NODAL COUNT OF GRAPH EIGENFUNCTIONS VIA MAGNETIC PERTURBATION

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Abstract. We establish a connection between the stability of an eigenvalue under a magnetic perturbation and the number of zeros of the corresponding eigenfunction. Namely, we consider an eigenfunction of discrete Laplacian on a graph and count the number of edges where the eigenfunction changes sign (has a “zero”). It is known that the $n$-th eigenfunction has $n-1+s$ such zeros, where the “nodal surplus” $s$ is an integer between 0 and the number of cycles on the graph.

We then perturb the Laplacian by a weak magnetic field and view the $n$-th eigenvalue as a function of the perturbation. It is shown that this function has a critical point at the zero field and that the Morse index of the critical point is equal to the nodal surplus $s$ of the $n$-th eigenfunction of the unperturbed graph.

1. Introduction

Studying zeros of eigenfunctions is a question with rich history. While experimental observations have been mentioned by Leonardo da Vinci [35], Galileo [23] and Hooke [9], and greatly systematized by Chladni [14], the first mathematical result is probably due to Sturm [44]. The Oscillation Theorem of Sturm states that the number of internal zeros of the $n$-th eigenfunction of a Sturm-Liouville operator on an interval is equal to $n-1$. Equivalently, the zero of the $n$-th eigenfunction divide the interval into $n$ parts. In higher dimensions, the latter equality becomes a one-sided inequality: Courant [17, 18] proved that the zero curves (surfaces) of the $n$-th eigenfunction of the Laplacian divide the domain into at most $n$ parts (called the “nodal domains”).

Recently, there has been a resurgence of interest in counting the nodal domains of eigenfunctions, with many exciting conjectures and rigorous results. The nodal count seems to have universal features [12, 13, 37], is conjectured to resolve isospectrality [24], and has connections to minimal partitions of the domain [27, 7], to name but a few. For a selection of research articles and historical reviews, see the dedicated volume [42].

On graphs, the question can be formulated regarding the signs of the eigenfunctions of the operator

$$H : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}, \quad H = Q - C,$$

where $V$ is the set of the vertices of the graph, $Q$ is an arbitrary real diagonal matrix and $C$ is the adjacency matrix of the graph. On a graph, by a “zero” we understand an edge on which the eigenfunction changes sign, and not the exceptional (with respect to perturbation of $Q$) situation of an eigenfunction having a zero entry.

The subject of sign changes and nodal domains (connected components of the graph left after cutting the above edges) was addressed in, among other sources, Fiedler [22] who showed the analogue of Sturm equality for tree graphs (see also [10]), Davies, Gladwell, Leydold and Stadler [19], who proved an analogue of Courant (upper) bound for the number of nodal domains, Berkolaiko [6], who proved a lower bound for graphs with cycles and Oren [38], who found a bound for the nodal domains in terms of the chromatic number of the graph. A number of predictions regarding the nodal count in regular graphs (assuming an adaptation of the random wave model) is put forward in [20]. For more information, the interested reader is referred to the book [11] and the review [3].
The study of magnetic Schrödinger operator on graphs has a similarly rich history. To give a sample, Harper [25] used the tight-binding model (discrete Laplacian) to describe the effect of the magnetic field on conduction (see also [30]). In mathematical literature, discrete magnetic Schrödinger operator was introduced by Lieb and Loss [34] and Sunada [45, 46], and studied, among other sources, in [41, 15, 16] (see also [47] for a review).

In this paper we present a surprising connection between the two topics, namely the number of sign changes of \(n\)-th eigenfunction and the behavior of the eigenvalue \(\lambda_n\) under the perturbation of the operator \(H\) by a magnetic field. To make a precise statement, we need to introduce some notation.

Consider a generic eigenfunction on the graph, that is an eigenfunction that corresponds to a simple eigenvalue and is nonzero at the vertices. We denote by \(\phi_n\) the number of sign changes (also called sign flips, hence the notation \(\phi\)) which are defined as the edges of the graph at whose endpoints the eigenfunction has different signs. Here \(n\) is the number of the eigenfunction in the sequence ordered according to increasing eigenvalue. The combined results of [22, 6, 8] bound the number \(\phi_n\) by

\[
    n - 1 \leq \phi_n \leq n - 1 + \beta,
\]

where \(\beta := |V| - |E| + 1\) is the first Betti number (number of independent cycles) of the graph. Here and throughout the manuscript we assume that the graph is connected. We will call the quantity

\[
    \sigma_n = \phi_n - (n - 1), \quad 0 \leq \sigma_n \leq \beta
\]

the nodal surplus. This is the extra number of sign changes that an eigenfunction has due to the graph’s non-trivial topology.

Magnetic field on discrete graphs has been introduced in, among other sources, [34, 46, 15]. Up to unitary equivalence, it can be specified using \(\beta\) phases \(\vec{\alpha} = (\alpha_j)_{j=1}^{\beta} \in (-\pi, \pi]^{\beta}\). We consider the eigenvalues of the graph as functions of the parameters \(\vec{\alpha}\). The zero phases, \(\vec{\alpha} = 0\), correspond to the graph \(\Gamma\) without the magnetic field. We are now ready to formulate our main result.

**Theorem 1.1.** The point \(\vec{\alpha} = 0\) is the critical point of the function \(\lambda_n(\vec{\alpha})\). If \(\lambda_n(0)\), the \(n\)-th eigenvalue of the non-magnetic operator on \(\Gamma\), is simple and the corresponding eigenfunction has no zero entries, its nodal surplus \(\sigma_n\) is equal to the Morse index — the number of negative eigenvalues of the Hessian — of \(\lambda_n(\vec{\alpha})\) at the critical point \(\vec{\alpha} = 0\).

An immediate consequence of this theorem is the following.

**Corollary 1.2.** The non-degenerate \(n\)-the eigenvalue of the discrete Schrödinger operator is stable with respect to magnetic perturbation of the operator if and only if the corresponding eigenfunction has exactly \(n - 1\) sign changes.

By “non-degenerate” we understand a simple eigenvalue whose eigenfunction does not vanish on vertices and by “stability” we mean that the eigenvalue has a local minimum at zero magnetic field.

Other possible consequences of our result and links to several other questions are discussed in Section 6. The rest of the paper is structured as follows. In Section 2 we provide detailed definitions. Section 3 is devoted to a duality between the magnetic perturbation and a certain perturbation to the potential, coupled with removal of edges. This leads to an alternative proof of the result in the case \(\beta = 1\) (subsection 3.3) which, although unnecessary for the general proof, provides us with some important insights. Section 4 collects the tools necessary for the proof of Theorem 1.1 while Section 5 contains the proof itself.

\(^1\)This is the generic situation with respect to the perturbation of the potential \(Q\).
2. Magnetic Hamiltonian on discrete graphs

Let $\Gamma = (V, E)$ be a simple finite graph with the vertex set $V$ and the edge set $E$. We define the Schrödinger operator with the potential $q : V \to \mathbb{R}$ by

$$H : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}, \quad (H\psi)_u = -\sum_{v \sim u} \psi_v + q_u \psi_u,$$

that is the matrix $H$ is

$$H = Q - C,$$

where $Q$ is the diagonal matrix of site potentials $q_u$ and $C$ is the adjacency matrix of the graph. It is perhaps more usual (and physically motivated) to represent the Hamiltonian as $H = Q + L$, where the Laplacian $L$ is given by $L = D - C$ with $D$ being the diagonal matrix of vertex degrees. But since we will not be imposing any restrictions on the potential $Q$, we absorb the matrix $D$ into $Q$.

