THE GROUP OF ISOMETRIES OF A LOCALLY
COMPACT METRIC SPACE WITH ONE END

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Abstract. In this note we study the dynamics of the natural
evaluation action of the group of isometries $G$ of a locally compact
metric space $(X, d)$ with one end. Using the notion of pseudo-
components introduced by S. Gao and A. S. Kechris we show that
$X$ has only finitely many pseudo-components exactly one of which
is not compact and $G$ acts properly on this pseudo-component. The
complement of the non-compact component is a compact subset of
$X$ and $G$ may fail to act properly on it.

1. Preliminaries and the main result

The idea to study the dynamics of the natural evaluation action of
the group of isometries $G$ of a locally compact metric space $(X, d)$
with one end, using the notion of pseudo-components introduced by
S. Gao and A. S. Kechris in [4], came from a paper of E. Michael [8].
In this paper he introduced the notion of a $J$-space, i.e. a topological
space with the property that whenever $\{A, B\}$ is a closed cover of
$X$ with $A \cap B$ compact, then $A$ or $B$ is compact. In terms of compactifications locally compact non-compact $J$-spaces are characterized
by the property that their end-point compactification coincides with
their one-point compactification (see [8, Proposition 6.2], [9, Theorem
6]). Recall that the Freudenthal or end-point compactification of a locally compact non-compact space $X$ is the maximal zero-dimensional
compactification $\varepsilon X$ of $X$. By zero-dimensional compactification of $X$ we here mean a compactification $Y$ of $X$ such that $Y \setminus X$ has a base of closed-open sets (see [7], [9]). The points of $\varepsilon X \setminus X$ are the
ends of $X$. From the topological point of view locally compact spaces

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with one end are something very general since the product of two non-compact locally compact connected spaces is a space with one end (see [9, Proposition 8], [8, Proposition 2.5]), so it is rather surprising that the dynamics of the action of the group of isometries \( G \) of a locally compact metric space \((X, d)\) with one end has a certain structure as our main result shows.

**Theorem 1.1.** Let \((X, d)\) be a locally compact metric space with one end and let \( G \) be its group of isometries. Then

(i) \( X \) has finitely many pseudo-components exactly one of which is not compact and \( G \) is locally compact.

(ii) Let \( P \) be the non-compact pseudo-component. Then \( G \) acts properly on \( P \), \( X \setminus P \) is a compact subset of \( X \) and \( G \) may fail to act properly on it.

Let us now recall some basic notions. Let \((X, d)\) be a locally compact metric space let \( G \) be its group of isometries. If we endow \( G \) with the topology of pointwise convergence then \( G \) is a topological group (see [2, Ch. X, §3.5 Corollary]). On \( G \) there is also the topology of uniform convergence on compact subsets which is the same as the compact–open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural evaluation action of \( G \) on \( X \), \( G \times X \to X \) with \((g, x) \mapsto g(x)\), \( g \in G \), \( x \in X \) is continuous (see [2, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]). An action by isometries is proper if and only if the limit sets \( L(x) = \{ y \in X \mid \text{there exists a net} \{ g_i \} \text{in} G \text{ with} g_i \to \infty \text{ and} \lim g_i x = y \} \) are empty for every \( x \in X \), where \( g_i \to \infty \) means that the net \( \{ g_i \} \) has no cluster point in \( G \) (see [5]). A few words about pseudo-components. They were introduced by S. Gao and A. S. Kechris in [4] and we used them in [5] to study the dynamics of the action of the group of isometries of a locally compact metric space. For the convenience of the reader we repeat what a pseudo-component is. For each point \( x \in X \) we define the radius of compactness \( \rho(x) \) of \( x \) as \( \rho(x) := \sup \{ r > 0 \mid B(x, r) \text{ has compact closure} \} \) where \( B(x, r) \) denotes the open ball centered at \( x \in X \) with radius \( r > 0 \). If \( \rho(x) = +\infty \) for some \( x \in X \) then every ball has compact closure (i.e. \( X \) has the Heine-Borel property), hence \( \rho(x) = +\infty \) for every \( x \in X \). In case where \( \rho(x) \) is finite for some \( x \in X \) then the radius of compactness is a Lipschitz function [4 Proposition 5.1]. It is also easy to see that \( \rho(gx) = \rho(x) \) for every \( g \in G \). We define next an equivalence relation \( \mathcal{E} \) on \( X \) as follows: Firstly we define a directed graph \( \mathcal{R} \) on \( X \) by \( xRy \) if and only if \( d(x, y) < \rho(x) \). Let \( \mathcal{R}^* \) be the transitive closure of \( \mathcal{R} \), i.e. \( xR^*y \) if and only if for some \( u_0 = x, u_1, \ldots, u_n = y \) we have \( u_i \mathcal{R} u_{i+1} \)
for every $i < n$. Finally, we define the following equivalence relation $E$ on $X$: $x E y$ if and only if $x = y$ or $(x R^* y$ and $y R^* x)$. We call the $E$-equivalence class of $x \in X$ the pseudo-component of $x$, and we denote it by $C_x$. It follows that pseudo-components are closed-open subsets of $X$, see [4, Proposition 5.3] and $gC_x = C_{gx}$ for every $g \in G$.

