Elasticity Theory in General Relativity

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The general relativistic theory of elasticity is reviewed from a Lagrangian, as opposed to Eulerian, perspective. The equations of motion and stress–energy–momentum tensor for a hyperelastic body are derived from the gauge–invariant action principle first considered by DeWitt. This action is a natural extension of the action for a single relativistic particle. The central object in the Lagrangian treatment is the Landau–Lifshitz radar metric, which is the relativistic version of the right Cauchy–Green deformation tensor. We also introduce relativistic definitions of the deformation gradient, Green strain, and first and second Piola–Kirchhoff stress tensors. A gauge–fixed description of relativistic hyperelasticity is also presented, and the nonrelativistic theory is derived in the limit as the speed of light becomes infinite.

I. INTRODUCTION

Elasticity theory in the nonrelativistic regime is a well–developed subject, with applications to many branches of engineering and science. (See, for example, Refs. [1–14].) The special relativistic theory dates back to Herglotz [5]. The work extended to general relativity by DeWitt [6], who tied extra degrees of freedom (a framework of “clocks”) to the material elements. The elastic material with clocks provides a natural coordinate system that DeWitt used to investigate a quantum theory of gravity. Some recent works on general relativistic elasticity include Carter and Quintana [7], Kijowski and Magli [8], Marsden and Hughes [9], Beig and Schmidt [10], Beig and Wernig–Pichler [11], Gundlach, Hawke and Erickson [12], Andersson [13], and Andersson, Oliynyk and Schmidt [14]. See also the extensive treatment by Wernig–Pichler [15].

Elastic materials differ from fluids in that they allow for the presence of shear stresses. In most astrophysical contexts, shear stresses are negligible. The material in most stars, jets, accretion disks and large planets are all well described as fluids. The interstellar and intergalactic media, and matter on cosmological scales, are modeled as fluids. One astrophysical context in which shear stress is important is the crust of a neutron star [16]. This is the main practical motivation for the development of a relativistic theory of elasticity. Of course, intellectual curiosity also serves as motivation. Can we compute, from first principles, the behavior of a rubber ball as it falls into a black hole?

An elastic body is described mathematically in terms of “matter space”, whose points coincide with the material elements (or particles) that make up the body. Coordinates on matter space serve as labels for the elements. The motion of the body through spacetime can be described using either the “Eulerian” or “Lagrangian” perspective. In the Eulerian approach, the independent variables are the matter space coordinates for the material element whose worldline passes through that event. With the Lagrangian approach, the independent variables are the matter space coordinates for a given element of the body, and a worldline parameter. The dependent variables are spacetime events.

The original works of Herglotz and DeWitt used the Lagrangian perspective. Many of the more recent studies of relativistic elasticity have focused on the Eulerian perspective [7, 8, 10, 12, 13]. Notable exceptions are found in Refs. [11, 14], where the Lagrangian description is used to address the existence and uniqueness of solutions to elastic body motion in general relativity.

This paper presents a detailed account of relativistic elasticity theory from the Lagrangian perspective. The approach is not mathematically rigorous and not mathematically sophisticated; basic tensor notation is used throughout. Many of the results can be found scattered throughout the literature [7–14], although these results can be difficult to recognize due to the variety of notations and the varying levels of mathematical rigor used by different authors. The goal here is to provide a comprehensive overview of the subject that is accessible to a wide audience.

One advantage of the Lagrangian approach is that it easily incorporates “natural” boundary conditions in which the surface of the body is free from physical constraints or forces. The Lagrangian description can also be more efficient computationally for a finite–size body, since the independent variables are the material points rather than the entire spacetime. A possible disadvantage of the Lagrangian description, as compared to the Eulerian description, is that it might be more difficult to treat material shocks and discontinuities.

In Sec. (II), we establish notation and introduce the kinematical relationships needed to describe relativistic elastic materials. Central to the description of elastic materials is the Landau–Lifshitz radar metric, discussed in Sec. (III). The radar metric defines the proper distance between neighboring material elements as measured in the local rest frame of an element. In Sec. (IV) we identify the radar metric as the right Cauchy–Green deformation tensor and define the Lagrangian (or Green) strain tensor. The action principle for a relativistic hyperelastic
body is presented in Sec. [IV]. The action principle determines the bulk equations of motion as well as the natural boundary conditions for the body. Section [VII] contains a detailed calculation of the stress–energy–momentum (SEM) tensor for the hyperelastic body. The spatial components of the SEM tensor are given by the second Piola–Kirchhoff stress tensor for the material. In Sec. [VII], we review the relativistic point particle. We begin the analysis using an arbitrary worldline parameter, then transform to the “gauge fixed” description by tying the parameter to a spacetime coordinate. Section [VII] repeats the analysis for the hyperelastic model, arriving at gauge fixed forms for the action principle and equations of motion. In Sec. [X] we obtain the nonrelativistic limit of the elastic body action and equations of motion. Section [XI] contains a discussion of various constitutive models for hyperelastic materials. Models are specified by giving the energy density as a function of either the Lagrangian strain, the Kirchhoff stress tensor for the material. In Sec. [XII], the stress invariants, or the principal stretches of the material, serve as coordinates in the three–dimensional matter space $\mathcal{S}$.

The sign conventions of Misner, Thorne and Wheeler [17] are used throughout.

II. KINEMATICS

Let $x^\mu$ denote the spacetime coordinates and $g_{\mu\nu}$ denote the spacetime metric. Indices on spacetime tensors are lowered and raised with $g_{\mu\nu}$ and it’s inverse $g^{\mu\nu}$, as usual.

A continuous body is a congruence of worldlines defined by $x^\mu = X^\mu(\lambda, \zeta)$ where the parameters $\zeta^i$ (with $i = 1, 2, 3$) label the continuum of material “particles” (or points, or elements) in the body and $\lambda$ parametrize the worldline of each particle. Typically the functions $X^\mu(\lambda, \zeta)$ are only defined for finite ranges of the labels $\zeta^i$. Correspondingly, the worldlines do not always fill the entire spacetime, but rather a subset of spacetime, the body’s “world tube.” The space of material particles (or elements) is called “matter space,” denoted $\mathcal{S}$. The labels $\zeta^i$ serve as coordinates in $\mathcal{S}$.

The functions $X^\mu(\lambda, \zeta)$ constitute a mapping from $\mathbb{R} \times \mathcal{S}$ to spacetime $\mathcal{M}$. That is, for each point $\zeta^i$ in matter space, $X^\mu(\lambda, \zeta)$ sweeps out a timelike worldline in spacetime as the parameter $\lambda$ ranges over real values; see Fig. (I). The inverse mappings are defined by $\lambda = \Lambda(x)$ and $\zeta^i = Z^i(x)$. Thus, given a spacetime event $x^\mu$ inside the world tube of the body, $\Lambda$ gives the parameter value $\lambda$ and $Z^i$ give the labels $\zeta^i$ of the point in the body that passes through that event. We therefore have the identities $\lambda = \Lambda(X(\lambda, \zeta))$ and $\zeta^i = Z^i(X(\lambda, \zeta))$, and differentiation with respect to $\lambda$ and $\zeta^i$ yields

$$X^\mu;_i Z^i_{,\mu} = \delta^\mu_i, \quad (II.1a)$$
$$\dot{X}^\mu \Lambda_{,\mu} = 1, \quad (II.1b)$$
$$X^\mu;_i \Lambda_{,\mu} = 0, \quad (II.1c)$$
$$\dot{X}^\mu Z^i_{,\mu} = 0. \quad (II.1d)$$

Here, “$\mu$” denotes $\partial/\partial x^\mu$, “$i$” denotes $\partial/\partial \zeta^i$, and the overdot denotes $\partial/\partial \lambda$. We can also use the identity $x^\mu = X^\mu(\Lambda(x), Z(x))$ to derive the useful relation

$$\dot{X}^\mu \Lambda_{,\mu} + X^\mu;_i Z^i_{,\mu} = \delta^\mu_i, \quad (II.2)$$

by differentiating with respect to the spacetime coordinates $x^\nu$ of events inside the body’s world tube.

The velocities of the material particles are defined by

$$U^\mu = \dot{X}^\mu/\alpha \quad (II.3)$$

where

$$\alpha = \sqrt{-\dot{X}^\mu \dot{X}_\mu}, \quad (II.4)$$

is the material lapse function. That is, $\alpha \, d\lambda$ is the interval of proper time along the material worldlines between $\lambda$ and $\lambda + d\lambda$. Equation (II.1I) tells us that the covectors $Z^i_{,\mu}$ are orthogonal to $U^\mu$:

$$U^\mu Z^i_{,\mu} = 0. \quad (II.5)$$

Thus, the vectors $Z^i_{,\mu} \equiv g^{\mu\nu} Z^i_{,\nu}$ are purely spatial as viewed in the rest frame of the material. That is, along a given material particle worldline, the vectors $Z^i_{,\mu}$ span the set of nearby events that are seen as simultaneous by an observer moving along that worldline.

