Sub-barrier fusion reactions

K. Hagino

Abstract The concept of compound nucleus was proposed by Niels Bohr in 1936 to explain narrow resonances observed in scattering of a slow neutron off atomic nuclei. A compound nucleus is a metastable state with a long lifetime, in which all the degrees of freedom are in a sort of thermal equilibrium. Fusion reactions are defined as reactions to form such compound nucleus by merging two atomic nuclei. Here a short description of heavy-ion fusion reactions at energies close the Coulomb barrier is presented. This includes: (i) an overview of a fusion process, (ii) a strong interplay between nuclear structure and fusion, (iii) fusion and multi-dimensional/multi-particle quantum tunneling, and (iv) fusion for superheavy elements.

Introduction

A general introduction to heavy-ion fusion reactions

A fusion reaction is defined as a reaction to form a compound nucleus, the concept of which was originally proposed by Niels Bohr in 1936 [1]. In the year earlier, Enrico Fermi performed experiments with slow neutrons and observed many narrow resonances in scattering cross sections. The width of those resonances was typically order of a few eV (see e.g., Ref. [2]), which is much smaller than a typical nuclear scale of order of MeV. This implies that the resonances formed by reactions of slow neutrons are very long-lived. Bohr considered that the energy of the incident neutron was distributed among the other nucleons in a nucleus after many collisions, and a kind of thermal equilibrium state was formed. This is the concept of compound nucleus which Bohr proposed. Since a nucleus is a finite system, the energy may
be concentrated once again to one of the neutrons in a nucleus, and that neutron is scattered off from a nucleus. This happens only at a long time after the compound nucleus is formed, leading to narrow resonance widths.

Similar compound nuclei are formed in heavy-ion fusion reactions by bombarding two heavy nuclei. Such fusion reaction plays an important role in several phenomena in nuclear physics, such as the energy production in stars, nucleosyntheses, and formations of superheavy elements. Theoretically, fusion, as well as fission, are large amplitudes motions of quantum many-body systems with a strong interaction, and their microscopic understanding is one of the ultimate goals of nuclear physics [3].

Figure 1 shows a schematic illustration of a fusion process. At first a projectile nucleus ("P") collides with a target nucleus ("T") and forms a unified nucleus ("P+T"), i.e., a compound nucleus. Since the projectile nucleus brings the energy and the angular momentum into a system, the compound nucleus is at high excitation energies with large angular momenta. For light compound nucleus with the mass number less than about 170, the compound nucleus decays mainly by emitting neutrons, protons, alpha particles, and gamma rays. Such process is called an evaporation process. For heavy compound nucleus with the mass number larger than about 220, fission dominates the decay process of the compound nucleus. This is particularly the case for fusion reactions relevant to superheavy nuclei. For compound nuclei with intermediate mass numbers, the evaporation and the fission processes compete with each other. In any case, fusion cross sections are measured by detecting residues of the evaporation process (called "evaporation residues") and/or fission fragments.

Figure 2 shows a typical potential between two nuclei as a function of the distance $r$ between them. Two different interactions are involved here. Firstly, a nucleus
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Fig. 2 An internucleus potential between two nuclei as a function of the distance between them (the solid line). The $^{16}$O+$^{144}$Sm reaction is considered as a typical example. The Coulomb and the nuclear contributions are denoted by the dotted and the dashed lines, respectively. $R_b$ and $V_b$ denote the position and the height of the Coulomb barrier, respectively. $R_{\text{touch}}$ is the distance at which two nuclei touch with each other.

has a positive charge, and the Coulomb interaction acts between two nuclei. This is a long-range and repulsive interaction. When the distance between the two nuclei gets smaller, an attractive short range nuclear interaction (i.e., the strong interaction) becomes active. Because of the cancellation of these two, a potential barrier, referred to as the Coulomb barrier, is formed at some distance $R_b$, which is usually larger than the touching radius $R_{\text{touch}}$ at which the two nuclei touch with each other. The height of the Coulomb barrier, $V_b$, specifies the energy scale of the reaction system.

In this article, we overview the fusion dynamics at energies around the Coulomb barrier, that is, subbarrier fusion reactions. There are two obvious reasons why the subbarrier region is important. One is related to fusion reactions to form superheavy elements. Usually such experiments are carried out at energies slightly above the Coulomb barrier. For instance, in the $^{209}$Bi($^{70}$Zn,n)$^{278}$Nh reaction to form the element 113 (Nihonium), the experiments were performed at $E_{\text{c.m.}} = 262$ MeV in the center of mass frame [4, 5, 6], while a barrier height for this reaction is around 260 MeV if the Bass potential [7] is employed. Fusion reactions for superheavy elements will be further discussed in a later section in this article. The second obvious reason to discuss the subbarrier region is in connection to nuclear astrophysical reactions. Nuclear fusion reactions in stars, such as the $^{12}$C+$^{12}$C reaction, take place at extremely low energies, for which direct measurements of fusion cross sections are difficult. One thus has to extrapolate measured fusion cross sections at higher
energies down to the energy region relevant to nuclear astrophysics. In order to do reliable extrapolations, deep understandings of the fusion dynamics in the subbarrier region are crucially important.

Besides these two obvious reasons, the reaction dynamics of subbarrier fusion is intriguing in its own. Firstly, it is known that nuclear structure affects significantly nuclear fusion, and thus there is a strong interplay between nuclear structure and nuclear reaction there. This is in contrast to high energy nuclear reactions, at which the reaction dynamics is much simpler. Secondly, subbarrier fusion reactions can be regarded as a typical example of many-particle tunneling phenomena. In order for fusion to take place, two nuclei have to get close at least to the touching radius, and thus fusion occurs only by quantum tunneling effect when the incident energy is below the Coulomb barrier (see Fig. 2). An interesting fact in atomic nuclei is that there are many types of intrinsic degrees of freedom which can affect quantum tunneling, that is, there are several types of collective vibrations as well as nuclear deformation with several multipolarities. Moreover, the energy dependence of the tunneling rate can be studied in fusion reactions by varying the incident energy, in a marked contrast to other tunneling phenomena in nuclear physics, such as alpha decays, for which the energy is basically fixed by the decay $Q$-value. Heavy-ion fusion reactions can be thus considered as an ideal playground to study many-particle quantum tunneling with many degrees of freedom.

