Noncommutative solitons and quasideterminants

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Abstract

We discuss an extension of soliton theory and integrable systems to noncommutative spaces, focusing on integrable aspects of noncommutative anti-self-dual Yang–Mills equations. We give a wide class of exact solutions by solving a Riemann–Hilbert problem for the Atiyah–Ward ansatz and present Bäcklund transformations for the $G = U(2)$ noncommutative anti-self-dual Yang–Mills equations. We find that noncommutative determinants of one kind, quasideterminants, play crucial roles in the construction of noncommutative solutions. We also discuss the reduction of a noncommutative anti-self-dual Yang–Mills equation to noncommutative integrable equations. This work is partially based on a collaboration with C R Gilson and J J C Nimmo (of Glasgow).

Keywords: anti-self-dual Yang–Mills equation, Bäcklund transformation, Penrose–Ward transformation, Riemann–Hilbert problem, quasideterminant

1. Introduction

The extension of integrable systems and soliton theories to noncommutative space–times\(^1\) has been studied by many authors over the past couple of years and various kinds of integrable-like properties have been revealed. (For reviews, see [1–7].) This is partially motivated by the recent developments of noncommutative gauge theories on D-branes. In the noncommutative gauge theories, the noncommutative extension corresponds to the presence of background flux, and in the effective theory of D-branes, noncommutative solitons can be identified with the lower-dimensional D-branes. (For reviews, see e.g. [8–11].) This makes it possible to reveal some aspects of D-brane dynamics, such as tachyon condensations, by constructing exact noncommutative solitons and studying their properties.

Most noncommutative integrable equations such as noncommutative Korteweg–de Vries (KdV) equations belong, apparently, not to gauge theories, but to scalar theories. However, it has now been proved that they can be derived from noncommutative anti-self-dual Yang–Mills equations by reduction (see e.g. [12, 13]), which was first conjectured explicitly by the author and Toda [14]. (The original commutative version was proposed by Ward [15] and hence this conjecture is sometimes called the noncommutative Ward conjecture.) As noncommutative anti-self-dual Yang–Mills equations belong to gauge theories, lower-dimensional many integrable equations must have physical correspondence (in the background flux), and therefore analysis of exact soliton solutions of noncommutative integrable equations could be applied to the corresponding physical situations in the framework of $N = 2$ string theories [16–19]. (In this context, the signature is not Euclidean (++++) but ultrahyperbolic (++−−) and the $N = 2$ string theory lives in this signature.) Furthermore, integrable aspects of the anti-self-dual Yang–Mills equation can be understood from the geometrical framework of the twistor theory. Via Ward’s conjecture, the twistor theory gives a new geometrical viewpoint on the lower-dimensional integrable equations and some classification can be carried out in such a way. These results are summarized elegantly in the book of Mason and Woodhouse [20]. (See also [21, 22].)

In this paper, we discuss integrable aspects of the noncommutative anti-self-dual Yang–Mills equations from the viewpoint of the noncommutative twistor theory. We give a series of noncommutative Atiyah–Ward ansatz solutions by solving a noncommutative version of the Riemann–Hilbert problem. The solutions include not only noncommutative instantons (with finite action) but also noncommutative nonlinear plane waves and so on (with infinite action). We also find that noncommutative determinants of a particular kind, the quasideterminants, play crucial roles in the construction of exact solutions and we present a direct proof of the results

\(^1\) In this paper, the word ‘noncommutative’ always refers to generalization to noncommutative spaces, not to non-Abelian ones and so on.
of the Bäcklund transformation and the solutions generated without a twistor framework. These are due to a collaboration with C Gilson and J Nimmo (of Glasgow) [23, 24].

Finally we give an example using the noncommutative Ward conjecture, reduction of the noncommutative anti-self-dual Yang–Mills equation to the noncommutative KdV equation via the noncommutative toroidal KdV equation. The reduced equation actually has integrable-like properties such as infinite conserved quantities, exact N-soliton solutions and so on. These results will lead to a kind of classification of noncommutative integrable equations from a geometrical viewpoint and to applications to the corresponding physical situations and geometry also.

2. Noncommutative anti-self-dual Yang–Mills equations

In this section, we review some aspects of the noncommutative anti-self-dual Yang–Mills equation and establish notation.

2.1. Noncommutative gauge theories

Noncommutative spaces are defined by the noncommutativity of the coordinates

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2.1} \]

where the \( \theta^{\mu\nu} \) are real constants called the noncommutative parameters. The noncommutative parameter is anti-symmetric with respect to \( \mu, \nu \): \( \theta^{\mu\nu} = -\theta^{\nu\mu} \), and the rank is even. This relation looks like the canonical commutation relation in quantum mechanics and leads to a ‘space–space uncertainty relation’. Hence singularities which exist on commutative spaces could be resolved on noncommutative spaces. This is one of the prominent features of noncommutative field theories and yields various new physical objects such as \( U(1) \) instantons.

Noncommutative field theories are given by the exchange of ordinary products in the commutative field theories for the star products and realized as deformed theories from the commutative ones. The orderings of nonlinear terms are determined in the Lax formalism. (For a review, see [3].)

