ON A NONLOCAL BOUNDARY-VALUE PROBLEM FOR TWO-DIMENSIONAL ELLIPTIC EQUATION

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — A boundary-value problem with a nonlocal integral condition is considered for a two-dimensional elliptic equation with constant coefficients and a mixed derivative. The existence and uniqueness of a weak solution of this problem are proved in a weighted Sobolev space. A difference scheme is constructed using the Steklov averaging operators. It is proved that the difference scheme converges in discrete $W^{1,2}_2(\omega, \rho)$ norm with the rate $O(h^{m-1})$, $m \in (1; 3]$, when the solution of the problem belongs to the space $W^m_2(\Omega)$.

2000 Mathematics Subject Classification: 65N10, 35J25.

Keywords: nonlocal boundary-value problem, difference scheme, elliptic equation, weighted spaces.

1. Introduction

Boundary-value problems for differential equations with a nonlocal condition occur in many applications. Problems with integral conditions were considered by various authors (see, e.g., [1, 8, 9]). In the present paper, a nonlocal boundary problem with integral restriction is considered in a domain $\Omega = (0, 1)^2$ for a second order elliptic equation with constant coefficients.

In Section 2, existence and uniqueness of a weak solution of this problem in the weighted Sobolev space $W^1_2(\Omega, \rho)$, $\rho(x) = x^\varepsilon$, $\varepsilon \in (0; 1)$ is proved.

In Section 3, the corresponding difference scheme is constructed. Under the assumption that the solution to the original problem belongs to Sobolev spaces, the estimate of convergence rate

$$||y - u||_{W^{1,2}_2(\omega, \rho)} \leq c h^{m-1} ||u||_{W^m_2(\Omega)}, \ m \in (1; 3]$$

is obtained, where $\omega$ is a uniform grid in $\Omega$ with the step $h$, $p = 2$ for $\varepsilon \in (0.5; 1)$, $p > 1/\varepsilon$ for $\varepsilon \in (0; 0.5]$.

2. Solvability of a nonlocal problem

Let $\Omega = \{(x_1, x_2) : 0 < x_k < 1, k = 1, 2\}$ be a unit square with a boundary $\Gamma$, and let $\Gamma_1 = \{(x_1, x_2) : 0 < x_2 < 1\}$, $\Gamma_* = \Gamma \setminus \Gamma_1$. 
Consider the nonlocal boundary-value problem with constant coefficients
\[ Lu = f(x), \ x \in \Omega, \ u(x) = 0, \ x \in \Gamma_*, \ l(u) = 0, \ 0 < x_2 < 1, \] (2)
where
\[ Lu = - \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a_0 u, \qquad l(u) = \int_{0}^{1} \beta(x_1) u(x_1) \, dx_1, \quad \beta(t) = \epsilon t^{\epsilon-1}, \ \epsilon \in (0;1) \]
and with the coefficients satisfying the following conditions:
\[ \sum_{i,j=1}^{2} a_{ij} t_i t_j \geq \nu_1 (t_1^2 + t_2^2), \quad \nu_1 > 0, \ a_0 \geq 0. \] (3)

Let
\[ (u, v) = \int_{\Omega} u(x) v(x) \, dx, \quad ||u|| = (u, u)^{1/2}. \]

By \( L_2(\Omega, \rho) \) we denote the weighted Lebesgue space of all real-valued functions \( u(x) \) on \( \Omega \) with the inner product and the norm
\[ (u, v)_{L_2(\Omega, \rho)} = \int_{\Omega} \rho(x) u(x) v(x) \, dx, \quad ||u||_{L_2(\Omega, \rho)} = (u, u)^{1/2}_{L_2(\Omega, \rho)}. \]

The weighted Sobolev space \( W^1_2(\Omega, \rho) \) is usually defined as a linear set of all functions \( u(x) \in L_2(\Omega, \rho) \), whose derivatives \( \partial u/\partial x_k, \ k = 1, 2 \) (in the generalized sense) belong to \( L_2(\Omega, \rho) \). It is a normed linear space if equipped with the norm
\[ ||u||_{W^1_2(\Omega, \rho)} = \left( ||u||^2_{L_2(\Omega, \rho)} + ||u|^{2}_{W^1_2(\Omega, \rho)} \right)^{1/2}, \quad ||u|^{2}_{W^1_2(\Omega, \rho)} = \left\| \frac{\partial u}{\partial x_1} \right\|_{L_2(\Omega, \rho)}^2 + \left\| \frac{\partial u}{\partial x_2} \right\|_{L_2(\Omega, \rho)}^2. \]

