Vanishing Chern classes for numerically flat Higgs bundles

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Abstract

I consider Higgs bundles satisfying a notion of ampleness that was introduce in [8, 5] and prove that the Chern classes of rank $r$ II-ample Higgs bundles over dimension $n$, polarized, smooth, complex, projective varieties are positive under opportune hypothesis. I extend this to non-negativeness of Chern classes of all numerically effective Higgs bundles; and use this condition to prove the vanishing of Chern classes of numerically flat Higgs bundles.

Introduction

Let $X$ be a smooth, complex, projective variety of dimension $n \geq 2$, and let $H$ be a polarization on $X$, that is an ample line bundle over $X$. A line bundle $L$ over $X$ is said numerically effective (nef, for short) if for every irreducible curve $C$ in $X$ the following inequality holds $\int_C c_1(L) \geq 0$; and a vector bundle $E$ over $X$ is said nef if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ on $\mathbb{P}(E^\vee)$ is nef.

In [11], the authors proved several properties of these vector bundles, in particular they proved that the numerically flat bundles, ($nflat$ being the vector bundles that are numerically effective together with their duals) have vanishing Chern classes (see [11] Corollary 1.19).

In [8, 5], the authors extended these notions to Higgs bundles setting, which is explained below.
Definition 0.1. A Higgs sheaf $\mathcal{E}$ is a pair $(E, \varphi)$ where $E$ is a $\mathcal{O}_X$-coherent sheaf equipped with a morphism $\varphi: E \to E \otimes \Omega^1_X$ such that the composition

$$\varphi \wedge \varphi: E \xrightarrow{\varphi} E \otimes \Omega^1_X \xrightarrow{\varphi \otimes \text{Id}} E \otimes \Omega^1_X \otimes \Omega^1_X \to E \otimes \Omega^2_X$$

vanishes. A Higgs subsheaf of a Higgs sheaf $(E, \varphi)$ is a $\varphi$-invariant subsheaf $G$ of $E$, i.e., $\varphi(G) \subset G \otimes \Omega^1_X$. A Higgs quotient of $\mathcal{E}$ is a quotient of $E$ such that the corresponding kernel is $\varphi$-invariant. A Higgs bundle is a Higgs sheaf whose underlying coherent sheaf is locally free.

One introduces suitable closed subschemes $\mathfrak{Gr}_s(\mathcal{E})$ which parameterizes the rank $s$ locally free Higgs quotient bundles of $\mathcal{E}$, from which the name of Higgs Graßmannians. These Higgs Graßmannian are used to give a notions of ample and numerically effective Higgs bundles ($H$-ample and $H$-nef, respectively, for short), which are sensible to the Higgs field. Of course, if $\mathcal{E}$ and $\mathcal{E}^\vee$ are both $H$-nef one says it numerically flat Higgs bundles ($H$-nflat, for short).

As for holomorphic vector bundles, one asks whether the following statement holds:

**Conjecture 0.2.** If $\mathcal{E} = (E, \varphi)$ is a $H$-nflat Higgs bundle, then $c_k(E) = 0$ for any $k > 0$.

The answer is positive almost in the following cases:

- $\varphi = 0$, of course;
- $\mathcal{E}$ has a filtration in Higgs subbundles whose quotients are Hermitian flat Higgs bundles ([7, Theorem 3.16]);
- $X$ is a simply-connected Calabi-Yau variety ([4, Theorem 4.1]).

On the other hand, the previous conjecture is equivalent to others interesting statements about Higgs bundles. Recall that the degree of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is the integer number

$$\text{deg}(\mathcal{F}) = \int_X c_1(\mathcal{F}) \cdot H^{n-1}$$

and if $\mathcal{F}$ has positive rank, its slope is defined as

$$\mu(\mathcal{F}) = \frac{\text{deg}(\mathcal{F})}{\text{rank}(\mathcal{F})}.$$  

Definition 0.3. Let $\mathcal{E} = (E, \varphi)$ be a torsion-free Higgs sheaf.

a) It is semistable (respectively, stable) if $\mu(G) \leq \mu(E)$ (respectively, $\mu(G) < \mu(E)$) for every Higgs subsheaf $\mathcal{G} = (G, \psi)$ of $\mathcal{E}$ with $0 < \text{rank} G < \text{rank} E$.

b) It is polystable if it is a direct sum of stable Higgs sheaves having the same slope.

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1 For the precise definition I remind to page [4].
c) It is curve semistable if for every morphism \( f: C \rightarrow X \), where \( C \) is a smooth projective irreducible curve, the pullback \( f^* \mathcal{E} \) is semistable as Higgs bundle.

**Theorem 0.4.** The Conjecture 0.2 is equivalent to the following statements:

a) if \( \mathcal{E} \) is curve semistable then \( \mathcal{E} \) is semistable with discriminant class \( \Delta(E) = \frac{1}{2r} c_2(\text{Ad}(E)) = 0 \in H^4(X, \mathbb{Q}) \) (see [2, Corollary 3.2]).

b) if \( \mathcal{E} \) is H-nflat then it has a filtration in Higgs subbundles whose quotients are Hermitian flat Higgs bundles (see [4, Theorem 5.2]).

**Remark 0.5.** a) If \( \mathcal{E} \) is curve semistable, by [7, Lemma 4.3.(i)] one can replace \( \mathcal{E} \) with its adjoint bundle \( \text{Ad}(\mathcal{E}) \); by [7, Lemma A.7] \( \mathcal{E} \) is H-nflat, by [7, Proposition A.8] \( \mathcal{E} \) is semistable.

b) If \( \mathcal{E} \) is H-nflat then it has a Jordan-Hölder filtration, i.e. a filtration whose quotients are locally free and stable, and moreover these quotients are H-nflat (see [4, Theorem 3.2]).

