Investigations on the thermal behavior and associated thermal stresses for short pulse laser heating

D W Xue¹, J B Wang¹ and G Y Xu¹²

¹School of Port and Transportation Engineering, Zhejiang Ocean University, Zhoushan 316022, Zhejiang, China

E-mail: xugy@zjou.edu.cn

Abstract. The fractional type Cattaneo heat conduction equation and the associated thermal stress equation are established for studying the thermal and related stress behavior of the short pulse laser heating. With the Laplace transform method, the analytical solution of the temperature distribution and thermal stress field are derived. Compared to the classical Fourier heat transfer model and the standard C-V one, the results of the present model show the thermal diffusion feature as well as the thermal wave behaviour, which is more realistic. Furthermore, the influences of the fractional order parameter on the thermal and related stress behavior are discussed. The thermal velocity, thermal variation rate and the peak value of the temperature and the thermal stress depend on the fractional order parameter. The fractional type Cattaneo heat model and associated thermal stress exhibit the non-local nature.

1. Introduction

Laser processing technology is in widespread application in manufacturing and modern medicine. In spite of many advantages, the non-uniform temperature field inside the material during laser processing will introduce extremely large temperature gradient, which leads to the generation of a great thermal stress. The clear understanding of the heat transfer process is of great importance to reveal the physical mechanism of the interaction between the pulse laser irradiation and the material.

The classical heat model, which is widely accepted, is based on the Fourier’s law. However, evidences [1,2] show that the Fourier heat conduction model fails to characterize the cases such as the short pulse laser heating because it implies an unphysically infinite thermal propagation velocity. In fact, the short pulse laser refers to the laser with very short pulse duration time, generally on the order of nanoseconds or even femtoseconds and meanwhile with high-intensity energy power of heat flux. In such cases, the short laser beam will generate a great heat gradient and high heating rate. Therefore, as mentioned in the previous researches [3,4], the Fourier heat conduction model is a standard diffusion model of equilibrium thermodynamics and is not valid to describe the related problems.

Since the thermal wave effects have been observed in experiments by Peshkov [5], it has been widely accepted that the thermal signal transfers at a finite speed like waves in the case of extreme low temperature and short pulse laser heating, etc. Almost at the same time, modifications of Fourier model have been made to describe such transient heat transfer process. Among the various modifications, the most widely accepted one is the Cattaneo-Vernotte (C-V) model [6,7] which considers the lagging effects. Lots of investigations [8-10] have proved the wave nature of the heat behavior and shown the advantage of C-V model, because the lagging phenomenon and the faster thermal rising rate, which are different from the nature of Fourier model, are always existing [11].
However, there are also some investigations show the experimental temperature distribution deviates from that predicted by C-V model and indicate the limitation of C-V model [12-15]. Besides, other researchers have shown that the C-V model may introduce other non-physical reality, such as the local temperature lower than absolute zero [16], the violation to the second law of thermodynamics [17].

As a matter of fact, the above physical process described by the C-V equation is a typical anomalous diffusion process. From the macroscopic aspect, the motion of the large number of particles in anomalous diffusion process is not in accordance with the standard statistical distribution. And the mean square displacement of the particles is not in accordance with the standard linear behavior, and as is known, the mean square displacement of the particles takes the form \( \langle x^2(t) \rangle \propto t^p \), where \( p \) refers to the anomalous diffusion exponent. For \( p = 1 \), it is the normal diffusion, and \( p \neq 1 \) corresponds to the anomalous diffusion.

Recently, as pointed out in [18-20], there is an inherent relation between the fractional differential operator and the anomalous diffusion. As is known, the integer order differential operator is a local operator while the fractional order one is non-local [21]. This means that the next state of a system depends on all of its historical states rather than its current state, which is obviously more realistic and it is one of the most important reasons why the fractional calculus has the advantage to describe the phenomena mentioned above [21,22]. Lots of different physical phenomena have proved that the fractional order calculus is efficient in modeling the anomalous process, and it has shown [23-26] that the fractional differential equation overcomes the defect induced by the traditional Fourier equation and the C-V equation when described the anomalous diffusion process. So far, many investigations have been focused on the study of non-equilibrium heating process, but most of the investigations are concerning on the thermal response and the temperature field, there are only a few investigations concerning on the associated thermal stresses, including the outstanding work of Povstenko [27-30], Youssef [31], and Sherief [32]. For the short pulse laser heating process, there are little investigation on both the thermal propagation and temperature field as well as the associated thermal stresses with the fractional type Cattaneo heat equation.

