Block-Randomized Gradient Descent Methods with Importance Sampling for CP Tensor Decomposition

Hanyu Li† and Yajie Yu‡

Abstract. This work considers the problem of computing the CANDECOMP/PARAFAC (CP) decomposition of large tensors. One popular way is to translate the problem into a sequence of overdetermined least squares subproblems with Khatri-Rao product (KRP) structure. In this work, for tensor with different levels of importance in each fiber, combining stochastic optimization with randomized sampling, we present a mini-batch stochastic gradient descent algorithm with importance sampling for those special least squares subproblems. Four different sampling strategies are provided. They can avoid forming the full KRP or corresponding probabilities and sample the desired fibers from the original tensor directly. Moreover, a more practical algorithm with adaptive step size is also given. For the proposed algorithms, we present their convergence properties and numerical performance. The results on synthetic data show that our algorithms outperform the existing algorithms in terms of accuracy or the number of iterations.

Key words. CANDECOMP/PARAFAC (CP), Khatri-Rao product, importance sampling, stochastic gradient descent, adaptive algorithm, randomized algorithm

AMS subject classifications. 15A69, 68W20, 90C52

1. Introduction. The CP decomposition of tensor means factorizing a tensor into a sum of rank-one tensors. That is, given an $N$-way tensor $X$ of size $I_1 \times I_2 \times \cdots \times I_N$, we wish to write it as:

$$X = [A^{(1)}, A^{(2)}, \cdots, A^{(N)}] = \sum_{r=1}^{R} a^{(1)}_r \circ a^{(2)}_r \circ \cdots \circ a^{(N)}_r,$$

where $R$ is a positive integer, $a^{(n)}_r \in \mathbb{R}^{I_n}$, and $A^{(n)} = [a^{(n)}_1, a^{(n)}_2, \cdots, a^{(n)}_R] \in \mathbb{R}^{I_n \times R}$ with $n = 1, \cdots, N$. Usually, $A^{(n)}$ is called the $n$-th factor matrix. The above tensor decomposition is an important tool for data analysis and has many important applications in many fields such as chemometrics, biogeochemistry, neuroscience, signal processing, cyber traffic analysis, and many others [1].

It is known that the computation of CP decomposition is a challenging problem. At present, there are many algorithms for CP decomposition. One popular one is the alternative least squares (ALS) method proposed in the original papers [2, 3]. Specifically, we translate the $n$-th factor matrix $A^{(n)}$ as the solution to the following least squares problem:

$$\min_{A \in \mathbb{R}^{I_n \times R}} \|Z^{(n)} A^\top - X^{(n)}\|^2_F,$$
where $Z^{(n)} = A^{(N)} \odot \cdots \odot A^{(n+1)} \odot A^{(n-1)} \odot \cdots \odot A^{(1)} \in \mathbb{R}^{I_n \times J_n}$ with the symbol $\odot$ denoting the KRP, also known as the matching columnwise Kronecker product, and $J_n = \prod_{m \neq n} I_m$, and $X^{(n)} \in \mathbb{R}^{I_n \times J_n}$ is the mode-$n$ unfolding of the input tensor. Here, the mode-$n$ unfolding of a tensor means aligning the mode-$n$ fibers as the columns of an $I_n \times J_n$ matrix and the relation between the index of the tensor entry $x_{i_1,i_2,\cdots,i_N}$ and the index of the matrix entry $x_{in,j}$ of $X^{(n)}$ is

$$j = 1 + \sum_{k=1,k \neq n}^{N} (i_k - 1)J'_k, \quad \text{where} \quad J'_k = \prod_{m=1,m \neq n}^{k-1} I_m.$$  

More symbol definitions are consistent with [1].

As we know, the ALS method is called the “workhorse” algorithm for CP decomposition (CP-ALS). However, for large-scale problems, the cost of the method is prohibitive. To reduce the cost, in 2018, Battaglino et al. [4] applied the randomized projection and randomized uniform sampling techniques of the regular least squares problem (see e.g., [5]) to (1.1) and designed the corresponding randomized algorithms. A main and attractive feature of the algorithms in [4] is that they never explicitly form the full KPR matrices when applying randomized projection and randomized uniform sampling. Later, to solve the least squares subproblem in CP-ALS, Fu et al. [6] considered the mini-batch stochastic gradient descent (SGD) method. The mini-batch in their algorithms is chosen by using the randomized uniform sampling technique from [4]. In 2020, Larsen and Kolda [7] performed the randomized leverage score sampling for (1.1) without forming the full KRP or the corresponding probabilities. By the way, the method for computing the leverage scores of the KRP matrix without forming it explicitly also appears in [8]. The randomized sampling methods introduced in [4, 6, 7] are built on the fiber sampling. Besides, there are also some other randomized sampling algorithms for CP decomposition built on the element sampling [9, 10, 11] or the sub tensor sampling [12, 13]. However, these two samplings are not suitable for the case of constraint [6].

Inspired by the methods in [6] and [7], in this work, we consider the mini-batch SGD methods with importance sampling for CP decomposition. Different from the uniform sampling used in [6], we use the importance sampling based on the leverage scores or the squared Euclidean norms of rows of the KRP matrices for choosing the mini-batch. As we know, the rows sampled by importance sampling contain more information compared to those by uniform sampling [14]. So our new algorithms converge faster than the one from [6]. Numerical results from Section 6 confirm this result. Besides, as done in [4], we don’t need to form the KRP matrices or the corresponding probabilities when sampling the mini-batch. As for the sampling strategies, we propose four different methods based on sampling the rows first or blocking the rows first. In comparison, the former has a better effect because it gets together the rows with high leverage scores or squared Euclidean norms and hence contains more information. The latter only selects the most important block from all the pre-divided blocks.

The remainder of this paper is organized as follows. Section 2 provides the preliminaries. In Section 3, we present the four different sampling methods. We describe our main algorithms in Section 4 and give the convergence properties in Section 5. Section 6 is devoted to numerical experiments to illustrate our algorithms.
2. Preliminaries. In this Section, we will mainly review the algorithm in [6] and the importance sampling technique in [7].

