On the shortfall risk control  
- a refinement of the quantile hedging method

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Abstract

The issue of constructing a risk minimizing hedge with additional constraints on the shortfall risk is examined. Several classical risk minimizing problems have been adapted to the new setting and solved. The existence and specific forms of optimal strategies in a general semimartingale market model with the use of conditional statistical tests have been proven. The quantile hedging method applied in [4] and [5] as well as the classical Neyman-Pearson lemma have been generalized. Optimal hedging strategies with shortfall constraints in the Black-Scholes and exponential Poisson model have been explicitly determined.

Key words: quantile hedging, Neyman-Pearson lemma, shortfall constraints, bankruptcy prohibition, conditional tests.

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1 Introduction

Let us briefly sketch the classical hedging problem in a stochastic model of financial market. The goal of an investor having an initial capital $x \geq 0$ is to hedge dynamically a given random variable $H$ which represents the payoff of a financial contract at some future date $T > 0$. He is looking for a trading strategy $\pi$ such that the related portfolio wealth $X_{T}^{x,\pi}$ at $T$ exceeds $H$ almost surely, i.e.

$$P(X_{T}^{x,\pi} \geq H) = 1.$$ (1.1)

A strategy $\pi$ satisfying (1.1) is called a hedging strategy for $H$ and it is well known that it exists if $x$ is greater then the price of $H$. In the opposite case each trading strategy is able to hedge the claim at most partially, i.e. $P(X_{T}^{x,\pi} \geq H) < 1$, and hence generates the shortfall $(H - X_{T}^{x,\pi})^{+}$ which is strictly positive with positive probability. The related shortfall risk which appears in that case should be minimized to protect the investor against the loss resulting from a law value of the portfolio. There are several classical approaches dealing with the shortfall risk minimization which differ in the measure of risk accepted by the investor. Three of them
listed below play a central role in our study. In the quantile hedging approach introduced in [4] the objective was to maximize the probability of a successful hedge, i.e.

$$P(X_T^{x,\pi} \geq H) \longrightarrow \max. \quad (1.2)$$

In a generalized version of (1.2) also the value of unsuccessful hedge was taken into account by involving the so called success ratio of the pair $(x, \pi)$ defined by

$$\varphi_{x,\pi} := 1_{\{X_T^{x,\pi} \geq H\}} + \frac{X_T^{x,\pi}}{H} 1_{\{X_T^{x,\pi} < H\}}. \quad (1.3)$$

Then the aim of the trader was the following

$$E[\varphi_{x,\pi}] \longrightarrow \max. \quad (1.4)$$

Another optimality criterion was to minimize the weighted expected shortfall, i.e.

$$E[l((H - X_T^{x,\pi})^+)] \longrightarrow \min, \quad (1.5)$$

where $l$ is a so called loss function. The case $l(z) = z$ has been studied in [2] and the general case in [5] and [6].

The motivation for the present paper arises from the fact that the profile of the shortfall in all the problems (1.2), (1.4) and (1.5) remains beyond the trader’s control. The preferences of the trader towards the size of the shortfall are not described sufficiently well by the risk measures mentioned above and consequently even a risk minimizing strategy may generate the loss which exceeds the solvency of the trader. So, even the best performance may lead to bankruptcy in finite time! This problem is apparent in the quantile hedging approach because, as shown in [4], the optimal strategy $\tilde{\pi}$ for (1.2) is such that

$$X_T^{x,\tilde{\pi}} = H 1_{A},$$

where $A$ is some subset of $\Omega$ which depends on $x$. It follows that the shortfall equals $H 1_{A^c}$ which means that the shortfall risk is completely unhedged on $A^c$. This depicts the quantile hedging method as a risky tool for minimizing risk. Although the risk measures in (1.4) and (1.5) are more involved, the problem of an uncontrolled shortfall profile appears there as well.

To illustrate that let us consider a call option $H = (S_T - K)^+$ on the underlying asset $S$ in the classical Black-Scholes model with drift $\alpha$ and volatility $\sigma > 0$. It was shown in [4] p.261 that in the case when $\alpha < \sigma^2$ the optimal strategy $\tilde{\pi}$ for (1.4) generates the wealth

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+ - (S_T - k)^+ - (k - K) 1_{\{S_T > k\}}.$$

where $k > K$ is a certain constant which depends on $x$. Thus the related shortfall equals

$$H - X_T^{x,\tilde{\pi}} = (S_T - k)^+ + (k - K) 1_{\{S_T > k\}}.$$

In particular, it is clear that the shortfall is unbounded on the set $\{S_T > k\}$ which implies a positive ruin probability for each investor regardless of the level of his solvency. An analogous example can be constructed for (1.5) with $l(z) = z$ and the claim $H := \frac{1}{S_T}$. 

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In this paper we show how to incorporate a relevant shortfall profile into the problems (1.2), (1.4) and (1.5) which in turn allows to obviate the drawbacks of optimal strategies mentioned above. The idea is to introduce a shortfall constraint $L$ which is a random variable acting as upper bound for the shortfall and study the problems (1.2), (1.4), (1.5) subject to the additional condition

$$P((H - X_T^{x,\tilde{\pi}})^+ \leq L) = 1.$$  \hspace{1cm} (1.6)

Since $L$ is of a fairly general form, this setting provides a flexible tool for managing hedging risk and allows to accommodate fully the risk preferences of the investor. In particular, an appropriate choice of a bounded shortfall constraint protects him against a bankruptcy threat.

Coming back to the example mentioned above of a call option in the B-S model assume that the trader wishes to keep the shortfall below a constant margin $c > 0$. Our general results applied to this particular situation yield explicit solutions to the problems (1.2), (1.4) and (1.5). It turns out that the portfolio wealth of an optimal strategy is a digital combination of two options: $(S_T - K)^+$ and $(S_T - (K + c))^+$. A precise form of the combination depends on the risk measure and the initial capital $x$. The optimal strategy for (1.2) satisfies

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+1_{\{S_T \leq k_1\}} \cup \{S_T \geq k_2\} + (S_T - (K + c))^+1_{\{k_1 < S_T < k_2\}},$$

with $K < k_1 \leq K + c$, $k_2 \geq K + c$. For (1.4) the optimal portfolio is such that

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+1_{\{S_T \leq k_3\}} + (S_T - (K + c))^+1_{\{S_T > k_3\}},$$

with $k_3 > K$ while for (1.5) with the loss function $l(z) = z$ we obtain

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+1_{\{S_T \leq k_4\}} + (S_T - (K + c))^+1_{\{S_T > k_4\}},$$

with $k_4 > K$. All the constants above depend on $x$.

In the paper we characterize optimal solutions for the problems (1.2), (1.4), (1.5) under (1.6) for the general forms of $H$ and $L$. Our assumptions concerning the market are minimal because we require only that the price process $(S_t)$ is a semimartingale and that the related set of martingale measures is not empty. The main results are Proposition 3.1, Theorem 3.2 and Theorem 3.3. Our investigation relies on a certain restriction of the success ratio (1.3) implied by (1.6). It allows to characterize the solutions of (1.2), (1.4), (1.5) in terms of certain statistical tests, that is $[0,1]$-valued random variables, which exceed a prespecified test $\varphi^*$. Each test of this form we call a conditional test with a rejection threshold $\varphi^*$. In the presented framework $\varphi^*$ is determined by $H$ and $L$. Our approach generalizes the celebrated quantile hedging method applied in [4] and [5] for the case when the shortfall profile is unconstrained. This particular situation corresponds to the condition $L = H$ which generates the trivial rejection threshold $\varphi^* \equiv 0$. In Lemma 4.1 we prove a generalized version of the Neyman-Pearson lemma for conditional statistical tests which is used then in dealing with complete markets. The explicit forms of optimal strategies are determined in the Black-Scholes and exponential Poisson model.

