Distributed Algorithms for Directed Betweenness Centrality and All Pairs Shortest Paths*

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Abstract. The betweenness centrality (BC) of a node in a network (or graph) is a measure of its importance in the network. BC is widely used in a large number of environments such as social networks, transport networks, security/mobile networks and more. We present an $O(n)$-round distributed algorithm for computing BC of every vertex as well as all pairs shortest paths (APSP) in a directed unweighted network, where $n$ is the number of vertices and $m$ is the number of edges. We also present $O(n)$-round distributed algorithms for computing APSP and BC in a weighted directed acyclic graph (dag). Our algorithms are in the CONGEST model and our weighted dag algorithms appear to be the first nontrivial distributed algorithms for both APSP and BC. All our algorithms pay careful attention to the constant factors in the number of rounds and number of messages sent, and for unweighted graphs they improve on one or both of these measures by at least a constant factor over previous results for both directed and undirected APSP and BC.

1 Introduction

There has been considerable research on designing distributed algorithms on networks for various properties of the graph represented by the network. The goal in these algorithms is to minimize the number of rounds used by the distributed algorithm.

In this paper we consider distributed algorithms for computing betweenness centrality (BC), a widely used measure of the importance of a node in a network (see definition below). We focus on directed graphs, and we use the CONGEST model (reviewed in Section 1.1). We present a $2n + O(D)$-round algorithm for computing BC in unweighted directed graphs, where $D$ is the (finite) directed diameter. The algorithm sends no more than $2mn + 2m$ messages. If $D$ is infinite (i.e., if the graph is not strongly connected), the algorithm runs in $4n$ rounds. Also, within our BC algorithm for unweighted directed graphs is an APSP algorithm that runs in $n + O(D)$ rounds when $D$ is finite, sending at most $mn + 2m$ messages. These algorithms work for undirected graphs with the same bounds.

Our distributed BC algorithm for unweighted directed graphs has been implemented and evaluated on the distributed platform D-Galois \cite{7}, and has been found to outperform earlier high-performance distributed BC implementations \cite{14}.

For weighted directed acyclic graphs (weighted dags) we present an $n + O(L)$-round algorithm for computing APSP and a $2n + O(L)$-round algorithm for BC, where $L$ is the length of a longest path from a source vertex in the dag to any other vertex.

Betweenness Centrality. Let $G = (V, E)$ be a directed graph with $|V| = n$, $|E| = m$, and with a positive edge weight $w(e)$ on each edge $e \in E$. Let $\sigma_{xy}$ denote the number of shortest paths (SPs)

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from $x$ to $y$ in $G$, and $\sigma_{xy}(v)$ the number of SPs from $x$ to $y$ in $G$ that pass through $v$, for each pair $x, y \in V$. Then, $BC(v) = \sum_{\substack{x \neq u, t \neq v}} \frac{\sigma_{st}(v)}{\sigma_{st}}$.

The measure $BC(v)$ is often used as a parameter that determines the relative importance of $v$ in $G$ relative to the presence of $v$ on shortest paths, and is computed for all $v \in V$. Some applications of BC include analyzing social interaction networks [21], identifying lethality in biological networks [31], identifying key actors in terrorist networks [622], and identifying and preventing security attacks on mobile networks [32]. BC is also used for identifying community structure in social and biological networks using the Girvan-Newman algorithm [13], and for understanding road network patterns of traffic analysis zones [35]. Many of the above systems are usually represented as directed networks (see Section 7 in [26]), and this motivates our interest in studying distributed solutions for computing BC in directed graphs. The widely used sequential algorithm for BC is the one by Brandes [4] but no nontrivial distributed algorithm was known for directed graphs prior to our results.

### 1.1 CONGEST Model

We start with some definitions. Let $G = (V, E)$ be a directed unweighted graph. For a node $u \in V$ we define $\Gamma_{\text{in}}(u) = \{v \in V \mid (u, v) \in E\}$ as the set of incoming neighbors of $u$ and $\Gamma_{\text{out}}(u) = \{v \in V \mid (v, u) \in E\}$ as the set of the outgoing neighbors of $u$. Let $G$ be the undirected version of $G$. A digraph $G$ is weakly connected if $G$ is connected. A digraph $G$ is strongly connected if it contains at least one directed path $u \rightarrow v$ and at least one directed path $v \rightarrow u$ for each pair of nodes $u, v \in V$. Similarly, we can define a weakly connected component (wcc) and strongly connected component (scc) in a digraph. For a path $\pi_{xy}$ from $x$ to $y$, the distance $d(x, y)$ is the sum of all edge weights in the path, while the length $\ell(\pi_{xy})$ is the number of edges in $\pi_{xy}$. For a dag $G$, we call $L$ the length of a longest (in terms of number of edges) path in $G$. We indicate the shortest path distance from $x$ to $y$ as $\delta(x, y)$, with $\delta(x, y) = \infty$ if there is no path. We use $D$ to denote the diameter of the directed graph $G$, while if the graph is undirected we use $D_u$.

In the CONGEST model a network of processors is generally modeled by an undirected graph $G = (V, E)$, with $|V| = n$ nodes and $|E| = m$ edges. If $G$ is weighted then each edge has a positive integer weight, which is often restricted to a poly($n$) value. Each node $v \in V$ has a unique ID in $\{1, \ldots, \text{poly}(n)\}$ and infinite computational power. If the graph $G = (V, E)$ is directed then it is assumed that the communication channels (edges in $G$) are bidirectional, i.e., the communication network is represented by the undirected graph $G$.

In the CONGEST model the size of a message is bounded: each node can send along each edge at most $O(B)$ bits in a given round. Usually, $B = \log n$ but sometimes $B = \log n + \log R$, where $R$ is an upper bound on the largest value that can naturally occur within the computation. Given the limit on the amount of data that can be transferred in a message, this model considers congestion issues, which occur when a long queue of messages (each of size at most $B$) is scheduled to be sent by the same node over the same edge. The performance of an algorithm is measured by the number of rounds it needs. In a single round a node $v \in V$ can receive a message of size $O(B)$ along each incoming edge $(u, v)$. Node $v$ processes its received messages (instantaneously, given its infinite computational power) and then sends a (possibly different) $O(B)$ bit message along each of its incident edges, or remains silent. The goal is to design distributed algorithms for the graph $G$ using a small number of rounds, and the round complexity in this model has been studied extensively [28]. For convenience, we assume that the vertices are numbered from 1 to $n$ and we denote the vertex $i$ by $v_i$. 

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2 Our Results

The following two theorems state our main results (see also Table 1). Here $D$ is the directed diameter in a directed graph and $D_u$ is the undirected diameter in an undirected graph.

- **Unweighted Directed Graphs:**

**Theorem 1.** On an unweighted graph $G$ with $n$ nodes and $m$ edges,

(I) Algorithm 5 computes directed APSP with the following bounds in the CONGEST model:

1. If $n$ is known, in $\min\{n + O(D), 2n\}$ rounds while sending $mn + O(m)$ messages in any graph.
2. If $n$ is known, in $2n$ rounds while sending at most $mn$ messages in any graph (by omitting Steps 4 and 10).
3. If $n$ is not known, in $n + O(D)$ rounds while sending at most $mn + O(m)$ messages if $G$ is strongly connected.

(II) Algorithm 5 computes BC values of all vertices with at most twice the number of rounds and messages as in part (I) for each of the three cases.

