Lattices of finite abelian groups

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Abstract. We study certain lattices constructed from finite abelian groups. We show that such a lattice is eutactic, thereby confirming a conjecture by Böttcher, Eisenbarth, Fukshansky, Garcia, Maharaj. Our methods also yield simpler proofs of two known results: First, such a lattice is strongly eutactic if and only if the abelian group has odd order or is elementary abelian. Second, such a lattice has a basis of minimal vectors, except for the cyclic group of order 4.

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1. Introduction

For each finite abelian group $A$, one can construct a certain lattice $L(A)$ of rank $|A| - 1$. (We recall the construction of $L(A)$ in Section 2 below.) For cyclic $A$, the lattice $L(A)$ probably appeared first in a paper by E. S. Barnes [2] and is therefore often called the 
Barnes lattice. The construction of $L(A)$ also generalizes a family of lattices coming from elliptic curves over finite fields [7]. For general abelian groups $A$, the lattice $L(A)$ has been studied by A. Böttcher, S. Eisenbarth, L. Fukshansky, S. R. Garcia, H. Maharaj [3, 4], and by R. Bacher [1], among others.

One of the results of Böttcher et al. [3] is that the lattice $L(A)$ is strongly eutactic if and only if $A$ has odd order or is an elementary abelian 2-group. They conjectured that $L(A)$ is always eutactic (except when $A \cong C_4$, the cyclic group of order 4). The main result of this paper, Theorem 5.10, is that this conjecture is true.

Eutactic lattices are interesting due to a connection to sphere packings. By a classical result of Voronoi [9, Theorem 4.6.3], a lattice is extreme, that is, a local maximum of the packing density, if and only if it is eutactic and perfect. Moreover, A. Schürmann [11] has shown that perfect, strongly eutactic lattices are periodic extreme. R. Bacher [1] has shown that $L(A)$ is perfect, except for some groups $A$ of order at most 8. Thus $L(A)$ is extreme, except for some $A$ with $|A| \leq 8$.

One key ingredient in our proof is the observation that the lattice $L(A)$ is in fact the square of the augmentation ideal of the group ring (see Lemma 2.2). From this observation, we derive in Section 3 a very useful parametrization of the set of minimal vectors of $L(A)$ (the set of nonzero vectors in $L(A)$ of minimal length). Böttcher, Fukshansky, Garcia, and Maharaj [4] showed that the lattice $L(A)$ is generated by minimal vectors (except

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when \( A \cong C_4 \), and has in fact a basis consisting of minimal vectors. In general, a lattice generated by minimal vectors need not have a basis of minimal vectors \([6]\). Such lattices exist in dimension \( \geq 10 \), but not in lower dimensions \([10]\). The lattices considered in this paper, however, do not exhibit this phenomenon. In Section 4, we use our methods to give a short and very elementary proof that \( L(\mathcal{A}) \) \( (\mathcal{A} \neq C_4) \) has a basis of minimal vectors. The proof is by direct computation in the group ring, and we indicate how to find different bases of minimal vectors. Section 4 is not needed for the final Section 5, where eutaxy is treated.

2. The lattice of a finite abelian group

Let \( \mathcal{A} \) be a finite abelian group, written multiplicatively. The group ring \( \mathbb{Z}\mathcal{A} \) of \( \mathcal{A} \) is by definition the set of formal sums
\[
\mathbb{Z}\mathcal{A} := \left\{ \sum_{a \in \mathcal{A}} r_a a : r_a \in \mathbb{Z} \right\}.
\]
Multiplication in \( \mathbb{Z}\mathcal{A} \) is induced by the multiplication in the group \( \mathcal{A} \), extended distributively. The augmentation ideal \( \Delta \mathcal{A} \) of the group ring is the kernel of the augmentation map (coefficient sum) \( \mathbb{Z}\mathcal{A} \to \mathbb{Z} \), that is,
\[
\Delta \mathcal{A} := \left\{ \sum_{a \in \mathcal{A}} r_a a : \sum_{a \in \mathcal{A}} r_a = 0, r_a \in \mathbb{Z} \right\}.
\]
We consider the exponential type homomorphism
\[
\psi_{\mathcal{A}}: \mathbb{Z}\mathcal{A} \to \mathcal{A}, \quad \psi_{\mathcal{A}} \left( \sum_{a \in \mathcal{A}} k_a a \right) = \prod_{a \in \mathcal{A}} a^{k_a}.
\]

2.1 Definition. The lattice \( L(\mathcal{A}) \) of the finite abelian group \( \mathcal{A} \) is defined as
\[
L(\mathcal{A}) := \ker(\psi_{\mathcal{A}}) \cap \Delta \mathcal{A}.
\]

This is the lattice that has been studied before by A. Böttcher, S. Eisenbarth, L. Fukshansky, S. R. Garcia, H. Maharaj \([3, 4]\), and R. Bacher \([1]\), with the difference that additive notation for the abelian group \( \mathcal{A} \) is used in the cited papers. In the present paper, we employ multiplicative notation for \( \mathcal{A} \) since then the notation for the group ring is more convenient.

