Research article

Bioperators on soft topological spaces

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Abstract: To contribute to soft topology, we originate the notion of soft bioperators $\tilde{\gamma}$ and $\tilde{\gamma}'$. Then, we apply them to analyze soft $(\tilde{\gamma}, \tilde{\gamma}')$-open sets and study main properties. We also prove that every soft $(\tilde{\gamma}, \tilde{\gamma}')$-open set is soft open; however, the converse is true only when the soft topological space is soft $(\tilde{\gamma}, \tilde{\gamma}')$-regular. After that, we define and study two classes of soft closures namely $Cl(\tilde{\gamma}, \tilde{\gamma}')$ and $\tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')-Cl$ operators, and two classes of soft interior namely $Int(\tilde{\gamma}, \tilde{\gamma}')$ and $\tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')-Int$ operators. Moreover, we introduce the notions of soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed sets and soft $(\tilde{\gamma}, \tilde{\gamma}')$-$T_2$ spaces, and explore their fundamental properties. In general, we explain the relationships between these notions, and give some counterexamples.

Keywords: bioperators $\tilde{\gamma}$ and $\tilde{\gamma}'$ on $\tilde{\tau}$; soft $(\tilde{\gamma}, \tilde{\gamma}')$-open sets; soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed sets; soft $(\tilde{\gamma}, \tilde{\gamma}')$-$T_2$ spaces

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1. Introduction

Vagueness and uncertainty occupied the human mind for centuries. In modern society, we face uncertainty and vagueness in different areas such as economics, engineering, medical science, sociality, and environmental sciences. Over the years, mathematicians, engineers, and scientists, particularly those who focus on artificial intelligence are seeking for approaches to solve the problems that contain uncertainty or vagueness. They established many tools for this purpose such as soft sets which are the most popular of all these.

The concept of soft sets was first constructed by Molodtsov [29] in 1999 as a general mathematical tool for dealing with uncertain objects. He successfully applied the soft set theory in several
directions of mathematics, such as smoothness of functions, game theory, operators research, Riemann integration, Perron integration, probability, theory of measurement, etc.

Maji et al. [27,28] presented an application of soft sets in decision making problems that is based on the reduction of parameters to keep the optimal choice objects. Chen [12] presented a new definition of soft set parametrization reduction and a comparison of it with attribute reduction in rough set theory. Pei and Miao [31] showed that soft sets are a class of special information systems. Kong et al. [26] introduced the notion of normal parameter reduction of soft sets and investigated the problem of sub-optimal choice. El-Shafei et al. [15] defined new relations between ordinary points and soft sets which leads to redefine many soft topological concepts. To keep more set-theoretic properties on soft set theory, Al-shami and El-Shafei [8] introduced the concepts of $T$-soft subset and $T$-equality relations. Also, they initiated soft linear system with respect to some soft equality relations. Al-shami [5] studied soft sets on ordered setting and applied to explore new types of compactness and expect missing values in the information systems.

The soft set theory has been applied to many different fields (for examples, [6–9, 14, 18, 23, 31, 36]). In 2011, Shabir and Naz [32] constituted the study of soft topological spaces. They defined a soft topology on the collection of soft sets over $X$. Consequently, they defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft closure, soft neighbourhood of a point, soft separation axioms, soft regular spaces and soft normal spaces and established several properties for them. Kharal and Ahmad [25] defined subspace, soft closure, soft neighbourhood of a point, soft separation axioms, soft regular spaces and they defined basic notions of soft topological spaces such as soft open and soft closed sets, soft

Kasahara [24] defined an operator associated with a topology, namely an $\alpha$ operator and initiated some definitions which are equivalent to their counterparts on topological spaces when the operator involved is the identity operator. He also studied $\alpha$-closed graphs of $\alpha$-continuous functions and $\alpha$-compact spaces. Later, Jankovic [20] employed $\alpha$ operator to introduce $\alpha$-closure of a set and give some characterizations on $\alpha$-closed graph of functions. Then, Ogata [30] defined the notion of $\gamma$-open sets to study operator-functions and operator-separation. Umehara et al. [33] defined the concept of bioperators on topological spaces, and studied some bioperators-separation axioms. Recently, some researchers defined $\tilde{\gamma}$ operator on the soft topology $\tilde{\gamma}$. By using this $\tilde{\gamma}$ operator, Benchalli et al. [11] and Kalaivani et al. [21] defined soft $\tilde{\gamma}$-open set via soft point $e_F$ and studied some of its properties. Also, Kalavathia [22] introduced soft $\tilde{\gamma}$-open set via an ordinary point and established some of its properties.

We aim through this paper to achieve three goals: (1) introduce and investigate the concept of soft bioperators $\tilde{\gamma}$ and $\tilde{\gamma}'$; (2) present and discuss two classes of soft closures namely $Cl_{(\tilde{\gamma},\tilde{\gamma}')}$, and $\tilde{\gamma}_{(\tilde{\gamma},\tilde{\gamma}')}\text{-}Cl$ operators, and two classes of soft interior namely $Int_{(\tilde{\gamma},\tilde{\gamma}')}$, and $\tilde{\gamma}_{(\tilde{\gamma},\tilde{\gamma}')}\text{-}Int$ operators; (3) formulate and explore the concepts of soft ($\tilde{\gamma},\tilde{\gamma}'$)-g.closed sets and soft ($\tilde{\gamma},\tilde{\gamma}'$)-$T_2$ spaces.

This research paper consists of five sections. Section 2 contains the concepts and findings from both soft set theory and soft $\tilde{\gamma}$ operator. Section 3 puts forward two novel soft topological concepts, namely, bioperators $\tilde{\gamma}$ and $\tilde{\gamma}'$ and soft ($\tilde{\gamma},\tilde{\gamma}'$)-regular spaces. Section 4 introduces and studies the concepts of soft $Cl_{(\tilde{\gamma},\tilde{\gamma}')}$, and $\tilde{\gamma}_{(\tilde{\gamma},\tilde{\gamma}')}\text{-}Cl$ operators, and soft $Int_{(\tilde{\gamma},\tilde{\gamma}')}$, and $\tilde{\gamma}_{(\tilde{\gamma},\tilde{\gamma}')}\text{-}Int$ operators. Section 5 presents soft ($\tilde{\gamma},\tilde{\gamma}'$)-g.closed sets and soft ($\tilde{\gamma},\tilde{\gamma}'$)-$T_2$ spaces, and discusses some of their characterizations. The goal of Section 6 is to outline our main findings and plan for future work.
2. Soft set and $\tilde{\gamma}$ operator

**Definition 2.1.** [29] Let $X$ be an initial universe and $E$ be a set of parameters. Let $P(X)$ denote the power set of $X$ and $A$ be a non-empty subset of $E$. A pair $(F,A)$ is called a soft set over $X$, where if $F$ is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parameterized family of subsets of the universe $X$. For a particular $e \in A$, $F(e)$ may be considered the set of $e$-approximate elements of the soft set $(F,A)$ and if $e \notin A$, then $F(e) = \emptyset$. The family of all these soft sets over the universal set $X$ is denoted by $SS(X)_{A}$.

We call $(F,A)$ a null soft set, denoted by $\tilde{\phi}$ if for all $e \in A$, $F(e) = \phi$, and we call it an absolute soft set, denoted by $\tilde{X}$ if for all $e \in A$, $F(e) = X$.

**Definition 2.2.** [16] For two soft sets $(F,A)$ and $(G,B)$ over a common universe $X$, we say that $(F,A)$ is a soft subset of $(G,B)$ (we write $(F,A) \subseteq (G,B)$) if $A \subseteq B$, and $F(e) \subseteq G(e)$ for all $e \in A$. We also say that these two soft sets are soft equal if $(F,A)$ is a soft subset of $(G,B)$ and $(G,B)$ is a soft subset of $(F,A)$.

**Definition 2.3.** [1] The complement of a soft set $(F,A)$, denoted by $(F,A)^{c}$ or $\tilde{X^{c}}(F,A)$, is defined by $(F,A)^{c} = (F^{c},A)$ where $F^{c} : A \rightarrow P(X)$ is a mapping given by $F^{c}(e) = X \setminus F(e)$ for all $e \in A$.

**Definition 2.4.** [28] The soft union of two soft sets $(F,A)$ and $(G,B)$ over the common universe $X$ is the soft set $(H,C)$, where $C = A \cup B$, denoted by $(H,C) = (F,A) \cup (G,B)$, and is defined as for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

**Definition 2.5.** [1] The soft intersection of two soft sets $(F,A)$ and $(G,B)$ over a common universe $X$ is the soft set $(H,C)$ where $C = A \cap B \neq \emptyset$, denoted by $(H,C) = (F,A) \cap (G,B)$, and is defined as $H(e) = F(e) \cap G(e)$ for all $e \in C$.

**Definition 2.6.** [32] The soft difference of two soft sets $(F,A)$ and $(G,A)$ over $X$ is the soft set $(H,A)$, denoted by $(H,A) = (F,A) \setminus (G,A)$, and is defined as $H(e) = F(e) \setminus G(e)$ for all $e \in A$.

**Definition 2.7.** [13] A soft set $(P,A) \in SS(X)_{A}$ is called a soft point in $\tilde{X}$, denoted by $P_{x}$, if there exist $e \in A$ and $x \in X$ such that $P(e) = \{x\}$ and $P(e') = \phi$ for every $e' \in A \setminus \{e\}$. We write $P_{x} \subseteq (F,A)$, if $x \in F(e)$.

**Definition 2.8.** [15, 32] Let $(F,A) \in SS(X)_{A}$, and let $x \in X$. We say that

1. $\exists e \in (F,A)$ whenever $x \in F(e)$ for all $e \in A$.
2. $\not\exists e \in (F,A)$ whenever $x \notin F(e)$ for some $e \in A$.

