Limits of Thompson’s group $F$

Roland Zarzycki

June 15, 2018

Abstract

Let $F$ be the Thompson’s group $\langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-i} x_1 x_0^i], i = 1, 2 \rangle$.

Let $G_n = \langle y_1, \ldots, y_m, x_0, x_1 | [x_0 x_1^{-1}, x_0^{-i} x_1 x_0^i], y_j^{-1} g_{j,n}(x_0, x_1), i = 1, 2, j \leq m \rangle$, where $g_{j,n}(x_0, x_1) \in F$, $n \in \mathbb{N}$, be a family of groups isomorphic to $F$ and marked by $m + 2$ elements. If the sequence $(G_n)_{n<\omega}$ is convergent in the space of marked groups and $G$ is the corresponding limit we say that $G$ is an $F$-limit group. The paper is devoted to a description of $F$-limit groups.

1 Preliminaries

The notion of limit group was introduced by Z. Sela in his work on characterization of elementary equivalence of free groups [12]. This approach has been extended in the paper of C. Champetier and V. Guirardel [7], where the authors look at limit groups as limits of convergent sequences in a space of marked groups. They have given a description of Sela’s limit groups in these terms (with respect to the class of free groups). This approach has been also applied by L. Guyot and Y. Stalder [10] to the class of Baumslag-Solitar groups.

Thompson’s group $F$ has remained one of the most interesting objects in geometric group theory. We study $F$-limit groups. We show in this paper, that among $F$-limit groups there are no free products of $F$ with any non-trivial group. Moreover, we prove that among $F$-limit groups there are no HNN-extensions over cyclic subgroups.

In the remaining part of the section we recollect some useful definitions and facts concerning limit groups and Thompson’s group $F$. In Section 2 we
present results concerning free products and in Section 3 results concerning HNN-extensions.

A marked group \((G, S)\) is a group \(G\) with a distinguished set of generators \(S = (s_1, s_2, \ldots, s_n)\). For fixed \(n\), let \(\mathcal{G}_n\) be the set of all \(n\)-generated groups marked by \(n\) generators (up to isomorphism of marked groups). Following [7] we put certain metric on \(\mathcal{G}_n\). We will say that two marked groups \((G, S), (G', S') \in \mathcal{G}_n\) are at distance less or equal to \(e^{-R}\) if they have exactly the same relations of length at most \(R\). The set \(\mathcal{G}_n\) equipped with this metric is a compact space [7]. Limit groups are simply limits of convergent sequences in this metric space.

**Definition 1.1** Let \(G\) be an \(n\)-generated group. A marked group in \(\mathcal{G}_n\) is a \(G\)-limit group if it is a limit of marked groups each isomorphic to \(G\).

To introduce the Thompson’s group \(F\) we will follow [5].

**Definition 1.2** Thompson’s group \(F\) is the group given by the following infinite group presentation:

\[
\langle x_0, x_1, x_2, \ldots | x_{j}x_{i} = x_{i}x_{j+1}(i < j) \rangle
\]

In fact \(F\) is finitely presented:

\[
F = \langle x_0, x_1 [x_0x_1^{-1}, x_0^{-i}x_1x_0^i], i = 1, 2 \rangle.
\]

Every non-trivial element of \(F\) can be uniquely expressed in the normal form:

\[
x_0^{b_0}x_1^{b_1}x_2^{b_2} \ldots x_n^{b_n}x_n^{-a_n} \ldots x_2^{-a_2}x_1^{-a_1}x_0^{-a_0},
\]

where \(n, a_0, \ldots, a_n, b_0, \ldots, b_n\) are non-negative integers such that:

i) exactly one of \(a_n\) and \(b_n\) is nonzero;

ii) if \(a_k > 0\) and \(b_k > 0\) for some integer \(k\) with \(0 \leq k < n\), then \(a_{k+1} > 0\) or \(b_{k+1} > 0\).