We first introduce the idea given in [4] on sampling the rows of a KRP matrix without forming it explicitly, that is, how to compute the sampled matrix $Z^{(n)}(\mathcal{F}_n,:) \in \mathbb{R}^{n \times R}$ without forming $Z^{(n)}$ explicitly, where $\mathcal{F}_n \subset \{1, \ldots, J_n\}$ with $|\mathcal{F}_n| = B$ contains the indices of the rows sampled from $Z^{(n)}$. Note that the sampled least squares problem of (1.1) is now:

\begin{equation}
(2.1) \quad \min_{A \in \mathbb{R}^{I_n \times R}} \|Z^{(n)}(\mathcal{F}_n,:) A^\top - X^{(n)}(\mathcal{F}_n,:))\|_F^2.
\end{equation}

Considering that, by (1.2), the index $j$ of the $j$-th row of $Z^{(n)}$ is mapped to the indices

$$(i_1, \ldots, i_{n-1}, i_{n+1}, \ldots, i_N),$$

the $j$th row of $Z^{(n)}$ is equal to the Hadamard product of the appropriate rows of the factor matrices, i.e.,

$$Z^{(n)}(j,:) = A^{(1)}(i_1,:) \odot \cdots \odot A^{(n-1)}(i_{n-1},:) \odot A^{(n+1)}(i_{n+1},:) \odot \cdots \odot A^{(N)}(i_N,:).$$

Based on this fact, Battaglino et al. [4] present the following Algorithm 2.1 for computing $Z^{(n)}(\mathcal{F}_n,:)$, i.e., the sampled Khatri-Rao (SKR) product. The notation $\text{idx}$ in Line 4 of Algorithm 2.1 denotes the set of tuples:

\begin{equation}
(2.2) \quad \{i_1^{(j)}, \ldots, i_{n-1}^{(j)}, i_{n+1}^{(j)}, \ldots, i_N^{(j)}\} \quad \text{with} \quad j \in \mathcal{F}_n.
\end{equation}

We assume that these tuples are stacked in matrix form for efficiency. Thus, each multiplicand $A\{m\}_{\mathcal{F}_n}$ is of size $|\mathcal{F}_n| \times R$.

\begin{verbatim}
Algorithm 2.1 SKR Product [4]
1: function Z^{(n)}(\mathcal{F}_n,:)= SKR(idx, A^{(N)}, \ldots, A^{(n+1)}, A^{(n-1)}, \ldots, A^{(1)})
2: \quad Z^{(n)}(\mathcal{F}_n,:) \leftarrow 1 \quad \triangleright 1 \in \mathbb{R}^{|\mathcal{F}_n| \times R}
3: \quad for m = 1, \ldots, n-1, n+1, \ldots, N do
4: \quad \quad A\{m\}_{\mathcal{F}_n} \leftarrow A\{m\}(\text{idx}(::,:)) \quad \triangleright \text{MATLAB-style indexing, the idx is as (3.1)}
5: \quad Z^{(n)}(\mathcal{F}_n,:) \leftarrow Z^{(n)}(\mathcal{F}_n,:) \odot A\{m\}_{\mathcal{F}_n}
6: \quad end for
7: return Z^{(n)}(\mathcal{F}_n,:)
8: end function
\end{verbatim}

Combining Algorithm 2.1 with uniform sampling and SGD, Fu et al. [6] proposed the minibatch SGD algorithm with uniform sampling for CP decomposition, i.e., the so called Block-Randomized SGD for CP Decomposition (B-CPD). Specifically, the CP decomposition of a tensor is first rewritten as the following optimization problem:

$$\min_{\{A^{(n)} \in \mathbb{R}^{I_n \times R}\}_{n=1}^N} f(A^{(1)}, \ldots, A^{(N)}),$$

where $f$ is the objective function.
We say $\alpha \sim \text{multinomial}(p)$ if $p \in [0, 1]^N$ is a probability distribution and $\Pr(\xi = i) = p_i$. The distribution selected for $p$ determines the quality of the estimate. One well-known selection is based on leverage scores.
Definition 2.3 (Leverage Scores [15]). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$, and let $\mathbf{Q} \in \mathbb{R}^{m \times n}$ be any orthogonal basis for the column space of $\mathbf{A}$. The leverage score of the $i$-th row of $\mathbf{A}$ is given by

$$\ell_i(\mathbf{A}) = \| \mathbf{Q}(i,:) \|_2^2.$$ 

Definition 2.4 (Leveraged-based Probability [16]). We say $\mathbf{q} = [q_1, \cdots, q_m]^{\top}$ is a leveraged-based probability distribution if $q_i \geq \beta p_i$ with $p_i = \ell_i(\mathbf{A}) / n$, $0 < \beta \leq 1$ and $i \in [m]$.

Another selection is based on the squared Euclidean norm of rows.

Definition 2.5 (Euclidean-based Probability). We say $\mathbf{p} = [p_1, \cdots, p_m]^{\top}$ is an Euclidean-based probability distribution if $p_i \geq \beta \| \mathbf{A}(i,:) \|_2^2 / \| \mathbf{A} \|_F^2$, $0 < \beta \leq 1$ and $i \in [m]$.

Since it is expensive to compute the leverage scores of a KRP matrix directly, Cheng et al. [8] presented their upper bounds.

Lemma 2.6 (Leverage Score Bounds for KRP [8]). For matrices $\mathbf{A}^{(k)} \in \mathbb{R}^{I_k \times R}$ with $I_k < R$ for $k = 1, \cdots, K$, let $\ell_i^{(k)}$ be the leverage score of the $i$-th row of $\mathbf{A}^{(k)}$. Then, for the $\prod_k I_k$-by-$R$ KRP matrix $\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \cdots \odot \mathbf{A}^{(K)}$ with leverage score $\ell_{i_1, \cdots, i_K}$ for the $j$-th row corresponding to $\{i_1, \cdots, i_K\}$, we have

$$\ell_{i_1, \cdots, i_K} \leq \prod_{k=1}^K \ell_i^{(k)}.$$ 

Hence, the leveraged-based sampling probability for the $j$-th row of the above KRP matrix is given by

$$p_j = \frac{1}{R^K} \prod_{k=1}^K \ell_i^{(k)}.$$ 

To further improve the performance, Larsen and Kolda [7] proposed to sample each mode independently instead of sampling from the above distribution explicitly based on the following fact.

