Small noise may diversify collective motion

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Abstract

Natural systems are undeniably affected by noises. How noise affects the collective behavior of self-organized systems has attracted wide interest from various fields in the past several decades. To describe the collective motion of multiple interacting particles, Vicsek et al. proposed a well-known self-propelled particle (SPP) system and conjectured it exhibited a second order phase transition from disordered to ordered motion by simulations. However, due to its nonequilibrium, randomness, and strong coupling nonlinear dynamics, the mathematical analysis of such system is still lack. To decouple the agent-based systems consisting of deterministic laws and randomness, this paper originally proposes a general method which builds a connection between these systems and cooperative control. Using our method we rigorously analyze the origin Vicsek model under both open and periodic boundary conditions for the first time, and also make some extensions to the inhomogenous SPP systems including the leader-follower models. Theoretic results show that the SPP systems will switch infinite times between ordered and disordered states for arbitrary small noise and large population density, which implies the phase transition should have new form differing from traditional senses. Also, the robust consensus and connectivity of these systems will be investigated. Moreover, our research shows the noise can lead to the diversity of collective motion of flocks, such as the appearance of turn, vortex, bifurcation and merger.

Keywords: collective behavior, multi-agent system, cooperative control, Vicsek’s model, self-propelled particles, consensus

1 Introduction

“Natural systems are undeniably subject to random fluctuations, arising from either environmental variability or thermal effects”[1]. In the past several decades,
how the noise affects the collective behavior of the self-organized systems which are shaped by the interplay of deterministic laws and randomness has attracted wide interest from various fields such as catalysis, cosmology, biology, reactive mixing, colloidal chemistry, geophysics, electronic engineering and statistical physics, see the review articles[1–5].

The collective motion of groups of animals is a common and spectacular scene of nature, for example, schools of fish, flocks of birds or groups of ants sometimes can move in a highly orderly fashion. To quantitatively describe such phenomenon a well-known self-propelled particle(SPP) model was proposed by Vicsek et al.[6]. This model contains only one basic rule: at each time step each agent is driven by a constant absolute velocity, where its heading is updated by the average value of the headings of agents in its neighborhood of radius $r$, with some random perturbation added. Using simulations Vicsek et al. mainly investigated the relation between the order and the noise and density, and conjectured that this model exhibited a second order phase transition from disordered to ordered motion concerning the noise and density under periodic boundary conditions[6].

The Vicsek model captures some common features of a number of actual systems, and can be considered as a starting point to the investigation of complex systems which shows the simple local interacting rule can lead to interesting global behaviors. Due to these fundamental importance, the Vicsek model and its variations have attracted much attention from various fields such as biology[7, 8, 36, 38–41], physics[9, 12, 33, 35, 37, 42], control theory and mathematics[13, 21, 23, 45, 50]. For example, a SPP system is adopted to predict the collective motion of the desert locusts[8]. By simulations it was shown that the system exhibited the large fluctuations of order parameter and the changes of group’s moving direction repeatedly when the density of the individuals was low and middle, and became highly ordered after a short time when the density was high[8]. The studies of SPP systems have been reviewed by[5, 43, 44].

There exist much literature on the investigation of the properties of the SPP systems using simulations or physical methods at present[6, 9, 12, 33, 35]. For mathematical analysis, the current work need to change the basic rule of the Vicsek model. The most common change of Vicsek model is to omit its noise, with which the consensus behavior is investigated under a joint connectivity assumption[13, 15, 17] or random framework[16, 18, 19]. Another change is to assume that each agent can communicate with all of others at any time[20, 21, 45–48]. There also exist some related works where the robust consensus is investigated by assuming the interaction between agents does not depend on the agents’ states[23–28]. However, these changes do not keep the basic features of the Vicsek model: local interaction and randomness. To our best knowledge, the mathematical analysis on the systems which can keep both the two features of the Vicsek model is still lack.

In fact, for many physical, biological and chemical systems including SPP systems, how the noise affects their collective behavior is still a key problem unsolved in mathematics. This paper will focus on this problem by carrying out the analysis on the inhomogenous SPP models. The main contribution of this paper can be listed as follows:

**Mathematics:** We originally propose a general method to decouple the agent-
based models formulated by deterministic laws and randomness. This method builds a connection between this type of model and cooperative control, though these models do not contain any control input. Using our method we strictly analyze the origin Vicsek model for the first time, and make some extensions to the inhomogenous SPP systems including the leader-follower models. Also, we give some clear answers to the problems of robust consensus and connectivity which are interested by the field of multi agent systems.

**Physics:** Our research reveals how local interactions on the micro level lead to global behavior on the macro level. It is shown that the SPP systems will switch infinite times between ordered and disordered states for arbitrary small noise and large population density, which indicates the small noise may break the order of the systems, and then strictly proves the viewpoint that the randomness can result in the nonequilibrium systems exhibiting anomalously large fluctuations\[32\]. Also, this result implies the phase transition should have new form differing from traditional senses such as in \[6,34,35\].

**Biology:** We show the noises in SPP systems can lead to the phenomena of turn, vortex, bifurcation and merger of flocks. Also, to some degree our results can give an explanation to the switches of group’s moving direction and the large fluctuations of order parameter in the locusts experiments for low and middle densities \[8\], and predict that these phenomena will still exist for high density when the time step grows large enough.

The rest of the paper is organized as follows: In Section 2, we will provide our model and give some notations. The main results under open and periodic boundary conditions are given in Section 3. In Section 4 we give the proofs of main results. Section 5 provides some simulations.

## 2 Models and definitions

### 2.1 Models

The origin Vicsek model consists of \(n\) autonomous agents moving in the plane with the same speed \(v(v \geq 0)\), where each agent \(i\) contains two state variables: \(X_i(t) = ((x_{i1}(t), x_{i2}(t)) \in \mathbb{R}^2\) and \(\theta_i(t) \in (-\pi, \pi]\), denoting its position and heading at time \(t\) respectively. Then the agent \(i\)’s velocity \(V_i(t)\) is \(v(\cos \theta_i(t), \sin \theta_i(t))\) at time \(t\). Each agent’s heading is updated according to a local rule based on the average direction of its neighbors, and two agents are called neighbors if and only if the distance between them is less than a pre-defined radius \(r(r > 0)\). Let

\[
N_i(t) := \{ j : \|X_i(t) - X_j(t)\|_2 \leq r \}
\]

denote the neighbor set of agent \(i\) at time \(t\), where \(\| \cdot \|_2\) is the Euclidean norm. Following \[6\], the dynamics of the origin Vicsek model can be formulated by

\[
\theta_i(t + 1) = \text{atan2} \left( \sum_{j \in N_i(t)} \sin \theta_j(t), \sum_{j \in N_i(t)} \cos \theta_j(t) \right) + \xi_i(t), \tag{2.1}
\]
\[ X_i(t+1) = X_i(t) + V_i(t+1) = X_i(t) + v(\cos \theta_i(t+1), \sin \theta_i(t+1)) \] (2.2)

for all \( i \in [1,n] \) and \( t \geq 0 \), where the function \( \text{atan2} \) is the arctangent function with two arguments\(^1\) and \( \xi_i(t) \) denotes a random noise independently and uniformly distributed in the interval \([-\eta,\eta]\). Here \( \eta \) is a positive constant. The system (2.1)-(2.2) is called as the origin Vicsek model. Let \( X(t) = (X_1(t), X_2(t), \ldots, X_n(t)) \) and \( \theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_n(t)) \). The origin Vicsek model is very complex to analyze in mathematics. An important step forward in analyzing this model was given by Jadbabaie, Lin, and Morse in \[13\], where they omitted the noise effect and linearized the heading updating rule (2.1) as follows:

\[ \theta_i(t+1) = \frac{1}{N_i(t)} \sum_{j \in N_i(t)} \theta_j(t). \] (2.3)

The origin Vicsek model assumes all the agents are homogenous. This paper will extend this model to the inhomogenous case: Assume each agent \( i \) can have different interaction radius \( r_i > 0 \), and its heading is updated by

\[ \theta_i(t+1) = \text{atan2} \left( \sum_{j=1}^{n} f_{ij}(t) \sin \theta_j(t), \sum_{j=1}^{n} f_{ij}(t) \cos \theta_j(t) \right) + \xi_i(t), \] (2.4)

where \( f_{ij}(t) \) is the non-negative weight function satisfying: (i) \( f_{ii}(t) > 0 \) for all \( i, t \), which means each agent has a certain inertia; (ii) \( f_{ij}(t) = 0 \) when \( \|X_i(t) - X_j(t)\|_2 > r_i \) for all \( i, j, t \), which indicates each agent cannot receive information directly from the ones out of its interaction radius. Define

\[ r_{\text{max}} := \max_{1 \leq i \leq n} r_i. \]

It is worth noting that this extension can satisfy the case of leader-follower relationships within a flock. For example, if agent \( i \) is the follower of agent \( j \), we can set \( f_{ij}(t) \) be a large value and \( f_{ik}(t) = 0 \) when \( k \neq i, j \). The leader-follower relationship has been observed in the actual experiments\[51\].

For more deep investigation we also consider the system whose heading is updated by

\[ \theta_i(t+1) = \frac{1}{\sum_{j=1}^{n} f_{ij}(t)} \sum_{j=1}^{n} f_{ij}(t) \theta_j(t) + \xi_i(t). \] (2.5)

For all \( i \in [1,n] \) and \( t \geq 0 \), we restrict the value of heading \( \theta_i(t) \) to the interval \([-\pi, \pi]\) by modulo \( 2\pi \) when it is out this interval. Differing from the existing changes for mathematical analysis, this system keeps the features of local interaction and randomness of the origin Vicsek model. Also, from simulations it will be

\(^1\)Literature\[6\] uses the \( \arctan \) function here, but it should be not correct because the quadrant information is lost.
shown that this system exhibits the similar properties to the origin Vicsek model, see Section 5.

To simplify the exposition we call the system evolved by (2.2) and (2.4) as system I, and call the system evolved by (2.2) and (2.5) as system II.

Let \((\Omega_{n,\eta}, \mathcal{F}_{n,\eta}, P_{n,\eta})\) denote the probability space of systems I and II, where the sample space \(\Omega_{n,\eta} = \prod_{t=0}^{\infty} [-\eta,\eta]^n\) and \(\mathcal{F}_{n,\eta}\) is the Borel \(\sigma\)-algebra of \(\Omega_{n,\eta}\). To simplify the exposition we write this probability space as \((\Omega, \mathcal{F}, P)\).

### 2.2 Order, robust consensus and connectivity

This paper will firstly investigate how the noise affects the order. Following [6], we define the order parameter

\[
\varphi(t) := \frac{\left\| \sum_{i=1}^{n} V_i(t) \right\|}{\sum_{i=1}^{n} \left\| V_i(t) \right\|} = \frac{1}{n} \left\| \sum_{i=1}^{n} (\cos \theta_i(t), \sin \theta_i(t)) \right\|_2
\]

for all \(t \geq 0\). It can be seen that \(\varphi(t)\) is close to its extreme value 1 indicating all the agents move in almost the same direction, whereas is close to 0 indicating an absence of any collective alignment. Naturally, we say the systems I and II is ordered at time \(t\) when \(\varphi(t)\) is close to 1, and is disordered when \(\varphi(t)\) is close to 0.

