lambda_3^{\gamma}\) is the heat relaxation time, \(Q_0\) heat source, \(E_0(t') = E_0 \sin \Omega t\), \(E_0\) is the amplitudes of AC electric field, \(\Omega\) the angular frequency. The relation between EDL potential \(\psi'\) and the linearized charge density \(\rho_e^{(I)}\) can be expressed by the Poisson equation:

\[
\nabla^2 \psi^{(I)} = - \frac{\rho_e^{(I)}}{\varepsilon} \tag{7}
\]

After using the Debye–Hückel approximation, Equation (7) becomes

\[
\nabla^2 \psi^{(I)} = \kappa^2 \psi^{(I)} \tag{8}
\]

where \(\kappa^2 = \frac{2 \varepsilon \alpha_m e^2}{\varepsilon_0} \), where \(\varepsilon\) is the dielectric permittivity, \(\alpha\) is the elementary charge, \(z\) is the valence of electrolyte, \(n_0\) is the ionic number concentration, \(B_i\) is the Boltzmann constant. Consider the initial, interface and boundary conditions in following form:

\[
\psi^{(I)} = \xi_1, \quad w^{(I)} = 0, \quad T^{(I)} = T_0, \quad \text{at} \ t' = 0,
\]

\[
\psi^{(II)} = \xi_2, \quad w^{(II)} = 0, \quad T^{(II)} = T_1, \quad \text{at} \ y' = H,
\]

\[
\psi^{(II)} = \xi_1, \quad w^{(II)} = 0, \quad T^{(II)} = T_0, \quad \text{at} \ y' = 0,
\]

\[
\left\{ \begin{array}{l}
\psi^{(I)} = \psi^{(II)}, \quad w^{(I)} = w^{(II)}, \quad T^{(I)} = T^{(II)}, \quad \psi^{(I)} = \psi^{(II)} \\
\frac{d\psi^{(I)}}{dy''} = \frac{d\psi^{(II)}}{dy''}, \quad \frac{d\theta^{(I)}}{dy''} = \frac{d\theta^{(II)}}{dy''}
\end{array} \right. \tag{9}
\]

where the liquid–liquid interface conditions included a velocity and temperature continuity, a potential continuity, a stresses balance, Gauss’s law for the electric displacement and the heat flux balance. Introducing the non-dimensional variables as follows:

\[
y' = \frac{y'}{H}, \quad \delta = \frac{\delta'}{H}, \quad (t, \lambda_1, \lambda_2, \lambda_3)
\]

\[
\frac{w'}{w_0}, \quad F = - \frac{H^2}{\mu_1 w_0} \frac{\partial \psi'}{\partial x'}, \quad \kappa = H \kappa', \quad \psi^{(I)} = \psi^{(II)}, \quad \frac{\Omega}{\mu_1}
\]

\[
\rho_e^{(I)} = \frac{H^2 \rho_e^{(II)}}{\xi_1 \varepsilon}, \quad \frac{\partial \psi^{(I)}}{\partial t'} = \frac{\partial \theta^{(II)}}{\partial t'} \tag{10}
\]

To continue, we non-dimensionalize Equations (4)–(6), & (8)–(9), this yields:

Region (I)

\[
\frac{\partial w^{(I)}}{\partial t} = F + \frac{\partial^2 w^{(I)}}{\partial y'^2} + \kappa^2 \psi^{(I)} \sin(\Omega t) + G \psi^{(I)} \tag{11}
\]

\[
Pr \left[ 1 + \lambda_3^{\gamma} \frac{\partial \psi}{\partial t} \right] \frac{\partial \theta^{(I)}}{\partial t} = \frac{\partial^2 \theta^{(I)}}{\partial y'^2} + Q_1 \left[ 1 + \lambda_3^{\gamma} \frac{\partial \psi}{\partial t} \right] \theta^{(I)} \tag{12}
\]

