Nonlinear Dynamic Analysis of Functionally Graded Timoshenko Beam fixed to a Rotating Hub

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Abstract. The present work accounts centrifugal stiffening effect on the nonlinear vibration response of an FGM Timoshenko beam. Analysis is carried out for a cantilever beam fixed with a rotating hub. Material is assumed to have a gradation relation along the depth of the beam. Centrifugal force and axial displacement raised due to the rotating hub is incorporated in the strain energy equations. Subsequent to this, an iterative technique is employed to obtain amplitude dependent vibration response of a rotating Timoshenko beam while material follows a gradation relation along the beam depth. Main objective of the work is to obtain the effects of rotational speeds, hub radius, and different gradation relations on the linear as well as nonlinear frequencies and mode shapes.

1. Introduction
Functionally graded materials (FGMs) have gained tremendous popularity in recent years. Use of these materials becomes more essential especially in situations where unevenly distributed thermal, chemical or mechanical loads are present. Such materials can be used in rotating machinery components for better performance. In order to avoid undesirable resonance phenomena and determination of suitable vibration control strategies, it becomes much more important to observe the vibration characteristics of such system. Numbers of investigations on vibration response of rotating beams have been proposed in recent times. Putter and Manor [1] and Kaya [2] found out natural frequencies of a rotating Timoshenko beam using differential quadrature method. Benerjee [3] obtained the effect of centrifugal stiffening on the dynamic response by deriving the dynamic stiffness matrix. Pohit et al. [4] used Perturbation technique to obtain free vibration response of a rotating beam with a nonlinear elastomeric constraint. Panigrahi and Pohit [5] proposed an iterative technique to study nonlinear free vibration of cracked FGM beams.

It is evident from the review that most of the literatures on the vibration of rotating beams have focused on linear vibration problem of isotropic beams. Literatures explaining the nonlinear effects and functionally graded material properties are rare. The present work accounts centrifugal stiffening effect on the nonlinear vibration response of a Timoshenko beam made of FGM. Main objective of the work is to obtain the effects of rotational speeds, hub radius and different gradation relations on the linear as well as nonlinear frequencies and mode shapes.

2. Solution methodology
Fig. 1 shows a cantilever beam of length L and thickness h, fixed end of which is attached to a hub of radius R and rotating with an angular speed of \( \Omega \) rad/sec. Midplane of the beam can be represented by \( z=0 \). Variation of the properties are assumed to vary exponentially in thickness direction as given in eq. 1, where \( E_t \) and \( \rho_t \) are effective young’s modulus and mass density of beam at top surface.
respectively and \( k \) is the material index \((E_2/E_1)\). Poisson’s ratio \( \nu \) is taken to be constant throughout the analysis.

\[
E(z) = E_1 \sqrt{k e} \left( \frac{z \ln(k)}{h} \right), \quad \rho(z) = \rho_1 \sqrt{k e} \left( \frac{z \ln(k)}{h} \right)
\]

(1)

In present work neutral plane is considered for the analysis purpose, in order to avoid bending-extension coupling [5]. Position of the neutral plane from the mid plane of the beam is \( C \) distance apart. Value of \( C \) can be easily obtained [5].

2.1 Timoshenko beam theory and energy equations

In fig. 1, \( z \) is the coordinate in the thickness direction, \( x_0 = R + x \) is the axial coordinate, where \( x \) is the span wise distance of the point from the hub edge. Displacement fields in space and time coordinates of Timoshenko beam can be given as follows.

\[
U(x, t) = U(x, t) + z(\Psi(x, t)), \quad W(x, t) = W(x, t)
\]

(2)

