THE VANISHING PRESSURE LIMITS OF RIEMANN SOLUTIONS TO THE CHAPLYGIN GAS EQUATIONS WITH A SOURCE TERM

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Abstract. We study the vanishing pressure limits of Riemann solutions to the Chaplygin gas equations with a source term. The phenomena of concentration and cavitation to Chaplygin gas equations with a friction term are identified and analyzed as the pressure vanishes. Due to the influence of source term, the Riemann solutions are no longer self-similar. When the pressure vanishes, the Riemann solutions to the inhomogeneous Chaplygin gas equations converge to the Riemann solutions to the pressureless gas dynamics model with a friction term.

1. Introduction. The Euler equations with a source term read

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + P)_x = \beta \rho,
\end{cases}
\]

where \(\beta\) is constant. The states \(\rho, u\) and \(P\) represent the density, the velocity and the pressure, respectively. The scalar pressure \(P\) is a function of the density \(\rho\) satisfying \(\lim_{\varepsilon \to 0} P(\rho, \varepsilon) = 0\), where \(\varepsilon > 0\) is a small parameter.

In this article, we consider the pressure function for Chaplygin gas:

\[
P = \varepsilon p, \quad p = -\frac{A}{\rho},
\]

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where $A > 0$ is constant. The equation (2) owns a negative pressure and allows mass concentrations in a finite time. Thus, the Chaplygin gas can be used to depict dark-energy models in cosmology [1, 7, 9, 11, 22].

For the homogeneous Chaplygin gas equations (1)–(2), Brenier [3] researched the Riemann problem:

$$(u, \rho)(x, 0) = \begin{cases} (u_-, \rho_-), & x < 0, \\ (u_+, \rho_+), & x > 0, \end{cases}$$

where $\rho_\pm, u_\pm$ are constants satisfying $\rho_\pm > 0$. He obtained the Riemann solutions with concentration when initial data belongs to a certain domain in phase plane. Guo, Sheng and Zhang [14] abandoned this constrain and obtained the general solutions. In 2014, Kong and Wei [16] identified and analyzed the phenomena of concentration and the formation of delta shock waves. Recently, Guo, Li, Pan and Han [13] found that the Chaplygin gas equations without a source term can be transformed to the equations of string moving in the Minkowski space $\mathbb{R}^2$ [15]. The delta shock waves can explain appearance of the singularities developed in the motion of the string. As for delta shock waves, we refer readers to [19, 20, 21, 30, 32], and the references cited therein for more details. We can also see such as [6, 17, 33] for the related results about the system (1)–(2) with $\beta = 0$.

In this paper, we are concerned with Chaplygin gas equations with a source term ($\beta \neq 0$) such as damping, friction and relaxation effect. Shen [26] studied the Riemann problem by using the new velocity

$$v(x, t) = u(x, t) - \beta t.$$  

The equation (4) was introduced by Faccanoni and Mangeney [10] to research the shock and rarefaction waves of the Riemann problem for the shallow water equations with a friction term. Recently, [13] researched the Riemann problem with delta initial data for system (1)–(2). The formula (4) was also used to solve the Riemann problem for the inhomogeneous generalized Chaplygin gas equations [31], while the generalized Chaplygin gas is defined by

$$p(\rho) = -\frac{A}{\rho^\alpha},$$

where $A > 0, 0 < \alpha < 1$.

As the pressure vanishes, the system (1)–(2) converges to the so-called pressureless gas dynamics model with body force as a source:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \beta \rho. \end{cases}$$

This system can be used to describe the motion process of free particles sticking under collision in the low temperature and the information of large-scale structures in the universe [12, 23]. The system (6) also can be derived from the isentropic gas dynamics model with a force term in which we take the pressure to be constant [8], where the force is assumed to be the gravity with $\beta$ being the gravity constant. Shen [27] studied the Riemann problem for system (6) and obtained the Riemann solutions, in which the delta shock wave and vacuum do occur. In addition, one can also see [2, 28] for the homogeneous pressureless gas dynamics model.

The method of vanishing pressure limit was first used by Li [18] in 2001. When pressure vanishes, the author identified and analyzed the formation of delta shock waves and vacuum states in the Riemann solutions with isothermal gases dynamics.
Then, in 2003 and 2004, Chen and Liu [4, 5] also obtain the same result for prototypical pressure functions for polytropic gases dynamics. We can see that the formation of the delta shock wave for system (1)–(2) which is the linearly degenerate system is different from those which are the genuinely nonlinear systems [4, 5, 18, 20] etc. For more results about vanishing pressure, we refer readers to [24, 25, 29, 34, 35].

In this article, we focus on the problem for the vanishing pressure Limits of Riemann Solutions to the inhomogeneous Chaplygin gas equations (1)–(3). The Riemann solutions are no longer self-similar due to the source term. Thus, we solve the limit behavior of Riemann solutions in the $ (x,t) $ plane. We give our main results.

