Logarithmic Jets and Hyperbolicity

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Abstract. We prove that the complement of a very generic curve of degree \( d \) at least equal to 15 in \( \mathbb{P}^2 \) is hyperbolic in the sense of Kobayashi (here, the terminology “very generic” refers to complements of countable unions of proper algebraic subsets of the parameter space). We first consider the Dethloff and Lu’s generalization to the logarithmic situation of Demailly’s jet bundles. We study their base loci for surfaces of log-general type in the same way as it was done in the compact case by Demailly and El Goul. With some condition on log-Chern classes, any entire holomorphic map to the surface can be lifted as a leaf of some foliation on a ramified covering. Then we obtain a logarithmic analogue of McQuillan’s result on holomorphic foliations which permits to conclude. Using the logarithmic formalism, we even obtain some simplifications of the original proof in McQuillan’s work.

0. Introduction

In 1970 S. Kobayashi [Ko70] posed the following problems: Is it true that the complement of generic hypersurface of degree \( d \geq e(n) \) in \( \mathbb{P}^n_{\mathbb{C}} \) is hyperbolic? Is this true for \( e(n) = 2n + 1 \)? Later Green [Gr75] for \( n = 2 \) and Zaidenberg [Za87] for arbitrary \( n \) proved that for \( d \leq 2n \) those complements contain \( \mathbb{C}^* \) and then are not hyperbolic.

In this paper we will study the case of complements of smooth curves in \( \mathbb{P}^2_{\mathbb{C}} \) where for \( d \geq 4 \) this is equivalent to the nonexistence of nonconstant entire curves by Brody Reparametrisation Lemma [Br78]. When the curve is supposed to have many components this had been studied by many authors, see [DSW92,94] for a complete bibliography and the study of the case of three components (see also the recent work [BD00]). In the smooth case, after an example of hyperbolic complement given by Nadel in [Na89] (for \( d \geq 21 \)), Zaidenberg [Za89] gave examples for all \( d \geq 5 \).

The first positive answer to this question was given in the work of Siu and Yeung [SY95], the bound they obtain is quite high. Their method consists of an explicit construction of special second order differential operators on an associated surface in \( \mathbb{P}^3_{\mathbb{C}} \) ramified over \( \mathbb{P}^2_{\mathbb{C}} \). This was done by an imitation of the construction of holomorphic 1-forms on Riemann surfaces and a clever reduction of the problem to a resolution of linear systems. Those operators are such that their pullbacks by the lifting of every entire curve must vanishes identically. This follows immediately from an Ahlfors type result.
In [DEG00], after studying the compact analogue of the above conjecture and proving that a generic surface of degree \(d \geq 21\) in \(\mathbb{P}^3_\mathbb{C}\) is hyperbolic, Demailly and the author, using the same covering trick, obtained the bound 21 also for complementary of curves in \(\mathbb{P}^2_\mathbb{C}\). This is made by using the whole force of Demailly’s jet bundles introduced in [De95] and a McQuillan’s result on holomorphic foliations [Mc98].

Here we obtain the bound 15. Our main theorem is the following

**Main Theorem.** — The complement of a very generic curve in \(\mathbb{P}^2_\mathbb{C}\) is hyperbolic and hyperbolically imbedded for all degrees \(d \geq 15\).

We follow almost the same strategy as in [DEG00] with the difference that we use an logarithmic analogous package introduced in [DL96], which introduces some additional technical complications. Dethloff and Lu’s jet bundles are a compactification à la Demailly of Noguchi’s logarithmic jet bundles introduced in [No84]. Using Riemann-Roch and a refined study of the base loci associated to those jet bundles, we reduce the problem to the study of holomorphic foliations on log-general type surfaces. We prove that such foliations do not admit a parabolic Zariski-dense leaf. We generalize, in particular, McQuillan’s result [Mc98] on Green-Griffiths conjecture to log-general type surfaces with \(c_1^2 > c_2\). This logarithmic point of view permits the observation that McQuillan’s refined tautological inequality is actually an easy consequence of a logarithmic tautological inequality obtained by [Vo99] and proved in the same way as the simple one (see also [Br00]).

The paper is organized as follows: In section 1 we first recall the main definitions and results in [DL96]. Then we introduce the 2-jet threshold of a log-general type surface \((X, C)\). We consider the case when the Picard group is \(\mathbb{Z}\) and we construct, with some conditions on log-Chern classes and the 2-jet threshold, a ramified cover \(\tilde{X}\) on \(X\) such that every entire curve in the complement \(X \setminus C\) could be lift as a leaf of a holomorphic foliation on \(\tilde{X}\). The last part of this section is devoted to estimate the 2-jet threshold in the case of \((\mathbb{P}^2_\mathbb{C}, C)\) where \(C\) a generic smooth plane curve.

In section 2 we study foliations on log-general type surfaces as in [Mc98]. The method we adopt is parallel to Brunella’s work [Br99]. We obtain that those foliations do not have a Zariski dense parabolic leaf.

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### 1. Logarithmic Demailly jet bundles

#### 1.1. Background material. — Here we will consider a logarithmic generalisation of Demailly’s invariant jets introduced in [De95], this is done by Dethloff and Lu in [DL96].
Let $X$ be an $n$-dimensional complex manifold with a normal crossing divisor $D$. According to Iitaka [Ii77], the logarithmic cotangent sheaf $\overline{T}_X^* = T_X^* (\log D)$ is defined to be the locally free sheaf generated by $T_X^*$ and the logarithmic differentials $ds_j/s_j$, where $s_j = 0$ is a local equation for the irreducible components of $D$. Its dual the logarithmic tangent sheaf is the sheaf of germs of vector fields tangent to $D$, denoted by $\overline{T}_X = T_X (\log D)$.

Recall from [GG80] that the $k$-jet bundle $J_kX$ is defined as the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if they have the same Taylor expansions of order $k$ in some local coordinate system of $X$ near $x$. We denote the equivalence class of $f$ by $j_k(f)$. In [No84], Noguchi generalised this object to the logarithmic situation as follows. Let $\omega \in H^0(U, T_X^*)$ be a holomorphic section over an open subset $U \subset X$. For a germ of a holomorphic map $f$ in $U$ we put $f^* \omega = Z(t)dt$. Then we have a well defined holomorphic mapping

$\tilde{\omega} : J_kX|_U \to \mathbb{C}^k ; j_k(f) \to (Z(j)(0))_{0 \leq j \leq k-1}$.

Now we say that a holomorphic section $s \in H^0(U, J_kX)$ is a logarithmic $k$-jet field if the map $\tilde{\omega} \circ s|_V : V \to \mathbb{C}^k$ is holomorphic for all $\omega \in H^0(V, \overline{T}_X)$ and for all open subset $V$ of $U$. The set of logarithmic $k$-jet fields over open subsets of $X$ defines a subsheaf of $J_kX$ called the logarithmic $k$-jet bundle of $(X, D)$, which we denote by $\overline{J}_kX$.

In [DL96], Dethloff and Lu constructed a more geometrically relevant jet bundles (in the same way as done in [De95] for the non-logarithmic case) by considering a suitable “quotient” of this bundle by the action of the group $\mathbb{G}_k$ containing all germs of $k$-jets biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k$, \quad a_1 \in \mathbb{C}^*, \ a_j \in \mathbb{C}, \ j \geq 2$.

As a generalization of Demailly’s directed jets to the logarithmic context, Dethloff and Lu defined a log-directed manifold to be the triple $(X, D, V)$ where $V$ is a holomorphic subbundle of $\overline{T}_X$ of rank $r$. To the log-directed manifold $(X, D, V)$, one associates inductively a sequence of directed manifolds $(\overline{X}_k, D_k, V_k)$ as follows. Starting with $(\overline{X}_0, D_0, V_0) = (X, D, V)$, one puts inductively $\overline{X}_k = P(V_{k-1})$ with its natural projection $\pi_k$ to $\overline{X}_{k-1}$ (where $P(V)$ stands for the projectivized bundle of lines in the vector bundle $V$), where $D_k = \pi_k^* (D_{k-1})$ and $V_k$ is the subbundle of $\overline{T}_{\overline{X}_k} (\log D_k)$ defined at any point $(x, [v]) \in \overline{X}_k$, $v \in V_{k-1,x}$, by

$V_{k,(x,[v])}^\cdot = \{ \xi \in \overline{T}_{\overline{X}_k,(x,[v])} (\log D_k) ; (\pi_k)_* \xi \in \mathbb{C} \cdot v \}$,

with $\mathbb{C} \cdot v \subset V_{k-1,x} \subset \overline{T}_{\overline{X}_{k-1},x} (\log D_{k-1})$.