In the eq. 2, \( U(x, t), W(x, t) \) and \( \Psi(x, t) \) represent longitudinal stretching, transverse and rotational displacement of neutral plane, respectively. Von Kerman type nonlinearity is incorporated in Strain-displacement relation. Motion is assumed to be harmonic motion of frequency \( \theta \) rad/sec. Following the energy method as expressed in [5] and incorporating the effect of centrifugal force and axial displacement raised due to rotating condition as potential of load (PL), Maximum form of Kinetic energy (KE_{max}), Potential energy (PE_{max}) and potential of load (PL_{max}) can be derived as

\[
KE_{\text{max}} = \frac{\theta^2}{2} \int_0^L \left( M_1 U'^2 + M_2 \Psi'^2 + M_4 W'^2 \right)dx, \quad PE_{\text{max}} = \frac{1}{2} \int_0^L \left( K_1 \left( \frac{\partial U}{\partial x} \right)^2 + K_3 \left( \frac{\partial \Psi}{\partial x} \right)^2 + K_4 \left( \frac{\partial W}{\partial x} + \Psi \right)^2 \right) dx
\]

\[
+ K_1 \left( \frac{\partial U}{\partial x} \right)^2 \left( \frac{\partial W}{\partial x} \right)^2 + K_1 \left( \frac{\partial W}{\partial x} \right)^4 \right)
\]

\[
PL_{\text{max}} = \frac{1}{2} \int_0^L M_1 \left( \frac{R}{L^2} \right) \left( \frac{R x}{L^2} \right)^2 + \frac{1}{2} \left( \frac{x^2}{2L^2} \right) \left( \frac{\partial W}{\partial x} \right)^2 dx
\]

(3)

Here \( K_1, K_2, K_3 \) and \( K_4 \) are stiffness parameter and \( M_1, M_2 \) and \( M_4 \) are inertial parameter as given by Panigrahi and Pohit [5]. Dimensionless parameters are taken as mentioned in eq. 4

\[
\xi = \frac{x}{L}, \quad u = \frac{U}{h}, \quad w = \frac{W}{h}, \quad \psi = \Psi, \quad \alpha = \frac{L}{h}, \quad \beta = \frac{R}{h}, \quad \delta = \frac{R}{L}, \quad \gamma = \frac{I}{AL^2} = \frac{h^2}{12L^2}, \quad \epsilon = \frac{m_3, k_3, k_4}{M_3, K_4}
\]

\[
\omega^2 = \frac{12 M_1 \theta^2 L^4}{K_4 h^2}, \quad \Omega^2 = \frac{12 M_1 \Omega^2 L^4}{K_4 h^2}
\]

(4)
Using eq. 3 and eq. 4 and multiplying each expression of eq. 3 by $12\alpha^2 / K_1 h$ dimensionless form of energy expression can be derived separately for linear and nonlinear potential energy as

$$KE_{\text{max}}^{*} = \frac{\omega^2}{2} \left[ \int_0^1 \left( u_1^2 + m_3 \psi_1^2 + w_1^2 \right) d\xi \right]$$

$$PE_{\text{linear}}^{*} = \frac{12\alpha^2}{2} \int_0^1 \left( \frac{\partial u}{\partial \xi} \right)^2 + k_3 \left( \frac{\partial \psi}{\partial \xi} \right)^2 + k_4 \left( \frac{\partial w}{\partial \xi} \right)^2 + k_5 \alpha^2 \psi^2 + k_6 \alpha \frac{\partial \psi}{\partial \xi} \right) d\xi$$

$$PE_{\text{nl}}^{*} = \frac{12\alpha^2}{2} \int_0^1 \left( \frac{\partial u}{\partial \xi} \right)^2 + \frac{1}{4\alpha^2} \left( \frac{\partial \psi}{\partial \xi} \right)^2 \right) d\xi$$

Energy functional can be obtained as -

$$\Pi = PE_{\text{linear}}^{*} + PE_{\text{nonlinear}}^{*} + PL_{\text{max}}^{*} - KE_{\text{max}}^{*}$$

