Estimation Methods for Cluster Randomized Trials with Noncompliance: A Study of A Biometric Smartcard Payment System in India

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Abstract

Many policy evaluations occur in settings with randomized assignment at the cluster level and treatment noncompliance at the unit level. For example, villagers or towns might be assigned to treatment and control, but residents may choose to not comply with their assigned treatment status. For example, in the state of Andhra Pradesh, the state government sought to evaluate the use of biometric smartcards to deliver payments from antipoverty programs. Smartcard payments were randomized at the village level, but residents could choose to comply or not. In some villages, more than 90% of residents complied with the treatment, while in other locations fewer than 15% of the residents complied. When noncompliance is present, investigators may choose to focus attention on either intention to treat effects or the causal effect among the units that comply. When analysts focus on effects among compliers, the instrumental variables framework can be used to evaluate identify causal effects. We first review extant methods for instrumental variable estimators in clustered designs which depend on assumptions that are often unrealistic in applied settings. In response, we develop a method that allows for possible treatment effect heterogeneity that is correlated with cluster size and uses finite sample variance estimator. We evaluate these methods using a series of simulations and apply them to data from an evaluation of welfare transfers via smartcard payments in India.

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1 Introduction

In many policy settings, randomized trials are used to evaluate policy. Randomized trials allow analysts to rule out that causal effect estimates are confounded with pretreatment differences or selection biases amongst subjects. In education and public health applications, investigators often use clustered randomized trials (CRTs), where treatments are applied to groups of individuals rather than individuals. While clustered treatment assignment reduces power, it allows for arbitrary patterns of treatment spillover for units within the same cluster (Imbens and Wooldridge 2008).

This design feature is critical in settings where interactions between units within clusters are difficult to prevent. The state of Andhra Pradesh conducted an intervention that is typical of many CRTs. The goal was to evaluate the effectiveness of using a biometric payment system to deliver social welfare payments to recipients in India (Muralidharan et al. 2016). The smartcard payment system used a network of locally hired, bank-employed staff to biometrically authenticate beneficiaries and make cash payments in villages. Such systems are designed to reduce the time it takes for payments to reach payees and reduce the chance that payments are siphoned off to someone other than the payee. The intervention was randomly assigned at the village level. Out of 157 villages, 112 were assigned to the intervention. Assignment at the village level prevented treatment spillovers that would have been difficult to prevent if assigned at the individual level.

For villages assigned to the control condition, smartcard payments were not available. For those in treated villages, recipients first had to enroll in the smartcard program by submitting biometric data (typically all ten fingerprints) and digital photographs. Beneficiaries were then issued a physical smartcard that included their photograph and an embedded electronic chip storing biographic and biometric data; the card was also linked to newly created bank accounts. Government officials conducted enrollment campaigns to collect the biometric data and issue the cards. The government contracted with banks to manage payments, and these banks in turn contracted with customer service providers (CSPs) to manage the accounts and
travel to villages to deliver payments. The system was used to deliver payments from two large welfare programs: the National Rural Employment Guarantee Scheme (NREGS), and Social Security Pensions (SSP). The first is a work-fare program that guarantees every rural household 100 days of paid employment each year (Dutta et al. 2010). SSP complements the first program by providing income support to those who are unable to work. Beneficiaries used the smartcards to collect payments from CSPs by inserting them into a point-of-service device. The device reads the card and retrieves account details. Payees were prompted for one of ten fingers, chosen at random, to be scanned. The device compares this scan with the records on the card, and authorizes a transaction if they match. Once authorized the amount of cash requested is disbursed, and the device prints out a receipt.

Muralidharan et al. (2016) conducted the original evaluation but did not account for noncompliance in the analysis. In the smartcard evaluation, as is true in many CRTs with human subjects, compliance with the treatment assignment varied. Beneficiaries were designated as compliant if they used the smartcard one or more times to collect payments. In some villages, 90% or more of the beneficiaries complied with the treatment, while in many villages classroom less 10% of the recipients complied with their assigned treatment assignment. While noncompliance is common in these designs, the statistical literature contains relatively few results on the analysis of CRTs with noncompliance. For example, two widely used texts on CRTs do not mention noncompliance at all (Hayes and Moulton 2009; Donner and Klar 2000). Noncompliance in CRTs can be analyzed within the instrumental variables (IV) framework. IV allows investigators to estimate the average causal effect among the subpopulation that complied with their assigned treatment (Angrist et al. 1996). Jo et al. (2008b) and Schochet (2013) both extended the IV estimation framework to CRTs.

In this paper, we study and develop methods of estimation and inference for CRTs within noncompliance. We review the two most commonly used methods of estimation and inference: one based on group level averages, and a second using unit level data. We then propose an alternatives. The first is designed to provide accurate finite sample inferences,
which may be useful given the relatively small sample sizes in many CRT applications. Using a simulation study, we compare these different methods of estimation and inference. We conclude with an analysis of the smartcard trial which motivates the study.

2 Review of Clustered Randomized Trials and Non-compliance

2.1 Notation and Setup

Suppose there are $J$ total clusters $j = 1, \ldots, J$. For each cluster $j$, there are $n_j$ units, indexed by $i = 1, \ldots, n_j$ and we have $n = \sum_{j=1}^{J} n_j$ total units. A fixed $m$ number of clusters are assigned treatment and the other $J - m$ clusters are assigned control. The treatment assignment is uniform within each cluster; if cluster $j$ is assigned treatment, all $n_j$ individuals in cluster $j$ are assigned treatment. Let $Z_j = 1$ indicate that cluster $j$ received treatment and $Z_j = 0$ indicates that cluster $j$ received control. Let $Z = (Z_1, \ldots, Z_J)$ be the collection of $Z_j$s and let $z = (z_1, \ldots, z_J)$ be one possible value of treatment assignment from a set of treatment assignments $\mathcal{Z} = \{z \in \{0, 1\}^J \mid \sum_{j=1}^{J} Z_j = m\}$. Let $D_{ji}$ indicate the observed compliance to treatment assignment for individual $i$ in cluster $j$ and $Y_{ji}$ denotes the observed outcome for individual $i$ in cluster $j$. We can also let $Z_{ji}$ to be the treatment assignment of individual $i$ in cluster $j$, but because treatment assignment is uniform within each cluster and to minimize notational burden, this notation is used sparingly.

We use the potential outcomes approach laid out in Neyman (1923) and Rubin (1974) to define causal effects. Let $D^{(1)}_{ji}$ and $D^{(0)}_{ji}$ indicate the potential compliances to treatment assignment of individual $i$ in cluster $j$ if cluster $j$ received treatment $Z_j = 1$ or control $Z_j = 0$, respectively. Let $Y_{ji}^{(1,D^{(1)}_{ji})}$ and $Y_{ji}^{(0,D^{(0)}_{ji})}$ indicate the potential outcomes of individual $i$ in cluster $j$ if cluster $j$ received treatment $Z_j = 1$ and control $Z_j = 0$, respectively. Because only one of the two potential outcomes can be observed for treatment assignment $Z_j$, the
the potential outcomes and observed values can be written as

\[ Y_{ji} = Z_j Y_{ji}^{(1,D_{ji}^{(1)})} + (1 - Z_j) Y_{ji}^{(0,D_{ji}^{(0)})}, \quad D_{ji} = Z_j D_{ji}^{(1)} + (1 - Z_j) D_{ji}^{(0)} \]

Let \( F = \left\{ Y_{ji}^{(z,D_{ji}^{(z)})}, D_{ji}^{(z)}, z = 0, 1, j = 1, \ldots, J, i = 1, \ldots, n_j \right\} \) be the set containing all values of the potential outcomes. Note that \( F \) forms a parameter space since knowing \( F \) allows us to characterize the distribution of the observables, \( Y_{ji} \) and \( D_{ji} \). We note that the potential outcomes notation implicitly assumes that there is no inter-cluster interference and the stable unit treatment value assumption (SUTVA) holds (Rubin 1986). That is, the treatment assignment or compliance of any individual in cluster \( j \) cannot affect the potential outcomes and the potential compliances of individuals in cluster \( k \) and is commonly assumed in clustered randomized trials; see Frangakis et al. (2002); Small et al. (2008b); Jo et al. (2008a); Imai et al. (2009); Schochet and Chiang (2011) and Middleton and Aronow (2015).

We also define summary measures for each cluster \( j \) in the form of sums and averages. Specifically, for each cluster \( j \), let \( Y_j = \sum_{i=1}^{n_j} Y_{ij} \) and \( D_j = \sum_{i=1}^{n_j} D_{ij} \) indicate the sums of individual outcomes and compliances, respectively. Similarly, for each cluster \( j \), let \( \bar{Y}_j = Y_j/n_j \) and \( \bar{D}_j = D_j/n_j \) indicate the average of individual outcomes and compliances, respectively.

For each cluster \( j \) and treatment indicator \( z \in \{0, 1\} \), let \( Y_j^{(z,D_{ji}^{(z)})} = \sum_{i=1}^{n_j} Y_{ij}^{(z,D_{ji}^{(z)})} \) and \( D_j^{(z)} = \sum_{i=1}^{n_j} D_{ji}^{(z)} \) indicate the sums of potential outcomes and compliance values, respectively, when cluster \( j \) is assigned treatment value \( z \).

### 2.2 Assumptions

We review the assumptions underlying CRTs with non-compliance and we begin by discussing assumptions specific to CRTs; see Hayes and Moulton (2009); Donner and Klar (2000). We assume that each cluster is randomly assigned to treatment \( Z_j = 1 \) or control \( Z_j = 0 \) and that the probability of receiving both is non-zero.

**Assumption 1** (Cluster Randomization). Given \( J \) clusters, \( m \) clusters \((0 < m < J)\) are
randomly assigned treatment and the other $J - m$ clusters are assigned control.

\[ P(Z = z \mid F, Z) = P(Z = z \mid Z) \quad \text{and} \quad P(Z = z \mid Z) = \binom{J}{m} \]

Assumption 1 holds by the design of CRTs since clusters are randomly assigned treatment and/or control. Also, with Assumption 1 and SUTVA, we can identify and unbiasedly estimate the average causal effect of treatment assignment on the outcome, denoted as $\mu_Y$ and commonly referred to as the intent-to-treat (ITT) causal effect, and the average causal effect of the treatment assignment on compliance, denoted as $\mu_D$ and commonly referred to as the average compliance rate.

\[ \mu_Y = \frac{1}{n} \sum_{j=1}^{J} \sum_{i=1}^{n_j} Y_{ji}^{(1)} - Y_{ji}^{(0)}, \quad \mu_D = \frac{1}{n} \sum_{j=1}^{J} \sum_{i=1}^{n_j} D_{ji}^{(1)} - D_{ji}^{(0)} \]

The unbiased estimators for the parameters $\mu_Y$ and $\mu_D$, denoted as $\hat{\mu}_Y$ and $\hat{\mu}_D$, respectively, are difference-in-means estimators of sums between treated and control clusters

\[ \hat{\mu}_Y = \frac{1}{n} \left( \frac{J}{m} \sum_{j=1}^{J} Z_j Y_j - \frac{J}{J - m} \sum_{j=1}^{J} (1 - Z_j) Y_j \right), \quad \hat{\mu}_D = \frac{1}{n} \left( \frac{J}{m} \sum_{j=1}^{J} Z_j D_j - \frac{J}{J - m} \sum_{j=1}^{J} (1 - Z_j) D_j \right) \]

where $E[\hat{\mu}_Y \mid F, Z] = \mu_Y$ and $E[\hat{\mu}_D \mid F, Z] = \mu_D$.

Next, we review terms and assumptions that are specific to noncompliance; see Imbens and Angrist (1994); Angrist et al. (1996); Hernán and Robins (2006), Baiocchi et al. (2014) and references therein for more information. We assume that the average compliance rate is non-zero and no individual systematically defies his/her treatment assignment, also referred to as monotonicity Imbens and Angrist (1994).

**Assumption 2** (Non-Zero Causal Effect). The treatment assignment $z_j$ induces, on average, changes in compliance $D_{ij}$, i.e. $\mu_D \neq 0$.

**Assumption 3** (Monotonicity). There is no individual who systematically defies the treatment assignment, i.e. $D_{ji}^{(0)} \leq D_{ji}^{(1)}$ for all $ij$. 
We briefly make the following comments about Assumptions 2 and 3. If Assumption 1 holds, Assumption 2 can be verified from data by using the estimator $\hat{\mu}_D$. However, Assumption 3 cannot be verified from data because it requires observing both potential compliance values $D_{ji}^{(0)}$ and $D_{ji}^{(1)}$ for every individual.

Under Assumption 3, each individual $i$ in cluster $j$ is categorized into four types, always-takers, never-takers, compliers, and defiers, based on his/her potential compliances $D_{ji}^{(0)}$ and $D_{ji}^{(1)}$ (Angrist et al. 1996). Always-takers are individuals where $D_{ji}^{(1)} = D_{ji}^{(0)} = 1$; these individuals would use the smartcard payment system irrespective of the treatment for his or her village. Never-takers are individuals where $D_{ji}^{(1)} = D_{ji}^{(0)} = 0$; these individuals would never use smartcard payments irrespective of the village treatment assignments. Compliers are individuals where $D_{ji}^{(1)} = 1$ and $D_{ji}^{(0)} = 0$; these individuals comply with their treatment assignments. That is, they use smartcard payments only when their village is assigned to treatment. Under this categorization of individuals, Assumption 3 states that there are no defiers in the study population: no one who uses the smartcard system when their village is not assigned or who does not use the smartcard system when their village is assigned. Assumption 3 can hold by experimental design if units receiving the control cannot obtain the treatment, i.e. $D_{ji}^{(0)} = 0$; this is known as one-sided noncompliance in the literature. In the smartcard intervention, noncompliance was one-sided, since the key technology for payment verification was not made available to control villages.

Next, Assumptions 2 and 3 do not imply each other. For instance, if Assumption 3 holds, Assumption 2 may fail to hold if all individuals have $D_{ji}^{(1)} = D_{ji}^{(0)}$. Conversely, if Assumption 2 holds, Assumption 3 may fail to hold if the proportion of individuals with $D_{ji}^{(1)} = 1$ and $D_{ji}^{(0)} = 0$ is higher than the proportion of individuals with $D_{ji}^{(1)} = 0$ and $D_{ji}^{(1)} = 1$.

Finally, we assume that conditional on compliance, the initial treatment assignment has no impact on the outcome. This is commonly known as the no direct effect or the exclusion restriction assumption.

