We investigate the $W_\infty$ algebra in the integer quantum Hall effects. Defining the simplest vacuum, the Dirac sea, we evaluate the central extension for this algebra. A new algebra which contains the central extension is called the $W_{1,\infty}$ algebra. The $W_{1,\infty}$ algebra is considered to play an important role for the incompressibility which is a property of the bulk of the electron liquid. We show that the $W_{1,\infty}$ algebra is crucial for the edge states and it is an origin of the Kac-Moody algebra which determines the behavior of edge states of the system.

§ 1. Introduction

In this paper, we discuss the $W_\infty$ algebra in the integer quantum Hall effect (IQHE). It has been shown that there is the $W_\infty$ algebra in the many-electron system constrained in the lowest Landau level (LLL).8,9 The operators contained in the $W_\infty$ algebra are defined in the bulk of the electron droplet and represent the incompressibility of the integer quantum Hall state. On the other hand, it is considered that edge states are described by the one-dimensional chiral fermion and the operators defined on the edge satisfy the Kac-Moody algebra.5-7 Relations between the $W_\infty$ algebra and the edge states have not been discussed clearly. In this paper, we show that the central extension for the $W_\infty$ algebra reveals the property of edge states. There are relations between the bulk and the edge.

When a charged particle moves on a plane in a perpendicular uniform magnetic field, discrete energy levels appear. They are called the Landau levels. There are degenerate states in each level. In the quantum Hall effect (QHE) whose filling factor $\nu$ is equal to or less than one, taking the limit of the large external magnetic field, electrons are constrained in the LLL.1,2 In the IQHE, the LLL is completely filled and liquid of electrons is incompressible without an interaction. The incompressibility is an important feature of the QHE.1,3 In an incompressible droplet, a local fluctuation of the density is forbidden. There are no gapless collective excitations in a droplet. A gapless excitation can occur only at a deformation of the edge. This excitation is called an edge state.4 To investigate an edge state, Wen and Stone and others considered edge-charge operators. It has been indicated that the dynamics of the edge is described by a one-dimensional chiral fermion and edge-charge operators satisfy the Kac-Moody algebra.5-7

While the Kac-Moody algebra is discussed for understanding the edge states, the $W_\infty$ algebra is introduced to understand the bulk properties. Although the wave functions in the LLL do not form a complete set, a second-quantized fermion field is constructed from them. Under unitary transformations of this field, a wave function remains in the LLL and the total fermion number does not change. The generators of these transformations form the $W_\infty$ algebra. The $W_\infty$ algebra is a quantized
version of the area-preserving diffeomorphisms. It represents the incompressibility which is one of the properties of the bulk.

In this paper, we relate the edge properties described by the Kac-Moody algebra and the bulk properties described by the \( W_\infty \) algebra. Since generators of the \( W_\infty \) algebra are products of fermion operators, they must be normal ordered and the commutation relations of them must acquire a central extension. We show that for a droplet of \( \nu=1 \) state, which we call the “vacuum” state, normal ordered generators get a central extension at the boundaries of the droplet. This central extended \( W_\infty \) algebra is called the \( W_{1+\infty} \) algebra and it contains the Kac-Moody generators for the edge states. There are infinitely many generators in the \( W_\infty \) algebra and this assures an existence of the Kac-Moody generators along any boundaries which we consider arbitrarily in the droplet. This consideration shows that the \( W_{1+\infty} \) algebra, which originally describes bulk properties of the IQHE, is essential even for edge properties of the IQHE.

Here, we summarize the contents of this paper. In § 2, we review a single-particle kinematics in the LLL. The degenerate wave function in the LLL can be labeled by an operator which commutes with the Hamiltonian. The wave functions in the LLL can be regarded as a one-dimensional system. Then, we review unitary transformations of the second-quantized field and the \( W_\infty \) algebra. Particularly we consider the meaning of the limit of the large external magnetic field \( B \). The large \( B \) limit is a sort of the classical limit.

In § 3, we discuss some properties of the \( W_\infty \) algebra. The \( W_\infty \) algebra (strictly speaking, the \( W_{1+\infty} \) algebra which has no central extension terms) discussed in this paper contains infinite operators whose conformal dimensions are equal to or above one. We arrange these operators in order. This \( W_\infty \) algebra contains the \( c=0 \) Virasoro algebra. We find primary fields which are made from the electron field. Because there are no central extension terms for the current algebra, we can hardly find a physical meaning in it. We notice that we had better define an appropriate vacuum and obtain a non-trivial current algebra.

In § 4, we discuss localized electron liquid. We regard this state as a vacuum. Defining the simplest vacuum, the Dirac sea, we derive the central extension and obtain an exact form of the \( W_{1+\infty} \) algebra. In this vacuum, the filling factor \( \nu \) is equal to one. It means that we are considering the IQHE. In the \( W_{1+\infty} \) algebra, there is a non-trivial current algebra. We can expect to extract some physical meanings from it. Taking the large \( B \) limit, we can regard electrons as a localized droplet whose density is uniform. Electrons behave as an incompressible droplet.

In § 5, we discuss the edge-charge operators. Wen and Stone and others regarded an edge wave as a one-dimensional chiral fermion model and claimed that edge-charge operators formed the Kac-Moody algebra. We show that the origin of this Kac-Moody algebra is the \( W_{1+\infty} \) algebra.

