

SECOND ORDER DENSITY PERTURBATIONS
FOR DUST COSMOLOGIES

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Abstract

We present simple expressions for the relativistic first and second order fractional density perturbations for FL cosmologies with dust, in four different gauges: the Poisson, uniform curvature, total matter and synchronous gauges. We include a cosmological constant and arbitrary spatial curvature in the background. A distinctive feature of our approach is our description of the spatial dependence of the perturbations using a canonical set of quadratic differential expressions involving an arbitrary spatial function that arises as a conserved quantity. This enables us to unify, simplify and extend previous seemingly disparate results. We use the primordial matter and metric perturbations that emerge at the end of the inflationary epoch to determine the additional arbitrary spatial function that arises when integrating the second order perturbation equations. This introduces a non-Gaussianity parameter into the expressions for the second order density perturbation. In the special case of zero spatial curvature we show that the time evolution simplifies significantly, and requires the use of only two non-elementary functions, the so-called growth suppression factor at the linear level, and one new function at the second order level. We expect that the results will be useful in applications, for example, studying the effects of primordial non-Gaussianity on the large scale structure of the universe.

PACS numbers: 04.20.-q, 98.80.-k, 98.80.Bp, 98.80.Jk

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1 Introduction

The increasingly accurate observations of the cosmic microwave background (CMB) and the large scale structure (LSS) of the universe that are becoming available are influencing theoretical developments in cosmology. Analyzing these observations necessitates the theoretical study of possible deviations from linearity, for example, how primordial non-Gaussianity affects the anisotropy of the CMB and the LSS, using nonlinear perturbations of Friedmann-Lemaître (FL) cosmologies (see for example [1, 2, 3]). Much of the theoretical work has dealt with nonlinear perturbations of flat FL cosmologies with dust (a matter-dominated universe, see for example [4, 5]) and more recently, also with a cosmological constant (a ΛCDM universe, see for example [6]–[9]). One aspect of the work is to provide an expression for the second order fractional density perturbation \( \delta^2 \) that can be used for comparison with observations, and this is the focus of the present paper.

In a recent paper [10], here referred to as UW, we derived a new expression for \( \delta^2 \) using the Poisson gauge, also including the effects of spatial curvature\(^1\) but subject to the restriction that the perturbation at linear order is purely scalar\(^2\). This expression for \( \delta^2 \) contains the integral of a complicated quadratic source term involving two arbitrary spatial functions, which makes it difficult to obtain a clear physical understanding. If one assumes that the decaying mode of the scalar perturbation at the linear order is zero, however, then the scalar perturbation depends on only one such function. In this case we showed that the temporal and spatial dependence of \( \delta^2 \) can be displayed explicitly, in a form that provides direct physical insight.

In the present paper we investigate what effect the choice of gauge has on the form of \( \delta^2 \). Specifically, we will use the expression for \( \delta^2 \) in UW and the change of gauge formulas in appendix B to calculate \( \delta^2 \) in three other commonly used gauges: the uniform curvature, the total matter\(^3\) and the synchronous gauges. We will use the notation \( \delta^2_\bullet \), where the bullet identifies the gauge. Our first main result is that \( \delta^2_\bullet \) has the following common structure for the Poisson, the uniform curvature and the total matter gauges:

\[
\delta^2_\bullet = \frac{A_1_\bullet}{\text{the super-horizon part}} \zeta^2 + \frac{A_2_\bullet}{\text{the super-horizon part}} \mathcal{D}(\zeta) + \frac{1}{3} m^2 x g \left[ A_{3_\bullet} (\mathcal{D}^2 \zeta)^2 + A_{4_\bullet} \mathcal{D}^2 \mathcal{D}(\zeta) + A_{5_\bullet} \mathcal{D}^2 \zeta^2 \right] \\
+ \frac{4}{7} m^2 x g^2 \left[ \mathcal{B}_3 \mathcal{D}^2 (\mathcal{D} \zeta)^2 + \mathcal{B}_4 \mathcal{D}^4 \mathcal{D}(\zeta) \right] \quad (1)
\]

Here \( m \) is a constant and the coefficients \( A_{i_\bullet}, i = 1, \ldots, 5, \mathcal{B}_3 \) and \( \mathcal{B}_4 \) are functions of the scale factor \( x \), while the spatial dependence is determined by the conserved quantity \( \zeta(x^i) \), which appears in seven quadratic differential expressions:

\[
\zeta^2, \quad \mathcal{D}(\zeta), \quad (\mathcal{D} \zeta)^2, \quad \mathcal{D}^2 \mathcal{D}(\zeta), \quad \mathcal{D}^2 \zeta^2, \quad \mathcal{D}^2 (\mathcal{D} \zeta)^2, \quad \mathcal{D}^4 \mathcal{D}(\zeta). \quad (2)
\]

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\(^1\)See equations (33)-(36) in UW.
\(^2\)For the motivation for imposing this restriction, we refer to [3], page 4.
\(^3\)The total matter gauge is referred to as the comoving gauge in [11] and in [12].
The spatial differential operators in (1) and (2) are defined in appendix A. We will refer to the group of terms in (1) whose differential expressions have zero weight in the spatial differential operator $D_i$ as the super-horizon part. The intermediate group of terms having weight two in $D_i$ (coefficient $m^{-2}$) is referred to as the post-Newtonian part. Finally, we refer to the group of terms in (1) having weight four in $D_i$ (coefficient $m^{-4}$) as the Newtonian part.

The common structure for $(2)^{\delta_k}$ in these three gauges is due to the fact that they all use the same spatial gauge. The differences thus depend on different temporal gauges, which affect the coefficients $A_i, i = 1, \ldots, 5,$ but not $B_3$ and $B_4$. On the other hand, the synchronous gauge uses a different spatial gauge, which has the effect of adding a term to the Newtonian part with spatial dependence given by $(D^2 \zeta)^2$, thereby adding a new quadratic differential expression to the set (2). For this reason we will treat this case separately in the paper.

The evolution of $(2)^{\delta_k}$ in the Poisson gauge is determined by four functions of time that are written as integrals, and these are the main source of the complexity of the expression. Our second main result is to show that if the background spatial curvature is zero, then three of these integral functions can be written in an explicit form. This fact enables us to give simple expressions for $(2)^{\delta_k}$ in all four gauges under consideration when the spatial geometry is flat.

The outline of the paper is as follows. In the next section we give a unified expression for the first order fractional density perturbation in the four gauges. In section 3 we derive the corresponding second order results, and address the issue of initial conditions. Section 4 deals exclusively with the spatially flat case $K = 0$. We show that the time dependence simplifies significantly, and then give a detailed comparison with previous work dealing with this case. Section 5 contains the concluding remarks. Finally, in appendix A we define the various spatial differential operators and in appendix B we derive the transformation laws that relate the density perturbations for the four gauges under consideration.

2 The density perturbation at first order

The background cosmology is an FL universe with scale factor $a$, Hubble scalar $H$ and curvature parameter $K$, containing dust with background matter density $(0)^{\rho_m}$ and a cosmological constant $\Lambda$. We introduce the usual density parameters:

$$\Omega_m := \frac{(0)^{\rho_m}}{3H^2}, \quad \Omega_k := -\frac{K}{H^2}, \quad \Omega_\Lambda := \frac{\Lambda}{3H^2},$$

which satisfy

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1.$$  \hspace{1cm} (3)

We use the dimensionless scale factor $x := a/a_0$, normalized at some reference epoch $t_0$, as time variable. The conservation law shows that $a^3(0)^{\rho_m}$ is constant, which we

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4 We use units such that $8\pi G = 1 = c$.

5 When $t_0$ is the present time, $x = (1 + z)^{-1}$, where $z$ is the redshift. We note that $x$ is related to the conformal time $\eta$, which we shall sometimes use, according to $\partial_\eta = Hx\partial_x$. 

write in terms of $\Omega_m$ and the dimensionless Hubble scalar $\mathcal{H} := aH$ as

$$\mathcal{H}^2 x \Omega_m = m^2. \quad (5)$$

Setting $x = 1$ shows that the constant $m$ is given by $m^2 = \mathcal{H}_0^2 \Omega_{m,0}$. It follows that

$$\mathcal{H}^2 = \mathcal{H}_0^2 (\Omega_{\Lambda,0} x^2 + \Omega_{k,0} + \Omega_{m,0} x^{-1}). \quad (6)$$

Equations (3)-(6) determine $\mathcal{H}$, $\Omega_m$, $\Omega_k$ and $\Omega_\Lambda$ explicitly in terms of $x$.

The gauge invariants that describe the scalar linear perturbations of the metric and matter in the Poisson gauge are the Bardeen potentials $(1)\psi_p$ and $(1)\Phi_p$, the velocity potential $(1)v_p$ and the fractional density perturbation $(1)\delta_p$. In the special case when the decaying mode of the scalar perturbation is set to zero we have the following expressions:

$$\Psi_p = \Phi_p = g(x)\zeta(x'), \quad \mathcal{H}v_p = -\frac{2}{3} \Omega_m^{-1} fg \zeta, \quad (7a)$$

$$\delta_p = -2\Omega_m^{-1} (f + \Omega_k) g \zeta + \frac{2}{3} m^{-2} x g D^2 \zeta, \quad (7b)$$

where

$$g(x) := \frac{3}{2} m^2 \frac{\mathcal{H}}{x^2} \int_0^x \frac{d\bar{x}}{\mathcal{H}(\bar{x})^3}, \quad f(x) := 1 + g^{-1} x \partial_x g. \quad (8)$$

Here the arbitrary spatial function $\zeta(x')$ equals the conserved quantity that we introduced in [12], denoted by $\zeta_v$, which can be written in the form

$$\zeta_v = (1 + \frac{2}{3} \Omega_k \Omega_m^{-1}) \Psi_p - \mathcal{H}v_p. \quad (9)$$

We emphasize that $\zeta_v$ is exactly conserved for dust (see [13], equation (B2), and [12] equations (69)-(71)). The fact that $\zeta = \zeta_v$ was established in UW (see equations (21)-(24)). We note in passing that if the background spatial curvature is zero ($\Omega_k = 0$) then $\zeta_v$ is the comoving curvature perturbation, often denoted by $\mathcal{R}$ (see [15], equation (7.46) in conjunction with (7.6) and (7.8)).

We can use (9) to simplify the expression for $\delta_p$, as follows. On substituting for $\Psi_p$ and $v_p$ from (7a) into (9) the function $\zeta$ cancels as a common factor and we obtain the following algebraic constraint relating $f$ to $g$:

$$(3\Omega_m + 2\Omega_k + 2f)g = 3\Omega_m. \quad (10)$$

Using this we can simplify the expression (7b) to obtain:

$$\delta_p = -3(1 - g)\zeta + \frac{2}{3} m^{-2} x g D^2 \zeta. \quad (11)$$

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6These gauge invariants are introduced in appendix B. Our strategy for working with gauge invariants is described in the first paragraph of that appendix.

7See equations (19b) and (30)-(32) in UW.

8We note in passing that the function $g(x)$, defined up to a constant factor, is sometimes referred to as the growth suppression factor. See for example [7], in the text following equation (2.3). Our function $f(x)$ equals their function $f(\eta)$ in equation (2.9), on noting that $g'(\eta) = \mathcal{H}x\partial_x g(x)$.

