Spin and Electromagnetic Duality: An Outline

Talk presented at the 15th Anniversary Meeting of Dirac Medallists, at Abdus Salam ICTP, Trieste, November 2000.

David I Olive,
Physics Department, University of Wales Swansea, Swansea SA2 8PP

Introduction

Quantum electromagnetic duality is a new sort of quantum symmetry principle that manifests itself in quantum field theories of the unified type. Certain physical quantities exhibit invariance or covariance under the action of an infinite discrete group, such as the modular group, acting on a dimensionless coupling constant. It seems to me to be important to clarify what is going on by studying various examples of this effect.

Rather than talk about the usual scenarios, supersymmetric gauge theories [1] or string theories, I shall talk about a different and relatively simple setup: a fixed four-dimensional background space-time that may be complicated topologically. Nevertheless it is assumed to be closed, compact, smooth and oriented. It supports non-singular Maxwell field strengths that may be dynamical and, in any case, have to allow the presence of complex quantum wave functions for charged particles. These wave functions may be either scalar or spinor, according to the spin of the corresponding charged particle, and the consequences will be distinguished and compared.

My discussion falls into two parts following the two papers written in collaboration with Marcos Alvarez [2], [3]. The first part deals with the Dirac quantisation condition for magnetic fluxes, that is, the consistency condition for the existence of the complex wave functions. There is an anomalous effect when complex spinor wave functions are considered which may lead to half-integral rather than integral fluxes through certain sorts of two-cycle. In four dimensional space-times when the fine structure constant is dimensionless and traditional electromagnetic duality holds, there is a simple characterisation of these anomalous two-cycles, namely that they are precisely the ones with odd self-intersection number. These results are already known to pure mathematicians but we shall find a more physical language for explaining them and talk about a slight generalisation, what we have called a “quantum Stokes’ theorem”, that holds when the Maxwell gauge potential or connection can only be defined locally, in patches, because of the complication of the background space-time topology.

In the second part, this effect is shown to tie in nicely with quantum electromagnetic duality. A sort of generalised partition function for the Maxwell theory can be evaluated using semiclassical methods, following E Witten [4] and E Verlinde [5]. The stationary points are labelled by the values of the quantised fluxes and so the sum over them yields a result proportional to a generalised theta function. This transforms nicely under the action of a subgroup of the modular group acting on the dimensionless coupling constant of the theory.
When the space-time background is such that the electromagnetic fields can support scalar and spinor wave functions simultaneously, the range of the sum is an even integral lattice. There is only one partition function and it transforms nicely under the full modular group of electromagnetic duality transformations.

When the background space-time possesses the aforementioned anomalous two-cycles there is a disagreement between the quantum consistency conditions for the fluxes needed to allow scalar and spinor wave functions. Therefore two distinct partition functions have to be considered. In one the range of summation is an odd unimodular lattice and the other this lattice displaced by one half of what is known as the characteristic vector of the odd unimodular lattice. By the Atiyah-Singer index theorem this encodes the self-intersection numbers mentioned above. Separately these two partition functions each transform nicely under a subgroup of the modular group. A simple and general construction with these lattices shows how the action of the full modular group is recovered by a mixing of these two, together with a third that arises naturally. It is this unexpected effect that explains the above title, spin and electromagnetic duality.

Fluxes and Homology

Faraday’s concept of electromagnetic fluxes through two-dimensional surfaces led mathematicians to a more precise language that will prove very relevant and useful physically and will fit in well with the ideas of Dirac that came later.

Open and closed surfaces are distinguished. An open surface $\Sigma$ has a boundary, denoted $\partial \Sigma$. If this boundary vanishes the surface is closed and called a cycle. Two two-cycles are equivalent, i.e. homologous, if they differ by the boundary of a three-dimensional object.

$$\Sigma \sim \Sigma' \leftrightarrow \Sigma - \Sigma' = \partial \alpha$$

These equivalence classes form a homology class and the set of these classes in a given background four-dimensional space-time, $\mathcal{M}_4$, is denoted $H_2(\mathcal{M}_4, \mathbb{Z})$. The electromagnetic field strength tensor in space-time defines a closed two-form, denoted $F$, where $dF = 0$. Then the magnetic flux through the two-cycle $\Sigma$ can be written as $\int_\Sigma F$. By virtue of Stokes’ theorem, this is unchanged with respect to alteration of $\Sigma$ to a homologous $\Sigma'$, as above, and $F$ to a cohomologous $F'$, that is such that $F' = F + dB$ and so is also closed as the exterior derivative, $d$, like the boundary operator, $\partial$, squares to zero.

