Pinsker estimators for local helioseismology: inversion of travel times for mass-conserving flows

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Abstract

A major goal of helioseismology is the three-dimensional reconstruction of the three velocity components of convective flows in the solar interior from sets of wave travel-time measurements. For small amplitude flows, the forward problem is described in good approximation by a large system of convolution equations. The input observations are highly noisy random vectors with a known dense covariance matrix. This leads to a large statistical linear inverse problem. Whereas for deterministic linear inverse problems several computationally efficient minimax optimal regularization methods exist, only one minimax-optimal linear estimator exists for statistical linear inverse problems: the Pinsker estimator. However, it is often computationally inefficient because it requires a singular value decomposition of the forward operator or it is not applicable because of an unknown noise covariance matrix, so it is rarely used for real-world problems. These limitations do not apply in helioseismology. We present a simplified proof of the optimality properties of the Pinsker estimator and show that it yields significantly better reconstructions than traditional inversion methods used in helioseismology, i.e. regularized least squares (Tikhonov regularization) and SOLA (approximate inverse) methods. Moreover, we discuss the incorporation of the mass conservation constraint in the Pinsker scheme using staggered grids. With this improvement we can reconstruct not only horizontal, but also vertical velocity components that are much smaller in amplitude.
Keywords: statistical inverse problem, pinsker estimator, helioseismology

(Some figures may appear in colour only in the online journal)

1. Introduction

Time–distance helioseismology aims at recovering the internal properties of the Sun from measurements of wave travel times between pairs of points [12]. The raw observations in helioseismology are time sequences of images of the line-of-sight velocity on the solar surface via Doppler shift measurements, for example from the solar dynamics observatory (45s cadence since 2010). These Doppler velocities contain information about the stochastic seismic wave field (acoustic waves and surface-gravity waves). Using a cross-correlation technique Duvall et al [12] showed that it is possible to measure the time it takes a wave packet to travel between any two points on the surface through the solar interior.

The wave travel times are linked to (perturbations of) physical quantities via a large system of convolution equations. In this paper we focus on the estimation of flows. The inversion is traditionally performed using Tikhonov regularization [35] or the method of approximate inverse [28, 31] that are respectively called in the helioseismology community, regularized least square (RLS) [25] and (subtractive) optimally localized averaging (OLA/SOLA) [21]. The latter goes back to the Backus–Gilbert method [1] and, as pointed out by Chavent [9], it is also closely related to the method of sentinels introduced by Lions for control problems (see [27]).

For overviews on linear statistical inverse problems we refer to [7, 16, 34]. Optimal rates of convergence for spectral regularization methods, in particular Tikhonov regularization, were shown in [3], and for the CG method in [4]. Pinsker-type estimator for deconvolution problems on the real line were studied theoretically in different degrees of generality in a series of papers by Ermakov (see e.g. [14, 15]). The case of periodic deconvolution problems with noise in the operator was treated in [8]. A minimax estimator for spherical deconvolution over a reduced class of estimators was developed in [22].

For linear inverse problems in Hilbert spaces with additive random noise Pinsker estimators are optimal in the following sense: for a given ellipsoid spanned by singular vectors of the forward operator, the Pinsker estimator minimizes the maximal risk (or expected square error) over this ellipsoid among all linear estimators. We point out that for deterministic inverse problems typically many optimal methods exist, e.g. Tikhonov regularization, some types of singular value decompositions (SVDs), the Showalter methods and (asymptotically) Landweber iteration and Lardy’s methods, each of course with an optimal choice of the regularization parameter or stopping index (see [33, 37]). In contrast, for statistical inverse problems, the Pinsker method is the only minimax linear estimator [26, 30]. Moreover, it was shown by Pinsker [30] under mild assumptions that it is even asymptotically optimal among all (not necessarily linear) estimators if the noise is Gaussian. In most real world applications this estimator cannot be applied for two main reasons: first, it requires the computation of a SVD of the forward operator which is often not affordable due to the size of the problem. Second, the noise covariance matrix has to be known while only a poor estimate is generally available. This explains why other methods such as Tikhonov regularization or conjugate gradient methods are more often used for real world applications. However, these limitations are not problematic for the helioseismology problem studied here since the forward operator separates into a collection of small matrices for each spatial frequencies, for which an SVD can be computed in reasonable time, and the noise covariance matrix is known [17, 19].
this paper we will study the implementation and performance of Pinsker estimators for such
problems.

A notorious difficulty in local helioseismology is the inversion for vertical velocity
components as their amplitude is much smaller than for horizontal velocities. The failure of
inversion was reported in several publications using synthetic data (see e.g. [11, 39]) and was
explained by the crosstalk between the variables. Here, we show that incorporation of the
mass conservation constraint in the Tikhonov or Pinsker methods allows to overcome these
difficulties. We will discuss the implementation of mass conservation constraints with the
help of staggered grids for the horizontal and the vertical velocity components.

The plan of this paper is as follows: after introducing the physical background and the
forward problem in section 2, we describe in section 3 the inversion methods that are
commonly used in this field so far. Then we introduce the Pinsker estimator in section 4 and
present a simple proof that it is the unique minimax linear estimator. Section 5 is devoted to
the incorporation of the mass conservation constraint into this regularization scheme. Finally,
numerical results demonstrating the advantages of Pinsker methods compared to the state-of-
the-art methods are discussed in section 6.

2. Estimating flows by local helioseismology

In local helioseismology, it is acceptable to consider small patches of the solar surface and to
neglect solar curvature. The domain of interest is approximated by a Cartesian box, with
horizontal coordinates \( r = (x, y) \) and vertical coordinate (height) \( z \). Let us denote this domain
by \( V \). Typically, \( x \) and \( y \) span several hundreds of megameters and \( z \) several tens of
megameters.

The observables are time series of the line-of-sight velocities \( \psi(r, t_j) \) at different points \( r \)
obtained from dopplergrams of the Sun’s surface taken by satellites at equidistant time points
\( t_j \). From these quantities, we compute averaged travel times \( \bar{\tau}^a(r) \) at different points \( r \) (and at
time \( t_0 \), but we assume the time series \( \psi \) to be stationary)

\[
\bar{\tau}^a(r) = \sum_j \int \text{Cov}(\psi(r, t_0), \psi(r + \vec{r}, t_0 + t_j))w^a(\vec{r}, t_j) \, d\vec{r}, \quad a = 1, \ldots, N_0.
\]

(We reserve the symbol \( \tau^a \) for differences of \( \bar{\tau}^a \) to a reference model.) The weights \( w^a \) are
chosen such that \( \bar{\tau}^a(r) \) approximates a spatial average of the times a certain type of wave
packet needs to travel from point \( r \) to points \( r + \vec{r} \), see [5, 12] and the end of this section for
more details. Hence, what will be called travel times in the following are linear functionals of
the covariance operator of the observable \( \psi \), written as \( \bar{\tau}^a = \mathcal{V}_a(\text{Cov}[\psi]) \) or
\( \bar{\tau} = \mathcal{V}(\text{Cov}[\psi]) \) for the vector \( \tau = (\tau^a)_{a=1 \ldots N_0} \) of all travel times.

The observable \( \psi \) is the image of the wave displacement \( \xi = \xi(r, z, t) \) under an
observation operator \( \mathcal{T} \), i.e. \( \psi = \mathcal{T}\xi \). Ideally, \( \psi(r, t) = \mathcal{T}(r, t, \xi(r, 0, t)) \) with the unit-
length line-of-sight vector \( \mathcal{T}(r, t) \) of the instrument. The wave displacement is linked to internal properties of the Sun via a PDE
describing the wave propagation in the Sun [6]:

\[
\mathcal{L}\xi := \rho(\partial_t + \Gamma + \mathbf{v} \cdot \nabla)\xi - \nabla(\rho c^2 \nabla \cdot \xi) + \nabla(\xi \cdot \nabla P) + \nabla \cdot (\rho g \xi) = f,
\]

where \( \rho \) is the density, \( c \) the sound speed, \( P \) the pressure, \( g \) the gravitational acceleration, \( \Gamma \)
the damping, \( \mathbf{v} \) the flow, and \( f \) (a stochastic) source term responsible for the excitation of the
seismic waves. Additional terms can be included to take into account the effects of rotation,
magnetic field or a more complex form of the gravitational term.
Our aim is to recover the 3D flow velocity field \( \mathbf{v} = (v^x, v^y, v^z) \) from observed travel times \( \bar{\tau} \). Inversion for other physical quantities can be performed analogously. We point out that actually computations are performed in the frequency domain, but at least formally we can write the forward operator as \( F(\mathbf{v}) = \mathcal{V}(T\mathcal{L}[\mathbf{v}]^{-1}\text{Cov}[\mathbf{f}](\mathcal{L}[\mathbf{v}]^{-1})^k \mathcal{F}^k) \), so we have to solve the nonlinear operator equation \( \bar{\tau} = F(\mathbf{v}) + n \) where \( n \) denotes noise. Under the assumption that \( \mathbf{v} \) is small compared to the local wave speed, which is true at least in quiet parts of the Sun, the Born approximation \( F(\mathbf{v}) \approx F(0) + F'(0)\mathbf{v} \) is sufficiently accurate [18], and we obtain the linear operator equation \( F'(0)\mathbf{v} = \bar{\tau} + n \) with \( \bar{\tau} \approx \bar{\tau} - F(0) \). The operator \( F'(0) \) can be written as an integral operator, and due to horizontal translation invariance the Schwartz kernel \( K \) only depends on the difference \( \mathbf{r} - \mathbf{r}' \), so

\[
\tau^a(\mathbf{r}) = \int_V \sum_{\beta \in \{x,y,z\}} K^{a,\beta}(\mathbf{r}' - \mathbf{r}, z)v^\beta(\mathbf{r}', z)d^2\mathbf{r}'dz + n^a(\mathbf{r}), \quad a = 1,\ldots,N_a
\]

(see [18]). The functions \( K^{a,\beta} \) are known as sensitivity kernels, but in contrast to the convention used in helioseismology where \( K^{a,\beta}(\mathbf{r}' - \mathbf{r}, z) \) is replaced by \( K^{a,\beta}(\mathbf{r}' + \mathbf{r}, z) \), we use a standard convolution integral as it is mathematically more convenient. The assumption that the kernels are invariant under horizontal translation is intimately connected to the assumption that we are modeling only a small patch on the solar surface.

