ADE Singularities and Coset Models

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We consider the compactification of the IIA string to (1 + 1) dimensions on non-compact 4-folds that are ALE fibrations. Supersymmetry requires that the compactification include 4-form fluxes, and a particular class of these models has been argued by Gukov, Vafa and Witten to give rise to a set of perturbed superconformal coset models that also have a Landau-Ginzburg description. We examine all these ADE models in detail, including the exceptional cosets. We identify which perturbations are induced by the deformation of the singularity, and compute the Landau-Ginzburg potentials exactly. We also show how the the Landau-Ginzburg fields and their superpotentials arise from the geometric data of the singularity, and we find that this is most naturally described in terms of non-compact, holomorphic 4-cycles.

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1. Introduction

The link between the classification of singularity types and quantum effective actions of field theories now has a fairly long and interesting history. The key to making this work is for the field theory to have enough supersymmetry so that a non-renormalization “theorem” protects the sector of the field theory that is determined by the classical singularity type. For theories with a mass gap there are generically BPS states whose spectrum can be computed semi-classically, and which can be used to identify the field theory. This was first used with considerable success in $\mathcal{N} = 2$ superconformal field theories, in which there is a Landau-Ginzburg superpotential \[1\]. Most particularly the ADE classification of complex singularities with no moduli was shown to correspond to precisely the modular invariant $\mathcal{N} = 2$ superconformal field theories with central charge $c < 3$. If one topologically twists these $\mathcal{N} = 2$ theories the result is topological matter \[2\], and the coupling to topological gravity could be naturally incorporated into the singularity theory \[3,4\].

The advent of string dualities gave rise to many new constructions of string and field theories, and in particular, low energy effective actions. An early offshoot of this general program was to realize that the complete Seiberg-Witten quantum effective action of $\mathcal{N} = 2$ super-Yang-Mills theories in four dimensions \[5,6\] could be obtained from the period integrals of $K3$-fibrations that were developing an $ADE$ singularity \[7\]. In this construction, the singularity type corresponds to the gauge group of the Yang-Mills theory.

This subject has now, in a sense, come full circle. In \[8\] it was shown how Landau-Ginzburg models in $(1 + 1)$-dimensions could be constructed using compactifications of type IIA strings on Calabi-Yau 4-folds. If the 4-fold in this construction is a $K3$-fibration, and if the fiber develops an $ADE$ singularity, then the $(1 + 1)$-dimensional field theory has been argued to be a massive perturbation of a coset model based upon the corresponding $ADE$ group. The mass scale is set by the deformation of the singularity, and a conformal field theory emerges when the fiber is singular.

Extracting field theories from singular limits of compactifications is often a subtle process, and there are sometimes multiple limits being taken at the same time. For example, to extract the Seiberg-Witten effective actions one needs to decouple gravity and yet maintain a cut-off whose dimensions are inherited from the string tension. This is done by scaling the size of the base of the fibration while scaling the $K3$ so as to isolate the singular fiber. This results in a non-trivial role for the fibration even in the field theory limit. A conformal field theory can occur in this context if the moduli of the singularity...
are fine-tuned to an Argyres-Douglas point, but the generic theory has a scale. Things are considerably simpler for the Landau-Ginzburg models that arise from singular $K3$ fibers in 4-folds: there is no cut-off, and the field theory is conformal when the $K3$ fiber is singular. All the essential physics comes from the isolated $ADE$ singularity in the fiber, and the base of the fibration only plays a trivial role.

While the singularity type determines the numerator of the coset conformal field theory, the denominator (and probably the level) of the model are determined by a background 4-form flux. Such fluxes are parameterized by weights of the Lie algebra, $G$, associated with the $ADE$ singularity. The Weyl group of the singularity acts on the flux, permuting it around a Weyl orbit. The denominator of the coset model is defined by the subgroup of the Weyl group that leaves a flux invariant, and the non-trivial Weyl images of each flux represent different ground states of the same model. Deformations of the singularity introduce masses, but in general the deformed theory does not have a mass gap: there are still massless excitations. However, for a certain class of minimal fluxes\textsuperscript{1}, the theory does have a mass gap, and the associated coset model was argued in \cite{8} to be (a perturbation of) the $\mathcal{N} = 2$, Kazama-Suzuki coset model based upon the level one, $A$, $D$ or $E$ hermitian symmetric space. This was checked in detail in \cite{8} for the $A$-type models, and here we will verify that the construction works for the $D$-type and the $E$-type singularities. It is not immediately clear what the corresponding models are for larger (non-minimal) fluxes, but it is tempting to try to identify the magnitude of the flux with the level of the coset model.

The identification of these models, even at level one, is a non-trivial problem. If one uses a non-compact Calabi-Yau 4-fold with an $ADE$ singularity as outlined above, it turns out that all the dynamics are frozen because the kinetic terms of the the model are not normalizable. On the other hand, in \cite{8} a Landau-Ginzburg superpotential, $W$, was conjectured for compact Calabi-Yau 4-folds, and for non-compact Calabi-Yau manifolds this yields expressions for the topological charges of solitons in terms of period integrals on the 4-fold. The problem is that for non-compact Calabi-Yau manifolds, there does not appear to be an obvious geometric characterization of the Landau-Ginzburg fields themselves: one does not, \textit{a priori}, know how many fields there are, let alone their $U(1)$ charges. Thus the singularity only yields topological data about the model, that is, one knows only the ground states and the topological charges of the solitons. On the other hand, knowing the coset conformal field theory determines the chiral primary fields and the

\textsuperscript{1} A minimal flux is one that corresponds to a miniscule weight of $G$. 
Landau-Ginzburg potential (if there is one). Such a theory also generically has many more deformations of the superpotential than there are deformations of the singularity of the 4-fold. Thus there is a special, “canonically” deformed Landau-Ginzburg superpotential whose deformations are precisely generated from the deformations of the 4-fold.

One of our purposes here is to characterize and compute all the canonically deformed superpotentials associated with the ADE singularities with minimal fluxes. We first do this rather abstractly by developing an algorithm for computing such superpotentials. In so doing, we find that there are obvious choices in the procedure, and that these choices correspond to determining the coset denominator, and hence the correct set of Landau-Ginzburg variables. We then look at the period integrals on the 4-fold, and find how to determine the superpotentials from such calculations. A direct computation appears to lead to only one superpotential for each $A$, $D$, or $E$ singularity. However, we also show that there are, once again, ambiguities in the calculation of the periods, and that different choices lead to the full set of canonically deformed superpotentials. We will show that the resolution of the ambiguity amounts to selecting holomorphic representatives of the non-compact homology of the singularity. These representatives are characterized by weights of $G$, and each choice is equivalent to selecting a Weyl orbit of fluxes. In doing the calculations of the period integrals, we also find the vestigial remnants of the Landau-Ginzburg variables, and this leads to some natural conjectures as to the role of the Landau-Ginzburg variables in the geometric picture on 4-folds.

The next two sections of this paper contain a detailed review of the $\mathcal{N} = 2$ superconformal coset models of Kazama and Suzuki, and the construction of associated Landau-Ginzburg superpotentials. In particular, in section 2 we will review that standard construction of the chiral rings and Landau-Ginzburg potentials of the simply laced, level one, Hermitian symmetric space (SLOHSS) models. We then generalize this to obtain the “canonically” deformed superpotentials. In section 3 we will construct some specific examples of these deformed superpotentials, and evolve an algorithm for generating all the superpotentials associated with the numerator Lie algebra, $G$, from any one such superpotential. In section 4 we first review the part of [8] that is relevant to the SLOHSS models, and then use the formula of [8] to calculate topological charges of solitons in terms of period integrals of the singularity. We then use this calculation, in combination with the algorithm developed in section 3, to reconstruct the Landau-Ginzburg superpotentials from the geometry of the singularity. Our methods of computing the period integrals can be more naturally cast in terms excising holomorphic surfaces, and computing intersection
numbers. We also discuss this in section 4, and show how the selection of the holomorphic surfaces is equivalent to choosing a flux at infinity. We also find that the Landau-Ginzburg variables appear very naturally in the parameterization of these surfaces. Finally, in section 5 we make some remarks about fermion numbers of solitons, and about generalizations of the results presented here.

2. The Landau-Ginzburg Potentials of Deformed SLOHSS Models

2.1. The conformal Landau-Ginzburg potential

The $\mathcal{N} = 2$, Kazama-Suzuki conformal coset models \[9\] have been extensively studied. They are based upon the coset construction using:

$$\mathcal{S} = \frac{G \times SO(dim(G/H))}{H},$$

where the $SO(dim(G/H))$ is a level one current algebra, and represents the bosonized fermions in a supersymmetrization of the coset model based upon $G/H$. In (2.1), $H$ is embedded diagonally into $G$ and $SO(dim(G/H))$, the embedding into $G$ is an index one embedding, and the embedding into $SO(dim(G/H))$ is a conformal embedding \[10\] of level $g - h$ where $g$ and $h$ are the dual Coxeter numbers of $G$ and $H$ respectively. If $G$ has a level $k$, then $H$ has a level $k + g - h$. If $G/H$ is Kähler, then $H = H_0 \times U(1)$, with $U(1)$ inducing the complex structure, and the corresponding coset model has an $\mathcal{N} = 2$ superconformal algebra.

The simply laced, level one, hermitian symmetric space (SLOHSS) models have been studied even more extensively. In these models, $G$ is represented by a level one current algebra, and $G/H$ is a hermitian symmetric space. These cosets are:

$$\begin{align*}
\frac{SU(n + m)}{SU(n) \times SU(m) \times U(1)}, & \quad \frac{SO(2n)}{SO(2n - 2) \times U(1)}, & \quad \frac{SO(2n)}{SU(n) \times U(1)}, \\
E_6 & \quad E_7 & \quad \frac{E_6}{SO(10) \times U(1)}.
\end{align*}$$

(2.2)

It was shown in \[11\] that the chiral ring of these coset models is isomorphic to the de Rham cohomology ring $H^*(\mathcal{S}, \mathbb{R})$. It has also been argued that these models have a Landau-Ginzburg formulation \[11\]. Most of the Landau-Ginzburg potentials have been determined.
The most direct way to compute the chiral ring and potential is to use the isomorphism with the cohomology ring of $S$. The latter can be generated by the Chern classes of $H$-bundles on $S$, and these can be generated from the irreducible $H$-representations. The vanishing relations of the ring are characterized by the trivial $H$-bundles, and the corresponding vanishing Chern classes are given by those combinations of $H$-representations that are actually $G$-representations, and hence generate trivial bundles. In more mechanistic terms, the chiral ring is generated by the Casimir invariants of $H$, and to find the vanishing relations one simply has to take all the Casimirs of $G$ and decompose them into the Casimirs of $H$.

In practice it is simplest to reduce this calculation to the Cartan subalgebra, $\mathcal{X}$, of $G$. (Since $G$ and $H$ have the same rank, this is also a Cartan subalgebra (CSA) of $H$.) We then parameterize $\mathcal{X}$ by variables, $\xi_j$, $j = 1, \ldots, r$, where $r$ is the rank of $G$. The Casimirs of $G$ and $H$ are then equivalent to $W(G)$ and $W(H)$ invariant polynomials, denoted $V_j$ and $x_i$ respectively, of the $\xi_j$, where $W(G)$ and $W(H)$ are the Weyl groups of $G$ and $H$ respectively. The inequivalent $W(H)$ invariant polynomials, modulo $W(G)$ invariant polynomials, are given by elements of $W(G)/W(H)$, and thus the chiral ring has the structure of this Weyl coset.

One should also observe that elements of the coset $W(G)/W(H)$, and hence the ground states, are in one-to-one correspondence with the weights of a “miniscule” representation of $G$. A miniscule representation, $\mathcal{R}$, is defined as one in which all the weights lie in a single Weyl orbit. The subgroup $H$ is then defined to be the one whose semi-simple part (not the $U(1)$) fixes the highest weight of $\mathcal{R}$. Since all other weights lie in the Weyl orbit of the highest weight, it follows that the representation is indeed in one-to-one correspondence with $W(G)/W(H)$. In this manner one can generate the list (2.2). We thus see that ground states of the SLOHSS models are naturally labeled by the weights of a miniscule representation of $G$. This will be important throughout this paper.

The “reduction of Casimirs” procedure was used in [11,12] to construct the chiral rings. The remarkable, and rather unexpected fact is that the vanishing relations appear to be integrable to produce a single superpotential for each SLOHSS model. In particular, the Landau-Ginzburg potentials for $E_6$ and $E_7$ were computed in [12].

The basic procedure is therefore as follows: Let $m_i$ and $\hat{m}_i$, $i = 1, \ldots, r$, be the exponents of $G$ and $H$ respectively. (We define the exponent of a $U(1)$ to be zero.) The

\footnote{This has not yet been proven for the general coset $SO(2n)/SU(n) \times U(1)$, but in fact it will be implicitly established in this paper.}
degrees of the Casimirs are thus $m_i + 1$ and $\hat{m}_i + 1$. (We define the Casimir of $U(1)$ by taking a trace: it thus has degree 1.) The generators of the chiral ring are thus variables $x_i$, $i = 1, \ldots, r$, of degrees $\hat{m}_i + 1$. The vanishing relations are polynomials $V_i(x_j)$, $i = 1, \ldots, r$, of degrees $m_i + 1$, obtained by the decomposition of Casimir invariants. To get the Landau-Ginzburg potential one wants to integrate the vanishing relations to obtain a potential $W(x_i)$ such that the set of equations $V_i = 0$ are equivalent to the set of equations $\partial W/\partial x_i = 0$. The only complexity in this task is that $\partial W/\partial x_j$ will generically be some constant multiple of $V_j$ plus $V_k$’s of lower degree multiplied by polynomials in the $x_i$’s. One has thus to find the proper combinations of $V_k$’s before integration is possible, and as we indicated earlier, it seems remarkable that such integration is possible.

2.2. The canonically deformed Landau-Ginzburg potential

The conformal Landau-Ginzburg potential is, of course, quasi-homogeneous, and therefore multi-critical. The index of the singularity, $\mu = |W(G)|/|W(H)|$, is the degeneracy of the Ramond ground states. We now wish to deform this potential in such a manner as to make a mass gap, and yield the corresponding theories associated with the $ADE$ singularities in [8]. The key to seeing how this must happen is to observe that the versal deformation of the singularity preserves the $W(G)$ symmetry of the singularity, and therefore so must the corresponding deformation of the coset model. In particular, the ground states of the deformed model must be a $W(G)$-invariant family. This leads to a unique versal deformation procedure for the coset model. Instead of setting the $G$-Casimirs, $V_i(x_j)$, to zero one can set them to constant values: $V_i = v_i$ for some $v_i$. This is manifestly $W(G)$ invariant, and represents a set of equations that define the ground states of the canonically deformed model.