In any case, since H-nflat Higgs bundles are semistable (see [7, Proposition A.8]) and have degree 0 (see Remark 3.2), then by [18, Corollary 3.10] to prove the conjecture is enough that the equality \( \int_X c_2(E) \cdot H^{n-2} = 0 \) holds.

\[ \diamond \]

The starting idea is to extend [2, Lemma 3.3] to H-ample Higgs bundles using the Harder-Narasimhan filtration for Higgs bundles over smooth complex projective varieties, which existence was proved in [19, Lemma 3.1]. And in the same way, I prove that the tensor product of H-ample Higgs bundles is H-ample as well.

This property enables us to extend [13, Theorem in Appendix B] partially to H-ample setting; and in particular, with a simple trick, to prove positiveness conditions on Chern classes of some H-ample Higgs bundles. Easily, applying some result in [5, 6] I prove non-negativeness conditions on Chern classes of all H-nef Higgs bundles; in particular, following the reasons in [11], I prove that Schur polynomials are non-negative for H-nef Higgs bundles.

This statement applied to H-nflat case proves the Conjecture 0.2.

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## 1 H-ample Higgs bundles

In this section I remind the main definitions concerning H-ample Higgs bundles. Let \( E \) be a rank \( r \) vector bundle over a smooth projective variety \( X \), and let \( 0 < s < r \) an integer
number. Let \( \pi_s : \text{Gr}_s(E) \to X \) be the Graßmann bundle parameterizing rank \( s \) locally free quotients of \( E \) (see [12]). In the short exact sequence of vector bundles on \( \text{Gr}_s(E) \)

\[
0 \longrightarrow S_{r-s,E} \xrightarrow{\psi} \pi_s^*E \xrightarrow{\eta} Q_{s,E} \longrightarrow 0
\]

\( S_{r-s,E} \) is the universal rank \( r - s \) subbundle of \( \pi_s^*E \) and \( Q_{s,E} \) is the universal rank \( s \) quotient. Let now \( \mathcal{E} = (E, \varphi) \) be a rank \( r \) Higgs bundle on \( X \). One defines closed sub-schemes \( \mathfrak{G}_s^r(\mathcal{E}) \subset \text{Gr}_s(E) \) (the Higgs-Graßmann scheme) as the zero loci of the composite morphisms

\[
(\eta \otimes \text{Id}) \circ \pi_s^*(\varphi) \circ \psi : S_{r-s,E} \to Q_{s,E} \otimes \pi_s^*\Omega_X^1.
\]

The restriction of previous sequence to \( \mathfrak{G}_s^r(\mathcal{E}) \) yields a universal short exact sequence

\[
0 \longrightarrow \mathfrak{G}_{r-s,E} \xrightarrow{\psi} \rho_s^*\mathcal{E} \xrightarrow{\eta} \mathfrak{Q}_{s,E} \longrightarrow 0,
\]

where \( \mathfrak{Q}_{s,E} = Q_{s,E}\!\! / \mathfrak{G}_s^r(\mathcal{E}) \) is equipped with the quotient Higgs field induced by the Higgs field \( \rho_s^*\varphi \) (here \( \rho_s = \pi_s|_{\mathfrak{G}_s^r(\mathcal{E})} : \mathfrak{G}_s^r(\mathcal{E}) \to X \)). The scheme \( \mathfrak{G}_s^r(\mathcal{E}) \) enjoys the usual universal property: a morphism of varieties \( f : Y \to X \) factors through \( \mathfrak{G}_s^r(\mathcal{E}) \) if and only if the pullback \( f^*\mathcal{E} \) admits a locally free rank \( s \) Higgs quotient. In that case the pullback of the above universal sequence on \( \mathfrak{G}_s^r(\mathcal{E}) \) gives the desired quotient of \( f^*\mathcal{E} \).

**Definition 1.1** (see Definition 2.3 of [5]). A Higgs bundle \( \mathcal{E} = (E, \varphi) \) of rank one is said to be Higgs ample (H-ample for short) if \( E \) is ample in the usual sense. If \( \text{rank}(\mathcal{E}) \geq 2 \), we inductively define H-ampleness by requiring that

a) all Higgs bundles \( \mathfrak{Q}_{s,E} \) are H-ample for all \( s \), and

b) the determinant line bundle \( \text{det}(E) \) is ample.

Definition [11] implies that the first Chern class of a H-ample Higgs bundle is positive. Note that if \( \mathcal{E} = (E, \varphi) \), with \( E \) ample in the usual sense, then \( \mathcal{E} \) is H-ample. If \( \varphi = 0 \), the Higgs bundle \( \mathcal{E} = (E, 0) \) is H-ample if and only if \( E \) is ample in the usual sense.

**Example 1.2.** Let \( X \) be a smooth projective curve of genus \( g \geq 2 \), let \( \mathcal{E} = (E, \varphi) \) be a rank 2 nilpotent\(^2\) Higgs bundle; for clarity, \( E = L_1 \oplus L_2 \) and \( \varphi : L_1 \to L_2 \otimes \Omega_X^1 \) and \( \varphi(L_2) = 0 \). By computations in [8 Section 3.4], \( \mathfrak{G}_1(\mathcal{E}) = \mathbb{P}(L_1) \cup \mathbb{P}(\mathcal{Q}) \) where

\[
\mathcal{Q} = \text{coker} \left( \varphi \otimes 1 : E \otimes (\Omega_X^1)^\vee \to E \right),
\]

then \( \mathfrak{Q} = (\mathcal{Q}, 0) \) is a Higgs quotient sheaf of \( \mathcal{E} \). This implies that \( \mathcal{E} \) has only two Higgs quotient bundles, which are \( L_1 \) and \( \mathfrak{Q} = \mathcal{Q}/\text{torsion} \). Noted that \( \deg(\mathfrak{Q}) \geq \deg(L_1) \); if one takes \( \deg(L_1) = 1 \) and \( \deg(L_2) = -2 \), then \( \mathcal{E} \) is a Higgs bundle such that \( \deg(E) = -1 \) and \( \mathfrak{Q}_{1,E} \) is an ample line bundle.