2.2 Lemma. \( L(\mathcal{A}) = (\Delta \mathcal{A})^2 \), the square of the augmentation ideal.

By definition, the square of an ideal \( I \) in a ring is the additive subgroup generated by all products \( ab \) with \( a, b \in I \).

Proof of Lemma 2.2. The elements of the form \( a - 1 \), \( 1 \neq a \in \mathcal{A} \), form a \( \mathbb{Z} \)-basis of \( \Delta \mathcal{A} \). Thus \( (\Delta \mathcal{A})^2 \) as \( \mathbb{Z} \)-module is spanned by elements of the form \( (a - 1)(b - 1) \) with \( a, b \in \mathcal{A} \). As these elements are in \( \ker \psi_{\mathcal{A}} \), it follows that \( (\Delta \mathcal{A})^2 \subseteq L(\mathcal{A}) \).
For the converse inclusion, we need the following congruence which seems to be well known [5, Lemma 2.1], [8, p. 7]:

\[ d \equiv \psi_A(d) - 1 \mod (\Delta A)^2 \text{ for all } d \in \Delta A. \]

This congruence follows from repeatedly applying the identities

\[
(a - 1) + (b - 1) = (ab - 1) - (a - 1)(b - 1) \quad \text{and} \\
(a - 1) - (b - 1) = (ab^{-1} - 1) - (a - b)(b^{-1} - 1).
\]

The congruence yields \( \psi_A(d) = 1 \iff d \in (\Delta A)^2 \) for \( d \in \Delta A \), and thus the lemma. \( \Box \)

3. Elements of minimal length

We consider \( \mathbb{Z}A \) as a subset of \( \mathbb{R}A \), and \( \mathbb{R}A \) as a Euclidean space with respect to the usual scalar product

\[
\left\langle \sum_{a \in A} x_a a, \sum_{a \in A} y_a a \right\rangle = \sum_{a \in A} x_a y_a.
\]

As usual, \( \|x\| := \sqrt{\langle x, x \rangle} \), this is called the length or Euclidean norm of \( x \in \mathbb{R}A \).

The minimum distance of a lattice \( \Lambda \) is defined as

\[ \text{MinDist}(\Lambda) := \min \{ \|x\| : x \in \Lambda, x \neq 0 \}. \]

As a \( \mathbb{Z} \)-module, \( (\Delta A)^2 \) is spanned by elements of the form

\[ (a - 1)(b - 1) = ab - a - b + 1, \quad a, b \in A. \]

We compute their length:

3.1 Lemma. Let \( a \neq 1 \neq b \in A \). Then

\[
\| (a - 1)(b - 1) \|^2 = \begin{cases} 
8, & \text{if } a = b = a^{-1}, \\
6, & \text{if } a = b \text{ or } a = b^{-1}, \text{ but } a^2 \neq 1, \\
4 & \text{else}.
\end{cases}
\]

Proof. Since \( a \neq 1 \neq b \), we also have \( a \neq ab \neq b \). Thus the only equalities that can occur among \( 1, a, b, ab \) are \( a = b \) and/or \( ab = 1 \). \( \Box \)

From the equality \( L(A) = \ker \psi_A \cap \Delta A \), it is immediate that every nonzero element in \( L(A) \) has length at least \( \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2 \), and this length is attained exactly at the elements of the form

\[ x + y - r - s \in \mathbb{Z}A \quad \text{such that } x, y, r, s \in A \text{ are distinct and } xy = rs. \]

We want to parametrize the vectors of length 2 in a different way:
3.2 Lemma. Write
\[ \Omega(A) := \{ (a, b) \in A \times A : a \neq 1 \neq b \neq a^{\pm 1} \}. \]
Define \( f : \Omega(A) \times A \to \mathbb{L}(A) \) by
\[ f(a, b, g) := (a - 1)(b - 1)g, \quad (a, b) \in \Omega(A), g \in A. \]
Then \( f \) defines a 4-to-1 map from \( \Omega(A) \times A \) onto the set
\[ \{ v \in \mathbb{L}(A) : \|v\| = 2 \}. \]