The operator $H$ has $|V|$ eigenvalues, which we number in increasing order, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|V|}$.

We define the magnetic Hamiltonian (magnetic Schrödinger operator) on discrete graphs as

$$(H\psi)_u = -\sum_{v \sim u} e^{iA_{u,v}} \psi_v + q_u \psi_u,$$

with the convention that $A_{u,v} = -A_{v,u}$, which makes $H$ self-adjoint. For further details, the reader should consult [34, 46, 15, 16].

A sequence of directed edges $C = [u_1, u_2, \ldots, u_n]$ is called a cycle if the terminus of edge $u_j$ coincides with the origin of the edge $u_{j+1}$ for all $j$ ($u_{n+1}$ is understood as $u_1$). The flux through the cycle $C$ by

$$\Phi_C = A_{u_1,u_2} + \ldots + A_{u_{n-1},u_n} + A_{u_n,u_1} \mod 2\pi$$

Two operators which have the same flux through every cycle $C$ are unitarily equivalent (by a gauge transformation). Therefore, the effect of the magnetic field on the spectrum is fully determined by $\beta$ fluxes through a chosen set of basis cycles of the cycle space. We denote them by $\alpha_1, \ldots, \alpha_\beta$ and consider the $n$-th eigenvalue of the graph as a function of $\vec{\alpha}$.

More precisely, fix an arbitrary spanning tree of the graph and let $S$ be the set of edges that do not belong to the chosen tree. Obviously, $S$ contains exactly $\beta$ edges.

**Lemma 2.1.** Any magnetic Schrödinger operator on the graph $\Gamma$ is unitarily equivalent to one of the operators of the type

$$(H_{u,v}) = \begin{cases} V_u, & u = v, \\
-1, & (u, v) \in E \setminus S, \\
-e^{\pm i\alpha_s}, & (u, v) = s \in S, 
\end{cases}$$

where the sign of the phase is plus if $u < v$ and minus if $u > v$.

**Example 2.2.** Consider the triangle graph — a graph with three vertices and tree edges connecting them. One of the equivalent forms of the magnetic Hamiltonian for this graph is

$$H(\Gamma_{\text{mag}}^\alpha) = \begin{pmatrix} q_1 & -e^{i\alpha} & -1 \\
-e^{-i\alpha} & q_2 & -1 \\
-1 & -1 & q_3 \end{pmatrix}.$$

The spectrum of $H(\Gamma_{\text{mag}}^\alpha)$ as a function of $\alpha \in (-\pi, \pi]$ is shown in Fig. 2.
3. A DUALITY BETWEEN A MAGNETIC PHASE AND A CUT

In this section we explore a simple result which shows a connection between two types of perturbations of the operator $H$ that will be used to prove the main theorem. It illustrates the duality between the perturbation of a discrete Schrödinger operator by a magnetic phase on a cycle and the operation of removing (“cutting”) an edge that lies on the cycle. The latter operation was used to prove the lower bound on the number of nodal domains in [6] and to study partitions on discrete graphs in [8].

3.1. Tools used. The result of this section (Theorem 3.3 below) is based on the following version of the Weyl’s inequality of linear algebra that can be obtained using the variational characterization of the eigenvalues (see [31, Chap. 4] for similar results).

**Theorem 3.1.** Let $A$ be a self-adjoint matrix and $B$ be a rank-one positive semidefinite self-adjoint matrix. Then

\[ \lambda_n(A - B) \leq \lambda_n(A) \leq \lambda_{n+1}(A - B), \]

where $\lambda_n$ is the $n$-th eigenvalue, numbered in increasing order, of the corresponding matrix. Moreover, the inequalities are strict if and only if $\lambda_n(A)$ is simple and its eigenvector is not in the null-space of $B$. 

Figure 1. The eigenvalues of the triangle graph as functions of a magnetic phase $\alpha$ (bold lines) and the eigenvalues of the unperturbed graph (horizontal lines).
Similarly, when $B$ is negative-definite, we have
\[
\lambda_{n-1}(A - B) \leq \lambda_n(A) \leq \lambda_n(A - B),
\]
with an analogous condition for strict inequalities.

Another useful result is the first term in the perturbation expansion of a parameter-dependent eigenvalue. Let $A(x)$ be a Hermitian matrix-valued analytic function of $x$. Let $\lambda(x)$ be an eigenvalue of the matrix $A$ that is simple in a neighborhood of a point $x_0$. We know from standard perturbation theory \cite{33} that $\lambda(x)$ is an analytic function. Denote by $u(x)$ the normalized eigenvector of the corresponding to the eigenvalue $\lambda$ and by $v$ all other normalized eigenvectors (in a slight abuse of notation). Then we have the following formula for the derivative of $\lambda$ evaluated at the point $x = x_0$.

\[
\frac{\partial}{\partial x} \lambda = \left\langle u, \frac{\partial A}{\partial x} u \right\rangle.
\]

3.2. Two operations on a graph. Let $\lambda_n$ be a simple eigenvalue and the corresponding eigenfunction $f$ be non-zero on vertices. Let $(u_1, u_2)$ be an edge that belongs to one of cycles of the graph. We allow the graph to have magnetic phases on some edges, but assume that there is no phase on the edge $(u_1, u_2)$. Then the operator $H = Q - C$ has the following subblock corresponding to vertices $u_1$ and $u_2$,

\[
H(\Gamma)[u_1, u_2] = \left( \begin{array}{cc} q_{u_1} - 1 & -1 \\ -1 & q_{u_2} \end{array} \right).
\]

We consider two modifications of the original graph. The first modification of the graph is a cut: we remove the edge $(u_1, u_2)$ and change the potential at sites $u_1$ and $u_2$. Namely, we change the $[u_1, u_2]$ subblock to
\[
H(\Gamma_{\text{cut}})[u_1, u_2] = \left( \begin{array}{cc} q_{u_1} - \gamma & 0 \\ 0 & q_{u_2} - 1/\gamma \end{array} \right),
\]
and leave the rest of the matrix $H$ intact. We denote this modification by $H(\Gamma_{\text{cut}})$. Note that this modification is a rank-one perturbation of the original operator $H(\Gamma)$. Let $B$ be the perturbation such that $H(\Gamma_{\text{cut}}) = H(\Gamma) - B^c$. Namely, the matrix $B^c$ has the $[u_1, u_2]$ subblock
\[
B^c_{[u_1, u_2]} = \left( \begin{array}{cc} \gamma & -1 \\ -1 & 1/\gamma \end{array} \right),
\]
and the rest of the elements are zero. Then $B^c$ is positive-definite if $\gamma > 0$ and negative-definite if $\gamma < 0$. Note that the cases $\gamma = \infty$ and $\gamma = 0$ can also be given meaning of removing (or imposing the Dirichlet condition at) the vertex $u_1$ or the vertex $u_2$ correspondingly. However, we will not dwell on this issue and exclude these cases from our consideration.

Notably, if $f$ is an eigenfunction of $H(\Gamma)$ and $\gamma = f_{u_2}/f_{u_1} \in \mathbb{R}$, then $f$ is also an eigenfunction of $H(\Gamma_{\text{cut}})$. Equivalently, $f$ is in the null-space of the perturbation $B^c$.