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\(^2\)A Higgs bundle is nilpotent if there is a decomposition \( E = \bigoplus_{i=1}^m E_i \) as direct sum of subbundles such that \( \varphi(E_i) \subset E_{i+1} \otimes \Omega_X^1 \) for \( i \in \{1, \ldots, m-1\} \) and \( \varphi(E_m) = 0 \) (see [8]).
I prove now some properties of H-ample Higgs bundles that they will be useful in the sequel. These extend the properties given in [5].

**Proposition 1.3.** a) Let \( f: Y \to X \) be a finite morphism of smooth projective varieties and let \( E \) be a Higgs bundle over \( X \). If \( E \) is H-ample then \( f^*E \) is H-ample. Moreover, if \( f \) is also surjective and \( f^*E \) is H-ample then \( E \) is H-ample.

b) Every quotient Higgs bundle of a H-ample Higgs bundle \( E \) over \( X \) is H-ample.

**Proof.** (a) In the rank one case, one applies [17, Proposition 1.2.9 and Corollary 1.2.24]. In the higher rank case, one first notes that \( f^* \det(E) = \det(f^*E) \), so that the condition on the determinant is fulfilled. By functoriality of Higgs-Graßmann schemes, \( f \) induces finite morphisms \( \overline{f}_s: \text{Gr}_s(f^*E) \to \text{Gr}_s(E) \) for any \( s \in \{1, \ldots, r-1\} \) such that \( \mathcal{Q}_{s,f^*E} \cong \overline{f}_s^*\mathcal{Q}_{s,E} \). By induction on the rank of \( E \), one concludes.

(b) Let \( \mathcal{F} = (F, \varphi_F) \) be a rank \( s \) Higgs quotient bundle of \( E \); the canonical projection \( \mathcal{E} \to \mathcal{F} \) induces the closed embeddings \( i_t: \text{Gr}_t(\mathcal{F}) \to \text{Gr}_t(\mathcal{E}) \) for any \( t \in \{1, \ldots, s-1\} \) where \( \mathcal{Q}_{t,\mathcal{F}} = i_t^*\mathcal{Q}_{t,\mathcal{E}} \) and, by the previous point, these are H-ample Higgs bundle. Considering the short exact sequence

\[
0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0
\]

one can assume that \( \mathcal{K} \) is a reflexive sheaf, so \( \det(\mathcal{K}) \) is a line bundle. Let \( C \subseteq X \) be an irreducible curve; since \( \det(E) = \det(F) \otimes \det(\mathcal{K}) \) is an ample line bundle, by Cartan-Serre-Grothendieck Theorem there exists \( N \gg 0 \) such that \( \det(E)^{\otimes N} \otimes \det(\mathcal{K}) \) is a globally generated line bundle, in particular it is nef, then:

\[
0 \leq \int_C c_1(\det(E)^{\otimes N} \otimes \det(\mathcal{K})) = \ldots = N \int_C c_1(\det(E)) + \int_C c_1(\det(\mathcal{K}))
\]

\[
\forall N \gg 0, \ \int_C c_1(\det(\mathcal{K})) \geq - \frac{1}{N} \int_C c_1(\det(E)) \Rightarrow \int_C c_1(\det(\mathcal{K})) \geq 0
\]

by definition \( \det(\mathcal{K}) \) is a nef line bundle. Let \( Y \subseteq X \) be an irreducible subvariety of dimension \( d \). If \( \deg \det(\mathcal{K}) > 0 \), since \( \det(E)^{\otimes N} \otimes \det(\mathcal{K}) \) is an ample line bundle (\([9, \text{Proposition 1.2.(2)}]\)), by \([13, \text{Lemma 1.1}]\):

\[
\forall M > 0, \ \exists f: \tilde{Y} \to Y \text{ finite surjective,} \quad L \text{ ample line bundle over } \tilde{Y}, \quad f^*(\det(E)^{\otimes N} \otimes \det(\mathcal{K})) = L^{\otimes M},
\]

and by the same reasoning, \( f^*(\det(F)) \otimes L \) is an ample line bundle. By Nakai-Moshezon
Criterion:

\[ 0 < \int_{\tilde{Y}} c_1 (f^* (\det (F)) \otimes L)^d = \ldots \]

\[ \ldots = \sum_{k=0}^{d} \binom{d}{k} \frac{1}{M^k} \int_{\tilde{Y}} c_1 (f^* (\det (F)))^{d-k} c_1 (f^* (\det (E) \otimes K))^{k} \mathop{\longrightarrow}_{M \to \infty} \int_{\tilde{Y}} c_1 (f^* (\det (F)))^d > 0 \Rightarrow \int_{Y} c_1 (\det (F))^d > 0. \]

If \( \deg \det (K) = 0 \) then \( c_1 (\det (K)) = 0 \), by [5, Remark 3.2.(ii) and Lemma 3.13], and:

\[ 0 < \int_{Y} c_1 (\det (E))^d = \int_{Y} c_1 (\det (F) \otimes K))^d = \ldots = \int_{Y} c_1 (\det (F))^d. \]

By Nakai-Moishezon Criterion \( \det (F) \) is an ample line bundle and \( \mathfrak{F} \) is H-ample by definition.

