TWO LIVES : COMPOSITIONS OF UNIMODULAR ROWS

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Abstract. The paper lays the foundation for the study of unimodular rows using Spin groups. We show that elementary orbits of unimodular rows (of any length \( n \geq 3 \)) are equivalent to elementary Spin orbits on the unit sphere. (This bijection is proved over all commutative rings). In the special case \( n = 3 \), this leads to an interpretation of the Vaserstein symbol using Spin groups.

In addition, we introduce a new composition law that operates on certain subspaces of the underlying quadratic space (using the multiplication in composition algebras). In particular, the special case of split-quaternions leads to the composition of unimodular rows (discovered by L. Vaserstein and later generalized by W. van der Kallen). Strikingly, with this approach, we now see the possibility of new orbit structures not only for unimodular rows (using octonion multiplication) but also for more general quadratic spaces.

1. Introduction

When multiple research areas evolve around the same object, one expects that there is a connection between them. The more distinct the methods are, the more fruitful the connection will be. In this paper, we will explore this double life for \textit{unimodular rows}. Though unimodular rows are primarily used as a tool to study Projective modules, we will see here that they can also be fruitfully employed from the perspective of Quadratic forms and Spin groups. One consequence of this approach is that it gives a neat interpretation of some surprising results like the Vaserstein symbol, through a simple composition law operating in the background. On the other hand, we arrive at new questions in this development via quadratic forms. We currently know (through the work of W. van der Kallen \([\text{vdk2}]\) and others) that the Vaserstein symbol can be generalized to a group law on certain (higher-dimensional) orbit-spaces of unimodular rows. But now, when interpreted as a result in quadratic forms, there is the exciting possibility that such group laws may generalize beyond hyperbolic quadratic spaces. In particular, we see (in Part C) that Vaserstein composition corresponds to the special case of split quaternions, and if we pick another composition algebra, we get a different composition rule.

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Let $R$ be any commutative ring. A vector $v = (a_1, \cdots, a_n) \in R^n$ is called a unimodular row (of length $n$) if $v \cdot w^\top = 1$ for some $w = (b_1, \cdots, b_n) \in R^n$. Let $Um_n(R)$ denote the set of unimodular rows of length $n$. Here are a few places where unimodular rows turn up.

1. **First life**: Cancellation of Projective modules. The study of projective modules is one of the primary motivations to investigate unimodular rows. Given a vector $v \in R^n$, consider the map $P_v : R^n \to R$, given by

$$P_v(w) = v \cdot w^\top.$$

Whenever $v$ is a unimodular row, the kernel of $P_v$ becomes a stably free module of rank $n - 1$ (and hence a finitely generated projective module). Note that the group $GL_n(R)$ (and hence any subgroup) acts on the right on the set of unimodular rows $Um_n(R)$. One way to show that the projective module $\ker(P_v)$ is free is to show that $v$ appears as a row in an invertible matrix. Thus the interest in unimodular rows began, and grew with the Quillen-Suslin theorem (also known as Serre’s problem) which states that finitely generated projective modules over polynomial-rings are free - Quillen received a Fields medal in 1978 in part for his proof of the theorem (see [Lam]). As one goes beyond polynomial rings, the orbits may not be trivial, leading naturally to the study of quotients such as $Um_n(R)/SL_n(R)$ and $Um_n(R)/E_n(R)$ and there is a rich array of results stating conditions under which these orbit spaces have an abelian group structure (see for example [DTZ, SV, vdk2]). As we will see, these same orbits can also be examined from a different point of view, as Spin-orbits on the unit sphere.

1.2. **Second life**: Group structures on spheres. Consider the space $H(R^n) = R^n \oplus (R^n)^*$, equipped with a quadratic form

$$q(x, y) = x \cdot y^\top.$$

Let $U_{2n-1}(R)$ denote the unit sphere, i.e., the set of all elements $(v, w) \in H(R^n)$ such that $v \cdot w^\top = 1$. Suppose there is another element $w' \in R^n$ such that $v \cdot w'^\top = v \cdot w^\top = 1$. Then it turns out (Theorem 4.1) that the two points on the unit sphere - $(v, w)$ and $(v, w')$ - lie on the same orbit under the action of $Epin_{2n}(R)$, a normal subgroup of $Spin_{2n}(R)$ called the elementary Spin group. (We will describe the elementary Spin group in Section 3).

This gives us the map

$$v \to \frac{(v, w)}{Epin_{2n}(R)}.$$

We will show that the kernel of the above map is the orbit of $v$ under the action of the elementary linear group $E_n(R)$.

$$v \quad E_n(R) \quad \leftrightarrow \quad \frac{(v, w)}{Epin_{2n}(R)}.$$
The main result of the paper (Theorem 4.4) is the following bijection

\[ \frac{U_{m_n}(R)}{E_n(R)} \longleftrightarrow \frac{U_{2n-1}(R)}{\text{Epin}_{2n}(R)} = \frac{U_{2n-1}(R)}{E\text{O}_{2n}(R)} \]

In short, orbits of unimodular rows can be studied as orbits of points on the unit sphere whose geometry is more familiar to us. The seminal paper of Suslin-Vaserstein [SV] contains some hints of the above bijection for \( n = 3 \) (though they don’t talk about Spin groups). In this paper, we will prove this bijection and generalize it to any \( n \geq 3 \) (Theorem 4.4).

The same paper of Suslin-Vaserstein [SV, Section 5] also introduced what is now referred to as the Vaserstein symbol, which revealed a surprising symplectic structure for the orbit spaces \( \frac{U_{m_3}(R)}{E_3(R)} \). When \( n = 3 \), we have \( \text{Epin}_6(R) \cong E_4(R) \) which will be used to show that there is a bijection between \( \frac{U_{m_3}(R)}{E_3(R)} \) and the \( E_4(R) \)-orbits of \( 4 \times 4 \) alternating matrices with Pfaffian 1 (where \( M \sim gMg^T \) for \( g \in E_4(R) \)). This gives an interpretation of the Vaserstein symbol using Spin groups (see Part B and Theorem 6.1).

The second contribution of this paper is the introduction of a new composition law that holds on certain subspaces of the hyperbolic space \( H(R^n) = R^n \oplus (R^n)^* \) (using quaternion multiplication). This composition law (on matrices) generalizes the Vaserstein composition for unimodular rows (see Part B and Remark 8.4 in Part C).

One quality of the composition law is that it is independent of the dimension of \( R \), whereas it seems necessary to place such restrictions on the base ring \( R \) to get group structures on orbit spaces of unimodular rows (in some situations). What is the reason for this dependence on the dimension of \( R \), and how do we arrive at this dependence: in this case, \( \dim(R) \leq 2n - 3 \)?

Looking back, there are two results that hint at a general composition law operating in the background - one is the Vaserstein symbol (see Part B), the other being the Mennicke-Newman Lemma (see [vdK3, Lemma 3.2]) that essentially says that, under the above dimension restrictions, one can project two points of the unit sphere \( U_{2n-1}(R) \) onto the same \((n + 1)\)-dimensional subspace where the composition law operates (see Remarks 5.4, 5.5).

This investigation using quadratic forms opens the door for research in two striking general directions:

a. In Section 10, we use the multiplication of split-octonions to define a (nonassociative) composition law on \((n + 4)\)-dimensional subspaces of \( H(R^n) \), suggesting that there may be a quasigroup structure on orbits of unimodular rows.

b. Let \((V, q)\) be any quadratic space, and \( U \) the set of unit vectors of \( V \) \((q(x) = 1)\). When is there a group structure on the orbit spaces \( \frac{U}{\text{Spin}(V)} \)?
1.3. The other lives of unimodular rows. The vector \( v = (a, b, c) \) also corresponds to coefficients of the quadratic form \( ax^2 + bxy + cy^2 \). The condition \( v \cdot w^\top = 1 \) can then be seen as a restriction to primitive quadratic forms. In the study of unimodular rows, one is mainly concerned with \( SL_3(R) \) orbits of \( Um_3(R) \), whereas Gauss’s composition gives a group structure on the \( SL_2(\mathbb{Z}) \) orbits of binary quadratic forms. It is known that Gauss’s composition extends to an arbitrary base ring (see [K] and [W]). It is also known that if \( \frac{1}{2} \in R \) and the discriminant \( b^2 - 4ac \) is a square, then the unimodular row \((a, b, c)\) is completable and the corresponding projective module is free (see [KM] or [Ko]). But it is not known whether there are deeper connections between projective modules and M. Bhargava’s work [Bh1] on the composition laws for quadratic and higher forms - an intriguing line to pursue, that hopefully future research can shed some light on.

Remark 1.4. There are many other active areas related to unimodular rows - notably, Euler class groups ([BRS, DTZ]), Grothendieck-Witt groups ([FRS]), \( A^1 \)-homotopy theory ([AF1, AF2, AF3]) and Suslin Matrices (see [RJ] for a survey).