III. RADAR METRIC

The radar metric is defined inside the world tube of the body by

$$f_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu. \quad (III.1)$$

This is sometimes called the Lagrangian metric [6]. The name radar metric stems from the analysis of Landau and Lifshitz [18], who show that this tensor defines the
distance between nearby objects by reflecting light rays between the objects’ worldlines\cite{Landau1975}.

The radar metric satisfies $f_{\mu \nu} U^\nu = 0$. It acts as a projection operator that projects tensors into the subspace orthogonal to the worldlines. For example, given a vector $V^\mu$ inside the body’s world tube, the vector $f^\mu_{\nu} V^\nu$ is orthogonal to the worldlines. Likewise, for any covector $W_\mu$ inside the body’s world tube, the covector $f^\mu_{\nu} W_\nu$ is orthogonal to the worldlines. A tensor is called “spatial” if it is orthogonal to the worldlines in each of its indices. If $V^\mu$ is a spatial vector, it satisfies $V^\mu = f^\mu_{\nu} V^\nu$. If $W_\mu$ is a spatial covector, it satisfies $W_\mu = f_{\mu \nu} W^\nu$.

The spacetime metric $g_{\mu \nu}$ defines the inner product between vectors. It also determines the spacelike or timelike separation between neighboring events, as follows. Consider a vector $V^\mu$ at some event $P$. We can construct a parametrized curve $X^\mu(\sigma) = V^\mu \sigma$ that passes through $P$ at $\sigma = 0$. The tangent to the curve at $P$ is $V^\mu = \partial X^\mu(\sigma)/\partial \sigma$. The coordinate separation between $P$ (at $\sigma = 0$) and the nearby event $\sigma = d\sigma$ is the “separation vector” $dx^\mu \equiv X^\mu(\sigma) - X^\mu(0) = V^\mu d\sigma$. The magnitude of the separation vector, given by the inner product of $V^\mu d\sigma$ with itself, defines the proper distance $ds^2 = g_{\mu \nu} V^\mu V^\nu d\sigma^2 = g_{\mu \nu} dx^\mu dx^\nu$.

If the separation vector $dx^\mu = V^\mu d\sigma$ at $P$ is a spatial vector, then the neighboring events are simultaneous as seen by the observer who is at rest with the material element that passes through $P$. For these events, the spacelike separation is given by $ds^2 = f_{\mu \nu} dx^\mu dx^\nu$. That is, at each event $P$ within the body’s world tube, the radar metric determines proper distances within the subspace orthogonal to the worldline passing through $P$.

The radar metric also acts as an inner product: $f_{\mu \nu} V^\mu W^\nu$. If $V^\mu$ and $W^\nu$ are non–spatial vectors, the radar metric eliminates the non–spatial components. Thus, the result $f_{\mu \nu} V^\mu W^\nu = (f^\mu_{\alpha} f^\alpha_{\nu}) g_{\mu \nu} (f^\alpha_{\beta} W^\beta)$ shows that $f_{\mu \nu} V^\mu W^\nu$ coincides with the inner product between spatial vectors $f^\mu_{\nu} V^\nu$ and $f^\mu_{\nu} W^\nu$, as defined by the spacetime metric.

Any spacetime tensor defined at a point in the world tube of the body can be mapped into the matter space $S$ by using the radar metric. For example, the spacetime vector $V^\mu$ is mapped to $S$ by $V^\mu = f^\mu_{\nu} Z_\nu$. The spacetime covector $W_\mu$ is mapped to $S$ by $W_\mu = f_{\mu \nu} X^\nu$. An important example is the radar metric and its inverse:

\begin{align*}
  f_{ij} &= X^\mu_i f_{\mu \alpha} X^\alpha_j, \quad \text{(III.2a)} \\
  f^{ij} &= Z^\mu_i f^{\mu \alpha} Z^\alpha_j. \quad \text{(III.2b)}
\end{align*}

Using Eqs. (III.1) and (III.2), one can verify that $f^{ij}$ is the inverse of $f_{ij}$. It is useful to note that $f^{ij} = Z^\mu_i g^{\mu \nu} Z^\nu_j$; however, $f_{ij} \neq X^\mu_i g_{\mu \nu} X^\nu_j$.

If $W_\mu$ is a spatial covector, then the definition $W_\mu = W_\mu X^\nu_i$ can be inverted by writing $W_\mu = W_i Z^\nu_i$. To verify this result, we use Eq. (III.2) and the fact that $X^\mu_i$ and $W_\mu$ are orthogonal. Note, however, that for a spatial vector $V^\mu$, the definition $V^\nu = V^\mu Z^\mu_i$ is not inverted in a similar way: $V^\mu \neq V^\nu X^\nu_i$. In particular, we have

\begin{align*}
  f_{\mu \nu} &= f_{ij} Z^\mu_i Z^\nu_j, \quad \text{(III.3a)} \\
  f^{\mu \nu} &= f^{ij} X^\nu_i X^\nu_j. \quad \text{(III.3b)}
\end{align*}

for the radar metric.

Generally, the mapping of tensors to $S$ is reserved for spatial tensors. For spatial tensors, the mapping preserves the raising and lowering of indices. Thus, consider $V^i = Z^i_\mu X^\mu V^\nu$ and $V_i = X^\mu_i V^\mu V^\nu$. If $V^i$ is a spatial vector, we can verify that $V_i = f_{ij} V^j$ and $V^i = f^{ij} V_j$. On the other hand, if $V^\nu$ is not spatial, the raising and lowering of indices is not preserved. Consider, for example, the material velocity $U^\nu$, which of course is not spatial. We have $U^i \equiv U^\mu Z^\mu_i = 0$ and $U_i \equiv U_\mu X^\mu_i$, which, in general, is not zero. Clearly $U_i \neq f_{ij} U^j$ and $U^i \neq f^{ij} U_j$.

The combination $U_\mu X^\mu_i$ appears sufficiently often that a shorthand notation is useful:

\begin{equation}
  v_i \equiv U_\mu X^\mu_i, \quad \text{(III.4)}
\end{equation}

These are the components of the spacetime velocity of the material, as a covector, projected into the subspace $\lambda = \text{const}$ and expressed in the coordinate system supplied by the matter space labels $\zeta^i$.

Consider the separation vector $dx^\mu = X^\mu(\lambda, \zeta + d\zeta) - X^\mu(\lambda, \zeta) = X^\mu_i d\zeta^i$ connecting nearby events $X(\lambda, \zeta)$ and $X(\lambda, \zeta + d\zeta)$. This separation vector is not spatial. However, as with any vector, we can construct its spatial component using the radar metric: $f^{\mu \nu} dx^\mu$. The magnitude of the spatial component of the separation vector is

\begin{equation}
  ds^2 = g_{\mu \nu} f^{\mu \nu} dx^\mu (f_{\beta \gamma} dx^\beta) = f_{\mu \nu} U^\mu_i X^\nu_j d\zeta^i d\zeta^j = f_{ij} d\zeta^i d\zeta^j. \quad \text{(III.5)}
\end{equation}

Thus, the radar metric on matter space, $f_{ij}$, defines the proper distance between the nearby events defined by projecting $X^\mu(\lambda, \zeta)$ and $X^\mu(\lambda, \zeta + d\zeta)$ into the subspace orthogonal to the particle worldline. That is, $ds^2 = f_{ij} d\zeta^i d\zeta^j$ is the (square of the) proper distance between the worldlines of the material particles $\zeta^i$ and $\zeta^i + d\zeta^i$, as seen in the rest frame of the material. See Fig. 2.

IV. STRAIN

In continuum mechanics\cite{Landau1975, Lifshitz1977}, the strain of the material is quantified by the deformation gradient. In the
II. RELATIVISTIC DESCRIPTION OF DYNAMIC DEFORMATION

III.1 Maxwellian Model

III.2 Boussinesq Model

IV.1 (IV.3b)

IV.10b

The radar metric at the event $X^\mu(\lambda, \zeta)$ defines the proper distance $ds$ between neighboring particles $\zeta^i$ and $d\zeta^i$ as measured in the rest frame of the material. The interval labeled $ds$ is orthogonal to the worldline. In general, the surfaces $\lambda = \text{const}$ are not orthogonal to the worldline.

