Can one extract source radii from transport theories?

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Abstract

To know the space time evolution of a heavy ion reaction is of great interest, especially in cases where the measured spectra do not allow to ascertain the underlying reaction mechanism. In recent times it became popular to believe that the comparison of Hanbury-Brown Twiss correlation functions obtained from classical or semiclassical transport theories, like Boltzmann Uehling Uhlenbeck (BUU), Quantum Molecular Dynamics (QMD), VENUS, RQMD or ARC, with experiments may provide this insight. It is the purpose of this article to show that this conjecture encounters serious problems. The models which are suited to be compared with the experiments at CERN and Brookhaven are not able to predict a correlation function. Any agreement with existing data has to be considered as accidental. The models suited for lower energies can in principle predict correlation functions. The systematic error may be
too large to be of use as far as quantitative conclusions are concerned.
I. INTRODUCTION

It is a common problem in heavy ion reactions between 25 MeV/N and 200 GeV/N that the single particle spectra do not allow to ascertain the underlying reaction mechanism. To mention only two examples: At low energies despite of many years of efforts the fragmentation of nuclei into many intermediate mass fragments remains still a process whose origin is heavily debated. At high energies it turned out to be very difficult to rule out a hadronic scenario which may produce the same spectra as those proposed as a signal for the creation of a quark gluon plasma.

In such a situation a search for experimental information beyond the single particle spectra is obvious. Most valuable would be an information on the spatial structure of the reaction. It would allow to calculate key quantities like densities or energy densities. This information is, however, hard to obtain.

The only promising method proposed up to now is based on the interferometry of identical particles. The interference of the amplitudes of two indistinguishable processes gives rise to a correlation function which in principle allows to extract the radius of the emitting source. This approach has been very successfully applied by Hanbury-Brown and Twiss [1] in astronomy to determine the angular radius of stars by measuring the spatial correlations between two photons. Later Goldhaber [2] and Kopylov and Podgoretsky [3] advanced its application in particle physics by showing that measurable momentum space correlations may contain information on the size of the emitting source.

In the ideal case of a large, randomly emitting source of known shape this method is indeed very powerful and the experimental results can be directly related to the source radius of the emitting object. In particle and heavy ion physics the situation is, however, much more difficult. There we encounter quite a number of problems. The size of the emitting sources is of the order of the radius of a nucleus and therefore
not small as compared to the size of the wave function of the emitted particles. This renders some approximations impossible. The signal may be distorted by final state interactions between the emitted particles or by the long range Coulomb force of the source. The emission time point cannot be defined unambiguously. The decay of resonances into identical particles or correlations between the momenta of the emitted particles and the coordinates of the emission point may pretend a wrong size of the source. For a discussion of these problems we refer to ref. \cite{5}, \cite{6}. Recently it has been discussed that the HBT correlation function for an expanding source, as encountered frequently in heavy ion reactions, yields a much more difficult relation between the space time structure of the emitting source and the correlation function as that for a static source \cite{7}.

Due to these problems the measured correlation function in heavy ion collisions cannot be directly related to the parameters of the emitting source even if its form were known. In this situation there are two possibilities. Either one assumes the form of the source and uses the measured correlation function to fix the source parameters. Unfortunately this procedure makes these parameters model dependent. Hence they cannot be used for more than a comparison between different experiments and yield little information on the actual source properties. Or one tries to describe the reaction in its entity. This turns out to be a quite complicated procedure but became very popular recently. In this approach one follows the time evolution of the system with help of one of the standard transport models. Unfortunately none of them propagates (anti)symmetrized wave functions but at most a direct product wave function. Since the HBT effect is based on the (anti)symmetrization of the wave function of identical particles the transport model themselves cannot predict the correlation function. Rather one assumes that each particle ”freezes out” at some time point. The freeze out time is different for each particle. At high energies it is assumed that the freeze
out time is that time at which the particle encounters its last collision with another particle of the system. At low energies, where potential interactions are important as well, the freeze out time cannot be unambiguously defined. The freeze out times as well as the particle momenta and positions form then the input for the subsequent calculation of the Hanbury-Brown and Twiss (HBT) correlation function [8] which is then compared with experiment.

Agreement is usually interpreted as a sign that the underlying transport model gives a realistic space time evolution of the different particles. Since these transport codes provide not only the momentum space coordinates of the particles but also that of the coordinate space they can then be used to calculate the time evolution of key quantities like the energy density or the density.

The weak point in this procedure is the transition between the transport model and the subsequent program which calculates the correlation function. Does the transport model provide the correct time evolution of those quantities which are essential for the calculation of the correlation function?

It is the purpose of this article to show that this is hardly the case. In order to understand the reason one has to understand in detail the derivation of the different transport models from the fundamental quantal equations as well as the derivation of the equation which is employed to determine the correlation function. We will perform this investigation for the three types of present day simulation programs: The Quantum Molecular Dynamics approach (QMD) [9], BUU type models like Boltzmann Uhling Uhlenbeck (BUU) [10]–[11], Vlasov Uhling Uhlenbeck (VUU) [12] or Landau Vlasov (LV) [13] and cascade models. The later class includes also the high energy simulation programs like VENUS [14], RQMD [15] and ARC [16].