Let us choose weight function \( \rho(x) \) in the following way: \( \rho(x) = \rho(x_1) = \int_{0}^{x_1} \beta(t) \, dt = x_1^\epsilon \).

It is well-known (see, e.g., [4, p.10], [5, Theorem 3.1]) that \( W^1_2(\Omega, \rho) \) is a Banach space and \( C^\infty(\bar{\Omega}) \) is dense in \( W^1_2(\Omega, \rho) \) and in \( L_2(\Omega, \rho) \). As an immediate consequence, we can define the space \( W^1_2(\Omega, \rho) \) as the closure of \( C^\infty(\bar{\Omega}) \) with respect to the norm \( ||\cdot||_{W^1_2(\Omega, \rho)} \), and these both definitions are equivalent.

Define the subspace of the space \( W^1_2(\Omega, \rho) \) which can be obtained by closing the set
\[ C^\infty(\bar{\Omega}) = \left\{ u \in C^\infty(\bar{\Omega}) : \text{supp} u \cap \Gamma_* = \emptyset, \ \int_{0}^{1} \beta(x_1) u(x_1) \, dx_1 = 0, \ 0 < x_2 < 1 \right\} \]
with the norm \( ||\cdot||_{W^1_2(\Omega, \rho)} \). Denote it by \( W^*_2(\Omega, \rho) \).

Let the right-hand side \( f(x) \) in equation (2) be a linear continuous functional on \( W^*_2(\Omega, \rho) \) which can be represented as
\[ f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \quad f_k(x) \in L_2(\Omega, \rho), \quad k = 0, 1, 2. \] (4)
We say that the function \( u \in W^{1}_{2}(\Omega, \rho) \) is a weak solution of problem (2)–(4), if the relation
\[
a(u, v) = \langle f, v \rangle, \quad \forall v \in W^{1}_{2}(\Omega, \rho)
\] (5)
holds, where
\[
a(u, v) = \int_{\Omega} \left( a_{11} x^{\rho}_{1} \frac{\partial u}{\partial x_{1}} \frac{\partial v}{\partial x_{1}} + (a_{12} + a_{21}) x^{\rho}_{1} \frac{\partial u}{\partial x_{2}} \frac{\partial v}{\partial x_{1}} + a_{22} \frac{\partial u}{\partial x_{2}} \frac{\partial v}{\partial x_{2}} + a_{0} u G v \right) \, dx,
\] (6)
\[
\langle f, v \rangle = \int_{\Omega} f_{0} G v \, dx - \int_{\Omega} x^{\rho}_{1} f_{1} \frac{\partial v}{\partial x_{1}} \, dx - \int_{\Omega} f_{2} \frac{\partial v}{\partial x_{2}} \, dx,
\] (7)
\[
G v(x) = \rho v(x) - \int_{0}^{x_{1}} \beta(t) v(t, x_{2}) \, dt.
\] (8)
Equality (5) formally is obtained from \((Lu - f, Gv) = 0\) by integration by parts.

To prove the existence of the unique solution of problem (5) (weak solution of problem (2)–(4)) we will apply the Lax-Milgram lemma [2]. First we will prove some auxiliary results.

