Tuenter polynomials and a Catalan triangle

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Abstract

We consider Tuenter polynomials as linear combinations of descending factorials and show that coefficients of these linear combinations are expressed via a Catalan triangle of numbers. We also describe a triangle of coefficients in terms of some polynomials.

1 Preliminaries. Tuenter polynomials

The polynomials we are going to study in this brief note are defined by a recursion [7]

\[ P_{k+1}(n) = n^2(P_k(n) - P_k(n - 1)) + nP_k(n - 1), \quad n \in \mathbb{N} \]

(1.1)

with initial condition \( P_0(n) = 1 \). The first few polynomials yielded by (1.1) are as follows.

\[ P_1(n) = n, \]
\[ P_2(n) = n(2n - 1), \]
\[ P_3(n) = n(6n^2 - 8n + 3), \]
\[ P_4(n) = n(24n^3 - 60n^2 + 54n - 17), \]
\[ P_5(n) = n(120n^4 - 480n^3 + 762n^2 - 556n + 155), \]
\[ P_6(n) = n(720n^5 - 4200n^4 + 10248n^3 - 12840n^2 + 8146n - 2073). \]

Let us refer to these polynomials as Tuenter ones. Introducing a recursion operator \( R := n^2(1 - \Lambda^{-1}) + n\Lambda^{-1} \), where \( \Lambda \) is a shift operator acting as \( \Lambda(f(n)) = f(n + 1) \), one can write \( P_k(n) = R^k(1) \). The sense of these polynomials is that they help to count the sum

\[ S_r(n) = \sum_{j=0}^{2n} \binom{2n}{j} \abs{n - j}^r \]

for odd \( r \).
Bruckman in [2] asked to prove that $S_3(n) = n^2 \binom{2n}{n}$. Strazdins in [6] solved this problem and conjectured that $S_{2k+1}(n) = \tilde{P}_k(n) \binom{2n}{n}$ with some monic polynomial $\tilde{P}_k(n)$ for any $k \geq 0$. Tuenter showed in [7] that it is almost true. More exactly, he proved that

$$S_{2k+1}(n) = P_k(n)n \binom{2n}{n} = P_k(n) \frac{(2n)!}{(n-1)!n!}.$$ 

One can see that polynomial $\tilde{P}_k(n)$ is monic only for $k = 0, 1$. The recursion (1.1) follows from [7]

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n-1)S_r(n-1).$$

Also, as was noticed in [7], polynomials $P_k(n)$ can be obtained as a special case of Dumont-Foata polynomials of three variables [3].

2 The Tuenter polynomials as linear combinations of descending factorials

Consider descending factorials

$$(n)_k := n(n-1)(n-2)\cdots(n-k+1).$$

It can be easily seen that

$$R((n)_k) = k^2(n)_k + (k+1)(n)_{k+1}. \quad (2.1)$$

Let us consider $P_k(n)$ as linear combinations of descending factorials

$$P_k(n) = \sum_{j=1}^{k} c_{j,k}(n)_j,$$

with some coefficients $c_{j,k}$ to be calculated. For example, for the first few $P_k(n)$ we get

$$P_1(n) = (n)_1,$$

$$P_2(n) = (n)_1 + 2(n)_2,$$

$$P_3(n) = (n)_1 + 10(n)_2 + 6(n)_3,$$

$$P_4(n) = (n)_1 + 42(n)_2 + 84(n)_3 + 24(n)_4,$$

$$P_5(n) = (n)_1 + 170(n)_2 + 882(n)_3 + 720(n)_4 + 120(n)_5,$$

$$P_6(n) = (n)_1 + 682(n)_2 + 8448(n)_3 + 15048(n)_4 + 6600(n)_5 + 720(n)_6.$$
With (2.1) we can easily derive recurrence relations for the coefficients $c_{j,k}$. Indeed, from

\[ P_{k+1}(n) = \sum_{j=1}^{k+1} c_{j,k+1}(n)_j \]

\[ = R(P_k(n)) \]

\[ = \sum_{j=1}^{k} c_{j,k} \left(j^2(n)_j + (j+1)(n)_{j+1}\right) \]

we get

\[ c_{j,k+1} = j^2c_{j,k} + jc_{j-1,k}, \quad j \geq 1, \quad k \geq j. \]  \hspace{1cm} (2.2)

To use (2.2), one must agree that $c_{0,k} = c_{k+1,k} = 0$ for $k \geq 1$. Then, starting from $c_{1,1} = 1$ we obtain the whole set $\{c_{j,k} : j \geq 1, k \geq j\}$. For example, $c_{1,k} = 1$ for all $k \geq 1$, while for $j = 2$ we obviously get a recursion

\[ c_{2,k+1} = 4c_{2,k} + 2, \quad c_{2,1} = 0. \]

As can be easily seen, a solution of this equation is given by

\[ c_{2,k} = \frac{1}{3}(2^{2k-1} - 2), \quad k \geq 2. \]  \hspace{1cm} (2.3)

**Remark 2.1.** It is interesting to note that integer sequence (2.3), known as A020988 in [5] gives $n$-values of local maxima for $s(n) := \sum_{j=1}^{n} a(j)$, where $\{a(n)\}$ is the Golay-Rudin-Shapiro sequence [1].

For the whole set of the coefficients $\{c_{j,k}\}$, we get the following.