Lemma 2.7 ([7]). Let $\mathbf{A}^{(k)} \in \mathbb{R}^{I_k \times R}$ for $k = 1, \cdots, K$, and let $\ell(\mathbf{A}^{(k)})$ be the vector of leverage scores for $\mathbf{A}^{(k)}$. Let

$$i_k \sim \text{MULTINOMIAL}(\ell(\mathbf{A}^{(k)}) / R) \quad \text{for} \quad k = 1, \cdots, K.$$ 

The probability of selecting the multi-index $\{i_1, \cdots, i_K\}$ is equal to

$$p_j = \frac{1}{R^K} \prod_{k=1}^K \ell_i^{(k)}.$$ 

3. Efficient Sampling for KRP matrices. In this Section, we provide four sampling strategies to choose the mini-batch of the mini-batch SGD method of CP-ALS. They can be classified into two classes. The first one is to sample rows wisely first and then to combine them into a block, and the second one is to block the rows first and then to sample one block. For the two classes, we will consider the sampling probabilities according to leverage scores and squared Euclidean norms of rows, respectively. So the two classes can be named as leverage/Euclidean-based method and block-leverage/Euclidean-based method, respectively.
3.1. Leverage/Euclidean-based method. In the following, by using Lemmas 2.6 and 2.7, we first introduce how to compute the sampling probabilities of the mini-batches based on leverage scores.

Suppose that we sample $B$ rows of $Z^{(n)}$ indexed by $F_n \subset \{1, \ldots, J_n\}$, where $F_n = \{j_1, j_2, \ldots, j_B\}$ and let $j = [j_1, j_2, \ldots, j_B]^T$. Then according to (2.2), we can get the set of tuples corresponding to $j$:

\[
\begin{pmatrix}
\ell_{i_1} & \ell_{i_2} & \cdots & \ell_{i_{n-1}} & \ell_{i_n} \\
\ell_{i_1} & \ell_{i_2} & \cdots & \ell_{i_{n-1}} & \ell_{i_n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\ell_{i_1} & \ell_{i_2} & \cdots & \ell_{i_{n-1}} & \ell_{i_n}
\end{pmatrix}.
\]

(3.1)

Applying Lemma 2.6, the leverage scores of $Z^{(n)}(F_n, \cdot)$ are bounded as

\[
\bar{\ell}_j(Z^{(n)}) = [\bar{\ell}_{j_1}(Z^{(n)}), \bar{\ell}_{j_1}(Z^{(n)}), \ldots, \bar{\ell}_{j_B}(Z^{(n)})]^T,
\]

where $\bar{\ell}_{j_k}(Z^{(n)}) = \prod_{k=1, k \neq n}^N \ell_{i_k}(A^{(k)}) \geq \ell_{j_k}(Z^{(n)})$. Let $\bar{\ell}(A^{(k)})$ be the vector of leverage scores for $A^{(k)}$, $p_k = \frac{\bar{\ell}(A^{(k)})}{R^N}$, and

\[
(3.2) \quad i_k = \text{randsample}(I_k, B, \text{true}, p_k) \in \mathbb{R}^B \quad \text{for} \quad k \in [N], k \neq n.
\]

Note that (3.2) means selecting $B$ indices with probability distribution $p_k$ from $I_k$ indices. Then according to Lemma 2.7, our sampling probability for rows $F_n$, i.e., the probability of selecting the multi-indices corresponding to $j$ as showed in (3.1) can be defined as

\[
(3.3) \quad p_j = [p_{j_1}, \cdots, p_{j_B}]^T = \frac{\bar{\ell}_j(Z^{(n)})}{R^{N-1}},
\]

where $p_{j_k} = \frac{\bar{\ell}_{j_k}(Z^{(n)})}{R^N}$. Thus, it is not difficult to see that the sampling probability of the mini-batch $F_n$ is $\sum_{k=1}^B p_{j_k}$. Once the index set $F_n$ is determined, the subproblem (2.1) is determined.

Now we consider computing the sampling probabilities of the mini-batches based on the squared Euclidean norm of rows.

From Definition 2.5, it follows that the squared Euclidean-based probability of $Z^{(n)}(F_n, \cdot)$ is

\[
(3.4) \quad p_j(Z^{(n)}) = [p_{j_1}(Z^{(n)}), p_{j_2}(Z^{(n)}), \cdots, p_{j_B}(Z^{(n)})]^T,
\]

where $p_{j_k}(Z^{(n)}) = \frac{||Z^{(n)}(j_k, \cdot)||^2_F}{||Z^{(n)}||^2_F}$. To avoid forming the KRP matrix $Z^{(n)}$, we can get an upper bound of the squared Euclidean norms after some simple algebraic operations. Thus, the sampling probability for rows $F_n$ can be computed by

\[
(3.4) \quad p_j = [p_{j_1}, \cdots, p_{j_B}]^T = p_j(Z^{(n)}),
\]
where \( p_{jb} = p_{jb}(Z^{(n)}) \leq \prod_{k=1, k \neq n}^N P_{k_{jb}}(A^{(k)}) = \frac{\prod_{k=1, k \neq n}^N \|A^{(k)}(i_{kj})\|_2^2}{\|A^{(k)}\|_{\infty}^N}, \) \( P_{k_{jb}}(A^{(k)}) \) denotes the Euclidean-based probability of the \( i_{kj} \)-th row of \( A^{(k)} \) corresponding to \( j_b \), and the sampling probability of the mini-batch \( F_n \) is \( \sum_{b=1}^B p_{jb} \).

Based on the above discussions, we can present the algorithm framework in Algorithm 3.1.