**Lemma 1.** Let \( u, v \in L^{2}(\Omega, \rho) \) and \( v \) satisfy the condition \( l(v) = 0 \). Then
\[
|(u, Gv)| \leq \frac{1 + \varepsilon}{1 - \varepsilon} ||u||_{L^{2}(\Omega, \rho)} ||v||_{L^{2}(\Omega, \rho)},
\] (9)
\[
||v||_{L^{2}(\Omega, \rho)} \leq (v, Gv),
\] (10)
\[
||v||_{L^{2}(\Omega, \rho^{2})} \leq ||Gv|| \leq (2 \varepsilon + 1) ||v||_{L^{2}(\Omega, \rho^{2})}.
\] (11)

**Proof.** Due to the density \( C^{\infty}(\bar{\Omega}) \) in \( L^{2}(\Omega, \rho) \) it suffices to prove the lemma for an arbitrary functions from the class \( C^{\infty}(\Omega) \). By virtue of the Cauchy inequality we have
\[
|(u, Gv)| \leq ||u||_{L^{2}(\Omega, \rho)} ||v||_{L^{2}(\Omega, \rho)} + \varepsilon J_{1}(v),
\] (12)
where
\[
J_{1}^{2}(v) = \int_{\Omega} x^{\rho}_{1} \left( \int_{0}^{x_{1}} t^{\rho_{1}} v(t, x_{2}) \, dt \right)^{2} \, dx = -\frac{2}{1 - \varepsilon} \int_{\Omega} v(x) \left( \int_{0}^{x_{1}} t^{\rho_{1}} v(t, x_{2}) \, dt \right) \, dx
\]
\[
\leq \frac{2}{1 - \varepsilon} ||v||_{L^{2}(\Omega, \rho)} \cdot J_{1}(v).
\]
Thus, \( J_{1}(v) \leq 2(1 - \varepsilon)^{-1} ||v||_{L^{2}(\Omega, \rho)} \) and the estimate (9) follows from (12).

Inequality (10) follows from the easily verifiable identity
\[
(v, Gv) = ||v||_{L^{2}(\Omega, \rho)}^{2} + \frac{\varepsilon(1 - \varepsilon)}{2} J_{1}^{2}(v).
\]

The first inequality in (11) is sequent of the identity
\[
||Gv||^{2} = \int_{\Omega} x^{2\rho} v^{2}(x) \, dx + (\varepsilon^{2} + \varepsilon) J_{2}(v), \quad J_{2}(v) = \int_{\Omega} \left( \int_{0}^{x_{1}} t^{\rho_{1}} v(t, x_{2}) \, dt \right)^{2} \, dx
\]
and in order to prove the second inequality of (11), it is enough to observe that

\[ J_2(v) = -2 \int \int_{x_1}^1 \int_0^{t-1} v(t, x_2) dt \, dx \leq 2 ||v||_{L^2(\Omega, \rho)}^2 (J_2(v))^{1/2}, \]

i.e., \( J_2(v) \leq 4 ||v||_{L^2(\Omega, \rho)}^2 \). This completes the proof of the lemma.

\[ \square \]

**Lemma 2.** Let \( u \in W_2^1(\Omega, \rho) \). Then

\[ ||u||_{W_2^1(\Omega, \rho)} \leq ||u||_{W_2^1(\Omega, \rho)} \leq c_1 ||u||_{W_2^1(\Omega, \rho)}, \quad c_1 = (4(1 + \varepsilon)^{-2} + 1)^{1/2}. \]

**Proof.** Due to the density \( \mathcal{C}_\infty^\ast(\overline{\Omega}) \) in \( W_2^1(\Omega, \rho) \), it is sufficient to prove the lemma for an arbitrary \( u \in \mathcal{C}_\infty^\ast(\overline{\Omega}) \). The first inequality of the lemma is obvious. Integrating by parts, we obtain

\[ \int \int_{x_1}^1 u^2(x) \, dx = - \int \int_{x_1}^{x_1+1} u(x) \left( \frac{\partial u}{\partial x_1} \right) \, dx. \]

Therefore,

\[ (1 + \varepsilon) \int \int_{x_1}^1 u^2(x) \, dx = -2 \int \int_{x_1}^{x_1+1} u(x) \left( \frac{\partial u}{\partial x_1} \right) \, dx \leq 2 ||u||_{L^2(\Omega, \rho)} \left( \int \int_{x_1}^{x_1+1} \left| \frac{\partial u}{\partial x_1} \right|^2 \, dx \right)^{1/2}, \]

that is

\[ ||u||_{L^2(\Omega, \rho)} \leq \frac{2}{1 + \varepsilon} \left( \int \int_{x_1}^{x_1+1} \left| \frac{\partial u}{\partial x_1} \right|^2 \, dx \right)^{1/2}, \]

which proves the lemma.