Region (II)

\[
\rho_2 \left[ 1 + \lambda_1^{\alpha} \frac{\partial \psi}{\partial t} \right] \frac{\partial w^{(II)}}{\partial t} = \mu_2 \left[ 1 + \lambda_2^{\beta} \frac{\partial \theta}{\partial t} \right] \frac{\partial^2 w^{(II)}}{\partial y'^2} + \frac{1 + \lambda_2^{\beta} \frac{\partial \psi}{\partial t}}{\partial t} \kappa^2 \psi^{(II)} \sin(\Omega t)
\]

\[
+ \rho_2 \gamma_2 \left[ 1 + \lambda_2^{\alpha} \frac{\partial \psi}{\partial t} \right] \psi^{(II)} \tag{13}
\]

\[
\rho_2 \left[ 1 + \lambda_3^{\gamma} \frac{\partial \psi}{\partial t} \right] \frac{\partial \theta^{(II)}}{\partial t} = \frac{\partial^2 \theta^{(II)}}{\partial t'} + Q_1 \left[ 1 + \lambda_3^{\gamma} \frac{\partial \psi}{\partial t} \right] \theta^{(II)}, \tag{14}
\]

Figure 1. Geometry of the problem.
the non-dimensional equation of the EDL potential \( \psi \) becomes:

\[
\frac{d^2 \psi}{dy^2} = \kappa^2 \psi, \tag{15}
\]

the non-dimensional initial, interface and boundary conditions will be:

\[
w^{(0)}(t) = \tilde{w}^{(0)} = \theta^{(0)} = 0, \quad \text{at} \quad t = 0,
\]

\[
\psi^{(0)}(y, t) = \tilde{\psi}^{(0)} = \psi^{(1)} = 1, \quad \text{at} \quad y = 1,
\]

\[
\psi^{(0)} = \psi^{(1)} = 0, \quad \text{at} \quad y = 0,
\]

\[
\left\{ 1 + \lambda^2 \frac{\partial^2 \psi}{\partial y^2} \right\} \frac{\partial \psi}{\partial y} = \mu_\lambda \left[ 1 + \lambda^2 \frac{\partial \theta^{(0)}}{\partial y} \right] \frac{\partial \theta^{(0)}}{\partial y} = k_1 \frac{\partial \theta^{(0)}}{\partial y},
\]

\[
\text{at} \quad y = \delta, \tag{16}
\]

where \( \rho_\lambda = \frac{\rho}{\rho_1} \) is density ratio, \( \mu_\delta = \frac{\mu_1}{\mu} \) is the viscosity ratio, \( k_2 \) is the thermal conductivities ratio, \( \kappa \) is the electrokinetic width, \( \text{Gr} = \frac{\alpha \mu \Theta (T_1 - T_0)}{\mu_1 \rho \kappa^4} \) is Grashof number, \( Pr = \frac{c_p \mu_1}{\rho_1} \) is Prandtl number of the fluid, \( Q_1 = \frac{h^2 \Theta_b}{\rho_1} \) is the non-dimensional heat source parameter and \( \xi = \frac{\rho_1}{\rho} \) is the zeta potentials ratio and \( \nu_1 = -\frac{\xi \xi_2}{\mu_1} \) is the Helmholtz–Smoluchowski velocity.

The time fractional derivatives \( \partial^\alpha_{\nu \xi} \) and \( \partial^\beta_{\nu \xi} \), will be replaced by the Caputo–Fabrizio fractional derivative [43]:

\[
\mathcal{D}^\nu_{\xi} f(t) = \frac{R^+ \left( \nu \right)}{1 - \nu} \int_0^t f(t) e^{\left\{ \frac{(t - \tau)}{\chi_1} \right\}} d\tau, \quad 0 < \nu < 1,
\]

where \( R^+ \left( \nu \right) \) represents a normalization function.