The second modification of the original graph is the introduction of a magnetic phase on the edge $(u_1, u_2)$. The $[u_1, u_2]$ subblock of the new operator $H(\Gamma_{\text{mag}})$ is
\[
H(\Gamma_{\text{mag}})[u_1, u_2] = \left( \begin{array}{cc} q_{u_1} - e^{-i\alpha} & -e^{i\alpha} \\ -e^{-i\alpha} & q_{u_2} \end{array} \right),
\]
while other entries coincide with those of $H(\Gamma)$. Note that $H(\Gamma_{\text{mag}})$ is not a rank-one perturbation of $H(\Gamma)$. However, it is a rank-one perturbation of the cut graph $H(\Gamma_{\text{cut}})$ for any values of $\alpha$ and $\gamma$. 

Namely, \( H(\Gamma_{\gamma}^\text{cut}) = H(\Gamma_{\text{mag}}^\alpha) - B_{mc}^{\alpha} \), where

\[
B_{mc}^{\alpha}_{[u_1,u_2]} = \begin{pmatrix}
\gamma & -e^{i\alpha_j} \\
-e^{-i\alpha_j} & 1/\gamma
\end{pmatrix},
\]

and all other entries of \( B_{mc}^{\alpha} \) are zero. Also, the spectrum of \( H(\Gamma_{\text{mag}}^\alpha) \) and \( H(\Gamma) \) coincide when \( \alpha = 0 \) since the operators coincide.

3.3. A duality between the two operations. We now want to apply Theorem 3.1 to the spectra of \( \Gamma, \Gamma_{\gamma}^\text{cut} \) and \( \Gamma_{\text{mag}}^\alpha \). However, we must take care to distinguish the two cases that correspond to equations (9) and (10) (\( \gamma > 0 \) and \( \gamma < 0 \) correspondingly).

Definition 3.2. The eigenvalues of \( \Gamma, \Gamma_{\gamma}^\text{cut} \) and \( \Gamma_{\text{mag}}^\alpha \) will be numbered in increasing order starting from 1. When we happen to index “nonexistent” eigenvalues we use the following convention:

\[
\lambda_j(\Gamma) = \begin{cases} 
-\infty, & j < 1, \\
\infty, & j > n.
\end{cases}
\]

Theorem 3.3. Let \( p(\gamma) \) be 1 if \( \gamma < 0 \) and 0 otherwise. Then the following inequalities hold

\[
\lambda_{n-p(\gamma)}(\Gamma_{\gamma}^\text{cut}) \leq \lambda_n(\Gamma_{\text{mag}}^\alpha) \leq \lambda_{n-p(\gamma)+1}(\Gamma_{\gamma}^\text{cut}),
\]

for all values of \( \alpha \) and \( \gamma \). Furthermore, for any fixed \( n \)

\[
\max_\gamma \lambda_{n-p(\gamma)}(\Gamma_{\gamma}^\text{cut}) = \min_\alpha \lambda_n(\Gamma_{\text{mag}}^\alpha)
\]

and

\[
\max_\alpha \lambda_n(\Gamma_{\text{mag}}^\alpha) = \min_\gamma \lambda_{n-p(\gamma)+1}(\Gamma_{\gamma}^\text{cut}).
\]

Finally, if there are no magnetic phases on the graph \( \Gamma \) (i.e. all entries of \( H(\Gamma) \) are real), then one of the extrema (18) or (19) is equal to \( \lambda_n(\Gamma) = \lambda_n(\Gamma_{\text{mag}}^{\alpha=0}) \), while the other is equal to \( \lambda_n(\hat\Gamma) := \lambda_n(\Gamma_{\text{mag}}^{\alpha=\pi}) \).

Remark 3.4. Note that at this point we don’t know which extremum, (18) or (19), is equal to \( \lambda_n(\Gamma) \). This information is related to the nodal surplus. We have also defined yet another modification of the graph \( \Gamma \), the graph \( \hat\Gamma \) whose adjacency matrix has \(-1\) in place of \(1\) for the entries \( C_{u_1,u_2} \) and \( C_{u_2,u_1} \).

Remark 3.5. Let \( \hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \) be the extended real line and \( \hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty = \infty\} \) be its projective (“wrapped”) version. The eigenvalue \( \lambda_{n-p(\gamma)}(\Gamma_{\gamma}^\text{cut}) \) is then a continuous function of \( \gamma \), considered as a function from \( \hat{\mathbb{R}} \) to \( \hat{\mathbb{R}} \). See Figure 2 for an example. Note that according to our definitions, \( \lambda_{n-p(\gamma)}(\Gamma_{\gamma}^\text{cut}) = -\infty \) for \( n = 1 \) and \( \gamma < 0 \).

Proof of Theorem 3.3 The inequalities follow directly from Theorem 3.1 since for any \( \alpha \) the graph \( \Gamma_{\text{mag}}^\alpha \) is a rank-one perturbation of \( \Gamma_{\gamma}^\text{cut} \). Whether it is positive- or negative-definite depends on the sign of \( \gamma \), and results in the shift by \( p \).

The properties of the extrema we can get as follows. First of all, observe that if \( \max \lambda_{n-p}(\Gamma_{\gamma}^\text{cut}) = \min \lambda_{n-p+1}(\Gamma_{\gamma}^\text{cut}) \), then \( \lambda_n(\Gamma_{\text{mag}}^\alpha) \) is constant and equal to the common value of \( \lambda_{n-p}(\Gamma_{\gamma}^\text{cut}) \) and \( \lambda_{n-p+1}(\Gamma_{\gamma}^\text{cut}) \).

Let now \( \max \lambda_{n-p}(\Gamma_{\gamma}^\text{cut}) < \min \lambda_{n-p+1}(\Gamma_{\gamma}^\text{cut}) \). The eigenvalues of a one-parameter family can always be represented as a set of analytic functions (that can intersect). Let \( \lambda'(\Gamma_{\gamma}^\text{cut}) \) be the analytic function
that achieves the maximum $\max\lambda_{n-p}(\Gamma_{\text{cut}})$ and $f$ be the corresponding eigenfunction. We will differentiate $\lambda'(\Gamma_{\text{cut}})$ using equation (11). At the maximum point $\gamma = \tilde{\gamma}$ we have (see equation (13)),

$$0 = \frac{d\lambda'}{d\gamma} = \left\langle f, \frac{dB}{d\gamma} f \right\rangle = -|f_{u_1}|^2 + |f_{u_2}|^2/\tilde{\gamma}^2.$$  

From here it follows that

$$\tilde{\gamma} = \pm \frac{|f_{u_2}|}{|f_{u_1}|} \quad \text{or, equivalently,} \quad |\tilde{\gamma} f_{u_1}/f_{u_2}| = 1.$$  

Let $\tilde{\alpha}$ be the solution of $e^{i\alpha} = \tilde{\gamma} f_{u_1}/f_{u_2}$. Direct calculation shows that the eigenfunction $f$ is in the null-space of the perturbation $B^{mc}$ of (16) with $\alpha = \tilde{\alpha}$ and therefore $f$ is both in the spectrum of $\Gamma_{\gamma}$ and in the spectrum of $\Gamma_{\text{mag}}$, implying equation (18) follows. Proof of equation (19) is completely analogous.

