Cooperative Global Robust Stabilization for a Class of Nonlinear Multi-Agent Systems and its Application *

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Abstract
This paper studies the cooperative global robust stabilization problem for a class of nonlinear multi-agent systems. The problem is motivated from the study of the cooperative global robust output regulation problem for the class of nonlinear multi-agent systems in normal form with unity relative degree which was studied recently under the conditions that the switching network is undirected and some nonlinear functions satisfy certain growth condition. We first solve the stabilization problem by using the multiple Lyapunov functions approach and the average dwell time method. Then, we apply this result to the cooperative global robust output regulation problem for the class of nonlinear systems in normal form with unity relative degree under directed switching network, and have removed the conditions that the switching network is undirected and some nonlinear functions satisfy certain growth condition.

Key words: Cooperative control, global robust stabilization, multi-agent systems, nonlinear systems, switched control.

1 Introduction
Consider a class of cascade-connected multi-agent nonlinear systems as follows:

\[ \dot{Z}_i = F_i(Z_i, e_i, d(t)) \]
\[ \dot{e}_i = G_i(Z_i, e_i, d(t)) + b_i u_i, \quad i = 1, \cdots, N \]  \tag{1}

where \((Z_i, e_i) \in \mathbb{R}^{n_i} \times \mathbb{R}\) is the state, \(u_i \in \mathbb{R}\) is the input, \(e_i\) is the output, and \(d : [0, \infty) \rightarrow \mathbb{D} \subset \mathbb{R}^n\) with \(\mathbb{D}\) a known compact subset represents external disturbance and/or parameter variations. It is assumed that \(b_m \leq |b_i| \leq b_M\) for some known positive real numbers \(b_m\) and \(b_M\), and the functions \(F_i : \mathbb{R}^{n_i} \times \mathbb{R} \rightarrow \mathbb{R}^{n_i}\), and \(G_i : \mathbb{R}^{n_i} \times \mathbb{R} \rightarrow \mathbb{R}\) are both sufficiently smooth and satisfy \(F_i(0, 0, d(t)) = 0\) and \(G_i(0, 0, d(t)) = 0\) for all \(d(t) \in \mathbb{R}^n\).

To describe our control law, let \(\sigma : [0, \infty) \rightarrow \mathcal{P} = \{1, 2, \cdots, n_0\}\) for some integer \(n_0 > 0\), be a piecewise constant switching signal, and \(S_{\text{ave}}[\tau_d, N_0]\) be the set of all signals possessing the property of average dwell-time \(\tau_d\) with chatter bound \(N_0\) \([8,9]\), \(H_p = [h_{ij}] \in \mathbb{R}^{N \times N}\), \(p = 1, \cdots, n_0\), be some \(\mathcal{M}\) matrices\footnote{A matrix \(M \in \mathbb{R}^{N \times N}\) is called an \(\mathcal{M}\) matrix if all of its non-diagonal elements are non-positive and all of its eigenvalues have positive real parts.}. Define a piecewise switching matrix \(H_{\sigma(t)} \in \mathbb{R}^{N \times N}\) such that, over each interval \([t_i, t_{i+1})\), \(H_{\sigma(t)} = H_p\) for some integer \(1 \leq p \leq n_0\). Denote the elements of \(H_{\sigma(t)}\) by \(h_{ij}(t)\), \(i, j = 1, \cdots, N\), and define the virtual output of (1) as

\[ e_{\sigma(t)i} = \sum_{j=1}^{N} h_{ij}(t) e_j, \quad i = 1, \cdots, N. \]  \tag{2}

Then, we describe our control law as follows:

\[ \bar{u}_i = k_i(e_{\sigma(t)i}), \quad i = 1, \cdots, N \]  \tag{3}

where the functions \(k_i(\cdot), \quad i = 1, \cdots, N\), are sufficiently smooth vanishing at the origin. Such a control law is called a distributed switched output feedback control law, since we can only use \(e_{\sigma(t)i}\) instead of \(e_i\) for feedback control due to the communication constraints, which will be further elaborated in Section 4.
Problem 1 Given the multi-agent system (1), a set of $\mathcal{M}$ matrices $H_p \in \mathbb{R}^{N \times N}$, $p = 1, \ldots, n_0$, and some compact subset $\mathbb{D} \subset \mathbb{R}^{n_d}$ with $0 \in \mathbb{D}$, find $\tau_d, N_0$, and a control law of the form (3) such that, for any $d(t) \in \mathbb{D}$, and any $\alpha \in \mathcal{SA}_\infty[\tau_d, N_0]$, the equilibrium point of the closed loop system composed of (1) and (3) at the origin is globally asymptotically stable.