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual; for example, \( \delta_\mu \star \delta_\nu = \delta_\mu \delta_\nu \), and the wedge product of \( \lambda = \lambda_\mu(x) \, dx^\mu \) and \( \rho = \rho_\nu(x) \, dx^\nu \) is \( \lambda_\mu \star \rho_\nu \, dx^\mu \wedge dx^\nu \).

Noncommutative gauge theories are defined in this way by imposing the noncommutative version of the gauge symmetry, where the gauge transformation is defined as follows:

\[ A_\mu \rightarrow g^{-1} \star A_\mu \star g + g^{-1} \star \delta_\mu g, \tag{2.5} \]

where \( g \) is an element of the gauge group \( G \). (The inverse is assumed to exist in the sense of the star product in this paper.) This is sometimes called the star gauge transformation. We note that because of the noncommutativity, the commutator terms in the field strength are always needed even when the gauge group is Abelian in order to preserve the star gauge symmetry. This \( U(1) \) part of the gauge group actually plays crucial roles in general. We note that because of the noncommutativity of matrix elements, cyclic symmetry of traces is broken in general

\[ \text{Tr} \, A \star B \neq \text{Tr} \, B \star A. \tag{2.6} \]

Therefore, gauge invariant quantities become hard to define on noncommutative spaces.

2.2. Noncommutative anti-self-dual Yang–Mills equations

Let us consider Yang–Mills theories in four-dimensional (4D) noncommutative spaces whose real coordinates of the space are denoted by \( (x^0, x^1, x^2, x^3) \), where the gauge group is \( GL(N, \mathbb{C}) \). Here, we follow the convention in [20].

First, we introduce double null coordinates of 4D space as follows:

\[ ds^2 = 2(dz \, d\bar{z} - dw \, d\bar{w}). \tag{2.7} \]

We can recover various kinds of real spaces by putting the corresponding reality conditions on the double null coordinates \( z, \bar{z}, w, \bar{w} \) as follows:
• Euclidean space ($\bar{w} = -w; \bar{z} = \bar{z}$). An example is

$$
\begin{pmatrix}
\bar{z} & w \\
\bar{w} & z
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
x_0 + \im x_1 & -(x_2 - \im x_3) \\
x_2 + \im x_3 & x_0 - \im x_1
\end{pmatrix}.
$$

(2.8)

• Minkowski space ($\bar{w} = \bar{w}; z$ and $\bar{z}$ are real). An example is

$$
\begin{pmatrix}
\bar{z} & w \\
\bar{w} & z
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
x_0 + \im x_1 & x_2 - \im x_3 \\
x_2 + \im x_3 & x_0 - \im x_1
\end{pmatrix}.
$$

(2.9)

• Ultrahyperbolic space ($\bar{w} = w; \bar{z} = \bar{z}$). Examples are

$$
\begin{pmatrix}
\bar{z} & w \\
\bar{w} & z
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
x_0 + \im x_1 & x_2 - \im x_3 \\
x_2 + \im x_3 & x_0 - \im x_1
\end{pmatrix}
$$

or $\bar{z}, \bar{z}, w, \bar{w} \in \mathbb{R}$.

(2.10)

The coordinate vectors $\partial_z, \partial_{\bar{z}}, \partial_w, \partial_{\bar{w}}$ form a null tetrad and are represented explicitly as

$$
\begin{align*}
\partial_z &= \frac{1}{\sqrt{2}} \begin{pmatrix}
\im \\
\partial x_0 + \im \partial x_1 \\
\partial x_2 + \im \partial x_3
\end{pmatrix}, \\
\partial_{\bar{z}} &= \frac{1}{\sqrt{2}} \begin{pmatrix}
\im \\
\partial x_0 - \im \partial x_1 \\
\partial x_2 - \im \partial x_3
\end{pmatrix}, \\
\partial_w &= \frac{1}{\sqrt{2}} \begin{pmatrix}
\im \\
\partial x_0 + \im \partial x_1 \\
\partial x_2 + \im \partial x_3
\end{pmatrix}, \\
\partial_{\bar{w}} &= \frac{1}{\sqrt{2}} \begin{pmatrix}
\im \\
\partial x_0 - \im \partial x_1 \\
\partial x_2 - \im \partial x_3
\end{pmatrix}.
\end{align*}
$$

(2.11)

For the Euclidean and ultrahyperbolic signatures, the Hodge dual operator $\ast$ satisfies $s^2 = 1$ and hence the space of two-forms $\beta$ decomposes into the direct sum of eigenvalues of $s$ with eigenvalues $\pm 1$, i.e. the self-dual (SD) part ($\ast \beta = \beta$) and the anti-self-dual part ($\ast \beta = -\beta$). From now on, we treat these two signatures.