Thus the notion of flux possesses a degree of invariance and moreover can be subjected to an operation of addition in a natural way:

$$\int_{\Sigma_1} F + \int_{\Sigma_2} F = \int_{\Sigma_1 + \Sigma_2} F.$$

So the set $H_2(\mathcal{M}_4, \mathbb{Z})$ forms an abelian group under addition that reflects the structure of the space-time. Although this group usually possesses an infinite number of elements it is finitely generated. The elements of finite order give rise to vanishing Maxwell fluxes.
and so are irrelevant for the present physical purposes. Because they form an invariant subgroup they can be divided out to leave a group described by a finite number, $b_2$, of copies of the integers:

$$H_2(M_4, \mathbb{Z})/FINITE \equiv \mathbb{Z}^{b_2}.$$ 

Thus the resultant group can be thought of as a lattice of dimension $b_2$. The positive integer $b_2$ is known as the second Betti number of $M_4$.

The other Betti numbers for $M_4$, $b_0, b_1, b_3$ and $b_4$ can be defined similarly, by adjusting the dimension of the cycles considered. One of the consequences of the Poincaré duality property of space-time $M_4$ is that the Betti numbers are equal in complementary pairs, $b_1 = b_3$ and $b_0 = b_4$. We shall assume space-time is connected so that $b_0 = b_4 = 1$ but not that it is simply connected (so $b_1$ need not vanish). A particular linear combination is familiar, $\chi(M_4) = b_0 - b_1 + b_2 - b_3 + b_4 = 2(1 - b_1) + b_2$ and is called the Euler number of $M_4$. Unlike the individual Betti numbers, it is local in the sense of being expressible as the integral of a local quantity over $M_4$. Hence it is likely to play a special role in local quantum field theory and this will be confirmed. There is one other such local quantity, called the Hirzebruch signature, denoted $\eta(M_4) = b_2^+ - b_2^-$ where $b_2^+ + b_2^- = b_2$.

Definitions of $b_2^\pm$ will be given two sections later.

**Dirac Quantisation Condition and the Quantum Stokes Relation**

We now seek electromagnetic field configurations on $M_4$ that permit a consistent definition of a complex scalar wave function, $\phi(x)$, there. Since this brings into play the gauge potential one-form, $A$, obtained from $F$ by integrating $F = dA$ we are forced to work in topologically trivial neighbourhoods covering $M_4$. In overlaps between a pair of neighbourhoods the two choices of $A$ and $\phi$ should be patched together by the $U(1)$ gauge transformations:

$$A \rightarrow A + d\chi, \quad \text{so} \quad F \rightarrow F,$$

$$\phi(x) \rightarrow e^{i\frac{q\chi(x)}{\hbar}} \phi(x)$$

where the gauge function $\chi$ is real and $q$ is the electric charge of the corresponding particle. Consistency conditions arise in triple overlap regions and imply, as shown by Orlando Alvarez [6], the Dirac quantisation conditions for the magnetic fluxes [7]:

$$q \int_\Sigma F \in 2\pi \hbar \mathbb{Z}.$$
Actually something more can then be shown; if \( \Sigma \) is now open:

\[
e^{\frac{i\pi}{4}} \int_{\Sigma} F = e^{\frac{i\pi}{4}} \int_{\partial\Sigma} A.
\]

The point here is that, although the exponent on the left hand side is unambiguous, that on the right hand side has a quantised ambiguity arising from choices made in the patching procedure that is precisely removed by taking the exponential. Because of the necessary presence of Planck’s constant we have referred to this as the quantum Stokes’ relation. Of course, if \( \Sigma \) is closed the right hand side equals unity and Dirac’s condition is recovered. Notice also that taking the charge \( q \) to vanish yields a trivial identity.

It is worth noting how economical this argument is. There is no need of a metric on space-time, nor any special dimension. No equations of motion are needed and there is no restriction on the topology of \( \Sigma \). If it is simply a two-sphere the argument reduces to the familiar one of Wu and Yang [8], involving two hemispherical neighbourhoods.

On the other hand the quantum description of charged spin \( 1/2 \) particles requires complex spinor wave functions. Then the fact that the spinor representation of the orthogonal or Lorentz group (whichever is appropriate) is two-valued introduces sign choices in the patching procedure. This follows through to yield an unexpected sign in the quantum Stokes’ relation which now reads:

\[
e^{\frac{i\pi}{4}} \int_{\Sigma} F = (-1)^{w(\Sigma)} e^{\frac{i\pi}{4}} \int_{\partial\Sigma} A.
\]

This extra sign, \((-1)^{w(\Sigma)}\), is intrinsic to \( \Sigma \), that is independent of the choice of neighbourhoods involved in the patching if \( \Sigma \) has an even boundary, and so, in particular if it is closed and hence a cycle. Comparison of the two versions of the quantum Stokes’ relation makes clear that the path dependent phase factor [9], such as occurs on the right hand side, is not an autonomous object but is tied to a particular sort of wave function.

