BOAS-TYPE FORMULAS IN BANACH SPACES WITH APPLICATIONS TO ANALYSIS ON MANIFOLDS

Dedicated to 85th Birthday of my teacher Paul Butzer
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Abstract. The paper contains Boas-type formulas for trajectories of one-parameter groups of operators in Banach spaces. The results are illustrated using one-parameter groups of operators which appear in representations of Lie groups.

1. Preface

My teachers were Vladimir Abramovich Rokhlin (my Master Thesis Advisor) and Selim Grigorievich Krein (my PhD Thesis Advisor). I first met Paul Butzer when I was about 50 years old but I also consider him as my teacher since his work had an enormous influence on my career and ultimately on my life.

After I graduated from university it was almost impossible for me to go straight to graduate school because of the Soviet discrimination towards Jews. However, I was trying to do some mathematics on my own. One day I came across a reference to the book by P. Butzer and H. Berens "Semi-Groups of operators and approximation", Springer, 1967. Since I had some background in Lie groups and Lie semi-groups and knew nothing about approximation theory the title sounded very intriguing to me. Unfortunately this excellent book had not been translated into Russian. Nevertheless I was lucky to get a microfilm of the book. Every day during a few months I was visiting a local library which had a special device to read microfilms.

By the time I finished reading the book I already knew what I was going to do: I decided to develop a similar "constructive theory of Interpolation Spaces" through replacing a single one-parameter semi-group of operators by a representation of a general Lie group in a Banach space.

I have to say that the book by P. Butzer and H. Berens is an excellent introduction to a number of topics in classical harmonic analysis. In particular it contains the first systematic treatment of the theory of Intermediate Spaces and a very detailed and application oriented treatment of the theory of Semi-Groups of Operators. Both these subjects were considered as "hot" topics at the end of 60s (see for example [13], [14]). In many ways this book is still up to date and I always recommend it to my younger colleagues.
Some time later I was influenced by the classical work of Paul with K. Scherer [5], [6] in which they greatly clarified the relationships between the Interpolation and Approximation spaces and by Paul’s pioneering paper with H. Berens and S. Pawelke [1] about approximation on spheres.

Mathematics I learned from Paul Butzer helped me to become a graduate student of Selim Krein another world level expert in Interpolation spaces and applications of semigroups to differential equations [13]-[15].

Many years later when I came to US and became interested in sampling theory I found out that Paul had already been working in this field for a number of years and I learned a lot from insightful and stimulating work written by P. Butzer, W. Splettstöser and R. Stens [7].

My interactions with Paul Butzer’s work shaped my entire mathematical life and the list of some of my papers [18]-[31] is the best evidence of it.

This is what I mean when I say that Paul Butzer is my teacher.

In conclusion I would like to mention that it was my discussions with Paul Butzer and Gerhard Schmeisser of their beautiful work with Rudolf Stens [9], that stimulated my interest in the topic of the present paper. I am very grateful to them for this.

2. Introduction

Consider a trigonometric polynomial $P(t)$ of one variable $t$ of order $n$ as a function on a unit circle $T$. For its derivative $P'(t)$ the so-called Bernstein inequality holds true

$$\|P'\|_{L_p(T)} \leq n\|P\|_{L_p(T)}, \quad 1 \leq p \leq \infty. \tag{2.1}$$

M. Riesz [33], [34] states that the Bernstein inequality is equivalent to what is known today as the Riesz interpolation formula

$$P'(t) = \frac{1}{4\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{\sin^2 \frac{\pi k}{2n}} P(t + t_k), \quad t \in S, \quad t_k = \frac{2k-1}{2n} \pi. \tag{2.2}$$

The next formula holds true for functions in the Bernstein space $B^\sigma_p$, $1 \leq p \leq \infty$ which is comprised of all entire functions of exponential type $\sigma$ which belong to $L_p(\mathbb{R})$ on the real line.

$$f'(t) = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} f\left(t + \frac{\pi}{\sigma}(k - 1/2)\right), \quad t \in \mathbb{R}. \tag{2.3}$$

This formula was obtained by R.P. Boas [2], [3] and is known as Boas or generalized Riesz formula. Again, like in periodic case this formula is equivalent to the Bernstein inequality in $L_p(\mathbb{R})$

$$\|f'\|_{L_p(\mathbb{R})} \leq \sigma \|f\|_{L_p(\mathbb{R})}. \tag{2.3}$$

Recently, in the interesting papers [35] and [9] among other important results the Boas-type formula (2.3) was generalized to higher order.
In particular it was shown that for \( f \in B_\sigma^\infty, \ \sigma > 0 \), the following formulas hold
\[
f^{(2m-1)}(t) = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} f \left( t + \frac{\pi}{\sigma} (k - \frac{1}{2}) \right), \ m \in \mathbb{N},
\]
\[
f^{(2m)}(t) = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} f \left( t + \frac{\pi}{\sigma} k \right), \ m \in \mathbb{N},
\]
where
\[
A_{m,k} = (-1)^{k+1} \text{sinc}^{(2m-1)} \left( \frac{1}{2} - k \right) =
\]
\[
\frac{(2m-1)!}{\pi(k - \frac{1}{2})^{2m}} \sum_{j=0}^{m-1} \frac{(-1)^j}{(2j)!} \left( \frac{\pi}{2} \right)^{2j}, \ m \in \mathbb{N},
\]
for \( k \in \mathbb{Z} \) and
\[
B_{m,k} = (-1)^{k+1} \text{sinc}^{(2m)}(-k) = \frac{(2m)!}{\pi k^{2m+1}} \sum_{j=0}^{m-1} \frac{(-1)^j(\pi k)^{2j+1}}{(2j+1)!}, \ m \in \mathbb{N}, \ k \in \mathbb{Z} \setminus 0,
\]
and
\[
B_{m,0} = (-1)^{m+1} \frac{\pi^{2m}}{2m+1}, \ m \in \mathbb{N}.
\]

Let us remind that \( \text{sinc}(t) \) is defined as \( \frac{\sin \pi t}{\pi t} \), if \( t \neq 0 \), and 1, if \( t = 0 \).

To illustrate our results let us assume that we are given an operator \( D \) that generates a strongly continuous group of isometries \( e^{tf} \) in a Banach space \( E \).

**Definition 2.1.** The subspace of exponential vectors \( E_\sigma(D) \), \( \sigma \geq 0 \), is defined as a set of all vectors \( f \in E \) which belong to \( D^\infty = \bigcap_{k \in \mathbb{N}} D^k \), where \( D^k \) is the domain of \( D^k \), and for which there exists a constant \( C(f) > 0 \) such that
\[
\|D^k f\| \leq C(f) \sigma^k, \ k \in \mathbb{N}.
\]

Note, that every \( E_\sigma(D) \) is clearly a linear subspace of \( E \). What is really important is the fact that union of all \( E_\sigma(D) \) is dense in \( E \) (Theorem 3.7).

**Remark 2.2.** It is worth to stress that if \( D \) generates a strongly continuous bounded semigroup then the set \( \bigcup_{\sigma \geq 0} E_\sigma(D) \) may not be dense in \( E \).