**Earlier review articles and textbooks**

A few review articles have been published on the subject of subbarrier fusion reactions. While Refs. [8, 9] discuss theoretical aspects of subbarrier fusion reactions, Ref. [10] summarizes experimental observations in subbarrier fusion reactions from a viewpoint of the so called fusion barrier distributions. Refs. [11, 12, 13] discuss heavy-ion fusion reactions at deep subbarrier energies, at which fusion cross sections appear to be hindered as compared to simple extrapolations of fusion cross sections at subbarrier energies. Subbarrier fusion reactions are discussed also in many textbooks of nuclear reactions, see e.g. Refs. [14, 15, 16, 17].

**Potential Model**

**Potential model and the Wong formula**

The simplest approach to fusion reactions is to employ the potential model, in which one considers inert projectile and target nuclei and assumes some potential between them. For fusion reactions of medium-heavy nuclei, it is considered to be a good approximation to assume that a compound nucleus is formed automatically once
the touching position is achieved. Fusion cross sections $\sigma_{\text{fus}}(E)$ are then given by

$$\sigma_{\text{fus}}(E) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l(E),$$

(1)

where $E$ is the bombarding energy in the center of mass frame and $k = \sqrt{2\mu E/\hbar^2}$ is the wave number for the relative motion between the two nuclei with the reduced mass $\mu$. $l$ is the orbital angular momentum for the relative motion, and $P_l(E)$ is the probability to reach the touching configuration. The factor $2l + 1$ is simply a statistical weight coming from the fact that the probability $P_l$ does not depend on the $z$-component of $l$. Notice that $P_l(E)$ is nothing but the penetrability of the Coulomb barrier. This can be evaluated e.g., by adding a short range absorbing potential to an internucleus potential. Such absorbing potential in general describes any process besides elastic scattering, but to a good approximation it simulates the compound nucleus formation as long as it is well confined inside the Coulomb barrier.

Based on this approach, Wong has derived a simple compact formula for fusion cross sections [18] (see also Ref. [19]). To this end, he first approximated the Coulomb barrier by an inverted parabola,

$$V(r) \sim V_b - \frac{1}{2}\mu \Omega^2 (r - R_b)^2,$$

(2)

for which the penetrability can be given analytically as

$$P_0(E) = \frac{1}{1 + \exp \left[ \frac{2\mu}{\hbar^2} (V_b - E) \right]}.$$

(3)

For non-zero partial waves, he considered $l$-independent barrier position and curvature, and replaced $P_l(E)$ with

$$P_l(E) \sim P_0 \left( E - \frac{l(l + 1)\hbar^2}{2\mu R_b^2} \right).$$

(4)

Finally, Wong replaced the sum in Eq. (1) by an integral

$$\sigma_{\text{fus}}(E) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l(E) \to \frac{\pi}{k^2} \int dl (2l + 1) P(l, E)$$

(5)

to obtain the so called Wong formula given by,

$$\sigma_{\text{fus}}(E) = \frac{\hbar \Omega}{2E} R_b^2 \ln \left[ 1 + \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \right].$$

(6)

Notice that the first energy derivative of $E \sigma_{\text{fus}}$ from this formula is proportional to the $s$-wave penetrability of the Coulomb barrier,
Fig. 3 A comparison of the fusion cross sections obtained with the Wong formula (the dashed lines) with the exact fusion cross sections (the solid lines) for the $^{16}\text{O} + ^{144}\text{Sm}$ system. The left and the right panels show the results with the linear and the logarithmic scales, respectively.

\[
\frac{d}{dE} [E\sigma_{\text{fus}}(E)] = \pi R_b^2 P_0(E),
\]

where $P_0(E)$ is given by Eq. (3).

Figure 3 shows calculated fusion cross sections for the $^{16}\text{O} + ^{144}\text{Sm}$ system. The dashed lines are obtained with the Wong formula, while the solid lines are obtained by numerically solving the Schrödinger equation for each partial wave to obtain $P_l(E)$. The left and the right panels show the results in the linear and the logarithmic scales, respectively. One can see that the Wong formula works well, except for the region well below the Coulomb barrier (the height of the Coulomb barrier is about $V_b = 61.25$ MeV for this system, see Fig. 2), at which the Wong formula overestimates the fusion cross sections. The overestimate of fusion cross sections at energies well below the barrier is because the parabolic approximation (2) used in the Wong formula underestimates the width of the potential barrier, which results in the overestimate of the penetrabilities.

**Comparisons with experimental data**

Figure 4 compares fusion cross sections obtained with the potential model to the experimental data for the $^{14}\text{N} + ^{12}\text{C}$ (the left panel) and the $^{16}\text{O} + ^{154}\text{Sm}$ (the right panel) systems. The height of the Coulomb barrier is around $V_b \sim 6.9$ MeV for the $^{14}\text{N} + ^{12}\text{C}$ system and $V_b \sim 59$ MeV for the $^{16}\text{O} + ^{154}\text{Sm}$ system. For the $^{14}\text{N} + ^{12}\text{C}$ system, one can see that the potential model works well. On the other hand, for the $^{16}\text{O} + ^{154}\text{Sm}$ system, the potential model largely underestimates the fusion cross sections at energies below the Coulomb barrier, even though it works well at energies above the barrier. This phenomenon is referred to as the subbarrier enhancement of
Fusion cross sections for the $^{14}\text{N}+^{12}\text{C}$ system (the left panel) and for the $^{16}\text{O}+^{154}\text{Sm}$ system (the right panel) obtained with the potential model. The arrows indicate the height of the Coulomb barrier for each system. The experimental data are taken from Refs. [22, 23].

The spectrum of the $^{154}\text{Sm}$ nucleus. Each level is specified by its angular momentum $I$ and parity $\pi$ as $I^\pi$.