Note that we could instead differentiate the eigenvalue of $\Gamma_{\text{mag}}$, leading to the condition

$$f_{u_2} f_{u_1} e^{i\alpha} \in \mathbb{R},$$
instead of equation (20). One then sets $\bar{\gamma} = e^{i\alpha} f_{u_2}/f_{u_1} \in \mathbb{R}$ to the same effect.

Finally, when the matrix $H(\Gamma)$ is real, the eigenfunctions of $\Gamma_{\gamma}^{\text{cut}}$, $\Gamma_{\text{mag}}^{\alpha=0}$ and $\Gamma_{\text{mag}}^{\alpha=\pi}$ are real-valued. When $\alpha = 0$ we can verify directly that the eigenfunction $f$ of $\Gamma_{\gamma}^{\text{cut}}$ is also an eigenfunction of $\Gamma_{\bar{\gamma}}^{\text{cut}}$ by setting $\bar{\gamma} = f_{u_2}/f_{u_1}$. When $\alpha = \pi$, we also set $\gamma = f_{u_2}/f_{u_1}$ and do the same.

Theorem 3.3 highlight a sort of duality between the two modifications of the graph $\Gamma$. The spectra of the graphs with a magnetic phase form bands (as the phase is varied) while the spectra of the graphs with the cut fill the gaps between this bands. Minimums of one correspond to maximums of the other and in half of the cases correspond to eigenvalues of the original graph.

We now explain how the $\beta = 1$ case of Theorem 4.1 follows from Theorem 3.3. While for general $\beta$ the proof is significantly different (it bypasses the interlacing inequalities and goes straight to the quadratic form), some key features are the same as in this simple case.

Starting with the eigenvalue $\lambda_n$ of $\Gamma$ and the corresponding eigenfunction $f$, we cut an edge on the only cycle of $\Gamma$ to obtain a family of trees $\Gamma_{\gamma}^{\text{cut}}$. For $\gamma = \bar{\gamma} = f_{u_2}/f_{u_1}$, we have either

$$\max_{\gamma} \lambda_{n-\rho(\gamma)}(\Gamma_{\gamma}^{\text{cut}}) = \lambda_{n-\rho(\bar{\gamma})}(\Gamma_{\bar{\gamma}}^{\text{cut}}) = \lambda_n(\Gamma) = \min_{\alpha} \lambda_n(\Gamma_{\text{mag}}^{\alpha}),$$

or

$$\max_{\alpha} \lambda_n(\Gamma_{\text{mag}}^{\alpha}) = \lambda_n(\Gamma) = \lambda_{n-\rho(\bar{\gamma})+1}(\Gamma_{\bar{\gamma}}^{\text{cut}}) = \min_{\gamma} \lambda_{n-\rho(\gamma)+1}(\Gamma_{\gamma}^{\text{cut}}).$$

In the first case, according to Fiedler theorem (equation 2 with $\beta = 0$), the function $f$ has $n-\rho(\bar{\gamma})-1$ sign changes with respect to the tree $\Gamma_{\gamma}^{\text{cut}}$. Adding back the removed edge $(u_1, u_2)$ adds another sign change if $\bar{\gamma} < 0$ and doesn’t change the number of sign changes otherwise. In other words, it adds $\rho(\bar{\gamma})$ sign changes. Thus, with respect to $\Gamma$, the function $f$ has $n-1$ sign change and $\sigma_n = 0$. In the second case, we similarly conclude that $f$ has $n-1$ sign changes with respect to $\Gamma_{\gamma}^{\text{cut}}$ and $n$ sign changes with respect to $\Gamma$. The nodal surplus is $\sigma_n = 1$.

On the other hand, in the first case $\lambda_n(\Gamma)$ is a minimum of $\lambda_n(\Gamma_{\text{mag}}^{\alpha})$ (Morse index 0), while in the second it is a maximum of $\lambda_n(\Gamma_{\text{mag}}^{\alpha})$ (Morse index 1), which shows that the Morse index coincides with $\sigma_n$ in the case $\beta = 1$.

4. Tools of the main proof

In this section we collect some basic facts that will be repeatedly used in the proof of Theorem 4.1.

4.1. Critical points of the quadratic form.

**Definition 4.1.** Let $F : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function. If $c$ is a critical point (i.e. $\nabla f(c) = 0$), the **inertia** of $c$ is the triple $(n_-, n_0, n_+)$ that counts the number of negative, zero and positive eigenvalues correspondingly of the Hessian (the matrix of second derivatives) at the point $c$. The number $n_-$ is called the **Morse index** (or simply **index**).

The next theorem is a reminder that the eigenvectors of a self-adjoint matrix are critical points of the quadratic form on the unit sphere.

**Theorem 4.2.** Let $A$ be an $d \times d$ real symmetric matrix and $h(x) = \langle x, Ax \rangle$, $x \in \mathbb{R}^d$, be the associated quadratic form. Then the (real) eigenvectors of the matrix $A$ are critical points of the function $h(x)$ on the unit sphere $\|x\| = 1$.

Let $\lambda_n$ be the $n$-th eigenvalue of $A$ and $f^{(n)}$ be the corresponding normalized eigenfunction. Define

$$n_- = \# \{ \lambda_m < \lambda_n \}, \quad n_0 = \# \{ \lambda_m = \lambda_n, \ m \neq n \}, \quad n_+ = \# \{ \lambda_m > \lambda_n \},$$

with $n_- + n_0 + n_+ = d - 1$. Then the inertia of the critical point $x = f^{(n)}$ is $(n_-, n_0, n_+)$. In particular, if $\lambda_n$ is a simple eigenvalue, the inertia is $(n-1, 0, d-n)$. 
Remark 4.3. The value of the quadratic form $h$ at the critical point $f^{(n)}$ is $\lambda_n$.

Proof. The idea is intuitively clear: $n_-$ — which is the Morse index — counts the number of directions in which the quadratic form decreases relative to the value at $x = f^{(n)}$. These directions are the eigenvectors corresponding to the eigenvalues that are less than $\lambda_n$. Similar characterizations are valid for $n_0$ and $n_+$.

We note that by Sylvester law of inertia, the inertia is invariant under the change of variables. Making the orthogonal change of coordinates to the eigen-basis of the matrix $A$, the quadratic form $h(a)$ becomes

$$h(a) = \lambda_1 a_1^2 + \lambda_2 a_2^2 + \ldots + \lambda_d a_d^2,$$

while the sphere is given by the equations

$$a_1^2 + a_2^2 + \ldots + a_d^2 = 1.$$

Thus, on the sphere, the quadratic form in terms of variables $a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_d$ is given by

$$h = \sum_{j \neq n} (\lambda_j - \lambda_n) a_j^2,$$

and the Hessian is a diagonal matrix with $\lambda_j - \lambda_n, j = 1, \ldots, d, j \neq n$. The statement of the Theorem follows immediately. \qed

In the more general case when the matrix is Hermitian we should consider the quadratic form on the space $\mathbb{C}^d$. However, we can also consider it on the real space of double dimension.

Theorem 4.4. Let $A$ be an $d \times d$ Hermitian matrix and $h(z) = \langle z, A z \rangle$, $z \in \mathbb{C}^d$, be the associated quadratic form. Consider $h$ as a function of $2d$ real variables $(x, y)$, where $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $z = x + iy$. Then the eigenvectors of the matrix $A$ are critical points of the function $h(x + iy)$ on the unit sphere $\|x + iy\| = 1$.