To understand the interrelationship between all the Landau-Ginzburg potentials for cosets $G/H$ with the same numerator $G$, but with different denominators $H$, it is important to understand how these ground states, and hence the canonical deformation, is realized in terms of the CSA variables, $\xi_j$. In terms of the Cartan subalgebra, $X$, setting $V_i = v_i$ uniquely defines a general point, $\xi_j = \xi_j^{(0)}$, in $X$ up to the action of $W(G)$. That is, the $v_j$ define a general point, $\xi_j^{(0)}$, in the fundamental Weyl chamber of $G$. Similarly, the values of the Casimirs, $x_j$, of $H$ uniquely specify a point in the fundamental Weyl chamber of $H$. The vacua of the deformed coset model are thus characterized by all the $W(G)$ images of $\xi_j^{(0)}$ that lie in the fundamental Weyl chamber of $H$. Non-zero values of the $v_j$ generically yield a massive theory with all vacua separated and $\xi_j^{(0)}$ an interior point of the Weyl
chamber. The theory becomes multi-critical, with massless solitons when the $v_j$ assume values at which $\xi_j^{(0)}$ goes to a wall of the Weyl chamber of $G$.

Remarkably enough, we find that the deformed “vanishing relations:” $V_i = v_i$ are still integrable to a Landau-Ginzburg potential $W(x_j; v_i)$. This will be proven in the next section. Based upon the foregoing observation we develop an algorithm for computing the desired Landau-Ginzburg potential for any SLOHSS model, $G/H$, given one such superpotential for any SLOHSS model with the same numerator, $G$. We then compute a superpotential for each choice of $G$, including the exceptional cosets, and we also give several examples of the application of the algorithm.

Finally, we note that the deformed potential is quasi-homogeneous:

$$W(\lambda^{m_j+1} x_j; \lambda^{m_i+1} v_i) = \lambda^{N+1} W(x_j; v_i),$$

where $N = m_r$ is the dual Coxeter number ($i.e.$ the degree of the highest Casimir) of $G$.

Once again, we stress that because we have preserved the Weyl symmetry of $G$, it is these deformed potentials that must be related to those of the versal deformations of the $ADE$ singularities.

3. Chiral rings and superpotentials

3.1. Grassmannians

Here one has $G = SU(m+n)$ and $H = SU(m) \times SU(n) \times U(1)$, but it is simpler to think of this coset as having $G = U(m+n)$ and $H = U(m) \times U(n)$. We will take $m \leq n$. The CSA of $U(m+n)$ can be parameterized by $(\xi_1, \ldots, \xi_{m+n})$, and the Casimirs of $G$ are the permutation invariants:

$$V_k \equiv \sum_{\ell=1}^{m+n} \xi_{\ell}^k, \quad k = 1, \ldots, m+n$$

while those of $H$ are:

$$x_k \equiv \sum_{\ell=1}^{m} \xi_{\ell}^k, \quad k = 1, \ldots, m; \quad \tilde{x}_k \equiv \sum_{\ell=m+1}^{m+n} \xi_{\ell}^k, \quad k = 1, \ldots, n.$$ (3.1)

It is convenient to introduce an equivalent set of Casimirs:

$$\tilde{V}_k \equiv (-1)^k \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq m+n} \xi_{j_1} \xi_{j_1} \ldots \xi_{j_k}, \quad k = 1, \ldots, m+n,$$
\[ z_k \equiv (-1)^k \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq m} \xi_{j_1} \xi_{j_2} \ldots \xi_{j_k}, \quad k = 1, \ldots, m. \quad \quad (3.2) \]

Since \( x_k + \tilde{x}_k = V_k = v_k \), we can use this linear equation to eliminate all the \( \tilde{x}_k \) in terms of \( x_k \) and \( v_k \). Note that for \( k > m \) one must write \( x_k \) as a polynomial in the \( x_j \) \( (j \leq m) \), and that for \( k > n \) one must write \( \tilde{x}_k \) in terms of a polynomial in the \( \tilde{x}_j \) \( (j \leq n) \). Thus the deformed Landau-Ginzburg potential is a function, \( W_{m,n} \), that depends upon \( x_j, j = 1, \ldots, m \) and \( v_k, k = 1, \ldots, m + n \).

For \( m = 1 \) one can set \( x = x_1 = \xi_1 \), and one has \( v_{n+1} = x^{n+1} + \tilde{x}_{n+1} \). One then writes \( \tilde{x}_{n+1} \) as a polynomial in \( \tilde{x}_j, j \leq n \) and then eliminates the \( \tilde{x}_j \) using \( \tilde{x}_k = v_k - x^k \). The result is a superpotential of the form:

\[ W \equiv W_{1,n} = \frac{1}{n+2} x^{n+2} + \sum_{k=2}^{n+2} \frac{1}{n+2-k} \hat{v}_k \ x^{n+2-k}, \quad \quad (3.3) \]

where \( \hat{v}_k \) are the values of \( \hat{V}_k \) given \( V_k = v_k \). The whole point is that one has

\[ \frac{dW}{dx} = \prod_{\ell=1}^{n+1} (x - \xi^{(0)}_{\ell}). \quad \quad (3.4) \]

The \( n + 1 \) critical points of this superpotential are then given by the components of the weight, \( \xi^{(0)}_{\ell} \), corresponding to the values, \( v_k \), of the Casimirs, \( V_k \).

Our purpose now is to show how to generate the superpotentials of all the Grassmannians from (3.3). For general, \( m \), the chiral ring is generated by the, \( x_k \), of (3.1) which are permutation invariants of the first \( m \) generators \( \xi_1, \ldots, \xi_m \). Then there are \( \binom{m+n}{m} \) ground states given by all choices of \( m \) of the \( \xi^{(0)}_{\ell} \). In particular, the lowest dimension chiral primary, \( x_1 \equiv \xi_1 + \ldots + \xi_m \) takes values \( \xi^{(0)}_{j_1} + \ldots + \xi^{(0)}_{j_m} \) for all choices of \( j_1, \ldots, j_m \).

One can thus extract the multi-variable potential from (3.3) by a rather simple algorithm. The ground states of \( W_{m,n} \) are characterized by subsets of \( m \) of the solutions to \( \frac{dW_{1,n}}{dx} = 0 \). The idea is to introduce an auxiliary equation that defines such a subset of \( m \) roots, and then use cross elimination between (3.3) or (3.4) and this auxiliary equation to reconstruct the superpotential \( W_{m,n} \). While the general superpotentials for the Grassmannians might be constructed more directly, the beauty of the foregoing algorithm is that it generalizes to other Lie algebras.
For the general Grassmannian model we need the \( m \) variables, \( x_k \), or equivalently and more conveniently, the \( z_k \) of (3.2). These characterize a subset, \( \xi_1, \ldots, \xi_m \), of the \( m+n \) roots of \( \tfrac{dW_{1,n}}{dx} = 0 \). Moreover, from (3.2) it follows that the individual roots, \( \xi_1, \ldots, \xi_m \), are related to the \( z_k \) as the \( m \) roots of the auxiliary equation:

\[
x^m + \sum_{k=1}^{m} z_k x^{m-k} = \prod_{j=1}^{m} (x - \xi_j) = 0. \tag{3.5}
\]

Thus, given a critical point of \( W_{m,n} \), one can use this formula to extract the set of \( m \) critical points, \( \{\xi^{(0)}_{j_1}, \ldots, \xi^{(0)}_{j_m}\} \) of (3.3), that characterize a single ground state of the Grassmannian model.

One now reverses this procedure: the multi-variable critical points are precisely characterized by adjusting the \( z_k \) (or \( x_k \)) so that all \( m \) roots of (3.5) are critical points of (3.3). The job is thus to properly recast (3.3) in terms of the \( z_k \) using the fact that we now wish to sum over subsets of \( m \) roots of \( \tfrac{dW_{1,n}}{dx} = 0 \).

One can do this by using (3.5) to eliminate all powers, \( x^k, k \geq m \), in \( \tfrac{dW_{1,n}}{dx} \). The result is a polynomial \( P(z_j; x) \) of overall degree \( n+1 \), but of degree \( m-1 \) in \( x \). If one makes the expansion:

\[
P(z_j; x) = \sum_{\ell=0}^{m-1} B_\ell(z_j) x^\ell,
\]

then this will vanish for a (generic) set of \( m \) roots of \( \tfrac{dW_{1,n}}{dx} = 0 \) if and only if all of the \( B_\ell(z_j) \) vanish. These must therefore be the deformed “vanishing relations” that characterize the chiral ring of the multi-variable model. It is these that must be integrated to give the multi-variable superpotential.

There is, however, a simpler way to get the Landau-Ginzburg superpotential: Perform the same elimination procedure, using (3.5), on the superpotential (3.3). The result is once again a function, \( W(z_j; x) \), that is a polynomial of degree at most \( m-1 \) in \( x \). Now replace \( x^j \) in this function by \( \tfrac{1}{m} x_j \), and rewrite everything in terms of either \( z_j \) or \( x_j \). We claim that this results in the requisite multi-variable potential, \( W_{m,n+1-m}(x_j; v_j) \). To see why this is so, observe that this prescription for replacing \( x^j \) is the same as summing \( \tfrac{1}{m} W(z_j; x) \) over the \( m \) roots of (3.5). Thus \( W_{m,n+1-m}(x_j; v_j) \) represents an average of the values of (3.3) over a set of roots of \( \tfrac{dW_{1,n}}{dx} = 0 \), and imposing \( \frac{\partial W_{m,n+1-m}(x_j; v_j)}{\partial x_j} = 0 \) implies that \( \tfrac{dW_{1,n}}{dx} = 0 \) on all \( m \) of these roots. Also recall that the values of the superpotential encodes the topological charge of solitons, and this averaging procedure reproduces the proper topological charge in the multi-variable case.

To illustrate this procedure we consider the potentials for which \( m = 2 \). Equation (3.3) implies:

\[
x = \tfrac{1}{2} z_1 \pm \tfrac{1}{2} \sqrt{z_1^2 - 4 z_2}. \tag{3.6}
\]
One now substitutes this into (3.3) and sums over both roots, which amounts to dropping all square-roots from the result. For \( n = 4 \) and \( n = 5 \) one obtains:

\[
W_{2,3}(y, z) = \frac{11}{12} z^6 - \frac{1}{7} y^3 + \frac{5}{4} y z^4 + \frac{1}{2} a_2 (y z^2 + z^4) - a_3 (y z + \frac{7}{6} z^3) \\
- (a_4 - \frac{1}{4} a_2^2) (y + z^2) + (a_5 - \frac{1}{2} a_3) z + \frac{1}{12} a_2^3 - \frac{1}{2} a_2 a_4 ,
\]

\[
W_{2,4}(z_1, z_2) = \frac{1}{7} z_1^7 - z_2^3 z_1 + 2 z_2^2 z_3 - z_2 z_1^5 + a_2 (z_2^2 - z_1^3 + \frac{1}{3} z_1^5) \\
+ a_3 (\frac{1}{2} z_2 - z_2 z_1 + \frac{1}{4} z_1^4) + a_4 (\frac{1}{3} z_1^3 - a_5 (\frac{1}{2} z_1^2 - z_2) + a_6 z_1 .
\]

In the first of the superpotentials we have made the change of variables: \( z = z_1, z_2 = y + \frac{3}{2} z^2 + \frac{1}{2} a_2 \). Note, in particular that the value of the modulus (the coefficient of \( y z^4 \), with \( y \) and \( z \) suitably normalized) in \( W_{2,3} \) is consistent with the results of [12, 13, 14].

3.2. The SO(2n) superpotentials

There are two infinite series of coset models with this numerator, and we begin with the \( SO(2n)/(SO(2n - 2) \times U(1)) \) coset model. The denominator group, \( SO(2n - 2) \times U(1) \), leaves an \( SO(2n) \) pair of vectors invariant, and so ground states of this model are characterized by the weights of the vector representation of \( SO(2n) \). This model may also be thought of as the \( D_{2n} \) minimal model, but we will see that the canonical deformation leads to a subset of the full set of deformations of the standard \( D_{2n} \) potential.

The Casimirs of \( SO(2n) \) are defined by

\[
V_{2k} = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1}^2 a_{i_2}^2 \cdots a_{i_k}^2 , \quad k = 1, 2, \ldots, n - 1 ,
\]

\[
\tilde{V}_n = a_1 a_2 \cdots a_n ,
\]

where the \( a_i \) are the skew eigenvalues of an \( SO(2n) \) matrix in the vector representation.

Under \( SO(2n) \supset SO(2n - 2) \times U(1) \) the Casimirs of \( SO(2n) \) are decomposed into the Casimirs \( x_j \) of \( SO(2n - 2) \times U(1) \),

\[
V_2 = x^2 + x_1^2 , \quad V_{2j} = x_{2j} + x_{2j-2} x_1^2 ; \quad j = 2, 3, \ldots, n - 2 ,
\]

\[
V_{2n-2} = x_{n-1}^2 + x_{n-4} x_1^2 , \quad \tilde{V}_n = \tilde{x}_{n-1} x_1 ,
\]

where \( x_1 \) is the degree 1 “Casimir” of \( U(1) \).

Set \( V_{2j} = v_{2j} \) with \( v_{2j} \) being some constant and eliminate \( x_2, \ldots, x_{2n-4} \) from the deformed relations \( V_{2j} = v_{2j} \) \( (1 \leq j \leq n - 1) \). We are then left with

\[
\tilde{x}_{n-1}^2 + (-1)^n x_1^{2(n-1)} + \sum_{j=1}^{n-1} \hat{v}_{2j} x_1^{2(n-1-j)} = 0 , \quad \hat{v}_{2j} = (-1)^{n+j} v_{2j} ,
\]
and \( \tilde{V}_n - \tilde{v}_n = 0 \). Introduce the superpotential so that
\[
\frac{\partial W}{\partial x_1} = \frac{1}{2} \left( x_{n-1}^2 + (-1)^n x_1^{2(n-1)} + \sum_{j=1}^{n-1} \hat{v}_{2j} x_1^{2(n-1-j)} \right),
\]
\[
\frac{\partial W}{\partial \tilde{x}_{n-1}} = \tilde{x}_{n-1} x_1 - \tilde{v}_n,
\]
then the desired \( D_{2n} \) potential is obtained as
\[
W = \frac{1}{2} y^2 x + \frac{(-1)^n}{2(2n-1)} x^{2n-1} + \sum_{j=1}^{n-1} \frac{\hat{v}_{2j}}{2(2n-2j-1)} x^{2n-2j-1} - \tilde{v}_n y, \tag{3.9}
\]
where we have set \( x_1 = x \) and \( \tilde{x}_{n-1} = y \).

Note that our procedure yields only an \( n \)-dimensional subset of the \( 2n \) possible relevant deformations of the conformal \( D_{2n} \) superpotential: The foregoing canonical deformation is odd under \( x \rightarrow -x, y \rightarrow -y \). Any other deformations would destroy the \( W(D_n) \) symmetry of the ground states.