Note that $\exists e \notin (F,A)$ if $x \notin F(e)$ for some $e \in A$, and $\not\exists e \notin (F,A)$ if $x \notin F(e)$ for all $e \in A$.

**Definition 2.9.** [32] Let $x \in X$. Then $(x,A)$ is the soft set over $X$ for which $x(e) = \{x\}$ for all $e \in A$.

**Definition 2.10.** [32] Let $\tilde{T}$ be the collection of soft sets over $X$. Then $\tilde{T}$ is said to be a soft topology on $X$ if it satisfies the following axioms:
Definition 2.11. [19, 32] Let \( (X, \bar{\tau}, A) \) be a soft topological space and let \( (F, A) \in SS(X)_A \).

1. The soft closure of \( (F, A) \) is the soft set defined by 
   \[ \text{Cl}(F, A) = \bar{\cap}(G, A) : (G, A) \text{ is a soft closed set and } (G, A) \subseteq (F, A) \]. Clearly \( \text{Cl}(F, A) \) is the smallest soft closed set over \( X \) which contains \( (F, A) \).
2. The soft interior of \( (F, A) \) is the soft set defined by 
   \[ \text{Int}(F, A) = \bigcup\{G, A) : (G, A) \text{ is a soft open set and } (F, A) \subseteq (G, A)\}. \] Thus \( \text{Int}(F, A) \) is the largest soft open set contained in \( (F, A) \).

Lemma 2.1. [19] Let \( (F, A) \in SS(X)_A \). Then \( \text{Cl}(F, A) = \bar{X}\setminus\text{Int}(\bar{X}\setminus(F, A)) \) and \( \text{Int}(F, A) = \bar{X}\setminus\text{Cl}(\bar{X}\setminus(F, A)) \).

Lemma 2.2. [34] \( \text{Cl}(F, A) \cap (U, A) \subseteq \text{Cl}((F, A) \cap (U, A)) \) for every soft open set \( (U, A) \) and every soft set \( (F, A) \) in \( (X, \bar{\tau}, A) \).

Definition 2.12. [11, 21, 22] Let \( (X, \bar{\tau}, A) \) be a soft topological space. An operator \( \bar{\gamma} \) on the soft topology \( \bar{\tau} \) is a mapping from \( \bar{\tau} \) into \( SS(X)_A \) such that \( (V, A) \subseteq \bar{\gamma}(V, A) \) for all \( (V, A) \in \bar{\tau} \), where \( \bar{\gamma}(V, A) \) denotes the value of \( \bar{\gamma} \) at \( (V, A) \). This operator will be denoted by \( \bar{\gamma} : \bar{\tau} \to SS(X)_A \).

The main definitions and results about \( \bar{\gamma} \) operator on the soft topology \( \bar{\tau} \) can be found in [11, 21, 22].

Now, we will define the soft \( \bar{\gamma} \)-open set with respect to a soft point \( P^\epsilon_\gamma \).

Definition 2.13. Let \( (X, \bar{\tau}, A) \) be a soft topological space and \( \bar{\gamma} : \bar{\tau} \to SS(X)_A \) be an operator on \( \bar{\tau} \). A soft set \( (F, A) \) of \( (X, \bar{\tau}, A) \) is said to be soft \( \bar{\gamma} \)-open if for each \( P^\epsilon_\gamma \in (F, A) \), there exists \( (V, A) \in \bar{\tau} \) with \( P^\epsilon_\gamma \in (V, A) \) and \( \bar{\gamma}(V, A) \subseteq (F, A) \).

\( \bar{\gamma} \) will be denoted by the class of all soft \( \bar{\gamma} \)-open sets of a soft topological space \( (X, \bar{\tau}, A) \). It is clear that \( \bar{\gamma} \subseteq \bar{\tau} \). The union of any soft \( \bar{\gamma} \)-open sets is soft \( \bar{\gamma} \)-open, but the intersection of any two soft \( \bar{\gamma} \)-open sets need not be soft \( \bar{\gamma} \)-open. Therefore, \( \bar{\gamma} \) is not a soft topology on \( \bar{\tau} \).

The definition of soft regular operator \( \bar{\gamma} \) on \( \bar{\tau} \) with respect to a soft point \( P^\epsilon_\gamma \) is as follows.

Definition 2.14. Let \( (X, \bar{\tau}, A) \) be any soft topological space. An operator \( \bar{\gamma} \) on \( \bar{\tau} \) is said to be soft regular if for every soft open neighborhoods \( (U, A) \) and \( (V, A) \) of each \( P^\epsilon_\gamma \in \bar{\tau} \), there exists a soft open neighborhood \( (W, A) \) of \( P^\epsilon_\gamma \) such that 
\[ \bar{\gamma}(W, A) \subseteq \bar{\gamma}(U, A) \cap \bar{\gamma}(V, A). \]

Proposition 2.1. Let \( \bar{\gamma} \) be a soft regular operator on \( \bar{\tau} \). If \( (F, A) \in \bar{\tau}_\gamma \) and \( (G, A) \in \bar{\tau}_\gamma \), then \( (F, A) \cap (G, A) \in \bar{\tau}_\gamma \). Thus, \( \bar{\tau}_\gamma \) is a soft topology on \( \bar{X} \).

Next, the definition of soft \( \bar{\gamma} \)-regular space \( (X, \bar{\tau}, A) \) with respect to a soft point \( P^\epsilon_\gamma \) is as follows.

Definition 2.15. A soft topological space \( (X, \bar{\tau}, A) \) with an operator \( \bar{\gamma} \) on \( \bar{\tau} \) is said to be soft \( \bar{\gamma} \)-regular if for every \( P^\epsilon_\gamma \in \bar{\tau} \) and for every \( (U, A) \in \bar{\tau} \) with \( P^\epsilon_\gamma \in (U, A) \), there exists \( (W, A) \in \bar{\tau} \) with \( P^\epsilon_\gamma \in (W, A) \) and \( \bar{\gamma}(W, A) \subseteq (U, A) \).
Proposition 2.2. A soft topological space $(X, \bar{\tau}, A)$ is soft \( \gamma \)-regular if and only if \( \bar{\tau} = \bar{\gamma} \).

Definition 2.16. Let $(X, \bar{\tau}, A)$ be any soft topological space. An operator \( \gamma \) on \( \bar{\tau} \) is said to be soft open if for each \( P^*_e \in \bar{X} \) and for each \( (U, A) \in \bar{\tau} \) with \( P^*_e \in (U, A) \), there exists \( (W, A) \in \bar{\tau} \) with \( P^*_e \in (W, A) \) and \( (W, A) \subseteq \gamma(U, A) \).

Definition 2.17. [22] Let \( (F, A) \in SS(X)_A \) and \( P^*_e \in \bar{X} \). A soft point \( P^*_e \in \bar{X} \) is in the soft \( \gamma \)-closure of \( (F, A) \) if \( \gamma(U, A) \cap (F, A) \neq \emptyset \) for every \( (U, A) \in \bar{\tau} \) with \( P^*_e \in (U, A) \). The set of all soft \( \gamma \)-closure points of \( (F, A) \) is called the soft \( \gamma \)-closure of \( (F, A) \) and it is denoted by \( Cl_{\gamma}(F, A) \).

Definition 2.18. [22] Let \( (F, A) \in SS(X)_A \). The soft set \( \tau_{\gamma} \)-\( Cl(F, A) \) denotes the soft intersection of all soft \( \gamma \)-closed sets of \( (X, \bar{\tau}, A) \) containing \( (F, A) \) and is defined as
\[
\tau_{\gamma} \text{-} Cl(F, A) = \bigcap \{(K, A) : (F, A) \subseteq (K, A) \text{ and } \bar{X} \cap (K, A) \subseteq \bar{\tau} \}.
\]

Definition 2.19. [22] Let \( (F, A) \in SS(X)_A \) and \( P^*_e \in \bar{X} \). A soft point \( P^*_e \in (F, A) \) is said to be soft \( \gamma \)-interior point of \( (F, A) \) if there exists a soft open neighborhood \( (U, A) \) of \( P^*_e \) such that \( \gamma(U, A) \subseteq (F, A) \). We denote the set of all soft \( \gamma \)-interior points of \( (F, A) \) by \( Int_{\gamma}(F, A) \). That is,
\[
\Int_{\gamma}(F, A) = \{P^*_e \in (F, A) : \gamma(U, A) \subseteq (F, A) \text{ for some } (U, A) \in \bar{\tau} \}
\]

Definition 2.20. [22] Let \( (F, A) \in SS(X)_A \). Denote \( \tau_{\gamma} \)-\( Int(F, A) \) by the soft union of all soft \( \gamma \)-open sets of \( (X, \bar{\tau}, A) \) contained in \( (F, A) \) and is defined as
\[
\tau_{\gamma} \text{-} Int(F, A) = \bigcup \{(U, A) : (U, A) \subseteq (F, A) \text{ and } (U, A) \in \bar{\tau} \}.
\]

Definition 2.21. [22] A soft set \( (F, A) \) of a soft space \( (X, \bar{\tau}, A) \) is said to be soft \( \gamma \)-g.closed if \( Cl_{\gamma}(F, A) \subseteq (U, A) \) whenever \( (F, A) \subseteq (U, A) \) and \( (U, A) \) is soft \( \gamma \)-open.

Definition 2.22. [22] A soft space \( (X, \bar{\tau}, A) \) is said to be soft \( \gamma \)-T\( _{\gamma} \) if every soft \( \gamma \)-g.closed set of \( (X, \bar{\tau}, A) \) is soft \( \gamma \)-closed.