This problem is independent of the relativistic or nonrelativistic nature of the approach. It also does neither depend on the time between the emissions of the
two identical particles nor on the presence of resonances. It is also independent of a possible final state interaction between the particles which is therefore omitted. The common demand on all transport programs is that an emission time point can be defined. Essential is the information the programs provide at that time point. This information is quite different for the three types of programs mentioned above and hence the systematic errors are specific to each of the different transport models. In two cases (QMD and BUU) this procedure implies systematic errors which question the usefulness of the approach for its original purpose: The discrimination between different reaction mechanisms which yield the same single particle spectra. For the high energy simulation programs the correlation function is completely artificial.

For clarity we limit our formalism to the simplest form possible by assuming that we are dealing with two bosons which are simultaneously emitted and can be treated nonrelativistically. For this simple case the formalism is very transparent. More realistic but also more complicated scenarios [7] may add additional problems but do not overcome the problems discussed here.

II. THE CORRELATION FUNCTION

We start with the derivation of the correlation function which relates the freeze out points with the measurable two body correlation function. We assume that a source, which is considered as classical, emits simultaneously two identical bosons with momenta $\vec{p}_1$ and $\vec{p}_2$. The differential two body probability $W$ reads then as follows [17] ($\hbar, c = 1$):

$$\frac{d^2W}{dp_1 dp_2} = |T_S(\vec{p}_1, \vec{p}_2, \alpha, \beta)|^2$$

(1)

$\alpha$ and $\beta$ characterize the emitting source. The (anti)symmetrized production amplitude $T_S$ is given by
\[ T_S(\vec{p}_1, \vec{p}_2, \alpha, \beta) = \frac{1}{\sqrt{2}}(T(\vec{p}_1, \vec{p}_2, \alpha, \beta) \pm T(\vec{p}_2, \vec{p}_1, \alpha, \beta)). \]  

where \( T(\vec{p}_1, \vec{p}_2, \alpha, \beta) \) is the Fourier transform of the wave function

\[ T(\vec{p}_1, \vec{p}_2, \alpha, \beta) = \int \frac{d^3x_1 d^3x_2}{(2\pi)^6} e^{-i(\vec{p}_1 \cdot \vec{x}_1 + \vec{p}_2 \cdot \vec{x}_2)} < \vec{x}_1, \vec{x}_2 | \psi(\alpha, \beta) >. \]

Introducing the Wigner density of the two body density matrix \( \rho_2 = |\psi_2 > < \psi_2| \)

\[ D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) = \frac{1}{(2\pi)^6} \int \prod_{i=1,2} d^3y_i e^{i\vec{p}_i \cdot \vec{y}_i} < \vec{x}_1 - \vec{y}_1/2, \vec{x}_2 - \vec{y}_2/2 | \psi_2 > < \psi_2 | \vec{x}_1 + \vec{y}_1/2, \vec{x}_2 + \vec{y}_2/2 > 
\]

we can express the probability as a function of the two particle Wigner density \[ 6 \]

\[ \frac{d^2W}{d\vec{p}_1 d\vec{p}_2} = \int d^3x_1 d^3x_2 [D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) \pm D(\vec{x}_1, \frac{\vec{p}_1 + \vec{p}_2}{2}, \vec{x}_2, \frac{\vec{p}_1 + \vec{p}_2}{2}) \cos(\vec{p}_1 - \vec{p}_2)(\vec{x}_1 - \vec{x}_2)]. \]

Hence for the calculation of this probability the transport theories have to provide the two body Wigner density. Unfortunately most of them do not permit to calculate this quantity. Therefore one has introduced an approximation, called smoothness assumption (SA) :

\[ D(\vec{x}_1, \frac{\vec{p}_1 + \vec{p}_2}{2}, \vec{x}_2, \frac{\vec{p}_1 + \vec{p}_2}{2}) \approx D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2). \]

We will discuss the limits of its validity which turns out to be crucial in the course of the article. Employing the smoothness assumption the two body probability reads as

\[ \frac{d^2W^{SA}}{d\vec{p}_1 d\vec{p}_2} = \int d^3x_1 d^3x_2 D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2)(1 \pm \cos(\vec{p}_1 - \vec{p}_2)(\vec{x}_1 - \vec{x}_2)). \]

This is the standard expression for the two particle probability employed in numerous publications to relate the measured cross section with the radius of the emitting source.
Depending on the available information, for actual calculations one may have to employ further approximations. For BUU type equations, which propagate the one particle Wigner density only, one assumes that the correlations between particles are negligible

\[ D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) \approx D(\vec{x}_1, \vec{p}_1) \cdot D(\vec{x}_2, \vec{p}_2) \]  

whereas for classical cascade calculations one assumes that the quantal two body Wigner density can be replaced by the classical 2 body phase space density \( F_{cl} \)

\[ D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) \approx F_{cl}(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2). \]  