**Theorem 1.1.** There are three cases for the vanishing pressure Limits of Riemann Solutions to (1)–(3).

1. In the case $ u_- > u_+ $, the Riemann solution converges to a delta shock wave when $ \varepsilon \to \varepsilon_2 $. The limit functions $ \rho $ and $ \rho u $ are the sums of a step function and a $ \delta $-measure with weights

\[
(u_d(t, \varepsilon_2)[\rho] - [\rho u])t \quad \text{and} \quad (u_d(t, \varepsilon_2)[\rho u] - [\rho u^2 + P])t,
\]

respectively, where $ u_d(t, \varepsilon) $ is the velocity of delta shock wave. If $ \rho_- \neq \rho_+ $, we have

\[
\frac{du(t, \varepsilon)}{d\varepsilon} < 0 \quad \text{and} \quad \frac{dw(t, \varepsilon)}{d\varepsilon} < 0,
\]

where $ w(t, \varepsilon) $ is the weight of delta shock wave. Eventually, when $ \varepsilon \to 0 $, the delta shock wave solution is exactly the solution to (6) and (3).

2. In the case $ u_- < u_+ $, as $ \varepsilon \to 0 $, the Riemann solution converges to two contact discontinuities connecting the states $ (u_+ \pm \beta t, \rho_\pm) $, where intermediate state is vacuum, which is consistent to Riemann solution to (6) and (3).

3. In the case $ u_- = u_+ $, as $ \varepsilon \to 0 $, the Riemann solution converges to a contact discontinuity connecting the states $ (u_\pm \pm \beta t, \rho_\pm) $, which is consistent to Riemann solution to (6) and (3).

The article is organized as follows. In section 2 and section 3, we review the Riemann solutions to (6) and (3), (1)–(3) respectively. In section 4, we study the vanishing pressure Limits of Riemann Solutions to (1)–(3). In section 5, we give our discussion.

2. **Riemann problem for pressureless gas dynamics model with a friction term (6).** In this section, we show some results briefly on the Riemann problem to the system (6) which was studied by [27].

By (4), the system (6) is reformulated in the follows:

\[
\begin{align*}
\rho_t &+ (\rho (v + \beta t))_x = 0, \\
(\rho v)_t &+ (\rho (v + \beta t))_x = 0.
\end{align*}
\]

Now, we consider system (8) with the following Riemann initial data:

\[
(v, \rho)(x, 0) = \begin{cases} 
(v_-, \rho_-), & x < 0, \\
(v_+, \rho_+), & x > 0,
\end{cases}
\]

where $ v_\pm = u_\pm $ due to (4). For the convenience, we use $ u_\pm $ to denote $ v_\pm $ throughout this paper.

The system (8) has a double eigenvalue $ \lambda_1 = \lambda_2 = \lambda = v + \beta t $. The corresponding right eigenvector is $ \vec{\tau} = (1,0)^\top $. The system (8) is linearly degenerate due to $ \nabla \lambda \cdot \vec{\tau} = 0 $. Thus, the elementary waves are contact discontinuities of one family.
Let $\sigma(t) = x'(t)$ be the speed of a bounded discontinuity $x = x(t)$. Then, the Rankine-Hugoniot conditions for the conservative system (8) holds:

$$
\begin{align*}
-\sigma(t) [\rho] + [\rho(v + \beta t)] &= 0, \\
-\sigma(t) [pv] + [pv(v + \beta t)] &= 0,
\end{align*}
$$

where $[\rho] = \rho - \rho_-$. From (10), we obtain contact discontinuity:

$$
\sigma(t) = v + \beta t = u_+ - \beta t.
$$

There are three cases for the Riemann solution to (8)–(9). Thus, we divide our discussion into three subsections.

2.1. **The Riemann solutions for $u_- > u_+$.** For the case $u_- > u_+$, singularity must happen. We use delta shock wave to connect states $(u_-, \rho_-)$ and $(u_+, \rho_+)$. The detail can be find in [27]. The solution can be expressed by:

$$
(u_-, \rho_-) + \delta S + (u_+, \rho_+),
$$

where “$+$” means “follow by”, i.e.,

$$
(v, \rho) = \begin{cases} 
(u_-, \rho_-), & -\infty < x < x(t), \\
(u_0, w(t)\delta(x - x(t))), & x = x(t), \\
(u_+, \rho_+), & x(t) < x < +\infty,
\end{cases}
$$

where $w(t)$ and $u_0(t) = v_0 + \beta t$ are weight and velocity of delta shock wave respectively, $v_0$ indicates the intermediate variable on this delta shock wave curve. This delta shock wave satisfies the generalized Rankine-Hugoniot conditions

$$
\begin{align*}
\frac{dx(t)}{dt} &= u_0(t), \\
\frac{dw(t)}{dt} &= u_0(t)[\rho] - [\rho(v + \beta t)], \\
\frac{d(w(t)v_0(t)))}{dt} &= u_0(t)[pv] - [pv(v + \beta t)],
\end{align*}
$$

where $(x, w)(0) = 0$.

