Thermal nonlocal Nambu–Jona-Lasinio model in the real time formalism

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The real time formalism at finite temperature and chemical potential for the nonlocal Nambu–Jona-Lasinio model is developed in the presence of a Gaussian covariant regulator. We construct the most general thermal propagator, by means of the spectral function. As a result, the model involves the propagation of massive quasiparticles. The appearance of complex poles is interpreted as a confinement signal, and, in this case, we have unstable quasiparticles with a finite decay width. An expression for the propagator along the critical line, where complex poles start to appear, is also obtained. A generalization to other covariant regulators is proposed.

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The Nambu–Jona-Lasinio model (NJL) has been vastly considered for studying nonperturbative aspects of QCD. Nowadays, it is mainly used to explore finite temperature and density effects in the frame of the mean-field approximation \textsuperscript{[1–3]}. One of the big challenges in QCD is to understand the confinement mechanism and the dynamics behind confinement. Perturbative QCD cannot describe confinement and, although lattice QCD is able to reproduce successfully hadron properties, like masses and coupling constants \textsuperscript{[4]}, it has problems when dealing with finite baryon chemical potential (the sign problem). However, there are effective models which include explicitly confinement, as, for example, different versions of the bag model \textsuperscript{[5–8]}, Dyson-Schwinger models \textsuperscript{[9–13]}, or the Polyakov loop effective action coupled to Dyson-Schwinger or NJL models \textsuperscript{[14–17]}.

The nonlocal NJL model (nNJL) is another attempt in this direction \textsuperscript{[18–21]}. When the gluon degrees of freedom are integrated out in the QCD action, a nonlocal quark action emerges and confinement should be hidden there. The idea of the nNJL approach is to incorporate nonlocal vertices through the presence of appropriate regulators.

Since the NJL model is nonrenormalizable, a momentum cutoff is needed in order to handle the UV divergences. The applicability of the model is, therefore, restricted to energy scales below the cutoff. Nonlocal extensions of the NJL model are designed to regularize the model in such a way that UV divergences are controlled, internal symmetries are preserved, and quark confinement is incorporated. The nNJL model has been extended to a finite temperature and density scenario \textsuperscript{[22–32]} through the usual Matsubara formalism \textsuperscript{[33–34]}. Here, nevertheless, the exact summation of Matsubara frequencies turns out to be cumbersome, due to the complicated shape of the regulators. In most cases, it is necessary to cut the series at some order.

The idea of this work is to develop the finite temperature real time formalism for the nNJL model. In this way, we are able to calculate temperature corrections, providing a physical picture in terms of quasiparticles. On the one hand, those states with real masses can propagate freely in the deconfined phase. On the other hand, the existence of complex poles of the propagator in the confined phase produces a strong damping avoiding the propagation of such states.

Real time formalisms, as thermo-field dynamics or the Schwinger-Keldysh formalism, can be constructed through the analytic continuation of the Euclidean action \textsuperscript{[33–39]}. Those formalisms double the number of degrees of freedom, providing the appearance of a $2 \times 2$ matrix propagator $S_{ij}$. In any version of the NJL model in the mean-field approximation, the gap equation corresponds to a one-loop self-consistent relation, so we only need to find the $S_{11}$ component \textsuperscript{[40, 41]}. The other matrix propagator components start to appear at the two-loop level. In the construction of the $S_{11}$ propagator, the main ingredient is given by the spectral density function. However, the construction of the spectral density function is not a simple task in a nonlocal frame, especially when dealing with a nontrivial analytic structure of the propagator.

In this article, we will develop the real time formalism for the Gaussian regulator in the nNJL model, which can easily be extended to other kind of regulators. As a result, we will get thermal propagators describing quasiparticles with temperature- and chemical-potential–dependent masses.

This paper is organized as follows: In Sec. II we introduce the nNJL model in the mean-field approximation at finite temperature and chemical potential. Section III is devoted to the discussion of the analytical structure of the nNJL model in the mean-field approximation with a Gaussian regulator. In Sec. III, the real time formalism is presented for the general case, constructing, then, in Sec. IV the corresponding real time propagator $S_{11}$ for the case of a Gaussian regulator. We explain how to obtain the gap equation and the thermodynamical potential in Sec. V and Sec. VI respectively. Finally, in Sec. VII...
we summarize our conclusions.