Fusion of deformed nuclei

The subbarrier enhancement of fusion cross sections for the $^{16}\text{O}+^{154}\text{Sm}$ system shown in Fig. 4 can be easily explained if one notices that the $^{154}\text{Sm}$ nucleus is
Fig. 6  A schematic view of nuclear deformation, see also Ref. [24].

a typical deformed nucleus. This nucleus exhibits characteristic rotational excitations, for which the energy of a state with the angular momentum \( I \) is proportional to \( I(I+1) \) (see Fig. 5). This is interpreted as that \(^{154}\text{Sm}\) is statically deformed in the ground state (see Fig. 6). For axially symmetric shape, the nuclear deformation is often characterized by the deformation parameters \( \{ \beta_\lambda \} \) defined as

\[
R(\theta) = R_0 \left( 1 + \sum_\lambda \beta_\lambda Y_{\lambda 0}(\theta) \right),
\]

where \( R(\theta) \) is the angle-dependent radius of a nucleus, \( R_0 \) is the radius in the spherical limit, \( \theta \) is the angle measured from the symmetric axis, and \( Y_{\lambda 0}(\theta) \) is the spherical harmonics (see the lower figure of Fig. 6). For the \(^{154}\text{Sm}\) nucleus, the deformation parameters are \( \beta_2 \sim 0.30 \) for \( \lambda=2 \) (the quadrupole deformation) and \( \beta_4 \sim 0.05 \) for \( \lambda=4 \) (the hexadecapole deformation) \([23, 25]\).

When a target nucleus is deformed, the internucleus potential depends on the orientation angle of the deformed nucleus. When the projectile nucleus approaches the target nucleus from the direction of the longer axis of the target, the Coulomb barrier is lowered as compared to the potential in the spherical limit. This is because the nuclear attraction acts from longer distances. The lowering of a barrier results in an enhancement of the penetrability. For a prolately deformed nucleus with \( \beta_2 > 0 \), this corresponds to the angle \( \theta = 0 \) (see Fig. 7). The opposite happens when
the projectile approaches from the direction of the shorter axis of the target. For a
prolately deformed nucleus with $\beta_2 > 0$, this corresponds to the angle $\theta = \pi/2$.

The total penetrability is computed by averaging the angle-dependent penetrabil-
ity as

$$ P(E) = \int_0^1 d(\cos \theta) P_0(E; \theta), $$

where $P_0(E; \theta)$ is the penetrability for the orientation angle $\theta$. The thick solid line in
the left panel of Fig. 8 is obtained in this way. Since the tunneling probability has an
exponential dependence on the energy, the enhancement of the penetrability due to
$\theta \sim 0$ leads to the main contribution to the total penetrability at energies below the
barrier. The total penetrability is thus enhanced at these energies as compared to the
penetrabilities in the spherical limit. This is the main mechanism for the subbarrier
fusion cross sections shown in Fig. 4. The formula (9) can be actually extended to
fusion cross sections $\sigma_{\text{fus}}$ as,

$$ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta) \sigma_{\text{fus}}^{(0)}(E; \theta). $$

The solid line in the right panel of Fig. 8 is obtained in this way. One can see that the
subbarrier enhancement of fusion cross sections for this system is well accounted for
by taking into account the deformation of the $^{154}$Sm nucleus. This clearly demon-
strates that there is a strong interplay between nuclear structure and heavy-ion fusion
reactions at subbarrier energies.
Fig. 8  (The left panel) The same as the right panel of Fig. 7 but with the penetrability obtained by averaging all the orientation angles (the thick solid line). (The right panel) The same as the right panel of Fig. 4 but with the fusion cross sections obtained by averaging all the orientation angles (the solid line). The fusion cross sections are plotted as a function of energy relative to the height of the Coulomb barrier.

Coupled-channels method: a quantal scattering theory with excitations

Coupled-channels approach

The subbarrier fusion enhancement discussed in the previous section has been observed also in systems with non-deformed target nuclei. It has been understood by now that the subbarrier fusion enhancement is caused by couplings of the relative motion between colliding nuclei to several low-lying collective excitations in the nuclei as well as particle transfer processes \[8, 9, 10\]. The deformation effect discussed...
in the previous section is a special case, in which the rotational excitations due to a nuclear deformation can be taken into account in terms of orientation-dependent internucleus potential \[9\]. In order to take into account such coupling effects, the coupled-channels approach has been developed \[9, 26, 27, 28, 29\]. This is a quantal reaction theory schematically illustrated in Fig. \[9\]. In this figure, couplings of the relative motion to a state in the target nucleus with the angular momentum \(I = 2\) and positive parity \(\pi = +\) are considered. At the initial stage of the reaction, the target nucleus is in the ground state, \(I^\pi = 0^+\). The wave function for the relative motion for this configuration is denoted by \(\psi_0(r)\). During the reactions, due to the interaction between the projectile and the target nuclei, the target nucleus may be excited to the \(I^\pi = 2^+\) state, and at the same time, the relative wave function is changed to \(\psi_1(r)\). The coupling is taken into account by the off-diagonal components of the potential \(V(r)\). The \(I^\pi = 0^+\) state may be de-excited to the \(0^+\) state, and thus the two wave functions \(\psi_0(r)\) and \(\psi_1(r)\) are coupled to each other. One then solves in a non-perturbative manner coupled Schrödinger equations, which are referred to as coupled-channels equations, to determine the \(S\)-matrix, from which several reaction observables can be constructed.

A few computer codes are available for coupled-channels calculations, such as \textsc{ecis} \[31, 32\], \textsc{fresco} \[16, 33\], and \textsc{ccfull} \[34\]. As an example, Fig. \[10\] shows fusion cross sections for the \(^{58}\text{Ni}+^{58}\text{Ni}\) system calculated with the code \textsc{ccfull}. Here, the quadrupole excitations up to the double phonon states are taken into account in each of the \(^{58}\text{Ni}\) nucleus. By taking into account the excitations of the \(^{58}\text{Ni}\) nuclei, the subbarrier enhancement of fusion cross sections is well reproduced. One may regard this as a clear example of coupling assisted tunneling phenomenon.
It is instructive to discuss how the subbarrier enhancement of fusion cross sections is realized using a schematic model. To this end, let us consider a two-channel problem for scattering in one-dimension \[35, 36\] and solve the coupled-channels equations in a form of

\[
\begin{bmatrix}
  \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + (V(x) F(x) V(x) + \varepsilon) & u_0(x) \\
  F(x) V(x) + \varepsilon & u_1(x)
\end{bmatrix} = 0.
\]  

Here, \(V(x)\) describes a potential barrier and \(F(x)\) denotes the coupling potential between the two channels. \(\varepsilon\) is the energy of the excited state relative to the ground state. Assuming that the particle is incident from the right hand side of the potential barrier, these equations are solved with the boundary conditions of

\[
u_0(x) \rightarrow e^{-ik_0x} - R_0 e^{ik_0x} \quad (x \rightarrow \infty),
\]

\[
u_0(x) \rightarrow T_0 e^{-ik_0x} \quad (x \rightarrow -\infty),
\]

and

\[
u_1(x) \rightarrow \sqrt{k_0/k_1} R_1 e^{ik_1x} \quad (x \rightarrow \infty),
\]

\[
u_1(x) \rightarrow \sqrt{k_0/k_1} T_1 e^{-ik_1x} \quad (x \rightarrow -\infty),
\]

with \(k_0 = \sqrt{2mE/\hbar^2}\) and \(k_1 = \sqrt{2m(E-\varepsilon)/\hbar^2}\). The penetrability of the barrier is then given by

\[
P(E) = |T_0|^2 + |T_1|^2.
\]

