CIRCULAR SPECTRUM AND BOUNDED SOLUTIONS OF PERIODIC EVOLUTION EQUATIONS

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Abstract. We consider the existence and uniqueness of bounded solutions of periodic evolution equations of the form $u' = A(t)u + \epsilon H(t, u) + f(t)$, where $A(t)$ is, in general, an unbounded operator depending 1-periodically on $t$, $H$ is 1-periodic in $t$, $\epsilon$ is small, and $f$ is a bounded and continuous function that is not necessarily uniformly continuous. We propose a new approach to the spectral theory of functions via the concept of “circular spectrum” and then apply it to study the linear equations $u' = A(t)u + f(t)$ with general conditions on $f$. For small $\epsilon$ we show that the perturbed equation inherits some properties of the linear unperturbed one. The main results extend recent results in the direction, saying that if the unitary spectrum of the monodromy operator does not intersect the circular spectrum of $f$, then the evolution equation has a unique mild solution with its circular spectrum contained in the circular spectrum of $f$.

1. Introduction

The main concern of this paper is the existence and uniqueness of bounded solutions to periodic evolution equations of the form

$$\frac{du}{dt} = A(t)u + f(t), \quad t \in \mathbb{R},$$

and nonlinear perturbed equations of the form

$$\frac{du}{dt} = A(t)u + \epsilon H(t, u) + f(t), \quad t \in \mathbb{R},$$

where $A(t)$ is, in general, an unbounded linear operator on a Banach space $X$, depending periodically on $t$, $H$ is periodic in $t$ with the same period as $A$, $\epsilon$ is small, and $f$ is in $L^\infty(\mathbb{R}, X)$. This is a central problem of the theory and applications of differential equations. The reader is referred to [1, 2, 9, 11, 14] and their references for more information.

Eq. (1.2) may serve as models for the following equations

$$\ddot{x} + \omega(t)x + \epsilon h(t, x, \dot{x}) = f(t),$$

where $\omega(t)$ is an 1-periodic continuous real function, $h(t, x, \dot{x})$ is real continuous, 1-periodic in $t$ and uniformly Lipschitz in $(x, \dot{x})$ such that $h(t, 0, 0) = 0$, and the forcing term $f(t)$ is almost periodic, or bounded.

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It also serves as an abstract setting for the following partial differential equations (the reader is referred to [1, 10, 28] for more details):

\[
\begin{align*}
\frac{\partial w}{\partial t}(x, t) &= w_{xx}(x, t) + a(t)w(x, t) + eb(t)w(x, t)^2 + f(x, t), \\
w(0, t) &= w(\pi, t) = 0, \quad 0 \leq s \leq \pi, \quad t \geq 0,
\end{align*}
\]

where \(a(t), b(t), w(x, t), f(x, t)\) are scalar-valued functions, \(a(t), b(t)\) are 1-periodic and continuous in \(t\), and \(f(\cdot, t)\) as an element in \(L^2[0, 1]\) is almost automorphic. We define the space \(X := L^2[0, \pi]\) and \(A_T : D(A_T) \subset X \rightarrow X\) by the formula

\[
\begin{align*}
A_T y &= y'' , \\
D(A_T) &= \{ y \in X : y, y' \text{ are absolutely continuous, } y'' \in X, \\
&\quad y(0) = y(\pi) = 0 \}.
\end{align*}
\]

We define \(A(t) := A_T + a(t)\), where \(a(t)\) is the operator of multiplication \(a(t)v(\cdot)\) in \(X\), and \(H(t, v)(\cdot) := b(t)v^2(\cdot)\) for each \(v \in X\). The evolution equation we are concerned with in this case is the following

\[
\frac{du(t)}{dt} = A(t)u(t) + \epsilon H(t, u(t)) + f(t), \quad u(t) \in X.
\]

As is well known (28), the linear part associated with this evolution equation is well posed, that is, \(A(t)\) generates an 1-periodic evolutionary process that is strongly continuous. Therefore, one may include this equation into our abstract model (1.2).

In [32] a conjecture on the existence and uniqueness of an almost periodic mild solution to (1.1) is stated when the unitary spectrum of the Poincare map \(P\) associated with (1.1) does not intersect the set \(e^\text{isp}(f)\), where \(\text{sp}(f)\) denotes the spectrum of \(f\) (see the definition below), and the overline means the closure in the topology of the complex plane. The evolution semigroup method, proposed in [22] (see also [3, 20, 11]) gives rise to a positive answer to the conjecture. To use the semigroup theory machinery a crucial requirement on the strong continuity of the associated evolution semigroup is made. For instance, \(f\) is a bounded, uniformly continuous function with pre-compact range. In particular, if \(f\) is almost periodic, the condition is automatically satisfied. In our more general setting \(f\) is merely a bounded and continuous function, so the strong continuity of the associated evolution semigroup is actually not assumed. Consequently, the beautiful results of Semigroup Theory do not apply. This general setting of the problem seems to be natural when one considers \(f\) from some frequently met classes of functions such as almost automorphic functions (see [4, 25, 35]). On the other hand, the above-mentioned requirement on \(f\) appears to be technical, and is an obstacle for potential applications of the results to other areas such as Control Theory. For complete accounts of results concerned with periodic evolution equations we refer the reader to [3, 6, 11, 13, 19, 31]. The asymptotic behavior of evolution equations and applications can be found in the monographs [11, 6, 10, 14, 23, 29]. For more information on the spectral theory of functions and its applications the reader is referred to the monographs [1, 13, 14, 11, 29] as well as papers [12, 17, 18, 33] and their references.
In this paper we propose a new approach that is based on a concept of the so-called circular spectrum of a bounded function $g$ (denoted by $\sigma(g)$). In turn, the circular spectrum of a bounded function $g$ is defined through a new transform of $g$, namely, $R(\lambda,S)g$, where $S$ is the translation to the period of the coefficient operator $A(\cdot)$ (that is assumed to be 1). When the function $g$ is bounded and uniformly continuous, the Weak Spectral Mapping Theorem in Semigroup Theory yields that $\sigma(g) = e^{is\rho(g)}$, where $\rho(g)$ is the Carleman spectrum of $g$. This makes our results new extensions of the previous ones to the general setting. Moreover, a perturbation theory of the linear equations is given.

Before concluding this introduction section we give an outline of the paper. We briefly list main notations in Section 2. This section also contains the definitions as well as properties of almost periodic and almost automorphic functions. Section 3 deals with a similar problem for difference equations with continuous time. Section 4 contains the main result of the paper that deals with the existence and uniqueness of bounded mild solutions of periodic evolution equations.

