Thermodynamic formalism and substitutions

Nicolas Bédaride∗ Pascal Hubert† Renaud Leplaideur‡

ABSTRACT

We consider the class of marked substitutions. For a substitution in this class we examine the renormalization operator determined by it. Then we study the thermodynamic formalism for potentials in some class, and prove they have a freezing phase transition, with ground state supported on the attracting quasi-crystal associated to the substitution.

1 Introduction

1.1 Background

Phase transitions are nowadays of prime interest in smooth ergodic theory (see [6, 13, 11, 18]). Given a dynamical system \((X, T)\) and \(V : X \to \mathbb{R}\), the pressure function is defined by

\[
P(\beta) := \sup \left\{ h_\mu + \beta \int V \, d\mu \right\},
\]

where the supremum is taken over \(T\)-invariant probabilities and \(h_\mu\) is the Kolmogorov entropy. A phase transition means that the pressure function is not analytic. Even if analyticity is rare, the usual way to study the pressure function makes it far from obvious to prove it is not analytic. In the setting of uniformly hyperbolic dynamical systems, and when the potential function \(V\) is Hölder-continuous, it is known that the logarithm of pressure is the single dominating eigenvalue of the transfer operator. Then, analyticity follows

∗Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France. Email: nicolas.bedaride@univ-amu.fr
†Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France. Email: pascal.hubert@univ-amu.fr
‡Département de Mathématiques. Université de Brest. 6 avenue Victor Le Gorgeu. C.S 93837, France. Email: renaud.leplaideur@univ-brest.fr
from the spectral gap between the dominating eigenvalue and the rest of
the spectrum [17]. With weaker assumptions (either weaker hyperbolicity
or weaker regularity for the potential), it turns out to be much harder to
study the spectrum of the transfer operator, thus difficult to prove some loss
of analyticity.

Here, we are interested in what people from Statistical Mechanics call a
freezing phase transition. That is that for \( \beta \) greater than some critical \( \beta_c \),
the pressure function is affine. Our goal is to study how substitutions in the
shift generate freezing phase transition. This paper is in the continuation
of [1, 3, 4]. It continues the investigation of links between some phase tran-
sitions, substitutions and renormalizations. Roughly speaking, the ideas of
the construction are the following: a substitution generates two objects.
On the one hand an attractor, say \( K \), on the other hand a renormalization
operator \( R \) acting on continuous potentials \( V \). This operator has a fixed
point, say \( V_R \), and a stable set, that is the set of potentials \( V \) such that
\( R^n(V) \) goes to \( V_R \) as \( n \) goes to +\( \infty \). Then, every potential in this stable
set has a freezing phase transition with ground state in \( K \), that is, after the
transition, the unique measure which realizes the maximum in the definition
of the pressure is the unique ergodic measure supported into \( K \).

In [1] the renormalization operator on potentials \( R \) was introduced. It
was related to a renormalization equation for the dynamics and then a link
between the two historical examples of phase transitions in ergodic theory
— the Hofbauer potential in the 2-full shift and the Manneville-Pomeau
map in the interval, see [10, 15]— and the operator \( R \) was made. In [3]
it was pointed out that substitutions with constant length 2 were natural
candidates to solve the renormalization equation for the dynamics and then
the Thue-Morse case was fully studied. In [4] the Fibonacci case was studied.
For that, the definition of the renormalization operator \( R \) had to be adapted
to take care that the Fibonacci substitution is not of constant length.

Here, we study the case of a family of substitutions and show the global
machinery described above works. The substitutions are not necessarily of
constant length and the main condition is that they are marked (see below),
which basically means that knowing the first or the last digit of \( H(a) \) implies
that we know who the digit \( a \) is.

In addition to this general result we improve the result of [3] on the
Thue-Morse substitution (which is in fact marked). In [3], the convergence
of potentials to the fixed point by iteration of the renormalization operator
\( R \) was proved only in the Cesaro sense. Moreover the existence of phase
transition for the good critical exponent (see below) was not proved. Here,
we get stronger convergence to the fixed point for iterations of \( R^n \) and also
phase transitions for the critical exponent $\alpha = 1$.

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1.2 Results

Let $\mathcal{A}$ be a finite set with cardinality $D \geq 2$ called the alphabet. Elements of $\mathcal{A}$ are called letters or digits. A word is a finite or infinite string of digits. If $v$ is the finite word $v = v_0 \ldots v_{n-1}$ then $n$ is called the length of the word $v$ and is denoted by $|v|$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$. If $u = u_0 \ldots u_{n-1}$ is a finite word and $v = v_0 \ldots$ is a word, the concatenation $uv$ is the new word $u_0 \ldots u_{n-1}v_0 \ldots$. If $v$ is a finite word, $v^n$ denotes the concatenated word

$$v^n = v \ldots v_{n \text{ times}}.$$

The shift map is the map defined on $\mathcal{A}^\mathbb{N}$ by $\sigma(u) = v$ with $v_n = u_{n+1}$ for all integer $n$. We endow $\mathcal{A}$ with the discrete topology and consider the product topology on $\mathcal{A}^\mathbb{N}$. This topology is compatible with the distance $d$ on $\mathcal{A}^\mathbb{N}$ defined by

$$d(x, y) = \frac{1}{D^n} \text{ if } n = \min\{i \geq 0, x_i \neq y_i\}.$$

**Definition 1.1.** An infinite word $u$ is said to be periodic (for $\sigma$) if it is the infinite concatenation of a finite word $v$, that is $u = vvvv\ldots$ In that case we set $u = v^\infty$.

A substitution $H$ is a map from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\epsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of $\mathcal{A}^*$ by concatenation, that is $H(uv) = H(u)H(v)$.

Several basic notions on substitutions are recalled in Section 2. We also refer to [16]. We recall here the notions we need to state our results.

**Definition 1.2.** If $H$ is a substitution, its incidence matrix is the $D \times D$ matrix $\mathcal{M}_H$ with entries $a_{ij}$ where $a_{ij}$ is the number of $j$’s in $H(i)$. Then, $H$ is said to be primitive if all entries of $\mathcal{M}_H^k$ are positive for some $k \geq 1$.

A $k$-periodic point of $H$ is an infinite word $u$ with $H^k(u) = u$ for some $k > 0$. If $k = 1$ the point is said to be fixed. Then, $H$ is said to be aperiodic if no fixed point for $H$ is a periodic sequence for $\sigma$. 

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We emphasize an equivalent definition for being primitive. The substitution $H$ is primitive if and only if there exists an integer $k$ such that for every couple of letters $(i, j)$, $j$ appears in $H^k(i)$.

Let $H$ be a substitution over the alphabet $A$, and $a$ be a letter such that $H(a)$ begins with $a$ and $|H(a)| \geq 2$. Then there exists a unique fixed point $u$ of $H$ beginning with $a$ (see [16, 1.2.6]). This infinite word is the limit of the sequence of finite words $H^n(a)$. Assume that $\omega$ is a fixed point for $H$, then we set

$$K := \{\sigma^n(\omega), n \in \mathbb{N}\}.$$ 

If $H$ is a primitive substitution, then $K$ does not depend on the fixed point $\omega$. If $H$ is aperiodic, then $K$ is uniquely ergodic but not reduced to a $\sigma$-periodic orbit. In that case, the unique $\sigma$-invariant probability is denoted by $\mu_K$.

We recall that the language of a primitive substitution is the set of finite words which appear in a fixed point. It is denoted by $L_H$.

**Definition 1.3.** A substitution is said to be 2-full if any word of length 2 in $A^*$ belongs to the language of the substitution. A substitution is said to be marked if the set of the first (and last) letters of the images of the letters by the substitution is in bijection with the alphabet.

Let $V : A^\mathbb{N} \to \mathbb{R}$ be a function called potential. For every integer $n$ and for every letter $a$, let us denote $t_n(x) = |H^n(a)|, x \in [a]$. Then we introduce the renormalization operator $R$ defined by:

$$R : C(A^\mathbb{N}, \mathbb{R}) \to C(A^\mathbb{N}, \mathbb{R})$$

$$V(x) \mapsto R(V)(x) = \sum_{i=0}^{t_1(x)-1} V \circ \sigma^i \circ H(x) \quad (1)$$

Then, our first main result deals with the existence of fixed points for $R$:

**Theorem 1.** Let $H$ be a 2-full, marked, aperiodic and primitive substitution, then there exists $U : A^\mathbb{N} \to \mathbb{R}$ continuous such that $R(U) = U$.

Moreover, if $V : A^\mathbb{N} \to \mathbb{R}$ is such that $V_K(x) \equiv 0$ and $V(x) = \frac{g(x)}{p^x} + \frac{h(x)}{p^x}$ if $d(x, K) = D^{-p}$, where $g$ is a continuous positive function and $h$ is continuous and satisfies $h_K \equiv 0$, then for every $x$ in $A^\mathbb{N}$,

$$\lim_{m \to +\infty} R^m V(x) = \begin{cases} 
0 & \text{if } \alpha > 1, \\
+\infty & \text{if } \alpha < 1, \\
\int g \, d\mu_K \cdot U(x) & \text{if } \alpha = 1.
\end{cases}$$
Remark 1. The expression of $U$ is explicit for a given substitution. It will be explained during the proof, and in Section 5.

In the following, we denote by $\Xi_\alpha$ the set of potentials of the form $V(x) = \frac{g(x)}{p^x} + \frac{h(x)}{p^x}$ as in Theorem 1.

We emphasize another byproduct of our study. It deals with theory of substitutions. Actually we present a new way to study substitutions, that is from the outside, on the contrary to the classical one which is from inside. From inside means that usually people study, for a given substitution $H$, its language $\mathcal{L}_H$ and get results on occurrence on some special words or family of words (see e.g. [5] or [9]). From outside means that we study how an orbit or a piece of orbit in the shift $x \ldots \sigma^n(x)$ behaves with respect to $K$ with $x \notin K$. In particular we emphasize the importance of the notion of *accident* (see Subsec. 2.3). Obviously, this new point of view uses tools from the study from inside, in particular we shall show how bispecial words are important.

Our second theorem deals with phase transition for potentials in $\Xi_1$. If $V$ is a potential and $\beta$ a non-negative real parameter, the pressure function for $V$ is defined by

$$P(\beta) := \sup_{\mu \sigma^{-\text{inv}}} \left\{ h_\mu + \beta \int V \, d\mu \right\},$$

where $h_\mu$ is the Kolmogorov entropy for the $\sigma$-invariant probability $\mu$. It is known that the pressure function is convex and admits an asymptote of the form $a + \beta b$ as $\beta$ goes to $+\infty$. If the phase transition reaches its asymptote we speak of *freezing phase transition*. Moreover, any measure realizing the supremum in the definition of $P(\beta)$ is called an *equilibrium state for $\beta.V.$*

**Theorem 2.** Consider a primitive, aperiodic and marked substitution, $K$ associated to $H$ as above and consider a potential $V := -\varphi$ with $\varphi$ in $\Xi_1$. Then there exists a positive number $\beta_c$ such that the pressure function has a freezing phase transition at $\beta_c$. More precisely:

- For $\beta < \beta_c$ the pressure function is analytic, there is a unique equilibrium state for $\beta.V$ and it has full support.

- For $\beta > \beta_c$ the pressure is equal to zero and $\mu_K$ is the unique equilibrium state for $\beta.V.$
We emphasized above that Theorem 2 extends the known results for Fibonacci and Thue-Morse substitutions to a family of substitutions. Actually, it turns out that our proof requires only ingredients which hold for a more general family of shift. We state here a more general theorem whose assumptions involve notions presented below. We point out that the attractor \( \mathbb{K} \) for a aperiodic primitive and marked substitution satisfies these assumptions.

**Theorem 3.** Consider a shift with finite alphabet which satisfies the following properties:

1. It is linearly recurrent (see [8, Sec7]).

2. The bispecial words (see Definition 2.1) are all of the length \( c \lambda^n + o(\lambda^n) \), where \( \lambda > 1 \) and \( c \) belongs to a finite set.

3. Bispecial words cannot overlap each other for more than a fixed proportion than the smaller one (see Lemma 4.6)

Then, every non-negative potential of the form \( \varphi(x) = 0 \) if and only if \( x \) belongs to \( \mathbb{K} \) and \( \varphi(x) = -\frac{1}{n} + o(\frac{1}{n}) \) if \( d(x, \mathbb{K}) = D^{-n} \) admits a freezing phase transition with ground state supported into \( \mathbb{K} \).