We study properties of \(F\)-limit groups. For this purpose let us consider a sequence, \((g_i,n)_{n<\omega}\), \(1 \leq i \leq t\), of elements taken from the group \(F\) and the corresponding sequence of limit groups marked by \(t + 2\) elements, \(G_n = (F, \langle x_0, x_1, g_{1,n}, \ldots, g_{t,n} \rangle), n \in \mathbb{N}\), where \(x_0\) and \(x_1\) are the standard generators of \(F\). Assuming that such a sequence is convergent in the space of groups marked by \(t + 2\) elements, denote by \(G = (\langle x_0, x_1, g_1, \ldots, g_t | R_F \cup R_G \rangle, (x_0, x_1, g_1, \ldots, g_t))\) the limit group formed in that manner; here \(x_0, x_1\)
are "limits" of constant sequences \((x_0)_{n<\omega}\) and \((x_0)_{n<\omega}\), \(g_i\) is the "limit" of \((g_{i,n})_{n<\omega}\) for \(1 \leq i \leq t\), \(R_F\) and \(R_G\) refer respectively to the set of standard relations taken from \(F\) and the set (possibly infinite) of new relations.

It has been shown in [7] that in the case of free groups some standard constructions can be obtained as limits of free groups. For example, it is possible to get \(\mathbb{Z}^k\) as a limit of \(\mathbb{Z}\) and \(\mathbb{F}_k\) as a limit of \(\mathbb{F}_2\). On the other hand, the direct product of \(\mathbb{F}_2\) and \(\mathbb{Z}\) can not be obtained as a limit group. HNN-extensions often occur in the class of limit groups (with respect to free groups). For example, the following groups are the limits of convergent sequences in the space of free groups marked by three elements: the free group of rank 3, the free abelian group of rank 3 or a HNN-extension over a cyclic subgroup of the free group of rank 2 ([6]). All non-exceptional surface groups form another broad class of interesting examples ([2], [3]).

In the case of Thompson’s group the situation is not so clear. Since the centrum of \(F\) is trivial it is surely not possible to obtain any direct product with the whole group as an \(F\)-limit group. In 1985 Brin and Squier [4] showed that Thompson’s group \(F\) does not satisfy any law (also see Abert’s paper [1] for a shorter proof). However, in this paper we show that there are certain non-trivial words with constants over \(F\) (which will be called later laws with constants), which are equal to the identity for each evaluation in \(F\). This implies that no free product of \(F\) with any non-trivial group is admissible as limit group with respect to \(F\) (see Section 2). Moreover, we prove that HNN-extensions over a cyclic subgroup are not admissible as limit groups with respect to \(F\) (see Section 3).

There are many geometric interpretations of \(F\), but here we will use the following one. Consider the set of all strictly increasing continuous piecewise-linear functions from the closed unit interval onto itself. Then the group \(F\) is realized by the set of all such functions, which are differentiable except at finitely many dyadic rational numbers and such that all slopes (derivatives) are integer powers of 2. The corresponding group operation is just the composition. For the further reference it will be useful to give an explicit form of the generators \(x_0, x_1, \ldots\) in terms of piecewise-linear functions:

\[
x_n(t) = \begin{cases} 
    t & , t \in \left[0, \frac{2^n-1}{2n} \right] \\
    \frac{t}{2} + \frac{2^n-1}{2n+1} & , t \in \left[\frac{2^n-1}{2n}, \frac{2^{n+1}-1}{2n+2} \right] \\
    t - \frac{1}{2^{n+2}} & , t \in \left[\frac{2^n+1}{2n+2}, \frac{2^{n+1}-1}{2n+3} \right] \\
    2t - 1 & , t \in \left[\frac{2^{n+2}-1}{2n+3}, 1 \right]
\end{cases}
\]
for $n = 0, 1, \ldots$.

For any diadic subinterval $[a, b] \subset [0, 1]$, let us consider the set of elements in $F$, which are trivial on its complement, and denote it by $F_{[a,b]}$. We know that it forms a subgroup of $F$, which is isomorphic to the whole group. Let us denote its standard infinite set of generators by $x_{[a,b],0}, x_{[a,b],1}, x_{[a,b],2}, \ldots$.

Let us consider an arbitrary element $g$ in $F$ and treat it as a piecewise-linear homeomorphism of the interval $[0, 1]$. Let $\text{supp}(g)$ be the set $\{x \in [0, 1] : g(x) \neq x\}$ and $\overline{\text{supp}(g)}$ the topological closure of $\text{supp}(g)$. We will call each point from the set $P_g = (\text{supp}(g) \setminus \text{supp}(g)) \cap \mathbb{Z} [\frac{1}{2}]$ a dividing point of $g$. This set is obviously finite and thus we get a finite subdivision of $[0, 1]$ of the form $[0 = p_0, p_1], [p_1, p_2], \ldots, [p_{n-1}, p_n = 1]$ for some natural $n$. It is easy to see that $g$ can be presented as $g = g_1g_2 \ldots g_n$, where $g_i \in F_{[p_{i-1}, p_i]}$ for each $i$. Since $g$ can act trivially on some of these subintervals, some of the elements $g_1, \ldots, g_n$ may be trivial. We call the set of all non-trivial elements from $\{g_1, \ldots, g_n\}$ the defragmentation of $g$.