**Assumption 4** (Exclusion Restriction). Conditional on the compliance value $d_{ji}$, the treat-
ment assignment has no effect on the outcome.

\[ \forall d_{ji} \in \{0, 1\}, \quad Y_{ji}^{(1,d_{ji})} = Y_{ji}^{(0,d_{ji})} = Y_{ji}^{(d_{ji})} \]

Similar to Assumption 3, Assumption 4 cannot be verified by data because it requires observing both potential outcomes \( Y_{ji}^{(1,d_{ji})} \) and \( Y_{ji}^{(0,d_{ji})} \). Also, designing an experiment where Assumption 4 holds can be difficult. For example, the exclusion restriction is often more credible in double-blinded trials in medicine (Hernán and Robins 2006). In general, Assumption 4’s validity is judged based on substantive knowledge about the treatment and the outcome. For example, in this study, for Assumption 4 to hold, it must be the case that being assigned to smartcard payments has no direct effect on the outcome except through exposure to the intervention. That is, being assigned to the smartcard payment system can only reduce payment delays through actual use of smartcard payments. This seems likely to hold, since the reduced payment times are directly facilitated by the technology that underlies the intervention.

### 2.3 Complier Average Treatment Effect

In CRTs with non-compliance, the causal estimand of interest is the population complier average causal effect (CACE), which is the average effect of actually taking the treatment (i.e. when \( D_{ji} \) is set to 1) versus not taking the treatment (i.e. when \( D_{ji} \) is set to 0) among compliers. We denote the population CACE as \( \tau \)

\[
\tau = \frac{\sum_{j=1}^{J} \sum_{i=1}^{n_j} (Y_{ji}^{(1)} - Y_{ji}^{(0)}) I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)}.
\]

The parameter \( \tau \) is also referred to as a local causal effect because it only describes the causal effect among a subgroup of individuals in the population, specifically the compliers. Thus, we can describe the effect of using smartcard payments among those recipient who were
monotonically induced to use smartcards when exposed to the intervention. Prior literature has shown that under assumptions $1$-$4$, $\tau$ is identified by taking the ratio of the ITT effect $\mu_Y$ with the compliance rate $\mu_D$ (Imbens and Angrist 1994; Angrist et al. 1996).

In the context of a CRT, we can decompose $\tau$ to understand how the CACE varies by cluster. Let $n_{CO,j} > 0$ be the number of compliers in cluster $j$, $n_{CO}$ be the total number of compliers across all clusters, and $\tau_j$ be the CACE for cluster $j$, i.e.

$$n_{CO,j} = \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0), \quad n_{CO} = \sum_{j=1}^{J} n_{CO,j}, \quad \tau_j = \frac{\sum_{i=1}^{n_j} (Y_{ji}^{(1)} - Y_{ji}^{(0)}) I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)}{n_{CO,j}}$$

Then, we can rewrite $\tau$ as a weighted average of cluster-specific CACEs where the weights, denoted as $(w_1, \ldots, w_J)$ are functions of the number of compliers in each cluster.

$$\tau = \sum_{j=1}^{J} w_j \tau_j, \quad w_j = \frac{n_{CO,j}}{\sum_{j=1}^{J} n_{CO,j}} \quad (2)$$

If the CACEs are constant across clusters so that $\tau_j = \tau_k$ for $j \neq k$, the population CACE $\tau$ is equal to the cluster CACE $\tau_j$. It is common to assume constant effects in econometrics where a structural model between the outcome $Y$ and the compliance $D$ is assumed, for instance $Y_{ji} = \beta_0 + \beta_1 D_{ji} + \epsilon_{ji}$ where $\beta_0, \beta_1$ are parameters, $\epsilon_{ji}$ is a mean-zero error term that is independent of the treatment assignment variable $Z_{ji}$, and algebra shows that $\beta_1 = \tau$ (Holland 1988; Hernán and Robins 2006; Wooldridge 2010; Kang et al. 2016). Finally, we make a technical note that the decomposition in equation $2$ will still hold if some clusters have no compliers, i.e. clusters $j$ where $n_{CO,j} = 0$; instead of summing all $J$ clusters in equation $2$, we would simply sum over clusters $j$ with $n_{CO,j} > 0$ and our results below will hold. For ease of exposition, we assume $n_{CO,j} > 0$ for every cluster, which is reasonable in practice and holds in our data.

We can formalize testing for $\tau$ as follows. For any $\tau_0 \in \mathbb{R}$, let $H_0$ denote the null hypothesis
for $\tau$

$$H_0 : \tau = \tau_0, \quad \tau_0 \in \mathbb{R} \tag{3}$$

The null hypothesis $H_0$ in equation (3) is a composite null hypothesis because there are several values of $F$ for which the null holds. In other words, $H_0$ is not a sharp null hypothesis (Fisher 1935); under a sharp null, we would be able to specify exactly the unobserved potential outcomes. In fact, the sharp null of no ITT effect, $Y_i^{(1,D_i^{(1)})} = Y_i^{(0,D_i^{(0)})}$ for all $i$ implies $H_0 : \tau = 0$, but the converse is not necessarily true; there are other values of $F$ that satisfies the null hypothesis $H_0 : \tau = 0$. Next, we describe extant procedures for learning about CACE in CRTs with non-compliance.

3 Extant Methods for Analysis of CRTs with Noncompliance

We review two popular methods of inferring CACE in CRTs with non-compliance. The first method outlined in Schochet (2013) and Hansen and Bowers (2009) aggregates individual observations at the cluster level and conducts a statistical analysis with the aggregate cluster-level quantities. However, the methods in Hansen and Bowers (2009) are only designed for binary outcomes. The second method applies two-stage least squares (TSLS) to the unit level data and uses a robust standard error to account for within cluster correlations. A third approach, which we do not explore, is to implement IV methods using random effects models (Jo et al. 2008b; Small et al. 2006). These methods are used more infrequently.

3.1 The Cluster-Level Method

Cluster-level methods analyzes CRTs with non-compliance analyze at the cluster level. Specifically, the investigator take averages or sums of individual outcomes and compliances within the clusters, and then treat the study as though it only took place at the cluster level.
and $J$ essentially servers as the effective population size (Schochet 2013; Hansen and Bowers 2009). Formally, given cluster-level average outcomes $\bar{Y}_j$ and compliances $\bar{D}_j$, we define the average outcomes and compliances by treatment status

$$
\bar{Y}_T = \frac{1}{m} \sum_{j=1}^{J} Z_j \bar{Y}_j, \quad \bar{Y}_C = \frac{1}{J - m} \sum_{j=1}^{J} (1 - Z_j) \bar{Y}_j
$$

$$
\bar{D}_T = \frac{1}{m} \sum_{j=1}^{J} Z_j \bar{D}_j, \quad \bar{D}_C = \frac{1}{J - m} \sum_{j=1}^{J} (1 - Z_j) \bar{D}_j
$$

Here, $\bar{Y}_T$ and $\bar{Y}_C$ represent the average $\bar{Y}_j$ among treated and control clusters, respectively. Similarly, $\bar{D}_T$ and $\bar{D}_C$ represent the average $\bar{D}_j$ among treated and control clusters. The four quantities, $\bar{Y}_T, \bar{Y}_C, \bar{D}_T$ and $\bar{D}_C$, are used to define a Wald-like estimator for $\tau$, which we denote as $\hat{\tau}_{CL}$:

$$
\hat{\tau}_{CL} = \frac{\bar{Y}_T - \bar{Y}_C}{\bar{D}_T - \bar{D}_C}. \tag{4}
$$

The numerator $\hat{\tau}_{cl}$ in equation (4) is intended to capture the ITT effect of the treatment assignment on the outcome. The denominator $\hat{\tau}_{cl}$ in equation (4) is intended to capture the effect of treatment assignment on compliance.

For testing the null hypothesis of $\tau$ in equation (3), Schochet (2013) use a Taylor approximation of the estimator $\hat{\tau}_{CL}$. Specifically, the estimated variance for $\bar{Y}_T - \bar{Y}_C$ and $\bar{D}_T - \bar{D}_C$ are

$$
\hat{\text{Var}}(\bar{Y}_T - \bar{Y}_C) = \frac{JS_Y^2}{m(J - m)}, \quad \hat{\text{Var}}(\bar{D}_T - \bar{D}_C) = \frac{JS_D^2}{m(J - m)}
$$

where

$$
S_Y^2 = \frac{\sum_{j=1}^{J} Z_j (\bar{Y}_j - \bar{Y}_T)^2 + \sum_{j=1}^{J} (1 - Z_j) (\bar{Y}_j - \bar{Y}_C)^2}{J - 2}
$$

$$
S_D^2 = \frac{\sum_{j=1}^{J} Z_j (\bar{D}_j - \bar{D}_T)^2 + \sum_{j=1}^{J} (1 - Z_j) (\bar{D}_j - \bar{D}_C)^2}{J - 2}
$$
Additionally, the estimated covariance between $Y_T - Y_C$ and $D_T - D_C$ is
\[
\hat{\text{Cov}}(Y_T - Y_C, D_T - D_C) = \frac{\sum_{j=1}^{J} Z_j (Y_j - Y_T)(D_j - D_T)}{m^2} + \frac{\sum_{j=0}^{J} (1 - Z_j)(\bar{Y}_j - \bar{Y}_C)(\bar{D}_j - \bar{D}_C)}{(J - m)^2}
\]

Using the estimated variances/covariances, Schochet (2013) proposes the following estimator for the variance of $\hat{\tau}_{CL}$ based on the Delta Method of estimating variance of ratios like $\hat{\tau}_{CL}$.
\[
\hat{\text{Var}}(\hat{\tau}_{CL}) = \frac{\text{Var}(Y_T - Y_C)}{(D_T - D_C)^2} + \hat{\tau}_{CL}^2 \frac{\text{Var}(D_T - D_C)}{(D_T - D_C)^2} - 2\hat{\tau}_{CL} \frac{\hat{\text{Cov}}(Y_T - Y_C, D_T - D_C)}{(D_T - D_C)^2}
\]

Also, for any $\alpha \in (0, 1)$, the corresponding $1 - \alpha$ confidence interval for $\tau$ is
\[
\hat{\tau}_{CL} \pm z_{1-\alpha/2} \sqrt{\hat{\text{Var}}(\hat{\tau}_{CL})} \quad (5)
\]

where $z_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of the standard Normal distribution. If $\tau_0$ in $H_0$ is included in the interval, then $H_0$ is retained at the $\alpha$ level. Otherwise, $H_0$ is rejected in favor of the two-sided alternative $H_1 : \tau \neq \tau_0$. While this approach to testing $H_0$ is straightforward, its validity, say whether the $1 - \alpha$ confidence interval in equation (5) actually covers the true value $\tau$ $1 - \alpha$ times, depends on the asymptotic argument where (i) the number of clusters $J$ is growing to infinity and (ii) the denominator of $\hat{\tau}_{CL}$ being far away from zero as $J \to \infty$.

It is well known that the Delta Method can perform poorly when compliance rates are low (Bound et al. 1995; Kang et al. 2017). See Hansen and Bowers (2009) for a sample theoretic approach that is closer to the methods we outline below.

### 3.2 The Unit-Level Method

Unit-level methods analyze CRTs with non-compliance at the unit level using individual measurements rather than cluster level using summary statistics in Section 3.1. For inference, unit level methods use robust variance estimation methods that take into account intra-cluster correlations. This approach is a generalization of clustered standard errors developed...
by Liang and Zeger (1986).

Here, point estimation of the CACE relies on two-stage least squares (TSLS) applied to the unit level outcomes, compliance indicators, and group-level treatment assignments. Formally, we define the the vectorized outcome variable \( Y = (Y_{11}, Y_{12}, \ldots, Y_{1n_1}, Y_{21}, \ldots, Y_{Jn_J}) \), the vectorized compliance variable \( D = (D_{11}, D_{12}, \ldots, D_{1n_1}, D_{21}, \ldots, D_{Jn_J}) \), and the vectorized treatment assignment \( Z = (Z_{11}, Z_{12}, \ldots, Z_{1n_1}, Z_{21}, \ldots, Z_{Jn_J}) \). We let \( \hat{D} \) be the predicted compliance vector \( D \) based on regressing \( Z \) on \( D \) with an intercept term. Then, the TSLS estimator of \( \tau \), denoted as \( \hat{\tau}_{\text{TSLS}} \), is the estimated coefficient of \( \hat{D} \) from running an ordinary least squares (OLS) regression between \( \hat{D} \) and \( Y \) with an intercept. Explicitly, this two-stage process can be expressed as:

\[
\hat{\tau}_{\text{TSLS}} = \frac{\sum_{j=1}^{J} Z_j Y_j - \sum_{j=1}^{J} (1-Z_j) Y_j}{\sum_{j=1}^{J} Z_j n_j - \sum_{j=1}^{J} (1-Z_j) n_j}
\]

For testing the null hypothesis of \( \tau \) in equation (3), let \( u_{ji} = Y_{ji} - D_{ji} \hat{\tau}_{\text{TSLS}} \) denote the residual and \( u_j = \sum_{i=1}^{n_j} u_{ji} \). Also, let \( \hat{D}_j = \sum_{i=1}^{n_j} \hat{D}_{ji} \), \( \hat{D}_j^2 = \sum_{i=1}^{n_j} \hat{D}_{ji}^2 \), and \( \hat{D}_j u_j = \sum_{i=1}^{n_j} \hat{D}_{ji} u_{ji} \). Then, following Liang and Zeger (1986); Wooldridge (2010) and Cameron and Miller (2015), the estimated cluster-robust standard error is

\[
\hat{\text{Var}}(\hat{\tau}_{\text{TSLS}}) = \frac{n \sum_{j=1}^{J} \hat{D}_j^2 (\sum_{j=1}^{J} u_{ji}^2) + n^2 (\sum_{j=1}^{J} \hat{D}_j u_j)^2 - 2n (\sum_{j=1}^{J} \hat{D}_j) \left( \sum_{j=1}^{J} u_j \hat{D}_j u_j \right)}{\left( n \sum_{j=1}^{J} \hat{D}_j^2 - (\sum_{j=1}^{J} \hat{D}_j)^2 \right)^2}
\]

Also, standard econometric arguments (see Chapter 5.2 of Wooldridge (2010)) can be used to construct a \( 1 - \alpha \) confidence interval for \( \tau \)

\[
\hat{\tau}_{\text{TSLS}} \pm z_{1-\alpha/2} \sqrt{\hat{\text{Var}}(\hat{\tau}_{\text{TSLS}})}
\]
An advantage of this approach is that standard statistical software can be used. However, the validity of the confidence interval in equation (8) relies on the same two sets of assumptions in Section 3.1: (i) the number of clusters $J$ is going to infinity and (ii) the denominator of $\hat{\tau}_{TSLS}$ is far away from zero as $J \to \infty$. Prior simulation studies suggest that $J \geq 40$ is necessary for the two set of assumptions to be reasonable (Angrist and Pischke 2009).