Let $\lambda_n$ be the $n$-th eigenvalue of $A$ and $f^{(n)}$ be a corresponding eigenfunction. Then the inertia of the critical point $x + iy = f^{(n)}$ with respect to the real space $\mathbb{R}^{2d}$ is $(2n_-, 2n_0 + 1, 2n_+)$, where $n_-, n_0$ and $n_+$ are defined by (23). In particular, if $\lambda_n$ is a simple eigenvalue, the inertia is $(2n - 2, 1, 2d - 2n)$.

Proof. We adopt the convention that in the scalar product $\langle z, w \rangle$ the conjugation is applied to the first argument. Also, when mapping a $z \in \mathbb{C}^d$ to $\mathbb{R}^{2d}$, we first list the real parts of the components of $z$ and then the imaginary parts. With this conventions, it’s an easy exercise to show that the quadratic form $h(z)$ in variables $(x, y)$ corresponds to the matrix

$$(24) \quad B = \begin{pmatrix} \text{Re} A & -\text{Im} A \\ \text{Im} A & \text{Re} A \end{pmatrix}.$$

Note that because $A$ is Hermitian, the matrix $\text{Im} A$ is skew-symmetric and, therefore, the matrix $B$ is real symmetric. Every eigenvector $z$ of the matrix $A$ corresponds to 2 real eigenvectors of $B$ with the same eigenvalue, namely $(\text{Re} z, \text{Im} z)$ and $(\text{Re}(iz), \text{Im}(iz)) = (-\text{Im} z, \text{Re} z)$. These eigenvectors are orthogonal to each other, and to other similarly obtained eigenvectors. We therefore conclude that the spectrum of $B$ is the same as the spectrum of $A$ with all multiplicities doubled. The statement now follows from Theorem 1.2. \qed

4.2. Reduction to critical manifold. The tool introduced in this section is a simple idea already used in [2, 7, 8]. If we have a function $f(x_1, \ldots, x_n)$ with a critical point $c$ and under some general conditions there is a $(n - 1)$-dimensional manifold around the point $c$ on which the local minimum of $f$ is achieved when we vary the variable $x_1$ and keep the others fixed. Then the Morse index of $f$ restricted to this manifold is the same as the Morse index of the unrestricted function. On the other hand, if the manifold is the locus of local maxima with respect to the variable $x_1$, the Morse index
on the manifold is one less than the unrestricted Morse index. The following theorem is a simple
generalization of this idea. The proof is adapted from [7].

**Theorem 4.5** (Reduction Theorem). Let \( X = Y \oplus Y^\perp \). Let \( f : X \to \mathbb{R} \) be a smooth functional such
that \((0, 0) \in X \) is its non-degenerate critical point with inertia \( I_X \). Further, let, for every \( y \in Y \)
locally around 0, the functional \( f(y, y') \) as a function of \( y' \) has a critical point at \( y' = 0 \) with inertia \( I_{Y^\perp} \),
that (locally) does not depend on \( y \). Then the Hessian of \( f \) is reduced by the decomposition
\( X = Y \oplus Y^\perp \) and the inertia of \( f \) with respect to the space \( Y \) is
\[
I_Y = I_X - I_{Y^\perp}.
\]

**Proof.** We calculate the mixed derivative of \( f \) with respect to one variable from \( Y \) and the other
from \( Y^\perp \). In a slight abuse of notation we denote these variables simply by \( y \) and \( y' \). We have
\[
\left. \frac{\partial^2 f}{\partial y \partial y'}(0, 0) = \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial y'}(y, 0) \right] \right|_{y=0} = 0,
\]
since \( y' = 0 \) is the critical point of \( f(y, y') \) as a function of \( y' \) for every \( y \). Thus the Hessian of \( f \) has
a block-diagonal form with two blocks that correspond to \( Y \) and \( Y^\perp \). The spectrum of the Hessian
is the union of the spectra of the blocks and the inertia is the sum of the inertias of the blocks,
\[
I_X = I_Y + I_{Y^\perp}.
\]
Equation (25) follows immediately. \( \square \)

The restriction of the function \( f \) to the subspace \( Y \) will be called the reduction of \( f \) with inertia \( I_{Y^\perp} \). More often than not, we will be concerned with only the index of a critical point (the entry
\( n_- \) of the inertia). In this case we will say “reduction with index \( m^\perp \)”. The space \( Y \) is the locus of
critical points of \( f \) with respect to the variable \( y' \) and we will refer to it as a critical manifold.

**Remark 4.6.** Theorem 4.5 can be simply extended to the case when, for every fixed \( y \), the critical
point with respect to \( y' \) is located at \( y' = q(y) \) (rather than \( y' = 0 \)). The function \( q(y) \) defines the
critical manifold \( \mathcal{Q} = (y, q(y)) \). If \( q(y) \) is a smooth function of \( y \) and \( q(0) = 0 \), the change of variables
\[
y \mapsto y, \quad y' \mapsto y' - q(y)
\]
is non-degenerate (its Jacobian is a triangular matrix with 1s on the diagonal) and makes \( f \) satisfy the
assumptions of Theorem 4.5. By Sylvester law of inertia, the conclusion of the Theorem is invariant
under the change of variables, that is inertia at point 0 of \( f \big|_{\mathcal{Q}} \) is
\[
I_{\mathcal{Q}} = I_X - I_{Y^\perp}.
\]

5. **Proof of the main theorem**

We prove the main result in three steps. First we show by an explicit computation that the point
0 is the critical point of the function \( \lambda_n(\alpha) \), where \( \alpha = (\alpha_1, \ldots, \alpha_\beta) \in (-\pi, \pi)^\beta \) are the magnetic
phases.

Then we fix an eigenpair \( \lambda = \lambda_n(\Gamma) \) and \( f \). We cut \( \beta \) edges of the graph turning it into a tree \( T \),
but modifying the potentials so that the eigenfunction \( f \) is also an eigenfunction of the tree \( T \). It
now corresponds to an eigenvalue number \( m \), that is \( \lambda_m(T) = \lambda \). Considering the eigenvalue \( \lambda_m(T) \)
as a function of the potentials, we find its inertia. This is done by considering the inertia of the
Corresponding quadratic form in the real space. The result of this step is related to the results on
critical partitions. [8].

Finally, we relate the inertia of the function \( \lambda_m(T) \) to the inertia of the function \( \lambda_n(\alpha) \) at the
corresponding critical points. This is done by relating the inertias of the quadratic forms, but now
in the complex space represented as a real space of double dimension.
We recall that $S$ is a set of $\beta$ edges whose removal turns the graph $\Gamma$ into a tree. By $\Gamma_{\text{mag}}^{\alpha_1, \ldots, \alpha_\beta}$ we denote the graph obtained from $\Gamma$ by introducing magnetic phases $\alpha_1, \ldots, \alpha_\beta$ on the edges from the set $S$. Similarly, by $\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}$ we denote the tree graph obtained by cutting every edge from $S$ in a manner described in section 3.2 (see equation (13) and around). For future reference we list the quadratic forms of the original graph, the graph $\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}$ and the graph $\Gamma_{\text{mag}}^{\alpha_1, \ldots, \alpha_\beta}$. We put them in the form that highlights the similarities between the three.

\begin{align}
    h(x) &= \sum_u q_u x_u^2 - \sum_{(u,v) \in E \setminus S} 2x_u x_v - \sum_{(u,v) \in S} 2x_u x_v, \\
    h_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}(x) &= \sum_u q_u x_u^2 - \sum_{(u,v) \in E \setminus S} 2x_u x_v - \sum_{e_j=(u,v) \in S} \left( -\gamma_j x_u^2 - x_v^2 / \gamma_j \right), \\
    h_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}(z) &= \sum_u q_u |z_u|^2 - \sum_{(u,v) \in E \setminus S} 2 \text{Re} \left( \overline{z_u} z_v \right) - \sum_{e_j=(u,v) \in S} 2 \text{Re} \left( \overline{z_u} e^{i\alpha_j} z_v \right).
\end{align}

It is essential that the last quadratic form be considered on the complex vector space $\mathbb{C}^d$ (that can be identified with a real space of double dimension). The first two forms can be considered on both real and complex spaces, with obvious modifications in the latter case.