(III) If $G$ is undirected the bounds for rounds and messages in parts (I) and (II) hold with $D$ replaced by $D_u$.

Parts I.1 and I.3 of Theorem 1 improve over the $2n$-round algorithm in [24] while sending a smaller number of messages. The number of messages sent is also improved for undirected graphs when compared to [24], where up to $2mn$ messages or $mn + O(m \cdot D_u)$ messages can be sent. Moreover, part I.3 of Theorem 1 computes APSP without knowing $n$ when $D$ is bounded: this case is not considered in [24] where knowledge of $n$ is needed for directed APSP. For message count, Part I.2 of Theorem 1 further reduces the number of messages to at most one message sent by each node for each source.

- **Weighted Directed Acyclic Graphs:**

**Theorem 2.** Let $L$ be the number of edges in a longest path in a directed acyclic graph (dag). Given a weighted dag on $n$ vertices,

1. If $n$ is known, Algorithm 7 computes APSP in $n + O(L)$ rounds in the CONGEST model. It sends at most $mn + m$ message.
2. If $n$ is known, using Algorithms 4 and 5, the BC values of all nodes can be computed in $2n + O(L)$ rounds in the CONGEST model while sending at most $2mn + m$ messages.
3. If $n$ is not known, Algorithm 7 computes APSP in $O(n)$ rounds in the CONGEST model. It sends at most $O(mn)$ message.
4. If $n$ is not known, using Algorithms 7 and 5, the BC values of all nodes can be computed in $O(n)$ rounds in the CONGEST model while sending at most $O(mn)$ messages.

Note that all our algorithms are in the broadcast CONGEST model and this further allows our results to map into the $k$-machine model [20].
| Graph Type                     | Problem | Previous Results | Our Results |
|-------------------------------|---------|------------------|-------------|
|                               | [Problem] | [Previous Results] | [Our Results] |
|                               |         | [Rounds] | [Messages] | [Req. n] | [Rounds] | [Messages] | [Req. n] |
| Unweighted directed graphs    | APSP    | $2n \ [24]$ | $\leq 2mn$ | yes | $> \min\{2n, n + 5D\}$ | $\leq mn + 4m$ | no |
|                               |         |         |         |     | $> 2n$ | $\leq mn$ | yes |
|                               | BC      | $O(m)$ (trivial) | $\leq m^2$ | no | $> \min\{4n, 2n + 7D\}$ | $\leq 2mn + 4m$ | no |
|                               |         |         |         |     | $> 4n$ | $\leq 2mn$ | yes |
| Unweighted undirected graphs  | APSP    | $n + O(D_u) \ [24]$ | $\leq mn + O(mD_u)$ | no | $> \min\{2n, n + 5D_u\}$ | $\leq mn + 4m$ | no |
|                               |         |         |         |     | $> 2n$ | $\leq mn$ | yes |
|                               | BC      | $O(n)$ (≥ 6n) \ [17] | $–$ | no | $> \min\{4n, 2n + 7D_u\}$ | $\leq 2mn + 4m$ | no |
|                               |         |         |         |     | $> 4n$ | $\leq 2mn$ | yes |
| Weighted dags                 | APSP    | $O(m)$ (trivial) | $\leq m^2$ | no | $> n + 2L$ | $\leq mn + m$ | yes |
|                               |         |         |         |     | $> O(n)$ | $O(mn)$ | no |
|                               | BC      | $O(m)$ (trivial) | $\leq m^2$ | no | $> 2n + 3L$ | $\leq 2mn + m$ | yes |
|                               |         |         |         |     | $> O(n)$ | $O(mn)$ | no |

Table 1. A summary of our results in the CONGEST model. Here $D$ ($D_u$) is the directed (undirected) diameter of a directed (undirected) graph (if it is finite), and $L$ is the longest length of a path in a dag. In our full graphs results, there are two bounds for each case. The first always refers to a weakly-connected directed graph without knowing $n$, the second to any directed graph knowing $n$. The columns ‘Req. n’ indicate if the result requires the knowledge of $n$ a priori.

**Undirected versus Directed APSP (and BC).** As noted earlier, the APSP algorithm in [24] is a correct $2n$-round algorithm for unweighted directed graphs even though it was presented as an undirected APSP algorithm. By using the height of a BFS-tree as a 2-approximation of $D_u$, an alternate $n + O(D_u)$-round bound is obtained in [24] for APSP in an undirected connected graph. However, this result does not hold for directed BFS and directed diameter. Instead, our Algorithm 4 uses a different method to achieve an $n + O(D)$-round bound for directed strongly-connected graphs. There are other $O(n)$-round undirected APSP algorithms [15,29] but these require bidirectional edges and do not work for directed graphs (for example, the use of distances along a pebble traversal of a BFS tree in the proof of Lemma 1 in [15]). Similarly, the undirected BC algorithm in [17] does not work for directed graphs even if we substitute a directed APSP algorithm since their method for the accumulation phase is tied to the undirected APSP method in [15].

In Section 4 we present the first nontrivial distributed algorithm for BC in unweighted directed graphs. At the same time we also improve the round and/or message complexity (by a constant factor) for APSP in both undirected and directed graphs and for BC in undirected graphs. Prior to our work, the best previous CONGEST algorithms for unweighted APSP were in [24] and the only nontrivial CONGEST algorithm for BC was the undirected unweighted BC algorithm in [17].

### 2.1 New Techniques

Our main contributions are in the introduction of new pipelining methods for orchestrating the message passing in the distributed network leading to new or improved algorithms in several settings.

- For dags, a new pipelining technique where global delays are computed using distances computed in a longest length tree (LLT) rooted at a node (see Section 5). This technique could be applicable to other classes of graphs where an LLT can be computed efficiently. We apply this technique to present the first $O(n)$ round APSP and BC algorithms for weighted dags, with $n + O(L)$ rounds for APSP and $2n + O(L)$ rounds for BC in a dag.
A simple timestamp pipelining technique based on reversing global delays that occur during a forward execution of a distributed algorithm. This general method is applicable when certain specific operations have to be back-propagated during a reverse pass of the algorithm. We use this technique in the Accumulation Phase for the BC scores following an APSP computation (Section 4.3).

Refining the pipelining technique in the Lenzen-Peleg algorithm [24] to obtain a faster (by a constant factor) and simpler algorithm for computing APSP in unweighted directed graphs (see Section 4.2). This refinement reduces the number of rounds to $n + O(D)$ and also reduces the number of messages sent to at most $mn + 2m$: essentially just one message for each source is sent from each vertex along its outgoing edges. We can similarly improve the bounds for the source detection task studied in [24]; we do not discuss this further in this paper. Building on our streamlined APSP algorithm, our BC algorithm runs in $2n + O(D)$ rounds and sends at most $2mn + 2m$ messages overall. Our directed APSP algorithm also gives the best bound for number of rounds and messages for undirected APSP, and with our reversed pipelining method we also improve on the bound in [17] for undirected BC by a constant factor.

### 2.2 Related Work

Distributed algorithms for undirected graphs in the CONGEST model have received considerable attention [28,29,15,27,12,24].

