ON UNITARY INVARIANTS OF QUOTIENT HILBERT MODULES ALONG SMOOTH COMPLEX ANALYTIC SETS

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Abstract. Let $\Omega \subset \mathbb{C}^m$ be an open, connected and bounded set and $\mathcal{A}(\Omega)$ be a function algebra of holomorphic functions on $\Omega$. In this article, we study quotient Hilbert modules obtained from submodules, consisting of functions in $\mathcal{M}$ vanishing to order $k$ along a smooth irreducible complex analytic set $Z \subset \Omega$ of codimension at least 2, of a quasi-free Hilbert module, $\mathcal{M}$. Our motive is to investigate unitary invariants of such quotient modules. We completely determine unitary equivalence of aforementioned quotient modules and relate it to geometric invariants of Hermitian holomorphic vector bundles. Then, as an application, we characterize unitary equivalence classes of weighted Bergman modules over $\mathcal{A}(D^m)$ in terms of those of quotient modules arising from the submodules of functions vanishing to order 2 along the diagonal in $D^m$.

1. Introduction

The basic problem alluded to the title is as follows: Given a Hilbert module $\mathcal{M}$ and a submodule $\mathcal{M}_0$ over the algebra of holomorphic functions $\mathcal{A}(\Omega)$ on a bounded domain $\Omega$ in $\mathbb{C}^m$, satisfying the exact sequence

$$0 \to \mathcal{M}_0 \xrightarrow{i} \mathcal{M} \xrightarrow{\pi} \mathcal{M}_q \to 0,$$

where $i$ is the inclusion map, $\pi$ is the quotient map and $\mathcal{M}_q$ is the quotient module $\mathcal{M} \ominus \mathcal{M}_0$, is it possible to determine $\mathcal{M}_q$ in terms of $\mathcal{M}$ and $\mathcal{M}_0$? One can make this general question more precise by asking if one can assign some computable invariants on $\mathcal{M}_q$ in terms of $\mathcal{M}$ and the submodule $\mathcal{M}_0$.

For a quasi-free Hilbert module $\mathcal{M}$ [9, Section 2], [10, Page 3] over $\mathcal{A}(\Omega)$, the quotient module $\mathcal{M}_q$ obtained from the submodule $\mathcal{M}_0$, where $\mathcal{M}_0$ is the maximal set of functions in $\mathcal{M}$ vanishing along a smooth hypersurface in $\Omega$, was first studied by R.G. Douglas and G. Misra in [7]. In fact, they considered a quasi-free Hilbert module $\mathcal{M}$ of rank 1 and described a geometric invariant of the quotient module $\mathcal{M}_q$, namely, the fundamental class of the variety $Z$ [16, Page 61] and they described the fundamental class $[Z]$ in terms of the curvatures of the line bundles (Remark 2.2) obtained from $\mathcal{M}$ and $\mathcal{M}_0$. Later in the paper [2], this result was extended to quotient modules corresponding to the submodules consisting of complex valued functions in $\mathcal{A}(\Omega)$ vanishing along a complex algebraic variety of complete intersection of finitely many smooth hypersurfaces. It was the paper [12] where quotient modules arising from submodules, $\mathcal{M}_0^k$, of functions in $\mathcal{A}(\Omega)$ vanishing on a smooth complex hypersurface of order $k \geq 2$. The module of jets corresponding to a Hilbert module was introduced by means of the jet construction [12, Page 372] and was showed that the quotient module considered there could be thought of as the module of jets restricted to the hypersurface. Thus, in the first half of [12] a model for the quotient module obtained from submodules $\mathcal{M}_0^k$ was provied while the later half was devoted in finding geometric invariants.

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of the quotient modules in terms of the fundamental class of the hypersurface generalizing the results of [7]. In [8] a complete set of unitary invariants for quotient modules with \( k = 2 \) was determined and in a subsequent paper [11] they described complete unitary invariants for quotient modules with an arbitrary \( k \). It was shown in [11] that two quotient modules \( Q := \mathcal{M} \ominus \mathcal{M}_0 \) and \( \tilde{Q} := \tilde{\mathcal{M}} \ominus \tilde{\mathcal{M}}_0 \) are unitarily equivalent if and only if the line bundles \( E_\mathcal{M} \) and \( E_{\tilde{\mathcal{M}}} \) arising from \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), respectively, are isomorphic in certain sense [11, Definition 4.2]. Moreover, for \( k = 2 \), a complete set of unitary invariants for quotient modules were obtained which are the tangential and transverse components of the curvature of the line bundle \( E_\mathcal{M} \) relative to hypersurface \( Z \) and the second fundamental form for the inclusion \( E_\mathcal{M} \subset J^{(2)}_1 \mathcal{M} \) where \( J^{(2)}_1 \mathcal{M} \) is the second order jet bundle of \( E_\mathcal{M} \) relative to \( Z \) [11, Section 3]. More recently, results in the paper [11] have been generalized to the quotient modules obtained from submodules of vector valued holomorphic functions on \( \Omega \) vanishing along a smooth complex hypersurface in \( \Omega \) by L. Chen and R.G. Douglas in [3]. So to this extent it is natural to consider the case where quotient modules are obtained from submodules of vector valued holomorphic functions on \( \Omega \subset \mathbb{C}^n \) vanishing of order \( k \geq 2 \) on a smooth complex analytic set of codimension at least 2.

In this article, we intend to study such quotient modules to obtain a canonical model for them. We then make use of these canonical models to describe the complete set of unitary invariants of those quotient modules. In order to accomplish our goal we consider a quasi-free Hilbert module \( \mathcal{M} \) (Section 2) over \( \mathcal{A}(\Omega) \) consisting of vector valued holomorphic functions on \( \Omega \) with the module action obtained by point wise multiplication and go on to describe submodules \( \mathcal{M}_0 \) of interest in Section 3. We first define the order of vanishing of a vector valued holomorphic function along a connected complex submanifold \( Z \) of arbitrary codimension which is the key ingredient of the definition of \( \mathcal{M}_0 \). Since the definition of the order of vanishing is not canonical it becomes difficult to calculate the same for a given holomorphic function. However, we provide an equivalent condition to define the order of vanishing which is easy to compute. We then, following the technique introduced in the paper [12], describe the jet construction on \( \mathcal{M} \) relative to the submanifold \( Z \) to identify the quotient module to the restriction of a Hilbert module \( J(\mathcal{M}) \) (we refer the readers to (4.3) for definition) to \( Z \).

The identification, mentioned above, enables us to associate a natural geometric object to the quotient module, namely, the \( k \)-th order jet bundle relative to \( Z \) (Section 5) of the vector bundle associated to the module \( \mathcal{M} \). Thus, we pass to the geometric counter part of our study of quotient modules to be able to provide some geometric invariants for them. These jet bundles are canonically associated to the module of jets \( J(\mathcal{M}) \) of the module \( \mathcal{M} \). Then using the technique of normalised frame [17, Lemma 2.3] of a Hermitian holomorphic vector bundle we successfully describe a complete set of unitary invariants for quotient modules. Since the jet bundles of our interest are holomorphically trivial (that is, they possess a global holomorphic frame) we always have a bundle isomorphism between any two of them which does not depend on the base manifold. But it is, a priori, not true that such an isomorphism also preserves the Hermitian metric of the jet bundle mentioned above. In our case, while the jet bundles are associated to equivalent quotient modules, we do have isometric bundle isomorphism between corresponding jet bundles which does not depend on the base manifold. More precisely, we show that two such quotient modules are unitarily equivalent if and only if there exists a constant isometric jet bundle isomorphism between the corresponding jet bundles restricted to the submanifold \( Z \) (Theorem 5.10) with the aid of normalized frame (we refer readers to Proposition 5.8 for definition). We then make use of this result to determine the unitary invariants of aforementioned quotient modules.
Finally, we describe the quotient module obtained from the submodule of functions in weighted Bergman module \( H^{(\alpha,\beta,\gamma)} \) over \( \mathcal{A}(\mathbb{D}^3) \) vanishing to order 2 along the diagonal set of \( \mathbb{D}^3 \). Furthermore, with the help of our main theorem (Theorem 5.13) we determine unitary equivalence classes of weighted Bergman modules over \( \mathcal{A}(\mathbb{D}^m) \) in terms of quotient modules arising from the submodules of functions vanishing to order 2 along the diagonal set of \( \mathbb{D}^m \).

Thus, the results presented in this article extend most of the results in the papers [11], [12], [8] and [3] to the case of quotient modules arising from submodules of vector valued holomorphic functions on a bounded domain in \( \mathbb{C}^m \) which vanishes along a smooth irreducible complex analytic set of order at least 2.

The present article is organized in the following way. In Section 2, we recall some basic definitions and introduce few notations which will be used throughout this note. Then a complete description of the submodule \( \mathcal{M}_0 \) of interest is presented in Section 3. Section 4 is devoted to study quotient modules obtained from submodules introduced in Section 3. There we provide a canonical model for such quotient modules and in the subsequent section, Section 5, describe the complete set of unitary invariants of those quotient modules. We then finish this article by presenting some examples and applications in Section 6.

2. Preliminaries

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^m \) and \( \mathcal{A}(\Omega) \) be the unital Banach algebra obtained as the norm closure with respect to the supremum norm on \( \overline{\Omega} \) of all functions holomorphic on a neighbourhood of \( \overline{\Omega} \). A complex Hilbert space \( \mathcal{H} \) is said to be a Hilbert module over \( \mathcal{A}(\Omega) \) with module map \( \mathcal{A}(\Omega) \times \mathcal{H} \xrightarrow{\cdot} \mathcal{H} \) by point wise multiplication such that the module action \( \mathcal{A}(\Omega) \times \mathcal{H} \xrightarrow{\cdot} \mathcal{H} \) is norm continuous. We say that a Hilbert module \( \mathcal{H} \) over \( \mathcal{A}(\Omega) \) is contractive if \( \pi \) is a contraction.

Suppose that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are two Hilbert modules over \( \mathcal{A}(\Omega) \) with module actions \((f, h_i) \mapsto M_f^{(i)}(h_i), i = 1, 2\). Then a Hilbert space isomorphism \( \Phi : \mathcal{H}_1 \to \mathcal{H}_2 \) is said to be module isomorphism if \( \Phi(M_f^{(1)}(h_1)) = M_f^{(2)}(\Phi(h_1)) \) and we denote \( \mathcal{H}_1 \simeq_{\mathcal{A}(\Omega)} \mathcal{H}_2 \).

In this article, we study quotient modules obtained from certain submodules of quasi-free Hilbert modules. So we recall that a Hilbert space \( \mathcal{H} \) is said to be a quasi-free Hilbert module over \( \mathcal{A}(\Omega) \) of rank \( r \), for \( 1 \leq r \leq \infty \) and a bounded domain \( \Omega \subset \mathbb{C}^m \), if \( \mathcal{H} \) is a Hilbert space completion of the algebraic tensor product \( \mathcal{A}(\Omega) \otimes_{alg} \mathbb{C}^r \) and following conditions happen to be true:

(i) the evaluation operators \( e_w : \mathcal{H} \to \mathbb{C}^r \) defined by \( h \mapsto h(w) \) are uniformly bounded on \( \Omega \),

(ii) a sequence \( \{h_k\} \subset \mathcal{A}(\Omega) \otimes_{alg} \mathbb{C}^r \) that is Cauchy in the norm in \( \mathcal{H} \) converges to 0 in the norm in \( \mathcal{H} \) if and only if \( e_w(h_k) \) converges to 0 in \( \mathbb{C}^r \) for \( w \in \Omega \), and

(iii) multiplication by functions in \( \mathcal{A}(\Omega) \) define a bounded operator on \( \mathcal{H} \).

The condition (i) and (ii) together make the completion \( \mathcal{H} \) a functional Hilbert space over \( \mathcal{A}(\Omega) \) [1, Page 347]. Moreover, condition (i) ensures that the Hilbert space \( \mathcal{H} \) possesses an \( r \times r \) matrix valued reproducing kernel thanks to Riesz representation theorem. Finally, condition (iii) along with (i) make \( \mathcal{H} \) a Hilbert module over \( \mathcal{A}(\Omega) \) in the sense of [13, Definition 1.2]. Thus, a quasi-free Hilbert module over \( \mathcal{A}(\Omega) \) of rank \( r \) gives rise to a reproducing kernel Hilbert module of \( \mathbb{C}^r \) valued holomorphic functions over \( \mathcal{A}(\Omega) \).

For \( \Omega \subset \mathbb{C} \), we recall the definition of the Cowen-Douglas class \( B_n(\Omega) \) consisting of operators \( T \) on a Hilbert space \( \mathcal{H} \) for which each \( w \in \Omega \) is an eigenvalue of uniform multiplicity \( n \) of \( T \), the eigenvectors span the Hilbert space \( \mathcal{H} \) and \( \text{ran}(T - wI_{\mathcal{H}}) \) is closed for \( w \in \Omega \). Later the
The definition was adapted to the case of an $m$-tuple of commuting operators $\mathbf{T}$ acting on a Hilbert space $\mathcal{H}$, first in the paper [5], and then in the paper [6] from slightly different point of view which emphasized the role of the reproducing kernel. Let us now define the class $B_n(\Omega)$ for $\Omega \subset \mathbb{C}^m$ a bounded domain.

**Definition 2.1.** The $m$-tuple $\mathbf{T} = (T_1, \ldots, T_m)$ is in $B_n(\Omega)$ if

(i) $\text{ran} D_{T-w}$ is closed for all $w \in \Omega$ where $D_T : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^n$ is defined by $D_T h = (T_1 h, \ldots, T_m h), h \in \mathcal{H};$

(ii) $\text{span}\{\ker D_{T-w} : w \in \Omega\}$ is dense in $\mathcal{H}$ and

(iii) $\dim \ker D_{T-w} = n$ for all $w \in \Omega$.

It was then shown that each of these $m$-tuples $\mathbf{T}$ determines a Hermitian holomorphic vector bundle $E$ of rank $n$ on $\Omega$ and that two $m$-tuples of operators in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding vector bundles are locally equivalent. In case of $n = 1$, this is a question of equivalence of two Hermitian holomorphic line bundles and hence is a question of equality of the curvatures of those line bundles. However, no such simple characterization is known if the rank of the bundle is strictly greater than 1.

**Remark 2.2.** Let us consider a quasi-free Hilbert module $\mathcal{H}$ of rank $r$ over the algebra $\mathcal{A}(\Omega)$. Then, as mentioned earlier, $\mathcal{H}$ is a reproducing kernel Hilbert module with reproducing kernel $K$ on $\Omega$. Let $\mathbf{M}$ be the $m$-tuple of multiplication operators $(M_1, \ldots, M_m)$ acting on $\mathcal{H}$ as multiplication by coordinate functions. It then follows from the reproducing property of $K$ that

\begin{equation}
M_i^* K(., w) \eta = \overline{w}_i K(., w) \eta, \quad \text{for } \eta \in \mathbb{C}^r, \ w \in \Omega, \ 1 \leq i \leq m.
\end{equation}

As a consequence, the dimension of the joint eigenspace of $\mathbf{M}^\sigma$ at $\overline{\mathbf{w}}$ is at least $r$. Moreover, from the definition of quasi-free Hilbert modules and holomorphic functional calculus we see that the joint eigenspace at $\overline{\mathbf{w}}$ must have dimension exactly $r$ for $w \in \Omega$. Thus, $s(\overline{\mathbf{w}}) := \{K(., w)\sigma_1, \ldots, K(., w)\sigma_r\}$ defines a global holomorphic frame for the vector bundle $E \to \Omega^*$ with fibre at $\overline{\mathbf{w}} \in \Omega^* := \{\overline{\mathbf{w}} : w \in \Omega\}$, $E_{\overline{\mathbf{w}}} := \text{span } s(\overline{\mathbf{w}}) = \ker D_{\mathbf{M}^\sigma - \overline{\mathbf{w}}}$ where $\{\sigma_j\}_{j=1}^r$ is the standard ordered basis for $\mathbb{C}^r$.

We also note that the third condition in the definition of quasi-free Hilbert modules implies that $\text{ran} D_{\mathbf{M} - \overline{\mathbf{w}}}$ is dense in $\mathcal{H}$. Thus, $\mathbf{M}$ satisfies every condition of the definition of $B_r(\Omega^*)$ except the first one. This difference was discussed in the paper [10] where the notion of a quasi-free Hilbert module was introduced. While most of our examples lie in the class $B_r(\Omega)$, our methods work even with the weaker hypothesis that the modules are quasi-free of rank $r$ over the algebra $\mathcal{A}(\Omega)$.

In this article, we are interested to present some geometric invariant of quotient modules obtained from certain class of submodules of a quasi-free Hilbert module over $\mathcal{A}(\Omega)$. We, therefore, need some geometric tools from complex differential geometry. For the sake of completeness let us recall some basic notions from complex differential geometry following [4] and Chapter 3 of [17].

Let $E$ be a Hermitian holomorphic vector bundle of rank $n$ over a complex manifold $M$ of dimension $m$ with the Chern connection $D$. Then a simple calculation shows that with respect to a local frame $s = \{e_1, \ldots, e_n\}$ of $E$ the Chern connection $D$ takes the form

\begin{equation}
D(s) = \partial H(s) \cdot H(s)^{-1}
\end{equation}

where $H(s)$ is the Grammian matrix of the frame $s$. From now on, by a connection on a Hermitian holomorphic vector bundle we will mean the Chern connection.
The curvature of $E$ is defined as $\mathcal{K} := D^2 = D \circ D$ and $\mathcal{K}$ is an element in $\mathcal{E}^2(M) \otimes \text{Hom}(E, E)$ where $\mathcal{E}^2(M)$ is the collection of smooth 2-forms on $M$. Consequently, in a local coordinate chart of $M$ one can write $\mathcal{K}$ as

$$\mathcal{K}(\sigma) = \sum_{i,j=1}^{m} \mathcal{K}^i_{\sigma} dz_i \wedge dz_j, \quad \sigma \in \mathcal{E}^0(M, E).$$

Since $\mathcal{K} \in \mathcal{E}^2(M) \otimes \text{Hom}(E, E)$ we have $\mathcal{K}^i_{\sigma}$ are also bundle map for $i, j = 1, \ldots, m$. As before we can also express the curvature tensor with respect to a local frame as follows

$$\mathcal{K}(s) = \bar{\partial}(\partial H(s) \cdot H(s)^{-1})$$

and equivalently, $\mathcal{K}^i_j(s) = \bar{\partial}_j(\partial_i H(s) \cdot H(s)^{-1})$

where $\partial_i = \frac{\partial}{\partial z_i}$ and $\bar{\partial}_j = \frac{\partial}{\partial z_j}$. It is well known that the curvature operator is self adjoint ([4, (2.15.4)]) in the sense that $\mathcal{K}^i_j = \mathcal{K}^j_i$.

Now following [4, Lemma 2.10], for a given local frame $s$ of $E$ and a $C^\infty$ bundle map $\Phi : E \to \widehat{E}$, we have that

$$\Phi_{z_i}(s) = \partial_i \Phi(s) - [\partial_i H(s) \cdot H(s)^{-1}, \Phi(s)]$$

where $\Phi(s), \Phi_{z_i}(s), \Phi_{z_j}(s)$ are matrix representations of $\Phi, \Phi_{z_i}, \Phi_{z_j}$, respectively, with respect to the local frame $s$ and $[A, B]$ denotes the commutator of matrices $A$ and $B$. In the following lemma we calculate the covariant derivatives of curvature tensor which will be useful in Section 5. The proof of the following lemma, for $d = 1$, is well known [4, Proposition 2.18]. Although the similar set of arguments used there with more than one variables yields the proof of the following lemma, we present a sketch of the proof for the sake of completeness.

**Lemma 2.3.** Let $E$ be a Hermitian holomorphic vector bundle over $\Omega$ in $\mathbb{C}^m$ with a fixed holomorphic frame $S := \{s_1, \ldots, s_r\}$ whose Grammian matrix is $H$. Then

(i) For $1 \leq d \leq m$, $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$, and $i, j = 1, \ldots, d$, the $r \times r$ matrices

$$\mathcal{K}^i_j(S)_{z_{\alpha_1} \cdots z_{\alpha_d} \bar{\sigma}_{\beta_1} \cdots \bar{\sigma}_{\beta_d}}$$

can be expressed in terms of $H^{-1}$ and $\partial_1^{p_1} \cdots \partial_d^{p_d} \bar{\partial}_1^{q_1} \cdots \bar{\partial}_d^{q_d} H$, $0 \leq p_l \leq \alpha_l + 1, 0 \leq q_l \leq \beta_l + 1, l = 1, \ldots, d$.

(ii) Given $1 \leq d \leq m$, $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$, $\partial_1^{p_1} \cdots \partial_d^{p_d} \bar{\partial}_1^{q_1} \cdots \bar{\partial}_d^{q_d} H$ can be written in terms of $H^{-1}$, $\partial_1^{p_1} \cdots \partial_d^{p_d} H$, $\bar{\partial}_1^{q_1} \cdots \bar{\partial}_d^{q_d} H$, $0 \leq p_l \leq \alpha_l, 0 \leq q_l \leq \beta_l$, and $(\mathcal{K}^i_j(S)_{z_{\alpha_1} \cdots z_{\alpha_d} \bar{\sigma}_{\beta_1} \cdots \bar{\sigma}_{\beta_d}}$, for $0 \leq r_l \leq \alpha_l - 1, 0 \leq s_l \leq \beta_l - 1, l = 1, \ldots, d, i, j = 1, \ldots, d$.

**Proof.** Let $E$, $S$ and $H$ be as above. Then, for $j = 1, \ldots, m$, we have

$$\partial_{z_j} H^{-1} = -H^{-1} \cdot \partial_{z_j} H \cdot H^{-1} \quad \text{and} \quad \partial_{z_j} H^{-1} = -H^{-1} \cdot \partial_{z_j} H \cdot H^{-1}. \quad (2.5)$$

Now from the definition of curvature we obtain, for $i, j = 1, \ldots, d$,

$$\mathcal{K}^i_j = \partial_{z_j}(H^{-1} \cdot \partial_{z_i} H) = -H^{-1} \cdot \partial_{z_j} H \cdot H^{-1} \cdot \partial_{z_i} H + H^{-1} \cdot \partial_{z_j} \partial_{z_i} H,$$

which also implies that

$$\partial_{z_j} \partial_{z_i} H = H \cdot \mathcal{K}^i_j + \partial_{z_j} H \cdot H^{-1} \cdot \partial_{z_i} H. \quad (2.6)$$

Then repeated application of Leibnitz rule together with the equations in (2.5) provide the desired expression in (i). Further, (ii) can also be obtained as before by using Leibnitz rule and formulas (2.5) and (2.6) repeatedly. \qed
2.1. Notations and Conventions. We finish this introduction with a list of notations and conventions those will be useful through out the paper.

