Equivalent Quantisations of (2+1)-Dimensional Gravity*

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Abstract
For spacetimes with the topology $\mathbb{R} \times T^2$, the action of (2+1)-dimensional gravity with negative cosmological constant $\Lambda$ is written uniquely in terms of the time-independent traces of holonomies around two intersecting noncontractible paths on $T^2$. The holonomy parameters are related to the moduli on slices of constant mean curvature by a time-dependent canonical transformation which introduces an effective Hamiltonian. The quantisation of the two classically equivalent formulations differs by terms of order $O(h^3)$, negligible for small $|\Lambda|$.

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Classical Theory

Amongst the various approaches to (2+1)-dimensional quantum gravity without matter couplings, there are two that have seen substantial development in the last few years: the second-order metric formalism of Moncrief [1] and the first-order holonomy algebra formalism of Nelson and Regge [2], inspired by Witten [3]. Both start with a spacetime with the topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is a closed Riemann surface. In the first-order formalism, the variables are two sets of $6g - 6$ ($g > 1$) time-independent traces of holonomies around intersecting noncontractible paths on $\Sigma$. Their Hamiltonian is zero, since the constraints have been solved exactly. In the second-order formalism the constraints are also solved exactly, but the remaining variables are $6g - 6$ time-dependent coordinates for the Teichmüller space of $\Sigma$ and their conjugate momenta, with a time development generated by a nontrivial effective Hamiltonian.

When the cosmological constant is zero and $\Sigma$ is a torus $T^2$ (for which there are only two degrees of freedom), it has been shown that the two approaches are classically equivalent, and are related through a time-dependent canonical transformation. The corresponding quantum theories are closely related, but are not quite equivalent in the most natural operator ordering [4]. The purpose of this letter is to affirm that the same is true when the cosmological constant is negative, and that the unique action of (2+1)-dimensional gravity can be written in the equivalent forms

$$
\int dt \int d^2x \, \pi^{ij} \dot{g}_{ij} = \int dt \int d^2x \, 2\epsilon^{ij} \epsilon_{abc} \dot{e}^c_j \dot{\omega}^a_i
$$

$$
= \int \frac{1}{2}(\dot{p}dm + pd\dot{m}) + Hd\tau - d(p^1m_1 + p^2m_2) \quad (1)
$$

$$
= \int \alpha(r_1^- dr_2^- - r_1^+ dr_2^+)
$$

In (1), $m = m_1 + im_2$ is the complex modulus of the torus, related to the spatial
components $g_{ij}$ of the metric tensor on a slice of constant mean curvature by

$$g^{-1/2}g_{ij} = \frac{1}{m_2} \begin{pmatrix} m_1^2 + m_2^2 & m_1 \\ m_1 & 1 \end{pmatrix}$$

(2)

and $p = p^1 + ip^2$ is the conjugate momentum, defined by [1]

$$p^a = \frac{1}{2}g_{ij}\pi - \pi_{ij}\frac{\partial}{\partial m_a}g^{ij} \quad (a = 1, 2)$$

(3)

The $r_{1,2}^\pm$ are the variables of the SL(2,\mathbb{R}) holonomies [2], expressed as

$$R_{1}^\pm = \cosh \frac{r_{1}^\pm}{2}$$

$$R_{2}^\pm = \cosh \frac{r_{2}^\pm}{2}$$

(4)

where the subscripts 1 and 2 in (4) refer to two intersecting paths $\gamma_1, \gamma_2$ on $\Sigma$ with intersection number +1. (A third holonomy, $R_{12}^\pm = \cosh \frac{r_{1}^\pm + r_{2}^\pm}{2}$, corresponds to the path $\gamma_1 \cdot \gamma_2$, which has intersection number −1 with $\gamma_1$ and +1 with $\gamma_2$.)

To arrive at (1), the first step is to choose spatial hypersurfaces $\Sigma$ labelled by $\text{Tr}K = -g^{-1/2}\pi = \tau = \text{const.}$ and to solve the constraints, which for negative cosmological constant $\Lambda = -1/\alpha^2$ read

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^c = \frac{1}{\alpha^2}e^a \wedge e^b$$

$$R^a = de^a - \omega^{ab} \wedge e_b = 0$$

(5)

with $a, b, c = 0, 1, 2$. In the time gauge $e_i^0 = 0$, these can be satisfied by the choice

$$e^0 = dt$$

$$e^1 = \frac{\alpha}{2} [(r_1^+ - r_1^-)dx + (r_2^+ - r_2^-)dy] \sin \frac{t}{\alpha}$$