### 3. Solution methodology

The approximation methods will be used to acquire the solutions for the current problem owing to the difficulty of the governing equations. The Laplace and finite Fourier sine transforms and their numerical inverses will be used to get the semi-analytical solutions for the velocity and temperature fields. The solutions of the EDL potential \( \psi \) with the corresponding boundary and interface conditions become:

\[
\hat{\psi} = \hat{\psi}^{II} = \frac{\xi \sinh(\kappa y) + \sinh((1 - y)\kappa)}{\sinh(\kappa)}. \tag{18}
\]

To proceed, we apply the Laplace transform to Equations (11)–(14) with the corresponding initial conditions Equation (16), we get the problem in the s-domain

**Region (I)**

\[
s \hat{w}^{(0)} = F \frac{s}{s + \lambda^2 \frac{\partial^2 \psi}{\partial y^2} + \kappa^2 \psi \Omega^2 + \Theta_2^2} + Gr\tilde{\theta}^{(0)}, \tag{19}
\]

\[
Pr \left[ s + \frac{\lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \right] \hat{\theta}^{(0)} = \hat{\theta}^{(0)} + \frac{\lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \hat{\theta}^{(0)}.
\]

**Region (II)**

\[
\hat{\rho}_s \left[ s + \frac{s^2 \lambda^2 \psi \Omega^2}{s + \alpha (1 - s)} \right] \hat{w}^{(0)} = \frac{F}{s} + \mu_\lambda \left[ 1 + \frac{s \lambda^2 \frac{\partial \tilde{\psi}}{\partial y}}{s + \beta (1 - s)} \right] \frac{\partial^2 \hat{\psi}^{(0)}}{\partial y^2}
\]

\[
+ \left[ 1 + \frac{s \lambda^2 \frac{\partial \tilde{\psi}}{\partial y}}{s + \alpha (1 - s)} \right] \frac{\kappa^2 \psi \Omega^2}{s^2 + \Omega_2^2}
\]

\[
+ \rho_\delta \gamma \tilde{\psi}^{(0)} \left[ 1 + \frac{s \lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \right] \hat{\psi}^{(0)},
\]

\[
\hat{\rho}_s \mu_\lambda \left[ s + \frac{s^2 \lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \right] \hat{\psi}^{(0)} = \hat{\psi}^{(0)} + \frac{s \lambda^2 \frac{\partial \tilde{\psi}}{\partial y}}{s + \gamma (1 - s)} \hat{\psi}^{(0)}.
\]

The solutions of the transformed heat Equations (20) and (22) with the corresponding boundary and interface conditions (Equation (23)) are

\[
\hat{\theta}^{(0)} = \frac{1}{\Gamma_{13}} \left[ k_3 \sqrt{\Gamma_{12}} \cos(\delta \sqrt{\Gamma_{12}}) \sin(\sqrt{\Gamma_{11}(y - \delta)}) + \sqrt{\Gamma_{11}} \cos(\sqrt{\Gamma_{11}(y - \delta)}) \sin(\sqrt{\Gamma_{12} \delta}) \right] \tag{24}
\]

\[
\hat{\theta}^{(0)} = \frac{1}{\Gamma_{14}} \left[ k_3 \sqrt{\Gamma_{12}} \cos(\sqrt{\Gamma_{11}(y - \delta)}) \sin(\sqrt{\Gamma_{12} \delta}) \sin(\sqrt{\Gamma_{11} y}), \tag{25}
\]

where

\[
\Gamma_{11} = \left[ 1 + \frac{s \lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \right] \left[ Q_1 - Pr s \right],
\]

\[
\Gamma_{11} = \left[ 1 + \frac{s \lambda^2 \psi \Omega^2}{s + \gamma (1 - s)} \right] \left[ Q_1 - \rho_\delta \gamma \tilde{\psi}^{(0)} \right],
\]

\[
\Gamma_{13} = k_3 \sqrt{\Gamma_{12}} \sin(\sqrt{\Gamma_{11}(y - \delta)}) \cos(\delta \sqrt{\Gamma_{12}}) \tag{26}
\]