(1) In this article, we are intended to study quotient Hilbert modules obtained from sub-modules of quasi-free Hilbert modules over $A(\Omega)$ for a bounded domain $\Omega \in \mathbb{C}^m$. So from now on we assume that our Hilbert modules are quasi-free unless and otherwise stated.

(2) Let $\mathcal{H}$ be a Hilbert module over $A(\Omega)$ consisting of holomorphic functions on $\Omega$ and $\mathcal{H}_0 \subset \mathcal{H}$ be a subspace which is also a Hilbert module over $\Omega$. Assume that $A(\Omega)$ acts on $\mathcal{H}$ by point wise multiplication and $\mathcal{H}_q$ be the quotient module $\mathcal{H} \ominus \mathcal{H}_0$. Let $U \subset \Omega$ be an open connected subset. Then from the identity theorem for holomorphic functions of several complex variables we have $\mathcal{H} \simeq A(\Omega)|_{\mathcal{H}_0|_{\mathcal{H}_q}}$, $\mathcal{H}_0 \simeq A(\Omega)|_{\mathcal{H}_0|_{\mathcal{H}_q}}$, and hence $\mathcal{H}_q \simeq A(\Omega)|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ where $\mathcal{H} \simeq A(\Omega)|_{\mathcal{H}_0|_{\mathcal{H}_q}} = \{ h|_U : h \in \mathcal{H} \}$. Indeed, the restriction map $R : \mathcal{H} \to \mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ defined by $f \mapsto f|_U$ is an an map whose kernel is trivial thanks to the identity theorem and hence the inner product $\langle R(f), R(g) \rangle := \langle f, g \rangle$ on $\mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ turns $R$ to a unitary map. Then one can make $\mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ to a Hilbert module by restricting the module action of $A(\Omega)$ to the open set $U$ and note that $R$ intertwines the module actions. Thus, $\mathcal{H}$ and $\mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ are unitarily equivalent as modules and we also have $\mathcal{H}_q \simeq \mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$, $\mathcal{H}_q \simeq \mathcal{H}|_{\mathcal{H}_0|_{\mathcal{H}_q}}$ as modules. We, therefore, may cut down the domain $\Omega$ to a suitable open subset $U$, if necessary, and pretend $U$ to be $\Omega$.

(3) Let $1 \leq d \leq m$, $k \in \mathbb{N}$, $N = \binom{d+k-1}{k-1} - 1$, $I_N := \{0,1,\ldots,N\}$ and $A := \{ \alpha = (\alpha_1,\ldots,\alpha_d) \in (\mathbb{N} \cup \{0\})^d : 0 \leq |\alpha| \leq k-1\}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Then consider the bijection $\theta : A \to I_N$ defined by

\begin{equation}
\theta(\alpha) := \sum_{j=1}^{d-1} \frac{1}{j!} \left( |\alpha| - \sum_{i=1}^{d-j} \alpha_i \right)_j + \frac{1}{d!}(|\alpha|)_d
\end{equation}

where $(z)_t$ is the Pochhammer symbols defined as, for any complex number $z$ and a natural number $t$, $(z)_t = z(z+1)\cdots(z+t-1)$. Then we put an order on $A$ by pulling back the usual order on $I_N$ via the bijection $\theta$, that is,

$$(\alpha_1,\ldots,\alpha_d) \leq (\alpha'_1,\ldots,\alpha'_d) \text{ if and only if } \theta(\alpha_1,\ldots,\alpha_d) \leq \theta(\alpha'_1,\ldots,\alpha'_d).$$

Note that the order induced by $\theta$ is nothing but the graded colexicographic ordering on $A$. Here we also point out that $A = \bigcup_{t=0}^{k-1} A_t$ where $A_t := \{ \alpha \in A : |\alpha| = t \}$. Therefore, one can have a natural bijection between $A_t$ and $I_{N_t}$ where $N_t$ is the cardinality of the set $A_t$, namely,

\begin{equation}
\theta_t := \theta|_{A_t} : A_t \to \theta(A_t).
\end{equation}

These new set of bijections will be useful in the next section.

(4) From now on, for $\alpha \in A$ and $\theta(\alpha_1,\ldots,\alpha_d) = l$, we use following notations

\begin{equation}
\partial^l (\text{respectively, } \bar{\partial}^l) := \partial^{|\alpha|} (\text{respectively, } \bar{\partial}^{|\alpha|})
:= \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}} \quad (\text{respectively, } \frac{\partial^{|\alpha|}}{\partial \bar{z}_1^{\alpha_1} \cdots \partial \bar{z}_d^{\alpha_d}})
\end{equation}

unless and otherwise stated, where $\partial_i = \frac{\partial}{\partial z_i}$, $i = 1,\ldots,d$. In this context, note that since $\theta$ is a bijection there exists unique $(\alpha_1,\ldots,\alpha_d) \in A$ for every $l \in I_N$, and we are denoting $\partial^{\theta^{-1}(l)}$ as $\partial^l$. 

3. The Submodule $\mathcal{M}_0$

Let $\mathcal{M}$ be a quasi-free Hilbert module of rank $r$ over $\mathcal{A}(\Omega)$ and denote the elements of $\mathcal{M}$ as $h = (h_1, \ldots, h_r)$ where $h_j \in \mathcal{A}(\Omega)$, $1 \leq j \leq r$. In this section, we define the submodule $\mathcal{M}_0$ of $\mathcal{M}$. So we begin by recalling some elementary definitions regarding complex analytic varieties.

**Definition 3.1.** Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. Then a subset $Z \subset \Omega$ is called an analytic set if, for any point $p \in \Omega$, there is a connected open neighbourhood $U$ of $p$ in $\Omega$ and finitely many holomorphic functions $\phi_1, \ldots, \phi_d$ on $U$ such that

$$U \cap Z = \{q \in U : \phi_j(q) = 0, \ 1 \leq j \leq d\}.$$

**Definition 3.2.** An analytic set $Z \subset \Omega$ is said to be regular of codimension $d$ at $p \in Z$ if there is an open neighbourhood $U_p \subset \Omega$ and holomorphic functions $\phi_1, \ldots, \phi_d$ on $U_p$ such that

(a) $Z \cap U_p = \{q \in U_p : \phi_1(q) = \cdots = \phi_d(q) = 0\}$,

(b) the rank of the Jacobian matrix of the mapping $q \mapsto (\phi_1(q), \ldots, \phi_d(q))$ at $p$ is $d$.

An analytic set is said to be irreducible if it cannot be decomposed as union of two analytic sets. It is known in literature that any smooth analytic set is irreducible if and only if it is connected with respect to the subspace topology [16, page 20].

Here we point out that such an analytic set $Z$ is a regular complex submanifold of codimension $d$ in $\Omega$ thanks to well known fact [15, page 161].

**Proposition 3.3.** An analytic set $Z$ is regular of codimension $d$ at $p \in M$ in a complex manifold $M$ of dimension $m$ if and only if there is a complex coordinate chart $(U, \phi)$ of $M$ such that $B := \phi(U)$ is an open subset of $\mathbb{C}^m$ with $\phi(p) = 0$ and $\phi(U \cap Z) = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in B : \lambda_1 = \cdots = \lambda_d = 0\}$.

**Remark 3.4.** In this article, we are interested in smooth irreducible analytic sets $Z$ of codimension $d$ in some bounded domain $\Omega$ in $\mathbb{C}^m$. So from the Definition 3.2 and the Proposition 3.3 we have, for each point $p \in Z$, there is a coordinate chart $(U, \phi)$ of $p$ at $\Omega$ satisfying following properties:

(a) $\phi(p) = 0$ with $\phi(U \cap Z) = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in B : \lambda_1 = \cdots = \lambda_d = 0\}$,

(b) the rank of the Jacobian matrix of the mapping $q \mapsto (\phi_1(q), \ldots, \phi_d(q))$ at $p$ is $d$.

We are now about to define the order of vanishing of a holomorphic function along a smooth analytic set. Our definition is essentially a direct generalization of the definition given in [12] to define the order of vanishing of a holomorphic function along a smooth complex hypersurface.

**Definition 3.5.** Let $\Omega$ and $Z$ be as above and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then $f$ is said to have zero of order $k$ at some point $p \in Z$ if there exists a coordinate chart $(U, \phi)$ at $p$ of $\Omega$ satisfying the properties (a) and (b) in the Remark 3.4 such that

$$[f] \in I_2^{k-1} \text{ but } [f] \notin I_2^{k}$$

where $[f]$ is the germ of $f$ at $p$ and $I_2$ is the ideal in $\mathcal{O}_{m,p}$ generated by $[\phi_1], \ldots, [\phi_d]$.

**Remark 3.6.** Note that the above definition is independent of the choice of coordinate chart at $p$. Indeed, for two such charts $(U_1, \phi_1)$ and $(U_2, \phi_2)$ with the properties listed in Remark 3.4, $\phi_1$ and $\phi_2 \circ \phi_1^{-1}$, respectively, induce isomorphisms $\Phi_1 : \mathcal{O}_{m,p} \to \mathcal{O}_{m,0}$ defined by $\Phi_1([g]) = [g \circ \phi_1^{-1}]$ and $\Phi : \mathcal{O}_{m,0} \to \mathcal{O}_{m,0}$ with $\Phi([g \circ \phi_1^{-1}]) = [g \circ \phi_2^{-1}]$. As a consequence, it turns out that $f$ satisfies (3.1) if and only if $[f \circ \phi_1^{-1}] \in I_1^{k-1}$ but $[f \circ \phi_1^{-1}] \notin I_1^{k}$ which is again equivalent to the fact that $[f \circ \phi_2^{-1}] \in I_2^{k-1}$ but $[f \circ \phi_2^{-1}] \notin I_2^{k}$ where $I_j$ is the ideal generated by the germs $[\lambda_j], \ldots, [\lambda_j]$, $j = 1, 2$, with local coordinates $\lambda_1, \ldots, \lambda_m$ of $\mathbb{C}^m$ corresponding to $\phi_j$. 

Definition 3.7. Let $\mathcal{M}$ be a quasi-free Hilbert module of rank $r$ over $A(\Omega)$. Then the submodule $\mathcal{M}_0$ is defined as

$$\mathcal{M}_0 := \{ h \in \mathcal{M} : h_j \text{ has zero of order } k \text{ at every } q \in \mathcal{Z}, \ 1 \leq j \leq r \}.$$ 

Lemma 3.8. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $\mathcal{Z}$ be a complex submanifold in $\Omega$ and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, for each point $p \in \mathcal{Z}$, $f$ vanishes to order $k$ at $p$ along $\mathcal{Z}$ if and only if $k$ is the largest integer such that

$$\partial^{|\alpha|}_h(f \circ \phi^{-1})|_{\phi(U \cap \mathcal{Z})} := \frac{\partial^{|\alpha|}}{\partial \lambda_1^{a_1} \cdots \partial \lambda_d^{a_d}}(f \circ \phi^{-1})|_{\phi(U \cap \mathcal{Z})} = 0 \text{ for } 0 \leq |\alpha| \leq k - 1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, for some coordinate chart $(U, \phi)$ as in the Remark 3.4.

In general, there are no global defining functions $\phi_1, \ldots, \phi_d$ for a smooth irreducible analytic set $\mathcal{Z}$. But since it has been shown at the end of the Section 2 that the modules and the submodules of interest can be localized we can work with a small enough open set $U \subset \Omega$ intersecting $\mathcal{Z}$. So from now on we consider a fixed neighbourhood $U \subset \Omega$ of $p$ with $U \cap \mathcal{Z} \neq \emptyset$ and defining functions $\phi_1, \ldots, \phi_d$ satisfying conditions (a) and (b) in Remark 3.4. Since the Jacobian matrix of the mapping $z \mapsto (\phi_1(z), \ldots, \phi_d(z))$ has rank $d$ at $p$, by rearranging the coordinates in $\mathbb{C}^n$, we can assume that $D_1(p) := ((\partial_j \phi_i)_p)_{i,j=1}^d$ is invertible. Then it is easily seen that $D_1(z)$ is invertible on some neighbourhood of $p$ in $U$. Abusing the notation, let us denote this neighbourhood by the same letter $U$. Now we consider the mapping $\phi : U \to \phi(U)$ defined as $\phi(z) = (\phi_1(z), \ldots, \phi_d(z), z_{d+1}, \ldots, z_m)$ and note that $\phi$ is a biholomorphism from $U$ onto $\phi(U)$ with $\phi(p) = 0$ and $\phi(U \cap \mathcal{Z}) = \{ \lambda = (\lambda_1, \ldots, \lambda_m) : \lambda = (\lambda_1, \ldots, \lambda_m) \in \phi(U) : \lambda_1 = \cdots = \lambda_d = 0 \}$. Thus, once we fix a chart as above and pretend that $U = \Omega$, the submodule $\mathcal{M}_0$ may be described as

$$\mathcal{M}_0 = \left\{ h \in \mathcal{M} : \frac{\partial^{|\alpha|}}{\partial \lambda_1^{a_1} \cdots \partial \lambda_d^{a_d}}(h \circ \phi^{-1})(\lambda)\big|_{\phi(\mathcal{Z})} = 0 \text{ for } 0 \leq |\alpha| \leq k - 1, \ 1 \leq j \leq r \right\}.$$ 

At this stage, we introduce a definition which separates out the coordinate chart described above and will be useful through out this article.

Definition 3.9. Let $\Omega$ be a domain in $\mathbb{C}^n$ and $\mathcal{Z} \subset \Omega$ be a complex submanifold of codimension $d$. Then, for any point $p \in \mathcal{Z}$, we call a coordinate chart $(U, \phi)$ of $\mathcal{Z}$ around $p$ an admissible coordinate chart if the biholomorphism $\phi : U \to \phi(U)$ takes the form $\phi(z) = (\phi_1(z), \ldots, \phi_d(z), z_{d+1}, \ldots, z_m)$ with $\phi(p) = 0$ and $\phi(U \cap \mathcal{Z}) = \{ \lambda = (\lambda_1, \ldots, \lambda_m) \in \phi(U) : \lambda_1 = \cdots = \lambda_d = 0 \}$ for some holomorphic functions $\phi_1, \ldots, \phi_d$ on $U$.

Now we should note that even in this local description of the submodule there is a choice of normal directions to the submanifold $\mathcal{Z}$ involved. The following proposition ensures that in this local picture two different sets of normal directions to $\mathcal{Z}$ give rise to equivalent submodules. At this point, let us recall some elementary definitions and properties of the ring of polynomial functions on a finite dimensional complex vector space which will be useful in the course of the proof of following proposition.

For any complex vector space $V$ of dimension $d$, we denote by $\mathbb{C}[V]$ the ring of polynomial functions on $V$. Let us recall that $f : V \to \mathbb{C}$ is an element of $\mathbb{C}[V]$ means that, for any basis $\{e_1, \ldots, e_d\}$ of $V$, there exists some polynomial $\phi \in \mathbb{C}[x_1, \ldots, x_d]$ such that $f(\alpha_1 e_1 + \cdots + \alpha_d e_d) = \phi(\alpha_1, \ldots, \alpha_d)$ for all $(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d$. In other words, $f$ is a polynomial into the elements $x_1 = e_1^*, \ldots, x_d = e^*_d$ of the dual basis. It is then clear that

\begin{equation}
\mathbb{C}[V] \simeq S(V^*) \simeq \mathbb{C}[x_1, \ldots, x_d]
\end{equation}
where $S(V^*)$ is the graded vector space of all symmetric tensors on $V^*$. Note that $\mathbb{C}[V]$ is an algebra over $\mathbb{C}$.

A polynomial function $f$ on $V$ is said to be homogeneous of degree $t$ if $f(\alpha v) = \alpha^t f(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. We denote $\mathbb{C}[V]_t$ the subspace of $\mathbb{C}[V]$ of homogeneous polynomial functions of degree $t$. In particular, $\mathbb{C}[V]_0 = \mathbb{C}$, $\mathbb{C}[V]_1 = V^*$ and $\mathbb{C}[V]_t$ is canonically identified in the first isomorphism in (3.2) with the $t$-th symmetric power $S^t(V^*)$, and it can also be identified with the subspace of $\mathbb{C}[x_1, \ldots, x_d]$ generated by the monomials $x_1^{t_1} \cdots x_d^{t_d}$ with $t_1 + \cdots + t_d = t$ via the second isomorphism of (3.2).

**Proposition 3.10.** Let $\Omega$ be a bounded domain in $\mathbb{C}^m$, $Z$ be a complex submanifold in $\Omega$ and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, for each point $p \in Z$ there exists an admissible coordinate chart $(U, \phi)$ of $\Omega$ at $p$ such that

$$\partial^\alpha (f \circ \phi^{-1})(\lambda)|_{\phi(p)} = 0, \quad 0 \leq |\alpha| \leq k - 1 \quad \text{if and only if} \quad \partial^\alpha f(z)|_p = 0, \quad 0 \leq |\alpha| \leq k - 1$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ denotes the standard coordinates on $\phi(U) \subset \mathbb{C}^m$, and $\partial^\alpha$ denotes the differential operator \(\frac{\partial^{\alpha}}{\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}}\).

**Proof.** Let us consider an admissible coordinate system $(U, \phi)$ (Definition 3.9) at $p \in Z \subset \Omega$ of $\Omega$, that is, $\phi : U \to \phi(U)$ defined as $\phi(z) = (\phi_1(z), \ldots, \phi_d(z), z_{d+1}, \ldots, z_m)$.

For $q \in U$, let $V_q$ and $V_{q(\phi)}$ be tangent spaces at $q$ and $\phi(q)$ to $(\mathbb{C}^d \times \{0\}) \cap U$ and $(\mathbb{C}^d \times \{0\}) \cap \phi(U)$, respectively. We denote the standard ordered basis of $V_q$ by $B_1(q) := \{ \frac{\partial}{\partial z_j} | q \}_{j=1}^d$ and that of $V_{q(\phi)}$ by $B_1(\phi(q)) := \{ \frac{\partial}{\partial \lambda_j} | \phi(q) \}_{j=1}^d$. Then it is easily seen that $\phi$ induces a linear transformation from $V_q^*$ onto $V_{q(\phi)}^*$, namely, $L_1(q) : V_q^* \to V_{q(\phi)}^*$ defined by

$$L_1(q)(dz_j) = \sum_{i=1}^d (\partial_j \phi_i(q)) d\lambda_i$$

where $\{dz_j\}_{j=1}^d$ and $\{d\lambda_j\}_{j=1}^d$ are dual bases of $B_1(q)$ and $B_1(\phi(q))$, respectively.

Now we consider the ring of polynomial functions $\mathbb{C}[V_q]$ and $\mathbb{C}[V_{q(\phi)}]$ on $V_q$ and $V_{q(\phi)}$, respectively, and observe, in view of the first isomorphism in (3.2), that $L_1(q)$ canonically induces linear mappings $L_t(q) : S^t(V_q^*) \to S^t(V_{q(\phi)}^*)$ defined by

$$L_t(q)(dz_1^{\alpha_1} \cdots dz_d^{\alpha_d}) = L_1(q)(dz_1)^{\alpha_1} \cdots L_1(q)(dz_d)^{\alpha_d}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d = t$ and by $L_1(q)(dz_j)^{\alpha_j}$ (respectively, by $L_1(q)(dz_j)$) we mean that the $\alpha_j$-th symmetric power of $dz_j$ (respectively, $L_1(q)(dz_j)$).

Let $B_t(q) := \{dz_1^{\alpha_1} \cdots dz_d^{\alpha_d} : |\alpha| = t\}$ and $B_t(\phi(q)) := \{d\lambda_1^{\alpha_1} \cdots d\lambda_d^{\alpha_d} : |\alpha| = t\}$ be bases for vector spaces $S^t(V_q^*)$ and $S^t(V_{\phi(q)}^*)$, respectively, and make them ordered bases with respect to the order induced by the bijection $\theta_t$ (2.8). We denote the matrix of $L_t(q)$ represented with respect to the basis $B_t(q)$ and $B_t(\phi(q))$ as $D_t(q)$, for $t \in \mathbb{N} \cup \{0\}$. Note that since $L_t(p)$ is a vector space isomorphism for each $t \in \mathbb{N} \cup \{0\}$ the matrices $D_t(p)$’s are invertible.
In this set up we claim, for \( z \in U \) with \( \phi(z) = \lambda \in \phi(U) \), that

\[
A_{k,\phi}(\phi) = \begin{pmatrix}
    f \circ \phi^{-1}(\lambda) \\
    \partial^1 f \circ \phi^{-1}(\lambda) \\
    \vdots \\
    \partial^N f \circ \phi^{-1}(\lambda)
\end{pmatrix}
= \begin{pmatrix}
    f(\phi(z)) \\
    \partial^1 f(\phi(z)) \\
    \vdots \\
    \partial^N f(\phi(z))
\end{pmatrix}
\]

where \( \partial^l \) stands for the differential operator \( \frac{\partial^{\lvert \alpha \rvert}}{\partial \lambda_1^{\alpha_1} \cdots \partial \lambda_d^{\alpha_d}} \) with \( (\alpha_1, \ldots, \alpha_d) = \theta^{-1}(t) \), \( A_{k,\phi}(\phi) \) is the block lower triangular matrix with 1, \( \mathcal{D}_1(z) \), \ldots, \( \mathcal{D}_{k-1}(z) \) as the diagonal blocks and \( N = \binom{d+k-1}{k-1} - 1 \).