We denote by $\mathcal{O}_{\overline{X}_k} (-1)$ the tautological line subbundle of $\pi_k^* V_{k-1}$, such that $\mathcal{O}_{\overline{X}_k} (-1)_{(x,[v])} = \mathbb{C} \cdot v$,

for all $(x, [v]) \in \overline{X}_k = P(V_{k-1})$. By definition, the bundle $V_k$ fits in an exact sequence

$0 \to \overline{T}_{\overline{X}_k/(\overline{X}_{k-1})} \to V_k \xrightarrow{\pi_k \cdot} \mathcal{O}_{\overline{X}_k} (-1) \to 0$. 

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and the Euler exact sequence of $T_{\overline{X}/X_{k-1}}$ yields

$$0 \rightarrow \mathcal{O}_{\overline{X}} \rightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{\overline{X}}(1) \rightarrow T_{\overline{X}/X_{k-1}} \rightarrow 0.$$ 

From these sequences, we infer

$$\text{rank } V_k = \text{rank } V_{k-1} = \cdots = \text{rank } V = r, \quad \dim \overline{X}_k = n + k(r - 1).$$

We note

$$\pi_{k,j} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : \overline{X}_k \rightarrow \overline{X}_j,$$

be the natural projection.

The canonical injection $\mathcal{O}_{\overline{X}}(-1) \hookrightarrow \pi_k^* V_{k-1}$ and the exact sequence

$$0 \rightarrow T_{\overline{X}_{k-1}/\overline{X}_{k-2}} \rightarrow V_{k-1} \rightarrow \mathcal{O}_{\overline{X}_{k-1}}(-1) \rightarrow 0$$

yield a canonical line bundle morphism

$$\mathcal{O}_{\overline{X}_k}(-1) \xrightarrow{(\pi_{k,k-2})^* \circ (\pi_{k-1})^*} \pi_k^* \mathcal{O}_{\overline{X}_{k-1}}(-1)$$

which admits precisely the hyperplane section $\Gamma_k := P(T_{\overline{X}_{k-1}/\overline{X}_{k-2}}) \subset \overline{X}_k = P(V_{k-1})$ as its zero divisor. Hence we find $\mathcal{O}_{\overline{X}_k}(-1) = \pi_k^* \mathcal{O}_{\overline{X}_{k-1}}(-1) \otimes \mathcal{O}(-\Gamma_k)$

and using the notation $\mathcal{O}_{\overline{X}_k}(a_1, a_2) := \pi_k^* \mathcal{O}_{\overline{X}_{k-1}}(a_1) \otimes \mathcal{O}_{\overline{X}_k}(a_2),$

$$\mathcal{O}_{\overline{X}_k}(-1, 1) \simeq \mathcal{O}(\Gamma_k)$$

is associated with an effective divisor in $\overline{X}_k$.

For simplicity let us consider the case $V = T_X$ and let $f : \Delta_r \rightarrow X \setminus D$ tangent to $V$ be a nonconstant trajectory. Then $f$ lifts to a well defined and unique trajectory $f'_k : \Delta_r \rightarrow \overline{X}_k \setminus D_k$ of $\overline{X}_k$ tangent to $V_k$. Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_r} \rightarrow f'_k \mathcal{O}_{\overline{X}_k}(-1).$$

With any section $\sigma$ of $\mathcal{O}_{\overline{X}_k}(m)$, $m \geq 0$, on any open set $\pi_{k,0}^{-1}(U), U \subset X \setminus D$, we can associate a holomorphic differential operator $Q$ of order $k$ acting on $k$-jets of germs of curves $f : (\mathbb{C}, 0) \rightarrow U$ tangent to $V$, by putting

$$Q(f)(t) = \sigma(f'_k(t)) \cdot f'_{[k-1]}(t)^{\otimes m} \in \mathbb{C}.$$

From [De95] this correspondence is, in fact, bijective. To see what happen with logarithmic jets recall the following characterisation of log-jet differentials in [DL96]:

**1.1.1. Proposition ([DL96])**. — A holomorphic function $Q$ on $J_k X|_U$ on some connected open subset $U \subset X$ which satisfies

$$(*) \quad Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \quad \forall j_k(f) \in J_k X|_V \quad \text{and} \quad \forall \phi \in \mathbb{G}_k$$

over some open subset $V$ of $U \setminus D$ defines a holomorphic section of $\mathcal{O}_{\overline{X}_k}(m)$ over $U$, and vice versa.
Now, using the characteristation above, the definition of $\mathcal{J}_k X$ (in fact, on the component $D_j$ of the divisor $D$ the $k$-th derivative of the function $\log s_j(f)$ is holomorphic on $\mathcal{J}_k X$) and the fact that holomorphic functions satisfying $(\ast)$ for all $\phi \in \mathbb{C}^* \subset \mathbb{G}_k$ are homogenous polynomials, we obtain

1.1.2. Proposition. — The direct image $(\pi_{k,0})_* \mathcal{O}_{\mathcal{X}_k}(m)$ coincides with the sheaf $\mathcal{O}(E_{k,m} \mathcal{T}^* X)$ of logarithmic jet differentials, that is, the locally free sheaf generated by all polynomial operators in the derivatives of order 1, 2, $\cdots$, $k$ of $f$, together with the extra function $\log s_j(f)$ along the $j$-th component of $D$, which are moreover invariant under arbitrary changes of parametrization: a germ of operator $Q \in E_{k,m} \mathcal{T}^*_X$ is characterized by the condition that, for every germ $f$ in $X \setminus D$ and every germ $\varphi$ of $k$-jet biomorphisms of $(\mathbb{C}, 0)$,

$$Q(f \circ \varphi) = \varphi^m Q(f) \circ \varphi.$$

A basic result from [DL96] relying on the Ahlfors-Schwarz lemma, is the following, for the 1-jet case see [No77] and [Lu91].

1.1.3. Theorem ([DL96]). — If $(X, D)$ has a $k$-jet metric $h_k$, i.e. a singular metric in the sens of Demailly on $\mathcal{O}_{\mathcal{X}_k}(-1)$ with negative curvature (along $V_k$), then every entire curve $f : \mathbb{C} \to X \setminus D$ is such that $f[k](\mathbb{C}) \subset \Sigma_{h_k}$, where $\Sigma_{h_k}$ denotes the singular set of $h_k$.

An important case where the previous theorem applies is when there are some integers $k, m > 0$ and an ample line bundle $A$ on $X$ such that

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes (\pi_{k,0})^* A^{-1}) \simeq H^0(X, E_{k,m} \mathcal{T}^*_X \otimes A^{-1})$$

has nonzero sections $\sigma_1, \ldots, \sigma_N$. Then, we can construct $k$-metric of negative curvature, singular on their base locus $Z \subset \mathcal{X}_k$.

By definition, a line bundle $L$ is big if there exists an ample divisor $A$ on $X$ such that $L^\otimes m \otimes \mathcal{O}(-A)$ admits a nontrivial global section when $m$ is large (then there are lot of sections, namely $h^0(X, L^\otimes m \otimes \mathcal{O}(-A)) \gg m^n$ with $n = \dim X$).

As a consequence, Theorem 1.1.3 can be applied when $\mathcal{O}_{\mathcal{X}_k}(1)$ is big or its restriction on some subvariety is big (See Theorem 4.3 in [DL96]).

In view of studying degeneration of entire curve drawn on a variety of log-general type and of Theorem 1.1.3, it is especially interesting to compute the base locus of the global sections of log-jet differentials, that is, the intersection

$$\mathcal{B}_k := \bigcap_{m > 0} \mathcal{B}_{k,m} \subset \mathcal{X}_k$$

of the base loci $\mathcal{B}_{k,m}$ of all line bundles $\mathcal{O}_{\mathcal{X}_k}(m) \otimes (\pi_{k,0})^* \mathcal{O}(-A)$, where $A$ is a given arbitrary ample divisor over $X$.