If \( \Sigma \) is a cycle, the path dependent phase factor on the right hand side collapses to unity leaving the modified flux quantisation condition

\[
q \int_{\Sigma} F \in 2\pi \hbar (\mathbb{Z} + \frac{w(\Sigma)}{2}).
\]

So, if \( w(\Sigma) \) is an odd integer, fluxes are half integer and, in particular, cannot vanish.

If the charge of the electrically charged spin \( 1/2 \) particle is taken to vanish, the quantum Stokes’ relation reduces to the identity \( 1 = (-1)^{w(\Sigma)} \). Thus the limit cannot be taken when \( w(\Sigma) \) is odd and this means that charge neutral spinor wave functions are forbidden on \( \mathcal{M}_4 \). Mathematicians recognise this as the Stiefel-Whitney obstruction. On four-dimensional space-times it is unnecessary to use this theory because something special happens there, namely that \( w(\Sigma) \) has a simple geometrical interpretation as being equal to the self-intersection number of \( \Sigma \), modulo 2. This will be shown to follow from something physicists would readily accept because of its relation to the theory of chiral anomalies, namely the Atiyah-Singer index theorem.
Intersection Numbers and Intersection Matrices

Two cycles of complementary dimension, that is summing to the dimension of the background space-time, generically intersect at a finite number of discrete points. Using the background orientation, these points can be assigned a sign. The algebraic sum of these signs over the points of intersection defines the intersection number in a way that it is unaffected by homology. It is instructive to visualise this in the case of the two-torus when a pair of one-cycles are complementary. As \( b_1 = 2 \) there are two natural one-cycles to consider and these intersect at one point. To define a self-intersection number two copies of the same one-cycle are considered. Of course these intersect at all their points but this is remedied by displacing one copy to a homologous cycle and considering points of intersection of this with the other copy. This can always be done in such a way that there are no intersections. In fact the self-intersection number of any one-cycle on a two-manifold always vanishes. On a four-manifold a pair of two-cycles are complementary and so self intersection numbers can be defined, and in this case do not necessarily vanish.

As explained above, the homology classes of the physically relevant two-cycles on a four-manifold space-time form a lattice. Choosing a basis for it, \( \Sigma_1, \Sigma_2, \ldots \Sigma_{b_2} \) we can define a \( b_2 \times b_2 \) matrix \( Q^{-1} \) formed of the intersection numbers

\[
(Q^{-1})_{ij} = I(\Sigma_i, \Sigma_j).
\]

It is yet another consequence of Poincaré duality that this matrix is unimodular, that is, has determinant equal to \( \pm 1 \). So both \( Q^{-1} \) and its inverse \( Q \) have integer entries. In addition \( Q \) is symmetric (whereas the \( b_1 \times b_1 \) analogue for two-manifolds is antisymmetric, thereby explaining why all self-intersection numbers vanish there). So the lattice of two-cycle homology classes on \( M_4 \) is furnished with a scalar product.

Furthermore, \( Q \) must be diagonalisable, being real and symmetric, and \( b_2^+ \) and \( b_2^- \) can be defined as the number of its positive and negative eigenvalues respectively. This completes the definition of the Hirzebruch signature as \( b_2^+ - b_2^- \), mentioned above.

Such unimodular matrices fall naturally into two classes, called even or odd, according as their diagonal entries are all even or not. Examples with \( b_2^- = 0 \) are, respectively, the Cartan matrix for the \( E_8 \) Lie algebra and the unit matrix. In fact, when neither of \( b_2^\pm \) vanish, all odd integer unimodular matrices are, after a change of basis, given by a diagonal matrix with \( b_2^+ \) entries 1 and \( b_2^- \) entries \( -1 \) on the diagonal. Even unimodular matrices only occur when the signature \( b_2^+ - b_2^- \) is a multiple of eight and can be constructed from the odd unimodular matrix there by a construction presented in the penultimate section below. When either of \( b_2^\pm \) vanishes no corresponding classification theorem is known.