**Algorithm 3.1** SKR product with importance probabilities for leverage/Euclidean-based method

1: function \([Z^{(n)}(F_{n,:}), p_j, idx]=\)KRPSAMPLE\(([F_{n,:}, \{p_k\}_{k=1,k \neq n}, \{A^{(N)}\}_{k=1,k \neq n})\)
2: \( for k = 1, \ldots, n - 1, n + 1, \ldots, N \) do
3: \( \text{idx}(k,:) = \text{randsample}(I_k, |F_n|, true, p_k) \) \( \triangleright \) Sample random index \( i \equiv (i_1, \cdots, i_k) \in [J_n] \)
4: \( end \) for
5: \( Z^{(n)}(F_{n,:})=\)SKR\((\text{idx}, A^{(N)}, \cdots, A^{(n+1)}, A^{(n-1)}, \cdots, A^{(1)})\)
6: \( p_j \leftarrow (3.3) \text{ or } (3.4) \)
7: \( \text{return } [Z^{(n)}(F_{n,:}), p_j, idx] \)
8: \( \text{end function} \)

### 3.2. Block-Leverage/Euclidean-based method.

Let \( D = [J_n/B] \) (for simplicity we assume that \( D = J_n/B \) is an integer) and let the \( D \) blocks of the matrix \( Z^{(n)} \) be \( \{r_1, r_2, \cdots, r_D\} \).

We first estimate the block partial leverage scores of \( Z^{(n)} \) by the block partial leverage scores of the corresponding rows of \( A^{(k)} \) with \( k = 1, \ldots, n - 1, n + 1, \ldots, N \), where the definition of the block partial leverage scores is given as follows.

**Definition 3.1** (Block Partial Leverage Scores [17]). Assume that \( \{r_{(d)}^{(k)}\}_{d=1}^{D} \) is a row index set of \( A^{(k)} \), corresponding to the \( d \)-th block of \( Z^{(n)} \). Let \( \ell_{r_{(d)}^{(k)}} \) be the leverage scores of the matrix \( A_{r_{(d)}}^{(k)} \) and define the block partial leverage scores for the \( d \)-th block as

\[
\ell_{r_{(d)}^{(k)}}(A_{r_{(d)}}^{(k)}) = \sum_{i_k \in [r_{(d)}^{(k)}]} \ell_{i_k}(A^{(k)}),
\]

where \( \ell_{i_k}(A^{(k)}) \) is the leverage scores of the \( i_k \)-th row of \( A^{(k)} \). Moreover, the block-leverage-based probability distribution is defined as

\[
P_{r_{(d)}^{(k)}}(A^{(k)}) = \frac{\ell_{r_{(d)}^{(k)}}(A_{r_{(d)}}^{(k)})}{\sum_{d=1}^{D} \ell_{r_{(d)}^{(k)}}(A_{r_{(d)}}^{(k)})} = \frac{\ell_{r_{(d)}^{(k)}}(A_{r_{(d)}}^{(k)})}{D}.
\]

Combining Lemma 2.6 and Definition 3.1, the leverage scores of \( Z^{(n)}(F_{n,:}) \) can be bounded by

\[
\tilde{\ell}_j(Z^{(n)}) = \sum_{\{r_d\}_{d=1}^{D}} \prod_{k=1, k \neq n}^N \ell_{i_{kj}}(A^{(k)}).
\]

Let \( \ell(A^{(k)}) \in \mathbb{R}^D \) be the vector of block partial leverage scores for \( A^{(k)} \), \( p_k = \frac{\ell(A^{(k)})}{D} \), and

\[
d = \text{randsample}(D, 1, true, p_k) \in \mathbb{R} \ \text{ for } \ k \in [N], k \neq n.
\]
Here, (3.5) means selecting one block with probability distribution \( p_k \) from \( D \) blocks. Then we get \( \{\tau^{(k)}_d\} \) and hence (3.1). Thus, according to Lemma 2.7, our sampling probability for rows \( \mathcal{F}_n \), i.e., the probability of selecting the multi-indices corresponding to \( j \), can be defined as

\[
p_j = \frac{\tilde{p}_j(Z^{(n)})}{R^{N-1}} \leq \sum_{\{\tau_d\}} \prod_{k=1, k \neq n}^N \frac{\ell_{i_n}(A^{(k)})}{R^{N-1}}.
\]

Once the block index set \( \mathcal{F}_n \) is determined, the subproblem (2.1) is determined.

Next, we consider the case where the sampling probability is based on the block squared Euclidean norms, whose definition is listed in the following:

**Definition 3.2 (Block Euclidean Norms [17])**. Assume that \( \{\tau^{(k)}_d\} (d = 1, \cdots, D) \) is a row index set of \( A^{(k)} \), corresponding to the \( d \)-th block of \( Z^{(n)} \). Let \( p_{\tau_d^{(k)}}(d = 1, \cdots, D) \) be the block-Euclidean-based probability of the matrix \( A^{(k)}_{\tau_d} \). Then the block-Euclidean-based probability of the \( d \)-th block is defined as

\[
p_{\tau_d^{(k)}}(A^{(k)}_{\tau_d}) = \frac{||A^{(k)}\|_F^2}{||A^{(k)}\|_F^2}.
\]

Thus, the row block squared Euclidean norm sampling probability of \( Z^{(n)}(\mathcal{F}_n, :) \) is

\[
p_j(Z^{(n)}) = ||Z^{(n)}(\mathcal{F}_n, :)||^2_F/||Z^{(n)}||^2_F.
\]

To avoid forming the KRP matrix \( Z^{(n)} \), we can also get a bound of the block squared Euclidean norms after some simple algebraic operations. Thus, our sampling probability for rows \( \mathcal{F}_n \) can be defined as:

\[
p_j = p_j(Z^{(n)}) \leq \sum_{\{\tau_d\}} \prod_{k=1, k \neq n}^N p_{i_n}(A^{(k)}) = \sum_{\{\tau_d\}} \prod_{k=1, k \neq n}^N \frac{||A^{(k)}(i_n, :)||^2}{(||A^{(k)}||)^{N-1}} (b = 1, \cdots, B),
\]

where \( p_j \) is the sampling probability of the mini-batch \( \mathcal{F}_n \).

The above discussions imply the following algorithm framework, i.e., Algorithm 3.2.

**4. Proposed Algorithm.** We first give two different forms of the gradient \( G^{(n)}(r) \), corresponding to the four kinds of sampling strategies described in Section 3.