9Referring to UW, use equations (11) and (12) to rewrite equation (21).
By combining this equation with the transformation formulas (101), (105) and (115) in appendix B, we can give a unified expression for \((1)\delta\), the gauge invariant associated with the first order density perturbation in the four gauges that we are considering in this paper:

\[
(1)\delta = A_\ast \zeta + \frac{2}{3} m^{-2} x g D^2 \zeta, \tag{12a}
\]

where

\[
A_p = -3(1 - g), \quad A_c = -3, \quad A_v = A_s = -2 \Omega^{-1} \Omega_k g. \tag{12b}
\]

Here the subscripts \(p, c, v\) and \(s\) identify the Poisson, uniform curvature, total matter and synchronous gauges, respectively.

The first term of \((1)\delta\) in (12), which has zero weight in the spatial differential operator \(D_i\), is referred to as the super-horizon term, while the second term, which has weight two, is referred to as the Newtonian term. We note that the super-horizon term is gauge-dependent, while the Newtonian part is gauge-independent. Finally, the form of the super-horizon term in \((1)\delta\), as given by (12), deserves comment: it is independent of time. This is to be expected since the quantity \(\zeta^\rho := -\frac{1}{3} (1)\delta\) satisfies the “conservation law" \(\partial_\eta \zeta^\rho = \frac{1}{2} D^2 v_p\), which suggests that \((1)\delta\) will be constant in a regime in which spatial derivatives can be neglected.

The expressions for the gauge invariants at first order given by equations (7), (8) and (9) provide the foundation for generalizing to second order perturbations. In particular, the constraint (10) will be used frequently in simplifying the expressions for the second order density perturbations.

3 The density perturbation at second order

In this section we derive explicit expressions for the second order fractional density perturbation \((2)\delta\), in the uniform curvature, total matter and synchronous gauges using the expression for \((2)\delta_p\) in the Poisson gauge given in UW, subject to the following restrictions: i) the perturbations at linear order are purely scalar, and ii) the decaying mode of the scalar perturbation is zero.

3.1 The Poisson gauge

The gauge invariants\(^{11}\) that describe the scalar second order perturbations of the metric and matter in the Poisson gauge are the Bardeen potentials \((2)\Psi_p\) and \((2)\Phi_p\), the velocity potential \((2)v_p\) and the fractional density perturbation \((2)\delta_p\). The equations that determine these gauge invariants in the case of dust cosmologies are given in a concise form in equations (14)-(16) in UW. A particular solution for \((2)\Psi_p\) is given by equations (26) in UW:

\[
(2)\Psi_p(x, x') = \frac{H}{x'} \int_0^{x'} \frac{\bar{\Omega}_m}{H} S(\bar{x}, x') d\bar{x}, \tag{13a}
\]

\(^{10}\)See [12], equations (65) and (66) specialized to a barotropic perfect fluid (\(\bar{\Gamma} = 0\)). See also [14], equation (18) in conjunction with (7)-(9).

\(^{11}\)These gauge invariants are introduced in appendix B.
where
\[ S(x, x^i) := m^{-2} \int_0^x S(\tilde{x}, x^i) d\tilde{x}, \] (13b)
and the source term \( S \) is given by equation (70a) in UW. \(^{12}\) With the solution for \((2)\Psi_p\) in the form (13), \((2)\delta_p\) and \((2)\nu_p\) are given by
\[ (2)\delta_p = \frac{2}{3} m^{-2} D^2(x(2)\Psi_p) + 3(2)\Psi_p - 2S + S_\delta, \] (14a)
\[ H(2)\nu_p = (1 + \frac{2}{3} \Omega_k \Omega_m^{-1})(2)\Psi_p - \frac{2}{3} S + S_v, \] (14b)
where
\[ S_\delta = S_D + 3S_\nu - 2(D^{(1)}\nu_p)^2, \] (15a)
\[ S_v = S_\nu - 2S^T \left( [f^{(1)}\delta_p - (1)\Psi]D_i(H^{(1)}\nu_p) \right), \] (15b)
and the source terms \( S_D \) and \( S_\nu \) are given by equations (70b) and (70c) in UW. These expressions follow from equations (15) and (16) in UW. It is necessary to eliminate the time derivative of \((2)\Psi_p\) using the following equation which is obtained by differentiating (13a) with respect to \( x \) and using (35a) and (35b):
\[ \partial_x (x(2)\Psi_p) = -(\frac{3}{2} \Omega_m + \Omega_k)(2)\Psi_p + \Omega_m S. \] (16)

Using the first order solution (7), we find that the source term \( S(x, x^i) \) in (13b) is given by \(^{13}\)
\[ S(x, x^i) = m^2 (T_1\zeta^2 + T_2D(\zeta) + m^{-2} [T_3(D\zeta)^2 + T_4D^2D(\zeta)]) , \] (17)
where
\[ T_1(x) := (x \Omega_m)^{-1} g^2((f - 1)^2 - 4\Omega_k), \] (18a)
\[ T_2(x) := -8(x \Omega_m)^{-1} g^2 \left( ((f - 1)^2 - \frac{1}{3}(1 - \Omega_m) + \Omega_k(1 + \frac{2}{3} \Omega_m^{-1} f^2) \right), \] (18b)
\[ T_3(x) := -\frac{1}{3} g^2 \left( 1 - \frac{4}{3} \Omega_m^{-1} f^2 \right), \] (18c)
\[ T_4(x) := \frac{4}{3} g^2 \left( 1 + \frac{2}{3} \Omega_m^{-1} f^2 \right). \] (18d)
Substituting (17) in (13b) leads to
\[ S(x, x^i) = \hat{T}_1\zeta^2 + \hat{T}_2D(\zeta) + m^{-2} \left[ \hat{T}_3(D\zeta)^2 + \hat{T}_4D^2D(\zeta) \right], \] (19)
where
\[ \hat{T}_A(x) := \int_0^x T_A(\tilde{x}) d\tilde{x}. \] (20)
Next, on substituting (19) in (13a) we obtain
\[ (2)\Psi_p = g \left( B_1(x)\zeta^2 + B_2(x)D(\zeta) + m^{-2}[B_3(x)(D\zeta)^2 + B_4(x)D^2D(\zeta)] \right), \] (21)

\(^{12}\) We have modified equations (26) in UW by using the constraint \( xH^2\Omega_m = m^2 \) to replace \( H^2 \) in the first integral and have rescaled \( S \) by a factor of \( m^{-2} \). In addition we set \( x_{initial} = 0 \), which is possible due to that we set the decaying mode to zero.

\(^{13}\) UW, equations (74)-(75). We have rescaled the \( T_A \) by multiplying by \( m^{-2} \) and have used (5).
where

\[ B_A(x) := \frac{\mathcal{H}}{x^2 g} \int_0^x \frac{\bar{\Omega}_m}{x} \bar{T}_A(\bar{x}) d\bar{x}. \] (22)

It is convenient to write equations (21) and (19) in the form

\[ (2) \psi_p = g(B_1 \zeta^2 + B_2 \mathcal{D}(\zeta) + \frac{2}{3} m^{-2} x g [B_3(D\zeta)^2 + B_4 D^2 D(\zeta)]), \] (23a)

\[ S = T_1 \zeta^2 + T_2 D(\zeta) + \frac{2}{3} m^{-2} x g (T_3(D\zeta)^2 + T_4 D^2 D(\zeta)), \] (23b)

where

\[ B_{1,2} := B_{1,2}, \quad B_{3,4} := (\frac{2}{3} x g)^{-1} B_{3,4}. \] (24a)

\[ T_{1,2} := T_{1,2}, \quad T_{3,4} := (\frac{2}{3} x g)^{-1} T_{3,4}. \] (24b)

We can now calculate \( (2) \psi_p \) and \( (2) \delta_p \) by substituting (23) into (14) and using the first order solution (7) to calculate the source terms (15). The results are as follows. For \( (2) \psi_p \) we obtain:

\[ \mathcal{H}(2) \psi_p = V_1 \zeta^2 + V_2 D(\zeta) + \frac{2}{3} m^{-2} g x [V_3(D\zeta)^2 + V_4 D^2 D(\zeta)], \] (25a)

where

\[ V_A(x) := -\frac{2}{3} T_A + (1 + \frac{2}{3} \Omega_k \Omega_m^{-1}) g B_A + \frac{2}{3} \Omega_m^{-1} g V_A, \] (25b)

with

\[ V_1 = g \left( 1 - 2 \Omega_m^{-1} f (f + \Omega_m + \Omega_k) \right), \] (25c)

\[ V_2 = 4 g \left( 1 + \frac{2}{3} \Omega_m^{-1} f (f - \Omega_k) \right), \] (25d)

\[ V_3 = -\frac{1}{3} f, \quad V_4 = \frac{1}{3} f. \] (25e)

For the density perturbation \( (2) \delta_p \) we obtain (1) with \( \star \) replaced by \( p \), with the coefficients \( A_{i,p} \) having the following form:

\[ A_{1,p} = -2 T_1 + 3 g B_1 + 2 \Omega_m^{-1} g^2 \left( (1 - f)^2 - 4 \Omega_k \right), \] (26a)

\[ A_{2,p} = -2 T_2 + 3 g B_2 + 8 \Omega_m^{-1} g^2 (1 + \frac{2}{3} \Omega_m^{-1} f^2), \] (26b)

\[ A_{3,p} = -2 T_3 + 3 g B_3 - g (5 + \frac{4}{3} \Omega_m^{-1} f^2), \] (26c)

\[ A_{4,p} = -2 T_4 + 3 g B_4 + B_2, \] (26d)

\[ A_{5,p} = B_1 + 4 g. \] (26e)

It should be noted that these coefficients give a particular solution for \( (2) \delta_p \) that corresponds to the particular solution (13) for \( (2) \psi_p \). We will give the general solution in section 3.5. The Einstein-de Sitter background arises as a special case \( (\Omega_m = 1, \Omega_A = 0, \Omega_k = 0) \). It follows from (8), in conjunction with (13) and (6), that \( g = \frac{2}{3}, f = 1 \). The definition (24) then yields \( B_1 = B_2 = 0, B_3 = \frac{1}{41}, B_4 = \frac{20}{31} \) and \( T_1 = T_2 = 0, T_3 = \frac{1}{10}, T_4 = 2 \). The expression for \( (2) \delta_p \) reduces to equation (44) in UW.
3.2 The uniform curvature and total matter gauges

The transformation formulas relating \((2)\delta_c\) and \((2)\delta_v\) to \((2)\delta_p\) are given by equations (101b) and (105b) in appendix B, which we repeat here:

\[
\begin{align*}
(2)\delta_c &= (2)\delta_p - 3(2)\Psi_p + S_\delta[(1)Z_{c,p}], \\
(2)\delta_v &= (2)\delta_p - 3H(2)\nu_p + S_\delta[(1)Z_{v,p}].
\end{align*}
\]