Due to mass conservation the flow velocity satisfies the equation

\[
\text{div}(\rho \mathbf{v}) = 0, \quad (2)
\]

where the mass density \( \rho \) is assumed to depend on \( z \) only. Note that this constraint reduces the effective number of unknowns of the inverse problem by about one third.

Besides the Born approximation we will use two further simplifying assumptions: The first approximation consists in imposing periodic boundary conditions in the horizontal variables. Since the kernels are localized, aliasing artifacts can be avoided by zero-padding, but, nevertheless, this approximation leads to a loss of information close to the boundaries. We may assume without further loss of generality that the periodicity cell is \([p, 2p] \) in dimensionless coordinates. The second approximation consists in a discrete treatment of the depth variable \( z \). For simplicity, we assume that the \( v^\beta(\mathbf{r}, \cdot) \) is represented by its values on a grid \( \{z_0,\ldots,z_N\} \) and define \( v^\beta(\mathbf{r}) = v^\beta(\mathbf{r}, z_j) \).

Then, (1) can be written as

\[
\tau^a(\mathbf{r}) = \sum_{j=0}^{N_z} \sum_{\beta \in \{x,y,z\}} (K^{a,\beta,z_j} * v^\beta,z_j)(\mathbf{r}) + n^a(\mathbf{r}), \quad a = 1,\ldots,N_a
\]

(3)

where * denotes periodic convolution. Denoting by

\[
v_k := (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\mathbf{r}) \exp(-i\mathbf{r} \cdot \mathbf{k})d\mathbf{r}, \quad \mathbf{k} \in \mathbb{Z}^2,
\]

the Fourier coefficients of a periodic function \( f : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{C} \), we can write (3) equivalently in Fourier space as

\[
\tau^a_k = \sum_{j=0}^{N_z} \sum_{\beta \in \{x,y,z\}} K^{a,\beta,z_j}_k v^\beta,z_j_k + n^a_k, \quad a = 1,\ldots,N_a, \quad \mathbf{k} \in \mathbb{Z}^2. \quad (4)
\]

The problem is now decoupled for each spatial frequency \( \mathbf{k} \) and can be written in a matrix form as

\[
\tau_k = K_k v_k + n_k, \quad \mathbf{k} \in \mathbb{Z}^2,
\]

(5)
where the quantities we want to recover have been reorganized in the column vectors 
\( \mathbf{v}_k = (v_k^{x,y,z})_{x,y,z} \in \mathbb{C}^{N_x} \), the observables are 
\( \tau_k = (\tau_k^{x,y,z})_{x,y,z} \in \mathbb{C}^{N_x} \), and the Fourier transformed 
convolution kernels are \( K_k = (K_k^{x,y,z}) \in \mathbb{C}^{N_x \times 3} \).

The noise is assumed to be translation invariant, so the noise covariance matrix 
\( \Lambda^{ab}(\mathbf{d}) = \text{Cov}[n^a(\mathbf{r}), n^b(\mathbf{r} + \mathbf{d})] \), \( a, b = 1, \ldots, N_0 \)
does not depend on \( \mathbf{r} \). As a consequence, noise vectors \( n_k, n_{k'} \) for different spatial frequencies 
\( k, k' \in \mathbb{Z}^2 \) are uncorrelated, and the covariance matrix of \( n_k \) is given by 
\( \Lambda_k = (\Lambda^{ab})_{ab} \in \mathbb{C}^{N_0 \times N_0} \). An expression for these matrices was first derived in [19] and 
generalized in [17] taking into account that the observation time is finite.

For our computations we will use the kernels \( K \) from [32], which we are going to 
describe briefly. We consider a Cartesian patch of the solar surface containing \( 200 \times 200 \) pixels with a spatial sampling width of \( h_x = 1.46 \) Mm. The vertical direction \( z \) is discretized 
with \( N_z = 89 \) points using a variable step size as the variations are stronger close to the 
surface due to the density profile. This variation of the mass density of several orders of 
magnitude near the surface is one of the difficulties to invert for velocities. The quantity 
\( \mathbf{v} = (v^x, v^y, v^z) \) we want to recover has thus \( 3N_z = 267 \) degrees of freedom for each spatial 
frequency \( k \).

In order to improve the signal-to-noise ratio, certain averages of point-to-point travel 
times are used, for example between the center of a disk and all the points located at a given 
radius of this disk. Such types of data are sensitive to in/out flows in this disk. Imposing other 
weights on the circle leads to data that are sensitive to East–West or North–South flows. 
Varying the center of this disk on the whole surface of the observational domain allow to 
build a map of observations. We use each of these three averaging schemes for 16 radii from 
5 Mm to 20 Mm. Moreover, we use filters for \( f, p1, p2, p3 \), and \( p4 \) waves. (The first one is a 
gravitational wave, and the latters are acoustic waves with 1, 2, 3 or 4 nodes.) This yields 
\( N_k = 3 \times 16 \times 5 = 240 \) travel time data for each of the \( 200^2 \) points on the solar surface. 
Thus, the kernels \( K_k \) are of the size \( 240 \times 267 \) for each of the \( 200^2 \) frequencies \( k \).

To provide some intuition for the problem we are solving, a representation of kernels for 
\( v_x \) and \( v_z \) using different filters is given in figure 1. The columns 2 and 4 represent cuts at 
\( z = 0 \) (surface of the Sun) for the part sensitive to \( v_x \) (column 2) and \( v_z \) (column 4). The
kernels are localized around the center indicating that the data are relatively close to the quantities we want to infer for. The columns 1 and 3 of figure 1 show the depth dependance of the kernels for different type of waves. One can see the importance of using different waves in order to probe different depths in the solar interior. However, all kernels are extremely sensitive to the surface making inversion at large depths highly ill-posed.

An example of travel time map for a given filter is given in figure 2 after adding the noise. The noise level corresponds to data averaged over 4 d with a temporal sampling of 45 s. Even with such a long averaging time, one can see that the noise is highly correlated, which underlines the importance of a good knowledge of the noise covariance matrix as computed in [17, 19].

3. Classical inversion methods used in local helioseismology

In order to solve (5) any regularization method for linear inverse problems could be used in principle (see [13]). In this section we present two techniques generally used in local helioseismology. Then, we introduce a new approach in the field based on the Pinsker estimator.

3.1. Regularized least squares

Tikhonov regularization is generally called RLS in the helioseismology community. Since the problem decouples for all \( k \) ([24]), we can compute

\[
\hat{v}_k^{\text{RLS}} = \arg\min_{v_k} \left[ \left\| \Lambda_k^{-\frac{1}{2}} (K_k v_k - \tau_k) \right\|^2 + \alpha \left\| L_k v_k \right\|^2 \right]
\]

\[
= (K_k^* \Lambda_k^{-1} K_k + \alpha L_k^* L_k)^{-1} K_k^* \Lambda_k^{-1} \tau_k
\]

independently for all spatial frequencies \( k \in \mathbb{Z}^2 \). Note that the first term \( \left\| \Lambda_k^{-\frac{1}{2}} (K_k v_k - \tau_k) \right\|^2 \) is the negative log-likelihood term if the noise is Gaussian. In a Bayesian framework the second term \( \alpha \left\| L_k v_k \right\|^2 \) corresponds to a Gaussian prior. Here we consider \( \alpha > 0 \) as a regularization parameter and \( L_k \) as a regularization matrix that can be the identity or the
discretized version of the gradient or the Laplacian in order to impose additional smoothness on the solution.

### 3.2. Optimally localized averaging

Different types of OLA methods are used in helioseismology. Recently, it was proposed to take advantage of the convolution in the horizontal space and to propose a multichannel OLA \[23\]. Similar to the previous approach the problem decouples for all frequencies and can be solved efficiently. We seek for a linear combination of travel times via weighting matrices

\[
W_k = (W_k^{\text{OLA}}) \in \mathbb{C}^{3N_k \times N_k} \quad \text{(the Fourier coefficients of weighting kernels)}
\]

\[
W(r) := \sum_{k \in \mathbb{Z}^2} W_k \exp(i r \cdot k)
\]

such that

\[
\hat{v}_k = W_k v_k, \quad k \in \mathbb{Z}^2
\]

is a good estimate of \(v_k\).

Note from the second line in (6) that RLS is also of this form with

\[
W_k^{\text{RLS}} = (K_k^* A_k^{-1} K_k + \alpha L_k^* L_k)^{-1} K_k^* A_k^{-1}.
\]

Inserting (4) into (7) yields

\[
\hat{v}_k = W_k K_k v_k + W_k a_k.
\]

**Definition 3.1.** For a regularization method of the form (7) the function

\[
K_k := W_k K_k
\]

and values in \(\mathbb{R}^{3N_k \times 3N_k}\) is called the *averaging kernel* of the method. (Often only specific rows of \(W\) and \(K\) corresponding to a specific depth \(z_j\) and a Cartesian component \(\beta\) are considered. We will denote them by \(W[\beta, z_j ;:](r)\) and \(K[\beta, z_j ;:](r)\).)

