A SEMIDEFINITE APPROACH FOR TRUNCATED K-MOMENT PROBLEMS

J. WILLIAM HELTON AND JIAWANG NIE

Abstract. A truncated moment sequence (tms) in \( n \) variables and of degree \( d \) is a finite sequence \( y = (y_\alpha) \) indexed by nonnegative integer vectors \( \alpha := (\alpha_1, \ldots, \alpha_n) \) such that \( \alpha_1 + \cdots + \alpha_n \leq d \). Let \( K \subseteq \mathbb{R}^n \) be a semialgebraic set. The truncated K-moment problem (TKMP) is: how to check if a tms \( y \) admits a \( K \)-measure (a nonnegative Borel measure supported in \( K \)) such that \( y_\alpha = \int_K x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, d\mu \) for every \( \alpha \)? This paper proposes a semidefinite programming (SDP) approach for solving TKMP. When \( K \) is compact, we get the following results: whether a tms admits a \( K \)-measure or not can be checked via solving a sequence of SDP problems; when \( y \) admits no \( K \)-measure, a certificate for the nonexistence can be found; when \( y \) admits one, a representing measure for \( y \) can be obtained from solving the SDP problems under some necessary and some sufficient conditions. Moreover, we also propose a practical SDP method for finding flat extensions, which in our numerical experiments always found a finitely atomic representing measure when it exists.

1. Introduction

A truncated moment sequence (tms) in \( n \) variables and of degree \( d \) is a finite sequence \( y = (y_\alpha) \) indexed by nonnegative integer vectors \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) with \( |\alpha| := \alpha_1 + \cdots + \alpha_n \leq d \). Let \( K \) be a semialgebraic set defined as
\[
K = \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0 \},
\]
with \( g_1, \ldots, g_m \) being polynomials in \( x \). We say a tms \( y \) admits a \( K \)-measure (a nonnegative Borel measure supported in \( K \)) if there exists a \( K \)-measure such that
\[
y_\alpha = \int_K x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, d\mu, \quad \forall \alpha \in \mathbb{N}^n : |\alpha| \leq d.
\]
(Here, denote \( x_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for \( x = (x_1, \ldots, x_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \).) If (1.2) holds, we say \( \mu \) is a representing measure for \( y \) and \( y \) admits the measure \( \mu \). The truncated K-moment problem (TKMP) is: How to check if a tms \( y \) admit a \( K \)-measure? If it admits one, how to get a representing measure? When \( K = \mathbb{R}^n \), TKMP is referred to as the truncated moment problem (TMP). Let
\[
\mathcal{M}_{n,d} := \{ y = (y_\alpha) : \alpha \in \mathbb{N}^n, |\alpha| \leq d \},
\]
and denote by \( \text{meas}(y, K) \) the set of all \( K \)-measures admitted by \( y \). Let
\[
\mathcal{R}_d(K) := \{ y \in \mathcal{M}_{n,d} : \text{meas}(y, K) \neq \emptyset \}.
\]
A measure with finite support is called a finitely atomic measure. A measure \( \mu \) is called \( r \)-atomic if \( |\text{supp}(\mu)| = r \).

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1.1. Background. Bayer and Teichmann \[1\] proved an important result: a tms \( y \in \mathcal{M}_{n,d} \) admits a \( K \)-measure \( \mu \) if and only if it admits a \( K \)-measure \( \nu \) with \( |\text{supp}(\nu)| \leq \binom{n+d}{d} \). Let \( \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n] \) be the ring of polynomials in \( (x_1, \ldots, x_n) \) with real coefficients, and \( \mathbb{R}[x]_d \) be the set of polynomials in \( \mathbb{R}[x] \) whose degrees are at most \( d \). Every tms \( y \in \mathcal{M}_{n,d} \) defines a Riesz functional \( \mathcal{L}_y \) acting on \( \mathbb{R}[x]_d \) as
\[
\mathcal{L}_y \left( \sum_{|\alpha| \leq d} p_\alpha x^\alpha \right) := \sum_{|\alpha| \leq d} p_\alpha y_\alpha.
\]
For convenience, sometimes we also denote \( \langle p, y \rangle := \mathcal{L}_y(p) \). Let \( P_d(K) \) denote the set of all polynomials in \( \mathbb{R}[x]_d \) that are nonnegative on \( K \). A necessary condition for \( y \) to belong to \( \mathcal{A}_d(K) \) is that \( \mathcal{L}_y \) is \( K \)-positive, i.e.,
\[
\mathcal{L}_y(p) \geq 0 \quad \forall p \in P_d(K).
\]
This is because
\[
\mathcal{L}_y(p) = \int_K p d\mu \geq 0 \quad \forall p \in P_d(K), \forall \mu \in \text{meas}(y, K).
\]
Indeed, when \( K \) is compact, \( \mathcal{L}_y \) being \( K \)-positive is also sufficient for \( y \) to belong to \( \mathcal{A}_d(K) \). This was implied by the proof of Tchakaloff’s Theorem \[30\]. A condition stronger than \( \mathcal{L}_y \) being \( K \)-positive is \( \mathcal{L}_y \) being \emph{strictly} \( K \)-positive, i.e.,
\[
\mathcal{L}_y(p) > 0 \quad \forall p \in P_d(K), p \neq 0 \text{ on } K.
\]
As will be shown in Lemma \[22\], when \( K \) has nonempty interior, a tms \( y \) belongs to the interior of \( \mathcal{A}_d(K) \) if and only if its Riesz functional \( \mathcal{L}_y \) is strictly \( K \)-positive. Typically, it is quite difficult to check whether \( \mathcal{L}_y \) is \( K \)-positive or strictly \( K \)-positive.

A weaker but more easily checkable condition is that the localizing matrix of a tms is positive semidefinite if it admits a \( K \)-measure. If a symmetric matrix \( X \) is positive semidefinite (resp., definite), we denote \( X \succeq 0 \) (resp., \( X \succ 0 \)). For a tms \( y \) of degree \( 2k \), let \( M_k(y) \) be the symmetric matrix linear in \( y \) such that
\[
\mathcal{L}_y(p^2) = p^T M_k(y)p \quad \forall p \in \mathbb{R}[x]_k.
\]
(For convenience of notation, we still use \( p \) to denote the vector of coefficients of \( p(x) \) in the graded lexicographical ordering.) The matrix \( M_k(y) \) is called a \emph{k-th order moment matrix}. For \( h \in \mathbb{R}[x]_{2k} \), define the new tms \( h \ast y \) such that
\[
\mathcal{L}_y(hq) = \mathcal{L}_{h \ast y}(q) \quad \forall q \in \mathbb{R}[x]_{2k-\deg(h)}.
\]
The tms \( h \ast y \) is known as a \emph{shifting} of \( y \). If \( \deg(hp^2) \leq 2k \), then it holds that
\[
\mathcal{L}_y(hp^2) = p^T \left( M_{k-\lceil \deg(h)/2 \rceil}(h \ast y) \right) p.
\]
The matrix \( M_{k-\lceil \deg(h)/2 \rceil}(h \ast y) \) is called a \emph{k-th order localizing matrix} of \( h \).

If a tms \( y \) belongs to \( \mathcal{A}_{2k}(K) \), then for \( i = 0, 1, \ldots, m \) (denote \( g_0 := 1 \))
\[
M_{k-d_i}(g_i \ast y) \succeq 0 \quad \text{where } d_i = \lceil \deg(g_i)/2 \rceil.
\]
This is because for every polynomial \( p \) with \( \deg(g_i p^2) \leq 2k \) we have
\[
p^T M_{k-d_i}(g_i \ast y)p = \mathcal{L}_y(g_i p^2) = \int_K g_i p^2 d\mu \geq 0
\]
for all $\mu \in \text{meas}(y, K)$. Thus, (1.7) is a necessary condition for $y$ to belong to $\mathcal{B}_{2k}(K)$. In general, (1.7) is not sufficient for $y$ to belong to $\mathcal{B}_{2k}(K)$. However, in addition to (1.7), if $y$ is also flat, i.e., $y$ satisfies the rank condition

$$\text{rank } M_{k-d_{g}}(y) = \text{rank } M_{k}(y),$$

then $y$ belongs to $\mathcal{B}_{2k}(K)$. The above integer $d_{g}$ is defined as

$$d_{g} := \max_{1 \leq i \leq m} \{1, \lceil \deg(g_{i})/2 \rceil\}.$$

The next important result is due to Curto and Fialkow.

**Theorem 1.1 ([8]).** Let $K$ be defined in (1.1) and $d$ be even. If a tms $y \in \mathcal{M}_{n,d}$ satisfies (1.7) and the flat extension condition (1.8) for $k = d/2$, then $y$ admits a unique $\text{rank } M_{k}$-atomic $K$-measure.

A nice exposition and proof of Theorem 1.1 can be found in Laurent [20]. Flat extensions are not only used in solving TKMPs, but also frequently used for solving polynomial optimization problems (cf. [14, 19, 20, 21]).

There are other necessary or sufficient conditions for a tms $y$ to belong to $\mathcal{B}_{d}(K)$, like recursively generated relations. Most of them are about the case of $M_{k}(y)$ being singular. We refer to [4, 5, 6, 7, 8, 9, 11, 12] for the work in this area. When $M_{k}(y)$ is positive definite, there is little work on TKMPs, except for the cases $n = 1$ (cf. [4]), or $d = 2$ or $(n, d) = (2, 4)$ (cf. [13]).

In solving TKMPs, flat extensions play an important role. It is interesting to know when and how a tms $y \in \mathcal{M}_{n,d}$ is extendable to a flat tms $w \in \mathcal{M}_{n,2k}$ with $2k \geq d$. We say that $y$ extends to $w$ (or $w$ is an extension of $y$) if $y_{\alpha} = w_{\alpha}$ for all $|\alpha| \leq d$. There is an important result due to Curto and Fialkow [8] about this.

**Theorem 1.2 ([8]).** Let $K$ be defined in (1.1). Then a tms $y \in \mathcal{M}_{n,d}$ admits a $K$-measure if and only if it is extendable to a flat tms $w \in \mathcal{M}_{n,2k}$ with $2k \geq d$ and $M_{k-d_{g}}(g_{i} \ast w) \succeq 0$ for $i = 0, 1, \ldots, m$.

In view of the above result, the following questions arise naturally: i) If a tms $y$ admits no $K$-measure, how do we get a certificate for the nonexistence? ii) If $y$ admits a $K$-measure, how do we get a representing measure? iii) Preferably, if a tms admits a $K$-measure, how can we get a finitely atomic representing measure?

### 1.2. Contributions

This paper focuses on the questions above. We propose a general semidefinite programming (SDP) approach for solving TKMPs. When $K$ is compact as in (1.1), we have the following results:

1. Whether a tms admits a $K$-measure or not can be checked via solving a sequence of semidefinite programs $\{(\text{SDP})k\}$ (see (3.2) or (3.5)).
   1. A tms $y$ admits a $K$-measure if and only if the optimal value of $\text{(SDP})k$ is nonnegative for all $k$. Consequently, when $y$ admits no $K$-measure, the optimal value of $\text{(SDP})k$ will be negative for some $k$, which gives a certificate for the nonexistence of a representing measure. See Theorems 3.2 and 3.3.
   2. When $y$ admits a $K$-measure, we show how to construct such a measure for $y$ via solving $\text{(SDP})k$ for a certain $k$ and using the flat extension condition. This works under some necessary and sufficient conditions (they are not far away from each other). See Theorems 3.2 and 3.6.
(2) We propose a practical SDP method for finding flat extensions, and thus provide a way for constructing finitely atomic representing measures. It is based on optimizing linear functionals (see [14]) on moment matrices and consists of a sequence of SDP problems. In our computational experiences, the method always produced a flat extension if it exists. We derive a bit of supporting theory for this fact. When a tms admits a $K$-measure, we prove that for a dense subset of linear functionals, the method asymptotically produces a flat extension. When a tms does not admit a $K$-measure, we prove that this sequence of SDP problems will become infeasible after some steps. This method is also applicable when $K$ is noncompact. See Subsection 4.1 and Theorem 4.3.

The results described in (1) are given in Section 3, and those in (2) are in Section 4. Section 2 presents some background for proving these results.

2. Preliminaries

2.1. Notation. The symbol $\mathbb{N}$ (resp., $\mathbb{R}$) denotes the set of nonnegative integers (resp., real numbers), and $\mathbb{R}_+^n$ denotes the nonnegative orthant of $\mathbb{R}^n$. For $t \in \mathbb{R}$, $[t]$ (resp., $[t]$) denotes the smallest integer not smaller (resp., the largest integer not greater) than $t$. The $[x]_d$ denotes the column vector of all monomials with degrees not greater than $d$:

$[x]_d = [1 \ x_1 \ \cdots \ x_n \ x_1^2 \ x_1 x_2 \ \cdots \ \cdots \ x_1^d \ x_1^{d-1} x_2 \ \cdots \ x_n^d]^T$.