For a static source without any correlation between the emission point and the momentum of the emitted particle the correlation function, the quantity one compares with experiment, is independent of the center of center of mass motion of the emitted pair and is given by

\[ C(\vec{p}) = \frac{\int d^2W \cdot d^3p}{\int d^2W \cdot d^3P} \]  

where \( \vec{P} = \frac{(\vec{p}_1 + \vec{p}_2)}{2} \) is the center of mass momentum, \( \vec{p} = \vec{p}_1 - \vec{p}_2 \) is the relative momentum and \( \frac{dW}{d\vec{p}_1} \) is the one particle momentum distribution

\[ \frac{dW}{d\vec{p}_1} = \int d^3x_1 D(\vec{x}_1, \vec{p}_1). \]  

In the general case, where correlations are present, \( C(\vec{p}) \) depends on the center of mass motion as well. As we will see the correlation function \( C(\vec{p}) \) contains the desired information about the spatial properties of the emitting source.

**III. CONSEQUENCES OF THE SMOOTHNESS ASSUMPTION FOR CASCADE CALCULATIONS**

One of the classes of models employed to extract source radii by comparing experimental results with model predictions are the so called cascade models. These
are classical n-body models which solve the Hamilton equations of a n-body system and are presently the only models available to simulate heavy ion reactions at CERN and Brookhaven energies. They include VENUS \[14\], RQMD \[15\] (in its usually employed cascade version) and ARC \[16\] as well as now less frequently employed programs for heavy ion reactions at an energy of around 1 GeV/N \[19\]. In these models the particles do not interact via potentials but suffer two body collisions if they come sufficiently close in coordinate space. In between the collisions the particles move on straight lines. In the computer programs they are treated as classical particles with a sharp momentum and a sharp position.

One may ask how classical models can be employed to calculate a correlation function which is solely based on the interference of amplitudes and hence a genuine quantal effect. For an understanding we have to make a detour. In order to employ eq. 5 we have to construct the Wigner density out of the classical two body phase space density. This is of course not unique but the approach

\[
F_{cl} = \delta(\vec{x}_1 - \vec{x}_\alpha)\delta(\vec{p}_1 - \vec{p}_\beta)\delta(\vec{x}_2 - \vec{x}_\beta)\delta(\vec{p}_2 - \vec{p}_\beta)
\]

\[
= \lim_{C \to \infty, D \to 0} \frac{C^3 D^3}{\pi^6} e^{-\left(\vec{P} - \vec{R}(t)\right)^2 C/4 - \left(\vec{X} - \vec{R}(t)\right)^2 D/4} e^{-\left(\vec{p} - \vec{k}(t)\right)^2 C - \left(\vec{x} - \vec{r}(t)\right)^2 D} \equiv D_{cl}(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2)
\]

serves our purpose. The expression in the last line will be considered as Wigner density. Here we have used the definitions

\[
\vec{P} = \vec{p}_1 + \vec{p}_2; \vec{K} = \vec{p}_\alpha + \vec{p}_\beta
\]

\[
\vec{X} = \frac{\vec{x}_1 + \vec{x}_2}{2}; \vec{R} = \frac{\vec{x}_\alpha + \vec{x}_\beta}{2}
\]

\[
\vec{\rho} = \frac{\vec{p}_1 - \vec{p}_2}{2}; \vec{k} = \frac{\vec{p}_\alpha - \vec{p}_\beta}{2}
\]

\[
\vec{x} = \vec{x}_1 - \vec{x}_2; \vec{r} = \vec{x}_\alpha - \vec{x}_\beta.
\]

The Wigner density \(D_{cl}(\vec{x}_1, \frac{\vec{p}_1 + \vec{p}_2}{2}, \vec{x}_2, \frac{\vec{p}_1 + \vec{p}_2}{2})\) is obtained by replacing \(\vec{p}_1\) and \(\vec{p}_2\) by \(\frac{\vec{p}_1 + \vec{p}_2}{2}\).
Please note that this Wigner density does not respect the uncertainty relation. Inserting this expression in eqs. 5 and 7 and performing the limit procedure we obtain for the two particle correlator without smoothness assumption
\[
\frac{d^2 W}{d\vec{p}_1 d\vec{p}_2} d^3 P = \int d^3 x_1 d^3 x_2 d^3 P \left[D_{cl}(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) \pm D_{cl}(\vec{x}_1, \frac{\vec{p}_1 + \vec{p}_2}{2}, \vec{x}_2, \frac{\vec{p}_1 + \vec{p}_2}{2}) \cos(\vec{p}_1 - \vec{p}_2)(\vec{x}_1 - \vec{x}_2)\right]
\]
\[= \delta(\vec{p} - \vec{k}(t)) \pm \delta(\vec{k}(t)) \cos(2\vec{p}\vec{r}). \tag{14}\]

This expression differs from
\[
\int \frac{dW}{d\vec{p}_1} \frac{dW}{d\vec{p}_2} d^3 P = \delta(\vec{p} - \vec{k}(t)) \tag{15}\]

only for the case that the relative momentum of the emitted classical particles is zero what in practical terms never happens. For all other cases we find
\[
C(\vec{p}) = 1. \tag{16}\]

Applying the smoothness assumption we find, however,
\[
\int \frac{d^2 W^{SA}}{d\vec{p}_1 d\vec{p}_2} d^3 P = \delta(\vec{p} - \vec{k}(t))(1 + \cos 2\vec{p}\vec{r}) \tag{17}\]

and hence
\[
C^{SA}(\vec{p}) = 1 + \cos 2\vec{p}\vec{r}. \tag{18}\]

Thus we observe that here the smoothness assumption creates correlations out of nothing. One faces the somewhat surprising result that the correlation function and hence the extracted radii are artificial and are only due to the differences between the approximate and the exact formula for the correlation function. Applying the correct formula the cascade calculations do not yield any correlation function, as the exact result shows.