Proof. It is clear that \( \|(a - 1)(b - 1)g\| = \|(a - 1)(b - 1)\| = 2 \) when \( (a, b) \in \Omega \). Conversely, suppose \( v \in \mathbb{L}(A) \) has length 2. Then \( v = x + y - r - s \) with distinct \( x, y, r, s \in A \) such that \( xy = rs \). The equation \( v = (a - 1)(b - 1)g \) yields
\[ g \in \{ x, y \} \quad \text{and} \quad \{ ag, bg \} = \{ r, s \}. \]
There are obviously four triples \((a, b, g)\) that fulfill these conditions, and in view of \( xy = rs \), every such triple solves \( v = (a - 1)(b - 1)g \). As \( \|v\| = 2 \), we must have \( (a, b) \in \Omega(A) \). \( \square \)

3.3 Corollary. Let \( A \) be an abelian group and let \( t \) be the order of the subgroup \( \{ a \in A : a^2 = 1 \} \leq A \). Then
\[ |\{ v \in \mathbb{L}(A) : \|v\| = 2 \}| = \frac{1}{4} |A|[|A| - 1](|A| - 3) + t - 1. \]

Proof. The size of \( \Omega(A) \) is \( |\Omega(A)| = (|A| - 1)(|A| - 3) + t - 1 \). \( \square \)

Together with Lemma 3.1, this yields:

3.4 Corollary. The minimum distance of \( \mathbb{L}(A) \) is \( \sqrt{5} \) for \( |A| = 2 \), is \( \sqrt{6} \) for \( |A| = 3 \) and is equal to \( \sqrt{4} = 2 \) for \( |A| \geq 4 \).

(This result is also contained in the paper by Böttcher et al. [4] as part of their Theorem 1.1.)

3.5 Remark. The equality \( \mathbb{L}(A) = (\Delta A)^2 \) suggests two natural generalizations, which can of course be combined. First, one may study the lattice theoretic properties of \( (\Delta G)^2 \) for a nonabelian group. Notice, however, that for a nonabelian group, the minimal distance of \( (\Delta G)^2 \) is \( \sqrt{2} \): Namely, if \( gh \neq hg \), then
\[ (g - 1)(h - 1) - (h - 1)(g - 1) = gh - hg \in (\Delta G)^2 \]
has length \( \sqrt{2} \).

Second, one may study higher powers \( (\Delta A)^r \). When \( A = C_n \) is cyclic of order \( n \), then \( (\Delta C_n)^r \) is isometric to the Craig lattice \( A_{n-1}^{(r)} \) [9, Prop. 5.4.5]. When \( n = p \) is prime, then it is known that \( \text{MinDist}(\Delta C_p)^r \geq \sqrt{2r} \) for \( r < (p - 1)/2 \), and equality holds when \( r \) is a proper divisor of \( p - 1 \) [9, Theorem 5.4.8]. Examples are known where the inequality is strict. When \( A \) is not cyclic of prime order, then in general we may have \( \text{MinDist}(\Delta A)^r < \sqrt{2r} \). For example, \( \text{MinDist}(\Delta C_6)^3 = \sqrt{4} = 2 \).
4. Basis of minimal vectors

One of the main results of Böttcher et al. [4] is that \( \mathbb{L}(A) = (\Delta A)^2 \) has a lattice basis of elements of minimal length, except when \( A \) is cyclic of order 4. We give an alternative proof of this result in this section.

It is elementary to see that \( \mathbb{L}(C_2) \) and \( \mathbb{L}(C_3) \) have a basis of minimal vectors, and that \( \mathbb{L}(C_4) \) has no such basis: for \( C_4 \), Corollary 3.3 yields that \( \mathbb{L}(C_4) \) has only four vectors of minimal length 2, and these come in pairs \( \pm v \).

We begin our proof with a (well-known) lemma:

4.1 Lemma. Let \( A = \langle a \rangle \) be cyclic of order \( n \). Then the ideal \( \Delta A = (a - 1)\mathbb{Z}A \) is principal, and the following set is a \( \mathbb{Z} \)-basis of \((\Delta A)^2\):

\[
\{ (a - 1)\beta \cdot (a - 1) : 1 \neq \beta \in A \}
= \{ (a - 1)(a - 1), (a - 1)(a^2 - 1), \ldots, (a - 1)(a^{n-1} - 1) \}.
\]

Proof. We have the identity

\[
a^k - 1 = (a - 1)(a^{k-1} + \cdots + a + 1).
\]

As the elements \( a^k - 1 \) for \( k = 1, \ldots, n-1 \) form a basis of \( \Delta A \), we see that \( \Delta A = (a - 1)\mathbb{Z}A \), and that the displayed set forms a basis of \( (\Delta A)^2 = (a - 1)\Delta A \).