Indeed, (see [17]) consider a strongly continuous bounded semigroup \( T(t) \) in \( L_2(0, \infty) \) defined for every \( f \in L_2(0, \infty) \) as \( T(t)f(x) = f(x-t) \), if \( x \geq t \) and \( T(t)f(x) = 0 \), if \( 0 \leq x < t \). If \( f \in E_\sigma(D) \) then for any \( g \in L_2(0, \infty) \) the function \( \langle T(t)f, g \rangle \) is analytic in \( t \) (see below section 3). Thus if \( g \) has compact support then \( \langle T(t)f, g \rangle \) is zero for all \( t \) which implies that \( f \) is zero. In other words in this case every space \( E_\sigma(D) \) is trivial.

One of our results is that a vector \( f \) belongs to \( E_\sigma(D) \) if and only if the following sampling-type formulas hold
\[
e^{tf}D^{2m-1}f = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{(t+\frac{\pi}{2}k(1/2))D} f, \ m \in \mathbb{N},
\]
\[
e^{tf}D^{2m}f = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{(t+\frac{\pi}{2}k)} D f, \ m \in \mathbb{N},
\]
Which are equivalent to the following Boas-type formulas

\[(2.10)\quad D^{2m-1}f = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{\left(\frac{\sigma}{\pi}(k-1/2)^2\right)} D^k f, \quad m \in \mathbb{N}, \quad f \in E_\sigma(D),\]

and

\[(2.11)\quad D^{2m}f = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{\pi k \frac{\sigma}{\pi} D^k f}, \quad m \in \mathbb{N} \cup 0, \quad f \in E_\sigma(D).\]

The formulas (2.8) and (2.9) are a sampling-type formulas in the sense that they provide explicit expressions for a trajectory \(e^{itD}D^k f\) with \(f \in E_\sigma(D)\) in terms of a countable number of equally spaced samples of trajectory of \(f\).

Note that since \(e^{itD}, \quad t \in \mathbb{R}\), is a group of operators any trajectory \(e^{itD}f, \quad f \in E\), is completely determined by any (single) sample \(e^{it_0D}f\), because for any \(t \in \mathbb{R}\)

\[e^{itD}f = e^{(t-t_0)D} (e^{it_0D}f).\]

The formulas (2.8) and (2.9) have, however, a different nature: they represent a trajectory as a "linear combination" of a countable number of samples.

It seems to be very interesting that an operator and the group can be rather sophisticated (think, for example, about a Schrödinger operator \(D = -\Delta + V(x)\) and the corresponding group \(e^{itD}\) in \(L_2(\mathbb{R}^d)\)). However, formulas (2.8)–(2.11) are universal in the sense that they contain the same coefficients and the same sets of sampling points.

We are making a list of some important properties of the Boas-type interpolation formulas (compare to [9]):

1. The formulas hold for vectors \(f\) in the set \(\bigcup_{\sigma \geq 0} E_\sigma(D)\) which is dense in \(E\) (see Theorem 3.7).
2. The sample points \(\frac{\sigma}{\pi}(k-1/2)\) are uniformly spaced according to the Nyquist rate and are independent on \(f\).
3. The coefficients do not depend on \(f\).
4. The coefficients decay like \(O(k^{-2})\) as \(k\) goes to infinity.
5. In formulas (2.10) and (2.11) one has unbounded operators (in general case) on the left-hand side and bounded operators on the right-hand side.
6. There is a number of interesting relations between Boas-type formulas, see below (3.9)–(3.10).

Our main objective is to obtain a set of new formulas for one-parameter groups which appear when one considers representations of Lie groups (see section 5). Note, that generalizations of (2.8) with applications to compact homogeneous manifolds were initiated in [22].

In our applications we deal with a set of non-commuting generators \(D_1, ..., D_d\). In subsection 5.1 these operators come from a representation of a compact Lie group and we are able to show that \(\bigcup_{\sigma \geq 0} \bigcup_{1 \leq j \leq d} E_\sigma(D_j)\) is dense in all appropriate Lebesgue spaces. We cannot prove a similar fact in the next subsection 5.2 in which we consider a non-compact Heisenberg group. Moreover, in subsection 5.3 in which the Schrödinger representation is discussed we note that this property does not hold in general.
3. Boas-type formulas for exponential vectors

We assume that $D$ is a generator of one-parameter group of isometries $e^{tD}$ in a Banach space $E$ with the norm $\| \cdot \|$.

**Definition 3.1.** The Bernstein subspace $B_\sigma(D)$, $\sigma \geq 0$, is defined as a set of all vectors $f$ in $E$ which belong to $D^\infty = \bigcap_{k \in \mathbb{N}} D^k$, where $D^k$ is the domain of $D^k$ and for which

$$\|D^k f\| \leq \sigma^k \|f\|, \quad k \in \mathbb{N}. \quad (3.1)$$

It is obvious that $B_\sigma(D) \subset E_\sigma(D)$, $\sigma \geq 0$. However, it is not even clear that $B_\sigma(D), \sigma \geq 0$, is a linear subspace. It follows from the following interesting fact.

**Theorem 3.2.** Let $D$ be a generator of an one parameter group of operators $e^{tD}$ in a Banach space $E$ and $\|e^{tD} f\| = \|f\|$. Then for every $\sigma \geq 0$

$$B_\sigma(D) = E_\sigma(D), \quad \sigma \geq 0,$$

*Proof.* If $f \in E_\sigma(D)$ then for any complex number $z$ we have

$$\|e^{zd} f\| = \left\| \sum_{r=0}^{\infty} (z^r D^r f) / r! \right\| \leq C(f) \sum_{r=0}^{\infty} |z|^r \sigma^r / r! = C(f) e^{z \sigma}.$$

It implies that for any functional $\psi^* \in E^*$ the scalar function $\langle e^{zd} f, \psi^* \rangle$ is an entire function of exponential type $\sigma$ which is bounded on the real axis by the constant $\|\psi^*\| \|f\|$. An application of the classical Bernstein inequality gives

$$\|\langle e^{zd} D^k f, \psi^* \rangle\|_{C(R)} = \left\| \left( \frac{d}{dt} \right)^k \langle e^{zd} f, \psi^* \rangle \right\|_{C(R)} \leq \sigma^k \|\psi^*\| \|f\|.$$

From here for $t = 0$ we obtain

$$\| \langle D^k f, \psi^* \rangle \| \leq \sigma^k \|\psi^*\| \|f\|.$$

Choice of $\psi^* \in E^*$ such that $\|\psi^*\| = 1$ and $\langle D^k f, \psi^* \rangle = \|D^k f\|$ gives the inequality $\|D^k f\| \leq \sigma^k \|f\|, \quad k \in \mathbb{N}$, which implies Theorem. \hfill $\Box$

**Remark 3.3.** We just mention that in the important case of a self-adjoint operator $D$ in a Hilbert space $E$ there is a way to describe Bernstein vectors in terms of a spectral Fourier transform or in terms of the spectral measure associated with $D$ (see [10], [22]-[31] for more details).