We prove this claim with the help of mathematical induction on \( k \). Here we note that the base case is the direct consequence of change of variables formula. So let the equation (3.3) hold true for \( t = l \), \( 1 \leq l \leq k - 1 \) and we need to prove that (3.3) holds for \( t = l + 1 \). Now, for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with \( \lvert \alpha \rvert = l \) and \( \theta_t(\alpha_1, \ldots, \alpha_d) = t \), the induction hypothesis yields that

\[
\partial^l f(\phi(z)) = \sum_{\lvert \beta \rvert = l} (\mathcal{D}_l(z))_{\beta} \partial^l \phi^{-1}(\lambda) + \text{other terms}
\]

where \( \mathcal{D}_l(z) \) is the matrix \( \begin{pmatrix} (\mathcal{D}_l(z))_{(\alpha) \theta_l(l)} \end{pmatrix}_{\lvert \alpha \rvert = l, \lvert \beta \rvert = l} \). Therefore, differentiating both sides of the above equation with respect to the \( z_j \)-th coordinate and using the Leibniz rule we have, for an arbitrary but fixed point \( q \in U \),

\[
\partial_j \partial^l f(z)_{z=q} = \sum_{\lvert \beta \rvert = l} (\mathcal{D}_l(z))_{\beta} \partial^l \phi^{-1}(\phi(q)) + \text{other terms}
\]

Let us now note that the rings of polynomial functions \( S(V^*_q) \) and \( S(V^*_{\phi(q)}) \) can be canonically identified with the algebras of linear partial differential operators with constant coefficients, namely, \( \Gamma_q : S(V^*_q) \simeq \{ \sum_{\alpha} a_{\alpha} \partial_{\lambda_1}^{\alpha_1} \cdots \partial_{\lambda_d}^{\alpha_d} : a_{\alpha} \in \mathbb{C} \} \) under the correspondence \( d z_d^{\alpha_1} \cdots d z_d^{\alpha_d} \mapsto \partial_{\lambda_1}^{\alpha_1} \cdots \partial_{\lambda_d}^{\alpha_d} \) and similarly, \( \Gamma_{\phi(q)} : S(V^*_{\phi(q)}) \simeq \{ \sum_{\alpha} a_{\alpha} \partial_{\lambda_1}^{\alpha_1} \cdots \partial_{\lambda_d}^{\alpha_d} : a_{\alpha} \in \mathbb{C} \} \) via the mapping \( d \lambda_1^{\alpha_1} \cdots d \lambda_d^{\alpha_d} \mapsto \partial_{\lambda_1}^{\alpha_1} \cdots \partial_{\lambda_d}^{\alpha_d} \). Then with respect to the above identification we have
\[ \partial^{\alpha + \varepsilon} f(q) = \sum_{|\beta| = l} (D_l(q))_{\theta_0(\beta)} \partial^\beta \left( \sum_{s=1}^d \partial_s \phi_s(q) \partial_s f \circ \phi^{-1}(\lambda) \right) \bigg|_{\lambda = \phi(q)} + \text{(other terms involving } \partial^\beta f \circ \phi^{-1}(\phi(q))\text{ with } |\alpha| \leq l) \]
\[ = (\Gamma_{\phi(q)} L_1(q) \Gamma_q^{-1}(\partial_1))^\alpha_1 \cdots (\Gamma_{\phi(q)} L_1(q) \Gamma_q^{-1}(\partial_j))^\alpha_j \cdots (\Gamma_{\phi(q)} L_1(q) \Gamma_q^{-1}(\partial_d))^\alpha_d \]
\[ \times (\Gamma_{\phi(q)} L_1(q) \Gamma_q^{-1}(\partial_j)) f \circ \phi^{-1}(\lambda) \bigg|_{\lambda = \phi(q)} + \text{(other terms involving } \partial^\beta f \circ \phi^{-1}(\phi(q))\text{ with } |\alpha| \leq l) \]
\[ = \sum_{|\beta| = l+1} (D_{l+1}(q))_{\theta(\alpha + \varepsilon)} \partial^\beta f \circ \phi^{-1}(\phi(q)) + \text{(other terms involving } \partial^\beta f \circ \phi^{-1}(\phi(q))\text{ with } |\alpha| \leq l). \]

Since \( q \) was chosen to be arbitrary in \( U \) we are done with the claim. Thus, \( A_{k,\phi}(z) \) is invertible if and only if \( D_1(z), \ldots, D_{k-1}(z) \) are simultaneously invertible which is the case for \( Z \in U \). Hence it completes the proof. \( \square \)

Thus, from the above proposition and Remark 3.6 we have another characterization of the submodules \( \mathcal{M}_0 \) as follows:
\[ \mathcal{M}_0 = \{ h \in \mathcal{M} : \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}(h_j) \big|_{Z} = 0, 0 \leq |\alpha| \leq k - 1, 1 \leq j \leq r \}. \]

**Remark 3.11.** Let \( U \) be an open subset of \( \Omega \) such that \( U \cap Z \) is non-empty. Then recall that the restriction map \( R : \mathcal{M} \rightarrow \mathcal{M}|_{\text{res} U} \) defined by \( f \mapsto f|_U \) is an unitary map with respect to the prescribed inner product on \( \mathcal{M}|_{\text{res} U} \) (2) in Section 2). Moreover, it is now clear from the definition of \( \mathcal{M}_0 \) that \( R|_{\mathcal{M}_0} \) and \( \mathcal{M}_0 \) are unitarily equivalent where \( R|_{\mathcal{M}_0} := \{ f \in R(\mathcal{M}) : f \text{ vanishes along } U \cap Z \text{ to order } k \} \). Consequently, \( R|_{\mathcal{M}} \) and \( \mathcal{M}_0 := R(\mathcal{M}) \cap R(\mathcal{M})_0 \) are also unitarily equivalent Hilbert modules. So restricting ourselves to an admissible coordinate chart \( (U, \phi) \) around some point \( p \in Z \subset \Omega \), it is enough to study these modules with respect to the new coordinate system obtained by \( \phi \). We now elaborate upon this fact.

So let us consider the module, \( \phi^*(\mathcal{M}|_{\text{res} U}) \) which is, by definition,
\[ \phi^*(\mathcal{M}|_{\text{res} U}) := \{ f|_U \circ \phi^{-1} : f \in \mathcal{M} \} \]
and note that it is a module over \( A(\Omega) \) with the module action \( g \cdot (f|_U \circ \phi^{-1}) := (gf)|_U \circ \phi^{-1} \), for \( g \in A(\Omega) \). Then it is evident that the modules \( \phi^*(\mathcal{M}|_{\text{res} U}) \) and \( \mathcal{M}_0 \) are isomorphic via the isomorphism \( \Phi : \mathcal{M} \rightarrow \phi^*(\mathcal{M}|_{\text{res} U}) \) defined by \( f \mapsto f|_U \circ \phi^{-1} \). So, defining an inner product as
\[ \langle f|_U \circ \phi^{-1}, g|_U \circ \phi^{-1} \rangle_{\phi^*(\mathcal{M}|_{\text{res} U})} := \langle f, g \rangle \mathcal{M} \]
we see that \( \phi^*(\mathcal{M}|_{\text{res} U}) \) is unitarily equivalent to \( \mathcal{M} \) as Hilbert modules. Since \( \mathcal{M} \) is a reproducing kernel Hilbert module with a reproducing kernel, say, \( K \) so is \( \phi^*(\mathcal{M}|_{\text{res} U}) \) with the kernel function \( K' \) defined by \( K'(u, v) = K(\phi^{-1}(u), \phi^{-1}(v)) \), for \( u, v \in \phi(U) \). It is also easily seen that the multiplication operators \( M_{z_1}, \ldots, M_{z_m} \) on \( \mathcal{M} \) are simultaneously unitarily
equivalent to \( M_{u_1}, \ldots, M_{u_m} \) on \( \phi^*(\mathcal{M}_{|\Omega}) \). Indeed, for \( \phi^{-1} = (\psi_1, \ldots, \psi_m) \), we note that 
\( \psi_i(\phi(z_1, \ldots, z_m)) = z_i, \ i = 1, \ldots, m \) and therefore, we have, for \( i = 1, \ldots, m \) and \( f \in \mathcal{M}, \)
\[
\Phi^{-1}M_u\Phi(f) = \Phi^{-1}(M_u(f|_U \circ \phi^{-1}))
\]
\[
= \Phi^{-1}((u_i \circ \phi^{-1}) \cdot (f|_U \circ \phi^{-1}))
\]
\[
= M_{z_i}f|_U.
\]
Furthermore, Proposition 3.10 together with the Remark 3.11 ensure that the submodules \( \mathcal{M}_0 \) and \( \phi^*R(\mathcal{M}_0) \) are also unitarily equivalent via the same map as mentioned earlier. As a consequence along with the help of the Remark 3.11 we have the following Proposition.

**Proposition 3.12.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^m \), \( Z \) be a complex connected submanifold in \( \Omega \) and \( \mathcal{M}_1, \mathcal{M}_2 \) be two quasi-free Hilbert modules of rank \( r \) over \( \mathcal{A}(\Omega) \). Let \( \mathcal{M}_0^1 \) and \( \mathcal{M}_0^2 \) be submodules of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), respectively, consisting of holomorphic functions vanishing of order \( k-1 \) along \( Z \). Assume that \( (U, \phi) \) is an admissible coordinate system around some point \( p \in Z \). Then \( \mathcal{M}_0^1 \) is unitarily equivalent to \( \mathcal{M}_0^2 \) as Hilbert modules if and only if \( \phi^*(\mathcal{M}_0^1_{|\Omega}) \) is unitarily equivalent to \( \phi^*(\mathcal{M}_0^2_{|\Omega}) \). In other words, the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{M}_0^1 & \xrightarrow{R} & \mathcal{M}_0^1_{|\Omega} \xrightarrow{\Phi} \phi^*(\mathcal{M}_0^1_{|\Omega}) \\
\downarrow & & \downarrow \\
\mathcal{M}_0^2 & \xrightarrow{R} & \mathcal{M}_0^2_{|\Omega} \xrightarrow{\Phi} \phi^*(\mathcal{M}_0^2_{|\Omega})
\end{array}
\]

In this article, we are interested in studying equivalence classes of quotient modules obtained from the aforementioned submodules of a quasi-free Hilbert modules. In the next section, we define the quotient modules which are the central object of this paper.

### 4. Quotient Module \( \mathcal{M}_q \)

We start with a quasi-free Hilbert module \( \mathcal{M} \) over \( \mathcal{A}(\Omega) \) of rank \( r \geq 1 \) and the submodule \( \mathcal{M}_0 \subset \mathcal{M} \) consisting of \( \mathbb{C} \)-valued holomorphic functions on \( \Omega \) vanishing to order \( k \) along an irreducible smooth complex analytic set \( Z \subset \Omega \) of codimension \( d \), \( d \geq 2 \). In this setting we are interested in studying the quotient module
\[
\mathcal{M}_q := \mathcal{M} / \mathcal{M}_0 = \mathcal{M} \ominus \mathcal{M}_0,
\]
in other words, we have following exact sequence
\[
0 \to \mathcal{M}_0 \xrightarrow{i} \mathcal{M} \xrightarrow{P} \mathcal{M}_q \to 0
\]  
(4.1)

where \( i \) is the inclusion map and \( P \) is the quotient map. Now, for \( f \in \mathcal{A}(\Omega) \) and \( h \in \mathcal{M} \), we define the module action on the quotient module \( \mathcal{M}_q \) as
\[
fP(h) = P(fh),
\]
(4.2)

here we mean \( (fh_1, \ldots, fh_r) \) by \( fh \).

In order to study quotient modules we first describe the jet construction relative to the submanifold \( Z \) following [12]. Suppose that \( \mathcal{H} \) is a reproducing kernel Hilbert space consisting of holomorphic functions on \( \Omega \) taking values in \( \mathbb{C}^r \) with a reproducing kernel \( K \). Let \( N = (d+k-1) - 1, \{\varepsilon_l\}_{l=0}^N \) be the standard ordered basis of \( \mathbb{C}^{N+1} \), \( \{\sigma_l\}_{l=1}^N \) be the standard ordered
basis of \( \mathbb{C}^{r} \) and recall that \( \partial_{1}, \ldots, \partial_{d} \) are the partial derivative operators with respect to \( z_{1}, \ldots, z_{d} \) variables, respectively. For \( h \in \mathcal{H} \), recalling the notations introduced (2.9) let us define

\[
h := \sum_{i=1}^{r} \left( \sum_{l=0}^{N} \partial^{l}_{h_{i}} \otimes \varepsilon_{l} \right) \otimes \sigma_{i}
\]

and we consider the space \( J(\mathcal{H}) := \{ h : h \in \mathcal{H} \} \subset \mathcal{H} \otimes \mathbb{C}^{(N+1)r} \). Consequently, we have the mapping

\[
J : \mathcal{H} \rightarrow J(\mathcal{H}) \text{ defined by } h \mapsto h.
\]

Since \( J \) is injective we define an inner product on \( J(\mathcal{H}) \) making \( J \) to be an unitary transformation as follows

\[
\langle J(h_{1}), J(h_{2}) \rangle_{J(\mathcal{H})} := \langle h_{1}, h_{2} \rangle_{\mathcal{H}}.
\]

Since \( \mathcal{H} \) is a reproducing kernel Hilbert space it is natural to expect that \( J(\mathcal{H}) \) is also a reproducing kernel Hilbert space. So we calculate the reproducing kernel of \( J(\mathcal{H}) \).

**Proposition 4.1.** The reproducing kernel \( JK : \Omega \times \Omega \rightarrow M_{(N+1)r}(\mathbb{C}) \) for the Hilbert space \( J(\mathcal{H}) \) is given by the formula

\[
(JK)_{ij}^{l}(z, w) = \partial^{k} \partial^{l} K_{ij}(z, w) \quad \text{for } 0 \leq l, k \leq N, \ 1 \leq i, j \leq r.
\]

**Proof.** Let \( \{e_{n}\}_{n \geq 1} \) be an orthonormal basis of \( \mathcal{H} \) with \( e_{n} = \sum_{i=1}^{r} e_{n}^{i} \otimes \sigma_{i} \) where for every \( n \) \( e_{n}^{i} \) is a holomorphic function on \( \Omega \). Since \( J \) is a unitary operator \( \{Je_{n}\}_{n \geq 1} \) is an orthonormal basis for \( J(\mathcal{H}) \). Thus, we have

\[
JK(z, w) = \sum_{n=1}^{\infty} Je_{n}(z)(Je_{n}(w))^{*}
\]

where \( Je_{n}(w)^{*} : \mathbb{C}^{(N+1)r} \rightarrow \mathbb{C} \) is defined by \( (Je_{n}(w))^{*}(\zeta) := \langle \zeta, Je_{n}(w) \rangle_{\mathbb{C}^{(N+1)r}} \) and \( Je_{n}(z) : \mathbb{C} \rightarrow \mathbb{C}^{(N+1)r} \) is defined as \( x \mapsto x \cdot Je_{n}(z) \). Then note that, for \( w \in \Omega \), \( Je_{n}(w)^{*}(\varepsilon_{l} \otimes \sigma_{j}) = \partial^{l} e_{n}^{j}(w) \) and hence we can write

\[
JK(z, w)\varepsilon_{l} \otimes \sigma_{j} = \sum_{n=1}^{\infty} Je_{n}(z)\partial^{l} e_{n}^{j}(w), \ 0 \leq l \leq N, 1 \leq j \leq r.
\]

Therefore, equation (4.5) together with the calculation below imply the identity in (4.4).

\[
\sum_{n=1}^{\infty} Je_{n}(z)\partial^{l} e_{n}^{j}(w)\varepsilon_{k} \otimes \sigma_{i} \rangle_{\mathbb{C}^{(N+1)r}} = \sum_{n=1}^{\infty} \partial^{l} e_{n}^{j}(w)\partial^{k} e_{n}^{i}(z) = \partial^{k} \partial^{l} K_{ij}(z, w).
\]

Since \( h \) can uniquely be expressed as \( h = \sum_{n=1}^{\infty} a_{n} Je_{n} \) with \( \{a_{n}\}_{n=1}^{\infty} \subset \mathbb{C} \), using (4.5), we have

\[
\langle h, JK(\, , w)\varepsilon_{l} \otimes \sigma_{j} \rangle_{J(\mathcal{H})} = \sum_{n=1}^{\infty} a_{n} \partial^{l} e_{n}^{j}(w) = \partial^{l} \left( \sum_{n=1}^{\infty} a_{n} e_{n}^{j}(w) \right) = \partial^{l} h_{j}(w).
\]

Here we note that the second equality in above two equations hold due to the fact that the series in first equality converges uniformly on compact subsets of \( \Omega \). This completes the proof. \( \square \)

Thus, we have shown that \( J(\mathcal{H}) \) is a reproducing kernel Hilbert space on \( \Omega \). Now we want to make \( J(\mathcal{H}) \) to be a Hilbert module over \( \mathcal{A}(\Omega) \). So let us define an action of \( \mathcal{A}(\Omega) \) on \( J(\mathcal{H}) \) so that
$J$ becomes a module isomorphism. We define, for $f \in \mathcal{A}(\Omega)$ and $h \in J(\mathcal{H})$, $J_f : J(\mathcal{H}) \to J(\mathcal{H})$ by $J_f(h) := J(f).h$ where $J(f)$ is an $(N + 1) \times (N + 1)$ complex matrix defined as follows

\[
J(f)_{ij} := \begin{pmatrix} \alpha \beta \end{pmatrix} \partial^{\alpha-\beta} f := \begin{pmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \vdots \\ \alpha_d \beta_d \end{pmatrix} \partial^{\alpha-\beta} f
\]

with $\alpha = (\alpha_1, \ldots, \alpha_d) = \theta^{-1}(i)$ and $\beta = (\beta_1, \ldots, \beta_d) = \theta^{-1}(j)$ and $h$ can be thought of an $(N + 1) \times r$ matrix with $h_i := \sum_{l=0}^{N} \partial^l h_i \otimes \varepsilon_l$, $1 \leq i \leq r$, as column vectors. Note that this is a lower triangular matrix and it takes the following matrix form

\[
J(f) = \begin{pmatrix} f \\ \vdots \\ J(f)_{ij} \\ \partial^N f \\ \cdots \\ f \end{pmatrix}
\]

Thus, with the above definition, $J(\mathcal{H})$ becomes a Hilbert module over $\mathcal{A}(\Omega)$ and $J$ is a module isomorphism between $\mathcal{H}$ and $J(\mathcal{H})$ as it is clear from following simple calculation which is essentially an application of Leibniz rule. For $1 \leq i \leq r$, we have

\[
J(f \cdot h_i) = \sum_{l=0}^{N} \partial^l (f \cdot h_i) \otimes \varepsilon_l = \sum_{l=0}^{N} \sum_{l_1=0}^{\alpha_1} \sum_{l_2=0}^{\alpha_2} \cdots \sum_{l_d=0}^{\alpha_d} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_d \end{pmatrix} \partial^{\alpha_1-\beta_1} \cdots \partial^{\alpha_d-\beta_d} f \cdot \partial^l h_i \otimes \varepsilon_l = J(f) \cdot h_i
\]

which shows that $J(f \cdot h) = J(f) \cdot h$.

Applying the above construction to the Hilbert module $\mathcal{M}$ we have the module of jets $J(\mathcal{M})$. As in the case of Hilbert submodule $\mathcal{M}_0$ of the Hilbert module $\mathcal{M}$ it is clear that the subspace

\[
J(\mathcal{M})_0 := \{ h \in J(\mathcal{M}) : h|_Z = 0 \}
\]

is a submodule of $J(\mathcal{M})$. Let $J(\mathcal{M})_q$ be the quotient module obtained by taking orthogonal complement of $J(\mathcal{M})_0$ in $J(\mathcal{M})$, that is, $J(\mathcal{M})_q := J(\mathcal{M}) \ominus J(\mathcal{M})_0$. The following theorem provides the equivalence of two quotient modules $\mathcal{M}_q$ and $J(\mathcal{M})_q$.

**Theorem 4.2.** $\mathcal{M}_q$ and $J(\mathcal{M})_q$ are isomorphic as modules over $\mathcal{A}(\Omega)$.

**Proof.** Let us begin with a naturally arising spanning set of $\mathcal{M}_q$. Since, for $w \in \Omega$ and $\zeta \in \mathcal{C}^r$, $K(., w)\zeta \in \mathcal{M}$ it is evident that $\bar{\partial}^\alpha K(., w)\zeta \in \mathcal{M}$ for $\zeta \in \mathcal{C}^r$ and $\alpha \in A$. Thus, using the reproducing property of $K$, we note that $\langle h, \bar{\partial}^\alpha K(., w)\zeta \rangle = \langle \partial^\alpha h(w), \zeta \rangle$ where $h(\cdot) = \sum_{n=1}^{\infty} a_n e_n(\cdot)$ is any vector in $\mathcal{M}$ and $\{e_n(\cdot)\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{M}$.

Then from the above calculation together with the identity obtained by differentiating both sides of the equation (2.1) we have, for $\zeta \in \mathcal{C}^r$, $w \in Z$, $\alpha \in A$, and $h \in \mathcal{M}_0$, that

\[
\langle h, \bar{\partial}^\alpha K(., w)\zeta \rangle = \langle \partial^\alpha h(w), \zeta \rangle = 0
\]

which in turn implies that $D := \{\bar{\partial}^\alpha K(., w)\sigma_i : w \in Z, 1 \leq i \leq r, \alpha \in A\}$ is contained in $\mathcal{M}_0^\perp$, that is, $(\text{span}D)^\perp \subset \mathcal{M}_0$ and $\mathcal{M}_0 \subset D^\perp = (\text{span}D)^\perp$. Consequently, $D$ is a spanning set for $\mathcal{M}_q$. 