Under certain smoothness conditions for a ring \( R \) with (Krull) dimension \( d \), J. Fasel has given an interpretation of \( \frac{Um_{d+1}(R)}{E_{d+1}(R)} \) in terms of cohomology (see [F1]). More recently this quotient space has been explicitly computed in [DTZ] for some rings. Ravi Rao and Selby Jose have written a series of papers ([JR1, JR2]) examining general quotients \( \frac{Um_n(R)}{E_n(R)} \) by studying the algebraic properties of Suslin matrices. As you can tell, the behaviour of the quotient \( \frac{Um_n(R)}{E_n(R)} \) depends on the base ring \( R \) (especially its Krull dimension), and there is a continuing trend simplifying the hypothesis on the base ring to construct and analyze the structure of the orbit spaces (see for example [FRS, GRK, GGR, Gu, SS] or Part II of the recent conference proceedings [AHS]).

The Vaserstein symbol gives a symplectic structure to the orbit-spaces of unimodular rows and plays an important role in the study of stably-free modules. It was first introduced in [SV] Section 5] where orbits of unimodular rows were investigated under the action of both linear and symplectic groups. Further investigation of the symplectic orbits can be found in [CR1, CR2, TS2]. The recent work of T. Syed [TS1] generalizes the Vaserstein symbol to study the orbit spaces \( \frac{Um(R+P)}{E(R+P)} \), where \( P \) is a rank-2 projective module with a fixed trivialization of its determinant.

A. Asok and J. Fasel have provided an interpretation of the Vaserstein symbol in terms of \( A^1 \)-homotopy theory (see [F2]) and we will explain this connection briefly in Part B. In [FRS Theorem 7.5] the Vaserstein symbol was used to prove that stably free modules of rank \( d - 1 \) are free under certain smoothness conditions (\( R \) is a smooth affine \( k \)-algebra of dimension \( d \geq 3 \), where \( k \) is an algebraically closed field and \( \frac{1}{(d-1)!} \in k \)), thus settling a long-standing question of A. Suslin.
Perhaps some day, another mathematician will write a “many lives” generalization of this paper.

Contents

Part A. The bijection between (elementary) Spin-orbits on the sphere and the elementary orbits of unimodular rows

Part B. Interpreting the Vaserstein symbol using Spin groups

Part C. A general Composition law

1.5. Overview. The paper is broken down into three parts and can be read non-linearly. A reader whose main interest is unimodular rows may begin with Part A, where the connection to Spin groups is explored in detail. Alternatively, a person who is curious about general quadratic forms may find it profitable to look at Part C first, where a new composition law is defined using the multiplication in composition algebras. Here the Vaserstein composition (for unimodular rows) corresponds to the special case of split quaternions. Finally, those who are comfortable with both the worlds and prefer to quickly know what is going on, may begin with Part B which acts as a bridge (examining the Vaserstein symbol), and then read around accordingly.

Essentially the paper makes two contributions. First, we look at (elementary) orbits of unimodular rows and prove that they correspond to (elementary) Spin-orbits on the unit sphere. Secondly, we introduce a composition law - that holds in certain \((n + 2)\)-dimensional subspaces of \(H(R^n)\). This composition (in terms of matrices) follows a simple recursive rule, starting with the multiplication of split-quaternions. When \(n = 3\), it has the same properties as the composition law (on unimodular rows) discovered by L. Vaserstein. For general \(n\), it describes in a simple matrix form, the composition introduced by van der Kallen using what are known as weak Mennicke symbols. This lays the foundation for the study of unimodular rows using Spin groups. The general formulation of the composition law also raises the possibility of new orbit structures using octonion multiplication.

1.6. Notation. All modules \(V\) defined in the paper are free \(R\)-modules over some commutative ring \(R\). No condition is placed on \(R\) so the results in the paper hold for all commutative rings. A quadratic form \(q\) on \(V\) is said to be non-degenerate when the corresponding bilinear form is non-degenerate, i.e., \(\langle x, v \rangle = 0\) holds for all \(v \in V\), if and only if \(x = 0\). (Here \(\langle v, w \rangle = q(v + w) - q(v) - q(w)\) for \(v, w \in V\).)
Part A. The bijection between (elementary) Spin-orbits on the sphere and the elementary orbits of unimodular rows

2. Preliminaries: From Clifford algebra to Suslin matrices

For general literature on Clifford algebras and Spin groups over a commutative ring, see [B2, Kn].

2.1. Clifford Algebras. Let $V$ be a free $R$-module where $R$ is any commutative ring. If we equip $V$ with a non-degenerate quadratic form $q$, then $(V, q)$ is called a quadratic space. The algebra $\Cl(V, q)$ is the “freest” algebra generated by $V$ subject to the condition $x^2 = q(x)$ for all $x \in V$. More precisely, $\Cl(V, q)$ is the quotient of the tensor algebra $T(V) = R \oplus V \oplus V \otimes R \oplus \cdots$ by the two sided ideal $I(V, q)$ generated by all the elements $x \otimes x - q(x)$ with $x \in V$.

For the purpose of this article, we only need to know the following basic properties of Clifford algebras:

- **$\mathbb{Z}_2$-grading**: Grading $T(V)$ by even and odd degrees, it follows that the Clifford algebra has a $\mathbb{Z}_2$-grading $\Cl(V, q) = \Cl_0 \oplus \Cl_1$ such that $V \subseteq \Cl_1$ and $\Cl_i \Cl_j \subseteq \Cl_{i+j}$ ($i, j \text{ mod } 2$).
- **Universal property**: Let $i : V \rightarrow \Cl(V, q)$ denote the inclusion map and $A$ be any associative algebra over $R$. Then any linear map $j : V \rightarrow A$ such that $j(x)^2 = q(x)$ for all $x \in V$, lifts to a unique $R$-algebra homomorphism $f : \Cl(V, q) \rightarrow A$ such that $f \circ i = j$.
- **Basis of $\Cl$**: The elements of $V$ generate the Clifford algebra. Furthermore, the following result implies that if $\text{rank}(V) = n$, then $\text{rank}(\Cl) = 2^n$.

**Theorem 2.2. (Poincaré-Birkhoff-Witt)**

Let $\{v_1, \ldots, v_n\}$ be a basis of $(V, q)$. Then $\{v_1^{e_1} \cdots v_n^{e_n} : e_i = 0, 1\}$ is a basis of $\Cl(V, q)$.

For a simple proof, see [Kn, Theorem IV. 1.5.1].

Let $\Cl$ denote the Clifford algebra of the quadratic space $H(R^n) := R^n \oplus R^{n*}$, with $q(v, w) = v \cdot w^t$. We will now give an explicit representation of $\Cl \cong M_{2^n}(R)$ using what are called Suslin matrices. For a detailed exposition, see [CV1].
2.3. Suslin matrices. For any two vectors $v = (a_1, \cdots, a_n)$ and $w = (b_1, \cdots, b_n)$ in $R^n$, the Suslin matrix $S_{n-1}(v, w)$ is defined as follows:

For $n = 2$, define

$$S_1(v, w) = \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}, \quad \overline{S}_1(v, w) = \begin{pmatrix} b_1 - a_2 \\ b_2 & a_1 \end{pmatrix}$$

For the general case, write $v = (a_1, v')$ and $w = (b_1, w')$ with $v', w' \in R^{n-1}$. Then

$$S_{n-1}(v, w) = \begin{pmatrix} a_1 & S_{n-2}(v', w') \\ -S_{n-2}(w', v') & b_1 \end{pmatrix}, \quad \overline{S}_{n-1}(v, w) = \begin{pmatrix} b_1 & -S_{n-2}(v', w') \\ S_{n-2}(w', v') & a_1 \end{pmatrix}$$

The matrix $S = S_{n-1}(v, w)$ has size $2^{n-1} \times 2^{n-1}$ and has the following properties:

a. $S(v, w) = \overline{S}(w, v)^\top$.

b. $SS = \overline{S}S = (v \cdot w^\top)I_{2^{n-1}}$.

In his paper [S], A. Suslin then describes a sequence of matrices $J_n \in M_{2^n}(R)$ by the recurrence formula

$$J_n = \begin{cases} 1 & \text{for } n = 0 \\ \begin{pmatrix} J_{n-1} & 0 \\ 0 & -J_{n-1} \end{pmatrix} & \text{for } n \text{ even} \\ \begin{pmatrix} 0 & J_{n-1} \\ -J_{n-1} & 0 \end{pmatrix} & \text{for } n \text{ odd} \end{cases}$$

One can check by induction that $J_nJ_n^\top = 1$. It follows that $M^* = J_nM^\top J_n^\top$ is an involution of $M_{2^n}(R)$. Importantly, their relation to Clifford algebras comes from the following equations:

$$S_{n-1}^*(v, w) = \begin{cases} S_{n-1}(v, w) & \text{for } n \text{ odd}, \\ \overline{S}_{n-1}(v, w) & \text{for } n \text{ even}. \end{cases}$$

(1)

We will sometimes omit the subscripts and simply write $J, S$ or $S(v, w)$ if the length of the vectors are clear.

