Generalized second law in modified theory of gravity

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Abstract
In the context of modified theory of gravity ($f(R)$ gravity) we try to study the conditions needed for validity of the generalized second law.

1 Introduction
Recently, motivated by astrophysical data which indicate that the expansion of the universe is accelerating [1], the modified theory of gravity (or $f(R)$ gravity) which can explain the present acceleration without introducing dark energy, has received intense attention [2].

Determining thermodynamic parameters of an (accelerated) expanding universe and verification of the first and the second law for different cosmological horizons [3]; investigating the relation between dynamics and thermodynamics of the universe [4]; studying the conditions required for validity of the generalized second law (GSL) [5], and so on, have also been the subjects of many researches in recent years.

In the modified theory of gravity, instead of Friedmann equations we must utilize modified Friedmann equations which may include the powers of Ricci scalar as well as its time derivatives. Besides, the relation of the entropy to the area of the horizon is also different with Einstein theory of gravity. So it is of interest to see how, in the framework of $f(R)$ gravity, thermodynamic properties of the universe may be modified.

In this paper we try to find necessary conditions for validity of GSL in the framework of $f(R)$ gravity in Friedmann Robertson Walker (FRW) universe. The future event horizon is taken as the horizon of the universe and the temperature is supposed to be proportional to the Gibbons-Hawking temperature [6]. The range of the proportionality constant will be determined through some examples. Using the relation between the entropy assigned to the horizon and its area derived from the Noether charge method,

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we study the behavior of horizon entropy with respect to the comoving time. We assume that the horizon is in thermal equilibrium with its environment which is filled with perfect fluids. Then from modified Friedmann equations and the first law of thermodynamics, the time derivative of the fluid entropy and subsequently the time derivative of the total entropy is determined. At the end we elucidate our results via two examples.

We use the units $\hbar = c = G = k_B = 1$.

2 Thermodynamics and GSL in FRW universe in modified gravity

The action of modified theory of gravity with the inclusion of matter is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) + S_m,$$

(1)

where $S_m$ is the matter action, $R$ is the Ricci scalar curvature and $f(R)$ is an arbitrary real function. Variation of the action with respect to the metric gives

$$R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f' = 8\pi T^m_{\mu\nu},$$

(2)

where the prime denotes the derivative with respect to $R$, and $T^m_{\mu\nu}$ is the energy-momentum tensor of the matter fields. For spatially flat FRW metric with scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

(3)

Eq. (2) yields

$$8\pi \rho = \frac{f(R)}{2} - 3(\dot{H} + H^2 - H \frac{d}{dt}) f'(R),$$

$$8\pi P = -\frac{f(R)}{2} + (\dot{H} + 3H^2 - \frac{d^2}{dt^2} - 2H \frac{d}{dt}) f'(R).$$

(4)

The Hubble parameter is given by $H = \dot{a}/a$, and the Ricci scalar is obtained as $R = 6\dot{H} + 12H^2$. The over dot indicates the derivative with respect to the comoving time $t$. $\rho$ and $P$ are the density and the pressure of the matter which behaves as a perfect fluid at large scale:

$$T^m_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu},$$

(5)

where $U_\mu$ is the four velocity of the fluid.

The radius of the future event horizon, $R_h$, is given by

$$R_h(t) = a(t) \int_t^\infty \frac{dt'}{a(t')}.$$

(6)
Note that the event horizon exists, \( R_h(t) \in \mathbb{R} \), when the above integral converges, i.e., \( \int_{t}^{\infty} \frac{dt'}{a(t')} \in \mathbb{R}^+ \). If at a time denoted by \( t_s \), the Big Rip singularity occurs, we must replace \( \infty \) by \( t_s \) in the integration. Using Eq. (6), we can verify that \( R_h \) satisfies the following equation
\[
\dot{R}_h = H R_h - 1.
\]

In the context of Einstein theory of gravity, the entropy of a black hole is given by Bekenstein-Hawking relation [7]
\[
S_h = \frac{A}{4},
\]
where \( A \) is the area of the event horizon. In the same way one can assign an entropy to the cosmological future event horizon whose area is \( A_h = 4\pi R_h^2 \). This entropy which is given by \( S_h = \frac{A_h}{4} \), may be regarded as a measure of information hidden behind the horizon. In de Sitter space-time \( R_H = \frac{1}{H} \) and the future event horizon becomes the same as the de Sitter (Hubble) horizon. In this space time the temperature, which is dubbed as Gibbon Hawking temperature [6], can be determined in terms of horizon radius as \( T = \frac{H}{2\pi} \).