In a relativistic context, we define the deformation gradient in terms of the radar metric and the mapping from $S$ to $M$ by

$$ F_{\mu i} \equiv f_{\mu i} X^\nu_{,i} . $$

The inverse of the deformation gradient is

$$ (F^{-1})^{\mu i} \equiv Z^i_{,\nu} g^{\nu\mu} . $$

One can check that

$$ F_{\mu i} (F^{-1})^{\nu i} = f^\nu_{\mu} $$

$$ (F^{-1})^{\mu i} F_{\nu j} = \delta^i_j $$

using the identities [IV.1]. The radar metric in matter space, $f_{ij}$, is called the right Cauchy–Green deformation tensor in continuum mechanics. We can write this in terms of the deformation gradient as

$$ f_{ij} = F_{\mu i} g^{\mu \nu} F_{\nu j} $$

Using the definitions [IV.1] for the deformation gradient and [II.1] for the radar metric, we find the previous definition [III.2a] of the radar metric, $f_{ij} = X^\mu_{,i} f_{\mu j} X^\nu_{,j}$.

The Lagrangian strain tensor (sometimes called the Green strain tensor) is

$$ E_{ij} = \frac{1}{2} (f_{ij} - \epsilon_{ij}) $$

where the relaxed metric $\epsilon_{ij}$ is the metric on matter space $S$ that characterizes the undeformed body. That is, when the body is relaxed (in flat spacetime with no vibrations, no rotations, and no bulk forces) the proper distance $ds^2 = \epsilon_{ij} d\zeta^i d\zeta^j$.

A number of other tensors appear in the literature on continuum mechanics. These tensors are not used in this paper, but we present them here for the sake of completeness.

The relativistic version of the left Cauchy–Green deformation tensor (sometimes called the Finger deformation tensor) is the spatial tensor

$$ B_{\mu \nu} = F_{\mu i} e^i_j F_{\nu j} = f_{\mu \alpha} X^\alpha_{,i} e^i_j f_{\nu \beta} X^\beta_{,j} , $$

where $e^{ij}$ is the inverse of $\epsilon_{ij}$. The inverse of the left Cauchy–Green deformation tensor is

$$ (B^{-1})^{\mu \nu} = g^{\mu \sigma} Z_{,\sigma}^{i} \epsilon_{ij} Z_{,\nu}^{j} g^{\nu \rho} , $$

so that

$$ B_{\mu \sigma} (B^{-1})^{\nu \rho} = f^\nu_{\mu} . $$

The Eulerian strain tensor (also called the Almansi strain tensor) is

$$ \epsilon_{\mu \nu} = \frac{1}{2} (f_{\mu \nu} - Z_{,\mu} Z_{,\nu}^{i} \epsilon_{ij}) . $$

Note that the Lagrangian and Eulerian strain tensors satisfy

$$ E_{ij} = X^\mu_{,i} \epsilon_{\mu \nu} X^\nu_{,j} , $$

$$ \epsilon_{\mu \nu} = Z_{,\mu} E_{ij} Z_{,\nu}^{j} . $$

Thus, the Lagrangian strain is just the Eulerian strain (which is a spatial tensor) mapped to matter space $S$.

V. ACTION AND EQUATIONS OF MOTION

A material whose behavior is only a function of the current state of deformation is called elastic. If the work done by stresses during the deformation process depends only on the initial and final configurations, the material is hyperelastic. In this case we can introduce an energy density as a function of the Lagrangian strain $E_{ij}$.

We will define the energy density $\rho(E)$ as the energy of a deformed material element per unit of physical volume occupied by the undeformed (relaxed) element. When the body is relaxed, the physical volume occupied by the coordinate cell $d^3\zeta$ is defined by $\sqrt{\epsilon} d^3\zeta$, where $\epsilon$ is the determinant of the relaxed metric $\epsilon_{ij}$. Thus, for the deformed body, the energy contained in the coordinate cell $d^3\zeta$ is given by $\sqrt{\epsilon} \rho(E) d^3\zeta$.

The relativistic action for a hyperelastic material is

$$ S[X, g] = -\int_{\lambda_1}^{\lambda_2} d\lambda \int_S d^3\zeta \sqrt{\epsilon} \alpha \rho(E) . $$

Here, $\alpha$ is the material lapse from Eq. [II.4], and $E_{ij}$ is the Lagrangian strain. This action defines the system from a Lagrangian perspective. That is, the dynamics are described using $x^\mu = X^\mu(\lambda, \zeta)$, with the matter space coordinates $\zeta^i$ as independent variables and the spacetime coordinates $x^\mu$ as dependent variables. Recent work on relativistic elasticity has focused on the Eulerian perspective. In that case the dynamics are described by $\zeta^i = Z^i(x)$, with the spacetime coordinates $x^\mu$ as independent variables and the matter space coordinates $\zeta^i$ as dependent variables.

The energy density $\rho(E)$ will typically depend on the relaxed metric $\epsilon_{ij}$ as well as $E_{ij}$. If the material properties are not uniform, the density will depend on the
matter space coordinates $\zeta^i$ as well. The energy density can also depend on other tensors in matter space, in addition to $E_{ij}$ and $\epsilon_{ij}$. For example, if the body has a crystal lattice structure, then the energy density will depend on a preferred frame (or vector fields) in $\mathcal{S}$. The energy density might also depend on the specific entropy density, which would appear as a scalar field in matter space. For notational simplicity, we won’t normally display the dependence of $\rho(E)$ on $\zeta^i$, $\epsilon_{ij}$, or any other matter space tensors.

The action \(\mathcal{V}_1\) depends on the dynamical variables $X^\mu$ directly, and also indirectly through the argument of the spacetime metric. Explicitly, the material lapse $\alpha$, as it appears in the action, is

$$\alpha = \sqrt{-X^\mu X^\nu g_{\mu\nu}(X)} .$$  \hspace{1cm} (V.2)

The Lagrangian strain $E_{ij}$ depends on the radar metric, which is explicitly given by

$$f_{ij} = X^\mu_{,i} \left( g_{\mu\nu}(X) + \frac{1}{\alpha^2} \hat{X}^\rho X^\tau g_{\rho\tau}(X) g_{\sigma\nu}(X) \right) X^\nu_{,j} .$$  \hspace{1cm} (V.3)

The specific properties of the hyperelastic material are determined by the functional form of the energy density $\rho(E)$, including its possible dependence on non-dynamical matter space tensors (such as $\epsilon_{ij}$) and coordinates $\zeta^i$.

The action \(\mathcal{V}_1\) is the natural extension of the action for a continuum of non–interacting (dust) particles. In that case the energy density $\rho(E)$ is a constant. We can specialize $S[X]$ to the action for the single relativistic particle by setting the density to $\rho = (m/\sqrt{\gamma}) \delta^3(\zeta - \zeta_0)$, with $\zeta_0$ some fixed point in $\mathcal{S}$. Then Eq. (V.1) reduces to

$$S_{\text{particle}}[X] = -m \int_{\lambda_1}^{\lambda_2} d\lambda \alpha$$  \hspace{1cm} (V.4)

with $\alpha$ defined in Eq. (V.2). The particle action is a functional of $X^\mu(\lambda, \zeta_0)$.

The equations of motion for the hyperelastic body follow from variation of $S[X, g]$ with respect to the fields $X^\mu(\lambda, \zeta)$. For this calculation, we need the results

$$\delta \alpha = -U_\mu \delta X^\mu - \frac{1}{2} \alpha U^\mu U^\nu \delta g_{\mu\nu} ,$$  \hspace{1cm} (V.5a)

$$\delta f_{ij} = \frac{2}{\alpha} F_{\mu(i} F^\mu_{j)} \delta X^\mu + 2 F_{\mu(i} \partial^\mu X^\nu_{,j)}$$

$$+ F^\mu_{i} F^\nu_{j} \delta g_{\mu\nu} .$$  \hspace{1cm} (V.5b)

Because the metric is evaluated at the spacetime event $x^\mu = X^\mu(\lambda, \zeta)$, it’s variation includes a contribution from the variation of the tensor components $g_{\mu\nu}(x)$ as well as a contribution from the variation of its argument:

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}(x)|_{x = X(\lambda, \zeta)}$$

$$+ \partial_\alpha g_{\mu\nu}(x)|_{x = X(\lambda, \zeta)} \delta X^\alpha(\lambda, \zeta) .$$  \hspace{1cm} (V.6)

The partial derivative of the metric can be written in terms of Christoffel symbols using $\partial_\sigma g_{\mu\nu} = 2\Gamma_{(\mu\nu)}^\sigma$.