3. Soft \((\gamma, \gamma')\)-open sets

Throughout this paper, let \( \gamma \) and \( \gamma' \) be given two operators on a soft topology \( \bar{\tau} \). That is, \( \gamma : \bar{\tau} \rightarrow SS(X)_A \) and \( \gamma' : \bar{\tau} \rightarrow SS(X)_A \) are mappings such that \( (U, A) \subseteq \gamma(U, A) \) and \( (V, A) \subseteq \gamma'(V, A) \) for all \( (U, A) \in \bar{\tau} \) and for all \( (V, A) \in \bar{\tau} \).

We begin this section by presenting the following definition:

Definition 3.1. Let \( (X, \bar{\tau}, A) \) be a soft topological space, and \( \gamma \) and \( \gamma' \) be operators on \( \bar{\tau} \). A non-null soft set \( (F, A) \) of \( (X, \bar{\tau}, A) \) is said to be soft \((\gamma, \gamma')\)-open if for each soft point \( P^*_e \in (F, A) \), there exist soft open neighborhoods \( (U, A) \) and \( (V, A) \) of \( P^*_e \) such that
\[
\gamma(U, A) \cup \gamma'(V, A) \subseteq (F, A).
\]

Suppose that the null soft set \( \emptyset \) is also soft \((\gamma, \gamma')\)-open for any operators \( \gamma \) and \( \gamma' \) on \( \bar{\tau} \).

\( \bar{\tau}_{(\gamma, \gamma')} \) will be denoted by the class of all soft \((\gamma, \gamma')\)-open sets of a soft topological space \( (X, \bar{\tau}, A) \).

Example 3.1. Let \( X = \{a_1, a_2, a_3\} \), \( A = \{e_1, e_2\} \) and \( \bar{\tau} = \{\emptyset, \bar{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\} \) be a soft topology on \( X \), where \( (F_1, A), (F_2, A), (F_3, A) \) and \( (F_4, A) \) defined as follows:
\[
(F_1, A) = \{(e_1, \{a_1\}), (e_2, \{a_1\})\},
(F_2, A) = \{(e_1, \{a_2\}), (e_2, \{a_2\})\},
\]
Remark 3.1. Consider the soft topological space converses of Proposition 3.1 are not true in general.

Thus, \( \tilde{\tau} = \{ \tilde{\phi}, \tilde{X}, (F_1, A), (F_4, A) \} \), \( \tilde{\tau}_\gamma = \{ \tilde{\phi}, \tilde{X}, (F_2, A), (F_4, A) \} \) and \( \tilde{\tau}_{(\gamma, \gamma')} = \{ \tilde{\phi}, \tilde{X}, (F_4, A) \} \).

Proposition 3.1. For any soft set \( (F, A) \) of \( (X, \tilde{\tau}, A) \), the following hold.

1. If \( (F, A) \) is soft \( (\tilde{\gamma}, \tilde{\gamma'}) \)-open, then \( (F, A) \) is soft \( \tilde{\gamma} \)-open for any operator \( \tilde{\gamma} \).
2. If \( (F, A) \) is soft \( (\tilde{\gamma}, \tilde{\gamma'}) \)-open, then \( (F, A) \) is soft \( \tilde{\gamma}' \)-open for any operator \( \tilde{\gamma}' \).
3. (a) \( (F, A) \) is soft \( (\tilde{\gamma}, \tilde{\gamma'}) \)-open if and only if \( (F, A) \) is soft \( \tilde{\gamma} \)-open and soft \( \tilde{\gamma}' \)-open.
   
   (b) \( \tilde{\tau}_{(\gamma, \gamma')} = \tilde{\tau}_\gamma \cap \tilde{\tau}_{\gamma'} \).
4. If \( (F, A) \) is soft \( (\tilde{\gamma}, \tilde{\gamma'}) \)-open, then \( (F, A) \) is soft open.

Proof. (1) Let \( P^\varepsilon_{\gamma} \in (F, A) \). Then there exist soft open neighborhoods \( (U, A) \) and \( (V, A) \) of \( P^\varepsilon_{\gamma} \) such that \( \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A) \). Hence, \( \tilde{\gamma}(U, A) \subseteq (F, A) \). This implies that \( (F, A) \) is soft \( \tilde{\gamma} \)-open.

(2) Let \( P^\varepsilon_{\gamma} \in (F, A) \). Then there exist soft open neighborhoods \( (U, A) \) and \( (V, A) \) of \( P^\varepsilon_{\gamma} \) such that \( \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A) \). Hence, \( \tilde{\gamma}'(V, A) \subseteq (F, A) \). This implies that \( (F, A) \) is soft \( \tilde{\gamma}' \)-open.

(3a) Necessity: Let \( P^\varepsilon_{\gamma} \in (F, A) \). It follows from assumptions that there exist soft open neighborhoods \( (U, A) \) and \( (V, A) \) of \( P^\varepsilon_{\gamma} \) such that \( \tilde{\gamma}(U, A) \subseteq (F, A) \) and \( \tilde{\gamma}'(V, A) \subseteq (F, A) \). Thus, \( \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A) \). Therefore, \( (F, A) \) is soft \( (\tilde{\gamma}, \tilde{\gamma'}) \)-open.

Sufficiency: Let \( P^\varepsilon_{\gamma} \in (F, A) \). Then from (1) and (2), we get \( \tilde{\gamma}(U, A) \subseteq (F, A) \) and \( \tilde{\gamma}'(V, A) \subseteq (F, A) \). Therefore, \( (F, A) \) is soft \( \tilde{\gamma} \)-open and soft \( \tilde{\gamma}' \)-open.

(3b) It is obvious.

(4) Since \( \tilde{\tau}_\gamma \subseteq \tilde{\tau} \) and \( (F, A) \) is soft \( \tilde{\gamma} \)-open (by (1)), \( (F, A) \) is soft open.

Remark 3.1. The following relations are shown by Proposition 3.1 (3).

\[
\tilde{\tau}_\gamma \cap \tilde{\tau}_{\gamma'} = \tilde{\tau}_{(\gamma, \gamma')} \subseteq \tilde{\tau}_\gamma \subseteq \tilde{\tau}
\]

and

\[
\tilde{\tau}_\gamma \cap \tilde{\tau}_{\gamma'} = \tilde{\tau}_{(\gamma, \gamma')} \subseteq \tilde{\tau}_{\gamma'} \subseteq \tilde{\tau}.
\]

The following example shows that the inverse inclusions of Remark 3.1 do not hold and the converses of Proposition 3.1 are not true in general.

Example 3.2. Consider the soft topological space \( (X, \tilde{\tau}, A) \) defined in Example 3.1. Then the soft set \( (F_1, A) \) is soft \( \tilde{\gamma} \)-open in \( \tilde{X} \), but \( (F_1, A) \) is neither soft \( \tilde{\gamma}' \)-open nor soft \( (\tilde{\gamma}, \tilde{\gamma}') \)-open. Whereas, the soft set \( (F_2, A) \) is soft \( \tilde{\gamma}' \)-open in \( \tilde{X} \), but \( (F_2, A) \) is neither soft \( \tilde{\gamma} \)-open nor soft \( (\tilde{\gamma}, \tilde{\gamma}') \)-open. Therefore, \( \tilde{\tau}_\gamma \nsubseteq \tilde{\tau}_{\gamma'} \), \( \tilde{\tau}_\gamma \nsubseteq \tilde{\tau}_{(\gamma, \gamma')} \) and \( \tilde{\tau}_{\gamma'} \nsubseteq \tilde{\tau}_{(\gamma, \gamma')} \). Also, \( (F_3, A) \) is soft open, but it is not soft \( (\tilde{\gamma}, \tilde{\gamma}') \)-open.
Proposition 3.2. For any soft set \((F, A)\) of \((X, \tilde{\tau}, A)\), the following statements are equivalent:

1. \((F, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma})\)-open.
2. \((F, A)\) is soft \(\gamma\)-open.
3. \((F, A)\) is soft \((\tilde{\gamma}, \tilde{id})\)-open, where \(\tilde{id} : \tilde{\tau} \to SS(X)_A\) is the identity operator, i.e. \(\tilde{\gamma}(F, A) = (F, A)\) for every \((F, A) \in \tilde{\tau}\).

Proof. (1) \(\iff\) (2) It is shown by setting \(\tilde{\gamma}' = \tilde{\gamma}\) in Proposition 3.1 (3a).

(2) \(\iff\) (3) It is shown by their definitions. \(\square\)

Lemma 3.1. If \((F, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open for every \(\lambda \in \Lambda\), then \(\bigcup\{(F, A) : \lambda \in \Lambda\}\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open.

Proof. Let \(P^e_{\tilde{\gamma}} \in \bigcup_{\Lambda \in \Lambda} (F, A)\). Then \(P^e_{\tilde{\gamma}} \in (F, A)\) for some \(\lambda_0 \in \Lambda\). Hence there exist soft open neighborhoods \((U, A)\) and \((V, A)\) of \(P^e_{\tilde{\gamma}}\) such that

\[
\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A) \subseteq \bigcup_{\lambda \in \Lambda} (F, A).
\]

Thus, \(\bigcup_{\lambda \in \Lambda} (F, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\). \(\square\)

Remark 3.2. The intersection of any two soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open sets need not be soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open as shown by the below example.

