p-Brane cosmology and phases of Brans-Dicke theory with matter

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Abstract

We study the effect of the solitonic degrees of freedom in string cosmology following the line of Rama. The gas of solitonic p-brane is treated as a perfect fluid in a Brans-Dicke type theory. In this paper, we find exact cosmological solutions for any Brans-Dicke parameter $\omega$ and for general parameter $\gamma$ of equation of state and classify the cosmology of the solutions on a parameter space of $\gamma$ and $\omega$.

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Recent developments of the string theory suggest that in a regime of Planck length curvature, quantum fluctuation is very large so that string coupling becomes large and consequently the fundamental string degrees of freedom are not a weakly coupled good ones. Instead, solitonic degrees of freedom like p-brane or D-p-branes are more important. Therefore it is a very interesting question to ask what is the effect of these new degrees of freedom to the space time structure especially whether including these degrees of freedom resolve the initial singularity, which is a problem in standard general relativity.

For the investigation of the p-brane cosmology, the usual low energy effective action coming from the beta function of the string worldsheet would not a good starting point. So there will be a difference from the string cosmology. We need to find the low energy effective theory that contains gravity and at the same time reveals the effect of these solitonic objects. But those are not known. Therefore one can only guess the answer at this moment. The minimum amount of the requirement is that it should give a gravity theory therefore it must be a generalization of general relativity. The Brans-Dicke (BD) theory is a generic deformation of the general relativity allowing variable gravity coupling. In fact low energy theory of the fundamental string contains the Brans-Dicke theory with a fine tuned deformation parameter (\(\omega=-1\)). Moreover Duff and et al. found that the natural metric that couples to the p-brane is the Einstein metric multiplied by certain power of dilaton field. In terms of this new metric, the action that gives the p-brane solution becomes Brans-Dicke action with definite deformation parameter \(\omega\) depending on \(p\). Using this action, Rama recently argued that the gas of solitonic p-brane treated as a perfect fluid type matter in a Brans-Dicke theory can resolve the initial singularity without any explicit solution. In a previous paper, we have studied this model and found several analytic solutions for a few values of parameter with which the coupled dilaton-graviton system could be decoupled by the simple “completing the square” method. In this paper, we give exact cosmological solutions for any Brans-Dicke parameter \(\omega\) and for general equation of state and classify the
cosmology of the solutions according to the range of parameters involved.

The rest of this paper is organized as follows. In section II, we set up the action for the p-brane cosmology. In section III, we find an analytic solution for the equation of motion and constraint equation for the general case. In section IV and V, we study the cosmology of the solution and classify them according to their behavior. In section IV, \( t \) as a function of the dilaton time \( \tau \) is studied and the behavior of the scale factor \( a \) with respect to the dilaton time \( \tau \) in the asymptotic region is studied in section V. In section VI, using the results in section IV and V, we classify the cosmology into several phases and investigate the behavior of the scale factor \( a \) as a function of the cosmic time \( t \).

In section VII, we summarize and conclude with some discussions.

II. CONSTRUCTION OF THE ACTION WITH THE SOLITONIC MATTER

We consider the bosonic part of the effective string action and analyse the evolution of a \( D \) dimensional homogeneous isotropic universe with the solitonic matter included. The action is given by

\[
S = \int d^D x \sqrt{-g} e^{-\phi} \left[ R - \omega \partial_\mu \phi \partial^\mu \phi \right] + S_m, \tag{1}
\]

where \( \phi \) is the dilaton field and \( S_m \) is the matter part of the action. In string theory the BD parameter \( \omega \) is fixed as \(-1\). In the high curvature regime, the string coupling is also big and the solitonic p-brane will be copiously produced since they become light and dominate the universe in that regime. Duff et al. \[5\] have shown that in terms of metric which couples minimally to p-brane\((p=d-1)\), the effective action can be written as Brans-Dicke theory with the BD parameter \( \omega \) given by

\[
\omega = \frac{(D - 1)(d - 2) - d^2}{(D - 2)(d - 2) - d^2}. \tag{2}
\]

In 4-dimension, the BD parameter is given by \( \omega = -\frac{4}{3} \) for the 0-brane \((p = 0)\) and \( \omega = -\frac{3}{2} \) for the instanton \((p = -1)\). Let’s assume that the gas of solitonic p-brane can be considered
as perfect fluid in the Brans-Dicke theory with the equation of state \( p = \gamma \rho, \gamma < 1 \). Therefore our starting point is the equation of the BD theory \[8\], \[9\]

\[
\mathcal{R}_{\mu \nu} - \frac{g_{\mu \nu}}{2} \mathcal{R} = \frac{\epsilon}{2} T_{\mu \nu} + \omega \{ \partial_\mu \phi \partial_\nu \phi - \frac{g_{\mu \nu}}{2} (\partial \phi)^2 \} + \{ -\partial_\mu \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi + g_{\mu \nu} D^2 \phi - g_{\mu \nu} (\partial \phi)^2 \},
\]

\[
\mathcal{R} - 2 \omega D^2 \phi + \omega (\partial \phi)^2 = 0
\]

(3)

where \( \phi \) is the dilaton and \( D \) means a covariant derivative. \( \mathcal{R} \) is the curvature scalar and cosmological metric is given as the following form

\[
ds_D^2 = -\frac{1}{\mathcal{N}} dt^2 + e^{2\alpha(t)} \delta_{ij} dx^i dx^j \quad (i, j = 1, 2, \ldots, D - 1),
\]