(q.e.d.) \( \Box \)

In order to generalize [2, Lemma 3.3] to H-ample case, from now on \( X \) is defined over the field of complex numbers. I recall the following notions.

**Definition 1.4.** A filtration

\[ \{0\} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_{m-1} \subset \mathcal{E}_m = \mathcal{E}, \]

whose successive Higgs quotient sheaves \( \mathcal{E}_i/\mathcal{E}_{i-1} \) are semistable and the sequence \( i \in \{1, \ldots, m\} \), \( \mu_i = \mu \left( \mathcal{E}_i/\mathcal{E}_{i-1} \right) \) is strictly decreasing, is called Harder-Narasimhan filtration of \( \mathcal{E} \).

Let \( \mu_{\min}(\mathcal{E}) = \mu_m \) and let \( \mu_{\max}(\mathcal{E}) = \mu_1 \); then [11, Proposition 2.9] holds for Higgs bundles.

**Proposition 1.5.** Let \( \mathcal{E}_1 = (E_1, \varphi_1) \) and \( \mathcal{E}_2 = (E_2, \varphi_2) \) be Higgs bundles over \( X \). Then

\[ \mu_{\min}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mu_{\min}(\mathcal{E}_1) + \mu_{\min}(\mathcal{E}_2), \mu_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mu_{\max}(\mathcal{E}_1) + \mu_{\max}(\mathcal{E}_2). \]

**Definition 1.6.** I denote:

a) \( N_1(X) = \frac{A_1(X)}{\equiv_{num}} \otimes \mathbb{R} \) and its dimension \( \rho(X) \) is the Picard number of \( X \);

b) \( NE(X) \subset N_1(X) \) the real cone generated by effective 1-cycles (the effective curve cone);

c) \( \overline{NE}(X) \) the closed cone of curves.
Lemma 1.7. Let $\mathcal{E}$ be a Higgs bundle over $X$. It is H-ample if and only if for any $f_i: C_i \to X$ smooth, complex, irreducible, projective curve and $\sum_{i=1}^{m} a_i C_i \in \overline{NE}(X) \setminus \{0\}$, one has $\sum_{i=1}^{m} a_i \mu_{\min}(f_i^*\mathcal{E}) > 0$.

**Proof.** By Proposition 1.3a, any $f_i^*\mathcal{E}$ is H-ample, by Proposition 1.3b and by Nakai-Moishezon criterion one has the claim.

Vice versa, by hypothesis one has:

$$\forall \sum_{i=1}^{m} a_i C_i \in \overline{NE}(X) \setminus \{0\}, \sum_{i=1}^{m} a_i \mu_{\min}(f_i^*\mathcal{E}) > 0 \Rightarrow \sum_{i=1}^{m} a_i \mu(f_i^*\mathcal{E}) > 0,$$

so $\det(E)$ is ample by Kleiman’s Criterion. Miming the reasoning explained in [7], the H-ampleness of $\mathcal{E}$ is equivalent to ampleness of a collection of line bundles $\mathcal{L}_S$ over a (possibly singular) scheme $S$ equipped with a projection $\pi_S: S \to X$ (these line bundles are obtained by successively taking the universal Higgs quotient until one reaches the rank one quotient bundles). Let $\beta_S: \tilde{S} \to S$ be the resolution of singularities, let $\beta_S^*\mathcal{L}_S = \mathcal{L}_{\tilde{S}}$, let $\gamma_S = \pi_S \circ \beta_S$ and let $\psi_S: \gamma_S^*\mathcal{E} \to \mathcal{L}_{\tilde{S}}$ be the quotient morphism. Let $g_i: C_i \to \tilde{S}$ be smooth complex irreducible projective curves such that

$$\sum_{i=1}^{m} a_i C_i \in \overline{NE}(X) \setminus \{0\};$$

the pull-back of $\psi_S$ to $C_i$ produces a quotient $f_i^*\mathcal{E} \to \mathfrak{F}_i$ on $C_i$, where $f_i = \gamma_S \circ g_i$. Let $\mathfrak{K}_i$ be the kernel of this quotient; the polygon corresponding to the filtrations $0 \subset \mathfrak{K}_i \subset f_i^*\mathcal{E}$ lies under the Harder-Narasimhan polygon of $f_i^*\mathcal{E}$ (see [16] for the definition of polygon of a filtration). Since

$$\sum_{i=1}^{m} a_i \mu_{\min}(f_i^*\mathcal{E}) > 0,$$

this implies that $\sum_{i=1}^{m} a_i \deg(\mathfrak{F}_i) > 0$, or equivalently

$$\forall \sum_{i=1}^{m} a_i C_i \in \overline{NE}(X) \setminus \{0\}, \sum_{i=1}^{m} a_i \deg(f_i^*\mathcal{L}_S) > 0$$

by Kleiman’s Criterion $\mathcal{L}_{\tilde{S}}$ is an ample line bundle. Since $\beta_S$ is a surjective finite morphism, then by [17, Corollary 1.2.24] also $\mathcal{L}_S$ is ample. This proves that $\mathcal{E}$ is H-ample.

(q.e.d.)

Using Proposition 1.5 one proves easily the following theorem.

**Theorem 1.8.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be H-ample Higgs bundles over $X$. Then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is H-ample.