Given a Maxwell field strength as a closed two-form \( F \), it is natural to form the exterior product \( F \wedge F \) which provides a closed four-form which can be integrated over \( M_4 \). For non-abelian gauge theories the result is familiar as the instanton number (once the necessary trace is taken). Maxwell theory has no instanton number and instead the result is quadratic in the magnetic fluxes:

\[
\int_{M_4} F \wedge F = \sum_{ij} \int_{\Sigma_i} F \cdot Q_{ij} \int_{\Sigma_j} F.
\]
The analogue of this for two-manifolds is known as the Riemann bilinear identity, and fundamental in the theory of Riemann surfaces. Further simplification follows on insertion of the quantised values of the fluxes according to whichever of the two conditions above, scalar or spinor, is appropriate. Notice how a nontrivial value requires $b_2$ to be nonzero, that is a space-time with non-trivial topological structure with which to capture the magnetic fluxes.

**Relation between $w(\Sigma)$ and $Q$ from the Index Theorem**

Four-dimensional space-times of the type considered possess yet another special property not valid in higher dimensions. Although it is not always possible to support charge neutral spinor wave functions it is always possible to support charged spinor wave functions providing the background electromagnetic field satisfies the flux quantisation conditions above. Mathematicians say that there always exists a spin$_C$ structure but not necessarily a spin structure.

Because of this it is always possible to formulate a Dirac operator $D_A$ including a minimal coupling to $A$, and try to solve the Dirac equation $D_A \psi = 0$. The index of $D_A$ is the difference between the numbers of solutions of opposite chirality, and according to the Atiyah-Singer index theorem, given by

$$\text{INDEX}(D_A) = \frac{1}{2} \left( \frac{q}{2\pi \hbar} \right)^2 \int_{\mathcal{M}_4} F \wedge F - \frac{\eta(M_4)}{8}. $$

We have already met both terms on the right hand side: the second involves the Hirzebruch signature and the first the integral just calculated. Inserting the spinor version of the Dirac quantisation conditions yields

$$\frac{1}{2} \sum_{ij} (m_i + \frac{w_i}{2}) Q_{ij} (m_j + \frac{w_j}{2}) - \frac{\eta}{8},$$

where $w_i = w(\Sigma_i)$ and the integer $m_i$ is determined by the flux through $\Sigma_i$. In view of what has been said this index expression has to be an integer for all values of the integers $m_i$ parametrising the consistent background fields, despite the non-integral nature of the two individual terms. By a trivial rearrangement and an abbreviation of notation

$$\frac{1}{2} (mQm + mQw) + \frac{wQw - \eta}{8} \in \mathbb{Z}.$$

Thus the integrality of the index reduces to the two conditions

$$wQw - \eta \in 8\mathbb{Z},$$

$$mQm + mQw \in 2\mathbb{Z}, \quad \text{for all} \quad m_i \in \mathbb{Z}.$$
means that \( wQw \) is unambiguous modulo 8, in accordance with the first condition. Of course if \( Q \) is even, the zero vector is a legitimate characteristic vector and the first condition reduces to a previous statement, that the signature of \( Q \) is a multiple of eight if it is an even unimodular matrix.

It is a trivial piece of algebra to verify that the solution of the second equation is \( w_i = (Q^{-1})_{ii} \), at least modulo two, which is precisely what we want. Thus the components of the characteristic vector of \( Q \) are given by the diagonal elements of its inverse, namely the self-intersection numbers of the \( \Sigma_i \). More generally, for any cycle \( \Sigma \),

\[
w(\Sigma) = I(\Sigma, \Sigma) + 2\mathbb{Z}.
\]

that is the self-intersection number as claimed earlier. More generally, the conclusion is that the matrix \( Q \), or equivalently \( Q^{-1} \), carries the essential topological information.

This sort of argument was introduced by S Hawking and C Pope [10] when they considered the four-manifold \( \text{CP}(2) \) which has \( b_2 = 1 \) and hence \( Q = \pm 1 \).

**Maxwell Partition Functions**

The way is ready for tests of quantum electromagnetic duality. It is familiar that the energy-momentum tensor and the equations of motion of Maxwell theory respect an \( SL(2, \mathbb{R}) \) group of symmetry transformations, even if the Lagrangian does not. Since this statement involves mixing \( F \) and its Hodge dual, \( *F \), it is required that \( \mathcal{M}_4 \) be endowed with a metric that is Minkowski in nature so that there is a single time, rather than a Euclidean metric. To find the quantum version it seems reasonable to consider the partition function \( Z = Tr(e^{-E}) \) as \( E \), the energy, is invariant classically. Indeed the first intimation of quantum duality, in this case a \( \mathbb{Z}_2 \) version, was found by H Kramers and G Wannier [11] by studying the partition function of the Ising model.