The paper is organized as follows. In Section 2 we describe the market model and formulate the optimization problems in a precise way. The main results characterizing optimal strategies
with shortfall constraints are proven in Section 3. The concept of a conditional statistical test is discussed in Section 4 where also a generalized version of the Neyman-Pearson lemma is proven. Examples concerning complete markets are presented in Section 5.

2 Formulations of the problems

We will consider a classical continuous time financial market with a stock price given by a semimartingale \((S_t)\) on a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathcal{F} = \mathcal{F}_T, P)\) with \(T < +\infty\). Investing possibilities are described by admissible trading strategies, that is predictable and \(S\)-integrable stochastic processes \((\pi_t)_{t \in [0,T]}\) such that the corresponding wealth process

\[
X^{x,\pi}_t = x + \int_0^t \pi(s) dS(s), \quad t \in [0,T],
\]

is nonnegative for each \(t \in [0,T]\). Above \(x\) stands for an initial capital of the investor. For the sake of simplicity we assume that the risk-free interest rate equals zero, so the value of 1 Euro on a savings account is constant in time. Let \(H\) be an \(\mathcal{F}_T\)-measurable, nonnegative random variable representing a contingent claim with payoff at time \(T\). A pair \((x, \pi)\) hedges \(H\) if

\[
P(X^{x,\pi}_T \geq H) = 1. \tag{2.1}
\]

We assume that the set of all martingale measures \(Q\) is not empty. Recall, that \(Q \in Q\) if and only if \(Q \sim P\) and the process \(S\) is a local martingale under \(Q\). Under each \(Q \in Q\) the wealth process of an admissible strategy is a supermartingale. The assumption \(Q \neq \emptyset\) precludes arbitrage opportunities from the market, which means that there is no admissible strategy \(\pi\) such that \(P(X^{0,\pi}_T > 0) > 0\). It is well known that then \(H\) can be hedged by some strategy \((x, \pi)\) if the initial capital \(x\) exceeds its price \(p(H)\), i.e.

\[
x \geq p(H) := \sup_{Q \in Q} E^Q[H]. \tag{2.2}
\]

This means that if (2.2) holds then there exists \(\pi\) such that the pair \((x, \pi)\) satisfies (2.1). In the particular case when \(Q\) is a singleton the price is given by the expectation of the claim under a unique martingale measure and the inequality in (2.1) can be replaced by equality. Then the market is complete and a hedging strategy satisfying \(X^{x,\pi}_T = H\) is called a replicating strategy.

If \(x < p(H)\) then (2.1) is violated and hence each admissible strategy \(\pi\) is biased by hedging risk with positive probability, that is

\[
P((H - X^{x,\pi}_T)^+) > 0.
\]

The problem of the investor is to minimize the risk which is quantified by a properly chosen risk measure. A component of the risk measures considered in the sequel is a condition describing a maximal size of the shortfall \((H - X^{x,\pi}_T)^+\) which is acceptable by the trader. A shortfall constraint is defined by an \(\mathcal{F}_T\)-measurable random variable \(L\) satisfying the technical condition

\[
0 \leq L \leq H, \tag{2.3}
\]
and it constitutes as acceptable these of the admissible strategies which satisfy

\[ P\left((H - X_T^{x,\pi})^+ \leq L\right) = 1. \quad (2.4) \]

The assumption (2.3) can be clearly interpreted. The positivity of \( L \) means that the trader is not interested in overperformance while the upper bound excludes strategies generating the shortfall which exceeds the value of the contract. It is intuitively clear that the initial capital and the shortfall constraint should be related to each other, that is portfolios with a restricted shortfall should keep the initial cost at a sufficiently high level. Indeed, under (2.3) the condition (2.4) is equivalent to \( P(X_T^{x,\pi} \geq H - L) = 1 \), which shows that the portfolio hedges the claim \( H - L \). This implies the relation

\[ x \geq p(H - L). \quad (2.5) \]

Below we give some natural examples of shortfall constraints corresponding to the various forms of the trader’s risk aversion.

**Examples**

a) If \( L = H \) then (2.4) boils down to the positivity of \( X_T^{x,\pi} \) and hence the profile of the shortfall remains unconstrained. This case corresponds to the classical framework considered in the literature.

b) For \( L = 0 \) the trader is expected to hedge the claim \( H \), so no shortfall is acceptable at all.

c) The trader can cover the arising portfolio loss providing that it lies below a fixed constant level \( c > 0 \). The maximal value of \( c \) is defined by the solvency of the trader. In this case we take

\[ L = c \wedge H. \]

d) Generalizing the previous example, the trader may want to keep the loss below \( c \) and simultaneously hedge \( H \) in some fixed price range \([a, b]\) of the underlying stock price. Then \( L \) is given by

\[ L = (c \wedge H)1_{\{S_T < a\}} + (c \wedge H)1_{\{S_T > b\}}. \]

e) In the subjective forecast of the trader the stock price range \((0, a), (b, +\infty)\) is viewed as unrealistic and hence ruled out as source of risk. The trader’s aim is to keep the shortfall below \( c \) only in the interval \([a, b]\). The related form of \( L \) is

\[ L = (c \wedge H)1_{\{S_T \in [a,b]\}} + H1_{\{S_T \notin [a,b]\}}. \]

f) Let \( \alpha \in [0, 1] \) describe a partial recovery of the claim, i.e. the claim which is to be hedged is \( \alpha H \). Then \( L \) is equal to

\[ L = (1 - \alpha)H. \]
Our aim is to solve the classical optimization problems (1.2), (1.4) and (1.5) mentioned in Introduction which are adapted to the new framework with a constrained shortfall profile. Taking into account (2.3), (2.4) and (2.5) the problems under consideration are the following

**Quantile hedging problem**

\[
\begin{align*}
(QH) & \quad \begin{cases} 
P(X_T^{x,\pi} \geq H) \to \max \pi \\
(i) & P((H - X_T^{x,\pi})^+ \leq L) = 1 \\
(ii) & p(H - L) \leq x < p(H),
\end{cases}
\end{align*}
\]

**Generalized quantile hedging problem**

\[
\begin{align*}
(GQH) & \quad \begin{cases} 
E[\varphi_{x,\pi}] \to \max \pi \\
(i) & P((H - X_T^{x,\pi})^+ \leq L) = 1 \\
(ii) & p(H - L) \leq x < p(H),
\end{cases}
\end{align*}
\]

**Weighted expected shortfall problem**

\[
\begin{align*}
(WES) & \quad \begin{cases} 
E[l((H - X_T^{x,\pi})^+)] \to \min \pi \\
(i) & P((H - X_T^{x,\pi})^+ \leq L) = 1 \\
(ii) & p(H - L) \leq x < p(H).
\end{cases}
\end{align*}
\]

A success ratio \(\varphi_{x,\pi}\) in (GQH) is defined by

\[
\varphi_{x,\pi} := \mathbb{1}_{\{x \geq H\}} + \frac{X_T^{x,\pi}}{H} \mathbb{1}_{\{x < H\}}. \tag{2.6}
\]

and the loss function \(l : [0, +\infty) \to [0, +\infty)\) in (WES) is increasing, convex and satisfies \(l(0) = 0\).

It is clear that for \(L = H\) the problems (QH), (GQH), (WES) amount to their counterparts (1.2), (1.4) and (1.5) with an unconstrained shortfall risk.