The variation of the action \(\mathcal{V}_1\) is

$$\delta S = - \int_{\lambda_1}^{\lambda_2} d\lambda \int_{\mathcal{S}} d^3\zeta \sqrt{\gamma} \left[ \rho \delta \alpha + \frac{\alpha}{2} S^{ij} \delta f_{ij} \right] ,$$  \hspace{1cm} (V.7)

where

$$S^{ij} = \partial_\rho / \partial E_{ij}$$  \hspace{1cm} (V.8)

is the second Piola–Kirchhoff stress tensor. (Stress is discussed in more detail in the next section.) Using the results for $\delta \alpha$, $\delta f_{ij}$, and $\delta g_{\mu\nu}$ from above, we find

$$\delta S = \int_{\lambda_1}^{\lambda_2} d\lambda \int_{\mathcal{S}} d^3\zeta \sqrt{\gamma} \left[ (\rho U_\mu - S^{ij} F_{\mu(i} v_{j)}) \delta X^\mu \right.$$

$$- \alpha S^{ij} F_{\mu(i} \delta X^\nu_{,j)}$$

$$+ \alpha (\rho U^\alpha U^\beta - S^{ij} F^\alpha_{i} F^\beta_{j}) \Gamma_{\alpha \beta \mu} \delta X^\mu \right] .$$  \hspace{1cm} (V.9)

where $F_{\mu\nu}$ is the deformation gradient (V.4). The next step in deriving the equations of motion is to remove the derivatives from $\delta X^\mu$ through integration by parts. This generates endpoint terms in $\delta S$ at the initial and final parameter values $\lambda_1$ and $\lambda_2$, as well as terms on the boundary of matter space $\partial S$. With the initial and final configurations of the elastic body fixed, the variations in $X^\mu$ vanish at $\lambda_1$ and $\lambda_2$. This ensures that the endpoint terms in $\delta S$ vanish. For the matter space boundary $\partial S$, we assume that the surface of the elastic body is free. These are referred to as “natural” boundary conditions since they arise naturally from the variational principle. In the language of continuum mechanics, these are called force/stress or traction boundary conditions, with the external force chosen to vanish. Thus, the variation of the action becomes

$$\delta S = \int_{\lambda_1}^{\lambda_2} d\lambda \int_{\partial S} d^2\eta \sqrt{\gamma} \alpha n_i S^{ij} F_{\mu(i} \delta X^\mu .$$  \hspace{1cm} (V.10)

Here, $\eta^A$ (with $A = 1, 2$) are coordinates on the matter space boundary $\partial S$. The metric on the boundary has determinant $\gamma$, and the unit normal to the boundary is $n_i$. These are defined using the relaxed metric $\epsilon_{ij}$.

Setting the variation of the action to zero, the volume integral term gives the “bulk” equation of motion

$$0 = -\sqrt{\gamma} \frac{D}{D\lambda} (\rho U_\mu - S^{ij} F_{\mu(i} v_{j)}) + \frac{D}{D\lambda} \left( \frac{\sqrt{\gamma} \alpha S^{ij} F_{\mu(i} v_{j)} \right) .$$  \hspace{1cm} (V.11)
where the covariant derivatives with respect to $\lambda$ and $\zeta^i$ are defined by

$$\frac{D}{D\lambda} = \chi^\mu \nabla_\mu,$$

(V.12a)

$$\frac{D}{D\zeta^i} = X^{\mu i} \nabla_\mu.$$

(V.12b)

With $\delta S = 0$, the integral over the matter space boundary gives

$$0 = -\eta_j S^{ij} F_{i\mu}|_{\partial S}.$$

(V.13)

These are the natural boundary conditions.

We can write these equations in slightly more compact form by introducing the first Piola–Kirchhoff stress:

$$P^i_{\mu} \equiv F_{\mu j} S^{ij}.$$

(V.14)

Then the bulk equations of motion and natural boundary conditions become

$$\sqrt{-\rho} \frac{D}{D\lambda} (\rho U_\mu - P^i_{\mu} v_j) = \frac{D}{D\zeta^i} (\sqrt{\alpha} P^i_{\mu}),$$

(V.15a)

$$P^i_{\mu} n^i_{\mu}|_{\partial S} = 0.$$  

(V.15b)

VI. STRESS, ENERGY AND MOMENTUM

The stress–energy–momentum (SEM) tensor for matter (non–gravitational) fields is defined by

$$T^{\mu\nu}(x) = \frac{2\alpha}{\sqrt{-g}} \delta S_{\text{matter}}.$$  

(VI.1)

The functional derivative of the matter action $S_{\text{matter}}$ is determined by the coefficient of $\delta g_{\mu\nu}(x)$ in the variation $\delta S_{\text{matter}}$. To apply this definition to the elastic material, we first write the action as an integral over the spacetime coordinates:

$$S[X, g] = -\int d^4 x \int_{\lambda_1}^{\lambda_2} d\lambda \int d\zeta \sqrt{\alpha} \rho \delta^4(x - X(\lambda, \zeta)).$$

(VI.2)

The Dirac delta function enforces the evaluation of the metric $g_{\mu\nu}(x)$ at the spacetime event $X^\mu(\lambda, \zeta)$. Using the results for $\delta \alpha$ and $\delta f_{ij}$ from Eqs. (V.5), we find

$$T^{\mu\nu}(x) = \int_{\lambda_1}^{\lambda_2} d\lambda \int_S d^3 \zeta \sqrt{\alpha} \frac{\delta}{\delta g_{\mu\nu}(x)} \left[ \rho U_\mu U_\nu - S^{ij} F^\mu_i F^\nu_j \right].$$

(VI.3)

We can evaluate the stress–energy–momentum tensor at the event $x^\mu = X^\mu(\lambda, \zeta)$. The Dirac delta function becomes

$$\delta^4(X(\lambda, \zeta) - X(\lambda, \zeta)) = \frac{1}{|\text{det}(X_\zeta)|} \delta(\lambda - \lambda) \delta^3(\zeta - \zeta),$$

(VI.4)

where $|\text{det}(X_\zeta)|$ is the determinant of the matrix formed from derivatives of $X^\mu$ with respect to $\lambda$ and $\zeta^i$. Then the SEM tensor becomes

$$T^{\mu\nu}(X(\lambda, \zeta)) = \frac{\alpha \sqrt{\zeta}}{\sqrt{-g} |\text{det}(X_\zeta)|} \left[ \rho U_\mu U_\nu - S^{ij} F^\mu_i F^\nu_j \right].$$

(VI.5)

where the right–hand side is evaluated at $\lambda$ and $\zeta^i$. We can, of course, drop the bars and rewrite this result by setting $T^{\mu\nu}(X(\lambda, \zeta))$ equal to the right–hand side above, with the right–hand side now evaluated at $\lambda, \zeta^i$.

The factor $|\text{det}(X_\zeta)|$ can be analyzed by considering the worldline parameter $\lambda$ and the matter space coordinates $\zeta^i$, together, to define a coordinate system on spacetime $\mathcal{M}$ within the world tube of the body. Denote this coordinate system by $x^\mu = (\lambda, \zeta^i)$, so the mapping from $\mathbb{R} \times S$ to $\mathcal{M}$ defines a coordinate transformation $x^\mu = X^\mu(x')$. The components of the metric in the primed coordinates are

$$g'_{\mu\nu} = \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\mu} g_{\alpha\beta}.$$

(VI.6)

Taking the determinant of this relation we find

$$|\text{det}(X_\zeta)| = \sqrt{-g}/\sqrt{-\rho}.$$  

(VI.7)

Now we can use the definitions from Sec. II for $\alpha, v_i$ and $f_{ij}$ to compute

$$g'_{\mu\nu} = \left( -\alpha^2 \alpha v_i \frac{\partial f_{ij}}{\partial x^\nu} - v_i v_j \right).$$

(VI.8)

The determinant of $g'_{\mu\nu}$ follows from the formula

$$\text{det} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{det}(A) \text{det}(D - CA^{-1}B).$$

(VI.9)

for the determinant of a block matrix. This yields

$$\text{det}(g') = -\alpha^2 f.$$  

(VI.10)

where $f$ is the determinant of the radar metric $f_{ij}$. Putting this together with Eq. (VI.7) gives the result

$$|\text{det}(X_\zeta)| = \alpha \sqrt{f}/\sqrt{-\rho}.$$  

(VI.11)

Then the SEM tensor (VI.5) becomes

$$T^{\mu\nu}(X(\lambda, \zeta)) = \frac{1}{f} \left[ \rho U_\mu U_\nu - S^{ij} F^\mu_i F^\nu_j \right].$$  

(VI.12)

where we have defined

$$J = \sqrt{f}/\sqrt{g}.$$  

(VI.13)
Recall that $\sqrt{\varepsilon d^3\zeta}$ is the physical volume occupied by the coordinate cell $d^3\zeta$ when the body is in its relaxed state. Similarly, $\sqrt{T d^3\zeta}$ is the physical volume occupied by the coordinate cell $d^3\zeta$ when the body is deformed. Thus, the factor $1/J$ in Eq. (VI.12) converts the energy per unit relaxed volume (the dimensions of $\rho$ and $S^{ij}$) into the energy per unit deformed volume (the dimensions of $T^{\mu\nu}$).