Typical examples of SD forms are

$$
a = dw \wedge dz, \quad \tilde{a} = d\bar{w} \wedge d\bar{z}, \quad \omega = dw \wedge d\bar{w} - dz \wedge d\bar{z}
$$

(2.12)

and typical examples of anti-self-dual forms are

$$
dw \wedge d\bar{z}, \quad d\bar{w} \wedge dz, \quad dw \wedge d\bar{w} + dz \wedge d\bar{z}.
$$

(2.13)

The noncommutative anti-self-dual Yang–Mills equation is derived from the compatibility condition of the following linear system:

$$
L \ast \psi := (D_w - \zeta D_{\bar{w}}) \ast \psi
$$

$$
= (\partial_w + A_w - \zeta (\partial_{\bar{w}} + A_{\bar{w}})) \ast \psi(x; \zeta)
$$

$$
= 0,
$$

$$
M \ast \psi := (D_z - \zeta D_{\bar{z}}) \ast \psi
$$

$$
= (\partial_z + A_z - \zeta (\partial_{\bar{z}} + A_{\bar{z}})) \ast \psi(x; \zeta)
$$

$$
= 0,
$$

(2.14)

where $A_z, A_w, A_{\bar{z}}, A_{\bar{w}}$ and $D_z, D_w, D_{\bar{z}}, D_{\bar{w}}$ denote gauge fields and covariant derivatives in the Yang–Mills theory, respectively. The constant $\zeta \in \mathbb{C} P_1$ is called the spectral parameter.

The compatibility condition $[L, M] = 0$ gives rise to a quadratic polynomial of $\zeta$ and each coefficient yields the following equations:

$$
F_{w\bar{w}} = \partial_w A_z - \partial_z A_w + [A_w, A_z],
$$

$$
F_{w\bar{z}} = \partial_w A_{\bar{z}} - \partial_{\bar{z}} A_w + [A_w, A_{\bar{z}}],
$$

$$
F_{z\bar{w}} = \partial_z A_{\bar{w}} - \partial_{\bar{w}} A_z + [A_z, A_{\bar{w}}],
$$

$$
F_{z\bar{z}} = \partial_z A_z - \partial_{\bar{z}} A_{\bar{z}} + [A_z, A_{\bar{z}}],
$$

(2.15)

which are equivalent to the noncommutative anti-self-dual Yang–Mills equations $F_{\mu\nu} = -\ast F_{\mu\nu}$ in the real representation.

Gauge transformations act on the linear system (2.14) as

$$
L \mapsto g^{-1} \ast L \ast g, \quad M \mapsto g^{-1} \ast M \ast g, \quad \psi \mapsto g^{-1} \ast \psi, \quad g \in G.
$$

(2.16)

We note that the solution $\psi$ ($N \times N$ matrix) of the linear system (2.14) is not regular at $\zeta = \infty$ because of Liouville’s theorem. (If it is regular, then the gauge fields become flat.) Hence we have to consider another linear system on another local patch in $\zeta \in \mathbb{C} P_1$ whose coordinate is $\tilde{\zeta} = 1/\zeta$ as

$$
\hat{L} \ast \psi := \hat{\zeta} D_w \ast \psi - D_{\bar{w}} \ast \psi = 0,
$$

$$
\hat{M} \ast \psi := \hat{\zeta} D_z \ast \psi - D_{\bar{z}} \ast \psi = 0.
$$

(2.17)

The compatibility condition of this system also gives rise to the anti-self-dual Yang–Mills equation.

2.3. Noncommutative Yang equations and $J, K$-matrices

Here we discuss the potential forms of the noncommutative anti-self-dual Yang–Mills equations such as noncommutative $J, K$-matrix formalisms and the noncommutative Yang equation, which is already presented by e.g. Takasaki [26].

Let us first discuss the $J$-matrix formalism of the noncommutative anti-self-dual Yang–Mills equation. The first equation of the noncommutative anti-self-dual Yang–Mills equation (2.15) is the compatibility condition of the linear system $D_w \ast h = 0, D_{\bar{w}} \ast \bar{h} = 0$, where $h$ is an $N \times N$ matrix. Hence we get

$$
A_z = - (\partial_z h) \ast h^{-1}, \quad A_w = - (\partial_w h) \ast h^{-1}.
$$

(2.18)

Similarly, the second equation of the noncommutative anti-self-dual Yang–Mills equation (2.15) leads to

$$
\hat{A}_{\bar{z}} = - (\partial_{\bar{z}} \bar{h}) \ast \bar{h}^{-1}, \quad \hat{A}_{\bar{w}} = - (\partial_{\bar{w}} \bar{h}) \ast \bar{h}^{-1},
$$

(2.19)

where $\bar{h}$ is also an $N \times N$ matrix. We note that $h(x) = \psi(x, \zeta = 0), \bar{h}(x) = \psi(x, \zeta = \infty)$.

On defining a new matrix $J = \bar{h}^{-1} \ast h$, the third equation of the noncommutative anti-self-dual Yang–Mills equation (2.15) becomes the noncommutative Yang equation

$$
\partial_z (J^{-1} \ast \partial_z J) - \partial_w (J^{-1} \ast \partial_w J) = 0
$$

(2.20)

or equivalently

$$
\partial (J^{-1} \ast \partial J) \wedge \omega = 0.
$$

(2.21)
where \( \tilde{\vartheta} = dw \partial_w + dz \partial_z \), \( \tilde{\bar{\vartheta}} = d\bar{w} \partial_{\bar{w}} + d\bar{z} \partial_{\bar{z}} \). Here \( \omega \) is the same as in (2.12).