Note from (8) that the expectation \(E[\hat{v}]\) and hence the bias \(E[\hat{v}] - v\) of the estimator \(\hat{v}\) is characterized by a convolution with the averaging kernel:

\[
E[\hat{v}] = K \ast v.
\]

To keep the bias small the diagonal entries (\(\alpha = \beta\)) of the averaging kernel \(K^{\alpha, z_j, \beta, z_l}(r)\) should be well concentrated around \(z_j \approx z_l\) and \(r = 0\). The off-diagonal entries (\(\beta \neq \alpha\)) measure the leakage from one Cartesian component \(\beta\) to another component \(\alpha\) and should be small.

The subtractive OLA (SOLA) methods aims at finding rows of a weighting kernel \(W\) indexed by \(\beta, z_j\) such that the corresponding rows of the averaging kernel \(K\) are as close as possible to rows of a prescribed target function \(T(r) \in \mathbb{R}^{3N_k \times 3N_k}\) while keeping the noise (last term in (8)) small. This can be achieved by setting

\[
W_k^{\text{OLA}} [\beta, z_j ;:] := \arg\min_{W \in \mathbb{C}^{3N_k \times 3N_k}} \| W K_k - T_k[\beta, z_j ;:] \|^2 + \mu W L_k W^* \]

(10)

(see [23]) where \(\mu > 0\) is a trade-off parameter. Other objective functional can be chosen, see e.g. [32]. The target function \(T^{\beta, z_j, \alpha, z_l}(r)\) for \(\alpha = \beta\) is generally chosen as a Gaussian in \((r, z_l)\) around the point \((0, z_j)\). For \(\alpha \neq \beta\) it is chosen as 0. Obviously, the convex quadratic minimization problem (10) can be solved by solving the linear first order optimality conditions. We also mention the multiplicative OLA (MOLA) [10] method which uses a product \(K T\) instead of the difference.

These methods involve the target functions \(T\) and the parameter \(\mu\) as parameters, the choices of which are not obvious and involve certain subjectivity. In the next section, we
propose to use the Pinsker estimator that is optimal in the sense that it minimizes the risk in a
given class of functions.

4. Pinsker estimator

The problem described in section 2 can be formulated as a linear operator equation

\[ \tau = Kv + n. \] (11)

in the Hilbert spaces \( \mathbb{X} = L^2([\pi, \pi]^N) \) and \( \mathbb{Y} = L^2([\pi, \pi]^N) \) with a compact, linear
operator \( K : \mathbb{X} \to \mathbb{Y} \) given by a matrix of convolution operators.

We assume that the noise \( n \) is a Hilbert space process in \( \mathbb{Y} \) with zero mean value and
known covariance operator \( \text{Cov}[n] \). The modeling errors that are ignored in the assumption
\( \mathbb{E}[n] = 0 \) and references for \( \text{Cov}[n] \) have been discussed in section 2.

An estimator is an operator \( \mathbb{W} : \mathbb{Y} \to \mathbb{X} \) that maps observations \( \tau \) to an approximation
\( \hat{v} \in \mathbb{X} \) of \( v \). The risk (or expected square error) of an estimator \( W \) at \( v \) is defined by

\[ R(W, v) = \mathbb{E}[\|W(Kv + n) - v\|^2]. \] (12)

If \( W \) is linear, the risk can be decomposed into a bias and a variance part using \( \mathbb{E}[n] = 0 \):

\[ R(W, v) = \| (WK - I)v \|^2 + \mathbb{E}[\|Wn\|^2]. \] (13)

The bias \( \| (WK - I)v \|^2 \) describes how far \( W \) is from the inverse of the forward operator while
the variance term \( \mathbb{E}[\|Wn\|^2] = \text{trace}(\text{Cov}[Wn]) = \text{trace}(W^*\text{Cov}[n]W) \) describes the
stochastic part of the error.

The maximal risk of an estimator \( W \) on a set \( \Theta \subset \mathbb{X} \) is defined as

\[ R^N(\Theta) = \sup_{v \in \Theta} R(W, v) = \sup_{v \in \Theta} \mathbb{E}[\|W(Kv + n) - v\|^2]. \] (14)

The minimax risk and the minimax linear risk on \( \Theta \) are obtained by taking the infimum over
all estimators (or all linear estimator, respectively) of (14)

\[ R^N(\Theta) = \inf_W R(W, \Theta), \quad R^L(\Theta) = \inf_{W \text{ linear}} R(W, \Theta). \] (15)

A linear estimator \( W \) that attains the infimum in (15) is called a minimax linear estimator. To construct such an estimator for (11) we first perform a whitening by multiplying (11) from the
left by \( \text{Cov}[n]^{-1/2} \) to obtain

\[ \tilde{\tau} = \tilde{K}v + \tilde{n}, \] (16)

where \( \tilde{\tau} := \text{Cov}[n]^{-1/2}\tau \) and \( \tilde{n} := \text{Cov}[n]^{-1/2}n \) is now a white noise process, i.e.
\( \text{Cov}[	ilde{n}] = I_l \). To ensure that \( \tilde{K} = \text{Cov}[n]^{-1/2}K \) is well defined, we assume that \( \text{Cov}[n] \) is
strictly positive definite, i.e. every linear functional of \( \tau \) contains a minimal fixed amount of
noise. Although this assumption could be relaxed, it is simple and intuitive, and also
guarantees compactness of \( \tilde{K} \). Hence, \( \tilde{K} \) admits a SVD \( \{(\sigma_l, \varphi_l, \psi_l) : l \in \mathbb{N}\} \). This allows us
to rewrite the operator equation (11) as a diagonal operator equation in sequence spaces given by

\[ y_l = \sigma_l v_l + n_l \] (17)

with observables \( y_l := (\text{Cov}[n]^{-1/2}\tau, \psi_l)_{\mathbb{Y}} \) and unknowns \( v_l := (\nu, \varphi_l)_{\mathbb{X}} \). Due to Gaussianity
the noise \( (n_l)_{l \in \mathbb{N}} \) is a sequence of uncorrelated \( N(0, 1) \) random variables.
Let us consider linear diagonal estimators of the form
\[
\hat{y}_l := \frac{\lambda}{\sigma} y_l, \quad W_\lambda := \sum_{l \in \mathbb{N}} \frac{\lambda}{\sigma^2} \langle \text{Cov}[n]^{-1/2}, \psi_l \rangle_y \varphi_l
\] (18)
with weights \(\lambda_l \in \mathbb{R}\). The risk \(R(\lambda, v) := R(W_\lambda, v)\) of such estimators is given by
\[
R(\lambda, v) = \sum_{l \in \mathbb{N}} \left[ (1 - \lambda_l)^2 v_l^2 + \frac{\lambda_l^2}{\sigma_l^2} \right].
\] (19)
We will consider ellipsoids of the form
\[
\Theta := \left\{ v \in \mathbb{X} : \sum_{l=1}^\infty a_l^2 v_l^2 \leq Q \right\}
\] (20)
with \(Q > 0\) and \(a_l > 0\) with \(a_l \to \infty\). Then the risk \(R(\lambda, \Theta) := R(W_\lambda, \Theta)\) is given by
\[
R(\lambda, \Theta) = Q \sup_{l \in \mathbb{N}} \frac{(1 - \lambda_l)^2}{a_l^2} + \sum_{l=1}^\infty \frac{\lambda_l^2}{\sigma_l^2}.
\] (21)

**Lemma 4.1.** Any minimax linear estimator must be of the diagonal form (18).

**Proof.** Note that since \(a_l \to \infty\), the supremum in (21) is attained at some index \(l_0 \in \mathbb{N}\), and \(R(\lambda, \Theta) = R(\lambda, \{v_{l_0}\})\) with \(v_{l_0} = (\sqrt{Q}/a_{l_0}) \varphi_{l_0} \in \Theta\). If a linear estimator \(W\) with a nondiagonal (infinite) matrix representation is replaced by its diagonal part \(\text{diag}(W)\), the bias part of \(R(W, \{v_{\text{diag}(W)}\})\) cannot increase and the variance part strictly decreases. Hence
\[
R(\text{diag}(W), \Theta) = R(\text{diag}(W), \{v_{\text{diag}(W)}\}) < R(W, \{v_{\text{diag}(W)}\}) \leq R(W, \Theta),
\]
which shows that \(W\) is not minimax.

Even though the following result is well-known, we would like to present a short proof since we consider it more instructive and simpler than other proofs, e.g. in [2, 30, 36] (all for the equivalent regression problems version of the theorem). In particular, we derive the formulas (22) and (23) and not just verify them, and it becomes apparent that \(\pi \sqrt{Q}\) is the bound on the bias.

**Theorem 4.2 (Pinsker estimator).** Consider a sequence \((a_l)_{l \in \mathbb{N}}\) such that \(a_l > 0\) and \(\lim_{l \to \infty} a_l = \infty\), and an ellipsoid of the form (20) with \(Q > 0\). Then there exists a unique minimax linear estimator on \(\Theta\). It is of the form (18), and its weights are given by
\[
\lambda_l = \max(1 - \pi a_l, 0),
\] (22)
where the constant \(\pi > 0\) is the unique solution of the equation
\[
\kappa Q = \sum_{l=1}^\infty \frac{a_l}{\sigma_l^2} \max(1 - \kappa a_l, 0) = 0.
\] (23)
The minimax linear risk is given by \(R^k(\Theta) = \sum_{l=1}^\infty \frac{1}{\sigma_l^2} \max(1 - \pi a_l, 0)\).

