Simultaneous Velocity and Position Estimation via Distance-Only Measurements With Application to Multi-Agent System Control

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Abstract—This paper proposes a strategy to estimate the velocity and position of neighbor agents using distance measurements only. Since with agents executing arbitrary motions, instantaneous distance-only measurements cannot provide enough information for our objectives, we postulate that agents engage in a combination of circular motion and linear motion. The proposed estimator can be used to develop control algorithms where only distance measurements are available to each agent. As an example, we show how this estimation method can be used to control the formation shape and secure velocity consensus of the agents in a multi agent system.

Index Terms—Cooperative control, discrete time Malkin theorem, distance-only measurements, distributed control, formation control, multi-agent systems, sensor networks.

I. INTRODUCTION

The performance of multi-agent systems in various tasks, e.g., consensus [1], formation shape control [2], cooperative geolocalization [3], etc., has been studied with increasing intensity over recent years. These tasks are usually required to be performed in a decentralized way [4] and using limited information. Examples of these tasks are retrieving information from an area covered by a sensor network (where the agents are sensors deployed in the area) [5], or moving together in a desired formation shape from one point to another.

In a formation control problem, which is the focus of this paper, each agent tries to contribute to achieve the global goal of the formation using measurements of, typically, relative position and velocity of its neighbors. Examples of such problems are given in [6], [7]. These problems become more challenging when the agents cannot instantaneously measure all the information required to apply motion corrections to achieve the final goal of the formation and have to estimate some of this information using their measurements.

An example of such a challenging problem is given in [8], where a formation (shape and translation motion) control method, called stop-and-go, has been devised to control the agents not able to measure the relative positions (both distance and angle) of their neighbors, but only able to measure the distances to their respective neighbors. This measurement restriction makes the control problem significantly harder.

This paper treats a related problem. Agents are required to estimate the relative position and velocity of their neighbor agents using only distance measurements to the neighbors, and achieve both velocity consensus and formation shape control. The key is to postulate that the motion of each agent comprises two parts: a translation and a circular motion. The circular motion is around a moving center (so it can appear sinusoidal), and it is the centers of each agent’s motion, rather than the agents themselves, which achieve velocity consensus. The purpose of the superimposed circular motion is to allow inter-agent localization and velocity estimation, not using instantaneous measurements, but using distance measurements collected over an interval. We postulate that neighbor agents remain in communication even if they initially have different velocities.

The notion of using deliberate motions of agents to assist in localization was suggested in [9], in relation to sensor network localization. However, the motions in [9] are random. This paper studies the localization problem using distance-only measurements when agents are executing independent circular motions. Note that the idea of introducing sinusoidal perturbation in formation control problems is not wholly novel: in [10], the authors have introduced sinusoidal perturbations to the usual gradient based control algorithm in order to achieve a different objective. An advantage of having a combination of linear and circular motion over only linear motion as in [8] is that the agents are less likely to travel out of communication range during the localization process.

An abbreviated conference version of this paper has been presented in [11]. The novel contributions of the paper, in comparison to the conference paper [11], are as follows (a) proposing a discrete-time control algorithm to achieve velocity consensus and simultaneous formation shape control, usable with distance-only measurements, and (b) introducing a major improvement by adaptively adjusting the circular motion radius.

The rest of this paper is organized as follows. Section II gives a solution to the location and velocity estimation problem using distance-only measurements when each agent is executing a combination of linear motion and circular motions. Section III discusses an improvement of the algorithm derived in the previous section involving adaptively adjusting the circular motion radius. Section IV discusses a discrete time control algorithm to achieve velocity consensus and formation shape control with distance-only measurements. Simulations are included in each section. Concluding remarks and directions for future research are given in Section V.

II. RELATIVE POSITION AND VELOCITY ESTIMATION USING SINUSOIDAL PERTURBATION

In the conference version of this paper [11], we gave detailed explanation on how to infer a neighboring agent’s relative position and
velocity. We have also discussed all special cases. Here we just give a
brief introduction to the ideas.

A. Problem Statement

Consider two point agents, 1 and 2. Each agent performs a combi-
nation of circular and rectilinear motion, so each has a certain radius,
direction and angular velocity for the circular motion and velocity for
the rectilinear motion. Agent 1 knows its own radius, angular velocity
and the translational velocity of its circle center and can only measure (continuously) the distance but not bearing of agent 2. Conversely,
agent 2 knows its radius, angular velocity and the velocity of its circle
center and can only measure the distance of agent 1. The goal is
for both agents to localize and sense the velocities of each other for
velocity consensus purposes.