The truth of this observation can even easily be verified without any calculation. If two particles with a sharp momentum are emitted from two localized sources one...
can measure the momentum sufficiently precise in order to identify the source from which each particle has been emitted provided that both momenta are not identical. Thus there are no alternative processes, hence no interference of their amplitudes and there is, as a consequence, no HBT correlation function. This has unfortunately the consequence that there is presently no microscopic model which may be used for the interpretation of the correlation data measured with ultrarelativistic heavy ion beams at CERN and AGS.

IV. QUANTUM MOLECULAR DYNAMICS (QMD)

The discussion of models which provide sufficient information to construct a correlation function we begin with the QMD approach because it is the only one which allows to calculate the 2 body Wigner density. Hence one can calculate \( \frac{d^2W}{d\vec{p}_1d\vec{p}_2} \) (eq.5) without any approximation. One can furthermore introduce the smoothness assumption and can calculate then \( \frac{d^2W_{SA}}{d\vec{p}_1d\vec{p}_2} \) applying eq. 7. This may serve as a test for the validity of this approximation in the situation of a heavy ion reaction and hence for the judgement of the predictive power of the correlation function calculated in the framework of the other models.

The QMD model is a n body theory which simulates heavy ion reactions between 30 MeV/N and 2 GeV/N on an event by event basis. Each nucleon is represented by a coherent state of the form

\[
\phi_\alpha(\vec{p}_1, t) = \left( \frac{L}{2\pi} \right)^{3/4} e^{-(\vec{p}_1 - \vec{p}_\alpha(t))^2 L/4} e^{-i\vec{p}_1 \vec{x}_\alpha(t)} e^{+i\vec{p}_\alpha^2(t)t/2m}
\] (19)

Thus the wave function has two time dependent parameters \( x_\alpha, p_\alpha \), \( L \) is fixed. As we will see this wave function serves as a test wave function for a variational principle. Hence it is an input of the calculation and not the result of the solution of the Schrödinger equation. It relies heavily on intuition; other test wave functions may
yield a different time evolution of the system. The total n body wave function is assumed to be the direct product of n coherent states

$$\phi = \phi_\alpha(\vec{x}_1, \vec{x}_\alpha, \vec{p}_\alpha, t) \phi_\beta(\vec{x}_2, \vec{x}_\beta, \vec{p}_\beta, t) \cdots,$$  \hspace{1cm} (20)

thus antisymmetrization is neglected. The initial values of the parameters are chosen in a way that the ensemble of $A_T + A_P$ nucleons gives a proper density distribution as well as a proper momentum distribution of the projectile and target nuclei. The time evolution of the system is calculated by means of a generalized variational principle: We assume that $\vec{p}_\alpha$ and $\vec{x}_\alpha$ contain the essential time dependence of the n-body wave function. The Lagrange function $\mathcal{L}$ can then be written as a functional of these parameters where $H$ is the n-body Hamiltonian.

$$\mathcal{L} = \left( \phi \left| i\hbar \left( \frac{\partial}{\partial t} + \frac{d\vec{p}_\alpha}{dt} \frac{d\vec{x}_\alpha}{dt} + \frac{d\vec{x}_\alpha}{dt} \frac{d\vec{p}_\alpha}{dt} \right) - H \right| \phi \right).$$ \hspace{1cm} (21)

The time evolution of the parameters is obtained by the requirement that the action

$$S = \int_{t_1}^{t_2} \mathcal{L}[\phi, \phi^*] dt$$ \hspace{1cm} (22)

is stationary under the allowed variation of the wave function. For the wave function of eq. 20 the Lagrange function is given up to a constant by

$$\mathcal{L} = \sum_\alpha (\vec{p}_\alpha \dot{\vec{x}}_\alpha - \vec{p}_\alpha \dot{\vec{p}}_\alpha + \frac{\vec{p}_\alpha^2}{2m} - \frac{1}{2} \sum_\beta V(\vec{x}_\alpha, \vec{x}_\beta)).$$ \hspace{1cm} (23)

$V(\vec{x}_\alpha, \vec{x}_\beta)$ is the expectation value of the (density dependent) 2 body potential. The variation of the Lagrange function gives Euler Lagrange equations for each of the 6 parameters

$$\dot{\vec{p}}_\alpha = -\vec{\nabla} \vec{x}_\alpha \sum_\beta V(\vec{x}_\alpha, \vec{x}_\beta)$$ \hspace{1cm} (24)

and

$$\dot{\vec{x}}_\alpha = \vec{p}_\alpha / m.$$ \hspace{1cm} (25)
With these two equations one has reduced the problem of solving a \( n \)-body Schrödinger equation to that of solving \( 6n \) ordinary differential equations. In reality \( V \) is a parametrization of the real part of the Brückner G-matrix. The imaginary part is approximated by the measured cross section. For details we refer to ref. \([9]\).