I. nNJL MODEL IN THE MEAN-FIELD APPROXIMATION AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

Nonlocal models with separable interactions have been considered several times. In the context of NJL, confinement is introduced through the inclusion of interacting nonlocal currents, which are extended regularized versions of the usual local currents.

The SU(2)\(f\) \(\otimes\) SU(3)\(_c\) nNJL Euclidean Lagrangian is given by

\[ \mathcal{L}_E = \bar{\psi}(-i\partial + m)\psi - \frac{G}{2} j_\alpha(x)j_\alpha(x), \] (1)

\(m\) being the current quark mass and \(j_\alpha(x)\) the nonlocal quark currents. Here, we are using the metric \(g_{\mu\nu} = \text{diag}(1,1,1,1)\).

There are two schemes on the market to introduce nonlocal effects in terms of extended currents [21]. We will use the one based on instanton liquid models:

\[ j_\alpha(x) = \int d^4y \ d^4z \ r(y - x)r(z - x)\bar{\psi}(y)\Gamma_\alpha\psi(z), \] (2)

where \(r(x - y)\) is the nonlocal regulator and \(\Gamma_\alpha = (1, i\gamma_5\tau)\), \(\tau\) being the Pauli matrices.

Since we want to deal with mesonic degrees of freedom, we will follow the standard bosonization procedure. This is realized through the introduction of auxiliary scalar and pseudoscalar fields. By integrating out the quark fields, an equivalent partition function, in terms of only bosonic degrees of freedom, is obtained:

\[ Z = \int D\sigma D^3\pi \ e^{-\Gamma[\sigma,\pi^\ast]}, \] (3)

We proceed within the mean-field approximation, keeping the mean values of the boson fields and neglecting their fluctuations. In this way, the partition function in Eq. (3) turns out to be \(Z \approx Z_{\text{MF}} = e^{-\Omega_{\text{MF}}(\sigma,T,\mu)}\), where the mean-field effective action in Euclidean momentum space is given by

\[ \Omega_{\text{MF}} = V_4 \left[ \frac{\bar{\sigma}^2}{2G} - 2N_c \int \frac{d^4q}{(2\pi)^4} \text{tr} \ln S^{-1}_E(q_E) \right], \] (4)

\(q_E = (q,q_4)\) being the four-momentum in Euclidean space and \(V_4\) the four-dimensional volume. The trace acts on the Euclidean Dirac matrices \(\gamma_E = (\gamma, i\gamma_5\tau)\) of the effective quark propagator

\[ S_E(q_E) = -\frac{q_E^2 + \Sigma(q_E^2)}{q_E^2 + \Sigma^2(q_E^2)}, \] (5)

where the running mass includes the Lorentz invariant regulator contribution in Euclidean momentum-space

\[ \Sigma(q_E^2) = m + \bar{\sigma}r^2(q_E^2), \] (6)

\(\bar{\sigma}\) being the mean-field value of the scalar field. The pseudoscalar mean-field value is zero due to isospin symmetry. Chiral symmetry, on the other hand, is explicitly broken through the current quark masses and spontaneously broken by a nonvanishing chiral condensate value.

Finite temperature \((T)\) and chemical potential \((\mu)\) effects are introduced in the standard way through the Matsubara formalism [33]. As a result, in Eq. (4), the four-dimensional volume, the momentum, and the integral in momentum turn out to be

\[ V_4 \rightarrow V/T, \]
\[ q_4 \rightarrow -q_n, \]
\[ \int \frac{d^4q}{2\pi} \rightarrow T \sum_n, \]

where \(q_n\) includes the Matsubara frequencies and the chemical potential

\[ q_n \equiv (2n + 1)\pi T + i\mu. \] (10)

The grand canonical thermodynamical potential is given by \(\Omega_{\text{MF}}(\bar{\sigma},T,\mu) = (T/V)\Gamma_{\text{MF}}[\bar{\sigma},T,\mu]\) [42]. The value of \(\bar{\sigma}\) is obtained then through the solutions of \(\partial\Omega_{\text{MF}}/\partial\bar{\sigma} = 0\), giving rise to the gap equation

\[ \frac{\bar{\sigma}}{G} = 2N_cT \sum_n \int \frac{d^3q}{(2\pi)^3} r^2(q_n^2 + q^2)\text{tr}S_E(q_n^2), \] (11)

This equation, which involves directly the propagator, will be our starting point in order to apply the real time formalism.