**Proof.** By previous lemma: for any $f_i: C_i \to X$ smooth complex irreducible projective curve and $\sum_{i=1}^{m} a_i C_i \in \overline{NE}(X) \setminus \{0\}$

$$\sum_{i=1}^{m} a_i \mu_{\min}(f_i^*(\mathcal{E}_1 \otimes \mathcal{E}_2)) = \sum_{i=1}^{m} a_i (\mu_{\min}(f_i^*\mathcal{E}_1) + \mu_{\min}(f_i^*\mathcal{E}_2)) > 0$$

Using Proposition 1.5 one proves easily the following theorem.
in other words one has the claim.
(q.e.d.) □

2 On the Chern classes of H-ample Higgs bundles

Following [13], I generalize the positiveness of Chern classes of ample vector bundles over $X$ to H-ample Higgs case.

Since in general, the Higgs-Graßmann schemes are neither smooth nor irreducible nor equidimensional, I need the following Hard Lesfschetz Theorem generalized, which works on irreducible equidimensional setting.

**Theorem 2.1** (Hard Lefschetz Theorem for Intersection Cohomology, cfr. [10] Theorem 2.2.3.(c)). Let $Z$ be a complex projective variety of pure complex dimension $d$, with $\xi \in H^2(Z, \mathbb{Q})$ the first Chern class of an ample line bundle over $Z$. Then there are isomorphisms

$$\xi^k \sim - : IH^{d-k}(Z, \mathbb{Q}) \rightarrow IH^{d+k}(Z, \mathbb{Q})$$

for any integer $k > 0$.

Now I am in position to prove the following first main result.

**Theorem 2.2.** Let $\mathcal{E}$ be a rank $r$ H-ample Higgs bundle over $(X, H)$, let $Q = \ker (\varphi \otimes 1 : E \otimes (\Omega^1_X)\vee \rightarrow E)$. If $\text{rank}(Q) = s \geq n$ then:

$$\int_X c_n(E) > 0$$

**Proof.** Let $U$ be the open dense subset of $X$ where $Q = Q/\text{torsion}$ is locally free, let $\mathbb{P} = \mathcal{G}r_1 (\mathcal{E}_|U)$ and let $\xi = c_1 (\mathcal{O}_X(1))$. Let $Z$ be the irreducible component of $\mathbb{P}$ containing $\mathcal{G}r_1 (\mathcal{O}_U)$; by construction:

$$\xi^s - \rho^*_Z c_1 (E|U) \xi^{s-1} + \rho^*_Z c_2 (E|U) \xi^{s-2} + \ldots + (-1)^n \rho^*_Z c_n (E|U) \xi^{s-n} = 0$$

where $\rho : \mathbb{P} \rightarrow U$ had been defined above; one has the canonical morphisms:

$$H^k(Z, \mathbb{Q}) \rightarrow IH^k(Z, \mathbb{Q}) \xrightarrow{\cdot \xi^s} H_{2d-k}(Z, \mathbb{Q})$$

compatible with multiplication by cohomology classes, where $\dim Z = d \geq n + s - 1$ and $[Z]$ is the fundamental class of $Z$. Consider the class

$$\alpha = \xi^{n-1}_Z - \rho^*_Z c_1 (E|U) \xi^{n-2}_Z + \ldots + (-1)^{n-1} \rho^*_Z c_{n-1} (E|U) \in H^{2(n-1)}(Z).$$

Since $\rho^*_Z (\cdot \xi^{s-n}_Z) = [U]$, the image of $\alpha$ in $IH^{2(n-1)}(Z, \mathbb{Q})$ is not $0$. But if $c_n (E|U) = 0$ then $\alpha \cdot \xi^{s-n}_Z = 0$, since by hypothesis $\mathcal{O}_X(1)$ is an ample line bundle, this
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contradicts the Hard Lefschetz Theorem for Intersection Cohomology. In particular one has $c_n(E) \neq 0$.

Let $n = 0$, then the statement is trivial. By induction, assume that the statement holds for any varieties of dimension less than $n$. Let $D$ be an ample $\mathbb{Q}$-Cartier divisor on $X$; then $mD \subseteq X$ for some integer $m \geq 1$. Let $L = \mathcal{O}_X(mD)$, one has:

$$
\int_X c_{n-1}(E \otimes L)c_1(L) = \int_{mD} c_{n-1}((E \otimes L)_{|mD}) > 0
$$

by hypothesis and by Theorem 1.8. In absurd, let $\int_X c_n(E \otimes L) < 0$ then there exists $N \gg 0$ such that

$$
\int_X c_n(E \otimes L) + \frac{1}{N} \int_X c_{n-1}(E \otimes L)c_1(L) = 0.
$$

Consider a Block-Gieseker covering $f : \tilde{X} \to X$ with $M = f^*(L^{\otimes (N-1)})$, by [6, Proposition 4.7] $f^*(E \oplus M, \varphi \oplus 0)$ is a H-nef Higgs bundle, by [5, Proposition 2.6.(i)] $F \otimes f^*L$ is a H-ample Higgs bundle, then

$$
\int_{\tilde{X}} c_n(F \otimes f^*L) = \int_{\tilde{X}} c_n((f^*E \otimes f^*L) \oplus (M \otimes f^*L)) =
$$

$$
= \int_{\tilde{X}} c_n(f^*E \otimes f^*L) + \int_{\tilde{X}} c_{n-1}(f^*E \otimes f^*L)c_1(M \otimes f^*L) =
$$

$$
= f^* \left( \int_X c_n(E \otimes L) + \frac{1}{N} \int_X c_{n-1}(E \otimes L)c_1(L) \right) = 0
$$

and this is in contradiction with previous part. Since $\int_X c_n(E \otimes \mathcal{O}_X(mD)) > 0$ for any $\mathbb{Q}$-Cartier divisor $D$, then $\int_X c_n(E) \geq 0$.

