Performance of Survey Propagation guided decimation algorithm for the random NAE-\(K\)-SAT problem

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Abstract

We show that the Survey Propagation-guided decimation algorithm fails to find satisfying assignments on random instances of the “Not-All-Equal-\(K\)-SAT” problem if the number of message passing iterations is bounded by a constant independent of the size of the instance and the clause-to-variable ratio is above \((1 + o(1)) \frac{2^{K-1}}{K} \log^2 K\) for sufficiently large \(K\). Our analysis in fact applies to a broad class of algorithms that may be described as “sequential local algorithms”. Such algorithms iteratively set variables based on some local information and/or local randomness, and then recurse on the reduced instance. Survey Propagation (SP)-guided as well as Belief Propagation (BP)-guided decimation algorithms — two widely studied message passing based algorithms, fall under this category of algorithms provided the number of message passing iterations is bounded by a constant. Another well-known algorithm falling into this category is the Unit Clause (greedy) algorithm. Our work constitutes the first rigorous analysis of the performance of the SP-guided decimation algorithm.

The approach underlying our paper is based on an intricate geometry of the solution space of random NAE-\(K\)-SAT problem. We show that above the \((1 + o(1)) \frac{2^{K-1}}{K} \log^2 K\) threshold, the overlap structure of \(m\)-tuples of satisfying assignments exhibit a certain clustering behavior expressed in the form of some constraints on pair-wise distances between the \(m\) assignments, for appropriately chosen positive integer \(m\). We further show that if a sequential local algorithm succeeds in finding a satisfying assignment with probability bounded away from zero, then one can construct an \(m\)-tuple of solutions violating these constraints, thus leading to a contradiction. Along with [GS14], where a similar approach was used by the authors in a (somewhat simpler) setting of non-sequential local algorithms, this result is the first work which directly links the clustering property of random constraint satisfaction problems to the computational hardness of finding satisfying assignments.

1 Introduction

In this work we study the behavior of some “natural”, statistical-physics-motivated, algorithms for constraint satisfaction problems on random instances. These algorithms, specifically BP-guided and SP-guided decimation algorithms, exhibited a spectacular performance empirically, capable of finding solutions very rapidly and very close to the thresholds, beyond which the satisfying assignments do not exist or are conjectured not to exist. A partial list of references documenting the performance of these algorithms includes the following papers [MPZ02], [BMZ05], [KMRT+07], [RTS09], [DRZ08], [KSS12] as well as the book by Mezard and Montanari [MM09]. At the same time, mathematically rigorous analysis of these algorithms is mostly lacking. Notable exceptions are the works of Coja-Oghlan [COT11] who analyzed the performance of the BP-guided decimation algorithm for random \(K\)-SAT problem, and of Maneva et al. [MMW07] who reformulate Survey Propagation algorithm as the Belief Propagation algorithm on a ”lifted” Markov Random Field. No rigorous results on the performance of the SP-guided algorithm is available, to the best of our knowledge.

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1.1 Our setting and results

In this work we consider a class of algorithms which we dub “sequential local algorithms” that capture natural local implementations of BP-guided and SP-guided decimation algorithms. We analyze their behavior on random instances of “Not-All-Equal-K-SAT (NAE-K-SAT)”. We describe the NAE-K-SAT problem, and our class of algorithms in that order below.

The NAE-K-SAT problem is a Boolean constraint satisfaction problem closely related to more commonly studied K-SAT problem. An instance of the NAE-K-SAT problem consists of a collection of $m$ $K$-clauses on $n$ Boolean variables $x_1, \ldots, x_n$. Each $K$-clause is given by $K$-literals, where each literal is either one of the variables or its negation. The clause is satisfied by a Boolean assignment to the variables if at least one of the literals is satisfied (set to 1) and at least one is unsatisfied (set to 0). (This symmetry between satisfied and unsatisfied literals lends a convenient symmetry to the NAE-K-SAT problem that is not shared by the K-SAT counterpart).

In this work we consider the ability to find satisfying assignments to random instances of the NAE-K-SAT problem. Here the $m$ clauses are chosen uniformly and independently from the collection of $2^K \cdot \binom{K}{r}$ possible $K$-clauses. In particular we consider the setting where $m = d \cdot n$ for some constant $d = d(K)$ which depends on $K$, but not $n$, and consider what is the largest $d$ for which there exists an efficient algorithm for identifying a satisfying assignment with probability approaching one as $n \to \infty$. The parameter $d$ is often referred to as the clause density. Of course, no algorithm can find a satisfying assignment if none exists; and the limit of when such an assignment exists is well-studied. In particular Coja-Oghlan and Panagiotou [COP12] have established that random instances of the NAE-K-SAT problem are satisfiable w.h.p. when the density $d$ is below $d_s \triangleq 2K^{-1} \ln 2 - \ln 2/2 - 1/4 - o_K(1)$, and is not satisfiable w.h.p. when $d > d_s$. Here $o_K(\cdot)$ denotes a function converging to zero as $K$ increases. (A similar convention is adopted for other notations for orders of magnitude). Our interest is in determining how qualitatively close to this threshold an efficient algorithm can get, i.e., how does the largest density at which the algorithm manages to find satisfying assignments compare with $d_s$.

The class of algorithms that we explore in this work are what we call “sequential local algorithms”. This is a class that abstracts algorithms such as the BP- and SP-guided decimation algorithms when the number of message passage iterations used in every decimation step is bounded by a constant $r$, independent of the size of the instance. (BP- and SP-guided decimation algorithms really form a very general class with many possible implementations and interpretations. In Section 1.3 we discuss the specific assumptions we make and their potential limitations.) A sequential local algorithm can be described roughly as follows. The algorithm works by assigning Boolean values to variables sequentially, where a chosen variable is assigned its value by a potentially probabilistic choice, which depends on the local neighborhood of the variable at the time the choice is made. The local neighborhood is defined to be the graph-theoretic $B(r)$ ball of constant $r$ radius with respect to the underlying factor graph on the set of variables and clauses, to be defined later. Once a variable is assigned a value, the formula is simplified (removing some clauses, and restricting others). This in turn may influence the local neighborhoods of other variables, and when the future variables are set to particular Boolean values, this is done with respect to thus possibly modified neighborhoods. The algorithm continues with its iterations till all variables are set. In the specific context of BP-guide decimation algorithm based on $r$ iterations, the local rule assigns value 1 to a variable $x$ with probability equal to the fraction of assignments in which $x$ is assigned value 1 among all assignments that satisfy all clauses in the local neighborhood $B(r)$. The SP-guided decimation algorithm uses a more complex rule for its assignments. It is based on lifting the Boolean constraint satisfaction problem to a constraint satisfaction problem involving three decisions, as opposed to two decisions, but otherwise follows the same spirit.

Our main contribution (Theorem 2.4) is to show that, with high probability (w.h.p.) as the size of the instance diverges to infinity, every “balanced” sequential local algorithm fails to produce a satisfying assignment when the ratio $d$ of the number of clauses to the number of variables exceeds
\[(1 + o_K(1)) \frac{2^{K-1}}{K} \log^2 K\] and clause size \(K\) is sufficiently large (but independent from the number of variables). “Balance” is a technical condition explained in Definition 2.3 which says that the local algorithm respects the inherent symmetry between 0 and 1. It is a condition satisfied by all known algorithms including BP- and SP-guided decimation.

Our bound on the ratio \(d\) is reasonably close to bounds at which simple algorithms actually work. In particular, it is well-known that a very simple Unit Clause algorithm is capable of finding satisfying assignments for this problem when \(d\) is below \(\rho \frac{2^K}{m}\) for some universal constant \(\rho\). [AKT+02] for \(K\) sufficiently large. The Unit Clause algorithm is a special case of the sequential local algorithm, as we will show in the paper, and is the best known algorithm for this problem. (A better algorithm is known for the random K-SAT problem which works up to clause to variables density \(1 - o_K(1)\) \(\frac{2^K}{m} \log K\) [CO10]. It is likely that a similar idea can be applied to the NAE-K-SAT setting, but such a result is not available to the best of our knowledge). One of the hopes was that BP- and SP-guided decimation algorithms may be able to bridge this factor of \(K\) between the unit clause algorithm and the satisfiability threshold \(d_s\) above. Our result however implies that, short of possibly a \(\log^2 K\) multiplicative factor, the ”infamous” factor \(O(K)\) gap between the satisfiability threshold and the region achievable by known algorithms cannot be broken by means of sequential local algorithms, in particular by BP- and SP-guided decimation algorithms with constant number of rounds of message passing iterations.

Previously, Coja-Oghlan [CO11] showed that the BP-guided decimation algorithm fails to find satisfying assignments for random K-SAT problem when \(d \geq \rho \frac{2^K}{m}\) for some universal constant \(\rho\), for an arbitrary number of iterations \(r\), which in particular might depend on the number of variables. (Here \(2^K\) factor is an ”appropriate” substitution for \(2^{K-1}\) when switching from NAE-K-SAT to the K-SAT problem. We maintain this distinction, even though technically it is eliminated by constant \(\rho\).) It is reasonable to expect that his result holds also for NAE-K-SAT problem using the same analysis. Thus our result reproduces the main result of [CO11] but only in the special case of bounded number of iterations (short of additional \(\log^2 K\) factor). At the same time, however, our result is applied in a ”blanket way” to a broad class of algorithms, including most notably SP-guided decimation algorithm, and, unlike the analysis of [CO11], our analysis is insensitive to the details of the algorithm.

1.2 Techniques

Our main proof technique relies on the intricate geometry of the solution space of the random NAE-K-SAT problem. Specifically it relies on the so-called overlap structure of satisfying assignments of random NAE-K-SAT, which was earlier established for random K-SAT problem, and several other related problems, including the problem of proper coloring of sparse random graphs. Roughly speaking, the property says that, above a certain density, the Hamming distance between every pair of satisfying assignments, commonly called overlap in statistical physics literature, normalized by the number of variables, is either smaller than a certain constant \(\delta_1\) or larger than some constant \(1 \geq \delta_2 > \delta_1\). As as result the solutions can be grouped into different subsets (clusters) based on their proximity to each other. For the case of NAE-K-SAT problem this 2-overlap property can be established for densities \(d\) exceeding approximately \(d > d_s/2\). (A weaker version of this result corresponding to ”almost” all pairs does hold at densities above \(O(\frac{d_s}{K} \log K)\) [MRTT].) Unfortunately, this is not strong enough to cover the regime of \(d > (d_s/K) \log^2 K\) claimed in our main theorem, so instead we have to establish a certain property regarding \(m\)-overlaps of satisfying assignments, for appropriately chosen \(m\). This is the essence of Theorem 11 which we prove in this paper. Roughly speaking this theorem says that when \(d \geq (1 + \epsilon) \frac{d_s}{K} \log^2 K\), and \(K\) is sufficiently large, one cannot find \(m \approx \epsilon K/\log K\) satisfying assignments such that the Hamming distance (overlap) between every pair of the assignment normalized by the number of variables is \(\approx \log K/K\). We then show that for every \(\beta \in (0, 1)\), if a sequential local algorithm was capable of finding a satisfying assignment, with probability bounded away from zero, then
by running the algorithm $m$ times and constructing a certain interpolation scheme, one can construct $m$ satisfying assignments such that the pairwise normalized distance between any pair of these assignments is $\approx \beta$ w.h.p., thus contradicting Theorem 4.1.

The link between the clustering property and the ensuing demise of local algorithms was recently established by authors [GS14] in a different context of finding a largest independent set in a random regular graph. There the argument was used to show that so-called i.i.d. factor based local algorithms are incapable of finding nearly largest independent sets in random regular graphs, refuting an earlier conjecture by Hatami, Lovász and Szegedy [HLS]. The result was further strengthened by Rahman and Virag [RV14], who obtained essentially the tightest possible result, using $m$-overlap structures of "large" independent set. Our use of $m$-overlaps is inspired by this work, though the set of restrictions on the $m$-overlaps implied by Theorem 4.1 is much simpler than the one appearing in [RV14].