### 3 Optimal strategies with shortfall constraint

In the following subsections we characterize optimal strategies for the problems (QH), (GQH) and (WES).

#### 3.1 Quantile hedging problem

The following result describes an optimal strategy for the problem (QH). Below \(A^c\) stands for the complement of a set \(A\).

**Proposition 3.1** Let \(p(H - L) \leq x < p(H)\). If there exists a set \(\tilde{A} \in \mathcal{F}\) solving the problem

\[
\begin{align*}
\begin{cases} 
P(A) \to \max \\
(i) & p(H - L\mathbb{1}_{\tilde{A}^c}) \leq x,
\end{cases}
\end{align*}
\]

then a hedging strategy \((\tilde{x}, \tilde{\pi})\) for the claim \(\tilde{H} := H - L\mathbb{1}_{\tilde{A}^c}\) with \(\tilde{x} = p(\tilde{H})\) solves (QH).
Proof: Let us define a success set of a strategy $(x, \pi)$ by

$$A_{x,\pi} := \{X^{x,\pi}_T \geq H\}.$$  

First we show that for any strategy $(x, \pi)$ satisfying (QH) (i), (ii) holds

$$P(X^{x,\pi}_T \geq H) = P(A_{x,\pi}) \leq P(\tilde{A}).$$

Since $X^{x,\pi}_T \geq H$ on $A_{x,\pi}$ and, by (QH) (i), $X^{x,\pi}_T \geq H - L$ a.s., it follows

$$H - L 1_{A^{x,\pi}} = H 1_{A_{x,\pi}} + (H - L) 1_{A^{x,\pi}} \leq X^{x,\pi}_T.$$  

Using the fact that $X^{x,\pi}$ is a $Q$-supermartingale for each martingale measure, we obtain

$$p(H - L 1_{A^{x,\pi}}) \leq p(X^{x,\pi}_T) \leq x,$$

which is (3.1) (i). Hence $P(A_{x,\pi}) \leq P(\tilde{A}).$

Now let us consider the strategy $(\tilde{x}, \tilde{\pi})$. Then the condition $X^{\tilde{x},\tilde{\pi}}_T \geq H - L 1_{\tilde{A}}$ implies that $X^{\tilde{x},\tilde{\pi}}_T \geq H$ on $\tilde{A}$ and further that

$$A^{\tilde{x},\tilde{\pi}} \supseteq \tilde{A}.$$  

Moreover, $(\tilde{x}, \tilde{\pi})$ satisfies (QH) (i), (ii) and thus $P(A^{\tilde{x},\tilde{\pi}}) \leq P(\tilde{A})$. Hence $A^{\tilde{x},\tilde{\pi}} = \tilde{A}$ and the optimality of $(\tilde{x}, \tilde{\pi})$ follows. □

3.2 Generalized quantile hedging problem

Let $\mathcal{R}$ stand for the family of all statistical tests, that is

$$\mathcal{R} := \{\varphi : \varphi \text{ is } \mathcal{F} \text{ - measurable and } 0 \leq \varphi \leq 1\}. \quad (3.2)$$

From (2.3) follows that $\frac{H - L}{H} \in \mathcal{R}$ provided that $\frac{H - L}{H}$, by definition, is equal to zero on the set $\{H = 0\}$. An optimal strategy for (GQH) is characterized by the following result.

Theorem 3.2 Let $x$ be an arbitrary initial capital satisfying $p(H - L) \leq x < p(H)$. Denote by $\hat{\varphi} \in \mathcal{R}$ a solution of the problem

$$\begin{align*}
E[\varphi] & \rightarrow \max \\
(i) & \varphi \geq \frac{H - L}{H}, \\
(ii) & p(H \varphi) \leq x.
\end{align*} \quad (3.3)$$

Then a hedging strategy $(\tilde{x}, \tilde{\pi})$ with $\tilde{x} = p(\tilde{H})$ for the payoff $\tilde{H} := H \hat{\varphi}$ is optimal for the problem (GQH) and $\varphi^{\tilde{x},\tilde{\pi}} = \hat{\varphi}$.

Proof: The existence of $\hat{\varphi}$ follows from the weak compactness of the set

$$\mathcal{R} \cap \left\{ \varphi : \varphi \geq \frac{H - L}{H} \right\}.$$
Consider any strategy \((x, \pi)\) satisfying (GQH) (i) and (ii). It follows from (GQH) (i) that
\[
\varphi_{x,\pi} = 1_{\{X_{T,x}^{x,\pi} \geq H\}} + \frac{X_{T,x}^{x,\pi}}{H}1_{\{X_{T,x}^{x,\pi} < H\}} \geq 1_{\{X_{T,x}^{x,\pi} \geq H\}} + \frac{H - L}{H}1_{\{X_{T,x}^{x,\pi} < H\}} \geq \frac{H - L}{H},
\]
which implies that \(\varphi_{x,\pi}\) satisfies (3.3) (i). Moreover, it follows from the inequality
\[
H \varphi_{x,\pi} = H1_{\{X_{T,x}^{x,\pi} \geq H\}} + X_{T,x}^{x,\pi}1_{\{X_{T,x}^{x,\pi} < H\}} \leq X_{T,x}^{x,\pi}
\]
and from the fact that \(X_{T,x}^{x,\pi}\) is a Q-supermartingale for each \(Q \in \mathcal{Q}\) that
\[
p(H \varphi_{x,\pi}) \leq p(X_{T,x}^{x,\pi}) \leq x.
\]
This means that \(\varphi_{x,\pi}\) satisfies (3.3) (ii). It follows that \(E[\varphi_{x,\pi}] \leq E[\tilde{\varphi}]\).

For the strategy \((\tilde{x}, \tilde{\pi})\) holds
\[
\varphi_{\tilde{x},\tilde{\pi}} = 1_{\{X_{T,x}^{\tilde{x},\tilde{\pi}} \geq H\}} + \frac{X_{T,x}^{\tilde{x},\tilde{\pi}}}{H}1_{\{X_{T,x}^{\tilde{x},\tilde{\pi}} < H\}} \geq 1_{\{X_{T,x}^{\tilde{x},\tilde{\pi}} \geq H\}} + \tilde{\varphi}1_{\{X_{T,x}^{\tilde{x},\tilde{\pi}} < H\}} \geq \tilde{\varphi}
\]
which, in view of the first part of the proof, implies \(\varphi_{\tilde{x},\tilde{\pi}} = \tilde{\varphi}\). Hence \((\tilde{x}, \tilde{\pi})\) solves (GQH). \(\square\)

### 3.3 Weighted expected shortfall problem

The theorem below characterizes an optimal strategy for (WES).

**Theorem 3.3** Assume that the initial capital \(x\) satisfies \(p(H - L) \leq x < p(H)\). Let \(\tilde{\varphi} \in \mathcal{R}\) be a solution of the problem
\[
\left\{ \begin{array}{l}
E[l((1 - \varphi)H)] \to \min \\
(i) \quad \varphi \geq \frac{H - L}{H}, \\
(ii) \quad p(H \varphi) \leq x.
\end{array} \right.
\]

Let \((\tilde{x}, \tilde{\pi}), \tilde{x} = p(H)\), be a hedging strategy for the claim \(H := H\tilde{\varphi}\). Then \((\tilde{x}, \tilde{\pi})\) solves (WES) and \(\varphi_{\tilde{x},\tilde{\pi}} = \tilde{\varphi}\).