### 1.3 Outline of the paper

First of all in Section 2 we recall some classical definitions and results on substitutions and symbolic dynamics. The last part of this section is devoted to some background on the notion of accidents, defined in [3].

Then in Section 3 we prove our first theorem. The proof can be decomposed in several parts. We obtain a formula for \( R^m V \) in Lemma 3.1. To study the convergence of this term we need to get good estimates for \( \delta^n_i(x) \) for \( i < t_n(x) \) and for any \( x \notin \mathbb{K} \). This is done in Corollary 3.8. Finally we compute the limit in two steps: one for the simplest case \( g = 1 \) and one for the general case, see Subsection 3.4.3.

In Section 4, we begin the proof of the second theorem. First of all we recall the method, based on the previous work [4], see Lemma 4.2. Now we begin by proving the result for a special potential. Then in Proposition 4.3 we obtain the first inequality. The second one is obtained in Equation (8). This inequality is improved by Proposition 4.5 and Equality (12). We obtain an explicit upper bound for \( C_{\mathcal{E}_F} \) in Proposition 4.7. Then Corollary 4.8 finishes the proof. Proposition 4.9 finishes the proof for a general potential.
At the end of this section we also summarize the main ingredients of the proof of Theorem 2 and explain why Theorem 3 holds. In Section 5, we give a concrete proof of the first theorem for the example of Thue Morse.

2 More definitions and tools

2.1 Words, languages and special words

For this paragraph we refer to [16].

Definition 2.1. A word $v = v_0 \ldots v_{r-1}$ is said to occur at position $m$ in an infinite word $u$ if there exists an integer $m$ such that for all $i \in [0; r-1]$ we have $u_{m+i} = v_i$. We say that the word $v$ is a factor of $u$.

For an infinite word $u$, the language of $u$ (respectively the language of length $n$) is the set of all words (respectively all words of length $n$) in $A^*$ which appear in $u$. We denote it by $L(u)$ (respectively $L_n(u)$). Then, the sequence of finite languages $(L_n(u))$ is said to be the factorial language for $L(u)$.

Definition 2.2. The dynamical system associated to an infinite word $u$ is the system $(K_u, \sigma)$ where $\sigma$ is the shift map and $K_u = \{\sigma^n(u), n \in \mathbb{N}\}$. An infinite word $u$ is said to be recurrent if every factor of $u$ occurs infinitely often.

Remark that $u$ is recurrent is equivalent to the fact that $\sigma$ is onto on $K_u$. Moreover we have equivalence between $\omega \in K_u$ and $L_\omega \subset L_u$. Thus the language of $K_u$ is equal to the language of $u$.

Definition 2.3. Let $\mathcal{L} = (\mathcal{L}_n)_{n \in \mathbb{N}}$ be a factorial and extendable language. The complexity function $p : \mathbb{N} \to \mathbb{N}$ is the function defined by $p(n) := \text{card}(\mathcal{L}_n)$. For $v \in \mathcal{L}_n$ let us define

\[
\begin{align*}
    m_l(v) &= \text{card}\{a \in A, av \in \mathcal{L}_{n+1}\}, \\
    m_r(v) &= \text{card}\{b \in A, vb \in \mathcal{L}_{n+1}\}, \\
    m_b(v) &= \text{card}\{(a,b) \in A^2, avb \in \mathcal{L}_{n+2}\}, \\
    i(v) &= m_b(v) - m_r(v) - m_l(v) + 1.
\end{align*}
\]

- A word $v$ is called right special if $m_r(v) \geq 2$. 

\[\text{7}\]
• A word \( v \) is called left special if \( m_l(v) \geq 2 \).

• A word \( v \) is called bispecial if it is right and left special.

**Definition 2.4.** A word \( v \) such that \( i(v) < 0 \) is called a weak bispecial. A word \( v \) such that \( i(v) > 0 \) is called a strong bispecial. A bispecial word \( v \) such that \( i(v) = 0 \) is called a neutral bispecial.

**Definition 2.5.** If \( u = u_0 \ldots u_{n-1} \) is a word, a prefix of \( u \) is any factor \( u_0 \ldots u_j \) with \( j \leq n - 1 \). A suffix of \( u \) is any word of the form \( u_j \ldots u_{n-1} \) with \( 0 \leq j \leq n - 1 \).

### 2.2 Substitutions

#### 2.2.1 Some more definitions

**Definition 2.6.** Let \( H \) be a substitution. The set of all prefixes and all suffixes for all the \( H(a) \), \( a \in A \), are respectively denoted by \( P \) and \( S \).

**Definition 2.7.** Let \( H \) be a substitution. We say that the word \( u \in \mathcal{L}_H \) is **uni desubstituable** if there exists only one way to write \( u = sH(v)p \) with \( p \in P, s \in S \) where

1. \( p \) is a prefix of \( H(\hat{p}) \),
2. \( s \) is a suffix of \( H(\hat{s}) \),
3. \( \hat{s}v\hat{p} \) is a word in \( \mathcal{L}_H \).

We recall the following theorem

**Theorem 2.8.** [14] Let \( H \) be a marked, primitive, aperiodic substitution. There exists a constant \( N_H \) such that for every word \( w \in \mathcal{L}_H \) the word \( w^{N_H} \) does not belong to this language.

**Remark 2.** Remark that \( N_H \) can be computed by an algorithm.

#### 2.2.2 Examples

• We recall that the following substitutions are called Thue Morse and Fibonacci substitutions:

\[
\begin{align*}
0 & \to 01 \\
1 & \to 10
\end{align*} \quad \begin{align*}
0 & \to 01 \\
1 & \to 0
\end{align*}
\]
Their incidence matrices are \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\] and \[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
\end{pmatrix}
\] respectively.

- The substitution \[
a \mapsto aba \\
b \mapsto bab
\]
has \(ababababab\cdots = (ab)\infty\) as fixed point. The substitution is thus not aperiodic, nevertheless, it is primitive.

- The next substitution is an example of 2-full marked substitution which is not of constant length:
\[
\begin{align*}
0 & \rightarrow 012 \\
1 & \rightarrow 10 \\
2 & \rightarrow 21 \\
\end{align*}
\]

2.2.3 Length of words in the language of a substitution

If \(H\) is a primitive substitution, the Perron Frobenius theorem shows that the incidence matrix admits a single and simple dominating eigenvalue. We denote it by \(\lambda\). It is a positive real number. The rest of the spectrum is strictly included into the disc \(D(0,\lambda)\).

Then, we emphasize that there exists a constant \(K\) such that the length of a word \(H^n(v)\) satisfies
\[
|H^n(v)| \leq K\lambda^n. \tag{2}
\]
Thus in all the following computations we will consider this upper bound.

2.3 Accidents

Let \(x\) be an element of \(A^\mathbb{N}\) which does not belong to \(K\), then we define and denote:

- The word \(w\) is the maximal prefix of \(x\) such that \(w\) belongs to the language of \(K\). Thus we denote \(d(x, K) = D^{-d}, x = x_1 \ldots x_d \ldots\) with \(w = x_1 \ldots x_d\). Let us denote \(\delta(x) = d\), and \(\delta_k^n = \delta(\sigma^k \circ H^n(x))\) for every integers \(k\) and \(n\). Note that \(\delta = \delta_0^0\).

- If there exists an integer \(b < d\) such that \(\delta_0^0(x) > d - b\) and \(\delta_i^0(x) = d - i\) for all \(i < b\), then we say that an accident appears at time \(b\). The prefix of \(\sigma^b(x)\) of length \(\delta_0^0\) is called the first accident of \(x\) of depth \(\delta_0^0\).

Remark that the word \(w\) is non-empty since every letter is in the language of \(K\) if the substitution is primitive. Then, \(w\) is the unique word such that
\[
x = wx', w \in L_H, wx'_0 \notin L_H.
\]
Figure 1 illustrates the next lemma which appears in [3].

**Lemma 2.9.** Let \( x \) be an infinite word not in \( K \). Assume that \( \delta(x) = d \) and that the first accident appears at time \( 0 < b \leq d \) then the accident \( x_b \ldots x_{d-1} \) is a bispecial word of \( \mathcal{L}_H \).

**Remark 3.** If \( A \) has cardinality two, then \( x_0 \ldots x_{d-1} \) is not right-special. Moreover, and always if \( A \) has cardinality two, if \( x = \sigma(z) \) and there is an accident at time 1 for \( z \), then \( x_0 \ldots x_{d-1} \) is not left-special. ■

![Figure 1: Accidents-dashed lines indicate infinite words in \( K \).](image)

**Definition 2.10.** We define inductiveley

\[
\begin{align*}
b_1 &= b = \min\{j \geq 1, d(\sigma^j x, K) \leq d(\sigma^j x, \sigma^j(y))\} \\
b_2 &= \min\{j \geq 1, d(\sigma^{j+b_1} x, K) \leq d(\sigma^{j+b_1} x, \sigma^j(y'))\} \\
b_3 &= \min\{j \geq 1, d(\sigma^{j+b_1+b_2} x, K) \leq d(\sigma^{j+b_1+b_2} x, \sigma^j(y''))\} \\
&\ldots
\end{align*}
\]

We also define \( B_j = b_0 + \ldots b_j \) where \( b_0 = 0 \), and let us denote \( d_j = \delta(\sigma^{B_{j-1}} x) \).

The integers \( B_i, i \geq 1 \) are called the times of accident, and the words \( x_{b_i} \ldots x_{b_i+b_{i+1}} \) are called **accidents**.

**Lemma 2.11.** Consider \( x \) such that \( \delta(x) = d \). Denote by \( B_1, B_2 \) the times of first and second accidents. Assume the two bispecial words defined by the accidents does not overlap, then we have:

\[
\begin{align*}
\delta_i(x) &= d - i, 0 \leq i < B_1 \\
\delta_i(x) &= d' - B_1 - i, B_1 \leq i < B_2
\end{align*}
\]
Proof. It is a simple application of the definition of accident. See also Figure 1 with \( B_1 = b \)

We recall that for \( x \in \mathcal{A}^N \) of the form \( x = a \ldots \) and for a primitive, 2-full and marked substitution \( H \), we have set \( t_n(x) = |H^n(a)| \). Then, we set:

Definition 2.12. We denote by \( \mathcal{B}_n(x) \) the set of times of accident lower than \( t_n(x) \).

3 Proof of Theorem 1

3.1 Renormalization operator and accidents

In order to prove Theorem 1 we need to compute \( R^nV \). We give here a formula for \( R^nV(x) \) and explain why \( \lim_{n \to +\infty} R^nV(x) \) only depends on the germ of \( V \) close to \( \mathcal{K} \).

3.1.1 A formula for \( R^nV \)

We emphasize that \( \sigma \) satisfies the following next renormalization equation (with respect to \( H \))

\[
H \circ \sigma(x) = \sigma^{t_1(x)} \circ H(x).
\]

This equality is the key point to prove the formula that gives an expression for \( R^n \):

Lemma 3.1. For every integer \( n \) and for every \( x \in \mathcal{A}^N \) we have

\[
R^nV(x) = \sum_{i=0}^{t_n(x)-1} V \circ \sigma^i \circ H^n(x).
\]

Proof. We make a proof by induction:
For \( n = 1 \) it is clear. Assume the result is true for \( n \). For all \( j \in [0 \ldots t_1(H(x)) - 1] \), and for all \( i \in [0 \ldots t_1(x) - 1] \) we have:

\[
\sigma^i \circ H = \sigma^{s(i,x)} \circ H^2, \quad \text{where} \quad s(i,x) = \sum_{j=1}^{i} t_1(x_j).
\]

By induction hypothesis we deduce

\[
R^{n+1}V(x) = R^n \circ RV(x) = \sum_{j=0}^{t_2(x)-1} \sum_{i=0}^{t_n(x)-1} V \circ \sigma^j \circ H \circ \sigma^i \circ H^n(x)
\]
\[ R^{n+1}V(x) = \sum_{j=0}^{t_1(x)-1} \sum_{i \leq t_n(x)-1} V \circ \sigma^j(x) \circ H^{n+1}(x) \]
\[ = \sum_{i=0}^{t_{n+1}(x)-1} V \circ \sigma^i(x) \circ H^{n+1}(x). \]

We used the fact that \( t_{n+1}(x) = |H^{n+1}(a)| = |H(H^n(a))| = \sum_{i=1}^{t_n(x)} t_1(i). \) The induction hypothesis is proved. \( \square \)

### 3.1.2 Distance between \( \sigma^i(H^n(x)) \) and \( K \)

Lemma 3.1 shows why it is so important to know the numbers \( \delta^n_k(x) = \delta(\sigma^k(H^n(x))) \) for every \( x \) and for \( k \leq t_n(x) - 1 \). We shall see below why accidents perturb the computation of \( R^n(V)(x) \). This explains why we need to control them.