For a set $S \subseteq \mathbb{R}^n$, $|S|$ denotes its cardinality. The symbol $\text{int}(\cdot)$ denotes the interior of a set. For a matrix $A$, $A^T$ denotes its transpose; if $A$ is symmetric, $\lambda_{\text{min}}(X)$ denotes its minimum eigenvalue. For $u \in \mathbb{R}^N$, $\|u\|_2 := \sqrt{u^T u}$ denotes the standard Euclidean norm, and $B(u, r) := \{x \in \mathbb{R}^n : \|x - u\|_2 \leq r\}$ denotes the closed ball with center $u$ and radius $r$. The $\bullet$ denotes the standard Frobenius inner product in matrix spaces. For a matrix $A$, $\|A\|_F$ denotes the Frobenius norm of $A$, i.e., $\|A\|_F = \sqrt{\text{Trace}(A^T A)}$. The zero set of a polynomial $q$ is denoted by $\mathcal{Z}(q)$. For a tms $w$, $w|_r$ denotes the subspace of $w$ whose indices have degrees not greater than $r$, i.e., $w|_r$ is a truncation of $w$ with degree $r$. For a measure $\mu$, $\text{supp}(\mu)$ denotes its support. A polynomial $f \in \mathbb{R}[x]$ is said to be sum of squares (SOS) if there exist $f_1, \ldots, f_k \in \mathbb{R}[x]$ such that $f = f_1^2 + \cdots + f_k^2$. The set of all SOS polynomials in $n$ variables and of degree $d$ is denoted by $\Sigma_{n,d}$. The symbol $\mathcal{C}(K)$ denotes the space of all functions that are continuous on a compact set $K$, and $\| \cdot \|_\infty$ denotes its standard $\infty$-norm.

2.2. Truncated moments. For a compact set $K$, a tms admits a measure supported in $K$ if and only if its Riesz functional is $K$-positive. This result is implied in the proof of Tchakaloff’s Theorem [30] (cf. [13]).

**Theorem 2.1** (Tchakaloff). Let $K$ be a compact set in $\mathbb{R}^n$. A tms $y$ admits a $K$-measure if and only if its Riesz functional $\mathcal{L}_y$ is $K$-positive.

In the following, we characterize the interior of $\mathcal{A}_d(K)$ via strict $K$-positivity of Riesz functionals.

**Lemma 2.2.** Let $K \subseteq \mathbb{R}^n$ be a set with $\text{int}(K) \neq \emptyset$ and $y$ be a tms in $\mathcal{M}_{n,d}$. Then $y$ belongs to $\text{int}(\mathcal{A}_d(K))$ if and only if $\mathcal{L}_y$ is strictly $K$-positive.
Proof. “⇒” Suppose $y$ belongs to $\text{int}(\mathcal{R}_d(K))$. Let $\zeta \in \mathcal{M}_{n,d}$ be the tms generated by the standard Gaussian measure restricted to $K$. Then, for every $p \in \mathcal{P}_d(K)$ with $p|_K \neq 0$, we must have $\mathcal{L}_y(p) > 0$ because $\text{int}(K) \neq \emptyset$. If $\epsilon > 0$ is small enough, the tms $v := y - \epsilon \zeta$ belongs to $\mathcal{R}_d(K)$ and it holds that

$$\mathcal{L}_y(p) = \mathcal{L}_v(p) + \epsilon \mathcal{L}_\zeta(p) \geq \epsilon \mathcal{L}_\zeta(p) > 0.$$ 

This means that $\mathcal{L}_y$ is strictly $K$-positive.

“⇐” Suppose $\mathcal{L}_y$ is strictly $K$-positive. Lemma 2.3 of [13] implies that, for some $\delta > 0$, the $\mathcal{L}_w$ is strictly $K$-positive for all $w \in \mathcal{M}_{n,d}$ with $\|w - y\|_2 < \delta$. Then, by Theorem 2.4 of [13], every such a tms $w$, including $y$, belongs to $\mathcal{R}_d(K)$. This implies that $y$ belongs to $\text{int}(\mathcal{R}_d(K))$. $\square$

A tms $w \in \mathcal{M}_{n,2k}$ generates the moment matrix $M_k(w)$. Recall that, for a polynomial $f$, we still denote by $f$ the vector of its coefficients. The vector $f$ is indexed by exponents of $x^\alpha$. The matrix vector product $M_k(w)f$ is defined in the usual way, i.e.,

$$M_k(w)f = \mathcal{L}_w(f[x]_k).$$

We say $p \in \ker M_k(w)$ if $M_k(w)p = 0$, i.e., $\mathcal{L}_w(px^\alpha) = 0$ for every $|\alpha| \leq k$.

An ideal of $\mathbb{R}[x]$ is a subset $I \subseteq \mathbb{R}[x]$ such that $I + I \subseteq I$ and $p \cdot q \in I$ for every $p$ and $q \in \mathbb{R}[x]$. Given $p_1, \ldots, p_m \in \mathbb{R}[x]$, denote by $\langle p_1, \ldots, p_m \rangle$ the ideal generated by $p_1, \ldots, p_m$. If $I = \langle p_1, \ldots, p_m \rangle$ and every $p_i \in \ker M_k(w)$, then $I \subseteq (\ker M_k(w))$, the ideal generated by polynomials in $\ker M_k(w)$.

Lemma 2.3. Let $w \in \mathcal{M}_{n,2k}, h, p \in \mathbb{R}[x]$ be such that $H := M_{k-\lceil \deg(h)/2 \rceil}(h \ast w) \geq 0$. If $p, q$ are polynomials with $p \in \ker H$ and

$$\deg(pq) \leq k - \lfloor \deg(h)/2 \rfloor - 1,$$

then we have $pq \in \ker H$.

Proof. Let $z = h \ast w$. Then $z$ is a tms in $\mathcal{M}_{n,2k-\deg(h)}$ and the moment matrix $M_{k-\lceil \deg(h)/2 \rceil}(z) \geq 0$. The conclusion is implied by Lemma 3.5 of [13] (also see Lemma 5.7 of [21] or Theorem 7.5 of [5]). $\square$

2.3. Quadratic module, preordering and semidefinite programming. For the semialgebraic set $K$ defined in [11], its $k$-th truncated quadratic module $Q_k(K)$ and truncated preordering $Pr_k(K)$ are respectively defined as

\begin{equation}
Q_k(K) := \left\{ \sum_{i=0}^{m} g_i \sigma_i \bigg| \begin{array}{c}
\text{each } \deg(\sigma_i g_i) \leq 2k \\
\text{and } \sigma_i \text{ is SOS}
\end{array} \right\},
\end{equation}

\begin{equation}
Pr_k(K) := \left\{ \sum_{v \in \{0,1\}^m} \sum_{u \in \{0,1\}^m} \sigma_v g_v \bigg| \begin{array}{c}
\text{deg}(\sigma_v g_v) \leq 2k \\
\text{each } \sigma_v \text{ is SOS}
\end{array} \right\}.
\end{equation}

$(g_v := g_1^v \cdots g_m^v.)$ The quadratic module and preordering of $K$ are then defined respectively as

$$Q(K) = \bigcup_{k \geq 0} Q_k(K), \quad Pr(K) = \bigcup_{k \geq 0} Pr_k(K).$$

The definitions of $Q_k(K)$ and $Pr_k(K)$ depend on the set of defining polynomials $g_1, \ldots, g_m$, which are not unique for $K$. Throughout the paper, when $Q_k(K)$ or $Pr_k(K)$ is used, we assume that $g_1, \ldots, g_m$ are clear in the context. In [11], if $K$
is defined by using polynomial equalities, like $h(x) = 0$, then it can be replaced by two inequalities $h(x) \geq 0$ and $-h(x) \geq 0$.

**Theorem 2.4.** Let $K$ be as in (1.1) and $p \in \mathbb{R}[x]$ be strictly positive on $K$.

(i) (Schmüdgen, [23]) If $K$ is compact, then we have $p \in \Pr(K)$.

(ii) (Putinar, [24]) If the archimedean condition holds for $K$ (a set $\{x : q(x) \geq 0\}$ is compact for some $q \in \mathbb{Q}(K)$), then we have $p \in \mathbb{Q}(K)$.

**Theorem 2.5** (Real Nullstellensatz). Let $K$ be defined in (1.1). If $f \in \mathbb{R}[x]$ vanishes identically on $K$, then $-f^{2\ell} \in \Pr_k(K)$ for some integer $\ell \geq 1$.

**Remark:** Theorem 2.5 is a special case of the so-called Positivstellensatz [27], and the set $K$ there does not need to be compact.

Let $I(K)$ be the vanishing ideal of $K$, i.e.,

$$I(K) := \{h \in \mathbb{R}[x] : h(u) = 0 \ \forall \ u \in K\}.$$ 

By Theorem 2.5 if $f \in I(K)$, then $-f^{2\ell} \in \Pr_k(K)$ for some $k, \ell$. This fact will be used in our proofs later.

The sets $Q_k(K)$ and $\Pr_k(K)$ are convex cones. Their dual cones lie in the space $\mathcal{M}_{n,2k}$. The dual cone of $\Pr_k(K)$ is defined as

$$\Pr_k(K)^* = \{y \in \mathcal{M}_{n,2k} : \langle p, y \rangle \geq 0 \ \forall p \in \Pr_k(K)\}.$$ 

The dual cone $Q_k(K)^*$ is defined similarly. It is known that (cf. [16, 19])

$$Q_k(K)^* = \{y \in \mathcal{M}_{n,2k} : M_{k-d_i}(g_i * y) \succeq 0 \ \text{for} \ i = 0,1,\ldots,m\},$$

$$\Pr_k(K)^* = \{y \in \mathcal{M}_{n,2k} : M_{k-d_\nu}(g_\nu * y) \succeq 0 \ \forall \ \nu \in \{0,1\}^m\}.$$ 

$d_\nu := \lceil \deg(g_\nu)/2 \rceil$. Given $\nu, a_1, \ldots, a_t \in \mathcal{M}_{n,d}$ and scalars $b_1, \ldots, b_t$, consider the linear conic optimization problem

$$\min_{p \in \mathcal{M}_{n,d}} \langle p, c \rangle \quad \text{s.t.} \quad \langle p, a_i \rangle = b_i (1 \leq i \leq t), \quad p \in \mathbb{R}[x]_d \cap \Pr_k(K).$$

Its dual optimization problem is

$$\max_{y \in \mathcal{M}_{n,2k}} \quad \text{s.t.} \quad b_1 \lambda_1 + \cdots + b_t \lambda_t$$

$$w|_d = c - \lambda_1 a_1 - \cdots - \lambda_t a_t,$$

$$M_{d-y}(g_{d} * w) \succeq 0 \ \forall \nu \in \{0,1\}^m.$$ 

The optimization problems (2.3) and (2.4) are reducible to SDP problems (cf. [16, 19, 21]). Any objective value of a feasible solution of (2.3) (resp., (2.4)) is an upper bound (resp., lower bound) for the optimal value of the other one (this is called weak duality). If one of them has an interior point (for (2.3) it means that there is a feasible $p$ lying in the interior of $\mathbb{R}[x]_d \cap \Pr_k(K)$, and for (2.4) it means that there is a feasible $w$ satisfying every $M_{d-y}(g_{d} * w) > 0$), then the other one has an optimizer and they have the same optimal value (this is called strong duality). Similar is true if $\Pr_k(K)$ is replaced by $Q_k(K)$. We refer to [16, 19, 21] for properties of SDPs arising from moment problems and polynomial optimization. In [17] Lasserre proposed a semidefinite programming approach for solving the generalized problem of moments. We refer to [31] for more about SDP.
3. Checking Existence of Representing Measures

This section discusses TKMPs when \( K \) is a compact semialgebraic set defined in (1.1). It gives a semidefinite approach for checking if a tms admits a representing measure and proves its properties alluded to in §1.2(1).

3.1. A certificate via semidefinite programming. Our semidefinite approach for solving TKMPs exploits the following basic fact.

Proposition 3.1. Let \( y \) be a tms in \( \mathcal{M}_{n,d} \) and \( K \) be a compact set defined by (1.1). Then \( y \) admits no \( K \)-measure if and only if there exists a polynomial \( p \) such that

\[
\langle p, y \rangle < 0, \quad p \in \mathbb{R}[x]_d \cap \text{Pr}(K).
\]

Under the archimedean condition for \( K \), the above is also true if \( \text{Pr}(K) \) is replaced by \( Q(K) \).