The upper panel of Fig. 11 shows the penetrability \(P(E)\) so obtained with a Gaussian barrier given by \(V(x) = V_0 e^{-x^2/2\sigma^2}\). The coupling potential is also assumed to have a Gaussian form, \(F(x) = F_0 e^{-x^2/2\sigma^2}\). The parameters are set to be \(V_0 = 100\ MeV\), \(\sigma = 3\ fm\), and \(F_0 = 3\ MeV\), together with \(\varepsilon = 1\ MeV\) and \(m = 29 \times 938\ MeV/c^2\). The dashed line shows the result without the coupling (i.e., the case with \(F_0 = 0\)), while the solid line is obtained by solving the coupled-channels equations. As in the subbarrier fusion enhancement, one can see that the penetrability is enhanced at energies below the barrier.

When the excitation energy \(\varepsilon\) is zero, the coupled-channels equations \[11\] can be transformed to two decoupled equations,

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \pm F(x) - E \right] u_{\pm}(x) = 0,
\]

with \(u_{\pm}(x) = (u_0(x) \pm u_1(x))/\sqrt{2}\). That is, the wave functions \(u_{\pm}(x)\) are governed by the potentials \(V(x) \pm F(x)\), one of which lowers the barrier and the other raises the barrier. The penetrability is then given by

\[
P(E) = |T_0|^2 + |T_1|^2.
\]
Fig. 11 (The upper panel) The penetrability of a one-dimensional Gaussian barrier in the presence of the channel coupling effects. The height of the barrier is set to be 100 MeV. The dotted line denotes the result without the channel coupling, while the solid line shows the result of the coupled-channels calculations. (The lower panel) The energy derivative of the penetrability shown in the upper panel.

\[
P(E) = \frac{1}{2} \left[ P_0(E; V(x) + F(x)) + P_0(E; V(x) - F(x)) \right],
\]

where \( P_0(E; V(x)) \) is the penetrability for a potential barrier \( V(x) \). Similar to fusion of deformed nuclei, the total penetrability is enhanced as compared to the penetrability for the no-coupling case, because of the contribution of the lowered barrier is much more significant than the contribution of the higher barrier. This remains the same even with a finite excitation energy, \( \varepsilon \) [37].

**Fusion Barrier Distributions**

Eq. (18) can be generalized to cases with more than two barriers as,

\[
P(E) = \sum_\alpha w_{\alpha} P_0(E; V_\alpha(x)),
\]
where \( \alpha \) denotes “eigen-channels” with the potential \( V_\alpha(x) \) and \( w_\alpha \) is the weight factor for each eigenchannel. That is, the penetrability is given as a weighted sum of the penetrability for each eigenchannel \( \alpha \). In this case, a single barrier is replaced by a set of distributed barrier. The corresponding formula for fusion cross sections reads
\[
\sigma_{\text{fus}}(E) = \sum_\alpha w_\alpha \sigma_{\text{fus}}^{(0)}(E; V_\alpha(r)).
\] (20)

In the case of fusion of deformed nuclei, Eq. (10), the eigen-channel \( \alpha \) corresponds to the orientation angle \( \theta \) with the weight factor \( w_\theta = 2\pi \sin \theta \).

Since the penetrability \( P \) varies from zero to one at energies around the barrier height, its energy derivative shows a Gaussian-like peak centered at the barrier height energy (see the dashed line in the lower panel of Fig. 11). This implies that the energy derivative of Eq. (19) shows many peaks centered at the barrier height for each eigenchannel, and that the height of each peak is proportional to the corresponding weight factor, \( w_\alpha \). This is demonstrated in the lower panel of Fig. 11 for a 2-channel case.

Noticing the relation given by Eq. (7), one finds that the corresponding quantity for fusion cross sections is the second energy derivative of \( E \sigma_{\text{fus}}(E) \) given by
\[
D_{\text{fus}}(E) = \frac{d^2(E \sigma_{\text{fus}}(E))}{dE^2}.
\] (21)

---

1 For a classical penetrability, \( P(E) = \theta(E - V_b) \), the energy derivative is given by a delta function, \( dP/dE = \delta(E - V_b) \).
This quantity is referred to as the fusion barrier distribution [10] [38], and has been experimentally extracted for several systems [10] [23]. Similar barrier distributions have been extracted using also quasi-elastic scattering (that is, a sum of elastic, inelastic, and transfer processes) at backward angles [39] [40]. The fusion barrier distribution converts the exponential behavior of fusion excitation functions to the linear scale, and is suitable to visualize details of the underlying dynamics of subbarrier fusion reactions. Fig. [12] shows the barrier distribution for the $^{16}\text{O}+^{154}\text{Sm}$ system as an example. The solid line shows the barrier distribution obtained with Eq. (10), while the dashed lines show the contribution of different orientation angles. The fusion barrier distribution is structured because of the distribution of many barriers. It has been shown that the shape of the fusion barrier distribution is sensitive to the deformation parameters used in the calculation [10] [23].

Using such sensitivity of the barrier distribution to the deformation parameters, the quadrupole and the hexadecapole deformation parameters of the $^{24}\text{Mg}$ nucleus have been extracted recently [41]. To this end, the barrier distribution extracted from the quasi-elastic scattering for the $^{24}\text{Mg}+^{90}\text{Zr}$ system was analyzed with the Bayesian statistics. From this analysis, the hexadecapole deformation parameter of $^{24}\text{Mg}$, $\beta_4 = -0.11 \pm 0.02$, has been determined precisely for the first time (see Fig. [13]).

**Deep Subbarrier Fusion Hindrance**

At energies well below the Coulomb barrier, that is, at $E \ll V_b$, the Wong formula (6) is reduced to
That is, fusion cross sections fall off exponentially. This has been generally observed experimentally. However, as the energy decreases further down, it has been systematically observed that fusion cross sections fall off much steeper [42, 11, 13]. This phenomenon has been referred to as deep subbarrier fusion hindrance. It has been considered that the hindrance is attributed to the dynamics after two colliding nuclei touch with each other [43, 13].

