The Cosmological Perturbation of Braneworld with An Anisotropic Bulk

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Abstract

The braneworld cosmological model was constructed by embedding a 3-brane into a higher dimensional bulk background geometry and the usual matter of our universe was assumed to be confined in the brane while the gravity can propagate in the bulk. By considering that the bulk geometry is anisotropic, after reviewed the separable solution for the bulk metric and the result that the anisotropic property of the bulk can support the perfect fluid kind of the matter in the brane, we develop a formalism of the cosmological perturbation for this kind of anisotropic braneworld model. As in isotropic case, we can also decompose the perturbation into scalar, vector and tensor modes, we find that the formalism for the anisotropic braneworld cosmological perturbation are very different from the isotropic case. The anisotropic effect can be reflected in the tensor modes which dominated the cosmological gravitation waves. Finally, we also discussed the perturbed Einstein equations governed the dynamics of the bulk geometry and the brane with the junction conditions.

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I. INTRODUCTION

The braneworld model with large extra dimensions has been studied for several years (for reviews see [1, 2, 3]). In braneworld cosmological scenarios, the ordinary Standard Model (SM) of particle physics matters of our "universe" are assumed to be confined in a lower dimensional brane which can move in a higher dimensional bulk spacetime, while gravity can propagate in the bulk. The braneworld model was first proposed by Randall and Sundrum [4, 5] as a possible solution to the hierarchy problem between the weak and Planck scales. In view of cosmology, the simplest braneworld model can be constructed by embedding the 4-D Friedmann-Robertson-Walker (FRW) cosmologies in a 5-D bulk spacetime. Here the FRW cosmology describes a homogenous and isotropic brane in which the matter contents are assumed as perfect fluid and the brane Friedmann equation can be derived out directly [6, 7, 8, 9, 10]. The homogenous and isotropic symmetries of the brane impose that the static bulk is necessarily the Schwarchild-Anti-de Sitter spacetime [11].

The astronomical observations suggest that the homogeneous and isotropic cosmological model is adequate to describe our universe. Although our universe seems homogeneous and isotropic today, it does not mean that the early era of our universe is necessarily homogeneous and isotropic and there are no observational data guaranteeing the isotropy in the era prior to the recombination. It was argued theoretically that the existence of an anisotropic phase can approach an isotropic one [12]. For this reason, as in the standard 4-D cosmology [13], it is worth studying the direct generalization of the homogeneous and isotropic braneworld cosmology, the homogeneous but anisotropic brane worlds [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

It is well known that the most information of the early stage of our universe in present, including the origin of the large scale structure of our universe is contained in the cosmological perturbations. The cosmological perturbation provide us with a window to understand the early universe. So it is very important to analyze the behave of the cosmological perturbation in the context of braneworld cosmology. That is to say it would be crucial to test whether the predictions in the braneworld cosmological perturbation are compatible with current astronomical observations, especially the observations of Cosmic Microwave Background Radiation (CMBR). It would also be very important to check whether the usual generation mechanism of cosmological perturbation would be still valid in brane cosmology (see Ref. [31]).
Recently, in [30], Fabbri, Langlois, Steer and Zegers gave out an explicit vacuum bulk solution for the spatially anisotropic braneworld cosmology with a negative cosmological constant. Then by embedding the $Z_2$ symmetric branes in this background, they found that for some bulk solutions, it is possible to embed a brane with perfect fluid in the bulk though the brane energy density and pressure are completely determined by the bulk geometry. That is to say, in the context of brane cosmology, the observed homogenous and isotropic matter distribution of our universe does not mean that our universe is necessarily isotropic, but may be anisotropic. For this reason, since the most information of the early era of our universe is contained in the cosmological perturbation, it is worth for us studying the anisotropic effect in the cosmological perturbation.

In this paper we will develop a formalism of the linear cosmological perturbation for the anisotropic braneworld models. First in section 2 we review the cosmological solution for the anisotropic braneworld obtained in [30]. Then in section 3 by using the method proposed in [33], we will first give out the the formalism that can describe the evolution of the linear perturbation in the bulk for the anisotropic braneworld cosmology. By taking into account the dynamics of the brane which was governed by the Einstein equations with the juncton conditions between the bulk and the brane matters, the relations between the perturbation of bulk metric and the perturbation of the matters in the brane are considered in section 4. Finally, in the last section we give some conclusions and remarks.

II. THE BRANEWORLD UNIVERSE WITH AN ANISOTROPIC BULK

The metric ansatz for the braneworld universe with an anisotropic bulk (B1) braneworld universe) can be constrcuted by generalizing the usual FRW isotropic braneworld directly to the spatially anisotropic case [30]

$$ds^2 = -n^2(t, y)dt^2 + \sum_{i=1}^{3} a_i^2(t, y)(dx^i)^2 + dy^2,$$

where the coordnates $x^i$ denotes the original three spatial dimensions and $y$ is the coordinate of extra dimension. Unlike the FRW braneworld cosmology which has the same scale factor for all spatial directions, the BI braneworld universe is constructed by assigning different scale factors for different spatial directions, thereby introducing the anisotropy to the system. Here the different scale factors are represented by three different functions $a_i(t, y), (i = 1, 2, 3)$ respectively.
For simplicity, the function form of the scale factors can be selected as

\[ n(t, y) = a_0(t, y) = e^{A_0(t, y)}, \quad a_i(t, y) = e^{A_i(t, y)}, \] (2)