§ 2. The many-body system in the lowest Landau level and the \( W_\infty \) algebra

At first, we review a single-particle quantum mechanics on a plane. When an electron moves on a plane with a perpendicular uniform magnetic field, energy levels
are quantized discretely. These energy levels are called the Landau levels. In the LLL, there are infinite degenerate states. We can label these states perfectly by an appropriate operator which commutes with the Hamiltonian.

Then we consider a many-fermion system in the LLL. We review the recent work that the $W_\infty$ algebra appears in unitary transformations of the second-quantized field. These transformations preserve the states in the LLL and keep the total fermion number invariant. Generators of these transformations are made from a density operator and the commutation relation of them is the $W_\infty$ algebra. We discuss the large $B$ limit, which is a sort of the classical limit.

The Hamiltonian of a single-particle quantum mechanics is given by

$$H_0 = \frac{1}{2m} (\vec{p} + eA)^2,$$

where $\vec{p} = -i\nabla$, $\nabla \times \vec{A} = B$. (2.1)

The unit is given by $c = \hbar = 1$. $B(>0)$ is a perpendicular uniform external magnetic field on the $xy$-plane. The momentum operators $\pi_i$ are given by

$$\pi_i = (p + eA)_i \quad \text{for } i = x, y, \quad [\pi_x, \pi_y] = -ieB.$$

Annihilation and creation operators are defined in the form,

$$a = \frac{1}{\sqrt{2eB}} (\pi_x - i\pi_y), \quad a^\dagger = \frac{1}{\sqrt{2eB}} (\pi_x + i\pi_y), \quad [a, a^\dagger] = 1. \quad (2.3)$$

Writing the Hamiltonian with the annihilation and creation operators, we can treat this system as a harmonic oscillator,

$$H_0 = \omega (a^\dagger a + \frac{1}{2}), \quad E_0 = \frac{1}{2} \omega, \quad \frac{3}{2} \omega, \cdots.$$  (2.4)

where $\omega = eB/m$. There are degenerate states in each level. In the massless limit, the electron occupies the lowest level. From now on we consider only the LLL. The condition that $\phi(\vec{x})$ is in the LLL is given by $a\phi(\vec{x}) = 0$.

Now, we define the guiding center coordinates,

$$\vec{X} = x - \frac{\pi_y}{eB}, \quad \vec{Y} = y + \frac{\pi_x}{eB}, \quad [\vec{X}, \vec{Y}] = \frac{i}{eB}. \quad (2.5)$$

There are the following relations, $[H_0, \vec{X}] = [H_0, \vec{Y}] = [H_0, \vec{X}^2 + \vec{Y}^2] = 0$. We can label the degenerate states in the LLL by eigenstates of $\vec{X}$, $\vec{Y}$ or $\vec{X}^2 + \vec{Y}^2$.

For example, let us consider wave functions in the LLL that diagonalize $\vec{X}^2 + \vec{Y}^2$. In the gauge $\vec{A} = (-By/2, Bx/2)$, we obtain an orthonormal basis,

$$\phi_n(\vec{x}) = \sqrt{\frac{eB}{2\pi 2^n n!}} (\vec{z} \sqrt{eB})^n \exp\left(-\frac{eB}{4} |\vec{z}|^2\right),$$  (2.6)

where $\vec{z} = x + iy$ and $n = 0, 1, 2, \cdots$. We can regard $\vec{X}^2 + \vec{Y}^2$ as an angular momentum operator.

Taking the gauge $\vec{A} = (-By, 0)$ and diagonalizing $\vec{Y}$, we obtain another orthonormal basis of wave functions in the LLL. The basis is given by
\[ \phi_r(x) = \sqrt{\frac{eB}{\pi}} \frac{1}{\sqrt{L}} \exp \left\{ ieB Y x - \frac{eB}{2} (y - Y)^2 \right\}, \]  

(2.7)

where \( 0 \leq x < L \). This wave function is localized around \( y = Y \). A density of states is equal to \( eB/2\pi \).

Let us consider a many-body problem. We define the filling factor by \( \nu = \frac{2\pi \rho}{eB} \), where \( \rho \) is a density of electrons. In the massless limit, if \( \nu \leq 1 \), all the electrons are constrained in the LLL.

We take the gauge \( \vec{A} = (-By/2, Bx/2) \) and use the basis of \( \{ \phi_n(x) \} \) defined in (2.6) for a while. We perform the second-quantization in the following way,

\[ \phi(x) = \sum_{n=0}^{\infty} \bar{C}_n \phi_n(x). \]  

(2.8)

\( \bar{C}_n \) and \( \bar{C}_n^\dagger \) obey the anti-commutation relation,

\[ \{ \bar{C}_n, \bar{C}_m^\dagger \} = \delta_{n,m}. \]  

(2.9)

(\( \{ \phi_n(x) \} \) does not form a complete set in a total Hilbert space and therefore, \( \phi(x) \) and \( \phi^\dagger(x) \) do not obey the usual anti-commutation relation.)

We consider a unitary transformation of \( \{ \bar{C}_n \} \),

\[ \bar{C}_n \to \bar{C}'_n = \sum_{m=0}^{\infty} u_{nm} \bar{C}_m. \]  

(2.10)

This transformation preserves the LLL condition, \( a\bar{C}'(x) = 0 \), and keeps the total fermion number invariant, \( \int d^2x \delta \rho(x) = 0 \), where \( \rho(x) = \phi^\dagger(x)\phi(x) \). The generator is given by the functional,

\[ \rho[\xi] = \int d^2x \rho(x) \xi(x), \quad [-i\rho[\xi], \phi(x)] = \delta \phi(x), \]  

(2.11)

where \( \xi(x) \) is an arbitrary real function.