Proof. (i) The “if” direction is obvious. It suffices to prove the “only if” direction. Suppose \( y \) does not belong to \( \mathcal{R}_d(K) \). Then, by Theorem 2.1, the Riesz functional \( L_y \) must achieve a negative value on \( \mathcal{P}_d(K) \), say \( \hat{p} \in \mathcal{P}_d(K) \), such that

\[
L_y(\hat{p}) < 0.
\]

Then, for a small enough \( \epsilon > 0 \), the polynomial \( p := \hat{p} + \epsilon \) also satisfies

\[
L_y(p) < 0.
\]

Since \( p \) is strictly positive on the compact set \( K \), by Theorem 2.4(i), we have \( p \in \text{Pr}(K) \). This \( p \) satisfies \( \langle p, y \rangle < 0 \). The proof is same when the archimedean condition holds (applying Theorem 2.4(ii)). \( \Box \)

In the following, we show how to apply Proposition 3.1 to check whether a tms \( y \) belongs to \( \mathcal{R}_d(K) \) or not. Choose a tms \( \xi \in \mathcal{R}_d(K) \) such that \( L_\xi \) is strictly \( K \)-positive. For an integer \( k \geq d/2 \), consider the optimization problem

\[
\lambda_k := \min_p \langle p, y \rangle \quad \text{s.t.} \quad \langle p, \xi \rangle = 1, \ p \in \mathbb{R}[x]_d \cap \text{Pr}_k(K).
\]

Its dual optimization problem is

\[
\lambda_k := \max_{\lambda \in \mathbb{R}, w \in \mathcal{M}_{n,2k}} \lambda \quad \text{s.t.} \quad w|_d = y - \lambda \xi, \quad M_{K-d}(g_\nu * w) \succeq 0 \ \forall \nu \in \{0,1\}^m.
\]

Both the primal (3.1) and dual (3.2) are reducible to SDP problems, and they are parameterized by an order \( k \). They can be solved efficiently, e.g., by software GloptiPoly [15] and SeDuMi [28].

A nonnegative Borel measure \( \mu \) is called \((K,d)\)-semialgebraic if \( \text{supp}(\mu) \subseteq K \) and there exists \( q \in \mathbb{R}[x]_d \cap \text{Pr}(K) \) such that

\[
\text{supp}(\mu) \subseteq \{ u \in K : q(u) = 0 \}, \quad q \neq 0 \text{ on } K.
\]

If a measure \( \mu \) is not \((K,d)\)-semialgebraic, then

\[
\text{supp}(\mu) \subseteq K \cap Z(q), \ q \in \mathbb{R}[x]_d \cap \text{Pr}(K) \quad \implies \quad q \equiv 0 \text{ on } K.
\]

Note that not every \( K \)-measure is \((K,d)\)-semialgebraic. For instance, for \( K \) being the unit ball, the probability measure uniformly distributed on \( K \) is not \((K,d)\)-semialgebraic, because every polynomial vanishing on it must be identically zero. So, being \((K,d)\)-semialgebraic is a bit restrictive for a measure \( \mu \), as it implies \( \text{supp}(\mu) \) has Lebesgue measure zero when \( K \) has nonempty interior.
Theorem 3.2. Assume the set $K$ defined in (1.1) is compact, the tms $\xi$ belongs to $\mathcal{R}_d(K)$ and its Riesz functional $\mathcal{Z}_\xi$ is strictly K-positive. Let $y, \lambda_k, \hat{\lambda}_k$ be as above. Then the sequence $\{\lambda_k\}$ is monotonically decreasing. Let $\lambda_\infty := \lim_{k \to \infty} \lambda_k \in \mathbb{R} \cup \{\infty\}$. We also have:

(i) For $k$ big enough, $\hat{\lambda}_k = \lambda_k$ and (3.2) has a maximizer.

(ii) The tms $y$ belongs to $\mathcal{R}_d(K)$ if and only if $\lambda_k \geq 0$ for all $k$. If $\lambda_\infty > -\infty$, then the shifted tms $\hat{y} := y - \lambda_\infty \xi$ belongs to $\mathcal{R}_d(K)$.

(iii) Assume $\text{int}(K) \neq \emptyset$. Then, $\lambda_\infty > -\infty$, and $\lambda_k = \lambda_\infty$ for some $k$ if and only if there exists a $(K,d)$-semialgebraic measure $\mu \in \text{meas}(\hat{y}, K)$.

(iv) Suppose, for some $k_0$, (3.2) has an optimizer $(\lambda_{k_0}, w^*)$ with $\lambda_{k_0} \geq 0$ and $w^*_1(t \geq d)$ flat. Then $y$ belongs to $\mathcal{R}_d(K)$ and $\lambda_k = \lambda_{k_0}$ for all $k \geq k_0$.

Proof. Since the feasible set of (3.2) is shrinking as $k$ increases, the sequence of optimal values $\{\lambda_k\}$ must be monotonically decreasing. So its limit $\lambda_\infty := \lim_{k \to \infty} \lambda_k$ exists (would possibly be $-\infty$).

(i) Since $K$ is compact, there exists $R > 0$ big enough such that, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$, the polynomial $R - 1 + x^\alpha$ is positive on $K$ and belongs to $Pr_k(K)$ for some $k$, by Theorem 2.31. For $\epsilon > 0$ small enough, we also have $R + c_0 + (1 + c_1)|\alpha| \leq Pr_k(K)$ whenever $|c_0|, |c_1| < \epsilon$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$. This implies that the polynomial $\hat{p} := \sum_{|\alpha| \leq d} (R + x^\alpha)$ lies in the interior of the cone $\mathbb{R}[x]_d \cap Pr_k(K)$. Thus, for $k$ big enough, the primal problem (3.1) has a feasible point lying in the interior of $\mathbb{R}[x]_d \cap Pr_k(K)$. Hence, the primal and dual optimization problems (3.1) and (3.2) have the same optimal value, and (3.2) has a maximizer (see the comments following (2.4)).

(ii) If the tms $y$ belongs to $\mathcal{R}_d(K)$, then $\lambda_k \geq 0$ for all $k$, because $\lambda = 0$ is feasible in (3.2) for every $k$. If $y$ does not belong to $\mathcal{R}_d(K)$, by Proposition 3.1 there exists $p \in \mathbb{R}[x]_d \cap Pr_k(K)$ such that $\langle p, y \rangle < 0$, for some $k$. Generally, we can assume $p \neq 0$ on $K$ (otherwise, replace $p$ by $p + \epsilon$ for a tiny $\epsilon > 0$). By the strict $K$-positivity of $\xi$, we can normalize $p$ as $\langle p, \xi \rangle = 1$. Hence, $\lambda_k \leq \hat{\lambda}_k < 0$. This shows that the first statement of item (ii) is true.

The second statement of item (ii) is implied by the first one. This is because, in (3.2), if $y$ is replaced by $\hat{y}$, then the resulting optimal values are all nonnegative.

(iii) By item (i), for big $k$, $\lambda_k = \hat{\lambda}_k$ is at least the optimal value of

$$\min_p \langle p, y \rangle \quad \text{s.t.} \quad \langle p, \xi \rangle = 1, p \in P_d(K).$$

Since $\text{int}(K) \neq \emptyset$, $p \neq 0$ on $K$ if and only if $p$ is not identically zero. Thus, the strict $K$-positivity of $\mathcal{Z}_\xi$ implies

$$1 = \langle p, \xi \rangle \geq \epsilon_1 \|p\|_2, \quad \text{for some } \epsilon_1 > 0.$$

So, the feasible set of (3.3) is compact and has a finite minimum value $\lambda > -\infty$. By the monotonicity of $\{\lambda_k\}$, we know $\lambda_k \geq \lambda$ for every $k$ and its limit $\lambda_\infty$ is finite.

"if" direction: If there is a $(K,d)$-semialgebraic measure $\mu \in \text{meas}(\hat{y}, K)$, then for some $k$ we can find $0 \neq q \in \mathbb{R}[x]_d \cap Pr_k(K)$ such that $\text{supp}(\mu) \subseteq K \cap \mathcal{Z}(q)$. Normalize $q$ as $\langle q, \xi \rangle = 1$, then

$$\langle q, \hat{y} \rangle = \langle q, y \rangle + \lambda_\infty = \int_K q \, d\mu + \lambda_\infty = \lambda_\infty.$$
“only if” direction: If \( \lambda_{k_0} = \lambda_\infty \) for some \( k_0 \), then \( \lambda_k = \lambda_\infty \) for all \( k \geq k_0 \). Fix such a \( k \). Since \( \text{int}(K) \neq \emptyset \), the feasible set of (3.1) is compact (by (3.4)) and has a maximizer \( p^* \neq 0 \). As shown in (i), if \( k \) is big enough, \( \tilde{\lambda}_k = \lambda_k \) and (3.2) has a maximizer, say, \((\lambda_k, w^*)\). Then, for every \( \mu \in \text{meas}(\tilde{y}, K) \), it holds that

\[
0 = \langle p^*, y - \lambda_k \xi \rangle = \langle p^*, \tilde{y} \rangle = \int_{K} p^* d\mu.
\]

Since \( p^* \) is nonnegative on \( K \), \( \text{supp}(\mu) \subseteq Z(p^*) \) and \( \mu \) is \((K, d)\)-semialgebraic.

(iv) Note that \( y = w^*|_{d} + \lambda_k \xi \) and \( \xi \) admits a \( K \)-measure, say, \( \nu \). If \( w^*|_{M}(t \geq d) \) is flat, then \( w^*|_{i} \) admits a finitely atomic \( K \)-measure \( \mu \), by Theorem 1.1. Since \( \lambda_{k_0} \geq 0 \), we know \( \mu + \lambda_k \nu \) is a representing \( K \)-measure for \( y \). So, \( y \) belongs to \( \mathcal{R}_d(K) \).

Clearly, for every \( k \), the projection of the feasible set of (3.2) into the \( \lambda \)-space contains the set

\[
 F = \{ \lambda : y - \lambda \xi \in \mathcal{R}_d(K) \}.
\]

Thus, every optimal value \( \lambda_k \) is greater than or equal to the optimal value

\[
 \lambda^* := \max \lambda \quad \text{s.t.} \quad \lambda \in F.
\]

So, we know \( \lambda_{k_0} \in F \) from the above. Hence, \( \lambda_k \geq \lambda^* \geq \lambda_{k_0} \) for all \( k \). By the decreasing monotonicity of \( \{\lambda_k\} \), we know \( \lambda_k = \lambda_{k_0} \) for all \( k \geq k_0 \). \( \square \)

When \( \text{int}(K) \neq \emptyset \), as we can see in the proof of item (iii) of Theorem 3.2, a measure in \( \text{meas}(\tilde{y}, K) \) is \((K, d)\)-semialgebraic if and only if all measures in \( \text{meas}(\tilde{y}, K) \) are \((K, d)\)-semialgebraic.

In (3.2), all the cross products \( g_{\nu} := g_{i_1}^{\nu_1} \cdots g_{i_m}^{\nu_m} \) are used, which might be inconvenient in applications. So, we consider a simplified version of (3.2):

\[
(\tilde{\lambda}_k) \quad \text{s.t.} \quad w|_{d} = y - \lambda \xi, \quad M_{k-d}(g_{i_1}^{\nu_1} \cdots g_{i_m}^{\nu_m}) \geq 0 \quad (0 \leq i \leq m).
\]

The dual optimization problem of (3.5) is

\[
(\tilde{\lambda}_k') \quad \text{s.t.} \quad \langle p, \xi \rangle = 1, \quad p \in \mathbb{R}[x]_{d} \cap Q_k(K).
\]

We have the following analogue to Theorem 3.2:

**Theorem 3.3.** Let \( K, \xi, y, \tilde{y}, \lambda, \lambda_\infty \) be same as in Theorem 3.2 and \( \tilde{\lambda}_k, \tilde{\lambda}_k' \) be as above. Suppose the archimedean condition holds for \( K \). Then, the sequence \( \{\lambda_k\} \) is monotonically decreasing. Let \( \lambda_\infty := \lim_{k \to \infty} \lambda_k \in \mathbb{R} \cup \{-\infty\} \). We also have:

(i) For \( k \) big enough, \( \tilde{\lambda}_k = \tilde{\lambda}_k' \) and (3.3) has a maximizer.

(ii) The tms \( y \) belongs to \( \mathcal{R}_d(K) \) if and only if \( \tilde{\lambda}_k \geq 0 \) for all \( k \).

(iii) It always holds that \( \tilde{\lambda}_\infty = \lambda_\infty \). If \( \text{int}(K) \neq \emptyset \), then \( \tilde{\lambda}_\infty > -\infty \), and \( \tilde{\lambda}_k = \tilde{\lambda}_\infty \) for some \( k \) if and only if there exists \( \mu \in \text{meas}(\tilde{y}, K) \) such that \( \text{supp}(\mu) \subseteq K \cap Z(q) \) for some \( q \in \mathbb{R}[x]_{d} \cap Q(K), \langle q \rangle_K \neq 0 \).

(iv) Suppose, for some \( k_0 \), the optimization problem (3.6) has an optimizer \((\tilde{\lambda}_{k_0}, w^*)\) with \( \lambda_{k_0} \geq 0 \) and \( w^*|_{M}(t \geq d) \) flat. Then, \( y \) belongs to \( \mathcal{R}_d(K) \) and \( \lambda_k = \lambda_{k_0} \) for all \( k \geq k_0 \).
Remark: By Theorem 3.3 (ii), if \( y \) does not belong to \( \mathcal{R}_d(K) \), then \( \hat{\lambda}_k < 0 \) for some \( k \), and for \( \lambda = 0 \) there is no \( w \in \mathcal{M}_{n,2k} \) satisfying

\[
(3.7) \quad w|_d = y, \quad M_k(w) \succeq 0, \quad M_{k-d_1}(g_1 \ast w) \succeq 0, \ldots, \quad M_{k-d_m}(g_m \ast w) \succeq 0.
\]

So, we have \( y \notin \mathcal{R}_d(K) \) if and only if (3.7) is infeasible for some \( k \).