\[ \square \]

Application of both lemmas 1, 2 and condition (3), (6) gives the continuity

\[ |a(u, v)| \leq c_2 ||u||_{W_2^1(\Omega, \rho)} ||v||_{W_2^1(\Omega, \rho)}, \quad c_2 > 0, \quad \forall u, v \in W_2^1(\Omega, \rho) \]

and \( W_2^1 \)-ellipticity

\[ a(u, u) \geq c_3 ||u||_{W_2^1(\Omega, \rho)}^2, \quad c_3 > 0, \quad \forall u \in W_2^1(\Omega, \rho) \]

of the bilinear form \( a(u, v) \).

By applying lemmas 1, 2 from (7) we obtain the continuity of linear form \( \langle f, v \rangle \):

\[ |\langle f, v \rangle| \leq c_4 ||v||_{W_2^1(\Omega, \rho)}, \quad c_4 > 0, \quad \forall v \in W_2^1(\Omega, \rho). \]

Thus, all conditions of the Lax-Milgram lemma are fulfilled. Therefore, the following theorem is true.

**Theorem 1.** The problem (2)–(4) has unique weak solution from \( W_2^1(\Omega, \rho) \).
3. Finite-difference scheme

Consider the following grid domains in $\Omega$: $\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h : i_\alpha = 0, 1, \ldots, n, \ h = 1/n\}$, $\omega_\alpha = \bar{\omega}_\alpha \cap (0, 1)$, $\omega_\alpha^* = \bar{\omega}_\alpha \cap [0, 1]$, $\omega__\alpha = \bar{\omega}_\alpha \cap [0, 1]$, $\alpha = 1, 2$, $\omega = \omega_1 \times \omega_2$, $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\gamma_s = \Gamma_s \cap \bar{\omega}$. Let us denote $h = h/2$ for $x_1 = 0$, and $h = h$ for $x_1 \neq 0$.

For grid functions and difference ratios, we use the standard notation from [6].

Define the following averaging operators:

\[ S_1^- u = \frac{1}{h} \int_{x_1}^{x_1 + h} u(t, x_2) dt, \quad S_1^+ u = \frac{1}{h} \int_{x_1 - h}^{x_1} u(t, x_2) dt, \quad T_1^- u = \frac{1}{2} (T_1^- + T_1^+) u, \]

\[ T_1^+ u = \frac{2}{h^2} \int_{x_1}^{x_1 + h} \int_{x_1 - h}^{x_1} (h + x_1 - t) u(t, x_2) dt, \quad T_1^- u = \frac{2}{h^2} \int_{x_1}^{x_1 + h} \int_{x_1 - h}^{x_1} (h - x_1 + t) u(t, x_2) dt. \]

The operators $S_2 \pm, T_2$ are defined likewise.

We introduce the notation

\[ \beta^+ = T_1^+ \beta, \quad \beta^- = T_1^- \beta, \quad \beta_k = \frac{1}{2} (\beta^+(kh) + \beta^-(kh)), \quad \beta_0 = \beta^+_0 = 0, \]

\[ \rho^+ = \rho + \frac{h}{2} \beta^+, \quad \rho^- = \rho - \frac{h}{2} \beta^-, \quad \rho_i = \sum_{k=0}^{i} h \beta_k - \frac{h}{2} \beta^+_i, \quad \bar{\rho} = \frac{1}{2} (\rho^+ + \rho^-). \]

It is not hard to check that

\[ \rho_i = \rho(ih), \quad \rho^+ = S_1^+ \rho, \quad \rho^- = S_1^- \rho, \quad \bar{\rho}_0 = \frac{h}{4} \beta^+_0. \]

We will define the difference analogue of the operator $G$ from (8) in the following way:

\[ G_h y = \bar{\rho} y - Py, \quad P y(ih, x_2) = \sum_{k=0}^{n} h \beta_k y(kh, x_2) - \frac{h}{2} \beta^+_y(ih, x_2). \quad (13) \]