For an unweighted undirected graph, a $\tilde{O}(n)$-round algorithms for approximate APSP can be found in [23] and [27]. Moreover, a lower bound of $\Omega\left(\frac{n}{\log n}\right)$ for computing diameter was established in [10], which implies a lower bound for solving APSP, and nearly optimal algorithms for this problem, running in $O(n)$ rounds, were given in [15] and [24]. The constant factor in the number of rounds was improved to $n + O(D_u)$ in [24]. Recently, a result matching the $\Omega(n/\log n)$ lower bound for APSP in unweighted undirected graphs was given in [18]. Additionally, for an unweighted undirected graph, $O(n)$ round APSP algorithms were given in [15,29]. The constant factor in the number of rounds was improved to $n + O(D_u)$ in [24]. Lower bounds of $\Omega(n/\log n)$ for computing diameter and APSP are given in [10,18].

For unweighted directed graphs, we became aware that the APSP algorithm claimed for undirected graphs in [24] in fact works for directed graphs. We observe that the bound on the number of rounds is $2n$, and the improved $n + O(D)$ bound obtained in [24] for undirected graphs does not hold for directed graphs (if all nodes need to know that the computation has terminated). Other distributed algorithms for path problems in directed graphs can be found in [27,12,5]. Very recently, a $\tilde{O}(n)$-round randomized APSP algorithm was announced in [3].

For a weighted directed (or undirected) graph, the exact APSP problem can be trivially solved in the CONGEST model in $O(m)$ rounds using an aggregation technique, where the entire network is aggregated at a single node. Even for weighted dags, prior to our results no exact algorithms were known except for the trivial one. Randomized algorithms solved exactly the APSP problem in weighted graphs in $\tilde{O}(n^{5/3})$ rounds [9], later improved to $\tilde{O}(n^{5/4})$ with a Las Vegas algorithm in [19]. A fully deterministic algorithm for solving APSP in weighted graphs in $\tilde{O}(n^{3/2})$ rounds has appeared in [2]. These deterministic results have been further improved for moderate edge-weights and distances in [1].

Distributed BC algorithms for BC from a a practical prospective are given in [33] and [34]. Recently, for unweighted undirected graphs an $O(n)$-round algorithm for computing BC in the
CONGEST model was given in [17], where they also show an $\Omega\left(\frac{n}{\log n} + D_u\right)$-round lower bound for computing BC and give a method to approximate an exponential number of shortest paths using log-size messages. An approximation algorithm for computing random walk BC in $O(n \log n)$ rounds in the CONGEST model was recently given in [16]. Distributed BC algorithms from a practical prospective are given in [33,34]. No $O(n)$-round BC algorithm was known for directed graphs prior to our algorithm.

Organization of the Paper. In Section 3 we review Brandes’ sequential algorithm for betweenness centrality [4]. In the following two sections, we present our new results: In Section 4 we describe our distributed BC algorithm for unweighted directed graphs as well as our improvement to the number of rounds for unweighted directed APSP. In Section 5 we present our APSP and BC algorithms for weighted dags.

3 Brandes’ Sequential Betweenness Centrality Algorithm

Brandes [4] noted that if the single source shortest path (SSSP) dags are available for each node in $G$ it is possible to compute BC values using a recursive accumulation technique.

$$BC(v) = \sum_{s \neq v} \delta_{sv}$$

where

$$\delta_{sv} = \sum_{t \in V \setminus \{v, s\}} \frac{\sigma_{sv} \cdot \sigma_{vt}}{\sigma_{st}}$$

(1)

where $\sigma_{st}$ is the number of shortest paths from $s$ to $t$, and $P_s(w)$ are all the predecessors of $w$ in the SSSP dag rooted at $s$. Moreover, $\delta_{sv}(v)$ can be recursively computed as

$$\delta_{sv}(v) = \sum_{w : v \in P_s(w)} \frac{\sigma_{sv}}{\sigma_{sw}} \cdot (1 + \delta_{sw}(w))$$

(2)

Brandes’ sequential BC algorithm is presented below and consists in the following steps: for each source $s$ compute the SSSP dag $DAG(s)$ rooted at $s$ (Alg. 1), for each $DAG(s)$ compute $\sigma_{sv}$ for each $v \in DAG(s)$ (Alg. 1) and, for each $DAG(s)$ starting from the leaves, apply equation 2 up to the root (Alg. 2).

Algorithm 1 Betweenness-centrality($G = (V, E)$) (from [4])

1: for every $v \in V$ do $BC(v) \leftarrow 0$
2: for every $s \in V$ do
3: \hspace{1em} run Dijkstra’s SSSP from $s$ and compute $\sigma_{st}$ and $P_s(t)$, $\forall t \in V \setminus \{s\}$
4: \hspace{1em} store the explored nodes in a stack $S$ in non-increasing distance from $s$
5: \hspace{1em} accumulate dependency of $s$ on all $t \in V \setminus s$ using Algorithm 2

The structure of the above algorithm can be naturally adapted into a distributed algorithm and was done so for undirected unweighted graphs in [17] (see Appendix). In the next section we present an efficient distributed algorithm for BC in directed unweighted graphs while also improving the round and/or message complexity (by a constant factor) for APSP in both undirected and directed graphs and for BC in undirected graphs.

The values of $\sigma_{uv}$ can be exponentially large in $n$. For computation of exact BC values we will assume that the value of $B$ in the CONGEST model is sufficiently large to allow transmitting any $\sigma_{uv}$ value. Alternatively, we can stay with $B = \log n$ and compute very good approximations to the BC values using a technique in [17] (see Appendix).
Algorithm 2 Accumulation-phase($s, S$) (from [4])

Require: $\forall t \in V$: $\sigma_{st}, P_t(t)$; a stack $S$ containing all $v \in V$ in non-increasing $d(s, v)$ value

1: for every $v \in V$ do $\delta_{s*}(v) \leftarrow 0$
2: while $S \neq \emptyset$ do
3: \hspace{1em} $w \leftarrow \text{pop}(S)$
4: \hspace{2em} for $v \in P_s(w)$ do $\delta_{s*}(v) \leftarrow \delta_{s*}(v) + \frac{1}{\sigma_{sv}} \cdot (1 + \delta_{s*}(w))$
5: \hspace{2em} if $w \neq s$ then $\text{BC}(w) \leftarrow \text{BC}(w) + \delta_{s*}(w)$

4 APSP and BC in Unweighted Directed Graphs

In this section, we present our algorithm for computing betweenness centrality in unweighted directed graphs in the CONGEST model. It is inspired by the Lenzen-Peleg distributed unweighted APSP algorithm [24], and contains new elements discussed in section 4.2. Section 4.3 gives our overall BC algorithm. In Brandes’ algorithm, and our simple distributed algorithm for the accumulation phase (Alg. 2) in Brandes’ algorithm, and our

4.1 The Lenzen-Peleg APSP Algorithm [24]

We start by reviewing some notation common to [24] and our directed APSP algorithm (Alg. 3). $L_v$ is an ordered list at node $v$ which stores pairs $(d_{sv}, s)$, where $s$ is a source and $d_{sv}$ is the shortest distance from $s$ to $v$. These pairs are stored on $L_v$ in lexicographically sorted order, with $(d_{sv}, r) < (d_{sv}, s)$ if either $d_{sv} < d_{sv}$, or $d_{sv} = d_{sv}$ and $r < s$.

In each round $r$ of the Lenzen-Peleg algorithm [24], every node $v$ sends along its outgoing edges the pair with smallest index in $L_v$ which has its status (a conditional flag) still set to ready, and then sets the status of this pair to sent. As noted in [24] this approach can result in multiple messages being sent from $v$ for the same source $s$. This is simplified in our algorithm, where only one correct message is sent from each node $v$ for each source, and this send is performed in a specific round without the need for the additional status flag.