Let’s introduce bounded operators

(3.2) \[ B_D^{(2m-1)}(\sigma) f = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{\frac{\sigma}{\pi} (k-1/2)D} f, \quad f \in E, \sigma > 0, \quad m \in \mathbb{N}, \]

(3.3) \[ B_D^{(2m)}(\sigma) f = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{\frac{\sigma}{\pi} D} f, \quad f \in E, \sigma > 0, \quad m \in \mathbb{N}, \]

where $A_{m,k}$ and $B_{m,k}$ are defined in [2]-[2.0]. Both series converge in $E$ due to the following formulas (see [9])
The proof of Theorem 3.2 shows that

\[(3.6) \quad \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} |A_{m,k}| = \sigma^{2m-1}, \quad \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} |B_{m,k}| = \sigma^{2m}.\]

Since \(\|e^{tD}f\| = \|f\|\) it implies that

\[(3.5) \quad \|B_D^{(2m-1)}(\sigma)f\| \leq \sigma^{2m-1}\|f\|, \quad \|B_D^{(2m)}(\sigma)f\| \leq \sigma^{2m}\|f\|, \quad f \in E.\]

**Theorem 3.4.** If \(D\) generates a one-parameter strongly continuous bounded group of operators \(e^{tD}\) in a Banach space \(E\) then the following conditions are equivalent:

1. \(f\) belongs to \(B_{\sigma}(D)\).
2. The abstract-valued function \(e^{tD}f\) is an entire function of exponential type \(\sigma\) which is bounded on the real line.
3. The following Boas-type interpolation formulas hold true for \(r \in \mathbb{N}\)

\[(3.6) \quad D^r f = B_D^{(r)}(\sigma)f, \quad f \in B_{\sigma}(D).\]

**Proof:** The proof of Theorem 3.2 shows that 1) \(\rightarrow\) 2). Then obviously for any \(\psi^* \in E^*\) the function \(F(t) = \langle e^{tD}f, \psi^* \rangle\) is of exponential type \(\sigma\) and bounded on \(\mathbb{R}\). Thus by [9] we have

\[
F^{(2m-1)}(t) = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} F \left(t + \frac{\pi}{\sigma}(k-1/2)\right), \quad m \in \mathbb{N},
\]

\[
F^{(2m)}(t) = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} F \left(t + \frac{\pi k}{\sigma}\right), \quad m \in \mathbb{N}.
\]

Together with

\[
\left(\frac{d}{dt}\right)^k F(t) = \langle D^k e^{tD}f, \psi^* \rangle,
\]

it shows

\[
\langle e^{tD} D^{2m-1} f, \psi^* \rangle = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} \langle e^{(t+\frac{\pi}{2}(k-1/2))D} f, \psi^* \rangle, \quad m \in \mathbb{N},
\]

and also

\[
\langle e^{tD} D^{2m} f, \psi^* \rangle = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} \langle e^{(t+\frac{\pi}{2})D} f, \psi^* \rangle, \quad m \in \mathbb{N}.
\]

Since both series (3.2) and (3.3) converge in \(E\) and the last two equalities hold for any \(\psi^* \in E\) we obtain the next two formulas

\[(3.7) \quad e^{tD} D^{2m-1} f = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{(t+\frac{\pi}{2}(k-1/2))D} f, \quad m \in \mathbb{N},
\]

\[(3.8) \quad e^{tD} D^{2m} f = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{(t+\frac{\pi}{2})D} f, \quad m \in \mathbb{N}.
\]

In turn when \(t = 0\) these formulas become formulas (3.6).

The fact that 3) \(\rightarrow\) 1) easily follows from the formulas (3.6) and (3.5). Theorem is proved.
Corollary 3.1. Every $B_\sigma(D)$ is a closed linear subspace of $E$.

Corollary 3.2. If $f$ belongs to $B_\sigma(D)$ then for any $\sigma_1 \geq \sigma$, $\sigma_2 \geq \sigma$ one has
\begin{equation}
B_D^{(r)}(\sigma_1)f = B_D^{(r)}(\sigma_2)f, \quad r \in \mathbb{N}.
\end{equation}

Let us introduce the notation
\[ B_D(\sigma) = B_1^{(1)}(\sigma). \]

One has the following "power" formula which easily follows from the fact that operators $B_D(\sigma)$ and $D$ commute on any $B_\sigma(D)$.

Corollary 3.3. For any $r \in \mathbb{N}$ and any $f \in B_\sigma(D)$
\begin{equation}
D^r f = B_D^{(r)}(\sigma)f = B_D^r(\sigma)f,
\end{equation}
where $B_D^r(\sigma)f = B_D(\sigma)\ldots B_D(\sigma)f$.

Let us introduce the following notations
\[ B_D^{(2m-1)}(\sigma, N) = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{|k| \leq N} (-1)^{k+1} A_{m,k} e^{\frac{\pi}{\sigma}(k-1/2)D} f, \]
\[ B_D^{(2m)}(\sigma, N) = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{|k| \leq N} (-1)^{k+1} B_{m,k} e^{\frac{k}{\pi}D} f. \]

One obviously has the following set of approximate Boas-type formulas.

Theorem 3.5. If $f \in B_\sigma(D)$ and $r \in \mathbb{N}$ then
\begin{equation}
D^{(r)} f = B_D^{(r)}(\sigma, N)f + O(N^{-2}).
\end{equation}

The next Theorem contains another Boas-type formula.

Theorem 3.6. If $f \in B_\sigma(D)$ then the following sampling formula holds for $t \in \mathbb{R}$ and $n \in \mathbb{N}$
\begin{equation}
e^{tD} f = Q_D^n(\sigma)f + O(N^{-2}).
\end{equation}

In particular, for $n \in \mathbb{N}$ one has
\begin{equation}
D^n f = Q_D^n(\sigma)f,
\end{equation}
where the bounded operator $Q_D^n(\sigma)$ is given by the formula
\begin{equation}
Q_D^n(\sigma)f = n \sum_k e^{\frac{k}{\sigma}Df - f} \left[ \sin^{(n-1)} \left( \frac{\sigma t}{\pi} - k \right) + \frac{\sigma t}{\pi} \sin^{(n)} \left( \frac{\sigma t}{\pi} - k \right) \right].
\end{equation}

Proof. If $f \in B_\sigma(D)$ then for any $g^* \in E^*$ the function $F(t) = \langle e^{tD} f, g^* \rangle$ belongs to $B_\sigma^{\infty}(\mathbb{R})$.