2. Preliminaries

2.1. Basic Notations. In the paper $X$ denotes a complex Banach space. The space of all bounded linear operators in $X$ is denoted by $L(X)$ with usual operator norm. If $A$ is a linear operator (not necessarily bounded) acting on $X$, $\sigma(A)$ ($\rho(A)$, respectively) denotes its spectrum (resolvent set, respectively). The part of spectrum of an operator $B \in L(X)$ on the unit circle is denoted by $\sigma_f(B)$. For $\lambda \in \rho(A)$, $R(\lambda,A)$ denotes $(\lambda - A)^{-1}$. If a mapping $T$ from a Banach space $X$ to another Banach space $Y$ is Lipschitz continuous, then

$$Lip(T) := \inf \{ L : \|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in X\}.$$  

In this paper we also use the following notations:

i) If $z$ is a complex number, then $\Re z$, $\Im z$ denote its real part and imaginary part, respectively;

ii) $BC(\mathbb{R}, X)$ is the space of all $X$-valued bounded and continuous functions on $\mathbb{R}$; $\text{BUC}(\mathbb{R}, X)$ is the function space of all $X$-valued bounded and uniformly continuous functions on $\mathbb{R}$; $\text{KBU C}(\mathbb{R}, X)$ is the function space of all $X$-valued bounded and uniformly continuous functions on $\mathbb{R}$ with pre-compact range;

iii) $L^\infty(\mathbb{R}, X)$ denotes the space of all measurable functions on $\mathbb{R}$ that are essentially bounded with usual norm $\|g\| := \text{ess sup}_{t \in \mathbb{R}} \|g(t)\|$;

iv) $\Gamma$ denotes the unit circle in the complex plane $\mathbb{C}$;

v) If $B(t)$ is a bounded linear operator on $X$ for each $t \in \mathbb{R}$ that is strongly continuous in $t$, then the operator of multiplication by $B(t)$ on $BC(J, X)$, denoted by $B$, is defined by $Bg = Bg(\cdot)$ for each $g \in BC(J, X)$;

vi) The translation operator $S(\tau)$ is defined to be $S(\tau)g(t) = g(t + \tau)$ for all $t \in \mathbb{R}, g \in L^\infty(\mathbb{R}, X)$; In particular, $S(1) := S$.

2.2. Function Spaces. The biggest function space we consider in this paper is $L^\infty(\mathbb{R}, X)$ of all measurable functions that are essentially bounded on $\mathbb{R}$ with ess sup norm. We will identify each element in $BC(\mathbb{R}, X)$ with its equivalence class in $L^\infty(\mathbb{R}, X)$, so we may think of $BC(\mathbb{R}, X)$ as a closed subspace of $L^\infty(\mathbb{R}, X)$. 


Definition 2.1. A function \( f \in BC(\mathbb{R}, X) \) is said to be almost automorphic if for any sequence of real numbers \((s'_n)\), there exists a subsequence \((s_n)\) such that

\[
\lim_{m \to \infty} \lim_{n \to \infty} f(t + s_n - s_m) = f(t)
\]

for any \( t \in \mathbb{R}. \)

The limit in (2.1) means

\[
g(t) = \lim_{n \to \infty} f(t + s_n)
\]

is well-defined for each \( t \in \mathbb{R} \) and

\[
f(t) = \lim_{n \to \infty} g(t - s_n)
\]

for each \( t \in \mathbb{R}. \) The reader is referred to [25, 24, 26, 19, 4] and their references for more information on this concept and results.

Definition 2.2. A function \( f \in BC(\mathbb{R}, X) \) is said to be almost periodic if for any sequence of real numbers \((s'_n)\), there exists a subsequence \((s_n)\) and a function \( g \in BC(\mathbb{R}, X) \) such that

\[
\lim_{n \to \infty} f(t + s_n) = g(t)
\]

uniformly in \( t \in \mathbb{R}. \)

It follows immediately from the definition that every almost periodic function is uniformly continuous. The space of all almost periodic functions on \( \mathbb{R} \) taking values in \( X \) is denoted by \( AP(X) \), so \( AP(X) \subset BUC(\mathbb{R}, X) \). For more information on almost periodic functions the reader is referred to [8, 14].

Remark 2.3. Because of pointwise convergence the function \( g \) is measurable but not necessarily continuous. It is also clear from the definition above that constant functions and almost periodic functions are almost automorphic.

If the limits in (2.2) and (2.3) are uniform on any compact subset \( K \subset \mathbb{R} \), we say that \( f \) is compact almost automorphic.

Theorem 2.4. Assume that \( f, f_1, \) and \( f_2 \) are almost automorphic and \( \lambda \) is any scalar, then the following hold true.

i) \( \lambda f \) and \( f_1 + f_2 \) are almost automorphic,
ii) \( f_\tau(t) := f(t + \tau), t \in \mathbb{R} \) is almost automorphic,
iii) \( f(t) := f(-t), t \in \mathbb{R} \) is almost automorphic,
iv) The range \( R_f \) of \( f \) is precompact, so \( f \) is bounded.

Proof. See [25, Theorems 2.1.3 and 2.1.4], for proofs.

Theorem 2.5. If \( \{f_n\} \) is a sequence of almost automorphic \( X \)-valued functions such that \( f_n \to f \) uniformly on \( \mathbb{R} \), then \( f \) is almost automorphic.

Proof. see [25, Theorem 2.1.10], for proof.

Remark 2.6. If we equip \( AA(X) \), the space of almost automorphic functions with the sup norm

\[
\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f(t)\|
\]
then it turns out to be a Banach space. If we denote $KAA(X)$, the space of compact almost automorphic $X$-valued functions, then we have

$$\text{(2.5)} \quad AP(X) \subset KAA(X) \subset AA(X) \subset BC(\mathbb{R}, X) \subset L^\infty(\mathbb{R}, X).$$

**Theorem 2.7.** If $f \in AA(X)$ and its derivative $f'$ exists and is uniformly continuous on $\mathbb{R}$, then $f' \in AA(X)$.

*Proof.* See [25, Theorem 2.4.1] for a detailed proof. \qed

**Theorem 2.8.** Let us define $F : \mathbb{R} \to X$ by

$$F(t) = \int_0^t f(s)ds$$

where $f \in AA(X)$. Then $F \in AA(X)$ iff $RF = \{F(t) | t \in \mathbb{R}\}$ is precompact.

*Proof.* See [25, Theorem 2.4.4] for a detailed proof. \qed

As a big difference between almost periodic functions and almost automorphic functions we remark that an almost automorphic function is not necessarily uniformly continuous, as shown in the following example due to B. M. Levitan:

**Example 2.9.** The following function

$$f(t) := \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right)$$

is almost automorphic, but not uniformly continuous. Therefore, it is not almost periodic.

### 3. A Spectral Theory of Functions

Below we will introduce a transform of a function $g \in L^\infty(\mathbb{R}, X)$ on the real line that leads to a concept of spectrum of a function. This spectrum coincides with the set of $\exp(g)$ if in addition $g$ is uniformly continuous. Recall that $\Gamma$ denotes the unit circle in the complex plane.

Let $g \in L^\infty(\mathbb{R}, X)$. Consider the complex function $Sg(\lambda)$ in $\lambda \in \mathbb{C}\setminus\Gamma$ defined as

$$\text{(3.1)} \quad Sg(\lambda) := R(\lambda, S)g, \quad \lambda \in \mathbb{C}\setminus\Gamma.$$ 

Since $S$ is a translation, this transform is an analytic function in $\lambda \in \mathbb{C}\setminus\Gamma$.

**Definition 3.1.** The circular spectrum of $g \in L^\infty(\mathbb{R}, X)$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $Sg(\lambda)$ has no analytic extension into any neighborhood of $\xi_0$ in the complex plane. This spectrum of $g$ is denoted by $\sigma(g)$ and will be called for short the spectrum of $g$ if this does not cause any confusion. We will denote by $\rho(g)$ the set $\Gamma\setminus\sigma(g)$.

**Lemma 3.2.** If $|\lambda| \neq 1$, then

$$\text{(3.2)} \quad \|R(\lambda, S)\| \leq \frac{1}{|1 - |\lambda||},$$

and if $|\lambda| = 1$, then $\lambda \in \sigma(S)$. 
Proof. If $|\lambda| \neq 1$, since $\|S\| = 1$, $\lambda \in \rho(S)$, so we have

\begin{equation}
I = (\lambda - S)R(\lambda, S) = \lambda R(\lambda, S) - SR(\lambda, S).
\end{equation}

Therefore, since $\|S\| = 1$

\begin{align*}
1 &= \|\lambda R(\lambda, S) - SR(\lambda, S)\| \\
&\geq |\lambda| \cdot \|R(\lambda, S)\| - \|S\| \cdot \|R(\lambda, S)\| \\
&= |\lambda| \cdot \|R(\lambda, S)\| - \|R(\lambda, S)\| \\
&= \|\lambda - 1\| \cdot \|R(\lambda, S)\|.
\end{align*}

This proves (3.2).