From (14), we obtain

$$
v_0 = \sqrt{\frac{\rho_0 - \rho_+}{\rho_+ + \rho_-}} + \frac{\sqrt{\rho_0 - \rho_+}}{\sqrt{\rho_+ + \rho_-}}, \quad x(t) = v_0 t + \frac{1}{2} \beta t^2, \quad w(t) = \sqrt{\rho_0 - \rho_+} (u_0 - u_+) t.
$$

In order to ensure the uniqueness of Riemann solution, the generalized entropy condition should be proposed as

$$
u_+ + \beta t < u_0(t) < u_- + \beta t.
$$

2.2. **The Riemann solutions for $u_- < u_+$.** For the case $u_- < u_+$, the Riemann solution for (8)–(9) consists of two contact discontinuities plus a vacuum state between them. The solution can be expressed by:

$$
(u_-, \rho_-) + J_1 + Vac + J_2 + (u_+, \rho_+),
$$

i.e.,

$$
(v, \rho)(x, t) = \begin{cases} 
(u_-, \rho_-), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \\
v_0, & u_- t + \frac{1}{2} \beta t^2 \leq x \leq u_+ t + \frac{1}{2} \beta t^2, \\
(u_+, \rho_+), & u_+ t + \frac{1}{2} \beta t^2 < x < +\infty,
\end{cases}
$$
where $u_- < v_m < u_+$.

2.3. The Riemann solutions for $u_- = u_+$. For the case $u_- = u_+$, the Riemann solution for (8)–(9) consists of one contact discontinuity. The solution can be expressed by:

$$ (u_-, \rho_-) + J + (u_+, \rho_+), \quad (19) $$
i.e.,

$$ (v, \rho)(x, t) = \begin{cases} 
(u_-, \rho_-), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \vspace{1ex} \\
(u_+, \rho_+), & u_- t + \frac{1}{2} \beta t^2 < x < +\infty. \end{cases} \quad (20) $$

From the above discussions, we obtain the Riemann solutions for the system (8)–(9). By using (4), we get the Riemann solutions to (6) and (3). There are three cases.

(1) For $u_- > u_+$, the Riemann solution can be expressed by

$$ (u_- + \beta t, \rho_-) + \delta S + (u_+ + \beta t, \rho_+). \quad (21) $$

(2) For $u_- < u_+$, the Riemann solution can be expressed by

$$ (u_- + \beta t, \rho_-) + J_1 + Vac + J_2 + (u_+ + \beta t, \rho_+). \quad (22) $$

(3) For $u_- = u_+$, the Riemann solution can be expressed by

$$ (u_- + \beta t, \rho_-) + J + (u_+ + \beta t, \rho_+). \quad (23) $$

3. Riemann problem for Chaplygin gas equations with a source term (1)–(2). In this section, we mainly consider the Riemann problem (1)–(3) which depends on the small parameter $\varepsilon$. We can see [13, 26] for the related study. Without loss of generality, we take $A = 1$ throughout this paper.

By using transform (4), we reformulate system (1)–(2) into

$$ \begin{cases} 
\rho_t + (\rho(v + \beta t))_x = 0, \\
(\rho v)_t + \left( \rho v(v + \beta t) - \frac{\varepsilon}{\rho} \right)_x = 0. \end{cases} \quad (24) $$

Then the system (1)–(2) becomes the homogeneous conservative form (24). It is interesting to note that $t$ appears in the equations.

The eigenvalues of system (24) are $\lambda_1 = v + \beta t - \frac{\sqrt{\varepsilon}}{\rho}$, $\lambda_2 = v + \beta t + \frac{\sqrt{\varepsilon}}{\rho}$. The corresponding right eigenvectors are $\vec{r}_1 = (1, -\frac{\sqrt{\varepsilon}}{\rho^2})^T$, $\vec{r}_2 = (1, \frac{\sqrt{\varepsilon}}{\rho^2})^T$. The system (24) is linearly degenerate due to $\nabla \lambda_i, r_i = 0$ ($i = 1, 2$). Thus, the elementary waves are contact discontinuities.
Let \( \sigma^\varepsilon(t) = x'(t, \varepsilon) \) be the speed of a bounded discontinuity \( x = x(t, \varepsilon) \). Then, the Rankine-Hugoniot conditions are:

\[
\begin{align*}
-\sigma^\varepsilon(t)[\rho] + [\rho(v + \beta t)] &= 0, \\
-\sigma^\varepsilon(t)[\rho v] + [\rho v(v + \beta t) - \varepsilon \rho] &= 0,
\end{align*}
\]  

(25)

From (25), we solve for two families of contact discontinuities:

1-contact discontinuity curve \( J^\varepsilon_1(u_-, \rho_-) \):

\[
\sigma^\varepsilon_1(t) = v^\varepsilon + \beta t - \sqrt{\varepsilon \rho} = u_- + \beta t - \sqrt{\varepsilon \rho}, \quad \rho > 0;
\]

(26)

2-contact discontinuity curve \( J^\varepsilon_2(u_-, \rho_-) \):

\[
\sigma^\varepsilon_2(t) = v^\varepsilon + \beta t + \sqrt{\varepsilon \rho} = u_- + \beta t + \sqrt{\varepsilon \rho}, \quad \rho > 0.
\]

(27)

In the phase plane \((v, \rho)\), the curve (26) has two asymptotic lines \( v^\varepsilon = u_- - \sqrt{\varepsilon \rho} \) and \( \rho = 0 \). The curve (27) has two asymptotic lines \( v^\varepsilon = u_- + \sqrt{\varepsilon \rho} \) and \( \rho = 0 \). Through the point \((u_- - 2\sqrt{\varepsilon \rho}, \rho_-)\), we draw the curve (27). This curve is denoted by \( S_\delta \). Then, the phase plane is divided into five regions (see Fig. 3.1).

