MASS, CAPACITARY FUNCTIONS, AND POSITIVE MASS THEOREMS WITH OR WITHOUT BOUNDARY

PENGZI MIAO

Abstract. We evaluate the three dimensional ADM mass via positive harmonic functions tending to zero at infinity. The resulting mass formulae reveal a family of monotone quantities and geometric inequalities associated to the harmonic function if the underlying manifold has simple topology and nonnegative scalar curvature.

We give three main applications. First, a few additional proofs of the three dimensional Riemannian positive mass theorem are observed. One proof leads to a positive mass theorem on manifolds with boundary, which asserts if a region enclosing the boundary does not have excess volume, then the mass is positive. We also find integral identities for the mass-to-capacity ratio. In another application, we use the inequalities to promote themselves so that they become equality on spatial Schwarzschild manifolds outside rotationally symmetric spheres.

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1. Introduction

There have been intriguing recent developments in finding monotone quantities associated to harmonic functions on 3-manifolds with nonnegative scalar curvature. In [18], Munteanu and Wang established sharp comparison results on complete, nonparabolic 3-manifolds via the discovery of a monotone quantity along level sets of the minimal positive Green’s function. In [2], Agostiniani, Mazzieri and Oronzio obtained a new proof of the Riemannian positive mass theorem through a different monotone quantity along level sets of the Green’s function on asymptotically flat 3-manifolds.

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On asymptotically flat 3-manifolds, another recent proof of the positive mass theorem was due to Bray, Kazaras, Khuri and Stern [6]. They found the mass can be detected by harmonic functions that are asymptotic to coordinate functions, and applied it to obtain an integral formula relating the mass and the scalar curvature.

In this paper, we consider harmonic functions $u$ on an asymptotically flat 3-manifold $(M, g)$ satisfying

$$u(x) = 1 - c|x|^{-1} + o(|x|^{-1}), \text{ as } x \to \infty$$

for some constant $c > 0$. We derive formulae that detect the mass of $(M, g)$ via the level sets of $u$, see Theorem 2.1. In particular, Theorem 2.1 (ii) shows

$$(1.1) \lim_{t \to 1} \frac{1}{1-t} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] = 4\pi \, m \, c^{-1}.$$ 

Here $\Sigma_t = u^{-1}(t)$ and $m$ is the mass of $(M, g)$.

Besides (1.1), in Theorem 2.1 (i), we find

$$(1.2) \lim_{t \to 1} \frac{1}{1-t} \left[ 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \right] = 12\pi \, m \, c^{-1}.$$ 

Here $H$ denotes the mean curvature of a regular level set $\Sigma_t$ with respect to $|\nabla u|^{-1} \nabla u$.

(1.1) and (1.2) suggest, as $t \to 1$,

$$8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| = 3 \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] + o((1-t)).$$

While this was observed only from information near $\infty$, we show in Theorem 3.1 that, if $M$ has simple topology and $g$ has nonnegative scalar curvature, then at each regular level set $\Sigma_t$,

$$(1.3) 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \leq 3 \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right],$$

and “=” holds if and only if $(M, g)$ outside $\Sigma_t$ is isometric to $\mathbb{R}^3$ minus a round ball.

Inequality (1.3) is derived via a monotone quantity along $\{\Sigma_t\}$ (see Lemma 3.1). Among other things, we apply (1.3) to find that the quantities in the mass formulae (1.1) and (1.2) are actually monotone non-decreasing, that is

$$(1.4) A(t) := \frac{1}{1-t} \left[ 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \right] \nearrow \text{ as } t \nearrow,$$

and

$$B(t) := \frac{1}{1-t} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \nearrow \text{ as } t \nearrow,$$

see Theorem 3.2. Moreover, in Theorem 3.2 we show that, if $u = 0$ at $\Sigma = \partial M$, then

$$(1.6) 8\pi - \int_{\Sigma} H|\nabla u| \leq 12\pi \, m \, c^{-1}.$$
and

\begin{equation}
4\pi - \int_{\Sigma} |\nabla u|^2 \leq 4\pi m c_{\Sigma}^{-1},
\end{equation}

where $c_{\Sigma}$ is the capacity of $\Sigma$ in $(M,g)$. Furthermore, “=” holds in any of these inequalities if and only if $(M,g)$ is isometric to $\mathbb{R}^3$ minus a round ball.

As an immediate application of (1.1) – (1.7), we observe several new arguments that prove the 3-dimensional Riemannian positive mass theorem which was originally proved by Schoen and Yau \[19\], and by Witten \[22\]. We discuss these in detail in Section 4.

Inequalities (1.6) and (1.7) also give rise to boundary conditions that imply the positivity of the mass. In Theorem 5.1 we show that if $M$ has simply topology and $g$ has nonnegative scalar curvature, then

\begin{equation}
H \leq \frac{8\pi L^2}{\text{Vol}(\Omega)} \implies m > 0.
\end{equation}

Here $\Omega$ is a region enclosed by a surface $S$ with the boundary $\Sigma$, $L$ is the distance from $S$ to $\Sigma$, and $\text{Vol}(\Omega)$ is the volume of $(\Omega, g)$. Such a result indicates, under the scalar curvature assumption, if a region enclosing the boundary does not have excess volume, then the mass is positive.

In \[2\], Agostiniani, Mazzieri and Oronzio showed, along $\{\Sigma_t\}$,

\begin{equation}
F(t) := \frac{1}{1-t} \left[ 4\pi - \frac{1}{1-t} \int_{\Sigma_t} H |\nabla u| + \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \not\to \text{ as } t \not\to \cdot
\end{equation}

We observe that $\mathcal{A}(t), \mathcal{B}(t)$ and $F(t)$ are related by

\begin{equation}
F(t) = \mathcal{A}(t) - \mathcal{B}(t).
\end{equation}

In \[A.14\] and \[A.15\] of the Appendix, we give integral identities for the differences

$\mathcal{B}(t_2) - \mathcal{B}(t_1), \mathcal{A}(t_2) - \mathcal{A}(t_1)$, for $t_1 < t_2$.

The monotonicity of $F(t)$ also follows from \[A.10\], \[A.14\] and \[A.15\]. Moreover, as a corollary of \[2.10\], \[2.11\] and \[2.2\], we have

\begin{equation}
\lim_{t \to 1} F(t) = 8\pi m c_{\Sigma}^{-1}.
\end{equation}

This limit was demonstrated in \[2\] in the case that $(M, g)$ is isometric to a spatial Schwarzschild manifold near infinity.

Making use of the limits of $\mathcal{A}(t), \mathcal{B}(t)$ as $t \to 1$ and the formulae of their differences at $t_1 < t_2$, we also derive integral identities for the mass-to-capacity ratio $m c_{\Sigma}^{-1}$ in Theorem 6.1. Such integral identities can be compared with the mass identity obtained by Bray, Kazarakas, Khuri and Stern \[6\] via harmonic functions having linear asymptotic.

In \[5\], Bray found a mass-capacity inequality for manifolds with minimal surface boundary, using the positive mass theorem. Inspired by Bray’s work, in Section 7 we apply inequalities (1.3), (1.6) and (1.7) to obtain inequalities that become equalities
in spatial Schwarzschild spaces. Among other things, we show in Corollary 7.1 that under the previous assumptions on \((M, g)\),

\[
(1.12) \quad \left( \frac{1}{\pi} \int_\Sigma |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}} + 1,
\]

where \(u\) is the harmonic function with \(u = 0\) at \(\Sigma\) and \(u \to 1\) at \(\infty\). Moreover, equality holds if and only if \((M, g)\) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

In Theorem 7.3, we show, given the same triple \((M, g, u)\),

\[
(1.13) \quad \frac{1}{2} \left( mc^{-1}_\Sigma \right) \geq 1 - \left( \frac{1}{4\pi} \int_\Sigma |\nabla u|^2 d\sigma \right)^{\frac{1}{2}},
\]

and equality holds if and only if \((M, g)\) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere. It follows from (1.12) and (1.13) that

\[
(1.14) \quad mc^{-1}_\Sigma \geq 1 - \left( \frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}},
\]

regardless of the mean curvature of \(\Sigma\) in \((M, g)\).

We finish this paper with an appendix, including regularization arguments that can be used to verify various monotonicity in Section 3.

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### 2. Detecting the Mass at \(\infty\)

Let \((M, g)\) denote an asymptotically flat 3-manifold (with one end) with boundary. By this, we mean there is a compact set \(K \subset M\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3\) minus a ball and, with respect to the standard coordinates on \(\mathbb{R}^3\), \(g\) satisfies

\[
(2.1) \quad g_{ij} = \delta_{ij} + O(|x|^{-\tau}), \quad \partial g_{ij} = O(|x|^{-\tau-1}), \quad \partial^2 g_{ij} = O(|x|^{-\tau-2})
\]

for some constant \(\tau > \frac{1}{2}\). The scalar curvature \(R\) of \(g\) is also assumed to be integrable so that the ADM mass \([3]\) of \((M, g)\), which we denote by \(m\), exists. By the results in \([4, 9]\), \(m\) is independent of coordinates satisfying (2.1).

Let \(\Sigma\) denote the boundary of \(M\). Let \(u\) be the function on \((M, g)\) given by

\[
(2.2) \quad \Delta u = 0 \text{ on } M, \quad u = 0 \text{ at } \Sigma, \quad \text{and } u \to 1 \text{ at } \infty.
\]

(The function \(\phi = 1 - u\) is often referred as the capacitary potential associated to \(\Sigma\).) Given any \(t \in [0, 1]\), let \(\Sigma_t = \{ x \in M \mid u(x) = t \}\) denote the level set of \(u\). Below, we collect some basic facts about \(u\) and \(\Sigma_t\).

By the maximum principle, \(\max_K u < 1\), hence \(|x|\) is defined on \(\Sigma_t\) for \(t\) close to 1; moreover, \(\min_{\Sigma_t} |x| \to \infty\) as \(t \to 1\). Now suppose \(\tau \in \left(\frac{3}{2}, 1\right)\). As \(x \to \infty\), it is known...
$u$ has an asymptotic expansion (see Lemma A.2 in [17] for instance)

$$u = 1 - c_\Sigma |x|^{-1} + O_2(|x|^{-1-\tau}).$$

Here $c_\Sigma > 0$ is a positive constant known as the capacity of $\Sigma$ in $(M,g)$. Let $\nabla$ and $\nabla^2 u$ denote the gradient and the Hessian on $(M,g)$, respectively. By (2.3),

$$|\nabla u|^2 = c_\Sigma^2 |x|^{-4} + O(|x|^{-4-\tau}),$$

(2.4)

$$(\nabla^2 u)_{ij} = c_\Sigma |x|^{-3} (-3|x|^{-2} x_i x_j + \delta_{ij}) + O(|x|^{-3-\tau}).$$

(2.5)

Thus, $t$ is a regular value if $t$ is close to 1 and the mean curvature $H$ of $\Sigma_t$ satisfies

$$H = \text{div} (|\nabla u|^{-1} \nabla u) = 2|x|^{-1} + O(|x|^{-1-\tau}).$$

(2.6)

As a result, for $t$ close to 1, $\Sigma_t$ has positive mean curvature and $\Sigma_t$ is area outer-minimizing as its exterior in $M$ is foliated by mean-convex surfaces $\{\Sigma_s\}_{s>t}$.

**Lemma 2.1.** Let $|\Sigma_t|$ be the area of $\Sigma_t$ in $(M,g)$ if $t$ is a regular value of $u$. Then, as $t \to 1$,

$$|\Sigma_t| = 4\pi c_\Sigma^2 (1-t)^{-2} + O((1-t)^{\tau-2}).$$

(2.7)

**Proof.** By (2.3), as $t \to 1$,

$$|x| = c_\Sigma (1-t)^{-1} + O((1-t)^{-\tau-1}).$$

(2.8)

Let $r_-(t) = \min_{\Sigma_t} |x|$ and $r_+(t) = \max_{\Sigma_t} |x|$. Since $\Sigma_t$ and the coordinate sphere $S_r := \{|x| = r\}$ are both area outer-minimizing in $(M,g)$, for $t$ close to 1 and large $r$, respectively, we have

$$|S_{r_-(t)}| \leq |\Sigma_t| \leq |S_{r_+(t)}|.\]$$

(2.9)

For large $r$, (2.1) implies

$$|S_r| = 4\pi r^2 + O(r^{2-\tau}).$$

(2.10)

Thus, (2.7) follows from (2.8) – (2.10).