If $|\lambda| = 1$, then there is a real $\xi$ such that $\lambda = e^{i\xi}$. It can be easily seen that the function $g_\xi(t) := e^{i\xi t} a$ for a fixed $0 \neq a \in \mathbb{X}$, $t \in \mathbb{R}$ is an eigenvector of $S$ in $L^\infty(\mathbb{R}, \mathbb{X})$, so $\lambda \in \sigma(S)$. \hfill \Box

Recall that if $A \in L(\mathbb{X})$, then $A$ denotes the operator of multiplication by $A$, given by $(Ag)(t) := Ag(t), \forall t \in \mathbb{R}$.

Proposition 3.3. Let \(\{g_n\}_{n=1}^\infty \subset L^\infty(\mathbb{R}, \mathbb{X})\) such that $g_n \to g \in L^\infty(\mathbb{R}, \mathbb{X})$, and let $\Lambda$ be a closed subset of the unit circle. Then the following assertions hold:

i) $\sigma(g)$ is closed.

ii) If $\sigma(g_n) \subset \Lambda$ for all $n \in \mathbb{N}$, then $\sigma(g) \subset \Lambda$.

iii) $\sigma(Ag) \subset \sigma(g)$ for all $A \in L(\mathbb{X})$.

iv) If $\sigma(g) = \emptyset$, then $g = 0$.

Proof. i) The first assertion follows immediately from the definition.

ii) The proof can be taken from that of [29, Theorem 0.8, pp.21-22]. In fact, assume that $\lambda_0 := e^{i\theta_0} \in \Gamma \setminus \Lambda$. Since $\Lambda$ is closed, we can choose $r > 0$ such that, if $|\xi - i\theta_0| < 4r$, then $Sg_n(e^{\xi}) = R(e^{\xi}, S)g_n$ is analytic for all $n$. Let us choose a positive $\delta$ such that $|x| < \delta$, where $x, y$ are reals, and $\lambda = e^{x+iy}$, then

\begin{equation}
\frac{1}{|1 - |\lambda||} \leq \frac{2}{|x|}.
\end{equation}

Notice that, if $0 < |\Re \xi| < \delta$, where $\delta$ is the number in the above,

\begin{equation}
\|R(e^{\xi}, S)g_n\| \leq \|R(e^{\xi}, S)\| \|g_n\| \leq \frac{1}{1 - |e^{\xi}|} \|g_n\| \leq \frac{2}{|\Re \xi|} \|g_n\|.
\end{equation}

We will show that, if $|\xi - i\theta_0| < r < \delta/4$,

\begin{equation}
\|Sg_n(e^{\xi})\| \leq \frac{16}{3r} \|g_n\|.
\end{equation}

Take the function $h(z) = (z - i\theta)(1 + (z - i\theta)^2/4r^2)$. By Cauchy’s theorem, we have that, if $|\xi - i\theta_0| < r$,

\begin{equation}
h(\xi)(Sg_n(e^{\xi})) = \frac{1}{2\pi i} \int_{|z - i\theta_0| = 2r} h(z)Sg_n(e^{\xi}) \frac{dz}{z - \xi},
\end{equation}

and that

\begin{equation}
\|h(\xi)(Sg_n(e^{\xi}))\| \leq \frac{1}{2\pi} \int_{|z - i\theta_0| = 2r} 2|\Re z| \|g_n\| \frac{|dz|}{|z - \xi|} \leq \frac{1}{2\pi} 4 \|g_n\| \frac{1}{r} 2\pi r = 8 \|g_n\|.
\end{equation}

\footnote{We thank the referee for his several detailed suggestions in the proof of part ii)}
This implies that

\[
\sup_{|\xi-i\theta_0| \leq r} \|Sg_n(e^\xi)\| = \sup_{|\xi-i\theta_0|=r} \|Sg_n(e^\xi)\| \leq 8\|g_n\| \sup_{|\xi-i\theta_0|=r} \left| \frac{1}{h(\xi)} \right| \\
\leq 8\|g_n\| \left( \frac{1}{(2r)^{3/4}} \right) \\
= \frac{16\|g_n\|}{3r}.
\]

Since \( g_n \to g \), we can take \( M \) such that \( \frac{16\|g_n\|}{3r} \leq M \) for \( n = 1, 2, \ldots \). Set \( U = \{ \lambda = e^\xi : |\xi-i\theta_0| < r \} \). Then \( U \) is a neighborhood of \( \lambda_0 = e^{i\theta_0} \) and for \( \lambda \in U \)

\[
(Sg_n(\lambda)) = \left( R(\lambda, S)g_n - g \right) \leq \frac{1}{|1-|\lambda||}\|g_n - g\|,
\]

so if \( \lambda \in U \setminus \Gamma \)

\[
\lim_{n \to \infty} Sg_n(\lambda) = Sg(\lambda).
\]

By Vitali’s Theorem [1], Theorem A.5, p. 458, the uniform boundedness (3.5) and (3.6) yield that \( Sg(\lambda) \) has an analytic extension to \( U \), that is, \( \lambda_0 \notin \sigma(g) \).

iii) The assertion is obvious.

iv) If \( \sigma(g) = \emptyset \), then \( Sg(\lambda) \) is analytic everywhere in \( \mathbb{C} \). Moreover, by Lemma 3.2 it should be bounded on \( \mathbb{C} \). This shows that \( Sg(\lambda) \) is a constant. Again using Lemma 3.2 we end up with this constant being zero. This yields that \( g = 0 \). \( \square \)

Below by \( F \) we denote one of the function spaces if not otherwise stated:

(3.7) \( AP(\mathbb{X}); \hfill KAA(\mathbb{X}); \hfill AA(\mathbb{X}); \hfill KBU C(\mathbb{R}, \mathbb{X}); \hfill BC(\mathbb{R}, \mathbb{X}); \hfill L^\infty(\mathbb{R}, \mathbb{X}) \).

**Lemma 3.4.** Let \( F \) be one of the function spaces \( AP(\mathbb{X}); \hfill KAA(\mathbb{X}); \hfill AA(\mathbb{X}); \hfill KBU C(\mathbb{R}, \mathbb{X}); \hfill BC(\mathbb{R}, \mathbb{X}); \hfill L^\infty(\mathbb{R}, \mathbb{X}) \). and let \( T \) be a bounded linear operator in \( BC(\mathbb{R}, \mathbb{X}) \) that commutes with all translations. Then, \( T \) leaves \( F \) invariant.

**Proof.** This lemma is obvious due to definitions of these function spaces \( F \). Therefore, the detailed proofs are omitted. \( \square \)

**Corollary 3.5.** Let \( \Lambda \) be a closed subset of the unit circle. Then, the set

(3.8) \( \Lambda_\mathcal{F}(\mathbb{X}) := \{ g \in \mathcal{F} | \sigma(g) \subset \Lambda \} \)

is a closed subspace of \( \mathcal{F} \).

