Invariant Hyperkähler Structures on the Cotangent Bundles of Hermitian Symmetric Spaces

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Abstract. Let \( G/K \) be an irreducible Hermitian symmetric space of compact type with the standard homogeneous complex structure. Then the real symplectic manifold \( (T^*(G/K), \Omega) \) has the natural complex structure \( J^- \). We construct all \( G \)-invariant Kähler structures \( (J, \Omega) \) on homogeneous domains in \( T^*(G/K) \) anticommuting with \( J^- \). Each such a hypercomplex structure, together with a suitable metric, defines a hyper-Kählerian structure. As an application, we obtain a new proof of the Harish-Chandra and Moore theorem for Hermitian symmetric spaces.

Bibliography: 13 titles.

§ 1. Introduction

Let \( M = G/K \) be an irreducible Hermitian symmetric space of compact type with a homogeneous metric \( g_M \). Since \( M \) is a homogeneous complex manifold, its cotangent bundle \( T^*M \) has a natural complex structure. Using \( g_M \) we can identify the cotangent and tangent bundles and thus obtain a complex structure on \( TM \), with respect to which the zero section \( M \subset TM \) is complex. This structure \( J^- \) is different from the standard complex structure \( J^+ \) on \( TM \) induced by that on \( M \).

On the other hand, the cotangent bundle \( T^*M \simeq TM \) is a symplectic manifold with the canonical symplectic form \( \Omega \). In this paper we make an explicit description of all \( G \)-invariant Kähler structures \( (J, \Omega) \) (with the Kähler form \( \Omega \)) on homogeneous domains \( D \subset TM \) anticommuting with \( J^- \) (Theorem 4.12). In fact, each resulting hypercomplex structure, together with the suitable metric \( g \), defines a hyper-Kählerian structure.

If the domain \( D \) contains the zero section \( M \), the restriction of the hyper-Kählerian metric \( g \) to \( M \) is the given homogeneous metric \( g_M \) up to a constant multiplier (one makes this multiplier \( = 1 \) using for the identification of \( T^*M \) and \( TM \) a homogeneous metric on \( M \) proportional to \( g_M \)). Such hyper-Kählerian metrics have been constructed in [Bu] using twistor methods and case by case the classification of symmetric spaces, in [Bi] using Nahm’s equations and in [DSz] (for spaces of classical groups) using deformation of the so-called adapted complex structure on \( TM \). In [BG1] Biquard and Gauduchon found explicit formulas for these hyper-Kählerian metrics in terms of some operator-functions \( P : m \to \text{End}(m) \) on the space \( m \simeq T_o(G/K) \), where \( o = \{ K \} \). These hyper-Kählerian structures are
global ones. Our additional structures are not defined on the zero section \( M \). So we cannot talk about a restriction of the corresponding hyper-Kählerian metric to \( M \). Nevertheless, our expressions for \( P \) and potential functions generalize the corresponding formulas of \([BG1,BG2]\).

For proofs in \([DSz,BG1,BG2]\) they used the decomposition of \( T(TM) \) between horizontal and vertical directions, induced by the Levi-Civita connection of \( M \). Our approach is based on the fact that \( T(G/K) \) is a reduced manifold for the (right) Hamiltonian action of \( K \) on \( TG \). We can substantially simplify matters by working as in \([My1,My2]\) in the trivial vector bundle \( G \times m \) which is a level surface for the corresponding moment map. So we use the natural homogeneous decomposition of \( T(G \times m) \) usual for the Lie algebras theory. As an application we obtain a new simple proof of the well-known Harish-Chandra and Moore theorem about restricted root systems of Hermitian symmetric spaces.

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§2. G-invariant Kähler structures on \( T(G/K) \)

2.1. Anticommuting structures. We recall some facts on hypercomplex and hyper-Kählerian structures (see for example \([BG1,Ob,Hi]\)). Let \( N \) be a smooth real manifold with a complex structure \( J \) and a symplectic 2-form \( \omega \) (all objects in this paper are smooth, unless otherwise indicated). For any vector bundle \( L \) on \( N \) denote by \( \Gamma L \) the set of its smooth sections. For the tensor \( J \) denote by \( F(J) \subset T^C N \) its (involutive) complex subbundle of \((0,1)\)-vectors, i.e. \( \Gamma F(J) = \{ X + iJX, X \in \Gamma(TM) \} \). We need some definitions.

Definition 2.1. The pair \((J,\omega)\) is a Kähler structure on \( N \) if

1. the closed 2-form \( \omega \) is invariant with respect to \( J \): \( \omega(JX,JY) = \omega(X,Y), \forall X,Y \in \Gamma(TM) \);
2. the bilinear form \( g = g(J,\omega) \), where \( g(X,Y) \overset{\text{def}}{=} \omega(JX,Y) \), is symmetric and positive definite.

We will denote such a Kähler structure also by the pair \((J,g)\) because \( \omega(X,Y) = g(-JX,Y), \) i.e. \( \omega = \omega(J,g) \).

Let \( \Pi : \tilde{N} \to N \) be a submersion of a manifold \( \tilde{N} \) onto \( N \) and \( \mathcal{K} \subset T\tilde{N} \) be the kernel of \( \Pi_* \). Let \( \mathcal{T} \) be some complementary subbundle to \( \mathcal{K} \) in \( T\tilde{N} \), i.e. \( \mathcal{K} \oplus \mathcal{T} = T\tilde{N} \). For the complex structure \( J \) on \( N \) there exists a unique (smooth) \((1,1)\)-tensor \( \tilde{J} \) on \( \tilde{N} \) such that

\[
\tilde{J}(\mathcal{T}) = \mathcal{T}, \quad \tilde{J}(\mathcal{K}) = 0, \quad \Pi_* \circ \tilde{J} = J \circ \Pi_* \quad \text{on} \quad \mathcal{T}.
\] (2.1)

Similarly, for the \((0,1)\)-subbundle \( F(J) \) there exists a unique complex subbundle \( \mathcal{F}(J) \subset T^C\tilde{N} \) containing the kernel \( \mathcal{K} \) and such that \( \Pi_*(\mathcal{F}(J)) = F(J) \). It is clear that this subbundle is involutive.

Lemma 2.2. The form \( \omega \) is invariant with respect to \( J \) iff the \((1,1)\)-tensor \( J \) is skew-symmetric with respect to the form \( \omega \): \( \omega(JX,Y) = \omega(X,-JY), \forall X,Y \in \Gamma(TM) \). The tensor \( J \) is skew-symmetric (with respect to \( \omega \)) iff \( \omega(F(J),F(J)) = 0 \), and symmetric iff \( \omega(F(J),\tilde{F}(J)) = 0 \).
Proof. Taking into account that \( J^2 = -1 \) and
\[
\omega(X + iJX, Y \pm iJY) = \pm i \left[ \omega(JX, Y) \pm \omega(X, JY) \right] + \left[ \omega(X, Y) \pm \omega(JX, JY) \right]
\]
we obtain the assertions of the lemma.

Observe the following fact:

**Corollary 2.2.1.** The tensor \( J \) is skew-symmetric with respect to the form \( \omega \) iff 
\[
(\Pi^* \omega)(\mathcal{F}(J), \mathcal{F}(J)) = 0,
\]
and symmetric iff 
\[
(\Pi^* \omega)(\mathcal{F}(J), \mathcal{F}(J)) = 0.
\]

The following assertion is well-known (see [GS, Lemma 4.3]).

**Lemma 2.3.** The pair \((J, \omega)\) is a Kähler structure on \( N \) iff \( \omega(F(J), F(J)) = 0 \) and 
\[ -i\omega(Z, \overline{Z}) > 0 \]
for all non-zero local vector-fields \( Z \in \Gamma_{loc}F(J) \).

**Definition 2.4.** A pair \((J_1, J_2)\) formed by two anticommuting complex structures \( J_1 \) and \( J_2 \) is a hypercomplex structure on \( N \). Then \( J_3 = J_1J_2 \) is also a complex structure on \( N \) (for a proof see [Ob]).

**Remark 2.5.** Almost-complex tensors \( J_1 \) and \( J_2 \) on \( N \) are anticommuting iff so are the tensors \( J_1^\prime \) and \( J_2^\prime \) on \( N \).

**Definition 2.6.** [BG1] A triple \((g, J_1, J_2)\) formed by a Riemannian metric \( g \) and two anticommuting complex structures \( J_1 \) and \( J_2 \) on \( N \) is a hyper-Kählerian structure on \( N \) whenever the pairs \((J_1, g)\) and \((J_2, g)\) are Kähler. Then the pair \((J_3 = J_1J_2, g)\) is a Kähler one as well.

For a hyper-Kählerian structure \((g, J_1, J_2)\), let us denote by \( \omega_j \) the Kähler form corresponding to \( J_j \), \( j = 1, 2, 3 \). It is clear that this hyper-Kählerian structure is determined by any pair \((J_k, (J_j, \omega_j))\), \( k \neq j \). Since \( J_k \) and \( J_j \) anticommute, the tensor \( J_k \) is symmetric with respect to the form \( \omega_j \) (while \( J_j \) is skew-symmetric).

It is clear that \( \omega_k(X, Y) = \omega_j(J_kJ_jX, Y) \), \( X, Y \in \Gamma(TN) \) \((k \neq j)\). The following simple lemma defines a hyper-Kählerian structure in these terms.

**Lemma 2.7.** Let \((J', (J, \omega))\) be a pair, where \( J', J \) are anticommuting complex structures and \((J, \omega)\) is a Kähler structure on \( N \) with the corresponding Hermitian metric \( g = g(J, \omega) \). Then the triple \((g, J', J)\) is a hyper-Kählerian structure on \( N \) iff

1. the tensor \( J' \) is symmetric with respect to \( \omega \);
2. the 2-form \( \omega' \), where \( \omega'(X, Y) \overset{\text{def}}{=} \omega(J'JX, Y) \forall X, Y \in \Gamma(TN) \), is closed.

**Proof.** Since \( J'J = -JJ' \), from (1) it follows that
\[
\omega'(X, Y) \overset{\text{def}}{=} \omega(J'JX, Y) = \omega(X, -JJ'Y) = \omega(JJ'Y, X) = -\omega'(Y, X),
\]
i.e. the bilinear form \( \omega' \) is skew-symmetric. Now to prove the lemma it is sufficient to verify that the form \( \omega' \) is \( J' \)-invariant and \( g(X, Y) = \omega'(J'X, Y) \). These properties follow from the following chains of equations:
\[
\omega'(J'X, J'Y) \overset{\text{def}}{=} \omega(J'J'J'X, J'Y) = \omega(JX, J'Y) = \omega(J'JX, Y) \overset{\text{def}}{=} \omega'(X, Y)
\]
and
\[
g(X, Y) \overset{\text{def}}{=} \omega(JX, Y) = \omega(J'JX, Y) \overset{\text{def}}{=} \omega'(J'X, Y).
\]
Since the kernels of the (1,1)-tensors $\tilde{J}$, $\check{J}$ and the forms $\Pi^*\omega$, $\Pi^*\omega$ coincide with $\mathcal{K}$, it follows from the definition of $\check{J}$ and $\tilde{J}$ (see (2.1)) that

$$(\Pi^*\omega)(\tilde{X}, Y) = (\Pi^*\omega)(\check{J}\check{X}, Y), \quad \forall \tilde{X}, Y \in \Gamma(T\tilde{N}).$$

(2.2)

**2.2. G-invariant Kähler structures** $(\mathbf{J(P)}, \Omega)$. Let $M = G/K$ be a symmetric space with a real reductive connected Lie group $G$ and a compact subgroup $K$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the groups $G$ and $K$ respectively,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$  

(2.3)

Suppose that there is a nondegenerate $\text{Ad}G$-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$ such that its restriction $\langle , \rangle|_\mathfrak{m}$ is a positive definite form and $\mathfrak{k} \perp \mathfrak{m}$. This form defines $G$-invariant Riemannian metric $g_M$ on $M = G/K$. The metric $g_M$ identifies the cotangent bundle $T^*M$ and the tangent bundle $TM$ and thus we can also talk about the canonical 1-form $\theta$ on $TM$. The form $\theta$ and the symplectic form $\Omega \overset{\text{def}}{=} \theta$ are $G$-invariant with respect to the natural action of $G$ on $TM$.

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is $\text{Ad}(K)$-invariant (orthogonal) splitting of $\mathfrak{g}$, we can consider a trivial vector bundle $G \times \mathfrak{m}$ with the two Lie group actions (which commute) on it: the left $G$-action, $l_h : (g, w) \mapsto (hg, w)$ and the right $K$-action $r_k : (g, w) \mapsto (gk, Ad_k^{-1}w)$. Let $\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$ be the natural projection. It is well known that $G \times_K \mathfrak{m}$ and $TM$ are isomorphic. Using the corresponding $G$-equivariant diffeomorphism $\varphi : G \times_K \mathfrak{m} \to TM$, $[(g, w)] \mapsto \frac{d}{dt}_{|t=0} g \exp(tw)K$ and the projection $\pi$ define the $G$-equivariant submersion $\Pi : G \times \mathfrak{m} \to TM$, $\Pi = \varphi \circ \pi$. Let $\xi^l$ be the left-invariant vector field on the Lie group $G$ defined by a vector $\xi \in \mathfrak{g}$. By [My1, Lemma 2.3] we have:

$$(\Pi^*\theta)(g, w)(\xi^l(g), u) = \langle w, \xi \rangle,$$

(2.4)

$$(\Pi^*\Omega)(g, w)((\xi^l_1(g), u_1), (\xi^l_2(g), u_2)) = \langle \xi^l_2, u_1 \rangle - \langle \xi^l_1, u_2 \rangle - \langle w, [\xi^l_1, \xi^l_2] \rangle,$$

(2.5)

where $g \in G$, $w \in \mathfrak{m}$, $\xi, \xi_1, \xi_2 \in \mathfrak{g}$, $u_1, u_2 \in \mathfrak{m} = T_w \mathfrak{m}$. Since $\Omega$ is a symplectic form, the kernel $\mathcal{K} \subset T(G \times \mathfrak{m})$ of the 2-form $\Pi^*\Omega$ is the kernel of $\Pi_\pi$.

