Matroids, Feynman categories, and Koszul duality

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Abstract

We show that various combinatorial invariants of matroids such as Chow rings and Orlik–Solomon algebras may be assembled into “operad-like” structures. Specifically, one obtains several operads over a certain Feynman category which we introduce and study in detail. In addition, we establish a Koszul-type duality between Chow rings and Orlik–Solomon algebras, vastly generalizing a celebrated result of Getzler. This provides a new interpretation of combinatorial Leray models of Orlik–Solomon algebras.

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1 Introduction

This story finds its origin in the celebrated work of De Concini and Procesi [6]. In this article the authors construct special compactifications for every projective complement \( \mathbb{P}(\mathbb{C}^n) \setminus \bigcup_{H \in \mathcal{H}} \mathbb{P}H \) of some hyperplane arrangement \( \mathcal{H} \). Those compactifications are called “wonderful” because the complement of \( \mathbb{P}(\mathbb{C}^n) \setminus \bigcup_{H \in \mathcal{H}} \mathbb{P}H \) in the compactification is a divisor with normal crossings. Wonderful compactifications are obtained by successively blowing up some of the intersections of hyperplanes of \( \mathcal{H} \). The choice \( \mathcal{G} \) of the intersections to blow up matters and the resulting blown up space \( \overline{\mathcal{Y}}_{H, \mathcal{G}} \) is a wonderful compactification when \( \mathcal{G} \) is a building set of the lattice \( \mathcal{L}_H = \{ \cap_{H \in I} H, \ I \subset \mathcal{H} \} \), ordered by reverse inclusion. The building set condition on \( \mathcal{G} \) ensures that all the non-transversal intersections will be blown up.

Wonderful compactifications are naturally stratified by the exceptional divisors obtained in the process of blowing up. There is one exceptional divisor \( D_G \) for each element \( G \) of \( \mathcal{G} \). Those divisors have normal crossings and an intersection of divisors \( D_{G_1}, \ldots, D_{G_n} \) is nonempty exactly when \( \{G_1, \ldots, G_n\} \) forms a nested set of \( \mathcal{G} \). The non-empty intersections of divisors are the closed strata of the stratification. De Concini and Procesi discovered that the closed strata are in fact isomorphic to products of “smaller” wonderful compactifications. With this in mind, simply considering inclusions of the strata in the compactifications leads to interesting additional structures.

For instance, when looking at the family of braid arrangements \( \text{Braid}_n \) consisting of the diagonal hyperplanes \( \{z_i = z_j\} \subset \mathbb{C}^n \), together with the unique minimal building set, this gives us the well-known structure of an operad on the Deligne–Mumford compactification of moduli spaces of pointed curves of genus zero. Passing to the homology gives the operad of hypercommutative algebras called \( \text{Hypercom} \). When looking at the family of boolean arrangements \( \text{Bool}_n \) consisting of coordinate hyperplanes in \( \mathbb{C}^n \), together with the unique maximal building set, we also get an operad-like structure, this time on the so-called Losev-Manin spaces.

The aim of this article is to develop a formalism enabling us to see the family of all possible wonderful compactifications of all possible hyperplane arrangements, with structural morphisms given by inclusions of strata, as one big “operad”. This way, we will be able to interpret commonalities between the above examples as mere consequences of the properties enjoyed by the unified structure. For instance, we shall be interested in finding a unified presentation by generators and quadratic relations of this operad, as well as a minimal model, which will be done by Koszul duality theory (see Loday-Valette [15] for an introduction). Fortunately, recent years have seen the advent of general theories that were successful in developing a language as well as methods to deal with “operad-like” structures. One of those theories is that of Feynman categories (Kaufmann-Ward [14]),
which we will use in this article to fulfil our goal.

Instead of dealing with hyperplane arrangements we shall be working at the more general level of matroids (see Welsh [21] for an introduction), which form a combinatorial abstraction of hyperplane arrangements. For our purposes it will be enough to consider only simple loopless matroids, which can be axiomatized by geometric lattices. This axiomatization is the most convenient to us and is the one we will be working with in this paper.

The wonderful compactifications do depend on the hyperplane arrangement itself, together with the choice of the building set, but their cohomology algebra only depend on the intersection lattice and its building set. In [10], Feichtner and Yuzvinsky introduced a generalization of those cohomology rings for every pair of a geometric lattice \( \mathcal{L} \) and a building set \( \mathcal{G} \subset \mathcal{L} \), which we will denote by \( \text{FY}(\mathcal{L}, \mathcal{G}) \). In the case where \( \mathcal{G} \) is equal to \( \mathcal{L} \setminus \{0\} \), the ring \( \text{FY}(\mathcal{L}, \mathcal{G}) \) is the combinatorial Chow ring of \( \mathcal{L} \). Those rings have been extensively studied and are known to satisfy very strong properties. For instance even though these rings are not necessarily cohomology rings of projective complex varieties, they all have a Hodge theory, meaning that they satisfy Poincaré duality as well as the Kähler package (see Adiprasito-Huh-Katz [1] and Pagaria-Pezzoli [17] for general building sets). In [4], Bibby, Denham and Feichtner show that the morphisms between cohomology algebras of wonderful compactification induced by the inclusions of strata can also be generalized to the purely combinatorial setting. This means that we have morphisms between the rings \( \text{FY}(\mathcal{L}, \mathcal{G}) \), indexed by nested sets.

In Section 3 we construct a Feynman category \( \mathcal{LB}S \) such that the family of algebras \( \{\text{FY}(\mathcal{L}, \mathcal{G})\}_{(\mathcal{L}, \mathcal{G})} \) together with the above morphisms forms an operad of type \( \mathcal{LB}S \), that is simply a monoidal functor from \( \mathcal{LB}S \) to some symmetric monoidal category (in the present case the category of graded commutative algebras). In short, the objects of \( \mathcal{LB}S \) will be the pairs \((\mathcal{L}, \mathcal{G})\) with \( \mathcal{G} \) a building set of \( \mathcal{L} \), and the morphisms will be given by the nested sets. The heart of the problem is to find a suitable “composition” of nested sets, which is given in Subsection 3.2.

Going back to braid arrangements, in [12] Getzler has shown that Hypercom is Koszul with Koszul dual the operad \( \text{Grav} \) consisting of the cohomology algebras of the projective complements of the braid arrangements, with operations given by residue morphisms. Cohomology algebras of projective complements of hyperplane arrangements are called projective Orlik–Solomon algebras and only depend on the intersection lattice. They can be generalized to arbitrary geometric lattices. We will denote those algebras by \( \text{OS}(\mathcal{L}) \), for \( \mathcal{L} \) any geometric lattice.

In this article we show that the family of projective Orlik–Solomon algebras \( \{\text{OS}(\mathcal{L})\} \)
also has an operadic structure over $\mathcal{LBS}$, with morphisms given by combinatorial generalizations of residue morphisms. When restricted to partition lattices (intersection lattices of braid arrangements) this gives back the linear dual of the operad $\text{Grav}$. Additionally, in [4] the authors show that the combinatorial Chow rings can be assembled to form a combinatorial model $B(\mathcal{L}, \mathcal{G})$ of $\overline{\text{OS}}(\mathcal{L})$:

$$\overline{\text{OS}}(\mathcal{L}) \xrightarrow{\sim} B(\mathcal{L}, \mathcal{G}).$$

In Section 6 we explain that $\{B(\mathcal{L}, \mathcal{G})\}$ is also an operad over $\mathcal{LBS}$ and is in fact the Bar construction of the operad $\{FY(\mathcal{L}, \mathcal{G})\}$, which immediately implies the corollary.

**Corollary 1.1.** The operad $\{FY(\mathcal{L}, \mathcal{G})\}$ is Koszul with Koszul dual $\{\overline{\text{OS}}(\mathcal{L})\}$.

We give an alternative proof of this fact by implementing a Gröbner bases machinery for operads over $\mathcal{LBS}$ and proving the following.

**Proposition 1.2.** If an operad over $\mathcal{LBS}$ admits a quadratic Gröbner basis then it is Koszul.

**Theorem 1.3.** The $\mathcal{LBS}$-operad $\{FY(\mathcal{L}, \mathcal{G})\}$ admits a quadratic Gröbner basis.

Finally, let us mention that evidence of “operad-like” structures related to combinatorics of building sets/nested sets were already highlighted by Forcey, Ronco [11] and Rains [18], in specific settings and using different formalisms.

Here is the general layout of the article.

Section 2 is devoted to introducing the combinatorial ingredients that will be used in the construction of the Feynman category $\mathcal{LBS}$.

Section 3 is the core of this paper. This is where we define the Feynman category $\mathcal{LBS}$. We also prove the important fact that $\mathcal{LBS}$ admits a graded presentation.

In Section 4 we show that the families $\{FY(\mathcal{L}, \mathcal{G})\}$, $\{\text{OS}(\mathcal{L})\}$, $\{\overline{\text{OS}}(\mathcal{L})\}$ form operads over the Feynman category $\mathcal{LBS}$.

In Section 5 we develop a theory of Gröbner bases for operads over $\mathcal{LBS}$ and prove that $\{FY(\mathcal{L}, \mathcal{G})\}$ admits a quadratic Gröbner basis.

In Section 6 we show that $\mathcal{LBS}$ is a cubical Feynman category which implies that there is a Koszul duality theory for operads over $\mathcal{LBS}$. We then show that $\{FY(\mathcal{L}, \mathcal{G})\}$ is Koszul via two different methods.
Finally, in Section 7, we give some last remarks toward possible generalizations and modifications of LBS.

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2 Combinatorial preliminaries

In this section we introduce the main combinatorial objects which will be used throughout this paper.

It is important to note that in this article we work at a strictly combinatorial level. However, many of the main protagonists in this story have a geometric origin and although the geometric picture is not formally required, it is our main source of inspiration and therefore we will try to draw this picture whenever possible.

2.1 Lattices, building sets and nested sets

Definition 2.1 (Lattice). A finite poset \( \mathcal{L} \) is called a lattice if every pair of elements in \( \mathcal{L} \) admits a supremum and an infimum.

The supremum of two elements \( G_1, G_2 \) is denoted by \( G_1 \vee G_2 \) and called their join, while their infimum is denoted by \( G_1 \wedge G_2 \) and called their meet.

Remark 2.2. Since \( \mathcal{L} \) is supposed to be finite, having supremums and infimums for pairs of elements implies having supremums and infimums for any subset \( S \) of \( \mathcal{L} \), which will be denoted by \( \bigvee S \) and \( \bigwedge S \) respectively. As a consequence, every lattice admits an upper bound (the supremum of \( S = \mathcal{L} \)) and a lower bound (the infimum of \( S = \mathcal{L} \)) which will be denoted by \( \hat{1} \) and \( \hat{0} \) respectively.

Definition 2.3 (Geometric lattice). A finite lattice \( (\mathcal{L}, \leq) \) is said to be geometric if it satisfies the following properties:

- For every pair of elements \( G_1 \leq G_2 \), all the maximal chains of elements between \( G_1 \) and \( G_2 \) have the same cardinal. (Jordan–Hölder property)

- The rank function \( \rho : \mathcal{L} \rightarrow \mathbb{N} \) which assigns to any element \( G \) of \( \mathcal{L} \) the cardinal of any maximal chain of elements from \( \hat{0} \) to \( G \) (not counting \( \hat{0} \)) satisfies the inequality

\[
\rho(G_1 \wedge G_2) + \rho(G_1 \vee G_2) \leq \rho(G_1) + \rho(G_2)
\]

for every \( G_1, G_2 \) in \( \mathcal{L} \). (Sub-modularity)

- Every element in \( \mathcal{L} \) can be obtained as the supremum of some set of atoms (i.e. elements of rank 1). (Atomicity)

One of the reasons to study this particular class of lattices is that the intersection poset of any hyperplane arrangement is a geometric lattice. In fact, one may think of geometric lattices as a combinatorial abstraction of hyperplane arrangements. In addition, this object is equivalent to the datum of a loopless simple matroid via the lattice of flats construction.
(see [21] for a reference on matroid theory) and therefore it has connections to many other areas in mathematics (graph theory for instance).

Here is a list of some important well-known geometric lattices.

**Example 2.4.**

- If $X$ is any finite set, the set $\mathcal{P}(X)$ of subsets of $X$ ordered by inclusion is a geometric lattice with join the union and meet the intersection. It is the intersection lattice of the hyperplane arrangement of coordinate hyperplanes in $\mathbb{C}^X$. Those geometric lattices are called boolean lattices and denoted by $B_X$.

- If $X$ is any finite set, the set $\Pi_X$ of partitions of $X$ ordered by refinement is a geometric lattice. It is the intersection lattice of the so-called braid arrangement which consists of the diagonal hyperplanes $\{z_i = z_j\}$ in $\mathbb{C}^X$. Those geometric lattices are called partition lattices.

- If $G = (V, E)$ is any graph one can construct the graphical matroid $M_G$ associated to $G$ and then consider $L_G$ the lattice of flats associated to $M_G$ (see [21] for the details of this construction). Those lattices are said to be graphical. This family of geometric lattices contains the two previous ones because $B_X$ is the lattice associated to any tree with edges $X$ and $\Pi_X$ is the lattice associated to the complete graph with vertices $X$. For any graph $G = (V, E)$ the geometric lattice $L_G$ is the intersection lattice of the hyperplane arrangement $\{\{z_u = z_v\}, (u, v) \in E\}$ in $\mathbb{C}^V$.

We have the following important fact about geometric lattices.

**Proposition 2.5 ([21]).** Let $(\mathcal{L}, \leq)$ be a geometric lattice. For every $G_1 \leq G_2 \in \mathcal{L}$, the interval $[G_1, G_2] = \{G \in \mathcal{L} | G_1 \leq G \leq G_2\}$ ordered by the restriction of $\leq$ is a geometric lattice.

In the rest of this article every lattice will be assumed to be geometric unless stated otherwise.

**Definition 2.6 (Building set).** Let $\mathcal{L}$ be a geometric lattice. A building set $\mathcal{G}$ of $\mathcal{L}$ is a subset of $\mathcal{L} \setminus \{\hat{0}\}$ such that for every element $X$ of $\mathcal{L}$ the morphism of posets

$$\prod_{G \in \text{max } \mathcal{G} \leq X} [\hat{0}, G] \xrightarrow{\gamma} [\hat{0}, X]$$

(1)

is an isomorphism (where $\text{max } \mathcal{G} \leq X$ is the set of maximal elements of $\mathcal{G} \cap [\hat{0}, G]$).

The elements of $\text{max } \mathcal{G} \leq X$ will be called the factors of $X$ in $\mathcal{G}$. In the rest of the paper we will prefer the more suggestive notation $\text{Fact}_\mathcal{G}(X)$ to refer to the set of those elements.

**Definition 2.7 (Built lattice).** The datum of a lattice $\mathcal{L}$ and a building set $\mathcal{G}$ of $\mathcal{L}$ will be called a built lattice. If $\mathcal{G}$ contains $\hat{1}$ we say that $(\mathcal{L}, \mathcal{G})$ is irreducible.
The definition of a building set makes sense for a larger class of posets, as shown in [10], but in this paper we will restrict ourselves to the case of geometric lattices. In this particular context, building sets are geometrically motivated by the construction of wonderful compactifications for hyperplane arrangement complements. In a nutshell, building sets are sets of intersections of a hyperplane arrangement that one can successively blow up in order to obtain a wonderful compactification of its complement (see [6] for more details). Each blowup creates a new exceptional divisor, so the wonderful compactification is equipped with a family of irreducible divisors indexed by \(G\). This family of divisors forms a normal crossing divisor when \(G\) is a building set.

There are a few key examples to keep in mind throughout this story.

Example 2.8.

- Every lattice \(\mathcal{L}\) admits \(\mathcal{L} \setminus \{\hat{0}\}\) as a building set.
- Every lattice \(\mathcal{L}\) also admits a unique minimal building set which consists of all the elements \(G\) of \(\mathcal{L}\) such that \([\hat{0}, G]\) is not a product of proper subposets.
- From the definition one can see that a building set of some lattice \(\mathcal{L}\) must contain all the atoms of \(\mathcal{L}\). If \(\mathcal{L}\) is a boolean lattice (see Example 2.4) then its set of atoms is in fact a building set (the minimal one). This fact characterizes boolean lattices.
- If \(\mathcal{L}\) is the lattice of partitions of some finite set (see Example 2.4) then the subset of partitions with only one block having more than two elements is a building set of \(\mathcal{L}\). This is the minimal building set of \(\mathcal{L}\).
- If \(\mathcal{L}\) is a graphical lattice (see Example 2.4) then the set of elements of \(\mathcal{L}\) corresponding to sets of edges which are connected forms a building set of \(\mathcal{L}\). This family of examples contains the two previous ones (by considering totally disconnected graphs for the former and complete graphs for the latter).
- Alternatively, if \(G = (V, E)\) is a graph one can consider the boolean lattice \(\mathcal{B}_V\). This lattice has a building set made up of the “tubes” of \(G\), that is sets of vertices of \(G\) such that the induced subgraph on those vertices is connected. This leads to the notion of graph associahedra introduced in [5].

A key fact about building sets is that any interval \([G_1, G_2]\) in some built lattice \((\mathcal{L}, \mathcal{G})\) admits an “induced” building set which we describe now. We start by introducing a useful notation.

Notation 2.9. For any element \(G\) of some lattice \(\mathcal{L}\) and a subset \(X\) of \(\mathcal{L}\), we denote by \(G \vee X\) the set of elements of \(\mathcal{L}\) which can be obtained as the join of \(G\) and some element of \(X\).
Definition 2.10 (Induced building set). Let $G_1 < G_2$ be two elements in some built lattice $(L, \mathcal{G})$. We denote by $\text{Ind}_{[G_1, G_2]}(\mathcal{G})$ the set $(G_1 \lor \mathcal{G}) \cap [G_1, G_2] \setminus \{G_1\}$ and we call it the induced building set on $[G_1, G_2]$.

Lemma 2.11 ([4] Lemma 2.8.5). The subset $\text{Ind}_{[G_1, G_2]}(\mathcal{G}) \subset [G_1, G_2]$ is a building set of $[G_1, G_2]$.

We will often write $\text{Ind}(\mathcal{G})$ instead of $\text{Ind}_{[G_1, G_2]}(\mathcal{G})$ if the interval can be deduced from the context. We have the obvious lemma.

Lemma 2.12. For any elements $X_1 \leq X_2 \leq X_3 \leq X_4$ in some lattice $L$ with building set $\mathcal{G}$, we have the equality of building set $\text{Ind}_{[X_2, X_3]}(\text{Ind}_{[X_1, X_4]}(\mathcal{G})) = \text{Ind}_{[X_2, X_3]}(\mathcal{G})$.

Definition 2.13 (Nested set). Let $(L, \mathcal{G})$ be a built lattice. A subset $S$ of $\mathcal{G}$ is called a nested set if for every set of pairwise incomparable elements $A$ in $S$, the join of the elements of $A$ does not belong to $\mathcal{G}$ whenever $A$ contains at least two elements. A nested set $S \subset \mathcal{G}$ is said to be irreducible if it contains $\max \mathcal{G}$.

A subset of pairwise incomparable elements in a poset will be called an antichain.

Example 2.14. A chain of elements in some building set $\mathcal{G}$ is always nested and those are the only nested sets of the maximal building set ($\mathcal{G} = L \setminus \{\hat{0}\}$).

Geometrically, nested sets correspond to sets of divisors in the wonderful compactification which have a nontrivial intersection. There are two crucial lemmas regarding nested sets.

Lemma 2.15 ([9] Proposition 2.8). Let $\mathcal{G}$ be a building set of a geometric lattice $L$ and let $X$ be any element of $L$. The subset $\text{Fact}_{\mathcal{G}}(X)$ is a nested antichain in $\mathcal{G}$ and furthermore it is the only nested antichain in $\mathcal{G}$ having join $X$.

Lemma 2.16 ([17] Proposition 2.4). A nested set of $L$ is a forest in the Hasse diagram of $L$. More precisely for every $K$ in $L$, $S_{> K}$ is either empty or has a unique minimal element.

We next introduce a map $\text{Comp}_{\mathcal{G}} : \text{Ind}_{[G, G']}(\mathcal{G}) \to \mathcal{G}$ which will help us define the composition of nested sets in further sections. Let $G''$ be an element of $\text{Ind}_{[G, G']}(\mathcal{G})$ and $F$ an element such that $G'' = G \lor F$ and which is maximal amongst elements satisfying this equality (such an $F$ exists by definition of $\text{Ind}(\mathcal{G})$ and by finiteness of $L$). Let us denote by $\{G_i, i \leq n\}$ the factors of $G$ in $\mathcal{G}$. We have equalities

$$G'' = G \lor F = \bigvee_{i \leq n} G_i \lor F = \bigvee_{i \leq n, G_i \not\subseteq F} G_i \lor F$$

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but the elements in the join on the right form an antichain which is nested by maximality of $F$ and therefore by Lemma 2.15 those elements are exactly the factors of $G''$ in $\mathcal{G}$. From this quick analysis it appears that such a maximal element $F$ is in fact unique and we can make the following definition.

**Definition 2.17.** For any element $G''$ in some induced building set $\text{Ind}_{\{G,G'\}}$, we define $\text{Comp}_G(G'')$ to be the unique maximal element of $\mathcal{G}$ satisfying

$$\text{Comp}_G(G'') \lor G = G''.$$  

If $G$ can be deduced from the context we will omit it. We have the simple lemma.

**Lemma 2.18.** The map $\text{Comp}_G$ is injective.

**Proof.** The map $\text{Comp}_G$ has a left inverse given by taking the join with $G$. \hfill \Box

### 2.2 Combinatorial invariants

To the objects introduced in the previous section one can associate various rings that generalize cohomology rings in the realizable case.

#### 2.2.1 The Feichtner–Yuzvinsky rings

**Definition 2.19.** For every built lattice $(L, \mathcal{G})$ we define the Feichtner–Yuzvinsky graded commutative ring $\text{FY}(L, \mathcal{G})$ by

$$\text{FY}(L, \mathcal{G}) = \mathbb{Z}[x_G, G \in \mathcal{G}]/\mathcal{I}_{\text{aff}},$$

with all the generators in degree 2, and $\mathcal{I}_{\text{aff}}$ the ideal generated by elements

$$\sum_{G \geq H} x_G$$

for every atom $H$, and elements

$$\prod_{G \in X} x_G$$

for every set $X \subset \mathcal{G}$ which is not nested.

In the realizable case, the ring $\text{FY}(L, \mathcal{G})$ is the cohomology ring of the wonderful compactification associated to the building set $\mathcal{G}$ (see [6] for the computation of the cohomology ring). Those rings were generalized to arbitrary built lattices by Feichtner and Yuzvinsky in [10].

The Feichtner–Yuzvinsky rings admit two other useful presentations.
Proposition 2.20. For every built lattice \((\mathcal{L}, \mathcal{G})\) we have the other classical presentation

\[
FY(\mathcal{L}, \mathcal{G}) \simeq \mathbb{Z}[x_G, \ G \in \mathcal{G} \setminus \{\hat{1}\}]/I_{\text{proj}}
\]

where \(I_{\text{proj}}\) is the ideal generated by elements

\[
\sum_{i > G \geq H_1} x_G - \sum_{i > G \geq H_2} x_G
\]

for every pair of atoms \(H_1\) and \(H_2\), and elements

\[
\prod_{G \in X} x_G
\]

for every set \(X \subset \mathcal{G} \setminus \{\hat{1}\}\) which is not nested.