A set of grid-functions given on $\bar{\omega}$ and satisfying the condition

\[ y = 0, \quad x \in \gamma_s, \quad l_h (y) \equiv \sum_{k=0}^{n} \beta_k y(kh, x_2) = 0, \quad x_2 \in \omega_2 \quad (14) \]

will be denoted by $H$. On the set $H$ let us introduce the inner product and the norm

\[ (y, v) = \sum_{\omega} h^2 yv, \quad ||y|| = (y, y)^{1/2}, \quad \bar{\omega} \subseteq \bar{\omega}. \]

Let, moreover,

\[ (y, v)_0 = \sum_{\omega_1 \times \omega_2} h \bar{\rho} yv, \quad ||y||_0 = (y, y)_0^{1/2}, \quad ||y||_{\rho} = \sum_{\omega_1 \times \omega_2} h \bar{\rho} yv, \quad ||y||_{\rho}^2 = \sum_{\omega_1 \times \omega_2} h \bar{\rho} yv, \]

\[ ||y||^2 = ||y||^2_0 + ||\nabla y||^2, \quad ||\nabla y||^2 = ||y_{x_1}||^2_{(1)} + ||y_{x_2}||^2_{(2)}, \quad ||y_{x_1}||^2_{(1)} = (\rho^+ y_{x_1}, y_{x_1})_{\omega_1^+ \times \omega_2}, \]

\[ ||y_{x_2}||^2_{(2)} = ||y_{x_2}||^2_{\rho}, \quad ||y||^2_{\rho} = \sum_{\omega_2} h y^2, \quad ||y||^2_{\rho} = \sum_{\omega_2} h y^2. \]
We approximate problem (2)–(4) by the difference scheme

$$L_h y = -a_{11} y_{\bar{x}_1 x_1} - 2a_{12} y_{\bar{x}_1 x_2} - a_{22} y_{x_1 x_2} + a_0 y = \varphi(x), \quad x \in \omega, \quad y \in H,$$

where

$$\varphi = T_1 T_2 f_0 + (S_1 T_2 f_1)_{x_1} + (T_1 S_2 f_2)_{x_2}. $$

**Lemma 3.** The estimates

$$(y, G_h y)_{\omega} \geq ||y||_1^2, \quad (y, G_h y)_{\omega_1 \times \omega_2^+} \geq ||y||_\rho^2$$

are true for grid functions $y(x)$, satisfying the conditions $l_h(y) = 0, \quad y(1, x_2) = 0, \quad x_2 \in \omega_2$.

**Proof.** It is not difficult to verify that

$$-\sum_{i=1}^{n-1} h y(ih, x_1) P y(ih, x_2) = \frac{1}{2} \left( \frac{h}{2} \beta_1^+ y(0, x_2) \right)^2 + J_3,$$

where

$$J_3 = 0, \quad n = 2, \quad J_3 = \frac{1}{2} \sum_{i=2}^{n-1} \left( \frac{1}{\beta_i} - \frac{1}{\beta_{i-1}} \right) \left( P y(ih, x_2) - h \beta_i y(ih, x_2) \right)^2, \quad n > 2.$$

Due to $J_3 \geq 0$ because of $(1/\beta_i) - (1/\beta_{i-1}) > 0$, and also $\beta_1^+ > \beta_1$, the validity of Lemma 3 follows from (16).

**Lemma 4.** For any $y \in H$ the inequality

$$(L_h y, G_h y)_{\omega} \geq c_5 ||y||_1^2, \quad c_5 = \nu/4$$

holds.

**Proof.** Using summation by parts, we get

$$\sum_{\omega_1} h v_{x_1} G_h y = -\sum_{\omega_1^+} h \rho^- v y_{x_1}, \quad \sum_{\omega_1} h v_{x_1} G_h y = -\sum_{\omega_1^-} h \rho^+ v y_{x_1},$$

where $v$ is an arbitrary grid function. Hence

$$-(y_{\bar{x}_1 x_1}, G_h y)_{\omega} = \frac{1}{2} \sum_{\omega_1^1 \times \omega_2} h^2 \rho^-(y_{x_1})^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho^+(y_{x_1})^2,$$

$$-(y_{\bar{x}_1 x_2}, G_h y)_{\omega} = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \rho^+ y_{x_1} y_{x_2} + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho^- y_{x_1} y_{x_2}.$$