Notice that all but two elements in the basis from Lemma 4.1 have squared length 4. The exceptions are the first and the last element, namely \( (a - 1)^2 \) and \( (a - 1)(a^{-1} - 1) \). We will see below how to replace these by shorter vectors when \( n \geq 5 \).

4.2 Lemma. Let \( A = B \times C \) be a direct product of finite, abelian groups. Suppose that \( D \) is a basis of \((\Delta B)^2\) and \( E \) is a basis of \((\Delta C)^2\). Define

\[
M(B, C) := \{ (b - 1)(c - 1) : b \in B \setminus \{1\}, c \in C \setminus \{1\} \}.
\]

Then \( D \cup E \cup M(B, C) \) is a basis of \((\Delta A)^2\).

Proof. The set \( D \cup E \cup M(B, C) \) has the right cardinality for a lattice basis. Thus it suffices to show that every generator of \( (\Delta A)^2 = (\Delta[B \times C])^2 \) is in the \( \mathbb{Z} \)-span of \( D \cup E \cup M(B, C) \). We know that \( (\Delta[B \times C])^2 \) is spanned by elements of the form

\[
(b_1 c_1 - 1)(b_2 c_2 - 1), \quad b_1, b_2 \in B, c_1, c_2 \in C.
\]

We have the following identity (which can be verified by direct computation):

\[
(b_1 c_1 - 1)(b_2 c_2 - 1) = (b_1 b_2 - 1)(c_1 c_2 - 1) - (b_1 - 1)(c_1 - 1) - (b_2 - 1)(c_2 - 1)
+ (b_1 - 1)(b_2 - 1) + (c_1 - 1)(c_2 - 1).
\]

Here, the first three summands are in \( M(B, C) \) (up to sign), and the last two are in the span of \( D \) or \( E \), respectively. Thus the lemma follows.
4.3 Theorem. Let $A$ be a finite abelian group.

(i) $(\Delta A)^2$ has a lattice basis of elements of the form $(a - 1)(b - 1)$ with $a, b \in A$.

(ii) [4, Theorem 1.2] If either $|A| > 4$ or $A \cong C_2 \times C_2$, then $(\Delta A)^2$ has a lattice basis of elements of length 2.

Proof. Write

$$A = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$$

as a direct product of cyclic groups. By Lemma 4.1, $(\Delta \langle a_i \rangle)^2$ has a basis $B_i$ containing elements of the form $(a_i - 1)(a_i^k - 1)$. Using Lemma 4.2, we can extend $B_1 \cup B_2$ to a basis $B_{1,2}$ of $(\Delta \langle a_1, a_2 \rangle)^2$. Using Lemma 4.2 again, we can extend $B_{1,2} \cup B_3$ to a basis $B_{1,2,3}$ of $(\Delta \langle a_1, a_2, a_3 \rangle)^2$. Repeating, we arrive at a basis of $(\Delta A)^2$ containing only elements of the form $(a - 1)(b - 1)$. This shows the first part.

To see the second part of the theorem, observe first that the set $M(B, C)$ defined in Lemma 4.2 contains only elements of squared length 4. However, each $B_i$ contains elements of larger squared length, namely $(a_i - 1)^2$ and $(a_i - 1)(a_i^{-1} - 1)$ (possibly equal). We need to replace these by smaller elements.

For the moment, write $a = a_i$. Choose an element $b \in A$ such that $b \notin \{1, a^\pm 1, a^2\}$. This is possible since we assume that $|A| > 4$ or $A \cong C_2 \times C_2$. We have

$$(a - 1)^2 = (a - 1)(a - b) + (a - 1)(b - 1).$$

The assumption on $b$ assures that both $(a - 1)(a - b)$ and $(a - 1)(b - 1)$ have squared length 4. Moreover, we can choose $b$ such that $(a - 1)(b - 1)$ is already part of our basis constructed in the first step: Namely, when $|\langle a \rangle| \geq 5$, we can take $b = a^2$. Otherwise, $A$ is not cyclic, and we can choose $b$ in either $\langle a_{i+1} \rangle$ or $\langle a_{i-1} \rangle$. (Recall that $a = a_i$.) Thus we can replace $(a - 1)^2$ by the element $(a - 1)(b - a)$ and still have a basis.