An important technical and conceptual difference between the present work and that of [GS14] and [RV14] is that algorithms considered in the aforementioned papers are not sequential. Instead the decision taken by each variable in those models are taken simultaneously for all variables. In the case of sequential local algorithms, since the variables are set sequentially, the decision for one variable can be non-localized for the remaining variables, this creating potential long-range dependencies. We deal with this potential long-range impact of decisions as follows. We associate variables with random i.i.d. weights chosen from an arbitrary continuous distribution, for example a uniform distribution. The weights are used solely to determine the order of fixing the values of the variables during the progression of the sequential local algorithm. Specifically largest weight first rule is used. The decision to fix the value of a particular variable then can only impact variables with lower weights. Specifically if the value of variable $x$ is fixed now, the value of variable $y$ can be impacted only if there exists a sequence $x_0 = x, x_1, \ldots, x_\ell = y$ such that the distance between $x_i$ and $x_{i+1}$ is at most $r$ (the radius of the decision making rule) and the weight of $x_i$ is larger than that of $x_{i+1}$ for all $i$. For a given set of variables $x_0, \ldots, x_\ell$ the likelihood of this total order of variables is $1/\ell!$ which decays faster than exponential function in $\ell$. This coupled with the fact that the growth rate of nodes at distance at most $r\ell$ from $x$ is at most exponential in $\ell$ (since $r$ is assumed to be constant), will allow us to control the range of influence of the variable $x$ when its value is set. A similar idea of controlling the range of influence is used in the analysis of local algorithms in several places, including [NO08]. Bounding the ranges of influences is a crucial idea in implementing the interpolation scheme and constructing $m$ assignments with "non-existence” normalized overlaps $\beta$.

### 1.3 Contrast with empirical studies of SP-guided decimation

The literature on BP-, and especially, SP-guided decimation (for instance [BMZ05], [KMRT+07], [RTS09], [MM09]) has shown that these algorithms perform well empirically on random instances of $K$-SAT for small values of $K$ ($K \leq 10$). There are several choices where these implementations differ (or may differ) from the setting we study: (1) They analyze SAT, as opposed to NAESAT; and the asymmetry in SAT may already make a difference for the algorithm. (2) They study 3SAT, so very local constraints, while we study $K$-clauses where $K$ is constant but large, and this increase in the locality of the constraints may make it harder for local algorithms to function effectively (even though the locality of the algorithm can be chosen to be arbitrarily after $K$ is fixed). (3) In the empirically analyzed algorithms, the order in which variables are set is not fixed a priori, but may depend on the probability estimates returned by the message passing iterations. While this could possibly also affect the ability of the algorithms to finding satisfying assignments, there appears to be no reasons based on the statistical physics theory which implies that such a presorting of variables is a crucial for SP-guided decimation algorithm to succeed. Size biasing rather appears to be a sensible implementation detail of the algorithm. (Some discussion of the accuracy of the size-biased version vs random order can
be found in \cite{KSS12}). (4) Finally, and probably most significantly, we analyze algorithms that work with a constant number of rounds of message passing iterations, and this allowed us to fit it within the framework of sequential local algorithms. In contrast the empirical studies suggest using message passing till the iterations converge and this may take more than linear or even exponential number of rounds. However it is believed that the message passing procedure do converge at some geometric rate $\gamma < 1$. Thus after $r$ iterations the “error” would be at most $\gamma^r$ - exponentially small in the number of iterations. Indeed this geometric rate of convergence is established in most examples where the model is amenable to analysis, including \cite{AS03, GNS06, BG08}, though not yet for NAE-$K$-SAT or 3SAT. However, if this belief about the convergence rate is correct also for NAE-$K$-SAT, then stopping after a large constant number of message passing iterations and using the estimates to guide the sequential decision process does seem like a reasonable heuristic.

Thus our work and setting makes a collection of choices that are different from some of the earlier works in the hope of getting some formal analysis. Unfortunately our result show that when the four choices are combined, it definitely produces a provable difference, and the algorithms fail to find satisfying assignments at densities that are qualitatively below the satisfiability threshold. Of course, it would be important to reduce the number of parameters in which the choices for the negative results differ from those used in the empirical setting (which yielded positive results) and we hope this will be a subject of future work.

### 1.4 Future work

As mentioned above it would be important to understand analytically what is the largest density at which the SP-guided decimation algorithm can find satisfying assignments, without the restriction of “constant number of message passing iterations”. In particular, it would be interesting to investigate whether the SP-guided algorithm is capable of breaking the ”$m$-overlap barrier” when the number of iterations is unbounded. Conversely, it remains open to get any analytic results showing that the more complex SP-guided decimation has an advantage over more conventional algorithms. Here it would be interesting to see if there is any implementation choice in the algorithm that offers a provable advantage.

Finally, going beyond specific classes of algorithms, a major challenge is to understand the intrinsic complexity of finding satisfying assignments in random instances of $K$-SAT and NAE-$K$-SAT problems. Given the repeated failure to produce polynomial time algorithms for, say NAE-$K$-SAT, above the density threshold of $\frac{1 + o_k(1)}{K} \log^2 K$ threshold it is plausible that the problem is actually average-case hard in this regime. The formalism of problems which are NP-hard on average is available \cite{Lev86}, however the problem which are known to be hard on average are not particularly natural and are quite distant from the types of problems considered here. Another problem that has defied designing a fast algorithm, and which is closer in spirit to the problems considered in this and related papers, is the problem of finding the largest independent set in a dense random graph. Specifically, consider the graph $G(n, 1/2)$ where every one of the $n(n - 1)/2$ of the undirected edges is present with probability $1/2$ independently for all edges. It is known that the largest independent set has size $2(1 + o(1)) \log_2 n$ w.h.p. At the same time the best known algorithm (greedy) finds only an independent set of size $(1 + o(1)) \log_2 n$ and bridging this gap has been a major open problem in the field of combinatorics and random graphs since Karp posed this as an open problem back in 1976 \cite{Kar76}. It is entirely plausible that this problem is NP-hard on average, and resolving this question one way or the other is a major open problem in theoretical computer science. It is furthermore worth noting that this problem does indeed exhibit the clustering property at the $(1 + o(1)) \log_2 n$ threshold, namely at the known algorithmic threshold. Specifically, fixing any $\beta \in (0, 1)$, one can show that there exists $0 < \delta_1(\beta) < \delta_2(\beta)$ such that for every pair of independent sets $I_1, I_2$ each with size $(1 + \beta + o(1)) \log_2 n$ (namely the size which is existentially achievable, but not achievable by known polynomial time algo-
rithms), it holds that $|I_1 \cap I_2|$ is either at most $(1 + \delta_1 + o(1)) \log_2 n$ or at least $(1 + \delta_2 + o(1)) \log_2 n$. Namely, the model exhibits the clustering property similar to the one considered for independent sets in $\text{COE11}$ and $\text{GS14}$. It is simple to show this fact by considering the expected values of the number of pairs of an independent sets with a given size and given overlap. By drawing an analogy with this, admittedly very different setup of dense random graph, and in light of 40 years old repeated failure to produce an algorithm for this problem, it is plausible to conjecture that NAE-K-SAT and related problems are NP-hard on average above thresholds corresponding to the emergence of clustering type properties. Shedding some light on this question is perhaps one of the most interesting problem in the area of random constraint satisfaction problems.

Organization and notational conventions. Our main result and applications to the BP-guided and SP-guided decimation algorithms are the subject of the next section. Some preliminary technical results are established in Section $\text{[3]}$. In particular, we establish bounds on the influence range of variables. The property regarding $m$-overlaps of satisfying assignments is established in Section $\text{[4]}$. The proof of the main result is in Section $\text{[5]}$.

Throughout the paper we use standard order of magnitude notations $O(\cdot), o(\cdot)$, for sequences defined in terms of the number of Boolean variables $n$. The constants hidden by this notation may depend on any other parameters of the model, such as $K$ and $d$. Similarly we use notations $O_K(\cdot)$ and $o_K(\cdot)$ to denote sequences indexed by $K$ as $K \to \infty$. The constants hidden in these notations are universal.

2 Formal statement of main result

In this section we formally present our main result. Before doing so we first introduce the mathematical notation and preliminaries needed to state our result.

2.1 Not-All-Equal-K-Satisfiability (NAE-K-SAT) problem

At the expense of being redundant, let us recall the NAE-K-SAT problem. An instance $\Phi$, of NAE-K-SAT problem is described as a collection of $n$ binary variables $x_1, \ldots, x_n$ taking values 0 and 1 and a collection of $m$ clauses $C_1, \ldots, C_m$ where each clause is given by a subset of $K$ literals. Each literal is a variable $x$ in $x_1, \ldots, x_n$ or negation $\bar{x}$ of a variable. An assignment is a function $\sigma : \{x_1, \ldots, x_n\} \to \{0, 1\}$. $\sigma$ satisfies $\Phi$ if in every clause, we have at least one literal valued 1 and at least one literal valued 0. For every assignment $\sigma = (\sigma(x_i), 1 \leq i \leq n)$, let $\bar{\sigma} = 1 - \sigma$ be the assignment given by $\bar{\sigma}(x_i) = 1 - \sigma(x_i), 1 \leq i \leq n$. Given a formula $\Phi$, denote by $\text{SAT}(\Phi) \subset \{0, 1\}^n$ the (possibly empty) set of satisfying assignments $\sigma$. The following “complementation closure” and resulting “balance” property of NAE-K-SAT are immediate (and do not hold for the K-SAT problem).

Observation 2.1. For every instance $\Phi$ of the NAE-K-SAT problem and assignment $\sigma$, we have that $\sigma$ satisfies $\Phi$ if and only if $\bar{\sigma}$ satisfies $\Phi$. Consequently, suppose $\text{SAT}(\Phi) \neq \emptyset$. Then if $\sigma$ is drawn uniformly from $\text{SAT}(\Phi)$, then for every $1 \leq i \leq n$ we have

$$P(\sigma(x_i) = 0) = P(\sigma(x_i) = 1) = 1/2.$$ 

Reduced Instances. We now introduce some notations for reduced instances of NAE-K-SAT. A clause of a reduced instance $C$ is given by a set of at most $K$ literals, along with a sign $\text{sign}(C) \in \{+,-,0\}$. Furthermore, $C$ has exactly $K$ literals if and only if $\text{sign}(C) = 0$. (Sometimes we refer to these signs as decorations.) An assignment $\sigma$ satisfies a reduced clause $C$ if one of the following occurs: $\text{sign}(C) = +$ and some literal in $C$ is assigned 0 by $\sigma$, OR $\text{sign}(C) = -$ and some literal in $C$ is assigned 1 by $\sigma$, OR $\text{sign}(C) = 0$ and there is at least one 0 literal and one 1 literal in $C$ under the assignment
A reduced NAE-$K$-SAT instance $\Phi$ consists of one or more reduced clauses, and $\sigma$ satisfies $\Phi$ if it satisfies all clauses in $\Phi$.

Note that Observation 2.1 does not necessarily hold for the reduced instances of NAE-$K$-SAT problem. Instances in which every clause has sign 0 will be called non-reduced instances.

Complements  Given a clause $C$ in a reduced instance of NAE-$K$-SAT, its complement, denoted $\overline{C}$, is the clause with the same set of literals, and its sign being flipped — so if $\text{sign}(C) = +$ then $\text{sign}(\overline{C}) = -$, if $\text{sign}(C) = -$ then $\text{sign}(\overline{C}) = +$, and if $\text{sign}(C) = 0$ then $\text{sign}(\overline{C}) = 0$. Given a reduced instance $\Phi$ of NAE-$K$-SAT, its complement $\overline{\Phi}$ is the instance with the complements of clauses of $\Phi$.

We now make the following observation, whose proof is immediate.

Observation 2.2. Given reduced instances $\Phi$ on variables $x_1, \ldots, x_n$ and $\Psi$ on variables $x_1, \ldots, x_{n+t}$ suppose $\Phi$ is the instance derived by reducing $\Psi$ with the assignment $\sigma : \{x_{n+1}, \ldots, x_{n+t}\}$. Then $\Phi$ is the reduced instance obtained by reducing $\overline{\Psi}$ with the assignment $\overline{\sigma}$, where $\overline{\sigma}(x_i) = 1 - \sigma(x_i)$.

Namely, whenever a reduced formula $\Phi$ is obtained from a non-reduced formula $\tilde{\Phi}$ by setting some variables of $\Psi$, setting the same variables to opposite values generates the complement $\overline{\Psi}$ of $\Psi$.