Hence in QMD the centroids of the Gaussians in momentum and coordinate space are the only quantities which change in time. The form of the wave function around the centroids is fixed. This is a consequence of the ansatz (eq. 19).

From the test wave function eq. (19) we calculate the Wigner density of a pair of particles

\[
D_{\text{QMD}}(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) = \frac{1}{\pi^6} e^{-\left(\vec{P} - \vec{K}(t)\right)^2 L^4/4 - \left(\vec{X} - \vec{R}(t)\right)^2 4L^2 L - \left(\vec{r} - \vec{r}(t)\right)^2/4L}. \tag{26}
\]

Inserting this Wigner density in eq. 5 one obtains after integration over the pair’s center of mass momentum

\[
\int \frac{d^2W}{d\vec{p}_1 d\vec{p}_2} d^3P = \left(\frac{L}{\pi}\right)^{3/2} \left(e^{-\left(\vec{p} - \vec{k}(t)\right)^2 L} \pm e^{-p^2 L - k(t)^2 L \cos 2\vec{r}}\right) \tag{27}
\]

where \( t \) is the (assumed common) freeze out time. This is the probability to find two particles with a relative momentum \( \vec{p} \), which have been emitted from two classical sources at a relative distance of \( \vec{r} \) and a relative momentum of \( \vec{k} \).

V. ONE BODY TRANSPORT THEORIES

In order to derive the equation for the time evolution of the one-body Wigner density of a particle moving in a selfconsistent potential \( V(\vec{x}) \) we start from that for the one body density matrix \( \rho_1 = |\psi_1 > < \psi_1| \)

\[
\dot{\rho}_1 = -i[H, \rho_1]. \tag{28}
\]

Applying to this equation the Wigner transformation for an operator \( O \)
\[ O_W(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^3} \int d^3y e^{i\vec{p}\vec{y}} < \vec{x} - \frac{\vec{y}}{2} | O | \vec{x} + \frac{\vec{y}}{2} > \]  

(29)

one obtains the differential equation

\[ \left( \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_{\vec{x}_1} \right) D(\vec{x}_1, \vec{p}_1, t) = \int d^3p' K_1(\vec{p}_1 - \vec{p}_1', \vec{x}_1) D(\vec{x}_1, \vec{p}_1', t) \]  

(30)

D being the Wigner density of the one body density operator and \( K_1 \) is defined as

\[ K_1(\vec{p}_1 - \vec{p}_1', \vec{x}_1) = \frac{1}{i\hbar} \int \frac{d^3y}{(2\pi \hbar)^3} e^{-i(\vec{p}_1 - \vec{p}_1')\vec{y}/\hbar} (V(\vec{x} + \vec{y}/2) - V(\vec{x} - \vec{y}/2)). \]  

(31)

We have restored \( \hbar \) here for reasons which will soon become obvious. One can expand the integration kernel around \( x_1 \)

\[ K_1(\vec{p}_1 - \vec{p}_1', \vec{x}_1) = \frac{2}{\hbar} \sin \left( \frac{\hbar \vec{\nabla}_{\vec{x}_1} \vec{\nabla}_{\vec{p}_1}}{2} V(\vec{x}_1) \right) \delta(\vec{p}_1 - \vec{p}_1'). \]  

(32)

We see that \( K_1 \) can be viewed as a series with the expansion coefficient \( \hbar \vec{\nabla}_{\vec{x}_1} \vec{\nabla}_{\vec{p}_1} \). Hence in the limit that the expansion can be terminated after the first term the Schrödinger equation in its Wigner representation is equivalent to the classical Vlasov equation:

\[ \left( \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_{\vec{x}_1} \right) D(\vec{x}_1, \vec{p}_1, t) = \left( \vec{\nabla}_{\vec{x}_1} V(\vec{x}_1) \right) \vec{\nabla}_{\vec{p}_1} D(\vec{x}_1, \vec{p}_1, t) \]  

(33)

The Vlasov equation describes the time evolution of the phase space density of particles which move on classical orbits specified by the Hamilton equations \( \frac{\partial \vec{x}_1}{\partial t} = \vec{p}_1/m \) and \( \frac{\partial \vec{p}_1}{\partial t} = -\vec{\nabla}_{\vec{x}_1} V \). As in QMD V presents the real part of the Brückner G- matrix and the imaginary part is added as a cross section.

