Collective excitations in Bose-Einstein condensates in triaxially anisotropic parabolic traps

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The wave equation of low-frequency density waves in Bose-Einstein condensates at vanishing temperature in arbitrarily anisotropic harmonic traps is separable in elliptic coordinates, provided the condensate can be treated in the Thomas-Fermi approximation. We present a complete solution of the mode functions, which are polynomials of finite order, and their eigenfrequencies which are characterized by three integer quantum numbers.

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I. INTRODUCTION

Bose-condensates differ from normal gases or fluids by the existence of a macroscopic wave-function, their order parameter. The macroscopic wave-function deeply influences the spectrum of low-lying elementary excitations in Bose-condensed systems, which become collisionless density waves with the velocity of sound. In the Bose-Einstein condensates of alkali-metal vapors in traps these collective sound waves have a discrete spectrum which is determined by the shapes of the trapping potential and of the condensate. Many experimental [1–3] and theoretical [4–12] studies have been devoted to their study. For Bose-Einstein condensates at zero temperature which are sufficiently large to validate the Thomas-Fermi approximation Stringari [9] found an analytical solution for the sound modes and their eigenfrequencies for the case of spherically symmetric parabolic traps. In the same work he even determined some of the eigenfrequencies for axially symmetric anisotropic traps. In subsequent works [10–12] the complete solution for the axially symmetric anisotropic case was given. In particular, it was demonstrated that the axially symmetric anisotropic problem forms a completely integrable system by exhibiting explicitly a third conserved operator \( \hat{B} \) besides the wave-operator \( \hat{G} \) and the axial angular momentum operator \( \hat{L}_z \). The eikonal or ‘classical limit’ of the sound-waves, determining their characteristic rays, was also studied in [12] in the axially symmetric case. In this ‘classical limit’ also the completely anisotropic case of a triaxial harmonic trap was investigated [12]. It was shown that even in this case the wave operator in eikonal approximation remains separable in elliptic coordinates. The complete integrability was demonstrated by exhibiting three phase-space functions \( G, B \) and \( A \) in involution. However, the solution for the mode-functions and eigenfrequencies was not yet given in [12].

In the present paper we wish to return to this completely anisotropic case. From a practical point of view this has become of interest, because the first completely anisotropic trap has now appeared on the experimental scene [13]. There the reported trap-frequency ratios are \( \omega_1^2 : \omega_2^2 : \omega_3^2 = 1 : 2 : 4 \). We shall return to this case when we give a numerical example at the end.

From a theoretical point of view the problem is also of considerable interest. The previous results on its classical limit suggest that also the full wave-operator remains separable in the general anisotropic case. This is indeed the case, as will be shown here. In fact we shall see that this problem can be related to a novel class of completely integrable elliptic billiards on an inhomogeneously curved space of arbitrary dimensionality. The sound modes in the Bose-Einstein condensate correspond to the 3-dimensional quantized version of such a billiard, their characteristics or rays are given by their classical limit. Furthermore it turns out that the classical limit of the billiards (in arbitrary dimension) can be connected mathematically to the equations of motion of an integrable system first studied by C. Neumann [14] 140 years ago: a mass point on the sphere \( |x| = 1 \) under the influence of an anisotropic harmonic force.

The problem of collective modes in Bose-Einstein condensates can be viewed as the problem of small perturbations of the macroscopic wave function around its static equilibrium. At zero temperature the macroscopic wave function satisfies the Gross-Pitaevskii equation [17].
\[
\left\{ -\frac{\hbar^2}{2m} \nabla^2 + U(x) - \mu + \frac{4\pi \hbar^2 a_0}{m} |\phi(x,t)|^2 \right\} \phi(x,t) = i\hbar \dot{\phi}(x,t)
\]  
(1.1)

where

\[
U(x) = \frac{1}{2} m \left( \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 \right)
\]  
(1.2)

is the anisotropic harmonic potential of the trap, \( \mu \) is the chemical potential, fixed by the requirement that \( N = \int d^3x |\phi(x,t)|^2 \), and \( a_0 \) is the s-wave scattering length, assumed to be positive throughout this paper. The macroscopic wave function (at zero temperature) is related to the number density

\[
\int \omega g = \text{constant}
\]

sometimes use the notation \( a \) without restriction of generality, i.e.

Here we introduce the characteristic lengths

\[
\text{eliminating } \delta v_s \text{ and with the ansatz } \delta n(t) = \psi(x)e^{-i\omega t} \text{ we are left with the wave equation, first derived along the present lines by Stringari} \]

\[
\frac{\partial n}{\partial t} + \nabla \cdot n v_s = 0
\]

We shall assume that the condition \( Na_0 \sqrt{m \bar{\omega}/\hbar} \gg 1 \) is satisfied, where \( \bar{\omega} = (\omega_1 \omega_2 \omega_3)^{1/3} \), so that the Thomas-Fermi approximation [10] can be applied to [12], where the term proportional to \( (\nabla^2 \sqrt{n})/\sqrt{n} \) is neglected. Then the equilibrium solution with \( \partial n/\partial t = 0 = \partial v_s/\partial t \) is given by \( v_s = 0, n = n_0(x) = \frac{n \sqrt{\mu}}{4\pi \hbar^2 a_0} (\mu - U(x)), \mu = (\bar{\omega}/2)(Na_0/\tilde{a})^{2/5}, \text{where } \tilde{a} = \sqrt{\hbar/m \bar{\omega}} \). Thus the condensate forms a triaxial ellipsoid. The collective excitations in the same approximation are now determined by eqs. [12], linearized around the equilibrium solution,

\[
\frac{\partial \delta n}{\partial t} + \nabla \cdot n_0(x) \delta v_s = 0 \quad \text{and} \quad \frac{\partial \delta v_s}{\partial t} + \frac{4\pi \hbar^2 a_0}{m^2} \nabla \delta n = 0.
\]

Eliminating \( \delta v_s \) and with the ansatz \( \delta n(t) = \psi(x)e^{-i\omega t} \) we are left with the wave equation, first derived along the present lines by Stringari [9]

\[
\frac{\omega^2}{c_0^2} \psi = -\nabla \cdot \left( 1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} \right) \nabla \psi.
\]

(1.5)

Here we introduce the characteristic lengths

\[
a = \sqrt{\frac{2\mu}{m \omega_1^2}}, \quad b = \sqrt{\frac{2\mu}{m \omega_2^2}}, \quad c = \sqrt{\frac{2\mu}{m \omega_3^2}}
\]

(1.6)

which are the three semi-axes of the condensate ellipsoid. We note that

\[
\omega_1^2 : \omega_2^2 : \omega_3^2 = \frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}.
\]

(1.7)

We also introduced the velocity of sound \( c_0 = \sqrt{\mu/m} \) in the center of the trap. In the following we assume \( a^2 \geq b^2 \geq c^2 \) without restriction of generality, i.e. \( \omega_1 \) is the smallest of the three trap frequencies and \( \omega_3 \) the largest. We shall sometimes use the notation \( a_1 \equiv a, a_2 \equiv b, a_3 \equiv c \). In this paper we shall be concerned with the solution of eq. [12].