Additionally, we have the presentation

\[
FY(\mathcal{L}, \mathcal{G}) \simeq \mathbb{Z}[h_G, \ G \in \mathcal{G}]/I_{\text{wond}}
\]

where \(I_{\text{wond}}\) is the ideal generated by relations

\[
h_H
\]

for every atom \(H\) and

\[
\prod_{G' \in A} (h_G - h_{G'})
\]

for every \(G \in \mathcal{G}\) and \(A\) an antichain in \(\mathcal{G}\) such that \(\bigvee A\) is equal to \(G\). The change of variable between the last presentation and the defining presentation is given by

\[
h_G = \sum_{G' \geq G} x_{G'}.
\]

The first (defining) presentation will be called the affine presentation, the second the projective presentation and the last one the wonderful presentation. The first two presentations appear in [10] (as a definition) while the second appeared first in [8] for the braid arrangement and in [2] for general maximal building sets. It is widely used in [17].

Proof. The proof can be found in [17] (Theorem 2.9).

In [10], the authors address the issue of finding a Gröbner basis for \(FY_{\text{aff}}(\mathcal{L}, \mathcal{G})\) (see [3] for a reference on Gröbner bases) and they show that when considering any linear order on generators refining the reverse order on \(\mathcal{G}\), although the elements defining \(I_{\text{aff}}\) do not form a Gröbner basis in general, one can still describe a fairly manageable Gröbner basis.
Theorem 2.21 ([10] Theorem 2). Elements of the form \((\prod_{G \in S} x_G)^{h_G(G') - \rho(\bigvee S)}\) with \(S\) any nested set and \(G'\) any element of \(G\) satisfying \(G' > \bigvee S\), together with the usual \(\prod_{G \in X} x_G\) for every non-nested set \(X\), form a Gröbner basis of \(FY_{\text{aff}}(L, G)\) for any linear order on generators refining the reversed order of \(L\). The normal monomials with respect to this Gröbner basis are monomials of the form
\[x_{G_1}^{\alpha_1} \cdots x_{G_n}^{\alpha_n}\]
where the \(G_i\)'s form a nested set \(S\) and for every \(i \leq n\) we have \(\alpha_i < \text{rk} \sqrt{S_{<G_i, G_i}}\).

Notice that \(S\) can be empty in which case we get the already known relations \(h_H = 0\) for any atom \(H\). Using that the above monomials form a linear basis of \(FY(L, G)\) one can see that every Feichtner–Yuzvinsky algebra is in fact of finite dimension and that the part of maximal grading, which is \(2(\text{rk}(L) - 1)\), has dimension one (generated by \(x_1^{\text{rk}(L) - 1}\)).

One can also find a Gröbner basis for the wonderful presentation.

Corollary 2.22. Elements of the form \((\prod_{G \in A} (h_G - h_{G'}))^{h_G^{(G') - \rho(A)}}\) for every antichain \(A \subset G\) without atoms and every \(G \geq \bigvee A\) form a Gröbner basis of \(FY_{\text{wond}}(L, G)\) for any linear order on generators refining the reversed order of \(L\).

Proof. Indeed one can see that the leading terms of those elements are terms of the form \((\prod_{G' \in A} h_{G'})^{h_G^{(G') - \rho(A)}}\) and therefore the normal monomials with respect to those relations are elements of the form \(\prod_{G \in S} h_G^{\alpha_G}\) for any nested set \(S\) and any positive integers \(\alpha_G\) satisfying the relations
\[\alpha_G < \rho(G) - \rho(\bigvee S_{<G})\]
for all \(G\) in \(S\), which are in obvious bijection with the normal monomials for the affine Gröbner basis. This proves that those monomials form a linear basis of \(FY_{\text{wond}}(L, G)\) (linearly independent with the right cardinality), which implies that the elements above form a Gröbner basis of \(I_{\text{wond}}\).

This Gröbner basis will be of use in subsequent sections. In the next and last preliminary subsection we introduce another important combinatorial invariant.

2.2.2 The Orlik–Solomon algebras

In this document “graded commutative” means with Koszul signs.

Definition 2.23 (Orlik–Solomon algebra). Let \(L\) be a geometric lattice. We define the Orlik–Solomon graded commutative algebra \(OS(L)\) by
\[OS(L) = \Lambda[e_H, H\ \text{atom of } L]/I\]
where \(I\) is the ideal generated by elements of the form \(\delta(e_{H_1} \wedge \cdots \wedge e_{H_n})\) for any circuit \(\{H_1, \ldots, H_n\}\) and \(\delta\) is the unique derivation of degree \(-1\) satisfying \(\delta(e_H) = 1\). All the generators \(e_H\) have degree 1.
A circuit is a notion coming from matroid theory. In the language of geometric lattices it is a set of atoms $C = \{H_i, i \leq n\}$ such that $\rho(\bigvee C)$ is $n - 1$ and $\rho(\bigvee X') = |X'|$ for all proper subsets $X' \subset X$.

In the complex realizable case this algebra is the cohomology ring of the complement of the hyperplane arrangement (see Orlik-Solomon [16]).

We denote by $\overline{\text{OS}}(\mathcal{L})$ the subalgebra of $\text{OS}(\mathcal{L})$ generated by elements of the form $e_H - e_{H'}$ for every pair of atoms $H, H'$. In the complex realizable case the algebra $\overline{\text{OS}}(\mathcal{L})$ is the cohomology ring of the projective complement. We have the following important lemma.

**Lemma 2.24** ([22] Section 2.4). For every geometric lattice $\mathcal{L}$ we have the equality

$$\overline{\text{OS}}(\mathcal{L}) = \ker \delta = \text{Im} \delta.$$

### 3 The Feynman category

In this section we show that the combinatorial objects introduced in the previous section (geometric lattices, building sets and nested sets) can be bundled up into a Feynman category.

#### 3.1 A short introduction to Feynman categories

The notion of Feynman categories was introduced by R. Kaufmann and B. Ward in [14]. Loosely speaking Feynman categories encode types of operadic structures.

**Notation.** Let $\mathcal{C}$ be a category. We denote by $\mathcal{C}^{\text{iso}}$ the subcategory of $\mathcal{C}$ having the same objects as $\mathcal{C}$ but only its isomorphisms as morphisms. We denote by $\text{Sym}(\mathcal{C})$ the free symmetric monoidal category generated by $\mathcal{C}$. For any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ a symmetric monoidal category, there is a unique induced strong monoidal functor $\text{Sym}(F) : \text{Sym}(\mathcal{C}) \to \mathcal{D}$ which restricts to $F$ on $\mathcal{C}$. If we are given a diagram of categories $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{E}$, the comma category $(F \downarrow G)$ is the category having for objects triples $(c \in \mathcal{C}, e \in \mathcal{E}, \phi : F(c) \to G(e))$ and for morphisms suitable commutative diagrams. If the functors $F$ and $G$ are clear from the context we will write instead $(\mathcal{C} \downarrow \mathcal{E})$.

**Definition 3.1** (Feynman category). A triple $\mathcal{F} = (\mathcal{V}, \mathcal{F}, i)$ is a Feynman category if $\mathcal{V}$ is a groupoid, $\mathcal{F}$ is a symmetric monoidal category and $i : \mathcal{V} \to \mathcal{F}$ is a functor such that:

1. The functor $i$ induces an equivalence of categories $\text{Sym}(i) : \text{Sym}(\mathcal{V}) \to \mathcal{F}^{\text{iso}}$.
2. The functor $i$ induces an equivalence of categories $\text{Sym}((\mathcal{F} \downarrow \mathcal{V})^{\text{iso}}) \to (\mathcal{F} \downarrow \mathcal{F})^{\text{iso}}$. 

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3. For every object $* \in \mathcal{V}$, the comma category $(\mathcal{F} \downarrow *)$ is essentially small (i.e. is equivalent to a small category).

A morphism in $\mathcal{F}$ which is not an isomorphism will be called a structural morphism.

**Definition 3.2** (Operad over a Feynman category). Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ be a Feynman category and $\mathcal{C}$ a symmetric monoidal category. An *operad over $\mathfrak{F}$* in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}$, and a *cooperad over $\mathfrak{F}$* in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}^{\text{op}}$. A *module over $\mathfrak{F}$* in $\mathcal{C}$ is a functor from $\mathcal{V}$ to $\mathcal{C}$.

(Co)operads (resp. modules) over $\mathfrak{F}$ will also be called $\mathfrak{F}$-(co)operads (resp. $\mathfrak{F}$-modules).

**Example 3.3.** As described in [14], there exists a Feynman category $\mathcal{Op}$ encoding classical operads i.e. such that operads over $\mathcal{Op}$ are classical operads and modules over $\mathcal{Op}$ are $\mathcal{S}$-modules. The underlying groupoid of $\mathcal{Op}$ is given by the category of finite sets with bijections and the structural morphisms are given by trees. Composition of trees is defined by the grafting of trees.

### 3.2 Construction of the Feynman category

Let us start by defining the underlying groupoid of our Feynman category.

**Definition 3.4.** We define $LBS$ to be the groupoid having as objects the built lattices and morphisms

$$\text{Mor}_{LBS}((\mathcal{L}, \mathcal{G}), (\mathcal{L}', \mathcal{G}')) = \{ f : \mathcal{L}' \xrightarrow{\sim} \mathcal{L} \text{ isomorphism of poset satisfying } f(\mathcal{G}') = \mathcal{G} \}.$$ 

We denote by $LBS_{\text{irr}}$ the full subcategory of $LBS$ having as objects the irreducible built lattices.

The groupoid $LBS_{\text{irr}}$ will play the role of $\mathcal{V}$ in Definition 3.1. We will add morphisms to $LBS$ in order to get the right category $\mathcal{F}$.

**Proposition 3.5.** The category $LBS$ admits a symmetric monoidal structure $\otimes$ given by

$$(\mathcal{L}, \mathcal{G}) \otimes (\mathcal{L}', \mathcal{G}') = (\mathcal{L} \times \mathcal{L}', \mathcal{G} \times \{\hat{0}\} \cup \{\hat{0}\} \times \mathcal{G}')$$

Furthermore the inclusion $\iota : LBS_{\text{irr}} \to LBS$ induces an equivalence of categories

$$\text{Sym}(\iota) : \text{Sym}(LBS_{\text{irr}}) \to LBS.$$ 

**Proof.** The fact that a product of two geometric lattices is again a geometric lattice is classical and the proof can be found in [21]. Additionally $\mathcal{G} \times \{\hat{0}\} \cup \{\hat{0}\} \times \mathcal{G}'$ is indeed a building
mands. This datum is exactly equivalent to an isomorphism in \( \text{Sym}(\text{irr}(X)) \) together with isomorphism between the maximal elements of the building set of the domain and the maximal elements by a bijection between the factors of both sides (the isomorphism induces a bijection between \( \text{Sym}(\text{irr}(X)) \)).

Besides, one can see that \( \otimes \) is functorial and satisfies the associativity/symmetry axioms of a symmetric monoidal product, the unit being \((\emptyset, \emptyset)\).

For the last claim we show that \( \text{Sym}(\iota) \) is essentially surjective and fully faithful. Let \((\mathcal{L}, \mathcal{G})\) be an object of \( \text{LBS} \). If we denote by \( \{G_i, i \leq n\} \) the factors of \( \mathcal{G} \) then we have an isomorphism \( \mathcal{G} \simeq \prod [\hat{0}, G_i]_{i \leq n} \) and \( \mathcal{G} \) is sent to \( \mathcal{G} \cap [\hat{0}, G_1] \cup \ldots \cup \mathcal{G} \cap [\hat{0}, G_n] \). In other words \((\mathcal{L}, \mathcal{G})\) is isomorphic to \((\mathcal{G} \cap [\hat{0}, G_1], \ldots, \mathcal{G} \cap [\hat{0}, G_n]) \otimes \mathcal{G} \cap [\hat{0}, G_n]) \) and \( \text{Sym}(\iota) \) is essentially surjective.

Finally, let \( \otimes_{i \leq n} (\mathcal{L}_i, G_i) \) and \( \otimes_{j \leq n'} (\mathcal{L}_j', G_j') \) be two elements of \( \text{Sym}(\text{LBS}_{\text{irr}}) \) (here \( \otimes \) denotes the free symmetric monoidal product in \( \text{Sym}(\text{LBS}_{\text{irr}}) \)) and let \( \phi \) be an isomorphism in \( \text{LBS} \) between \( \otimes_{i \leq n} (\mathcal{L}_i, G_i) \) and \( \otimes_{j \leq n'} (\mathcal{L}_j', G_j') \). Such an isomorphism is given by a bijection between the factors of both sides (the isomorphism induces a bijection between the maximal elements of the building set of the domain and the maximal elements of the building set of the target) together with isomorphisms between corresponding summands. This datum is exactly equivalent to an isomorphism in \( \text{Sym}(\text{LBS}_{\text{irr}}) \) between \( \otimes_{i \leq n} (\mathcal{L}_i, G_i) \) and \( \otimes_{j \leq n'} (\mathcal{L}_j', G_j') \) (via \( \iota \)), which proves that \( \text{Sym}(\iota) \) is fully faithful.

\[ \square \]

We now add structural morphisms to \( \text{LBS} \) to get our Feynman category. Let \((\mathcal{L}, \mathcal{G})\) be an object of \( \text{LBS}_{\text{irr}} \) and \( \mathcal{S} = \{G_i, i \leq n\} \) an irreducible linearly ordered nested set of \( \mathcal{G} \). For any \( G \in \mathcal{S} \) we define \( \tau_{\mathcal{S}}(G) := \bigvee_{S \leq G} S \) and we set

\[ (\mathcal{S}, \mathcal{G}_{\mathcal{S}}) := \bigotimes_{i} ([\tau_{\mathcal{S}}(G_i), G_i], \text{Ind}_{[\tau_{\mathcal{S}}(G_i), G_i]}(\mathcal{G})) \]

which is an object of \( \text{LBS} \). We will view \( \mathcal{S} \) as a new formal morphism \( (\mathcal{S}, \mathcal{G}_{\mathcal{S}}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G}) \) that we will add by hand to \( \text{LBS} \). However, to this end one must specify how those new morphisms compose with each other and with the isomorphisms as well. This is the object of the next definition/lemma.

From now on, every nested set is assumed to be irreducible unless stated otherwise. If \( \mathcal{S} \) is a nested set, the intervals \([\tau_{\mathcal{S}}(G), G] \) for \( G \) any element of \( \mathcal{S} \) will be called the “local
intervals of $S$.

Let $S = \{G_i, i \leq n\}$ be a nested set in $(\mathcal{L}, \mathcal{G})$ and let there be given additional linearly ordered nested sets $S_i$’s in each irreducible built lattice $([\tau_S(G_i), G_i], \text{Ind}_{[\tau_S(G_i), G_i]}(\mathcal{G}))$. We define

$$S \circ (S_i)_i := S \cup \bigcup_i \{\text{Comp}_{\tau_S(G_i)}(K), K \in S_i\} \quad (2)$$

which comes naturally equipped with a linear order (by concatenating the linear orders of the $S_i$’s). We have the key lemma.

**Lemma 3.6.** $S \circ (S_i)_i$ is a nested set of $(\mathcal{L}, \mathcal{G})$.

**Proof.** By Remark 2.18 and Lemma 2.16 we have that all the elements of the form $\text{Comp}(K)$ with $K$ in some $S_i$ are distinct and not in $S$, and they are partitioned according to the unique minimal element of $S$ above them.

Let $A = \{G_i | i \in I\} \sqcup \bigsqcup_{j \in J} A_j$ be an antichain in $S \circ (S_i)_i$, partitioned according to the previous remark, such that the join of $A$ belongs to $\mathcal{G}$. Let us prove that $A$ is a singleton. Since $A$ is an antichain, $I$ and $J$ are disjoint. Let $M$ be the set of maximal elements of $\{G_i, i \in I\} \cup \{G_j, j \in J\}$. Since $\bigvee A$ belongs to $\mathcal{G}$ and $S$ is a nested set, $M$ is a singleton. If this singleton belongs to $\{G_i, i \in I\}$, then by the fact that $A$ is an antichain we must have $A = M$. If this singleton belongs to $\{G_j, j \in J\}$, let us denote it $\{G_j\}$. By nestedness of $S_j$ we see that $A_j$ is a singleton, which we denote by $\{\text{Comp}(K)\}$ with $K$ some element in $S_j$. We have

$$\text{Comp}(K) \lor \tau_S(G_j) = \bigvee A \lor \tau_S(G_j) = K.$$

By maximality of $\text{Comp}(K)$ (see Definition 2.17), this means that $\bigvee A$ is equal to $\text{Comp}(K)$ and by the fact that $A$ is an antichain, $A$ must be equal to the singleton $\{\text{Comp}(K)\}$.

**Lemma 3.7.** We have an isomorphism of built lattices

$$\bigotimes_i (\mathcal{L}_{S_i}, \mathcal{G}_{S_i}) \xrightarrow{\Phi} (\mathcal{L}_{S \circ (S_i)_i}, \mathcal{G}_{S \circ (S_i)_i}).$$

**Proof.** We just need to show that the irreducible built lattices appearing on the left are canonically isomorphic to the irreducible built lattices appearing on the right. Let $K$ be some element in some $S_i$ and let $K_1, \ldots, K_p$ be the maximal elements of $(S_i)_<K$. We need to find an isomorphism of built lattice

$$([\tau_{S_i}(K), K], \text{Ind}_{[\tau_{S_i}(K), K]}(\text{Ind}_{[\tau_{S}(G_i), G_i]}(\mathcal{G}))) \xrightarrow{\Phi}$$

$$([\bigvee_i \text{Comp}(K_i) \lor \bigvee_{G_j < \text{Comp}(K)} G_j, \text{Comp}(K)], \text{Ind}(\mathcal{G})).$$

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Such an isomorphism is given by taking the join with $\tau_S(G_i)$. The fact that this is indeed an isomorphism of poset comes from the building set isomorphism

$$[\hat{0}, K] \simeq [\hat{0}, \text{Comp}(K)] \times \prod_{G_j \in \max S \leq G, G_j \notin \text{Comp}(K)} [\hat{0}, G_j].$$

The fact that it sends building set to building set comes from Lemma 2.12.

Finally, we have to prove that the operation $\circ$ on nested sets is "associative".

**Lemma 3.8.** Let $S$ be a nested set in $(\mathcal{L}, \mathcal{G})$. Let $(S_i)_i$ be nested sets in every local interval of $S$ and for every $i$ let $(S_j)_j$ be nested sets in every local interval of $S_i$. The nested sets $S \circ (S_i \circ (S_j)_j)_j$ and $(S \circ (S_i)_i) \circ (S_j)_j$ are equal (the last composition being performed via the isomorphism $\Phi$ of Lemma 3.7).

**Proof.** This is a statement about the Comp operation. When necessary we put the lattice in which we are doing the Comp operation in superscript. We denote $S = \{G_i, i \leq n\}$ and choose some $i_0 \leq n$. We then denote $S_{i_0} = \{K_j, j \leq m\}$ and choose some $j_0 \leq m$. We also define $I_0 := \{i \mid G_i \in \max S_{i \leq i_0}\}$ and $J_0 := \{j \mid K_j \in \max(S_{i_0} \leq K_{j_0})\}$.

We partition $I_0$ into the following subsets:

$$I_{0}^{\text{ext}} := \{i \in I_0 \mid G_i \notin \text{Comp}(K_{j_0})\},
\quad I_{0}^{\text{int}} := \{i \in I_0 \mid G_i \leq \text{Comp}(K_{j_0})\} \text{ and } \forall j \in J_0, G_j \notin \text{Comp}(K_j)\},
\quad I_{0}^{j} := \{i \in I_0 \mid G_i \leq \text{Comp}(K_j)\}.$$

It is a partition because the set $\{K_j, j \in J_0\}$ is nested. The map $\Phi$ is taking the join with $\bigvee_{i \in I_{0}^{\text{ext}}} G_i$.

The lemma amounts to showing that for any $L$ in $\text{Ind}_{\tau_{S_{i_0}}(K_{j_0})}(\mathcal{G})$ we have the equality

$$\text{Comp}_{\tau_S(G_{i_0})}(\text{Comp}_{\tau_{S_{i_0}}(K_{j_0})}(L)) = \text{Comp}_{\bigvee_{\{\text{Comp}(K_j), j \in J_0\} \lor \bigvee_{\{G_i, i \in I_{0}^{\text{int}}\}} (\Phi^{-1}(L))}.\]

We have

$$\text{Comp}_{\tau_S(G_{i_0})}(\text{Comp}_{\tau_{S_{i_0}}(K_{j_0})}(L)) \lor \bigvee_{j \in J_0} \text{Comp}(K_j) \lor \bigvee_{i \in I_{0}^{\text{int}}} G_i \lor \bigvee_{i \in I_{0}^{\text{ext}}} G_i = \text{Comp}_{\tau_{S_{i_0}}(K_{j_0})}(L) \lor \bigvee_{j \in J_0} \text{Comp}(K_j) \lor \bigvee_{i \in I_{0}^{\text{int}}} G_i \lor \bigvee_{i \in I_{0}^{\text{ext}}} G_i = L.$$
Applying $\Phi^{-1}$ to both sides we get

$$\text{Comp}_\mathcal{L}^{\mathcal{C}}(\text{Comp}_{\tau S(G_{i_0})}^{[\tau S(G_{i_0})),G_{i_0}] (L)) \lor \bigvee_{j \in J_0} \text{Comp}(K_j) \lor \bigvee_{i \in I_0^{\text{int}}} G_i = \Phi^{-1}(L).$$

We need to prove that $\text{Comp}_\mathcal{L}^{\mathcal{C}}(\text{Comp}_{\tau S(G_{i_0})}^{[\tau S(G_{i_0})),G_{i_0}] (L))$ is the biggest element in $\mathcal{G}$ which satisfies this equation.

Let $G' \in \mathcal{G}$ such that we have

$$G' \lor \bigvee_{j \in J_0} \text{Comp}(K_j) \lor \bigvee_{i \in I_0^{\text{int}}} G_i = \Phi^{-1}(L).$$

Applying $\Phi$ on both sides we get

$$G' \lor \bigvee_{j \in J_0} \text{Comp}(K_j) \lor \bigvee_{i \in I_0^{\text{int}}} G_i \lor \bigvee_{i \in I_0^{\text{ext}}} G_i = L. \quad (3)$$

The element $G' \lor \bigvee_{i \in I} G_i$ is below $L$ and belongs to $\text{Ind}_{[\tau S(G_{i_0})),G_{i_0}] (\mathcal{G})$ so it is below one of the factors of $L$ in $\text{Ind}_{[\tau S(G_{i_0})),G_{i_0}] (\mathcal{G})$. Those factors are $\text{Comp}_{\tau S(G_{i_0})}^{[\tau S(G_{i_0})),G_{i_0}] (L)$ or $K_j$ for some $j$ in $J$. By equation (3), $G' \lor \bigvee_{i \in I} G_i$ cannot be below any $K_j$ so we have

$$G' \lor \bigvee_{i \in I} G_i \leq \text{Comp}_{\tau S(G_{i_0})}^{[\tau S(G_{i_0})),G_{i_0}] (L),$$

which implies

$$G' \leq \text{Comp}_\mathcal{L}^{\mathcal{C}}(\text{Comp}_{\tau S(G_{i_0})}^{[\tau S(G_{i_0})),G_{i_0}] (L)).$$

We are now in position to make the following definition.