\[
- \sqrt{\Gamma_{11}} \cos(\sqrt{\Gamma_{11}(y - \delta)}) \sin(\sqrt{\Gamma_{12} \delta}) \tag{27}
\]

\[
\Gamma_{14} = k_3 \sin(\sqrt{\Gamma_{11}(y - \delta)}) \cos(\sqrt{\Gamma_{12} \delta}) \sin(\sqrt{\Gamma_{11} y}) \tag{28}
\]
\[ + k_s \sqrt{\Gamma_{12}} \cot(\sqrt{\Gamma_{12}} \delta) \]
\[ + s \sin(\sqrt{\Gamma_{11}} \delta)(\sqrt{\Gamma_{11}} - k_s \sqrt{\Gamma_{12}}) \cot(\sqrt{\Gamma_{11}} \cot(\delta \sqrt{\Gamma_{12}})). \]

Applying the finite sine-Fourier transform to Equations (19) and (21), which defined by

\[ F\{\bar{\chi}(y)\} = \tilde{\bar{\chi}}(\xi_m) = \int_a^b \bar{\chi}(y) \sin(\xi_m(y-a)) \, dy, \quad (26) \]

where \( \xi_m = \frac{mn}{b-a}, \, y \in [a,b], \, a < b \) and the inverse defined by

\[ \delta^{-1}\{\hat{\tilde{\chi}}(\xi_m)\} = \frac{2}{(b-a)} \sum_{m=1}^{\infty} \hat{\tilde{\chi}}(\xi_m) \sin(\xi_m(y-a)). \quad (27) \]

On using the finite Fourier sine transform Equation (26) to Equations (19) and (21) along with the boundary and interface conditions Equation (23), the transformed velocities take the form

**Figure 2.** Variation of the velocity \( w \) for different values of \( \kappa \) (Panel (a) \( \delta = 0.5 \), Panel (b) \( \delta = 0.2 \), Panel (c) \( \delta = 0.8 \)) at \( \alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, \, Gr = 0, \, \xi = 1, \Omega = \frac{\pi}{\pi}, \rho_s = 1, \mu_s = 1, Q_1 = 1, \Pr = 6.2, F = 0, t = 2. \)

**Figure 3.** Variation of the velocity \( w \) versus \( y \) for different values of \( Gr \) (Panel (a) \( \delta = 0.5 \), Panel (b) \( \delta = 0.2 \), Panel (c) \( \delta = 0.8 \)) at \( \alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, \kappa = 2, \xi = 1, \Omega = 2\pi, \rho_s = 0.5, \mu_s = 1, Q_1 = 1, \Pr = 6.2, F = 2, \tau = 0.5. \)
Region (I)

\[
\tilde{w}^{(I)}(\delta) = \frac{\tilde{w}^{(I)}(\delta)}{\xi(1,m)} + \frac{\sin(\xi(1,m)(y-\delta))}{s + \xi^2(1,m)} + \frac{1}{(s + \xi^2(1,m))} \int_{-\delta}^{1} \sin(\xi(1,m)(y-\delta)) \, dy 
\]

Region (II)

\[
\tilde{w}^{(II)}(\delta) = \frac{(-1)^{m+1} \tilde{w}^{(II)}(\delta)}{\xi(2,m)} + \frac{(-1)^m \mu_s A \tilde{w}^{(II)}(\delta)}{\xi(2,m)\mu_s B(\frac{\rho_s A_s}{\mu_s B} + \xi^2(2,m))} + \frac{\mu_s B \tilde{\theta}^{(II)}}{\mu_s B(\frac{\rho_s A_s}{\mu_s B} + \xi^2(2,m))}
\]

where \( \xi(1,m) = \frac{m \pi_1}{1-\delta} \).