Then by introducing the the average scale factor \( a \equiv \exp(\sum_i A_i) \) with the definition

\[ a = e^A, \quad A = \frac{1}{3} \sum_{i=1}^{3} A_i, \] (3)

the anisotropic property of the bulk metric can be represented by a vector \( N \) which is defined as

\[ N_i \equiv A_i - A \] (4)

and satisfy

\[ \sum_i N_i = 0. \] (5)

A. Vacuum solutions for the bulk Einstein equations

By using this separation of the scale factor to the isotropic and anisotropic parts in the metric (1), the vacuum Einstein equations in the bulk with a negative cosmological constant \( \Lambda \equiv -6/\ell^2 \)

\[ R_{AB} = \frac{2}{3} \Lambda g_{AB} \quad (A, B = 0, \cdots, 4) \] (6)

can then be reexpressed in terms of the isotropic and anisotropic quantities defined by (3) - (4) as

\[ A''_0 + A'_0(A'_0 + 3A') + e^{-2A_0} \left[ -3\ddot{A} - 3\dot{A}^2 - \dot{N}^2 + 3A_0\dot{A} \right] = \frac{4}{\ell^2} \] (7)

\[ A'' + A'(A'_0 + 3A') + e^{-2A_0} \left[ -\ddot{A} - 3\dot{A}^2 + \dot{A}_0\dot{A} \right] = \frac{4}{\ell^2} \] (8)

\[ H'e^{-4A} + \frac{1}{3} \dot{N}^2 A' + \frac{1}{3} e^{-2A_0} \left( \dot{N}^2 A' - 2\dot{A}\dot{N} \cdot N' \right) = \frac{4}{\ell^2} \] (9)

\[ \dot{H}e^{-4A} - \frac{1}{3} \dot{N}^2 \dot{A} - \frac{1}{3} e^{-2A_0} \dot{N}^2 \dot{A} + \frac{2}{3} A'\dot{N} \cdot N' = \frac{4}{\ell^2} \] (10)

and

\[ \left[ N' e^{3A + A_0} \right]' = \left[ \dot{N} e^{3A - A_0} \right]' \] (11)

where a prime denotes the derivatives with respect to the extra dimensional coordinate \( y \), and a dot one with respect to the time \( t \). The quantity \( H \) is defined by

\[ H \equiv e^{4A} \left( A'^2 - e^{-2A_0} \ddot{A}^2 \right) . \] (12)

When \( N = 0 \), we can recover the vacuum Einstein equations for the isotropic braneworld models.
The difficulties of integrating above equations indicate that in order to find the explicit solutions we have to consider some special cases. Here we assume that the metric is separable:

\[ A_\mu(t, y) = \alpha_\mu(t) + A_\mu(y), \]  

then an analytic solution can be obtained by integrating the Einstein equations directly. The solution is

\[ ds^2 = \text{sinh}^{1/2} \left( \frac{4y}{\ell} \right) \left[ -\tanh^{2q_0} (2y/\ell) dt^2 + \sum_i \tanh^{2p_i} (2y/\ell) t^{2p_i} \left( dx^i \right)^2 \right] + dy^2 \]  

where the seven coefficients \( q_\mu \) and \( p_i \) satisfy the following constraints

\[ \sum_\mu q_\mu = 0, \quad \sum_\mu q_\mu^2 = \frac{3}{4}, \quad \sum_i p_i = 1, \quad \sum_i p_i^2 = 1, \quad \sum_i q_i(p_i + 1) = 0. \]  

This metric ”mixes” the 5-D static solution and the well-known 4-D Kasner solution.

**B. The embedded brane and the junction conditions**

When an infinite thin brane was embedded in the bulk, the brane dynamics was governed by the bulk Einstein equations with the junction conditions. Here the junction conditions can be obtained by imposing a \( Z_2 \) symmetry on the brane configurations.

For an embedded brane in the bulk, in case of the \( Z_2 \) symmetries, the Israel junction conditions can be expressed as

\[ K_a^b = -\kappa^2 \left( T_a^b - \frac{T}{3} \delta_a^b \right), \]  

where \( T_a^b \) denote the energy-momentum stress tensor on the brane and \( K_a^b \) is the extrinsic curvature on one side of the brane which is defined by

\[ K_{ab} = X_A^A X_B^B D_A n_B = \frac{1}{2} \left[ g_{AB} \left( X_a^A \partial_a n_B + X_b^B \partial_a n_B \right) + X_a^A X_b^B n^C g_{AB,C} \right] \]  

where \( D_A \) is the covariant derivatives associated with the bulk metric \( g_{AB} \), \( n^A \) is the unit vector normal to the brane. Here the geometry of the brane is defined by its embedding in the bulk, i.e. \( X^A = X^A(x^a) \), where \( x^a \) are the intrinsic coordinates on the brane and \( X_a^A \equiv \partial X^A / \partial x^a \).