4 Identification of CACE with Existing Methods

4.1 Fixed Population

While the two methods in Sections 3.1 and 3.2 with their respective point estimators, $\hat{\tau}_{CL}$ and $\hat{\tau}_{TSLS}$, claim to identify the CACE, a closer examination reveals that this is only true in very narrow settings where (i) the number of units in all clusters are the same or (ii) cluster-level CACE is identical across all clusters. In this section, we explore the identifiability of $\hat{\tau}_{CL}$ and $\hat{\tau}_{TSLS}$ when the population $F$ is assumed to be fixed.

The two propositions below study the identification of $\hat{\tau}_{CL}$, the estimator of $\tau$ based on looking at the CRT data at the cluster level, and $\hat{\tau}_{TSLS}$, the estimator of $\tau$ based on using TSLS.

Proposition 1. For each cluster $j$, define the following weights

$$w_{CL,j} = \frac{n_{CO,j}}{\sum_{j=1}^{J} \frac{n_{CO,j}}{n_j}},$$

where $n_{CO,j}$ is the number of compliers in cluster $j$. If Assumptions 1-4 hold, $\hat{\tau}_{CL}$ identifies a weighted complier average treatment effect

$$\tau_{CL} = \frac{E[\bar{Y}_T - \bar{Y}_C \mid F, Z]}{E[D_T - D_C \mid F, Z]} = \sum_{j=1}^{J} w_{CL,j} \cdot \tau_j$$

where $\tau_j$ is the cluster-level complier average treatment effect.
Proposition 2. For each cluster \( j \), define the following weights

\[
    w_{\text{TSLS},j} = \frac{n_{\text{CO},j}(n - n_j)}{\sum_{j=1}^{J} n_{\text{CO},j}(n - n_j)}
\]

where \( n_{\text{CO},j} \) is the number of compliers in cluster \( j \). If Assumptions 1-4 hold, \( \hat{\tau}_{\text{tsls}} \) identifies a weighted complier average treatment effect.

\[
    \tau_{\text{TSLS}} = \frac{E \left[ \sum_{j=1}^{J} (1 - Z_j) n_j \sum_{j=1}^{J} Z_j Y_j - \sum_{j=1}^{J} Z_j n_j \sum_{j=1}^{J} (1 - Z_j) Y_j \mid F, Z \right]}{E \left[ \sum_{j=1}^{J} (1 - Z_j) n_j \sum_{j=1}^{J} Z_j D_j - \sum_{j=1}^{J} Z_j n_j \sum_{j=1}^{J} (1 - Z_j) D_j \mid F, Z \right]} = \sum_{j=1}^{J} w_{\text{TSLS},j} \tau_j
\]

where \( \tau_j \) is the cluster-level complier average treatment effect.

Propositions 1 and 2 show that the two primary methods of analyzing CRT data identify \( \tau_{\text{CL}} \) and \( \tau_{\text{TSLS}} \), which may not always equal the CACE because of how each method weights cluster-level complier average treatment effects. The cluster-level \( \hat{\tau}_{\text{CL}} \) averages the cluster-level CACE by the proportion of compliers, \( n_{\text{CO},j}/n_j \); it does not take into consideration the size of each cluster. The individual-level \( \hat{\tau}_{\text{TSLS}} \) averages the cluster-level CACE by the product of the number of compliers per cluster and the number of individuals in other clusters, via \( n_{\text{CO},j}(n - n_j) \).

Tables 1 and 2 provide numerical demonstration of the consequences of the weights on identifying \( \tau \) in a CRT with \( J = 3 \) clusters. In Table 1, the compliance rates across the three clusters are identical and set to 50% whereas in Table 2, the compliance rates differ across the three clusters, ranging from 20% to 80%. In both tables, the size of the clusters \( n_j \) vary and the complier average treatment effect \( \tau_j \) vary.

In Table 1, when the compliance rates are identical, the cluster-based method \( \hat{\tau}_{\text{CL}} \) gives disproportionately large weights to small clusters of size 10, \( j = 2, 3 \), and consequently, all clusters, despite their differences in size, are weighted equally with weights \( w_{\text{CL},1} = w_{\text{CL},2} = w_{\text{CL},3} = 1/3 \). The TSLS method \( \hat{\tau}_{\text{TSLS}} \) also gives disproportionately large weights to small clusters. But, unlike the cluster-based \( \hat{\tau}_{\text{CL}} \), it still takes the size of each cluster
into consideration when weighing cluster-level CACE and thus, each cluster gets slightly different weights, $w_{TSLS,1} \approx 0.47$, and $w_{TSLS,2} = w_{TSLS,3} \approx 0.26$. Finally, the true CACE $\tau$ weights the cluster-level CACE proportional to the number of compliers in each cluster with weights $w_1 = 0.8, w_2 = w_3 = 0.1$. Both the cluster-level method and the TSLS method under-estimate the true CACE. However, under the scenario where the compliance rates are identical across clusters, the two-stage least squares estimator preserves some elements of the weights associated with the true CACE $\tau$ in that clusters of identical size receive identical weights and that the absolute number of units in each cluster are taken into account. In contrast, the cluster-level method ignores the size of the cluster and weights each cluster equally.

Table 1: A numerical example of a CRT with three clusters $J = 3$ and identical compliance rates.

| Cluster Size: $n_j$ | Number of Compliers: $n_{CO,j}$ (% compl. rate) | CACE per Cluster: $\tau_j$ |
|-------------------|-------------------------------|--------------------------|
| $j = 1$ | $n_1 = 80$ | $n_{CO,1} = 40$ (50%) | $\tau_1 = 1$ |
| $j = 2$ | $n_2 = 10$ | $n_{CO,2} = 5$ (50%) | $\tau_2 = 2$ |
| $j = 3$ | $n_3 = 10$ | $n_{CO,3} = 5$ (50%) | $\tau_3 = 1.5$ |
| $J = 3$ | $n = 100$ | $n_{CO} = 50$ | $\tau = 1\left(\frac{40}{50}\right) + 2\left(\frac{5}{50}\right) + 1.5\left(\frac{5}{50}\right) = 1.15$ |
| | | | $\tau_{CL} = 1\left(\frac{4.5}{1.5}\right) + 2\left(\frac{4.5}{1.5}\right) + 1.5\left(\frac{4.5}{1.5}\right) = 1.5$ |
| | | | $\tau_{TSLS} = 1\left(\frac{800}{1700}\right) + 2\left(\frac{450}{1700}\right) + 1.5\left(\frac{450}{1700}\right) \approx 1.4$ |

Note: $\tau$ is the true CACE, $\tau_{CL}$ is the identified value based on the cluster-level estimator $\hat{\tau}_{CL}$, and $\tau_{TSLS}$ is the identified value based on TSLS $\hat{\tau}_{TSLS}$.

Table 2 examines the same scenario as Table 1 except that the compliance rates vary across cluster. In particular, small clusters have higher compliance rates than large clusters compliance rates (80% versus 10%), but the number of compliers remain identical across clusters. Similar to Table 1, we see that both the cluster-based method $\hat{\tau}_{CL}$ and the TSLS $\hat{\tau}_{TSLS}$ tend to up-weigh the smaller clusters and down-weigh the large clusters in comparison to $\tau$. We also see that similar to Table 1, the TSLS method tends to be closer to the true
CACE \( \tau \) than the cluster-based method; however both methods tend to overestimate the true CACE.

Table 2: A numerical example of a CRT with three clusters \( J = 3 \) and different compliance rates.

| Cluster Size: \( n_j \) | Number of Compliers: \( n_{CO,j} \) (% compl. rate) | CACE per Cluster: \( \tau_j \) |
|------------------------|---------------------------------|-----------------|
| \( j = 1 \) | \( n_1 = 80 \) \( n_{CO,1} = 8 \) (10%) | \( \tau_1 = 1 \) |
| \( j = 2 \) | \( n_2 = 10 \) \( n_{CO,2} = 8 \) (80%) | \( \tau_2 = 2 \) |
| \( j = 3 \) | \( n_3 = 10 \) \( n_{CO,3} = 8 \) (80%) | \( \tau_3 = 1.5 \) |
| \( J = 3 \) | \( n = 100 \) \( n_{CO} = 24 \) | \( \tau = 1\left(\frac{5}{24}\right) + 2\left(\frac{5}{24}\right) + 1.5\left(\frac{8}{24}\right) = 1.5 \) |
| | \( \tau_{CL} = 1\left(\frac{0.1}{17}\right) + 2\left(\frac{0.8}{17}\right) + 1.5\left(\frac{0.8}{17}\right) \approx 1.71 \) |
| | \( \tau_{TSLS} = 1\left(\frac{320}{1760}\right) + 2\left(\frac{720}{1760}\right) + 1.5\left(\frac{720}{1760}\right) \approx 1.68 \) |

Note: \( \tau \) is the true CACE, \( \tau_{CL} \) is the identified value based on the cluster-level estimator \( \hat{\tau}_{CL} \), and \( \tau_{TSLS} \) is the identified value based on TSLS \( \hat{\tau}_{TSLS} \).

The following Corollary shows that we can identify the desired complier average treatment effect \( \tau \) using the cluster-based method \( \hat{\tau}_{CL} \) and the TSLS method \( \hat{\tau}_{TSLS} \) if either the clusters are of equal size or if the cluster-level CACE is identical across clusters.

**Corollary 1.** Suppose either condition below hold:

1. The number of units in each cluster \( n_j \) is identical across all clusters.
2. The complier average treatment effect per cluster \( \tau_j \) is homogeneous/identical across all clusters.

Then, \( \hat{\tau}_{CL} \) and \( \hat{\tau}_{TSLS} \) identify the complier average treatment effect \( \tau \).
econometrics in the form of a linear structural modeling assumption between $Y$ and $D$ where the coefficient associated with $D$ and $Y$ are assumed to be constant (Wooldridge 2010). We also note that the conditions stated in Corollary 1 are sufficient conditions for $\hat{\tau}_{CL}$ and $\hat{\tau}_{TSLS}$ to identify $\tau$.

In summary, under a fixed population $F$, the two popular methods for analyzing CRT data with non-compliance may not always identify the target causal parameter of interest, CACE ($\tau$). This can be seen by looking at the weights that make up both $\hat{\tau}_{CL}$, $w_{CL,j}$ and $\hat{\tau}_{TSLS}$, $w_{TSLS,j}$, which differ from the weights of $\tau$ in equation (2). Indeed, only under the restrictive conditions stated in Corollary 1 can the two popular methods actually identify the CACE.

4.2 Growing Population

The previous section discussed identification of $\tau$ under the case where the population $F$ remained fixed. However, for testing hypothesis about $\tau$ in equation (3), often one assumes the sample size $n \to \infty$ and consequently, $F$ is growing. Here, we consider whether the cluster-based and unit-based methods identifies the true $\tau$ in this asymptotic regime, even though these methods generally do not in the finite sample case.

In the CRT setting, there are primarily two ways in which we might imagine $n \to \infty$. The first way is where the number of clusters $J$ is getting larger, but the units per cluster $n_j$ remains bounded. This regime is the basis for justifying the testing properties of many estimators in the CRT literature, including both $\hat{\tau}_{CL}$ and $\hat{\tau}_{TSLS}$. This regime also serves as the basis for justifying the inferential properties of many causal estimators outside of the CRT literature; see Li and Ding (2017) for a recent review. In practice, this regime is relevant in CRT designs that use household as the cluster and the number of households is large compared to the number of individuals that make up the household. For example, this type of design is common in the political science literature on the effectiveness of voter mobilization strategies. See Gerber et al. (2008) for an example and Green et al. (2013) for
an overview.

The second was is a setting where the number of clusters \( J \) remains bounded (often fixed), but the units per clusters \( n_j \) is growing. This regime is not as common as the first type and inferential properties (e.g. confidence intervals) for many estimators in CRT, including both \( \hat{\tau}_{\text{CL}} \) and \( \hat{\tau}_{\text{TSLS}} \), typically break down under this regime. In practice, this regime might occur under an experimental design that was implemented at the state or county level. If each state in the U.S. serves as a cluster and we’re studying individuals within each state, typically the number of individuals that reside in a state is larger than the number of states in the U.S. Some CRTs are close to this regime. For example, [Hayes et al. (2014)] outlines a CRT design with 21 clusters with approximately 55,000 units in each cluster. However, for analysis 2,500 units are randomly sampled from each cluster. [Solomon et al. (2015)] describe a design with 12 clusters and 1000 units per cluster. A design of this type is more common in clustered observational studies, see [Acemoglu and Angrist (2000)] for one example.

Let \( p_{\text{CO},j} = n_{\text{CO},j}/n_j \) be the proportion of compliers in cluster \( j \). Proposition 3 examines the identification properties of both the cluster-level and unit-level methods, under the asymptotic regime of the first type where the number of clusters \( J \to \infty \) and the number of units per cluster remain \( n_j \leq B \) for all \( j \). Under this asymptotic regime, only the individual-level method always identify the CACE at the limit whereas the cluster-level method may fail to do so.