Consequently, in the present study, the heat transfer process and the associated thermal stresses due to the interaction between the pulse laser irradiation are investigated. The fractional type Cattaneo heat equation and the thermal stress equation are derived first. With the Laplace transform method, the analytical solutions of above equations are then obtained. At last, an example is given to study the effects of the fractional parameters on the transient temperature and thermal stress field and the heat conduction mechanism during the anomalous diffusion process is discussed further.

2. Mathematical model

2.1. Fractional type Cattaneo heat transfer equation

Considering the metal material irradiating by a short pulse laser, the extensively used model is the well-known standard C-V model with the constitutive relation expressed as

\[
q(r,t) + \tau \frac{\partial q}{\partial t} = -k\nabla T
\]

(1)

The standard C-V model is in the form of integer order differential operator, and using the fractional Taylor series expansion, the fractional form of equation (1) is obtained

\[
q(r,t) + \frac{\tau^p}{\Gamma(p+1)} \frac{\partial^p q}{\partial t^p} = -k\nabla T, \quad 0 < p < 1
\]

(2)

Where \( p \) refers to the fractional order of differentiation, \( \partial / \partial t^p \) stands for the Caputo’s operator of fractional derivative.

The energy conservation equation is
The fractional type Cattaneo heat transfer equation with equations (2) and (3) as:

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot q + Q$$

One can get the fractional type Cattaneo heat transfer equation with equations (2) and (3) as:

$$\frac{\partial T}{\partial t} + \frac{\tau^p}{\Gamma(p+1)} \frac{\partial^{p+1} T}{\partial t^{p+1}} = a \nabla^2 T + \frac{1}{\rho c_p} \left( 1 + \frac{\tau^p}{\Gamma(p+1)} \frac{\partial^p}{\partial t^p} \right) Q, \quad 0 \leq p \leq 1$$

Where $\rho$ is the material density and $T$ is the temperature, $c_p$ is the specific heat capacity, $a = k/(\rho c_p)$ stands for the thermal conductivity, $Q$ is the heat resource, $\nabla^2$ is the Laplace operator.

It is noted that as $\tau = 0$, equation (4) is changing in the form of the classical Fourier heat transfer equation and in the form of the standard C-V heat transfer equation with integer order as $p = 1$.

For the heating source of the short pulse laser, the non-Gaussian type laser with the following temporal profile is given:

$$I(t) = I_0 f(t)$$

where $I_0$ is the amplitude of laser peak power density, and $f(t)$ is expressed as:

$$f(t) = \exp(-\beta t) - \exp(-\mu t)$$

And it denotes the pulse laser intensity in which $\beta$ and $\mu$ denote the laser pulse rise-time and the laser pulse fall-time parameter. The temporal variations of $f(t)$ corresponding to different parameters of $\beta$ and $\mu$, and in the present paper, the parameters are chosen as $\beta = 0.5$, $\mu = 2.5$, shown in Figure 1, which is referred to [33,34].

![Figure 1. Dimensionless temporal distribution of the laser pulse intensity $f(t)$ with $\beta=0.5$, $\mu=2.5$.](image)

Considering the surface reflection and absorption, the internal heat source is

$$Q = (1 - r_f) I(t) \delta \exp(-\alpha t)$$

Where $r_f$ and $\delta$ stand for the reflection coefficient, and the absorption coefficient.