For BI braneworld cosmological models, the embedded brane must respect the BI symmetries. The anisotropic property of the bulk imply that the brane energy-momentum stress tensor is necessarily to take the form:

\[ T_a^b = \text{diag} \left( -\rho, P_1, P_2, P_3 \right), \]
Here in different spatial directions, the pressure of the matter takes different value and all off-diagonal elements in the stress tensor are necessarily zero, i.e. the stress tensor of the brane matter is still anisotropic. When all $P_i$ ($i = 1, 2, 3$) take the same value, i.e. $P_1 = P_2 = P_3$, this stress tensor form go back to the isotropic case, i.e. the perfect fluid form.

As the metric form of the anisotropic bulk, the brane pressure can also be separated into isotropic and anisotropic part:

$$P_i = P + \pi_i$$  \hspace{1cm} (19)

where the anisotropic property of the brane pressure are represented by the vector $\pi_i$ which satisfy: $\sum_i \pi_i = 0$.

By this separation, the spatial components of the junction condition (16) can also be seperated into an isotropic part:

$$e^{-A_0} y_b A'_b |_b + \sqrt{1 + \dot{y}_b^2} A'_b |_b = \frac{\kappa^2}{6} \rho,$$

and an anisotropic part

$$e^{-A_0} y_b N'_i |_b + \sqrt{1 + \dot{y}_b^2} N'_i |_b = \frac{\kappa^2}{2} \pi_i$$  \hspace{1cm} (21)

This last equation tells us that the anisotropic property of the bulk and the stress on the brane must be consistent.

It should be noticed here that the brane position in the extra dimension $y_b$ is not assumed to be fixed. In this sense, the coordinate system is not Gaussian Normal.

For the separable metric solution (14), it is easy to find that when

$$q_0 = \pm \frac{\sqrt{3}}{4},$$  \hspace{1cm} (22)

the brane can support perfect fluid type of matter, i.e. $\pi_i = 0$. That is to say that in an anisotropic background bulk geometry, we still can embed in a brane with isotropic perfect fluid as matter. This implies that in the sense of braneworld cosmology, the isotropic brane matter distribution does not imply that the background bulk geometry is necessarily isotropic, an anisotropic background bulk geometry is also possible, while the trajectory of brane in the bulk can no longer be choosed arbitrarily in constrast with the isotropic case, it must satisfy the first integration of (21). The type of the perfect fluid matter in the brane must also be compatible with the given bulk geometry.
III. THE METRIC PERTURBATION

In the previous section we have reviewed the results obtained in [30], we see that in an anisotropic background bulk geometry, it is also possible to embed an brane with perfect fluid type of matter. That is to say in the context braneworld cosmology, the observed homogeneous and isotropic matter distribution can not tell us that the background geometry is necessarily isotropic.

In this section we will develop a linear cosmological perturbation formalism for the anisotropic bulk system. As in [33] where the formalism for the usual isotropic braneworld cosmological perturbation was given out, here we still work in the Gaussian Normal (GN) system of coordinates adapted to the embedded brane in which the brane is localized at \( y = 0 \).

The most general perturbed form for the metric (1) can be write as

\[
ds^2 = (g_{AB} + h_{AB})\,dx^A\,dx^B
= -n^2(1 + 2\psi)\,dt^2 + 2\sum_i B_i dx^i\,dt + \sum_{ij} a_i a_j \left( \delta_{ij} + \hat{h}_{ij} \right) dx^i dx^j + dy^2. \tag{23}
\]

Due to the complexity of the index system for the anisotropic bulk metric, in the context following we will always write the summation symbol explicitly. As in [34] (see also [35]), the perturbation field in the perturbed metric (23) can also be classified further into scalar, vector and tensor quantities by according their transformation properties with respect to the three spatial directions. So we have

\[
B_i = \nabla_i B - S_i \tag{24}
\]

and

\[
\hat{h}_{ij} = 2R\delta_{ij} + \nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + s_{ij} \tag{25}
\]

where \( \nabla_i \) denotes the covariant derivatives associated with the \( 3 \times 3 \) anisotropic spatial metric. In this definition of the perturbed metric, \( \psi, B, R \) and \( E \) are 4 scalar perturbation fields, \( S_i \) and \( F_i \) are 2 divergence-free 3-vectors perturbation fields and \( s_{ij} \) is a transverse and traceless 3-tensor perturbation field. Here we should notice that different with the usual isotropic case, the index of the vector and tensor perturbation fields can not upper and lower directly by the anisotropic spatial metric, but the index of their product with the anisotropic scale factor together can be upper and lower by the anisotropic spatial metric. This is because that for the perturbed anisotropic bulk metric, all the spatial perturbed fields have to be multiplied by the anisotropic scale factors. Then by this
definition of the perturbed metric, the divergence-free constraint conditions for the vector perturbations fields $S_i$ becomes

$$\sum_i \nabla^i S_i = \sum_i \frac{1}{a_i^2} S_{i,i} = 0.$$  \hspace{1cm} (26)

While the divergence-free constraint conditions for the vector perturbation fields $F_i$ are same as in the isotropic case

$$\sum_i a_i^2 \nabla^i F_i = \sum_i F_{i,i} = 0.$$ \hspace{1cm} (27)

The transverse traceless constraint conditions for the tensor perturbation fields $s_{ij}$ become

$$\sum_i s_{ii} = 0,$$ \hspace{1cm} (28)

and

$$\sum_i a_i a_j \nabla^i s_{ij} = \sum_i \frac{a_i}{a_j} s_{ij,i} = 0.$$ \hspace{1cm} (29)

Here a comma denotes the differentiation with respect to the corresponding spatial coordinates. From these constraints conditions we know that the vector metric perturbation of the bulk has 4 degree of freedom, while the tensor perturbation has 2 degree of freedom.