Similarly, we have

$$(a - 1)(a^{-1} - 1) = (a^{-1} - 1)(ab - 1) + (a - 1)(b - 1).$$

Here we choose $b$ such that $b \notin \{1, a^\pm 1, a^{-2}\}$ and $(a - 1)(b - 1)$ is part of our basis already constructed. Then we can replace $(a - 1)(a^{-1} - 1)$ by $(a^{-1} - 1)(ab - 1)$. This finishes the proof. \hfill \Box

4.4 Remark. We can even fulfill both conditions of the theorem simultaneously: of course, the element $(a - 1)(a - b)$ in the above proof is not of the form $(x - 1)(y - 1)$ with $x, y \in A$. But in view of the equations

$$(a - 1)^2 = (a - 1)(a^{-1} - 1) - (a^{-1} - 1)(a^2 - 1) = (a - 1)(b - 1) - (a^2 - 1)(b - 1) + (a - 1)(ab - 1),$$

we can also replace $(a - 1)^2$ by $(a^{-1} - 1)(a^2 - 1)$ when $|\langle a \rangle| \geq 4$, or by an element of the form $(a - 1)(ab - 1)$ when $A$ is not cyclic.
In particular, for $A = \langle a \rangle$ cyclic of order $n \geq 5$, we have the following basis of minimal elements:
\[
\{ (a - 1)(a^2 - 1), (a - 1)(a^3 - 1), \ldots, (a - 1)(a^{n-2} - 1) \} \\
\cup \{ (a^{-1} - 1)(a^2 - 1), (a^{-1} - 1)(a^3 - 1) \}.
\]
This basis of $L(A) = (\Delta A)^2$ (up to an automorphism) was given by M. Sha [13, Theorem 3.2], with an entirely different proof.

4.5 Remark. Another way of constructing a basis of minimal vectors goes as follows: Suppose that $A = \langle a \rangle = \langle b \rangle$. Then by Lemma 4.1, $(\Delta A)^2 = (a - 1)(b - 1)Z.A$, and thus
\[
\{ (a - 1)(b - 1)a^k : k = 0, 1, \ldots, n - 2 \}
\]
is a basis of $(\Delta A)^2$, where $n = |A|$ as before. When $\varphi(n) > 2$ (that is, $n \notin \{1, 2, 3, 4, 6\}$), then we can choose $b \neq a^\pm 1$, so that all elements in this basis have squared length 4. (When $n$ is odd, we can choose $b = a^2$, and get the basis given by Martinet [9, Prop. 5.3.5].)

The basis of this remark is contained in a single $A$-orbit, while the basis of the previous remark contains elements from $n - 3$ different $A$-orbits.

5. Eutaxy

5.1 Definition. Let $(V, \langle , \rangle)$ be a Euclidean space. For $s \in V$, let $\pi_s : V \to V$ be the linear map defined by
\[
v \pi_s = \langle v, s \rangle s.
\]
A finite subset $S \subset V$ is called eutactic (in $V$), when there are positive scalars $r_s > 0$ such that
\[
\text{Id}_V = \sum_{s \in S} r_s \pi_s.
\]
Equivalently, for all $v \in V$ we have
\[
v = \sum_{s \in S} r_s \langle v, s \rangle s.
\]
The subset $S$ is called strongly eutactic, when all $r_s$ can be chosen to be equal.

When $S \subset V$ is eutactic in $V$, then $S$ spans $V$. So it is often more interesting to know whether a set $S$ is eutactic in its $\mathbb{R}$-linear span $\text{span}_\mathbb{R}(S)$. We have the following simple observation, which follows from $\text{Ker}(\pi_s) = \langle s \rangle^\perp$:

5.2 Lemma. A subset $S \subset V$ is eutactic in its span $U$ if and only if there are $r_s > 0$ such that
\[
\sum_{s \in S} r_s \pi_s = p_U,
\]
where $p_U : V \to V$ is the orthogonal projection from $V$ onto $U$, and $\pi_s : V \to V$ is as in Definition 5.1.
5.3 Definition. A lattice $\Lambda \subset V$ is called (strongly) eutactic in $V$ if the set

$$\text{MinVecs}(\Lambda) := \{ s \in \Lambda : \|s\| = \text{MinDist}(\Lambda) \}$$

is (strongly) eutactic in $V$.

Let $G$ be a finite group. We recall some facts on the group algebra $\mathbb{R}G$ as a Euclidean space, which we will need. The group algebra $\mathbb{R}G$ has an involution, given by

$$\left( \sum_{g \in G} r_g g \right)^* = \sum_{g \in G} r_g g^{-1}.$$

The canonical inner product on $\mathbb{R}G$ can be defined as $\langle r, s \rangle = \varepsilon(rs^*) = \varepsilon(s^*r)$, where $\varepsilon : \mathbb{R}G \to \mathbb{R}$ sends $\sum_{g \in G} r_g g$ to $r_1$. Let us write

$$e_G := \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{R}G.$$

The next result is easy to derive from standard results in representation theory (cf. [12, § 2.6, Thm. 8]), but for the sake of completeness, we include a short proof here.