(1) When we use the sampling strategies in Subsection 3.1, we have

\[
G^{(n)}(r) = \frac{1}{|\mathcal{F}_n| \times J_n} \left( A^{(n)}(r)(DZ^{(n)}(\mathcal{F}_n, :)^T)Z^{(n)}(\mathcal{F}_n, :) - (DX^{(n)}(:, \mathcal{F}_n))Z^{(n)}(\mathcal{F}_n, :) \right),
\]

where \( D = \text{diag}[\frac{1}{p_{j_1}}, \cdots, \frac{1}{p_{j_B}}] \) is from (3.3) or (3.4).

(2) When we use the sampling strategies in Subsection 3.2, we have

\[
G^{(n)}(r) = \frac{1}{p_j \times J_n} \left( A^{(n)}(r)Z^{(n)}(\mathcal{F}_n, :)^TZ^{(n)}(\mathcal{F}_n, :) - X^{(n)}(:, \mathcal{F}_n)Z^{(n)}(\mathcal{F}_n, :) \right),
\]
Algorithm 3.2 SKR Product with importance probabilities for block-leverage/Euclidean-based method

1: function $[Z^{(n)}(F_n,::), p_j, \text{idx}]$ = KRPSAMP2($(F_n, \{p_k\}_{k=1,k\neq n}^N, \{A^{(k)}(n)\}_{k=1,k\neq n}^N$)
2: for $k = 1, \ldots, n-1, n+1, \ldots, N$ do
3: blocking $A^k$ into $D$ block($\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_D^{(k)}$)
4: $d = \text{randsample}(D, 1, \text{true}, p_k) \in \mathbb{R}$ \rhd Sample random index $d \in [D]$
5: $\text{idx}(::,::,k) = \tau_d^{(k)} = (i^{(1)}_{k}, i^{(2)}_{k}, \ldots, i^{(n)}_{k})$
6: end for
7: $Z^{(n)}(F_n,::) = \text{SKR}(\text{idx}, A^{(n)}, \ldots, A^{(n+1)}, A^{(n-1)}, \ldots, A^{(1)})$
8: $p_j \leftarrow (3.6)$ or (3.7)
9: return $[Z^{(n)}(F_n,::), p_j, \text{idx}]$
10: end function

where $p_j$ is from (3.6) or (3.7).

Thus, the latent factor matrices are updated by

$$(4.3) \quad A^{(n)}_{(r+1)} \leftarrow A^{(n)}_{(r)} - \alpha(\cdot) G^{(n)}_{(r)}, \quad n = 1, \ldots, N.$$  

To compute the gradient, we need to compute $Z^{(n)}(F_n,::)$ and $X^{(n)}(:,F_n)$ efficiently and fastly. The former is discussed in Section 3, i.e., Algorithms 3.1 and 3.2. Now we show how to compute $X^{(n)}(:,F_n)$ efficiently in Algorithm 4.1:

Algorithm 4.1 Sampled tensor fibers with importance probabilities

1: function $X^{(n)}(:,F_n)$ = TENSORsamp(\text{idx}) \rhd the \text{idx} form like (3.1)
2: for $k = 1, \ldots, n-1, n+1, \ldots, N$ do
3: $J_k = \prod_{m=1,m\neq n}^{k-1} I_m$
4: end for
5: for $b = 1, \ldots, B$ do
6: for $k = 1, \ldots, n-1, n+1, \ldots, N$ do
7: $J_k \leftarrow 1 + \sum_{k=1,k\neq n}^{N}(i^{(k)}_{b} - 1)J_k$ \rhd (1.2)
8: end for
9: end for
10: $j = [j_1, \ldots, j_N]^T$
11: $X^{(n)}(:,F_n) = X^{(n)}(:,j)$
12: return $X^{(n)}(:,F_n)$
13: end function

Putting the above discussions together, we propose the following Algorithm 4.2. Like [6], we call it Block-Randomized Weighted SGD for CPD (BrawsCPD). Similarly, we call BrawsCPD with the sampling strategies in Subsections 3.1 and 3.2 the LBrawsCPD $^1$, EBrawsCPD $^2$, BLBrawsCPD $^3$, and EBEBrawsCPD $^4$, respectively.

**Remark 1.** To avoid running the step size schedule, similar to [6], we also give an adaptive
Algorithm 4.2  BrawsCPD (Main Algorithm)

1: function \((A^{(n)})_{n=1}^{N} = \text{BrawsCPD}(X, R, B, \{A^{(n)}_{(0)}\}, \{o^{(r)}\}) \quad \triangleright \text{N-way tensor } X \in \mathbb{R}^{I_1 \times \cdots \times I_N}; \text{ rank } R;
\)

\[ \text{sample size } B; \text{ initialization } \{A^{(n)}_{(0)}\}, \text{ step size } \{o^{(r)}\}_{r=0,1, \ldots} \]

2: \(r \leftarrow 0\)
3: \(\text{repeat}\)
4: \(\quad \text{uniformly sample } n \text{ from } \{1, \ldots, N\}\)
5: \(\quad [Z^{(n)}(F_{n}, i), p_{i}, \text{idx}] = \text{KrpSamp1}(\{p_{k}\}_{k=1,k\neq n}^{N}, \{A^{(N)}\}_{k=1,k\neq n}^{N})\)
6: \(\quad \text{or } [Z^{(n)}(F_{n}, i), p_{i}, \text{idx}] = \text{KrpSamp2}(\{p_{k}\}_{k=1,k\neq n}^{N}, \{A^{(N)}\}_{k=1,k\neq n}^{N})\)
7: \(\quad X_{(n)}(i, F_{n}) = \text{TensorSamp}(\text{idx})\)
8: \(\quad \text{form the stochastic gradient } G^{(n)}_{(r)} ← (4.1)/(4.2)\)
9: \(\quad \text{update } A^{(n)}_{(r+1)} ← (4.3), A^{(n')}_{(r+1)} ← A^{(n')}_{(r)} \text{ for } n' \neq n\)
10: \(r ← r + 1\)
11: \(\text{until some stopping criterion is reached}\)
12: \(\text{return } \{A^{(n)}\}_{n=1}^{N}\)
13: \(\text{end function}\)

step size scheme with the following updating rule:

\[
\begin{align*}
(4.4a) & \quad \left[\eta^{(n)}_{(r)}\right]_{i,r'} \leftarrow \frac{\eta}{\left(b + \sum_{i=1}^{r} G^{(n)}_{(r)}\right)^{1/2}}, \quad i \in [I_n], \ r' \in [R] \\
(4.4b) & \quad A^{(n)}_{(r+1)} \leftarrow A^{(n)}_{(r)} - \eta^{(n)}_{(r)} \odot G^{(n)}_{(r)} \\
(4.4c) & \quad A^{(n')}_{(r+1)} \leftarrow A^{(n')}_{(r)}, \ n' \neq n,
\end{align*}
\]

where \(\eta, b, \epsilon > 0\). Here, \(b > 0\) and \(\epsilon > 0\) are technical conditions used for establishing theoretical convergence. In practice, setting \(b = \epsilon = 0\) does not hurt the performance. We summary the corresponding algorithm in Algorithm 4.3, which is named \textit{AdawsCPD}. And we named \textit{AdawsCPD} with the sampling strategies in Subsections 3.1 and 3.2 the \textit{LAdawsCPD}, \textit{BLAdawsCPD}, \textit{EAdawsCPD}, and \textit{EAdawsCPD}, respectively.

5. Convergence Properties. We first show that the stochastic gradients in \textit{BrawsCPD} and \textit{AdawsCPD} are unbiased. For simplicity, let us define \(\xi_{(r)} \in \{1, \ldots, N\}\) and \(\zeta_{(r)} \subseteq \{1, \ldots, J_{\xi(r)}\}\) as the random variables (r.v.s) responsible for selecting the mode and fibers in iteration \(r\), respectively. These r.v.s are distributed as

\[
\begin{align*}
\text{Pr}(\xi_{(r)} = n) &= \frac{1}{N}, \\
\text{Pr}(\xi_{(r)} = S \mid \xi_{(r)} = n) &= \sum_{j} p_{j}, \quad \text{for leverage/Euclidean-based method}, \\
\text{Pr}(\zeta_{(r)} = S \mid \xi_{(r)} = n) &= p_{j}, \quad \text{for block-leverage/Euclidean-based method},
\end{align*}
\]

where \(n \in \{1, \ldots, N\}\), \(S \in \Sigma\) such that \(\Sigma\) is the set of all subsets of \(\{1, \ldots, J_{\xi_{(r)}}\}\) with size \(B\), and \(D = [J_n/B]\).

Theorem 5.1 (Unbiased Gradient). Denote \(\mathcal{B}_{(r)}\) as the filtration generated by the r.v.s \(\{\xi_{(1)}, \xi_{(1)}', \ldots, \xi_{(r-1)}, \xi_{(r-1)}'\}\)
Algorithm 4.3 AdawsCPD (Main Algorithm)

1: function \( \{A^{(n)}\}_{n=1}^{N} = \text{ADAWSCPD}(X, R, B, \{A_{(0)}^{(n)}\}) \) \( \triangleright \) N-way tensor \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \); rank \( R \); sample size \( B \), initialization \( \{A_{(0)}^{(n)}\} \)
2: \( r \leftarrow 0 \)
3: repeat
4: uniformly sample \( n \) from \( \{1, \ldots, N\} \)
5: \( [Z]^{(n)}(F_n, \cdot, p_l, \text{idx}) = \text{KrpSamp1}(\{p_{k}\}_{k=1}^{N} \setminus \{p_{n}\}) \)
6: or \( [Z]^{(n)}(F_n, \cdot, p_l, \text{idx}) = \text{KrpSamp2}(\{p_{k}\}_{k=1}^{N} \setminus \{p_{n}\}) \)
7: \( X_{(n)}(, F_n) = \text{TensorSamp}(\text{idx}) \)
8: form the stochastic gradient \( G_{(r)}^{(n)} \leftarrow (4.1)/(4.2) \)
9: determine the step size \( \eta_{(r)}^{(n)} \leftarrow (4.4a) \)
10: update \( A_{(r+1)}^{(n)} \leftarrow (4.4b), A_{(r+1)}^{(n')} \leftarrow A_{(r)}^{(n')} \) for \( n' \neq n \)
11: \( r \leftarrow r + 1 \)
12: until some stopping criterion is reached
13: return \( \{A_{(0)}^{(n)}\}_{n=1}^{N} \)
14: end function

such that the \( r \)-th iteration \( \theta_{(r)} \) is determined conditioned on \( B_{(r)} \). The stochastic gradients in (4.1) and (4.2) are unbiased estimates for the full gradient w.r.t. \( A_{(\xi_{(r)})} \):

\[
\mathbb{E}_{\xi_{(r)}} \left[ G_{(r)}^{(\xi_{(r)})} \mid B_{(r)}, \xi_{(r)} \right] = \nabla_{A_{(\xi_{(r)})}} f(\theta_{(r)}).
\]

Proof. We show the desired results for the general optimization problem

\[
f(\theta) = \frac{1}{J_n} \sum_{i=1}^{J_n} f_i(\theta) = \mathbb{E} f_i(\theta).
\]

We sample \( f_{j_b}(\theta) \) with the probability \( p_{j_b} \), and we use \( |F_n| = B \) gradients to estimate the full gradient at \( r \)-th iteration. Denote \( \nabla_{A_{(\xi_{(r)})}} f(\theta_{(r)}) = G_{(\xi_{(r)})} = G_{(\theta_{(r)})} \). Then we have

\[
\mathbb{E}
\left( \frac{1}{|F_n| \times J_n} \sum_{b=1}^{B} \frac{1}{p_{j_b}} G_{j_b}(\theta_{(r)}) \right) = \frac{1}{|F_n| \times J_n} \sum_{b=1}^{B} \mathbb{E}
\left( \frac{1}{p_{j_b}} G_{j_b}(\theta_{(r)}) \right)
= \frac{1}{|F_n| \times J_n} \sum_{b=1}^{B} \sum_{b_{j_b} \in [J_n]} \left( \frac{1}{p_{j_b}} \cdot G_{j_b}(\theta_{(r)}) \right)
= \frac{1}{J_n} \sum_{j_b \in [J_n]} G_{j_b}(\theta_{(r)}) = G(\theta_{(r)}).
\]

Hence, \( \frac{1}{|F_n| \times J_n} \sum_{b=1}^{B} \frac{1}{p_{j_b}} G_{j_b}(\theta_{(r)}) \) is an unbiased estimate of the full gradient \( G(\theta_{(r)}) \). So, for the problem (2.3) with two different sampling probabilities, we have that the stochastic gradient (4.1) is unbiased.