As shown in Fig. 1, we set up a global coordinate system with origin
at agent 1’s circle center and agent 2’s circle center on the x axis when
t = 0. Suppose r_1 is the radius of agent i’s motion, \( \omega_i \) is the angular velocity of agent i, z(t) is the distance at time t between agent 1 and 2 and \( \delta \) is the distance between the two circle centers. The coordinate
system is defined by the agent pair, and is used for analysis purposes
by us. Its orientation with respect to agent 1’s local coordinate basis
is not known by agent 1 at this stage though the orientation can be
obtained after that agent learns \( \phi_i \). In addition, let \( v_{xi} \) be the velocity
of agent i’s circle center, \( v_{yi} \) be the relative velocity of agent j’s circle
center with respect to agent i’s circle center, \( v_x \) be the x component of
the velocity \( v_{21} \), and \( v_y \) be the y component of the velocity \( v_{21} \). The
positive direction of angular velocities is counter-clockwise.

We assume in this paper that \( v_x \) and \( v_y \) are constant for\( kT < t < (k+1)T \), \( T > 0 \), \( k = 0, 1, 2, \ldots \) and may only change at time
instants \( kT \), perhaps reflecting a discrete-time consensus algorithm.
We explain later how to choose \( T \).

There holds
\[
z^2(t) = [d + v_x t + r_2 \cos(\omega_2 t + \phi_2) - r_1 \cos(\omega_1 t + \phi_1)]^2
+ [v_y t + r_2 \sin(\omega_2 t + \phi_2) - r_1 \sin(\omega_1 t + \phi_1)]^2.
\]

Let \( d_x = d + v_x t \) and \( d_y = v_y t \) and rewrite (1) using easy algebra as
\[
z^2(t) = (d_x^2 + d_y^2 + r_1^2 + r_2^2) + 2d_x r_2 \cos(\omega_2 t + \phi_2)
+ 2d_y r_2 \sin(\omega_2 t + \phi_2) - 2d_x r_1 \cos(\omega_1 t + \phi_1)
- 2d_y r_1 \sin(\omega_1 t + \phi_1) - 2r_1 r_2 \cos[(\omega_1 - \omega_2) t + (\phi_1 - \phi_2)].
\]

B. Finding Relative Position and Velocity of a Neighbor

In the system comprising a pair of agents 1 and 2, without loss of
generality, we only show how agent 1 can localize and estimate the
relative velocity of agent 2. The first step is for agent 1 to identify the
angular velocity of agent 2, using a Fourier representation of \( z^2(t) \) for
\( t \in [0, T] \). When \( \|v_{2j}\|^2 \|T \) is sufficiently small in comparison to \( r_1 \), \( r_2 \),
and \( d \), four distinct peaks will show up at 0, \( |\omega_1|, |\omega_2| \), and \( |\omega_1 - \omega_2| \) in
frequency domain. This allows agent 1 to pick up the angular velocity
of agent 2. More insights about this assumption will be discussed in
Section III.

In order to identify the value of \( d \), \( \phi_1 \), \( v_x \), and \( v_y \), we allow agent 1 to
measure the distance between the two agents \( z(t) \) and then analyze
the Fourier series of the periodic extension of \( z^2(t) \). Theorem 1 gives
the procedure to identify \( d \), \( \phi_1 \), \( v_x \) and \( v_y \), detailed explanation about
this process is given in [11].

**Theorem 1:** For a pair of point agents in \( \mathbb{R}^2 \), if each agent
is executing a combination of circular motion and linear motion [as
described by (1) and (2)] and the associated angular frequencies are
commensurate, each agent can find the position and translational
velocity of the other agent by distance-only measurements over an
interval, which equals to an integer multiple of a common period for
the agent’s circular motions.¹

**Proof:** The definitions of \( r_1 \), \( r_2 \), \( \omega_1 \), \( \omega_2 \), \( d \), \( z \), \( v_x \), \( v_y \), \( \phi_1 \), and
\( \phi_2 \) are the same as in Section II-A. We choose \( T \) so that there exist
integers \( k_1 \), \( k_2 \) defining the multiple which \( T \) represents of the periods
associated with the two angular velocities, i.e. \( k_1 = (\omega_1 T/2\pi) \)
and \( k_2 = (\omega_2 T/2\pi) \). The existence of \( k_1 \) and \( k_2 \) relates to the concept
of commensurable numbers, see [11, Remark 3].

Suppose one continuously measures \( z \) for a time period \( T \) and
finds the Fourier series of the periodic extension of \( z^2 \). Consider (2)
and suppose \( c_n \) are the coefficients of Fourier series using powers
of \( e^{j(2\pi n/T)} \) of the periodic extension of \( z^2 \). Similarly, \( s_n \), \( u_n \) and
\( w_n \) are the coefficients of the Fourier series of the periodic extension
of \( (d^2_1 + d^2_2 + r^2_1 + r^2_2) \), \( -2d_x r_1 \cos(\omega_1 t + \phi_1) - 2d_x r_2 \sin(\omega_1 t + \phi_1) \)
and \( 2d_r r_2 \cos(\omega_2 t + \phi_2) + 2d_r r_2 \sin(\omega_2 t + \phi_2) \) respectively.