(4)

where \( e^{\alpha(t)}(= a(t)) \) is the scale factor and \( \mathcal{N} \) is the (constant) lapse function. Now, we assume that all variables are the functions of time only. The curvature scalar \[10\] in \( D \) dimension is given by

\[
\mathcal{R} = g^{00} \mathcal{R}_{00} + g^{ij} \mathcal{R}_{ij},
\]

\[
g^{00} \mathcal{R}_{00} = \frac{D - 1}{\mathcal{N}} [\dot{\alpha} + \dot{\alpha}^2],
\]

\[
g^{ij} \mathcal{R}_{ij} = \frac{D - 1}{\mathcal{N}} [\dot{\alpha} + (D - 1)\dot{\alpha}^2]
\]

(5)

where \( \dot{\alpha} \) means the time derivative of \( \alpha \).

The energy-momentum tensor of the solitonic matter is given by

\[
T_{\mu \nu} = p g_{\mu \nu} + (p + \rho) U_\mu U_\nu
\]

(6)

where \( U_\mu \) is the fluid velocity. The hydrostatic equilibrium condition of energy-momentum conservation is

\[
\dot{\rho} + (D - 1)(p + \rho) \dot{\alpha} = 0.
\]

(7)

Using \( p = \gamma \rho \), we get the solution

\[
\rho = \rho_0 e^{-\alpha(D-1)(1+\gamma)}. \quad (8)
\]
The parameters $\gamma$ and $\omega$ expressed in eq.(1) and eq.(8) are free parameters. Our goal is to study how the metric variables change their behavior for various values of $\gamma$ and $\omega$.

If we consider only the time dependence, the action can be brought to the following form

$$S = \int dt \ e^{(D-1)\alpha - \phi} \left[ \frac{1}{\sqrt{N}} \left\{ - (D-2)(D-1)\dot{\alpha}^2 + 2(D-1)\dot{\phi}^2 + \omega \phi^2 \right\} - \sqrt{N} \rho_0 e^{-(D-1)(1+\gamma)\alpha + \phi} \right].$$

(9)

where we eliminated $p$ and $\rho$ by eq.(7) and eq.(8). The variation over the constant lapse function, which is only $g_{00}$, gives a constraint equation. When we set the lapse function $N$ to be 1 after varying of the action over the lapse function $N$, this constraint equation is the equation of motion of the $g_{00}$ component in eq.(5).

III. ANALYTIC SOLUTION

Now, we introduce a new time variable $\tau$ by

$$dt = e^{(D-1)\alpha - \phi} d\tau.$$ 

(10)

Then the action becomes

$$S = \int d\tau \left[ \frac{1}{\sqrt{N}} \left\{ (D-1)\kappa Y^2 + \mu X^2 \right\} - \sqrt{N} \rho_0 e^{-2X} \right],$$

(11)

where new variables presented in the action are given by

$$\kappa = (D-1)(1-\gamma)^2(\omega - \omega\kappa),$$

$$\nu = 2(1-\gamma)(\omega - \omega\nu),$$

$$\mu = -\frac{4(D-2)}{\kappa}(\omega - \omega_{-1}),$$

$$-2X = (D-1)(1-\gamma)\alpha - \phi,$$

$$Y = \alpha + \frac{\nu}{\kappa} X,$$

$$\omega\kappa = -\frac{D-2D\gamma + 2\gamma}{(D-1)(1-\gamma)^2},$$

$$\omega\nu = -\frac{1}{1-\gamma},$$

$$\omega_{-1} = -\frac{D-1}{D-2}.$$  

(12)
The constraint equation is written as

\[ 0 = (D - 1)\kappa \dot{Y}^2 + \mu \dot{X}^2 + \rho_0 e^{-2X}, \]  

(13)

where \( \rho_0 \) is a positive real constant. The equations of motion are written as

\[ 0 = \ddot{Y}, \]

\[ 0 = \ddot{X} - \frac{\rho_0}{\mu} e^{-2X}. \]  

(14)

Note that \( \omega_{-1} \) in eq.(12) happens to be the value of the instanton. If \( \omega \) is less than \( \omega_{-1} \), the kinetic term of the dilaton has a negative energy in Einstein frame. So we will consider the case where \( \omega \) is larger than \( \omega_{-1} \). According to the sign of \( \kappa \), the types of solutions are very different.

When \( \kappa \) is negative, an exact solution becomes

\[ X = \ln \left[ \frac{q}{c} \cosh(c\tau) \right], \]

\[ Y = A\tau + B, \]  

(15)

where \( c, A, B \) and \( q = \sqrt{\frac{\mu}{|\mu|}} \) are arbitrary real constants. Using the constraint equation, we determine \( A \) in terms of other variables

\[ A = \frac{c}{\delta}, \quad \text{with} \quad \delta = \sqrt{-\frac{(D - 1)\kappa}{\mu}} = \frac{|\kappa|}{2\sqrt{1 + \omega D - 2D-1}}. \]  

(16)

If \( \kappa \) is zero, then we can obtain a solution of the equations of motion, but it does not satisfy the constraint equation. If \( \kappa \) is positive, the solution is

\[ X = \ln \left[ \frac{q}{c} \sinh(c\tau) \right], \]

\[ Y = \frac{c}{\delta}\tau + B. \]  

(17)