**Proof.** The infimum of \(R(\lambda, \Theta)\) over all sequences \(\lambda\) can be reduced to the set \(\Lambda := \{ \lambda \in l^2(\mathbb{N}) : \|\lambda\|_\infty \leq 1 \}\) since \(R(\lambda, \Theta) = \infty\) if \(\lambda \notin l^2(\mathbb{N})\) and \(R(\lambda, \Theta)\) strictly decreases if some \(\lambda_l \notin [-1, 1]\) is replaced by its metric projection onto \([-1, 1]\). Let us...
introduce the decomposition

\[ \Lambda = \bigcup_{\kappa \in \mathbb{N}} \Lambda_\kappa \quad \text{with} \quad \Lambda_\kappa = \left\{ \lambda \in \Lambda : \sup_{j \in \mathbb{N}} \frac{|1 - \lambda_j|}{a_j} = \kappa \right\} \]

with \( g := \min_j a_j \). In view of (21) we have \( R(\lambda, \Theta) = \kappa^2 Q + \sum_{j=1}^{\infty} (\lambda_j/\sigma_j)^2 \) for \( \lambda \in \Lambda_\kappa \), so the infimum over \( \lambda \in \Lambda_\kappa \) is attained if and only if \( \lambda_j = \arg\min_{1-\lambda_j \leq 0} x_j^2 = \max(1 - \lambda a_j, 0) \) for all \( j \in \mathbb{N} \). Note that this is (22) if \( k = \pi \). Using this formula for the minimizer we find that

\[ \inf_{\lambda \in \Lambda_\kappa} R(\lambda, \Theta) = \varphi(\kappa) \quad \text{with} \quad \varphi(\kappa) = \kappa^2 Q + \sum_{j=1}^{\infty} \max(1 - \kappa a_j, 0)^2. \]

Therefore \( \inf_{\lambda \in \Lambda} R(\lambda, \Theta) = \inf_{0 < \kappa \leq 1/g} \varphi(\kappa) \). Note that \( \varphi \) is strictly convex and differentiable with \( \varphi'(\kappa) \) given by the left-hand side of (23) since the sum is finite in a neighborhood of any \( \kappa > 0 \). Moreover, \( \lim_{\kappa \to 0} \varphi(\kappa) = \infty \) and \( \varphi'(1/g) = Q/g > 0 \). Therefore, \( \varphi \) attains its infimum on \([0, 1/g]\) at the unique solution \( \kappa \) to \( \varphi'(\kappa) = 0 \).

Instead of the implicit equation (23) for \( \pi \) there is also the following explicit formula if the sequence \((a_j)\) is non-decreasing (see [36]):

\[ \pi = \frac{\sum_{j=1}^{N} a_j}{Q + \sum_{j=1}^{N} a_j^2} \quad \text{with} \quad N := \max\left\{ n \in \mathbb{N} : \sum_{j=1}^{n} \frac{1}{\sigma_j^2} (a_j - a_i) < Q \right\}. \]

From a practical point of view, this formula is only useful if \( Q \) is known exactly. But this is a rather unrealistic assumption. \( Q \) should rather be seen as a regularization parameter. But since there is a one-to-one correspondence between \( Q \) and \( \pi \) via (23), it is much simpler to consider \( \pi \) as regularization parameter. The choice of regularization parameters is an important and well-studied problem, but since the focus of this paper is on the comparison of regularization methods, we do not further discuss it here.

A comparison of the linear minimax risk \( R^L \) with the nonlinear one was given in [30]. Under the additional assumptions that the noise is Gaussian and that

\[ \sup_{j \in \mathbb{N}} \frac{\sum_{j=1}^{\infty} \sigma_j^4}{\sup_{j \in \mathbb{N}} (\sigma_j^4)^{1/2}} < \infty, \tag{24} \]

then \( R^L(\Theta) \sim R^N(\Theta) \) as the noise level tends to 0. Assumption (24) was later relaxed to \( \sup_{j \in \mathbb{N}} (\sigma_j^4)^{1/2} < \infty \) [20]. This assumption is very plausible in the context of our problem.

It remains to discuss the choice of the ellipsoid \( \Theta \). Without depth inversion, i.e. for \( N_c = 1 \) and a scalar physical quantity, it is natural to define \( \Theta \) in terms of some bound on the power spectrum of the form

\[ \sum_{k \in \mathbb{Z}^d} \gamma(k) |v_k|^2 \leq Q. \]

E.g. for the choice \( \gamma(k) = (1 + |k|^2)^\gamma \) the ellipsoids \( \Theta \) are balls in the periodic Sobolev \( H^\gamma([-\pi, \pi]^2) \). In depth direction admissible choices of \( \Theta \) are more difficult to interpret since the axes of the ellipsoid must coincide with the singular vectors of the forward operator.

We choose the weights \( a_l \) such that \( a_l^2 \) grows asymptotically as the weights \( \gamma(k) = \gamma(k(l)) \) on the \( L^2 \)-Fourier coefficients defining an \( H^\gamma(V) \)-ball in a cuboid \( V \subset \mathbb{R}^3 \) as \( l \to \infty \), i.e.
Here $l_k$ denotes an enumeration of the three-dimensional spatial frequencies such that $|l_k|$ is non-decreasing. Empirically, we observe that the singular values $s_l$ of our forward operator decay exponentially, i.e. $s_l \sim \alpha^{-l}$ for some $\alpha > 0$, and their ordering at least roughly corresponds to the ordering described by $l_k$ (see figure 3).

5. Mass conservation constraint

In this section we discuss how the mass conservation constraint $\text{div}(\rho v) = 0$ mentioned in (2) can be incorporated into Tikhonov regularization and the Pinsker method. The minimization problems are similar to the ones presented in sections 3 and 4, but now the reconstructed velocity is subject to mass conservation. An extension of SOLA respecting mass conservation is not straightforward as the velocity does not appear explicitly in the minimization problem. Therefore, SOLA is not considered in this section.

We will assume that $0 < \rho_{\text{min}} \leq \rho \leq \rho_{\text{max}} < \infty$, that $\rho$ is smooth and depends only on $z$. Then the inverse problem can be formulated as

$$Kv = \tau \quad \text{subject to } \text{div}(\rho v) = 0. \quad (26)$$

In an abstract Hilbert space setting an equality constraint $Bv = 0$ with a bounded linear operator $B : \mathcal{H} \to \mathcal{Z}$ does not change much since we can simply replace $\mathcal{H}$ by the null-space $\mathcal{N}(B)$ of $B$. However, as it is often inconvenient to explicitly construct a basis of $\mathcal{N}(B)$, it is preferable to work in the larger space $\mathcal{H}$.

E.g. statistical Tikhonov regularization with noise covariance operator $\Lambda$ and a (differential) operator $L$ mapping to a Hilbert space $\mathcal{V}$ applied to (26) reads

$$v_\alpha = \arg\min_{Bv = 0} \{ \| \Lambda^{-1/2}Kv - \tau \|^2_\mathcal{V} + \alpha \|Lv\|^2_\mathcal{V} \}.$$  

To treat the side condition we consider the corresponding Lagrange function $L(v, \mu) := \| \Lambda^{-1/2}Kv - \tau \|^2_\mathcal{V} + \alpha \|Lv\|^2_\mathcal{V} + \langle \mu, \alpha Bv \rangle_\mathcal{Z}$ with a Lagrange multiplier $\mu \in \mathcal{Z}$.

Here $B$ has been multiplied by the regularization parameter $\alpha$ to improve the condition number of the optimality conditions $\frac{\partial L}{\partial v} = 0$ and $\frac{\partial L}{\partial \mu} = 0$. These then lead to the saddle point equation...
\[
\left( \frac{K^* \Lambda^{-1} K + \alpha L^a L}{\alpha B} \right) \left( \begin{array}{c}
\gamma \\
\mu
\end{array} \right) = \left( \frac{K^* \Lambda^{-1} \tau}{0} \right).
\]

5.1. Fully continuous setting

In this subsection we discuss a continuous treatment of the depth variable \(z\). If \(V = [-\pi, \pi]^2 \times [z_N, z_0]\) is the domain of interest, we may choose

\[\mathcal{X} = \{ \mathbf{v} \in H^1(V)^3 : \mathbf{v}(\cdot, z) \text{ periodic}, \mathbf{v}^\tau(\cdot, z_0) = \mathbf{v}^\tau(\cdot, z_N) = 0 \}.\]

This choice of boundary conditions rules out coronal mass ejections, which are very simple to detect and for which the Born approximation used in the derivation of the forward operator breaks down anyways.

We equip \(\mathcal{X}\) with the norm \(\| \mathbf{v} \|_\mathcal{X} = (\rho \mathbf{v}, \rho \mathbf{v})^{1/2}\) where

\[\langle \rho \mathbf{v}, \rho \mathbf{w} \rangle_{H^1} = \sum_{\beta \in \{1,2,3\}} \langle \rho \mathbf{v}^\beta, \rho \mathbf{w}^\beta \rangle_{L^2(V)} + \langle \text{grad} \rho \mathbf{v}^\beta, \text{grad} \rho \mathbf{w}^\beta \rangle_{L^2(V)}.\]

Under our assumptions on \(\rho\) the norms \(\| \rho \mathbf{v} \|_{H^1}\) and \(\| \mathbf{v} \|_{H^1}\) are equivalent, but since \(\rho\) varies over several orders of magnitude, the incorporation of \(\rho\) in the norm makes a significant difference.

Let us introduce the operators \(\text{grad}_x \mathbf{v} := \text{grad}(\rho \mathbf{v}), \text{curl}_x \mathbf{v} := \text{curl}(\rho \mathbf{v})\), and \(\text{div}_x \mathbf{v} := \text{div}(\rho \mathbf{v})\). The following lemma summarizes the properties of the subspace \(\mathcal{N}(\text{div}_x) \subset \mathcal{X}\) and will be proved in an appendix.