By the same reasoning, for any ample $\mathbb{Q}$-Cartier divisor $\tilde{D}$ on $\tilde{X}$, $(f^*E \oplus \mathcal{O}_{\tilde{X}}(\tilde{m}\tilde{D})) \otimes f^*L$ is a H-ample Higgs bundle, and one has:

$$
0 \leq \int_{\tilde{X}} c_n(f^*E \oplus \mathcal{O}_{\tilde{X}}(\tilde{m}\tilde{D})) = \ldots = \int_{\tilde{X}} c_n(f^*E) + \int_{\tilde{m}\tilde{D}} c_{n-1}(f^*E)
$$

by the same reason above $\int_{\tilde{m}\tilde{D}} c_{n-1}(f^*E) > 0$; then, since this holds for any ample $\mathbb{Q}$-Cartier divisor on $\tilde{X}$ and by non-negativeness of $c_n(f^*E)$, one has

$$
\int_{\tilde{X}} c_n(f^*E \oplus \mathcal{O}_{\tilde{X}}(\tilde{m}\tilde{D})) > 0.
$$

Repeating the same reasoning, one has:
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\[ f^* \left( \int_X c_n(E) \right) = \int_{\tilde{X}} c_n(f^*E) > 0 \Rightarrow \int_X c_n(E) > 0. \]

(q.e.d.) □

The previous theorem can be improved as follows.

**Corollary 2.3.** Let \( \mathcal{E} \) be a H-ample Higgs bundle over \((X, H)\) which satisfies the hypothesis above. Then:

\[ \forall k \in \{1, \ldots, \min\{n, r\}\}, \int_X c_k(E) \cdot H^{n-k} > 0 \]

**Proof.** Let \( n \geq 2 \). For \( m_1 \) large enough there exists a smooth divisor \( Y_1 \) in the linear system \( |m_1H| \); let \( H_1 \) be the restriction of \( H \) to \( Y_1 \) and let \( i_1 : Y_1 \hookrightarrow X \) be the inclusion. By Proposition 1.3.a, \( i_1^* \mathcal{E} \) is H-ample and:

\[ \int_X c_k(E) \cdot H^{n-k} = \int_{Y_1} c_k(i_1^*E) \cdot H_1^{(n-1)-k}. \]

Repeating this reasoning until one finds a smooth subvarieties \( Y_k \) of \( X \) of dimension \( k \); let \( H_k \) be the restriction of \( H \) to \( Y_k \) and let \( i_k : Y_k \hookrightarrow X \) be the inclusion. By previous reasoning and theorem:

\[ \int_X c_k(E) \cdot H^{n-k} = \int_{Y_k} c_k(i_k^*E) > 0. \]

(q.e.d.) □

3 On the Chern classes of H-nef Higgs bundles

Miming Definition 1.1, one has the definitions of H-nef and H-nflat Higgs bundles. For clarity.

**Definition 3.1** (see Definition 2.3 of [5]). A Higgs bundle \( \mathcal{E} = (E, \varphi) \) of rank one is said to be Higgs numerically effective (H-nef for short) if \( E \) is numerically effective in the usual sense. If \( \text{rank}(\mathcal{E}) \geq 2 \), we inductively define H-nefness by requiring that

a) all Higgs bundles \( \mathcal{Q}_{s,\mathcal{E}} \) are H-nef for all \( s \), and

b) the determinat line bundle \( \text{det}(E) \) is nef.

\( \mathcal{E} \) is Higgs-numerically flat (H-nflat, for short) if \( \mathcal{E} \) and \( \mathcal{E}^\vee \) are both H-nef.

Also for H-nflat Higgs bundles hold statements analogous to H-ample case; for more examples one can consult [5] Examples 2.4 and 2.5.

**Remark 3.2.** By [6] Remark 4.2.(ii) and Lemma 3.13], the first Chern class of a H-nflat Higgs bundle vanishes.

◊
3.1 First inequalities

Using the previous theorem, one has the following lemma, generalization of [9, Proposition 1.2.11] to H-nef case.

Lemma 3.3. Let $\mathcal{E}$ be a rank $r$ H-nef Higgs bundle over $(X, H)$. Then:

$$\forall k \in \{1, \ldots, r\}, \int_X c_k(E) \cdot H^{n-k} \geq 0.$$ 

Proof. With the same notation of Theorem 2.2 if $s < n$, by [5, Proposition 2.6.(iii)] and by [6, Proposition 4.7] one can replace $E$ with $E \oplus \mathcal{O}_{X}^{-r}$, where $\mathcal{O}_{X} = (\mathcal{O}_{X}, 0)$ is the trivial Higgs line bundle over $X$. By [5, Proposition 2.6.(i)], for any ample $\mathbb{Q}$-Cartier divisor $D$ on $X$, $\mathcal{E} \otimes \mathcal{O}_{X}(D)$ is H-ample; by Corollary 2.3 $\int_X c_k (E \otimes \mathcal{O}_{X}(D)) \cdot H^{n-k} > 0$. By generality of $D$, one has the claim.

(q.e.d.) □

Previous lemma can be improved as follow.

Proposition 3.4. Let $\mathcal{E} = (E, \varphi)$ be a H-nef Higgs bundle over $(X, H)$; let $i : Y \hookrightarrow X$ be an $d$-dimension subvariety with $n > d \geq r$. Then $\int_Y c_r(E) \cdot H^{d-r} \geq 0$.