5.4 Lemma. The map $r \mapsto (1 - e_G)r$ is the orthogonal projection from $\mathbb{R}G$ onto $\text{span}_\mathbb{R}(\Delta G) = \left\{ \sum_{g \in G} r_g g \in \mathbb{R}G : \sum_{g \in G} r_g = 0 \right\}$.

Proof. Write $\iota : \mathbb{R}G \to \mathbb{R}$ for the augmentation map (coefficient sum), $\iota \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g$, so that $\text{Ker} \iota = \text{span}_\mathbb{R}(\Delta G)$. Clearly $ge_G = e_G = e_G g$ for all $g \in G$. Thus $e_G^2 = e_G$ and $e_Gr = \iota(r)e_G$ for all $r \in \mathbb{R}G$. We can write $r = e_G r + (1 - e_G) r$ for any $r \in \mathbb{R}G$. We have $\langle e_G r, (1 - e_G) r \rangle = \varepsilon(r^*(1 - e_G)e_G r) = \varepsilon(0) = 0$, and $(1 - e_G) r$ is in $\text{Ker} \iota = \text{span}_\mathbb{R}(\Delta G)$. The result follows. \qed

5.5 Lemma. Let $s \in \mathbb{R}G$ and let $\pi_s : \mathbb{R}G \to \mathbb{R}G$ be defined as in Definition 5.1, with respect to the canonical inner product on $\mathbb{R}G$. Then

$$r \sum_{g \in G} \pi_{sg} = ss^*r$$

for all $r \in \mathbb{R}G$, that is, $\sum_g \pi_{sg}$ is left multiplication with $ss^*$.

Proof. Write $r = \sum_{x \in G} r_x x$ and $s = \sum_{y \in G} s_y y$. Then

$$r \sum_{g \in G} \pi_{sg} = \sum_{g \in G} \langle r, sg \rangle sg = \sum_{g \in G} r_g s_y g = s \sum_{x, y \in G} r_x y^{-1} x = ss^*r.$$ \qed
When $A$ is a finite abelian group and $A \not\cong C_2$ or $C_3$, then

$$\text{MinVecs}(\mathbb{L}(A)) = S(A) := \{ v \in \mathbb{L}(A) : \|v\| = 2 \}.$$

To show that $\mathbb{L}(A)$ is eutactic in its span for $A \not\cong C_4$, we have to show that the set $S(A)$ is eutactic in its span, when $A \not\cong C_2$, $C_3$ or $C_4$. (The lattices $\mathbb{L}(C_2)$ and $\mathbb{L}(C_3)$ are also eutactic, but MinVecs($\mathbb{L}(A)$) $\neq S(A)$ for these groups, and $\mathbb{L}(C_4)$ is not eutactic, since $S(A) = \text{MinVecs}(\mathbb{L}(C_4))$ spans only a subspace of the span of $\mathbb{L}(C_4)$.)

The group $A$ acts by right multiplication on $\mathbb{R}A$, and this action preserves the scalar product. The set $S(A)$ is invariant and decomposes into $A$-orbits. In Lemma 3.2, we have described a parametrization of $S(A)$, namely

$$S(A) = \{ (1 - a)(1 - b)g : (a, b) \in \Omega(A), g \in A \},$$

where

$$\Omega(A) = \{ (a, b) \in A \times A : a \neq 1 \neq b \neq a^{\pm 1} \}.$$

(For each element $s \in S(A)$, there are actually exactly four triples $(a, b, g) \in \Omega(A)$ such that $s = (1 - a)(1 - b)g$.) By Lemma 5.5, the problem of showing that $\mathbb{L}(A)$ is eutactic is reduced to a problem in the group algebra:

\[ 1 - e_A = \sum_{(a, b) \in \Omega(A)} \gamma_{a, b} m(a, b)m(a, b)^*. \]

Then $S(A)$ is eutactic. Moreover, $S(A)$ is strongly eutactic if and only if there is a $\gamma > 0$ such that

$$1 - e_A = \gamma \sum_{(a, b) \in \Omega(A)} m(a, b)m(a, b)^*.$$

**Proof.** If the condition holds, then

$$r \sum_{(a, b) \in \Omega(A)} \sum_{g \in G} \gamma_{a, b} \pi_m(a, b)g = \sum_{(a, b) \in \Omega(A)} \gamma_{a, b} m(a, b)m(a, b)^* r$$

$$= (1 - e_A) r.$$}

On the other hand, the double sum on the left can be rewritten as $\sum_{s \in S(A)} \lambda_s \pi_s$, with

$$\lambda_s := \sum_{(a, b) \in \Omega(A) : \exists g \in A : s = m(a, b)g} \gamma_{a, b} > 0.$$