**Lemma 5.1.**

(i) For all \(\mathbf{v}, \mathbf{w} \in \mathcal{X}\) we have

\[
\sum_{\beta \in \{1,2,3\}} \langle \text{grad}_x \mathbf{v}^\beta, \text{grad}_x \mathbf{w}^\beta \rangle_{L^2(V)} = \langle \text{curl}_x \mathbf{v}, \text{curl}_x \mathbf{w} \rangle_{L^2(V)} + \langle \text{div}_x \mathbf{v}, \text{div}_x \mathbf{w} \rangle_{L^2(V)}. \tag{27}
\]

(ii) There exists a constant \(c > 0\) such that the inequalities

\[
c \| \rho \mathbf{v} \|_{H^1}^2 \leq \| \text{curl}_x \mathbf{v} \|_{L^2}^2 + \frac{1}{|V|} \sum_{\beta \in \{1,2,3\}} \left( \int_V \rho \mathbf{v}^\beta \text{d}(\mathbf{r}, x) \right)^2 \leq \| \mathbf{v} \|_{H^1}^2
\]

hold true for all \(\mathbf{v} \in \mathcal{X}\) with \(\text{div}_x \mathbf{v} = 0\).

(iii) \(\mathcal{X}_0 := \{ \mathbf{v} \in \mathcal{X} : \int_V \rho \mathbf{v} \text{d}(\mathbf{r}, x) = \int_V \rho \mathbf{v} \text{d}(\mathbf{r}, x) = 0 \}\) has the Helmholtz decomposition

\[\mathcal{X}_0 = \mathcal{N}_0(\text{div}_x) \oplus \mathcal{N}_0(\text{curl}_x),\]

with \(\mathcal{N}_0(\text{div}_x) := \{ \mathbf{v} \in \mathcal{X}_0 : \text{div}_x \mathbf{v} = 0 \}\) and \(\mathcal{N}_0(\text{curl}_x) := \{ \mathbf{v} \in \mathcal{X}_0 : \text{curl}_x \mathbf{v} = 0 \}\). These subspaces are orthogonal both with respect to the \(\mathcal{X}\) inner product and the inner product \((\rho \mathbf{v}, \rho \mathbf{w})_{L^2(V)}\).

We will choose \(L \mathbf{v} := \text{curl}(\rho \mathbf{v})\). This means we do not incorporate the means of the horizontal velocity components into the penalty term, which are needed to obtain a norm on \(\{ \mathbf{v} \in \mathcal{X} : \text{div}_x \mathbf{v} = 0 \}\). This is justified as the data are sensitive to constant horizontal velocities, i.e. \(K\) restricted to \(\text{span} \{ (1/\rho, 0, 0), (0, 1/\rho, 0) \}\) is bounded from below (see [13, 8.2]).
5.2. (Semi-) discrete approximation

In this subsection we discuss a discrete approximation of the \(z\)-variable which inherits the essential properties of the continuous setting. We found this crucial for good numerical results. Since \(\rho\) depends only on \(z\), the constraint \(\text{div}(\rho \mathbf{v}) = 0\) separates into

\[
  i k_x \rho v^x_k + i k_y \rho v^y_k + \frac{\partial \rho v^z_k}{\partial z} = 0, \quad k = (k_x, k_y) \in \mathbb{Z}^2.
\]

Hence the only difference between a continuous and a discrete treatment of the (periodic) horizontal variables \(x\) and \(y\) is that in the former case infinitely many spatial frequencies must be considered, and in the latter case only finitely many.

It will be essential to use different grids for the horizontal and the vertical velocities to preserve the most important properties of the continuous setting as summarized in lemma 5.1 in the discrete setting. For a given grid \(z_0 > z_1 > \ldots > z_N\) in vertical direction we introduce the midpoints \(z_{j+1/2} = \frac{1}{2}(z_j + z_{j+1})\). The horizontal velocities will be represented on \([z_1/2, \ldots, z_{N-1/2}]\) whereas the vertical velocities will be represented by their values on \([z_0, \ldots, z_N]\). Here the points \(z_0\) and \(z_N\) have been omitted due to the Dirichlet boundary conditions for \(v^z\) such that

\[
v^x, v^y \in \mathbb{V} := \mathbb{C}^{N_x}, \quad v^z \in \mathbb{W} := \mathbb{C}^{N_z}.
\]

These quantities will be indexed by \(v^x_k = (v^x_{k,1}, \ldots, v^x_{k,N_{z}-1})^T\) and \(v^y_k = (v^y_{k,1/2}, \ldots, v^y_{k,N_{z}-1/2})^T\), \(v^z_k = (v^z_k)_{j=0,1,2,1,1}^\beta\) for \(j = 1, \ldots, N_z - 1\) and \(\delta_{j+1/2} = z_j + 1/2\) for \(j = 0, \ldots, N_z - 1\). Then we introduce Gram matrices

\[
G_{\mathbb{V}} := \text{diag}(\delta_{1/2}, \ldots, \delta_{N_{z}-1/2}) \quad \text{and} \quad G_{\mathbb{W}} := \text{diag}(\delta_{1/2}, \ldots, \delta_{N_{z}-1})
\]

defining inner products \(\langle v_1, v_2 \rangle_{\mathbb{V}} := v^x_k G_{\mathbb{V}} v_1^x\) on \(\mathbb{V}\) and \(\langle v_1, v_2 \rangle_{\mathbb{W}} := v^y_k G_{\mathbb{V}} v_1^y\) on \(\mathbb{W}\). Similarly, we define \(\rho_j := \rho(z_j)\) for \(j \in \{0, 1/2, 1, \ldots, N_z\}\) and the matrices \(M^y_\rho = \text{diag}(\rho_{1/2}, \ldots, \rho_{N_z-1/2})\) and \(M^y_\rho = \text{diag}(\rho_{1/2}, \ldots, \rho_{N_z-1})\). We approximate derivatives by the finite differences

\[
\frac{\partial v^z_k}{\partial z}(z_{j+1/2}) \approx \frac{v^z_{k,j} - v^z_{k,j+1}}{\delta_{j+1/2}}, \quad \frac{\partial v^y_k}{\partial z}(z_j) \approx \frac{v^y_{k,j} - v^y_{k,j+1/2}}{\delta_{j}}
\]

for \(\beta = x, y\) with \(D^z_\rho = \text{diag}(\rho_{1/2}, \ldots, \rho_{N_z-1/2})\) and \(D^z_\rho = \text{diag}(\rho_{1/2}, \ldots, \rho_{N_z-1})\). We approximate derivatives by the finite differences

\[
D^z_\rho := G_{\mathbb{W}}^{-1} \begin{pmatrix} 1 & 1 & \cdots & 1 \ 0 & 1 & \cdots & 1 \ \vdots & \ddots & \ddots & \vdots \ 0 & \cdots & 0 & 1 \end{pmatrix}
\]

These matrices are skew-adjoint with respect to the inner products in \(\mathbb{V}\) and \(\mathbb{W}\) since

\[
\langle D^z_\rho w, v \rangle_{\mathbb{V}} = v^y G_{\mathbb{V}} D^z_\rho w = -v^x (G_{\mathbb{W}} D^z_\rho) w = -(D^z_\rho v)^x G_{\mathbb{W}} w = -(w, D^z_\rho v)_{\mathbb{V}}.
\]

Now we introduce the following approximations to the div, grad, curl, and curl\(^3\) for the spatial frequency \(k \in \mathbb{Z}^2\):
\[
\begin{aligned}
\text{div}_k &:= (i_kI^V, i_kI^V, D_z^W) : \mathbb{V} \times \mathbb{V} \times \mathbb{W} \rightarrow \mathbb{V}, \\
\text{grad}_k &:= (i_kI^V, i_kI^V, (D_z^W)^T) : \mathbb{V} \rightarrow \mathbb{V} \times \mathbb{V} \times \mathbb{W} \\
\text{curl}_k &:= \begin{pmatrix} 0 & -D_z^W & i_kI^W \\ D_z^W & 0 & -i_kI^V \\ -i_kI^V & i_kI^V & 0 \end{pmatrix} : \mathbb{V} \times \mathbb{V} \times \mathbb{W} \times \mathbb{W} \times \mathbb{V}, \\
\text{curl}^\#_k &:= \begin{pmatrix} 0 & -D_z^W & i_kI^V \\ D_z^W & 0 & -i_kI^V \\ -i_kI^V & i_kI^V & 0 \end{pmatrix} : \mathbb{W} \times \mathbb{V} \rightarrow \mathbb{V} \times \mathbb{V} \times \mathbb{W}.
\end{aligned}
\]

Let us introduce the spaces \( \mathbb{X}_k := \mathbb{V} \times \mathbb{V} \times \mathbb{W} \) and \( \mathbb{Y}_k := \mathbb{W} \times \mathbb{W} \times \mathbb{V} \), the multiplication operator \( \text{M}_p^X := \text{blockdiag}(M_p^X, M_p^Y, M_p^W) \), and the mappings

\[
\begin{aligned}
\text{div}_{\lambda,k} &:= \text{div}_k M_p^X, \\
\text{curl}_{\lambda,k} &:= -\text{curl}_k M_p^X, \\
\text{curl}^\#_{\lambda,k} &:= (M_p^X)^{-1} \text{curl}^\#_k, \\
\text{grad}_{\lambda,k} &:= (M_p^X)^{-1} \text{grad}_k.
\end{aligned}
\]

The Gram matrices in \( \mathbb{X} \) and \( \mathbb{Y} \) are

\[
G_X := (M_p^X)^2 \text{ blockdiag}(G_V, G_V, G_W) \quad \text{and} \quad G_Y := \text{ blockdiag}(G_W, G_W, G_Y).
\]

These matrices have the following properties:

**Lemma 5.2.**

(i) \( \text{div}_{\lambda,k} \text{ curl}^\#_{\lambda,k} = \text{div}_k \text{ curl}_k = 0 \) and \( \text{curl}_{\lambda,k} \text{ grad}_{\lambda,k} = \text{curl}_k \text{ grad}_k = 0 \).

(ii) \( \mathcal{N}(D_z^W) = \{0\} \).

