THE SEGAL CONJECTURE FOR INFINITE GROUPS

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Abstract. We formulate and prove a version of the Segal Conjecture for infinite groups. For finite groups it reduces to the original version. The condition that $G$ is finite is replaced in our setting by the assumption that there exists a finite model for the classifying space $EG$ for proper actions. This assumption is satisfied for instance for word hyperbolic groups or cocompact discrete subgroups of Lie groups with finitely many path components. As a consequence we get for such groups $G$ that the zero-th stable cohomotopy of the classifying space $BG$ is isomorphic to the $I$-adic completion of the ring given by the zero-th equivariant stable cohomotopy of $EG$ for $I$ the augmentation ideal.

0. Introduction

We first recall the Segal Conjecture for a finite group $G$. The equivariant stable cohomotopy groups $\pi^n_G(X)$ of a $G$-CW-complex are modules over the ring $\pi^0_G = \pi^0_G(\{\bullet\})$ which can be identified with the Burnside ring $A(G)$. The augmentation homomorphism $A(G) \to \mathbb{Z}$ is the ring homomorphism sending the class of a finite set to its cardinality. The augmentation ideal $I_G \subseteq A(G)$ is its kernel. Let $\pi^n_G(X)_{\hat{I}_G}$ be the $I_G$-adic completion $\invlim_{n \to \infty} \pi^n_G(X)/I^n_G \cdot \pi^n_G(X)$ of $\pi^n_G(X)$.

The following result was formulated as a conjecture by Segal and proved by Carlsson [6].

Theorem 0.1 (Segal Conjecture for finite groups). For every finite group $G$ and finite $G$-CW-complex $X$ there is an isomorphism

$$\pi^n_G(X)_{\hat{I}_G} \cong \pi^n(EG \times_G X).$$

In particular we get for $X = \{\bullet\}$ and $m = 0$ an isomorphism

$$A(G)_{\hat{I}_G} \cong \pi^0_0(BG).$$

The purpose of this paper is to formulate and prove a version of it for infinite (discrete) groups, i.e., we will show

Theorem 0.2. (Segal Conjecture for infinite groups). Let $G$ be a (discrete) group. Let $X$ be a finite proper $G$-CW-complex. Let $L$ be a proper finite dimensional $G$-CW-complex with the property that there is an upper bound on the order of its isotropy groups. Let $f : X \to L$ be a $G$-map.

Then the map of pro-$\mathbb{Z}$-modules

$$\lambda^n_G(X) : \{\pi^n_G(X)/I^n_G \cdot \pi^n_G(X)\}_{n \geq 1} \cong \{\pi^n(EG \times_G X)_{(n-1)}\}_{n \geq 1}$$

defined in (3.14) is an isomorphism of pro-$\mathbb{Z}$-modules.

In particular we obtain an isomorphism

$$\pi^n_G(X)_{\hat{I}_G(L)} \cong \pi^n(EG \times_G X).$$
If there is a finite $G$-CW-model for $EG$, we obtain an isomorphism
$$\pi^m_G(EG)_{\mathcal{I}(EG)} \cong \pi^m_0(BG).$$

Here $EG$ is the classifying space for proper $G$-actions and $\pi^*_G(X)$ is equivariant stable cohomotopy as defined in [10, Section 6]. The ideal $\mathcal{I}_G(L)$ is the augmentation ideal in the ring $\pi^*_G(L)$ (see Definition 3.1). We view $\pi^*_G(X)$ as $\pi^*_G(L)$-module by the multiplicative structure on equivariant stable cohomotopy and the map $f$. We denote by $\pi^*_G(X)_{\mathcal{I}(L)}$ its $\mathcal{I}_G(L)$-completion. More explanations will follow in the main body of the text.

In [10] various mutually distinct notions of a Burnside ring of a group $G$ are introduced which all agree with the standard notion for finite $G$. If there is a finite $G$-CW-model for $EG$, then the homotopy theoretic definition is $A(G) := \pi^*_0(EG)$, we define the ideal $\mathcal{I}_G \subseteq A(G)$ to be $\mathcal{I}_G(EG)$, and we get in this notation from Theorem 0.2 an isomorphism
$$A(G)_{\mathcal{I}_0} \cong \pi^*_0(BG).$$

We will actually formulate for every equivariant cohomology theory $H_G^*$ with multiplicative structure a Completion Theorem (see Problem 5.3). It is not expected to be true in all cases. We give a strategy for its proof in Theorem 1.1. We show that this applies to equivariant stable cohomotopy, thus proving Theorem 0.2. It also applies to equivariant topological $K$-theory, where the Completion Theorem for infinite groups has already been proved in [15].

If $G$ is finite, we can take $L = EG = \{\bullet\}$ and then Theorem 0.2 reduces to Theorem 0.1. We will not give a new proof of Theorem 0.1 but use it as input in the proof of Theorem 1.2.

This paper is part of a general program to systematically study equivariant homotopy theory, which is well-established for finite groups and compact Lie groups, for infinite groups and non-compact Lie groups. The motivation comes among other things from the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

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1. Equivariant Cohomology Theories with Multiplicative Structure

We briefly recall the axioms of a (proper) equivariant cohomology $H_G^*$ theory with values in $R$-modules with multiplicative structure. More details can be found in [11].

Let $G$ be a (discrete) group. Let $R$ be a commutative ring with unit. A (proper) $G$-cohomology theory $H_G^*$ with values in $R$-modules assigns to any pair $(X,A)$ of (proper) $G$-CW-complexes $(X,A)$ a $Z$-graded $R$-module $H_G^n(X,A)$ such that $G$-homotopy invariance holds and there exists long exact sequences of pairs and long exact Mayer-Vietoris sequences. Often one also requires the disjoint union axiom what we will need not here since all our disjoint unions will be over finite index sets.

A multiplicative structure is given by a collection of $R$-bilinear pairings
$$\cup: H^m_G(X,A) \otimes_R H^n_G(X,B) \to H^{m+n}_G(X,A \cup B).$$

This product is required to be graded commutative, to be associative, to have a unit $1 \in H_G^0(X)$ for every (proper) $G$-CW-complex $X$, to be compatible with boundary homomorphisms and to be natural with respect to $G$-maps.

Let $\alpha: H \to G$ be a group homomorphism. Given an $H$-space $X$, define the induction of $X$ with $\alpha$ to be the $G$-space $\text{ind}_\alpha X$ which is the quotient of $G \times X$
Example 1.5. (Equivariant Stable Cohomotopy). In [10, Section 6] equivariant stable cohomotopy $\pi^*_{G}$ where $G$ is a finite group. It has the property that for any finite subgroup $H \subseteq G$ and a (proper) $H$-CW-pair $(X, A)$ there are for each $n \in \mathbb{Z}$ natural homomorphisms

\[
\text{ind}_n : H^n_G(ind_n(X, A)) \to H^n_H(X, A)
\]

If $\ker(\alpha)$ acts freely on $X$, then $\text{ind}_n : H^n_G(ind_n(X, A)) \to H^n_H(X, A)$ is bijective for all $n \in \mathbb{Z}$. The induction structure is required to be compatible with the boundary homomorphisms, to be functorial in $\alpha$ and to be compatible with inner automorphisms.

Sometimes we will need the following lemma whose elementary proof is analogous to the one in [11, Lemma 1.2].

**Lemma 1.2.** Consider finite subgroups $H, K \subseteq G$ and an element $g \in G$ with $gHg^{-1} \subseteq K$. Let $R_{g^{-1}} : G/H \to G/K$ be the $G$-map sending $g'H$ to $g'g^{-1}K$ and $e(g) : H \to K$ be the group homomorphism sending $h$ to $ghg^{-1}$. Denote by $\text{pr} : (\{\bullet\}) \to (\{\bullet\})$ the projection to the one-point space $(\{\bullet\})$.