**Proof.** It is easy to check that this is a linear subspace of \( \mathcal{F} \). Moreover, by (ii) of the above proposition, this space is closed. \( \square \)

**Lemma 3.6.** Let \( \Lambda \) be a closed subset of the unit circle. Then, the translation operator \( S \) leaves the space \( \Lambda_\mathcal{F}(\mathbb{X}) \) invariant. Moreover,

(3.9) \( \sigma(S|_{\Lambda_\mathcal{F}(\mathbb{X})}) = \Lambda \).
Proof. Since the function \( g_\mu(t) := \mu^t \) is an eigenvector of \( S|_{\Lambda}(X) \) for each \( \mu \in \Lambda \), it is clear that \( \sigma(S|_{\Lambda}(X)) \supset \Lambda \). Now we prove the converse. Let \( \mu \in \Gamma \) but \( \mu \not\in \Lambda \). We will show that \( \mu \in \rho(S|_{\Lambda}(X)) \). That is, for each \( g \in \Lambda(X) \) the equation
\[
\mu y - Sy = g,
\]
has a unique solution in \( \Lambda(X) \).

Obviously, \( R(\lambda, S)g \) has an analytic extension in a neighborhood of \( \mu \). Moreover, note that \( R(\lambda, S)g \in \Lambda(X) \) whenever \( g \in \Lambda(X) \). Therefore, (3.10) has a solution \( y_1 := \lim_{\lambda \to \mu} R(\lambda, S)g \in F \). This equation has a unique solution in \( F \). In fact, suppose the homogeneous equation
\[
\mu y - Sy = 0
\]
has a solution \( y_0 \in \Lambda(X) \). Then, since \( \mu y_0 = Sy_0 \),
\[
R(\lambda, S)y_0 = \mu^{-1} R(\lambda, S)Sy_0 \quad = \mu^{-1}(\lambda R(\lambda, S)y_0 - y_0),
\]
so
\[(1 - \mu^{-1} \lambda)R(\lambda, S)y_0 = -\mu^{-1}y_0.
\]
Therefore,
\[
R(\lambda, S)y_0 = \frac{-\mu^{-1}y_0}{1 - \mu^{-1} \lambda} = \frac{y_0}{\lambda - \mu}.
\]
This shows that \( \sigma(y_0) \subset \{\mu\} \). And hence, \( \sigma(y_0) \subset \{\mu\} \cap \Lambda = \emptyset \). By iv) of Proposition 3.3, \( y_0 = 0 \), that is the uniqueness of the solution of the homogeneous equation. This proves that \( \mu \in \rho(S|_{\Lambda}(X)) \), and so the lemma is proved. \( \square \)

Recall that for \( u \in L^\infty(\mathbb{R}, X) \), \( sp(u) \) stands for the Carleman spectrum, which consists of all \( \xi \in \mathbb{R} \) such that the Carleman transform of \( u \), defined by
\[
\hat{u}(\lambda) := \begin{cases} 
\int_{0}^{\infty} e^{-\lambda t} u(t) dt & (Re\lambda > 0) \\
-\int_{0}^{\infty} e^{\lambda t} u(-t) dt & (Re\lambda < 0), 
\end{cases}
\]
has no holomorphic extension to any neighborhoods of \( i\xi \). For each \( u \in BUC(\mathbb{R}, X) \) we denote \( \mathcal{M}_u := \text{span}\{S(\tau)u, \tau \in \mathbb{R}\} \subset BUC(\mathbb{R}, X) \). If \( u \in BUC(\mathbb{R}, X) \), the Carleman spectrum of \( u \) coincides with its Arveson spectrum, defined by (see [1 Lemma 4.6.8])
\[
i sp(u) = \sigma(D_u),
\]
where \( D_u \) is the infinitesimal generator of the restriction of the group of translations \( (S(t)|_{\mathcal{M}_u})_{t \in \mathbb{R}} \) to the closed subspace \( \mathcal{M}_u \).

The following lemma relates the spectrum \( \sigma(g) \) with Carleman spectrum of a uniformly continuous and bounded function.

**Lemma 3.7.** Let \( g \in BUC(\mathbb{R}, X) \). Then
\[
\sigma(g) = \sigma(S|_{\mathcal{M}_g}).
\]
Proof. The definition of $\sigma(g)$ yields that
\begin{equation}
\sigma(g) \subset \sigma(S|_{M_g}).
\end{equation}
Now we prove the inverse inclusion. Let $\lambda_0 \in \Gamma$ such that $\lambda_0 \notin \sigma(g)$. By Lemma 3.6 $\lambda_0 \in \rho(S|_{A_F(X)})$, where $A := \sigma(g)$ and $F := \text{BUC}(R, X)$. From the definition of $M_g$ we can show that $M_g \subset A_F(X)$, and that
\[ R(\lambda_0, S|_{A_F(X)})M_g \subset M_g. \]
And thus $\lambda_0$ is in $\rho(S|_{M_g})$. This proves the inverse inclusion of (3.13). The lemma is proved.

Corollary 3.8. Let $g \in \text{BUC}(R, X)$. Then
\begin{equation}
\sigma(g) = e^{\text{isp}(g)}.
\end{equation}
Proof. Since the translation group is bounded and strongly continuous in $\text{BUC}(R, X)$, by the Weak Spectral Mapping Theorem (see e.g. [7])
\[ \sigma(S|_{M_g}) = e^{\sigma(D|M_g)}. \]
Therefore, the corollary follows immediately from the above lemma.

Remark 3.9. In general, for $g \in L\infty(R, X)$ we do not know the relation between the circular spectrum $\sigma(g)$ and its Carleman spectrum $\text{sp}(g)$. We guess that $\sigma(g)$ may be larger than the set $e^{\text{isp}(g)}$.

4. Bounded Solutions of Difference Equations

In this section we consider the existence of solutions in $F$ as one of the function spaces listed in (3.7) to difference equations with continuous time of the form
\begin{equation}
u(t) = B(t)u(t - 1) + f(t),
\end{equation}
where $B(t)$ is a linear operator in a Banach space $X$ that is 1-periodic, strongly continuous in $t$, and $f$ is in $F$. We are interested in finding conditions for the existence and uniqueness of solutions in $F$ to (4.1).

Below we assume that $F$ is one of the function spaces listed in (3.7). Then, under the above assumption on $B(t)$ the operator of multiplication by $B(t)$ in $L\infty(R, X)$, denoted by $B$, leaves $F$ invariant. Therefore, it make sense to consider the restriction of $B$ to $F$, and to denote by $\sigma_F(B)$ and $\rho_F(B)$ the spectrum and resolvent set of this restriction, respectively. For simplicity we introduce e new notation:
\begin{equation}
\sigma_F(B) \cap \Gamma =: \sigma_{\Gamma,F}(B).
\end{equation}
When $F$ is $L\infty(R, X)$ we may use $\sigma_{\Gamma}(B)$ instead of $\sigma_{\Gamma,F}(B)$ if it does not cause any confusion.

Lemma 4.1. Let $f \in F$, and let $u \in F$ be a bounded solution of Eq. (4.1). Then, the following holds:
\begin{equation}
\sigma(u) \subset \sigma_{\Gamma,F}(B) \cup \sigma(f).
\end{equation}
Proof. We consider the restrictions of $S$ and $B$ to $\mathcal{F}$. First we prove the following identity for each $\lambda \notin \Gamma$

$$
(4.4) \quad \lambda R(\lambda, S)(-1) = R(\lambda, S) + S(-1).
$$

In fact, we have

$$
\begin{align*}
R(\lambda, S)(\lambda - S)S(-1) &= S(-1) \\
R(\lambda, S)(\lambda S(-1) - I) &= S(-1) \\
\lambda R(\lambda, S)S(-1) - R(\lambda, S) &= S(-1),
\end{align*}
$$

so

$$
\lambda R(\lambda, S)S(-1) = R(\lambda, S) + S(-1).
$$

Therefore, $(4.4)$ is valid.