We now turn to the \( SO(2n)/SU(n) \times U(1) \) coset model, whose central charge is \( c = \frac{3n(n-1)}{2(2n-1)} \). The \( SU(n) \) denominator factor fixes a (complex) pair of \( SO(2n) \) spinors, and so the ground states of this model are labeled by a set of spinor weights of \( SO(2n) \). The Landau-Ginzburg variables are, \textit{a priori}, the Casimir invariants of \( U(n) \). Let \( b_1, b_2, \ldots, b_n \) be the parameters of the \( SU(n) \) CSA with \( \sum_{i=1}^{n} b_i = 0 \). A set of \( SU(n) \) Casimirs can then be defined by
\[
z_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} b_{i_1} b_{i_2} \cdots b_{i_k}, \quad k = 2, 3, \ldots, n.
\]

In view of the decomposition \( 2n = n_2 + \bar{n}_2 \) under \( SO(2n)/SU(n) \times U(1) \), the \( SO(2n) \) eigenvalues \( a_i \) are expressed as \( a_i = \pm(b_i - 2z_1) \), \( i = 1, 2, \ldots, n \), where \( z_1 \) denotes the degree 1 Casimir of \( U(1) \). Then one can rewrite the \( SO(2n) \) Casimirs \([8]\) in terms of \( z_j, j = 1, 2, \ldots, n \). Inspecting the Casimir decomposition we see that the variables, \( z_j \), of even degree and the variable \( z_n \) can be immediately eliminated using the vanishing relations. The \( SO(2n)/SU(n) \times U(1) \) model can thus be characterized by \([\frac{n}{2}]\) Landau-Ginzburg variables \([9] z_{2j-1} \) of degree \( 2j - 1 \) with \( j = 1, 2, \ldots, [\frac{n}{2}] \) and the superpotential is of degree \( 2n - 1 \). Evaluating the central charge we obtain the correct value
\[
c = 3 \sum_{j=1}^{[n/2]} \left( 1 - \frac{2(2j-1)}{2n-1} \right) = \frac{3n(n-1)}{2(2n-1)}.
\]

\(^3\) Here \([\frac{n}{2}]\) stands for the integral part of \( \frac{n}{2} \).
For $n = 2, 3, 4$ the coset models are identified with the $A_2$, $A_4$ and $D_8$ minimal models, respectively. It is easily checked that the present method produces the corresponding Landau-Ginzburg superpotentials with deformations\footnote{In particular, the deformed superpotential for the $SO(8)/SU(4) \times U(1)$ model is converted into that for the $SO(8)/SO(6) \times U(1)$ model through the $SO(8)$ triality transformation.}

The foregoing procedure is, in general, very hard to integrate directly to obtain the general Landau-Ginzburg superpotential, and so we illustrate it in the first non-trivial example: $n = 5$. This model has $c = \frac{10}{3} > 3$ and turns out to be instructive since its shares interesting properties with the Grassmannian as well as the $E_6$ model. To decompose the Casimirs $V_i$ of $SO(10)$ we take $a_i = -(b_i - 2z_1)$. One finds

\begin{align*}
V_2(z_j) &= 20 z_1^2 - 2 z_2, \\
V_4(z_j) &= 160 z_1^4 - 16 z_2 z_1^2 + 12 z_3 z_1 + 2 z_4 + z_2^2, \\
\widetilde{V}_5(z_j) &= 32 z_1^5 + 8 z_2 z_1^3 - 4 z_3 z_1^2 + 2 z_4 z_1 - z_5, \\
V_6(z_j) &= 640 z_1^6 + 80 z_1^3 z_3 - (40 z_4 - 12 z_2^2) z_1^2 - (4 z_2 z_3 + 20 z_5) z_1 - 2 z_2 z_4 + z_3^2, \\
V_8(z_j) &= 1280 z_1^8 + 256 z_1^5 z_2 + 64 z_1^3 z_3 + (48 z_2^2 - 160 z_4) z_1^4 + (160 z_5 - 32 z_2 z_3) z_1^3 \\
& \quad + 8 z_3^2 z_1 + (-4 z_3 z_4 + 12 z_2 z_5) z_1 - 2 z_3 z_5 + z_4^2, \\
(3.10)
\end{align*}

Once again, setting $V_i$ to constant values, $v_i$ ($\tilde{v}_5$ for $\widetilde{V}_5$), we solve the first three equations of (3.10) for $z_2$, $z_4$ and $z_5$. We then substitute these values of $z_j$, $j = 2, 4, 5$ into equations for $V_6$ and $V_8$. With the notation $z_1 = x/5, z_3 = y$, the equation for $V_6$ reads

\begin{align*}
V_6 &= \frac{184}{625} x^6 + \frac{144}{25} x^3 y + y^2 - \frac{38}{125} v_2 x^4 + (-2 v_4 + \frac{33}{50} v_2^2) x^2 + (4 \tilde{v}_5 - \frac{4}{5} v_2 y) x + \frac{1}{2} v_2 v_4 - \frac{1}{8} v_3^2.
\end{align*}

Similarly the equation for $V_8$ becomes

\begin{align*}
V_8 &= \frac{36}{625} x^8 - \frac{24}{25} x^5 y + 4 x^2 y^2 - \frac{12}{125} v_2 x^6 + \left(\frac{6}{25} v_4 - \frac{1}{50} v_2^2\right) x^4 + \left(-\frac{56}{25} \tilde{v}_5 + \frac{4}{5} v_2 y\right) x^3 \\
& \quad + \left(-\frac{1}{5} v_2 v_4 + \frac{1}{20} v_2^2\right) x^2 + \left(\frac{1}{2} v_2^2 - 2 v_4\right) y + \frac{8}{5} v_2 \tilde{v}_5 x + 2 \tilde{v}_5 y \\
& \quad - \frac{1}{8} v_2^2 v_4 + \frac{1}{4} v_4^2 + \frac{1}{64} v_4^4.
\end{align*}

In order to obtain the Landau-Ginzburg superpotential we set

\begin{align*}
\frac{\partial W}{\partial y} &= \frac{1}{2} (V_6 - v_6), \\
\frac{\partial W}{\partial x} &= (V_8 - v_8) + \frac{1}{2} \left(\frac{16}{25} x^2 - \frac{2}{5} v_2\right) (V_6 - v_6).
\end{align*}

\footnote{This choice corresponds to the $16$ of $SO(10)$ as will be seen later. The other choice corresponds to the $16$.}
The second term of \( \frac{\partial W}{\partial x} \) is introduced so that the integrability condition is satisfied

\[
\frac{\partial}{\partial x} \left( \frac{\partial W}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial W}{\partial x} \right). 
\]

By integrating (3.11) we obtain the deformed Landau-Ginzburg superpotential for the SO(10)/SU(5) × U(1) coset model

\[
W = \frac{2372}{140625} x^9 + \frac{92}{625} x^6 y + \frac{36}{25} x^3 y^2 + \frac{1}{6} y^3 - \frac{788}{21875} v_2 x^7 + (\frac{63}{1250} v_2^2 - \frac{2}{25} v_4) x^5 \\
+ (-\frac{19}{125} v_2 y - \frac{6}{25} \tilde{v}_5) x^4 + (-\frac{8}{75} v_6 + \frac{3}{25} v_2 v_4 - \frac{61}{1500} v_2^3) x^3 \\
+ ((-v_4 + \frac{33}{100} v_2^2 y + \frac{1}{5} v_2 \tilde{v}_5) x^2 \\
+ (-\frac{1}{3} v_2 y^2 + 2 \tilde{v}_5 y - v_8 - \frac{9}{40} v_2^2 v_4 + \frac{1}{4} v_4^2 + \frac{13}{320} v_2^4 + \frac{1}{8} v_2 v_6) x \\
+ (\frac{1}{4} v_2 v_4 - \frac{1}{16} v_2^3 - \frac{1}{2} v_6) y. 
\]

This procedure does not expose any obviously generalizable structure, and so it seems rather hard to use it to extract the superpotential for general \( n \). Fortunately, it is possible to generalize the algorithm developed for Grassmannians. The crucial observation is again that the canonical deformation defines a point in the Cartan subalgebra, while the ground states of the SO(2n)/(SO(2n−2) × U(1)) and SO(2n)/(SU(n) × U(1)) models are respectively characterized in terms of vector and spinor weights. We can thus map the Landau-Ginzburg potential (3.3) onto the Landau-Ginzburg potential for SO(2n)/(SU(n) × U(1)) by finding the auxiliary equation that relates the vector and spinor weights, and then cross-eliminating in the proper manner.

We start by eliminating, or integrating out, the variable \( y \) in (3.9) using \( \frac{\partial W}{\partial y} = 0 \) to get:

\[
\tilde{W}(x) = \frac{(-1)^n}{2 (2n-1)} x^{2n-1} + \sum_{j=1}^{n-1} \frac{\tilde{v}_{2j}}{2 (2n-2j-1)} x^{2n-2j-1} - \frac{\tilde{v}_n^2}{2}.
\]

The 2n roots of this equation are, by construction, the 2n vector weights, \( x = \pm a_j \). The critical points of the SO(2n)/(SU(n) × U(1)) model are determined by the 2^{n−1} spinorial combinations: \( \pm \frac{1}{2} a_1 \pm \ldots \pm \frac{1}{2} a_n \), with an even (or odd) number of − signs. The simplest auxiliary equation has roots \( x = a_j, j = 1, \ldots, n \) and so determines the vector weights up to a sign. One now rewrites this polynomial equation in terms of the \( z_j \): That is, we define the polynomial

\[
R(x) = \prod_{i=1}^{n} (x - a_i) = \prod_{i=1}^{n} (x + b_i - 2 z_1) \\
= x^n - 2 n z_1 x^{n-1} + \sum_{k=2}^{n} c_k(z_j) x^{n-k}.
\]
When calculating the coefficients $c_k(z_j)$ we utilize the deformed relations $V_{2i}(z_j) = v_{2i}$ to eliminate $z_{2i}$ ($i = 1, 2, \ldots$) in favor of $z_{2j-1}$ with $j = 1, 2, \ldots, [\frac{n}{2}]$. Then we have

$$c_k(z_j) = c_k(z_1, z_3, \ldots; v_{2j}), \quad k = 2, 3, \ldots, n - 1, \quad (3.14)$$

whereas the relation $\tilde{V}_n(z_j) = \tilde{v}_n$ yields

$$c_n(z_j) = (-1)^n \tilde{v}_n. \quad (3.15)$$

For example, for $n = 5$ we find:

$$c_2 = 50 z_1^2 - \frac{1}{2} v_2, \quad c_3 = -140 z_1^3 + 3 v_2 z_1 + z_3,$$
$$c_4 = 150 z_1^4 - 5 v_2 z_1^2 - 10 z_3 z_1 - \frac{1}{8} v_2^2 + \frac{1}{2} v_4, \quad c_5 = -\tilde{v}_5.$$

The Landau-Ginzburg superpotential for the $SO(2n)/(SU(n) \times U(1))$ coset model is then computed by summing $\tilde{W}(x)$ over all the roots of $R(x) = 0$:

$$W(z_1, z_3, \ldots, z_{2[\frac{n}{2}]-1}; v_{2j}, \tilde{v}_n) = \sum_{i=1}^{n} \tilde{W}(a_i), \quad (3.16)$$

where the sums of powers of the $a_j$ that appear on the right-hand side are evaluated by making use of the auxiliary equation $R(x) = 0$. In particular, the sum of the last term of $\tilde{W}(x)$ becomes

$$-\frac{\tilde{v}_n^2}{2} \sum_{i=1}^{n} \frac{1}{a_i} = \frac{(-1)^n}{2} \tilde{v}_n c_{n-1}(z_j),$$

where (3.15) and (3.14) have been used.

We have confirmed that this method correctly recovers the deformed superpotentials obtained by the procedure of linear elimination for the models with $n = 2, 3, 4$ and 5.

3.3. The canonically deformed $E_6$ superpotential

There is only one SLOHSS model involving each of $E_6$ and $E_7$, and so there are no shortcuts: we need to use the Casimir decompositions. Here we present the computation for $E_6$. The details for $E_7$ are similar, but more complicated, and so they have been included in an appendix.

The denominator of the $E_6$ model involves $SO(10)$, which fixes a weight in the $27$ of $E_6$, and there are thus $27$ ground states. The first step to obtaining the superpotential is to decompose the six Casimirs, $V_j$, of $E_6$ into the Casimirs, $x_j$, of $SO(10) \times U(1)$. This was
done explicitly in [12], but we reproduce the result here since there was a typographical error in the printed version.

\[ V_2(x_j) = 12x_1^2 + x_2 , \]
\[ V_5(x_j) = 48x_1^5 - 8x_1^3x_2 + x_1x_2^2 - 2x_1x_4 + 4x_5, \]
\[ V_6(x_j) = -4680x_1^6 - 1062x_1^4x_2 - \frac{177}{2}x_1^2x_2^2 - \frac{23}{8}x_2^3 - 15x_1^2x_4 + \frac{5}{4}x_2x_4 - 60x_1x_5 - x_6, \]
\[ V_8(x_j) = 25830x_1^8 + 7098x_1^6x_2 + \frac{3027}{4}x_1^4x_2^2 + \frac{363}{8}x_1^2x_2^3 + \frac{171}{128}x_2^4 + \frac{555}{2}x_1x_4 \]
\[ + \frac{105}{4}x_2x_2x_4 - \frac{15}{32}x_2^2x_4 - \frac{35}{32}x_4^2 + 1740x_1x_5 + 75x_1x_2x_5 - 6x_1^2x_6 - \frac{1}{2}x_2x_6 + \frac{15}{8}x_8, \]
\[ V_9(x_j) = 28560x_1^9 - 1008x_1^7x_2 + 42x_1^5x_2^2 + 35x_1^3x_2^3 + \frac{105}{16}x_1x_2^4 - 924x_1^5x_4 - 70x_1^3x_2x_4 - \frac{105}{4}x_1x_2^2x_4 + \frac{35}{4}x_1x_4^2 + 840x_1^4x_5 + 420x_1^2x_2x_5 + \frac{35}{2}x_2x_5 - 7x_4x_5 - 112x_1^3x_6 + 28x_1x_2x_6 - 21x_1x_8, \]
\[ V_{12}(x_j) = 177660x_1^{12} + 97902x_1^{10}x_2 + \frac{36063}{4}x_1^8x_2^2 + \frac{1635}{4}x_1^6x_2^3 + \frac{7559}{64}x_1^4x_2^4 - \frac{577}{128}x_1^2x_2^5 - \frac{15}{1024}x_2^6 + \frac{15705}{2}x_1^8x_4 + \frac{6087}{2}x_1^6x_2x_4 - \frac{7299}{16}x_1^4x_2^2x_4 \]
\[ + \frac{1741}{32}x_1^2x_2^3x_4 + \frac{1307}{1536}x_1^2x_2^4x_4 + \frac{7557}{16}x_1^4x_4^2 - \frac{795}{32}x_1^2x_2x_4^2 - \frac{429}{256}x_2^2x_4^2 + \frac{85}{128}x_4^3 + 94104x_1^7x_5 + 13050x_1^5x_2x_5 + \frac{1551}{2}x_1^3x_2^2x_5 - \frac{59}{8}x_1x_2^3x_5 + 219x_1^3x_4x_5 + \frac{243}{4}x_1x_2x_4x_5 + 6x_1^2x_5^2 - \frac{10}{2}x_2x_5^2 - 948x_1^6x_6 \]
\[ + 1041x_1^4x_2x_6 - \frac{313}{4}x_1^2x_2^2x_6 - \frac{61}{48}x_2^3x_6 - \frac{25}{2}x_1^2x_4x_6 + \frac{25}{24}x_2x_4x_6 - 50x_1x_5x_6 + \frac{1}{3}x_2^6 - \frac{4257}{4}x_1^4x_8 + \frac{561}{8}x_1^2x_2x_8 + \frac{97}{64}x_2^2x_8 - \frac{45}{32}x_4x_8. \]

We then solve the equations \( V_i = v_i \), where \( v_i \) are constants, for \( i = 2, 5, 6, 8 \). These are the equations that are linear in \( x_i \), and so we use them to directly eliminate \( x_2, x_5, x_6 \) and \( x_8 \). This leaves the equations: \( V_9 = v_9 \) and \( V_{12} = v_{12} \). These are integrable, and former is proportional to \( \frac{\partial W}{\partial x_4} \), and the latter must be proportional to \( \frac{\partial W}{\partial x_1} + p_3(x_j, v_j) \frac{\partial W}{\partial x_4} \), where \( p_3(x_j, v_j) \) is a homogeneous polynomial of degree 3.

