Quantum Mechanical Probability Interpretation In The Mini-superspace Model Of Higher Order Gravity Theory

Abhik Kumar Sanyal

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Dept. of Physics, Jangipur College, Murshidabad, India - 742213
and
Relativity and Cosmology Research Centre
Dept. of Physics, Jadavpur University
Calcutta - 700032, India
e-mail : aks@juphys.ernet.in

Abstract
It has been shown that introduction of higher order curvature invariant terms like \( R^2 \) or \( R_{\mu \nu} R^{\mu \nu} \) in the Robertson-Walker minisuperspace model of the Einstein-Hilbert action leads to Schrödinger-like equation, whose corresponding effective Hamiltonian is hermitian. Thus, it is possible to write the continuity equation in a straight forward manner which reveals a quantum mechanical probability interpretation of the theory.

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1 Introduction

In a couple of recent communications [1] and [2], it has been observed that if the Einstein-Hilbert action is replaced or modified by the introduction of a curvature squared term in the Robertson-Walker minisuperspace model, then one of the true degrees of freedom is distangled from the kinetic part of the canonical variables. This kinetic term behaves as a time parameter in the quantum domain and the corresponding Wheeler-deWitt equation looks like Schrödinger equation. This result is interesting since it tends to resolve the long standing disagreement between ‘General Theory of Relativity’ and ‘Quantum Mechanics’. The disagreement is, in GTR time stands on the same footing as the space coordinates and it does not act as an external parameter, while ‘Quantum Mechanics’ knows to predict uniquely only when time is separated from the rest of the variables and acts as an external parameter. To resolve the contradiction Wheeler [3] proposed that dynamical objects are space and not space-time, and the geometrical configuration of space evolves with time. This proposal requires a (3+1) decomposition of the metric and one can canonize the theory to get Wheeler-deWitt equation. However it is a very special feature of gravitation that it does not naturally lead to a Hamiltonian, but only to the Hamiltonian constraint. So it remained obscure to construct a satisfactory Hilbert space, on which the probability interpretation is based. In the quantum domain of the modified (by the curvature squared term) cosmological model, one of the canonical variables is automatically selected as the time parameter, resolving the disagreement.

This essentially is not a new result. Horowitz [4] and Pollock [5] observed it two decades back. What new is, a proper choice of the canonical auxiliary variable turns out the effective Hamiltonian hermitian and thus it becomes possible to construct the equation of continuity and hence to establish the quantum mechanical idea of the probability and the current densities at least in the periphery of the Robertson-Walker minisuperspace model. Further, it gives rise to an effective potential whose extremization yields classical constraint equation [1] and [2]. These results have not been observed earlier, since it requires as mentioned, the correct choice of the auxiliary variable. The question thus naturally arises is, why a definite choice of such variable is required and how can it be chosen?

Canonical quantization of a theory whose action contains higher order terms is far from being trivial. It is required to choose an auxiliary variable, and to write the action in the canonical form. In the context of quantum cosmology, introduction of \( R^2 \) term in the action, requires such technique. Auxiliary variables can be chosen in
many different ways \[1\], \[3\] and \[10\]. For all such variables it is possible to express the action in the canonical form, leading to the same classical field equations. However quantum dynamics turns out to be different for different auxiliary variable. Though such an important fact has only been observed fairly recently, it is not surprising. The reason being that, though all canonically related variables lead to the same classical field equations, there is no counterpart of such result in the quantum domain and hence should not be extended naturally. Rather, as mentioned, one has to choose such auxiliary variable in a unique manner so that correct quantum dynamics emerges. Boulware et-al \[7\] prescribed a technique to pick up canonical variables for higher order theory. The prescription can be summarised in the following three steps. First, take the first derivative of the action with respect to the highest derivative of the field variable present in the action and idetify it as the auxiliary variable. Second, write the action in the canonical form after introducing the new variable and finally, quantize with basic variables.

The above technique was taken up by Hawking and Luttrell \[8\], Horowitz \[4\] and Pollock \[5\] to identify the auxiliary variable. Interestingly enough, Pollock \[4\] showed that if one introduces auxiliary variable in the induced theory of gravity, where it is not at all required, one still ends up with a Schrödinger like equation. Sanyal and Modak \[1\] have shown that auxiliary variable can even be introduced in the vacuum Einstein-Hilbert action, yielding completely wrong quantum dynamics. Auxiliary variable can be introduced in such models if one simply does not remove removable total derivative terms from the action. In Boulware et-al’s \[7\] prescription one has to quantize with basic variables. The basic variables are \(\alpha, \alpha’\), if auxiliary variable is introduced in the vacuum Einstein-Hilbert action, while truely the basic variable in this situation is \(\alpha\). Effectively, introduction of auxiliary variable in such situations, where it is not required, means to entertain redundant degree of freedom. So it was concluded that Boulware et-al’s \[7\] prescription should be taken up only after removing removable total derivative terms from the action. Unless this is done, wrong quantum dynamics would emerge. In this sense all the auxiliary variables identified by Hawking and Luttrell \[8\], Horowitz \[4\] and Pollock \[5\] are wrong in the quantum domain.