According to old ideas of R Feynman [12] and M Kac the partition function \( Z \) can be expressed as a Feynman path integral integrated over the field degrees of freedom. But in this representation time is automatically imaginary because of the form of the exponential \( e^{-E} \), and furthermore, possesses periodic boundary conditions because of the trace. So, now the partition function reads

\[
Z(\tau) = Tr(e^{-E}) = \int \cdots \int \delta A e^{i \int_{EUCLIDEAN} W_{EUCLIDEAN}(S_1 \times \mathcal{M}_3)},
\]

where the Euclidean action \( W_{EUCLIDEAN} \) is integrated over the four-manifold \( S_1 \times \mathcal{M}_3 \) with periodic time on the circle, all equipped with a Euclidean metric. This is what may be called the strict partition function. It is a popular procedure to consider a more general object, no longer necessarily real and positive, that may be called the extended partition function. In this the Euclidean action is obtained by integration over any (Poincaré dual) four-manifold, \( \mathcal{M}_4 \), equipped with a Euclidean metric,

\[
\int \cdots \int \delta A e^{i \int_{EUCLIDEAN} W_{EUCLIDEAN}(\mathcal{M}_4)}.
\]
The exponent, \(i/\hbar\) times the Euclidean action, \(W_{\text{EUCLIDEAN}}\), is actually a complex number. The real part is always negative, thereby guaranteeing a good convergence of the integration, whilst the imaginary part is proportional to precisely the integral that has already been evaluated to yield a quadratic form in magnetic fluxes.

To see this in detail it is preferable to work in terms of a field strength rescaled so as to have dimensionless fluxes; \(f = qF/\hbar\). In terms of this

\[
\frac{1}{\hbar}W_{\text{EUCLIDEAN}}(\mathcal{M}_4) = \frac{1}{4\pi} \int_{\mathcal{M}_4} f \wedge \hat{\tau}f,
\]

where \(\hat{\tau} = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{2i\pi\hbar}{q^2}\) and \(\ast\) is again the Hodge dual. Since the metric involved is now Euclidean \(\ast\) has square plus one. Taking the eigenvalue +1 yields the complex variable

\[
\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{2i\pi\hbar}{q^2}
\]

which encodes the dimensionless couplings which parametrise the theory. The imaginary part \(\tau_2\) is the inverse of the fine structure constant and \(\theta\) the vacuum angle. The dependence of the partition functions as functions of this variable can be made remarkably explicit by an argument based on the semi-classical approximation.

**Semi-Classical Evaluation**

Since the partition functions are expressed as integrals of phase factors, they can be evaluated by semi-classical methods as a sum of contributions from points of stationary phase. As the integral is Gaussian the results can be expected to be exact as E Verlinde [5] and E Witten [4] first pointed out.

The stationary points of the action are given by solutions to Maxwell’s equations, \(df = 0 = d\ast f\), where the rescaled field strengths are constrained to possess fluxes quantised in accordance with the conditions appropriate to the support of scalar or spinor wave functions.

Since the metric in the integral is Euclidean, Hodge’s theorem applies and states that the number of linearly independent solutions (in the sense of real coefficients) equals the second Betti number \(b_2\). The following normalisation determines a basis \(f^1, f^2, \ldots f^{b_2}\):

\[
\int_{\Sigma_j} f^i = \frac{q}{\hbar} \int_{\Sigma_j} F^i = 2\pi \delta^i_j.
\]

Solutions respecting the two flux quantisation conditions are, respectively,

\[
f = \sum_i m_i f^i \quad \text{or} \quad \sum_i \left( m_i + \frac{w_i}{2} \right) f^i, \quad m_i \in \mathbb{Z}.
\]

The values of the \(f \wedge f\) term in the action action can be found immediately, by inserting the bilinear identity above. To evaluate the other term it is necessary to realise that
*f^i also satisfies Maxwell’s equations and hence must be expressible in terms of the basis of solutions as

\[ *f^i = (GQ^{-1})_{ij} f^j \]

where \((GQ^{-1})^2 = 1\)
as the Hodge * squares to unity. \(G\) is a symmetric, positive definite matrix depending on the conformal class of the Euclidean metric.