We consider $F_1 \in B_\sigma^2(\mathbb{R})$, which is defined as follows. If $t \neq 0$ then
\begin{equation}
F_1(t) = \frac{F(t) - F(0)}{t} = \frac{e^{tD} f - f}{t} + O(t^{-2}).
\end{equation}
and if \( t = 0 \) then \( F_1(t) = \frac{d}{dt} F(t)|_{t=0} = \langle Df, g^* \rangle \). We have

\[
F_1(t) = \sum_k F_1 \left( \frac{k \pi}{\sigma} \right) \text{sinc} \left( \frac{\sigma t}{\pi} - k \right).
\]

From here we obtain the next formula

\[
\left( \frac{d}{dt} \right)^n F_1(t) = \sum_k F_1 \left( \frac{k \pi}{\sigma} \right) \text{sinc}^{(n)} \left( \frac{\sigma t}{\pi} - k \right)
\]

and since

\[
\left( \frac{d}{dt} \right)^n F(t) = n \left( \frac{d}{dt} \right)^{n-1} F_1(t) + t \left( \frac{d}{dt} \right)^n F_1(t)
\]

we obtain

\[
\left( \frac{d}{dt} \right)^n F(t) = n \sum_k F_1 \left( \frac{k \pi}{\sigma} \right) \text{sinc}^{(n-1)} \left( \frac{\sigma t}{\pi} - k \right) + \frac{\sigma t}{\pi} \sum_k F_1 \left( \frac{k \pi}{\sigma} \right) \text{sinc}^{(n)} \left( \frac{\sigma t}{\pi} - k \right)
\]

Since \( \left( \frac{d}{dt} \right)^n F(t) = \langle D^n e^{tD} f, g^* \rangle \), and

\[
F_1 \left( \frac{k \pi}{\sigma} \right) = \left( \frac{e^{k \pi D} f - f}{k \pi} , g^* \right)
\]

we obtain that for \( t \in \mathbb{R}, \ n \in \mathbb{N}, \)

\[
D^n e^{tD} f = \sum_k \frac{e^{k \pi D} f - f}{k \pi} \left[ n \text{sinc}^{(n-1)} \left( \frac{\sigma t}{\pi} - k \right) + \frac{\sigma t}{\pi} \text{sinc}^{(n)} \left( \frac{\sigma t}{\pi} - k \right) \right].
\]

Theorem is proved. \( \square \)

The next Theorem shows that Boas-type formulas make sense for a dense set of vectors.

**Theorem 3.7.** The set \( \bigcup_{\sigma \geq 0} B_\sigma(D) \) is dense in \( E \).

**Proof.** Note that if \( \phi \in L_1(\mathbb{R}), \ ||\phi||_1 = 1 \), is an entire function of exponential type \( \sigma \) then for any \( f \in E \) the vector

\[ g = \int_{-\infty}^{\infty} \phi(t) e^{tD} f dt \]

belongs to \( B_\sigma(D) \). Indeed, for every real \( \tau \) we have

\[ e^{\tau D} g = \int_{-\infty}^{\infty} \phi(t) e^{(t+\tau)D} f dt = \int_{-\infty}^{\infty} \phi(t-\tau) e^{tD} f dt. \]

Using this formula we can extend the abstract function \( e^{\tau D} g \) to the complex plane as

\[ e^{z D} g = \int_{-\infty}^{\infty} \phi(t-z) e^{tD} f dt. \]

Since by assumption \( h \) is an entire function of exponential type \( \sigma \) and \( ||\phi||_{L_1(\mathbb{R})} = 1 \) we have

\[ ||e^{z D} g|| \leq ||f|| \int_{-\infty}^{\infty} |\phi(t-z)| dt \leq ||f|| e^{\sigma |z|}. \]

This inequality implies that \( g \) belongs to \( B_\sigma(D) \).
Let
\[ h(t) = a \left( \frac{\sin(t/4)}{t} \right)^4 \]
and
\[ a = \left( \int_{-\infty}^{\infty} \left( \frac{\sin(t/4)}{t} \right)^4 dt \right)^{-1}. \]
Function \( h \) will have the following properties:

1. \( h \) is an even nonnegative entire function of exponential type one;
2. \( h \) belongs to \( L_1(\mathbb{R}) \) and its \( L_1(\mathbb{R}) \)-norm is 1;
3. the integral
\[ \int_{-\infty}^{\infty} h(t)|t|dt \]
is finite.
Consider the following vector
\[ R_{h}^\sigma(f) = \int_{-\infty}^{\infty} h(t)e^{\xi D}f dt = \int_{-\infty}^{\infty} h(t\sigma)e^{t\sigma D}f dt, \]
Since the function \( h(t) \) has exponential type one the function \( h(t\sigma) \) has the type \( \sigma \).
It implies (by the previous) that \( R_{h}^\sigma(f) \) belongs to \( B_\sigma(D) \).

4. Analysis on compact homogeneous manifolds
Let \( M, \dim M = m \), be a compact connected \( C^\infty \)-manifold. One says that a compact Lie group \( G \) effectively acts on \( M \) as a group of diffeomorphisms if the following holds true:

1. Every element \( g \in G \) can be identified with a diffeomorphism \( g : M \to M \) of \( M \) onto itself and \( g_1g_2 \cdot x = g_1 \cdot (g_2 \cdot x) \), \( g_1, g_2 \in G, x \in M \), where \( g_1g_2 \) is the product in \( G \) and \( g \cdot x \) is the image of \( x \) under \( g \).
2. The identity \( e \in G \) corresponds to the trivial diffeomorphism \( e \cdot x = x \).
3. For every \( g \in G, g \neq e \), there exists a point \( x \in M \) such that \( g \cdot x \neq x \).
A group $G$ acts on $M$ transitively if in addition to 1)- 3) the following property holds: 4) for any two points $x, y \in M$ there exists a diffeomorphism $g \in G$ such that $g \cdot x = y$.

A homogeneous compact manifold $M$ is an $C^\infty$-compact manifold on which transitively acts a compact Lie group $G$. In this case $M$ is necessary of the form $G/K$, where $K$ is a closed subgroup of $G$. The notation $L_p(M)$, $1 \leq p \leq \infty$, is used for the usual Banach spaces $L_p(M, dx)$, $1 \leq p \leq \infty$, where $dx$ is the normalized invariant measure.

Every element $X$ of the Lie algebra of $G$ generates a vector field on $M$ which we will denote by the same letter $X$. Namely, for a smooth function $f$ on $M$ one has

$$Xf(x) = \lim_{\tau \to 0} \frac{f(\exp \tau X \cdot x) - f(x)}{\tau}$$

for every $x \in M$. In the future we will consider on $M$ only such vector fields. Translations along integral curves of such vector field $X$ on $M$ can be identified with a one-parameter group of diffeomorphisms of $M$ which is usually denoted as $\exp \tau X$, $-\infty < \tau < \infty$. At the same time the one-parameter group $\exp \tau X$, $-\infty < \tau < \infty$, can be treated as a strongly continuous one-parameter group of operators in a space $L_p(M)$, $1 \leq p \leq \infty$ which acts on functions according to the formula $f \to f(\exp \tau X \cdot x)$, $\tau \in \mathbb{R}$, $f \in L_p(M)$, $x \in M$. The generator of this one-parameter group will be denoted as $D_{X,p}$ and the group itself will be denoted as $e^{\tau D_{X,p}} f(x) = f(\exp \tau X \cdot x)$, $\tau \in \mathbb{R}$, $f \in L_p(M)$, $x \in M$.

According to the general theory of one-parameter groups in Banach spaces [4], Ch. 1, the operator $D_{X,p}$ is a closed operator in every $L_p(M)$, $1 \leq p \leq \infty$. In order to simplify notations we will often use notation $D_X$ instead of $D_{X,p}$.