Moreover, \( R^n(V)(x) \) involves a Birkhoff sum at point \( H^n(x) \) which changes if \( n \) increases. Clearly, \( H^n(x) \) converges to a fixed point of \( H \), thus goes to \( K \) if \( n \) increases. But this convergence may be faster than what we could expect, just knowing for how many digits \( x \) coincides with \( K \). We give here two examples illustrating this point:

**Example.** Consider \( H : \{ \begin{array}{c} a \rightarrow abbaaa \\ b \rightarrow baaaab \end{array} \} \). The word \( bbb \) does not belong to the language. Nevertheless \( H(bbb) \) belongs to \( \mathcal{L} \) as seen by the computation of

\[ H(aaaa) = abbaaabbaaaabbaaaabbaaa = abH(bbb)aa \]

Here, for \( x = bbb \ldots \) we have \( \delta(x) = 2 \) and \( \delta(H(x)) = \delta_1(x) \geq 3 \ast 6 > 2 \ast 6. \)

Consider

\[ H : \{ \begin{array}{c} a \rightarrow aaab \\ b \rightarrow abaa \end{array} \} \]

We have \( H(a^3) = a^3ba^3b = a^2H(bb)ab, \) thus \( bb \) does not belong to the language, and \( H \) is not 2-full. Nevertheless we have \( H(bb) = aba^3ba^2, \) which is a factor of \( H(aaa). \) Now let \( x = b^3H^\infty(a), \) then we obtain \( x = bba^3ba^3babab^5ba^3b \ldots \) Remark that \( \delta(x) = 1. \) Moreover \( H(x) = aba^3ba^5b \ldots, \) thus we obtain \( \delta_1(x) = 7. \)
3.1.3 Necessity of 2-full hypothesis. Germ of a potential close to $K$

We can now explain why knowing the germ close to $K$ is sufficient to determine $\lim_{n \to +\infty} R^n V(x)$. Note that $H$ is 2-full which means that for every $x$, $\delta(x) \geq 2$. Set $x = ab \ldots$, it follows that $\delta_n^a(x)$ is bigger than $t_n(a) + t_n(b)$, and then for every $k \leq t_n(a) - 1$

$$\delta_n^a(x) \geq t_n(b) + t_n(a) - k. \quad (3)$$

Remember that $t_n(b)$ has a length of order $\lambda^n$. This computation shows that among all the points $\sigma_k(H^n(x))$, the farthest from $K$ is at distance at most $D - t_n(b) - 1 \sim D - \lambda^n$. It thus makes sense to replace $V(\sigma^k(H^n(x)))$ by $g(\sigma^k(H^n(x)))/(\delta_k^a(x))^\alpha$.

**Counter-example** On the contrary, consider the following substitution

$$H = \begin{cases} a \to abba \\ b \to bab \end{cases}$$

This substitution is primitive, marked but is not 2-full since $aa$ does not belong to the language.

Then consider $x = aa \ldots$ we have $\delta(x) = 1$. Therefore, $H^n(x) = H^n(a)H^n(a) \ldots$. Note that $H^n(a)$ finishes and starts with $a$ and then $H^n(a)H^n(a)$ contains the word $aa$ in its middle. Furthermore, any suffix of $H^n(a)$ is in the language but no suffix of $H^n(a)a$ belongs to the language. Therefore, for any $i \leq n \delta_n^a(x) = |H^n(a)| - i$ and thus $\delta_n^{n-1}(x) = 1$. This shows that knowing the germ close to $K$ is not sufficient to determine the limit for $R^n(V)(x)$. Furthermore, we will see later that $R^n(V)(x)$ does not converge.

3.2 Bispecial words for marked substitutions

As we have seen above, it is important to detect accidents. We also pointed out that accidents are related to occurrences of bispecial words in the language. It is therefore of prime importance to study these bispecial words. We prove here a strong version of Theorem 2.8 in Theorem 3.4. This allows us to get a complete description of the set of bispecial words (see Proposition 3.5).

**Lemma 3.2.** Assume that $H$ is a marked substitution. If $z = H(x) = SH(y)$ is an infinite word and $S$ a finite word in $A^*$, then either $S$ is empty and $x = y$ or the word $z$ is ultimately periodic.
Proof. If $S$ is the empty word, then the left marking proves the result. If not, then let us denote by $t$ the length of $S$. Denote $x = x_1x_2\ldots$. The infinite word $H(x)$ can be cut by construction into words corresponding to the images of the letters by $H$, i.e $H(x) = H(x_1)H(x_2)\ldots$. Let us do the same thing for $H(y)$. Since $H$ is left marked, the first letters of the image are in bijection with the alphabet, thus we can assume that $H(x_i)$ begins with $x_i$ for every integer. We denote by $t' = |H(x_1)| - t$, see the Fig. 2.

\[ \begin{array}{c}
H(x_1) \quad H(x_2) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
H(y_1) \quad H(y_2) \\
\uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
t \quad \quad \quad \quad \quad t' 
\end{array} \]

Figure 2: $\sigma^t H(x) = H(y)$

First of all assume that $t + |H(y_1)| = |H(x_1)| + |H(x_2)|$. Then we have $SH(y_1) = H(x_1x_2)$, the hypothesis of right marking allows us to deduce $y_1 = x_2$ and $S = H(x_1)$ which is impossible.

Thus we can define a function $\psi$ on $A^2 \times [0\ldots\max|H(a)|]$ by the formula

\[ (x_1,y_1,t) \mapsto \psi(x_1,y_1,t) = \begin{cases} (x_2,y_1,t') & t < |H(x_1)| \\ (y_1,x_2,t) & t > |H(x_1)| \end{cases} \]

This function is defined on a finite set and can be iterated by the previous argument, thus $\psi$ is ultimately periodic. This implies that the word $z$ is ultimately periodic by the pigeonhole principle.

From Lemma 3.2 we deduce a very important result. If $x$ belongs to $A^N \setminus K$, then so does $H(x)$:

Corollary 3.3. For each word $x = wx'$ with $w \in L_H$ and $wx' \notin L_H$, for every integer $s$ there exists $m < \infty$ such that $\delta[H^s(x)] = m$.

Proof. The proof is by contradiction and by induction. Assume $H(x) \in K$ thus it can be written $SH(y)$ with $y \in K$. Then we apply Lemma 3.2. If $S = \epsilon$ (the empty word) then, $x = y$ and it is a contradiction with our
assumption. If \( S \neq \epsilon \), then \( y \) is ultimately periodic which is in contradiction with Theorem 2.8. This shows
\[
x \notin K \implies H(x) \notin K.
\]

Then, the result follows by induction.

**Theorem 3.4.** Consider a primitive, aperiodic and marked substitution. There exists \( l(H) > 0 \) such that for every \( z \in \mathcal{L}_H \) with \( |z| > l(H) \) there exists a unique decomposition \( z = SH(x)P \) with \( (S, P) \in \mathcal{S} \times \mathcal{P} \), \( S = H(s) \), \( P = H(p) \) and \( sxp \in \mathcal{L}_H \).

**Proof.** The existence of the decomposition is clear because \( K = \{ \sigma^n(v), n \in \mathbb{N} \} \) where \( v \) is any fixed point for \( H \). Now assume we have two decompositions
\[
SH(x)P = S'H(y)P'.
\]
We will apply an effective version of the proof of Lemma 3.2. Let us denote \( s = \max_a |H(a)| \). The same proof can be applied, it suffices to remark that the period and the pre-period are bounded by the cardinality \( D \) of the finite alphabet \( A \). Consider the minimum \( p_0 \) of the integers \( p \) such that \( (D^2s)^p + sD^2 > N_H \). The proof is done with \( l(H) = (D^2s)^{p_0} + sD^2 \). We deduce \( S = S' \), then the same argument shows that \( P = P' \).

**Remark 4.** We emphasize that Theorem 3.4 is false without the marked assumption. Consider \( H : \begin{cases} a \rightarrow aba \\ b \rightarrow ab \end{cases} \) which is not marked. Note that both \( aa \) and \( ab \) belong to the language. We thus claim that there exists a sequence of right special words with length going to infinity. Let \( u \) be a right-special word with length as big as wanted. Then we have \( H(ua) = H(u)H(a) = H(u)aba = H(u)H(b)a = H(ub)a \). This contradicts uniqueness of the decomposition \( H(ua) \).

**Proposition 3.5.** Let \( H \) be a primitive, aperiodic and marked substitution. Let \( \mathcal{W}_b \) be the set of bispecial words of length less than \( l(H) \). Then every bispecial word can be written \( H^n(v) \) with \( v \in \mathcal{W}_b \) and \( n \) some integer.

**Proof.** Consider a bispecial word \( u \). By Theorem 3.4 we can write \( u = SH(v)P \) where \( v \) has maximal length, \( v, S \) and \( P \) are unique.

We claim that \( S \) is empty. Indeed, since \( u \) is a bispecial word, there exist two letters such that \( au \) and \( bn \) belong to the language. If \( S \) is non-empty, then \( aS, bS \) are the suffixes with the same length of \( H(c) \) where \( c \) is a letter.
(unique by assumption on $H$). We deduce $a = b$, which is impossible. The same argument applies for $P$.

Now we prove that $v$ is a bispecial word. If $aH(v)$ belongs to the language $L_H$, the properties of $H$ show that it is the suffix of a unique word $H(c)H(v)$. The same argument works for $bH(v)$ the other left extension of $H(v)$. The two left extensions of $v$ are different by assumption on $H$. By the same argument $v$ is right special. The proof finishes by an iteration of this process.

We recall that $\lambda$ is the dominating eigenvalue for the incidence matrix of $H$. Then Proposition 3.5 yields:

**Corollary 3.6.** There exist $0 < \theta < \lambda$ and a finite set of positive numbers $c$, such that the lengths of the bispecial words of $L_H$ are of the form $c\lambda^n + O(\theta^n)$, $n \in \mathbb{N}$.

Note that the numbers $c$ are the lengths of the words in $W_h$.

### 3.3 Crucial Proposition

By Lemma 3.1, $R^n(V)(x) = S_{tn(x)}(V) \circ H^n(x)$. To study the convergence of this term we need to get good estimates for $\delta^i_0(x)$ for $i < t_n(x)$ and for any $x \not\in \mathbb{K}$ (see also the discussion after Lemma 3.1). We have an easy bound from above:

$$\delta^i_0(x) \geq \delta^n_0(x) - i,$$

but we need a sharper estimate. For that purpose, we need to know the times of accidents $B_n(x)$ (recall 2.12). The following main proposition shows that accidents do not occur randomly.

**Proposition 3.7.** Let $H$ be a 2-full, marked, aperiodic and primitive substitution. Let $x \not\in \mathbb{K}$ and $p$ be such that $\delta^0_0(x) = p$. Set $x = w_1 \ldots w_p x_{p+1} \ldots \notin \mathbb{K}$ and let $k$ be such that $|H^k(w_2 \ldots w_p)| \geq l(H)$. Then

$$B_n(x) = H^{n-k}(B_k(x)) \text{ for } n \geq k.$$

**Proof.** Note that $x = wx_{p+1} \ldots$ and $w \in L_H$. Let us write $H^k(x) = e_1 \ldots e_{mk} e_{mk+1} \ldots$ with $m_k = \delta^k_0(x)$. Corollary 3.3 shows that $m_k$ is finite.