There are two approaches to solve the above equation. Either one solves the differential equation directly or one creates a swarm of test particles which are subject to the Hamilton equations and fulfill the initial condition \( D(\vec{x}_1, \vec{p}_1, t_0) \). One propagates this swarm with help of the Hamilton equations until a time \( t \) and then constructs the Wigner density \( D(\vec{x}_1, \vec{p}_1, t) \) by coarse graining. The latter solution method is called test particle method and is employed in the BUU, VUU and LV approaches.
When calculating the observables, i.e. the expectation values of operators, the
transition from the first to the second method corresponds to the replacement of an
analytical integration by a Monte Carlo procedure. Using the swarm of test particles
the analytical solution

\[ \langle O(t) \rangle = \int D(\vec{x}_1, \vec{p}_1, t)O(\vec{x}_1, \vec{p}_1) d^3x_1 d^3p_1 \]  

is replaced by the corresponding Monte Carlo type integral

\[ \langle O(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} O(\vec{r}_i(t), \vec{k}_i(t)) \]  

where the \( \vec{r}_i(t) \) and \( \vec{k}_i(t) \) are the phase space coordinates of the \( N \) test particles prop-
agated with the Hamilton equations. As said, they are distributed like \( D(\vec{x}_1, \vec{p}_1, t) \).
According to the theory of the Monte Carlo integration both integration procedures
yield the same result in the limit of an infinite number of test particles. In practice
one has to verify that the results do not depend on this number. Usually 100 test
particles per physical nucleon in the system are considered as sufficient. It is very
important to realize that these test particles have nothing to do with physical nucle-
ons. They serve only as a representation of the one body Wigner den-
sity \( D(\vec{x}_1, \vec{p}_1, t) \).
All observables which require more than its knowledge are beyond the scope of ap-
PLICABILITY of these theories. Hence the possibility to extract source radii and hence
correlation functions from the one body theories requires:

- The smoothness assumption is valid

\[ D(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2, t) \approx D(\vec{x}_1, \vec{p}_1, t)D(\vec{x}_2, \vec{p}_2, t) \]

They are a consequence of the impossibility to create two body Wigner densities or
Wigner densities of two body observables like \( \vec{p}_1 + \vec{p}_2 \) from the swarm of test particles
defined as above. If both approximation were valid the correlation function is given
by
\[
\frac{d^2W}{dp_1 dp_2} = \int d^3x_1 d^3x_2 D(x_1, p_1) D(x_2, p_2) (1 \pm \cos(p_1 - p_2)(x_1 - x_2))
\]
\[
= \frac{1}{N(N-1)} \sum_{i \neq j} (1 \pm \cos(p_i(t_0) - p_j(t_0))(x_i(t_0) - x_j(t_0)))
\]

(36)

t_0 is the (assumed common) freeze out time. The second approximation, the absence of two body correlations, is hard to control. The importance of many particle correlations for the fragment formation has been discussed in [18] but its relevance for the proton or pion emission has not yet been investigated.

VI. RESULTS FOR A GIVEN SOURCE DISTRIBUTION

To interpret the different results given above it is useful to apply them to a situation where the source is known. To keep the things simple we assume a completely chaotic source without any correlation between coordinate and momentum space:

\[
S(\vec{k}, \vec{r}) = \left( \frac{B}{A \pi^2} \right)^{3/2} e^{-\vec{k}^2/2 - \vec{r}^2/2A}.
\]

(37)

We start out from the Wigner density (eq.26) for a pair of particles as given in the QMD simulation. Averaging over the Gaussian source distribution we obtain for the correlation function as in eq.5

\[
C(\vec{p}) = \frac{\int d^2W \int d^3PS(\vec{k}, \vec{r})d^3kd^3r}{\int d^2W \int d^3PS(\vec{k}, \vec{r})d^3kd^3r} = 1 \pm e^{-p^2(L+A - \frac{L_B}{L+B})}
\]

(38)

where \(S(\vec{k}, \vec{r}) = \int S(\vec{k}_1, \vec{r}_1) \cdot S(\vec{k}_2, \vec{r}_2)d^3Kd^3R\). If we apply the smoothness assumption (eq.6) we obtain

\[
C^{SA}(\vec{p}) = \frac{\int d^2W^{SA} \int d^3PS(\vec{k}, \vec{r})d^3kd^3r}{\int d^2W^{SA} \int d^3PS(\vec{k}, \vec{r})d^3kd^3r} = 1 \pm e^{-p^2(L+A)}.
\]

(39)

If we assume as in BUU, VUU or LV that the one body Wigner density is not given by Gaussians but as a sum over test particles (TP) each represented by a delta function in coordinate and momentum space
\[ D^{TP}(\vec{x}_1, \vec{p}_1, t) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(\vec{x}_1 - \vec{x}_\alpha(t))\delta(\vec{p}_1 - \vec{p}_\alpha(t)) \]  

where the \( p_\alpha \)'s and \( x_\alpha \)'s are distributed according to our source function we obtain as a correlation function

\[ C^{TP}(\vec{p}) = 1 \pm e^{-p^2 A}. \]

Defining the square of the source radii as \( \int \frac{C(\vec{p}) d^3 p}{\int C(\vec{p}) d^3 p} \) and comparing eqs. 38,39,41 we observe that for the same measured correlation function \( C(\vec{p}) \) we obtain different source radii depending on the simulation programs and the approximations used. From a mathematical point of view the difference between \( C^{TP}(\vec{p}) \) and \( C^{SA}(\vec{p}) \) is easy to understand. Because the wave function used for the calculation of \( C^{SA}(\vec{p}) \) has a width of \( L \), the true distribution of the source is the convolution of the distribution of the centers given by \( S(\vec{r}, \vec{k}) \) with the distribution of the one particle density around the centers. For \( C^{TP}(\vec{p}) \) one assumes that the true source distribution is given by \( S(\vec{r}, \vec{k}) \). In order to make both quantities comparable, both mean square radii have to be the same and hence one has to replace in \( C^{TP}(\vec{p}) \) \( A \) by \( A' = L + A \).