Example 3.3. Let \((X, \tilde{\tau}, A)\) be same as given in Example 3.1. Define operators \(\tilde{\gamma} : \tilde{\tau} \to SS(X)_A\) and \(\tilde{\gamma}' : \tilde{\tau} \to SS(X)_A\) as follows: For all \((F, A) \in \tilde{\tau}\)

\[
\tilde{\gamma}(F, A) = \begin{cases} (F, A) & \text{if } P^e_{\tilde{\gamma}} \in (F, A) \\ \text{Cl}(F, A) & \text{if } P^e_{\tilde{\gamma}} \notin (F, A) \end{cases}
\]

and

\[
\tilde{\gamma}'(F, A) = \begin{cases} (F, A) & \text{if } P^e_{\tilde{\gamma}'} \in (F, A) \\ \tilde{\tilde{X}} & \text{if } P^e_{\tilde{\gamma}'} \notin (F, A) \end{cases}
\]

Thus, \(\tilde{\tau}_\gamma = \{\tilde{\phi}, \tilde{\tilde{X}}, (F, A), (F_3, A), (F_4, A)\}\), \(\tilde{\tau}'_\gamma = \{\tilde{\phi}, \tilde{\tilde{X}}, (F_3, A), (F_4, A)\}\) and \(\tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}') = \{\tilde{\phi}, \tilde{\tilde{X}}, (F, A), (F_3, A), (F_4, A)\}\).

Then, \((F_3, A)\) and \((F_4, A)\) are soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open sets. However, their intersection \((F_3, A) \cap (F_4, A) = (F_2, A)\) is not a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open set in \(\tilde{\tilde{X}}\).

Remark 3.3. It follows that \(\tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\) may not be a soft topology on \(\tilde{\tilde{X}}\). According to Lemma 3.1 \((X, \tilde{\tau}, A)\) is a supra soft topological space.

Proposition 3.3. Let \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) be soft regular operators on \(\tilde{\tau}\).

1. If \((F, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\) and \((G, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\), then \((F, A) \cap (G, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\).
2. \(\tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\) is a soft topology on \(\tilde{\tilde{X}}\).

Proof. (1) Assume that \((F, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\) and \((G, A) \in \tilde{\tau}(\tilde{\gamma}, \tilde{\gamma}')\). Let \(P^e_{\tilde{\gamma}} \in (F, A) \cap (G, A)\). Then \(P^e_{\tilde{\gamma}} \in (F, A)\) and \(P^e_{\tilde{\gamma}} \in (G, A)\). So, there exist soft open neighborhoods \((U_1, A), (V_1, A), (U_2, A)\) and \((V_2, A)\) of \(P^e_{\tilde{\gamma}}\) such that

\[
\tilde{\gamma}(U_1, A) \cup \tilde{\gamma}'(V_1, A) \subseteq (F, A)
\]
\[ \tilde{\gamma}(U_2, A) \cup \tilde{\gamma}'(V_2, A) \subseteq (G, A). \]

Since \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) are soft regular operators on \( \tilde{r} \), there exist soft open neighborhoods \((W_1, A)\) and \((W_2, A)\) of \( P_e^\gamma \) such that
\[ \tilde{\gamma}(W_1, A) \subseteq \tilde{\gamma}(U_1, A) \cap \tilde{\gamma}(V_1, A) \]
and
\[ \tilde{\gamma}'(W_2, A) \subseteq \tilde{\gamma}'(U_2, A) \cap \tilde{\gamma}'(V_2, A). \]

So,
\[
\tilde{\gamma}(W_1, A) \cup \tilde{\gamma}'(W_2, A) \\
\subseteq \left[ \tilde{\gamma}(U_1, A) \cap \tilde{\gamma}(V_1, A) \right] \cup \left[ \tilde{\gamma}'(U_2, A) \cap \tilde{\gamma}'(V_2, A) \right] \\
\subseteq \left[ \tilde{\gamma}(U_1, A) \cup \tilde{\gamma}(U_2, A) \right] \cap \left[ \tilde{\gamma}'(V_1, A) \cup \tilde{\gamma}'(V_2, A) \right] \\
\subseteq (F, A) \cap (G, A).
\]

Thus, \((F, A) \cap (G, A) \subseteq \tilde{r}_{(\tilde{\gamma}, \tilde{\gamma}')}\).

(2) Since \( \tilde{\phi} \) and \( \tilde{X} \) are soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open sets, it is proved by (1) and Lemma 3.1 that \( \tilde{r}_{(\tilde{\gamma}, \tilde{\gamma}')} \) is a soft topology on \( \tilde{X} \).

\( \square \)

The following example shows that the soft regularity on \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) of Proposition 3.3 cannot be removed in general.

**Example 3.4.** Consider the soft topological space \((X, \tilde{r}, A)\) defined in Example 3.3. Then \( \tilde{r}_{(\tilde{\gamma}, \tilde{\gamma}')} \) is not a soft topology on \( \tilde{X} \) since \( \tilde{\gamma} \) is not a soft regular operator on \( \tilde{r} \). If we take soft open neighborhoods \((F_2, A)\) and \((F_3, A)\) of a soft point \( P_e^\gamma \), then \( \tilde{\gamma}(F_3, A) \cap \tilde{\gamma}(F_4, A) = (F_3, A) \cap \text{Cl}(F_4, A) = (F_3, A) \cap (F_4, A) = (F_2, A) \). But we cannot find a soft open neighborhood \((W, A)\) of this soft point \( P_e^\gamma \) such that \( \tilde{\gamma}(W, A) \subseteq (F_2, A) \).

**Definition 3.2.** A soft topological space \((X, \tilde{r}, A)\) is said to be soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular if for every \( P_e^\gamma \in \tilde{X} \) and for every \((U, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (U, A) \), there exist \((V, A)\) in \( \tilde{r} \) and \((W, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (V, A) \), \( P_e^\gamma \in (W, A) \) and \( \tilde{\gamma}(V, A) \cup \tilde{\gamma}'(V, A) \subseteq (U, A) \).

**Theorem 3.1.** Let \((X, \tilde{r}, A)\) be a soft topological space. Then the following statements are equivalent:

1. \( \tilde{r} = \tilde{r}_{(\tilde{\gamma}, \tilde{\gamma}')} \).
2. \((X, \tilde{r}, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular.
3. For each \( P_e^\gamma \in \tilde{X} \) and for each \((U, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (U, A) \), there exists \((W, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (W, A) \) and \( \tilde{\gamma}(W, A) \cup \tilde{\gamma}'(W, A) \subseteq (U, A) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( P_e^\gamma \in \tilde{X} \) and \((U, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (U, A) \). It follows from assumption that \((U, A)\) in \( \tilde{r}_{(\tilde{\gamma}, \tilde{\gamma}')} \). This implies that there exist \((W, A)\) in \( \tilde{r} \) and \((V, A)\) in \( \tilde{r} \) with \( P_e^\gamma \in (W, A) \), \( P_e^\gamma \in (V, A) \) and
\[ \tilde{\gamma}(W, A) \cup \tilde{\gamma}'(V, A) \subseteq (U, A). \]
Thus, \((X, \tilde{\tau}, A)\) is a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular space.

\((2) \Rightarrow (3)\) Let \(P_e^\prime \in \tilde{X}\) and \((U, A) \in \tilde{\tau}\) with \(P_e^\prime \in (U, A)\). Then by \((2)\), there exist \((V_1, A) \in \tilde{\tau}\) and \((V_2, A) \in \tilde{\tau}\) with \(P_e^\prime \in (V_1, A), P_e^\prime \in (V_2, A)\) and \(\tilde{\gamma}(V_1, A) \cup \tilde{\gamma}'(V_2, A) \subseteq (U, A)\). Again, by using \((2)\) for the soft sets \((V_1, A)\) and \((V_2, A)\), there exist soft open neighborhoods \((S_1, A), (S_2, A), (T_1, A)\) and \((T_2, A)\) of \(P_e^\prime\) such that

\[ \tilde{\gamma}(S_1, A) \cup \tilde{\gamma}'(S_2, A) \subseteq (V_1, A) \]

and

\[ \tilde{\gamma}(T_1, A) \cup \tilde{\gamma}'(T_2, A) \subseteq (V_2, A) \]

This means that \((V_1, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) and \((V_2, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\). Take \((W, A) = (V_1, A) \cup (V_2, A)\). Thus, by Lemma 3.1, \((W, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) with \(P_e^\prime \in (W, A)\) and \((W, A) \subseteq (U, A)\).

\((3) \Rightarrow (1)\) By using \((3)\) and Lemma 3.1, it follows that \((U, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\). That is, \(\tilde{\tau} \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\). Since \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \subseteq \tilde{\tau}\) (by Remark 3.1), \(\tilde{\tau} = \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}.\)

**Example 3.5.** Since \(\tilde{\tau} \neq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) as shown in Example 3.3, \((X, \tilde{\tau}, A)\) is not a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular space.

**Lemma 3.2.** \((X, \tilde{\tau}, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular if and only if \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} = \tilde{\tau}_{\gamma} = \tilde{\tau}_{\gamma'} = \tilde{\tau}\).

**Proof.** The proof follows from Theorem 3.1 and Remark 3.1.

**Proposition 3.4.** \((X, \tilde{\tau}, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular if and only if \((X, \tilde{\tau}, A)\) is both soft \(\tilde{\tau}\)-regular and soft \(\tilde{\gamma}'\)-regular.

**Proof.** The proof follows from Lemma 3.2 and Proposition 2.2.

**Proposition 3.5.** The following statements are equivalent:

1. \((X, \tilde{\tau}, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular.
2. \((X, \tilde{\tau}, A)\) is soft \(\tilde{\gamma}\)-regular.
3. \((X, \tilde{\tau}, A)\) is soft \((\tilde{\gamma}, id)\)-regular.

**Proof.** Since \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} = \tilde{\tau}_{\tilde{\gamma}} = \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \subseteq \tilde{\tau}\) holds in general, the equivalences are proved by using Theorem 3.1.

**4. Between \(Cl_{(\tilde{\gamma}, \tilde{\gamma}')}\) and \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\)-\(Cl\) operators**

In this section, we introduce two closure operators, namely, \(Cl_{(\tilde{\gamma}, \tilde{\gamma}')}\) and \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\)-\(Cl\), and two interior operators, namely, \(Int_{(\tilde{\gamma}, \tilde{\gamma}')}\) \(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\)-\(Int\). We illustrate the relationships between them and discuss main properties.