The source terms \(S_\delta[(1)Z_{c,p}]\) and \(S_\delta[(1)Z_{v,p}]\) are given by (103) and (107) in terms of \((1)\Psi_p, (1)\nu_p\) and \((1)\delta_p\), which in the case of zero decaying mode are given by (7). It follows that the source terms are a linear combination of the \(\zeta\)-expressions in (2) of weights zero and two in \(D_i\), as are \((2)\Psi_p\) and \((2)\nu_p\) which are given by (23a) and (25). Equations (27) thus imply that \((2)\delta_c\) and \((2)\delta_v\) are of the canonical form (1), with the Newtonian terms being unchanged. The coefficients \(A_{i,c}\) and \(A_{i,v}\) are obtained by appropriately collecting terms on the right side of equations (27), leading to

\[
\begin{align*}
A_{1,c} &= -2T_1 + 2\Omega^{-1}m^2 g^2 \left(1 + 4f + f^2 + 2\Omega_k + \frac{4}{3}\Omega_m \right), \\
A_{2,c} &= -2T_2 + 8\Omega^{-1}m^2 g^2 \left(1 + \frac{2}{3}\Omega_m^2 f^2\right), \\
A_{3,c} &= -2T_3 + g \left(1 - 2f - \frac{2}{3}\Omega_m - \frac{4}{3}\Omega_m f^2\right), \\
A_{4,c} &= -2T_4 + B_2 + \frac{3}{2}g \Omega_m, \\
A_{5,c} &= B_1 + g(1 + f),
\end{align*}
\]

and

\[
\begin{align*}
A_{1,v} &= 2\kappa x g \left[B_1 - \frac{2}{3}\Omega_m^{-1} g(f^2 - 3f - 6\Omega_m)\right], \\
A_{2,v} &= 2\kappa x g \left[B_2 - \frac{8}{3}\Omega_m^{-1} g f\right], \\
A_{3,v} &= 2\kappa x g B_3 - \frac{5}{3}\Omega_m^{-1} g (2f + 3\Omega_m), \\
A_{4,v} &= 2\kappa x g B_4 + B_2 - \frac{5}{3}\Omega_m^{-1} g f, \\
A_{5,v} &= B_1 - \frac{2}{3}\Omega_m^{-1} g(f^2 - 3f - 6\Omega_m).
\end{align*}
\]

Here we have used the fact that

\[
\Omega_k\Omega_m^{-1} = -\kappa x.
\]

where the constant \(\kappa\) is given by

\[
\kappa := \frac{K}{m^2},
\]

3.3 The synchronous gauge

The second order density perturbation \((2)\delta_s\) in the synchronous gauge is related to \((2)\delta_v\) according to equation (115) in appendix B, which we repeat here:

\[
(2)\delta_s = (2)\delta_v - \frac{4}{3}xm^{-2}(D^i\delta_v)(D_i\Psi_p).
\]
On evaluating the source term using (7a), (12) and the identity (73c) in appendix A and noting that \( (2) \delta_v \) has the general form (1), we obtain:

\[
(2) \delta_s = A_{1,v} \zeta^2 + A_{2,v} D(\zeta) \\
+ \frac{2}{3}m^{-2}xg \left[A_{3,v}(D\zeta)^2 + A_{4,v} D^2 D(\zeta) + A_{5,v} D^2 \zeta^2 \right] \\
+ \frac{4}{3}m^{-4}x^2 g^2 \left[(B_3 + \frac{1}{3}) D^2(D\zeta)^2 + (B_4 - \frac{4}{3}) D^4(D\zeta) + 2(D^2 \zeta)^2 \right],
\]

(32a)

where

\[
A_{3,v} = A_{3,v} - 4\kappa xg, \\
A_{4,v} = A_{4,v} - \frac{8}{3}\kappa xg,
\]

(32b)

and the \( A_{i,v} \) coefficients are given by (29). Note the appearance of the new quadratic differential expression \((D^2 \zeta)^2\) in the Newtonian part.

### 3.4 A second order conserved quantity \((2) \delta_c\)

We have shown earlier that the super-horizon term in \((1) \delta_c\) is independent of time. This reflects the fact that \(\zeta_\rho := -\frac{1}{3}(1) \delta_c\) satisfies a "conservation law" that reduces to \(\partial_x \zeta_\rho = 0\) when spatial derivative terms can be neglected. At second order, we conjecture that \((2) \zeta_\rho := -\frac{1}{3}(2) \delta_c\) has a similar property, in other words that the super-horizon terms in \((2) \delta_c\), namely \(A_{1,c}\) and \(A_{2,c}\), as given by (28a) and (28b), are independent of time. We can confirm this by showing directly that

\[
\partial_x A_{1,c} = 0, \quad \partial_x A_{2,c} = 0.
\]

(33)

This calculation uses the fact that \(\partial_x T_{1,2} = T_{1,2}\), the constraint (10) and the following derivatives:

\[
x \partial_x \Omega_m = (2q - 1) \Omega_m, \\
x \partial_x g = g(f - 1), \\
x \partial_x f = (1 + f)(q - 2 - f) + 2f + 3 - \Omega_k,
\]

(34a-34c)

where the deceleration parameter \(q\) is defined by

\[
x \partial_x H = -qH.
\]

(35a)

It follows from (38) that for dust

\[
q = \frac{3}{2} \Omega_m + \Omega_k - 1.
\]

(35b)

We can then determine the constant values of \(A_{1,c}\) and \(A_{2,c}\) by evaluating the limit of the expressions (28a) and (28b) as \(x \to 0\), leading to

\[
A_{1,c} = \frac{27}{5}, \quad A_{2,c} = \frac{24}{5}.
\]

(36)

When these values are substituted into (28a) and (28b) we obtain

\[
T_1 = \Omega_m^{-1} g^2 \left[1 + 4f + f^2 + 2\Omega_k + \frac{2}{5} \Omega_m\right] - \frac{27}{10}, \\
T_2 = 4\Omega_m^{-1} g^2 \left(1 + \frac{3}{2} \Omega_m f^2 \right) - \frac{12}{5}.
\]

(37a-37b)

\textsuperscript{14}Equation (8) gives (34b), apply \(\partial_x\) to (5) and use (35a) to get (34b), and finally apply \(\partial_x\) to (10) to get (34c).

\textsuperscript{15}This is equivalent to \(q = -\frac{\ddot{a}}{(a)}\).
3.5 Initial conditions

The solution (13) for \( (2)\Psi_p \) is a particular solution that satisfies \( \lim_{x \to 0} (2)\Psi_p = 0 \). The general solution for \( (2)\Psi_p \) for a zero decaying scalar mode at the linear level is given by

\[
(2)\Psi_p|_{gen} = (2)\Psi_p + C(x^i)g(x),
\]

where \( C(x^i) \) is an arbitrary function. Note that the second term on the right side of (38) is the general solution of the homogeneous equation for \( (2)\Psi_p \) (see UW, equation (37)). The corresponding general expression for \( (2)\delta_\bullet \) for the Poisson, the uniform curvature, the total matter and the synchronous gauges is given by

\[
(2)\delta_\bullet|_{gen} = (2)\delta_\bullet + A_\bullet C + \frac{2}{3}xg^{-2}D^2C,
\]

where \( A_\bullet \) is the coefficient in the expression (12) for \( (1)\delta_\bullet \). Note that the extra terms on the right side of (39) take the same form as the first order density perturbation, but with \( \zeta(x^i) \) replaced by the arbitrary function \( C(x^i) \).

In applications the arbitrary function \( C(x^i) \) is usually determined by using the metric and matter perturbations at the end of inflation as initial conditions. Various theories of inflation predict that these perturbations will not be purely Gaussian i.e. there will be a certain level of primordial non-Gaussianity. It is convenient to use the first and second order conserved quantities \( (1)\zeta_{mw} := -(1)\Psi_\rho \) and \( (2)\zeta_{mw} := -(2)\Psi_\rho \) to parameterize this primordial non-Gaussianity on super-horizon scales. Specifically, it is assumed that

\[
(2)\zeta_{mw} = 2a_{nl} ((1)\zeta_{mw})^2,
\]

where \( a_{nl} \) is a parameter that depends on the physics of the model of inflation. It has been shown that primordial non-Gaussianity in the CMB temperature anisotropy at second order is represented by the quantity \( (2)\zeta_{mw} - 2(1)\zeta_{mw}^2 \) (16, equation (8)). It follows that the absence of primordial non-Gaussianity corresponds to \( a_{nl} = 1 \).

It follows from equations (109) in appendix B in conjunction with (7a) and (12) that

\[
\begin{align*}
(1)\Psi_\rho &= \zeta - \frac{2}{3}xg m^{-2} D^2 \zeta, \\
(2)\Psi_\rho &= -\frac{1}{5}[4\zeta^2 + 8D(\zeta)] + C(x^i) + (D_i \text{ terms up to order 6}).
\end{align*}
\]

These equations and the restriction (40) determine the arbitrary function \( C(x^i) \) in terms of \( \zeta \) and \( D(\zeta) \). The resulting function is denoted by \( C_{nl} \):

\[
C_{nl}(x^i) = \frac{4}{5}[(1 - \frac{5}{2}a_{nl})\zeta^2 + 2D(\zeta)].
\]

\[\text{16}\text{See [15], equations (7.61) and (7.71). The subscript }_\rho \text{ stands for the uniform density gauge, and the subscript }_{mw} \text{ stands for Malik-Wands.}\]
\[\text{17}\text{See for example [16], page 4, and [8], equation (9), and references given in these papers. When making comparisons with CMB observations it is customary to use a non-linearity parameter } f_{nl}, \text{ which takes into account that the nonlinear gravitational dynamics after inflation contributes to the non-Gaussianity. This parameter has the form } f_{nl} = \frac{2}{3}a_{nl} - 1 + \ldots, \text{ where } + \ldots \text{ refers to terms that describe the effect of the post-inflation nonlinear gravitational dynamics on the primordial non-Gaussianity. See for example [16], equation (9), [8], equation (31) and [1], section 8.4.2.}\]
We will denote the density perturbation \((2)\delta\) corresponding to this choice of initial condition, which is determined by substituting the expression (42) into (39), by 

\[
(2)\delta |_{nl} = (2)\delta + A_n C_{nl} + \frac{2}{3} x g m^{-2} D^2 C_{nl}.
\]

It follows from (1), (43) and (42) that the coefficients \(A_i |_{nl}\) are given by

\[
\begin{align*}
A_{1,1} |_{nl} &= A_{1,1} + \frac{4}{5} (1 - \frac{5}{2} a_{nl}) A_1, \\
A_{2,1} |_{nl} &= A_{2,1} + \frac{8}{5} A_1, \\
A_{3,1} |_{nl} &= A_{3,1}, \\
A_{4,1} |_{nl} &= A_{4,1} + \frac{8}{5}, \\
A_{5,1} |_{nl} &= A_{5,1} + \frac{4}{5} (1 - \frac{5}{2} a_{nl}),
\end{align*}
\]

where \(A_1\) is given by (12).

## 4 The specialization to a flat background

In the previous section we showed that the time dependence of \((2)\delta\) is determined by the linear perturbation function \(g(x)\) and the background functions \(\Omega_m\) and \(H\), either algebraically, or as the integrals \(B_A\) given by (22) and (24a), and \(T_A\) given by (20) and (24b), with \(A = 1, \ldots, 4\). Subsequently, we showed that \(T_1\) and \(T_2\) could be written algebraically, as in (37). In this section we show that if the spatial curvature is zero, a significant simplification occurs: only one integral function is required.