For (4.2), we write the above optimization problem as

\[
f(\theta) = \frac{1}{D} \sum_{d=1}^{D} g_{r_d}(\theta) = \mathbb{E} g_{r_d}(\theta), \quad \text{where} \quad g_{r_d}(\theta) = \frac{1}{B} \sum_{j_b \in r_d} f_{j_b}(\theta).
\]
We sampling $g_r (\theta)$ with the probability $p_j$. Denote $\nabla_{A(\xi)} f(\theta) = G(\theta)$ and $\nabla g_r (\theta) = G_r (\theta)$. Then we have

$$\mathbb{E}(\frac{1}{p_j \times J_n} \sum_{b=1}^B G_{jb}(\theta)) = \frac{1}{D} \mathbb{E}(\frac{1}{p_j} G_{r}(\theta)) = \frac{1}{D} \mathbb{E}(\frac{1}{p_{r}} G_{r}(\theta)) = \frac{1}{D} \sum_{d=1}^D (\frac{1}{p_{r}} \cdot p_{r} \cdot G_{r}(\theta))$$

$$= \frac{1}{D} \sum_{d=1}^D G_{r}(\theta) = G(\theta).$$

Hence, $\frac{1}{p_j \times J_n} \sum_{b=1}^B G_{jb}(\theta)$ is an unbiased estimate for the full gradient $G(\theta)$. So, for the problem $(2.3)$ with two different sampling probabilities, we have that the stochastic gradient $(4.2)$ is unbiased.

Before giving the main theorem, we also need the following assumptions:

**Assumption 1 ([6]).** The stepsize schedule follows the Robbins-Monro rule [18]:

$$\sum_{r=0}^\infty \alpha_r = \infty, \quad \sum_{r=0}^\infty (\alpha_r)^2 < \infty.$$

**Assumption 2 ([6]).** The updates $A^{(n)}$ are bounded for all $n, r$.

We now present the convergence properties for BraawsCPD and AdawsCPD.

**Theorem 5.2 (Main Theorem).** Assume that Assumption 1 and Assumption 2 hold. The solution sequence produced by BraawsCPD satisfies:

$$\lim_{r \to \infty} \mathbb{E}[\|\nabla f(\theta)\|^2] = 0.$$

**Proof.** The proof can be completed easily by combining the proof of Proposition 1 in [6] and Theorem 5.1.

**Theorem 5.3 (Main Theorem).** Assume that $Pr(\xi = n) = 1/N$ for all $r$ and $n$. Under the Assumption 1 and Assumption 2, the solution sequence produced by AdawsCPD satisfies

$$\Pr\left(\lim_{r \to \infty} \|\nabla f(\theta)\|^2 = 0\right) = 1.$$

**Proof.** The proof can be completed easily by combining the proof of Proposition 3 in [6] and Theorem 5.1.

6. Numerical Results. In this Section, we use synthetic data to test the effectiveness of the proposed algorithms. Because of the maneuverability, we only perform AdawsCPD with four sampling strategies to avoid tuning the step size manually. In addition, we mainly compare our algorithm with AdasCPD from [6] since our algorithms are built on this method and many other randomized algorithms for CP decomposition are not iterative methods.
### Experimental setup

| Name       | Size     | Tol   | \(|\mathcal{F}_n|\) factor matrices | R |
|------------|----------|-------|-----------------------------------|---|
| create-system | \(I \times I \times 15\) | \(10^{-4}\) | uniform on \([0,1]\) | 25 |

### Result

| Algorithm               | Iterations | Seconds |
|-------------------------|------------|---------|
| AdasCPD (uniform) \([6]\) | 189        | 0.32466 |
| EAdawsCPD (euclidean)   | 95         | 0.21996 |
| LAdawsCPD (leverage)    | 78         | 0.20081 |
| BEAdawsCPD (beuclidean) | 106        | 0.24994 |
| BLAdawsCPD (bleverage)  | 462        | 1.2345  |

| Algorithm               | Iterations | Seconds |
|-------------------------|------------|---------|
| AdasCPD (uniform) \([6]\) | 189        | 0.32466 |
| EAdawsCPD (euclidean)   | 95         | 0.21996 |
| LAdawsCPD (leverage)    | 78         | 0.20081 |
| BEAdawsCPD (beuclidean) | 106        | 0.24994 |
| BLAdawsCPD (bleverage)  | 462        | 1.2345  |

### 6.1 Environment setup

The experiments were carried out by using the Tensor Toolbox for MATLAB (Version 2018a) \([19]\), and we used a 2.3 GHz 8-Core Intel Core i9 CPU with 16 GB 2400 MHz DDR4 memory.

### 6.2 Data generation

The synthetic \(I \times I \times J\) tensors are used in our experiments, and, to show the advantages of importance sampling, we use the method from \([7]\) to generate the data.

Specifically, two \(I\)-by-\(R\) and one \(J\)-by-\(R\) factor matrices with independent standard Gaussian entries are first generated. Then the first three columns of each factor matrix are set to be 0. Following this, a data-generating function with two parameters spread and magnitude is applied to the new factor matrices. The first parameter is used to control the number of non-zero elements in those three columns, and the second one is used to control the size of the non-zero elements. Thus, three factor matrices with very high leverage scores and hence a desired tensor are generated. Finally, we also add 5% Gaussian noise to the elements of the tensor.

Note that the above way for generating data is not unique. As long as the data has the characteristic of different importance between rows, our algorithm will have better performance in terms of accuracy or the number of iterations compared with AdasCPD from \([6]\).

### 6.3 Results

Table 6.1 shows the number of iterations of the algorithms with the same accuracy \(Tol = 10^{-4}\), and Table 6.2 shows the relative errors of the algorithms with the same number of iterations. In both tables, we show the running time of the algorithms.