From (2) we know that for any \( n > 0 \cap n \neq |k_1 - k_2| \) there holds
\( c_n = s_n + u_n + w_n \). Further by calculating the Fourier Series of (2)
we know for any \( n > 0 \cap n \neq |k_1 - k_2| \) or \( |k_1 - k_2| \) there holds
\[
c_n = \frac{1}{n^2} R + \frac{1}{n} I j + \frac{1}{n - k_1} + \frac{1}{n + k_1}
\times 2U + \left( \frac{1}{n - k_2} + \frac{1}{n + k_2} \right) \cdot 2W
\]
where
\[
U = r_1 \left( \frac{j v_x T}{4 \pi} + \frac{v_y T}{4 \pi} \right) e^{j(\phi_1 + \pi)}
\]
\[
W = r_2 \left( \frac{j v_x T}{4 \pi} + \frac{v_y T}{4 \pi} \right) e^{j(\phi_2)}
\]
\[
R = \frac{(v_x^2 + v_y^2)}{2\pi^2} T^2
\]
\[
I = \frac{(v_x^2 + v_y^2)}{2\pi^2} T^2 + 2v_x dT
\]

From (3) and the non-singular condition [11, Lemma 3] we know that
if we have four values of \( c_n \), \( n > 0 \cap n \neq |k_1 - k_2| \) or \( |k_1 - k_2| \),
we are able to find the unique solutions of \( R, I, U \) and \( W \).

Now we have the value of \( U \) and \( W \) and can obtain \( u_{k_1} \), from the
Fourier Series of (2). Furthermore, from (4) and the Fourier Series of
(2) we know that
\[
u_{k_1} = \frac{U}{2k_1} = r_1 \left( \frac{d + \frac{1}{2} v_x T - \frac{1}{2} v_y T j}{4 \pi} \right) e^{j(\phi_1 + \pi)}
\]
\[
u = r_1 \left( \frac{j v_x T}{4 \pi} + \frac{v_y T}{4 \pi} \right) e^{j(\phi_1 + \pi)}
\]

¹ Refer to [11, Remark 3] for treatment of incommensurate periods.
and \( d, v_x, v_y, \) and \( \phi_i \) can be found from these equations. They are 
\[
d = (2/r_1)||u_{\hat{k}_1} - (U/2k_1) + \pi j U||, \quad \phi_1 = \arg(u_{\hat{k}_1} - (U/2k_1) + \pi j U) \quad \pi \quad \nu_x = \text{Im}(4\pi U/T r_1 e^{i(\phi_1 + \pi)}), \quad \nu_y = \text{Re}(4\pi U/T r_1 e^{i(\phi_1 + \pi)}).
\]

### III. Adaptive Radius Assignment

In the above sections, we let each agent infer the position and relative velocity information of neighboring agents by 1) carrying out a Fourier transform and then 2) identifying peaks to estimate \( \omega \) of neighboring agents 3) solving the set of linear equations (3). In step 2) if \( ||v_{t12}/\pi|| \) is sufficiently small in comparison to \( r_1, r_2, \) we can show that there are always peaks at \( k_1 \) and \( k_2, \)

**Lemma 1:** Adopt the hypothesis in Theorem 1. If \( d > r_1, r_2 \) and \( ||v_{t1}/\pi|| \) is sufficiently small in comparison to \( r_1, r_2, \) then \( c_n \) (regarded as a function of the integer \( n \)) has peaks at \( n = k_1 \) and \( n = k_2, \)

**Proof:** When \( n \neq 0, k_1, k_2, |k_1 - k_2|, \) the dependence of \( ||c_n|| \) on \( ||v_{t1}/\pi|| \) can be expressed as follows:

\[
||c_n|| = ||h_1(k_1, k_2, T)||v_{t1}||^2 + h_2(k_1, k_2, T)||v_{t1}||d + h_3(k_1, k_2, T)||v_{t1}||r_1 + h_4(k_1, k_2, T)||v_{t1}||r_2||v_{t1}||v_{t2}||(10)
\]

On the other hand, when \( n = k_1, \) there holds

\[
||c_n|| = ||h_5(k_1, k_2, T)||v_{t1}||^2 + h_6(k_1, k_2, T)||v_{t1}||d + h_7(k_1, k_2, T)||v_{t1}||r_1 + h_8(k_1, k_2, T)dr_1||v_{t1}||v_{t2}||(11)
\]

where \( h_1, h_2, ..., h_8(k_1, k_2, T) \) are all bounded functions for integer \( k_1, k_2 \) and \( T > 0. \)

Now compare (10) and (11). Suppose \( d > r_1, r_2. \) For a sufficiently small \( ||v_{t1}/\pi|| \) the term \( h_6(k_1, k_2, T)dr_1 \) will be dominant and thus there will be peak recognized at \( n = k_1. \) Similarly there will also be peak recognized at \( n = k_2. \)

According to Lemma 1, the proposed algorithm works if \( d > r_1, r_2 \) and \( ||v_{t1}/\pi|| \) is sufficiently small with respect to \( r_1, r_2. \) In reality, \( d > r_1, r_2 \) is automatically satisfied if we aim to avoid collision. Furthermore, in order to ensure that \( ||v_{t1}/\pi|| \) is sufficiently small with respect to \( r_1, r_2 \) for each agent pair, we propose an adaptive radius algorithm whereby \( r_1 \) of each agent is re-set at the end of each \( T \) second intervals as follows:

\[
r_1((k+1)T) = \alpha \cdot \max_j(||v_{ij}(kT)||)
\]

where \( j \) denotes the indices of neighboring agents of \( i \) and \( \alpha \) is a sufficient large value depend on \( T. \) Note that because \( r_1 \) only changes at the end of each interval \( T, \) the radius is fixed within each interval.