Now we claim that $J(D)$ spans $J(\mathcal{M})_q$. In order to establish the claim we follow the same strategy as before, in other words, we first show that $J(D)^\perp = J(\mathcal{M})_0$. So let us recall the definition of the operator $J$ and try to understand the set $J(D)$. Note that, by definition of the map $J$, we have

$$J(D) = \{ J(\bar{\partial}^\alpha K(.,w)\sigma_i) : w \in \mathbb{Z}, 1 \leq i \leq r, \alpha \in A \} = \{ JK(.,w)\varepsilon_j \otimes \sigma_1 : 0 \leq j \leq N, 1 \leq i \leq r, w \in \mathbb{Z} \}.$$

As before, for $h \in J(\mathcal{M})$, $1 \leq i \leq r$, and $w \in \Omega$, from the reproducing property of $JK$ we have

$$\langle h, JK(.,w)\varepsilon_j \otimes \sigma_i \rangle_{J(\mathcal{M})} = \langle h(w),\varepsilon_j \otimes \sigma_i \rangle_{\mathbb{C}^{(N+1)r}} = \partial^i h_i(w), \ 0 \leq j \leq N,$$

which justifies our claim. Hence we conclude that $J(D)$ spans the quotient space $J(\mathcal{M})_q$, that is,

$$J(\mathcal{M})_q = J(\text{span}D) = \text{span} J(D) = J(\mathcal{M})_q.$$

Now in course of completion of our proof it remains to check that $J$ is a module isomorphism from $\mathcal{M}$ onto $J(\mathcal{M})_q$. In other words, we need to verify the following identity

$$J \circ P \circ M_f = (JP) \circ J_f \circ J,$$

for $f \in A(\Omega)$, which is equivalent to show that

$$JM_f^* P = J_f^*(JP)J$$

where $JP : J(\mathcal{M}) \to J(\mathcal{M})_q$ is the orthogonal projection operator. Since it amounts to show that $J$ intertwines the module actions on $D$ and both $P$ and $JP$ are identity on $D$ and $J(D)$, respectively, it is enough to prove that

$$JM_f^* = J_f^* J, \text{ for } f \in A(\Omega), \text{ on } D.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in A, 1 \leq i \leq r$ and $\bar{\partial}^\alpha K(.,w)\sigma_i \in D$ (we refer the readers (2.9) for the notation $\bar{\partial}^\alpha$). For $f \in A(\Omega)$, $w \in \mathbb{Z} \subset \Omega$, we have

$$M_f^* K(.,w)\sigma_i = \bar{f}(w) K(.,w)\sigma_i.$$

Then differentiating both sides of the above equation and using induction on degree of the differentiation and adopting the notation introduced in (2.9) we obtain

$$M_f^* \bar{\partial}^\alpha K(.,w)\sigma_i = \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_d=0}^{\alpha_d} \bar{\partial}_{1}^{\alpha_1-\beta_1} \cdots \bar{\partial}_{d}^{\alpha_d-\beta_d} \overline{f(w)\partial_{1}^{\beta_1} \cdots \partial_{d}^{\beta_d} K(.,w)}\sigma_i.$$

Therefore,

$$J(M_f^* \bar{\partial}^\alpha K(.,w)\sigma_i) = \sum_{l=0}^{N} \partial^l \left[ \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_d=0}^{\alpha_d} \bar{\partial}_{1}^{\alpha_1-\beta_1} \cdots \bar{\partial}_{d}^{\alpha_d-\beta_d} \overline{f(w)\partial_{1}^{\beta_1} \cdots \partial_{d}^{\beta_d} K(.,w)} \right] \otimes \varepsilon_l \otimes \sigma_i$$

$$= \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_d=0}^{\alpha_d} \bar{\partial}_{1}^{\alpha_1-\beta_1} \cdots \bar{\partial}_{d}^{\alpha_d-\beta_d} \overline{f(w) JK(.,w)(\varepsilon_{\theta(\beta)} \otimes \sigma_i)} \quad \beta = (\beta_1, \ldots, \beta_d)$$

$$= JK(.,w) \bar{f}(w) J \langle J(f)(w) \rangle^* (\varepsilon_{\theta(\alpha)} \otimes \sigma_i).$$

Thus, for $h \in J(\mathcal{M})$, $\zeta \in \mathbb{C}^{(N+1)r}$ and $w \in \Omega$, we have

$$\langle h, J_f^* JK(.,w) : \zeta \rangle_{J(\mathcal{M})} = \langle h(w), J(f)(w)^* \zeta \rangle_{\mathbb{C}^{(N+1)r}} = \langle h, J K(.,w) J(f)(w)^* \zeta \rangle_{J(\mathcal{M})}.$$

This completes the roof. \qed
Remark 4.3. Note that as mentioned in [12] the Theorem 4.2 is equivalent to the fact that the following diagram of exact sequences is commutative:

\[
\begin{array}{ccccccccc}
& 0 & \longrightarrow & \mathcal{M}_0 & i & \rightarrow & \mathcal{M} & P & \rightarrow & \mathcal{M}_q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \longrightarrow & J(\mathcal{M})_0 & i & \rightarrow & J(\mathcal{M}) & JP & \rightarrow & J(\mathcal{M})_q & \longrightarrow & 0
\end{array}
\]

In [1] it was shown that for a reproducing kernel Hilbert space \( \mathcal{H} \) with scalar valued reproducing kernel \( K \) on some set \( W \), the restriction of \( K \) on a subset \( W_1 \) of \( W \) is also a reproducing kernel and restriction of \( K \) to \( W_1 \) constitutes a reproducing kernel Hilbert space which is isomorphic to the quotient space \( \mathcal{H} \oplus \mathcal{H}_0 \) where \( \mathcal{H}_0 := \{ f \in \mathcal{H} : f|_{W_1} = 0 \} \). Here, adopting the proof from [1] for our case with vector valued kernel, we have the following theorem. Since this result is well known (Theorem 3.3, [12]) for the case while the codimension of the submanifold, \( Z \), is one and using the techniques used in that proof in a similar way one can the following theorem, we omit the proof.

Theorem 4.4. The normed linear space \( J(\mathcal{M})|_{\text{res}Z} \) is a Hilbert space and the Hilbert spaces \( J(\mathcal{M})_q \) and \( J(\mathcal{M})|_{\text{res}Z} \) are unitarily equivalent. Consequently, the reproducing kernel \( K_1 \) for \( J(\mathcal{M})|_{\text{res}Z} \) is the restriction of the kernel \( JK \) to the submanifold \( Z \). Moreover, \( J(\mathcal{M})_q \) and \( J(\mathcal{M})|_{\text{res}Z} \) are isomorphic as modules over \( \mathcal{A}(\Omega) \).

Theorem 4.5. The quotient module \( \mathcal{M}_q \) is equivalent to the module \( J(\mathcal{M})|_{\text{res}Z} \) over \( \mathcal{A}(\Omega) \).

Proof. It is obvious from Theorem 4.2 and Theorem 4.4. \( \square \)

We now provide a necessary condition for equivalence of two quotient modules in the following theorem.

Theorem 4.6. Let \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) be Hilbert modules in \( B_1(\Omega) \) and \( \mathcal{M}_0 \) and \( \tilde{\mathcal{M}}_0 \) be the submodules of functions in \( \mathcal{A}(\Omega) \) vanishing along \( Z \) to order \( k \). If \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are equivalent as Hilbert modules over \( \mathcal{A}(\Omega) \) then the corresponding quotient modules \( \mathcal{M}_q \) and \( \tilde{\mathcal{M}}_q \) are also equivalent as Hilbert modules over \( \mathcal{A}(\Omega) \).

Proof. Let us begin with a unitary module map \( T : \mathcal{M} \rightarrow \tilde{\mathcal{M}} \). Then, following [6], we have that there is a non-vanishing holomorphic function \( \psi : \Omega \rightarrow \mathbb{C} \) such that \( T = T_\psi \) where \( T_\psi : \mathcal{M} \rightarrow \tilde{\mathcal{M}} \) defined by \( T_\psi f = \psi f \).

Now recalling the definition (4.3) of the unitary operator \( J \) we note that \( T_\psi \) gives rise to the module map \( J_\psi : J(\mathcal{M}) \rightarrow J(\tilde{\mathcal{M}}) \) by the formula \( J_\psi := J \circ T_\psi \circ J^* \) and

\[ J_\psi(h) = J \circ T_\psi \circ J^*(h) = J(\psi h) = \mathcal{J}(\psi)h \]

which is actually a unitary module map. Since \( \psi \) is non-vanishing the definition (4.6) of \( \mathcal{J}(\psi) \) ensures that \( J(\mathcal{M})_0 \) gets mapped onto \( J(\tilde{\mathcal{M}})_0 \) by \( J_\psi \) and hence \( J(\mathcal{M})_q \) is equivalent to \( J(\tilde{\mathcal{M}})_q \). Thus, we are done thanks to Theorem 4.2. \( \square \)

Remark 4.7. Let us now clarify the module action of \( \mathcal{A}(\Omega) \) on the quotient module \( \mathcal{M}_q \) before proceeding further. To facilitate this action we, following [12, page 384], consider the algebra of holomorphic functions on \( \Omega \) taking values in \( \mathbb{C}^{N+1} \) with \( N = \binom{d+k-1}{k-1} - 1 \),

\[ J\mathcal{A}(\Omega) := \{ Jf : f \in \mathcal{A}(\Omega) \} \subset \mathcal{A}(\Omega) \otimes \mathbb{C}^{(N+1) \times (N+1)} \]

with the multiplication defined by the usual matrix multiplication, namely, \( (Jf \cdot Jg)(z) := Jf(z)Jg(z) \). Then from (4.6) it is clear that \( J(\mathcal{M})|_{\text{res}Z} \) is a module over the algebra
\( J\mathcal{A}(\Omega)|_{resZ} \) obtained by restricting \( J\mathcal{A}(\Omega) \) to the submanifold \( Z \). Note that \( J \) defines an algebra isomorphism from \( \mathcal{A}(\Omega) \) onto \( J\mathcal{A}(\Omega) \) and intertwines the restriction operators \( R_1 : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)|_{resZ} \) and \( R_2 : J\mathcal{A}(\Omega) \to J\mathcal{A}(\Omega)|_{resZ} \). Consequently, \( J : \mathcal{A}(\Omega)|_{resZ} \to J\mathcal{A}(\Omega)|_{resZ} \) is also an algebra isomorphism. So \( J(\mathcal{M})|_{resZ} \) can be thought of as a Hilbert module over \( J\mathcal{A}(\Omega)|_{resZ} \).

On the other hand, considering the inclusion \( i : Z \to \Omega \) we see that \( i \) induces a map \( i^* : J\mathcal{A}(\Omega) \to J\mathcal{A}(\Omega)|_{resZ} \) defined by \( i^*(\mathcal{J}f)(z) = \mathcal{J}f(i(z)) \), for \( z \in Z \). Now one can make \( J(\mathcal{M})|_{resZ} \) to a module over the algebra \( J\mathcal{A}(\Omega) \) by pushing it forward under the map \( i^* \), that is,

\[
\mathcal{J}f \cdot h|_Z := i^*(\mathcal{J}f)h|_Z.
\]

Thus, recalling the fact that \( J \) defines an algebra isomorphism between \( \mathcal{A}(\Omega) \) and \( J\mathcal{A}(\Omega) \), we can think of \( J(\mathcal{M})_q \) as a module over \( \mathcal{A}(\Omega) \).

Since the similar construction can be done for the Hilbert modules \( \mathcal{M} \in B_r(\Omega) \) with submodules \( \mathcal{M}_0 \) consisting of holomorphic functions \( \mathcal{A}(\Omega) \) vanishing along \( Z \) to order \( k \), one can also ask whether the quotient modules arising from such submodules are in \( B_{(N+1)r}(Z) \). In the following theorem, we give an affirmative answer of this for a simple class of Hilbert modules in \( B_1(\Omega) \).

**Theorem 4.8.** Let \( \Omega \subset \mathbb{C}^m \) be a bounded domain containing the origin and \( Z \subset \Omega \) be the coordinate plane defined by \( Z := \{z = (z_1, \ldots, z_m) \in \Omega : z_1 = \cdots = z_d = 0\} \). We also assume that \( \mathcal{M} \) is a reproducing kernel Hilbert space with the property that the reproducing kernel \( K \) has diagonal power series expansion, that is, for \( z, w \in \Omega \)

\[
K(z, w) = \sum_{\alpha \geq 0} a_\alpha (z-z_0)^\alpha \overline{(w-w_0)}^\alpha,
\]

for some \( z_0, w_0 \in Z \). Then the quotient module \( \mathcal{M}_q \) restricted to a module over \( \mathcal{A}(Z) \) lies in \( B_{N+1}(Z) \) provided \( \mathcal{M} \in B_1(\Omega) \).

**Proof.** In view of Theorem 4.5 and Remark 4.7, it is enough to prove that the module of jets, \( J(\mathcal{M}) \) restricted to \( Z \) is in \( B_{(N+1)}(Z) \). Let \( JK|_Z \) be the reproducing kernel of \( J(\mathcal{M})|_{resZ} \). Then \( JK|_Z \) has the following power series expansion at \( (z_0, w_0) \in Z \):

\[
JK|_Z(\tilde{z}, \tilde{w}) = \sum_{\lambda, \mu \geq 0} A_{\lambda\mu}(\tilde{z} - z_0)^\lambda \overline{(\tilde{w} - w_0)}^\mu,
\]

where \( \tilde{z}, \tilde{w} \in Z, \lambda, \mu \in (\mathbb{N} \cup \{0\})^{m-d} \), and \( A_{\lambda\mu} \in M_{N+1}(\mathbb{C}) \) are defined by the following formula

\[
A_{\lambda\mu} = \partial_{d+1}^{\lambda_1} \cdots \partial_{d+1}^{\lambda_{m-d}} \partial_{d+1}^{\mu_1} \cdots \partial_{d+1}^{\mu_{m-d}} JK|_Z(z_0, w_0).
\]

Therefore, using the definition of \( JK|_Z \) we get, for \( 0 \leq l, k \leq N \), that

\[
(A_{\lambda\mu})_{lk} = \partial_{d+1}^{\lambda_1} \cdots \partial_{d+1}^{\lambda_{m-d}} \partial_{d+1}^{\mu_1} \cdots \partial_{d+1}^{\mu_{m-d}} \partial_{d+1}^{\alpha_1} \cdots \partial_{d+1}^{\alpha_1} \cdots \partial_{d+1}^{\beta_{l-1}} \cdots \partial_{d+1}^{\beta_{l-1}} \partial_{d+1}^{\beta_1} \cdots \partial_{d+1}^{\beta_1} K(z_0, w_0)
\]

where \( (A_{\lambda\mu})_{lk} \) is the \( lk \)-th entry of the matrix \( A_{\lambda\mu} \) and \( \theta^{-1}(l) = (\alpha_1, \ldots, \alpha_d), \theta^{-1}(k) = (\beta_1, \ldots, \beta_d) \).

Since \( K \) has a diagonal power series expansion it is clear, from the equation (4.10), that \( A_{\lambda\mu} = 0 \) unless \( \lambda = \mu \). Moreover, in a similar way the same equation also shows that \( (A_{\lambda\lambda})_{lk} = 0 \) if \( l \neq k \) and

\[
(A_{\lambda\lambda})_{ll} = a_{\alpha_1, \ldots, \alpha_d, \lambda_1, \ldots, \lambda_{m-d}}.
\]

Now from the hypothesis we have that the Taylor coefficients, \( a_\alpha \) satisfy the inequality stated in part (b) of the Theorem 5.4 in [6]. As a consequence, a straightforward calculation using the equation (4.11) shows that the matrices \( A_{\lambda\lambda} \) also satisfy the same inequality in [6].
but with matrix valued constants. Furthermore, it follows, from the equation (4.11), that the coordinate functions of \( Z \) act on \( J(\mathcal{M})_{|\text{res}Z} \) by weighted shift operators with weights determined by matrices \( A_{\lambda \lambda} \). Therefore, by the same theorem in [6] the Hilbert module, \( J(\mathcal{M})_{|\text{res}Z} \), as a module over \( \mathcal{A}(Z) \) is in \( B_{N+1}(Z) \).

We note that the above theorem provides examples of quotient modules which are in the Cowen-Douglas class. Thus, it motivates to consider the following class of Hilbert modules.

**Definition 4.9.** Let \( \Omega \subset \mathbb{C}^m \) be bounded domain and \( Z \subset \Omega \) be a connected complex submanifold of codimension \( d \). Then we say that the pair of Hilbert modules \( (\mathcal{M}, \mathcal{M}_q) \) over the algebra \( \mathcal{A}(\Omega) \) is in \( B_{r,k}(\Omega, Z) \) if

1. \( \mathcal{M} \in B_r(\Omega) \);
2. there exists a resolution of the module \( \mathcal{M}_q \) as in (4.1) where the module \( \mathcal{M} \) appearing in the resolution is quasi-free of rank \( r \) over the algebra \( \mathcal{A}(\Omega) \);
3. for \( f \in \mathcal{A}(\Omega) \), the restriction of the map \( J_f \) to the submanifold defines the module action on \( J(\mathcal{M})_{|\text{res}Z} \) which is an isomorphic copy of \( \mathcal{M}_q \); and
4. the quotient module \( \mathcal{M}_q \) as a module over the algebra \( \mathcal{A}(\Omega)_{|\text{res}Z} \) is in \( B_{(N+1)r}(Z) \) where \( N = \left( d + k - 1 \right) - 1 \).

5. **Jet Bundle**

This section is devoted to provide geometric invariants of quotient modules introduced in the previous section. Suppose, to begin with, we have the Hilbert module \( \mathcal{M} \) in \( B_r(\Omega) \) with the submodule \( \mathcal{M}_0 \) and quotient module \( \mathcal{M}_q \), as introduced in Section 4, satisfying the exact sequence (4.1). Then we have, following Remark 2.2, that \( \mathcal{M} \) gives rise to a Hermitian holomorphic vector bundle \( E \) with the frame \( \{ K(\cdot, \overline{\pi})\sigma_1, \ldots, K(\cdot, \overline{\pi})\sigma_r : w \in \Omega^* \} \) on \( \Omega^* \). Now to make calculations simpler let us consider the map \( c : \Omega \to \Omega^* \) defined by \( w \mapsto \overline{\pi} \) and pull back the bundle \( E \) to a vector bundle over \( \Omega \). Then we denote this new bundle with the same letter \( E \) and note that \( E \) is a Hermitian holomorphic vector bundle over \( \Omega \) with the global holomorphic frame \( s := \{ s_1(w), \ldots, s_r(w) : w \in \Omega \} \) with \( s_j(w) := K(\cdot, \overline{\pi})\sigma_j, 1 \leq j \leq r \). Correspondingly, we have \( \partial^j s_j(w) = \partial^j K(\cdot, \overline{\pi})\sigma_j, 1 \leq j \leq r, 0 \leq l \leq N \), where \( N = \left( d + k - 1 \right) - 1 \).

The jet bundle construction of a line bundle relative to a hypersurface, introduced in the paper [12], involves the frame of the line bundle and directional derivatives of the frame in the normal direction to the hypersurface. We have generalized in previous section this notion of jet construction for a Hilbert module relative to a smooth irreducible complex analytic set of arbitrary codimension. In this section, we attempt to describe the same for vector bundles obtained above, that is, the jet bundle construction for such a trivial vector bundle relative to a connected complex submanifold of codimension \( d \geq 1 \).

We start with a Hermitian holomorphic vector bundle \( E \) over \( \Omega \) corresponding to the Hilbert module \( \mathcal{M} \in B_r(\Omega) \) described above and \( Z \subset \Omega \) is a connected complex submanifold of codimension \( d \). Without loss of generality assume that \( 0 \in Z \) and let \( (U, \phi) \) be an admissible coordinate chart (Definition 3.9) at 0 of \( Z \). So pretending \( U \) as \( \Omega \) we have that \( \phi(\Omega \cap Z) = \{ w \in \phi(\Omega) : w_1 = \cdots = w_d = 0 \} \). Since we are interested to investigate unitary invariants of the quotient module \( \mathcal{M}_q \) with \( (\mathcal{M}, \mathcal{M}_q) \in B_{r,k}(\Omega, Z) \), following the Proposition 3.12, it is enough to consider the submanifold \( \phi(\Omega \cap Z) \subset \phi(\Omega) \). Therefore, pretending \( \phi(\Omega) \) as \( \Omega \), we consider the submanifold \( Z \) defined as

\[
Z := \{ z = (z_1, \ldots, z_m) \in \Omega : z_1 = \cdots = z_d = 0 \}.
\]

We then define the jet bundle \( J^kE \) of order \( k \) of \( E \) relative to the submanifold \( Z \) on \( \Omega \) by declaring \( \{ s, \partial^1 s, \ldots, \partial^N s \} \) as a frame for \( J^kE \) on \( \Omega \) where the differential operators \( \partial^j \),
0 ≤ j ≤ N are as introduced in (2.9), and by \( \partial^k s \) we mean the ordered set of sections \( \{ \partial^k s_1, \ldots, \partial^k s_r \} \), 0 ≤ l ≤ N. Since we have a global frame on \( J^k E \) we do not need to worry about the transition rule.

At this point, we should note that our construction depends on the choice of the normal direction to \( \mathcal{Z} \) which is, a priori, not unique. Nevertheless one way to show that our construction is essentially unambiguous is the following proposition.

Proposition 5.1. Let \((U_1, \phi_1)\) and \((U_2, \phi_2)\) be two admissible coordinate charts of \( \Omega \) around some point \( p \in \mathcal{Z} \). Then two jet bundles \( J^k_1 E \) and \( J^k_2 E \) obtained as above with respect to \((U_1, \phi_1)\) and \((U_2, \phi_2)\), respectively, are equivalent holomorphic vector bundles over \( U_1 \cap U_2 \).

Proof. In fact, from Proposition 3.10 it is clear, for a frame \( s = \{ s_1, \ldots, s_r \} \) of \( E \) on \( U_1 \cap U_2 \), that on a small enough neighbourhood \( U \) of \( p \) in \( U_1 \cap U_2 \) we have, for \( i = 1, 2 \),

\[
A_{k,\phi_i}(z) \cdot \left( \begin{array}{c} s_{01}^i(\lambda) \\ \vdots \\ s_{N1}^i(\lambda) \end{array} \right) = \left( \begin{array}{c} s_1(z) \\ \vdots \\ s_N(z) \end{array} \right),
\]

for \( z \in U \) and \( \lambda_i = \phi_i(U) \), where \( \lambda_i = (\lambda_{i1}, \ldots, \lambda_{id}) \), \( (\alpha_1, \ldots, \alpha_d) = \theta^{-1}(l) \), and \( s_{ij}^i = \frac{\partial^{\alpha_1}_{\lambda_{i1}} \cdots \partial^{\alpha_d}_{\lambda_{id}}}{\partial^{\alpha_1}_{\lambda_{i1}} \cdots \partial^{\alpha_d}_{\lambda_{id}}} (s_j \circ \phi_i^{-1}) \), for \( 1 \leq j \leq r \). Since \( A_{k,\phi_i}(z) \), for \( i = 1, 2 \) and \( z \in U \), are invertible (Proposition 3.10) we can see that \( (A_{k,\phi_1}(z) \circ A_{k,\phi_2}(z)^{-1}) \otimes I_r \) is the desired bundle map where \( I_r \) is the identity matrix of order \( r \).