For any given right state \((u_+, \rho_+) \in (I \cup II \cup III \cup IV) \cup (u_-, \rho_-)\), the Riemann solution of system (24) can be expressed by

\[
(u_-, \rho_-) + J^\varepsilon_1 + (v^\varepsilon_*, \rho^\varepsilon_*) + J^\varepsilon_2 + (u_+, \rho_+).
\]

(28)

The intermediate constant state \((v^\varepsilon_*, \rho^\varepsilon_*)\) is given by

\[
\begin{align*}
\sigma^\varepsilon_* &= \frac{1}{2}(u_+ + \sqrt{\varepsilon \rho_+}) + \frac{1}{2}(u_- - \sqrt{\varepsilon \rho_-}), \\
\rho^\varepsilon_* &= \frac{1}{2}(u_+ + \sqrt{\varepsilon \rho_+}) - \frac{1}{2}(u_- - \sqrt{\varepsilon \rho_-}).
\end{align*}
\]

(29)

When \((u_+, \rho_+) \in V \cup S_\delta\), the singularity must happen. We use delta shock wave to connect states \((u_-, \rho_-)\) and \((u_+, \rho_+).\) The detail study can be find in [13, 26]. In this situation, the Riemann solution for system (24) can be expressed by

\[
(u_-, \rho_-) + \delta S + (u_+, \rho_+),
\]

(30)

i.e.,

\[
(v^\varepsilon, \rho)(x, t) = \begin{cases} 
(u_-, \rho_-), & x < x(t, \varepsilon), \\
(v^\varepsilon_*, \rho^\varepsilon_*)\delta(x - x(t, \varepsilon)), & x = x(t, \varepsilon), \\
(u_+, \rho_+), & x > x(t, \varepsilon),
\end{cases}
\]

(31)
where \( w(t, \varepsilon) \) and
\[
\frac{dx(t, \varepsilon)}{dt} = u_{\delta}(t, \varepsilon) = v_{\delta} + \beta t,
\]
are weight and velocity of delta shock wave respectively, \( v_{\delta} \) indicates the intermediate variable on this delta shock wave curve. This delta shock wave satisfies the generalized Rankine-Hugoniot conditions
\[
\begin{aligned}
dx(t, \varepsilon) dt &= u_{\delta}(t, \varepsilon) = v_{\delta} + \beta t, \\
dw(t, \varepsilon) dt &= u_{\delta}(t, \varepsilon)[\rho] - [\rho(v + \beta t)], \\
dw(t, \varepsilon) v_{\delta} dt &= u_{\delta}(t, \varepsilon)[\rho v] - \left[ \rho v(v + \beta t) - \frac{\varepsilon}{\rho} \right],
\end{aligned}
\]
with initial data: \( w(0) = 0, \ x(0) = 0. \)

From (33), we obtain
\[
\begin{aligned}
w(t, \varepsilon) &= \sqrt{\rho_{-} - \rho_{+}} \left( (u_{+} - u_{-})^2 - \varepsilon \left( \frac{1}{\rho_{+}} - \frac{1}{\rho_{-}} \right)^2 \right) t, \\
v_{\delta} &= \frac{\rho_{+} u_{+} - \rho_{-} u_{-} + \frac{dW(t, \varepsilon)}{dt}}{\rho_{+} - \rho_{-}}, \\
x(t, \varepsilon) &= v_{\delta} t + \frac{1}{2} \beta t^2,
\end{aligned}
\]
for \( \rho_{+} \neq \rho_{-} \) and
\[
\begin{aligned}
\omega(t, \varepsilon) &= (\rho_{-} u_{-} - \rho_{+} u_{+}) t, \\
v_{\delta} &= \frac{1}{2} (u_{+} + u_{-}), \\
x(t, \varepsilon) &= \frac{1}{2} (u_{+} + u_{-}) t + \frac{1}{2} \beta t^2,
\end{aligned}
\]
for \( \rho_{+} = \rho_{-} \).

Therefore, by using (4), we obtain the Riemann solution to (1)–(3). If \( (u_{+}, \rho_{+}) \in (I \cup II \cup III \cup IV)(u_{-}, \rho_{-}) \), combining (4) and (28), the Riemann solution to (1)–(3) can be expressed by
\[
(u_{-} + \beta t, \rho_{-}) + J_{\delta} + (v_{\delta} + \beta t, \rho_{\delta}) + J_{\delta} + (u_{+} + \beta t, \rho_{+}),
\]
where \( (v_{\delta}, \rho_{\delta}) \) is given in (29). If \( (u_{+}, \rho_{+}) \in V \cup S_{\delta} \), from (4) and (30), the Riemann solution to (1)–(3) can be expressed by
\[
(u_{-} + \beta t, \rho_{-}) + \delta S + (u_{+} + \beta t, \rho_{+}).
\]

4. The Limits of Riemann solutions to (1)–(3) as pressure vanishes. According to the relations of \( u_{-} \) and \( u_{+} \), we divide our discussion into three cases.
4.1. The Limit of Riemann solutions for $u_- > u_+$. In this subsection, we study the limit behavior of Riemann solutions when $u_- > u_+$ (see Fig. 4.1(a)). We divide our discussion into two steps. Firstly, we identify and analyze the formation of delta shock wave when $\varepsilon$ tends to certain value. Next, we display how the strength and propagation speed of the delta shock wave change when $\varepsilon$ tends to zero.