The map $\phi : H(R^n) \to M_{2^n}(R)$ defined by $\phi(v, w) = \left( \begin{smallmatrix} 0 & \overline{S}_{n-1}(v, w) \\ S_{n-1}(v, w) & 0 \end{smallmatrix} \right)$ induces an $R$-algebra homomorphism $\phi : Cl \to M_{2^n}(R)$. In fact $\phi$ is an isomorphism (see [CV1, Section 3.1]); the elements $\phi(v, w)$ give a set of generators of the Clifford algebra. In addition, the involution $M^* = JMJ^\top J^\top$ turns out to be what is called the standard involution of the Clifford algebra [CV1, Theorem 4.1]. Note that the quadratic form is $q(v, w) = S(v, w)\overline{S}(v, w)$. For $S_i = S(v, i)$, the corresponding bilinear form is

$$\langle S_1, S_2 \rangle = S_1\overline{S}_2 + \overline{S}_2S_1 = v_1 \cdot w_2^\top + v_2 \cdot w_1^\top.$$
2.4. **Properties of the basis vectors.** Let \( e_i \) and \( f_i \) denote the standard basis vectors of \( \mathbb{R}^n \) and \( \mathbb{R}^{n*} \) respectively. Define 

\[
E_i = S_{n-1}(e_i, 0), \quad F_i = S_{n-1}(0, f_i).
\]

Notice that \( E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( F_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). For \( i > 1 \) the matrices \( E_i, F_i \) are of the form \( \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} \) for some Suslin matrix \( X \) with \( XX^* = 0 \).

It is easy to check that the elements \( E_i, F_i \) satisfy the following elementary properties.

**Lemma 2.5.** Let \( X_k \in \{E_k, F_k\} \) for \( 1 \leq k \leq n \). Let \( i \neq 1 \). Then

a. \( X_i^2 = X_1 \) and \( X_1 + X_i = 1 \)

b. \( \overline{X_i} = -X_i \) and \( X_i^2 = 0 \).

c. \( X_iX_1 = \overline{X_1}X_i \).

**Theorem 2.6.** Let \( X_k \in \{E_k, F_k\} \) for \( 1 \leq k \leq n \). For any \( \lambda \in \mathbb{R} \) and \( 1 < i, j \leq n \), we have the following commutator relations:

\[
1 + \lambda X_iX_j = [1 + \lambda X_iX_1, 1 + X_1X_j]
\]

**Proof.** It follows from Lemma 2.5 that the inverse of \( 1 + \lambda X_iX_1 \) is \( 1 - \lambda X_iX_1 \).

Similarly the inverse of \( 1 + X_1X_j \) is \( 1 - X_1X_j \). Moreover, since \( X_i^2 = X_j^2 = 0 \) and \( X_iX_j + X_jX_i = \langle X_i, X_j \rangle = 0 \), any term where \( X_i \) or \( X_j \) appears twice is zero. Thus we are left with

\[
[1 + \lambda X_iX_1, 1 + X_1X_j] = 1 - \lambda X_1X_jX_iX_1 + \lambda X_iX_1X_j
\]

\[
= 1 + \lambda X_iX_j(X_1 + \overline{X_1})
\]

Since \( X_1 + \overline{X_1} = 1 \) we are done. \( \square \)

3. **The Elementary Spin Group**

As stated earlier, the Clifford algebra is a \( \mathbb{Z}_2 \)-graded algebra \( Cl = Cl_0 \oplus Cl_1 \). Under the isomorphism \( \phi : Cl \cong M_{2^n}(R) \), the elements of \( Cl_0 \) correspond to matrices of the form \( \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \).

The Spin group is defined as

\[
\text{Spin}_{2n}(R) := \{ x \in Cl_0 \mid xx^* = 1 \text{ and } xH(R^n)x^{-1} = H(R^n) \}.
\]

Just like we have the elementary group \( E_n(R) \) corresponding to \( SL_n(R) \), we have similar analogues for the orthogonal and Spin groups.

**Definition 3.1.**

a. Let \( e_{ij} \) denote the matrix with 1 in the \((i, j)\) position and zeroes everywhere else. For \( i \neq j \), define

\[
E_{ij}(\lambda) = 1 + \lambda e_{ij}
\]
The matrices $E_{ij}(\lambda)$ are called elementary matrices and the group generated by $n \times n$ elementary matrices is called the elementary group $E_n(R)$.

b. Let $\vartheta$ denote the permutation $(1\ n+1)\ldots(n\ 2n)$. We define for $1 \leq i \neq j \leq 2n$, $\lambda \in R$,

$$E_{ij}^\vartheta(\lambda) = I_{2n} + \lambda(e_{ij} - e_{\vartheta(j)\vartheta(i)}).$$

We call these the elementary orthogonal matrices and the group generated by them is called the elementary orthogonal group $EO_{2n}(R)$.

c. From the definition of the Spin group, we have the map $\pi : Spin_{2n}(R) \to O_{2n}(R)$ given by

$$\pi(g) : (v, w) \mapsto g \cdot (v, w) \cdot g^{-1} \text{ for } g \in Spin_{2n}(R).$$

We denote by $Epin_{2n}(R)$ the inverse image of $EO_{2n}(R)$ under $\pi$.

The group $Epin_{2n}(R)$ satisfies the following exact sequence (see [B2] p. 189)

$$1 \to \mu_2(R) \to Epin_{2n}(R) \to EO_{2n}(R) \to 1$$

where $\mu_2(R) = \{x \in R : x^2 = 1\}$.

Let $U_{2n-1}(R)$ denote the unit sphere in $H(R^n)$. The action of $O_{2n}(R)$ on $U_{2n-1}(R)$ induces an action of $Spin_{2n}(R)$ on $U_{2n-1}(R)$ (via $\pi$). Since $\pi : Epin_{2n}(R) \to EO_{2n}(R)$ is surjective, it follows that

$$\frac{U_{2n-1}(R)}{Epin_{2n}(R)} = \frac{U_{2n-1}(R)}{EO_{2n}(R)},$$

(2)

**Lemma 3.2.** There is a homomorphism $H : E_n(R) \to EO_{2n}(R)$ given by $\varepsilon \mapsto \left(\begin{array}{cc} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{array}\right) \in EO_{2n}(R)$.

**Proof.** The lemma follows from the observation that $H(E_{ij}(\lambda)) = E_{ij}^\varepsilon(\lambda)$. \qed

### 3.3. Generators of $Epin_{2n}(R)$.

Let $V = R^n$ with standard basis $e_1, \ldots, e_n$ and dual basis $f_1, \ldots, f_n$ for $V^*$. We will identify $H(V)$ with the corresponding matrices in the Clifford algebra. In terms of Suslin matrices,

$$e_i = \begin{bmatrix} 0 & s_{n-1}(e_i, 0) \\ s_{n-1}(e_i, 0) & 0 \end{bmatrix}, \quad f_i = \begin{bmatrix} 0 & s_{n-1}(0, f_i) \\ s_{n-1}(0, f_i) & 0 \end{bmatrix}$$

It can be proved (see [B2] Section 4.3) that $Epin_{2n}(R)$ is generated by elements of the form $1 + \lambda e_i e_j, 1 + \lambda e_i f_j, 1 + \lambda f_i f_j$ with $\lambda \in R$, $1 \leq i, j \leq n$, $i \neq j$.

Let $(x_k, \bar{x}_k) \in \{(e_k, \overline{e_k}), (f_k, \overline{f_k})\}$. Then the generator $1 + \lambda x_i x_j$ corresponds to the matrix

$$\phi(1 + \lambda x_i x_j) = \begin{bmatrix} 1 + \lambda \overline{x}_i \overline{x}_j & 0 \\ 0 & 1 + \lambda \overline{x}_i \overline{x}_j \end{bmatrix}.$$
Since $e_i, e_1$ are orthogonal we have $e_i e_1 = -e_1 e_i$. Similarly $f_i f_1 = -f_1 f_i$. By also taking into account the commutator relations in Theorem 2.6, we find that $E_{\text{pin}}(R)$ is generated by the (smaller) set of elements of the type

$$1 + \lambda e_1 e_i, \quad 1 + \lambda e_1 f_i, \quad 1 + \lambda f_1 e_i, \quad 1 + \lambda f_1 f_i. \quad (1 < i \leq n) \quad (3)$$

Since $X_i = -X_i$ for $i > 1$, these generators correspond to matrices of the form

$$\phi(1 + \lambda X_1 x_1) = \begin{bmatrix} 1 - \lambda X_1 x_1 & 0 \\ 0 & 1 - \lambda X_1 x_1 \end{bmatrix}.$$ 

3.4. The action of the Epin group.

So how do the generators of $E_{\text{pin}}(R)$ act on the quadratic space? Let $X_k \in \{X_k, T_k\}$ for $1 \leq k \leq n$.

