G/G - Topological Field Theories by Cosetting $G_k$.

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G/G topological field theories based on $G_k$ WZW models are constructed and studied. These coset models are formulated as Complex BRST cohomology in $G_k^c$, the complexified level $k$ current algebra. The finite physical spectrum corresponds to the conformal blocks of $G_k$. The amplitudes for $G/G$ theories are argued to be given in terms of the $G_k$ fusion rules. The $G_k/G_k$ character is the Kac-Weyl numerator of $G_k$ and is interpreted as an index. The Complex BRST cohomology is found to contain states of arbitrary ghost number. Intriguing similarities of $G/G$ to $c \leq 1$ matter systems coupled to two dimensional gravity are pointed out.

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1. Introduction and Summary.

Recently “matrix models” were applied to the solution of two dimensional gravity performing the sum over triangulations. This converges to the sum over surfaces in the “double scaling limit” [1] [2] [3]. The amplitudes were found to obey the KdV flows [4]. The amplitudes of topological two dimensional gravity [5] [6] [7] were shown to obey similar relations. The equality between amplitudes in matrix models and in topological gravity was explicitly verified as well, for small genera [8]. Thus, the matrix models, while combinatorically performing the sum over all surfaces, expose the topological features revealed by topological gravity. In the framework of multi-matrix models one can describe gravity coupled to \((p,q)\) minimal model, by going to higher critical points in the “double scaling limit” [9] [10] [11] [4]. There are other ways to discuss the coupling of matter to 2d gravity, like the traditional Liouville approach [12], in which the amplitudes were recently calculated [13] [14], or the light-cone gauge approach implying \(SL(2,\mathbb{R})\) structure [15].

Topological matter can be naturally coupled to topological 2d gravity. An important example is given by the Topological Sigma Models [16]. In this work, we construct a new class of topological matter theories associated with Rational Conformal Field Theories (RCFTs). We start with theories based on current algebras, as those are believed to be the basic building blocks of RCFTs [17]. It should be noted that topological matter systems, which are not based on CFTs, can be constructed and coupled to two dimensional (topological) gravity, like the \(CP^1\) sigma model in [1]. Here we focus, nonetheless, on the derivation of topological field theories from CFTs, a choice motivated by the possible insight into string theory they can provide and even more, by their inherent simplicity. The coupling of topological CFTs to gravity becomes simpler due to the vanishing trace of the energy momentum tensor, \(T_{\mu\nu} = 0\) (\(T_{\mu\nu}\) is just BRST exact in the case of a general topological field theory [18]).

Rational Conformal Field Theories have been under intensive investigation recently. Here we construct and analyze their topological counterparts employing tools inherited from the study of RCFTs [17]. While studying these topological theories we will unravel the features of their ancestor RCFTs which they encode, in particular conformal blocks and their fusion rules. These features were recognized as being among the essentials of RCFTs, starting with the observation of E. Verlinde [19] through the RCFT endeavor summarized in [17]. They are also central to the profound relation which exists between RCFTs (which are based on current algebras) and the topological 2+1 dimensional Chern-Simons Gauge Theory [20].
The coupling of these simple topological systems to (topological) gravity, may give us a deeper insight into the structure of gravity and into the way it couples to matter. What does gravity couple to, is an important question in the study of quantum gravity. $G/G$ topological theories provide interesting matter systems which enables us to address this question applying ideas and techniques developed in the study of (R)CFTs. $G/G$ theories allow for the algebraic study of two dimensional gravity. Recently, it was argued by Witten [21] (through a stringy black hole coset construction) and emphasized by Eguchi [22] that space-time singularities in general relativity have a natural description in terms of topological field theories.

In the present paper we construct the $G_k/G_k$ topological models. The structure of these theories is intimately connected to the level $k$ current algebra $G_k$ [23] [24] (based on the compact Lie group $G$). In particular the physical spectrum of the $G_k/G_k$ topological model consists of the $G_k$ highest weight states. The structure of $G_k/G_k$ is in fact inherited from the corresponding $G_k$ Wess-Zumino-Witten model [25] and the $G_k/G_k$ physical states are in one to one correspondence with the conformal blocks of the WZW model. Moreover, we provide evidence for our conjecture [26] that the amplitudes in the $G_k/G_k$ topological theory are given by the fusion rules of the corresponding $G_k$ WZW model. This was recently proven by Witten [27].

We start our investigation with the $G_k$ WZW model and construct the $G_k/G_k$ topological model out of it. We follow the usual gauged WZW approach to $G/H$ based on gauging the $H \subset G$ subgroup in the $G_k$ WZW model [28] [29] [30] [31]. By employing the BRST variant of this construction [32] [33] we develop a BRST approach to the $G/G$ models similar to the one Witten advocated for topological field theories [34] and for topological $\sigma$ models in particular [16]. We would like to impose the naive $G_k$ constraints, which turn out however, to be second class constraints. This difficulty is overcome by enlarging the theory, promoting it to a complex theory by adding $G_c/G$ degrees of freedom. This extra $G_c^C/G$ degrees of freedom form a WZW-like model with a coefficient $\bar{k} = k + 2c_G$, which can be viewed as its “level” ($c_G$ is the quadratic Casimir in the adjoint representation of $G$). We denote it by $(G_c^C/G)_\bar{k}$, noticing the formal analogy to the $G$ WZW model at level $-(k + 2c_G)$. We are, therefore, led to the study of the Complex BRST cohomology of $G_k$. This constitutes the CBRST approach to $G_k/G_k$.

1 The Virasoro algebra, which undoubtedly has an important rôle in the coupling of matter to 2d gravity, is present in the universal enveloping algebra of the current algebra and therefore, does not require any special attention.
Complexification is also required in the *the gauged WZW approach*. It turns out that in this approach, complex gauge transformations are both natural and useful. Their gauge fixing introduces a complex ghost system \((\rho_z, \chi)\) of spin \((1,0)\). Complex gauge transformation allows us to set (locally) the field strength to zero. Hence, they leave the stage clear to the moduli of the flat gauge connections which are associated with the holonomies around handles and holes (*i.e.* insertions) in the world-sheet \(\Sigma\). In both approaches \(G^c\), the complexified version of the group \(G\), provides the natural setting for our investigation.

The paper is organized as follows: In section 2 we construct the \(\mathbb{R}/\mathbb{R}\) topological model. Although it is based on a free scalar-field in \(\mathbb{R}\) which is not a RCFT, it does have the virtue of simplicity. We elaborate there on both the gauged WZW and the CBRST approaches and demonstrate their equivalence. The abelian \(\mathbb{R}/\mathbb{R}\) model contains the original scalar field \(X\), an additional scalar field \(Y\) combining with \(X\) into a complex scalar field and a complex ghost pair \((\rho_z, \chi)\). This field content allows us to realize \(\mathbb{R}/\mathbb{R}\) as (the linear version of) the topological \(\sigma\) model \([16]\) on \(\mathbb{C}\). Alternatively, \(\mathbb{R}/\mathbb{R}\) follows from the additively gauged Lagrangian of the free scalar field \(X\), upon complexification and gauge fixing. The translational gauge symmetry of \(X\) can be fixed in this complex setting, eliminating the propagating degrees of freedom of the theory. Fixing the \(\mathbb{C}\) gauge algebra introduces the complex \((\rho_z, \chi)\) ghost system.

Section 3 is devoted to the construction of topological theories based on RCFTs. We start with a discussion of the spectrum, and demonstrate that the result of the CBRST cohomology is equivalent to the simple argument of \([36]\) describing the \(G/H\) fields as the fields of the \(G\) theory which are primary with respect to the chiral algebra of \(H\). Both arguments prescribe the \(G_k\) highest weight states as the \(G_k/G_k\) physical states demonstrating that \(G_k/G_k\) is the theory of \(G_k\) conformal blocks. We then move to a simple RCFT of the rational torus or circle of radius \(r_K = \sqrt{K}\) and construct \(U(1)_K/U(1)_K\). In this \(U(1)/U(1)\) case, CBRST is the more tricky approach and we therefore, resort to the gauged WZW approach. Associating physical operators with gauge holonomies around their insertion points indicates that the amplitudes of \(U(1)_K/U(1)_K\) are given in terms of the \(U(1)_K\) fusion rules. This result is further investigated and generalized in \([26]\). We conclude this section with the construction of \(G_k/G_k\) for a general non-abelian current algebra \(G_k\). Gauging the WZW model according to the Gawędzki Kupiainen approach \([31]\), to \(G/H\) construction, leads us to consider the complexified algebra \(G^c\) in our \(H = G\) case. This recasts the action as a sum of three independent parts: a WZW model for \(G\) at
level $k$ (analogous to $X$ in the $\mathbb{R}$ case), the $(G^c/G)_{\bar{k}}$ WZW model with $\bar{k} = k + 2c_G$ (analogous to $Y$ in the $\mathbb{R}$ case) and the complex $(\rho_z, \chi)$ spin $(1, 0)$ ghost system in the adjoint representation. The $G^c$ WZW model produced, has thus, two sets of bosonic currents. When properly supplemented by the ghost contribution, they deserve the name “complexified current algebra”. Following the standard BRST approach we construct the BRST operator $Q$. The currents $J^a = [Q, \rho^a]_+$ satisfy the Kac-Moody algebra with vanishing anomaly, $c^{tot} = 0$. The energy momentum $T_{zz}$ is also a BRST anti-commutator leading to $c^{tot} = 0$.

In section 4 we investigate the mechanism of BRST cohomology applied to current algebras. After discussing the cancellation of zero and negative norm states in this complex BRST cohomology [37] [38], we turn to the calculation of characters, employing the technique of ref. [31]. The $\lambda$ representation of $G$ leads to a $G_k$ conformal block (all the representations we discuss are integrable) whose Kac-Weyl [23] character is $\chi_{\lambda}$. $G_k$ is then complemented by $(G^c/G)_{\bar{k}}$ along with the $(\rho, \chi)$ ghost system to generate $G_k/G_k$. We find the $G_k/G_k$ character $M_{k,\lambda}$, to be given by the numerator of $\chi_{k,\lambda}$. $M_{k,\lambda}$ plays the rôle of “boundary” [39] for the $\lambda$ representation of $G_k$. $M_{k,\lambda}(q, u) \ (q = e^{2\pi i \tau}, \tau$ is the modular parameter for the torus and $u$, the moduli of flat gauge connections on it) is useful in extracting two important results about $G_k/G_k$. The torus partition function turns out to be simply the number of $G_k$ conformal blocks ($k + 1$ for the $SU(2)_k$ case). More detailed information follows by expanding $M_{k,\lambda}(q, u)$ in powers of $q$. We interpret $M_{k,\lambda}(q, u)$ as $\text{Tr} (-1)^{N_g} q^N e^{i\theta J_3}$ i.e. the index or Euler number for the BRST complex [38]. $N_g$ denotes the ghost number and $N$ is the excitation level for the states contributing to this index, namely, states in the $Q$ cohomology. We will be able to read the whole cohomology from $M_{k,\lambda}(q, u)$. We find infinitely many states at all values of $N_g$, in addition to the $N_g = 0$ physical state. Using an educated guess in order to construct the cohomology, supplemented by demonstrating a correspondence between the cohomology and algebraic relations in the $\lambda$ multiplet, we argue that, in fact, the cohomology contains one state at each $N_g$ sector. Those $N_g \neq 0$ states are null states of the $\lambda$ representation of $G_k$, “dressed” by the ghosts and $(G^c/G)_{\bar{k}}$.

The BRST cohomology which we find, reveals some analogies to the discrete states in two dimensional gravity, coupled to $c \leq 1$ matter. In the $c = 1$ system, states in the $^2 M_{k,\lambda}$ contains no states produced by current excitations. It is analogous to the Weyl numerator serving as “boundary” for a representation of a Lie Algebra.
Virasoro cohomology with “non-physical” ghost number, has been recently emphasized [10][11]. There is also a structural analogy between the $G/G$ system and two dimensional Liouville gravity. The $G_k$ WZW model plays the rôle of the matter system, while $(G^c/G)_{\bar{k}}$ plays the rôle of the Liouville field, as hinted by general properties the two have in common. $(G^c/G)_{\bar{k}}$ is pretty universal, it adapts to the $G_k$ “matter” system (resulting in $k^{\text{tot}} = 0$) and does not introduce topological features of its own. The ghost system has the spin of the symmetry generators, $(1, 0)$ in the $G_k$ current algebra case and $(2, -1)$ in the Virasoro gravitational case [12]. Various facets of this analogy between $G/G$ and two dimensional gravity are pointed out and discussed throughout the paper.

We conclude in section 5 with further remarks on the $G/G$ topological theories. In particular, we put them in the context of Landau-Ginzburg theories and 2+1 dimensional Chern-Simons theories. We also briefly comment on their coupling to topological gravity in two dimensions.

In the appendix we review complex gauge transformations which are an important tool in discussing the topology of gauge configurations [35].

### 2. Complex BRST Cosetting - the Abelian Case.

The $G/G$ theories are naturally formulated as gauge theories, particularly, gauged WZW models in the non-abelian case. An alternative formulation follows from the BRST gauge fixed variant. In this section we introduce these two formulations for the abelian $\mathbb{R}/\mathbb{R}$ theory. We start by twisting the free $N = 2$ supersymmetric theory ($c = 3$) and subsequently present it as a Complex BRST (CBRST) gauge fixed theory. This complex field theory consists of a complex free scalar field $u(z)$ ($\mathcal{C}$ being the target space), along with a complex ghost system $(\rho_z, \chi)$, of spin $(1, 0)$. In this linear version of the two dimensional $\sigma$ model [16] we will realize that the CBRST mechanism is the gauge fixing of the translational symmetry of $X(z) = \Re(u(z))$. Once this symmetry is identified, we can write down the gauge theory Lagrangian for our model,

$$\mathcal{L}_G = (\partial_z X - A_z)(\partial_{\bar{z}} X - \bar{A}_\bar{z})$$  \hspace{1cm} (2.1)

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3 This is apparent from the expression of the $(G^c/G)_{\bar{k}}$ contribution to the character [31], which as seen in section 4, is independent of $k$ and $\lambda$ apart from normalization.
taking the gauge field $A_z$ to be complex. By fixing this complex gauge algebra we derive the same theory, as formulated in the CBRST approach. We thus, show explicitly the equivalence of the two formulations, in the simple case of $\mathbb{R}/\mathbb{R}$. 