Lemma 4.1. Suppose that $u_- > u_+$, then, if $\rho_+ \neq \rho_-$, there exist two certain values $\varepsilon_1, \varepsilon_2 > 0$, such that $(u_+, \rho_+) \in IV(u_-, \rho_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$; $(u_+, \rho_+) \in V(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_2$. If $\rho_+ = \rho_-$, there exist certain value $\varepsilon_2 > 0$ such that $(u_+, \rho_+) \in IV(u_-, \rho_-)$ when $\varepsilon > \varepsilon_2$; $(u_+, \rho_+) \in V(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_2$.

Proof. Suppose that $u_- > u_+$ and $(u_+, \rho_+) \in IV(u_-, \rho_-)$, then $(u_+, \rho_+)$ satisfies the following conditions (see Fig. 4.1(a)):

$$u_+ < u_- - \frac{\sqrt{\varepsilon}}{\rho_-} + \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ > \rho_-,$$

(38)

$$u_+ < u_- + \frac{\sqrt{\varepsilon}}{\rho_-} - \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ < \rho_-,$$

(39)

and

$$u_+ > u_- - \frac{\sqrt{\varepsilon}}{\rho_-} - \frac{\sqrt{\varepsilon}}{\rho_+}.$$

(40)

If $\rho_+ \neq \rho_-$, from (38) and (39), we derive

$$\sqrt{\varepsilon} \left| \frac{1}{\rho_+} - \frac{1}{\rho_-} \right| < u_- - u_+,$$

(41)

i.e.,

$$\varepsilon < \left( \frac{(u_- - u_+)(\rho_- - \rho_+)}{\rho_- - \rho_+} \right)^2.$$

(42)

Setting

$$\varepsilon_1 = \left( \frac{(u_- - u_+)(\rho_- - \rho_+)}{\rho_- - \rho_+} \right)^2,$$

(43)

we have $(u_+, \rho_+) \in IV \cup V(u_-, \rho_-)$ when $\varepsilon < \varepsilon_1$. It follows from (40) that

$$\varepsilon > \left( \frac{\rho_- - \rho_+ (u_- - u_+)}{\rho_- + \rho_+} \right)^2.$$

(44)

Setting

$$\varepsilon_2 = \left( \frac{(u_- - u_+)(\rho_- - \rho_+)}{\rho_- + \rho_+} \right)^2,$$

(45)

we obtain $(u_+, \rho_+) \in IV(u_-, \rho_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$, and $(u_+, \rho_+) \in V(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_2$.

If $\rho_+ = \rho_-$, It follows from (38) and (39) that $(u_+, \rho_+) \in IV \cup V(u_-, \rho_-)$ for any $\varepsilon > 0$. By using (40), we get

$$\varepsilon > \frac{\rho_-^2 (u_- - u_+)^2}{4} = \left( \frac{(u_- - u_+)(\rho_- - \rho_+)}{\rho_- + \rho_+} \right)^2.$$

(46)

Thus, we have $(u_+, \rho_+) \in IV(u_-, \rho_-)$ when $\varepsilon > \varepsilon_2$ and $(u_+, \rho_+) \in V(u_-, \rho_-)$ when $\varepsilon < \varepsilon_2$. The proof is completed.
From Lemma 4.1, we find that there is no delta shock wave when $\varepsilon > \varepsilon_2$. As $\varepsilon$ decreases, from (26)–(27), we find that the tangent of two contact discontinuities $J_1^*$ and $J_2^*$ become steeper.

Firstly, we discuss the situation $\varepsilon_2 < \varepsilon < \varepsilon_1$ for $\rho_+ \neq \rho_-$. As $\varepsilon$ decreases, we find that the tangent of two contact discontinuities $J_1^*$ and $J_2^*$ become steeper.

Letting $\varepsilon \rightarrow \varepsilon_2$, we obtain

$$J_1^* : \left\{ \begin{array}{l} v_1^* = u_+ + \sqrt{\varepsilon} \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right), \\ \sigma_1^*(t) = u_+ + \beta t - \sqrt{\varepsilon} = v_1^* + \beta t - \sqrt{\varepsilon}, \quad \rho_1^* > \rho_-, \end{array} \right. \quad (47)$$

$$J_2^* : \left\{ \begin{array}{l} u_+ = v_2^* - \sqrt{\varepsilon} \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right), \\ \sigma_2^*(t) = u_+ + \beta t + \sqrt{\varepsilon} = v_2^* + \beta t + \sqrt{\varepsilon}, \quad \rho_2^* > \rho_+, \end{array} \right. \quad (48)$$

the intermediate state $(v_1^*, \rho_1^*)$ is given by (29) (see Fig. 4.1(a)). From the second equation of (29), letting $\varepsilon \rightarrow \varepsilon_2$, we derive

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \frac{u_+ + \sqrt{\varepsilon}}{\rho_+} = \lim_{\varepsilon \rightarrow \varepsilon_2} \left( \frac{1}{2} \left( u_+ + \sqrt{\varepsilon} \right) - \frac{1}{2} \left( u_- - \sqrt{\varepsilon} \right) \right) = 0. \quad (49)$$

It indicates that

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_1^* = \infty. \quad (50)$$

Thus we have the following result.