Consider a comoving observer, that is, an observer whose worldline coincides with a particular particle $\zeta^i$ in the body. The observer’s velocity is $U^\mu$ and their spatial directions are spanned by $Z^i_\mu$. The energy density as seen by this observer is

$$T^{\mu\nu} U_\mu U_\nu = \rho / J .$$  
(VI.14)

The energy flux (momentum density) for this observer vanishes: $T^{\mu\nu} U_\mu Z^i_\nu = 0$. The momentum flux (spatial stress) is

$$T^{\mu\nu} Z^i_\mu Z^j_\nu = -S^{ij} / J .$$  
(VI.15)

Note the relative minus sign between the spatial components of the SEM tensor and the second Piola–Kirchhoff stress tensor. This arises because the stress components of $T^{\mu\nu}$ give the $i$-component of force that the material on the $\zeta^i > \text{const}$ side of the surface $\zeta^j = \text{const}$ exerts on the $\zeta^j > \text{const}$ side. The second Piola–Kirchhoff stress is defined with the opposite convention, as the force that the $\zeta^j > \text{const}$ side exerts on the $\zeta^j < \text{const}$ side.

Of course the elastic body stress–energy–momentum tensor must satisfy the local conservation law $\nabla_\mu T^{\mu\nu} = 0$. We can verify this by explicit calculation using Eq. (VI.13). First, expand the covariant derivative as

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + T^{\nu\alpha} \partial_\mu T^{\alpha\mu} .$$  
(VI.16)

The partial derivative with respect to $x^\mu$ acts on the Dirac delta function in Eq. (VI.3) to give $\partial_\mu \delta^4(x - X(\lambda, \zeta))$. The index $\mu$ is contracted with either $U^\mu$ or $F^\mu_i$. Recall that the spacetime velocity $U^\mu$ is proportional to $\dot{X}^\mu$. Using the definitions of Sec. II, the deformation gradient (VI.1) can be rewritten as

$$F^\mu_i = f^\mu_\alpha X^\alpha_i = X^\nu_i + \frac{1}{\alpha} v_i \dot{X}^\mu .$$  
(VI.17)

Thus, $F^\mu_i$ is a combination of terms that are proportional to $\dot{X}^\mu$ and $X^\nu_i$. So the $\mu$ index in $\partial_\mu \delta^4(x - X(\lambda, \zeta))$ is always contracted with either $X^\mu$ or $X^\nu_i$. We can rewrite these expressions using the following identities:

$$\dot{X}^\mu \partial_\mu \delta^4(x - X(\lambda, \zeta)) = \dot{X}^\mu \left[ \frac{\partial}{\partial X^\mu} \delta^4(x - X) \right]_{X = X(\lambda, \zeta)} = -\frac{\partial}{\partial \lambda} \delta^4(x - X(\lambda, \zeta)) ,$$  
(VI.18a)

$$X^\mu_i \partial_\mu \delta^4(x - X(\lambda, \zeta)) = X^\mu_i \left[ -\frac{\partial}{\partial X^\mu} \delta^4(x - X) \right]_{X = X(\lambda, \zeta)} = -\frac{\partial}{\partial \zeta^i} \delta^4(x - X(\lambda, \zeta)) .$$  
(VI.18b)

The next step in evaluating $\nabla_\mu T^{\mu\nu}$ is to integrate by parts to shift the derivatives $\partial / \partial \lambda$ and $\partial / \partial \zeta^i$ away from the Dirac delta function. The endpoint and boundary terms can be discarded if we choose the spacetime point $x^\mu$ inside the world tube of the body. The result that follows from Eq. (VI.16) is

$$\nabla_\mu T^{\mu\nu}(x) = \frac{1}{\sqrt{-g}} \int_{\lambda_1}^{\lambda_2} \, d\lambda \int_{S} \, d^3\zeta \left\{ \sqrt{\varepsilon} \frac{D}{D\lambda} (\rho U^{\mu} - S^{ij} F^\nu_i v_j) - \frac{D}{D\zeta^i} (\sqrt{\varepsilon S^{ij} F^\nu_j}) \right\} \delta^4(x - X(\lambda, \zeta)) .$$  
(VI.19)

We can evaluate this expression at the point $X^\mu(\bar{\lambda}, \bar{\zeta})$ inside the world tube of the body, then rewrite the Dirac delta function as in Eqs. (VI.4) and (VI.11). Carrying out the integrals over $\lambda$ and $\zeta^i$, then dropping the bars on $\lambda$ and $\zeta^i$, we have

$$\nabla_\mu T^{\mu\nu}(X(\lambda, \zeta)) = \frac{1}{\alpha \sqrt{\varepsilon}} \left\{ \sqrt{\varepsilon} \frac{D}{D\lambda} (\rho U^{\nu} - S^{ij} F^\nu_i v_j) - \frac{D}{D\zeta^i} (\sqrt{\varepsilon S^{ij} F^\nu_j}) \right\} .$$  
(VI.20)

The term in curly brackets vanishes when the bulk equations of motion (VI.11) are satisfied, so the equations of motion insure that the local conservation law $\nabla_\mu T^{\mu\nu} = 0$ holds. More than that, we see that the bulk equations of motion for a hyperelastic material coincide with the conservation law $\nabla_\mu T^{\mu\nu} = 0$.

VII. POINT PARTICLE

It will be useful to review the simple example of a relativistic point particle. If we choose the energy density as $\rho = (m / \sqrt{\varepsilon}) \delta^4(\zeta - \zeta_0)$, then the elastic body action (VI.4) reduces to the action (VI.4) for a single particle of mass $m$ (located at the point $\zeta_0$ in matter space).
The particle action is invariant under reparametrizations of the worldline [21]. This is a type of gauge symmetry, which can be understood as follows. Consider worldline parameters $\lambda_1$ and $\lambda_2$ related by $\lambda_1 = \Lambda(\lambda_2)$ \footnote{Do not confuse $\Lambda(\lambda)$ with the function $\Lambda(x)$ from Sec. II.}.

Given a history $X^\mu(\lambda)$, define a new history $\tilde{X}^\mu(\lambda)$ by

$$\tilde{X}^\mu(\lambda) = X^\mu(\Lambda(\lambda)) \quad (VII.1)$$

Derivatives of these histories are related by

$$\frac{\partial \tilde{X}^\nu}{\partial \lambda} = \frac{\partial X^\nu}{\partial \lambda} \bigg|_{\lambda = \Lambda(\lambda)} \quad (VII.2)$$

From this result, we find that the action for the history $\tilde{X}^\mu(\lambda)$ is

$$S_{\text{particle}}[\tilde{X}] = -m \int_{\lambda_1}^{\lambda_2} d\lambda \alpha \bigg|_{\lambda = \Lambda(\lambda)} \quad (VII.3)$$

If the reparametrization is the identity at the endpoints, so that $\Lambda(\lambda_1) = \lambda_1$ and $\Lambda(\lambda_2) = \lambda_2$, then a simple change of integration variables from $\lambda$ to $\Lambda$ shows that $S[\tilde{X}] = S[X]$. That is, the action is the same for any two histories that are related by the reparametrization \footnote{We are not allowed to set the parameter $\lambda$ equal to proper time in the action because this would fix the proper time interval between initial and final configurations. The variational principle must allow for histories with differing proper time intervals.} of the worldline. The action is gauge invariant.

For the single point particle, the equations of motion \footnote{See Sec. VII.1} reduce to the geodesic equations, $DU^\mu/\partial \Lambda = 0$, where $U^\mu \equiv \dot{X}^\mu/\alpha$. Using the identity $\alpha DU^\mu/\partial \lambda = (\delta_{\mu}^a + U^\mu U_\nu)DX^\nu/\partial \lambda$, the equations of motion become

$$\left(\delta_{\mu}^a + U^\mu U_\nu\right) \frac{D}{D\lambda} \dot{X}^\nu = 0 \quad (VII.4)$$

Although there are four equations of motion, only three are independent. In particular, the linear combination obtained by contracting Eqs. (VII.4) with $U_\mu$ vanishes. Said another way, the equations of motion cannot be solved for all of the $\dot{X}^\mu$’s as functions of $X^\mu$ and $\dot{X}^\mu$. Given initial data $X^\mu(0)$ and $\dot{X}^\mu(0)$, the future evolution is not fully determined because the worldline parameter is arbitrary.