Let us consider the commutation relation of \( \rho[\xi] \). Because it is difficult to calculate the commutation relation of \( \rho[\xi] \) directly, we derive the commutation relation of the Fourier component of \( \rho(x) \) first. Fourier components of the density operator \( \rho(x) \) are given by

\[ \rho(k) = \int d^2x \rho(x) \exp(i k \cdot x) \]

\[ = \sum_{n,m=0}^{\infty} \bar{C}_n^\dagger \bar{C}_m \left( -i \sqrt{2eB} \right)^{n+m} \left( \frac{\partial}{\partial k} \right)^n \left( \frac{\partial}{\partial \bar{k}} \right)^m \exp \left( -\frac{|k|^2}{2eB} \right). \]  

(2.12)

Using this representation, we obtain

\[ [\bar{\rho}(k), \bar{\rho}(k')] = -2i \sin \left( \frac{k \times k'}{2eB} \right) \bar{\rho}(k + k') \exp \left( -\frac{|k|^2}{2eB} \right). \]  

(2.13)

Here, we define \( W(k) \) in the form, \( W(k) = \bar{\rho}(k) \exp(|k|^2/4eB) \). The commutation relation of \( W(k) \) is given by
This commutation relation is called the $W_\infty$ algebra or the Fairlie-Fletcher-Zachos algebra.\textsuperscript{8-11)}

In the large $B$ limit, we obtain

\[ [W(\vec{k}), W(\vec{k}')] = -\frac{i}{eB} (\vec{k} \times \vec{k}') W(\vec{k} + \vec{k}'). \]  

This algebra is called the $w_\infty$ algebra or the area-preserving diffeomorphisms.\textsuperscript{14)} We can regard (2.14) as a quantized version of the area-preserving diffeomorphisms. Here we must pay attention to the following. We can always eliminate the appearance of the factor $eB$ by scaling the coordinates, $\vec{x}' = \sqrt{eB}/2 \vec{x}$. Therefore, strictly speaking, there is no meaning in taking the large $B$ limit. But, we can interpret this limit as the classical limit which reduces (2.14) to the $w_\infty$ algebra. The large $B$ limit sometimes reveals important information contained in the theory. (The classical limit is discussed in Ref. 15, too.)

A commutation relation of $\rho[\xi]$ is obtained,

\[ [\rho[\xi_1], \rho[\xi_2]] = \rho[[\{\xi_1, \xi_2\}]], \]  

where

\[ \{\{\xi_1, \xi_2\}\} = - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{2}{eB} \right)^n \{\partial^n\xi_1(\vec{x}) \partial^n\xi_2(\vec{x}) - \partial^n\xi_1(\vec{x}) \partial^n\xi_2(\vec{x})\}. \]  

$\{\{\xi_1, \xi_2\}\}$ is called the Moyal bracket.\textsuperscript{12)} In the large $B$ limit, the Moyal bracket becomes the Poisson bracket,

\[ \{\{\xi_1, \xi_2\}\} \to \frac{i}{eB} [\partial_\xi_1 \partial_- \xi_2 - \partial_\xi_1 \partial_+ \xi_2] = \{\xi_1, \xi_2\}_{PB}. \]  

Even if we take the gauge $\vec{A} = (-By, 0)$ and use the $\{\phi_Y(\vec{x})\}$ defined in (2.7), we obtain the same commutation relations.

§ 3. Some properties of the $W_\infty$ algebra

Let us investigate some properties of the $W_\infty$ algebra obtained in (2.14). This algebra contains infinite operators. Labelling these operators with the conformal dimensions, we arrange them in order. We show that (2.14) contains the Virasoro algebra. Then, we look for primary fields.

We define operators,

\[ \tilde{L}^{(m)}(k_x) = \exp\left(\frac{k_x^2}{4eB}\right) \int d^2x \ y^m \exp(ik_x x) \rho(\vec{x}) \]  

From (2.14), we obtain commutation relations of $\tilde{L}^{(m)}(k_x)$,

\[ [\tilde{L}^{(m)}(k_x), \tilde{L}^{(n)}(k_x)] = \]
\[ \begin{align*}
&= n! m! (-i)^{n+m} (-2i)^{\sum_{p,q,r=0}^{2p+1} \sum_{j=0}^{r} \delta_{n,2p+1-j+q+k} \delta_{m,j+q+r-h}} \\
&\times (-1)^{p+1-j} (i)^{r} \frac{1}{(2eB)^{2p+1+q}} \frac{1}{j! (2p+1-j)! q! (r-h)! h!} \\
&\times k_x k_z k^{2p+1-j} L^{(r)}(k_x + k_z). 
\end{align*} \]

We redefine \( L^{(n)}(k_x) \) by the form,
\[ L^{(n)}_m = \left( \frac{eBL}{2\pi} \right)^n \tilde{L}^{(n)} \left( -\frac{2\pi m}{L} \right) , \quad m \in \mathbb{Z} . \]