Let $D$ be an open connected $G$-invariant subset of $TM$. Denote by $W$ a unique $\text{Ad}(K)$-invariant open subset of $\mathfrak{m}$ such that $\Pi^{-1}(D) = G \times W$. Let $\text{Eqv}(W)$ be the set of all smooth $K$-equivariant mappings $A : W \to \text{End}(\mathfrak{m}^C)$, $w \mapsto A_w$, i.e. for which

$$\text{Ad}_k \circ A_w \circ \text{Ad}_{k^{-1}} = A_{\text{Ad}_k w} \quad \text{on} \quad \mathfrak{m} \quad \text{for all} \quad w \in W, \quad k \in K.$$  

(2.6)

Denote by $\text{Alm}(W)$ the set of all $P \in \text{Eqv}(W)$ such that the operator $P_w : \mathfrak{m}^C \to \mathfrak{m}^C$ and its real part $\text{Re}P_\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$ are nondegenerate for each $w \in W$. Such a $K$-equivariant mapping $P \in \text{Alm}(W)$ determines a complex (left) $G$-invariant subbundle $\mathcal{F}(P) \subset T^C(G \times W)$ generated by (left) $G$-invariant vector fields $\xi^L$, $\xi \in \mathfrak{m}$ and $\xi^L$, $\xi \in \mathfrak{k}$ on $G \times W$, where

$$\xi^L(g, w) = (\xi^l(g), iP_\mathfrak{m}(\xi)), \quad \xi^L(g, w) = (\xi^l(g), [w, \xi]).$$

The subbundle $\mathcal{F}(P)$ is (right) $K$-invariant by (2.6) and because the vector fields $\xi^L$, $\xi \in \mathfrak{k}$ span the (right) $K$-invariant subbundle (kernel) $\mathcal{K}$. Therefore $\mathcal{F}(P) \overset{\text{def}}{=} \Pi_\pi(\mathcal{F}(P))$ is a well-defined (smooth) complex subbundle of $T^C \mathcal{K}$ such that $\mathcal{F}(P) + \overline{\mathcal{F}(P)} = T^C \mathcal{K}$ $\mathcal{F}(P) \cap \overline{\mathcal{F}(P)} = 0$. In other words, the mapping
$P$ determines an almost-complex structure $J(P)$ on $D \subset TM$ with $F(P)$ as the subbundle of its $(0,1)$-vectors.

For a vector field $X \in \Gamma(TW)$ and $A \in \text{Equiv}(W)$ denote by $L_X A$ the derivative of $A$ along $X$, i.e. $(L_X A)_w \overset{\text{def}}{=} (d/dt)_{t=0} A_{w+tX(w)} \in \text{End}(m^C)$. We extend this definition on complex vector fields using linearity. Each vector $\xi \in m$ defines the vector field $P\xi$ on $W$ by $(P\xi)_w = P_w(\xi)$. Now we want to present a result which will be effectively used in a remaining part of the paper.

**Proposition 2.8.** [My1] Let $M = G/K$ be a Riemannian symmetric space. The almost-complex structure $J(P)$ on the domain $D = \Pi(G \times W) \subset TM$ is

1. integrable iff for all (fixed) vectors $\xi, \eta \in m$ and $w \in W$

\[
(L_{P\xi}P)\eta(w) - (L_{P\eta}P)\xi(w) = -[w, [\xi, \eta]]; \quad (2.7)
\]

2. a Kähler structure with the Kähler form $\Omega$ iff (1) holds and for each $w \in W$ the endomorphism $P_w$ is symmetric with positive-definite real part $\text{Re} P_w$ (with respect to the bilinear form $\langle , \rangle$ on $m$).

For any $G$-invariant Kähler structure $J$ on $D$ with $\Omega$ as a Kähler form there exists a unique mapping $P \in \text{Alm}(W)$ such that $J = J(P)$.

Observe the following fact:

**Corollary 2.8.1.** If the structure $J(P)$ is integrable then so are $J(-P)$ and $J(\overline{P})$.

**Corollary 2.8.2.** Suppose that $(J(P), \Omega)$ is a Kähler structure on $D$. If $0 \in W$ and the Lie algebra $\mathfrak{g}$ is simple then $P_0 = \psi_0 \cdot \text{Id}_m$, where $\text{Re} \psi_0 \in \mathbb{R}^+$.

**Proof.** If $\mathfrak{g}$ is a simple algebra, $m$ is a simple $\text{Ad}(K)$-module [GG, (8.5.1)]. Since $P_0$ is a symmetric endomorphism which commutes with all endomorphisms $\text{Ad}_k |m$, $k \in K$ (condition (2.6)), $P_0 = \psi_0 \cdot \text{Id}_m$ for some $\psi_0 \in \mathbb{C}$.

**§3. Invariant Kähler structures on Hermitian symmetric spaces**

We continue with the previous notations but in this section it is assumed in addition that $G/K$ is an Hermitian symmetric space, i.e. there exists an endomorphism $I : m \rightarrow m$ such that

1. $I^2 = -\text{Id}_m$;
2. $\text{Ad}_k I = I \text{Ad}_k$ on $m$, $\forall k \in K$;
3. the form $\langle , \rangle|m$ is $I$-invariant; and

\[
[I\xi, I\eta] = [\xi, \eta], \quad I[\xi, \eta] = [I\xi, I\eta] \quad \text{for all} \ \xi, \eta \in m, \ \zeta \in \mathfrak{k}. \quad (3.1)
\]

Such a triple $(\mathfrak{g}, \mathfrak{k}, I)$ we will call an Hermitian orthogonal symmetric Lie algebra. It follows from (3.1) that $[\xi^l + i(I\xi)^l, \eta^l + i(I\eta)^l] = 0$, $\forall \xi, \eta \in m$. Thus the complex subbundle of $T^C G$, generated by vector-fields $\xi^l + i(I\xi)^l$, $\xi \in m$, is involutive. Since this subbundle is left $G$-invariant and right $K$-invariant, its image under the canonical projection $G \rightarrow G/K$ defines a $G$-invariant complex structure on $M = G/K$.

**3.1. Hypercomplex structures on the tangent bundles of Hermitian symmetric spaces.** Here we apply the general results of the previous Section 2 in the special situation here. The $G$-invariant complex structure on $M$ induces the $G$-invariant complex structures $J^+$ and $J^-$ on $TM$ and $T^*M \simeq TM$ respectively.
It is clear that the subbundle \( F^\pm \subset T^C(TM) \) of \((0, 1)\) vectors of \( J^\pm \) coincide with the subbundle \( \Pi_\ast (\mathcal{F}^\pm) \), where \( \mathcal{F}^\pm \) is the (left) \( G \)-invariant and (right) \( K \)-invariant subbundle of \( T^C(G \times \mathfrak{m}) \):

\[
F^\pm(g, w) = \{(\xi^l(g) + i(I\xi)^l(g), u \pm i(Iu)), \xi, u \in \mathfrak{m}\} \oplus \mathcal{K}(g, w). \tag{3.2}
\]

Fix some mapping \( P \in \text{Alm}(W) \). It is clear that the mapping \( PI, (PI)_w \overset{\text{def}}{=} P_wI \) is also an element of the set \( \text{Alm}(W) \) because the group \( \text{Ad}(K)|\mathfrak{m} \) commutes elementwise with \( I \).

**Lemma 3.1.** If \( J(P), P \in \text{Alm}(W) \) is a complex structure then so is \( J(PI) \).

**Proof.** Since \( F(P) \) is an involutive subbundle and \( I \) is independent of \( w \), from Proposition 2.8 and relations (3.1) it follows that

\[
(L_{P(I\xi)}P)_w(I\eta) - (L_{P(I\eta)}P)_w(I\xi) = -[w, [I\xi, I\eta]] = -[w, [\xi, \eta]]
\]

on \( W \) for any vectors \( \xi, \eta \in \mathfrak{m} \), i.e. the subbundle \( F(PI) \) is also involutive. \( \square \)

In order to describe the defined above complex structures in terms of their almost-complex tensors, consider two (left) \( G \)-invariant and (right) \( K \)-invariant subbundles \( \mathcal{T}_h \) and \( \mathcal{T}_v \) of the tangent bundle \( T(G \times W) \) given by

\[
\mathcal{T}_h(g, w) = \{(\xi^l(g), 0), \xi \in \mathfrak{m}\}, \quad \mathcal{T}_v(g, w) = \{(0, u), u \in \mathfrak{m} = T_wW\}.
\]

Put \( \mathcal{T} = \mathcal{T}_h \oplus \mathcal{T}_v \). Then \( T(G \times W) = \mathcal{K} \oplus \mathcal{T} \). The mapping \( (\xi^l(g), 0) \mapsto (0, \xi) \) determines the canonical isomorphism of the spaces \( \mathcal{T}_h(g, w) \) and \( \mathcal{T}_v(g, w) \). Using this isomorphism, we obtain that the \((1, 1)\)-tensors \( \tilde{J}^\pm \) and \( J(P) \) on \( G \times W \) (see (2.1)) at the point \( (g, w) \) are given by

\[
\tilde{J}^\pm_{(g, w)}|\mathcal{T} = \begin{pmatrix} I & 0 \\ 0 & \pm I \end{pmatrix}, \quad \tilde{J}_{(g, w)}(P)|\mathcal{T} = \begin{pmatrix} -R^{-1}w^{-1}S_w & -R^{-1}w^{-1} \\ R_w + S_wR_w^{-1}S_w & S_wR_w^{-1} \end{pmatrix}, \tag{3.3}
\]

where \( R = \text{Re} P, S = \text{Im} P \). Now it is easy to verify that the tensors \( \tilde{J}^- \) and \( \tilde{J}(P) \) are anticommuting iff \( RI = IR \) and \( SI = -IS \), i.e. by Remark 2.5

\[
J^-J(P) = -J(P)J^- \iff RI = IR, SI = -IS \iff IP = \overline{PI}. \tag{3.4}
\]

Considering now the almost complex structures \( J(P) \) and \( J(PI) \) with a real mapping \( P \in \text{Alm}(W) \) (i.e. with \( \text{Im} P = 0 \)), we obtain that

\[
\tilde{J}_{(g, w)}(P)|\mathcal{T} = \begin{pmatrix} 0 & -P^{-1}_w \\ P_w & 0 \end{pmatrix}, \quad \tilde{J}_{(g, w)}(PI)|\mathcal{T} = \begin{pmatrix} 0 & IP^{-1}_w \\ P_wI & 0 \end{pmatrix}.
\]

In this case \( J(P)J(PI) = -J(PI)J(P) \). We have established the following result.

**Proposition 3.2.** Let \( J(P), P \in \text{Alm}(W) \) be a complex structure on \( D \) such that \( \text{Im} P = 0 \). Then the pair \((J^{-}, (J(P), \Omega))\), where \((J(P), \Omega)\) is a Kähler structure on the domain \( D \subset TM \).

**3.2. Hyperkähler structures on the tangent bundles of Hermitian symmetric spaces.** In this subsection we study properties of the pair \((J^{-}, (J(P), \Omega))\), where \((J(P), \Omega)\) is a Kähler structure on the domain \( D \subset TM \).
Therefore Ω′ of Lemma 2.7 hold, and we are done.

Π where

Applying the well-known formula

\begin{align*}
J \text{ structure }
\end{align*}

Theorem 3.3. Let \((J(P), \Omega), P \in \text{Alm}(W)\) be a Kähler structure on \(D\) with the Hermitian metric \(g = g(J(P), \Omega)\). Then the triple \((g, J^-, J(P))\) is a hyper-Kählerian structure (on \(D\)) iff \(IP = \overline{PJ}\) on \(W\).

Proof. By (3.4) the complex structures \(J^-\) and \(J(P)\) anticommute iff \(IP = \overline{PJ}\).

For the pair \((J^-, J(P))\) of anticommuting complex structures the almost-complex structure \(J^J = J^- J(P)\) is integrable [Ob]. Therefore the triples \((g, J^-, J(P))\) and \((g, J^J, J(P))\) are hypercomplex structures simultaneously. Thus to prove the theorem it is sufficient to show that for the pair \((J^J, (J, \Omega))\), where \(J = J(P)\), conditions (1) and (2) of Lemma 2.7 hold.

Since by definition the form \(\Omega\) is \(J\)-invariant and \(JJ^J = -J^J\), we derive the identities \(\Omega(JJ^JX, Y) = \Omega(J^JX, Y)\) and \(\Omega(X, J^J(JY)) = \Omega(X, JY)\), where \(X, Y \in \Gamma(TD)\).

In other words, the tensor \(J^J\) is symmetric with respect to \(\Omega\) iff so is \(J^J\). But the tensor \(JJ^J = J^-\) is symmetric with respect to \(\Omega\) because \((\Pi^*\Omega)(\overline{F^-}, \overline{F^-}) = 0\) (see Corollary 2.2.1). Indeed, using the relation \([m, m] \subseteq m\), property (3) of \(I\) and definitions (2.5), (3.2) of \(\Pi^*\Omega\) and \(F^-\), we obtain that for all \(\xi_1, \xi_2, u_1, u_2 \in m\)

\[
(\Pi^*\Omega)((\xi_1^I + i(I\xi_1^I), u_1 - i(Iu_1)), (\xi_2^I - i(I\xi_2^I), u_2 + i(Iu_2)) = 0. \tag{3.5}
\]

Define the 1-form \(\theta^J\) and the tensor \(\Omega\) on \(D\) putting for \(X, Y \in \Gamma(TD)\)

\[
\theta^J(X) \overset{\text{def}}{=} \theta(-J^-X) \quad \text{and} \quad \Omega(X, Y) \overset{\text{def}}{=} \Omega(-J^-X, Y) = \Omega(J^JX, Y). \tag{3.5}
\]

Since we already have proved that \(J^-\) is symmetric with respect to \(\Omega\), we have \(\Omega^J(X, Y) = -\Omega^J(Y, X)\). To prove that the form \(\Omega^J\) is closed we will show that \(\Omega^J = d\theta^J\). By (3.3) and by definition of the form \(\theta\)

\[
(\Pi^*\theta^J)_{(g, w)}((\xi^I, u_1)) = \langle w, -I\xi \rangle = \langle Iw, \xi \rangle, \quad \xi \in g, \ u \in m.
\]

Applying the well-known formula \(d\theta^J(X, Y) = X\theta^J(Y) - Y\theta^J(X) - \theta^J([X, Y])\) to the form \(\Pi^*\theta^J\) we obtain that for \(\xi_1, \xi_2, u_1, u_2 \in m\)

\[
d(\Pi^*\theta^J)_{(g, w)}((\xi_1^I, u_1), (\xi_2^I, u_2)) = \langle \xi_2, Iu_1 \rangle - \langle \xi_1, Iu_2 \rangle, \tag{3.6}
\]

because \(\langle Iw, [\xi_1, \xi_2] \rangle = 0\). But using expression (3.3) for almost-complex tensor \(J^-\), we derive from (2.2) the following formula for \(\Pi^*\theta^J\)

\[
(\Pi^*\Omega^J)((\xi_1^I, u_1), (\xi_2^I, u_2)) = (\Pi^*\Omega)((-I\xi_1^I, Iu_1), (\xi_2^I, u_2)) = \langle \xi_2, Iu_1 \rangle - \langle -I\xi_1, u_2 \rangle = \langle \xi_2, Iu_1 \rangle - \langle \xi_1, Iu_2 \rangle, \tag{3.7}
\]

where \(\xi_1, \xi_2, u_1, u_2 \in m\). From (3.6) and (3.7) it follows that the forms \(d(\Pi^*\theta^J) = \Pi^*(d\theta^J)\) and \(\Pi^*\Omega^J\) coincide when restricted to \(T_h \oplus T_v\) and, consequently, on the whole tangent bundle \(T(G \times W) = K \oplus T_h \oplus T_v\) because \(K\) is the kernel of \(\Pi\). Therefore \(\Omega^J = d\theta^J\). Thus for the pair \((J^-J(P), (J(P), \Omega))\) conditions (1) and (2) of Lemma 2.7 hold, and we are done.