It is known [12], Ch. V, that on every compact homogeneous manifold $M = G/K$ there exist vector fields $X_1, X_2, ..., X_d$, $d = \dim G$, such that the second order differential operator

$$X_1^2 + X_2^2 + ... + X_d^2, \quad d = \dim G,$$

commutes with all vector fields $X_1, ..., X_d$ on $M$. The corresponding operator in $L_p(M)$, $1 \leq p \leq \infty$,

$$(4.1) \quad -\mathcal{L} = D_1^2 + D_2^2 + ... + D_d^2, \quad D_j = D_{X_j}, \quad d = \dim G,$$

commutes with all operators $D_j = D_{X_j}$. This operator $\mathcal{L}$ which is usually called the Laplace operator is involved in most of constructions and results of our paper.

The operator $\mathcal{L}$ is an elliptic differential operator which is defined on $C^\infty(M)$ and we will use the same notation $\mathcal{L}$ for its closure from $C^\infty(M)$ in $L_p(M)$, $1 \leq p \leq \infty$. In the case $p = 2$ this closure is a self-adjoint positive definite operator in the space $L_2(M)$. The spectrum of this operator is discrete and goes to infinity $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$, where we count each eigenvalue with its multiplicity. For eigenvectors corresponding to eigenvalue $\lambda_j$ we will use notation $\varphi_j$, i. e.

$$\mathcal{L}\varphi_j = \lambda_j \varphi_j.$$

The spectrum and the set of eigenfunctions of $\mathcal{L}$ are the same in all spaces $L_p(\mathbb{R}^d)$.

Let $\varphi_0, \varphi_1, \varphi_2, ...$ be a corresponding complete system of orthonormal eigenfunctions and $E_\sigma(\mathcal{L}), \sigma > 0$, be a span of all eigenfunctions of $\mathcal{L}$ whose corresponding eigenvalues are not greater $\sigma$. 
In the rest of the paper the notations $\mathbb{D} = \{D_1, ..., D_d\}$, $d = \dim G$, will be used for differential operators in $L_p(M)$, $1 \leq p \leq \infty$, which are involved in the formula (4.1).

**Definition 4.1** ([24], [28]). We say that a function $f \in L_p(M)$, $1 \leq p \leq \infty$, belongs to the Bernstein space $B_\sigma^p(M)$, $\mathbb{D} = \{D_1, ..., D_d\}$, $d = \dim G$, if and only if for every $1 \leq i_1, ..., i_k \leq d$ the following Bernstein inequality holds true

\[ \|D_{i_1}...D_{i_k}f\|_p \leq \sigma^k \|f\|_p, k \in \mathbb{N}. \]  

We say that a function $f \in L_p(M)$, $1 \leq p \leq \infty$, belongs to the Bernstein space $B_\sigma^p(\mathcal{L})$, if and only if for every $k \in \mathbb{N}$ the following Bernstein inequality holds true

\[ \|\mathcal{L}^k f\|_p \leq \sigma^k \|f\|_p, k \in \mathbb{N}. \]

Since $\mathcal{L}$ in the space $L_2(M)$ is self-adjoint and positive-definite there exists a unique positive square root $\mathcal{L}^{1/2}$. In this case the last inequality is equivalent to the inequality $\|\mathcal{L}^{k/2} f\|_2 \leq \sigma^{k/2} \|f\|_2, k \in \mathbb{N}$.

Note that at this point it is not clear if the Bernstein spaces $B_\sigma^p(\mathbb{D})$, $B_\sigma^p(\mathcal{L})$ are linear spaces. The facts that these spaces are linear, closed and invariant (with respect to operators $D_j$) were established in [28].

It was shown in [28] that for $1 \leq p, q \leq \infty$ the following equality holds true

\[ B_{\sigma}^p(\mathbb{D}) = B_{\sigma}^q(\mathbb{D}) = B_{\sigma}(\mathbb{D}), \quad \mathbb{D} = \{D_1, ..., D_d\}, \]

which means that if the Bernstein-type inequalities (4.2) are satisfied for a single $1 \leq p \leq \infty$, then they are satisfied for all $1 \leq p \leq \infty$.

**Definition 4.2.** $\mathcal{E}_\lambda(\mathcal{L}), \lambda > 0$, be a span of all eigenfunctions of $\mathcal{L}$ whose corresponding eigenvalues are not greater $\lambda$.

The following embeddings which describe relations between Bernstein spaces $B_\sigma$ and eigen spaces $\mathcal{E}_\lambda(\mathcal{L})$ were proved in [28]

\[ B_{\sigma}^p(\mathbb{D}) \subset \mathcal{E}_{\sigma^2d}(\mathcal{L}) \subset B_{\sigma\sqrt{\sigma}}(\mathbb{D}). \]

These embeddings obviously imply the equality

\[ \bigcup_{\sigma > 0} B_{\sigma}(\mathbb{D}) = \bigcup_{\lambda > 0} \mathcal{E}_\lambda(\mathcal{L}), \]

which means that a function on $M$ satisfies a Bernstein inequality (4.3) in the norm of $L_p(M)$, $1 \leq p \leq \infty$, if and only if it is a linear combination of eigenfunctions of $\mathcal{L}$.

As a consequence we obtain [28] the following Bernstein inequalities for $k \in \mathbb{N}$,

\[ \|D_{i_1}...D_{i_k}f\|_p \leq (d\lambda^2)^k \|f\|_p, \quad d = \dim G, \quad f \in \mathcal{E}_\lambda(\mathcal{L}), \quad 1 \leq p \leq \infty. \]

One also has [28] the Bernstein-Nikolskii inequalities

\[ \|D_{i_1}...D_{i_k}\varphi\|_q \leq C(M)(\lambda^+)^k \|\varphi\|_q, \quad d = \dim M, \quad \varphi \in \mathcal{E}_\lambda(\mathcal{L}), \quad k \in \mathbb{N}, \quad 1 \leq q \leq \infty. \]

where $1 \leq q \leq p \leq \infty$ and $C(M)$ is constant which depends just on the manifold.

It is known [30], Ch. IV, that every compact Lie group $G$ can be identified with a subgroup of orthogonal group $O(N)$ of an Euclidean space $\mathbb{R}^N$. It implies that every compact homogeneous manifold $M$ can be identified with a submanifold which is
trajectory of a unit vector $\mathbf{e} \in \mathbb{R}^N$. Such identification of $M$ with a submanifold of $S^{N-1}$ is known as the equivariant embedding into $\mathbb{R}^N$.

Having in mind the equivariant embedding of $M$ into $\mathbb{R}^N$ one can introduce the space $P_n(M)$ of polynomials of degree $n$ on $M$ as the set of restrictions to $M$ of polynomials in $\mathbb{R}^N$ of degree $n$. The following relations were proved in [23]:

$$P_n(M) \subset B_n(\mathbb{D}) \subset \mathcal{E}_{n,d}(\mathcal{L}) \subset B_{n+1}(\mathbb{D}), \quad d = \dim G, \ n \in \mathbb{N},$$

and

$$(4.5) \quad \bigcup_{n \in \mathbb{N}} P_n(M) = \bigcup_{\sigma \geq 0} B_\sigma(\mathbb{D}) = \bigcup_{j \in \mathbb{N}} \mathcal{E}_{\lambda_j}(\mathcal{L}).$$

The next Theorem was proved in [11], [32].