Now in course of completing our construction to make the jet bundle \( J^k E \) a Hermitian holomorphic vector bundle we need to put a Hermitian metric on \( J^k E \) compatible with the metric on \( E \). To this extent, if \( H(w) = ((s_i(w), s_j(w)))_{i,j=1}^r \) is the metric on \( E \) over \( \Omega \) then the Hermitian metric on \( J^k E \) with respect to the frame \( \{ s, \partial s, \ldots, \partial^N s \} \) is given by the Grammian \( JH := ((JH_{il}))_{i,j=0}^{N} \) with \( r \times r \) blocks

\[
(JH_{il}(w)) := (((\partial^l s_i(w), \partial^l s_j(w))))_{i,j=1}^r \text{ for } 0 \leq l, t \leq N, w \in \Omega.
\]

This completes our construction of the jet bundle.

Remark 5.2. Note that, for the Hilbert module \( \mathcal{M} \) over \( \mathcal{A}(\Omega) \) with the corresponding Hermitian holomorphic vector bundle \( E \) over \( \Omega \), the Hermitian holomorphic vector bundle \( \mathcal{E} \) obtained from \( J(\mathcal{M}) \) is equivalent to the jet bundle \( J^k E|_{\mathcal{Z}} \rightarrow \mathcal{Z} \) of \( E \) relative to \( \mathcal{Z} \). To facilitate, let \( \mathcal{M} \) be a reproducing kernel Hilbert module over \( \mathcal{A}(\Omega) \) which is in \( \mathcal{B}_r(\Omega) \). Let \( K = ((K_{ij}))_{i,j=1}^r \) be the reproducing kernel of \( \mathcal{M} \). Then from the preceding construction we have that the metric for the jet bundle is given by the formula

\[
\langle \partial^l K(., w)\sigma_i, \partial^l K(., w)\sigma_j \rangle = \partial^l \partial^l K_{ij}(w, w) \text{ for } w \in \Omega, \ 0 \leq l, t \leq N, \ 1 \leq i, j \leq r.
\]

On the other hand, the jet construction presented in Section 4 gives rise to the Hilbert module \( J(\mathcal{M}) \) where \( J \) is the unitary module map \( J : \mathcal{M} \rightarrow J(\mathcal{M}) \). Therefore, the vector bundle \( \mathcal{E} \) is unitarily equivalent to \( J^k E \).

Note that the action of the algebra \( \mathcal{A}(\Omega) \) on the module \( J(\mathcal{M}) \) defines, for every \( f \in \mathcal{A}(\Omega) \), a holomorphic bundle map \( \Psi_f : J^k E \rightarrow J^N E \) whose matrix representation with respect to the frame \( J(s) := \{ \sum_{i=1}^N \partial^i s_1 \otimes \varepsilon_i, \ldots, \sum_{i=1}^N \partial^i s_r \otimes \varepsilon_i \} \) is the matrix \( J(f) \otimes I_r \), where \( J(f) \) is as in (4.6) and \( I_r \) is the identity matrix of order \( r \). Thus, \( \Psi_f \) induces an action of \( \mathcal{A}(\Omega) \) on the holomorphic sections of the jet bundle \( J^k E \) defined by

\[
f \cdot \sigma(w) := \Psi_f(\sigma)(w),
\]
for \( f \in \mathcal{A}(\Omega) \), \( w \in \Omega \) and \( \sigma \) is a holomorphic section of \( J^k E \).

Therefore, we observe that the question of determining the equivalence classes of modules \( J(\mathcal{M}) \) is same as understanding the equivalence classes of the jet bundles \( J^k E \) with an additional assumption that the equivalence bundle map is also a module map on holomorphic sections over \( \mathcal{A}(\Omega) \). Hence it is natural to give the following definition (Definition 4.2, [11]).

**Definition 5.3.** Two jet bundles are said to be equivalent if there is an isometric holomorphic bundle map which induces a module isomorphism of the class of holomorphic sections.

5.1. **Main results from Jet bundle.** In order to find geometric invariants of quotient modules we first investigate the simple case, \( d = k = 2 \). We show here that the curvature is the complete set of unitary invariants of the quotient module \( \mathcal{M}_q \) for a quasi-free Hilbert module \( \mathcal{M} \) of rank 1. For this case, we give a computational proof to depict the actual picture behind the general result which we will prove later in this subsection. Although the line of idea of the proof for \( k = 2 \) essentially is same as in [11], in our case calculations become more complicated as here we have to deal with more than one transversal directions to \( Z \). Thus, our results extend most of the results of the paper [12], [11] as well as those from a recent paper [3].

Without loss of generality, under some suitable change of coordinates, we can assume that \( 0 \in \Omega \) and \( U \) is a neighbourhood of 0 such that \( U \cap Z = \{(z_1, \ldots, z_m) \in \Omega : z_1 = z_2 = 0 \} \). Consequently, \((0, 0, z_3, \ldots, z_m) \) is the coordinates of \( Z \) in \( U \). Now let us begin with a line bundle \( E \) over \( U^* \) with the real analytic metric \( G \) which possesses the following power series expansion

\[
G(z', z'') = \sum_{\alpha, \beta = 0}^{\infty} G_{\alpha \beta}(z'') z^{\alpha} \overline{z}^{\beta}
\]

where \((z', z'') \in U^*, \alpha, \beta \) are multi-indices, \( z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2}, \overline{z}^{\beta} = \overline{z}_1^{\beta_1} \overline{z}_2^{\beta_2} \) and \( z'' = (z_3, \ldots, z_m) \).

**Lemma 5.4.** Let \( \Omega \subset \mathbb{C}^m \) be a bounded domain and \( Z \) be a complex connected submanifold of \( \Omega \) of codimension 2. Suppose that \( \mathcal{K} \) and \( \bar{\mathcal{K}} \) are the curvature tensors of line bundles \( E \) and \( \bar{E} \) with respect to the Hermitian metric \( p \) and \( \bar{p} \) of \( E \) and \( \bar{E} \), respectively. Then \( \mathcal{K} \) and \( \bar{\mathcal{K}} \) are equal on \( Z \) if and only if there exists holomorphic functions \( \psi_{00}, \psi_{10}, \psi_{01} \) on \( Z \) such that

\[
((\rho_{\theta-1(i)\theta-1(j)})_{i,j=0}^2 = \Psi \cdot ((\rho_{\theta-1(i)\theta-1(j)})_{i,j=0}^2 \cdot \Psi^*
\]

on \( Z \) where \( \theta \) is as in (2.7) and \( \Psi \) is the \( 3 \times 3 \) matrix

\[
\Psi = \begin{pmatrix}
\psi_{00} & 0 & 0 \\
\psi_{10} & \psi_{00} & 0 \\
\psi_{01} & 0 & \psi_{00}
\end{pmatrix}
\]

Before going into the proof of the lemma let us give an application of it as follows.

**Theorem 5.5.** Suppose that \( \mathcal{M} \) and \( \bar{\mathcal{M}} \) are pair of quasi-free Hilbert modules of rank 1 over \( \mathcal{A}(\Omega) \) and that \( E \) and \( \bar{E} \) are the line bundles corresponding to \( \mathcal{M} \) and \( \bar{\mathcal{M}} \), respectively. Let \( \mathcal{M}_q = \mathcal{M} \otimes \mathcal{M}_0 \) and \( \bar{\mathcal{M}}_q = \bar{\mathcal{M}} \otimes \bar{\mathcal{M}}_0 \) be a pair of quotient modules of Hilbert modules \( \mathcal{M} \) and \( \bar{\mathcal{M}} \), respectively, over \( \mathcal{A}(\Omega) \). Assume that \((\mathcal{M}, \mathcal{M}_q)\) and \((\bar{\mathcal{M}}, \bar{\mathcal{M}}_q)\) are in \( B_{1,2}(\Omega, Z) \). Then the quotient modules \( \mathcal{M}_q \) and \( \bar{\mathcal{M}}_q \) are isomorphic if and only if the corresponding curvature tensors \( \mathcal{K} \) and \( \bar{\mathcal{K}} \) of the line bundles \( E \) and \( \bar{E} \), respectively, are equal on \( Z \).

**Proof.** In fact, Theorem 4.5 provides that equivalence of \( \mathcal{M}_q \) and \( \bar{\mathcal{M}}_q \) is same as the equivalence of \( J(\mathcal{M})|_{res Z} \) and \( J(\bar{\mathcal{M}})|_{res Z} \). So let us begin with an isometric module map \( \Psi : J(\mathcal{M})|_{res Z} \rightarrow \)
$J(\mathcal{M})_{\text{res} Z}$. Since $\Psi$ intertwines the module action $\Psi$ is of the form given in (5.4). Moreover, being an isometry, $\Psi$ satisfies

\begin{equation}
JK|_Z = \Psi \cdot J\tilde{K}|_Z \cdot \Psi^*
\end{equation}

which is equivalent to saying that $\Psi$ satisfies the identity (5.3) on $Z$ as, for $z \in Z$, $\rho(z)$ is nothing but $K(z, z)$. Then the Lemma 5.4 proves the necessity part.

Conversely, following the Lemma 5.4 the equality of curvature tensors $\mathcal{K}$ and $\tilde{\mathcal{K}}$ on $Z$ implies that $\Psi$ is of the form given in (5.4) and satisfies (5.5) which in turn yields that $\Psi$ is an isometry from $J(\mathcal{M})_{\text{res} Z}$ onto $J(\mathcal{M})_{\text{res} Z}$ and intertwines the module action. Hence this completes the proof. \hfill \square

Proof of Lemma 5.4. Let us begin with the assumption that there exists holomorphic functions $\psi_{00}, \psi_{10}, \psi_{01}$ on $Z$ such that (5.3) holds on $Z$. Then we wish to show that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are equal restricted to $Z$. We have, by the local expression of the curvature (2.3), that $\mathcal{K}_{1i} = \partial_i \bar{\partial}_j \log \rho$ and $\tilde{\mathcal{K}}_{1j} = \partial_i \bar{\partial}_j \log \tilde{\rho}$, for $i, j = 1, \ldots, m$. So let us calculate $\tilde{\mathcal{K}}_{1j}(z)$ for any point $z \in Z$ and $i, j = 1, \ldots, m$.

$$\tilde{\mathcal{K}}_{1j}(z) = \frac{\tilde{\rho} \partial_i \partial_j \tilde{\rho} - \partial_j \tilde{\rho} \partial_i \tilde{\rho}}{\tilde{\rho}^{\alpha}}$$

We also find that $\tilde{\mathcal{K}}_{1j}(z) = \mathcal{K}_{2i}(z)$ for $z \in Z$ by doing a similar calculation as in the case of $\mathcal{K}_{1i}(z)$. Now we calculate $\tilde{\mathcal{K}}_{12}(z)$.

$$\tilde{\mathcal{K}}_{12}(z) = \frac{\tilde{\rho} \partial_1 \partial_2 \tilde{\rho} - \partial_2 \tilde{\rho} \partial_1 \tilde{\rho}}{\tilde{\rho}^{\alpha}}$$

Since $\tilde{\mathcal{K}}_{1j}(z) = \mathcal{K}_{2i}(z)$ we have $\tilde{\mathcal{K}}_{2i}(z) = \mathcal{K}_{1i}(z)$ for $z \in Z$. Finally, let us calculate $\tilde{\mathcal{K}}_{11}(z)$, for $z \in Z$ and $2 < i \leq m$.

$$\tilde{\mathcal{K}}_{11}(z) = \frac{\tilde{\rho} \partial_i \partial_i \tilde{\rho} - \partial_i \tilde{\rho} \partial_i \tilde{\rho}}{\tilde{\rho}^{\alpha}}$$

Similarly one can show that $\tilde{\mathcal{K}}_{1i}(z) = \mathcal{K}_{1i}(z)$ for $z \in Z$ and $1 \leq i \leq m$. Thus, we are done with the converse part using the skew symmetry property of the matrix $(\mathcal{K}_{1j}(z))_{1 \leq j \leq 1}$. Now let us prove the forward direction, namely, assuming that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are equal along $Z$ we want to find $\psi_{00}, \psi_{10}, \psi_{01}$ holomorphic on $Z$ such that (5.3) holds.

Let $\tilde{\rho} = r \cdot \rho$, and $\Gamma = \log r$. Then $\Gamma$ is real analytic function on $\Omega$. We can, therefore, expand $\Gamma$ in power series, that is,

$$\Gamma(z', z'') = \sum_{\alpha, \beta = 0}^{\infty} \Gamma_{\alpha\beta}(z'') z'^\alpha \bar{z}'^\beta$$

where $\alpha, \beta$ are multi-indices, $z'^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$, $z'^\beta = \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$ and $z'' = (z_3, \ldots, z_m)$.

We have, from our assumption, that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are equal along $Z$ which is equivalent to the fact that $\partial_i \partial_j \Gamma = 0$, for $1 \leq i, j \leq m$, along $Z$. We separate out this into following three different cases.
Case I: $\partial_i \partial_j \Gamma = 0$ along $\mathcal{Z}$, for $i = 1, 2, j = 3, \ldots, m$.

For $i = 1$ we have $\partial_1 \partial_j \Gamma|_\mathcal{Z} = 0$, for $j = 3, \ldots, m$, which is, from (5.6), equivalent to $\partial_j \Gamma|_{\mathcal{Z}} = 0$ on $\mathcal{Z}$. In other words, $\Gamma|_{\mathcal{Z}}$ is holomorphic on $\mathcal{Z}$. Similarly considering the case with $i = 2$, we get $\Gamma|_{\mathcal{Z}}$ is also holomorphic on $\mathcal{Z}$.

Case II: $\partial_i \partial_j \Gamma = 0$ along $\mathcal{Z}$, for $i, j = 1, 2$.

In this case, we are considering the equation $\partial_i \partial_j \Gamma|_\mathcal{Z} = 0$, for $i, j = 1, 2$. For $i = j = 1$, we have $\partial_1 \partial_1 \Gamma|_\mathcal{Z} = 0$, that is, $\Gamma|_{\mathcal{Z}} = 0$ on $\mathcal{Z}$ and, for $i = j = 2, \Gamma|_{\mathcal{Z}} = 0$ on $\mathcal{Z}$. Finally, for $i = 1, j = 2$, it is easy to verify from the equation (5.6) that $\Gamma|_{\mathcal{Z}} = 0$ on $\mathcal{Z}$ and doing the same calculation with $i$ and $j$ interchanged we have $\Gamma|_{\mathcal{Z}} = 0$ on the submanifold $\mathcal{Z}$.

Case III: $\partial_i \partial_j \Gamma = 0$ along $\mathcal{Z}$, for $i, j = 3, \ldots, m$.

In this last case, we have $\partial_i \partial_j \Gamma|_\mathcal{Z} = 0$, for $i, j = 3, \ldots, m$, on $\mathcal{Z}$. Since $\mathcal{Z}$ is a complex submanifold with coordinates $z = (0, 0, z_3, \ldots, z_m) \in \mathcal{Z}$ the above equations together imply that $\Gamma|_{\mathcal{Z}} = 0$ on $\mathcal{Z}$ and some holomorphic functions $\psi_1, \psi_2$ on $\mathcal{Z}$.

Now, substituting the above coefficients in the equation (5.6) and noting that $\Gamma$ is real valued, we have
\[
\Gamma(z', z'') = \psi_1 + \beta_1 z_1 + \eta_1 z_2 + \bar{\psi}_2 + \bar{\beta}_2 z_1 + \bar{\eta}_2 z_2 + \text{ (terms of degree } \geq 3)\]
where $\psi_1, \beta_1, \eta_1, i = 1, 2$, are holomorphic functions on $\mathcal{Z}$. Since $\Gamma$ is a real valued function $\Gamma = \Gamma_{\mathcal{Z}}$ and hence we have
\[
(5.7) \quad \Gamma(z', z'') = \psi + \beta z_1 + \eta z_2 + \bar{\psi} + \bar{\beta} z_1 + \bar{\eta} z_2 + \text{ (terms of degree } \geq 3)\]
where $\psi = \frac{\psi_1 + \psi_2}{2}, \beta = \frac{\beta_1 + \beta_2}{2}$ and $\eta = \frac{\eta_1 + \eta_2}{2}$. So from the definition of $\Gamma$ we can write
\[
r = \exp \Gamma = |\exp \psi^2 \cdot (1 + \beta z_1 + \bar{\beta} z_1 + \beta^2 z_1 z_1 + \cdots) |^2 \cdot |(1 + \eta z_2 + \bar{\eta} z_2 + \eta^2 z_2 z_2 + \cdots)|^2 \cdots
\]
\[
= |\exp \psi^2 \cdot (1 + \beta z_1 + \eta z_2 + |\beta|^2 z_1 z_2 + |\eta|^2 z_2 z_2 + \cdots)\]
Thus, putting the above expression of $r$ in $\tilde{\rho} = r \cdot \rho$ and equating the coefficients of $\tilde{\rho}$ and $\rho$ we see that $\psi_0$, $\psi_10$ and $\psi_01$ with
\[
\psi_0 = \exp \psi, \psi_{10} = \exp \psi \beta, \psi_{01} = \exp \psi \eta
\]
yield our desired matrix in (5.4). 

It would be nice if one could carry forward the arguments used in the proof of Lemma 5.4 in order to achieve similar results in the case of arbitrary order of vanishing of vector valued functions. However, for general $k$, it would be cumbersome to continue the calculation done in the above Lemma. On the other hand, application of normalized frames makes the calculations simpler and enables us to get a conceptual proof in the general case as well. We adopt the idea of using normalized frame from the paper [3] in our case to provide the geometric invariants for quotient modules using jet bundle construction relative to a smooth complex submanifold of codimension $d$. To this extent the following theorem provides the required dictionary between the analytic theory and geometric theory for quotient modules obtained from submodules consisting of vector valued holomorphic functions on $\Omega$ vanishing along a smooth complex submanifold of codimension $d$. 

\[\square\]
Theorem 5.6. Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $Z$ be the complex submanifold in $\Omega$ of codimension $d$. Suppose that $(\mathcal{M}, \mathcal{N})$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ are in $B_{l,k}(\Omega, Z)$. Then the quotient modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are equivalent as modules over $A(\Omega)$ if and only if the jet bundles $J^lE|_{\text{res}Z}$ and $J^k\tilde{E}|_{\text{res}Z}$ are equivalent where $E$ and $\tilde{E}$ are the Hermitian holomorphic vector bundles over $\Omega$ corresponding to Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$, respectively.

Proof. Proof follows from Theorem 4.5 and Remark 5.2.

Thanks to Theorem 5.6 we are now prepared to determine geometric invariants of quotient modules $\mathcal{M}$ by studying the geometry of the jet bundles $J^lE|_{\text{res}Z}$, for $0 \leq l \leq k$. Before proceeding further, let us recall a fact from complex analysis.

Lemma 5.7. Let $\Omega \subset \mathbb{C}^m$ be a domain and $f(z, w)$ be a function on $\Omega \times \Omega$ which is holomorphic in $z$ and anti-holomorphic in $w$. If $f(z, z) = 0$ for all $z \in \Omega$, then $f(z, w) = 0$ identically on $\Omega$.

Since this lemma is well known [14, Proposition 1] we omit the proof. We use the lemma several times in the proof of following theorems.

Note also that a Hermitian holomorphic vector bundle can not have holomorphic orthonormal frame in general. Instead one can have (Lemma 2.4 of [4]) a holomorphic frame on a neighbourhood of a point which is orthonormal at that point. Then using the technique of the proof of Lemma 2.4 in [4] in a similar way, we have the following existence of normalized frame of a Hermitian holomorphic vector bundle over $\Omega$ along a submanifold of codimension at least $d$ in $\Omega$. In the following proposition we use the notation $z = (z', z'')$ where $z' = (z_1, \cdots, z_d)$ and $z'' = (z_{d+1}, \ldots, z_m)$.

Proposition 5.8. Let $E$ be a Hermitian holomorphic vector bundle of rank $r$ over a bounded domain $\Omega \subset \mathbb{C}^m$. Assume that $0 \in \Omega$ and $Z \subset \Omega$ is the submanifold defined by $z_1 = \cdots = z_d = 0$. Then there is a holomorphic frame $s(z', z'') = \{s_1(z', z''), \ldots, s_r(z', z'')\}$ on a neighbourhood of the origin in $\Omega$ such that $\{(\partial^ls_i(0, z''), \partial s_i(0, 0))\}_{i,l=1}^r$ is the identity matrix for any integer $l$ and $\{(s_i(0, z''), s_j(0, 0))\}_{i,j=1}^r$ is the zero matrix on $Z$.

We say a frame is normalized at origin if it satisfies the properties in the above proposition.

Theorem 5.9. Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $Z$ be the complex submanifold in $\Omega$ of codimension $d$ defined by $z_1 = \cdots = z_d = 0$. Assume that pair of Hilbert modules $(\mathcal{M}, \mathcal{N})$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ are in $B_{l,k}(\Omega, Z)$. Then $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are unitarily equivalent as modules over $A(\Omega)$ if and only if $\partial^l\bar{\partial}^j\|\tilde{s}\|^2 = \partial^l\bar{\partial}^j\|s\|^2$ on $Z$ for all $0 \leq l, j \leq N$ where $\{s(z)\}$ and $\{\tilde{s}(z)\}$ are frames of the line bundles $E$ and $\tilde{E}$ on $\Omega$ associated to the Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$, respectively, normalized at origin.

Proof. We begin with the observation, following Theorem 5.6, that the quotient modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are equivalent as modules over $A(\Omega)$ if and only if the jet bundles $J^lE|_{\text{res}Z}$ and $J^k\tilde{E}|_{\text{res}Z}$ are equivalent where $E$ and $\tilde{E}$ are the Hermitian holomorphic vector bundles over $\Omega$ corresponding to Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$, respectively.