Gauge transformations act on \( h \) and \( \tilde{h} \) as
\[
h \mapsto g^{-1} h, \quad \tilde{h} \mapsto g^{-1} \tilde{h}, \quad g \in G. \tag{2.22}
\]
Hence Yang’s \( J \)-matrix is gauge invariant while the matrices \( h \) and \( \tilde{h} \) are gauge dependent. In this paper, we sometimes use the following gauge for \( G = GL(2) \):
\[
h_{MW} = \begin{pmatrix} f & 0 \\ e & 1 \end{pmatrix}, \quad \tilde{h}_{MW} = \begin{pmatrix} 1 & g \\ 0 & 0 \end{pmatrix},
\]
then \( J = \tilde{h}_{MW}^{-1} \cdot h_{MW} = \begin{pmatrix} f - g \cdot b^{-1} \cdot e & -g \cdot b^{-1} \\ b^{-1} \cdot e & b^{-1} \end{pmatrix} \).

(2.23)
which is called the Mason–Woodhouse gauge.

There is another potential form of the noncommutative anti-self-dual Yang–Mills equation, known as the \( K \)-matrix formalism. In the gauge \( A_w = A_{\bar{z}} = 0 \), the third equation of (2.15) becomes \( \partial_w A_z - \partial_z A_w = 0 \). This implies the existence of a potential \( K \) such that \( A_z = \partial_w K, A_{\bar{w}} = \partial_{\bar{z}} K \). Then the second equation of (2.15) becomes
\[
\partial_w \partial_z K - \partial_w \partial_{\bar{z}} K + [\partial_w K, \partial_{\bar{z}} K] = 0. \tag{2.24}
\]
Then, we get
\[
\psi = 1 + \xi K + O(\xi^2), \quad \tilde{\psi} = J^{-1} + O(\tilde{\xi}) \tag{2.25}
\]
and \( A_{\bar{w}} = J^{-1} \cdot \partial_w J = \partial_{\bar{z}} K, \quad A_{\bar{w}} = J^{-1} \cdot \partial_{\bar{z}} J = \partial_w K \). This gauge is suitable for the discussion of the (binary) Darboux transformations for the (noncommutative) anti-self-dual Yang–Mills equations [27–29].

### 3. The twistor description of noncommutative anti-self-dual Yang–Mills equations

In this section, we construct a wide class of exact solutions of the noncommutative anti-self-dual Yang–Mills equations from the geometrical viewpoint of the noncommutative twistor theory. The noncommutative twistor theory has been developed by several authors and mathematical foundations are established [26, 30–32].

The twistor theory is based on a correspondence between (complexified) space–time coordinates \((z, \bar{z}, w, \bar{w})\) and twistor coordinates \((\lambda, \mu, \zeta)\) which are local coordinates of a three-dimensional complex projective space (twistor space). The explicit relation is called the incidence relation, and represented as follows:
\[
\lambda = \zeta w + \bar{z}, \quad \mu = \zeta z + \bar{w} \tag{3.1}
\]
which implies that for any twistor function \( f(\lambda, \mu, \zeta) \)
\[
lf(\lambda, \mu, \zeta) := (\partial_w - \zeta \partial_z) f(\lambda, \mu, \zeta) = 0,
\]
\[
mf(\lambda, \mu, \zeta) := (\partial_{\bar{w}} - \zeta \partial_{\bar{z}}) f(\lambda, \mu, \zeta) = 0. \tag{3.2}
\]

#### 3.1. The noncommutative Penrose–Ward transformation

For the anti-self-dual Yang–Mills theory, there is a one-to-one correspondence between solutions of the anti-self-dual Yang–Mills equation and holomorphic vector bundles on the twistor space. The former is given by solutions \( \psi, \tilde{\psi} \) of the linear systems (2.14) and (2.17). The latter is given by patching matrices \( P \) of the holomorphic vector bundles. The explicit correspondence is called the Penrose–Ward correspondence.

Here we just need the Moyal-deformed Penrose–Ward correspondence between the anti-self-dual Yang–Mills solution \( \psi, \tilde{\psi} \) and the patching matrix \( P \).

From given \( \psi \) and \( \tilde{\psi} \), the patching matrix \( P \) is constructed as
\[
P(\zeta w + \bar{z}, \zeta z + \bar{w}, \zeta) = \tilde{\psi}^{-1}(\zeta w; \zeta) \cdot \psi(\zeta; \zeta). \tag{3.3}
\]
(Here we note that \( \psi(x; \zeta) \) is regular w.r.t. \( \zeta \) around \( \zeta = 0 \) and \( \tilde{\psi}(x; \zeta) \) is regular w.r.t. \( \zeta \) around \( \zeta = 0 \) or equivalently \( \zeta = \infty \).) Conversely, if there exists the factorization (3.3) into \( \psi \) and \( \tilde{\psi} \) for a given \( P \) where \( \psi(x; \zeta) \) is regular w.r.t. \( \zeta \) around \( \zeta = 0 \) and \( \tilde{\psi}(x; \zeta) \) is regular w.r.t. \( \zeta \) around \( \zeta = 0 \), then the \( \psi \) and \( \tilde{\psi} \) are solutions of linear systems (2.14) and (2.17) for the noncommutative anti-self-dual Yang–Mills equations. (This factorization problem is called the Riemann–Hilbert problem and solved formally [26]. Noncommutativity can be introduced into only two variables \( \zeta w + \bar{z} \) and \( \zeta z + \bar{w} \). Then \( \zeta \) is a commutative variable and the ways of solving the Riemann–Hilbert problem become similar to commutative ones.)