The Lenzen-Peleg algorithm [24] completes in $n + O(D_n)$ rounds and correctly computes shortest path distances to $v$ from each vertex $s$ that has a path to $v$ (the undirected diameter, which we denote by $D_n$, here, is called $D$ in [24] because they only consider undirected graphs). Although this is claimed in [24] only for undirected APSP, their techniques can be adjusted to work for directed APSP as well. In particular, if the total number of vertices $n$ is known (or computed), the undirected APSP algorithm in [24] can be modified to terminate in $2n$ rounds and compute APSP in a directed graph.

In Section 4.2 we present a method to improve the number of rounds from $2n$ to $\min\{2n, n + O(D)\}$. Our algorithm terminates in $n + 5D$ rounds on strongly connected graphs without knowing $n$; if $n$ is known, it terminates in $2n$ rounds in any directed graph. Moreover, our algorithm reduces the total number of messages sent to $mn + 2m$ even for the undirected case. Further, since we are interested in computing BC, our new algorithm also computes for each node $v$ the set $P_s(v)$ of predecessors of $v$ in the shortest path dag rooted at each source $s$, and the number of shortest paths $\sigma_{sv}$ from $s$ to $v$.

In [24], since only APSP is of interest, a node forwards only the first shortest path message it receives from a predecessor in its shortest path dag. But here we need to monitor messages from all incoming edges to identify all shortest path predecessors and to compute the number of shortest paths for each source. These enhancements appear in our new Algorithm 3 together with a call to
In our directed APSP algorithm (Alg. 3) initially each node \( v \) has just the pair \((0, v)\) in \( L_v \) (Step 3, Alg. 3). Let \( d \) be the pair \((d_{sv}, s)\) in \( L_v \). If there is an entry on \( L_v \) with \( d_{sv} + \ell_v^r(d_{sv}, s) = r \) (and there can be at most one), then this value is sent out along with the associated \( \sigma_{sv} \) value (Steps 8-9), otherwise \( v \) does not send out anything in round \( r \). A received message for source \( s \) is either added to \( L_v \) (or updates an existing value for \( s \) in \( L_v \) if it improves the distance value for its source). If new shortest paths from \( s \) to \( v \) are added by this received message, the \( \sigma_{sv} \) value and \( P_s(v) \) are updated to reflect this (Steps 11-17). Steps 11 and 10 are used to reduce the number of rounds from \( 2n \) to \( n + O(D) \) and are discussed in Section 4.2.4.

Algorithm 3 may need to send more than one value from a vertex \( v \) in a round because of the parallel computation of Step 1, but it never sends more than a constant number of values. In this case, \( v \) will combine all these values into a single \( O(B) \)-bit message.

**Algorithm 3 Directed-APSP(G)**

1: compute (in parallel with Step 7) a BFS tree \( B \) rooted at vertex \( v_1 \) (node with smallest ID); each vertex \( u \) computes its set of children \( C_u \) and its parent \( p_u \) in \( B \) \hfill \triangleright \) This will be used in Alg. 4
2: for each vertex \( v \) in \( G \) do
3: \( L_v \leftarrow ((0, v)); \) set flag \( f_v \leftarrow 0 \) \hfill \triangleright \) Initialize
4: for each source \( s \) in \( G \) do if \( s = v \) then \( \sigma_{sv} \leftarrow 1 \) else \( \sigma_{sv} \leftarrow 0 \); \( P_s(v) \leftarrow \emptyset \)
5: if \( n \) is not known then \hfill \triangleright \) Assumes \( G \) is weakly-connected
6: compute and broadcast \( n \) to every node in at most \( 2 \cdot D_u \) rounds, where \( D_u \) is the diameter of \( U_G \)
7: for \( \text{rounds} \ 1 \leq r \leq 2n \) do \hfill \triangleright \) Step 10 could cause termination before round \( 2n \) when \( G \) is strongly connected
8: if \( r = d_{sv} + \ell_v^r(d_{sv}, s) \) then
9: \( \tau_{sv} \leftarrow r \); send \((d_{sv}, s, \sigma_{sv})\) to all vertices in \( \Gamma_{\text{out}}(v) \) \hfill \triangleright \) Timestamp \( \tau_{sv} \) will be used in Alg. 4
10: run APSF-Predecessor(v, ps, C, n) \hfill \triangleright \) See Alg. 4
11: for a received \((d_{sv}, s, \sigma_{sv})\) from an incoming neighbor \( u \) do
12: if \( \exists (d_{sv}, s) \in \Gamma_u \) then
13: vertex \( v \) adds \((d_{sv}, s)\) in \( L_v \) with \( d_{sv} = d_{su} + 1 \), sets \( \sigma_{sv} \leftarrow \sigma_{su} \); \( P_s(v) \leftarrow \{u\} \)
14: else if \( \exists (d_{sv}, s) \in \Gamma_u \) with \( d_{sv} = d_{su} + 1 \) then
15: vertex \( v \) updates \( \sigma_{sv} \leftarrow \sigma_{sv} + \sigma_{su} \); \( P_s(v) \leftarrow P_s(v) \cup \{u\} \)
16: else if \( \exists (d_{sv}, s) \in \Gamma_u \) with \( d_{sv} > d_{su} + 1 \) then
17: vertex \( v \) replaces \((d_{sv}, s)\) in \( L_v \) with \((d_{su} + 1, s)\); vertex \( v \) sets \( \sigma_{sv} \leftarrow \sigma_{su} \); \( P_s(v) \leftarrow \{u\} \)

We now establish the correctness of Algorithm 3. We start by showing that every \( d_{sv} \) value arrives at \( v \) before the round in which it will need to be sent by \( v \) in Step 8.

**Lemma 1.** If an entry \((d_{sv}, s)\) is inserted in \( L_v \) at position \( k \) in round \( r \) then \( d_{sv} + k > r \).

**Proof.** In round \( r = 1 \) any entry \((d_{sv}, s)\) inserted in \( L_v \) has \( d_{sv} = 1 \) and the minimum value for \( k \) is 1. Hence \( d_{sv} + k \geq 2 > 1 \) so the lemma holds for round 1.

If the lemma does not hold, consider the first round \( r \) in which an entry \((d_{sv}, s)\) is inserted in \( L_v \) at a position \( k \) with \( d_{sv} + k \leq r \). Let this \( d_{sv} \) be inserted due to a message \((d_{su}, s, \sigma_{su})\) received
by \( v \) in round \( r \) in Step 11. Then, if \((d_{sv}, u)\) was in position \( i \) in \( L_u \) in round \( r \), \( r = d_{su} + i \) and the entries in \( L_u \) in positions 1 to \( i - 1 \) must have been sent to \( v \) in rounds earlier than \( r \). Each of these entries correspond to a different source, and a corresponding entry for that source will be present at a position less than \( k \) in \( L_v \) (either because a corresponding entry was inserted at \( L_v \) when the message for it from \( u \) was received or an entry with an even smaller value for \( d_{su} \) was already present in \( L_v \)). Hence \( k \geq i \). But for the values in round \( r \), \( d_{su} + k = d_{su} + 1 + k \geq d_{su} + i + 1 \) since \( d_{su} = d_{su} + 1 \) and \( k \geq i \) in round \( r \). Since \( r = d_{su} + i \) we have \( d_{sv} + k \geq r + 1 \). This gives the desired contradiction and the lemma is established.