We start with the necessity. Let $\Phi : J^lE|_{\text{res}Z} \to J^k\tilde{E}|_{\text{res}Z}$ be a jet bundle isomorphism. Consequently, by Definition 5.3, $\Phi$ intertwines the module actions on the space of holomorphic sections and preserves the Hermitian metrics. The isomorphism $\Phi$ can be represented by an $(N+1) \times (N+1)$ complex matrix $((\phi_{ij})_{i,j=0}^N$ with respect to the frames $\{s(0, z''), \partial s(0, z''), \ldots, \partial^Ns(0, z'')\}$ and $\{\tilde{s}(0, z''), \partial \tilde{s}(0, z''), \ldots, \partial^N\tilde{s}(0, z'')\}$ where $\phi_{ij}$ are holomorphic functions on $Z$.
Then in terms of matrices the fact that $\Phi$ is an isomorphism of two jet bundles $J^kE|_{\text{res}Z}$ and $J^k\tilde{E}|_{\text{res}Z}$ translates to the following two matrix equations on $Z$:

\[
(\mathbf{5.8}) \quad \mathcal{J}(\mathbf{s})^N_{j=0} = ((\mathbf{\phi}_{ij})^N_{i,j=0}((\mathbf{\partial^i s}, \mathbf{\partial^j z})))^N_{i,j=0}((\mathbf{\phi}_{ij})^N_{i,j=0})^* \\
(\mathbf{5.9}) \quad (\mathbf{\phi}_{ij})^N_{i,j=0}((\mathbf{J}(\mathbf{f}))^N_{i,k=0} = ((\mathbf{J}(\mathbf{f}))^N_{i,k=0}((\mathbf{\phi}_{ij})^N_{i,j=0})^N_{i,j=0}.
\]

Then the proof of the forward direction easily follows from the following claims.

**Claim 1.**

(a) For $0 \leq k \leq N$ and $z = (0, z'') \in Z$, $\phi_{kk}(0, z'') = \phi_{00}(0, z'')$.

(b) Let $1 \leq i, j \leq N$, $\alpha = (\alpha_1, \ldots, \alpha_d) = \theta^{-1}(i)$, $\beta = (\beta_1, \ldots, \beta_d) = \theta^{-1}(j)$. Then for $z \in Z$ we have

\[
(\mathbf{5.10}) \quad \phi_{ij}(0, z'') = \begin{cases} \alpha - \beta \phi_{\theta(\alpha - \beta)0}(0, z'') & \text{if } \alpha_t \geq \beta_t \forall t = 1, \ldots, d, \\
0 & \text{otherwise.} \end{cases}
\]

Before going into the proof of Claim 1, let us make some observations about the matrix $\mathcal{J}(\mathbf{f}) = ((\mathcal{J}(\mathbf{f})_{ik})^N_{i,k=0}$ which will be used in the proof. So to begin with, let $\gamma = (\gamma_1, \ldots, \gamma_d)$ with $\theta(\gamma) = l$, $0 \leq l \leq N$, and $\gamma_j \in \mathbb{N} \cup \{0\}$, $1 \leq j \leq d$. We also denote the subdiagonals of the matrix $\mathcal{J}(\mathbf{f})$ as $S_0, \ldots, S_N$, that is, $S_j$ is the set

\[
S_j := \{\mathcal{J}(\mathbf{f})_{ij} : j \leq l \leq N, 0 \leq t \leq N - j + 1, l - t = j - 1\}
\]

Thus, $S_0$ consists of all diagonal entries and $S_N$ is the singleton set $\{\mathcal{J}(\mathbf{f})_{NN}\}$. Then a simple calculation using the definition (4.6) of $\mathcal{J}(\mathbf{f})$ yields the following property.

**Claim 1.**

- For $f(z', z'') = z_1^{\gamma_1} \cdots z_d^{\gamma_d}$ with $\theta(\gamma_1, \ldots, \gamma_d) = l$, the matrix $\mathcal{J}(\mathbf{f})|_z$ has only non-zero entries along the subdiagonals, $S_j$, for $l \leq j \leq N$.

Now we note, for $l \leq i \leq N$, that common entries of $i$-th row of $\mathcal{J}(\mathbf{f})$ with subdiagonals $S_i, \ldots, S_N$ are $\mathcal{J}(\mathbf{f})_{ii}, \ldots, \mathcal{J}(\mathbf{f})_{ii-l+1}$, respectively. Therefore, for $l \leq i \leq N$ with $\theta^{-1}(i) = \alpha = (\alpha_1, \ldots, \alpha_d)$ and $f(z', z'') = z_1^{\gamma_1} \cdots z_d^{\gamma_d}$, the above property, (P1), shows that non-zero entries of $i$-th row of the matrix $\mathcal{J}(\mathbf{f})|_z$ must live in the set $\{\mathcal{J}(\mathbf{f})_{ij} : 0 \leq j \leq i - l + 1\}$. On the other hand, for $0 \leq j \leq i - l + 1$ with $\theta^{-1}(j) = \beta = (\beta_1, \ldots, \beta_d)$,

\[
\mathcal{J}(z_1^{\gamma_1} \cdots z_d^{\gamma_d})_{ij}|_z \neq 0 \iff \partial^{\alpha - \beta}(z_1^{\gamma_1} \cdots z_d^{\gamma_d})|_z \neq 0 \iff \beta = \alpha - \gamma.
\]

Thus, we have the following property of $\mathcal{J}(z_1^{\gamma_1} \cdots z_d^{\gamma_d})$.

**Claim 2.**

- For $\alpha, \gamma, i, l$ as above and $f(z', z'') = z_1^{\gamma_1} \cdots z_d^{\gamma_d}$, $\mathcal{J}(f)_{\theta(\alpha - \gamma)0}(0, z'')$ is the only non-zero entry of $i$-th row of $\mathcal{J}(\mathbf{f})|_z$. In particular, we observe that $\mathcal{J}(\mathbf{f})_{ij}(0, z'') = 0$, for any $j$ with $0 \leq j \leq N$ whenever $\alpha_t < \gamma_t$ for some $t \in \{1, \ldots, d\}$.

Since $\Phi$ is a jet bundle isomorphism between $J^kE|_{\text{res}Z}$ and $J^k\tilde{E}|_{\text{res}Z}$, by the definition (Definition 5.12) $\Phi$ commutes with the module action of $\mathcal{A}(\Omega)$ on the sections of the above jet bundles, namely, from (5.1) we have the equation (5.9) holds on $Z$ for all $f \in \mathcal{A}(\Omega)$. Let $g(z', z'') = z_1^{\eta_1} \cdots z_d^{\eta_d}$, for given $0 \leq k \leq N$ with $\theta^{-1}(k) = (\eta_1, \ldots, \eta_d)$. Then from (P2) we have that $\mathcal{J}(g)_{k0}(0, z'')$ is the only non-zero entry of $k$-th row of $\mathcal{J}(g)|_z$. Therefore, equating $k0$-th entry of matrices in (5.9) with $f = g$ we obtain, for $z \in Z$, that

\[
\phi_{kk}(0, z'')\mathcal{J}(g)_{k0}(0, z'') = \mathcal{J}(g)_{k0}(0, z'')\phi_{00}(0, z'')
\]

which proves (a) of the claim above as $\mathcal{J}(g)_{k0} = \eta!$.

To prove the part (b) let $0 \leq i, j \leq N$, $\theta^{-1}(i) = (\alpha_1, \ldots, \alpha_d)$, $\theta^{-1}(j) = (\beta_1, \ldots, \beta_d)$ and $g(z', z'') = z_1^{\beta_1} \cdots z_d^{\beta_d}$. We also assume that $i < j$. Then, using (P2) with $f = g$ and $l = j$,
we see that $\mathcal{J}(g)_{\, j_0}(0,z'')$ is the only non-zero entry of the matrix $\mathcal{J}(g)(0,z'')$ for $(0,z'') \in \mathbb{Z}$. Therefore, only $i_0$-th entry of the matrix in left hand side of (5.9) contains $\phi_{ij}$. On the other hand, the $i_0$-th entry of the matrix in other side in (5.9) is 0 as $i < j$, thanks to the property (P1). Thus, comparing the $i_0$-th entry of matrices in (5.9) we conclude that $\phi_{ij}(0,z'') = 0$ on $\mathbb{Z}$, for $1 \leq i < j \leq N$.

Now we assume that $i > j$. Since only non-zero entry of $j$-th row of the matrix $\mathcal{J}(g)|_{\mathbb{Z}}$ is $\mathcal{J}(g)_{\, j_0}(0,z'')$ it is clear that only $i_0$-th entry of the matrix in left hand side of (5.9) contains $\phi_{ij}$. The $i_0$-th entry of this matrix is $\beta!\phi_{ij}(0,z'')$ as $\mathcal{J}(g)_{\, j_0}(0,z'') = \beta!$, for $(0,z'') \in \mathbb{Z}$. So, as before in order to calculate $\phi_{ij}$, we need to compare $i_0$-th entry of the matrices in (5.9). From (P2) we have that only non-zero entry of $i$-th row of $\mathcal{J}(g)|_{\mathbb{Z}}$ is $\mathcal{J}(g)_{\, i_0(\alpha - \beta)}(0,z'')$ which, by definition (4.6) of $\mathcal{J}(g)$ with $g(z',z'') = z_1^{\beta_1} \cdots z_d^{\beta_d}$, is

$$\left(\frac{\alpha}{\alpha - \beta}\right) \partial^{\alpha - (\alpha - \beta)} g(z',z'')|_{\mathbb{Z}} = \left(\frac{\alpha}{\alpha - \beta}\right) \partial^{\beta} g(z',z'')|_{\mathbb{Z}} = \left(\frac{\alpha}{\alpha - \beta}\right) \beta!,$$

provided $\alpha_t \geq \beta_t$, for all $t = 1, \ldots, d$. Furthermore, if $\alpha_t < \beta_t$ for some $t \in \{1, \ldots, d\}$, it follows from (P2) that every entry of $i$-th row is zero. Therefore, equating $i_0$-th entry of matrices in (5.9) we get

$$\phi_{ij}(0,z'') = \left\{ \begin{array}{ll} \left(\frac{\alpha}{\alpha - \beta}\right) \phi_{\theta(\alpha - \beta)0}(0,z'') & \text{for } \alpha_t \geq \beta_t \quad \forall \ t = 1, \ldots, d, \\ 0 & \text{otherwise} \end{array} \right.$$  

(5.11)

which completes the proof of Claim 1.

Thus, Claim 1 shows that the matrix $((\phi_{ij}(0,z'')))_{i,j=0}^N$ is a lower triangular matrix. Consequently, we have that $\Phi$ induces bundle morphisms $\Phi|_{J^l E|_{\text{res} \mathbb{Z}}} : J^l E|_{\text{res} \mathbb{Z}} \rightarrow J^l \tilde{E}|_{\text{res} \mathbb{Z}}$, for $0 \leq l \leq k$.

**Claim 2.** $\phi_{00}$ is a constant function and $\phi_{ii} = \phi_{00}$, for $i = 0, \ldots, N$, on $\mathbb{Z}$.

Note that it is enough to show that $\phi_{00}$ is a constant function on $\mathbb{Z}$ thanks to Claim 1. In fact, from the equation (5.8) we have

$$\langle s(0,z'),s(0,z'') \rangle = \phi_{00}(0,z'') \langle \tilde{s}(0,z''), \tilde{s}(0,z'') \rangle \varphi_{00}(0,z'').$$

Consequently, Lemma 5.7 and Proposition 5.8 together yield that

$$\phi_{00}(0,z'') \varphi_{00}(0,0) = 1.\]$$

Hence we are done with Claim 2.

**Claim 3.** $((\phi_{ij}(0,z'')))_{i,j=0}^N$ is a constant diagonal matrix with diagonal entries $\phi_{ii} = \phi_{00}$, for $0 \leq i \leq N$, that is, $((\phi_{ij}(0,z'')))_{i,j=0}^N = \phi_{00} \cdot I$ where $I$ is the $(N + 1) \times (N + 1)$ identity matrix.

In view of Claim 1 and Claim 2, it is enough to show that $\phi_{i0} = 0$, for $0 < l \leq N$, on $\mathbb{Z}$. So calculating $l_0$-th entry of the matrices in the equation (5.8) and using the Lemma 5.7 we have

$$\langle \partial^{l_0} s(0,z''), s(0,w'') \rangle = \left( \sum_{j=0}^{l_0} \phi_{ij}(0,z'') \langle \partial^{l_0} \tilde{s}(0,z''), \tilde{s}(0,w'') \rangle \right) \varphi_{00}$$

and consequently, after putting $w'' = 0$ and applying the Proposition 5.8 to the frames $\{s\}$ and $\{\tilde{s}\}$ at origin we get $\phi_{00}(0,z'') = 0$ on $\mathbb{Z}$.

Thus, Claim 1, Claim 2, Claim 3 and the equation (5.8) together yield that

$$\partial^{l_0} \tilde{s}(0,z'') \parallel s(0,z'') \parallel^2 = \phi_{00} \partial^{l_0} \tilde{s}(0,z'') \parallel \tilde{s}(0,z'') \parallel^2 \varphi_{00} = \partial^{l_0} \tilde{s}(0,z'') \parallel \tilde{s}(0,z'') \parallel^2$$

(5.12)
on $Z$, for $0 \leq l, j \leq N$.

The converse statement is easy to see. Indeed, if the equation (5.12) happens to be true then the desired jet bundle isomorphism $\Phi$ is given by the constant matrix $I$ with respect to the frames $\{s, \partial s, \ldots, \partial^N s\}$ and $\{\tilde{s}, \partial \tilde{s}, \ldots, \partial^N \tilde{s}\}$ where $I$ is the identity matrix of order $N + 1$. □

**Theorem 5.10.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $Z$ be the complex submanifold in $\Omega$ of codimension $d$ defined by $z_1 = \cdots = z_d = 0$. Assume that $(\mathcal{M}, \mathcal{M}_q)$ and $(\mathcal{N}, \mathcal{N}_q)$ are pair of Hilbert modules in $B_{r,k}(\Omega, Z)$. Then $\mathcal{M}_q$ and $\mathcal{N}_q$ are unitarily equivalent as modules over $A(\Omega)$ if and only if there exists a constant unitary matrix $D$ such that $\tilde{\partial} \tilde{\partial}^* H = D(\partial^j \partial^j H)D^*$ on $Z$, for all $0 \leq l, j \leq N$ where $H(z)$ and $\tilde{H}(z)$ are the Grammian matrices for the holomorphic frames $s$ and $\tilde{s}$ of the Hermitian holomorphic vector bundles $E$ and $\tilde{E}$ on $\Omega$ associated to the Hilbert modules $\mathcal{M}$ and $\mathcal{N}$, respectively, normalized at origin.

**Proof.** We note, as in Theorem 5.9, that it is enough to show that the jet bundles $J^k E|_{\res Z}$ and $J^k \tilde{E}|_{\res Z}$ are equivalent according to the Definition 5.3 if and only if there exists a constant unitary matrix $D$ such that $\tilde{\partial} \tilde{\partial}^* H = D(\partial^j \partial^j H)D^*$ on $Z$, for all $0 \leq l, j \leq N$, where $H(z)$ and $\tilde{H}(z)$ are as above.

So, to begin with, let $\Phi : J^k E|_{\res Z} \rightarrow J^k \tilde{E}|_{\res Z}$ be a jet bundle isomorphism. Then the isomorphism $\Phi$ can be represented by an $(N + 1) \times (N + 1)$ block matrix $(\Phi_{lt})_{l,t=0}^N$ with respect to the frames $\{s, \partial s, \ldots, \partial^N s\}$ and $\{\tilde{s}, \partial \tilde{s}, \ldots, \partial^N \tilde{s}\}$ where $\Phi_{lt}$ are holomorphic $r \times r$ matrix valued functions on $Z$. Then in terms of matrices the fact that $\Phi$ is an isometry of two jet bundles $J^k E|_{\res Z}$ and $J^k \tilde{E}|_{\res Z}$ translates to the following matrix equation on $Z$:

\[(\partial^j \partial^j H)^{N}_{l,t=0} = (\Phi_{lt})^N_{l,t=0} (\partial^j \partial^j H)^{N}_{l,t=0} (\Phi_{lt})^N_{l,t=0} \ast.
\]

(5.13)

Let $E|_{\res Z}$ and $\tilde{E}|_{\res Z}$ be line bundles determined by the frames $\{s_i\}$ and $\{\tilde{s}_i\}$ respectively, on $Z$, for $1 \leq i \leq r$. Then we can have the decomposition $E|_{\res Z} = \oplus_{i=1}^r E|_{\res Z}$ and $J^k E|_{\res Z} = \oplus_{i=1}^r J^k E|_{\res Z}$ with $\{s_i, \partial s_i, \ldots, \partial^N s_i\}$ as a frame on $Z$. Further, let $P_i : J^k E|_{\res Z} \rightarrow J^k E|_{\res Z}$ and $\tilde{P}_i : J^k \tilde{E}|_{\res Z} \rightarrow J^k \tilde{E}|_{\res Z}$ be projection morphisms where the frame $\{\tilde{s}_i, \partial \tilde{s}_i, \ldots, \partial^N \tilde{s}_i\}$ defines the jet bundle $J^k \tilde{E}|_{\res Z}$. Then note that the matrix of $\Phi$ with respect to the frames $J(s) = \{\sum_{l=0}^N \partial^l s_{j1} \otimes \varepsilon_l, \ldots, \sum_{l=0}^N \partial^l s_{j_{\ell}} \otimes \varepsilon_l\}$ and $J(\tilde{s}) = \{\sum_{l=0}^N \partial^l \tilde{s}_{j1} \otimes \varepsilon_l, \ldots, \sum_{l=0}^N \partial^l \tilde{s}_{j_{\ell}} \otimes \varepsilon_l\}$ is $(\Phi_{ij})_{i,j=1}^\ell$ where $[P_{ij}]$ represents the matrix of $\tilde{P}_i \Phi P_j^*$ with respect to the frames $\{s_j, \partial s_j, \ldots, \partial^N s_j\}$ and $\{\tilde{s}_i, \partial \tilde{s}_i, \ldots, \partial^N \tilde{s}_i\}$. Since $\Phi$ is a jet bundle isomorphism (Definition 5.3) it intertwines the module action on the class of holomorphic sections of $J^k E|_{\res Z}$ and $J^k \tilde{E}|_{\res Z}$. As a consequence, we have

\[((\Phi_{ij}))_{i,j=1}^\ell ((J(f) \otimes I_r) = (J(f) \otimes I_r) ((\Phi_{ij}))_{i,j=1}^\ell ,
\]

for $f \in A(\Omega)$, which is equivalent to the fact that the bundle morphisms $\tilde{P}_i \Phi P_j^*$ intertwine the module action (5.1) on holomorphic sections of $J^k E|_{\res Z}$ and $J^k \tilde{E}|_{\res Z}$. Thus, $\tilde{P}_i \Phi P_j^*$ defines a jet bundle morphism from $J^k E|_{\res Z}$ onto $J^k \tilde{E}|_{\res Z}$.

We, therefore, can apply Claim 1 in Theorem 5.10 to $\tilde{P}_i \Phi P_j^*$ to conclude, for $0 \leq l, t \leq N$, that

\[ [P_{ij}]_{lt} = \begin{cases} \alpha & \text{if } \alpha \neq \beta \\ \alpha - \beta & \text{if } \alpha = \beta \end{cases} \]

for $0 \leq l, t \leq N$.
otherwise, $[P_{ij}]_t$ is the zero matrix. Then it follows that the matrix of $\Phi(0, z'')$ with respect to the frames $\{s, \partial s, \ldots, \partial^N s\}$ and $\{\tilde{s}, \partial \tilde{s}, \ldots, \partial^N \tilde{s}\}$ is a lower triangular block matrix with \(\Phi_t(0, z'') = \Phi_{00}(0, z'')\) for $0 \leq l \leq N$ and, for $0 \leq l, t \leq N$, $\alpha = (\alpha_1, \ldots, \alpha_d) = \theta^{-1}(t), \beta = (\beta_1, \ldots, \beta_d) = \theta^{-1}(l)$ and $1 \leq i, j \leq r$,

$$\left(\Phi_{lt}(0, z'')\right)_{ij} = \left(\frac{\alpha}{\alpha - \beta}\right)\left(\Phi_{\theta(\alpha - \beta)0}(0, z'')\right)_{ij} \text{ if } (\alpha - \beta) \in (\mathbb{N} \cup \{0\})^d,$$

and is zero, otherwise, on $Z$. Now a similar proof as in Claim 2 in Theorem 5.9 with matrix valued holomorphic functions $H, \tilde{H}$ and $\Phi_{00}$ on $Z$ yields that $\Phi_{00}$ is a constant unitary matrix. Thus, the proof will be done once we prove that $\Phi_{lt}(0, z'') = 0$ for $0 < l \leq N$, on $Z$. So calculating the $l$-th block of the matrices in the equation (5.13) and using the Lemma 5.7 we get

$$\left(\left(\partial^s l_i(0, z''), s_j(0, w'')\right)\right)_{i,j=1}^t = \left(\sum_{t=0}^l \Phi_{lt}(0, z'')\left(\left(\partial^s l_i(0, z''), \tilde{s}_j(0, w'')\right)\right)\right)_{i,j=1}^t \Phi_{00}^*$$

and consequently, after putting $w'' = 0$ and applying the Proposition 5.8 to the frames $s$ and $\tilde{s}$ at origin we get $\Phi_{l0}(0, z'') = 0$ on $Z$. Thereby from (5.13) we have

$$\partial^s l_i(0, z'') = D(\partial^s l_i \tilde{H}(0, z''))D^*$$

on $Z$ for all $0 \leq l, j \leq N$ where $D = \Phi_{00}$.

For the converse direction, note that the equation (5.15) canonically gives rise to the jet bundle isomorphism $\Phi$ by prescribing the matrix of $\Phi$ as $D \otimes I$ with respect to the frames $J(s)$ and $J(\tilde{s})$ where $I$ is the identity matrix of order $N + 1$.