**IV. COSMOLOGY OF THE SOLUTION**

Now, we investigate the relation between the cosmic time \( t \) and the dilaton time \( \tau \). Since the solutions of the equations of motion have different forms, we study the behavior of \( t \) as a function of \( \tau \) case by case.
A. $\kappa < 0$ case

In this region, $\omega < \omega_{\kappa}$. $\gamma$ is always less than 1. We find the relation between $t$ and $\tau$ using eq.(10)

$$t - t_0 = \int_{\tau_0}^{\tau} d\tau' \exp \left[ \frac{(D - 1)\gamma c}{\delta} \tau' - (2 + \frac{(D - 1)\gamma \nu}{\kappa}) \ln \left\{ \frac{q}{c} \cosh(c\tau') \right\} \right] + (D - 1)\gamma B,$$

(18)

where $(D - 1)\gamma B$ is a constant. This constant can be ignored in the limit $\tau \to \pm \infty$. Because $\frac{dt}{d\tau}$ is always positive definite, $t$ is a monotonic function of $\tau$. The behavior of $a(t)$ as a function of $t$ depends crucially on the relation between $t$ and $\tau$. When $\tau$ goes to $\pm \infty$, $t$ is reduced to

$$t - t_0 \approx \frac{1}{T_\pm} \left( e^{T_\pm \tau} - e^{T_\pm \tau_0} \right),$$

(19)

where

$$T_\pm = \frac{2c}{|\kappa|} \left[ (D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} \pm \{\kappa + (D - 1)\gamma (1 + \omega (1 - \gamma))\} \right].$$

(20)

We define a new concept for our purpose: $t$ is super-monotonic function of $\tau$ if it is monotonic and $t$ runs entire real line when $\tau$ does so. When $t$ is super-monotonic function of $\tau$, the universe evolves from infinite past to infinite future. Otherwise the scale factor $a(t)$ has a starting (ending) point at a finite cosmic time $t_i$ ($t_f$) which corresponds to initial (final) singularity. As a mapping, $t$ maps the real line of $\tau$ to

$$(-\infty, \infty) \text{ if } T_- < 0 < T_+,$$

$$(-\infty, t_f) \text{ if } T_- < 0 \text{ and } T_+ < 0,$$

$$(t_i, \infty) \text{ if } T_- > 0 \text{ and } T_+ > 0,$$

$$(t_i, t_f) \text{ if } T_+ < 0 < T_-.$$

In the limit $\tau \to \pm \infty$, the condition $T_\pm < 0$ is expressed as

$$(D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} < \mp[\kappa + (D - 1)\gamma (1 + \omega (1 - \gamma))].$$

(21)
This inequality is divided into two cases according to the sign of $\gamma$. In each case we obtain the different region of $\omega$ satisfying the condition $T_{\pm} < 0$.

1. $\gamma > 0$ case

Because we have considered only the case $\omega > \omega_{-1}$, so the left hand side in eq.(21) is positive definite. To satisfy the inequality $T_- < 0$, the conditions

$$\kappa + (D-1)\gamma\{1+\omega(1-\gamma)\} > 0 \quad \text{and} \quad \left( (D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}} \right)^2 < (\kappa + (D-1)\gamma\{1+\omega(1-\gamma)\})^2,$$

must be satisfied. If they don’t, we know that $T_-$ is positive. The first inequality in eq.(22) is reduced to the following inequality

$$\omega > \omega_{-\infty} := -\frac{D - (D-1)\gamma}{(D-1)(1-\gamma)}.$$  \hspace{1cm} (23)

It is remarkable that the second inequality in eq.(22) is written as

$$(\omega - \omega_0)(\omega - \omega_\kappa) > 0,$$  \hspace{1cm} (24)

where $\omega_\kappa$ appeared in the definition of $\kappa$ and

$$\omega_0 = -\frac{D}{D-1},$$  \hspace{1cm} (25)

is the value of $\omega$ for the 0-brane. See eq.(2).

Figure 1.

As shown in Figure 1, $\omega_\kappa < \omega_0$ ($\omega_\kappa > \omega_0$) in the region $0 < \gamma < \frac{2}{D}$ ($\gamma > \frac{2}{D}$). Therefore the solution of eq.(24) becomes

$$\omega_{-1} < \omega < \omega_\kappa \quad \text{for} \quad 0 < \gamma < \frac{2}{D},$$

$$\omega_{-1} < \omega < \omega_0 \quad \text{for} \quad \gamma > \frac{2}{D}.$$  \hspace{1cm} (26)
Combining eq. (23) and eq. (26), we find the region of \( \omega \) satisfying the condition \( T_- < 0 \) as the following

\[
\omega_1 < \omega < \omega_\kappa \quad \text{for} \quad \frac{1}{D-1} < \gamma < \frac{2}{D}, \\
\omega_1 < \omega < \omega_0 \quad \text{for} \quad \gamma > \frac{2}{D}. \tag{27}
\]

The condition \( T_+ < 0 \) is

\[
(D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} < -[\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\}]. \tag{28}
\]

For this, two conditions

\[
\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\} < 0 \quad \text{and} \\
\left( (D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} \right)^2 < (-[\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\}])^2, \tag{29}
\]

must be satisfied at the same time. In eq. (29), the first inequality gives

\[
\omega < \omega_{-\infty} \tag{30}
\]

and the second inequality gives eq. (24) again. Therefore using eq. (26) and eq. (30), we find the region of \( \omega \) satisfying \( T_+ < 0 \)

\[
\omega_1 < \omega < \omega_\kappa \quad \text{for} \quad 0 < \gamma < \frac{1}{D - 1}. \tag{31}
\]