**Fact 1.3 (Corollary 15.36 in [8], Proposition 3.2 in [11])** The centralizer of any element $g \in F$ is the direct product of finitely many cyclic groups and finitely many groups isomorphic to $F$.

Moreover if the element $g \in F$ has the defragmentation $g = g_1 \ldots g_n$, then some roots of the elements $g_1, \ldots, g_n$ are the generators of cyclic components of the decomposition of the centralizer above. The components of this decomposition which are isomorphic to $F$ are just the groups of the form $F_{[a,b]}$, where $[a, b]$ is one of the subintervals $[p_{i-1}, p_i] \subset [0, 1]$, which are stabilized pointwise by $g$. Generally, if we interpret the elements of $F$ as functions, the relations occurring in the presentation of $F$, $[x_0x_1^{-1}, x_0^{-i}x_1x_0^i]$ for $i = 1, 2$, have to assure, that two functions, which have mutually disjoint supports except of finitely many points, commute. In particular, these relations imply analogous relations for different $i > 2$. According to the fact that $x_0^{-i}x_1x_0^i = x_{i+1}$, we conclude that all the relations of the form $[x_0x_1^{-1}, x_M], M > 1$, hold in Thompson’s group $F$. We often refer to these geometrical observations.

I am grateful to the referee for his helpful remarks.

## 2 Free products

Brin and Squier have shown in [4] that the Thompson’s group $F$ does not satisfy any group law. In this section we show how to construct words with
constants from \( F \), which are equal to the identity for any substitution in \( F \).

**Definition 2.1** Let \( w(y_1, \ldots, y_t) \) be a non-trivial word over \( F \), reduced in the group \( \mathbb{F}_t \ast F \) and containing at least one variable. We will call \( w \) a law with constants in \( F \) if for any \( \bar{g} = (g_1, \ldots, g_t) \in F^t \), the value \( w(\bar{g}) \) is equal to \( 1_F \).

The following proposition gives a construction of certain laws with constants in \( F \).

**Proposition 2.2** Consider the standard action of Thompson’s group \( F \) on \([0,1] \). Suppose we are given four pairwise disjoint closed diadic subintervals \( I_i = [p_i, q_i] \subset [0,1], 1 \leq i \leq 4 \), and assume that \( p_1 < p_2 < p_3 < p_4 \). Then for any non-trivial \( h_1 \in F_{I_1} \), \( h_2 \in F_{I_2} \), \( h_3 \in F_{I_3} \) and \( h_4 \in F_{I_4} \), the word \( w \) obtained from

\[
[y^{-1}h_1^{-1}yh_4^{-1}y^{-1}h_1y, y^{-1}h_2^{-1}yh_3^{-1}y^{-1}h_2yh_3]
\]

by reduction in \( \mathbb{Z} \ast F \) (we treat the variable \( y \) as a generator of \( \mathbb{Z} \)) is a law with constants in \( F \).

**Proof.** We will use the following notation: \( w_{14} = y^{-1}h_1^{-1}yh_4^{-1}y^{-1}h_1y \) and \( w_{23} = y^{-1}h_2^{-1}yh_3^{-1}y^{-1}h_2yh_3 \). It is easy to see that \( w \) cannot be reduced to a constant.

We claim that

for any any \( g \in F \) satisfying \( g(q_1) < p_4 \) and \( g(p_4) > q_1 \) the word \( w_{14}(g) \) is equal to the identity.