For reasons that will become apparent in sections 3.4 and 4.2, we replace the \( v_j \) with some other constants, \( w_\alpha \) according to:

\[
\begin{align*}
    v_2 &= -2w_1, & v_5 &= -4w_2, & v_6 &= w_3 + 20w_1^3, \\
    v_8 &= 5w_4 + 4w_1w_3 + 15w_4^4, & v_9 &= -28(w_5 + 2w_2w_2), \\
    v_{12} &= 5w_6 + \frac{26}{3}w_1^2w_4 + \frac{1}{3}w_3^2 + 4w_1^3w_3 + 19w_1^2w_2 + w_1^6.
\end{align*}
\]
Finally, setting \( x_1 = \frac{1}{4} x \) and \( x_4 = \frac{60}{13} z + \frac{117}{16} x^4 + 5w_1 x^2 + 2w_1^2 \) one arrives at the superpotential:

\[
W = x^{13} - \frac{25}{169} x z^3 + \frac{5}{26} z^2 w_2 \\
+ z \left( x^9 + x^7 w_1 + \frac{1}{3} x^5 w_1^2 - x^4 w_2 - \frac{1}{3} x^2 w_1 w_2 + \frac{1}{12} x^3 w_3 - \frac{1}{6} x w_4 + \frac{1}{3} w_5 \right) \\
+ \frac{247}{165} x^{11} w_1 + \frac{13}{15} x^9 w_1^2 - \frac{39}{20} x^8 w_2 + \frac{169}{945} x^7 w_1^3 + \frac{13}{105} x^7 w_3 - \frac{26}{15} x^6 w_1 w_2 \\
+ \frac{13}{220} x^5 w_1 w_3 - \frac{13}{50} x^5 w_4 - \frac{91}{180} x^4 w_1^2 w_2 + \frac{13}{30} x^4 w_5 + \frac{13}{15} x^3 w_2^2 - \frac{13}{90} x^3 w_1 w_4 \\
- \frac{13}{120} x^2 w_2 w_3 + \frac{13}{90} x^2 w_1 w_5 - \frac{13}{270} x w_6 - \frac{13}{360} w_1^4 w_2 + \frac{13}{90} w_1^2 w_5 .
\]

This potential then has the property that:

\[
\frac{\partial W}{\partial z} = \frac{1}{84} (V_9 - v_9) ; \quad \frac{\partial W}{\partial x} = \frac{13}{1350} (V_{12} - v_{12}) + \frac{13}{210} (x^3 - \frac{1}{6} w_1 x) (V_9 - v_9) .
\]

3.4. Single variable potentials

One can always partially eliminate variables from the Landau-Ginzburg potential by using some of the equations, \( \frac{\partial W}{\partial x_j} = 0 \). Indeed we have already done this for all the quadratic variables in \( W \): this is what is meant by using the linear vanishing relations. One can go further and eliminate all variables except the one of lowest degree, the Casimir, \( x \), of degree 1. The result, \( W(x; v_j) \), is defined to be the single variable potential. To be more explicit, let \( W(x, y, z, \ldots ; v_i) \) be the deformed superpotential, where \( x, y, z, \ldots \) are the Landau-Ginzburg variables. One can solve the equations of motion

\[
\frac{\partial W}{\partial y} = \frac{\partial W}{\partial z} = \ldots = 0 ,
\]

and expressing \( y, z, \ldots \) in terms of \( x \) and \( v_i \), and substitute the result back into \( W \) to obtain \( W(x; v_j) \). This one variable potential has the property that all the ground states are determined from the solutions of:

\[
\frac{d W}{d x} = \frac{d}{d x} W(x, y_{cl}(x), z_{cl}(x), \ldots ; v_i) = 0 .
\]

In general one will not be able to solve (3.20) explicitly, and so the one variable potential is generically implicit. On the other hand, for several important examples, the equations can be explicitly solved and one obtains a polynomial or irrational potential.

There is also another more direct way to get the single variable potential, and this approach is more directly related to the applications considered in [13,4]. In this approach one takes the characteristic polynomial, \( P(x; v_j) \), of a general CSA matrix in the
representation of $G$ that corresponds to the ground state of the SLOHSS model of interest. One then shifts the Casimir of highest degree according to $v_r \rightarrow v_r + \tau$, and solves $P(x; v_1, \ldots, v_{v-1}, v_r + \tau) = 0$ for $\tau(x; v_j)$. The roots of the equation $\tau(x; v_j) = 0$ are precisely the ground states of the SLOHSS model, and so the single variable potential in terms of the $U(1)$ Casimir, $x$, is given by:

$$W(x; v_j) = \int \tau(x; v_j) \, dx.$$  \hspace{1cm} (3.21)

We thus refer to $\tau(x; v_j)$ as the pre-potential. This function is discussed further in the Appendix.

Given an irrational or implicit potential in a single variable, one can reconstruct the multi-variable, polynomial potentials in a relatively straightforward manner: One replaces every algebraically independent irrational or implicit form by a new chiral primary field. Algebraic vanishing relations are then introduced so that their solution yields the relationships between the new variables and the original irrational or implicit forms. One then integrates these vanishing relations (where possible) to obtain a Landau-Ginzburg potential. For all the SLOHSS potentials that we have studied in detail, we have found that the irrational, single variable potentials are equivalent to the full algebraic, multi-variable potentials.

There are several reasons why the single variable potentials are useful. First, we will find that the results of the period integrals in the 4-fold will most directly reduce to the single variable potential. Knowing the single variable potentials is thus valuable data in the reconstruction process. More generally, one finds such single variable potentials also naturally arise in the construction of Seiberg-Witten Riemann surfaces from period integrals on Calabi-Yau 3-folds \cite{15,7,16}.

There is a further application of the single variable potentials in the coupling of topological matter to topological gravity \cite{3,4}. In this context, the physical operators and their correlators are most directly expressed in terms of a single variable potential and residue formulae.

We will therefore illustrate both approaches to computing the single variable superpotentials. We will use the first method on the $E_6$ superpotential, and arrive at results closely related to those of \cite{16,4}. We will then use the second approach to rederive the result from section 3.2 for $SO(2n)/U(n)$. 

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For $E_6$, one can eliminate the variable $z$ from (3.18) by solving $\frac{\partial W}{\partial z} = 0$. One finds that: $z = \frac{13}{30} (w_2 \pm \sqrt{p_2})$ where,

$$p_2 \equiv 12 x^{10} + 12 w_1 x^8 + 4 w_1^2 x^6 - 12 w_2 x^5 + w_3 x^4 - 4 w_1 w_2 x^3 - 2 w_4 x^2 + 4 w_5 x + w_2^2.$$ (3.22)

Substituting this into the superpotential one obtains:

$$W = \frac{13}{270} \left( q_0 \pm \frac{1}{2 x^2} (\sqrt{p_2})^3 \right),$$ (3.23)

where

$$q_0 \equiv \frac{270}{13} x^{13} + \frac{342}{11} w_1 x^{11} + 18 w_1^2 x^9 - \frac{63}{2} w_2 x^8 + \frac{2}{7} (13 w_1^3 + 9 w_3) x^7 - 27 w_1 w_2 x^6 + \frac{3}{2} (2 w_1 w_3 - 9 w_4) x^5 - \frac{3}{2} (5 w_1^2 w_2 - 6 w_5) x^4 + 3 (3 w_2^2 - w_1 w_4) x^3 - \frac{3}{2} (w_2 w_3 - 2 w_1 w_5) x^2 - (3 w_1 w_2^2 + w_6) x - \frac{3}{4} (w_1^4 w_2 + 2 w_2 w_4 - 4 w_1^2 w_5) + 3 \frac{w_2 w_5}{x} + \frac{1}{2} \frac{w_3^2}{x^2}.$$ (3.24)

This is precisely the one-variable $E_6$ potentials discussed in [16,14,17]. In particular, one has:

$$\frac{dq_0}{dx} = \frac{1}{x^3} q_1 - w_6, \quad \frac{d}{dx} \left[ \frac{1}{2 x^2} (\sqrt{p_2})^3 \right] = \frac{1}{x^3} p_1 \sqrt{p_2},$$ (3.25)

where

$$p_1 \equiv 78 x^{10} + 60 w_1 x^8 + 14 w_1^2 x^6 - 33 w_2 x^5 + 2 w_3 x^4 - 5 w_1 w_2 x^3 - w_4 x^2 - w_5 x - w_2^2,$$

$$q_1 \equiv 270 x^{15} + 342 w_1 x^{13} + 162 w_1^2 x^{11} - 252 w_2 x^{10} + (26 w_1^3 + 18 w_3) x^9 - 162 w_1 w_2 x^8 + (6 w_1 w_3 - 27 w_4) x^7 - (30 w_1^2 w_2 - 36 w_5) x^6 + (27 w_2^2 - 9 w_1 w_4) x^5 - (3 w_2 w_3 - 6 w_1 w_5) x^4 - 3 w_1 w_2^2 x^3 - 3 w_2 w_5 x - w_2^3.$$ (3.26)

The pre-potential:

$$\tau \equiv \frac{dW}{dx} = \frac{1}{x^3} (q_1 \pm p_1 \sqrt{p_2}) - w_6$$ (3.27)

is precisely the one that plays a crucial role in defining the Seiberg-Witten Riemann surface for $E_6$ [16].
We now start from the other end for the $SO(10)/(SU(5) \times U(1))$ model: The ground states are classified by the $16$ of $SO(10)$, and the relevant characteristic polynomial is therefore given by:

$$P_{SO(10)}^{16}(x) = \prod_{i=1}^{16} \left( x - (\pm \frac{1}{2} a_1 \pm \frac{1}{2} a_2 \cdots \pm \frac{1}{2} a_5) \right)$$

with an even number of "−" signs. Expanding this we have

$$P_{SO(10)}^{16}(x) = \frac{1}{16} \left( q_0^2 - \frac{1}{64} q_1^2 q_2 - 2 v_8 q_0 + v_8^2 \right)$$

$$= x^{16} - 2 v_2 x^{14} + \left( \frac{7}{4} v_2^2 - v_4 \right) x^{12} - 12 v_5 x^{11} + \cdots ,$$

where we have put the Casimirs to $V_{2i} = v_{2i}$ and $\tilde{V}_5 = \tilde{v}_5$. Here $q_0, q_1$ and $q_2$ are polynomials

$$q_0 = 68 x^8 - 20 v_2 x^6 - 8 \left( -\frac{7}{4} v_4 + \frac{5}{16} v_2^2 \right) x^4 - 24 \tilde{v}_5 x^3 - 8 \left( -\frac{3}{32} v_2^3 + \frac{3}{8} v_2 v_4 - \frac{1}{2} v_6 \right) x^2$$

$$+ 2 \tilde{v}_5 v_2 x + \frac{1}{64} v_4^2 + \frac{1}{4} v_2^4 - \frac{1}{8} v_2^2 v_4 ,$$

$$q_1 = 48 x^5 - 8 v_2 x^3 - v_2^2 x + 4 v_4 x - 4 \tilde{v}_5 ,$$

$$q_2 = 2 \left( 64 x^6 - 16 v_2 x^4 + (-4 v_2^2 + 16 v_4 x^2) x^2 - 32 \tilde{v}_5 x + v_2^3 - 4 v_2 v_4 + 8 v_6 \right) .$$

Note that

$$\frac{dq_2}{dx} - 16 q_1 = 0 . \quad (3.28)$$

Then we obtain the pre-potential, $\tau$, by solving $P_{SO(10)}^{16}(x; v_8 + \tau) = 0$:

$$\tau_{16}(x) = -v_8 + q_0 \pm \frac{1}{8} q_1 \sqrt{q_2} .$$

This expression can then be integrated with the use of (3.28) to give:

$$\mathcal{W}(x) \equiv \int \tau_{16}(x) \, dx = \int \left( -v_8 + q_0 \right) dx \pm \frac{1}{192} (\sqrt{q_2})^3 .$$

To get to a multi-variable, rational potential one simply has to replace $\sqrt{q_2}$ by a new Landau-Ginzburg variable. It is convenient to mix this with other degree 3 terms in $x$ and $v_j$ to arrive at the definition:

$$y = \pm \sqrt{q_2} - \frac{7}{25} x^3 + \frac{2}{5} v_2 x . \quad (3.29)$$

One now replaces all occurrences of $\sqrt{q_2}$ in $\mathcal{W}(x)$ using (3.29), and the result is precisely $W(x,y)$ of (3.12). Moreover, the equation (3.29) is exactly the solution of $\frac{\partial W}{\partial y} = 0$. 

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4. Landau-Ginzburg Models and the IIA theory on 4-folds

4.1. Review

The construction of [8] starts by identifying the vacuum states of the IIA theory that correspond to the Landau-Ginzburg vacua. These IIA vacua are obtained by first choosing a Calabi-Yau 4-fold, $Y$, and then choosing a set of fluxes for the 4-form field strength, $G$, on $Y$. There is a quantization condition on the fluxes which requires that the quantity:

$$ N = \frac{\chi}{24} + \frac{1}{2} \int \frac{G \wedge G}{(2\pi)^2}, $$

be an integer. This means that the characteristic class $\xi \equiv [\frac{G}{2\pi}]$ satisfies a shifted Dirac quantization condition in that the difference between any two fluxes $G$ and $G'$ must satisfy $\xi - \xi' \in H^4(Y; \mathbb{Z})$. The integer $N$ is then the number of “world-filling” fundamental strings in the vacuum state of the theory, and these fundamental strings are to be located at some points, $P_i$, chosen in $Y$. For the vacuum to have a mass gap one must thus have $N = 0$.

Domain walls, or solitons are represented by kinks that interpolate between spatial regions in which the $G$-fluxes are different. In particular, for any element $S$ of the integral homology $H_4(Y; \mathbb{Z})$, we may choose a D4-brane that wraps this cycle and appears as a “particle” in the $\mathbb{R}^{1,1}$ “world”. Such a brane is a source of $G$-flux, and indeed across this domain wall one has $\xi - \xi' = [S]$ where $[S] \in H^4(Y; \mathbb{Z})$ is the Poincaré dual of $S$. Thus the model is specified by the family of $G$-fluxes obtained that are mapped into one another via such solitonic D4-branes.