Commutation relations for \( L^{(0)}_n, L^{(1)}_n, L^{(2)}_n \) can be written in the forms,
\[ \begin{align*}
[L^{(0)}_n, L^{(0)}_m] &= 0 , \\
[L^{(0)}_n, L^{(1)}_m] &= n L^{(0)}_{n+m} , \\
[L^{(1)}_n, L^{(1)}_m] &= (n-m) L^{(1)}_{n+m} , \\
[L^{(0)}_n, L^{(2)}_m] &= 2n L^{(1)}_{n+m} , \\
[L^{(1)}_n, L^{(2)}_m] &= 2(n-m) L^{(2)}_{n+m} - n \left\{ \frac{nm}{4} + \left( \frac{L}{2\pi} \right)^2 eB \right\} L^{(1)}_{n+m} , \\
[L^{(2)}_n, L^{(2)}_m] &= 2(n-m) L^{(3)}_{n+m} + (n-m) \left\{ \frac{nm}{2} + \left( \frac{L}{2\pi} \right)^2 2eB \right\} L^{(1)}_{n+m} .
\end{align*} \]

In these relations, \( L^{(0)}_n \) looks like a lowering operator. It is important that \( L^{(0)}_n \) and \( L^{(1)}_n \) form a closed algebra. \( (3\cdot5) \) is the Virasoro algebra.

The conformal dimension is decided in the following way. If an operator \( A_m \) satisfies
\[ [L^{(1)}_n, A_m] = (n(h-1) - m) A_{n+m} + \cdots , \]
\( A_m \) is the Fourier mode of the field whose conformal dimension is \( h \). We notice that \( L^{(n)}_m \) is a component of the conformal dimension \( (n+1) \) field.

The algebra defined in \( (2\cdot14) \) contains the current operator \( L^{(0)}_n \). Commonly, the \( W_\infty \) algebra contains infinite operators whose conformal dimensions are equal to or above two. The general \( W_\infty \) algebra does not contain current operators. Therefore, strictly speaking, the algebra defined in \( (2\cdot14) \) is not the \( W_\infty \) algebra. It is the \( W_{1+\infty} \) algebra which has no central extension terms. In particular, the algebra of the current operator \( (3\cdot4) \) is trivial. If we define a vacuum and consider a central extension term, there must be an interesting physics in the current algebra. In § 4, we define the simplest vacuum and derive the central extension terms.

Next, we look for primary fields of \( L^{(1)}_n \). At first, we consider a field written in the form,
\[ \xi(x) = \int dy \, \phi(\vec{x}) . \]

Let us take the gauge \( \vec{A} = (-By, 0) \) and use the eigenfunctions of \( \vec{\gamma} \) defined in \( (2\cdot7) \).
We impose a periodic boundary condition, $0 \leq x < L$. $\xi(X)$ is given in the form,

$$\xi(x) = \sum_n \tilde{C}_n \sqrt{\frac{2}{L}} \sqrt{\frac{\pi}{eB}} \exp(i e B Y_n x),$$

(3.8)

where $Y_n = 2\pi n/ebL$, $n = 0, \pm 1, \pm 2, \ldots$. $\tilde{L}^{(1)}(k_x)$ is given in the form,

$$\tilde{L}^{(1)}(k_x) = \frac{2\pi a L}{eBL} \sum_n \tilde{C}_n \tilde{C}_{n-Lk_x/2\pi\pi} \frac{1}{2} \left( 2n - \frac{Lk_x}{2\pi} \right).$$

(3.9)

The commutation relations of $\tilde{L}^{(1)}(k_x)$ and $\xi(x)$, $\xi^*(x)$ are given by

$$[\tilde{L}^{(1)}(k_x), \xi(x)] = \frac{1}{eB} \exp(ik_x x) \left( i \delta_x - \frac{k_x}{2} \right) \xi(x),$$

(3.10)

$$[\tilde{L}^{(1)}(k_x), \xi^*(x)] = \frac{1}{eB} \exp(ik_x x) \left( i \delta_x - \frac{k_x}{2} \right) \xi^*(x).$$

(3.11)

Now, we define a new variable and new field operators,

$$w = \exp\left(-i \frac{2\pi x}{L}\right), \quad \eta(w) = \frac{\xi(x)}{w}, \quad \bar{\eta}(w) = \frac{\xi^*(x)}{w}.$$  

(3.12)

We can express the commutation relations in the form,

$$[L_n^{(1)}, \eta(w)] = w^{n+1} \partial_w \eta(w) + \frac{1}{2} (n+1) w^n \eta(w),$$

$$[L_n^{(1)}, \bar{\eta}(w)] = w^{n+1} \partial_w \bar{\eta}(w) + \frac{1}{2} (n+1) w^n \bar{\eta}(w).$$

(3.13)

$\eta(w)$ and $\bar{\eta}(w)$ are primary fields. Because we are considering free fermions, it is quite reasonable that both of their conformal dimensions are equal to $1/2$.

§ 4. The central extension of the $W_\infty$ algebra for the $\nu=1$ Dirac sea vacuum

In this section, we consider the state of the localized electron liquid. We define a vacuum and derive a central extension. We obtain an exact form of the $W_\infty$ algebra. The central extension creates the Kac-Moody current. To make the problem easy, we assume the simplest vacuum, the Dirac sea. (Cappelli and others considered the vacuum only for the Virasoro algebra and the Kac-Moody algebra. They did not consider the vacuum for all the generators of the $W_\infty$ algebra, therefore they did not derive the complete form of the $W_\infty$ algebra.) We take the large $B$ limit of this vacuum. We notice that the Dirac sea vacuum can be regarded as a liquid of electrons filled in the area $y \leq 0$. The filling factor $\nu$ is equal to 1 and the liquid of electrons shows incompressibility.