**Definition 3.9.** $\text{LBS}$ is the monoidal category defined as follow.

- The objects of $\text{LBS}$ are built lattices.
- The morphisms of $\text{LBS}$ are generated (via composition and tensoring) by
  1. Structural morphisms
     
     $$(\mathcal{L}_S, \mathcal{G}_S) \overset{\mathcal{S}}{\rightarrow} (\mathcal{L}, \mathcal{G})$$
     
     for every totally ordered nested set in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$. The composition of those morphisms is given by $\mathcal{F}.$
2. Isomorphisms between built lattices

\[(L, G) \xrightarrow{\sim} (L', G')\]

for each isomorphism of poset \(f : L' \to L\) such that \(f(G') = G\),

quotiented by relations

\[
(L_{f(S)}, G_{f(S)}) \xrightarrow{f(S)} (L', G') \xrightarrow{f} (L, G) \sim (L_{f(S)}, G_{f(S)}) \xrightarrow{\otimes f} (L_S, G_S) \xrightarrow{S} (L, G)
\]  

(4)

for any isomorphism of irreducible built lattice \(f : (L', G') \to (L, G)\), and for every permutation \(\sigma\) of \(S\):

\[
(L_S, G_S) \xrightarrow{\sigma} (L_{S_{\sigma}}, G_{S_{\sigma}}) \xrightarrow{S_{\sigma}} (L, G) \sim (L_S, G_S) \xrightarrow{S} (L, G)
\]  

(5)

where \(S_{\sigma}\) denotes the nested set equipped with the new linear order given by \(\sigma\).

Finally we also impose the relation

\[
\{\hat{1}\} \sim \text{Id}_{(L, G)}.
\]

• The monoidal structure is the same as the one on \(LBS\), which in addition acts on nested sets (which are now considered as morphisms) by disjoint union.

**Proposition 3.10.** The triple \(\mathcal{LBS} = (LBS_{\text{irr}}, LBS, \iota)\) with \(\iota\) the obvious inclusion is a Feynman category.

**Proof.** This is a consequence of Proposition 3.5 and of the construction itself. \(\Box\)

We conclude this subsection by proving a general lemma on the composition of nested sets which will be important later on.

**Lemma 3.11.** Let \(S\) be some irreducible nested set in some irreducible built lattice and \(S' \subset S\) a subset containing \(\hat{1}\). For any \(G'\) in \(S'\) we put \(S_{G'} := (S \lor \tau_{S'}(G')) \cap (\tau_{S'}(G'), G')\). For any \(G'\) in \(S', S_{G'}\) is a nested set in \([(\tau_{S'}(G'), G'), \text{Ind}(G)]\) and we have the equality between nested sets:

\[
S = S' \circ (S_{G'})_{G' \in S'}.
\]

(6)

**Proof.** Let \(G'\) be any element of \(S'\) and \(G'_1, ..., G'_{n}\) the maximal elements of \(S'_{G'}\). Let \(G\) be some element in \(S\) which is below \(G'\) and not below any of the \(G'_i\)'s. We denote \(K := \tau_{S'}(G') \lor G\). By the fact that \(S\) is nested, \(G\) is the maximal element of \(G\) satisfying the equality

\[
G \lor \tau_{S'}(G') = K.
\]

This proves the equality

\[
G = \text{Comp}_{\tau_{S'}(G')}(\tau_{S'}(G') \lor G),
\]
which implies equality (6).

For the nestedness, assume we have $G_1, ..., G_k$ some elements of $S$ such that $\tau_{S'}(G') \lor G_1, ..., \tau_{S'}(G') \lor G_k$ are elements of $(\tau_{S'}(G'), G')$ and such that $\bigvee_i \tau_{S'}(G') \lor G_i$ belongs to $\text{Ind}_{[\tau_{S'}(G'), 1]}(G_i)$. The factors of $\bigvee_i \tau_{S'}(G') \lor G_i$ in $G_i$ are some of the $(G'_i)'s$ and the element $\text{Comp}_{\tau_{S'}(G')}(\bigvee_i \tau_{S'}(G') \lor G_i)$. Since $\text{Comp}_{\tau_{S'}(G')}$ is increasing (it has a left inverse given by taking the join with $\tau_{S'}(G')$), by the first part of this proof we have

$$G_i = \text{Comp}_{\tau_{S'}(G')}(\tau_{S'}(G') \lor G_i) \leq \text{Comp}_{\tau_{S'}(G')}(\bigvee_i \tau_{S'}(G') \lor G_i)$$

for all $i \leq k$ and by the building set isomorphism we must have

$$\bigvee_i G_i = \text{Comp}_{\tau_{S'}(G')}(\bigvee_i \tau_{S'}(G') \lor G_i).$$

By nestedness of $S$ the $G_i$’s do not form an antichain and therefore the $\tau_{S'}(G') \lor G_i$’s do not either. This proves that $(\tau_{S'}(G') \lor S) \cap (\tau_{S'}(G'), 1]$ is a nested set.

### 3.3 Presentation of $\mathcal{LBS}$

In this section we give a presentation of the category $\mathcal{LBS}$, that is a set of structural morphisms that together with isomorphisms generate every other morphisms in $\mathcal{LBS}$, via composition and tensoring. In general, having a presentation of some category $\mathcal{F}$, part of a Feynman category $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \mathfrak{i})$, is useful from a practical point of view because it makes defining operads over $\mathfrak{F}$ a lot easier (one just needs to specify the operad on the generators and check that those morphisms satisfy the right relations).

**Example 3.12.** In the case of classical operads, the Feynman category $\mathcal{Op}$ has a natural set of generating structural morphisms given by rooted trees with two inner vertices. This translates into the fact that operads can be defined by their partial compositions (composition of two operations).

**Proposition 3.13.** Every morphism in $\mathcal{LBS}$ can be obtained as a composition and tensoring of morphisms of the form $\{G, 1\}$ and isomorphisms.

From now on since all our nested sets must contain $\hat{1}$ we will omit it (for instance the above generators will be written $\{G\}$).

**Proof.** Iterate Lemma 3.11 with $S'$ of the form $\{G\}$. □

**Proposition 3.14.** Relations between compositions of generators $\{G\}$ are all generated by the relations

$$\{G_1\} \circ (\{G_2\} \otimes \text{Id}) = \{G_2\} \circ (\text{Id} \otimes \{G_1\})$$

(7)
for every pair $G_1 < G_2$, relations
\[
\{G_1\} \circ (\{G_1 \lor G_2\} \otimes \text{Id}) = \{G_2\} \circ (\{G_1 \lor G_2\} \otimes \text{Id}) \circ \sigma^{2,3}
\] (8)

for every pair $G_1, G_2$ of non-comparable elements forming a nested set (with $\sigma^{2,3}$ the transposition swapping the two last summands) and relations
\[
f \circ f(\{G\}) = \{G\} \circ (f_{[G,i]} \otimes f_{[0,G]})
\] (9)

for every isomorphism $f : (\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')$ in LBS and every element $G \in \mathcal{G} \setminus \{\hat{1}\}$.

Proof. One can check that those relations are indeed satisfied in LBS. In order to prove that they generate all relations we start with the following lemma.

**Lemma 3.15.** Let $(\mathcal{L}, \mathcal{G})$ be a built lattice and $\prec$ a linear order on $\mathcal{L}$. Any morphism in LBS
\[
F : \bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})
\]
can be uniquely written as a composition
\[
\bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \xrightarrow{\otimes f_i} (\mathcal{L}', \mathcal{G}') \xrightarrow{\sigma} (\mathcal{L}'_{\sigma(i)}, \mathcal{G}'_{\sigma(i)}) \xrightarrow{S} (\mathcal{L}, \mathcal{G}),
\] (10)

where

- The $f_i$'s are isomorphisms in LBS_{irr}.
- $\sigma$ is a permutation of the summands.
- $S$ is an irreducible nested set of $(\mathcal{L}, \mathcal{G})$ with total order given by restriction of $\prec$.

Proof. By iteration of relations (4) and (5) one can see that every morphism can be written in the form (10). For the unicity we define an invariant $\iota(F) : \bigsqcup_i (\mathcal{L}_i \setminus \{\hat{0}\}) \to \mathcal{L} \setminus \{\hat{0}\}$ for every morphism $F : \bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$ in LBS by setting

- For any nested set in some built lattice $(\mathcal{L}, \mathcal{G})$:
\[
\iota(S) : \bigsqcup_G ([\tau_S(G), G] \setminus \{\tau_S(G)\}) \to \mathcal{L} \setminus \{\hat{0}\}
\]
is the obvious inclusion.
- For any isomorphism $(\mathcal{L}', \mathcal{G}') \xrightarrow{f} (\mathcal{L}, \mathcal{G})$ in LBS, $\iota(f)$ is equal to $f^{-1}$. 

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and then extending to all morphisms by composition. One can see that \( \iota \) preserves the relations (4) and (5) and therefore it passes to the quotient and gives a well-defined invariant for every morphism in \( LBS \).

Now if \( F : \bigotimes_i (L_i, G_i) \to (L, G) \) can be written as a composition
\[
\bigotimes_i (L_i, G_i) \xrightarrow{\otimes f_i} (L'_i, G'_i) \xrightarrow{\sigma} (L'_{\sigma(i)}, G'_{\sigma(i)}) \xrightarrow{S} (L, G),
\]
then we see that \( S \) and the \( f'_i \)s can be extracted from \( \iota(F) \) (\( S \) is just the image by \( \iota(F) \) of \( \bigsqcup_i \{1_{L_i}\} \) in \( L \setminus \{0\} \) and the \( f'_i \)s are just restrictions of \( \iota(F) \) to the suitable subsets) and \( \sigma \) is the only permutation that permutes the summands in the right order when \( S \) is given the order \( \triangleleft \). This proves the unicity.

Assume now that we have two sequences of morphisms in \( LBS \)
\[
\varphi = A \to X_1 \to ... \to X_n \to (L, G) \quad \text{and} \quad \psi = A \to X'_1 \to ... \to X'_n' \to (L, G),
\]
such that every morphism in \( \psi \) or \( \psi' \) is either a generator of the form \( \text{Id} \otimes ... \otimes \text{Id} \otimes \{G\} \otimes \text{Id} \otimes ... \otimes \text{Id} \) or an isomorphism, and \( (L, G) \) is some irreducible built lattice. We want to prove that if the composition of the morphisms of \( \varphi \) is equal to the composition of the morphisms of \( \psi \) then there is a chain of equivalences of the form (7), (8) or (9) (possibly tensored and composed with other common morphisms) between \( \varphi \) and \( \psi \).

First, by iteration of relation (9) one can assume that the only morphisms in \( \varphi \) and \( \psi \) which are not isomorphisms are the first morphisms of \( \varphi \) and \( \psi \) respectively. By Lemma 3.15 we can assume that those two isomorphisms are only permutations of summands and that the nested obtained by composition of the other morphisms of \( \varphi \) is the same as the one obtained by composition of the other morphisms of \( \psi \). We will denote this nested set by \( S \). We also denote by \( S_i \) (resp. \( S'_i \)) the nested set obtained by composing the last \( i \) morphisms of \( \varphi \) (resp. \( \psi \)). By construction of the composition of nested sets (see Section 3.2) we have
\[
S_1 \subset ... \subset S_n = S
\]
and
\[
S'_1 \subset ... \subset S'_{n'} = S,
\]
and the cardinal of the nested sets increases exactly by one at each step. Let us denote \( S_1 = \{G_\varphi\} \). By the equations above there exist some \( j \leq n' \) such that we have
\[
S'_j \setminus S'_{j-1} = \{G_\varphi\}.
\]

One can find a chain of relations of the form (7) and (8) between \( \psi \) and some \( \psi' \) such that the first morphism of \( \psi' \) is \( \{G_\varphi\} \) (applying relations (7) and (8) allows one to swap
successively the morphism corresponding to $G_\varphi$ in $\psi$ with the morphism after, until it reaches the end).

This means that we can assume that the last morphism of $\psi$ is $\{G_\varphi\}$. We denote by $S^{<G_\varphi}$ (resp. $S^{>G_\varphi}$) the nested set obtained by composing the morphisms of $\varphi$ which correspond to generators in $[\hat{0}, G]$ (resp. $[G, \hat{1}]$), and we denote similarly $S'^{<G_\psi}$, $S'^{>G_\psi}$ the same constructions but with $\psi$. We have

$$\{G_\varphi\} \circ (S^{<G_\varphi}, S^{>G_\varphi}) = S = \{G_\varphi\} \circ (S'^{<G_\psi}, S'^{>G_\psi})$$

which implies that we have

$$S^{<G_\varphi} = S'^{<G_\psi}$$
$$S^{>G_\varphi} = S'^{>G_\psi}$$

and we conclude by induction.

\[ \square \]

### 3.4 LBG is a graded Feynman category

An important feature of $\text{LBG}$ is that morphisms of $\text{LBS}$ can be graded in the following sense.

**Definition 3.16.** A degree function on a Feynman category $(V, F, \iota)$ is a map

$$\deg : \text{Mor}(F) \to \mathbb{N}$$

such that

- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- Morphisms of degree 0 and 1 generate $\text{Mor}(F)$ by compositions and tensor products.

Furthermore the degree function is said to be proper if the degree 0 morphisms are exactly the isomorphisms. A graded Feynman category is a Feynman category equipped with a degree function.

**Example 3.17.** The Feynman category $\mathcal{Op}$ encoding classical operads has a proper grading given by defining the degree of a tree $t$ as the number of inner vertices of $t$ minus one.

For $\text{LBG}$ we define a proper degree function by setting

$$\deg(f) = 0 \text{ for all isomorphisms } f,$$
$$\deg(\{G\}) = 1 \text{ for all } G \text{ different from } \hat{1}.$$

One can see that the relations introduced in Proposition 3.14 are homogeneous with respect to this grading and therefore we can extend this grading to every morphism in $\text{LBG}$, which makes $\text{LBG}$ a properly graded Feynman category.
4 (Co)operads over $\mathcal{LBS}$

In this section we show that the algebraic invariants introduced in Section 2 (Feichtner–Yuzvinsky algebras, Orlik–Solomon algebras) can be bundled up to form various (co)operads over $\mathcal{LBS}$ (see Definition 3.2).

4.1 The Feichtner–Yuzvinsky cooperad

In this subsection we define and study an $\mathcal{LBS}$-cooperadic structure on the family of Feichtner–Yuzvinsky rings.

4.1.1 Definition of the cooperad

Lemma 4.1. The map $(\mathcal{L}, \mathcal{G}) \to \text{FY}((\mathcal{L}, \mathcal{G})$ can be upgraded to a (strong) monoidal functor from $\mathcal{LBS}$ to $\text{grComRing}^{op}$ where $\text{grComRing}$ is the symmetric monoidal category of graded commutative rings.

Proof. Let $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{L}', \mathcal{G}')$ be two built lattices and let $(\mathcal{L}'', \mathcal{G}'') = (\mathcal{L}, \mathcal{G}) \otimes (\mathcal{L}', \mathcal{G}')$. We have an isomorphism of algebras

$$
\text{FY}((\mathcal{L}, \mathcal{G}) \otimes \text{FY}((\mathcal{L}', \mathcal{G}')) \xrightarrow{\sim} \text{FY}((\mathcal{L}'', \mathcal{G}''))
$$

with inverse

$$
\text{FY}((\mathcal{L}'', \mathcal{G}'')) \to \text{FY}((\mathcal{L}, \mathcal{G}) \otimes \text{FY}((\mathcal{L}', \mathcal{G}'))
$$

If $\phi$ is an isomorphism $(\mathcal{L}', \mathcal{G}') \sim (\mathcal{L}, \mathcal{G})$ in $\mathcal{LBS}$, it induces the isomorphism of algebras

$$
\text{FY}((\mathcal{L}, \mathcal{G}) \to \text{FY}((\mathcal{L}', \mathcal{G}'))
$$

which is compatible with composition.

Next we upgrade $\text{FY}$ into a monoidal functor from $\mathcal{LBS}$ to $\text{grComRing}^{op}$. Thanks to the presentation of $\mathcal{LBS}$ given in Subsection 3.3 we only need to specify the action of $\text{FY}$ on nested sets of cardinal one and then check that it satisfies the right relations. For every $G \in \mathcal{G} \setminus \{\hat{1}\}$ we set (using this time the wonderful presentation)

$$
\text{FY}((\{G\}) : \text{FY}((\mathcal{L}, \mathcal{G}) \to \text{FY}([G, \hat{1}], \text{Ind}(\mathcal{G})) \otimes \text{FY}([\hat{0}, G], \text{Ind}(\mathcal{G}))
$$

$$
h_G' \to \left\{ \begin{array}{ll}
1 \otimes h_G' & \text{if } G' \leq G \\
h_G' \otimes 1 & \text{otherwise.}
\end{array} \right.
$$
In the realizable case this morphism of algebra is induced by the inclusion of the stratum $\mathbf{Y}_{\{G, 1\}}$ in the wonderful compactification $\mathbf{Y}_{L, G}$.

Let us quickly justify why this map passes to the quotient. If $H$ is an atom of $L$ which is below $G$ then $h_H$ is sent to $h_H \otimes 1$ which is zero in the target algebra. On the contrary if $H$ is an atom of $L$ which is not below $G$ then this means by sub-modularity of $L$ that $H \lor G$ is an atom of $[G, \hat{1}]$ and thus $h_H$ is sent to zero again. Notice that this is the first time we have actually used the geometricity of our lattices. Now let $X = \{G_1, ..., G_n\}$ be some elements in $G$ having join $G' \in G$. Let us assume that the first $kG_i$'s are the elements of $X$ below $G$. If $G' \leq G$ then $\prod_i (h_{G_i} - h_{G_i})$ is sent to $1 \otimes \prod_i (h_{G_i} - h_{G_i})$, which is zero in the target algebra. Otherwise if $G' > G$ then $\prod_{i > k} (h_{G_i} - h_{G_i})$ is sent to $\prod_{i > k} (h_{G_i} \lor G - h_{G_i} \lor G) \otimes 1$, which is also zero in the target algebra.

**Proposition 4.2.** The maps $FY(\{G\})$ satisfy the relations given in Proposition 3.14.

*Proof.* Let $G_1 < G_2$ be two comparable elements in $G \setminus \{\hat{1}\}$. We have to check the equality of algebra morphisms

$$(FY(\{G_2\}) \otimes \text{Id}) \circ FY(\{G_1\}) = (\text{Id} \otimes FY(\{G_1\}) \circ FY(\{G_2\})),$$

and it is enough to check it on generators. Let $G$ be any element in $G$. If $G \leq G_1$ one can check that both morphisms send $h_G$ to $1 \otimes 1 \otimes h_G$. If $G \leq G_2$ and $G \not\leq G_1$ one can see that both morphisms send $h_G$ to $1 \otimes h_{G_1 \lor G} \otimes 1$. Lastly, if $G \not\leq G_2$ one can check that both morphisms send $h_G$ to $h_{G_2 \lor G} \otimes 1 \otimes 1$.

Let $G_1, G_2$ be two non-comparable elements of $G \setminus \{\hat{1}\}$ forming a nested set. We need to check the equality of algebra morphisms

$$(FY(\{G_2 \lor G_2\}) \otimes \text{Id}) \circ FY(\{G_1\}) = \sigma^{2, 3} \circ (FY(\{G_1 \lor G_2\}) \otimes \text{Id}) \circ FY(\{G_2\}).$$

It amounts again to a simple verification on generators with a small dichotomy. Let $G$ be any element in $G$. If $G \leq G_1$ this means that $G$ cannot be below $G_2$ and we see that both morphisms send $h_G$ to $1 \otimes 1 \otimes h_G$, if $G \leq G_2$ by a similar argument both morphisms send $h_G$ to $1 \otimes h_G \otimes 1$. Finally if $G$ is neither below $G_1$ nor below $G_2$ then both morphisms send $h_G$ to $h_{G_1 \lor G_2} \otimes 1 \otimes 1$. 

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We can give an explicit formula for general morphisms $FY(S)$. If $G$ is any element in $\mathcal{G}$ let $G'$ be the unique minimal element of $S_{>G}$. We then have

$$FY(S)(h_G) = 1^\otimes \otimes h_{\tau_S(G') \lor G} \otimes 1^\otimes,$$

where $1^\otimes$ means that we put a 1 in every interval which is not $[\tau_S(G_i), G_i]$. 

Finally, we need to check that the morphisms $FY(\{G\})$ satisfy relation (9). Let $(L, G)$ and $(L', G')$ be two built lattices and $f : (L', G') \sim (L, G)$ an isomorphism in $\mathcal{LBS}$, i.e. an isomorphism of poset $f : L \sim L'$ such that $f(G)$ is equal to $G'$. Let $G$ be an element in $\mathcal{G} \setminus \{1\}$. We have to check the equality between algebra morphisms

$$FY(\{f(G)\}) \circ FY(f) = (FY(f|_{[G, 1]} \otimes FY(f|_{[0, G]})) \circ FY(\{G\}).$$

Let $h_K$ be some generator in $FY(L, G)$. If $K \leq G$ one can check that both morphisms send $h_K$ to $1 \otimes h_{f(K)}$. Otherwise if $K \not\leq G$ one can check that both morphisms send $h_K$ to $h_{f(G) \lor f(K)} \otimes 1$. \hfill $\square$

In the sequel we will write $FY$ when referring to the $\mathcal{LBS}$-cooperad and not just the algebras. We also write $FY^\vee$ for the (linear) dual operad (apply the duality functor to all objects and morphisms). This is an $\mathcal{LBS}$-operad in the category of graded coalgebras.

4.1.2 A quadratic presentation for $FY^\vee$

In this section we exhibit a quadratic presentation for the operad $FY^\vee$. Let us first quickly recall what this means in the context of operads over general Feynman categories. This is all part of the theory developed by R. Kaufmann and B. Ward in [14].

Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, i)$ be a Feynman category and $M$ an $\mathfrak{F}$-module in some monoidal category $\mathcal{C}$ (see Definition 3.2). If $\mathcal{C}$ is cocomplete there exists a “free” $\mathfrak{F}$-operad in $\mathcal{C}$ generated by $M$ denoted by $\mathfrak{F}(M)$. The $\mathfrak{F}$-operad $\mathfrak{F}(M)$ satisfies the universal property that for any morphism of $\mathfrak{F}$-module between $M$ and some $\mathfrak{F}$-operad $P$ (viewed as an $\mathfrak{F}$-module), there exists a unique morphism of $\mathfrak{F}$-operad between $\mathfrak{F}(M)$ and $P$ which extends the first morphism. Concretely, $\mathfrak{F}(M)$ is given by the left Kan extension of $M : \text{Sym}(\mathcal{V}) \to \mathcal{C}$ along $\text{Sym}(i)$. The left Kan extension universal property is exactly the freeness universal property.

If furthermore $\mathfrak{F}$ is assumed to be a graded Feynman category and $\mathcal{C}$ is a category of modules for instance, then free operads are naturally graded (i.e. components in all arity are graded and structural morphisms preserve this grading).