Substituting the heat source (equation (7)) into the heat transfer equation (equation (4)), the one-dimensional fractional type Cattaneo heat equation is obtained.
\[
\frac{1}{a} \left( \frac{\partial T}{\partial t} + \frac{\tau_p}{1 + p} \frac{\partial^{p+1} T}{\partial t^{p+1}} \right) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial_1 e^{-\delta t}}{k} \left[ f(t) + \frac{\tau_p}{1 + p} \frac{d^p f}{dt^p} \right]
\]

(8)

Where \( I_1 = (1 - r_f) I_0 \)

The initial condition is

\[
T(x, t = 0) = T_0, \frac{\partial}{\partial t} T(x, t) = 0, x > 0, t = 0
\]

(9)

Correspondingly, the boundary conditions are,

\[
\frac{\partial}{\partial x} T(x, t) = 0, x = 0, t > 0
\]

(10)

\[
T(x, t) = T_0, x \to \infty, t = 0
\]

(11)

2.2. Thermal stresses equation

The components of displacement for the half space with the traction free surface is as follows:

\[
\begin{align*}
u_x &= u(x, t), u_y = u_z = 0
\end{align*}
\]

(12)

Taking count into the stress-strain relation,

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \varepsilon_{yy} &= \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{zx} = \varepsilon_{zy} = 0
\end{align*}
\]

(13)

One can get

\[
\begin{align*}
\sigma_{xx} &= (\lambda + 2\kappa) \frac{\partial u}{\partial x} - (3\lambda + 2\kappa) \alpha_x (T - T_0) \\
\sigma_{yy} &= \lambda \frac{\partial u}{\partial x} - (3\lambda + 2\kappa) \alpha_x (T - T_0)
\end{align*}
\]

(14)

(15)

Where \( \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \) is the Lame coefficient, \( E \) the elastic modulus, \( \kappa = \frac{E}{2(1 + \nu)} \) is the shear elastic modulus, \( \nu \) is the Poisson ratio. The differential equation of one-dimensional thermal elastic motion is expressed as

\[
\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

(16)

Combining equations (14) and (16) with eliminating displacement \( u \) yields the differential equation of thermal stress

\[
\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{1}{V_e^2} \frac{\partial^2 \sigma_{xx}}{\partial t^2} = \frac{1 + \nu}{1 - \nu} \rho \alpha_x \frac{\partial^2 T}{\partial t^2}
\]

(17)

where \( V_e = \sqrt{(\lambda + 2\mu)/\rho} \) is the dilatation wave velocity.

The initial and boundary conditions are as follows

\[
\sigma_{xx} = 0, \frac{\partial \sigma_{xx}}{\partial t} = 0, t = 0, x > 0
\]

(18)
\[ \sigma_{xx} = 0, \ x = 0, \ t > 0 \] (19)
\[ \sigma_{xx} = 0, \ x \to \infty, \ t > 0 \] (20)

3. Analytical solutions

For simplicity, the following non-dimensional quantities are introduced

\[ 10^{*} I T T k T - \delta \] (10)
\[ t a t 2* \delta = \] (22), (23)
\[ I a k T xx \delta \rho \alpha \sigma \mu + - = \] (24)

3.1. Solution of the temperature field

Equations (8)-(11) will change into the following forms after dimensionalization (dropping the superscript for convenience).

\[ \frac{\partial T}{\partial t} + \frac{\tau}{\Gamma(p+1)} \frac{\partial^{p+1} T}{\partial t^{p+1}} = \frac{\partial^2 T}{\partial x^2} + e^{-x}(1 + \frac{\tau}{\Gamma(p+1)} \frac{\partial^p}{\partial t^p})f(t) \] (21)

\[ T = 0, \ \frac{\partial T}{\partial t} = 0, \ x > 0, \ t = 0 \] (22)
\[ \frac{\partial T}{\partial x} = 0, \ x = 0, \ t > 0 \] (23)
\[ T = 0, \ x \to \infty, t > 0 \] (24)

Equations (21)-(24) will change into the following form after the application of Laplace transform

\[ \frac{d^2 \bar{T}}{dx^2} - \left(1 + \tau^p s^p / \Gamma(1 + p)\right)\bar{T} = -\exp(-x)(1 + \tau^p s^p / \Gamma(1 + p))F(s) \] (25)

\[ \frac{d\bar{T}}{dx} = 0 \ \text{at} \ x = 0 \] (26)
\[ \bar{T} = 0 \ \text{at} \ x \to \infty \] (27)