Before we give the proof of Theorem 1.1 we need some results that may be of independent interest.

**Lemma 1.2.** Let $X$ be a non-compact $J$-space and let $\mathcal{A} = \{A_i; i \in I\}$ be a partition of $X$ with closed-open non-empty sets. Then $\mathcal{A}$ contains only finitely many sets exactly one of which is not compact; its complement is a compact subset of $X$.

**Proof.** We show firstly that there exists a set in $\mathcal{A}$ which is not compact. We argue by contradiction. Assume that every set $B \in \mathcal{A}$ is compact. Then $\mathcal{A}$ contains infinitely many distinct sets because otherwise $X$ must be a compact space. Let $\{B_n, n \in \mathbb{N}\} \subset \mathcal{A}$ with $B_n \neq B_k$ for $n \neq k$ (i.e. $B_n \cap B_k = \emptyset$). The sets $D := \bigcup_{n=1}^{+\infty} B_{2n-1}$ and $X \setminus D$ are open (since $X \setminus D$ is a union of elements of $\mathcal{A}$) and disjoint so they form a closed partition of $X$. Hence, one of them must be compact. This is a contradiction because both $D$ and $X \setminus D$ are an infinite disjoint union of open sets.

Fix a non-compact $P \in \mathcal{A}$. Since $P$ is a closed-open subset of $X$ then $\{P, X \setminus P\}$ is a closed partition of $X$. Hence $P$ or $X \setminus P$ must be compact. But $P$ is non-compact so $X \setminus P$ is compact. If $K \in \mathcal{A}$ with $K \neq P$ then $K \subset X \setminus P$. Therefore, $K$ is compact. Moreover $\mathcal{A}$ contains finitely many sets, since $X \setminus P$ is compact and $\mathcal{A}$ is a partition of $X$ with closed-open non-empty sets. \qed

The previous lemma makes $X$ a second countable space (i.e. $X$ has a countable base):

**Proposition 1.3.** A metrizable locally compact $J$-space has a countable base.

**Proof.** Sierpinski has proved in [10] that every metrizable locally separable space $X$ can be represented as a disjoint union of open separable subsets. Then Lemma 1.2 implies that we have here only finitely many of these sets, and hence, $X$ is second countable. \qed

The proof of Theorem 1.1 is heavily based on the next proposition. Its proof can be found in [5, Theorem 1.3] but we repeat it here for the convenience of the reader.
Proposition 1.4. Let \((X, d)\) be a locally compact metric space and let \(G\) denote its group of isometries. Let \(x, y \in X\) and a net \(\{g_i\}\) in \(G\) such that \(g_i x \to y\). Then there exist a subnet \(\{g_j\}\) of \(\{g_i\}\) and a map \(f : C_x \to X\) which preserves the distance such that \(g_j \to f\) pointwise on \(C_x\), \(f(x) = y\) and \(f(C_x) = C_{f(x)}\), where \(C_x\) and \(C_y\) denote the pseudo-components of \(x\) and \(y\) respectively. In the case where \(X\) has, moreover, a countable base and \(\{g_i\}\) is a sequence, then there exist a subsequence \(\{g_{i_k}\}\) of \(\{g_i\}\) and a map \(f : C_x \to X\) which preserves the distance such that \(g_{i_k} \to f\) pointwise on \(C_x\), \(f(x) = y\) and \(f(C_x) = C_{f(x)}\).

Proof. Let \(F\) be a subset of \(G\). We define \(K(F)\) to be the set
\[
K(F) := \{x \in X \mid \text{the set } Fx \text{ has compact closure in } X\}.
\]
Each \(K(F)\) is a closed-open subset of \(X\) (see \cite{6} Lemma 3.1, \cite{11}).