II. EIKONAL APPROXIMATION

To get the eikonal approximation to the wave equation [12] we first return to its explicitly time-dependent form replacing \( \omega^2 \rightarrow -\partial^2/\partial t^2 \), and replace space and time derivatives via \( i \frac{\partial}{\partial t} \rightarrow H, -i\hbar \nabla \rightarrow p \). This leaves us with the Hamiltonian

\[
H = c_0 \sqrt{p^2 \left( 1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} \right)}
\]

(2.1)

whose trajectories describe the characteristics of the wave equation from which it was derived. Some features of the classical dynamics described by eq. [2.1] where studied in ref. [12]. Here we wish to make a number of additional points.
a. Connection to Neumann’s system

The Hamiltonian equations of motion following from eq. \((2.1)\) (with time now measured as a length by taking units with \(c_0 = 1\))

\[
\dot{p}_i = \frac{x_i}{a_i^2} \sqrt{\frac{p}{1 - \frac{x_i^2}{a_i^2} - \frac{x_j^2}{a_j^2} - \frac{x_k^2}{a_k^2}}} , \quad \dot{x}_i = \frac{p_i}{p} \sqrt{1 - \frac{x_i^2}{a_i^2} - \frac{x_j^2}{a_j^2} - \frac{x_k^2}{a_k^2}} \quad (2.2)
\]

have the interesting property, remarked in \([12]\), that the dynamics of the unit vector \(\mathbf{\hat{p}} = p/p\) can be decoupled from the dynamics of \(p\). In fact, eliminating \(x_j\) from eqs. \((2.2)\) we obtain

\[
\dot{\mathbf{\hat{p}}}_i = \left( \frac{x_i}{a_i^2} - \mathbf{\hat{p}}_i \sum_j \frac{\dot{\mathbf{\hat{p}}}_j x_j}{a_j^2} \right) \sqrt{1 - \sum_k x_k^2/a_k^2}
\]

and the equations of motion

\[
\ddot{\mathbf{\hat{p}}}_i = \dot{p}_i \left( \frac{1}{a_i^2} - \sum_{j=1}^3 \left( \frac{\dot{\mathbf{\hat{p}}}_j^2}{a_j^2} + \dot{\mathbf{\hat{p}}}_j^2 \right) \right), \quad (2.3)
\]

with the constraints \(\mathbf{\hat{p}}^2 = 1, \ \mathbf{\hat{p}} \cdot \mathbf{\hat{p}} = 0\). These are formally the equations of motion of a particle with unit mass with coordinates \(\mathbf{\hat{p}}\) on a sphere \(\mathbf{\hat{p}}^2 = 1\) under the influence of the force \(F\) with components \(F_i = \dot{\mathbf{\hat{p}}}_i/a_i^2\) which were studied by Neumann \([14]\). A discussion of this problem within the modern mathematical theory of integrable systems has been given by Moser \([17]\). He derives its conservation laws by constructing a matrix whose eigenvalues are preserved under the dynamics \((2.3)\). The conserved quantities (see also \([8]\) are

\[
M_k = -\dot{\mathbf{\hat{p}}}^2_k + \sum_{i \neq k} \frac{a_i^2 a_k^2}{a_i^2 - a_k^2} (\dot{\mathbf{\hat{p}}}_i \dot{\mathbf{\hat{p}}}_k - \dot{\mathbf{\hat{p}}}_k \dot{\mathbf{\hat{p}}}_i)^2 \quad (2.4)
\]

They satisfy \(\sum_k M_k = -\dot{\mathbf{\hat{p}}}^2 = 1\). We note that the Hamiltonian \(H \quad (2.3)\) is no longer among these integrals because the absolute value \(p\) of the momentum was eliminated in the derivation of \((2.3)\) from \((2.2)\). However the conservation of \((2.1)\) can be used to recover the motion of \(p\) from the solution of \((2.3)\). There is a new obvious ‘energy’-integral of eq. \((2.3)\) which is given by

\[
E_N = \frac{1}{2} \sum_i \frac{p_i^2}{a_i^2} = \frac{1}{2} \sum_i M_i \quad (2.5)
\]

It can be expressed in the original \(x, p\) variables as

\[
E_N = -\frac{1}{2H^2} \sum_i \left\{ \frac{\dot{p}_i^2}{a_i^2} \left( 1 - \sum_k \frac{x_k^2}{a_k^2} \right) - \frac{x_i^2}{a_i^2} p_i^2 + \frac{x_i p_i}{a_i^2} \sum_k x_k p_k \right\} \quad (2.6)
\]

i.e. it is now a quite complicated looking and far from obvious first integral of eqs. \((2.2)\). With some labor it can be expressed in terms of the first integral \(A\) which will be introduced in sections 2c and 3 (see \((3.16)\) as

\[
E_N = \frac{1}{2} \left( \frac{A}{a_1^2 a_2^2 a_3^2 H^2} - \sum_{i=1}^3 \frac{1}{a_i^2} \right) \quad (2.7)
\]

Another simple linear combination of the \(M_k\) is

\[
B_N = -\sum_k a_k^2 M_k = \sum_k a_k^2 \dot{p}_k^2 + \frac{1}{2} \sum_{i,k} a_i^2 a_k^2 (\dot{\mathbf{\hat{p}}}_i \dot{\mathbf{\hat{p}}}_k - \dot{\mathbf{\hat{p}}}_k \dot{\mathbf{\hat{p}}}_i)^2 .
\]

It can be expressed in terms of the first integral \(B\) introduced in section 2c and 3 (see \((3.16)\) as

\[
B_N = \frac{1}{H^2} \left( \sum_k a_k^2 p_k^2 + (x \cdot p) \right)^2 = \frac{B}{H^2} .
\]
b. **Connection to a billiard on a curved space**

The Hamiltonian $H$ describes the geodesic motion of a particle in a space with the metric

$$\frac{ds^2}{1 - \sum_j \frac{x_j^2}{a_j^2}} \sum_i dx_i^2. \quad (2.8)$$

This metric is inhomogeneous and conformal to the Euclidean metric. Its Riemann scalar curvature is given by

$$R = \sum_i \frac{2}{a_i^2} \left( 2 + \frac{5}{1 - \sum_j \frac{x_j^2}{a_j^2}} \right)$$

and singular on the surface-ellipsoid. The distance to the surface-ellipsoid $\sum_j x_j^2/a_j^2 = 1$ from any point inside the surface is finite. Furthermore, the metric velocity $ds/dt$ is constant and simply given by $ds/dt = 1$ in our present units. The billiard particle always reaches the surface-ellipsoid perpendicularly and with diverging orthogonal and finite tangential components of the momentum and is reflected with conserved tangential momentum component \[^12\].