Having chosen a $G$-flux, the complex and Kähler structures on $Y$ are required to satisfy constraints [13] in order to preserve the requisite supersymmetry with a zero cosmological constant. Specifically, if $G_{p,q}$ are the $(p, q)$ parts of $G$ one requires that $G_{0,4} = G_{1,3} = 0$, and if $K$ is the complexified Kähler form, one requires that $G \wedge K = 0$. These imply that $G$ must be a self-dual (2,2)-form. In [8] a superpotential was proposed for the complex structure and Kähler moduli. We will only consider the former here, and it is given by:

$$ W(T_i) = \frac{1}{2\pi} \int_Y \Omega \wedge G. $$

The variation of Hodge structure means that $W$ and $dW$ vanish in the vacuum states, and the result is the Landau-Ginzburg theory at a conformal point. More generally, one can seek vacua of massive Landau-Ginzburg theories, and then one only impose the condition that $dW = 0$ for a vacuum state.
A Landau-Ginzburg soliton is a BPS state whose mass is equal to its topological charge, and the latter is given by the change in the value of the superpotential along the soliton. Since the solitons are represented by $D4$-branes wrapping integral 4-cycles, one thus concludes that:

$$\Delta W = \frac{1}{2\pi} \int_Y \Omega \wedge (G - G') = \int_{[S]} \Omega,$$

where $[S]$ is (the class of) the 4-cycle dual to the class $[(G - G')/2\pi]$.

For non-compact Calabi-Yau manifolds, the quantization condition, (4.1), is replaced by a boundary condition at infinity. That is, there is a new conserved quantity, the “flux at infinity:”

$$\Phi = N + \frac{1}{2} \int \frac{G \wedge G}{(2\pi)^2},$$

and the value of $\Phi$ must be given in order to specify the model. Going to non-compact Calabi-Yau manifolds also affects the dynamics in other non-trivial ways. In particular, some of the complex structure moduli will give rise to scalar fields whose kinetic terms are non-normalizable. These moduli will thus have to have zero kinetic energy, and thus their dynamics is frozen for the non-compact manifold. These complex structure moduli thus become true moduli, or coupling constants of the Landau-Ginzburg model.

In this paper we will follow [8] and consider only non-compact Calabi-Yau manifolds that are fibrations of $ALE$ singularities over some complex 2-dimensional base. Specifically, we consider non-compact 4-folds defined by the equation $P(z_1, \ldots, z_5) = 0$ where $P(z_1, \ldots, z_5) = H(z_1, z_2) - z_3^2 - z_4^2 - z_5^2$ and:

$$H(z_1, z_2) = z_1^{n+1} + z_2^2 + \ldots \quad A_n,$$
$$H(z_1, z_2) = z_1^{n-1} + z_1 z_2^2 + \ldots \quad D_n,$$
$$H(z_1, z_2) = z_1^3 + z_2^4 + \ldots \quad E_6,$$
$$H(z_1, z_2) = z_1^3 + z_1 z_2^3 + \ldots \quad E_7,$$
$$H(z_1, z_2) = z_1^3 + z_2^5 + \ldots \quad E_8.$$

In these expressions $+\ldots$ indicates the addition of all possible relevant deformations of the singularity. The number of monomials involved in these deformations is the rank of the $ADE$ group, and the natural degrees of the coefficients of the monomials are the degrees of the corresponding Casimir invariants. In [8] it was shown that for these singularities the
condition $G \wedge K = 0$ is always satisfied, and all the moduli of the singularity are coupling constants, and indeed have no dynamics.

The homology, $H_4(Y, \mathbb{Z})$, of a surface determined by (4.5) is naturally identified with the root lattice, $\Gamma$, of the corresponding group, and the intersection form is the Cartan matrix. The Poincaré duals of these cycles are the elements of the compact cohomology, $H^4_{cpt}(Y; \mathbb{Z})$, while the full set of non-trivial $G$-fluxes, $\xi$, are classified by $H^4(Y; \mathbb{Z})$, which can be identified with the weight lattice, $\Gamma^*$, of the group. The fluxes are thus characterized by weights of the group, monodromies of the singularity will permute these by the action of the Weyl group, and solitons can add or subtract root vectors.

A model is thus specified by a weight, $\xi$, in $\Gamma^*$. Conservation of $\Phi$ means that adding or subtracting a root either acts as a Weyl reflection, preserving the magnitude of the flux, or it shortens the weight, trading some flux for a string in the ground state. As was argued in [8], a state with $N \neq 0$ has massless excitations coming from motions of the string. Thus a non-trivial model with a mass gap must have $N = 0$ in all states, and the weight, $\xi$, must therefore be miniscule. (Non-miniscule weights can be shortened by adding or subtracting roots.) In terms of the “flux at infinity,” $\Phi$, the models with a mass gap are those with minimum value of $\Phi$ in each of the classes $\Gamma^*/\Gamma$.

Thus the models with mass gaps are classified by miniscule representations of the underlying $ADE$ group, and as we saw earlier, the choice of a miniscule weight determines the denominator group of the coset model, and leads us to the hermitian symmetric space models (2.2).

We now wish to examine how the Landau-Ginzburg potentials of the SLOHSS models emerge from the period integrals of the surfaces defined by (4.5). From the foregoing discussion there is an obvious problem that will arise: all the dynamics is frozen. The singularity, and its period integrals encode only topological data about the Landau-Ginzburg theory, and do not contain dynamical fields. It turns out that we will still be able to extract the Landau-Ginzburg potentials from this topological data, and thus implicitly find some Landau-Ginzburg fields. While we cannot give definitive geometric characterizations of these Landau-Ginzburg fields, we find that they emerge in the calculation in a very interesting and natural manner. We will also find that the evaluation of the period integrals has implicit ambiguities whose resolution corresponds to selecting the miniscule weight, or flux at infinity, and thus determines the Landau-Ginzburg variables and chiral ring. We will remark further upon this in sections 4.4, and we will begin by direct evaluation of periods.
4.2. Integrating over 4-cycles

The integral of the 4-form, \( \Omega \), over a cycle, \( S \), of the non-compact surfaces defined by (4.5) can be written:

\[
\int_{[S]} \Omega = \int \frac{dz_1 \, dz_2 \, dz_3 \, dz_4}{\sqrt{H(z_1, z_2) - z_3^2 - z_4^2}},
\]

where the 4-cycle is defined as follows: \( z_3 \) runs around branch-cut of the square-root; \( z_4 \) runs between \( z_4 = \pm \sqrt{H(z_1, z_2)} \), i.e. points at which the branch cut shrinks to a point; the variables \( z_1, z_2 \) are then integrated over a 2-cycle, \( S_2 \), of the singularity \( H(z_1, z_2) - z_5^2 = 0 \). The first two integrals are elementary, and reduce to the following (up to an overall normalization):

\[
\int_{S_2} dz_1 \, dz_2 \sqrt{H(z_1, z_2)}. \tag{4.7}
\]

For the \( A_n \) singularity, one has \( H(z_1, z_2) = P(z_1; a_j) + z_2^2 \), where \( P(z_1; a_j) \) is a polynomial of degree \( n + 1 \). As was discussed in [8], the integral (4.7) then reduces to:

\[
\int dz_1 \, P(z_1; a_j), \tag{4.8}
\]

evaluated between the zeroes of \( P(z_1; a_j) \). That is, we have recovered the superpotential, \( W \), of (3.3), with the integral (\( i.e. \) topological charge of the soliton) expressed in terms of the values of \( W \) at its critical points.

It is almost as elementary to perform the integrals for the \( D_n \) singularity. In this instance one has: \( H(z_1, z_2) = P(z_1; a_j) + z_1 z_2^2 + a_0 z_2 \), where \( P(z_1; a_j) = z_1^{n-1} + \sum_{j=0}^{n-2} a_{n-1-j} z_1^j \). One makes the change of variable \( z_1 = x^2 \), and one performs the elementary integral over \( z_2 \) to obtain:

\[
W = \int dx \left( P(x^2; a_j) - \frac{a_0^2}{4x^2} \right) = \int P(x^2; a_j) \, dx + \frac{a_0^2}{4x}, \tag{4.9}
\]

which is to be evaluated between the zeroes of the integrand.

This is the single variable potential (3.13) for the \( SO(2n) / SO(2n-2) \times U(1) \) coset model. As we remarked earlier, it is obtained from (3.9) by using \( \frac{\partial W}{\partial y} = 0 \) to eliminate \( y \). The superpotential, (3.3), can be recovered by removing the singular term by introducing the variable \( y = \frac{a_0}{2x} \). Also observe that (4.9) is an odd function and contains only half of the general versal deformations of the \( D_n \) singularity.

The calculation for the \( E_6 \) singularity is essentially the same as that performed in [10], and so we will only sketch the details here.
One starts with the singularity:

\[
H_{E_6}(z_i) = z_1^3 + z_2^4 + \frac{1}{2} w_1 z_1 z_2^2 - \frac{1}{4} w_2 z_1 z_2 + \frac{1}{96} (w_3 - w_1^3) z_2^2
\]

\[
+ \frac{1}{96} (w_4 + \frac{1}{4} w_1 w_3 - \frac{1}{8} w_1^4) z_1 - \frac{1}{48} (w_5 - \frac{1}{4} w_1^2 w_2) z_2
\]

\[
+ \frac{1}{3456} \left( \frac{1}{16} w_1^6 - \frac{3}{16} w_1^3 w_3 + \frac{3}{32} w_3^2 - \frac{3}{4} w_1^2 w_4 + w_6 \right),
\]

where we have made a convenient choice for the deformation parameters, \(w_j\). One reparameterizes the singularity by setting

\[
z_1 = xy + \alpha(x); \quad z_2 = y + \beta(x),
\]

and the result is a quartic in \(y\). One now sets

\[
\beta(x) = -\frac{1}{4} \left( x^3 + \frac{1}{2} w_1 x \right),
\]

\[
\alpha(x) = \frac{1}{48} (2 w_1 x^2 + w_1^2) + \frac{1}{24} \left( w_2 \pm \sqrt{p_2} \right),
\]

where \(p_2\) is given by (3.22). The functions \(\beta\) and \(\alpha\) are chosen so that the \(y^3\) and \(y^1\) terms in the quartic vanish respectively. The integral (4.7) now takes the form:

\[
\int dx \, \sqrt{A(x) y^2 + B(x)},
\]

for some functions, \(A(x)\) and \(B(x)\). By taking a contour at large \(y\) this integral reduces to:

\[
\int dx \left( \frac{1}{4} (A(x))^2 - B(x) \right) = \frac{1}{3456} \int dx \left[ \frac{1}{x^3} (q_1 \pm p_1 \sqrt{p_2}) - w_6 \right]
\]

\[
= \frac{1}{3456} \left( q_0 \pm \frac{1}{2x^2} (\sqrt{p_2})^3 \right),
\]

and thus we regenerate the one variable superpotential (3.23) with all the correct moduli. As we remarked earlier, one can reconstruct (3.18) by replacing the singular irrational part using:

\[
\frac{13}{30x} (w_2 \pm \sqrt{p_2}) \equiv z.
\]

### 4.3. Other superpotentials

Our integration procedure appears to have led us directly to a single superpotential for the \(A_n\) and \(D_n\) models. The reason for this is that we must have implicitly chosen a flux, or boundary condition at infinity. Indeed, the sleight-of-hand occurred when we passed from the definite integral to the indefinite integral with a single end-point to the integration.
We know from the algorithms of section 3 that we can get the multi-variable potentials for any of the Grassmannian models and for the $SO(2n)/U(n)$ series by summing the potentials (3.3) and (3.9) over carefully selected sets of their critical points. Exactly the same choice emerges in the period integrals. For example, in the Grassmannian we could take the end-points of the integration to be a subset of $m$ solutions of $\frac{dW}{dx} = 0$, with this subset defined by (3.5). The result would be an indefinite integral that depends upon $z_1, \ldots, z_m$, and reproduces the multi-variable potential for the Grassmannian.

We therefore see that the selection of the $G$-flux in $H^4(Y, \mathbb{Z})$, or the choice of the flux at infinity, amounts to selecting a set of critical points of the single variable potential. In the next subsection we will demonstrate how this works more explicitly by showing that the “boundaries” of the integration procedure, and in particular these critical points, correspond to special holomorphic divisors of the surfaces defined by (4.5).

It is also interesting to note that if we now consider the period integral defined by two critical points of the Grassmannian potential, $W_{m,n}(z_1, \ldots, z_m)$, then it will decompose into a difference of sums of critical values of the superpotential $W_{1,m+n-1}(x)$. Generically this will represent the sum of topological charges of a multi-soliton (and thus non-BPS) state. Thus not all pairs of ground states represent boundary conditions for BPS solitons. This fact was first discovered by considering the integrable Landau-Ginzburg models, where it was seen that one could not make a factorizable $S$-matrix involving only solitons for certain classes of integrable model. Some boundary conditions could only give rise to multi-soliton states \[19,12\]. Indeed, based on the conserved quantities of the $E_6$ model it was shown in \[12\] that for the $E_6$ integrable model, the correct soliton spectrum was exactly given by the prescription that two ground states labeled by weights $\lambda_1$ and $\lambda_2$ would give rise to a single, fundamental BPS soliton if and only if $\lambda_1 - \lambda_2$ was exactly a root. It is now very satisfying to have a simpler, and far more general string theoretic explanation of this result: A wrapped of $D4$-branes on the ADE singularity is a fundamental BPS soliton if and only if it wraps a cycle that is represented by a root of the Lie algebra.

4.4. Intersection forms and holomorphic 4-cycles

We have just seen how the Landau-Ginzburg potentials emerge from some kind of “semi-periods” of the singularity: That is, we get the potentials by making the indefinite integral of the single variable potential, and then parameterizing families of its critical points. This suggests that one should try to characterize these families of critical points and the “semi-periods” more geometrically. As we will see, this is indeed possible since
they represent non-compact homology cycles of the singularity. Ideally we would like to integrate $\Omega$ over these non-compact cycles, but such an integral will diverge. This can be fixed by compactifying the singularity. Our approach is based upon the techniques used in section 17 of [3], and we will therefore only summarize the key steps here.

The first step is to introduce a new coordinate and make the equation of the singularity homogeneous (see [3] for details). The resulting compact manifold will not be a Calabi-Yau manifold and so $\Omega$ will have some kind of problem at infinity: If one wants to preserve holomorphicity, then it will be singular. Since we wish to consider period integrals, we wish to keep $\Omega$ regular, but the cost is the loss of holomorphy in the patch at infinity. This will not affect the computation of topological charges since they only involve integrals of $\Omega$ over compact cycles of the original singularity. This means that such a compactification will only affect the Landau-Ginzburg superpotential by some additive constant.

The basic idea now is to try to write $\Omega = d\lambda$ for some 3-form, $\lambda$. This is not possible precisely because $\Omega$ is a non-trivial element of cohomology. However, if one excises all the non-trivial homology from the compactified singularity then it is possible to write $\Omega = d\lambda$. Let $C_a$ be a basis of the non-trivial 4-cycles. One can then write

$$\Omega = d\lambda + \sum_a m_a [C_a]$$

where $[C_a]$ denotes the Poincaré dual of the cycle $C_a$, and is a 4-form with delta-function support on $C_a$. The parameters, $m_a$ are determined from the integral of $\Omega$ over $C_a$: Specifically, if $M_{ab} = \mathcal{I}(C_a, C_b)$ is the intersection form of the basis of cycles, then:

$$m_a = \sum_b M_{ab}^{-1} \int_{C_b} \Omega.$$

Given any other cycle, $S$, one then has:

$$\int_S \Omega = \sum_a m_a \mathcal{I}(S, C_a),$$

where $\mathcal{I}(S, C_a)$ denotes the intersection numbers of the cycle $S$ with $C_a$.