Applying the inverse of the finite sine-Fourier transform, we get

Region (I)

\[
\tilde{w}^{(I)} = \frac{(1-y)}{1-\delta} \tilde{w}^{(I)}(\delta) - \frac{2 \tilde{w}^{(I)}(\delta)}{(1-\delta)} \sum_{m=1}^{\infty} \frac{s \sin(\xi(1,m)(y-\delta))}{\xi(1,m)\mu_s B(\xi^2(1,m))} + \frac{2Gr}{(1-\delta)} \sum_{m=1}^{\infty} \frac{\tilde{\theta}^{(I)}(\xi(1,m)(y-\delta))}{(s + \xi^2(1,m))}
\]

where \( \xi(2,m) = \frac{m \pi_2}{1-\delta} \), \( A = [1 + \frac{\mu_s^2}{s + \alpha(1-\delta)}] \), \( B = [1 + \frac{\mu_s^2}{s + \beta(1-\delta)}] \).

Figure 4. Variation of the velocity \( w \) versus \( y \) for different values of \( \mu_s \) (Panel (a) \( \delta = 0.5 \), Panel (b) \( \delta = 0.2 \), Panel (c) \( \delta = 0.8 \)) at \( \alpha = 0.5 \), \( \beta = 0.5 \), \( \gamma = 0.5 \), \( \lambda_1 = 0.5 \), \( \lambda_2 = 0.5 \), \( \lambda_3 = 0.5 \), \( k = 2 \), \( \xi = 1 \), \( \Omega = 2\pi \), \( \rho_s = 0.5 \), \( Gr = 2 \), \( Q_1 = 1 \), \( Pr = 6.2 \), \( F = 2 \), \( t = 0.5 \).
Figure 5. Variation of the velocity $w$ versus $y$ for different values of $\rho_s$ (Panel (a) $\delta = 0.5$, Panel (b) $\delta = 0.2$, Panel (c) $\delta = 0.8$) at $\alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, \kappa = 2, \xi = 1, \Omega = 2\pi, \mu_s = 1, Gr = 2, Q_1 = 1, Pr = 6.2, F = 2, t = 0.5$.

Figure 6. Variation of the velocity $w$ (Panel (a)) and flow rate $Q_T$ (Panel (b)) versus $t$ for different values of $\Omega$ at $\alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, \kappa = 2, \xi = 1, \rho_s = 1, \mu_s = 1, Gr = 2, Q_1 = 1, Pr = 6.2, F = 2$. 
\begin{equation}
+ \frac{2}{(1-\delta)} \sum_{m=1}^{\infty} \frac{\sin(\xi_{(1,m)}(y-\delta))}{(s+\xi_{(1,m)}^2)} \int_{\delta}^{1} \left[ \frac{F}{s} + \frac{\kappa^2 \psi^{(i)}}{s^2 + \Omega^2} \right] \sin(\xi_{(1,m)}(y-\delta)) \, dy.
\end{equation}

Region (II)

\begin{equation}
\bar{W}^{(II)} = \frac{y\bar{W}^{(II)}(\delta)}{\delta} + \frac{2\bar{w}^{(II)}(\delta)}{\delta} \sum_{m=1}^{\infty} \frac{(-1)^m \rho_s As \sin(\xi_{(2,m)}y)}{\xi_{(2,m)} \mu_s B \left( \frac{\rho_s A_s}{\mu_s} + \frac{\xi_{(2,m)}^2}{s^2 + \Omega^2} \right) + \bar{W}^{(II)}} \int_{\delta}^{1} \left[ \frac{F}{s} + \frac{\kappa^2 \psi^{(i)}}{s^2 + \Omega^2} \right] \sin(\xi_{(2,m)}(y)) \, dy,
\end{equation}

by using the interface conditions Equation (23) and after tedious calculations, we get the following expiration of

Figure 7. Variation of the velocity \( w \) versus \( t \) for different values of \( \lambda_2 \) at \( \alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \lambda_3 = 0.5, \Omega = 0, \kappa = 2, \xi = 1, \rho_s = 1, \mu_s = 1, \Gr = 2, Q_1 = 1, Pr = 6.2, F = 2. \)