2. \( \gamma < 0 \) case

In this case, the condition \( T_- < 0 \) is written as

\[
(D - 1) | \gamma | \sqrt{1 + \omega \frac{D - 2}{D - 1}} > -[\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\}]. \tag{32}
\]

For this, we need

\[
\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\} > 0 \quad \text{or} \\
\left( (D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} \right)^2 > (-[\kappa + (D - 1)\gamma\{1 + \omega(1 - \gamma)\}])^2. \tag{33}
\]
Eq.(33) can be simplified as

\[ \omega_{-\infty} < \omega \text{ or } \omega_0 < \omega < \omega_\kappa. \] (34)

Thus the solution, which is the sum of two regions in eq.(34), is reduced to

\[ \omega_0 < \omega < \omega_\kappa \text{ for } \gamma < 0. \] (35)

This solution includes the region of the first inequality of eq.(34).

Similarly, the condition \( T_+ < 0 \) is written as

\[ (D - 1) | \gamma | \sqrt{1 + \omega \frac{D - 2}{D - 1}} > \kappa + (D - 1)\gamma \{1 + \omega(1 - \gamma)\}, \] (36)

which gives

\[ \kappa + (D - 1)\gamma \{1 + \omega(1 - \gamma)\} < 0 \text{ or } \left( (D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1}} \right)^2 > (\kappa + (D - 1)\gamma \{1 + \omega(1 - \gamma)\})^2. \] (37)

The solution of these can be written as \( \omega < \omega_{-\infty} \) or \( \omega_0 < \omega < \omega_\kappa \). From these, the region satisfying \( T_+ < 0 \) is

\[ \omega_{-1} < \omega < \omega_\kappa \text{ for } \gamma < 0. \] (38)

**B. \( \kappa > 0 \text{ case} \)**

Now we consider positive \( \kappa \), which means

\[ \omega > \omega_\kappa. \] (39)

Since the solution \( X(\tau) \) has a singularity at \( \tau = 0 \), we have to treat carefully the behavior of \( t \) near \( \tau = 0 \). The relation between \( t \) and \( \tau \) is given by

\[
t - t_0 = \int_{\tau_0}^\tau d\tau' \exp \left[ \frac{(D - 1)\gamma c}{\delta} \tau' - (2 + \frac{(D - 1)\gamma v}{\kappa}) \ln \left\{ \frac{q}{c} \sinh(c\tau') \right\} \right] + (D - 1)\gamma B \right]. \] (40)
In the limit $\tau \to 0$, the above equation is reduced to

$$t - t_0 \approx \text{sign}(\tau) \frac{q^{-\eta}e^{(D-1)\gamma B}}{1 - \eta} \left[ |\tau|^{1-\eta} - |\tau_0|^{1-\eta} \right],$$

(41)

where $\eta = 2 + \frac{(D-1)\nu}{\kappa}$ and $\tau_0$ and $t_0$ are real constants. In case of $\eta > 1$, $t$ has a singularity at $\tau \to 0$. In the other case, $t$ has no singularity. So we consider two cases $\eta < 1$ and $\eta > 1$.

1. $\eta < 1$ case

In this case, $t$ has no singularity at $\tau = 0$. So we investigate the behavior of $t$ at $\tau \to \pm \infty$ only.

i) $\gamma > 0$ case

In the case $\kappa > 0$, eq.(40) is reduced to

$$t - t_0 \approx \frac{1}{T_\pm} (e^{T_\pm \tau} - e^{T_\pm \tau_0}),$$

(42)

where

$$T_\pm = \frac{2c}{|\kappa|} \left[ (D - 1)\gamma \sqrt{1 + \omega \frac{D - 2}{D - 1} + \{\kappa + (D - 1)\gamma(1 + \omega(1 - \gamma))\}} \right].$$

(43)

The condition $T_- < 0$ is written as eq.(28) and gives the solution

$$\omega < \omega_{-\infty} \text{ and } \omega > \omega_0 \text{ for } 0 < \gamma < \frac{2}{D},$$

$$\omega < \omega_{-\infty} \text{ and } \omega > \omega_\kappa \text{ for } \gamma > \frac{2}{D},$$

(44)

where we use $\omega > \omega_\kappa$. As shown Figure 1, $\omega_0 > \omega_{-\infty}$ for $0 < \gamma < \frac{2}{D}$ and $\omega_\kappa > \omega_{-\infty}$ for $\gamma > \frac{2}{D}$. Therefore there is no solution satisfying the condition $T_- < 0$. Hence $T_-$ is positive.

Now we investigate the behavior of $t$ at $\tau \to +\infty$. The condition $T_+ < 0$ is written like eq.(22). Applying the similar method used in the above analysis, the region of $\omega$ satisfying $T_+ < 0$ is summarized as the following

$$\omega_0 < \omega \text{ for } 0 < \gamma < \frac{2}{D},$$

$$\omega_\kappa < \omega \text{ for } \gamma > \frac{2}{D}.$$
ii) $\gamma < 0$ case

Through the same calculation, we can show that $T_-$ is positive and $T_+$ is negative for all negative $\gamma$.