**A. Gauge transformation of the metric perturbation**

In different coordinate systems, the metric perturbation defined above can be different quantitatively, i.e. there exist some gauge freedoms for the metric perturbation. In order to distinguish the gauge effect and physical degrees of freedom, we need to study the coordinate transformation effect on all metric perturbation fields defined above.

The infinitesimal changes of coordinates

$$x^A \rightarrow \tilde{x}^A = x^A + \xi^A,$$ \hspace{1cm} (30)

can induce the transformations for the bulk metric perturbations:

$$h_{AB} \rightarrow \tilde{h}_{AB} = h_{AB} - D_A \xi_A - D_B \xi_A,$$ \hspace{1cm} (31)

here $D_A$ is the covariant derivatives associated with the unperturbed bulk metric.

Now we parametrize the infinitesimal coordinate transformation by the vector $\xi^A = (\xi^0, \xi^i, \xi^5)$. As discussed in (3). for the isotropic case, in a GN coordinate system, the metric perturbation components related to the extra dimension $h_{55}$, $h_{05}$ and $h_{i5}$ vanished,
while their transformation do not vanish. In order to bring any coordinate system into a
GN coordinate system, one has to choose $\xi^0, \xi^i$ and $\xi^5$ appropriately to adjust the position
of the brane and to set $h_{55}, h_{05}$ and $h_{i5}$ to zero. The remaining gauge freedoms for the
GN system make it possible to redefine the coordinates inside the brane worldsheet. This
discussion is also valid for our anisotropic case now.

The spatial vector $\xi^i$ can be decomposed further into

$$\xi^i = \nabla^i \xi + \tilde{\xi}^i.$$  \hspace{1cm} (32)

Here the vector $\tilde{\xi}^i$ is transverse, i.e. $\nabla_i \tilde{\xi}^i = 0$ and the index is upper and lower by the
anisotropic spatial $3 \times 3$ metric. With this decomposition, from the transformations of the
bulk metric perturbation (31), one can easily find the transformations for the perturbation
fields respectively.

1. **Scalar gauge transformation**

The scalar perturbations transform as

$$\psi \rightarrow \tilde{\psi} = \psi - \dot{\xi}^0 - \frac{n}{n} \xi^0 - \frac{n'}{n} \xi^5,$$  \hspace{1cm} (33)

$$R \rightarrow \tilde{R} = R,$$  \hspace{1cm} (34)

$$E \rightarrow \tilde{E} = E,$$  \hspace{1cm} (35)

$$B \rightarrow \tilde{B} = B + n^2 \xi^0 - \xi.$$  \hspace{1cm} (36)

2. **Vector gauge transformation**

The vector perturbations transform as

$$F_i \rightarrow \tilde{F}_i = F_i,$$  \hspace{1cm} (37)

$$S_i \rightarrow \tilde{S}_i = S_i + \hat{\xi}_i.$$  \hspace{1cm} (38)

3. **Tensor gauge transformation**

The tensor perturbation transforms as

$$s_{ij} \rightarrow \tilde{s}_{ij} = s_{ij} - \frac{2}{a_i a_j} \nabla_i \nabla_j \xi - \frac{1}{a_i a_j} \left( \nabla_i \tilde{\xi}_j + \nabla_j \tilde{\xi}_i \right) - 2 \left( \frac{a_i}{a_j} \xi^0 + \frac{a_j}{a_i} \xi^5 \right) \delta_{ij}.$$  \hspace{1cm} (39)

In the above transformation for the perturbation fields, we see that they are very dif-
ferent from the isotropic case. In the isotropic braneworld cosmological perturbation,
the tensor perturbation fields remain unchanged under the gauge transformation. While
here we see that the tensor perturbation fields changed under gauge transformation, and
the scalar and vector perturbation fields in the 3 spatial perturbed metric remain un-
changed. This can be interpreted as the anisotropic effect in the braneworld cosmological
perturbation. In the braneworld cosmological perturbation in an anisotropic background
bulk geometry, the anisotropic effect can be reflected in the gauge transformations of the
perturbation fields. Different gauge choice will give different anisotropic effect. Since the
tensor modes of the cosmological perturbations are the gravitational waves which can
propagate independently, the anisotropic effect must can be reflected in the evolution of
the cosmological gravitational waves.

In the following we will restrict ourselves only to consider the system inside a subset
of GN coordinate system, then \( \xi^5 = 0 \) and \( \xi^0, \xi^i \) are not depend on the extra
dimension.