Let $g = \begin{bmatrix} 1 - \lambda X_1 x_i & 0 \\ 0 & 1 - \lambda X_1 x_i \end{bmatrix}$ for some $i > 1$. Since $X_i = -X_i$ and $X_i X_i = X_i X_1$, we have

$$g \begin{bmatrix} 0 & S(v, w) \\ S((v, w) & 0 \end{bmatrix} g^{-1} = \begin{bmatrix} 0 & S(v', w') \\ S((v', w') & 0 \end{bmatrix},$$

where

$$S(v', w') = (1 - \lambda X_1 x_i) \cdot S(v, w) \cdot (1 + \lambda X_1 x_i) = (1 - \lambda X_1 x_i) \cdot S(v, w) \cdot (1 - \lambda X_1 x_i).$$

Recall that for $i > 1$ the matrices $E_i, F_i$ are of the form $\begin{bmatrix} a & \tau \\ -\tau & b \end{bmatrix}$ for some Suslin matrix $\mathcal{X}$ with $\mathcal{X} \mathcal{X} = 0$. There are two cases:

- $X_1 = E_1$: Then $1 - \lambda E_1 X_i$ and $1 - \lambda X_1 E_1$ will be equal to $\begin{bmatrix} 1 - \lambda X \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 - \lambda X \\ 0 \end{bmatrix}$ respectively.
- $X_1 = F_1$: Then $1 - \lambda F_1 X_i$ and $1 - \lambda X_1 F_1$ will be equal to $\begin{bmatrix} 1 - \lambda X \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 - \lambda X \\ 0 \end{bmatrix}$ respectively.

The next two lemmas calculate the action of the generators $g \in E_{\text{pin}}(R)$. They imply that $g(v, w)g^{-1} = (v \sigma, w \sigma^{-1})$ for some $\sigma \in E_n(R)$ (Theorem 4.1). This plays a crucial part in showing that there is a bijection between $E_{\text{pin}}(R)$ orbits of the unit sphere and $E_n(R)$ orbits of unimodular rows for any $n \geq 3$ (Theorem 4.4).

**Lemma 3.5.** Let $\mathcal{X}, \mathcal{F} \in M_2(R)$ be two Suslin matrices and $\mathcal{X} \mathcal{X} = 0$. Let $S = \begin{bmatrix} a & \tau \\ -\tau & b \end{bmatrix}$. Then

$$\begin{cases} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} = \begin{bmatrix} a - (X, T) + bX \\ -\tau - bX \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} S \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a \tau + aX \\ -\tau - aX \end{bmatrix}. \end{cases}$$
Proof. Let $y = x + T$. Since $yT = T y$, note that

$$\langle x, T \rangle = x T + T x = x T + T x = \langle x, T \rangle.$$ 

The proof follows by straightforward matrix multiplication. \( \square \)

**Lemma 3.6.** Let $x \in \{-\lambda \mathcal{E}_k, -\lambda \mathcal{F}_k\}$ where $1 \leq k \leq n$ and $\lambda \in \mathbb{R}$. Suppose $v \cdot w^T = 1$ for two vectors $v, w \in \mathbb{R}^{n+1}$ with $n \geq 2$. Then

$$(\begin{pmatrix} x_0 \cr 0 \cr 1 \end{pmatrix}) S(v, w) \begin{pmatrix} 1 \\ 0 \\ -x_1 \end{pmatrix} = S(v\varepsilon, w\varepsilon^T)$$

for some $\varepsilon, \sigma \in E_{n+1}(\mathbb{R})$.

Proof. Let $x = -\lambda \mathcal{E}_k$. The proof is similar in the other case. Write $v = (a_0, \cdots, a_n)$ and $w = (b_0, \cdots, b_n)$.

From Lemma 3.5, we have $$(\begin{pmatrix} x_0 \\ 0 \\ 1 \end{pmatrix}) S(v, w) \begin{pmatrix} 1 \\ 0 \\ -x_1 \end{pmatrix} = S(v', w'),$$

where

$$\begin{bmatrix} v' \\ w' \end{bmatrix} = \begin{bmatrix} a_0 + \lambda b_k, \cdots, a_k - \lambda b_0, \cdots, a_n \\ w \end{bmatrix}.$$ 

Since $v' \cdot w^T = v \cdot w^T = 1$, it follows from [S Corollary 2.7] that the matrices

$$\varepsilon = I_n + w^T(v' - v), \quad (\varepsilon^T)^{-1} = I_n - (v' - v)^T w$$

are in $E_n(\mathbb{R})$. We have $(v\varepsilon, w\varepsilon^T) = (v', w)$. (The hypothesis $n \geq 2$ is needed to use [S Corollary 2.7] where the length of the unimodular rows must be at least 3).

For the second part, we use Lemma 3.5 and get $$(\begin{pmatrix} -x_1 \\ 0 \\ 1 \end{pmatrix}) S(v, w) \begin{pmatrix} 1 \\ 0 \\ x_1 \end{pmatrix} = S(v'', w''),$$

where

$$\begin{bmatrix} v'' \\ w'' \end{bmatrix} = \begin{bmatrix} (a_0, \cdots, a_k - \lambda a_0, \cdots, a_n) \\ (b_0 + \lambda b_k, \cdots, b_n) \end{bmatrix}.$$ 

Clearly $(v'', w'') = (v\sigma, w\sigma^{-1})$ where $\sigma = E_{1,k+1}(-\lambda)$. \( \square \)

4. The Bijection between $\text{Epin}_{2n}(\mathbb{R})$ and $E_n(\mathbb{R})$ Orbits

We are now ready to prove the bijection between $E_n(\mathbb{R})$-orbits of unimodular rows and $\text{Epin}_{2n}(\mathbb{R})$-orbits on the unit sphere in $H(\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^{n*}$. We will break it down into simple parts with each part explaining one aspect of the bijection.

**Theorem 4.1.** Let $n \geq 3$. Let $g(v, w) = 1$ and $g \in \text{Epin}_{2n}(\mathbb{R})$. Then

$$g(v, w)g^{-1} = (v\sigma, w\sigma^{-1})$$

for some $\sigma \in E_n(\mathbb{R})$. 

Theorem 4.4. Let \( n \) \( v \)plies that Suppose for the two unimodular rows

\[
\text{Proof.} \quad \text{It is enough to prove the theorem for the generators of } \text{Epin}_2(R) \text{ (given in Section 3.3) and this was done in Section 3.4 and Lemma 3.6} \quad \Box
\]

Remark 4.2. There are two papers in the literature which prove some variation of the above theorem, though neither of them discuss Spin groups. The special case \( n = 3 \) was considered in the proof of [SV Corollary 7.4], and an alternate approach can be found in [JR1, Lemma 3.2]. Both the papers study different group structures and connect them to the elementary-group actions on unimodular rows. In Part B, we will analyze the case \( n = 3 \) and use it to explain the Vaserstein symbol.

Theorem 4.3. Let \( n \geq 3 \). If \( q(v, w_1) = q(v, w_2) = 1 \), then \((v, w_1)\) and \((v, w_2)\) are in the same \( EO_{2n}(R) \) and \( \text{Epin}_{2n}(R) \) orbits.

\[
\text{Proof.} \quad \text{By our hypothesis, we have } v \cdot w_1^\top = v \cdot w_2^\top = 1. \text{ Then it follows, from [S Corollary 2.7], that the matrix }
\]

\[
\varepsilon := I_n + v^\top(w_2 - w_1) \in E_n(R).
\]

Since \( w_1 \cdot \varepsilon = w_2 \), both \( w_1, w_2 \) lie in the same \( E_n(R) \) orbit.

By Lemma 3.2 we have \( H : \varepsilon^\top \to (\varepsilon^\top^{-1} 0 0 \varepsilon) \in EO_{2n}(R) \). Since \( \varepsilon^\top^{-1} = I_n - (w_2 - w_1)^\top v \), it is easy to check that

\[
w_1 \varepsilon = w_2,
\]

\[
v \varepsilon^\top^{-1} = v.
\]

Therefore \((v, w_1)\) and \((v, w_2)\) lie in same \( EO_{2n}(R) \) orbit, and so by Equation (2) in Section 3, they lie on the same \( \text{Epin}_{2n}(R) \) orbit. \( \Box \)

Let \( U_{2n-1}(R) \) be the unit sphere in \( H(R^n) \). By the above theorem, the map \( Um_n(R) \to \frac{U_{2n-1}(R)}{\text{Epin}_{2n}(R)} \) given by \( v \to (v, v) \) is well defined.

Theorem 4.4. Let \( (v_1, w_1), (v_2, w_2) \) be two points on the unit sphere \( U_{2n-1}(R) \), where \( n \geq 3 \). Then \( (v_1, w_1) \underset{E_n(R)}{\sim} (v_2, w_2) \) if and only if \( v_1 \underset{E_n(R)}{\sim} v_2 \).