The above cooperative global robust stabilization problem is of interest on its own and has not been studied before, since (3) is a switched control law which results in a switched closed loop system. On the other hand, it is motivated from the study of the cooperative global robust output regulation problem for a class of nonlinear multi-agent systems with switching network. In fact, we will show that our main result is applicable to the cooperative global robust output regulation problem for the unity relative degree nonlinear multi-agent systems with switching network in Section 4. It is also noted that the result in 4 includes some existing results as special cases. If $n_0 = 1$, then $H_1$ reduces to a constant matrix. For this special case, the above problem has been studied in [2] under the assumption that $H_1$ is a symmetric $\mathcal{M}$ matrix. More recently, for the case where $n_0$ is any positive integer, the above problem was studied in [9] under the assumption that $H_p, p = 1, \ldots, n_0$, are all symmetric $\mathcal{M}$ matrices and the nonlinear function $G_i$ satisfy certain growth condition. In this paper, we will further study the above problem for the more general case where the nonlinear function $G_i$ does not satisfy any growth condition, and $H_p, p = 1, \ldots, n_0$, are any $\mathcal{M}$ matrices, i.e., do not need to be symmetric. As an application of our main result, we will obtain the solution of the same problem studied in [9] without assuming that the nonlinear functions $G_i$ satisfy any growth condition, and the communication graph of the multi-agent system is undirected for all $t \geq 0$.

Without assuming the growth condition, and the undirectedness of the communication graph, the problem in this paper is technically much more challenging than the one in [9]. To overcome these difficulties, we need to develop a changing supply pair technique for exponentially input-to-state stable (exp-ISS) nonlinear systems to remove the growth condition of the nonlinear function $G_i$, and we need to employ a non-quadratic function for the closed-loop system in Section 3 to handle the directed communication graph.

Notation. For any column vectors $a_i, i = 1, \ldots, s$, denote $\text{col}(a_1, \ldots, a_s) = [a_1^T, \ldots, a_s^T]^T$. A function $\bar{\rho}(\cdot)$ is called $SN$ function if it is a smooth non-decreasing function $\bar{\rho} : [0, \infty) \to [0, \infty)$ satisfying $\bar{\rho}(s) > 0$ for all $s > 0$. $\bar{\alpha}(s) = O(\alpha(s))$ as $s \to 0^+$ means $\lim \sup_{s \to 0^+} \frac{\bar{\alpha}(s)}{\alpha(s)} < \infty$. The notation $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a symmetric real matrix $A$.

2 See Assumption 8 for the definition of growth condition.

2 Preliminaries

In this section, we will establish a technical lemma. Consider a general nonlinear system:

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and $f(0, 0) = 0$. It is known from [11] that system (4) is said to be input-to-state stable (ISS) if there exists a $C^1$ function $V : \mathbb{R}^n \to [0, \infty)$ such that, for all $x, u$,

$$\alpha_1(\parallel x \parallel) \leq V(x) \leq \alpha_2(\parallel x \parallel)$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\parallel x \parallel) + \gamma(\parallel u \parallel)$$

for some class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$, and some class $\mathcal{K}$ function $\gamma(\cdot)$. The pair of functions $(\gamma, \alpha)$ is called a supply pair for system (4), and $V(x)$ is called an ISS Lyapunov function of (4). Moreover, if system (4) is ISS, then for any class $\mathcal{K}_\infty$ function $\bar{\alpha}(\cdot)$ satisfying $\bar{\alpha}(s) = O(\alpha(s))$ as $s \to 0^+$, there exists $C^1$ function $V(x)$ such that, for all $x, u$,

$$\dot{\bar{\alpha}_1}(\parallel x \parallel) \leq \dot{V}(x) \leq \bar{\alpha}_2(\parallel x \parallel)$$

$$\frac{\partial \bar{V}}{\partial x} f(x, u) \leq -\bar{\alpha}(\parallel x \parallel) + \bar{\gamma}(\parallel u \parallel)$$

for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1(\cdot), \bar{\alpha}_2(\cdot)$ and some class $\mathcal{K}$ function $\bar{\gamma}(\cdot)$ [11]. This result is called the changing supply pair technique which plays a key role in finding a suitable Lyapunov function for a nonlinear system to conclude the asymptotic stability of its origin. However, as will be explained in Section III, this version of the changing supply pair technique is not adequate for handling the stability of switched systems. We need to further establish a lemma for the following class of nonlinear systems:

$$\dot{x} = f(x, u, d(t))$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $d : [t_0, \infty) \to \mathbb{D} \subset \mathbb{R}^{n_d}$ with $\mathbb{D}$ some non-empty set, represents external unpredictable disturbance and/or internal parameter variation, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{D} \to \mathbb{R}^n$ is piecewise continuous in $d(t)$ and locally Lipschitz, $d(t)$ is piecewise continuous in $t$ and $f(0, 0, d(t)) = 0$ for any $d(t) \in \mathbb{R}^{n_d}$.

Lemma 2.1 Suppose that there exists a $C^1$ function $V(x, t)$ such that, for any $d(t) \in \mathbb{D}$,

$$\alpha_1(\parallel x \parallel) \leq V(x, t) \leq \alpha_2(\parallel x \parallel), \ \forall \ x$$

$$\dot{V}(x, t) \leq -\lambda V(x, t) + \beta(u), \ \forall \ x, u$$
where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are some known class $K_{\infty}$ functions, $\lambda$ is some known positive real number and $\beta(\cdot)$ is some known smooth positive definite function. Then,

(i) For any function $\bar{\rho} \in SN$, the following $C^1$ function

$$\dot{V}(x, t) = \int_0^{\rho(x,t)} \bar{\rho}(s) ds, \quad (10)$$

is an ISS Lyapunov function in the sense that for any positive real number $0 < \lambda < \frac{\lambda}{2}$ and any real number $c = \frac{\lambda}{2} - \lambda$,

$$\dot{V}(x, t) \leq -\lambda V(x, t) + \bar{\rho}(\beta(u)), \quad \forall x, u, \forall d(t) \in D \quad (12)$$

for some class $K_{\infty}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, some smooth positive definite function $\beta(\cdot)$.

(ii) For any class $K_{\infty}$ function $\bar{\alpha}(\cdot)$ satisfying $\bar{\alpha}(s) = O(\alpha_1(s))$ as $s \to 0^+$, there exists some function $\bar{\rho} \in SN$ such that, for any $d(t) \in D$, $\dot{V}(x, t)$ satisfies (11), and the following

$$\dot{V}(x, t) \leq -\lambda V(x, t) + \alpha_\bar{\alpha}(\beta(u)), \quad \forall x, u \quad (13)$$

for any positive real number $0 < \lambda < \frac{\lambda}{2}$ and positive real number $c = \frac{\lambda}{2} - \lambda$.

Proof: Part (i): It is easy to see that

$$\dot{V}(x, t) \leq \bar{\rho}(V(x, t))V(x, t), \quad (14)$$

and equations (8) and (10) imply that

$$\bar{\alpha}_1(\|x\|) \leq \int_0^{\alpha_1(\|x\|)} \bar{\rho}(s) ds \leq \bar{\alpha}_2(\|x\|), \quad \forall x \quad (15)$$

for some class $K_{\infty}$ functions $\bar{\alpha}_1(\cdot)$ and $\bar{\alpha}_2(\cdot)$.

We now show that along the trajectory of $\dot{x} = f(x, u, d(t))$

$$\dot{V}(x, t) \leq \bar{\rho}(V(x, t))( -\lambda V(x, t) + \beta(u) )$$

$$\leq -\frac{\lambda}{2} \bar{\rho}(V(x, t))V(x, t) + \bar{\rho}(\frac{2}{\lambda} \beta(u)) \beta(u). \quad (16)$$

For this purpose, consider the following two cases.

1. $\frac{\lambda}{2} V(x, t) \geq \beta(u)$: In this case,

$$\dot{V}(x, t) \leq \bar{\rho}(V(x, t))( -\lambda V(x, t) + \beta(u) )$$

$$\leq \bar{\rho}(V(x, t))( -\lambda V(x, t) + \bar{\rho}(\frac{2}{\lambda} \beta(u)) \beta(u) )$$

$$= -\frac{\lambda}{2} \bar{\rho}(V(x, t))V(x, t) + \bar{\rho}(\frac{2}{\lambda} \beta(u)) \beta(u).$$