### 4.1 The flatness conditions

We here show that if the background is flat then \(T_{3,4}\) and \(B_{1,2}\) are algebraic expressions in \(g\) and \(\Omega_m\), and in addition \(B_3 + B_4 = 1\). For convenience we define:

\[
\begin{align*}
T_3 := T_3 - \frac{1}{2} g + \frac{3}{4} \Omega_m g^{-1} (1 - g)^2, \\
T_4 := T_4 - 3 + 2 g - \frac{3}{4} \Omega_m g^{-1} (1 - g)^2, \\
B_1 := B_1 - \frac{1}{5} + g - \frac{3}{4} \Omega_m g^{-1} (1 - g)^2, \\
B_2 := B_2 - \frac{12}{5} + 4 g, \\
B_{3+4} := x g (B_3 + B_4 - 1).
\end{align*}
\]

Then the result can be expressed as follows: if \(\Omega_k = 0\) then \(T_{3,4} = 0\), \(B_{1,2} = 0\), and \(B_{3+4} = 0\).

These results can be proved by differentiation, as follows. First we show that if \(K = 0\) then \(\partial_x (x g T_{3,4}) = 0\). This calculation requires \(\partial_x (x g T_{3,4}) = \frac{3}{2} T_{3,4}\), as follows from (20) and (24b), and also equations (34b) and (34a). It follows that \(x g T_{3,4} = C_{3,4}\), a constant. Since \(T_{3,4}\) is bounded as \(x \to 0\) we conclude that \(C_{3,4} = 0\), which gives the desired result.

Second we show that if \(K = 0\) then the quantities \(B_1, B_2\) and \(B_{3+4}\) satisfy

\[
x \partial_x B_* = -\frac{3}{2} \Omega_m g^{-1} B_*, \quad \lim_{x \to 0} B_* = 0,
\]

(46)
Since $g > 0$ it follows from (40) that $(B_\cdot)^2$ is monotone decreasing or identically zero. The limit condition then implies that $B_\cdot \equiv 0$, which gives the desired relations $B_{1,2} = 0$, and $B_{3+4} = 0$. This calculation requires

$$\partial_x (x^2 g \mathcal{H}^{-1} B_A) = x\Omega_m \mathcal{H}^{-1} \dot{T}_A, \quad \partial_x (x^2 g \mathcal{H}^{-1}) = \frac{3}{2} x\Omega_m \mathcal{H}^{-1}. \quad (47)$$

The first of these follows from (22), together with the definitions (24a) and (24b). The second follows from (34a), (35a) and (11). Note that the constraint (10) can be written in the form $2g(f + q + 1) = 3\Omega_m$, using (35b).

### 4.2 Alternate expressions for $^{(2)}\delta_\cdot$

We now cast the expressions for $^{(2)}\delta_\cdot$ into a form in which the role played by the spatial curvature becomes clear. We use (37) to eliminate $T_{1,2}$ in the expressions for $^{(2)}\delta_\cdot$ and use (45) to express $T_{3,4}$ and $B_{1,2}$ in terms of $T_{3,4}$ and $B_{1,2}$. We also use the constraint (10) to eliminate $f$ in favour of $g$, and use (41) to introduce the non-Gaussianity initial condition. The coefficients $A_{i,\cdot}$ in equations (26), (28) and (29) assume the form

$$A_{1,\cdot} = 3(1-g) (1+2a_{nl} - 4g + \frac{3}{2} \Omega_m (1-g)) + 3g B_1, \quad (48a)$$

$$A_{2,\cdot} = 12g(1-g) + 3g B_2, \quad (48b)$$

$$A_{3,\cdot} = 3g(B_3 - 2) - \frac{3}{2} \Omega_m g^{-1}(1-g)^2 + 4\Omega_k(1-g + \frac{1}{3} \kappa x g) - 2T_3, \quad (48c)$$

$$A_{4,\cdot} = -2 + 3gB_1 - \frac{3}{2} \Omega_m g^{-1}(1-g)^2 - 2T_4 + B_2, \quad (48d)$$

$$A_{5,\cdot} = 1 - 2a_{nl} + 3g + \frac{3}{2} \Omega_m g^{-1}(1-g)^2 + B_1, \quad (48e)$$

$$A_{1,c} = 3(1+2a_{nl}), \quad (49a)$$

$$A_{2,c} = 0, \quad (49b)$$

$$A_{3,c} = -\frac{3}{2} \Omega_m g^{-1} + 2\Omega_k (2-g + \frac{2}{3} \kappa x g) - 2T_3, \quad (49c)$$

$$A_{4,c} = -2 - \frac{3}{2} \Omega_m g^{-1}(1-2g) - 2T_4 + B_2, \quad (49d)$$

$$A_{5,c} = 1 - 2a_{nl} + \frac{3}{2} \Omega_m g^{-1}(1-g) - \Omega_k g + B_1, \quad (49e)$$

$$A_{1,v} = 2\kappa x g \left[ B_1 + \frac{14}{3} + 2\kappa x g(1 + \frac{1}{3} \Omega_k) + 2\Omega_k (1-g) \right], \quad (50a)$$

$$A_{2,v} = 2\kappa x g \left[ B_2 - \frac{8}{3} - \frac{8}{3} \kappa x g \right], \quad (50b)$$

$$A_{3,v} = 2\kappa x g (B_3 - \frac{5}{3}) - 5, \quad (50c)$$

$$A_{4,v} = 2\kappa x g (B_4 - \frac{4}{3}) + B_2, \quad (50d)$$

$$A_{5,v} = 2(2-a_{nl}) + 2\kappa x g(1 + \frac{1}{3} \Omega_k) + 2\Omega_k (1-g) + B_1, \quad (50e)$$

where the constant $\kappa$ is defined by (39). For the synchronous gauge it follows from (32) that

$$(A_1, A_2, A_3, A_4, A_5)_s = (A_1, A_2, A_3, A_4, A_5)_v - 4\kappa x g(0,0,0,1,\frac{2}{3},0), \quad (51)$$

with the Newtonian part unchanged.
4.3 Zero spatial curvature

In the case of zero spatial curvature we have $\Omega_k = 0$, $\kappa = 0$, $T_{3,4} = 0$, $B_{1,2} = 0$ and $B_3 + B_4 = 1$. We write $B_3 = B$ and $B_4 = 1 - B$, and can express the scalar $B$ as a standard integral involving $g$, $\Omega_m$ and $H$:

$$B(x) = \frac{\mathcal{H}(x)}{2x^3 g(x)^2} \int_0^x \frac{x^2 \Omega_m}{\mathcal{H}} \left( g^2 - \frac{3}{2} \Omega_m (1 - g)^2 \right) dx,$$

where $g$, $\Omega_m$ and $\mathcal{H}$ inside the integral are functions of $\bar{x}$. This result follows from equations (22) and (24) with $A = 3$, when one uses the expression for $T_3$ given by (24b) and (45a) with $T_3 = 0$.

With these simplifications the expressions (48)-(50) for the coefficients in (2)$\delta_i$ for the Poisson, uniform curvature and total matter gauges reduce to those in [17], where the present results are summarized. The full expression is given by (1), with the Newtonian part given by

$$\left(2\delta_i\right)_{\text{Newtonian}} = \frac{4}{9} m^{-4} x g^2 \left[ B D^2(D\zeta)^2 + (1 - B) D^4 D(\zeta) \right].$$

For the reader’s convenience, we give the coefficients $A_{i,p}$ in the Poisson gauge when $K = 0$, obtained by specializing (48):

$$A_{1,p} = 3(1 - g) \left( 1 + 2a_{nl} - 4g + \frac{3}{2} \Omega_m (1 - g) \right),$$

$$A_{2,p} = 12g(1 - g),$$

$$A_{3,p} = 3g(B - 2) - \frac{3}{2} \Omega_m g^{-1}(1 - g)^2,$$

$$A_{4,p} = -2 + 3g(1 - B) - \frac{3}{2} \Omega_m g^{-1}(1 - g)^2,$$

$$A_{5,p} = 1 - 2a_{nl} + 3g + \frac{3}{2} \Omega_m g^{-1}(1 - g)^2.$$

We also give the full expression for the synchronous gauge:

$$\left(2\delta_i\right)_s = \frac{2}{3} m^{-2} x g \left[ -5(D\zeta)^2 + 2(2 - a_{nl}) D^2 \zeta^2 \right]$$

$$+ \frac{4}{9} m^{-4} x^2 g^2 \left[ (B + \frac{1}{3}) \left( D^2(D\zeta)^2 - D^4 D(\zeta) \right) + 2(D^2 \zeta)^2 \right],$$

as follows from (50) and (51).

4.4 Relation with the literature

In the course of doing the research reported in this paper we made a detailed comparison of our expressions for (2)$\delta_i$ with those in the literature, which deal solely with the case where the background spatial curvature is zero. In addition the expressions for (2)$\delta_i$ with $\Lambda > 0$ are restricted to the synchronous and Poisson gauges. In [17] we gave a brief overview of the results in the literature. In this section we describe in detail the relation between our results and the papers in the literature, focussing in particular on the work of Tomita [6] and Bartolo and collaborators [8].

Comparing the different results is not straightforward since there are many different ways of representing the spatial dependence in the expression for (2)$\delta_i$, which involves an arbitrary spatial function and the spatial differential operator $D_x$. We thus begin by describing the various quadratic differential expressions that have been used in the literature and showing how they are related to our canonical set (2).
4.4.1 Spatial quadratic differential expressions

In discussing our canonical set of quadratic differential expressions \( \mathcal{D}(A) \), as defined by (71) and (72), plays a key role. We begin with the zero order derivative expressions \( \zeta^2 \) and \( \mathcal{D}(\zeta) \) that determine the super-horizon terms in \( (2) \delta_\bullet \). Two of the three second order derivative expressions that determine the post-Newtonian terms are obtained by acting with \( \mathcal{D}^2 \) on the zero order expressions, while the third, \( (\mathcal{D}\zeta)^2 \equiv \mathcal{D}^i\zeta\mathcal{D}_i\zeta \), is a new expression. Finally, the two fourth order derivative expressions that determine the Newtonian terms are obtained by acting with \( \mathcal{D}^2 \) on two of the second order expressions. Before continuing we mention that the appearance of \( \mathcal{D}(\zeta) \) in the second order density perturbation has its origin in the quadratic source term in the evolution equation for the second order Bardeen potential (2)\( \Psi_p \) (see equations (61b) and (61f) in \[18\]), through the use of the mode extraction operator \( \mathbf{S}^{ij} \), as defined by (72).

We now list the various other spatial quadratic differential expressions that have appeared in the literature:

\[
\begin{align*}
AD^2 A, & \quad D^iD^j(D_iAD_jA), & \quad (D^iD^jA)(D_iD_jA), & \quad (56a) \\
(D^iA)(D_iD^2 A), & \quad D^i(D_iAD^2 A), & \quad D^iD^j(AD_iD_jA), & \quad (56b)
\end{align*}
\]

sometime with \( D^{-2} \) acting on the left. Each of these expressions can be written as a linear combination of our canonical set \( \mathcal{D}(A) \), augmented by the terms \( \mathcal{D}^2A \) using the identities (73) in the Appendix. Here we use the generic symbol \( A = A(x^i) \) to denote the arbitrary spatial function. Although there is no consensus for this function the various choices differ only by an overall numerical factor.