Then the following diagram commutes

\[
\begin{array}{ccc}
H^n_G(G/K) & \overset{H^n_G(R_{g^{-1}})}{\longrightarrow} & H^n_G(G/H) \\
\text{ind}_K^n \downarrow & & \downarrow \text{ind}_H^n \\
H^n_K(\{\bullet\}) & \overset{\text{ind}_n(\alpha) \circ H^n_K(\text{pr})}{\longrightarrow} & H^n_H(\{\bullet\})
\end{array}
\]

Let $H^n$ be a (proper) equivariant cohomology theory. A multiplicative structure on it assigns a multiplicative structure to the associated (proper) $G$-cohomology theory $H^n_G$ for every group $G$ such that for each group homomorphism $\alpha : H \to G$ the maps given by the induction structure of (1.1) are compatible with the multiplicative structures on $H^n_G$ and $H^n_H$.

**Example 1.3. Equivariant cohomology theories coming from non-equivariant ones.** Let $K^n$ be a (non-equivariant) cohomology theory with multiplicative structure, for instance singular cohomology or topological $K$-theory. We can assign to it an equivariant cohomology theory with multiplicative structure $H^n_G$ in two ways. Namely, for a group $G$ and a pair of $G$-CW-complexes $(X, A)$ we define $H^n_G(X, A)$ by $K^n(G\backslash(X, A))$ or by $K^n(EG \times_G (X, A))$.

**Example 1.4. (Equivariant topological $K$-theory).** In [15] equivariant topological $K$-theory is defined for finite proper equivariant $CW$-complexes in terms of equivariant vector bundles. It reduces to the classical notion which appears for instance in [2]. Its relation to equivariant $KK$-theory is explained in [16]. This definition is extended to (not necessarily finite) proper equivariant $CW$-complexes in [15] in terms of equivariant spectra using $\Gamma$-spaces and yields a proper equivariant cohomology theory $K^n_G$ with multiplicative as explained in [11, Example 1.7].

It has the property that for any finite subgroup $H$ of a group $G$ we have

\[
K^n_G(G/H) = K^n_H(\{\bullet\}) = \begin{cases} R_\mathbb{C}(H) & n \text{ even}; \\ \{0\} & n \text{ odd}, \end{cases}
\]

where $R_\mathbb{C}(H)$ denote the complex representation ring of $H$.

**Example 1.5. (Equivariant Stable Cohomotopy).** In [10, Section 6] equivariant stable cohomotopy $\pi^*_G$ is defined for finite proper equivariant $CW$-complexes in
terms of maps of sphere bundles associated of equivariant vector bundles. For finite $G$ it reduces to the classical notion. This definition is extended to arbitrary proper $G$-CW-complexes by Degrijse-Hausmann-Lück-Patchkoria-Schwede [2], where a systematic study of equivariant homotopy theory for (not necessarily compact) Lie groups and proper $G$-CW-complexes is developed.

Let $H \subseteq G$ be a finite subgroup. Recall that by the induction structure we have $\pi_n^0(G/H) = \pi^n_H(\{\bullet\})$. The equivariant stable homotopy groups $\pi^n_H$ are computed in terms of the splitting due to Segal and tom Dieck (see [17, Proposition 2] and [18, Theorem 7.7 in Chapter II on page 154]) by

$$\pi^n_H = \pi^n_{-n} = \bigoplus_{(K)} \pi^s_{-n}(BW_H K),$$

where $\pi^s_{-n}$ denotes (non-equivariant) stable homotopy and $(K)$ runs through the conjugacy classes of subgroups of $H$. In particular we get

$$|\pi^0_n(G/H)| < \infty \quad n \leq -1;$$
$$\pi^0_n(G/H) = A(H);$$
$$\pi^0_n(G/H) = \{0\} \quad n \geq 1,$$

where $A(H)$ is the Burnside ring.

2. Some Preliminaries about Pro-Modules

It will be crucial to handle pro-systems and pro-isomorphisms and not to pass directly to inverse limits. In this section we fix our notation for handling pro-$R$-modules for a commutative ring with unit $R$. For the definitions in full generality see for instance [3, Appendix] or [4, §2].

For simplicity, all pro-$R$-modules dealt with here will be indexed by the positive integers. We write $\{M_n, \alpha_n\}$ or briefly $\{M_n\}$ for the inverse system

$$M_0 \leftarrow^\alpha_1 M_1 \leftarrow^\alpha_2 M_2 \leftarrow^\alpha_3 M_3 \leftarrow \ldots .$$

and also write $\alpha^m_n := \alpha_{m+1} \circ \cdots \circ \alpha_n: M_n \to M_m$ for $n > m$ and put $\alpha^0_n := id_{M_n}$.

For the purposes here, it will suffice (and greatly simplify the notation) to work with “strict” pro-homomorphisms $\{f_n\}: \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$, i.e., a collection of homomorphisms $f_n: M_n \to N_n$ for $n \geq 1$ such that $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$ holds for each $n \geq 2$. Kernels and cokernels of strict homomorphisms are defined in the obvious way, namely levelwise.

A pro-$R$-module $\{M_n, \alpha_n\}$ will be called pro-trivial if for each $m \geq 1$, there is some $n \geq m$ such that $\alpha^m_n = 0$. A strict homomorphism $f: \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ is a pro-isomorphism if and only if ker($f$) and coker($f$) are both pro-trivial, or, equivalently, for each $m \geq 1$ there is some $n \geq m$ such that $\text{im}(\beta^m_n) \subseteq \text{im}(f_m)$ and $\ker(f_n) \subseteq \ker(\alpha^m_n)$. A sequence of strict homomorphisms

$$\{M_n, \alpha_n\} \xrightarrow{(f_n)} \{M_n', \alpha_n'\} \xrightarrow{g_n} \{M_n'', \alpha_n''\}$$

will be called exact if the sequences of $R$-modules $M_n \xrightarrow{f_n} N_n \xrightarrow{g_n} M_n''$ is exact for each $n \geq 1$, and it is called pro-exact if $g_n \circ f_n = 0$ holds for $n \geq 1$ and the pro-$R$-module $\{\ker(g_n)/\text{im}(f_n)\}$ is pro-trivial.

The elementary proofs of the following two lemmas can be found for instance in [13, Section 2].
Lemma 2.1. Let \(0 \rightarrow \{M''_n, \alpha''_n\} \xrightarrow{(f_n)} \{M_n, \alpha_n\} \xrightarrow{(g_n)} \{M'_n, \alpha'_n\} \rightarrow 0\) be a pro-exact sequence of pro-R-modules. Then there is a natural exact sequence

\[
0 \rightarrow \text{invlim}_{n \geq 1} M'_n \xrightarrow{\text{invlim}_{n \geq 1} f_n} \text{invlim}_{n \geq 1} M_n \xrightarrow{\text{invlim}_{n \geq 1} g_n} \text{invlim}_{n \geq 1} M''_n \xrightarrow{\delta} \text{invlim}_{n \geq 1} M''_n \rightarrow 0.
\]

In particular a pro-isomorphism \(f_n\) \(\{M_n, \alpha_n\} \rightarrow \{N_n, \beta_n\}\) induces isomorphisms

\[
\text{invlim}_{n \geq 1} f_n : \text{invlim}_{n \geq 1} M_n \xrightarrow{\cong} \text{invlim}_{n \geq 1} N_n;
\]

\[
\text{invlim}_{n \geq 1} g_n : \text{invlim}_{n \geq 1} N_n \xrightarrow{\cong} \text{invlim}_{n \geq 1} M_n.
\]

Lemma 2.2. Fix any commutative Noetherian ring \(R\), and any ideal \(I \subseteq R\). Then for any exact sequence \(M' \rightarrow M \rightarrow M''\) of finitely generated \(R\)-modules, the sequence

\[
\{M'/I^nM'\} \rightarrow \{M/I^nM\} \rightarrow \{M''/I^nM''\}
\]

of pro-\(R\)-modules is pro-exact.

3. The Formulation of a Completion Theorem

Consider a proper equivariant \(G\)-cohomology theory \(H^*_G\) with multiplicative structure. In the sequel \(H^*_G\) is the non-equivariant cohomology theory with multiplicative structure given by \(H^*_G\) for \(G = \{1\}\). Notice that \(H^0((\{\bullet\})\) is a commutative ring with unit and \(H^*_G(X)\) is a \(H^0((\{\bullet\})\)-module. In most applications \(H^0((\{\bullet\})\) will be \(\mathbb{Z}\). In the sequel \([Y, X]^G\) denotes the set of \(G\)-homotopy classes of \(G\)-maps \(Y \rightarrow X\). Notice that evaluation at the unit element of \(G\) induces a bijection \([G, X]^G \xrightarrow{\cong} \pi_0(X)\).