#### 3.2. Noncommutative Atiyah–Ward ansatz solutions for \( G = GL(2) \)

From now on, we restrict ourselves to \( G = GL(2) \). For this gauge group, we can take a simple ansatz for the patching matrix \( P \), which is called the Atiyah–Ward ansatz in the commutative case [33]. Noncommutative generalization of this ansatz is straightforward and actually leads to a solution of the factorization problem. The 4th-order noncommutative Atiyah–Ward ansatz is specified by the following form of the patching matrix up to constant matrix actions from both sides \((l = 0, 1, 2, \ldots)\):
\[
P_l(\zeta; x) = \begin{pmatrix} 0 & \zeta^{-l} \\ \zeta^l & \Delta(x; \zeta) \end{pmatrix}. \tag{3.4}
\]
We note that \( P \) satisfies (3.2) and hence, the Laurent expansion of \( \Delta \) w.r.t. \( \zeta \)
\[
\Delta(x; \zeta) = \sum_{i=-\infty}^{\infty} \Delta_i(x) \zeta^{-i} \tag{3.5}
\]
gives rise to the following recurrence relations in the coefficients:
\[
\frac{\partial \Delta_j}{\partial z} = \frac{\partial \Delta_{j+1}}{\partial \bar{w}}, \quad \frac{\partial \Delta_j}{\partial \bar{w}} = \frac{\partial \Delta_{j+1}}{\partial \bar{z}}. \tag{3.6}
\]
The wavefunctions $\psi$ and $\tilde{\psi}$ can be expanded in terms of $\xi$ and $\tilde{\xi} = 1/\xi$, respectively:

$$
\psi = h + \mathcal{O}(\xi) = \left( \hat{h}_{11} + \sum_{i=1}^{\infty} a_i \xi^i, \hat{h}_{21} + \sum_{i=1}^{\infty} b_i \xi^i \right), \quad \tilde{\psi} = \tilde{h} + \mathcal{O}(\tilde{\xi}) = \left( \tilde{\hat{h}}_{11} + \sum_{i=1}^{\infty} \tilde{a}_i \tilde{\xi}^i, \tilde{\hat{h}}_{21} + \sum_{i=1}^{\infty} \tilde{b}_i \tilde{\xi}^i \right).
$$

(3.7)

Now let us solve the factorization problem $\tilde{\psi} \ast P = \psi$ for the noncommutative Atiyah–Ward ansatz. This is concretely written down as:

$$
\left( \begin{array}{c} \tilde{\psi}_{11} \\ \tilde{\psi}_{21} \\ \tilde{\psi}_{22} \end{array} \right) \ast \left( \begin{array}{c} \xi^i \\ \Delta(\xi) \end{array} \right) = \left( \begin{array}{c} \psi_{11} \\ \psi_{21} \\ \psi_{22} \end{array} \right),
$$

(3.8)
i.e.

$$
\tilde{\psi}_{12} \xi^i = \psi_{11}, \quad \tilde{\psi}_{22} \xi^i = \psi_{21}, \quad \tilde{\psi}_{11} \xi^{-i} + \tilde{\psi}_{12} \Delta = \psi_{12}, \quad \tilde{\psi}_{21} \xi^{-i} + \tilde{\psi}_{22} \Delta = \psi_{22}.
$$

(3.9)

(3.10)

From (3.7) and (3.9) we find that some entries become polynomials w.r.t. $\xi$:

$$
\psi_{11} = h_{11} + ai \xi + a_2 \xi^2 + \cdots + a_{i-1} \xi^{i-1} + \hat{h}_{12} \xi^i, \quad \psi_{21} = h_{21} + b_1 \xi + b_2 \xi^2 + \cdots + b_{i-1} \xi^{i-1} + \tilde{h}_{22} \xi^i, \quad \psi_{12} = h_{12} + a_1 \xi^{-1} + a_{2} \xi^{-2} + \cdots + a_i \xi^{-i} + h_{11} \xi^{-1}, \quad \psi_{22} = h_{22} + b_1 \xi^{-1} + b_2 \xi^{-2} + \cdots + b_i \xi^{-i} + h_{22} \xi^{-1}
$$

(3.11)

and so on. By substituting these relations into (3.10), we get sets of equations for $h$ and $\tilde{h}$ in the coefficients of $\xi^0, \xi^{-1}, \ldots, \xi^{-i}$:

$$(h_{11}, a_i, \ldots, a_{i-1}, \tilde{h}_{12})_{D_{i+1}} = (-\tilde{h}_{11}, 0, \ldots, 0, h_{12}), \quad (h_{21}, a_i, \ldots, c_{i-1}, \tilde{h}_{22})_{D_{i+1}} = (-\tilde{h}_{21}, 0, \ldots, 0, h_{22}),$$

(3.12)

where

$$
D_i := \left( \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-i} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{i-1} & \Delta_{i-2} & \cdots & \Delta_0 \end{array} \right).
$$

(3.13)