Random NAE-$K$-SAT problem  We denote by $\Phi(n, dn)$ a random (non-reduced) instance of NAE-$K$-SAT problem on variables $x_1, \ldots, x_n$ and $[dn]$ clauses $C_1, \ldots, C_m$ generated as follows. The variables in each clause $C_j$ are chosen from $x_1, \ldots, x_n$ uniformly at random without replacement, independently for all $j = 1, 2, \ldots, m$. Furthermore, each $x$ variable is negated (namely appears as $\overline{x}$) with probability 1/2 independently for all variables in the clause and for all clauses. We are interested in the regime when $n \to \infty$ and $d$ is constant. $d$ is called the density of clauses to variables.

Graphs associated with NAE-$K$-SAT instances. Two graphs related to an instance $\Phi$ of the NAE-$K$-SAT problem are important to us. The first is the so-called factor graph, denoted $\mathbb{F}(\Phi)$, which is a bipartite undirected graphs with left nodes corresponding to the variables and right nodes corresponding to the clauses. A clause node is connected to a variable node if and only if this variable appears in this clause. The edges are labelled positive or negative to indicate the polarity of the literal corresponding to the clauses. A clause node is connected to a variable node if and only if this variable which is a bipartite undirected graphs with left nodes corresponding to the variables and right nodes corresponding to the clauses. A clause node is connected to a variable node if and only if this variable appears in this clause. The edges are labelled positive or negative to indicate the polarity of the literal corresponding to the clauses. Thus the factor graph of a NAE-$K$-SAT instance uniquely defines this instance.

The second graph we associate with $\Phi$ is the variable-to-variable graph of $\Phi$, denoted $\mathbb{G}(\Phi)$, which has nodes corresponding to the variables and two nodes are adjacent if they appear in the same clause. Note that in contrast to the factor graph, the variable-to-variable graph loses information about the NAE-$K$-SAT instance $\Phi$.

Local neighborhoods  Given a (possibly reduced) instance $\Phi$ of a NAE-$K$-SAT problem, a variable $x$ in this instance, and an even integer $r \geq 1$, we denote by $B_\Phi(x, r)$ the corresponding depth-$r$ neighborhood of $x$ in $\mathbb{F}(\Phi)$, the factor graph of $\Phi$. When the underlying formula $\Phi$ is unambiguous, we simply write $B(x, r)$. We restrict $r$ to be even so that for every clause appearing in $B(x, r)$ all of its associated variables also appear in $B(x, r)$. Abusing notation slightly we also use $B(x, r)$ to denote the reduced instance of NAE-$K$-SAT induced by the clauses in $B(x, r)$ alone. Since $r$ is even we have that the factor graph of this induced instance is $B(x, r)$. In light of this, observe that $B(x, r)$ is also a reduced instance of a NAE-$K$-SAT problem.

2.2 Sequential local algorithms for NAE-$K$-SAT problem and the main result

We now define the notion of sequential local algorithms formally and describe our main result.
Fix a positive even integer \( r \geq 0 \). Denote by \( \mathcal{SAT}_r \) the set of all NAE-\(K\)-SAT reduced and non-reduced instances \( \Psi \) with a designated (root) variable \( x \) such that the distance from \( x \) to any other variable in \( \Psi \) is at most \( r \) in \( F(\Psi) \). We note that \( \mathcal{SAT}_r \) is an infinite set, since even though the depth of the factor graph \( F(\Psi) \) of any \( \Psi \in \mathcal{SAT}_r \) is bounded by \( r \), the degree is not. The set \( \mathcal{SAT}_r \) is the set of all instances \( \Psi \) which can be observed as depth \( r \) neighborhood \( B_\Phi(x, r) \) of an arbitrary variable \( x \) in an arbitrary reduced and non-reduced NAE-\(K\)-SAT instance \( \Phi \).

Consider any function \( \tau : \mathcal{SAT}_r \rightarrow [0, 1] \) which takes as an argument an arbitrary member \( \Psi \in \mathcal{SAT}_r \) and outputs a value (probability) in \([0, 1]\). We now describe a sequential local algorithm, which we refer to as the \( \tau \)-decimation algorithm, for solving NAE-\(K\)-SAT problem. Given a positive even integer \( r \), the depth-\(r \) neighborhood \( B(x_i, r) = B_{\Phi(n, d_n)}(x_i, r) \in \mathcal{SAT}_r \) of any fixed variable \( x_i \in [n] \) in the formula \( \Phi(n, d_n) \), rooted at \( x_i \), is a valid argument of the function \( \tau \), when the root of the instance \( B(x_i, r) \) is assigned to be \( x_i \). This remains the case when some of the variables \( x_1, \ldots, x_n \) are set to particular values and all of the satisfied and violated clauses are removed. In this case \( B(x_i, r) \) is a reduced instance. In either case, the value \( \tau(B(x_i, r)) \) is well defined for every variable \( x_i \) which is not set yet. The value \( \tau(B(x_i, r)) \) is intended to represent the probability with which the variable \( x_i \) is set to take value 1 when its neighborhood is a reduced or non-reduced instance \( B(x_i, r) \), according to the local algorithm. Specifically, we now describe how the function \( \tau \) is used as a basis of a local algorithm to generate an assignment \( \sigma : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\} \).

**\( \tau \)-decimation algorithm**

**INPUT:**
an instance \( \Phi \) of a NAE-\(K\)-SAT formula on binary variables \( x_1, \ldots, x_n \), a positive even integer \( r \), function \( \tau \).

Set \( \Phi_0 = \Phi \).

FOR \( i = 1 : n \)

Set \( \sigma(x_i) = 1 \) with probability \( \tau(B_{\Phi_{i-1}}(x_i, r)) \)

Set \( \sigma(x_i) = 0 \) with the remaining probability \( 1 - \tau(B_{\Phi_{i-1}}(x_i, r)) \).

Set \( \Phi_i \) to be the reduced instance obtained from \( \Phi_{i-1} \) by fixing the value of \( x_i \) as above, removing satisfied and violated clauses and decorating newly generated partially satisfied clauses with + and − appropriately.

**OUTPUT** \( \sigma(x_1), \ldots, \sigma(x_n) \).

In particular, even if at some point a contradiction is reached and one of the clauses is violated, the algorithm does not stop but proceeds after the removing violated clauses from the formula. This is assumed for convenience so that the output \( \sigma(x_i) \) is well defined for all variables \( x_i, 1 \leq i \leq n \) even if the assignment turns out to be not satisfying. We denote by \( \sigma_{\Phi, \tau} \) the (random) output \( \sigma(x_1), \ldots, \sigma(x_n) \) produced by the \( \tau \)-decimation algorithm above. We say that \( \tau \)-decimation algorithm solves instance \( \Phi \) if the output \( \sigma_{\Phi, \tau} \) is a satisfying assignment, namely \( \sigma_{\Phi, \tau} \in \mathcal{SAT}(\Phi) \).

We now formally define the following important symmetry condition.

**Definition 2.3.** We say that a local rule \( \tau : \mathcal{SAT}_r \rightarrow [0, 1] \) is balanced if for every instance \( \Phi \in \mathcal{SAT}_r \), we have \( \tau(\Phi) = 1 - \tau(\Phi) \).

The balance condition above basically says that the \( \tau \)-decimation algorithm does not have a prior bias in setting variables to 1 vs 0. In particular, when the instance is non-reduced, \( \tau \)-decimation algorithm
sets variable values equi-probably, consistently with Observation 2.1. This condition will allow us to take advantage of Observation 2.2 when applying the rule $\tau$ to reduced instances.

We now state the main result of the paper.

**Theorem 2.4.** For every $\epsilon > 0$ there exists $K_0$ such that for every $K \geq K_0$, $d > (1 + \epsilon)2^{K-1}\ln^2 K/K$, every even $r > 0$ and every balanced local rule $\tau : SAT_r \rightarrow [0, 1]$ the following holds:

$$\lim_{n \rightarrow \infty} \mathbb{P}(\sigma_{\Phi(n, dn), \tau} \in SAT(\Phi(n, dn))) = 0.$$  

Namely, with overwhelming probability, $\tau$-decimation algorithm fails to find a satisfying assignment. As we have mentioned above, the threshold for satisfiability is $d_s = 2^{K-1}\ln 2 - \ln 2/2 - 1/4 - o_K(1)$. Thus our theorem implies that sequential local algorithms fail to find a satisfying assignment at densities approximately $(d_s/K)\ln^2 K$.

### 2.3 BP-guided and SP-guided decimation algorithms as local sequential algorithms

We now show that BP-guided decimation and SP-guided decimation algorithms are in fact special cases of $\tau$-decimation algorithms as described in the previous section, when the number of message passing iterations is bounded by a constant independent from $n$. As a consequence we have that the negative result given by Theorem 2.4 applies to these algorithms as well.

The BP and SP algorithms are designed to compute certain marginal values associated with a NAE-$K$-SAT instance $\Phi$ and reduced instances obtained after some of the variables are set. The natural interpretation of these marginals is that variables may be set according to these marginals sequentially, while refining the marginals as decisions are made. It is common to call such algorithms BP-guided decimation algorithm and SP-guided decimation algorithms. We now describe these algorithms in detail, starting from the BP and BP-guided decimation algorithms.

**Belief Propagation.** The BP algorithm is a particular message-passing type algorithm based on variables and clauses exchanging messages on the bi-partite factor graph $F(\Phi(n, dn))$. After several rounds of such exchange of messages, the messages are combined in a specific way to compute marginal probabilities.

However, the relevant part for us is the fact that if the messages are passed only a constant $r$ number of rounds, then for every variable $x_i$ such that the neighborhood $B(x_i, r)$ is in fact a tree, the computed marginals $\mu(x_i)$ are precisely the ratio of the number of assignments satisfying NAE-$K$-SAT formula $B(x_i, r)$ which set $x_i$ to one to the number of such assignments which set this variable to zero. A standard fact is that for the majority of variables $B(x_i, r)$ is indeed a tree. Thus most of the times BP iterations compute marginal values corresponding to the ratio described above. These marginals are then used to design the BP-guided decimation algorithm as follows. Variable $x_1$ is selected and BP algorithm is used to compute its marginal $\mu(x_1)$ with respect to the neighborhood tree $B(x_1, r)$. Then the decision $\sigma(x_1)$ for this variable is set to $\sigma(x_1) = 1$ with probability $\mu(x_1)/(\mu(x_1) + 1)$ and $\sigma(x_1) = 0$ with probability $1/(\mu(x_1) + 1)$. Namely, the variable is set probabilistically proportionally to the ratio of the number of solutions setting it to one vs the number of solutions setting it to zero. After the decision for variable $x_1$ is set in the way described above, the variable $x_2$ is selected from the reduced formula on variables $x_2, \ldots, x_n$. The marginal $\mu(x_2)$ with respect to the neighborhood $B(x_2, r)$ for this reduced formula is computed and the value $\sigma(x_2)$ is determined based on $\mu(x_2)$ similarly, and so on. The procedure is called BP-guided decimation algorithm. It is thus parametrized by the computation depth $r$.

It is clear that the such a BP-guided decimation algorithm is precisely the $\tau$-decimation algorithm where $\tau(B(x_i, r)) = \mu(x_i)/(\mu(x_i) + 1)$ - the marginal probability of the variable corresponding to the
reduced formula $B(x_i, r)$. Furthermore, such a rule satisfies the balance condition described in Definition 2.3. Thus, as an implication of our main result, Theorem 2.4, we conclude that BP-guided decimation algorithm fails to find a satisfying assignment for $\Phi(n, dn)$ in the regime where our result on $\tau$-decimation algorithms applies:

**Corollary 2.5.** There exists $K_0$ such that for every $K \geq K_0$, $d > 2^{K-2}\ln 2$ and $r > 0$

$$\lim_{n \to \infty} \mathbb{P}(\text{BP-guided decimation algorithm with } r \text{ iterations solves } \Phi(n, dn)) = 0.$$

**Survey Propagation.** We now describe Survey Propagation-guided decimation algorithm in similar level of detail. The algorithm is significantly more complex to describe, but we will show again that it is a $\tau$-decimation algorithm when the number of message passing rounds is bounded by a constant independent from $n$, and that $\tau$ is a balanced rule. As a consequence we will conclude that SP-guided decimation algorithm also fails to find satisfying assignments for instances with density larger than $(d_s/K) \ln^2 K$ when the number of rounds is bounded by a constant. This is summarized in Corollary 2.7 below.