As an immediate consequence of the proof we obtain
Corollary 3.3.1. Let $P, g$ be as in Theorem 3.3 and $IP = \overline{PI}$. Let $\theta'$ and $\Omega'$ be the forms given by (3.5). Then $\Omega'$ is the Kähler form of the Kähler structure $(J^−J(P), g)$ and $\Omega' = d\theta'$.

It is evident that integrability condition (2.7) for $P$ is equivalent to a pair of real equations for its real and imaginary parts $R$ and $S$. The following proposition establishes more restrictive conditions for $R$ and $S$ if the pair $(J^−(J(P), \Omega))$ defines a hyper-Kählerian structure.

Proposition 3.4. Let $(J(P), \Omega)$ be a Kähler structure on $D$ such that $IP = \overline{PI}$ on $W$. Then

\[
(L_\xi(R + SR^{-1}S))_w(\eta) = [w, IR^{-1}_w\xi], \quad w \in W, \xi, \eta \in m; \quad (3.8)
\]

\[
L_\xi(SR^{-1})\eta = L_\eta(SR^{-1})\xi, \quad w \in W, \xi, \eta \in m. \quad (3.9)
\]

Locally the mapping $w \mapsto (SR^{-1})_w$ is a tangent one of some vector-function with values in $m$.

Proof. Put $J = J(P)$. Let $g$ be the Hermitian metric corresponding to $(J, \Omega)$. By Theorem 3.3 the triple $(g, J^−, J)$ is a hyper-Kählerian structure, in particular, the 2-form $\omega_1 = \Omega^−, \Omega^−(\cdot, \cdot) \overset{\text{def}}{=} \Omega(J^−J\cdot, \cdot)$ on $D$ is closed and so is the form $\Pi^*\Omega^-$ on $G \times W$. By (2.2) the form $(\Pi^*\Omega^-)(\cdot, \cdot) = (\Pi^*\Omega)(J^−J\cdot, \cdot)$. Therefore for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $G \times W$ we have

\[
\sum_{\tilde{X}\tilde{Y}\tilde{Z}} \tilde{X}((\Pi^*\Omega)(J^−J\tilde{Y}, \tilde{Z})) + \sum_{\tilde{X}\tilde{Y}\tilde{Z}} (\Pi^*\Omega)(\tilde{J}^−J\tilde{X}, [\tilde{Y}, \tilde{Z}]) = 0 \quad (3.10)
\]

(we sum here over the cyclic permutations of $\tilde{X}, \tilde{Y}, \tilde{Z}$).

Putting in (3.10) $\tilde{X} = (0, \xi)$, $\tilde{Y} = (\eta^t, 0)$ and $\tilde{Z} = (\chi^t, 0)$ for (fixed) $\xi, \eta, \chi \in m$, we obtain only two non-zero terms in the left-hand side of (3.10) (here all objects are left-$G$-invariant) and, consequently,

\[
(L_\xi\langle -I(R + SR^{-1}S)\eta, \chi \rangle)_w - \langle w, [-IR^{-1}\xi, [\eta, \chi]] \rangle = 0,
\]

because $\tilde{J}^−J_0(0, \xi) = ((-IR^{-1}\xi)^t, -ISR^{-1}\xi)$. Since the form $\langle \cdot, \cdot \rangle$ is Ad-$G$-invariant and $IR = RI, IS = -IS$, we derive condition (3.8).

To prove (3.9), put in (3.10) $\tilde{X} = (0, \xi)$, $\tilde{Y} = (0, \eta)$ and $\tilde{Z} = (\chi^t, 0)$. Then we have only two non-zero terms on the left in (3.10), i.e. the following equality

\[
L_\xi\langle -ISR^{-1}\eta, \chi \rangle - L_\eta\langle \xi, -IR^{-1}S\chi \rangle = 0.
\]

By Proposition 2.8 the endomorphisms $R_w$ and $S_w$ are symmetric and by definition $I$ is skew-symmetric (with respect to the form $\langle \cdot, \cdot \rangle|m)$. Using (3.4) we obtain that $IR^{-1}S = -R^{-1}SI$ and, consequently, $L_\xi(SR^{-1})\eta = L_\eta(SR^{-1})\xi$. This identity then gives the latter assertion of the proposition. □

Remark 3.5. It is easy to verify that conditions (3.8) and (3.9) are equivalent to condition (1.8) in [BG1].
Lemma 3.6. Let $P \in \text{Alm}(W)$ and $IP = \overline{PI}$. Assume that for $P$ conditions (3.8) and (3.9) hold. Then the almost complex structure $J(P)$ on $D$ is integrable iff for all $w \in W$

$$(SR^{-1})_w([w, [\xi, \eta]]) = [[w, (R^{-1}S)_w \xi], \eta] - [[w, (R^{-1}S)_w \eta], \xi], \quad \xi, \eta \in \mathfrak{m}. \quad (3.11)$$

Proof. Since $R + SR^{-1}S = (1 - iSR^{-1})(R + iS)$, we have

$$L_X(R + SR^{-1}S) = -iL_X(SR^{-1}) \cdot P + (1 - iSR^{-1}) \cdot L_XP$$

for $X \in \Gamma(TW)$ and, consequently, by (3.8) and (3.9)

$$(1 - iSR^{-1})_w\left(L_{P \xi}P(\eta) - L_{P \eta}P(\xi)\right)_w
= i\left((L_{P \xi}(SR^{-1}))(P \eta) - (L_{P \eta}(SR^{-1}))(P \xi)\right)_w
+ \left(L_{P \xi}(R + SR^{-1}S)(\eta) - L_{P \eta}(R + SR^{-1}S)(\xi)\right)_w
= \left[[w, I(R^{-1}P)_w \xi], -I\eta\right] - \left[[w, I(R^{-1}P)_w \eta], -I\xi\right].$$

Since the operator $1 - iSR^{-1} = \overline{PR}^{-1}$ is invertible and $IP = \overline{PI}$, integrability condition (2.7) holds iff

$$(PR^{-1})_w([w, [\xi, \eta]]) = [[w, (R^{-1}\overline{P})_w I \xi], I \eta] - [[w, (R^{-1}\overline{P})_w I \eta], I \xi]. \quad (3.12)$$

Using the Jacobi identity and properties (3.1) of $I$, we obtain that real part of the right-hand side of (3.12) is equal to $[[w, [I \xi, I \eta]] = [w, [\xi, \eta]]$. The imaginary part of (3.12) is equivalent to (3.11). \hfill \square

§4. Hyperkähler structures on irreducible Hermitian symmetric spaces

In this section the main theorem describing Kähler structures anticommuting with $J^+$ on homogeneous domains in $T(G/K)$ is proved. We give a general formula for the Kähler potential. We seek this formula in the form as in [BG1].

We continue with the previous notations but in this section it is assumed in addition that the Hermitian symmetric space $G/K$ is irreducible, has compact type and the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is positive-definite. In particular, then $\mathfrak{g}$ is a simple compact Lie algebra of rank $l$ and the subgroup $K$ is connected.

4.1. Root theory of Hermitian symmetric spaces. Here we will review few facts about Hermitian symmetric spaces [He, Ch.VIII, §84–7]. The compact Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ has the one-dimensional center $\mathfrak{z}$ and coincides with the centralizer $\mathfrak{g}^\mathfrak{z}$ of $\mathfrak{z}$ in $\mathfrak{g}$, in particular, $\text{rk} \mathfrak{g} = \text{rk} \mathfrak{k} = l$. Let $\mathfrak{t}$ be some Cartan subalgebra of $\mathfrak{k}$. Then $\mathfrak{z} \subset \mathfrak{t}$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. The complex space $\mathfrak{t}^\mathbb{C}$ is a Cartan subalgebra of the simple complex Lie algebra $\mathfrak{g}^\mathbb{C}$. Let $\Delta$ be the root system of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{t}^\mathbb{C}$. Denote by $\Delta_\mathfrak{t}$ the set of roots in $\Delta$ which vanish identically on $\mathfrak{z}$. This is the root system of $(\mathfrak{t}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$. Put $\Delta_\mathfrak{m} = \Delta \setminus \Delta_\mathfrak{t}$. Then we have the direct decompositions

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha, \quad \mathfrak{t}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \sum_{\alpha \in \Delta_\mathfrak{t}} \mathfrak{g}^\alpha, \quad \mathfrak{m}^\mathbb{C} = \sum_{\alpha \in \Delta_\mathfrak{m}} \mathfrak{g}^\alpha.$$
The algebra $\mathfrak{t}$ is a maximal subalgebra of $\mathfrak{g}$ [GG, (8.5.1)]. Since this subalgebra is ad $t$-invariant, according to [GG, (8.3.7)] there exists a system of simple roots $\pi = \{\alpha_1, \ldots, \alpha_l\} \subset \Delta$ such that for any $\alpha \in \Delta$ we have $n_{\alpha} = 0 \in \{0, 1, -1\}$, where $\alpha = n_{\alpha_1}^1 \alpha_1 + \cdots + n_{\alpha_l}^l \alpha_l$. Then $\Delta = \{\alpha \in \Delta : n_{\alpha} = 0\}$. Denote by $\Delta^+ \subset \Delta$ the corresponding set of positive roots.

Choose for any $\alpha \in \Delta^+$ a triple $(H_\alpha, E_\alpha, E_{-\alpha}) \in t \times \mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}$ such that $[H_\alpha, E_{\pm \alpha}] = \pm 2E_{\pm \alpha}$ and $[E_\alpha, E_{-\alpha}] = -H_\alpha$. This choice can be made so that $\mathfrak{g}$ has a basis consisting of a basis of $t$ and $X_\alpha = \frac{1}{2}(E_\alpha + E_{-\alpha})$, $Y_\alpha = \frac{1}{2}(E_\alpha - E_{-\alpha})$, $\alpha \in \Delta^+$ (the space $t$ is spanned by the vectors $T_\alpha = \frac{1}{2}H_\alpha$, $\alpha \in \Delta^+$). One has

$$[X_\alpha, Y_\alpha] = T_\alpha, \quad [T_\alpha, X_\alpha] = Y_\alpha, \quad [T_\alpha, Y_\alpha] = -X_\alpha, \quad \alpha \in \Delta^+.$$

Putting $\Delta_\pm^+ = \Delta_\pm \cap \Delta^+$ and $\Delta_\pm = \Delta_\pm \cap \Delta^+$, we obtain

$$\mathfrak{t} = t \oplus \sum_{\alpha \in \Delta_+^+} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha), \quad \mathfrak{m} = \sum_{\alpha \in \Delta_-^+} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha).$$

Since for each pair $\alpha, \beta \in \Delta_\pm^+$ the sum $\alpha + \beta$ is not a root, $-2i\alpha(T_\beta) = \alpha(H_\beta) \geq 0$. There exists a unique element $Z_\alpha \in \mathfrak{z}$ such that $\alpha_1(Z_\alpha) = i(\alpha_2|_\beta = 0$ for $j = \overline{1, \ell})$. Putting $I = \text{ad}Z_\alpha \mid \mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$, we see that $IX_\alpha = Y_\alpha$ and $IY_\alpha = -X_\alpha$ for all $\alpha \in \Delta_\pm^+$. Then

$$[T, \xi_\alpha] = -i\alpha(T)\xi_\alpha, \quad \text{where } T \in \mathfrak{t}, \ \xi_\alpha \in (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha) \subset \mathfrak{m}.$$

Hence $I^2 = -\text{Id}_\mathfrak{m}$. Moreover, $I$ and the automorphism $\exp(\frac{\pi}{2} \text{ad}Z_\alpha) \in \text{Ad}(K)$ coincide when restricted to $\mathfrak{m}$. Since the Lie group $K$ is connected, the group $\text{Ad}(K)$ commutes elementwise with $\exp(\frac{\pi}{2} \text{ad}Z_\alpha)$.

Whereas all Cartan subspaces (maximal abelian subalgebras) of $\mathfrak{m}$ are conjugate under the linear isotropy group $\text{Ad}(K)$ it is possible in the special situation here to select such a Cartan subspace $\mathfrak{a}$ with particular reference to $\Delta$. Two roots $\alpha, \beta \in \Delta$ are called strongly orthogonal if $\alpha \pm \beta \not\in (\Delta \cup \{0\})$. There exists a subset of $\Delta_\pm^+$ consisting of $r = \text{rk}(G/K)$ strongly orthogonal roots $\beta_1, \ldots, \beta_r$ [He, Ch.VIII, Prop.7.4]. Then the subspaces $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_{\beta_j}$ and $\mathfrak{l} = \sum_{j=1}^r \mathbb{R}Y_{\beta_j}$ are Cartan subspaces of $\mathfrak{m}$; the Lie subalgebra of $\mathfrak{g}$ generated by subspaces $\mathfrak{a}$ and $\mathfrak{l}$ is isomorphic to the semisimple compact Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{g}_{\beta_j}$, where each $\mathfrak{g}_{\beta_j} = (\mathbb{R}X_{\beta_j} \oplus \mathbb{R}Y_{\beta_j} \oplus \mathbb{R}T_{\beta_j}) \simeq su(2)$. Then $-i\beta_k(T_{\beta_j}) = \frac{1}{2}\beta_k(H_{\beta_j}) = \delta_{jk}$. We have

**Proposition 4.1.** [He] Any Cartan subspace $\mathfrak{a}$ of $\mathfrak{m}$ has the form $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_{\beta_j}$. The Lie subalgebra of $\mathfrak{g}$ generated by subspaces $\mathfrak{a}$ and $\mathfrak{l}$ is isomorphic to the semisimple compact Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{g}_{\beta_j}$, where each $\mathfrak{g}_{\beta_j} \simeq su(2)$.

Denote by $\mathfrak{a}^a$ and $\mathfrak{t}^a$ the centralizers of the Cartan subspace $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{t}$ respectively. By (2.3), $\mathfrak{g}^a = \mathfrak{a} \oplus \mathfrak{t}^a$. In particular, $\text{rk} \mathfrak{t}^a = \text{rk} \mathfrak{g} - r$ and by (3.1) $[\mathfrak{l}, \mathfrak{t}^a] = I[a, \mathfrak{t}^a] = 0$.

**Proposition 4.2.** Let $\mathfrak{a} \subset \mathfrak{m}$ be a Cartan subspace. For the irreducible symmetric space $G/K$ either $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ (and $\mathfrak{t}^a = 0$) or the centralizer of the algebra $\mathfrak{t}^a$ in $\mathfrak{m}$ coincides with the space $\mathfrak{a} \oplus Ia$.