B. Perturbed Einstein equations in the bulk

The evolution of the bulk metric perturbation is governed by the perturbed Einstein
equations in the bulk. Outside the embedded brane, the Einstein equations read

\[
R_{AB} = \kappa^2 \left( \bar{T}_{AB} - \frac{1}{3} \bar{T} g_{AB} \right),
\]

where \( \bar{T}_{AB} \) is the energy-momentum tensor of the bulk matter and \( \bar{T} = \sum_{AB} g^{AB} \bar{T}_{AB} \).

Then the form of the perturbed Einstein equations in the bulk are

\[
\delta R_{AB} = \kappa^2 \left( \delta \bar{T}_{AB} - \frac{1}{3} \delta \bar{T} g_{AB} - \frac{1}{3} \delta h_{AB} \right).
\]

In previous section we have reviewed the solutions for the braneworld in an anisotropic
bulk background. In this solutions, the empty bulk with a cosmological constant is of par-
ticular important. The perturbed Einstein equations for a vacuum bulk with cosmological
constant \( \Lambda \) are simply

\[
\delta R_{AB} = \frac{2}{3} \Lambda h_{AB}.
\]

With the definition of the metric perturbation \( h_{AB} \), the general form of the curvature
perturbations can be expressed as

\[
\delta R_{AB} = -\frac{1}{2} D_A D_B h - \frac{1}{2} \sum_C D^C D_C h_{AB} + \frac{1}{2} \sum_C D^C D_A h_{BC} + \frac{1}{2} \sum_C D^C D_B h_{AC},
\]

\[43\]
where \( h \) denotes the trace of the metric perturbation, namely

\[
h = \sum_{AB} g^{AB} h_{AB}.
\]  

(44)

Now substituting the definition of the metric perturbation (23) into (43), after some tedious but straightforward calculations, we can obtain the explicit expressions of the curvature perturbation which can also be separated into scalar, vector and tensor parts.

1. Scalar components

\[
\delta R_{00}^S = -3 \ddot{R} - \triangle \ddot{E} + \sum_i \frac{\dot{a}_i}{a_i} \psi + \left( \frac{\dot{n}}{n} - 2 \sum_i \frac{\dot{a}_i}{a_i} \right) \ddot{R} + \frac{\dot{n}}{n} \triangle \ddot{E} - 2 \sum_i \frac{\dot{a}_i}{a_i} \ddot{E}_{,ii}
\]

\[
+ \sum_i \frac{1}{a_i^2} \ddot{B}_{,ii} - \frac{\dot{n}}{n} \sum_i \frac{1}{a_i^2} B_{,ii} + n^2 \sum_i \frac{1}{a_i^2} \psi_{,ii}
\]

\[
+ n^2 \left[ \psi'' + \left( 2 \frac{n'}{n} + \sum_i \frac{a_i'}{a_i} \right) \psi' + 3 \frac{n'}{n} R' + \frac{n'}{n} \triangle E' + \left( 2 \frac{n''}{n} + 2 \frac{n'}{n} \sum_i \frac{a_i'}{a_i} \right) \psi \right]
\]  

(45)

\[
\delta R_{0i}^S = -2 \ddot{R}_i + \left( \frac{3}{n} \frac{\dot{a}_i}{a_i} - \sum_j \frac{\dot{a}_j}{a_j} \right) R_i + \left( \sum_j \frac{\dot{a}_j}{a_j} - \frac{\dot{a}_i}{a_i} \right) \psi_{,i}
\]

\[
- \frac{1}{2} B_{,ii}'' + \frac{1}{2} \left( \frac{n'}{n} + 2 \frac{a_i'}{a_i} - \sum_j \frac{a_j'}{a_j} \right) B_{,i}'
\]

\[+ \left[ \frac{1}{n^2} \left( \frac{\dot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \left( \sum_j \frac{\dot{a}_j}{a_j} - \frac{\dot{a}_i}{a_i} \right) - \frac{\dot{n} a_i}{n a_i} \right) - 2 \frac{n a_i'}{n a_i} \right] B_{,i}
\]

\[+ \sum_j \left\{ \left( \frac{a_i}{a_j} - 1 \right) \dot{E}_{,jij} + \left[ \frac{\dot{a}_j}{a_j} \left( \frac{a_i}{a_j} - 1 \right) + \frac{\dot{a}_i}{a_i} - \frac{\dot{a}_j}{a_j} \right] E_{,jji} \right\}
\]  

(46)

\[
\delta R_{ij}^S = \left\{ \frac{a_i^2}{n^2} \left[ \ddot{R} + \left( \frac{3}{n} \frac{\dot{a}_i}{a_i} + \sum_k \frac{\dot{a}_k}{a_k} - \frac{\dot{n}}{n} \right) \dot{R} - \frac{\dot{a}_i}{a_i} \psi + \frac{\dot{a}_i}{a_i} \triangle \dot{E} - \frac{\dot{a}_i}{a_i} \sum_k \frac{1}{a_k^2} B_{,kk} \right.
\]

\[+ 2 \left( \frac{\dot{n} a_i}{n a_i} - \frac{\dot{a}_i}{a_i} \left( \sum_k \frac{\dot{a}_k}{a_k} - \frac{\dot{a}_i}{a_i} \right) \right) \psi \right] \left[ R'' + \frac{a_i'}{a_i} \psi' \right]
\]