**Proof:** Using the same type arguments as in the proof of Proposition 3.1 in [5] we prove that \(\tilde{\varphi}\) exists. Let \(\{\varphi\}_n\) be a minimizing sequence satisfying (3.4) (i), (ii). There exists a new minimizing sequence
\[
\tilde{\varphi}_n \in \text{conv}\{\varphi_n, \varphi_{n+1}, \ldots\}
\]
which converges almost surely to a limit \(\tilde{\varphi}\). It is easy to see that \(\tilde{\varphi}\) also satisfies (i) and (ii), so it solves (3.4).

Let \((x, \pi)\) be a strategy satisfying the conditions (WES)(i) and (WES)(ii). Repeating the first part of the proof of Theorem 3.2 we get
\[
\varphi_{x,\pi} \geq \frac{H - L}{H}, \quad \text{and} \quad p(H - L) \leq p(H \varphi_{x,\pi}) \leq x,
\]
and thus \( \varphi_{x,\pi} \) satisfies (3.4) (i), (ii). Further, we have
\[
E[l((H - X_T^{x,\pi})^+)] = E[l((1 - \varphi_{x,\pi})H)] \geq E[l((1 - \tilde{\varphi})H)].
\] (3.5)

For the strategy \((\tilde{x}, \tilde{\pi})\) holds \( \varphi_{\tilde{x},\tilde{\pi}} \geq \tilde{\varphi} \) and from the monotonicity of \( l \) we obtain
\[
E[l((H - X_T^{\tilde{x},\tilde{\pi}})^+)] = E[l((1 - \varphi_{\tilde{x},\tilde{\pi}})H)] \leq E[l((1 - \tilde{\varphi})H)].
\] (3.6)
The result follows from (3.5) and (3.6).

\[\square\]

4 Conditional statistical tests and generalized Neyman-Person lemma

In this section we analyse the conditions describing the success ratios of optimal strategies. Notice that using (2.2), which defines the price of \( H \), the problem (3.3) can be written in the form
\[
\begin{cases}
(i) & E^P[\varphi] \to \max \\
(ii) & \varphi^* \leq \varphi \leq 1, \\
(iii) & \sup_{Q \in \hat{Q}} E^Q[\varphi] \leq x.
\end{cases}
\] (4.1)

where \( \varphi^* := \frac{H-L}{H} \) and \( \hat{Q} \) is the family of finite measures defined by \( d\hat{Q} := H dQ, Q \in Q \). The conditions (4.1) (i) and (4.1) (iii) correspond to the classical problem of testing a null composite hypothesis represented by the family \( \hat{Q} \) against a simple alternative hypothesis given by the measure \( P \). More precisely, (4.1) (iii) is a constraint for the type I statistical error while (4.1) (i) describes minimization of the type II statistical error. The non-standard condition is (4.1) (ii) which tells that each test must exceed the minimal threshold \( \varphi^* \) of rejecting the null hypothesis. We call tests satisfying (4.1) (ii) and (4.1) (iii) conditional tests with a rejection threshold \( \varphi^* \). The rejection threshold affects of course both statistical errors. The error of the first kind is bounded from below, i.e.
\[
\sup_{Q \in \hat{Q}} E^Q[\varphi] \geq \sup_{Q \in \hat{Q}} E^Q[\varphi^*],
\]
while the error of the second kind is bounded from above, i.e.
\[
E^P[1 - \varphi] \leq E^P[1 - \varphi^*].
\]

It follows, in particular, that (4.1) is well posed if \( x \geq \sup_{Q \in \hat{Q}} E^Q[\varphi^*] \). The special case when \( \hat{Q} \) is a singleton is of prime importance because it corresponds to complete markets which are analytically tractable. If this is the case and \( \varphi^* = 0 \) then (4.1) becomes a classical testing problem with simple hypotheses and its solution is described by the Neyman-Pearson lemma. There are several results in the literature which extend the classical Neyman-Pearson lemma to composite hypotheses, see [3], [7], [8], [10]. The result proven below set up a new kind of generalization concerned with conditional tests for simple hypotheses.

Recall, \( \mathcal{R} \) stands for the family of statistical tests, see (3.2).
Lemma 4.1 Let $P$ and $Q$ be any two equivalent probability measures. For given $\varphi^* \in \mathcal{R}$ and $\alpha \in [E^Q[\varphi^*], 1]$ a solution $\hat{\varphi}$ of the problem

$$
\begin{cases}
E^P[\varphi] \to \max \\
(i) \quad \varphi^* \leq \varphi \leq 1, \\
(ii) \quad E^Q[\varphi] \leq \alpha,
\end{cases}
$$

(4.2)

has the form

$$
\hat{\varphi} = 1_{\{\varphi^* = 1\}} \cup \left\{ \varphi^* + (1 - \varphi^*) \right\} = 1_{\{\varphi^* < 1\}} + \gamma(1 - \varphi^*)1_{\{\varphi^* < 1\}},
$$

(4.3)

where $k \geq 0, \gamma \in [0, 1]$ are constants such that $E^Q[\hat{\varphi}] = \alpha$.

Proof: It is clear that $\hat{\varphi} = \varphi^* = 1$ on the set $\{\varphi^* = 1\}$. On the set $\{\varphi^* < 1\}$ the optimal solution $\hat{\varphi}$ must solve the problem

$$
\begin{cases}
E^P[\varphi 1_{\{\varphi^* < 1\}}] \to \max \\
(i) \quad \varphi^* 1_{\{\varphi^* < 1\}} \leq \varphi 1_{\{\varphi^* < 1\}} \leq 1, \\
(ii) \quad E^Q[\varphi 1_{\{\varphi^* < 1\}}] \leq \alpha - P(\varphi^* = 1).
\end{cases}
$$

(4.4)

For any $\varphi$ such that $\varphi^* \leq \varphi \leq 1$ consider the transformation

$$
\Phi = \Phi(\varphi) := \frac{\varphi 1_{\{\varphi^* < 1\}} - \varphi^* 1_{\{\varphi^* < 1\}}}{(1 - \varphi^*)1_{\{\varphi^* < 1\}},
$$

(4.5)

which defines a random variable on the set $\hat{\Omega} := \{\varphi^* < 1\}$. The problem (4.4) can be transformed with the use of two auxiliary probability measures on $\hat{\Omega}$ with densities

$$
\frac{d\hat{P}}{dP} := \frac{(1 - \varphi^*)1_{\{\varphi^* < 1\}}}{E^P[1_{\{\varphi^* < 1\}}]}, \quad \frac{d\hat{Q}}{dQ} := \frac{(1 - \varphi^*)1_{\{\varphi^* < 1\}}}{E^Q[1_{\{\varphi^* < 1\}}]},
$$

to the form

$$
\begin{cases}
E^\hat{P}[\Phi] \to \max \\
(i) \quad 0 \leq \Phi \leq 1, \\
(ii) \quad E^\hat{Q}[\Phi] \leq \frac{\alpha - Q(\varphi^* = 1) - E^Q[1_{\{\varphi^* < 1\}}]}{E^Q[1_{\{\varphi^* < 1\}}]}.
\end{cases}
$$

(4.6)

The problem (4.6) is a standard testing problem and the classical Neyman-Pearson lemma provides its solution

$$
\tilde{\Phi} = 1_{\{d\hat{P}/d\hat{Q} > k\}} + \gamma 1_{\{d\hat{P}/d\hat{Q} = k\}},
$$

(4.7)

where $k \geq 0, \gamma \in [0, 1]$ are constants such that (4.6) (ii) holds as equality. Since

$$
\frac{d\hat{P}}{d\hat{Q}} = \text{const.} \frac{dP}{dQ} 1_{\{\varphi^* < 1\}}, \quad \text{const.} > 0,
$$