**Lemma 4.2.** Setting $u_3(t, \varepsilon_2) = \frac{\rho_+ - \rho_- + \rho_+ u_+}{\rho_- + \rho_+} + \beta t$, we have

$$\lim_{\varepsilon \rightarrow \varepsilon_2} u_1^* = \lim_{\varepsilon \rightarrow \varepsilon_2} \left( v_1^* + \beta t \right) = \lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_1^*(t) = \lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_2^*(t) = u_3(t, \varepsilon_2), \quad (51)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{x_1(t, \varepsilon)}^{x_2(t, \varepsilon)} \rho_1^* dx = \left( u_3(t, \varepsilon_2)[\rho] - [\rho(v + \beta t)] \right) t, \quad (52)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{x_1(t, \varepsilon)}^{x_2(t, \varepsilon)} \rho_1^* v_1^* dx = \left( u_3(t, \varepsilon_2)[\rho v] - \left[ \rho v(v + \beta t) - \frac{\varepsilon}{\rho} \right] \right) t, \quad (53)$$

where $[\rho] = \rho_+ - \rho_-,$

$$x_1(t, \varepsilon) = \int_0^t \sigma_1^*(\tau) d\tau = \left( u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right) t + \frac{1}{2} \beta t^2, \quad (54)$$

$$x_2(t, \varepsilon) = \int_0^t \sigma_2^*(\tau) d\tau = \left( u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) t + \frac{1}{2} \beta t^2. \quad (55)$$

**Proof.** Letting $\varepsilon \rightarrow \varepsilon_2$, from the first equation of (29) and (4), we get

$$\lim_{\varepsilon \rightarrow \varepsilon_2} u_1^* = \lim_{\varepsilon \rightarrow \varepsilon_2} \left( v_1^* + \beta t \right) = \lim_{\varepsilon \rightarrow \varepsilon_2} \left( \frac{1}{2} \left( u_+ + \sqrt{\varepsilon} \right) + \frac{1}{2} \left( u_- - \sqrt{\varepsilon} \right) + \beta t \right). \quad (56)$$

Combining (45) and (56), we obtain

$$\lim_{\varepsilon \rightarrow \varepsilon_2} u_1^* = \frac{\rho_+ - \rho_- + \rho_+ u_+}{\rho_- + \rho_+} + \beta t = u_3(t, \varepsilon_2). \quad (57)$$

It follows from (45), (47) and (48) that

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_1^*(t) = u_- - \frac{\rho_+ - \rho_- (u_- - u_+)}{\rho_+ + \rho_-} \frac{1}{\rho_-} + \beta t = u_3(t, \varepsilon_2), \quad (58)$$
Therefore, by using (58)-(59), we see that the two contact discontinuities \( J_1^2 \) and \( J_2^2 \) will coincide when \( \varepsilon \) arrives at \( \varepsilon_2 \).

Using the first equation of Rankine-Hugoniot conditions (25) for both \( J_1^2 \) and \( J_2^2 \), we get
\[
\begin{align*}
&\sigma_1^2(\rho_+^2 - \rho_-) = \rho_-^2 \left( v_+^2 + \beta t \right) - \rho_-^2 \left( u_+ - \beta t \right), \\
&\sigma_2^2(\rho_+ - \rho_-) = \rho_+^2 \left( u_+ + \beta t \right) - \rho_-^2 \left( v_+^2 + \beta t \right).
\end{align*}
\]
(60)

From (58)-(60), we have
\[
\lim_{\varepsilon \to \varepsilon_2} \left( \sigma_2^2 - \sigma_1^2 \right) \rho_+^2 = \delta_s(t, \varepsilon_2)[\rho] - \left[ \rho(v + \beta t) \right].
\]
(61)

Combining (47)-(48), (54)-(55) and (61), we derive
\[
\lim_{\varepsilon \to \varepsilon_2} \int_{x_1(t, \varepsilon)}^{x_2(t, \varepsilon)} \rho_+^2 \, dx = \lim_{\varepsilon \to \varepsilon_2} \left( \sigma_2^2 - \sigma_1^2 \right) \rho_+^2 t = (u_3(t, \varepsilon_2)[\rho] - \left[ \rho(v + \beta t) \right]) t.
\]
(62)

Using the second equation of Rankine-Hugoniot conditions (25) for both \( J_1^2 \) and \( J_2^2 \), we obtain
\[
\begin{align*}
&\sigma_1^2(\rho_+^2 v_+^2 - \rho_- u_-) = \rho_-^2 v_+^2 \left( v_+^2 + \beta t \right) - \rho_-^2 \left( u_+ - \beta t \right) - \frac{\varepsilon}{\rho_-^2} + \frac{\varepsilon}{\rho_+}, \\
&\sigma_2^2(\rho_+ u_+ - \rho_-^2 v_+^2) = \rho_+ u_+ \left( u_+ + \beta t \right) - \rho_-^2 v_+^2 \left( v_+^2 + \beta t \right) - \frac{\varepsilon}{\rho_+} + \frac{\varepsilon}{\rho_-^2}.
\end{align*}
\]
(63)

From (58)-(59) and (63), we obtain
\[
\lim_{\varepsilon \to \varepsilon_2} \left( \sigma_2^2 - \sigma_1^2 \right) \rho_+^2 v_+^2 = u_3(t, \varepsilon_2)[\rho v] - \left[ \rho v(v + \beta t) - \frac{\varepsilon}{\rho} \right].
\]
(64)

Using (47)-(48), (54)-(55) and (64), we have (53). The proof is completed. \( \square \)

From Lemma 4.2, we find that the two contact discontinuities \( J_1^2 \) and \( J_2^2 \) coincide as \( \varepsilon \) tends to \( \varepsilon_2 \) (see Fig. 4.1(b)). The equalities (50) and (52) show that \( \rho_+^2 \) possesses the singularity, which is a weighed Dirac delta function with speed \( u_3(t, \varepsilon_2) \). Thus, we conclude that there appears a delta shock wave when \( \varepsilon \) arrives at \( \varepsilon_2 \), which is consistent to the result in [13].