- First we prove $\delta^0_0(x) = |H^{n-k}(e_1 \ldots e_{mk})|$. Note that we have $H^n(x) = H^{n-k}H^k(w_1 \ldots w_p \ldots) = H^{n-k}(e_1 \ldots e_{mk} e_{mk+1} \ldots)$ holds, which shows that $\delta^0_0(x)$ is bigger than $|H^{n-k}(e_1 \ldots e_{mk})|$ because $e_1 \ldots e_{mk}$ belongs to $L_H$. Actually, the proof is also done by induction.
Assume by contradiction that $\delta_0^{k+1}(x)$ is strictly bigger than the number $|H(e_1 \ldots e_{m_k})|$. This means that there exists a letter $a$ such that $H(e_1 \ldots e_{m_k})a \in \mathcal{L}_H$. Note that $|H(e_1 \ldots e_{m_k})| > |H^k(w_2 \ldots w_p)| \geq l(H)$, we can thus apply Theorem \ref{thm:main} to the word $H(e_1 \ldots e_{m_k})a$. By the left marking of $H$ we deduce that $e_1 \ldots e_{m_k}e \in \mathcal{L}_H$ with letter $e$ such that $H(e)$ begins with $a$, as $H(e_{m_{k+1}})$. This is a contradiction with the definition of $m_k$. We then iterate this argument, noting that $|H^j(e_1 \ldots e_{m_k})|$ increases in $j$ and is thus bigger than $l(H)$.

• Now consider the first time of accident of $H^k(x)$ and denote it by $j_1$. We argue by contradiction and prove that $H^n(x)$ cannot have an accident for $i < |H^{n-k}(e_1 \ldots e_{j_1})|$. By definition we have $j_1 < t_k(x) \leq m_k$ and $\delta_{j_1}^k(x) > m_k - j_1$ whereas $\delta_{j_1-1}^k(x) = m_k - j_1 + 1$.

Pick $0 < i < |H^{n-k}(e_1 \ldots e_{j_1})|$ and assume that $\delta_{j_1}^n(x) > \delta_{j_1}^n(x) - i$. We have $H^n(x) = H^{n-k}(e_1)H^{n-k}(e_2) \ldots$. Let us introduce $l$ the smallest integer such that $i < |H^{n-k}(e_1 \ldots e_l)|$. A prefix of $\sigma^lH^n(x)$ can be written $SH^{n-k}(e_{l+1} \ldots e_{m_k})a \in \mathcal{L}_H$ with $S$ suffix of $H^{n-k}(e_1)$ and $a \in A$. Note that $l \leq j_1 < t_k(x)$, which yields that $H^n(x) = H^{n-k}(H^k(w_2 \ldots w_p))$ is a factor of $H^{n-k}(e_{l+1} \ldots e_{m_k})$. We can thus apply Theorem \ref{thm:main} and by the right marking of $H^k$, we obtain a word suffix of $e_1 \ldots e_{m_k}e \in \mathcal{L}_H$. This means that $H^k(x)$ has an accident at time $l - 1 < j_1$ and this is a contradiction with the definition of $j_1$.

Finally we have proven

$$\delta_{j_1}^n(x) = \delta_{j_1}^0(x) - i, 0 \leq i < |H^{n-k}(e_1 \ldots e_{j_1})| - 1.$$ 

• We claim that $\delta_{j_1}^n(x) > \delta_{j_1}^n(x) - j_1$. By definition of an accident we know that $e_{j_1} \ldots e_{m_k}e \in \mathcal{L}_H$ for some letter $e$. Then by application of $H^{n-k}$ we deduce that $H^{n-k}(e_{j_1} \ldots e_{m_k})a \in \mathcal{L}_H$.

• Let us denote by $j_2$ the second time of accident of $H^k(x)$. Note that $H^n(w_2 \ldots w_p)$ has length bigger than $l(H)$ and is still a factor of $H^{n-k}(e_{j_2} \ldots e_{m_k})$ because $j_2 < t_k(x)$. Note also that $\sigma^{j_2}(H^k(x))$ coincides with a word of $K$ for at least $m_k - j_1 + 1$ digits. In other words, $H^{n-k}(e_{j_1} \ldots e_{m_k}e_{m_{k+1}})$ is a suffix of the coincidence of $\sigma^{j_2}(H^n(x))$ coincides with $K$. This suffix contains $H^{n-k}(e_{j_2} \ldots e_{m_k})$, thus it also contains $H^n(w_2 \ldots w_p)$. We can thus repeat the same process to $j_2$ and more generally to each accident of $H^k(x)$.

$\square$

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Corollary 3.8. Denote the times of accidents of $H^k(x)$ by $j_1, j_2, \ldots, j_s$, and their lengths by $\Delta_{j_1}, \ldots, \Delta_{j_s}$. We have:

- The accidents of $H^n(x)$ appear at times $t_{i,n-k} := \lambda^{n-k} j_i + O(\theta^n), i \leq s$.
- Their lengths are equal to $\Delta_{i,n-k} := \lambda^{n-k} \Delta_{j_i} + O(\theta^n), i \leq s$.

where $0 < \theta < \lambda$.

Proof. This a a direct corollary of the previous proposition. Note that $\Delta_{i,0} = \Delta_{j_i}$.

Remark 5. In Section 2 the numbers $\Delta_i$ coincide with the numbers $d_i - b_i + 1$.

3.4 Proof of Theorem 1

3.4.1 Preliminary lemma

Lemma 3.9. Let $a, \lambda$ be some positive real numbers and $f$ a Lipschitz function defined on a neighborhood of $[0, a]$. Let $\phi : \mathbb{N} \to \mathbb{R}$ be a real sequence such that $|\phi(n)| \leq C\theta^n$ with $C > 0$ and $0 < \theta < \lambda$. We have

$$\lim_{n \to +\infty} \frac{1}{\lambda^n} \sum_{k=0}^{a\lambda^n} f \left( \frac{k + \phi(n)}{\lambda^n} \right) = \int_{0}^{a} f(x) dx.$$ 

Proof. Let us denote $S_n$ the sum and $K$ the Lipschitz constant of the function $f$. We obtain

$$\left| S_n - \frac{1}{\lambda^n} \sum_{k=0}^{a\lambda^n} f \left( \frac{k}{\lambda^n} \right) \right| \leq \frac{1}{\lambda^n} \sum_{k=0}^{a\lambda^n} |f \left( \frac{k + \phi(n)}{\lambda^n} \right) - f \left( \frac{k}{\lambda^n} \right)|$$

$$\leq \frac{1}{\lambda^n} a\lambda^n K \frac{|\phi(n)|}{\lambda^n} \leq Ka \frac{|\phi(n)|}{\lambda^n}.$$

The upper bound term converges to zero as $n$ goes to infinity. The term $\frac{1}{\lambda^n} \sum_{k=0}^{a\lambda^n} f \left( \frac{k}{\lambda^n} \right)$ is a Riemann sum, thus we deduce the result.

Remark 6. The same type of proof works if $f$ is an uniformly continuous function. It also holds if the sum is done up to $a\lambda^n + o(\lambda^n)$ instead of $a\lambda^n$. ■
3.4.2 Computation of the $\lim_{m \to +\infty} R^m V$: the case $g \equiv 1$

We want to compute $\lim_{m \to +\infty} R^m(V)$. By Lemma 3.1 we have

$$R^m V(x) = \sum_{i=0}^{t_m(x)-1} V \circ \sigma^i \circ H^m(x).$$

The potential $V$ has the following form $V(x) = \frac{1}{p^\alpha} + o(\frac{1}{p^\alpha}).$

First of all consider the case $\alpha = 1$. Since $V(x) = \frac{1}{p} + o(\frac{1}{p})$ if $\delta(x) = p$, we obtain

$$R^m V(x) = \sum_{j=0}^{t_{m(x)}-1} \frac{1}{\delta^m_j(x)} + o(\ldots).$$

We emphasize that the term $o(\ldots)$ is actually a negligible term with respect to the first summand. Therefore, it does not influence the limit for $R^m V(x)$ and we shall forget it in the rest of our proof.

We pick some $x \not\in \mathbb{K}$ and reemploy notations from Corollary 3.8. Let $p = \delta(x)$ and $k$ be such that $|H^k(x_2 \ldots x_p)| > l(H)$. Let $j_1, j_2, \ldots, j_s$ be the times of accidents of $H^k(x)$, $\Delta_{j_i}$ their respective length. The accidents of $H^m(x)$ appear at times $t_{i,m-k} := \lambda^{m-k} j_i + O(\theta^{m-k})$ with lengths $\Delta_{i,m-k} = \lambda^{m-k} \Delta_{j_i} + O(\theta^{m-k})$.

Moreover, by Lemma 2.11

$$\delta^m_j(x) = \Delta_{i,m-k} - (j - t_{i,m-k}) \quad t_{i,m-k} \leq j < t_{i+1,m-k}$$

holds.

We split the sum $\sum_{j=0}^{t_{m(x)}-1} \delta^m_j(x)$ into the sums $\sum_{j=t_{i,m-k}}^{t_{i+1,m-k}-1}$ with the convention $t_{0,m-k} = 0$ and $t_{s+1,m-k} = t_{m(x)}$. To make notations consistent we also set $j_0 = 0$, $\Delta_0 = \delta^k_0(x)$ and $j_{s+1} = t_k(x) - 1$. Then we have
\[ R^m V(x) = \sum_{l=0}^{t_1,m-k-1} \frac{1}{\Delta_{0,m-k} - l} + \sum_{l=t_1,m-k}^{t_2,m-k-1} \frac{1}{\Delta_{1,m-k} - l + t_1,m-k} + \cdots + \sum_{l=t_s,m-m}^{t_m(x)-1} \frac{1}{\Delta_{s,m-k} - l + t_s,m-k} + o(...) \]

\[ = \sum_{i=0}^{s} \left( t_{i+1,m-k} - t_i,m-k-1 \sum_{l=0}^{1} \frac{1}{\Delta_{j_i} \lambda^{m-k} - l + \phi_i(m - k)} \right) \]

where \( \phi_i(m - k) \) and \( \phi_i'(m - k) \) are in \( O(\theta^{m-k}) \) with \( 0 < \theta < \lambda \). The computation of the sums is made with Lemma 3.9. We finally obtain

\[ U(x) = \lim_{+\infty} R^m(x) = \sum_{i=0}^{s} \log \left( \frac{\Delta_{j_i}}{\Delta_{j_i} - (j_{i+1} - j_i)} \right). \]

Note that this last quantity only depends on how close \( H^k(x) \) is to \( K \). This shows that \( U \) is continuous.

It remains to consider the cases \( \alpha \neq 1 \). The proof is simpler and is based on convergence of Riemann sums. In all the cases, the renormalization term to get a Riemann sum is \( \lambda^{-\alpha(m-k)} \) and the sums have \( \lambda^{m-k} \) summands.

For \( \alpha > 1 \), the renormalization term is too heavy and the sum goes to 0. For \( \alpha < 1 \) the renormalization term is too light and the sum goes to \(+\infty\).

We left the exact computations to the reader and refer to [3, 4] for similar computations.

### 3.4.3 Limit for \( R^m V(x) \). The general case

We consider \( V \) of the form \( V(x) = \frac{g(x)}{p^x} + o(\frac{1}{p^x}) \) if \( \delta(x) = p \) and with \( g \) a positive and continuous function. First, we emphasize that continuity and positiveness for \( g \) imply that \( g \) is bounded from above and from below away from zero. Therefore, the proof for \( \alpha \neq 1 \) is the same. We can thus focus on \( \alpha = 1 \). Note that the same computation works whatever \( o(\frac{1}{p^x}) \) is.

In that case we need to compute

\[ R^m V(x) = \sum_{j=0}^{t_m(x)-1} \frac{g \circ \sigma^j(H^m(x))}{\delta_j^m(x)} + o(...). \]
There are two main arguments to deal with these extra terms. First, we show that the terms $g \circ \sigma^j(H^n(x))$ can be exchanged by terms $g \circ \sigma^k(H^n(y_{k,j}))$ with $y_{k,j} \in K$. Then, we use a technical lemma to show the convergence to the desired quantity.

Replacing $g \circ \sigma^j(H^n(x))$. We reemploy notations from above. Let $j_1, \ldots, j_s$ the times of accidents for $H^k(x)$. We also set $j_0 = 0$ and $j_{s+1} = t_k(x) - 1$. We have defined $t_{i,m-k}$ and $\Delta_{i,m-k}$.

There exist points $y^0, \ldots, y^s$ in $K$ such that $d(\sigma^j(H^m(x)), y^i) = d(\sigma^j(H^k(x)), K)$. In other words, the $y^i$'s are points in $K$ and coincide with $\sigma^j(H^k(x))$ for exactly $\delta^k_j(x)$-digits.