The issue now is to find a good basis, $C_a$; and there is a particularly nice way to do this using holomorphic lines in the ADE singularity in two complex dimensions, or using holomorphic planes in four complex dimensions. The important point is that $\Omega = \frac{dz_1 dz_2 dz_3 dz_4}{z_5}$ is odd under any reflection $z_j \rightarrow -z_j$, and so any holomorphically defined dual homology cycle must similarly be odd.
In practice, given the singularity type:

\[ z_5^2 = H(z_1, z_2) - z_3^2 - z_4^2, \quad (4.14) \]

one starts by looking for complex “2-planes” of the form

\[ z_1 = a \zeta + b, \quad z_2 = c \zeta + d, \quad z_3 = \pm i z_4, \quad (4.15) \]

where \( \zeta \) is a complex variable, and \( a, b, c \) and \( d \) are constants. For generic \( a, b, c, d \) \((4.14)\) and \((4.15)\) define an irreducible rational surface that is even under \( z_5 \to -z_5 \). At special values of \( a, b, c, d \) one finds that \( H(a \zeta + b, c \zeta + d) \) is a perfect square and the rational surface degenerates to two intersecting planes: The odd 4-cycles that we seek are the differences of such pairs of 2-planes. More generally, one can seek holomorphic 2-surfaces of higher degree (see, for example, [20]). Such holomorphic surfaces form Weyl orbits whose order depends upon the degree of the surface. Here we are focussing on “planes,” or surfaces of lowest degree so as to recover the miniscule orbits. One could also go to surfaces of higher degree, and these are presumably relevant to the physics of more general fluxes at infinity.

Going back to the earlier parts of this section, one sees that the planes described above played a crucial role in the explicit evaluation of the integrals. The cycles were defined by families of contour integrals and the planes defined where these contours collapse to points, that is, the planes defined the limits of the integration. In simple terms, a 4-cycle is essentially an \( S^4 \) and its intersection points with 2-planes define the extremities, or “north” and “south” poles of the \( S^4 \). Thus the difference of two lines, \( C_a - C_b \) defines the compact cycle over which we integrate, and the fact that integration is the difference in values of the superpotential at the two extremities simply reflects the fact that the integral is given by \( m_a - m_b \) and that \( m_d \) is the value of the superpotential at the line \( C_d \).

The Grassmannian and \( SO(2n)/U(n) \) models are obtained by considering families of critical points of the simplest potentials, \( i.e. \) families of critical points for either \( U(n)/(U(n-1) \times U(1)) \) or \( SO(2n)/(SO(2n) \times U(1)) \), and then summing these simple potentials over such families. This means that the more general superpotentials must correspond to taking periods of \emph{sums} of holomorphic 2-surfaces in the singularity. The correspondence with the flux at infinity is now much more transparent: The Poincaré duals of these surfaces can be taken to be \( \delta \)-functions of their volume forms. Since the \( 6 \) Technically they will not strictly be 2-planes since \( z_5 \) is generically going to be polynomial in the other variables.
surfaces are holomorphic, their volume forms are necessarily \((2, 2)\)-forms. We thus see a very explicit correspondence between the “non-compact” \((2, 2)\)-fluxes of the singularity, and the ground states of the Landau-Ginzburg superpotential.

Having compactified the singularity, we might hope to have unfrozen the dynamics and recover some information about the dynamical Landau-Ginzburg fields. In compactifying we have made some choices about how to regularize \(\Omega\) at infinity. We expect these choices to be related to the choice of the “irrelevant” \(D\)-terms, and the unrenormalized dynamical Landau-Ginzburg variables should emerge in terms of properties that can be expressed entirely in terms of the non-compact singularity. Indeed, by examining the holomorphic 2-surfaces more explicitly for the examples above, we find a much more direct role for the Landau-Ginzburg variables.

For the \(A_n\) singularity the zeroes of \(P(z_1; a_j)\) are precisely the points where the singularity contains the planes:

\[
z_5 = \pm i z_2, \quad z_3 = \pm i z_4.
\]

For the \(D_n\) singularity the solutions of \(P(x^2; a_j) = \frac{a_0^2}{4x^2}\) define precisely the points where the singularity contains the planes:

\[
z_5 = \pm x \left( z_2 + \frac{a_0}{2x^2} \right), \quad z_3 = \pm i z_4.
\]

Indeed, more generally, the \(D_n\) superpotential is given by \(W(x, y) \equiv \int P(x^2; a_j)dx - xy^2 + a_0 y\), and reduces to (4.9) upon eliminating \(y\) via: \(\frac{\partial W}{\partial y} = 0\). The critical points of the full superpotential \(W(x, y)\) precisely define the planes:

\[
z_5 = \pm (x z_2 + y), \quad z_3 = \pm i z_4,
\]

lying in the \(D\)-type singularity.

The story is similar for the \(E_6\) and \(E_7\) singularities. The details for the \(E_7\) singularity may be found in Appendix A. For the \(E_6\), the surfaces are quadratic in \(z_5\), but linear in the other \(z_j\). Once again take \(z_3 = \pm i z_4\) and introduce \(\zeta \in \mathbb{C}\) with

\[
z_1 = x \zeta + \frac{5}{52} z + \frac{1}{48} (2 w_1 x^2 + w_1^2), \quad z_2 = \zeta - \frac{1}{8} (2 x^3 + w_1 x),
\]
The parameters of these surfaces are $x$ and $z$. When the superpotential $W(x, z)$ of (3.18) has a critical point then the foregoing defines a “plane” in the $E_6$ singularity (4.10) with:

$$z_5 = \pm \left( \zeta^2 + \frac{3}{16} x^6 + \frac{1}{8} w_1 x^4 + \frac{5}{192} w_1^2 x^2 - \frac{1}{8} w_2 x + \frac{1}{192} w_3 + \frac{15}{104} (x^2 + \frac{1}{6} w_1) z \right).$$

Indeed, the 27 critical points of $W(x, z)$ corresponding to the weights of the fundamental of $E_6$ define the celebrated 27 lines in a cubic hypersurface in $\mathbb{P}^3$.

We therefore see that the Landau-Ginzburg variables naturally emerge in the parameterization of representatives of the non-compact homology cycles. In retrospect this is rather natural: we know that the Landau-Ginzburg variables must parameterize a family of fluxes, and have non-normalizable kinetic terms in the non-compact singularity. Dual to such fluxes are non-compact homology 4-cycles, and so the Landau-Ginzburg dynamics can be converted into a description of these 4-cycles: they intersect the compact cycles of the singularity in a manner determined by the inner products of the corresponding weight and root vectors, and the dynamics is such that these supersymmetric ground states correspond holomorphic surfaces. It is therefore tempting to think of the ground-state flux as being created by a non-compact $D4$-brane that threads the singularity along one of these non-compact cycles. The solitons then intersect these non-compact branes and then combine with them to yield another non-compact $D4$-brane that threads the compact part of the singularity with different set of intersection numbers.

5. Conclusions

The $\mathcal{N} = 2$ superconformal models that arise from the non-compact ADE singularities are determined by a flux at infinity. For the minimal, or miniscule, fluxes the versal deformations of the singularity lead to a theory with a mass gap. It was argued in [8] that these particular models should be perturbations of the SLOHSS models. Here we observed that the perturbations of the singularity respect the underlying Weyl symmetry, and thus the corresponding perturbations of the Landau-Ginzburg superpotential must do the same.

We then used this to either explicitly construct, or provide a precise computational algorithm for the construction of the Landau-Ginzburg superpotentials of the SLOHSS models with the most general perturbations consistent with the Weyl symmetry. We then showed that exactly the same superpotentials and algorithms emerged from the topological data and period integrals of corresponding ADE singularities. This provides some extremely detailed checks on the results of [8]. There are several interesting by-products of our work.
which may find application in the study of Landau-Ginzburg models and in topological matter coupled to topological gravity.

There are also quite a number of interesting open questions. Our work suggests an interpretation of the Landau-Ginzburg variables in terms of moduli of non-compact cycles that thread the singularity. It would be interesting to verify these ideas more explicitly by considering nearly singular, compact Calabi-Yau manifolds in more detail.

Here we have considered only the theories with a mass gap. It would be interesting to find out exactly what perturbed conformal theories emerge for non-miniscule fluxes. It is tempting to conjecture that the flux at infinity, $\Phi$, encodes the level of the numerator current algebra of the coset model. Since versal deformation of the singularity is not supposed to yield a mass gap, the corresponding perturbed Landau-Ginzburg models will have to remain multi-critical. This is certainly possible, but there is a potentially interesting conundrum here: We know from the results in [21] that simple perturbative attempts at adding marginally relevant operators can create non-trivial higher order corrections, and indeed result in a theory with a mass gap. For example, the $\mathcal{N} = 2$ minimal models perturbed by the least relevant chiral primary results in a theory with a Chebyshev superpotential. Indeed, the such a perturbation would be a natural candidate for the $A_1$ singularity with higher flux. It would thus be interesting to see if the resulting theory really does have massless modes, or if, somehow the theory does indeed develop a Chebyshev superpotential and hence have a mass gap. It is also quite possible that the conformal field theories associated with the singularities with non-miniscule fluxes are not the various coset models at higher levels and with different denominators.

One way to approach this problem that might be particularly interesting would be to find a method of computing the elliptic genus of the conformal field theory using only the topological data of the singularity and its flux at infinity. For Landau-Ginzburg theories this is equivalent to finding the fundamental Landau-Ginzburg fields and determining their $U(1)$ charges. For more general theories, the elliptic genus gives a lot of valuable information about the partition function.

Finally, there are some interesting properties of Landau-Ginzburg solitons that may have interesting consequences for $D4$-branes more generally. Most notable, but probably least relevant is the fact that some very special perturbations lead to quantum integrable models, with factorizable $S$-matrices. This sort of property is not at all robust, and so it likely to be only a property of the field theory in the near singular limit. Integrability will very likely be spoiled in the full string theory.
More interesting is the fact that solitons have fractional fermion number \[22,23\]. In particular, for a Landau-Ginzburg soliton running between two vacua, the fermion number is given by \[23\]:

\[
f = -\frac{1}{2\pi} \Delta \text{Im} \left( \log \left( \det(\partial_i \partial_j W) \right) \right), \tag{5.1}
\]

where \(\Delta\) indicates the change in the value of the quantity between the two critical points.

Note that this quantity is, in principle, a very complicated function of the moduli and can presumably assume arbitrary (even irrational) values. However, for singularities with discrete geometric symmetries (like the Coxeter resolution), this fermion number will take simple fractional values. One might naturally wonder whether this fractional fermion number might also be an artifact of the field theory that emerges near the singularity, and that this fermion number would disappear in the full string theory. There are no global conserved currents in string theory, and so one might expect that fermion number would not persist in string theory. However, the formula (5.1) only defines the fermion number mod 1, and indeed fermion number mod 2 is a well defined concept in string theory. The fact that (5.1) follows from an index calculation gives some further hope that fractional fermion number might persist in string theory. Finally, it seems particularly likely that fractional fermion number will persist if the fractional fermion number is fixed by discrete geometric symmetries of the singularity.

If some version of fractional fermion number persists in string theory then it would be very interesting to find the geometric meaning of (5.1). As regards physical consequences: there are the obvious phases that would arise in scattering, but perhaps more interesting would be the consequences for quantization conditions under compactification, and thus for partition functions.

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Appendix A. The $E_7$ Singularity and the Deformed Coset Model

In this appendix we begin by deriving the single-variable version of the Landau-Ginzburg potential for the $E_7$ singularity closely following [16]. We then obtain the multi-variable Landau-Ginzburg potential for the deformed $E_7/(E_6 \times U(1))$ coset model using the method developed in the text. Finally we discuss the integral over 4-cycles in a non-compact 4-fold with a singularity of type $E_7$.

A.1. The single-variable $E_7$ superpotential

The $E_7$ singularity with versal deformations is described by

$$W_{E_7}(z_1, z_2, z_3) = 0,$$  \hspace{1cm} (A.1)

where

$$W_{E_7}(z_1, z_2, z_3) = z_1^3 + z_1 z_2^3 + z_3^2 - w_2 z_1^2 z_2 - w_6 z_1 z_2^2 - w_8 z_1 z_2 - w_{10} z_2^2 - w_{12} z_1 - w_{14} z_2 - w_{18},$$ \hspace{1cm} (A.2)

where $w_q$ are the deformation parameters. To obtain the single-variable potential we determine the lines for the generic $E_7$ singularity. The process is very similar to that of the $E_6$ singularity. We first make a change of variables from $(z_1, z_2)$ to $(x, y)$, where:

$$z_1 = x^2 y + \alpha(x), \quad z_2 = y + \beta(x),$$ \hspace{1cm} (A.3)

where $\alpha(x)$ and $\beta(x)$ will be fixed momentarily. With these substitutions, $W_{E_7}$ becomes a quartic in $y$. We then set $\alpha(x) = -x^6 + w_2 x^4 - 3 \beta(x) x^2$ so as to eliminate the $y^3$ term in this quartic. Eliminating the term linear in $y$ gives rise to a cubic equation for $\beta(x)$. It is convenient to introduce a function $Y(x)$, where $\beta(x) = \frac{Y(x)}{4x} + x^4$, and then the cubic for $Y(x)$ reads

$$Y^3 + 3q(x)Y - 2r(x) = 0,$$ \hspace{1cm} (A.4)

where $q(x)$ and $r(x)$ are polynomials in $x$ given by:

$$q = -28 x^{10} - \frac{44}{3} v_2 x^8 - \frac{8}{3} v_2^2 x^6 - \frac{1}{3} v_6 x^4 + \frac{2}{9} v_8 x^2 - \frac{1}{3} v_{10},$$

$$r = 148 x^{15} + 116 v_2 x^{13} + 36 v_2^2 x^{11} + (4 v_2^3 + \frac{8}{3} v_6) x^9 + \left(\frac{2}{3} v_2 v_6 - 2 v_8\right) x^7 + \left(2 v_10 - \frac{2}{3} v_2 v_8\right) x^5 + \left(\frac{1}{81} v_6^2 - \frac{2}{27} v_2 v_{12}\right) x^3 + \left(\frac{109}{8613} v_2 v_6^2 - \frac{2}{3} v_{14}\right) x.$$ \hspace{1cm} (A.5)
In writing the foregoing we have reparameterized the deformations of the $E_7$ singularity as follows:

$$w_2 = -v_2, \quad w_6 = \frac{1}{12} v_6, \quad w_8 = -\frac{1}{6} v_8, \quad w_{10} = -\frac{1}{4} v_{10},$$

$$w_{12} = \frac{1}{105} (2 v_{12} - \frac{1}{3} v_2^2), \quad w_{14} = \frac{1}{12} (2 v_{14} - \frac{109}{2871} v_2 v_0^2), \quad w_{18} = -\frac{1}{30} v_{18}. \quad (A.6)$$