A quantity closely related to our \( D(A) \) in (71) has been defined by several authors as follows. Let

\[
\begin{align*}
\Psi_0 & := \frac{1}{2}\lambda D^{-2} (D^iD^jA D_iD_jA - (D^2A)^2), & \quad (57a) \\
\Theta_0 & := D^{-2} (\Psi_0 - \frac{1}{3}\lambda(DA)^2), & \quad (57b)
\end{align*}
\]

where \( \lambda \) is a numerical factor that we have introduced to accommodate different scalings. It follows from (73d) that

\[
\begin{align*}
\Psi_0 & := -\frac{1}{3}\lambda (D^2A - (DA)^2), & \quad \Theta_0 & = -\frac{1}{3}\lambda D(A). & \quad (58)
\end{align*}
\]

This makes clear that \( \Theta_0 \) corresponds to our \( \mathcal{D}(A) \), while \( \Psi_0 \) is closely related to our \( \mathcal{D}^2\mathcal{D}(A) \). These quantities were used in [4] and [5] with \( A = \varphi = \frac{3}{5}\zeta \) and \( \lambda = 1 \) (see equations (4.36) and (6.6) in [4], and following equation (9) in [5]). They were also used in [6] with \( A = F = -2\zeta \) and \( \lambda = \frac{9}{100} \) (see equations (4.11), noting that a factor of \( D^{-2} \) is missing in the first equation). Furthermore, Bartolo et al [8] come close to introducing our \( \mathcal{D}(A) \). They define (see their equation (18))

\[
\alpha_0 := D^{-2}(D\varphi_0)^2 - 3D^{-4}D^iD^j(D_i\varphi_0D_j\varphi_0), & \quad (59)
\]

where \( \varphi_0 = \frac{3}{5}\zeta \). It follows from (73d) that

\[
\alpha_0 = -2\mathcal{D}(\varphi_0). & \quad (60)
\]
4.4.2 Synchronous gauge with $K = 0, \Lambda \geq 0$

The first expression for $\delta s$ with $\Lambda > 0$ was given by Tomita [6] (see his equation (2.22)). The time dependence in his expression is described by two functions $P(\eta)$ and $Q(\eta)$ that satisfy second order differential equations:

\[ (\partial_\eta^2 + 2\mathcal{H}\partial_\eta) P = 1, \quad (\partial_\eta^2 + 2\mathcal{H}\partial_\eta) Q = P - \frac{5}{2}(\partial_\eta P)^2, \]  

(61)

and the spatial dependence is described by a function $F(x^i)$ and its first and second partial derivatives, including the following quadratic differential expressions:

\[ FD^2F, \quad D_iD_jF D^iD^jF, \]

(62)

in addition to the ones in our canonical list [2]. We use the identities (73a) and (73d) to relate these expressions to our canonical expressions. To match the density perturbations requires

\[ g = 1 - \mathcal{H}\partial_\eta P, \quad F(x^i) = -2\zeta(x^i), \]

(63a)

at linear order and

\[ \partial_\eta Q = \frac{1}{3}m^{-2}xg, \]

(63b)

\[ \partial_\eta Q = \frac{1}{3}m^{-2}x \left[ 21g^2B - g(9g - 2) + \frac{21}{2}\Omega_m(1 - g)^2 \right], \]

(63c)

at second order. With these equations it follows that Tomita’s expression (2.22) is transformed into our expression for $\delta s$ given by (55), but with $a_{nl} = 0$.

4.4.3 Poisson gauge with $K = 0, \Lambda \geq 0$

The first expression for $\delta p$ with $\Lambda > 0$ was given by Tomita [6] (see his equation (4.16)). As with $\delta s$ the time dependence is described by $P$ and $Q$ and the spatial dependence is described by the quadratic differential expressions in our canonical list [2] together with the expressions (62) and

\[ (D^iF)D^2D_iF. \]

(64)

In particular the combinations $\Psi_0$ and $\Theta_0$, as defined by (57a) and (57b), are used with $F = -2\zeta$ and $\lambda = \frac{9}{100}$. We write these combinations in the form (58), and use the identity (73c) for the expression in (64). Using equations (63) we can now show that Tomita’s expression (4.16), with a few minor typos corrected\(^{19}\) is transformed into our expression for $\delta p$ given by (1), (53) and (54), but with $a_{nl} = 0$.

An expression for $\delta p$ with $\Lambda > 0$ has more recently been derived by Bartolo and collaborators [8] (see their equation (29)). In order to make a comparison with our expression which has the general form (1), we first consider their linear perturbation. The time dependence of the linear perturbation is described by a function $g$, which

\(^{18}\)Here $\eta$ denotes conformal time. Note that $\partial_\eta = \mathcal{H}x\partial_x$.

\(^{19}\)In line 2 in Tomita’s equation (4.16), $-\frac{2a'}{a}PP'$ should be $\frac{2a'}{a}PP'$, in line 3, $-\frac{1}{2}P'$ should be $-P'$, and in line 4, $-\frac{1}{2}P$ should be $-P$. In addition the sign of $Q$, which appears in lines 1 and 4, should be reversed.
we denote by $g_b$ to distinguish it from our $g$, and the spatial dependence is described by a function $\varphi(x')$ which is a constant multiple of our $\zeta$. Since $g_b = 1$ and $g = \frac{3}{5}$ when $\Lambda = 0$ it follows by comparing their equation (11) with our (7a) that

$$g_b = \frac{5}{8} g, \quad \varphi_0 = \frac{3}{5} \zeta.$$  \hfill (65)

We next consider the Bardeen potential $(2)\Psi_p$ (equation (20) in [8]) whose time dependence is described by $g_b$ and four functions $B_A, A = 1, 2, 3, 4$. Our expression for $(2)\Psi_p$, including the non-Gaussianity initial condition, is given by equations (23a), (38) and (42). In order to match the spatial dependence terms we note that their $\alpha_0$ is given by (60). We also need to use the identities (73a) and (73b). Comparing the two expressions for $(2)\Psi_p$ leads to the following relation between our $B_A$ and the quantities $B_A$ in [8]:

$$(B_1, B_2, B_3, B_4) = \frac{9}{25} g^{-1} \left( B_1, -2B_2, m^2(\frac{1}{3}B_3 + B_4), \frac{2}{5} m^2 B_3 \right).$$  \hfill (66)

To establish consistency we need to show that the definition of the $B_A$ in [8] (see equations (22)-(26)) translates into our definition of the $B_A$ in (22) under the transformation (66). The definitions of the $B_A$ in [8] can be collectively written in the form

$$\frac{9}{25} g^{-1} B_A = \left( \frac{5}{3} g_0 \right) \frac{\mathcal{H}}{x^2 g} \int_0^x (I(x) - I(\bar{x})) g(\bar{x})^2 T_A(\bar{x}) d\bar{x},$$  \hfill (67)

where $I(x)$ is expressed in terms of $g(x)$ by (8). The functions $T_A$ are related to our functions $T_A$ according to

$$m^2(T_1, T_2, T_3, T_4) = g^2 (T_1, -2T_2, m^2(\frac{1}{3} T_3 + T_4), \frac{2}{5} m^2 T_3),$$  \hfill (68)

where we note that our variable $f$ coincides with the $f$ in [8]. On the other hand our equation (22) expressing $B_A$ in terms of $T_A$, can be written in the equivalent form

$$B_A = \frac{\mathcal{H}}{x^2 g} \int_0^x (I(x) - I(\bar{x})) m^2 T_A(\bar{x}) d\bar{x}, \quad I(x) := \int_0^x \frac{d\bar{x}}{\mathcal{H}(\bar{x})^2}$$  \hfill (69)

The common structure of (66) and (68) ensures that (67) translates into equation (69), provided that $g_0 = \frac{3}{5}.

We are now in a position to show that the expression for $(2)\delta_p$ in [8] can be transformed into our expression. We first convert the quadratic differential expressions in [8] in $\varphi_0$ into our canonical expressions (2) in $\zeta$ using (73a), (73b), (60) and (65). We then express $g_b$ and $B_A$ in the time-dependent coefficients in terms of $g$ and $B_A$, using (65) and (66). It is necessary to use (16) in order to eliminate the derivatives $\partial_x B_A$. We find agreement except with the coefficient of $\zeta^2$, which corresponds to the coefficient of $\varphi^2$ in equation (29) in [8].

We conclude that the term $(f-1)^2 - 1$ in this coefficient should be replaced by $2(f-1)^2$. The $\varphi^2$ term then correctly specializes to $-\frac{3}{\delta}(1 - \frac{2}{5} a_n) \varphi^2$ when one restricts consideration to the Einstein-de Sitter universe ($\Lambda = 0$), in agreement with equation (8) in [5].

\hfill \footnote{Here we have used (10) and $m^2 = \Omega_m \mathcal{H}_0^2$ to write the equation $\tilde{B}_A = \mathcal{H}_0^2 (f_0 + \frac{3}{2} \Omega_m, 0) B_A$ in [8] in the form $\tilde{B}_A = \frac{3}{5} m^2 g_0^{-1} \tilde{B}_A$.}

\hfill \footnote{Equation (22), together with (20), is an iterated integral. Use $x \Omega_m \mathcal{H}^2 = m^2$ in the integrand, reverse the order of integration and make use of the definition of $I(x)$.}
5 Concluding remarks

The results in this paper fall under three headings. First, we have presented exact expressions for the second order fractional density perturbation for dust, a cosmological constant and spatial curvature in a simple and physically transparent form in four popular gauges: the Poisson, the uniform curvature, the total matter and the synchronous gauges. Our results unify and generalize all the known results in the literature, which are confined to the case of zero spatial curvature and, when $\Lambda > 0$, to the Poisson and synchronous gauges. Our approach has two novel features. We have introduced a canonical way of representing the spatial dependence of the perturbations at second order which makes clear how the choice of gauge affects the form of the expressions. In addition we have formulated the time dependence in such a way that the dynamics of the perturbations and the effect of spatial curvature can be read off by inspection. In particular, in the special case of zero spatial curvature we have shown that the time evolution simplifies significantly, and requires the use of only two non-elementary functions, the so-called growth suppression factor $g$ that arises at the linear level, and one new function $B$ at the second order level. We emphasize that the assumption of zero decaying mode underlies the simple expressions for $(2)\delta_\ast$ that we have presented. This assumption is usually made in cosmological perturbation theory, presumably on the grounds that the decaying mode will become negligible. However, if $\Lambda > 0$ the name "decaying mode" is a misnomer since this mode, after decaying in the matter-dominated epoch ($\Omega_m \approx 1$), increases when $\Omega_\Lambda$ becomes significant and contributes to the density perturbation on an equal footing with the growing mode in the de Sitter regime. This is made clear by the asymptotic expressions given in UW (see equation (66a)). Into the past the decaying mode grows without bound on approach to the initial singularity. On the other hand, if the decaying mode is set to zero, the perturbations remain finite into the past and one is essentially considering perturbations in a universe with an isotropic singularity [19].