Next we show that the position of an entry for a source \( s \) in \( L_v \) can never decrease unless its value is changed.

**Lemma 2.** If an entry \((d_{sv}, s)\) in \( L_v \) remains unchanged at \( v \) between rounds \( r \) and \( r' \), with \( r' > r \), then \( e_v^{(r')}(d_{sv}, s) \geq e_v^{(r)}(d_{sv}, s) \).

**Proof.** Once an entry is added to the list \( L_v \) it can only be replaced by a lexicographic smaller one but it never disappears. Thus, every entry in \( L_v \) that is below \((d_{sv}, s)\) in round \( r \) either remains in its position or moves to an even lower position in subsequent rounds. Hence if \( d_{sv} \) does not change between \( r \) and \( r' \), every entry below \((d_{sv}, s)\) in round \( r \) remains below it until round \( r' \). It is possible that new entries could be added below the position of \((d_{sv}, s)\) in \( L_v \) but this can only increase the position of \((d_{sv}, s)\) in round \( r' \).

Lemmas 1 and 2 establish that every entry that remains in \( L_v \) at the end of the algorithm was sent out at a prescribed round number (Step 6, Alg. 3) since the entry was placed at its assigned spot before that round number is reached and after it was placed in \( L_v \) its position can only increase and hence it will be available to be sent out at the round corresponding to its new higher position.

**Lemma 3.** At each vertex \( v \), the distance values in the sequence of messages sent by \( v \) are non-decreasing.

**Proof.** Suppose \( v \) sends a message with value \( d_{sv} \) in round \( r \) and then sends a message with a smaller \( d \) value in a later round. Then this smaller \( d \) value must be received by \( v \) in round \( r \) or later since otherwise it would have been placed in \( L_v \) (and thus sent) before \( d_{sv} \).

Let \( k = e_v^{(r)}(d_{sv}, s) \). Let \( d_{sv}' \) be the first \( d \) value smaller than \( d_{sv} \) that is inserted in \( L_v \) in a round \( r' \geq r \). Then, \( d_{sv}' \) is inserted in a position \( k' \leq k \) since the \( d \) values are in non-decreasing order on \( L_v \). But then \( d_{sv}' + k' < d_{sv} + k = r \leq r' \). But this contradicts Lemma 1.

Lemma 3 shows that the distance messages are sent out in non-decreasing order, and hence at most one message is sent by each vertex for each source. Finally, the next lemma shows that the shortest path counts \( \sigma_{sv} \) and the predecessor lists are also correctly computed.

**Lemma 4.** During the execution of Algorithm 3 a vertex \( v \) sends out the correct shortest path distance \( d_{sv} = \delta(s, v) \) and path count \( \sigma_{sv} \) for each source from which it is reachable. Also, \( P_s(v) \) contains exactly the predecessors of \( v \) in \( s \)'s SP dag when the message \((\delta(s, v), s, \sigma_{sv})\) is sent by \( v \) in Step 4.

**Proof.** Let \( D_{sv} \) denote the dag of shortest paths from \( s \) to \( v \). We use induction on the number of hops \( h \) from \( s \) to \( v \) in \( D_{sv} \).
Base case: \( h = 0 \). The initializations in Steps 3 and 4 correctly set \((d_{su} = 0, s), \sigma_{su} = 1\) and \(P_s(s) = \emptyset\). For \( h = 1 \), the dag consists of a single edge \((s, v)\) and the three values are set correctly to \(d_{sv} = 1, \sigma_{su} = 1\) and \(P_s(v) = \{s\}\) in Step 12.

Induction step: Assume that lemma holds for all \( s, u \) such that \( D_{su} \) has at most \( h - 1 \) hops and let \( D_{sv} \) have \( h \) hops. Consider any predecessor \( u \) of \( v \) in \( D_{sv} \). By the induction hypothesis \( u \) will send out the message \((\delta(s, u), u, \sigma_{su})\) at the designated round \( r \) in Step 6 and by Lemma 1 \( v \) will insert the value \((\delta(s, v), s)\) at a position \( k \) with \( r < \delta(s, v) + k \) (if \( \delta(s, v), v \) is not already in \( L_v \)), in either case, \( \sigma_{su} \) is updated correctly with the value of \( \sigma_{su} \) and \( P_s(v) \) is updated with \( u \) in Step 12, 14, or 17). The same process occurs with every predecessor of \( v \) in \( D_{sv} \). Finally, by Lemma 1 all of these updates occur before the round in which the message for source \( s \) is sent from \( v \) in Step 8. This establishes the induction step and the lemma.

\((S, h, r)\)-detection and \((h, k)\)-SSP Problems. Algorithm 3 computes APSP, the predecessors and number of shortest paths to each vertex since these are the parameters of interest for betweenness centrality. However, our techniques are applicable to related problems in the literature such as the source detection or \((S, h, r)\)-detection task [24] and the \((h, k)\)-SSP problem [2]. In both of these problems, a subset \( S \) of \( k \) nodes is designated as the source set, and a hop length \( h \) specifies that only paths with at most \( h \) edges are to be considered. In the \((S, h, r)\)-detection task, \( r \) is at most \( k \) and each node \( v \) needs to compute the shortest path distance to \( v \) from the up to \( r \) nearest sources in \( S \), all with hop length at most \( h \). In the \((h, k)\)-SSP problem each node \( v \) needs to compute the shortest path distance to \( v \) from every source in \( S \) with hop length at most \( h \).

The following results are readily obtained by simple adaptations of the above lemmas for APSP. Here \( D \) is the directed or undirected diameter of the graph \( G \), according to whether \( G \) is directed or undirected. To obtain these results, we modify Algorithm 3 so that the initialization in Step 1 applies only to source nodes (with \( L_v \) set to \( \emptyset \) for all other nodes), and during a general round, at each node \( v \) we keep in \( L_v \) only those entries that are relevant to the problem being considered.

Lemma 5. The \((S, h, r)\)-detection problem can be computed in \( r + h \) rounds, and the \((h, k)\)-SSP problem can be computed in \( k + h \) rounds. In both bounds the second term can be improved to \( \min\{h, D\} \) where \( D \) is the diameter of the graph, if knowledge of global termination is not required.

4.2.1 Improving the Round Complexity We now describe Algorithm 4 which guarantees that Algorithm 3 will terminate in \( \min\{2n, n + O(D)\} \) rounds. More precisely, Alg. 4 terminates the computation before \( n + 5D \) rounds provided \( G \) is strongly connected with \( D < n/5 \). Otherwise, the computation terminates necessarily within \( 2n \) rounds because of step 7 of Alg. 3. We now focus on the non-trivial case where \( G \) is strongly connected and \( D \) is bounded.