In other words, there is a bijection between the sets (of orbits)

\[
\frac{Um_n(R)}{E_n(R)} \leftrightarrow \frac{U_{2n-1}(R)}{\text{Epin}_{2n}(R)} = \frac{U_{2n-1}(R)}{EO_{2n}(R)}.
\]

\[
\text{Proof.} \quad \text{Suppose for the two unimodular rows } v_1, v_2, \text{ we have } v_1 \cdot \varepsilon = v_2 \text{ for some } \varepsilon \in E_n(R). \text{ Then Theorem 4.3 implies that}
\]

\[
(v_1, w_1) \underset{\text{Epin}_{2n}(R)}{\sim} (v_1 \cdot \varepsilon, w_1 \cdot \varepsilon^\top^{-1}) \underset{\text{Epin}_{2n}(R)}{\sim} (v_2, w_2).
\]

On the other hand, suppose \( (v_1, w_1) \underset{\text{Epin}_{2n}(R)}{\sim} (v_2, w_2) \). Then Theorem 4.3 implies that \( v_1 \underset{E_n(R)}{\sim} v_2 \). Therefore we have a bijection

\[
\frac{Um_n(R)}{E_n(R)} \leftrightarrow \frac{U_{2n-1}(R)}{\text{Epin}_{2n}(R)} \quad \Box
\]
Corollary 4.5. Let \((v_1, w_1), (v_2, w_2)\) be two points on the unit sphere \(U_{2n-1}(R)\), where \(n \geq 3\). Then \((v_1, w_1) \sim_{EO_{2n}(R)} (v_2, w_2)\) if and only if \(v_1 \sim_{E_n(R)} v_2\).

The above bijection says that for any \(g \in E_{2n}(R)\) and a point \((v, w)\) on the unit sphere,
\[
g(v, w)g^{-1} = (v\sigma, w\sigma^\top^{-1})
\]
for some \(\sigma \in E_n(R)\). Here, the element \(\sigma \in E_n(R)\) may vary with the choice of \((v, w)\). It should be stressed that the above bijection does not imply that the groups \(E_n(R)\) and \(E_{2n}(R)\) are isomorphic. Only the corresponding orbit spaces are in bijection.

Part B. Interpreting the Vaserstein symbol using Spin groups

In this part, we will focus on the case \(n = 3\) and examine the Vaserstein symbol.

5. The Vaserstein symbol

A matrix \(A\) is said to be alternating if \(A^\top = -A\) and the diagonal elements of \(A\) are all zero. It is well known that there exists a polynomial \(pf\) called the Pfaffian (in the matrix elements with integral coefficients, depending only on the size of the matrix) such that \(\det(A) = (pf(A))^2\) for all alternating matrices \(A\) (see [1], Ch. XV, §9).

**Definition 5.1.** ([SV, p. 945]) The elementary symplectic-Witt group \(W_E(R)\) is an abelian group consisting of (equivalent classes of) alternating matrices with Pfaffian 1. For alternating matrices \(\alpha_r \in M_r(R)\) their sum is defined as
\[
\alpha_r \perp \alpha_s := \left( \begin{array}{cc} \alpha_r & 0 \\ 0 & \alpha_s \end{array} \right) \in M_{r+s}(R).
\]
The identity element is \(\psi_r = \psi_{r-1} \perp \psi_1\) where \(\psi_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)\). Two matrices \(\alpha_r, \alpha_s\) are said to be equivalent if \(\alpha_r \perp \psi_{s+l} = \varepsilon(\alpha_s \perp \psi_{r+l})\varepsilon^\top\), for some \(l \geq 0\) and \(\varepsilon \in E(R)\).

The Vaserstein symbol is a map \(\frac{Um_3(R)}{E_3(R)} \to W_E(R)\), giving a symplectic structure on orbits of unimodular rows. This is done by identifying a point on the unit sphere \((v, w) \in H(R^3)\) with a \(4 \times 4\) alternating matrix. Here we will use Suslin matrices which helps us to see the connection to Clifford algebras and Spin groups.

Let \(v = (a_1, a_2, a_3)\) and \(w = (b_1, b_2, b_3)\). Recall from Section 2.3 that
\[
S_2(v, w) = \left( \begin{array}{ccc} a_1 & 0 & a_2 \\ 0 & a_1 & -b_3 \\ -b_2 & a_3 & b_1 \end{array} \right), \quad J_2 = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right)
\]
and $JS^TJ^T = S$ (from Equation (1)). Since $J^{-1} = J^T = -J$, this can be rewritten as

$$(SJ)^T = -SJ.$$ 

We need one more matrix $\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. Note that $\beta^T = \beta^{-1}$.

Define

$$V(v, w) := \beta S_2(v, w) J_2 \beta^T = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}.$$ 

The matrix $V(v, w)$ is alternating with Pfaffian $v \cdot w^T = a_1b_1 + a_2b_2 + a_3b_3$. Thus when $v \cdot w^T = 1$, the matrix $V(v, w)$ represents an element of $W_E(R)$. It is the same matrix corresponding to the Vaserstein symbol (in [SV, §5]). In the next section we will break down the Vaserstein symbol into two parts and interpret it using Spin groups:

a. Let $A_4(R)$ denote the set of $4 \times 4$ alternating matrices with Pfaffian $1$. First, we will show that there is a bijection $\frac{Um_3(R)}{E_3(R)} \leftrightarrow \frac{A_4(R)}{E_4(R)}$. As we will see, this follows from the isomorphism $Epin_6 R \cong E_4(R)$ and then utilizing the results from Part A to get

$$\frac{Um_3(R)}{E_3(R)} \leftrightarrow \frac{U_5(R)}{Epin_6(R)} \leftrightarrow \frac{A_4(R)}{E_4(R)}.$$ 

b. Then the obvious inclusion map gives us $A_4(R) \rightarrow W_E(R)$, thus revealing the Witt-group structure on orbits of unimodular rows.

**Remark 5.2.** The Vaserstein symbol was introduced in [SV, §5] to study orbits of unimodular rows. Suslin and Vaserstein studied the injectivity and surjectivity of the Vaserstein symbol and proved that it is a bijection if $R$ is a commutative noetherian ring of Krull dimension two (see [SV, Corollary 7.4]). The recent paper [GRK] gives a survey of the non-injectivity of the Vaserstein symbol in dimension three.

Aravind Asok and Jean Fasel have provided an interpretation of the Vaserstein symbol using $\mathbb{A}^1$-homotopy theory. The paper [AP] explains this connection in detail (also see [AF2, Theorem 4.3.1]). Following [AP], let $k$ be a perfect field and $Q_5$ be the smooth affine quadric with $k[Q_5] = k[x_1, x_2, x_3, y_1, y_2, y_3]/(\sum x_iy_i - 1)$. Let $[X, Y]_{\mathbb{A}^1}$ denote the set of all morphisms from $X$ to $Y$ in the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ as defined by Morel-Voevodsky.

For any smooth affine $k$-scheme $X = Spec(R)$, there is a natural bijection $[X, Q_5]_{\mathbb{A}^1} = [X, \mathbb{A}^3 \setminus 0]_{\mathbb{A}^1} = Um_3(R)/E_3(R)$. Moreover $Q_5$ is isomorphic to the quotient of algebraic varieties $SL_4/Sp_4$ giving us the composite map $Q_5 \rightarrow SL_4/Sp_4 \rightarrow SL/Sp$. It turns out that the quotient $SL/Sp$ represents the (reduced) higher Grothendieck-Witt group $GW_{1,red}(X)$ which coincides
with $W_E(R)$ for any smooth affine variety $X = \text{Spec}(R)$. Thus one has the following interpretation of the Vaserstein symbol

$$Um_3(R)/E_3(R) = [X, Q_5]_{A^1} \to [X, SL/Sp]_{A^1} = W_E(R).$$

6. The dictionary between Vaserstein and Suslin matrices

We will borrow results from [CV1] on the connection between Suslin matrices and Clifford algebras. Specifically we need the well-known exceptional isomorphisms $\text{Spin}_6(R) \cong SL_4(R)$ and $\text{Epin}_6(R) \cong E_4(R)$. (For a proof using Suslin matrices, see [CV1, Theorems 7.1, 8.4]. For another proof, see [Kn, Ch. 5, §5.6]).

Define $*$ to be the involution on $M_4(R)$ given by $M^* = J M^T J$ where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ Note that $*$ is an involution because $J^T = -J = J^{-1}$.

Let’s identify the Suslin matrix $S(v, w)$ with the element $(v, w)$ in the quadratic space $H(R^3)$. Under the isomorphism $\psi : \text{Spin}_6(R) \cong SL_4(R)$, the Spin group behaves as follows: for $g \in SL_4(R)$, the action is given by $g \cdot S(v, w) = gS(v, w)g^*$. Simplifying the notation, we will sometimes write $S, V$ instead of $S(v, w), V(v, w)$.