Now, we refer the reader to Figure 3.4.3 for the next discussion. We claim that Proposition 3.7 implies that for every $m \geq k$, for every $t_{i,m-k} \leq j < t_{i+1,m-k}$

$$\delta^m_j(x) = d(\sigma^j(H^m(x)), K) = d(\sigma^j(H^m(x)), H^{m-k}(y^i)).$$

(4)

As $H$ is 2-full, for every $i$, $\delta^k_j(x) \geq j_{i+1} - j_i + 1$ (otherwise $j_{i+1} - 1$ would be an accident) and then for $0 \leq j \leq t_{i+1,m-k} - t_{i,m-k}$

$$d(\sigma^{t_{i,m-k+j}}(H^m(x)), \sigma^j(H^{m-k}(y^i))) = D^{-\Delta_{i,m-k+j}} \leq D^{-\lambda^{m-k} + O(\delta^{m-k})}. $$

(5)

Figure 3: $H^{m-k}$ renormalization

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This shows that replacing $\sigma^j(H^m(x))$ by $\sigma^{j-t_i,m-k}(H^{m-k}(y))$ for $t_i,m-k \leq j < t_i+1,m-k$ just add an error in $o(D^{-\lambda m-k})$ and thus does not influence the limit. Then we have

$$R^mV(x) = \sum_{j=0}^{t_m(x)-1} \frac{g \circ \sigma^j(H^m(x))}{\delta^m_j(x)} + o(\ldots)$$

$$= \sum_{i=0}^{s} \sum_{l=0}^{t_i+1,m-k-t_i,m-k-1} \frac{g \circ \sigma^l \circ \sigma^{t_i,m-k}H^m(x)}{\Delta_{i,m-k} - l} + o(\ldots)$$

$$= \sum_{i=0}^{s} \sum_{l=0}^{t_i+1,m-k-t_i,m-k-1} \frac{g \circ \sigma^lH^{m-k}(y)}{\Delta_{i,m-k} - l} + o(\ldots).$$

A technical lemma. First we prove

Lemma 3.10. Let $(X,\sigma)$ be an uniquely ergodic subshift. Let $f$ be a continuous integrable function on $(0,1)$, let $g : X \to \mathbb{R}$ be a continuous function on $X$. Then we have uniformly in $x \in X$:

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)g(\sigma^kx) = \int_0^1 f(x)dx \int_X g \, d\mu$$

Proof. Let us define $a_k = f(\frac{k}{n})$ and the Birkhoff sum $S_n(x) = \sum_{k=0}^{n-1} g(\sigma^kx)$ with $S_0 = 0$. Finally denote $X_n = \frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)g(\sigma^kx)$. We have

$$X_n = \frac{1}{n} \sum_{k=0}^{n} a_k(S_{k+1}(x) - S_k(x)) = \frac{1}{n} \left[ \sum_{k=1}^{n+1} a_{k-1}S_k(x) - \sum_{k=0}^{n} a_kS_k(x) \right]$$

$$= \frac{1}{n} \sum_{k=1}^{n} (a_{k-1} - a_k)S_k(x) + \frac{a_nS_{n+1}(x) - a_0S_0}{n}$$

Now by unique ergodicity we have $\lim \frac{S_n(x)}{n} = \int_X g(x) \, d\mu$ uniformly in $x$. Thus for all $\varepsilon > 0$, there exists $N$ such that for $n \geq N$ we have $S_n(x) = n \int_X g \, d\mu + n\varepsilon(n)$ with $\varepsilon(n) \leq \varepsilon$.

First of all assume $f \in C^1([0,1])$.

$$X_n = \frac{1}{n} \sum_{k=1}^{n} (a_{k-1} - a_k)S_k(x) + \frac{a_nS_{n+1}(x) - a_0S_0}{n}$$
\[ X_n = \frac{1}{n} \sum_{k=1}^{n} (a_{k-1} - a_k)(k \int_X gd\mu + k\varepsilon(k)) + \frac{a_n S_{n+1}(x) - a_0 S_0}{n} \]

\[ X_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k \int_X gd\mu - a_0 + \frac{na_n}{n} \int_X gd\mu + \frac{1}{n} \sum_{k=1}^{n} (a_{k-1} - a_k)k\varepsilon(k) + a_n \frac{S_{n+1}(x)}{n} - \int_X gd\mu - \frac{a_0 S_0}{n} - \frac{a_0}{n} \]

Then there exists \( c_k \in \left[ \frac{k-1}{n}, \frac{k}{n} \right] \) such that \( a_k - a_{k-1} = \frac{f'(c_k)}{n} \). Now by property of \( f \), there exists \( c_k \) such that \( a_k - a_{k-1} = \frac{f'(c_k)}{n} \)

\[ X_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k \int_X gd\mu + \frac{1}{n^2} \sum_{k=1}^{n} f'(c_k)k\varepsilon(k) + a_n \frac{S_{n+1}(x)}{n} - \int_X gd\mu - \frac{a_0 S_0}{n} - \frac{a_0}{n} \]

We deduce there exists a constant \( C > 0 \) such that

\[ \left| \frac{1}{n^2} \sum_{k=1}^{n} f'(c_k)k\varepsilon(k) \right| \leq \frac{1}{n^2} \sum_{k=1}^{N} Ck\varepsilon(k) + \frac{n^2 - N}{n^2} \varepsilon \leq C\varepsilon \]

Thus \( X_n \) converges to \( \int_0^1 f(t)dt \int_X gd\mu \) uniformly in \( x \).

Now if \( f \) is only a continuous function. It is a uniform limit of \( C^1 \) functions. We apply the previous proof.

**Corollary 3.11.** We consider \( V \) of the form \( V(x) = \frac{g(x)}{\delta^p} + o(\frac{1}{\delta^p}) \) if \( \delta(x) = \delta \) and with \( g \) a positive and continuous function. Then we have

\[ \lim_{+\infty} \int X V(x) = \int_X gd\mu \log \left( \frac{\Delta_{j_i}}{\Delta_{j_i} - (j_{i+1} - j_i)} \right). \]

**Proof.** We apply the previous lemma to \( H^n(x) \), which is possible due to the uniform convergence, and use the computation in the case \( g \equiv 1 \).
3.4.4 Back to 2-full assumption

We gave an example above (see page 13) where the substitution is not 2-full. We can now complete this example and check that for any \( m \),

\[
R^m V(x) = \sum_{k=1}^{\lfloor H^m(a) \rfloor - 1} \frac{1}{k}
\]

which diverges.

We emphasize that the 2-full assumption is important to guaranty some fast convergence to \( K \) iterating \( H^m \) and taking the images by \( \sigma^j \). For instance, we used the assumption in the previous proof to check that \( \Delta_i - j_i + 1 \) is positive, which is a crucial point to exchange the \( \sigma^j(H^m(x)) \) by the \( \sigma^j(H^{m-k}(y)) \).

4 Proof of Theorem 2

4.1 Induced transfer operator

The Thermodynamic Formalism was introduced in dynamical systems by Sinai, Ruelle and Bowen ([19, 17, 2]). The main tool for uniformly hyperbolic systems and Hölder continuous potentials is the transfer operator:

\[
T(g)(x) := \sum_{y \in \sigma^{-1}(x)} e^{V(y)} g(y).
\]

Hyperbolicity and Hölder continuity combine themselves to give nice spectral properties to this operator. The main issue is that the pressure for \( V \) is the logarithm of the spectral radius for \( T \) and it is a single dominating eigenvalue.

For systems with weaker hyperbolicity or potentials with weaker continuity, it may be harder to get the same spectral properties. A way to recover them is to consider an inducing scheme. Several methods exist in the literature. We shall use here the one summarized in [12]. Actually this method turned out to be extremely powerful to detect some phase transitions as the ones we want to detect here.

Let \( V : A^\mathbb{N} \to \mathbb{R} \) be some potential function and let \( w_J \) a finite word outside \( \mathcal{L}_H \) which defines a cylinder \( J = [w_J] \). Consider the first return map to \( J \), with return time \( \tau(x) = \min\{n \geq 1, \sigma^n(x) \in J\} \). Then we define, for
each $\beta > 0$ and $Z \in \mathbb{R}$, an induced transfer operator by:

$$L_{Z,\beta}(g)(x) = \sum_{n \in \mathbb{N}} \sum_{\tau(y)=n} e^{\beta S_n(V)(y) - nZ} g(y)$$

where $S_N(V)(y) = \sum_{k=0}^{N-1} V \circ \sigma^k(y)$ and $g$ is a continuous function from $J$ to $\mathbb{R}$.

For a given function $V$, it is a power series in $e^{-Z}$. We recall the following results of [12].

**Proposition 4.1.** We have with the previous notations:

- For every $\beta \geq 0$, there exists a minimal $Z_c(\beta) \in \mathbb{R} \cup \{-\infty\}$ such that for every $Z > Z_c(\beta)$, $L_{Z,\beta}$ acts on $C^0(J)$. In particular, for every $Z > Z_c(\beta)$, for every $x \in J$ and for every $g \in C^0(J)$, $L_{Z,\beta}(g)(x)$ converges.

- $P(\beta) \geq Z_c(\beta)$.

- Let $\lambda_{Z,\beta}$ be the spectral radius for $L_{Z,\beta}$ and for $Z > Z_c(\beta)$. Then $Z \mapsto \log \lambda_{Z,\beta}$ is a decreasing function and we have three possible cases given by Figure 4.

- If case 1 holds, then $\log \lambda_{Z,\beta} = 0$ if and only if $Z = P(\beta)$ and there is a unique equilibrium state for $\beta.V$; it is a fully supported measure in $\mathcal{A}^N$. Moreover, $Z_c(\beta) < P(\beta)$ and $\beta \to P(\beta)$ is analytic on the largest open interval where case 1 holds.

- If case 3 holds, then no equilibrium state gives positive weight to $J$. 

---

Figure 4: The tree possible graphs for $\log \lambda_{Z,\beta}$. 

[Figure 4]
We consider a potential $\varphi$ such that $\varphi(x) = \log(1 + \frac{1}{n})$ if $d(x, K) = D^{-n}$. We will compute $L_{0, \beta}(1_J)(x)$ for a particular potential $V = -\varphi$ and deduce the result for this potential. With Lemma 4.10 we will deduce the result for every potential in $\Xi_1$. For our special $V$, the next lemma is the key point to detect phase transition.

Lemma 4.2. Assume there exists $\beta_0$ such that for $x \in J$ $L_{0, \beta_0}(1_J)(x) < 1$ holds. Then for every $\beta \geq \beta_0$ $P(\beta) = Z_c(\beta) = 0$ and $\mu_K$ is the unique equilibrium state.

Proof. We use here other results summarized in [12]. Actually, using previous notations, we denote by $\Sigma_J$ the invariant set of points whose trajectories never visit $J$. If $P(\beta, \Sigma_J)$ denotes the pressure function for this new system and for the potential $\beta V$, then

$$Z_c(\beta) = P(\beta, \Sigma_J)$$

holds. In our case, we claim that $P(\beta, \Sigma_J) \geq 0$ because $\Sigma_J$ contains $K$ and

$$0 = h_{\mu_K} + \beta \int V d\mu_K.$$

On the other hand, we claim that $L_{Z, \beta}(\Pi_J)(x)$ converges for some $x$ implies that for every $g \in C^0(J)$ and for every $y \in J$, $L_{Z, \beta}(g)(y)$ converges. In other words, $L_{Z, \beta}(\Pi_J)(x) < +\infty$ (note it is a positive series) implies $Z \geq Z_c(\beta)$. Then, our assumption shows that $Z_c(\beta_0)$ is equal to 0. For convexity reason this also holds for every $\beta > \beta_0$.

Finally, we claim that $\lambda_{Z, \beta} = L_{Z, \beta}(\Pi_1)(x)$ for every $x \in J$ and this is due to the special form of our potential (which only depends on the distance to $K$). This shows that for $\beta_0$ we are in case 3, and this also holds for every $\beta > \beta_0$.

The last point to check is that $\mu_K$ is the unique equilibrium state. Actually, we claim that for every cylinder $J'$ with empty intersection with $K$ the same conclusion holds than for $J$ holds. Indeed, otherwise, there would be some fully supported equilibrium state (see [12]) and then it would give positive weight to $J$. Therefore, no equilibrium state gives positive weight to any cylinder which does not intersect $K$, which means that any equilibrium state is supported into $K$. Now, we recall that $K$ is uniquely ergodic.