Proof. Let $\left(\tilde{Y}, \beta\right)$ be the strong resolution of $Y$: it is a compact, smooth, complex analytic space $\tilde{Y}$, and $\beta$ is a projective holomorphic map (see [15, Theorem 3.27]). Let $f = i \circ \beta$, and let $\omega$ be a Kähler form on $X$ determined by $H$; by compactness of $\text{Sing}(Y)$ there exists a real positive number $C_0$ such that for any $C \geq C_0$, $\tilde{\omega}_Y = \omega_{\beta} + C f^* \omega$, where $\omega_{\beta}$ is the Kähler form on $\beta^{-1}(\text{Sing}(Y)) = D$ determined by $\mathcal{O}_{\tilde{Y}}(-D)|_D$; from all this, $C^{-1} \tilde{\omega}_Y$ is another Kähler form on $\tilde{Y}$. By previous proposition, one has:

$$0 \leq \int_{\tilde{Y}} c_r (f^* E) \wedge C^{-1} \tilde{\omega}_Y^{d-r} = \int_{\tilde{Y}} c_r (f^* E) \wedge (C^{-1} \omega_{\beta} + f^* \omega)^{d-r} \xrightarrow{C \rightarrow +\infty} 0$$

$$\xrightarrow{C \rightarrow +\infty} \int_Y c_r(E) \cdot H^{d-r} = \int_Y c_r(E) \wedge \omega^{d-r} = \int_{\tilde{Y}} c_r (f^* E) \wedge (f^* \omega)^{d-r} \geq 0.$$ 

(q.e.d.) □

Remark 3.5. I ignore whether an analogous statement holds for H-ample Higgs bundles.

◊
3.2 Preliminaries on cones in vector bundles

Let $\overline{E} = \mathbb{P}(E \oplus O_X)$ be the compactification of $E$ by the hyperplane at infinity, let $C$ be a cone in $E$, that is a $\mathbb{C}^\times$-invariant analytic subset of $E$, and let $C$ be the closure of $C$ in $\overline{E}$. Let $\Omega_{\overline{E}} = \mathbb{P}^n(E \oplus O_X)/\mathcal{O}_{\overline{E}}(-1)$, where $\mathbb{P} : \overline{E} \to X$ is the canonical projection, $\mathcal{O}_X = (X \times \mathbb{C}, 0)$ and $\mathcal{O}_{\overline{E}}(-1) = (\mathcal{O}_{\overline{E}}(-1), 0)$. By [5, Proposition 2.6.(ii) and (iii)] and [6, Proposition 4.7] $\Omega_{\overline{E}}$ is a rank $r$ H-nef Higgs bundle over $\overline{E}$.

Let $0_{E} : X \to E$ be the zero section of $E$, I set $Z_E = 0_{E}(X)$ the image in $E$; let $[Z_E]$ be the closed positive current on $E$ associated to $Z_E$, its (real) dimension is $2n$ or equivalently its (real) degree is $2r$; let $i : E \hookrightarrow E$ be the inclusion, one has

$$i_*([Z_E]) \in H^{2r}(\overline{E}, \mathbb{R}), [C] \in H^{2p}(\overline{E}, \mathbb{R}),$$

where $p = \text{codim}_C C$. Let $0 \oplus 1 : Q_{\overline{E}} \to \mathbb{P}^n(E \oplus O_X)$ be the canonical section of underlying vector bundle to $\Omega_{\overline{E}}$; because the zero locus of $0 \oplus 1$ is transversal to $i(Z_E)$ then $c_r(Q_{\overline{E}}) = i_*([Z_E])$.

Now I am in position to state the following proposition, which is a consequence of Proposition 3.4.

**Proposition 3.6.** Let $\mathfrak{E} = (E, \varphi)$ be a H-nef Higgs bundles. For any cone $C$ in $E$ of dimension $d = n + r - p \geq r$, the following inequality holds:

$$i_*([Z_E]) \cdot [C] \cdot [(\mathbb{P}^r H)^{d-r}] \overset{\text{def.}}{=} \int_C c_r(Q_{\overline{E}}) \wedge (\mathbb{P}^r H)^{d-r} \geq 0.$$

3.3 On Schur polynomials of H-nef Higgs bundles

Let $\lambda$ be a $r$-partition of a non negative integer number $k$ by non negative integer numbers, that is

$$\lambda \in \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{N}_{\geq 0}^r | \lambda_1 + \ldots + \lambda_r = k, r \geq \lambda_1 \geq \ldots \geq \lambda_r\};$$

to any $\lambda$ one associates the Schur (weighted) polynomial $P_\lambda \in \mathbb{Z}[c_1, \ldots, c_r]$ as following

$$P_\lambda(c) = \det(c_{\lambda_i-i+j})_{1 \leq i, j \leq r}$$

where:

- $c_0 = 1$;
- for any $h \notin \{0, \ldots, r\}$, $c_h = 0$;
- the weighted degree of $c_k$ is $2k$, where $k \in \{0, \ldots, r\}$.

By [11, Formula 2.4], Propositions 3.4 and 3.6 one can extend [11, Theorem 2.5] to H-nef Higgs bundles.
Theorem 3.7. Let \( \mathcal{E} = (E, \varphi) \) be a H-nef Higgs bundle over \( (X, H) \), let \( Y \) be a \( d \)-dimensional analytic subset of \( X \). For any Schur polynomial \( P_\lambda (\mathcal{c}(E)) \) of weighted degree \( 2k \), with \( d \geq k \), one has
\[
\int_Y P_\lambda (\mathcal{c}(E)) \cdot H^{d-k} \geq 0.
\]

Remark 3.8. Let \( \mathcal{E} = (E, \varphi) \) be a H-nef Higgs bundle over \( (X, \omega) \).