Second, we have made a detailed comparison of our results with the known expressions for $(2)\delta$ in different gauges when the background spatial curvature is zero. Our canonical representation of the spatial dependence has enabled us to unify seemingly disparate results, while at the same time revealing a number of errors in the expressions in the literature. For example, two expressions for $(2)\delta_p$ with $\Lambda > 0$ have been given. The first, by Tomita [6], was derived by solving the perturbation equations at second order in the synchronous gauge and then transforming to the Poisson gauge. The second, by Bartolo et al [8], was derived by solving the perturbation equations directly in the Poisson gauge. The two expressions appear to be completely different. However, by simplifying the $B$-functions of Bartolo and introducing our canonical representation of the spatial dependence we have been able to show, after correcting some typos, that both of these expressions can be written in our canonical form for $(2)\delta_p$, which is given by (1), (53) and (54).

Third, we have given a systematic procedure for performing a change of gauge for second order perturbed quantities. The derivation of our expressions for $(2)\delta_\ast$ relied on solving the perturbation equations in the Poisson gauge as done in UW, and then using our change of gauge procedure to calculate $(2)\delta_\ast$ in the other gauges. The
procedure is easy to implement in this application since the change of gauge induces a simple change in the time-dependent coefficients \( A_{i*} \) in (1), while preserving the overall structure of \((2\delta_*)\). However, we anticipate that the generality of our procedure will make it useful in other contexts.

## Acknowledgements

We thank Marco Bruni and David Wands for helpful correspondence concerning their recent paper [9]. CU also thanks the Department of Applied Mathematics at the University of Waterloo for kind hospitality. JW acknowledges financial support from the University of Waterloo.

## A Spatial differential operators

The definitions of the spatial differential operators that we use are as follows. First, the second order spatial differential operators are defined by

\[
D^2 := \gamma^{ij} D_i D_j, \quad D_{ij} := D_{(i} D_{j)} - \frac{1}{3} \gamma_{ij} D^2,
\]

where \( D_i \) denotes covariant differentiation with respect to the conformal background spatial metric \( \gamma_{ij} \). Second, we use the shorthand notation

\[
(D A)^2 := (D^k A)(D_k A), \quad D(A) := S^{ij}(D_i A)(D_j A),
\]

where \( A \) is a scalar field and \( S^{ij} \) is defined in (72). Finally, we define the mode extraction operators (see [18], equations (85)):

\[
\begin{align*}
S^i &= D^{-2} D^i, \\
S^{ij} &= \frac{3}{2} D^{-2} (D^2 + 3K)^{-1} D^{ij}, \\
V^i_k &= \delta^i_k - D_i S^j, \\
V^{ij}_k &= (D^2 + 2K)^{-1} V^i_j D^k, \\
T^{km}_{ij} &= \delta_i^{(k} \delta_j^{m)} - D_{(i} V_{j)km} - D_{ij} S^{km}.
\end{align*}
\]

If some expression \( L(D_i) \) involving \( D_i \) scales as \( L(\lambda D_i) = \lambda^p L(D_i) \) under a rescaling of coordinates \( x^i \to \lambda^{-1} x^i, \eta \to \lambda^{-1} \eta \), we say that \( L(D_i) \) has weight \( p \) in \( D_i \). It follows that the canonical differential expressions in (2) have the following weights: \( \zeta^2 \), \( D(\zeta) \) are of weight zero, \( (D \zeta)^2 \), \( D^2 D(\zeta) \), \( D^2 \zeta^2 \) are of weight two and \( D^2 (D \zeta)^2 \), \( D^4 D(\zeta) \) are of weight four.

We now give identities involving the spatial differential operators that we use to relate results in the literature to our results:

\[
\begin{align*}
AD^2 A &= \frac{1}{2} D^2 A^2 - (DA)^2, \\
D^j (D_i A D_j A) &= \frac{1}{3} D^2 \left[ (DA)^2 + 2(D^2 + 3K)D(A) \right], \\
D^j (D_i A^2 D_j A) &= -\frac{1}{6} D^2 \left[ (DA)^2 - 4(D^2 + 3K)D(A) \right], \\
(D^i D^j) (D_i D_j A) &= \frac{2}{3} \left[ (D^2 - 3K)(DA)^2 - (D^2 + 3K)D^2 D(A) \right] + (D^2 A)^2, \\
D^i D^j (A D_i D_j A) &= -\frac{1}{6} D^2 \left[ 2(DA)^2 + 4(D^2 + 3K)D(A) - 3(D^2 + 2K)A^2 \right].
\end{align*}
\]

Note that \( \mathcal{H} \to \lambda \mathcal{H} \) and \( K \to \lambda^2 K \).
B Transformation laws for gauge invariants

The first purpose of this appendix is to define the gauge invariants that are associated with the perturbed metric and matter distribution. We do not, however, write out the expressions for the gauge invariants in terms of gauge-variant quantities since our strategy is to work solely with gauge invariants. First we require the governing equations that determine the gauge invariants in the Poisson gauge, and these are given in UW. Second we require a framework for determining how gauge invariants transform under a change of gauge at second order. For example, given \( \delta^\text{P} \) (Poisson gauge) how can one calculate \( \delta^\text{C} \) (uniform curvature gauge) efficiently? The framework that we present in this appendix is based on the transformation law for the perturbations of a given tensor field up to second order under a gauge transformation, first given by Bruni et al [20]. This transformation law has been used for this purpose in a number of specific cases (for example, synchronous to Poisson [4, 6], Poisson to uniform curvature [21], synchronous to total matter [9] and Poisson to total matter [22].) Our goal is to give a general framework that is valid for a specific gauge invariant and two chosen gauges. In the body of the paper we consider pressure-free matter, but in this appendix we assume that the matter content is a perfect fluid with equation of state \( p = w \rho, w = \text{constant} \), in order to increase the applicability of the results.

B.1 Gauge invariants associated with an arbitrary tensor field

In cosmological perturbation theory a second order gauge transformation can be represented in coordinates as follows:

\[
\tilde{x}^a = x^a + \epsilon(1)\xi^a + \frac{1}{2} \epsilon^2 \left( (2)\xi^a + (1)\xi^a,_{b} (1)\xi^b \right),
\]

where \( (1)\xi^a \) and \( (2)\xi^a \) are independent dimensionless background vector fields. We consider a 1-parameter family of tensor fields \( A(\epsilon) \), which we assume can be expanded in powers of \( \epsilon \), i.e. as a Taylor series:

\[
A(\epsilon) = (0)A + \epsilon(1)A + \frac{1}{2} \epsilon^2 (2)A + \ldots,
\]

where \( (0)A \) is called the unperturbed value, \( (1)A \) is called the first order (linear) perturbation and \( (2)A \) is called the second order perturbation of \( A(\epsilon) \). Such a transformation induces a change in the first and second order perturbations of \( A(\epsilon) \) according to

\[
\begin{align*}
(1)A[\xi] &= (1)A + \mathcal{L}(\xi)(0)A, \\
(2)A[\xi] &= (2)A + \mathcal{L}(\xi)(0)A + \mathcal{L}(\xi)(2)A + \mathcal{L}(\xi)(1)A,
\end{align*}
\]

where \( \mathcal{L} \) is the Lie derivative (see [20], equations (1.1)–(1.3)). One fixes a gauge by requiring some components of the perturbations of some tensor fields \( \langle r \rangle A[\xi] \), \( \langle r \rangle B[\xi] \), etc, with \( r = 1, 2 \), to be zero, thereby determining unique values for \( (1)\xi^a \) and \( (2)\xi^a \) which we denote by \( (1)\xi^a_* \) and \( (2)\xi^a_* \). Since there is no remaining gauge freedom, the non-zero components \( \langle r \rangle A[\xi]* \), obtained by replacing \( \xi \) by \( \xi_* \) in (76), are gauge
invariants. We refer to Malik and Wands \[15\] (see pages 18-20) for an illustration of this process using the Poisson gauge. When uniquely determined, the vector fields \((1)\xi^a\) and \((2)\xi^a\) will be referred to as \textit{gauge fields}. \[23\]

In order to derive a transformation law for gauge invariants under a change of gauge we consider two gauge fields \((r)\xi^a\) and \((r)\xi'^a\) and define

\[
\begin{align*}
(1)Z^a[\xi,\xi_0] &:= (1)\xi^a - (1)\xi_0^a, \\
(2)Z^a[\xi,\xi_0] &:= (2)\xi^a - (2)\xi_0^a + [(1)\xi, (1)\xi_0]^a.
\end{align*}
\]

We now consider \((76)\) with \(\xi = \xi\) and \(\xi_0 = \xi_0\) and form the difference of the two sets of equations. This leads to the following transformation law relating the gauge invariants \((r)A[\xi]\) and \((r)A[\xi_0]\):

\[
\begin{align*}
(1)A[\xi] &= (1)A[\xi_0] + \mathcal{L}(1)_Z (0)A, \\
(2)A[\xi] &= (2)A[\xi_0] + \mathcal{L}(2)_Z (0)A + \mathcal{L}(1)_Z (2)A[\xi_0] + \mathcal{L}(1)_Z (0)A.
\end{align*}
\]

where \((r)Z \equiv (r)Z^a[\xi,\xi_0]\). We shall refer to the functions \((r)Z^a[\xi,\xi_0]\), which are gauge invariants, as the \textit{transition functions}. They are determined by the conditions that specify the gauge fields \((r)\xi^a\) and \((r)\xi_0^a\). We note the formal similarity between \((76)\) and \((78)\). In going from \((76)\) to \((78)\) one replaces gauge-variant quantities by gauge-invariant quantities: \((r)A[\xi]\) by \((r)A[\xi]\), \((r)A\) by \((r)A[\xi_0]\), \((r)A\) and \((r)\xi\) by \((r)Z\).

### Shorthand notation for gauge invariants and transition functions

The full notation for the first and second order gauge invariants associated with a tensor \(A(\epsilon)\) is \((r)A[\xi]\), \(r = 1, 2\), where \(\xi\) is a gauge field. If there is no danger of confusion we will use a subscript notation:

\[
(r)A_r \equiv (r)A[\xi_r].
\]

The full notation for the transition functions is \((r)Z^a[\xi,\xi_0]\) where \(\xi\) and \(\xi_0\) are the two gauge fields. In general we will use the kernel \(Z\) as shorthand for \(Z[\xi,\xi_0]\). If specific gauge fields are used, for example, \(\xi = \xi_p\) and \(\xi_0 = \xi_c\), we will use subscripts:

\[
Z \equiv Z[\xi,\xi_0], \quad Z_{c,p} \equiv Z[\xi_c,\xi_p].
\]

The source terms in the transformation laws, which have the general form \(\mathcal{F}^{(1)}Z\) or \(S^{(1)}Z\), are quadratic in first order gauge invariants. Specific source terms of the form \(\mathcal{F}^{(1)}Z_{c,p}\) or \(S^{(1)}Z_{c,p}\) are quadratic in the first order gauge invariants \((1)\Psi, (1)\delta\) and \((1)\nabla\). We will omit the superscript \((1)\) and the subscript \(p\) when there is no danger of confusion.

\[23\] In previous papers \[23, 12, 15\] and UW, influenced by the approach of Nakamura \[24, 25\] to cosmological perturbations, we used the kernel \(-X\) to denote a gauge field. Here we use the kernel \(\xi\) but with a subscript, to indicate that the arbitrary vector field \(\xi^a\) has been uniquely determined, thereby fixing a gauge.
B.2 The metric gauge invariants

The gauge invariants \((r)g_{ab}[^{\xi}]\) associated with the metric \(g_{ab}\) are given by (76) with the arbitrary tensor \(A\) chosen to be \(g_{ab}\). Since \(a^{-2}g_{ab}\) is dimensionless we can define dimensionless gauge invariants by

\[
(r)f_{ab}[^{\xi}] := a^{-2}(r)g_{ab}[^{\xi}].
\]

We choose the tensor \(A\) in equation (78) to be \(g_{ab}\) and use (80) to obtain the following transformation law for \((r)f_{ab}[^{\xi}]\):

\[
\begin{align*}
(1)f_{ab}[^{\xi}] &= (1)f_{ab}[0] + a^{-2}L_{a\hat{\nu}Z} (a^{2}\gamma_{ab}), \\
(2)f_{ab}[^{\xi}] &= (2)f_{ab}[0] + a^{-2}L_{(2)Z} (a^{2}\gamma_{ab}) + F_{ab}[^{(1)}Z],
\end{align*}
\]

where

\[
F_{ab}[^{(1)}Z] := a^{-2}L_{a\hat{\nu}Z} (2a^{2}(1)f_{ab}[0] + L_{a\hat{\nu}Z} (a^{2}\gamma_{ab})).
\]

Here \(\gamma_{ab}\) is the conformally related background metric, given by \((0)g_{ab} = a^{2}\gamma_{ab}\).