In addition, if we are given an $\mathfrak{F}$-operad $P$ and $M$ a sub $\mathfrak{F}$-module of $P$ then one can define the ideal $\langle M \rangle$ generated by $M$ in $P$ by considering all possible elements in $P$ which
can be obtained as the composition (along some structural morphism in $\mathfrak{F}$) of an element of $M$ with elements of $P$. Finally, we can define the quotient of an operad by an ideal in an obvious way (just take the quotient in each arity and notice that the morphisms pass to the quotient).

With all those notions at hand we can define a quadratic operad over a graded Feynman category to be the quotient of a free operad by an ideal generated by degree 1 elements.

Since we have proved that $\mathfrak{LBS}$ is a graded Feynman category in Section 3.4 this vocabulary applies to $\mathfrak{LBS}$-operads.

Let $\text{Gen}$ be the $\mathfrak{LBS}$-module in the category of graded $\mathbb{Z}$-modules defined by

$$\text{Gen}(\mathcal{L}, \mathcal{G}) = \mathbb{Z}[2(\text{rk}(\mathcal{L}) - 1)]$$

and

$$\text{Gen}(f) = \text{Id}$$

for every irreducible built lattice $(\mathcal{L}, \mathcal{G})$ and all isomorphism of built lattice $f$.

If $(\mathcal{L}, \mathcal{G})$ is an irreducible built lattice we denote by $\Psi_{(\mathcal{L}, \mathcal{G})}$ the canonical generator of $\text{Gen}(\mathcal{L}, \mathcal{G})$. We also denote for all irreducible nested sets $\mathcal{S}$ of $(\mathcal{L}, \mathcal{G})$:

$$\Psi_{\mathcal{S}} := \mathfrak{LBS}(\text{Gen})(\mathcal{S})((\Psi_{([r_s(\mathcal{G}), \text{Ind}(\mathcal{G})])})_{G \in \mathcal{S}}),$$

which is an element of $\mathfrak{LBS}(\text{Gen})(\mathcal{L}, \mathcal{G})$. We are now able to state the main result of this section.

**Proposition 4.3.** The $\mathfrak{LBS}$-operad $\mathbb{FY}^\vee$ is isomorphic to the quotient of $\mathfrak{LBS}(\text{Gen})$ by the ideal generated by the elements

$$\sum_{1 > G \geq H_1} \Psi_{\{G\}} - \sum_{1 > G \geq H_2} \Psi_{\{G\}}, \quad (11)$$

for all atoms $H_1, H_2$ in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$.

**Proof.** We have a map of $\mathfrak{LBS}$-modules $\text{Gen} \xrightarrow{\pi} \mathbb{FY}^\vee$ sending $\Psi_{(\mathcal{L}, \mathcal{G})}$ to the linear form which is zero in all degrees except degree $2(\text{rk}(\mathcal{L}) - 1)$ where it takes value 1 on $x_1^{\text{rk}(\mathcal{L}) - 1}$ (which linearly generates $\mathbb{FY}^{2(\text{rk}(\mathcal{L}) - 1)}(\mathcal{L}, \mathcal{G})$).

This map is a natural transformation since for any isomorphism $f : (\mathcal{L}, \mathcal{G}) \to (\mathcal{L}', \mathcal{G}')$ between built lattices, $f$ preserves the top element and therefore $\mathbb{FY}(f)$ preserves $h_1^{\text{rk}(\mathcal{L}) - 1} =$
\[ x_1^{\text{rk}(L)-1} \] which implies that \( FY(f)^{\vee} \) sends \( \pi(\Psi_{(L,G)}) \) to \( \pi(\Psi_{(L',G')}) \).

This map extends to an \( \mathcal{LBG} \)-operadic map \( \mathcal{LBG}(\text{Gen}) \to FY^{\vee} \) (by universal property of free operads). Our goal is to prove that this map passes to the quotient by the elements (11) and that the induced morphism is an isomorphism. The proof splits into three steps.

**Step 1:** The map \( \hat{\pi} \) is surjective.

Going back to explicit formulas for general left Kan extensions in cocomplete categories yields

\[
\mathcal{LBG}(\text{Gen})(L,G) = \bigoplus_{\otimes_i(L_i,G_i) \to (L,G)} \times \text{Gen}((L_i,G_i))/\sim ,
\]

where the sum is over all possible maps \( \otimes_i(L_i,G_i) \to (L,G) \) in \( \mathcal{LBG} \) and the equivalence relation \( \sim \) identifies components corresponding to equivalent maps (two maps that can be obtained from one another by precomposing with isomorphisms).

More explicitly if we have two maps \( \otimes_i(L_i,G_i) \xrightarrow{\psi} (L,G) \) and \( \otimes_i(L'_i,G'_i) \xrightarrow{\phi} (L,G) \) such that there exist isomorphisms \( f_i : (L_i,G_i) \xrightarrow{\sim} (L'_i,G'_i) \) satisfying

\[
\psi = \phi \circ (\otimes_i f_i),
\]

then we have

\[
\otimes_i \alpha_i \sim \otimes_i \text{Gen}(f_i)(\alpha_i)
\]

for every element \( \otimes \alpha_i \) in \( \otimes_i \text{Gen}((L_i,G_i)) \). Likewise, if we have an equality of the form

\[
\psi = \phi \circ \sigma
\]

with \( \sigma \) some permutation of the \( (L_i,G_i) \)'s then we have

\[
\otimes_i \alpha_i \sim \otimes_i \alpha_{\sigma(i)}.
\]

Finally, replacing \( \text{Gen}(L,G) \) by its definition we get (for every irreducible built lattice \( (L,G) \)):

\[
\mathcal{LBG}(\text{Gen})((L,G)) = \mathbb{Z}(\{\otimes_i(L_i,G_i) \to (L,G)\}/\sim)
\]

(generators are equivalence classes of maps in \( \mathcal{LBG} \) having target \( (L,G) \), with the equivalence relation being the precomposition with isomorphisms). With this identification the map \( \hat{\pi} \) is given by

\[
\mathcal{LBG}(\text{Gen})(L,G) \to FY^{\vee}(L,G)
\]

\[
[\mu : \otimes_i(L_i,G_i) \to (L,G)] \mapsto (\alpha \mapsto \otimes_i \pi(\Psi_{(L_i,G_i)})(FY(\mu)(\alpha))).
\]
Let us fix an irreducible built lattice \((L, G)\) and some linear order extending the order on \(L\). Amongst all maps of the form \(\otimes_i (L_i, G_i) \to (L, G)\) we have the maps given by linearly ordered nested sets \((L_S, G_S) \to (L, G)\) whose linear order is given by our chosen global linear order on \(L\). It is enough to prove the surjectivity of \(\hat{\pi}\) restricted to equivalence classes of those morphisms. Passing to the dual we must prove the injectivity of the map

\[
FY(L, G) \to \mathbb{Z}[\{\text{irreducible nested sets of } (L, G)\}] \to (\pi(S)(FY(S)(\alpha)))_S
\]

where \(\pi(S)\) denotes the tensor product of maps \(\otimes_{G \in S} \pi(\Psi_{[rS(G), G, \text{Ind}(G)]})\).

Let \(\alpha\) be an element of \(FY(L, G)\) which is sent to zero by the above map, meaning that for every nested set \(S\) in \((L, G)\) we have \(\pi(S)(FY(S)(\alpha)) = 0\). We can assume that \(\alpha\) is homogeneous and by Corollary 2.22 we can write it uniquely as a sum of normal monomials

\[
\alpha = \sum_{S \text{ nested } \mu \text{ admissible}} \lambda_{S, \mu} h^\mu_S
\]

where \(h^\mu_S\) denotes the monomial \(\prod_{G \in S} h^\mu_G\). We have to prove that all the \(\lambda_{S, \mu}\)'s are zero.

Arguing by contradiction, let \(G_0\) some element of \(G\) which is minimal amongst elements belonging to some nested set \(S\) such that there exist some \(S\)-admissible index \(\mu\) satisfying \(\lambda_{S, \mu} \neq 0\). We denote by \(S_0\) and \(\mu_0\) the corresponding nested set and index for \(G_0\).

For any irreducible built lattice \((L, G)\) and any \(k < rkL\), one can construct a nested set \(S(L, G, k)\) having only local intervals with rank 1 except the top interval having rank \(k\), by picking any maximal chain \(\hat{0} < X_1 < \ldots < X_n < \hat{1}\) in \(L\) and putting

\[
S(L, G, k) = \{X_1\} \circ \{X_2\} \circ \ldots \circ \{X_{n-k+1}\}.
\]

The formula makes sense because each \(X_i\) is an atom in \([X_{i-1}, \hat{1}]\) and must therefore belong to \(\text{Ind}_{[X_{i-1}, \hat{1}]}(G)\). Notice that we have

\[
\hat{\pi}(\Psi_{S(L, G, k)})(h^k_1) = 1.
\]

For any nested set \(S'\) in \(([G_0, \hat{1}], \text{Ind}(G))\) and any nested set \(S\) in \(G\) with admissible index \(\mu\) we have

\[
\hat{\pi}(\Psi_{\{G\} \circ (S', S([\hat{0}, G], \text{Ind}(G), \mu_0(G_0)+1)})(h^\mu_S) = 0
\]
if $S$ does not contain some element with strictly positive index and which is below $G_0$. By minimality of $G_0$ this means that we have

$$\hat{\pi}(\Psi_G \circ (S', S([0, G], \text{Ind}(G), \mu_0(G_0) + 1))(\alpha) = \sum_{S \text{ nested}} \lambda_{S, \mu} \hat{\pi}(\Psi_G \circ (S'[\mu_0(G_0) + 1]))(h^\mu_S)$$

$$= \sum_{S \text{ nested}} \lambda_{S, \mu} \hat{\pi}(\Psi_S)(h^\mu_{G_0 \lor S} / h^\mu_{G_0})$$

$$= 0.$$ 

One can check that the monomials $h^\mu_{G_0 \lor S} / h^\mu_{G_0}$ are normal monomials in the irreducible built lattice $([G, 1], \text{Ind}(G))$. By induction we get that $\lambda_{S_0, \mu_0}$ is equal to zero which is a contradiction.

**Remark 4.4.** Let us remark that for any irreducible nested set $S$ the linear form

$$\hat{\pi}(\Psi_S) \circ \text{FY}(S) : \text{FY}(L, G) \to \mathbb{Z}$$

is zero on $\text{FY}^k(L, G)$ for $k$ different from $2(\text{rk}(L) - \#S)$, and for $k = 2(\text{rk}(L) - \#S)$ one can check that it is equal to the multiplication by $x^{S \setminus \{1\}}$, via the identification

$$\text{FY}(L, G)^{2(\text{rk}(L)-1)} \simeq \mathbb{Z}$$

(given by choosing the generator $x^{\text{rk}(L)-1}$). The injectivity of the map (12) can be rewritten as the condition that for all element $\alpha \in \text{FY}^{2k}(L, G)$, if $\alpha x_S = 0$ for all nested set $S \subset G \setminus \{1\}$ of cardinal $\text{rk}(L) - 1 - k$, then $\alpha = 0$. In other words the Feichtner–Yuzvinsky algebras satisfy Poincare duality. In other words, Poincare duality of Feichtner–Yuzvinsky algebras can be explained at the operadic level.

**Step 2:** The map $\hat{\pi}$ sends the elements (11) to zero.

We must check that the linear forms

$$\phi_H = \hat{\pi} \left( \sum_{1 > G \geq H} \text{MB}(\text{Gen})(\{G\}) \left( \Psi_{([G, 1], \text{Ind}(G))} \cdot \Psi_{([0, G], \text{Ind}(G))} \right) \right)$$

indexed by atoms $H$ are all equal and it is enough to check it on normal monomials. Those linear forms are zero in degree other than $2(\text{rk}(L) - 2)$. Fortunately, normal monomials of degree $2(\text{rk}(L) - 2)$ are rather simple.
Lemma 4.5. For any irreducible built lattice \((\mathcal{L}, \mathcal{G})\), the only degree \(2(\text{rk}(\mathcal{L}) - 2)\) normal monomials are the monomials \(h_G^{\text{rk}(G) - 1} h_1^{\text{rk}(1) - \text{rk}(G) - 1}\) with \(G\) any element of \(\mathcal{G}\) (when \(G\) is an atom we get the monomial \(h_{\text{rk}(\mathcal{L}) - 2}\)).

Proof. The proof hinges on the following result.

Lemma 4.6. Let \(S\) be an irreducible nested set in some irreducible built lattice \((\mathcal{L}, \mathcal{G})\). We have the equality

\[
\sum_{G \in S} \text{rk}([\tau_S(G), G]) = \text{rk}(\mathcal{L}).
\]

Proof. The proof goes by induction on the rank of \(\mathcal{L}\). Let \(G_0\) be a minimal element in \(S, G_0 \lor \{-G_0\}\) a nested set in \((\mathcal{L}, \text{Ind}(\mathcal{G}))\) so by induction hypothesis the sum of the rank of its intervals is the rank of \([G_0, 1]\) but by nestedness of \(S\) taking the join with \(G_0\) establishes a bijection between intervals of \(\bigvee S\) which are not the interval \([\hat{1}, G_0]\) and intervals of \(G_0 \lor \{-G_0\}\), and this bijection preserves the rank of the intervals. Consequently, by induction we have

\[
\sum_{G \in S} \text{rk}([\tau_S(G), G]) = \text{rk}([\hat{0}, G_0]) + \sum_{G \in S \setminus \{G_0\}} \text{rk}([\tau_S(G), G])
\]

\[
= \text{rk}([\hat{0}, G_0]) + \text{rk}([G_0, \hat{1}])
\]

\[
= \text{rk}(\mathcal{L}).
\]

Now if \(S\) is any irreducible nested set then the degree of any normal monomial of the form \(h_{S, \mu(S)}^{\text{rk}(S) - 1} h_1^{\text{rk}(1) - \text{rk}(S) - 1}\) for some \(S\)-admissible index is at most \(\sum_{G \in S} \text{rk}([\tau_S(G), G]) - 1\) which is equal by the previous lemma to \(\text{rk}(\mathcal{L}) - |S|\). This means that to have degree \(2(\text{rk}(\mathcal{L}) - 2)\) the cardinality of \(S\) must be at most two (counting \(\hat{1}\)) which proves the result. For normal monomials with underlying nested set not containing \(\hat{1}\) we just add \(\hat{1}\) to the nested set and follow the same line of argument.

Let \(h_G^{\text{rk}(G) - 1} h_1^{\text{rk}(1) - \text{rk}(G) - 1}\) be any degree \(2(\text{rk}(\mathcal{L}) - 2)\) normal monomial and \(H\) some atom. Going back to the definition of \(\phi_H\) we get

\[
\phi_H(h_G^{\text{rk}(G) - 1} h_1^{\text{rk}(1) - \text{rk}(G) - 1}) = \\
\sum_{G' \geq H} (\pi([G', \hat{1}]) \otimes \pi([0, G'])) \left( \mathcal{P}(\{G\})(h_G^{\text{rk}(G) - 1} h_1^{\text{rk}(1) - \text{rk}(G) - 1}) \right).
\]

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If $H \leq G$ then the only term which is not zero in this sum is the term $G' = G$ which gives
\[
\phi_H(h_{G}^{\mathrm{rk}(G)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) = (\pi([G,\hat{1}]) \otimes \pi([\hat{1},G])) \left( \mathbb{F} \mathcal{Y}(\{G\}) (h_{G}^{\mathrm{rk}(G)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) \right)
\]
\[
= (\pi([G,\hat{1}]) \otimes \pi([\hat{1},G])) \left( h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1} \otimes h_{G}^{\mathrm{rk}(G)-1} \right)
\]
\[
= 1.
\]

If $H \nleq G$ the only term which is not zero in this sum is the term $G' = H$ which gives
\[
\phi_H(h_{G}^{\mathrm{rk}(G)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) = (\pi([H,\hat{1}]) \otimes \pi([\hat{1},H])) \left( \mathbb{F} \mathcal{Y}(\{H\}) (h_{G}^{\mathrm{rk}(G)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) \right)
\]
\[
= (\pi([H,\hat{1}]) \otimes \pi([\hat{1},H])) \left( h_{G \lor H}^{\mathrm{rk}(G \lor H)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G \lor H)-1} \otimes 1 \right)
\]
\[
= \pi([H,\hat{1}]) \left( h_{G \lor H}^{\mathrm{rk}(G \lor H)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G \lor H)-1} \right). \tag{13}
\]

The monomial in the last equation is not a normal monomial in the irreducible built lattice $([H, \hat{1}], \mathrm{Ind}(\mathcal{G}))$ and therefore we must rewrite it. By geometricity of $\mathcal{L}$ there exist atoms $H_1, ..., H_{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}$ such that $\hat{1}$ is equal to the join of $G \lor H$ with those atoms. This means that we have the relation
\[
(h_1 - h_{G \lor H})(h_1 - h_{H_1})...(h_1 - h_{H_{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}}) = 0
\]
in the algebra $\mathbb{F} \mathcal{Y}([H, \hat{1}], \mathrm{Ind}(\mathcal{G}))$, and replacing the $h_{H_i}$'s by zero we get
\[
h_1^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)} = h_1^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}h_{G \lor H}
\]
which implies
\[
h_1^{\mathrm{rk}(G)-1}h_1^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1} = h_1^{\mathrm{rk}(\mathcal{L})-2}
\]
This equality together with equation (13) leads directly to
\[
\phi_H(h_{G}^{\mathrm{rk}(G)-1}h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) = 1
\]
as in the case $H \leq G$, which proves that $\phi_H$ does not depend on $H$.

**Step 3:** The kernel of $\hat{\pi}$ is generated by the relations (11).

We postpone the proof of this last step to Subsection 5.5, where it will be an application of our theory of Gröbner bases for $\mathbb{L}\mathbb{B}\mathbb{G}$-operads. \hfill $\blacksquare$
4.1.3 The Feichtner–Yuzvinsky operad

One can define another operad out of the Feichtner–Yuzvinsky algebras as follow. Let \((\mathcal{L}, \mathcal{G})\) be any irreducible built lattice. We put

\[
\text{FY}^{PD}(\mathcal{L}, \mathcal{G}) := \text{FY}(\mathcal{L}, \mathcal{G}).
\]

For any isomorphism of built lattice \(f : (\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')\) we define

\[
\text{FY}^{PD}(f) : \text{FY}(\mathcal{L}, \mathcal{G}) \to \text{FY}(\mathcal{L}', \mathcal{G}'),
\]

and for any \(G \in \mathcal{G} \setminus \{\hat{1}\}\) we put

\[
\text{FY}^{PD}(\{G\}) : \text{FY}^{PD}([G, \hat{1}], \text{Ind}(\mathcal{G})) \otimes \text{FY}^{PD}([\hat{0}, G], \text{Ind}(\mathcal{G})) \to \text{FY}^{PD}(\mathcal{L}, \mathcal{G})
\]

\[
\prod_{i} x_{G_{i}}^{\alpha_{i}} \prod_{j} x_{G_{j}}^{\alpha_{j}} \to x_{G} \prod_{i} x_{\text{Comp}_{G}(G_{i})}^{\alpha_{i}} \prod_{j} x_{G_{j}}^{\alpha_{j}}
\]

if all the \(G_{j}\)'s are different from \(G\).

Let us show that this morphism is well-defined. For any antichain \(\{G_{i}\}\) below \(G\) and such that \(\bigvee G_{i}\) belongs to \(\mathcal{G}\) we have

\[
\text{FY}^{PD}(1 \otimes \prod_{i} x_{G_{i}}) = x_{G} \prod_{i} x_{G_{i}} = 0.
\]

For any antichain \(\{G \vee G_{i}\}\) in \(\text{Ind}_{[G, \hat{1}]}(\mathcal{G})\) having join \(G'\) in \(\text{Ind}_{[G, \hat{1}]}(\mathcal{G})\) we have

\[
\text{FY}^{PD}(\prod_{i} x_{G \vee G_{i}} \otimes 1) = x_{G} \prod_{i} x_{\text{Comp}_{G}(G \vee G_{i})} = 0
\]

because either \(G'\) belongs to \(\mathcal{G}\) and the elements \(\{G, \text{Comp}_{G}(G \vee G_{i})\}\) have join \(G'\) in \(\mathcal{G}\),

either \(G'\) does not belong to \(\mathcal{G}\) and the elements \(\text{Comp}_{G}(G \vee G_{i})\) have join \(\text{Comp}_{G}(G')\).

For any atom \(H\) below \(G\) we have

\[
\text{FY}^{PD}(1 \otimes \sum_{G' \geq H \atop G' < G} x_{G'}) = \sum_{G' \geq H \atop G' < G} x_{G} x_{G'}
\]

\[
= x_{G}(h_{H} - \sum_{G' \geq G} x_{G'} - \sum_{G', G \text{ incomparables}} x_{G'})
\]

\[
= x_{G}( - \sum_{G' \geq G} x_{G'} - \sum_{G', G \text{ incomparables}} x_{G'}),
\]

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which does not depend on \( H \).

Finally, if \( G \lor H \) is an atom in \([G, \hat{1}]\), either \( G \lor H \) belongs to \( \mathcal{G} \) in which case we have
\[
\mathcal{F}_Y^{PD}([G]) (h_H \otimes \hat{1}) = x_{G \lor H} = 0,
\]
either \( G \lor H \) does not belong to \( \mathcal{G} \) in which case we have
\[
\mathcal{F}_Y^{PD}([G]) (h_H \otimes \hat{1}) = x_{G H} = 0.
\]

Our next order of business is to prove that those morphisms satisfy the relations in Proposition [3.14]. Let \((L, \mathcal{G})\) be some irreducible built lattice. If \( G_1 \) and \( G_2 \) are two non comparable elements of \( \mathcal{G} \setminus \{\hat{1}\} \) forming a nested set, one can check that both
\[
\mathcal{F}_Y^{PD}([G_1]) \circ (\mathcal{F}_Y^{PD}([G_1 \lor G_2]) \otimes Id)
\]
and
\[
\mathcal{F}_Y^{PD}([G_2]) \circ (\mathcal{F}_Y^{PD}([G_1 \lor G_2]) \otimes Id) \circ \sigma_{2,3}
\]
send \( \prod_i x_{G_i} \otimes \prod_j x_{G'_j} \otimes \prod_k x_{G''_k} \) to \( x_{G_1} x_{G_2} \prod_i x_{\text{Comp}_{G_1 \lor G_2}(G_i)} \prod_j x_{G'_j} \prod_k x_{G''_k} \).

If \( G_1 < G_2 < \hat{1} \) one can check that both
\[
\mathcal{F}_Y^{PD}([G_1]) \circ (\mathcal{F}_Y^{PD}([G_2]) \otimes Id)
\]
and
\[
\mathcal{F}_Y^{PD}([G_2]) \circ (Id \otimes \mathcal{F}_Y^{PD}([G_1]))
\]
send \( \prod_i x_{G_i} \otimes \prod_j x_{G'_j} \otimes \prod_k x_{G''_k} \) to \( x_{G_1} x_{G_2} \prod_i x_{\text{Comp}_{G_1 \lor G_2}(G_i)} \prod_j x_{\text{Comp}_{G_1 \lor G'_2}(G'_j)} \prod_k x_{G''_k} \).