1.1.4. Remark. — The 1-jet case was studied in [Lu91]. Using Riemann-Roch [Hi66], we prove that if $(X, D)$ is a nonsingular surface of log-general type with logarithmic Chern classes $\tau_1 > \tau_2$ then there is a lot of log-symmetric differentials,
i.e. sections in $E_{1,m}^{\mathcal{T}} = S^{m}T_{X}^{*}$, and the base locus $B_{1}$ is 2-dimensionnal. Unfortunately, the “order 1” techniques are insufficient to deal with complement of smooth curve $C$ of degree $d$ in $\mathbb{P}^{2}$, because in this case
\[ \overline{\tau}_{1}^{2} = (d - 3)^{2} < \overline{\tau}_{2} = (d^{2} - 3d + 3). \]

Lemma 1.4.1 below shows in fact that $H^{0}(X, S^{m}T_{X}^{*}) = 0$ for all $m > 0$.

1.2. Base locus of logarithmic 2-jets. — From now on, we suppose that $(X, D)$ is a nonsingular minimal surface of log-general type (i.e. with $K_{X} := K_{X} \otimes \mathcal{O}(D)$ big and nef) and let us study the base locus $B_{2}$ in $X$. As in the non-logarithmic case the bundle of log-jet differentials of order 2 has the following filtration
\[ \text{Gr}^{\bullet} E_{2,m}^{\mathcal{T}} = \bigoplus_{0 \leq j \leq m/3} S^{m-3j}T_{X}^{*} \otimes K_{X}^{\otimes j}. \]

This filtration consists in writing an invariant polynomial log-differential operator outside $D$ as
\[ Q(f) = \sum_{0 \leq j \leq m/3} \sum_{\alpha \in \mathbb{N}^{2}, |\alpha| = m-3j} a_{\alpha,j}(f) (f')^{\alpha} (f' \wedge f'')^{j} \]
where
\[ f = (f_{1}, f_{2}), \quad (f')^{\alpha} = (f_{1}')^{\alpha_{1}} (f_{2}')^{\alpha_{2}}, \quad f' \wedge f'' = f_{1}' f_{2}'' - f_{1}'' f_{2}'. \]

On a component $D_{j}$ of $D$ given in local coordinate by $z_{1} = 0$ we replace only $f_{1}$ by log $f_{1}$ in this expression. A calculation based on the above filtration of $E_{2,m}^{\mathcal{T}}$ and Riemann-Roch yields
\[ \chi(X, E_{2,m}^{\mathcal{T}}) = \frac{m^{4}}{648} (13 \overline{\tau}_{1}^{2} - 9 \overline{\tau}_{2}) + O(m^{3}). \]

On the other hand,
\[ H^{2}(X, E_{2,m}^{\mathcal{T}} \otimes \mathcal{O}(-A)) = H^{0}(X, K_{X} \otimes E_{2,m}^{\mathcal{T}} \otimes \mathcal{O}(A)) \]
by Serre duality. From the filtration above, $K_{X} \otimes (E_{2,m}^{\mathcal{T}}) \otimes \mathcal{O}(A)$ admits a filtration with graded pieces
\[ S^{m-3j}T_{X}^{*} \otimes K_{X}^{\otimes -j} \otimes K_{X} \otimes \mathcal{O}(A). \]

Recall now that $T_{X}$ is semi-stable (see [KR85] and [TY87]) so by Bogomolov’s vanishing theorem [Bo79], we have $h^{0}(X, S^{p}T_{X}^{*} \otimes K_{X}^{\otimes q}) = 0$, $p - 2q > 0$. This implies that
\[ h^{2}(X, E_{2,m}^{\mathcal{T}} \otimes \mathcal{O}(-A)) = 0 \]
for $m$ large. Consequently we get the following

1.2.1. Theorem. — If $(X, D)$ is an algebraic surface of general type and $A$ an ample line bundle over $X$, then
\[ h^{0}(X, E_{2,m}^{\mathcal{T}} \otimes \mathcal{O}(-A)) \geq \frac{m^{4}}{648} (13 \overline{\tau}_{1}^{2} - 9 \overline{\tau}_{2}) - O(m^{3}). \]
In particular: If $13\overline{c}_1^2 - 9\overline{c}_2 > 0$, then $\overline{B}_2 \neq \overline{X}_2$.

In the special case when $X = \mathbb{P}^2$ and $D = C$ is a smooth plane curve of degree $d$, we take $A = \mathcal{O}_{\mathbb{P}^2}(1)$. Then we have $\overline{c}_1 = (3-d)h$ and $\overline{c}_2 = (d^2 - 3d + 3)h^2$ where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, $h^2 = 1$, thus

$$\chi(E_{2,m}T_X^* \otimes \mathcal{O}(-A)) = (4d^2 - 51d + 90)\frac{m^4}{648} + O(m^3).$$

A straightforward computation shows that the leading coefficient $4d^2 - 51d + 90$ is positive if $d \geq 11$. Thus, we obtain

1.2.2. Corollary. — For every smooth curve of degree $d \geq 11$ the associated log-surface $(\mathbb{P}^2, C)$ has its 2-jets base locus $\overline{B}_2 \neq \overline{X}_2$.

1.2.3. Remark. — As a consequence of the calculus above and with the previous condition on Chern classes, every holomorphic entire curve $f$ into $X \setminus D$ could be lifted in $\overline{X}_2$ and its image is contained in an irreducible component $Z$ of $\overline{B}_2$. We have to distinguish three cases

(a) $\pi_{2,1}(Z) = \overline{X}_1$, then $Z$ is three dimensional and called horizontal, in this case the 2-jet lifting of $f$ is a leaf of a foliation by curves on $Z$. In fact the lifting of $f$ is tangent to $T_Z \cap V_2$ which defines a distribution of lines on a Zariski open subset of $Z$ which is obviously integrable.

(b) $\pi_{2,0}(Z) = X$ and we are not in case (a), then our curve $f$ could be lifted to $\overline{X}_1$ as a leaf of foliation by curves on the surface $Y := \pi_{2,1}(Z)$ (defined by the distribution $T_Y \cap V_1$)

(c) The curve $f$ is degenerated.

The difficulty, in the case (a), is to study (singular) foliation by curves on a variety of dimension bigger than two. Actually, we have no reasonable model with reducible foliated singularities in this case until now (see however [Mc99]). So the next step is to show that with a slightly stronger condition on Chern classes, in the case of $(\mathbb{P}^2, C)$, we have to consider only foliations on surfaces.

1.3. Existence of the multi-foliation. — Our aim now is to study the restriction of the tautological line bundle on 2-jets on a 3-dimensionnal horizontal component $Z$ of $\overline{B}_2$. Let us first make the following useful definition (as in [DEG00])

1.3.1 Definition. — Let $(X, D)$ be a nonsingular projective variety of log-general type. We define the $k$-jet log-threshold $\overline{\theta}_k$ of $(X, D)$ to be the infimum

$$\overline{\theta}_k = \inf_{m > 0} \overline{\theta}_{k,m} \in \mathbb{R},$$

where $\overline{\theta}_{k,m}$ is the smallest rational number $t/m$ such that there is a non zero section in $H^0(X, E_{k,m}T_X^* \otimes \mathcal{O}(tK_X))$ (assuming that $tK_X$ is an integral divisor, $t \in \mathbb{Q}$).
In the case when the Picard group equals \( \mathbb{Z} \) we have a more clear idea about the jet log-threshold. In fact we have

**1.3.2. Lemma.** — Let \((X,D)\) be a nonsingular surface of log-general type with \(\text{Pic}(X) = \mathbb{Z}\). Suppose that \(\theta_1 \geq 0\) and \(\theta_2 < 0\). Then

\[
\overline{B}_2 \subset Z_\sigma = Z \cup \Gamma_2,
\]

where \(Z\) is an irreducible section and \(Z_\sigma\) is the set of zeros of a section \(\sigma\) in \(H^0(X, \mathcal{O}_X(m_0) \otimes \mathcal{O}(t_0 \mathcal{K}_X))\). Moreover, in the case \(\overline{B}_2 = Z_\sigma\), we have \(\theta_2 = t_0/m_0\).