2. $\eta > 1$ case

In this case, the behavior of $t$ is singular at $\tau = 0$. $\tau_0$ and $t_0$ can be ignored due to the divergence of $|\tau|^{1-\eta}$. From eq.(40) or eq.(41), we know that $\frac{dt}{d\tau}$ is always positive definite except a singular point $\tau = 0$. The condition $\eta > 1$ is reduced to

$$\omega > -\frac{D}{(D-1)(1-\gamma^2)} := \omega_\eta.$$  (46)

Under this condition, the region of $\tau$ is divided into $-\infty < \tau < 0$ and $0 < \tau < \infty$. Near $\tau = 0$, we obtain the behavior of $t$ characterized by the sign of $\tau$. When $\tau$ goes to zero from below, $t$ in eq.(41) is written as

$$t \approx \frac{q^{-\gamma}e^{(D-1)\gamma B}}{(\eta - 1)} \frac{1}{(-\tau)^{\eta-1}}.$$  (47)

When $\tau$ goes to zero from above, $t$ is reduced as the following

$$t \approx -\frac{q^{-\gamma}e^{(D-1)\gamma B}}{(\eta - 1)} \frac{1}{\tau^{\eta-1}}.$$  (48)

Thus $t \to +\infty$ as $\tau \to -0$ but $t \to -\infty$ as $\tau \to +0$. We also have to examine the behaviors of $t$ at $\tau \to \pm\infty$. However these have been already described when we discussed the case $\eta < 1$.

To describe the behavior of $t$ as a function of $\tau$ at $\tau \to \pm\infty$ and $\tau \to 0$, we classify the parameter space of $\gamma$ and $\omega$ using all results obtained in this section. These are shown in Figure 2.

Figure 2
The behavior of the $t$ as a function of $\tau$ in Figure 2 is summarized as the followings:

1) In region I, $T_- > 0$ and $T_+ < 0$ ($\omega < \omega_\kappa$).

$t$ evolves from finite initial time $t_i$ to finite final time $t_f$ as $\tau$ runs $(-\infty, +\infty)$.

2) In region II, $T_- < 0$ and $T_+ < 0$ ($\omega < \omega_\kappa$).

$t$ evolves from negative infinity to finite final time $t_f$ as $\tau$ runs $(-\infty, +\infty)$.

3) In region III, $T_- > 0$ and $T_+ < 0$ ($\omega > \omega_\kappa$ and $\omega > \omega_\eta$).

In this region, because $t$ has a singular behavior at $\tau = 0$, the region of $\tau$ divided into $-\infty < \tau < 0$ and $0 < \tau < \infty$. Therefore $t$ has two branches for any given values of $\gamma$ and $\omega$. For $-\infty < \tau < 0$, $t$ evolves from finite initial time $t_i$ to positive infinity. For $0 < \tau < \infty$, $t$ evolves from negative infinity to finite final time $t_f$.

4) In region IV, $T_- > 0$ and $T_+ > 0$ ($\omega > \omega_\kappa$ and $\omega < \omega_\eta$).

In this region, because $t$ has no singularity, $t$ evolves from finite initial time $t_i$ to positive infinity as $\tau$ runs $(-\infty, +\infty)$.

5) In region V, $T_- > 0$ and $T_+ > 0$ ($\omega > \omega_\kappa$ and $\omega > \omega_\eta$).

By the same reason that explained in region III, $t$ evolves from finite initial time $t_i$ to positive infinity for $-\infty < \tau < 0$ and $t$ evolves from negative infinity to positive infinity for $0 < \tau < \infty$.

6) In region VI, $T_- < 0$ and $T_+ > 0$ ($\omega < \omega_\kappa$).

$t$ evolves from negative infinity to positive infinity as $\tau$ runs $(-\infty, +\infty)$.

7) In region VII, $T_- > 0$ and $T_+ > 0$ ($\omega < \omega_\kappa$).

$t$ evolves from finite initial time $t_i$ to positive infinity as $\tau$ runs $(-\infty, +\infty)$.

V. THE BEHAVIOR OF THE SCALE FACTOR

Now we study the behavior of the scale factor $a$ as a function of $\tau$.

A. $\kappa < 0$ case

We consider the exponent of the scale factor $\alpha(\tau)$. Using eq.(10), $\alpha(\tau)$ is given by
\[ \alpha(\tau) = \frac{2}{|\kappa|} \left[ \sqrt{1 + \omega \frac{D-2}{D-1} c^2} + \{1 + \omega(1 - \gamma)\} \ln \left\{ \frac{q}{c} \cosh(c\tau) \right\} \right] + B. \]  

In the limit \( \tau \to \pm \infty \), the scale factor \( a(\tau) = e^{\alpha(\tau)} \) is rewritten as the following

\[ a(\tau) \approx e^{H_{\pm} \tau}, \]  

where \( H_{\pm} \) is defined as

\[ H_{\pm} = \frac{2c}{|\kappa|} \left[ \sqrt{1 + \omega \frac{D-2}{D-1}} \pm \{1 + \omega(1 - \gamma)\} \right]. \]  

Just as \( t(\tau) \), the behavior of \( a(\tau) \) at \( \tau \to \pm \infty \) is determined by the sign of \( H_{\pm} \). Using this and the sign of \( T_{\pm} \), we can read the behavior of the scale factor \( a(t) \) as a function of \( t \) in the asymptotic regions.