Where \[ \bar{T} = \int_0^\infty \exp(-st)f(x,t)dt \] , \[ F(s) = \frac{1}{s + \beta} - \frac{1}{s + \mu} \]

Laplace transformed solution of equation (27) combining equations (28) and (29) can be expressed

\[ \bar{T}(x,s) = \bar{\phi}(x,s)F(s) \] (28)

Where
\[ \bar{\phi}(x,s) = \phi(s)\left[\exp(-x) - \phi_s(x,s)\right] \] (29)
\[ \bar{\phi}(s) = \frac{(1 + \tau^p s^p / \Gamma(1 + p))}{s(1 + \tau^p s^p / \Gamma(1 + p)) - 1} \] (30)
The analytical solution of the equation (28) is

\[ T(x,t) = \int_0^t \phi(x,\xi)f(t-\xi)d\xi \]  

(32)

In which,

\[ \phi(x,t) = \phi_1(t) \exp(-x) - \int_0^t \phi_1(\xi)\phi_2(x,t-\xi)d\xi \]  

(33)

\[ \phi_1(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{(-1)^j}{\tau^{n+j+1}} \Gamma(n+1)(1+p)\phi^{(n)}_{p,n+1}\phi^{(p)}_{n+1+p} \]  

(34)

According to inverse Laplace transform formula in [24]

\[ \phi_2(x,t) = L^{-1}\left[ \frac{\exp\left(-x\sqrt{s(1+\tau^p s^p/\Gamma(1+p))}\right)}{\sqrt{s(1+\tau^p s^p/\Gamma(1+p))}} \right] = \]  

(35)

\[ \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \Gamma(n+1/2)(1+p)\left(-1\right)^n \frac{t^{n+1/2}}{n!} \frac{\phi^{(n+1/2)}}{\tau^{n+1/2}} H_{0.2}^{2,0} \left[ \frac{x^p t^p}{4 t^{1+p} \Gamma(1+p)} \right] \]  

(36)

where \( E^{(n)}_{a,\beta}(z) \) is two parameters Mittag-Leffler function [21]. The fractional derivative of \( E^{(n)}_{a,\beta}(z) \) is defined as

\[ \frac{d^n}{dz^n} E^{(n)}_{a,\beta}(z) = \sum_{j=0}^{n} \frac{(j+n)!}{j! \Gamma(aj + \alpha n + \beta)} \]  

(37)

It is noted that one can get the analytical solution of the standard C-V heat equation as \( p = 1 \), and similarly, the analytical solution of the classical Fourier heat equation as \( \tau = 0 \).

3.2. Solution of the thermal stress field

By introducing the above non-dimensional quantities into equations (17)-(20) and yields

\[ \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \Lambda^2 \frac{\partial^2 \sigma_{xx}}{\partial t^2} = \frac{\partial^2 T}{\partial t^2} \]  

(38)

\[ \sigma_{xx} = 0, \quad \frac{\partial \sigma_{xx}}{\partial t} = 0, \quad x > 0, \quad t = 0 \]  

(39)

where the dimensionless parameter \( \Lambda = a \delta/V_e \).

Applying the Laplace transform to equations (37) and (39) yields...
\[
\frac{d^2 \bar{\sigma}_{xx}}{dx^2} - \Lambda^2 s^2 \bar{\sigma}_{xx} = s^2 \bar{T}
\]  
(40)

where \( \bar{\sigma}_{xx} = \int_0^\infty \exp(-st)\sigma_{xx}(x,t)dt \).

Solving equations (40) and (41) leads to the Laplace transformed solution

\[
\bar{\sigma}_{xx} = \bar{\Sigma}_{xx} F(s)
\]  
(42)

Where

\[
\bar{\Sigma}_{xx} = \bar{\phi}_1(s) \left[ A_1 \exp(-\Lambda x) + A_2 \exp(-x) + A_3 \bar{\phi}_2(x,s) \right]
\]  
(43)

\[
A_1(s) = \frac{s^2}{\Lambda^2 s^2 - 1} + \frac{s}{1 + \tau^p s^p / \Gamma(1 + p)} - \Lambda^2 s \sqrt{s(1 + \tau^p s^p / \Gamma(1 + p))}
\]  
(44)

\[
A_2(s) = -\frac{s^2}{\Lambda^2 s^2 - 1}, \quad \bar{A}_3(s) = -\frac{s}{1 + \tau^p s^p / \Gamma(1 + p)} - \Lambda^2 s
\]  
(45)