Proof of Theorem 3.3. The decreasing monotonicity of \( \{\hat{\lambda}_k\} \) holds because the feasible set of (3.5) shrinks as \( k \) increases.

(i) As in Theorem 3.2, under the archimedean condition on \( K \) of Theorem 3.2. The only difference is to apply Proposition 3.1 with \( K \) and \( 3.3 \) provide a certificate for checking the membership \( y \in \mathcal{R}_d(K) \), under the archimedean condition on \( K \).

(ii) This can be proved in a way similar to item (ii) of Theorem 3.2, under the archimedean condition on \( K \).

(iii) We first show that \( \hat{\lambda}_\infty = \lambda_\infty \). Clearly, \( \hat{\lambda}_k \geq \lambda_k \) for all \( k \). Thus \( \hat{\lambda}_\infty \geq \lambda_\infty \). If \( \hat{\lambda}_\infty = -\infty \), then we are done. If \( \lambda_\infty > -\infty \), we need to show \( \hat{\lambda}_\infty = \lambda_\infty \). If not, seeking a contradiction, we suppose otherwise \( \hat{\lambda}_\infty > \lambda_\infty \). Let \( y := y - \hat{\lambda}_\infty \xi \). If we replace \( y \) by \( y \) in (3.5), then all its optimal values are nonnegative, and thus we have \( \tilde{y} \in \mathcal{R}_d(K) \) by item (ii). However, by Theorem 3.2 (ii), we get \( \tilde{y} \notin \mathcal{R}_d(K) \), a contradiction. This is because if we replace \( y \) by \( y \) in (3.2), then its optimal value is \( \lambda_k - \hat{\lambda}_\infty \) which is negative for big \( k \). So, we must have \( \hat{\lambda}_\infty = \lambda_\infty \).

The second statement of item (iii) can be shown in a similar way as for item (iii) of Theorem 3.2. The only difference is to apply Proposition 3.1 with \( Q(K) \), under the archimedean condition on \( K \).

(iv) The proof is the same as for item (iv) of Theorem 3.2. \( \square \)

Example 3.4. Consider the tms \( y \in \mathcal{M}_{2,6} \).

\[
(28; 0, 0; 0, 1; 1, 0, 3.4; 0, 0, 0; 0, 1, 0, 1.2, 0, 1.6; 0, 0, 0, 0, 0, 0; 28, 0, 3.4, 0, 1.6, 0, 1.2).
\]

Its 3rd order moment matrix \( M_3(y) \) is positive definite. Let \( K = \{x \in \mathbb{R}^2 : \|x\|^2_2 \leq 25\} \). Choose \( \xi \) to be the tms in \( \mathcal{M}_{2,6} \) induced by the probability measure uniformly distributed on the unit ball. For \( K = 3, 4, 5 \), we solve (3.3) (which is same as (3.2), since \( K \) is defined by a single inequality) using GloptiPoly [14], and get optimal values \( \lambda_k \) numerically as

\[
\lambda_3 \approx 0.3702 > 0, \quad \lambda_4 \approx 0.0993 > 0, \quad \lambda_5 \approx -0.2370 < 0.
\]

By Theorem 3.2 or 3.3, we know this tms does not admit a \( K \)-measure. \( \square \)

By Theorem 3.2, when \( K \) is compact, a tms \( y \) belongs to \( \mathcal{R}_d(K) \) if and only if \( \lambda_k \) is nonnegative for all \( k \), which is difficult to check in applications. Similarly, it is also difficult to check \( \hat{\lambda}_k \geq 0 \) for all \( k \). However, the items (iv) of Theorems 3.2 and 3.3 provide a certificate for checking the membership \( y \in \mathcal{R}_d(K) \). This is because we can easily check the flat extension condition (1.8) as follows. Let \( \lambda(w^*) \) be an optimal pair for (3.2) or (3.3). If a truncation \( w^*|_d (t \geq d) \) is flat and \( \lambda_k \geq 0, \) a \( K \)-measure representing \( y \) can be constructed from the relation \( y = w^*_d + \lambda_k \xi \), because the tms \( \xi \) admits a measure by its choice. If we have \( \mu \in \text{meas}(w^*, K) \) and \( \nu \in \text{meas}(\xi, K) \), then \( \mu + \lambda_k \nu \) belongs to \( \text{meas}(y, K) \).

\[1\] Throughout the paper, the moments of a tms are listed in the graded lexicographical ordering, and moments of different degrees are separated by semicolons.
Example 3.5. Consider the tms \( y \in \mathcal{M}_{2,6} \):

\[
(1; 7/50, 1/50; 2/5, 0, 2/5; 91/1250, -6/625, 42/625, 37/1250; \\
6973/25000, -42/3125, 1777/25000, 42/3125, 6973/25000; 1267/31250, \\
-222/15625, 504/15625, 72/15625, 546/15625, 781/31250; 23713/100000, \\
-42/3125, 2929/100000, 0, 2929/100000, 42/3125, 23713/100000).
\]

Let \( K \) be the 2-dimensional unit ball; thus \( d_g = 1 \). The 3rd order moment matrix \( M_3(y) \) is positive definite. Let \( \xi \) be the tms in \( \mathcal{M}_{2,6} \) induced by the uniform probability measure \( \nu \) on the unit ball. We solve (3.2) by GloptiPoly [15]. For \( k = 5 \) we get the optimal value \( \lambda_5 \approx 1.2 \cdot 10^{-7} \). The computed optimal \( w^* \) is flat, because \( \text{rank} M_3(w^*) = \text{rank} M_5(w^*) = 10 \). By Theorem 1.1, we know \( w^* \) admits a unique 10-atomic measure \( \mu \).

For a flat tms, a numerical algorithm is given in [14] to find the support of its finitely atomic representing measure. In Example 3.5 by using this algorithm, we get the support of the 10-atomic measure \( \mu \) admitted by \( w^* \) there:

\[
\{ \pm(1, 0), \pm(0, 1), (\pm 1/2, \pm 1/2), (4/5, -3/5), (3/5, 4/5) \}.
\]

From \( y = w^*|_6 + \lambda_5 \xi \), we know \( \mu + \lambda_5 \cdot \nu \) is a ball-measure representing the tms \( y \) there.

3.2. Finding representing measures via shifting. The sign of \( \lambda_\infty \) is critical in checking the membership in the membership in \( \mathcal{R}_d(K) \). When \( \text{int}(K) \neq \emptyset \), by Lemma 2.2 a tms \( y \) belongs to \( \mathcal{R}_d(K) \) if and only if its Riesz functional \( L_y \) is strictly K-positive.

By Theorem 2.1 for compact \( K \), we know \( \mathcal{R}_d(K) \) is closed. Thus, for compact \( K \) with \( \text{int}(K) \neq \emptyset \), we can check the membership in \( \mathcal{R}_d(K) \) as:

- If \( \lambda_\infty > 0 \), \( y \) lies in the interior of \( \mathcal{R}_d(K) \);
- if \( \lambda_\infty = 0 \), \( y \) lies on the boundary of \( \mathcal{R}_d(K) \);
- if \( \lambda_\infty < 0 \), \( y \) lies outside \( \mathcal{R}_d(K) \).

When \( \lambda_\infty > -\infty \), the shifted tms \( \hat{y} := y - \lambda_\infty \xi \) always admits a \( K \)-measure. When \( \lambda_\infty \geq 0 \), if we have \( \mu \in \text{meas}(\hat{y}, K) \) and \( \nu \in \text{meas}(\xi, K) \), then \( \mu + \lambda_\infty \nu \) is a \( K \)-measure representing \( y \), because of the relation \( y = \hat{y} + \lambda_\infty \xi \). Thus, it is enough to investigate how one can get a \( K \)-measure representing \( \hat{y} \).

When \( \text{int}(K) \neq \emptyset \), Theorem 3.2(iii) implies that \( \lambda_\infty > -\infty \) and \( \lambda_k = \lambda_\infty \) is achieved at some step \( k \) if and only if a measure representing \( \hat{y} \) is \( (K, d) \)-semialgebraic. We are mostly interested in this case, since it is not possible to solve (3.2) for infinitely many \( k \)'s. Therefore, in the following, we assume \( \hat{y} \) admits a \( (K, d) \)-semialgebraic measure.

For an optimizer \( w^* \) of (3.2), the kernel of the moment matrix \( M_k(w^*) \) is a useful tool in constructing supports of representing measures and their vanishing ideals. In particular, the kernel \( \text{ker} M_k(w^*) \) is of the greatest interest when \( M_k(w^*) \) achieves the maximum rank over the set of all optimizers of (3.2). This approach was introduced by Lasserre, Laurent and Rostalski [18] when they compute zero-dimensional real radical ideals via semidefinite relaxations. For more details, we refer to the surveys [21] by Laurent and [22] by Laurent and Rostalski. In the following, we use this approach.

For each \( k \), denote by \( \text{Opt}_k(K) \) the set of optimizers of (3.2).

2The ranks here are evaluated numerically. We ignore singular values smaller than \( 10^{-6} \) when evaluating ranks. The same procedure is applied in computing ranks throughout this paper.
**Theorem 3.6.** Let $K$ be defined in (1.1) (not necessarily compact), and $y, \xi, \eta, \lambda_k, \lambda_\infty$ be same as in Theorem 3.2. Suppose $\lambda_{k_0} = \lambda_\infty$ and there exist $\mu \in \text{meas}(\tilde{y}, K)$ and $0 \neq q \in \mathbb{R}[x]d \cap Pr_{k_1}(K)$, for some $k_0, k_1$, such that

\[ (3.9) \quad \text{supp}(\mu) \subseteq V := \{ u \in K : q(u) = 0 \}. \]

Let $k \geq \max(k_0, k_1)$. If $\text{Opt}_k(K) \neq \emptyset$, then for each $(\lambda, w) \in \text{Opt}_k(K)$ we have:

(i) For $k$ sufficiently large, it holds that $I(V) \subseteq \langle \ker M_k(w) \rangle$.

(ii) Let $r = \max \{ \text{rank} M_k(v) : (\lambda, v) \in \text{Opt}_k(K) \}$. If $\text{rank} M_k(w) = r$, then $\langle \ker M_k(w) \rangle \subseteq I(\text{supp}(\mu))$.

(iii) If $|V| < \infty$, then $w|_{2k-2}$ is flat for $k$ sufficiently large.

(iv) If $|\text{supp}(\mu)| = \infty$ and $\text{rank} M_k(w) = r$, then $w|_{2t}$ can not be flat for all $0 < t \leq k$.

**Proof.** Since $q$ is in $Pr_{k_1}(K)$, there exist SOS polynomials $\sigma_\nu$ such that

\[ q = \sum_{\nu \in \{0,1\}^m} \sigma_\nu g_\nu. \]

Write each $\sigma_\nu = \sum_j \theta_{\nu,j}^2$. For every $k \geq \max(k_0, k_1)$, it holds that (note $w|_{d} = \tilde{y}$)

\[ 0 = \int_K q d\mu = \langle g, \tilde{y} \rangle = \sum_{\nu \in \{0,1\}^m} \sum_j \theta_{\nu,j}^T M_{k-d_\nu}(g_\nu * w) \theta_{\nu,j}. \]

Because $M_{k-d_\nu}(g_\nu * w) \preceq 0$ for all $\nu$, we have $M_{k-d_\nu}(g_\nu * w) \theta_{\nu,j} = 0$ for all $\nu, j$.

(i) Let $\{h_1, \ldots, h_r\}$ be a Grobner basis of the vanishing ideal $I(V)$, under a total degree ordering. Then, each $h_i$ vanishes on $V$. By Theorem 2.3 for each $h_i$, there exist an integer $\ell \geq 1$, a polynomial $\phi$ and SOS polynomials $\varphi_\nu$ such that

\[ (3.9) \quad h_i^{2\ell} + q\phi + \sum_{\nu \in \{0,1\}^m} \varphi_\nu g_\nu = 0. \]

Write $\phi = \phi_1^2 - \phi_2^2$ for some $\phi_1, \phi_2 \in \mathbb{R}[x]$. From $\theta_{\nu,j} \in \ker M_{d-d_\nu}(g_\nu * w)$, by Lemma 2.3 we must have $\theta_{\nu,j} \phi_\nu \in \ker M_{d-d_\nu}(g_\nu * w)$ if

\[ \deg(\theta_{\nu,j} \phi_\nu) + \lfloor \deg(g_\nu)/2 \rfloor \leq k - 1 (s = 1, 2). \]

This implies that

\[ \langle q\phi, w \rangle = \sum_{\nu \in \{0,1\}^m} \left( (\theta_{\nu,j} \phi_1)^T M_{k-d_\nu}(g_\nu * w) (\theta_{\nu,j} \phi_1) - (\theta_{\nu,j} \phi_2)^T M_{k-d_\nu}(g_\nu * w) (\theta_{\nu,j} \phi_2) \right) = 0. \]

Combined with (3.9), this gives

\[ \langle M_k(w) h_i^{\ell}, h_i^{\ell} \rangle + \sum_{\nu \in \{0,1\}^m} \langle \varphi_\nu g_\nu, w \rangle = 0. \]

Since $M_{k-d_\nu}(g_\nu * w) \preceq 0$ for all $\nu$, we know $\langle \varphi_\nu g_\nu, w \rangle \geq 0$ for all $\nu$, which implies $\langle M_k(w) h_i^{\ell}, h_i^{\ell} \rangle = 0$. By an induction on $\ell$, we can get $h_i \in \ker M_k(w)$ (cf. Lemma 3.9 of [13]). So, $I(V) \subseteq \langle \ker M_k(w) \rangle$ for $k$ big enough.