For negative \( \kappa \) (\( \omega < \omega_{\kappa} \)), \( H_{-} > 0 \) can be written as

\[ \sqrt{1 + \omega \frac{D-2}{D-1}} > 1 + \omega(1 - \gamma). \]  

This means

\[ 1 + \omega(1 - \gamma) < 0 \quad \text{or} \quad \left( \sqrt{1 + \omega \frac{D-2}{D-1}} \right)^2 > (1 + \omega(1 - \gamma))^2. \]  

The first inequality in eq.(53) is equivalent to

\[ \omega < \omega_{\nu}, \]  

where \( \omega_{\nu} \) has been already written in eq.(12). As one can see in Figure 3, \( \omega_{\kappa} < \omega_{\nu} \) if \( \gamma < \frac{1}{D-1} \) and \( \omega_{\kappa} > \omega_{\nu} \) if \( \gamma > \frac{1}{D-1} \). Together with \( \omega < \omega_{\kappa} \), the first inequality condition gives the region of \( \omega \) satisfying \( H_{-} > 0 \)

\[ \omega_{-1} < \omega < \omega_{\kappa} \quad \text{for} \quad \gamma < \frac{1}{(D-1)}. \]  

The second inequality in eq.(53) is rewritten as

\[ \omega(\omega - \omega_{\kappa}) < 0. \]
Notice that $\omega_\kappa < 0$ if $\gamma < \frac{D}{2(D-1)}$ and $\omega_\kappa > 0$ if $\gamma > \frac{D}{2(D-1)}$. Since $\omega < \omega_\kappa$, we can rewrite eq.(56) as the following

$$0 < \omega < \omega_\kappa \text{ for } \gamma > \frac{D}{2(D-1)}.$$  \hspace{1cm} (57)

As a result, eq.(55) and eq.(57) are the regions of $\omega$ satisfying the condition $H_- > 0$.

Figure 3

Now we consider the condition $H_+ > 0$

$$\sqrt{1 + \omega} \frac{D - 2}{D - 1} > -[1 + \omega(1 - \gamma)].$$ \hspace{2cm} (58)

Like the case $H_- > 0$, this inequality is divided into two inequalities

$$1 + \omega(1 - \gamma) > 0 \text{ or }$$

$$\left(\sqrt{1 + \omega} \frac{D - 2}{D - 1}\right)^2 > (1 + \omega(1 - \gamma))^2. \hspace{3cm} (59)$$

The first inequality gives the region of $\omega$ satisfying the condition $H_+ > 0$

$$\omega > \omega_\nu. \hspace{3cm} (60)$$

Using $\omega > \omega_- \text{ and } \kappa < 0$, eq.(60) is rewritten as the following

$$\omega_- < \omega < \omega_\kappa \text{ for } \gamma > \frac{1}{D - 1}. \hspace{3cm} (61)$$

The second inequality in eq.(59) has the same region of $\omega$ that appeared in eq.(57). Because the region of $\omega$ in eq.(61) contains the region of $\omega$ in eq.(57), eq.(61) is the solution satisfying the condition $H_+ > 0$.

**B. $\kappa > 0$ case**

The exponent of the scale factor $\alpha(\tau)$ is given by

$$\alpha(\tau) = \frac{2}{|\kappa|} \left[ \sqrt{1 + \omega} \frac{D - 2}{D - 1} c \tau - \{1 + \omega(1 - \gamma)\} \ln\left\{\frac{q}{c} | \sinh(c\tau) | \right\} \right] + B. \hspace{3cm} (62)$$
1. $\eta < 1$ case

In this case, the scale factor $a(\tau)$ has no singular behavior. So we investigate the behavior of $a$ at $\tau \to \pm \infty$.

In the limit $\tau \to \pm \infty$, $a(\tau)$ is given by

$$a(\tau) \approx e^{H_\pm \tau},$$

(63)

where $H_\pm$ is defined as

$$H_\pm = \frac{2c}{|\kappa|} \left[ \sqrt{1 + \omega \frac{D - 2}{D - 1}} \mp \{1 + \omega(1 - \gamma)\} \right].$$

The condition $H_- > 0$ is exactly equal to eq.(58) due to the sign of $\kappa$. When we solve eq.(58) under the condition $\kappa > 0, \omega > \omega_\kappa$ instead of $\omega < \omega_\kappa$ must be applied to the solution. Then we obtain the region of $\omega$ satisfying the condition $H_- > 0$

$$\omega > \omega_\kappa \quad \text{for all} \quad \gamma.$$  

(64)

The condition $H_+ > 0$ described by eq.(52) and $\omega > \omega_\kappa$ gives the region of $\omega$

$$\omega_\kappa < \omega < 0 \quad \text{for} \quad \gamma < \frac{D}{2(D - 1)}.$$  

(65)

2. $\eta > 1$ case

In this case, we need to investigate the behavior of $a(\tau)$ at $\tau \to 0$ because $a(\tau)$ has a singular behavior at $\tau = 0$. In the limit $\tau \to 0$, $a(\tau)$ is written as

$$a(\tau) \approx e^B(q | \tau |)^{-2(1-\gamma)(\omega - \omega_\nu)} |\kappa|.$$  

(66)

For $\omega > \omega_\nu$, where $-(1 - \gamma)(\omega - \omega_\nu)$ is negative, $a(\tau)$ goes to infinite at $\tau \to 0$. And for $\omega < \omega_\nu$, $a(\tau)$ goes to zero at $\tau \to 0$. The behavior of $a(\tau)$ at $\tau \to \pm \infty$ has been described already when we discussed the case $\eta < 1$. 

15
From these studies, we classify the behavior of $a(\tau)$ on the parameter space of $\gamma$ and $\omega$. This is shown in Figure 4.