Besides, applying Lemma 3, we have

$$-(y_{\bar{x}_2 x_2}, G_h y)_{\omega} \geq ||y_{\bar{x}_2}||_{(2)}^2,$$

Let

$$\hat{\rho} = \rho + \frac{h}{2} \beta^+ - \frac{h}{4} \beta_0^+, \quad \bar{\rho} = \rho - \frac{h}{2} \beta^- + \frac{h}{4} \beta_0^+.$$
Then $\bar{\rho} = \frac{1}{2}(\hat{\rho} + \hat{\rho})$, $\bar{\rho}_0 = \frac{h}{2}\partial_{\varphi}^+$, and after some transformations we obtain

$-(y_{x_1x_1}, G_hy)_\omega = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2^-} h^2 \hat{\rho}(y_{x_1})^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2^{-}} h^2 \hat{\rho}(y_{x_1})^2,$

(21)

$-(y_{x_1 x_2}, G_hy)_\omega = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2^+} h^2 \hat{\rho}y_{x_1} y_{x_2} + \frac{1}{2} \sum_{\omega_1^- \times \omega_2^+} h^2 \hat{\rho}y_{x_1} y_{x_2},$

(22)

$-(y_{x_2 x_2}, G_hy)_\omega \geq \frac{1}{2} \sum_{\omega_1^+ \times \omega_2^+} h^2 \hat{\rho}(y_{x_2})^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2^+} h^2 \hat{\rho}(y_{x_2})^2,$

(23)

from (18), (19), and (20) respectively.

Taking into account (21)–(23), from (15) we have

$$4(L_h y, G_h y)_\omega \geq \sum_{\omega_1^+ \times \omega_2^-} h^2 \hat{\rho} F(y_{x_1}, y_{x_2}) + \sum_{\omega_1^- \times \omega_2^+} h^2 \hat{\rho} F(y_{x_1}, y_{x_2})$$

$$+ \sum_{\omega_1^- \times \omega_2^-} h^2 \hat{\rho} F(y_{x_1}, y_{x_2}) + \sum_{\omega_1^+ \times \omega_2^+} h^2 \hat{\rho} F(y_{x_1}, y_{x_2}) + a_0(y, G_h y)_\omega,$$

(24)

where $F(t_1, t_2) = a_{11}t_1^2 + 2a_{12}t_1t_2 + a_{22}t_2^2.$

Taking into account

$$\hat{\rho} = \frac{1}{h} \int_{x_1-h}^{x_1} \rho(t) dt + \frac{1}{2h} \int_0^h \rho(t) dt > 0,$$

$$\bar{\rho} = \frac{1}{h} \int_{x_1-h}^{x_1+h} \rho(t) dt - \frac{1}{2h} \int_0^h \rho(t) dt > 0,$$

due to the condition of ellipticity the estimate

$$(L_h y, G_h y)_\omega \geq \nu_1 ||\nabla y||^2$$

follows from (24), which together with (see [1])

$$||y||_0^2 \leq 4 ||y_{\bar{x}}||^2 \leq 4 ||\nabla y||^2$$

prove Lemma 4.

Thus, if $\varphi(x) = 0$, $x \in \omega$, then $y(x) = 0$, $x \in \bar{\omega}$ and, consequently, the solution of difference scheme (15) exists and it is unique.

**Lemma 5.** If the grid function $y$ defined on $\bar{\omega}$ satisfies the conditions $l_h(y) = 0$, $y(1, x_2) = 0$, $x_2 \in \omega_2$, then

$$\left|\sum_{\omega_1} hv G_h y\right| \leq c \left(\sum_{\omega_1} \hat{\rho} v^2\right)^{1/2} \left(\sum_{\omega_1} \hat{\rho} y^2\right)^{1/2},$$

where $v(x)$ is an arbitrary grid function.
Proof. By the definition of the operator $G_h$, we have

$$\left| \sum \omega \ h v G_h y \right| \leq \left( \sum \omega \ h \tilde{p} v^2 \right)^{1/2} \left[ \left( \sum \omega \ h \tilde{p} y^2 \right)^{1/2} + J_4(y) \right],$$  \hspace{0.5cm} (25)

where

$$J_4^2(y) = \sum \omega \ h (\tilde{p})^{-1}(Py)^2.$$

Let

$$2(\tilde{P}y)_i = \sum_{k=0}^{i} h \beta_k y(kh, x_2), \quad \sigma_i = \sum_{k=1}^{i} \frac{h}{\rho_k}, \quad \sigma_0 = 0.$$