In a nutshell, in order to uniquely identify the auxiliary variable and to obtain the correct quantum description, we emphasize on the fact that Boulware et-al’s \[7\] prescription should be taken up only after removing the total derivative terms from the action. If this is done one can never get such variables to introduce in situations like vacuum Einstein-Hilbert action or induced theory of gravity and to end up with wrong quantum equations. On the other hand such variables can be introduced in the higher order gravity theory uniquely. Further, this method of identifying the auxiliary variable has got extremely interesting consequences as already mentioned at the beginning. It gives birth to a Schrödinger-like equation where one of the variables being distangled from the kinetic part of the canonical variables act as the time parameter. Further an effective Hamiltonian emerges which is hermitian and as a result quantum mechanical idea of current and probability densities is realized. In addition the effective Hamiltonian contains an effective potential whose extremization yields classical constraint equation \[1\] and \[3\]. Horowitz \[4\] claimed that the action for the higher order gravity theory does not require to have surface terms. This as a result of identifying the new variable properly, turns out to be wrong. This is due to the fact that removal of total derivative terms from the action leads to surface terms. One can get rid of these surface energy terms only if such counter terms are already included in the action. Tomboulis \[9\] stated that higher order gravity theory does not require to have surface terms from the action. If this is done one can never get such variables to introduce in situations like vacuum Einstein-Hilbert action, while truely the basic variable in this situation is \(\alpha\). Effectively, introduction of auxiliary variable in such situations, where it is not required, means to entertain redundant degree of freedom. So it was concluded that Boulware et-al’s \[7\] prescription should be taken up only after removing removable total derivative terms from the action. Unless this is done, wrong quantum dynamics would emerge. In this sense all the auxiliary variables identified by Hawking and Luttrell \[8\], Horowitz \[4\] and Pollock \[5\] are wrong in the quantum domain.

So far, there has been different strong motivations to modify Einstein-Hilbert action by higher order curvature invariant terms. To state a few, Stell \[11\] has shown that the action \(\int d^4x \sqrt{-g}[AC_{ijkl} + BR + CR^2]\) is renormalizable in 4-dimensions. \(C_{ijkl}\) is the Weyl tensor, \(R\) is the Ricci scalar and \(A, B, C\) are the coupling constants. Starobinsky \[12\] obtained an inflationary solution without invoking phase transition in the very early Universe, from a field equation containing only geometric terms. Later, Starobinsky and Schmidt \[13\] have shown that the inflationary phase is an attractor in the neighbourhood of the fourth order gravity theory. Hawking and Luttrell \[8\] observed that Einstein-Hilbert action being modified by a curvature squared term is equivalent to the Einstein-Hilbert action being coupled to a massive scalar field. The Euclidean form of the Einstein-Hilbert action is not bounded from below, as a result the ground state wave function proposed by Hartle and Hawking \[14\] diverges badly. A remedy to this undesirable feature was suggested by Horowitz \[4\], who proposed a positive definite action in the form

\[
S = \int d^4x \sqrt{-g}[AC_{ijkl}^2 + B(R - 4\Lambda)^2]
\]  

(1)

A being the cosmological constant. Not only that this action leads to a convergent functional integral, it also reduces to the Einstein-Hilbert action in the weak field limit. A very recent and perhaps one of the most important motivation to modify Einstein-Hilbert action by the introduction of the curvature squared term is being put
forward by Sanyal and Modak [1], [2]. The motivation is to obtain a quantum mechanical interpretation of quantum cosmology. It is therefore required to extend our previous work by applying it to different situations to convincingly prove that our method is correct.

In the present paper, we first take up the Induced Theory of Gravity in the following section and show that the corresponding Wheeler-deWitt equation is not satisfied by the Schrödinger like equation obtained by Pollock [5], by introducing auxiliary variable. This situation is considered to prove that auxiliary variable, if introduced in situations where it is not required, will yield wrong quantum dynamics. The reason behind is that, in the Boulware-et-al’s [7] proposal one has to finally quantize with basic variables. Introduction of auxiliary variables in such situations means to introduce redundant degrees of freedom. It further shows that once removable total derivative terms are removed from the action, auxiliary variables can not be introduced any more in such cases, and there is no possibility of emergence of wrong quantum description.

We then take up the modified Induced Theory of Gravity, by introducing $R^2$ term in the action, in section 3. This case was again studied by Pollock [5]. We have shown that, following the method mentioned it is possible to distangle a combination of the expansion parameter and the scalar field from the kinetic part of the canonical variables that starts behaving as the time parameter of the corresponding quantum equation. Further, the effective Hamiltonian turns out to be hermitian and the quantum mechanical idea of current and probability densities follows naturally. These results are completely different from that obtained by Pollock [5] and obcourse extremely illuminating.

In section 4 The positive definite action proposed by Horowitz [4] has been considered. In the Robertson-Walker mini-superspace model the Weyl tensor does not contribute. The quantum version of this model is found tally with $R + R^2$ theory.

In section 5, the next higher order curvature invariant term, viz., $R_{\mu\nu}R^{\mu\nu}$ is taken up to modify Einstein-Hilbert action. Interestingly enough, it has been found to yield the same type of classical and quantum results as was obtained in the $R^2$ case [5]. It therefore appears that the hermiticity of the effective Hamiltonian operator and the quantum mechanical probabilistic interpretation perhaps are the generic feature of higher order gravity theory.

Concluding remarks are made in section 6.