As an example, Fig. 14 shows the fusion cross sections for the $^{64}$Ni+$^{64}$Ni system. The results of the standard coupled-channels calculations are denoted by the dashed and the dot-dashed lines. These calculations well reproduce the experimental data at energies larger than about 89 MeV (see the arrow). However, at lower energies the experimental data show hindrance as compared to the standard coupled-channels calculation and fall off much steeper. The solid line models the deep subbarrier hindrance by quenching the coupling strengths after two nuclei touch each other [44] (see Ref. [45] for another modelling of deep subbarrier fusion hindrance, which introduces a repulsive core to an internucleus potential). This calculation well accounts for the data, clearly indicating an importance of the dynamics of a transition from a two-body system with two separate nuclei to a one-body mono-nuclear system after the touching. See Ref. [13] for a recent review article on this topic.
Fig. 15. A schematic illustration of fusion dynamics in the presence of breakup of the projectile. CF and ICF refer to the complete fusion and the incomplete fusion, respectively. The complete fusion is further subdivided into the direct complete fusion (DCF) and the sequential complete fusion (SCF) processes. Taken from Ref. [46].

**Fusion of Neutron-Rich Nuclei**

One of the main research fields in modern nuclear physics is physics of unstable nuclei, especially neutron-rich nuclei far from the stability line. Those nuclei are weakly bound and are characterized by a spatially extended density distribution. It is likely that excited states of those nuclei are in the continuum spectrum and thus the breakup process plays an important role when such nuclei are used either as a projectile or as a target in nuclear reactions.

In fusion of weakly-bound nuclei, several effects may interplay with each other. Those are:

1. a lowering of the Coulomb barrier due to the extended density distribution [47],
2. the breakup process, which may hinder fusion cross sections since the lowering of the Coulomb barrier disappears. At the same time, it may also enhance fusion cross sections if couplings to a breakup channel dynamically lowers the Coulomb barrier [48, 49, 50],
3. the transfer processes. Since the $Q$-value is positive for neutron-rich nuclei, it may significantly affect the fusion process [51, 52].
Furthermore, the breakup process significantly complicates the reaction dynamics of complete fusion and incomplete fusion in a non-trivial way (see Fig. 15). Here, the complete fusion is the process in which all the breakup fragments are absorbed by a target nucleus, while the incomplete fusion refers to the process in which only a part of the breakup fragments is absorbed. A theoretical model which coherently incorporates all of these effects has still yet to be developed, even though the continuum discretized coupled-channels (CDCC) method has been developed for the breakup process [53] (notice also that there have been recent developments in theoretical descriptions of inclusive breakup processes [54, 55, 56, 57]). See Refs. [26, 46, 58] for review articles on fusion of weakly bound nuclei. It is worth noticing that fusion of neutron-rich nuclei is important for nuclear astrophysics [59, 60] as well as for superheavy elements [61].

**Fusion Reactions for Superheavy Nuclei**

*Superheavy Nuclei*

The elements heavier than plutonium (the atomic number $Z = 94$) are all unstable and do not exist in nature. Yet, one can artificially synthesize them using nuclear reactions. There have been continuous efforts since the 1950s, and the elements up to $Z = 118$ have been synthesized by now. The transactinide elements, that is, the elements with $Z \geq 104$, are referred to as superheavy elements and have attracted lots of attention in recent years [62, 63, 64, 65, 66, 67, 68].

One of the main motivations to study superheavy elements, in addition to synthesizing new elements, is to explore the island of stability, which was theoretically predicted some 50 years ago [69, 70]. While heavy nuclei in the transactinide region are unstable against alpha decay and spontaneous fission, the shell effect due to magic numbers can stabilize a certain number of nuclei in that region. The predicted proton and neutron magic numbers are $Z = 114$ and $N = 184$, respectively [69, 70]. The region around these magic numbers is referred to as the island of stability, where nuclei may have a life time as long as $10^3$ years [71]. More modern Hartree-Fock calculations have also predicted $(Z, N) = (114, 184), (120, 172),$ and $(126, 184)$ for candidates for the next double magic nucleus beyond $^{208}$Pb [72].

The island of stability has not yet been reached experimentally. In fact, the heaviest FI element ($Z = 114$) synthesized so far is $^{289}_{175}$Fl [73], which is 9 neutrons less from the the predicted magic number, $N = 184$. This implies that neutron-rich beams are indispensable in order to reach the island of stability. An experimental technique has yet to be developed to deal with low intensity of such beams.
Heavy-Ion Fusion Reactions for Superheavy Nuclei

Heavy-ion fusion reactions at energies around the Coulomb barrier have been used as a standard tool to synthesize those superheavy elements [62, 63]. Figure 16 schematically illustrates fusion reactions to form superheavy nuclei (see also Fig. 1). In the first phase of reaction, two nuclei approach to each other to reach the touching configuration after the Coulomb barrier is overcome. A compound nucleus is formed almost automatically for medium-heavy systems once the touching configuration is achieved. In contrast, in the superheavy region, there is a huge probability for the touching configuration to reseparate due to a strong Coulomb repulsion between the two nuclei. This process is referred to as quasi-fission. Furthermore, even if a compound nucleus is formed with a small probability, it decays most likely by fission, again due to the strong Coulomb interaction. For heavy systems, quasi-fission characteristics significantly overlap with fission of the compound nucleus, and a detection of fission events itself does not guarantee a formation of the compound nucleus. Therefore, a formation of superheavy elements has been identified by measuring evaporation residues. These are extremely rare events, in which a compound nucleus is survived against fission.

As an example, Fig. 17 shows the measured cross sections for the $^{48}$Ca+$^{238}$U reaction forming the Cn ($Z = 112$) element. The filled circles show the capture cross sections [74] to form the touching configuration shown in Fig. 16. On the other hand, the filled squares and triangles denote the evaporation residue cross sections for emissions of 3 and 4 neutrons, respectively [75, 76]. One can observe that the evaporation residue cross sections are indeed much smaller than the capture cross sections, by about 11 orders of magnitude.
Fig. 17 The experimental evaporation residue cross sections for the $^{48}\text{Ca} + ^{238}\text{U}$ reaction leading to the formation of Cn ($Z = 112$) element. The filled circles denote the capture cross sections to form the touching configuration. The filled squares and triangles show the evaporation residue cross sections, for which the former and the latter correspond to the $3n$ (emission of 3 neutrons) and the $4n$ (emission of 4 neutrons) channels, respectively.