It is compatible with the left \(G\)-actions, which are on the source induced by pre-composing with right multiplication \(r_g : G \rightarrow G, g' \mapsto g'g\) and on the target by the given left \(G\)-action on \(X\).

So we can represent elements in \(G\backslash\pi_0(X)\) by classes \(\overline{x}\) of \(G\)-maps \(x : G \rightarrow X\), where \(x : G \rightarrow X\) and \(y : G \rightarrow X\) are equivalent, if for some \(g \in G\) the composite \(y \circ r_g\) is \(G\)-homotopic to \(x\).

Definition 3.1 (Augmentation ideal). For any proper \(G\)-CW-complex \(X\) the augmentation module \(\mathbb{I}_G(X) \subseteq H^*_G(X)\) is defined as the kernel of the map

\[
H^*_G(X) \xrightarrow{\prod_{x \in G \backslash \pi_0(X)} \text{ind}_{(1) \rightarrow G} \circ H^*_G(x)} \prod_{x \in G \backslash \pi_0(X)} H^0((\{\bullet\}))
\]

(The composite above is independent of the choice of \(x \in \overline{x}\) by \(G\)-homotopy invariance and Lemma [1.2].) If \(n = 0\), the map above is a ring homomorphism and \(I_G(X) := \mathbb{I}_G(X)\) is an ideal called the augmentation ideal.

Given a \(G\)-map \(f : X \rightarrow Y\), the induced map \(H^n_G(f) : H^n_G(Y) \rightarrow H^n_G(X)\) restricts to a map \(\mathbb{I}_G(Y) \rightarrow \mathbb{I}_G(X)\).

We will need the following elementary lemma:

Lemma 3.2. Let \(X\) be a \(CW\)-complex of dimension \((n-1)\). Then any \(n\)-fold product of elements in \(\mathbb{I}_G(X)\) is zero.

Proof. Write \(X = Y \cup A\), where \(Y\) and \(A\) are closed subsets, \(Y\) contains \(X^{(n-2)}\) as a homotopy deformation retract, and \(A\) is a disjoint union of \((n-1)\)-disks. Fix elements \(v_1, v_2, \ldots, v_n \in \mathbb{I}_G(X)\). We can assume by induction that \(v_1 \cdots v_{n-1}\) vanishes after restricting to \(Y\), and hence that it is the image of an element \(u \in H^*_G(X, Y)\). Also, \(v_n\) clearly vanishes after restricting to \(A\), and hence is the image of an element \(v \in H^*_G(X, A)\). The product of \(v_1 \cdots v_{n-1}\) and \(v_n\) is the image in \(H^*_G(X)\) of the element \(uv \in H^*_G(X, Y \cup A) = H^*_G(X, X) = 0\) and so \(v_1 \cdots v_n = 0\). \(\Box\)
Now fix a map \( f : X \to L \) between \( G \)-CW-complexes. Consider \( \mathcal{H}_G^0(X) \) as a module over the ring \( \mathcal{H}_G^0(L) \). Consider the composition

\[
\mathcal{H}_G^0(L)^n \cdot \mathcal{H}_G^m(X) \xrightarrow{\mathcal{H}_G^m(pr)} \mathcal{H}_G^m(\mathcal{E} \times X) \xrightarrow{\mathcal{H}_G^m(i_{(n-1)})^{-1}} \mathcal{H}_G^m(\mathcal{E} \times \mathcal{E} \times X) \xrightarrow{\mathcal{H}_G^m(j)} \mathcal{H}_G^m(\mathcal{E} \times \mathcal{E} \times X)_{(n-1)}.
\]

where \( i \) and \( j \) denote the inclusions, \( pr \) the projection and \( (\mathcal{E} \times X)_{(n-1)} \) is the \((n-1)\)-skeleton of \( \mathcal{E} \times X \). This composite is zero because of Lemma 3.2 since its image is contained in \( \mathcal{E} \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \). Thus we obtain a homomorphism of \( \mathcal{H}_G^0(\bullet) \)-modules

\[
(3.3) \quad \lambda_G^n(f : X \to L) : \{ \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \}_{n \geq 1} \\
\to \{ \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \}_{n \geq 1}.
\]

We will sometimes write \( \lambda^n_G \) or \( \lambda^n_G(X) \) instead of \( \lambda^n_G(f : X \to L) \) if the map \( f : X \to L \) is clear from the context. Notice that the target of \( \lambda^n_G(f : X \to L) \) depends only on \( X \) but not on the map \( f : X \to L \), whereas the source does depend on \( f \).

**Problem 3.4 (Completion Problem).** Under which conditions on \( \mathcal{H}_G^* \) and \( L \) is the map of \( \mathcal{H}_G^0(\bullet) \)-modules \( \lambda_G^n(f : X \to L) \) defined in (3.3) an isomorphism of \( \mathcal{H}_G^0(\bullet) \)-modules?

**Remark 3.5 (Consequences of the Completion Theorem).** Suppose that the map of \( \mathcal{H}_G^0(\bullet) \)-modules \( \lambda^n_G(X) \) defined in (3.3) is an isomorphism of \( \mathcal{H}_G^0(\bullet) \)-modules. Obviously the pro-module \( \{ \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \}_{n \geq 1} \) satisfies the Mittag-Leffler condition since all structure maps are surjective. This implies that its \( \lim^1 \)-term vanishes. We conclude from Lemma 2.21

\[
\text{in} \lim^1_{n \to \infty} \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) = 0;
\]

\[
\text{in} \lim_{n \to \infty} \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \cong \text{in} \lim_{n \to \infty} \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) / \mathcal{H}_G^m(\mathcal{E} \times X).
\]

Milnor’s exact sequence

\[
0 \to \text{in} \lim_{n \to \infty} \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \to \mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \to 0
\]

implies that we obtain an isomorphism

\[
\mathcal{H}_G^m((\mathcal{E} \times X)_{(n-1)}) \cong \text{in} \lim_{n \to \infty} \mathcal{H}_G^m(\mathcal{E} \times X) / \mathcal{H}_G^m(\mathcal{E} \times X).
\]

**Remark 3.6 (Taking \( L = \mathcal{E} \mathcal{G} \)).** The classifying space \( \mathcal{E} \mathcal{G} \) for proper \( G \)-actions is a proper \( G \)-CW-complex such that the \( H \)-fixed point set is contractible for every finite subgroup \( H \subseteq G \). It has the universal property that for every proper \( G \)-CW-complex \( X \) there is up to \( G \)-homotopy precisely one \( G \)-map \( f : X \to \mathcal{E} \mathcal{G} \). Recall that a \( G \)-CW-complex is proper if and only if all its isotropy groups are finite and it is finite if and only if it is cocompact. There is a cocompact \( G \)-CW-model for the classifying space \( \mathcal{E} \mathcal{G} \) for proper \( G \)-actions for instance if \( G \) is word-hyperbolic in the sense of Gromov, if \( G \) is a compact subgroup of a Lie group with finitely many path components, if \( G \) is a finitely generated one-relator group, if \( G \) is an arithmetic group, a mapping class group of a compact surface or the group of outer automorphisms of a finitely generated free group. For more information about \( \mathcal{E} \mathcal{G} \) we refer for instance to [5] and [12].