2 Induced Theory Of Gravity - A Toy Model

As mentioned in the introduction, Pollock [5] introduced an auxiliary variable in the induced theory of gravity to show that the corresponding Wheeler-deWitt equation can well be written as Schrödinger like equation. In this section our aim is to show that not only it is unnecessary to introduce auxiliary variable in such a theory but also it leads to wrong quantum dynamics. For this purpose we shall deal with the same action and write the corresponding Wheeler-deWitt equation in the standard procedure without invoking an auxiliary variable, obtain the semi–classical solution and compare it with that obtained by Pollock [5]. Let us start with the following action,

$$A = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \epsilon \phi^2 R - \frac{1}{4} \lambda \phi^4 \right].$$

(2)

In the Robertson-Walker mini-superspace model

$$ds^2 = \epsilon^{2\alpha(\eta)} \left[ -d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

(3)

the above action takes the following form

$$A = -2\pi^2 \int \left[ 3\epsilon \phi^2 (1 + \alpha'^2 + \alpha'')e^{2\alpha} - \frac{1}{4} \lambda \phi^4 e^{4\alpha} \right] d\eta$$

(4)

where dash(’) stands for derivative with respect to the conformal time $\eta$. It further reads,

$$A = 2\pi^2 \int \left[ 3\epsilon (\phi^2 (\alpha'^2 - 1) + 2\phi\alpha' \phi') + \frac{\lambda \phi^4 e^{2\alpha}}{4} \right] e^{2\alpha} d\eta$$

(5)

apart from a total derivative term. Corresponding classical field equations are

$$\alpha'' + \alpha'^2 + 2\alpha' \frac{\phi'}{\phi} + \frac{\phi''}{\phi} + \frac{\phi^2}{\phi^2} - \frac{\lambda \phi^2 e^{2\alpha}}{6\epsilon} + 1 = 0$$

(6)

$$\alpha'' + \alpha'^2 - \frac{\lambda \phi^2 e^{2\alpha}}{6\epsilon} + 1 = 0$$

(7)
\[ \alpha^2 + 2\alpha' \frac{\partial}{\partial \phi} - \frac{\lambda \phi^2 e^{2\alpha}}{12\epsilon} + 1 = 0 \]  

(8)

In the phase space variable the Hamiltonian is

\[ H = \frac{1}{36 \epsilon^2 \phi^2 e^{4\alpha}} \left[ -\dot{\phi}^2 + \frac{2 \rho a \rho}{\phi} - 3 \epsilon \phi^4 e^{6\alpha} + 36 \epsilon^2 \phi^2 e^{4\alpha} \right] = 0 \]  

(9)

The corresponding Wheeler-deWitt equation is

\[ [\hbar^2 (\partial^2 / \partial \phi^2) + \frac{q}{\phi} \frac{\partial}{\partial \phi} - \frac{1}{\phi} (\partial^2 / \partial \alpha \partial \phi)] + 3 \epsilon \phi^2 e^{4\alpha} (12\epsilon - \phi^2 e^{2\alpha})] \psi = 0 \]  

(10)

where \( q \) is the operator ordering index. The above Wheeler-deWitt equation admits the following semiclassical solution for \( q = 0 \).

\[ \psi = C \exp \left[ \frac{i}{\hbar} (m \phi^{l+1} + m) e^{\lambda x} \right] \]  

(11)

where \( C, l, m, n \) are constants.

Now we compare our result with that obtained by Pollock for the same action. Instead of removing surface term from the action, Pollock introduced an auxiliary variable in the action (4) as prescribed by Boulware et-al. Indeed the classical field equations remain unchanged because the variables are canonical. The Wheeler-deWitt equation obtained by Pollock is

\[ i \frac{\partial \psi}{\partial \alpha} = \frac{1}{m x \partial x^2} + \frac{i \partial (1 + x^2) \psi}{x \partial x} + i (p - 2) \psi. \]  

(12)

where \( x = \alpha' \). This equation looks like Schrödinger equation which admits a semiclassical solution as was obtained by Pollock for \( p = 2 \), which is equivalent to our \( q = 0 \) situation, viz.,

\[ \psi(\alpha, x) = \exp \left[ -i m x \right] \]  

(13)

where \( m = \frac{72 \pi^2}{\lambda} \). It is not difficult to see that this Wheeler-deWitt equation is different from (10) with a different solution altogether. So, the question is what went wrong? This was suggested in a couple of recent communications. The fact that all canonical variables lead to the same classical field equations has no analogue in quantum domain. So to obtain the correct quantum description one has to choose canonical variables carefully and perhaps uniquely. This is achieved if such variables are chosen only after removing total derivative terms from the action. Unless otherwise redundant degrees of freedom are taken into account, since in the Boulware-et-al’s prescription one has to finally quantize with basic variables. We observe from action (5) that the basic variables are \( \alpha \) and \( \phi \), while Pollock considered \( \alpha, \alpha' \) and \( \phi \) to be the basic variables which is definitely not true. As it is evident, that removal of total derivative term from action (4) yields action (5) where there is no scope to introduce any auxiliary variable and as such unique quantum description emerges for a particular action.

### 3 Induced Theory Of Gravity Modified By Curvature Squared term

We now extend our work by introducing higher order term in the above action (2) along with a kinetic and potential term for the coupling scalar field. The action now reads,

\[ A = \int \sqrt{-g} \left[ -\frac{1}{2} \epsilon \phi^2 R - \frac{1}{2} \phi, \phi, \phi, \phi, - V(\phi) - \frac{\beta}{24} R^2 \right] d^4 x \]  

(14)

which in the Robertson-Walker model takes the following form,

\[ A = 2 \pi^2 \int \left[ (3 \epsilon \phi^2 e^{2\alpha} - 3 \phi^2 - 3 \beta) \phi'' + 3 \epsilon \phi^2 (\phi'' + 1) e^{2\alpha} - \frac{3}{2} \beta (\phi'^2 + \phi'^4 + 2 \phi'^2 + 1) + \frac{1}{2} \phi'^2 e^{2\alpha} - V(\phi) e^{4\alpha} \right] d\eta \]  

(15)