First, we define soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed set.

**Definition 4.1.** A soft subset \((K, A)\) of a soft space \((X, \tilde{\tau}, A)\) is said to be soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed if its complement \(\tilde{X}\setminus(K, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-open in \((X, \tilde{\tau}, A)\).

Next, two classes of soft closures via bioperators \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) are investigated.

**Definition 4.2.** Let \((F, A) \in SS(X)_A\) and \(P_e^\prime \in \tilde{X}\).
1. A soft point \( P_e^* \) is said to be a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closure point of \((F, A)\) if \((\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A)) \cap (F, A) \neq \emptyset\) for every \((U, A) \in \tilde{\tau}\) and \((V, A) \in \tilde{\tau}\) with \( P_e^* \in (U, A) \cap (V, A) \). The set of all soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closure points of \((F, A)\) is called soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closure of \((F, A)\) and it is denoted by \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \).

2. \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \) denotes the soft intersection of all soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed sets of \((X, \tilde{\tau}, A)\) containing \((F, A)\) and is defined as

\[
\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) = \bigcap \{ (K, A) : (F, A) \subseteq (K, A) \text{ and } \tilde{X}(K, A) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \}
\]

**Theorem 4.1.** Let \((F, A) \in SS(X, A)\) and \( P_e^* \in \tilde{X}\). Then \( P_e^* \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \) if and only if \((F, A) \cap (U, A) \neq \emptyset\) for each \((U, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) with \( P_e^* \in (U, A) \).

**Proof.** Necessity: Let \( P_e^* \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \) and assume that \((F, A) \cap (U, A) = \emptyset\) for some \((U, A) \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) with \( P_e^* \in (U, A) \). Then \((F, A) \subseteq \tilde{X}(U, A)\) and \( \tilde{X}(U, A)\) is a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed set in \( \tilde{X}\). Hence \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \subseteq \tilde{X}(U, A)\). Thus, \( P_e^* \in \tilde{X}(U, A)\). This is a contradiction. So, the proof is completed.

Sufficiency: Let \( P_e^* \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)\). So there exists a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed set \((K, A)\) containing \((F, A)\) with \( P_e^* \notin (K, A)\). Hence, \( \tilde{X}(K, A) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')}\) with \( P_e^* \notin \tilde{X}(K, A)\) and \([\tilde{X}(K, A)] \cap (F, A) = \emptyset\), which is a contradiction by hypothesis. Thus, \( P_e^* \in \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)\). \(\square\)

**Lemma 4.1.** Let \((F, A), (G, A) \in SS(X, A)\). Then the following statements are true:

1. \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed in \( \tilde{X}\) and \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \) is soft closed in \( \tilde{X}\).
2. (a) \( (F, A) \subseteq Cl(F, A) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \).
   (b) \( (F, A) \subseteq Cl(F, A) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \).
3. The following are equivalent:
   (a) \( (F, A) \) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed.
   (b) \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) = (F, A)\).
   (c) \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) = (F, A)\).
4. If \((F, A) \subseteq (G, A)\), then \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(G, A)\) and \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (G, A)\).
5. (a) \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl((F, A) \cap (G, A)) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \cap \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(G, A)\).
   (b) \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \cap (G, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \cap Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (G, A)\).
6. (a) \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl((F, A) \cup (G, A)) \subseteq \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A) \cup \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(G, A)\).
   (b) \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \cup (G, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) \cup Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (G, A)\).
7. \( \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)) = \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)\).
8. \( Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (\tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)) = \tilde{\tau}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-} Cl(F, A)\).

**Proof.** Straightforward. \(\square\)

**Theorem 4.2.** Let \((F, A) \in SS(X, A)\). Then,

\[
Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A) = Cl_{\tilde{\gamma}} (F, A) \cup Cl_{\tilde{\gamma}'} (F, A).
\]

**Proof.** We start by their definitions,

\[
P_e^* \notin Cl_{(\tilde{\gamma}, \tilde{\gamma}')} (F, A).
\]

\[
\Leftrightarrow \text{ There exist } (U, A) \in \tilde{\tau} \text{ and } (V, A) \in \tilde{\tau} \text{ with } P_e^* \in (U, A) \text{ and } P_e^* \in (V, A) \text{ such that } \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \nabla (F, A) = \emptyset.
\]

\[
\Leftrightarrow \text{ There exist } (U, A) \in \tilde{\tau} \text{ and } (V, A) \in \tilde{\tau} \text{ with } P_e^* \in (U, A) \text{ and } P_e^* \in (V, A) \text{ such that } \tilde{\gamma}(U, A) \nabla (F, A) = \emptyset \text{ and } \tilde{\gamma}'(V, A) \nabla (F, A) = \emptyset.
\]

\[
\Leftrightarrow P_e^* \notin Cl_{\tilde{\gamma}} (F, A) \text{ and } P_e^* \notin Cl_{\tilde{\gamma}'} (F, A).
\]
\[ P_e^* \notin Cl_{\tilde{\gamma}}(F, A) \cup Cl_{\tilde{\gamma}'}(F, A). \]
So, we get the proof. \[ \square \]

**Proposition 4.1.** Let \((F, A) \in SS(X)_A\). If \((X, \tilde{\tau}, A)\) is a soft regular space, then \(Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}}Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}'}Cl(F, A) = Cl(\tilde{\gamma}_{\tilde{\gamma}'})(F, A) = \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A)\).

**Proof.** By using Theorem 3.1, \(\tilde{\tau} = \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A)\). Hence \(Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}}Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}'}Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A)\). It follows from Lemma 4.1 (2) that \(Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}}Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}'}Cl(F, A) = \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A)\). \[ \square \]

**Lemma 4.2.** Let \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) be soft regular operators on \(\tilde{\tau}\). For any \((F, A), (G, A) \in SS(X)_A\), we have

1. \(\tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\).
2. \(Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\).

**Proof.** (1) It follows directly from Lemma 4.1 (6) that \(\tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A)) \subseteq \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A)\). Then it is enough to prove that \(\tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A) \subseteq \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A)\). Let \(P_e^* \notin \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A)\). Then there exist \((U, A), (V, A) \in \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\) with \(P_e^* \notin (U, A), P_e^* \notin (V, A)\). Then \((U, A) \cap (V, A) = \emptyset\) and \((G, A) \cap (V, A) = \emptyset\). Since \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) are soft regular operators on \(\tilde{\tau}\), by Proposition 3.3 (1), \((U, A) \cap (V, A) \notin \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). This means that \(P_e^* \notin \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). Hence,

\[ \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A) \subseteq \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl(F, A) \cup (G, A). \]

(2) Let \(P_e^* \notin Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). Then there exist soft open sets \((U, A), (U, A), (V, A) \cup (V, A) \cup (V, A) \subseteq Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). Then there exist soft open sets \((U, A), (V, A) \in \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). Then there exist \((W, A) \in \tilde{\tau}_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\) such that \(\tilde{\gamma}(U, A) \subset \tilde{\gamma}(U, A) \cap \tilde{\gamma}(V, A)\).

and

\[ \tilde{\gamma}'(W, A) \subset \tilde{\gamma}'(W, A) \cap \tilde{\gamma}'(V, A). \]

Thus, we have

\[ \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(U, A) \supseteq [\tilde{\gamma}(U, A) \cap \tilde{\gamma}(V, A)] \cup [\tilde{\gamma}'(U, A) \cap \tilde{\gamma}'(V, A)] \]

\[ \supseteq [\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(U, A)] \cap [\tilde{\gamma}(V, A) \cup \tilde{\gamma}'(V, A)]. \]

Hence, \([F(A) \cup (G, A)] \cap [\tilde{\gamma}(W, A) \cap \tilde{\gamma}'(W, A)] \subseteq ([F(A) \cup (G, A)] \cap [\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(U, A)] \cap [\tilde{\gamma}(V, A) \cup \tilde{\gamma}'(V, A)]\).

The disjoint of \([F(A) \cup (G, A)] \cap [\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(U, A)] \cap [\tilde{\gamma}(V, A) \cup \tilde{\gamma}'(V, A)]\) leads to

\[ ([F(A) \cup (G, A)] \cap [\tilde{\gamma}(W, A) \cup \tilde{\gamma}'(W, A)] = \emptyset. \]

This means that \(P_e^* \notin Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A))\). Therefore, \(Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A)) \subseteq Cl_{\tilde{\gamma}_{\tilde{\gamma}'}-}\!Cl((F, A) \cup (G, A)). \)

From Lemma 4.1 (6), we obtain the equality. \[ \square \]

**Lemma 4.3.** Let \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) be soft regular operators on \(\tilde{\tau}\), and let \((F, A) \in SS(X)_A\). Then
\[ \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A) \subseteq \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A) \]

holds for each \((U, A) \in \tilde{\tau}_{(\gamma, \gamma')}\).

**Proof.** Suppose that \(P_e^x \in \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A)\) for each \((U, A) \in \tilde{\tau}_{(\gamma, \gamma')}\), then \(P_e^x \in \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\) and \(P_e^x \in (U, A)\). Let \((V, A) \in \tilde{\tau}_{(\gamma, \gamma')}\) with \(P_e^x \in (V, A)\). Since \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) are soft regular operators on \(\tilde{\tau}\), by Proposition 3.3 (1), \((U, A) \cap (V, A) \in \tilde{\tau}_{(\gamma, \gamma')}\) with \(P_e^x \in (U, A) \cap (V, A)\). Since \(P_e^x \in \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\), by Theorem 4.1, we find that \((F, A) \cap (U, A) \cap (V, A) \notin \tilde{\phi}\). Therefore, \((F, A) \cap (U, A) \cap (V, A) \notin \tilde{\phi}\). Thus, by Theorem 4.1, we have that \(P_e^x \in \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A)\). Hence,
\[ \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A) \subseteq \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \cap (U, A) \]

\[ \square \]

**Theorem 4.3.** Let \((F, A) \in S S(X)_A\), then the following properties are equivalent:

1. \((F, A) \in \tilde{\tau}_{(\gamma, \gamma')}\)
2. \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((X \setminus (F, A))) = \tilde{X}(F, A)\)
3. \(\tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(X \setminus (F, A)) = \tilde{X}(F, A)\)
4. \(\tilde{X}(F, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed.