### 5.1. Critical points

Let $f$ be an eigenfunction of the graph $\Gamma$. We have seen in Theorem 3.3 and its proof that the points $\alpha = 0$ and $\gamma = \tilde{\gamma}$ (see equation (21)) are special: at these points $f$ is an eigenfunction of the graphs $\Gamma_{\text{mag}}^{\alpha}$ and $\Gamma_{\text{cut}}^{\gamma}$. Moreover, they are critical points of the corresponding eigenvalues considered as function of the parameters $\alpha$ and $\gamma$ respectively. The result of this section generalizes this observation.

**Theorem 5.1.** Let $f$ be an eigenfunction of $H(\Gamma)$ that corresponds to a simple eigenvalue $\lambda = \lambda_n(\Gamma)$. Assume $f$ is non-zero on vertices of the graph $\Gamma$. For every edge $(u_j, v_j) \in S$, $j = 1, \ldots, \beta$, let

\begin{equation}
    \tilde{\gamma}_j = f_{v_j} / f_{u_j}.
\end{equation}

Let $p$ denote the number of negatives among the values $\tilde{\gamma}_j$,

\begin{equation}
    p = \# \{ \tilde{\gamma}_j < 0, \; j = 1, \ldots, \beta \}.
\end{equation}

Then

\begin{equation}
    \lambda_n(\Gamma) = \lambda_{\phi_n - p + 1}(\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}),
\end{equation}

where $\phi_n$ is the number of sign changes of $f$ with respect to the graph $\Gamma$. Moreover, the point $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\beta)$ is a critical point of the function $\lambda_{\phi_n - p + 1}(\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta})$.

Similarly for $\Gamma_{\text{mag}}^{\alpha_1, \ldots, \alpha_\beta}$,

\begin{equation}
    \lambda_n(\Gamma) = \lambda_n(\Gamma_{\text{mag}}^{0, \ldots, 0})
\end{equation}

and $(0, \ldots, 0)$ is a critical point of the function $\lambda_n(\Gamma_{\text{mag}}^{\alpha_1, \ldots, \alpha_\beta})$.

**Proof.** It can be verified directly that $f$ is an eigenfunction of the graph $\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}$. The nodal bound [2] with $\beta = 0$ (proved by Fiedler in [22], see also [6]) shows that the eigenvalue corresponding to the function $f$ has number $\mu' + 1$ in the spectrum of the tree $\Gamma_{\text{cut}}^{\gamma_1, \ldots, \gamma_\beta}$. Here $\mu'$ is the number of sign changes of $f$ with respect to the tree. In general, this number is different from $\phi_n$ because we might have cut some of the edges on which $f$ was changing sign. However, according to (30), these edges gave rise to negative values of $\tilde{\gamma}_j$, therefore $\mu' = \phi_n - p$, proving equation (31). Equation (32) is trivial since $\Gamma_{\text{mag}}^{0, \ldots, 0} = \Gamma$. 

To prove criticality of the points, we calculate the derivatives. Because the corresponding eigenvalue is simple (for the tree, this follows from a theorem of Fiedler [22] and the fact that $f$ does not vanish on vertices), the corresponding eigenvalues are analytic functions of the parameters and can be differentiated.

Derivative of $\lambda_{\phi_{n-p+1}}\left(\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}\right)$ with respect to $\gamma_j$ has been calculated in equation (20), resulting in

$$\frac{\partial}{\partial \gamma_j} \lambda_{\phi_{n-p+1}}\left(\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}\right) = -|f_{u_j}|^2 + |f_{v_j}|^2/\bar{\gamma}_j^2 = 0,$$

where we used the definition of $\bar{\gamma}$, equation (30).

The derivative of $\lambda_n\left(\Gamma_{\text{mag}}^{\alpha_1,\ldots,\alpha_\beta}\right)$ can be evaluated similarly using (11), leading to

$$\frac{\partial}{\partial \alpha_j} \lambda_n\left(\Gamma_{\text{mag}}^{\alpha_1,\ldots,\alpha_\beta}\right) = -iF_{u_j}f_{v_j} + if_{u_j}F_{v_j} = \text{Im}(F_{u_j}f_{v_j}) = 0,$$

since the eigenfunction $f$ is real-valued. □

5.2. Index of the eigenvalue on the tree. In this section we elaborate on the first part of the result of Theorem 5.1 namely that $(\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ is a critical point of the function $\lambda_{\phi_{n-1}}\left(\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}\right)$.

**Theorem 5.2.** Let $f$ be an eigenfunction of $H(\Gamma)$ that corresponds to a simple eigenvalue $\lambda = \lambda_n(\Gamma)$. Assume $f$ is non-zero on vertices of the graph $\Gamma$ and has $\phi_n$ sign changes. For every edge $(u_j, v_j) \in S$, $j = 1, \ldots, \beta$, let

$$\bar{\gamma}_j = f_v/f_u.$$

As before, $p$ denotes the number of negatives among the values $\bar{\gamma}_j$. Then the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ as a critical point of the function $\lambda_{\phi_{n-p+1}}\left(\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}\right)$ has index $n - 1 + \beta - \phi_n$.

**Proof.** Denote by $d$ the number of vertices of the graph $\Gamma$. Consider $h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(\bar{x})$, which is the quadratic form on the Hamiltonian of $\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}$, as a function of $d + \beta$ real variables $(x_1, \ldots, x_d, \gamma_1, \ldots, \gamma_\beta)$ on the manifold $x_1^2 + \ldots + x_d^2 = 1$. We note that the point $(f_1, \ldots, f_d, \bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ is a critical point of $h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(\bar{x})$, as can be easily shown by explicit computation. Indeed, the value of the Lagrange multiplier is the eigenvalue $\lambda_n$ and the gradient of

$$F(x_1, \ldots, x_d, \gamma_1, \ldots, \gamma_\beta) = h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(\bar{x}) - \lambda_n(x_1^2 + \ldots + x_d^2)$$

is zero: the first $d$ equations become the eigenvalue condition $Hf = \lambda_n f$ and the last $\beta$ are the same as (33).

Denote the index of the point $(f_1, \ldots, f_d, \bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ by $M$. For every value of $(\gamma_1, \ldots, \gamma_\beta)$ locally around the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ the $(\phi_n - p + 1)$-th eigenvector $f_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}$ of $\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}$ is a critical point of $h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(\bar{x})$ as a function of $\bar{x}$. According to Theorem 4.2 the critical point has index $\phi_n - p$.