The first three terms in the action (15) are removable total derivative terms, which as pointed out earlier leads to wrong Wheeler-deWitt equation when not removed. So our first task is to integrate out these terms and then introduce auxiliary variable and write the action in the canonical form as suggested by Boulware et-al. So in the first step we obtain the following action

\[ A = 2 \pi^2 \int \left[ -3 \epsilon \phi \alpha' (\phi' \alpha' + 2 \phi') e^{2\alpha} - \frac{3}{2} \beta (\phi'' + \phi'^4 + 2 \phi'^2 + 1) + 3 \epsilon \phi^2 e^{2\alpha} - V(\phi) e^{4\alpha} \right] d\eta + \Sigma_1 \]  

(16)
where $\Sigma_1 = 2\pi^2 [3\epsilon\phi^2\alpha'\epsilon^{2\alpha} - \beta\alpha'^3 - 3\beta\alpha']$ is the surface term. In the next step we introduce the auxiliary variable, which is

$$Q = -\frac{1}{2\pi^2} \frac{\partial A}{\partial \alpha''} = 3\beta\alpha''.$$  \hspace{1cm} (17)

In the following step we write the action (15) in the canonical form where again one encounters another integrable term which when integrated one obtains the following action.

$$A = 2\pi^2 \int \left[ \alpha'Q' - \frac{3}{2}\beta\alpha'^2 - 3\epsilon\phi^2\epsilon^{2\alpha} + \beta\alpha'^2 - 6\epsilon\phi\alpha'\epsilon^{2\alpha} + \frac{1}{2}\epsilon\phi^2\epsilon^{2\alpha} + \frac{Q^2}{6\beta} + 3\epsilon\phi^2\epsilon^{2\alpha} - \frac{3}{2}\beta - Ve^{4\alpha} \right]d\eta + \Sigma_2(18)$$

where $\Sigma_2 = \Sigma_1 - 2\pi^2\alpha'Q$ is the modified surface term. One can now write the classical field equations and verify after substituting the definition of $Q$ from equation (17) that the equations are the same even if one would have introduced the auxiliary variable in action (15) where total derivative terms are not removed. However, since we are only interested in the quantum dynamics for such an action, therefore we are writing only the Hamiltonian constraint equation, which is

$$H = p_\alpha p_\alpha + \frac{p_\phi^2}{2\epsilon^{2\alpha}} + 6\epsilon\phi p_\phi p_\alpha + \frac{3}{2}\beta p_\phi^2 + 3(\beta + \epsilon\phi^2\epsilon^{2\alpha} + 6\epsilon^2\phi^2\epsilon^{2\alpha})p_\phi^2 Q^2 - \frac{Q^2}{6\beta} - 3\epsilon\phi^2\epsilon^{2\alpha} + \frac{3}{2}\beta + V(\phi)e^{4\alpha}.$$  \hspace{1cm} (19)

In the above, $p_\alpha = \frac{\partial L}{\partial \dot{\alpha}}$, $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \alpha'$ and $p_\phi = \frac{\partial L}{\partial \dot{\phi}}$. Now as suggested by Boulware et-al [7], we quantize with basic variables, which are $\alpha, \alpha'$ and $\phi$. To avoid confusion we choose $\alpha' = x$ and replace $p_\alpha = \alpha'$ by $x$ and $Q$ by $-p_x$. The Hamiltonian thus reads,

$$H = xp_\alpha + \frac{p_\phi^2}{2\epsilon^{2\alpha}} + 6\epsilon\phi xp_\phi - \frac{p_\phi^2}{6\beta} + \frac{3}{2}\beta x^4 + 3[\beta + \epsilon\phi^2(1 + 6\epsilon)e^{2\alpha}]x^2 - 3\epsilon\phi^2\epsilon^{2\alpha} + \frac{3}{2}\beta + V(\phi)e^{4\alpha}.$$  \hspace{1cm} (20)

This Hamiltonian which is constrained to vanish is now ready for quantization. The corresponding Wheeler-deWitt equation is

$$i\hbar(\frac{\partial}{\partial \alpha} + 6\epsilon\phi \frac{\partial}{\partial \phi})\psi = \hbar^2 (\frac{1}{x} \frac{\partial^2}{\partial x^2} + \frac{n}{x^2} \frac{\partial}{\partial x})\psi - \frac{\hbar^2}{2xe^{2\alpha}} (\frac{\partial^2}{\partial \phi^2} + \frac{p}{\phi} \frac{\partial}{\partial \phi})\psi + V_c \psi,$$  \hspace{1cm} (21)

where, $n$ and $p$ are operator ordering indices. The effective potential $V_c$ is given by

$$V_c = \frac{3\beta}{2x}(x^2 + 1)^2 + \frac{3\epsilon\phi^2\epsilon^{2\alpha}}{x} - \frac{2(2\epsilon\phi^2 + 2\alpha) - 1}{x} + \frac{2\epsilon\phi^2\epsilon^{2\alpha}}{x}.$$  \hspace{1cm} (22)

Choosing the time variable as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + 6\epsilon\phi \frac{\partial}{\partial \phi}$$  \hspace{1cm} (23)

the above Wheeler-deWitt equation can clearly be expressed as

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}_0 \psi.$$  \hspace{1cm} (24)

This equation looks very similar to the Schrödinger equation, where the Hamiltonian operator $\hat{H}_0$, given by the right hand side of equation (20) is clearly hermitian. This implies that we are now dealing with observables for which a probability interpretation exits. For $n = -1$ and $p = 0$ the continuity equation is expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$  \hspace{1cm} (25)

where, $\rho = \psi^*\psi$, is the probability density and $\mathbf{J} = (j_x, j_\phi, 0)$, is the current density, with $j_x = \frac{\partial}{\partial x}(\psi^*\psi, x - \psi\psi^*, x)$ and $j_\phi = -\frac{\partial}{\partial \phi}((\psi^*\psi, \phi - \psi\psi^*, \phi)$. It is possible to write the continuity equation for other operator ordering indices also but that with respect to a different variable which is only functionally related to $x$. It is to be noted that for $\epsilon = 0$ one can obtain the results corresponding to an action containing $R^2$ term minimally coupled with a scalar field $[\phi]$. The form of the continuity equation (25) convincingly suggests that the choice of the time parameter in equation (23) is correct. Thus time parameter arises naturally in the quantum dynamics of the alternative theory of gravitation containing curvature squared term.