Figure 8. Variation of the velocity \( w \) versus \( t \) for different values of \( \lambda_1 \) at \( \alpha = 0.5, \beta = 0.5, \gamma = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, \Omega = 0, \kappa = 2, \xi = 1, \rho_s = 1, \mu_s = 1, \Gr = 2, Q_1 = 1, Pr = 6.2, F = 2. \)
the velocity at the interface ($y = \delta$),

$$\tilde{w}^{(II)} = \tilde{w}^{(I)} = \frac{-1}{A \frac{s}{s + \xi_{1,1}^{(I)}}} + \frac{\mu_s B}{s + \xi_{1,1}^{(I)}} + \frac{\mu_s B}{s + \xi_{1,1}^{(I)}} \left[ \frac{1}{2} \sum_{m=1}^{\infty} \left( \frac{(-1)^m \rho_s \mu_s \xi_{1,1}^{(I)}}{s + \xi_{1,1}^{(I)}} \right) \right]$$

As shown above, it is difficult to evaluate the inverse Laplace transform analytically. Therefore, it is more adequate to use the numerical Laplace inversion method.
Figure 11. Variation of the heat transfer $\theta$ versus $t$ for different values of $\lambda_3$ (Panel (a) $\delta = 0.5$, Panel (b) $\delta = 0.2$, Panel (c) $\delta = 0.8$) at $\gamma = 0.5$, $\rho_s = 0.5$, $k_s = 0.5$, $Gr = 2$, $Q_1 = 2$, $Pr = 6.2$.

Figure 12. Variation of the heat transfer $\theta$ versus $t$ for different values of $k_s$ (Panel (a) $\delta = 0.5$, Panel (b) $\delta = 0.2$, Panel (c) $\delta = 0.8$) at $\gamma = 0.5$, $\rho_s = 1$, $\lambda_3 = 0.7$, $Gr = 2$, $Q_1 = 2$, $Pr = 6.2$. 
to get the accurate approximation for the transformed solutions. For the current problem, Durbin’s method [53] has been used to determine the numerical Laplace inversion of the velocities and heat fields.

4. Results and discussion

In the previous section semi-analytical solutions were derived for the AC EOF of two layers of immiscible fluids through a vertical microchannel with heat transfer. The solutions depend mainly on density ratio \( \rho_s \), viscosity ratio \( \mu_s \), the electrokinetic width \( \kappa \), the angular frequency \( \Omega \), the relaxation time \( \lambda_1 \), the retardation time \( \lambda_2 \), Grashof number \( Gr \), the heat source parameter \( Q_1 \) and normalized wall zeta potential \( \xi \). According to the previous studies the dimensionless parameters in the range of [26,27, 29]: 0 < \( \rho_s \) ≤ 1, 0 < \( \mu_s \) ≤ 2, 0 < \( \kappa \) ≤ 1, 0 < \( \kappa_\| \) ≤ 40, 0 ≤ \( \Omega \) ≤ 2\( \pi \), 0 ≤ \( \lambda_1 \) ≤ 1, 0 ≤ \( \lambda_2 \) ≤ 1, 0 ≤ \( Gr \) ≤ 4, \( Q_1 \) ≥ 0 and 0 < \( \xi \) ≤ 1.

4.1. Electroosmotic velocity

In this subsection, the influences of the above-mentioned parameters on electroosmotic velocity are examined.