II. POLES OF THE EFFECTIVE QUARK PROPAGATOR WITH A GAUSSIAN REGULATOR IN MINKOWSKI SPACE

Among the different kind of existent covariant regulators, we will use here the Gaussian one due to its simplicity [13]. In Euclidean momentum space, the regulator is given by

\[ r(q_E^2) = \exp\left(-\frac{q_E^2}{2\Lambda^2}\right), \] (12)

where \(\Lambda\) is a free parameter that has to be chosen from phenomenological considerations. Thus, the effective model depends on three parameters: the current quark mass \(m\), the effective coupling \(G\), and the scale parameter \(\Lambda\), the last one being associated to the cutoff in the usual NJL model by setting the regulator as \(r = \theta(\Lambda^2 - q^2)\). These parameters are fixed in order to get the physical values of the pion mass \(m_\pi = 139\) MeV, the pion decay constant \(f_\pi = 92.4\) MeV, and the chiral condensate \(-\langle\bar{q}q\rangle^{1/3} \approx 200-260\) MeV [21]. Following [22], we will use two sets of parameters. Set I is given by \(\langle\bar{q}q\rangle = -(200\text{ MeV})^3\), \(m = 10.5\) MeV, \(G = 50\ \text{GeV}^{-1}\) and \(\Lambda = 627\) MeV.
MeV. Set II is given by (\bar{q}q) = -(220 \text{ MeV})^3, m = 7.7 \text{ MeV}, G = 30 and \Lambda = 760 \text{ MeV}.

To start our analysis in the real time formalism, let us explore more in detail the analytic structure of the propagator by setting \( q_4 = i q_0 \). The Euclidean propagator in Eq. (5) turns out to be \( S_E \rightarrow i S_0 \), where the propagator in Minkowski space is defined as

\[
S_0(q) = i \frac{g + \Sigma(-q^2)}{q^2 - \Sigma^2(-q^2)}, \tag{13}
\]

and where now \( q = (q_0, \vec{q}) \) and \( \gamma = (\gamma_0, \vec{\gamma}) \), with the metric \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \). This propagator, as shown in Fig. 1, presents three different kind of poles which depend on \( \bar{\sigma} \). For Low values of \( \bar{\sigma} \), there are two real poles associated to real masses that we denote as \( M_1 \) and \( M_2 \). In this case we will say that the system is in the deconfined phase since we have freely propagating quasiparticle states. As \( \bar{\sigma} \) grows up to a certain critical value, the two masses join into a single real mass \( M_c \). For higher values of \( \bar{\sigma} \), this critical mass splits into complex poles. There exist an infinite number of those poles for the Gaussian regulator. Nevertheless, it can be shown that the only relevant poles correspond to the first pair \[23\], where the real part is associated with a mass \( M_0 \) and the imaginary part is related to a decay width factor \( \Gamma_0 \). The other complex poles involved have considerable higher values for the masses and decay widths, and these masses are not continuously connected with the critical mass \( M_c \). The appearance of complex poles is interpreted as a signal of confinement \[18, 19\], since the corresponding quasiparticles do not propagate freely. As soon as the poles becomes complex, we will be in the confined phase.

The poles of the propagator in Eq. \[13\] are plotted in Fig. 2 for both sets of parameters as a function of the self-energy at zero 4th-momentum \( \Sigma(0) = m + \bar{\sigma} \). When \( \Sigma(0) = \Sigma_c \), there is only one positive real pole \( q^2 = M^2_c \), which is determined from the condition \( \partial_{q^2}(q^2 - \Sigma^2(-q^2)) = 0 \).