One can now extremize the effective potential, which is equivalent to extremize the effective action when Kinetic energy is neglected. the effective potential given by equation (22) is asymmetric with respect to the
expansion parameter $x = \alpha'$, i.e. $V_e = -V_e$. Further it is unsatable at the short and long range limits of $x = \alpha'$. However, the classical cosmological models depict that the initial epoch of the evolution of the Universe, where quantum physics plays the dominating role, starts with a large expansion rate $x$. It slows down as Universe evolves and the infinity of the effective potential at small $x$ has nothing to do with the physical world since Universe by that time has entered the classical regime and the effective potential has no role at that stage.

Extremization of the effective potential (22) with respect to $x$ yields, keeping $\alpha$ and $\phi$ fixed

$$
\beta(3x^4 + 2x^2 - 1) + 2\epsilon \phi^2(2\epsilon x^2 + x^2 + 1)e^{2\alpha} - \frac{2}{3}V(\phi)e^{4\alpha} = 0 \quad (26)
$$

In the absence of the coupling parameter $\epsilon$, i.e. the situation where the action is given by the curvature squared term minimally coupled with a scalar field, the above equation reads

$$
\beta(3x^4 + 2x^2 - 1) = \frac{2}{3}V(\phi)e^{4\alpha} \quad (27)
$$

This is the classical constraint equation at a regime much below the Planck scale, where the Hamiltonian is mostly dominated by the potential energy, kinetic energy being much small.

Further in the absence of the scalar potential $V(\phi)$ the equation (27) becomes

$$
3x^4 + 2x^2 - 1 = 0 \quad (28)
$$

ie. either

$$
x^2 + 1 = 0 \quad (29)
$$

which is the vacuum Einstein’s equation admitting Euclidean wormhole solution, or,

$$
3x^2 - 1 = 0 \quad (30)
$$

which implies that the extremum admits a solution $a = \frac{e^{-ln(t)}}{\sqrt{3}}$, $a$ being the scale factor of the Universe in proper time $t$. For this solution the horizon radius $r_H$ is proportional to $ln(t)$. So at $t \to 0$, $r_H \to \infty$ and thus the horizon problem is solved. These results are obtained previously in [1] and [2]. It should be noted that $\beta$ can not be made zero at any stage after equation (16). This removes any possibilities of including redundant degrees of freedom, as already mentioned.

Now extremum of the effective potential $V_e$ with respect to $\phi$ gives, keeping $\alpha$ and $x$ fixed

$$
6\epsilon \phi(2\epsilon x^2 + x^2 - 1) + \frac{dV(\phi)}{d\phi}e^{2\alpha} = 0 \quad (31)
$$

If the scalar field potential $V(\phi)$ has got an extremum at $\phi = 0$ then the above equation is trivially satisfied, for which $\frac{\partial V_e}{\partial x} = 0$ gives

$$
\beta(x^2 + 1)(3x^2 - 1) = \frac{2}{3}e^{4\alpha}V(\phi)|_{extremum} \quad (32)
$$

and

$$
\frac{\partial^2 V_e}{\partial x^2} = 6\beta(3x^2 + 1) \quad (33)
$$

which is positive definite, implying extremum of $V_e$ with respect to $x$ has a minimum. Further,

$$
\frac{\partial^2 V_e}{\partial \phi^2} = \frac{6\epsilon e^{2\alpha}}{x}(2\epsilon x^2 + x^2 + 1) + \frac{e^{4\alpha}}{x} \frac{d^2 V(\phi)}{d\phi^2}|_{extremum} \quad (34)
$$

and finally,

$$
\frac{\partial^2 V_e}{\partial x \partial \phi} = 0 \quad (35)
$$

at the extremum. The condition that $V_e$ has an extremum being a function of two variables $x$ and $\phi$ is

$$
\frac{\partial^2 V_e}{\partial x^2} \frac{\partial^2 V_e}{\partial \phi^2} - \frac{\partial^2 V_e}{\partial x \partial \phi} > 0 \quad (36)
$$
The left hand side turns out to be
\[ 6\beta \frac{(3x^2 + 1)}{x} [6\epsilon(2\epsilon x^2 + x^2 + 1) + e^{2\alpha} \frac{d^2 V(\phi)}{d\phi^2} |_{\text{extremum}}]e^{2\alpha} \]  
(37)
which is positive definite if the scalar field potential \( V(\phi) \) has got a minimum, since \( \epsilon > 0 \), as long as \( x > 0 \) and real. It should be noted that \( \alpha \) has not been considered as a variable, since as \( \epsilon \to 0 \), as well as when \( \phi \to 0 \), \( \alpha \) alone acts as the time parameter. As the condition for the extremum has been fulfilled, if we now consider that the minimum of the scalar field potential \( V(\phi) = 0 \), then equation (32) reduces to either
\[ x^2 + 1 = 0 \]  
(38)
or
\[ 3x^2 - 1 = 0 \]  
(39)
The same conditions that we have already visited. However, the first condition implies that \( x \) is imaginary for which the condition for extremum of \( V_e \) given in equation (37) is not satisfied. Hence, second condition viz. equation (39) should only be considered keeping \( x > 0 \), i.e.
\[ x = \frac{1}{\sqrt{3}} \]  
(40)
whose solution has already been given. We can thus conclude that the Universe is sitting at the extremum of the effective potential \( V_e \), at the location of the configuration space variables \( \phi = 0 \) and \( x = \frac{1}{\sqrt{3}} \), which satisfies a solution \( a = \frac{t-t_0}{\sqrt{3}} \), that solves the horizon problem, as already noticed.