Starting with the CBRST approach, the $\mathbb{R}/\mathbb{R}$ case involves the complex plane $\mathbb{C}$ bosonic as target space for this twisted $N = 2$ supersymmetric theory. Thus, the field $X(z) \in \mathbb{R}$ is promoted to a complex scalar field $u(z) = X(z) + iY(z)$ (and its complex conjugate $\bar{u}(z) = X(z) - iY(z)$). We also introduce a complex ghost system $\rho_z$ and $\chi$ of spins 1 and 0 respectively. $\bar{\rho}_z$ and $\bar{\chi}$ are their complex conjugates. The Lagrangian is

$$L = \frac{1}{2} \partial_{\bar{z}} \bar{u} \partial_z u - i \rho_z \partial_{\bar{z}} \bar{\chi} - i \bar{\rho}_z \partial_z \chi$$

(2.2) (the complex structure of $u$ is thus, the symplectic form). The equations of motion render $\rho_z$ and $\bar{\chi}$ holomorphic whereas $\bar{\rho}_z$ and $\chi$ turn out anti-holomorphic. The operator product expansion (OPE) are

$$<\bar{u}(z, \bar{z}) u(w, \bar{w})> = \ln |z - w|^2, \quad <\rho_z(z) \bar{\chi}(w)> = \frac{i}{(z - w)},$$

$$<\bar{\rho}_z(\bar{z}) \chi(\bar{w})> = \frac{-i}{(\bar{z} - \bar{w})}.$$  

(2.3)

In order to have a BRST symmetry, satisfying $\delta^2 = 0$ off-shell, a complex spin 1 auxiliary field $H_z$ was introduced in ref. [16]. The Lagrangian

$$L_H = -2 \bar{H}_z H_z - \bar{H}_z \partial_z u - H_z \partial_{\bar{z}} \bar{u} - i \rho_z \partial_{\bar{z}} \bar{\chi} - i \bar{\rho}_z \partial_z \chi$$

(2.4) is invariant under the BRST transformations $\delta u = i \chi$, $\delta \bar{u} = i \bar{\chi}$, $\delta \rho_z = H_z$, $\delta \bar{\rho}_z = \bar{H}_z$ and $\delta \chi = \delta \bar{\chi} = \delta H = 0$. $L_H$ is in fact a BRST anti-commutator

$$L_H = \delta (- \bar{\rho}_z \partial_z u - \bar{\rho}_z H_z - \rho_z \partial_{\bar{z}} \bar{u} - \rho_z \bar{H}_z)$$

(2.5) and is hence a topological Lagrangian. We will work on-shell, since holomorphic fields on the world sheet are sufficient when we consider the current algebra and the energy momentum tensor. Then, $\delta \rho_z = \partial_z u$ and $\delta^2 = 0$ follows with the use of the $\chi$ equation of motion. The energy momentum tensor is also a BRST anti-commutator

$$T_{zz} = \partial_z u \partial_{\bar{z}} \bar{u} + i \rho_z \partial_{\bar{z}} \bar{\chi} = \delta b_{zz}.$$  

(2.6) $b_{zz} = \rho_z \partial_z \bar{u}$ is the fermionic partner of $T_{zz}$ and also the Virasoro ghost as the fermionic partner of $T_{zz}$. 

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The BRST transformation $\delta$ is generated by the BRST operator $Q = \oint J^u_z dz$ with the BRST current given by $J^u_z = \bar{\chi} \partial_z u$. Calculating the cohomology of $Q$ we impose the constraint $J_z(z) = \partial_z u = 0$, which is a first class constraint since it is a BRST commutator. This makes the reason for complexification now apparent. Starting with a real free scalar field Lagrangian, $L_X = \partial_z X \bar{\partial}_z X$, we could naively try to impose the current $j_z(z) = \partial_z X$ as the constraint. However, it is not a first class constraint (since $< j_z(z) j_z(w) > = 1/(z-w)^2$). We resort to complexifying $X$ and impose the current $J_z$ encouraged by $< J_z(z) J_z(w) > = \text{regular}$. By adding ghosts, we get a twisted $N = 2$, supersymmetry or a BRST symmetry and can employ BRST cohomology to impose $J_z = 0$. This algebraic argument shows that the topological theory in hand, is actually a scalar field theory, which has its translational symmetry $X \rightarrow X + \alpha$ gauged and fixed by complexification. In short, these symmetry considerations imply that this theory, presented so far as a $\sigma$ model, is actually the gauged $\mathbb{R}$ model which is identified with $\mathbb{R}/\mathbb{R}$.

Let us show explicitly the equivalence of the CBRST approach presented so far, to the more conventional Gauged Wess-Zumino-Witten model approach for coset construction [28] [29] [30] [31]. Additively gauging $L_X$ gives:

$$
L_G = (\partial_z X - A_z) (\bar{\partial}_z X - \bar{A}_\bar{z}).
$$

We are now interested in complex gauge transformations, $\delta A_z = \partial_z \lambda$, $\delta \bar{A}_\bar{z} = \partial_{\bar{z}} \bar{\lambda}$ and $\delta X = i \text{Re} \lambda$. Although the action (2.7) is only invariant under real gauge transformations, there are two reasons for the use of complex gauge transformations. First, complex gauge transformations can be used to set the field strength $F = \partial_z \bar{A}_\bar{z} + \bar{\partial}_z A_z$ to zero (for a compact world-sheet $\Sigma$, a necessary condition would be the vanishing of the first Chern class $c(F) = \int_\Sigma F ds$ [26]). This keeps the stage clear for the topology of the gauge configurations, i.e. holonomies around handles and holes of $\Sigma$. Hence, complex gauge algebra provides a tool for studying topological aspects of gauge configurations [35]. This is discussed further in the appendix. The second reason to employ complex gauge transformations is that they are required to fix the gauge properly, by CBRST techniques. Unable to impose $\partial_z X = 0$, we resort to complexifying the currents along with the gauge fields coupled to them.

For a topologically trivial $\Sigma$ ($\mathbb{C}$ or a disk), we can see two ways to trade $A_z$ for the scalar field $Y$. First, considering only real gauge transformations (which are the actual symmetries of the action) we make the change of variables $A_z = i \partial_z Y$, $\bar{A}_\bar{z} = -i \partial_{\bar{z}} Y$. The $\rho, \chi$ ghost system accounts for the Jacobian resulting from the change in the functional
integration measure \[31\]. Alternatively, we consider complex gauge transformations with 
\[ A_z = i \partial_y Y, \quad \bar{A}_z = -i \partial_{\bar{y}} Y \]
as the gauge condition. Since the action is not invariant under all the complex gauge transformations it does depend upon the gauge choice \( Y \). In particular, the action depends on the complex transformations that change the curvature and are exactly described by the field \( Y \). In other words, \( Y \) accounts for the curvature, the local information encoded in the gauge configuration (see the appendix for more details). From this point of view \( \rho_z \) and \( \chi \) are the Fadeev-Popov ghosts associated with this complex gauge fixing. Either ways (2.2) follows from (2.7).

We have formulated the \( \mathbb{R}/\mathbb{R} \) coset model both as a gauged abelian WZW model and as a topological theory in the CBRST approach. In the latter, complex formulation for both matter and ghost systems is required. The need for a complex formulation is less apparent when \( \mathbb{R}/\mathbb{R} \) is written as a gauged WZW model. However, in order to avoid a gauge anomaly, complex gauge algebra is the proper way to BRST quantize the model. This promotes the bosonic degrees of freedom to a complex scalar. Gauge fixing requires a complex ghost system showing that the two formulations are indeed equivalent.

The next section leads us to the nonabelian case and following \[31\] we will define the \( G/G \) coset model as a gauged WZW model. The gauge symmetry is fixed yielding a non-abelian generalization of (2.2). The result involves \( G^c_k \) and may be formally described as a new kind of a \( G^c_k \) WZW model with complex ghosts and current algebra. In the nonabelian case \( G^c_k \) takes the rôle of \( \mathfrak{c} \) in the \( \mathbb{R}/\mathbb{R} \) case.

3. BRST Cosetting in Rational Field Theories.

In the previous section the abelian coset model \( \mathbb{R}/\mathbb{R} \) was recasted as a topological 2d theory, formulated in complex BRST terms. We would now like to discuss non-abelian theories as well. We thus proceed to discuss Rational Conformal Field Theories having finite number of highest weight states. We concentrate on the cosetting of the level \( k \) WZW model which has the Kac-Moody algebra \( G_k \) as its chiral algebra. The highest weight states of \( G_k \) are of particular interest. As we will soon argue, the physical spectrum of the \( G/G \) theory consists exactly of these \( G_k \) highest weight states \[31\]. We will then turn to a detailed construction of the simplest rational \( G_k/G_k \) model, namely the topological model of \( U(1)_K/U(1)_k \), following from the abelian RCFT of the rational torus. We will dwell on some of its peculiarities and then spend the rest of this section on the construction of the general non-abelian \( G_k/G_k \) theory.
3.1. The Spectrum of $G/G$

We wish to determine the spectrum of the $G/G$ model by imposing the $G_k$ constraints via BRST cohomology. It is useful to compare the BRST approach to a simpler argument for the spectrum of coset models. We are restricting the scope of the discussion to the holomorphic part of the theory (We can always glue it to the anti-holomorphic part and get the diagonal modular invariant. Other invariants are beyond the scope of our discussion here, c.f. [43]). The operator content of the coset model is then easily determined. $G/H$ contains the operators in the $G$ theory, which are primary with respect to the chiral algebra of $H$ (in ref. [36] this observation serves as a definition of the $G/H$ model and could be easily generalized to $(G/H)/(G/H)$ as will be discussed in section 5).

Let us show how this spectrum of $H$ highest weight states in the $G$ theory follows in the language of BRST cohomology. The constraints we should impose are $j^H_z$, the currents of $H$. As explained in the $\mathbb{R}/\mathbb{R}$ case, these constraints are second class and cannot be imposed quantum mechanically. As in the abelian case, first class constraints $\mathcal{J}^H_z(z)$ follow from $j^H_z$, once extra bosonic degrees of freedom are added. These $H^c/H$ degrees of freedom were introduced by [31] and further studied by Karabali and Schnitzer [32]. They give rise to currents which combine with $j^H_z$ into the complex currents $\bar{\mathcal{J}}^H_z(z)$ and $\mathcal{J}^H_z(z)$. $\mathcal{J}^H_z(z)$ are the constraints imposed by the BRST cohomology. These complex currents also contain a contribution from the spin $(1,0)$ $(\rho_z, \chi)$ ghost system, as expected in a non-abelian algebra of constraints. Quantum mechanically, by imposing the current algebra $\mathcal{J}^H_z$ via the cohomology of the BRST operator $Q_H = \oint dz \chi \mathcal{J}^H_z + a\ ghost\ part^4$, we cause the negatively indexed modes of the current $\mathcal{J}^H_z$ (as well as those of the complex conjugate $\bar{\mathcal{J}}^H_z$) to vanish on the physical Hilbert space. As a matter of fact, the physical Hilbert space is exactly the $N_g = 0$ sector of this CBRST cohomology and is free of all the $H$ excited states. It is only due to the zero modes of $H$. In section 4 we will explicitly see the cancellation of the $H$ excitations in pairs and find the cohomology. Therefore, the physical spectrum consists of the $H$ highest weight states in the $G$ theory. The CBRST has removed $H$ descendants from the cohomology producing the expected $G/H$ spectrum.

The same procedure applies to the construction of $G/G$ theories. Imposing the chiral algebra $G_k$ leaves only the highest weight states of the $G_k$ theory in the physical Hilbert space. It should be noted that all the highest weight states (or the zero modes which give

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4 $Q_H$ contains a ghost part, although we have included ghosts in $\mathcal{J}^H_z(z)$, since $Q_H$ requires only $\frac{1}{2}\chi$ times the ghost part of the currents.
rise to them) are inherited from the original \( G_k \) theory and are not produced through the CBRST construction. The extra \( H^c/H \) and ghost degrees of freedom, added in the CBRST constructing of \( G/H \) (and in particular \( G^C/G \) which is added in the \( G/G \) construction), serve to eliminate the \( H \) descendant states.\(^5\) They do not contribute extra states to the spectrum. Hence, the topological structure exposed in the \( G/G \) topological field theory resides in the original \( G_k \) theory and not in the topologically trivial \( G^c/G \) extension. The \( G/G \) theory is the theory of the \( G_k \) conformal blocks. It is therefore of little surprise that \( Z \), the torus partition function in this theory just counts those conformal blocks. The amplitudes turn out to be generated from the \( G_k \) fusion rules [26], in analogy with the \( S^2 \times S^1 \) Chern-Simons amplitudes [20].

3.2. The Rational Torus.

Although the \( \mathbb{R} \) model of the previous section is not a RCFT, it could still be claimed that the spectrum of the \( \mathbb{R}/\mathbb{R} \) model consists of the highest weight states \( i.e. \) the Fock ground states, given as eigenstates of the momentum \( p \). The \( U(1)_K \) case of the rational torus provides us with an instructive debut into the construction of topological theories based on RCFTs. Its formulation relies on flat \( U(1) \) gauge connections [31] and hence differs from the CBRST formulation, which is applied in the subsequent nonabelian case.