The analytical solution of dimensionless thermal stress is obtained

\[
\sigma_{xx} = \int_0^\infty \Sigma_{xx}(x, \xi)f(t - \xi)d\xi
\]  
(46)

In which

\[
\Sigma_{xx}(x, t) = \Sigma_{xx1}(x, t) + \Sigma_{xx2}(x, t) + \Sigma_{xx3}(x, t)
\]  
(47)

\[
\Sigma_{xx1} = \frac{1}{A^2} \left[ \phi_1(t - Ax) + \frac{1}{A} \int_0^{t-Ax} \phi_1(\xi) \sinh \left( \frac{t - \xi}{A} \right) d\xi - \int_0^t \phi_1(t - \xi) \phi_1(\xi) d\xi \right]
\]  
(48)

\[
\Sigma_{xx2} = -\frac{e^{-x}}{A^2} \left[ \phi_1(t) + \frac{1}{A} \int_0^t \phi_1(\xi) \sinh \left( \frac{t - \xi}{A} \right) d\xi \right]
\]  
(49)

\[
\Sigma_{xx3} = -\frac{1}{A} \left[ \int_0^t \phi_1(t - \tau) \phi_2(x, \tau) d\tau + \frac{1}{A} \int_0^t \phi_1(\tau) \phi_2(x, \tau) d\tau \right]
\]  
(50)

\[
\phi_1(t) = \int_0^t \phi_1(t - \tau) \phi_1(\tau) d\tau
\]  
(51)

\[
\phi_2(t) = \frac{A^{-2m} \tau^{pm}}{m!} \left[ \tau^{-m} \phi_1^{(m)}(t) \phi_2^{(m)}(A^{-2}t) + \phi_1^{(m)}(A^{-2}t) \right]
\]  
(52)
4. Results and discussions

As mentioned above, equations (32) and (46) are the analytical solutions of the temperature and associated stress. To investigate the effect of the fractional order $\mathcal{P}$ on the thermal and associated stress behavior, the specific material copper (Cu) is chosen. The material parameters are listed in Table 1 and the corresponding constants are obtained as $\tau = 4.25\,\text{ms}, \Lambda = 1.66, V_{\text{e}} = 4163\,\text{m/s}$.

| $\delta$ (m/°C) | $A$ (m/s) | $C_{\text{e}}$ (J/kg.°C) | $\rho$ (kg/m$^3$) | $k$ (W/m.K) | $\tau$ (s) | $\lambda$ (N/m$^2$.K) | $\mu$ (N/m$^2$) |
|-----------------|-----------|-----------------|-----------------|-------------|-----------|-----------------|-------------|
| 6.16            | 1.12      | 385             | 8930            | 385         | 1         | 7.76            | 3.86        |

4.1. The dimensionless temperature and thermal stress variation

Figure 2 presents the dimensionless temperature and thermal stress variations with time. It can be found from figure 2(a) that the temperature rising and falling rate are different for different positions. The temperature response at $x=0$ and 2 is obviously faster than that of $x=4$ and 6, and the farther away from the surface, the slower the temperature response, which indicates that the thermal wave propagates at a finite speed. Moreover, the peak value of the temperature decreases from $x=0$ to $x=6$ because of the decaying of the intensity of the pulse laser. Figure 2(b) gives the corresponding thermal stress fields. From figure 2(b), with the material surface($x=0$) going through the stage of heating($0<t<5$) and cooling($t>5$), the inner part of the substrate material undergoes compression and tension. With the thermal wave signal transfers, the associated stress at different positions appears the similar distribution. That is to say, the thermal stress is first negative because of the compression and positive because of the tension, and the absolute peak thermal stress value for different positions occurs at the stage of heating. And the peak value takes place at different time for different positions as the thermal stress wave transfers.