The system considered now can be regarded as a one-dimensional fermion system. In this system, $\hat{X}$ plays the role of a coordinate and $\hat{Y}$ plays the role of a momentum, $[\hat{X}, \hat{Y}] = i/eB$. We use a set of eigenstates defined in (2.7), $\hat{Y} \phi_{Yn}(x) = Y_n \phi_{Yn}(x)$. The vacuum state $|G\rangle$ is defined,
\[ |G\rangle = \prod_{n=0}^{\infty} \hat{C}_n^\dagger |0\rangle = \hat{C}_0^\dagger \hat{C}_1^\dagger \hat{C}_2^\dagger \cdots |0\rangle, \]

(4.1)

where \( \hat{C}_n |0\rangle = 0 \), for \( \forall n \in \mathbb{Z} \). In the vacuum state \( |G\rangle \), all states with non-positive momenta are filled. (See Fig. 1.)

The normal ordering \( \, : \, \) is defined by the following. \( \hat{C}_n(n > 0) \) or \( \hat{C}_n^\dagger(n \leq 0) \) are put on the right of \( \hat{C}_i(n > 0) \) or \( \hat{C}_n(n \leq 0) \). Whenever operators are exchanged, they are multiplied by \((-1)\).

Using Wick's theorem, we can derive a new commutation relation,

\[ \{ : W(\vec{k}) : , : W(\vec{k}'') : \} = -2i \sin \left( \frac{\vec{k} \times \vec{k}'}{2eB} \right) \{ : W(\vec{k} + \vec{k}'') : \} + \langle W(\vec{k} + \vec{k}'') \rangle, \]

(4.2)

where \( \langle \rangle \) is a vacuum expectation. The second term on the right-hand side of (4.2) is the central extension.

Let us derive the vacuum expectation. We impose the periodic boundary condition, \( 0 \leq X < L \). \( \langle W(\vec{k}) \rangle \) is given by the form,

\[ \langle W(\vec{k}) \rangle = 2\pi \delta_{k_x,0} \frac{1}{L} \left[ 1 - \exp \left( -i 2\pi k_y/LB \right) \right]. \]

(4.3)

In the limit \( L \to \infty \), it becomes

\[ \langle W(\vec{k}) \rangle = \delta_{k_x,0} \frac{eB}{ik_y}. \]

(4.4)

Therefore, we obtain a new commutation relation,

\[ \{ : L^{(n)}(k_x) : , : L^{(m)}(k_x) : \} = -2i \sin \left( \frac{\vec{k} \times \vec{k}'}{2eB} \right) \{ : L^{(n)}(k_x + k_y) : \} - \delta_{k_x+k_y,0} \frac{2eB}{k_y+k_y} \sin \left( \frac{\vec{k} \times \vec{k}'}{2eB} \right). \]

(4.5)

To understand (4.5) more clearly, we derive commutation relations of \( :L^{(n)}(k_x) : \),

\[ \{ :L^{(n)}(k_x) : , :L^{(m)}(k_x) : \} = \{ :\tilde{L}^{(n)}(k_x) : , :\tilde{L}^{(m)}(k_x) : \} = \]

\[ -\delta_{k_x+k_y,0} n! m! (-i)^{n+m} \sum_{r=0}^{\infty} \sum_{j=0}^{r} \sum_{h=0}^{2q} (-1)^{r+q-r} (2eB)^{-2q-r} k_x^{2q+1} \]

\[ \times \frac{1}{j! (r-j)! (2q-h)!(2q+1)} \delta_{n,2j+h} \delta_{m,2(r-j)+h}. \]

(4.6)

We obtain a new closed algebra of \( :L^{(n)} : \) and \( :L^{(n)} : \).
\[ [L_n^{(0)}, L_n^{(0)}] = n \delta_{n+m,0}, \quad [L_n^{(0)}, L_m^{(0)}] = n:L_n^{(0)}, \]

\[ [L_n^{(1)}, L_n^{(1)}] = (n - m):L_n^{(1)} + \delta_{n+m,0} \frac{1}{12} n(n^2 - 1). \]

(We redefined \( L_0^{(1)} \) in the form, \( L_0^{(1)} \rightarrow L_0^{(1)} - 1/24 \)). These are the \( c=1 \) Virasoro algebra and the \( U(1) \) Kac-Moody algebra. The algebra defined in (4·5) is called the \( W_{1+n} \) algebra. (The \( W_{1+n} \) algebra is a closed algebra which contains infinite operators whose conformal dimensions are equal to or above one.)

We can rewrite (4·5) by the functional,

\[ \langle \rho| :\rho[\xi_1], :\rho[\xi_2] :\rangle \]

\[ = \rho\langle \{\xi_1, \xi_2\} \rangle + \frac{1}{2\pi} \int d^2 x \int d^2 x' \delta(x-x') \delta(y) \delta(y') \frac{i}{2} (\partial_x - \partial_{x'}) \]

\[ \times \left\{ 1 + \frac{1}{2eB} \frac{1}{2} \left( \nabla^2 + \nabla'^2 \right) + \left( \frac{1}{2eB} \right)^2 \left[ \frac{1}{8} \left( \nabla^2 + \nabla'^2 \right)^2 - \frac{1}{3!} \left( \nabla \times \nabla' \right)^2 \right] + \ldots \right\} \]

\[ \times \xi_1(x) \xi_2(x') \]

\[ = \rho\langle \{\xi_1, \xi_2\} \rangle + \frac{1}{2\pi} \int d^2 x \int d^2 x' \delta(x-x') \delta(y) \delta(y') \frac{i}{2} (\partial_x - \partial_{x'}) \]

\[ \times \int du \delta\left( u - \frac{\nabla \times \nabla'}{2eB} \right) \frac{\sin u}{u} \exp \left( \frac{\nabla^2 + \nabla'^2}{4eB} \right) \xi_1(\bar{x}) \xi_2(\bar{x}'). \]  

(4·7) is the most important equation in this paper. Look at the \( \delta(x-x') \delta(y) \delta(y') \) in the central extension term. We understand that the central extension term is crucial only on the edge. Before deriving the central extension, the \( W_{1+n} \) algebra describes only the state of the bulk. However, if this algebra acquires the central extension, it describes not only the bulk but also the edge.