Finally, if \( f : (L, \mathcal{G}) \sim (L', \mathcal{G}') \) is an isomorphism of irreducible built lattice and \( G' \) is some element in \( \mathcal{G}' \), one can check that both
\[
\mathcal{F}_Y^{PD}(f) \circ \mathcal{F}_Y^{PD}([f(G')])
\]
and
\[
\mathcal{F}_Y^{PD}([G']) \circ (\mathcal{F}_Y^{PD}(f|[G', \hat{1}]) \otimes \mathcal{F}_Y^{PD}(f|[0, G'])))
\]
send \( \prod_i x_{G_i} \otimes \prod_j x_{G'_j} \) to \( x_{G'} \prod_i x_{f^{-1}(\text{Comp}_f(G_i))} \prod_j x_{f^{-1}(G'_j)} \).

The operads \( \mathcal{F}_Y \) and \( \mathcal{F}_Y^{PD} \) are strongly related via Poincaré duality. If we denote by \( PD \) the isomorphism \( \mathcal{F}_Y(L, \mathcal{G}) \sim \mathcal{F}_Y(L, \mathcal{G})^\vee \) given by Poincare duality we have the equality between morphisms
\[
\mathcal{F}_Y^{PD}([G]) = PD^{-1} \circ \mathcal{F}_Y^\vee([G]) \circ (PD \otimes PD).
\]
4.2 The affine Orlik–Solomon cooperad

In this section we introduce an $\mathcal{LS}$-cooperadic structure on the Orlik–Solomon algebras, which extends the linear dual of the Gerstenhaber operad (defined on partition lattices with minimal building set).

4.2.1 Definition

Let $(\mathcal{L}, \mathcal{G})$ be an irreducible built lattice. For all $G \in \mathcal{G} \setminus \{\hat{1}\}$ we define a morphism of algebras $\text{OS}(\{G\}) : \text{OS}(\mathcal{L}) \to \text{OS}([G, \hat{1}]) \otimes \text{OS}([\hat{0}, G])$ by

$$\text{OS}(\{G\})(e_H) = \begin{cases} 1 \otimes e_H & \text{if } H \leq G \\ e_{G \vee H} \otimes 1 & \text{otherwise}, \end{cases}$$

for every generator $e_H$.

Lemma 4.7. The morphism $\text{OS}(\{G\})$ is well defined.

Proof. We have to check that $\text{OS}(\{G\})$ vanishes on elements of the form $\delta$ circuit. Let $C = \{H_i\} \cup \{H'_j\}$ be a circuit with the $H_i$’s below $G$ and the $H'_j$’s not below $G$. We have

$$\text{OS}(\{G\})(\delta(\prod C)) = \text{OS}(\{G\})(\delta(\prod e_{H_i} \land \prod e_{H'_j})) = \text{OS}(\{G\})(\delta(\prod e_{H_i} \land \prod e_{H'_j} \pm \prod e_{H_j} \land \delta(\prod e_{H'_j})) = \delta(\prod e_{H_i}) \otimes \prod e_{G \vee H'_j} \pm \prod e_{H_j} \otimes \delta(\prod e_{G \vee H'_j}).$$

(14)

If the $H_i$’s form a set of dependent atoms of $\mathcal{L}$ then we have $\delta(\prod e_{H_i}) = \prod e_{H_i} = 0$ (these identities holding in both $\text{OS}(\mathcal{L})$ and $\text{OS}([\hat{0}, G])$). If on the contrary the $H'_j$’s form a set of independent atoms, by the fact that $C$ is dependent there exists an atom $H'_j_0$ in $C$ which is below $\bigvee H_i \lor \bigvee_{j<j_0} H'_j$. By taking the join with $G$ in this relation we obtain $G \lor H'_j_0 \leq G \lor \bigvee_{j<j_0} H'_j$ which implies that the $G \lor H'_j$’s form a set of dependent atoms of $[G, \hat{1}]$ which shows that we have $\delta(\prod e_{G \lor H'_j}) = \prod e_{G \lor H'_j} = 0$ in the algebra $\text{OS}([G, \hat{1}])$. □

Lemma 4.8. $\text{OS}$ extends to an $\mathcal{LS}$-cooperad of graded commutative algebras.

Proof. On objects we set $\text{OS}(\mathcal{L}, \mathcal{G}) = \text{OS}(\mathcal{L})$ for every built lattice $(\mathcal{L}, \mathcal{G})$. For structural morphisms we use the morphisms $\text{OS}(\{G\})$ introduced above on each generator of $\mathcal{LS}$ (one-element nested sets). On isomorphisms the action is

$$\text{OS}(\mathcal{L}, \mathcal{G}) \to \text{OS}(\mathcal{L}', \mathcal{G}')$$

$$e_H \to e_{f(H)}$$

for any isomorphism $f : (\mathcal{L}', \mathcal{G}') \sim (\mathcal{L}, \mathcal{G})$ in $\mathcal{LS}$ (an isomorphism of posets must preserve the atoms).
We have to check that those morphisms satisfy the relations given in Proposition 3.14. Let \( G_1 \leq G_2 < \hat{1} \) be two comparable elements in the building set \( \mathcal{G} \) of a lattice \( \mathcal{L} \). Let us prove the equality of morphisms

\[
(Id \otimes \mathcal{OS} (\{G_1\})) \circ \mathcal{OS} (\{G_2\}) = (\mathcal{OS} (\{G_2\}) \otimes Id) \circ \mathcal{OS} (\{G_1\}).
\]

Since we are dealing with morphisms of algebras it is enough to prove the equality on generators which amounts to a simple verification. For any atom \( H \) in \( \mathcal{L} \), if \( H \leq G_1 \) then both morphisms send \( e_H \) to \( 1 \otimes 1 \otimes e_H \). If \( H \leq G_2 \) and \( H \not\leq G_1 \) then both morphisms send \( e_H \) to \( 1 \otimes e_{G_1 \lor H} \otimes 1 \). Lastly, if \( H \not\leq G_2 \) then both morphisms send \( e_H \) to \( e_{G_2 \lor H} \otimes 1 \otimes 1 \).

Let \( G_1 \) and \( G_2 \) be two non-comparable elements of \( \mathcal{G} \setminus \{\hat{1}\} \) forming a nested set. We have to show the equality of morphisms

\[
(\mathcal{OS} (\{G_1 \lor G_2\}) \otimes Id) \circ \mathcal{OS} (\{G_2\}) = \sigma^{2,3} \circ (\mathcal{OS} (\{G_1 \lor G_2\}) \otimes Id) \circ \mathcal{OS} (\{G_1\}).
\]

For any atom \( H \) in \( \mathcal{L} \), if \( H \leq G_1 \) then by nestedness \( H \not\leq G_2 \) and consequently both morphisms send \( e_H \) to \( 1 \otimes 1 \otimes e_H \). If \( H \leq G_2 \) then by the same argument both morphisms send \( e_H \) to \( 1 \otimes 1 \otimes e_H \). Finally if \( H \not\leq G_1 \) and \( H \not\leq G_2 \) then both morphisms send \( e_H \) to \( e_{G_1 \lor G_2 \lor H} \otimes 1 \otimes 1 \) which finishes the proof.

Lastly, we need to prove the equality

\[
\mathcal{OS} (\{f(G)\}) \circ \mathcal{OS}(f) = (\mathcal{OS} (f_{|[G,G]})) \otimes \mathcal{OS} (f_{|[\hat{0},G]})) \circ \mathcal{OS} (\{G\})
\]

for every isomorphism \( f : (\mathcal{L}', \mathcal{G}') \rightarrow (\mathcal{L}, \mathcal{G}) \) in \( \mathcal{LBS} \) and \( G \in \mathcal{G} \setminus \{\hat{1}\} \). For any atom \( H \) of \( \mathcal{L} \), if \( H \leq G \) then both morphisms send \( e_H \) to \( 1 \otimes e_{f(H)} \). If on the contrary \( H \not\leq G \) then both morphisms send \( e_H \) to \( e_{f(G) \lor f(H)} \otimes 1 \).

Finally, one can check that \( \mathcal{OS} \) is strong monoidal, which finishes the proof.

5 Gröbner bases for operads over \( \mathcal{LBS} \)

In this section we develop a theory of Gröbner bases for operads over \( \mathcal{LBS} \).

Classical Gröbner bases \( \mathcal{B} \) are a computational tool which is used to work out quotients of free associative algebras. The general idea is to start by introducing an order on generators of the free algebra. This order is then used to derive an order on all monomials, which is compatible in some sense with the multiplication of monomials (we call such orders “admissible”).
We then use this order to rewrite monomials in the quotient algebra:

\[
\text{greatest term} \rightarrow \sum \text{lower terms},
\]

for every relation \( R = \text{greatest term} - \sum \text{lower terms} \) in some subset \( B \) of the quotient ideal (usually the greatest term is called the “leading term” and we will use this denomination).

The subset \( B \) is called a Gröbner basis when it contains “enough” elements. To be precise we want that every leading term of some relation in the quotient ideal is divisible by the leading term of some element of \( B \).

The goal is to find a Gröbner basis as little as possible so that the rewriting is as easy as possible. At the end of the rewriting process (which stops if the monomials are well-ordered) we are left with all the monomials which are not rewritable i.e. which are not divisible by a leading term of some element of \( B \). Those monomials are called “normal” and they form a linear basis of our algebra exactly when \( B \) is a Gröbner basis. This basis comes with multiplication tables given by the rewriting process.

It turns out that this general strategy can be applied to structures which are much more general and complex than associative algebras, such as operads for instance. Loosely speaking, all we need in order to implement this strategy is to be able to make (reasonable) sense of the key words used above, such as “monomials”, “admissible orders” and “divisibility between monomials”.

For operads over a Feynman category, the only non-trivial part is to construct admissible orders on monomials out of orders on generators. The main issue comes from the symmetries, because usually the compatibility with symmetries is too strong and prevents us from finding any admissible order. In order to circumvent this problem, drawing inspiration from the case of classical operads which was sorted out by Dotsenko and Khoroshkin in [7], we introduce a notion of a “shuffle” operad over \( \mathcal{LBS} \).

5.1 Shuffle operads over \( \mathcal{LBS} \)

As it turns out we could directly define a shuffle \( \mathcal{LBS} \)-operad to be an \( \mathcal{LBS} \)-operad without symmetries but this definition would not be completely satisfactory, in particular when trying to find an admissible order on nested sets. Essentially, we would like to have a bit more data than just built lattices to construct those orders in a functorial way (for instance in the case of shuffle operads the new important data is the linear ordering of the entries, see [7]).

In view of what has just been said we make the following definition.
**Definition 5.1.** A directed built lattice is a triple \((L, G, <)\) where \((L, G)\) is a built lattice and \(<\) is a linear order on atoms of \(L\). A directed built lattice \((L, G, <)\) is said to be irreducible if \((L, G)\) is irreducible.

We are going to do the same construction all over again but with directed built lattices instead of built lattices.

**Definition 5.2.** Let \((L, G, <)\) be a directed built lattice and \([G_1, G_2]\) be an interval of \(L\). The interval \([G_1, G_2]\) admits an induced directed built lattice structure given by the building set \(\text{Ind}_{[G_1, G_2]}(\mathcal{G})\) and the linear order \(\triangleleft_{\text{ind}}\) defined by

\[
K_1 \triangleleft_{\text{ind}} K_2 \iff \min \{H \mid G_1 \lor H = K_1, H \text{ atom}\} < \min \{H \mid G_1 \lor H = K_2, H \text{ atom}\} \tag{15}
\]

for any pair of elements \(K_1, K_2\) covering \(G_1\) (i.e. atoms of \([G_1, G_2]\)).

Both minima in (15) are well defined by geometricity of \(L\). As in the case of built lattices, doing a double induction is the same as doing a single induction directly on the smallest interval (Lemma 2.12).

**Definition 5.3.** The monoidal product \(\otimes\) on directed built lattices is defined by

\[
(L_1, G_1, <_1) \otimes (L_2, G_2, <_2) = (L_1 \times L_2, G_1 \times \{0\} \sqcup \{0\} \times G_2, <)
\]

where \(<\) is defined by putting all the atoms of \(L_1\) before the atoms of \(L_2\). As a quick reminder, for any nested set \(S\) and \(G\) an element of \(S\) we have defined in Subsection 3.2 the notation \(\tau_S(G) := \bigvee_{S < G}\). If \(S\) is an (ordered) nested set in a directed built lattice \((L, G, <)\) we denote

\[
(L_S, G_S, <_S) := \bigotimes_{G \in S} ([\tau_S(G), G], \text{Ind}(G), \triangleleft_{\text{ind}}).
\]

We are now able to make the following definition.

**Definition 5.4.** The Feynman category \(\mathcal{LBS}_{III}\) is the triple \((\mathcal{V}, \mathcal{F}, i)\) where

- The objects of the groupoid \(\mathcal{V}\) are the directed irreducible lattices and its morphisms are defined by

\[
\text{Mor}_{\mathcal{V}}((L, G, <), (L', G', <')) = \{f : L' \to L \mid f \text{ poset isomorphism, increasing with respect to } < \text{ and } <' \text{ and such that } f(G') = G\}.
\]

- The objects of the category \(\mathcal{F}\) are monoidal products of directed built lattices and its morphisms are given by structural morphisms \((L_S, G_S, <_S) \overset{S}{\to} (L, G, <)\) together

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with the tensored isomorphisms of $V$ and the permutations of monoidal summands, quotiented by the relations

$$S \circ \sigma \sim \sigma \cdot S$$

for every ordered nested set $S$ in some directed irreducible nested set and $\sigma$ some permutation of the summands of $(L_S, G_S, \prec_S)$ acting on $S$ by changing the order of the elements.

- The functor $i$ is the obvious inclusion.

We define the composition of nested sets in this context exactly as it was defined in Section 3.2. In subsequent sections a shuffle $LBS$-operad will mean an operad over the Feynman category $LBS\llbracket X \rrbracket$.

### 5.2 Monomials and divisibility

In order to be able to talk about Gröbner bases one needs a suitable notion of “monomials” in a free $LBS\llbracket X \rrbracket$-operad as well as a notion of divisibility between those monomials. Let $M$ be a module over $LBS\llbracket X \rrbracket$ in the category of vector spaces over some fixed arbitrary field $\mathbb{K}$. We have defined in Section 4.1.2 the free $LBS\llbracket X \rrbracket$-operad $LBS\llbracket (M) \rrbracket$ generated by the module $M$. By the same analysis conducted in the latter section we have the explicit formula for each directed irreducible built lattice $(L, G, \prec)$:

$$LBS\llbracket (M) \rrbracket(L, G, \prec) = \bigoplus_{S \subset G \text{ nested set}} \bigotimes_{G \in S} M([\tau_S(G), G], \text{Ind}(G), \prec_{\text{Ind}}). \quad (16)$$

If furthermore we are given a basis $B(L, G, \prec)$ of every vector space $M(L, G, \prec)$ then we can make the following definition.

**Definition 5.5 (Monomial).** A monomial in $LBS\llbracket (M) \rrbracket(L, G, \prec)$ is an element which is a tensor of elements of the basis $\bigcup_{(L', G', \prec')} B(L', G', \prec')$.

In other words a monomial corresponds to the datum $(S, (e_G)_{G \in S})$ of a nested set $S$ and elements of the basis $e_G \in B([\tau_S(G), G], \text{Ind}(G), \prec_{\text{Ind}})$ for each $G$ in $S$. Monomials are stable under composition in $LBS\llbracket (M) \rrbracket$ and by (16) they form a basis of every vector space $LBS\llbracket (M) \rrbracket(L, G, \prec)$. Additionally, we have a notion of divisibility between monomials.

**Definition 5.6 (Division between monomials).** Let $m_1$ and $m_2$ be two monomials in $LBS\llbracket (M) \rrbracket(L, G, \prec)$. We say that $m_1$ divides $m_2$ if $m_2$ can be expressed as a composition:

$$m_2 = LBS\llbracket (M) \rrbracket(S)((\alpha_G)_{G \in S})$$

for some nested set $S$, where one of the $\alpha_G$’s is $m_1$ and the rest are elements of the basis $\bigcup_{(L, G, \prec)} B(L, G, \prec)$.
5.3 An admissible order on monomials

Let \( M \) be an \( \mathfrak{B} \mathfrak{G}_{\mathfrak{III}} \)-module, with a basis \( B(\mathcal{L}, \mathcal{G}, \prec) \) of \( M(\mathcal{L}, \mathcal{G}, \prec) \) for each \( (\mathcal{L}, \mathcal{G}, \prec) \). Assume that we have a total order \( \prec \) of those bases in each arity. In this section we will construct an order on monomials induced by \( \prec \), which is compatible with the composition of monomials in the sense of Proposition 5.12. We start by defining a total order \( \prec^* \) on \( \mathcal{L} \), induced by the direction \( \prec \). For any element \( G \) in \( \mathcal{L} \) we denote by \( w(G) \) the word in the alphabet \( \text{At}(\mathcal{L}) \) (the set of atoms of \( \mathcal{L} \)) given by the list of atoms below \( G \) in increasing order.

**Definition 5.7.** Given two elements \( G_1, G_2 \) in \( \mathcal{L} \), we say \( G_1 \prec^* G_2 \) if \( w(G_2) \) is an initial subword of \( w(G_1) \), or if \( w(G_1) \) is less than \( w(G_2) \) for the lexicographic order.

There are a few important lemmas/remarks to be made about this order. First we prove that \( \prec^* \) is compatible with the already existing order on \( \mathcal{L} \).

**Lemma 5.8.** The total order \( \prec^* \) extends the reversed lattice order on \( \mathcal{L} \).

*Proof.* Let \( G_1 \prec G_2 \) be two comparable elements in \( \mathcal{G} \), such that \( w(G_1) \) is not an initial subword of \( w(G_2) \). One can write

\[
\begin{align*}
w(G_1) &= uH_1w_1 \\
w(G_2) &= uH_2w_2
\end{align*}
\]

with \( u, w_1, w_2 \) some words in the alphabet \( \text{At}(\mathcal{L}) \) and \( H_1, H_2 \) two different atoms. Since we have \( \text{At}_{\prec}(G_1) \subset \text{At}_{\prec}(G_2) \) we immediately get \( H_2 \prec H_1 \) which shows that \( w(G_2) \) is smaller than \( w(G_1) \) for the lexicographic order.

Next we prove that \( \prec^* \) behaves well with respect to restriction to intervals.

**Lemma 5.9.** Assume we are given an interval \( [\mathcal{G}_1, \mathcal{G}_2] \) in some irreducible directed built lattice \( (\mathcal{L}, \mathcal{G}, \prec) \). The two total orders \( \prec_{\text{ind}}^* \) and \( \prec_{[\mathcal{G}_1, \mathcal{G}_2]} \) on \( [\mathcal{G}_1, \mathcal{G}_2] \) are the same.

*Proof.* Since we are comparing two total orders we only need to prove one implication, for instance \( K \prec_{[\mathcal{G}_1, \mathcal{G}_2]} K' \Rightarrow K \prec_{\text{ind}}^* K' \). If \( w(K') \) is included in \( w(K) \) then this means that we have \( K' \leq K \) which proves that we have \( K \prec_{\text{ind}}^* K' \) by the previous lemma. If not then one can write \( w(K) = uHw \) and \( w(K') = uH'w' \) with \( u, w, w' \) some words and \( H \prec H' \) two atoms of \( \mathcal{L} \). The atom \( H \) cannot be below \( G_1 \) otherwise \( H \) would also be a letter in \( w(K') \). If \( H' \) is below \( G_1 \) then let \( H'' \) be the first letter bigger than \( H' \) in \( w(K') \) which is not below \( G_1 \) and such that \( G_1 \lor H'' \) does not belong to \( G_1 \lor u \) (such a letter exists because we have assumed that \( w(K') \) is not included in \( w(K) \)). In this case one can write

\[
\begin{align*}
w_{[\mathcal{G}_1, \mathcal{G}_2]}(K) &= v(G \lor H)t \\
w_{[\mathcal{G}_1, \mathcal{G}_2]}(K') &= v(G \lor H'')t'
\end{align*}
\]

with \( v, t, t' \) some words in \( \text{At}([\mathcal{G}_1, \mathcal{G}_2]) \). This implies that we have \( K \prec_{\text{ind}}^* K' \).

\[ \square \]
Finally, we prove that $\,<^*$ is compatible with the join in $\mathcal{L}$.

**Lemma 5.10.** Let $G, G_1$ and $G_2$ be three elements in $\mathcal{L}$ such that $\text{Fact}_G(G_1)$ and $\text{Fact}_G(G_2)$ are both disjoint from $\text{Fact}_G(G)$ and such that $\text{Fact}_G(G_1) \cup \text{Fact}_G(G)$ and $\text{Fact}_G(G_2) \cup \text{Fact}_G(G)$ are both nested antichains. Then we have the equivalence

$$G_1 \,<^* G_2 \iff G \lor G_1 \,<^* G \lor G_2.$$  

**Proof.** Let us start by proving the direct implication. If $G_1 > G_2$ then we have $G \lor G_1 > G \lor G_2$ (the strictness coming from the nestedness condition) which proves the result by Lemma 5.8. Otherwise write $w(G_1) = uH_1w_1$ and $w(G_2) = uH_2w_2$ where $u, w_1$ and $w_2$ are some words and $H_1$ is strictly smaller than $H_2$. By the nestedness condition in the proposition we have

$$w(G \lor G_1) = \text{sh}(w(G), w(G_1)),$$

$$w(G \lor G_2) = \text{sh}(w(G), w(G_2)),$$

where $\text{sh}(\cdot, \cdot)$ is the operation which merges two given words with increasing letters into a word with increasing letters. From this we see that one can write

$$w(G \lor G_1) = u'H_1w_1',$$

$$w(G \lor G_2) = u'H_2w_2',$$

with $w', w_1', w_2'$ some words and $H_2'$ an atom of $\mathcal{L}$. If $H_2'$ is in $w(G)$ then $H_1$ is strictly smaller than $H_2'$ (otherwise $H_2'$ would belong to $w'$). If on the contrary $H_2'$ belongs to $w(G_2)$ then $H_2'$ is equal to $H_2$ and $H_1$ is strictly smaller than $H_2'$.

For the converse, assume we have $G \lor G_1 \,<^* G \lor G_2$. If $G \lor G_2 < G \lor G_1$ then $G_2 < G_1$ by nestedness which implies $G_1 \,<^* G_2$ by Lemma 5.8. Otherwise, write $w(G \lor G_1) = uH_1w_1$ and $w(G \lor G_2) = uH_2w_2$ for $u, w_1, w_2$ some words and $H_1$ strictly smaller than $H_2$. One can check that $H_1$ necessarily belongs to $w(G_1)$ which means that we can write $w(G_1) = u'H_1w_1'$ and $w(G_2) = u'H_2w_2'$ where $H_2'$ is the first letter of $w(G \lor G_2)$ which belongs to $w(G_2)$ and which comes after $H_2$. We immediately get $H_1 < H_2 \leq H_2'$ which finishes the proof.

For any monomial $m = (S, (e_G)_G)$ and $G_0$ some element in $\min_S S$, we denote

$$G_0 \lor m := (G_0 \lor (S \setminus \{G_0\}), (e_{\text{Comp}_{G_0}}(G))_G),$$

which is a well-defined monomial in $\mathcal{LEB}_{\text{III}}(M)((G_0, \hat{1}), \text{Ind}(G), <_{\text{ind}})$, by Lemma 5.11. For any nested set $S$ we denote $\text{MM}(S) := \min_S \min_S S$. 