Figure 4.

As shown in Figure 4, we summarize the behavior of $a(\tau)$ as the followings:

1) In the region I, $H_- > 0$ and $H_+ < 0$ ($\omega < \omega_{\kappa}$).
   In the limit $\tau \to \pm\infty$, $a(\tau)$ goes to a zero size.

2) In the region II, $H_- < 0$ and $H_+ > 0$ ($\omega < \omega_{\kappa}$).
   In the limit $\tau \to \pm\infty$, $a(\tau)$ goes to an infinite size.

3) In the region III, $H_- > 0$ and $H_+ > 0$ ($\omega < \omega_{\kappa}$).
   In the limit $\tau \to -\infty$, $a(\tau)$ goes to a zero size. And in the limit $\tau \to \infty$, $a(\tau)$ goes to an infinite size.

4) In the region IV, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_{\kappa}$ and $\omega < \omega_{\eta}$).
   In this region, the behavior of $t$ is not singular. So we need not to consider the behavior of $a(\tau)$ at $\tau = 0$. In the limit $\tau \to -\infty$, $a(\tau)$ goes to a zero size. And in the limit $\tau \to \infty$, $a(\tau)$ goes to an infinite size.

5) In the region V, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_{\kappa}$, $\omega > \omega_{\eta}$ and $\omega < \omega_\nu$).
   In this region, because $t$ has a singular behavior at $\tau = 0$, we interpret the behavior of $a(\tau)$ as the following.
   For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and $\tau \to 0$. For $0 < \tau < \infty$, $a(\tau)$ goes to a zero size at $\tau \to 0$ and goes to an infinite size at $\tau \to \infty$.

6) In the region VI, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_{\kappa}$, $\omega > \omega_{\eta}$ and $\omega > \omega_\nu$).
   For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and goes to an infinite size at $\tau \to 0$.
   For $0 < \tau < \infty$, $a(\tau)$ goes to an infinite size at $\tau \to 0$ and at $\tau \to \infty$.

7) In the region VII, $H_- > 0$ and $H_+ < 0$ ($\omega > \omega_{\kappa}$, $\omega > \omega_{\eta}$ and $\omega > \omega_\nu$).
   For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and goes to an infinite size at $\tau \to 0$. 

16
For $0 < \tau < \infty$, $a(\tau)$ goes to an infinite size at $\tau \to 0$ and goes to a zero size at $\tau \to \infty$.

**VI. THE PHASE OF COSMOLOGY**

Using all results obtained from the section IV and V, we now classify the parameter space of $\gamma$ and $\omega$ into several phases and find the behavior of $a(t)$. These phases are characterized according to the behavior of $a(t)$.

Using eq.(19) and eq.(50), in the limit $\tau \to \pm \infty$, $a(t)$ is written as

$$a(t) \approx [T_{-}(t - t_{i})]^{H_{-}/T_{-}} \text{ at } \tau \to -\infty,$$

$$a(t) \approx [T_{+}(t - t_{f})]^{H_{+}/T_{+}} \text{ at } \tau \to \infty,$$

where $t_{i}$ and $t_{f}$, which are defined in section IV, are real constant. Notice that $t_{i}$ ($t_{f}$) becomes the starting point (the ending point) in the case $T_{-} > 0$ ($T_{+} < 0$) and that $t_{i}$ ($t_{f}$) can be neglected in the case $T_{-} < 0$ ($T_{+} > 0$) because $t \to \pm \infty$ as $\tau \to \pm \infty$.

Now, we explain two examples:

i) For $T_{-} < 0$ and $H_{-}/T_{-} > 0$, $T_{-}(t - t_{i})$ is positive and $a(t)$ goes to positive infinite at $t \to -\infty$.

ii) For $T_{-} > 0$ and $H_{-}/T_{-} > 0$, $t$ can be defined in the region $t > t_{i}$ only. So $t - t_{i}$ is positive and $a(t)$ goes to zero at $t \to t_{i}$.

Other cases can be analysed through the same method.

In the region $\omega > \omega_{k}$ and $\eta > 1$, we must investigate the behavior of $a(t)$ at $\tau \to 0$. From eq.(47), eq.(48) and eq.(66), $a(t)$ is obtained as the following

$$a(t) \approx E \times \left| t \right|^{\frac{2(1-\gamma)(\omega-\omega_{k})}{(\eta-1)|\kappa|}},$$

where

$$E = [q(\eta - 1)]^{\frac{2(1-\gamma)(\omega-\omega_{k})}{(\eta-1)|\kappa|}} e^{\frac{1 - 2(D-1)\gamma(1-\gamma)(\omega-\omega_{k})}{(\eta-1)|\kappa|}}.$$
is a positive value because $q$ and $(1 - \gamma)$ are positive in the previous definition. $a(t)$ goes to zero at $t \to \pm\infty$ ($\tau \to \pm0$) in the case $\omega < \omega_\nu$ and $a(t)$ goes to infinite at $t \to \pm\infty$ in the case $\omega > \omega_\nu$.

Figure 5

As shown in Figure 5, using the sign of $T_\pm$ and $H_\pm$ with the consideration of the behavior of $a(t)$ at $\tau \to 0$, the behavior of $a(t)$ in each region is characterized as the followings:

1) In region I, $T_- > 0$, $T_+ < 0$, $H_- > 0$, and $H_+ < 0$.
The universe evolves from a zero size at finite initial time $t_i$ to a zero size at finite final time $t_f$.