In \( f(R) \) gravity, Noether charge method can be used to obtain the horizon entropy [8]
\[
S_h = \frac{1}{4} \int_A f'(R) dA,
\]
where the integration is taken over the surface of the horizon, \( A \). So in FRW universe, where the scalar curvature is spatially constant, the entropy is obtained as
\[
S_h = \frac{A f'(R)}{4}.
\]

In the following we choose the future event horizon as the horizon of the universe. Differentiating Eq. (10) with respect to the comoving time gives
\[
\dot{S}_h = 2\pi \dot{R}_h R_h F + \pi R_h^2 \dot{F}.
\]

We have defined \( F = f'(R) \). In Einstein theory of gravity the above equation reduces to \( \dot{S}_h = 2\pi R_h \dot{R}_h \). Note that in a super-accelerated universe defined by \( \dot{H} > 0 \), \( R_h \) is decreasing; \( \dot{R}_h < 0 \) [9], therefore \( \dot{S}_h < 0 \). But in modified theory this is not the case and, depending on the function \( f(R) \), one may have \( \dot{S}_h > 0 \). Consider the model \( f(R) = \alpha R^m \), \( \alpha, m \in \mathbb{R} \). For \( \dot{H} > 0 \) we have \( R > 0 \), hence \( \dot{S}_h \geq 0 \) leads to
\[
\alpha \left( \frac{\dot{R}_h}{R_h} + \frac{(m-1)\dot{R}}{2R} \right) \geq 0.
\]

It is clear that for \( \alpha = m = 1 \) the above inequality cannot be satisfied in a super-accelerated universe.
If $S_{\text{in}}$ is the entropy of the matter inside the horizon, then the first law of thermodynamics states

$$TdS_{\text{in}} = dE + PdV = (P + \rho)dV + Vd\rho.$$  \hspace{1cm} (13)

By taking $V = \frac{4}{3}\pi R_h^3$, we arrive at

$$TS_{\text{in}}' = 4\pi(P + \rho)R_h^2\dot{R}_h + \frac{4}{3}\pi R_h^3 \dot{\rho}.$$  \hspace{1cm} (14)

From Eq.\,(7), and energy conservation relation

$$\dot{\rho} + 3H(P + \rho) = 0,$$  \hspace{1cm} (15)

we can write Eq.\,(14) in the form

$$TS_{\text{in}}' = -4\pi(P + \rho)R_h^2.$$  \hspace{1cm} (16)

$S_{\text{in}}$ is a decreasing (increasing) function of time when $w > (\leq) -1$, where $w = \frac{P}{\rho}$ is the effective equation of state (EOS) parameter of the perfect fluid filling the universe. In the model $f(R) = R$, Eq. \,(10) becomes $TS_{\text{in}}' = \dot{H}R_h^2$, hence $\dot{S}_{\text{in}} > 0$ is satisfied when $\dot{H} > 0$. Note that in $f(R)$ theory of gravity we have

$$w = -1 + \frac{-4\dot{H}F + 2\dot{H}\ddot{F} - 2\dddot{F}}{f(R) - 6(\dddot{H} + H^2 - H\frac{d\dot{H}}{dt})F}.$$  \hspace{1cm} (17)

Hence in contrast to Einstein theory of gravity, in $f(R)$ models, $\dot{H} > 0$ does not requires $w < -1$ and $\dot{H} > 0$ is not a necessary condition for $\dot{S}_{\text{in}} > 0$.