The adaptive radius law will ensure that \( r_1, r_2 \geq \alpha \cdot ||v_{ij}/\pi|| \) holds for each agent pair. Furthermore, as velocity consensus is being achieved, \( ||v_{ij}(kT)|| \) will approach zero and so will \( r_1. \) It is noticeable that the accuracy of estimation of \( ||v_{ij}(kT)|| \) is independent of the value of \( \phi_i. \) Each agent can estimate the norm of velocities of neighbors’ circle centers via \( R \) according to (6), even if no peaks are identified. This phenomenon is consistent with the paper [8], which shows that without circular motions, for agents only doing linear motions, it is possible to estimate the norm of relative velocities of neighbors, even though the directions are left unknown.

When there are sudden changes in velocities of agents due to e.g., wind or deliberate change of course by a leader agent, an already achieved consensus and formation may be broken. In this case, even if the radius of the circle of each agent has already approached to zero, each agent can still obtain a good estimate of the absolute value of velocities of its neighbors’ circle centers. This can result in an increase of radius of circular motions in response to the broken consensus, which allows the agents to achieve velocity consensus and formation shape control again.

Similarly to the setting in [11, Section V-B], consider a multi-agent system shown in Fig. 2, suppose \( \omega_i \) is a circular radius of agent \( i, \) \( T \) is the sampling time interval, \( (v_{xi}, v_{yi}) \) is the translational velocity of agent \( i \) and \( (p_{xi}, p_{yi}) \) is the position of circle center of agent \( i. \) The in simulation, we set \( \omega_1 = \omega_3 = 5, \omega_2 = -3, \) \( T = 2\pi. \) When \( t = 0, (v_{x1}, v_{y1}) = (-4, 2), \) \( (v_{x2}, v_{y2}) = (3, -2), \) \( (v_{x3}, v_{y3}) = (2, 4), \) \( (p_{x1}, p_{y1}) = (70, 30), \) \( (p_{x2}, p_{y2}) = (0, 50), \) \( (p_{x3}, p_{y3}) = (0, 0) \) and \( \varepsilon = 0.35. \) Figs. 3 and 4 show the simulation result, where the velocity of agent 2 changes suddenly at \( t = 20T. \)
IV. COMBINING VELOCITY CONSENSUS AND FORMATION SHAPE CONTROL

A. Stability of Discrete Time Control Algorithm

To the best of our knowledge, there is so far no discrete-time algorithm that combines velocity consensus with formation shape control for a multi-agent system. Although there is a continuous-time version algorithm proposed in [12], it cannot be implemented directly when we let each agent measure distance for a time period $T$ and then make a velocity adjustment at the end of each such interval. This section proposes a discrete-time algorithm that combines velocity consensus with formation shape control, where the sampling time interval is not required to be sufficiently small.

We start with the continuous-time algorithm as stated in [12]

$$
\dot{p}_i = v_i
$$

$$
\dot{v}_i = \sum_{j \in N_i} (v_j - v_i) + 2 \sum_{j \in N_i} (d_{ij}^2 - d_{ij}^2) (p_i - p_j)
$$

(13)

where $p_i$ is the position of the $i$th agent, $i = 1, \ldots, N$, $v_i$ is the velocity of the $i$th agent and $N_i$ is the set of neighboring agents of agent $i$. Further, $d_{ij}^2$ is the desired distance between agent $i$ and $j$ and $d_{ij}$ is the current distance between agent $i$ and $j$. In our context, agent positions and velocities refer to the center of the circular motion. The system (13) can be written in the matrix form

$$
\dot{p} = v
$$

$$
\dot{v} = - (L \otimes I_2) v + f(p)
$$

(14)

where $p \in \mathbb{R}^{2N}$ is the vector of all $p_i$, stacked together, $L$ denotes the Laplacian matrix which is positive semi-definite and has one zero eigenvalue when the graph is connected and undirected, and $f(p)$ is a vector with the entries $2 \sum_{j \in N_i} (d_{ij}^2 - d_{ij}^2) (p_i - p_j), i = 1, \ldots, N$.