Since $B(t)$ is 1-periodic in $t$, we have $R(\lambda, S)B = BR(\lambda, S)$. As $u$ is a bounded solution, by $(4.4)$ we have

$$
\begin{align*}
R(\lambda, S)u &= R(\lambda, S)BS(-1)u + R(\lambda, S)f \\
&= BR(\lambda, S)S(-1)u + R(\lambda, S)f \\
&= B(\lambda^{-1} R(\lambda, S) + \lambda^{-1} S(-1))u + R(\lambda, S)f,
\end{align*}
$$

so, for each $\lambda \notin \Gamma$,

$$
\lambda R(\lambda, S)u = B(R(\lambda, S) + S(-1))u + \lambda R(\lambda, S)f.
$$

Therefore, for each $\lambda \notin \Gamma$,

$$
\lambda R(\lambda, S)u - BR(\lambda, S)u = BS(-1)u + \lambda R(\lambda, S)f, \quad (\lambda - B)R(\lambda, S)u = BS(-1)u + \lambda R(\lambda, S)f.
$$

From this it follows that if $\lambda_0 \notin \sigma_{\Gamma,\mathcal{F}}(B)$ and $\lambda_0 \notin \sigma(f)$, then $R(\lambda, S)$ has an analytic extension around a neighborhood of $\lambda_0$. This shows $(4.3)$. $\square$

**Corollary 4.2.** Let $u$ and $f$ be in $\mathcal{F}$ that is one of the function spaces listed in $(3.7)$, and let $\sigma_{\Gamma,\mathcal{F}}(B) \cap \sigma(f) = \emptyset$. Then, Eq. $(4.1)$ has no more than one solution $u \in \Lambda_{\mathcal{F}}(X)$, where $\Lambda := \sigma(f)$.

Proof. It suffices to show that the homogeneous equation associated with $(4.1)$ has no more than one solution in $\Lambda_{\mathcal{F}}(X)$. Let $u$ be such a solution of the homogeneous equation. By the above lemma $\sigma(u) \subset \sigma_{\Gamma,\mathcal{F}}(B)$. Therefore, $\sigma(u) \subset \sigma_{\Gamma,\mathcal{F}}(B) \cap \sigma(f) = \emptyset$, so $u$ is the zero function. $\square$

**Lemma 4.3.** Let $Q(t)$ be 1-periodic strongly continuous in $t$, $u \in L^\infty(\mathbb{R}, X)$, and let $Q$ be the operator of multiplication by $Q(t)$. Then

$$
(4.5) \quad \sigma(Qg) \subset \sigma(g).
$$

Proof. Since $Q$ commutes with $R(\lambda, S)$, we have

$$
(4.6) \quad R(\lambda, S)Qg = QR(\lambda, S)g,
$$

so, $R(\lambda, S)Qg$ has an analytic extension to a neighborhood of $\lambda_0 \in \Gamma$ whenever so does $R(\lambda, S)g$. $\square$
Remark 4.4. By Lemma 3.8 yields in particular that if \( g \in \text{BUC}(\mathbb{R}, X) \), then
\[
\sigma_{sp}(Q_g) \subset \overline{\sigma_{sp}(g)}.
\]
This spectral estimate was given in [3].

Recall that \( \sigma_F(\mathcal{B}) \) denotes the spectrum of the restriction of \( \mathcal{B} \) to \( F \), where \( F \) is one of the function spaces listed in (3.7).

Lemma 4.5. Let \( F \) be one of the function spaces listed in (3.7), and let \( \Lambda \) be a closed subset of unit circle \( \Gamma \). Then, under the above assumption on \( B(t) \) the operator \( \mathcal{B} \) leaves \( \Lambda_F(X) \) invariant. Moreover,
\[
(4.7) \quad \sigma_B(\Lambda_F(X)) \subset \sigma_F(\mathcal{B}).
\]

Proof. For the first assertion we note that \( \mathcal{B} \) leaves \( F \) invariant and commutes with \( S \). Therefore, for \( |\lambda| \neq 1 \) and \( g \in \Lambda_F(X) \) we have
\[
R(\lambda, S)Bg = BR(\lambda, S)g,
\]
so, \( \sigma_B(g) \subset \sigma(g) \subset \Lambda \), that is, \( Bg \in \Lambda_F(X) \).

For the last assertion suppose that \( \lambda_0 \in \rho_F(\mathcal{B}) \). We will show that \( \lambda_0 \in \rho(\mathcal{B}|_{\Lambda_F(X)}) \). In fact, we have to show that for each \( g \in \Lambda_F(X) \) the equation
\[
(4.8) \quad \lambda_0 u - Bu = g
\]
has a unique solution \( u \) in \( \Lambda_F(X) \). First, since \( \lambda_0 \in \rho_F(\mathcal{B}) \), in \( F \) there exists a unique solution \( R(\lambda_0, \mathcal{B})g = u \) of (4.8). Therefore, if Eq. (4.8) has a solution in \( \Lambda_F(X) \), it cannot have more than one. Next, since \( \mathcal{B} \) commutes with \( S \), we can prove that \( R(\lambda_0, \mathcal{B}) \) commutes with \( R(\lambda, S) \) for all \( |\lambda| \neq 1 \). And hence, this yields that for all \( |\lambda| \neq 1 \),
\[
R(\lambda, S)u = R(\lambda, S)R(\lambda_0, \mathcal{B})g = R(\lambda_0, \mathcal{B})R(\lambda, S)g.
\]
This shows that whenever \( R(\lambda, S)g \) has an analytic extension into a neighborhood of a complex number \( \lambda_1 \in \Gamma \), so does \( R(\lambda, S)u \). This yields that \( \sigma(u) \subset \sigma(g) \), that is, \( u \in \Lambda_F(X) \). The lemma is proven.

Remark 4.6. As will be shown in the next section if \( B(t) \) is good enough, e.g. the monodromy operator of a periodic evolution equation, the spectrum and resolvent \( \sigma_F(\mathcal{B}) \) and \( \rho_F(\mathcal{B}) \) can be estimated independently of \( F \) (see Lemma 5.2 below).

Theorem 4.7. Let \( f \) be in \( F \), where \( F \) is any of the function spaces listed in (3.7). If
\[
(4.9) \quad \sigma_F(\mathcal{B}) \cap \sigma(f) = \emptyset,
\]
then, Eq. (4.9) has a unique solution \( u \) in \( F \) such that
\[
(4.10) \quad \sigma(u) \subset \sigma(f).
\]

Proof. Consider the equation
\[
(4.11) \quad u = BS(-1)u + f
\]
in \( \Lambda_F(X) \), where \( \Lambda := \sigma(f) \). This equation is equivalent to the following due to the commutativity of \( B|_{\Lambda_F(X)} \) and \( S|_{\Lambda_F(X)} \)
\[
(4.12) \quad (S|_{\Lambda_F(X)} - B|_{\Lambda_F(X)})u = Sf.
\]
Moreover, the commutativeness of $\mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})}$ and $S|_{\Lambda_{\mathcal{F}}(\mathcal{X})}$ yields (see [30])

$$\sigma(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})}) - \mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})} \subset \sigma(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})}) - \sigma(\mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})}).$$

By Lemmas 3.6 and 4.5

$$\sigma(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})}) = \Lambda := \sigma(f), \quad \sigma(\mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})}) \subset \sigma(\mathcal{F}).$$

Therefore, by the theorem’s assumption

$$0 \notin \sigma(f) - \sigma(\mathcal{F}) \supset \sigma(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})}) - \mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})}$$

This shows that $0 \in \rho(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})} - \mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})})$, that is, the operator $(S|_{\Lambda_{\mathcal{F}}(\mathcal{X})} - \mathcal{B}|_{\Lambda_{\mathcal{F}}(\mathcal{X})})$ is invertible. In particular, this yields (4.12), and thus (4.11) has a unique solution in $\Lambda_{\mathcal{F}}(\mathcal{X})$. This proves the theorem. \(\square\)

5. **Bounded mild solutions of periodic evolution equations**

As an application of the above result we consider the existence and uniqueness of different classes of bounded mild solutions of evolution equations of the form

$$\frac{du(t)}{dt} = A(t)u(t) + f(t), \quad t \in \mathbb{R},$$

where $u(t) \in \mathcal{X}$, $\mathcal{X}$ is a complex Banach space, $A(t)$ is a (unbounded) linear operator acting on $\mathcal{X}$ for every fixed $t \in \mathbb{R}$ such that $A(t) = A(t + 1)$ for all $t \in \mathbb{R}$, $f : \mathbb{R} \to \mathcal{X}$ is a bounded function. Under suitable conditions the homogeneous equation associated with Eq. (5.1) is well-posed (see e.g. [28]), i.e., one can associate with this equation an evolutionary process $(U(t,s))_{t \geq s}$ which satisfies, among other things, the conditions in the following definition.