Let \(x, y \in X\) and \(\{g_i\}\) be a net in \(G\) with \(g_i x \to y\). Since \(X\) is locally compact there exists an index \(i_0\) such that the set \(F(x)\), where \(F := \{g_i \mid i \geq i_0\}\), has compact closure. We claim that for every \(z \in C_x\) the set \(F(z)\) has, also, compact closure in \(X\), hence \(C_x \subset K(F)\). The strategy is to start with an open ball \(B(x, r)\) centered at \(x\) with radius \(r < \rho(x)\), where \(\rho(x)\) is the radius of compactness of \(x\) and prove that \(F(z)\) has compact closure for every \(z \in B(x, r)\). Then, our claim follows just from the definition of the pseudo-component of \(x\). To prove the claim take a sequence \(\{g_n z\}\), with \(g_n \in F\) for every \(n \in \mathbb{N}\). Since the closure of \(F(x)\) is compact we may assume, without loss of generality, that \(g_n x \to w\) for some \(w\) in the closure of \(F(x)\). Assume that \(\rho(x)\) is finite and take a positive number \(\varepsilon\) such that \(r + \varepsilon < \rho(x)\). Then for \(n\) big enough
\[
d(g_n z, w) \leq d(g_n z, g_n x) + d(g_n x, w) = d(z, x) + d(g_n x, w) < r + \varepsilon < \rho(x).
\]
Recall that the radius of convergence is a continuous map, and since \(g_n x \to w\) then \(\rho(x) = \rho(w)\). So, the sequence \(\{g_n z\}\) is contained eventually in a ball of \(w\) with compact closure, hence it has a convergence subsequence. The same also holds in the case where \(\rho(x) = +\infty\) and the claim is proved.

Set \(A := K(F)\). By \cite{6} Lemma 3.1 \(A\) is a closed-open subset of \(X\). If \(g_i|_A\) denotes the restriction of each \(g_i\) on \(A\), then the Arzela-Ascoli theorem implies that the set \(\{g_i|_A : A \to X \mid i \geq i_0\}\) has compact closure in \(C(A, X)\) (this the set of all continuous maps from \(A\) to \(X\)). Thus, there exist a subnet \(\{g_j\}\) of \(\{g_i\}\) and a map \(f : A \to X\) with \(f(x) = y\) which preserves the distance such that \(g_j \to f\) pointwise on \(A\). Hence, \(g_j \to f\) pointwise on \(C_x\). If, moreover, \(X\) has a countable base then it is \(\sigma\)-compact, i.e. it can be written as a countable union of compact subsets. Since \(A = K(F)\) is a closed-open subset of \(X\)
then it also a \( \sigma \)-compact locally compact metrizable space. Hence, by [3] Theorem 5.2 p. 265 and 8.5 p. 272, \( C(A, X) \) is a metrizable space with a countable base. So, if \( \{g_i\} \) is a sequence there exist a subsequence \( \{g_{i_k}\} \) of \( \{g_i\} \) and a map \( f : C_x \to X \) which preserves the distance such that \( g_{i_k} \to f \) pointwise on \( C_x \).

Let us show that \( f(C_x) = C_{f(x)} \). Since \( d(x, g_j^{-1}f(x)) = d(g_jx, f(x)) \) and \( d(g_jx, f(x)) \to 0 \) it follows that \( g_j^{-1}f(x) \to x \). Repeating the previous procedure, there exist a subnet \( \{g_k\} \) of \( \{g_j\} \) and a map \( h : C_{f(x)} \to X \) which preserves the distance such that \( g_k^{-1} \to h \) pointwise on \( C_{f(x)} \) and \( h(f(x)) = x \). Note that since \( g_kx \to f(x) \) and the pseudo-component \( C_{f(x)} \) is a closed-open subset of \( X \) then \( g_kx \in C_{f(x)} \) eventually for every \( k \). Therefore, \( g_kC_x = C_{g_kx} = C_{f(x)} \). Take a point \( z \in C_x \). Then \( g_kz \to f(z) \) and since the pseudo-component \( C_{f(x)} \) is a closed-open subset of \( X \) then \( f(z) \in C_{f(x)} \), so \( f(C_x) \subset C_{f(x)} \). In a similar way and repeating the same arguments as before it follows that \( hC_{f(x)} \subset C_x \). Take now a point \( w \in C_{f(x)} \). Then \( h(w) \in C_x \), hence \( g_k^{-1}(w) \in C_x \) eventually for every \( k \). So, \( w = g_kg_k^{-1}(w) \to f(h(w)) \in f(C_x) \) from which follows that \( C_{f(x)} \subset f(C_x) \).