To show this we consider the action of \( w_{14}(g) \) on each point from \([0,1] \). Assume, that \( t \in [0, g^{-1}(q_1)) \). Since \( t \notin \text{supp}(h_4) \) we have:

\[
w_{14}(g)(t) = g^{-1}h_1^{-1}gh_4^{-1}g^{-1}h_1g(h_4(t)) = g^{-1}h_1^{-1}gh_4^{-1}g^{-1}(h_1(g(t))).
\]

By \( g^{-1}(h_1(g(t)))) < g^{-1}(h_1(q_1)) = g^{-1}(q_1) < p_4 \) we see \( h_4^{-1}(g^{-1}(h_1(g(t)))) = g^{-1}(h_1(g(t))) \). Thus:

\[
w_{14}(g)(t) = g^{-1}h_1^{-1}gg^{-1}(h_1(g(t))) = t.
\]

If \( t \in [g^{-1}(q_1), 1] \) then since \( h_4^{-1}(t) \geq \min(t, p_4) \), we have \( g(h_4^{-1}(t)) \geq q_1 \) and hence:

\[
w_{14}(g)(t) = g^{-1}h_1^{-1}gh_4^{-1}g^{-1}h_1g(h_4(t)) = g^{-1}h_1^{-1}gh_4^{-1}g^{-1}g(h_4(t)) =
\]
Now assume that \( g \) such that \( g(q_1) < p_4 \) and \( g(p_4) > q_1 \), \( w_{14}(g) = 1_F \) and hence \( w(g) = [w_{14}(g), w_{23}(g)] = [1_F, w_{23}(g)] = 1_F \). Thus we are left with the case when \( g(q_1) \geq p_4 \) (the proof of the case \( g(p_4) \leq q_1 \) uses the same argument). Now we will prove that

for any \( g \in F \) satisfying \( g(q_1) \geq p_4 \) the word \( w_{23}(g) \) is equal to the identity.

Assume that \( t \in [0, g^{-1}(p_2)] \). Since \( g(q_1) \geq p_4 \), we have \( q_1 \geq g^{-1}(p_4) > g^{-1}(p_2) \geq t \). Thus:

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2gh_3(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2g(t) = \]

\[
= g^{-1}h_2^{-1}gh_3^{-1}g^{-1}g(t) = t.
\]

Now assume that \( t \in (g^{-1}(p_2), g^{-1}(q_2)) \). Then since again \( q_1 \geq g^{-1}(p_4) > g^{-1}(q_2) \geq t \) we obtain:

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2gh_3(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2g(t).
\]

Since \( h_2(g(t)) \in (p_2, q_2) \) we have \( g^{-1}(h_2(g(t))) < g^{-1}(q_2) < q_1 \) and:

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}(g^{-1}h_2g(t)) = g^{-1}h_2^{-1}g(g^{-1}h_2g(t)) = t.
\]

Assume that \( t \in [g^{-1}(q_2), g^{-1}(p_3)] \). Since we still have \( g^{-1}(p_3) < q_1 \), we see that:

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2gh_3(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2g(t) = \]

\[
= g^{-1}h_2^{-1}gh_3^{-1}g^{-1}g(t) = t.
\]

Let \( t \in (g^{-1}(p_3), g^{-1}(q_3)) \). Then since \( g(p_3) > q_2 \) and \( h_3(t) \neq t \Rightarrow h_3(t) > p_3 \):

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2gh_3(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}gh_3(t) = \]

\[
= g^{-1}h_2^{-1}g(t) = t.
\]

Finally assume \( t \in [g^{-1}(q_3), 1] \) (and then \( g(t) > q_2 \)). Similarly as above:

\[
w_{23}(g)(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}h_2gh_3(t) = g^{-1}h_2^{-1}gh_3^{-1}g^{-1}g(h_3(t)) =
\]
Now we see that for $g$ such that $g(q_1) \geq p_4$, we have $w_{23}(g) = 1_F$ and hence $w(g) = [w_{14}(g), w_{23}(g)] = [w_{14}(g), 1_F] = 1_F$. The proof is finished.

\[= g^{-1}h_2^{-1}g(t) = t.\]

We now apply the construction from Proposition 2.2 to limits of Thompson’s group $F$.

**Theorem 2.3** Suppose we are given a convergent sequence of marked groups $((G_n, (x_0, x_1, g_{n,1}, \ldots, g_{n,s}))_{n<\omega},$ where $G_n = F$, $(g_{n,1}, \ldots, g_{n,s}) \in F$, $n \in \mathbb{N}$, and denote by $\mathbb{G}$ its limit. Then $\mathbb{G} \neq F * G$ for any non-trivial $G$.