Figure 2. The dimensionless temperature and thermal stress variations at different positions.

Figure 3 shows the dimensionless temperature and associated stress distribution at different time. In figure 3(a), it is obvious that the heating area enlarges with heating time from $t=2$ to $t=4$. As

\[
\phi_{\delta}(t) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} \int_{0}^{t} \phi_{\delta}(t-\tau)\phi_{\delta}(\tau) d\tau
\]

\[
\phi_{\delta}(t) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} \int_{0}^{t} \phi_{\delta}(t-\tau)\phi_{\delta}(\tau) d\tau
\]
mentioned above, the material surface ($x=0$) is heated by the laser source until $t=5$, the dimensionless temperature at $t=6$ is lower than that of $t=4$. It is noted that the heating and cooling process both exists at $t=6$. The non-synchronous heating and cooling is consistent with finite thermal wave velocity of the physical mechanism for non-equilibrium heat conduction. Figure 3(b) gives the corresponding thermal stress distribution at different time. It can be found that the all parts of the material is undergoing compressive deformation at $t=2$, while part of the material is in tension and part under compression at $t=4$ and $t=6$. As mentioned above, the heating stage leads to the compression and cooling generates the tension. The peak value of the stress is results from the large temperature gradient at different locations.

![Temperature and Stress Distribution](image)

**Figure 3.** The dimensionless temperature and associated stress distribution at different times.

## 4.2. The influences of the fractional order $p$ on the temperature and associated stress field

![Temperature and Stress Variations](image)

**Figure 4.** The influences of the fractional order $p$ on the temperature and associated stress variations.

Figure 4 shows the influences of the fractional order $p$ on the dimensionless temperature and thermal stress variations at $x=3$. Figure 4(a) depicts the temperature variation. It is clear that the fractional order $p$ influences the rising and falling rates of temperature directly. The larger the fractional order $p$, the faster the rising rate of the temperature. Moreover, the corresponding temperature ($0 < p < 1$) always lies between that of $p=1$ (the C-V model) and $r=0$ (the Fourier model). From figure 4(b), the thermal stress variations are similar for the five different kinds of situations. But the peak value of the
thermal stress gets larger with the fractional order gets bigger. The same as the temperature, the thermal stress predicted by fractional equation always lies between that of \( p=1 \) and \( \tau=0 \).

![Figure 5](image.png)

**Figure 5.** The effects of the fractional order \( p \) on the temperature and thermal stress distributions.

Figure 5 shows the effects of the fractional order \( p \) on the dimensionless temperature and thermal stress distributions at \( t=3 \). Figure 5(a) represents the temperature distribution and figure 5(b) shows the corresponding thermal stress distribution. From figure 5(a), the temperature distribution from the fractional Cattaneo heat transfer equation stands between that of the classical Fourier and the standard C-V one. With \( p \) increasing, the surface temperature gets higher and the affected area is closer to the surface. Consequently, the larger the fractional order \( p \) is, the smaller the thermal wave velocity is. From figure 5(b), the thermal stress distribution is similar for the five different kinds of situations. But the peak value of thermal stress becomes smaller as the fractional order gets smaller.

5. **Conclusion**

The authors aim to introduce a mathematical model of fractional type Cattaneo heat conduction to reveal the heat transfer mechanism and the related thermal stress induced by short pulse laser heating, in which case, the thermal behavior exhibits both thermal diffusion and thermal wave. The heat transfer predicted by the present model shows the thermal behavior between thermal diffusion and the thermal wave feature. The non-local nature of the fractional type heat transfer model is more realistic.

The transient heat behavior is dependent on the fractional order parameter. Thermal wave mechanism is dominant as the fractional order parameter is larger, while thermal diffusion mechanism is dominant for the smaller one. With the fractional order parameter increasing, the thermal wave velocity decreases while the peak value of the temperature increases.

With the surface of the material undergoing the heating and cooling stage irradiated by the short pulse laser, a compressive stress develops first and is followed by a tensile stress. The associated thermal stress induced by transient heat behavior exhibits different features from the classical Fourier and standard C-V model. The peak value of the associated stress increases with the fractional order increasing.

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