Equation (A.1) now reduces to:

$$z_3^2 = -(x^2 y^4 + A(x) y^2 + B(x)). \quad (A.7)$$

The “lines” are determined by requiring that the right-hand side of (A.7) be a perfect square, which means that the discriminant, $\Delta$, vanishes:

$$0 = \Delta = A^2 - 4 x^2 B$$

$$= \frac{1}{16} (p_1 Y^2 + p_2 Y + p_3 - \frac{16}{9} v_{18} x^2). \quad (A.8)$$

The polynomials $p_1$, $p_2$ and $p_3$ are

$$p_1 = 1596 x^{10} + 88 v_2^2 x^6 + 7 v_6 x^4 + 660 v_2 x^8 - 2 v_8 x^2 - v_{10},$$

$$p_2 = 16872 x^{15} + 11368 v_2 x^{13} + 2952 v_2^2 x^{11} + (176 v_6 + 264 v_2^3) x^9$$

$$+ (-100 v_8 + \frac{100}{3} v_2 v_0) x^7 + (-\frac{68}{3} v_2 v_8 + 68 v_{10}) x^5$$

$$+ (\frac{2}{5} v_8 - \frac{4}{3} v_{12}) x^3 + (\frac{218}{3613} v_2 v_0^2 - \frac{4}{3} v_{14}) x,$$

$$p_3 = 44560 x^{20} + 41568 v_2 x^{18} + 16080 v_2^2 x^{16} + (2880 v_2^3 + \frac{2216}{3} v_0) x^{14}$$

$$+ (312 v_2 v_6 + 192 v_2^2 - \frac{1552}{3} v_8) x^{12} + (32 v_2^2 v_6 - 40 v_2 v_{10} - \frac{64}{3} v_{12} + \frac{11}{3} v_0^2) x^8$$

$$+ (-\frac{416}{3} v_{14} - 16 v_2 v_{10} - \frac{4}{9} v_2 v_8 - \frac{32}{9} v_2 v_{12} + \frac{27776}{3613} v_2 v_0^2) x^6$$

$$+ (\frac{3488}{3613} v_2^2 v_0^2 + \frac{4}{3} v_8^2 - \frac{64}{3} v_2 v_{14} - \frac{2}{3} v_6 v_{10}) x^4 + \frac{4}{3} v_8 v_{10} x^2 + v_{10}. \quad (A.9)$$

The polynomials $q, r, p_1, p_2$ and $p_3$ play an important role in our calculations. In particular they obey remarkable identities

$$x \frac{dq}{dx} = \frac{1}{2} q - \frac{1}{6} p_1, \quad x \frac{dr}{dx} = \frac{3}{4} r + \frac{1}{8} p_2. \quad (A.10)$$

The condition $\Delta = 0$ yields a single-variable version of the pre-potential, $\tau_{E_7}(x; v_i)$, for $E_7$,

$$\tau_{E_7}(x; v_i) = -v_{18} + \frac{9}{16 x^2} (p_1 Y^2 + p_2 Y + p_3), \quad (A.11)$$

33
where $Y$, a root of the cubic (A.4), is given by

$$Y = \{s_+ + s_-, \omega s_+ + \omega^2 s_-, \omega^2 s_+ + \omega s_-\}$$

with $\omega = e^{2\pi i/3}$ and $s_\pm = (r \pm \sqrt{q^3 + r^2})^{1/3}$. Using the relation between the roots and the coefficients of the cubic equation it is easy to verify from (A.11) that

$$(\tau_{E_7} + v_{18})^3 + A_2(x)(\tau_{E_7} + v_{18})^2 + A_1(x)(\tau_{E_7} + v_{18}) + A_0(x) = 0,$$  \hspace{1cm} (A.12)

where

$$A_2 = \frac{9}{16 x^2} (6 q p_1 - 3 p_2),$$
$$A_1 = (\frac{9}{16 x^2})^2 (9 q^2 p_1^2 - 6 r p_1 p_2 - 12 q p_1 p_3 + 3 q p_2^2 + 3 p_3^2),$$
$$A_0 = - (\frac{9}{16 x^2})^3 (4 r^2 p_1^3 + 6 q r p_1^2 p_2 + 9 q^2 p_1^2 p_3 - 6 r p_1 p_2 p_3 - 6 q p_1 p_2^2 + 2 r p_2^3 + 3 q p_2^2 p_3 + p_3^3).$$

The single-variable pre-potential, $\tau_{E_7}(x; v_i)$, can be integrated with respect to $x$ to yield the single variable potential, $W_{E_7}(x; v_i)$. This integral has two parts:

$$W_{E_7}(x; v_i) = \int dx \tau_{E_7}(x; v_i) = I_1 + I_2,$$

where

$$I_1 = \int dx \left( - v_{18} + \frac{9}{16 x^2} p_3 \right), \quad I_2 = \frac{9}{16} \int dx \frac{1}{x^2} (p_1 Y^2 + p_2 Y).$$

The integral $I_1$ is easily performed. We only note that since $p_3$ has no linear term in $x$ there appears no logarithm of $x$. To evaluate $I_2$, on the other hand, we first use (A.10) to derive

$$I_2 = \frac{9}{16} \int dx \left( \frac{d}{dx} \left( - \frac{6 q}{x} Y^2 + \frac{8 r}{x} Y \right) + (3 q Y - 2 r) \left( \frac{4}{x} \frac{dY}{dx} - \frac{Y}{x^2} \right) \right).$$

By virtue of (A.4) the second term is reduced to

$$(3 q Y - 2 r) \left( \frac{4}{x} \frac{dY}{dx} - \frac{Y}{x^2} \right) = - \frac{d}{dx} \left( \frac{Y^4}{x} \right).$$

Hence we find

$$\int dx \tau_{E_7}(x; v_i) = - \frac{27}{16 x} (q Y^2 - 2 r Y) + \int dx \left( - v_{18} + \frac{9}{16 x^2} p_3 \right). \hspace{1cm} (A.13)$$
Before moving on to the multi-variable potential, we wish to describe how this single-variable potential is related to the characteristic polynomial of the 56 of $E_7$. Generally the characteristic polynomial of a representation $\mathcal{R}$ of the Lie algebra $G$ is defined by

$$P^\mathcal{R}_G(x; V_i) = \det(x - \vec{a} \cdot \vec{H})$$

which is of degree $\dim \mathcal{R}$ in $x$. Here $\vec{a}$ is an $r$-dimensional vector in the Cartan subspace spanned by $\vec{H}$ and the $V_i$ ($i = 1, \ldots, r = \text{rank} G$) are the Casimirs built out of $\vec{a}$ among which $V_r$ is the top Casimir whose degree equals the Coxeter number $h$ of $G$. Let $\vec{\lambda}$ be the weight of $\mathcal{R}$, then we have

$$P^\mathcal{R}_G(x; V_i) = \prod_{\vec{\lambda} \in \mathcal{R}} (x - \vec{a} \cdot \vec{\lambda}).$$

For the 56 of $E_7$ we obtain

$$P^\text{56}_{E_7}(x; v_i) = x^{56} + 12 v_2 x^{54} + 66 v_2^2 x^{52} + (2 v_6 + 220 v_2^3) x^{50}
+ (10 v_8 + 495 v_2^4 + 20 v_2 v_6) x^{48}
+ (-126 v_10 + 792 v_2^5 + 90 v_2^2 v_6 + 84 v_2 v_8) x^{46}
+ (10 v_12 + 924 v_2^6 + 240 v_2^3 v_6 + \frac{934}{3} v_2^2 v_8 + 86 v_2 v_{10}) x^{44} + \cdots, \quad (A.14)$$

where the Casimirs have been set to $V_i = v_i$. Note that this has the degree 3 in the top Casimir $v_{18}$, which is analogous to (A.12). In fact, one can show explicitly that

$$P^\text{56}_{E_7}(x; v_i) = -\frac{x^2}{36} \left(v_{18}^3 + A_2(x) v_{18}^2 + A_1(x) v_{18} + A_0(x)\right).$$

Hence, (A.12) for $\tau_{E_7}$ may be expressed as

$$P^\text{56}_{E_7}(x; v_2, \ldots, v_{14}, v_{18} + \tau_{E_7}) = 0.$$ 

This also makes it clear that each of the lines in the $E_7$ singularity we have constructed is in correspondence with a weight of the 56 of $E_7$.

More generally, for any $ADE$ group, the single-variable version of the Landau-Ginzburg potential can be obtained from $\int \tau(x) dx$, where $\tau(x)$ is obtained by solving

$$P^\mathcal{R}_G(x; v_1, \ldots, v_r + \tau(x)) = 0.$$
Thus the pre-potential, $\tau$, depends upon the representation, $\mathcal{R}$, and takes the form:

$$\tau_{\mathcal{R}}(x; v_i) = -v_r + F_{\mathcal{R}}(x; v_1, \ldots, v_{r-1}),$$

where $F_{\mathcal{R}}$ is some irrational, or implicit function of $x$ whose expansion at $x = \infty$ starts with the $x^h$ term and does not carry the pole term $\frac{1}{x}$ as can be seen by degree counting.

A.2. The Landau-Ginzburg potential for the deformed $E_7/(E_6 \times U(1))$ coset model

The calculation is also parallel to that for the $E_6/(SO(10) \times U(1))$ model. We start with the decomposition of the seven Casimirs, $V_j$, of $E_7$ into the Casimirs $x_j$ of $E_6 \times U(1)$. Under $E_7 \supset E_6 \times U(1)$, the 56 of $E_7$ branches as

$$56 = 27_1 + 27_{-1} + 13 + 1_{-3}.$$ 

Accordingly the characteristic polynomial for the 56 of $E_7$ is factorized

$$P_{E_7}^{56}(x) = P_{E_6}^{27}(x + 1) \cdot P_{E_6}^{27}(x - 1) \cdot (x + 3x_1) \cdot (x - 3x_1),$$

from which we read off the Casimir decomposition as follows:

$$V_2(x_j) = x_2 - 3x_1^2,$$

$$V_6(x_j) = -60x_5x_1 + x_6 - 12x_2^2x_1^2 - 24x_2x_1^4 - 72x_1^6,$$

$$V_8(x_j) = 18x_2x_5x_1 - 24x_2x_1^6 + x_8 - 12x_5x_1^3 - 54x_1^8 + 3x_6x_1^2,$$

$$V_{10}(x_j) = x_6x_1^4 - 4x_2x_5x_1^3 + 12x_2x_1^8 + 4x_2x_1^6 + x_5^2 + 12x_1^{10}$$

$$- 12x_5x_1^5 + 4x_9x_1 - 2x_8x_1^2,$$

$$V_{12}(x_j) = -12x_2^3x_1^6 + 12x_6x_1^6 + 102x_2x_5x_1^5 + 36x_2x_1^10 + 114x_2x_1^8 + x_{12} + \frac{1}{6}x_6^2$$

$$+ 792x_5x_1^7 - 35x_2x_6x_1^4 + 24x_2x_1^4 + 270x_1^{12} - 135x_8x_1^4 + 132x_2x_5x_1^3$$

$$+ 222x_5^3x_1 - 144x_9x_1^3 - 18x_2x_8x_1^2 - 18x_2x_9x_1^3 - 4x_2x_6x_1^2 - 2x_5x_6x_1.$$ 

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7 Although $\tau_{\mathcal{R}}(x)$ depends on the representation, it turns out that the description of $ADE$ topological matter theories in terms of $\tau_{\mathcal{R}}$ is independent of $\mathcal{R}$, but depends only on the singularity type $G$. This is deeply related with the universality of the special Prym variety in the theory of spectral curves of periodic Toda lattice, which plays a fundamental role in formulating Seiberg-Witten solution of four-dimensional $\mathcal{N} = 2$ Yang-Mills theory.
\[ V_{14}(x_j) = \frac{5402}{319} x_2^3 x_1^4 + \frac{2616}{319} x_6 x_1^8 - 18 x_2 x_5 x_1^7 - \frac{10338}{319} x_2 x_1^{12} - \frac{11358}{319} x_2^2 x_1^4 \]
\[ - \frac{164616}{319} x_5 x_1^9 + \frac{109}{5742} x_2^2 x_6^2 x_2 + x_5 x_9 - \frac{436}{957} x_2 x_6 x_1^2 - \frac{66357}{319} x_5 x_1^4 + x_1 x_2^2 \]
\[ + x_2 x_6 x_1^4 - \frac{76950}{319} x_1^4 - \frac{2180}{957} x_2 x_5 x_6 x_1 - 2 x_5 x_8 x_1 + \frac{872}{319} x_2^2 x_1^6 \]
\[ - \frac{10634}{319} x_2^2 x_5 x_1^9 - 36 x_9 x_1^4 + 9 x_8 x_1^6 + \frac{17972}{319} x_2 x_5 x_1^2 - 14 x_2 x_9 x_1^3 \]
\[ + \frac{8720}{319} x_2 x_5 x_1^9 + x_2 x_8 x_1^4 - \frac{109}{1914} x_2 x_6 x_1^2 + \frac{436}{957} x_2 x_6 x_1^4 \]
\[ + \frac{1542}{319} x_6 x_1^9 + \frac{872}{319} x_2^2 x_1^4, \]
\[ V_{18}(x_j) = x_0^2 + 252 x_2 x_1^6 + 396 x_2 x_1^{16} + 288 x_2^2 x_1^{14} + 144 x_5 x_1^{13} + 144 x_2 x_1^{12} + 288 x_2 x_5 x_1^{11} \]
\[ + 24 x_2 x_6 x_1^{10} + 216 x_2^2 x_5 x_1^9 + 21 x_2 x_6 x_1^8 + 114 x_2 x_8 x_1^7 + 72 x_3 x_5 x_1^7 \]
\[ - 24 x_5 x_6 x_1^7 + 24 x_9 x_1^7 - 12 x_6 x_1^{12} + 36 x_4 x_1^{10} + 168 x_8 x_1^{10} - 156 x_9 x_1^9 \]
\[ + 225 x_2 x_1^{12} + 4 x_6 x_1^6 + 36 x_1 x_2 x_1^6 + 108 x_3 x_1^2 + 2 x_1 x_3 x_1^2 + 12 x_5 x_2 x_1^3 \]
\[ - 9 x_5 x_6 x_1^5 - 2 x_1 x_2 x_1^5 + 2 x_8 x_1^5 + 12 x_2 x_8 x_1^5 + 180 x_2 x_5 x_1^6 \]
\[ + 84 x_5 x_8 x_1^5 + 12 x_2 x_9 x_1^5 - 4 x_6 x_1^4 + 102 x_5 x_9 x_1^4 + 12 x_2 x_5 x_6 x_1^5 \]
\[ + 36 x_2^2 x_5 x_1^4 + 10 x_1 x_2 x_1^4 - 4 x_6 x_9 x_1^3 + 12 x_2 x_5 x_9 x_1^2. \]