(iii) \( \text{curl}^\#_{\lambda,k} \) is the adjoint of \( \text{curl}_{\lambda,k} \) with respect to the Gram matrices \( G_X \) and \( G_Y \), i.e. \( G_X \text{ curl}^\#_{\lambda,k} = (G_Y \text{ curl}_{\lambda,k})^\# \), and similarly \( G_Y \text{ div}_{\lambda,k} = -(G_X \text{ grad}_{\lambda,k})^\# \).

(iv) With respect to the Gram matrix \( G_X \) we have the orthogonal decomposition

\[
\mathbb{X}_k \approx \mathcal{N}(\text{div}_{\lambda,k}) \oplus \mathcal{N}(\text{curl}_{\lambda,k}) \quad \text{and} \quad \mathcal{N}(\text{div}_{\lambda,k}) = \mathcal{R}(\text{curl}^\#_{\lambda,k}) \quad \text{for} \quad k \neq 0.
\]

**Proof.** Part (i) can be verified by straightforward computations. Part (ii) is also easy to see, and part (iii) follows from (30).

To show part (iv) we first demonstrate that

\[
\mathcal{N}(\text{curl}_{\lambda,k}) \cap \mathcal{N}(\text{div}_{\lambda,k}) = \{0\}.
\]

Let \( v_k := (v_k^x, v_k^y, v_k^z) \in \mathcal{N}(\text{curl}_{\lambda,k}) \cap \mathcal{N}(\text{div}_{\lambda,k}) \). We only treat the case \( k_z = 0 \) as the case \( k_z \neq 0 \) is analogous. The last line in \( \text{curl}_{\lambda,k} v_k = 0 \) implies that

\[
k_z M_p^X v_k^x = k_z M_p^X v_k^x.
\]

Together with the relation \( \text{div}_{\lambda,k} v_k = 0 \) this yields

\[
(i_k D_z^W M_p^W v_k^x) = k_z^2 M_p^X v_k^x + k_z k_x M_p^X v_k^x = |k|^2 M_p^X v_k^x.
\]
From the second line in \( \text{curl}_v M_{v_k} = 0 \) we obtain
\[ D^W_x D^Y_{x} M_{v_k} = k^2 M_{v_k}. \]
Together with (30) we find that \((D^W_x G^Y_{x} D^Y_{x} + [k]^2 G^{-1}_{T}) M_{v_k} = 0 \). Since the matrix on the left-hand side is strictly positive definite, it follows that \( v_k = 0 \). Now it follows from part (ii), (35), (36) and \( k \neq 0 \) that \( v_k = 0 \) and \( v_k = 0 \), completing the proof of (34).

From parts (i) and (iii) we obtain
\[ \mathcal{N}(\text{curl}_{v,k}) = \mathcal{R}(\text{curl}_{v,k}^\#) \subset \mathcal{N}(\text{div}_{v,k}) \]
with orthogonality with respect to the inner product generated by \( G_X \). Together with (34) this implies (33).

**Remark 5.3.** Let us discuss the case \( k = 0 \). We claim that in analogy to the continuous situation we have
\[ \mathcal{N}(\text{curl}_{v,0}) \cap \mathcal{N}(\text{div}_{v,0}) = \{(c_x (M^Y_{v})^{-1}, c_y (M^Y_{v})^{-1}, 0) : c_x, c_y \in \mathbb{C}\}, \]
where \( e \in \mathbb{V} \) is the vector with all entries equal to 1. In fact, for \( v_0 \in \mathcal{N}(\text{curl}_{v,0}) \cap \mathcal{N}(\text{div}_{v,0}) \) it follows from part (ii) and \( \text{div}_{v,0} v_0 = 0 \) that \( v_0 = 0 \). Note that \( \mathcal{N}(D^Y_{v}) = \{e (M^Y_{v})^{-1} : e \in \mathbb{C}\} \). Now (37) follows from \( \text{curl}_{v,0} v_0 = 0 \).

The projection matrices onto \( \mathcal{N}(\text{curl}_{v,k}) \) and \( \mathcal{N}(\text{div}_{v,k}) \) can be computed using a QR-decomposition of \( G^\frac{1}{2}_X \text{curl}_{v,k}^\# \):
\[ G^\frac{1}{2}_X \text{curl}_{v,k}^\# = (Q_k \hat{Q}_k) \begin{pmatrix} R_k \\ 0 \end{pmatrix}, \quad P_k = G^{-\frac{1}{2}}_X Q_k Q_k^* G^\frac{1}{2}_X, \quad k \neq 0. \]  

Here \( R_k \) has full row rank \( p \), \( [Q_k \hat{Q}_k] \) is unitary, and \( Q_k \) has \( p \) columns. We summarize the properties of \( P_k \):

**Lemma 5.4.** Let \( k \neq 0 \). Then \( P_k \) is a projection onto \( \mathcal{N}(\text{div}_{v,k}) \) (i.e. \( P_k^2 = P_k \) and \( \mathcal{R}(P_k) = \mathcal{N}(\text{div}_{v,k}) \), and\( I - P_k \) is a projection onto \( \mathcal{N}(\text{curl}_{v,k}) \). \( P_k \) is orthogonal both with respect to the inner product induced by \( G_X \) (i.e. \( P_k^* G_X = G_X P_k \)) and the semi-definite inner product induced by the (Hermitian) Gram matrix
\[ G_{k,\nu} = G_X \text{curl}_{v,k}^\# G_X \text{curl}_{v,k} - G_X \text{grad}_{v,k} G_X \text{div}_{v,k} \]
(i.e. \( P_k^* G_{k,\nu} = G_{k,\nu} P_k \)).

**Proof.** The identity \( P_k^* G_X = G^\frac{1}{2}_X Q_k Q_k^* G_X^\frac{1}{2} = G_X P_k \) is obvious from the definition. We have \( P_k^2 = G^\frac{1}{2}_X Q_k (Q_k^* Q_k) G^\frac{1}{2}_X = P_k \), so \( P_k \) is a projection, which implies that \( I - P_k \) is a projection as well. Using lemma 5.2, parts (iii) and (i) we obtain
\[ \mathcal{R}(P_k) = \mathcal{R}(G^{-\frac{1}{2}}_X Q_k) = \mathcal{R}(\text{curl}_{v,k}^\#) = \mathcal{N}(\text{div}_{v,k}). \]

Moreover, \( \mathcal{R}(I - P_k) = \mathcal{R}(P_k)^\perp = \mathcal{N}(\text{div}_{v,k})^\perp = \mathcal{N}(\text{curl}_{v,k}^\#) \) using 5.2(iv) and the self-adjointness of \( P_k \) in \( X \). By lemma 5.2(iii) we have \( G^H_{k,\nu} = \text{curl}_{v,k}^\# G^H_{k,\nu} \text{curl}_{v,k} + \text{div}_{v,k} G^H_{k,\nu} \text{div}_{v,k} \), so \( G^H_{k,\nu} \) is Hermitian and positive semi-definite. Moreover, since \( \text{div}_{v,k} P_k = 0 \) we have
\[ P_k^v G_{k,p}^H = (P_k^v G_x^{1/2})(G_x^{1/2} \text{curl}_{j,k}^\#) G_Y \text{curl}_{j,k} = (G_x^{1/2} Q_k Q_k^* (Q_k R_k) G_Y) \text{curl}_{j,k} = G_x^{1/2} (Q_k R_k) G_Y \text{curl}_{j,k} = G_x^{1/2} \text{curl}_{j,k}^\# G_Y \text{curl}_{j,k} = \text{curl}_{j,k}^\# G_x^{1/2} \text{curl}_{j,k}. \]

Since the right-hand side of this equation is Hermitian, so is the left-hand side, which implies \( P_k^v G_{k,p}^H = G_{k,p}^H P_k \).

For \( k = 0 \) we define \( R_k \) as follows:

\[
\begin{pmatrix}
G_x^{1/2} \text{curl}_{j,0}^\#

((M^Y)^{-1} e, 0, 0)

(0, (M^Y)^{-1} e, 0)
\end{pmatrix} = (Q_0 \hat{Q}_0) \begin{pmatrix} R_0 \\ 0 \end{pmatrix}, \quad P_0 := G_x^{1/2} Q_0 Q_0^* G_x^{1/2}. \tag{39}
\]

see remark 5.3. In this case \( P_0 \) is the orthogonal projection onto \( \mathcal{N}(\text{div}_{j,0}) \oplus \{(c_j (M^Y_p)^{-1} e, c_i (M^Y_p)^{-1} e, 0) : c_i, c_j \in \mathbb{C}\} \).

5.3. Implementation of the Pinsker estimator with mass conservation constraint

Let us recall of the definition of the generalized singular value decomposition GSVD (see [38]): let \( A \in \mathbb{R}^{m \times n} \) and \( L \in \mathbb{R}^{r \times q} \) be matrices with \( m \geq n \) and \( \text{rank}(L) = p \). Then there exist unitary matrices \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{r \times q} \) and an invertible matrix \( X \in \mathbb{R}^{n \times n} \) such that

\[
A = USX^{-1} \quad \text{and} \quad L = VCX^{-1}, \tag{40}
\]

where \( S = \text{diag}(s_1, \ldots, s_q) \in \mathbb{R}^{m \times q} \) and \( C = \text{diag}(c_1, \ldots, c_{\text{min}(q,n)}) \in \mathbb{R}^{q \times q} \) with \( 1 \geq c_1 \geq \ldots \geq c_p > c_{p+1} = \ldots = 0 \). The generalized singular values \( \sigma_i \) of \((A, L)\) are \( \sigma_i = s_i/c_j \) for \( i = 1, \ldots, p \), and the generalized right singular vectors of \((A, L)\) are the first \( p \) columns \( x_1, \ldots, x_p \) of \( X \). They satisfy the orthogonality relations \( \lambda_i^A L^* L x_k = c_k^2 \delta_{jk} \) for \( j, k \in \{1, \ldots, n\} \). If \( L = I \) then the GSVD and the SVD coincide (except for that the ordering of the singular values).