(ii) Let $v \in \mathcal{M}_{2k}$ be the tms induced by $\mu$, i.e., $v = \int [x]_{2k} d\mu$. Then $(\lambda_k, v)$ belongs to $\text{Opt}_k(K)$. The rank $M_k(w)$ being maximum over $\text{Opt}_k(K)$ implies

\[ \text{rank} M_k(w) \geq \text{rank} M_k((w + v)/2) = \text{rank} M_k(w + v). \]
Since both $M_k(w) \geq 0$ and $M_k(v) \geq 0$, it holds that
\[ \ker M_k(w + v) = \ker M_k(w) \cap \ker M_k(v). \]
The relation $\text{rank } M_k(w) \geq \text{rank } M_k(w + v)$ and the above imply
\[ \ker M_k(w) = \ker M_k(w + v) \subseteq \ker M_k(v). \]
For all $f \in \ker M_k(w)$, we have $f \in \ker M_k(v)$ and
\[ \langle x^\alpha, f \rangle = \int_K f^2 d\mu. \]
Thus, $f$ vanishes on $\text{supp}(\mu)$, and $(\ker M_k(w)) \subseteq I(\text{supp}(\mu))$.

(iii) When $|V| < \infty$, the quotient space $\mathbb{R}[x]/I(V)$ is finitely dimensional. Let \( \{b_1, \ldots, b_t\} \) be a standard basis for it. Then, every monomial $x^\alpha$ can be written as
\[ x^\alpha = r(\alpha) + \sum \phi_i h_i, \quad \deg(\phi_i h_i) \leq |\alpha|, \quad r(\alpha) \in \text{span}\{b_1, \ldots, b_t\}. \]
If $k$ is big enough, we have $h_i \in \ker M_k(w)$ for all $i$, by item (i). We also have $\phi_i h_i \in \ker M_k(w)$ for all $i$ if $|\alpha| > k - 1$, by Lemma 4.3. Thus, $x^\alpha - r(\alpha)$ belongs to $\ker M_k(w)$ for all $|\alpha| = k - 1$. So, if
\[ k - 1 - d_g > \max\{\deg(b_1), \ldots, \deg(b_t)\} \]
is big enough, every $\alpha$-th ($|\alpha| = k - 1$) column of $M_k(w)$ is a linear combination of the $\beta$-columns of $M_k(w)$ with $|\beta| < k - 1 - d_g$. This means that the tms generating the moment matrix $M_{k-1}(w)$ is flat, i.e., the truncation $w|_{2k-2}$ is flat.

(iv) For a contradiction, suppose $w|_{2t}$ is flat for some $0 < t < k$. Then, for every $|\alpha| = t$, there exists a polynomial $\phi_\alpha \in \mathbb{R}[x]_{t-1}$ such that $x^\alpha - \phi_\alpha$ belongs to $\ker M_t(w)$. This implies that
\[ 0 = (x^\alpha - \phi_\alpha)^T M_t(w)(x^\alpha - \phi_\alpha) = (x^\alpha - \phi_\alpha)^T M_k(w)(x^\alpha - \phi_\alpha). \]
Since $M_k(w) \geq 0$, we have $x^\alpha - \phi_\alpha \in \ker M_k(w)$. By item (ii), we have $x^\alpha - \phi_\alpha \in I(\text{supp}(\mu))$. Then a simple induction on $|\alpha|$ shows that
\[ x^\alpha \equiv \psi_\alpha \mod I(\text{supp}(\mu)) \]
for some $\psi_\alpha \in \mathbb{R}[x]_{t-1}$, whenever $|\alpha| > t$. That is, every high degree polynomial is equivalent to a polynomial of degree not bigger than $t - 1$ modulo $I(\text{supp}(\mu))$. So, $I(\text{supp}(\mu))$ is zero-dimensional, i.e., $\dim \mathbb{R}[x]/I(\text{supp}(\mu)) < \infty$, which contradicts $|\text{supp}(\mu)| = \infty$ (cf. Proposition 2.1 of Sturmfels [29]).

Remarks: a) In Theorem 3.6, we do not need to assume $K$ is compact. b) The condition \( 3.8 \) implies $L_\beta(q) = 0$ and the Riesz functional $L_\beta$ is not strictly $K$-positive. Thus, the tms $\hat{y}$ lies on the boundary of $\mathcal{R}_d(K)$. This is an advantage of using $\hat{y}$ instead of $y$. c) When \( 3.8 \) holds and $|V| < \infty$, the truncation $w^*|_{2k-2}$ is flat for $k$ big enough, but $w^*$ itself might not be flat.

3.3. Tms with a finite $K$-variety. In this subsection, we consider tms’ whose associated algebraic varieties have finite intersection with $K$. Let $d_0 = \lfloor d/2 \rfloor$. The algebraic variety associated to a tms $y \in M_{n,d}$ is defined as (cf. [13])
\[ (3.10) \quad V(y) := \bigcap_{p \in \ker M_{d_0}(y)} \{ x \in \mathbb{R}^n : p(x) = 0 \}. \]
Then, we define the $K$-variety associated to $y$ as
\[ (3.11) \quad V_K(y) := V(y) \cap K. \]
When the moment matrix $M_{d_0}(y)$ is singular (thus the variety $\mathcal{V}(y)$ is a proper subset of $\mathbb{R}^n$, like a finite set or a curve, etc.), there exists work on discussing whether $y$ admits a measure. We refer to [7, 9, 11, 12].

Clearly, for $\mathcal{V}_K(y)$ to be a proper subset of $K$, the moment matrix $M_{d_0}(y)$ must be singular, and any $\lambda > 0$ is not feasible in (3.2). To see this, suppose otherwise a pair $(\lambda, w)$ with $\lambda > 0$ is feasible in (3.2). When $\mathcal{V}_K(y) \neq K$, there must exist $p \in \ker M_{d_0}(y)$ such that $p \neq 0$ on $K$. The relation $w|_d = y - \lambda \xi$ implies

$$M_{d_0}(y) = M_{d_0}(w) + \lambda M_{d_0}(\xi).$$

The above moment matrices are all positive semidefinite. Thus, from $p \in \ker M_{d_0}(y)$ and $\lambda > 0$, we know $p \in \ker M_{d_0}(\xi)$ and $\mathcal{L}_\xi(p^2) = 0$. This contradicts the strict $K$-positivity of $\xi$. Therefore, when $\mathcal{V}_K(y) \neq K$, the optimal value $\lambda_0$ of (3.2) must be zero if $y$ belongs to $\mathcal{R}_d(K)$, and (3.2) is equivalent to the problem

$$\begin{align*}
\text{(3.12)} & \quad \text{find } w \in \mathcal{M}_{n,2k} \quad \text{s.t. } \quad w|_d = y, \quad M_{K - d_{\nu}}(g_{\nu} * w) \geq 0 \forall \nu \in \{0, 1\}^n.
\end{align*}$$

**Theorem 3.7.** Let $K$ be defined in (1.4) (not necessarily compact), and $y$ be a tms in $\mathcal{M}_{n,d}$. Suppose $y$ belongs to $\mathcal{R}_d(K)$ and $|\mathcal{V}_K(y)| < \infty$. If $k$ is sufficiently large, then every $w$ satisfying (3.12) has a flat truncation $w|_{2k-2}$. 

**Proof.** As we have seen in the above, every optimal $\lambda_k$ in (3.2) is zero. Thus, $\lambda_\infty = 0$ and $\hat{y} = y - \lambda_\infty \xi = y$. Choose a measure $\mu \in \text{meas}(y, K)$ and a basis $\{p_1, \ldots, p_r\}$ of $\ker M_{d_0}(y)$. Let $q = p_1^2 + \cdots + p_r^2 \in \mathbb{R}[x]_d$. From

$$\int_K q d\mu = \sum_{i=1}^r \int_K p_i^2 d\mu = \sum_{i=1}^r p_i^T M_{d_0}(y) p_i = 0,$$

we know $\text{supp}(\mu) \subseteq Z(y) \cap K$. The finiteness of $\mathcal{V}_K(y)$ implies the set $Z(y) \cap K$ is finite. Then, this theorem follows from item (iii) of Theorem 3.6. \hfill \Box

**Remark:** As one can see in the proof, Theorem 3.7 is an application of Theorem 3.6. If the moment matrix $M_{d_0}(y)$ is singular and the tms $y$ belongs to $\mathcal{R}_d(K)$ and $\mathcal{V}_K(y) \neq K$, we have already seen that the optimal value of (3.5) must be zero, and hence (3.5) is equivalent to the feasibility problem (3.12) (see the argument preceding Theorem 3.7).

**Example 3.8.** Consider the tms $y \in \mathcal{M}_{2,4}$:

$$(1; 0, 0; 1, 0; 1; 0, 0, 0; 0; 1, 0, 1, 0, 1).$$

The set $K = \mathbb{R}^2$, $d_0 = 2$, and the moment matrix $M_{d_0}(y)$ is singular (it has rank 4 and size 6 × 6). The $K$-variety is defined by $x_1^2 = x_2^2 = 1$. For $k = 4$, we solve (3.12) and get a feasible tms $\hat{w} \in \mathcal{M}_{2,8}$. Its truncation $\hat{w}|_6$ is flat (while $\hat{w}$ itself is not), and it, as well as $y$, admits a 4-atomic measure supported on $\{(\pm 1, \pm 1)\}$. \hfill \Box

**3.4. The case $K = \mathbb{R}^n$.** When $K = \mathbb{R}^n$ is the whole space, we can similarly apply the approach described in the preceding subsections. However, since $\mathbb{R}^n$ is not compact, weaker conclusions can be made.

Let $y$ be a tms in $\mathcal{M}_{n,d}$ and $d_0 = \lfloor d/2 \rfloor$. If $y$ is extendable to a flat tms $w \in \mathcal{M}_{n,2k}$, then we can get a finitely atomic representing measure for $y$, by Theorem 1.1. In analogue to (3.2), for an integer $k \geq d_0$, consider the optimization problem

$$\begin{align*}
\text{(3.13)} & \quad \max_{\eta \in \mathbb{R}, w \in \mathcal{M}_{n,2k}} \quad \eta \\
& \quad \text{s.t. } \quad w|_d = y - \eta \xi, \quad M_k(w) \succeq 0.
\end{align*}$$
Similarly, we choose \( \xi \) to be a tms in \( \mathcal{R}_d(\mathbb{R}^n) \) with \( M_{d_0}(\xi) \succ 0 \). Clearly, if \( \eta \) is feasible in (3.13), then

\[
(3.14) \quad \eta \leq \eta^* := \lambda_{\min} \left( M_{d_0}(\xi)^{-1/2} M_{d_0}(y) M_{d_0}(\xi)^{-1/2} \right).
\]

When \( \eta < \eta^* \), the moment matrix \( M_{d_0}(y - \eta \xi) \) is positive definite, and the tms \( y - \eta \xi \) is extendable to a tms \( w \in \mathcal{M}_{n,2k} \) with \( M_k(w) \succ 0 \). So the optimal value of (3.13) is equal to \( \eta^* \) for every \( k \). Clearly, we have \( \eta^* \geq 0 \) if and only if the matrix \( M_{d_0}(y) \) is positive semidefinite, and \( \eta^* = 0 \) if and only if \( \lambda_{\min}(M_{d_0}(y)) = 0 \).

**Theorem 3.9.** Let \( y \) be a tms in \( \mathcal{M}_{n,d} \) and \( \xi \) be a tms in \( \mathcal{R}_d(\mathbb{R}^n) \) with \( M_{d_0}(\xi) \succ 0 \). Suppose \( (\eta,w) \) is feasible for (3.13). Let \( \tilde{y} := y - \eta^* \xi \) and \( k \geq d_0 \).

(i) If \( \eta \geq 0 \) and \( w|_t(t \geq d) \) is flat, then \( y \) belongs to \( \mathcal{R}_d(\mathbb{R}^n) \).

(ii) If \( \mu \) is a measure in \( \text{meas}(\tilde{y},\mathbb{R}^n) \), then there exists \( p \in \Sigma_{n,2d_0} \) such that \( \text{supp}(\mu) \subseteq Z(p) \).

In the following, suppose \( \mu,p \) satisfy (ii) and \( (\eta,w) \) is optimal for (3.13).