The setup is similar to the one for BP. In particular in steps $i = 1, 2, \ldots, n$ certain marginal value is computed and the decision for $x_i$ is again based on this marginal value, except now the marginal values do not correspond to the ratio of the number of assignments, but rather correspond to ratios when the problem is lifted to a new certain constraint satisfaction problem with decision variables $0, 1, \ast$. We do not describe here the rationale for this lifting procedure, as this has been documented in many papers, including [BMZ05], [MMW07], [MPZ02], [MM09]. Instead we simply formally present the SP algorithm and SP-guided decimation algorithm, following closely [MM09] with the appropriate adjustment from the K-SAT problem to the NAE-K-SAT problem. We will convince ourselves that SP-guided decimation algorithm is again the special case of a balanced $\tau$-decimation algorithm. We will then be able to conclude that SP-guided decimation algorithm fails to find a satisfying assignment with probability approaching unity, in the regime outlined in our main result, Theorem 2.4.

The SP algorithm is an iterative scheme described as follows. The details and notations are very similar to the ones described in [MM09]. Specifically, iterations 11-15 below correspond to iterations (20.17)-(20.20) in this book. Consider an arbitrary reduced or non-reduced NAE-K-SAT formula $\Phi$ on variables $x_1, \ldots, x_N$. For each iteration $t = 0, 1, \ldots$, each variable/clause pair $(x, C)$ such that $x$ appears in $C$ (namely there is an edge between $x$ and $C$ in the bi-partite factor graph representation) is associated with five random variables $Q^t_{x,C,U}$, $Q^t_{x,C,S}$, $Q^t_{x,C,*}$, $Q^t_{C,x,S}$ and $Q^t_{C,x,U}$. Here is the interpretation of these variables. Each of them is a message send from variable to a clause containing this variable, or a message from a clause to a variable which belongs to this clause. Specifically, $Q^t_{x,C,U}$ ($Q^t_{x,C,S}$) is interpreted as the probability computed at iteration $t$ that the variable $x$ is forced by clauses $D$ other than $C$ to take value which does not (does) satisfy $C$. $Q^t_{x,C,*}$ is interpreted that none of these forcing takes place. $Q^t_{C,x,S}$ is interpreted as probability computed at iteration $t$ that all variables $y \in C$ other than $x$ do not satisfy $C$, and thus the only hope of satisfying $C$ is for $x$ to do so. Similarly, $Q^t_{C,x,U}$ is the probability that all variables $y$ in $C$ other than $x$ do satisfy $C$ and thus the only hope of satisfying clause $C$ is for $x$ to violate it. The latter case is an artifact of the NAE variant of the problem and need not be introduced in the SP iterations for the K-SAT problem.

The variables $Q^t_{C,x,U}$ are then computed as follows. At time $t = 0$ the variables are generated uniformly at random from $[0, 1]$ independently for all five variables. Then they are normalized so that $Q^0_{x,C,U} + Q^0_{x,C,S} + Q^0_{x,C,*} = 1$, which is achieved by dividing each term by the sum $Q^0_{x,C,U} + Q^0_{x,C,S} + Q^0_{x,C,*}$. Similarly, variables $Q^t_{C,x,S}$ and $Q^t_{C,x,U}$ are normalized to sum to one.

Now we describe the iteration procedures at times $t \geq 0$. For each such pair $x, C$, let $S_{x,C}$ be the set of clauses containing $x$ other than $C$, in which $x$ appears in the same way as in $C$. Namely if $x$ appears
in $C$ without negation, it appears without negation in clauses in $S_{x,C}$ as well. Similarly, if $x$ appears as $\bar{x}$ in $C$, the same is true for clauses in $S_{x,C}$. Let $U_{x,C}$ be the remaining set of clauses containing $x$, namely clauses, where $x$ appears opposite to the way it appears in $C$. Now for each $t = 0, 1, 2, \ldots$ assume $Q_{t,x,U}^t, Q_{t,x,C,U}^t, Q_{t,x,C}^t, Q_{t,x,C,U}^t, Q_{t,x,S}^t, Q_{t,x,C,S}^t, Q_{t,x,C,S}^t$ and $Q_{t,x,C,*,*}^t$ are defined. Define the random variable $Q_{t,x,C}^t$ and $Q_{t,x,C,U}^t$ as follows. Suppose $C$ is unsigned in $\Phi$. Then

$$Q_{t,x,C,S}^t = \prod_{y \in C \setminus x} Q_{t,y,C,U}, \tag{1}$$

and

$$Q_{t,x,C,U}^t = \prod_{y \in C \setminus x} Q_{t,x,C,S}. \tag{2}$$

Here $C \setminus x$ is the set of variables in clause $C$ other than $x$. The interpretation for this identities is as follows. When $C$ is not signed, the clause $C$ forces its variable $x$ to satisfy it if all other variables $y$ in $C$ where forced not to satisfy $C$ at previous iteration due to other clauses. The first identity is the probability of this event assuming the events $"y is forced not to satisfy $C"$ are independent. The second identity is interpreted similarly, though it is only relevant only for NAE-$K$-SAT problem and does not appear for the corresponding iterations for the K-SAT problem.

If the clause $C$ is signed $+$, then we set $Q_{t,x,C,S}^t = 0$ and

$$Q_{t,x,C,U}^t = \prod_{y \in C \setminus x} Q_{t,x,C,S}. \tag{3}$$

The interpretation is that if $C$ is signed $+$, then one of the variables was already set to satisfy it. Thus the only way the clause $C$ can force $x$ to violate it is when all other variables $y$ are forced to satisfy $C$. Again this is only relevant for the NAE-$K$-SAT problem. Similarly, if $C$ is signed $-$, then $Q_{t,x,C,U}^t = 0$ and

$$Q_{t,x,C,S}^t = \prod_{y \in C \setminus x} Q_{t,x,C,U}. \tag{4}$$

Next we define variables $R_{t,x,C,S}^t, R_{t,x,C,U}^t$ and $R_{t,x,C,*}^t$ which stand for $Q_{t,x,C,S}^t, Q_{t,x,C,U}^t$ and $Q_{t,x,C,*}^t$ before the normalization. These random variables are computed using the following rules:

$$R_{t,x,C,S}^t = \prod_{D \in U_{x,C}} (1 - Q_{t,D,x,S}^t) \prod_{D \in S_{x,C}} (1 - Q_{t,D,x,U}^t) - \prod_{D \in U_{x,C}} (1 - Q_{t,D,x,*,*}^t) \prod_{D \in S_{x,C}} (1 - Q_{t,D,x,*,*}^t), \tag{5}$$

which is interpreted as follows. The first term in the right-hand side of the expression above is interpreted as the probability that none of the clauses $D$ in $U_{x,C}$ force $x$ to take value which satisfies $D$ and therefore violates $C$ (since otherwise a contradiction would be reached) and none of the clauses $D$ in $S_{x,C}$ force $x$ to take value which violates $D$ and therefore violates $C$ (since otherwise a contradiction would be reached). The second term term in the right-hand side is interpreted as the probability variable $x$ is not forced to take any particular value by clauses it belongs to other than $C$. The difference of the two terms is precisely the probability that $x$ is forced to take value satisfying $C$ and is not forced to take value contradicting this choice.

Similarly, define

$$R_{t,x,C,U}^t = \prod_{D \in U_{x,C}} (1 - Q_{t,D,x,U}^t) \prod_{D \in S_{x,C}} (1 - Q_{t,D,x,S}^t) - \prod_{D \in U_{x,C}} (1 - Q_{t,D,x,*}^t) \prod_{D \in S_{x,C}} (1 - Q_{t,D,x,*}^t). \tag{6}$$
The interpretation for $R^{t+1}_{x,C,U}$ is similar: it is the probability that $x$ is forced to take value violating $C$ and is not forced a contradicting value of satisfying $C$. Next, define

$$R^{t+1}_{x,C,*} = \prod_{D \in S_x, \tau} (1 - Q_{D,x,S}^t - Q_{D,x,U}^t).$$

$R^{t+1}_{x,C,*}$ is interpreted as the probability that $x$ is not forced in either way by clauses other than $C$. Finally, we let $Q^{t+1}_{x,C,S}, Q^{t+1}_{x,C,U}$ and $Q^{t+1}_{x,C,*}$ to be quantities $R^{t+1}_{x,C,S}, R^{t+1}_{x,C,U}$ and $R^{t+1}_{x,C,*}$, respectively, normalized by their sum $R^{t+1}_{x,C,S} + R^{t+1}_{x,C,U} + R^{t+1}_{x,C,*}$, so that the three variables sum up to one. The iterations (1)-(5) are conducted for some number of steps $t = 0, 1, \ldots, r$. Next variables $W_x(1)$ and $W_x(0)$ and $W_x(*)$ are computed for all variables $x$ as follows. Let $S_x$ be the set of clauses where $x$ appears without negation and let $U_x$ be the set of clauses where $x$ appears with negation. Then set

$$W_x(1) = \prod_{D \in U_x} (1 - Q_{D,x,S}^t) \prod_{D \in S_x} (1 - Q_{D,x,U}^t) - \prod_{D \in U_x} (1 - Q_{D,x,S}^t) \prod_{D \in S_x} (1 - Q_{D,x,U}^t).$$

$W_x(1)$ is interpreted as probability (after normalization) that variable $x$ is forced to take value 1, but is not forced to take value zero by all of the clauses containing $x$. Similarly, we set

$$W_x(0) = \prod_{D \in S_x} (1 - Q_{D,x,S}^t) \prod_{D \in U_x} (1 - Q_{D,x,U}^t) - \prod_{D \in S_x} (1 - Q_{D,x,S}^t) \prod_{D \in U_x} (1 - Q_{D,x,U}^t).$$

with a similar interpretation. Then set

$$W_x(*) = \prod_{D \in S_x, U_x} (1 - Q_{D,x,S}^t - Q_{D,x,U}^t),$$

which is interpreted as the probability (after normalization) that $x$ is not taken to be either 0 or 1. Finally, the values $W_x(0), W_x(1), W_x(*)$ are normalized to sum up to one. For simplicity we use the same notation for these quantities after normalization.

The random variables $W_x(0), W_x(1), W_x(*)$ are used to guide the decimation algorithm as follows. Given a random formula $\Phi(n,dn)$, variable $x_1$ is selected. The random quantities $W_{x_1}(0), W_{x_1}(1)$ and $W_{x_1}(*)$ are computed and $x_1$ is set to 1 if $W_{x_1}(1) > W_{x_1}(0)$ and set it to zero otherwise. The formula is now reduced and contains variables $x_2, x_3, \ldots, x_n$. Variable $x_2$ is then selected and the random quantities $W_{x_2}(0), W_{x_2}(1)$ are computed with respect to the reduced formula. Then $W_{x_2}$ is computed and $x_2$ is set to 1 if $W_{x_2}(1) > W_{x_2}(0)$, and set to zero otherwise. The procedure is repeated until all variables are set. This defines the SP-guided decimation algorithm.

It is clear again that the SP-guided decimation algorithm is the special case of $\tau$-decimation algorithm, where $\tau$ function corresponds to the probability of the event $W_x(1) > W_x(0)$, when it applies to a reduced instance $B(x,r)$ with $x$ as its root. The depth $r$ of the instance corresponds to the number of iterations of the SP procedure. Furthermore, we claim that this $\tau$ rule is balanced.