Then the contribution to \(iW_{EUCLIDEAN}/\hbar\) labelled by the integers \(m_i\) is either \(i\pi m^T \Omega(\tau)m\) or \(i\pi (m + w/2)^T \Omega(\tau)(m + w/2)\), where \(\Omega(\tau) = \tau_1 Q + i\tau_2 G\). Putting everything together the two possible partition functions associated with scalar and spinor wave functions are respectively (apart from a constant factor),

\[
Z_0(\tau) = \tau_2^{-b_1 - 1/2} \sum_{m_i \in \mathbb{Z}} e^{i\pi m^T \Omega(\tau)m},
\]

\[
Z_w(\tau) = \tau_2^{-b_1 - 1/2} \sum_{m_i \in \mathbb{Z} + w/2} e^{i\pi m^T \Omega(\tau)m}.
\]
The prefactor is the contribution of Gaussian fluctuations and is the same for all stationary points [4]. It is these expressions that are sufficiently explicit that the response to modular transformations can be found.

**Modular Group Action**

It is immediate that, when \(Q\) is even, \(Z_0(\tau)\) is invariant under the effect of \(T: \tau \rightarrow \tau + 1\) whereas, if \(Q\) is odd, it is not but instead is invariant under the effect of \(T^2\). The effect of \(S: \tau \rightarrow -1/\tau\) on \(Z_0\) can be calculated using the Poisson summation formula and exploiting the unimodular nature of \(Q\) that stems from Poincaré duality:

\[
Z_0(-1/\tau) = e^{-\frac{2i\pi m}{8\tau}} \frac{\chi_{-\eta}}{\chi_{-\eta}} Z_0(\tau).
\]

Thus when \(Q\) is even, both \(S\) and \(T\) act nicely on \(Z_0\). Since the whole modular group consisting of fractional linear transformations,

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1,
\]
is generated by \(S\) and \(T\), \(Z_0\) is then covariant under this action. This is like the original \(SL(2,\mathbb{R})\) action in the classical theory, mentioned above, broken to a discrete subgroup by quantum effects. However when \(Q\) is odd the nice action on \(Z_0\) is generated by \(S\) and \(T^2\), which yields a subgroup of index three in the full modular group. This is how the matter was left by E. Verlinde and E. Witten.

With the extra work above concerning fractional fluxes when \(Q\) is odd, the remedy is apparent. When \(Q\) is even, \(w\) vanishes and the two partition functions are the same. When \(Q\) is odd they differ and both come into play. A rather general argument is
given in the next section to show that when both are taken into account together with a third, \((Z_0(\tau + 1))\), the whole modular group action is again realised.

Notice that the \(S\) action shows that the action of the modular group is accompanied by factors with “modular weights” \(\frac{\chi}{\eta^2}\), where \(\chi\) and \(\eta\) are the Euler number and Hirzebruch signature of the four-manifold \(M_4\) entering the extended partition function. These are precisely those topological numbers, independent of any choice of metric, which are local.

All this applies to the two extended partition functions. The strict partition function, \(\text{Tr}(e^{-E})\), is given as the special case when \(M_4\) factorises as \(S_1 \times M_3\). Then both of the numbers, \(\chi\) and \(\eta\), vanish and \(Q\) is even so \(Z_0\) and \(Z_w^2\) coincide. Furthermore this strict partition function, which is the physically meaningful quantity in Minkowski space, is indeed invariant under the full modular group.

**Theta Functions and Even Integral Lattices**

In order to understand how the action of the modular group relates the two versions of the partition function which are relevant when \(Q\) is odd, some ideas are developed that seem to have some intrinsic interest.

Suppose a slightly more abstract language is introduced and a lattice \(\Lambda\) is endowed with a non-singular scalar product. Then the reciprocal lattice \(\Lambda^*\) can be defined (so that \(\Lambda \cdot \Lambda^* \in \mathbb{Z}\)). \(\Lambda\) is said to be integral if \(\Lambda \subset \Lambda^*\). In this case \(\Lambda^*\) decomposes into cosets with respect to \(\Lambda\). These cosets form a finite abelian group denoted \(\Lambda^*/\Lambda \equiv Z(\Lambda)\), say. If \(Z(\Lambda)\) possesses only one element then \(\Lambda = \Lambda^*\) and is unimodular.