With the help of this lemma, we deduce our strategy of proof: We will compute $L_{0, \beta}(1_J)(x)$ and show that for $\beta$ large enough, $L_{0, \beta}(\Pi)(x) = \lambda_{0, \beta}$ is strictly smaller than 1.
4.2 Excursion-free sequences and operator

Let us recall that $w_J$ is the word which defines the cylinder $J$. For $x \in J$ we have to compute

$$L_{0,\beta}(1_J)(x) = \sum_{n \in \mathbb{N}} \sum_{\tau(y)=n, \sigma^n(y)=x, y \in J} e^{-\beta(S_n\varphi)(y)}.$$

Note that such a point $y$ is of the form $y = ux$, where $u = u_0 \ldots u_{n-1}$ and $w_J$ is a prefix of $u$. Now, due to the form of our potential, we claim that $S_n(\varphi)(y)$ does only depend on $u$. In other words, each summand in $L_{0,\beta}(1)(x)$ does only depend on the path from $J$ to $J$. We shall also say that $u$ is a return word.

Now let $N$ be the integer such that $d(J, \mathbb{K}) > D^{-N}$, and consider

$$A_N = -\log (1 + \frac{1}{N}).$$

The integer $N$ is a parameter that can be fixed as big as needed. For a fixed $N$, we define two classes of integers for each return word $u$: the $u$-free and the $u$-excursions. An integer $k \in [0, n-1]$ is $u$-free if $\delta(u_k \ldots u_{n-1}w_J) \leq N$. The integers between two consecutive $u$-free integers are called $u$-excursions. Remark that 0 is $u$-free by definition of $N$.

We fix $N > l(H)$, where $l(H)$ was defined in Theorem 3.4. Then, every bispecial word that appear during an excursion has length bigger than $l(H)$.

Remark 7. The terminology free and excursion words are used in order to have in mind some points far from $\mathbb{K}$ and some points close to $\mathbb{K}$. Actually, when points are far from $\mathbb{K}$ the digits may appear randomly as we are in the full shift $\mathcal{A}^N$. On the contrary, when points are close to $\mathbb{K}$ the digits must obey for a while to the language $\mathcal{L}_H$. ■

A word $w$ is said to be excursion free if we can write $w = EF$ such that the integers inside $[0, |E|]$ are $w$-excursions and those inside $[|E|+1, |E|+|F|]$ are $w$-free. The set of all these words is denoted by $\mathcal{E}F$.

Let us denote the following quantity \footnote{In all the following we make computations in $\mathbb{K}$ since we have positive terms. It allows us to avoid problems of convergence of series.}

$$C_{\mathcal{E}F} = \sum_{w \in \mathcal{E}F} e^{-\beta S_{|w|\varphi}(w)}.$$

(6)
Proposition 4.3. Let $J$ be a cylinder outside $\mathbb{K}$ defined by the word $w_J$ et $x \in J$. Assume that $C_{\mathcal{E}F}$ (cf Equation (6)) is finite, then we have

$$\mathcal{L}_{0,\beta}(1_J)(x) \leq \sum_{k \geq 0} C_{\mathcal{E}F}^k \sum_{n \geq 0} e^{(n+1)(\beta A_N + \log D)}.$$

Proof. The proof needs several steps:

1. By definition of the transfer operator we have:

$$\mathcal{L}_{0,\beta}((1_J)(x) = \sum_{n} \sum_{u, |u| = n} e^{-\beta(S_n \varphi)(u..)}$$

Recall that $S_n \varphi(y) = \sum_{k=0}^{n-1} \varphi(\sigma^k(y))$.

2. To the word $u$ we associate the sequence of integers $u$-free and $u$-excursions. We denote the first $u$-free integer by $k_0$, then the following by $k_0 + k_1$ and so on...

Let us remark the following facts implied by decreasing properties of $-\varphi$:

- If $k$ is a free integer then $-\varphi(\sigma^k(uw_J)) \leq -\log (1 + \frac{1}{N})$.
- If $k_0$ is the biggest integer such that everything before is a free integer, then we have $-S_{k_0+1} \varphi(uw_J) \leq (k_0 + 1)A_N$. 

Figure 5: Path with free moments and excursions

Recall that $S_n \varphi(y) = \sum_{k=0}^{n-1} \varphi(\sigma^k(y))$. 

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3. Consider a path starting from $J$, free at the beginning and at the end. In between it alternates the words excursion-free, see Figure 5. Let us denote these excursions-free words by $E_i F_i$, $i \leq k$:

$$u = F_0(E_1 F_1)(E_2 F_2) \ldots (E_k F_k).$$

**Remark 8.** We emphasize that a return word $u$ must start and finish by $u$-free moments. This is due to our assumption on $N$. ■

We denote $E_0 = \emptyset$, then the Birkhoff sums can be written:

$$S_{|u|} \varphi(u) = \sum_{i=0}^{k} S_{|E_i F_i|} \varphi(E_i F_i)$$

Remark that

$$\sum_{F_0 E_1 F_1 E_2 F_2} e^{-\beta S_{|E_1 F_1|} \varphi(u)} e^{-\beta S_{|E_2 F_2|} \varphi(u)} \leq \left[ \sum e^{-\beta S_{|E_F|} \varphi(u)} \right]^2 \quad (\ast)$$

Thus we deduce

$$L_{0,\beta}(1_J)(x) = \sum_{n} \sum_{k \geq 0} \sum_{u, \; |u| = n, \; \text{k excursions}} e^{-\beta S_{|u|} \varphi(u)}$$

$$L_{0,\beta}(1_J)(x) = \sum_{n} \sum_{k \geq 0} \sum_{u, \; |u| = n, \; \text{k excursions}} e^{-\beta \left( \sum_{i=0}^{k} S_{|E_i F_i|} \varphi(E_i F_i) \right)}$$

$$L_{0,\beta}(1_J)(x) = \sum_{n} \sum_{k \geq 0} \sum_{u, \; |u| = n, \; \text{k excursions}} \prod_{i=0}^{k} e^{-\beta S_{|E_i F_i|} \varphi(u)}.$$

With the help of $(\ast)$ we deduce, after separating the case $i = 0$

$$L_{0,\beta}(1_J)(x) \leq \sum_{M \geq 0} C_{E_F}^k \sum_{n} \sum_{u, \; |u| = n, \; \text{free}} e^{-\beta S_{|u|} \varphi(u)}.$$

If $u$ is a word such that all the integers in $[0, |u|]$ are $u$-free, then we obtain: $-S_{|u|} \varphi(u) \leq (n + 1)A_N$. We conclude that
\[ \mathcal{L}_{0,\beta}(1)(x) \leq \sum_{M \geq 0} C_{EF}^k \sum_n \sum_{u, |u| = n, \text{free}} e^{\beta(n+1)A_N} \]
\[ \leq \sum_{k \geq 0} C_{EF}^k \sum_n D^n e^{\beta(n+1)A_N}. \]

We let the reader check that the inequality \( \mathcal{L}_{0,\beta}(1)(x) \geq e^{w_J|x| \beta A_N} C_{EF} \) holds. Therefore, if \( C_{EF} = +\infty \), \( \lambda_{0,\beta} = +\infty \). We also emphasize that
\[ \sum_{n \geq 0} D^n e^{\beta(n+1)A_N} = \frac{e^{\beta A_N}}{1 - De^{\beta A_N}} \]
(if \( De^{\beta A_N} < 1 \)) which can be made as small as wanted by increasing \( \beta \). This shows that a sufficient condition to get \( \lambda_{0,\beta} < 1 \) for \( \beta \) large enough is to show that for large enough \( \beta \), \( C_{EF} < 1 \) holds. We point out that we will show even better since we will prove that \( C_{EF} \to 0 \) if \( \beta \to +\infty \).

4.3 Packing excursion-free with respect to their number of accidents

In view to estimate on \( C_{EF} \) we need to sum the contribution of each excursion-free word. For this we shall pack them in function of their respective number of accidents and also in function of the depth of each accident. For that goal, we introduce some notations related to those in Section 2.3

Let \( M \) be a positive integer, denote by \( W_M \) the set of excursion free words which contains \( M \) accidents. To each word of \( W_M \) is associated \( M \) accidents which times are denoted \( (B_i)_{1 \leq i \leq M} \), see Definition 2.10. Let us denote \( W^{(i)} \) the bispecial word of \( L_H \) which defines the accident at time \( B_i \). Remark that \( B_{i+1} = B_i + b_{i+1} \), and \( b_{i+1} \) represents exactly the length between occurrences of \( W^{(i)} \) and \( W^{(i+1)} \). We also emphasize that an excursion-free word must start by a bispecial word \( W^{(0)} \) because any return word starts with free moments.

For each accident, we consider its depth \( d_i \). We refer to Figure 6 for a picture describing these notations.

We recall
\[ d_i - b_{i+1} = |W^{(i+1)}|, \]
\[ d_i - b_{i+1} < d_{i+1}. \]
By definition of excursions all the $W^{(i)}$ for $i > 0$ must have length bigger than $N > l(H)$. On the other hand there is no condition on the length of $W^{(0)}$.

**Lemma 4.4.** Let $C_{EF}(M)$ be the sum of all the contributions of excursion-free paths with $M$ accident (not counting the one at time 0). Then,

$$C_{EF}(M) \leq \frac{e^{\beta AN}}{1 - e^{\beta AN}} \sum_{d_i, b_i \text{ possible}} \prod_{i=0}^{M-1} \left( \frac{d_i + 1 - b_{i+1}}{d_i + 1} \right) ^ \beta \left( \frac{N + 1}{d_M + 1} \right) ^ \beta . \quad (8)$$

**Proof.** Consider an excursion-free path with length $n$. Say it is the orbit of a point $y$. We split the interval $[0, n-1]$ into the $M + 1$ intervals defined by the accidents: Thus we write :

$$W^{(0)} T_0 W^{(1)} T_1 W^{(2)} T_2 W^{(3)} \ldots \quad (9)$$

where

- The word $W^{(i)}$ is a bispecial word of the language.
- The word $W^{(i)} T_i W^{(i+1)}$ is in the language.
- The word $W^{(i)} T_i W^{(i+1)} a$ is not in the language, where $a$ is the first letter of $T_{i+1}$.

By definition of being the last accident, there exists $l \leq B_M + d_M$ such that for every $l \leq j < n$, $j$ is a free moment. In other words, $l - 1$ is the end of the excursion.

By Lemma 2.9 we know that if $k \in [B_i, B_{i+1}]$ we have $\delta(\sigma^k(y)) = d_i - k$. Thus we obtain
Taking the opposite of this last inequality, multiplying by $\beta$, taking the exponential and then doing the sum over all possibilities (noting that $n - l$ is at least 1) we get the result. 

4.4 Occurrences of bispecial words

In order to estimate the upper bound in (8) we need to estimate what the possible $d_i$’s and $b_i$’s are. We point out that $d_i$ is the length of the word $W(1)T_iW(i+1)$ (where we reemploy notations from (9) and we refer to Figure 6). Recall that $d_i - b_i+1$ is the length of the $(i + 1)^{th}$ bispecial word $W(i+1)$ (see Equation (7a)), and that the lengths of bispecial words $W(1)\ldots W(M)$ are denoted by $K_n$.

Therefore, the set of possible $d_i$’s and $b_i$’s is exactly determined by the set of possible lengths $K_n$’s and possible lengths for the $T_i$’s.

Now, it turns out that the set of possible lengths for $T_i$ is completely determined by $W(i)$ and $W(i+1)$. This yields some estimations for each possible $d_i$. To fix notation, let us thus consider the set of possible depths for the $i^{th}$ accident. Then, we prove:

**Proposition 4.5.** Assume we consider $M$ bispecial words $W(1),\ldots, W(M)$. Then,

1. for every $0 \leq i \leq M - 1$, there exists $d_i(0)$ such that for every $j \geq 0$
   
   \[ d_i(j) \geq d_i(0) + jK_{n_i+1}, \]

2. The reason we write the length with a double subscript will be clearer below.
2. There exists $0 < c < 1$ such that for every $1 \leq i \leq M - 1$,
\[
d_i(0) \geq K_{n_{i+1}} + \begin{cases} cK_{n_i} & \text{if } K_{n_i} < K_{n_{i+1}} \\ K_{n_i} - cK_{n_{i+1}} & \text{otherwise}. \end{cases}
\]

Proof. We start by recalling equality $d_i - b_{i+1} = K_{n_{i+1}}$. Therefore for each possible depth $d_i(j)$, we associate a possible occurrence $b_{i+1}(j)$. We also recall that as long as there is no accident, the word is admissible for the language of the substitution.