Theoretical modelings

Based on the time-scale of each process, the formation process of evaporation residues can be conceptually divided into a sequence of the following three processes (see Fig. 16). The first phase is a process in which two separate nuclei form the touching configuration after overcoming the Coulomb barrier. After two nuclei touch with each other, a huge number of nuclear intrinsic motions are activated and the energy for the relative motion of the colliding nuclei is quickly dissipated to internal energies. Because of the strong Coulomb interaction, the touching configuration appears outside a fission barrier, which has to be thermally activated to form a compound nucleus against a severe competition to the quasi-fission process. The Langevin approach has often been used to describe this process [77, 78, 79, 80, 81, 82]. The third process is a statistical decay of the compound nucleus [83], with strong competitions between fission and particle emissions (i.e., evaporations). Here, the fission barrier height is one of the most important parameters which significantly affect evaporation residue cross sections [84].
For a given partial wave \( l \), the probability for a formation of an evaporation residue is given as a product of the probability for each of the three processes, \( P_l \), \( P_{CN} \), and \( W_{sur} \), that is,

\[
P_{ER}(E, l) = P_l(E)P_{CN}(E, l)W_{sur}(E^*, l),
\]

where \( E \) and \( E^* \) are the bombarding energy in the center of mass frame and the excitation energy of the compound nucleus, respectively. Cross sections for a formation of evaporation residues are then given by

\[
\sigma_{ER}(E) = \frac{\pi}{k^2} \sum_l (2l + 1)P_l(E)P_{CN}(E, l)W_{sur}(E^*, l).
\]

For medium-heavy systems, the probability for the second phase, \( P_{CN} \), is almost unity, and Eq. (24) is reduced to Eq. (1) when fission cross sections are added to it. In contrast, for superheavy nuclei, \( P_{CN} \) is significantly smaller than unity. As has been mentioned, this quantity cannot be determined experimentally, which causes large ambiguities in theoretical calculations.
**Hot versus Cold Fusion Reactions**

Since a formation probability of evaporation residues is extremely small, it is important to choose appropriate combinations of the projectile and the target nuclei in order to efficiently synthesize superheavy elements. For this purpose, two different experimental strategies have been employed. In the so called *cold fusion reactions*, $^{208}$Pb and $^{209}$Bi are used for the target nuclei so that the compound nucleus is formed with relatively low excitation energies. The competition between neutron emissions and fission can be minimized, which in turn maximizes $W_{\text{sur}}$ in Eq. (24) [62, 63]. An advantage of this strategy is that alpha decays of the evaporation residues end up in the known region of nuclear chart, and thus superheavy elements can be identified unambiguously. On the other hand, in the so called *hot fusion reactions*, the neutron-rich double magic nucleus $^{48}$Ca has been used as a projectile to optimize the formation probability of the compound nucleus, $P_{\text{CN}}$ [62, 63]. This strategy has been successfully employed by the experimental group at Dubna, led by Oganessian, to synthesize superheavy elements up to $Z = 118$.

Figure 18 shows the measured evaporation residue cross sections for the hot fusion reactions (the filled circles) and for the cold fusion reactions (the open circles). For the cold fusion reactions, the cross sections drop rapidly as a function of $Z$ of the compound nucleus. It would therefore be difficult to go beyond Nihonium using this strategy. In contrast, for the hot fusion reactions, the cross sections remain relatively large between $Z = 113$ and 118. This may be due to the fact that the survival probability, $W_{\text{sur}}$, is increased because the compound nuclei formed are in the proximity of the predicted island of stability [69, 70]. An increase of nuclear dissipation at high temperatures may also play a role [87].

**Role of deformation in hot fusion reactions**

The incident energy for fusion reactions to synthesize superheavy nuclei is usually taken at energies slightly above the Coulomb barrier. This is because the compound nucleus formed has to be as cold as possible, but yet the capture probability, $P_f(E)$, has to be large enough.

In the hot fusion reactions, with the $^{48}$Ca projectile, the corresponding target nuclei are in the actinide region, in which the nuclei are well deformed in the ground state. It has been argued that the collision with $\theta = \pi/2$ (i.e., the “side collision”, see Fig. 7) plays an important role in the hot fusion, since the touching configuration is more compact than that formed with the “tip collision” with $\theta = 0$, and thus the effective barrier height for the diffusion process is low [88, 89].

Recently, barrier distributions for the capture process have been extracted for several hot fusion systems and the notion of compactness has been confirmed experimentally for the first time [90, 92]. As an example, the top panel of Fig. 19 shows the experimental barrier distribution for the $^{48}$Ca+$^{248}$Cm system [90] and its comparison to the coupled-channels calculations which take into account the defor-
Fig. 19 (Upper panel) The barrier distribution for the capture process for the $^{48}\text{Ca}+^{248}\text{Cm}$ system extracted from quasi-elastic scattering at backward angles. The solid line is obtained with the coupled-channels calculation which takes into account the deformation of the $^{248}\text{Cm}$ target. The octupole phonon excitation of $^{48}\text{Ca}$ and a one-neutron transfer process are also taken into account. The dashed line shows the contribution of the side collision with $\theta = \pi/2$. The experimental data are taken from Ref. [90]. (Lower panel) The evaporation residue cross sections for the same system. The experimental data are taken from Ref. [76, 91].