## 4 Quantum Cosmology Of A Positive Definite Action

As already mentioned in the introduction, it is well known that the Euclidean form of the Gravitational Einstein-Hilbert action is not bounded from below. As a result the ground state wave function \( \psi_0[h_{ij}] = N \int \delta g e^{x(-I_E[g])} \) proposed by Hartle-Hawking [13] diverges. A remedy is to suggest a positive definite Euclidean action. Horowitz [4] suggested an action (1) whose pseudo-Riemannian form is,
\[ A = \int \sqrt{-g} d^4x [AC_{ijkl} + B(R - 4\lambda)^2]. \]  
(41)
For the Robertson-Walker metric \( C_{ijkl} \), the Weyl tensor vanishes and hence the above action can be expressed in the form,
\[ A = -\frac{\beta}{24} \int [R - 4\lambda]^2 \sqrt{-g} d^4x. \]  
(42)
Our interest is to study the quantum dynamics of such a theory. In the Robertson-Walker minisuperspace model this action takes the following form
\[ A = -\frac{m}{4} \int [\alpha'^2 + (1 + \alpha'^2)^2 + \frac{4\lambda}{3}(1 - \alpha'^2)e^{2\alpha} + \frac{4}{9}\lambda^2 e^{4\alpha}] \, d\eta \]  
(43)
apart from a surface term \( \Sigma_1 = -\frac{1}{m} [6\alpha' + 2\alpha'^3 - 4\lambda\alpha'e^{2\alpha}] \), where \( m = 12\beta\pi^2 \). Now we define the auxiliary variable
\[ Q = -\frac{4}{m} \frac{\delta A}{\delta \alpha''} = 2\alpha'' \]  
(44)
and write the action in the canonical form after introducing the auxiliary variable as,
\[ A = \frac{m}{4} \int [\alpha'Q' - (\alpha'^2 + 1)^2 + \frac{4\lambda}{3}(\alpha'^2 - 1)e^{2\alpha} + \frac{Q^2}{4} - \frac{4}{9}\lambda^2 e^{4\alpha}] \, d\eta \]  
(45)
apart from a modified surface term \( \Sigma_2 = -\frac{1}{6} [3\alpha' + \alpha'^3 - 2\lambda\alpha'e^{2\alpha} + 3\alpha'\alpha''] \). The Hamiltonian in the phase space variable can be written as
\[ P_{\alpha}P_Q + \frac{16}{m^2} P_Q^4 + 2(1 - \frac{2\lambda}{3}e^{2\alpha})P_{Q^2} + m^2[\frac{\lambda^2}{36} e^{4\alpha} + \frac{\lambda}{12} e^{2\alpha} - \frac{1}{64} Q^2 + \frac{1}{16}] = 0 \]  
(46)
We now quantize the theory with basic variables $\alpha$ and $\alpha' = x$ (say). Therefore we replace as before, $Q$ by $-\frac{a}{m}P_x$ and $P_Q$ by $\frac{P}{x}$. With such replacement the Hamiltonian constraint equation reads

$$xP_\alpha = \frac{1}{m}P_x^2 - \frac{m}{4}(x^2 + 1)^2 + \frac{m\lambda}{x^3}(x^2 - 1)e^{2\alpha} - \frac{m}{9}\lambda^2 e^{4\alpha}. \quad (47)$$

The corresponding Wheeler-deWitt equation is thus,

$$i\hbar \frac{\partial \psi}{\partial \alpha} = \frac{\hbar^2}{mx} \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2 n}{m^2x^2} \frac{\partial \psi}{\partial x} + \frac{m}{4x}(x^2 + 1)^2 - \frac{m\lambda}{3x}(x^2 - 1)e^{2\alpha} + \frac{m\lambda^2}{9x^2}e^{4\alpha} \psi \quad (48)$$

where $n$ is the operator ordering index. As before the Wheeler-deWitt equation can be written as

$$i\hbar \frac{\partial \psi}{\partial \alpha} = \hat{H}_0 \psi, \quad (49)$$

where, $\hat{H}_0 \psi$ is the right hand side of the Wheeler-de-Witt equation (48). The probability interpretation follows naturally from the continuity equation, which can be written for $n = -1$ as,

$$\frac{\partial \rho}{\partial \alpha} + \nabla \cdot J = 0, \quad (50)$$

where $\rho = \psi^*\psi$ is the probability density and $J = (j_x, 0, 0)$ is the current density, with $j_x = \frac{\hbar}{mx}(\psi^*\psi_{,x} - \psi\psi_{,x}^*)$. The continuity equation (50) proves that there is nothing wrong in considering $\alpha$ as the time parameter, and it appears naturally in the quantum dynamics of alternative theory of gravitational action containing curvature squared term.