We also give an intuitive definition concerning the order:

**Definition 2.1** For any heading vector \(\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in [-\pi, \pi]^n\), define the length of the minimal interval which can cover it as

\[
d_\theta := \inf \{ l \in [0, 2\pi] : \text{there exists a constant } c \in [-\pi, \pi] \text{ such that } \theta_i \in [c, c + l] \text{ for all } 1 \leq i \leq n \},
\]

where \([c, c + l] := [c, \pi] \cup [0, c + l - \pi]\) for the case of \(c + l > \pi\).

This definition can also be understood to be the maximum headings’ difference in the flocks. Obviously, \(d_\theta\) is close to 0 when all the agents move with almost same directions.

The robust consensus has attracted much attention in the research of multi-agent systems [22-28]. In [23] Wang and Liu provided a definition of robust consensus for the systems whose topologies does not couple their states. We adapt this definition to our model as follows:

**Definition 2.2** The system I (or II) achieves robust consensus if there exists a function \(g(\cdot)\) satisfying \(\lim_{x \to 0^+} g(x) = 0\), such that for any \(\eta > 0\) and \(\omega \in \mathcal{F}\),

\[
\lim_{t \to \infty} \sup_{t} d_{\theta}(t) \leq g(\eta).
\]

This paper will try to study whether the robust consensus can be reached.

The connectivity of the underlying graphs is a key issue for consensus of multi-agent systems. For system I (or II), let \(G(t) = G(\mathcal{X}, \mathcal{E}(t))\) denote its underlying graph at time \(t\), where the vertex set \(\mathcal{X}\) is the \(n\) agents, and the edge set \(\mathcal{E}(t) = \)
\{(j, i) : \|X_i(t) - X_j(t)\|_2 \leq r_i\}. Let \(\tilde{G}(t)\) denote the graph obtained by replacing all directed edges of \(G(t)\) with undirected edges. Obviously, the graph \(\tilde{G}(t)\) is undirected. An undirected graph is said to be connected if there exists at least one path between its any two vertices. Given two graphs \(G(\mathcal{X}, \mathcal{E}_1)\) and \(G(\mathcal{X}, \mathcal{E}_2)\), define \(G(\mathcal{X}, \mathcal{E}_1) \cup G(\mathcal{X}, \mathcal{E}_2) := G(\mathcal{X}, \mathcal{E}_1 \cup \mathcal{E}_2)\). Following [24], we give the definition of uniformly joint weak connectivity as follows:

**Definition 2.3** The graph sequence \(\{G(t)\}_{t=0}^{\infty}\) is said to be uniformly jointly weakly connected if there exists an integer \(T > 0\) such that \(\bigcup_{k=t}^{t+T} \tilde{G}(k)\) is connected for any \(t > 0\).

The assumption of uniformly joint connectivity is widely used as a necessary condition of consensus in multi-agent systems [13, 15, 17, 23–26, 28, 29]. For the systems whose topologies coupled with states, whether this assumption can be satisfied remains a quite interesting problem. This paper will provide an answer to this problem.

### 2.3 Turn, vortex, bifurcation and merger

The turn, bifurcation and merger of flocks are very common phenomena in the nature. These phenomena have been studied by the well-known Boid model using simulations [30]. Saber provided a specific flocking algorithm which could produce bifurcation and merger behavior by adding a global leader and some obstacles [31]. We will show that the SPP models can spontaneously produce these phenomena.

The phenomena of turn, bifurcation and merger of flocks are hard to be precisely defined. We give their descriptive definitions as follows:

**Turn** and **vortex**: All agents of a flock gradually change their headings from one angle to another in a finite time, where the difference of the two angles is larger than a certain value (for example, \(\pi/2\)), and during this time all the agents keep almost synchronization, i.e. the headings of all the agents are almost same at each time. A turn with angle change exceeding \(2\pi\) is called a vertex.

**Bifurcation**: A group of agents with the almost same direction separates into two groups with different directions, while in each group all the agents keep almost synchronization.

**Merge**: Two groups of agents with different directions merge into one group with the almost same direction.

### 3 Main results

#### 3.1 Results under open boundary conditions

This subsection will give some results under open boundary conditions of positions of agents, which indicates that all the agents can move on \(\mathbb{R}^2\) without boundary limitation. Throughout this subsection we assume that the population size \(n \geq 3\), the parameters \(\eta \in (0, \pi)\), \(v > 0\), \(r_i \geq 0\), \(1 \leq i \leq n\), and the initial positions \(X(0) \in \mathbb{R}^{2n}\) and headings \(\theta(0) \in [-\pi, \pi)^n\) are arbitrarily given.
We firstly give the following theorem concerning with the heading difference of the agents for both systems I and II:

**Theorem 3.1** For system I (or II), with probability 1 there exists an infinite time sequence \( t_1 < t_2 < \cdots \) such that \( d_{\theta(t_i)} \geq \pi \) for all \( i \geq 1 \); moreover, let \( \tau_0 = 0 \) and \( \tau_{i+1} \) denote the stopping time as

\[
\tau_{i+1} := \min\{t > \tau_i : d_{\theta(t)} \geq \pi\},
\]

then for all \( i \geq 1 \) and \( t \geq 0 \),

\[
P(\tau_i - \tau_{i-1} > t) \leq (1 - c)^{\lfloor t/T \rfloor}, \tag{3.1}
\]

where \( c \in (0, 1) \) and \( T > 0 \) are constants depending on \( n, r_{\max}, \eta, v \) only.

The proof of Theorem 3.1 will be given in Subsection 4.2.

Together with Definition 2.2 and Theorem 3.1, the following corollary can be deduced immediately.

**Corollary 3.2** The robust consensus cannot be achieved for both systems I and II.

For system II, we will give the following two theorems concerning with the order parameter and connectivity respectively.

**Theorem 3.3** Let \( \varepsilon \in (0, 1) \) be arbitrarily given. Then for system II, with probability 1 there exists an infinite time sequence \( t_1 < t_2 < \cdots \) such that

\[
\varphi(t_i) \begin{cases} 
\geq 1 - \varepsilon & \text{if } i \text{ is odd,} \\
\leq \varepsilon & \text{if } i \text{ is even.}
\end{cases}
\]

Moreover, let \( \tau_0 = 0 \) and \( \tau_i \) denote the stopping time as

\[
\tau_i = \begin{cases} 
\min\{t > \tau_{i-1} : \varphi(t) \geq 1 - \varepsilon\} & \text{if } i \text{ is odd} \\
\min\{t > \tau_{i-1} : \varphi(t) \leq \varepsilon\} & \text{if } i \text{ is even}
\end{cases}
\]

for \( i \geq 1 \), then for all \( k \geq 0 \) and \( t \geq 0 \),

\[
P(\tau_{2k+2} - \tau_{2k} > t) \leq (1 - c)^{\lfloor t/T \rfloor}, \tag{3.2}
\]

where \( c \in (0, 1) \) and \( T > 0 \) are constants depending on \( n, r_{\max}, \eta, v \) only.

**Theorem 3.4** For system II, \( \{G(t)\}_{t=0}^{\infty} \) is not uniformly jointly weakly connected with probability 1.

The proof of Theorems 3.3 and 3.4 is put in Subsection 4.2 and Appendix C respectively.

We also give a theorem concerning with turn, bifurcation and merge for system II:

**Theorem 3.5** For system II, the events of turn, bifurcation and merge will happen infinite times with probability 1.

The proof of Theorem 3.5 will be given in Appendix C.


### 3.2 Results under periodic boundary conditions

The system studied by Vicsek et al. in [6] assumes all the agents move in the rectangular area $[0, L)^2$ with periodic boundary conditions, which indicates that if an agent hits the boundary of the rectangular area, it will enter this rectangle from the opposite boundary with the same velocity and heading. In mathematics, these conditions contain two meanings: (i) For all $i \in [1, n]$ and $t \geq 1$ we restrict $x_{i1}(t)$ and $x_{i2}(t)$ to the interval $[0, L)$ by modulo $L$ when they are out this interval; (ii) For all $i, j \in [1, n]$ and $t \geq 0$,

$$
\|X_i(t) - X_j(t)\|^2 = \min\{|x_{i1}(t) - x_{j1}(t)|, |x_{i1}(t) - x_{j1}(t) \pm L|\}^2 \\
+ \min\{|x_{i2}(t) - x_{j2}(t)|, |x_{i2}(t) - x_{j2}(t) \pm L|\}^2.
$$

Similar to subsection 3.1, throughout this subsection we assume that the population size $n \geq 3$, the parameters $\eta \in (0, \pi)$, $v > 0$, $r_i \geq 0$, $1 \leq i \leq n$, and the initial positions $X(0) \in [0, L)^{2n}$ and headings $\theta(0) \in [-\pi, \pi)^n$ are arbitrarily given. Additionally, we also assume all the agents move in $[0, L)^2$ with periodic boundary conditions.

In the following part we will give Theorems 3.6, 3.7, 3.8 and 3.9 similar to Theorems 3.1, 3.3, 3.4 and 3.5 respectively.

**Theorem 3.6** For system I(or II), if

$$
L > 2r_{\max} + 2v \sum_{k=0}^{\left\lfloor \frac{\pi}{\eta} - \frac{1}{2} \right\rfloor} \sin\left(\frac{\eta}{2} + k\eta\right),
$$

then all the results of Theorem 3.1 still hold with $c$ and $T$ depending on $L$ additionally.

**Theorem 3.7** For system II, if

$$
L > \begin{cases} 
2r_{\max} + 2v \sum_{k=0}^{\left\lfloor \frac{\pi}{\eta} - \frac{1}{2} \right\rfloor} \sin\left(\frac{\eta}{2} + k\eta\right) & \text{if } n \text{ is even or } \varepsilon > \frac{1}{n}, \\
3r_{\max} + 2v \sum_{k=0}^{\left\lfloor \frac{\pi}{\eta} + \frac{1}{2} \right\rfloor} \arcsin\left(\frac{1}{n-1} - \frac{1}{2}\right) \sin\left(\frac{\eta}{2} + k\eta\right) & \text{otherwise},
\end{cases}
$$

then all the results of Theorem 3.3 still hold with $c$ and $T$ depending on $L$ additionally.

The proofs of Theorems 3.6 and 3.7 will be given in Subsection 4.3.

The following theorem provides a theoretic result for the problem of how the noise affects the connectivity.

**Theorem 3.8** For protocol II, if $L > 2r_{\max}$ then $\{G(t)\}_{t=0}^\infty$ is not uniformly jointly weakly connected with probability 1.

**Theorem 3.9** For system II, the event of turn will happen infinite times with probability 1. Additionally, if (3.3) is satisfied, the events of bifurcation and merge will also happen infinite times with probability 1.