One way to choose the parametrization (to “fix the gauge”) is to set $\lambda$ equal to proper time. Then $\alpha = 1$ and $\dot{X}^\mu$ equals the spacetime velocity $U^\mu$. Note that in this gauge, $\dot{X}^\mu$ is normalized: $\dot{X}^\mu g_{\mu\nu} \dot{X}^\nu = -1$. The four second derivatives, $\ddot{X}^\mu$, are now determined by the three independent equations \footnote{For the covariant and contravariant components of the spacetime velocity.} plus the covariant $\lambda$–derivative of the normalization condition:

$$\dot{X}_\mu \frac{D}{D\lambda} \dot{X}^\mu = 0 \quad (VII.5)$$

Together, these equations yield the geodesic equations $D\ddot{X}^\mu/\partial \lambda = 0$ with affine parametrization.

The worldline parameter can be chosen in a convenient way if one of the spacetime coordinates, say $t = x^0$, has the property that the $t = \text{const}$ surfaces are spacelike. That is, the coordinate basis vectors $\partial/\partial x^a$ with $a = 1, 2, 3$ are spacelike. (Note that the coordinate basis vector $\partial/\partial t$ need not be timelike.) In this case, we can choose $X^0(\lambda) = \lambda$. It follows that $\dot{X}^0 = 1$. In this gauge the dynamical variables are the spatial components $X^a(t)$ of the particle worldline, where $a = 1, 2, 3$.

The action in this gauge is most conveniently expressed using the ADM decomposition of the spacetime metric:

$$g_{\mu\nu} = \left(\begin{array}{cc} N_a N_a & N^2 \\ N_b & g_{ab} \end{array}\right), \quad (VII.6)$$

where $N$ is the spacetime lapse function (not to be confused with the material lapse function $\alpha$) and $N^a$ is the shift vector. Indices on $N^a$ are raised and lowered with the spatial metric $g_{ab}$. In the $\lambda = t$ gauge the material lapse \footnote{See Sec. VII.4} becomes

$$\alpha = \left[N^2 - (\dot{X}^a + N^a)g_{ab}(\dot{X}^b + N^b)\right]^{1/2}. \quad (VII.7)$$

This can be written more simply by defining

$$V^a \equiv (\dot{X}^a + N^a)/N, \quad (VII.8)$$

which are the spatial components (in the coordinate basis $\partial/\partial x^a$) of the particle velocity as seen by observers at rest in the $t = \text{const}$ surfaces. Then $\alpha = N\sqrt{1 - V^a V_a}$ and we see that

$$\gamma \equiv N/\alpha = 1/\sqrt{1 - V^a V_a}. \quad (VII.9)$$

is the relativistic gamma factor between the particle and the observers at rest in $t = \text{const}$ surfaces. Note that indices on $V^a$ are lowered with the spatial metric $g_{ab}$.

The action \footnote{See Sec. VII.4} in the $\lambda = t$ gauge reduces to

$$S_{\text{particle}}[X] = -m t_{\lambda_1}^{\lambda_2} dt N \sqrt{1 - V^a V_a}. \quad (VII.10)$$

The spacetime metric components $N$, $N^a$ and $g_{ab}$, as they appear in the action, are functions of $t$ and $X^a(t)$. The action is a functional of $X^a(t)$.

In the gauge $\lambda = t$, the equations of motion are

$$D_t(\gamma V_a) + \gamma \partial_a N - \gamma V_b D_a N^b = 0, \quad (VII.11)$$

where $D_t$ and $D_a$ are covariant derivatives compatible with the spatial metric $g_{ab}$. Explicitly, we have $D_t(\gamma V_a) = \partial_t(\gamma V_a) - (\Gamma^c_{ab}(\gamma V_c)) \dot{X}^b$ where $\Gamma^c_{ab}$ are the Christoffel symbols constructed from $g_{ab}$. The result \footnote{Note that this is most easily obtained by extremizing the action \footnote{See Sec. VII.4} of the worldline.} is most easily obtained by extremizing the action \footnote{Alternatively, we can set $\lambda = t$ in the gauge invariant equations \footnote{See Sec. VII.4} and make use of the results.

\begin{align*}
U^a &= \dot{X}^a/\alpha, \quad U_a = \gamma V_a, \quad (VII.12a) \\
U^0 &= 1/\alpha, \quad U_0 = \gamma(N_a V^a - N), \quad (VII.12b)
\end{align*}

for the covariant and contravariant components of the spacetime velocity.}.
VIII. GAUGE FIXED THEORY

The formal structure of the relativistic elastic theory is closely analogous to that of the relativistic particle. Let two worldline parameters \( \lambda_1 \) and \( \lambda_2 \) be related by

\[
\lambda_1 = \Lambda(\lambda_2, \zeta, t) .
\]

(VIII.1)

Note that \( \Lambda \) depends on \( \zeta \); the parameter for each particle in the body can be changed independently from one another, restricted only by continuity. Define a new history \( \tilde{X}^\mu(\lambda, \zeta) \) related to the old history \( X^\mu(\lambda, \zeta) \) by

\[
\tilde{X}^\mu(\lambda, \zeta) = X^\mu(\Lambda(\lambda, \zeta)) .
\]

(VIII.2)

Derivatives of these histories are related by

\[
\frac{\partial \tilde{X}^\mu(\lambda, \zeta)}{\partial \lambda} = \frac{\partial X^\mu(\lambda, \zeta)}{\partial \lambda} \bigg|_{\lambda = \Lambda}, \quad \frac{\partial \tilde{X}^\mu(\lambda, \zeta)}{\partial \zeta} = \frac{\partial X^\mu(\lambda, \zeta)}{\partial \zeta} \bigg|_{\lambda = \Lambda} \tag{VIII.3a}
\]

\[
\frac{\partial \tilde{X}^\mu(\lambda, \zeta)}{\partial \zeta} = \frac{\partial X^\mu(\lambda, \zeta)}{\partial \zeta} + \frac{\partial X^\mu(\lambda, \zeta)}{\partial \lambda} \bigg|_{\lambda = \Lambda} \bigg|_{\lambda = \Lambda} \tag{VIII.3b}
\]

with \( \Lambda \equiv \Lambda(\lambda, \zeta) \). From these results we find that the action for \( \tilde{X}^\mu(\lambda, \zeta) \) is

\[
S[\tilde{X}] = - \int_S \lambda_1 d\lambda \int d^3\zeta \sqrt{\epsilon \alpha \rho} \bigg|_{\lambda = \Lambda(\lambda, \zeta)} .
\]

(VIII.4)

Let the reparametrization become the identity at the endpoints: \( \Lambda(\lambda_i, \zeta) = \lambda_i \) and \( \Lambda(\lambda_f, \zeta) = \lambda_f \). Then a simple change of integration variables, with \( d\lambda d^3\zeta = d\lambda d^3\zeta (d\lambda / d\lambda) \), shows that \( S[\tilde{X}] = S[X] \). Thus the action for the elastic body is the same for any two histories that are related by the reparametrization (VIII.1) of the worldlines. The action is gauge invariant.

The equations of motion that follow from the gauge invariant action are listed in Eqs. (VIII.6). We want to show that the linear combination of bulk equations (VII.15) obtained by contracting with \( U^\mu \) is vacuous; that is, simply \( 0 = 0 \). Begin by contracting both sides with \( U^\mu \) and use the fact that \( D / D\lambda \) and \( D / D\zeta \) obey the product rule of differentiation. Since \( U^\mu P_{ij}^\mu = 0 \), we find (after dropping an overall factor of \( -\sqrt{\epsilon} \))

\[
\frac{D\rho}{D\lambda} + \frac{DU^\mu}{D\lambda}(\rho U_\mu - P_{ij}^\mu v_{ij}) = \frac{DU^\mu}{D\zeta} = \alpha P_{ij}^\mu .
\]

(VIII.5)

From the definitions \( S_{ij} = \partial \rho / \partial E_{ij} \) and \( E_{ij} = (f_{ij} - \epsilon_{ij})/2 \) for the second Piola–Kirchhoff stress and the Lagrangian strain, the first term above becomes

\[
\frac{D\rho}{D\lambda} = \frac{1}{2} S_{ij} \frac{\partial f_{ij}}{\partial \lambda} .
\]

(VIII.6)