Choose a smooth positive definite function $\bar{\beta}(\cdot)$ such that

$$\bar{\beta}(u) \geq \bar{\rho}(\frac{2}{\lambda} \beta(u)) \beta(u)$$

From (14), (16), and (17), we have

$$\dot{V}(x, t) \leq -\frac{\lambda}{2} \bar{\rho}(V(x, t))V(x, t) + \bar{\beta}(u)$$

$$\leq -\frac{\lambda}{2} V(x, t) + \bar{\beta}(u). \quad (18)$$

Part (ii): Since $\bar{\alpha}(s) = O(\alpha_1(s))$ as $s \to 0^+$, by Lemma 2 of [11], it is always possible to find a function $\bar{\rho} \in SN$ such that, for all $x$,

$$\bar{\rho}(\alpha_1(\|x\|) \|x\|) \leq \bar{\alpha}(\|x\|). \quad (19)$$

Thus

$$\bar{\rho}(V(x, t))V(x, t) \geq \bar{\rho}(\alpha_1(\|x\|) \|x\|) \geq \bar{\alpha}(\|x\|).$$

Choose a real number $\bar{\lambda}$ satisfying $0 < \bar{\lambda} < \frac{\lambda}{2}$ and let

$$c = \frac{\lambda}{2} - \bar{\lambda}. \quad \text{Then, from (14), (18) and (20), we have}$$

$$\dot{V}(x, t) \leq -\frac{\lambda}{2} \bar{\rho}(V(x, t))V(x, t) + \bar{\beta}(u)$$

$$= (\bar{\lambda} + c) \bar{\rho}(V(x, t))V(x, t) + \bar{\beta}(u)$$

$$\leq - (\bar{\lambda} V(x, t) + c\alpha(\|x\|)) + \bar{\beta}(u).$$
Thus the proof is completed. □

Remark 2.1 Lemma 2.1 and its proof can be viewed as an extension of the main result in [11]. If we let \( \tilde{\alpha}(\cdot) \) be any smooth positive definite function \( \tilde{\alpha}(\cdot) \) in Lemma 2.1, then there exists a class \( K_\infty \) function \( \tilde{\alpha}(\cdot) \) satisfying \( \tilde{\alpha}(s) = O(s^2) \) as \( s \to 0^+ \), and \( \tilde{\alpha}([x]) \geq \tilde{\alpha}(x) \) for any \( x \in \mathbb{R}^n \). Thus, if \( \alpha_1(\cdot) \) satisfies \( \lim_{s \to 0^+} \sup \frac{s^2}{\alpha_1(s)} < \infty \), then \( \tilde{\alpha}(s) = O(\alpha_1(s)) \) as \( s \to 0^+ \). Then we conclude that if \( \alpha_1(\cdot) \) satisfies \( \lim_{s \to 0^+} \sup \frac{s^2}{\alpha_1(s)} < \infty \), then, for any smooth positive definite function \( \tilde{\alpha}(\cdot) \), by Part (ii) of Lemma 2.1, we have

\[
\dot{V}(x, t) \leq -(\lambda V(x, t) + \tilde{\alpha}(\|x\|)) + \tilde{\beta}(u), \quad \forall x, u.
\]  

(21)

Remark 2.2 From [10], a system of the form (7) that admits a \( C^1 \) function \( V \) satisfying the inequalities (8) and (9) is called exp-ISS, and the function \( V \) is called an exp-ISS Lyapunov function of (7). Moreover, by Proposition 8 of [10] or Theorem 3 of [12], system (4) is ISS if and only if it is exp-ISS. In this paper, we further call (7) strong exp-ISS if it admits a \( C^1 \) function \( \tilde{V} \) satisfying the inequalities (11) and (13), and call \( \tilde{V} \) a strong exp-ISS Lyapunov function of (7). Thus, Lemma 2.1 shows that exp-ISS is equivalent to strong exp-ISS. It will be seen in the proof of Theorem 3.1 that Lemma 2.1 plays the key role to eliminate the growth condition of the nonlinear functions \( G_i(\bar{Z}_i, e_i, d(t)) \).

3 Main Result

In this section, by combining Lemma 2.1 with the multiple Lyapunov functions and average dwell time method, we will design a distributed switched output feedback control law to solve the cooperative global robust stabilization problem for system (1).