We will set \( A := \Lambda_k^{-1/2} K_p P_k \) with \( P_k \) defined in (38) and \( L := \begin{pmatrix} G_x^{1/2} \text{curl}_{j,k}^\# \\ G_x^{1/2} \text{div}_{j,k}^\# \end{pmatrix} \) for \( k \neq 0 \) this yields vectors \( x_1, \ldots, x_{\dim(X_k)}, v_1, \ldots, v_{\dim(X_k)} \) and \( u_1, \ldots, u_p \) and numbers \( s_1, \ldots, s_p, c_1, \ldots, c_p > 0 \) such that

\[
\Lambda_k^{-1/2} K_p x_j = s_j u_j, \quad j = 1, \ldots, p
\]

\[
L x_j = c_j v_j, \quad j = 1, \ldots, \dim(X_k).
\]

For \( j \leq p \) we have

\[
x_j \in \mathcal{N}(\Lambda_k^{-1/2} K_p P_k)^\perp \subset \mathcal{N}(P_k)^\perp = \mathcal{N}(\text{div}_{j,k})
\]

with orthogonality w.r.t. the \( L \)-induced inner product and \( x_j^\# G_{k,p}^H x_j = \|L x_j\|^2 = c_j^2 \). Therefore \( x_j/c_j \) are the singular vectors of \( A \) w.r.t. this inner product, and the \( k \) th Fourier coefficient of the Pinsker estimator is

\[
W_k x_k = \sum_{j=1}^p \frac{1}{\sigma_j} (u_j, \Lambda_k^{-1/2} x_j) = \sum_{j=1}^p \frac{1}{\sigma_j} (u_j, \Lambda_k^{-1/2} x_j) x_j.
\]
Algorithm 1.  Pinsker algorithm with mass conservation constraint.

**Data** • kernels $K_k \in \mathbb{C}^{M \times N}$ and noise covariance matrices $\Lambda_k \in \mathbb{C}^{N \times N}$ for all frequencies $k$;
• regularization parameter $\pi$.

**Result:** linear estimator $W_k$ for all frequencies $k$

set up Gram matrices $G_V$, $G_{UU}$, $G_{US}$, and $G_Y$ (equations (29), (32));

**For** $k \in [-N_k/2, \ldots, N_k/2 - 1]^2$ **do**

- set up matrices $\text{div}_{V,k}$, $\text{curl}_{V,k}$ and $P_k$ (equations (31), (38), (39));

- $[U_k, X_k, V_k, s_k, c_k] = \text{svd}\left(\Lambda_k^{-1/2} K_k P_k \left( \begin{array}{c} G_Y^{1/2} \text{curl}_{V,k} \\ G_Y^{1/2} \text{div}_{V,k} \end{array} \right) \right)$;

- $(U_k, X_k, V_k$ are matrices with columns $u_{k,j}, s_{k,j}, v_{k,j};$

- $s_k, c_k$ are vectors with entries $s_{k,j}, c_{k,j}$);

end

Find bijective ordering $l: \left[\frac{-N_k}{2}, \ldots, \frac{N_k}{2} - 1\right]^2 \times \{1, \ldots, M\} \rightarrow N_k^2 N_k$ such that $\frac{c_{k,j}}{c_{k,l}} \geq \frac{c_{k,j}}{c_{k,l}}$ if $l(k, j) \leq l(k, j), c_{k,j}, c_{k,l} > 0, l(k, j) \geq l(k, j)$ if $c_{k,j} = 0$ and $c_{k,l} > 0$;

**For** $k \in [-N_k/2, \ldots, N_k/2 - 1]^2$ **do**

- $p_k = \max \{ j : c_{k,j} > 0 \}$;

- For $j = 1, \ldots, p_k$ **do**

  - $a_{k,j} := l(k, j)^{1/3}$;

  - $\lambda_{k,j} := \max (1 - \pi a_{k,j}, 0)$;

end

- $W_k = X_k (:, 1: p_k) \text{diag}(\frac{\lambda_{k,1}}{a_{k,1}}, \ldots, \frac{\lambda_{k,p_k}}{a_{k,p_k}}) U_k (:, 1: p_k) \Lambda_k^{-1/2}$;

end

6. Numerical results

In the following we will compare RLS, SOLA and Pinsker methods for recovering three-dimensional velocity fields from travel time measurements on the solar surface.

To compare the different inversion methods on synthetic data, we use the velocity model presented in [11] which reproduces an average supergranule. Supergranulation is a convection pattern with an average life time of about 1 d and a characteristic length of around 30 Mm that is observed at the surface of the Sun. A representation of the velocity field $v^{x}$ and $v^{z}$ is given in figures 4 and 6 (top rows). This velocity is built such that mass is conserved, which explains the decrease of the amplitude with depth due to the strong density gradient. These velocities are then convolved with the kernels, and noise is added according to (1) in order to obtain travel time maps as shown in figure 2.
6.1. Reconstruction without mass conservation

In the RLS method we have chosen the regularization term as $H^1$ norm in horizontal and vertical directions, and in the Pinsker method the ellipsoid $\Theta$ was chosen according to (25) to approximate a ball in the Sobolev space $H^1(V)$. The regularization parameters $\alpha$ and $\pi$ have been chosen by the discrepancy principle. Although the discrepancy principle performs poorly for high dimensional white noise (and is not even well-defined in the infinite-dimensional case), here the noise is sufficiently correlated for the discrepancy principle to work reasonably well. The SOLA weighting kernels are obtained by minimizing (10) with a target function.

Figure 4. Vertical velocities $v^z(x, y, z_t)$ in m s$^{-1}$ of a supergranule model from [11] (top) and their reconstructions with RLS (2nd row), SOLA (3rd row) and Pinsker (bottom) at three different depths $z_t \in [-0.9 \text{ Mm}, -3.5 \text{ Mm}, -5.5 \text{ Mm}]$. The circles at radii 10 Mm and 20 Mm represent zero level lines of the exact solution.
where \( s_h \) and \( s_v \) determine the localization of the averaging kernels in the horizontal and vertical directions. As usual we added a constraint for \( k = 0 \) via Lagrange multipliers to ensure that the integrals over the averaging kernels for horizontal velocities are 1. This is not possible for the vertical velocities since constant vertical flows are in the nullspace of \( K \). To allow a fair comparison with RLS and Pinsker, we did not impose a strong additional penalty to suppress cross-talk as in [32] since we found that this induces a significant loss of resolution.

It is well-known in helioseismology that RLS and SOLA can reconstruct horizontal velocity components \( v^x, v^y \) fairly well, but perform poorly for the reconstruction of vertical velocity components. Figure 4 shows the reconstruction of \( v^z \) for the different methods without mass conservation. As expected, the results for Tikhonov regularization are poor except close to the surface. The SOLA method is a bit better at larger depths and the Pinsker estimator leads to a clear improvement with almost correct reconstructions at \( z = -3.5 \) Mm and a detection of a positive value of the velocity close to the center at \( z = -5.5 \) Mm. However, the amplitudes of the reconstructed velocities both at \(-3.5 \) Mm and in particular at \(-5.5 \) Mm are too small.

To understand the difficulties of RLS with the reconstruction of \( v^z \), we look at the depth localisation of the averaging kernels. To compare the different estimators \( W^\beta \) for some velocity component \( v^\beta, \beta \in \{x, y, z\} \), we choose the parameters in these methods such that the variance \( \mathbb{E}[|\mathbb{E}(W^\beta_n(r, z_j))|^2] \) at the target depth \( z_j \) has the same value for all the methods. (Due to translation invariance of the noise covariance structure this value is independent of \( r \).) Then we compare the corresponding averaging kernels \( K^\beta_{z_j; \alpha; z_f} \) describing the bias (see definition 3.1). In figure 5, we represented the horizontal \( L^2 \) norm of \( K^\beta_{z_j; \alpha; z_f} \) as a function of the depth \( z_j \) for RLS, SOLA and Pinsker methods at two different target depths \( z_f \). One can see that the averaging kernel for the RLS method is mostly localized close to the surface rather than at the target depth. In contrast, the averaging kernel of Pinsker is much better.
localized at $z_t$, but still exhibits some sensitivity to the values close to the surface. Intermediately, the SOLA averaging kernel is localized at the correct depth, but is extremely broad, so the reconstruction of $v_z$ at the target depth $z_t$ is greatly influenced by the other depths.

The reconstructions of the horizontal velocity $v^x$ by Tikhonov, SOLA and Pinsker methods are shown in figure 6. As expected, all methods perform well. Surprisingly, from visual inspection Pinsker seems slightly less accurate than Tikhonov regularization at $z_t = -5.5 \text{ Mm}$. 

Figure 6. Horizontal velocities $v^y(x, y, z)$ in m s$^{-1}$ of a supergranule model from [11] (top) and their reconstruction with different methods: RLS, SOLA and Pinsker (from top to bottom) at three different depths $z_t = -0.9 \text{ Mm}$ (left), $-3.5 \text{ Mm}$ (middle), and $-5.5 \text{ Mm}$ (right). The circles at 20 Mm and 30 Mm indicate the zero level lines of the exact horizontal velocity component.
To get a better insight into the reconstructions, we can again look at the averaging kernels with the same choice of parameters as described above. Figure 7 shows $K(x, z_j; x_0)$ (characterizing the $v^i$ influence on the bias of the $v^i$ estimators) as functions of $x$ and $z_j$ for target depths $z_t \in \{-0.9 \text{ Mm}, -3.5 \text{ Mm}, -5.5 \text{ Mm}\}$ for the estimators RLS, SOLA, and Pinsker. The variances of these estimators for each target depth are chosen to be of the same size.