The proofs of Theorems 3.8 and 3.9 are put in Appendix C.
3.3 Results under an assumption

Naturally the origin SPP system can be evolved from disordered to ordered states. This nature can be verified by the simulations[6], however, is quite hard to be proved in mathematics. If this bottleneck can be broken, we can get that the results in Theorems 3.3, 3.5, and 3.7-3.9 still hold for system I. This fact can be formulated as the following theorem:

**Theorem 3.10**  For system I, assume there exists a finite time $T > 0$ depending on the system parameters only such that with a positive probability $\min_{1 \leq t \leq T} d_{\theta(t)} \leq \pi$ for arbitrary initial states. Then the results in Theorems 3.3, 3.4, 3.5, 3.7, 3.8 and 3.9 also hold for system I; moreover, with probability 1 system I exist vortices whose duration can be arbitrary long under both open and periodic boundary conditions.

The proof of this theorem is put in Appendix C.

3.4 Notations for our results

We remark that the above theorems do not depend the population density. By Theorems 3.1, 3.3, 3.6 and 3.7, it can be seen that for arbitrary small noise and large population density, the order of the SPP systems will be broken when the time grows large. This result coincides the phenomenon that randomness may make the equilibrium systems exhibiting anomalously giant fluctuations, which has been observed in many actual systems such as glassy systems, granular packings, active colloids, etc[4, 32]. Also, these results indicate the order parameter does not exhibit the simple phase transition concerning the noise and population density described in [6, 35]. Combined our previous work[19], it is suggested that the switch time between ordered and disordered states may exhibit phase transition concerning the noise and population density.

Because our systems have the similar rules and features with the locusts model of [8], to some degree Theorems 3.7, 3.9 and 3.10 can explain the switches of group’s moving direction and the large fluctuations of order parameter for low and middle densities in [8], and can predict that these behaviors still exist for high density when the time step grows large enough.

In [13] Jadbabaie et al. analyzed the system II without noise, and mentioned that to understand the effect of additive noise, one should focus on how noise affected the connectivity of the associated neighbor graphs. Later, Tahbaz-Salehi and Jadbabaie investigated the origin Vicsek model without noise and claimed that the neighbor graphs were jointly connected over infinitely many time intervals for almost all initial states under periodic boundary conditions[14]. Theorems 3.4 and 3.8 provide an answer of how noise affects the connectivity under the open and periodic boundary conditions respectively.

As mentioned in the Subsection 2.2, the robust consensus has been interested by many papers[22, 28]. Corollary 3.2 says for arbitrary initial state and system parameter, the robust consensus cannot be reached under the open boundary conditions. For the periodic boundaries, the problem whether the robust consensus can be reached is still unresolved. In [23] it was shown that for the systems whose topologies did not
couple their states, the uniformly joint connectivity of the associated graphs was a necessary and sufficient condition for robust consensus. However, such result cannot be adapted to our models.

4 Proof of results

4.1 Transform to robust cooperative control

This paper will use the method of robust cooperative control to analyze the systems I and II. Firstly we need to construct a new system of robust control as follows: Let \( \delta_i(t) \in (0, \eta) \) be the arbitrarily given numbers; for \( i = 1, \ldots, n \) and \( t \geq 0 \), let

\[
\begin{align*}
\theta_i(t + 1) &= \text{atan2}(\sum_{j=1}^{n} f_{ij}(t) \sin \theta_j(t), \sum_{j=1}^{n} f_{ij}(t) \cos \theta_j(t)) \\
& \quad + u_i(t) + b_i(t), \\
X_i(t + 1) &= X_i(t) + v(\cos \theta_i(t + 1), \sin \theta_i(t + 1)),
\end{align*}
\]

(4.1)

where \( F_i[X(t), \theta(t)] \) represents \( \text{atan2}(\sum_{j=1}^{n} f_{ij}(t) \sin \theta_j(t), \sum_{j=1}^{n} f_{ij}(t) \cos \theta_j(t)) \) for system I and \( \frac{1}{\sum_{j=1}^{n} f_{ij}(t)} \sum_{j=1}^{n} f_{ij}(t) \theta_j(t) \) for system II respectively, \( u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)] \) is the bounded control input and \( b_i(t) \in [-\delta_i(t), \delta_i(t)] \) denotes the parameter uncertainty.

Let \( \Omega^* := \mathbb{R}^{2n} \times [-\pi, \pi]^n \) (or \( [0, L]^{2n} \times [-\pi, \pi]^n \) for the periodic boundary case) be the state space of \( (X(t), \theta(t)) \) for all \( t \geq 0 \). Given \( \Omega_1 \subset \Omega^* \), we say \( \Omega_1 \) is reached at time \( t \) if \( (X(t), \theta(t)) \in \Omega_1 \), and is reached in the time \([t_1, t_2]\) if there exists \( t' \in [t_1, t_2] \) such that \( \Omega_1 \) is reached at time \( t' \).

**Definition 4.1** Let \( \Omega_1, \Omega_2 \subset \Omega^* \) be two state sets. Under protocol (4.2), \( \Omega_1 \) is said to be finite-time robustly reachable from \( \Omega_2 \) if there exist constants \( T > 0 \) and \( \varepsilon \in (0, \eta) \) such that for any \( (\theta(0), X(0)) \in \Omega_2 \) we can find \( \delta_i(t) \in [\varepsilon, \eta] \) and \( u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)] \), \( 1 \leq i \leq n \), \( 0 \leq t < T \) which guarantees that \( \Omega_1 \) is reached in the time \([1, T]\) for arbitrary \( b_i(t) \in [-\delta_i(t), \delta_i(t)] \), \( 1 \leq i \leq n \), \( 0 \leq t < T \).

The following lemma establishes a connection between systems I and (4.1), and also systems II and (4.2).

**Lemma 4.2** Let \( \Omega_1, \ldots, \Omega_k \subset \Omega^* \), \( k \geq 1 \) be the state sets. If under protocol (4.1) (or (4.2)) they are finite-time robustly reachable from \( \Omega^* \), then for system I (or II):

(i) For any initial state, with probability 1 there exists an infinite sequence \( t_1^j < t_1^k < t_2^j < \ldots < t_2^k < \ldots \) such that \( \Omega_j \) is reached at time \( t_l^j \), \( j = 1, \ldots, k, l \geq 1 \).

(ii) Let \( \tau_0 = 0 \) and \( \tau_i := \min\{t_k : \text{there exist } \tau_{i-1} < t_1 < t_2 < \ldots < t_k \text{ such that} \}

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\(\Omega_j\) is reached at time \(t_j, 1 \leq j \leq k\) for \(i \geq 1\). Then there exist constants \(T > 0\) and \(c \in (0, 1)\) satisfying

\[
P\left(\tau_i - \tau_{i-1} > t | \forall (X(0), \theta(0)) \in \Omega^* \right) \leq c^{t/T}, \forall i, t \geq 1.
\]

**Proof** (i) Under protocol (4.1) (or (4.2)), because \(\Omega_1, \ldots, \Omega_k\) are finite-time robustly reachable, for each \(j \in [1, k]\) there exists constants \(T_j > 0\) and \(\delta_j \in (0, \eta)\) such that for any \(t \geq 0\) and arbitrary \(X(t)\) and \(\theta(t)\), we can find \(\delta_i^j(t') \in [\delta_j^j(t'), \eta]\) and \(u_i^j(t') \in [-\delta_i^j(t'), \eta - \delta_i^j(t')]\), \(1 \leq i \leq n, t \leq t' < t + T_j\) with which \(\Omega_j\) is reached in the time \([t + 1, t + T_j]\) for arbitrary \(b_i^j(t') \in [-\delta_i^j(t'), \delta_i^j(t')]\), \(1 \leq i \leq n, t \leq t' < t + T_j\). Thus, for any \(j \in [1, k]\) and \(t \geq 0\), the system I or II has

\[
P\left(\{\Omega_j \text{ is reached in } [t + 1, t + T_j]\} | \forall (X(t), \theta(t)) \in \Omega^* \right) \geq P\left(\bigcap_{t \leq t' < t + T_j} \bigcap_{1 \leq i \leq n} \{\xi_i(t') \in [u_i^j(t') - \delta_i^j(t'), u_i^j(t') + \delta_i^j(t')]\} \right)
\]

\[
= \prod_{t \leq t' < t + T_j} \prod_{i=1}^{n} \frac{\delta_i^j(t')}{\eta} \geq \left(\frac{\delta_j}{\eta}\right)^{nT_j}, \quad (4.3)
\]

where the first equality uses the fact that \(\xi_i(t')\) is independently and uniformly distributed in \([-\delta_i^j(t'), \eta\] \(\delta_i^j(t')\). Let \(T = T_1 + T_2 + \ldots + T_k\). For any integers \(m \geq 0\) and \(j \in [1, k]\), define the event

\[
E_m^j := \left\{\Omega_j \text{ is reached in } [mT + \sum_{i=1}^{j-1} T_i + 1, mT + \sum_{i=1}^{j} T_i] \right\}.
\]

By Bayes’ theorem and (4.3) we can get for any \((X', \theta') \in \Omega^*\),

\[
P\left(\bigcap_{1 \leq j \leq k} E_m^j | X(mT) = X', \theta(mT) = \theta'\right)
\]

\[
= P\left(E_m^1 | X(mT) = X', \theta(mT) = \theta'\right)
\]

\[
\prod_{j=2}^{k} P\left(E_m^j | \bigcap_{1 \leq t < j} E_m^t, X(mT) = X', \theta(mT) = \theta'\right)
\]

\[
\geq \prod_{j=1}^{k} \left(\frac{\delta_j}{\eta}\right)^{nT_j} := c. \quad (4.4)
\]
Let \( E_m = \bigcap_{j=1}^{k} E_j^c \). By Bayes’ theorem and (4.4), for any \( M > 0 \) we have
\[
P \left( \bigcap_{m=M}^{\infty} E_m^c | X(0) = X', \theta(0) = \theta' \right) \\
= P \left( E_M^c | X(0) = X', \theta(0) = \theta' \right) \\
\prod_{m=M+1}^{\infty} P \left( E_m^c | \bigcap_{m'<m} E_{m'}^c, X(0) = X', \theta(0) = \theta' \right) \\
\leq \prod_{m=M}^{\infty} (1 - c) = 0,
\]
which indicates that \( E_m \) infinitely occurs (i.o.) with probability 1 for any initial state.

From this our first result is obtained.