**Proof.** Let $\mathfrak{n}(\mathfrak{t}^a)$ be the normalizer of the algebra $\mathfrak{t}^a$ in $\mathfrak{g}$. Since $\mathfrak{t}^a$ is an ideal of the compact algebra $\mathfrak{n}(\mathfrak{t}^a)$, we obtain the following splitting $\mathfrak{n}(\mathfrak{t}^a) = \mathfrak{t}^a \oplus \mathfrak{g}_a$, where
\( \mathfrak{g}_* = \{ X \in \mathfrak{g} : [X, \mathfrak{t}^0] = 0, \langle X, \mathfrak{t}^0 \rangle = 0 \} \). It is clear that the rank of the Lie algebra \( \mathfrak{g}_* \) does not exceed \((\text{rk } \mathfrak{g} - \text{rk } \mathfrak{t}^0) = r \). By the Jacobi identity \([ [a, Ia], \mathfrak{t}^0 ] = 0 \). Therefore \([ \mathfrak{g}_*, \mathfrak{t}^0 ] = 0 \) and \( \langle \mathfrak{g}_*, \mathfrak{t}^0 \rangle = 0 \). Hence \( \mathfrak{g}_* \) is a semisimple algebra of rank \( r \) containing the semisimple subalgebra \( \tilde{\mathfrak{g}} \) of maximal rank. Since \( Z_0 \) is an element of the center of the subalgebra \( \mathfrak{t} \), we obtain that \( \text{ad}_{Z_0} (\mathfrak{g}_*) \subset \mathfrak{g}_* \) and the triple \((\mathfrak{g}_*, \mathfrak{t}, I_* )\), where \( \mathfrak{t}_* = \mathfrak{t} \cap \mathfrak{g}_* \), \( \mathfrak{m}_* = \mathfrak{m} \cap \mathfrak{g}_* \), \( I_* = I|_{\mathfrak{m}_*} \), is an Hermitian orthogonal symmetric Lie algebra. Taking into account that \( \mathfrak{a} \subset \mathfrak{m}_* \), we conclude that this orthogonal Lie algebra has maximal possible rank \( r = \text{rk } \mathfrak{g}_* \). Its each irreducible component also has maximal possible rank \( \mathfrak{a} \) is a Cartan subalgebra of \( \mathfrak{g}_* \), i.e. corresponds to the compact symmetric space \( Sp(\mathfrak{n})/U(\mathfrak{n}) \) for appropriate \( n \geq 1 \) [He, Ch.X, §6]. Therefore \( \mathfrak{g}_* \) is a semisimple Lie algebra of type \( C_n \oplus C_{n_2} \oplus \cdots \). We claim that \( n_1 = n_2 = \ldots = n_r = 1 \) or \( \mathfrak{t}^0 = 0 \).

Indeed, by construction the algebra \( \mathfrak{g}_* \oplus \mathfrak{t}^0 \) is a subalgebra of \( \mathfrak{g} \) of maximal rank. Hence \( \mathfrak{g}_* \) is a regular subalgebra of \( \mathfrak{g} \), i.e. \([ \mathfrak{t}_1, \mathfrak{g}_* ] \subset \mathfrak{g}_* \) for some Cartan subalgebra \( \mathfrak{t}_1 \) of \( \mathfrak{g} \). But the algebra \( \mathfrak{g} \) is a compact Lie algebra from the following list \( A_l, B_l, C_l, D_l, E_l, F_7 \) [He, Ch.X, §6]. Since for algebras \( A_l, D_l, E_l, F_7 \) all roots of their root systems have the same length, these algebras do not contain regular subalgebras of type \( C_n, n \geq 2 \). So we have to consider only two cases when \( G/K \) is the symmetric space \( SO(2l + 1)/(SO(2) \times SO(2l - 1)) \) with \( l \geq 3 \) or \( Sp(l)/U(l) \) with \( l \geq 2 \). In the first case \( \mathfrak{t}^0 \simeq so(2l - 3) \) and \( \mathfrak{g}_* \simeq so(4) \simeq C_1 \oplus C_1 \) [He, Ch.X]. In the second case ranks of \( G \) and \( G/K \) coincide, i.e. \( \mathfrak{t}^0 = 0 \). So the claim is proved.

\( \square \)

4.2. Invariant mappings and root theory of Hermitian symmetric spaces. In this subsection the transformation of the restricted root system of \( (\mathfrak{g}, \mathfrak{a}) \) induced by the action of \( I \) on \( \mathfrak{m} \) is studied.

Let \( \mathfrak{a} \) be some Cartan subspace of \( \mathfrak{m} \). For each \( \lambda \) in the dual space of \( \mathfrak{a}^\mathbb{C} \) let \( \tilde{\mathfrak{g}}_\lambda = \{ X \in \mathfrak{g}^\mathbb{C} : [H, X] = \lambda(H), \forall H \in \mathfrak{a}^\mathbb{C} \} \). Then \( \lambda \) is called a restricted root if \( \lambda \neq 0 \) and \( \tilde{\mathfrak{g}}_\lambda \neq 0 \). The set of all such \( \lambda \) is denoted by \( \Sigma \). The simultaneous diagonalization of \( \text{ad}(\mathfrak{a}^\mathbb{C}) \) in \( \mathfrak{g}^\mathbb{C} \) gives the decomposition \( \mathfrak{g}^\mathbb{C} = \tilde{\mathfrak{g}}_0 \oplus \sum_{\lambda \in \Sigma^+} (\tilde{\mathfrak{g}}_\lambda \oplus \tilde{\mathfrak{g}}_{-\lambda}) \), where \( \Sigma^+ \) is an arbitrary subset of positive restricted roots in \( \Sigma \).

For each linear form \( \lambda \) on \( \mathfrak{a}^\mathbb{C} \) put

\[
\mathfrak{m}_\lambda \overset{\text{def}}{=} \{ \eta \in \mathfrak{m} : \text{ad}_w^2(\eta) = \lambda^2(\lambda) \eta, \forall w \in \mathfrak{a} \},
\]

\[
\mathfrak{t}_\lambda \overset{\text{def}}{=} \{ \zeta \in \mathfrak{t} : \text{ad}_w^2(\zeta) = \lambda^2(\zeta) \zeta, \forall w \in \mathfrak{a} \}.
\]

Then \( \mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}, \mathfrak{t}_\lambda = \mathfrak{t}_{-\lambda}, \mathfrak{m}_0 = \mathfrak{a} \) and \( \mathfrak{t}_0 = \mathfrak{t}^0 \), the centralizer of \( \mathfrak{a} \) in \( \mathfrak{t} \). By [He, Lemma 11.3, Ch.VII] the following decompositions are direct and orthogonal:

\[
\mathfrak{m} = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda, \quad \mathfrak{t} = \mathfrak{t}^0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{t}_\lambda. \tag{4.2}
\]

We need the following lemma which is a weak generalization of Lemma 2.3 in [He, Ch.VII].

**Lemma 4.3.** For any vector \( \xi_\lambda \in \mathfrak{m}_\lambda, \lambda \in \Sigma^+ \) there exists a unique vector \( \zeta_\lambda \in \mathfrak{t}_\lambda \) such that

\[
[w, \xi_\lambda] = i \lambda(w) \zeta_\lambda, \quad [w, \zeta_\lambda] = -i \lambda(w) \xi_\lambda \quad \text{for all } w \in \mathfrak{a}. \tag{4.3}
\]
In particular, \( \dim m_\lambda = \dim \mathfrak{f}_\lambda \) and
\[
\text{ad}_{w'} \text{ad}_{w''}(\xi_\lambda) = \lambda(w')\lambda(w'')\xi_\lambda, \quad \text{where} \quad w', w'' \in \mathfrak{a}.
\]

**Proof.** For completeness and mainly to fix the notation we shall prove this lemma. It is clear that \((m_\lambda \oplus \mathfrak{f}_\lambda)^C = (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})\). Therefore \(\text{ad}_{w'}(m_\lambda) \subset \mathfrak{f}_\lambda \) and \(\text{ad}_w(\mathfrak{f}_\lambda) \subset m_\lambda\) for \(w \in \mathfrak{a}\). So the endomorphisms \(\text{ad}_w \) and \(\text{ad}_w^2\), when restricted to \(m_\lambda \oplus \mathfrak{f}_\lambda\), are nondegenerate or degenerate simultaneously. Hence the subspace \([w, \lambda_\lambda], \ w \in \mathfrak{a}\) \(\subset \mathfrak{f}_\lambda\) is one-dimensional. Since \(\lambda(a) \in i\mathbb{R}\), there is the element \(\xi_\lambda \in \mathfrak{f}_\lambda\) such that for the pair \(\{\lambda_\lambda, \lambda_\zeta\}\) condition (4.3) holds. Now the latter assertion of the lemma is evident. \(\square\)

Let \(f : W \to m\) be a mapping. Identifying the tangent spaces \(T_w m\) and \(T_{f(w)} m\) with \(m\), we can consider the tangent mapping \(f_* : T_w m \to T_{f(w)} m\) as an endomorphism on \(m\). We say \(f\) is \(K\)-equivariant if \(\text{Ad}_k \circ f = f \circ \text{Ad}_k\) on \(W\) for all \(k \in K\). For such a mapping its tangent map \(f_* : w \mapsto f_* w \in \text{End}(m)\) is also \(K\)-equivariant, i.e. satisfies (2.6). Denote by \(\text{EC}(W)\) the set of all \(K\)-equivariant mappings \(f\) (on \(W\)) which leave some (and, consequently, each) Cartan subspace \(\mathfrak{a}\) of \(m\) invariant, i.e. \(f(W \cap \mathfrak{a}) \subset \mathfrak{a}\).

Let \(f \in \text{EC}(W)\). By \(K\)-equivariance, \(f_* ([\xi, w]) = [\xi, f(w)]\) for any \(\xi \in \mathfrak{f}\). Hence by (4.3) for any \(w \in W \cap \mathfrak{a}\) and \(\lambda_\lambda, \lambda \in \Sigma^+\)
\[
f_* (i\lambda(w)\xi_\lambda) = f_* ([\xi_\lambda, w]) = [\xi_\lambda, f(w)] = i\lambda(f(w))\xi_\lambda. \quad (4.4)
\]

For future use we next prove two lemmas.

**Lemma 4.4.** Let \(f \in \text{EC}(W)\). If \(w \in W \cap \mathfrak{a}\) then \(f_* (\mathfrak{a}) \subset \mathfrak{a}\) and for each \(\lambda \in \Sigma^+\) the subspace \(m_\lambda\) is an eigenspace of \(f_* w\) with the eigenvalue \(\lambda(f(w))/\lambda(w)\). Moreover, for all \(w \in W, \xi, \eta \in m\)
\[
f_* ([w, [\xi, \eta]]) = [[w, f_* (\xi)], \eta] - [[w, f_* (\eta)], \xi]. \quad (4.5)
\]

**Proof.** Let \(w \in W \cap \mathfrak{a}\). Suppose that \(\xi \in \mathfrak{a}\). Since by definition \(f_* (\mathfrak{a}) \subset \mathfrak{a}\), the first term on the right in (4.5) vanishes. But \([\text{ad}_w, \text{ad}_\xi] = 0\). Therefore (4.5) is equivalent to the relation
\[
([\text{ad}_w \text{ad}_\xi] m, f_* w) = 0,
\]
which holds, because \(\text{ad}_w \text{ad}_\xi (\mathfrak{a}) = 0\) and each subspace \(m_\lambda, \lambda \in \Sigma^+\) is an eigenspace of endomorphisms \(\text{ad}_w \text{ad}_\xi\) and \(f_* w\).

Since relation (4.5) is skew-symmetric for exchanges of two variables \(\xi\) and \(\eta\), it remains to prove (4.5) if \(\xi = \xi_\lambda \in m_\lambda\) and \(\eta = \xi_\nu \in m_\nu\) for \(\lambda, \nu \in \Sigma^+\). There are vectors \(\xi_\lambda \in \mathfrak{f}_\lambda\) and \(\xi_\nu' \in \mathfrak{f}_\nu\) such that for the pairs \((\xi_\lambda, \xi_\lambda)\) and \((\xi_\nu', \xi_\nu')\) condition (4.3) holds for all \(w \in W \cap \mathfrak{a}\). But
\[
[\mathfrak{f}_\lambda, m_\nu] + [m_\lambda, \mathfrak{f}_\nu] \subset m_{\lambda+\nu} + m_{\lambda-\nu} \quad \text{and} \quad [m_\lambda, m_\nu] \subset \mathfrak{f}_{\lambda+\nu} + \mathfrak{f}_{\lambda-\nu}
\]
(see [He, Ch.VII, Lemma 11.4]). In particular, \([\xi_\lambda, \xi_\nu'] = \xi_\pm + \xi_{-\pm}\), where \(\xi_{\pm} \in \mathfrak{f}_{\lambda_{\pm}}\). If \(\lambda - \nu = 0\) then \([\mathfrak{a}, \xi_{-\pm}] = 0\). Therefore there exist vectors \(\xi_{\pm} \in m_{\lambda_{\pm}}\) such that \([w', \xi_{\pm}] = -i(\lambda \pm \nu)(w')\xi_{\pm}\) for all \(w' \in \mathfrak{a}\). Now from the Jacobi identity
\[
[w', [\xi_\lambda, \xi_\nu']] = [[w', \xi_\lambda], \xi_\nu'] - [[w', \xi_\nu'], \xi_\lambda]
\]
12
it follows that
\[-i(\lambda + \nu)(w')\xi_+ - i(\lambda - \nu)(w')\xi_- = i\lambda(w')[\xi_+][\xi_-] - iv(w')[\xi_+][\xi_-].\] (4.6)

Taking into account that 
\[f_{sw}(\xi_{\pm}) = \frac{(\lambda \pm \nu)(f(w))}{(\lambda \pm \nu)(w)}\xi_{\pm}\] and similar relations hold for 
\[\xi_+, \xi_-\], we obtain (4.5) replacing \(w\) by \(f(w)\) in identity (4.6). Noting that \(m\) is a union of its Cartan subspaces, we complete the proof. \(\square\)

**Lemma 4.5.** Let \(f \in EC(W)\) and let \(\sigma\) be a 1-form on \(W\) such that \(\sigma_w(\cdot) = \langle f(w), \cdot \rangle\). Then

1. the form \(\sigma\) is \(Ad(K)\)-invariant;
2. \(\sigma\) is closed iff so is its restriction to the set \(W \cap a\).

**Proof.** It is immediate that \(\sigma\) is invariant. By definition
\[d\sigma_w(\xi, \eta) = \langle f_{sw}(\xi), \eta \rangle - \langle f_{sw}(\eta), \xi \rangle, \quad \xi, \eta \in m.\]

For \(w \in a\), by Lemma 4.4 \(f_{sw}(a) \subset a\), \(f_{sw}(a) \subset a^\perp\) and the restriction \(f_{sw}|_{a^\perp}\) is a symmetric operator. Therefore \(d\sigma_w = 0\) iff \(d\sigma_w(a, a) = 0\) for all \(w \in W \cap a.\) \(\square\)

Let \(O \subset \mathbb{R}\) be a domain containing spectrums of all operators \((-\text{ad}_{w}^2 - \text{ad}_{w}^2)|m\), \(w \in W\). For a real-analytic function \(q\) on \(O\), define a mapping \(\hat{q} : W \rightarrow m\) by
\[\hat{q}(w) = q(-\text{ad}_{w}^2 - \text{ad}_{w}^2)(w).\]

**Proposition 4.6.** The mapping \(\hat{q} : W \rightarrow m\) is \(K\)-equivariant and each Cartan subspace \(a\) of \(m\) is invariant with respect to \(\hat{q}\), i.e. \(\hat{q} \in EC(W)\). Moreover,
\[m = \sum_{\lambda \in \Sigma^+} \mathbb{R} X_j \oplus \sum_{\lambda \in \Sigma^+} m_\lambda, \quad \text{where} \ X_j = X_{\beta_j},\] (4.7)

is the orthogonal eigenspace splitting for all \(\hat{q}_{sw}, w \in W \cap a\).