Figure 2 shows the effect of the electrokinetic width \( \kappa \) in the two regions on the velocity with different layer thicknesses. It is clear that the elevation of the electrokinetic width leads to accelerate the electroosmosis velocity. Moreover, with higher values of electrokinetic width, the electrostatic force is confined near the walls, while near the interface, the electrostatic effect takes negligible effect outside the thin EDL region. For different layer thicknesses and Grashof number, the variation of the velocity is shown through Figure 3. It is worth mentioning that the buoyancy force \( (Gr > 0) \) supports the electroosmosis velocity in the two layers except for the small thickness of the Non-Newtonian layer. The electroosmotic velocity is influenced in the two layers by varying the viscosity ratio due to its resistance to deformation as shown in Figure 4. Moreover, the steeper electroosmosis velocity depended on the viscosity ratio in the two layers. The effect of the variation of density ratio on the electroosmosis velocity is shown in Figure 5. The figure shows that for large thicknesses of the non-Newtonian layer (II) the effect of density ratio appears, but in the small thickness of the non-Newtonian layer (II), the effect of density ratio disappears in the two layers. Figure 6 depicts the variation of the electroosmosis velocity for DC \( (\Omega = 0) \)/AC \( (\Omega > 0) \)

\[ \text{Figure 13. Variation of the heat transfer } \theta \text{ versus } t \text{ for different values of } \rho_s \text{ (Panel (a) } \delta = 0.5, \text{ Panel (b) } \delta = 0.2, \text{ Panel (c) } \delta = 0.8) \text{ at } \gamma = 0.5, k_\| = 1, \lambda_\| = 0.7, Gr = 2, Q_1 = 2, Pr = 6.2. \]
electric field. It is clear that for the DC case, the velocity increases rapidly and goes to its steady-state after a short time. While for AC case, the electrophoresis velocity oscillates and increases by elevating the angular frequency. The figure also shows that the electrophoresis velocity and the flow rate take the same wave shape of the AC electric field. Figure 7 depicts the variation of the electrophoresis velocity for DC ($\Omega = 0$) electric field under the effect of the retardation time ($\lambda_2$) in the two layers. The figure shows that as the retardation time increases, the electrophoresis velocity decreases until a certain time after this time the effect dwindles. Physically, a viscoelastic fluid with a longer retardation time requires more time for the stress to respond to deformation. Moreover, the retardation time not only affects the non-Newtonian layer but extends to the Newtonian layer. Figure 8 shows the impact of the relaxation time (\(\lambda_3\)) on the electrophoresis velocity for DC ($\Omega = 0$) electric field in the two layers. It is clear that the electrophoresis velocity delays by increasing the value of the relaxation time until a critical time the variation of the electrophoresis velocity disappears. The figure also shows that the relaxation time effect is clear in the non-Newtonian layer than in the Newtonian layer. Figures 9 and 10 show the influence of the fractional-orders ($\alpha$ & $\beta$) on the electrophoresis velocity in the two regions. The figures show that the electrophoresis velocity is highly affected by the variation of the fractional orders.

4.2. Heat characteristics

In this subsection, the effects of the paramount parameters on heat transfer have been discussed.

Figure 11 depicts the variation of the heat transfer in the two layers under the effect of the heat relaxation time ($\lambda_3$) at a certain time. It is noticed that the heat transfer delays by elevating the value of the heat relaxation time in the two layers except the case when the thickness of the Newtonian layer shrinks. The effect of the thermal conductivity ratio ($k_s$) on the heat transfer in the two layers is shown in Figure 12. The figure shows that as the thermal conductivity ratio ($k_s$) increases the heat transfer increases in the non-Newtonian layer while it decreases in the Newtonian layer. However, when the thickness of the non-Newtonian layer shrinks then there is no heat transfer variation noticed. The

Figure 15. Variation of the shear stress $\tau$ versus $t$ for different values of $\kappa$ at $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.5$, $Gr = 1$, $\xi = 1$, $\Omega = 2\pi$, $\rho_s = 0.5$, $\mu_s = 0.5$, $Q_1 = 1$, $Pr = 6.2$, $F = 2$.