**Definition 5.1.** A family of bounded linear operators $(U(t,s))_{t \geq s}$, $(t,s \in \mathbb{R})$ from a Banach space $\mathcal{X}$ to itself is called 1-periodic strongly continuous evolutionary process if the following conditions are satisfied:

i) $U(t,t) = I$ for all $t \in \mathbb{R}$,

ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,

iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathcal{X}$,

iv) $U(t + 1, s + 1) = U(t, s)$ for all $t \geq s$,

v) $\|U(t,s)\| \leq N e^{\omega(t-s)}$ for some positive $N, \omega$ independent of $t \geq s$.

Recall that for a given 1-periodic evolutionary process $(U(t,s))_{t \geq s}$ the following operator

$$P(t) := U(t, t - 1), t \in \mathbb{R}$$

is called monodromy operator (or sometime period map, Poincaré map). Thus we have a family of monodromy operators. We will denote $P := P(0)$. The nonzero eigenvalues of $P(t)$ are called characteristic multipliers. An important property of monodromy operators is stated in the following lemma whose proof can be found in [10] [11].

**Lemma 5.2.** Under the notation as above the following assertions hold:

i) $P(t + 1) = P(t)$ for all $t$; characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of $P(t)$ coincide with those of $P$,

ii) $\sigma(P(t)) \setminus \{0\} = \sigma(P) \setminus \{0\}$, i.e., it is independent of $t$,

iii) If $\lambda \in \rho(P)$, then the resolvent $R(\lambda, P(t))$ is strongly continuous,
iv) If $\mathcal{P}$ denotes the operator of multiplication by $P(t)$ in any one of the function spaces $\mathcal{F}$ listed in (3.7), then
\[(5.3) \quad \sigma_{\mathcal{F}}(\mathcal{P}) \setminus \{0\} \subset \sigma(P) \setminus \{0\}.
\]

Proof. For (i)-(iii) the proofs can be found in [10, 11]. For (iv), for a fixed function space $\mathcal{F}$, note that by (i)-(iii), if $\lambda_0 \in \rho(P)$, then the operator of multiplication by $R(\lambda_0, P(t))$ leaves $\mathcal{F}$ invariant, so $R(\lambda_0, P)$ can be determined by $R(\lambda_0, P(t))$. Therefore, (5.3) holds. □

We note that in the infinite dimensional case there does not always exist a Floquet representation of the monodromy operator $P$. And in general we do not know if by a "change of variables" Eq. (5.4) can be reduced to an autonomous equation. In the finite dimensional case, this can be done in the framework of the Floquet Theory. If the Poincare map $P$ is compact, a partial Floquet representation of $P$ may be used as in [10, 21]. When $f$ is almost periodic it was conjectured in [32] that the condition $\sigma_{\Gamma}(P) \cap \sigma(f)$ is a sufficient condition for the existence and uniqueness of an almost periodic mild solution $u$ to Eq. (5.1) such that $\sigma(u) \subset \sigma(f)$. The evolution semigroup method proposed in [22] shows to be working well to give a positive answer to the conjecture. For more information about this we refer the reader to [22, 3, 20, 11].

Recall that given a 1-periodic evolutionary process $(U(t, s))_{t \geq s}$, the following formal semigroup associated with it
\[(5.4) \quad (T^h u)(t) := U(t, t-h)u(t-h), \forall t \in \mathbb{R},
\]
where $u$ is an element of some function space, is called evolutionary semigroup associated with the process $(U(t, s))_{t \geq s}$. As is known, this evolution semigroup is strongly continuous at each almost periodic function, or more generally at each bounded and uniformly continuous function with pre-compact range. The strong continuity of the evolution semigroup is essential in the evolution semigroup method. However, it may not be strongly continuous at any bounded and continuous function. Since an almost automorphic function may not be uniformly continuous the extended conjecture of Vu in [32] with almost automorphic $f$ is still open.

Below we will give a positive answer to the extended conjecture of Vu with general $f \in BC(\mathbb{R}, X)$ by applying the results in the previous section.

Let $U(t, s)$ be a 1-periodic strongly continuous evolutionary process. We note that all results can be adjusted if the process is $\tau$-periodic with any positive $\tau$. For each fixed positive $h$ let us define an operator $G$ as follows
\[(5.5) \quad Gg(t) := \int_{t-h}^{t} U(t, \xi)g(\xi)d\xi, \quad g \in L^\infty(\mathbb{R}, X), t \in \mathbb{R}.
\]

Note that this operator $G$ is well defined because of the strong continuity of the process $(U(t, s))_{t \geq s}$. Moreover, $Gg \in BC(\mathbb{R}, X)$ for each $g \in L^\infty(\mathbb{R}, X)$.

**Lemma 5.3.** Let $G$ be defined as above. Then the following assertions hold:

i) If $\mathcal{F}$ is one of the function spaces [3.7], then $G$ leaves $\mathcal{F}$ invariant;

ii) For each $g \in L^\infty(\mathbb{R}, X)$,
\[(5.6) \quad \sigma(Gg) \subset \sigma(g).
\]
Proof. By the 1-periodicity of \((U(t, s))_{t, s \in \mathbb{R}}\), for all \(t \in \mathbb{R}\) and all \(f \in BC(\mathbb{R}, X)\) we have

\[
[SGg](t) = \int_{t-h}^{t+1} U(t + 1, \xi)g(\xi)d\xi = \int_{t-h}^{t} U(t + 1, \eta + 1)g(\eta + 1)d\eta = \int_{t-h}^{t} U(t, \eta)g(\eta + 1)d\eta = [GSg](t),
\]

so \(S\) commutes with \(G\). This yields in particular that if \(F\) is one of the function spaces \(AP(X), KAA(X), AA(X)\) the operator \(G\) leaves \(F\) invariant. When \(F\) is one of the function spaces \(KBU(\mathbb{R}, X)\) or \(BC(\mathbb{R}, X)\), the invariance under \(G\) can be easily checked. Therefore, the first assertion follows.

For the second assertion, the commutativeness of \(G\) and \(S\) yields

\[
R(\lambda, S)Gg = GR(\lambda, S)g,
\]

and \(R(\lambda, S)Gg\) has an analytic extension into a neighborhood of \(\lambda_0 \in \Gamma\) whenever so does \(R(\lambda, S)g\). Finally, this yields \((5.6)\).