(4.8)
the optimal solution of (4.6) can be written in the form
\[ \tilde{\Phi} = 1_{\{dP \wedge \tilde{\Phi} \} > k} + \gamma 1_{\{dP \wedge \tilde{\Phi} \} = k}, \tag{4.8} \]
where the constant \( k \) in (4.7) and (4.8) may differ. Coming back to (4.5) we determine \( \tilde{\phi} 1_{\{\tilde{\phi} < 1\}} \)
from the equation
\[ \tilde{\Phi} = \tilde{\Phi}(\tilde{\phi}) = 1_{\{dP \wedge \tilde{\Phi} \} > k} + \gamma 1_{\{dP \wedge \tilde{\Phi} \} = k}, \]
which gives
\[ \tilde{\phi} 1_{\{\tilde{\phi} < 1\}} = 1_{\{dP \wedge \tilde{\Phi} \} > k} + \left[ \tilde{\phi} + \gamma (1 - \tilde{\phi}) \right] 1_{\{dP \wedge \tilde{\Phi} \} = k} + \tilde{\phi} 1_{\{dP \wedge \tilde{\Phi} \} < k}. \]
This, in view of the decomposition \( \tilde{\phi} = \tilde{\phi} 1_{\{\tilde{\phi} = 1\}} + \tilde{\phi} 1_{\{\tilde{\phi} < 1\}} \), yields (4.3). \( \square \)

One can check that Lemma 4.1 with \( \varphi^* = 0 \) boils down to the classical Neyman-Pearson lemma.

5 Complete markets

Our aim now is to minimize the hedging risk of a call option \((S_T - K)^+\), \( K > 0 \) in the class of strategies subject to the shortfall constraint \( L = c \wedge (S_T - K)^+ \) with \( c \geq 0 \). The Black-Scholes and exponential Poisson models will be examined. The initial capital of the investor is assumed to satisfy the restriction (2.5), which amounts to
\[ p\left( (S_T - (K + c))^+ \right) \leq x < p\left( (S_T - K)^+ \right). \]
This means that \( x \) is less than the replicating cost of the option but is also greater than the replicating cost of the call option with the strike \( K + c \). Combining Proposition 3.1, Theorem 3.2, Theorem 3.3 and Lemma 4.1 we show in the following paragraphs that an optimal strategy hedges always an option which is a sum of two knock-out options, i.e. it has the form
\[ \tilde{H} = (S_T - K)^+ 1_A + (S_T - (K + c))^+ 1_{A^c}, \tag{5.1} \]
where \( A \in \mathcal{F}_T \) is a set which depends on the initial capital \( x \) and the risk measure of the investor. For the exponential Poisson model an additional term in (5.1) appears which is related to the presence of jumps of the price process, see formula (5.14) in the sequel.

5.1 Black-Scholes model

Let us recall some basics concerning the Black-Scholes model. The asset price dynamics has the form
\[ dS_t = S_t(\alpha dt + \sigma dW_t), \quad S_0 = s_0, \quad t \in [0, T], \quad \alpha \in \mathbb{R}, \sigma > 0. \]
The unique martingale measure \( Q \) is given by
\[ \frac{dQ}{dP} = Z = e^{-\theta W_T - \frac{1}{2} \theta^2 T}, \]
with \( \theta = \frac{\alpha}{\sigma} \). Under \( Q \) the process \( \tilde{W}_t := W_t + \theta t \) is a Wiener process and the dynamics of \( S \) under \( Q \) has the form \( dS_t = \sigma d\tilde{W}_t \). The price of the call option is given by

\[
C_{BS}(K) := p\left((S_T - K)^+\right) = E^Q[(S_T - K)^+] = s_0 \phi(d_1) - K \phi(d_2),
\]

where

\[
d_1 := \frac{\ln \left( \frac{S_0}{K} \right) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}, \quad d_2 := d_1 - \sigma \sqrt{T},
\]

and \( \Phi \) stands for the \( N(0,1) \)-cumulative distribution function.

Below we solve the problems (QH), (GQH) and (WES) explicitly, mainly in the case when the parameters satisfy \( 0 < \alpha < \sigma^2 \). Another cases can be treated, however, in a similar way.

**Quantile hedging problem**

**Proposition 5.1** Let \( C_{BS}(K + c) \leq x < C_{BS}(K) \). An optimal strategy for a call option \((S_T - K)^+\) with the shortfall constraint \( L = c \wedge (S_T - K)^+ \) in the Black-Scholes model with parameters satisfying \( 0 < \alpha < \sigma^2 \) is a replicating strategy for the payoff

\[
\tilde{H} = (S_T - K)^+ 1_{\{S_T \leq I(k)\}} + (S_T - K)^+ 1_{\{S_T \geq J(k)\}} + (S_T - (K + c))^+ 1_{\{I(k) < S_T < J(k)\}},
\]

where

\[
I(k) := \hat{y}(k) \wedge (K + c), \quad J(k) := (Ck) \frac{\alpha^2}{\sigma} \vee (K + c),
\]

with \( C := \left( \frac{1}{s_0} e^{-\frac{1}{2}(\alpha + \sigma^2)T} \right)^{-\frac{\alpha^2}{\sigma^2}} \) and \( \hat{y}(k) \) being the unique solution of the equation

\[
\frac{1}{Ck} y^\frac{\alpha^2}{\sigma^2} = y - K, \quad y \geq 0.
\]

The constant \( k \) in (5.2) is uniquely defined by the relation

\[
C_{BS}(K) + C_{BS}(K + c) - C_{BS}(I(k)) - (I(k) - K)(1 - \Phi(e_1(k))) + c(1 - \Phi(e_2(k))) = x, \quad (5.3)
\]

where

\[
e_1(k) := \frac{\ln \left( \frac{I(k)}{s_0} \right) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}, \quad e_2(k) := \frac{\ln \left( \frac{J(k)}{s_0} \right) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}.
\]

In particular, the shortfall of the optimal strategy equals \([S_T \wedge (K + c) - K] 1_{\{I(k) < S_T < J(k)\}}\).

**Proof:** It follows from Proposition 3.1 that the optimal payoff has the form

\[
\tilde{H} = (S_T - K)^+ - L1_{\tilde{A}^c},
\]

where \( \tilde{A} \) solves (3.1). It is clear that (3.1) amounts to the problem

\[
\left\{ \begin{array}{l}
P(A) \to \max \\
E^Q[L1_A] \leq x - E^Q[H - L],
\end{array} \right.
\]

where
and thus can be transformed, with the use of the auxiliary measure $d\hat{Q} = \frac{P}{E^Q[H-L]}dQ$, to the problem

\[
\begin{cases}
P(A) \to \max \\
\hat{Q}(A) \leq \frac{x-E^Q[H-L]}{E^Q[L]},
\end{cases}
\]

From the classical Neyman-Pearson lemma follows the form of its solution

\[
\hat{A} = \left\{ \frac{dP}{dQ} \geq a \right\} = \left\{ \frac{1}{Z} \geq kL \right\} = \left\{ \frac{1}{Z} \geq k((S_T - K)^+ \land c) \right\}, \quad a, k \geq 0,
\]

if only $k$ is such that $\hat{Q}(\hat{A}) = \frac{x-E^Q[H-L]}{E^Q[L]}$. The set $\hat{A}$ can be represented as

\[
\hat{A} = \left\{ \frac{1}{k c} \geq Z, (S_T - K)^+ \geq c \right\} \cup \left\{ \frac{1}{k Z} \geq (S_T - K)^+, (S_T - K)^+ < c \right\}.
\]