Suppose that there is a finite model for the classifying space of proper \( G \)-actions \( \overline{\mathcal{E} \mathcal{G}} \). Then we can apply this to \( id : \overline{\mathcal{E} \mathcal{G}} \to \overline{\mathcal{E} \mathcal{G}} \) and obtain an isomorphism

\[
\mathcal{H}_G^m(\mathcal{E} \mathcal{G}) \cong \text{in} \lim_{n \to \infty} \mathcal{H}_G^m(\mathcal{E} \mathcal{G}) / \mathcal{H}_G^m(\mathcal{E} \mathcal{G}) \cong \mathcal{H}_G^m(\mathcal{E} \mathcal{G}).
\]
Remark 3.7 (The free case). The statement of the Completion Theorem as stated in Problem 3.4 is always true for trivial reasons if \( X \) is a free finite \( G \text{-CW} \)-complex. Then induction induces an isomorphism

\[
\text{ind}_{G \to (1)} : \mathcal{H}^m(\partial G \setminus X) \xrightarrow{\cong} \mathcal{H}^m_G(X).
\]

Since \( I(\partial G \setminus X)^n = 0 \) for large enough \( n \) by Lemma 3.2, the canonical map

\[
\{ \mathcal{H}^m(\partial G \setminus X) \}_{n \geq 1} \xrightarrow{\cong} \{ \mathcal{H}^m(\partial G \setminus X)/I_G(L)^n \cdot \mathcal{H}^m(\partial G \setminus X) \}_{n \geq 1}
\]

with the constant pro-\( \mathcal{H}^0(\{\bullet\}) \)-module as source is an isomorphism. Hence the source of \( \lambda^n_{\mathcal{H}^0}(f : G \to X) \) can be identified with constant pro-\( \mathcal{H}^0(\{\bullet\}) \)-module \( \{ \mathcal{H}^m(\partial G \setminus X) \}_{n \geq 1} \).

The projection \( E \mathcal{G} \times_G X \to \partial G \setminus X \) is a homotopy equivalence and induces an isomorphism pro-\( \mathbb{Z} \)-modules

\[
\{ \mathcal{H}^m(\partial G \setminus X)_{(n-1)} \}_{n \geq 1} \xrightarrow{\cong} \{ \mathcal{H}^m((E \mathcal{G} \times_G X)_{(n-1)}) \}_{n \geq 1}.
\]

Since \( \partial G \setminus X \) is finite dimensional, the canonical map

\[
\{ \mathcal{H}^m(\partial G \setminus X) \}_{n \geq 1} \xrightarrow{\cong} \{ \mathcal{H}^m((\partial G \setminus X)_{(n-1)}) \}_{n \geq 1}
\]

is an isomorphism of pro-\( \mathbb{Z} \)-modules. Hence also the target of \( \lambda^n_{\mathcal{H}^0}(f : G \to X) \) can be identified with constant pro-\( \mathcal{H}^0(\{\bullet\}) \)-module \( \{ \mathcal{H}^m(\partial G \setminus X) \}_{n \geq 1} \). One easily checks that under these identifications \( \lambda^n_{\mathcal{H}^0}(f : G \to X) \) is the identity.

Hence the Completion Theorem is only interesting in the case, where \( G \) contains torsion.

4. A Strategy for a Proof of a Completion Theorem

Theorem 4.1. (Strategy for the proof of Theorem 1.) Let \( \mathcal{H} \) be an equivariant cohomology theory with values in \( R \)-modules with a multiplicative structure.

Let \( L \) be a proper \( G \text{-CW} \)-complex. Suppose that the following conditions are satisfied, where \( \mathcal{F}(L) \) is the family of subgroups of \( G \) given by \( \{ H \subseteq G \mid L^H \neq \emptyset \} \).

1. The ring \( \mathcal{H}^0(\{\bullet\}) \) is Noetherian;
2. For any \( H \in \mathcal{F}(L) \) and \( m \in \mathbb{Z} \) the \( \mathcal{H}^0(\{\bullet\}) \)-module \( \mathcal{H}^m_H(\{\bullet\}) \) is finitely generated;
3. Let \( H \in \mathcal{F}(L) \), let \( \mathcal{P} \subseteq \mathcal{H}^0_H(\{\bullet\}) \) be a prime ideal, and let \( f : G/H \to L \) be any \( G \)-map. Then the augmentation ideal

\[
I_H(\{\bullet\}) = \ker \left( \mathcal{H}^0_H(\{\bullet\}) \xrightarrow{\mathcal{H}^0_H(\text{pr})} \mathcal{H}^0_H(H) \xrightarrow{\text{ind}_{G \to H}} \mathcal{H}^0_H(\{\bullet\}) \right)
\]

is contained in \( \mathcal{P} \) if \( \mathcal{H}^0_G(L) \xrightarrow{\mathcal{H}^0_G(f)} \mathcal{H}^0_G(G/H) \xrightarrow{\text{ind}_{G \to H}} \mathcal{H}^0_H(\{\bullet\}) \) maps \( I_G(L) \) into \( \mathcal{P} \);
4. The Completion Theorem is true for every finite group \( H \) with \( H \in \mathcal{F}(L) \) in the case, where \( X = L = \{\bullet\} \) and \( f = \text{id} : \{\bullet\} \to \{\bullet\} \), i.e., for every finite group \( H \) with \( L^H \neq \emptyset \) the map of pro-\( \mathcal{H}^0(\{\bullet\}) \)-modules

\[
\lambda^n_{\mathcal{H}^0}(\{\bullet\}) : \{ \mathcal{H}^m_H(\{\bullet\})/I_H(\{\bullet\}) \}_{n \geq 1} \to \{ \mathcal{H}^m((BH)_{(n-1)}) \}_{n \geq 1}
\]

is an isomorphism of pro-\( \mathcal{H}^0(\{\bullet\}) \)-modules.

Then the Completion Theorem is true for \( \mathcal{H}^* \) and every \( G \)-map \( f : X \to L \) from a finite proper \( G \text{-CW} \)-complex \( X \) to \( L \), i.e., the map of pro-\( \mathcal{H}^0(\{\bullet\}) \)-modules

\[
\lambda^n_G(X) : \{ \mathcal{H}^m_G(X)/I_G(L)^n \cdot \mathcal{H}^m_G(X) \}_{n \geq 1} \to \{ \mathcal{H}^m_G((E \mathcal{G} \times_G X)_{(n-1)}) \}_{n \geq 1}
\]

is an isomorphism of pro-\( \mathcal{H}^0(\{\bullet\}) \)-modules.
Proof. We first prove the Completion Theorem for $X = G/H$, i.e., for any a $G$-map $f: G/H \to L$. Obviously $H$ belongs to $\mathcal{F}(L)$. The following diagram of pro-modules commutes

\[
\begin{array}{c}
\{\mathcal{H}_G^0(G/H)/\mathbb{L}_G(L)^n \cdot \mathcal{H}_G^0(G/H)\}_{n \geq 1} \\
\xrightarrow{\lambda_H^n(f: G/H \to L)} \\
\{\mathcal{H}_H^m((\bullet))/\mathbb{L}_H((\bullet))^n \cdot \mathcal{H}_H^m((\bullet))\}_{n \geq 1} \\
\xrightarrow{\lambda_H^n(\text{pr})}_{n \geq 1}
\end{array}
\]

where $\text{pr}$ denotes the obvious projection. The lower horizontal arrow is an isomorphism of pro-modules by condition (4). The right vertical arrow and the upper left vertical arrow are obviously isomorphisms of pro-modules. Hence the upper horizontal arrow is an isomorphism of pro-modules if we can show that the lower left vertical arrow is an isomorphism of pro-modules.

Let $I_f$ be the image of $\mathbb{L}_G(L)$ under the composite of ring homomorphisms

\[
\mathcal{H}_G^0(L) \xrightarrow{\mathcal{H}_G^0(f)} \mathcal{H}_G^0(G/H) \xrightarrow{\text{ind}_{H \to G}} \mathcal{H}_H^0((\bullet)).
\]

Let $J_f$ be the ideal in $\mathcal{H}_H^0((\bullet))$ generated by $I_f$. Obviously $I_f \subseteq J_f \subseteq \mathbb{L}_H((\bullet))$. Then the left lower vertical arrow is the composite

\[
\mathcal{H}_H^m((\bullet))/\mathbb{L}_H((\bullet))^n \cdot \mathcal{H}_H^m((\bullet)) \to \mathcal{H}_H^m((\bullet))/(J_f)^n \cdot \mathcal{H}_H^m((\bullet)) \to \mathcal{H}_H^m((\bullet))/\mathbb{L}_H((\bullet))^n \cdot \mathcal{H}_H^m((\bullet))
\]

where the first map is already levelwise an isomorphisms and in particular an isomorphism of pro-modules. In order to show that the second map is an isomorphism of pro-modules, it remains to show that $\mathbb{L}_H((\bullet))^k \subseteq J_f$ for an appropriate integer $k \geq 1$. Equivalently, we want to show that the ideal $\mathbb{L}_H((\bullet))/J_f$ of the quotient ring $\mathcal{H}_H^0((\bullet))/J_f$ is nilpotent. Since $\mathcal{H}_H^0((\bullet))$ is Noetherian by conditions (1) and (2), the ideal $\mathbb{L}_H((\bullet))/J_f$ is finitely generated. Hence it suffices to show that $\mathbb{L}_H((\bullet))/J_f$ is contained in the nilradical, i.e., the ideal consisting of all nilpotent elements, of $\mathcal{H}_H^0((\bullet))/J_f$. The nilradical agrees with the intersection of all the prime ideals of $\mathcal{H}_H^0((\bullet))/J_f$ by [3] Proposition 1.8. The preimage of a prime ideal in $\mathcal{H}_H^0((\bullet))/J_f$ under the projection $\mathcal{H}_H^0((\bullet)) \to \mathcal{H}_H^0((\bullet))/J_f$ is again a prime ideal. Hence it remains to show that any prime ideal of $\mathcal{H}_H^0((\bullet))$ which contains $I_f$ also contains $\mathbb{L}_H((\bullet))$. But this is guaranteed by condition (3). This finishes the proof in the case $X = G/H$.