**Proposition 3.** Suppose Assumptions 1-4 hold. Consider the asymptotic regime for \( \mathcal{F} \) where (i) \( J \to \infty \), (ii) \( n_j \)s are bounded, (iii) \( \tau_j \)s are bounded. If \( \bar{p}_{\text{CO}} = \lim_{J \to \infty} \sum_{j=1}^{J} p_{\text{CO},j}/J \), the limiting average proportion of compliers across all clusters and \( \bar{n}_{\text{CO}} = \lim_{J \to \infty} \sum_{j=1}^{J} n_{\text{CO},j}/J \), the limiting average number of compliers across all clusters, exist, then \( \tau_{\text{CL}} \) has the following
asymptotic property.

\[
\lim_{J \to \infty} |\tau_{CL} - \tau| = \frac{1}{\hat{p}_{CO}n_{CO}} \lim_{J \to \infty} \frac{1}{J^2} \sum_{j<l} \hat{p}_{CO,j}\hat{p}_{CO,l}(n_l - n_j)(\tau_j - \tau_l)
\]

\[
\lim_{J \to \infty} \sup_{\mathcal{F}} |\tau_{CL} - \tau| > K
\]

for some constant \(K > 0\). Also, \(\tau_{TSLS}\) has the following (rate-sharp) asymptotic property.

\[
\sup_{\mathcal{F}} |\tau_{TSLS} - \tau| = O\left(\frac{1}{J}\right) \to 0
\]

Unlike the results in Propositions 1 and 2 where in finite samples, the two methods may not identify the true population CACE, Proposition 3 states that asymptotically, the identifying values \(\tau_{TSLS}\) always converge to the limiting \(\tau\) (if it exists). However, \(\tau_{CL}\) may not converge to the limiting \(\tau\) and in the worst case, it will never converge to it. One example non-convergence that may arise in practice is in household surveys where each household defines a cluster and the total number of people per household, \(n_j\), is bounded by \(B\); this type of design is studied under the growing \(J\) asymptotic regime in Proposition 3. Specifically, if \(B = 4\) and the distribution of the cluster size \(n_j\) is uniform between values 2, 4 with 50% compliers across all clusters and there are \(B\) distinct cluster-specific CACE where smaller clusters have higher CACE than larger cluster, say \(\tau_j = 2\) if \(n_j = 4\), \(\tau_j = 4\) if \(n_j = 2\), then

\[
\lim_{J \to \infty} |\tau_{CL} - \tau| = \frac{2}{3} \lim_{J \to \infty} \frac{1}{J^2} \sum_{j<l,n_l \neq n_j} 1 \approx \frac{1}{6}
\]

We can make the difference between \(\tau_{CL}\) and \(\tau\) arbitrary large by selecting different values of \(\tau_j\). However, \(\tau_{CL}\) will converge to \(\tau\) if conditions in Corollary 1 are satisfied. In short, using cluster-based methods, are sensitive to the design as well as the underlying (unobserved) heterogeneity of CACE across clusters.
Additionally, the result in Proposition 1 also has broader implications. First, the result in Proposition 3 underscores the role of asymptotic assumptions for identifying the target causal parameter \( \tau \) of interest, especially for \( \tau_{TSLS} \). Second, because there may be clustered study designs for which \( \tau_{CL} \) will never identify the CACE, this suggests that doing asymptotic inference using clustered-based method, say computing p-values or confidence intervals in Section 3.1, may be invalid (i.e. inflated Type I errors or incorrect coverage) since the target parameter \( \tau \) is never identified in the first place.

Next, Proposition 4 examines the identification properties of the cluster-based method and the unit-level method, in the asymptotic regime of the second type where the number of clusters \( J \) remains bounded and the number of units per cluster follows \( n_j \to \infty \).

**Proposition 4.** Suppose we are in the asymptotic regime where (i) \( J \) remains bounded, (ii) for every \( j \), \( n_j, n_{CO,j} \to \infty \) where \( n_{CO,j}/n_j \to p_{CO,j} \in (0, 1) \) and \( n_j/n_k \to \rho_{jk} \in [0, \infty) \) for every \( j \neq k \), and (iii) \( \tau_j \to \tau_{j,\infty} \). Then, the identifying values from the two methods, \( \tau_{CL} \) and \( \tau_{TSLS} \) converge to the following values

\[
\lim_{n \to \infty} |\tau_{CL} - \tau| = \sum_{j=1}^{J} \frac{\tau_{j,\infty} \sum_{l \neq j} p_{CO,j} p_{CO,l} (\rho_{lj} - 1)}{\left( \sum_{l=1}^{J} p_{CO,l} \right) \left( p_{CO,j} + \sum_{l \neq j} \rho_{lj} p_{CO,l} \right)},
\]

and

\[
\lim_{n \to \infty} |\tau_{TSLS} - \tau| = \sum_{q=1}^{L} \tau_{q,\infty} \sum_{k \neq q} \frac{p_{CO,q} p_{CO,k} (\rho_{kq} - 1)}{\left( \sum_{l \neq j} p_{CO,l} \rho_{lk} \rho_{jq} \right) \left( p_{CO,q} + \sum_{l \neq q} \rho_{lq} p_{CO,l} \right)}.
\]

Furthermore, we have that \( \lim_{n \to \infty} \sup_F |\tau_{CL} - \tau| > K \) and \( \lim_{n \to \infty} \sup_F |\tau_{TSLS} - \tau| > K \) for some constant \( K > 0 \).

Proposition 4 states that under the growing \( n_j \) asymptotic regime, both \( \tau_{CL} \) and \( \tau_{TSLS} \) may not converge to \( \tau \). For example, if we have \( J \) clusters where \( n_1 < n_2 < \ldots < n_J \) and \( \tau_{1,\infty} > \tau_{2,\infty} > \ldots > \tau_{J,\infty} \), the limiting values will be both positive and away from zero. However, following Corollary 1, if the cluster sizes are asymptotically identical so that \( \rho_{lj} = 1 \) for all \( l \neq j \), the two estimators will converge to the limiting values of \( \tau \). Also, in
the worst case, i.e. under the supremum limit, there is a data generating process whereby neither methods converge to the limiting value of \( \tau \).

In summary, both Propositions 3 and 4 demonstrate that in asymptotic settings where the number of clusters is growing and the number of units within each cluster are bounded, the finite-sample identification problems of the cluster-based and unit-level methods laid out in Propositions 1 and 2 go away uniformly across all data generating processes of this type. Also, the unit-level method converges to the true \( \tau \) faster than the cluster-based method. In contrast, in the asymptotic setting where the number of units within each cluster grow and the number of clusters remain fixed, the finite sample problems may remain, and the two methods may not converge to the true CACE.

We conclude the section by making a few technical points. First, condition (iii) in Proposition 3 can be relaxed so that the cluster-level complier average treatment \( \tau_j \) is growing on the order of \( J^p \) where \( 0 \leq p < 1 \) for the cluster-level method and \( 0 \leq p < 2 \) for the TSLS method. We do not believe this scenario is realistic in practice and hence, we state a more simpler condition for \( \tau_j \) in Proposition 3. Second, for clarity of exposition, we avoid a more technically accurate exposition where we place subscript \( J \) in \( F \) and the parameters \( \tau_{CL} \), \( \tau_{TSLS} \), and \( \tau \) should be functions of both \( F \) and \( J \). Third, for the interested reader, in the appendix, we derive a more precise upper and lower bounds, including constants, for the convergences \( |\tau_{CL} - \tau| \) and \( |\tau_{TSLS} - \tau| \).

### 4.3 Inference

We summarize the inferential properties of these two methods, which have been well-established in the literature [Schochet 2013; Wooldridge 2010]. Suppose that the estimators \( \hat{\tau}_{cl} \) and \( \hat{\tau}_{tsls} \) are identifying the CACE, whether by conditions stated in Corollary 1 or in an asymptotic sense in Proposition 3. Both rely on asymptotic approximations for inference, specifically a variation of the Delta method where the denominators of \( \hat{\tau}_{cl} \) and \( \hat{\tau}_{tsls} \) are assumed to be away from zero as \( J \to \infty \), i.e. a similar asymptotic regime described in Proposition 3.
In particular, both confidence intervals derived from the point estimators rely on the treatment assignment having a strong effect on compliance, or, equivalently, having experiments with uniformly high compliance rates. If the experiment has low compliance rates, then the asymptotic arguments used to construct these confidence intervals no longer remains valid and will often be shorter and/or off from the true target value. In general, we would argue that the conditions of the smartcard intervention do not closely hew to any of the asymptotic templates. The number of clusters is not particularly large, the number of units vary from cluster to cluster, and the compliance rates vary from cluster to cluster.

5 An Almost Exact Approach

The methods outlined above suffer from two particular weaknesses. First, these methods do not always identify the CACE, especially in finite samples, and asymptotics assumptions have to be used if uniform identification is desired. This is a particular concern, since CRTs often have use much small sample sizes than non-clustered experimental designs. Second, both methods rely on inferential techniques that require the compliance rate to be high. Next, we propose solutions that always identifies CACE and can produce confidence intervals that remain valid even if the compliance rate is low.

To allow for finite sample inference, we adapt almost exact methods from non-clustered experimental designs. The almost exact approach approximates exact inference but uses closed-form expressions for interval estimation. See Kang et al. (2017) for a review of both approaches with unclustered data. More specifically, we rely on “finite sample asymptotics” to approximate the exact null distribution (Hájek 1960; Lehmann 2004).

First, we outline a point estimator, by defining a series of adjusted responses. Adjusted responses are observed responses that have been adjusted to be consistent with the null hypothesis. Let $A_j(\tau_0) = Y_j - D_j \tau_0$ be the adjusted response, $A_j^{(1)}(\tau_0) = Y_j^{(1,D_j^{(1)})} - D_j^{(1)} \tau_0$ is the adjusted response if cluster $j$ was treated, and $A_j^{(0)}(\tau_0) = Y_j^{(0,D_j^{(0)})} - D_j^{(0)} \tau_0$ is the adjusted
response if cluster $j$ was not treated. Next, $A_T(\tau_0) = \frac{1}{m} \sum_{j=1}^{J} Z_j(Y_j - D_j \tau_0)$ is the mean of the adjusted response among the treated, and $A_C(\tau_0) = \frac{1}{J-m} \sum_{j=1}^{J} (1 - Z_j)(Y_j - D_j \tau_0)$ is the mean of the adjusted response among the control. Then, consider the test statistic $T(\tau_0)$ for the hypothesis in equation (3)

$$T(\tau_0) = \frac{1}{m} \sum_{j=1}^{J} Z_j(Y_j - D_j \tau_0) - \frac{1}{J-m} \sum_{j=1}^{J} (1 - Z_j)(Y_j - D_j \tau_0)$$

$$= \frac{1}{m} \sum_{j=1}^{J} Z_j A_j(\tau_0) - \frac{1}{J-m} \sum_{j=1}^{J} (1 - Z_j) A_j(\tau_0)$$

$$= A_T(\tau_0) - A_C(\tau_0)$$

Then, we have the following statement about the property of $T(\tau_0)$.

**Proposition 5.** Suppose Assumptions (A1)-(A4) hold and the null hypothesis $H_0 : \tau = \tau_0$ holds. Then, as $J, J-m \to \infty$ where $m/J \to p \in (0, 1)$, suppose (i) $\mu_Y$ and $\mu_D$ are constants so that $\mu_Y/\mu_D = \tau_0$ and (ii) the following growth condition holds

$$\max_j \left( \frac{A_j^{(1)}}{m} + \frac{A_j^{(0)}}{J-m} - \left( \frac{1}{J} \sum_{j=1}^{J} \frac{A_j^{(1)}}{m} + \frac{A_j^{(0)}}{J-m} \right) \right) \to 0$$

$$\sum_{j=1}^{J} \left( \frac{A_j^{(1)}}{m} + \frac{A_j^{(0)}}{J-m} - \left( \frac{1}{J} \sum_{j=1}^{J} \frac{A_j^{(1)}}{m} + \frac{A_j^{(0)}}{J-m} \right) \right) \to 0$$

Then, we have

$$\frac{T(\tau_0)}{\sqrt{\text{Var}(T(\tau_0) \mid \mathcal{F}, \mathcal{Z})}} \to N(0, 1)$$

A consequence of Proposition 5 is that we can construct a Hodges-Lehmann type of point estimator (Hodges and Lehmann 1963) based on the testing properties of $T(\tau_0)$. Specifically, our estimate is the value of $\tau$ that satisfies the equation $T(\tau) = 0$. Algebraic manipulation reveals that the $\tau$ that satisfies $T(\tau) = 0$ is

$$\hat{\tau}_{AE} = \frac{1}{m} \sum_{j=1}^{J} Z_j Y_j - \frac{1}{J-m} \sum_{j=1}^{J} (1 - Z_j) Y_j$$

$$= \frac{1}{m} \sum_{j=1}^{J} Z_j D_j - \frac{1}{J-m} \sum_{j=1}^{J} (1 - Z_j) D_j$$

(9)
The following corollary shows that the identifying value of the proposed point estimator \( \hat{\tau}_{AE} \) is the CACE in finite samples, irrespective of the number of units within the cluster or cluster-level CACE homogeneity; this is in contrast to the two extant methods where either (i) the number of clusters had to be identical, (ii) the cluster-level CACE had to be identical across clusters, or (iii) for TSLS, asymptotic argument involving growing \( J \) had to be invoked. Note that one can also derive the same estimator using the methods in [Middleton and Aronow (2015)] as plug-in estimates for the terms in equation (9).

**Corollary 2.** If Assumptions 1-4 hold, \( \hat{\tau}_{AE} \) in equation (9) identifies the complier average treatment effect, i.e.

\[
E\left[ \frac{1}{m} \sum_{j=1}^{J} Z_j Y_j - \frac{1}{J-m} \sum_{j=1}^{J} (1-Z_j) Y_j \mid \mathcal{F}, \mathcal{Z} \right] = \tau
\]

Another consequence of Proposition 5 is that for any \( \alpha, 0 < \alpha < 1 \), we can construct a two-sided \( 1 - \alpha \) confidence interval for \( \tau \) by inverting the test and using the asymptotic Normal distribution under the null hypothesis. This requires specifying an estimator for \( \text{Var}(T(\tau_0) \mid \mathcal{F}, \mathcal{Z}) \) and one such estimator is the classic sum of variance between the treated and control units (Neyman 1923; Imbens and Rubin 2015):

\[
S^2(\tau_0) = \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j (A_j(\tau_0) - A_T(\tau_0))^2 + \frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J} (1-Z_j)(A_j(\tau_0) - A_C(\tau_0))^2.
\]

The variance estimator \( S^2(\tau_0) \) is well-known to be conservative and generally speaking, there does not exist a consistent nor unbiased estimator for \( \text{Var}(T(\tau_0) \mid \mathcal{F}, \mathcal{Z}) \) (Neyman 1923). Recent proposals by Robins (1988) and Aronow et al. (2014) provide sharper estimates of \( \text{Var}(T(\tau_0) \mid \mathcal{F}, \mathcal{Z}) \), which can also be used in our context. For example, we can replace our variance estimate \( S^2(\tau_0) \) by the upper bound \( \hat{V}_N^H \) in equation (9) of Aronow et al. (2014). Regardless, once we have selected an estimator for variance, say \( S^2(\tau_0) \), a two-sided \( 1 - \alpha \)
confidence interval of \( \tau \) is the set of \( \tau_0 \) that is accepted under the null hypothesis.

\[
\left\{ \tau_0 : P_{H_0} \left( \frac{T(\tau_0)}{S(\tau_0)} \leq z_{1-\alpha/2}|F, \mathcal{Z} \right) \right\} \tag{10}
\]

where \( z_{1-\alpha/2} \) is the \( 1 - \alpha/2 \) quantile of the standard Normal distribution. The interval will necessarily be conservative because \( S^2(\tau_0) \) is a conservative estimator of the variance \( \text{Var}(T(\tau_0) \mid F, \mathcal{Z}) \); but since no unbiased or consistent estimator for the variance generally exist, most two-sided \( 1 - \alpha \) interval will be conservative. But, the interval produces valid statistical inference in that the interval in equation (10) will cover \( \tau \) with at least \( 1 - \alpha \) probability.