(ii) For any \( t > 0 \) and \( m \geq 0 \), define
\[
E_{m,t} := \bigcap_{j=1}^{k} \{ \Omega_j \text{ is reached in } [t + mT + \sum_{l=1}^{j-1} T_l + 1, t + mT + \sum_{l=1}^{j} T_l] \}.
\]
Then together with (4.5), for any \( M \geq 0 \) and \( i \geq 0 \) we have
\[
P \left( \tau_i - \tau_{i-1} > t | X(0) = X', \theta(0) = \theta' \right) \\
\leq P \left( \bigcap_{m=0}^{M-1} E_{m,\tau_{i-1}}^c | X(0) = X', \theta(0) = \theta' \right) \\
\leq (1 - c)^{M},
\]
so
\[
P \left( \tau_i - \tau_{i-1} > t | X(0) = X', \theta(0) = \theta' \right) \\
\leq P \left( \tau_i - \tau_{i-1} > \left\lceil \frac{t}{T} \right\rceil T | X(0) = X', \theta(0) = \theta' \right) \\
\leq (1 - c)^{\left\lceil \frac{t}{T} \right\rceil}.
\]

\[\Box\]

**Lemma 4.3** Let \( \Omega \subseteq \Omega^* \) be a state set. If under protocol (4.1) (or (4.2)) \( \Omega \) is finite-time robustly reachable from \( \Omega^c \), then for system I (or II):

(i) With probability 1 \( \Omega \) will be reached for infinite times provided any initial state.

(ii) Let \( \tau_0 = 0 \) and \( \tau_i := \min \{ t > \tau_{i-1} : \Omega \text{ is reached at time } t \} \) for \( i \geq 1 \). Then there exist constants \( T > 0 \) and \( c \in (0, 1) \) such that
\[
P \left( \tau_i - \tau_{i-1} > t \right) | (X(0), \theta(0)) \in \Omega^* \right) \leq c^{\left\lceil \frac{t}{T} \right\rceil}, \forall i, t \geq 1.
\]

**Proof** (i) Because \( \Omega \) is finite-time robustly reachable from \( \Omega^c \), similar to (4.3) there must exist constants \( T > 0 \) and \( \varepsilon \in (0, \eta) \) such that for any \( t \geq 0 \),
\[
P \left( \{ \Omega \text{ is reached in } [t + 2, t + T] \} | (X(t+1), \theta(t+1)) \in \Omega^c \right) \\
\geq \left( \frac{\varepsilon}{\eta} \right)^{n(T-1)}. \tag{4.6}
\]
Let $E_t$ denote the event of $\Omega$ is reached in $[t + 1, t + T]$. For any $(X', \theta') \in \Omega^*$, by total probability formula and (4.6) we can get
\[
\begin{align*}
P(E_t | X(t) = X', \theta(t) = \theta') &= P \{ (X(t + 1), \theta(t + 1)) \in \Omega | X(t) = X', \theta(t) = \theta' \} \\
&\quad \cdot P \{ E_t | (X(t + 1), \theta(t + 1)) \in \Omega, X(t) = X', \theta(t) = \theta' \} \\
&\quad + P \{ E_t | (X(t + 1), \theta(t + 1)) \in \Omega | X(t) = X', \theta(t) = \theta' \} \\
&\quad \cdot P \{ (X(t + 1), \theta(t + 1)) \in \Omega^c, X(t) = X', \theta(t) = \theta' \} \\
&\geq P \{ E_t | (X(t + 1), \theta(t + 1)) \in \Omega^c, X(t) = X', \theta(t) = \theta' \} \\
&\geq (\varepsilon / \eta)^n(T - 1) := c,
\end{align*}
\]
where the last inequality uses the fact of $E_t$ includes the event of $\Omega$ is reached in $[t + 2, t + T]$, and the latter only depends on the state at time $t + 1$ but does not depend on the earlier states. With the similar process from (4.4) to (4.5) we have $\Omega$ will be reached for infinite times with probability 1.

(ii) With the similar method to the proof of Lemma 4.2(ii) yields this result. \(\square\)

**Remark 4.4** The results in Lemmas 4.2 and 4.3 can be easily extended to other agent-based systems with deterministic rules and randomness, by transforming the origin system to a robust control system with the similar method from system I to system (4.7). This idea may simplify the analysis of the complex agent-based models greatly, specially on the properties of convergence and convergent rate.

### 4.2 Proofs of Theorems 3.1 and 3.3

For any $t \geq 0$ and $1 \leq i \leq n$, set
\[
\bar{\theta}_i(t) = \begin{cases} 
\text{atan2}(\sum_{j=1}^{n} f_{ij}(t) \sin \theta_j(t), \sum_{j=1}^{n} f_{ij}(t) \cos \theta_j(t)), & \text{for system I and protocol (4.1)} \\
\sum_{j=1}^{n} f_{ij}(t) \theta_j(t), & \text{for system II and protocol (4.2)}
\end{cases}
\]

For any $\alpha > 0$, define
\[
\Omega^1 \alpha := \left\{ (X_1, \ldots, X_n), (\theta_1, \ldots, \theta_n) \in \Omega^* : \max_{1 \leq i \leq n} |\bar{\theta}_i| \leq \frac{\alpha}{2} \right\}.
\]

The following Lemmas 4.5-4.8 are all under the open boundary conditions, and assume the population size $n \geq 3$, the parameters $\eta \in (0, \pi)$, $\nu > 0$, $r_i \geq 0$, $1 \leq i \leq n$ are arbitrarily given. The proofs of all of them are put in Appendix A.

**Lemma 4.5** For any $\alpha > 0$, $\Omega^1 \alpha$ is finite-time robustly reachable from $\Omega^*$ under protocol (4.2).
Lemma 4.6 For any $\varepsilon \in (0, 1)$ and $\theta(t) \in [-\pi, \pi]^n$, if $d_{\theta(t)} \leq \arccos(1 - \varepsilon)^2$ then the order function $\varphi(t) \geq 1 - \varepsilon$.

For any $\varepsilon > 0$, define

$$\Omega^2_{\varepsilon} := \left\{ ((X_1, \ldots, X_n), (\theta_1, \ldots, \theta_n)) \in \Omega^* : \frac{1}{n} \left\| \sum_{i=1}^{n} (\cos \theta_i, \sin \theta_i) \right\|_2 \leq \varepsilon \right\}.$$ 

Lemma 4.7 For any $\varepsilon > 0$, $\Omega^2_{\varepsilon}$ is finite-time robustly reachable from $\Omega^1_\eta$ under protocol (4.2).

For any $\alpha > 0$, set

$$\Omega^3_{\alpha} := \{ (X, \theta) \in \Omega^* : d_{\theta} < \alpha \}.$$ 

Lemma 4.8 Under protocol (4.1) (or (4.2)), $(\Omega^3_{\pi})^c$ is finite-time robustly reachable from $\Omega^3_{\pi}$.

Proof of Theorem 3.1 Immediate from Lemmas 4.3 and 4.8.

Proof of Theorem 3.3 Firstly by Lemmas 4.5 and 4.7 we can get $\Omega^2_{\varepsilon}$ is finite-time robustly reachable for any initial states. Also, define

$$\Omega^2_{\varepsilon} := \left\{ ((X_1, \ldots, X_n), (\theta_1, \ldots, \theta_n)) \in \Omega^* : \frac{1}{n} \left\| \sum_{i=1}^{n} (\cos \theta_i, \sin \theta_i) \right\| \geq 1 - \varepsilon \right\}.$$ 

By Lemmas 4.5 and 4.6 we have $\Omega^2_{\varepsilon}$ is also finite-time robustly reachable for any initial states. Using Lemma 4.2 our results can be obtained by taking $\Omega_1 = \overline{\Omega}_\varepsilon$ and $\Omega_2 = \Omega^2_{\varepsilon}$.

4.3 Proofs of Theorems 3.6 and 3.7

Both of the following Lemmas 4.9 and 4.10 are under the periodic boundary conditions of the positions of the agents, and assume the population size $n \geq 3$, the parameters $\eta \in (0, \pi)$, $\nu > 0$, $r_i \geq 0$, $1 \leq i \leq n$ are arbitrarily given. The proofs of them are put in Appendix B.

Lemma 4.9 Assume (3.4) is satisfied. Then under protocol (4.2) with periodic boundary conditions, $\Omega^2_{\varepsilon}$ is finite-time robustly reachable from $\Omega^1_\eta$ for any $\varepsilon > 0$.

Lemma 4.10 Assume (3.3) is satisfied. Then under protocol (4.1) (or (4.2)) with periodic boundary conditions, $(\Omega^3_{\pi})^c$ is finite-time robustly reachable from $\Omega^3_{\pi}$.

Proof of Theorem 3.6 Immediate from Lemmas 4.3 and 4.10.

Proof of Theorem 3.7 With the same process of the proof of Lemma 4.5 we can get under protocol (4.2) with periodic boundary conditions, $\Omega^1_\eta$ is finite-time robustly reachable from $\Omega^*$. Using this instead of Lemma 4.5 and Lemma 4.9 instead of Lemma 4.7 our result can be obtained with the same argument as the proof of Theorem 3.3.
5 Simulations

To illustrate our results better this section we will provide some simulations for the homogenous versions of our models under periodic boundary conditions. Set $r = 1, \eta = 0.6, v = 0.01, L = 5$ and choose

$$f_{ij}(t) = \begin{cases} 
1 & \text{if } \|X_i(t) - X_j(t)\|_2 \leq r, \\
0 & \text{else.}
\end{cases}$$

We firstly simulate the system II by choosing $n = 10, 25, 40$, which represents the low density, middle density and high density respectively. The initial headings and positions are random selected in $[-\pi, \pi)$ and $[0, L)^2$ respectively with independently and uniformly distribution. The maximum time step is set to be $10^6$. The value of the order function $\varphi(t)$ is shown in Figure 1.

![Figure 1](image.png)

**Figure 1:** The order function $\varphi(t)$ of the system II with low density, middle density and high density from top to bottom

By this simulation it can be observed that from low density to high density, the system will exhibit ordered state at some moments and disordered state at some other
moments when the time grows large. Such observation is completely in conformity with our theoretical results.

We also give the simulations for the origin Vicsek model with the same configurations as the system II, see Figure 2. In these simulations the origin Vicsek model exhibits the similar behaviors of the order function as Figure 1, which implies our theoretical results on how the noise affects the order should be still adopted to this model.