The general case of a $G$-map $f: X \to L$ from a finite $G$-CW-complex $X$ to a $G$-CW-complex $L$ is done by induction over the dimension $r$ of $X$ and subinduction over the number of top-dimensional equivariant cells. For the induction step we write $X$ as a $G$-pushout

\[
\begin{array}{ccc}
G/H \times S^{r-1} & \xrightarrow{q} & Y \\
\downarrow f & & \downarrow k \\
G/H \times D^r & \xrightarrow{Q} & X
\end{array}
\]

In the sequel we equip $G/H \times S^{r-1}$, $Y$ and $G/H \times D^r$ with the maps to $L$ given by the composite of $f: X \to L$ with $k \circ q$, $k$ and $Q$. The long exact Mayer-Vietoris sequence of the $G$-pushout above is a long exact sequence of $\mathcal{H}_G^0(L)$-modules and
looks like

\[ \ldots \rightarrow H^{m-1}(G/H \times D^r) \oplus H^{m+1}_G(Y) \rightarrow H^{m-1}_G(G/H \times S^{r-1}) \rightarrow H^m_G(X) \]

\rightarrow H^m_G(G/H \times D^r) \oplus H^{m+1}_G(Y) \rightarrow H^m_G(G/H \times S^{r-1}) \rightarrow \ldots. \]

Condition \textsuperscript{2} implies that \( H^m_G(G/H) \) and \( H^m_G(G/H \times D^r) \) are finitely generated as \( H^0(\{\bullet\}) \)-modules. Since \( H^0(\{\bullet\}) \) is Noetherian by condition \textsuperscript{1}, the \( H^0(\{\bullet\}) \)-module \( H^m_G(X) \) is finitely generated provided that the \( H^0(\{\bullet\}) \)-module \( H^m_G(Y) \) is finitely generated. Thus we can show inductively that the \( H^0(\{\bullet\}) \)-module \( H^m_G(X) \) is finitely generated for every \( m \in \mathbb{Z} \). In particular the ring \( H^0_G(X) \) is Noetherian.

Let \( J \subseteq H^0_G(X) \) be the ideal generated by the image of \( H_G(L) \) under the ring homomorphism \( H^0_G(L) \rightarrow H^0_G(X) \). Then for every \( H^0_G(X) \)-module the obvious map \( \{M/I \mid M \in H^0_G(L)\}_{n \geq 1} \rightarrow \{M/J \mid M \in H^0_G(L)\}_{n \geq 1} \) is levelwise an isomorphism and in particular an isomorphism of \( H^0_G(X) \)-modules. We conclude from Lemma \textsuperscript{2.2} that the following sequence of pro-\( H^0(\{\bullet\}) \)-modules is exact, where \( M/I \) stands for \( M/I \times M \).

\[ H^{m-1}(G/H \times D^r)/I \rightarrow H^{m-1}(G/H \times S^{r-1})/I \rightarrow H^m_G(X)/I \rightarrow H^{m+1}_G(Y)/I \rightarrow \ldots \]

Applying \( EG_{m-1} \times G -\) to the \( G \)-pushout above yields a pushout and thus a long exact Mayer-Vietoris sequence

\[ \ldots \rightarrow H^{m-1}((EG_{m-1} \times G(EG_{m-1} \times G Y)) \rightarrow H^{m-1}((EG_{m-1} \times G(G/H \times S^{r-1}))) \rightarrow H^m_G((EG_{m-1} \times G X)) \rightarrow H^m_G((EG_{m-1} \times G(G/H \times D^r))) \rightarrow H^m_G((EG_{m-1} \times G Y)) \rightarrow \ldots \]

The obvious map
\[ \{H^m_G((EG_{m-1} \times G Z)) \}_{n \geq 1} \xrightarrow{\sim} \{H^m_G((EG \times G Z)) \}_{n \geq 1} \]

is an isomorphism of pro-\( H^0(\{\bullet\}) \)-modules for any finite dimensional \( G \)-CW-complex \( Z \). Hence we obtain a long exact sequence of pro-\( H^0(\{\bullet\}) \)-modules

\[ \ldots \rightarrow \{H^{m-1}_G((EG \times G(G/H \times D^r)))_{n \geq 1} \oplus \{H^{m}_G((EG \times G Y))_{n \geq 1} \} \rightarrow \{H^{m-1}_G((EG \times G(G/H \times S^{r-1})))_{n \geq 1} \}
\]

\[ \rightarrow \{H^{m}_G((EG \times G X))_{n \geq 1} \} \rightarrow \{H^{m}_G((EG \times G(G/H \times D^r)))_{n \geq 1} \} \oplus \{H^{m}_G((EG \times G Y))_{n \geq 1} \} \rightarrow \{H^{m}_G((EG \times G(G/H \times S^{r-1})))_{n \geq 1} \} \rightarrow \ldots \]

Now the various maps \( \lambda^m_G \) induce a map from the long exact sequence of pro-\( H^0(\{\bullet\}) \)-modules \textsuperscript{1.2} to the long exact sequence of pro-\( H^0(\{\bullet\}) \)-modules \textsuperscript{1.3}. The maps for \( G/H \times S^{r-1} \), \( G/H \times D^r \) and \( Y \) are isomorphisms of pro-\( H^0(\{\bullet\}) \)-modules by induction hypothesis and by \( G \)-homotopy invariance applied to the \( G \)-homotopy equivalence \( G/H \times D^r \rightarrow G/H \). By the Five-Lemma for maps of pro-modules the map
\[ \lambda^m_G(X): \{H^m_G(X)/I \}_{n \geq 1} \rightarrow \{H^m_G((EG \times G X))_{n \geq 1} \} \rightarrow \ldots \]
is an isomorphism of pro-$\mathcal{H}^0(\bullet)$-modules. This finishes the proof of Theorem 4.1. □

The next lemma will be needed to check condition 3 appearing in Theorem 4.1.

Given an $G$-cohomology theory $\mathcal{H}^*_G$, there is an equivariant version of the Atiyah-Hirzebruch spectral sequence of $\mathcal{H}^0(\bullet)$-modules which converges to $\mathcal{H}^{p+q}(L)$ in the usual sense provided that $L$ is finite dimensional, and whose $E_2$-term is

$$E_2^{p,q} := \mathcal{H}^p_G(L; \mathcal{H}^q_G(G/\mathcal{D})),$$

where $\mathcal{H}^p_G(X; \mathcal{H}^q_G(G/\mathcal{D}))$ is the Bredon cohomology of $L$ with coefficients in the $\text{Zor}(G)$-module sending $G/H$ to $\mathcal{H}^q_G(G/H)$. If $\mathcal{H}^*_G$ comes with a multiplicative structure, then this spectral sequence comes with a multiplicative structure.

**Lemma 4.4.** Suppose that $L$ is a 1-dimensional proper $G$-CW-complex for some positive integer $l$. Suppose that for $r = 0, 1, \ldots, l$ the differential appearing in the Atiyah-Hirzebruch spectral sequence for $L$ and $\mathcal{H}^*_G$

$$d_r^{0,0} : E_r^{0,0} \to E_r^{1-r},$$

vanishes rationally.