For convenience, let \( Z = \text{col}(Z_1, \cdots, Z_N) \), \( e = \text{col}(e_1, \cdots, e_N) \), \( e_{\sigma(t)} = \text{col}(e_{\sigma(t)}, \cdots, e_{\sigma(t)}) \). From (2), we have \( e_{\sigma(t)} = H_{\sigma(t)}e \) for all \( t \geq 0 \). Then we can put the closed loop system composed of (1) and (3) into the following form

\[
\dot{Z} = F(Z, e, d(t))
\]

(22)

\[\dot{e} = G_{\sigma(t)}(Z, e, d(t))\]

where \( F(Z, e, d(t)) = \text{col}(F_1(Z_1, e_1, d(t)), \cdots, F_N(Z_N, e_N, d(t))) \), \( G_{\sigma(t)}(Z, e, d(t)) = \text{col}(G_1(Z_1, e_1, d(t)) + b_{1k_1}(e_{\sigma(t)}), \cdots, G_N(Z_N, e_N, d(t)) + b_{Nk_N}(e_{\sigma(t)})) \). Let \( f_{\sigma(t)}(x_c, d(t)) = \text{col}(F(Z, e, d(t)), G_{\sigma(t)}(Z, e, d(t))) \) and \( x_c = \text{col}(Z, e) \). Then we can further put system (22) into the following compact form

\[
x_c = f_{\sigma(t)}(x_c, d(t)).
\]  

(23)

Since (3) is a switched control law, the closed loop system (23) is a switched nonlinear system. To analyze the stability of system (23), we will resort to the multiple Lyapunov functions and average dwell time method [3,8]. From Theorem 4 of [3], if the closed loop system (23) admits multiple \( C^1 \) Lyapunov functions \( U_p(x_c), p \in P \), satisfying

\[
\dot{\alpha}_1(\|x_c\|) \leq U_p(x_c) \leq \dot{\alpha}_2(\|x_c\|), \quad \forall x_c, \forall p \in P
\]  

(24)

\[\frac{\partial U_p}{\partial x_c} f_{\sigma(t)}(x_c, d(t)) = -\lambda_0 U_p(x_c), \quad \forall x_c, \forall p \in P
\]  

(25)

for some class \( K_\infty \) functions \( \dot{\alpha}_1(\cdot) \) and \( \dot{\alpha}_2(\cdot) \), and some positive numbers \( \lambda_0 \), then the origin of (23) is globally asymptotically stable for every \( \sigma \in S_{\text{ave}}[\tau_d, N_0] \) with \( \tau_d > \frac{\ln \mu_0}{\chi_0} \) and arbitrary \( N_0 \), where \( \mu_0 = \sup_{x_c \neq 0} \frac{\dot{\alpha}_2(\|x_c\|)}{\dot{\alpha}_1(\|x_c\|)} \).

Assumption 1 For a given compact subset \( D \subset \mathbb{R}^{n_d} \), the subsystem \( \bar{Z}(t) = F_i(Z_i, e_i, d(t)) \) admits a \( C^1 \) exp-ISS Lyapunov function \( V_i(Z_i) \) such that, for any \( d(t) \in D \),

\[
\dot{\alpha}_1(\|Z_i\|) \leq V_i(Z_i) \leq \dot{\alpha}_2(\|Z_i\|), \quad \forall Z_i
\]  

(26)

\[
\dot{V}_i(Z_i) \leq -\lambda_2 V_i(Z_i) + \pi_i(e_i), \quad \forall Z_i, e_i
\]  

(27)

where \( \dot{\alpha}_1(\cdot) \) and \( \dot{\alpha}_2(\cdot) \) are some class \( K_\infty \) functions with \( \dot{\alpha}_1(\cdot) \) satisfying \( \lim_{s \to 0^+} \sup \frac{s^2}{\dot{\alpha}_1(s)} < \infty \), \( \lambda_2 \) is some positive real number, and \( \pi_i(\cdot) \) is some smooth positive definite function.

Remark 3.1 Assumption 1 implies that the subsystem \( \bar{Z}_i = F_i(Z_i, e_i, d(t)) \) is exponentially input-to-state stable with \( e_i \) as the input and \( V_i(Z_i) \) is an exp-ISS Lyapunov function of the subsystem \( \bar{Z}_i = F_i(Z_i, e_i, d(t)) \).

Now we describe our main result as follows.

Theorem 3.1 Under Assumption 1, for every \( \sigma \in S_{\text{ave}}[\tau_d, N_0] \) with \( \tau_d > \frac{\ln \mu_0}{\chi_0} \) and arbitrary \( N_0 \), there exists a distributed switched output feedback control law of the form

\[
\bar{u}_i = -p_i(e_{\sigma(t)}e_{\sigma(t)})e_{\sigma(t)}, \quad i = 1, \cdots, N
\]  

(28)

where \( p_i(\cdot), i = 1, \cdots, N \), are some sufficiently smooth positive functions, \( \lambda_0 \) and \( \mu_0 \) are some positive real numbers, that solves the cooperative global robust stabilization problem of system (1).