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**Appendices**

**Appendix A** Proofs of Lemmas 4.5-4.8

**Proof of Lemma 4.5** Without loss of generality we assume \( \alpha \in (0, \eta] \). For \( t \geq 0 \) and \( 1 \leq i \leq n \), we choose

\[
(\delta_i(t), u_i(t)) = \begin{cases} 
(\eta/4, -3\eta/4) & \text{if } \tilde{\theta}_i(t') > \eta - \alpha/2, \\
(\alpha/2, -\tilde{\theta}_i(t')) & \text{if } \tilde{\theta}_i(t') \in [\alpha/2 - \eta, \eta - \alpha/2], \\
(\eta/4, 3\eta/4) & \text{if } \tilde{\theta}_i(t') < \alpha/2 - \eta.
\end{cases}
\]  \( .(1) \)

Then it can be computed that

\[
 u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)], \quad \forall 1 \leq i \leq n, t \geq 0.
\]  \( .(2) \)

Define

\[
\theta_{\max}(t) := \max_{1 \leq i \leq n} \theta_i(t) \quad \text{and} \quad \theta_{\min}(t) := \min_{1 \leq i \leq n} \theta_i(t).
\]

If \( \theta_{\max}(t) > \alpha/2 + \eta/2 \) we can get

\[
\theta_{\max}(t + 1) \leq \theta_{\max}(t) - \frac{\eta}{2}.
\]  \( .(3) \)

That is because if there exists \( i \in [1, n] \) such that

\[
\theta_i(t + 1) > \theta_{\max}(t) - \frac{\eta}{2} \geq \frac{\alpha}{2},
\]  \( .(4) \)
by $\theta_i(t+1) = \tilde{\theta}_i(t) + u_i(t) + b_i(t)$ and \textbf{[1]} we have $\tilde{\theta}_i(t) > \eta - \alpha/2$ and $u_i(t) + b_i(t) \in [-\eta, -\eta/2]$. But at the same time, by the definition of $\tilde{\theta}_i(t)$ we have $\tilde{\theta}_i(t) \leq \theta_{\text{max}}(t)$, so

$$\theta_i(t+1) \leq \theta_{\text{max}}(t) + u_i(t) + b_i(t) \leq \theta_{\text{max}}(t) - \frac{\eta}{2},$$

which is contradictory with the first inequality of \textbf{[4]}. Similar to \textbf{[3]}, we can get that if $\theta_{\text{min}}(t) < -\alpha/2 - \eta/2$ then

$$\theta_{\text{min}}(t+1) \geq \theta_{\text{min}}(t) + \frac{\eta}{2}. \quad (5)$$

Combined this with \textbf{[3]} we have if $\max_{1 \leq i \leq n} |\theta_i(t)| > \alpha/2 + \eta/2$ then

$$\max_{1 \leq i \leq n} |\theta_i(t+1)| \leq \max_{1 \leq i \leq n} |\theta_i(t)| - \frac{\eta}{2}. \quad (6)$$

Also, if $\max_{1 \leq i \leq n} |\theta_i(t)| \leq \alpha/2 + \eta/2$, by \textbf{[1]}

$$\max_{1 \leq i \leq n} |\theta_i(t+1)| \leq \alpha/2. \quad (7)$$

Let $t_1 := \lceil \frac{2\pi - \alpha}{\eta} \rceil$. By \textbf{[6], [7]} and with the fact of $\max_{1 \leq i \leq n} |\theta_i(t)| \leq \pi$, we can get

$$\max_{1 \leq i \leq n} |\theta_i(t_1)| \leq \alpha/2. \quad (8)$$

Combining \textbf{[8], [11]} and \textbf{[2]} we have $\Omega_\alpha$ is robustly reached at time $t_1$ from any initial states under protocol \textbf{[4.2]}. \hfill \Box

**Proof of Lemma 4.6** By the definition of $\varphi(t)$ we have

$$\varphi(t) = \frac{1}{n} \left\| \left( \sum_{i=1}^{n} \cos \theta_i(t), \sum_{i=1}^{n} \sin \theta_i(t) \right) \right\|_2$$

$$= \frac{1}{n} \sqrt{\sum_{i,j} \cos [\theta_i(t) - \theta_j(t)]}$$

$$\geq \sqrt{\cos (\arccos(1 - \varepsilon)^2)} \geq 1 - \varepsilon.$$

\hfill \Box

**Proof of Lemma 4.7** Without loss of generality we assume $\varepsilon \in (0, 1)$. Define the constant $\beta := \min \{ \frac{\pi}{2}, 2 \arcsin \frac{\varepsilon}{2} \}$. We will prove this our result with the following two cases respectively:

**Case I**: $n$ is even. We separate the $n$ agents into two disjoint sets $A_1$ and $A_2$ with $|A_1| = |A_2| = \frac{n}{2}$, and $x_{i2}(0) \geq x_{j2}(0)$ for any agent $i \in A_1$, $j \in A_2$. Here we recall that $x_{i2}(0)$ denotes the second coordinate of $X_i(0)$. Let

$$t_1 := \left\lfloor \frac{r_{\text{max}}}{2v \sin(\eta/4)} \right\rfloor + 1. \quad (9)$$
For $0 \leq t < t_1$, we choose

$$\delta_i(t) = \frac{\eta}{8}, \quad \forall 1 \leq i \leq n,$$

(10)

and set

$$u_i(t) = \begin{cases} 
\frac{3\eta}{8} - \bar{\theta}_i(t) & \text{if } i \in A_1, \\
-\frac{3\eta}{8} - \bar{\theta}_i(t) & \text{if } i \in A_2.
\end{cases}$$

(11)

From this we can get for all $t \in [0, t_1)$,

$$\theta_i(t + 1) \in \begin{cases} 
[\eta/4, \eta/2] & \text{if } i \in A_1, \\
[-\eta/2, -\eta/4] & \text{if } i \in A_2.
\end{cases}$$

(12)

From this for any $i \in A_1$ and $j \in A_2$, we have

$$x_{i2}(t_1) - x_{j2}(t_1) = x_{i2}(0) + \sum_{0 < t \leq t_1} v \sin \theta_i(t) - x_{j2}(0) - \sum_{0 < t \leq t_1} v \sin \theta_j(t) \geq v \sum_{0 < t \leq t_1} 2 \sin \frac{\eta}{4} = 2vt_1 \sin \frac{\eta}{4} > r_{\max}.$$  

(13)

which indicates that there exists no edges between $A_1$ and $A_2$ at time $t_1$. Also, by (11), (12) and the condition of $\max_{1 \leq i \leq n} |\theta_i(0)| \leq \frac{\eta}{2}$ we have

$$u_i(t) \in \left[-\frac{\eta}{8}, \frac{\eta}{8}\right] = [-\eta + \delta_i(t), \eta - \delta_i(t)], \quad \forall 1 \leq i \leq n, 0 \leq t < t_1.$$  

(14)

Next we will give the control algorithm to minimize the value of the order function. Set

$$t_2 := \max \left\{ t_1 + \left[ \frac{\pi - 2\beta}{\eta} - \frac{1}{2} \right], t_1 + 1 \right\}.$$  

For $t \in [t_1, t_2)$, we choose

$$(\delta_i(t), u_i(t)) = \begin{cases} 
\left( \frac{\eta}{4}, \frac{3\eta}{4} \right) & \text{if } \bar{\theta}_i(t) < \frac{\pi}{2} + \beta - \eta \\
\left( \beta, \frac{\pi}{2} - \bar{\theta}_i(t) \right) & \text{otherwise}
\end{cases}$$

(15)

for $i \in A_1$, and choose

$$(\delta_i(t), u_i(t)) = \begin{cases} 
\left( \frac{\eta}{4}, -\frac{3\eta}{4} \right) & \text{if } \bar{\theta}_i(t) > -\frac{\pi}{2} - \beta + \eta \\
\left( \beta, -\frac{\pi}{2} - \bar{\theta}_i(t) \right) & \text{otherwise}
\end{cases}$$

(16)

for $i \in A_2$. From (12), (15) and (16) it can be computed that

$$u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)], \quad \forall 1 \leq i \leq n, t_1 \leq t < t_2.$$  

(17)
If the sets $A_1$ and $A_2$ are disconnected at time $t$, then with the similar methods to (3) and (5) we can get

$$\min_{i \in A_1} \theta_i(t+1) \begin{cases} \in \left[ \frac{\pi}{2} - \beta, \frac{\pi}{2} + \beta \right] & \text{if } \min_{i \in A_1} \theta_i(t) \geq \frac{\pi}{2} - \beta - \frac{\eta}{2}, \\ \geq \frac{\eta}{2} + \min_{i \in A_1} \theta_i(t) & \text{otherwise}, \end{cases}$$

and

$$\max_{i \in A_2} \theta_i(t+1) \begin{cases} \in \left[ -\frac{\pi}{2} - \beta, -\frac{\pi}{2} + \beta \right] & \text{if } \max_{i \in A_2} \theta_i(t) \leq -\frac{\pi}{2} + \beta + \frac{\eta}{2}, \\ \leq -\frac{\eta}{2} + \max_{i \in A_2} \theta_i(t) & \text{otherwise}. \end{cases}$$

So by (13) and induction we can get $A_1$ and $A_2$ are always disconnected in the time $[t_1, t_2]$. Then, similar to (8) we have

$$\theta_i(t_2) \in \begin{cases} \left[ \frac{\pi}{2} - \beta, \frac{\pi}{2} + \beta \right] & \text{if } i \in A_1, \\ \left[ -\frac{\pi}{2} - \beta, -\frac{\pi}{2} + \beta \right] & \text{if } i \in A_2, \end{cases} \quad (18)$$

which is followed by

$$\varphi(t_2) = \frac{1}{n} \left\| \sum_{i \in A_1} (\cos \theta_i(t_2), \sin \theta_i(t_2)) \right\|$$

$$= \frac{1}{n} \left\| \sum_{i \in A_1} (\cos \theta_i(t_2), \sin \theta_i(t_2) - 1) \\ + \sum_{i \in A_2} (\cos \theta_i(t_2), \sin \theta_i(t_2) + 1) \right\|$$

$$\leq \left\| (\cos \frac{\pi}{2} - \beta), \sin \left( \frac{\pi}{2} - \beta \right) - 1 \right\|$$

$$= \sqrt{2 - 2 \cos \beta} = 2 \sin \frac{\beta}{2} \leq \varepsilon. \quad (19)$$

Together this with (14) and (17) we have $\Omega^2$ is robustly reachable at time $t_2$.

**Case II:** $n$ is odd. We separate the $n$ agents into three disjoint sets $A_1$, $A_2$ and $A_3$ which satisfy that $|A_1| = |A_2| = \frac{n-1}{2}$, $|A_3| = 1$, and $x_{i2}(0) \geq x_{j2}(0) \geq x_{k2}(0)$ for any agent $i \in A_1$, $j \in A_3$ and $k \in A_2$. Let

$$t_3 := \left\lfloor \frac{r_{\max}}{v(\sin \frac{\pi}{2} - \sin \frac{\eta}{2})} \right\rfloor + 1.$$

For $0 \leq t < t_3$, we choose $\delta_i(t) = \frac{\eta}{2}$, $1 \leq i \leq n$, and set

$$u_i(t) = \begin{cases} \frac{3n}{2} - \tilde{\theta}_i(t) & \text{if } i \in A_1, \\ -\frac{3n}{2} - \tilde{\theta}_i(t) & \text{if } i \in A_2, \\ -\tilde{\theta}_i(t) & \text{if } i \in A_3. \end{cases} \quad (20)$$
Similar to (12) and (14), we can get for all \( t \in [0, t_3) \),
\[
\theta_i(t + 1) \in \begin{cases} 
[\eta/4, \eta/2] & \text{if } i \in A_1, \\
[-\eta/2, -\eta/4] & \text{if } i \in A_2, \\
[-\eta/8, \eta/8] & \text{if } i \in A_3,
\end{cases}
\]
and
\[
u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)], \quad \forall 1 \leq i \leq n.
\]
Also, similar to (13) we can get the sets \( A_1, A_2 \) and \( A_3 \) are mutually disconnected at time \( t_3 \).