c. **Separation in ellipsoid coordinates**

Let us introduce elliptical coordinates \[^{19}\] $\lambda, \mu, \nu$ as the three roots of

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \quad (2.9)$$

We order these roots $\rho = \lambda, \mu, \nu$ according to

$$-a^2 \leq \nu \leq -b^2 \leq \mu \leq -c^2 \leq \lambda \leq 0. \quad (2.10)$$

Geometrically surfaces $\lambda = \text{const}$ are ellipsoids, $\mu = \text{const}$ are one-sheeted hyperboloids, and $\nu = \text{const}$ are two-sheeted hyperboloids, all confocal to the basic ellipsoid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$ 

Explicitly, the $x_i$ are given in terms of the new variables by

$$x_1 = \pm \sqrt{(a^2 + \lambda)(b^2 + \mu)(c^2 + \nu)} \quad (2.11)$$

The Hamiltonian-Jacobi equation

$$\omega = H(\nabla S, x) \quad (2.12)$$

is then separable, as shown in \[^{12}\]. Written in elliptical coordinates it takes the form

$$0 = \left\{ (\mu - \nu) \left[ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \left( \frac{\partial S}{\partial \lambda} \right)^2 + \frac{a^2 b^2 c^2 \omega^2}{4\lambda} \right] + \text{cyclic} \right\} \quad (2.13)$$

which is satisfied only if the angular bracket is equal to a linear function of $\lambda$, i.e.

$$\left[ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \left( \frac{\partial S}{\partial \lambda} \right)^2 + \frac{a^2 b^2 c^2 \omega^2}{4\lambda} \right] = \frac{1}{4} (-A - B\lambda) \quad \text{and cyclic} \quad (2.14)$$

where $A$ and $B$ are separation constants. Eqs. \[^{2,14}\] can be solved for $A, B, \omega^2 = H^2$ as functions of the coordinates $\lambda, \mu, \nu$ and the canonically conjugate momenta $p_\lambda = \partial S/\partial \lambda, p_\mu = \partial S/\partial \mu, p_\nu = \partial S/\partial \nu$. The results, expressed in terms of the Cartesian coordinates and momenta have been given in \[^{12}\] and need not to
be given here. In any case they are easily recovered from the operators $\hat{A}$, $\hat{B}$ derived in section III upon taking the classical limit $-i\hbar \nabla \rightarrow \mathbf{p}$, $\hbar \rightarrow 0$. Here we wish to remark on a direct geometrical significance of the values of the three conserved quantities $H^2$, $A$, and $B$. Let us introduce to this end the end the $\lambda_1 > \lambda_2$

$$\lambda_{1,2} = -\frac{A}{2B} \pm \sqrt{\frac{A^2}{4B^2} - \frac{C}{B}}$$

(2.15)

of the quadratic equation

$$B\lambda^2 + A\lambda^2 + C = 0$$

(2.16)

where $C = a^2b^2c^2\omega^2 > 0$. The equations for the momenta $p_\lambda$, $p_\mu$, $p_\nu$ now take the form

$$p_\lambda = \pm \sqrt{-\frac{B(\lambda - \lambda_1)(\lambda - \lambda_2)}{\lambda(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}}$$

and cyclic.

(2.17)

For a physical motion inside the ellipsoid each of the momenta $p_\lambda$, $p_\mu$, $p_\nu$ needs to have an outer and an inner turning point. Thus there must be six turning points, which, according to eq. (2.17) can only be at values $\lambda_1$, $\lambda_2$, $0$, $-a^2$, $-b^2$, $-c^2$. Of these the turning points at $-a^2$, $-b^2$, $-c^2$ correspond to coordinate singularities, the turning point at 0 is the surface of the condensate and the turning points at $\lambda_1$, $\lambda_2$ correspond to caustic surfaces. Therefore $\lambda_1$, $\lambda_2$ must be real, and negative in order to be in the range of $\lambda$, $\mu$, $\nu$, which imposes the conditions

$$A^2 > 4BC, \quad A > 0, \quad B > 0.$$  

(2.18)

There are then four possible cases for the roots $\lambda_1$, $\lambda_2$ in which $p_\lambda$, $p_\mu$, $p_\nu$ are real. These are:

1) $-a^2 < \nu < -b^2 < \lambda_2 < -c^2 < \lambda_1 < \lambda < 0$
   In this case $p_\lambda$ turns at $p_\lambda = 0$ on a surface of inner turning points forming the ellipsoid $\lambda = \lambda_1$ and similarly $p_\mu$ turns at $p_\mu = 0$ on a second surface of inner turning points forming a one-sheeted hyperboloid $\mu = \lambda_2$.
2) $-a^2 < \nu < -b^2 < \lambda_2 < \mu < \lambda_1 < -c^2 < \lambda < 0$
   Here $p_\mu$ turns at an outer surface $\mu = \lambda_1$ and an inner surface $\mu = \lambda_2$ which are both one-sheeted hyperboloids.
3) $-a^2 < \nu < \lambda_2 < -b^2 < \mu < -c^2 < \lambda_1 < \lambda < 0$
   Here $p_\nu$ turns on an inner ellipsoid $\lambda = \lambda_1$ and $p_\nu$ turns on an outer two-sheeted hyperboloid $\nu = \lambda_2$.
4) $-a^2 < \nu < \lambda_2 < -b^2 < \mu < \lambda_1 < -c^2 < \lambda < 0$
   Here $p_\mu$ turns on an outer one-sheeted hyperboloid $\mu = \lambda_1$ and $p_\nu$ turns on an outer two-sheeted hyperboloid.

The conservation of $A$ and $B$ for given $\omega$ and the particular value taken by these quantities thus is reflected in the geometry of the two caustic surfaces occurring in each case. Similar results have been obtained for billiards in Euclidean space with ellipsoidal boundaries [20].

d.\textbf{Semiclassical quantum numbers}

A ‘semiclassical’ approach to the solution of the wave equation could be the application of the Bohr-Sommerfeld rule

$$I_\lambda = \frac{1}{2\pi} \oint p_\lambda d\lambda = n_\lambda + 1/2$$

and cyclic.

(2.19)

See ref. [21] for the case of isotropic traps. Here the quantum numbers $n_\lambda$, $n_\mu$, $n_\nu$ can be interpreted to count the number of nodal surfaces with $\lambda = \text{const}$, $\mu = \text{const}$, $\nu = \text{const}$, respectively, which are ellipsoids, one-sheeted hyperboloids, and two-sheeted hyperboloids respectively. The original conserved quantities $A$, $B$ and $\omega^2$ can be expressed in terms of these quantum numbers and are thereby quantized in terms of the three independent integers $n_\lambda$, $n_\mu$, $n_\nu$. In section IV these quantum numbers will reappear in the exact solution of the wave equation in the slightly different notation $n_3$, $n_2$, $n_1$, the correspondence being $n_\lambda = n_3$, $n_\mu = n_2$, $n_\nu = n_1$. 