**Proposition 2.6.** The local rule $\tau$ corresponding to the Survey Propagation iterations is balanced.

**Proof.** Recall that at the iteration $t = 0$, the variables $Q^t$ are chosen independently uniformly at random from $[0,1]$, normalized appropriately. The main idea of the proof is to use the symmetry of the uniform distribution. Given a formula $\Phi$, we claim that if we initialize random variables $Q^t$ with variables $Q^0_{x,C,U}$ and $Q^0_{x,C,S}$ swapped, variables $Q^0_{C,x,S}$ and $Q^0_{C,x,U}$ swapped, variables $Q^0_{x,C,*}$ left intact, and apply it to formula $\Phi$ instead of $\Phi$, we obtain values $W_x(0), W_x(1)$ and $W_x(*)$ such that under this initialization $W_x(1) > W_x(0)$ holds if $W_x(0) < W_x(1)$ under the original initialization for the original formula $\Phi$. The claim of the proposition then follows.
We now establish the claim by a simple inductive reasoning. As suggested above, given $Q^0_{x,C,U}$, $Q^0_{x,C,S}$, $Q^0_{x,C,*}$, $Q^0_{C,x,S}$ and $Q^0_{C,x,U}$ (after normalization for concreteness), define

$$
\begin{align*}
P^0_{x,C,U} &= Q^0_{x,C,S}, \\
P^0_{x,C,S} &= Q^0_{x,C,U}, \\
P^0_{x,C,*} &= Q^0_{x,C,*}, \\
P^0_{C,x,S} &= Q^0_{C,x,U}, \\
P^0_{C,x,U} &= Q^0_{C,x,S}.
\end{align*}
$$

Then define variables $P^t_{x,C,U}$, $P^t_{x,C,S}$, $P^t_{x,C,*}$, $P^t_{C,x,S}$ and $P^t_{C,x,U}$ with respect to the formula $\Phi$ similarly to the way variables $Q^t_{x,C,U}$, $Q^t_{x,C,S}$, $Q^t_{x,C,*}$, $Q^t_{C,x,S}$ and $Q^t_{C,x,U}$ are defined with respect to the formula $\Phi$. We now prove by induction that the identities (11) hold for general $C$. The case when $C$ is non-reduced $\text{NAE-SAT}$ instance: specifically we prove two structural results about the $\text{SP-guided decimation algorithm}$ with $r$ iterations solves $\Phi(n, dn)$.

\textbf{Theorem 2.4 then becomes applicable and we conclude:}

\textbf{Corollary 2.7. There exists $K_0$ such that for every $K \geq K_0$, $d > 2^{K-2}\ln 2$ and $r > 0$}

$$
\lim_{n \to \infty} \mathbb{P}(\text{SP-guided decimation algorithm with } r \text{ iterations solves } \Phi(n, dn)) = 0.
$$

\section{Local algorithms and long-range independence}

In this section we obtain some preliminary results needed for the proof of our main result, Theorem 2.4. Specifically we prove two structural results about the $\tau$-decimation algorithm for a local rule $\tau$.

The first result is simple to state - we show that balanced local rules lead to unbiased decisions for every non-reduced $\text{NAE-K-SAT}$ instance: specifically the marginal probability that a variable is set to 1 is $1/2$. More generally we show that the probability that a variable is set to 1 in any reduced or non-reduced instance $\Phi$ equals the probability that the same variable is set to 0 in the complementary instance $\bar{\Phi}$. (See Lemma 3.1.) This lemma later allows us to find satisfying assignments with small overlap in random instances $\Phi(n, dn)$.

Next, we consider the “influence” of a decision $\sigma(x_i) \in \{0, 1\}$ and ask how many other variables are affected by this decision. In particular, we show that the decisions $\sigma$ assigned to a pair of fixed variables $x_i$ and $x_j$ are asymptotically independent as $n \to \infty$. Namely, the decisions exhibit a long-range independence. Such a long-range independence is not a priori obvious, since setting a value of a variable
x_i can have a downstream implications for setting variables \( x_j, j \geq i \). We will show, however, that the chain of implications, appropriately defined is typically short. Definition 3.2 and Proposition 3.4 formalize these claims.

In what follows, we first introduce some notation that makes the decisions of the randomized algorithm more formal and precise. We then prove the two main claims above in the following subsections.

### 3.1 Formalizing random choices of a \( \tau \)-decimation algorithm

The \( \tau \)-decimation algorithm described in the previous section is based on the ordering of the variables \( x_i \), since the values \( \sigma(x_i) \) are set in the order \( i = 1, 2, \ldots, n \). In the case of the random NAE-\( K \)-SAT formula \( \Phi(n, dn) \), due to symmetry we may assume, without the loss of generality, that the ordering is achieved by assigning random i.i.d. labels chosen uniformly from \([0,1]\) and using order statistics for ordering of variables. (This is equivalent to renaming the variables at random and this renaming will be convenient for us.) Specifically, let \( \mathbf{Z} = (Z_i, 1 \leq i \leq n) \) be the i.i.d. sequence of random variables with uniform in \([0,1]\) distribution. Let \( \pi : [n] \to [n] \) be the permutation induced by the order statistics of \( \mathbf{Z} \). Namely \( Z_{\pi(1)} > Z_{\pi(2)} > \cdots > Z_{\pi(n)} \). We now assume that when the \( \tau \)-decimation algorithm is performed, the first variable selected is \( x_{\pi(1)} \) (as opposed to \( x_1 \)), the second variable selected is \( x_{\pi(2)} \) (as opposed to \( x_2 \)), etc. Namely, we assume that \( \tau \)-decimation algorithm performed on a random instance of the NAE-\( K \)-SAT problem \( \Phi(n, dn) \) is conducted according to this ordering.

To facilitate the randomization involved in selecting randomized decisions based on the \( \tau \) rule, consider another i.i.d. sequence \( \mathbf{U} = (U_i, 1 \leq i \leq n) \) of random variables with the uniform in \([0,1]\) distribution, which is independent from the randomness of the instance \( \Phi \) and sequence \( \mathbf{Z} \). The purpose of the sequence is to serve as random seeds for the decision \( \sigma(x_i) \) based on \( \tau \). Specifically, when the value \( \sigma(x_i) \) associated with variable \( x_i \) is determined, it is done so according to the rule \( \sigma(x_i) = 1 \) if \( U_i < \tau(B(x_i, r)) \) and \( \sigma(x_i) = 0 \) otherwise, where \( B(x_i, r) = B_{\Phi, \tau \pi}(x_i, r) \) is the reduced NAE-\( K \)-SAT instance rooted at \( x_i \), observed at a time when the decision for \( x_i \) needs to be made. Namely, the \( \tau \)-decimation algorithm is faithfully executed. Conditioned on \( \mathbf{Z}, \mathbf{U} \) and \( \Phi \), the output \( \sigma : [n] \to \{0,1\} \) is uniquely determined. We denote by \( \sigma_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i), 1 \leq i \leq n \) the output of the \( \tau \)-decimation algorithm conditioned on the realizations \( \Phi, \mathbf{Z}, \mathbf{U} \) of the random instance \( \Phi(n, dn) \), vector \( \mathbf{Z} \) and vector \( \mathbf{U} \), respectively. Similarly, we denote by \( B_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i, r), 1 \leq i \leq n \) the (possibly) reduced NAE-\( K \)-SAT instance corresponding to the \( r \)-depth neighborhood of variable \( x_i \) at the time when the value of \( x_i \) is determined by the \( \tau \)-decimation algorithm. In particular, \( \sigma_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i) = 1 \) if \( u_i \in [0, \tau(B_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i, r))] \) and \( \sigma_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i) = 0 \) if \( u_i \in (\tau(B_{\Phi, \mathbf{Z}, \mathbf{U}}(x_i, r)), 1] \).

### 3.2 Implications of balance

We now establish the following implication of the the Definition 2.3 of balanced local rules.

**Lemma 3.1.** For every formula \( \Phi \), and vectors \( \mathbf{z}, \mathbf{u} \), the following identities hold for every variable \( x_i, 1 \leq i \leq n \):

\[
B_{\Phi, \mathbf{z}, \mathbf{u}}(x_i, r) = B_{\Phi, \mathbf{z}, \mathbf{u}}(x_i, r),
\]

\[
\sigma_{\Phi, \mathbf{z}, \mathbf{u}}(x_i) = 1 - \sigma_{\Phi, \mathbf{z}, \mathbf{u}}(x_i),
\]

where \( \mathbf{\bar{u}} \) is defined by \( \bar{u}_i = 1 - u_i, 1 \leq i \leq n \). As a result, when \( \mathbf{U} \) is a vector of i.i.d. random variables chosen uniformly from \([0,1]\), for \( \Phi \) and \( \mathbf{z} \), the following holds for all \( i = 1, 2, \ldots, n \):

\[
P(\sigma_{\Phi, \mathbf{z}, \mathbf{U}}(x_i) = 0) = 1/2.
\]

Note, that the randomness in the probability above is with respect to \( \mathbf{U} \) only and the claim holds for every formula \( \Phi \) and every vector \( \mathbf{z} \).
Proof. We prove the claim by induction on $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}$, where $\pi$ is the permutation generated by $z$, that is $z_{\pi(1)} > z_{\pi(2)} > \ldots > z_{\pi(n)}$. Specifically, we will show by induction that for every $i = 0, 1, 2, \ldots, n$, just before the value of variable $x_{\pi(i)}$ is determined, the identity (13) holds for all variables $x_{\pi(j)}$ where $i \leq j - 1$ (namely for variables for which the value is already determined at time $i$), and the identity (12) in fact holds for all neighborhoods $B_{\Phi, z, u}(x_{\pi(k)}, r)$, $i \leq k \leq n$ and $B_{\Phi, z, \bar{u}}(x_{\pi(k)}, r)$, $i \leq k \leq n$, and not just for $B_{\Phi, z, u}(x_{\pi(i)}, r)$ and $B_{\Phi, z, \bar{u}}(x_{\pi(i)}, r)$.

For the base of the induction corresponding to $i = 1$, no variables are set yet and all the neighborhoods $B_{\Phi, z, u}(x_k, r), B_{\Phi, z, \bar{u}}(x_k, r), 1 \leq k \leq n$ correspond to non-reduced instances, for which by our convention, its symmetric complement is the instance itself. Namely $B_{\Phi, z, u}(x_k, r) = B_{\Phi, z, \bar{u}}(x_k, r)$, and thus (12) is verified.

Fix $i \geq 1$ and assume now the inductive hypothesis holds for $j \leq i$. In particular, the values $\sigma(x_{\pi(j)})$ are determined for $j = 1, \ldots, i - 1$ under $u$ and $\bar{u}$. Now consider the step of assigning the value of $x_{\pi(i)}$. We have $\sigma_{\Phi, z, u}(x_{\pi(i)}) = 1$ iff $u_{\pi(i)} \leq \tau(B_{\Phi, z, u}(x_{\pi(i)}, r))$ and $\sigma_{\Phi, z, u}(x_{\pi(i)}) = 1$ iff $\bar{u}_{\pi(i)} \leq \tau(B_{\Phi, z, \bar{u}}(x_{\pi(i)}, r))$. By the inductive assumption we have that $B_{\Phi, z, u}(x_{\pi(i)}, r) = \hat{B}_{\Phi, z, u}(x_{\pi(i)}, r)$. Since $\tau$ is balanced, we have $\tau(\hat{B}_{\Phi, z, u}(x_{\pi(i)}, r)) = 1 - \tau(B_{\Phi, z, u}(x_{\pi(i)}, r))$. Since $\bar{u} = 1 - u$, we conclude that $\sigma_{\Phi, z, u}(x_{\pi(i)}) = 1$ iff $\sigma_{\Phi, z, \bar{u}}(x_{\pi(i)}) = 0$ and vice versa. Namely, $\sigma_{\Phi, z, u}(x_{\pi(i)}) = 1 - \sigma_{\Phi, z, u}(x_{\pi(i)})$ and identity (13) is verified.

It remains to show that identity (12) still holds for all variables after the value $\sigma(x_{\pi(i)})$ is determined. All neighborhoods $B(x_k, r)$ which do not contain $x_{\pi(i)}$ are not affected by fixing the value of $x_{\pi(i)}$ and thus the identity holds by the inductive assumption. Suppose $B(x_k, r)$ contains $x_{\pi(i)}$. This means this neighborhood contains one or several clauses which contains $x_{\pi(i)}$. Fix any such clause $C$. If this clause was unsigned under $u$, then by the inductive assumption it was also unsigned under $\bar{u}$ (as the instances under $u$ and $\bar{u}$ are complements of each other). The clause then becomes signed after fixing the value of $x_{\pi(i)}$, and, furthermore, the signs will be opposite under $u$ and $\bar{u}$, since (12) holds for $x_{\pi(i)}$ as we have just established.