Before the proof we formulate a general statement, which exposes the main point of our argument.

**Proposition 2.4** Let $H = \langle h_1, \ldots, h_m \rangle$ be a finitely generated torsion-free group, which satisfies a one variable law with constants and does not satisfy any law without constants. Let $\mathbb{G}$ be the limit of a convergent sequence of marked groups $((G_n, (h_1, \ldots, h_m, g_{n,1}, \ldots, g_{n,t}))_{n<\omega},$ where $(g_{n,1}, \ldots, g_{n,t}) \in H$, $G_n = H$, $n \in \mathbb{N}$. Then $\mathbb{G} \neq H * K$ for any non-trivial $K < \mathbb{G}$.

**Proof.** It is clear that $\mathbb{G}$ is torsion-free. To obtain a contradiction suppose that $\mathbb{G} = H * K$, $K \neq \{1\}$, and $\mathbb{G}$ is marked by a tuple $(h_1, \ldots, h_m, f_1, \ldots, f_t)$. Let $f = u(h, f)$ be an element of $K \setminus \{ 1 \}$ and let $w(y)$ be a law with constants in $H$. Obviously $w(u(h, g_{n,1}, \ldots, g_{n,t})) = 1_H$ for all $n < \omega$. It follows from the definition of an $H$-limit group that $w(u(h, f)) = 1_G$. Since $w$ was chosen to be non-trivial, with constants from $H$ and $|f| = \infty$, we obtain a contradiction with the fact that $\mathbb{G}$ is the free product of $H$ and $K$.

\[\square\]

**Proof of Theorem 2.3.** It follows directly from Proposition 2.2 that there is some word $w(y)$, which is a law with constants in $F$, and hence we just apply Proposition 2.4 for $H = F$, $h_1 = x_0$ and $h_2 = x_1$.

\[\square\]
3 HNN-extensions

Now we proceed to discuss the case of HNN-extensions. For this purpose we consider a sequence of groups marked by three elements, \((G_n)_{n<\omega}\), and the corresponding limit group \(G = \langle x_0, x_1, g | R_F \cup R_G, (x_0, x_1, g) \rangle\). The following theorem is the main result of the section.

**Theorem 3.1** Let \((G_n)_{n<\omega}\) be a convergent sequence of groups, where \(G_n = (F, (x_0, x_1, g_n))\), and let \(G = \langle x_0, x_1, g | R_F \cup R_G, (x_0, x_1, g) \rangle\) be its limit. Then \(G\) is not a HNN-extension of Thompson’s group \(F\) of the following form \(\langle x_0, x_1, g | R_F, ghg^{-1} = h' \rangle\) for some \(h, h' \in F\).

In what follows we will need two easy technical lemmas:

**Lemma 3.2** Suppose \(g \in F\) and let \(x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_0^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0}\) be its normal form. There is \(M \in \mathbb{N}\) such that for all \(m > M\):

\[
g^{-1} x_m g = x_{m+t} \quad \text{or} \quad gx_m g^{-1} = x_{m+t},
\]

where \(t = |\sum_{i=0}^{n} (a_i - b_i)|\).

**Proof.** Consider the case when \(\sum_{i=0}^{n} (a_i - b_i) \geq 0\). Then for sufficiently large \(m\):

\[
x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_0^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0} =
= x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_m x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_0^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0} =
= x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_m + \sum_{i=0}^{n} a_i x_n^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0} =
= x_m + \sum_{i=0}^{n} (a_i - b_i)
\]

In the case when \(\sum_{i=0}^{n} (a_i - b_i) < 0\) we consider the symmetric conjugation and apply the same argument.

\(\square\)

**Lemma 3.3** Under the assumptions of Lemma 3.2, the numbers \(M\) and \(t\) defined in that lemma, additionally satisfy the property that for all \(m > M\) and \(k > 0\):

\[
g^{-k} x_m g^k = x_{m+kt} \quad \text{or} \quad g^k x_m g^{-k} = x_{m+kt}.
\]
Proof. If \( g^{-1}x_mg = x_{m+t} \) holds (one of possible conclusions of Lemma 3.2) then \( M \leq m + t \) and applying Lemma 3.2 \( k \) times we obtain the result. The case \( gx_mg^{-1} = g_{m+t} \) is similar.

\[ \square \]

Proof of Theorem 3.4 First we prove the theorem in the case of centralized HNN-extensions.