Due to the fact that $Z = C S_T^{-\frac{\sigma^2}{2}}$ and $\alpha > 0$ we have

\[
\left\{ \frac{1}{k c} \geq Z \right\} = \left\{ \frac{1}{C k c} \geq S_T^{-\frac{\sigma^2}{2}} \right\} = \left\{ (k c C)^{\frac{\sigma^2}{2}} \leq S_T \right\}.
\]

By the definition of $\hat{y}(k)$ and the fact that $\alpha < \sigma^2$ one obtains

\[
\left\{ \frac{1}{k Z} \geq (S_T - K)^+ \right\} = \left\{ \frac{1}{C k} S_T^{\frac{\sigma^2}{2}} \geq (S_T - K)^+ \right\} = \{ S_T \leq \hat{y}(k) \}.
\]

Using the relation $\{ (S_T - K)^+ \geq c \} = \{ S_T \geq K + c \}$ we get the required form of $\hat{A}$

\[
\hat{A} = \{ S_T \geq (k c C)^{\frac{\sigma^2}{2}} \lor (K+c) \} \cup \{ S_T \leq \hat{y}(k) \land (K+c) \} = \{ S_T \leq I(k) \} \cup \{ S_T \geq J(k) \}
\]

which together with (5.4) gives the optimal payoff

\[
\bar{H} = \bar{H}(k) = (S_T - K)^+ 1_{\bar{A}} + ((S_T - K)^+ - (S_T - K)^+ \land c) 1_{\bar{A}^c} \\
= (S_T - K)^+ 1_{\{ S_T \leq I(k) \}} + (S_T - K)^+ 1_{\{ S_T \geq J(k) \}} + (S_T - (K+c))^+ 1_{\{ I(k) < S_T < J(k) \}}.
\]

The function $z \to E^Q[\bar{H}(z)]; z \geq 0$ is continuous, monotone and satisfies

\[
E^Q[\bar{H}(z)] \xrightarrow{z \to 0} C_{BS}(K), \quad E^Q[\bar{H}(z)] \xrightarrow{z \to +\infty} C_{BS}(K+c),
\]

so the existence and uniqueness of the constant $k$ satisfying $E^Q[\bar{H}(k)] = x$ follows. Now let us characterize $k$ more precisely. One can check the following equalities for $a, b \in \mathbb{R}^+$

\[
(x - a)^+ 1_{\{ x \geq b \}} = (x - b)^+ + (b - a) 1_{\{ x \geq b \}}, \quad \text{if} \quad a \leq b,
\]

\[
(x - a)^+ 1_{\{ x \geq b \}} = (x - a)^+, \quad \text{if} \quad a > b,
\]

\[
(x - a)^+ 1_{\{ x < b \}} = (x - a)^+ - (x - b)^+ - (b - a) 1_{\{ x \geq b \}}, \quad \text{if} \quad a \leq b,
\]

\[
(x - a)^+ 1_{\{ x < b \}} = 0, \quad \text{if} \quad a > b.
\]
Application of them in (5.5) yields
\[ \tilde{H} = (S_T - K)^+ + (S_T - (K + c))^+ - (S_T - I(k))^+ - (I(k) - K)1_{\{S_T > I(k)\}} + c1_{\{S_T > J(k)\}}, \]
which allows to determine the price of \( \tilde{H}(k) \) in terms of the Black-Scholes call prices. Finally, \( k \) must solve the equation
\[ C_{BS}(K) + C_{BS}(K + c) - C_{BS}(I(k)) - (I(k) - K)Q(S_T > I(k)) + cQ(S_T > J(k)) = x \]
which leads directly to (5.3).

**Generalized quantile hedging problem**

**Proposition 5.2** Let \( C_{BS}(K + c) \leq x < C_{BS}(K) \). An optimal strategy for a call option \((S_T - K)^+\) with the shortfall constraint \( L = c \wedge (S_T - K)^+ \) in the Black-Scholes model with parameters satisfying \( 0 < \alpha < \sigma^2 \) is a replicating strategy for the payoff
\[ \tilde{H} = (S_T - K)^+1_{\{S_T \leq \hat{y}(k)\}} + (S_T - (K + c))^+1_{\{S_T > \hat{y}(k)\}}, \]
where \( \hat{y}(k) \) is defined in Proposition 5.1 and \( k \) is a solution of the equation
\[ C_{BS}(K) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - K)(1 - \Phi(e(k))) + C_{BS}(K + c) - \left[ C_{BS}(K + c) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - (K + c))(1 - \Phi(e(k))) \right]1_{\{K + c \leq \hat{y}(k)\}} = x \] (5.6)
where \( e(k) := \frac{\ln(\frac{\hat{y}(k)}{s_0}) + \frac{1}{2}\sigma^2T}{\alpha} \).

In particular, the shortfall of the optimal strategy equals \( \{S_T \wedge (K + c) - K\}1_{\{S_T > \hat{y}(k)\}} \).

**Proof:** Motivated by Theorem 5.2 we solve first the problem
\[ \begin{align*}
E[\varphi] & \rightarrow \text{max} \\
(i) & \quad \varphi \geq \frac{H - L}{H}, \\
(ii) & \quad E^Q[\varphi] \leq \frac{x}{E^Q[H]},
\end{align*} \] (5.7)
with the measure \( \hat{Q} \) given by \( \frac{d\hat{Q}}{dQ} = \frac{H}{E^Q[H]} \). In view of Lemma 4.1 we search for its solution in the form
\[ \hat{\varphi} = 1_{\{\frac{d\hat{Q}}{dQ} > a\}} + \frac{H - L}{H}1_{\{\frac{d\hat{Q}}{dQ} < a\}} = 1_{\{\frac{y}{x} > ZH\}} + \frac{H - L}{H}1_{\{\frac{y}{x} < ZH\}}, \]
where \( k \) above is such that the optimal payoff \( \tilde{H} = H\hat{\varphi} = H - L1_{\{\frac{y}{x} < ZH\}} \) satisfies
\[ E^Q[\tilde{H}] = x \] (5.8)
Following the arguments in the proof of Proposition 5.1, we obtain
\[
\left\{ \frac{1}{k} < ZH \right\} = \{ S_T > \hat{y}(k) \},
\]
which provides the representation of the optimal payoff
\[
\tilde{H} = \tilde{H}(k) = H \tilde{\varphi} = (S_T - K)^+ - ((S_T - K)^+ \land l)1_{\{S_T > \hat{y}(k)\}}
\]
\[
= (S_T - K)^+ 1_{\{S_T \leq \hat{y}(k)\}} + (S_T - (K + c))^+ 1_{\{S_T > \hat{y}(k)\}}.
\]
The function \( z \to E^Q[\tilde{H}(z)] \), \( z \geq 0 \) is continuous, monotone and satisfies
\[
\lim_{z \to 0} E^Q[\tilde{H}(z)] = C_{BS}(K), \quad \lim_{z \to +\infty} E^Q[\tilde{H}(z)] = C_{BS}(K + c),
\]
so (5.8) has a unique solution \( k \). Now let us characterize the constant \( k \). In view of the decompositions
\[
(x - a)^+ 1_{\{x \leq b\}} = (x - a)^+ - (x - b)^+ - (b - a) 1_{\{x > b\}}, \quad a \leq b,
\]
\[
(x - a)^+ 1_{\{x > b\}} = (x - a)^+ - (x - a)^+ 1_{\{x \leq b\}},
\]
\[
= (x - a)^+ - [(x - a)^+ - (x - b)^+ - (b - a) 1_{\{x > b\}}] 1_{\{a \leq b\}} \quad a, b \in \mathbb{R}, \quad (5.9)
\]
we obtain
\[
\tilde{H} = (S_T - K)^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - K) 1_{\{S_T > \hat{y}(k)\}} + (S_T - (K + c))^+
\]
\[
- [(S_T - (K + c))^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - (K + c)) 1_{\{S_T > \hat{y}(k)\}}] 1_{\{K + c \leq \hat{y}(k)\}}.
\]
The price of \( \tilde{H} \) is thus equal to
\[
E^Q[\tilde{H}] = C_{BS}(K) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - K)(1 - \Phi(c(k))) + C_{BS}(K + c)
\]
\[
- [C_{BS}(K + c) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - (K + c))(1 - \Phi(c(k)))] 1_{\{K + c \leq \hat{y}(k)\}},
\]
and in view of (5.8) we obtain (5.6). \( \square \)