**Lemma 2.2.** As $t \to 1$,

$$\frac{1}{1-t} \int_{\Sigma_t} H |\nabla u| = 8\pi + O((1-t)\tau)$$

and

$$\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 = 4\pi + O((1-t)\tau).$$

**Proof.** By (2.6) and (2.8),

$$H = 2c_\Sigma^{-1} (1-t) + O((1-t)^{1+\tau}).$$

(2.11)

Therefore, using the fact $\int_{\Sigma_t} |\nabla u| = 4\pi c_\Sigma$, one has

$$\left( \frac{1}{1-t} \int_{\Sigma_t} H |\nabla u| \right) - 8\pi = \int_{\Sigma_t} \left( H \left( \frac{1}{1-t} - 2 c_\Sigma \right) \right) |\nabla u| = O((1-t)\tau).$$
Similarly, by (2.4) and (2.8),
\begin{equation}
|\nabla u| = c_{\Sigma}^{-1}(1 - t)^2 + O((1 - t)^{2+\tau}).
\end{equation}

Therefore,
\begin{equation}
\left(\frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right) - 4\pi = \left(\int_{\Sigma_t} \frac{|\nabla u|}{(1 - t)^2} - \frac{1}{c_{\Sigma}}\right)|\nabla u| = O((1 - t)^\tau).
\end{equation}

\[\square\]

**Lemma 2.3.** As \( t \to 1 \), the gradient of \( |\nabla u| \) on \( \Sigma_t \) satisfies
\begin{equation}
|\nabla_{\Sigma_t} |\nabla u|| = O(|x|^{-3-\tau}).
\end{equation}

**Proof.** Write \( \nabla u = (\nabla u)^i \partial_i \). By (2.3),
\begin{equation}
(\nabla u)^j = c_{\Sigma} |x|^{-2}|x|^{-1}x_j + O(|x|^{-2-\tau}).
\end{equation}

Let \( V = V^i \partial_i \) denote any unit vector tangent to \( \Sigma_t \). Then \( V^i = O(1) \) and the fact \( \langle V, \nabla u \rangle = 0 \) shows
\begin{equation}
\sum_i V^i (\nabla u)^i = O(|x|^{-2-\tau}), \text{ and hence } \sum_i V^i x_i = O(|x|^{1-\tau}).
\end{equation}

Therefore, by (2.3) and (2.14),
\begin{equation}
V(|\nabla u|^2) = 2(\nabla^2 u)(V, \nabla u)
\end{equation}
\begin{equation}
= 2c_{\Sigma} |x|^{-3} \left[ -3|x|^{-2}x_i V^i x_j (\nabla u)^j + \delta_{ij} V^i (\nabla u)^j \right] (1 + O(|x|^{-\tau}))
\end{equation}
\begin{equation}
= O(|x|^{-5-\tau}).
\end{equation}

Thus, (2.13) follows from (2.15) and (2.4). \[\square\]

**Lemma 2.4.** As \( t \to 1 \), the traceless part of the second fundamental form \( \mathbb{II} \) of \( \Sigma_t \), denoted by \( \mathbb{II} \), satisfies
\begin{equation}
|\mathbb{II}| = O(|x|^{-1-\tau}),\end{equation}
and the Gauss curvature \( K \) of \( \Sigma_t \) satisfies \( K = |x|^{-2} + O(|x|^{-2-\tau}) \).

**Proof.** Let \( V = V^i \partial_i \) and \( W = W^j \partial_j \) be any two unit vectors tangent to \( \Sigma_t \) at a given point. Then \( \delta_{ij} V^i W^j = g(V, W) + O(|x|^{-\tau}) \). As \( \nabla^2 u(V, W) = |\nabla u| \mathbb{II}(V, W) \), one has
\begin{equation}
|\nabla u| \mathbb{II}(V, W) = (\nabla^2 u)_{ij} V^i W^j
\end{equation}
\begin{equation}
= c_{\Sigma} |x|^{-3} \left( -3|x|^{-2}x_i V^i x_j W^j + \delta_{ij} V^i W^j \right) (1 + O(|x|^{-\tau}))
\end{equation}
\begin{equation}
= c_{\Sigma} |x|^{-3} g(V, W) + O(|x|^{-3-\tau}),
\end{equation}
where one used (2.5) and (2.14). Therefore, by (2.4),
\begin{equation}
\mathbb{II}(V, W) = |x|^{-1} g(V, W) + O(|x|^{-1-\tau}).
\end{equation}

This combined with (2.6) shows
\begin{equation}
\mathbb{II}(V, W) = \mathbb{II}(V, W) - \frac{1}{2} H g(V, W) = O(|x|^{-1-\tau}),
\end{equation}
\[\square\]
which proves (2.16). The conclusion on the Gauss curvature follows from (2.18), (2.6) and the Gauss equation.

Lemma 2.5. If \((M,g)\) satisfies \(\partial\partial g_{ij} = O(|x|^{-3-\tau})\) in (2.1), then
\[
|D \mathcal{H}| = O(|x|^{-2-\tau}). \tag{2.19}
\]

Here \(D\) denote covariant differentiation on \(\Sigma_t\).

Proof. If \(g\) satisfies the higher order derivatives decay assumption, then \(u\) satisfies
\[
u = 1 - c|\Sigma|^{-1} + O_{3}(1 - 1 - \tau)
\]
(see the proof of Lemma A.2 in [17] for instance). The terms \(O(|x|^{-3-\tau})\) in (2.3) and \(O(|x|^{-1-\tau})\) in (2.6) are then replaced by \(O_{1}(1 - 3 - \tau)\) and \(O_{1}(1 - 1 - \tau)\), respectively.

To prove (2.19), let \(\{V_{\alpha}\}_{\alpha=1,2}\) be a local orthonormal frame around a given point \(p\) on \(\Sigma_t\). By definition,
\[
(D_{V_{\mu}} \mathcal{H})(V_{\alpha}, V_{\beta}) = (D_{V_{\mu}} \mathcal{H})(V_{\alpha}, V_{\beta}) - \frac{1}{2} V_{\mu}(H) \delta_{\alpha\beta}.
\]
By (2.6) and (2.14),
\[
V_{\mu}(H) = 2(-1)|x|^{-2}V_{\mu}(1) + O(|x|^{-2-\tau}) = O(|x|^{-2-\tau}).
\]

To estimate \(D\mathcal{H}\), one may assume \(\{V_{\alpha}\}\) is normal at \(p\) so that
\[
(D_{V_{\mu}} \mathcal{H})(V_{\alpha}, V_{\beta}) = V_{\mu}(\mathcal{H}(V_{\alpha}, V_{\beta})) = V_{\mu}(\nabla u)^{-1} \left(\nabla^2 u\right)_{\alpha\beta} + |\nabla u|^{-1} V_{\mu} \left(\nabla^2 u\right)_{\alpha\beta}.
\]
By (2.13) and (2.17),
\[
V_{\mu}(\nabla u)^{-1} \left(\nabla^2 u\right)_{\alpha\beta} = O(|x|^{-2-\tau}).
\]
By (2.17) and (2.14),
\[
|\nabla u|^{-1} V_{\mu} \left(\nabla^2 u\right)_{\alpha\beta} = |\nabla u|^{-1} O(|x|^{-4+\tau}) = O(|x|^{-2+\tau}).
\]
Thus, \((D_{V_{\mu}} \mathcal{H})(V_{\alpha}, V_{\beta}) = O(|x|^{-2+\tau})\). This proves (2.19).

Let \(m_{\mu}(\Sigma_t)\) denote the Hawking mass [11] of \(\Sigma_t\) if \(t\) is a regular value of \(u\). That is
\[
m_{\mu}(\Sigma_t) = \frac{r_t}{2} \left(1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2\right).
\]

Here \(r_t = \sqrt{\frac{\Sigma_{\tau}}{4\pi}}\) is the area radius of \(\Sigma_t\). By Lemma 2.1
\[
r_t = c_{\Sigma}(1 - t)^{-1} + O((1 - t)^{\tau-1}). \tag{2.21}
\]

Proposition 2.1. If \(\lim_{t\to 1} m_{\mu}(\Sigma_t) = m\), where \(m\) is the mass of \((M,g)\), then
\[
\lim_{t\to 1} \frac{1}{1 - t} \left[\frac{1}{8\pi} - \frac{1}{1 - t} \int_{\Sigma_t} H |\nabla u|\right] = 12\pi m c_{\Sigma}^{-1} \tag{2.22}
\]
and
\[
\lim_{t\to 1} \frac{1}{1 - t} \left[\frac{1}{4\pi} - \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u| \right] = 4\pi m c_{\Sigma}^{-1}. \tag{2.23}
\]
Proof. For regular values $t$, define

$$A(t) = 8\pi - \frac{1}{1 - t} \int_{\Sigma_t} H|\nabla u|.$$  

Then

$$-A'(t) = \frac{1}{1 - t} \left[ -A(t) + 8\pi + \left( \int_{\Sigma_t} H|\nabla u| \right)' \right].$$

Direct calculation gives

$$\left( \int_{\Sigma_t} H|\nabla u| \right)' = \int_{\Sigma_t} H'|\nabla u| + H|\nabla u'| + H|\nabla u|H|\nabla u|^{-1}$$

$$= \int_{\Sigma_t} -|\nabla u'|^2|\nabla_{\Sigma_t} |\nabla u||^2 + K - \frac{3}{4}H^2 - \frac{1}{2}|\mathbb{II}|^2,$$

where $|\nabla u'| = -H$ and $H' = -\Delta_{\Sigma_t}|\nabla u|^{-1} - (\text{Ric}(\nu, \nu) + |\mathbb{II}|^2)|\nabla u|^{-1}$.

By (2.20) and the Gauss-Bonnet theorem,

$$8\pi + \left( \int_{\Sigma_t} H|\nabla u| \right)' = \frac{24\pi m_{\mu}(\Sigma_t)}{r_t} - E(t),$$

where $E(t) = \int_{\Sigma_t} |\nabla u|^{-2} |\nabla_{\Sigma} |\nabla u||^2 + \frac{1}{2}|\mathbb{II}|^2$. Therefore,

$$-A'(t) = -\frac{A(t)}{1 - t} + \frac{24\pi m_{\mu}(\Sigma_t)}{(1 - t)r_t} - \frac{1}{1 - t} E(t).$$

By Lemma 2.2, $\lim_{t \to 1} A(t) = 0$. Hence,

$$A(t) = \frac{1}{1 - t} \int_t^1 \left[ \frac{24\pi m_{\mu}(s)}{r_s} - E(s) \right].$$

As $t \to 1$, $m_{\mu}(\Sigma_t) = m + o(1)$ by the assumption. Thus, by (2.21),

$$\frac{m_{\mu}(\Sigma_t)}{r_t} = m c_{\Sigma}^{-1}(1 - t) + (1 - t)o(1).$$

Consequently,

$$\int_t^1 \frac{m_{\mu}(\Sigma_s)}{r_s} = \frac{1}{2} m c_{\Sigma}^{-1}(1 - t)^2 + o((1 - t)^2).$$

To estimate $E(t)$, we note that Lemmas 2.3 and 2.4 combined with (2.8), show

$$|\nabla u|^{-2} |\nabla_{\Sigma} |\nabla u||^2 + \frac{1}{2}|\mathbb{II}|^2 = O(|x|^{-2-2\tau}) = O((1 - t)^{2+2\tau}).$$

Thus, by Lemma 2.1

$$\int_{\Sigma_t} |\nabla u|^{-2} |\nabla_{\Sigma} |\nabla u||^2 + \frac{1}{2}|\mathbb{II}|^2 = O((1 - t)^{2\tau}).$$

As a result,

$$\int_t^1 \int_{\Sigma_s} |\nabla u|^{-2} |\nabla_{\Sigma} |\nabla u||^2 + \frac{1}{2}|\mathbb{II}|^2 = O((1 - t)^{1+2\tau}).$$
It follows from (2.28), (2.30) and (2.32) that
\begin{equation}
\frac{1}{1-t} A(t) = 12\pi \mathbf{m} c^{-1}_\Sigma + o(1) + O \left( (1-t)^{2\tau - 1} \right).
\end{equation}

Since \( \tau > \frac{1}{2} \), this proves (2.22).

Similarly, define
\begin{equation}
B(t) = 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2.
\end{equation}

At any regular value \( t \),
\begin{equation}
-B'(t) = \frac{1}{1-t} \left[ 2(-B(t) + 4\pi) + \frac{1}{1-t} \left( -\int_{\Sigma_t} H|\nabla u| \right) \right]
= \frac{1}{1-t} \left[ -2B(t) + A(t) \right].
\end{equation}

By Lemma 2.2, \( \lim_{t \to 1} B(t) = 0 \). Thus,
\[ B(t) = \frac{1}{(1-t)^2} \int_t^1 (1-s)A(s). \]

Therefore, as \( t \to 1 \), by (2.33),
\[ \frac{1}{1-t} B(t) = 4\pi \mathbf{m} c^{-1}_\Sigma + o(1) + O \left( (1-t)^{2\tau - 1} \right). \]

This proves (2.23). \( \square \)

**Theorem 2.1.** Let \((M, g)\) be an asymptotically flat 3-manifold with boundary \( \Sigma \), with \( \partial \partial \partial g_{ij} = O(|x|^{-3-\tau}) \) at \( \infty \). Let \( u \) be the harmonic function that tends to 1 at \( \infty \) and vanishes at \( \Sigma \). Then
\begin{enumerate}
\item \( \lim_{t \to 1} \frac{1}{1-t} \left[ 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \right] = 12\pi \mathbf{m} c^{-1}_\Sigma; \)
\item \( \lim_{t \to 1} \frac{1}{1-t} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] = 4\pi \mathbf{m} c^{-1}_\Sigma. \)
\end{enumerate}

Here \( \mathbf{m} \) is the mass of \((M, g)\) and \( c_\Sigma \) is the capacity of \( \Sigma \) in \((M, g)\).