**Proof.** — As \(\theta_2 < 0\) we have a nontrivial section

\[
s \in H^0(X, E_{2,m}\mathcal{H}^*_X \otimes \mathcal{O}(t \mathcal{K}_X)), \quad m > 0, \ t \in \mathbb{Q}_-
\]

Let \(u_1 = \pi_{2,1}^* \mathcal{O}_X(1)\) and \(u_2 = \mathcal{O}_X(1)\), then its zero divisor

\[
Z_s = m u_2 + t \pi_{2,0}^* \mathcal{K}_X \quad \text{in} \ \text{Pic}(X),
\]

Let \(Z_s = \sum p_j Z_j\) be the decomposition of \(Z_s\) in irreducible components. From the equality \(\text{Pic}(X) = \mathcal{O}(X) \oplus \mathcal{O}(u_1) \oplus \mathcal{O}(u_2)\) and the assumption \(\text{Pic}(X) \simeq \mathbb{Z}\), we find

\[
Z_j \sim a_{1,j} u_1 + a_{2,j} u_2 + t_j \pi_{2,0}^* \mathcal{K}_X,
\]

for suitable integers \(a_{1,j}, a_{2,j} \in \mathbb{Z}\) and rational numbers \(t_j \in \mathbb{Q}\). We can prove that (see Lemma 3.3 in [DEG00]), as \(Z_j\) is effective, we must have one of the following three disjoint cases:

- \((a_{1,j}, a_{2,j}) = (0,0)\) and \(Z_j \in \pi_{2,0}^* \text{Pic}(X), t_j > 0\);
- \((a_{1,j}, a_{2,j}) = (-1,1)\), then \(Z_j\) contains \(\Gamma_2\), so \(Z_j = \Gamma_2\) and \(t_j = 0\);
- \(a_{1,j} \geq 2a_{2,j} \geq 0\) and \(m_j := a_{1,j} + a_{2,j} > 0\).

We can suppose \(t_0/m_0 = \min \frac{t_j}{m_j}\) then \(t_0\) is clearly negative. Now we have \(a_{2,0} \neq 0\) because \(\theta_1 \geq 0\) and then \(Z_0\) gives a section

\[
\sigma \in H^0(X, \mathcal{O}_X(m_0) \otimes \mathcal{O}_X(0))
\]

(we use the identity \(\mathcal{O}_X(a_1, a_2) = \mathcal{O}_X(a_1 + a_2) \otimes \mathcal{O}_X(-a_1 \mathcal{K}_2)\). Then, by definition, we obtain \(\overline{B}_2 \subset Z_\sigma\) and we have equality if and only if \(Z\) is the unique irreducible section with \(t < 0\). As \(t_j/m_j \leq t/m\) we conclude, in this case that \(\theta_2 = t_0/m_0\). \(\square\)

As a generalisation to the log-case of the main theorem of [DEG00] we have the following

**1.3.3. Theorem.** — Let \((X,D)\) be a nonsingular surface of log-general type with \(\text{Pic}(X) = \mathbb{Z}\). Suppose that \(\theta_2 < 0\) and that the log-Chern numbers of \(X\) satisfy

\[
(13 + 12 \theta_2) \sqrt{c_1} > 9 \sqrt{c}_2.
\]

Then every Zariski-dense holomorphic map \(f : \mathbb{C} \to X \setminus D\) is a leaf of an algebraic multi-foliation on \(X\).
Proof. — If \( \theta_1 < 0 \) then \( \overline{B}_1 \neq X_1 \) and we conclude by a direct application of Theorem 1.1.3. (the foliation is defined by the intersection of \( V_1 \) and the tangent of the irreducible component of \( \overline{B}_1 \) which contains the lifting of \( f \) to 1-jets) so we suppose that \( \theta_1 \geq 0 \). As \( \theta_2 < 0 \), then \( \overline{B}_2 \neq X_2 \) and the discussion made in Remark 1.2.3. shows that we have to consider only the case when the lifting of \( f \) to 2-jets is contained in a horizontal irreducible divisor \( Z \) in \( X_2 \). By Lemma 1.3.2 we have

\[
Z \sim a_1 u_1 + a_2 u_2 + t_0 \pi_{2,0}^* K_X \quad \text{in } \text{Pic}(X_2), \quad t_0 \in \mathbb{Q}, \quad t_0 < 0, \quad a_1 + a_2 = m_0,
\]

where \( t_0/m_0 = \theta_2 \).

Our aim now is to prove that the restriction of the tautological line bundle to \( Z \) is big. First, we have the following intersection equalities

\[
\begin{align*}
  u_1^4 &= 0, \quad u_1^3 u_2 = \overline{c}_2 - \overline{c}_2, \quad u_1^2 u_2^2 = \overline{c}_2, \quad u_1 u_2^3 = \overline{c}_1^2 - 3\overline{c}_2, \quad u_2^4 = 5\overline{c}_2 - \overline{c}_1^2, \\
  u_1^3 \cdot F &= 0, \quad u_1^2 u_2 \cdot F = -\overline{c}_1 \cdot F, \quad u_1 u_2^2 \cdot F = 0, \quad u_2^3 \cdot F = 0,
\end{align*}
\]

where \( F \) is any divisor in \text{Pic}(X).

Using this table, we obtain easily

\[
(2u_1 + u_2)^3 \cdot Z = m_0(13\overline{c}_1^2 - 9\overline{c}_2) + 12t_0 \overline{c}_1^2,
\]

moreover we have \( t_0/m_0 = \theta_2 \), hence

\[
(2u_1 + u_2)^3 \cdot Z = m_0((13 + 12\theta_2)\overline{c}_1^2 - 9\overline{c}_2) > 0.
\]

As in [DEG00] Proposition 3.4., we conclude that the restriction \( \mathcal{O}_{X_2}(1)|_Z \) is big. Consequently, by Theorem 1.1.3, every nonconstant entire curve \( f : \mathbb{C} \to X \) is such that \( f[2](\mathbb{C}) \) is contained in the base locus of \( \mathcal{O}_{X_2}(l) \otimes \pi_{2,0}^* \mathcal{O}(-A)|_Z \) for \( l \) large. This base locus is at most 2-dimensional, and projects onto a proper algebraic subvariety \( Y \) of \( X_1 \). Therefore \( f[1](\mathbb{C}) \) is contained in \( Y \), and the Theorem is proved. \( \square \)

1.4. Complement of curves in \( \mathbb{P}^2 \). — In this section we will consider the case \((\mathbb{P}^2, C)\) where \( C \) is a plane curve of degree \( d \). We will estimate the associated 2-jet log-threshold. We start with a vanishing theorem of log-symmetric differentials (similar of that of Sakai [Sa78]).

1.4.1. Lemma. — Let \( C \) be a smooth curve of degree \( d \) in \( \mathbb{P}^2 \), \( m \) a nonnegative integer and \( k \in \mathbb{Z} \). Then

\[
H^0(\mathbb{P}^2, S^m T^*_x \otimes \mathcal{O}(k)) = 0 \quad \text{for all } k \leq \min(m - 1, d - 2). \quad \text{In particular, for } d \geq 4, (\mathbb{P}^2, C) \text{ is of log-general type and we have the estimate}
\]

\[
(d - 3) \theta_{1,m} \geq \min(1, (d - 1)/m).
\]

Proof. — We consider the natural ramified covering \( X \subset \mathbb{P}^3 \) over \( \mathbb{P}^2 \) associated to \( C \) (if \( C \) is given by \( P(z_0, z_1, z_2) = 0 \), then \( X \) is defined by \( z_3^d = P(z_0, z_1, z_2) \)),
let $L$ be the hyperplane section in $X$ over $C$. Then we have an injective morphism (by taking pullbacks)

$$H^0(\mathbb{P}^2, S^m \Omega_{\mathbb{P}^2}(\log C)(k)) \hookrightarrow H^0(X, S^m \Omega_X(\log L)(k)),$$

the last group is contained in $H^0(X, S^m \Omega_X(m + k))$ which vanishes for $k \leq \min(m - 1, d - 2)$ by Lemma 5.1 in [DEG00].