**Proof.** Since the endomorphism \(I\) belongs to the center of the group \(\text{Ad}(K)|m\), it follows that \(\text{Ad}_k \circ \text{ad}_{I_w} = \text{ad}_{(\text{Ad}_k)w} \circ \text{Ad}_k\) and, consequently, \(\text{Ad}_k \circ \hat{q} = \hat{q} \circ \text{Ad}_k\) on \(W\) for all \(k \in K\). Now fix some Cartan subspace \(a = \sum_{j=1}^r \mathbb{R} X_{\beta_j}\) of \(m\) (as in subsection 4.1) and relabel \(X_{\beta_j}, Y_{\beta_j}, T_{\beta_j}\) to read \(X_j, Y_j, T_j, j = 1, \ldots, r\). One has
\[\begin{align*}
[X_j, Y_k] &= \delta_j^k T_j, \quad [T_j, X_k] = \delta_j^k Y_j, \quad [T_j, Y_k] = -\delta_j^k X_j, \quad j, k = 1, \ldots, r.
\end{align*}\] (4.8)

In particular, \(I X_j = Y_j\) and \(I Y_j = -X_j\). Since \(\text{ad}_{I X_j}(X_k) = -\delta_j^k X_k\) and \([a, a] = 0\), for any \(w = \sum_{j=1}^r x_j X_j \in W \cap a\) we have
\[\hat{q}(w) = \sum_{j=1}^r x_j q(x_j^2) X_j \quad \text{and} \quad \hat{q}_{sw}(X_j) = (q(x_j^2) + 2x_j^2 q'(x_j^2)) X_j, j = 1, \ldots, r,\] (4.9)
i.e. \(\hat{q}(W \cap a) \subset a\). The latter assertion follows immediately from (4.4) and (4.9). \(\square\)

Fix in the Cartan subspace \(a \subset m\) a basis \(\{X_j\}_{j=1}^r\) (4.8). Let the set of restricted roots \(\Sigma\) of \((g, a)\) be ordered lexicographically with respect to the basis \(\{-iX_j\}_{j=1}^r\) in \(ia \subset a^C\) (all \(\lambda \in \Sigma\) are real on the subspace \(ia \subset g^C\)). Denote by \(\Sigma^+\) the corresponding system of positive restricted roots. Choose the basis \(\{e_j\}_{j=1}^r\) in the (complex) space \((a^C)^\ast\) dual to the basis \(\{X_j\}_{j=1}^r\) of \(a^C\).
Proposition 4.7. For any vector $\xi \in m_\lambda$, $\lambda \in \Sigma^+$ the vector $I\xi$ belongs to the subspace $m_{\lambda_1}$, $\lambda_1 \in \Sigma^+ \cup \{0\}$, i.e. $\text{ad}_{\lambda_1}^2(\xi) = \lambda^2(w)\xi$, $\forall w \in a$ implies $\text{ad}_{\lambda}^2(I\xi) = \lambda_1^2(w)(I\xi)$. The set $\Sigma^+$ is a subset of the set

$$(BC)_+^+ = \left\{ \frac{1}{2} \varepsilon_j, \ v = \frac{1}{2} \varepsilon_j, \ 1 \leq p < k \leq r \right\},$$

and the set $\{(\lambda, \lambda_1), \lambda \in \Sigma^+\}$ is a subset of the set

$$\left\{ \left( \frac{1}{2} \varepsilon_j, \frac{1}{2} \varepsilon_j \right), \ (v, 0), \ 1 \leq p < k \leq r \right\}.$$

Proof. Fix $\lambda \in \Sigma^+$. Then for any vector $w = \sum_{j=1}^{r} x_j X_j \in a$: $\lambda(w) = \sum_{j=1}^{r} ic_j x_j$, where $c_j \in \mathbb{R}$. Applying (4.4) and (4.9) to the function $q$ with $q(z) = z^n$, $n \in \mathbb{N}$, we obtain that if $\lambda(x_1, \ldots, x_r) = 0$ then $\lambda(x_{2n+1}, \ldots, x_{2n+1}) = 0$. Therefore

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_1^3 & x_2^3 & \cdots & x_r^3 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2r-1} & x_2^{2r-1} & \cdots & x_r^{2r-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} = 0$$

for all real vectors $(x_1, \ldots, x_r) \in \ker \lambda$. Since the column $(c_1, \ldots, c_r)$ is nonzero, the determinant $\pm \prod_{j=1}^{r} x_j \prod_{1 \leq p < k \leq r} (x_p^2 - x_k^2)$ of the matrix above $= 0$ at these points. In other words, $\ker \lambda = \bigcup_{\sigma \in \Sigma} (\ker \lambda \cap \ker \sigma)$. Hence $\ker \lambda = \ker \sigma$ for some $\sigma \in (BC)_+^+$ because $(BC)_+^+$ is a finite set, or equivalently, $a^{-1}_\lambda \lambda \in (BC)_+^+$ for some $a_\lambda > 0$. Now put $q(z) = -z$. Then $\hat{q}(w) = [Iw, [Iw, w]]$ and by (3.1)

$$\hat{q}_{w}(\eta) \overset{\text{def}}{=} [I\eta, [Iw, w]] + [Iw, [Iw, w]] + [Iw, [Iw, \eta] = [I\eta, [Iw, w]] - 2[Iw, [Iw, \eta]] = \text{ad}_{[Iw, w]}(I\eta) - 2(\text{ad}_{w}^2(I\eta)$$

for each $\eta \in m$. Using the Jacobi identity for the vectors $I\eta, Iw, w$ and relations (3.1) again, we calculate the vector $\text{ad}_{[Iw, w]}(I\eta)$:

$$[[w, Iw], I\eta] = [w, [Iw, I\eta]] - [Iw, [w, I\eta]] = [w, [w, \eta]] - I[w, [w, I\eta]] = [w, [w, \eta]] - I[w, [w, I\eta]]. \quad (4.10)$$

So that $\hat{q}_{w}(\eta) = \text{ad}_{w}^2(\eta) - 3(I \text{ad}_{w}^2)(I\eta)$. Now from (4.4) for given $\xi \in m_\lambda$ it follows that

$$\lambda^2(x_1, \ldots, x_r)\xi_\lambda - 3(\text{ad}_{w}^2)(I\xi_\lambda) = \frac{\lambda(-x_1^3, \ldots, -x_r^3)}{\lambda(x_1, \ldots, x_r)} \xi_\lambda. \quad (4.11)$$

Applying $I$ to equation (4.11) we obtain that $I\xi_\lambda$ is a common eigenvector of all endomorphisms $\text{ad}_{w}^2, w \in a$. Hence there is a unique element $\lambda_1 \in \Sigma^+ \cup \{0\}$ such that $\text{ad}_{w}^2(I\xi_\lambda) = \lambda_1^2(w)I\xi_\lambda$ and

$$\lambda^2(x_1, \ldots, x_r) + 3\lambda_1^2(x_1, \ldots, x_r) = -\frac{\lambda(x_1^3, \ldots, x_r^3)}{\lambda(x_1, \ldots, x_r)}. \quad (4.12)$$
It is easy to verify that if \( \lambda = a_\lambda \cdot \frac{1}{i}(\epsilon_p \pm \epsilon_k) \), \( p \neq k \), then the pair \((\lambda, \lambda_I)\) satisfies equation (4.12) iff \( a_\lambda = 1 \) and \( \lambda_I = \frac{1}{i}(\epsilon_p \mp \epsilon_k) \).

Since \( \text{ad}_X^2(IX_k) = -\delta^k_h(IX_k) \), the covectors \( i\epsilon_j, j = 1, \ldots, r \) are positive restricted roots from \( \Sigma^+ \). Therefore if the restricted roots \( \sigma, \lambda \in \Sigma^+ \) are proportional, then \( \sigma = i\epsilon_j \) for some \( j \in \{1, \ldots, r\} \) and \( \lambda \) equals \( \frac{1}{2}\epsilon_j \) or \( 2i\epsilon_j \). In this case all possible solutions \((\lambda, \lambda_I)\), \( \lambda \in \Sigma^+ \) of (4.12) are pairs \((i\epsilon_j, 0)\) and \((\frac{1}{2}\epsilon_j, \frac{1}{2}\epsilon_j)\), \( j = 1, \ldots, r \). \( \square \)

Let 

\[
\Sigma^{++} = \Sigma^+ \cap \{ \frac{1}{2}\epsilon_j, i\epsilon_j, j = 1, \ldots, r; \ \frac{1}{2}(\epsilon_p + \epsilon_k), 1 \leq p < k \leq r \}. 
\]

So

\[
m = \sum_{\lambda \in \Sigma^{++}} M_\lambda, \quad \text{where} \quad M_\lambda = m_\lambda + Im_\lambda, \quad (4.13)
\]

is an orthogonal splitting of \( m \). Since \( a = m_0 \) and \( \dim a = r \), we have

**Corollary 4.7.1.** The covector \( \lambda = i\epsilon_j \), \( 1 \leq j \leq r \) is a positive restricted root from \( \Sigma^+ \) with multiplicity one and \( m_\lambda = \mathbb{R}(IX_j) \). If \( \lambda \in \Sigma^+ \setminus \{i\epsilon_1, \ldots, i\epsilon_r\} \) then \( Im_\lambda = m_\lambda \).

Let \( t' \) be the subspace of \( t \) spanned by \( \{T_j\}_{j=1}^r \). Choose the basis \( \{\epsilon'_j\}_{j=1}^r \) in the (complex) space \((t'^C)^*\) dual to the basis \( \{T_j\}_{j=1}^r \) of \( t'^C \). Let \( \rho \) denote the restriction mapping \((t'^C)^* \rightarrow (t'^C)^*\) and let \( \rho_m \) be the mapping from \((BC)^+_r \subset (a^C)^* \rightarrow (t'^C)^*\) with the graph given by

\[
\{ (\frac{1}{2}\epsilon_j), i\epsilon_j, i\epsilon'_j, j = 1, \ldots, r; \ \frac{1}{2}(\epsilon_p + \epsilon_k), 1 \leq p < k \leq r \}. \quad (4.14)
\]

**Corollary 4.7.2.** For each \( j = 1, \ldots, r \), \( \rho(\beta_j) = i\epsilon'_j \). The set \( \rho(\Delta_m^+) \) is a subset of the set

\[
\{ \frac{1}{2}\epsilon'_j, i\epsilon'_j, j = 1, \ldots, r; \ \frac{1}{2}(\epsilon'_p + \epsilon'_k), 1 \leq p < k \leq r \} \quad (4.15)
\]

For each \( \lambda \in \Sigma^+ \),

\[
m_\lambda + Im_\lambda = \sum_{\alpha \in \rho^{-1}(\rho_m(\lambda))} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha).
\]

In particular, the space \( m_\lambda + Im_\lambda \) is an eigenspace of \( I \text{ad}_T \), \( T \in t' \) with the eigenvalue \( i\rho_m(\lambda)(T) \).

**Proof.** Choose \( \lambda \in \Sigma^+ \) and \( \xi_\lambda \in m_\lambda \). By (4.8) \( T_j = [X_j, IX_j] \). Putting in relation (4.10) \( w = X_j \) and \( \eta = -I\xi_\lambda \) we obtain that \( \text{ad}_{T_j}(\xi_\lambda) = -(\lambda^2(X_j) + \lambda^2(X_j))I\xi_\lambda \). But \( [I, \text{ad}_{T_j}] | m = 0 \). Therefore

\[
[T, \xi] = -\sum_{j=1}^r (\lambda^2(X_j) + \lambda^2(X_j)) \epsilon'_j(T) \cdot I\xi, \quad \forall T \in t', \ \xi \in (m_\lambda + Im_\lambda). \quad (4.16)
\]

Thus \( \xi \pm iI\xi \in m'^\mathbb{C} \) are the root vectors corresponding to the roots \( \pm i \sum_{j=1}^r (\lambda^2(X_j) + \lambda^2(X_j)) \epsilon'_j \) from \( \rho(\Delta_m^+) \). Now from splitting (4.13) and Proposition 4.7 it follows that \( \rho(\Delta_m^+) \subset Q \cup (-Q) \), where \( Q \) is set (4.15). Taking into account that \( -i\alpha(T_{\beta_j}) \geq 0 \) for \( \alpha \in \Delta_m^+ \) and \( -i\beta_k(T_{\beta_j}) = \delta^k_i \) (see subsection 4.1), we complete the proof. \( \square \)

Let \( \rho_t \) be the mapping from \((BC)^+_r \subset (a^C)^* \rightarrow (t'^C)^*\) with the graph given by

\[
\{ (\frac{1}{2}\epsilon_j, i\epsilon'_j), (\epsilon_j, 0), j = 1, \ldots, r; \ \frac{1}{2}(\epsilon_p + \epsilon_k), \frac{1}{2}(\epsilon'_p - \epsilon'_k), 1 \leq p < k \leq r \}. \]
Corollary 4.7.3. Let $\lambda \in \Sigma^+$ be a restricted root. Suppose that $\lambda_I \neq 0$. Let $\xi_\lambda \in m_\lambda$ and $\xi'_I = I \xi_\lambda \in m_I$. Then for any $T \in t'$ and $\xi_\lambda \in t_\lambda$, $\xi'_I \in t'_I$ with the notations of Lemma 4.3

$$\text{ad}_T(\xi_\lambda) = i\rho_T(\lambda)(T)\xi'_I.$$  

If $\lambda_I = 0$ then $[t', t_\lambda] = 0$.

Proof. Using the notations of Lemma 4.3 and the Jacobi identity for the vectors $X_j, IX_j$ and $\xi_\lambda \in t_\lambda$ we obtain that

$$[[X_j, IX_j], \xi_\lambda] = [X_j, [IX_j, \xi_\lambda]] - [IX_j, [X_j, \xi_\lambda]] = 2[X_j, I[IX_j, \xi_\lambda]] = -2i\lambda(X_j)[X_j, I\xi_\lambda] = 2\lambda(X_j)\lambda_I(X_j)\xi'_I.$$  

If $\lambda_I = 0$, in the chain of equations above $[X_j, I\xi_\lambda] = 0$ because $I\xi_\lambda \in a = m_0$. Now the assertion of the corollary comes from Proposition 4.7.

By (3.1) $\text{ad}_{Iw}^2(I\xi_\lambda) = I\text{ad}_w^2(\xi_\lambda)$ and by (4.10) $I\text{ad}_{[w, Iw]}(\xi_\lambda) = (\text{ad}_w + \text{ad}_{Iw})^2(\xi_\lambda)$ (for all $\xi \in m$). So as an immediate consequence of Proposition 4.7 we obtain

Corollary 4.7.4. Sum (4.7) (resp. (4.13)) is an orthogonal eigenspace splitting of $m$ for all operators $\text{ad}_w^2, \text{ad}_{Iw}^2, w \in a$ (resp. $I\text{ad}_{[w, Iw]}, \text{ad}_w + \text{ad}_{Iw}, w \in a$). In particular, for each $w \in m, [\text{ad}_w, \text{ad}_{Iw}]|m = 0$ and $I\text{ad}_{[w, Iw]} = \text{ad}_w + \text{ad}_{Iw}$ on $m$.