5
III. SEPARATION OF THE WAVE EQUATION

Let us now return to the wave equation

\[ \omega^2 \psi = -\nabla \cdot \left( 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \frac{x_3^2}{c^2} \right) \nabla \psi \]  

(3.1)

where we again adopt units with \( c_0 = 1 \). We are interested in the solutions \( \psi \) of eq. (3.1) in the Hilbert space with the scalar product

\[ \langle \psi_i | \psi_j \rangle = \int_{TF} d^3x \psi_i^*(x) \psi_j(x) \]  

(3.2)

where the integration is extended over the interior of the Thomas-Fermi ellipsoid. With this choice of the scalar product the operator \( \hat{G} = -\nabla \cdot \left( 1 - \sum_i x_i^2/a_i^2 \right) \nabla \) is self-adjoint if we pose as boundary condition

\[ \left( 1 - \sum_i x_i^2/a_i^2 \right) \partial_n \psi = 0 \]

on the boundary, i.e. the normal derivative \( \partial_n \psi \) should diverge there less than \( \left( 1 - \sum_i x_i^2/a_i^2 \right)^{-1} \). This can be satisfied by choosing the Hilbert space of polynomials of finite order with the scalar product (3.2). These polynomials can be divided into eight different parity classes depending on whether they are even or odd under the inversion of any of the three coordinates \( x_1, x_2, x_3 \). To be specific we put

\[ \psi = x_1^\alpha x_2^\beta x_3^\gamma P_m(x_1^2, x_2^2, x_3^2) \]

where \( \alpha, \beta, \gamma = 0, 1 \) determine the parity class and \( P_m \) is a polynomial of order \( m \) in \( x_1^2, x_2^2, x_3^2 \). Eq. (3.1) is now transformed to elliptical coordinates where it takes the form

\[ a^2 b^2 c^2 \omega^2 = -\left( 1 - \sum_i x_i^2/a_i^2 \right) \nabla \cdot \left( \frac{x_i^2}{a_i^2} \right) \nabla \]

(3.3)

with

\[ F(\varrho) = (a^2 + \varrho)(b^2 + \varrho)(c^2 + \varrho). \]

(3.4)

Eq. (3.3) can be solved by separation of variables via

\[ \psi = \varphi_\lambda(\lambda) \varphi_\mu(\mu) \varphi_\nu(\nu) \]

(3.5)

which leads to

\[ 0 = (\mu - \nu) g_\lambda(\lambda) + (\nu - \lambda) g_\mu(\mu) + (\lambda - \mu) g_\nu(\nu) \]

(3.6)

with

\[ g_\varrho(\varrho) = \frac{1}{\varphi_\varrho(\varrho)} \left\{ -F(\varrho) \frac{d^2}{d\varrho^2} + \left( \frac{F(\varrho)}{\varrho} + \frac{1}{2} F'(\varrho) \right) \frac{d}{d\varrho} + \frac{a^2 b^2 c^2 \omega^2}{4 \varrho} \right\} \varphi_\varrho(\varrho) \]

(3.7)

If eq. (3.6) is rewritten as

\[ g_\lambda(\lambda) = \frac{\mu g_\nu(\nu) - \nu g_\mu(\mu)}{\mu - \nu} + \lambda \frac{g_\mu(\mu) - g_\nu(\nu)}{\mu - \nu} \]

(3.8)

it becomes apparent that it can hold as an identity in \( \lambda, \mu, \nu \) only if

\[ g_\lambda(\lambda) = -\frac{A}{4} - \frac{B}{4} \lambda \]

\[ g_\mu(\mu) = -\frac{A}{4} - \frac{B}{4} \mu \]

\[ g_\nu(\nu) = -\frac{A}{4} - \frac{B}{4} \nu \]

(3.9)
where $A$ and $B$ are the same constants in all three of eqs. (3.9). From eq. (3.4) and (3.7) the three constants $\omega^2$, $A$, $B$ can be expressed as eigenvalues of certain corresponding operators $\hat{G}$, $\hat{A}$, $\hat{B}$. To do this explicitly we define the operator $\hat{F}_\varphi$ for arbitrary $\varphi$ as

$$\hat{F}_\varphi \psi = 4 \left[ \varphi F(\varphi) \frac{\partial^2}{\partial \varphi^2} + \left( F(\varphi) + \frac{1}{2} \varphi F'(\varphi) \right) \frac{\partial}{\partial \varphi} \right] \psi $$

(3.10)

because then eqs. (3.9) with (3.7) can be rewritten simply as

$$\hat{F}_\varphi \psi = (a^2 b^2 c^2 \omega^2 + A\varphi + B\varphi^2) \psi $$

(3.11)

where $\psi$ is the total wave function and $\varphi = \lambda, \mu, \nu$. Solving for $\omega^2 \psi$, $A\psi$, $B\psi$ we find the simultaneous eigenvalue equations for $\omega^2$, $B$, $A$, namely eq. (3.9) which we abbreviate as $\omega^2 \psi = \hat{G} \psi$ and

$$\hat{B} \psi = B \psi, \quad \hat{A} \psi = A \psi $$

(3.12)

with

$$\hat{B} = -\frac{4}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)} \left\{ (\mu - \nu) \left[ \lambda F(\lambda) \frac{\partial^2}{\partial \lambda^2} + \left( F(\lambda) + \frac{1}{2} \lambda F'(\lambda) \right) \frac{\partial}{\partial \lambda} \right] \right\} + \text{cyclic} $$

$$\hat{A} = \frac{4}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)} \left\{ (\mu^2 - \nu^2) \left[ \lambda F(\lambda) \frac{\partial^2}{\partial \lambda^2} + \left( F(\lambda) + \frac{1}{2} \lambda F'(\lambda) \right) \frac{\partial}{\partial \lambda} \right] \right\} + \text{cyclic} $$

(3.13)

(3.14)

A lengthy but straightforward calculation then yields the operators $\hat{A}$ and $\hat{B}$ in Cartesian coordinates

$$\hat{A} = \left\{ [(b^2 + c^2)(x_1^2 - a^2) + a^2(x_2^2 + x_3^2)] \frac{\partial^2}{\partial x_1^2} + 2a^2 x_2 x_3 \frac{\partial^2}{\partial x_2 \partial x_3} + 3(b^2 + c^2) x_1 \frac{\partial}{\partial x_1} + \text{cyclic} \right\} $$

$$\hat{B} = (x \cdot \nabla)(x \cdot \nabla + 3) - a^2 \frac{\partial^2}{\partial x_1^2} - b^2 \frac{\partial^2}{\partial x_2^2} - c^2 \frac{\partial^2}{\partial x_3^2} $$

(3.15)

(3.16)