Now we have $\overline{K}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(d - 3)$, consequently, there are no nonzero sections in $H^0(\mathbb{P}^2, S^m \mathcal{T}^*_{\mathbb{P}^2} \otimes \mathcal{O}(\mathcal{K}_{\mathbb{P}^2}))$ unless $t(d - 3) \geq \min(m, d - 1)$, whence the lower bound for $\overline{\vartheta}_{1,m}$. □

Using the above vanishing lemma we obtain a lower bound on the 2-jet log-threshold

1.4.2. Lemma. — Let $C$ be a curve of degree $d \geq 4$ in $\mathbb{P}^2$. Suppose that the 2-jet base locus $\overline{B}_2$ associated with $(X, D) = (\mathbb{P}^2, C)$ is of the form $Z_\sigma = Z_0 \cup \Gamma_2$, where $Z_0$ an irreducible section and $Z_\sigma$ is the set of zeros of a section $\sigma \in H^0(\mathcal{X}_2, \mathcal{O}_{\mathcal{X}_2}(m_0) \otimes \mathcal{O}(t_0 \mathcal{K}_X))$. Then for $m_0 \geq 6$ we have the estimate

$$\overline{\vartheta}_2 = \overline{\vartheta}_{2,m_0} \geq \max\left(-\frac{1}{m_0}; \min\left(\frac{1}{2(d - 3)} - \frac{1}{6}; \frac{d - 1}{2m_0(p_0 - 1)(d - 3)} - \frac{1}{6}\right)\right),$$

where $p_0 = \left\lfloor \frac{m_0}{d} \right\rfloor$.

Proof. — Observe that $\sigma$ can be considered as a global holomorphic section of the bundle $E_{2,m_0} \mathcal{T}^*_X \otimes \mathcal{O}(t_0 \mathcal{K}_X)$. By the filtration of $E_{2,m_0} \mathcal{T}^*_X$, we have a short exact sequence

$$0 \to S^m \mathcal{T}^*_X \to E_{2,m_0} \mathcal{T}^*_X \to E_{2,m_0-3} \mathcal{T}^*_X \otimes \mathcal{O}(\mathcal{K}_X) \to 0.$$

Multiply all terms by $\mathcal{O}(t_0 \mathcal{K}_X)$ and consider the associated sequence in cohomology. As $t_0 < 0$ and by Lemma 1.4.1, the first $H^0$ group vanishes and we get an injection

$$H^0(X, E_{2,m_0} \mathcal{T}^*_X \otimes \mathcal{O}(t_0 \mathcal{K}_X)) \hookrightarrow H^0(X, E_{2,m_0-3} \mathcal{T}^*_X \otimes \mathcal{O}(t_0 + 1 \mathcal{K}_X)).$$

By assumption on $\overline{B}_2$ we must have $t_0 + 1 \geq 0$, this gives the first part of the estimate. Let now $m_0 = 3p_0 + q_0$, $0 \leq q_0 \leq 2$ a positive integer. Then there is a (nonlinear) discriminant mapping

$$\Delta : E_{2,m_0} \mathcal{T}^*_X \otimes \mathcal{O}(t_0 \mathcal{K}_X) \to S^{(p_0-1)(3p_0+2q_0)} \mathcal{T}^*_X \otimes \mathcal{O}((p_0 + 2t_0)(p_0 - 1) \mathcal{K}_X).$$

In fact, we write an element of $E_{2,m} \mathcal{T}^*_X$ in the form

$$P(f) = \sum_{0 \leq j \leq p} a_j \cdot f^{3(p-j)+q} W^j$$

where the $a_j$ is viewed as an element of $S^{3(p-j)+q} \mathcal{T}^*_X \otimes \mathcal{K}_X$, and $W \in \mathcal{K}_X^{-1}$. The discriminant $\Delta(P)$ is calculated by interpreting $P$ as a polynomial in the indeterminate $W$. 
Applying this to $\sigma$, we obtain $2(p_0 - 1)t + p_0(p_0 - 1) \geq (p_0 - 1)(3p_0 + 2q_0)\theta_{1,(p_0-1)(3p_0+2q_0)}$ and this implies
\[ \frac{t}{m} \geq \frac{3p_0 + 2q_0}{2m_0} \theta_{1,(p_0-1)(3p_0+2q_0)} - \frac{p_0}{2m_0}. \]
Using Lemma 1.4.1 again we obtain the remainder part of the estimate. \( \square \)

We now turn to the question of the existence of 2-jet differentials of small degree. Recall from the filtration of the bundle of 2-jet differentials that we have an exact sequence
\[ 0 \to S^mT^*_X \to E_{2,m}T^*_X \xrightarrow{\Phi} E_{2,m-3}T^*_X \otimes K_X \to 0. \]
We have the following “proportionality” Lemma

**1.4.3. Lemma.** — Let $(X, D)$ be a nonsingular surface of log-general type. Then, for all sections
\[ P_i \in H^0(X, E_{2,m_i}T^*_X \otimes \mathcal{O}_X(t_i\bar{K}_X)) \]
with $m_i = 3, 4, 5$ and $t_i \in \mathbb{Q}$, the section $\beta_1P_2 - \beta_2P_1$ associated with $\beta_i = \Phi(P_i)$ can be considered as a section in
\[ H^0(X, S^{m_1+m_2-3}T^*_X \otimes \mathcal{O}_X((1 + t_1 + t_2)\bar{K}_X)), \]
and it vanishes when $1 + t_1 + t_2 \leq (m_1 + m_2 - 3)\bar{\theta}_{1,m_1+m_2-3}$.

**Proof.** — The section $\beta_1P_2 - \beta_2P_1$ is contained in $H^0(X, E_{2,m_1+m_2-3}T^*_X \otimes \mathcal{O}_X((1 + t_1 + t_2)\bar{K}_X))$, its image by $\Phi$ is zero. Then it can be considered as a section in $H^0(X, S^{m_1+m_2-3}T^*_X \otimes \mathcal{O}_X((1 + t_1 + t_2)\bar{K}_X))$. \( \square \)

Now we have the following application of the proportionality Lemma

**1.4.4. Lemma.** — Let $C$ be a generic curve of degree $d \geq 6$. Then
\[ \overline{\theta}_{2,m} \geq -\frac{1}{2m} \left( \frac{1 - (3 + \epsilon)/2m}{d - 3} \right) \quad \text{for } m = 3, 4, 5, \]
where $\epsilon := d \mod 2$.

**Proof.** — We consider the curve
\[ C_a = \left\{ z_0^{k_0}(z_0^{d-k_0} + a z_1 z_2^k) + z_1^d + z_2^d = 0 \right\}, \]
where $k_0, k_1, k_2 \geq 0$ are integers with $\sum k_i = d$ and $a$ a complex number such that $C_a$ is non singular. We put
\[ s_0 = z_0^{k_0}(z_0^{d-k_0} + a z_1 z_2^k), \quad s_i = z_i^d, \quad i = 1, 2. \]
Using Nadel’s method [Na89], we solve the linear system
\[ \sum_{0 \leq k_2 \leq 2} \tilde{g}_{ij}^k \frac{\partial s_{\ell}}{\partial z_k} = \frac{\partial^2 s_{\ell}}{\partial z_i \partial z_j}, \quad 0 \leq i, j, \ell \leq 2, \]
and get in this way a homogeneous meromorphic connection of degree $-1$ on $\mathbb{C}^3$. One can check that this connection descends to a partial projective meromorphic
connection $\nabla = (\Gamma^k_{ij})$ on $\mathbb{P}^2$ such that $C_a$ is totally geodesic (see [EG96] and [DEG97]), the pole divisor of the connection $\nabla$ is given by

$$B = \{ z_0 z_1 z_2 (d^{k_1+k_2}_0 + ak_0 z_1 z_2^k) = 0 \}.$$ 

Then, this connection can be seen as a meromorphic connection on $\mathcal{T}_{\mathbb{P}^2}$. In fact, if we take two tangents vector fields $u$ and $v$ to $C_a$, the vector field $\nabla_u v$ is also tangent to $C_a$ by construction.