A discrete version of (13) for our use is given by

$$
\dot{p}_i = v_i
$$

$$
v_i((k+1)T) = v_i(kT) + \epsilon_1 T \sum_{j \in N_i} (v_j(kT) - v_i(kT))
$$

$$
+ 2 \epsilon_2 T \sum_{j \in N_i} (d_{ij}^2 - d_{ij}^2) (p_i(kT) - p_j(kT))
$$

(15)

where $\epsilon_1, \epsilon_2$ are suitably small positive constants. Note the first equation remains in continuous time. However, since $v_i(t)$ is constant over an interval $T$, it follows that the discretization of the first equation, viz.

$$
p_i((k+1)T) = p_i(kT) + T v_i(kT)
$$

exactly interpolates the continuous function $p_i(t)$ for $t = kT$ with integer $k$.

To show (15) yields convergence to the desired shape with velocity consensus, we start with the continuous-time system and transform

$$
\tilde{p}_r = R p, \quad \tilde{v}_r = R v
$$

(16)

where $R$ is an orthonormal matrix whose first two rows are $(1 \otimes I_2)^T / \sqrt{N}$, $p_0 := [p_{i_0}^{\top} \tilde{p}_0^{\top}]^T$ with $p_0 \in \mathbb{R}^2$ and $v_0 := [v_{i_0}^{\top} \tilde{v}_0^{\top}]^T$. Then $v_0 = 0$, i.e., the center of mass of the agents in $\tilde{p}$-coordinates is constant, and the system equations in $\tilde{p}$ and $\tilde{v}$ are

$$
\dot{\tilde{p}} = \tilde{v}
$$

$$
\dot{\tilde{v}} = L \tilde{v} + \tilde{f}(\tilde{p})
$$

(17)

where $L$ is the $(2N - 2) \times (2N - 2)$ nonzero block of $-R(L \otimes I_2) R^T$ which is negative definite and $\tilde{f}((\tilde{p})$ contains the nonzero entries of $R f(R^T \tilde{p}) = R f(p)$.

A convergence property for (15) or equivalently the discrete version of (17) is as follows: first we define a Malkin structure in Definition 1. After that we show in Lemma 2 that (17) is transformable to a Malkin structure. Then we develop in Theorem 2 a discrete-time version of the continuous-time Malkin’s theorem as invoked by Krick [13]. Finally, we use these results and show in Theorem 3 that (15) yields convergence to the correct shape with velocity consensus for sufficiently small values of $\epsilon_1$ and $\epsilon_2$.

**Definition 1:** (Malkin structure).

A system has Malkin structure if it is in the form

$$
\dot{r} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} r + g(\theta, \rho), \quad r = \begin{bmatrix} \theta \\ \rho \end{bmatrix}, \quad g = \begin{bmatrix} \Theta(\theta, \rho) \\ P(\theta, \rho) \end{bmatrix}
$$

(18)

where $A$ is constant with negative real eigenvalues. Furthermore, $g(\theta, \rho)$ is a second order term satisfying the following conditions

i) $g(\theta, 0) = 0$, ii) there exists:

$$
h_1(\theta) = \lim_{\rho \to 0} \frac{\Theta(\theta, \rho)}{\rho}, \quad h_2(\theta) = \lim_{\rho \to 0} \frac{P(\theta, \rho)}{\rho}
$$

$$
\begin{cases} 
\phi(\theta, \rho) & \text{if } \rho \neq 0, \\
h_1(\theta) & \text{if } \rho = 0
\end{cases}, \quad b_2 = \begin{cases} 
\frac{P(\theta, \rho)}{\rho} & \text{if } \rho \neq 0, \\
h_2(\theta) & \text{if } \rho = 0
\end{cases}
$$

such that $b_1$ and $b_2$ are bounded smooth functions and $b_2(0) = 0$.

**Lemma 2:** The system equations in (17) can be transferred to a Malkin structure through a local diffeomorphism around the equilibrium point of (17).

**Proof:** Suppose there are $N$ agents in a formation. Consider the single-integrator formation shape control system [14]

$$
\dot{p} = f(p)
$$

(19)

where $f(p)$ is a vector with entries $\sum_{j \in N_i} (d_{ij}^2 - d_{ij}^2) (p_i - p_j), i = 1, \ldots, N$, and $d_{ij}, d_{ij}', p_i, p_j$ are as defined in (13). It is shown in [13] that there is a local diffeomorphism around the equilibrium point that transfers (19) to a Malkin structure. Suppose the diffeomorphism is

$$
r = \phi(p), \quad \tilde{p} = \psi(r)
$$

(20)

such that $\tilde{p} = \tilde{f}(\tilde{p})$, where $\tilde{p}$ and $\tilde{f}$ are defined in (16) and (17), transfers to a Malkin structure

$$
\dot{r} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} r + g(\theta, \rho), \quad r = \begin{bmatrix} \theta \\ \rho \end{bmatrix}, \quad g = \begin{bmatrix} \Theta(\theta, \rho) \\ P(\theta, \rho) \end{bmatrix}
$$

(21)

where $A$ has eigenvalues with negative real parts and $g(\theta, \rho)$ fulfills the conditions of the second order term $g(\cdot)$ in Definition 1. Let $n_\theta$ denote the number of elements in $\theta$ and $n_\rho$ denote the number of elements in $\rho$.