The spacetime velocity is defined by \( U^a = \tilde{X}^\mu / \alpha \) with \( \alpha = \sqrt{-\tilde{X}^\mu \tilde{X}_\mu} \); from this we find

\[
\frac{DU^\mu}{D\lambda} = \frac{1}{\alpha} f_{ij} \frac{DU^\nu}{D\lambda} , \quad \frac{DU^\mu}{D\zeta} = \frac{1}{\alpha} f_{ij} \frac{DU^\nu}{D\zeta} .
\]

(VIII.7a)

(VIII.7b)

With the result (VIII.7b), we can compute the derivative of the radar metric (VIII.2):

\[
\frac{\partial f_{ij}}{\partial \lambda} = 2X^\mu, f_{ij} \frac{DU^\nu}{D\lambda} + \frac{2}{\alpha} v_i X^\mu, f_{ij} \frac{DU^\nu}{D\lambda} .
\]

(VIII.8)

Using the results (VIII.7) and (VIII.8), we find that Eq. (VIII.5) simplifies to

\[
f_{ij} X^\mu, S_{ij} \frac{DU^\nu}{D\lambda} = f_{ij} X^\mu, S_{ij} \frac{DU^\nu}{D\zeta} .
\]

(VIII.9)

It is not difficult to verify that the \( \lambda \) and \( \zeta \) derivatives acting on \( X^\nu \) commute: \( DX^\nu, f_{ij} / D\lambda = DX^\nu / D\zeta \). Therefore, the equation of motion (VIII.5) is indeed vacuous; it reduces to \( 0 = 0 \). In turn, this tells us that only three of the four elastic body equations of motion (VII.15) are independent.

The gauge (reparametrization) invariance can be fixed by setting \( \lambda \) equal to proper time. Then \( \alpha = 1 \) and \( \tilde{X}^\mu = U^\mu \). The four second derivatives \( \tilde{X}^\mu \) are determined by the three independent equations (VIII.15) plus the derivative of the normalization condition, Eq. (VIII.4).

As with the relativistic particle, we can fix the gauge by setting \( \lambda = t \) where the coordinate \( t = x^0 \) has the property that the \( t = \text{const} \) surfaces are spacelike. Then \( X^0 = 1 \) and \( X^0 = 0 \). The evolution of the elastic body is described by \( X^a(t, \zeta) \).

With the ADM metric splitting (VII.6), the material lapse \( \alpha \) is given by Eq. (VIII.7). Using the definition

\[
V^a = (\tilde{X}^a + N^a) / N ,
\]

(VIII.10)

we find \( \alpha = N \sqrt{1 - V^a V_a} \) and the relativistic gamma factor is \( \gamma = N / \alpha \). The expressions (VIII.12) for the covariant and contravariant components of the spacetime velocity hold in this case as well.

In this \( \lambda = t \) gauge for the elastic body, we have the following useful results for the matter space velocity \( v_i \) and the radar metric \( f_{ij} \):

\[
v_i = \gamma V_a X^a, \quad f_{ij} = X^a, (g_{ab} + \gamma^2 V_a V_b) X^b, .
\]

(VIII.11a)

(VIII.11b)

Note that \( f_{ab} = (g_{ab} + \gamma^2 V_a V_b) \) are the spatial components of \( f_{ij} \). The action in this gauge is

\[
S[X] = - \int_S dt \int d^3\zeta \sqrt{\epsilon} \sqrt{1 - V^a V_a} \rho(E) ,
\]

(VIII.12)

where \( E_{ij} = (f_{ij} - \epsilon_{ij})/2 \) is the Lagrangian strain. This action is a functional of \( X^a(t, \zeta) \).

The elastic body equations of motion in this gauge are most easily obtained by extremizing the action. Using the relations above, we find
\[ \sqrt{c} D_t (\gamma \rho V^a - v_i P^i_a) - D_j (\sqrt{c} \alpha P^j_a) + \sqrt{c} \gamma (\rho - S^{ij} v_i v_j) \partial_a N - \sqrt{c} (\gamma \rho V^b - v_i P^i_b) D_a N^b = 0 \]  

(VIII.13)

where \( D_a \) is the covariant derivative compatible with the spatial metric \( g_{ab} \) and

\[
\begin{align*}
D_t &= X^a D_a, \\
D_i &= X^{a_i} D_a.
\end{align*}
\]

(VIII.14a)

(VIII.14b)

We also make use of the definition

\[ P^i = (g_{ab} + \gamma^2 V^a V^b) X^{b,j} S^{ij} ; \]

(VIII.15)

these are the spatial components of the first Piola–Kirchhoff stress \( \sqrt{c} \). Note that the equations of motion \( \text{(VIII.13)} \) reduce to the single particle equations \( \text{(VIII.11)} \) when \( \sqrt{c} \rho = m \delta^3(\zeta - \zeta_0) \).

IX. NONRELATIVISTIC LIMIT

Consider the nonrelativistic limit of the elastic theory in the \( \lambda = t \) gauge, as defined by the action \( \text{(VIII.12)} \) and equations of motion \( \text{(VIII.13)} \).

Let square brackets denote the dimensions of a quantity, where \( L \) is length, \( T \) is time and \( M \) is mass. For example, the speed of light and Newton's gravitational constant have dimensions \([c] = L/T\) and \([G] = L^3/(MT^2)\).

We will assume that the spacetime coordinates are \( t \) and \( x^a \), with dimensions \([\ell] = T\) and \([a^a] = L\). Let the matter space coordinates have dimensions \([\zeta'] = L\). With these choices, the spatial metric \( g_{ab} \), radar metric \( f_{ij} \), relaxed metric \( \epsilon_{ij} \) and Lagrangian strain \( E_{ij} \) are all dimensionless. The spacetime lapse function \( N \) and shift vector \( N^a \) are defined by the ADM splitting \( \text{(VII.6)} \), with \( N \) replaced by \( c N \) and \( N^a \) replaced \( c N^a \). The factors of \( c \) compensate for the change in the “time” coordinate from \( x^0 \) (with dimensions \( L \)) to \( t \) (with dimensions \( T \)). Then the lapse \( N \) and shift \( N^a \) are dimensionless.

With the above choices, \( V^a \) and \( v_i \) have dimensions of velocity, \([V^a] = [v_i] = L/T\). The definition \( \text{(VIII.10)} \) becomes \( V^a = (X^a + c N^a)/N \). The relativistic gamma factor is defined by \( \gamma = 1/\sqrt{1 - V^a V^a/c^2} \), and the factor \( V_a V_b \) in the result \( \text{(VIII.11)} \) for \( f_{ij} \) must be divided by \( c^2 \).

The energy density has dimensions \([\rho(E)] = M/(LT^2)\). The second and first Piola–Kirchhoff stress tensors, \( S^{ij} \) and \( P^i_a \), have dimensions \( M/(LT^2) \) as well.

Inserting the appropriate factors of \( c \) into the elastic body equations of motion \( \text{(VIII.13)} \), we have

\[
\frac{1}{c^2} \sqrt{c} D_t (\gamma \rho V^a - v_i P^i_a) - D_j (\sqrt{c} \alpha P^j_a) + \sqrt{c} \gamma (\rho - S^{ij} v_i v_j/c^2) \partial_a N - \frac{1}{c} \sqrt{c} (\gamma \rho V^b - v_i P^i_b) D_a N^b = 0 .
\]

IX.1

The nonrelativistic limit is obtained by writing the spacetime metric as

\[
g_{\mu \nu} dx^\mu dx^\nu = -(c^2 + 2 \Phi) dt^2 + g_{ab} dx^a dx^b ,
\]

(XI.2)

setting the matter density to

\[
\rho(E) = \rho_0 c^2 + W(E) .
\]

(XI.3)

and letting \( c \rightarrow \infty \). Here, \( \Phi \) is the Newtonian gravitational potential (with dimensions \( L^2/T^2 \)) and \( g_{ab} \) is the flat spatial metric. Also, \( \rho_0 \) is the rest mass per unit undeformed volume (with dimensions \( M/L^3 \)) and \( W(E) \) is the potential energy per unit undeformed volume (with dimensions \( M/(LT^2) \)).

For the spacetime metric above, the spacetime lapse is

\[
N = \sqrt{1 + 2 \Phi/c^2} \]

(XI.4)

and the shift vanishes: \( N^a = 0 \). Inserting these into Eqs. \( \text{[IX.1]} \) and letting \( c \rightarrow \infty \), we find

\[
\sqrt{c} D_t (\rho_0 \dot{X}^a) - D_j (\sqrt{c} P^j_a) + \sqrt{c} \rho_0 \partial_a \Phi = 0 .
\]

(XI.5)

The second Piola–Kirchhoff stress \( S^{ij} = \partial W(E)/\partial E_{ij} \) is defined in terms of the Lagrangian strain \( E_{ij} = (f_{ij} - \epsilon_{ij})/2 \), where the radar metric reduces to

\[
f_{ij} = X^{a_i} g_{ab} X^{b,j} .
\]

(XI.7)

in the \( c \rightarrow \infty \) limit. Equations \( \text{[IX.5]} \) are the equations of motion for a nonrelativistic elastic body.