FIG. 1. Schematic description of the poles of the propagator in Minkowski space. \( M_1 \) and \( M_2 \) correspond to the deconfined masses, \( M_c \) is the critical mass and \( M_0 \) and \( \Gamma_0 \) are the mass and the decay width in the confined phase, respectively.

FIG. 2. Poles of the propagator as a function of the self-energy at zero 4th-momentum for set I (upper graph) and set II (lower graph). Defining the complex poles as \( q^2 = M_0^2 \pm i M_0 \Gamma_0 \), the solid, dashed, and dot-dashed lines correspond to the masses, and the dotted line corresponds to the decay width.

\[
\Sigma(0) = m + (M_c - m)e^{-M^2_c/\Lambda^2}, \tag{14}\]

\[
M_c = \frac{1}{2} (m + \sqrt{m^2 + 2\Lambda^2}). \tag{15}\]

The corresponding values are \( \Sigma_c = 273 \text{ MeV} \) for set I and \( \Sigma_c = 329 \text{ MeV} \) for set II. The previous analysis can be easily verified by plotting \( q^2 - \Sigma^2(-q^2) \) as a function of \( q^2 \).

The values of \( \Sigma(0) \) at zero temperature and chemical potential are \( \bar{\Sigma} = 350 \text{ MeV} \) for set I and \( \bar{\Sigma} = 300 \text{ MeV} \) for set II, respectively. The tendency for the \( \bar{\sigma} \) parameter is to decrease as temperature and chemical potential increase. Then, if at zero temperature and chemical potential the corresponding poles are real, they will still be real at finite \( T \) and \( \mu \). On the other hand, if the poles are complex at zero temperature and chemical potential, they will become real at some finite values of \( T \) and \( \mu \). As \( \Sigma > \Sigma_c \) in set I and \( \bar{\Sigma} < \Sigma_c \) in set II, they are called confining and nonconfining sets, respectively.

Now we proceed to extend this formalism to a finite temperature and density scenario in the frame of real time formalism.
III. REAL TIME FORMALISM

It is well-known that when formulating quantum field theory at finite temperature in the real time formalism, we have to double the number of degrees of freedom. The new fields that appear are called thermal ghosts. As a consequence, the new thermal propagators are given by $2 \times 2$ matrices $S_{ij}$. However, for one-loop calculations, we only need the term $S_{11}$. The other components of the thermal matrix propagator participate only in higher loop calculations, since the thermal ghosts do not couple directly to external physical lines. The treatment of the gap equation, in a self-consistent way, is equivalent to a one-loop calculation. The same happens with the calculation of the chiral condensate and the number density. For our purpose, we only need to obtain the $S_{11}$ component of the propagator matrix.

Following the construction of $S_{11}$, the general $S_{11}$ propagator is obtained in terms of the spectral density function (SDF): \[ S_{11}(q) = \int \frac{dk_0}{2\pi i} \frac{\rho(k_0, q)}{k_0 - q_0 - i\epsilon} - n_F(q_0 - \mu)\rho(q). \] \[ \text{(16)} \]

The SDF is related to the real time propagator in Eq. \[ S_{0}(iq_n, q) = \int \frac{dk_0}{2\pi i} \frac{\rho(k_0, q)}{k_0 - iq_n}. \] \[ \text{(17)} \]

where the connection between both formalisms, real time and imaginary time, is realized through the analytic extension of $iq_n \rightarrow z$. What we need now is to obtain the SDF. In the case of free particles, the SDF can be gotten from the relation $\rho = S_{0}(q_0 + i\epsilon, q) - S_{0}(q_0 - i\epsilon, q)$. However, in this case where we have a nontrivial propagator, we need a more general prescription to extract the SDF. This can be achieved by defining \[ S_{\pm}(q) \equiv \pm \int \frac{dz}{2\pi i} \frac{S_{0}(z \mp i\epsilon, q)}{z - q_0 + i\epsilon}. \] \[ \text{(18)} \]

where the integration path is shown in Fig. 3. With this definition, the SDF can be written as \[ \rho(q) = S_{+}(q) - S_{-}(q). \] \[ \text{(19)} \]