(1) Then we can find for a given $x \in \mathcal{H}^0_G(L; \mathcal{H}^0_G(G/\mathcal{D}))$ a positive integer $k$ such that $x^k$ is contained in the image of the edge homomorphism

$$\text{edge}^{0,0} : \mathcal{H}^0_G(L) \to \mathcal{H}^0_G(L; \mathcal{H}^0_G(G/\mathcal{D}));$$

(2) Let $H \in \mathcal{F}(L)$, let $\mathcal{P} \subseteq \mathcal{H}^0_G((\bullet))$ be a prime ideal and let $f : G/H \to L$ be any $G$-map. Suppose that the augmentation ideal

$$\mathfrak{i}_H((\bullet)) := \ker \bigg( \mathcal{H}^0_G((\bullet)) \xrightarrow{\mathcal{H}^0_G(f_H)} \mathcal{H}^0_G(H) \xrightarrow{\text{ind}(1) - H} \mathcal{H}^0_G((\bullet)) \bigg)$$

is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the structure map for $H$ of the inverse limit over the orbit category $\text{Or}(G; \mathcal{F}(L))$ associated to the family $\mathcal{F}(L)$

$$\phi_H : \underset{K \in \text{Or}(G; \mathcal{F}(L))}{\varprojlim} \mathcal{H}_K((\bullet)) \to \mathbb{I}_H((\bullet)).$$

Then condition 3 appearing in Theorem 4.1 is satisfied for $H$, $\mathcal{P}$ and $f$.

**Proof.** Consider $x \in \mathcal{H}^0_G(L; \mathcal{H}^0_G(G/\mathcal{D}))$. We construct inductively positive integers $k_1, k_2, \ldots, k_l$ such that

$$x^k \in E_r^{0,0}$$

for $r = 1, 2, \ldots, l$.

Put $k_1 = 1$. We have $\mathcal{H}^0_G(L; \mathcal{H}^0_G(G/\mathcal{D})) = E_0^{0,0}$ and hence $x = x^1 = x^\prod_{i=1}^1 k_i = E_2^{0,0}$. This finishes the induction beginning $r = 1$.

In the induction step from $(r - 1)$ to $r \geq 2$ we can assume that we have already constructed $k_1, \ldots, k_{r-1}$ and shown that $x^\prod_{i=1}^{r-1} k_i$ belongs to $E_r^{0,0}$. Now choose $k_r$ such that $k_r \cdot d_r^{0,0} \left( x^\prod_{i=1}^{r-1} k_i \right) = 0$. This is possible since by assumption $d_r^{0,0} \otimes_\mathbb{Z} \text{id}_\mathbb{Q} = 0$.

For any element $y \in E_r^{0,0}$ one checks inductively for $j = 1, 2, \ldots$

$$d_r^{0,0}(y) = j \cdot d_r^{0,0}(y) \cdot y^{j-1}. $$

This implies

$$d_r^{0,0} \left( x^\prod_{i=1}^{r-1} k_i \right) = d_r^{0,0} \left( x^\prod_{i=1}^{r-1} k_i \right) = k_r \cdot d_r^{0,0} \left( x^\prod_{i=1}^{r-1} k_i \right) = 0.$$

Since $E_r^{0,0}$ is the kernel of $d_r^{0,0} : E_r^{0,0} \to E_r^{0,0}$, we conclude $x^\prod_{i=1}^r k_i \in E_r^{0,0}$. Since $L$ is $l$-dimensional, we get for $k = \prod_{i=1}^l k_i$ that $x^k \in E_\infty^{0,0}$. Since $E_\infty^{0,0}$ is the image of the edge homomorphism $\text{edge}^{0,0}$, assertion 1 follows.
Consider the following commutative diagram

\[
\begin{array}{cccc}
H_G^0 \left( E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/H) \right) & \xrightarrow{\alpha} & \invlim_{G/K \in \text{Or}(G, \mathcal{F}(L))} \mathcal{H}_K^0 \left( \{ \bullet \} \right) \\
\mathcal{H}_G^0(L) & \xrightarrow{\text{edge}^{0,0}} & H_G^0 \left( L; \mathcal{H}_G^0(G/H) \right) & \xrightarrow{H_G^0(u)} \mathcal{H}_K^0 \left( \{ \bullet \} \right) \\
\mathcal{H}_G^0(G/H) & \xrightarrow{\text{edge}^{0,0}} & H_G^0 \left( G/H; \mathcal{H}_G^0(G/H) \right) & \xrightarrow{H_G^0(f)} \mathcal{H}_K^0 \left( \{ \bullet \} \right) \\
\end{array}
\]

Here \( \alpha \) is the isomorphism, which sends \( v \in H_G^0 \left( E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/H) \right) \) to the system of elements that is for \( G/K \in \text{Or}(G, \mathcal{F}(L)) \) the image of \( v \) under the homomorphism

\[
H_G^0 \left( E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/H) \right) \xrightarrow{H_G^0(u)} H_G^0 \left( G/K; \mathcal{H}_G^0(G/H) \right)
\]

for the up to \( G \)-homotopy unique \( G \)-map \( i_K: G/K \to E_{\mathcal{F}(L)}(G) \). The \( G \)-map \( u: L \to E_{\mathcal{F}(L)}(G) \) is the up to \( G \)-homotopy unique \( G \)-map from \( L \) to the classifying space of the family \( \mathcal{F}(L) \), and \( \Phi_H \) is the structure map of the inverse limit for \( H \).

We have to prove that \( I_H(\{ \bullet \}) \) is contained in the prime ideal \( \mathcal{P} \) provided that \( \mathcal{P} \) contains the image of \( I_G(L) \) under the composite \( \mathcal{H}_G^0(L) \xrightarrow{\Phi_L(f)} \mathcal{H}_G^0(G/H) \xrightarrow{\text{ind}_G \circ \alpha} \mathcal{H}_K^0(\{ \bullet \}) \).

Consider \( a \in \invlim_{G/K \in \text{Or}(G, \mathcal{F}(L))} \mathcal{I}_K(\{ \bullet \}) \). Let \( x \in H_G^0 \left( L; \mathcal{H}_G^0(G/H) \right) \) be the image of \( a \) under the composite

\[
\invlim_{G/K \in \text{Or}(G, \mathcal{F}(L))} \mathcal{I}_K(\{ \bullet \}) \xrightarrow{\alpha^{-1}} H_G^0 \left( E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/H) \right) \xrightarrow{H_G^0(u; \mathcal{H}_G(f))} H_G^0 \left( L; \mathcal{H}_G^0(G/H) \right).
\]

We conclude from assertion 14 that for some positive number \( k \) there is an element \( y \in \mathcal{H}_G^0(L) \) with \( \text{edge}^{0,0}(y) = x^k \). One easily checks that \( y \) belongs to \( I_G(L) \), just inspect the diagram above for \( H = \{1\} \). Hence the composite

\[
\mathcal{H}_G^0(L) \xrightarrow{\Phi_L(f)} \mathcal{H}_G^0(G/H) \xrightarrow{\text{ind}_G \circ \alpha} \mathcal{H}_K^0(\{ \bullet \})
\]

maps \( y \) to \( \mathcal{P} \) by assumption. An easy diagram chase shows that

\[
\phi_H: \invlim_{G/K \in \text{Or}(G, \mathcal{F}(L))} \mathcal{I}_K(\{ \bullet \}) \to \mathcal{I}_H(\{ \bullet \})
\]

maps \( a^k \) to \( \mathcal{P} \). Since \( \mathcal{P} \) is a prime ideal and \( \phi_H \) is multiplicative, \( \phi_H \) sends \( a \) to \( \mathcal{P} \).

Hence the image of \( \phi_H : \invlim_{G/K \in \text{Or}(G, \mathcal{F}(L))} \mathcal{I}_K(\{ \bullet \}) \to \mathcal{I}_H(\{ \bullet \}) \) lies \( \mathcal{P} \). Hence we get by assumption \( \mathcal{I}_H(\{ \bullet \}) \subseteq \mathcal{P} \). This finishes the proof of Lemma 4.4.
5. The Segal Conjecture for Infinite Groups

In this section we prove the Segal Conjecture \([0.2]\) for infinite groups. It is just the Completion Theorem formulated in Problem \([3.4]\) for equivariant stable cohomotopy \(\mathcal{H}_t = \pi_t^G\) under the condition that there is an upper bound on the orders of finite subgroups on \(L\) and \(L\) has finite dimension.