After some algebra, the \( 1 - \alpha \) confidence interval in equation (10) can be greatly simplified to be the solution to a quadratic equation. Consider the following quantities.

\[
\hat{s}_{Yr}^2 = \frac{1}{m-1} \sum_{j=1}^{J} Z_j \left( Y_j - \frac{\sum_{j=1}^{J} Z_j Y_j}{m} \right)^2
\]

\[
\hat{s}_{Yc}^2 = \frac{1}{J-m-1} \sum_{j=1}^{J} (1 - Z_j) \left( Y_j - \frac{\sum_{j=1}^{J} (1 - Z_j) Y_j}{J - m} \right)^2
\]

\[
\hat{s}_{Dr}^2 = \frac{1}{m-1} \sum_{j=1}^{J} Z_j \left( D_j - \frac{\sum_{j=1}^{J} D_j Y_j}{m} \right)^2
\]

\[
\hat{s}_{Dc}^2 = \frac{1}{J-m-1} \sum_{j=1}^{J} (1 - Z_j) \left( D_j - \frac{\sum_{j=1}^{J} (1 - Z_j) D_j}{J - m} \right)^2
\]

\[
\hat{s}_{YD_T}^2 = \frac{1}{m-1} \sum_{j=1}^{J} Z_j \left( Y_j - \frac{\sum_{j=1}^{J} Z_j Y_j}{m} \right) \left( D_j - \frac{\sum_{j=1}^{J} Z_j D_j}{m} \right)
\]

\[
\hat{s}_{YD_C}^2 = \frac{1}{J-m-1} \sum_{j=1}^{J} (1 - Z_j) \left( Y_j - \frac{\sum_{j=1}^{J} (1 - Z_j) Y_j}{J - m} \right) \left( D_j - \frac{\sum_{j=1}^{J} (1 - Z_j) D_j}{J - m} \right)
\]

The terms \( \hat{s}_{Yr}^2 \) and \( \hat{s}_{Dr}^2 \) are the estimated variances of \( Y \) and \( D \) among treated clusters. The terms \( \hat{s}_{Yc}^2 \) and \( \hat{s}_{Dc}^2 \) are the estimated variances of \( Y \) and \( D \). The terms \( \hat{s}_{YD_T}^2 \) and \( \hat{s}_{YD_C}^2 \) are the estimated covariances between \( Y \) and \( D \) among the treated and control clusters, respectively. Using these estimated variances, the interval in equation (10) is a solution to
the quadratic equation

\[ \{ \tau_0 \mid a\tau_0^2 + 2b\tau_0 + c \leq 0 \} \]

(11)

where the coefficients \(a, b, \) and \(c\) in equation (11) are

\[
\begin{align*}
a &= \hat{\mu}_D^2 - z_{1-\alpha/2}^2 \left( \frac{s^2_{Dx}}{m} + \frac{s^2_{DC}}{J-m} \right) \\
b &= - \left[ \hat{\mu}_Y \hat{\mu}_D - z_{1-\alpha/2}^2 \left( \frac{s^2_{YDx}}{m} + \frac{s^2_{YDC}}{J-m} \right) \right] \\
c &= \hat{\mu}_Y^2 - z_{1-\alpha/2}^2 \left( \frac{s^2_{Yx}}{m} + \frac{s^2_{YC}}{J-m} \right)
\end{align*}
\]

The \(1-\alpha\) confidence interval of \(\tau\) based on equation (11) is simple, requiring a quadratic solver based on coefficients \(a, b,\) and \(c\). A notable feature of this confidence interval is that it will produce an infinite confidence intervals when an instrument is weak. An infinite confidence interval is a warning that the data contain little information [Rosenbaum 2002, p. 185] and is a theoretically necessary requirement for confidence interval in IV settings [Dufour 1997]. Also, this quadratic confidence interval in equation (11) can never be empty; see the supplementary materials for details. However, this may not be generally true for different variance estimators \(\text{Var}(T(\tau_0) \mid \mathcal{F}, \mathcal{Z})\) which is used to construct confidence intervals of the type in equation (10).

We also propose an exact method for confidence interval construction under a more narrow null hypothesis than in equation (3). Specifically, consider the null \(H_{00}\) where the cluster-specific CACE \(\tau_j\) are equal to \(\tau_0\) for all \(j\), i.e. there is cluster-level CACE homogeneity.

\[ H_{00} : \tau_j = \tau_0, \quad \forall j = 1, \ldots, J \]

(12)

The null \(H_{00}\) is a narrower hypothesis than \(H_0\) in that the parameters \(\mathcal{F}\) that satisfy \(H_{00}\) also satisfies \(H_0\), but the converse is not true. However, and most importantly, \(H_{00}\) is not a strict sharp null in the sense that we still cannot specify all the potential outcomes, say \(Y_{ji}^{(z, D_{ji})}\) for any \(z \in \{0, 1\}\) even under \(H_{00}\). Note that under \(H_{00}\), the population CACE is
equal to $\tau_0$.

A benefit of deriving inferential properties under a restrictive $H_{00}$ is that we can obtain exact, non-parametric, finite-population confidence intervals instead of the asymptotic ones in Proposition 5. In particular, the $1 - \alpha$ confidence interval derived under $H_{00}$ covers $\tau$ with probability at least $1 - \alpha$ in any finite sample.

**Proposition 6.** Suppose Assumptions (A1)-(A4) hold and the null hypothesis $H_{00}$ in equation (12) holds. Then, for any $t \in \mathbb{R}$ and for any $\tau_0$, the null distribution of $T(\tau_0)$ under $H_{00}$ is the permutation distribution.

$$P (T(\tau_0) \leq t \mid \mathcal{Z}, \mathcal{F}) = \left\{ z \in \mathcal{Z} \mid \frac{1}{m} \sum_{j=1}^{J} z_j A_j(\tau_0) - \frac{1}{J-m} \sum_{j=1}^{J} (1 - z_j) A_j(\tau_0) \leq t \right\}$$  \hspace{1cm} (13)

Using the duality between testing and confidence intervals, one can construct a $1 - \alpha$ confidence interval of $\tau$ by finding values of $\tau_0$ for which $H_{00}$ is accepted at the $\alpha$ level. For example, let $q_{1-\alpha/2}$ be the $1 - \alpha/2$ quantile of the null distribution in equation (13). Then, a two-sided $1 - \alpha$ confidence interval of $\tau$ based on Proposition 6 is the following set of $\tau_0$

$$\left\{ \tau_0 : P \left( |T(\tau_0)| \leq q_{1-\alpha/2} \mid \mathcal{Z}, \mathcal{F} \right) \right\}$$  \hspace{1cm} (14)

where we assumed, for simplicity, $T(\tau_0)$ is symmetric around 0; if the latter does not hold, we can take the union of two one-sided confidence intervals to form the desired two-sided confidence interval. Unlike the $1 - \alpha$ confidence interval in equation (14), the $1 - \alpha$ confidence interval in equation (14) is exact in finite-samples, not relying on any asymptotic arguments. Also, like the confidence interval in equation (10), if the confidence interval in equation (14) is infinite, is a warning that the data contain little information about $\tau_0$. Also, if the confidence interval in equation (14) is empty, it suggests either that the instrumental variables assumptions 2-4 are not met or, most likely, that the CACE homogeneity assumption $H_{00}$ is not met since technically speaking, the permutation null distribution in equation (14)
remains valid even if assumptions 2-4 fails to hold; see the supplementary materials for technical details.

A caveat of the interval in equation (14) is that one has to compute \( q_{1-\alpha/2} \), which depending on the size of \( J \) and \( m \), may be computationally burdensome. In practice, we recommend using the confidence interval in (14) if \( J \) is small, roughly under 40 depending on one’s computational power. Next, we evaluate our proposed methods against current practice using a simulation study.

6 Simulation Study

We now use simulations to compare the performance of current methods of estimation and inference to the almost exact approach. As we outlined above, extant methods may not recover the target causal estimand when complier effects vary with cluster size. In the following simulations, we focus on how performance of the various methods change as complier effects vary with cluster sizes. First, we describe general aspects of the simulation. In general, we designed the simulations to closely replicate the conditions of the smartcard intervention.

We assume non-compliance is one-sided. Next, \( \pi \) is the probability a unit within a cluster is a complier. Let \( P_i \) denote the compliance class, which only includes compliers and never-takers in the one-sided compliance setting, since always-takers and defiers are excluded by design. We designate a unit as complier, \( P_i = co \), by random sampling units from each cluster with probability \( \pi \) and \( 1 - \pi \). We generated outcomes using a linear mixed model (LMM) of the form

\[
Y_{ji} = \alpha + \tau I(P_i = co)Z_j + \beta n_i + \gamma Z_j n_i + c_i + \epsilon_{ij}.
\]

where \( co \) indicates that a unit is a complier. Under this model, \( Z_j = 1 \) if cluster \( i \) was assigned to treatment and \( Z_j = 0 \) if the cluster was assigned to control, so \( \tau \) is a measure of the individual level treatment effect if the unit is a complier. In the model, \( c_i \) is a cluster-level
random effect, $\beta$ is a specific cluster-level effect that varies with the size of the cluster, and $\gamma$ allows the treatment effect to vary by cluster size. In the simulations that follow $\gamma$ is the key parameter that we vary in the simulations. In the simulations, we used a $t$-distribution with five degrees of freedom for the error terms $c_i$ and $\epsilon_{ij}$.

For this data generating process, we define the intraclass correlation, $\lambda$, as $\text{Var}(c_i)/\{\text{Var}(c_i) + \text{Var}(\epsilon_{ij})\}$ when $c_i$ and $\epsilon_{ij}$ have finite variance. We can adjust the scale of the cluster distribution errors $c_i$, so that we can control the value of $\lambda$ in the simulations. In all the simulations, we set $\lambda$ to 0.28, which is equal to the estimated intraclass correlation in the smartcard data. This is a fairly large ICC value. Hedges and Hedberg (2007) using ICC estimates from clustered randomized experiments in education find that $\lambda$’s range from 0.07 to 0.31, with an average value of 0.17. Small et al. (2008a) note that $\lambda$ values in the range of .002 to 0.03 are more typical in public health interventions that target clusters such as hospitals, clinics or villages.

We repeated the simulation for differing numbers of clusters. We used cluster sizes of 20, 30, 50, 80, 100, and 200. In most CRTs with noncompliance, the number of units per cluster and the compliance rate tend to vary from cluster to cluster. In our simulations, we used the units per cluster and the compliance rates from the smartcard data. We did this by sampling from the cluster sizes and compliers rates from the data. That is, when the number of cluster is 50, we took a random sample of 50 cluster sizes and compliance rates from the data. This allowed us to vary cluster sizes and compliance rates in a fashion that mimics the typical data structure in a CRT with noncompliance.

As we noted above, the key parameter in the simulation is $\gamma$. As such, we conducted three different sets of simulations. In the first, we set $\gamma = 0$, which implies that complier effects do not vary with cluster size. In the second and third set of simulations, we set $\gamma$ to -0.03 and 0.03. For these two simulations, the complier effects are negatively and positively correlated with cluster sizes respectively.

Table 3 contains the results from the first simulation study. First, we observe that when
complier effects do not vary with cluster sizes all the methods perform the same in terms of bias. Here, even with as few as 20 clusters all three methods recover the true complier average causal effect. However, if complier average causal effects are inversely correlated with cluster sizes, the estimates based on cluster level averages underestimates the true complier causal effect by 30 to 34%. This bias does not vary with the number of clusters. Finally, when true complier causal effects are positive correlated, again the cluster level averages underestimate the true effect by 6%. As predicted in the analytical result, using TSLS recovers so long as $J$ is large enough, although we are surprised by how fast the TSLS converges to the CACE, roughly $J \geq 20$ in the simulation. In more limited simulation work, we found the performance of TSLS suffer with a greater spread in cluster sizes.

Next, we focus on inferential properties. In Table 4 we record coverage rates for 95% confidence intervals, and Table 5 contains the average length of the 95% confidence intervals. As we might expect, the cluster level method under covers the nominal 95% rate when complier effects are nonconstant. TSLS overcovers when complier effects are negatively correlated with cluster size. The almost exact methods performs well in all situations. The almost exact method also produces a substantially longer confidence interval. Especially, when the number of clusters is small. For example when there are 30 clusters the almost exact confidence intervals are nearly twice as long as those based on TSLS. Even with 200 clusters, the almost exact confidence interval tends to be about 40% longer than those based on asymptotic approximations. Moreover, we found that when there were only 20 clusters, the almost exact method produces an infinite confidence interval approximately one percent of the time. Thus in some instances, the almost exact method provides a clear warning that the instrument is weak.