(iii) If \( |Z(p)| < \infty \), then \( w|_{2k-2} \) is flat for \( k \) sufficiently large.

(iv) If \( |\text{supp}(\mu)| = +\infty \) and \( \text{rank} M_k(w) \) is maximum, then \( w|_{2k} \) can not be flat for all \( 0 < t \leq k \).

**Proof.** (i) is implied by \( y = \eta \xi + w|_d \) and Theorem 1.1.

(ii) The dual optimization problem of (3.13) is

\[
(3.15) \quad \min_f \quad \langle f,y \rangle \quad \text{s.t.} \quad \langle f,\xi \rangle = 1, \quad f \in \Sigma_{n,2d_0}.
\]

Since (3.13) has a feasible \( w \) with \( M_k(w) \succ 0 \), the optimization problem (3.15) has a minimizer \( p \in \Sigma_{n,2d_0} \), and its optimal value equals \( \eta^* \). So

\[
\int_{\mathbb{R}^n} p d\mu = \langle p,\tilde{y} \rangle = \langle p,y \rangle - \eta^* = 0.
\]

The nonnegativity of \( p \) in \( \mathbb{R}^n \) implies \( \text{supp}(\mu) \subseteq Z(p) \).

(iii) It can be proved in a way similar to items (iii)-(iv) of Theorem 3.6. Write \( p = p_1^T + \cdots + p_L^T \). Then \( \text{supp}(\mu) \subseteq Z(p) \) implies

\[
\sum_{i=1}^L p_i^T M_{d_0}(\tilde{y}) p_i = \langle p,\tilde{y} \rangle = \int_{\mathbb{R}^n} p d\mu = 0.
\]

Let \( \{h_1,\ldots,h_r\} \) be a Grobner basis for the vanishing ideal \( I(Z(p)) \), under a total degree ordering. Similarly, by Theorem 2.20, for each \( h_i \), there exist \( \ell \geq 1 \) and \( f_1,\ldots,f_r \in \mathbb{R}[x] \) satisfying \( h_i^{\ell} + f_1 p_1 + \cdots + f_r p_r = 0 \). As in the proof of Theorem 3.6, we can similarly prove \( h_i \in \ker M_k(w) \) and \( \langle f_j p_j, w \rangle = 0 \) for \( j = 1,\ldots,r \), when \( k \) is sufficiently large. When \( Z(p) \) is finite, we can similarly prove \( w|_{2k-2} \) is flat as for (iii) of Theorem 3.6.

(iv) The proof is same as for (iv) of Theorem 3.6. \( \square \)

**Example 3.10.** Consider the same tms \( y \) as in Example 3.8. Take \( K = \mathbb{R}^2 \); thus \( d_y = 1 \) and \( d_0 = 2 \). The moment matrix \( M_1(y) \) has rank 3 and \( M_2(y) \) has rank 4, so \( y \) is not flat. The \( 6 \times 6 \) moment matrix \( M_2(y) \) is singular, so \( \eta^* = 0 \). Choose \( \xi \) in (3.13) to be the tms in \( \mathcal{M}_{2,4} \) generated by the standard Gaussian measure with mass one. For \( k = 4 \), solve the optimization problem (3.13). The computed optimal tms \( w^* \in \mathcal{M}_{2,8} \) is not flat, but its truncation \( w^*|_6 \) is flat and admits a
unique 4-atomic measure with support \(\{(\pm 1, \pm 1)\}\). The tms \(y\) admits the same measure. □

Unlike for the case that \(K\) is compact, the shifted tms \(\hat{y} := y - \eta^* \xi\) does not always admit a measure, even when \(n = 1\). For instance, for the tms \((1, 1, 1, 1, 2) \in \mathcal{M}_{1,4}\), its moment matrix \(M_2(y)\) is positive semidefinite but singular. So, \(\eta^* = 0\) but \(\hat{y} = y\) does not admit a measure (cf. [10, Example 2.1]).

4. A Practical Method For Finding Flat Extensions

The preceding section discusses how to check if a tms \(y\) admits a \(K\)-measure. The limit \(\lambda_\infty\) of the optimal values of (3.2) or (3.5) plays a critical role. For a compact set \(K\), if \(\lambda_\infty \geq 0\), then \(y\) admits a \(K\)-measure; otherwise, it does not. When \(\lambda_\infty \geq 0\), a representing measure for \(y\) is also constructible from the relation \(y = \hat{y} + \lambda_\infty \xi\) if \(\hat{y}\) admits a \(K\)-measure. In the case \(\lambda_\infty > 0\), this approach typically does not give a finitely atomic representing measure, because \(\xi\) is usually generated by a measure with infinite support.

In many applications, one is interested in getting a finitely atomic representing measure. By Theorem 1.2 of Curto and Fialkow, a tms \(y\) admits a \(K\)-measure if and only if it is extendable to a flat tms \(w\) of higher degree. This means that if \(y\) admits a \(K\)-measure, then a finitely atomic measure for \(y\) can be obtained by finding a flat extension. In some special cases (e.g., a moment matrix is singular, or its associated variety is finite, etc), there exists work investigating when and how a flat extension could be found (cf. [7, 9, 11, 12]). For the general case, especially when a moment matrix is positive definite, methods for constructing a flat extension and determining if it exists are relatively unexplored.

This section presents a practical SDP method for this purpose, alluded to in §1.2(2). Our numerical experiments show that it often finds a flat extension of \(y\) when it exists.

4.1. TKMP. Suppose a tms \(w \in \mathcal{M}_{n,2k}\ (k \geq d/2)\) is an extension of \(y\). Clearly, if a truncation \(w|_t (t \geq d)\) is flat, then the finitely atomic measure admitted by \(w|_t\) is a representing measure for \(y\). To find such an extension, for an integer \(k \geq d/2\), consider the semidefinite optimization problem:

\[
\begin{array}{l}
\min_{w \in \mathcal{M}_{n,2k}} c_k^T w \\
\text{s.t.} \\
\quad \quad w|_d = y, M_{k-1}(\rho \ast w) \succeq 0, \\
\quad \quad M_{k-d_i}(g_i \ast w) \succeq 0, \ i = 0, 1, \ldots, m.
\end{array}
\]

(4.1)

In the above, \(c_k\) is a generic vector, and the polynomial \(\rho(x) = R^2 - ||x||_2^2\) is such that \(K \subseteq B(0, R)\). Recall that \(g_0 = 1\) and \(M_k(w) = M_{k-d_0}(g_0 \ast w)\).

(1) If \(w^*\) is an optimizer of (4.1), and \(w^*|_t (t \geq d)\) is flat, then the finitely atomic measure admitted by \(w^*|_t\) also represents \(y\).

(2) If \(y\) does not admit a \(K\)-measure, then (4.1) will not be feasible for \(k\) sufficiently large (see the remark after Theorem 3.3).

We would like to remark that the optimization problem (4.1), as well as (3.12), is an obvious and natural way to check Theorem 1.2 of Curto and Fialkow.

We show some examples of illustrating this method.
Table 1. Inside the parentheses are the numbers of randomly generated instances we tested for each pair \((n,d)\).

| n  | d       | d  | d   | d   | d    |
|----|---------|----|-----|-----|------|
| 2  | 4(100)  | 6(100) | 8(50) | 10(50) | 12(50) |
| 3  | 4(100)  | 5(100) | 6(50) | 7(50) | 8(50) |
| 4  | 3(100)  | 4(100) | 5(50) | 6(20) | 7(10) |

Example 4.1. Let \(K = \left\{ x \in \mathbb{R}_+^4 : \sum_{i=1}^4 x_i \leq 1 \right\} \). Consider the tms \(y \in \mathcal{M}_{4,2}^\ast\): \((1, 1/5, 1/5, 1/5, 1/10, 1/10, 1/20, 0, 0, 1/10, 1/20, 0, 1/10, 1/20, 1/10)\).

Its moment matrix \(M_k(y)\) is positive definite. Choose \(c_k\) to be the vector of all ones. Solve \((4.1)\) for \(k = 4\), and get an optimizer \(w^*\). The tms \(w^*\) and its truncations \(w^*|_4, w^*|_6\) are all flat. They admit a 5-atomic measure supported on the points:
\[(0.5,0,0,0), (0,0,0,0.5), (0,0,5,0,5), (0,0,0,5,0), (0,0,5,0,0).\]

Since \(w^*\) is an extension of \(y\), the tms \(y\) admits the same 5-atomic measure. \(\square\)

Next we describe an experimental test of using \((4.1)\) to find flat extensions for randomly generated tms’. It produced a flat extension for all the instances we tested.

Example 4.2 (random problems). Let \(K = [-1,1]^n\) be the unit hypercube. We generate testing instances as follows. In \((4.1)\), choose \(c_k\) to be a Gaussian random vector, and \((n,d)\) from Table 1. By the theorem of Bayer and Teichmann [1], a tms \(y\) admits a \(K\)-measure if and only if \(y\) admits a \(N\)-atomic \(K\)-measure with \(N := \binom{n+d}{d}\). So, we randomly generate \(N\) points \(u_1, \ldots, u_N\) from \([-1,1]^n\), and set \(y = a_1[u_1]_d + \cdots + a_N[u_N]_d\) for random positive numbers \(a_i > 0\). In each case, we solve \((4.1)\) starting with \(k = d\). If an optimal \(w^*\) of \((4.1)\) has a flat truncation \(w^*|_t(t \geq d)\), we stop; otherwise, increase \(k\) by one, and then solve \((4.1)\) again. Repeat this process until we get a flat extension of \(y\). For each \((n,d)\), the number of randomly generated instances is listed in the parenthesis after \(d\) in Table 1. In all the tested instances, we got a flat truncation \(w^*|_t\), for some \(k, t \geq d\). Furthermore, the supports of the finitely atomic measures we obtained often have cardinalities smaller than \(N\) (we are not sure whether they are minimum or not). \(\square\)

Our conclusion is that solving \((4.1)\) is a practical method for finding a flat extension of a tms \(y\) when it admits a \(K\)-measure.

4.2. A theoretical analysis. While we do not understand well why solving \((4.1)\) always produced a flat extension in our numerical experiments. Here, we present a bit of theoretical analysis suggesting that this will often produce a flat extension. Our analysis is to set up a linear-convex optimization problem for which \((4.1)\) is an approximation. Then, under some assumptions, we prove that \((4.1)\) asymptotically produces a flat extension. We start with some definitions. Denote
\[
E_k(y) := \{ w \in \mathcal{M}_{n,2k} : w \text{ is feasible for } (4.1) \},
\]
\[
E_\infty(y) := \{ z \in \mathbb{R}_\infty(K) : z|_d = y \}.
\]

Embed \(E_k(y)\) into \(\mathcal{M}_{n,\infty}\) by the mapping \(w \mapsto (w,0,\ldots)\) of adding zeros. Thus, \(E_k(y)\) and \(E_\infty(y)\) can be thought of as convex subsets of \(\mathcal{M}_{n,\infty}\). For every \(w \in\)
$\mathcal{M}_{n,\infty}$, define $\|w\|_2$ as $\|w\|_2^2 = \sum_{n \in \mathbb{N}_0} w_n^2$. Define the Hilbert space
\begin{equation}
\mathcal{M}_{n,\infty}^2 := \{w \in \mathcal{M}_{n,\infty} : \|w\|_2 < \infty\}.
\end{equation}
Up to a shuffling and scaling, we can generally assume $K \subseteq B(0, R)$ with $R < 1$.

We assume the vectors $c_k$ in (4.1) for different $k$ are consistent, i.e., each $c_k$ is a leading subvector of $c_k$. Write $c_k(x) = c_k^T [x]_{2k}$. Then, its limit is a real analytic function, which is denoted as $c(x) = \Sigma_\alpha c_\alpha x^\alpha$. The analytic function $c(x)$ can also be thought of as a vector $c \in \mathcal{M}_{n,\infty}$. If $\|c\|_2 < \infty$, then $c$ is a consistent linear functional acting on $\mathcal{M}_{n,\infty}^2$ as
\begin{equation}
\langle c, w \rangle := \sum_\alpha c_\alpha w_\alpha.
\end{equation}
Now we consider the optimization problem:
\begin{equation}
\min_{\mu} \int_K c(x) d\mu \quad \text{s.t.} \quad \mu \in \text{meas}(y, K).
\end{equation}

When $y$ belongs to $\mathcal{R}_d(K)$, its feasible set is nonempty. The objective is a linear functional acting on the convex set $\text{meas}(y, K)$. Note that a vector $w$ belongs to $E_\infty(y)$ if and only if it admits a measure in $\text{meas}(y, K)$. Thus, the optimization problem (4.3) is also equivalent to
\begin{equation}
\min_w \langle c, w \rangle \quad \text{s.t.} \quad w \in E_\infty(y).
\end{equation}

If the optimization problem (4.3) has a unique minimizer $\mu^*$, then $\mu^*$ must have finite support (see Appendix). Theorem 4.3 below shows that if the biggest support of minimizing measures of (4.3) is finite, then (4.1) asymptotically yields a flat extension as $k$ goes to infinity. Theorem A.2 in Appendix shows that for a dense subset of analytic functions $c$ in $\mathcal{C}(K)$ (the space of all continuous functions defined on $K$), this condition (see (4.5)) holds.