According to standard perturbation theory (see, e.g., [33]) the eigenvector $f_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}$ is a smooth (indeed, analytic) function of $(\gamma_1, \ldots, \gamma_\beta)$. This allows us to use Theorem 4.5 via Remark 4.6 concluding that the critical point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta)$ of $h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(f_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}})$ has index $M - (\phi_n - p)$. At this point we observe (see Remark 4.3) that

$$h_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}(f_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}) = \lambda_{\phi_{n-p+1}}\left(\Gamma_{\gamma_1,\ldots,\gamma_\beta}^{\text{cut}}\right),$$

which is the function whose index we strive to evaluate.
Figure 3. Schematic diagram of the proof of Theorem 5.2. The reductions are indicated by arrows, with the description of the parameters that are being reduced and the index of the reduction. Since we know the index of the critical point of $h(\vec{x})$, we can follow the diagram, applying the Reduction Theorem, to calculate the index of $\lambda_{\phi_n-p+1}(\Gamma_{\gamma_1,...,\gamma_\beta}^\text{cut})$.

On the other hand, consider $\vec{x}$ varying locally around the point $f$, so that the elements of $\vec{x}$ remain bounded away from zero. For each fixed $\vec{x}$ we look for a critical point with respect to the variables $(\gamma_1, ..., \gamma_\beta)$. The terms of $h_{\gamma_1,...,\gamma_\beta}^\text{cut}(\vec{x})$ that depend on a given $\gamma$ have the form

$$T(\gamma) = -\gamma x_u^2 - x_v^2/\gamma.$$  

The critical point is $g = g(\vec{x}) = x_v/x_u$, which is a smooth function of $\vec{x}$. The points $(x_1, ..., x_d, g_1, ..., g_{\beta})$ define another critical manifold to apply Theorem 4.5 to. Note that the critical manifold includes the point $(f_1, ..., f_d, \tilde{\gamma}_1, ..., \tilde{\gamma}_\beta)$. Moreover, the critical point with respect to a given $\gamma$ is a maximum if $g > 0$ and a minimum if $g < 0$. Each point is nondegenerate and, moreover, the sign of $g_j$ is locally the same as the sign of $\tilde{\gamma}_j$ for all $j$. Different variables $\gamma$ are not coupled, thus the Hessian is diagonal. Therefore, the inertia of the points on the critical manifold is $(\beta - p, 0, p)$ — it is a minimum with respect to $p$ variables and maximum with respect to $\beta - p$.

Consider now the function $h_{\gamma_1,...,\gamma_\beta}^\text{cut}(\vec{x})$ on the critical manifold. When $\gamma = g$, the term (36) evaluates to

$$T(g) = -2x_u x_v$$

and we find that, on the critical manifold, the function $h_{\gamma_1,...,\gamma_\beta}^\text{cut}(\vec{x})$ coincides with the quadratic form of the original graph, $h(\vec{x})$. The point $\vec{x} = f$, being the $n$-th eigenfunction of the graph is a critical point of $h(\vec{x})$ and has index $n - 1$. Applying Theorem 4.5 we obtain

$$n - 1 = M - (\beta - p).$$

Coming back to $(\tilde{\gamma}_1, ..., \tilde{\gamma}_\beta)$ as a critical point of $\lambda_{\phi_n-p+1}(\Gamma_{\gamma_1,...,\gamma_\beta}^\text{cut})$ we conclude that its index is

$$M - (\phi_n - p) = (n - 1) + (\beta - p) - (\phi_n - p) = n - 1 + \beta - \phi_n.$$  

The steps of the proof are summarized in Fig. 5.2.

Remark 5.3. In [8] the eigenvalue of the tree graph $\Gamma_{\gamma_1,...,\gamma_\beta}^\text{cut}$ was interpreted as the energy of the “partition” with the given number of domains. Theorem 5.2 gives another route for the proof of the results of [8].

5.3. Index of the eigenvalue as a function of the magnetic field. Now we move from the critical point on the the tree to the critical point of the eigenvalue of the graph with magnetic phases. We apply the same method, practically retracing our steps, but now the quadratic form is a function of complex variables.
Theorem 5.4. Let \( f \) be an eigenfunction of \( H(\Gamma) \) that corresponds to a simple eigenvalue \( \lambda = \lambda_n(\Gamma) \). Assume \( f \) is non-zero on vertices of the graph \( \Gamma \) and has \( \phi_n \) sign changes. Let \( \Gamma^{\alpha_1, \ldots, \alpha_\beta} \) be the graphs with the magnetic phases \( \alpha_1, \ldots, \alpha_\beta \) introduced on the edges from the set \( S \). Then the index of \((0, \ldots, 0)\) as a critical point of the function \( \lambda_n(\Gamma^{\alpha_1, \ldots, \alpha_\beta}) \) is the nodal surplus \( \phi_n - (n - 1) \).

Proof. Let \( z \) be the \( d \)-dimensional vector of complex numbers \( z_j = x_j + i y_j \). We consider \( h_{\gamma_1, \ldots, \gamma_\beta}^{\text{cut}}(z) \), the quadratic form of the Hamiltonian of \( \Gamma^{\gamma_1, \ldots, \gamma_\beta} \), as a function of \( 2d + \beta \) real variables

\[
x_1, \ldots, x_d, y_1, \ldots, y_d, \gamma_1, \ldots, \gamma_\beta
\]
on the manifold \( |z_1|^2 + \ldots + |z_d|^2 = 1 \). We also consider \( h_{\alpha_1, \ldots, \alpha_\beta}^{\text{mag}}(z) \), the quadratic form of the Hamiltonian of \( \Gamma^{\alpha_1, \ldots, \alpha_\beta} \), as a function of \( 2d + \beta \) real variables on the same manifold. See equations (28) and (29) for explicit formulas.

As before, the point \( z_j = f_j, \gamma_k = \bar{\gamma}_k \) is a critical point of the function \( h_{\gamma_1, \ldots, \gamma_\beta}^{\text{cut}}(z) \). Similarly, \( z_j = f_j, \alpha_k = 0 \) is a critical point of the function \( h_{\alpha_1, \ldots, \alpha_\beta}^{\text{mag}}(z) \).

For a fixed vector \( z \) in the vicinity of the eigenfunction \( f \) we find the critical points of functions \( h_{\gamma_1, \ldots, \gamma_\beta}^{\text{cut}}(z) \) and \( h_{\alpha_1, \ldots, \alpha_\beta}^{\text{mag}}(z) \) with respect to the corresponding set of parameters. We choose the critical points close to \( (\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta) \) and \((0, \ldots, 0)\) correspondingly.

Concentrating first on \( h_{\gamma_1, \ldots, \gamma_\beta}^{\text{cut}}(z) \) we observe that now the terms that depend on a \( \gamma \) have the form

\[
T(\gamma) = -\gamma|z_u|^2 - |z_v|^2 / \gamma.
\]
The critical point is \( g(z) = |z_u|/|z_u| \) and it is a maximum or minimum depending whether \( \gamma > 0 \) or \( \gamma < 0 \) (the sign of \( \gamma \) is determined by the sign of the corresponding \( \bar{\gamma} \)). Similarly to the proof of Theorem 5.2, the index of the critical points that are close to \( (\bar{\gamma}_1, \ldots, \bar{\gamma}_\beta) \) is \( \beta - p \), where \( p \) is the number of negatives among \( \bar{\gamma}_j \). On the critical manifold the term (37) evaluates to

\[
T(g) = -2 \text{sign}(\bar{\gamma}) |z_u||z_v|.
\]

Moving on to \( h_{\alpha_1, \ldots, \alpha_\beta}^{\text{mag}}(z) \) we first investigate its dependence on a single phase \( \alpha \). The terms involving \( \alpha \) are of the form

\[
T_m(\alpha) = -z_u e^{i\alpha} z_2 - z_v e^{-i\alpha} z_u = -2|z_u||z_v| \cos(\alpha - \theta_{uv}),
\]
where \( \theta_{uv} = \arg(z_u/z_v) \). Since \( z_u, z_v \in \mathbb{C} \) are close to \( f_u, f_v \in \mathbb{R} \) correspondingly, the angle \( \theta_{uv} \) is close to 0 if \( \tilde{\gamma} = f_v/f_u > 0 \) and close to \( \pi \) if \( \tilde{\gamma} = f_v/f_u < 0 \). The corresponding critical point \( a = a(z) \) is then minimum (\( \tilde{\gamma} > 0 \)) or maximum (\( \tilde{\gamma} < 0 \)).