In sum, the simulations reveal two important results. First, the estimator based on cluster level averages produces poor results. Unless the complier effects do not vary with cluster size, this method always was farther from the truth than the either TSLS or the method we propose. Second, while TSLS performs nearly the same in terms of bias, it
Table 3: Bias in Estimators Three Different IV Methods for CRTs

| Number of Clusters | Gen. Effect Ratio Bias | Cluster-level Averages Bias | TSLS Bias |
|-------------------|------------------------|----------------------------|-----------|
|                   |                        |                            |           |
| **Constant Complier Effects** |                        |                            |           |
| 20                | 1.01                   | 1.00                       | 1.01      |
| 30                | 0.99                   | 1.00                       | 0.99      |
| 50                | 1.02                   | 1.01                       | 1.01      |
| 80                | 1.00                   | 1.00                       | 1.00      |
| 100               | 1.00                   | 1.00                       | 1.00      |
| 200               | 0.99                   | 1.00                       | 0.99      |
|                   |                        |                            |           |
| **Nonconstant Complier Effects: Negative Correlation** |                        |                            |           |
| 20                | 1.01                   | 0.69                       | 1.01      |
| 30                | 1.01                   | 0.68                       | 1.00      |
| 50                | 1.05                   | 0.69                       | 1.02      |
| 80                | 1.03                   | 0.67                       | 1.01      |
| 100               | 1.01                   | 0.66                       | 1.00      |
| 200               | 1.01                   | 0.66                       | 1.00      |
|                   |                        |                            |           |
| **Nonconstant Complier Effects: Positive Correlation** |                        |                            |           |
| 20                | 1.00                   | 0.94                       | 1.00      |
| 30                | 1.01                   | 0.94                       | 1.01      |
| 50                | 1.00                   | 0.94                       | 1.00      |
| 80                | 1.00                   | 0.94                       | 1.00      |
| 100               | 1.00                   | 0.94                       | 1.00      |
| 200               | 1.00                   | 0.94                       | 1.00      |

Note: Cell entries are ratio of average estimate to true effect size.
| Number of Clusters | Gen. Effect Ratio | Cluster-level Averages | TSLS |
|--------------------|------------------|------------------------|------|
|                    | Constant Complier Effects |                    |      |
| 20                 | 0.92             | 0.97                   | 0.94 |
| 30                 | 0.93             | 0.95                   | 0.94 |
| 50                 | 0.96             | 0.97                   | 0.96 |
| 80                 | 0.95             | 0.96                   | 0.96 |
| 100                | 0.95             | 0.96                   | 0.95 |
| 200                | 0.95             | 0.95                   | 0.94 |
| Non-constant Complier Effects: Negative Correlation | |
| 20                 | 0.94             | 0.94                   | 0.97 |
| 30                 | 0.97             | 0.92                   | 0.98 |
| 50                 | 0.96             | 0.88                   | 0.98 |
| 80                 | 0.96             | 0.78                   | 0.98 |
| 100                | 0.96             | 0.70                   | 0.98 |
| 200                | 0.97             | 0.44                   | 0.98 |
| Non-constant Complier Effects: Positive Correlation | |
| 20                 | 0.94             | 0.95                   | 0.94 |
| 30                 | 0.94             | 0.95                   | 0.95 |
| 50                 | 0.95             | 0.94                   | 0.96 |
| 80                 | 0.96             | 0.89                   | 0.96 |
| 100                | 0.95             | 0.86                   | 0.96 |
| 200                | 0.96             | 0.73                   | 0.96 |

Note: Cell entries are ratio of average estimate to true effect size.
Table 5: Confidence Interval Length of Three Different CRT IV Methods

| Number of Clusters | Gen. Effect Ratio | Cluster-level Averages | TSLS |
|--------------------|-------------------|------------------------|------|
|                    | Constant Complier Effects |                   |      |
| 20                 | 4.65              | 1.07                   | 0.93 |
| 30                 | 1.54              | 0.84                   | 0.75 |
| 50                 | 1.03              | 0.62                   | 0.57 |
| 80                 | 0.78              | 0.48                   | 0.45 |
| 100                | 0.69              | 0.43                   | 0.40 |
| 200                | 0.47              | 0.30                   | 0.28 |
|                    | Nonconstant Complier Effects: Negative Correlation |                   |      |
| 20                 | 2.51              | 1.04                   | 0.94 |
| 30                 | 1.42              | 0.82                   | 0.76 |
| 50                 | 0.91              | 0.61                   | 0.56 |
| 80                 | 0.69              | 0.48                   | 0.45 |
| 100                | 0.61              | 0.43                   | 0.40 |
| 200                | 0.42              | 0.30                   | 0.29 |
|                    | Nonconstant Complier Effects: Positive Correlation |                   |      |
| 20                 | 4.15              | 1.61                   | 1.28 |
| 30                 | 1.96              | 1.26                   | 1.07 |
| 50                 | 1.34              | 0.93                   | 0.83 |
| 80                 | 1.01              | 0.72                   | 0.66 |
| 100                | 0.90              | 0.64                   | 0.59 |
| 200                | 0.61              | 0.45                   | 0.42 |

Note: Cell entries are ratio of average estimate to true effect size.
generally understates statistical uncertainty. This is not surprising in that it is well known that asymptotic approximations for IV methods do not perform well in many settings.

7 Analysis of The Smartcard Intervention

Finally, we turn to an analysis of the data from the smartcard intervention. As we noted above, the intervention was designed to reduce inefficiency in the payment of welfare benefits. In the original study, data were collected on two different aspects of the payment experience. First, they measured the time, in days, between when the work was completed and the payment was collected. Second, they measured the time in minutes for recipients to collect payments on the day they were disbursed. These data were collected using surveys in each village after the smartcard intervention was in effect. The original study only focused on intention to treat effects. Here, we report IV estimates using both the two extant methods, and the almost exact method.

First, we review some basic descriptive statistics. As we noted above, a total of 157 villages participated in the study. Of these, 112 were randomly assigned to treatment. Across the 157 villages, 6,891 people were surveyed, such that cluster sizes varied from six to 85. The key issue we highlighted above is whether complier effects vary with cluster size. Given that the intervention was implemented across a wide geographic area and that village sizes varied significantly, there is little reason to believe effects are constant across clusters.

| Table 6: Analysis of Smartcard Payment System - Complier Average Causal Effects |
|---------------------------------|-----------------|-----------------|-----------------|
|                                   | Gen. Effect Ratio | Cluster-level Averages | TSLS            |
| Payment Delay                    |                  |                  |                 |
| Point Estimate                   | -2.72            | -9.14            | -10.15          |
| 95% Confidence Interval         | [-21.87, 13.56]  | [-19.16, 0.88]   | [-21.17, 0.87]  |
| Time to Collect Payment         |                  |                  |                 |
| Point Estimate                   | -27.03           | -57.98           | -60.06          |
| 95% Confidence Interval         | [-82.26, 26.36]  | [-98.61, -17.35] | [-100.05, -20.06] |
Table 6 contains the results from the three methods we outlined above. For both outcomes, cluster-level averages and TSLS produce very similar results for both the point estimates and the confidence intervals. However, the estimate based on the generalized effect ratio is much smaller in magnitude and the confidence intervals are much wider. Thus there are striking similarities and contrasts between the empirical results and the simulations. First, as was the case in the simulations, confidence intervals based on the almost exact method are notably longer. However, in the simulations the generalized effect ratio and TSLS produced very similar point estimates. Here, we find that the generalized effect ratio differs from the other two methods which are very similar. In fact, in the data, we find the pattern suggested by the analytic results: the extant methods substantially overestimate the CACE. What might explain the difference between the simulations and the empirical results? Most likely, the pattern of complier effects varies in a more complex fashion than was true in the simulation. Thus the empirics are more closely aligned with the analytic results which suggest that TSLS does not recover the causal estimand of interest in settings of this type. To explore this possibility, we plotted the compliance rate against the cluster size. This plot is contained in the first panel of Figure 1. There appears to be little relationship between cluster size and compliance. Next, we calculated the average outcome for each treated cluster and subtracted the average control outcome from each of the treated averages. We plotted these adjusted treated outcomes against cluster size. This plot is in the second panel of Figure 1. We find a weakly positive relationship between outcomes and cluster size.

8 Discussion

Assessment of policy interventions is often done using cluster randomized trials. This study design has two key advantages. First, it relies on randomization to remove both hidden and overt bias in the estimation of treatment effect. Second, it allows for natural patterns of
Figure 1: Scatterplots between cluster size, compliance rates and treated cluster outcome deviated from average control outcome for time to collect outcome.

interaction within clusters to reduce the likelihood of spillovers from treatment to control. The use of smartcards to deliver welfare payments in India follows such a template.

We demonstrated through analytic results, simulations, and empirics that extant methods do not recover the target causal estimand when investigators are interested in the causal effect among those that actually complied with the treatment. These methods impose the strong assumption that complier effects do not vary with cluster size. Moreover, both methods rely on asymptotic assumptions for interval estimation. Many CRTs rely on relatively small sample sizes, which makes such assumptions less justifiable. Here, we have developed methods that allow for consistent estimates of the CACE when complier effects vary with cluster size. We introduce the generalized effect ratio coupled it with an approach to inference that provides correct confidence intervals when instruments are weak. Our method approximate an exact result, but has a closed-form solution.
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A Cluster-Robust Variance Formula

Let \( W = [1, \hat{D}] \) be the concatenation of the vector of ones 1 and the vector \( \hat{D} \) and let \( W_j \) refer to the submatrix of \( W \) that only consists of rows of individuals in cluster \( j \). Let \( \hat{\Omega}_j = (u_{j1}^2, \ldots, u_{jn_j}^2) \) be the vector of square residuals from cluster \( j \). Then, the matrix version of the variance of \( \hat{\tau}_{TSLS} \) is

\[
\hat{\tau}_{TSLS} = \left[(W^tW)^{-1} \left( \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} W_j u_{ji} \right) \left( \sum_{i=1}^{n_j} W_j^t u_{ji} \right) \right) \right] \left(W^tW\right)^{-1}
\]

where the subscript \([22]\) indicates the value corresponding to the 2nd row and column of the matrix. Simplifying the inner sum gives us

\[
\left( \sum_{i=1}^{n_j} W_j u_{ji} \right) \left( \sum_{i=1}^{n_j} W_j^t u_{ji} \right) = \begin{bmatrix}
(\sum_{i=1}^{n_j} u_{ji})^2 & (\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) \\
(\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) & \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right)^2
\end{bmatrix}
\]

Taking the outer sum of the above expression gives us

\[
\sum_{j=1}^{J} \begin{bmatrix}
(\sum_{i=1}^{n_j} u_{ji})^2 & (\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) \\
(\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) & \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right)^2
\end{bmatrix} = \sum_{j=1}^{J} \begin{bmatrix}
(\sum_{i=1}^{n_j} u_{ji})^2 & \sum_{j=1}^{J} (\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) \\
\sum_{j=1}^{J} (\sum_{i=1}^{n_j} u_{ji}) \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right) & \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right)^2
\end{bmatrix}
\]

Also, the inverse \( W^tW \) simplifies to

\[
(W^tW)^{-1} = \frac{1}{n \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji}^2 - \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right)^2} \begin{bmatrix}
\sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji}^2 & -\sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \\
-\sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} & n
\end{bmatrix}
\]
Combining the simplifications together, we obtain the expression in the main text.

\[
\hat{\tau}_{\text{TSLS}} = \left( \frac{1}{n \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji}^2 - \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right)^2} \right)^2 
\cdot \left[ - \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right) \left( \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \right)^2 \right) + n \left( \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \right) \left( \sum_{i=1}^{n_j} \hat{u}_{ji} \hat{D}_{ji} \right) \right) \right] 
\cdot \left[ - \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right] 
\cdot \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right)^2 
\cdot \left( \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \right)^2 \right) 
- 2 n \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right) \left( \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \right) \left( \sum_{i=1}^{n_j} \hat{u}_{ji} \hat{D}_{ji} \right) \right) 
+ n^2 \sum_{j=1}^{J} \left( \sum_{i=1}^{n_j} u_{ji} \hat{D}_{ji} \right)^2 \right] \left( \frac{1}{n \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji}^2 - \left( \sum_{j=1}^{J} \sum_{i=1}^{n_j} \hat{D}_{ji} \right)^2} \right)^2
\]

\[
B \quad \text{Quadratic Form and Proposed Confidence Interval}
\]

This section outlines the algebra that transforms the confidence interval expression of our proposed method into a quadratic expression.

We begin by noticing that \( \frac{|T(\tau_0)|}{S(\tau_0)} \leq z_{1-\alpha/2} \) is equivalent to \( T^2(\tau_0) - z_{1-\alpha/2}^2 S^2(\tau_0) \leq 0 \).

The term \( \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j(A_j(\tau_0) - A_T(\tau_0))^2 \) in the expression \( S^2(\tau_0) \) can be re-expressed as
follows.

\[
\frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j (A_j(\tau_0) - A_T(\tau_0))^2
\]

\[
= \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j \left( (Y_j - \tau_0D_j)^2 - 2(Y_j - \tau_0D_j) \sum_{j=1}^{J} \frac{Z_j(Y_j - \tau_0D_j)}{m} + \left( \sum_{j=1}^{J} \frac{Z_j(Y_j - \tau_0D_j)}{m} \right)^2 \right)
\]

\[
= \frac{1}{m(m-1)} \left[ \sum_{j=1}^{J} Z_j(Y_j - \tau_0D_j)^2 - \frac{\left( \sum_{j=1}^{J} Z_j(Y_j - \tau_0D_j) \right)^2}{m} \right]
\]

\[
= \frac{1}{m(m-1)} \left[ \frac{m-1}{m} \sum_{j=1}^{J} Z_j(Y_j - \tau_0D_j)^2 - \sum_{j=1}^{J} Z_j \frac{Z_jZ_k(Y_j - \tau_0D_j)(Y_k - \tau_0D_k)}{m} \right]
\]

\[
= \frac{1}{m^2} \sum_{j=1}^{J} Z_j(Y_j - \tau_0D_j)^2 - \frac{1}{m^2(m-1)} \sum_{j=1}^{J} Z_j \frac{Z_jZ_k(Y_j - \tau_0D_j)(Y_k - \tau_0D_k)}{m}
\]

\[
= \frac{1}{m^2} \sum_{j=1}^{J} Z_jY_j^2 - \frac{1}{m^2(m-1)} \sum_{j=1}^{J} Z_j \frac{Z_jZ_kY_jY_k}{m} + \frac{\tau_0^2}{m^2} \sum_{j=1}^{J} Z_jD_j^2 - \frac{\tau_0^2}{m^2(m-1)} \sum_{j=1}^{J} Z_jZ_kD_jD_k
\]

\[- \frac{2\tau_0}{m^2} \sum_{j=1}^{J} Z_jY_jD_j + \frac{2\tau_0}{m^2(m-1)} \sum_{j=1}^{J} Z_jZ_kY_jD_k
\]

\[
= \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j \left( Y_j - \frac{\sum_{j=1}^{J} Z_jY_j}{m} \right)^2 + \frac{\tau_0^2}{m(m-1)} \sum_{j=1}^{J} Z_j \left( D_j - \frac{\sum_{j=1}^{J} Z_jD_j}{m} \right)^2
\]

\[- \frac{2\tau_0}{m(m-1)} \left( \sum_{j=1}^{J} Z_jY_jD_j - \frac{\left( \sum_{j=1}^{J} Z_jY_j \right) \left( \sum_{j=1}^{J} Z_jD_j \right)}{m} \right)
\]

The last equality is based on the observation that

\[
\frac{1}{m^2} \sum_{j=1}^{J} Z_jY_jD_j - \frac{1}{m^2(m-1)} \sum_{j=1}^{J} Z_jZ_kY_jD_k
\]

\[
= \frac{1}{m(m-1)} \left[ \frac{m-1}{m} \sum_{j=1}^{J} Z_jY_jD_j - \frac{1}{m} \sum_{j=1}^{J} Z_jZ_kY_jD_k \right]
\]

\[
= \frac{1}{m(m-1)} \left[ \sum_{j=1}^{J} Z_jY_jD_j - \frac{1}{m} \left( \sum_{j=1}^{J} Z_jY_j \right) \left( \sum_{j=1}^{J} Z_jD_j \right) \right]
\]

\[
= \frac{1}{m(m-1)} \left[ \sum_{j=1}^{J} Z_j \left( Y_j - \frac{\sum_{j=1}^{J} Z_jY_j}{m} \right) \left( D_j - \frac{\sum_{j=1}^{J} Z_jD_j}{m} \right) \right]
\]