Temperature of the horizon, $T$, which is taken the same as the fluid temperature, is independent of the gravity theory that leads to the horizon geometry (see e.g. Ref. [10]). In $f(R)$ gravity, like Einstein theory of gravity, and in the absence of a well defined temperature for cosmological horizon, we assume that $T$ is proportional to Gibbons-Hawking temperature

$$T = \frac{bH}{2\pi}.$$  \hspace{1cm} (18)

$b$ is a real constant. In a de Sitter space we must take $b = 1$. Indeed the parameter $b$ shows the deviation from Gibbons-Hawking temperature. Inserting the modified Einstein-Friedmann equations (Eqs. (4)), and (18), into Eq. \,(16), results in

$$\dot{S}_{\text{in}} = \frac{2\pi R_h^2}{bH}(\dddot{H} + \frac{1}{2}\frac{d^2}{dt^2} - \frac{1}{2}\dot{H}\frac{d}{dt})F.$$  \hspace{1cm} (19)

The total entropy of the universe, denoted by $S$, is the sum of the matter entropy inside the horizon, $S_{\text{in}}$, and $S_h$: $S = S_{\text{in}} + S_h$. The generalized
second law (GSL) states that the total entropy is not a decreasing function of time:

$$\dot{S} = \dot{S}_{\text{in}} + \dot{S}_h \geq 0. \quad (20)$$

Following our previous results, this leads to

$$\dot{S} = \pi R_h^2 \left( (1 - \frac{1}{b}) \dot{F} + 2 \left( \frac{\dot{R}_h}{R_h} + \frac{\dot{H}}{bH} \right) F + \frac{1}{bH} \ddot{F} \right) \geq 0, \quad (21)$$

which is more complicated with respect to the model $f(R) = R$ in which Eq. (21) reduces to the following simple inequality

$$\left( H^b R_h \right) \dot{\geq} 0. \quad (22)$$

To illustrate our results let us consider some examples. As the first example consider a quasi de Sitter FRW space time, defined by

$$H = H_0 + H_0 \epsilon t + O(\epsilon^2), \quad \epsilon := \frac{\dot{H}}{H^2} \ll 1, \quad \dot{\epsilon} = O(\epsilon^2). \quad (23)$$

In this space time the future event horizon is given by

$$R_h = \frac{1}{H} (1 - \frac{\dot{H}}{H^2}) + O(\epsilon^2), \quad (24)$$

leading to

$$\dot{R}_h \simeq - \frac{\dot{H}}{H^2}. \quad (25)$$

Therefore $\frac{\ddot{R}_h}{R_h} \simeq - \frac{\dddot{H}}{H^2}$ and $\dot{S} \geq 0$ reduces to

$$\dot{S} \simeq \pi R_h^2 \left( 1 - \frac{1}{b} \right) \left( \dot{F} - \frac{2\dot{H}}{H} F \right) \simeq 0. \quad (26)$$

Consider the model

$$f = \beta R + \alpha R^m, \quad \beta, \alpha, m \in \mathbb{R}. \quad (27)$$

By using $\dddot{R} \simeq 24H\dddot{H}$ and $\dddot{R} \simeq 24\dot{H}^2$, one can verify that the GSL is satisfied up to the order $O(\epsilon^2)$, provided that

$$(1 - \frac{1}{b}) \dot{H} \left( \alpha m (m-2)(12H^2)^{m-1} - \beta \right) \simeq 0. \quad (28)$$

For $\alpha = 0$ and $\beta = 1$, corresponding to Einstein theory of gravity, the above equation reduces to

$$-(1 - \frac{1}{b}) \frac{\dot{H}}{H} \simeq 0, \quad (29)$$

showing that for a super-accelerated universe ($\dot{H} > 0$), $\dot{S} \simeq 0$ is satisfied if $b \gtrsim 1$ and for quintessence phase ($\dot{H} < 0$), GSL is respected when $b \gtrsim 1$. 

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If $\beta = 0$ and $\alpha > 0$, $\dot{S} \gtrless 0$ is satisfied when $\alpha m(m-2)(1-\frac{1}{b})\dot{H} \gtrless 0$. So if $\dot{H} > 0$ and $m > 2$, we must have $b \gtrless 1$. Therefore in modified gravity in contrast to the Einstein theory of gravity, a super-accelerated FRW model, which departs slightly from de Sitter space, may have temperature greater than Gibbons-Hawking temperature and meanwhile respects the GSL.