a. Let \( \lambda = (k, 0, \ldots, 0) \), then \( P_\lambda (\mathcal{c}(E)) = c_k(E) \); let \( \lambda = (1, 1, \ldots, 1, 0, \ldots, 0) \), then \( P_\lambda (\mathcal{c}(E)) = s_k(E) \) the \( k \)-th Segre class of \( E \). In other words, the previous theorem states non-negative conditions for Chern and Segre classes of \( \mathcal{E} \) over \( X \) and over its analytical subset too.

b. (cfr. \[11, Corollary 2.6\]) For any \( k \in \{1, \ldots, r\} \) let \( \lambda = (k-1, 1, 0, \ldots, 0) \), then
\[
P_\lambda (\mathcal{c}(E)) = c_k(E) - c_{k-1}(E) - c_k(E) = \sum_{j=2}^{k} c_{k-j}(c_1c_j - c_j).
\]

By Littlewood-Richardson rule, \( c_k(E) - c_k(E) \) is a linear combination of Schur polynomials which scalars are non negative integer numbers. By this statement:
\[
\int_X (c_k(E) - c_k(E)) \cdot H^{n-k} \geq 0 \iff \int_X c_k(E) \cdot H^{n-k} \geq \int_X c_k(E) \cdot H^{n-k} \geq 0.
\]

Generalizing previous reasoning, one has
\[
\int_X c_{k}(E) \cdot H^{n-k} \geq \int_X c_{k}(E) \cdot H^{n-k} \geq \int_X c_{k}(E) \cdot H^{n-k} \geq 0
\]
where \( c_{k}(E) = c_1^r(E) \ldots c_r^r(E) \) is the generic Chern monomial of degree \( 2k \).

By previous Remark 3.8b, the following corollary holds:

Corollary 3.9. Let \( \mathcal{E} = (E, \varphi) \) be a H-nef Higgs bundle over \( (X, H) \), such that \( c_1(E)^n = 0 \). One has \( \int_X c_n(E) = 0 \).
4 On the Chern classes of H-nflat Higgs bundles and applications

Let $\mathcal{E} = (E, \varphi)$ be a Higgs bundle, and let $h$ be an Hermitian metric on $E$, one defines the Hitchin-Simpson connection of the pair $(\mathcal{E}, h)$ as
$$D_{(h, \varphi)} = Dh + \varphi + \overline{\varphi}$$
where $D_h$ is the Chern connection of the Hermitian bundle $(E, h)$, and $\overline{\varphi}$ is the metric adjoint of $\varphi$ defined as
$$h(s, \varphi(t)) = h(\overline{\varphi}(s), t)$$
for all sections $s, t$ of $E$. The curvature $R_{(h, \varphi)}$ and the mean curvature $\mathcal{K}_{(h, \varphi)}$ of $D_{h, \varphi}$ are defined as usual; in particular, if $R_{(h, \varphi)} = 0$ then $\mathcal{E}$ is called Hermitian flat.

By [18, Theorem 1.(2)], the following theorem holds.

**Theorem 4.1.** A Higgs bundle $\mathcal{E} = (E, \varphi)$ over $(X, H)$ is polystable if and only if it admits a Hermitian metric $h$ such that it satisfies the Hermitian Yang-Mills condition
$$\mathcal{K}_{(h, \varphi)} = \frac{2n\pi \mu(E)}{n! \text{Vol}(X)} \cdot \text{Id}_E \equiv \kappa \cdot \text{Id}_E,$$
where $\text{Vol}(X) = \frac{1}{n!} \int_X H^n$. If $\kappa = 0$ then $h$ is called harmonic.

With all this, I prove the second main theorem of this paper, i.e. I prove the Conjecture 0.2.

**Theorem 4.2.** The Chern classes of a H-nflat Higgs bundle $\mathcal{E} = (E, \varphi)$ over $(X, H)$ vanish.

**Proof.** By [4, Theorem 3.2], $\mathcal{E}$ admits a Jordan-Hölder filtration whose quotients are H-nflat. One can replace $\mathcal{E}$ with the graded module associated to this filtration; in other words, one can assume $\mathcal{E}$ polystable and H-nflat. By Remark 3.2 and Corollary 3.9 $c_1(E) = 0$, $\int_X c_2(E) \cdot H^{n-2} = 0$; by Theorem 4.1 $\mathcal{E}$ admits a Hermitian Yang-Mills metric $h$ which is harmonic. Let $\mathcal{R}_{(h, \varphi)}$ be the curvature form of the Hitchin-Simpson connection associated to $h$, by hypothesis and [14] formulas in Chapter 4.4:
$$0 = -4\pi^2 \int_X \left[c_1^2(E) - 2c_2(E)\right] \cdot H^{n-2} = \int_X \text{Tr} \left(\mathcal{R}_{(h, \varphi)} \wedge \mathcal{R}_{(h, \varphi)}\right) \cdot H^{n-2} = \\
\gamma_1 \left\|\mathcal{R}_{(h, \varphi)}\right\|^2_{L^2} - \gamma_2 \left\|\mathcal{K}_{(h, \varphi)}\right\|^2_{L^2} = \gamma_1 \left\|\mathcal{R}_{(h, \varphi)}\right\|^2_{L^2}$$
for some positive real numbers $\gamma_1$ and $\gamma_2$, so $\mathcal{E}$ is a flat vector bundle.

(q.e.d.)

Trivially, by remarks [13] the following corollaries hold.

**Corollary 4.3.** Let $\mathcal{E} = (E, \varphi)$ be a curve semistable Higgs bundle over $(X, H)$. Then $\mathcal{E}$ is semistable and $\Delta(E) = 0 \in H^4(X, \mathbb{Q})$.

**Corollary 4.4.** A Higgs bundle over $(X, H)$ is H-nflat if and only if it has a filtration in Higgs subbundles whose quotients are stable and Hermitian flat Higgs bundles.
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