The Lagrangian \( \mathcal{L} \) is given by (2.7)

\[
\mathcal{L} = (\partial_z X - A_z) (\partial_{\bar{z}} X - \bar{A}_z).
\]

However, \( X \) is now periodic and we choose its period to be \( \frac{2\pi}{\sqrt{K}} \). The possible \( A_z \) holonomies are then \( \frac{2\pi n}{\sqrt{K}} \), exactly like flux quantization in a superconductor or in the presence of a charge \( \sqrt{K} \) Higgs field. So far, everything is independent of the period of \( A_z \) itself. Choosing \( A_z \) as a \( U(1) \) field of \( 2\pi\sqrt{K} \) period\(^6\), makes its holonomies close into a \( Z_K \) group, since those are now additive modulo \( K \). \( K \) is an integer and we are actually dealing to with the Rational Torus Model with radius \( r_K = \frac{\sqrt{K}}{\sqrt{K}} \). In other words, the possible holonomies around a contour \( \mathcal{C} \) are \( H[A^n_z; \mathcal{C}] = \oint_{\mathcal{C}} A^n_z \cdot d\ell = \frac{2\pi n}{\sqrt{K}} \), \( 0 < n \leq K \) and \( A^n_z \) denotes a gauge

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\(^5\) The (C)BRST mechanism eliminates states in “splitted-pairs” as we will discuss in section 4. In \( G/H \) the pair is splitted between \( J^H_z \) and \( \bar{J}^H_{\bar{z}} \) excitations. Therefore, on the cohomology, the excitations of \( J^H_{\bar{z}} \) and \( \bar{J}^H_{\bar{z}} \) vanish as well as those of \( j^H_z \) and the \( H^c/H \) currents.

\(^6\) The normalization of the fundamental charge is \( \frac{1}{\sqrt{K}} \), chosen to comply with the standard presentation of the Rational Torus [14] with \( K \) states.
configuration with the holonomy $\frac{2\pi n}{\sqrt{K}}$. These $K$ possible holonomies correspond to the $K$ primary fields in the $r_K$ rational torus model [44].

Indeed, considering the construction of amplitudes in the $U(1)_K/U(1)_K$ model [26], we find the possible insertions to consist of the $K$ holonomies presented above. Moreover, the spectrum of the $K$ possible insertions or vertex operators in the $U(1)_K/U(1)_K$ model, along with their $Z_K$ structure, corresponds to the primary operators of the $r_K$ rational torus in agreement with the discussion we had in subsection (3.1).

Introducing operators by holonomies around the insertion points raises an interesting issue. In order to calculate a Wilson loop observable $W_q[A_z;C]$, we should calculate the exponent of a holonomy, $W_q[A_z;C] = \exp(iqH[A_z;C])$. This is the “trace” in the $q$ representation of $U(1)$, with the charge $q/\sqrt{K}$ running around the Wilson loop $C$. $W_q[A^n_z;C] = S_{nq} = e^{i\frac{2\pi}{K}nq}$ is the transformation matrix between two bases in our Hilbert space of $U(1)_K$ conformal blocks. $S_{nq}$ transforms the basis spanned by $|n>$ with $n$ specifying the inserted flux at the puncture; into the basis spanned by $|q>$ specifying the charge (which is used to probe the insertion in the trace leading to $W_q[A_z;C]$). The transformation matrix $S_{nq}$ is actually the Hartle-Hawking wave function generated by the insertion of the operator $n$ into the path integral, in the $|q>$ basis (specified on $C$). $S_{nq}$ is also the matrix representing the modular transformation $S(\tau \rightarrow -1/\tau)$ on the world-sheet torus, for the original $U(1)_K$ model. In [26] $S_{nq}$ is used to calculate the amplitudes in this model (the arguments based on [20] apply to general $G/G$ models as well). The amplitudes, following from $S_{nq}$, are given by products of the $U(1)_K$ fusion rules.

It should be noted that we have formulated the $U(1)_K/U(1)_K$ case and determined the spectrum through studying the topology of the $U(1)$ field configurations, while we avoided gauge fixing. Now we are going to attempt to gauge fix $U(1)_K/U(1)_K$, via CBRST and point out the difficulties. We try to use the $\rho$, $\chi$ ghosts to fix the $U(1)$ gauge symmetry, expecting $\mathcal{L} = \frac{1}{2}\partial\bar{\rho}\partial\rho - i\bar{\rho}\partial\bar{\chi} - i\rho\partial\chi$, like in (2.2), to be the gauge fixed Lagrangian. A puzzling problem arises concerning the periodicity appropriate for $Y$. The options seem to be $\frac{2\pi}{\sqrt{K}}$ — the period of the field $X$ and $2\pi\sqrt{K}$ — the period of $A_z$ gauge fixed by $Y$. However, it turns out that no single field $Y$ is sufficient.

In the CBRST cohomology the ghosts $\rho$, $\chi$ serve to eliminate the excitations created by the $X$ and $Y$ oscillators. They leave the spectrum, which is associated with the zero modes of $X$ i.e. the (quantized) momentum and winding number of $X$. This infinite set of states is in contradiction with the expected finite dimensional Hilbert space. The latter
space contains $K$ states, in correspondence with the $K$ primary states of $U(1)_K$. We conclude, therefore, that the cancellation of more states is required in order to get a finite dimensional BRST cohomology. More ghost states are needed, which could pair with the extra momentum states and cancel in quartets [33] [32]. However, we seem to have used up all the symmetry of the $U(1)$ theory with all the ghosts it implies and still were not able to produce the expected finite dimensional Hilbert space of $U(1)_K/U(1)_K$.

Looking for a resolution for this puzzling situation of missing symmetries and ghosts, we focus for a moment on the $K = 2$ case. The equivalence of $U(1)_2$ to $SU(2)_1$ gives us a clue. While imposing constraints we have overlooked the $j^\pm_z$ part of the chiral algebra $SU(2)_1$ (and for $K \neq 2$ we did not impose the $M^\pm = \exp(\pm i\sqrt{K}X)$ operators in the chiral algebra). Extra ghosts, $\rho^\pm$ and $\chi^\pm$, would be necessary to fix this additional part of the chiral algebra. Moreover, the bosonic degrees of freedom were also not properly taken into account. Although $SU(2)_1$ is easily bosonized and described by a single periodic scalar, its complexification is not readily described this way. Thus, we have to complexify $SU(2)_1$ without relying on its realization as a scalar field on a circle. Three additional scalar fields are, therefore, required to promote $SU(2)_1$ to $SL(2,\mathbb{C})$ (as discussed later in this section) rather than a single scalar $Y$. The full set of the extra bosonic degrees of freedom needed to complexify the current algebra oughts, therefore, to be identified in order to proceed with a CBRST formulation of $U(1)/U(1)$. The addition of a single field does not seem sufficient, even though the original $SU(2)_1$ (or $U(1)_K$) can be formulated in terms of a single periodic field. The general $U(1)_K$ case is harder to formulate using CBRST, since the chiral algebra contains operators of spin $K$ that result in ghosts with varying spins. In this $U(1)_K$ example, the gauged WZW approach seems simpler. We, therefore, leave the proper complexification of the rational torus along with the corresponding CBRST approach, for a future work.

We would like to point out that our attempt above at applying a CBRST construction for $U(1)_K$ gives rise to a RCFT which is interesting in its own right and also in the context of two dimensional gravity. We managed to cancel the oscillator excitations in the rational torus. We were left with the excitations due to the quantized values of the momentum.  

7 It is well known [17] that the $U(1)_K$ primary states are associated with the momentum values taken modulo the winding. This follows from the exponential operators $M^\pm = e^{\pm iX\sqrt{K}}$, which together with the momentum current $\partial_z X$, generate the $U(1)_K$ chiral algebra. The possible primary operators are, therefore, $e^{ip_nX}$ with $p_n = \frac{n}{\sqrt{K}}$. Only $-\frac{K}{2} < n \leq \frac{K}{2}$ yield states which are also primary under $M^\pm$. 

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Those values were not bounded by the winding convincing us to dismiss this (C)BRST construction for $U(1)/U(1)$. By a slight abuse of notation, we could denote the theory we have found, of $X$ primaries in $U(1)_K$, by $U(1)_K/\mathbb{R}$. It is different from the topological theories we have discussed so far, since it has an infinite dimensional (although discrete) Hilbert space.

For the $U(1)_2 = SU(2)_1$ case, this CBRST construction yields the discrete $c = 1$ Virasoro states. This follows upon the examination of the $X$ primary states in $SU(2)_1$. Then, the application of $J^\pm_X = \oint j^\pm dz$ on these states, accounts for all the Virasoro primaries in the $X(z)$ Fock space or equivalently the zeros of the Kac determinant (with the right degeneracy). Recently, the discrete Virasoro states have drawn much attention \[40\] \[41\], especially those generating the ground ring for $c = 1$ two dimensional gravity. Here they are shown to constitute the $SU(2)/\mathbb{R}$ modules. This observation merits further study. It reveals an additional facet of the relation between the WZW model and two dimensional gravity, a theme which is developed throughout the paper.

The discussion above is easy to generalize to tori in higher dimensions. If we choose a torus generated by the root lattice of the simply laced simple Lie algebra $G$, we will have the interesting description of the topological model $G_1/G_1$ using abelian gauge fields. These cases could also be described in the CBRST formulation following from the nonabelian description of $G_1/G_1$ which is the subject of the coming subsection. It should be noted, that in this topological context (or in the CS context), we have a theory with both an abelian and a nonabelian description.

3.3. The Nonabelian Case.

We would like now to construct the $G/G$ theory by imposing the currents of a nonabelian Kac-Moody algebra $G_k$ as constraints on the $G_k$ theory. Like in the abelian case, we will employ a complex ghost system $\rho_z, \chi$ and $\bar{\rho}_z, \bar{\chi}$, taken in the adjoint representation of $G$. From $\mathbb{R}/\mathbb{R}$ we know that we ought to complexify the Kac-Moody currents, prior to imposing them, since the $G_k$ currents are not first class for $k \neq 0$. The extra bosonic

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8 A short demonstration: In the one dimensional realization of $SU(2)_1$ where $j^3_z = \partial_z X$, the $X$ primaries are also Virasoro primaries. Upon the application of $J^\pm_X$ those states stay Virasoro primaries (since the Virasoro generators are scalar under the global $SU(2)$ algebra, generated by the zero modes of the $SU(2)_1$ currents including $J^\pm_X$). We thus have produced Virasoro primaries in the Fock space created by the oscillators of $X$. 

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degrees of freedom required to “complexify the model”, are found through the gauged WZW model. The WZW action is

\[
S_k(g) = \frac{k}{8\pi} \int_{\Sigma} \text{tr} \left( g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g \right) + \frac{k}{12\pi} \int_B \text{tr} \left( \tilde{g}^{-1} d\tilde{g} \right)^3; \tag{3.2}
\]

\(\tilde{g}\) extends \(g\) into \(B\), such that \(\partial B = \Sigma\).

In [31] the \(G/H\) coset model was formulated as the gauged WZW model

\[
S_k(g; A) = S_k(g) + \frac{k}{4\pi} \int_{\Sigma} \left( A_z g^{-1} \partial_{\bar{z}} g g^{-1} - \bar{A}_{\bar{z}} g^{-1} \partial_z g + A_z g \bar{A}_{\bar{z}} g^{-1} - A_{\bar{z}} \bar{A}_{\bar{z}} \right), \tag{3.3}
\]

where \(g \in G\) and the (anomaly free) subgroup \(H \subset G\) is gauged. We will be interested in the case where \(H = G\). For \(\Sigma\) topologically trivial (a disk or \(\mathbb{C}\) with no insertions), we can choose the gauge \(A_z = i h^{-1} \partial_z h\), \(\bar{A}_{\bar{z}} = -i h^* \partial_{\bar{z}} h^*^{-1}\). The complex gauge transformation \(h(z) \in G^c\) takes into account local features of the gauge configuration, namely, curvature or holonomies around shrinkable loops. When the world-sheet \(\Sigma\) is compact there is the restriction of trivial first Chern class, or else there is a global obstruction for solving the equation \(A_z = i h^{-1} \partial_z h\). We restrict ourselves to gauge configurations with zero first Chern class and overlook this restriction as long as we are working locally.

Following [31] we change variables from \(A_z\) to \(h\). Among the complex gauge transformations \(h(z) \in G^c\) the unitary transformations are really gauge symmetries and \(S_k(g; A)\) is independent of them. \(S_k(g; A)\) only depends on \(hh^*\) and the Polyakov-Wiegmann formula gives

\[
S_k(g; A) = S_k(g) - S_k(hh^*), \tag{3.4}
\]

which seems like the required “complexification”. In the second term, \(h\) can be taken in \(G^c/G\). The bosonic action is therefore, a WZW action for the group \(G^c\). From the first term one derives the original \(G_k\) currents while the second term is producing their \(G^c/G\) counterparts. (3.4) thus provides the \(G^c\) current algebra.

Further evidence for complexification in \(G/G\) theories is found for \(G = SU(N)\) through bosonization [46] (which offers an additional approach to the study of \(G/G\)). Expressing the

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9 In the abelian case the corresponding statement is that solving the Poisson equation for the gauge condition, with the magnetic field \(F\) being the source, requires that the monopole number is zero.
it is straightforward to check that bosonic fields, originating from the two terms, naturally combine into complex fields.

The ghost system is introduced by the Fadeev-Popov determinant upon gauge fixing or, alternatively, as the Jacobian resulting from the change of variable from $A_z$ to $h(z)$. The ghost field $\chi$ is a fermionic parameter for the infinitesimal gauge transformation. The anti-ghost $\rho$ has the following origin (in analogy with an argument for the Virasoro anti-ghost of the bosonic string [47]). Topological reasons may prevent having $A_z = i h^{-1} \partial_z h$. We then settle for the closest $A_z$, with minimal difference from $i h^{-1} \partial_z h$. Thus, we minimize the “action”

$$ S = \text{tr} \left( A_z - i h^{-1} \partial_z h \right) (\bar{A}_z + i \partial_z h^* h^{*-1}). \tag{3.5} $$

The anti-ghost $\rho_z$ denotes the difference $A_z - i h^{-1} \partial_z h$ (more precisely, this difference is $H_z$, the bosonic counterpart of $\rho_z$, introduced in section 2). It satisfies the Euler-Lagrange equation $\mathcal{D}_z \rho_z = 0$, $\mathcal{D}_z = \partial_z + \bar{A}_z$. The zero modes of $\rho_z$ are associated with the moduli of the flat gauge configuration corresponding to $A_z$ (via a complex gauge transformation). This argument, which actually determines the classical degrees of freedom left in the theory (we will soon argue that non-zero modes are canceled), has its counterpart for the bosonic string. The zero modes of the Virasoro anti-ghost $b$ in the bosonic string case are the moduli of the complex structure on the Riemann surface. They are given by the two dimensional metric up to diffeomorphisms. In our case, the gauge moduli, or the zero modes of the anti-ghosts, correspond to gauge configurations up to complex gauge transformations.