In the special case of free fermions with mass $M$, the corresponding spectral density function will be \[ \rho_{\text{free}}(q) = 2\pi \text{sign}(q_0)(q + M)\delta(q^2 - M^2). \] \[ \text{(20)} \]

By replacing $\rho_{\text{free}}$ in Eq. \[ S_{\text{Dolan-Jackiw}}(q; M) = (q + M) \left[ \frac{i}{q^2 - M^2 + i\epsilon} - 2\pi N(q_0)(q + M)\delta(q^2 - M^2) \right]. \] \[ \text{(21)} \]

IV. DRESSED PROPAGATORS FOR THE GAUSSIAN REGULATOR

Following the procedure described above, we will start from the deconfined region, where there are two simple real poles. The corresponding $S_{11}$ propagator is \[ S_{11\text{dec}}(q) = Z(M_1)S_{\text{Dolan-Jackiw}}(q; M_1) + Z(M_2)S_{\text{Dolan-Jackiw}}(q; M_2). \] \[ \text{(24)} \]
\( S_{\text{DJ}}(M) \) being the DJ propagator with mass \( M \) and where \( Z(M) \) is the field-strength renormalization constant, defined as

\[
Z(M) = \left[ \partial_{M^2} \{ M^2 - \Sigma^2(-M^2) \} \right]^{-1} = [1 - 2M(m - M)]^{-1}. \tag{25}
\]

In the last step, we used the pole relation \( M = \Sigma(-M^2) \).

Now, this propagator describes the quasiparticles involved, and the nonlocal interaction is enclosed in the effective masses and the field-strength renormalization constants.

The critical case that separates the real from the complex pole regions is a little bit different from the case we already discussed. As we can see from Fig. 2, the two real poles converge into a single real pole at \( \Sigma(0) = \Sigma_c \). However, this particular case corresponds to a second-order pole. In addition, the new relation \( \partial_{q^2}[q^2 - \Sigma^2(-q^2)] = 0 \) must also be satisfied. Following the Cauchy theorem, by expanding in a Laurent series and using the general expression that provides SDF, we find the critical propagator

\[
S^{\text{crit}}_{11}(q) = \left( Z_c + Z_c' \frac{\partial}{\partial M_c^2} \right) S_{\text{DJ}}(q; M_c), \tag{26}
\]

with the constants

\[
Z_c = \frac{2M_c(4M_c - 3m)}{3(2M_c - m)^2}, \tag{27}
\]

\[
Z_c' = -\frac{4M_c^2(M_c - m)}{2M_c - m}, \tag{28}
\]

and where \( M_c \) was already defined in Eq. 15.

In both cases described above (deconfined and critical), the finite temperature and chemical potential effects appear separated from the \( T, \mu = 0 \) terms. Notice that the finite temperature and chemical potential terms are on mass shell due to the delta function in the propagators. This fact does not imply any difficulty when integrating in \( q_0 \). The case of complex poles needs more attention. The spectral density function in this case is not proportional to a delta function, but it is given by a Breit-Wigner distribution:

\[
\rho_{\text{cont}}(q) = \frac{M_0^2 \Gamma_0(A_+ + A_-) - i(q^2 - M_0^2)(A_+ - A_-)}{(q^2 - M_0^2)^2 + M_0^2 \Gamma_0^2}, \tag{29}
\]

with

\[
A_{\pm}(q) \equiv \frac{2Z(M_\pm)}{\sqrt{q^2 + M_\pm^2}} \left[ q_0 \left( q + M_\pm \right) - \gamma_0 \left( q^2 - M_\pm^2 \right) \right], \tag{30}
\]

and where

\[
M_\pm \equiv \sqrt{M_0^2 \pm iM_0 \Gamma_0} \tag{31}
\]

are the first complex solutions of the equation \( M = \Sigma(-M^2) \). \( Z(M) \) was already defined in Eq. (25). The \( S^{\text{cont}}_{11} \) propagator can be easily calculated from Eq. (16), but we will skip this in order to avoid long expressions. In fact, the full \( S^{\text{cont}}_{11} \) will not be necessary for the rest of this article.