The effective potential in this case is

$$V_\epsilon = \frac{m}{4x}[(x^2 + 1)^2 - \frac{4\lambda}{3}(x^2 - 1)e^{2\alpha} + \frac{4}{9}\lambda^2 e^{4\alpha}] \quad (51)$$

The extremum of the effective potential $V_\epsilon$ with respect to $x$ yields

$$(x^2 + 1)(3x^2 - 1 - \frac{4\lambda}{3}e^{2\alpha}) - \frac{4}{9}\lambda^2 e^{4\alpha} = 0 \quad (52)$$

As it is evident, this is essentially the classical constraint equation at the epoch when the Universe is dominated by the potential energy term, kinetic energy being neglected. For vanishing $\lambda$ the extremum is either at

$$x^2 + 1 = 0 \quad (53)$$

which has the classical wormhole solution, or at

$$3x^2 - 1 = 0 \quad (54)$$

whose solution is $\alpha = \ln(\frac{1}{\sqrt{3}})$. The consequences of these solutions are already discussed. All these results for $\lambda = 0$ are already obtained by Sanyal and Modak [2].

5 Einstein-Hilbert Action Modified By $R_{\mu\nu}R^{\mu\nu}$ Term

In this section we couple the Einstein-Hilbert action with the next higher order curvature invariant term, viz. $R_{\mu\nu}R^{\mu\nu}$. Thus the action becomes

$$A = \int -\frac{\sqrt{-g}}{16\pi G}[R + \frac{\gamma}{2}R_{\mu\nu}R^{\mu\nu}]d^4x \quad (55)$$

where $\gamma$ is the coupling constant, the Ricci scalar $R = -6(\alpha'' + \alpha'^2 + 1)$ and $R_{\mu\nu}R^{\mu\nu} = 3(3\alpha'' + \alpha'' + 2\alpha'^2 + 2)e^{-4\alpha}$. Introducing $R$ and $R_{\mu\nu}R^{\mu\nu}$ and removing removable total derivative terms from the action, as proposed, we arrive at

$$A = M \int [-\gamma\alpha'' - (\gamma(\alpha'' + 1)^2 + (\alpha'' - 1)e^{2\alpha})]d\eta + \Sigma_1 \quad (56)$$

where $M = \frac{2\pi}{\alpha''}$ and $\Sigma_1 = \frac{2\pi}{\alpha''}(\alpha'e^{2\alpha} - \gamma\alpha' - \frac{\gamma}{3}\alpha'^3)$ is the surface term.
Now introducing the new variable
\[ Q = \frac{1}{M} \frac{\partial A}{\partial \alpha'} = -2\gamma \alpha' \]  
we express the action (56) in the following canonical form
\[ A = M \int \left[ -\alpha' Q' + (1 - \alpha'^2)e^{2\alpha} - \gamma (1 + \alpha')^2 + \frac{Q^2}{4\gamma} \right] + \Sigma_2 \]  
Where \( \Sigma_2 = \Sigma_1 + M\alpha'Q \) is the modified surface term. The classical field equations are
\[ \gamma(\alpha'''' - 6\alpha'' \alpha' + 2\alpha'') - (\alpha'' + \alpha'^2 + 1)e^{2\alpha} = 0 \]  
and
\[ \gamma[2\alpha''(\alpha' - \alpha'') - (1 + \alpha'^2)(3\alpha'^2 - 1)] - (1 + \alpha'^2)e^{2\alpha} \]  
The last equation is essentially the Hamiltonian constraint equation. It can easily be seen that both the equations reduce to the vacuum Einstein equations for the coupling constant \( \gamma = 0 \). Further, most interesting is the fact that the classical field equations for \( R + R_{\mu\nu}R^{\mu\nu} \) action turns out the same as that for \( R + \alpha R^2 \) action, only having the difference in the coupling constant.

To quantize the system we express the Hamiltonian in the phase space variables as
\[ H = \frac{1}{M} \left[ -p_\alpha p_Q - \frac{M^2}{4\gamma}Q^2 + \frac{\gamma}{M^2}p_Q^4 + 2\gamma p_Q^2 + M^2\gamma + (p_Q^2 - M^2)e^{2\alpha} \right] = 0 \]  
since the Hamiltonian is constrained to vanish. As already mentioned we quantize with the basic variables \( \alpha \) and \( \alpha' = x \) (say). Thus the Hamiltonian Constraint equation takes the following look in terms of the basic variables
\[ -xp_\alpha = -\frac{1}{4M\gamma}p_x^2 + \gamma M(x^2 + 1) + M(x^2 - 1)e^{2\alpha} = 0 \]  
Hence the quantum dynamics of the system is determined by the following equation
\[ i\hbar \frac{\partial \psi}{\partial \alpha} = \frac{\hbar^2}{4M\gamma x} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{n}{x} \frac{\partial \psi}{\partial x} \right) + V_\psi. \]  
where the effective action
\[ V_\psi = \frac{M}{x} [\gamma(x^2 + 1)^2 + (x^2 - 1)e^{2\alpha}]. \]  
Now equation (63) can be written as
\[ i\hbar \frac{\partial \psi}{\partial \alpha} = \hat{H}_0 \psi \]  
Where, \( \hat{H}_0 \), the effective Hamiltonian operator operating on \( \psi \), being given by the right hand side of equation (63), is again found to be hermitian. As a result one can easily construct the continuity equation
\[ \frac{\partial \rho}{\partial \alpha} + \nabla \cdot J \]  
where, the current density \( J = (J_x, 0, 0) \), \( J_x = \frac{i\hbar}{4M\gamma x} (\psi^* \psi_{,x} - \psi \psi^*_{,x}) \) and the probability density \( \rho = \psi^* \psi \). Though the above continuity equation is formed for \( n = -1 \), yet it is not difficult to see that it is true in general for arbitrary operator ordering index but then with respect to a new variable which is functionally related to \( x \). In analogy to quantum mechanics it is now quite trivial to identify \( \alpha \) as the time parameter in quantum cosmology while the variable \( x = \alpha' \) acts as the spatial coordinate, \( \alpha \) being the expansion parameter. It appears that the effective Hamiltonian operator \( \hat{H}_0 \) diverges at the bounce of the Universe, ie. at \( \alpha' = 0 \). It can be interpreted as the Universe enters into the Classical regime much before \( \alpha' = 0 \) and so effective Hamiltonian has no role to play at that regime. It can also be interpreted as the arrow of time reverses it’s direction at the bounce.