If \(\Lambda\) is even, that is \(\ell^2\) is an even integer for any element \(\ell\) of \(\Lambda\), and the scalar product is positive definite, then there is known to be a nice construction for \(|Z(\Lambda)|\) theta functions, one for each coset of \(\Lambda\) in \(\Lambda^*\), denoted \(\lambda_j + \Lambda\) where \(\lambda_j\) is a representative element, by

\[
\theta_j(\tau) = \sum_{\ell \in \lambda_j + \Lambda} e^{\pi i \tau \ell^2}.
\]

These are holomorphic in the upper half plane of the complex variable \(\tau\) and support an action of the modular group acting on \(\tau\) by the usual fractional linear transformations. For example, the effect of \(T: \tau \to \tau + 1\) is simply

\[
\theta_j(\tau + 1) = e^{\pi i \ell^2} \theta_j(\tau).
\]

This would not work if the lattice \(\Lambda\) were odd. But whether it is odd or even it is not difficult to use the Poisson summation formula to find that \(S: \tau \to -1/\tau\) has nice action on these theta functions. This is sufficient as \(S\) and \(T\) generate the whole modular group. These results are well known and explained in the book by Green, Schwarz and Witten [13], for example.

The magnetic charge lattice defined previously with scalar product coming from the intersection matrix, \(Q\), differs in that it is actually unimodular, and hence integral, but more importantly in that the scalar product need not be positive definite. Indeed \(\eta\) measures the signature. That means that the corresponding theta functions have to
be more complicated, exactly as was found above. Nevertheless the same sort of coset construction can again be performed, now for even integral indefinite lattices. The corresponding theta functions are no longer holomorphic in the upper half $\tau$ plane but they are still highly convergent there (because of the structure coming from $G$). The action of the modular group can be evaluated in detail and is very similar to the case of positive definite $Q$.

The $S$ and $T$ actions are not totally independent because of the relation within the modular group, $(ST)^3 = -1$. The consistency of this leads to an identity, known as Milgram's formula in the context of the theory of even lattices [14],

$$|Z(\Lambda)| \sum_{j=1}^{\mid Z(\Lambda) \mid} e^{i\pi \lambda_j^2} = \sqrt{|Z(\Lambda)|} e^{\frac{2i\pi \eta}{8}}.$$

Now return to the situation that $Q$ is an odd unimodular matrix. Corresponding to it is an odd unimodular lattice, $\Lambda$ and two partition functions, $Z_0$ and $Z_{w/2}$, relevant to the fluxes supporting complex scalar and spinor wave functions respectively. They involve sums over $\Lambda$ and $\Lambda + \frac{w}{2}$ in the new notation. Consider the union of these two sets:

$$\Lambda_{TOTAL} = \Lambda \cup (\Lambda + \frac{w}{2}).$$

This is closed under addition as $\Lambda$ is and $w$, being its characteristic vector, lies in it. This means $\Lambda_{TOTAL}$ is a lattice. $\Lambda$ itself can be split into two pieces, $\Lambda_{EVEN}$ and $\Lambda_{ODD}$ according as the scalar product with $w$ is even or odd. Then $\Lambda_{EVEN}$ is an even lattice and is reciprocal to $\Lambda_{TOTAL}$

$$\Lambda_{EVEN}^* = \Lambda_{TOTAL}.$$ 

So we have an example of the situation considered above with an even lattice whose reciprocal could be decomposed into $|Z(\Lambda_{EVEN})| = 4$ cosets, each with their own theta function. Here

$$\Lambda_{TOTAL} = \Lambda_{EVEN} \cup \Lambda_{ODD} \cup (\Lambda_{EVEN} + \frac{w}{2}) \cup (\Lambda_{ODD} + \frac{w}{2}).$$

Corresponding to this are four theta functions spanning the space upon which the full modular group acts. Two linear combinations yield the two partition functions $Z_0$ and $Z_{w/2}$ and that is why the modular group acts on them. Actually a certain linear combination of the four theta functions vanishes, leaving just three needed to support the modular group action, as claimed earlier.

Milgram's formula above can be specialised to $\Lambda_{EVEN}$ and implies that $w^2 - \eta$ should vanish, mod 8, the result that was previously deduced from the index theorem.

**More on Odd Unimodular Lattices**

Although not strictly relevant to the main argument it is interesting to develop the above construction of an even integral lattice $\Lambda_{ODD}$ from an odd unimodular
lattice $\Lambda$ a step further. Consider two other subsets of the above coset decomposition of $\Lambda_{\text{TOTAL}}$,

$$\Lambda' \equiv \Lambda_{\text{EVEN}} \cup \left( \Lambda_{\text{EVEN}} + \frac{w}{2} \right) \quad \text{and} \quad \Lambda'' \equiv \Lambda_{\text{EVEN}} \cup \left( \Lambda_{\text{ODD}} + \frac{w}{2} \right).$$

By Milgram’s formula, when the signature, $\eta$, is even, the characteristic vector $w$ lies in $\Lambda_{\text{EVEN}}$, and as a consequence, $\Lambda'$ and $\Lambda''$ are both lattices. Furthermore they are integral, unimodular lattices if $\eta$ is a multiple of 4. These are even when $\eta$ is a multiple of 8 and odd otherwise. This procedure is interesting because it can generate non-trivial unimodular lattices from trivial ones, for example, the $E_8$ root lattice from the hypercubic lattice $\mathbb{Z}^8$.