These linear equations can be solved by the taking inverse matrix of $D_{i+1}$ from the right, and can be rewritten in terms of quasideterminants (for a brief review, see the appendix):

$$
\begin{align*}
    h_{11} &= \hat{h}_{11} \star \left[ D_{i+1} \right]_{i+1,i+1}^{-1} \hat{h}_{11} \star \left[ D_{i+1} \right]_{i+1,i+1}, \\
    h_{21} &= \hat{h}_{22} \star \left[ D_{i+1} \right]_{i+1,i+1}^{-1} \hat{h}_{22} \star \left[ D_{i+1} \right]_{i+1,i+1}, \\
    \tilde{h}_{12} &= \tilde{h}_{12} \star \left[ D_{i+1} \right]_{i+1,i+1}^{-1} \tilde{h}_{11} \star \left[ D_{i+1} \right]_{i+1,i+1}, \\
    \tilde{h}_{22} &= \tilde{h}_{22} \star \left[ D_{i+1} \right]_{i+1,i+1}^{-1} \tilde{h}_{22} \star \left[ D_{i+1} \right]_{i+1,i+1}.
\end{align*}
$$

(3.14)

If we take the Mason–Woodhouse gauge (2.24), equation (3.14) can be solved for $h$ and $\tilde{h}$ in terms of quasideterminants of $D_{i+1}$:

$$
\begin{align*}
    f &= h_{11} = - \left[ \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-i} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{i-1} & \Delta_{i-2} & \cdots & \Delta_0 \end{array} \right]^{-1}, \\
    e &= h_{21} = - \left[ \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-i} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{i-1} & \Delta_{i-2} & \cdots & \Delta_0 \end{array} \right]^{-1}, \\
    g &= \tilde{h}_{12} = - \left[ \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-i} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{i-1} & \Delta_{i-2} & \cdots & \Delta_0 \end{array} \right]^{-1}, \\
    b &= \tilde{h}_{22} = - \left[ \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-i} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{i-1} & \Delta_{i-2} & \cdots & \Delta_0 \end{array} \right]^{-1}.
\end{align*}
$$

(3.15)

This is the $i$th-order noncommutative Atiyah–Ward ansatz solution. For $i = 0$, the noncommutative anti-self-dual Yang–Mills equation becomes a noncommutative linear equation $(\partial_i \partial_{\bar{j}} - \partial_{\bar{j}} \partial_i) \Delta_{\alpha \beta} = 0$. (We note that for the Euclidean space, this is the noncommutative Laplace equation because of the reality condition $\bar{w} = -\bar{w}$. The fundamental solutions lead to noncommutative instanton solutions [34].) The plane wave solutions yield a noncommutative version of the nonlinear plane wave solutions [35]. Other scalar functions $\Delta_i(\xi)$ are determined explicitly by the recurrence relation (3.6) from the solution $\Delta_i(\xi)$ of this linear equation up to integral constants. Hence the noncommutative Atiyah–Ward ansatz solutions are exact.

### 3.3. The Bäcklund transformation for the noncommutative Atiyah–Ward ansatz solutions

Finally let us discuss an adjoint action for the patching matrices $\alpha : P_i \mapsto P_{i+1} = A^{-1} P_i A$ in the twistor side, which leads to a Bäcklund transformation for the noncommutative anti-self-dual Yang–Mills equation in the Yang–Mills side. This is a noncommutative generalization of the Corrigan–Fairlie–Yates–Goddard transformation [20, 36, 37].

The adjoint action is defined by the following two kinds of adjoint actions:

$$
\alpha = \beta \circ \psi_0, \quad \beta : P \mapsto P^{\text{new}} = B^{-1} P B,
$$

(3.16)

where

$$
A = BC, \quad B = \left( \begin{array}{cc} 0 & 1 \\ \xi^{-1} & 0 \end{array} \right), \quad C_0 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
$$

(3.17)

In order to find the corresponding transformations in the Yang–Mills side, we have to observe how the adjoint actions...
The explicit calculation gives
\[ \bar{\psi}^\text{new} = s \cdot \bar{\psi} \quad \text{and} \quad \psi^\text{new} = s \cdot \psi \]
(3.18)
where the singular gauge transformation is
\[ s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}. \]
(3.19)
The beta transformation is generated by the two\[ \Delta_0 \Delta_{-1} \cdots \Delta_{-l} , \quad \Delta_1 \Delta_0 \cdots \Delta_{1-l} \]
and
\[ \Delta_l \Delta_{l-1} \cdots \Delta_0 \]
(3.19)
where the singular gauge transformation is
\[ s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}. \]
(3.19)
The beta transformation gives
\[ \psi^\text{new} = \begin{pmatrix} b^{-1} \psi_{22} \\ -\zeta f^{-1} \psi_{12} \end{pmatrix} + \begin{pmatrix} \zeta b^{-1} \psi_{21} \\ -f^{-1} \psi_{11} \end{pmatrix}, \]
(3.20)
where \( \psi_{ij} \) is the \((i, j)\)th element of \( \psi \). In the \( \zeta \to 0 \) limit, this reduces to the Mason–Woodhouse gauge
\[ h^\text{new} = \begin{pmatrix} f^\text{new} \\ e^\text{new} \end{pmatrix} = \begin{pmatrix} b^{-1} \\ -f^{-1} \end{pmatrix}, \]
(3.21)
where \( \psi = h + j \gamma + O(\zeta^{-2}) \).