Consequently, by taking the Wronskian operator

$$W_{\nabla}(f) = f' \wedge f''_v, \quad f'' = \nabla_{f'} f',$$

we get a section

$$P_1 \in H^0 E_{2,3} \mathcal{T}_{\mathbb{P}^2}^* \otimes \mathcal{O}(-B_2) \otimes \mathcal{O}(B) = H^0 E_{2,3} \mathcal{T}_{\mathbb{P}^2}^* \otimes \mathcal{O}(t_1 B_{\mathbb{P}^2}),$$

where $t_1 = \frac{3+k+k_2}{d-3} - 1$. Remark that $p = 3 + k_1 + k_2$ can be taken equal to any integer in $[3,d+3]$.

We take $p = \left[ \frac{d-1}{2} \right]$ (the biggest integer less or equal to $\frac{d+1}{2}$), so that

$$\frac{1}{2} + t_1 = \frac{3+\varepsilon}{2(d-3)}, \quad \text{where} \quad \varepsilon = d \mod 2, \quad \varepsilon \in \{0,1\}.$$

The integer $p$ must be at least equal to 3, thus our choice is permitted if $d \geq 6$. We claim that $\mathbb{P}^2$ has no non trivial section in

$$H^0(\mathbb{P}^2, E_{2,m} B_{\mathbb{P}^2} \otimes \mathcal{O}(B)), \quad m = m_2 = 3, 4, 5,$$

if $\frac{1}{2} + t < \frac{m-3/2-\varepsilon/2}{d-3}$. We assume the contrary, so that $P_2 \in H^0(\mathbb{P}^2, E_{2,m} B_{\mathbb{P}^2} \otimes \mathcal{O}(B)), P_2 \neq 0$. Then, for $m_1 = 3, m_2 = m$ and $t_2 = t$, our choices imply

$$1 + t_1 + t_2 < \frac{m}{d-3} \leq (m_1 + m_2 - 3)\theta_{1,m_1+3m_2-3},$$

as $\theta_{1,m} \geq \frac{1}{d-3}$, for all $m = 3, 4, 5$ and $d \geq 6$ (by 1.4.1). By Lemma 1.4.3, we get a meromorphic connection with logarithmic pole on $C_a$ associated with a Wronskian operator $P_2/\beta_2 = P_1/\beta_1$. As $P_1/\beta_1$ is an irreducible fraction with $\text{div} \beta_1 = B$, we conclude that $\beta_2/\beta_1 \in H^0(\mathbb{P}^2, S^{m-3} B_{\mathbb{P}^2} \otimes \mathcal{O}(B_{\mathbb{P}^2}))),$ must be holomorphic, hence

$$t_2 \geq t_1 + (m-3)\theta_{1,m-3} \geq t_1 + \frac{(m-3)}{d-3},$$

On the other hand

$$t_2 = t < -\frac{1}{2} + \frac{m-3/2-\varepsilon/2}{d-3} = t_1 + \frac{m-3-\varepsilon}{d-3},$$

which yields a contradiction. By the Zariski semicontinuity of cohomology, the group

$$H^0(\mathbb{P}^2, E_{2,m} B_{\mathbb{P}^2} \otimes \mathcal{O}(B_{\mathbb{P}^2})))$$

vanishes for a generic curve $C$ of degree $d \geq 6$, unless

$$\frac{t}{m} \geq -\frac{1}{2m} + \frac{1-(3+\varepsilon)/2m}{d-3}.$$
this yields the estimate. □

□

As a corollary we obtain the main result of this first part

1.4.5. Theorem.

Every non degenerate holomorphic entire map into the complement of a generic curve of degree \( d \geq 15 \) in \( \mathbb{P}^2 \) is a leaf of a multi-foliation on \( \mathbb{P}^2 \).

Proof. — Let \( C \) be a generic curve of degree \( d \geq 15 \) in \( \mathbb{P}^2 \). If \( \overline{\mathcal{E}}_2 \) is not as in Lemma 1.4.2, then we are done (we have two independent sections), so suppose that \( \overline{\mathcal{E}}_2 = Z \cup \Gamma_2 \) (with the notations of lemma 1.4.2). If \( m_0 = 3, 4, 5, 6, 7 \) (we have clearly \( m_0 \geq 3 \)) we apply 1.4.4 and 1.4.2 to get the estimate on the 2-jet threshold

\[
\overline{\theta}_2 \geq \frac{(3 - \epsilon)}{6(d - 3)} - \frac{1}{6},
\]

where \( \epsilon = d \mod 2 \). According to this estimate,

\[
(13 + 12\overline{\theta}_2)c_1^2 - 9c_2 \geq (d - 3)(2d - 27 - 2\epsilon) - 27
\]

this is positive when \( d \geq 15 \) and we can apply Theorem 1.3.2 to obtain the statement.

When \( m_0 \geq 8 \) we apply the estimate in lemma 1.4.2, we obtain \( \overline{\theta}_2 \geq -\frac{1}{8} \). It is easy to verify that \((13 - 3/2)c_1^2 - 9c_2 = (d - 3)(2, 5d - 34, 5) - 27\) is positive for \( d \geq 15 \) and we can again apply Theorem 1.3.2.

□

2. Entire leaves of foliations on log-general type surfaces

In this part we will generalize the main result in [Mc98], we will follow basically the strategy in [Br99] with a little improvement due to the ”convenience” of the logarithmic formalism.

2.1. Singularities of foliations on surfaces. — Let \( X \) be a compact complex surface. Recall from [GM87] that we have a bijective correspondence between a holomorphic foliation \( \mathcal{F} \) on the surface \( X \) with isolated singularities and a locally free subsheaf of the tangent sheaf noted \( T_{\mathcal{F}} \). In this case we have an exact sequence of sheaves

\[
0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_X \longrightarrow N_{\mathcal{F}}.I_Z \longrightarrow 0,
\]

where \( N_{\mathcal{F}} \) is called the normal bundle of the foliation and \( I_Z \) an ideal supported on the singularity set \( Z \) of \( \mathcal{F} \).

The elements of \( Z \) are the points where local vector fields defining \( \mathcal{F} \) vanish. Suppose that \( \mathcal{F} \) is given around a singular point \( p \) by a vector field \( v \), then we note by \( \lambda_1 \) and \( \lambda_2 \) the eigenvalues of the linear part of \( v \) and we make the following
2.1.1 Definition. — The singularity \( p \) is called reduced if the linear part \((Dv)(p)\) is nonzero (say \( \lambda_2 \neq 0 \)) and the quotient \( \lambda = \frac{\lambda_1}{\lambda_2} \) is not a positive rational number.

A reduced singularity at \( p \) is called nondegenerate if \( \lambda_1 \lambda_2 \neq 0 \) and a saddle-node otherwise. The importance of those singularities comes from the following well-known theorem

2.1.2 Theorem([Se68]). — There exist a sequence of blow-ups \( \sigma : \tilde{X} \to X \) such that the foliation \( \sigma^* \mathcal{F} \) has only reduced singularities.

Now let \( D \) be a normal crossing divisor on \( X \), we say that the foliation \( F \) defines a logarithmic foliation on \((X,D)\) if \( F \) is tangent to each component of \( D \).

The sheaf injection from \( \mathcal{T}
\mathcal{F}
\hookrightarrow \mathcal{T}
\mathcal{X} \) factors to a sheaf injection
\[
0 \to \mathcal{T}
\mathcal{F}
\hookrightarrow \mathcal{T}
\mathcal{X}(-\log D) \to \mathcal{N}
\mathcal{F}(-D)I_{Z'}, \to 0,
\]
where \( Z' \) is the set of logarithmic singularities of \( \mathcal{F} \) which is obviously contained in \( Z \). The bundle \( \mathcal{N}
\mathcal{F}(-D) \) will be called the logarithmic normal bundle of \( \mathcal{F} \), and denoted by \( \mathcal{N}
\mathcal{F} \).