Below we always assume that \(F\) is one of the function spaces in \((3.7)\). We consider the following semigroup \((T_f^h)_{h \geq 0}\) of affine operators in \(L^\infty(\mathbb{R}, X)\): for each \(h \geq 0\) and \(f, g \in L^\infty(\mathbb{R}, X)\),

\[
(T_f^h) (t) := U(t, t - h)g(t - h) + \int_{t-h}^{t} U(t, \xi)f(\xi)d\xi, \quad \text{for almost all } t \in \mathbb{R}.
\]

Let \(\Lambda\) be a closed subset of \(\Gamma\), and let \(f \in \Lambda_F(X)\). By Lemmas \((4.3)\) and \((5.3)\) \(T_f^h\) leaves \(\Lambda_F(X)\) invariant. Moreover, it forms a semigroup of operators in \(\Lambda_F(X)\). In fact, we will show that for any nonnegative \(h, k, T_f^{h+k} = T_f^h T_f^k\). To this end,

\[
(T_f^{h+k})(t) = U(t, t-h-k)g(t-h-k) + \int_{t-h-k}^{t} U(t, \xi)f(\xi)d\xi
= U(t, t-h) \left(U(t-h, t-h-k)g(t-h-k) + \int_{t-h-k}^{t-h} U(t, \xi)g(\xi)d\xi\right)
+ \int_{t-h}^{t} U(t, \xi)f(\xi)d\xi
= U(t, t-h)T_f^h g(t-h) + \int_{t-h}^{t} U(t, \xi)f(\xi)d\xi
= (T_f^h T_f^k)(t).
\]

Recall that for a given \(f \in BC(\mathbb{R}, X)\), a function \(u \in BC(\mathbb{R}, X)\) is a mild solution of a well-posed Eq. \((5.1)\) if

\[
u(t) = U(t, s)u(s) + \int_{s}^{t} U(t, \xi)f(\xi)d\xi, \quad \text{for all } t \geq s.
\]
Lemma 5.4. Let \( f \in BC(\mathbb{R}, X) \). Then, \( u \in BC(\mathbb{R}, X) \) is a mild solution of Eq. (5.1) if and only if it is a common fixed point for all operators of the semigroup \((T^h_f)_{h \geq 0}\) in \( BC(\mathbb{R}, X) \).

Proof. This lemma is obvious. \( \square \)

We are now ready to prove the main result of the paper:

Theorem 5.5. Let the homogeneous equation associated with Eq. (5.1) generate a 1-periodic evolutionary process with monodromy operator \( P \), and let \( f \in \mathcal{F} \), where \( \mathcal{F} \) is one of the function spaces \( AP(X), KAA(X), AA(X), KBUC(\mathbb{R}, X), BC(\mathbb{R}, X) \). Then, Eq. (5.1) has a unique mild solution \( u \in \mathcal{F} \) such that

\[
\sigma(u) \subset \sigma(f),
\]

provided that

\[
\sigma(P) \cap \sigma(f) = \emptyset.
\]

Proof. By the above lemma, it suffices to show that the semigroup \((T^h_f)_{h \geq 0}\) has a unique common fixed point in \( \Lambda_{\mathcal{F}(X)} \), where \( \Lambda = \sigma(f) \). By Theorem 4.7, the operator \( T^1_f \) has a unique fixed point in \( \Lambda_{\mathcal{F}(X)} \). We are going to show that it should be common for all operators in the semigroup. In fact, let \( u \) be the unique fixed point for \( T^1_f \). Then, for each \( h \geq 0 \),

\[
T^1_f T^h_f u = T^h_f T^1_f u = T^h_f u,
\]

so, \( T^h_f u \) is another fixed point of \( T^1_f \) such that \( \sigma(T^h_f u) \subset \sigma(f) \). By the uniqueness of the fixed point of \( T^1_f \) this yields that \( T^h_f u = u \). And hence, \( u \) is a common fixed point (in \( \Lambda_{\mathcal{F}(X)} \)) for the whole semigroup. Therefore, \( u \) is a mild solution of Eq. (5.1) such that \( \sigma(u) \subset \sigma(f) \). The uniqueness follows from the uniqueness of the fixed point of \( T^1_f \). \( \square \)

Consider autonomous equations of the form

\[
\frac{du(t)}{dt} = Au(t) + f(t), \quad t \in \mathbb{R},
\]

where \( A \) generates a \( C_0 \)-semigroup \((T(t))_{t \geq 0}\), \( f \) is an \( X \)-valued bounded and continuous function in \( \mathcal{F} \) that is defined in Theorem 5.5. The case when \( f \) is uniformly continuous is well studied in [29, 32, 33, 11]. Under the assumption, the Poincare operator \( P \) is nothing but \( T(1) \). Note that in the autonomous case the operator of multiplication by a linear bounded operator \( B \) in \( BUC(\mathbb{R}, X) \) leaves this space invariant, so in this case in addition to the function spaces listed in (3.7) we can add \( BUC(\mathbb{R}, X) \). Therefore, the following corollary is valid.

Corollary 5.6. Let \( f \) be in \( \mathcal{F} \) that is any function space in (3.7) or \( BUC(\mathbb{R}, X) \). Then, Eq. (5.11) has a unique mild solution \( u \in \mathcal{F} \) such that \( \sigma(u) \subset \sigma(f) \) provided that

\[
\sigma_T(T(1)) \cap \sigma(f) = \emptyset.
\]
Let us consider the perturbed equation (1.2). We will fix a closed subset $\Lambda \subset \Gamma$ and a function space $\mathcal{F}$ being one of the function spaces
\begin{equation}
AP(\mathbb{X}), KAA(\mathbb{X}), AA(\mathbb{X}), KBU C(\mathbb{R}, \mathbb{X}), BC(\mathbb{R}, \mathbb{X}).
\end{equation}
We assume that
\begin{enumerate}
\item[(H1)] $H(t, 0) = 0$, and $H(t, x)$ is 1-periodic;
\item[(H2)] There exists an increasing function $l: \mathbb{R}^+ \to \mathbb{R}^+$ such that for each positive $r$ and for all $x, y \in \{\xi \in \mathbb{X} : \|\xi\| \leq r\}$ and $t \in \mathbb{R}$, the following holds
\begin{equation}
\|H(t, x) - H(t, y)\| \leq l(r)\|x - y\|;
\end{equation}
\item[(H3)] The Nemytsky operator $\mathcal{H}$ acting in $\mathcal{F}$ induced by $H$, that is, $\mathcal{H}g : t \mapsto H(t, g(t))$ leaves $\Lambda_{\mathcal{F}}(\mathbb{X})$ invariant.
\end{enumerate}

Before we proceed we recall an operator associated with the linear equation (1.1). The operator $L$ associated with (1.1) is defined on $BC(\mathbb{R}, \mathbb{X})$ with domain consisting of all $u \in BC(\mathbb{R}, \mathbb{X})$ such that there exists such a function $f \in BC(\mathbb{R}, \mathbb{X})$ for which
\begin{equation}
(5.8)
Lu := f.
\end{equation}
As is well known (see e.g. [22, 11, 15]), $L$ is a closed, single-valued operator acting on $BC(\mathbb{R}, \mathbb{X})$.

\begin{lemma}
Let $\mathcal{F}$ be one of the function spaces in (5.13), and let $\Lambda$ be a closed subset of the unit circle. Then the restriction of the operator $L$ to $\Lambda_{\mathcal{F}}(\mathbb{X})$ (denoted by $L_{\mathcal{F}, \Lambda}$) is closed.
\end{lemma}

\begin{proof}
Let $u_n \in D(L) \to u \in \Lambda_{\mathcal{F}}(\mathbb{X})$ such that $Lu_n = f_n \to f \in \Lambda_{\mathcal{F}}(\mathbb{X})$. By definition, for each $n \in \mathbb{N}$,
\begin{equation}
(5.15)
u_n(t) = U(t, s)u_n(s) + \int_s^t U(t, \xi)f_n(\xi)d\xi, \text{ for all } t \geq s.
\end{equation}
For every fixed $(t, s)$ let $n$ tend to infinity, so we have
\begin{equation}
(5.16)
u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi)d\xi, \text{ for all } t \geq s.
\end{equation}
Since $u \in \Lambda_{\mathcal{F}}(\mathbb{X})$ this shows that $u$ is in the domain of the restriction of $L$ to $\Lambda_{\mathcal{F}}(\mathbb{X})$. Therefore, the restriction of $L$ to $\Lambda_{\mathcal{F}}(\mathbb{X})$ is closed. \hfill \square
\end{proof}