**Theorem 4.4.** Let \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) be soft open operators on \(\tilde{\tau}\), and let \((F, A) \in S S(X)_A\). If \((X, \tilde{\tau}, A)\) is a soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular space, then the following hold:

1. \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\)
2. \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')} \text{-Cl}(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))) = \text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')} ((F, A))\)
3. \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed in \(\tilde{X}\).

**Proof.** (1) First we need to show that \(\tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \subseteq \text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))\). By Lemma 4.1 (2), we have \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\). Now let \(P_e^x \notin \text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))\), then there exist \((U, A) \in \tilde{\tau}\) and \((V, A) \in \tilde{\tau}\) with \(P_e^x \notin (U, A)\) and \((\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A)) \cap (F, A) = \tilde{\phi}\). Since \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) are soft open operators on \(\tilde{\tau}\), there exist \((W_1, A) \in \tilde{\tau}\) and \((W_2, A) \in \tilde{\tau}\) with \(P_e^x \notin (W_1, A)\), \(P_e^x \notin (W_2, A)\) such that \((W_1, A) \subseteq \tilde{\gamma}(U, A)\) and \((W_2, A) \subseteq \tilde{\gamma}'(V, A)\). So \((F, A) \cap ((W_1, A) \cup (W_2, A)) = \tilde{\phi}\). Since \((X, \tilde{\tau}, A)\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-regular, by Lemma 3.2, \((W_1, A) \in \tilde{\tau}_{(\gamma, \gamma')}\) and \((W_2, A) \in \tilde{\tau}_{(\gamma, \gamma')}\). Hence by Lemma 3.1, \((W_1, A) \cup (W_2, A)) \subseteq \tilde{\tau}_{(\gamma, \gamma')}\) with \(P_e^x \notin ((W_1, A) \cup (W_2, A))\). Thus, by Theorem 4.1, \(P_e^x \notin \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\). Therefore, \(\tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A) \subseteq \text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))\). Hence \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \tilde{\tau}_{(\gamma, \gamma')} \text{-Cl}(F, A)\).

(2) The proof follows from part (1) and Lemma 4.1 (7).

(3) By part (2) and Lemma 4.1 (3), we get \(\text{Cl}_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))\) is soft \((\tilde{\gamma}, \tilde{\gamma}')\)-closed in \(\tilde{X}\).

\[ \square \]

In the end of this section, we introduce two classes of soft interior via bioperators \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) called \(\text{Int}_{(\gamma, \gamma')}\) and \(\tilde{\tau}_{(\gamma, \gamma')} - \text{Int}\) operators.

**Definition 4.3.** Let \((F, A) \in S S(X)_A\) and \(P_e^x \notin \tilde{X}\).

1. A soft point \(P_e^x \in (F, A)\) is said to be soft \((\tilde{\gamma}, \tilde{\gamma}')\)-interior point of \((F, A)\) if there exist soft open neighborhoods \((U, A)\) and \((V, A)\) of \(P_e^x\) such that \(\tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A)\). We denote the set of all soft \((\tilde{\gamma}, \tilde{\gamma}')\)-interior points of \((F, A)\) by \(\text{Int}_{(\gamma, \gamma')}((F, A))\). That is,
   \[\text{Int}_{(\gamma, \gamma')}((F, A)) = \{ P_e^x \in (F, A) : \tilde{\gamma}(U, A) \cup \tilde{\gamma}'(V, A) \subseteq (F, A) \text{ for some } (U, A), (V, A) \in \tilde{\tau} \text{ with } P_e^x \in (U, A) \text{ and } P_e^x \notin (V, A) \}\]

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2. $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$ denotes the soft union of all soft $(\tilde{\gamma},\tilde{\gamma}')$-open sets of $(X,\tilde{\tau},A)$ contained in $(F,A)$ and is defined as

$$\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = \bigcup\{(U,A) : (U,A) \subseteq (F,A) \text{ and } (U,A) \in \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')\}.$$

**Lemma 4.4.** Let $(F,A),(G,A) \in S S(X)_A$. Then the following statements are true:

1. $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$ is soft $(\tilde{\gamma},\tilde{\gamma}')$-open in $\tilde{\tau}$ and $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$ is soft open in $\tilde{\tau}$.

2. (a) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq (F,A)$.

(b) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq (F,A)$.

3. The following are equivalent:

(a) $(F,A)$ is soft $(\tilde{\gamma},\tilde{\gamma}')$-open.

(b) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = (F,A)$.

(c) $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = (F,A)$.

4. If $(F,A) \subseteq (G,A)$, then $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A)$ and $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \subseteq \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A)$.

5. (a) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A) \cap (G,A) \subseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cap (G,A)$.

(b) $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A) \cap (G,A) \subseteq \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cap (G,A)$.

6. (a) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cup (G,A) \supseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cup \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A)$.

(b) $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cup (G,A) \supseteq \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cup \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A)$.

7. (a) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$.

(b) $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$.

8. (a) $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$.

(b) $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) = \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A)$.

**Proof.** Straightforward. □

**Lemma 4.5.** Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be soft regular operators on $\tilde{\tau}$. For any $(F,A),(G,A) \in S S(X)_A$, we have

1. $\tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cap (G,A) \subseteq \tilde{\tau}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(G,A) \cap (G,A)$.

2. $\text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cap (G,A) = \text{Int}(\tilde{\gamma},\tilde{\gamma}')-\text{Int}(F,A) \cap (G,A)$.

**Definition 4.4.** Let $(X,\tilde{\tau},A)$ be a soft space and a soft point $P_e \in \tilde{\tau}$. Then a soft $(\tilde{\gamma},\tilde{\gamma}')$-neighbourhood (soft $(\tilde{\gamma},\tilde{\gamma}')$-nbd, in short) of a soft point $P_e$ is a soft set $(N,A)$ which contains a soft $(\tilde{\gamma},\tilde{\gamma}')$-open set $(U,A)$ in $\tilde{\tau}$ such that $P_e \subseteq (U,A)$. Evidently, a soft set $(N,A)$ is a soft $(\tilde{\gamma},\tilde{\gamma}')$-nbd of $P_e$ if $P_e \subseteq \text{Int}(X\setminus\tilde{\tau})(N,A)$.

The class of all soft $(\tilde{\gamma},\tilde{\gamma}')$-nbd of $P_e$ is called the soft $(\tilde{\gamma},\tilde{\gamma}')$-nbd system at $P_e$ and is denoted by $(N_{P_e},A)$.

**Theorem 4.5.** The soft $(\tilde{\gamma},\tilde{\gamma}')$-nbd system $(N_{P_e},A)$ at $P_e$ in a soft space $(X,\tilde{\tau},A)$ has the following properties:

1. If $(N,A) \in (N_{P_e},A)$, then $P_e \in (N,A)$.

2. If $(N,A),(M,A) \in (N_{P_e},A)$, then $(N,A) \cap (M,A) \in (N_{P_e},A)$, where $\tilde{\gamma}$ and $\tilde{\gamma}'$ are soft regular operators on $\tilde{\tau}$.

3. If $(N,A) \in (N_{P_e},A)$, then there is $(U,A) \in (N_{P_e},A)$ such that $(N,A) \in (N_{P_e},A)$ for each $P_e \subseteq (U,A)$ such that $y \neq x$.

4. If $(N,A) \in (N_{P_e},A)$ and $(N,A) \subseteq (M,A)$, then $(M,A) \in (N_{P_e},A)$.  


Suppose that there exists a soft point

\textbf{Proposition 5.1.} A soft set \( \sim \) set containing

\textbf{Theorem 4.6.} A soft set \((N, A)\) is a soft \((\tilde{y}, \tilde{y}')\)-open set in \(\tilde{X}\) if and only if \((N, A)\) is a soft \((\tilde{y}, \tilde{y}')\)-nbd of each of its soft points.

\textbf{Proof.} Necessity: If \((N, A)\) is soft \((\tilde{y}, \tilde{y}')\)-open, then \((N, A) = \text{Int}_{(\tilde{y}, \tilde{y}')}((N, A))\) (by Lemma 4.4 (3)). Therefore, \((N, A)\) is a soft \((\tilde{y}, \tilde{y}')\)-nbd of each of its soft points.

Sufficiency: Let \((N, A)\) be a soft \((\tilde{y}, \tilde{y}')\)-nbd of each of its soft points. Then, \((N, A)\) contains a soft \((\tilde{y}, \tilde{y}')\)-open set \((U, A)\) in \(\tilde{X}\) such that \(P e \in (U, A)\) for each \(P e \in (N, A)\). Therefore, \((N, A) = \bigcup_{P e \in (N, A)} (U, A)\) is a union of soft \((\tilde{y}, \tilde{y}')\)-open sets and hence by Lemma 3.1, \((N, A)\) is soft \((\tilde{y}, \tilde{y}')\)-open.

\textbf{5. Soft \((\tilde{y}, \tilde{y}')\)-g.closed sets and soft \((\tilde{y}, \tilde{y}')\)-T\(_1\) spaces}

In this section, we introduce soft \((\tilde{y}, \tilde{y}')\)-g.closed sets and soft \((\tilde{y}, \tilde{y}')\)-T\(_1\) spaces, and study some of their characterizations.