In what follows, we are interested to separate the thermal and density effects. Then, we can write the propagator as

\[
S_{11}(q) = S_0(q) + \tilde{S}(q; T, \mu), \tag{32}
\]

where \( \tilde{S}(q; 0, 0) = 0 \). In the case of the deconfined and critical propagators, the \( \tilde{S} \) term can be easily identified since it is proportional to \( N(q_0) \).

For the confined phase, we need to find a procedure to calculate integrals in the complex energy plane that involve the confined propagator, keeping in mind that when \( T \) and \( \mu \) vanish we have to recover the \( S_0 \) propagator inside the integral. The finite temperature and chemical potential contribution must be an analytic function. We can see from the general expression of the \( S_{11} \) propagator that the last term in Eq. (10) contains all the thermal and density information. Nevertheless, it does not vanish when \( T, \mu \to 0 \), but \( n_F(q_0 - \mu) \to 0 \) for \( q_0 \), which is not an analytic function. However, we can integrate the last term in Eq. (16), which is analytic, proceeding then to remove constants independent of \( T \) and \( \mu \). Such constants then can be joined into the rest of the integral. This can be summarized by writing

\[
\tilde{S}_{\text{conf}}(q) = -n_F(q_0 - \mu)\rho_{\text{cont}}(q) + C\delta^4(q), \tag{33}
\]

where \( C \) is a divergent factor that will be fixed after integration in order to produce a vanishing function when \( T, \mu \to 0 \), being absorbed into the \( T, \mu = 0 \) contribution. Now we proceed to calculate these integrals in order to obtain the gap equation, the thermodynamic potential, and all the other relevant quantities.

V. GAP EQUATION

The interesting quantities we want to calculate, such as the thermodynamical potential, chiral condensate, number density, susceptibilities, etc., require momentum integration. In particular, the thermodynamical potential at zero temperature and chemical potential has to be calculated in the Euclidean formulation. However, the thermal and density corrections can be handled in the frame of the real time formalism.

The Gaussian regulator is constructed with the aim of regularizing UV divergences in the Euclidean formulation. Nevertheless, this regulator induces a divergence in Minkowski space when integrating in \( q_0 \). The finite temperature and chemical potential part of the dressed propagators \( \tilde{S} \), for the deconfined and critical case, involves a delta function which has the effect of evaluating the regulator at the effective masses \( r(-q^2) \to r(-M^2) \). The zero temperature and chemical potential term in the
dressed propagator $S_0$ can be Wick rotated, turning back to Euclidean space, where the regulator produces finite integrals.

We start first with the gap equation, which gives us the value of $\tilde{\sigma}$. Following the previous section, we apply the replacements described in Eq. (32) to the gap equation in (11), obtaining

$$\tilde{\sigma} = 2N_c \int \frac{d^4 q}{(2\pi)^4} r^2(-q^2) {\rm tr} S_{11}(q).$$  

(34)

Now, expressing the dressed propagator in the form of Eq. (33), the gap equation turns out to be

$$\frac{\partial \Omega_{MF}}{\partial \tilde{\sigma}} = g_0(\tilde{\sigma}) + \tilde{g}(\tilde{\sigma}, T, \mu) = 0,$$  

(35)

where turning back to Euclidean space, the $T, \mu = 0$ contribution gives

$$g_0(\tilde{\sigma}) = \frac{\tilde{\sigma}}{G} - \frac{N_c}{\pi^2} \int_0^\infty \frac{dpp^3r^2(p^2)^2}{p^2 + \Sigma(p^2)^2},$$  

(36)

where $p = \sqrt{q^2_E}$ and the angular integral has already been done. The finite $T, \mu$ contribution in Minkowski space gives

$$\tilde{g}(\tilde{\sigma}, T, \mu) = -2N_c \int \frac{d^4 q}{(2\pi)^4} r^2(-q^2) {\rm tr} \tilde{S}(q; T, \mu).$$  

(37)

Notice that the $\tilde{\sigma}$ variable and the different mass terms are related through $M = \Sigma(-M^2)$ or, more explicitly,

$$\tilde{\sigma} = (M - m)e^{-M^2/\Lambda^2},$$  

(38)

where $M$ stands for $M_{1,2}$ in the deconfined phase, and $M_c$ in the critical phase. In the confined phase, $M$ stands for all the complex poles in the confined phase, $M_{\pm}$ in Eq. (33) being the numerically relevant ones.