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Proposition 5.12. The order on monomials $\triangleleft^*$ on monomials in the following inductive way. For $m_1 = (S_1, (e_G^1)_{G \in S_1})$ and $m_2 = (S_2, (e_G^2)_{G \in S_2})$ we put $m_1 \triangleleft^* m_2$ if there exists some $G$ in $\min_{\prec} S_1 \cap \min_{\prec} S_2$ such that $e_G^1 = e_G^2$ and $G \triangleright m_1 \triangleleft^* G \triangleright m_2$, or if there is no such $G$ and $\MM(S_1) \triangleleft^* \MM(S_2)$ or $\MM(S_1) = \MM(S_2)$ and $e_{\MM(S_1)}^1 \triangleright e_{\MM(S_2)}^2$.

One can check that this definition does not depend on the choice of the element of $\triangleleft^*$ because if we have two different elements $G$ and $G'$ in $\min_{\prec} S_1 \cap \min_{\prec} S_2$ such that $e_G^1 = e_G^2$ and $e_{G'}^1 = e_{G'}^2$ then we have the equality of monomials

\[(G \triangleright G') \triangleright (G \triangleright m_i) = (G \triangleright G') \triangleright (G' \triangleright m_i)\]

for $i = 1, 2$. If there is no ambiguity on the order $\triangleright$ we write $\triangleleft^*$ instead of $\triangleleft^*_i$.

Definition 5.11. We define a total order $\triangleleft_*$ on monomials in the following inductive way. For $m_1 = (S_1, (e_G^1)_{G \in S_1})$ and $m_2 = (S_2, (e_G^2)_{G \in S_2})$ we put $m_1 \triangleleft_* m_2$ if there exists some $G$ in $\min_{\prec} S_1 \cap \min_{\prec} S_2$ such that $e_G^1 = e_G^2$ and $G \triangleright m_1 \triangleleft_* G \triangleright m_2$, or if there is no such $G$ and $\MM(S_1) \triangleleft_* \MM(S_2)$ or $\MM(S_1) = \MM(S_2)$ and $e_{\MM(S_1)}^1 \triangleright e_{\MM(S_2)}^2$.

Definition 5.12. The order on monomials $\triangleleft^*$ is compatible with the composition of monomials. More precisely, if $S$ is any nested set in some directed irreducible built lattice $(\mathcal{L}, G, \prec)$ and we have some generators $e_G \in \mathbb{B}(\mathcal{L}, G, \prec)$ for all $G$ in $S$ except for $G_0$ in $S$ where we have monomials $m_1 = (S_1, (e_G^1)_{G \in S_1})$, $m_2 = (S_2, (e_G^2)_{G \in S_2}) \in \mathcal{L}_S \mathcal{B}_{\mathcal{S}}([\tau_S(G_0), G_0], \Ind(G), \triangleleft_\Ind)$, with $\#S_1 = \#S_2$, then we have

\[m_1 \triangleleft_* m_2 \Rightarrow \mathcal{L}_S \mathcal{B}_{\mathcal{S}}(M)(S)((e_G)_{G \in S_1}, m_1) \triangleleft_* \mathcal{L}_S \mathcal{B}_{\mathcal{S}}(M)(S)((e_G)_{G \in S_2}, m_2)\]

Proof. The proof goes by induction on $\#S + \#S_1$. The initialization at $\#S = 0$ or $\#S_1 = 0$ is obvious. The induction step is a consequence of Lemma 5.10. 

Since every element in a free $\mathcal{L}_S \mathcal{B}_{\mathcal{S}}$-operad can be uniquely written as a sum of monomials, we can make the following definition.

Definition 5.13 (Leading term). If $f$ is an element in some free $\mathcal{L}_S \mathcal{B}_{\mathcal{S}}$-operad with total order on generators $\triangleright$ then the leading term of $f$, denoted by $\text{Lt}(f)$, is the biggest monomial with respect to $\triangleleft_*$ which has a non-zero coefficient in $f$.

At last, everything has been leading to the following definition.

Definition 5.14 (Gröbner basis). Let $\mathcal{I}$ be an operadic ideal in some free $\mathcal{L}_S \mathcal{B}_{\mathcal{S}}$-operad $\mathcal{L}_S \mathcal{B}_{\mathcal{S}}(M)$, where $M$ is some $\mathcal{L}_S \mathcal{B}_{\mathcal{S}}$-module in some category of vector spaces which is endowed with a basis and a well-order $\triangleright$ of this basis in each arity. A Gröbner basis of $\mathcal{I}$ relative to $\triangleright$ is a subset $\mathcal{B}$ of $\mathcal{I}$ such that every leading term relative to $\triangleleft_*$ of some element of $\mathcal{I}$ is divisible by the leading term of some element of $\mathcal{B}$.

A Gröbner basis is said to be quadratic if it contains only degree 1 elements.

Definition 5.15 (Normal monomial). A normal monomial with respect to some set of elements $\mathcal{B}$ is a monomial which is not divisible by the leading term of some element in $\mathcal{B}$. 

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**Proposition 5.16.** The set of normal monomials with respect to some set of elements $B$ in an ideal $I \subset \mathcal{LBS}_\text{III}(M)$ linearly generates $\mathcal{LBS}_\text{III}(M)/I$ in every arity. This set of monomials is linearly independant if and only if $B$ is a Gröbner basis of $I$.

**Proof.** The proof is the same as in every other context where we have a notion of Gröbner basis. Let us just point out that since the basis is well ordered one can see that the monomials are well ordered as well. \qed

### 5.4 Relating $\mathcal{LBS}$-operads and shuffle $\mathcal{LBS}$-operads

There is an obvious functor between Feynman categories:

$$\text{III} : \mathcal{LBS}_\text{III} \rightarrow \mathcal{LBS}$$

$$(\mathcal{L}, \mathcal{G}, <) \rightarrow (\mathcal{L}, \mathcal{G})$$

$S \rightarrow S.$

This allows us to define a forgetful functor from $\mathcal{LBS}$-operads/modules to shuffle $\mathcal{LBS}$-operads/modules by precomposition.

**Definition 5.17.** For any $\mathcal{LBS}$-operad (resp. module) $\mathcal{P}$ we define the shuffle $\mathcal{LBS}$-operad (resp. module) $\mathcal{P}\text{III}$ by

$$\mathcal{P}\text{III} := \mathcal{P} \circ \text{III}.$$

As in the classical operadic case this functor enjoys very nice properties which are listed and proved below.

**Proposition 5.18.**

1. For any $\mathcal{LBS}$-module $M$ we have an isomorphism of shuffle operads:

$$\mathcal{LBS}_\text{III}(M\text{III}) \simeq \mathcal{LBS}(M)\text{III}. \quad (17)$$

2. Let $R$ be a sub-$\mathcal{LBS}$-module in some free $\mathcal{LBS}$-operad $\mathcal{LBS}(M)$ with $M$ some $\mathcal{LBS}$-module. The $\mathcal{LBS}$-module $R\text{III}$ can be identified with a sub-$\mathcal{LBS}_\text{III}$-module of the free $\mathcal{LBS}_\text{III}$-operad $\mathcal{LBS}_\text{III}(M\text{III})$. The $\mathcal{LBS}_\text{III}$-module $\langle R \rangle\text{III}$ can be identified with an ideal of $\mathcal{LBS}(\text{Gen})\text{III}$. The isomorphism $(17)$ sends (via identifications) the (shuffle) ideal $\langle R \rangle\text{III}$ to the shuffle ideal $\langle R \rangle$. As a consequence we have the isomorphism of $\mathcal{LBS}_\text{III}$-operads

$$\mathcal{LBS}_\text{III}(M\text{III})/\langle R \rangle\text{III} \simeq \mathcal{LBS}(M)\text{III}/\langle R \rangle.$$

3. Lastly, we have an isomorphism of shuffle operads

$$\mathcal{LBS}(M)\text{III}/\langle R \rangle\text{III} \simeq (\mathcal{LBS}(M)/\langle R \rangle)\text{III}.$$
Proof. 1. By the definition of left Kan extensions we have a morphism of $\mathcal{LBS}$-modules $M \to \mathcal{LBS}(M)$.
Applying the forgetful functor we get a morphism of $\mathcal{LBS}_{\mathcal{III}}$-modules $M_{\mathcal{III}} \to \mathcal{LBS}(M)_{\mathcal{III}}$.
By universal property of free $\mathcal{LBS}_{\mathcal{III}}$-operads (which comes from the universal property of left Kan extensions) we get a morphism of $\mathcal{LBS}_{\mathcal{III}}$-operads $\mathcal{LBS}_{\mathcal{III}}(M_{\mathcal{III}}) \to \mathcal{LBS}(M)_{\mathcal{III}}$.
Let us take a closer look at this morphism.

By unpacking the construction of free operads (left Kan extensions) we get the following formula for every irreducible directed built lattice $(\mathcal{L}, \mathcal{G}, <)$:
\[
\mathcal{LBS}_{\mathcal{III}}(M_{\mathcal{III}})(\mathcal{L}, \mathcal{G}, <) \simeq \bigoplus_{S \in \mathcal{G}} M_S.
\]
We also have
\[
\mathcal{LBS}(M)_{\mathcal{III}}(\mathcal{L}, \mathcal{G}, <) = \mathcal{LBS}(M)(\mathcal{L}, \mathcal{G}) = \left( \bigoplus_{\phi_i : (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})} \bigotimes_i M(\mathcal{L}_i, \mathcal{G}_i) \right) / \sim
\]
where $\sim$ identifies components corresponding to equivalent maps via isomorphisms, as explained at the beginning of the proof of Theorem 4.3. The above morphism sends the component $M(S)$ to the equivalence class of the component $M(S)$.

However by Lemma 3.15 the nested sets in $S$ with linear order $<^*$ form a system of representatives for the equivalence classes of morphisms which means that the above morphism is indeed a linear isomorphism in each arity.

2. For the first identification if we start with the injective morphism $R \hookrightarrow \mathcal{LBS}(M)$, then apply the forgetful functor (which preserves injective morphisms since it is the right adjoint to left Kan extension), and then compose with isomorphism (17) we get an injective morphism $R_{\mathcal{III}} \hookrightarrow \mathcal{LBS}_{\mathcal{III}}(M_{\mathcal{III}})$. For the second identification we have an injective morphism $\langle R \rangle \hookrightarrow \mathcal{LBS}(M)$ and applying the forgetful functor gives us an injective morphism $\langle R \rangle_{\mathcal{III}} \hookrightarrow \mathcal{LBS}(M)_{\mathcal{III}}$. By unraveling again the explicit description of isomorphism (17) we see that it sends the shuffle ideal $\langle R \rangle_{\mathcal{III}}$ to the shuffle ideal $\langle R \rangle_{\mathcal{III}}$.

3. We have an obvious identification of components in each arity between the two shuffle operads and one can check that this identification is operadic. 

\[\square\]
5.5 Application: the example of $\mathbb{P}Y^V$

We have proved in Subsection 4.1.2 that if $\text{Gen}$ is the $\mathcal{LBS}$-module with one generator $\Psi_{(L,G)}$ of degree $2(rk(L) - 1)$ in each arity $(L, G)$, and $I$ is the ideal generated by the elements

$$
\sum_{G \geq H_1} \mathcal{LBS}(\text{Gen})(\{G\})(\Psi_{([G,1],\text{Ind}(G))}, \Psi_{([0,G],\text{Ind}(G))}) - \\
\sum_{G \geq H_2} \mathcal{LBS}(\text{Gen})(\{G\})(\Psi_{([G,1],\text{Ind}(G))}, \Psi_{([0,G],\text{Ind}(G))})
$$

(18)

for each pair of atoms $H_1$ and $H_2$, then we have a surjective morphism of $\mathcal{LBS}$-operads:

$$
\mathcal{LBS}(\text{Gen})/I \xrightarrow{\pi} \mathbb{P}Y^V.
$$

Let us denote by $R$ the linear span of the elements (18), which is a sub $\mathcal{LBS}$-module of $\mathcal{LBS}(\text{Gen})$. By Proposition 5.18 we have an isomorphism of shuffle $\mathcal{LBS}$-operads

$$
\mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})/\langle R_{\text{III}} \rangle \xrightarrow{\sim} \mathcal{LBS}(\text{Gen})_{\text{III}}/\langle R \rangle_{\text{III}}.
$$

Let us use our theory of Gröbner bases for shuffle $\mathcal{LBS}$-operads to study the operad $\mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})/\langle R_{\text{III}} \rangle$. Notice that $R_{\text{III}}$ is just the linear span of elements of the form

$$
\sum_{G \geq H_1} \mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})(\{G\})(\Psi_{([G,1],\text{Ind}(G))}, \Psi_{([0,G],\text{Ind}(G))}) - \\
\sum_{G \geq H_2} \mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})(\{G\})(\Psi_{([G,1],\text{Ind}(G))}, \Psi_{([0,G],\text{Ind}(G))}).
$$

We denote by $B$ the set of those elements, and we put

$$
\Psi_S := \mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})(S)(\Psi_{([\tau_S(G),G],\text{Ind}(G))})_{G \in S}.
$$

By applying the same arguments as in the proof of Theorem 4.3 we get a surjective morphism of $\mathcal{LBS}_{\text{III}}$-operads.

$$
\mathcal{LBS}_{\text{III}}(\text{Gen}_{\text{III}})/\langle R_{\text{III}} \rangle \twoheadrightarrow \mathbb{P}Y^V_{\text{III}}.
$$

(19)

We will compute the normal monomials associated to $B$ and find that they have the desired cardinality, which will prove that $B$ forms a Gröbner basis of $\langle R_{\text{III}} \rangle$, and that morphism (19) is an isomorphism.

To describe those monomials in a natural way we introduce a classical tool in poset combinatorics called “EL-labeling”.

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**Definition 5.19 (EL-labeling).** Let $P$ be a finite poset with set of covering relations $\text{Cov}(P)$. An EL-labeling of $P$ is a map $\lambda : \text{Cov}(P) \to \mathbb{N}$ such that for any two comparable elements $X < Y$ in $P$ there exists a unique maximal chain going from $X$ to $Y$ which has increasing $\lambda$ labels (when reading the covering relations from bottom to top) and this unique maximal chain is minimal for the lexicographic order on maximal chains (comparing the words given by the successive $\lambda$ labels from bottom to top).

We refer the reader to [19] for more details on this notion. The main result we will use about EL-labelings is the following.

**Proposition 5.20.** Let $\mathcal{L}$ be a geometric lattice. Any linear ordering $H_1 \preceq \cdots \preceq H_n$ of the atoms of $\mathcal{L}$ induces an EL-labeling $\lambda_\mathcal{L}$ of $\mathcal{L}$ defined by

$$\lambda_\mathcal{L}(X \prec Y) = \min\{i \mid X \lor H_i = Y\}$$

for any covering relation $X \prec Y$ in $\mathcal{L}$.

**Proof.** The proof of this result can be found in [19].

If $\lambda$ is an EL-labelling of some poset $P$ and $X < Y$ are two comparable elements we denote by $\omega_{X,Y,\lambda}$ the unique maximal chain from $X$ to $Y$ which has increasing $\lambda$ labels. If the EL-labelling can be deduced from the context we will drop it from the notation. We also define $\omega_{X,Y,\lambda}^k$ to be the the chain $\omega_{X,Y,\lambda}$ truncated at height $k$ for any positive integer $k$ which is less than the length of $\omega_{X,Y,\lambda}$. More precisely if $\omega_{X,Y,\lambda} = \{X_0 = X \prec X_1 \prec \cdots \prec X_n = Y\}$ then $\omega_{X,Y,\lambda}^k := \{X_0 \prec \cdots \prec X_k\}$. We will also need a new definition in nested set combinatorics.

**Definition 5.21 (Cluster).** An irreducible nested set $S$ is called a cluster if all its intervals except the top one have rank 1. A cluster is said to be proper if its top interval has rank strictly greater than 1.

Clusters can be constructed out of truncated maximal chains in the following way. If $\omega$ is a chain $\omega = \{X_0 = 0 \prec X_1 \prec \cdots \prec X_n\}$ in some built geometric lattice $\mathcal{L}$ then we put:

$$S(\omega) := \{X_1\} \circ \{X_2\} \circ \cdots \circ \{X_n\},$$

which is a cluster. This formula makes sense even if the $X_i$’s do not belong to $\mathcal{G}$ because each $X_i$ covers $X_{i-1}$ and therefore is an atom in $[X_{i-1}, 1]$ which must belong to the induced building set. We can finally state the main result of this section.

**Proposition 5.22.** The normal monomials with respect to $\mathcal{B}$ in arity $(\mathcal{L}, \mathcal{G}, \prec)$ are the monomials of the form

$$\mathcal{B}\mathcal{G}\mathcal{H} \left( \Psi_S(\omega_{\tau_S(G),G,\lambda}) \right)$$

where $S$ is some nested set without any rank 1 intervals and the $k_G$’s are integers strictly less than $\text{rk}(\tau_S(G,G)) - 1$, except $k_1$ which can be equal to $\text{rk}(\tau_S(1,1)) - 1$. Furthermore this decomposition is unique.
Proof. We start with the following lemma.

**Lemma 5.23.** Any irreducible nested set $S$ in some irreducible built lattice can be written as

$$S = S' \circ (S' G')_{G' \in S'}$$

where $S'$ is an irreducible nested set with no rank 1 intervals and the $S' G'$s are proper clusters except $S^\hat{1}$ which is any cluster.

The nested set $S'$ will be called the frame of $S$ and denoted by $\text{fr}(S)$.

Proof. We put $\text{fr}(S) = \{ G \in S \text{ s.t. } \text{rk}([\tau_S(G), G]) > 1 \} \cup \{ \hat{1} \}$ and conclude by Lemma 3.11.

Now let us proceed with the proof of the statement. We denote by $\mathcal{M}(L, G, \triangleleft)$ the normal monomials with respect to $B$ in arity $(L, G, \triangleleft)$ and $\mathcal{M}'(L, G, \triangleleft)$ the monomials of the form

$$\mathcal{LBS}_{\text{III}}(\text{Gen})(S) \left( \Psi \omega^{k \lambda} \right).$$

Our goal is to show the equality $\mathcal{M}(L, G, \triangleleft) = \mathcal{M}'(L, G, \triangleleft)$ for all irreducible directed built lattice $(L, G, \triangleleft)$. However we have a bijection between $\mathcal{M}'(L, G, \triangleleft)$ and normal monomials of $\text{FY}(L, G)$ with respect to the Gröbner basis introduced in Theorem 2.21 given by

$$\mathcal{LBS}_{\text{III}}(\text{Gen})(S) \left( \Psi \omega^{k \lambda} \right) \rightarrow \prod_{G \in S} x^{\text{rk}([\tau_S(G), G]) - k \lambda - G},$$

and we have the surjective morphism of $\mathcal{LBS}_{\text{III}}$-operads (19). This means that it is enough to prove the inclusion $\mathcal{M}(L, G, \triangleleft) \subset \mathcal{M}'(L, G, \triangleleft)$ (see Proposition 5.16). By Lemma 5.23 it is enough to prove that for any cluster $S$, if $\Psi_S$ is a normal monomial then $S$ is of the form $\omega^{k \lambda} \hat{0}, \hat{1}, \lambda \triangleleft$.

Leading terms of elements of $B$ are monomials of the form $\Psi_H$ where $H$ is not the minimal atom. Let $S$ be any cluster such that $\Psi_S$ is a normal monomial, i.e. is not divisible by any $\Psi_H$ with $H$ not minimal. For any $G \in S \setminus \hat{1}$, let us denote by $H_G$ the smallest atom such that $\tau_{S \setminus \{G\}}(G) \lor H_G = G$.

**Lemma 5.24.** The map $G \rightarrow H_G$ is increasing (with respect to the order $\leq$ on the domain and the order $\triangleleft$ on the codomain).

Proof. Let $G_1 < G_2$ be two elements in $S$ such that $\text{rk}([\tau_S(G_i), G_i]) = 1$ for $i = 1, 2$ and such that there is no element in $S$ strictly between $G_1$ and $G_2$. By Lemma 3.11 we can write $S = (S \setminus \{G_1\}) \circ [\tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_1}]$. Since $\Psi_S$ is not divisible by any monomial $\Psi_H$ where $H$ is not minimal this means that $\tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_1}$ is the minimal atom in $[\tau_{S \setminus \{G_1\}}(G_2), G_2]$, but this interval contains the atom $\tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_2}$ so we have
\( \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_1} \triangleq \text{ind} \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_2} \).

Technically this means

\[
\min \{ H \mid \tau_{S \setminus \{G_1\}}(G_2) \lor H = \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_1} \} < \min \{ H \mid \tau_{S \setminus \{G_1\}}(G_2) \lor H = \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_2} \}.
\]

(20)

By nestedness of \( S \) we have

\[
\{ H \mid \tau_{S \setminus \{G_1\}}(G_2) \lor H = \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_1} \} = \{ H \mid \tau_S(G_1) \lor H = G_1 \}
\]

which has minimum \( H_{G_1} \), and

\[
\{ H \mid \tau_{S \setminus \{G_1\}}(G_2) \lor H = \tau_{S \setminus \{G_1\}}(G_2) \lor H_{G_2} \} = \{ H \mid \tau_S(G_2) \lor H = G_2 \}
\]

which has minimum \( H_{G_2} \). Inequality (20) concludes the proof.

Let us denote \( S = \{ G_1, \ldots, G_n \} \cup \{ \hat{1} \} \) with \( H_{G_1} \triangleq \ldots \triangleq H_{G_n} \). By the previous lemma and successive applications of Lemma 3.11 we get

\[
S = \{ H_{G_1} \} \circ \{ H_{G_1} \lor H_{G_2} \} \circ \ldots \circ \{ H_{G_1} \lor \ldots \lor H_{G_n} \}.
\]

What is left to prove is that the chain \( H_{G_1} \triangleq \ldots \triangleq H_{G_n} \lor H_{G_n} \) is exactly the chain \( \omega^n_{\hat{0}, \hat{1}, \lambda_d} \).

We consider the concatenation of chains

\[
H_{G_1} \triangleq \ldots \triangleq H_{G_n} \lor H_{G_n} < \omega_{H_{G_1} \lor \ldots \lor H_{G_n} \triangleright \lambda_{\triangleq \text{ind}}}.
\]

This chain has increasing labels everywhere except possibly at \( H_{G_1} \lor \ldots \lor H_{G_n} \).

By Lemma 3.11 we have \( S = (S \setminus \{ G_n \}) \circ \{ \tau_{S \setminus \{G_n\}}(\hat{1}) \lor H_{G_n} \} \) so if \( \Psi_S \) is a normal monomial this means that \( \tau_{S \setminus \{G_n\}}(\hat{1}) \lor H_{G_n} \) is the minimal atom in \( [\tau_{S \setminus \{G_n\}}(\hat{1}), \hat{1}] \). This means that \( H_{G_n} \) is smaller than all the atoms which are not below \( \tau_{S \setminus \{G_n\}}(\hat{1}) \) and consequently the maximal chain introduced previously also has increasing labels at \( H_{G_1} \lor \ldots \lor H_{G_n} \).