Remark 4.8. The well-known Harish-Chandra and Moore theorem for Hermitian symmetric spaces (see [Wo]) describes the restricted root system of $(g, t')$ and, using the Cayley transform, such a system for $(g, a)$ (the spaces $t'$ and $a$ are conjugated in $g$ as Cartan subalgebras of the same compact Lie algebra $a \oplus Ia \oplus [a, Ia]$). This theorem follows from Corollaries 4.7.2 and 4.7.3. But the mapping $\Sigma^+ \rightarrow \Sigma^+ \cup \{0\}$, $\lambda \mapsto \lambda_I$ of Proposition 4.7 allows us to find direct connection between the restricted root decompositions of $(g, a)$ and $(g, t')$.

4.3. The main theorem. Here using the result of previous subsection we construct all antiholomorphic $K$-equivariant mappings on homogeneous domains in $m$ and prove the main theorem.

Let $\sigma : T(G/K) \rightarrow T(G/K)$ be the involution which maps any tangent vector $Y$ at $gK$ onto $-Y$ at $gK$ and let $P \in \text{Alm}(W)$, where $W = -W$. It is easy to see that $\sigma$ is an antiholomorphic involution for the almost complex structure $J(P)$, i.e. $\sigma_*(F(P)) = F(P)$, iff $P_w = P_{-w}$ for all $w \in W$. But $I \in \text{Ad}(K)|m$ (see subsection 4.1), so by (2.6) $IP_w = P_{Iw}I$ and, consequently, $P_w = P_{-w}$. We have proved

Lemma 4.9. Let $P \in \text{Alm}(W)$. The mapping $\sigma$ is an antiholomorphic involution for $J(P)$ iff $P = Iw = P_{-w}$.  

The following proposition and lemma will be crucial for the subsequent part of the paper.

Proposition 4.10. Let $P = (R + IS) \in \text{Alm}(W)$ and $RI = IR, SI = -IS$. Suppose that $L_{\xi}(SR^{-1})\eta = L_{\eta}(SR^{-1})\xi$ for all (fixed) $\xi, \eta \in m$. Then $SR^{-1} = (a_1 + a_2 I)\Upsilon$ on $W$, where $a_1, a_2 \in \mathbb{R}$ are some numbers and $\Upsilon$ is a rational $K$-equivariant mapping on $m$ given by

$$\Upsilon(w) = (- \text{ad}_w^2 - \text{ad}_{Iw}^2)^{-1}(w).$$
The endomorphism $\Upsilon_w : m \to m$, where $w$ belongs to the set $W_\gamma$ of all regular points of $\Upsilon$, anticommute with $I$ iff all restricted roots from $\Sigma$ are indivisible, i.e. $\Sigma$ has type $C_r$.

Proof. Fix the Cartan subspace $a = \sum_{j=1}^r \mathbb{R}X_{\beta_j}$ of $m$. Let $\mathfrak{a} = a \oplus Ia$. Consider $(m, I)$ as a space over $\mathbb{C}$ with fixed basis $\{X_\alpha, \alpha \in \Delta_m^+\}$. For each complex number $z_0 = x + iy$ put $z_0X_\alpha \overset{\text{def}}{=} xX_\alpha + yIX_\alpha = xX_\alpha + yY_\alpha \in m$ (here $i^2 = -1$, $x, y \in \mathbb{R}$). Denote by $C : m \to m$ the corresponding complex conjugation mapping, i.e. $C(z_0X_\alpha) = \overline{z_0}X_\alpha$, where $\overline{z_0} = x - iy$. For any complex vector $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ we denote by $zX$ the vector $\sum_{j=1}^r z_jX_j \in \mathfrak{a}$. Put $Z = \{z \in \mathbb{C}^r : zX \in W \cap \mathfrak{a}\}$. Since $ICS^{-1} = C^{-1}I$, there exists the complex matrix-function $z \mapsto (h_{\alpha\beta}(z))$, $\alpha, \beta \in \Delta_m^+$ on $Z$ such that the $\mathbb{C}$-linear mapping $C(S^{-1})_zX$ takes each $\sum_{\beta} v_{\beta}X_\beta \in m$ to $\sum_{\alpha, \beta} h_{\alpha\beta}(z)v_{\alpha}X_\alpha \in m$. Considering $C$ as an $\mathbb{R}$-linear mapping on the set $W \cap \mathfrak{a}$ and using relation (3.9), we conclude that locally each function $h_{\alpha\beta} : Z \to \mathbb{C}$ is a partial derivative of some holomorphic function, i.e. $h_{\alpha\beta}$ is holomorphic on the set $Z \subset \mathbb{C}^r$. In particular, each holomorphic $1$-form $\sum_{j=1}^r h_{\alpha\beta}(z) \cdot dz_j$ is closed. From (4.1) it follows that for any vector $T \in t'$, $\exp(\text{ad}_T)(X_\alpha) = e^{i\alpha(T)}X_\alpha$, where $\alpha' = -i\alpha|t'$ is a real linear function on $t'$. Because of $K$-equivariance of $S^{-1}$

$$\text{Ad}_k(C^{-1})_{w} \text{Ad}_{k-1} = C(C^{-1})_{w} \text{Ad}_{k-1}, \quad \forall w \in W, k \in K,$$

and, consequently,

$$e^{-i(\alpha' + \beta')(T)}h_{\alpha\beta}(z_1, \ldots, z_r) = h_{\alpha\beta}(e^{i\beta'_{\alpha}(T)}z_1, \ldots, e^{i\beta'_{\alpha}(T)}z_r), \quad \forall z \in Z, T \in t'.$$

Therefore $h_{\alpha\beta}(z_1, \ldots, z_r) = a_{\alpha\beta}z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ if $\alpha' + \beta' = -\sum_{k=1}^r p_k\beta'_{k}$ for some integers $p_k \in \mathbb{Z}$ else $h_{\alpha\beta}(z) = 0$. Here $a_{\alpha\beta} \in \mathbb{C}$ is some constant. Then by Corollary 4.7.2

$$h_{\alpha\beta}(z) = \begin{cases} \alpha_{\alpha\beta} \cdot z_1^{\alpha_1} z_j^{\alpha_j}, & \text{if } \alpha = \beta, \beta = \beta_j, k, j = 1, r; \\ \alpha_{\alpha\beta} \cdot z_1^{\alpha_1} z_{k}^{\alpha_k}, & \text{if } \alpha' = \beta' = \frac{1}{2}(\epsilon_p' + \epsilon_k'), p \neq k; \\ \alpha_{\alpha\beta} \cdot z_1^{\alpha_1} z_j^{\alpha_j}, & \text{if } \alpha' = \beta' = \frac{1}{2}\epsilon_j', j = 1, r; \\ \end{cases}$$

and otherwise $h_{\alpha\beta}(z) = 0$.

Let $m_1$ be a subspace of $m$ spanned by vectors $X_\alpha, Y_\alpha$ with $\alpha' = \frac{1}{2}(\epsilon_p' + \epsilon_k')$, $1 \leq p \leq k \leq r$. Denote by $m_{1/2}$ the orthogonal complement to the space $m_1$ in $m$. Using the expressions for the matrix element $h_{\alpha\beta}$ of $S^{-1}$, we obtain that the subspaces $m_1$ and $m_{1/2}$ are invariant with respect to all operators $(C^{-1})_w$ with $w \in W \cap a$ and

$$\frac{d}{dt}|_0(C^{-1})_{w+tw}m_1 = -2(C^{-1})_wm_1;$$

$$\frac{d}{dt}|_0(C^{-1})_{w+tw}m_{1/2} = -(C^{-1})_wm_{1/2}.$$

Consider the mapping

$$\tilde{\Upsilon} : W \to m, \quad \tilde{\Upsilon}(w) = -(S^{-1})_w \cdot w$$

This mapping is $K$-equivariant because the mapping $w \mapsto (S^{-1})_w$ satisfies condition (2.6). By (3.9) for $\xi \in m$

$$\tilde{\Upsilon}_w(\xi) = -(L_{\xi}(S^{-1}))_w w - (S^{-1})_w \xi = -(\frac{d}{dt}|_0(S^{-1})_{w+tw})\xi - (S^{-1})_w \xi.$$
Thus for all $w \in W \cap \mathfrak{a}$

$$\tilde{\Upsilon}_{sw}|m_1 = (SR^{-1})_w|m_1 \quad \text{and} \quad \tilde{\Upsilon}_{sw}|m_{1/2} = 0.$$  

(4.17)

Let us find this function $\tilde{\Upsilon}$. Since the holomorphic form $\sum_{j=1}^r h_{\beta_j} \cdot dz_j$ is closed, $h_{\beta_j} = 0$ if $k \neq j$. Relabel the function $h_{\beta_j}$ by $h_j$. Since $G/K$ is an irreducible hermitian symmetric space, the restricted root system of $(g, \mathfrak{a})$ has type $C_r$ or $(BC)_r$ [12, Ch.X, §6]. Therefore the restricted Weyl group of $(g, \mathfrak{a})$ induces all signed permutations $X_j \mapsto \pm X_j$. Taking into account $K$-equivariance (2.6) again, we obtain that $h_{\beta_j}(z_1, \ldots, z_r) = h_j(z_{s(1)}, \ldots, z_{s(r)})$, where $s$ is any permutation. But the function $h_j(z)$ depends only on the $j$-th coordinate $z_j$ of $z$. So there is a unique function $h(t) = z_0 t^{-2}, z_0, t \in \mathbb{C}$ such that $h_j(z) = h(z_j)$.

Then $(SR^{-1})_{zX}$ takes each $v_j X_j \in \mathfrak{a}$ to $z_0 z_j^{-2} v_j \cdot X_j \in \mathfrak{a}$, i.e. $\tilde{\Upsilon}(\sum_j x_j X_j) = (a_1 + a_2 I) \sum j x_j^{-1} X_j$, where $-z_0 = a_1 + a_2$, $x_j \in \mathbb{R}$.

The restriction of the mapping $w \mapsto (- \operatorname{ad}_w^\mathfrak{a} - \operatorname{ad}^2_w)^{-1}(w)$ to $\mathfrak{a}$ takes each $\sum_j x_j X_j$ to $\sum_j x_j^{-1} X_j$. Thus the mappings $\tilde{\Upsilon}$ and $(a_1 + a_2 I) \Upsilon$ coincide on the set $W \cap \mathfrak{a}$ and because of $K$-equivariance of $\tilde{\Upsilon}$ and $\Upsilon$

$$\tilde{\Upsilon}(w) = (a_1 + a_2 I) \Upsilon(w) \quad \text{for all} \quad w \in W. \quad (4.18)$$

Splitting (4.7) is the common eigenspace splitting for all $\Upsilon_{sw}, w \in \mathfrak{a}$ (see Proposition 4.6). The restrictions of $\Upsilon_{sw}$ and $I$ to $\mathfrak{a} = \mathfrak{a} \oplus I \mathfrak{a}$ anticommute because

(4.4) and (4.9)

$$\Upsilon_{sw}(x_j) = -x_j^{-2} X_j \quad \text{and} \quad \Upsilon_{sw}(I X_j) = x_j^{-2} I X_j, \quad w \in W \cap \mathfrak{a}, \quad j = \overline{1, r}.$$  

Suppose that $\lambda^\pm_{pk} = i(\epsilon_p \pm \epsilon_k)/2, \; p \neq k, \; \lambda_{j/2} = i \epsilon_j/2, \; 1 \leq j \leq r$ are restricted roots and $\xi_{pk}^\pm \in \mathfrak{m}_{\lambda^\pm_{pk}}, \; \xi_{j/2} \in \mathfrak{m}_{\lambda_{j/2}}$. Then applying (4.4) to the $K$-equivariant mapping $\Upsilon$, we obtain that

$$\Upsilon_{sw}(\xi_{pk}^\pm) = \frac{-x_p^{-1} \pm x_k^{-1}}{x_p \pm x_k} \xi_{pk}^\pm = \pm \frac{1}{x_p x_k} \xi_{pk}^\pm \quad \text{and} \quad \Upsilon_{sw}(\xi_{j/2}) = \frac{1}{x_j} \xi_{j/2}, \quad (4.19)$$

where $w = \sum_{j=1}^r x_j X_j$. But $\tilde{\Upsilon}_{sw}|m_{1/2} = 0$ (see (4.17)). Therefore by (4.18) and (4.19) $a_1 = a_2 = 0$ if $m_{1/2} \neq 0$, i.e. $S = 0$. Thus always $\tilde{\Upsilon}_s = SR^{-1}$. Taking into account that $I(m_{\lambda^\pm_{pk}}) = m_{\lambda^\mp_{pk}}$ and $I(m_{\lambda_j}) = m_{\lambda_j}$, we complete the proof. $\Box$

We can supplement Proposition 4.10 with the following simple statement.

**Corollary 4.10.1.** Suppose that the restricted root system $\Sigma$ of $(g, \mathfrak{a})$ has type $C_r$. Then for arbitrary $a_1, a_2 \in \mathbb{R}$ the mapping $(a_1 + a_2 I) \circ \Upsilon$ satisfies (4.5).

**Proof.** Since $\Upsilon \in \text{EC}(W_T)$, it only remains to prove (4.5) for the mapping $I \circ \Upsilon$. But $I \Upsilon_{sw} = \Upsilon_{sw} I$ because the mapping $\Upsilon$ is $K$-equivariant and $I$ and the automorphism $\exp \frac{\pi}{2} \text{ad} z_0 \in \text{Ad}(K)$ coincide when restricted to $\mathfrak{m}$ (see subsection 4.1). By Proposition 4.10 $I \Upsilon_{sw} = -\Upsilon_{sw} I$. So that $\Upsilon_{sw} I = -\Upsilon_{sw}$. Taking into account already proved identity (4.5) for $\Upsilon$ at $Iw$ and properties (3.1) of $I$, we obtain

$$I \Upsilon_{sw}([w, [\xi, \eta]]) = \Upsilon_{sw} ([Iw, [\xi, \eta]]) = [[Iw, \Upsilon_{sw}(\xi)], \eta] - [Iw, \Upsilon_{sw}(\eta)], \xi$$

$$= [w, -I \Upsilon_{sw}(\xi)], \eta] - [w, -I \Upsilon_{sw}(\eta)], \xi]$$

$$= [w, I \Upsilon_{sw}(\xi)], \eta] - [w, I \Upsilon_{sw}(\eta)], \xi].$$
Lemma 4.11. Let $(J(P), \Omega)$ be a Kähler structure on $D$ such that $IP = \overline{IP}$ on $W$. Then for each Cartan subspace $a = \sum_{j=1}^{r} \mathbb{R} X_j \subset m$, $R_w(a) = a$, where $w \in W \cap a$. Moreover, $R_w(X_j) \in \mathbb{R} X_j$, $j = 1, r$ and $R_w(\mathfrak{M}_\lambda) \subset \mathfrak{M}_\lambda$ for all roots $\lambda \in \Sigma^{++}$.