Let \( c_n := \frac{\pi}{2} + \arcsin \frac{1}{n-1} \) and set
\[
t_4 := \max \left\{ t_3 + \left[ \frac{2c_n - 2\beta}{\eta} - \frac{1}{2} \right], t_3 + 1 \right\}.
\]
For all \( t \in [t_3, t_4) \), similar to (15) and (16) we choose
\[
(\delta_i(t), u_i(t)) = \begin{cases} 
\left( \frac{\eta}{4}, \frac{3\eta}{4} \right) & \text{if } \tilde{\theta}_i(t) < c_n + \beta - \eta \\
(\beta, c_n - \tilde{\theta}_i(t)) & \text{otherwise}
\end{cases}
\]
for \( i \in A_1 \), and choose
\[
(\delta_i(t), u_i(t)) = \begin{cases} 
\left( \frac{\eta}{4}, -\frac{3\eta}{4} \right) & \text{if } \tilde{\theta}_i(t) > -c_n - \beta + \eta \\
(\beta, -c_n - \tilde{\theta}_i(t)) & \text{otherwise}
\end{cases}
\]
for \( i \in A_2 \). Also, for \( i \in A_3 \), set \( \delta_i(t) = \beta \) and \( u_i(t) = -\tilde{\theta}_i(t) \). Similar to (17) we can get
\[
u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)], \quad \forall 1 \leq i \leq n, t_3 \leq t < t_4.
\]
Also, similar to Case I we have \( A_1, A_2 \) and \( A_3 \) are always mutually disconnected in the time \([t_3, t_4)\). Thus, similar to (18) we can get
\[
\theta_i(t_4) \in \begin{cases} 
[c_n - \beta, c_n + \beta] & \text{if } i \in A_1 \\
[-c_n - \beta, -c_n + \beta] & \text{if } i \in A_2 \\
[-\beta, \beta] & \text{if } i \in A_3
\end{cases}
\]
which indicates that
\[
\varphi(t_4) = \frac{1}{n} \left\| \sum_{i \in A_1} \left( \cos \theta_i(t_4) - \cos c_n, \sin \theta_i(t_4) - \sin c_n \right) \\
+ \sum_{i \in A_2} \left( \cos \theta_i(t_4) - \cos c_n, \sin \theta_i(t_4) + \sin c_n \right) \\
+ \sum_{i \in A_3} \left( \cos \theta_i(t_4) - 1, \sin \theta_i(t_4) \right) \right\|
\]
\[
\leq \sqrt{2 - 2 \cos \beta} = 2 \sin \frac{\beta}{2} \leq \varepsilon.
\]
Together this with (21) and (22) we have $\Omega^2_\varepsilon$ is robustly reachable at time $t_4$. □

**Proof of Lemma 4.8** We will discuss protocol (4.1) firstly. Because the system I has the isotropic property under open boundary conditions, without loss of generality we assume the initial headings $\theta_i(0)$, $1 \leq i \leq n$ are distributed in the interval $[-\pi/2, \pi/2)$. Thus we can get

$\theta_{\min}(0) \leq \tilde{\theta}_i(0) \leq \theta_{\max}(0), \ \forall 1 \leq i \leq n. \quad (25)$

For $t \geq 0$ and $1 \leq i \leq n$, we choose $(\delta_i(t), u_i(t))$ as same as (4.1) but using $\eta$ instead of $\alpha$. With the almost same process of (2)-(.8) we have

$\max_{1 \leq i \leq n} |\theta_i(t_1')| \leq \eta/2, \quad (26)$

where $t_1' := \lceil \frac{n}{\eta} \rceil - 1$.

Similar to the case II of the proof of Lemma 4.7 we separate the $n$ agents into three non-empty disjoint sets $A_1, A_2$ and $A_3$ with $x_{i2}(t_1') \geq x_{j2}(t_1') \geq x_{k2}(t_1')$ for any agent $i \in A_1$, $j \in A_3$ and $k \in A_2$. Let

$t_3' := t_1' + \left\lfloor \frac{r_{\max}}{v (\sin \frac{\pi}{4} - \sin \frac{\eta}{8})} \right\rfloor + 1.$

For $t_1' \leq t < t_3'$, we choose $\delta_i(t) = \frac{\eta}{8}, 1 \leq i \leq n$, and set $u_i(t)$ as same as (4.20). With the similar discussion to the case II of the proof of Lemma 4.7 we have the sets $A_1, A_2$ and $A_3$ are mutually disconnected at time $t_3'$.

Let $t_4' := t_3' + \lceil 6n/\eta \rceil - 1$. For all $t \in [t_3', t_4')$ we set $\delta_i(t) = \eta/8$, and choose

$u_i(t) = \begin{cases} \frac{3\eta}{4} & \text{if } \tilde{\theta}_i(t) < \frac{3\pi}{4} - \frac{3\eta}{4} \\ \frac{3\pi}{4} - \tilde{\theta}_i(t) & \text{otherwise} \end{cases}$

for $i \in A_1$,

$u_i(t) = \begin{cases} -\frac{3\eta}{4} & \text{if } \tilde{\theta}_i(t) > -\frac{3\pi}{4} + \frac{3\eta}{4} \\ -\frac{3\pi}{4} - \tilde{\theta}_i(t) & \text{otherwise} \end{cases}$

for $i \in A_2$, $u_i(t) = -\tilde{\theta}_i(t)$ for $i \in A_3$. Similar to (23) we can get

$\theta_i(t_4') \in \begin{cases} \left[ \frac{3\pi}{4} - \frac{\eta}{8}, \frac{3\pi}{4} + \frac{\eta}{8} \right] & \text{if } i \in A_1 \\ \left[ -\frac{3\pi}{4} - \frac{\eta}{8}, -\frac{3\pi}{4} + \frac{\eta}{8} \right] & \text{if } i \in A_2 \\ \left[ -\frac{\eta}{8}, \frac{\eta}{8} \right] & \text{if } i \in A_3 \end{cases}. \quad (27)$

With the fact of $\eta \in (0, \pi)$ we have $d_{\theta_i(t_4')} > \pi$.

For protocol (4.2), with the same process as (25)-(27) our result follows. □
Appendix B  Proofs of Lemmas 4.9 and 4.10

Proof of Lemma 4.9  This proof partly takes the ideas of the proof of Lemma 4.7. Given a large integer $K > 0$, throughout this proof we choose $\tilde{\delta}_i(t) = \frac{\eta}{2K}$ for $i = 1, \ldots, n$ and $t \geq 0$. Set $t_0 := \left\lceil \frac{L}{2v \sin \frac{\eta}{K}} \right\rceil$. For $i = 1, \ldots, n$ and $t \in [0, t_0)$, we choose

$$u_i(t) = \begin{cases} -\frac{3\eta}{2K} - \tilde{\delta}_i(t) & \text{if } x_{i2}(t) \in \left[ \frac{L}{2}, L \right), \\ \frac{3\eta}{2K} - \tilde{\delta}_i(t) & \text{if } x_{i2}(t) \in \left[ 0, \frac{L}{2} \right). \end{cases} \tag{.28}$$

Under protocol (4.2), for $t \in [0, t_0)$, in the case of $x_{i2}(t) \geq L/2$, we have $\tilde{\theta}_i(t + 1) \in [-2\eta/K, -\eta/K]$ and

$$x_{i2}(t + 1) = x_{i2}(t) + v \sin \tilde{\theta}_i(t + 1)$$

$$\in [x_{i2}(t) - v \sin \frac{2\eta}{K}, x_{i2}(t) + v \sin \frac{\eta}{K}],$$

and in the case of $x_{i2}(t) < L/2$, we have $\tilde{\theta}_i(t + 1) \in [\eta/K, 2\eta/K]$ and

$$x_{i2}(t + 1) \in [x_{i2}(t) + v \sin \frac{\eta}{K}, x_{i2}(t) + v \sin \frac{2\eta}{K}].$$

From these and with the condition $\max_{1 \leq i \leq n} |\theta_i(0)| \leq \eta/2$ we have

$$u_i(t) \in [-\eta + \tilde{\delta}_i(t), \eta - \tilde{\delta}_i(t)], \quad \forall 1 \leq i \leq n, 0 \leq t < t_0, \tag{.29}$$

and can compute that

$$\max_{1 \leq i \leq n} |\theta_i(t_0)| \leq \frac{2\eta}{K} \quad \text{and} \quad \max_{1 \leq i \leq n} |x_{i2}(t_0) - \frac{L}{2}| \leq v \sin \frac{2\eta}{K}. \tag{.30}$$

Next we proceed the proof for the following two cases respectively:

Case 1: $n$ is even or $\varepsilon > 1/n$. We separate the $n$ agents into two disjoint sets $A_1$ and $A_2$ with $|A_1| = \left\lfloor \frac{n}{2} \right\rfloor$, $|A_2| = \left\lceil \frac{n}{2} \right\rceil$, and $x_{i2}(t) \geq x_{j2}(t)$ for any agent $i \in A_1, j \in A_2$. Let

$$t_1 := t_0 + \left\lceil \frac{r_{\max}}{2v \sin (\frac{\eta}{2} - \frac{\eta}{K})} \right\rceil + 1.$$

For $t_0 \leq t < t_1$, we choose

$$u_i(t) = \begin{cases} \frac{\eta}{2} - \frac{\eta}{2K} - \tilde{\delta}_i(t) & \text{if } i \in A_1, \\ -\frac{\eta}{2} + \frac{\eta}{2K} - \tilde{\delta}_i(t) & \text{if } i \in A_2. \end{cases} \tag{.31}$$

From this and the protocol (4.2) we can get

$$\theta_i(t + 1) \in \begin{cases} \left[ \frac{\eta}{2} - \frac{\eta}{2K} + \frac{\eta}{2} \right] & \text{if } i \in A_1, \\ \left[ -\frac{\eta}{2} + \frac{\eta}{2K} + \frac{\eta}{2} \right] & \text{if } i \in A_2. \end{cases} \tag{.32}$$

Thus, similar to (4.13) we have

$$x_{i2}(t_1) - x_{j2}(t_1) > r_{\max}, \quad \forall i \in A_1, j \in A_2, \tag{.33}$$

26
and together (30) and (32) we can compute

\[ |x_{i2}(t_1) - \frac{L}{2} + v(t_1 - t_0)\sin\frac{\eta}{2} - (\eta_1^2 + v(t_1 - t_0)\sin\frac{\eta}{2} + \eta_1 | \forall 1 \leq i \leq n. \quad (34) \]

Also, combining (31), (32) and the first inequality of (30) we have

\[ u_i(t) \in [\eta + \delta_i(t), \eta - \delta_i(t)], \forall 1 \leq i \leq n, t_0 \leq t < t_1. \quad (35) \]

Next we will give the control algorithm to minimize the value of the order function. Set

\[ t_2 := \max \left\{ t_1 + \left[ \frac{(\pi - \eta)K + \eta}{2(K - 1)\eta} \right], t_1 + 1 \right\}. \]