![Figure 7. Averaging kernels $K(x, z_j; x_0)$ (characterizing the $v^i$ influence on the bias of the $v^i$ estimators) as functions of $x$ and $z_j$ for three different depths $z_t \in \{-0.9 \text{ Mm}, -3.5 \text{ Mm}, -5.5 \text{ Mm}\}$ for the estimators RLS, SOLA, and Pinsker. The variances of these estimators for each target depth are chosen to be of the same size.](image)

The differences between the three methods are the more pronounced the greater the target depth $z_t$, i.e. the greater the ill-posedness. The Pinsker averaging kernels turn out to be the most localized, in particular in $z$ direction while the SOLA averaging kernels are the least localized. RLS and Pinsker produce similar averaging kernels for the $v^i$ estimators, which is consistent with the observed reconstructions. However, it is surprising that the reconstruction with the Pinsker method is not the best at $-5.5 \text{ Mm}$ as the averaging kernels are the most

...
Cross-talk averaging kernels $K^{\xi \rightarrow z \rightarrow y}$ and $K^{\xi \rightarrow z \rightarrow x}$ for $z_i = -3.5$ Mm for the RLS, SOLA, and Pinsker methods. These kernels characterize the influence of the variables $v^y$ and $v^x$ on the bias of the $v^y$-estimators.
localized. To explain this apparent inconsistency, we need to look at the cross-talk, i.e. how \( v^y \) and \( v^z \) influence the estimator of \( v^x \). Figure 8 shows the averaging kernels \( K^{x,z,y,z} \) and \( K^{x,z,z,y} \) at a target depth of \( z_t = -3.5 \) Mm. The cross talk is rather strong for Pinsker where the maximum value of the off-diagonal averaging kernels is only 50% smaller than the maximum \( K^{x,z,z,y} \), as opposed to around 10% for RLS and 5% for SOLA.

6.2. Incorporation of the divergence constraint

Figure 9 shows the reconstruction of the vertical component of the velocity for the Tikhonov and Pinsker methods with mass conservation constraint. It underlines the importance of incorporating the constraint into the inversion process. The vertical velocity is now properly reconstructed by both methods.

To better compare all the methods, figure 10 represents a cut of the vertical velocity at \( x = 0 \) and \( z_t \in \{-3.5 \) Mm, \(-5.5 \) Mm\}. Incorporating mass conservation into Tikhonov leads to a quite good reconstruction with an amplitude of about 70% of the true one. Finally, Pinsker with mass conservation is almost perfect at the depths up to \(-5.5 \) Mm with correct shape and amplitude.

Finally, we also study averaging kernels. Note that in the case of divergence constraints we have to think again about the definition of such kernels as \( \delta \)-peaks are not divergence free. We redefine the Fourier coefficients of the averaging kernel as

\[
K_{k,\text{div}} = (I - P_k) + P_k K_k P_k,
\]

where \( P_k \) denotes the \( L^2 \)-orthogonal projection on the nullspace of \( \text{div}_a \). This type of kernel still characterizes the bias of regularization methods if they are applied to solutions satisfying
the mass conservation constraint and if $P_k$ is applied as a postprocessing step. Note that this definition implies that the Fourier coefficients of the averaging kernel are non-zero even at high frequencies due to the identity term. Thus, these averaging kernels cannot be directly compared to the ones of section 3 (and thus to the ones classically used in helioseismology), but their definition using (41) is natural as the convolution of $K_{\text{div}}'$ with the velocities characterizes the bias of the method.

In figure 11 we plot at each voxel the Frobenius norm of the $3 \times 3$ matrix of averaging kernels $K_{\text{div}}'$ for the Pinsker method with mass conservation at three different depths $z_t \in \{-0.9 \text{ Mm}, -3.5 \text{ Mm}, -5.5 \text{ Mm}\}$.

Figure 10. Comparison of the different methods to reconstruct the vertical velocity $v_z(x, 0, z_t)$ at $z_t = -3.5 \text{ Mm}$ (left) and $z_t = -5.5 \text{ Mm}$ (right). The Pinsker method with mass conservation provides the best reconstruction.

Figure 11. Pointwise Frobenius norm of the $3 \times 3$ averaging kernels $K_{\text{div}}'$ for the Pinsker method with mass conservation at three different depths $z_t \in \{-0.9 \text{ Mm}, -3.5 \text{ Mm}, -5.5 \text{ Mm}\}$.

7. Conclusions

We have shown that Pinsker estimators yield significantly better reconstructions of vertical velocities from travel time maps than Tikhonov regularization, and is also superior to Subtractive OLA. This is consistent with theoretical optimality properties of these estimators.
However, as soon as depth inversion is involved, no simple, precise characterization of the ellipsoids on which Pinsker method is optimal is available. This is the usual situation for all spectral regularization methods such as Tikhonov regularization, Showalter’s method, Landweber iteration, and many others in a deterministic context. As opposed to many other real-world problems, Pinsker estimators are computationally efficient and easy to implement in the context of local helioseismology.

The mass conservation constraint can be incorporated naturally into the Pinsker estimator leading to another significant improvement of accuracy and resolution. Under realistic noise levels this yields reliable estimators of vertical velocity components up to a depth of \(-5.5\) Mm using travel times from \(f\) and \(p_1\) to \(p_4\) modes.

Alternatively, one may study an adaptive, data-driven choice of the size of the ellipsoid in the Pinsker method, which may be interpreted as a regularization parameter. We plan to address this as well as the application to real data in future work.

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Appendix

This appendix contains the proof of lemma 5.1.

Proof. We make the substitutions \(p = \rho v\) and \(q = \rho w\). To prove (i), note that

\[
\sum_{\beta \in \{x,y,z\}} \langle \text{grad } p^\beta, \text{grad } q^\beta \rangle_{L^2(V)^d} = \sum_{\beta, \gamma \in \{x,y,z\}} \frac{\partial p^\beta}{\partial \gamma} \frac{\partial q^\beta}{\partial \gamma}
\]

and

\[
\langle \text{curl } p, \text{curl } q \rangle_{L^2(V)^d} + \langle \text{div } p, \text{div } q \rangle_{L^2(V)}
\]

\[= \sum_{\beta, \gamma \in \{x,y,z\}} \int_V \frac{\partial p^\beta}{\partial \gamma} \frac{\partial q^\beta}{\partial \gamma} \, dx + \sum_{\beta, \gamma \in \{x,y,z\}} \left[ \frac{\partial p^\beta}{\partial \beta} \frac{\partial q^\beta}{\partial \gamma} - \frac{\partial p^\beta}{\partial \gamma} \frac{\partial q^\beta}{\partial \beta} \right] \, dx.
\]

For all terms in the second sum (coming among other terms from the curl part) we can perform partial integrations without boundary terms to see that these terms vanish. (Note that this would not work without the Dirichlet boundary conditions for the z-components, e.g. for \(\beta = x\) and \(\gamma = z\).) Therefore, the left-hand sides of the last two equations are equal.

Part (ii): as

\[
\|p\|_{H^1(V)^d} = \sum_{\beta \in \{x,y,z\}} \|p^\beta\|^2_{L^2(V)} + \|\text{grad } p^\beta\|^2_{L^2(V)^d}
\]

the second inequality in (28) follows from (27) and the Cauchy–Schwarz inequality

\[
\int_V p^\beta d(r, z) \leq \|p^\beta\|_{L^2(V)^d}^{1/2} \|V\|^{1/2} \text{ for } \beta \in \{x, y\}.
\]
To prove the first inequality in (28) it suffices to show that there exists a constant $C \geq 0$ such that
$$\|p|^\beta|_{L^2} \leq C \|\text{grad } p|^\beta|_{L^2} + \frac{C(1 - \delta_{L^2})}{|V|} \int_V |p|^\beta d(r, z)$$
for all $\beta \in \{x, y, z\}$

and all $p \in \mathcal{X}$. For $\beta = z$ this follows from the Poincaré inequality due to the Dirichlet boundary conditions, and for $\beta \in \{x, y\}$ it is a consequence of the Poincaré–Wirtinger inequality.

Part (iii): to show orthogonality w.r.t. the inner product $\langle \rho v, \rho w \rangle_{L^2(V)}$, let $v \in \mathcal{N}_0(\text{curl}_\rho)$. Then by potential theorems (see e.g. [29, theorem 3.37]) we have $\rho v = \text{grad } f$ for some $f \in H^1(V)$. It follows by partial integration that $\langle \rho v, \rho w \rangle_{L^2(V)} = \int_V \text{div}_r w \, dx = 0$ for all $w \in \mathcal{N}_0(\text{div}_\rho)$ where the boundary terms vanish due to the boundary conditions. Hence, $v \perp \mathcal{N}_0(\text{div}_\rho)$ w.r.t. the weighted $L^2$ inner product. Together with (27) we also obtain orthogonality w.r.t. the $\mathcal{X}$ inner product.

Let $v \in \mathcal{X}_0$ satisfy $\langle \rho v, \rho w \rangle_{L^2(V)} = 0$ for all $w \in \mathcal{N}_0(\text{div}_\rho)$ and all $w \in \mathcal{N}_0(\text{curl}_\rho)$. We aim to show that $v = 0$. Since $\text{div}_r(\rho^{-1} \text{curl } g) = 0$ and $\text{curl}_r(\rho^{-1} \text{grad } f) = 0$ for all smooth $f$ and $g$ vanishing at the boundaries, we may choose $w = \rho^{-1} \text{curl } g$ or $w = \rho^{-1} \text{grad } f$ and perform partial integrations to obtain $\langle \text{curl}_r v, g \rangle_{L^2(V)} = 0$ and $\langle \text{div}_r v, f \rangle = 0$. Therefore $\text{curl}_r v = 0$ and $\text{div}_r v = 0$, and so $v = 0$ from part (ii). This shows that the sum of the nullspaces is dense in $L^2(V)$. Since the nullspaces are closed and orthogonal in $\mathcal{X}_0$, their sum equals $\mathcal{X}_0$.

\[\square\]

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