Now suppose the clause was signed + under $u$. Then again by the inductive assumption it was signed − under $\bar{u}$. In this case if the assignment $\sigma_{\Phi, z, u}(x_{\pi(i)})$ satisfies $C$, then the clause remains signed + after setting the value of $x_{\pi(i)}$. At the same time this means that $\sigma_{\Phi, z, u}(x_{\pi(i)}) = 1 - \sigma_{\Phi, z, u}(x_{\pi(i)})$ does not satisfy $C$ and the clause remains signed − after setting the value of $x_{\pi(i)}$. In both cases the variable $x_{\pi(i)}$ is deleted and the identity (12) still holds. On the other hand if $\sigma_{\Phi, z, u}(x_{\pi(i)})$ does not satisfy $C$ when $u$ is used, then (since it was signed +) the clause $C$ is now satisfied and disappears from the formula. But at the same time this means $\sigma_{\Phi, z, u}(x_{\pi(i)})$ satisfies $C$, since it was signed − under $\bar{u}$, and therefore $C$ is satisfied again and disappears from the formula. The variable $x_{\pi(i)}$ is deleted in both cases and again (12) is verified.

The case when clause $C$ is signed − under $u$ and signed + under $\bar{u}$ is considered similarly. Finally, suppose $\sigma_{\Phi, z, u}(x_{\pi(i)})$ violates a clause $C$ containing $x_{\pi(i)}$. This means that $C$ contains only this variable when setting this variable to $\sigma_{\Phi, z, u}(x_{\pi(i)})$. By inductive assumption we see that the same is true under $\bar{u}$. In both cases both the variable and clause are removed from the formula. This completes the proof of the inductive step.

Finally, since the distribution of $U$ and $\bar{U}$ is identical for i.i.d. sequences chosen uniformly at random from $[0, 1]$, we obtain (13). □

3.3 Influence ranges

We now define the notion of influence (which depends on the formula $\Phi(n, dn)$ and ordering $Z$, but not on random choices of the $\tau$-decimation algorithm). We introduce the following relationship between the variables $x_1, \ldots, x_n$ of our formula.

Definition 3.2. Given a random formula $\Phi(n, dn)$ and random sequence $Z$ we say that $x_i$ influences $x_j$ if either $x_j = x_i$ or in the underlying node-to-node graph $G = G(\Phi(n, dn))$ there exists a sequence of
Given a variable $u$, for some fixed index $i$, such a variable exists if either

(i) $y_0 = x_i$ and $y_{l} = x_j$.

(ii) $y_l$ and $y_{l+1}$ are connected by a path of length at most $r$ in graph $G$ for all $l = 0, 1, \ldots, t - 1$.

(iii) $Z_{y_l} > Z_{y_l}$ for $l = 1, 2, \ldots, t$. In particular, $Z_{x_i} > Z_{x_j}$.

In this case we write $x_i \leadsto x_j$. We denote by $\mathcal{IR}_{x_i}$ the set of variables $x_j$ influenced by $x_i$ and call it influence range of $x_i$.

Note that indeed the randomness underlying the sets $\mathcal{IR}_{x_i}, 1 \leq i \leq n$ as well as the relationship $\leadsto$ is the function of the randomness of the formula $\Phi(n, dn)$ and vector $Z$, but is independent from the random vector $U$.

While the definition above is sound for every constant $r > 0$, we will apply it in the case where $r$ is the parameter appearing in the context of $\tau$-decimation algorithm. Namely, in the context of the $\tau$ function defined the set of rooted instances $\mathcal{SAT}_r$ introduced above. In this case the notion of influence range is justified by the following observation.

**Proposition 3.3.** Given realizations $\Phi$ and $z$ of the random formula $\Phi(n, dn)$ and random ordering $Z$, suppose $u = (u_i, 1 \leq i \leq n)$ and $u' = (u'_i, 1 \leq i \leq n)$ are such that $u_{i_0} = u'_{i_0}$ and $u_i = u'_i$ for all $i \neq i_0$, for some fixed index $i_0$. Then $\sigma_{\Phi, z, u}(x) = \sigma_{\Phi, z, u'}(x)$ for every $x \notin \mathcal{IR}_{i_0}$. That is, changing the values of $u$ at $i_0$ may impact the decisions associated with only variables $x$ in $\mathcal{IR}_{i_0}$.

**Proof.** Given a variable $x_i, i \neq i_0$, in order for its decision $\sigma_{\Phi, z, u}(x_i)$ to be affected by switching from $u$ to $u'$, there must exist a variable $x_{i_1}$ with distance at most $r$ (with respect to the node-to-node graph $G = G(\Phi)$) from $x_i$ such that $z_{x_{i_1}} > z_{x_i}$ and such that the decision for $x_{i_1}$ is affected by the switch. In its turn such a variable exist if either $i_1 = i_0, z_{i_1} = z_{i_0} > z_i$ and $x_{i_0} \in B(x_i, r)$ (in particular $x_{i_0} \leadsto x_i$), or if there exists $x_{i_2} \in B(x_{i_1}, r)$ such that $z_{i_2} > z_{i_1}$ and $x_{i_2}$ is affected by the switch. In this case again $x_{i_2} \leadsto x_i$. Continuing, we see that in order for $x_i$ to be affected by the switch, it must be the case that $x_{i_0} \leadsto x_i$. 

We now obtain a probabilistic bound on the size of a largest in cardinality of the influence range classes $\mathcal{IR}_{x_i}, 1 \leq i \leq n$.

**Proposition 3.4.** The following holds

$$\lim_{n \to \infty} \mathbb{P}(\max_{1 \leq l \leq n} |\mathcal{IR}_{x_i}| \geq n^{\frac{1}{3}}) = 0.$$  

The choice of exponent $1/3$ is somewhat arbitrary here. In fact the bound holds for any exponent in $(0, 1)$, and for our purposes, as we are about to see in Section 5, any constant in the range $(0, 1/2)$ suffices.

**Proof.** Fix a variable $x$ in $\Phi(n, dn)$. We first establish an upper bound on the number of variables in a neighborhood $B(x, t)$ of $x$ in the node-to-node graph $G(\Phi(n, dn))$ when $t$ is moderately growing.

**Lemma 3.5.** There exists $\delta > 0$ and $\epsilon = \epsilon(\delta) < 1/3$ such that for all sufficiently large $n$

$$\mathbb{P}(|B(x, t)| \geq n^\epsilon) \leq \frac{1}{n^{t}},$$

when $t \leq \delta \ln n$. 

16
From the proof below it will be clear that the bound $1/n^2$ is very crude and in fact a bound $\exp(-n^{\epsilon/5})$ can be established. But a cruder bound suffices for our purposes.

**Proof.** It is well known that for small enough $\delta > 0$ and $t = \delta \ln n$, the $B(x, t)$ is distributed approximately as a Poisson branching process with the off-spring distribution being Poisson with parameter $\beta \triangleq dK$. Furthermore, by increasing the number of clauses by $o(n)$ the Poisson branching process stochastically dominates the distribution of $B(x, t)$. Thus we will obtain instead an upper bound on the number of off-springs in the $t$ generations of a Poisson branching process with parameter $\beta$. Letting $W_i$ denote the size of the $l$-th generation, our goal is then to obtain a bound on $\sum_{l \leq t} W_i$. We claim that for some $\epsilon = \epsilon(\delta)$ satisfying $\epsilon(\delta) \to 0$ as $\delta \to 0$, the following upper bound holds for each $W_i, l \leq t = \delta \ln n$:

$$\Pr(W_i > n^{t/2}) \leq \exp(-n^{\epsilon/4}),$$

(15)

from which the claim of the lemma follows by a union bound. To establish this bound we rely on the following known representation of the probability generating function of $W_i$. That is, let $G(\theta) = \mathbb{E}[\theta^{W_i}]$ for $\theta > 0$, where $W_i$ has Poisson mean $\beta$ distribution. Then $G(\theta) = \exp(\beta \theta - \beta)$ and $\mathbb{E}[\theta^{W_i}] = G^{(l)}(\theta)$ - the $l$-th iterate of function $G(\theta)$. Now we let $\theta = 1 + \frac{1}{(e\beta)^l}$. Define $\gamma_l = 1/(e\beta)^l$. We now obtain an upper bound on $G^{(l)}(\theta)$. We have

$$G^{(l)}(\theta) = \exp(\beta \theta - \beta) = \exp(\beta \gamma_l) \leq 1 + \gamma_{l-1},$$

where we have used $\beta \gamma_l < 1$ and inequality $e^z \leq 1 + ez$ for $z \leq 1$. Then

$$G^{(2)}(\theta) = \exp(\beta G^{(1)}(\theta) - \beta) \leq \exp(\beta \gamma_{l-1}) \leq 1 + \gamma_{l-2},$$

since $\beta \gamma_{l-1} < 1$. Continuing, we obtain $G^{(l)}(\theta) \leq 1 + \gamma_{l-1}, 1 \leq l \leq t$. Applying this bound

$$\Pr(W_i \geq n^{t/2}) = \Pr(\theta^{W_i} \geq \theta^{n^{t/2}})
\leq \theta^{-n^{t/2}} \mathbb{E}[\theta^{W_i}]
\leq \theta^{-n^{t/2}} (1 + \gamma_{l-1})
\leq 2\theta^{-n^{t/2}}.$$

Now

$$\theta^{-n^{t/2}} = \exp(-n^{t/2} \ln(\theta))
= \exp(-n^{t/2}(\gamma_l + o(\gamma_l))).$$

Now since $t = \delta \ln n$, then $\gamma_l = (e\beta)^{-t} = n^{-\ln(e\beta)\delta}$, implying the upper bound $\exp(-n^{\epsilon/4})$ for large enough $n$ when $\epsilon(\delta) > 2\ln(e\beta)\delta$. This completes the proof of the bound (15) and of lemma.

Now we complete the proof of the Proposition. Applying union bound we have that that for every $\epsilon > 0$, $|B(x_i, t)| \leq n^\epsilon$ for all $i = 1, \ldots, n$ with probability approaching unity as $n \to \infty$. Given two variables $x_i$ and $x_j$ if $x_i \sim x_j$ and the distance in $G(\Phi(n, dn))$ between $x_i$ and $x_j$ is at least $t$, then there must exist $x_k \in B(x_i, t) \setminus B(x_i, t - 1)$ such that $x_i \sim x_k$. Given a sequence $y_0 = x_i, y_1, \ldots, y_t = x_k$, with $x_k \in B(x_i, t) \setminus B(x_i, t - 1)$, the probability of an event $Z_{y_l} > Z_{y_{l+1}}, 0 \leq l \leq t - 1$ is $1/(t + 1)!$. The total number of paths between $x_i$ and variables in $B(x_i, t) \setminus B(x_i, t - 1)$ is trivially at most $B(x_i, t)$, since $B(x_i, t)$ is tree. Thus, conditioned on $B(x_i, t)$, the expected number of variables in $B(x_i, t)$ is at most $B(x_i, t)r^t/(t + 1)!$, where the extra factor $r^t$ is due to choices of points $y_1, \ldots, y_t$ on a given path. When $t = \epsilon \ln n$, the expected number is $B(x_i, t)n^{-\Omega(\ln \ln n)}$. Applying the bound Lemma 3.5 and a union bound over $x_i$ we obtain the result. 

□
4 The clustering property of random NAE-\( K \)-SAT problem

In this section we establish the clustering property of random NAE-\( K \)-SAT problem when \( d \) is large enough (in terms of \( K \)). Recall that the random NAE-\( K \)-SAT formula \( \Phi(n,dn) \) is satisfiable with probability approaching unity as \( n \to \infty \), when \( d \leq d_s \), where \( d_s = 2^{K-1} \ln 2 - \ln 2/2 - 1/4 - f(K) \) for some function \( f(K) \) satisfying \( \lim_{K \to \infty} f(K) = 0 \). Recalling our notation for the set of satisfying assignment \( \text{SAT}(\Phi) \) of a formula \( \Phi \), we have \( \mathbb{P}(\text{SAT}(\Phi(n,dn)) \neq \emptyset) \to 1 \) as \( n \to \infty \) for every \( d < d_s \).