On the other hand, for \( \rho_+ \neq \rho_- \), combining (34) and (45), we derive
\[
\omega(t, \varepsilon_2) = \frac{2\rho_- \rho_+ (u_- - u_+)}{\rho_- + \rho_+} = (u_3(t, \varepsilon_2)[\rho] - \left[ \rho(v + \beta t) \right]) t,
\]
(65)

\[
u_3(t, \varepsilon_2) = v_3^2 + \beta t = \frac{\rho_- u_- + \rho_+ u_+}{\rho_- + \rho_+} + \beta t.
\]
(66)

From the above discussions, we see that the quantities \( \omega(t, \varepsilon) \), \( u_3(t, \varepsilon) \) and the limits of \( u_3^2 \), \( \sigma_1^2 \) and \( \sigma_2^2 \) are consistent with (34) when \( \varepsilon \) arrives at \( \varepsilon_2 \). For \( \rho_+ = \rho_- \), we can get same conclusion.

Therefore, when \( \varepsilon \to \varepsilon_2 \), the limit of the Riemann solutions to (1)-(3) is the delta shock solution of (1)-(3) in the boundary case \( (u_+, \rho_+) \in S_\delta \) (see Fig. 4.1(a)).

Next, we discuss the situation \( 0 < \varepsilon < \varepsilon_2 \), in which \( (u_+, \rho_+) \in V(u_-, \rho_-) \). In this situation, the Riemann solution to (1)-(3) is (37) which consists of a delta shock wave. The strength and propagation speed of this delta shock wave are given in (34) for \( \rho_+ \neq \rho_- \). From (34), we obtain
\[
\frac{d\omega(t, \varepsilon)}{d\varepsilon} < 0 \quad \text{and} \quad \frac{du_3(t, \varepsilon)}{d\varepsilon} < 0.
\]
(67)
The formula (67) indicates that both the strength and the propagation speed of the delta shock wave increase when ε decreases. Furthermore, taking \( \varepsilon \to 0 \) in (34), we get

\[
\lim_{\varepsilon \to 0} w(t, \varepsilon) = \sqrt{\rho_+ \rho_-} (u_- - u_+ t),
\]

\[
\lim_{\varepsilon \to 0} u_\delta(t, \varepsilon) = \frac{\sqrt{\rho_+ u_+} + \sqrt{\rho_- u_-}}{\sqrt{\rho_+} + \sqrt{\rho_-} + \beta t}.
\]

From (68)–(69), we derive the delta shock solution to (1)–(3) converges to the delta shock solution to (6) and (3) as \( \varepsilon \to 0 \). For \( \rho_+ = \rho_- \), we can obtain same result.

From the above discussion, we conclude that the two contact discontinuities of the Riemann solution to (1)–(3) converge to a delta shock solution to (1)–(3) at \( \varepsilon = \varepsilon_2 \). As \( \varepsilon \) continues to decrease, both the strength and propagation speed of this delta shock are becoming stronger. Eventually, when \( \varepsilon \to 0 \), the delta shock solution converges to the Riemann solution to (6) and (3) for the corresponding case in (21). Therefore, we have proved the first part of Theorem 1.1.

4.2. Limit of Riemann solutions for \( u_- < u_+ \). In this subsection, we study the limit behavior of Riemann solutions to (1)–(3) when \( \varepsilon \) tends to zero (see Fig. 4.2(a)).

Lemma 4.3. Suppose that \( u_- < u_+ \), then, if \( \rho_+ \neq \rho_- \), there exists \( \varepsilon_3 > 0 \) such that \( (u_+, \rho_+) \in I(u_-, \rho_-) \) when \( 0 < \varepsilon < \varepsilon_3 \); If \( \rho_+ = \rho_- \), for each \( \varepsilon > 0 \), we have \( (u_+, \rho_+) \in I(u_-, \rho_-) \).

Proof. Suppose that \( u_- < u_+ \) and \( (u_+, \rho_+) \in I(u_-, \rho_-) \), then \( (u_+, \rho_+) \) satisfies following conditions (see Fig. 4.2(a)):

\[
u_+ > u_- - \sqrt{\varepsilon} - \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ < \rho_-,
\]

\[
u_+ > u_- + \frac{\sqrt{\varepsilon}}{\rho_-}, \quad \rho_+ > \rho_-.
\]

If \( \rho_+ \neq \rho_- \), from (70) and (71), we derive

\[
\varepsilon < \frac{1}{\rho_+} - \frac{1}{\rho_-} < u_+ - u_-,
\]

i.e.,

\[
\varepsilon < \left( \frac{(u_+ - u_-) \rho_- \rho_+}{\rho_- - \rho_+} \right)^2.
\]
Setting

$$\varepsilon_3 = \left( \frac{(u_+ - u_-)\rho_- - \rho_+}{\rho_- - \rho_+} \right)^2,$$  \hspace{1cm} (74)

we have \((u_+, \rho_+) \in \mathcal{I}(u_-, \rho_-)\) when \(\varepsilon < \varepsilon_3\). The proof is completed.