**Corollary 5.11.** Let $T = (T_1, \ldots, T_m)$ and $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_m)$ be two operator tuples in $B_1(\Omega)$. Then $T$ and $\tilde{T}$ are unitarily equivalent if and only if there are jet bundle isomorphisms $\Phi_k : J^k E|_{\text{res} Z} \to J^k \tilde{E}|_{\text{res} Z}$, for every $k \in \mathbb{N} \cup \{0\}$ where $Z$ is any singleton set $\{p\}$, for $p \in \Omega$.

**Proof.** The necessity part is trivial and so we only show that $T$ and $\tilde{T}$ are unitarily equivalent assuming that there are jet bundle isomorphisms $\Phi_k : J^k E|_{\text{res} Z} \to J^k \tilde{E}|_{\text{res} Z}$, for every $k \in \mathbb{N} \cup \{0\}$.

Let $E$ and $\tilde{E}$ be vector bundles over $\Omega$ corresponding to operator tuples $T$ and $\tilde{T}$, respectively, and $Z = \{0\}$. Thus, the codimension of $Z$ is $m$. We now wish to apply the previous theorem for each non-negative integer $k$. So let us start with frames $s$ and $\tilde{s}$ for $E$ and $\tilde{E}$, respectively, normalized at origin. Then by Theorem 5.9 we have, for every $k \in \mathbb{N} \cup \{0\}$, $\partial^s \partial^{\tilde{s}} \|s\|^2 = \partial^s \partial^{\tilde{s}} \|\tilde{s}\|^2$ at $0$ for all $0 \leq l, j \leq N(k)$ where $N(k) = \left(\frac{d+k-1}{k-1}\right) - 1$. In other words, translating the notations used in the above equation we get

$$\partial^{\alpha_1} \ldots \partial^{\alpha_m} \partial^{\beta_1} \ldots \partial^{\beta_m} \|s(0)\|^2 = \partial^{\alpha_1} \ldots \partial^{\alpha_m} \partial^{\tilde{\beta}_1} \ldots \partial^{\tilde{\beta}_m} \|\tilde{s}(0)\|^2$$

for all $\alpha, \beta \in (\mathbb{N} \cup \{0\})^m$.

Now since $s$ and $\tilde{s}$ both are holomorphic on their domains of definition $\|s\|^2$ and $\|\tilde{s}\|^2$ are real analytic there. Consequently, using the power series expansion of $\|s\|^2$ and $\|\tilde{s}\|^2$ together with the equation (5.16) we obtain that

$$\|s(z)\|^2 = \|\tilde{s}(z)\|^2$$

on some open neighbourhood, say $\Omega_0$, of the origin in $\Omega$. Thus, the bundle map $\Phi : E \to \tilde{E}$ determined by the formula $\Phi(s(z)) = \tilde{s}(z)$ defines an isometric bundle isomorphism between $E$ and $\tilde{E}$ over $\Omega_0$. Then our result is a direct consequence of the Rigidity theorem in [4].
Remark 5.12. Note that the above theorem shows that the unitary equivalence of local operators (1.5 in [4]) $N_{\omega_0}^{(k)}$ and $\tilde{N}_{\omega_0}^{(k)}$ corresponding to $T$ and $\tilde{T}$, respectively, for all $k \geq 0$ but at a fixed point $\omega_0 \in \Omega$ implies the unitary equivalence of $T$ and $\tilde{T}$. In other words, any tuples of operators $T \in B_1(\Omega)$ enjoy the "Taylor series expansion" property. Moreover, following the technique used in Theorem 18 in [3], it is seen that the same property is also enjoyed by any $T \in B_r(\Omega)$, $r \geq 1$.

The following theorem is one of the main results in this article which generalizes the study of quotient modules done in the paper [11] to the case of arbitrary codimension. For the definition of bundle maps used in the following theorem we refer the readers to (2.4).

Theorem 5.13. Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $Z \subset \Omega$ be the complex manifold of codimension $d$ defined by $z_1 = \cdots = z_d = 0$. Suppose that pair of Hilbert modules $(\mathcal{M}, \mathcal{N})$ and $(\mathcal{M}, \mathcal{N}_q)$ are in $B_{r,k}(\Omega, Z)$. Then $\mathcal{N}$ and $\mathcal{N}_q$ are isomorphic as modules over $\mathcal{A}(\Omega)$ if and only if following conditions hold:

(i) There exists holomorphic isometric bundle map $\Phi : E_{\text{res}|Z} \to \tilde{E}_{\text{res}|Z}$ where $E$ and $\tilde{E}$ are Hermitian holomorphic vector bundles over $\Omega$ corresponding to the Hilbert modules $\mathcal{M}$ and $\mathcal{M}$ over $\mathcal{A}(\Omega)$.

(ii) The transverse curvature of $E$ and $\tilde{E}$ as well as their covariant derivatives of order at most $k - 2$, along the transverse directions to $Z$, are intertwined by $\Phi$ on $\Omega$.

(iii) The bundle map $\Phi$ intertwines the bundle maps $\mathcal{J}_l^i(H) := \partial_i(H^{-1}\partial^l H)$ and $\tilde{\mathcal{J}}_l^i(\tilde{H}) := \partial_i(\tilde{H}^{-1}\partial^l \tilde{H})$, $d + 1 \leq i \leq m$, holds on $Z$ where $s = \{s_1, \ldots, s_r\}$ and $\tilde{s} = \{s_1, \ldots, s_r\}$ are frames of $E$ and $\tilde{E}$, respectively, for $0 \leq l \leq N$, and $H$ and $\tilde{H}$ are Grammians of $s$ and $\tilde{s}$, respectively.

Remark 5.14. At this point although it seems that the condition (iii) in the above theorem depends on the choice of a frame it is not the case. For instance, if $t$ is another frame normalized at origin we have $t = sA$ for some holomorphic function $A : Z \to GL_r(\mathbb{C})$. Now since both $s$ and $t$ are normalized at origin the same proof as in Claim 2 in Theorem (5.9) with matrix valued holomorphic functions shows that $A$ is a constant unitary matrix. Thus, we have $H = AHA^*$ and hence it follows that

$$\mathcal{J}_l^i(G) = A\mathcal{J}_l^i(H)A^{-1}, d + 1 \leq i \leq m,$$

where $G$ is the Gramm matrix of the frame $t$.

Proof. Let $\Omega \subset \mathbb{C}^m$ and $Z \subset \Omega$ be as given. Suppose that $\mathcal{M}$ and $\mathcal{M}_q$ are equivalent as modules over $\mathcal{A}(\Omega)$. Then by Theorem 5.10 there exists a constant unitary matrix $D$ such that

$$(5.17) \quad \partial_l^j \partial_j H(0, z''') = D(\partial_l^j \partial_j \tilde{H}(0, z'''))D^*, \text{ for } (0, z''') \in Z \text{ and } 0 \leq l, j \leq N,$$

where $H(z)$ and $\tilde{H}(z)$ are the Grammian matrices for holomorphic frames $s = \{s_1, \ldots, s_r\}$ and $\tilde{s} = \{s_1, \ldots, s_r\}$ of the Hermitian holomorphic vector bundles $E$ and $\tilde{E}$ on $\Omega$ associated to Hilbert modules $\mathcal{M}$ and $\mathcal{M}$, respectively, normalized at origin. In particular, for $l = j = 0$,

(5.17) becomes

$$H(0, z''') = D\tilde{H}(0, z''')D^*, \text{ for } (0, z''') \in Z.$$

Let $\Phi : E_{\text{res}|Z} \to \tilde{E}_{\text{res}|Z}$ be the bundle morphism whose matrix representation with respect to the frames $s$ and $\tilde{s}$ is $D$. Then $\Phi$ is the desired isometric bundle map in (i). Further, the equation (5.17) together with (i) of Lemma 2.3 yields (ii), and since $D$ is a constant unitary matrix on $Z$, (iii) is an easy consequence of (5.17) with $j = 0$. 

Now let us prove the converse direction. To do so we show that the condition (i), (ii), (iii) in the statement together imply the condition of the Theorem 5.10, that is, we need to show that there exists a constant unitary matrix $D$ on $Z$ such that the equation (5.18) holds on the submanifold $Z$ for $0 \leq l, t \leq N$ and frames $s$ and $\tilde{s}$ of $E$ and $\tilde{E}$, respectively, normalized at origin.

We first extend the holomorphic isometric bundle map $\Phi : E|_{\text{res}Z} \to \tilde{E}|_{\text{res}Z}$, obtained from condition (i), to a family of linear isometries $\hat{\Phi}_{z_0} : J^k E|_{z_0} \to J^k \tilde{E}|_{z_0}$ for every $z_0 \in Z$. Then we show that this extension is actually a jet bundle isomorphism providing our desired matrix. So let us begin with frames $s$ and $\tilde{s}$ for $E$ and $\tilde{E}$, respectively, normalized at $z_0 \in Z$. Condition (i) then yields an isometric holomorphic bundle map $\Phi : E|_{\text{res}Z} \to \tilde{E}|_{\text{res}Z}$ and consequently, we have a holomorphic $r \times r$ matrix valued function $\phi$ on $Z$ such that

$$H(0, z'') = \phi(0, z'') \tilde{H}(0, z'') \phi(0, z'')^*$$

where $\phi$ represents $\Phi$ with respect to frames $s$ and $\tilde{s}$. Since both $s$ and $\tilde{s}$ are normalized at $z_0$, the above equation (5.18) shows that $\phi(0, z'')$ is a unitary matrix. Furthermore, from condition (ii) of our hypothesis along with second statement of Lemma 2.3 we have, for $0 \leq \alpha_1 + \cdots + \alpha_d \leq k - 1, 0 \leq \beta_1 + \cdots + \beta_d \leq k - 1$,

$$\partial^{\alpha_1} \cdots \partial^{\alpha_d} \tilde{\partial}^{\beta_1} \cdots \tilde{\partial}^{\beta_d} H(0, z'') = \phi(0, z'') (\partial^{\alpha_1} \cdots \partial^{\alpha_d} \tilde{\partial}^{\beta_1} \cdots \tilde{\partial}^{\beta_d} \tilde{H}(0, z''))$$

as $\partial^{\alpha_1} \cdots \partial^{\alpha_d} H(0, z'')$ (respectively, $\partial^{\alpha_1} \cdots \partial^{\alpha_d} \tilde{H}(0, z'')$) and $\tilde{\partial}^{\beta_1} \cdots \tilde{\partial}^{\beta_d} \tilde{H}(0, z'')$ (respectively, $\tilde{\partial}^{\beta_1} \cdots \tilde{\partial}^{\beta_d} \tilde{H}(0, z'')$) are zero matrices for any $\alpha_i, \beta_i \geq 0, i = 1, \ldots, d$. Thus, the equations above (5.18, 5.19) lead to the following natural isometric extension, $\hat{\Phi}_{z_0} : J^k E|_{z_0} \to J^k \tilde{E}|_{z_0}$ defined by

$$\hat{\Phi}_{z_0} (\partial^l s_j (0, z'')) = \sum_{i=1}^{r} \phi_{ji} (0, z'') \partial^l \tilde{s}_i (0, z''), \quad 0 \leq l \leq N, \quad 1 \leq j \leq r.$$ 

Then we note, for $0 \leq l \leq N, 1 \leq j \leq r$, $z_0 = (0, z'') \in Z$ and $f \in A(\Omega)$, that

$$\hat{\Phi}_{z_0} (\partial^l s_j (0, z'')) = \sum_{i=1}^{r} \phi_{ji} (0, z'') \partial^l \tilde{s}_i (0, z'') = \sum_{i=1}^{r} \phi_{ji} (0, z''), \quad 0 \leq l \leq N, \quad 1 \leq j \leq r.$$ 

Thus, the above extension (5.20) intertwines the module action (5.1) on the sections of $J^k E$ and $J^k \tilde{E}$ over $Z$. From now on, in the rest of the proof, we denote this extension by $\hat{\Phi}$. We also note that $\hat{\Phi}$ satisfies the equation (5.13).

Let us now work with frames $s$ and $\tilde{s}$ of $E$ and $\tilde{E}$, respectively, normalized at origin, and $D(0, z'') := ((\hat{\Phi}_{ij} (0, z'')))_{i,j=0}^N$ be the matrix of $\hat{\Phi}$ with respect to the frames $\{s, \partial s(0, z''), \ldots, \partial^N s(0, z'')\}$ and $\{\tilde{s}, \partial \tilde{s}(0, z''), \ldots, \partial^N \tilde{s}(0, z'')\}$. Then we wish to show that $D$ is a constant matrix on $Z$.

We point out that the $(1, 1)$-th block of $D(0, z'')$, namely, $\hat{\Phi}_{00} (0, z'')$ is the matrix representation of $\hat{\Phi} : E|_{\text{res}Z} \to \tilde{E}|_{\text{res}Z}$ with respect to the above frames, $s$ and $\tilde{s}$, and hence $\hat{\Phi}_{00} (0, z'')$ is holomorphic on $Z$. So recalling the proof of Claim 2 in Theorem 5.9 with matrix valued holomorphic functions, we conclude that $\hat{\Phi}_{00} (0, z'')$ is a constant unitary matrix, say, $\Phi_{00}$ on $Z$.

We also note, from the construction of $\hat{\Phi}$ above, that $\hat{\Phi} (J^l E|_{\text{res}Z}) \subset J^l \tilde{E}|_{\text{res}Z}$, for $0 \leq l < k$. Consequently, $D(0, z'')$ is a lower triangular matrix. Moreover, since $\hat{\Phi}$ commutes with the module action on the sections of $J^k E|_{\text{res}Z}$ and $J^k \tilde{E}|_{\text{res}Z}$, the same proof as in Theorem 5.10 shows that entries of the matrix satisfy the properties stated in (5.14). So it is enough to show that $\Phi_{0l} (0, z'') = 0$ on $Z$ for $0 < l \leq N$. In order to show this, we first need to establish
that \( \hat{\Phi}_{10}(0, z'') \) is a holomorphic function on \( Z \) so that we can invoke the argument used in the Theorem 5.10 to show \( \hat{\Phi}_{10}(0, z'') = 0 \) on \( Z \) using the Lemma 5.7 and Proposition 5.8.

Here we prove our claim by using mathematical induction. For the base case, let us calculate \( \Phi_{10}(0, z'') \) from the equation (5.13) by equating 10-th block and we have

\[
\hat{\Phi}_{10}(0, z'') = (\partial_1 H(0, z'') \hat{\Phi}_{00} - \hat{\Phi}_{00} \partial_1 H(0, z'')) \hat{H}^{-1}(0, z'').
\]

For \( d + 1 \leq i \leq m \), differentiating both sides of the above equation with respect to \( \Xi_i \) we obtain

\[
\tilde{\partial}_i \hat{\Phi}_{10}(0, z'') = \partial_1 \partial_i H(0, z'') \hat{\Phi}_{00} \hat{H}^{-1}(0, z'') - \hat{\Phi}_{00} \partial_1 \partial_i H(0, z'') \hat{H}^{-1}(0, z'')
\]

where the second inequality holds as \( \hat{\Phi}_{00} \) is a constant unitary matrix satisfying the equation (5.13). Then using condition (iii) with \( l = 1 \) we have \( \tilde{\partial}_l \hat{\Phi}_{10}(0, z'') = 0 \) on \( Z \), for \( d + 1 \leq i \leq m \), which completes the proof of the base case. Now let \( \tilde{\partial}_l \hat{\Phi}_{00}(0, z'') = 0 \) on \( Z \), for \( 0 < j \leq l, d + 1 \leq i \leq m \) and for some \( 1 \leq l \leq N \). Since \( \hat{\Phi}_{00} \) is holomorphic on \( Z \), for \( 0 \leq j \leq l \), recalling the equation (5.14), and using Lemma 5.7 and Proposition 5.8 as in the proof of Claim 2 in Theorem 5.9 with matrix valued holomorphic functions, we conclude that the \((l + 1)\text{-th}\) row of \( D \) contains only two non-zero blocks, namely, \( \hat{\Phi}_{l+10} \) and \( \hat{\Phi}_{l+1l+1} \). Therefore, from the equation (5.13) by equating \( l + 10\text{-th}\) block we have

\[
\hat{\Phi}_{l+10}(0, z'') = \partial^{l+1} H(0, z'') \hat{\Phi}_{l+1l+1}(0, z'') \partial^{l+1} H(0, z'') \hat{H}^{-1}(0, z'').
\]

Now as before, for \( d + 1 \leq i \leq m \), applying the differential operator \( \tilde{\partial}_i \) both sides of the equation above and recalling that \( \hat{\Phi}_{l+1l+1}(0, z'') = \Phi_{00} \), for \( (0, z'') \in Z \), we get

\[
\tilde{\partial}_i \hat{\Phi}_{l+10}(0, z'') = \partial^{l+1} H(0, z'') \hat{\Phi}_{00} \hat{H}^{-1}(0, z'') \partial^{l+1} H(0, z'') \hat{H}^{-1}(0, z'')
\]

Then by condition (iii) we conclude that \( \tilde{\partial}_i \hat{\Phi}_{l+10}(0, z'') = 0 \) on \( Z \) for \( d + 1 \leq i \leq m \). So by mathematical induction \( \hat{\Phi}_{10}(0, z'') \) is holomorphic on \( Z \) for \( 0 \leq l \leq N \). Now, as in the proof of Theorem 5.10, Proposition 5.8 and Lemma 5.7 together yield our desired conclusion. \( \square \)

**Remark 5.15.** From the proof of Theorem 5.13, it is clear that, for \( r = 1 \) and \( d = k = 2 \), condition (i), (ii) and (iii) of Theorem 5.13 together yield that the curvatures of the bundles \( E|_{\text{res} Z} \) and \( E|_{\text{res} Z} \) are equal. Further, the matrix in (5.4) turns out to be the diagonal matrix \( \psi_{00} I \) with respect to a normalized frame at origin where \( I \) is the identity matrix of order 3. Moreover, following the proof of Claim 2 in Theorem 5.9 we see that \( \psi_{00} \) is a constant function on \( Z \) with \( |\psi_{00}| = 1 \). Thus, Theorem 5.13 is exact generalization of Lemma 5.4.

We also note that three conditions (i), (ii) and (iii) listed in the theorem above correspond to the condition that the metric of \( E \) and \( \overline{E} \) are equivalent to order \( k \), in the sense of the paper [11], on \( Z \) while the codimension of \( Z \) is 1. Consequently, following [11, Remark 6.1], we see that the condition (iii) in the above theorem corresponds to equality of the second fundamental forms for the inclusion \( E|_{\text{res} Z} \subset J^2 E|_{\text{res} Z} \) and \( E|_{\text{res} Z} \subset J^2 \overline{E}|_{\text{res} Z} \), for \( k = 2 \).

6. Examples and Application

For \( \lambda \geq 0 \), let \( H^{(\lambda)} \) be the Hilbert space of holomorphic functions on \( \mathbb{D} \) with the reproducing kernel \( K^{(\lambda)}(z, w) = (1 - z \overline{w})^{-\lambda} \) for \( z, w \in \mathbb{D} \). It is then evident that the set \( \{ e_n^{(\lambda)}(z) := c_n^{-\frac{1}{2}} z^n : n \geq 0 \} \)
n \geq 0 \} forms a complete orthonormal set in \( \mathcal{H}(\lambda) \) where \( c_n \) are the \( n \)-th coefficient of the power series expansion of \((1 - |z|^2)^{-\lambda}\), in other words,

\[
c_n = \left( -\frac{\lambda}{n!} \right) = \frac{\lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1)}{n!}.
\]

Let us recall that for \( \lambda \geq 0 \), the natural action of polynomial ring \( \mathbb{C}[z] \) on each Hilbert space \( \mathcal{H}(\lambda) \) makes it into a Hilbert module over \( \mathbb{C}[z] \). We also point out that, for \( \lambda > 1 \), \( \mathcal{H}(\lambda) \) becomes a Hilbert module over the disc algebra \( \mathcal{A}(\mathbb{D}) \).

It is well known that product of two reproducing kernels is also a reproducing kernel \([1, 8]\). So, for \( \alpha = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i \geq 0, i = 1, \ldots, m \), let us consider the Hilbert space \( \mathcal{H}(\alpha) := \mathcal{H}(\alpha_1) \otimes \cdots \otimes \mathcal{H}(\alpha_m) \) with the natural choice of complete orthonormal set \( \{ e_{i_1}^{(\alpha_1)}(z) \otimes \cdots \otimes e_{i_m}^{(\alpha_m)}(z) : i_j \geq 0, j = 1, \ldots, m \} \).

Then under the identification of the functions \( z_1^{i_1} \cdots z_m^{i_m} \) on \( D^m := \mathbb{D} \times \cdots \times \mathbb{D} \), \( \mathcal{H}(\alpha) \) naturally possesses an obvious reproducing kernel

\[
K^{(\alpha)}(z, w) := (1 - z_1 \overline{w_1})^{-\alpha_1} \cdots (1 - z_m \overline{w_m})^{-\alpha_m}
\]
on \( D^m \). Furthermore, the natural action of \( \mathbb{C}[z] \) on \( \mathcal{H}(\alpha) \) makes it a Hilbert module over \( \mathbb{C}[z] \), for \( \alpha_i \geq 0, i = 1, \ldots, m \), where by \( \mathbb{C}[z] \) we mean \( \mathbb{C}[z_1, \ldots, z_m] \).

Let us now consider the subspace \( \mathcal{H}_0^{(\alpha)} \) consisting of holomorphic functions in \( \mathcal{H}(\alpha) \) which vanish to order 2 along the diagonal \( \Delta := \{(z_1, \ldots, z_m) \in D^m : z_1 = \cdots = z_m \} \), that is, following the definition given in Section 3,

\[
\mathcal{H}_0^{(\alpha)} = \{ f \in \mathcal{H}(\alpha) : f = \partial_1 f = \cdots = \partial_m f = 0 \text{ on } \Delta \}.
\]

We are now interested in describing the quotient space \( \mathcal{H}_q^{(\alpha, \beta, \gamma)} := \mathcal{H}(\alpha, \beta, \gamma) / \mathcal{H}_0^{(\alpha, \beta, \gamma)} \) in case of \( m = 3 \), where \( \alpha, \beta, \gamma \geq 0 \).