Suppose that \( h = h' \neq 1 \) in the formulation, i.e. the limit group has a relation of the form \( ghhg^{-1} = h \) and denote by \( H \) the corresponding HNN-extension of Thompson’s group \( \langle x_0, x_1, g | R_F, ghg^{-1} = h \rangle \). Assume that \( ghhg^{-1} = h \) is satisfied in \( G \). From the definition of a limit group it follows, that \( ghngh^{-1} =_{F} h \) for almost all \( n \). Denote by \( C(h) \) the centralizer of \( h \) and by \( C_1 \oplus \ldots \oplus C_m \) its decomposition taken from Fact 1.3. As almost all \( g_0 \) commute with \( h \), almost all \( g_n \) have a decomposition of the form \( g_n = g_{n,1} \ldots g_{n,m} \), where \( g_{n,i} \in C_i \).

As \( h \neq 1 \), at least one of the factors \( C_1 \oplus \ldots \oplus C_m \) is isomorphic to \( \mathbb{Z} \), say \( C_{i_0} \). Denote by \([a,b] \) the support of elements taken from the subgroup \( C_{i_0} \). It follows from the construction of this decomposition, that \( h \) can only fix finitely many points in \([a,b]\).

Let us consider the sequence \( (g_{n,i_0})_{n<\omega} \). Wlog we may suppose that \( (g_{n,i_0})_{n<\omega} \) consists of powers of some element of \( F \) (which is a generator of \( C_{i_0} \)). Consider the case when it has infinitely many occurences of the same element. If \( g' \) occurs infinitely many times in this sequence, then infinitely many \( g_n(g')^{-1} \) commute with \( x_{[a,b],0}, x_{[a,b],1} \). That gives us a subsequence \( (g_{n,i_0})_{n<\omega} \) for which the relation \( [g_n(g')^{-1}, f] \) holds for all \( n \) and for all \( f \in \langle x_{[a,b],0}, x_{[a,b],1} \rangle \). As \( \langle x_{[a,b],0}, x_{[a,b],1} \rangle \) is isomorphic to Thompson’s group \( F \), we will find a word of the form \( g'f^{-1}y^{-1}g'(f^{-1}) \) with \( f \in \langle x_{[a,b],0}, x_{[a,b],1} \rangle \), which is trivial for \( y = \lim_{n \to \infty} g_{k_n} \) in the limit group corresponding to the sequence \( (g_{k_n,i_0})_{n<\omega} \) and non-trivial for \( y = g \) in the group \( H \). Indeed, it follows from Britton’s Lemma on irreducible words in an HNN-extension ([9], page 181), that the considered word can be reduced in \( H \) only if \( f^{-1} \) lies in the cyclic subgroup generated by \( h \). But \( f \in \langle x_{[a,b],0}, x_{[a,b],1} \rangle \) can be easily choosen outside \( \langle h \rangle \).

Let us now assume that the sequence \( (g_{n,i_0})_{n<\omega} \) is not stabilizing. By the discussion from the end of Section 1 we see that

\[
(\dagger) \quad \text{for all } m > 1, [x_{[a,b],0}x_{[a,b],1}^{-1}, x_{[a,b],m}] = 1.
\]

9
On the other hand any \( g_{n,i} \) is a power of some fixed element from \( C_{i_0} \). Thus we see by Lemma 3.2 that for \( M \) found for the generator of \( C_{i_0} \) as in Lemma 3.3 (\forall n) \( g_{n,i_0}^{-1} x_{[a,b],M} g_n i_0 = x_{[a,b],M+t_n} \) or (\forall n) \( g_{n,i_0} x_{[a,b],M} g_n^{-1} = x_{[a,b],M+t_n} \), \( t_n \geq 0 \).