By construction the eigenvalue equations for $\hat{A}$, $\hat{B}$ and $\hat{G}$ can be satisfied simultaneously, i.e. these operators must commute, as one may also check by explicit calculation

$$[\hat{G}, \hat{A}] = [\hat{B}, \hat{A}] = [\hat{G}, \hat{B}] = 0. $$

In the axially symmetric case, e.g. $b^2 = c^2$, the operator $\hat{A}$ may be expressed in terms of the angular momentum $\hat{L}_z$ around the axis of symmetry, here chosen as the 1-axis, according to

$$\hat{A} = c^2 \hat{B} + (a^2 - c^2) \hat{L}_z^2 + a^2 c^2 \hat{G}.$$ 

IV. SOLUTION OF THE WAVE EQUATION

After the separation of variables the equation to be solved follows from eq. (3.7) with (3.9) as

$$\left[ - \varphi F(\varphi) \frac{d^2}{d\varphi^2} - (F(\varphi) + \frac{1}{2} \varphi F'(\varphi)) \frac{d}{d\varphi} + \frac{1}{4} (a^2 b^2 c^2 \omega^2 + A\varphi + B\varphi^2) \right] \varphi(\varphi) = 0 $$

(4.1)

for $\varphi = \lambda, \mu, \nu$, i.e. precisely the same equation appears in all three elliptical coordinates. We now restrict the solutions of (4.1) to the space of polynomials in Cartesian coordinates. This means that in elliptical coordinates they must be of the form

$$\varphi(\varphi) = [a^2 + \varphi^\alpha] [b^2 + \varphi^\beta] [c^2 + \varphi^\gamma] \mathcal{P}_m(\varphi) $$

(4.2)

where $\mathcal{P}_m(\varphi)$ is a polynomial of order $m$ and the exponents $\alpha, \beta, \gamma$ can take on the values 0 and 1. Inserting this ansatz in eq. (4.1) results in an expression containing the same prefactors as eq. (4.2) but multiplied with a polynomial of order $m + 2$ whose coefficients must all vanish, yielding $m + 2$ equations from which the three eigenvalues $A, B, \omega^2$ and
the $m-1$ unknown coefficients of $P_m$ (one coefficient is fixed by normalization) must be determined. The vanishing of the coefficient of highest order fixes the value of $B$ as

$$B = (2m + \alpha + \beta + \gamma)(2m + \alpha + \beta + \gamma + 3).$$

(4.3)

The condition that the coefficient of the next to highest order term vanishes in principle fixes the value of $A$. However, it turns out that $A$, and also $\omega^2$, cannot be determined without at the same time calculating all the coefficients of the polynomial $P_m$ in eq. (4.2). Simple results are therefore not obtained in this way, except for a few of the lowest lying modes.

Therefore a different procedure is used. It is a generalization of the analysis used in the solution of the Lamé equation \cite{19}. We shall assume now, and make plausible at the end of this section that the eigenvalues of the three separation constants $\omega^2$, $A$, $B$ uniquely specify the corresponding eigenfunction, up to an arbitrary multiplicative factor. Then one can choose

$$\varphi(\theta) = \varphi(\theta) = \varphi(\theta).$$

(4.4)

In the polynomial ansatz (4.2) we write

$$P_m(\theta) = \prod_{i=1}^{m}(\theta - \theta_i)$$

(4.5)

where $\theta_i$ are the (possibly complex) roots of the polynomial $P_m$. The ansatz for the total wave function $\psi = \varphi(\lambda)\varphi(\mu)\varphi(\nu)$ then becomes

$$\psi = \text{const}|(a^2 + \lambda)(a^2 + \mu)(a^2 + \nu)|^{\alpha/2}|(b^2 + \lambda)(b^2 + \mu)(b^2 + \nu)|^{\beta/2}

[(c^2 + \lambda)(c^2 + \mu)(c^2 + \nu)]^{\gamma/2} \prod_{i=1}^{m}(\lambda - \theta_i)(\mu - \theta_i)(\nu - \theta_i)$$

(4.6)

Using eqs. (2.11) and the identity

$$\frac{x_1^2}{a^2 + \theta_i} + \frac{x_2^2}{b^2 + \theta_i} + \frac{x_3^2}{c^2 + \theta_i} - 1 = \frac{(\lambda - \theta_i)(\mu - \theta_i)(\nu - \theta_i)}{(a^2 + \theta_i)(b^2 + \theta_i)(c^2 + \theta_i)}$$

(4.7)

the wave function (4.8) can be written rather simply in Cartesian coordinates as

$$\psi(x) = x_1^{\alpha}x_2^{\beta}x_3^{\gamma} \prod_{i=0}^{m}\left(\frac{x_1^2}{a^2 + \theta_i} + \frac{x_2^2}{b^2 + \theta_i} + \frac{x_3^2}{c^2 + \theta_i} - 1\right)$$

(4.8)

where for $i = 0$ the factor under the product is defined as 1. In (4.8) and in the following we omit a normalization constant and work with unnormalized wave functions. They are completely parametrized by the yet unknown parameters $\theta_i$ which determine the nodal surfaces. As can be seen directly from (4.7) the remarkably simple form of the wave function (4.8) is a direct consequence of the separation ansatz. For all of the following considerations the form (4.8) of the wave function will be used. It follows from (4.8) that the nodal surfaces are quadrics confocal to the Thomas-Fermi ellipsoid. In order to relate the eigenvalues of $G$, $A$, $B$ to the $\theta_i$ we insert the ansatz (4.8) in the eigenvalue equations. The calculations become simpler with the use of the intermediate variables

$$\varphi_i = \frac{x_1^2}{a^2 + \theta_i} + \frac{x_2^2}{b^2 + \theta_i} + \frac{x_3^2}{c^2 + \theta_i} - 1$$

$$\varphi_0 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} - 1$$

$$\Pi = \prod_{i}^{m}\varphi_i$$

(4.9)

Then

$$\frac{\partial \Pi}{\partial \varphi_j} = \Pi, \quad \frac{\partial^2 \Pi}{\partial \varphi_i \partial \varphi_j} = \frac{\partial \Pi}{\partial \varphi_i}$$

(4.10)
From $\hat{G}\psi = \omega^2 \psi$ with $\psi = x_1^a x_2^b x_3^\gamma \Pi$ we obtain

$$
\hat{G}\psi = x_1^a x_2^b x_3^\gamma \left\{ \frac{2\alpha}{a^2} + \frac{2\beta}{b^2} + \frac{2\gamma}{c^2} - 4 \sum_{i=1}^{m} \frac{1}{\theta_i} \Pi 
+ \varphi_0 \sum_{i=1}^{m} \frac{\partial \Pi}{\partial \varphi_i} \left[ 4 \frac{\theta_i}{\theta_i + \theta_j} \right] \right\}. 
$$

Thus the $\theta_i$ have to satisfy

$$
G_i(\theta) = \frac{4}{\theta_i} + \frac{4\alpha + 2}{a^2 + \theta_i} + \frac{4\beta + 2}{b^2 + \theta_i} + \frac{4\gamma + 2}{c^2 + \theta_i} + \sum_{j=1}^{m} \frac{8}{\theta_i - \theta_j} = 0 \quad (4.11)
$$

for $i = 1, 2, \ldots, m$. Here and in the following $\sum'$ denotes the sum without the diagonal term. The eigenvalues of $\hat{G}$ become

$$
\frac{\omega^2}{c_0^2} = \frac{2\alpha}{a^2} + \frac{2\beta}{b^2} + \frac{2\gamma}{c^2} - 4 \sum_{i=1}^{m} \frac{1}{\theta_i}. \quad (4.12)
$$