Figure 16. Variation of the shear stress $\tau$ versus $t$ for different values of $Gr$ at $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.5$, $\kappa = 4$, $\xi = 1$, $\Omega = 2\pi$, $\rho_s = 0.5$, $\mu_s = 1$, $Q_1 = 1$, $Pr = 6.2$, $F = 2$. 
effect of the density ratio \((\rho_s)\) on the heat transfer in the two layers is shown in Figure 13. It is evident that as the density ratio increases the heat transfer highly dwindling, especially in the non-Newtonian layer. Moreover, for the small thickness of the non-Newtonian layer, the heat transfer variation is not noticeable in the two layers. The influence of the fractional-order parameter \((\gamma)\) on the heat transfer in the two regions is shown in Figure 14. It is evident that the heat transfer fluctuates by elevating the heat fractional-order parameter and this fluctuation decreases with time. An important observation here is that the amount of the heat transfer in the non-Newtonian layer is small compared with the Newtonian layer.

### 4.3. Shear stress

In this subsection, the effects of the relevant parameters on the shear stress at the walls have been discussed. Figure 15 reveals that the magnitude of the shear stress at the walls boosts by elevating the value of the electrokinetic width \((\kappa)\). Moreover, the shear stress oscillates with time taking the same waveshape as the AC electric field. The effect of the Grashof number on the shear stress at the walls is shown in Figure 16. Clearly, the shear stress at the walls is influences by increasing the Grashof number, and a high variation is noticed at the heated wall. Figure 17 shows the variation of the shear stress at the walls under the influence of electric field angular frequency \(\Omega\). It is noticed that for the DC case \((\Omega = 0)\) the shear stress at the walls increases rapidly and goes to its steady-state after a short time. Moreover, for the AC case \((\Omega > 0)\) the shear stress at the walls fluctuates with time taking the same waveshape as the AC electric field. Figure 18 exhibits the impacts of the retardation time \((\lambda_2)\) (Panel (a)), the relaxation time \((\lambda_1)\) (Panel (b)) on the shear stress at the walls. It is evident that the amount of the shear stress at the wall \((y = 1)\) decreases by increasing the retardation time and increases by increasing the relaxation time. While the shear stress at the wall \((y = 0)\) has an inverse result under the influence of the retardation time and the relaxation time parameters. The shear stress at the
walls affected by the variation of the fractional orders ($\alpha$&$\beta$) as shown in Figure 19.

4.4. Validation of results

Figure 20 shows that the non-Newtonian layer in the current model in the case of fractional second-grade fluid ($\lambda_2 = 0$) and DC electric field can be compared with that of the single-layer fractional second-grade model discussed by Abdellateef et al. [29] for verification purposes.

4.5. Conclusions

A semi-analytical solution of the unsteady EOF of two layers of immiscible fluids through a vertical microchannel with heat transfer has been investigated. The channel is subjected to an alternating current (AC) electric field. The overall flow is divided into two layers: layer (I) is filled with Newtonian fluid, while layer (II) is filled with fractional Jeffreys fluid (or fractional Oldroyd-B fluid). The fractional derivatives replaced by the well-known Caputo–Fabrizio fractional derivatives. The Laplace and finite Fourier sine transforms and their numerical inverses have been used to get the semi-analytical solutions for the current problem. The electroosmosis velocity and heat transfer expressions of two layers were obtained. The numerical computations show the following conclusions:

- At higher values of electrokinetic width, the electrostatic force is confined near the walls, while near the interface, the electrostatic effect takes negligible effect outside the thin EDL region.
- The results show that the relaxation time effect is clear in the non-Newtonian layer than in the Newtonian layer.
- The convection force boosts the electroosmosis velocity in the two layers except for the case when the non-Newtonian layer shrinks.
- The results show that the viscosity and density ratio are important key factors of immiscible fluids.
- For the DC case, the velocity increases rapidly and goes to its steady-state after a short time. While for AC case, the electroosmosis velocity oscillates and increases by elevating the angular frequency.
- In the presence of the convection force, there is a noticeable fluctuation in the shear stress between the channel walls.
- The heat transfer delays by elevating the value of the heat relaxation time in the two layers.
A high thermal conductivity ratio results in larger heat transfer within the non-Newtonian layer.

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