2) In region II, $T_- < 0$, $T_+ < 0$, $H_- > 0$, and $H_+ < 0$.
The universe evolves from a zero size at negative infinity to a zero size at finite final time $t_f$.

3) In region III, $T_- > 0$, $T_+ > 0$, $H_- > 0$, and $H_+ > 0$.
The universe evolves from a zero size at finite initial time $t_i$ to an infinite size at positive infinity.

4) In region IV, $T_- < 0$, $T_+ > 0$, $H_- < 0$, and $H_+ > 0$.
The universe evolves from an infinite size at negative infinity to an infinite size at positive infinity.

5) In region V, $T_- > 0$, $T_+ > 0$, $H_- < 0$, and $H_+ > 0$.
The universe evolves from an infinite size at finite initial time $t_i$ to an infinite size at positive infinity.

6) In region VI, $T_- > 0$, $T_+ > 0$, $H_- > 0$, and $H_+ > 0$.
The universe evolves from a zero size at finite initial time $t_i$ to an infinite size at positive infinity.

From 7) to 11), we consider the case $\eta > 1$ and $\omega > \omega_\kappa$ in which $t$ has a singular behavior at $\tau \to 0$. In these cases, we can divide the region of $\tau$ into $-\infty < \tau < 0$ and $0 < \tau < \infty$. 

18
Therefore we obtain two branches of $a(t)$ having the different behaviors in each region of $\tau$.

7) In region VII, $T_- > 0$, $T_+ > 0$, $H_- > 0$, and $H_+ > 0$.

In the region $-\infty < \tau < 0$, the universe evolves from a zero size at finite initial time $t_i$ to a zero size at positive infinity.

In the region $0 < \tau < \infty$, the universe evolves from a zero size at negative infinity to an infinite size at positive infinity.

8) In region VIII, $T_- > 0$, $T_+ < 0$, $H_- > 0$, and $H_+ > 0$.

In the region $-\infty < \tau < 0$, the universe evolves from a zero size at finite initial time $t_i$ to a zero size at positive infinity.

In the region $0 < \tau < \infty$, the universe evolves from a zero size at negative infinity to an infinite size at finite final time $t_f$.

9) In region IX, $T_- > 0$, $T_+ > 0$, $H_- > 0$, and $H_+ > 0$.

In the region $-\infty < \tau < 0$, the universe evolves from a zero size at finite initial time $t_i$ to an infinite size at positive infinity.

In the region $0 < \tau < \infty$, the universe evolves from an infinite size at negative infinity to an infinite size at positive infinity.

10) In region X, $T_- > 0$, $T_+ < 0$, $H_- > 0$, and $H_+ > 0$.

In the region $-\infty < \tau < 0$, the universe evolves from a zero size at finite initial time $t_i$ to an infinite size at positive infinity.

In the region $0 < \tau < \infty$, the universe evolves from an infinite size at negative infinity to an infinite size at finite final time $t_f$.

11) In region XI, $T_- > 0$, $T_+ < 0$, $H_- > 0$, and $H_+ < 0$.

In the region $-\infty < \tau < 0$, the universe evolves from a zero size at finite initial time $t_i$ to an infinite size at positive infinity.

In the region $0 < \tau < \infty$, the universe evolves from an infinite size at negative infinity to a zero size at finite final time $t_f$. 

19
VII. DISCUSSION AND CONCLUSION

In this paper we studied the effect of the gas of solitonic p-brane by treating them as a perfect fluid in the Brans-Dicke theory. We found exact cosmological solutions for any Brans-Dicke parameter $\omega$ and for general constant $\gamma$ and classified the cosmology of the solutions according to the parameters involved. We assumed that the universe is dominated by one kind of p-brane and they are treated as perfect fluid. We found the analytic solution which is singularity free for some $\gamma$ and $\omega$. It is very interesting that $a(t)$ has no initial and final singularities at finite initial and final cosmic time in region IV and in region VII. $a(t)$ has also an inflation behavior in region VII. So we need to study more intensively the behavior of $a(t)$ in these regions.

Presumably the value of $\gamma$ as well as $\omega$ should be fixed once $p$ is fixed. Without knowing the value of $\gamma$ for a given $p$, the classification was the best thing we could do. It would be very interesting to determine the parameter $\gamma$ for the given $p$. Also we need more rigorous justification of our basis for the p-brane cosmology. If what we took as basis goes wrong, then what we have done is just Brans-Dicke cosmology in the presence of some perfect fluid type matter. We wish that more study of the effect of the solitons in the string cosmology be done in the future.

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Figure 1. In 4 dimension ($D = 4$), all functions ($\omega_\kappa, \omega_\eta, \cdots$), defined by the relation between $t$ and $\tau$, are presented on a parameter space of $\gamma$ and $\omega$.

Figure 2. In 4 dimension ($D = 4$), the parameter space is classified by the relation between $t$ and $\tau$.

Figure 3. In 4 dimension ($D = 4$), all functions ($\omega_\kappa, \omega_\nu, \cdots$), defined by the relation between $a(\tau)$ and $\tau$, are presented on a parameter space of $\gamma$ and $\omega$.

Figure 4. In 4 dimension ($D = 4$), the parameter space is classified by the behavior of $a(\tau)$.

Figure 5. In 4 dimension ($D = 4$), the parameter space is classified by the behavior of $a(t)$.