(where we momentarily restored $c_0^2$ for later convenience). For $m = 0$ the sum $\sum_{i=1}^{m} 1/\theta_i$ has to be interpreted as 0. Similarly we obtain for $\hat{B}$

$$
\hat{B}\psi = x_1^a x_2^b x_3^\gamma \left\{ (2m + \alpha + \beta + \gamma)(2m + \alpha + \beta + \gamma + 3)\Pi + \sum_{i=1}^{m} \frac{\partial \Pi}{\partial \varphi_i} G_i(\theta) \right\} \quad (4.13)
$$

which leads again to the condition $G_i(\theta) = 0$ and otherwise gives back the eigenvalue \([3,4]\) for $B$, which, in particular turns out to be independent of the $\theta_i$. Finally applying $\hat{A}$ to $\psi$ we obtain

$$
\hat{A}\psi = x_1^a x_2^b x_3^\gamma \left\{ 2\beta\gamma a^2 + 2\gamma\alpha b^2 + 2\alpha\beta c^2 
+ (4m + 3)(\alpha b^2 + c^2 + \beta c^2) + \gamma(a^2 + b^2)
+ 4(a^2 + b^2 + c^2) m(m + 1)
+ (4\alpha + 4\beta + 4\gamma + 8m + 2) \sum_{i=1}^{m} \theta_i \Pi
+ \sum_{i=1}^{m} \frac{\partial \Pi}{\partial \varphi_i} \left[ (a^2 + b^2 + c^2 - x_1^2 - x_2^2 - x_3^2) \theta_i + \theta_i^2 \right] G_i(\theta) \right\}. \quad (4.14)
$$

Again $G_i(\theta) = 0$ must be satisfied, and the eigenvalues of $\hat{A}$ then can be read off eq. \([4,14]\) in the form

$$
A = 4 (\alpha + \beta + \gamma + 2m + \frac{1}{2}) \sum_{i=1}^{m} \theta_i 
+ [2\alpha\beta c^2 + (4m + 3)\alpha (b^2 + c^2) + 4m (m + 1) a^2 + cyclic]. \quad (4.15)
$$

It remains to determine the $\theta_i$ from the $m$ equations \([1.1]\) $G_i(\theta) = 0$. First we show that all $\theta_i$ are real. To prove this let us suppose they are complex, in which case they also satisfy $G_i(\theta^*) = 0$. Hence

$$
0 = \sum_i (\theta_i - \theta_i^*)(G_i(\theta) - G_i(\theta^*)) \quad (4.16)
$$

which, after some algebra, and writing $\theta_i = |\theta_i| e^{i \phi_i}$, leads to

$$
0 = \sum_i \left\{ \frac{4}{|\theta_i|^2} + \frac{4\alpha + 2}{|a^2 + \theta_i|^2} + \frac{4\beta + 2}{|b^2 + \theta_i|^2} + \frac{4\gamma + 2}{|c^2 + \theta_i|^2} \right\} |\theta_i|^2 \sin^2 \phi_i
+ \sum_{ij} \frac{4}{|\theta_i - \theta_j|^2} (|\theta_i| \sin \phi_i - |\theta_j| \sin \phi_j)^2 \quad (4.17)
$$
This condition is only satisfied if all phases $\vartheta_i$ equal either 0 or $\pi$, i.e., if the $\vartheta_i$ are real. A permutation of two $\vartheta_i$ does not lead to a new eigenfunction. Hence the $\vartheta_i$ can be assumed ordered according to

$$\vartheta_m \leq \vartheta_{m-1} \leq \ldots \leq \vartheta_1.$$  

(4.18)

Let us consider the cases $m = 0, m = 1$. For $m = 0$ we obtain from (4.12) eight eigenvalues

$$\omega^2 = \frac{a^2}{c^2} \left( \frac{2\alpha}{a^2} + \frac{2\beta}{b^2} + \frac{2\gamma}{c^2} \right)$$  

(4.19)

for $\alpha, \beta, \gamma = 0, 1$. These are the frequencies $\omega^2 = \alpha w_1^2 + \alpha w_2^2 + \alpha w_3^2 + \beta w_1^2 + \beta w_2^2 + \beta w_3^2 + \gamma w_1^2 + \gamma w_2^2 + \gamma w_3^2$. For $m = 1$ one has to solve just a single cubic equation

$$0 = \frac{4}{b_1^2} + \frac{4\alpha + 2}{a^2 + b_1^2} + \frac{4\beta + 2}{b^2 + b_1^2} + \frac{4\gamma + 2}{c^2 + b_1^2}$$  

(4.20)

to find three different values for $\vartheta_1$, which we order according to

$$-a^2 < \vartheta_{1,3} < -b^2 < \vartheta_{1,2} < -c^2 < \vartheta_{1,1} < 0$$  

(4.21)

which correspond, for each of the eight choices of the triple $(\alpha, \beta, \gamma)$, to three independent solutions. Taken together the case $m = 1$ therefore gives 24 different frequencies

$$\omega^2 = \alpha \omega_1^2 + \beta \omega_2^2 + \gamma \omega_3^2 - \frac{4c_0^2}{\vartheta_{1,k}} \quad k = 1, 2, 3.$$  

(4.22)

For the special case $\alpha, \beta, \gamma$ all vanishing the result (4.22) with the cubic equation (4.20) was already obtained by Stringari [22]. The three roots $\vartheta_{1,k}$ correspond to wave functions with a single nodal surface, which for $k = 3$ is a two-sheeted hyperboloid, for $k = 2$ a one-sheeted hyperboloid and for $k = 1$ an ellipsoid.