**Proof.** It suffices to show \( \lim_{t \to 1} \mathbf{m}_H(\Sigma_t) = \mathbf{m} \). For \( t \) close to 1, let \( r_-(t) = \min_{\Sigma_t} |x| \) and \( r_+(t) = \max_{\Sigma_t} |x| \). By (2.8), \( r_+(t) \leq Cr_-(t) \). Here and below, \( C > 0 \) denotes some constant independent on \( t \). By Lemma 2.1, \( |\Sigma_t| \leq C r^2 \). By Lemma 2.4, \( K \geq C r^{-2} \), hence \( \text{diam}(\Sigma_t) \leq C r_\cdot \). By Lemma 2.4 and Lemma 2.3, \( |\bar{\mathbf{m}}| \leq C r^{-1-\tau} \) and \( |\bar{D\mathbf{m}}| \leq C r^{-2-\tau} \). Hence, \( \{\Sigma_t\} \) is a family of nearly round surfaces near \( \infty \) in \((M, g)\) according to Definition 1.3 in [20]. By Theorem 2 in [20], \( \lim_{t \to 1} \mathbf{m}_H(\Sigma_t) = \mathbf{m} \).

Theorem 2.1 now follows from Proposition 2.1. \( \square \)
We can indeed interpret the mass-to-capacity ratio as the derivatives at $\infty$ of the two functions

$$A(t) = 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \quad \text{and} \quad B(t) = 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2.$$  

**Corollary 2.1.** Let $(M, g)$ be an asymptotically flat 3-manifold with boundary $\Sigma$, with $\partial\Omega g_{ij} = O(|x|^{-3-\tau})$ at $\infty$. Let $u$ be the harmonic function that tends to 1 at $\infty$ and vanishes at $\Sigma$. Then the functions $A(t)$ and $B(t)$ have $C^1$ extensions to $t = 1$ with

$$A(1) = 0, \quad A'(1) = -12\pi \, m \, c_{\Sigma}^{-1}$$

and

$$B(1) = 0, \quad B'(1) = -4\pi \, m \, c_{\Sigma}^{-1}.$$  

**Proof.** By Lemma 2.2, $A(t)$ and $B(t)$ extend continuously to $t = 1$ with $A(1) = 0$ and $B(1) = 0$.

By Theorem 2.1 (i), (2.27), (2.29) and (2.31),

$$\lim_{t \to 1} A'(t) = \lim_{t \to 1} \left[ \frac{1}{1-t} A(t) - \frac{24\pi m_u(\Sigma_t)}{(1-t)\tau} + \frac{1}{1-t} E(t) \right]$$

$$= 12\pi \, m \, c_{\Sigma}^{-1} - 24\pi \, m \, c_{\Sigma}^{-1}$$

$$= \lim_{t \to 1} \frac{1}{t-1} A(t).$$

Similarly, by Theorem 2.1 (i), (ii) and (2.35),

$$\lim_{t \to 1} B'(t) = \lim_{t \to 1} \frac{1}{1-t} [2B(t) - A(t)] = -4\pi \, m \, c_{\Sigma}^{-1} = \lim_{t \to 1} \frac{1}{t-1} B(t).$$

This shows $A'(t)$ and $B'(t)$ are continuous at $t = 1$ with $A'(1) = -12\pi \, m \, c_{\Sigma}^{-1}$ and $B'(1) = -4\pi \, m \, c_{\Sigma}^{-1}$. \qed

### 3. Inequalities along the level sets

In this section, we establish a family of geometric inequalities along $\{\Sigma_t\}$ under assumptions that $g$ has nonnegative scalar curvature and $M$ has simple topology.

We first compare

$$A(t) = 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \quad \text{and} \quad B(t) = 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2.$$  

**Theorem 3.1.** Let $(M, g)$ be a connected, orientable, asymptotically flat 3-manifold with boundary $\Sigma$. Suppose $\Sigma$ is connected and $H_2(M, \Sigma) = 0$. Let $u$ be the harmonic function that tends to 1 at $\infty$ and vanishes at $\Sigma$. If $g$ has nonnegative scalar curvature, then

$$4\pi + \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \geq \frac{3}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2$$

for all regular values $t$, and equality holds at some $t$ if and only if $(M, g)$, outside $\Sigma_t$, is isometric to $\mathbb{R}^3$ minus a round ball.
In particular, at $\Sigma$,

$$ 4\pi + \int \Sigma \left( H|\nabla u| - 3 \frac{1}{1 - t} \int \Sigma_{t_s} |\nabla u|^2 \right), $$

and equality holds if and only if $(M, g)$ is isometric to $\mathbb{R}^3$ minus a round ball.

Remark 3.1. Inequality (3.1) is equivalent to

$$ A(t) \leq 3B(t). $$

We will use Theorem 3.1 in this form later to derive other inequalities along $\{\Sigma_t\}$.

To prove Theorem 3.1, we begin with a lemma which may be derived directly from the work of Stern in [21].

Lemma 3.1. Let $(\Omega, g)$ be a compact, orientable, Riemannian 3-manifold with nonnegative scalar curvature, with boundary $\partial \Omega$. Suppose $\partial \Omega$ has two connected components $S_1$ and $S_2$. Let $u$ be a harmonic function on $(\Omega, g)$ such that $u = c_i$ on $S_i$, $i = 1, 2$, where $c_1, c_2$ are constants with $c_1 < c_2 < 1$. If the level set $\Sigma_s := u^{-1}(s)$ is connected for $s \in [c_1, c_2]$, then

$$ \Psi(t) := 4\pi(1 - t) + \int_{\Sigma_{t}} H|\nabla u| - \frac{3}{1 - t} \int_{\Sigma_{t}} |\nabla u|^2 \xrightarrow{t \to t'} , $$

i.e. $\Psi(t)$ is monotone nonincreasing. Here $t \in [c_1, c_2]$ denotes a regular value of $u$ and $H$ is the mean curvature of $\Sigma_s$ with respect to the unit normal $\nu = |\nabla u|^{-1}\nabla u$.

Proof. Let $t_1 < t_2$ be two regular values of $u$. On $\Omega_{[t_1, t_2]} := \{x \in \Omega \mid t_1 \leq u(x) \leq t_2\}$, one has

$$ \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u| \geq \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{1}{2} \left( \frac{|\nabla^2 u|^2}{|\nabla u|^2} + R \right) - 2\pi \int_{t_1}^{t_2} \chi(\Sigma_t). $$

Here $\nabla^2 u$, $\nabla u$ denote the Hessian, the gradient of $u$ on $(M, g)$, respectively, $R$ is the scalar curvature of $g$, and $\chi(\Sigma_t)$ is the Euler characteristic of $\Sigma_t$. Relation (3.4) is a direct consequence of Stern’s computations in Section 2 of [21], and can also be found explicitly from (4.7) in [6] and (2.18) in [12].

Let $\mathbb{II}$ denote the second fundamental form of $\Sigma_t$ w.r.t. $\nu$. Along $\Sigma_t$,

$$ \nabla^2 u(X, Y) = |\nabla u| \mathbb{II}(X, Y), \quad \nabla^2 u(X, \nu) = X(|\nabla u|), \quad \nabla^2 u(\nu, \nu) = -H|\nabla u|, $$

where $X, Y$ denote vectors tangent to $\Sigma_t$ and the last equation follows from $\Delta u = 0$. Thus,

$$ |\nabla u|^{-2} |\nabla^2 u|^2 = |\mathbb{II}|^2 + 2|\nabla u|^{-2} |\nabla_{\Sigma_t} \nabla u|^2 + H^2. $$

Here $\nabla_{\Sigma_t}$ denotes the gradient on $\Sigma_t$. Under the assumption $\Sigma_t$ is connected, it follows from (3.4) and (3.6) that

$$ 4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u| \geq \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{1}{2} |\mathbb{II}|^2 + |\nabla u|^{-2} |\nabla_{\Sigma_t} \nabla u|^2 + \frac{3}{4} H^2 + \frac{1}{2} R, $$

(3.7)
where $\mathcal{I}$ denotes the traceless part of $\mathcal{I}$.

To handle the term of $H^2$ in (3.7), we follow the idea in [18, 2] to replace it with $(H - 2|\nabla u|(1 - u)^{-1})^2$. A motivation to this may be seen in the model case in which $\Omega = \{R_1 \leq |x| \leq R_2\} \subset \mathbb{R}^3$ and $u = 1 - |x|^{-1}$. In this special setting, $H$ and $|\nabla u|$ satisfy $H = 2|\nabla u|(1 - u)^{-1}$ along any level set sphere.

Thus, one can rewrite (3.7) as

$$4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u|$$

$$+ 3 \int_{t_1}^{t_2} \left[ -\frac{1}{1 - t} \int_{\Sigma_t} H|\nabla u| + \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right]
$$

$$\geq \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{1}{2} |\mathcal{I}|^2 + |\nabla u|^{-2}|\nabla_{\Sigma_t} |\nabla u||^2 + \frac{3}{4} \left( H - \frac{2|\nabla u|}{1 - u} \right)^2 + \frac{1}{2} R.$$

At each regular value $t$, one has $\left( \int_{\Sigma_t} |\nabla u|^2 \right)' = -\int_{\Sigma_t} H|\nabla u|$, and therefore,

$$-\frac{1}{1 - t} \int_{\Sigma_t} H|\nabla u| + \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 = \frac{d}{dt} \left( \frac{1}{1 - t} \int_{\Sigma_t} |\nabla u|^2 \right).$$

Thus, if $[t_1, t_2]$ has no critical values, the above directly shows

$$\int_{t_1}^{t_2} \left( -\frac{1}{1 - t} \int_{\Sigma_t} H|\nabla u| + \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right)
$$

$$= \frac{1}{1 - t_2} \int_{\Sigma_{t_2}} |\nabla u|^2 - \frac{1}{1 - t_1} \int_{\Sigma_{t_1}} |\nabla u|^2.$$

In general, if $[t_1, t_2]$ has critical values, one may use a regularization argument to still obtain (3.9). For instance, applying Lemma A.1 of the Appendix to $u$ on $\Omega[t_1, t_2]$, one has

$$\frac{1}{1 - t_2} \int_{\Sigma_{t_2}} |\nabla u|^2 - \frac{1}{1 - t_1} \int_{\Sigma_{t_1}} |\nabla u|^2
$$

$$= \int_{\Omega[t_1, t_2]} |\nabla u|^2 + \int_{\Sigma_{t_1} \cup \Sigma_{t_2}} \frac{1}{1 - u} |\nabla u|^{-1}\nabla^2 u(\nabla u, \nabla u).$$

This, together with the coarea formula and (3.5), gives (3.9).

By (3.5) and (3.9),

$$\Psi(t_1) - \Psi(t_2) \geq \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{1}{2} |\mathcal{I}|^2 + |\nabla u|^{-2}|\nabla_{\Sigma_t} |\nabla u||^2 + \frac{3}{4} \left( H - \frac{2|\nabla u|}{1 - u} \right)^2 + \frac{1}{2} R.$$

For the later purpose in Section A, we note that (3.11) holds without assumptions on the scalar curvature $R$.

If the scalar curvature $R$ is nonnegative, then (3.11) implies $\Psi(t_1) \geq \Psi(t_2)$, which proves the Lemma. \qed
In the context of Theorem 3.1, the assumption \( \Sigma \) is connected and \( H_2(M, \Sigma) = 0 \) is a sufficient condition to ensure \( \chi(\Sigma_t) \leq 2 \) for a regular \( \Sigma_t \). Under this condition, \( u \) being harmonic and the maximum principle guarantee \( \Sigma_t \) is connected. (The same assumption was used by Bray and the author [17] in estimating the capacity of \( \Sigma \) in \((M,g)\) via the solution to the weak inverse mean curvature \((1/H)\) flow [13]. In that setting, a different reasoning shows the level set of the \( 1/H \) flow is connected.)

**Proof of Theorem 3.1.** Let \( \Psi(t) \) be given from Lemma 3.1. On an asymptotically flat \((M,g)\), a corollary of Lemma 2.2 shows

\[
\lim_{t \to 1} \Psi(t) = 0.
\]

Thus, letting \( t_2 \to 1 \) in (3.11) gives

\[
\Psi(t) \geq \int_{t}^{1} \int_{\Sigma_\sigma} \frac{1}{2} \| \Pi \|^2 + |\nabla u|^{-2} |\nabla_{\Sigma_t} u|^{-2} \right) + \frac{3}{4} \left( H - \frac{2|\nabla u|}{1-u} \right)^2 + \frac{1}{2} R
\]

for every regular value \( t \). In particular, if \( R \geq 0 \), then \( \Psi(t) \geq 0 \).

Inequality (3.11) follows from (3.12) by noting that

\[
\frac{1}{1-t} \Psi(t) = 3B(t) - A(t).
\]

To show the rigidity case of (3.1), it suffices to establish it for the case \( t = 0 \). Suppose the equality in (3.2) holds, then, by (3.12) and its proof, for every regular value \( t \in [0,1] \), \( \Sigma_t \) is connected (orientable) with \( \chi(\Sigma_t) = 2 \), hence \( \Sigma_t \) is a 2-sphere; moreover, \( R = 0 \), \( |\nabla u| \) only depends on \( t \), \( \Sigma_t \) is totally umbilic, and \( H = \frac{2}{1-t} |\nabla u| \).