**Proof of Theorem \([0.2]\).** We want to apply Theorem \([1.1]\) and therefore have to prove conditions \([1]\), \([2]\), \([3]\) and \([4]\) appearing there.

Condition \([1]\) appearing there is satisfied because of \(\pi^G_0(\{\bullet\}) = \mathbb{Z}\).

Condition \([2]\) is satisfied because of Example \([1.5]\).

Next we prove condition \([3]\). Recall the assumption that there is an upper bound on the orders of finite subgroups of \(G\) and that \(G\) is finite dimensional. Recall that \(\mathcal{F}(L)\) denotes the family of finite subgroups \(H \subseteq G\) with \(L^H \neq \emptyset\). We can find by Example \([1.5]\) for every \(q \in \mathbb{Z}\) with \(q \neq 0\) a positive integer \(C(q)\) such that the order of \(\pi^G_q(\{\bullet\})\) divides \(C(q)\) for every \(H \in \mathcal{F}(L)\). Furthermore recall that \(G\) is finite dimensional. Consider the equivariant cohomological Atiyah-Hirzebruch spectral sequence converging to \(\pi^G_{p+q}(L)\). Its \(E_2\)-term is given by

\[ E_{2}^{p,q} = H^{p}_{\mathbb{Z}}(L; \pi^{G}_{q}(\{\bullet\})) \]

Therefore \(E^{r,1-r}_{r} = 0\) annihilated by multiplication with \(C(1-r)\) and hence rationally trivial for \(r \geq 2\). Hence for \(r \geq 2\) the differential

\[ d^{r,0}_{r}: E^{r,0}_{r} \to E^{r,1-r}_{r} \]

vanishes rationally. We have shown that the conditions appearing in Lemma \([4.4]\) are satisfied. Hence in order to verify condition \([5]\), it suffices to prove for any family \(\mathcal{F}\) of subgroups of \(G\) with the property that there exists an upper bound on the orders of subgroups appearing \(\mathcal{F}\), any \(H \in \mathcal{F}\) and any prime ideal \(\mathcal{P}\) of the Burnside ring \(A(H)\) that \(\mathcal{P}\) contains the augmentation ideal \(\mathbb{H}_H\) provided \(\mathcal{P}\) contains the image of the structure map for \(H\) of the inverse limit

\[ \phi_H : \text{invlim}_{G/K \in \text{Or}(G,\mathcal{F})} \mathbb{H}_K \to \mathbb{H}_H. \]

Fix a finite group \(H\). We begin with recalling some basics about the prime ideals in the Burnside ring \(A(H)\) taken from \([5]\). In the sequel \(p\) is a prime number or \(p = 0\). For a subgroup \(K \subseteq H\) let \(\mathcal{P}(K,p)\) be the preimage of \(p \cdot \mathbb{Z}\) under the character map for \(K\)

\[ \text{char}^H_K : A(H) \to \mathbb{Z}, \quad [S] \mapsto |S^K|. \]

This is a prime ideal and each prime ideal of \(A(H)\) is of the form \(\mathcal{P}(K,p)\). If \(\mathcal{P}(K,p) = \mathcal{P}(L,q)\), then \(p = q\). If \(p\) is a prime, then \(\mathcal{P}(K,p) = \mathcal{P}(L,p)\) if and only if \((K[p]) = (L[p])\), where \(K[p]\) is the minimal normal subgroup of \(K\) with a \(p\)-group as quotient. Notice for the sequel that \(K[p] = \{1\}\) if and only if \(K\) is a \(p\)-group. If \(p = 0\), then \(\mathcal{P}(K,p) = \mathcal{P}(L,p)\) if and only if \((K) = (L)\).

Fix a prime ideal \(\mathcal{P} = \mathcal{P}(K,p)\). Choose a positive integer \(m\) such that \(|H|\) divides \(m\) for all \(H \in \mathcal{F}\). Fix \(H \in \mathcal{F}\). Choose a free \(H\)-set \(S\) together with a bijection \(u : S \xrightarrow{\sim} [m]\), where \([m] = \{1, 2, \ldots, m\}\). Such \(S\) exists since \(|H|\) divides \(m\) and we can take for \(S\) the disjoint union of \(m\) copies of \(H\). Thus we obtain an injective group homomorphism

\[ \rho_u : H \to S_m, \quad h \mapsto u \circ l_h \circ u^{-1}, \]

where \(l_h : S \to S\) is given by left multiplication with \(h\) and \(S_m = \text{aut}([m])\) is the group of permutations of \([m]\). Let \(S_m[\rho_u]\) denote the \(H\)-set obtained from \(S_m\) by the \(H\)-action \(h : \sigma \mapsto \rho_u(h) \circ \sigma\). Let \(\text{Syl}_p(S_m)\) be the \(p\)-Sylow subgroup of \(S_m\). Let \(S_m / \text{Syl}_p(S_m)[\rho_u]\) denote the \(H\)-set obtained from the homogeneous space
$S_m / \text{Syl}_p(S_m)$ by the $H$-action given by $h \cdot \tau = \rho_p(h) \circ \tau$. The $H$-action on $S_m[\rho_u]$ is free. If for $K \subseteq H$ we have $(S_m / \text{Syl}_p(S_m))[\rho_u]^K \neq \emptyset$, then for some $\sigma \in S_m$ we get $\rho_u(K) \subseteq \sigma \cdot \text{Syl}_p(S_m) \cdot \sigma^{-1}$ and hence $K$ must be a $p$-group.

Suppose that $T$ is another free $H$-set together with a bijection $v : T \xrightarrow{\cong} [m]$. Then we can choose an $H$-isomorphism $w : S \xrightarrow{\cong} T$. Let $\tau \in S_m$ be given by the composition $v \circ w \circ u^{-1}$. Then $c(\tau) \circ \rho_u = \rho_v$ holds, where $c(\tau) : S_m \to S_m$ sends $\sigma$ to $\tau \circ \sigma \circ \tau^{-1}$. Moreover, left multiplication with $\tau$ induces isomorphisms of $H$-sets

$$S_m[\rho_u] \cong_H S_m[\rho_v];$$
$$S_m / \text{Syl}_p(S_m)[\rho_u] \cong_H S_m / \text{Syl}_p(S_m)[\rho_v].$$

Hence we obtain elements in $A(H)$

$$[S_m] := [S_m[\rho_u]];$$
$$[S_m / \text{Syl}_p(S_m)] := [S_m / \text{Syl}_p(S_m)[\rho_u]],$$

which are independent of the choice of $S$ and $u : S \xrightarrow{\cong} [m]$. If $i : H_0 \to H_1$ is an injective group homomorphisms between elements in $\mathcal{F}$, then one easily checks that the restriction homomorphism $A(i) : A(H_1) \to A(H_0)$ sends $[S_m]$ to $[S_m]$ and $[S_m / \text{Syl}_p(S_m)]$ to $[S_m / \text{Syl}_p(S_m)]$. Thus we obtain elements

$$[[S_m]], [[S_m / \text{Syl}_p(S_m)]] \in \text{invlim}_{G/K \in \text{Or}(G; F)} A(K)$$

Define elements

$$[S_m] \cdot 1, [S_m / \text{Syl}_p(S_m)] \cdot 1 \in \text{invlim}_{G/K \in \text{Or}(G; F)} A(K)$$

by the collection of elements $[S_m] \cdot [K/K]$ and $[S_m / \text{Syl}_p(S_m)] \cdot [K/K]$ in $A(K)$ for $K \in \mathcal{F}$. Thus we get elements

$$[[S_m]] - [S_m] \cdot 1, [[S_m / \text{Syl}_p(S_m)]] - [S_m / \text{Syl}_p(S_m)] \cdot 1 \in \text{invlim}_{G/K \in \text{Or}(G; F)} A(K).$$