Proof: First note that, since \( G_i(Z_i, e_i, d(t)) \) is smooth and \( G_i(0, 0, d(t)) = 0 \) for all \( d(t) \in \mathbb{R}^{n_d} \), by Lemma 7.8 of [6], there exist some smooth positive definite functions \( \gamma_i(Z_i), \chi_i(e_i) \), such that, for all \( Z_i \in \mathbb{R}^n, e_i \in \mathbb{R} \) and \( d(t) \in D \),

\[
|G_i(Z_i, e_i, d(t))|^2 \leq \gamma_i(Z_i) + \chi_i(e_i).
\]  

(29)
Without loss of generality, assume $b_m \leq b_1 \leq b_M$. By Lemma 2.1 and Remark 2.1, under Assumption 1, the subsystem $\dot{Z}_i = F_i(Z_i, e_i, d(t))$ admits a $C^1$ strong explicit Lyapunov function $\bar{V}_i(Z_i)$ such that, for any $d(t) \in \mathbb{D}$, \[
abla_i \leq \bar{V}_i(Z_i) \leq \tilde{\alpha}_2 i(Z_i), \quad \forall \ Z_i (30)
\]

For each $p \in P$, let $e_{vp} = H_pe$ and $G(Z, e, d(t)) = \text{col}(G_1(Z_1, e_1, d(t)), \ldots, G_N(Z_N, e_N, d(t)))$. Then, from (1), we have \[
\begin{align*}
\dot{e}_{vp} &= H_p \dot{e} = H_p G(Z, e, d(t)) + H_p B \dot{\bar{u}} \\
&= G_p(Z, e_{vp}, d(t)) + H_p B \tilde{u}
\end{align*}
(32)
\]

where $B = \text{diag}(b_1, \ldots, b_N)$, $\tilde{u} = \text{col}(u_1, \ldots, \tilde{u}_N)$, and $G_p(Z, e_{vp}, d(t)) = \text{col}(G_{p1}(Z, e_{vp}, d(t)), \ldots, G_{pN}(Z, e_{vp}, d(t))) = H_p G(Z, H_p^{-1} e_{vp}, d(t))$. Since $H_p = [h_{ij}^p]$ is a constant $M$ matrix, then by Lemma 2.5.3 of [4], there exists a positive definite diagonal matrix $D_p = \text{diag}(d_{p1}, \ldots, d_{pN})$ such that $D_p H_p + H_p^T D_p$ is positive definite. Thus $B(D_p H_p + H_p^T D_p)B$ is also a positive definite matrix. Define $d_M = \max_{p \in P} \{ \max_{i = 1, \ldots, N} d_{pi} \}$, $d_m = \min_{p \in P} \{ \min_{i = 1, \ldots, N} d_{pi} \}$ and $\hat{\lambda}_i = \min_{p \in P} \{ \lambda_{\min}(B(D_p H_p + H_p^T D_p)) \}$. Let $\rho_i(e_{vp}) = k_i \omega_i(e_{vp})$, where $k_i$ is a positive real number and $\omega_i(\cdot) \geq 1$, $i = 1, 2, \ldots, N$, are some smooth non-decreasing functions to be determined later. Let \[
V_{vp}(e) = \sum_{i=1}^N d_{pi} b_i \int_0^{e_{vp}} \omega_i(s) ds.
(33)
\]

Then, it can be seen that $V_{vp}(e)$ is positive definite and radially unbounded. Thus, there exist some class $\mathcal{K}_\infty$ functions $\hat{\beta}_{1p}(\cdot)$ and $\hat{\beta}_{2p}(\cdot)$ such that \[
\hat{\beta}_{1p}(\|e\|) \leq V_{vp}(e) \leq \hat{\beta}_{2p}(\|e\|), \quad \forall e, \forall p \in P.
(34)
\]

Choose two class $\mathcal{K}_\infty$ functions $\hat{\beta}_1(\cdot)$ and $\hat{\beta}_2(\cdot)$ such that, for all $p \in P$ and all $e \in \mathbb{R}^N$, $\hat{\beta}_1(\|e\|) \leq \hat{\beta}_{1p}(\|e\|)$ and $\hat{\beta}_2(\|e\|) \geq \hat{\beta}_{2p}(\|e\|)$. Then \[
\hat{\beta}_1(\|e\|) \leq V_{vp}(e) \leq \hat{\beta}_2(\|e\|), \quad \forall \ e, \forall \ p \in \mathcal{P}.
(35)
\]