Let us now turn to the case of general order $m$ of the polynomial mode function. It is quite remarkable that in this case the $m$ equations $G_i(\theta) = 0$ can be derived as the extrema of a single potential $V(\theta)$

$$G_i(\theta) = -\frac{\partial V(\theta)}{\partial \vartheta_i} = 0$$  

(4.23)

where

$$8V(\theta) = -\frac{1}{2} \sum_{i=1}^{m} \ln |\vartheta_i| - \left( \frac{\alpha}{2} + \frac{1}{4} \right) \sum_{i=1}^{m} \ln |a^2 + \vartheta_i| - \left( \frac{\beta}{2} + \frac{1}{4} \right) \sum_{i=1}^{m} \ln |b^2 + \vartheta_i|$$

$$- \left( \frac{\gamma}{2} + \frac{1}{4} \right) \sum_{i=1}^{m} \ln |c^2 + \vartheta_i| - \sum_{i=1}^{m} \sum_{j=i+1}^{m} \ln |\vartheta_i - \vartheta_j|$$  

(4.24)

Thus the problem has now become the following exercise in statics: In a one-dimensional space one has four fixed positive fictitious point-charges aligned along the negative $\theta$-axis, namely

- a charge $1/2$ at $\theta = 0$,
- a charge $\frac{a}{2} + 1/4$ at $\theta = -a^2$,
- a charge $\frac{b}{2} + 1/4$ at $\theta = -b^2$,
- a charge $\frac{c}{2} + 1/4$ at $\theta = -c^2$,

between these fixed charges $m$ movable positive point-charges of unit strength, and interacting among themselves and with the fixed charges with 1-dimensional inverse-distance forces, are to be distributed in such a way that a force-equilibrium (4.23) may result. It is clear from the form of the potential (4.24) that the movable charges can be distributed arbitrarily over the three intervals, namely

- $n_3$ charges in $-c^2 < \theta < 0$,
- $n_2$ charges in $-b^2 < \theta < -c^2$,
- $n_1$ charges in $-a^2 < \theta < -b^2$.

with $n_1 + n_2 + n_3 = m$. There are $\binom{m+2}{2}$ ways to make this distribution, each leading to a different unique equilibrium configuration for the movable charges at positions $\vartheta_i$. It is immediately clear from the mechanical analogy
that all \( \theta_i \) must be different. The three integer numbers \( n_1, n_2, n_3 \) are therefore the natural quantum numbers of the problem. As the eigenvalue \( B \) is independent of the positions of the movable charges, it must be \( (m^2)^2 \)-fold degenerate for each of the eight choices of the triple \((\alpha, \beta, \gamma)\). The uniqueness of the equilibrium distribution of the \( \theta_i \) for given \( n_1, n_2, n_3 \), which determine uniquely the eigenvalues \( \omega^2 \), \( A, B \), justifies a posteriori the uniqueness assumption made after eq. (1.3). It is very easy to find the minima \( \theta_i \) of the potential (1.24) numerically for any desired triple of integers \( n_1, n_2, n_3 \) and to determine thereby the mode frequencies (1.12) and the corresponding mode functions (1.18). In table 1 we give the 20 lowest lying mode frequencies for the experimentally realized case \( \alpha = 1 \), \( \beta = 1 \), \( \gamma = 1 \), and the quantum numbers \( n_3, n_2, n_1 \). The latter give, respectively, the numbers of nodal surfaces of ellipsoidal, one-sheeted hyperbolic and two-sheeted hyperbolic form, all confocal to the Thomas-Fermi ellipsoid. The ellipsoidal nodal surfaces counted by \( n_3 \) are more elongated ellipsoids inside the Thomas-Fermi ellipsoid and would correspond to radial waves in the spherically symmetric case. As the \( \theta \)-values counted by \( n_3 \) are in the group \(-c^2 < \theta < 0 \) with the smallest absolute values, it is clear from (1.12) that the ellipsoidal nodal surfaces lead to the highest frequencies. The one-sheeted hyperbolic nodal surfaces counted by \( n_2 \) are ellipsoidal hyperboloids around the \( x_3 \)-axis which intersect planes \( x_3 = \text{const} \) in ellipses and planes \( x_2 = \text{const} \cdot x_1 \) in the two branches of hyperbolas, the hyperbolas opening up in all directions orthogonal to the \( x_3 \)-axis. Finally, the two-sheeted hyperbolic nodal surfaces counted by \( n_1 \) are formed by the two branches of ellipsoidal hyperboloids around the \( 1 \)-axis opening up in the positive and negative \( x_1 \)-direction. They are cut in ellipses by planes \( x_1 = \text{const} \) and in hyperbolas by planes \( x_2 = \text{const} \) and \( x_3 = \text{const} \). The quantum numbers \( n_2 \) and \( n_1 \) are clearly the analogues of angular momentum quantum numbers (i.e. the quantum numbers of spherical harmonics) for the elliptic geometry.

V. SYMMETRIC TRAPS AS LIMITING CASES

The cases of axially symmetric and isotropic traps must of course be contained in our results as limiting cases. Let us see how.

a. Axially-symmetric trap

Let us suppose we have axial symmetry of the trap around the \( x_1 \)-axis. In this case we should study the limit \( b \to c \) from above. This limit has to be taken in the expression for the wave function (4.3), in the force equation (1.11) and in the result for the mode frequencies (1.12). The positions \( \theta_i \) of the \( n_2 \) charges which have been distributed in the interval \(-b^2 < \theta < -c^2 \) all approach \(-c^2 \) in the limit. Therefore the mode frequencies in the limit become

$$\frac{\omega^2}{c_0^2} = \frac{2\alpha}{a^2} + \frac{2\beta + 2\gamma + 4n_2}{c^2} - 4 \left( \sum_{i=1}^{n_1} \sum_{i=n_1+n_2+1}^{m} \right) \frac{1}{\theta_i}. \quad (5.1)$$

The force-equilibrium for the \( n_1 + n_2 \) charges outside the interval \([-b^2, -c^2]\), which alone enter the sums in eq. (5.1), becomes

$$0 = \frac{4}{\theta_i} + \frac{4\alpha + 2}{a^2 + \theta_i} + \frac{4\beta + 4\gamma + 8n_2 + 4}{c^2 + \theta_i} + \left( \sum_{j=1}^{n_1} \sum_{j=n_1+n_2+1}^{m} \right) \frac{8}{\theta_i - \theta_j} \quad (5.2)$$

for \( i = 1, 2, \ldots, n_1; n_1 + n_2 + 1, \ldots, m \). The solution of eq. (5.2) is sufficient to determine the mode frequencies. However, the wave functions still depend on the asymptotics of the distribution of the \( n_2 \) charges in \(-b^2 < \theta < -c^2 \). The force equations for \( i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 \) have of course to be handled with some care, as the interaction terms between these charges, which approach each other arbitrarily closely, diverge. To isolate and divide out the diverging prefactor, which would be automatically cancelled if we worked with normalized wave functions, we define parameters \( t_i \) with \(-1 \leq t_i \leq 0 \) by \( \theta_{n_1+i} = -c^2 + t_i(b^2 - c^2) \). Then the limiting term of the force equations in question for \( b \to c \) reads

$$0 = \frac{4\gamma + 2}{t_i} + \frac{4\beta + 2}{1 + t_i} + \frac{n_2}{\sum_{j=1}^{n_2} t_i - t_j} \quad i = 1, 2, \ldots, n_2. \quad (5.3)$$