\[+ \left( \frac{3}{n} \frac{\dot{a}_i}{a_i} + \sum_k \frac{\dot{a}_k}{a_k} + \frac{n'}{n} \right) R' + \frac{\dot{a}_i}{a_i} \triangle E' + \sum_k \frac{1}{a_k^2} R_{kk}'' \right) \delta_{ij} + a_i^2 \mathcal{M}_{ij} \delta_{ij} + \partial_i \partial_j \left( a_i a_j \left[ \frac{1}{n^2} \left( \ddot{E} + \left( \sum_k \frac{\dot{a}_k}{a_k} - \frac{n}{n} \right) \dot{E} \right) - \left( E'' + \sum_k \frac{a_k'}{a_k} + \frac{n'}{n} \right) E' \right]
\]

\[+ a_i a_j \mathcal{M}_{ij} E + \delta_{ij} \left[ \sum_k \left( \frac{1}{a_i a_k} + \frac{1}{a_j a_k} - \frac{1}{a_k^2} \right) E_{,kk} - \frac{1}{a_i a_j} \triangle E \right] - \psi - R
\]

\[+ \frac{1}{n^2} \left[ \ddot{B} + \left( \frac{n}{n} + \frac{\dot{a}_i}{a_i} + \frac{\dot{a}_j}{a_j} - \sum_k \frac{\dot{a}_k}{a_k} \right) B \right]
\]  

(47)
2. Vector components

\[ \delta R_{i5}^S = -\psi_i' - 2R_i + \left( \frac{a_i'}{a_i} - \frac{n'}{n} \right) \psi_i + \left( 3 \frac{a_i'}{a_i} - \sum_j \frac{a_j'}{a_j} \right) R_i \]
\[ + \frac{1}{n^2} \left\{ - \frac{1}{2} \dot{B}_i' + \frac{a_i'}{a_i} \dot{B}_i + \frac{1}{2} \left( \frac{n}{n} - \sum_j \frac{a_j'}{a_j} \right) B_i' \right\} \]
\[ + \sum_j \left\{ \left( \frac{a_i}{a_j} - 1 \right) E_{i,ijj} + \left[ \frac{a_i'}{a_i} + \frac{1}{2} \left( \frac{a_i}{a_j} - 1 \right) - \frac{a_j'}{a_j} \right] E_{i,ii} \right\} \]  
\[ \text{(48)} \]

\[ \delta R_{35}^S = -\psi'' - 3R'' - \Delta E'' - 2 \frac{n'}{n} \psi' - 2 \sum_i \frac{a_i'}{a_i} R' - 2 \sum_i \frac{a_i'}{a_i} E_{i,ii} \]  
\[ \text{(49)} \]

\[ \delta R_{05}^S = -3 \dot{R}' - \Delta \dot{E}' + \sum_i \left( \frac{n'}{n} - \frac{a_i'}{a_i} \right) (\dot{R} + \dot{E}_{i,ii}) + \sum_i \frac{\dot{a}_i}{a_i} (\psi' - R' - E_{i,ii}) \]
\[ + \frac{1}{2} \sum_i \frac{1}{a_i^2} B_{i,ii} - \frac{n'}{n} \sum_i \frac{1}{a_i^2} B_{ii} \]  
\[ \text{(50)} \]

2. Vector components

\[ \delta R_{00}^V = -2 \sum_i \frac{\dot{a}_i}{a_i} \dot{F}_{i,i} \]  
\[ \text{(51)} \]

\[ \delta R_{0i}^V = \sum_j \left\{ \frac{a_i}{2a_j} \dot{F}_{i,j} + \frac{a_i}{2a_j} \dot{F}_{j,ji} + \frac{1}{2} \left( \frac{a_i \dot{a}_j}{a_j^2} - \frac{\dot{a}_i}{a_j} \right) F_{i,j} \right\} \]
\[ + \sum_j \left\{ \frac{\dot{a}_j}{a_j} \left( \frac{a_i}{2a_j} - 1 \right) + \frac{\dot{a}_i}{a_i} - \frac{\dot{a}_j}{2a_j} \right\} F_{j,ji} \]
\[ + \frac{1}{2} S_{ii}' - \frac{1}{2} \left( \frac{n'}{n} + 2 \frac{a_i'}{a_i} - \sum_j \frac{a_j'}{a_j} \right) S_{ii}' \]
\[ - \left[ \frac{1}{n^2} \left( \frac{\dot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \left( \sum_j \frac{a_j'}{a_j} - \frac{\dot{a}_i}{a_i} \right) \right) - \frac{n' a_i'}{n a_i} \right] S_{ii} + \frac{1}{2} \sum_j \frac{1}{a_j^2} S_{ij,ji} \]  
\[ \text{(52)} \]