In the large \( B \) limit, (4·7) becomes simple. We neglect \( O(1/B) \) terms. We obtain

\[ [\rho[\xi_1], :\rho[\xi_2] :] \equiv \rho\langle \{\xi_1, \xi_2\} \rangle + \frac{eB}{2\pi} \int d^2 x \theta(-y) \rho(\xi_1, \xi_2) \rho. \]  

(4·8)

This result is interesting. Because of (4·2), in the large \( B \) limit, the commutation relation of \( :\rho[\xi] : \) is given by

\[ [\rho[\xi_1], :\rho[\xi_2] :] \rightarrow \rho\langle \{\xi_1, \xi_2\} \rangle + \int d^2 x \langle \rho(\bar{x}) \rangle \{\xi_1, \xi_2\} \rho. \]  

(4·9)

Comparing (4·8) with (4·9), we notice that \( \langle \rho(\bar{x}) \rangle \) is equal to \( eB\theta(-y)/2\pi \) in this limit. Let us see Fig. 1. In the large \( B \) limit, \( X \) and \( Y \) are approximately equal to \( x \) and \( y \),

\[ \bar{X} = x - \frac{\pi_y}{eB} \approx x, \quad \bar{Y} = y + \frac{\pi_x}{eB} \approx y. \]

This result means the following. In the large \( B \) limit, the electrons are localized in the area \( y \leq 0 \). We can conclude that the Fermi liquid in this system is approximately
an incompressible fluid in the large $B$ limit. In particular, $eB/2\pi$ is the density of the LLL. In this system, the filling factor $\nu$ is equal to one.

§ 5. The edge-charge operators and the Kac-Moody algebra

In this section we discuss the edge-charge operator. The edge-charge operator has been discussed by Stone, Wen and others. These authors have claimed that because the edge wave propagates along the edge of the droplet in one direction, quantizing the classical equation of the surface wave on the droplet, they obtain a one-dimensional chiral fermion. The commutation relation of the edge-charge operators is the Kac-Moody algebra. We show that the edge states are related deeply with the $W_{1+\infty}$ algebra, which includes the Kac-Moody algebra. We show that the origin of the edge-charge operators' commutation relation is the $W_{1+\infty}$ algebra by using the result obtained in § 4. The IQHE system is essentially a one-dimensional system in the whole of the edge and the bulk.

Let us consider electrons not only in an external magnetic field but also in a weak electrostatic potential $V(\vec{x})$,

$$V(\vec{x}) = \begin{cases} \epsilon A, & \Lambda < y, \\ \epsilon y, & -\Lambda \leq y \leq \Lambda, \\ -\epsilon A, & y < -\Lambda, \end{cases}$$

where $\Lambda > 0$, $\Lambda = \text{const}$, and $\epsilon > 0$, $\epsilon = \text{const}$. In this case, the electrostatic potential keeps the electrons inside the area, $y \leq 0$. In the large $B$ limit, the ground state of this system becomes approximately the Dirac sea vacuum. Therefore we can use the results which are obtained in § 4.

Using Stone's notation, an edge-charge operator is defined,

$$j(x) = \int dy \, g_A(y) :\rho(\vec{x}) :,$$

where $g_A(y) = \begin{cases} 1 & -\Lambda \leq y \leq \Lambda, \\ 0 & y < -\Lambda, \Lambda < y. \end{cases}$ (5·1)

We consider that $\Lambda$ is small but it is larger than the magnetic length, $\Lambda \gg \ell_0$ where $\ell_0 = 1/\sqrt{\epsilon B}$. $j(x)$ is the charge operator which lies on the edge $(x, 0)$. (See Fig. 1.) A functional is defined,

$$j[f] = \int dx \, f(x) j(x) = :\rho[f \cdot g_A] :,$$ (5·2)

where $f(x)$ is any non-singular function. Using (4·8), in the large $B$ limit, the commutation relation of $j[f]$ is given by

$$[j[f_1], j[f_2]] = :[\rho[f_1 g_A], \rho[f_2 g_A]] :$$

$$\approx -\frac{i}{2 e B} \int_{-\infty}^{\infty} dx \int_{-\Lambda}^{\Lambda} dy \, \frac{\partial}{\partial y} :\rho(\vec{x}) : (f_1 f_2 - f_2 f_1)$$

$$+ \frac{i}{4 \pi} \int dx \, (f_1 f_2 - f_2 f_1) .$$ (5·3)
Let us evaluate the first term on the right-hand side of this equation. Using the basis defined in (2·7), the first term is written explicitly in the form,

$$\frac{i}{2\sqrt{\frac{eB}{\pi}}} \frac{1}{L_{n,m}} \sum_{\mathcal{C}_n} \mathcal{C}_n: \int_{-\infty}^{+\infty} dx (f'_i f_2 - f_i f'_2) \exp\left(-ieB(Y_n - Y_m)x\right)$$

$$\times \int_{-\Delta}^{\Delta} dy [2y - (Y_n + Y_m)] \exp\left(-eB\left[y - \frac{1}{2}(Y_n + Y_m)\right]^2 - \frac{eB}{4}(Y_n - Y_m)^2\right).$$