The notion of “clustering” we consider is with respect to the Hamming distance where the Hamming distance between two assignments \( \sigma^1 \) and \( \sigma^2 \), denoted \( \rho(\sigma^1,\sigma^2) \), is the number of variables \( x_i \) with different assignments according to \( \sigma^1 \) and \( \sigma^2 \). A simplistic notion of clustering might say that the “satisfaction graph”, the graph on satisfying assignments where two assignments are deemed adjacent if the Hamming distance between them is \( o(n) \), has many connected components. A condition in turn that implies this simple notion is that for every pair of satisfying assignment \( \sigma^1 \) and \( \sigma^2 \), it is the case that \( \rho(\sigma^1,\sigma^2)/n \not\in (\beta-\eta,\beta) \) for some \( \eta > 0 \). Note that this implies that for any pair of satisfying assignments \( \sigma^1 \) and \( \sigma^3 \) with \( \rho(\sigma^1,\sigma^3) > \beta n \) they must be disconnected in the satisfaction graph, or else there will be a point \( \rho(\sigma^1,\rho^2) \) on the path between them with \( \rho(\sigma^1,\rho^2)/n \in (\beta-\eta,\beta) \).

Working purely with this notion we only get a clustering result for very high densities \( d \), specifically for \( d \) at least \( d_s/2 \). (We skip details since it is not needed for our main result). To get a result for smaller densities we work with a more sophisticated notion of clustering inspired by [RV14]. Informally, this notion may be seen to allow paths between any pair of vertices in the graph on satisfying assignments mentioned above. However (again informally) it restricts the number of “fundamentally” different paths to be small. Formally, we insist that there can not be many satisfying assignments \( \sigma^1,\ldots,\sigma^m \) with all pairwise distance being between \( (\beta-\eta)n \) and \( \beta n \). We give the formal definition below.

Fix \( \beta,\eta \in [0,1] \) and a positive integer \( m \). Given an NAE-\( K \)-SAT formula \( \Phi \), denote by \( \text{SAT}(\Phi;\beta,\eta,m) \) the set of all \( m \)-tuples \( (\sigma^1,\ldots,\sigma^m) \) of assignments \( \sigma^j : \{x_1,\ldots,x_n\} \to \{0,1\} \), \( 1 \leq j \leq m \) satisfying the following properties:

(a) Every \( \sigma^j, 1 \leq j \leq m \) is a satisfying assignment. Namely \( \text{SAT}(\Phi;\beta,\eta,m) \subset \text{SAT}^m(\Phi) \).

(b) For every \( j,k \), \( (\beta-\eta)n \leq \rho(\sigma^j,\sigma^k) \leq \beta n \). In our application we will choose \( \eta \) to be much smaller than \( \beta \). In this case the pairwise distances \( \rho(\sigma^j,\sigma^k) \) are nearly \( \beta n \). Thus we may think of the such an \( m \)-tuple as a set of \( m \) equidistant points in the Hamming cube \( (0,1)^n \) with pairwise distances nearly \( \beta n \).

We now state the main result of this section. Intuitively it states for certain choices of \( \beta, \eta \) and \( m \) which depend on \( K \) only, there are no such \( m \) equidistant points when \( d \) crosses the threshold \( (d_s/K) \ln^2 K \).

**Theorem 4.1.** Fix arbitrary \( 0 < \epsilon < 1 \), and let \( \beta = \ln K / K, \eta = (\ln K)^2 / K, m = \lfloor 2^{K-1} \ln^2 K / K \rfloor \). Then there exists \( K_0 = K_0(\epsilon) \), such that for all \( K \geq K_0 \) and \( d \geq (1+\epsilon)2^{K-1} \ln^2 K / K \), the following holds

\[
\lim_{n \to \infty} \mathbb{P}(\text{SAT}(\Phi(n,dn),\beta,\eta,m) = \emptyset) = 1.
\]

**Proof.** The proof is based on the application of the first moment argument. We consider the expected number of \( m \)-tuples satisfying the conditions (a)-(b), and show that this expectation converges to zero exponentially fast as \( n \to \infty \). Applying Markov’s inequality the result then will follow.

We begin by computing asymptotically the number of \( m \)-tuples \( \sigma^1,\ldots,\sigma^m \) satisfying condition (b) only. We have \( 2^n \) choices for \( \sigma^1 \). For any fixed choice of \( \sigma^1 \), and any fixed \( j = 2,\ldots,m \) the number of choices for \( \sigma^j \) is

\[
\sum_{n(\beta-\eta) \leq r \leq n\beta} \binom{n}{r},
\]
by considering all the subsets of variables $x_1, \ldots, x_n$ where $\sigma^1$ and $\sigma^j$ disagree. Since this applies for every $j$, we obtain the following upper bound on the number of $m$-tuples satisfying (b):

$$2^n \left( \sum_{n(\beta - \eta) \leq r \leq n \beta} \binom{n}{r} \right)^{m-1}.$$ 

This bound is clearly loose, since it ignores the constraints on $\rho(\sigma^j, \sigma^k)$ for $j, k \geq 2$. Nevertheless, it suffices for our purposes. We now obtain an asymptotic upper bound on this expression in terms of $\epsilon, K$ and $n$.

Using Stirling’s approximation and since the function $-x \ln x$ is increasing in the range $x < e^{-1}$, and decreasing in the range $x > e^{-1}$, the expression is at most

$$\exp(n \ln 2 - nm\beta \ln \beta - nm(1 - \beta) \ln(1 - \beta) + o(n)).$$

Here we use $\beta = \ln K/K < e^{-1}$, for sufficiently large $K$. Further, the same asymptotics gives $-\ln \beta = \ln K + O_K(\ln \ln K)$, implying

$$-m\beta \ln \beta = m \beta \ln K + O_K(\ln \ln K)$$

$$= \epsilon^2 \ln K + O_K(\ln \ln K).$$

Next, we have for sufficiently large $K$

$$-m(1 - \beta) \ln(1 - \beta) \leq m((\ln K/K) + o_K(\ln K/K))$$

$$\leq \epsilon^2 + o_K(1).$$

We conclude that for sufficiently large $K$, the term (16) is at most

$$\exp(n\epsilon^2 \ln K + nO_K(\ln \ln K) + o(n)).$$

We now compute an upper bound on the probability that a given $m$-tuple $\sigma^1, \ldots, \sigma^m$ satisfying (b), consists of satisfying assignments. Let $C$ be a clause generated uniformly at random from the space of all clauses (a generic element of the formula $\Phi(n, dn)$). Applying the truncated exclusion-inclusion principle, the probability that $C$ is satisfied by every assignment $\sigma^1, \ldots, \sigma^m$ is

$$\mathbb{P}(C \text{ satisfied by } \sigma^j, \forall j = 1, \ldots, m) = 1 - \mathbb{P}(\exists j: C \text{ is not satisfied by } \sigma^j, 1 \leq j \leq m)$$

$$\leq 1 - \sum_{1 \leq j \leq m} \mathbb{P}(C \text{ is not satisfied by } \sigma^j)$$

$$+ \sum_{1 \leq j_1 < j_2 \leq m} \mathbb{P}(C \text{ is not satisfied by } \sigma^{j_1}, \sigma^{j_2}).$$

Now $\mathbb{P}(C \text{ is not satisfied by } \sigma^j) = 2^{-K+1}$. Also for every two assignments $\sigma^1$ and $\sigma^2$ which disagree in $n_0 \leq n$ variables

$$\mathbb{P}(C \text{ is not satisfied by } \sigma^1, \sigma^2) = 2^{-K+1} \left( \left( \frac{n_0}{n} \right)^K + \left( 1 - \frac{n_0}{n} \right)^K \right).$$

We conclude that for every $m$-tuple $\sigma_1, \ldots, \sigma_m$ satisfying (b)

$$\mathbb{P}(\sigma^1, \ldots, \sigma^m \in \text{SAT}(\Phi(n, dn)) \leq \left( 1 - m2^{-K+1} + (m(m-1)/2)2^{-K+1}(\beta^K + (1 - \beta + \eta)^K) \right)dn$$

$$\leq \left( 1 - \epsilon^2 K(\ln K)^{-1/2}2^{-K+1} + \epsilon^4 K^2(\ln K)^{-2}2^{-K+2}(K^{-1} + o_K(K^{-1})) \right)dn.$$
Here we used the fact that for $\beta = \ln K/K$ and $\eta = (\ln K/K)^2$, we have

$$\beta^K + (1 - \beta + \eta)^K = K^{-1} + o_K(K^{-1}).$$

The upper bound then simplifies to

$$(1 - \epsilon^2 K(\ln K)^{-1}2^{-K+1} + o_K(2^{(\ln K)^{-1}2^{-K}}))^d_n,$$

which applying the lower bound $d \geq (1 + \epsilon)(2^{-K}/K)^2$ leads to a bound

$$\exp \left( -n(1 + \epsilon^2 \ln K + n o_K(\ln K)) \right).$$

Now combining with (17), we conclude that the expected number of $m$-tuples satisfying conditions (a) and (b) is at most

$$\exp(-ne^3 \ln K + no_K(\ln K)),$$

and the proof of the theorem is complete. \hfill \square

5 Proof of Theorem 2.4

The main result of this section states that if a $\tau$-decimation algorithm works well on random instances of NAE-$K$-SAT, then it can be run several times to produce several satisfying assignments, and in particular that their overlaps (Hamming distances) satisfy properties (a),(b) described in the previous sections with parameters $\alpha, \eta$ and $m$ given in Theorem 4.1. Since such overlaps are "forbidden" by this theorem, we will obtain a contradiction. We state our main proposition below and show how Theorem 2.4 follows almost immediately. The rest of this section is devoted to the proof of the proposition.

We first recall some notation from Section 3. Given a local rule $\tau : SAT_r \to [0,1]$, let $\sigma_{\Phi,Z,U}$ denote the assignment produced by the $\tau$-decimation algorithm on input $\Phi$, ordering given by $Z$, and using $U$ to determine the rounding of the probabilities given by $\tau$. Recall that $\rho(\sigma^1, \sigma^2)$ denotes the Hamming distance between assignments $\sigma^1$ and $\sigma^2$. Let $\alpha_n$ denote the probability that $\tau$-decimation algorithm finds a satisfying assignment in a random formula $\Phi(n,dn)$. Namely, $\alpha_n = P(\sigma_{\Phi(n,dn),Z,U} \in SAT(\Phi(n,dn)))$ and the claim of Theorem 2.4 is that $\lim_n \alpha_n = 0$.