With the ghosts included, the action for the $G/G$ model is

$$ S_k(g; h, \rho, \chi) = S_k(g) - S_k(hh^*) - i \int_\Sigma \left( (\rho_z \mathcal{D}_z \bar{\chi}) + (\bar{\rho}_z \mathcal{D}_z \chi) \right). \tag{3.6} $$

To get rid of the remaining $A_z$ dependence in $\mathcal{D}_z$ and $\mathcal{D}_\bar{z}$, we perform a gauge transformation on the determinant which gives rise to the chiral anomaly $\exp(S_{2c_G}(hh^*)$. $c_G$ is the Casimir of the adjoint representation of $G$ or its dual Coxeter number. We thus get

$$ S_k(g; h, \rho, \chi) = S_k(g) - S_{k+2c_G}(hh^*) - i \int_\Sigma \left( (\rho_z \partial_z \bar{\chi}) + (\bar{\rho}_z \partial_z \chi) \right). \tag{3.7} $$

\footnote{Some of the bosons can be replaced by the appropriate $(\beta, \gamma)$ bosonic ghost system of spin $(1,0)$.}
We see that actually $S_k(g) - S_{k+2c_G}(hh^*)$ should be considered as the “complexified” WZW action for $G_c$. The significance of the particular coefficient in $S_{k+2c_G}(hh^*)$ will be discussed shortly.

Let us suggest a convenient gauge choice for $h \in G_c$. This relies on the fact that the action $S_k(g; h, \rho, \chi)$ depends only on the combination $hh^*$ and thus, $h$ can be taken in $G_c/G$. We can multiply $h(z)$ by $f(z)$, a (unitary) element of $G$. This is a gauge transformation that will not change the action. We can choose $f(z) = \sqrt{h(z)h(z)^*h(z)^{-1}}$ with $f(z)h(z)$ hermitian. A hermitian field $h(z)$ is our gauge choice in analogy to $A_z = i\partial_z Y$, in the abelian case (where $Y$ is real). For topologically non-trivial $\Sigma$, phases associated with the non-trivial holonomies, should be introduce. We will sum over these holonomies in the next section, where $\Sigma$ is the torus. We will further discuss holonomies and the gauge choices associated with them in \[27\].

Let us now demonstrate that we have really constructed a topological matter system by presenting the energy momentum tensor $T_{zz}$ as a BRST commutator, as we did in the abelian case. Again we find the on-shell version of the BRST transformations sufficient for the discussion of the chiral part of the model. The currents are: $j_z = kg^{-1}\partial_z g$, $j_\bar{z} = -k\partial_z gg^{-1}$ and $i_z = -(k+2c_G)h^{-1}\partial_z h$, $i_\bar{z} = (k+2c_G)\partial_\bar{z} hh^{-1}$. We can now construct the bosonic parts of the complexified currents $J_z = j_z + i_z$, $\bar{J}_z = j_\bar{z} - i_\bar{z}$, $J_\bar{z} = j_\bar{z} + i_\bar{z}$ and $\bar{J}_\bar{z} = j_z - i_z$. The contribution of the nonabelian ghosts to the currents (written momentarily with indices, to display the structure constants $f^{abc}$) is $J_z^{gh} = if^{abc}\rho^b_z\chi^c$, $J_\bar{z}^{gh} = if^{abc}\rho^b_\bar{z}\chi^c$. The chirality of the ghost system leaves $J_z$ and $J_\bar{z}$ with no ghosts contribution. This reveals an interplay between “world-sheet” and “space-time” holomorphicity, which is typical to $N = 2$ supersymmetric two dimensional theories (playing an important rôle in $(2,0)$ constructions \[18\]). These combinations of $G$ and $G^c/G$ currents, have the appropriate complex structure required to be related to the currents following from the $(\chi, \rho)$ ghost system. Therefore, they enter naturally into the BRST transformations.

The BRST currents are $J_z^B = \bar{\chi} (J_z + \frac{1}{2}J_\bar{z}^h)$ for the left movers and $J_\bar{z}^B = \chi (\bar{J}_\bar{z} + \frac{1}{2}\bar{J}_z^h)$ for the right movers. They give rise to the following BRST transformation laws

$$\delta J_z = i \partial_z \chi + i[\chi, J_z], \quad \delta \bar{J}_\bar{z} = i \partial_\bar{z} \bar{\chi} + i[\bar{\chi}, \bar{J}_\bar{z}]$$

$$\delta \rho_z = J_z + J_\bar{z}^h \equiv J_z^{tot}, \quad \delta \rho_\bar{z} = \bar{J}_\bar{z} + J_\bar{z}^h \equiv \bar{J}_\bar{z}^{tot} \quad \text{and} \quad \delta \chi = \delta \bar{\chi} = 0.$$  \hspace{1cm} (3.8)

By the $\chi$ equations of motion, $\delta \bar{J}_z^{tot} = 0$ and $\delta J_z^{tot} = 0$ and $\delta^2 = 0$ ensues.
$T_{zz}$ is expressed in the Sugawara form in terms of the complex currents $J$ and $\bar{J}$. This is a consequence of the particular values of $k$ and $\bar{k}$, the Kac-Moody levels, in the two parts of the bosonic Lagrangian. The $hh^*$ part looks formally like an ordinary $G$ Kac-Moody at level $-\bar{k} = -k - 2c_G$. This sets $N_i$, the normalization of the bilinear $i_z i_{\bar{z}}$ in $T_{zz}$, to be

$$N_i = \frac{1}{-k + c_G} = -\frac{1}{k + c_G} = -N_j,$$

(3.10)

where $N_j$ is the Sugawara normalization of $j_z j_{\bar{z}}$. Consequently

$$T_{zz} = N_j \text{tr}(J_z J_{\bar{z}}) + \text{tr}(\rho_z \partial_z \bar{\chi}) = \delta(G_{zz}) = \delta(\text{tr}(\rho_z J_z)).$$

(3.11)

$\rho_z J_{\bar{z}}$ is actually $b_{zz}$, the Virasoro anti-ghost. This is an example of the Super-Sugawara construction giving the supersymmetric partner of $T_{zz}$ as the product of the current and its anti-ghost. Since the Virasoro algebra is in the enveloping algebra of $G_k$, its anti-ghost is a composite operator as well. A similar understanding of the Virasoro ghost $c^z$ is still missing in this picture (understanding $c^z$ could be helpful in coupling $G/G$ to gravity). Since both $\mathcal{J}_z^{tot}$ and $T_{zz}$ are BRST anti-commutators, they have zero central charges $k^{tot} = 0$ and $c^{tot} = 0$, which can also be explicitly seen from

$$k^{tot} = k \left[G_k + k \left[G^c/G_{(k+2c_G)}\right]\right] \quad k^{gh} = \quad k \quad - \quad (k + 2c_G) \quad + \quad 2c_G = 0$$

and

$$c^{tot} = c \left[G_k + c \left[G^c/G_{(k+2c_G)}\right]\right] \quad c^{gh} = \quad \frac{k d_G}{k + c_G} + \frac{(k + 2c_G) d_G}{k + c_G} \quad - \quad 2d_G = 0.$$  

(3.12)

d$_G$ is the dimension of $G$ and also the number of the ($\rho$, $\chi$) ghost-pairs. Their zero modes, $L_0$ and $\mathcal{J}_0^{tot}$ (the index $u$ runs over the Cartan sub-algebra of $G$), are by themselves BRST anti-commutators and therefore, annihilate the states in the $Q$ cohomology.\footnote{The general argument is the following \cite{13}: if $X = [Q, \phi]_+$ is some $N_g = 0$ hermitian operator ($\phi$ is consequently $N_g = -1$ operator) then $X$ commutes with the BRST operator $Q$ and we can diagonalize $X$ while taking the $Q$ cohomology. Let $|x\rangle$ be an $X$ eigenstate with the eigenvalue $x$. If $|x\rangle$ is in the $Q$ cohomology, $Q|x\rangle = 0$ is required. Then $x|x\rangle = X|x\rangle = Q\rho|x\rangle$. Thus, $|x\rangle$ is in the image of $Q$, unless $x = 0$.} This provides an important guide for working out this cohomology in the next section.

Another remark is related to the Sugawara normalization factors in (3.10). $N_j^{-1} = k + c_g$ has an additional meaning as the periodicity of the $G_k$ conformal blocks (and their fusion) on the weight lattice of $G$ \cite{50}. We notice that $N_i^{-1} = -(k + c_g)$ in the $(G^c/G)_{k+2c_g}$ part. There is no apparent meaning to conformal blocks in this part, as we have noticed it is pretty much structureless. However, it has the formal appearance of $G_{-k-2c_g}$ and fusion may well have meaning for it. Although a lot have to be understood about this formal analogy, it is reassuring to notice that the periodicities match.
4. The Spectrum of $G/G$.  

We have formulated $G/G$ theories via cohomology in a complex BRST construction associated with the $G_k$ constraints. Imposing the constraints reduces substantially the large number of states typical to the $G_k$ WZW field theory. We have seen that a topological theory emerges, with no propagating fields, which describes merely global degrees of freedom. Quantum mechanically we have a finite spectrum (or discretely infinite spectrum, as in the $SU(2)/\mathbb{R}$ case we mentioned in the previous section) since local fields, the usual cause for the enormous Hilbert space of field theory, are missing in $G/G$. The excited states of the $G_k$ theory disappear from the $G_k/G_k$ spectrum via the mechanism of Complex BRST cohomology. This gets rid of states in complex pairs\textsuperscript{12} and is actually a quartet mechanism\textsuperscript{13}. CBRST was described as twisted $N = 2$ supersymmetry in section 2. The negative-norm decoupling mechanism provides, therefore, another argument for the complex $G^c$ formulation, since it always involves complex pairs. In YM theories and in string theory the non light-cone bosonic degrees of freedom (those which eventually disappear via the BRST cohomology) always form complex pairs \textsuperscript{18}. CBRST (or twisted $N = 2$ supersymmetry) is thus a well-known mechanism which we apply here for the case of current algebras.

4.1. The Physical States.  

We are going now to study the physical spectrum of the $G/G$ model by calculating the characters, which lead to the torus partition function. We are going to see that the physical spectrum, reflected in the partition function, consists of a finite number of states. We also wish to study the BRST cohomology. We will find it to be much bigger than the physical spectrum. The cohomology space contains infinitely many states, which will be argued to be spread over all possible values of the ghost number $N_g$. The physical spectrum on the other hand, is restricted to the $N_g = 0$ sector of this cohomology space which is finite.

\textsuperscript{12} The “split-a-pair” mechanism implies that two non-orthogonal zero norm states, schematically called $|a_\pm>\rangle$, result from a negative norm state $|a_0>\rangle$. $|a_+>\rangle$ is excluded by $Q|a_+>\rangle \neq 0$. $|a_->\rangle$ is then excluded from the cohomology, being an image of $Q$. The pair $|a_\pm>\rangle$ can be viewed as a complex pair.

\textsuperscript{13} The quartet mechanism implies that the a “split-pair” with ghost number $N_g$, combines with two other excitations of ghost numbers $N_g \pm 1$ to form a twisted $N = 2$ multiplet, which in turn disappears from the physical spectrum.
All these states are zero eigenstates of the total $L_0$, as follows from $L_0 = [Q, b_0]_+$ and the arguments in the last section. They are also zero eigenstates of the total current $J_0^{3\,\text{tot}} = [Q, \rho^3_0]_+$ and thus singlets of total $G$. Although the whole cohomology exceeds the physical spectrum, we will find its calculation challenging. Moreover, the results have interesting physical implications and show an intriguing analogy with the Liouville theory of two dimensional gravity coupled to matter. Both the physical spectrum and the cohomology follows from the calculation of the $G/G$ characters which we obtain following the work of Gawędzki and Kupiainen \[31\].

In \[31\] the torus partition function was calculated for a general coset model $G/H$, starting with the gauged WZW model and trading the gauge field $A_z$ for a complex gauge transformation $h$ (see the previous section for more details). We will restrict our review of this calculation to the simpler $G/G$ case ($H/H$ is however, a crucial step towards $G/H$ \[31\]). We follow the framework presented in the previous section. An essential difference is that we have so far discussed the case of a topologically trivial world-sheet $\Sigma$. We will indicate how to modify those arguments to the case of a torus, which allows for non-trivial holonomies along the two basic homology cycles. These holonomies can be viewed as Aharonov-Bohm fluxes running inside (and outside) the torus. They prevent the gauge configuration $A_z$ from being written as a complex gauge transform of a zero gauge field.

In order to perform the functional integration, the holonomies will be expressed by writing $A_z$ as the complex gauge transform of a non-trivial standard gauge configuration $C_z$, i.e.

$$A_z = i h^{-1} \partial_z h + C_z, \quad \bar{A}_{\bar{z}} = -i h^* \partial_{\bar{z}} h^{*\!-1} + \bar{C}_{\bar{z}}.$$ 

We will have to include integration over the holonomies (parametrizing the $C_z$’s) as a part of the functional integration. To be more specific, let us recall from the previous section that $h(z)$ can account for all the local features of the gauge configurations (like curvature). A map from the fundamental group $\pi_1(\Sigma)$ to $G$ accounts for the holonomies \[35\]. This map is specified through a flat gauge configuration (which fixes our standard for $C_z$) or equivalently, by two commuting elements of $G$ to express the two holonomies. Therefore, the $d[A]$ functional integral turns into the $d[h]$ functional integral plus an ordinary $du$ integral over a complex parameter residing in the Cartan subalgebra of $G$, which accounts for the two holonomies \[31\].