For the deconfined and critical phases, the integration can be done in a straightforward way. However, as we mentioned in the last section, the situation is not as simple when we try to integrate the thermal part of the propagator given in Eq. (33). In Minkowski space, the regulator in the gap equation diverges in some regions of the complex $q_0$ plane. However, we will see that finally all the divergent terms do not depend on temperature and chemical potential and, therefore, can be removed, as will be explained in the last section. Replacing $\tilde{S}$ from Eq. (33) into Eq. (37), we get

$$\tilde{g}_{\text{conf}} = N_c \int \frac{d^4 q}{(2\pi)^4} r^2(-q^2) [n_F(q_0 - \mu) + n_F(q_0 + \mu)] \times \text{tr}_{\text{conf}}(q) - \frac{2N_c C}{(2\pi)^4},$$  

(39)

where, in the last equation, we have used the fact that $\text{tr}_{\text{conf}}$ is an odd function of $q_0$. Now we integrate along the path shown in Fig. 4. If we consider only the two first poles of the propagator, $M_{\pm}$, the poles inside the closed path will be $\sqrt{q^2 + M_{\pm}^2}$ and $-\sqrt{q^2 + M_{\pm}^2}$. The same results can be obtained if we integrate in the lower half plane. The integrand vanishes along the upper line ($\text{Im } q_0 \to \infty$), and the contribution from the lines surrounding the poles from the Fermi-Dirac factor (crosses) cancel each other. The sum of the left and right straight lines ($\text{Re } q_0 \to \mp \infty$) leaves a divergent contribution to the integral which, however, is independent of temperature and chemical potential. Therefore, this term, which is independent of $T$ and $\mu$, is canceled with the constant factor $C$ in Eq. (39).

The deconfined and critical gap equations are obtained immediately, due to the presence of the delta function in the propagators. The resulting expressions for $\tilde{g}$ are

$$\tilde{g}(\tilde{\sigma}, T, \mu) = \frac{4N_c}{\pi^2} \sum_M Z(M) \int_0^\infty \frac{dk k^2}{2E} \frac{r^2(-M^2)M [n_F(E - \mu) + n_F(E + \mu)]}{M^2}.$$  

(40)

$$\tilde{g}_{\text{crit}}(\tilde{\sigma}, T, \mu) = \frac{4N_c}{\pi^2} \left( Z_c + Z'_c \frac{\partial}{\partial M_c^2} \right) \int_0^\infty \frac{dk k^2}{2E} \frac{r^2(-M_c^2)M_c [n_F(E_c - \mu) + n_F(E_c + \mu)]}{M_c^2},$$  

(41)
where $k = |q|$, $E_c = \sqrt{k^2 + M_c^2}$, $E = \sqrt{k^2 + M^2}$, and where the sum in the first equation is performed over all the mass poles involved in the confined phase or the deconfined phase. For the case of a complex mass term, the gap equation is real and can be easily written in terms of real components.

As an example, we calculate the behavior of $M_1$ in terms of the temperature and chemical potential. This mass term will produce the main contribution to the dynamics in the deconfined phase because it is the lower one. The other mass term $M_2$ turns out to be relevant only for values near the critical mass $M_c$. Figure 5 shows the evolution of the mass $M_1$ as a function of the chemical potential for different values of the temperature. The transition observed at low temperature is obtained by analyzing the minimum of the thermodynamical potential as a function of $\bar{\sigma}$. In the next section, we will show how to calculate the thermodynamical potential in the real time formalism through the gap equation.