Any $4 \times 4$ alternating matrix is of the form $V(v, w)$, corresponding to the element $(v, w) \in H(R^3)$. Let $A_4(R)$ denote the set of all such matrices with $v \cdot w^T = 1$ (the unit sphere in $H(R^3)$). The group $SL_4(R)$ acts on the matrices $V(v, w)$ as $(g, V) \to gV g^T$. Recall that the unit sphere in $H(R^3)$ is denoted by $U_5(R)$.

**Theorem 6.1.** We have the bijection $\frac{U_5(R)}{\text{Spin}_6(R)} \leftrightarrow \frac{A_4(R)}{SL_4(R)}$. Therefore,

$$\frac{Um_3(R)}{E_3(R)} \leftrightarrow \frac{U_5(R)}{EO_6(R)} = \frac{U_5(R)}{\text{Epin}_6(R)} \leftrightarrow \frac{A_4(R)}{E_4(R)}$$

where $v \to S(v, w) \to V(v, w)$ for any element $(v, w) \in U_5(R)$.

**Proof.** The bijection between the $E_3(R)$-orbits of unimodular rows and $\text{Epin}_6(R)$-orbits on the unit sphere follows from Theorem 4.4 in Part A. For the second part, note that $A_4(R)$ corresponds to the unit sphere in $H(R^3)$. We will now show that the group actions on $H(R^3)$ are the same. Remember that
\[ V = \beta S J^\top. \] Let \( S(v', w') = g \cdot S(v, w) \). Writing \( S = S(v, w) \), we have

\[
V(v', w') = \beta (g \cdot S) J^\top \\
= \beta (g S g^*) J^\top \\
= \beta (g S J g^T J^T) J^\top \\
= \beta g S J g^T \beta^T \\
= (\beta g \beta^T)(\beta S J \beta^T)(\beta g^T \beta^T) \\
= g'(\beta S J \beta^T)g^T \\
= g' V(v, w) g^T
\]

where \( g' = \beta g \beta^T = \beta g \beta^{-1} \). Since \( E_n(R) \) is a normal subgroup of \( GL_n(R) \) for \( n \geq 3 \) (see [S2, Corollary 1.4]), it follows that \( g' \in E_4(R) \) whenever \( g \in E_4(R) \). Therefore we have the bijection between the respective (elementary) orbit spaces

\[
\frac{U_5(R)}{E_{pin_6}(R)} \leftrightarrow \frac{A_4(R)}{E_4(R)}. \]

The above correspondence gives another proof of the following well-known exceptional isomorphism.

**Theorem 6.2.**

\[ \text{Spin}_5(R) \cong Sp_4(R). \]

**Proof.** The proof follows by identifying \( \text{Spin}_5(R) \) as a subgroup of \( \text{Spin}_6(R) \) which fixes \((v, w) = (1, 0, 0, 1, 0, 0)\). The elements of \( \text{Spin}_5(R) \) then correspond to matrices \( g \in SL_4(R) \) such that \( gg^* = 1 \). In other words, \( g J g^T = J \). Since \( J \) is (canonically) isometric to \( \psi_2 \), it follows that \( \text{Spin}_5(R) \cong Sp_4(R) \).

In light of Theorem 6.1, the Vaserstein symbol \( \frac{U_{m_3}(R)}{E_3(R)} \to W_E(R) \) can now be decomposed as

\[
V : \frac{U_{m_3}(R)}{E_3(R)} \cong \frac{A_4(R)}{E_4(R)} \to W_E(R).
\]

The injectivity (surjectivity) of the Vaserstein symbol boils down to the injectivity (surjectivity) of the map \( \frac{A_4(R)}{E_4(R)} \to W_E(R) \), which is defined naturally via the inclusion map. The interpretation in terms of Spin groups is summarized in the table below:
The Spin group interpretation

$X g X$ Suslin matrix $S_2(v, w)$, $(V = \beta S_2 J_2 \beta^\top)$

Action of $SL_4(R), E_4(R)$: 

$(g, V) \rightarrow g V g^\top$

Action of Spin$_6(R), \text{Epin}_6(R)$: 

$g \cdot S = g S g^\top$

$A_4(R)$ $U_5(R)$

Orbits of Unimodular rows

$\frac{U^m_3(R)}{E_3(R)} \cong \frac{A_4(R)}{E_4(R)}$

Orbits on the sphere $(v \cdot w^\top = 1)$

$\text{Spin}_6(R), \text{Epin}_6(R)$

$\text{Spin}_5(R)$

### 6.3. A question about $K\text{Spin}_1(R)$.

It turns out that if $\mathcal{X}, \mathcal{Y} \in M_{2k}(R)$ are two Suslin matrices, then $\mathcal{X} \mathcal{Y} \mathcal{X}$ is also a Suslin matrix and $\overline{\mathcal{X} \mathcal{Y} \mathcal{X}} = \mathcal{X} \mathcal{Y} \mathcal{X}$ (see [CV1], Theorem 3.4). Now suppose $\overline{\mathcal{X}} = 1$, and take $g = \begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{X} \end{pmatrix}$. Then $g^{-1} = \begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{X} \end{pmatrix}$ and

$$g \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & \mathcal{X} y \mathcal{X} \\ \mathcal{X} y \mathcal{X} & 0 \end{pmatrix}.$$  \hspace{1cm} (4)

Moreover, as $\overline{\mathcal{X}} = 1$, we have $\mathcal{X} \mathcal{Y} \cdot \overline{\mathcal{X} \mathcal{Y} \mathcal{X}} = \mathcal{Y} \mathcal{Y}$. Recall that an element $(v', w') \in H(R^n)$ corresponds to the matrix $\phi(v', w') = \begin{pmatrix} 0 & y' \\ y' & 0 \end{pmatrix}$, where $y' = S_{n-1}(v', w')$. Therefore it follows from Equation (4) that there is a map $U_{2n-1}(R) \rightarrow \text{Spin}_{2n}(R)$ given by $(v, w) \rightarrow \begin{pmatrix} S_{n-1}(v, w) & 0 \\ 0 & S_{n-1}(v, w) \end{pmatrix}$.

This induces a map $U_{2n-1}(R) \rightarrow \text{Spin}_{2n}(R) \rightarrow \text{KSpin}_1(R)$. What is the relation between $W_E(R)$ and the abelian group $K\text{Spin}_1(R)$?

### 6.4. Vaserstein composition.

The paper [SV] also introduced a composition law on unimodular rows ([SV], Theorem 5.2). The composition law was later generalized to $Um_n(R)$ by W. van der Kallen using what are called weak Mennicke symbols as follows (see [vdK2], Lemma 3.4)

Let $v_1 = (a_1, a_2, a_3, \ldots, a_n)$ and $v_2 = (c_1, c_2, a_3, \ldots, a_n)$ be two unimodular rows and choose $d_1, d_2$ such that the determinant of $\beta = \begin{pmatrix} -d_2 & c_2 \\ d_1 & -c_1 \end{pmatrix}$ has image 1 in $R/(a_3, \ldots, a_n)$. Then

$$\text{wms}(v_1) \text{wms}(v_2) = \text{wms}(p, q, a_3, \ldots, a_n)$$

where $(p, q) = (a_1, a_2) \beta$.

In Part C, we will introduce a new composition law on certain subspaces of $H(R^n)$ satisfying the same properties. (See Remark [S.4]). Moreover this law has the nice feature that it is expressed recursively using matrices. It turns out that this composition of unimodular rows is a special case of a more general law, which acts on certain subspaces of $A \oplus H(R^n)$ where $A$ is
a composition algebra. The Vaserstein composition corresponds to the case where the composition algebra is the algebra of split quaternions.

As an illustration of the results in Part C, we will now interpret Vaserstein’s composition rule using Suslin matrices for the case \( n = 3 \).

Fix \( a_1 \in R \). Let \( v = (a_1, a_2, a_3) \) and \( v' = (a_1', a_2', a_3') \) be two unimodular rows (with \( a_1 = a_1' \)). Suppose \( v \cdot w^\tau = v' \cdot w'^\tau = 1 \) for some \( w = (b_1, b_2, b_3) \) and \( w' = (b_1', b_2', b_3') \). The Suslin matrices corresponding to \((v, w)\) and \((v', w')\) are

\[
X = S(v, w) = \left( \begin{array}{cc} a_1 & \alpha \\ \frac{a_2}{-b_3} & b_1 \end{array} \right) \quad \text{and} \quad Y = S(v', w') = \left( \begin{array}{cc} a_1 & \beta \\ \frac{a_2'}{-b_3'} & b_1' \end{array} \right),
\]

where \( \alpha = \left( \begin{array}{cc} a_2 & a_3 \\ -b_3 & b_2 \end{array} \right) \) and \( \beta = \left( \begin{array}{cc} a_2' & a_3' \\ -b_3' & b_2' \end{array} \right) \).