To a logarithmic foliation \((\mathcal{F}, D)\) we associate a (singular) surface \( \tilde{X} \) in the 1-jet logarithmic space \( \overline{X}_1 \), called the logarithmic graph of \((\mathcal{F}, D)\), which consists of the adherence of the liftings of all leaves or equivalently the blow-up of \( X \) along the ideal \( I_{Z'} \).

When \( \mathcal{F} \) has only reduced singularities, and if \( p \) is a nondegenerate singularity in \( Z' \), then the surface \( \tilde{X} \) is smooth around \( p \) and isomorphic around \( \pi^{-1}(p) \) to the blow up of \( X \) at \( p \). In the case \( p \) is a saddle node of multiplicity \( d \) (i.e., the first terms of \( v \) in local coordinates are \( z \frac{\partial}{\partial z} + w^d \frac{\partial}{\partial w} \)), then \( \tilde{X} \) has a singular point of type \( A_{d-1} \) (i.e., it is given in local coordinates by \( z^d = xy \)).

We finish this section by the following useful natural formula
\[
\mathcal{O}
\overline{X}_1(-1)|\tilde{X} = \pi^*(\mathcal{T}
\mathcal{F}) \otimes \mathcal{O}(\sum_{p \in Z'} d_p E_p),
\]
where \( d_p \) is the multiplicity of \( p \) and \( E_p \) the fibre \( \pi^{-1}(p) \).

2.2. The log-tautological inequality and consequences. — Recall from [Mc98] that to a holomorphic curve \( f : \mathbb{C} \to X \), where \( X \) is supposed to be endowed with a Kähler form \( \omega \), we can associate a closed positive current in the following way: for every 2-form \( \eta \) and \( r > 0 \) we define
\[
T_r(\eta) = \int_0^r \frac{dt}{t} \int_{D(t)} f^* \eta,
\]
where \( D(t) \) is the disc of radius \( t \). Then we consider the positive currents \( \Phi_r \) defined by
\[
\Phi_r(\eta) := \frac{T_r(\eta)}{T_r(\omega)} \quad \forall \eta \in A^2(X).
\]
The family \( \{\Phi_r\}_{r > 0} \) is bounded and we can see easily that there is a closed positive current \( \Phi \) in its adherence. When \( f(\mathbb{C}) \) is not contained in a hypersurface \( Y \), we
prove, using the Lelong-Poincaré formula, that $\Phi$ has positive intersection with $Y$. As a consequence, if $f$ is nondegenerated, then the current $\Phi$ is actually numerically effective.

Remark that this construction is independent from the dimension of $X$. Then, we can associate a positive current $\Phi_1$ on the $\overline{X}_1$ to the curve $f_1$ (the lifting of $f$). Now if we suppose that $f$ intersects $D$ in finite set, we have the following logarithmic tautological inequality (see [Vo99] and [Mc99])

$$\mathcal{O}_{\overline{X}_1}(-1).\Phi_1 \geq 0,$$

As a consequence of this inequality we have $\pi_*\Phi_1 = \Phi$. From now on we will suppose that there is a logarithmic foliation $\mathcal{F}$ on $(X,D)$ with reduced singularities such that $D$ is the union of its algebraic leaves and $f$ is a Zariski dense entire leaf. Let $\nu(\Phi, p) = [\Phi_1],[d_pE_p]$, where $d_p$ and $E_p$ are defined in the previous section, then we have the following

2.2.1. Observation. — The logarithmic tautological inequality applied to the triple $(X, \mathcal{F}, D)$ implies the refined tautological inequality.

Proof. — We apply the logarithmic tautological inequality to $f_1$ which gives

$$\pi^*(\mathcal{T}_\mathcal{F}) \otimes \mathcal{O}(\sum_{p \in Z'} d_pE_p).\Phi_1 \geq 0.$$

As $\pi_*\Phi_1 = \Phi$, then we obtain

$$T_{\mathcal{F}}.\Phi \geq -\sum_{p \in Z'} \nu(\Phi, p).$$

This last inequality is exactly the refined tautological inequality because the intersection of algebraic leaves are not counted in $Z'$, in fact those points are smooth from the logarithmic point of view. □

Now we iterate this construction: $(X,D)$ will be replaced by $(\tilde{X}, \tilde{D})$, where $\tilde{D}$ is the union of all algebraic leaves of the induced foliation on $\tilde{X}$ (we replace $\tilde{X}$ by its desingularisation if necessary). The last step is almost the same as in [Mc98] and [Br99]. Let $(X^{(n)}, \mathcal{F}^{(n)})$ be the foliated surface obtained after $n$ iterations of this process and let $\pi^{(n)}$ denotes the canonical morphism from $X^{(n)}$ to $X^{(0)} = X$. As the singularities are reduced we have

$$\mathcal{F}^{(n)} = (\pi^{(n)})^*(\mathcal{F}).$$

Suppose (for simplicity) that we start from $Z' = \{p\}$ only one point of multiplicity $d_p$, then $\mathcal{F}^{(n)}$ have at most two logarithmic singularities: surely $q_1^n$ of multiplicity $d_p$ and probably $q_2^n$ of multiplicity 1 so the log-taualogical inequality gives

$$T_{\mathcal{F}}.\Phi = T_{\mathcal{F}^{(n)}}.\Phi^{(n)} \geq -\nu(\Phi^{(n)}, q_1^{(n)}) - \nu(\Phi^{(n)}, q_2^{(n)}).$$
Now we prove that $\nu(\Phi^{(n)}, q_i^{(n)})$ tends to zero as $n$ tends to infinity, in fact, by comparing $\Phi^{(n)}$ and $\Phi$, we obtain the inequality

$$0 \leq [\Phi^{(n)}]^2 \leq [\Phi]^2 - \sum_{j=0, i=1, 2}^{n-1} \nu(\Phi^{(n)}, q_i^{(n)}).$$

Consequently, we obtain

**2.2.2. Theorem ([Mc98]).** — Let $\mathcal{F}$ a holomorphic foliation (with reduced singularities) on a compact surface $X$ and $\Phi$ the current associated to a Zariski dense entire leaf. Then we have the intersection inequality

$$T_{\mathcal{F}}.\Phi \geq 0.$$

**2.3. Positivity of the log-normal bundle on leaves.** — Recall first the following result from [Br99]

**2.3.1. Theorem ([Br99]).**

Let $\mathcal{F}$ be a holomorphic foliation (with reduced singularities) on a compact surface $X$ and $\Phi$ be an diffuse current (i.e., with zero Lelong numbers except at a finite set of points. Suppose that $\Phi$ is $\mathcal{F}$-invariant. Then, we have the intersection inequality

$$N_{\mathcal{F}}(-D).\Phi \geq 0,$$

forall invariant divisor $D$.

The proof of the previous theorem consists of an explicit construction of a closed 2-form which represents the Chern class of $N_{\mathcal{F}}(-D)$. The intersection is computed by integrating this form along the support of $\Phi$ which is a union of leaves. This integration is concentrated around singularities for which structure and holonomy are well understood.

Now using this, the generalisation of the positivity of the normal bundle on the current associated with a Zariski dense leaf to the log-case is immediate from the following

**2.3.2. Corollary.**

Let $(\mathcal{F}, D)$ a holomorphic log-foliation with reduced singularities and $\Phi$ an $\mathcal{F}$-invariant current such that the support of $\Phi_{\text{alg}}$ is contained in $D$. Then we have the following intersection inequality for the logarithmic normal bundle

$$\overline{N}_{\mathcal{F}}.\Phi \geq 0.$$

**Proof.** — By Theorem 2.3.1, it remains to prove that $\overline{N}_{\mathcal{F}}.\Phi_{\text{alg}} \geq 0$. We use the same observation as in [Br99]: let $C$ be a component of $\Phi_{\text{alg}}$, then

$$N_{\mathcal{F}}(-D).C = C.C + Z(C, \mathcal{F}) - D.C,$$
where $Z(C, \mathcal{F})$ is the total multiplicity of the singularities of $\mathcal{F}$ along $C$ (cf [Br97] lemme 3). This number is at least equal to the intersectin $(D - C).C$, so we obtain $N_{\mathcal{F}}(-D).C \geq 0$. This is True for every component in the support of $\Phi_{\text{alg}}$, which concludes the proof.