To show \((M,g)\) is isometric to \( \mathbb{R}^3 \) minus a round ball, we start from a neighborhood of the boundary \( \Sigma \). For convenience, we normalize \((M,g)\) so that \( |\Sigma| = 4\pi \). It follows from the equality

\[
4\pi + \int_{\Sigma} H |\nabla u| = 3 \int_{\Sigma} |\nabla u|^2
\]

that \( |\nabla u| = 1 \) and \( H = 2 \) at \( \Sigma = \Sigma_0 \). Locally, \( g \) takes the form of \( g = \eta(t)^{-2} dt^2 + \gamma_t \) near \( \Sigma_0 \), where \( t = u \), \( \eta(t) = |\nabla u| \) and \( \gamma_t \) denotes the induced metric on \( \Sigma_0 \), which satisfies \( \partial_t \gamma_t = 2\eta(t)^{-1} \Pi u = \eta(t)^{-1} H \gamma_t = 2(1-t)^{-1} \gamma_t \). Thus, \( (1-t)^2 \gamma_t = \) a fixed metric. Similarly, since \( |\nabla u'| = -H \), \( \eta(t) \) satisfies \( \eta'(t) = -\frac{1}{\gamma_t} \eta(t) \). Hence, \( (1-t)^{-2} \eta = \) a constant. As \( |\nabla u| = 1 \) at \( \Sigma \), we thus have \( \eta = (1-t)^{-2} \) and \( g = (1-t)^{-4} dt^2 + (1-t)^{-2} \sigma_o \) for some fixed metric \( \sigma_o \) on the 2-sphere \( \Sigma \). Invoking the fact \( R = 0 \) near \( \Sigma \), we see \( \sigma_o \) is a round metric with Gauss curvature 1 on \( \Sigma \).

Now, if \( u \) has a critical value, let \( t_0 \in (0,1) \) be the smallest critical value of \( u \). The above argument then shows \( u^{-1}([0,t_0]) \) is isometric to

\[
(\Sigma \times [0,t_0], (1-t)^{-4} dt^2 + (1-t)^{-2} \sigma_o).
\]

In particular, this implies \( |\nabla u| = (1-t_0)^2 \neq 0 \) on the set \( \partial \{ u < t_0 \} = \partial \{ u \geq t_0 \} \). As a result, \( \partial \{ u \geq t_0 \} \) is an embedded surface in \( M \). Therefore, \( \partial \{ u \geq t_0 \} = \{ u = t_0 \} \) by the strong maximum principle. In summary, this shows \( \nabla u \neq 0 \) on the set \( \{ u = t_0 \} \),
which contradicts to the assumption $t_0$ is a critical value. Hence, $u$ has no critical values. We conclude $(M, g)$ is isometric to 
\[
(\Sigma \times [0, 1), (1 - t)^{-4} dt^2 + (1 - t)^{-2}\sigma_0),
\]
which, upon a change of variable $1 - t = r^{-1}$, is isometric to $\mathbb{R}^3$ minus a unit ball. \□

Theorem 3.1 implies an upper bound of $\frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2$ by the Willmore functional of $\Sigma_t$ in $(M, g)$.

**Corollary 3.1.** Let $(M, g)$ be a connected, orientable, asymptotically flat 3-manifold with boundary $\Sigma$. Suppose $\Sigma$ is connected and $H_2(M, \Sigma) = 0$. Let $u$ be the harmonic function such that $u = 0$ at $\Sigma$ and $u \to 1$ at $\infty$. If $g$ has nonnegative scalar curvature, then
\[
\frac{1}{4\pi} \int_\Sigma |\nabla u|^2 \leq \frac{1}{9} \left[ 2W + 2\sqrt{W^2 + 3W} + 3 \right],
\]
where $W = \frac{1}{16\pi} \int_\Sigma H^2$, and equality holds if and only if $(M, g)$ is isometric to $\mathbb{R}^3$ minus a round ball.

**Proof.** Let $z = (\int_\Sigma |\nabla u|^2)^{\frac{1}{2}}$. By Theorem 3.1 and Hölder’s inequality,
\[
4\pi + \sqrt{16\pi W} z \geq 3z^2.
\]
This implies the bound of $z$ in (3.14) by elementary reason. The equality case follows from the equality case in Theorem 3.1. \□

We next apply Theorem 3.1 to show that the quantities in Theorem 2.1, which approach to constant multiples of $mc^{-1}_{\Sigma}$ at $\infty$, are actually monotone.

**Theorem 3.2.** Let $(M, g)$ be a connected, orientable, asymptotically flat 3-manifold with boundary $\Sigma$. Suppose $\Sigma$ is connected and $H_2(M, \Sigma) = 0$. Let $u$ be the harmonic function such that $u = 0$ at $\Sigma$ and $u \to 1$ at $\infty$. If $g$ has nonnegative scalar curvature, then
\begin{enumerate}
  
(i) $A(t) := \frac{1}{1 - t} \left[ 8\pi - \frac{1}{1 - t} \int_{\Sigma_t} H|\nabla u| \right]$ \uparrow as $t \nearrow$, i.e. $A(t)$ is monotone non-decreasing in $t$. As a result,
\[
A(t) \leq 12\pi m c^{-1}_{\Sigma}.
\]
In particular, at $\Sigma$,
\[
8\pi - \int_{\Sigma} H|\nabla u| \leq 12\pi m c^{-1}_{\Sigma},
\]
and equality holds if and only if $(M, g)$ is isometric to $\mathbb{R}^3$ minus a round ball.

(ii) $B(t) := \frac{1}{1 - t} \left[ 4\pi - \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right]$ \uparrow as $t \nearrow$, i.e. $B(t)$ is monotone non-decreasing in $t$. As a result,
\[
B(t) \leq 4\pi m c^{-1}_{\Sigma}.
\]
\]
In particular, at $\Sigma$,

$$
4\pi - \int_{\Sigma} |\nabla u|^2 \leq 4\pi m c^{-1}_{\Sigma},
$$

and equality holds if and only if $(M, g)$ is isometric to $\mathbb{R}^3$ minus a round ball.

Proof. We first show (ii) as it is more straightforward. By (2.35) and (3.3), at every regular value $t$, we have

$$
-B'(t) = \frac{1}{1-t} [-2B(t) + A(t)] \leq \frac{1}{1-t} B(t).
$$

Therefore,

$$
\left[ \frac{1}{1-t} B(t) \right]' \geq 0,
$$

which implies the monotonicity of $B(t) = \frac{1}{1-t} B(t)$ in the case $u$ has no critical values. If $u$ has critical values, we may again apply a regularization argument to show that $B(t_2) - B(t_1) \geq 0$ for $t_2 > t_1$, see Proposition A.1 in the Appendix for details.

By Theorem 2.1 (ii),

$$
\lim_{t \to 1} B(t) = 4\pi m c^{-1}_{\Sigma}.
$$

Therefore, the monotonicity of $B(t)$ shows

$$
B(t) \leq 4\pi m c^{-1}_{\Sigma}.
$$

At $t = 0$, this gives

$$
B(0) = 4\pi - \int_{\Sigma} |\nabla u|^2 \leq 4\pi m c^{-1}_{\Sigma}.
$$

The rigidity part follows from the rigidity part of Theorem 3.1.

To show (i), we calculate

$$
\mathcal{A}'(t) = \frac{1}{(1-t)^2} \left[ A(t) - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| - \left( \int_{\Sigma_t} H|\nabla u| \right) \right].
$$

By (2.25) and the Gauss-Bonnet theorem,

$$
\mathcal{A}'(t) \geq \frac{1}{(1-t)^2} \left[ A(t) - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| + \int_{\Sigma_t} \left( -K + \frac{3}{4} H^2 \right) \right]
\geq \frac{1}{(1-t)^2} \left[ A(t) - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| - 4\pi + \frac{3}{4} \int_{\Sigma_t} H^2 \right]
= \frac{1}{(1-t)^2} \left[ 4\pi + \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| - \frac{3}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right.
\left. \left( \frac{2}{1-u} \right)^2 \right] + \frac{3}{4} \int_{\Sigma_t} \left( H - \frac{2|\nabla u|}{1-u} \right)^2 .
$$
By (3.3),
\[ I(t) = 3B(t) - A(t) \geq 0. \]
Therefore, \( A'(t) \geq 0 \), which implies the monotonicity of \( A(t) \) in the absence of critical values. The general case can be again handled by a regularization argument that shows \( A(t_2) - A(t_1) \geq 0 \) for \( t_2 > t_1 \), see Proposition A.1 in the Appendix.

The remaining conclusions in (i) follow from Theorem 2.1 (i) and Theorem 3.1. □

**Remark 3.2.** We need the technical assumption \( \partial \partial g_{ij} = O(|x|^{-\tau - 3}) \) in obtaining (3.15) and (3.16) as Theorem 2.1 is used in that step. For convenience, we include this assumption in the asymptotic flatness description (2.1) henceforth.

**Remark 3.3.** Comparing (3.2), (3.15) and (3.16), we have (3.2) + (3.16) \( \Rightarrow \) (3.15).

### 4. Proofs of the positive mass theorem

The 3-dimensional Riemannian positive mass theorem, first proved by Schoen-Yau [19] and later by Witten [22], asserts that if \((M, g)\) is a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, then \( m \geq 0 \) and \( m = 0 \) if and only if \((M^3, g)\) is isometric to \( \mathbb{R}^3 \).

Since the work of Schoen-Yau and Witten, other proofs of this theorem have been given by Huisken-Ilmanen [13], by Li [16], by Bray-Kazaras-Khuri-Stern [6], and by Agostiniani-Mazzieri-Oronzio [2]. (Agostiniani-Mantegazza-Mazzieri-Oronzio [4] also gave a new proof of the Riemannian Penrose inequality, first proved by Bray [5] and Huisken-Ilmanen [13].)

As applications of Theorems 2.1 and 3.2, we observe a few additional arguments that prove the positive mass theorem. We first outline the tools and features of the proofs to be given:

- **Proof I** uses Theorem 2.1 (ii) and a result of Munteanu-Wang [18].
- **Proof II** is self-contained. It uses Theorem 2.1 and the monotonicities in Theorems 3.2.
- **Proof III** is self-contained. It uses the inequalities in Theorem 3.2. Proof III leads to a positive mass theorem with boundary in Section 5.

**Proof I.** Let \((M, g)\) be a complete, asymptotically flat 3-manifold without boundary, with nonnegative scalar curvature. Suppose \( M \) is topologically \( \mathbb{R}^3 \).

Take \( p \in M \). Let \( G(x) \) be the minimal positive Green’s function with a pole at \( p \), with \( G(x) \to 0 \) as \( x \to \infty \). Let \( u = 1 - G \). By Theorem 1.1 of Munteanu-Wang [18],
\[ 4\pi(1 - t) - \frac{1}{1 - t} \int_{\Sigma_t} |\nabla u|^2 \, dV \quad \text{as} \quad t \to 1, \]
i.e. it is monotone non-increasing in \( t \).

As \( t \to 1 \), \( \frac{1}{1 - t} \int_{\Sigma_t} |\nabla u|^2 \to 0 \) by Lemma 2.2. Hence, \( 4\pi(1 - t) - \frac{1}{1 - t} \int_{\Sigma_t} |\nabla u|^2 \geq 0 \). Consequently,
\[ \frac{1}{(1 - t)} \left[ 4\pi - \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \geq 0. \]
It follows from Theorem 2.1 (ii) and the fact $u = 1 - \frac{1}{4\pi} |x|^{-1} + O(|x|^{-1-\gamma})$ that

$$(4\pi)^2 m = \lim_{t \to 1} \frac{1}{(1-t)} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \geq 0.$$ 

□

**Remark 4.1.** To prove the 3-dimensional positive mass theorem, it is known it suffices to assume $M$ is topologically $\mathbb{R}^3$, see Section 2 in [6] for instance. For this reason, we make such an assumption in all the proofs. It is also known the rigidity case $m = 0$ in the theorem follows from the inequality $m \geq 0$ by a variational argument, see [19].

**Proof II.** Take $p \in M$. Let $G(x)$ be the minimal positive Green’s function with a pole at $p$. Let $d(x)$ denote the distance from $x$ to $p$ in $(M,g)$. As $x \to p$, it is known

$$(4.1) \quad G(x) = \frac{1}{4\pi} d(x)^{-1} + o(d(x)^{-1}), \quad |\nabla G(x)| = \frac{1}{4\pi} d(x)^{-2} + o(d(x)^{-2}).$$

Consider $u = 1 - G$. By Theorem 3.2 (ii),

$$(4.2) \quad B(t) = \frac{1}{1-t} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \nearrow \text{ as } t \nearrow,$$

i.e. it is monotone non-decreasing in $t$. Note this is different from the monotonicity of Munteanu-Wang [18]. The latter asserts $(1-t)^2 B(t)$ is monotone non-increasing.

As $t \to -\infty$, by (4.1), $\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2$ is bounded, hence $\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \to 0$. Thus,

$$(4.3) \quad \lim_{t \to -\infty} B(t) = 0.$$ 

Hence, by (4.2) and (4.3), $B(t) \geq 0$. By Theorem 2.1 (ii),

$$(4\pi)^2 m = \lim_{t \to 1} B(t) \geq 0.$$ 

□

**Remark 4.2.** Proof II has a similar spirit as the proof of Agostiniani-Mazzieri-Oronzo [2]. The difference is the uses of different monotone quantities, i.e. $B(t)$ and $F(t)$.

**Remark 4.3.** In Proof II, one can also work with $A(t)$, and apply Theorem 3.2 (i) and Theorem 2.1 (i). In this case, one needs to check $\lim_{t \to -\infty} \frac{1}{(1-t)^2} \int_{\Sigma_t} H|\nabla u| = 0$. But this is true and follows from the known estimate on $\nabla^2 G$ near the pole (see [18] and [2] for instance).