By Lemmas 5.2 and 5.4, $S(A)$ is eutactic. If the $\gamma_{a, b}$’s are all equal, then the $\lambda_s$’s are all equal and $S(A)$ is strongly eutactic. Conversely, when $S(A)$ is strongly eutactic, then there is $\lambda > 0$ such that

$$r \sum_{s \in S(A)} \pi_s = \lambda (1 - e_A) r$$
for all \( r \in \mathbb{R}G \). It follows from Lemmas 3.2 and 5.5 that

\[
    r \sum_{s \in S(A)} \pi_s = \left( \frac{1}{4} \right) \sum_{(a, b) \in \Omega(A)} m(a, b) m(a, b)^* r.
\]

Thus the result. \( \square \)

5.7 Lemma. Let \( A \) be a finite abelian group, and let \( T := \{ a \in A : a^2 = 1 \} \) and \( S := \{ a^2 : a \in A \} \). Write \( m(a, b) = (1 - a)(1 - b) \) as before. Then

\[
    \sum_{(a, b) \in \Omega(A)} m(a, b) m(a, b)^* = 4|A|(|A| - 4)(1 - e_A) + 4|A|(1 - e_S) + 8|T|(1 - e_T).
\]

Proof. Notice that \( m(1, b) = m(a, 1) = 0 \) and \( m(a, a) m(a, a)^* = (1 - a)^2(1 - a^{-1})^2 = m(a, a^{-1}) m(a, a^{-1})^* \). This yields

\[
    \sum_{(a, b) \in \Omega(A)} m(a, b) m(a, b)^* = \sum_{a, b \in A} m(a, b) m(a, b)^* - \sum_{a \in A} m(a, a) m(a, a)^* - \sum_{a \in A} m(a, a^{-1}) m(a, a^{-1})^* + \sum_{t \in T} m(t, t) m(t, t)^* = \left( \sum_{a \in A} (1 - a)(1 - a^{-1}) \right)^2 - 2 \sum_{a \in A} (1 - a)^2(1 - a^{-1})^2 + \sum_{t \in T} (1 - t)^4.
\]

We have \( (1 - a)(1 - a^{-1}) = 2 - a - a^{-1} \) and thus the first sum on the last line equals

\[
    \sum_{a \in A} (2 - a - a^{-1}) = 2|A|(1 - e_A).
\]

Next, we have

\[
    \sum_{a \in A} (1 - a)^2(1 - a^{-1})^2 = \sum_{a \in A} (2 - a - a^{-1})^2 = \sum_{a \in A} (4 + a^2 + a^{-2} - 4a - 4a^{-1} + 2) = 6|A| - 8|A|e_A + 2|A|e_S = 8|A|(1 - e_A) - 2|A|(1 - e_S).
\]

Finally, for \( t^2 = 1 \) we have \( (1 - t)^4 = 1 - 4t + 6 - 4t + 1 = 8(1 - t) \) and thus

\[
    \sum_{t \in T} (1 - t)^4 = 8|T|(1 - e_T).
\]
Therefore,
\[
\sum_{(a,b) \in \Omega(A)} m(a,b)m(a,b)^* = 4|A|^2(1 - e_A)^2 - 16|A|(1 - e_A) + 4|A|(1 - e_S) + 8|T|(1 - e_T)
= 4|A|(|A| - 4)(1 - e_A) + 4|A|(1 - e_S) + 8|T|(1 - e_T).
\]

As a corollary, we get an alternative proof of the following result which was first proved by Böttcher et al. [3]:

**5.8 Corollary.** Let \( A \) be a finite abelian group of size at least 4. Then \( S(A) \) is strongly eutactic in its span if and only if \( A \) has odd order or \( A \) is elementary abelian.

**Proof.** By Lemma 5.6, \( S(A) \) is strongly eutactic if and only if
\[
\sum_{(a,b) \in \Omega(A)} m(a,b)m(a,b)^*
\]
is a positive scalar multiple of \( (1 - e_A) \). Now when \( A \) is odd, then \( S = A \) and \( T = \{1\} \), so Lemma 5.7 yields
\[
\sum_{(a,b) \in \Omega(A)} m(a,b)m(a,b)^* = |A|(|A| - 3)(1 - e_A).
\]
When \( A \) is elementary 2-abelian, then \( S = \{1\} \) and \( T = A \), and we get
\[
\sum_{(a,b) \in \Omega(A)} m(a,b)m(a,b)^* = |A|(|A| - 2)(1 - e_A).
\]
But in all other cases, we have \( \{1\} < S, T < A \), and the right hand side in Lemma 5.7 is not a scalar multiple of \( (1 - e_A) \).