By Proposition 5.20 the chain \( H_{G_1} \triangleq \ldots \triangleq H_{G_n} \lor H_{G_n} < \omega_{H_{G_1} \lor \ldots \lor H_{G_n} \triangleright \lambda_{\triangleq \text{ind}}} \) must be the chain \( \omega^n_{\hat{0}, \hat{1}, \lambda_d} \) and therefore \( H_{G_1} \triangleq \ldots \triangleq H_{G_n} \lor H_{G_n} \) is the chain \( \omega^n_{\hat{0}, \hat{1}, \lambda_d} \) which concludes the proof.

**Corollary 5.25.** The morphism

\[
\mathcal{LBS}(\text{Gen})/L \xrightarrow{\pi} \mathcal{FY}^V
\]

is an isomorphism and the shuffle \( \mathcal{LBS} \)-operad \( \mathcal{FY}^V_{\text{III}} \) admits a quadratic Gröbner basis.
6 Koszulness of $\mathcal{LBS}$-operads

In [14], R. Kaufmann and B. Ward constructed a Koszul duality theory for operads over certain well-behaved Feynman categories called “cubical”. This cubicality condition is what allows us to define odd operads and a cobar construction on odd operads, which is central in Koszul duality theory.

In the first subsection we prove that $\mathcal{LBS}$ is cubical. Then we unpack Koszulness for operads over $\mathcal{LBS}$, following the definitions in [14]. Finally, we prove that having a quadratic Gröbner basis implies being Koszul and we apply this result to $\mathcal{FY}^\vee$.

6.1 $\mathcal{LBS}$ is cubical

Let us start with some reminders on the notion of “cubicality” (we refer to [14] for more details). Given $(\mathcal{V}, \mathcal{F}, \iota)$ a graded Feynman category (see Section 3.4) and $A, B$ two objects in $\mathcal{F}$ we denote by $C^+_n(A, B)$ the set of composable chains of morphisms of degree less than 1 having exactly $n$ morphisms of degree 1, quotiented by relations:

\[ A \rightarrow ... \rightarrow X_{i-1} \xrightarrow{f} X_i \xrightarrow{g} X_{i+1} \rightarrow ... \rightarrow B \sim A \rightarrow ... \rightarrow X_{i-1} \xrightarrow{g \circ f} X_{i+1} \rightarrow ... \rightarrow B \]  

(21)

provided $f$ or $g$ has degree 0. There is a composition map going from $C^+_n(A, B)$ to $\text{Hom}_\mathcal{F}(A, B)$ given by composing all the morphisms of the chain (the equivalence relation preserves this composition). This map will be denoted by $c_{A,B}$.

Definition 6.1 (Cubical Feynman category). A graded Feynman category $(\mathcal{V}, \mathcal{F}, \iota)$ is called cubical if the degree function is proper and if for every $A, B$ objects of $\mathcal{F}$ there is a free $S_n$ action on $C^+_n(A, B)$ such that

- The composition map is invariant over the action of $S_n$.
- The composition map defines a bijection $c_{A,B} : C^+_n(A, B)_{S_n} \xrightarrow{\sim} \text{Hom}_\mathcal{F}(A, B)$.
- The $S_n$ action is compatible with concatenation of sequences (considering the inclusion $S_p \times S_q \subset S_{p+q}$).

Proposition 6.2. The Feynman categories $\mathcal{LBS}$ and $\mathcal{LBS}_{\text{III}}$ are cubical.

Proof. We only prove the result for $\mathcal{LBS}$, the same arguments also work for $\mathcal{LBS}_{\text{III}}$. By Section 3.3, degree 0 and degree 1 morphisms generate every morphism in $\mathcal{LBS}$. Let us now define an explicit faithful symmetric action on $C^+_n(A, B)$ for every $A, B$ in $\mathcal{LBS}$. It is enough to define it for $B$ an irreducible built lattice.
Using relation (4) and relation (21) we can see that every chain \( \psi \) in \( C^+_n(A, B) \) has a representative of the form

\[
A \xrightarrow{(\otimes, f_i \otimes (G_1 \otimes g_1) \otimes \otimes_j \otimes_j f_j)_{i\nu}} A' \xrightarrow{\psi} B,
\]

where the \( f_i \)'s, \( f_j \)'s and \( g_1, g_2 \) are isomorphisms in \( \text{LBS}_{ir} \), \( \nu \) is a permutation of the summands of \( A \) and \( \phi \) is an element of \( C^+_n(A', B) \) which contains only degree 1 morphisms which are nested sets of cardinality one. By Lemma 3.15 this representative is in fact unique. We denote \( \psi' = [A'' \xrightarrow{\text{Id} \otimes \{G_1 \otimes \text{Id}\}} A' \xrightarrow{\phi} B] \). The composition of the chain \( \phi \) is a nested set \( S \), we have a linear ordering \( G_1, G_2, ..., G_n \) of \( S \) given by reading the chain \( \phi \) say from right to left. Let \( \sigma \) be any element of \( S_n \). We define

\[
\sigma \cdot \psi' := [A'' \xrightarrow{\sigma'} A''' \xrightarrow{\text{Id} \otimes \{G_\sigma(1) \lor \ldots \lor G_\sigma(n)\} \otimes \text{Id} \otimes \ldots} \xrightarrow{\{G_\sigma(1)\} B}]
\]

where

- The formula \( \text{Id} \otimes \{G_\sigma(1) \lor \ldots \lor G_\sigma(n)\} \otimes \text{Id} \otimes \ldots \) means that we tensor the morphism \( \{G_\sigma(1) \lor \ldots \lor G_\sigma(n)\} \) by the identities of the summands of the codomain not containing \( G_\sigma(1) \lor \ldots \lor G_\sigma(n) \).

- \( \sigma' \) is the only permutation of the summands of \( A'' \) which gives us \( A''' \).

Finally, we put

\[
\sigma \cdot \psi := [A \xrightarrow{\otimes_i f_i \otimes g_1 \otimes g_2 \otimes \otimes_j f_j} A'' \xrightarrow{\sigma \cdot \psi'} B],
\]

First, let us prove that this defines an action of \( S_n \). Assume \( \sigma \) is a product \( \sigma_1 \sigma_2 \) with \( \sigma_1, \sigma_2 \in S_n \). We have

\[
\sigma_1 \cdot (\sigma_2 \cdot \psi) = \sigma_1 \cdot [A \xrightarrow{\otimes_i f_i \otimes g_1 \otimes g_2 \otimes \otimes_j f_j} A'' \xrightarrow{\sigma_2 \cdot \psi'} B] = [A \xrightarrow{\otimes_i f_i \otimes g_1 \otimes g_2 \otimes \otimes_j f_j} A'' \xrightarrow{\sigma_2' \otimes \sigma_2''} A'' \xrightarrow{\text{Id} \otimes \{G_\sigma(1) \lor \ldots \lor G_\sigma(n)\} \otimes \text{Id} \otimes \ldots} \xrightarrow{\{G_\sigma(1)\} B}]
\]

Second, let us remark that by relation (5), we have \( c_{A'',B}(\sigma \cdot \psi') = c_{A'',B}(\psi') \) which implies \( c_{A,B}(\sigma \cdot \psi) = c_{A,B}(\psi) \) i.e. the composition map is invariant by the action of \( S_n \).

Third, we see that the action is free because of the unicity of decomposition (22).
Fourth and lastly, the composition map is bijective after passing to the quotient by the action of $S_n$. The surjectivity immediately comes from the fact that morphisms of degree 0 and degree 1 generate every morphism in $\mathcal{LBS}$. The injectivity is a consequence of the unicity of decomposition (22).

6.2 Definition of Koszulness for $\mathcal{LBS}$-operads

6.2.1 Odd operads over cubical Feynman categories

Let $\mathcal{F} = (\mathcal{F}, V, i)$ be cubical category. An odd operad over $\mathcal{F}$ is an operad over the Feynman category $\mathcal{F}^{\text{odd}} = (\mathcal{F}^{\text{odd}}, V, i^{\text{odd}})$ where $\mathcal{F}^{\text{odd}}$ is the category enriched in abelian groups having the same objects as $\mathcal{F}$ andmorphisms $\mathcal{F}^{\text{odd}}(X,Y) = \mathbb{Z} < C_n^+(X,Y) > / \sigma.\phi - \epsilon(\sigma)\phi.$ with composition given by concatenating chains of morphisms. Since $V$ only has isomorphisms it is clear that $V$ is also embedded in $\mathcal{F}^{\text{odd}}$ and we call this embedding $i^{\text{odd}}$.

In the case of $\mathcal{LBS}$, the category $\mathcal{LBS}^{\text{odd}}$ is generated by isomorphisms and generators $\{G\}^{\text{odd}}$ for each element $G$ which is not the maximal element in some building set of some lattice, quotiented by relations

$$\{G_1\}^{\text{odd}} \circ (\{G_1 \lor G_2\}^{\text{odd}} \otimes \text{Id}) = -\{G_2\}^{\text{odd}} \circ (\{G_1 \lor G_2\}^{\text{odd}} \otimes \text{Id}) \circ \sigma_{2,3} \quad (23)$$

for every nested antichain $\{G_1 \neq \hat{1}, G_2 \neq \hat{1}\}$ in some building set, relations

$$\{G_1\}^{\text{odd}} \circ (\{G_2\}^{\text{odd}} \otimes \text{Id}) = -\{G_2\}^{\text{odd}} \circ (\text{Id} \otimes \{G_1\}^{\text{odd}}) \quad (24)$$

for every chain $G_1 < G_2 < \hat{1}$ in some building set, as well as relations

$$f \circ f(\{G\}^{\text{odd}}) = \{G\}^{\text{odd}} \circ (f\{G,1\} \otimes f\{0,G\}) \quad (25)$$

for every isomorphism $f$ between built lattices.

6.2.2 An example of an odd $\mathcal{LBS}$ cooperad

The family of projective Orlik–Solomon algebras $\{\overline{OS}(L)\}_{(L,G)}$ has an odd cooperadic structure over $\mathcal{LBS}$ which extends the dual of the odd operad $\text{Grav}$. It will be denoted by $\overline{OS}$ and defined as follow.

- For any built lattice $(L, G)$ we define

$$\overline{OS}(L, G) := \overline{OS}(L).$$

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• For any element \( G \in G \setminus \{1\} \) in some irreducible built lattice \((\mathcal{L}, \mathcal{G})\), we define

\[
\overline{\text{OS}}(\{G\}^{\text{odd}}) : \overline{\text{OS}}(\mathcal{L}) \to \overline{\text{OS}}([G, 1]) \otimes \overline{\text{OS}}([0, G])
\]

\[
\prod_i e_{H_i} \prod_j e_{H'_j} \to \delta(\prod_i e_{G \vee H_i}) \otimes \prod_j e_{H'_j}
\]

where the \( H_i \)'s are atoms not below \( G \) and the \( H'_j \)'s are atoms below \( G \).

Let us check that the image belongs to the tensored projective Orlik–Solomon algebras. By Lemma 2.24 it is enough to show that \( \overline{\text{OS}}(\{G\}^{\text{odd}})(\delta(\prod_{H \in \mathcal{H}} e_{H})) \) belongs to \( \overline{\text{OS}}([G, 1]) \otimes \overline{\text{OS}}([0, G]) \) for all sets of atoms \( \mathcal{H} \). We partition \( \mathcal{H} \) into \( \{H_i\} \sqcup \{H'_j\} \) where the \( H_i \)'s are atoms not below \( G \) and the \( H'_j \)'s are atoms below \( G \). We then have

\[
\overline{\text{OS}}(\{G\}^{\text{odd}})(\delta(\prod_{H \in \mathcal{H}} e_{H})) = \overline{\text{OS}}(\{G\}^{\text{odd}})(\delta(\prod_i e_{H_i} e_{H'_j} \pm \prod_i e_{H_i} \delta(\prod_j e_{H'_j})))
\]

\[
= \delta(\prod_i e_{G \vee H_i}) \otimes \prod_j e_{H'_j} \pm \delta(\prod_i e_{G \vee H_i}) \otimes \delta(\prod_j e_{H'_j})
\]

\[
= \pm \delta(\prod_i e_{G \vee H_i}) \otimes \delta(\prod_j e_{H'_j}) \in \overline{\text{OS}}([G, 1]) \otimes \overline{\text{OS}}([0, G])
\]

• For any isomorphism of built lattice \( f : (\mathcal{L}', \mathcal{G}') \simto (\mathcal{L}, \mathcal{G}) \) we define \( \overline{\text{OS}}(f) \) as the restriction of \( \text{OS}(f) \) to the projective subalgebra.

We must check that the morphisms \( \overline{\text{OS}}(\{G\}^{\text{odd}}) \) and \( \overline{\text{OS}}(f) \) satisfy relations (23), (24) and (25) above. Let \( \{G_1 \neq 1, G_2 \neq 1\} \) be a nested antichain in some irreducible built lattice \((\mathcal{L}, \mathcal{G})\) and let \( \alpha = \delta(\prod_{i \leq n} e_{H_i} \prod_{j \leq n'} e_{H'_j} \prod_{k \leq n''} e_{H''_k}) \) be an element in \( \overline{\text{OS}}(\mathcal{L}) \) where the \( H_i \)'s are atoms below neither \( G_1 \) nor \( G_2 \), the \( H'_j \)'s below \( G_2 \) and the \( H''_k \)'s below \( G_1 \) (by nestedness of \( \{G_1, G_2\} \) there can be no atom below both \( G_1 \) and \( G_2 \)). In this case one can check that the morphism

\[
(\overline{\text{OS}}(\{G_1 \lor G_2\}^{\text{odd}}) \otimes \text{Id}) \circ \overline{\text{OS}}(\{G_1\}^{\text{odd}})
\]

sends \( \alpha \) to \( \delta(\prod_i e_{G_1 \lor G_2 \vee H_i}) \otimes \delta(\prod_j e_{H'_j}) \otimes \delta(\prod_k e_{H''_k}) \) whereas the morphism

\[
\sigma_{2,3} \circ (\overline{\text{OS}}(\{G_1 \lor G_2\}^{\text{odd}}) \otimes \text{Id}) \circ \overline{\text{OS}}(\{G_2\}^{\text{odd}})
\]

sends \( \alpha \) to the opposite. Let \( G_1 < G_2 < 1 \) be a chain in some irreducible built lattice \((\mathcal{L}, \mathcal{G})\) and let \( \alpha = \delta(\prod_{i \leq n} e_{H_i} \prod_{j \leq n'} e_{H'_j} \prod_{k \leq n''} e_{H''_k}) \) be an element in \( \overline{\text{OS}}(\mathcal{L}) \) where the \( H_i \)'s are atoms not below \( G_2 \), the \( H'_j \)'s are atoms below \( G_2 \) and not below \( G_1 \) and the \( H''_k \)'s are atoms below \( G_1 \). In this case one can check that the morphism

\[
(\overline{\text{OS}}(\{G_2\}^{\text{odd}}) \otimes \text{Id}) \circ \overline{\text{OS}}(\{G_1\}^{\text{odd}})
\]

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sends \( \alpha \) to \( \delta(\prod_i e_{G2\lor H_i}) \otimes \delta(\prod_j G_1 \lor e_{H'_j}) \otimes \delta(\prod_k e_{H''_k}) \) whereas the morphism

\[
(\text{Id} \otimes \overline{OS}(\{G_2\}^{\text{odd}})) \circ \overline{OS}(\{G_2\}^{\text{odd}})
\]

sends \( \alpha \) to the opposite. At last, equation (25) is also easily verified.

To conclude, we have shown that \( \overline{OS} \) is an odd \( \mathcal{LG} \)-cooperad (in graded abelian groups).

### 6.2.3 The bar/cobar construction

Let \( \mathcal{C} \) be some complete cocomplete symmetric monoidal abelian category. We denote by \( \text{Ch}\mathcal{C} \) the category of chain complexes over \( \mathcal{C} \). Let \( \mathfrak{F} \) be a cubical Feynman category.

In [14], R. Kaufmann and B. Ward define a bar operator

\[
\mathcal{B} : \mathfrak{F} - \text{Ops}_{\text{Ch}\mathcal{C}} \to \mathfrak{F}^{\text{odd}} - \text{Ops}_{\text{Ch}\mathcal{C}}^{\text{op}}
\]

and a cobar operator

\[
\Omega : \mathfrak{F}^{\text{odd}} - \text{Ops}_{\text{Ch}\mathcal{C}}^{\text{op}} \to \mathfrak{F} - \text{Ops}_{\text{Ch}\mathcal{C}}
\]

which form an adjunction \( \Omega \dashv \mathcal{B} \) and such that the counit

\[
\Omega \mathcal{B} \Longrightarrow \text{Id}
\]

is a level-wise quasi-isomorphism. Informally those functors are defined by taking free constructions together with a differential constructed using the degree 1 generators. Let us describe \( \Omega \) explicitly in our case.

Let \( \mathcal{P} \) be an \( \mathcal{LG}^{\text{odd}} \) cooperad in \( \text{Ch}\mathcal{C}^{\text{op}} \). We have

\[
\Omega(\mathcal{P}) = (\mathcal{LG}((\mathcal{P}), d_\Omega + d_\mathcal{P}),
\]

where \( d_\mathcal{P} \) is the obvious differential coming from \( \mathcal{P} \) and \( d_\Omega \) is defined as follow. Recall the explicit formula

\[
\mathcal{LG}(\mathcal{P})(\mathcal{L}, \mathcal{G}) = \bigoplus_{\otimes (\mathcal{L}, \mathcal{G}) \xrightarrow{f} (\mathcal{L}, \mathcal{G})} \bigotimes_{i} \mathcal{P}(\mathcal{L}_i, \mathcal{G}_i) / \sim
\]

where \( \sim \) identifies components corresponding to equivalent maps (maps that can be obtained from each other by precomposition of isomorphisms). For any \( \otimes p_i \in \bigotimes_{i \leq n} \mathcal{P}(\mathcal{L}_i, \mathcal{G}_i) \) we put

\[
d_\Omega(\otimes p_i, f : \bigotimes_{i} (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})) = \sum_{j \leq n} \left[ (\text{Id} \otimes \mathcal{P}(\{G\}) \otimes \text{Id})(\otimes p_i), f \circ (\text{Id} \otimes \{G\} \otimes \text{Id}) \right].
\]
The cubicality condition and the fact that $\mathbb{P}$ is an odd cooperad ensures that $d_\Omega + d_\mathbb{P}$ is indeed a differential. Let us now describe $B$ explicitly for $\mathcal{LBS}$-operads. Let $\mathbb{P}$ be an $\mathcal{LBS}$-operad in $\text{Ch} \, C$. We have

$$B(\mathbb{P}) = (\mathcal{LBS}^{\text{odd}}(\mathbb{P}), d_B + d_\mathbb{P}),$$

where $d_\mathbb{P}$ is the obvious differential coming from $\mathbb{P}$ and $d_B$ is defined as follows. We have the formula

$$\mathcal{LBS}^{\text{odd}}(\mathbb{P})(\mathcal{L}, G) = \bigoplus_{n \in \mathbb{N}} \mathbb{P}(\mathcal{L}', G')/\sim,$$

where the equivalence relation $\sim$ is given by

$$(\mathbb{P}(f)(\alpha), \psi) \sim (\alpha, \psi \circ f)$$

for every isomorphism $f$ and

$$(\alpha, \psi) \sim e(\sigma)(\alpha, \sigma \psi)$$

for every permutation $\sigma$. Let $\psi$ be an element in $C^+_n((\mathcal{L}', G'), (\mathcal{L}, G))$, and let $\alpha$ be an element of $\mathbb{P}(\mathcal{L}', G')$. We have

$$d_B((\alpha, [\psi])) := \sum_{\phi : (\mathcal{L}'', G'') \to (\mathcal{L}, G)} e(\phi)((\text{Id} \otimes \mathbb{P}(\{G\}) \otimes \text{Id})(\alpha), \phi),$$

where $c$ is the composition map and $e(\phi)$ is the signature of the permutation sending $\psi$ to $\phi \circ \{G\}$. One can check that this descends to the quotient by the equivalence relation $\sim$.

### 6.2.4 Quadratic duality and Koszul duality

In this subsection we assume that $C$ is a category of vector spaces over some field. For any graded Feynman category $\mathfrak{F}$, an $\mathfrak{F}$-quadratic data is a pair $(M, R)$ with $M$ an $\mathfrak{F}$-module and $R$ a submodule of $\mathfrak{F}_1(M)$, where $\mathfrak{F}_1(M)$ denotes the part of grading 1 in the free $\mathfrak{F}$-operad $\mathcal{LBS}(M)$. Notice that an $\mathfrak{F}$-quadratic data can also be seen as an $\mathfrak{F}^{\text{odd}}$-quadratic data since $\mathfrak{F}$ and $\mathfrak{F}^{\text{odd}}$ have the same modules and we have $\mathfrak{F}^{\text{odd}}_1(M) = \mathfrak{F}_1(M)$.

If $\mathbb{P}$ is an $\mathfrak{F}$-operad which is a quotient $\mathfrak{F}(M)/(R)$ for some $\mathfrak{F}$-quadratic data $(M, R)$, we define

$$\mathbb{P}^\flat := \mathfrak{F}^{\text{odd}}(M^\vee)/(R_{\perp}),$$

which is an $\mathfrak{F}^{\text{odd}}$-operad. We have a morphism of differential graded $\mathfrak{F}^{\text{odd}}$-operads

$$(\mathbb{P}^\flat)^\vee \to B\mathbb{P}, \quad (26)$$

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which is induced by the morphism of $\mathfrak{g}$-modules given by the composition

$$(\mathbb{P}^1)^{\vee} \to M^{\vee} \hookrightarrow \mathbb{P}.$$  

We say that $\mathbb{P}$ is Koszul with Koszul dual $(\mathbb{P}^1)^{\vee}$ if morphism (26) is a quasi-isomorphism. We refer to [14] and [20] for more details. This coincides with the classical Koszul duality theories (Koszul duality for operads for instance).

In addition to having a homological degree (given by the grading of $\mathfrak{LBS}_{\mathfrak{III}}$), the odd cooperad $\mathbb{B}\mathbb{P}$ has a weight grading coming from the grading of $\mathbb{P}$. The differential preserves this weight grading. One can check that the map

$$(\mathbb{P}^1)^{\vee} \to \mathbb{B}\mathbb{P}$$

is injective and its image is exactly the kernel of $d_{\mathbb{B}}$ in the diagonal $\{\text{weight grading} = \text{degree}\}$, which is also the homology of the diagonal since the elements on the diagonal are the highest degree elements in their respective weight component. As a consequence $\mathbb{P}$ is Koszul if and only if the homology of $\mathbb{B}\mathbb{P}$ is concentrated on the diagonal.

6.3 Koszulness of $\mathbb{F}_Y^{PD}$ using the projective combinatorial Leray model

In [4] the authors define a differential bigraded algebra $B(\mathcal{L}, \mathcal{G})$ as follow.

**Definition 6.3** (Projective combinatorial Leray model [4]). Let $(\mathcal{L}, \mathcal{G})$ be an irreducible built lattice. The differential bigraded algebra $B(\mathcal{L}, \mathcal{G})$ is defined as the quotient of the free commutative algebra $\mathbb{Q}[e_G, x_G, G \in \mathcal{G}]$ by the ideal $\mathcal{I}$ generated by

1. The elements $e_Sx_T$ with $S \cup T$ not nested.
2. The elements $\sum_{G \geq H} x_G$ for all atoms $H$ of $\mathcal{L}$.
3. The element $e_1$.