**Remark 5.3** It follows from the proofs of Propositions 5.1 and 5.2 that optimal payoffs for (QH) and (GQH) are of the forms
\[
(S_T - K)^+ 1_{\{ \frac{1}{k_1} \geq ZL \}} + (S_T - (K + c))^+ 1_{\{ \frac{1}{k_1} < ZL \}}, \quad k_1 \geq 0,
\]
\[
(S_T - K)^+ 1_{\{ \frac{1}{k_2} \geq ZH \}} + (S_T - (K + c))^+ 1_{\{ \frac{1}{k_2} < ZH \}}, \quad k_2 \geq 0,
\]
respectively, even for a general form of the shortfall constraint \( L \). It follows that, in general, the solutions of (QH) and (GQH) differ except the case \( L = H \) which corresponds to the case with no shortfall constraint studied in [4].
Weighted expected shortfall problem

**Proposition 5.4** Let $C_{BS}(K + c) \leq x < C_{BS}(K)$ and $l(z) = z$. An optimal strategy for a call option $(S_T - K)^+$ with the shortfall constraint $L = c \wedge (S_T - K)^+$ in the Black-Scholes model with parameters satisfying $\alpha > 0$, $\sigma^2 > 0$ is a replicating strategy for the payoff

$$\tilde{H} = (S_T - K)^+1_{\{S_T > k\}} + (S_T - (K + c))1_{\{S_T \leq k\}}$$

with the constant $k \geq 0$ solving

$$C_{BS}(K) - [C_{BS}(K) - C_{BS}(k) - (k - K)(1 - \Phi(f(k)))]1_{\{k > K\}} + [C_{BS}(K + c) - C_{BS}(k) - (k - (K + c))(1 - \Phi(f(k)))]1_{\{c > K + l\}} = x,$$

where

$$f(k) := \frac{\ln(k/s_0) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}.$$

In particular, the shortfall of the optimal strategy equals $\{S_T \wedge (K + c) - S_T \wedge K\}1_{\{S_T \leq k\}}$.

**Proof:** The problem (3.4) amounts to

$$\begin{cases} 
\{ \varphi \geq \frac{H - L}{H}, \\
E[\varphi H] \to \max 
\} 
\end{cases}$$

and, in view of Lemma 4.1, can be explicitly solved by using the auxiliary measures

$$d\hat{P} := \frac{H}{E^Q[H]}, \quad d\hat{Q} := \frac{H}{E^Q[H]}.$$

We search for a solution in the form

$$\hat{\varphi} = 1_{\{\frac{d\hat{P}}{dQ} > a\}} + \frac{H - L}{H}1_{\{\frac{d\hat{P}}{dQ} \leq a\}} = 1_{\{S_T > k\}} + \frac{H - L}{H}1_{\{S_T \leq k\}}$$

where $a$ and $k$ are constants such that the corresponding optimal payoff

$$\tilde{H} = \tilde{H}(k) = H\hat{\varphi} = H - L1_{\{S_T \leq k\}} = (S_T - K)^+1_{\{S_T > k\}} + (S_T - (K + c))^+1_{\{S_T \leq k\}},$$

satisfies $E^Q[\tilde{H}] = x$. The existence and uniqueness of the constant $k$ can be argued as before. Its direct characterization we obtain by decomposing the optimal payoff $\tilde{H}$ to the combination of the call options with maturities $K, K + c$ and applying the Black-Scholes formula. This leads to (5.10). \hfill $\Box$

### 5.2 Exponential Poisson model

Let the asset price be given by

$$S_t = e^{N_t - \gamma t}, \quad t \in [0, T],$$

where $N_t$ is a Poisson process with intensity $\gamma$. The dynamics of the asset price are given by

$$dS_t = \gamma S_t dt + S_t dN_t,$$

where $dN_t$ is the infinitesimal increase in the number of events. The risk-neutral measure $Q$ is defined by

$$d\tilde{P} := \frac{S_t}{E^Q[S_t]} dP = e^{-\gamma t} dP.$$
where $N$ is a Poisson process with intensity $\lambda > 0$ under the measure $P$ and $\gamma > 0$. The paths of $S$ are neither increasing nor decreasing almost surely, so the model is arbitrage-free, see Theorem 3.2 in [9] or Proposition 9.9 in [1]. It is known that any equivalent measure $Q$ is characterized by a new intensity $\lambda_Q(t) = \lambda_Q(\omega, t) \geq 0$ of $N$ which means that the process

$$N_t^Q := N_t - \int_0^t \lambda_Q(s) \mathrm{d}s, \quad t \in [0, T]$$

is a $Q$-martingale. Using the jump measure language it means that the jump measure of $N$ defined by

$$\pi(t, A) := \mathbb{P}\{s \in [0, t] : \Delta N_s \in A\}, \quad A \subseteq \mathbb{R}, \quad 0 \notin A,$$

has a compensating measure under $Q$ of the form

$$\nu_Q(dt, dy) := \lambda_Q(t)1_{\{y = 1\}}(y)dt \delta_{y},$$

that is $\tilde{\pi}_Q(dt, dy) := \pi(dt, dy) - \nu_Q(dt, dy)$ is a compensated measure under $Q$. The corresponding density of $Q$ with respect to $P$, given by the Girsanov theorem, equals

$$\frac{dQ}{dP} = e^{\int_0^T \ln\left(\lambda_{Q}(s)\right) \mathrm{d}N_s - \int_0^T (\lambda_{Q}(s)-\lambda) \mathrm{d}s}. \quad (5.11)$$

Let us determine the process $\lambda_Q$ so that $Q$ is a martingale measure. The Itô formula provides

$$S_t = 1 + \int_0^t S_{s-} \mathrm{d}N_s - \gamma \int_0^t S_{s-} \mathrm{d}s + \sum_{s \in [0, t]} S_{s-} (e^{\Delta N_s} - 1 - \Delta N_s)$$

$$= 1 - \gamma \int_0^t S_{s-} \mathrm{d}s + \sum_{s \in [0, t]} S_{s-} (e^{\Delta N_s} - 1)$$

$$= 1 - \gamma \int_0^t S_{s-} \mathrm{d}s + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \tilde{\pi}_Q(ds, dy) + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \nu_Q(ds, dy)$$

$$= 1 + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \tilde{\pi}_Q(ds, dy) + \int_0^t \int_{\mathbb{R}} S_{s-} (e - 1) \lambda_Q(s) - \gamma) \mathrm{d}s,$$

and it follows that $S$ is a local martingale under $Q$ if and only if

$$\lambda_Q(t) \equiv \lambda_Q = \frac{\gamma}{e - 1}. \quad (5.12)$$

Hence from (5.11) and (5.12) it follows that the model admits only one martingale measure $Q$ with the density of the form

$$\frac{dQ}{dP} = Z = e^{\int_0^T \ln\left(\lambda_{Q}(s)\right) \mathrm{d}N_s - (\lambda_Q - \lambda)T} = C \cdot S_T^{\frac{\gamma}{e-1}},$$

where

$$C := e^{T \left(\lambda - \frac{\gamma}{e-1} - T \gamma \ln\left(\frac{\lambda (e-1)}{\gamma}\right)\right)}. \quad (5.13)$$

It follows that the price of a call option $(S_T - K)^+$, $K \geq 0$ is equal to

$$C_{EP}(K) := E^Q[(e^{N_T - \gamma T} - K)^+] = \sum_{n = \lfloor \ln K + \gamma T \rfloor}^{+\infty} (e^{-\gamma T} - K) \frac{\left(\sqrt{n} T\right)^n}{n!},$$

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where \([a] := \inf\{n \in \mathbb{N} : n \geq a\}\).