Then

$$(\tilde{P}y)_i + (\tilde{P}y)_{i-1} = (Py)_i, \quad (\tilde{P}y)_i - (\tilde{P}y)_{i-1} = \frac{h \beta_i}{2} y(ih, x_2), \quad (\tilde{P}v)_{n-1} = 0, \quad \sigma_i - \sigma_{i-1} = \frac{h}{\rho_i}$$

and we will have

$$J_4^2(y) \leq 2 \sum_{i=1}^{n-1} (\sigma_i - \sigma_{i-1})((\tilde{P}y)_i^2 + (\tilde{P}y)_{i-1}^2) = -2 \sum_{i=1}^{n-1} (\sigma_i + \sigma_{i-1})((\tilde{P}y)_i^2 - (\tilde{P}y)_{i-1}^2)$$

$$= - \sum_{i=1}^{n-1} (\sigma_i + \sigma_{i-1}) h \beta_i y(ih, x_2)(Py)_i.$$  \hspace{0.5cm} (26)

It is possible to show that $(\sigma_i + \sigma_{i-1})\beta_i \leq c$. Consequently, the inequality

$$J_4^2(y) \leq c \sum \omega \ h |y Py| \leq c \left( \sum \omega \ h \tilde{p} y^2 \right)^{1/2} J_4(y), \quad \text{i.e.} \quad J_4(y) \leq c \left( \sum \omega \ h \tilde{p} y^2 \right)^{1/2}$$

follows from (26). This together with (25) completes the proof of Lemma 5. \qed

To investigate the convergence and accuracy of scheme (15), we consider the error of the method $z = y - u$, where $y$ is a solution to problem (15) and $u = u(x)$ is a solution to problem (2)–(4). Substituting $y = u + z$ into (15), we obtain the problem

$$L_h z = \psi, \quad z = 0, \quad l_h(z) = \chi(x_2), \quad x_2 \in \omega_2,$$  \hspace{0.5cm} (27)

where

$$\psi = a_{11} \eta_{11} x_1^{x_1} + a_{12} \eta_{12} x_1 x_2 + a_{22} \eta_{22} x_2 x_2 + a_0 \eta_0,$$

$$\eta_0 = T_1 T_2 u - u, \quad \eta_{aa} = u - T_3 - \alpha u, \quad \alpha = 1, 2,$$

$$\eta_{12} = \frac{1}{2}(u + u(-11) + u(-12) + u(-11, -12)) - 2 T_1 S_2 u(x), \quad \chi = l(u) - l_h(u).$$

If we notice that

$$l_h(u) = \sum \omega \int_{x_1 - h}^{x_1} \beta(t) \left( \frac{x_1 - t}{h} u(x_1 - h, x_2) + \frac{t - x_1 + h}{h} u(x_1, x_2) \right) dt,$$
then we can write the error $\chi$ as follows:

$$
\chi = \sum_{\omega_i^+} \eta, \quad \eta = \int_{x_{1-h}}^{x_1} \beta(t) \frac{t-x_1}{h} \int_{x_{1-h}}^{t} (\xi - x_1 + h) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2} \, d\xi \, dt
$$

$$
+ \int_{x_{1-h}}^{x_1} \beta(t) \frac{t-x_1 + h}{h} \int_{t}^{x_1} (\xi - x_1) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2} \, d\xi \, dt.
$$