For \( t_1 \leq t < t_2 \), we choose

\[ u_i(t) = \begin{cases} \eta - \frac{\eta}{2K} & \text{if } \tilde{\theta}_i(t) < \frac{\pi}{2} + \frac{\eta}{2K} - \eta \\ \frac{\pi}{2} - \theta_i(t) & \text{otherwise} \end{cases} \]

for \( i \in A_1 \), and

\[ u_i(t) = \begin{cases} -\eta + \frac{\eta}{2K} & \text{if } \tilde{\theta}_i(t) > -\frac{\pi}{2} - \frac{\eta}{2K} + \eta \\ -\frac{\pi}{2} - \theta_i(t) & \text{otherwise} \end{cases} \]

for \( i \in A_2 \). From these and the fact of \( \max_{1 \leq i \leq n} |\theta_i(t_1)| \leq \frac{\eta}{2} \) it can be obtained that

\[ u_i(t) \in [-\eta + \delta_i(t), \eta - \delta_i(t)], \forall 1 \leq i \leq n, t_1 \leq t < t_2. \quad (36) \]

Also, if the sets \( A_1 \) and \( A_2 \) are disconnected at time \( t \), with the similar methods to (3) and (5) we can get

\[ \min_{i \in A_1} \theta_i(t + 1) \begin{cases} \in \left[ \frac{\pi}{2} - \frac{\eta}{2K}, \frac{\pi}{2} + \frac{\eta}{2K} \right] & \text{if } \min_{i \in A_1} \theta_i(t) \geq \frac{\pi}{2} + \frac{\eta}{2K} - \eta, \\ \geq \frac{(K - 1)\eta}{K} + \min_{i \in A_1} \theta_i(t) & \text{otherwise}, \end{cases} \quad (37) \]

and

\[ \max_{i \in A_2} \theta_i(t + 1) \begin{cases} \in \left[ -\frac{\pi}{2} - \frac{\eta}{2K}, -\frac{\pi}{2} + \frac{\eta}{2K} \right] & \text{if } \max_{i \in A_2} \theta_i(t) \leq \eta - \frac{\eta}{2K} - \frac{\pi}{2}, \\ \leq -\frac{(K - 1)\eta}{K} + \max_{i \in A_2} \theta_i(t) & \text{otherwise}. \end{cases} \]

Therefore, for any \( t_1 < t < t_2 \), if there exists no edge between \( A_1 \) and \( A_2 \) at every time in \( [t_1, t) \) we can get that: for all \( i \in A_1 \), together (32), (37) and the fact of \( u_i(t) + b_i(t) \leq \eta \) we have

\[ \frac{\eta}{2} - \frac{\eta}{K} + (t - t_1)(1 - \frac{1}{K})\eta \leq \theta_i(t) \leq \frac{\eta}{2} + (t - t_1)\eta, \]
so we can get \( x_{i2}(t) \geq x_{i2}(t_1) \) and

\[
x_{i2}(t) = x_{i2}(t_1) + \sum_{t_1 < k \leq t} v \sin \theta_i(k)
\]

\[
\leq \frac{L}{2} + v \sin \frac{2\eta}{K} + v(t - t_0) \sin \frac{\eta}{2}
\]

\[
+ v \sum_{k=t_1+1}^{t_2-1} \max_{\alpha \in [-\frac{\eta}{\pi} + (t-t_0)(1-\frac{1}{K})\eta, (t-t_0)\eta]} \sin \left( \frac{\eta}{2} + \alpha \right)
\]

\[
\leq \frac{L}{2} + \frac{r_{\max}}{2} + v \sum_{k=0}^{\left\lfloor \frac{\pi r_{\max}}{2} \right\rfloor} \sin \left( \frac{\eta}{2} + k\eta \right) \quad \text{as } K \to \infty,
\]

where the first inequality uses \([34]\); symmetrically, for \( j \in A_2 \) we can get \( x_{j2}(t) \leq x_{j2}(t_1) \) and

\[
x_{j2}(t) \geq \frac{L}{2} - \frac{r_{\max}}{2} - v \sum_{k=0}^{\left\lfloor \frac{\pi r_{\max}}{2} \right\rfloor} \sin \left( \frac{\eta}{2} + k\eta \right) \quad \text{as } K \to \infty.
\]

Thus, together these with \([33]\) and the condition

\[
L > 2r_{\max} + 2v \sum_{k=0}^{\left\lfloor \frac{\pi r_{\max}}{2} \right\rfloor} \sin \left( \frac{\eta}{2} + k\eta \right),
\]

by induction we can get \( A_1 \) and \( A_2 \) are always disconnected during the time interval \([t_1, t_2]\) for large \( K \). Using this and the similar method to \([19]\) we have \( \varphi(t_2) \leq \varepsilon \) for large \( K \). Combined with \([29], [35], [36]\) this yields our result.

**Case II:** \( n \) is odd and \( \varepsilon \leq \frac{1}{n} \). We separate the \( n \) agents into three disjoint sets \( A_1, A_2 \) and \( A_3 \) which satisfy that \( |A_1| = |A_2| = \frac{n-1}{2}, |A_3| = 1 \), and \( |X_i(t)|_2 \geq |X_j(t)|_2 \geq |X_k(t)|_2 \) for any agent \( i \in A_1, j \in A_3 \) and \( k \in A_2 \). Let

\[
t_3 := t_0 + \left\lfloor \frac{r_{\max}}{v(\sin(\frac{\eta}{2} - \frac{\eta}{K}) - \sin \frac{\eta}{2K})} \right\rfloor + 1.
\]

For \( t_0 \leq t \leq t_3, \) we choose \( u_i(t) \) to be the same values as \([31]\) when \( i \in A_1 \cup A_2 \), and to be \(-\theta_i(t)\) when \( i \in A_3 \), which indicates that

\[
\theta_i(t+1) \in \begin{cases} 
\left[ \frac{\eta}{2} - \frac{\eta}{K}, \frac{\eta}{2} \right] & \text{if } i \in A_1, \\
\left[ -\frac{\eta}{2} - \frac{\eta}{2K}, -\frac{\eta}{2} \right] & \text{if } i \in A_2, \\
\left[ -\frac{\eta}{2K}, \frac{\eta}{2K} \right] & \text{if } i \in A_3.
\end{cases}
\]

Then with the similar argument to \([13]\) we can get the sets \( A_1, A_2 \) and \( A_3 \) are mutually disconnected at time \( t_3 \).

Let \( c_n := \frac{\pi}{2} + \arcsin \frac{1}{n-1} \) and set

\[
t_4 := \max \left\{ t_3 + \left\lfloor \frac{(c_n - \frac{\eta}{2})K + \frac{\eta}{2}}{(K-1)\eta} \right\rfloor, t_3 + 1 \right\}.
\]

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For $t_3 \leq t < t_4$, we choose

\[
u_i(t) = \begin{cases} 
\frac{\eta}{2} - \frac{\eta}{4K} & \text{if } \tilde{\theta}_i(t) < c_n + \frac{\eta}{2K} - \eta \\
c_n - \theta_i(t) & \text{otherwise}
\end{cases}
\]

for $i \in A_1$, and

\[
u_i(t) = \begin{cases} 
-\frac{\eta}{2} + \frac{\eta}{4K} & \text{if } \tilde{\theta}_i(t) > -c_n - \frac{\eta}{2K} + \eta \\
-c_n - \theta_i(t) & \text{otherwise}
\end{cases}
\]

for $i \in A_2$, and $u_i(t) = -\tilde{\theta}_i(t)$ for $i \in A_3$. With the similar argument to Case I and using the condition of

\[L > 2r_{\text{max}} + 2v \left\lfloor \frac{c_n}{\eta} - \frac{1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{c_n}{\eta} - \frac{1}{2} \right\rfloor} \sin \left( \frac{\eta}{2} + k\eta \right)\]

we can get $A_1$, $A_2$, and $A_3$ are mutually disconnected at every time from $t_3$ to $t_4$, and so $\varphi(t_4) \leq \varepsilon$ by the similar method of (4.24). Also, similar to Case I we can get

\[
u_i(t) \in \left[ -\eta + \delta_i(t), \eta - \delta_i(t) \right], \quad \forall 1 \leq i \leq n, t_0 \leq t < t_4.
\]

Together these with (29) our result is obtained.

**Proof of Lemma 4.10** We consider the protocol (4.1) firstly. Let $b$ be the middle value of the minimal interval contains all the initial headings of the agents. Without loss of generality we assume $b \in [0, \pi/4)$. Let $t_0 := \left\lceil \frac{\pi}{4\eta} \right\rceil - 1$. For $t \in [0, t_0)$ and $1 \leq i \leq n$, we choose

\[
(\delta_i(t), u_i(t)) = \begin{cases} 
(\eta/4, -3\eta/4) & \text{if } \tilde{\theta}_i(t) > b + \eta/2, \\
(\eta/2, b - \tilde{\theta}_i(t)) & \text{if } \tilde{\theta}_i(t) \in [b - \eta/2, b + \eta/2], \\
(\eta/4, 3\eta/4) & \text{if } \tilde{\theta}_i(t) < b - \eta/2.
\end{cases}
\]

Similar to (26), we can get

\[
\max_{1 \leq i \leq n} |\theta_i(t_0) - b| \leq \frac{\eta}{2}.
\]

Set $t_1 := t_0 + \left\lceil \frac{\pi}{2\eta} \right\rceil - 1$. For $t \in [t_0, t_1)$ and $1 \leq i \leq n$, we choose

\[
(\delta_i(t), u_i(t)) = \begin{cases} 
(\eta/4, -3\eta/4) & \text{if } \tilde{\theta}_i(t) > \eta/2, \\
(\eta/2, -\tilde{\theta}_i(t)) & \text{if } \tilde{\theta}_i(t) \in [-\eta/2, \eta/2].
\end{cases}
\]

With the similar method to (26) again we have

\[
\max_{1 \leq i \leq n} |\theta_i(t_1)| \leq \frac{\eta}{2}.
\]
Set \( t_2 := t_1 + \left\lceil \frac{L}{2v \sin \frac{\eta}{K}} \right\rceil \). For \( i = 1, \ldots, n \) and \( t \in [t_1, t_2) \), we choose \( \delta_i(t) = \frac{\eta}{2K} \) and \( u_i(t) \) as (.28). Similar to (.30) we have

\[
\max_{1 \leq i \leq n} |\theta_i(t_2)| \leq \frac{2\eta}{K} \quad \text{and} \quad \max_{1 \leq i \leq n} |x_{i2}(t_2) - \frac{L}{2}| \leq v \sin \frac{2\eta}{K}.
\]

Next we separate the \( n \) agents into four disjoint nonempty sets \( A_i, i = 1, 2, 3, 4 \), which satisfy that \([X_i(t)]_2 \geq [X_j(t)]_2 \geq [X_k(t)]_2 \geq [X_l(t)]_2 \) for any agent \( i \in A_1, j \in A_2, k \in A_3 \) and \( l \in A_4 \). Let

\[
t_3 := t_2 + \left\lceil \frac{r_{\max}}{2v \sin(\frac{\pi}{2} - \frac{\eta}{K})} \right\rceil + 1.
\]