Corresponding to this geometry the modular behaviour of the theta functions simplifies as explained in [3].

Discussion

The arguments presented to support the physical ideas of quantum electromagnetic duality have illustrated what seems to be a beautiful interplay between different mathematical ideas: homology theory, spin obstructions, index theorems, theory of integral lattices, theta functions and the modular group, and so on. This is reassuring but it indicates that there is a deeper underlying structure still to be found.

Unfortunately the test of quantum electromagnetic duality considered is rather crude and it is therefore desirable to formulate more stringent tests. It is odd that the most interesting calculations involve what were called the extended partition functions. Except for the strict partition function which occurred as a special case, these have a rather unclear physical interpretation. For example, what is the physical significance of the Euclidean metric used in the construction of the Euclidean action? It would be wrong to think of it as being obtained from a Minkowski metric by some sort of Wick rotation. Indeed the Euler number $\chi(M_4)$ need not vanish, and this would leave no possibility for a Minkowski metric on $M_4$.

The work has relied on special features of four-manifolds not valid in higher dimensions, but these have been equipped with Maxwell fields which, being two-forms, are mid-forms. It can be anticipated that something similar happens in space-times of higher even dimension when, again, mid-forms are considered together with their putative coupling to the appropriate branes, instead of particles. But this needs checking and, indeed, requires new mathematics as a good way of considering brane wave functions seems to be lacking so far.

In particular, the success of the work so far has depended on careful attention to signs associated with spinor structures and it is important to understand how the simple geometric interpretation valid on four-manifolds can be extended to higher dimensions. It is even more difficult to see how to extend the analysis to superstring theories with the same level of precision.

I am grateful to Miguel Virasoro for organising such a pleasant meeting. I wish to thank Marcos Alvarez for his collaboration in respect of the work in [2] and [3].
also wish to repeat my thanks to those cited there. Finally my thanks for their helpful
comments on the present manuscript go to Stephen Howes, Luis Miramontes and Ani
Sinkovics.

[1] DI Olive; “Exact Electromagnetic Duality” Nucl. Phys.(Proc Suppl) 45A (1996)
88-102.
[2] M Alvarez and DI Olive; “The Dirac quantisation condition for fluxes on Four-
manifolds”, Commun. Math. Phys.210 (2000) 13-28, hep-th/9906093.
[3] M Alvarez and DI Olive; “Spin and Abelian Electromagnetic Duality on Four-
manifolds”, Commun. Math. Phys.217 (2001) 331-356, hep-th/0003155.
[4] E Witten; “On S-duality in abelian gauge theory”, Selecta Math (NS) 1 (1995),
383-410. hep-th/9505186.
[5] E Verlinde; “Global aspects of electric-magnetic duality”, Nucl. Phys. B455 (1995),
211-228.
[6] O Alvarez; “Topological Quantisation and Cohomology”, Commun. Math. Phys.
100 (1985) 279-309.
[7] PAM Dirac; “Quantised singularities in the electromagnetic field”, Proc. Roy. Soc.
A133 (1931) 60-72.
[8] TT Wu and CN Yang; “Concept of non-integrable phase factors and global formul-
ation of gauge fields”, Phys. Rev. D12 (1975), 3845-3857.
[9] PAM Dirac; “Gauge invariant formulation of Quantum Electrodynamics”, Cana-
dian Journal of Physics 33 (1955), 650-660.
[10] SW Hawking and CN Pope; “Generalised spinor structures in quantum gravity”,
Phys Lett B73 (1978) 42-44.
[11] HA Kramers and GH Wannier; “Statistics of the Two-Dimensional Ferromagnet
I”, Phys. Rev.60 (1941), 252-276.
[12] RP Feynman and AR Hibbs; Quantum Mechanics and Path Integrals, Chapter
10, McGraw-Hill, 1965,
RP Feynman; Statistical Mechanics, Chapter 3, Benjamin/Cummings, 1972.
[13] M Green, J Schwarz and E Witten; Superstring Theory, Vol. 2, Appendix 9B,
Cambridge University Press, 1987.
[14] J Milnor, D. Husemoller; Symmetric bilinear forms, Ergebnisse der Mathematik
und ihrer Grenzgebiete, Band 73, Springer-Verlag, 1973.