Finally define the composition

\[
X \odot Y := \left( \begin{array}{cc} a_1 & \alpha \beta \\ \frac{-\alpha \beta}{a_1} & b_1 + b_1' - a_1 b_1 b_1' \end{array} \right).
\]

The element \( X \odot Y \) is also a Suslin matrix and \( q(X \odot Y) = q(X)q(Y) \). Notice that the product \( X \odot Y = S(v'', w'') \) is similar to the Vaserstein composition, as we get \( v'' = (a_1, (a_2, a_3)\beta) \).

Part C. A general Composition law

7. Preliminaries on Composition algebras

7.1. Notation. Let \( R \) be a commutative ring and \( V \) be a free \( R \)-module equipped with a quadratic form \( q \) and basis \( \{v_1, \cdots, v_n\} \). Let \( B = ((v_i, v_j)) \) be the matrix corresponding to the bilinear form \( \langle v, w \rangle = q(v+w) - q(v) - q(w) \), for \( v, w \in V \).

A quadratic form \( q \) is said to be non-degenerate if \( \langle x, v \rangle = 0 \) for all \( v \in V \) implies that \( x = 0 \). Writing \( x = (x_1, \cdots, x_n) \), this means that the equation \( xB = 0 \) has only the trivial solution, or equivalently, \( \det(B) \) is a non-zero divisor [MD, Corollary 1.30]. We say that \( q \) is non-singular if \( B \) is invertible.

One advantage with non-singular quadratic spaces \((V, q)\) is that the Clifford algebra \( Cl(V, q) \) will have the structure of a graded Azumaya algebra [HM, Corollary 3.7.5] and this can be a useful tool in proving results. In this paper, we are only assuming that the quadratic form is non-degenerate.

Remark 7.2. For general theory of quadratic forms and Clifford algebras over commutative rings, see [B2, HM, Kn]. The book [HM] uses a slightly different terminology by referring to \( q \) as weakly non-degenerate if \( \det(B) \) is a non-zero divisor, and non-degenerate if \( B \) is invertible.

7.3. Composition algebras. A composition algebra \((A, q)\) over \( R \) is a (not necessarily associative) \( R \)-algebra such that \( (A \times A, q) \) is a non-degenerate composition algebra.

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• $A$ has an identity element and contains a copy of $R$, i.e., $R \hookrightarrow A$ (mapping $1 \in R$ to $1 \in A$).

• $A$ is equipped with a non-degenerate quadratic form $q$ satisfying $q(xy) = q(x)q(y)$ for all $x, y \in A$.

For any composition algebra $(A, q)$, there is an involution $\alpha \rightarrow \overline{\alpha}$ such that $q(\alpha) = \alpha\overline{\alpha} = \overline{\alpha}\alpha = q(\overline{\alpha})$, for all $\alpha \in A$. The quadratic form on $A$ is sometimes referred to as the norm of $A$. It is known that $\text{rank}(A)$ has to be 1, 2, 4, or 8 (see [Kn, V. 7.1.6]).

Composition algebras of rank 4 and 8 are called *quaternions* and *octonions* respectively. Most of their applications arise when the base ring is a field. The new book [Vo] captures some of the wide-ranging connections of quaternions to different branches of Mathematics. Some applications of octonions can be found in [Ba, CS].

When the base ring is a field, a scalar multiple of the norm of the composition algebra $A$ determines $A$ up to isomorphism [SVe, §1.7]. For non-singular quadratic forms, this result has been extended to quaternion algebras over commutative rings by Knus-Ojanguren-Sridharan [KOS, Prop 4.4]. On the other hand, octonion algebras (over commutative rings) are not determined by their norm, and a counterexample has been provided by P. Gille [G].

8. Composition law

The following construction is inspired by the construction of Suslin matrices. Let $(A, q)$ be any composition algebra. Consider the quadratic space $A \oplus H(R)$, where $H(R)$ is a hyperbolic plane. For each element $(\alpha, a, b) \in A \oplus H(R)$, the quadratic form is given by

$$q(\alpha, a, b) = a\overline{\alpha} + ab.$$ \[8.1.1\]

One can represent $(\alpha, a, b)$ as a matrix $Z = \left( \begin{array}{cc} a & \alpha \\ -\overline{\alpha} & b \end{array} \right)$. Define $\overline{Z} = \left( \begin{array}{cc} b & -\alpha \\ \overline{\alpha} & a \end{array} \right)$. Then

$$q(Z) = ZZ = Z\overline{Z} = a\overline{\alpha} + ab.$$ \[8.1.2\]

For any such matrix, we will sometimes write $q_Z$ instead of $q(Z)$. One of the reasons we rewrite the elements as $2 \times 2$ matrices is that it is easier to express the composition law and generalize the analysis to $A \oplus H(R^n)$. In addition, as we shall see later, this matrix representation gives a simple description of the corresponding Clifford algebra.

8.1. Composition law for hyperplanes of $A \oplus H(R)$.

Let $X = \left( \begin{array}{cc} a & \alpha \\ -\overline{\alpha} & b \end{array} \right)$ and $Y = \left( \begin{array}{cc} a & \beta \\ -\overline{\beta} & b' \end{array} \right)$. When $q_X = q_Y = 1$, define

$$X \odot Y := \left( \begin{array}{cc} a & \alpha\beta \\ -\overline{\alpha}\beta & b + b' - ab' \end{array} \right).$$ \[8.1.3\]

Then $q(X \odot Y) = 1$. 

For general $X,Y$ define

$$X \odot Y := \begin{pmatrix} a & \alpha \beta \\ -\alpha \beta & b q Y + b' q X - a b b' \end{pmatrix}.$$  

From the equations

$$\alpha \bar{\alpha} = q X - a b, \quad \beta \bar{\beta} = q Y - a b',$$

it follows that

$$q(X \odot Y) = q(X) q(Y).$$  

When the underlying composition algebra is associative, the operation $\odot$ is also associative with the identity element $\left( \begin{array}{c} a \\ 1 \\ -1 \end{array} \right).$

If we take $a = b = b' = 0$, then $X \odot Y = \begin{pmatrix} 0 & \alpha \beta \\ -\alpha \beta & 0 \end{pmatrix}$ corresponds to the multiplication in the composition algebra. When $A \cong M_2(R)$ is the algebra of split quaternions, then the above composition law gives us the Vaserstein composition on unimodular rows stated in Part B (see Remark 8.4).

8.2. The quadratic space $A \oplus H(R^n)$.

Consider next the quadratic space $A \oplus H(R^n)$, where $H(R^n) = R^n \oplus R^{n*}$. For each element $(\alpha, v, w) \in A \oplus H(R^n)$, the quadratic form is given by

$$q(\alpha, v, w) = \alpha \bar{\alpha} + v \cdot w^t.$$  

Here $\alpha$ is an element of the composition algebra $A$ and $v, w \in R^n$. By fixing a basis of $R^n$, let us write $v = (a_1, \cdots, a_n)$ and $w = (b_1, \cdots, b_n)$.

Let $Z_1(\alpha, v, w) = \left( \begin{array}{c} a_1 \\ -a \end{array} \right)$ and $\overline{Z_1(\alpha, v, w)} = \left( \begin{array}{c} b_1 \\ -a \end{array} \right)$.

For $i > 1$, define recursively the matrices $Z_i(\alpha, v, w) := \left( \begin{array}{c} a_i \\ -\overline{Z_i-1(\alpha, v, w)} \end{array} \right)$ and $\overline{Z_i(\alpha, v, w)} := \left( \begin{array}{c} b_i \\ -\overline{Z_i-1(\alpha, v, w)} \end{array} \right)$.

Then $Z_i$ is a $2^i \times 2^i$ matrix and

$$q(Z_i) = Z_i \overline{Z_i} = \bar{Z_i} Z_i = \alpha \bar{\alpha} + a_1 b_1 + \cdots + a_i b_i.$$

8.3. Composition law for certain subspaces of $A \oplus H(R^n)$.

Fix $v = (a_1, \cdots, a_n) \in R^n$. Let $\alpha, \beta \in A, w = (b_1, \cdots, b_n)$ and $w' = (b'_1, \cdots, b'_n)$.

Write $X_i = Z_i(\alpha, v, w)$ and $Y_i = Z_i(\beta, v, w')$. By definition, we have

$$qX_i = a_i b_i + qX_{i-1} \quad \text{and} \quad qY_i = a_i b'_i + qY_{i-1}.$$  

Define the composition $X_i \odot Y_i$ recursively as

$$X_i \odot Y_i := \begin{pmatrix} a_i & X_{i-1} \odot Y_{i-1} \\ -X_{i-1} \odot Y_{i-1} & b_i q Y_i + b'_i q X_i - a_i b_i b'_i \end{pmatrix}.$$
By induction, it follows that
\[ qX_n \circ Y_n = qX_n qY_n. \]

**Remark 8.4.** As promised in Section 6.4, we will now interpret (van der Kallen’s) composition of unimodular rows (see [vdk2, Lemma 3.4]) using \( \circ \).