Proof of Theorem 1.1. (i) Since every pseudo-component is a closed-open subset of \( X \) we can apply Lemma 1.2 for the family of the pseudo-components of \( X \). Hence, \( X \) has finitely many pseudo-components exactly one of which, say \( P \), is not compact and its complement \( X \setminus P \) is a compact subset of \( X \). Take any \( g \in G \). Then \( gP \) is a non-compact pseudo-component hence \( gP = P \). This shows that \( P \) is \( G \)-invariant. Then, by [4] Corollary 6.2, \( G \) is locally compact since \( X \) has finitely many pseudo-components.

(ii) We shall show that \( G \) acts properly on \( P \). By Proposition 1.3, the space \((X, d)\) has a countable base. Hence, as we mentioned in the proof of Proposition 1.4 by [3] Theorem 5.2 p. 265 and 8.5 p. 272, \( G \) is a metrizable locally compact group with a countable base. So, if we would like to check if \( G \) acts properly on \( P \) it is enough to consider sequences in \( G \) instead of nets. Assume that there are points \( x, y \in P \) and a sequence \( \{g_n\} \) in \( G \) with \( g_nx \to y \). Let us denote by \( \{P, C_1, C_2, \ldots, C_k\} \) the pseudo-components of \( X \). Each pseudo-component \( C_i, i = 1, \ldots, k \) is compact. Choose points \( x_i \in C_i, i = 1, \ldots, k \). Since \( X \setminus P \) is compact we may assume that there exist points \( y_i \in X \setminus P, i = 1, \ldots, k \) and a subsequence \( \{g_{n_i}\} \) of \( \{g_n\} \) such that \( g_{n_i}x_i \to y_i \) for every \( i = 1, \ldots, k \). Since by Proposition 1.3 \( X \) has a countable base then, by Proposition 1.4 there is a subsequence of \( \{g_{n_m}\} \) of \( \{g_n\} \) and a map \( f : X \to X \) which preserves the distance such that \( g_{n_m} \to f \) pointwise on \( X \). Note that \( g_{n_m}^{-1}y \to x \in P \), since
\[ d(g_{n_m}^{-1}y, x) = d(y, g_{n_m}x). \] Repeating the previous arguments we conclude that there exists a map \( h : X \to X \) and a subsequence \( \{g_{n_{m_p}}\} \) of \( \{g_{n_m}\} \) such that \( g_{n_{m_p}}^{-1} \to h \) pointwise on \( X \) and \( h \) preserves the distance. Obviously \( h \) is the inverse map of \( f \), hence \( f \in G \) and \( G \) acts properly on \( P \). The group \( G \) may fail to act properly on \( X \setminus P \). As an example we may take as \( X = P \cup S \subset \mathbb{R}^3 \), where \( P \) is the plane \( \{(x, y, 0) | x, y \in \mathbb{R}\} \) and \( S \) is the circle \( \{(x, y, 2) | x^2 + y^2 = 1\} \). We endow \( X \) with the metric \( d = \min\{d_E, 1\} \), where \( d_E \) is the usual Euclidean metric on \( \mathbb{R}^3 \). Then the action of \( G \) on \( S \) is not proper, since for a point \( x \in S \) the isotropy group \( G_x := \{g \in G | gx = x\} \) is not compact. 

\[ \square \]

**Remark 1.5.** If \( G \) does not act properly on \( X \setminus P \) one may ask if the orbits on \( X \setminus P \) are closed or if the isotropy groups of points \( x \in X \setminus P \) are non-compact. The answer is negative in general. As an example we may consider the example in [1]. In this paper we constructed a one-dimensional manifold with two connected components, one compact isometric to \( S^1 \), and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. We can regard the real line as a distorted helix with a locally Euclidean metric. The problem is that this manifold has two ends. But this is not really a problem. Following the same arguments as in [1] we can replace the distorted helix by a small distorted helix-like stripe and have a space with one end and two connected components, one compact isometric to \( S^1 \), and one non-compact with a locally Euclidean metric so that the group of isometries has non-closed dense orbits on the compact component.

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