\textbf{Definition 5.1.} A soft set \((F, A)\) of a soft space \((X, \tilde{\tau}, A)\) is said to be soft \((\tilde{y}, \tilde{y}')\)-g.closed if \(\text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A)) \subseteq (U, A)\) whenever \((F, A) \subseteq (U, A)\) and \((U, A)\) is soft \((\tilde{y}, \tilde{y}')\)-open.

\textbf{Remark 5.1.} Every soft \((\tilde{y}, \tilde{y}')\)-closed set in \((X, \tilde{\tau}, A)\) is soft \((\tilde{y}, \tilde{y}')\)-g.closed, but its converse is not true as may be shown from the following example.

\textbf{Example 5.1.} Consider the soft topological space \((X, \tilde{\tau}, A)\) defined in Example 3.3. Take \((F, A) \in SS(X, A)\) such that \((F, A) = \{e_1, \{a_1, a_2\}, e_2, \{a_2, a_3\}\}\). Then, \(\text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A)) = \tilde{X}\), and \((F, A)\) is not soft \((\tilde{y}, \tilde{y}')\)-closed. However, \((F, A)\) is soft \((\tilde{y}, \tilde{y}')\)-g.closed in \((X, \tilde{\tau}, A)\), because \(\tilde{X}\) is the only soft \((\tilde{y}, \tilde{y}')\)-open set containing \((F, A)\).

\textbf{Proposition 5.1.} A soft set \((F, A)\) of a soft space \((X, \tilde{\tau}, A)\) is soft \((\tilde{y}, \tilde{y}')\)-g.closed if and only if \((F, A)\) is soft \(\tilde{y}\)-g.closed.

\textbf{Proof.} The proof is immediate consequence of Proposition 3.1 (3).

The following results characterize soft \((\tilde{y}, \tilde{y}')\)-g.closed sets.

\textbf{Lemma 5.1.} A soft set \((F, A)\) of a soft space \((X, \tilde{\tau}, A)\) is soft \((\tilde{y}, \tilde{y}')\)-g.closed if and only if \((F, A) \subseteq \text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\), for every \(P e \in \text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\).

\textbf{Proof.} Necessity: Suppose that there exists a soft point \(P e \in \text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\) such that \((F, A) \subseteq \text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\). Since \(\tilde{X}\text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\) is soft \((\tilde{y}, \tilde{y}')\)-closed, \(\tilde{X}\text{Cl}_{(\tilde{y}, \tilde{y}')}((F, A))\)
Proof. Necessity only if $\tilde{\gamma}, \tilde{\gamma}'$-open. Now, soft $(\tilde{\gamma}, \tilde{\gamma}')$-closedness of $(F, A)$ in $\tilde{X}$ implies that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq \tilde{X} \setminus \tilde{\gamma}' - Cl(P_e')$. Therefore, $P_e' \notin Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))$. This is a contradiction. Thus, $(F, A) \cap \tilde{\gamma}' - Cl(P_e') = \emptyset$.

Sufficiency: Let $(U, A)$ be a soft $(\tilde{\gamma}, \tilde{\gamma}')$-open set in $\tilde{X}$ such that $(F, A) \subseteq (U, A)$. To show that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (U, A)$, let $P_e' \notin Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))$. Then by hypothesis, $(F, A) \cap \tilde{\gamma}' - Cl(P_e') = \emptyset$. So, let $P_e' \in (F, A) \cap \tilde{\gamma}' - Cl(P_e')$ for some soft point $P_e' \in \tilde{X}$ such that $y \neq x$. Thus, $P_e' \in (F, A) \subseteq (U, A)$ and $P_e' \notin \tilde{\gamma}' - Cl(P_e')$. By Theorem 4.1, $P_e' \cap (U, A) = \emptyset$ and so, $P_e' \in (U, A)$. This implies that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (U, A)$. Thus, $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed.

$\blacksquare$

**Theorem 5.1.** Let $(F, A) \in SS(X)_A$. Then the following hold:

1. If $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed, then $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ does not contain any non-null soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed set in $(X, \tilde{\tau}, A)$.

2. If both $\tilde{\gamma}$ and $\tilde{\gamma}'$ are soft open operators on $\tilde{\tau}$, and the soft space $(X, \tilde{\tau}, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-regular; then the converse of (1) is true.

**Proof.**

(1) Suppose that $(E, A) \notin \phi$ is a soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed set in $\tilde{X}$ such that $(E, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$. Then $(E, A) \subseteq \tilde{X}(E, A)$ and so, $(F, A) \subseteq \tilde{X}(E, A)$. Since $\tilde{X}(E, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-open and $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed, $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq \tilde{X}(E, A)$. That is, $(E, A) \subseteq \tilde{X}(E, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))$. Therefore, $(E, A) \subseteq \tilde{X}(E, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \cap Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \cap Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \emptyset$. Thus, $(E, A) = \emptyset$. But this is a contradiction. Hence, $(E, A) \notin Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))$.

(2) Let $(U, A)$ be soft $(\tilde{\gamma}, \tilde{\gamma}')$-open such that $(F, A) \subseteq (U, A)$. So, by hypothesis and Theorem 4.4 (3), $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A))$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed in $\tilde{X}$. Thus, we have $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq \tilde{X}(U, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed in $\tilde{X}$. Since $\tilde{X}(U, A) \subseteq \tilde{X}(U, A) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \cap Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \cap Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \emptyset$. This implies that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (U, A)$. Thus, $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed in $(X, \tilde{\tau}, A)$.

$\blacksquare$

**Corollary 5.1.** Let $(F, A)$ be soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed of $(X, \tilde{\tau}, A)$. Then $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed if and only if $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed in $(X, \tilde{\tau}, A)$.

**Proof.**

Necessity: Let $(F, A)$ be $(\tilde{\gamma}, \tilde{\gamma}')$-closed in $(X, \tilde{\tau}, A)$. It follows from Lemma 4.1 (3) that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = (F, A)$ and hence $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \emptyset$ which is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed.

Sufficiency: Suppose $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed and $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed. It follows from Theorem 5.1 (1) that $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ does not contain any non-null soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed set in $(X, \tilde{\tau}, A)$. Since $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ is a soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed subset of itself, $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = \emptyset$ implies $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \cap \tilde{X}(F, A) = \emptyset$. Hence, $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) = (F, A)$. Therefore, by Lemma 4.1 (3), we obtain $(F, A)$ is a soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed set in $(X, \tilde{\tau}, A)$.

$\blacksquare$

**Proposition 5.2.** If $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed and soft $(\tilde{\gamma}, \tilde{\gamma}')$-open set of $\tilde{X}$, then $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed.

**Proof.** Since $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed and soft $(\tilde{\gamma}, \tilde{\gamma}')$-open set in $\tilde{X}$, $Cl_{(\tilde{\gamma}, \tilde{\gamma}')}((F, A)) \subseteq (F, A)$ and hence by Lemma 4.1 (3), $(F, A)$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed.

$\blacksquare$

**Proposition 5.3.** For each $P_e' \in \tilde{X}$, $P_e'$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed or $\tilde{X}P_e'$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed.

**Proof.** Suppose that $P_e'$ is not soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed. Then $\tilde{X}P_e'$ is not soft $(\tilde{\gamma}, \tilde{\gamma}')$-open. So, $\tilde{X}$ is the only soft $(\tilde{\gamma}, \tilde{\gamma}')$-open set containing $\tilde{X}P_e'$. Thus, $\tilde{X}P_e'$ is soft $(\tilde{\gamma}, \tilde{\gamma}')$-g.closed.

$\blacksquare$
Definition 5.2. Let \((F, A) \in SS(X)_A\). Then the \(\bar{\tau}_{(\bar{y}, \bar{y}')}\)-kernel of \((F, A)\), denoted by \(\bar{\tau}_{(\bar{y}, \bar{y}')}\)-\(\ker(F, A)\), is defined as follows:

\[
\bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A) = \bar{\cap} \{(U, A) : (F, A) \subseteq (U, A) \text{ and } (U, A) \in \bar{\tau}_{(\bar{y}, \bar{y}')}\}
\]

That is, \(\bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A)\) is the intersection of all soft \((\bar{y}, \bar{y}')\)-open sets of \((X, \bar{\tau}, A)\) containing \((F, A)\).

Theorem 5.2. Let \((F, A) \in SS(X)_A\). Then \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-g.closed if and only if \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq \bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A)\).

Proof. Necessity: Suppose that \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-g.closed. Then \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq (U, A)\), whenever \((U, A) \supseteq (F, A)\) and \((U, A)\) is soft \((\bar{y}, \bar{y}')\)-open. Let \(P^*_{e} \in Cl_{(\bar{y}, \bar{y}')}(F, A)\). Hence, by Lemma 5.1, \((F, A) \ni \bar{\tau}_{(\bar{y}, \bar{y}')\ker(F, A)}\). So, there exists a soft point \(P^*_e \in \bar{X}\) such that \(z \neq x\) and \(P^*_e \in (F, A) \ni \bar{\tau}_{(\bar{y}, \bar{y}')\ker(F, A)}\). It follows from Theorem 4.1 that \(P^*_e \ni (U, A) \neq \emptyset\). Hence we show that \(P^*_e \in \bar{\tau}_{(\bar{y}, \bar{y}')\ker(F, A)}\). Thus, \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq \bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A)\).