As a consequence we have the following

2.3.3. Theorem. — Let $(X, D)$ be a surface of log-general type with a logarithmic foliation $(\mathcal{F}, D)$, then an entire leaf of $\mathcal{F}$ must be degenerated.

Proof. — By the Seidenberg theorem we can suppose that $\mathcal{F}$ has only reduced singularities. Suppose that $\mathcal{F}$ has a Zariski dense entire leaf. Then, by theorem 2.2.2 and corollary 2.3.2, as $K_X^{-1} = T_{\mathcal{F}} \otimes N_{\mathcal{F}}$, we obtain

$$K_X \otimes \mathcal{O}(D').\Phi \leq 0,$$

where $D'$ is the union between $D$ and the support of $\Phi_{\text{alg}}$. As $K_X \otimes \mathcal{O}(D) \hookrightarrow K_X \otimes \mathcal{O}(D')$, the latter bundle is big and, hence, has the decomposition $\mathcal{O}(A + E)$, where $A$ is ample and $E$ is effective. But $\Phi$ is numerically effective, so we obtain an obvious contradiction.

2.4. Entire leaves on a surface of log general type. — In this section we will generalise the main theorem in [Mc98]. We must consider now non necessarily logarithmic foliation.

2.4.1. Lemma. — Let $\mathcal{F}$ be a foliation on a surface $X$ with a non-invariant curve $C$. Suppose that there is a Zariski-dense entire curve $f : \mathbb{C} \to X \setminus C$. Then there exists a sequence of blow ups $\sigma : \tilde{X} \to X$ such that if we denote by $\tilde{C}$ the strict transform of $C$ and $\tilde{f}$ the lifting to $\tilde{X}$ of $f$, the support of an associated current $\tilde{\Phi}$ to $\tilde{f}$ is disjoint from $C$. In particular, $\tilde{\Phi}.\tilde{C} = 0$.

Proof. — By the Seidenberg theorem, using a sequence of blow ups $\tilde{X} \to X$, we can reduce the singularities of $\mathcal{F}$ and suppose that the leaves of the induced foliation $\tilde{\mathcal{F}}$ are smooth, we can also suppose that these leaves are transverse to $\tilde{C}$ by blowing up the tangency points. Let $\mathcal{L}$ be the leaf containing the image of $\tilde{f}$, then as $f$ is Zariski-dense, $\mathcal{L}$ intersects $\tilde{C}$ on at most one point ($\mathcal{L}$ is parametrised by $\mathbb{C}$ or $\mathbb{C}^*$). Blowing up this point if it exists, we can suppose that $\mathcal{L}$ does not intersect $\tilde{C}$. We will prove that the topological closure of $\mathcal{L}$, which we denote by $K$, does not intersect $\tilde{C}$. Remark that $K$ is a union of leaves and let $p$ a point on $\tilde{C} \cup K$, so there is a leaf $\mathcal{L}_p$ in $K$ passing by $p$. Now $\mathcal{L}$ accumulates on $\mathcal{L}_p$ and this leaf is transverse to $\tilde{C}$, as consequence in a neighbourhood of $p$ the number of intersection points of $\mathcal{L}$ with $C$ is infinite, this is a contradiction. Finally, the support of $\tilde{\Phi}$ is contained in this closure so does not intersect $\tilde{C}$. □

As a consequence of Theorem 2.3.3 and the previous Lemma we obtain the following generalisation of the main theorem of [Mc98]

2.4.2. Theorem. — Let $X$ be a surface with a foliation $\mathcal{F}$ and a divisor $D$ such
that \((X, D)\) is of log-general type. Then every entire curve \(f : \mathbb{C} \to X \setminus D\) contained in a leaf of \(\mathcal{F}\) is degenerated.

**Proof.** — Suppose that we have a Zariski-dense entire curve contained in a leaf of \(\mathcal{F}\). We make a sequence of blow-ups to reduce the singularities of \(\mathcal{F}\) and to make the non-invariant components of \(D\) verify the conclusion of Lemma 2.4.1. We will use the same notation on the obtained surface. Let \(D_1 \subset D\) be the union of the components of \(D\) transverse to \(\mathcal{F}\), and \(D_2\) the union of algebraic leaves, then, using Lemma 2.4.1, we obtain
\[
(K_X + D_1 + D_2)\cdot \Phi = (K_X + D_2)\cdot \Phi.
\]
As \(\mathcal{F}\) induce a logarithmic foliation on \((X, D_2)\), using the proof of Theorem 2.3.3 we obtain
\[
(K_X + D_1 + D_2)\cdot \Phi = 0,
\]
which implies
\[
(K_X + D_1 + D_2)\cdot \Phi \leq 0.
\]
Now the divisor \(K_X + D_1 + D_2\) is big and \(\Phi\) is nef, so we have a contradiction. \(\Box\)

As a consequence, we obtain the following

**2.4.3 Corollary.** — Let \((X, D)\) be a log-surface of log-general type such that its logarithmic Chern classes verify \(\tau_1^2 > \tau_2\). Then every entire curve \(f : \mathbb{C} \to X \setminus D\) is degenerated.

**Proof.** — We can suppose that \((X, D)\) is minimal. Then we apply Riemann-Roch to symmetric powers of \(\overline{T}_X\), the Euler characteristic is positive with our condition on Chern classes. Using Serre duality and nefness of \(K_X + D\), the \(h^2\) term is bounded by the \(h^0\) term. As a consequence we get \(\overline{B}_1 \neq \overline{X}_1\), and we apply Theorem 1.1.3 to get a foliation \(\mathcal{F}\) on a (singular) surface \(\tilde{X}\) in \(X_1\) which is ramified over \(X\) (the foliation is defined by the intersection of \(V_1\) and the tangent of the irreducible component of \(\overline{B}_1\) which contains the lifting of \(f\) to 1-jets). Let \(\tilde{D}\) be the divisor on \(\tilde{X}\) over \(D\), then \((\tilde{X}, \tilde{D})\) is of log-general type and \(f\) can be lifted in \(\tilde{X} \setminus \tilde{D}\) as a leaf of the foliation \(\mathcal{F}\). \(\Box\)

### 3. Proof of the Main Theorem

By the results of Theorem 1.4.5, if \(C\) is a generic plane curve of degree \(d \geq 15\), then there is a ramified cover \(\tilde{X} \subset \overline{X}_1\) over \(X = \mathbb{P}_C^2\) with a foliation \(\mathcal{F}\) such that every entire curve \(f \in \mathbb{P}_C^2 \setminus C\) is such that \(f[2]\) is contained in \(\tilde{X}\) as leaf of \(\mathcal{F}\). Moreover, \(f[2]\) is contained in \(\tilde{X} \setminus \tilde{C}\) where \(\tilde{C}\) is the (reduced) divisor in \(\tilde{X}\) over \(C\).

A log model of \((\tilde{X}, \tilde{C})\) is clearly of log-general type. In fact, recall that given two log-manifolds \((X, D_X)\) and \((Y, D_Y)\), a holomorphic map \(\psi : X \to Y\) such that \(\psi^{-1}D_Y \subset D_X\) (in the geometric sense) is called a log-morphism. Such a morphism induces (see [Ii77]) a vector bundle morphism
\[
\psi^* : \psi^*T_Y^\vee \to T_X^\vee.
\]
If $\psi$ is birational, then this morphism is clearly injective. Thus we have a natural injection of sheaves $\psi^*\mathcal{K}_Y \hookrightarrow \mathcal{K}_X$.

Now, by Theorem 2.4.2 every entire curve $f$ in $X \setminus C$ has its lifting in $\tilde{X} \setminus \tilde{C}$ degenerated and then itself has its image contained in an algebraic plane curve. Now every algebraic curve in $\mathbb{P}^2$ intersects a very generic curve of degree $d \geq 5$ in at least 3 point (see [Si95]) and then $f$ is constant and $\mathbb{P}^2_{\mathbb{C}} \setminus C$ is hyperbolic and hyperbolically embedded in $\mathbb{P}^2_{\mathbb{C}}$ (see [Gr77]).

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