The $G/G$ torus partition function $Z_{G/G}$ is expressed in \[31\] as

$$Z_{G/G} = C \tau^{-2} \int Z^g(\tau, u) Z^{hh^*}(\tau, u) F(\tau, u) \, du, \quad (4.1)$$
where $du$ is the measure over the moduli space of flat $G$ connections on the torus and $r$ is the rank of $G$. $Z^g$ is the (torus) partition function for the WZW model based on $G_k$. It is given by the sum over the $G_k$ conformal blocks:

$$Z^g(\tau, u) = (q\bar{q})^{-c/24} \sum_{\lambda_L, \lambda_R} N_{\lambda_L, \lambda_R} \chi_{k, \lambda}(\tau, u) \chi_{k, \lambda}(\tau, u).$$  \hfill (4.2)

In (4.2) $q = e^{2\pi i \tau}$, $\tau$ being the modular parameter of the torus. $c = c[G_k]$ is the Virasoro central charge for $G_k$, given by (3.12). $\lambda_L$ and $\lambda_R$ are $G_k$ highest weights and each term in the sum (4.2), is a product of a right and a left $G_k$ Kac-Weyl characters written as $\chi_{k, \lambda}(\tau, u)$.

$$\chi_{k, \lambda}(\tau, u) = \frac{M_{k, \lambda}(\tau, u)}{M_{0,0}(\tau, u)},$$ \hfill (4.3)

$N_{\lambda_L, \lambda_R}$ is the positive and symmetric “mass matrix” for the WZW model which is $N_{\lambda_L, \lambda_R} = \delta_{\lambda_L, \lambda_R}$ for the diagonal modular invariants of a simply-connected group $G$ (we will restrict ourselves to those cases). $Z^{hh^*}(\tau, u)$ in (4.1) is the contribution of $h \in G^c/G$, at level $k+2c_G$, to the partition function ($c_G$ is the dual Coxeter number of $G$). This was calculated in [31], using the iterated Gaussian path integration technique. $G^c/G$ is topologically trivial with only one conformal block.

$$Z^{hh^*}(\tau, u) \propto |M_{0,0}(\tau, u)|^{-2}$$ \hfill (4.4)

$M_{0,0}(\tau, u)^{-1}$ thus gives the level $k+2c_G$ character of $G^c/G$ (which can also be viewed as the level $-k - 2c_G$ character of $G$) up to normalization. $F(\tau, u)$ in (4.1) is the Fadeev-Popov ghost determinant which also factorizes holomorphically. Conformal blocks appear only in the $G_k$ sector which is also the only source for a numerator in $Z_{G/G}$ (the Kac-Weyl numerator $M_{k,\lambda}$). This is due to the presence of algebraic relations corresponding to null states only in the $G_k$ sector of the $G_k/G_k$ theory. $G^c/G$ is free of algebraic relations and their manifestation in the form of null states and thus results in a single Verma module (this is for the case where $k$ is an integer, restricting ourselves to integrable representations. A richer structure is revealed [52] when these restrictions are lifted). Therefore, when in a short while, we will study the cohomology, the important algebraic arena will still be $G_k$ itself and will employ mainly representation theory for $G_k$.

\footnote{It would be nice to relate the appearance of more than one conformal block in the $G_k$ sector, as well as other algebraic features of $G_k$ (like null states), resulting in a non-trivial numerator, to the topology which distinguishes it from the $(G^c/G)_k$ sector.}
We learn from $Z_{G/G}$ about the spectrum. More detailed information about it, follows from a careful examination of the part holomorphic in $q$, which we call the $G_k/G_k$ character. $\Theta = \text{Re } u$ will be sufficient for our study of the spectrum and analyzing it in terms of $\hat{G}$ multiplets (the global $\hat{G}$ subgroup is generated by $j^a_0$, the zero modes of the $G_k$ currents). This actually turns the torus into a cylinder, having one holonomy $\Theta$ around it and also one representation $\lambda$ along it. Following [31] the $G_k/G_k$ character is given by multiplying 

$$\chi_{k,\lambda}(\tau, \Theta) \big( \text{the } G_k \text{ character} \big), \quad M_{0,0}(\tau, \Theta)^{-1} \big( \text{for } G_c/G \big) \text{ and } F(\tau, \Theta) \big( \text{the ghost contribution} \big).$$

Apart from a power of $q$ normalizing the character, $G_c/G$ contributes $M_{0,0}(\tau, \Theta)^{-1}$. $F(\tau, \Theta) = M_{0,0}(\tau, \Theta)^2$ (again, up to normalization), is the Fadeev-Popov determinant of the differential operator $\partial_z$ in the $\bar{\rho}_z \partial_z \chi$ term of (3.7). Therefore, the $G_k/G_k$ character is given by $M_{k,\lambda}(\tau, \Theta)$, the numerator of $\chi_{k,\lambda}$, the corresponding $G_k$ character.

This $G/G$ character, given by the Kac-Weyl numerator $M_{k,\lambda}(\tau, \Theta)$, has an interpretation as a Kac-Weyl “boundary” (the numerator of a character, which can be viewed as a “reduction” of a representation which is reconstructed by “filling”), introduced in [34]. $\chi_{k,\lambda}$, the original character of $G_k$, decomposes into various $\hat{G}$ multiplets, with each power of $q$ expressed as a sum of $\hat{G}$ representations. Similarly, the expansion of the numerator $M_{k,\lambda}(\tau, u)$ contains the “boundaries” of $\hat{G}$ representations. These $\hat{G}$ “boundaries” are numerators of the Weyl character formula (expressed as exponentials in $\Theta$ summed over the Weyl group of $\hat{G}$). The Kac-Weyl “boundary”, given by the Kac-Weyl numerator, amounts to a more radical reduction since we have imposed the whole $G_k$ algebra in addition to $\hat{G}$. The states accounted by this $G/G$ character contain neither descendant states present in the original $G_k$ representation, nor excitations of ghosts and $G_c/G$. The cohomology described by this character contains the primary states. It does not contain any of their excitations, due to “denominator cancellation”. The other states which do appear in the $G_k/G_k$ cohomology are associated with the null states of the original $G_k$ theory, “dressed” by the ghosts and $G_c/G$, as is evident from the $q$ expansion of $M_{k,\lambda}(\tau, \Theta)$ (the “ghost dressing” accounts for the alternating signs in the $q$ expansion). We will further elaborate on these extra states through this section.

A notable similarity exists between the Kac-Weyl “denominator cancellation” in $G/G$ and the mechanism canceling oscillator excitations in characters of $c = 1$ matter systems coupled to two dimensional Liouville gravity [13] ($c < 1$ systems also follow along similar lines). In the Liouville case too, the contributions from the oscillator excitations to the character cancel between the three constituent of the theory \textit{i.e.} the $c = 1$ matter, the Liouville field and the $(b, c)$ ghosts. The structural analogy to $G/G$ which we have pointed
out in the introduction, is strengthened when realizing that “denominator cancellation” is responsible for the disappearance of excitations in the Liouville case as well. For a $c = 1$ matter system coupled to Liouville let $\eta(q)$ denote $\prod_{n=1}^{\infty} (1 - q^n)$. $\eta(q)$ serves as the denominator in the character of the matter system, where it accounts for the excitations due to oscillators. Combined with another $\eta(q)^{-1}$ factor from the Liouville character it cancels against the ghost character, $\eta(q)^2$. Thus, once the contribution due to the oscillator excitations is canceled, the stage is left clear for the topological properties of the matter system manifested in the $c = 1$ numerator, such as winding. Note that the analogy we are pursuing here between the $G/G$ theory and the $c = 1$ matter coupled to gravity goes one step further. Both the Liouville and the $G^c/G$ sectors show no null states and thus contribute simple factors ($\eta(q)^{-1}$ and $M_{0,0}(\tau, \Theta)^{-1}$ respectively) to the total character.

Now, we would like to employ the character found above to extract the spectrum of the $G/G$ theory. For the sake of simplicity we will concentrate in our study on the $SU(2)_k$ case, although the conclusions will be general. $\Theta$ is then simply an angle $\theta$ and integrable multiplets (or conformal blocks) are characterized by the spin $j$ of the global $SU(2)$. The spectrum can be read from the character in two ways. The first one is by calculating the partition function from the characters [31]. This gives the trace over the physical space, or its dimension. Since the $G/G$ characters are orthonormal in the $du$ measure $\left( \int du M_{k,j}(\tau, u) \overline{M_{k,j'}}(\tau, u) = \delta_{jj'} \right)$ where $j$ and $j'$ denote multiplets of $SU(2)_k$, $Z_{SU(2)_k/SU(2)_k} = k + 1$. This is the number of conformal blocks of $SU(2)_k$ confirming that there is one physical state per block.

4.2. More Physical States—The BRST Cohomology.

We can, however, get more detailed information about the CBRST cohomology by a closer look at the $G/G$ character, which was found above to be the Kac-Weyl numerator $M_{k,j}(\tau, \theta)$. We will see that $M_{k,j}(\tau, \theta)$ serves as an index, or the Euler number, for the current-algebra cohomology in our CBRST complex [38]. This follows from realizing that the $SU(2)_k$ ghost contribution $F(\tau, \theta)$ is actually calculated with twisted boundary conditions in the “time direction” (a common practice in supersymmetric theories). Thus, the trace expressed in $F(\tau, \theta)$ contains $(-1)^{N_g}$, as implied in the product

$$F(\tau, \theta) = M_{0,0}(\tau, \theta)^2 = \prod_{n=1}^{\infty} \left( (1 - q^n)(1 - q^n e^{i\theta})(1 - q^n e^{-i\theta}) \right)^2. \quad (4.5)$$
\((-1)^{N_g}\) is responsible for the signs of \(q^n\) in (4.3). It amounts to the change \(q^n \rightarrow -q^n\), transforming the ordinary fermionic partition function into (4.5). The index \(M_{k,j}(\tau, \theta)\) is particularly useful as a formal power series in \(q\) (and \(e^{i\theta}\)); with the term \(a_N q^N\) giving rise to \(a_N\), the Euler number for the subspace of states of excitation level \(N\). The formal power series allows us to cover all the excitation levels \(N\) at once. We note that the \(G/G\) physical state \(|m_{k,j}\rangle\), corresponding to the \(j\)'s conformal block, is the highest \(SU(2)\) weight component of the \(N = 0\) excitation level in the cohomology. It is represented by the term with the highest \(e^{i\theta}\) power and the lowest \(q\) power in \(M_{k,j}(\tau, \theta)\). It has the physical value for the ghost number \(i.e. N_g = 0\) (or a zero degree form, if we use the de-Rahm analogy [47]). This is the \(G_k\) highest weight state discussed in section 3 and counted by \(Z_{G_k}/G_k\). We will see how a richer \(G_k/G_k\) cohomology is unfolding via the index \(M_{k,j}(\tau, \theta)\).

For \(SU(2)_k/SU(2)_k\) the character for the \(j\)'s multiplet is

\[
M_{k,j}(\tau, \theta) = \sum_{\ell = -\infty}^{\infty} q^{(k+2)(\ell+\frac{2j+1}{2(k+2)})^2} \sin((k+2)\ell + \frac{2j+1}{2})\theta. \quad (4.6)
\]

It combines “boundaries” of spin \((k+2)\ell + j\) \(SU(2)\) multiplets which result from \(\ell > 0\) and negative “boundaries” of spin \((k+2)\ell + j - 2\) from \(\ell < 0\). We are trying to read, as much as possible, about the cohomology by interpreting \(M_{k,j}(\tau, \theta)\) as the index. Clearly, the mere fact that \(M_{k,j}\) is given as an infinite power series tells us that each physical highest weight state gives rise to an infinite number of states in the cohomology.

We have denoted the excitation level \(i.e. the power of q\) by \(N\). \(N\) is the contribution to \(L_0\) from the non-zero mode excitations of the currents and the ghost fields:

\[
N = \sum_{m \neq 0} \left( N_j : j^a_m j^a_{-m} : + N_i : i^a_m i^a_{-m} + \rho^a_m \chi^a_{-m} \right). \quad (4.7)
\]

\(N\) thus includes contributions from the \(m \neq 0\) modes of \(j^a_m\) (the excitations of the \(SU(2)\) currents), \(i^a_m\) (the excitations of the \((SL(2,\mathbb{C})/SU(2))_{k+4}\) currents) and the ghosts \((\rho^a_m, \chi^a_{-m})\). \(N\) is devoid of of the zero mode contributions \(j^a_0\), \(i^a_0\), \(\rho^a_0\) and \(\chi^a_0\). Their inclusion renders \(L_0 = 0\) and \(\mathcal{J}^{3tot}_0 = 0\) on the \(G/G\) cohomology. \(L_0 = 0\) results here in a close analogy to the way it results on the Virasoro algebra cohomology (as employed for the no-ghost theorem [38] ) in the critical strings case. For critical strings, the total \(L_0\) receives contributions from the string excitations \(N_{str}\) and from the zero modes. \(L_0 = 0\) on
the string physical states results from the cancellation of these two contributions.\textsuperscript{15} $N_{\text{str}}$ is the excitation level in the index found in \[38\], like $N$ in the $G/G$ case.