Paying attention to the $y$-integral, we notice that we can neglect this term. We obtain

$$[j[f_i], j[f_2]] \approx \frac{i}{4\pi} \int dx [f_i(x)f_2(x) - f_2(x)f_i(x)]. \quad (5·4)$$

(Stone obtained this relation, too.\textsuperscript{9}) He did not consider the central extension of (4·5). He substituted the mean value of $\rho(x)$ on the right-hand side of (2·16.)

We can write the commutation relation formally in another form,

$$[j[f_i], j[f_2]] = \int dx \int dx' f_i(x)f_2(x')[j(x), j(x')]. \quad (5·5)$$

Hence, we obtain the commutation relation of $j(x)$ in the large $B$ limit,

$$[j(x), j(x')] = -\frac{i}{2\pi} \partial_x \delta(x - x'). \quad (5·6)$$

There is the Kac-Moody algebra in (5·6). Let us decompose $j(x)$ in the Fourier components,

$$\tilde{j}_n = \int dx \exp\left(-i\frac{2\pi}{L} nx\right) j(x). \quad (5·7)$$

From (5·6), we obtain the Kac-Moody algebra, $[\tilde{j}_n, \tilde{j}_m] = n \delta_{n+m,0}$.

We can obtain the following relation easily,

$$\tilde{j}_n = \exp\left(-\frac{1}{4eB} \left(\frac{2\pi n}{L}\right)^2\right) :L_n^{(0)}:. \quad (5·8)$$

In the large $B$ limit, $\tilde{j}_n$ is identified with $:L_n^{(0)}:$. Let us consider the physical meaning of $\tilde{j}_n$. For example, in the large $B$ limit, $\tilde{j}_{-n}(n > 0)$ is given by $\tilde{j}_{-n} \approx \sum_{i=-\infty}^{+\infty} \mathcal{C}_i \mathcal{C}_{i-n}$: This operator acts on the droplet in the following way. (See Fig. 2.) $\mathcal{C}_l \mathcal{C}_{i-n}$: $(n \geq l > 0)$ annihilates the electron at $Y = Y_{l-n}$ and creates the electron at $Y = Y_l$. Therefore, $\tilde{j}_{-n}$ shifts electrons along the direction of $Y$ by a distance $2\pi n/eBL$.

Because the droplet is incompressible, $\tilde{j}_{-n}$ is an operator which deforms the edge of the droplet. This deformation propagates with the velocity $v_F = E/B$ along the edge. There is the current which is not parallel to the external electric field $E$. This is the IQHE. The electric current $\tilde{j}$ is given by

$$\tilde{j} = \begin{pmatrix} 0 & e^2/2\pi \beta \\ -e^2/2\pi & 0 \end{pmatrix} \frac{0}{E}.$$


Generally we can conjecture the following. We assume an arbitrary vacuum. (See Fig. 3.) A droplet of electrons is localized in the area, $S$. $\partial S$ is an edge of $S$. Taking the classical limit, we obtain

$$
[\rho(\xi_1), \rho(\xi_2)] \approx \int d^2 x \langle \rho(\vec{x}) \rangle \left\{ \xi_1, \xi_2 \right\}_{\rho}
$$

$$
\approx \frac{i}{2\pi} \oint_S d^2 x \nabla \times \vec{\xi}
$$

$$
\approx \frac{i}{2\pi} \int_{\partial S} d\vec{x} \cdot \vec{\xi},
$$

(5.9)

where $\rho(\vec{x}) = \begin{cases} 
\frac{eB}{2\pi} & \vec{x} \in S, \\
0 & \vec{x} \notin S, 
\end{cases}$

$$
\vec{\xi} = (\xi_2, \xi_1).
$$

(5.10)

From this equation, we notice that this commutation relation reflects a behavior of the edge.

§ 6. Discussion

It is indicated that there is a close relation between the QHE and a one-dimensional system. We have presented a simple example of that relation. The $W_{1+\infty}$ algebra is constructed from the operators of the bulk. We show that the $W_{1+\infty}$ algebra reflects the properties of not only the bulk but also the edge. The central extension term is crucial for the edge states. In this paper, we have considered the $W_{\infty}$ algebra in $\nu=1$ case.

The $W_{\infty}$ algebra in $\nu=1/m$ fractional quantum Hall (FQH) states has been discussed by Karabali. Karabali showed that there was an underlying $W_{\infty}$ algebra in the space of the Laughlin wave functions. She considered that the $W_{\infty}$ generators in the FQH states act only on a restricted Hilbert space whose wave functions can be constructed by multiplying polynomials to the Laughlin state. It is not clear whether such a restriction can be justified and the full algebraic structure of the FQHE is still
unsolved.