By the result of [7] we know that the subshift defined by a primitive substitution is linearly recurrent. It means that for every word of this language its sequence of return times is linear as a function of the length of the word. We apply this result to the bispecial word $W^{(i+1)}$ and its first occurrence $b_{i+1}(0)$. Then we have
\[
b_{i+1}(j) \geq b_{i+1}(0) + jK_{n_{i+1}}
\]
and the first point of Proposition 4.5 holds.

The proof of the second point is based on a key lemma which gives a bound for overlap of bispecial words. Actually, it could happen that the bispecial word $W^{(i+1)}$ appears before the end of the bispecial word $W^{(i)}$. In that case the image of Figure 6 is false, and if it happens, we say that there is an overlap. Next lemma allows to control the length of an overlap.

Lemma 4.6. Assume two bispecial words overlap, then the size of the overlap is uniformly bounded by a ratio of the smallest one. In other terms: there exists $0 < C < 1$ such that for all bispecial words $U, V$ we have
\[
\frac{|U \cap V|}{|U|} \leq C < 1.
\]

Proof of Lemma 4.6. Consider two bispecial words $U, V$. We assume that the size of $U$ is less than the size of $V$ and that it appears before in $x$. By Proposition 3.5 we have $U = H^n(W)$ and $V = H^m(X)$. The overlap $U \cap V$ is also a bispecial word: indeed it suffices to look at the right extensions of $V$ and the left extensions of $U$. Thus we deduce $U \cap V = H^k(Y)$. By Proposition 3.5 the word $Y$ belongs to the finite set $E$.

We consider two cases:

- If $n > k$. We obtain $U = H^k(H^{n-k}(W)) = SH^k(Y)$. Theorem 3.4 shows that $Y$ is a suffix of $H^{n-k}(W)$ and that $S = H^k(Z)$. Remark
that $Z$ does not necessarily belong to the finite set $E$, since it is not a bispecial word. We deduce, with Corollary 3.6, that the ratio of the lengths of $U \cap V$ and $U$ is bounded by:

$$\frac{|U \cap V|}{|U|} = \frac{\lambda^k |Y| + o(\lambda^k)}{\lambda^k |Z| + \lambda^k |Y| + o(\lambda^k)} \leq \frac{|Y|}{|Z| + |Y|} + o_k(1).$$

Thus there exists $0 < c < 1$ such that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\max_{Y \in E} \frac{|Y|}{1 + |Y|} + o_k(1) \leq c < 1$.

Now if $k < k_0$ then we also have $\frac{|U \cap V|}{|U|} \leq \frac{|H^k(Y)|}{1 + |H^k(Y)|} \leq c' < 1$ for a constant $c' > 0$ since $Y$ belongs to a finite set. Thus there exists a constant $C > 0$ such that for any integer $k$ we have $\frac{|U \cap V|}{|U|} \leq C < 1$.

\[\text{Now assume } n \leq k. \text{ The same method applies: Let us denote } M = \max_{u \in E} |u|; m = \min_{u \in E} |u|. \text{ We have}\]

$$\lambda^k m \leq |U \cap V| \leq |U| \leq M \lambda^n.$$

We deduce $k \leq n + \frac{1}{\log \lambda} \log \left(\frac{M}{m}\right)$. Now let us consider $\alpha = \frac{1}{\log \lambda} \log \left(\frac{M}{m}\right)$. We obtain $H^k(Y) = H^n(H^{k-n}(Y))$, thus $U = S H^n(H^{k-n}(Y))$ and finally $S = H^n(T)$. We conclude with the inequality

$$\frac{|U \cap V|}{|U|} = \frac{\lambda^k |Y|}{\lambda^n |T| + \lambda^k |Y|} \leq \frac{M}{m + \frac{m}{\lambda^n}} < 1.$$

By Lemma 4.6 there exists $0 < c < 1$ such that for each integer $i$ we have

$$b_{i+1}(0) \geq \begin{cases} cK_n & \text{if } K_n < K_{n+1} \\ K_n - cK_{n+1} & \text{otherwise.} \end{cases}$$

This finishes the proof of Proposition 4.5.

\[\square\]
4.5 Computations

4.5.1 Computation I

Our goal is to give a as sharp as possible upper bound for Inequality (8):

\[ C_{EF}(M) \leq \frac{e^{\beta A_N}}{1 - e^{\beta A_N}} \sum_{d_i, b_i \text{ possible}} \prod_{i=0}^{M-1} \left( \frac{d_i + 1 - b_{i+1}}{d_i + 1} \right)^{\beta} \left( \frac{N + 1}{d_M + 1} \right)^{\beta}. \]

We recall (7a) which states \( d_i - b_{i+1} = |W^{(i+1)}| = K_{n_i} \) and (7b) which states \( d_i - b_{i+1} < d_{i+1} \). This yields that \( d_M \) is strictly bigger than \( K_{n,M} \). We also recall that \( A_N = -\log \left( 1 + \frac{1}{N} \right) \). Therefore we get:

\[ C_{EF}(M) \leq \frac{e^{\beta A_N}}{1 - e^{\beta A_N}} \sum_{d_i, b_i \text{ possible}} \prod_{i=0}^{M-1} \left( \frac{d_i + 1 - b_{i+1}}{d_i + 1} \right)^{\beta} \frac{1}{\sum_{j=1}^{\infty} \left( \frac{N + 1}{K_{n,M} + j + 1} \right)^{\beta}} \]

\[ \leq \frac{N^\beta}{(\beta - 1)(K_{n,M} + 2)^{\beta - 1}} \sum_{d_i, b_i \text{ possible}} \prod_{i=0}^{M-1} \left( \frac{d_i + 1 - b_{i+1}}{d_i + 1} \right)^{\beta}. \quad (10) \]

To get this series of inequalities, the step from line 1 to line 2 is obtained by the comparison \( \sum_{j=1}^{\infty} \frac{1}{(K_{n,M} + j + 1)^{\beta}} \leq \int_2^{\infty} \frac{1}{(K_{n,M} + x)^{\beta}} dx \).

Now, from Proposition 4.5 we get that the sum \( \sum_{d_i, b_i \text{ possible}} \prod_{i=0}^{M-1} \) in the righthand side term in (10) is equal to the sum \( \sum_{d_i(0), K_{n_{i+1}} \text{ possible}} \prod_{i=0}^{M-1} \sum_{j_i=0}^{+\infty} \). Namely we have

\[ C_{EF}(M) \leq \frac{N^\beta}{(\beta - 1)(K_{n,M} + 2)^{\beta - 1}} \sum_{d_i(0), K_{n_{i+1}} \text{ possible}} \prod_{i=0}^{M-1} \sum_{j_i=0}^{+\infty} \left( \frac{K_{n_{i+1}} + 1}{d_i(0) + j_i K_{n_{i+1}} + 1} \right)^{\beta} \]

\[ \leq \frac{N^\beta}{(\beta - 1)^{M+1}(K_{n,M} + 2)^{\beta - 1}} \sum_{d_i(0), K_{n_{i+1}} \text{ possible}} \prod_{i=0}^{M-1} \frac{(K_{n_{i+1}} + 1)^{\beta}}{K_{n_{i+1}}(d_i(0) + 1)^{\beta - 1}}, \quad (11) \]

where we again compared a discrete sum with an integral.

It is now necessary to make precise what the possible \( K_{n_i} \)'s are. We recall that \( K_{n_i} \) is the length for the bispecial word \( W^{(i)} \). By Proposition 3.5
and Corollary 3.6, and by assumption on \( N \), all the \( W^{(i)} \) are of the form \( H^{n_i}(v_i) \), where \( v_i \) is a bispecial word in \( \mathcal{W}_b \). This explains why lengths are of the form \( K^{n_i} \).

Hence, we may denote by \( a_i \) the length of \( v_i \). This yields that our condition is

\[
a_i \lambda^{n_i} > N.
\]

**Some notations.** We denote by \( P \) the smallest integer such that \( \lambda^P > N \). Then for each \( a_i = |v_i| \) with \( v_i \in \mathcal{W}_b \) we set \( p_i := \frac{\log a_i}{\log \lambda} \). Finally, we set \( \eta := (\beta - 1) \log \lambda \).

Note that \( a_i \lambda^{n_i} > N \) is equivalent to \( n_i \geq P - p_i \). Similarly, \( K^{n_i} > K^{n_{i+1}} \) is equivalent to \( n_i + p_i > n_{i+1} + p_{i+1} \). Now, Proposition 4.5 shows that for \( i \geq 1 \),

\[
d_i(0) \geq K^{n_{i+1}} + \begin{cases} cK^{n_i} & \text{if } n_i + p_i < n_{i+1} + p_{i+1} \\ K^{n_i} - cK^{n_{i+1}} & \text{otherwise.} \end{cases}
\]

For \( d_0(0) \), we recall that \( d_0 - b_1 = K^{n_1} \). By definition of excursion, the path starts by a bispecial word \( W^{(0)} \) of length smaller or equal to \( N \). Lemma 4.6 still holds in that case, and as we have

\[
|W^{(0)}| \leq N < K^{n_1},
\]

then \( d_0(0) \geq K^{n_1} + (1 - C)N \) holds. Moreover there are only finitely many possibilities for \( W^{(0)} \), say \( C(N) \). Hence, inequality 11 yields

\[
C_{\mathcal{E}, \mathcal{F}}(M) \leq C(N) \frac{N^\beta}{(\beta - 1)^{M+1}} \frac{1}{\left(\lambda^{n_M} + p_M + 2\right)^{\beta - 1}} \sum_{n_i \geq P - p_i} \prod_{i=1}^{M-1} \frac{1}{(1 + X(i))^{\beta - 1}},
\]

where

\[
X(i) = \begin{cases} c\lambda^{n_i + p_i - n_{i+1} - p_{i+1}} & \text{if } n_i + p_i < n_{i+1} + p_{i+1} \\ \lambda^{n_i + p_i - n_{i+1} - p_{i+1}} - c & \text{otherwise} \end{cases}
\]

**4.5.2 Computation II**

**Proposition 4.7.** There exists \( A_\beta \) satisfying \( \lim_{+\infty} A_\beta = 0 \) such that

\[
C_{\mathcal{E}, \mathcal{F}} \leq \frac{NC(N)}{\beta - 1} \sum_{n \geq 1} \lambda^{-n(\beta - 1)} \sum_{M \geq 0} A_\beta^M \sum_{i=0}^{M} \frac{n_i}{i!}.
\]

\[
(13)
\]
Proof. The proof is an adaptation of the computation done in [4 Sec. 4]. We just explain here the main steps.

Note that \( C_{EF} = \sum_M C_{EF}(M) \), this yields the sum over \( M \) in \[13\]. To get a bound from above for \( C_{EF}(M) \) we use Inequality \[12\]. This inequality involves quantities \( n_i + p_i \) and we can thus change them to \( n_i \) to get simpler expression, which is also closer to the one obtained in [4 Sec. 4].

The sum over the \( n_i \)'s is done by induction, using the following arguments:

1. If \( n_i < n_{i+1} \), the quantity \( \frac{1}{(1+c\lambda^{n_i-n_{i+1}})^{\beta-1}} \) is upper bounded by 1.
2. If \( n_i \geq n_{i+1} \), then quantity \( \frac{1}{(1-c+\lambda^{n_i-n_{i+1}})^{\beta-1}} \) is upper bounded by \( e^{-\eta(n_i-n_{i+1})} \).
3. discrete summations are upper bounded by integrals (continuous summations).
4. Equality \( \int_y^\infty x^ne^{-\eta(x-y)}dx = \sum_{j=0}^{n} \frac{n!y^j}{j!(n+1-j)} \) holds and allows to get the proof by induction.