Now we are going to show that the velocity and formation shape control problem in (17)

$$
\dot{\tilde{p}} = L \tilde{p} + \tilde{f}(\tilde{p}), \quad L = LT^T < 0
$$

(22)

is also transferred to a Malkin structure by the same diffeomorphism. Observe

$$
\dot{r} = \frac{\partial \phi}{\partial \tilde{p}} \dot{\tilde{p}} + \frac{\partial \phi}{\partial \tilde{p}} \tilde{f}(\tilde{p}) = \left( \frac{\partial \psi}{\partial r} \right)^{-1} \tilde{f}(\psi(r))
$$

(23)

2There are minor differences in the definition of Malkin structure in different references. We use the definition in [13] here.
Now the right sides of (21) and (23) are the same. Next
\[ \dot{r} = \Psi(r, \dot{r}) + \frac{\partial \phi}{\partial p} \dot{p} \]
(24)
where the row \( i \) column \( j \) entry of \( \Phi(p, \dot{p}) \) takes the form
\[ \sum_k \frac{\partial^2 \phi_j}{\partial p_i \partial p_k} \dot{p}_i \dot{p}_k. \]
Combining (24) and (22), we obtain
\[ \dot{r} = \Psi(r, \dot{r}) + \frac{\partial \phi}{\partial p} L \dot{p} + \frac{\partial \phi}{\partial p} (\dot{p} \dot{r}) \]
(25)
where \( \Psi(r, \dot{r}) = \Phi(\psi(r), (\partial \psi / \partial r) \dot{r}) \) is \( O(\| \dot{r} \|^2) \). Hence
\[ \dot{r} = \Psi(r, \dot{r}) + h(r) \dot{r} + g(\theta, \rho). \]
(26)
Now we have the system equation
\[ \frac{d}{dt} \begin{bmatrix} r \\ \dot{r} \end{bmatrix} = C \begin{bmatrix} r \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 \\ \Psi(r, \dot{r}) + h \cdot \dot{r} + g(\theta, \rho) \end{bmatrix} \]
(27)
where
\[ C = \begin{bmatrix} 0 & D \\ 0 & E \end{bmatrix} \quad \text{and} \quad h = - \left( \frac{\partial \phi}{\partial r} \right)^{-1} L \left( \frac{\partial \psi}{\partial r} \right) + \left( \frac{\partial \phi}{\partial r} \right)^{-1} L \left( \frac{\partial \psi}{\partial r} \right) \]
with \( D \) a \((n_\theta + 2n_\phi) \times (n_\theta + 2n_\phi)\) nonsingular square matrix.

Consider the nonsingular similarity transformation \( T = \begin{bmatrix} I & -DE^{-1} \\ 0 & I \end{bmatrix} \) and define \( \begin{bmatrix} \bar{r} \\ \dot{\bar{r}} \end{bmatrix} = T \begin{bmatrix} r \\ \dot{r} \end{bmatrix} \). Note \( \dot{\bar{r}} = \bar{\theta} / \bar{p} \). There holds
\[ \frac{d}{dt} \begin{bmatrix} \bar{r} \\ \dot{\bar{r}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \bar{r} \\ \dot{\bar{r}} \end{bmatrix} + T \cdot o(\bar{r}, \dot{\bar{r}}) \]
(29)
with \( \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \).

Krick shows in (13) that first applying the diffeomorphism \( r = \phi(p) \) and then linearizing the system is equivalent to first linearizing the system and then applying the diffeomorphism. It is shown in (15) that the system (22) locally converges exponentially fast to a point on a center manifold; therefore the system matrix of the linearization of (22) at a point on the center manifold
\[ \left( \begin{array}{cc} 0 & I \\ \frac{\partial \phi}{\partial p} & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{\partial \phi}{\partial p} \end{array} \right) \]
has eigenvalues with non-positive real parts. Furthermore, because the local diffeomorphism is smooth, its linearization around the equilibrium \( \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{\partial \phi}{\partial p} \end{array} \right) \) is a nonsingular similarity transformation. Therefore, the linearization of the system equation after applying the diffeomorphism
\[ C = \left[ \begin{array}{cc} \frac{\partial \phi}{\partial p} & 0 \\ \frac{\partial \phi}{\partial p} & 0 \end{array} \right] \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{\partial \phi}{\partial p} \end{array} \right)^{-1} \]
also has eigenvalues with non-positive real parts. With (28) and the fact that \( A \) and \( L \) are both full rank, we know \( E \) is a non-singular square matrix. Thus \( E \) has eigenvalues with negative real parts.

Define
\[ l(\bar{\theta}, \bar{p}, \bar{r}) = o(\bar{r}, \dot{\bar{r}}) = \Psi(r, \dot{r}) + h \cdot \dot{r} + g(\theta, \rho) \]
Because \( \Psi = O(\| \dot{r} \|^2) \), \( h = 0 \) when \( \bar{p} = 0 \) and \( \dot{\bar{r}} = 0 \), and \( g(\theta, \rho) \) fulfills the conditions of second order term \( g(\cdot) \) in Definition 1, we can conclude that \( l(\bar{\theta}, \bar{p}, \bar{r}) \) fulfills the conditions of second order term \( g(\cdot) \) in Definition 1. Therefore, we have completed the proof.