\section{VI. THERMODYNAMICAL POTENTIAL}

The inverse propagator in the real time formalism does not carry information about temperature and density \cite{40}. This can be seen in the case of a free fermion, whose thermal propagator is the Dolan-Jackiw one in Eq. (21), where the thermal information is enclosed in the on mass shell term. However, the inverse of the full propagator, $-i(q - M)$, does not have any information on temperature. Nevertheless, once we have obtained the expression for the gap equation, we can reintegrate it, getting

$$\Omega_{MF}(\bar{\sigma}; T, \mu) = \Omega_0(\bar{\sigma}) + \tilde{\Omega}(\bar{\sigma}; T, \mu),$$ \hspace{1cm} (43)

where the temperature and chemical potential contributions to the thermodynamical potential in the confined phase as

$$\tilde{\Omega}_{conf}(\bar{\sigma}; T, \mu) = \frac{4N_c}{\pi^2} \int_0^{\infty} dk k^2 T \times \left[ \ln(1 + B_+) + \ln(1 + B_-) \right],$$ \hspace{1cm} (49)

With these definitions, we can write the finite $T, \mu$ contribution to the thermodynamical potential in the confined phase as

$$\Omega_0(\bar{\sigma}) = \frac{\bar{\sigma}^2}{2G} - \frac{N_c}{2\pi^2} \int_0^{\infty} dp \sqrt{p^2 + \Sigma^2(p^2)}.$$

The finite temperature and chemical potential expressions for the gap equation, however, are not defined in terms of $\bar{\sigma}$, but in terms of the effective masses and the decay constant. This relation is given in Eq. (38), giving, as a result,

$$\Omega_0(\bar{\sigma}) = \frac{\bar{\sigma}^2}{2G} - \frac{N_c}{2\pi^2} \int_0^{\infty} dp \sqrt{p^2 + \Sigma^2(p^2)}.$$ \hspace{1cm} (44)

FIG. 5. $M_1$ as a function of the chemical potential for different values of the temperature. The end point, where the first-order phase transition turns to a crossover, occurs about $(T, \mu) \approx (70, 180)$ MeV.
with the functions $B_\pm$ defined as
\[
B_\pm = 2 \cos \left( \frac{M_0 \Gamma}{2 T \omega} \right) e^{-(\omega \pm \mu)/T} + e^{-2(\omega \pm \mu)/T}.
\]

Our construction was compared with \cite{23}, obtaining the same results. The cosine term in Eq. (50) can produce the regulated thermodynamic potential \cite{11}. Note that now, we can also obtain other relevant quantities directly from the thermodynamical potential already calculated in the real time formalism. The quark number density function is obtained by taking the derivative with respect to the chemical potential: $n = -\partial \Omega/\partial \mu$. The chiral condensate is calculated through the derivative of the regulated thermodynamic potential with respect to the mass: $\langle \bar{q} q \rangle = \partial \Omega'/\partial m$. Here, the current mass is related to the effective mass terms through Eq. \cite{35}, and the partial derivative is done by taking $\tilde{\sigma}$ and $\Lambda$ as constants.

\section{Conclusions}

In the present work, we have constructed the real time formalism for the nonlocal Nambu–Jona-Lasinio model at finite temperature and chemical potential, in the particular case where the nonlocal term is given by a Gaussian regulator. Following the general construction of the $S_{11}$ propagator through the spectral density function \cite{33,40}, we generalize the procedure to get the spectral density function. This generalization allows us to deal with dressed propagators whose analytical structure includes complex poles. With this, we obtain different propagators if the quasiparticles are deconfined (real poles), confined (complex poles), and also the critical case which separates both regimes. Once we have obtained the real time propagators, we find the gap equation which provides the value of $\tilde{\sigma}$ and, consequently, the value of the quasiparticles effective mass terms. The thermodynamical potential is obtained from the gap equation, and it is given in terms of simple expressions as a function of the effective mass terms. We verified that our results coincide with the ones reported in \cite{24}.

This procedure gives an intuitive phenomenological description in terms of quasiparticles. Its generalization to other regulators is, in principle, straightforward. In fact, our prescription for obtaining the spectral density function is well-defined, even if the propagator presents a cut along the real axis, which is the case for some type of Lorentzian regulator with fractional power.

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