Substituting \( x_{[a,b],M+t_n} \) into (\dagger) instead of \( x_{[a,b],m} \) we have:

\[
[x_{[a,b],0} x_{[a,b],1}^{-1}, g_{n,i_0} x_{[a,b],M} g_{n,i_0}^{-1}] = 1 \quad \text{or} \quad [x_{[a,b],0} x_{[a,b],1}^{-1}, g_{n,i_0} x_{[a,b],M} g_{n,i_0}] = 1.
\]

Thus we see that one of the following relations holds for all \( n \):

\[
[x_{[a,b],0} x_{[a,b],1}^{-1}, g_n^{-1} x_{[a,b],M} g_n] = 1 \quad \text{or} \quad [x_{[a,b],0} x_{[a,b],1}^{-1}, g_n x_{[a,b],M} g_n^{-1}] = 1.
\]

Suppose that the first relation holds for all \( n \)’s. Consider the corresponding word in the group \( H \):

\[
w = x_{[a,b],1} x_{[a,b],0}^{-1} g_n^{-1} x_{[a,b],M} g_n x_{[a,b],0} x_{[a,b],1}^{-1} g_n, M g_n^{-1} x_{[a,b],M} g_n.
\]

We claim that \( w \neq 1 \) in this HNN-extension. Once again, it follows from Britton’s Lemma on irreducible words in an HNN-extension, that we can reduce \( w \) if \( x_{[a,b],M} \) is a power of \( h \) or \( x_{[a,b],1} x_{[a,b],1}^{-1} \) is a power of \( h \). We know that \( x_{[a,b],0}, x_{[a,b],1}, \ldots, x_{[a,b],m}, \ldots \) generate the group \( F_{[a,b]} \), which is isomorphic to \( F \). From the properties of \( F \) we know that for different \( m, m’ > M \), \( x_{[a,b],m} \) and \( x_{[a,b],m’} \) do not have a common root. Thus, possibly increasing the number \( M \), we can assume that \( x_{[a,b],M} \) is not a power of \( h \). If \( x_{[a,b],0} x_{[a,b],1}^{-1} = h^d \) for some integer \( d \), then \( h^d \) fixes pointwise the segment \( \left[\frac{1}{4}a + \frac{3}{4}b, b\right] \subset [a, b] \). Hence \( h \) also fixes some final subinterval of \([a, b]\). This gives a contradiction as \( h \) was chosen to fix only finitely many points in \([a, b]\). This finishes the case of centralized HNN-extensions.

Generally, let us consider the situation, where in the limit group we have one relation of the form \( ghg^{-1} = h’ \) for some \( h, h’ \in F \). By the construction of limit groups \( h’ = h’ \) for some element \( f \in F \). Indeed, if \( h \) and \( h’ \) are not conjugated in \( F \), then there is no sequence \( (g_n)_{n<\omega} \) in \( F \) with \( g_n h g_n^{-1} = h’ \) for almost all \( n \). We now apply the argument above: let \( (fg_n)_{n<\omega} \) be a sequence convergent to the element \( fg \). It obviously commutes with \( h \), so we can repeat step by step the proof above. That completes the proof.

\[\square\]
References

[1] M. Abert, Group laws and free subgroups in topological groups, Bull. London Math. Soc. 37 (2005), 525-534.

[2] B. Baumslag, Residually free groups, Proc. London Math. Soc. (3), 17:402-418, 1967.

[3] G. Baumslag, On generalised free products, Math. Z., 78:423-438, 1962.

[4] M.G. Brin, C.C. Squier, Groups of piecewise linear homeomorphisms of the real line, Invent. Math. 79 (1985), 485-498.

[5] J.W. Cannon, W.J. Floyd, W.R. Parry, Introductory notes on Richard Thompson’s groups, Enseign. Math. (2) 42 (1996), 215-256.

[6] B. Fine, A. M. Gaglione, A. Myasnikov, G. Rosenberger, D. Spellman, A classification of fully residually free groups of rank three or less, J. Algebra, 200 (2):571-605, 1998.

[7] C. Champetier, V. Guirardel, Limit groups as limits of free groups: compactifying the set of free groups, Israel J. Math. 146 (2005), 1-76.

[8] V. Guba, M. Sapir, Diagram groups, Memoirs of the American Mathematical Society, Volume 130, Number 620, November 1997.

[9] R. Lyndon, P. Schupp, Combinatorial group theory, Springer, Berlin 1977.

[10] L. Guyot, Y. Stalder, Limits of Baumslag-Solitar groups and other families of marked groups with parameters, eprint arXiv:math/0507236.

[11] M. Kassabov, F. Matucci, The simultaneous conjugacy problem in Thompson’s group F, eprint arXiv:math/0607167.

[12] Z. Sela, Diophantine geometry over groups I: Makanin-Razborov diagrams, Publications Mathematiques de l’IHES 93(2001), 31-105.