**Proof III.** Take $p \in M$. Given a small $r > 0$, let $B_r$ denote the geodesic ball of radius $r$ centered at $p$. Let $\Sigma_r = \partial B_r$ and $u = u_r$ be the harmonic function with $u = 0$ at $\Sigma_r$ and $u \to 1$ at $\infty$. Let $c_r$ be the capacity of $\Sigma_r$ in $(M,g)$.

Applying (3.15) of Theorem 3.2 (i) to $(M \setminus B_r, g)$, we have

$$8\pi - \int_{\Sigma_r} H|\nabla u| \leq 12\pi m c_r^{-1}.$$
Since $c_r > 0$, this is equivalent to
\begin{equation} 
(4.4) \quad c_r \left( 8\pi - \int_{\Sigma_r} H|\nabla u| \right) \leq 12\pi \ m. 
\end{equation}

It remains to check, as $r \to 0$,
\begin{equation} 
(4.5) \quad c_r = O(r) \quad \text{and} \quad \int_{\Sigma_r} H|\nabla u| = O(1). 
\end{equation}

A conclusion $m \geq 0$ will follow from (4.4) and (4.5).

To estimate $c_r$, we may use the variational characterization of the capacity, i.e.
\[ c_r = \inf_f \left\{ \frac{1}{4\pi} \int_{M \setminus B_r} |\nabla f|^2 \right\}, \]
where $f$ is a Lipschitz function with $f = 0$ at $\Sigma_r$ and $f \to 1$ at $\infty$. Consider a test function $f(x) = r^{-1} (d(x) - r)$ in $B_{2r} \setminus B_r$ and extend $f$ to be 1 outside $B_{2r}$. Here $d(x)$ is the distance from $x$ to $p$. Then
\begin{equation} 
(4.6) \quad c_r \leq \frac{1}{4\pi} \int_{B_{2r} \setminus B_r} |\nabla f|^2 = \frac{1}{4\pi r^2} \text{Volume}(B_{2r} \setminus B_r) = O(r). 
\end{equation}

For $\int_{\Sigma_r} H|\nabla u|$, we have
\begin{equation} 
(4.7) \quad \left| \int_{\Sigma_r} H|\nabla u| \right| \leq \max_{\Sigma_r} |H| \int_{\Sigma_r} |\nabla u| = \max_{\Sigma_r} |H| \ c_r = O(1) 
\end{equation}
by (4.6) and the fact $H = 2r^{-1} + O(r)$ (see (3.34) in [10] for instance).

This verifies (4.5) and completes the proof. \hfill \square

Remark 4.4. In the above proof, we estimated $c_r$ by the so-called relative capacity of $\Sigma_r$ in $B_{2r}$. By a result of Jauregui [14], one can check
\[ \limsup_{r \to 0} \left( 8\pi - \int_{\Sigma_r} H|\nabla u| \right) \geq 0. \]

Remark 4.5. Alternatively one may use (3.16) of Theorem 3.2 (ii) to have
\[ c_r \left( 4\pi - \int_{\Sigma_r} |\nabla u|^2 \right) \leq 4\pi \ m. \]

It then remains to show $\int_{\Sigma_r} |\nabla u|^2 = O(1)$. By the maximum principle, $|\nabla u| \leq |\nabla v|$ at $\partial B_r$, where $v$ is the harmonic function with $v = 0$ at $\partial B_r$ and $v = 1$ at $\partial B_{2r}$. By scaling and applying elliptic boundary estimates, one has $\int_{\partial B_r} |\nabla v|^2 = O(1)$, and hence $\int_{\Sigma_r} |\nabla u|^2 = O(1)$. 
5. A POSITIVE MASS THEOREM WITH BOUNDARY

Proof III in the preceding section suggests the following theorem.

**Theorem 5.1.** Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary \(\Sigma\). Suppose \(\Sigma\) is connected and \(H_2(M, \Sigma) = 0\). Let \(S\) be a closed surface enclosing \(\Sigma\) in \(M\). Let \(\Omega\) denote the region bounded by \(\Sigma\) and \(S\). Let Vol\((\Omega)\) be the volume of \((\Omega, g)\) and \(L\) be the distance from \(S\) to \(\Sigma\). Then

\[ H \leq \frac{8\pi L^2}{\text{Vol}(\Omega)} \implies m > 0. \]

Here \(H\) is the mean curvature of \(\Sigma\) in \((M, g)\).

**Proof.** Let \(u\) be the harmonic function with \(u = 0\) at \(\Sigma\) and \(u \to 1\) at \(\infty\). Then

\[ \int_{\Sigma} H|\nabla u| \leq 4\pi c_\Sigma \max_{\Sigma} H, \]

where \(c_\Sigma\) is the capacity of \(\Sigma\) in \((M, g)\). If \(H \leq 0\), (3.15) implies \(m > 0\). Below we assume \(\max_{\Sigma} H > 0\).

Consider a test function \(f(x)\) which is \(L^{-1}d(x)\) on \(\Omega\) and equals 1 outside \(S\). Here \(d(x)\) denotes the distance from \(x\) to \(\Sigma\). Then

\[ c_\Sigma < \frac{1}{4\pi} \int_M |\nabla f|^2 = \frac{1}{4\pi L^2} \text{Vol}(\Omega). \]

Thus, if

\[ \max_{\Sigma} H \leq \frac{8\pi L^2}{\text{Vol}(\Omega)}, \]

then

\[ \int_{\Sigma} H|\nabla u| < 8\pi. \]

We conclude \(m > 0\) by (3.15) of Theorem 3.2 (i).

**Remark 5.1.** Given any \(L > 0\), by considering the test function \(f_L(x) = L^{-1}d(x)\) if \(d(x) \leq L\) and \(f_L(x) = 1\) if \(d(x) \geq L\), we may choose \(\Omega = \Omega_L := \{x \in M \mid d(x) \leq L\}\). Thus, the condition in (5.1) is a condition on the volume growth of \(\Omega_L\). In particular, Theorem 5.1 shows

\[ m \leq 0 \implies \text{Vol}(\Omega_L) > \frac{8\pi L^2}{\max_{\Sigma} H}, \quad \forall L. \]

We may also seek potential applications of (3.16) in Theorem 3.2 (ii). The following proposition is known and was proved previously via the weak inverse mean curvature \((1/H)\) flow developed by Huisken-Ilmanen [13]. We include it here only to show that the result can also be proved using harmonic functions.
Proposition 5.1. Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with boundary \(\Sigma\). Suppose \(\Sigma\) is connected and \(H_2(M, \Sigma) = 0\). If \(g\) has nonnegative scalar curvature, then
\[
\int_{\Sigma} H^2 \leq 16\pi \implies m \geq 0,
\]
and \(m = 0\) if and only if \((M, g)\) is isometric to \(\mathbb{R}^3\) minus a round ball.

Proof. By Corollary 3.1
\[
\int_{\Sigma} H^2 \leq 16\pi \implies \int_{\Sigma} |\nabla u|^2 \leq 4\pi.
\]
Hence, \(m \geq 0\) by (3.16). The rigidity case follows from that of Corollary 3.1 \(\square\)

It is conceivable that Theorem 5.1 may be used to study the mass of incomplete asymptotically flat 3-manifolds. Recently Cecchini-Zeidler \(8\) and Lee-Lesourd-Unger \(15\) have given sufficient conditions, involving a positive lower bound of the scalar curvature on suitable regions in a manifold \((M^n, g)\) that is spin or of dimension \(3 \leq n \leq 7\), which guarantee the positivity of the mass. If such conditions can be viewed as shielding the incomplete part by a region with sufficiently positive scalar curvature, then the condition in Theorem 5.1 may be thought as a shielding condition by a region that does not have excess volume.

6. Integral identities for the mass-to-capacity ratio

In [6], Bray-Kazaras-Khuri-Stern found an integral identity for the mass of an asymptotically flat manifold. More precisely, if \((E, g)\) denotes the exterior region of a complete, asymptotically flat Riemannian 3-manifold \((M, g)\) with mass \(m\), then
\[
(6.1) \quad 16\pi m \geq \int_E \left( \frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u| \right),
\]
where \(u\) is a harmonic function on \((E, g)\) satisfying Neumann boundary conditions at \(\partial E\), and which is asymptotic to one of the asymptotically flat coordinate functions at \(\infty\). In particular, if the scalar curvature is nonnegative, then \(m \geq 0\).

In this section, we derive mass identities analogous to (6.1) with \(u\) being a harmonic function that equals 0 at the boundary and is asymptotic to 1 at \(\infty\).

Theorem 6.1. Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with boundary \(\Sigma\). Suppose \(\Sigma\) is connected and \(H_2(M, \Sigma) = 0\). Let \(u\) be the harmonic function such that \(u = 0\) at \(\Sigma\) and \(u \to 1\) at \(\infty\). Let \(\Phi_u\) be a symmetric \((0, 2)\) tensor given by
\[
\Phi_u = \frac{|\nabla u|^2}{1-u} g - \frac{3du \otimes du}{1-u}.
\]
Let $m$ be the mass of $(M, g)$ and $c_\Sigma$ be the capacity of $\Sigma$ in $(M, g)$. Then

\[
mc_\Sigma^{-1} - \left(1 - \frac{1}{4\pi} \int_\Sigma |\nabla u|^2\right)
\geq \frac{1}{16\pi} \int_M \left[\frac{1}{(1-u)^2} - 1\right] \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R|\nabla u|\right).
\]

and

\[
mc_\Sigma^{-1} - \frac{2}{3} \left(1 - \frac{1}{8\pi} \int_\Sigma H|\nabla u|\right)
\geq \frac{1}{16\pi} \int_M \left[\frac{1}{(1-u)^2} - \frac{1}{3}\right] \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R|\nabla u|\right).
\]

Proof. By (3.5), along a regular level set $\Sigma_t$, $(\nabla^2 u - \Phi_u)$ satisfies

\[
(\nabla^2 u - \Phi_u) (\nu, \nu) = -H|\nabla u| + \frac{2|\nabla u|^2}{1-u},
\]

\[
(\nabla^2 u - \Phi_u) (\nu, \cdot)|_{\Sigma_t} = \langle \nabla_{\Sigma_t} |\nabla u|, \cdot \rangle,
\]

\[
(\nabla^2 u - \Phi_u) (\cdot, \cdot)|_{\Sigma_t} = |\nabla u| \left(\mathbb{I} - \frac{|\nabla u|}{1-u} \gamma\right),
\]

where $\gamma$ denotes the induced metric on $\Sigma_t$. Therefore,

\[
|\nabla u|^{-2} |\nabla^2 u - \Phi_u|^2 = \frac{3}{2} \left(H - \frac{2|\nabla u|^2}{1-u}\right)^2 + 2|\nabla u|^{-2} |\nabla_{\Sigma_t} |\nabla u|^2 + |\mathbb{II}|^2.
\]

Given two regular values $t_1 < t_2$, by (A.14) in Proposition A.1 of the Appendix, we have

\[
\mathcal{B}(t_2) - \mathcal{B}(t_1) = \int_{t_1}^{t_2} \frac{1}{(1-t)^2} [3B(t) - A(t)].
\]

By (3.13) and (3.12),

\[
3B(t) - A(t) = \frac{1}{1-t} \Psi(t) \quad \text{and} \quad \Psi(t) \geq \int_t^1 \psi(s),
\]

where

\[
\psi(t) = \int_{\Sigma_t} \left[\frac{3}{4} \left(H - \frac{2|\nabla u|^2}{1-u}\right)^2 + |\nabla u|^{-2} |\nabla_{\Sigma_t} |\nabla u|^2 + \frac{1}{2} |\mathbb{II}|^2 + \frac{1}{2} R\right]
\]

\[
= \frac{1}{2} \int_{\Sigma_t} \left(\frac{|\nabla u - \Phi_u|^2}{|\nabla u|^2} + R\right).
\]
Taking $t_1 = 0$ and letting $t_2 \to 1$, by Theorem 2.1 we hence have

\[
4\pi mc_{\Sigma_1}^{-1} - B(0) = \int_0^1 \frac{1}{(1 - t)^2} [3B(t) - A(t)]
\]

(6.8)

\[
= \int_0^1 \frac{1}{(1 - t)^3} \Psi(t)
\]

\[
\geq \int_0^1 \frac{1}{(1 - t)^3} \left( \int_t^1 \Psi(s) \right).
\]

Integration by parts gives

\[
\int_0^1 \frac{1}{(1 - t)^3} \left( \int_t^1 \Psi(s) \right) \]

(6.9)

\[
= \frac{1}{2} \left[ \lim_{t \to 1} \int_0^1 \frac{1}{(1 - t)^2} \Psi(s) - \int_0^1 \Psi(s) + \int_0^1 \frac{\psi(t)}{(1 - t)^2} \right].
\]

We claim

(6.10)

\[
\lim_{t \to 1} \frac{1}{(1 - t)^2} \int_t^1 \Psi(s) = 0.
\]

This can be seen as follows. First, in (2.32), we have shown

(6.11)

\[
\int_t^1 \int_{\Sigma_s} |\nabla u|^2 |\nabla^2 u|^2 + \frac{1}{2} |\Sigma_1|^2 = O((1 - t)^{1+2\tau}).
\]

Also, by (2.11), (2.12) and Lemma 2.1,

(6.12)

\[
\int_t^1 \int_{\Sigma_s} \left( H - \frac{2|\nabla u|}{1 - u} \right)^2 = O((1 - t)^{1+2\tau}).
\]

Therefore, it suffices to check

(6.13)

\[
\lim_{t \to 1} \frac{1}{(1 - t)^2} \int_t^1 \int_{\Sigma_s} |R| = 0.
\]

But this follows from the coarea formula and the fact that $R$ is integrable. We have

\[
\int_{\Sigma_s} |R| = \int_{\Sigma_s} |R||\nabla u|^{-1} |\nabla u| = \left( \int_{\Sigma_s} |R||\nabla u|^{-1} \right) O((1 - s)^2)
\]

by (2.12). Hence,

\[
\int_t^1 \int_{\Sigma_s} |R| = O((1 - t)^2) \left( \int_t^1 \int_{\Sigma_s} |R||\nabla u|^{-1} \right).
\]

This combined with

\[
\lim_{t \to 1} \left( \int_t^1 \int_{\Sigma_s} |R||\nabla u|^{-1} \right) = \lim_{t \to 1} \int_{u \geq t} |R| = 0
\]

shows (6.13).
Now it follows from (6.8) – (6.10) that
\[
4\pi mc^{-1} - B(0) \\
\geq \frac{1}{2} \int_0^1 \left[ \frac{1}{(1-t)^2} - 1 \right] \psi(t) \\
= \frac{1}{4} \int_0^1 \left[ \frac{1}{(1-t)^2} - 1 \right] \int_{\Sigma_t} \left( \frac{|\nabla^2 u - \Phi u|^2}{|\nabla u|^2} + R \right) \\
= \frac{1}{4} \int_M \left[ \frac{1}{(1-u)^2} - 1 \right] \left( \frac{|\nabla^2 u - \Phi u|^2}{|\nabla u|^2} + R|\nabla u| \right).
\]
(6.14)

This proves (6.2).