By Lemma 4.3, for any given \(\varepsilon \in (0, \varepsilon_3)\), the Riemann solution to (1)-(3) is (36), where

\[
J_{\varepsilon 1}^r : \begin{cases} 
 v_{\varepsilon}^r = u_- + \sqrt{\varepsilon} \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right), \\
 \sigma_{\varepsilon 1}^r(t) = u_- + \beta t - \sqrt{\varepsilon} \frac{\rho_-}{\rho_+}, \quad \rho_{\varepsilon}^r < \rho_-;
\end{cases}
\]

\[
J_{\varepsilon 2}^r : \begin{cases} 
 u_+ = v_{\varepsilon}^r - \sqrt{\varepsilon} \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right), \\
 \sigma_{\varepsilon 2}^r(t) = u_+ + \beta t + \sqrt{\varepsilon} \frac{\rho_-}{\rho_+}, \quad \rho_{\varepsilon}^r < \rho_+;
\end{cases}
\]

the intermediate state \((v_{\varepsilon}^r, \rho_{\varepsilon}^r)\) is given by (29) (see Fig. 4.2(a)). From the second equation of (29), we get

$$\rho_{\varepsilon}^r = \frac{2\sqrt{\varepsilon}}{u_+ - u_- + \sqrt{\varepsilon} \left( \frac{1}{\rho_-} + \frac{1}{\rho_+} \right)}.$$ \hspace{1cm} (77)

From (75)-(77), we have

$$\lim_{\varepsilon \to 0} \rho_{\varepsilon}^r = 0, \quad \lim_{\varepsilon \to 0} \sigma_{\varepsilon 1}^r(t) = u_- + \beta t, \quad \lim_{\varepsilon \to 0} \sigma_{\varepsilon 2}^r(t) = u_+ + \beta t.$$ \hspace{1cm} (78)

Thus, when \(u_- < u_+\), for the Riemann solution to (1)-(3), as \(\varepsilon \to 0\), we obtain that \(\rho_{\varepsilon}^r\) vanishes and two contact discontinuities \(J_{\varepsilon 1}^r\) and \(J_{\varepsilon 2}^r\) converge to two contact discontinuities connecting the states \((u_+ + \beta t, \rho_+)\) and the vacuum \((\rho_+ = 0)\), which is one kind of Riemann solution to (6) and (3) for the corresponding case (22) (see Fig. 4.2(b)). Therefore, we have proved the second part of Theorem 1.1.
4.3. Limit of Riemann solutions for \( u_- = u_+ \). The Riemann solution to (1)–(3) is (36), where the intermediate state \((v_\varepsilon^*, \rho_\varepsilon^*)\) is given by (29). Combining (4), (29) and \( u_- = u_+ \), we have

\[
\lim_{\varepsilon \to 0} u_\varepsilon^* = \lim_{\varepsilon \to 0} (v_\varepsilon^* + \beta t) = u_- + \beta t. \tag{79}
\]

It follows from (26)–(27) and \( u_- = u_+ \) that

\[
\lim_{\varepsilon \to 0} \sigma_1^\varepsilon(t) = \lim_{\varepsilon \to 0} \sigma_2^\varepsilon(t) = u_- + \beta t. \tag{80}
\]

From the identities (79), (80), as \( \varepsilon \) goes to zero, we conclude that the two contact discontinuities \( J_1^\varepsilon \) and \( J_2^\varepsilon \) converge to one contact discontinuity with the propagation speed \( u_- + \beta t \), which is the Riemann solution to (6) and (3) for the corresponding case (23) (see Fig. 4.3(b) or Fig. 4.4(b)). Therefore, we have proved the third part of Theorem 1.1.

5. Discussion. We have identified and analyzed the phenomena of concentration and cavitation to the Chaplygin gas equations with a friction term as the pressure vanishes. For \( u_- > u_+ \), we show that, as \( \varepsilon \) decreases to \( \varepsilon_2 \), the Riemann solution converges to a delta shock wave solution of the same system (1)–(2). As \( \varepsilon \) continues to decrease, we discover that both the strength and propagation speed of this delta shock are becoming stronger. When \( \varepsilon \to 0 \), the delta shock wave solution converges to the Riemann solution of the pressureless gas dynamics model with a friction term. For \( u_- < u_+ \), as \( \varepsilon \to 0 \), we obtain that the two contact discontinuities converge to two contact discontinuities connecting the states \((u_\pm + \beta t, \rho_\pm)\) and the vacuum. For \( u_- = u_+ \), the two contact discontinuities converge to one contact discontinuity with the propagation speed \( u_- + \beta t \).

The behavior of Riemann solutions for generalized Chaplygin gas equations with a source term (1) and (5) when pressure vanishes is an interesting problem. We leave this problem for our future work.

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