**Theorem 2—[Discrete Time Malkin Theorem]:** Consider the time-discretized version (30) below of the Malkin structure (18) in Definition 1, where \( \theta_k \) and \( p_k \) are the \( k \)th sample of the quantities \( \theta \) and \( \rho \) in Definition 1. Then there exists a sufficiently small sampling time interval \( \epsilon \) (certainly with \( \epsilon < 1 \), and a sufficiently small open ball \( V \) around the origin such that if \( (\theta_0, p_0) \) lies in this open ball, then \( (\theta_k, p_k) \) lies in the ball for all \( k \) and \( p_k \rightarrow 0 \) exponentially fast and \( \theta_k \) approaches a limit exponentially fast.

**Proof:** The time-discretized version of Malkin structure takes the following form:
\[ \begin{bmatrix} \theta_{k+1} \\ p_{k+1} \end{bmatrix} = \begin{bmatrix} 1 + \epsilon & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \theta_k \\ p_k \end{bmatrix} + \epsilon \begin{bmatrix} \Theta(\theta_k, p_k) \\ P(\theta_k, p_k) \end{bmatrix} \]
(30)
where \( A \) has eigenvalues with negative real parts, \( \Theta(\theta, 0) = 0 \) and \( P(\theta, 0) = 0 \). Define
\[ b_1(\theta_k) = \lim_{\rho_k \to 0} \frac{\Theta(\theta_k, p_k)}{\| p_k \|} \quad b_2(\theta_k, p_k) = \lim_{\rho_k \to 0} \frac{P(\theta_k, p_k)}{\| p_k \|} \]
(31)
Because \( \lim_{\rho_k \to 0} (P(0, \rho)/\| \rho \|) = 0 \), we know that \( b_2(0) = 0 \). Since \( A \) has eigenvalues with negative real parts, for all sufficiently small \( \tau > 0 \), the matrix \( A_{d, \tau} = I + \tau A \) will have eigenvalues inside the unit circle. Without loss of generality, we may assume (using a nonsingular similarity transformation \( T \) if necessary, corresponding to a replacement of \( p_k \) by \( T p_k \)) that for some \( \gamma > 0 \), there holds
\[ I - A_{d, \tau} \geq \gamma I \]
(32)
Now set \( V(\rho_k) = \rho_k^2 \). Also, note that given any \( \sigma > 0 \), there exists \( \eta(\sigma) \) and a closed ball \( B_\eta \), without loss of generality contained in \( V \), such that
\[ \| b_1(\rho_k, \theta_k) \| \leq \sigma \quad \forall (\rho_k, \theta_k) \in B_\eta \]
(33)
Now observe that for \( (\rho_k, \theta_k) \in B_\eta \) there holds
\[ V(\rho_{k+1}) - V(\rho_k) = \rho_k^2 (I - A_{d, \tau}^T A_{d, \tau}) \rho_k + 2\tau \rho_k^2 A_{d, \tau}^T P(\rho_k, \theta_k) + \tau^2 \| P(\rho_k, \theta_k) \|^2 \]
\[ \leq -\gamma \rho_k^2 \rho_k + 2\tau \| A_{d, \tau} \| \rho_k^2 \| b_1(\rho_k, \theta_k) \| + \tau^2 \| \rho_k \|^2 \| b_1(\rho_k, \theta_k) \|^2 \]
\[ \leq -\gamma + 2\tau \sigma + \tau^2 \sigma^2 \| \rho_k \|^2 \]
(34)
Restrict \( \sigma \) to be small enough that \( 2\tau + \sigma^2 < \gamma/2 \). Then we achieve
\[ V(\rho_{k+1}) - V(\rho_k) \leq \gamma - (\gamma/2) \| \rho_k \|^2 \]
(35)
Provided that the sequence \((\rho_k, \theta_k)\) remains in \(B_{\eta_0}\), exponential convergence to zero of \(\rho_k\) is achieved. We shall now argue that this can be assured through appropriate selection of the initial condition. Suppose to obtain a contradiction that there exists a finite \(K\) such that \((\rho_k, \theta_k) \in B_{\eta_0} \forall k \in [0, K]\) but the condition fails for \(k = K + 1\). Suppose that the function \(b_2\), which is continuous, attains an upper bound of \(\bar{m}\) on \(B_{\eta_0}\). Observe that for all \(k \in [0, K]\)
\[
\|\Theta(\rho_k, \theta_k)\| = \|\rho_k\| \|b_2(\rho_k, \theta_k)\| \leq \bar{m}\|\rho_0\| \left(1 - \left(\frac{\gamma}{2}\right)^k\right) \tag{36}
\]
which implies by summation that
\[
\|\theta_{k+1}\| \leq \bar{m}\|\rho_0\| \frac{1}{1 - (\gamma/2)} + \|\theta_0\| \tag{37}
\]
Now restrict the initial condition \((\rho_0, \theta_0)\) to lie-in a smaller ball than \(B_{\eta_0}\). Define a \(\eta_0 < \eta\) as a positive quantity satisfying
\[
\eta_0 + \bar{m} \frac{1}{1 - (\gamma/2)} \eta_0 < \eta \tag{38}
\]
and suppose that \((\rho_k, \theta_k) \in B_{\eta_0}\). Then while the trajectory \((\rho_k, \theta_k)\) remains in \(B_{\eta_0}\), i.e. for all \(k \in [0, K]\) with \(K\) maximal, we know using (35), (37) that
\[
\|\rho_{k+1}\| \leq \|\rho_k\| + \|\theta_{k+1}\|
\leq \eta_0 + \bar{m} \frac{1}{1 - (\gamma/2)} \eta_0 + \eta_0 < \eta. \tag{39}
\]
This shows that \((\rho_{K+1}, \theta_{K+1})\) \(\in B_{\eta_0}\), and that \(K\) is not maximal, i.e. there cannot be a finite \(K\). Hence exponential convergence of the sequence \(\rho_k\) to zero and the sequence \(\theta_k\) to a constant is established. \(\square\)