Similarly, by (A.17) in Proposition A.1 of the Appendix,
\[
[A(t_2) - B(t_2)] - [A(t_1) - B(t_1)] \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \psi(t).
\]
(6.15)

Taking \( t_1 = 0 \), letting \( t_2 \to 1 \) and applying Theorem 2.1 we have
\[
8\pi mc^{-1} - (A(0) - B(0)) \\
\geq \frac{1}{2} \int_0^1 \frac{1}{(1-t)^2} \int_{\Sigma_t} \left( \frac{|\nabla^2 u - \Phi u|^2}{|\nabla u|^2} + R \right) \\
= \frac{1}{2} \int_M \left[ \frac{1}{(1-u)^2} \left( \frac{|\nabla^2 u - \Phi u|^2}{|\nabla u|^2} + R|\nabla u| \right) \right].
\]
(6.16)

This together with (6.14) proves (6.3).

\[ \Box \]

Remark 6.1. Suppose the scalar curvature \( R \) is nonnegative in Theorem 6.1 then (6.2) implies (3.16), (6.3) implies (3.15), and (6.16) implies
\[
4\pi - \int_{\Sigma} H|\nabla u| + \int_{\Sigma} |\nabla u|^2 \leq 8\pi mc^{-1}.
\]
(6.17)

For manifolds that are spatial Schwarzschild manifolds near infinity, (6.17) followed from the work of Agostiniani-Mazzieri-Oronzio [2]. On the other hand, one can see (6.17) is a direct algebraic consequence of (3.2) and (3.16).

Remark 6.2. If \( M \) in Theorem 6.1 has no boundary, then taking \( u = 1 - 4\pi G \), where \( G \) is the minimal positive Green’s function with a pole at some \( p \in M \), letting \( t_2 \to 1 \) and \( t_1 \to -\infty \) in (6.15), one has
\[
m \geq \frac{1}{(8\pi)^2} \int_M \frac{1}{G^2} \left( \frac{|\nabla^2 G + \Phi_G|^2}{|\nabla G|^2} + R|\nabla G| \right),
\]
(6.18)

where \( \Phi_G \) is the \((0, 2)\) tensor given by
\[
\Phi_G = \frac{|\nabla G|^2}{G} g - \frac{3dG \otimes dG}{G}.
\]
(6.18) is the integral version of the proof of the 3-d positive mass theorem in [2].
7. Promoting inequalities via Schwarzschild models

Inequalities in Section 3 are derived via monotone quantities that become constant in Euclidean spaces outside round balls. As a result, they are strict inequalities when evaluated in spatial Schwarzschild manifolds with nonzero mass outside rotationally symmetric spheres.

Inspired by Bray’s proof of the Riemannian Penrose inequality [5], in this section we apply results from the previous sections to derive inequalities that become equality in Schwarzschild spaces.

We first outline the idea. Given a tuple \((M, g, u)\) satisfying assumptions in Theorem 3.1 (or equivalently in Theorem 3.2), let \(v\) be any other harmonic function on \((M, g)\) with \(v \to 1\) at \(\infty\) and \(v > 0\) at \(\Sigma\). The following facts hold:

1. the metric \(\bar{g} := v^4 g\) is asymptotically flat, with nonnegative scalar curvature;
2. the function \(\bar{u} := v^{-1} u\) is a harmonic function with respect to the metric \(\bar{g}\), and satisfies \(\bar{u} = 0\) at \(\Sigma\) and \(\bar{u} \to 1\) at \(\infty\).

Thus, results from the previous sections are applicable to \(M\) with the conformally deformed metric \(\bar{g}\) and the \(\bar{g}\)-harmonic function \(\bar{u}\).

To proceed, we compute the quantities involved. Let \(\bar{\nabla}\) denote the gradient on \((M, \bar{g})\), let \(\bar{H}\) be the mean curvature of \(\Sigma\) in \((M, \bar{g})\) with respect to the \(\infty\)-pointing normal. Let \(d\sigma\), \(d\bar{\sigma}\) denote the surface measure on \(\Sigma\) in \((M, g)\), \((M, \bar{g})\), respectively. As \(\Sigma\) has dimension two, it can be checked

\[
\int_{\Sigma} |\bar{\nabla}\bar{u}|^2_{\bar{g}} d\bar{\sigma} = \int_{\Sigma} |\nabla\bar{u}|^2 d\sigma.
\]

(We omitted writing the area and volume measures in previous integrals as there was only one metric \(g\) involved therein.) The mean curvature \(H\) of \(\Sigma\) in \((M, g)\) via \(H = v^{-2}(4v^{-1}\partial_{\nu}v + H)\). Thus,

\[
\int_{\Sigma} \bar{H} |\nabla\bar{u}|_{\bar{g}} d\bar{\sigma} = \int_{\Sigma} \left(4v^{-1}\partial_{\nu}v + H\right) |\nabla\bar{u}| d\sigma.
\]

(7.2)

Let \(\bar{m}\) denote the mass of \((M, \bar{g})\). \(\bar{m}\) and \(m\) are related by

\[
\bar{m} = m - 2c_v,
\]

(7.3)

where \(c_v\) is the constant in the expansion

\[
v = 1 - \frac{c_v}{|x|} + o(|x|^{-1}),
\]

as \(x \to \infty\). Since \(\bar{u} = v^{-1} u\), \(\bar{u}\) satisfies

\[
\bar{u} = 1 - \frac{(c_{\Sigma} - c_v)}{|x|} + o(|x|^{-1}),
\]

where \(c_{\Sigma} > c_v\) by the fact \(v > u\) and the maximum principle. The capacity of \(\Sigma\) in \((M, \bar{g})\), which we denote by \(\bar{c}_{\Sigma}\), is then given by

\[
\bar{c}_{\Sigma} = c_{\Sigma} - c_v.
\]

(7.4)
Finally, we note, as $u = 0$ at $\Sigma$,

\begin{equation}
|\nabla u| = v^{-1}|\nabla u| \text{ at } \Sigma.
\end{equation}

We want to seek implications of the inequalities (3.2), (3.16), (3.15) and (6.17), i.e.

\begin{equation}
4\pi + \int_{\Sigma} H|\nabla u| \geq 3 \int_{\Sigma} |\nabla u|^2, 
\end{equation}

\begin{equation}
4\pi - \int_{\Sigma} |\nabla u|^2 \leq 4\pi m \epsilon^{-1},
\end{equation}

\begin{equation}
8\pi - \int_{\Sigma} H|\nabla u| \leq 12\pi m \epsilon^{-1},
\end{equation}

\begin{equation}
4\pi - \int_{\Sigma} H|\nabla u| + \int_{\Sigma} |\nabla u|^2 \leq 8\pi m \epsilon^{-1},
\end{equation}

when they are applied to the conformally deformed triple $(M, \bar{g}, \bar{u})$. As mentioned in Remark 3.3 and Remark 6.17, one knows

\begin{equation}
(7.6) + (7.7) \implies (7.8) \text{ and } (7.9).
\end{equation}

For this reason, we focus on the use of (7.6) and (7.7) below.

**Theorem 7.1.** Let $(M, g)$ be a connected, orientable, asymptotically flat 3-manifold with boundary $\Sigma$. Suppose $\Sigma$ is connected and $H_2(M, \Sigma) = 0$. Let $u$ be the harmonic function such that $u = 0$ at $\Sigma$ and $u \to 1$ at $\infty$. If $g$ has nonnegative scalar curvature, then, for any constant $k > 0$,

\begin{equation}
4\pi + k \int_{\Sigma} H|\nabla u| \geq k(4 - k) \int_{\Sigma} |\nabla u|^2.
\end{equation}

Moreover, equality in (7.10) holds for some $k$ if and only if $(M, g)$ is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere, that is, up to isometry,

\begin{equation}
(M, g) = \left( \mathbb{R}^3 \setminus \{|x| < r\}, \left(1 + \frac{m}{2|x|}\right)^4 g_E \right),
\end{equation}

where $r > 0$ is a constant, $g_E = \delta_{ij}dx^i dx^j$ is the Euclidean metric, and $m$, $k$, $r$ are related by $m = 2r(k - 1)$.

**Proof.** Given any positive harmonic function $v$ on $(M, g)$, let $\bar{g} = v^4g$ and $\bar{u} = v^{-1}u$. Applying (3.2) in Theorem 3.1 to the triple $(M, \bar{g}, \bar{u})$, we have

\begin{equation}
4\pi + \int_{\Sigma} \bar{H}|\nabla \bar{u}|_{\bar{g}} d\bar{\sigma} \geq 3 \int_{\Sigma} |\nabla \bar{u}|_{\bar{g}}^2 d\bar{\sigma}.
\end{equation}

By (7.1) – (7.5), (7.11) shows

\begin{equation}
4\pi + \int_{\Sigma} (4v^{-1}\partial_v v + H) v^{-1}|\nabla u| d\sigma \geq 3 \int_{\Sigma} v^{-2}|\nabla u|^2 d\sigma.
\end{equation}
Given any constant $k > 0$, choose
\begin{equation}
    v = u + \frac{1}{k} (1 - u).
\end{equation}
It follows from (7.12) and the fact $\partial_\nu u = |\nabla u|$ at $\Sigma$ that
\begin{equation}
    4\pi + k \int_{\Sigma} H |\nabla u| d\sigma \geq k (4 - k) \int_{\Sigma} |\nabla u|^2 d\sigma,
\end{equation}
which proves (7.10).

The above also shows equality in (7.10) holds for some $k$ if and only if equality in (7.11) holds for the corresponding $(M, \bar{g}, \bar{u})$. By Theorem 3.1, this occurs if and only if $(M, \bar{g})$ is isometric to $(\mathbb{R}^3 \setminus B_r, g_E)$, where $B_r = \{ x \in \mathbb{R}^3 \mid |x| < r \}$ for some constant $r > 0$. In this case,
\begin{equation}
    \bar{u} = 1 - \frac{r}{|x|}.
\end{equation}
This combined with (7.13) and the fact $\bar{u} = v^{-1} u$ shows
\begin{equation}
    v^{-1} = 1 + \frac{r(k - 1)}{|x|}.
\end{equation}
As a result,
\begin{equation}
    g = v^{-4} g_E = \left( 1 + \frac{r(k - 1)}{|x|} \right)^4 \delta_{ij} \, dx^i \, dx^j,
\end{equation}
which is a spatial Schwarzschild metric with mass $m = 2r(k - 1)$.

Theorem 7.1 implies a sharp bound of $\int_{\Sigma} |\nabla u|^2$ by the Willmore functional of $\Sigma$, with the bound achieved by Schwarzschild spaces outside mean-convex round spheres.