However, being purely mathematical, this argument has an essential drawback. We have started out from the approximation (eq.1) that the source can be treated classically, and hence that the distance between two sources is large as compared to \( L \). Hence, either the difference between \( A' \) and \( A \) is small and can be neglected. Then our approximation is valid. Or this difference is not negligible. Then our classical source approximation breaks down. That the wave function plays indeed a nontrivial role can be seen if one compares \( C^{SA}(\vec{p}) \) and \( C(\vec{p}) \). In both cases the same single particle wave function has been employed. The result for the correlation function is, however, different. Only if \( L << A \) the difference between both is negligible. Hence our quantitative result confirms the well known qualitative argument that the smoothness assumption is only valid if the source can be assumed to be classical, i.e. if
the width of the wave function is small as compared to the size of the emitting system. Opposite, if $L$ is of the same order as $A$ as in all presently employed simulation models, the difference becomes important as we will see below and hence the smoothness assumption will break down. Hence we are confronted with the fact that present day simulation programs use a value of $L$ which neither justifies the classical treatment of the source nor confirms the validity of the smoothness assumption.

Nevertheless it seems that the community has agreed upon a pragmatic point of view in pretending that at least the classical treatment of the source is acceptable in modeling heavy ion collisions although a proof has not be given yet. Hence it may be useful to see whether under this assumption a quantitative prediction is possible. This includes the answer to two questions: To what precision we desire to measure the density and is the systematic error of the correlation function sufficiently small to obtain the desired precision.

The study of the space time correlation is born out of the demand to measure the size of the system at the moment when the particles are emitted. If we study nucleons of the fireball, the density has to be in between twice and half normal nuclear matter density, because if the expanding fireball passes the latter density, there are no interactions anymore and hence the emission of particles defined as the time point of the last collision has ceased. For nucleons emitted from the spectator matter which remains at normal nuclear matter density one would like to know the source size. At lower energies the interest is to study whether the emitted nucleons come from a compound nucleus or whether they are emitted from a subsystem called hot spot. Also here the density varies little around normal nuclear matter density. Whereas in the first case an uncertainty of the density determination of about 20% may be tolerable, the latter two require a precision of the determination if the source radius by about 3% (and hence of the mass number of about 10%) if one
would like to avoid that the uncertainty is already as large as the possible variation of the size of the system under investigation.

In order to see whether this precision can be obtained we have to calculate the values for A, L and B for the cases of interest. If we assume that the rms radius of the source corresponds to the size of a nucleus at normal nuclear matter density $R_0 = 1.2A_M^{1/3}$ where $A_M$ is the atomic number of the nucleus we obtain

$$A = A_M^{2/3}[fm^2]$$

i.e. $A = 21.5 \text{ fm}^2$ for $A_M = 100$ and $A = 34 \text{ fm}^2$ for $A_M = 200$. Because our source emits particles according to a Maxwell Boltzmann distribution we can relate the slope in momentum space with the temperature of the emitting source and find that

$$B = \frac{40}{T[\text{MeV}]}[fm^2]$$

In the standard versions of QMD resp. IQMD the parameter L has the value 4.33 and 8.66 $\text{fm}^2$, respectively. Hence first of all we observe that L is not at all negligible as compared to A. However, as mentioned above, accepting a classical treatment of the source we can correct for this. It remains to be seen whether the smoothness assumption can be justified. Comparing the mean square radii obtained with and without the smoothness assumption

$$F = \frac{R}{R^{SA}} = \frac{(L + A - \frac{LB}{L+B})}{(L + A)} = 1 - \frac{LB}{(L + B)(L + A)}$$

we find that the smoothness assumption pretends a larger radius of the system. The value of F ranges between .87 for small systems at low temperature (5 MeV) and .98 for large systems at high temperature (80 MeV). Hence for particles emitted from a
compound nucleus or from the spectator matter the error in the determination of the mass number due to the smoothness approximation is of the order of 20% even if the source is completely chaotic and of known form and the classical approximation of the source remains valid. For fireball nucleons the smoothness assumption produces an error of about 4% on the density. Of course if we were sure that we have a source of a given temperature we could also correct for the temperature, however such a source is not encountered in heavy ion physics where the excitation energy and hence the temperature changes in the course of time.