Proof. For arbitrary mapping $A \in E_{QW}^1$ it follows from (2.6) that $[\text{ad}_\zeta, A_w] = (L_{[\zeta, w]} A)_w$, where $\zeta \in \mathfrak{t}$. Then by (3.8)

$$[\text{ad}_\zeta, (1 + (SR^{-1})_a^2)R_w](\eta) = I[\eta, [w, R^{-1}_w I[\zeta, w]]], \quad \forall \zeta \in \mathfrak{t}, \eta \in m. \quad (4.20)$$

We first shall prove that $R_w(a) \subset \mathfrak{A}$, where $\mathfrak{A} = a \oplus Ia$. To see this, denote by $K^a$ the connected subgroup of $K$ with the Lie algebra $\mathfrak{t}^a$ (the centralizer of the Cartan subspace in $\mathfrak{t}$). All automorphisms $\text{Ad}_k$, $k \in K^a$ leave the space $a \oplus Ia$ pointwise fixed. Then by (2.6) all these automorphisms leave $R_w(a)$ pointwise fixed. Now it follows from Proposition 4.2 that $R_w(a) \subset \mathfrak{A}$ if $\mathfrak{t}^a \neq 0$.

It remains to consider the case when $\mathfrak{t}^a = 0$, i.e. $a \subset m$ is a Cartan subalgebra of $\mathfrak{g}$ and $\Sigma$ is a root system of $(\mathfrak{g}, a)$. Then $t' = t$ and by Corollary 4.7.2 for each root $\alpha \in \Delta^+_+\alpha$ there is a unique root $\lambda^+ \in \Sigma^{++}$ such that $\mathbb{R} X_\alpha \oplus \mathbb{R} Y_\alpha = m_+ \oplus \mathfrak{m}_{\lambda^+}$. Here $\alpha = \frac{i}{t}(\epsilon'_p + \epsilon_k)$ and $\lambda^+ = \frac{1}{t}(\epsilon_p + \epsilon_k)$, where $1 \leq p \leq k \leq r$. If $\lambda^+ \neq i \epsilon_p$ then $I\mathfrak{m}_{\lambda^+} = \mathfrak{m}_{\lambda^+}$, where $\lambda^- = \frac{1}{t}(\epsilon_p - \epsilon_k)$, else $I\mathfrak{m}_{\lambda^+} = \mathbb{R} X_p$. In this one-dimensional subspace $I\mathfrak{m}_{\lambda^+}$ fix a non-zero vector $X^-_\alpha$ assuming that $X^-_\beta, j = 1, r$. Then for $\alpha = \frac{i}{t}(\epsilon'_p + \epsilon_k)$, $T = \sum_{j=1}^{r} t_j T_j \in t'$, $w = \sum_{j=1}^{r} x_j X_j \in a$

$$\text{ad}_T(X^-_\alpha) = \frac{1}{t}(t_p + t_k)I X^-_\alpha,$$

$$\text{ad}_w^2(X^-_\alpha) = -\frac{1}{t}(x_p - x_k)^2 X^-_\alpha,$$

$$\text{ad}_w^2(I X^-_\alpha) = -\frac{1}{t}(x_p + x_k)^2 X^-_\alpha. \quad (4.21)$$

Now consider $(\mathfrak{m}, I, \langle, \rangle)$ as an Hermitian space over $\mathbb{C}$ with fixed orthogonal basis $\{X_j, \beta \in \Delta^+_+\}$. Assume that all base vectors have the same length which is equal to length of (conjugated) vectors $\{X_j\}_{j=1}^r$. Since each operator $R_w : m \to m$ is symmetric and commuting with $I$, the corresponding complex matrix is Hermitian. Let $(R_{\beta j})$ and $(\bar{r}_{\beta j})$, $\beta \in \Delta^+_+$, $j = 1, r$ be two complex matrix-functions corresponding to the operator-functions $R|\mathfrak{t}$ and $R^{-1}|\mathfrak{t}$. Then $\sum_{\beta \in \Delta^+_+} R_{\beta p} \bar{r}_{\beta k} = \delta^p_k$.

Put $\eta = X_n \in a$ and $\zeta = T \in t'$ in (4.20). By Proposition 4.10 $(SR^{-1})_w^2 = (a_1^2 + a_2^2)(Y_n)_{\alpha_0}$ and taking into account relations (4.21) and (4.19), we obtain from (4.20) the following equation

$$\begin{align*}
\left(\frac{1}{t} t_p + \frac{1}{t} t_k - t_n\right) \left(1 + (a_1^2 + a_2^2)x_p^2 x_k^{-2}\right) R_{\alpha n} = & \\
= & \sum_{j=1}^{r} t_j x_j \left(\frac{1}{t} (\delta_p^n - \delta_k^n)(x_p - x_k) \bar{r}_{\alpha j} + \frac{1}{t} (\delta_p^n + \delta_k^n)(x_p + x_k) \bar{r}_{\alpha j}\right)
\end{align*} \quad (4.22)$$

Here $\alpha = \frac{i}{t}(\epsilon'_p + \epsilon_k)$, $\bar{r}_{\alpha j}$ and $\bar{r}_{\alpha j}$ are real and imaginary parts of the complex function $r_{\alpha j}$. This equation then gives that $R_{\alpha n} = r_{\alpha n} = 0$ if $n \neq p$ and $n \neq k$.

Assume now that $p \neq k$. It follows from (4.22) that $R_{\alpha p} = \overline{R_{\alpha k}}$ and

$$\begin{align*}
R_{\alpha p} \left(1 + (a_1^2 + a_2^2)x_p^2 x_k^{-2}\right) = & \frac{1}{t} x_p \left((x_p - x_k) \bar{r}_{\alpha p} + i(x_p + x_k) \bar{r}_{\alpha p}\right) \\
= & \frac{1}{t} x_k \left((x_p - x_k) \bar{r}_{\alpha k} + i(x_p + x_k) \bar{r}_{\alpha k}\right).
\end{align*} \quad (4.23)$$
But

\[ \sum_{\beta \in \Delta^+_m} R_{\beta p} r_{\beta k} = R_{\alpha p} r_{\alpha k} = \delta^p_k = 0, \]

i.e. either \( R_{\alpha p} \) or \( r_{\alpha k} \) equal zero. By (4.23) these two functions on \( W \cap \mathfrak{a} \) equal zero simultaneously. Thus \( R_w(\mathfrak{a}) \subset \mathfrak{a} \).

Turning to the general case, consider again relation (4.20). Using the basis \( \{X_j\}, \quad j = 1, r \) in \( \mathfrak{a} \), we deduce the following equations for the matrix elements \( R_{pn} \) and \( r_{pn} \) of \( R, R^{-1} : \mathfrak{a} \to \mathfrak{a} \).

\[
(t_p - t_n)(1 + (a_1^2 + a_2^2)x_p^{-4}) R_{pn} = \delta^0_p \sum_{j=1}^r t_j x_j x_p (\text{Im} r_{pj}), \quad \forall t_j \in \mathbb{R}, \quad j = 1, r.
\]

Thus \( R_{pn} = 0 \) if \( n \neq p \) and \( r_{pp} \in \mathbb{R} \), i.e. \( R_w(\mathbb{R} x_p) \subset \mathbb{R} x_p \). But \( I[t', \mathfrak{a}] \subset \mathfrak{a} \). Therefore for any \( \zeta \in t' \) the expression on the right in (4.20) vanishes and, consequently, \( (1 + (SR^{-1})^2) R_w(\mathfrak{m}_\Lambda) \subset \mathfrak{m}_\Lambda \). □

We showed above that a structure of the mapping \( P \) depends on a type of the restricted root system \( \Sigma \) of the symmetric space \( G/K \). For each type we define a set \( \mathcal{A}^\Sigma \subset \mathbb{R}^3 \times \{ \pm 1 \} \) by

\[
\mathcal{A}^C = \mathbb{R}^3 \times \{ 1 \} \quad \text{and} \quad \mathcal{A}^{BC} = \mathbb{R}^+ \times \{ 0 \} \times \{ 0 \} \times \{ \pm 1 \} \cup \{ (0, 0, 0, -1) \}. \quad (4.24)
\]

For an element \( a \in \mathcal{A}^C \), let \( a_\uparrow = \frac{1}{2} \left( \sqrt{a_1^2 + 4a_2^2} + 4a_1^2 - a_0 \right) \) if \( a_1^2 + a_2^2 > 0 \) and \( a_\uparrow = -a_0 \) if \( a_1 = a_2 = 0 \). Put \( a_\downarrow = \{ \sum_{j=1}^r x_j \in \mathfrak{a} : x_j^2 > a_\downarrow \} \). This subset of the Cartan subspace \( \mathfrak{a} \) defines a unique Ad(\( K \))-invariant open connected subset \( W_{a_\uparrow}^w \) of \( \mathfrak{m} \) with \( W_{a_\uparrow} \cap \mathfrak{a} = a_\uparrow \). For \( a \in \mathcal{A}^{BC} \), let \( W_{a_\uparrow}^{BC} = \mathfrak{m} \) if \( \varepsilon = 1 \) and \( W_{a_\uparrow}^{BC} = \mathfrak{m} \setminus \{ 0 \} \) if \( \varepsilon = -1 \). Set \( D_{a_\uparrow}^w = \Pi(G \times W_{a_\uparrow}^w) \).

The central result of this paper is the following.

**Theorem 4.12.** Let \( (J(P), \Omega) \) be a G-invariant Kähler structure on the G-invariant domain \( D \subset T(G/K) \), where \( G/K \) is the irreducible Hermitian symmetric space of compact type. Suppose that \( IP = T^\perp I \) on \( W \). Then there exists a unique quadruple \( (a_0, a_1, a_2, \varepsilon) \in \mathcal{A}^\Sigma \) such that \( W \subset W_{a_\uparrow}^w \) and

\[
P_w = (1 + i(a_1 + a_2 I) \Upsilon_w) \cdot (1 + (a_1^2 + a_2^2)(\Upsilon_w)^2)^{-1} \cdot B_w, \quad w \in W \subset \mathfrak{m}, \quad (4.25)
\]

\[
B_w = (I \cdot \text{ad}_{[\hat{\phi}(w), w]} + \varepsilon \sqrt{|a_0|} \cdot \text{Id}) |w|, \quad \hat{\phi}(w) = \phi(- \text{ad}_{w}^2 - \text{ad}_{I_w}^2) w, \quad (4.26)
\]

\[
\phi(t) = \frac{\sqrt{a_0 + t - (a_1^2 + a_2^2)t^{-1}} - \varepsilon \sqrt{|a_0|}}{t}.
\]

For arbitrary \( (a_0, a_1, a_2, \varepsilon) \in \mathcal{A}^\Sigma \) the operator-function \( P \) (4.25) determines a Kähler structure \( (J(P), \Omega) \) on the G-invariant domain \( D_{a_\uparrow}^w \subset T(G/K) \). This structure anticommutes with \( J \).

Moreover, if \( a_2 = 0 \), this Kähler structure \( (J(P), \Omega = d\theta) \) (4.25) admits a potential function \( Q \), i.e. \( \Omega = 2i\partial \partial Q \); if, in addition, \( a_1 = 0 \), then \( \theta = 2 \text{Im} \partial Q \). The function \( (\Pi^*Q)(g, w) = \langle q(- \text{ad}_{w}^2 - \text{ad}_{I_w}^2) w, w \rangle \), where \( q(t) = \frac{1}{2t} \int \frac{dt}{\sqrt{a_0 + t - a_1^2 t^{-1}}} \).

**Proof.** By Proposition 4.10 \( SR^{-1} = (a_1 + a_2 I) \Upsilon_w \) on \( W \). Define the K-equivariant mapping \( B \in \text{Eqv}(W) \) putting \( B_w = (1 + c^2(\Upsilon_w)^2) R_w, \quad c^2 = a_1^2 + a_2^2 \). A change \( R_w \mapsto (1 + c^2(\Upsilon_w)^2)^{-1} B_w \) converts (3.8) into

\[
(L_\xi) w(\eta) = -[[w, IB_w^{-1}(1 + c^2(\Upsilon_w)^2) \xi], I\eta]], \quad w \in W, \quad \xi, \eta \in \mathfrak{m}. \quad (4.27)
\]
By Lemma 4.11 \( B_w(X_j) = b_j(w)X_j \) for all \( w \in W \cap A \). Putting in (4.27) \( \eta = X_j \) and \( \xi = X_k \), we obtain that the function \( b_j \) on \( W \cap A \) depends only on the \( j \)-th coordinate of the vector \( w = \sum_{j=1}^{r} x_j X_j \). Taking into account the action of the restricted Weyl group of \((g,A)\) on \( A \) and \( K \)-equivariance of \( B \), we conclude that all \( b_j \) coincide as functions on some subset of \( \mathbb{R} \) (see the proof of Proposition 4.10 and (2.6)). This unique function will be denoted by \( b \), i.e. \( b_j(w) = b(x_j) \). Solving equation (4.27) for \( \xi = \eta = X_j \), i.e. \( b'(x) = b^{-1}(x)(1 + c^2x^4)x \), we find

\[
b(x) = \sqrt{a_0 + x^2 - (a_1^2 + a_2^2) x^2 - 2}
\]

for some constant \( a_0 \in \mathbb{R} \). Also \( B_w(IX_j) = b_j(w)IX_j \) because \( B_w I = IB_w \).

By Lemma 4.11 \( B_w(\mathfrak{m}_\lambda) = \mathfrak{m}_\lambda \) for each \( \lambda \in \Sigma^{++} \). Since \([a, Ia] \subset \mathfrak{t}'\), from (4.27) and Corollary 4.7.2 it follows that (4.13) is the orthogonal eigenspace splitting for all operators \((L_\xi B)_w\) with \( w \in W \cap A \), \( \xi \in A \), and, consequently, there exists a constant operator \( B_I : \mathfrak{m} \to \mathfrak{m} \) such that \((B_w - B_I)\mathfrak{m}_\lambda = b_\lambda(w) \cdot \text{Id}_{\mathfrak{m}_\lambda} \). We can choose the operator \( B_I \) with trace = 0 when restricted to each \( \mathfrak{m}_\lambda \).