**Theorem 4.3.** If $M = G/K$ is a compact homogeneous manifold and $\mathcal{L}$ is defined as in (4.1), then for any $f$ and $g$ belonging to $\mathcal{E}_\omega(\mathcal{L})$, their pointwise product $fg$ belongs to $\mathcal{E}_{d\omega}(\mathcal{L})$, where $d$ is the dimension of the group $G$.

Using this Theorem and (4.3) we obtain the following

**Corollary 4.1.** If $M = G/K$ is a compact homogeneous manifold and $f, g \in B_\sigma(\mathbb{D})$ then their product $fg$ belongs to $B_{2d\sigma}(\mathbb{D})$, where $d$ is the dimension of the group $G$.

**An example. Analysis on $S^d$**

We will specify the general setup in the case of standard unit sphere. Let

$$S^d = \{ x \in \mathbb{R}^{d+1} : \|x\| = 1 \}.$$ 

Let $\mathcal{P}_n$ denote the space of spherical harmonics of degree $n$, which are restrictions to $S^d$ of harmonic homogeneous polynomials of degree $n$ in $\mathbb{R}^d$. The Laplace-Beltrami operator $\Delta_S$ on $S^d$ is a restriction of the regular Laplace operator $\Delta$ in $\mathbb{R}^d$. Namely,

$$\Delta_S f(x) = \Delta \bar{f}(x), \quad x \in S^d,$$

where $\bar{f}(x)$ is the homogeneous extension of $f$: $\bar{f}(x) = f(\|x\|)$. Another way to compute $\Delta_S f(x)$ is to express both $\Delta_S$ and $f$ in a spherical coordinate system.

Each $\mathcal{P}_n$ is the eigenspace of $\Delta_S$ that corresponds to the eigenvalue $-n(n+d-1)$. Let $Y_{n,d}, \ l = 1, \ldots, l_n$ be an orthonormal basis in $\mathcal{P}_n$.

Let $e_1, \ldots, e_{d+1}$ be the standard orthonormal basis in $\mathbb{R}^{d+1}$. If $SO(d+1)$ and $SO(d)$ are the groups of rotations of $\mathbb{R}^{d+1}$ and $\mathbb{R}^d$ respectively then $S^d = SO(d+1)/SO(d)$.

On $S^d$ we consider vector fields

$$X_{i,j} = x_j \partial_{x_i} - x_i \partial_{x_j}$$

which are generators of one-parameter groups of rotations $\exp t X_{i,j} \in SO(d+1)$ in the plane $(x_i, x_j)$. These groups are defined by the formulas

$$\exp \tau X_{i,j} \cdot (x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_i \cos \tau - x_j \sin \tau, \ldots, x_i \sin \tau + x_j \cos \tau, \ldots, x_{d+1})$$

Let $e^{\tau X_{i,j}}$ be a one-parameter group which is a representation of $\exp \tau X_{i,j}$ in a space $L_p(S^d)$. It acts on $f \in L_p(S^d)$ by the following formula

$$e^{\tau X_{i,j}} f(x_1, \ldots, x_{d+1}) = f(x_1, \ldots, x_i \cos \tau - x_j \sin \tau, \ldots, x_i \sin \tau + x_j \cos \tau, \ldots, x_{d+1}).$$
Let $D_{i,j}$ be a generator of $e^{r X_{i,j}}$ in $L_p(S^d)$. In a standard way the Laplace-Beltrami operator $L$ can be identified with an operator in $L_p(S^d)$ for which we will keep the same notation. One has

$$\Delta = L = \sum_{(i,j)} D_{i,j}^2.$$ 

5. Applications

5.1. Compact homogeneous manifolds. We return to the setup of subsection 5.1. Since $D_j, 1 \leq j \leq d$ generates a group $e^{r D_j}$ in $L_p(M)$ the formulas (3.6) give for $f \in B_\sigma(D)$

$$D_{2m-1}^j f(x) = B_j^{(2m-1)}(\sigma)f(x) = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{\pi(k-1/2)D_j} f(x), m \in \mathbb{N},$$

$$D_{2m}^j f = B_j^{(2m)}(\sigma)f = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{\pi k D_j} f, m \in \mathbb{N} \cup \{0\}.$$ 

Note, that every vector field $X$ on $M$ is a linear combination $\sum_{j=1}^d a_j(x)D_j, x \in M$. Thus we can formulate the following fact.

Theorem 5.1. If $f \in B_\sigma(D)$ then for every vector field $X = \sum_{j=1}^d a_j(x)D_j$ on $M$

$$Xf = \sum_{j=1}^d a_j(x)B_j(\sigma)f,$$

where $B_j(\sigma) = B_j^{1}(\sigma)$.

Moreover, every linear combination $X = \sum_{j=1}^d a_jX_j$ with constant coefficients can be identified with a generator $D = \sum_{j=1}^d a_jD_j$ of a bounded strongly continuous group of operators $e^{tD}$ in $L_p(M), 1 \leq p \leq \infty$.

A commutator

$$[D_l, D_m] = D_l D_m - D_m D_l = \sum_{j=1}^d c_j D_j$$

where constant coefficients $c_j$ here are known as structural constants of the Lie algebra is another generator of an one-parameter group of translations. Formulas (5.1) imply the following relations.

Theorem 5.2. If $D = \sum_{j=1}^d a_jD_j$ then for operators $B_D(\sigma) = B_D^1(\sigma)$ and all $f \in B_\sigma(D)$

$$B_D(\sigma)f = \sum_{j=1}^d a_j B_j(\sigma)f.$$ 

In particular

$$[D_l, D_m] f = B_l(\sigma)B_m(\sigma)f - B_m(\sigma)B_l(\sigma)f = \sum_{j=1}^d c_j B_j(\sigma)f.$$
Moreover,
(5.7) \[ D_{j_1} \cdots D_{j_k} = B_{j_1}(\sigma) \cdots B_{j_k}(\sigma)f. \]

Clearly, for any two smooth functions \( f, g \) on \( M \) one has
\[ D_j(fg)(x) = f(x)D_jg(x) + g(x)D_jf(x). \]

If \( D = \sum_{j=1}^d a_j(x)D_j \) then for \( f, g \in B_{\sigma}(\mathcal{D}) \) the following equality holds
\[ D(fg)(x) = \sum_{j=1}^d a_j(x) \{ f(x)B_j(\sigma)g(x) + g(x)B_j(\sigma)f(x) \}. \]

It is the Corollary 4.1 which allows to formulate the following result.

**Theorem 5.3.** If \( f, g \in B_{\sigma}(\mathcal{D}) \) and \( D = \sum_{j=1}^d a_jD_j \), where \( a_j \) are constants then
\[ B_D(2d\sigma)(fg)(x) = \sum_{j=1}^d a_j \{ f(x)B_j(\sigma)g(x) + g(x)B_j(\sigma)f(x) \}. \]

The formula \(-\mathcal{L} = D_1^2 + D_2^2 + \ldots + D_d^2\) implies the following result.