The group structure in $G/G$ yields more information which allows us to arrange the states appearing in the $q$ expansion of the index, into multiplets of $\bar{G}$. This information is provided by the $e^{i\theta J_3}$ factor included in the index. $J_3$ is the sum of the zero modes of two out of the three contributions to the total current.\textsuperscript{14} namely, $J_3 = J_0^3 + J_0^{3gh}$ with $j_0^3$ due to $SU(2)$ and $J_0^{3gh}$ due to the ghosts. Now we are in the position to regard $M_{k,j}(\tau, \theta)$ as the index $\text{Tr} (-1)^{Ng} q^N e^{i\theta J_3}$ and read the cohomology from it.\textsuperscript{38} Here we will outline, how this interpretation, which we offer for the power expansion of $M_{k,j}(\tau, \theta)$, correctly identifies the $G/G$ cohomology, with the details supplied in \[54\]. We will also give a heuristic argument, taking stock of the algebraic relations available for the $G/G$ models, that we have identified the \textbf{whole} cohomology. It is reassuring that this argument seems to agree with the rigorous result for this semi-infinite cohomology.\textsuperscript{55}

The interpretation of the index $M_{k,j}(\tau, \theta) = \text{Tr} (-1)^{Ng} q^N e^{i\theta J_3}$ as the Euler number allows us to study the states in the cohomology. The values of $N$ gives the excitation level, $J_3$ gives the “angular momentum” and $N_g$ gives the ghost number (only its parity, strictly speaking) of these states. Usually, the Euler index does not determine the states and due to cancellations it only indicates few of the states in the cohomology (a very small fraction in most cases, which usually prevents us from deducing the Betti numbers from the Euler characteristics). Here, there seems to be an exception to this. We will see that a lot can be learned about the cohomology from the index. It is clear from (4.6) that \textbf{there are infinitely many states in the cohomology for each conformal block $j$,} at least one state per term in the sum over $\ell$, since all the contributions to the index are due to states in the cohomology. In the next subsection we will use this interpretation to delve deeper into the structure of the whole cohomology. We will suggest an iterative procedure for the actual construction of the states. Although we do not carry the full construction, there will be enough indications to provide compelling arguments supporting the conclusion that for each conformal block there is one state in the cohomology for every possible $N_g$ value.

\textsuperscript{15} In the bosonic string, for instance, $L_0 = -(p^2 + 1) + N_{\text{str}} = 0$ is the mass shell condition. $p^2$ which is associated with the bosonic zero modes (and 1 which can be traced to the ghost zero modes) cancel against the contribution of the excitations $N_{\text{str}}$.

\textsuperscript{16} $J_0^{3\text{tot}} = 0$, follows from the cancellation of $J_3$ against $i_0^3$. $i_0^3$ generates the Cartan subalgebra of the $SL(2,\mathbb{C})/SU(2)$ part. Its analogy to $p$, which renders $L_0 = 0$ in the bosonic string case will become apparent soon. We will see in the following $SU(2)/SU(2)$ examples that $i_0^3 = -J^3$ also ensures $L_0 = 0$, thus providing an independent check for our construction.
4.3. States at all Levels—Cohomology from Algebraic Relations.

We start our construction of the CBRST cohomology by using information which is encoded in the index $M_{k,j}(\tau, \theta) = \text{Tr}(-1)^{N_g}q^N e^{i \theta J_3}$ to guess the “leading” states $|f^j_\ell >$ ($j$ is the $SU(2)_k$ representation while $\ell$ refers to the $\ell$’th term in the expansion of $M_{k,j}$ in (L0)). The “leading” states, which we define, are the “leading components” (LCs) of the actual states $|\psi^j_\ell >$ in the cohomology. The LCs constitute the starting points for an iterative procedure for the construction of $|\psi^j_\ell >$. “Leading” states are constructed by hitting the ground state $|m_{k,j} >$ (with $j_0^3 = j$) by $o^j_\ell$, a minimal degree monomial in the currents and ghost excitations. Recall that for a physical highest weight state $|m_{k,j} >$, encoded in the $\ell = 0$ term, $J^3$ is saturated by $j$. $J^3_{\text{tot}} = 0$ determines the $i_3$ eigenvalue to be $-j - 1$ (the shift by one unit, is due to the ghost zero mode involved in $|m_{k,j} >$). For the higher terms, $o^j_\ell$ is constructed to be of minimal degree saturating the leftover $J^3 - j$ and the level $N$, as read from the index. This choice allows, therefore, only the negative modes of the $SU(2)_k$ currents $j^+_m$, ghosts $\chi^+_m$ and anti-ghosts $\rho^+_m$ to appear in $o^j_\ell$. It excludes zero modes and $i^-_m$ components (those would increase $N$ with no effect on $J^3$). Moreover, we would like $o^j_\ell$, to be iterative i.e. to have $o^j_\ell$ as a factor in $o^j_{\ell+1}$. To account for the alternating parity of the ghost number (indicated by $(-1)^{N_g}$ in the index) we choose to have in $o^j_\ell$ only anti-ghost modes for the $N_g < 0$ states and only ghost modes for the “dual” $N_g > 0$ states. This implies that that one “leading” state appears for each value of $N_g$.

There is a lot of arbitrariness in our “educated guess” for the “leading” states. We do not have an a-priori justification for all the assumptions we made in this construction. The true check would be provided by producing the states $|\psi^j_\ell >$ in the cohomology via an iterative procedure, starting from these LCs. This task of building the states $|\psi^j_\ell >$ as polynomials in excitations (i.e. as sum over states generated by non-minimal monomials acting on various ground states, characterized by different eigenvalues of the zero modes) and checking that they are in the cohomology, is pretty tedious and is therefore, deferred to [54]. Here, we will just discuss the crucial point of using the algebraic relations, (that shape the $j$ representation of $SU(2)_k$ by supplying null states for it) to verify that the $|\psi^j_\ell >$ states in the cohomology could be constructed iteratively, starting from the guessed “leading” states $|f^j_\ell >$. We observe that there is an algebraic relation corresponding to every state $|f^j_\ell >$. This relation should be employed to check that $Q$ annihilates $|\psi^j_\ell >$. Since we did not construct $|\psi^j_\ell >$, we will only do the first stage of this verification and
examine $Q |f^j_\ell \rangle$. We will see that the LC ("leading component") of $Q |f^j_\ell \rangle$ vanishes, due to the algebraic relation. There is a "non-leading" part to $Q |f^j_\ell \rangle$ which is canceled by the "non-leading" terms of $|\psi^j_\ell \rangle$. Those can be constructed iteratively with the details deferred to [54]. The check that $|\psi^j_\ell \rangle$ is not in the image of $Q$ is easy. We will see in a while, that there could be no state whose $Q$ image is $|f^j_\ell \rangle$. Thus also $|\psi^j_\ell \rangle$ can not occur in the image of $Q$. For this argument, choosing $o^j_\ell$ to contain only anti-ghost modes is crucial. We should comment, that for the “dual” $N_g > 0$ states with only ghosts in their corresponding $o^j_\ell$’s, this picture reverses. While by construction the $N_g > 0$ states are annihilated by $Q$, the algebraic relations are needed to guarantee that these states are not in the $Q$ image.

The fact that algebraic relations and the associated null states are present in the representation of $SU(2)_k$ is helpful in two ways. It verifies that the “leading” states are in the cohomology, in the “leading” sense (i.e. up to “non-leading corrections it is in the kernel of $Q$ and not in its image), thus offering the first stage of an iterative process of corrections. This process results in the state $|\psi^j_\ell \rangle$ belonging to the cohomology. This state has $|f^j_\ell \rangle$ as its LC and the appropriate corrections involve combinations of $i^a_m$ and the zero modes. The lack of relations in the $SL(2, \mathbb{C})/SU(2)$ sector seems to allow this iterative process to go unstopped, canceling “non-leading” contributions in stages. This brings us to the observation that the cohomology is a manifestation of the algebraic relations in the $j$ multiplet of $SU(2)_k$. This observation implies that having utilized all the algebraic relations, we have found the whole cohomology from the Euler index. From our construction it is evident that we will not be able to find a state in the cohomology unless it is ensured by the algebraic relations. The same holds for the “dual” states (with ghosts rather than anti-ghosts in their $o^j_\ell$ monomials). These are trivially annihilated by $Q$. The algebraic relations are essential in verifying that they are not in its image.

As our first simple example, we take the case of complex cohomology for the Lie group SU(2). The Weyl numerator for the multiplet $j$ of $SU(2)$ is

$$M_j(\theta) = \sin(j + \frac{1}{2})\theta = \frac{1}{2i} e^{i(j+\frac{1}{2})\theta} \left(1 - e^{-i(2j+1)\theta}\right).$$

(4.8)

We are studying the complex generated by the $SU(2)$ generators $j^+, j^-, j^0$, by their $SL(2, \mathbb{C})/SU(2)$ counterparts $i^+, i^-, i^0$, by the anti-commuting ghosts $\chi^+, \chi^-, \chi^0$ and the anti-ghosts $\rho^+, \rho^-, \rho^0$.

Interpreting $M_j(\theta)$ as $\text{Tr} (-1)^N \theta e^{i\theta J_0}$ would lead us to the conclusion that the first term in $(1 - e^{-i(2j+1)\theta})$ is due to the ground state $|-j, j \rangle$, which is a lowest weight state.
of \( j \) as well as a highest weight state of \( i \) (\( j^0|a, b > = a|a, b > \) and \( i^0|a, b > = b|a, b > \)). The second term stands for a state with \( N_g = -1 \). The “leading” state is \( |N_g = -1 > = \rho^+(j^+)^{2j} |-j, -(j + 1) > \) in this case (to keep \( J^0_{\text{tot}} = 0 \), the \( i^0 \) value changes by \((2j + 1)\)). Verifying that \( Q|N_g = -1 > = 0 \), relies on \( Q|-j, j > = 0 \) and on \([Q, \rho^+]_+ = j^+ + i^+\). The LC of \( Q|N_g = -1 > \) is \((j^+)^{2j+1} |-j, -(j + 1) > = 0 \) (ignoring the \( i^+ \) from \([Q, \rho^+]_+\), which results in a “non-leading” contribution). The last equality is simply the defining relation for the \( j \)’s \( SU(2) \) multiplet (defined via a lowest weight state annihilated by \((j^+)^{2j+1}\)).

Denoting all the states in the cohomology generically by \( |\psi^j > \) and its LC by \( |f^j > \), we can now state our main result: The “\( G/G \) character” given by the Weyl numerator for the representation \( j \), encodes all the states \( |\psi^j > \) in the cohomology. \( |\psi^j > \) are constructed to satisfy \( Q|\psi^j > = 0 \), in an iterative process starting from the “leading components” \( |f^j > \).

The “leading” parts of \( Q|\psi^j > \) are the “leading” parts of \( Q|f^j > \) and vanish due to the algebraic relations satisfied in the representation \( j \). Since \( Q|\psi^j > = 0 \) have consumed all the algebraic relations, they constitutes all the \( G/G \) cohomology.

Returning to complex current algebra cohomology, we are going to see a similar picture in the \( SU(2)_2 \) example. Let us write down the monomials in currents and ghosts excitations \( a^\dagger \), for the LCs of the states in the CBRST cohomology implied by \( M_{2,j}(\tau, \theta) \)

\[
M_{2,0}(\tau, \theta) \propto \sum_{\ell = -\infty}^{\infty} q^\ell (4\ell + 1) \sin (4\ell + \frac{1}{2}) \theta = \quad (4.9)
\]

\[
= 0 - q^3 3 + q^5 4 - q^{14} 7 + q^{18} 8 - + \quad \ldots
\]

LCs:
\[
\eta_1 l_1^2 \quad \eta_1 \eta_2 l_1^2 \quad \eta_1 \eta_2 \eta_3 l_1^2 l_3^2 \quad \eta_1 \eta_2 \eta_3 \eta_4 l_1^3 l_3^2 \quad \ldots
\]

For the sake of concise notation, we use \( \eta_n \) to denote \( \rho^+_n \), the anti-ghost creation operator and \( l_n \) to denote \( j^+_n \). Note that \( l_n \) is the LC of \([Q, \eta_n]_+\) (ignoring the \( i^+_n \) and the ghost parts which are “non-leading”). We use short hand notation for \(|\sin(4\ell + \frac{1}{2})\theta|\) (\(i.e.\) \( \sin(4\ell + \frac{1}{2})\theta \) for \( \ell \geq 0 \) and \( \sin(4\ell - \frac{1}{2})\theta \) for \( \ell < 0 \)). These are viewed as Weyl numerators, or “boundaries” of the \( SU(2) \) multiplets, whose spin is \( 4|\ell| \) (for \( \ell \geq 0 \)) and \( 4|\ell| - 1 \) (for when \( \ell < 0 \)). Thus \( 3 \), for instance, in the \( \ell = -1 \) term of \((4.3)\), denotes the term \( \sin \frac{3}{2} \theta \) arising from the \( SU(2) \) multiplet of spin 3. Decoding the cohomology from the \( j = \frac{1}{2}, 1 \) blocks:

\[\eta_n = \chi^+ \] is a dual choice producing states with positive ghost number, which will be explored later in this section.
\[
M_{2, \frac{1}{2}}(\tau, \theta) \propto \sum_{\ell=-\infty}^{\infty} q^{\ell(4\ell+2)} \sin(4\ell + 1) \theta = \\
= \frac{1}{2} - q^2 \frac{2}{1} + q^6 \frac{4}{1} - q^{12} \frac{6}{12} + q^{20} \frac{8}{2} + \ldots
\]

LCs: \[\eta_1 l_1 \quad \eta_1 \eta_2 l_1 l_2 \quad \eta_1 \eta_2 \eta_3 l_1 l_2 l_3 \quad \eta_1 \eta_2 \eta_3 \eta_4 l_1 l_2 l_3 l_4 \quad \ldots\]

\[
M_{2,1}(\tau, \theta) \propto \sum_{\ell=-\infty}^{\infty} q^{\ell(4\ell+3)} \sin(4\ell + \frac{3}{2}) \theta = \\
= 1 - q^2 + q^7 \frac{5}{1} - q^{10} \frac{6}{1} + q^{22} \frac{9}{1} + \ldots
\]

LCs: \[\eta_1 \quad \eta_1 \eta_2 l_2^2 \quad \eta_1 \eta_2 \eta_3 l_2^2 \quad \eta_1 \eta_2 \eta_3 \eta_4 l_2^2 l_4^2 \quad \ldots\]

Let us be more explicit about the states which we have decoded above; with \(|f^j_\ell\rangle\) denoting the LC of the state\(^{18}\) resulting from the \(\ell\)'th term of the index \(M_{2,j}\). We will demonstrate that the LC of \(Q|f^j_\ell\rangle\), which we denote by \(|g^j_\ell\rangle\), vanishes indicating that the whole state \(|\psi^j_\ell\rangle\) could be constructed in the kernel of \(Q\). \(|g^j_\ell\rangle = 0\) follows from relations satisfied in the \(j\) multiplet (highest weight representation) of \(SU(2)_2\). The verification that \(|\psi^j_\ell\rangle\) is in the kernel of \(Q\), thus reduces into an algebraic problem in \(G_k\) representation theory (followed by an iterative construction for \(|\psi^j_\ell\rangle\), which is deferred to \([54]\)). For \(\ell = -1\) the analysis of \(|g^j_\ell\rangle\) is simple and will be presented here for the general \(SU(2)_k\), as well. From the \(\ell = -1\) term we decode the LCs \(|f^j_{-1}\rangle = \eta_1 l_1^{k-2j}|j\rangle\), where \(j^3|j\rangle = j|j\rangle\). \(|j\rangle\) denotes the ground state of all non-zero modes and a highest weight state with respect to \(SU(2)\) \((|j\rangle\) is also a \(i_0^3\) highest weight. The \(i_0^3\) value will be discussed below). \(|g^j_{-1}\rangle\), the “leading part” of \(Q|f^j_{-1}\rangle = Q\eta_1 l_1^{k-2j}|j\rangle\), is now \((j^+_{-1})^{k+1-2j}|j\rangle = 0\). This follows from a general algebraic relation in the \(j\)'s representation of \(SU(2)_k\), asserting that \(|j\rangle\) is also a lowest weight state in a multiplet of \(S \subset SU(2)_k\) as well.\(^{19}\) \(S\) is the \(SU(2)\) subgroup generated by \(s^+ = j^+_1\), \(s^- = j^1_1\) and \(s^0 = k/2 - j^0_0\). Since \(s^0|j\rangle = (k/2 - j)|j\rangle\), the \(S\) lowest weight state \(|j\rangle\) is annihilated exactly by \((j^+_{-1})^{k+1-2j}\).