To show that \( \mathbb{L}(A) \) is eutactic in general, the strategy is to express the annoying term \( 4|A|(1 - e_S) + 8|T|(1 - e_T) \) in Lemma 5.7 as \( \mathbb{R} \)-linear combinations of some of the elements \( m(a,b)m(a,b)^* \), with coefficients strictly smaller than 1. By subtracting, we get an expression of \( 4|A|(|A| - 4)(1 - e_A) \) as a positive combination of the elements \( m(a,b)m(a,b)^* \), as required by Lemma 5.6.

**5.9 Lemma.** Let \( B < A \) be a subgroup and \( g \in A \setminus B \). Write \( m(a,b) = (1 - a)(1 - b) \). Then
\[
\sum_{a,b \in B} m(a, bg)m(a, bg)^* = 4|B|^2(1 - e_B)
\]

**Proof.** We have
\[
\sum_{a,b \in B} m(a, bg)m(a, bg)^* = \sum_{a,b \in B} (1 - a)(1 - bg)(1 - g^{-1}b^{-1})(1 - a^{-1})
= \sum_{a \in B} (2 - a - a^{-1}) \sum_{b \in B} (2 - bg - g^{-1}b^{-1})
= 2|B|(1 - e_B)|B|(2 - e_Bg - g^{-1}e_B)
= 2|B|^2(2 - e_Bg - g^{-1}e_B - 2e_B + e_Bg + g^{-1}e_B)
= 4|B|^2(1 - e_B). \]
5.10 Theorem. For each abelian group $A$ not isomorphic to $C_4$, the lattice $\mathbb{L}(A)$ is eutactic in its span $\text{span}_\mathbb{R}(\mathbb{L}(A))$.

Proof. The cases where $|A| \leq 4$ have already been dealt with, so we may assume that $|A| > 4$, and $\text{MinVecs}(\mathbb{L}(A)) = S(A)$. By Corollary 5.8, we can assume that $A$ is neither odd nor elementary 2-abelian, and thus both $S$ and $T$ from Lemma 5.7 are proper subgroups of $A$.

Let $B < A$ be a proper subgroup and suppose $1 \neq b \in B$ and $a \in A \setminus B$. Then $(a, b), (b, a) \in \Omega(A)$, with $\Omega(A)$ as above and defined in Lemma 3.2. It follows from Lemma 5.9 that

$$\sum_{a \in A \setminus B, b \in B} m(a, b)m(a, b)^* = \sum_{a \in A \setminus B, b \in B} m(b, a)m(b, a)^* = (|A : B| - 1)|B|^2(1 - e_B).$$

It follows that (with notation as in Lemma 5.7) we can write

$$4|A|(1 - e_S) + 8|T|(1 - e_T) = \frac{4|A|}{8|S|^2(|A : S| - 1)} \sum_{(a, b) \in S \times (A \setminus S) \cup (A \setminus S) \times S} m(a, b)m(a, b)^* + \frac{8|T|}{8|T|^2(|A : T| - 1)} \sum_{(a, b) \in T \times (A \setminus T) \cup (A \setminus T) \times T} m(a, b)m(a, b)^*.$$

Here, an element $m(a, b)m(a, b)^*$ appears with a coefficient at most

$$\frac{4|A|}{8|S|^2(|A : S| - 1)} + \frac{8|T|}{8|T|^2(|A : T| - 1)} \leq \frac{1}{|S|} + \frac{1}{|T|} < 1,$$

since $|S||T| = |A| > 4$. Now it follows from Lemma 5.7 that we can write

$$4|A|(|A| - 4)(1 - e_A) = \sum_{(a, b) \in \Omega(A)} \gamma_{a, b} m(a, b)m(a, b)^*, \quad 0 < \gamma_{a, b} \leq 1.$$

Then, by Lemma 5.6, we get that $S(A)$ is eutactic, which was to be proved. \qed

R. Bacher [1, § 5] has shown that $\mathbb{L}(A)$ is perfect for $|A| \geq 7$, except when $A \cong C_4 \times C_2$. Together with Voronoi’s classical criterion, we get:

5.11 Corollary. Let $A$ be an abelian group with $|A| \geq 7$ and $A \not\cong C_4 \times C_2$. Then $\mathbb{L}(A)$ is extreme.
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