A similar algebra for the term \(\frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1 - Z_j)(A_j(\tau_0) - A_C(\tau_0))^2\) in the expression
Then, we have

\[
S^2(\tau_0) \text{ leads us to }
\]

\[
\frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1-Z_j)(A_j(\tau_0) - A^C(\tau_0))^2 = \sum_{j=1}^{J}
\]

\[
= \frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1-Z_j) \left( Y_j - \frac{\sum_{j=1}^{J}(1-Z_j)Y_j}{J-m} \right)^2
\]

\[
+ \frac{\tau_0^2}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1-Z_j) \left( D_j - \frac{\sum_{j=1}^{J}(1-Z_j)D_j}{J-m} \right)^2
\]

\[
- \frac{2\tau_0}{(J-m)(J-m-1)} \left( \sum_{j=1}^{J}(1-Z_j)Y_jD_j - \frac{(\sum_{j=1}^{J}(1-Z_j)Y_j)(\sum_{j=1}^{J}(1-Z_j)D_j)}{J-m} \right)
\]

Thus, \(S^2(\tau_0)\) simplifies to

\[
S^2(\tau) = \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j \left( Y_j - \frac{\sum_{j=1}^{J} Z_j Y_j}{m} \right)^2 + \frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1-Z_j) \left( Y_j - \frac{\sum_{j=1}^{J}(1-Z_j)Y_j}{J-m} \right)^2
\]

\[
- 2\tau_0 \left[ \frac{1}{m(m-1)} \left( \sum_{j=1}^{J} Z_j Y_j D_j - \frac{(\sum_{j=1}^{J} Z_j Y_j)(\sum_{j=1}^{J} Z_j D_j)}{m} \right) \right]
\]

\[
+ \frac{1}{(J-m)(J-m-1)} \left( \sum_{j=1}^{J}(1-Z_j)Y_jD_j - \frac{(\sum_{j=1}^{J}(1-Z_j)Y_j)(\sum_{j=1}^{J}(1-Z_j)D_j)}{J-m} \right)
\]

\[
+ \tau_0^2 \left[ \frac{1}{(J-m)(J-m-1)} \sum_{j=1}^{J}(1-Z_j) \left( D_j - \frac{\sum_{j=1}^{J}(1-Z_j)D_j}{J-m} \right)^2 + \frac{1}{m(m-1)} \sum_{j=1}^{J} Z_j \left( D_j - \frac{\sum_{j=1}^{J} Z_j D_j}{m} \right)^2 \right]
\]

\[
= \frac{s_Y^2}{m} + \frac{s_{Y_C}^2}{J-m} - 2\tau_0 \left( \frac{s^2_{YDx}}{m} + \frac{s_{YDC}^2}{J-m} \right) + \tau_0^2 \left( \frac{s^2_{Dx}}{m} + \frac{s^2_{DC}}{J-m} \right)
\]

Next, \(T^2(\tau_0)\) simplifies to

\[
T^2(\tau_0) = \hat{\mu}_Y^2 - 2\tau_0 \hat{\mu}_Y \hat{\mu}_D + \tau_0^2 \hat{\mu}_D^2
\]

Then, we have \(T^2(\tau_0) - z_{1-\alpha/2}^2 S^2(\tau_0) \leq 0\) implies

\[
T^2(\tau_0) - z_{1-\alpha/2}^2 S^2(\tau_0)
\]

\[
= \hat{\mu}_Y - z_{1-\alpha/2}^2 \left( \frac{s_{Y_P}^2}{m} + \frac{s_{Y_{PC}}^2}{J-m} \right) - 2\tau_0 \left[ \hat{\mu}_Y \hat{\mu}_D - z_{1-\alpha/2}^2 \left( \frac{s_{YDx}^2}{m} + \frac{s_{YDC}^2}{J-m} \right) \right]
\]

\[
+ \tau_0^2 \left[ \hat{\mu}_D^2 - z_{1-\alpha/2}^2 \left( \frac{s_{Dx}^2}{m} + \frac{s_{DC}^2}{J-m} \right) \right] \leq 0
\]
C Non-Empty Quadratic Confidence Interval

Here, we show that the confidence interval constructed based on the quadratic equation can never be empty. To show this, we simply have to show that the roots of the quadratic equation in (11) always exist, or equivalently that the determinant is not negative, i.e.

\[(2b^2) - 4ac = 4(b^2 - ac) \geq 0\]

We see that

D Proofs

Proof of Proposition 1. We first take the expectation of the numerator of \( \hat{\tau}_{CL} \) under Assumption 1

\[ E[\hat{Y}_T - \hat{Y}_C \mid \mathcal{F}, Z] = \frac{1}{J} \sum_{j=1}^J \frac{1}{n_j} \sum_{i=1}^{n_j} Y_{ij}^{(1,D_j(1))} - Y_{ij}^{(0,D_j(0))} \]

Under Assumptions 3 and 4, the above sum simplifies to

\[ \sum_{j=1}^J \frac{1}{n_j} \sum_{i=1}^{n_j} Y_{ij}^{(1,D_j(1))} - Y_{ij}^{(0,D_j(0))} = \sum_{j=1}^J \frac{1}{n_j} \sum_{i=1}^{n_j} (Y_{ji}^{(1)} - Y_{ji}^{(0)}) I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0) \]

Similar algebra with the denominator of \( \hat{\tau}_{CL} \) along with Assumption 3 reveals

\[ E \left[ \sum_{j=1}^J (1 - Z_j)n_j \sum_{j=1}^J Z_jD_j - \sum_{j=1}^J Z_jn_j \sum_{j=1}^J (1 - Z_j)D_j \mid \mathcal{F}, Z \right] = \frac{1}{J} \sum_{j=1}^J \frac{1}{n_j} \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0) \]

Under Assumption 2, we can take the ratio between the numerator and denominator, which then leads to the desired expression.

Proof of Proposition 2. We first take the expectation of the numerator of \( \hat{\tau}_{TSLS} \) under As-
Proof of Corollary 1: If \( n_j \) are identical to each other, then \( w_{\text{CL},j} \) trivially simplifies to \( n_{\text{CO},j}/\sum_{j=1}^{J} n_{\text{CO},j} \). Also, if \( n_j \) are identical, then \( n - n_j \) are identical for every \( j \) and thus, \( w_{\text{TSLS},j} \) simplifies to \( n_{\text{CO},j}/\sum_{j=1}^{J} n_{\text{CO},j} \). Similarly, if the weights are different, but the complier average treatment effect per cluster is identical across all strata and equal to \( C \), then the complier average treatment effect across all clusters is \( C \). Also, the estimators \( \hat{\tau}_{\text{CL}} \) or \( \hat{\tau}_{\text{TSLS}} \) simplify to
\[
\sum_{j=1}^{J} w'_j \cdot \frac{\sum_{i=1}^{n_j} (Y_{j_i}^{(1)} - Y_{j_i}^{(0)}) I(D_{j_i}^{(1)} = 1, D_{j_i}^{(0)} = 0)}{n_{\text{CO},j}} = \sum_{j=1}^{J} w'_j = C
\]
where \( w'_j \) are either \( w_{\text{CL},j} \) or \( w_{\text{TSLS},j} \) with the property that \( \sum_{j=1}^{J} w'_j = 1 \). \( \square \)
Proof of Proposition First, we decompose the difference between $\tau_{CL}$ and $\tau$

$$\tau_{CL} - \tau = \sum_{j=1}^{J} (w_{CL,j} - w_j) \tau_j$$

$$= \sum_{j=1}^{J} \tau_j \left( \frac{n_{CO,j}}{n_j} \frac{\sum_{l=1}^{J} n_{CO,j} \sum_{l=1}^{J} n_{CO,l}}{\sum_{l=1}^{J} n_{CO,l}} \left( \sum_{l=1}^{J} n_{CO,l} \right) \right)$$

$$= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} \right)^{\frac{1}{n_j}}} \sum_{j=1}^{J} \tau_j \left( \frac{n_{CO,j}^{2}}{n_j} + \sum_{l\neq j} \frac{n_{CO,j}n_{CO,l}}{n_j} - \frac{n_{CO,j}^{2}}{n_j} - \sum_{l\neq j} \frac{n_{CO,j}n_{CO,l}}{n_l} \right)$$

$$= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} \right)^{\frac{1}{n_j}}} \sum_{j=1}^{J} \tau_j \left( \sum_{l\neq j} \frac{n_{CO,j}n_{CO,l}}{n_j} (n_l - n_j) \right)$$

Replacing each terms in the summand with $p_{CO,j}$ gives you the desired limiting result.

To prove the uniform convergence result, consider a set $A$ of pairs of clusters $j, l$ for which $n_l - n_j \neq 0$, $A = \{(j, l) | j < l, n_l \neq n_j\}$. Then, starting from the equality expression above, we have

$$\tau_{CL} - \tau = \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} \right)^{\frac{1}{n_j}}} \sum_{j<l} \frac{n_{CO,j}n_{CO,l}}{n_j n_l} (n_l - n_j) (\tau_j - \tau_l)$$

Taking the absolute value and supremum over $F$, we get

$$\sup_{F} |\tau_{CL} - \tau| = \sup_{F} \left( \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} \right)^{\frac{1}{n_j}}} \sum_{(j, l) \in A} \frac{n_{CO,j}n_{CO,l}}{n_j n_l} (n_l - n_j) (\tau_j - \tau_l) \right)$$

To prove the limiting lower bound, we consider a large $J > B^2$ so that the division of integers $J/B$, can be written uniquely as $J/B = q+r/B$ where $q$ and $r$ are unique integers and $q \geq 1$. Then, consider $J$ clusters where (a) the first $q$ clusters have size $n_1 = \ldots = n_q = 1$, the next $q$ clusters have size $n_{q+1} = \ldots = n_{2q} = 2$, and so forth, (b) $n_{CO,j} = n_j$ for all $j$ and (b) there are $B$ distinct cluster-specific CACE values $\{1, \ldots, B\}$ where the first $q$ $\tau_j$s, $B = \tau_1 = \ldots = \tau_q$ are identical, the next $q$ clusters have $B - 1 = \tau_{q+1} = \ldots = \tau_{2q}$, and so forth. Note that by
condition (a) and (c), we have \( n_j \leq n_l \) and \( \tau_j \geq \tau_l \) for \( j < l \). Then, for some constant \( K > 0 \)

\[
\sup_{\mathcal{F}} |\tau_{CL} - \tau| = \sup_{\mathcal{F}} \frac{1}{\left( \sum_{l=1}^{J} \frac{n_{CO,l}}{n_l} \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \left| \sum_{(j,l) \in A} n_{CO,j} n_{CO,l} (n_l - n_j) (\tau_j - \tau_l) \right| \\
\geq \frac{1}{(J) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \sum_{(j,l) \in A} (n_l - n_j) (\tau_j - \tau_l) \\
\geq \frac{1}{J^2 B} \sum_{(j,l) \in A} (n_l - n_j) (\tau_j - \tau_l) \\
\geq \frac{1}{J^2 B} J^2 K \min_{(j,l) \in A} (n_l - n_j) (\tau_j - \tau_l) \\
= \frac{K}{B} > 0
\]

Second, we decompose the difference between \( \tau_{TSLS} \) and \( \tau \).

\[
\tau_{TSLS} - \tau = \sum_{j=1}^{J} (w_{TSLS,j} - w_j) \tau_j \\
= \sum_{j=1}^{J} \tau_j \left( \frac{n_{CO,j} (n - n_j) \sum_{l=1}^{J} n_{CO,j} - n_{CO,j} \sum_{l=1}^{J} n_{CO,l} (n - n_l)}{\left( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \right) \\
= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \sum_{j=1}^{J} \tau_j \left( \sum_{l \neq j} n_{CO,l} n_{CO,j} (n - n_j) - \sum_{l \neq j} n_{CO,j} n_{CO,l} (n - n_l) \right) \\
= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \sum_{j=1}^{J} \tau_j \left( \sum_{l \neq j} n_{CO,j} n_{CO,l} (n_l - n_j) \right) \\
= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \sum_{j=1}^{J} n_{CO,j} n_{CO,l} (n_l - n_j) (\tau_j - \tau_l) \\
= \frac{1}{\left( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \right) \left( \sum_{l=1}^{J} n_{CO,l} \right)} \sum_{(j,l) \in A} n_{CO,j} n_{CO,l} (n_l - n_j) (\tau_j - \tau_l)
\]

The denominator term \( \sum_{l=1}^{J} n_{CO,l} (n - n_l) \) has the form

\[
\sum_{l=1}^{J} n_{CO,l} (n - n_l) = \sum_{l=1}^{J} n_{CO,l} \left( \sum_{j=1}^{J} n_j - n_l \right) = \sum_{l=1}^{J} n_{CO,l} \sum_{j \neq l} n_j = \sum_{l \neq j} n_{CO,l} n_j
\]
Thus, taking the absolute value of the expression above, we have for some constant \( K > 0 \),

\[
\sup_{\mathcal{F}} |\tau_{\text{TLS}} - \tau| \leq \sup_{\mathcal{F}} \frac{1}{\left( \sum_{l \neq j} n_{\text{CO},l} n_{j} \right) \left( \sum_{l=1}^{J} n_{\text{CO},l} \right)} \sup_{(j,l) \in \mathcal{A}} n_{\text{CO},j} n_{\text{CO},l} |n_{l} - n_{j}| |\tau_{j} - \tau_{l}|
\]

\[
\leq \sup_{\mathcal{F}} \frac{1}{\left( \sum_{l \neq j} n_{\text{CO},l} n_{j} \right) \left( \sum_{l=1}^{J} n_{\text{CO},l} \right)} B^{2}(B - 1)^{2} \max_{(j,l) \in \mathcal{A}} |\tau_{j} - \tau_{l}| \sum_{(j,l) \in \mathcal{A}} 1
\]

\[
\leq \sup_{\mathcal{F}} \frac{1}{\left( \sum_{l \neq j} n_{\text{CO},l} n_{j} \right) \left( \sum_{l=1}^{J} n_{\text{CO},l} \right)} J^{2} K B^{2}(B - 1)^{2} \max_{(j,l) \in \mathcal{A}} |\tau_{j} - \tau_{l}|
\]

Then, so long as the product of \( \sum_{l \neq j} n_{\text{CO},l} n_{j} \) and \( \sum_{l=1}^{J} n_{\text{CO},l} \) grow to infinity faster than \( J^{k} \), \( k > 2 \), we get \( |\tau_{\text{TLS}} - \tau| \to 0 \). For example, under the assumption that \( n_{\text{CO},j} > 0 \), the smallest possible value for the said product is \( J(J - 1)J \). Therefore, we obtain the bound

\[
|\tau_{\text{TLS}} - \tau| \leq \frac{J^{2} K B^{2}(B - 1)^{2}}{J^{2}(J - 1)} \max_{(j,l) \in \mathcal{A}} |\tau_{j} - \tau_{l}| \to 0
\]