**Corollary 7.1.** Let $(M, g)$ be a connected, orientable, asymptotically flat 3-manifold with boundary $\Sigma$. Suppose $\Sigma$ is connected and $H_2(M, \Sigma) = 0$. Let $u$ be the harmonic function such that $u = 0$ at $\Sigma$ and $u \to 1$ at $\infty$. If $g$ has nonnegative scalar curvature, then
\begin{equation}
    \left( \frac{1}{\pi} \int_{\Sigma} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{16\pi} \int_{\Sigma} H^2 \right)^{\frac{1}{2}} + 1.
\end{equation}
Moreover, equality holds if and only if $(M, g)$ is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

**Proof.** Consider the following quadratic form of $k$,
\begin{equation}
    Q(k) := \alpha(u) k^2 + \beta(u) k + 4\pi,
\end{equation}
where
\begin{align*}
    \alpha(u) &= \int_{\Sigma} |\nabla u|^2, \quad \beta(u) = \int_{\Sigma} H |\nabla u| - 4 \int_{\Sigma} |\nabla u|^2.
\end{align*}
We have $Q(0) = 4\pi$, and Theorem 7.1 shows
\begin{equation}
    Q(k) \geq 0, \quad \forall k > 0.
\end{equation}
Thus, by elementary reasons, either
\begin{equation}
\beta(u)^2 - 16\pi\alpha(u) \leq 0
\end{equation}
or
\begin{equation}
\beta(u)^2 - 16\pi\alpha(u) > 0 \quad \text{and} \quad -\beta(u) + \sqrt{\beta(u)^2 - 16\pi\alpha(u)} < 0.
\end{equation}
The latter case is equivalent to
\begin{equation}
\beta(u) > \sqrt{16\pi\alpha(u)},
\end{equation}
that is
\begin{equation}
\frac{\int_{\Sigma} H|\nabla u|}{(\int_{\Sigma} |\nabla u|^2)^{\frac{1}{2}}} - 4\left(\int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}} > \sqrt{16\pi}.
\end{equation}
If (7.21) holds, then, by Hölder’s inequality,
\begin{equation}
\left(\frac{1}{16\pi} \int_{\Sigma} H^2\right)^{\frac{1}{2}} > 1 + \left(\frac{1}{\pi} \int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}}.
\end{equation}
If (7.19) holds, then
\begin{equation}
\left|\int_{\Sigma} H|\nabla u| - 4\int_{\Sigma} |\nabla u|^2\right| \leq 4\left(\pi \int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}},
\end{equation}
which in particular implies
\begin{equation}
4\int_{\Sigma} |\nabla u|^2 \leq 4\left(\pi \int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}} + \int_{\Sigma} H|\nabla u|.
\end{equation}
Combined with Hölder’s inequality, this shows
\begin{equation}
\left(\frac{1}{\pi} \int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}} \leq 1 + \left(\frac{1}{16\pi} \int_{\Sigma} H^2\right)^{\frac{1}{2}}.
\end{equation}
Therefore, in either case, we conclude (7.17) holds.

If equality in (7.17) holds, then (7.20) does not hold; (7.23) holds with equality; and $H = c|\nabla u|$ for some constant $c \geq 0$. In particular, this gives
\begin{equation}
-\beta(u) = 4\int_{\Sigma} |\nabla u|^2 - \int_{\Sigma} H|\nabla u| = \sqrt{16\pi\alpha(u)} > 0.
\end{equation}
As a result, $Q(k_0) = 0$ at
\begin{equation}
k_0 = -\frac{\beta(u)}{2\alpha(u)} = 2\left(\frac{1}{\pi} \int_{\Sigma} |\nabla u|^2\right)^{-\frac{1}{2}} = \frac{2}{1 + \left(\frac{1}{16\pi} \int_{\Sigma} H^2\right)^{\frac{1}{2}}} > 0.
\end{equation}
By Theorem 7.1, $(M, g)$ is isometric to a spatial Schwarzschild manifold
\begin{equation}
(M, g) = \left(\mathbb{R}^3 \setminus \{|x| < r\}, \left(1 + \frac{m}{2|x|}\right)^4 g_\varepsilon\right),
\end{equation}
where \( r > 0 \) and \( m = 2r(k_0 - 1) \). As \( k_0 \leq 2 \), the boundary \( \{|x| = r\} \) has nonnegative mean curvature in \((M, g)\).

On such an \((M, g)\), direct calculation shows
\[
\left( \frac{1}{16\pi} \int \Sigma H^2 \right)^{\frac{1}{2}} = \frac{2}{k} - 1 \quad \text{and} \quad \left( \frac{1}{\pi} \int \Sigma |\nabla u|^2 \right)^{\frac{1}{2}} = \frac{2}{k}.
\]
As \( k \leq 2 \), equality holds in (7.17). This completes the proof. \( \square \)

An immediate application of Corollary 7.1 yields a result of Bray and the author \cite{7} on the estimate of the capacity-to-area-radius ratio.

**Theorem 7.2** \((\cite{7})\). Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with boundary \( \Sigma \). Suppose \( \Sigma \) is connected and \( H_2(M, \Sigma) = 0 \). If \( g \) has nonnegative scalar curvature, then

\[
(7.25) \quad \frac{2c_\Sigma}{r_\Sigma} \leq \left( \frac{1}{16\pi} \int \Sigma H^2 \right)^{\frac{1}{2}} + 1.
\]

Here \( c_\Sigma \) is the capacity of \( \Sigma \) in \((M, g)\) and \( r_\Sigma = \left( \frac{|\Sigma|^2}{4\pi} \right)^{\frac{1}{2}} \) is the area-radius of \( \Sigma \). Moreover, equality holds if and only if then \((M, g)\) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

**Proof.** This follows directly from
\[
\left( \int \Sigma |\nabla u|^2 \right)^{\frac{1}{2}} \geq \frac{\int \Sigma |\nabla u|}{|\Sigma|^{\frac{1}{2}}} = \sqrt{\frac{2\pi}{r_\Sigma}} c_\Sigma
\]
and Corollary 7.1. \( \square \)

Next, we proceed to find implications of (3.16) in Theorem 3.2.

**Theorem 7.3.** Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with boundary \( \Sigma \). Suppose \( \Sigma \) is connected and \( H_2(M, \Sigma) = 0 \). Let \( u \) be the harmonic function such that \( u = 0 \) at \( \Sigma \) and \( u \to 1 \) at \( \infty \). If \( g \) has nonnegative scalar curvature, then

\[
(7.26) \quad \frac{m}{2c_\Sigma} \geq 1 - \left( \frac{1}{4\pi} \int \Sigma |\nabla u|^2 \, d\sigma \right)^{\frac{1}{2}}.
\]

Moreover, equality holds if and only if \((M, g)\) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere.

**Proof.** Given any positive harmonic function \( v \) on \((M, g)\), let \( \bar{g} = v^4 g \) and \( \bar{u} = v^{-1} u \). Applying (3.16) in Theorem 3.2 to \((M, \bar{g}, \bar{u})\), we have

\[
(7.27) \quad 4\pi - \int \Sigma |\nabla \bar{u}|^2 \, d\bar{\sigma} \leq 4\pi m c_\Sigma^{-1}.
\]
By (7.1) – (7.5), (7.27) becomes

\[
(7.28) \quad 4\pi \int_{\Sigma} v^{-2} |\nabla u|^2 \, d\sigma \leq 4\pi \frac{m - 2c_v}{c_\Sigma - c_v}.
\]

Given any constant \( k > 0 \), choose

\[
(7.29) \quad v = u + \frac{1}{k}(1 - u).
\]

Then \( v = k^{-1} \) at \( \Sigma \), \( c_v = (1 - k^{-1})c_\Sigma \), and (7.28) shows

\[
(7.30) \quad \frac{m}{c_\Sigma} \geq 2 - \frac{1}{k} - \frac{k}{4\pi} \int_{\Sigma} |\nabla u|^2 \, d\sigma.
\]

Maximizing the right side of (7.30) over all \( k > 0 \), we have

\[
(7.31) \quad \frac{m}{2c_\Sigma} \geq 1 - \left( \frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2 \, d\sigma \right)^{\frac{1}{2}},
\]

which proves (7.26).

If equality in (7.26) holds, then equality in (7.27) holds for \( v = u + k^{-1}(1 - u) \) with the constant \( k \) given by

\[
(7.32) \quad k = \left( \frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2 \, d\sigma \right)^{-\frac{1}{2}}.
\]

By Theorem 3.2 \((M, \bar{g})\) is isometric to \((\mathbb{R}^3 \setminus B_r, g_E)\), where \( B_r = \{ x \in \mathbb{R}^3 \mid |x| < r \} \) for some \( r > 0 \), and

\[
(7.33) \quad \bar{u} = 1 - \frac{r}{|x|}.
\]

This combined with \( \bar{u} = v^{-1}u \) and (7.29) shows

\[
(7.34) \quad v^{-1} = 1 + \frac{r(k - 1)}{|x|}.
\]

As a result,

\[
(7.35) \quad g = v^{-4}g_E = \left( 1 + \frac{r(k - 1)}{|x|} \right)^4 \delta_{ij} \, dx^i dx^j,
\]

which is a spatial Schwarzschild metric with the mass \( m = 2r(k - 1) \).

On any such an \((M, g)\), direct calculation shows

\[
\frac{m}{2c_\Sigma} = 1 - \frac{1}{k} \quad \text{and} \quad \left( \frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2 \right)^{\frac{1}{2}} = \frac{1}{k},
\]

which verifies equality in (7.26). This completes the proof. \( \square \)

We now have a succinct lower bound of the mass-to-capacity ratio by the Willmore functional.
Theorem 7.4. Let \((M, g)\) be a connected, orientable, asymptotically flat 3-manifold with boundary \(\Sigma\). Suppose \(\Sigma\) is connected and \(H_2(M, \Sigma) = 0\). If \(g\) has nonnegative scalar curvature, then

\[
\frac{m}{c_\Sigma} \geq 1 - \left( \frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}}.
\]

Moreover, equality holds if and only if \((M, g)\) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

Proof. This is a direct consequence of Corollary 7.1 and Theorem 7.3. 

We give a few remarks.

Remark 7.1. Theorem 7.4 gives another way to see the 3-dimensional positive mass theorem. In the context of Proof III in Section 4, Theorem 7.4 shows

\[
\frac{m}{c_r} \geq 1 - \left( \frac{1}{16\pi} \int_{\Sigma_r} H^2 \right)^{\frac{1}{2}} = o(1), \text{ as } r \to 0,
\]

where \(c_r\) is the capacity of a small geodesic ball of radius \(r\) centered at a point \(p \in M\). Hence, \(m \geq 0\).

Remark 7.2. Under an additional assumption of \(m_\Sigma(\Sigma) \geq 0\), (7.36) was found in [7] via the method of \(1/H\) flow. The argument [7] was to apply the relation

\[
m \geq m_\Sigma(\Sigma)
\]

(assuming \(\Sigma\) is outer-minimizing) and to convert the capacity estimate (7.25) to a Hawking mass estimate

\[
m_\Sigma(\Sigma) \geq \left[ 1 - \left( \frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}} \right] \frac{2c_\Sigma}{r_\Sigma}.
\]

To obtain (7.37) from (7.25), one used the assumption \(\int_\Sigma H^2 \leq 1\).

In our current derivation of (7.36), we directly find a lower bound of the ratio \(m c_{\Sigma}^{-1}\) via \(\int_\Sigma |\nabla u|^2\) and obtain an upper bound of \(\int_\Sigma |\nabla u|^2\) via \(\int_\Sigma H^2\), therefore bypassing the use of \(m_\Sigma(\Sigma)\) in relating \(m\) and \(c_\Sigma\).

Remark 7.3. One may equivalently view the result in Theorem 7.4 as

\[
\mathcal{M}(g) := \frac{m}{c_\Sigma} + \left( \frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}} - 1 \geq 0.
\]

This interpretation gives a nonnegative quantity \(\mathcal{M}(g)\) on any asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary (assuming the topological condition on \(M\)). \(\mathcal{M}(g)\) vanishes precisely if \((M, g)\) is rotationally symmetric.
APPENDIX A. REGULARIZATION AND INTEGRATION

In this appendix, we give the regularization arguments that can be used to verify the monotonicity of $\Psi(t)$, $A(t)$ and $B(t)$ in Section 3.

**Lemma A.1.** Let $u$ be a harmonic function on a compact Riemannian manifold $(\Omega, g)$ with boundary $\partial \Omega$. Suppose $\max_\Omega u < 1$. Then

\begin{equation}
(A.1) \quad \int_{\partial \Omega} \frac{1}{1 - u} \frac{\partial u}{\partial \zeta} = \int_{\Omega} \frac{|\nabla u|^3}{(1-u)^2} + \int_{\{u \neq 0\} \subset \Omega} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1-u)|\nabla u|}
\end{equation}

and

\begin{equation}
(A.2) \quad \int_{\partial \Omega} \frac{1}{1 - u^3} \frac{\partial u}{\partial \zeta} = \int_{\Omega} \frac{3|\nabla u|^3}{(1-u)^4} + \int_{\{u \neq 0\} \subset \Omega} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3|\nabla u|}.
\end{equation}

Here $\zeta$ denotes the unit normal to $\partial \Omega$ pointing out of $\Omega$.

**Proof.** Given any constant $\epsilon > 0$, one has

$$\text{div} \left( \frac{\sqrt{|\nabla u|^2 + \epsilon}}{1 - u} \nabla u \right) = \frac{\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} |\nabla u|^2 + \frac{1}{1-u} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.$$ 

Therefore,

\begin{equation}
(A.3) \quad \int_{\partial \Omega} \frac{\sqrt{|\nabla u|^2 + \epsilon}}{1 - u} \frac{\partial u}{\partial \zeta} = \int_{\Omega} \frac{\sqrt{|\nabla u|^2 + \epsilon}}{1 - u} |\nabla u|^2 + \int_{\Omega} \frac{1}{1-u} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.
\end{equation}

For the third term in (A.3), one notes

$$\int_{\Omega} \frac{1}{1-u} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}} = \int_{\{u \neq 0\} \subset \Omega} \frac{1}{1-u} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.$$

Thus, taking $\epsilon \to 0$ in (A.3) proves (A.1).