**Theorem 3**: Consider the system of equations in \((15)\) and suppose the graph associated with the velocity measurements is connected and undirected. Then \(\forall T \geq 0, \exists \epsilon_1, \epsilon_2 > 0\) (being sufficiently small) such that if the velocity is kept constant in every time interval \((kT, (k+1)T)\), then the system \((15)\) converges to the manifold \(d_i^2 - d_{ij}^2(kT) = 0, \forall i, j\) as \(k \to \infty\).

**Proof**: Lemma 2 shows that \((17)\) can be transformed to a Malkin structure and therefore the variation of it with positive \(\alpha, \beta < 1\)
\[
\begin{align*}
\dot{\bar{p}} &= \bar{v} \\
\dot{\bar{v}} &= \alpha L\bar{v} + \beta \bar{f}(\bar{p}) \tag{40}
\end{align*}
\]
can also be transformed to a Malkin structure. Furthermore, Theorem 2 shows that the discretization of this Malkin structure with a sufficiently small sampling interval will converge to some limit exponentially fast. In fact the operations of coordinate basis change through a diffeomorphism to a Malkin equation and time-discretization commute (see [16, Appendix III]). Hence the discretization of (40)
\[
\begin{align*}
\bar{p}_i ((k + 1)\delta) &= \bar{p}_i (k\delta) + \delta \bar{v}_i (k\delta) \\
\bar{v}_i ((k + 1)\delta) &= \bar{v}_i (k\delta) + \delta \alpha \sum_{j \in N_i} (\bar{v}_j (k\delta) - \bar{v}_i (k\delta)) \\
&+ 2\delta \beta \sum_{j \in N_i} (d_{ij}^2 - d_{ij}^2(k\delta)) (\bar{p}_i (k\delta) - \bar{p}_j (k\delta)) \tag{41}
\end{align*}
\]
with sufficiently small sampling interval \(\delta\) converges exponentially fast to a point on the manifold \(d_i^2 - d_{ij}^2(kT) = 0, \forall i, j\) as \(k \to \infty\). \(\square\)

### B. Simulation Results Combining Velocity Consensus and Formation Shape Control

Consider a three-agent system where each agent can measure its distance to the other two agents. The goal is to achieve velocity consensus and form a triangular formation. Suppose \(\omega_i\) is the angular velocity of agent \(i\) and \(T\) is the sampling time interval, \((v_{x1}, v_{y1})\) is the translational velocity of agent \(i\), \((p_{x1}, p_{y1})\) is the position of circle center of agent \(i\). In the simulation, we set \(\omega_1 = 5, \omega_2 = -3, \omega_3 = 7, T = 2\pi\). When \(t = 0\), \((v_{x1}, v_{y1}) = (-4, 1.5), (v_{x2}, v_{y2}) = (3, -3.5), (v_{x3}, v_{y3}) = (2, 3.5), (p_{x1}, p_{y1}) = (100, 50), (p_{x2}, p_{y2}) = (0, 80), (p_{x3}, p_{y3}) = (0, 0), \epsilon_1 = 5 \times 10^{-2} \) and \(\epsilon_2 = 7 \times 10^{-7}\). The desired distance between each pair of agents in the formation is \(20\). Agents’ trajectories are shown in Fig. 5.
V. CONCLUSION

In this paper, we proposed a strategy to achieve velocity consensus and formation control using distance-only measurements for multiple agents. Given the fact that for agents to execute arbitrary motions, instantaneous distance-only measurements cannot provide enough information, we studied agents performing a combination of circular motion and linear motion.

In further research, we are looking to achieve formation control and velocity consensus using agent perturbations, where agents are not limited to performing a combination of circular motion and linear motion. In addition, it appears very likely that the same strategy as we proposed in this paper can be used in velocity consensus using bearing-only measurements.

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