These equations can now be solved by similar techniques as used before. Fortunately, however, it will be sufficient to use (5.3) without explicit knowledge of its solutions. Turning now to the wave functions we define cylinder
coordinates around the $x_1$-axis by $x_1 = z$, $x_2 = \rho \sin \varphi$, $x_3 = \rho \cos \varphi$. Using the variables $t_i$ and multiplying by a factor $(b^2 - c^2)^{n_2}$ to eliminate the divergence the dominant term of the wave function for $b \rightarrow c$ becomes

$$\psi \sim x_1^\alpha x_2^\beta x_3^\gamma \prod_{i=1}^{m} \left( \frac{x_1^2}{a^2 + \theta_i} + \frac{x_2^2}{b^2 + \theta_i} + \frac{x_3^2}{c^2 + \theta_i} \right)$$

$$\sim \varrho^{n_2 + \beta + \gamma} \phi^{n_2} \prod_{j=1}^{m_1} \prod_{i=m-n_3+1}^{m} \left( \frac{\varrho^2}{c^2 + \theta_i} + \frac{z^2}{a^2 + \theta_i} - 1 \right).$$

Using the force equilibrium (5.3) for the $n_2$ charges pinned between $-b^2$ and $-c^2$ it can be shown that

$$\sin^\beta \phi \cos^\gamma \phi \prod_{i=1}^{n_2} \left( \cos^2 \varphi + t_i \right) \sim \cos \left( (2n_2 + \beta + \gamma)\varphi - \frac{b\pi}{2} \right).$$

As a result the wave function (5.4) for $b \rightarrow c$ goes over to

$$\psi \sim \varrho^{2n_2 + \beta + \gamma} \cos \left( (2n_2 + \beta + \gamma)\varphi - \frac{b\pi}{2} \right)$$

$$\cdot z^n \prod_{i=1}^{n_1} \prod_{i=m-n_3+1}^{m} \left( \frac{\varrho^2}{c^2 + \theta_i} + \frac{z^2}{a^2 + \theta_i} + 1 \right).$$

The quantum number $\ell_z$ for the conserved angular momentum around the $z$-axis can be read off from eq. (5.6)

$$|\ell_z| = 2n_2 + \beta + \gamma.$$  

Clearly one may take linear combinations of the wave functions (5.4) for $|\ell_z|$ fixed but suitably changing $\beta$, $\gamma$ or $n_2$ to form eigenstates $\sim e^{ \pm i |\ell_z| \varphi}$ of $L_z = -i\partial / \partial \varphi$.

b. Isotropic traps:

We can finally take the further limit $c \rightarrow a$ in the results of the preceding section. By similar considerations as described there we obtain for the mode frequencies

$$\frac{\omega^2}{c_0^2} = \frac{2\alpha + 2\beta + 2\gamma + 4n_1 + 4n_2}{a^2} - 4 \sum_{i=m-n_3+1}^{m} \frac{1}{\theta_i}. $$

The positions of the $n_3$ free charges satisfy the force equilibrium

$$0 = \frac{4}{\theta_i} + \frac{4\alpha + 4\beta + 4\gamma + 8n_1 + 8n_2 + 6}{a^2} + \sum_{j=m-n_3+1}^{m} \frac{8}{\theta_i - \theta_j} \quad \text{for} \quad i = m - n_3 + 1, \ldots, m.$$  

Introducing $\varrho = r \sin \theta$, $z = r \cos \theta$ the wave functions take the form

$$\psi \sim r^\ell P_{\ell}(|\ell_z| \cos \theta) e^{i \ell \varphi} \prod_{i=m-n_3+1}^{m} \left( \frac{r^2}{a^2 + \theta_i} - 1 \right)$$

where

$$\ell = \alpha + \beta + \gamma + 2n_1 + 2n_2 = \alpha + \beta + \gamma + 2(m - n_3)$$

is the total angular momentum quantum number. Clearly $n_3$ now is the radial quantum number.
Using the force equilibrium (5.8) it can be shown that
\[
\prod_{i=m-n_3+1}^{m} \left( \frac{r^2}{a^2 + \theta_i} - 1 \right) = (-1)^{n_3} 2 F_1 \left( -n_3, n_3 + \ell + \frac{3}{2}; \ell + \frac{3}{2}, \frac{r^2}{a^2} \right)
\]
(5.11)
and, furthermore, that
\[
\sum_{i=m-n_3+1}^{m} \frac{1}{\theta_i} = -\left( \ell + \frac{3}{2} \right) \sum_{i=m-n_3+1}^{m} \frac{1}{a^2 + \theta_i} = n_3 \left( n_3 + \ell + \frac{3}{2} \right).
\]
(5.12)
This simplifies the result (5.7) for the mode frequencies which now becomes
\[
\omega^2 = \omega_0^2 \left[ 2n_3^2 + 2n_3\ell + 3n_3 + \ell \right]
\]
(5.13)
and is indeed the result originally derived by Stringari [9], of course in a much more direct way.

VI. CONCLUSION

We have provided a complete solution for the eigenmodes and eigenfrequencies of the anisotropic wave equation governing the low-frequency collisionless density waves in Bose-Einstein condensates in harmonic oscillator traps with arbitrary anisotropy in the Thomas-Fermi limit. The eigenfrequencies are given by eq.(1.12), the mode functions are given by eq.(1.3), where the parameters \( \theta_i \) are the solutions of eqs.(1.23), (1.24). The solution was possible, because the system was found to be completely integrable, with three mutually commuting operators \( \hat{G}, \hat{A}, \hat{B} \). However, unlike in many more familiar examples in quantum mechanics, the solution was not constructed by directly solving the simultaneous eigenvalue equations for the three commuting operators, because their eigenvalues, apart from that of \( \hat{B} \), turned out to be rather complicated, not providing us with simple quantum numbers. Rather our solution proceeded by first constructing a simple form of the total wave function which followed from the separation ansatz. The solution to the equations fixing the free parameters \( \theta_i \) in the wave function then provided us with the natural simple integer quantum numbers \( n_1, n_2, n_3 \) of the problem, on which the eigenvalues of \( \hat{A} \) and the mode frequencies depend indirectly and in a complicated way. We never even had to determine the eigenvalues of \( \hat{A} \) explicitly. The form (4.8) of the wave function is remarkably reminiscent of the Bethe-ansatz. It might be interesting to follow this connection further as it might shed more light on the mathematical structure of the problem we have solved in this paper, and might e.g. allow us to gain a deeper understanding of the degeneracy of the operator \( \hat{B} \). The method of solution, and in fact even the detailed structure of the solution, generalizes directly to the analogous problem in an arbitrary number of dimensions. Thus the physical problem we have considered is found to be a member of a whole family of integrable systems with connections, as we have discussed, to billiards on a curved space conformal to Euclidean space, and the class of integrable systems discussed in the memoir of Moser [17].

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TABLE I. Mode frequencies for the trap $\omega_1^2 : \omega_2^2 : \omega_3^2 = 1 : 2 : 4$ in units of $\omega_1$. $\alpha, \beta, \gamma$ are the parity quantum numbers, $n_1, n_2, n_3$ are the three positive integer quantum numbers which label uniquely each mode function (see text). An accidental degeneracy occurs for the states $(0, 0, 0, 0, 1, 0)$ and $(1, 0, 1, 0, 0, 0)$ where $\omega/\omega_1 = \sqrt{5}$.