**Theorem 4.3.** Assume $K \subseteq B(0, R)$ with $R < 1$, the $c_k$’s are consistent, and $c$ is in $\mathcal{M}_{n,\infty}^2$. Let $\text{Sol}$ be the set of all optimizers of (4.3). Suppose there exists $N > 0$ such that
\begin{equation}
\left| \bigcup_{\mu^* \in \text{Sol}} \{\text{supp}(\mu^*)\} \right| \leq N.
\end{equation}
Let $w(k)$ be an optimizer of (4.3). Then, for every $r > d_g N$ ($d_g$ is given by (1.9)) and every $\mu^* \in \text{Sol}$, the tms $z^* := \int_K [x]_{2r} d\mu^*$ is flat and
\begin{equation}
\lim_{k \to \infty} \text{dist}(w(k)|_{2r}, U) = 0 \quad \text{where} \quad U = \left\{ \int_K [x]_{2r} d\mu^* : \mu^* \in \text{Sol} \right\}.
\end{equation}

In particular, if $\text{Sol}$ is a singleton, then $\{w(k)|_{2r}\}$ converges to a flat tms.

**Proof.** Let $r_0 := d_g N$. If $z_0^* = 0$, then $z^*$ is a zero vector, and it is clearly flat. So, we consider the general case $z_0^* > 0$. We claim that a truncation $z^*_l$ of $z^*$ must be flat for some $0 < l \leq r_0$. Otherwise, if not, then we must have
\begin{equation}
1 = \text{rank} M_0(z^*) < \text{rank} M_{d_g}(z^*) < \text{rank} M_{2d_g}(z^*) < \cdots < \text{rank} M_{r_0}(z^*).
\end{equation}
The above then implies the contradiction:
\begin{equation}
|\text{supp}(\mu^*)| \geq \text{rank} M_{r_0}(z^*) \geq 1 + r_0/d_g > N.
\end{equation}
So, $z^*|_l$ is flat for some $l \leq r_0$. By Theorem 1.3, we know $\mu^*$ is the unique representing measure of $z^*|_l$. Since $z^*$ is an extension of $z^*|_l$ and admits the same measure $\mu^*$, we know that $z^*$ must also be flat.

We prove (4.6) by contradiction. Suppose otherwise it is false, then

$$ \text{dist}(w^{(k)}|_{2r}, U) \geq a \quad \text{for some} \ a > 0. $$

By Lemma A.3, we know the sequence $\{w^{(k)}\}$ is bounded in the Hilbert space $\mathcal{H}_{n,\infty}^2$. By the Alaoglu Theorem (cf. [2, Theorem V.3.1]), $\{w^{(k)}\}$ has a subsequence that is convergent in the weak-* topology. That is, it has a subsequence, which we denote again by $\{w^{(k)}\}$ for convenience, such that for some $w^* \in \mathcal{H}_{n,\infty}^2$

$$ \ell(w^{(k)}) \to \ell(w^*) $$

for every continuous linear functional in $\mathcal{H}_{n,\infty}^2$ (the space $\mathcal{H}_{n,\infty}^2$ is self-dual). This implies (e.g., choose $\ell$ as $\ell(w) = w_\alpha$ for each $\alpha \in \mathbb{N}^n$) that for all $t$

$$ w^{(k)}|_{2r} \to w^*|_{2r}, \quad \text{and} \quad w^*|_d = y. $$

Since $w^{(k)}$ is feasible in (4.1), from the above we get that

$$ M_{t-d}(g_i * w^*) \geq 0 \quad \text{(i = 0, 1, \ldots, m),} \quad M_{t-1}(\rho * w^*) \geq 0. $$

Hence, the vector $w^*$ is a full moment sequence whose localizing matrices of all orders are positive semidefinite (because we have $w^*|_{2r} \in Q_k(K)^\ast$ for all $t$). By Lemma 3.2 of Putinar [24], we know $w^*$ admits a $K$-measure and $w^*$ belongs to $E_\infty(y)$. Note that $c$ is a continuous linear functional acting on $\mathcal{H}_{n,\infty}^2$ as $\langle c, w \rangle$. So,

$$ \langle c, w^* \rangle = \lim_{k \to \infty} \langle c, w^{(k)} \rangle. $$

By Lemma A.4, we know that $w^*$ is also an optimizer of (4.1) and $w^*|_{2r}$ belongs to $U$. However, (4.8) implies the convergence $w^{(k)}|_{2r} \to w^*|_{2r}$, and (4.7) implies that $w^*|_{2r}$ does not belong to $U$, a contradiction. Therefore, (4.6) must be true.

When $\mathcal{S}_01$ is a singleton, $\{w^{(k)}|_{2r}\}$ clearly converges to a flat tms. 

By Theorem 4.3, we know that every limit point of the sequence $\{w^{(k)}|_{2r}\}$ is flat. We would like to remark that the conditions in Theorem 4.3 are only for theoretical analysis. In practical implementations, they are not required.

4.3. The case $K = \mathbb{R}^n$. When $K = \mathbb{R}^n$, we propose a similar method like (4.1) for solving TMPs. To find a finitely atomic representing measure for a tms, it is enough to find one of its flat extensions. Like (4.1), a practical method is that for an order $k \geq d/2$ we solve the SDP problem:

$$ \min_{w \in \mathcal{H}_{n,2k}} \quad c_k^T w $$

$$ \text{s.t.} \quad w|_d = y, \quad M_k(w) \succeq 0. $$

For $k > d/2$, the above feasible set is typically unbounded. To guarantee (4.9) has a minimizer, we usually choose $c_k$ as $c_k^T [x]|_{2k} = [x]^T_k C_k[x]|_k$ with $C_k \succeq 0$. A simple choice for $C_k$ is the identity matrix.

**Example 4.4.** Consider the tms $y \in \mathcal{H}_{2,4}$:

$$ (1; -1/6, 1/2; 3/2, 0, 3/2; -7/6, 1, 1/3, 3/2; 7/2, -1, 2, 1, 7/2). $$

Its moment matrix $M_2(y)$ is positive definite. By Theorem 3.3 of [13], this tms $y$ admits a measure. Choose $c_k$ as $c_k^T [x]|_{2k} = [x]^T_k [x]|_k$, and solve (4.9) for $k = 4$. The
computed optimal \( w^* \in \mathcal{M}_{2,8} \) is flat. It admits a 7-atomic measure supported on the points\(^3\):

\[
(-1.1902, -1.4545), (-0.2725, -1.4977), (1.4406, -0.9396), (0.2150, -0.4613),
\]

\[
(-1.7147, 0.1952), (-1.6942, 1.1593), (0.9645, 1.7098).
\]

The tms \( y \) admits the same 7-atomic measure. \( \square \)

We would like to remark that solving (4.11) or (4.9) might not give a representing measure for \( y \) whose support has minimum cardinality. For instance, in Example 4.4, the tms \( y \) there admits a 6-atomic measure (with support \( \pm (1, 1), \pm (1, -1), (1, 2), (-2, 1) \)), but we got a 7-atomic measure by solving (4.9).

**Example 4.5** (random examples). Now we test the performance of solving (4.9) for finding flat extensions. We choose \( c_k \) such that \( c_k^T w = C_k \bullet M_k(w) \) with \( C_k = L L^T \) and \( L \) being a Gaussian random matrix. Select \((n, d)\) from Table 1. Like in Example 4.2, for each pair \((n, d)\), randomly generate \( N = (n+d) \) points \( u_1, \ldots, u_N \) obeying a Gaussian distribution, and set \( y = a_1[u_1]_d + \cdots + a_N[u_N]_d \) (\( a_i > 0 \) is randomly chosen). For each instance, solve (4.9) starting with \( k = d \). If an optimal \( w^* \) of (4.9) has a flat truncation \( w^*|_{t \geq d} \), we stop; otherwise, increase \( k \) by one, and solve (4.9) again. Repeat this process until a flat extension of \( y \) is found. In all the tested instances, we were able to find a flat extension of \( y \) for some \( k \geq d \). In the computations, we sometimes obtained \(|\text{supp}(\mu)| < N \) and sometimes \(|\text{supp}(\mu)| > N \), while the former was gotten more often. \( \square \)

The above random example leads us to believe that (4.9) is practical for solving TMPs. We now give a sufficient condition for (4.9) to produce a flat extension.

**Theorem 4.6.** In (4.9), let \( d = 2d_0 + 1 \), \( k = d_0 + 1 \) and \( c_k^T[x]_{2k} = [x]^T_k C_k[x]_k \) be such that \( C_k > 0 \). If the tms \( y \in \mathcal{M}_{n,d} \) admits a rank \( M_{d_0}(y) \)-atomic measure, then the optimizer \( w^* \) of (4.9) is flat.

**Proof.** Let \( r = \text{rank} \ M_{d_0}(y) \) and \( \mu \) be a \( r \)-atomic measure admitted by \( y \). Set

\[
z = \int [x]_d+1 \ d\mu, \quad M_{d_0+1}(z) = \begin{bmatrix} M_{d_0}(y) & B \end{bmatrix}, \quad C_k = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}.
\]

The matrix \( B \) consists of moments \( y_\alpha \) with \(|\alpha| \leq d \). Note that

\[
r = |\text{supp}(\mu)| \geq \text{rank} \ M_{d_0+1}(z) \geq \text{rank} M_{d_0}(y) = r.
\]

So \( \text{rank} M_{d_0+1}(z) = r \). Then \( M_{d_0+1}(z) \succeq 0 \) implies

\[
Z = B^T M_{d_0}(y)^+ B \succeq 0,
\]

where the superscript \( + \) denotes the Moore-Penrose Pseudo inverse of a matrix. For every \( w \) that is feasible for (4.9), write

\[
M_{d_0+1}(w) = \begin{bmatrix} M_{d_0}(y) & B \end{bmatrix}, \quad C_k \bullet M_{d_0+1}(w) = C_{11} \bullet M_{d_0}(y) + 2C_{12} \bullet B + C_{22} \bullet W.
\]

Thus, the objective of (4.9) can be equivalently replaced by \( C_{22} \bullet W \). If \( M_{d_0+1}(w) \succeq 0 \), then \( W - Z \succeq 0 \) and \( C_{22} \bullet W \geq C_{22} \bullet Z \). So, \( C_{22} \bullet Z \) is a lower bound of \( C_{22} \bullet W \) for all feasible \( w \). This implies that \( z \) is an optimizer of (4.9). Indeed, \( z \) is the unique optimizer of (4.9). Suppose \( w \) is another optimizer, then \( C_{22} \bullet (W - Z) = 0 \).
The fact that $W \succeq Z$ and $C_{22} \succ 0$ implies $W = Z$. Hence $w = z$. Clearly, $z$ is flat, and the proof is complete. □

Remark: When $K$ in (1.1) is noncompact, we can similarly solve the SDP problem:

$$\begin{align*}
\min_{w \in \mathcal{M}_{n \times 2k}} & \quad c_k^T w \\
\text{s.t.} & \quad w|_d = y, M_{t-d}(g_i \ast w) \succeq 0, i = 0, 1, \ldots, m
\end{align*}$$

in analogue to (4.10) and (4.9). To guarantee it always has a minimizer $w^*$, we typically choose $c_k$ as $c_k^T |x|_{2k} = |x|^T C_k |x|_k$ with $C_k \succ 0$, like in (4.9). If a truncation $w^*|_t (t \geq d)$ is flat, we get a flat extension of $y$.

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**Appendix A. Optimization with measures**

We prove that for a dense set of optimization problems (4.3), the optimal solutions are measures having finite supports. This justifies condition (4.5) in §4.2.

**Lemma A.1.** Let $y$ be a tms in $\mathcal{M}_{n,d}$ and $K$ be a subset of $\mathbb{R}^n$. Assume the set $\text{meas}(y, K)$ is nonempty. If a measure $\mu$ is an extreme point of $\text{meas}(y, K)$, then $\mu$ is finitely atomic and $|\text{supp}(\mu)| \leq \binom{n+d}{d}$.

**Proof.** We prove by contradiction. Suppose $\mu$ is extreme and $|\text{supp}(\mu)| > N := \binom{n+d}{d}$. Then we can decompose $\text{supp}(\mu)$ in a way such that

$$\text{supp}(\mu) = \bigcup_{j=1}^{N+1} S_j, \quad S_i \cap S_j = \emptyset, \quad \mu(S_j) > 0, \quad \forall i \neq j.$$

For each $j$, let $\mu_j$ be the restriction of $\mu$ on $S_j$, i.e., $\mu_j = \mu|_{S_j}$. Note that $\dim \mathbb{R}[|x|_d] = N$. So there exists $(a_1, \ldots, a_{N+1}) \neq 0$ satisfying

$$\sum_{j=1}^{N+1} a_j \int_K x^n d\mu_j = 0, \quad \forall |\alpha| \leq d.$$

Define a measure $\nu$ (possibly non-positive) such that

$$\nu = \nu|_{S_1} + \cdots + \nu|_{S_{N+1}}, \quad \nu|_{S_j} = a_j \mu_j, \quad j = 1, \ldots, N + 1.$$

Note that $\text{supp}(\nu) \subseteq \text{supp}(\mu)$. Then, for $\epsilon > 0$ sufficiently small, $\mu \pm \epsilon \nu$ are nonnegative Borel measures supported on $K$, representing the same tms $y$ and $\mu = (\mu + \epsilon \nu)/2 + (\mu - \epsilon \nu)/2$.