This yields an inequality of the form

\[
C_{EF}(M) \leq C(N) \frac{N}{(\beta - 1)} \left( \frac{K}{(\beta - 1)(1-\eta)} \right)^M \sum_{n_M > P} \sum_{j=0}^{M-1} \frac{(n_M - P)^j}{j!} \frac{N^\beta - 1}{(\lambda^{n_M} + 2)^{\beta-1}},
\]

where \( K \) is a constant which cares about the fact that they may be several bispecial words in \( W_b \) with the same length. In this last inequality, we exchange \( N^{\beta - 1} \) by \( \lambda^P(\beta - 1) \) and get a sum over \( n_M \) for \( n_M > P \) which only involves \( n_M - P \). Again, we can change the variable by \( n \) and we get the result.

\[\Box\]

Corollary 4.8. With the same notations we obtain \( \lim_{\beta \to +\infty} C_{EF} = 0. \)

Proof. We have by Proposition 4.7

\[
C_{EF} \leq \frac{N.C(N)}{\beta - 1} \sum_{n \geq 1} \lambda^{-n(\beta - 1)} \sum_{M \geq 0} A^M \sum_{i=0}^{M} \frac{n^i}{i!}.
\]

First remark that \( \sum_{M \geq 0} A^M \sum_{j=0}^{M} \frac{n^j}{j!} \leq \frac{e^n}{1 - A} \). This shows that the series

\[\sum_{n \geq 1} \lambda^{-n(\beta - 1)} \sum_{M \geq 0} A^M \sum_{i=0}^{M} \frac{n^i}{i!} \]

converges if \( \beta \) is big enough. Then we deduce the result. \[\Box\]
4.6 Last step in the proof of Theorem 2

**Proposition 4.9.** Let \( \varphi \) be the potential defined by \( \varphi(x) = \log (1 + \frac{1}{n}) \) if \( d(x, \mathcal{K}) = D^{-n} \). If \( H \) is a substitution which satisfies our hypothesis, then there exists \( \beta_0 \) such that for \( \beta > \beta_0 \) and \( x \in J \) we have:

\[
\mathcal{L}_{0,\beta}(1_J)(x) < 1.
\]

**Proof.** We apply Proposition 4.3 and Corollary 4.8.

\[
\mathcal{L}_{0,\beta}(1_J)(x) \leq \sum_{M \geq 0} C_{\mathcal{L}_F}^{M} \sum_{n \geq 0} e^{(n+1)(\beta A_N + \log D)}.
\]

Thus Theorem 2 is proved for this particular potential by using Lemma 4.2. We conclude the proof with the next lemma:

**Lemma 4.10.** Assume Theorem 2 is true for the potential \( \varphi \), then it is true for every potential \( V \in \Xi \).

**Proof.** If \( V \in \Xi \) then there exists \( k, k' > 0 \) such that \( k' \varphi \leq -V \leq k \varphi \). We deduce that the pressure function of the potential \( V \) vanishes for \( \beta \geq \frac{\beta_0}{k} \). Since this function is continuous, convex and decreasing there exists \( \beta'_c \) such that \( P(\beta) > 0 \) for \( 0 \leq \beta \leq \beta'_c \) and \( P(\beta) = 0, \beta \geq \beta'_c \). The rest of the proof is similar. For \( \beta < \beta'_c \) there exists an unique equilibrium state since \( P(\beta) > 0 \) by Proposition 4.1. It has full support and \( P(\beta) \) is analytic on this interval. For \( \beta > \beta'_c \) we use Lemma 4.2.

4.7 Proof of Theorem 3

We refer the reader to [8, Sec. 7] for a definition of linearly recurrent shift. We recall that these shifts are uniquely ergodic. Now, we emphasize that the ingredients for the proof of Theorem 2 are exactly the ones involved in Theorem 3.

1. Proposition 4.5 exactly means that \( \mathcal{K} \) is linearly recurrent.

2. The bispecial words are of length \( c \lambda^n \) where \( c \) belongs to a finite set (see Corollary 3.6).

3. The bispecial words cannot overlap more than a fixed proportion of the smaller one (see Lemma 4.6).

This shows that Theorem 3 holds.
5 Example of Thue Morse with explicit computations

Consider the Thue Morse substitution $H: \begin{cases} 0 \mapsto \rightarrow 01 \\ 1 \mapsto \rightarrow 10 \end{cases}$ For this example we rephrase the proof of Theorem 5.1 and give an explicit form for the potential $U$.

**Theorem 5.1.** For the Thue Morse substitution there exists a unique function $U$ such that for all $x \in \mathcal{A}^\mathbb{N}$ we have $U(x) = \lim_{m \to \infty} R^m V(x)$ for all potential $V: \mathcal{A}^\mathbb{N} \to \mathbb{R}$ such that $V(x) = \frac{1}{p} + o\left(\frac{1}{p}\right)$ if $d(x, K) = 2^{-p}$. Moreover if we denote $p = \delta(x)$ we obtain

$$U(x) = \begin{cases} \ln\left(\frac{p}{p-1}\right) & p \geq 3 \\ \frac{1}{2} \ln\left(\frac{4}{3}\right) & p = 2 \end{cases}$$

5.1 Technical lemmas

**Lemma 5.2.** The Thue Morse substitution and its language $\mathcal{L}$ fulfill:

- $H$ is 2-full and marked.
- The non uniquely desubstituable words of $\mathcal{L}$ are $010, 101$.
- Every word of length at least 5 in $\mathcal{L}$ is uniquely desubstituable inside the language.
- The fixed point which begins by 0 can be written
  $$u = 01.10.10.01.10.01.01.10.10.01.10. \ldots$$

- The language contains the words
  $$\begin{cases} 0, 1 \\ 00, 01, 10, 11 \\ 001, 010, 011, 100, 101, 110 \end{cases}$$

**Proof.** We refer to [16] and [5].

Let $x$ be a sequence outside $\mathcal{L}$ which begins by a word $w$ of the language. We can always assume that $x = w1 \ldots$. We denote $x = w_1 \ldots w_p1 \ldots$ where $p = \delta(x)$. We obtain

$$H^n(x) = H^n(w_1) \ldots H^n(w_p)H^n(1) \ldots$$

Let us consider different cases:
First case: \( p \geq 3 \)

**Proposition 5.3.** For all infinite word \( x \) with \( \delta(x) \geq 3 \) we have

\[
\delta(\sigma^k \circ H^n(x)) = p2^n - k,
\]

for all \( k \in [0, 2^n - 1] \).

**Proof.** 

- We begin by the case \( k = 0 \): The substitution has constant length, thus the length of \( H^n(w) \) is equal to \( p2^n \), thus we have \( \delta_0 \geq p2^n \). Remark that \( H^n(x) = H^{n-1}(H(w))H^n(1) \ldots \). The word \( H(w) \) belongs to \( \mathcal{L} \) and its length is equal to \( 2p > 4 \). Assume \( \delta_0 > p2^n \), then \( H(w)1 \in \mathcal{L} \) by Lemma 5.2. We deduce \( w1 \in \mathcal{L} \): this yields a contradiction. Thus we have \( \delta_0^u = p2^n \).

- Assume \( 1 \leq k \leq 2^{n-1} - 1 \). Let us denote \( H(w) = u_1 \ldots u_{2p} \). We have

\[
\sigma^k(H^n(x)) = \sigma^k H^{n-1}(u_1).H^{n-1}(u_2 \ldots u_{2p})H^{n-1}(1) \ldots
\]

First of all remark that \( \sigma^k(H^n(x)) \) begins with a strict suffix of \( H^{n-1}(u_1) \). We know that \( \delta(\sigma^k(H^n(x)) \geq p2^n - k \).

Assume that the word \( \sigma^k H^{n-1}(u_1).H^{n-1}(u_2 \ldots u_{2p})1 \) belongs to \( \mathcal{L} \). We apply Lemma 5.2 with the remark that the word \( \sigma^k H^{n-1}(u_1) \) is non empty and that \( p \geq 3 \), thus we have \( 2p - 1 \geq 5 \). We deduce that \( w1 \) belongs to the language: contradiction. Thus we obtain \( \delta_n^u = p2^n - k \).

- Now assume \( k = 2^{n-1} + l \) with \( 0 \leq l < 2^{n-1} \), then we have

\[
\sigma^k H^n(x) = \sigma^l(H^{n-1}(u_2)).H^{n-1}(u_3 \ldots u_{2p})H^{n-1}(10) \ldots
\]

The shift acts at most on the image of \( u_2 \). We know \( \delta_n^u \geq p2^n - k \), and \(|u_3 \ldots u_{2p}| = 2p - 2 > 3 \). The same argument goes on: If \( H^{n-1}(u_2 \ldots u_{2p})1 \) belongs to \( \mathcal{L} \), the same is true for \( u_2 u_3 \ldots u_{2p}1 \). It is equal to \( u_2 H(w_2 \ldots w_p)1 \), by Lemma 5.2 since \( 2p - 1 \geq 3 \). Thus it is the unique suffix of \( H(w_1 w_2 \ldots w_p)1 \): contradiction. We deduce that \( \delta_k^u = p2^n - k \). \( \square \)

Second case: \( p < 3 \)  
First of all the case \( p = 1 \) is impossible, because the substitution is 2-full. By Lemma 5.2 the word \( w \) is not right special thus it is equal either to 11 or to 00. The word 001 belongs to \( \mathcal{L} \), thus the only possibility is \( w = 11 \) (and 111 \( \notin \mathcal{L} \)).

**Proposition 5.4.** Let \( x \) be an infinite word with \( \delta(x) \leq 2 \), we obtain

\[
\delta(\sigma^k \circ H^n(x)) = \begin{cases} 
2 \cdot 2^n - k & k < 2^{n-1} \\
2^n - k & k = 2^{n-1} + l, 0 \leq l \leq 2^{n-1} - 1
\end{cases}
\]

There is an accident.
Proof. The argument before the proof shows that $x = 111\ldots$

- First assume $k = 0$. We have

\[
H^n(x) = H^n(1)H^n(1)H^n(1)\ldots
\]

\[
= H^{n-1}(1010)H^{n-1}(10)\ldots
\]

Remark that $\delta^n_0 \geq 2.2^n$. Assume that $H^n(11)1$ belongs to $\mathcal{L}$. The word $1010$ has length 4, we apply Lemma 5.2, we deduce that $10101$ belongs to $\mathcal{L}$. Since $10101 = H(11)1$ we deduce that $111$ belongs also to $\mathcal{L}$: contradiction. We have proved $\delta^n_0 = 2.2^n = 2^n+1$.

- Now assume $1 \leq k < 2^n - 1$, then we have

\[
\sigma^k H^n(x) = \sigma^k(H^{n-1}(1010))H^n(1)\ldots
\]

\[
\sigma^k H^n(x) = \sigma^k[H^{n-1}(1)]H^{n-1}(010)H^n(1)\ldots
\]

We prove by contradiction that $\delta^n_k = 2^n+1 - k$. Since $k < 2^n-1$ the last letter of $H^{n-1}(1)$ is not shifted by $\sigma$: we denote it $a$. The word $aH^{n-1}(010)1$ belongs to the language. Once again we apply Lemma 5.2, we deduce $a'0101 \in \mathcal{L}$: contradiction whatever the value of $a$ is.

- Now assume $k = 2^n-1$. We obtain

\[
\sigma^k H^n(x) = H^{n-1}(010)1.
\]

The word $0101$ belongs to the language, thus we obtain $\delta^n_{2^n-1} \geq 2^n+1$. There is an accident. Assume $\delta^n_{2^n-1} > 2^n+1$. This implies that $H^{n-1}(010)0$ also belongs to $\mathcal{L}$, and the same for $01010$: contradiction since $01010 = H(00)0 = 0H(11)$. Thus we have $\delta^n_{2^n-1} = 2^n+1$.

- The last case is identical and left to the reader: For $k = 2^n-1+l$, we obtain $\delta^n_k = 2^n+1 - l$.

\[\Box\]

5.2 Proof of Theorem 5.1

Consider $V(x) = \frac{1}{p} + o(1/p)$ with $d(x, K) = 2^{-p}$.

- If $p \leq 2$ the last proposition shows:

\[
R^n V(x) = 2 \sum_{k=0}^{2^n-1-1} \frac{1}{2^n - k} = \frac{1}{2^n-2} \sum_{k=0}^{2^n-1-1} \frac{1}{4 - k/2^{n-1}}
\]

It converges to $\frac{1}{2} \int_0^1 \frac{dx}{3-x} = \frac{1}{2} \ln \left(\frac{4}{3}\right)$.

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• If \( p \geq 3 \), then we deduce

\[
\mathcal{R}^n V(x) = \sum_{k=0}^{2^n-1} \frac{1}{p,2^n-k} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{1}{p-k/2^n}
\]

It converges to \( \ln \left( \frac{p}{p-1} \right) \).

Finally, with the notation \( p = \delta(x) \), the limit is equal to:

\[
U(x) = \begin{cases} 
\ln \left( \frac{p}{p-1} \right) & p \geq 3 \\
\frac{1}{2} \ln \left( \frac{4}{3} \right) & p = 2 
\end{cases}
\]

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