Similarly, to show (A.2), one has

$$\text{div} \left( \frac{\sqrt{|\nabla u|^2 + \epsilon}}{1 - u^3} \nabla u \right) = 3\frac{\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^4} |\nabla u|^2 + \frac{1}{(1-u)^3} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.$$ 

Consequently,

\begin{equation}
(A.4) \quad \int_{\partial \Omega} \frac{\sqrt{|\nabla u|^2 + \epsilon}}{1 - u^3} \frac{\partial u}{\partial \zeta} = \int_{\Omega} 3\frac{\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^4} |\nabla u|^2 + \int_{\Omega} \frac{1}{(1-u)^3} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.
\end{equation}

The third term above satisfies

$$\int_{\Omega} \frac{1}{(1-u)^3} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}} \leq \int_{\{u \neq 0\} \subset \Omega} \frac{1}{(1-u)^3} \frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2 + \epsilon}}.$$ 

Thus, (A.2) follows by taking $\epsilon \to 0$ in (A.4). \qed
Lemma A.2. Let \( u \) be a harmonic function on a compact, orientable, Riemannian 3-manifold \((\Omega, g)\) with boundary \( \partial \Omega \). Suppose \( \operatorname{max}_\Omega u < 1 \) and \( u \) equals a constant on each connected component of \( \partial \Omega \). Then

\[
(A.5) \quad \int_{\partial \Omega} H |\nabla u|^2 \leq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left\{ \int_{\Sigma_t} \left[ 2H |\nabla u| - \frac{1}{2} \left( \frac{|\nabla^2 u|^2}{|\nabla u|^2} + R \right) \right] + 2\pi \chi(\Sigma_t) \right\}.
\]

Here the mean curvature \( H \) of \( \partial \Omega \) is taken with respect to the unit normal \( \zeta \) pointing out of \( \Omega \), the mean curvature \( H \) of a regular level set \( \Sigma_t \) is taken with respect to \( |\nabla u|^{-1} |\nabla u| \), \( \chi(\Sigma_t) \) is the Euler characteristic of \( \Sigma_t \), \( t_1 = \min_\Omega u \), and \( t_2 = \max_\Omega u \).

Proof. For any constant \( \epsilon > 0 \), one has

\[
\operatorname{div} \left[ \frac{\nabla \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \right] = \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} + \frac{2 \nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 \sqrt{|\nabla u|^2 + \epsilon}}.
\]

Therefore,

\[
(A.6) \quad \int_{\partial \Omega} \partial_\zeta \sqrt{|\nabla u|^2 + \epsilon} = \int_\Omega \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} + \int_\Omega \frac{2 \nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 \sqrt{|\nabla u|^2 + \epsilon}}.
\]

As \( u \) is constant on each connected component of \( \partial \Omega \), direct calculations gives

\[
\partial_\zeta \sqrt{|\nabla u|^2 + \epsilon} = -\frac{|\nabla u|^2}{\sqrt{|\nabla u|^2 + \epsilon}} H.
\]

(See Lemma 2.1 in [12] for instance.) Thus,

\[
(A.7) \quad \lim_{\epsilon \to 0} \int_{\partial \Omega} \partial_\zeta \sqrt{|\nabla u|^2 + \epsilon} = -\int_{\partial \Omega} \frac{H |\nabla u|}{(1-u)^2}.
\]

As in the proof of the previous lemma, taking \( \epsilon \to 0 \) in the third term in \( (A.6) \) gives

\[
\lim_{\epsilon \to 0} \int_\Omega \frac{2 \nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 \sqrt{|\nabla u|^2 + \epsilon}} = \int_{\{\nabla u \neq 0\} \subset \Omega} \frac{2 \nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 |\nabla u|}
\]

\[
(A.8) \quad = -\int_{t_1}^{t_2} \frac{2}{(1-t)^3} \int_{\Sigma_t} H |\nabla u|,
\]

where the second equation follows from the coarea formula and \( (3.5) \).

To deal with the second term in \( (A.6) \), we follow an argument of Stern [21]. Let \( \mathcal{C} \) denote the set of critical values of \( u \) in \( [t_1, t_2] \). Let \( W \) denote an open set of \( [t_1, t_2] \) such that \( W \) contains \( \mathcal{C} \). Let \( D \) be the complement of \( W \) in \( [t_1, t_2] \).

On \( u^{-1}(D) \), \( \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \) is integrable. By coarea formula,

\[
\int_{u^{-1}(D)} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} = \int_D \int_{\Sigma_t} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2 |\nabla u|}.
\]
Along $\Sigma_t$ which a regular level set of $u$, by equation (14) in [21],

\begin{equation}
\Delta \sqrt{|\nabla u|^2 + \epsilon} \geq \frac{1}{2\sqrt{|\nabla u|^2 + \epsilon}} \left[ |\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2 \right],
\end{equation}

where $K_{\Sigma_t}$ is the Gauss curvature of $\Sigma_t$. Thus,

\begin{equation}
\int_{u^{-1}(D)} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \geq \int_D \int_{\Sigma_t} \frac{1}{(1-u)^2} |\nabla u| \left[ |\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2 \right] 2\sqrt{|\nabla u|^2 + \epsilon}.
\end{equation}

With $W$ fixed, letting $\epsilon \to 0$ in (A.10) gives

\begin{equation}
\liminf_{\epsilon \to 0} \int_{u^{-1}(D)} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \geq \int_D \int_{\Sigma_t} \frac{|\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2}{(1-u)^2 2|\nabla u|^2} \geq \int_D \frac{1}{(1-t)^2} \left[ \int_{\Sigma_t} \frac{1}{2} (|\nabla u|^{-2}|\nabla^2 u|^2 + R) - 2\pi \chi(\Sigma_t) \right],
\end{equation}

where one also used the Gauss-Bonnet theorem.

To estimate the integral on $u^{-1}(W)$, one notes

\begin{align*}
\Delta \sqrt{|\nabla u|^2 + \epsilon} &= \frac{1}{\sqrt{|\nabla u|^2 + \epsilon}} \left[ |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) - \frac{1}{4(|\nabla u|^2 + \epsilon)|\nabla|\nabla u|^2| \right] \\
&\geq \frac{1}{\sqrt{|\nabla u|^2 + \epsilon}} \text{Ric}(\nabla u, \nabla u).
\end{align*}

This implies

\begin{equation}
\int_{u^{-1}(W)} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \geq - \max_{\Omega} |\text{Ric}| \int_{u^{-1}(W)} \frac{|\nabla u|}{(1-u)^2} \\
= - \max_{\Omega} |\text{Ric}| \int_W \int_{\Sigma_t} \frac{1}{(1-u)^2}.
\end{equation}

It follows from (A.11) and (A.12) that

\begin{align*}
\lim_{\epsilon \to 0} \int_{\Omega} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \geq \int_D \frac{1}{(1-t)^2} \left[ \int_{\Sigma_t} \frac{1}{2} (|\nabla u|^{-2}|\nabla^2 u|^2 + R) - 2\pi \chi(\Sigma_t) \right] \\
- \max_{\Omega} |\text{Ric}| \int_W \int_{\Sigma_t} \frac{1}{(1-u)^2}.
\end{align*}

As $\int_{\Omega} \frac{|\nabla u|}{(1-u)^2} < \infty$, by choosing the measure of $W$ to be arbitrarily small, one has

\begin{equation}
\lim_{\epsilon \to 0} \int_{\Omega} \frac{\Delta \sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left[ \int_{\Sigma_t} \frac{1}{2} (|\nabla u|^{-2}|\nabla^2 u|^2 + R) - 2\pi \chi(\Sigma_t) \right].
\end{equation}

The lemma now follows from (A.6), (A.7), (A.8) and (A.13). $\square$

**Remark A.1.** (A.5) may be viewed as a weighted version of the identity (3.24).
Proposition A.1. Let \((\Omega, g)\) be a connected, compact, orientable, Riemannian 3-manifold with boundary \(\partial \Omega\). Suppose \(\partial \Omega\) is the disjoint union of two nonempty pieces \(S_1\) and \(S_2\). Let \(u\) be a harmonic function on \((\Omega, g)\) such that \(u = c_i\) on \(S_i\), \(i = 1, 2\), where \(c_1, c_2\) are constants with \(c_1 < c_2 < 1\). For regular values \(t\), let

\[
A(t) = 8\pi - \frac{1}{(1 - t)} \int_{\Sigma_t} H |\nabla u|, \quad \mathcal{A}(t) = \frac{A(t)}{1 - t},
\]

\[
B(t) = 4\pi - \frac{1}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2, \quad \mathcal{B}(t) = \frac{B(t)}{1 - t}.
\]

Then, for any \(t_1 < t_2\),

\[
\mathcal{B}(t_2) - \mathcal{B}(t_1) = \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} [3B(t) - A(t)], \quad \text{(A.14)}
\]

and

\[
\mathcal{A}(t_2) - \mathcal{A}(t_1) \geq \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} [3B(t) - A(t) + 2\pi (2 - \chi(\Sigma_t)) + \psi(t)], \quad \text{(A.15)}
\]

where

\[
\psi(t) = \int_{\Sigma_t} \left[ \frac{3}{4} \left( H - \frac{2|\nabla u|}{1 - u} \right)^2 + |\nabla u|^{-2} |\nabla u|_\Sigma |\nabla u|^2 + \frac{1}{2} |\mathcal{II}|^2 + \frac{1}{2} R \right].
\]

As a result, if

1. \((M, g)\) is a connected, orientable, asymptotically flat 3-manifold with connected boundary \(\Sigma\) and \(H_2(M, \Sigma) = 0\);
2. \(u\) is the harmonic function on \((M, g)\) with \(u = 0\) at \(\Sigma\) and \(u \to 1\) at \(\infty\); and
3. \(g\) has nonnegative scalar curvature,

then \(\Sigma_t\) is connected, \(3B(t) - A(t) \geq 0\) by \([33]\), and consequently,

\[
\mathcal{B}(t_2) - \mathcal{B}(t_1) \geq 0 \quad \text{and} \quad \mathcal{A}(t_2) - \mathcal{A}(t_1) \geq 0, \quad \forall\ t_2 > t_1.
\]

Proof. Applying (A.2) in Lemma A.1 to \(\Omega_{[t_1, t_2]} = \{x \mid t_1 \leq u(x) \leq t_2\}\), one has

\[
\int_{\Sigma_{t_2}} \frac{|\nabla u|^2}{(1 - u)^3} - \int_{\Sigma_{t_1}} \frac{|\nabla u|^2}{(1 - u)^3} = \int_{\Omega_{[t_1, t_2]}} \frac{3|\nabla u|^3}{(1 - u)^4} + \int_{\{u \neq 0\} \subset \Omega_{[t_1, t_2]}} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1 - u)^3|\nabla u|} = \int_{t_1}^{t_2} \int_{\Sigma_t} \left[ \frac{3|\nabla u|^2}{(1 - t)^4} - \frac{H|\nabla u|}{(1 - t)^3} \right],
\]

This, combined with \(\frac{1}{1 - t_2} - \frac{1}{1 - t_1} = \int_{t_1}^{t_2} \frac{1}{(1 - t)^2}\), shows

\[
\mathcal{B}(t_2) - \mathcal{B}(t_1) = \int_{t_1}^{t_2} \left[ \frac{4\pi}{(1 - t)^2} + \int_{\Sigma_t} \frac{H|\nabla u|}{(1 - t)^3} - \int_{\Sigma_t} \frac{3|\nabla u|^2}{(1 - t)^4} \right] = \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} [3B(t) - A(t)], \quad \text{(A.16)}
\]

which proves (A.14).
Similarly, applying Lemma A.2 to \( u \) on \( \Omega_{[t_1, t_2]} \) and using (3.6), one has

\[
\frac{1}{(1 - t_2)^2} \int_{\Sigma_{t_2}} H |\nabla u| - \frac{1}{(1 - t_1)^2} \int_{\Sigma_{t_1}} H |\nabla u| \\
\leq \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} \left[ 2\pi \chi(\Sigma_t) + \int_{\Sigma_t} 2H |\nabla u| \right. \\
- \left. \int_{\Sigma_t} \frac{3}{4} H^2 - \int_{\Sigma_t} \left( |\nabla u|^2 |\nabla_{\Sigma_t} |\nabla u||^2 + \frac{1}{2} \frac{\|\mathbb{II}\|^2}{H} + \frac{1}{2} R \right) \right] \\
= \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} \left[ 2\pi \chi(\Sigma_t) - \psi(t) - \frac{1}{1 - t} \int_{\Sigma_t} H |\nabla u| + \frac{3}{(1 - t)^2} \int_{\Sigma_t} |\nabla u|^2 \right].
\]

Therefore,

\[
\mathcal{A}(t_2) - \mathcal{A}(t_1) \geq \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} \left[ 3B(t) - A(t) + 2\pi (2 - \chi(\Sigma_t)) + \psi(t) \right],
\]

which proves (A.15). \( \square \)

**Remark A.2.** As a corollary of (A.14) and (A.15),

\[
[A(t_2) - B(t_2)] - [A(t_1) - B(t_1)] \\
\geq \int_{t_1}^{t_2} \frac{1}{(1 - t)^2} \left[ 2\pi (2 - \chi(\Sigma_t)) + \psi(t) \right].
\]

(A.17)

This corresponds to the monotonicity of \( F(t) = A(t) - B(t) \) in [2].

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(Pengzi Miao) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FL 33146, USA

*Email address:* pengzim@math.miami.edu