We now perform a mode decomposition of \((r)f_{ab}[^{\xi}]\) as follows:

\[
\begin{align*}
(\eta)f_{00}[^\xi] &= -2(\eta)\Phi[^\xi], \\
(\eta)f_{0i}[^\xi] &= D_i(\eta)B[^\xi] + (\eta)B_i[^\xi], \\
(\eta)f_{ij}[^\xi] &= -2(\eta)\Psi[^\xi][\eta] + 2D_iD_j(\eta)C[^\xi] + 2D_i(\eta)C_{ij}[^\xi] + 2(\eta)C_{ij}[^\xi].
\end{align*}
\]

We can apply the mode extraction operators defined in equations (72) in appendix A to (81) to obtain the transformation laws for the individual gauge invariants on the right side of (82), obtaining:

\[
\begin{align*}
(\eta)\Phi[^\xi] &= (\eta)\Phi[0] + (\partial_\eta + \mathcal{H})(\eta)Z^0 - \frac{1}{2}F_{00}(\eta)Z, \\
(\eta)B[^\xi] &= (\eta)B[0] - (\eta)Z^0 + \partial_\eta (\eta)Z + S^iF_{0i}(\eta)Z, \\
(\eta)\Psi[^\xi] &= (\eta)\Psi[0] - \mathcal{H}(\eta)Z^0 - \frac{1}{6}(\mathcal{F}^k_k - D^2 S_{ij}\hat{F}_{ij})(\eta)Z, \\
(\eta)B_i[^\xi] &= (\eta)B_i[0] + \partial_\eta (\eta)\hat{Z}_i + \mathcal{V}^i_{\eta}(\eta)F_0(\eta)Z, \\
(\eta)C[^\xi] &= (\eta)C[0] + (\eta)Z + \frac{1}{2}S_{ij}\hat{F}_{ij}(\eta)Z, \\
(\eta)C_{ij}[^\xi] &= (\eta)C_{ij}[0] + (\eta)\hat{Z}_i + \mathcal{V}_j^{\eta}(\eta)F_{0j}(\eta)Z, \\
(\eta)C_{ij}[^\xi] &= (\eta)C_{ij}[0] + \frac{1}{2}\mathcal{T}^{km}_{ij}(\eta)F_{km}(\eta)Z,
\end{align*}
\]

where \(r = 1, 2\). The source terms \(F_{ab}(\eta)Z\), which are given by (81d), do not appear when \(r = 1\). We will give explicit expressions for them later. Here we have decomposed the transition functions \((\eta)Z^a \equiv (\eta)Z[^{\xi}][\eta][0][\xi] \) according to

\[
(\eta)Z^a = (\eta)Z^0, (\eta)Z^i, \quad (\eta)Z^i = D^i(\eta)Z + (\eta)\hat{Z}_i, \quad D_i(\eta)\hat{Z}_i = 0.
\]

---

24 We use notation that is compatible with the notation in [18]. See equations (24) and (88).

25 Here \(\eta\) is conformal time, and \(\partial_\eta = \mathcal{H}x\partial_x\).
B.3 Density gauge invariants

We choose $A = \rho$, the matter density scalar in eq. (78). On evaluating the Lie derivatives we obtain

\[
(1)\rho[\xi] = (1)\rho[\xi_0] + (1)Z^0(0)\rho',
\]

\[
(2)\rho[\xi] = (2)\rho[\xi_0] + (2)Z^0(0)\rho' + \left((1)Z^0_\eta + (1)Z^i D_i\right) \left((2)\rho[\xi_0] + (1)Z^0(0)\rho'\right),
\]

where $'$ denotes differentiation with respect to conformal time $\eta$. Here and in the rest of this section the kernel $Z$ is shorthand for $Z[\xi_0, \xi_0]$. We introduce dimensionless gauge invariants by normalizing with the inertial mass density:

\[
(\rho^r)\delta[\xi] := \rho[\xi] \left(\rho[0] + (0)p\right),
\]

which in the case of dust is just the usual fractional density perturbation. Then (85) leads to the following transformation law for the density gauge invariants:

\[
(1)\delta_\bullet = (1)\delta_\circ - 3H(1)Z^0,
\]

\[
(2)\delta_\bullet = (2)\delta_\circ - 3H(2)Z^0 + F(1)\delta[1]Z,
\]

where

\[
F(1)\delta[1]Z := (Z^0(\partial_\eta - 3(1 + w) \mathcal{H}) + Z^i D_i) \left(2\delta[\xi_0] - 3H(0)Z^0\right),
\]

and we are using the shorthand notation (79). Here we have dropped the superscript (1) on the first order quantities on the right hand side of this equation. In deriving equations (87) we used the following background equations for a perfect fluid:

\[
(0)\rho' = -3H((0)p + (0)p), \quad (0)p' = w(0)p',
\]

where $w$ is the constant equation of state parameter.

B.4 Velocity gauge invariants

The gauge invariants $(v^r)a[\xi]$ associated with the covariant unit vector field $u_a$ are given by (79) with the arbitrary tensor $A$ chosen to be $u_a$. Since $a^{-1}u_a$ is dimensionless we can define dimensionless gauge invariants by

\[
(v^r)a[\xi] := a^{-1}(v^r)a[\xi],
\]

Equation (78), with the tensor $A$ chosen to be $u_a$, in conjunction with (89), then leads to the following transformation law for $(v^r)a[\xi]$:

\[
(1)v_a[\xi] := (1)v_a[\xi_0] + a^{-1} L_{v\xi}(a^0)v_a,
\]

\[
(2)v_a[\xi] := (2)v_a[\xi_0] + a^{-1} L_{v\xi}(a^0)v_a + (F_v)a[1]Z,
\]

where

\[
(F_v)a[1]Z := a^{-1} L_{v\xi}(2a(1)v_a[\xi_0] + L_{v\xi}(a^0)v_a),
\]
and $a^{(0)}_a \equiv (^{(0)}u_a)$. Evaluating the Lie derivatives and restricting to the spatial components yields the following
\begin{align*}
^{(1)}v_i[\xi] &= ^{(1)}v_i[\xi] - D_i^{(1)Z^0}, \\
^{(2)}v_i[\xi] &= ^{(2)}v_i[\xi] - D_i^{(2)Z^0} + (F_v)_i^{[1]Z},
\end{align*}
(91a) (91b)
where
\begin{align*}
(F_v)_i^{[1]Z} := 2Z^0(\partial_\eta + \mathcal{H})v_i[\xi] - 2\Phi[\xi]D_iZ^0 - \frac{1}{2}D_i(\partial_\eta + 2\mathcal{H})(Z^0)^2 \nonumber \\
- \frac{1}{2}D_i(Z^iD_jZ^0) + 2(Z^jD_jv_i[\xi] + (D_jZ^j)v_j[\xi]).
\end{align*}
(91c)

We now mode decompose $(^r)v_i[\xi]$ into a scalar and vector part according to $v_i = D_i\nu + \tilde{v}_i$, $D^i\tilde{v}_i = 0$. On restricting to the purely scalar case at linear order (i.e. $v_i = D_i\nu$, $Z^i = D^iZ$), (91) reduces to
\begin{align*}
^{(1)}v_\bullet &= ^{(1)}v_o^{\bullet} - ^{(1)}Z^0, \\
^{(2)}v_\bullet &= ^{(2)}v_o^{\bullet} - ^{(2)}Z^0 + F_v^{[1]Z},
\end{align*}
(92a) (92b)
where
\begin{align*}
F_v^{[1]Z} &= 2S^i(Z^0(\partial_\eta + \mathcal{H})D_i\nu[\xi] - \Phi[\xi]D_iZ^0) \\
- \frac{1}{2}(\partial_\eta + 2\mathcal{H})(Z^0)^2 - (D_jZ^j)(Z^0 - 2v[\xi]),
\end{align*}
(92c)
and we are using the shorthand notation (79).

### B.5 Transformation laws between the Poisson, the uniform curvature and the total matter gauges

The Poisson, uniform curvature and total matter gauges all satisfy the following conditions on the metric gauge invariants:
\begin{align*}
^{(r)}C[\xi] &= 0, \\
^{(r)}C_i[\xi] &= 0,
\end{align*}
(93)
for $r = 1, 2$, where $\xi$ is any of the gauge fields $\xi_p, \xi_c$ and $\xi_v$. It follows from (83c) and (83f) with $r = 1$ that the spatial part of the first order transition function $(^1Z^a$ relating these three gauges will be zero:
\begin{align*}
^{(1)}Z[\xi_\bullet, \xi] &= 0, \\
^{(1)}\tilde{Z}_i[\xi_\bullet, \xi] &= 0,
\end{align*}
(94)
where $\xi_\bullet$ and $\xi$ can be chosen to be any two of the gauge fields $\xi_p, \xi_c$ and $\xi_v$. On the other hand, these three gauges are distinguished by the specification of the temporal gauge, as follows:
\begin{align*}
^{(r)}B[\xi_p] &= 0, \\
^{(r)}\Psi[\xi_c] &= 0, \\
^{(r)}\nu[\xi_v] &= 0,
\end{align*}
(95)
for $r = 1, 2$ respectively.