Sub-barrier fusion reactions

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References

1. N. Bohr. Nature, 137:351, 1936.
2. M. Asghar et al. Nucl. Phys., 85:305, 1966.
3. M. Bender et al. J. of Phys. G, 47:113002, 2002.
4. K. Morita et al. J. of Phys. Soc. Jpn., 73:2593, 2004.
5. K. Morita et al. J. of Phys. Soc. Jpn., 76:5001, 2007.
6. K. Morita et al. J. of Phys. Soc. Jpn., 81:3201, 2012.
7. R. Bass. Nuclear Reactions with Heavy-Ions. Springer-Verlag, Berlin, 1980.
8. A. B. Balantekin and N. Takigawa. Rev. Mod. Phys., 70:77, 1998.
9. K. Hagino and N. Takigawa. Prog. Theo. Phys., 128:1061, 2012.
10. M. Dasgupta, D. J. Hinde, N. Rowley, and A. M. Stefanini. Annu. Rev. Nucl. Part. Sci., 48:401, 1998.
11. B. B. Back, H. Esbensen, C. L. Jiang, and K. E. Rehm. Rev. Mod. Phys., 86:317, 2014.
12. G. Montagnoli and A. M. Stefanini. Eur. Phys. J. A, 53:169, 2017.
13. C. L. Jiang, B. B. Back, K. E. Rehm, K. Hagino, G. Montagnoli, and A. M. Stefanini. Eur. Phys. J. A, 57:235, 2021.
14. P. Fröbrich and R. Lipperheide. Theory of Nuclear Reactions. Clarendon Press, Oxford, 1996.
15. C. A. Bertulani and P. Danielewicz. Introduction to Nuclear Reactions. IOP Publishing, Bristol, UK, 2004.
16. I. J. Thompson and F. M. Nunes. Nuclear Reactions for Astrophysics. Cambridge University Press, Cambridge, 2009.
17. L. F. Canto and M. S. Hussein. Scattering Theory of Molecules, Atoms and Nuclei. World Scientific Publishing Co. Pte. Ltd., Singapore, 2013.
18. C. Y. Wong. Phys. Rev. Lett., 31:766, 1973.
19. N. Rowley and K. Hagino. Phys. Rev. C, 91:044617, 2015.
20. M. Beckerman. Phys. Rep., 129:145, 1985.
21. M. Beckerman. Rep. Prog. Phys., 51:1047, 1988.
22. Z. E. Switkowski, R. G. Stokstad, and R. M. Wieland. Nucl. Phys. A, 279:502, 1977.
23. J. R. Leigh, M. Dasgupta, D. J. Hinde, J. C. Mein, C. R. Morton, R. C. Lemmon, J. P. Lestone, J. O. Newton, H. Timmers, J. X. Wei, and N. Rowley. Phys. Rev. C, 52:3151, 1995.
24. T. Naito, S. Endo, K. Hagino, and Y. Tanimura. J. of Phys. B, 54:165201, 2021.
25. J. R. Leigh, N. Rowley, R. C. Lemmon, D. J. Hinde, J. O. Newton, J. X. Wei, J. C. Mein, C. R. Morton, S. Kayucak, and A. T. Kruppa. Phys. Rev. C, 47:R437, 1993.
26. K. Hagino, K. Ogata, and A. M. Moro. to be published, 2021.
27. T. Tamura. Rev. Mod. Phys., 37:679, 1965.
28. G. R. Satchler. Direct Nuclear Reactions. Clarendon Press, Oxford, 1983.
29. R. A. Broglia and A. Winther. Heavy-ion Reactions. Westview Press, Cambridge, MA, 2004.
30. M. Beckerman, J. Ball, H. Enge, M. Salomaa, A. Sperduto, S. Gazes, A. DiRienzo, and J. D. Molitoris. Phys. Rev. C, 23:1581, 1981.
31. J. Raynal. Saclay Report No. DP/T 69/42 (unpublished).
32. A. Lépine-Szily and R. Lichtenháler. Euro. Phys. J. A, 57:99, 2021.
33. I. J. Thompson. Comput. Phys. Rep., 7:167, 1988.
34. K. Hagino, N. Rowley, and A. T. Kruppa. Comput. Phys. Comm., 123:143, 1999.
35. C. H. Dasso, S. Landowne, and A. Winther. Nucl. Phys. A, 405:381, 1983.
36. C. H. Dasso, S. Landowne, and A. Winther. Nucl. Phys. A, 407:221, 1983.
37. K. Hagino, N. Takigawa, and A. B. Balantekin. *Phys. Rev. C*, 56:2104, 1997.
38. N. Rowley, G. R. Satchler, and P. H. Stelson. *Phys. Lett. B*, 254:25, 1991.
39. H. Timmers, J.R. Leigh, M. Dasgupta, D.J. Hinde, R.C. Lemmon, J.C. Mein, C.R. Morton, J.O. Newton, and N. Rowley. *Nucl. Phys. A*, 584:190, 1995.
40. K. Hagino and N. Rowley. *Phys. Rev. C*, 69:054610, 2004.
41. Y.K. Gupta, B.K. Nayak, U. Garg, K. Hagino, K.B. Howard, N. Sensharma, M. Şenyiğit, W.P. Tan, P.D. O’Malley, M. Smith, Ramandeeep Gandhi, T. Anderson, R.J. deBoer, B. Frentz, A. Gyorjinyan, O. Hall, M.R. Hall, J. Hu, E. Lamere, Q. Liu, A. Long, W. Lu, S. Lyons, K. Ostdiek, C. Seymour, M. Skulski, and B. Vande Kolk. *Phys. Lett. B*, 806:135473, 2020.
42. C.L. Jiang, H. Esbensen, K. E. Rehm, B. B. Back, R. V. F. Janssens, J. A. Caggiano, P. Collon, J. Greene, A. M. Heinz, D. J. Henderson, I. Nishinaka, T. O. Pennington, and D. Seweryniak. *Phys. Rev. Lett.*, 89:052701, 2002.
43. C. L. Jiang, H. Esbensen, K. E. Rehm, B. B. Back, R. V. F. Janssens, J. A. Caggiano, P. Collon, J. Greene, A. M. Heinz, D. J. Henderson, I. Nishinaka, T. O. Pennington, and D. Seweryniak. *Phys. Rev. Lett.*, 89:052701, 2002.
44. C. L. Jiang, H. Esbensen, K. E. Rehm, B. B. Back, R. V. F. Janssens, J. A. Caggiano, P. Collon, J. Greene, A. M. Heinz, D. J. Henderson, I. Nishinaka, T. O. Pennington, and D. Seweryniak. *Phys. Rev. Lett.*, 89:052701, 2002.
45. C. H. Dasso and A. Vitturi. *Phys. Rev. C*, 50:R12, 1994.
46. N. Takigawa and H. Sagawa. *Phys. Lett. B*, 265:23, 1991.
47. H. Timmers, J.R. Leigh, M. Dasgupta, D.J. Hinde, R.C. Lemmon, J.C. Mein, C.R. Morton, J.O. Newton, and N. Rowley. *Nucl. Phys. A*, 584:190, 1995.
48. C. H. Dasso and A. Vitturi. *Phys. Rev. C*, 50:R12, 1994.
49. K. Hagino, A. Vitturi, C. H. Dasso, and S. M. Lenzi. *Phys. Rev. C*, 61:037602, 2000.
50. A. Diaz-Torres and I. J. Thompson. *Phys. Rev. C*, 92:064604, 2015.
51. K. Hagino, N. Takigawa, and A. B. Balantekin. *Phys. Rev. C*, 56:2104, 1997.
52. N. Rowley, G. R. Satchler, and P. H. Stelson. *Phys. Lett. B*, 254:25, 1991.
53. H. Timmers, J.R. Leigh, M. Dasgupta, D.J. Hinde, R.C. Lemmon, J.C. Mein, C.R. Morton, J.O. Newton, and N. Rowley. *Nucl. Phys. A*, 584:190, 1995.
54. K. Hagino and N. Rowley. *Phys. Rev. C*, 69:054610, 2004.
74. E. M. Kozulin, G. N. Knyazheva, I. M. Itkis, M. G. Itkis, A. A. Bogachev, E. V. Chernysheva, L. Krupa, F. Hanappe, O. Dorvaux, L. Stuttgé, W. H. Trzaska, C. Schmitt, and G. Chubarian. *Phys. Rev. C*, 90:054608, 2014.