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18 Note that the choice we had for \(J_3\) looks now more natural. Like \(N\) in the index \(\text{Tr} (-1)^N q^N e^{i\theta J_3}\), \(J_3\) also counts the excitations of ghosts and currents in \(\sigma^j_\ell\) (apart from \(j\), a \(j^3_0\) contribution, which is constant on a given \(j\) multiplet). \(i_0^3\) is purely due to zero modes as we are going to discuss soon.

19 \(S\) is the same algebra used to find the integrable representations of \(SU(2)_k\) in \([23, 24]\).
From the $\ell = 1$ terms in the expansion of $M_{2,j}$, we find the $N_g = -2$ states in the cohomology, $|\psi_1^j\rangle >$ having their LC equal to $|f_1^j\rangle > = o_1^j|j\rangle >$ with $o_1^0 = \eta_1\eta_2l_2^2, o_1^{1/2} = \eta_1\eta_2l_1l_2$ and $o_1^1 = \eta_1\eta_2l_2^2$. We wish to show the vanishing of $|g_1^j\rangle >$, the “leading” part of $Q|f_1^j\rangle >$. We consider here only the $SU(2)_2$ case utilizing the fact that the $o_1^j$ are iteratively constructed. Notice that the state $|j\rangle >$ is similar to to the state $|j\rangle >$ in the $\ell = -1$ case (differing only by the implicit value for $i_0^3$) and $o_1^j$ still contains a $l_1^{k-2j}$ factor. Concerned only with the “leading” $j_+^1$ part of $[Q, o_1^j]_+$, no extra $l_1$ can be generated (by the above $S$ argument). Therefore, an extra $l_2$ is generated in the anti-commutation. This amounts to a power of $l_2$ which annihilates $|j\rangle >$, by carrying it “outside the $j$’s representation”. In short, we have nailed down the relations\(^{20}\) $l_1^2l_2|j\rangle = 0 > = l_1l_2^2|j\rangle = \frac{1}{2} > = l_2^3|j\rangle = 1 > = 0$, which ensure $|g_1^j\rangle > = 0$. Similar relations appear for all values of $\ell$ and ensure the vanishing of the “leading” part of $Q|f_\ell^j\rangle >$. This in turn allows for an iterative construction of the state $|\psi_\ell^j\rangle >$ satisfying $Q|\psi_\ell^j\rangle > = 0[^{54}]$. $|\psi_\ell^j\rangle >$ is the state in the cohomology suggested by the $\ell$’th term in the expansion of $M_{2,j}$\(^{21}\).

The argument presented here is by no means complete. It does illustrate the availability of the algebraic relations needed to ensure that $Q$ annihilates the states in the cohomology; states enfolding through the index given by the Kac-Weyl numerator. The precise way the $i$’s and the zero modes of the ghosts enter, will be discussed appropriately in \(^{54}\). Here we only wish to comment about the value of $i_0^3$, which ensures that both $J_3^{\text{tot}}$ and $L_0$ vanish on the states that we have found. For this sake, we have to assign the appropriate $i_0^3$ eigenvalue to each state (that we have decoded from the $\ell$’th term) in (4.9), (4.10) and (4.11). $i_0^3$ is taken to be $-J_3$, i.e. $i_0^3 = -(k+2)\ell-(j+\frac{1}{2})$\(^{22}\). In addition to $J_3^{\text{tot}} = 0$, this value of $i_0^3$ renders the total $L_0$, independent of both $\ell$ and the block $j$. The

\(^{20}\) These relations describe the “envelopes” of the $j$’s multiplets of $SU(2)_2$. In terms of the $S$ highest weight state $|s^0 = 1 - j\rangle > = l_1^{2-2j}|j\rangle >$, the relations $l_2^{2j+1}|s^0 = 1 - j\rangle > = 0$ in dicate the power of $l_2$ “missing” the $j$’s multiplet. $|Q, o_1^j\rangle_+$ are “trying” to increase $j_0^3$ too much. $j_0^3 = j + 3$ would have been reached, which is just not present at the $SU(2)_2$ excitation level of $N-3 = 2j+4$.

\(^{21}\) Checking that these states are not in the $Q$ image is easy and as argued above, can be done for their LC. The $N_g = r$ LC $|N_g = r\rangle >$ contains the $\eta_1, \ldots, \eta_r$ excitations and no $l_i$ for $r < t$. If $|N_g = r\rangle > = Q|N_g = r - 1\rangle >, |N_g = r - 1\rangle >$ which is lower by one $N_g$ unit, must include an extra anti-ghost creation operator $\eta_u$. For $u > r$, $Q \eta_u \ldots$ should contain $i_{u>}$, missing from $|N_g = r\rangle >$. For $u \leq r$ on the other hand, $\eta_u$ would render $|N_g = r - 1\rangle > = 0$ by the Pauli principle.

\(^{22}\) The shift in $i_0^3$ by half a unit, is accounted by the choice of the ground state for the ghost zero modes.
fact that choosing the value of $i_0^3$, took care of both $J^{3 \text{tot}}$ and $L_0$ provides an independent check of our evaluation of the cohomology through the interpretation we offered for $M_{j,k}$. Moreover, $i_0^3$ turns out to be quantized, although a-priori it can be continuous. In this, the $SU(2)/SU(2)$ cohomology is similar to the Virasoro discrete states for $c = 1$, which exhibit quantized momentum eigenvalues. Another facet of the similarity between $SU(2)/SU(2)$ and the $c = 1$ Virasoro discrete states was seen by the $SU(2)_{1}/\mathbb{R}$ interpretation of the latter, offered in section 3. We should comment that the $SU(2)_{1}/SU(2)_{1}$ cohomology looks much like a refinement of $SU(2)_{1}/\mathbb{R}$.

We have already commented that we can produce the LCs $|\phi_{j}^{J} \rangle$, for a similar family of states with $N_{g} > 0$, by taking $\eta_{n}$ to be the ghost creation operator $\chi^{+}_{n}$ and a suitable modification of $l_{n}$. Checking that $|\phi_{j}^{J} \rangle$ are annihilated by $Q$ proceeds by a ghost counting argument, like in footnote 21. The algebraic relations are then used to show that $|\phi_{j}^{J} \rangle$ are not in the image of $Q$. There are thus, states in the cohomology for all values of the ghost number $N_{g}$ and they are paired between $N_{g}$ and $-N_{g}$.

4.4. “Dressed Null States”– Constituents of the Cohomology.

We have discussed here the spectrum of the $G/G$ theory and explained how the cohomology works to cancel the excited states. The character for $G_k/G_k$ is the Kac-Weyl numerator $M_{k, \lambda}(\tau, \Theta)$. It was also found to be the index $\text{Tr} (-1)^{N_{g}} q^{N_e} e^{i \Theta \cdot J_{h}} (\Theta \cdot J_{h} = \theta J_{3} \text{ for the } SU(2) \text{ case, or generally, a scalar product in weight space of } G)$ and as such has been used to identify the BRST cohomology. We have established that even though, there is a single physical state per conformal block (corresponding to $\lambda$, the highest weight representation of $G_k$), there are infinitely many states in the $G/G$ CBRST cohomology which belong to this block. With the help of some guesswork, we managed to argue that there is precisely one state in the cohomology with a given $N_{g}$ value. These states are in one to one correspondence with the various $\ell$ terms in the expansion of $M_{k, \lambda}(\tau, \Theta)$. Their actual construction follows through an iterative procedure starting from the “leading” components for states with $N_{g} > 0$ (and a dual construction for states with $N_{g} < 0$). The physical state is unique in the conformal block and is found in the $N_{g} = 0$ sector. The multitude of $N_{g} \neq 0$ states is nonetheless very intriguing in its close similarity to the discrete states which constitutes an interesting part of $c = 1$ two dimensional gravity. These extra $N_{g} \neq 0$ states, encoded in the index $M_{k, \lambda}$, have their origin at the null states of $G_k$, which are appropriately “dressed” by ghost and $(G^c/G)_{k}$ excitations. Recall that the standard rôle of the null states is to shape and bound the $G_k$ multiplets (the same holds for
Lie algebras). They correspond to the algebraic relations defining the \( \lambda \) representation of \( G_k \). Once the “denominator cancellation” takes place, the “dressed” null states reappear as genuine contributions to the \( G_k/G_k \) character or to the cohomology.

The cohomology, containing “dressed null” states, is also typical to the Liouville theory of two dimensional gravity. The matter system is chosen to be a \( c < 1 \) minimal model, or a \( c = 1 \) RCFT. In the gravitational case, the discrete states in the cohomology can be traced to the matter null states \( |\delta, c > \) “dressed” by the Liouville state \( |1 - \delta, 26 - c > \) and by ghost excitations. The relation between the matter null state \( |\delta, c > \) (of anomalous dimension \( \delta \)) and its Liouville “dressing” \( |1 - \delta, 26 - c > \), is a manifestation of the “duality” pointed out in [40]. We expect a similar duality to hold between \( G_k \) and \( (G^c/G)_{k+2c} \) (or \( G_{-(k+2c)} \)). We have noted the matching periodicities in the previous section. We expect that this duality as well as the analogy to [40], can be drafted for the construction of representatives for the cohomology classes and facilitate the tedious iterative construction.

5. Discussion and Outlook.

In the present paper we have constructed and investigated the \( G/G \) topological theories. In our analysis we have employed both the CBRST approach as well as a more direct approach based on the gauged WZW model. The need for complexification was quite apparent in both approaches. The topological theory based on \( G_k \) contains a finite number of physical states–\( N_B \) which is the number of \( G_k \) conformal blocks, as reflected in the calculation of the torus partition function. These physical states are understood as the highest weight states of the \( G_k \) theory. They are related to the classical theory of flat \( G \) gauge configurations [26] (quantized with \( 1/k \) playing the rôle of Planck’s constant). For \( SU(2) \) \( N_B = k + 1 \).

This physical space of states is actually the zero ghost number sector of the BRST cohomology discussed in section 4. For \( N_g \neq 0 \) the rest of the cohomology can be viewed as extra discrete states, encoded in the Kac-Weyl numerators which serve as the characters of \( G_k/G_k \). These states are one facet of the analogy between \( G/G \) and two dimensional gravity coupled to matter (\( c = 1 \) matter was recently receiving a great deal of attention [40],[41]). Another facet of the analogy is the content of these theories. They both contain three parts. The components of \( G_k/G_k \) are \( G_k \) WZW model, \( (G^c/G)_k \) model (which is formally also the level \(- (k+2c_g) \) \( G \) WZW model) and spin (1, 0) complex ghosts; whereas matter, Liouville and (2, \(- 1 \)) ghosts are the corresponding ingredients of a two dimensional gravity
system. To discuss this analogy further, we have investigated the cohomology of $G/G$. An intriguing issue is the correspondence between the $G_k$ representation and the particular $(G^c/G)_k$ representation which combines with it into a $N_g \neq 0$ state in the cohomology. We have explicitly analyzed here the $G = SU(2)$ case. We would clearly like to verify this correspondence for a general group $G$ and also to understand the general principle behind it along with its counterpart in the case of gravity. An approach to the Liouville theory which relies on current algebra may be of help [15] [12] [21]. An investigation, along these lines, of the cohomology versus the physical space for $G/G$ as well as for two dimensional gravity coupled to matter is under its way.

Amplitudes in the $G_k/G_k$ theory factorize in terms of the three point functions. Those are in turn the $N_{ijl}$ i.e. the fusion rules for the $G_k$ WZW model. From this view point (employed in [26] to calculate the $G/G$ amplitudes), $G/G$ theories are the two dimensional counterparts of the three dimensional Chern-Simons Gauge-Theories [20]. In CS the $N_{ijl}$ also give physical amplitudes, or actually the partition functions in the $S^2 \times S^1$ three dimensional topology. The $i$, $j$ and $l$ representations of $G$ are located on three unlinked Wilson loops which run parallel to the $S^1$. The $G/G$ amplitudes follow upon dimensional reduction, ignoring the $S^1$. This comes as no surprise since the CS theories are known to reduce to the fully gauged WZW model [57] [27].