This contradicts the extremality of $\mu$ in $\text{meas}(y, K)$. □

**Theorem A.2.** Let $y$ be a tms in $\mathcal{M}_{n,d}$. For any real analytic function $f$ on a compact set $K$ and $\epsilon > 0$, there exists an real analytic function $\hat{f}$ on $K$ such that $||\hat{f} - f||_\infty \leq \epsilon$ and every minimizer of

$$\begin{align*}
\min_{\mu} & \quad \int_K \hat{f} d\mu \\
\text{s.t.} & \quad \mu \in \text{meas}(y, K)
\end{align*}$$

in $\text{meas}(y, K)$.
has cardinality at most \( \binom{n+d}{d} \). Let \( \text{Sol} \) be the set of all optimizers above. Then
\[
(A.2) \quad \left| \bigcup_{\mu \in \text{Sol}} \{\text{supp}(\mu)\} \right| \leq \binom{n+d}{d}.
\]

**Proof.** For a given real analytic function \( f \) on \( K \), consider the optimization problem
\[
(A.3) \quad \min_{\mu} \int_K f d\mu \quad \text{s.t.} \quad \mu \in \text{meas}(y,K).
\]

The set of positive measures on \( K \) whose total masses equal \( y_0 \) is compact in the weak-* topology (denote this topology by \( \mathcal{T} \)) by the Alaoglu Theorem (cf. \[2\], Theorem V.3.1). Recall that the weak-* topology is the topology on the measures regarded as the dual space of the Banach space \( \mathcal{C}(K) \). It is the weakest topology for which the convergence of a sequence of measures \( \mu_n \to \mu \) implies that for every \( h \in \mathcal{C}(K) \)
\[
(A.4) \quad \int_K h d\mu_n \to \int_K h d\mu.
\]

This implies that every moment of \( \mu_n \) converges to the corresponding one of \( \mu \). Hence, \( \text{meas}(y,K) \) is \( \mathcal{T} \)-closed inside a compact set, and it is also \( \mathcal{T} \)-compact.

Thus, by compactness of its feasible set, the optimization problem \( (A.3) \) has a minimizer, which is a generalized convex combination of certain extreme points, by the Choquet-Bishop-de Leeuw Theorem (cf. \[23\]). To be more precise, this means that for every \( \hat{\mu} \in \text{meas}(y,K) \), there exists a probability measure \( \Gamma \) on the set \( E \) of extreme points of \( \text{meas}(y,K) \) such that
\[
\ell(\hat{\mu}) = \int_E \ell(\mu) \, d\Gamma(\mu)
\]
for every affine function \( \ell \) on \( \text{meas}(y,K) \).

Let \( \gamma^* \) be the optimal value of \( (A.3) \) and \( \hat{\mu} \) be a minimizer, and let \( \Gamma \) denote the probability measure on \( E \) representing \( \hat{\mu} \) and let \( \bar{E} \) denote the support of \( \Gamma \). Then
\[
\gamma^* = \int_K f d\hat{\mu} = \int_{\bar{E}} \left( \int_K f d\mu \right) d\Gamma(\mu).
\]

By optimality of \( \gamma^* \), \( \int_K f d\mu \geq \gamma^* \) for all \( \mu \in \bar{E} \). Indeed, \( \int_K f d\mu = \gamma^* \) for all \( \mu \in \bar{E} \). Otherwise, suppose \( \int_K f d\mu > \gamma^* \) on a set of \( \mu \) having positive \( \Gamma \) measure. Then
\[
\gamma^* = \int_{\bar{E}} \left( \int_K f d\mu \right) d\Gamma(\mu) > \gamma^*,
\]
which yields a contradiction. This implies \( \int_K f d\mu = \gamma^* \) on \( \bar{E} \). Choose a measure \( \mu^* \in \bar{E} \). It is extreme and the optimum of \( (A.3) \) is attained at \( \mu^* \).

By Lemma \[A.1\] we have \( |\text{supp}(\mu^*)| \leq N := \binom{n+d}{d} \). Without loss of generality, we can normalize \( y_0 = \int_K d\mu^* = 1 \) and denote \( \text{supp}(\mu^*) = \{u_1, \ldots, u_r\} \). Let \( e(x) \) be the exponential function defined as
\[
e(x) = \epsilon \cdot \exp(-\|x - u_1\|_2^2 \cdots \|x - u_r\|_2^2).
\]

Clearly, \( e(x) = \epsilon \) for all \( x \in \text{supp}(\mu^*) \) and \( 0 < e(x) < \epsilon \) for all \( x \notin \text{supp}(\mu^*) \). This implies that
\[
\max_{\mu \in \text{meas}(y,K)} \int_K e \, d\mu = \int_K e \, d\mu^* = \epsilon.
\]
Set \( \tilde{f} := f - e \) and
\[
\beta := \min_{\mu \in \text{meas}(y,K)} \int_K \tilde{f} \, d\mu.
\]

Then
\[
\beta \geq \min_{\mu \in \text{meas}(y,K)} \int_K f \, d\mu - \max_{\mu \in \text{meas}(y,K)} \int_K e \, d\mu = \int_K f \, d\mu^* - \epsilon.
\]

On the other hand
\[
\int_K f \, d\mu^* - \epsilon = \int_K \tilde{f} \, d\mu^* \geq \min_{\mu \in \text{meas}(y,K)} \int_K \tilde{f} \, d\mu = \beta
\]
because \( \mu^* \) belongs to \( \text{meas}(y,K) \). Thus
\[
\beta = \int_K f \, d\mu^* - \epsilon = \int_K \tilde{f} \, d\mu^*
\]
and \( \mu^* \) is a minimizer of \((A.5)\) as well as \((A.1)\).

Suppose \( \tilde{\mu}^* \) is another minimizer of \((A.5)\). We wish to show that
\[
\text{suppt}(\tilde{\mu}^*) \subseteq \{u_1, \ldots, u_r\}.
\]
If this is not true, then \( \int e \, d\tilde{\mu}^* < \epsilon \) and
\[
\beta < \int_K f \, d\mu^* - \epsilon < \min_{\mu \in \text{meas}(y,K)} \int_K f \, d\mu - \int_K e \, d\tilde{\mu}^*.
\]
This implies
\[
\beta < \int_K f \, d\mu^* - \int_K e \, d\tilde{\mu}^* = \int_K \tilde{f} \, d\mu^* = \beta,
\]
which is a contradiction.

The inequality \((A.2)\) follows from the first part, because the average of all the minimizers is still a minimizer. \(\square\)

Remark: As we can see in the above proof, if \( \mu^* \) is a unique minimizer of \((4.3)\), then \( \mu^* \) must have finite support and \( |\text{supp}(\mu^*)| \leq \binom{n+d}{d} \).

The following lemmas are used in the proof of Theorem 4.3.

**Lemma A.3.** Suppose \( K \subseteq B(0,R) \) with \( R < 1 \) and \( e \) is in \( \mathcal{M}_{n,\infty}^2 \). If a tms \( w \) belongs to \( E_k(y) \cup E_\infty(y) \), then
\[
\|w\|_2 \leq y_0/(1 - R^2),
\]
and for any integer \( t > 0 \) it holds that
\[
(A.8) \quad \left| \sum_{|\alpha| > 2t} c_\alpha w_\alpha \right| \leq \|c\|_2 \cdot y_0 \cdot R^{2t+2}/(1 - R^2).
\]

**Proof.** First, we consider the case that \( w \in E_k(y) \). The condition \( M_{t-1}(\rho \ast w) \geq 0 \) in \((4.1)\) implies that for every \( t = 1, \ldots, k \)
\[
R^2 \mathcal{L}_w(||x||_2^{2t-2}) - \mathcal{L}_w(||x||_2^{2t}) \geq 0.
\]
A repeated application of the above gives
\[
\mathcal{L}_w(||x||_2^{2t}) \leq R^{2t} y_0.
\]
Since $M_k(w)$ is positive semidefinite, we can see that

$$\|w\|_2 \leq \|M_k(w)\|_F \leq \text{Trace}(M_k(w)) = \sum_{t=0}^{k} \sum_{|\alpha|=t} w_{2\alpha},$$

$$\sum_{|\alpha|=t} w_{2\alpha} = \mathcal{L}_w(\sum_{|\alpha|=t} x^{2\alpha}) \leq \mathcal{L}_w(\|x\|_2) \leq R^t y_0.$$  

Thus, it holds that

$$\|w\|_2 \leq y_0(1 + R^2 + \cdots + R^{2k}) \leq y_0/(1 - R^2).$$

When $k > t$, we have

$$\left| \sum_{2t<|\alpha|\leq 2k} c_\alpha w_\alpha \right|^2 \leq \left( \sum_{2t<|\alpha|\leq 2k} c_\alpha^2 \right)^{1/2} \leq \|M_{t,k}\|_F \leq \text{Trace}(M_{t,k})$$

$$= \sum_{j=t+1}^{k} \sum_{|\alpha|=j} w_{2\alpha} \leq \sum_{j=t+1}^{k} \mathcal{L}_w(\|x\|_2^2) \leq \sum_{j=t+1}^{k} y_0 R^{2j}.$$  

Combining all of the above , we get

$$\left| \sum_{2t<|\alpha|\leq 2k} c_\alpha w_\alpha \right| \leq \|c\|_2 \cdot y_0 \cdot (R^{2t+2} + \cdots + R^{2k}) \leq \|c\|_2 \cdot y_0 \cdot R^{2t+2}/(1 - R^2).$$

Note that $w_\alpha = 0$ if $|\alpha| > 2k$. So, (A.8) is true.

Second, we consider the case that $w \in E_\infty(y)$. The sequence $w$ admits a $K$-measure $\mu$ such that $w_\alpha = \int_{K} x^\alpha d\mu$ for every $\alpha$. For every $k$, consider the truncation $z = \int_{K} \langle x \rangle \|x\|_2^2 d\mu$. Clearly, the tms $z$ is feasible for (4.1). By part (i), we have

$$\|z\|_2 \leq y_0/(1 - R^2), \quad \sum_{2t<|\alpha|\leq 2k} c_\alpha z_\alpha \leq \|c\|_2 \cdot y_0 \cdot R^{2t+2}/(1 - R^2).$$

The above is true for all $k$, and implies (A.8) by letting $k \to \infty$. □

**Lemma A.4.** Assume $K \subseteq B(0,R)$ with $R < 1$, and $c$ is in $\mathcal{M}_{n,\infty}^2$. Let $\gamma_k, \gamma$ be the optimal values of (4.1) and (4.3) respectively. Then, we have the convergence $\gamma_k \to \gamma$ as $k \to \infty$.

**Proof.** By Lemma A.3 for an arbitrary $\epsilon > 0$, there exists $t$ such that

$$\sum_{|\alpha|>t} c_\alpha w_\alpha < \epsilon, \quad \forall w \in E_k(y) \cup E_\infty(y).$$

This implies that for every $w \in E_k(y) \cup E_\infty(y)$ with $k \geq t$, it holds that

$$\|\langle c, w \rangle - c^T_i w\|_{2t} < \epsilon.$$

Now we consider the truncated optimization problems

$$\min_{w \in \mathcal{M}_{n,2k}} c^T_i w |_{2t} \quad s.t. \quad w \in E_k(y).$$
and
\[
\min_w c^T w |_{2t} \quad \text{s.t.} \quad w \in E_\infty(y).
\]

Let \( \tau_k \) and \( \tau \) be the optimal values of (A.11) and (A.12) respectively. Note that (A.11) is a semidefinite relaxation of (A.12). Thus, we have \( \tau_k \to \tau \), by Theorem 1 of Lasserre [17]. The inequality (A.10) implies that for all \( k \geq t \)
\[
|\gamma_k - \tau_k| \leq \epsilon \quad \text{for all} \quad k \geq t.
\]

This shows that \( |\gamma_k - \tau_k| \leq \epsilon \) for \( k \geq t \). Applying a similar argument to (A.12), we can get \( |\gamma - \tau| \leq \epsilon \). Note that
\[
|\gamma_k - \gamma| \leq |\gamma_k - \tau_k| + |\tau_k - \tau| + |\tau - \gamma|.
\]

Because \( \tau_k \to \tau \), one can get that
\[
0 \leq \limsup_{k \to \infty} |\gamma_k - \gamma| \leq 2\epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we must have \( \gamma_k \to \gamma \) as \( k \to \infty \).

\[ \square \]

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E-mail address: helton@math.ucsd.edu

E-mail address: njw@math.ucsd.edu