In this model the quantile hedging problem does not have a solution because the set \(\hat{A}\) described by Proposition 3.1 may not exist. The reason is that the density of the martingale measure is not absolutely continuous and consequently the candidate for \(\hat{A} = \{\frac{1}{e} \geq kL\}\) given by the classical Neyman-Pearson lemma does not satisfy, in general, the condition

\[
E^Q[L1_{\hat{A}}] = x - E^Q[H - L],
\]

for any value of \(k\). Below we solve the generalized quantile hedging problem for a call option when the coefficients satisfy \(1 < \frac{\lambda(e-1)}{c} < \gamma\). Another cases can be treated in a similar manner.

**Proposition 5.5** Let \(H = (S_T - K)^+\) and \(L = c \wedge (S_T - K)^+\) with \(c, K \geq 0\). If

\[
1 < \frac{\lambda(e-1)}{c} < \gamma,
\]

then an optimal solution to the generalized quantile hedging problem with initial capital \(x\) satisfying

\[
C_{EP}(K + c) \leq x < C_{EP}(K)
\]

is a replicating strategy for the payoff

\[
\hat{H} = (S_T - K)^+ 1_{\{S_T < \hat{y}(k)\}} + (S_T - (K + c))^+ 1_{\{S_T \geq \hat{y}(k)\}} + \gamma \left( c \wedge (S_T - K)^+ \right) 1_{\{S_T = \hat{y}(k)\}}.
\]

Above \(\hat{y}(k)\) stands for the unique solution of the equation

\[
y - K = \frac{1}{CK} y^{\ln\left(\frac{\lambda(e-1)}{\gamma}\right)}; \quad y \geq 0,
\]

and the constants \(k, \gamma\) are determined as follows

\[
k := \inf \left\{ u \geq 0 : f(u) := C_{EP}(K) - C_{EP}(\hat{y}(u)) - (\hat{y}(u) - K) Q(S_T > \hat{y}(u)) + C_{EP}(K + c) - [C_{EP}(K + c) - C_{EP}(\hat{y}(u)) - (\hat{y}(u) - (K + c)) Q(S_T \geq \hat{y}(u))] 1_{\{K + c \leq \hat{y}(u)\}} \leq x \right\},
\]

and \(\gamma\) solves the equation

\[
f(k) + \gamma \left( c \wedge (\hat{y}(k) - K) \right) Q(S_T = \hat{y}(k)) = x.
\]

The corresponding shortfall equals

\[
\left( c \wedge (S_T - K)^+ \right) 1_{\{S_T > \hat{y}(k)\}} + (1 - \gamma) \left( c \wedge (\hat{y}(k) - K)^+ \right) 1_{\{S_T = \hat{y}(k)\}}.
\]

**Proof:** By Theorem 3.2 we know that an optimal solution is of the form \(\hat{H} = H \hat{\varphi}\), where \(\hat{\varphi}\) is characterized by 3.3. Application of Lemma 4.1 with \(\varphi^* = \frac{H - L}{H}\) yields an explicit form of the optimal payoff

\[
\hat{H} = H \hat{\varphi} = H 1_{\{\frac{dP}{d\varphi} > a\}} + (H - L + \gamma L) 1_{\{\frac{dP}{d\varphi} = a\}} + (H - L) 1_{\{\frac{dP}{d\varphi} < a\}}
\]

\[
= H - L 1_{\{\frac{dP}{d\varphi} \leq a\}} + \gamma L 1_{\{\frac{dP}{d\varphi} = a\}}
\]

\[
= (S_T - K)^+ - \left( c \wedge (S_T - K)^+ \right) 1_{\{\frac{1}{e} \leq Z(S_T - K)^+\}} + \gamma \left( (c \wedge (S_T - K)^+) 1_{\{\frac{1}{e} = Z(S_T - K)^+\}}
\]

\[\]
where \( \frac{dQ}{dP} = \frac{H}{E^Q[H]} \) and the constants \( a, k \geq 0, \gamma \in [0, 1] \) should be such that \( E^Q[\tilde{H}] = x \). Using (5.13) we obtain
\[
\left\{ \frac{1}{k} \leq Z(S_T - K)^+ \right\} = \{ S_T \geq \hat{y}(k) \}
\]
and further
\[
\tilde{H} = \tilde{H}(k, \gamma) = (S_T - K)^+ - \left( c \wedge (S_T - K)^+ \right) 1_{\{ S_T \geq \hat{y}(k) \}} + \gamma \left( c \wedge (S_T - K)^+ \right) 1_{\{ S_T = \hat{y}(k) \}}.
\]
To characterize \( k \) and \( \gamma \) let us notice that the function
\[
f(z) := E^Q \left[ (S_T - K)^+ - \left( c \wedge (S_T - K)^+ \right) 1_{\{ S_T \geq \hat{y}(z) \}} \right]; \quad z \geq 0,
\]
is decreasing, càdlàg and satisfies
\[
\lim_{z \to 0} f(z) = C_{EP}(K), \quad \lim_{z \to +\infty} f(z) = C_{EP}(K + c), \quad (5.17)
\]
\[
\triangle f(z) = -\left( c \wedge (\hat{y}(z) - (K + c))^+ \right) Q(S_T = \hat{y}(z)). \quad (5.18)
\]
Since \( C_{EP}(K) \leq x < C_{EP}(K + c) \) and (5.17) holds, the constant \( k \)
\[
k := \inf \{ z \geq 0 : f(z) \leq x \},
\]
is well defined. Moreover, there exists \( \gamma \in [0, 1] \) such that
\[
x - f(k) = \gamma (-\triangle f(k)).
\]
In view of (5.18) this yields
\[
x = f(k) + \gamma \left( c \wedge (\hat{y}(k) - (K + c))^+ \right) Q(S_T = \hat{y}(k)),
\]
which means that \( E^Q[\tilde{H}(k, \gamma)] = x \) as required. To obtain (5.15) and (5.16) one decomposes \( \tilde{H} \) into the form
\[
\tilde{H} = (S_T - K)^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - K) 1_{\{ S_T > \hat{y}(k) \}} + (S_T - (K + c))^+
\]
\[
- \left[ (S_T - (K + c))^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - (K + c)) 1_{\{ S_T \geq \hat{y}(k) \}} \right] 1_{\{ K + c \leq \hat{y}(k) \}}
\]
\[
+ \gamma \left( c \wedge (S_T - K)^+ \right) 1_{\{ S_T = \hat{y}(k) \}}.
\]
and notices that \( E^Q[\tilde{H}] = f(k) + \gamma \left( c \wedge (\hat{y}(k) - K) \right) Q(S_T = \hat{y}(k)) \).
\]

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