We still miss a genuine two dimensional CBRST understanding of the amplitudes. This was clearly demonstrated in section 3, where we had hard time studying the rational torus. Instead of constructing $U(1)_K/U(1)_K$ via CBRST cohomology, we had to resort to a description in terms of gauge configurations and their holonomies. Hopefully, a thorough understanding of the $G_k$ current algebra cohomology and its $N_g \neq 0$ richness, would create a promising alternative approach for calculating the amplitudes. This hope is based on the analogy with the superstring amplitudes which were properly calculated only after the introduction of the ghost system and picture changing [58]. It is tempting to regard the states in the cohomology, with various values of $N_g$ as the same state in different “pictures”. If true, insertions in various “pictures” should be used in the calculation of an amplitude. Some evidence for this view is given by the apparent equivalence of the different $N_g$ states in the cohomology, where the difference is in the choice of the state being called physical.

The close analogy between $G/G$ theories and matter coupled to two dimensional gravity in the Liouville approach, suggests to look for a $2 + 1$ dimensional setting for the

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As a curiosity, this attempted construction produced $SU(2)_1/\mathbb{R}$, a WZW setting for the Virasoro discrete states.
latter. Three dimensional CS gravity is related to $SL(2, \mathbb{R})$ [59] (which tend to appear in the Liouville [12] and other two dimensional gravity contexts [13] ) and other subgroups of $SL(2, \mathbb{C})$. The recent discovery of area preserving diffeomorphisms in the study of $c = 1$ matter systems coupled to two dimensional gravity [10] [56], is another indication for a $2 + 1$ dimensional description. Higher dimensional behavior also seems to appear near the singularity of two dimensional black hole [21].

The topological $G/G$ theories can also be placed in the space of topological two dimensional Landau-Ginzburg (LG) theories [43]. They appear at the special points of the parameter space of twisted $N = 2$ models, where the structure constants are totally symmetric as it is expected for the fusion rules $N_{ijl}$. The fusion ring is a deformation of the LG chiral ring [30]. Another curiosity pointed out in [43] is that different $G/G$ theories reside in the same topological LG (TLG) space, like $SU(2)_k/SU(2)_k$ and $U(1)_{k+1}/U(1)_{k+1}$. We would like to use this LG singularity approach to access non-diagonal ADE invariants in $G/G$.

The $G/G$ points may turn out to be helpful for the understanding of Topological LG theories and their TLG space. A particularly important issue is the one of “untwisting” i.e. relating the topological theories to ordinary $N = 2$ supersymmetric theories. This challenge is easily met by the twisted $N = 2$ super-conformal (ADE) theories at “the origin” of the TLG space [31] [18]. At this point the “untwisted” super-conformal theory, is given as a conformal “fixed point”. Its vicinity in the TLG space, seems to be the twisting of the “flow” into the $N = 2$ superconformal theory. The theories close to “the origin”, seem therefore, to untwist into the non-conformal theories obtained by the deformations of the $N = 2$ superconformal theories. We can now turn to the $G/G$ points, to look for more hints concerning the nature of these non-conformal theories. $G_k/G_k$ is a conformal topological field theory. In its CBRST formulation, $G/G$ looks like the twisting of the $N = 2$ super-conformal theory of $G^c$. It would be instructive to find directly the $N = 2$ supersymmetric theory, involving the $G^c_k$ degrees of freedom, which is twisted into $G_k/G_k$. (The complexified $G^c$ is the right arena for such an $N = 2$ theory rather than the compact group $G$ although some $N = 2$ cosets are fine [12] ). The detailed construction is, however, still missing. The only way we know, so far, to untwist $G_k/G_k$ is by first bosonizing it [40]. It would be interesting to probe the TLG space around its $G/G$ points and to relate it to the “flow” into the fixed point of the $N = 2 G^c_k$ theory. The whole subject of the relationship

\[24\] A complexified framework could be useful for this $SL(2, \mathbb{R})/U(1)$ coset model as well.
between “flows” in the topological space (as described in the LG approach) and the flows associated with perturbations of the $N = 2$ superconformal theories is an important open problem. In particular, it would be important to determine those deformations which are integrable. Some work in this direction was carried out recently by Nemeschansky and Warner [63]. The $G/G$ theories may provide a good starting point for finding the integrable deformations. The observation of Witten that deformations of CS theories in three dimensions lead to integrable models in two dimensions [64], combined with the close relation of $G/G$ to CS theories, may provide an important clue for locating the integrable deformations.

It is interesting to note that $G_k/G_k$ amplitudes can be obtained also from a discretized approach starting with a triangulation and going over to the dual $\phi^3$ graph. We associate with each vertex of this graph the numbers $\{N_{ijkl}\}$ satisfying duality $N_{ijk} N_{kml} = N_{iln} N_{njm}$ and completeness $N_{iik} N_{klj} = \delta_{ij}$. For a fixed area (and genus) all the triangulations are obtained by the “flip” operation. Moreover, since duality and completeness allow to take away palquettes from the graph, all configurations with the same genus are equivalent. We are left, therefore, with one simple graph per genus. In particular there is no dependence on the area, as indeed should be expected for a topological theory. Specifying the boundary of the triangulated graph or equivalently the set of incoming legs $\{i_1, i_2, \ldots, i_n\}$ in the dual $\Phi^3$ graph, leads (on the sphere) to the $n$ point function

$$A_n = \sum_{\{k_i\}} N_{i_1i_2k_1} N_{k_1i_3k_2} \cdots N_{k_{n-3}i_{n-1}i_n}, \quad (5.1)$$

which is precisely the result of $G_k/G_k$ [20]. The fusion rules of $G_k$ provide an example for $N_{ijkl}$’s which satisfy the duality and completeness requirements mentioned above and are symmetric as well.

Actually, the $N_{ijkl}$’s associated with any CFT could do just the same job. We focus our attention to the RCFT case since in this case the states associated with the links belong to a finite set (the range of the index $i$). In particular we look at $G/H$ as a generic example. We would thus expect to be able to define a topological theory based on the coset model $G/H$ which would naturally be called $(G/H)/(G/H)$. The formalism for this kind of iterated cosetting was presented in [36]. It has the form of an inclusion-exclusion procedure, which is cohomological in its nature. We are now looking for a framework,

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25 This observation was made in discussions with A.A. Migdal and Y. Sonnenschein.
based on CBRST cohomology, for \((G/H)/(G/H)\) theories. We also try to formulate them
as gauge theories. We hope to be able to show that the amplitudes are given in terms
of the \(N_{ijl}\)'s following from the \(G/H\) model, as expected. A better understanding of the
amplitudes directly from the CBRST cohomology will be most useful here, as well.

Next, we would like to discuss some applications of \(G/G\) theories. Our investigation
of topological cosets is helpful for general \(G/H\) coset models (by far the most common
construction for RCFTs). The mechanism of the \(G/H\) cosetting can be presented along the
lines of a \(H/H\) theory for \(H_k' \subset G_k\) (and \(k'\) properly taken). The \(H_k'\) degrees of freedom
are provided by \(H \subset G\) \([31]\) \([32]\) \([33]\). Adding to the \(G\) theory the extra degrees of freedom –
the \(H^c/H\) current algebra and the complex ghost system in the adjoint represen
tation of \(H\), turns the \(H \subset G\) part of the \(G\) model into a standard \(H/H\). The CBRST coho
mology (with respect to the \(H\) current algebra), leads to \(G/H\).

It should be also noted that \(H/H\) theories present a potential ambiguity in the context
of \(G/H\) coset models (and in principle, may find use in the understanding of \(G_k\) itself and
the Quantum Group structure found in its amplitudes \([66]\) \([42\]) \([42\]). A demonstration for this
ambiguity can be found in the context of the stringy two dimensional black hole \([21]\) \([67]\).
An extra set of winding around a black hole was found for the string, following from
a \(U(1)/U(1)\) ambiguity in the \(SL(2, \mathbb{R})/U(1)\) (or the \(SU(2)/U(1)\) for region III) coset
model. In \([67]\), the analogy to the statistical mechanics \(Z_k\) clock model was used to sort
this ambiguity.

We have already mentioned in section 3 \(SU(2)/\mathbb{R}\) as the way to get the discrete states
of \(c_m = 1\) \([26]\). We have also discussed the analogy of \(G/G\) to this theory, both in the formal
structure (discussed in section 1) and in the appearance of “discrete” states in \(N_g \neq 0\)
sectors of the cohomology (in section 4). We would also like to address briefly the issue of
coupling the topological \(G/G\) theories themselves to gravity. We have started this project
motivated by the hope to find topological \(c = 0\) matter systems which when coupled to
topological gravity, give the \(c_m < 1\) (in particular, unitary) matter theories coupled to
gravity. This question is still under investigation. Other examples of this type were given

\(^{26}\) \(c_m\) denotes here, the Virasoro anomaly of the matter system coupled to two dimensional
gravity. This system may be described by a topological matter system with \(c = 0\) coupled to
topological gravity. This \(c = 0\) system may be in its turn, a twisted version of an \(N = 2\)
superconformal theory with \(c_u > 0\). In the example of the twisted minimal \(N = 2\) models
\([68]\) \(c_u = \frac{3k}{k+2}\) These models correspond to the one-matrix models, which in turn describe two
dimensional gravity coupled to the \((2, 2k+1)\) minimal model matter system with \(c_m = 1 - \frac{3(2k-1)^2}{2k+1}\)
in ref. [68], where the topological theories given by twisted minimal $N = 2$ models [61] [18] were coupled to topological gravity and found to describe some minimal models coupled to gravity. It should be noted that the models of [68] are actually in the same LG space as $SU(2)_k/SU(2)_k$ and $U(1)_{k+1}/U(1)_{k+1}$ [43]. We would like to couple $G/G$ to topological gravity and understand it as some $c_m$ matter system coupled to two dimensional gravity. This may be very tricky, since enumerating $G/G$ (and $(G/H)/(G/H)$ as well) theories, seems to show more theories than needed to represent $c_m \leq 1$ matter coupled to gravity if one would expect to find minimal models (of course there many more models, since we could always tensor whatever $c_m < 1$ model we have with our favorite $c = 0$ model without changing its $c_m$). Should this be the case, one is led to suspect that either some of the theories represent $c_m > 1$ matter when they couple to gravity (and hence, their simplicity and tight algebraic structure, could bring some insight to the difficult $c_m > 1$ domain of two dimensional gravity) or that some different topological theories become identical when coupled to gravity (the LG spaces may hint in this direction). Hopefully understanding this issue of $G/G$ theories coupled to topological gravity as matter coupled to gravity, will bring some insight into the question—what does gravity couples to, after all.

We would like to abstract from the structure of the $n$ points functions in $G/G$ theories, an algebraic concept which is related to the topology of the world-sheet. The formal structure of the cohomology, is presented in terms of “boundaries” [39] of multiplets. The amplitudes on the other hand, are given by the fusion rules which are calculated using “boundaries” and their “filling”. It takes $n - 3$ filling to add $n$ spins or to get the fusion rules relevant to the $n$ points function (as seen in (5.1)). “Filling” is also employed for higher genus amplitudes (applied $2(g - 1)$ time) and constitutes, therefore, the (non-abelian) group theoretical concept corresponding to the topology of the world-sheet. It remains to be seen whether “filling” could be a clue for the coupling the $G/G$ theory to geometry and gravity.

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Appendix  Complex Gauge Transformations in Real Notation.

We want to examine the significance of complex gauge transformations and provide an introduction to the subject \[35\]. We start with a real abelian gauge field \( \overrightarrow{\mathcal{A}} = (A_x, A_y) \). We previously used \( A_z = A_x + iA_y \) and \( \overrightarrow{\overline{\mathcal{A}}} = A_x - iA_y \). In a topologically trivial situation (taking here the world-sheet to be \( \Sigma = \mathbb{R}^2 \)) let us choose the gauge \( \nabla \cdot \overrightarrow{\mathcal{A}} = 0 \). \( \overrightarrow{\mathcal{A}} \) could now be written as \((\partial_y \Im, -\partial_x \Im)\). For a given field strength \( F \), \( \Im \) satisfies \( \nabla^2 \Im = F \) (when \( \Sigma \) is compact, the obstruction for the solution of this Poisson equation is the first Chern class of the connection \( \overrightarrow{\mathcal{A}} \)). If we do not wish to choose a particular gauge we could take \( \overrightarrow{\mathcal{A}} = (\partial_x \Re + \partial_y \Im, \partial_y \Re - \partial_x \Im) \). In complex notation we take \( \lambda = \Re + i\Im \), \( \overline{\lambda} = \Re - i\Im \) and have \( A_z = \partial_z \lambda \) and \( \overrightarrow{\overline{A}} = \partial_{\overline{z}} \overline{\lambda} \). This allows to write the gauge configuration \( \overrightarrow{\mathcal{A}} \) as a complex gauge transformation of \( \overrightarrow{\mathcal{A}} = 0 \). We use \( \Im \), the imaginary part of the gauge transformation, to encode \( F \). In the nonabelian case \( h(z) \) or more precisely \( hh^* \), plays the same rôle. In that case \( hh^* \) is analogous to \( \Im = \text{Im} \lambda \). The “rest” of \( h(z) \) (encoded less canonically in \( f = \sqrt{hh^*}h^{-1} \)) is a real gauge transform on \( g \) and \( A_z \) much like \( \Re = \text{Re} \lambda \).

We were careful to use the term gauge transformations rather than gauge symmetries since only \( \Re = \text{Re} \lambda \) generates a symmetry of the action. \( \Im = \text{Im} \lambda \) becomes a dynamical variable, enabling us to trade the gauge field physical degrees of freedom for the scalar field \( \Im \), denoted by \( Y \) in section 2 and given by \( hh^* \) in the WZW case. The matter fields “feel” only the real gauge transformations. This sets the stage for nontrivial \( \Sigma \)’s with topologically non-trivial gauge configurations, resulting from holonomies around handles and holes. All this global information is encoded in \( \Re \), the real part of the gauge transformations. The allowed \( \Re \) depends on the matter system coupled to \( \overrightarrow{\mathcal{A}} \) and should be taken to be multivalued in order to change holonomies. The topological information is insensitive to imaginary gauge transformations, which merely deals with the local curvature. This separation between local and global information, makes complex gauge transformations so useful in setting the stage for the study of the topology hiding in gauge configurations. A topologically equivalent flat gauge configuration can be reached from any gauge configuration via a complex gauge transformation.
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