---

26 Apply the mode extraction operator $S^i$ to the second of equations (91) to get the second of equations (92). We introduce the shorthand notation $F_v \equiv S^i(F_v)_i$. 
We now give the components of the source terms $\mathcal{F}_{ab}[Z]$ in the transformation laws (83), assuming that the linear metric perturbation is purely scalar, and that the condition (94) is satisfied. We calculate the Lie derivatives in (81c), making use of (82), (93) and (94), which leads to

\[
\mathcal{F}_{00}^{[1]}[Z] = -2[Z^0(\partial_\eta + 2\mathcal{H})] (2\partial_\eta + \mathcal{H})Z^0 + 2\Phi[\xi],
\]

(96a)

\[
\mathcal{F}_{0i}^{[1]}[Z] = -\left[Z^0(\partial_\eta + 2\mathcal{H}) (\partial_\eta + \mathcal{H})Z^0 - 2B[\xi]\right] - 2(D_iZ^0) (2(D_\eta Z^0) + 2\Phi[\xi]),
\]

(96b)

\[
\mathcal{F}_{ki}^{[1]}[Z] = 6Z^0(\partial_\eta + 2\mathcal{H})(\mathcal{H}Z^0 - 2\Psi[\xi]) - 2(D_kZ^0)(2(D_\eta Z^0)(\mathcal{H}Z^0 - 2B[\xi]),
\]

(96c)

\[
\hat{\mathcal{F}}_{ij}^{[1]}[Z] = -2D_i(Z^0 - 2B[\xi])(D_jZ^0),
\]

(96d)

where $Z^0 \equiv Z^0[\xi, \xi]$, and $\xi$, and $\xi$ can be chosen to be any two of the gauge fields $\xi_p, \xi_c$ and $\xi_v$.

When evaluating the source terms (87c), (92c) and (96) in the following sections it is convenient to eliminate the temporal derivatives of the first order gauge invariants $\delta, \nu$ and $\Psi$ in the Poisson gauge. To do this we use the linearized conservation equations for a perfect fluid in the following form:

\[
x \partial_x (\delta^p - 3\Psi^p) + \mathcal{H}^{-2}D^2(\mathcal{H}\nu_p) = 0,
\]

(97a)

\[
\mathcal{H}(x \partial_x + 1)\nu_p + \Psi_p + w(\delta^p - 3\mathcal{H}\nu_p) = 0,
\]

(97b)

and the velocity equation in the form

\[
(x \partial_x + 1)\Psi_p = -\frac{A}{2\mathcal{H}^2}(\mathcal{H}\nu_p),
\]

(97c)

(see equations (15b) and (16b) in UW). Here the scalar $A$ is given by

\[
A = 2(-\partial_\eta \mathcal{H} + \mathcal{H}^2 + K) = 3(1 + w)\mathcal{H}^2\Omega_m,
\]

(98)

the second equality holding for a perfect fluid with linear equation of state. In addition (97b) and (98) lead to

\[
x \partial_x (\mathcal{H}\nu_p) = -\frac{A - 2K}{2\mathcal{H}^2}(\mathcal{H}\nu_p) - \Psi_p - w(\delta^p - 3\mathcal{H}\nu_p),
\]

(99)

on noting that $\partial_\eta = \mathcal{H}x\partial_x$.

B.5.1 Transforming from the Poisson to the uniform curvature gauge

The transition quantities $(r)^0Z^0_{c,p} \equiv (r)^0Z^0[\xi_c, \xi_p]$ are obtained by choosing $\xi_p = \xi_c$ and $\xi_o = \xi_p$ in (83c) and using the second of equations (95). This leads to

\[
\mathcal{H}^{(1)}Z^0_{c,p} = (1)\Psi_p,
\]

(100a)

\[
\mathcal{H}^{(2)}Z^0_{c,p} = (2)\Psi_p - \frac{1}{6} \left( \mathcal{F}_{kj} - D^2S^i_{ij}\hat{\mathcal{F}}_{ij} \right) [^{(1)}Z_{c,p}].
\]

(100b)

---

27 Choose $\xi = \xi_p$ in equations (43) in [12], and specialize to a perfect fluid by setting $\Gamma = 0, \Xi = 0$ and noting that $\Phi_p = \Psi_p$. Also note that $\nu_p = \nu_p$ and $\mathcal{H}_\nu = \delta^p - 3\mathcal{H}\nu_p$.

28 See equations (16) and (36) in [12], noting that the background Einstein equations imply $\mathcal{A}_G = \mathcal{A}_T$, or equation (6) in UW, but note the typo: the signs on $\mathcal{H}'$ and $\mathcal{H}^2$ are reversed.
Next, we substitute (100) in (87) with \( \bullet \) and \( \circ \) replaced by \( c \) and \( p \), and obtain

\[
\begin{align*}
(1) \delta_c &= (1) \delta_p - 3(1) \Psi_p, \\
(2) \delta_c &= (2) \delta_p - 3(2) \Psi_p + S_\delta^{[1]} Z_{c,p},
\end{align*}
\]

where

\[
S_\delta^{[1]} Z_{c,p} := \frac{1}{2} (2 F_\delta + F^k_k - D^2 S^i_j \hat{f}_{ij})^{[1]} Z_{c,p}.
\]

We now use (87c), (96), the first of equations (95), and (100a) to show that

\[
S_\delta^{[1]} Z_{c,p} = 3 \Psi [(1 + 3w) \Psi - 2(1 + w) \delta] + 2 \Psi x \partial_x (\delta - 3 \Psi) - \mathcal{H}^{-2} \left[ (D \Psi)^2 - D^2 \mathcal{D}(\Psi) \right],
\]

where we for brevity drop the subscript \( p \) on the first order quantities in the source terms. Finally we use (97a) to eliminate the temporal derivative, obtaining

\[
S_\delta^{[1]} Z_{c,p} = 3 \Psi [(1 + 3w) \Psi - 2(1 + w) \delta] - \mathcal{H}^{-2} \left[ 2 \Psi D^2 (\mathcal{H} \nu) + (D \Psi)^2 - D^2 \mathcal{D}(\Psi) \right].
\]

In summary, equation (101b), with the source term given by (103), is the transformation law that relates \((2) \delta_c\) to \((2) \delta_p\).

### B.5.2 Transforming from the Poisson to the total matter gauge

The transition quantities \((^r Z^0)_{v,p} = (^r Z^0)_{v,\xi_p} \) are obtained by replacing \( \bullet \) and \( \circ \) with \( \nu \) and \( p \) in (92) and using the third of equations (95). This leads to

\[
\begin{align*}
(1)^r Z^0_{v,p} &= (1)^r v_p, \\
(2)^r Z^0_{v,p} &= (2)^r v_p + F_v^{[1]} Z_{v,p},
\end{align*}
\]

It follows from (87) that \((^r \delta) \nu \) is related to \((^r \delta) p \) according to

\[
\begin{align*}
(1) \delta_v &= (1) \delta_p - 3 \mathcal{H}(1) v_p, \\
(2) \delta_v &= (2) \delta_p - 3 \mathcal{H}(2) v_p + S_\delta^{[1]} Z_{v,p},
\end{align*}
\]

where

\[
S_\delta^{[1]} Z_{v,p} := (F_\delta - 3 \mathcal{H} F_v)^{[1]} Z_{v,p}.
\]

Then we use (87c), (92c), the first of equations (95), and (104a) to calculate an explicit expression for the source term:

\[
\begin{align*}
F_\delta^{[1]} Z_{v,p} &= \mathcal{H} v (x \partial_x - 3(1 + w)) (2 \delta - 3 \mathcal{H} v), \\
F_v^{[1]} Z_{v,p} &= 2 S^i [ \mathcal{H} v D_i (x \partial_x v) - \Phi D_i v] - \mathcal{H} v x \partial_x v.
\end{align*}
\]

Eliminating the time derivatives using equations (97) and (99) and making use of (3) and (98) we obtain

\[
S_\delta^{[1]} Z_{v,p} = \mathcal{H} v [ -6 \delta + 3 \left( 3 - \frac{3}{2} (1 + w) \Omega_m + \Omega_k \right) \mathcal{H} v - 2 \mathcal{H}^{-2} D^2(\mathcal{H} v)] - 6w S^i (\delta D_i (\mathcal{H} v)).
\]

In summary, equation (105b), with the source term given by (107), is the transformation law that relates \((2) \delta_v\) to \((2) \delta_p\).
B.6 Transforming from the Poisson to the uniform density gauge

The uniform density gauge is defined by

\[(r)\delta[\xi_p] = 0,\]  \hspace{1cm} (108)

with the spatial part of the gauge field fixed as for the Poisson gauge in \[26\]. We specialize \[33c\] by choosing \(\xi_s = \xi_p\) and \(\xi = \xi_p\), which relates \((r)\Psi_p\) to \((r)\Psi_p\). Substituting equation (108) in (87) with \(\cdot\) and \(\circ\) replaced by \(\rho\) and \(p\), yields expressions for the required transition quantities \((r)Z_{\rho,p}^0 \equiv (r)\Phi_0[\xi_p,\xi_p]\), which when substituted in \[33c\] lead to

\[\begin{align*}
(1)\Psi^s_{\rho} &= (1)\Psi^s_p - \frac{1}{3}(1)\delta_p, \\
(2)\Psi^s_{\rho} &= (2)\Psi^s_p - \frac{1}{3}(2)\delta_p + S(\Psi)[Z_{\rho,p}],
\end{align*}\]  \hspace{1cm} (109a, 109b)

where

\[S(\Psi)[Z_{\rho,p}] := -\frac{1}{6} \left(2\Phi_\delta + F^k_k - D^2 S \hat{F}_{ij} \right)[Z_{\rho,p}].\]  \hspace{1cm} (109c)

Finally it follows from (87c), (96) and (97a) that

\[S(\Psi)[Z_{\rho,p}] = \frac{1}{9} \delta((1 + 3w)\delta + 12\Psi) + \frac{1}{27}H - \frac{2}{3}(D\delta)^2 - 6\delta D^2 H v.\]  \hspace{1cm} (109d)

B.7 Transforming from the total matter to the synchronous gauge

The synchronous gauge is defined by

\[\begin{align*}
(r)B[\xi_s] &= 0, \\
(r)\Phi[\xi_s] &= 0, \\
(r)\psi[\xi_s] &= 0, \\
& r = 1, 2, \hspace{1cm} (110)
\end{align*}\n
where the subscript \(s\) stands for synchronous. We have to determine the transition quantities \((r)Z_{s,v}^0 \equiv (r)\Phi[\xi_s,\xi_v]\) and \((r)\psi[\xi_s,\xi_v]\). First, since \((r)\psi[\xi_v] = 0\), it follows from (110) and (92) with \(\cdot\) and \(\circ\) replaced by \(\cdot\) and \(\circ\) that

\[\psi[\xi_s,\xi_v] = \frac{1}{3} \delta((1 + 3w)\delta + 12\Psi) + \frac{1}{27}H - \frac{2}{3}(D\delta)^2 - 6\delta D^2 H v.\]  \hspace{1cm} (112)

A comparison with (112) leads to an exact temporal differential, which when integrated yields a spatial function. Setting this function to zero fixes the residual gauge freedom in the synchronous gauge and leads to a one-to-one gauge invariant relationship with the total matter gauge determined by

\[\begin{align*}
(1)\delta^s_{\rho} &= (1)\delta^s_{\rho}, \\
(2)\delta^s_{\rho} &= (2)\delta^s_{\rho} - \frac{4}{3}x m^{-2}(D\psi_0)D\psi_0. \\
& (113)
\end{align*}\n
Using \[111\] and \[114\] it follows from \[87\] that

\[\begin{align*}
(1)\delta^s &= (1)\delta^s, \\
(2)\delta^s &= (2)\delta^s - \frac{4}{3}x m^{-2}(D\psi_0)(D\psi_0). \\
& (115)
\end{align*}\n
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