Displacements (admissible functions) are chosen as $u = \sum_{j=1}^{\infty} A_j \xi^j, w = \sum_{j=1}^{\infty} B_j \xi^j, u = \sum_{j=1}^{\infty} C_j \xi^j$. Minimizing the energy functional by taking derivative with unknown coefficients, sets of nonlinear governing equations can be derived in the form of

$$[K_{\text{linear}}] \{U\} + [K_{\text{nonlinear}}] \{U\} = \omega^2 [M] \{U\}$$

Here UC is unknown coefficients. Elements of nonlinear stiffness matrix contain terms of unknown coefficients as well. In order to solve nonlinear problem, an iterative technique is employed with the following steps: First all the nonlinear terms are neglected and a linear set of unknown coefficients are obtained. Linear set of unknown coefficients are then scaled up in such a manner that the maximum nonlinear displacement occurs at the free end and is equal to the assumed amplitude of vibration ($w_{\text{max}}$). Using the scaled unknown vectors nonlinear set of vectors are found out. Nonlinear set of unknown coefficient vectors are again scaled up. Repeating the above process until relative error between two consecutive values of the non-linear frequency is less than the tolerance.

### 3. Results and discussions

A matlab code is developed for analysis purpose. Number of Polynomial terms is taken as eight.

#### 3.1 Comparison study

Results obtained from the present analysis are compared with the results available in the literatures. For comparison purpose geometric and material properties are taken as $\delta=0, E_2/\kappa G_1 = 3.059, r=1/30$, where $\kappa$ is shear correction factor. Otherwise for all the results $r$ is taken as 0.1 and other properties remain same until and unless specified. Table 1 show that the results from present analysis matched fairly well with that of Benerjee [3].

| $\Omega$ | 0 | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|---|
| Dimensionless First linear frequency (present) | 3.4798 | 3.6451 | 4.0995 | 4.7562 | 5.5383 |
| Benerjee[3] | 3.4798 | 3.6445 | 4.0971 | 4.7516 | 5.5314 |

#### 3.2 Effect of material gradient on linear frequencies

Fig. 2 shows the variation of the linear frequency with respect to the variation in the material gradient index (k). Ordinate is taken as the ratio of linear natural frequency of FGM material to the same of isotropic material and abscissa contains the variation of k. Results for first three modes of vibration are shown.

#### 3.3 Effect of material gradient, rotational speed and hub radius on non-linear frequencies

Fig. 3 represents the backbone curve for 1st mode, which shows the effect of k (1, 0.2 and 2) and rotational speeds. Nonlinear frequency ratios are comparatively higher for k=0.2, whereas a very small
difference in nonlinear frequency ratio is observed for \(k=1\) and \(k=2\). Rotational speed causes the centrifugal stiffening effect. Stiffening effect makes the beam stiffer and linear as well as nonlinear frequency increases. Therefore Backbone curves become straighter as the rotational speed increases.

Table 2 shows the variation in the nonlinear frequency ratios with the hub radius \(\delta\) for \(\Omega=4\). With increase in hub radius nonlinear frequency ratios are found to be reduced.

![Figure 2. Variation of linear frequency with material index k.](image)

**Table 2.** Effect of hub radius.

| Hub radius | \(\delta=0\) | \(\delta=0.1\) | \(\delta=0.2\) | \(\delta=0.3\) | \(\delta=0.4\) |
|------------|--------------|--------------|--------------|--------------|--------------|
| \(\omega_1/\omega_{1l}\) | 1.072 | 1.015 | 1.008 | 1.005 | 1.002 |
| \(\omega_2/\omega_{2l}\) | 1.099 | 1.057 | 1.043 | 1.036 | 1.032 |

**4. Conclusion**

An iterative technique is employed in order to capture the nonlinear dynamic response of a rotating FGM beam. Methodology is computationally efficient and accurate. Effects of centrifugal stiffening arise due to rotation of hub is incorporated and effects are captured effectively. Effects of different gradation relation on linear as well as nonlinear vibration response are captured successfully.

**References**

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