Kac and Radul investigated the representation theory of the $W_{1+\infty}$ algebra. Its application to the QHE was discussed by Cappelli, Trugenberger and Zemba. Kac and Radul showed that the unitary, irreducible, highest-weight representation of the $W_{1+\infty}$ algebra, whose anomaly $c$ was given by $c=m, m\in \mathbb{Z}_+$, was characterized by an $m$-dimensional real vector $\vec{s}$. Cappelli and others insisted that all the two-dimensional incompressible fluid with the edge excitations in the QHE corresponded to the irreducible highest-weight representation of the $W_{1+\infty}$ algebra. When the incompressible fluid is considered, the vortex excitation is important. In the QHE, a vortex excitation is interpreted as a quasi-particle excitation. They intended to classify the two-dimensional incompressible fluids by the filling factor $\nu$ and the quantum numbers (the charge and the spin) of the quasi-particle excitations.

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Appendix A

——The Free Fermion Representation of the $W_\infty$ Algebra——

Let us show that the $W_\infty$ algebra can be constructed from a one-dimensional free fermion field. Then, we will show that the $W_{1+\infty}$ algebra is also constructed in a similar way.

At first, we obtain an explicit form of $L_j^{(l)}$. Taking the gauge $\vec{A}=(-By, 0)$ and using wave functions defined in (2.7), we obtain general forms of $\{L_j^{(l)}\}$,

$$L_j^{(2l)}=\sum_{n}^{\infty} C_n^j \bar{C}_{n+j} \sum_{p=0}^{l} \left( \frac{2l}{2l+1} \right) \left( \frac{2p-1}{2p+1} \right)^{p} \left( \frac{L}{\pi} \right)^{2p} (eB)^p (2n+j)^{2l+1}$$

for $l=0, 1, 2, \ldots$, (A.1)

$$L_j^{(2l+1)}=\sum_{n}^{\infty} C_n^j \bar{C}_{n+j} \sum_{p=0}^{l} \left( \frac{2l+1}{2l+2} \right) \left( \frac{2p-1}{2p+1} \right)^{p} \left( \frac{L}{\pi} \right)^{2p} (eB)^p (2n+j)^{2l+2}$$

for $l=0, 1, 2, \ldots$. (A.2)

Because these are complicated, we present some examples,

$$L_j^{(0)}=\sum_{n}^{\infty} C_n^j \bar{C}_{n+j}, \quad L_j^{(1)}=\sum_{n}^{\infty} C_n^j \bar{C}_{n+j} \frac{1}{2} (2n+j).$$

We define operators,

$$T^{(l)}(x)\equiv\left(\frac{1}{L}\right)^{l+1} \sum_{j} L_j^{(l)} \exp \left( i \frac{2\pi}{L} j x \right).$$

(A.4)

$T^{(l)}(x)$ is an energy-momentum tensor in the conformal field theory. We notice that $T^{(0)}(x)$ can be written in the form,

$$T^{(0)}(x)=\frac{1}{2} \sqrt{\frac{eB}{\pi}} \xi^{*}(x)\xi(x),$$

(A.5)
where $\zeta(x)$ is the primary field defined in (3·8). Using $\zeta(x)$, we can rewrite $T^{(1)}(x)$ in the form,

$$T^{(1)}(x) = \frac{1}{2} \sqrt{\frac{eB}{\pi}} \zeta^*(x) \left( -\frac{i}{4\pi} \right) (\partial_x - \bar{\partial}_x) \zeta(x). \quad (A·6)$$

The general form of $T^{(l)}(x)$ is given by

$$T^{(2l)}(x) = \frac{1}{2} \sqrt{\frac{eB}{\pi}} \zeta^*(x) \sum_{p=0}^{l} \left( \frac{2l}{2p} \right) \frac{(2p-1)!!}{2^{2l+p}} \left( \sqrt{\frac{eB}{\pi}} \right)^{2p}$$

$$\times \left\{ -\frac{i}{2\pi} (\partial_x - \bar{\partial}_x) \right\}^{2(l-p)} \zeta(x) \quad \text{for} \quad l = 0, 1, 2, \ldots, \quad (A·7)$$

$$T^{(2l+1)}(x) = \frac{1}{2} \sqrt{\frac{eB}{\pi}} \zeta^*(x) \sum_{p=0}^{l+1} \left( \frac{2l+1}{2p} \right) \frac{(2p-1)!!}{2^{2l+p+1}} \left( \sqrt{\frac{eB}{\pi}} \right)^{2p}$$

$$\times \left\{ -\frac{i}{2\pi} (\partial_x - \bar{\partial}_x) \right\}^{2(l-p)+1} \zeta(x), \quad \text{for} \quad l = 0, 1, 2, \ldots \quad (A·8)$$

Conversely, if we prepare a one-dimensional free fermion field $\zeta(x)$, we can always construct $\{T^{(l)}(x)\}$ and $\{L^{(l)}_i\}$. We can construct the $W_\infty$ algebra from $\zeta(x)$.

These results are true for the $W_{1+\infty}$ algebra. We can construct $\{L^{(l)}_i\}$ in a similar way. For example, we define $:T^{(0)}(x):$ by the form,

$$:T^{(0)}(x): = \frac{1}{2} \sqrt{\frac{eB}{\pi}} \zeta^*(x) \zeta(x). \quad (A·9)$$

We can construct the $W_{1+\infty}$ algebra from a one-dimensional fermion field $\zeta(x)$.

Here, we consider the following. We can construct the energy-momentum tensor, from the Kac-Moody current $:T^{(0)}(x):$ as the Sugawara form. Decomposing this energy-momentum tensor, we obtain the Virasoro operators. This new Virasoro operator corresponds to $:L^{(1)}_i:$.  

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