Put \( B^*_w = B_w|_{\mathfrak{m}} \) for \( w \in W \cap A \). Fix an element \( \lambda \in \Sigma^{++} \) and a vector \( \xi_0 \in \mathfrak{m}_\lambda \). There is a vector \( \zeta \in \mathfrak{t}\lambda \) satisfying (4.3). The change \( R_w \mapsto (1 + c^2(\text{Tr}(w))B_w) \) converts (4.20) into

\[
[w, B_w^{-1}(1 + c^2(\text{Tr}(w))^2) I[\zeta, w]] \quad \forall \zeta \in \mathfrak{t}, \eta \in \mathfrak{m}.
\]

We claim that \( B_w(\mathfrak{m}_\lambda) \subset \mathfrak{m}_\lambda \). Indeed, if \( \lambda = \lambda_I \) then \( \mathfrak{m}_\lambda = \mathfrak{m}_\lambda \), i.e. the claim holds. Suppose now that \( \lambda \neq \lambda_I \) and \( \lambda_I \neq 0 \). Then \( \mathfrak{m}_\lambda = \mathfrak{m}_\lambda \oplus \mathfrak{m}_{\lambda_I} \). Put in (4.29) \( \zeta = \zeta_0 \) and \( \eta = w' \in A \). In view of (4.19) \( \text{Tr}(w) = u_\lambda(w)\zeta_0 \). Because of Lemma 4.3 equation (4.29) gives

\[
\lambda(B_w w')\zeta_0 - \lambda(w')B_w(\zeta_0) = -\lambda(w)\lambda_I(w)\lambda_I(w')(1 + c^2u_\lambda^2(w)) \cdot (B_w^{-1}\zeta_0)_{\mathfrak{m}_{\lambda_I}}
\]

\[
- \lambda(w)\lambda_I(w)\lambda_I(w')(1 + c^2u_\lambda^2(w)) \cdot (B_w^{-1}\zeta_0)_{\mathfrak{m}_{\lambda_I}}.
\]

Setting equal in (4.30) the components belonging to the subspace \( \mathfrak{m}_{\lambda_I} \), we obtain that

\[
(B_w\zeta_0)_{\mathfrak{m}_{\lambda_I}} = \lambda^2(w)(1 + c^2u_\lambda^2(w)) \cdot (B_w^{-1}\zeta_0)_{\mathfrak{m}_{\lambda_I}}.
\]

Similarly for the vector \( \xi_{\lambda_I} = I_0 \zeta_0 \in \mathfrak{m}_I \), we have

\[
(B_w\zeta_0)_{\mathfrak{m}_I} = \lambda^2(w)(1 + c^2u_\lambda^2(w)) \cdot (B_w^{-1}\zeta_0)_{\mathfrak{m}_I}.
\]

But \( (B_w I_0 \zeta_0)_{\mathfrak{m}_I} = I(B_w \zeta_0)_{\mathfrak{m}_I} \) and an analogous relation holds for the operator \( B_w^{-1} \). Note also that \( u_\lambda^2 = u_{\lambda_I}^2 \) and \( \lambda^2 \neq \lambda_I^2 \). Therefore \( (B_w \zeta_0)_{\mathfrak{m}_{\lambda_I}} = 0 \). The claim is proved.

Turning to the general case, we obtain for arbitrary \( \lambda \in \Sigma^{++} \) that

\[
\lambda(B_w w')\zeta_0 - \lambda(w')B_w(\zeta_0) = -\lambda(w)\lambda_I(w)\lambda_I(w')(1 + c^2u_\lambda^2(w))B_w^{-1}(\zeta_0).
\]

Moreover, we can replace the vector \( \zeta_0 \) in this equation by the vector \( I_0 \zeta_0 \) because \( B_w I = IB_w \). Thus

\[
\lambda(B_w w') \cdot B_w^\lambda - \lambda(w') \cdot (B_w^\lambda)^2 = -\lambda(w)\lambda_I(w)\lambda_I(w')(1 + c^2u_\lambda^2(w)) \cdot \text{Id}_{\mathfrak{m}_\lambda}.
\]
Since \( B_w w' = \sum_{j=1}^{r} b(x_j) x_j' X_j \), the linear functionals \( \lambda(B_w w') \) and \( \lambda(w') \) on \( \mathfrak{a} \) are linearly independent for any \( w \) in general position. Thus \( B_+ = 0 \) and \( B^\lambda \) is a scalar operator. Then the real function \( b_\lambda(w) \) satisfies the equation

\[
\lambda(B_w w') \cdot b_\lambda(w) - \lambda(w') \cdot b_\lambda^2(w) = -\lambda(w) \lambda_I(w) \lambda_I(w')(1 + c^2 u^2_\lambda(w)).
\]

This equation has the following solutions

\[
b_\lambda = \frac{i}{2} (b_1 + b_k) \text{ if } \lambda = \frac{i}{2} (\epsilon_p + \epsilon_k) \text{ and } b_\lambda = \frac{i}{2} (b_j + \epsilon \sqrt{a_0}) \text{ if } \lambda = \frac{i}{2} \epsilon_j,
\]

where \( \epsilon = \pm 1 \). In other words, with restrictions (4.24) on the parameters \( a_0, a_1, a_2, \epsilon \)

\[
B_w = (I \text{ ad}_{T(w)} + \epsilon \sqrt{|a_0|} \text{ Id})|\mathfrak{m}, \text{ where } T(w) = \sum_{j=1}^{r} -(b_j(w) - \epsilon \sqrt{|a_0|}) T_j. \tag{4.32}
\]

Since \( [IX_j, X_j] = -T_j \) and \( -\text{ad}_{T(w)}^2(X_j) = x_j^2 X_j \), expressions (4.26) and (4.32) for \( B_w, w \in W \cap \mathfrak{a} \) coincide. The equivariance of the mapping \( B : w \mapsto B_w \) (4.26) proves the first assertion of the theorem.

Since \( \Upsilon_{sw} I = -IT_{sw} \), each operator \( 1 + i(a_1 + a_2 I) \Upsilon_{sw} \) is invertible. So by construction the operator-function \( P \) (4.25) determines an almost complex structure on \( D^\Sigma_{\mathfrak{a}} \) anticommuting with \( J^- \). Therefore to prove the second assertion of the theorem it suffices to show that the almost complex structure \( J(P) \) with \( P \) (4.25) is integrable. The following lemma generalizes some assertions of Lemma 6 of [BG1], where the case \( \text{Im} P = 0 \) was considered.

**Lemma 4.13.** Let \( P \in \text{Alm}(W) \), \( RS = SR \) and \( IP = \overline{PI} \). Suppose that \( P \) satisfies conditions (3.8) and (3.9). Then the almost complex structure \( J(P) \) on \( D \) is integrable.

**Proof.** The lemma follows immediately from Lemma 3.6, Proposition 4.10 and its Corollary 4.10.1.

From Proposition 4.10 and the proof above it follows that (4.7) is eigenspace splitting for all operators \( (SR^{-1})_w B_w \), \( w \in W \cap \mathfrak{a} \) and, consequently, for all \( S_w \) and \( R_w \). By equivariance \( R_w, S_w \) is a pair of commuting operators for all \( w \in W \).

Thus to prove integrability of \( J(P) \), we have to verify only condition (4.27). Without loss of generality, we may assume that \( w \in W \cap \mathfrak{a} \). By equivariance of \( B \), this condition is equivalent to (4.27) with \( \xi \in \mathfrak{a} \) and (4.29).

Considering equation (4.27) with \( \eta = \xi_\lambda, \lambda \in \Sigma^{++} \) and the restriction \( b_\lambda|\mathfrak{a} \) as a function of \( x_1, ..., x_r \), we obtain

\[
(\partial b_\lambda/\partial x_j) \cdot \xi_\lambda = -I \text{ ad}_{T_j}(\xi_\lambda) \cdot (1 + c^2 x_j^{-4}) x_j b_j^{-1}, \quad j = 1, r. \tag{4.33}
\]

By Corollary 4.7.2 \( I \text{ ad}_{T_j}(\xi_\lambda) = i \rho_\mathfrak{a}(\lambda)(T_j)\xi_\lambda \), so (4.33) is a linear combination of equations for \( b_j \) with the solutions \( b(x_j) \) (4.28).

Since \( I \) commutes with \( \text{ad}_\mathfrak{c} \), from (4.32) it follows that the left-hand side of (4.29) equals \( I \text{ ad}_{[\xi, T(w)]} \). But the \( \text{ad} \)-representation of \( \mathfrak{k} \) in \( \mathfrak{m} \) is faithful (and irreducible) [GG, (8.5.1)]. Hence (4.29) is equivalent to the equation

\[
[T(w), \zeta] = [w, B_w^{-1}(1 + c^2 (\Upsilon_{sw}^2) I[\zeta, w]), \quad \forall \zeta \in \mathfrak{k}. \tag{4.34}
\]
For $\zeta \in \mathfrak{k}_0 = \mathfrak{k}^a$ the left and right sides of (4.34) vanish, because $w \in \mathfrak{a}$ and $T(w) \in \mathfrak{t} = [\mathfrak{a}, \mathfrak{I}a]$. Similarly we have zeroes for $\zeta \in \mathfrak{k}_\lambda$, $\lambda \in \Sigma$, where $\lambda_l = 0$, because $[\mathfrak{t}^l, \mathfrak{k}_\lambda] = 0$ by Corollary 4.7.3 and $I[\mathfrak{a}, \mathfrak{k}_\lambda] \subset I\mathfrak{m}_\lambda \subset \mathfrak{a}$.

Suppose now that $\zeta = \zeta_\lambda \in \mathfrak{k}_\lambda$ and $\lambda_l \neq 0$. Applying Corollary 4.7.3 again we obtain that

$$i\rho_\ell(\lambda)(T(w)) \cdot \zeta'_\lambda = -\lambda(w)\lambda_l(w)(1 + c^2w^2_\lambda)b^{-1}_\lambda \cdot \zeta'_\lambda, \tag{4.35}$$

Here $\lambda = \frac{1}{2}(\epsilon_p \pm \epsilon_k), p \neq k$ or $\lambda = \frac{1}{2}\epsilon_j$. It is easy to verify this algebraic identity using (4.19), (4.28), (4.31) and the expressions for $\rho_\ell(\lambda)$ (4.14). Thus the almost complex structure $J(P)$ is integrable.

To prove the last assertion of the theorem for the complex structure $(J(P), \Omega)$ consider its subbundle $F(P)$ of $(0, 1)$-vectors. By definition $d\Phi|F(P) = d\Phi|F(P) = 0$. Denote by $\Phi$ the one-form $\Pi^*(\Phi)$ on $G \times W$. Then for any $\xi \in \mathfrak{m}, \zeta \in \mathfrak{k}, w \in \mathfrak{m}$:

$$\Phi_{(g,w)}(\xi^l(g), -iF_w(\xi)) = 0, \quad \Phi_{(g,w)}(\xi^l(g), [w, \zeta]) = 0$$

and

$$\Phi_{(g,w)}(\xi^l(g), iP_w(\xi)) = i\langle \hat{q}(w), P_w(\xi) \rangle + i\langle \hat{q}_{sw}(P_w(\xi), w).$$

Fix $w \in W \cap \mathfrak{a}$. Then $\hat{q}(w) \in \mathfrak{a}, \hat{q}_{sw}(\mathfrak{a}) \subset \mathfrak{a}$ and $P_w(\mathfrak{a}) \subset \mathfrak{a}^C$ because $a_2 = 0$. Now using the invariance of the space $\mathfrak{a}^\perp \subset \mathfrak{m}$ with respect to $P_w$ and $\hat{q}_{sw}$, we obtain that $\Phi_{(e,w)}(\xi \in \mathfrak{a}^\perp, \mathfrak{a}^\perp) = 0 ([w, \xi] \in \mathfrak{a})$. Since the endomorphisms $P_w$ and $\hat{q}_{sw}$ commute and are symmetric, we have for $\xi_0 \in \mathfrak{a}$

$$\Phi_{(e,w)}(\xi_0, -i(R - iS)w\xi_0) = 0$$

and

$$\Phi_{(e,w)}(\xi_0, i(R + iS)w\xi_0) = i\langle (R + iS)w(\hat{q}(w) + \hat{q}_{sw}(w)), \xi_0 \rangle.$$

Thus for any $\eta \in \mathfrak{g}, u \in \mathfrak{m}$ we have

$$\Phi_{(e,w)}(\eta, u) = \frac{1}{2}\langle (1 + (1 + iSR^{-1})wR_w(w'), \eta \rangle + \frac{1}{2}\langle (1 + iSR^{-1})w(w'), u \rangle. \tag{4.35}$$

Here $w'$ is the vector $\hat{q}(w) + \hat{q}_{sw}(w) \in \mathfrak{a}$. It is clear that $d\Phi = \frac{1}{2}id\tilde{\theta}$ if the first term in the right-hand side of (4.35) equals $\frac{1}{2}\tilde{\theta}$ and the second one determines a closed 1-form $\sigma$ on $W^\Sigma_\mathfrak{a}$. In other words, if $(1 + iSR^{-1})wR_w(w') = w$, i.e.

$$b(x)(2xq(x^2) + 2x^3q'(x^2)) = b(x)(x^2q(x^2))' = x$$

because of relation (2.4) and (4.9). Next, the form

$$\sigma_w(u) = 2^{-1}\langle (1 + iSR^{-1})(\hat{q}(w) + \hat{q}_{sw}(w)), u \rangle$$

is closed. Indeed, $SR_{w}^{-1} = a_1\Upsilon_{sw}$. The mapping $w \mapsto (1 + i\mathfrak{a}_1\Upsilon_{sw})(\hat{q}(w) + \hat{q}_{sw}(w))$ is $K$-equivariant because so are the mappings $\hat{q}, w \mapsto \hat{q}_{sw}$ and $w \mapsto \Upsilon_{sw}$. Now by Lemma 4.5, $d\sigma = 0$ because a restriction $\sigma$ to $W \cap \mathfrak{a}$ is a linear combination of the closed forms $(1 - i/x^2)\partial(x^2q(x^2))$. \qed
Example 4.14. Hyperkähler structures on homogeneous domains in $T(SU(2)/U(1))$. Let $G/K = SU(2)/U(1)$. The Lie algebra $g = su(2)$ has the basis $X_1, Y_1, T_1$ \((4.8)\).

For each complex number $z = x + iy$ put $zX_1 \overset{\text{def}}{=} xX_1 + yIX_1 = xX_1 + yY_1 \in m$. Let $(J(R + iS), \Omega)$ be a Kähler structure anticommuting with $J^-$. Then for each $w = zX_1$, $|z|^2 > a_1$

$$R_w = \psi(|z|) \cdot \text{Id}_m, \quad \psi(x) = \frac{\sqrt{x^6 \cdot (x^4 + a_0 x^2 - (a_1^2 + a_2^2))}}{x^4 + (a_1^2 + a_2^2)},$$

and

$$S_w(vX_1) = \psi(|z|)(a_1 + ia_2)z^{-2}v \cdot X_1 \quad \text{for all} \quad v \in \mathbb{C}.$$

Remark 4.15. The global hyper-Kählerian structure $(g, J^-, J(P))$, where $J(P)$ is defined by \((4.25)\) with $a_0 > 0, \varepsilon = 1$ and $a_1 = a_2 = 0$, coincides with the structure constructed in [BG1, Theorem 1]. Here $g$ is the corresponding to the pair $(J(P), \Omega)$ (hyper)Kähler metric. This is a unique metric for which its restriction to $G/K$ coincides with the metric $\sqrt{a_0} \cdot g_M$ on $M = G/K$ (because $P_0 = \sqrt{a_0} \cdot \text{Id}_m$). Our formula for a potential generalizes such a formula in [BG2].

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