**Theorem 2.2.** A solution of equation (2.2) with $c_{0,k} = c_{k+1,k} = 0$ for $k \geq 1$ and $c_{1,1} = 1$ is given by

\[ c_{j,k} = \frac{j!}{(2j-1)!} \left( \sum_{q=1}^{j} (-1)^{q+j} B_{j,q} q^{2k-1} \right), \quad \forall j \geq 1, \quad k \geq j, \]  \hspace{1cm} (2.4)

where the numbers

\[ B_{j,q} := \frac{q}{j} \left( \begin{array}{c} 2j \\ j-q \end{array} \right) \]

constitute a Catalan triangle [4].

**Proof.** Substituting (2.4) into (2.2) and collecting terms at $q^{2k-1}$, we obtain that sufficient condition for (2.4) to be a solution of (2.2) is that the numbers $B_{j,q}$ enjoy the relation

\[ \frac{q^2j!}{(2j-1)!} B_{j,q} = \frac{j^2j!}{(2j-1)!} B_{j,q} - \frac{j!}{(2j-3)!} B_{j-1,q}, \quad \forall q = 1, \ldots, j - 1. \]
Simplifying the latter we get the relation
\[(j - q)(j + q)B_{j,q} = (2j - 1)(2j - 2)B_{j-1,q}\]
which can be easily verified. Therefore the theorem is proved. □

The set \(\{c_{j,k}\}\) can be presented as the number triangle
\[
\begin{array}{cccc}
  & c_{1,1} & & \\
  c_{1,2} & c_{2,2} & & \\
  c_{1,3} & c_{2,3} & c_{3,3} & \\
  & \ddots & \ddots & \ddots
\end{array}
\]
whose description is given by theorem 2.2.

**Remark 2.3.** From [4] one knows that the number \(B_{j,q}\) can be interpreted as the number of pairs of non-intersecting paths of length \(j\) and distance \(q\). The Catalan numbers itself (A000108) are
\[C_j := B_{j,1} = \frac{1}{j} \binom{2j}{j-1}.\]

Therefore, we got an infinite number of integer sequences each of which is defined by numbers from the Catalan triangle and begins from \(c_{j,j} = j!\). Let us list the first few ones. For example, one has,
\[c_{1,k} = 1,\]
\[c_{2,k} = \frac{1}{3} \left(2^{2k-1} - 2\right),\]
\[c_{3,k} = \frac{1}{20} \left(3^{2k-1} - 4 \cdot 2^{2k-1} + 5\right),\]
\[c_{4,k} = \frac{1}{210} \left(4^{2k-1} - 6 \cdot 3^{2k-1} + 14 \cdot 2^{2k-1} - 14\right),\]
\[c_{5,k} = \frac{1}{3024} \left(5^{2k-1} - 8 \cdot 4^{2k-1} + 27 \cdot 3^{2k-1} - 48 \cdot 2^{2k-1} + 42\right),\]
\[c_{6,k} = \frac{1}{55440} \left(6^{2k-1} - 10 \cdot 5^{2k-1} + 44 \cdot 4^{2k-1} - 110 \cdot 3^{2k-1} + 165 \cdot 2^{2k-1} - 132\right), \ldots\]

All these sequences are indeed integer because they are solutions of (2.2).

Let us replace \(k \mapsto j + k\) in (2.2) and seek its solution in the form \(c_{j,j+k} = F_k(j) j!\). Substituting the latter in (2.2) we come to the recurrence relation
\[F_k(j) - F_k(j - 1) = j^2 F_{k-1}(j)\]  
(2.5)
with conditions \(F_0(j) = 1\) and \(F_k(1) = 1\). A solution of (2.5) is
\[F_k(j) = 1 + \sum_{2 \leq \lambda_1 \leq j} \lambda_1^2 + \sum_{2 \leq \lambda_1 \leq \lambda_2 \leq j} \lambda_1^2 \lambda_2^2 + \cdots + \sum_{2 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq j} \lambda_1^2 \lambda_2^2 \cdots \lambda_k^2.\]
In particular,
\[ F_1(j) = 1 + 2^2 + \cdots + j^2 = \frac{1}{6}j(j+1)(2j+1), \]
that is, \(F_1(j)\) yields A000330 sequence of the square pyramidal numbers. The next two polynomials \(F_k(j)\) are
\[ F_2(j) = \frac{1}{360}j(j+1)(j+2)(2j+1)(2j+3)(5j-1) \]
and
\[ F_3(j) = \frac{1}{45360}j(j+1)(j+2)(j+3)(2j+1)(2j+3)(2j+5)(35j^2 - 21j + 4). \]
Looking at these examples and others we can suppose that
\[ F_k(j) = \prod_{q=0}^{k} (j+q) \prod_{q=0}^{k-1} (2j + 2q + 1) \tilde{F}_k(j), \]
where \(\tilde{F}_k(j)\) is a polynomial of \((k - 1)\) degree, which satisfy
\[ \tilde{F}_k(1) = \frac{1}{(k+1)!(2k+1)!}. \]

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**References**

[1] J. Brillhart and P. Morton, A case study in mathematical research: the Golay-Rudin-Shapiro sequence, The American Mathematical Monthly, 103 (1996), 854-869.

[2] P. S. Bruckman, Problem B-871, Fibonacci Quarterly, 37 (1999), 85.

[3] D. Dumont and D. Foata, Une propriété de symétrie des nombres de Genocchi, Bulletin de la Société Mathématique de France 104 (1976), 433-451.

[4] L. W. Shapiro, A Catalan triangle, Discrete Math., 14 (1976), 83-90.

[5] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[6] I. Straždins, Solution to problem B-871, Fibonacci Quarterly, 38.1 (2000), 86-87.

[7] H. J. H. Tuenter, Walking into an absolute sum, Fibonacci Quarterly, 40 (2002), 175-180.