Sufficiency: Let \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq \bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A)\). Let \((U, A) \supseteq (F, A)\) where \((U, A)\) is soft \((\bar{y}, \bar{y}')\)-open in \(\bar{X}\). Let \(P^*_e\) be a soft point in \(X\) such that \(P^*_e \in Cl_{(\bar{y}, \bar{y}')}(F, A)\). Then \(P^*_e \in \bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A)\). So, we have \(P^*_e \in (U, A)\), because \((U, A) \supseteq (F, A)\) and \((U, A) \in \bar{\tau}_{(\bar{y}, \bar{y}')}\). That is, \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq \bar{\tau}_{(\bar{y}, \bar{y}')}\ker(F, A) \subseteq (U, A)\). Thus, \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-g.closed in \(\bar{X}\).

Definition 5.3. A soft set \((K, A)\) of a soft space \((X, \bar{\tau}, A)\) is said to be soft \((\bar{y}, \bar{y}')\)-g.open if its complement \(\bar{X}\setminus(K, A)\) is soft \((\bar{y}, \bar{y}')\)-g.closed in \((X, \bar{\tau}, A)\).

Proposition 5.4. A soft set \((K, A)\) of a soft space \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-g.open if and only if \((E, A) \subseteq Int_{(\bar{y}, \bar{y}')}(K, A)\) whenever \((E, A) \subseteq (K, A)\) and \((E, A)\) is soft \((\bar{y}, \bar{y}')\)-closed.

In the end of this section, we introduce the notion of soft \((\bar{y}, \bar{y}')\)-\(T^*_2\) space and investigate some of its properties.

Definition 5.4. A soft space \((X, \bar{\tau}, A)\) is said to be soft \((\bar{y}, \bar{y}')\)-\(T^*_2\) if every soft \((\bar{y}, \bar{y}')\)-g.closed set of \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-closed.

Theorem 5.3. A soft space \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-\(T^*_2\) if and only if for each \(P^*_e \in \bar{X}\), the soft set \(P^*_e\) is soft \((\bar{y}, \bar{y}')\)-closed or soft \((\bar{y}, \bar{y}')\)-open.

Proof. Necessity: Suppose that \(P^*_e\) is not soft \((\bar{y}, \bar{y}')\)-closed in \((X, \bar{\tau}, A)\). By Proposition 5.3, we have \(\bar{X}\setminus P^*_e\) is soft \((\bar{y}, \bar{y}')\)-g.closed. Since \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-\(T^*_2\), \(\bar{X}\setminus P^*_e\) is soft \((\bar{y}, \bar{y}')\)-closed and hence \(P^*_e\) is soft \((\bar{y}, \bar{y}')\)-open.

Sufficiency: Let \((F, A)\) be any soft \((\bar{y}, \bar{y}')\)-g.closed. Then, we claim that \(Cl_{(\bar{y}, \bar{y}')}(F, A) = (F, A)\) holds. It is sufficient to show that \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq (F, A)\). Let \(P^*_e \in Cl_{(\bar{y}, \bar{y}')}(F, A)\). By the assumption, \(P^*_e\) is soft \((\bar{y}, \bar{y}')\)-closed or soft \((\bar{y}, \bar{y}')\)-open. So there are two cases:

1st Case: If \(P^*_e\) is soft \((\bar{y}, \bar{y}')\)-closed and \(P^*_e \notin (F, A)\), then \(P^*_e \notin Cl_{(\bar{y}, \bar{y}')}(F, A)\) contains a non-null soft \((\bar{y}, \bar{y}')\)-closed set \(P^*_e\). Since \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-g.closed and according to Theorem 5.1 (1), we obtain a contradiction. Hence, \(P^*_e \in (F, A)\). Thus, \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq (F, A)\) and \(Cl_{(\bar{y}, \bar{y}')}(F, A) = (F, A)\). Hence by Lemma 4.1 (3), \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-closed in \((X, \bar{\tau}, A)\). Therefore, \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-\(T^*_2\).

2nd Case: If \(P^*_e\) is soft \((\bar{y}, \bar{y}')\)-open, then by Theorem 4.1, \((F, A) \ni P^*_e \neq \emptyset\) because \(P^*_e \in \bar{\tau}_{(\bar{y}, \bar{y}')\ker(F, A)}\). This implies that \(P^*_e \in (F, A)\). So \(Cl_{(\bar{y}, \bar{y}')}(F, A) \subseteq (F, A)\). Thus by Lemma 4.1 (3), \((F, A)\) is soft \((\bar{y}, \bar{y}')\)-closed. Thus, \((X, \bar{\tau}, A)\) is soft \((\bar{y}, \bar{y}')\)-\(T^*_2\).
The following corollary follows directly from Theorem 5.3, Proposition 5.1 and Proposition 3.1 (3).

**Corollary 5.2.** For any soft space \((X, \bar{\tau}, A)\), the following are equivalent:

1. \((X, \bar{\tau}, A)\) is soft \((\bar{\gamma}, \bar{\gamma}^\prime)-T_{\frac{1}{2}}^1\).
2. \((X, \bar{\tau}, A)\) is soft \(\bar{\gamma}-T_{\frac{1}{2}}^1\).
3. For each \(P^\varepsilon \in \bar{X}\), the soft set \(P^\varepsilon\) is soft \(\bar{\gamma}\)-closed or soft \(\bar{\gamma}\)-open.

**Proposition 5.5.** If \((X, \bar{\tau}, A)\) is soft \((\bar{\gamma}, \bar{\gamma}^\prime)-T_{\frac{1}{2}}^1\), then it is soft \(\bar{\gamma}-T_{\frac{1}{2}}^1\) and soft \(\bar{\gamma}^\prime-T_{\frac{1}{2}}^1\).

**Proof.** It follows from Theorem 5.3, Corollary 5.2 and Proposition 3.1 (3). \(\square\)

The following example shows that the converse of Proposition 5.5 is not true in general.

**Example 5.2.** Let \(X = \{a_1, a_2\}\), \(A = \{e_1, e_2\}\) and \(\bar{\tau} = \{\bar{\phi}, \bar{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A), (F_7, A), (F_8, A)\}\) where

\[
\begin{align*}
(F_1, A) &= \{(e_1, \{a_1\}, \{e_2, \phi\}\}, \\
(F_2, A) &= \{(e_1, \{a_2\}, \{e_2, \phi\}\}, \\
(F_3, A) &= \{(e_1, \phi), (e_2, \{a_2\}\}, \\
(F_4, A) &= \{(e_1, \{a_1\}), (e_2, \{a_2\}\}, \\
(F_5, A) &= \{(e_1, \{a_2\}), (e_2, \{a_2\}\}, \\
(F_6, A) &= \{(e_1, \{a_2\}), (e_2, X)\}, \\
(F_7, A) &= \{(e_1, X), (e_2, \{a_2\}\}, \\
(F_8, A) &= \{(e_1, X), (e_2, \phi)\}.
\end{align*}
\]

Then \((X, \bar{\tau}, A)\) is a soft topological space over \(X\). Let \(\bar{\gamma}: \bar{\tau} \rightarrow SS(X)_A\) and \(\bar{\gamma}^\prime: \bar{\tau} \rightarrow SS(X)_A\) be operators defined as follows: For all \((F, A) \in \bar{\tau},

\[
\bar{\gamma}(F, A) = \begin{cases} (F, A) & \text{if } P^\varepsilon = (F, A) \\
\text{Int}(Cl(F, A)) & \text{if } P^\varepsilon \notin (F, A) \\
\bar{X} & \text{if } (F, A) = (F_6, A) \end{cases}
\]

and

\[
\bar{\gamma}^\prime(F, A) = \begin{cases} (F, A) & \text{if } (F, A) = (F_2, A) \text{ or } (F, A) = (F_3, A) \\
\bar{X} & \text{otherwise.} \end{cases}
\]

It is clear that \(\bar{\tau}_\gamma = \bar{\tau}\backslash\{(F_5, A), (F_6, A)\}\) and

\[
\bar{\tau}_{\gamma^\prime} = \{\bar{\phi}, \bar{X}, (F_2, A), (F_3, A), (F_6, A), (F_7, A)\}.
\]

So, \((X, \bar{\tau}_\gamma, \bar{\tau}_\gamma^\prime)\). Therefore, \((X, \bar{\tau}, A)\) is both soft \(\bar{\gamma}-T_{\frac{1}{2}}^1\) and soft \(\bar{\gamma}^\prime-T_{\frac{1}{2}}^1\).

However, \((X, \bar{\tau}, A)\) is not soft \((\bar{\gamma}, \bar{\gamma}^\prime)-T_{\frac{1}{2}}^1\) because the soft set \(P^\varepsilon = (F_1, A)\) is not soft \((\bar{\gamma}, \bar{\gamma}^\prime)-closed\), nor soft \((\bar{\gamma}, \bar{\gamma}^\prime)\)-open.

### 6. Conclusions

Researchers and scientists proposed different approaches to handle problems of uncertainty. Among them, soft set theory has received the attention of the topologists who always seek to generalize and apply the topological notions on different structures.
As a contribution to this area, we have presented and studied the concepts of bioperators $\tilde{\gamma}$ and $\tilde{\gamma}'$ on soft topology $\tilde{\tau}$, and the notion of soft $(\tilde{\gamma}, \tilde{\gamma}')$-open sets. Then, we have defined two soft closure and two soft interior operators, and elucidated the relationships between them. Finally, we have initiated the concepts of soft $(\tilde{\gamma}, \tilde{\gamma}')$-closed sets and soft $(\tilde{\gamma}, \tilde{\gamma}')$-$T^*_2$ spaces and investigated main properties.

It was investigated in [10] the interchangeable property of soft interior and closure operators between soft sets and and their components. In the upcoming work, we will study, by making use of this property, the transmission of the concepts given herein from soft topology to its parametric topology and vise versa. Also, we will investigate this work in the contents of supra soft topology and fuzzy soft topology.

Conflict of interest

The authors declare that they have no competing interest.

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