(6a)

$$e^2 = \frac{\alpha}{2} [(r_1^+ + r_1^-)dx + (r_2^+ + r_2^-)dy] \cos \frac{t}{\alpha}$$

$$\omega^{12} = 0$$

$$\omega^{01} = -\frac{1}{2} [(r_1^+ - r_1^-)dx + (r_2^+ - r_2^-)dy] \cos \frac{t}{\alpha}$$

(6b)

$$\omega^{02} = \frac{1}{2} [(r_1^+ + r_1^-)dx + (r_2^+ + r_2^-)dy] \sin \frac{t}{\alpha}$$
where $x$ and $y$ have period one and the time coordinate $t$ is determined by $\tau = -\frac{2}{\alpha} \cot \frac{2\tau}{\alpha}$. The components (6a–b) can be used in two ways. Firstly, we calculate the metric tensor $g_{ij} = e^a_i e^b_j \eta_{ab}$ of $\Sigma$, and thence from (2) and (3) the modulus*

$$m = \left( r_1^+ e^{it/\alpha} + r_1^- e^{-it/\alpha} \right) \left( r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha} \right)^{-1}$$

and its conjugate momentum

$$p = -\frac{i\alpha}{2 \sin \frac{2\tau}{\alpha}} \left( r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha} \right)^2$$

These variables satisfy the Poisson bracket algebra

$$(\tilde{m}, p) = (m, \tilde{p}) = -2, \quad (m, p) = (\tilde{m}, \tilde{p}) = 0$$

Secondly, we compute the traces of the $\text{SL}(2,\mathbb{R})$ holonomies corresponding to the two generators of the fundamental group $\pi_1(\Sigma)$, using the decomposition of the spinor group of $\text{SO}(2,2)$ as a tensor product $\text{SL}(2,\mathbb{R}) \otimes \text{SL}(2,\mathbb{R})$. This approach recovers the traces (4), which satisfy the nonlinear classical Poisson bracket algebra [2]

$$(R_1^\pm, R_2^\pm) = \mp \frac{1}{4\alpha} (R_{12}^\pm - R_1^\pm R_2^\pm)$$

consistent with the brackets

$$(e^a_i(x), \omega^{bc}_j(y)) = -\frac{1}{2} \epsilon_{ij} \epsilon^{abc} \delta^2(x - y)$$

obtained from (1). Equation (10) implies that the holonomy parameters satisfy

$$(r_1^\pm, r_2^\pm) = \mp \frac{1}{\alpha}, \quad (r^+, r^-) = 0$$

* This result is essentially equivalent to that of Fujiwara and Soda [1], although some care must be taken in the conversion; in particular, as they note in section 5, their spatial coordinates need not have period one.
It is easily checked that the brackets (12) induce (9) and (10). The holonomy parameters \( r_{1,2}^\pm \) are thus related to the modulus \( m \) and momentum \( p \) through a (time-dependent) canonical transformation, as expressed in (1).

The Hamiltonian in equation (1) now takes the form

\[
H = g^{1/2} = \frac{\alpha^2}{4} \sin \frac{2t}{\alpha} \left( r_1^- r_2^+ - r_1^+ r_2^- \right)
\]  

which generates the development of the modulus (7) and momentum (8) through

\[
\frac{dp}{d\tau} = (p, H), \quad \frac{dm}{d\tau} = (m, H)
\]  

Alternatively, the Hamiltonian

\[
H' = \frac{d\tau}{dt} H = \frac{4}{\alpha^2} \csc^2 \frac{2t}{\alpha} H
\]  

generates evolution in coordinate time \( t \) by

\[
\frac{dp}{dt} = (p, H'), \quad \frac{dm}{dt} = (m, H')
\]  

The time-dependent moduli \( m_1, m_2 \) lie on a semicircle,

\[
\left( m_1 - \frac{r_1^+ r_2^+ - r_1^- r_2^-}{r_2^+ - r_2^-} \right)^2 + m_2^2 = \left( \frac{r_1^+ r_2^+ - r_1^- r_2^-}{r_2^+ - r_2^-} \right)^2
\]  

as can be seen from (7), again agreeing with Fujiwara and Soda [1].