In the sequel we will need the Inverse Function Theorem for Lipschitz continuous mappings, that is the following lemma that can be found as a slight modification of a well known result in [16, 27].

\begin{lemma}
Let $T$ be a bounded operator from a Banach space $\mathbb{X}$ onto another Banach space $\mathbb{Y}$ such that $T^{-1}$ exists as a bounded operator, and let $\varphi : \mathbb{X} \to \mathbb{Y}$ is a Lipschitz continuous operator with
\begin{equation}
\text{Lip}(\varphi) < \|T^{-1}\|^{-1}.
\end{equation}
Then, $(T + \varphi)$ is invertible with a Lipschitz continuous inverse, and
\begin{equation}
\text{Lip}((T + \varphi)^{-1}) \leq \frac{1}{\|T^{-1}\|^{-1} - \text{Lip}(\varphi)}.
\end{equation}
\end{lemma}

Below we assume that $\mathcal{F}$ is one of the function spaces in (5.13).
Theorem 5.9. Let the homogeneous equation associated with Eq. (5.1) generate a 1-periodic evolutionary process with monodromy operator \( P \). \( \Lambda \) be a closed subset of \( \Gamma \) such that

\[
\sigma(P) \cap \Lambda = \emptyset,
\]

\( \mathcal{F} \) be a any fixed space from (5.13), and let \( f \in \Lambda_\mathcal{F}(\mathbb{X}) \). Assume further that \( H \) in (1.2) satisfies all conditions (H1), (H2), (H3). Then, there exists a positive constant \( \epsilon_0 \) such that if \( \epsilon < \epsilon_0 \), the perturbed equation (1.2) has a bounded mild solution in \( \Lambda_\mathcal{F}(\mathbb{X}) \) that is locally unique.

Proof. We will use Lemma 5.8. Let

\[
M := \sup_{t \in \mathbb{R}} \|f(t)\|.
\]

As shown in Lemma 5.7 and in Theorem 5.5, the restriction of \( L \) to \( \Lambda_\mathcal{F}(\mathbb{X}) \) is closed and invertible. Therefore, if we equip \( \mathbb{X}_1 := D(L_{\mathcal{F},\Lambda}) \) with its graph norm, then \( L_{\mathcal{F},\Lambda}^{-1}: \mathbb{X}_2 := \Lambda_\mathcal{F}(\mathbb{X}) \to \mathbb{X}_1 := D(L_{\mathcal{F},\Lambda}) \) is bounded. Let us denote

\[
\rho := \|L_{\mathcal{F},\Lambda}^{-1}\|.
\]

We define the cut-off mapping

\[
\mathcal{H}_M(\phi) = \begin{cases} 
\mathcal{H}(\phi), & \forall \phi \text{ with } \|\phi\| \leq 2\rho M \\
\mathcal{H}_{\frac{2\rho M}{\|\phi\|}}(\phi), & \forall \phi \text{ with } \|\phi\| > 2\rho M.
\end{cases}
\]

As is shown in [34, Proposition 3.10, p.95], in \( B(2\rho M) := \{\phi \in \mathbb{X}_1 : \|\phi\| \leq 2\rho M\} \), \( \mathcal{H}_M(\cdot) \) is globally Lipschitz continuous (in the new graph norm) with

\[
\text{Lip}(\mathcal{H}_M) \leq 2\text{Lip}(\mathcal{H}|_{B(2\rho M)}),
\]

where \( \text{Lip}(R) \) denotes the Lipschitz coefficient of an operator \( R : \mathbb{X}_1 \to \mathbb{X}_2 \), so \( \mathcal{H}_M \) satisfies

\[
\text{Lip}(\mathcal{H}_M) \leq 2\text{l}(2\rho M).
\]

Since

\[
\lim_{\epsilon \downarrow 0} \frac{\rho}{1 - \epsilon \rho} = \rho > 0
\]

we can choose \( \epsilon_1 \) so that

\[
\frac{\rho}{1 - \epsilon \rho 2\text{l}(2\rho M)} < 2\rho
\]

for all \( \epsilon < \epsilon_1 \). By Lemma 5.8 if we choose \( \epsilon_2 \) such that

\[
\epsilon_2 = \frac{\rho}{2\text{l}(2\rho M)}
\]

then, since \( \text{Lip}(\epsilon \mathcal{H}_M) < \rho = \|L_{\mathcal{F},\Lambda}^{-1}\| \), the operator \( L_{\mathcal{F},\Lambda} + \epsilon \mathcal{H}_M \) is invertible for all \( \epsilon < \epsilon_2 \). Finally, if we choose \( \epsilon_0 = \min(\epsilon_1, \epsilon_2) \), then \( (L_{\mathcal{F},\Lambda} + \epsilon \mathcal{H}_M)^{-1} \) exists and (5.18) holds. Note that \( \mathcal{H}_M(0) = 0 \). Therefore, if we let \( T := L_{\mathcal{F},\Lambda}, \varphi = \epsilon \mathcal{H}_M \) for
\[ \varepsilon < \varepsilon_0, \text{ then, by the above corollary} \]
\[ \|(L_{\mathcal{F},\Lambda} - \epsilon \mathcal{H}_M)^{-1} f\| = \|(T + \varphi)^{-1} f\| \]
\[ = \|(T + \varphi)^{-1} f - (T + \varphi)^{-1}(0)\| \]
\[ \leq \frac{M}{\|T^{-1}\|^{-1} - \text{Lip}(\varphi)} \]
\[ = \frac{M}{\rho - \epsilon_0 2l(2\rho M)} \]
\[ = \frac{M\rho}{1 - \rho \epsilon_0 2l(2\rho M)} \]
\[ \leq 2\rho M. \]

This shows that if \( w := (L_{\mathcal{F},\Lambda} - \epsilon \mathcal{H}_M)^{-1} f \), then \((L_{\mathcal{F},\Lambda} - \epsilon \mathcal{H}_M)w = f\), and \( \|w\| \leq 2\rho M \). By the definition of \( \mathcal{H}_M \), if \( \|w\| \leq 2\rho M \), then \( \mathcal{H}_M w = \mathcal{H} w \). Finally, this yields that

\[ (L_{\mathcal{F},\Lambda} - \epsilon \mathcal{H})w = f, \]

that is, \( w \) is a mild solution of \((1.2)\). The theorem is proved. \( \square \)

6. Concluding remarks

The choice of 1-periodicity for the evolution equations in this paper does not restrict the generality of the obtained results. However, when dealing with \( \tau \)-periodic evolution equations the concept of circular spectrum should be adjusted. Instead of using the transform \( R(\lambda, S) \) we use \( R(\lambda, S(\tau)) \). If we denote this spectrum by \( \sigma^\tau(g) \), then the relation between the Carleman spectrum and this circular spectrum can be established in the following for \( g \in \text{BUC}(\mathbb{R}, X) \) via the Weak Spectral Mapping Theorem

\[ \sigma^\tau(g) = e^{i\tau \text{sp}(g)}. \]

We may extend a little the statements of the results by refining the classes of functions for \( \mathcal{F} \) to be taken as in \[3, 11, 20, 22, 33\].

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