**Theorem 5.4.** If \( f \in B_{\sigma}(\mathcal{D}) \) then
\[ (5.8) \quad \mathcal{L}f = B_2^2(\sigma)f = \sum_{j=1}^d B_j^2(\sigma)f. \]

**Remark 5.5.** Note that it is not easy to find "closed" formulas for groups like \( e^{it\mathcal{L}} \). Of course, one always has a representation
\[ (5.9) \quad e^{it\mathcal{L}} f(x) = \int_M K(x, y)f(y)dy, \]
with \( K(x, y) = \sum_{\lambda} e^{it\lambda} \langle u_j(x), u_j(y) \rangle \), where \( \{ u_j \} \) is a complete orthonormal system of eigenfunctions of \( \mathcal{L} \) in \( L^2(M) \) and \( \mathcal{L}u_j = \lambda_j u_j \). But the formula (5.9) doesn’t tell much about \( e^{it\mathcal{L}} \). In other words the explicit formulas for operators \( B_j^2(\sigma) \) usually unknown. At the same time it is easy to understand the right-hand side in (5.8) since it is coming from translations on a manifold in certain basic directions (think for example of a sphere).

Let us introduce the notations
\[ B_j^{(2m-1)}(\sigma, N)f = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{|k| \leq N} (-1)^{k+1} A_{m,k} e^{\frac{\sigma}{\pi}(k-1/2)D_j} f, \]
\[ B_j^{(2m)}(\sigma, N)f = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{|k| \leq N} (-1)^{k+1} B_{m,k} e^{\frac{\sigma}{\pi}D_j} f. \]

The following approximate Boas-type formulas hold.

**Theorem 5.6.** If \( f \in B_{\sigma}(\mathcal{D}) \) then
\[ (5.10) \quad \sum_{j=1}^d \alpha_j(x)D_jf = \sum_{j=1}^d \alpha_j(x)B_j(\sigma, N)f + O(N^{-2}), \]
and if $a_j$ are constants then

\begin{equation}
B_D(\sigma)f = \sum_{j=1}^{d} a_j B_j(\sigma,N)f + O(N^{-2}), \quad D = \sum_{j=1}^{d} a_j D_j.
\end{equation}

Moreover,

\begin{equation}
[D_1,D_m]f = \sum_{j=1}^{d} c_j B_j(\sigma,N)f + O(N^{-2}),
\end{equation}

\begin{equation}
L f = \sum_{j=1}^{d} B_j^2(\sigma,N)f + O(N^{-2}).
\end{equation}

5.2. The Heisenberg group. In the space $\mathbb{R}^{2n+1}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n, t)$ we consider vector fields

\[X_j = \partial x_j - \frac{1}{2} y_j \partial t, \quad Y_j = \partial y_j + \frac{1}{2} x_j \partial t, \quad T = \partial t, \quad 1 \leq j \leq n.\]

As operators in the regular space $L_p(\mathbb{R}^{2n+1})$, $1 \leq p \leq \infty$, they generate one-parameter bounded strongly continuous groups of operators. In fact, these operators form in $L_p(\mathbb{R}^{2n+1})$ a representation of the Lie algebra of the so-called Heisenberg group $\mathbb{H}^n$. The corresponding one-parameter groups are

\[e^{\tau X_j} f(x_1, ..., t) = f(x_1, ..., x_j + \tau, ..., x_n, y_1, ..., y_n, t - \frac{1}{2} y_j \tau)\]

\[e^{\tau Y_j} f(x_1, ..., t) = f(x_1, ..., x_n, y_1, ..., y_j + \tau, ..., y_n, t + \frac{1}{2} x_j \tau).\]

As we already know for every $\sigma > 0$ there exists a non-empty set $\mathcal{B}_\sigma(X_j)$ such that their union is dense in $L_p(\mathbb{R}^{2n+1}), 1 \leq p \leq \infty$, and for which the following formulas hold with $m \in \mathbb{N}$

\[X_j^{2m-1} f = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_m,k e^{\frac{\sigma^2}{\pi^2}(k-1/2)X_j} f,
\]

\[X_j^{2m-1} f = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{|k| < N} (-1)^{k+1} A_m,k e^{\frac{\sigma^2}{\pi^2}(k-1/2)X_j} f + O(N^{-2}),
\]

for every $f \in \mathcal{B}_\sigma(X_j)$. One can easily obtain a number of similar formulas for $Y_j$ and $T$.

5.3. The Schrödinger representation. Take 3-dimensional Heisenberg group and consider what is known as its Schrödinger representation in the regular space $L_2(\mathbb{R})$. The infinitesimal operators of this representation are differential operators $D = \frac{1}{2\pi i} \frac{d}{dx}$ and multiplication by independent variable $x$ which will be denoted as $X$. Every linear combination $pd + qX$ with constant coefficients $p, q$ generates a unitary group in $L_2(\mathbb{R})$ according to the formula

\begin{equation}
e^{2\pi i(pD+qx)} f(x) = e^{2\pi iqx + \pi ip} f(x + p),
\end{equation}

and in particular

\begin{equation}
e^{2\pi iqx} f(x) = e^{2\pi iqx} f(x), \quad e^{2\pi ipD} f(x) = f(x + p).
\end{equation}
For every $\sigma > 0$ one can consider corresponding spaces $\mathcal{B}_\sigma(pD+qX)$, $\mathcal{B}_\sigma(D)$, $\mathcal{B}_\sigma(X)$ and corresponding operators $\mathcal{B}$ and $\mathcal{Q}$. One of possible Boas-type formulas would look like
\[
(pD + qX)^m f(x) = \mathcal{B}_\sigma^m(pD + qX)f(x)
\]
and holds for $f \in \mathcal{B}_\sigma(pD + qX)$ where $\bigcup_{\sigma \geq 0} \mathcal{B}_\sigma(pD + qX)$ is dense in $L_2(\mathbb{R})$.

**Remark 5.7.** As it was noticed in [22] the intersection of $\mathcal{B}_\sigma(D)$ and $\mathcal{B}_\sigma(X)$ contains only 0. It follows from the fact that $\mathcal{B}_\sigma(D)$ is the regular Paley-Wiener space and $\mathcal{B}_\sigma(X)$ is the space of functions whose support is in $[-\sigma, \sigma]$, $\sigma > 0$. As a result a formula like
\[
(pD + qX)f(x) = p\mathcal{B}_\sigma D f(x) + q\mathcal{B}_\sigma X f(x)
\]
holds only for $f = 0$ (unlike similar formulas in Theorem 5.1).

**Remark 5.8.** We note that the operator $D^2 + X^2$ is self-adjoint and the set $\bigcup_{\sigma \geq 0} \mathcal{B}_\sigma(D^2 + X^2)$ is a span of all Hermite functions.

**References**

[1] H. Berens, P.L. Butzer, S. Pawelke, Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten, (German) Publ. Res. Inst. Math. Sci. Ser. A 4 1968/1969 201-268.