Let $e_{vp}^* = \text{col}(\omega_1(e_{vp}^*)^2, e_{vp1}^*, \ldots, \omega_N(e_{vpN}^*)^2)$. Then, by (28), (29), and (32), for any $\varepsilon > 0$ and any $d(t) \in \mathbb{D}$, the time derivative of $V_{vp}(e)$ is given by \[
\begin{align*}
\frac{\partial V_{vp}}{\partial e}(G(Z, e, d(t)) + B \bar{u}) &= \frac{\partial V_{vp}}{\partial e}(\hat{G}_p(Z, e_{vp}, d(t)) + H_p B \tilde{u}) \\
&= 2 \sum_{i=1}^N d_{pi} b_i \omega_i(e_{vp}) e_{vp}^* \left( \hat{\bar{G}}_p(Z, e_{vp}, d(t)) - \sum_{j=1}^N h_{ij}^p b_j \omega_j(e_{vp}) e_{vp}^* \right) \\
&\leq \sum_{i=1}^N \sum_{j=1}^N d_{pi}^2 b_i^2 \omega_i^2(e_{vp}) e_{vp}^2 + \frac{1}{\varepsilon^2} \sum_{i=1}^N |G_{pi}(Z, e, d(t))|^2 \\
&\quad - k(e_{vp})^T B(D_p H_p + H_p^T D_p) B e_{vp}^* \\
&\leq \sum_{i=1}^N \sum_{j=1}^N d_{pi}^2 b_i^2 \omega_i^2(e_{vp}) e_{vp}^2 + \frac{1}{\varepsilon^2} \sum_{i=1}^N |G_{pi}(Z, e, d(t))|^2 \\
&\quad - \sum_{i=1}^N (k \hat{\lambda}_i - \varepsilon^2 d_M^2 b_i^2 \omega_i^2(e_{vp}) e_{vp}^2) \\
&\quad + \varepsilon^2 \sum_{i=1}^N \sum_{j=1}^N \gamma_i(Z_i) + \chi_i(e_i))
(36)
\]

Let $V(Z) = \sum_{i=1}^N \bar{V}_i(Z_i)$. By inequality (31) we have \[
\dot{V}(Z) \leq - \sum_{i=1}^N \left( \lambda_0 \bar{V}_i(Z_i) + c_0 \gamma_i(Z_i) \right) + \sum_{i=1}^N \bar{\pi}_i(e_i).
(37)
\]

Finally, let $U_p(x_c) = \dot{V}(Z) + V_{vp}(e)$. Clearly, there exist two class $\mathcal{K}_\infty$ functions $\hat{\alpha}_1(\cdot)$ and $\hat{\alpha}_2(\cdot)$ such that the condition (24) is satisfied for all $p \in \mathcal{P}$. Also, according to (36) and (37), we have \[
\begin{align*}
\frac{\partial U_p}{\partial x_c}(x_c, d(t)) &= \frac{\partial U_p}{\partial x_c}(f(X_c, d(t))) \\
&\geq \sum_{i=1}^N \lambda_0 \bar{V}_i(Z_i) - \sum_{i=1}^N (c_0 - \frac{\|H_p\|^2}{\varepsilon^2}) \gamma_i(Z_i) \\
&\quad - \sum_{i=1}^N (k \hat{\lambda}_i - \varepsilon^2 d_M^2 b_i^2 \omega_i^2(e_{vp}) e_{vp}^2 + \hat{\bar{\rho}}(e))
(38)
\]
with $\hat{\bar{\rho}}(e) = \sum_{i=1}^N \left( \frac{d_{pi}^2 b_i^2}{\varepsilon^2} \chi_i(e_i) + \bar{\pi}_i(e_i) \right)$. By lemma 7.8
of [6] again, for each $p \in \mathcal{P}$, there exist some smooth positive functions $\hat{p}_p(e_{vpi}) \geq 1$, $i = 1, \ldots, N$ such that $\hat{p}(e) = \hat{p}(H^{-1}e_{vpi}) \leq \sum_{i=1}^{N} \hat{p}_p(e_{vpi})e_{vpi}^2$. Choose some smooth functions $\hat{p}_i(e_{vpi})$ such that $\hat{p}_i(e_{vpi}) \geq \hat{p}_p(e_{vpi})$ for all $p \in \mathcal{P}$. Then $\hat{p}(e) \leq \sum_{i=1}^{N} \hat{p}_i(e_{vpi})e_{vpi}^2$ for all $p \in \mathcal{P}$. Define a function $\Delta_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\Delta_i(s) = \max_{0 \leq |e_{vpi}| \leq s} \hat{p}_i(e_{vpi})$. Clearly, $\Delta_i(\cdot)$ is non-decreasing and $\Delta_i(|e_{vpi}|) \geq \hat{p}_i(e_{vpi}) \geq 1$ for any $e_{vpi} \in \mathbb{R}$. Then we choose the positive real number $k$ such that