Here we note that the linear systems can be represented in terms of \( b, f, e, g \) as
\[ L \cdot \psi = (\partial_w - \zeta \partial_{\omega}) \cdot \psi \quad \text{and} \quad M \cdot \psi = (\partial_{\omega} - \zeta \partial_w) \cdot \psi \]
(3.22)
By picking the first-order term of \( \zeta \) in the 1–2 component of the first equation, we get
\[ \partial_w (f^{-1} \cdot j_{12}) = -f^{-1} \cdot g_z \cdot b^{-1}. \]
(3.23)
Hence from the 1–2 and 2–1 components of (3.21), we have
\[ f^\text{new} = b^{-1}, \quad \partial_w e^\text{new} = \partial_w (f^{-1} \cdot j_{12}) = -f^{-1} \cdot g_z \cdot b^{-1}. \]
(3.24)
Noncommutative Atiyah–Ward ansatz solutions $R'_i$: noncommutative Atiyah–Ward ansatz solutions $R'_i$ are represented by the explicit form of elements $b'_i$, $e'_i$, $f'_i$, $g'_i$ as quasideterminants of $l \times l$ matrices

$$g_i = \begin{bmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{bmatrix}$$

$$\tilde{h}_i^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{bmatrix}$$

In the Mason–Woodhouse gauge

$$h_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-l} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$h_i^{-1} = \begin{bmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$J_i = \begin{bmatrix} 0 \end{bmatrix} -1 \begin{bmatrix} 0 \end{bmatrix} -1 \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

$$J_i^{-1} = \begin{bmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$
In the Mason–Woodhouse gauge

\[
\begin{align*}
\tilde{h}^l_i &= \begin{bmatrix} \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_{2} & \Delta_{1-l} & 0 \\
\Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_{2} & \Delta_{1-l} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_{2} & \Delta_{1-l} & 0 \\
\Delta_l & 0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\
0 & 0 & \cdots & 0 & -1 & 0
\end{bmatrix}, \\
\tilde{h}^{-1}_i &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 & 0 \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\
\Delta & 0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\
0 & 0 & \cdots & 0 & -1 & 0
\end{bmatrix}, \\
\tilde{h}^{-1}_j &= \begin{bmatrix} 0 & \Delta_1 & \cdots & \Delta_{3-l} & \Delta_{2-l} \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \Delta_1 & \cdots & \Delta_{3-l} & \Delta_{2-l} \\
\Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
0 & 0 & \cdots & 0 & -1 & 0
\end{bmatrix}, \\
\tilde{h}^{-1}_j &= \begin{bmatrix} 0 & \Delta_1 & \cdots & \Delta_{3-l} & \Delta_{2-l} \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
0 & 0 & \cdots & 0 & -1 & 0
\end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} \\
\end{bmatrix}
&= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} \\
\end{bmatrix}
\end{align*}
\]

Because \( J \) is gauge invariant, this shows that the present Bäcklund transformation is not just a gauge transformation but a nontrivial one.

The proof of these results can be given directly by using identities of quasideterminants only, such as the noncommutative Bäcklund transformation is not just a gauge transformation but a nontrivial one.

Here we briefly discuss reductions of the noncommutative anti-self-dual Yang–Mills equation into lower-dimensional noncommutative integrable equations such as the noncommutative KdV equation. The reductions are specified by a choice of gauge group, symmetry, gauge fixing and so on. Gauge groups are in general \( GL(N) \). We have to take the \( U(1) \) part of the gauge group into account in the noncommutative case. The noncommutativity in the reduced directions is assumed to be eliminated because of compatibility with the symmetry. (Hence within the reduced directions, the symmetry is the same as the commutative one.) The residual gauge symmetry sometimes shows equivalence of a few reductions.

Here, we present nontrivial reductions of the noncommutative anti-self-dual Yang–Mills equation with \( G = GL(2) \) to the noncommutative KdV equation via a \((2 + 1)\)-dimensional integrable equation.

Let us start with the standard anti-self-dual Yang–Mills equation (2.15) with \( G = GL(2, \mathbb{C}) \) and impose the following translational invariance:

\[
Y = \partial_z, \quad (4.1)
\]

and impose the following nontrivial reduction conditions for the gauge fields:

\[
\begin{align*}
A_\tilde{w} &= 0, \\
A_z &= \begin{bmatrix} 0 & 0 \\
1 & 0
\end{bmatrix}, \\
A_w &= \begin{bmatrix} q & -q \\
q & -1
\end{bmatrix}, \\
A_\tilde{z} &= \begin{bmatrix} (1/2) q_{\tilde{w} w} + q \tilde{w} q + \alpha \\
-q \tilde{w} \alpha
\end{bmatrix},
\end{align*}
\]

where

\[
\alpha = \tilde{\partial}^{-1}_w [q_w, q_\tilde{w}], \quad \tilde{\partial}^{-1}_w f(w) := \int_w^\infty dw' f(w'),
\]

where \( [A, B] := A \star B + B \star A \).

Then we get a noncommutative version of the toroidal KdV equation [14] by making the identification \( 2q_w = u \)

\[
\begin{align*}
u_z &= \frac{1}{4} u_{w w \tilde{w}} + \frac{1}{2} \{ u, u \tilde{w} \} + \frac{1}{2} \{ u_{\tilde{w}}, \tilde{u} \tilde{w}^{-1} u \tilde{w} \}, \\
+ \frac{1}{2} \tilde{\partial}^{-1}_w [u, \tilde{\partial}^{-1}_w u_{\tilde{w}}].
\end{align*}
\]

This equation has hierarchy and \( N \)-soliton solutions in terms of quasideterminants of the Wronskian [42]. We note that under the ultrahyperbolic signature \((+, +, +, -)\), all remaining coordinates among \( z, w, \tilde{w} \) can be set to be real [20].