As another example consider again the $f(R)$ gravity model defined by Eq. (27). Assume

$$a = a_0(t_s - t)^{-n}, n > 0. \quad (30)$$

In this model

$$H = \frac{n}{t_s - t}, \quad R = \frac{12n^2 + 6n}{(t_s - t)^2}. \quad (31)$$

Hence Eq. (30) describes a super accelerated FRW universe, $\dot{H} > 0$, with a Big Rip singularity at $t = t_s$. The future event horizon radius is

$$R_h = \frac{t_s - t}{n + 1}. \quad (32)$$

To show that Eq. (30) may be a solution of Eq. (4) in $f(R)$ gravity, one can assume that the perfect fluid inside the event horizon includes two components with EOS: $P_1 = \gamma_1 \rho_1$ and $P_2 = \gamma_2 \rho_2$, where the constant $\gamma$'s are the EOS parameters. These components satisfy

$$8\pi \rho_1 = \frac{\beta R^2}{2} - 3\beta(\dot{H} + H^2)$$

$$8\pi P_1 = -\frac{\beta R^2}{2} + \beta(\dot{H} + 3H^2), \quad (33)$$

and

$$8\pi \rho_2 = \frac{\alpha R^m}{2} - 3m\alpha(\dot{H} + H^2 - H \frac{d}{dt}R^{m-1})$$

$$8\pi P_2 = -\frac{\alpha R^m}{2} + m\alpha(\dot{H} + 3H^2 - \frac{d^2}{dt^2} - 2H \frac{d}{dt}R^{m-1}). \quad (34)$$

Besides, each component ($i = 1, 2$) satisfies the energy conservation equation

$$\dot{\rho}_i + 3H(\gamma_i + 1)\rho_i = 0. \quad (35)$$

Using Eqs. (33), (34) and Eq. (35) and after some calculations it may be verified that Eq. (4) is satisfied in the case (27) provided that:

$$\frac{1}{1 + \gamma_1} = \frac{m}{1 + \gamma_2} = -\frac{3n}{2}, \quad (36)$$

which results in $\gamma_1 < 0$, and $\gamma_2 < (>) - 1$ if $m > (<) 0$. Using Eq. (21), one can verify that in this model GSL is respected only for times satisfying

$$-m\alpha \left( (m - 2) (b - 1) n + 2m^2 + 1 - 3m \right) \left( 6n(1 + 2n) \right)^{m-1} \times (t_s - t)^{2m-2} + \beta n(b - 1) \leq 0. \quad (37)$$
In Einstein theory of gravity ($\alpha = 0, \beta = 1$), $\dot{S} > 0$ is satisfied only when $b < 1$, in this case $\dot{S} > 0$ holds $\forall t < t_s$. In the modified theory, depending on the values of $m, n, \alpha$, and $\beta$, $\dot{S} > 0$ holds only for times belonging to special domain specified by the Eq. (37) and we may have $\dot{S} > 0$ while $b > 1$. We can also restrict the values of the parameters to specific domains such that GSL holds $\forall t < t_s$, e.g., for $m\alpha \left[ (m - 2)(b - 1)n + 2m^2 + 1 - 3m \right] > 0$, GSL holds $\forall t < t_s$ provided that $\beta(b - 1) < 0$.

It seems that GSL does not hold near the Big Rip singularity, $t \simeq t_s$, for $\beta(b - 1) > 0$. This may be related to the fact that in our classical computation we have ignored the contribution of the radiation energy density, generated by semiclassical particle creation from Rindler horizon near the Big Rip [12], in the total entropy.

For an adiabatic expansion ($\dot{S} = 0$), the following equations hold:

$$-m\alpha \left( (m - 2)(b - 1)n + 2m^2 + 1 - 3m \right) \left( 6n(1 + 2n) \right)^{m-1} = 0,$$

$$\beta n(b - 1) = 0,$$

which in Einstein theory of gravity ($\beta = 1, \alpha = 0$), infers $b = 1$. Therefore, in this theory, during an adiabatic expansion the temperature is the same as Gibbons-Hawking temperature. While in the modified theory of gravity ($\alpha \neq 0$), in order to have $\dot{S} = 0$, $(m - 2)(b - 1)n + 2m^2 + 1 - 3m = 0$ and $\beta(b - 1) = 0$ must be satisfied. Thereby $\beta \neq 0$ leads to $b = 1$ and $m = 1/2$; and $\beta = 0$ implies $b = 1 + \frac{1 - m(2m - 1)}{n(m - 2)}$, so depending on the values of $m$ and $n$, in adiabatic expansion, the temperature may be less or more than the Gibbons-Hawking temperature.

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