For \( t_2 \leq t < t_3 \), we choose

\[
u_i(t) = \begin{cases} \frac{\eta}{2} + \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{if } i \in A_1 \cup A_2, \\ \frac{\eta}{2} - \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{if } i \in A_3 \cup A_4. \end{cases}
\]

Set

\[
t_4 := \max \left\{ t_3 + \left\lceil \frac{(\pi - \eta)K + 2\eta}{2(K - 1)\eta} \right\rceil, t_3 + 1 \right\}.
\]

In the time \( t \in [t_3, t_4) \), we choose

\[
u_i(t) = \begin{cases} \frac{\eta}{2} - \frac{\eta}{2K} & \text{if } \bar{\theta}_i(t) < \frac{\pi}{2} - \frac{\eta}{K}, \\ \frac{\eta}{2} + \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{otherwise} \end{cases}
\]

for \( i \in A_1 \), and

\[
u_i(t) = \begin{cases} \frac{\eta}{2} - \frac{\eta}{2K} & \text{if } \bar{\theta}_i(t) < \frac{\pi}{2} - \eta, \\ \frac{\eta}{2} + \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{otherwise} \end{cases}
\]

for \( i \in A_2 \), and

\[
u_i(t) = \begin{cases} -\frac{\eta}{2} + \frac{\eta}{2K} & \text{if } \bar{\theta}_i(t) > -\frac{\pi}{2} + \eta, \\ -\frac{\eta}{2} - \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{otherwise} \end{cases}
\]

for \( i \in A_3 \), and

\[
u_i(t) = \begin{cases} -\frac{\eta}{2} - \frac{\eta}{2K} & \text{if } \bar{\theta}_i(t) > -\frac{\pi}{2} - \eta + \frac{\eta}{K}, \\ -\frac{\eta}{2} - \frac{\eta}{2K} - \bar{\theta}_i(t) & \text{otherwise} \end{cases}
\]

for \( i \in A_4 \). With the similar discuss to the Case I of the proof of Lemma 4.9 we can get \( d_{\theta(t_4)} \geq \pi \) under condition (5.3).

For protocol (4.2), combined (.30) and the process of the above paragraph our result follows.
Appendix .C  Proofs of Theorems 3.4-3.5 and 3.8-3.10

Proof of Theorem 3.4  For any \( t \geq 0 \), if \( \max_{1 \leq i \leq n} |\theta_i(t)| \leq \frac{\eta}{2} \), similar to the proof of Lemma 4.7 we separate the \( n \) agents into two disjoint sets \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) with \( |\mathcal{S}_1| = \lfloor \frac{\eta}{2} \rfloor \), \( |\mathcal{S}_2| = \lceil \frac{\eta}{2} \rceil \), and \( x_{ij}(0) \geq x_{ji}(0) \) for any agent \( i \in \mathcal{A}_1, j \in \mathcal{A}_2 \). Let \( T_1 := \left\lfloor \frac{-\max_{1 \leq i \leq n} \{ |\theta_i(t)| \}}{2\sin(\eta/4)} \right\rfloor + 1 \) and \( T \) be an arbitrary large integer. Under protocol (4.2), for \( t' \in [t, t + T_1 + T) \), we choose \( \delta_i(t') \) and \( u_i(t') \) as same as (4.10) and (4.11) respectively. Then by (4.12) and (4.13) we can get that there is always no edge between \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) in the time \([t + T_1, t + T_1 + T]\). With this and the method of (4.3) we have for system II,

\[
P\left( \bigcup_{t'=t+T_1}^{t+T_1+T} G(t') \text{ is not weakly connected} \Big| \forall (X(t), \theta(t)) \in \Omega_{\eta}^1 \right) \geq \frac{1}{8^n(T_1+T)} . \tag{39}
\]

Also, for any \( t \geq 0 \) and \((X', \theta') \in \Omega^*, \) together (4.1), (4.2), (4.8) and the method of (4.3) we can get for system II,

\[
P\left( (X(t+T_2), \theta(t+T_2)) \in \Omega_{\eta}^1 | X(t) = X', \theta(t) = \theta' \right) \geq \frac{1}{4^nT_2} , \tag{40}
\]

where \( T_2 := \lfloor \frac{2(\pi - \eta/4)}{\eta} \rfloor = \left\lfloor \frac{2\pi}{\eta} - \frac{1}{4} \right\rfloor \). By (39), (40) and Bayes’ theorem we have

\[
P\left( \bigcup_{t'=t+T_1+T_2}^{t+T_1+T_2+T} G(t') \text{ is not weakly connected} | X(t) = X', \theta(t) = \theta' \right) \geq P\left( (X(t+T_2), \theta(t+T_2)) \in \Omega_{\eta}^1 | X(t) = X', \theta(t) = \theta' \right) .
\]

\[
P\left( \bigcup_{t'=t+T_1+T_2}^{t+T_1+T_2+T} G(t') \text{ is not weakly connected} | (X(t+T_2), \theta(t+T_2)) \in \Omega_{\eta}^1, X(t) = X', \theta(t) = \theta' \right) \geq \frac{1}{4^nT_2 8^n(T_1+T)} .
\]

Similar to (4.5) with probability 1 there is a time \( t^* > 0 \) such that \( \bigcup_{t'=t^*+T_1+T_2}^{t^*+T_1+T_2+T} G(t') \) is not weakly connected. \( \square \)

Proof of Theorem 3.5  For any \( t \geq 0 \), if \( \max_{1 \leq i \leq n} |\theta_i(t)| \leq \frac{\eta}{2} \), with \( \varepsilon \) being a small positive constant, then we set \( T := \left\lfloor \frac{(\pi - \varepsilon)K + \eta}{2(K-1)\eta} \right\rfloor \) with \( K \) being a large integer, and choose \( \delta_i(t') = \frac{\eta}{2K} \) and

\[
u_i(t') = \begin{cases} \frac{\eta}{2} - \frac{\eta}{2K} & \text{if } \theta_i(t') < \frac{\pi}{2} + \frac{\eta}{2K} - \eta \\ \frac{\pi}{2} - \theta_i(t') & \text{otherwise} \end{cases}
\]

for \( t' \in [t, t + T) \) and \( i = 1, \ldots, n \). Under this process we can get that \( \theta_i(t + T) \in \left[ \frac{\pi}{2} - \frac{\eta}{2K}, \frac{\pi}{2} + \frac{\eta}{2K} \right] \) for \( 1 \leq i \leq n \), and during the time \([t, t + T]\) all the agents keep almost synchronization, which indicate the event of turn has happened. Using
Lemmas 4.2 and 4.5 we can get the event of turn will happen infinite times with probability 1.

Similarly, combing (1.18), Lemmas 4.2 and 4.5 we can get the events of bifurcation and merge will happen infinite times with probability 1.

**Proof of Theorem 3.8** Given \( t > 0 \), suppose \( \max_{1 \leq i \leq n} |\theta_i(t)| \leq \frac{\eta}{2} \). We separate the \( n \) agents into two disjoint sets \( A_1 \) and \( A_2 \) with \( |A_1| = \lceil \frac{n}{2} \rceil \), \( |A_2| = \lceil \frac{n}{2} \rceil \), and \( x_{i2}(t) \geq x_{j2}(t) \) for any agent \( i \in A_1, j \in A_2 \). Set \( T_1 := \left\lfloor \frac{r_{\max}}{2\pi \sin(\eta/4)} \right\rfloor + 1, \) where \( K \) is an integer not smaller than 4, and set \( T \) be an arbitrary large integer. Under protocol (4.2), for \( t' \in [t, t + T + T] \) we choose \( \delta_i(t') = \frac{\eta}{2K} \) for \( 1 \leq i \leq n \),

\[
\begin{cases}
\frac{-3n}{2K} - \bar{\theta}_i(t') & \text{if } x_{i2}(t') \geq \frac{3L}{4} \\
\frac{3n}{2K} - \bar{\theta}_i(t') & \text{if } x_{i2}(t') < \frac{3L}{4},
\end{cases}
\]

for \( i \in A_1 \), and

\[
\begin{cases}
\frac{-3n}{2K} - \bar{\theta}_i(t') & \text{if } x_{i2}(t') \geq \frac{L}{4} \\
\frac{3n}{2K} - \bar{\theta}_i(t') & \text{if } x_{i2}(t') < \frac{L}{4},
\end{cases}
\]

for \( i \in A_2 \). Similar to (3.30) we can get for all \( t' \in [t + T_1, t + T_1 + T] \),

\[
\max_{i \in A_1} |x_{i2}(t') - \frac{3L}{4}| \leq v \sin \frac{2\eta}{K} \quad \text{and} \quad \max_{i \in A_2} |x_{i2}(t') - \frac{L}{2}| \leq v \sin \frac{2\eta}{K},
\]

which indicates that if \( L > 2r_{\max} \) then \( \bigcup_{t'=t_0}^{t+T_1} G(t') \) is not weakly connected for large \( K \). Under Protocol (2.5), similar to (4.39) we have

\[
P \left( \bigcup_{t'=t+T_1}^{t+T_1+T} G(t') \text{ is not weakly connected} \mid \forall (X(t), \theta(t)) \in \Omega^1_\eta \right) \geq (2K)^{-n(T_1+T)}.
\]

Because \( \Omega^1_\eta \) is also finite-time robustly reachable from \( \Omega^* \) under the periodic boundary conditions, with the similar procedure from (4.40) to the end of the proof of Theorem 3.4 we can get our result.

**Proof of Theorem 3.9** With the same discussion as the first paragraph of the proof of Theorem 3.5 we can get the event of turn will happen infinite times with probability 1.

Given a time \( t_1 \), suppose \( \max_{1 \leq i \leq n} |\theta_i(t_1)| \leq \frac{\xi}{2} \) for a small constant \( \xi > 0 \). Under the similar process from (3.38) to the end of the proof of Lemma 4.9 we can get the event of bifurcation happens in the time \( [t_1, t_4] \), where \( t_4 \) is the same constant in the proof of Lemma 4.9. Also, with the similar process as the proof of Lemma 4.5 we can get there exist a time \( t_5 > t_4 \) such that the event of merge happens in the time \( [t_4, t_5] \). Using Lemmas 4.2 and 4.5 we can get the events of bifurcation and merge will happen infinite times with probability 1.

**Proof of Theorem 3.10** The condition of this theorem equals to \( \Omega^3_\pi \) is finite-time robustly reachable from \( \Omega^* \) under protocol (4.1). Also, for any \( \alpha > 0 \), by (2.6) we
can get $\Omega_1^\alpha$ is finite-time robustly reachable from $\Omega_3^\beta$ under protocol (4.1). Thus, similar to Lemma 4.5, we have $\Omega_1^\alpha$ is finite-time robustly reachable from $\Omega_1^* \pi$ under protocol (4.1). With the almost same methods to the proofs of Theorems 3.3, 3.5 and 3.7-3.9 but using this instead of Lemma 4.5, our results can be obtained. Also, because system I has the property of isotropy, we can get it exist vortices with arbitrary long duration by adding the turning angles in the proofs of Theorems 3.5 and 3.9. □