[2] R. Boas, Entire Functions, Academic Press, New York (1954).

[3] R. Boas, The derivative of a trigonometric integral, J. Lond. Math. Soc. 1937, 1-12, 164.

[4] P. L. Butzer, H. Berens, Semi-Groups of operators and approximation, Springer, Berlin, 1967.

[5] P. L. Butzer, K. Scherer, Jackson and Bernstein-type inequalities for families of commutative operators in Banach spaces, JAT 5, 308-342, 1972.

[6] P.L. Butzer, K. Scherer, Approximation theorems for sequences of commutative operators in Banach spaces, Constructive theory of functions (Proc. Internat. Conf., Varna, 1970) (Russian), pp. 137-146. Izdat. Bolgar. Akad. Nauk, Sofia, 1972.

[7] P.L. Butzer, W. Splettstößer, R. Stens, The sampling theorem and linear prediction in signal analysis, Jahresber. Deutsch. Math.-Verein. 1988, 90, 1-70.

[8] P.L. Butzer, P.J.S.G. Ferreira, J.R. Higgins, G. Schmeisser, R.L. Stens, The sampling theorem, Poisson's summation formula, general Parseval formula, reproducing kernel formula and the Paley-Wiener theorem for bandlimited signals- their interconnections, Applicable Analysis Vol. 90, Nos. 3-4, March-April 2011, 431-461.

[9] P.L. Butzer, G. Schmeisser, R. Stens, Shannons Sampling Theorem for Bandlimited Signals and Their Hilbert Transform, Boas-Type Formulas for Higher Order Derivatives-The Aliasing Error Involved by Their Extensions from Bandlimited to Non-Bandlimited Signals, Entropy 2012, 14, 2192-2226; doi:10.3390/e14112192.

[10] G. Folland, Harmonic analysis in phase space, Princeton University Press, Princeton, NJ, 1989.

[11] D. Geller, I. Pesenson, Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds, J. Geom. Anal. 21 (2011), no. 2, 334-37.

[12] S. Helgason, Groups and Geometric Analysis, Academic Press, 1984.

[13] S. G. Krein, Ju. I. Petunin, Scales of Banach spaces, (Russian) Uspehi Mat. Nauk 21 1966 no. 2 (128), 89-168.

[14] S. G. Krein, Lineiiske differentsialnye uravneniya v Banakhovom prostranstve, (Russian) [Linear differential equations in a Banach space] Izdat. “Nauka”, Moscow 1967 464 pp. English translation: Linear differential equations in Banach space. Translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 29. American Mathematical Society, Providence, R.I., 1971. vi+390 pp.

[15] S. G. Krein, Ju. I. Petunin, E. M. Semenov, Interpoljatsiya lineinyh operatorov, (Russian) [Interpolation of linear operators] “Nauka”, Moscow, 1978. 400 pp. English translation: S. Krein, Y. Petunin, E. Semenov, Interpolation of linear operators, Translations of Mathematical Monographs, 54. AMS, Providence, R.I., 1982.
[16] S. Krein, I. Pesenson, *Interpolation Spaces and Approximation on Lie Groups*, The Voronezh State University, Voronezh, 1990.

[17] E. Nelson, *Analytic vectors*, Ann. of Math. (2) 70 (1959) 572-615.

[18] I. Pesenson, *Moduli of continuity on Lie groups*, (Russian) Collection of articles on applications of functional analysis (Russian), pp. 115-121. Voronezh. Tekhnolog. Inst., Voronezh, 1975.

[19] I. Pesenson, *Interpolation of noncommuting operators*, (Russian) Uspehi Mat. Nauk 33 (1978), no. 3(201), 183-184.

[20] I. Pesenson, *Interpolation spaces on Lie groups*, (Russian) Dokl. Akad. Nauk SSSR 246 (1979), no. 6, 1208-1303.

[21] I. Pesenson, *Nikolskii-Besov spaces connected with representations of Lie groups*, (Russian) Dokl. Akad. Nauk SSSR 273 (1983), no. 1, 45-49.

[22] I. Pesenson, *The Best Approximation in a Representation Space of a Lie Group*, Dokl. Acad. Nauk USSR, v. 302, No 5, pp. 1055-1059, (1988) (Engl. Transl. in Soviet Math. Dokl., v.38, No 2, pp. 384-388, 1989.)

[23] I. Pesenson, *On the abstract theory of Nikolskii-Besov spaces*, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1988, no. 6, 59-68; translation in Soviet Math. (Iz. VUZ) 32 (1988), no. 6, 80-92.

[24] I. Pesenson, *The Bernstein Inequality in the Space of Representation of Lie group*, Dokl. Acad. Nauk USSR 313 (1990), 86-90; English transl. in Soviet Math. Dokl. 42 (1991).

[25] I. Pesenson, *Approximations in the representation space of a Lie group*, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1990, no. 7, 43-50; translation in Soviet Math. (Iz. VUZ) 34 (1990), no. 7, 49-57.

[26] I. Pesenson, *A sampling theorem on homogeneous manifolds*, Trans. Amer. Math. Soc., Vol. 352(9), (2000), 4257-4269.

[27] I. Pesenson, *Sampling of Band limited vectors*, J. of Fourier Analysis and Applications 7(1), (2001), 93-100.

[28] I. Pesenson, *Bernstein-Nikolski inequalities and Riesz interpolation formula on compact homogeneous manifolds*, J. of Approx. Theory 150, (2008), no. 2, 175-198.

[29] I. Pesenson, *Paley-Wiener approximations and multiscale approximations in Sobolev and Besov spaces on manifolds*, J. Geom. Anal. 19 (2009), no. 2, 390-419.

[30] I. Pesenson, A. Zayed, *Paley-Wiener subspace of vectors in a Hilbert space with applications to integral transforms*, J. Math. Anal. Appl. 353 (2009) 566-582.

[31] I. Pesenson, M. Pesenson, *Approximation of Besov vectors by Paley-Wiener vectors in Hilbert spaces*, Approximation Theory XIII: San Antonio 2010 (Springer Proceedings in Mathematics), by Marian Neamtu and Larry Schumaker, 249–263.

[32] I. Pesenson, D. Geller, *Cubature Formulas and Discrete Fourier Transform on Compact Manifolds*, in “From Fourier Analysis and Number Theory to Radon Transforms and Geometry: In Memory of Leon Ehrenpreis” (Developments in Mathematics 28) by Hershel M. Farkas, Robert C. Gunning, Marvin I. Knopp and B. A. Taylor, Springer NY 2013.

[33] M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen f?r Polynome*, Jahresber. Deutsch. Math.-Verein. 1914, 23, 354368.

[34] M. Riesz, *Les fonctions conjuguées et les series de Fourier*, C.R. Acad. Sci. 178, 1924, 1464-1467.

[35] G. Schmeisser, *Numerical differentiation inspired by a formula of R. P. Boas*, J. Approx. Theory 2009, 160, 202-222.

[36] D. Zelobenko, *Compact Lie groups and their representations*, Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, R.I., 1973. viii+448 pp.