If we take the further reduction \( \partial_\tilde{w} = \partial_w \), that is, dimensional reduction to the \( X = \tilde{w} - \partial_w \) direction, then the reduced equation coincides with the noncommutative KdV equation

\[
\dot{u} = \frac{1}{2} u'' + \frac{1}{3} (u' \star u + u u').
\]
is not traceless. This implies that the $U(1)$ part of the gauge group plays a crucial role in the reduction process also.

This noncommutative KdV equation has been studied by several authors and proved to possess infinite conserved quantities [43] in terms of Strachan’s [44] products and exact multi-soliton solutions in terms of quasideterminants [42, 45]. (See also [46].)

5. Conclusion and discussion

In this paper, we have presented the Bäcklund transformations for the noncommutative anti-self-dual Yang–Mills equation with $G = GL(2)$ and constructed a series of the exact noncommutative Atiyah–Ward ansatz solutions in terms of quasideterminants.

The quasideterminants play important roles in the construction of noncommutative soliton solutions not only for the noncommutative anti-self-dual Yang–Mills equation, but also for various lower-dimensional noncommutative integrable equations [47–62].

Such common properties have been revealed in the study of the noncommutative extension; however, even within the commutative limit, this gives us new insight. Various properties and identities of the quasideterminants are actually very useful and suitable for the noncommutative soliton theory. Surprisingly, obtaining a proof by using the quasideterminants is sometimes easier than achieving the same end by using the commutative determinants! This suggests that the quasideterminants might be more essential than the usual determinants in the context of soliton theories (even within the commutative limit!). In Sato’s theory of solitons, the Plücker relations of the Wronskian play crucial roles. The present results would suggest the possibility of both noncommutative extension and higher-dimensional extension of his theory. It might be time to reconsider a formulation of Sato’s theory of (noncommutative) anti-self-dual Yang–Mills equations from the viewpoint of quasideterminants. (For commutative discussions, see e.g. [63, 64].)

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Appendix. A brief review of quasideterminants

In this section, we give a brief introduction to quasideterminants, introduced by Gelfand and Retakh [65] in 1991, and present a few of their properties which play important roles in section 5. A good survey is e.g. [66] and the relation between quasideterminants and noncommutative symmetric functions is summarized in e.g. [67]. (See also [68, 69].)

Quasideterminants are not just a noncommutative generalization of usual commutative determinants, but rather related to inverse matrices.

Let $A = (a_{ij})$ be an $n \times n$ matrix and $B = (b_{ij})$ be the inverse matrix of $A$. Here all matrix elements are supposed to belong to a (noncommutative) ring with an associative product. This general noncommutative situation includes the Moyal or noncommutative deformation which we discuss in the main sections.

Quasideterminants of $A$ are defined formally as the inverses of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}. \quad (A.1)$$

In the commutative limit, this is reduced to

$$|A|_{ij} \longrightarrow (-1)^{i+j} \frac{\det A}{\det A'} \quad (A.2)$$

where $A'$ is the matrix obtained from $A$ by deleting the $i$th row and the $j$th column.

We can write down more explicit forms of quasideterminants. In order to see this, let us recall the following formula for a square matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$ \quad (A.3)

where $A$ and $D$ are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a $2 \times 2$ matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed $2 \times 2$ matrix. So the explicit forms of quasideterminants are given iteratively by the following formula:

$$|A|_{ij} = a_{ij} - \sum_{i' \neq i, j' \neq j} a_{i'i}(A')^{-1}_{i'j'}a_{j'j} \quad (A.4)$$

It is sometimes convenient to represent the quasideterminant as follows:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \quad (A.5)$$
Examples of quasideterminants include, for a $1 \times 1$ matrix $A = a$,

$$|A| = a$$

(A.6)

for a $2 \times 2$ matrix $A = (a_{ij})$,

$$|A|_{11} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{21}^{-1}a_{21},$$

$$|A|_{12} = \begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{11} - a_{12}a_{21}^{-1}a_{22},$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{21} - a_{22}a_{11}^{-1}a_{11},$$

$$|A|_{22} = \begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{22} - a_{21}a_{11}^{-1}a_{12}$$

(A.7)

and for a $3 \times 3$ matrix $A = (a_{ij})$,

$$|A|_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \begin{vmatrix} a_{22} & a_{23} \ a_{32} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} a_{21} \\ a_{31} \end{vmatrix}$$

$$= a_{11} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}a_{31}$$

$$- a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}a_{31}$$

(A.8)

and so on.

Quasideterminants have various interesting properties similar to those of determinants. Among them, the following ones play important roles in this paper. In the block matrices given in these results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions, such that the overall matrix is square. (By using boxes, it becomes easier to calculate various identities. Such calculations are fully presented in e.g. [49, 59].)

- The noncommutative Jacobi identity [49, 65]

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix}$$

- Homological relations [65]

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \\ E & h \end{vmatrix}$$

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \\ E & h \end{vmatrix}$$

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \\ E & h \end{vmatrix}$$

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