Set \( V_i = v_i \) with the \( v_i \) being arbitrary parameters. Since \( V_j \) is linear in \( x_j \) for \( j = 2, 6, 8, 12 \), we can express as \( x_j = x_j(x_1, x_5, x_9; v_\ell) \) by solving \( V_j = v_j \). Substituting these \( x_j \) into \( V_k \) with \( k = 10, 14, 18 \) we are left with three relations

\[ V_k(x, y, z; v_\ell) - v_k = 0, \quad k = 10, 14, 18. \quad (A.15) \]

where, for convenience, we have set \( x = 3 x_1, y = x_5, z = x_9 \). As we found for \( E_6 \), these relations are integrable. In order to construct the superpotential, particular linear combinations of the relations \( (A.15) \) must be chosen so that they can be integrated to a superpotential. These linear combinations are determined using the integrability conditions: \( \frac{\partial}{\partial x} \left( \frac{\partial W}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial W}{\partial x} \right) \), etc.. We then find that there is a superpotential, \( W(x, y, z) \), with:

\[ \frac{\partial W}{\partial z} = \frac{3}{2} (V_{10} - v_{10}), \quad \frac{\partial W}{\partial y} = 3 (V_{14} - v_{14}) + \left( \frac{43}{9} x^4 + 4 v_2 x^2 \right) (V_{10} - v_{10}), \]
\[ \frac{\partial W}{\partial x} = (V_{18} - v_{18}) + \left( \frac{26}{27} x^4 + \frac{4}{3} v_2 x^2 \right) (V_{14} - v_{14}) \]
\[ + \left( \frac{173}{243} x^8 + \frac{172}{9} x^3 y - v_8 + \frac{16}{9} v_2 x^4 + \frac{508}{243} v_2 x^6 + 8 v_2 x y + \frac{2}{3} v_6 x^2 \right) (V_{10} - v_{10}), \]

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The explicit form of the superpotential reads:

\[
W = \frac{1016644}{817887699} x^{10} + \frac{33326}{177147} x^{14} y + \frac{266}{6561} x^{10} z + \frac{16850}{2187} x^9 y^2 + \frac{80}{27} x^5 y z + \frac{124}{9} x^4 y^3 \\
+ x z^2 + \frac{3}{2} y^2 z + \frac{753964}{81310473} v_2 x^{17} + \frac{45392}{1594323} v_2^2 x^{15} + \left( \frac{18566}{2302911} v_6 + \frac{96064}{2302911} v_2^3 \right) x^{13} \\
+ \frac{48826}{59049} v_2 x^{12} y + \left( \frac{6532}{216513} v_2 v_6 + \frac{1816}{7217} v_4^2 - \frac{4640}{216513} v_8 \right) x^{11} + \frac{962}{729} v_2^2 x^{10} y \\
+ \left( -\frac{173}{2187} v_{10} + \frac{239}{6561} v_2 v_6 - \frac{178}{2187} v_2 v_8 \right) x^9 + \left( \frac{464}{729} v_6 + \frac{418}{729} v_2^3 \right) y + \frac{110}{729} v_2 z \right) x^8 \\
+ \left( \frac{904}{81} v_2 y^2 - \frac{508}{1701} v_2 v_10 - \frac{508}{5103} v_2^2 v_8 - \frac{2}{1701} v_2 v_6 + \frac{23}{1701} v_2^2 \right) x^7 \\
+ \left( \frac{109}{243} v_2 v_6 - \frac{145}{81} v_8 \right) y + \frac{44}{243} v_2^2 z \right) x^6 \\
+ \left( \frac{140}{27} v_2^2 y^2 - \frac{26}{135} v_14 + \frac{1736}{9575585} v_2 v_6^2 - \frac{2}{405} v_2 v_{12} - \frac{16}{45} v_2^2 v_10 - \frac{29}{405} v_8 v_6 \right) x^5 \\
+ \left( \frac{41}{27} v_2 v_8 - \frac{43}{27} v_10 \right) y + \frac{3}{8} v_6 z \right) x^4 \\
+ \left( \frac{11}{9} v_6 y^2 + \frac{16}{9} v_2 y z - \frac{2}{9} v_6 v_{10} + \frac{1}{9} v_2^3 - \frac{4}{9} v_2 v_{14} + \frac{218}{255839} v_2^2 v_6^2 \right) x^3 \\
+ \left( 4 v_2 y^3 + \left( -\frac{1}{3} v_{12} - 4 v_2 v_{10} + \frac{1}{18} v_2^3 \right) y - \frac{1}{3} v_8 z \right) x^2 + \left( -v_8 y^2 + v_8 v_{10} - v_{14} \right) x \\
+ (-3 v_{14} + \frac{109}{1914} v_2 v_6^2 - \frac{2}{9} v_10 \right) y + \frac{1}{3} v_8 z .
\]

(A.16)

Putting \( v_i = 0 \) and making a change of variables \( x = X_1, y = 2 \left( \frac{2791}{19} \right)^{1/4} X_5 + \frac{416}{81} X_5, z = 6 \left( \frac{2791}{19} \right)^{1/2} X_9 - \frac{593188}{6561} X_9 - \frac{496}{27} \left( \frac{2791}{19} \right)^{1/4} X_1 X_5 \), we obtain the superpotential at criticality

\[
W = \frac{100476}{19} \left( X_1^{19} + X_1 X_9^2 + X_2^2 X_9 + 37 \left( \frac{19}{2791} \right)^{3/4} X_1^{14} X_5 - 21 \left( \frac{19}{2791} \right)^{1/2} X_1^{10} X_9 \right)
\]

which agrees with the result of [12].

The multi-variable superpotential \( W(x,y,z) \) does indeed reduce to the single-variable potential, \( W_{E_7}(x) \) of (A.13) by eliminating \( y \) and \( z \) from \( W \) with the aid of the equations of motion. First, solving \( \frac{\partial W}{\partial z} = 0 \) we find:

\[
z_{cl} = -\frac{1}{x} \left( \frac{133}{6561} x^{10} + \frac{40}{27} x^5 y + \frac{3}{4} y^2 + \frac{55}{729} v_2 x^8 + \frac{22}{243} v_2^2 x^6 + \frac{7}{108} v_6 x^4 \\
+ \frac{8}{9} v_2 x^3 y - \frac{1}{9} v_8 x^2 - \frac{3}{4} v_10 \right).
\]

Substituting this into \( \frac{\partial W}{\partial y} \) and letting \( y = Y + \frac{416}{81} x^5 + \frac{32}{27} v_2 x^3 \) lead to

\[
\left[ \frac{\partial W}{\partial y} \right]_{z=z_{cl}} = -\frac{9}{4} x (Y^3 + 3qY - 2r).
\]

Thus \( \frac{\partial W}{\partial y} = 0 \) is equivalent to the cubic equation (A.4). After some algebra we finally arrive at:

\[
\left[ \frac{\partial W}{\partial x} \right]_{z=z_{cl}, y=y_{cl}} = \tau_{E_7}(x; v_i).
\]

Therefore,

\[
\int dx \tau_{E_7}(x; v_i) = W(x, y_{cl}(x), z_{cl}(x); v_i).
\]

(A.17)
A.3. Integrating over 4-cycles in the $E_7$ singularity

We consider a non-compact 4-fold defined by (4.5) of type $E_7$, where $H(z_1, z_2) = W_{E_7}(z_1, z_2, z_3 = 0)$, and evaluate the period integral (4.6). As for $E_6$, after the change of variables (A.3), $H(z_1, z_2)$ takes the form of (A.7), and hence the integral (4.7) becomes

$$\int dx \oint dy (2xy + \alpha'(x) - x^2 \beta'(x)) \sqrt{x^2 y^4 + A(x) y^2 + B(x)}.$$

Performing the integral over $y$ along a contour at large $|y|$ we obtain

$$\int dx \left( \frac{1}{4x} (A(x)^2 - B(x)) \right) = -\frac{1}{36} \int dx \tau_{E_7}(x; v_i),$$

where $\tau_{E_7}(x)$ is given by (A.11). The remaining integral has already been evaluated. The result is (A.13) and (A.17).

In section 4.4 we have seen how holomorphic 2-surfaces of the form (4.15) lying in the $A_n$, $D_n$ and $E_6$ singularities are determined by the critical points of the deformed superpotentials. Turning to $E_7$ we may take:

$$z_1 = x^2 \zeta + \frac{3}{133} \left( 1787 xy + 243 \frac{y^2}{x^4} + 324 \frac{z}{x^3} \right) + \frac{1}{133} \left( \frac{527}{9} v_2 x^4 + 88 v_2^2 x^2 + 63 v_6 + 864 v_2 \frac{y}{x} - 162 \frac{v_8}{x^2} - 729 \frac{v_{10}}{x^4} \right), \quad (A.18)$$

$$z_2 = \zeta - \frac{23}{81} x^4 + \frac{1}{4} \frac{y}{x} - \frac{8}{27} v_2 x^2,$$

where $\zeta \in \mathbb{C}$ and $(x, y, z)$ are parameters. The superpotential $W(x, y, z)$ of (A.16) for the $E_7/(E_6 \times U(1))$ model has 56 critical points corresponding to the 56 of $E_7$. At a critical point, (A.18) and $z_3 = \pm iz_4$ describe the holomorphic 2-surfaces, i.e. $H(z_1, z_2)$ with (A.18) becomes a perfect square, yielding

$$z_5 = \pm \left( x \zeta^2 - \frac{869}{26344} x^9 - \frac{19}{27} x^4 y + \frac{1}{4} z - \frac{211}{2916} v_2 x^7 - \frac{1}{18} v_2^2 x^5 - \frac{11}{432} v_6 x^3 - \frac{1}{3} v_2 x^2 y + \frac{1}{24} v_8 x - \frac{1}{16} \frac{v_{10}}{x} \right).$$

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References

[1] E. Martinec, *Algebraic Geometry and Effective Lagrangians*, Phys. Lett. **217B** (1989) 431;
C. Vafa and N.P. Warner, *Catastrophes and the Classification of Conformal Theories*, Phys. Lett. **218B** (1989) 51.
[2] E. Witten, *Topological Quantum Field Theory*, Comm. Math. Phys. **117** (1988) 353;
T. Eguchi and S.-K. Yang, *N=2 Superconformal Models As Topological Field Theories*, Mod. Phys. Let. **A5** (1990) 1693.
[3] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Topological Strings in D < 1*, Nucl. Phys. **B352** (1991) 59;
K. Li, *Recursion Relations in Topological Gravity with Minimal Matter*, Nucl. Phys. **B354** (1991) 725;
T. Eguchi, Y. Yamada and S.-K. Yang, *Topological field theories and the period integrals*, Mod. Phys. Let. **A5** (1993) 1627, hep-th/9304121;
T. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, *Topological Strings, Flat Coordinates and Gravitational Descendants*, Phys. Lett. **305B** (1993) 235, hep-th/9302048.
[4] T. Eguchi and S.-K. Yang, *A New Description of the E6 Singularity*, Phys. Lett. **394B** (1997) 315; hep-th/9612086.
[5] N. Seiberg and E. Witten, *Electric-Magnetic Duality, Monopole Condensation, and Confinement in N=2 Supersymmetric Yang-Mills Theory*, Nucl. Phys. **B426** (1994) 19, hep-th/9407087.
[6] N. Seiberg and E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD*, Nucl. Phys. **B426** (1994) 19, Nucl. Phys. **B431** (1994) 484, hep-th/9408099.
[7] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Non-perturbative Results on the Point Particle Limit of N=2 Heterotic String Compactifications*, Nucl. Phys. **B459** (1996) 537, hep-th/9508155;
A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner *Self-Dual Strings and N=2 Supersymmetric Field Theory*, Nucl. Phys. **B477** (1996) 746, hep-th/9604034.
[8] S. Gukov, C. Vafa and E. Witten, *CFT’s From Calabi-Yau Four-folds*, Nucl. Phys. **B584** (2000) 69; hep-th/9906070.
[9] Y. Kazama and H. Suzuki, *Characterization of N=2 Superconformal Models Generated by Coset Space Method*, Phys. Lett. **216B** (1989) 112; *New N=2 Superconformal Field Theories and Superstring Compactification* Nucl. Phys. **B321** (1989) 232.
[10] A.N. Schellekens and N. P. Warner, *Conformal Subalgebras of Kac-Moody Algebras*, Phys. Rev. **D34** (1986) 3092.
[11] W. Lerche, C. Vafa and N. P. Warner, *Chiral Rings in N = 2 Superconformal Theories*, Nucl. Phys. **B324** (1989) 427.
[12] W. Lerche and N. P. Warner, Polytopes and Solitons in Integrable, N = 2 Supersymmetric Landau-Ginzburg Theories, Nucl. Phys. B358 (1991) 571.

[13] D. Nemeschansky and N. P. Warner, Refining the Elliptic Genus, Phys. Lett. 329B (1994) 53; hep-th/9403047.

[14] W. Lerche and A. Sevrin, On the Landau-Ginzburg Realization of Topological Gravities, Nucl. Phys. B428 (1994) 259; hep-th/9403183.

[15] E. Martinec and N.P. Warner, Integrable systems and supersymmetric gauge theory, Nucl. Phys. B459 (1996) 97, hep-th/9509161.

[16] W. Lerche and N. P. Warner, Exceptional SW geometry from ALE fibrations, Phys. Lett. 423B (1998) 79; hep-th/9608183.

[17] K. Ito and S.-K. Yang, Flat Coordinates, Topological Landau-Ginzburg Models and the Seiberg-Witten Period Integrals, Phys. Lett. 415B (1997) 45; hep-th/9708017; A-D-E Singularity and Prepotentials in N=2 Supersymmetric Yang-Mills Theory, Int. J. Mod. Phys. A13 (1998) 5373; hep-th/9712018.

[18] K. Becker and M. Becker, M-Theory on Eight-Manifolds, Nucl. Phys. B477 (1996) 155; hep-th/9605053.

[19] P. Fendley, S. D. Mathur, C. Vafa and N. P. Warner, Integrable Deformations and Scattering Matrices for the N=2 Supersymmetric Discrete Series, Phys. Lett. 243B (1990) 257;
A. LeClair, D. Nemeschansky and N. P. Warner, S-matrices for Perturbed N=2 Superconformal Field Theory from Quantum Groups, Nucl. Phys. B390 (1993) 653, hep-th/9206041.

[20] J.A. Minahan and D. Nemeschansky, Superconformal Fixed Points with E_n Global Symmetry, Nucl. Phys. B489 (1997) 24, hep-th/9610076;
J. A. Minahan, D. Nemeschansky and N. P. Warner, Investigating the BPS spectrum of Non-Critical E_n Strings, Nucl. Phys. B508 (1997) 64, hep-th/9705237.

[21] R. Dijkgraaf, E. Verlinde and H. Verlinde, Topological Strings in D < 1, Nucl. Phys. B352 (1991) 59.

[22] R. Jackiw and C. Rebbi, Solitons with Fermion Number 1/2, Phys. Rev. D13 (1976) 3398;
J. Goldstone and F. Wilczek, Fractional Quantum Numbers on Solitons, Phys. Rev. Lett. 47 (1981) 986.

[23] P. Fendley and K. Intriligator, Scattering and Thermodynamics of Fractionally-Charged Supersymmetric Solitons, Nucl. Phys. B372 (1992) 533; hep-th/9111014;
Scattering and Thermodynamics of Integrable N = 2 Theories, Nucl. Phys. B380 (1992) 265; hep-th/9202011.