Let \( B \) be a BFS tree rooted at \( v_1 \) (node with smallest ID) and created in Step 1, Alg. 3. Also, let \( C_v \) be the set of children of \( v \) in \( B \). Note that, if \( n \) is not known, Step 5 of Alg. 3 computes it in at most \( 2D_u \leq 2D \) rounds. Thus, \( n \) is always available during the execution of Alg. 4. The special vertex \( v_1 \) is used only to uniquely select a source node for the BFS (as in [24]). If we omit Alg. 4 (and terminate in \( 2n \) rounds), or if the unique BFS source vertex can be efficiently selected in some other way, there is no need to identify vertex 1, or to assume that vertices are numbered from 1 to \( n \). Note that \( B \) will be completely defined after \( D \) rounds, and the activity of Alg. 4 for a node \( v \) becomes relevant only after \( n \) rounds. In the first step, the algorithm checks if \( v \) has received the diameter \( D \) from its parent \( p_v \) in \( B \). In this case, \( v \) broadcasts \( D \) to all its children
in \( C_v \) and it stops. Otherwise, the algorithm checks if \( v \) has received one finite distance estimate from every node in \( G \) (Step 2 Alg. 4). (The flag \( f_v \) is initialized in Step 3 of Algorithm 3 and is used to ensure that steps 4, 5 are performed only once.) These distances will be correct when round \( r \geq \max_s(d_{sv} + \ell_v^r(d_{sv}, s)) \) (see Lemma 4), and Algorithm 4 proceeds by distinguishing two cases: if a node \( v \) is a leaf in the tree \( B \) (Step 3 Alg. 4), it computes the maximum shortest distance \( d_v^* \) from any other node \( s \) and broadcasts \( d_v^* \) to its parent \( p_v \) in \( B \) (Step 4 Alg. 4). Then, \( v \) will wait up to round \( 2n \) to receive the diameter \( D \) from its parent \( p_v \) in \( B \) (because of the check in step 1 Alg. 4).

In the second case, when \( v \) is not a leaf (and not \( v_1 \)), if it has collected (for the first time) the distances \( d_v^* \) from all its children in \( C_v \) (Step 6 Alg. 4), it will execute the following steps only once (thanks to the flag \( f_v \) initialized to 0 in Alg. 3 and updated to 1 in Step 8 Alg. 4): \( v \) computes the maximum shortest distance \( d_v^* \) from any source \( s \) (Step 7 Alg. 4) and the largest distance value \( d_{C_v}^* \) received from its children in \( C_v \) (Step 8 Alg. 4). Then \( v \) sends the larger of \( d_v^* \) and \( d_{C_v}^* \) to its parent \( p_v \) (Step 8 Alg. 4), and it waits for \( D \) from \( p_v \) as in the first case. Finally, when \( v \) is in fact \( v_1 \), after receiving the distances from all its children, it broadcasts the diameter \( D \) to its children in \( C_{v_1} \) (Step 9 Alg. 4).

Algorithm 4 APSP-Finalizer\((v, f_v, p_v, C_v, n)\) \(\triangleright p_v, C_v, n \) computed in Step 1 Alg. 4

Ensure: Compute and broadcast the network directed diameter \( D < \infty \)

1: if \( v \) receives diameter \( D \) from parent \( p_v \) in round \( r < 2n \), it broadcasts \( D \) to all vertices in \( C_v \) and stops
2: if \( |L_v^r| = n \) and \( f_v = 0 \) then
3: if \( r = \max_x(d_{sv} + \ell_v^r(d_{sv}, s)) \) and \( C_v = \emptyset \) then \(\triangleright v \) is a leaf in the BFS tree \( B \)
4: \( d_v^* \leftarrow \max_x(d_{sv}) \); send \( d_v^* \) to parent \( p_v \); \( f_v \leftarrow 1 \)
5: if \( r \geq \max_x(d_{sv} + \ell_v^r(d_{sv}, s)) \) then \(\triangleright \) completed only once
6: if \( v \) has received \( d_v^* \) from all children \( x \in C_v \) then
7: \( d_v^* \leftarrow \max_x(d_{sv}) \); \( d_{C_v}^* \leftarrow \max_{x \in C_v}(d_v^*) \)
8: if \( v \neq v_1 \) then send \( \max(d_v^*, d_{C_v}^*) \) to parent \( p_v \); \( f_v \leftarrow 1 \)
9: else broadcast \( D = \max(d_v^*, d_{C_v}^*) \) to \( C_{v_1} \); stop

It is readily seen that Algorithm 4 broadcasts the correct diameter to all nodes in \( G \) since after round \( r = \max_x(d_{sv} + \ell_v^r(d_{sv}, s)) \) the \( d_{sv} \) values at \( v \) are the correct shortest path lengths to \( v \) (by Lemma 4). Moreover, since \( \max_x(d_{sv} + \ell_v^r(d_{sv}, s)) > n \) when \( |L_v^r| = n \), Step 1 of Alg. 4 is completed and each node \( v \) knows its parent and its children in \( B \). Thus, the value sent by \( v \) to its parent in Step 8 of Alg. 4 is the largest shortest path length to any descendant of \( v \) in \( B \), including \( v \) itself. Thus, node \( v_1 \) computes the correct diameter of \( G \) in Step 9 of Alg. 4.

Lemma 6. The execution of Algorithm 4 requires at most \( \min\{2n, n + 5D\} \) rounds.

Proof. Step 1 of Alg. 3 can be completed in \( D \) rounds using standard techniques, and it is executed in parallel with the loop in step 7, Alg. 4. If \( n \) is not known, Step 6 of Alg. 3 computes it in at most \( 2D_n \leq 2D \) rounds. Moreover, when \( D = \infty \) each vertex stops after \( 2n \) rounds because of step 7 of Alg. 4.

When \( D \) is bounded, each \( v \in V \) will have \( |L_v^r| = n \) at some round \( r \). In Alg. 4 (called in Step 10, Alg. 4), using the parent pointers of the BFS tree \( B \) already computed (Step 11, Alg. 3), the longest shortest path value reaches \( v_1 \) within \( D \) rounds after the last vertex computes its local maximum value. At this point \( v_1 \) computes the diameter \( D \) and broadcasts it to all vertices \( v \) in at most \( D \)
Algorithm 5 BC(G)

1: run Algorithm 3 (Directed-APSP(G)) on G; let \( R \) be the termination round for Alg 3
2: {Recall that \( \tau_{sv} \) is the round when \( v \) broadcasts \( (d_{sv},\sigma_{sv}) \) to \( F_{\text{out}}(v) \) in Step 6 Alg. 3}
3: set absolute time to 0
4: for each vertex \( v \) in \( G \) do
5:   for all \( s \) do \( A_{sv} = R - \tau_{sv} \)
6:   for a round \( 0 \leq r \leq R \) do
7:     if \( r = A_{sv} \) then send \( m = \frac{1+\delta_s(v)}{\sigma_{sv}} \) to \( v \)'s predecessors
8:     for a received \( m \) from an outgoing neighbor in \( F_{\text{out}}(v) \) do
9:       \( \delta_s(v) \leftarrow \delta_s(v) + \sigma_{sv} \cdot m \)

Steps. Since \( \max_s, \max_s\{d_{sv} + \delta_s(v) (d_{sv}, s)\} \leq n + D \), the total number of rounds is at most \( n + 5D \) (including \( 2D_u \leq 2D \) rounds for computing \( n \)). The lemma is proved.

4.3 Accumulation Technique and BC Computation

In Algorithm 5 we present a simple distributed algorithm to implement the accumulation phase in the Brandes algorithm (Alg. 2). Recall that in Algorithm 5 in the round when node \( v \) broadcasts its finalized message \( (d_{sv}, s, \sigma_{sv}) \) on its outgoing edges in step 6 it also notes the absolute time of this round in \( \tau_{sv} \). Also, by Lemma 6 Alg. 3 completes in round \( R = \min\{n + 3D, 2n\} \). Alg. 5 sets the global clock to 0 in Step 3 after these \( R \) rounds complete in Alg. 3. In Step 5 each node \( v \) computes its accumulation round \( A_{sv} \) as \( R - \tau_{sv} \). Then, \( v \) computes \( \delta_s(v) \) and broadcasts \( \frac{1+\delta_s(v)}{\sigma_{sv}} \) to its predecessors in \( P_s(v) \) in round \( A_{sv} \) (Steps 5–9 Alg. 5).