75. Yu. Ts. Oganessian, V. K. Utyonkov, Yu. V. Lobanov, F. Sh. Abdullin, A. N. Polyakov, I. V. Shirokovsky, Yu. S. Tsyganov, G. G. Gulbekian, S. L. Bogomolov, B. N. Gikal, A. N. Mezentsev, S. Iliev, V. G. Subbotin, A. M. Sukhov, O. V. Ivanov, G. V. Buklanov, K. Subotic, M. G. Itkis, K. J. Moody, J. F. Wild, N. J. Stoyer, M. A. Stoyer, R. W. Lougheed, C. A. Laue, Ye. A. Karelín, and A. N. Tatarinov. *Phys. Rev. C*, 63:011301, 2000.

76. Yu. Ts. Oganessian, V. K. Utyonkov, Yu. V. Lobanov, A. N. Polyakov, I. V. Shirokovsky, Yu. S. Tsyganov, G. G. Gulbekian, S. L. Bogomolov, B. N. Gikal, A. N. Mezentsev, S. Iliev, V. G. Subbotin, A. M. Sukhov, A. A. Voïnov, G. V. Buklanov, K. Subotic, V. I. Zagrebaev, M. G. Itkis, J. B. Patin, K. J. Moody, J. F. Wild, M. A. Stoyer, N. J. Stoyer, D. A. Shaughnessy, J. M. Kenneally, P. A. Wilk, R. W. Lougheed, R. I. Il’kaev, and S. P. Vesnovskii. *Phys. Rev. C*, 70:064609, 2004.

77. W. J. Świątacki, K. Siwek-Wilczyńska, and J. Wilczyński. *Phys. Rev. C*, 71:014602, 2005.

78. Y. Abe, D. Boilley, B. G. Giraud, and T. Wada. *Phys. Rev. E*, 61:1125, 2000.

79. C. W. Shen, G. Kosenko, and Y. Abe. *Phys. Rev. C*, 66:061602, 2002.

80. W. J. Świątacki, K. Siwek-Wilczyńska, and J. Wilczyński. *Acta Phys. Pol. B*, 34:2049, 2003.

81. Y. Aritomo and M. Ohta. *Nucl. Phys. A*, 744:3, 2004.

82. V. I. Zagrebaev and W. Greiner. *Nucl. Phys. A*, 944:257, 2015.

83. H. Lü, A. Marchix, Y. Abe, and D. Boilley. *Comput. Phys. Commun.*, 200:381, 2016.

84. H. Lü, D. Boilley, Y. Abe, and C. Shen. *Phys. Rev. C*, 94:034616, 2016.

85. “Nuclear Reaction Video”. http://nrv.jinr.ru/nrv/.

86. Yu. Ts. Oganessian, V. K. Utyonkov, F. Sh. Abdullin, S. N. Dmitriev, R. Graeger, R. A. Henderson, M. G. Itkis, Yu. V. Lobanov, A. N. Mezentsev, K. J. Moody, S. L. Nelson, A. N. Polyakov, M. A. Ryabinin, R. N. Sagaidak, D. A. Shaughnessy, I. V. Shirokovsky, M. A. Stoyer, N. J. Stoyer, V. G. Subbotin, K. Subotic, A. M. Sukhov, Yu. S. Tsyganov, A. Türler, A. A. Voïnov, G. K. Vostokin, P. A. Wilk, and A. Yakushev. *Phys. Rev. C*, 87:034605, 2013.

87. R. Yanez, W. Loveland, L. Yao, J. S. Barrett, S. Zhu, B. B. Back, T. L. Khoo, M. Alcorta, and M. Albers. *Phys. Rev. Lett.*, 112:152702, 2014.

88. D. J. Hinde, M. Dasgupta, J. R. Leigh, J. P. Lestone, J. C. Mein, C. R. Morton, J. O. Newton, and H. Timmers. *Phys. Rev. Lett.*, 74:1295, 1995.

89. K. Hagino. *Phys. Rev. C*, 98:014607, 2018.

90. T. Tanaka, Y. Narikiyo, K. Morita, K. Fujita, D. Kaji, K. Morimoto, S. Yamaki, Y. Wakahayashi, K. Tanaka, M. Takeyama, A. Yoneda, H. Haba, Y. Komori, S. Yanou, B. J.-P. Gull, Z. Asfari, H. Faure, H. Hasebe, M. Huang, J. Kanaya, M. Murakami, A. Yoshiida, T. Yamaguchi, F. Tokanai, T. Yoshida, S. Yamamoto, Y. Yamano, K. Watanabe, S. Ishizawa, M. Asai, R. Aono, S. Goto, K. Katori, and K. Hagino. *J. Phys. Soc. Japan*, 87:014201, 2018.

91. S. Hofmann et al. *Euro. Phys. J. A*, 48:62, 2012.

92. T. Tanaka, K. Morita, K. Morimoto, D. Kaji, H. Haba, R. A. Boll, N. T. Brewer, S. Van Cleve, D. J. Dean, S. Ishizawa, Y. Ito, Y. Komori, K. Nishio, T. Niwase, B. C. Rasco, J. B. Roberto, K. P. Rykaczewski, H. Sakai, D. W. Stracener, and K. Hagino. *Phys. Rev. Lett.*, 124:052502, 2020.