VII. REALISTIC SIMULATIONS

We have seen that for the most favourable condition (chaotic source of known form without any momentum space coordinate space correlation) the smoothness assumption enlarges the apparent source size by about 20%. If one applies now the simulation programs to real experiments one has to inspect the consequences of two facts:

1) Nature most probably does not keep the rms radius of a nuclear wave function constant during a heavy ion reactions, QMD does. For observables which do not depend on the width of the wave function explicitly this may be of minor importance, the influence on observables which depend explicitly on the width, like the correlation function, is hard to judge since no calculations are available for a more sophisticated treatment of the reaction as done in QMD. If the width of the wave function has changed in the course of the reaction the difference between $C^{TP}(\vec{p})$ and $C^{SA}(\vec{p})$ cannot be corrected anymore by use of the known initial density distribution.

In the QMD calculations the width of the wave function $L$ serves two purposes. First it is used to have the proper one body density distribution when one initialize the nuclei. This is, however, a very weak condition because with much larger widths than
that actually employed one can obtain the same one particle density distribution. Second, it appears in the time evolution equations but only in form of the expectation value of the potential. Thus what counts for the time evolution is the convolution of the potential range and the width of the wave functions. Hence one can obtain the same expectation value of the potential for a smaller width and a larger potential range. Hence there is no need for an exact determination of the width $L$ in the QMD calculation or, vice versa, the success of these calculations cannot be used to determine $L$.

2) The source is as simulation programs show not at all chaotic and shows strong correlations between momenta and positions.

Momentum space coordinate space correlations decrease the source size extracted from the correlation function as compared to the geometrical size of the source. This can be easily understood if one goes to the extreme. If the momentum is a monotonic function of the position, two particles with a small relative momentum have to come from the places very close in coordinates space. Thus the correlation function measures only that region in coordinate space from where these particles can come. Hence the stronger the momentum space coordinate space correlations are the smaller is the source size measured by the correlation function. As a consequence the value of $A$ becomes smaller and the importance of the width of the wave function increases. Thus the stronger these correlations are the larger becomes the difference between $R^{SA}$ and $R$ (eq. 44).

Thus for realistic calculations the situation becomes worse as compared to a static source. For a given size of the system correlations make $A$ smaller and hence increase the importance of the width of the wave function if one compares eqs. 39 and 41. They also do not allow to corrected for the smoothness assumption (eqs. 38 and 39) because the temperature is not anymore a global variable. The calculation of the
systematic error of the value of the radius determined by eqs. 39 or 41 requires more than the present models can predict, however the above arguments show that it will be larger than that for a static source.

VIII. CONCLUSION

We have discussed the possibility to extract source radii by comparing the experimental results with the prediction of simulation programs. There is no doubt that the experimental results indeed show momentum space correlations caused by the bosonic or fermionic nature of the observed hadrons. These correlations carry information about the space time structure of the reaction. The goal to relate the observed correlation functions in momentum space with physical parameters in coordinate space like source radii, densities or energy densities can presently only achieved by use of transport theories.

None of these transport models takes the bosonic or fermionic nature of the hadrons into account. Whereas this may be no essential drawback for many observables it makes it impossible to calculate the correlation function in a straightforward manner. Every model requires the introduction of the (anti)symmetrization of the wave function in an ad hoc fashion in order to predict a correlation function.

We have found that for all presently existing models, which can be subdivided into three classes, this introduction poses problems.

Cascade models, in which classical particles are propagated, do not allow the calculation of a correlation function. The quoted values of source radii are totally artificial being a consequence of the employed approximation and not of physical origin.

QMD, LV, BUU and VUU models allow the calculation of a correlation function. We find, however, that the basic approximation of the whole approach, namely that
the source can be considered as classical, is not fulfilled. Even if it were fulfilled, the
systematic error of the extracted density introduced by the smoothness approxima-
tion is for the most favourable case of a chaotic source of known form up to 20%. For
realistic cases where space momentum space correlations are present and where we
do not know the form of the source we have shown that the error will increase. This
questions the possibility that in nuclear physics the HBT method allows a determi-
nation of the density to a precision which allows to discriminate between different
proposed interaction mechanisms.

Of course this raises the question how to proceed. As we have seen we are plagued
with systematic errors of the order L/A. There is first of all the open question whether
the wave function of a emitted nucleon is smaller than L. Mean field calculation yield
a much broader wave function and consequently the approximation of a classical
source cannot be justified any more. Short range correlations, however, may distort
this wavefunction. Hence it may be justified to address the question if there is a
possibility to construct dynamical theories which can provide a prediction for the
correlation function? Either one can try to decrease L or to avoid the systematic
errors.

The first suggestion implies a localization of the particles with a precision of about
1fm. This will be hardly possible. Not only because in a nuclear environment the root
mean square radius of the wave function of the nucleon is considerably larger than
the radius of a free nucleon but also because it implies an uncertainty of 200 MeV/c
for the momentum of the nucleons which poses several severe technical problems for
transport theories:

- How to propagate particles in semiclassical theories whose velocity uncertainty
  is about 0.2c is unknown.

- The sequence of collisions becomes undetermined
• The applied scattering cross sections have to be modified because the scattering partners are asymptotically not in a plane wave state.

The second suggestion implies the construction of transport theories which propagates at least two particle wave functions and not parameters of the wave function. Presently such an approach is not available.

Before a solution to these problems has been found the Hanbury Brown Twiss effect is a very nice quantal effect. Its application in nuclear physics to study the space time structure remains, however, premature.

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