It is evident that $\chi = 0$ for $u(x) = 1 - x_1$. Consequently, $l_h(1-x_1) = l(1-x_1) = 1/(1+\varepsilon)$ and the substitution

$$
z(x) = \tilde{z}(x) + \frac{1 - x_1}{1 + \varepsilon} \chi(x_2)
$$

turns problem (27) (in which the nonlocal condition is not homogeneous) into the problem with the homogeneous conditions

$$
L_h \tilde{z} = \tilde{\psi}, \quad \tilde{z} = 0, \quad \tilde{z} = 0, \quad l_h(\tilde{z}) = 0, \quad x_2 \in \omega_2,
$$

where

$$
\tilde{\psi} = \psi + 2a_{12} \left( \frac{1 - x_1}{1 + \varepsilon} \chi \right)_{\bar{x}_1 \bar{x}_2} + a_{22} \left( \frac{1 - x_1}{1 + \varepsilon} \chi \right)_{\bar{x}_2} - a_0 \frac{1 - x_1}{1 + \varepsilon} \chi.
$$

Applying Lemma 4 to the solution of problem (29) we come to

$$
||\tilde{z}||_1^2 \leq c(\tilde{\psi}, G_h \tilde{z})_\omega.
$$

Using Lemma 5 gives

$$
||\tilde{z}||_1 \leq c(||\eta_{1\bar{x}_1}||_{\omega_2^+ \times \omega_2} + ||\eta_{2\bar{x}_2}||_{\omega_1^+ \times \omega_2} + ||\eta_{2\bar{x}_2}||_{\omega_1^+ \times \omega_2^+} + ||\eta_{0}\omega + ||\chi||_\ast + ||\chi_{\bar{x}_2}||_\ast). \tag{30}
$$

For the error of the method, according to (28), we can write

$$
||z||_1 \leq ||\tilde{z}||_1 + c(||\chi||_\ast + ||\chi_{\bar{x}_2}||_\ast)
$$

which together with (30) gives

$$
||z||_1 \leq c(||\eta_{1\bar{x}_1}||_{\omega_2^+ \times \omega_2} + ||\eta_{2\bar{x}_2}||_{\omega_1^+ \times \omega_2} + ||\eta_{2\bar{x}_2}||_{\omega_1^+ \times \omega_2^+} + ||\eta_{0}\omega + ||\chi||_\ast + ||\chi_{\bar{x}_2}||_\ast). \tag{31}
$$

In order to estimate the convergence rate of finite-difference scheme (15), it is enough to estimate the norm of error functionals on the right-hand side of (31). For this we apply the standard technique (see, e.g., [3, 7]).

First, for each summands of $\chi_{\bar{x}_2}$ we write

$$
||\eta_{\bar{x}_2}|| \leq c h^{-1} \int_{x_{1-h}}^{x_1} \beta(t) \, dt \, h^{m-2/p} |u|_{W^m_p(e)}, \quad pm > 1, \quad m \in (1; 3), \quad e = (x_1-h, x_1) \times (x_2-h, x_2).
$$

Next,

$$
||\eta_{\bar{x}_2}|| \leq c \left( \int_{x_{1-h}}^{x_1} t^{(\varepsilon-1)p/(p-1)} \, dt \right)^{(p-1)/p} |u|_{W^m_p(e)} h^{m-1-1/p},
$$
therefore,

\[
|\chi_{x_2}| \leq ch^{m-1-1/p} \left( \int_0^1 t^{(e-1)p/(p-1)} \, dt \right)^{(p-1)/p} |u|_{W^m_p(\bar{e})}, \quad \bar{e} = (0; 1) \times (x_2 - h; x_2).
\]

Taking into account the inequality

\[
\sum_{\omega_2} |u|_{W^m_p(\bar{e})}^2 \leq ch^{-1+2/p} |u|_{W^m_p(\Omega)}^2,
\]

we will have

\[
|\chi_{x_2}| \leq ch^{m-1-1/p} |u|_{W^m_p(\Omega)}.
\]

The analogous estimate is obtained for \( |\chi|_* \).

With the well-known estimates for \( \eta_{11}, \eta_{12}, \eta_{22}, \eta_0 \) (see [3,7]), (31) yields the convergence theorem.

**Theorem 2.** The finite-difference scheme (15) converges and the convergence rate estimate (1) holds.

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*Received 1 Dec. 2002*

*Revised 10 Dec. 2002*