Proposition 5.1. Fix $r < \infty$ and let $\tau : SAT_r \to [0,1]$ be any balanced local rule. Suppose $\limsup_n \alpha_n > 0$. Then for every $0 < \eta < \beta$ such that $[\beta - \eta, \beta] \subset [0,1/2]$ and every positive integers $m, K$ and $d$,

$$\liminf_n P(\Phi(n,dn) \in SAT(\Phi(n,dn)); \beta, \eta, m) > 0.$$ 

Proof of Theorem 2.4. The result follows immediately from Theorem 4.1 and Proposition 5.1 by setting $\beta, \eta$ and $m$ exactly as in Theorem 4.1 and noting that $[\beta - \eta, \beta] \subset [0,1/2]$ is satisfied for sufficiently large $K$. \hfill \square

5.1 Proof of Proposition 5.1

Proof of Proposition 5.1. Given a random formula $\Phi(n,dn)$ and a random sequence $Z$ generating the order of setting the variables, let us consider $m$ independent vectors $U^0, \ldots, U^{m-1}$ which can be used to generate assignments. By definition we have

$$P(\sigma_{\Phi(n,dn),Z,U} \in SAT(\Phi(n,dn))) = \alpha_n,$$
for \( j = 0, \ldots, m - 1 \). We now construct a sequence of vectors \( V^{t,j}, 0 \leq t \leq n, 0 \leq j \leq m - 1 \), where for each \( j = 1, \ldots, m - 1 \), the sequence \( V^{t,j} \) will interpolate between vectors \( U^0 \) and \( U^j \). Specifically, let \( V^{t,j} = (V^{t,j}_1, \ldots, V^{t,j}_n) \) where \( V^{t,j}_i = U^j_i, i \leq t \) and \( V^{t,j}_i = U^0_i, t < i \leq n \). Note that for every \( t = 0, 1, \ldots, n \), \( V^{t,j} \) is a vector of i.i.d. random variables with the uniform in \([0,1]\) distribution. Furthermore, \( V^{0,j} = U^0, V^{t,0} = U^0 \), and \( V^{n,j} = U^j \). Recall the notation \( \mathcal{IR}_{x_t} \) for the influence region of variable \( x_t \), i.e., all variables whose decision is potentially influenced by assignment of \( x_t \) by the \( \tau \)-decimation algorithm. Observe that given any realizations \( u^j, 0 \leq j \leq m - 1 \) of vectors \( U^j \), and the corresponding realizations \( v^{t,j} \) of \( V^{t,j} \), we have

\[
\rho(\sigma_{\Phi, z, V^{t+1,j}}, \sigma_{\Phi, z, V^{t,j}}) \leq |\mathcal{IR}_{x_{t+1}}|, \quad 0 \leq t \leq n - 1,
\]

since, \( v^{t,j} \) and \( v^{t+1,j} \) differ only in one coordinate \( t + 1 \), and by Proposition 3.3 changing the value of \( u_{t+1} \) impacts only the decisions for variables in \( \mathcal{IR}_{x_{t+1}} \). We now consider a realization \( \Phi \) of a formula \( \Phi(n, dn) \) and realization \( z \) of the order \( Z \). \( \Phi \) and \( z \) uniquely determine sets \( \mathcal{IR}_{x_i}, 1 \leq i \leq n \). Let \( E_n \) denote the event (the set of \( \Phi \) and \( z \)) that \( \max_{1 \leq i \leq n} |\mathcal{IR}_{x_i}| \leq n^{1/3} \). By Proposition 3.3 we have

\[
\lim_{n \to \infty} \mathbb{P}(E_n) = 1.
\]

We assume without the loss of generality that \( n \) is large enough so that \( n^{1/3} < (\beta - \eta)n \). We have by property (14) of Lemma 3.1 that for every \( \Phi \) and \( z \),

\[
\mathbb{E}[\rho(\sigma_{\Phi, z, U^0}, \sigma_{\Phi, z, U^j})] = n/2,
\]

for each \( j = 1, \ldots, m - 1 \).

We first suppose that \( \Phi \) and \( z \) are realizations such that the event \( E_n \) takes place. Then, we can find \( t_0 = t_0(\Phi, z) \) such that

\[
\mathbb{E}[\rho(\sigma_{\Phi, z, U^0}, \sigma_{\Phi, z, V^{t_0,j}})] \in \left[(\beta - \eta/2)n, (\beta - \eta/2)n + n^{1/3}\right],
\]

for all \( j = 1, \ldots, m - 1 \), as by (18) the increments \( \rho(\sigma_{\Phi, z, V^{t+1,j}}, \sigma_{\Phi, z, V^{t,j}}) \) are bounded by \( n^{1/3} \) with probability one with respect to the randomness of \( V^{t,j} \). Note that \( t_0 \) does not depend on \( j \) since \( V^{t,j} \) are identically distributed for \( 1 \leq j \leq m - 1 \). Furthermore, since \( U^0 \) and \( U^j \) are identical in distribution, we also have for every \( 0 \leq j_1 < j_2 \leq m - 1 \)

\[
\mathbb{E}[\rho(\sigma_{\Phi, z, V^{t_0,j_1}}, \sigma_{\Phi, z, V^{t_0,j_2}})] \in \left[(\beta - \eta/2)n, (\beta - \eta/2)n + n^{1/3}\right].
\]

We now fix \( j_1 \neq j_2 \) and argue that in fact \( \rho(\sigma_{\Phi, z, V^{t_0,j_1}}, \sigma_{\Phi, z, V^{t_0,j_2}}) \) is concentrated around its mean as \( n \to \infty \). The distance is a function of \( n + t_0 \) i.i.d. random variables \( U^{j_1}_1, U^{j_1}_0, U^{j_2}_1, \ldots, U^{j_2}_0, U^{t_0+1}_0, \ldots, U^{t_0}_n \). Further, changing any one of these \( n + t_0 \) random variables changes the distance \( \rho \) by at most \( 2n^{1/3} \) again by Proposition 3.3 and by our assumption that \( \Phi \) and \( z \) are realizations such that the event \( E_n \) holds. Applying Azuma’s inequality

\[
\mathbb{P} \left( \left| \rho(\sigma_{\Phi, z, V^{t_0,j_1}}, \sigma_{\Phi, z, V^{t_0,j_2}}) - (\beta - \eta/2)n \right| \geq \frac{\eta n}{4} \right) \leq 2 \exp \left( -\frac{(\frac{\eta n}{4} - 2n^{1/3})^2}{2(n + t_0)n^{1/3}} \right) = \exp(-\delta n^{1/3} + o(n^{1/3})))
\]
for some constant \( \delta > 0 \), and the concentration is established. The event

\[
\left| \rho(\sigma_{\Phi, z}, V_{t_0, j_1}, \sigma_{\Phi, z}, V_{t_0, j_2}) - (\beta - \eta/2) n \right| < \frac{\eta}{4} n
\]

implies the event

\[
\rho(\sigma_{\Phi, z}, V_{t_0, j_1}, \sigma_{\Phi, z}, V_{t_0, j_2}) \in [(\beta - \eta)n, \beta n].
\]

We conclude that for every \( \Phi \) and \( z \) such that the event \( \mathcal{E}_n \) takes place, we have

\[
\lim_{n} \mathbb{P} \left( \rho(\sigma_{\Phi, z}, V_{t_0, j_1}, \sigma_{\Phi, z}, V_{t_0, j_2}) \in [(\beta - \eta)n, \beta n] \right) = 1.
\]

(20)

Since \( m \) does not depend on \( n \), we obtain by union bound

\[
\lim_{n} \mathbb{P} \left( \forall 0 \leq j_1 \neq j_2 \leq m - 1, \ \rho(\sigma_{\Phi, z}, V_{t_0, j_1}, \sigma_{\Phi, z}, V_{t_0, j_2}) \in [(\beta - \eta)n, \beta n] \right) = 1.
\]

(21)

For completion, let us set \( t_0 = 0 \) when \( \Phi \) and \( z \) are such that the event \( \mathcal{E}_n \) does not take place. Let now \( T = t_0(\Phi(n, dn), z) \) to be thus defined random variable. This way we have assignments \( \sigma_{\Phi, z, V_{T, j}}, 0 \leq j \leq m - 1 \) defined for all realizations of \( \Phi \) and \( z \), in particular whether the event \( \mathcal{E}_n \) takes place or not. Since the former is the high probability event, we conclude from above that

\[
\lim_{n} \mathbb{P} \left( \forall 0 \leq j_1 \neq j_2 \leq m - 1, \ \rho(\sigma_{\Phi(n, dn), z, V_{T, j_1}}, \sigma_{\Phi(n, dn), z, V_{T, j_2}}) \in [(\beta - \eta)n, \beta n] \right) = 1.
\]

(22)

Thus, we established that with high probability as \( n \to \infty \) there exist a sequence of assignments \( \sigma^j \triangleq \sigma_{\Phi(n, dn), z, V_{T, j}}, 0 \leq j \leq m - 1 \) satisfying property (b) of the definition of \( \text{SAT}(\Phi; \beta, \eta, m) \).

Our next goal is to show that the assignments \( \sigma^j, 0 \leq j \leq m - 1 \) above are also satisfying formula \( \Phi(n, dn) \) with probability bounded away from zero as \( n \to \infty \). To be exact we claim

\[
\lim_{n} \inf \mathbb{P}(\sigma_{\Phi(n, dn), z, V_{T, j}} \in \text{SAT}(\Phi), 0 \leq j \leq m - 1) > 0.
\]

(23)

namely property (a) holds with probability bounded away from zero, and thus the set \( \text{SAT}(\Phi; \beta, \eta, m) \) is non-empty with probability bounded away from zero, as claimed. Observe that \( \sigma_{\Phi(n, dn), z, V_{T, j}} \) have identical distribution for all \( j \). Furthermore, each of them individually is distributed as \( \sigma_{\Phi(n, dn), z, U_{j}} \), \( 0 \leq j \leq m - 1 \) since the random variable \( T \) only affects the indices \( i \) for which we switch from \( U_{i}^0 \) vs \( U_{i}^1 \), and since each vector \( U^j \) is an i.i.d. vector of random variables. Therefore,

\[
\mathbb{P}(\sigma_{\Phi(n, dn), z, V_{T, j}} \in \text{SAT}(\Phi(n, dn))) = \alpha_n,
\]

for each \( j \). Suppose \( \Phi, z \) are such that the event \( \mathcal{E}_n \) takes place and fix the corresponding deterministic value \( t_0 = t_0(\Phi, z) \). In the derivation below we use notation \( \mathbb{P}_Z \) to indicate probability with respect random variable \( Z \). We have

\[
\begin{align*}
\mathbb{P}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}(\sigma_{\Phi, z, V_{T, j}} \in \text{SAT}(\Phi), 0 \leq j \leq m - 1) &= \mathbb{E}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}[1(\sigma_{\Phi, z, V_{t_0, j}} \in \text{SAT}(\Phi), 0 \leq j \leq m - 1)] \\
&= \mathbb{E}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}[\mathbb{E}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}[1(\sigma_{\Phi, z, V_{t_0, j_1}} \in \text{SAT}(\Phi), 0 \leq j_1 \leq m - 1 \ | \ U_{t_0}^0, \ldots, U_{t_0}^{m-1})] \\
&= \mathbb{E}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}[\mathbb{E}_{U_{t_0}^0, \ldots, U_{t_0}^{m-1}}[1(\sigma_{\Phi, z, U_{t_0}^0} \in \text{SAT}(\Phi)) \ | \ U_{t_0}^0, \ldots, U_{t_0}^{m-1}]] \\
&= \mathbb{E}_{U_{t_0}^0}[1(\sigma_{\Phi, z, U_{t_0}^0} \in \text{SAT}(\Phi))] \\
&= \mathbb{P}_{U_{t_0}^0}(\sigma_{\Phi, z, U_{t_0}^0} \in \text{SAT}(\Phi)).
\end{align*}
\]

(24)
Here (a) follows since \( U_j \) are independent vectors of i.i.d. random variables and (b) follows by applying Jensen’s inequality and the convexity of the polynomial function \( t^m \) on \( t \in [0, \infty) \) for all positive integers \( m \).

Suppose now \( \Phi \) and \( z \) are such that the event \( \mathcal{E}_n \) does not take place. Then \( \sigma_{\Phi, z, V^T, j} = \sigma_{\Phi, z, U^0} \), implying

\[
P_{U^0, \ldots, U^{m-1}}(\sigma_{\Phi, z, V^T, j} \in \text{SAT}(\Phi), \ 0 \leq j \leq m - 1) = P_{U^0}(\sigma_{\Phi, z, U^0} \in \text{SAT}(\Phi)) \\
\geq P_{U^0}(\sigma_{\Phi, z, U^0} \in \text{SAT}(\Phi)).
\]

Combining with (24) we conclude that for every \( \Phi, z \) we have

\[
P_{U^0, \ldots, U^{m-1}}(\sigma_{\Phi, z, V^T, j} \in \text{SAT}(\Phi), \ 0 \leq j \leq m - 1) \geq P_{U^0}(\sigma_{\Phi, z, U^0} \in \text{SAT}(\Phi)).
\]

Since \( P_{U^0}(\sigma_{\Phi, z, U^0} \in \text{SAT}(\Phi)) = \alpha_n \), then integrating over \( \Phi(n, dn) \) and \( Z \), we obtain

\[
P(\sigma_{\Phi(n, dn), z, V^T, j} \in \text{SAT}(\Phi(n, dn)), \ 0 \leq j \leq m - 1) \geq P(\sigma_{\Phi(n, dn), z, U^0} \in \text{SAT}(\Phi(n, dn))) = \alpha_m^n,
\]

implying

\[
\liminf_n P(\sigma_{\Phi(n, dn), z, V^T, j} \in \text{SAT}(\Phi(n, dn)), \ 0 \leq j \leq m - 1) \geq \liminf_n \alpha_m^n > 0,
\]

and (23) is established.

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