\[ \delta R_{ij}^V = \frac{a_i a_j}{2} \sum_k \left[ \frac{1}{a_j a_k} (F_{i,j,kk} + F_{k,kj}) + \frac{1}{a_i a_k} (F_{j,i,kk} + F_{k,kj}) - \frac{1}{a_k^2} (F_{i,j,kk} + F_{j,i,kk}) \right] \]
\[ + \frac{a_i a_j}{2} \left\{ \mathcal{M}_{ij} (F_{i,j} + F_{j,i}) + \frac{1}{n^2} \left[ \dot{F}_{i,j} + \dot{F}_{j,i} + \left( \sum_k \frac{\dot{a}_k}{a_k} - \frac{n'}{n} \right) \left( \dot{F}_{i,j} + \dot{F}_{j,i} \right) \right] \right\} \]
\[ - \left[ F_{i,j}'' + F_{j,i}'' + \left( \sum_k \frac{a_k'}{a_k} + \frac{n'}{n} \right) (F_{i,j}'' + F_{j,i}'') \right] \]
\[ + \frac{1}{2n^2} \left[ \ddot{S}_{i,j} + \ddot{S}_{j,i} + \left( \sum_k \frac{\ddot{a}_k}{a_k} - \frac{2 a_j'}{a_j} - \frac{n'}{n} \right) S_{i,j} + \left( \sum_k \frac{\ddot{a}_k}{a_k} - \frac{2 a_i'}{a_i} - \frac{n'}{n} \right) S_{j,i} \right] \]  
\[ \text{(53)} \]
\[ \delta R_{i5}^V = + \sum_j \left\{ \frac{a_i}{2a_j} F_{i,jj} + \frac{a_i}{2a_j} F_{j,ji} + \frac{1}{2} \left( \frac{a_i a_j'}{a_j} - \frac{a_j'}{a_i} \right) F_{i,jj} \right\} 
\quad + \left[ \frac{a_i'}{a_i} + \frac{a_j'}{a_j} \left( \frac{a_i}{2a_j} - 1 \right) - \frac{a_j'}{2a_j} \right] F_{j,ji} \right\} 
\quad - \frac{1}{n^2} \left\{ -\frac{1}{2} \frac{\dot{\gamma}'}{\gamma} + \frac{\dot{\gamma}'}{\gamma} + \frac{1}{2} \left( \frac{n}{\sum_j a_j} - \frac{\dot{\gamma}}{\gamma} \right) S_i' \right\} 
\quad + \left[ \frac{\dot{a}_i'}{a_i} + \frac{\dot{a}_j'}{a_j} \left( \sum_j \frac{\dot{a}_j - \dot{a}_i}{a_j} \right) - \frac{n\dot{a}_i'}{\gamma a_i} \right] S_i \right\} 
\] 
\[ \delta R_{55}^V = -2 \sum_i \frac{a_i'}{a_i} F_{i,i} \] 
\[ \delta R_{05}^V = - \sum_i \frac{a_i'}{a_i} F_{i,i} - \sum_i \frac{a_i}{a_i} F_{i,i} - \frac{1}{2} \sum_i \frac{1}{a_i} S_i' \] 

3. Tensor components

\[ \delta R_{00}^T = - \sum_i \frac{\dot{a}_i}{a_i} s_{ii} \] 
\[ \delta R_{0i}^T = + \frac{1}{2} \sum_j \left[ \frac{a_j}{a_j} \frac{\dot{s}_{ij}}{a_j} + \left( \frac{\dot{a}_j}{a_j} - \frac{\dot{a}_i}{a_i} \right) s_{jj,i} + \left( \frac{a_i a_j'}{a_j} - \frac{a_j'}{a_i} \right) s_{ij} \right] \] 
\[ \delta R_{ij}^T = - \frac{1}{2} a_i a_j \sum_k \frac{1}{a_k^2} s_{ij,kk} + \frac{1}{2} \frac{a_i a_j}{n^2} \left[ \ddot{s}_{ij} + \left( \sum_k \frac{\dot{a}_k}{a_k} + \frac{n'}{n} \right) \dot{s}_{ij} \right] - \frac{a_i a_j}{2} \left[ s_{ij}'' + \left( \sum_k \frac{a_k'}{a_k} + \frac{n'}{n} \right) s_{ij}' \right] + \frac{a_i a_j}{2} M_{ij} s_{ij} \] 
\[ \delta R_{i5}^T = + \frac{1}{2} \sum_j \left[ \frac{a_j}{a_j} s_{ij,j} + \left( \frac{a_i'}{a_i} - \frac{a_j'}{a_j} \right) s_{jj,i} + \left( \frac{a_i a_j'}{a_j^2} - \frac{a_j'}{a_j} \right) s_{ij,j} \right] \] 
\[ \delta R_{55}^T = - \sum_i \frac{a_i'}{a_i} s_{ii}' \] 
\[ \delta R_{05}^T = - \frac{1}{2} \sum_i \frac{a_i'}{a_i} s_{ii} - \frac{1}{2} \sum_i \frac{\dot{a}_i}{a_i} s_{ii} \]
Here $\Delta f = \sum_i f_{i,ii}$ and the quantity $\mathcal{M}_{ij}$ was defined as

$$\mathcal{M}_{ij} = \frac{1}{n^2} \left[ a_i \ddot{a}_i - \dddot{a}_i + \frac{a_j^2}{a_i} \dddot{a}_j + \sum_k \frac{a_k}{a_i} \right] \left( \dddot{a}_i + \frac{\dot{a}_j}{a_j} \right) - \left[ \frac{a_i''}{a_i} - \frac{a_j''}{a_j} - \frac{a_i'^2}{a_i} + \frac{a_j'^2}{a_j} + \sum_k \frac{a_k'}{a_i} \right] \left( \frac{a_i'}{a_i} + \frac{a_j'}{a_j} \right).$$ (65)

If we take $a_1 = a_2 = a_3 = a$, i.e. we go back to the isotropic cosmological perturbation, then the expressions we obtained above can reduce to all the results obtained by Langlois in [33].