The image of $[[S_m]] - [S_m] \cdot 1$ and $[[S_m / \text{Syl}_p(S_m)]] - [S_m / \text{Syl}_p(S_m)] \cdot 1$ respectively under the structure map of the inverse limit $\text{invlim}_{G/K \in \text{Or}(G; F)} A(K)$ for the object $G/H \in \text{Or}(G; F)$ is $[S_m] - [S_m] \cdot [H/H]$ and $[S_m / \text{Syl}_p(S_m)] - [S_m / \text{Syl}_p(S_m)] \cdot [H/H]$. Hence by assumption

$$[S_m] - [S_m] \cdot [H/H] \in \mathcal{P}(K, p);$$
$$[S_m / \text{Syl}_p(S_m)] - [S_m / \text{Syl}_p(S_m)] \cdot [H/H] \in \mathcal{P}(K, p).$$

Therefore $\text{char}^H_K : A(H) \to \mathbb{Z}$ sends both $[S_m] - [S_m] \cdot [H/H]$ and $[S_m / \text{Syl}_p(S_m)] - [S_m / \text{Syl}_p(S_m)] \cdot [H/H]$ to elements in $p\mathbb{Z}$. Since $\text{char}^H_K([S_m] - [S_m] \cdot [H/H]) = 0 - [S_m]$ for $K \neq \{1\}$, we conclude that $K = \{1\}$ or that $p \neq 0$. If $K = \{1\}$, then $I_H(\{\bullet\}) = \mathcal{P}(\{1\}, 0)$ is contained in $\mathcal{P}(K, p)$. Suppose that $K \neq \{1\}$. Then $p$ is a prime. We have

$$\text{char}^H_K ([S_m / \text{Syl}_p(S_m)] - [S_m / \text{Syl}_p(S_m)] \cdot [H/H])$$

$$= \left([S_m / \text{Syl}_p(S_m)]^K - [S_m / \text{Syl}_p(S_m)]\right).$$

Since this integer must belong to $p\mathbb{Z}$ and $[S_m / \text{Syl}_p(S_m)]$ is prime to $p$, we get $(S_m / \text{Syl}_p(S_m))^K \neq \emptyset$. Hence $K$ must be a $p$-group. This implies $\mathcal{P}(K, p) = \mathcal{P}(\{1\}, p)$ and therefore $I_H(\{\bullet\}) = \mathcal{P}(\{1\}, 0) \subseteq \mathcal{P}(K, p)$. This finishes the proof of condition (3).

Condition (4) follows from the proof of the Segal Conjecture for a finite group $H$ due to Carlsson [6]. This finishes the proof of Theorem 12.
6. An improved Strategy for a Proof of a Completion Theorem

The next result follows from Theorem 4.3 Lemma 4.3 and a construction of a modified Chern character analogous to the one in Theorem 4.6 and Lemma 6.2 which will ensure that the condition about the differentials in the equivariant Atiyah-Hirzebruch spectral sequence appearing in Lemma 4.4 is satisfied. We do not give more details here, since the interesting case of the Segal Conjecture and of the Atiyah-Segal Completion Theorem are already covered by Theorem 0.2 and by [15].

Let $G$ be a (discrete) group. Let $\mathcal{F}$ be a family of subgroups of $G$ such that there is an upper bound on the orders of the subgroups appearing $\mathcal{F}$. Let $\mathcal{H}_+^*$ be an equivariant cohomology theory with values in $R$-modules which satisfies the disjoint union axiom. Define a contravariant functor

\[
\mathcal{H}_+^*(\{\cdot\}) : \text{FGINJ} \to \text{R-MODULES}
\]

with the category FGINJ of finite groups with injective group homomorphism as source by sending an injective homomorphism $\alpha : H \to K$ to the composite

\[
\mathcal{H}_+^*(\{\cdot\}) \xrightarrow{\mathcal{H}_+^*(pr)} \mathcal{H}_+^*(K/H) \xrightarrow{\text{ind}_\alpha} \mathcal{H}_+^*(\{\cdot\}),
\]

where $\text{pr} : H/K = \text{ind}_\alpha(\{\cdot\}) \to \{\cdot\}$ is the projection and $\text{ind}_\alpha$ comes from the induction structure of $\mathcal{H}_+^*$. Assume that $\mathcal{H}_+^*$ comes with a multiplicative structure.

**Theorem 6.2** (Improved Strategy for the proof of Theorem 0.2). Suppose that the following conditions are satisfied.

1. The ring $\mathcal{H}_+^0(\{\cdot\})$ is Noetherian;
2. Let $H \subseteq G$ be any finite subgroup and $m \in \mathbb{Z}$ be any integer. Then the $\mathcal{H}_+^0(\{\cdot\})$-module $\mathcal{H}_+^m(\{\cdot\})$ is finitely generated, there exists an integer $C(H, m)$ such that multiplication with $C(H, m)$ annihilates the torsion-submodule $\text{tors}_\mathbb{Z}(\mathcal{H}_+^m(\{\cdot\}))$ of the abelian group $\mathcal{H}_+^m(\{\cdot\})$ and the $R$-module $\mathcal{H}_+^m(\{\cdot\})/\text{tors}_\mathbb{Z}(\mathcal{H}_+^m(\{\cdot\}))$ is projective;
3. Let $H$ be any element of $\mathcal{F}$. Let $\mathcal{P} \subseteq \mathcal{H}_+^0(\{\cdot\})$ be any prime ideal. Then the augmentation ideal

\[
\mathbb{I}_H(\{\cdot\}) = \ker \left( \mathcal{H}_+^0(\{\cdot\}) \to \mathcal{H}_+^0(H) \xrightarrow{\phi_H} \mathcal{H}_+^0(\{\cdot\}) \right)
\]

is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the structure map for $H$ of the inverse limit

\[
\phi_H : \text{invlim}_{G/K \in \text{Or}(G, \mathcal{F})} \mathbb{I}_K(\{\cdot\}) \to \mathbb{I}_H(\{\cdot\});
\]
4. The Completion Theorems is true for every finite group $H$ in the case $X = L = \{\cdot\}$ and $f = \text{id} : \{\cdot\} \to \{\cdot\}$, i.e., for every finite group $H$ the map of pro-$\mathcal{H}_+^0(\{\cdot\})$-modules

\[
\lambda_H^m(\{\cdot\}) : \{\mathcal{H}_+^m(\{\cdot\})/\mathcal{H}_+^m(\{\cdot\})\} \to \{\mathcal{H}_+^m((BH)_{(n-1)})\}_{n \geq 1}
\]

defined in (5) is an isomorphism of pro-$\mathcal{H}_+^0(\{\cdot\})$-modules;
5. The covariant functor (5.1) extends to a Mackey functor.

Then the Completion Theorem is true for $\mathcal{H}_+^*$ and every $G$-map $f : X \to L$ from a finite proper $G$-CW-complex $X$ to a proper finite dimensional $G$-CW-complex $L$ with the property that there is an upper bound on the order of its isotropy groups. $L$, i.e., the map of pro-$\mathcal{H}_+^0(\{\cdot\})$-modules

\[
\lambda_G^m(X) : \{\mathcal{H}_+^m(G)(X)/\mathcal{H}_+^m(G)(X)\} \to \{\mathcal{H}_+^m((EG \times_G X)_{(n-1)})\}_{n \geq 1}
\]

defined in (5.3) is an isomorphism of pro-$\mathcal{H}_+^0(\{\cdot\})$-modules.
Remark 6.3. The advantage of Theorem 6.2 in comparison with Theorem 4.1 is that the conditions do not involve \( L \) and \( f : X \to L \) anymore and do only depend on the functor \( H^q(\bullet) : \text{FGINJ} \to \text{Z-MODULES} \). If one considers the case \( R = \text{Z} \) and assumes \( H^0(\bullet) = \text{Z} \), then condition \( \text{1} \) is obviously satisfied and condition \( \text{2} \) reduces to the condition that for any finite subgroup \( H \subseteq G \) and any integer \( m \in \text{Z} \) the abelian group \( H^m(\bullet) \) is finitely generated.

Remark 6.4 (Family version). We mention without proof that there is also a family version of Theorem 0.2. Its formulation is analogous to the one of the family version of the Atiyah-Segal Completion Theorem for infinite groups, see [14, Section 6].

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