In order to achieve our goal we first compute an orthonormal basis for \( \mathcal{H}_q^{(\alpha, \beta, \gamma)} \) with the help of which we find the desired expression of the reproducing kernel \( K_q^{(\alpha, \beta, \gamma)} \) of \( \mathcal{H}_q^{(\alpha, \beta, \gamma)} \) obtained from the Theorem 4.5. We then present an application of the Theorem 5.13 showing that the unitary equivalence classes of weighted Hardy modules are precisely determined by those of the quotient modules obtained from the submodules of functions vanishing to order at least 2 along the diagonal set \( \Delta := \{(z_1, z_2, z_3) \in D^3 : z_1 = z_2 = z_3 \} \).

6.1. **Examples.** We start with the submodule \( \mathcal{H}_0^{(\alpha, \beta, \gamma)} \) which is the closure of the ideal \( I := \langle (z_1 - z_2)^2, (z_1 - z_2)(z_1 - z_3), (z_1 - z_3)^2 \rangle \) in the Hilbert space \( \mathcal{H}(\alpha, \beta, \gamma) \). It then follows that \( B := \{ z_1^i z_2^j z_3^k (z_1 - z_2)^2, z_1^i z_2^j z_3^k (z_1 - z_2)(z_1 - z_3), z_1^i z_2^j z_3^k (z_1 - z_3)^2 : i, j, k \in \mathbb{N} \cup \{0\} \} \) is a spanning set for the submodule \( \mathcal{H}_0^{(\alpha, \beta, \gamma)} \). So to calculate an orthonormal basis for \( \mathcal{H}_q^{(\alpha, \beta, \gamma)} \) it is enough to find an orthonormal basis for the orthogonal complement of \( B \). It is easily verified that \( \{ g_1(p), g_2(p), g_3(p) : p \in \mathbb{N} \cup \{0\} \} \) forms a basis for \( B^\perp \) where

\[
g_1^{(p)} := \sum_{l=0}^{p} \left( \sum_{k=0}^{p-l-k} \frac{z_1^{p-l-k} z_2^k}{\|z_1^{p-l-k} z_2^k\|^2} \right) \frac{z_1^l}{\|z_1\|^2},
\]

\[
g_2^{(p)} := \sum_{l=0}^{p} \left( \sum_{k=0}^{p-l-k} \frac{l_1 z_1^{p-l-k} z_2^k}{\|z_1^{p-l-k} z_2^k\|^2} \right) \frac{z_1^l}{\|z_2\|^2},
\]

\[
g_3^{(p)} := \sum_{l=0}^{p} \left( \sum_{k=0}^{p-l-k} \frac{z_1^{p-l-k} z_2^k}{\|z_1^{p-l-k} z_2^k\|^2} \right) \frac{z_1^l}{\|z_3\|^2}.
\]

Then note that the corresponding orthogonal basis for \( B^\perp \) is \( \{ f_1^{(p)}, f_2^{(p)}, f_3^{(p)} : p \in \mathbb{N} \cup \{0\} \} \) with
\[ f_1^{(p)} := g_1^{(p)}, \quad f_2^{(p)} := \langle g_1^{(p)}, g_2^{(p)} \rangle g_1^{(p)} = \|g_1^{(p)}\|^2 g_2^{(p)}, \quad f_3^{(p)} := \langle \tilde{f}_3^{(p)}, f_2^{(p)} \rangle f_2^{(p)} = \|f_2^{(p)}\|^2 \tilde{f}_3^{(p)} \]

where \( \tilde{f}_3^{(p)} \) is orthogonal to \( g_1^{(p)} \) and is given by the following formula

\[ \tilde{f}_3^{(p)} = \langle g_1^{(p)}, g_3^{(p)} \rangle g_1^{(p)} - \|g_1^{(p)}\|^2 g_3^{(p)}. \]

Thus, our required orthonormal set of vectors in the quotient module \( \mathcal{H}^{(\alpha, \beta, \gamma)}_q \) is

\[ \mathcal{B} := \left\{ e_1^{(p)} = \frac{f_1^{(p)}}{\|f_1^{(p)}\|}, e_2^{(p)} = \frac{f_2^{(p)}}{\|f_2^{(p)}\|}, e_3^{(p)} = \frac{f_3^{(p)}}{\|f_3^{(p)}\|} \right\}_{p=0}^{\infty}. \]

Following the Theorem 4.5 to calculate the reproducing kernel we need to describe the unitary map

\[ (6.1) \quad h \mapsto h|_\Delta := \sum_{l=0}^{N} \partial^l h \otimes \varepsilon_l|_\Delta \]

for \( h \in \mathcal{H}^{(\alpha, \beta, \gamma)} \) where \( N = \frac{k(k-1)}{2} \). In our calculation, for \( k = 2 \), it is enough to determine the action of this map on the orthonormal basis \( \mathcal{B} \). In this context the following calculations provide us the necessary ingredients to compute the action of this unitary map.

We first note that

\[
(1 - |z_1|^2)^{-(\alpha + \beta + \gamma)} = (1 - |z_1|^2)^{-\alpha} (1 - |z_2|^2)^{-\beta} (1 - |z_3|^2)^{-\gamma} |\Delta| \\
= \sum_{p=0}^{\infty} \left[ \sum_{l=0}^{p} \sum_{k=0}^{p-l} \frac{1}{\|z_1^p - l - k\|^2 \|z_2^k\|^2} \right] \frac{1}{|z_3|^{2p}} \cdot |z_1|^{2p}
\]

which implies that \( \|f_1^{(p)}\|^2 \) is the \( p \)-th coefficient of the power series expansion of \( (1 - |z_1|^2)^{-(\alpha + \beta + \gamma)} \).

Thus, we have \( \|f_1^{(p)}\|^2 = \left( -\frac{\beta(1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 1)}}{p} \right) \). Further we point out that

\[
\beta(1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 1)} = (1 - |z_1|^2)^{-\alpha} (1 - |z_3|^2)^{-\gamma} \frac{d}{d|z_2|^2} (1 - |z_2|^2)^{-\beta} |\Delta| \\
= \sum_{p=0}^{\infty} \left[ \sum_{l=0}^{p} \sum_{k=0}^{p-l} \frac{1}{\|z_1^p - l - k\|^2 \|z_3^l\|^2} \right] \frac{l}{|z_2|^{2(p-1)}} \cdot |z_1|^{2(p-1)}
\]

which, as before, together with a similar calculation for \( \gamma(1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 1)} \) ensure that

\[ \langle g_1^{(p)}, g_2^{(p)} \rangle = \beta \binom{-(\alpha + \beta + \gamma + 1)}{p-1} \quad \text{and} \quad \langle g_1^{(p)}, g_3^{(p)} \rangle = \gamma \binom{-(\alpha + \beta + \gamma + 1)}{p-1}. \]

Now to calculate the inner product of \( g_2^{(p)} \) and \( g_3^{(p)} \) we consider the following power series
\[
\beta \gamma (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} = (1 - |z_1|^2)^{-\alpha} \frac{d}{d|z_2|^2} (1 - |z_2|^2)^{-\beta} \frac{d}{d|z_3|^2} (1 - |z_3|^2)^{-\gamma}|\Delta
\]
\[
= \sum_{p=0}^{\infty} \sum_{l=0}^{p} \left( \sum_{k=0}^{p-l} \frac{k(p-l-k)}{|z_p^2|^2 \|z_k^2\|^2} \right) \frac{1}{\|z_l^2\|^2} \cdot |z_1|^{2(p-2)}
\]
which shows that \( \langle g_2^{(p)}, g_3^{(p)} \rangle = \beta \gamma^{-(\alpha + \beta + \gamma + 2)} \). Furthermore, we have
\[
\beta (1 + |z_1|^2)(1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} = (1 - |z_1|^2)^{-\alpha} (1 - |z_3|^2)^{-\gamma} \left( \frac{d}{d|z_2|^2} (1 - |z_2|^2)^{-\beta} \right)|\Delta
\]
\[
= \sum_{p=0}^{\infty} \sum_{l=0}^{p} \left( \sum_{k=0}^{p-l} \frac{1}{|z_1| \|z_2^2\|^2 \|z_k^2\|^2} \right) |z_1|^{2(p-1)}
\]
and hence with the help of a similar calculation we find that
\[
\|g_2^{(p)}\|^2 = \beta \left( \left( -\alpha + \beta + \gamma + 2 \right) \right) \text{ and } (p-1)
\]
\[
\|g_3^{(p)}\|^2 = \gamma \left( \left( -\alpha + \beta + \gamma + 2 \right) \right) \text{ and (p-1)}
\]

Thus, the above calculations lead us to compute the norms of the vectors \( \{f_1^{(p)}, f_2^{(p)}, f_3^{(p)} : p \in \mathbb{N} \cup \{0\} \} \). We have already showed that \( \|f_1^{(p)}\|^2 \) is \( -\alpha + \beta + \gamma \) and recalling the definition of \( f_2^{(p)} \) and \( f_3^{(p)} \) it is easily seen that
\[
\|f_2^{(p)}\|^2 = \|g_1^{(p)}\|^2 \left( \|g_1^{(p)}\|^2 \|g_2^{(p)}\|^2 - \langle g_1^{(p)}, g_2^{(p)} \rangle^2 \right)
\]
\[
\frac{\beta (\alpha + \gamma)}{\left( -\alpha + \beta + \gamma \right)} \left( -\alpha + \beta + \gamma \right) \left( -\alpha + \beta + \gamma + 2 \right) \|g_2^{(p)}\|^2
\]
and a similarly one can find, from the definition of \( f_3^{(p)} \), that
\[
\|f_3^{(p)}\|^2 = \alpha \beta^2 \gamma (\alpha + \gamma) \left( -\alpha + \beta + \gamma \right) \left( p \right) \left( -\alpha + \beta + \gamma + 2 \right) \left( p - 1 \right)
\]
We also point out that similar computations as above give rise to following identities:
\[
\partial_1 g_1^{(p)} = \alpha \left( -\alpha + \beta + \gamma + 1 \right) \text{, } \partial_2 g_1^{(p)} = \beta \left( -\alpha + \beta + \gamma + 1 \right) \text{, } (p-1)
\]
\[
\partial_1 g_2^{(p)} = \alpha \beta \left( -\alpha + \beta + \gamma + 2 \right) \text{, } \partial_2 g_2^{(p)} = \beta \left( -\alpha + \beta + \gamma + 2 \right) + \beta \left( -\alpha + \beta + \gamma + 2 \right) \text{, } (p-2)
\]
\[
\partial_1 g_3^{(p)} = \alpha \gamma \left( -\alpha + \beta + \gamma + 2 \right) \text{ and } \partial_2 g_3^{(p)} = \beta \gamma \left( -\alpha + \beta + \gamma + 2 \right) \text{, } (p-2)
\]
So we are now in position to calculate the orthonormal basis for the quotient module \( \mathcal{H}_q^{(\alpha,\beta,\gamma)} \) and their derivatives along \( z_1 \) and \( z_2 \) direction restricted to the diagonal set \( \Delta \) which we exactly require to compute the reproducing kernel of the quotient module. Let us begin, from (6.1), by pointing out that

\[
e^{(p)}_1 \mapsto \begin{pmatrix} \alpha \sqrt{\frac{p}{\alpha + \gamma + \gamma}} \left( -\frac{\alpha + \gamma + 1}{\alpha + \gamma + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \\ \beta \sqrt{\frac{p}{\alpha + \beta + \gamma}} \left( -\frac{\alpha + \gamma + 1}{\alpha + \beta + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \end{pmatrix},
\]

\[
e^{(p)}_2 \mapsto \begin{pmatrix} \frac{\alpha \beta}{\sqrt{\alpha + \gamma}} \frac{1}{\sqrt{\alpha + \beta + \gamma}} \left( -\frac{\alpha + \gamma + 2}{\alpha + \beta + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \\ \frac{\beta \gamma}{\sqrt{\alpha + \gamma}} \frac{1}{\sqrt{\alpha + \beta + \gamma}} \left( -\frac{\alpha + \gamma + 2}{\alpha + \beta + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \end{pmatrix},
\]

and \( e^{(p)}_3 \mapsto \begin{pmatrix} 0 \\ \sqrt{\frac{\alpha^{\gamma}}{\alpha + \gamma}} \left( -\frac{\alpha + \beta + \gamma + 2}{\alpha + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \\ -\sqrt{\frac{\alpha^{\gamma}}{\alpha + \gamma}} \left( -\frac{\alpha + \beta + \gamma + 2}{\alpha + \gamma} \right)^{\frac{1}{2}} \frac{z_1^{p-1}}{z_1} \end{pmatrix} \).

This allows us to compute the reproducing kernel of the quotient module \( \mathcal{H}_q^{(\alpha,\beta,\gamma)} \) as follows

\[
K_q(z, w) = \sum_{p=0}^{\infty} e^{(p)}_1(z) \cdot e^{(p)}_1(w)^* + e^{(p)}_2(z) \cdot e^{(p)}_2(w)^* + e^{(p)}_3(z) \cdot e^{(p)}_3(w)^*, \quad z, w \in \mathbb{D}^3
\]

which is a 3 \times 3 matrix-valued function \( ((K_q(z, z))_{ij})_{i,j=1}^{3} \) on \( \mathbb{D}^3 \) as expected. To compute the kernel \( K_q(z, z) \) for \( z \in \Delta \) we note, for \( z = (z_1, z_1, z_1) \) in \( \Delta \), that

\[
\begin{align*}
K_q(z, z)_{11} &= K^{(\alpha,\beta,\gamma)}(z, z), \\
K_q(z, z)_{12} &= \partial_1 K^{(\alpha,\beta,\gamma)}(z, z), \\
K_q(z, z)_{13} &= \partial_2 K^{(\alpha,\beta,\gamma)}(z, z), \\
K_q(z, z)_{23} &= \frac{\alpha \beta}{\alpha + \beta + \gamma} \frac{d}{d|z_1|^2} \left( |z_1|^2 (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 1)} \right) \\
&\quad + \frac{\alpha \beta \gamma}{(\alpha + \gamma)(\alpha + \beta + \gamma)} \frac{1}{\alpha + \gamma} (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} \\
&= \alpha |z_1|^2 (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} \\
&\quad + \partial_1 \partial_2 K^{(\alpha,\beta,\gamma)}(z, z), \\
K_q(z, z)_{22} &= \frac{\alpha^2}{\alpha + \beta + \gamma} \frac{d}{d|z_1|^2} \left( |z_1|^2 (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 1)} \right) \\
&\quad + \frac{\alpha^2 \beta}{(\alpha + \gamma)(\alpha + \beta + \gamma)} \frac{1}{\alpha + \gamma} (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} \\
&= [\alpha + \alpha^2 |z_1|^2] (1 - |z_1|^2)^{-(\alpha + \beta + \gamma + 2)} \\
&\quad + \partial_1 \partial_2 K^{(\alpha,\beta,\gamma)}(z, z),
\end{align*}
\]

and similar calculations also yield that \( K_q(z, z)_{21} = \partial_1 K^{(\alpha,\beta,\gamma)}(z, z), \) \( K_q(z, z)_{31} = \partial_2 K^{(\alpha,\beta,\gamma)}(z, z), \) \( K_q(z, z)_{32} = \partial_2 \partial_1 K^{(\alpha,\beta,\gamma)}(z, z), \) and \( K_q(z, z)_{33} = \partial_2 \partial_2 K^{(\alpha,\beta,\gamma)}(z, z). \) Thus, we have

\[
K_q(z, z)|_\Delta = JK^{(\alpha,\beta,\gamma)}(z, z)|_\Delta
\]

which verifies the Theorem 4.5.
6.2. Application. Let us consider the family of Hilbert modules \( Mod(\mathbb{D}^m) := \{ \mathcal{H}^{(\alpha)} : \alpha = (\alpha_1, \ldots, \alpha_m) \geq 0 \} \) over the polydisc \( \mathbb{D}^m \) in \( \mathbb{C}^m \). In this subsection we prove that for any pair of tuples \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \alpha' = (\alpha'_1, \ldots, \alpha'_m) \), the unitary equivalence of two quotient modules \( \mathcal{H}^{(\alpha)}_q \) and \( \mathcal{H}^{(\alpha')}_q \), obtained from the submodules of functions vanishing of order 2 along the diagonal set \( \Delta \), implies the equality of the Hilbert modules \( \mathcal{H}^{(\alpha)} \) and \( \mathcal{H}^{(\alpha')} \). In other words, the restriction of the curvature of the jet bundle \( J^2 E^{(\alpha)} \) to the diagonal \( \Delta \) is a complete unitary invariant for the class \( Mod(\mathbb{D}^m) \) where the jet bundle \( J^2 E^{(\alpha)} \) is defined by the global frame \( \{ K^{(\alpha)}(., w), \partial_1 K^{(\lambda)}(., w), \ldots, \partial_m K^{(\lambda)}(., w) \} \) where \( \partial_j \) are the differential operators with respect to the variable \( z_j \), for \( j = 1, \ldots, m \).

**Theorem 6.1.** For \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \alpha' = (\alpha'_1, \ldots, \alpha'_n) \) with \( \alpha_i, \alpha'_i \geq 0 \), for all \( i = 1, \ldots, n \), the quotient modules \( \mathcal{H}^{(\alpha)}_q \) and \( \mathcal{H}^{(\alpha')}_q \) are unitarily equivalent if and only if \( \alpha_i = \alpha'_i \), for all \( i = 1, \ldots, n \).

**Proof.** The proof of sufficiency is trivial. So we only prove the necessity. Let us begin by pointing out that the diagonal set \( \Delta \) in \( \mathbb{D}^m \) can be described as the zero set of the ideal \( I := \langle z_1 - z_2, \ldots, z_i - z_{i+1}, \ldots, z_{m-1} - z_m \rangle \). Then it is easy to verify that \( \phi : U \to \mathbb{C}^m \) defined by

\[
\phi(z_1, \ldots, z_m) = (z_1 - z_2, \ldots, z_i - z_{i+1}, \ldots, z_{m-1} - z_m, z_m)
\]

yields an admissible coordinate system (Definition 3.9) around the origin. We choose \( U \) small enough so that \( \phi(U) \subset \mathbb{D}^m \). A simple calculation then shows that \( \phi^{-1} : \phi(U) \to U \) takes the form

\[
\phi^{-1}(u_1, \ldots, u_m) = \left( \sum_{j=1}^{m} u_j, \ldots, \sum_{j=1}^{m} u_j, u_{m-1} + u_m, u_m \right).
\]

For rest of the proof we pretend \( U \to \mathbb{D}^m \) thanks to the Remark 3.11. Now recalling the Proposition 3.12 it is enough to prove that \( \alpha_i = \alpha'_i, i = 1, \ldots, m \), provided \( \phi^* \mathcal{H}^{(\alpha)}_q \) is unitarily equivalent to \( \phi^* \mathcal{H}^{(\alpha')}_q \) where \( \phi^* \mathcal{H}^{(\alpha)}_q \) and \( \phi^* \mathcal{H}^{(\alpha')}_q \) are the quotient modules obtained from the submodules \( \phi^* \mathcal{H}^{(\alpha)}_0 \) and \( \phi^* \mathcal{H}^{(\alpha')}_0 \) of the Hilbert modules \( \phi^* \mathcal{H}^{(\alpha)} \) and \( \phi^* \mathcal{H}^{(\alpha')} \), respectively.

We now note that \( \phi^* \mathcal{H}^{(\alpha)} \) and \( \phi^* \mathcal{H}^{(\alpha')} \) are reproducing kernel Hilbert modules with reproducing kernels

\[
K(u) = \prod_{i=1}^{m} \left( 1 - \sum_{j=1}^{m} u_j \right)^{-\alpha_i} \quad \text{and} \quad K'(u) = \prod_{i=1}^{m} \left( 1 - \sum_{j=1}^{m} u_j \right)^{-\alpha'_i},
\]

respectively, where \( u = (u_1, \ldots, u_m) \in \phi(U) \). We also point out that the submodules \( \phi^* \mathcal{H}^{(\alpha)}_0 \) and \( \phi^* \mathcal{H}^{(\alpha')}_0 \) consists of functions in \( \phi^* \mathcal{H}^{(\alpha)} \) and \( \phi^* \mathcal{H}^{(\alpha')} \), respectively, vanishing along the submanifold \( Z := \{ (0, \ldots, 0, u_m) : u_m \in \mathbb{D} \} \cap \phi(U) \) of order 2.

Since \( \phi^* \mathcal{H}^{(\alpha)}_q \) and \( \phi^* \mathcal{H}^{(\alpha')}_q \) are unitarily equivalent recalling the Theorem 5.13 we conclude that

\[
(6.2) \quad K|_Z = K'|_Z
\]

where \( K \) and \( K' \) are the curvature matrix for the vector bundles \( E \) and \( E' \) over \( \phi(U) \) obtained from the Hilbert modules \( \phi^* \mathcal{H}^{(\alpha)} \) and \( \phi^* \mathcal{H}^{(\alpha')} \), respectively. Now we have, by definition, \( K(u) = ((K_{ij}(u)))_{i,j=1}^{m} \) where

\[
K_{ij}(u) = \frac{\partial^2}{\partial u_i \partial u_j} \log K(u, u),
\]
for \( u = (u_1, \ldots, u_m) \in \phi(U) \). Thus, for \( 1 \leq i \leq m \) and \( u \in \phi(U) \),

\[
K_{ii}(u) = \frac{\partial^2}{\partial u_i \partial u_i} \log K(u, u) = \sum_{l=1}^{i} \alpha_l \left( 1 - \left| \sum_{j=l}^{m} u_j \right|^2 \right)^{-1}.
\]

A similar computation also yields, for \( i = 1, \ldots, m \) and \( u \in \phi(U) \), that

\[
K'_{ii}(u) = \sum_{l=1}^{i} \alpha'_l \left( 1 - \left| \sum_{j=l}^{m} u_j \right|^2 \right)^{-1}.
\]

We now note that, for \( u \in Z \),

\[
K_{ii}(u) = \frac{\sum_{l=1}^{i} \alpha_l}{(1 - |u_m|^2)} \quad \text{and} \quad K'_{ii}(u) = \frac{\sum_{l=1}^{i} \alpha'_l}{(1 - |u_m|^2)}.
\]

Thus, by using the equality in equation (6.2) it is not hard to see that \( \alpha_i = \alpha'_i \), \( i = 1, \ldots, m \). \( \square \)

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