IV. PERTURBATION OF THE JUNCTION CONDITIONS

Having obtained the formalism of the metric perturbation in the bulk, this section we will discuss the perturbation of the matter in the brane.

In subsection II B we have discussed the junction conditions which govern the dynamics of the brane. In the anisotropic background bulk geometry, the matter of the brane can be some kind of the isotropic perfect fluid matter. In the following we will mainly discuss the matter perturbation of this kind of brane matter.

The energy-momentum tensor of the matter in the brane is

$$T^\mu_\nu = \delta(y) S^\mu_\nu.$$ (66)

Here we assume that the brane is located at the position of $y = 0$. For the perfect fluid kind of matter, we have

$$S^\mu_\nu = (\rho + P) u^\mu u_\nu + Pg^\mu_\nu.$$ (67)

From the junction condition (16) with $Z_2$ symmetry, we can easily get the jump of the extrinsic curvature across the brane

$$[K_{\mu\nu}] = -\kappa^2 \left( S_{\mu\nu} - \frac{1}{3} S g_{\mu\nu} \right).$$ (68)

Then the perturbed junction conditions are

$$[\delta K_{\mu\nu}] = \kappa^2 \left( -S_{\mu\nu} + \frac{1}{3} g_{\mu\nu} \delta S + \frac{1}{3} Sh_{\mu\nu} \right).$$ (69)

In case of the GN coordinate system, the extrinsic curvature can be expressed simply as

$$K_{\mu\nu} = \frac{1}{2} \partial_y g_{\mu\nu}.$$ (70)
Using this simple form of the extrinsic curvature, the perturbed junction conditions become

$$h'_{\mu\nu}|_{y=0^+} = \kappa^2 \left( -S_{\mu\nu} + \frac{1}{3} g_{\mu\nu} \delta S + \frac{1}{3} S h_{\mu\nu} \right).$$

(71)

From the normalization condition satisfied by the 4-velocity, we can get the perturbations of the unit 4-velocity

$$\delta u^\mu = \{-n^{-1} \psi, a_i^{-1} v^i\}.$$  

(72)

Then the components of the perturbation of the energy-momentum tensor are

$$\delta S_{00} = n^2 \delta \rho + 2n^2 \psi \rho,$$

(73)

$$\delta S_{0i} = -(\rho + P) na_i v_i + PB_i$$

(74)

$$\delta S_{ij} = a_i \delta_{ij} \delta P + Ph_{ij} + \pi_{ij},$$

(75)

(76)

where the index of $v^i$ is still lower and upper by the anisotropic spatial 3-metric: $v_i = \sum_j a_j \delta_{ij} v^j$, $\pi_{ij}$ is the anisotropic stress tensor.

As in the metric perturbation, it is also possible to decompose above expressions further into scalar, vector and tensor components by using the decomposition

$$v^i = \nabla^i v + \bar{v}^i$$

(77)

with the divergence-free condition for $\bar{v}^i$: $\sum_i \nabla_i \bar{v}^i = 0$ and

$$\pi_{ij} = \left( \nabla_i \nabla_j - \frac{1}{3} \sum_k \frac{1}{a_k^2} \nabla_k \nabla_k \delta_{ij} \right) E_\pi + \nabla_i F_{\pi j} + \nabla_j F_{\pi i} + s_{\pi ij}$$

(78)

with $F_{\pi i}$ anisotropic transverse and $s_{\pi ij}$ anisotropic transverse traceless.

Substituting these expressions and the background quantities into (71), we can get the explicit form of the perturbed junction conditions directly.

V. CONCLUSIONS

In this paper, we have developed a formalism of the cosmological perturbation for the anisotropic braneworld. First we reviewed the solutions of the Einstein equations for the braneworld model in an anisotropic background bulk geometry. Since the anisotropic background bulk geometry can support a still anisotropic brane but with perfect kind of matter in it, in the sense of brane cosmology, this means that the homogeneous and isotropic matter distributions of our universe do not imply our universe (the brane embedded in the bulk) is necessarily isotropic.
Then we turned to calculate the cosmological perturbation of this kind of anisotropic braneworld model. The anisotropic braneworld model is natural generalization of the usual homogeneous and isotropic braneworld model, the formalism of the cosmological perturbation obtained in present paper is also a direct generalization of the isotropic braneworld cosmological perturbation. When we go back to the isotropic case by taking all the spatial scale factor to be same: $a_1 = a_2 = a_3 = a$, we can recovery all the cosmological perturbation results for the isotropic case.

The difference of the cosmological perturbation between the anisotropic and isotropic braneworld models is caused by the anisotropic effect reflected in the braneworld cosmological perturbation. It is well known that the cosmological perturbation plays a key role in the study of the CMBR, in the context of braneworld with anisotropic background, the anisotropic effect can also be reflected in the anisotropic spectrum of the CMBR. This will be considered in our future work.

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