In the weak energy limit, the contribution of kinetic energy is pretty small with respect to the potential energy in the Hamiltonian \( \hat{H}_0 \). At that regime the Hamiltonian is almost dominated by the potential energy, and so one can extremize the effective potential for \( k = 1 \) to get either
\[ x^2 + 1 = 0 \]
which is the vacuum Einstein’s equation admitting Euclidean wormhole solution as already discussed, or
\[\gamma(3x^2 - 1) + e^{2\alpha} = 0\]  
which has a solution
\[e^\alpha = a = \sqrt{\gamma \sin(t - t_0 / \sqrt{3\gamma})}\]
where \(a\) is the scale factor in proper time \(t\). It is interesting to note that all these results are at par with those obtained by Sanyal and Modak [2] for the Einstein-Hilbert action modified by the curvature squared term ignoring the scalar field. Altogether, we observe that in the Robertson-Walker minisuperspace model the Einstein-Hilbert action modified by the next higher order curvature invariant term \(R_{\mu\nu}R^{\mu\nu}\) contributes nothing more than that modified by the \(R^2\) term both at the classical and the quantum regime. However, it has been interestingly observed that the inclusion of such higher order terms in the Einstein-Hilbert action naturally gives rise to a Schrödinger like equation with an effective Hamiltonian which is hermitian. As a result, quantum mechanical probability interpretation of quantum cosmology is available.

6 Concluding Remarks

The aim of this paper is to show how Gravitational action containing higher order terms or being modified by such terms leads to a quantum mechanical probability interpretation of quantum cosmology. In a couple of recent publications [1] and [2] it has been shown that to quantize Gravitational action containing curvature squared term or being modified by such term, Boulware et-al’s [3] prescription can be taken up only after removing removable total derivative terms from the action. In [1] a toy model viz. Einstein-Hilbert action was considered to show that if removable total derivative terms are kept in the action, one can introduce auxilliary variable, that leads to wrong quantum dynamics. Here again, it has been shown that it is possible to introduce auxilliary variable in the induced theory of gravity which ultimately leads to wrong quantum description. Here it has been pointed out that, introduction of auxilliary variables in situations where it is not required, effectively implies to consider redundant degrees of freedom during quantization. This is because, in Boulware et-al’s [3] prescription one finally has to quantize with basic variables. It has been shown in section 3 that, while induced theory of gravity has actually a pair of basic variables viz. \(\alpha, \phi\), introduction of auxilliary variable demands basic variables to be three, viz. \(\alpha, \phi, \alpha'\). Hence we conclude that all earlier works of choosing auxilliary variables arbitrarily or through some definite prescription lead to wrong quantum dynamics unless it has been introduced only after removing removable total derivative terms from the action.

We have shown that once removable total derivative terms are removed from the action, there is no chance of introducing such auxilliary variables in situations where not required. Further, such variables are thus chosen uniquely in situations where required, leading to unique quantum description of the theory.

Such method of choosing auxilliary variables gives rise to a Schrödinger like equation , where a time parameter comes out automatically. Further, it gives rise to an effective Hamiltonian which is hermitian and thus one can write the continuity equation in view of the probability and current densities leading to a quantum mechanical probability interpretation of the theory.

The same type of result has been obtained by introducing the next higher order curvature invariant term \(R_{\mu\nu}R^{\mu\nu}\) in the Einstein-Hilbert action. This implies that quantum mechanical probability interpretation of quantum cosmology might be a generic feature of higher order curvature terms.

It should be mentioned that all these results are valid only in the isotropic and homogeneous minisuperspace models. How to extend our work in the anisotropic models is under investigation.

References

[1] A.K.Sanyal and B.Modak, Phys. Rev. D63 064021 (2001)
[2] A.K.Sanyal and B.Modak, Class. Quan. Gravit. 19, 515(2002)
[3] J.A.Wheeler, in Relativity Groups and Topology, Ed. C.de-Witt and B.de-Witt, Les Houches, Gordon and Breach science Publishers. (1963)
[4] G.T.Horowitz, Phys. Rev. D (1988)
[5] M.D.Pollock, Nucl.Phys.B306 931 (1988)

[6] H.-J. Schmidt, Phys. Rev. D49, 6354 (1994)

[7] D.Boulware, A Strominger and E.T.Tomboulis, Quantum theory of gravity, ed. S.Christensen (Adam higler, Bristol, 1984)

[8] S.Hawking and J.C.Luttrell, Nucl. Phys. B306, 931 (1988)

[9] E.T.Tomboulis, Phys. Lett. 70B, 361 (1977)

[10] K.S.Stell, Phys. Rev. D16, 953 (1977)

[11] A.A.Starobinsky, Phys. Lett. 91B, 99 (1980)

[12] A.A.Starobinsky and H.-J.Schmidt, Class. Quan. Gravit. 4, 695 (1987)

[13] J.B.Hartle and S.Hawking, Phys.Rev. D28, 2960 (1983)