Considering now the critical point of \( h^{\alpha_1,\ldots,\alpha_\beta}(\gamma) \) as a function of all parameters \( \alpha_1, \ldots, \alpha_\beta \) (the variable \( z \) is fixed), we find that the Hessian is diagonal and therefore the index is the number of coordinate-wise maxima, that is \( p \).

Thus indices of the point \( z = f \) with respect to the corresponding critical manifold of \( h^\text{cut}_{\gamma_1,\ldots,\gamma_p}(\rho) \) and \( h^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}(\alpha) \) are equal. This establishes a bridge between the two forms. Note that the bridging function \( \tilde{h}(z) \) is different from the form \( h(z) \) of the original graph \( \Gamma \). The relations between the quadratic forms and the eigenvalues as functions of parameters are sketched in Fig. 5.3. We will now be repeatedly applying the reduction of critical manifolds, Theorem 4.4. The indices of the reductions are noted along the arrows in Fig. 5.3 (see below for explanations).

Fixing the parameters \( \gamma_1, \ldots, \gamma_p \), we find the critical point of \( h^\text{cut}_{\gamma_1,\ldots,\gamma_p}(\rho) \) as a function of \( z \). Locally around the point \( z_j = f_j, \gamma_k = \bar{\gamma}_k \), the critical point is the \( (\phi_n - p + 1) \)-th eigenvector of the graph \( \Gamma^\text{cut}_{\gamma_1,\ldots,\gamma_p} \). According to Theorem 4.4, its index is \( 2(\phi_n - p) \). We apply the reduction theorem to the result of Theorem 5.2 to conclude that the index of the point \( z = f, \gamma = \bar{\gamma} \) with respect to all parameters is

\[
(n - 1 + \beta - \phi_n) + 2(\phi_n - p) = n - 1 + \beta + \phi_n - 2p.
\]

Now we go down to the “bridge” critical point and then back up to the critical point \( z = f, \alpha = 0 \) of the function \( h^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}(\alpha) \), applying the reduction theorem both ways. For the latter critical point we obtain the index

\[
(n - 1 + \beta + \phi_n - 2p) - (\beta - p) + p = n - 1 + \phi_n.
\]

Finally, for every choice of parameters \( \alpha_1, \ldots, \alpha_\beta \), we find the critical point with respect to \( z \) that is close to \( z = f \). This critical point \( f^\text{mag}_{\alpha_1,\ldots,\alpha_\beta} \) is the \( n \)-th eigenvector of the graph \( \Gamma^\text{mag}_{\alpha_1,\ldots,\alpha_\beta} \) (since \( f \) is the \( n \)-th eigenvector of \( \Gamma^\text{mag}_{\alpha_1,\ldots,\alpha_\beta} \)). According to Theorem 4.4, it has index \( 2(n - 1) \). We apply the reduction theorem one last time to conclude that the point \((0, \ldots, 0)\) as a critical point of \( h^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}(f^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}) \) has index

\[
n - 1 + \phi_n - 2(n - 1) = \phi_n - (n - 1).
\]

Since \( h^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}(f^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}) = \lambda_n(\Gamma^\text{mag}_{\alpha_1,\ldots,\alpha_\beta}) \), this concludes the proof of the theorem.

6. Discussion

Perhaps the most important feature of Theorem 4.4 is that it allows to access some of the features of the eigenfunction via the behavior of the corresponding eigenvalue under perturbation. It is known that the eigenvalues of the Laplacian are connected to the statistics of the closed paths on the graph. The connection is given through the so-called “trace formulae”, which can be obtained from a graph analogue of the Selberg zeta function, the Ihara zeta function. An extension by Bartholdi was used in [33] to obtain a family of trace formulae including the ones for the magnetic Laplacian. Thus, the closed paths on the graph determine the spectrum of the magnetic Laplacian which, in turn, determines the nodal count. This, in principle, establishes the existence of a general connection between the nodal count and the closed paths. However, we are not aware...
of any concrete general formulas. We note that such a connection has been earlier conjectured by Smilansky, with special cases reported in [24, 1]. We note that the eigenvalue $\lambda_n(\Gamma_{mag}^{\alpha_1,\ldots,\alpha_\beta})$ featuring in this paper is a well-studied object. It is the dispersion relation for the maximal Abelian cover of the graph $\Gamma$. One of the interesting questions regarding this object is the “full spectrum property” [29, 28, 47]: whether the continuous spectrum of the cover graph of a regular graph — in our terms, the union of ranges of the functions $\lambda_n(\Gamma_{mag}^{\alpha_1,\ldots,\alpha_\beta})$ — contains no gaps. This question can be reformulated in terms of eigenfunctions of graphs $\Gamma_{mag}^{\alpha_1,\ldots,\alpha_\beta}$ with all $\alpha_j = 0$ or $\pi$ that have minimal and maximal number of sign changes.

This, in turn, is related to the question of whether the extrema of the dispersion relation are always achieved at the symmetry points (namely, all $\alpha_j = 0$ or $\pi$). Examples to the contrary have been put forward in [26, 21]. However, an important question remains, how can one characterize the extremal points that are not points of symmetry? In this direction, the duality with the cut graphs (Section 3) might provide some answers. One can speculate that critical points of dispersion relation correspond to critical points of the eigenvalues of the cut graph $\Gamma_{cut}^{\gamma_1,\ldots,\gamma_\beta}$ that do not give rise to the eigenfunction of the graph $\Gamma$. Further, we conjecture that these “unclaimed” critical points correspond to eigenfunctions of $\Gamma$ modified by enforcing Dirichlet conditions at some vertices.

The results of the present paper are derived under the assumption that the eigenvalue is non-degenerate. While this is the generic situation with respect to the change in the potential $Q$, it is also interesting to consider what happens in the degenerate case. Linear Zeeman effect (the magnetic perturbation splits eigenvalues) suggests that the singularities of $\lambda_n(\Gamma_{mag}^{\alpha_1,\ldots,\alpha_\beta})$ are conical. It should be possible to define the index of the singularity point that does not rely on differentiability.

Finally, it would be most interesting to generalize the results of the present paper to domains of $\mathbb{R}^d$. However, we immediately encounter a conceptual problem — the “number” of zeros is infinite. Still, some measure of instability of the eigenvalue under magnetic perturbation should be related to some measure of the zero set of the corresponding eigenfunction. This can be intuitively visualized by approximating the domain eigenfunction by eigenfunctions of a discrete mesh.

7. Acknowledgment

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