When \( A \cong M_2(R) \) is the algebra of split quaternions, the matrices \( Z(\alpha, v, w) \) are Suslin matrices. Let \( v_1 = (a_1, a_2, a_3, \cdots, a_n) \) and \( v_2 = (c_1, c_2, a_3, \cdots, a_n) \) be two unimodular rows such that \( v_i \cdot w_i^T = 1 \). Suppose \( S(v_1, w_1) \circ S(v_2, w_2) = S(v_3, w_3) \). Then
\[ v_3 = (p, q, a_3, \cdots, a_n) \]
where \( (p, q) = (a_1, a_2) \beta \). Here \( \beta = \left( \begin{array}{cc} c_1 & c_2 \\ -d_2 & d_1 \end{array} \right) \) where \( w_2 = (d_1, d_2, \cdots, d_n) \). Clearly the determinant of \( \beta \) has image 1 in \( R/\langle a_3, \cdots, a_n \rangle \) as \( v_2 \cdot w_2^T = 1 \).

**Remark 8.5.** Suppose \( R \) is a commutative Noetherian ring of stable dimension \( d \) with \( d \leq 2n-3 \). Let \( v_1, v_2 \in \text{Um}_n(R) \) such that \( v_i \cdot w_i^T = 1 \). Then the Mennicke-Newman lemma says that one can find \( \varepsilon_i \in \text{E}_n(R) \) such that \( v_2 \varepsilon_i = (x_1, a_2, \cdots, a_n) \) (for some \( x_1, a_k \in R \)). See [vdk3, Lemma 3.2].

Together with Theorem 4.4, this means that \( (v_1, w_1), (v_2, w_2) \) are \( \text{Epind}_n(R) \)-equivalent to the points \( (v_1 \varepsilon_i, w_2 \varepsilon_i^{-1}) \) which lie on the same \((n+1)\)-dimensional subspace (determined by the elements \( a_k \)) where the composition \( \circ \) can operate.

9. The Clifford algebra of \( A \oplus H(R^n) \): the quaternion case

Here we will consider the case when \( A \) is a quaternion algebra over \( R \). Let \( V = A \oplus H(R^n) \) and we will continue representing its elements \((\alpha, v, w)\) as a matrix \( Z_n(\alpha, v, w) \). Notice that \( Z_n(\alpha, v, w) \in M_{2n}(A) \).

Consider the map \( \phi : V \to M_{2n+1}(A) \) given by
\[ (\alpha, v, w) \mapsto \begin{bmatrix} 0 & Z_n(\alpha, v, w) \\ Z_n(\alpha, v, w) & 0 \end{bmatrix} \]
Since \( \phi(\alpha, v, w)^2 = q(\alpha, v, w) \), by the universal property of Clifford algebras the map lifts to an \( R \)-algebra homomorphism
\[ \phi : \text{Cl}(V) \to M_{2n+1}(A). \]
The map \( \phi \) is in fact a \( \mathbb{Z}_2 \)-graded homomorphism, where the even and odd elements of \( M_{2n+1}(A) \) correspond to matrices of the form \( \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \) and \( \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \).

**Theorem 9.1.** The map \( \phi : \text{Cl}(V) \to M_{2n+1}(A) \) is injective. Moreover, \( \text{rank}[\text{Cl}(V)] = \text{rank}[M_{2n+1}(A)] \).

**Proof.** Let \( \ker(\phi) \) denote the kernel of \( \phi \). Since \( \phi \) restricts to an injective map on \( V \) and \( R \hookrightarrow A \), we have \( \ker(\phi) \cap R = \{0\} \). Then it follows from [CV2, Theorem 2.7] that \( \phi \) is injective.

Since \( M_{2n+1}(A) = M_{2n+1}(R) \otimes A \), one can see that its rank is \( 2^{2n+4} \), the same as \( \text{rank}(\text{Cl}) = 2^{\text{rank}(V)}. \)
Remark 9.2. If the quadratic space $(A, q_A)$ (and thus $(V, q)$) is non-singular, then both $M_{2n+1}(A)$ and the Clifford algebra $Cl(V)$ are graded Azumaya algebras of the same rank (see [HM, Theorem 3.6.8, Corollary 3.7.5]). In that case, $\phi : Cl(V) \to M_{2n+1}(A)$ will be an isomorphism (see [HM, Lemma 6.7.10]).

10. Clifford algebra: the octonion case

10.1. The embedding in the endomorphism ring. Let $O$ be an octonion algebra over $R$. The problem here is that the matrix algebra $M_{2n+1}(O)$ is not associative anymore. However the octonion algebra $O$ has the interesting property that $\alpha(\alpha\beta) = \alpha(\beta\alpha) = q(\alpha)\beta$ for all $\alpha, \beta \in O$. (See [Kn, Ch. V, §7]).

Putting it another way, consider the left multiplication map $L : O \to \text{End}(O)$ where $L_\alpha$ is left-multiplication by $\alpha$. Since $q(\alpha) = q(\alpha)$, these maps satisfy the property that $L_\alpha L_\beta = L_\beta L_\alpha = L_{q(\alpha)}$. We will modify the matrices $Z_i(\alpha, v, w)$ by replacing $\alpha$ with $L_\alpha$ in the matrix.

Define $Z'_1(\alpha, v, w) = \begin{pmatrix} a_1 & L_\alpha \\ -L_\alpha & b_1 \end{pmatrix}$ and $Z'_1(\alpha, v, w) = \begin{pmatrix} b_1 & -L_\alpha \\ L_\alpha & a_1 \end{pmatrix}$.

For $i > 1$, define recursively the matrices $Z'_i(\alpha, v, w) := \begin{pmatrix} a_i & Z'_{i-1} \\ -Z'_{i-1} & b_i \end{pmatrix}$ and $Z'_i(\alpha, v, w) := \begin{pmatrix} b_i & Z'_{i-1} \\ Z'_{i-1} & a_i \end{pmatrix}$.

10.2. The Clifford algebra. We have the map $\phi : Cl(V) \to M_{2n+1}(\text{End}(O))$ given by

$$(\alpha, v, w) \mapsto \begin{bmatrix} 0 & Z'_n(\alpha, v, w) \\ Z'_n(\alpha, v, w) & 0 \end{bmatrix}$$

Theorem 10.3. The map $\phi : Cl(V) \to M_{2n+1}(\text{End}(O))$ is injective. Moreover, $\text{rank}(Cl(V)) = \text{rank}(M_{2n+1}(\text{End}(O)))$.

Proof. The proof is similar to Theorem 9.1 Since $\phi$ restricts to an injective map on $V$ and $R \hookrightarrow A$, we have $\ker(\phi) \cap R = \{0\}$. Then it follows from [CV2, Theorem 2.7] that $\phi$ is injective.

Note that $\text{rank}(\text{End}(O)) = 64$ because $\text{rank}(O) = 8$. Since $M_{2n+1}(\text{End}(O)) = M_{2n+1}(R) \otimes \text{End}(O)$, one can see that its rank is $2^{2n+8}$ which is the same as $\text{rank}(Cl) = 2^{\text{rank}(V)}$.

The algebra $M_{2n+1}(\text{End}(O))$ has a $\mathbb{Z}_2$-grading where the even and odd elements are matrices of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. The homomorphism $\phi$ is therefore a $\mathbb{Z}_2$-graded homomorphism. Here too, if we restrict ourselves to non-singular quadratic forms, then the map $\phi$ (being a graded-homomorphism between graded Azumaya algebras of the same rank) is an isomorphism [HM, Lemma 6.7.10]. The paper [CV2] analyzes such embeddings for general quadratic spaces, in particular describing the structure of the Clifford algebra and Spin groups.
Composition in $H(R^5)$ using Octonion multiplication. Let $v = (a, v_1)$ and $w = (b, w_1)$, where $(v_1, w_1) \in H(R^4)$. Let us identify elements of $H(R^4)$ with the elements of the split octonion algebra - write $O_i = (v_i, w_i)$ with $q(O_i) = O_i O_i = v_i \cdot w_i^\top$. Let $X = \left( \frac{a}{-O_1} O_1 b \right)$ and $Y = \left( \frac{a}{-O_2} O_2 b' \right)$. When $q_X = q_Y = 1$, we have
\[
X \odot Y = \left( \begin{array}{c}
\frac{a}{-O_1 O_2} O_1 O_2 \\
\frac{a}{b+b'-abb'}
\end{array} \right)
\]
The product $X \odot Y$ corresponds to another pair $(v', w') \in H(R^5)$ and $q(X \odot Y) = v' \cdot w'^\top = 1$. This composition is obviously non-associative.

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