Finally, to prove rate-sharpness, we use the same specific case for the lower bound of \( \tau_{\text{CL}} - \tau \) except we set \( n_{\text{CO},j} = 1 \). Then, for some constant \( K > 0 \), we have

\[
\sup_{\mathcal{F}} |\tau_{\text{TLS}} - \tau| = \sup_{\mathcal{F}} \frac{1}{\left( \sum_{l \neq j} n_{\text{CO},l} n_{j} \right) \left( \sum_{l=1}^{J} n_{\text{CO},l} \right)} \left| \sum_{(j,l) \in \mathcal{A}} n_{\text{CO},j} n_{\text{CO},l} (n_{l} - n_{j})(\tau_{j} - \tau_{l}) \right|
\]

\[
\geq \frac{1}{\left( \sum_{l \neq j} n_{j} \right) \left( J \right) \left( (j,l) \in \mathcal{A} \right)} \sum_{(j,l) \in \mathcal{A}} (n_{l} - n_{j})(\tau_{j} - \tau_{l})
\]

\[
\geq \frac{1}{\left( \sum_{l \neq j} n_{j} \right) \left( J \right) \left( (j,l) \in \mathcal{A} \right)} \sum_{(j,l) \in \mathcal{A}} 1
\]

\[
\geq \frac{1}{J^{2}(J - 1)K J^{2}} = \frac{K}{J - 1}
\]
Proof of Proposition 4. First, we work on \( \tau_{\text{CL}} \). From the proof in Proposition 4 we have

\[
\tau_{\text{CL}} - \tau = \left( \sum_{l=1}^{J} \frac{n_{\text{CO,}l}}{n_i} \right) \left( \sum_{j=1}^{J} \tau_j \right) \left( \sum_{l \neq j} \frac{n_{\text{CO,}j} n_{\text{CO,}l}}{n_i n_j} (n_l - n_j) \right)
\]

\[
= \tau_1 \sum_{l \neq 1} \frac{n_{\text{CO,}1} n_{\text{CO,}l}}{n_i n_1} (n_l - n_1) + \ldots + \tau_J \sum_{l \neq J} \frac{n_{\text{CO,}J} n_{\text{CO,}l}}{n_i n_j} (n_l - n_j)
\]

\[
= \left( \sum_{l=1}^{J} \frac{n_{\text{CO,}l}}{n_i} \right) \left( \sum_{j=1}^{J} \frac{n_{\text{CO,}1}}{n_1} + \sum_{l=2}^{J} \frac{n_l n_{\text{CO,}l}}{n_i n_l} \right) + \ldots + \left( \sum_{l=1}^{J} \frac{n_{\text{CO,}l}}{n_i} \right) \left( \sum_{j=1}^{J} \frac{n_{\text{CO,}J}}{n_j} + \sum_{l \neq J} \frac{n_{\text{CO,}l} n_l}{n_i n_j} \right)
\]

\[
\rightarrow \tau_{1,\infty} \sum_{l \neq 1} p_{\text{CO,}1} p_{\text{CO,}l} (\rho_{l1} - 1) + \ldots + \tau_{J,\infty} \sum_{l \neq J} p_{\text{CO,}J} p_{\text{CO,}l} (\rho_{lJ} - 1)
\]

Then, taking the absolute value of this limit gets us the desired result.

For the proof of the constant lower bound, we consider the case where (a) the first cluster has \( n_1 \) units and the rest have \( 0 < n_1 < n_2 = \ldots = n_J \) units, (b) \( n_{\text{CO,}1} < n_{\text{CO,}2} = \ldots = n_{\text{CO,}J} \) and (c) the cluster-specific CACE is \( \tau_1 > \tau_2 = \ldots = \tau_J \). Then, we have

\[
\sup_{F} |\tau_{\text{CL}} - \tau| = \sup_{F} \left( \sum_{l=1}^{J} \frac{n_{\text{CO,}l}}{n_i} \right) \left( \sum_{j \neq l} \frac{n_{\text{CO,}j} n_{\text{CO,l}}}{n_i n_j} (n_l - n_j) (\tau_j - \tau_l) \right)
\]

\[
\geq \left( \frac{n_{\text{CO,}1}}{n_1} + (J - 1) \frac{n_{\text{CO,}2}}{n_1} \right) \left( n_{\text{CO,}1} + (J - 1) n_{\text{CO,}2} \right) \frac{n_{\text{CO,}1} n_{\text{CO,}2}}{n_1 n_2} (n_2 - n_1) (\tau_1 - \tau_2) (J - 1)
\]

\[
= \left( \frac{n_{\text{CO,}1}}{n_1} + (J - 1) \frac{n_{\text{CO,}2}}{n_1} \right) \left( n_{\text{CO,}1} \frac{1}{n_1} + \frac{n_2}{n_1} (J - 1) \frac{n_{\text{CO,}2}}{n_2} \right) \frac{n_{\text{CO,}1} n_{\text{CO,}2}}{n_1 n_2} \left( \frac{n_2}{n_1} - 1 \right) (\tau_1 - \tau_2) (J - 1)
\]

\[
\rightarrow (p_{\text{CO,}1} + (J - 1) p_{\text{CO,}2}) (p_{\text{CO,}1} + \rho_{21} (J - 1) p_{\text{CO,}2}) p_{\text{CO,}1} p_{\text{CO,}2} (\rho_{21} - 1) (\tau_{1,\infty} - \tau_{2,\infty}) (J - 1)
\]

\[
> 0
\]

Second, we work on \( \tau_{\text{TSLs}} \). Again, from the proof in Proposition 4 we have

\[
\tau_{\text{TSLs}} - \tau = \left( \sum_{l \neq j} n_{\text{CO,}l} n_j \right) \left( \sum_{l=1}^{J} \tau_j \right) \left( \sum_{l \neq j} n_{\text{CO,}j} n_{\text{CO,}l} (n_l - n_j) \right)
\]

\[
= \tau_1 \left( \sum_{l \neq 1} n_{\text{CO,}1} n_{\text{CO,}l} (n_l - n_1) \right) + \ldots + \tau_J \left( \sum_{l \neq J} n_{\text{CO,}J} n_{\text{CO,}l} (n_l - n_J) \right)
\]
Each of the terms above can be further decomposed into the following:

\[
\tau_1 \left( \sum_{t \neq 1} n_{CO,1} n_{CO,1} (n_t - n_1) \right) /
\left( \sum_{t \neq j} n_{CO,j} n_j \right) 
\left( n_{CO,1} + \sum_{t \neq 1} n_{CO,1} \right)
\]

\[
= \tau_1 \frac{n_{CO,1} n_{CO,2}}{n_1} \frac{n_1}{n_2} \left( \frac{n_2}{n_1} - 1 \right) + \ldots + 
\tau_1 \frac{n_{CO,1} n_{CO,1}}{n_1} \frac{n_1}{n_2} \left( \frac{n_2}{n_1} - 1 \right)
\]

\[
\rightarrow \frac{\tau_{1,\infty} PCO_1 PCO_2 (p_{21} - 1)}{
\left( \sum_{t \neq j} PCO_1 PCO_2 (p_{2j} - 1) \right) 
\left( p_{CO,1} + \sum_{t \neq 1} \rho_{1t} PCO_i \right)
\}
\]

Thus, we have

\[
\tau_{TSL} - \tau \rightarrow \sum_{q=1}^{L} \tau_q \sum_{k \neq q} PCO_q PCO_k \left( \rho_{kq} - 1 \right) 
\left( p_{CO,q} + \sum_{t \neq q} \rho_{tq} PCO_i \right)
\]

For the proof of the constant lower bound, we consider the same identical case as \( \tau_{CL} \).

Then,

\[
\sup_{\mathcal{F}} |\tau_{TSL} - \tau|
\]

\[
= \sup_{\mathcal{F}} \left\{ \frac{1}{\sum_{t \neq j} n_{CO,t} n_j} \left( \sum_{t=1}^{J} n_{CO,1} \right) \right. 
\left( \sum_{(j,l) \in \mathcal{A}} n_{CO,j} n_{CO,l} (n_t - n_j) (\tau_{j} - \tau_l) \right)
\]

\[
\geq \frac{1}{(n_{CO,1} n_2 J (J - 1)/2 + n_{CO,2} n_1 J (J - 1)/2) (n_{CO,1} + (J - 1) n_{CO,2})} 
\left( \frac{(J - 1) n_{CO,1} n_{CO,2} (n_2 - n_1)}{n_1} (\tau_1 - \tau_2) \right)
\]

\[
= \frac{1}{n_1 n_2} J \left( n_{CO,1} n_2 + n_{CO,2} n_1 \right) \frac{1}{n_1} (n_{CO,1} + (J - 1) n_{CO,2}) 
\frac{n_{CO,1} n_{CO,2} n_2 - n_1}{n_1} (\tau_1 - \tau_2)
\]

\[
\rightarrow \frac{2}{(p_{CO,1} + p_{CO,2}) (p_{CO,1} + p_{21} p_{CO,2})} p_{CO,1} PCO_2 (p_{21} - 1) (\tau_{1,\infty} - \tau_{2,\infty}) > 0
\]

□

**Proof of Proposition**  Let \( v_j(\tau_0) = \frac{A_j^{(1)}(\tau_0)}{m} + \frac{A_j^{(0)}(\tau_0)}{J-m} \) for \( j = 1, \ldots, J \). Then, our test statistic simplifies to

\[
T(\tau_0) = \sum_{j=1}^{J} Z_j v_j(\tau_0) - \frac{1}{J-m} \sum_{j=1}^{J} A_j^{(0)}(\tau_0)
\]

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Note that \( \text{Var}[T(\tau_0) \mid \mathcal{F}, \mathcal{Z}] = \text{Var}[\sum_{j=1}^J Z_j v_j(\tau_0) \mid \mathcal{F}, \mathcal{Z}] \). Also, \( \text{E}[\sum_{j=1}^J Z_j v_j(\tau_0) \mid \mathcal{F}, \mathcal{Z}] = \frac{1}{J-m} \sum_{j=1}^J A_j^{(0)}(\tau_0) \). Thus,

\[
\frac{T(\tau_0) - \text{E}[T(\tau_0) \mid \mathcal{F}, \mathcal{Z}]}{\sqrt{\text{Var}[T(\tau_0) \mid \mathcal{F}, \mathcal{Z}]}} = \frac{\sum_{j=1}^J Z_j v_j(\tau_0) - \text{E}[\sum_{j=1}^J Z_j v_j(\tau_0) \mid \mathcal{F}, \mathcal{Z}]}{\sqrt{\text{Var}[\sum_{j=1}^J Z_j v_j(\tau_0) \mid \mathcal{F}, \mathcal{Z}]}}
\]

By Theorem 2.8.2 of Lehmann (2006), our test statistic meets the criterion for Normal convergence.

**Proof of Corollary 3.** We first take the expectation of the numerator of \( \hat{\tau}_{AE} \) under Assumption 3.

\[
\text{E} \left[ \frac{1}{m} \sum_{j=1}^J Z_j Y_j - \frac{1}{J-m} \sum_{j=1}^J (1-Z_j) Y_j \mid \mathcal{F}, \mathcal{Z} \right] = \frac{1}{J} \sum_{j=1}^J \sum_{i=1}^{n_j} (Y_{ij}^{(1,D_{ji}^{(1)})} - Y_{ij}^{(0,D_{ji}^{(0)})})
\]

Under Assumptions 3 and 4, the above sum simplifies to

\[
\frac{1}{J} \sum_{j=1}^J \sum_{i=1}^{n_j} Y_{ij}^{(1,D_{ji}^{(1)})} - Y_{ij}^{(0,D_{ji}^{(0)})} = \frac{1}{J} \sum_{j=1}^J \sum_{i=1}^{n_j} (Y_{ij}^{(1)} - Y_{ij}^{(0)}) I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)
\]

Similar algebra with the denominator of \( \hat{\tau}_{AE} \) along with Assumption 3 reveals

\[
\text{E} \left[ \frac{1}{m} \sum_{j=1}^J Z_j D_j - \frac{1}{J-m} \sum_{j=1}^J (1-Z_j) D_j \mid \mathcal{F}, \mathcal{Z} \right] = \frac{1}{J} \sum_{j=1}^J \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)
\]

Under Assumption 2, we can take the ratio between the numerator and denominator, which then leads to the desired expression.

**Proof of Proposition 6.** First, we prove that under \( H_{00} \), \( A_j^{(1)}(\tau_0) = A_j^{(0)}(\tau_0) \). Based on As-
Using Assumptions 2-4 and the definition of $\tau_j$, we see that

$$A_j^{(1)}(\tau_0) - A_j^{(0)}(\tau_0) = \sum_{i=1}^{n_j} Y_{ji}^{(1)} - \tau_0 D_{ji}^{(1)} - \sum_{i=1}^{n_j} Y_{ji}^{(0)} - \tau_0 D_{ji}^{(0)}$$

$$= \sum_{i=1}^{n_j} Y_{ji}^{(1)} - Y_{ji}^{(0)} - \tau_0 \sum_{i=1}^{n_j} D_{ji}^{(1)} - D_{ji}^{(0)}$$

$$= \sum_{i=1}^{n_j} (Y_{ji}^{(1)} - Y_{ji}^{(0)}) I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0) - \tau_0 \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)$$

$$= \tau_j \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0) - \tau_0 \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)$$

$$= (\tau_j - \tau_0) \sum_{i=1}^{n_j} I(D_{ji}^{(1)} = 1, D_{ji}^{(0)} = 0)$$

$$= 0$$

where the last equality uses the fact that we are under $H_{00}$. Thus, $A_j^{(1)}(\tau_0) = A_j^{(0)}(\tau_0)$ for all $j$ under $H_{00}$. Also, note that even if Assumptions 2-4 did not hold, the argument $A_j^{(1)}(\tau_0) = A_j^{(0)}(\tau_0)$ will still hold under a more general definition of $\tau_j$, where $\tau_j$ is defined as the value that satisfies the equation

$$\sum_{i=1}^{n_j} Y_{ji}^{(D_{ji}^{(1)})} - Y_{ji}^{(D_{ji}^{(0)})} = \tau_j \sum_{i=1}^{n_j} D_{ji}^{(1)} - D_{ji}^{(0)}$$

Then, since under the null hypothesis, $A_j(\tau_0) = A_j^{(1)}(\tau_0) = A_j^{(0)}(\tau_0)$ remain the same for all clusters and $A_j(\tau_0)$ remains fixed. Thus, the null distribution of $T(\tau_0)$ is simply the permutation distribution of the test statistic $T(\tau_0)$ where we fix the adjusted value $A_j(\tau_0)$ at the observed value and we permute $z \in Z$. 

$\square$