The standard action of the modular group on the torus modulus,

\[
S : m \to -\frac{1}{m}, \quad p \to m^2 p
\]

\[
T : m \to m + 1, \quad p \to p
\]

preserves the brackets (9). The same group acts on the holonomy parameters as

\[
S : r_{1,2}^\pm \to r_{2,1}^\pm, \quad r_{2,1}^\pm \to -r_{1,2}^\pm
\]

\[
T : r_{1,2}^\pm \to r_{1,2}^\pm + r_{2,1}^\pm, \quad r_{2,1}^\pm \to r_{2,1}^\pm
\]  

preserving the brackets (12) and the Hamiltonians (13) and (15). On the traces (4), the group action is

\[
S : R_1^\pm \rightarrow R_2^\pm, \quad R_2^\pm \rightarrow R_1^\pm, \quad R_{12}^\pm \rightarrow 2R_1^\pm R_2^\pm - R_{12}^\pm
\]

\[
T : R_1^\pm \rightarrow R_{12}^\pm, \quad R_2^\pm \rightarrow R_2^\pm, \quad R_{12}^\pm \rightarrow 2R_{12}^\pm R_2^\pm - R_1^\pm
\]

(20)
corresponding to the intersection number preserving exchanges

\[
S : \gamma_1 \rightarrow \gamma_2^{-1}, \quad \gamma_2 \rightarrow \gamma_1
\]

\[
T : \gamma_1 \rightarrow \gamma_1 \cdot \gamma_2, \quad \gamma_2 \rightarrow \gamma_2
\]

(21)

It is easy to verify from (7–8) that the transformations (19) of \( r^\pm \) induce the correct transformations (18) of \( m \) and \( p \).

The relationship between the metric and the holonomies can be further checked by means of a quotient space construction. The holonomies (4) generate a subgroup

\[
\Gamma = \langle R_1^+ \otimes R_1^-, R_2^+ \otimes R_2^- \rangle
\]

(22)
of \( \text{SL}(2,\mathbb{R}) \otimes \text{SL}(2,\mathbb{R}) \), which acts on three-dimensional anti-de Sitter space as a group of isometries. It may be shown that the quotient of anti-de Sitter space by this group is a spacetime with topology \( \mathbb{R} \times T^2 \) whose induced metric is precisely the metric \( g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \) obtained from the triads (6a). The holonomies thus describe a geometric structure of the type discussed by Mess [5].

Quantisation

It is easy to quantise the system described so far. The quantisation of (10) gives the weighted algebra [2]

\[
R_1^\pm R_2^\pm e^{\pm i\theta} - R_2^\pm R_1^\pm e^{\mp i\theta} = \pm 2i \sin \theta R_{12}^\pm
\]

(23)

with \( \tan \theta = -\hbar /8\alpha \), consistent with the commutators

\[
[r_1^\pm, r_2^\mp] = \pm 8i\theta, \quad [r^\pm, r^\mp] = 0
\]

(24)
when the holonomies (4) are represented by the operators

\[ R^\pm_a = \sec \theta \cosh \frac{r^\pm_a}{2} \quad (a = 1, 2) \tag{25} \]

With the ordering of (7–8) and (13) it follows that

\[ [\bar{m}, p] = [m, \bar{p}] = 16i\alpha \theta, \quad [m, p] = [\bar{m}, \bar{p}] = 0 \]

\[ [p, H'] = -8i\alpha \theta \frac{dp}{dt}, \quad [m, H'] = -8i\alpha \theta \frac{dm}{dt} \tag{26} \]

which, with \( \theta = -\tan^{-1}(\hbar/8\alpha) \), differ from the direct quantisation of (12), (9) and (16) by terms of order \( O(\hbar^3) \), small when \( |\Lambda| = 1/\alpha^2 \) is small.

Full details, including the action of the quantum modular group, the precise relationship with Fujiwara and Soda’s results [1], the quantum group construction of reference [2], and the limit \( \Lambda \to 0 \), will be discussed elsewhere [6].

References

[1] V. Moncrief, J. Math. Phys. 30 (1989) 2907; A. Hosoya and K. Nakao, Class. Quantum Grav. 7 (1990) 163, Prog. Theor. Phys. 84 (1990) 739; Y. Fujiwara and J. Soda, Prog. Theor. Phys. 83 (1990) 733.

[2] J. E. Nelson and T. Regge, Phys. Lett. B272 (1991) 213, Nucl. Phys. B328 (1989), Commun. Math. Phys. 141 (1991) 211, Commun. Math. Phys. 155 (1993) 561; J. E. Nelson, T. Regge and F. Zertuche, Nucl. Phys. B339 (1990) 516.

[3] E. Witten, Nucl. Phys. B311 (1988/89) 46-78.

[4] S. Carlip, Phys. Rev. D42 (1990) 2647, Phys. Rev. D45 (1992) 3584, Phys. Rev. D47 (1993) 4520.

[5] G. Mess, “Lorentz Spacetimes of Constant Curvature,” Institut des Hautes Etudes Scientifiques preprint IHES/M/90/28 (1990).

[6] S. Carlip and J. E. Nelson, in preparation.