The nonrelativistic equations can also be obtained from the \( c \rightarrow \infty \) limit of the action \( \text{[VIII.12]} \). Inserting the appropriate factors of \( c \) and using the energy density \( \text{[IX.3]} \) and lapse \( \text{[IX.4]} \), we find

\[
S[X] = \int_{t_i}^{t_f} dt \int d^3 \xi \sqrt{c} \left[ \frac{1}{2} \rho_0 X^a \dot{X}_a - W(E) - \rho_0 \Phi \right] \]

(XI.8)

in the limit \( c \rightarrow \infty \). Note that the additive constant \( - \int dt \int d^3 \xi \sqrt{c} \rho_0 c^2 \) has been dropped from the action.
Extremization of this action gives the result \( P^i_q n_i |_{\partial S} = 0 \) (IX.9), as well as the natural boundary condition on the boundary of matter space.

For the nonrelativistic elastic body, we can define the Cauchy stress tensor (true stress tensor) \( \sigma^{ab} \) with dimensions \( [\sigma^{ab}] = M/(LT^2) \). That is, \( \sigma^{ab} n_b \) is the force per unit of deformed area acting across a surface with unit normal \( n_a \) in the deformed body. The Cauchy stress is related to the second Piola–Kirchhoff stress \( S^{ij} \) by

\[
\sigma^{ab} = \frac{1}{J} X^a_{,i} S^{ij} X^b_{,j}, \tag{IX.10}
\]

with \( J = \sqrt{\sqrt{\varepsilon}/\sqrt{\epsilon}} \). In terms of the first Piola–Kirchhoff stress tensor, \( \sigma^{ab} = X^a_{,i} g^{bc} F^i_{,c}/J \).

### X. ISOTROPIC HYPERELASTIC MODELS

The energy density \( \rho(E) \) is a scalar on matter space. For isotropic hyperelastic materials, \( \rho \) depends only on the Lagrangian strain \( E_{ij} \) and relaxed metric \( \epsilon_{ij} \). For example, the Saint Venant–Kirchhoff model is defined by the potential energy density

\[
W(E) = \frac{\lambda}{2} (\epsilon^{ij} E_{ij})^2 + \mu (\epsilon_{ik} \epsilon^{ij} f_{ijk} E_{kt}) , \tag{X.1}
\]

where \( \epsilon^{ij} \) is the inverse of the relaxed metric \( \epsilon_{ij} \) on \( S \). Here, \( \lambda \) and \( \mu \) are the Lamé constants with dimensions \( [\lambda] = [\mu] = M/(LT^2) \). The second Piola–Kirchhoff stress tensor \( S^{ij} = \partial W/\partial E_{ij} \) for the Saint Venant–Kirchhoff model is

\[
S^{ij} = \lambda (\epsilon^{kt} E_{kt}) \epsilon^{ij} + 2 \mu \epsilon_{ik} \epsilon^{ij} E_{kt} . \tag{X.2}
\]

Note that \( S^{ij} \) is a linear function of \( E_{ij} \); this is not physically realistic for large stress.

Isotropic models of hyperelastic materials are often defined in terms of the type \( (\lambda) \) matter space tensor \( \epsilon^{ik} f_{kij} \). Recall that \( f_{ij} \) is the right Cauchy–Green deformation tensor, which we refer to as the radial metric. The scalars built from \( \epsilon^{ik} f_{kij} \) are

\[
I_1 = 1/2 \left[ (\epsilon^{ij} f_{ij})^2 - \epsilon^{ik} \epsilon^{ij} f_{ijk} f_{ikt} \right] , \tag{X.3a}
\]

\[
I_2 = \det(\epsilon^{ik} f_{kij}) = f/\epsilon , \tag{X.3b}
\]

where \( f = \det(f_{ij}) \) and \( \epsilon = \det(\epsilon_{ij}) \). These are the first, second and third stress invariants. Note that

\[
I_3 = J^2 \tag{X.4}
\]

with \( J = \sqrt{\sqrt{\varepsilon}/\sqrt{\epsilon}} \) as defined in Eq. (VI.13).

With the notation above, the Saint Venant–Kirchhoff model becomes

\[
W(E) = \frac{1}{8} (\lambda + 2\mu) (I_1 - 3)^2 + \mu (I_1 - 3) - \frac{\mu}{2} (I_2 - 3) . \tag{X.5}
\]

Another common model for an isotropic, hyperelastic body is the Mooney–Rivlin material \([22]\), defined by

\[
W(E) = \frac{\mu_1}{2} (I_1 - 3) + \frac{\mu_2}{2} (I_2 - 3) + \kappa (J - 1)^2 , \tag{X.6}
\]

where \( I_1 = I_1/J^{2/3} \) and \( I_2 = I_2/J^{4/3} \). For small deformations, the material parameter \( \kappa \) coincides with the bulk modulus and \( \mu_1 + \mu_2 \) coincides with the shear modulus. A special case of the Mooney–Rivlin model is the neo–Hookean model, in which \( \mu_2 = 0 \).

It is common practice to define a hyperelastic model in terms of the principal stretches, denoted \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). These are defined as the square roots of the eigenvalues of \( \epsilon^{ik} f_{kij} \). In terms of the principal stretches, the stress invariants can be written as

\[
I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 , \tag{X.7a}
\]

\[
I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 , \tag{X.7b}
\]

\[
I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2 . \tag{X.7c}
\]

As an example, the potential energy for the Ogden model \([23]\) is

\[
W(E) = \sum_{\rho=1}^N \frac{\mu_p}{\alpha_p} \left[ I_1^{\alpha_p} + I_2^{\alpha_p} + I_3^{\alpha_p} - 3 \right] \tag{X.8}
\]

where \( \mu_p \) and \( \alpha_p \) are material parameters. This model is used to describe rubbers, polymers and biological tissues with large stress.

Obviously, these constitutive models were developed for the purpose of describing ordinary materials (rubber, steel, etc.) in a nonrelativistic setting. We can use these same models in the relativistic regime by choosing \( \rho(E) = \rho_0 c^2 + W(E) \).

Finally, we note that a perfect fluid is a special case of an elastic material in which the energy density is a function of \( J \) only: \( \rho = \rho(J) \). In this case the second Piola–Kirchhoff stress is

\[
S^{ij} = \frac{\partial \rho}{\partial E_{ij}} = 2 \rho' \frac{\partial J}{\partial f_{ij}} = J \rho' f^{ij} . \tag{X.9}
\]

where \( \rho' = \partial \rho/\partial J \), and the stress–energy–momentum tensor \([12]\) becomes

\[
T^{\mu\nu}(X(\lambda, \zeta)) = \frac{1}{J} \left[ \rho U^\mu U^\nu - J \rho' f^{ij} F_i^\mu F_j^\nu \right] . \tag{X.10}
\]

Using the result \([12]\) and the identity \([12]\), along with the definition \( F_i^\mu = f_{\nu i} X^{\nu \alpha} \) for the deformation gradient, this simplifies to

\[
T^{\mu\nu}(X(\lambda, \zeta)) = \frac{\rho}{J} U^\mu U^\nu - \rho' f^{\mu\nu} . \tag{X.11}
\]
This is the SEM tensor for a perfect fluid with energy density $\rho/J$ and pressure $-\rho'$. Recall that $\rho$ is the energy per unit undeformed volume and $J$ is the ratio of deformed to undeformed volume. Thus, $\rho/J$ is the usual rest energy density for a perfect fluid. The identification $P = -\rho'$ for pressure comes from the first law of thermodynamics. Let $V = d^3\zeta$ denote a coordinate volume in matter space $S$ occupied by an element of fluid. Since $\rho' = \partial \rho / \partial J$, we can rewrite $P = -\rho'$ as
\[
d(\sqrt{\epsilon \rho} V) = -P d(\sqrt{f} V) . \tag{X.12}
\]
On the left-hand side, $\sqrt{\epsilon \rho} V$ is the energy of the fluid element. On the right-hand side, $\sqrt{V}$ is the physical volume occupied by the fluid element. Equation (X.12) is a statement of the first law of thermodynamics applied to the fluid element, relating the change in energy to the change in volume and the pressure $P$.

XI. ACKNOWLEDGMENTS

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