Although we have described Alg. 5 specifically as a follow-up to Algorithm 3 it is a general method that works for any distributed BC algorithm where each node can keep track of the round in which step 3 in Algorithm 1 (Brandes’ algorithm) is finalized for each source. This is the case not only for Algorithm 3 for both directed and undirected unweighted graphs, but also for our BC algorithm for weighted dags in the next section, and for the BC algorithm in [17] for undirected unweighted graphs (though our Algorithm 5 uses a smaller number of rounds). In contrast the distributed accumulation phase in [17] is tied to the start times of the shortest path computations at each node in the first phase of the undirected APSP algorithm used there, and hence is specific to that method.

Lemma 7. In Algorithm 5 each node \( v \) computes the correct value of \( \delta_s(v) \) at round \( A_{sv} = R - \tau_{sv} \), and the only message it sends in round \( A_{sv} \) is \( m = \frac{1+\delta_s(v)}{\sigma_{sv}} \), which it sends to its predecessors in the SSSP dag for \( s \).

Proof. We first show that at time \( A_{sv} \), node \( v \) has received all accumulation values from its successors in \( \text{DAG}(s) \). This follows from the fact that, in the forward phase, each successor \( w \) of \( v \) will send its message for source \( s \) to nodes in \( F_{\text{out}}(w) \) in round \( \tau_{sw} \), which is guaranteed to be strictly greater than \( \tau_{sv} \). Thus, since \( A_{sw} < A_{sv} \), node \( v \) will receive the accumulation value from every successor in the dag for \( s \) before time \( A_{sv} \), and hence computes the correct values of \( \delta_s(v) \) and \( \frac{1+\delta_s(v)}{\sigma_{sv}} \). Further, since the timestamps \( A_{sw} \) are different for different sources \( s \), only the message for source \( s \) is sent out by \( v \) in round \( A_{sv} \).

Although we have described Alg. 5 specifically as a follow-up to Algorithm 3 it is a general method that works for any distributed BC algorithm where each node can keep track of the round
in which step 3 in Algorithm 1 (Brandes’ algorithm) is finalized for each source. This is the case not only for Algorithm 3 for both directed and undirected unweighted graphs, but also for our BC algorithm for weighted dag in the next section, and for the BC algorithm in [17] for undirected unweighted graphs (though our Algorithm 3 uses a smaller number of rounds). In contrast the distributed accumulation phase in [17] is tied to the start times of the shortest path computations at each node in the first phase of the undirected APSP algorithm used there, and hence is specific to that method.

5 APSP and BC in Weighted DAGs

We now consider the case when the input graph \( G = (V, E) \) is a directed acyclic graph (dag), where each edge \((x, y)\) has weight \( w(x, y) \), and the number of vertices \( n \) is known. For simplicity, we will assume that the dag has a single source \( s \). If the dag contains multiple sources \((s_1, \ldots, s_k)\), we will assume a virtual source \( \hat{s} \) which is connected with a direct edge to the real sources. The procedures we present can be readily adapted to the multiple sources using such a virtual source.

We start with some definitions. Given a path \( \pi \) in \( G \), the length \( \ell(\pi) \) will denote the number of edges on \( \pi \) and the weight \( w(\pi) \) will denote the sum of the weights on the edges in \( \pi \). The shortest path weight from \( x \) to \( y \) will be denoted by \( \delta(x, y) \). Also, here we assume that the \( n \) nodes have unique IDs between 1 and \( n \), strengthening our earlier assumption that the unique node IDs are between 1 and \( \text{poly}(n) \). If this condition is not initially satisfied, we can have an initial \( O(n) \)-round phase to compute these IDs as follows. Each node broadcasts its ID to all other nodes. The node with the minimum ID then locally relabels the IDs from 1 to \( n \) and broadcasts these values to the other nodes. The initial broadcast from all nodes can be performed in \( O(n) \) rounds by piggy-backing on our APSP algorithm in the previous section, and the final broadcast can be performed in \( O(n) \) rounds by pipelining the \( n \) values along an SSSP tree rooted at the vertex with minimum ID.

**Definition 1.** A longest length tree (LLT) \( T_s \) for a dag \( G \) is a directed spanning tree rooted at its source \( s \) where, for each node \( v \), the path in \( T_s \) from \( s \) to \( v \) has the maximum length (number of edges) of any path from \( s \) to \( v \) in \( G \). The level \( \ell(v) \) of a node \( v \) is the length of the path from \( s \) to \( v \) in \( T_s \).

In this section, we focus only on computing APSP, the \( \sigma_{sv} \) values and the \( P_s(v) \) sets in a weighted dag. After this we can reverse the timings \( \tau_{xy} \) (obtained in Step 5, Alg. 7) and then use Algorithm 5 to compute the BC values.

We first describe our distributed algorithm to construct LLT(\( G \)) (Alg. 6). It uses a delayed-BFS algorithm on dag \( G \). It starts a BFS from the source \( s \) (Step 2, Alg. 6), and it delays the BFS extension from each node \( v \) until each incoming node \( u \in \Gamma_{\text{in}}(v) \) has propagated its longest length \( \ell(u) \) from \( s \) to \( v \). Then node \( v \) will finalize the longest length received, \( \max_{u \in \Gamma_{\text{in}}(v)}(\ell(u) + 1) \), as its level \( \ell(v) \) (Steps 4 – 6, Alg. 6) and it will broadcast \( \ell(v) \) to all outgoing nodes \( x \in \Gamma_{\text{out}}(v) \) (Step 8, Alg. 6).

The proofs of the following lemma and observation are straightforward and are omitted.

**Lemma 8.** Algorithm LLT computes the parent pointers \( \pi() \) for an LLT tree \( T_s \) of dag \( G \) in \( L \) rounds, where \( L \) is the length of a longest finite directed path in \( G \).

**Observation 3** If \( T_s \) is an LLT of dag \( G \) then every edge \((u, v)\) in \( G \) has \( \ell(u) < \ell(v) \).
Proof. Since every edge \((u, v)\) of the number of shortest paths \((1)\), is sent to each outgoing node \(v\) has length \(\ell\) is at most \(\ell\) (Lemma 9). The value \(\sigma\) 10: \(\sigma_{xy} \leftarrow \sum_{u \in P_u(y)} \sigma_{xu}\) 11: send message \((x, \delta(x, y), \sigma_{xy})\) to \(z\) for each \(z \in \Gamma_{\text{out}}(y)\) 14

Complexity. We now establish correctness and round complexity of Algorithm 7.

Lemma 9. The value \(\delta(x, y)\) computed in Step 7 is the correct shortest path distance from \(x\) to \(y\).

Proof. Since every edge \((u, v)\) in \(G\) has \(\ell(u) < \ell(v)\) (see Observation 3), any path from \(x\) to \(y\) in \(G\) has length at most \(\ell(y) - \ell(x)\). The SSSP \((x)\) starts at \(x\) at absolute time \(ID_x + \ell(x)\), and hence the
value sent on every path from $x$ to $y$ in $G$ arrives at $y$ at absolute time $ID_x + \ell(y)$ or less. Since $\delta(x,y)$ is computed as the minimum of the values received at time $ID_x + \ell(y)$, this is the correct $x$-$y$ shortest path weight.