The generators $e_G$ have bidegree $(0, 1)$ and the generators $x_G$ have bidegree $(2, 0)$. The differential $d$ of this algebra has bidegree $(2, -1)$ and is defined by

$$d(e_G) = x_G$$

$$d(x_G) = 0.$$  

The authors of [4] have shown that we have isomorphisms of graded vector spaces

$$B^{*,d}(\mathcal{L}, \mathcal{G}) \simeq \bigoplus_{\text{\#S=d+1}} \mathbb{F}_Y^{PD}(S)$$

for irreducible nested set of $(\mathcal{L}, \mathcal{G})$.
for every integer \(d\) (Proposition 5.1.4). Those isomorphisms give an isomorphism of complexes between \(B(L, G)\) and \(BFYPD(L, G)\). For each irreducible built lattice \((L, G)\) we have a morphism of differential graded algebras

\[
\overline{OS}(L) \to B^\bullet(L, G),
\]

induced by the map \(e_H \to \sum_{G > H} e_G\). One can check that this is a morphism of \(\mathcal{LBS}^{\text{odd}}\)-cooperads. The main result of [4] is the following.

**Theorem 6.4** ([4], Theorem 5.5.1). The morphism \(\overline{OS}(L) \sim \to B^\bullet(L, G)\) is a quasi-isomorphism for every pair \((L, G)\).

This immediately implies:

**Corollary 6.5.** The operad \(FYPD\) is Koszul with Koszul (co)dual \(OS\).

**Remark 6.6.** The algebra structure on \(BFYPD(L, G)\) coming from the isomorphism

\[
BFYPD(L, G) \simeq B(L, G)
\]

can be defined purely operadically as follow. Let \(\alpha\) be some element in \(FYPD(S)\) for some nested set \(S\) in some built lattice \((L, G)\) and \(\beta\) some element of \(FYPD(S')\) for some nested set \(S'\) in the same built lattice. The product of \(\alpha\) and \(\beta\) in \(BFYPD(L, G)\) is given by

\[
\alpha \cdot \beta = \begin{cases} 
FY(S')(\alpha)FY(S)(\beta) & \text{if } S \cap S' = \{1\} \text{ and } S \cup S' \text{ is a nested set.} \\
0 & \text{otherwise.}
\end{cases}
\]

In the first row, \(S'\) is viewed as a nested set of \((L_S, G_S)\) and \(S\) is viewed as a nested set of \((L_{S'}, G_{S'})\) via Lemma 3.11. The product takes place in the algebra \(FY(L_{S\cup S'}, G_{S\cup S'})\). It is interesting to note that we have used the operadic structure of \(FY\) and not that of \(FYPD\). This shows that Poincaré duality plays an important role when trying to relate the properties of \(FY\) and the properties of the Feichtner–Yuzvinsky algebras.

### 6.4 Koszulness and the affine Leray model

In [4] the authors also define a Leray model \(\hat{B}(L, G)\) for the (affine) Orlik-Solomon algebras, just by taking out the relation \(e_1 = 0\) in \(B(L, G)\). One can also interpret this affine Leray model as a bar construction in a larger Feynman category \(\mathcal{LBS}^{\text{mod}}\) defined as follow. The set of objects of the underlying groupoid of \(\mathcal{LBS}^{\text{mod}}\) is

\[
\text{Ob}(\mathcal{LBS}_{\text{irr}}) \sqcup \text{Ob}(\mathcal{LBS}_{\text{irr}}).
\]

For each irreducible built lattice \((L, G)\) we will denote the two copies of \((L, G)\) in \(\mathcal{LBS}^{\text{mod}}\) by \((L, G)^{\text{proj}}\) and \((L, G)^{\text{aff}}\), for reasons which will be clear later. If \((L, G)\) is an irreducible
built lattice, the structural morphisms of $\mathcal{LBS}\mathfrak{mod}$ with target $(\mathcal{L}, \mathcal{G})^{\text{proj}}$ are labelled by irreducible nested sets

$$\bigotimes_{G \in S} ([\tau_S(G), G], \text{Ind}(G))^{\text{proj}} \xrightarrow{S} (\mathcal{L}, \mathcal{G})^{\text{proj}},$$

with composition as in $\mathcal{LBS}$. In other words when restricting $\mathcal{LBS}\mathfrak{mod}$ to the “projective” arities we get the Feynman category $\mathcal{LBS}$. The structural morphisms of $\mathcal{LBS}\mathfrak{mod}$ with target $(\mathcal{L}, \mathcal{G})^{\text{aff}}$ are labelled by nested sets which can either contain $\hat{1}$ or not. If $S$ contains $\hat{1}$ then we have the morphism

$$\bigotimes_{G \in S} ([\tau_S(G), G], \text{Ind}(G))^{\text{proj}} \xrightarrow{S^{\text{aff}}} (\mathcal{L}, \mathcal{G})^{\text{aff}},$$

and if $S$ does not contain $\hat{1}$ we have the morphism

$$\bigotimes_{G \in S} ([\tau_S(G), G], \text{Ind}(G))^{\text{proj}} \otimes ([\tau_S(\hat{1}), \hat{1}], \text{Ind}(G))^{\text{aff}} \xrightarrow{S^{\text{aff}}} (\mathcal{L}, \mathcal{G})^{\text{aff}}.$$

The composition of those morphisms is defined as in $\mathcal{LBS}$. This Feynman category encodes pairs $(\mathcal{P}, \mathcal{M})$ with $\mathcal{P}$ an $\mathcal{LBS}$-operad and $\mathcal{M}$ a “$\mathcal{P}$-module” ($\mathcal{P}$ is the restriction to the projective part and $\mathcal{M}$ the restriction to the affine part). A set of generating morphisms of $\mathcal{LBS}\mathfrak{mod}$ is given by

$$\{G, \hat{1}\}^{\text{proj}} (G \neq \hat{1}) \text{ and } \{G\}^{\text{aff}}.$$

One can define an odd $\mathcal{LBS}\mathfrak{mod}$-cooperad $\mathcal{O}S_{\text{tot}}$ by setting

$$\mathcal{O}S_{\text{tot}}((\mathcal{L}, \mathcal{G})^{\text{proj}}) = \overline{\mathcal{O}S(\mathcal{L})}, \quad \mathcal{O}S_{\text{tot}}((\mathcal{L}, \mathcal{G})^{\text{aff}}) = \mathcal{O}S(\mathcal{L}),$$

for each irreducible built lattice $(\mathcal{L}, \mathcal{G})$, and

$$\mathcal{O}S_{\text{tot}}(\{G, \hat{1}\}^{\text{proj}}) = \overline{\mathcal{O}S(\{G, \hat{1}\})}$$

together with

$$\mathcal{O}S_{\text{tot}}(\{G\}^{\text{aff}}) = (\delta \otimes \text{Id}) \circ \mathcal{O}S(\{G\}).$$

In this case the odd $\overline{\mathcal{O}S}$-comodule structure on $\mathcal{O}S$ comes from the morphism of $\mathcal{LBS}$-cooperads

$$\mathcal{O}S \xrightarrow{\delta} \overline{\mathcal{O}S}.$$

On the other hand one can define an $\mathcal{LBS}\mathfrak{mod}$-operad $\mathcal{F}Y_{\text{tot}}^{\text{PD}}$ by setting

$$\mathcal{F}Y_{\text{tot}}^{\text{PD}}((\mathcal{L}, \mathcal{G})^{\text{proj}}) = \mathcal{F}Y(\mathcal{L}, \mathcal{G}), \quad \mathcal{F}Y_{\text{tot}}^{\text{PD}}((\mathcal{L}, \mathcal{G})^{\text{aff}}) = \mathcal{F}Y(\mathcal{L}, \mathcal{G}),$$

and

$$\mathcal{F}Y_{\text{tot}}(\{G, \hat{1}\}^{\text{proj}}) = \mathcal{F}Y^{\text{PD}}(\{G, \hat{1}\}^{\text{proj}}),$$

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together with
\[ FY_{\text{tot}}(\{G\}) = FY(\{G, \hat{1}\}) \]
for \( G \neq \hat{1} \), and finally \( FY_{\text{tot}}(\{1\}) \) is set to be the multiplication by \( x_1 \). Exactly as for the projective part, one can use the results of \[4\] to see that \( \hat{B}(\mathcal{L}, \mathcal{G}) \) is isomorphic to \( BF^\text{PD}_{\text{tot}} \) and the morphism \( e_H \to \sum_{G \geq H} e_G \) induces a quasi-isomorphism of odd \( \mathcal{L}B\mathcal{G}\mathcal{O}d\mathcal{M}\text{-cooperads} \)
\[ \mathcal{O}S_{\text{tot}} \overset{\sim}{\to} BF^\text{PD}_{\text{tot}}, \]
which implies Koszulness of the \( \mathcal{L}B\mathcal{G}\mathcal{O}d\mathcal{M}\text{-operad} FY_{\text{tot}}^{\text{PD}}. \)

### 6.5 Koszulness via shuffle operads

As in the case of classical operads and their shuffle counterpart, we have the key proposition.

**Proposition 6.7.** Let \( \mathcal{P} \) be an \( \mathcal{L}B\mathcal{G}\)-operad. \( \mathcal{P} \) is Koszul if and only if \( \mathcal{P}_{\text{III}} \) is Koszul.

**Proof.** By Proposition \[5.18\] we have isomorphisms of shuffle \( \mathcal{L}B\mathcal{G}\text{odd}\)-operads
\[ (\mathcal{P}_{\text{III}})^! \overset{\sim}{\to} (\mathcal{P}^!)_{\text{III}} \]
which gives an isomorphism of shuffle \( \mathcal{L}B\mathcal{G}\text{odd}\)-cooperad (in \( \text{Ch} \) \( \mathcal{C} \)):
\[ ((\mathcal{P}_{\text{III}})^!)^! \overset{\sim}{\to} ((\mathcal{P}^!)^!)_{\text{III}}. \]

On the other hand, we have
\[ (\mathcal{L}B\mathcal{G}\text{odd}(\mathcal{P}))_{\text{III}} \cong \mathcal{L}B\mathcal{G}\text{odd}_{\text{III}}(\mathcal{P}_{\text{III}}). \]

By going back to explicit formulas one can check that those isomorphisms are compatible with the Bar differential and that we have a commutative diagram
\[
\begin{array}{c}
(\mathcal{P}^!)_{\text{III}} \quad \xrightarrow{\sim} \quad (\mathcal{P}_{\text{III}})^!
\end{array}
\]
\[
\begin{array}{c}
(\mathcal{L}B\mathcal{G}(\mathcal{P}))_{\text{III}} \quad \xrightarrow{\sim} \quad (\mathcal{B}(\mathcal{P}))_{\text{III}}
\end{array}
\]
but of course in every arity \((\mathcal{L}, \mathcal{G}, <)\) we have the commutative diagram of complexes
\[
\begin{array}{c}
(\mathcal{P}^!)_{\text{III}}(\mathcal{L}, \mathcal{G}) \quad \xrightarrow{\sim} \quad (\mathcal{B}(\mathcal{P}))_{\text{III}}(\mathcal{L}, \mathcal{G}, <)
\end{array}
\]
Combining the two diagrams in every arity finishes the proof. \( \square \)

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6.6 Koszulness and Gröbner bases

As in the case of classical shuffle operads, we have the key proposition.

**Proposition 6.8.** Let \( P \) be a shuffle \( \mathcal{LBS} \)-operad. If \( P \) admits a quadratic Gröbner basis then \( P \) is Koszul.

**Proof.** This is just an adaptation of the proof given in [13] to our setting, and translating Gröbner basis language in “PBW basis” language. We denote \( P = \mathcal{LBS}_{\text{III}}(M)/\langle R \rangle \).

Let us use our total well-order on monomials to construct a filtration on \( B^P \). We will denote this total order by “\(<". Let \( m = (S, (e_G)_{G \in S}) \) be some monomial. We define

\[
F_m B^P = \langle \{ m_1 \otimes \ldots \otimes m_n \in P(S') \mid m_1, \ldots, m_n \text{ monomials s.t. } \mathcal{LBS}_{\text{III}}(S')((m_i)_i) \leq m \} \rangle
\]

where the brackets \( \langle, \rangle \) denote the linear span. The bar differential preserves this filtration. We will now show that the associated spectral sequence collapses at the first page and its homology is concentrated on the diagonal. The complex \( E^0_n B^P \) is spanned by elements of the form

\[
P(S_1)((e_G)_{G \in S_1}) \otimes \ldots \otimes P(S_n)((e_G)_{G \in S_n}),
\]

where the nested sets \( S_i \) are such that there exist some nested set \( S' \) satisfying \( S = S' \circ (S_i)_i \), and such that the monomials \( P(S_i)((e_G)_{G \in S_i}) \) are all normal.

For any \( G \in S \setminus \{ 1 \} \) we denote by \( n(G) \) the unique minimum of \( S_{>G} \). We also denote by \( \text{Adm}(m) \) the set of elements of \( S \setminus \{ 1 \} \) such that \( P(\{ G \})((e_G, e_{n(G)})) \) is a normal monomial. By the fact that our Gröbner basis is quadratic we see that \( E^0_m B^P \) is isomorphic to the augmented dual of the combinatorial complex \( C_*(\Delta_{\text{Adm}(m)}) \), which has trivial homology except when \( \text{Adm}(m) = \emptyset \), in which case the complex is reduced to \( \mathbb{K} \) on the diagonal (with generator given by \( \otimes G \in G \)). By a standard spectral sequence argument this concludes the proof.

As a corollary of this proposition and 5.25 we get

**Corollary 6.9.** The operad \( \mathcal{FY}^\vee \) is Koszul.

7 Further directions

In this section we highlight some possible ways to extend/refine \( \mathcal{LBS} \) which seem natural to us and may lead to further applications.
7.1 Working with matroids instead of geometric lattices

One possible refinement of $\mathcal{LBS}$ would be to do everything with matroids instead of geometric lattices, which would allow us to take loops and parallel elements into account. For now this refinement is useless because all the operads we know (Feichtner–Yuzvinsky rings, Orlik–Solomon algebras) do not “see” the loops and parallel elements (i.e. factor through the lattice of flat construction). However, it may happen that some finer invariants of matroids which detect loops and parallel elements may also have an operadic structure. In order to implement this refinement it will be beneficial to have a purely matroidal axiomatization of building sets. Let us describe a possible way to obtain that. Recall that a matroid can be defined by its rank function as follow.

**Definition 7.1** (Matroid, via rank function). Let $E$ be a finite set. A matroid structure on $E$ is the datum of a map

$$
rk : \mathcal{P}(E) \to \mathbb{N}
$$
called the rank function, satisfying the following properties.

1. The rank function takes value 0 on the empty set.
2. For every $A, B \in \mathcal{P}(E)$ we have

$$
\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B).
$$
3. For every $A \in \mathcal{P}(E)$ and $x \in E$ we have

$$
\text{rk}(A \cup \{x\}) \leq \text{rk}(A) + 1.
$$

Here is a possible way of axiomatizing a building set in terms of the rank function.

**Definition 7.2** (Building decomposition). A building decomposition of a matroid $(E, \text{rk})$ is a function $\nu$ which assigns to every subset $X \subseteq E$ a partition of $X$ and which satisfies the following axioms.

1. If $X \subseteq Y$ are two subsets of $E$, then $\nu(X)$ refines the restriction of $\nu(Y)$ to $X$.
2. If $\nu(X)$ is the partition with blocks $P_1|...|P_n$ then for all $i \leq n$ the partition $\nu(P_i)$ is the partition with only one block.
3. For all $X \subseteq E$, if $\nu(X)$ is the partition $P_1|...|P_n$ then $\text{rk}(X) = \text{rk}(P_1) + ... + \text{rk}(P_n)$.

On simple loopless matroids the datum of a building decomposition is equivalent to the datum of a building set on the lattice of flats. One can construct a building set out of a building decomposition by considering the flats which have a partition with only one block. On the other hand, one can construct a building decomposition out of a building set by setting $\nu(X)$ to be the partition induced by the factor decomposition of $\sigma(X)$.
**Example 7.3.** Let $G = (V, E)$ be a graph and $M_G$ its cycle matroid. $M_G$ admits a building decomposition given by the partitions into connected components for each subset of $E$. Naturally if we look at the induced building set on the lattice of flats this gives the graphical building set introduced in Example 2.8.

We also have a natural notion of induced building decomposition on restrictions and contractions of matroids.

**Definition 7.4.** Let $M = (E, \text{rk})$ be a matroid with building decomposition $\nu$ and $S$ a subset of the ground set $E$. The contraction $M^S$ admits a building decomposition $\text{Ind}_S^S(\nu)$ given by $\text{Ind}_S^S(\nu)(X) = \nu(X \cup S)_{|X}$ for every $X \subset E \setminus S$. The restriction $M_S$ admits a building decomposition $\text{Ind}_S^S(\nu)$ given by $\text{Ind}_S^S(\nu)(X) = \nu(X)$ for every $X \subset S$.

Working with those definitions, we are fairly certain everything should work in the same fashion as in Section 3 by just replacing the built lattices $(\mathcal{G}, \text{Ind}(G)), ([G, \hat{1}], \text{Ind}(G))$ by the matroidal restrictions/contractions $(M_G, \text{Ind}(\nu)), (M^G, \text{Ind}(\nu))$, for $G$ such that $\nu(G)$ is the trivial partition (a priori we would not even need $G$ to be closed, which would give additional structural morphisms).

### 7.2 The polymatroidal generalization

One can also naturally consider an extension of LBS to polymatroids, which form a combinatorial abstraction of subspace arrangements. This is justified by the fact that the wonderful compactification story also works for subspace arrangements and the cohomology algebras give us a natural candidate for an operad over this bigger Feynman category. It has been shown by Pagaria and Pezzoli [17] that those cohomology rings also admit natural generalizations to arbitrary polymatroids and that they also have a Hodge theory. Here are some reminders on polymatroids.

**Definition 7.5 (Polymatroid).** Let $E$ be a finite set. A polymatroid structure on $E$ is the datum of a function

$$\text{cd} : \mathcal{P}(E) \rightarrow \mathbb{N}$$

satisfying

1. $\text{cd}(\emptyset) = 0$.

2. For any subsets $A \subset B$ of $E$ we have $\text{cd}(A) \leq \text{cd}(B)$.

3. For any subsets $A, B$ of $E$ we have

$$\text{cd}(A \cap B) + \text{cd}(A \cup B) \leq \text{cd}(A) + \text{cd}(B).$$
The letters “cd” stand for codimension. If we ask that cd take value 1 on singletons we get a classical matroid. We can define the lattice of flats of a polymatroid by considering the subsets $F$ of $E$ such that $cd(F \cup \{x\}) > cd(F)$ for all $x \not\in F$. However, for general polymatroids the lattice of flats does not contain enough information and needs to be considered together with $cd$ to recover the polymatroid (for matroids “cd” is just the rank function of the lattice of flats and does not add any information). In [17] the authors introduced a notion of building set for polymatroids.

**Definition 7.6.** Let $P = (E, cd)$ be a polymatroid with lattice of flats $\mathcal{L}$. A building set of $P$ is a subset $\mathcal{G}$ of $\mathcal{L} \setminus \{\hat{0}\}$ such that for any $X$ in $\mathcal{L}$ the join gives an isomorphism of posets

$$\prod_{G \in \text{Fact}_\mathcal{G}(X)} [\hat{0}, G] \cong [\hat{0}, X]$$

and we additionally have

$$cd(X) = \sum_{G \in \text{Fact}_\mathcal{G}(X)} cd(G).$$

Notice that the last condition is automatically verified for matroids ($cd = \text{rk}$). The authors also give suitable generalizations of nested sets, and they show that for any $G$ in $\mathcal{L}$, the (polymatroidal) contraction $([G, \hat{1}], \text{Ind}(cd))$ has an induced (polymatroidal) building set given (as in the matroidal case) by

$$\text{Ind}_{[G, \hat{1}]}(\mathcal{G}) = (\mathcal{G} \lor G) \cap (G, \hat{1}],$$

and the same goes for (polymatroidal) restrictions. With those definitions we are fairly certain one can readily extend LBS to polymatroids.

In [17], the authors also introduce a generalization of the Feichtner–Yuzvinsky algebras to the polymatroidal setting as follows.

**Definition 7.7.** Let $(\mathcal{L}, cd)$ be a polymatroid with some building set $\mathcal{G}$. The algebra $\text{FY}(\mathcal{L}, \mathcal{G}, cd)$ is defined by

$$\text{FY}(\mathcal{L}, \mathcal{G}, cd) = \mathbb{Q}[x_{G}, G \in \mathcal{G}]/\mathcal{I},$$

with all the generators in degree 2 and $\mathcal{I}$ the ideal generated by elements

$$\prod_{i \leq n} x_{G_i}$$

where $\{G_1, ..., G_n\}$ is not nested and elements

$$\left( \sum_{G \geq H} x_{G} \right)^{cd(H)}$$

for any atom $H$. 

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As in the matroidal case, one can get another presentation by considering the change of variable $h_G := \sum_{G' \geq G} x_{G'}$. The algebra morphisms

$$FY(L, cd, \mathcal{G}) \longrightarrow FY([G, \hat{1}], \text{Ind}(cd), \text{Ind}_{[G, \hat{1}]}(\mathcal{G})) \otimes FY(\hat{0}, G, \text{Ind}(cd), \text{Ind}_{[0,G]}(\mathcal{G}))$$

$$h_{G'} \longrightarrow \begin{cases} h_{G \vee G'} \otimes 1 & \text{if } G' \nleq G \\ 1 \otimes h_G & \text{otherwise.} \end{cases}$$

are well-defined and give us an operadic structure on the family of generalized Feichtner–Yuzvinsky algebras.

### 7.3 Adding morphisms to $\mathcal{LBS}$

One could also consider adding more morphisms of degree 0 in $\mathcal{LBS}$. This is justified by the fact that the Feichtner–Yuzvinsky algebras have a lot more functoriality than what we have in $\mathcal{LBS}$. More precisely let $(L, \mathcal{G})$ and $(L', \mathcal{G}')$ be two built lattices and let $f : L \rightarrow L'$ be a poset morphism which sends $\mathcal{G}$ to $\mathcal{G}'$, atoms of $L$ to atoms of $L'$ and which is compatible with the join on both sides, i.e.

$$f(G_1 \vee G_2) = f(G_1) \vee f(G_2)$$

for all $G_1, G_2$ in $L$. With those hypotheses the map induced by

$$FY(L, \mathcal{G}) \xrightarrow{h_{\mathcal{G}}} FY(L', \mathcal{G}')$$

is a well-defined map of algebras. This incentivizes us to formally add such morphisms in $\mathcal{LBS}$. Some of those morphisms are very natural to add in their own right. For instance if $\mathcal{G} \subset \mathcal{G}'$ are two building sets of some lattice $L$ then the identity of $L$ satisfies the above conditions. In the realizable case the corresponding map $FY(f)$ is induced by the blow down

$$\Upsilon_{L, \mathcal{G'}} \rightarrow \Upsilon_{L, \mathcal{G}}.$$
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