From Tables 6.1 and 6.2, we can see that EAdawsCPD and LAdawsCPD always exhibit very competitive performance. They need less time and fewer iterations under the same accuracy. However, BEAdawsCPD is not always satisfactory due to more time or more iterations. For BLAdawsCPD, it performs worse than AdasCPD in most of cases. Comparing the Euclidean/leverage-based and block-Euclidean/leverage-based methods, we can observe that
**Table 6.2:** Performance of the algorithms with the same number of iterations.

| Algorithm          | 1=100, it=55 | 1=200, it=80 | 1=300, it=200 |
|-------------------|---------------|---------------|---------------|
|                   | Relative error | Relative error | Relative error |
|                   | Seconds       | Seconds       | Seconds       |
| AdasCPD (uniform) | 0.14263       | 0.12494       | 0.13429       |
|                   | 0.095582      | 0.12701       | 0.094841      |
| EAdawsCPD (euclidean) | 0.92162       | 0.97152       | 0.94841       |
|                   | 0.41568       | 0.1417        | 0.1414       |
|                   | 0.41568       | 0.52798       | 0.54091      |
| LAdawsCPD (leverage) | 0.00060437    | 0.00020604    | 0.00090031   |
|                   | 0.48464       | 0.79444       | 4.5048       |
|                   | 0.00020604    | 0.54091       | 4.5048       |
|                   | 0.00090031    | 0.79444       | 4.5048       |

*Figure 6.1:* Number of iterations v.s. Relative errors output by the algorithms with $R = 25$, $\text{Tol} = 10^{-4}$, $I = 300$, $|F_n| = 18$, Spread = 15, and Magnitude = 24.

The performance of the former is better than that of the latter. For the Euclidean-based and leverage-based algorithms, the effect of the former is a little better than that of the latter in most of cases.

*Figure 6.1* shows how the relative errors change along with the number of iterations or runtime, from which one can see that our algorithms reduces the relatives quickly.

**REFERENCES**

[1] Tamara G. Kolda and Brett W. Bader. Tensor Decompositions and Applications. *SIAM Review*, 51(3):455–500, 2009.

[2] J. Douglas Carroll and Jih-Jie Chang. Analysis of individual differences in multidimensional scaling via an n-way generalization of “Eckart-Young” decomposition. *Psychometrika*, 35(3):283–319, 1970.

[3] R. A. HARSHMAN. Foundations of the PARAFAC procedure: Models and conditions for an *UCLA*
Working Papers in Phonetics, 16:1–84, 1970.

[4] Casey Battaglino, Grey Ballard, and Tamara G. Kolda. A Practical Randomized CP Tensor Decomposition. *SIAM Journal on Matrix Analysis and Applications*, 39(2):876–901, 2018.

[5] Petros Drineas, Michael W. Mahoney, S. Muthukrishnan, and Tamás Sarlós. Faster least squares approximation. *Numerische Mathematik*, 117(2):219–249, 2011.

[6] X. Fu, S. Ibrahim, H. Wai, C. Gao, and K. Huang. Block-Randomized Stochastic Proximal Gradient for Low-Rank Tensor Factorization. *IEEE Transactions on Signal Processing*, 68:2170–2185, 2020.

[7] Brett W. Larsen and Tamara G. Kolda. Practical Leverage-Based Sampling for Low-Rank Tensor Decomposition. *arXiv:2006.16438*, 2020.

[8] Dehua Cheng, Richard Peng, Joakim Perros, and Yan Liu. SPALS: Fast alternating least squares via implicit leverage scores sampling. In *Proceedings of the 30th International Conference on Neural Information Processing Systems*, NIPS’16, pages 721–729, Red Hook, NY, USA, 2016. Curran Associates Inc.

[9] N. Vervliet, O. Debals, L. Sorber, and L. De Lathauwer. Breaking the Curse of Dimensionality Using Decompositions of Incomplete Tensors: Tensor-based scientific computing in big data analysis. *IEEE Signal Processing Magazine*, 31(5):71–79, 2014.

[10] Srinadh Bhojanapalli and Sujay Sanghavi. A New Sampling Technique for Tensors. *arXiv:1502.05023*, 2015.

[11] X. T. Vu, S. Maire, C. Chaux, and N. Thirion-Moreau. A New Stochastic Optimization Algorithm to Decompose Large Nonnegative Tensors. *IEEE Signal Processing Letters*, 22(10):1713–1717, 2015.

[12] Alex Beutel, Partha Pratim Talukdar, Abhimanyu Kumar, Christos Faloutsos, Evangelos E. Papalexakis, and Eric P. Xing. FlexiFaCT: Scalable Flexible Factorization of Coupled Tensors on Hadoop. In *Proceedings of the 2014 SIAM International Conference on Data Mining*, Proceedings, pages 109–117. Society for Industrial and Applied Mathematics, 2014.

[13] N. Vervliet and L. De Lathauwer. A Randomized Block Sampling Approach to Canonical Polyadic Decomposition of Large-Scale Tensors. *IEEE Journal of Selected Topics in Signal Processing*, 10(2):284–295, 2016.

[14] Deanna Needell, Nathan Srebro, and Rachel Ward. Stochastic gradient descent, weighted sampling, and the randomized Kaczmarz algorithm. *Mathematical Programming*, 155(1):549–573, 2016.

[15] Petros Drineas, Malik Magdon-Ismail, Michael W. Mahoney, and David P. Woodruff. Fast approximation of matrix coherence and statistical leverage. *The Journal of Machine Learning Research*, 13(1):3475–3506, 2012.

[16] David P. Woodruff. Sketching as a Tool for Numerical Linear Algebra. *Foundations and Trends® in Theoretical Computer Science*, 10(1–2):1–157, 2014.

[17] Peng Xu, Jiyan Yang, Farbod Roosta-Khorasani, Christopher Ré, and Michael W. Mahoney. Sub-sampled Newton Methods with Non-uniform Sampling. *arXiv:1607.00559*, 2016.

[18] Herbert Robbins and Sutton Monro. A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400–407, 1951.

[19] Tamara G. Kolda and Brett W. Bader. MATLAB Tensor Toolbox. Technical Report TensorToolbox; 001963MLTPL00, Sandia National Laboratories, 2006.