Lemma 10. Each node transmits a message for at most one SSSP in each round.

Proof. Consider a node $x$ and let $u$ and $v$ be any two nodes from which $x$ is reachable. Node $x$ will transmit the message for SSSP($u$) in round $ID_u + \ell(x)$, and the message for SSSP($v$) in round $ID_v + \ell(x)$. Hence, these messages will be transmitted on different rounds. Hence the message for at most one SSSP dag will be sent out by $x$ in each round.

Finally, since $ID_x + \ell(y) \leq n + L$ for all $x,y \in V$, the round complexity of computing APSP in a weighted dag is $n + O(L)$.

![Fig. 1](image.jpg)

**Fig. 1.** An example of distributed SSSP execution from $u$ and $v$ in Alg. 7. Snake lines represent the LLT $T_s$ path. Dotted lines are shortest paths. SSSP($u$) and SSSP($v$) will leave $w$ at different times $ID_u + 8$ and $ID_v + 8$. Moreover the two SSSP($u$) paths $u \sim w$ and $u \sim v \sim w$ could reach $w$ at different time steps, but they will be processed (only the shortest path will be propagated) at the same absolute time $ID_u + 8$.

$\ell(s) = 0$
$\ell(u) = 4$
$\ell(v) = 5$
$\ell(w) = 8$

**Correctness.** The correctness follows from the execution of a BFS procedure from each node.

### 5.1 BC in a Weighted DAG

We can now use Algorithm 5 to compute betweenness centrality in a weighted dag $G$ using the round numbers $\tau_{xy}$ computed in Algorithm 7 to schedule the accumulation step at node $y$ for source $x$. As seen from step 5 of Algorithm 7, $\tau_{xy}$ is the round when node $y$ broadcasts $\delta(x,y)$ and $\sigma_{xy}$ to all its outgoing nodes (Steps 5–11, Alg. 7). Thus, similarly to Alg. 5, in the accumulation round $A_{xy} = 2n - \tau_{xy}$ (Step. 5, Alg. 5), node $y$ will receive all the accumulation values from every successor in the dag for $x$, and it will compute the correct value of $\delta_{x\bullet}(y)$. The overall dag BC algorithm is in Algorithm 8. It uses double the number of rounds as the dag APSP algorithm, and hence runs in $2n + O(L)$ rounds.

**Algorithm 8** Weighted dag-BC($G$)

1: run Algorithm 2 on $G$
2: for all $s$ and $v$, $\tau_{sv}$ computed in Step 5 Alg. 7 will be used in Alg. 5
3: set absolute time to 0
4: run steps 3–9 of Algorithm 5 on $G$

Thus, BC for a weighted dag can be computer in no more than $2n + O(L)$ rounds in the CONGEST model.
6 Conclusion

We have presented several distributed algorithms in the CONGEST model for computing BC and path problems in directed graphs. The sub-area of distributed algorithms for directed graphs is still in early development, and our work has presented several new results and techniques. A useful observation highlighted by our research is that global delay pipelining techniques can in fact cooperate to improve the efficiency of distributed algorithms for directed graphs. Moreover, they can be used to reduce the number of messages used by the algorithm. A distributed implementation of our BC algorithm for unweighted directed graphs on the distributed platform D-Galois [7] has been found to outperform earlier high-performance distributed BC implementations [4].

A major open question left by our work is to obtain improved deterministic algorithms for APSP and BC in weighted graphs, both directed and undirected. The current best deterministic bounds for general weighted graphs are in [2] and [1].

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7 Appendix

7.1 BC in Undirected Unweighted Graphs [17]

Recently, a distributed BC algorithm for unweighted undirected graphs which terminates in $O(n)$ rounds in the CONGEST model was presented in [17], together with a lower bound of $\Omega(n/\log n)$ rounds for computing BC. This algorithm computes the predecessor lists and the number of shortest paths (Step 3 in Alg. 1) by a natural extension of the unweighted undirected APSP algorithm in [15] (see also [29]). The undirected APSP algorithm in [15] starts concurrent BFS computations from different sources scheduled by a pebble that performs a DFS traversal of a spanning tree for $G$. Each time the pebble reaches a new node $v$, it pauses for one round before activating $BFS(v)$ and then proceeds to the next unexplored node. At each node $v$, all messages for a given BFS (say started at source $s$) reach $v$ at the same round, and the updated distance is sent out from $v$ in the next round. Hence, before $v$ broadcasts its distance from $s$ to adjacent nodes, it can readily compute and store $P_s(v)$ and $\sigma_{sv}$ using the incoming messages related to $BFS(s)$ in this round. It is well known that this approach does not work in directed graphs, since the APSP algorithm in [15] could create congestion (see Figure 2).
Fig. 2. Counterexample for the APSP algorithm in [15] for directed graphs. Here BFS(v) and BFS(w) will congest at node u4. Value $t_j$ represents the round when the pebble $P$ starts the BFS from the corresponding node, with $t_i < t_j$ iff $i < j$. In this example BFS(v) will start at round $t_6 = 21$, while BFS(w) will start at round $t_7 = 24$. They will both reach $u_4$ at the beginning of round 25 creating a congestion for the next round.

Since the pebble pauses at each node and a DFS traversal backtracks over $\Theta(n)$ nodes before activating the last BFS, this distributed algorithm for step 3 in Algorithm 1 completes in $3n + O(D)$ rounds.

The distributed algorithm in [17] for Algorithm 2 is described in the next section. It uses the triangle inequality for its proof of correctness, which does not apply to the directed case (since the pebble backtracks along DFS edges in this algorithm). The algorithm in [17] also handles the issue that a graph could have an exponential number of shortest paths which would cause the $\sigma_{st}$ values to have a linear number of bits. Since the CONGEST model allows only messages of size $O(\log n)$, they use a floating point representation with $O(\log n)$ bits to approximate the $\sigma_{st}$ values. We review this method in Section 7.1.2. We will use this same method in our algorithms since it works without change for directed graphs and for weighted graphs.

7.1.1 Accumulation Phase for Undirected BC The distributed method for Algorithm 2 in [17] first computes and broadcasts the diameter $D_u$ of the network during the APSP algorithm. Then, each node $v$ sets its accumulation broadcast time for each source $s$ to $T_s(v) = T_s + D_u - d(s,v)$, where $T_s$ is the absolute time when BFS(s) started in the APSP algorithm. The global clock is reset to 0 and each node $v$ sends its accumulation value for $s$ at time $T_s(v)$. Since $D_u - d(s,v) \geq 0$, this approach completes in at most $3n$ rounds. Thus, overall BC algorithm in [17] runs in $6n + O(D_u)$ rounds.

In Section 4 we present another simple method which works for our algorithm and can also replace the above algorithm in [17].

7.1.2 Handling Exponential Values Given the $O(\log n)$-bit restriction in the CONGEST model, [17] maintains approximate values of the $\sigma_{st}$ values using a floating point representation, and guarantee a relative error for the computed BC which is only $O(n^{-c})$ (where $c$ is a constant). Since the technique in [17] works for both undirected and directed graphs (weighted or unweighted), we will use the same method in our algorithms in order to handle exponential counts of paths.