$$k \geq \frac{1}{\lambda_1}(e_2d_M^2b_M^2 + \lambda_0d_Mb_M + 1)$$

and choose $\varepsilon$ such that $c_0 = \frac{\|H_{\varepsilon}^p\|^2}{\varepsilon^2} \geq 0$. Then,

$$\frac{\partial U_p}{\partial x_c}f_{ep}(x_c, d(t)) \leq -\lambda_0 \sum_{i=1}^{N} V_i(Z_i) - \lambda_0 \sum_{i=1}^{N} dMb_M\omega_i^2(e_{vpi})e_{vpi}^2$$

(40)

Finally, choose the smooth non-decreasing function $\omega_i(\cdot)$ such that

$$\omega_i(e_{vpi}) \geq \frac{1 + e_{vpi}^2}{2}.$$ (41)

Then, since $\Delta_i(1 + e_{vpi}^2) \geq \Delta_i(e_{vpi}) \geq \hat{p}_i(e_{vpi}) \geq 1$, we have

$$\omega_i^2(e_{vpi}) \geq \omega_i(e_{vpi}) \geq \Delta_i(\frac{1 + e_{vpi}^2}{2}) \geq \hat{p}_i(e_{vpi})$$ (42)

and thus

$$\frac{\partial U_p}{\partial x_c}f_{ep}(x_c, d(t)) \leq -\lambda_0 \sum_{i=1}^{N} V_i(Z_i) - \lambda_0 \sum_{i=1}^{N} dMb_M\omega_i(e_{vpi})e_{vpi}^2$$

(43)

$$\leq -\lambda_0 \sum_{i=1}^{N} V_i(Z_i) - \lambda_0 \sum_{i=1}^{N} dMb_M\omega_i(e_{vpi})e_{vpi}^2$$

(44)

$$\leq -\lambda_0 \sum_{i=1}^{N} V_i(Z_i) - \lambda_0 V_{ep}(e) = -\lambda_0 U_p, \forall p \in \mathcal{P}.$$ (45)

Let $\mu_0 = \sup_{x, \neq 0} \frac{\Delta_i(\|e_{vpi}\|)}{\omega_i(\|e_{vpi}\|)}$. The proof is thus completed by invoking Theorem 4 of [3] as rephrased at the beginning of this section. \(\square\)

Remark 3.2 From the proof of Theorem 3.1, we know that if we replace the exp-ISS condition (27) in Assumption 1 by the strong exp-ISS condition (31), then the condition $\lim_{t \to 0^+} \sup_{x \neq 0} \frac{\|e(t)\|^2}{\omega_i(\|e(t)\|)} < \infty$ in Assumption 1 can be removed.

Remark 3.3 The recent paper [14] handled the cooperative output regulation problem for nonlinear systems with static directed graph. The Lyapunov function (33) here is similar to that used in Lemma 5 of [14].

Example 1 Consider the following controlled Lorenz multi-agent systems taken from [2]:

$$\dot{z}_{1i} = -L_{1i}z_{1i} + L_{1i}e_i$$

$$\dot{z}_{2i} = L_{2i}z_{2i} + z_{1i}e_i$$

$$\dot{e}_i = L_{3i}z_{1i} - e_i - z_{1i}z_{2i} + b_i\bar{u}_i, \ i = 1, 2, 3$$

where $b_i = 1$, $L_i = \text{col}(L_{1i}, L_{2i}, L_{3i})$ is a constant parameter vector that satisfies $L_{1i} > 0$, $L_{2i} < 0$, and $L_{3i} > 0$. Clearly, system (44) is in the form of (1) with $Z_i = \text{col}(z_{1i}, z_{2i})$. To account for the uncertainty, for $i = 1, 2, 3$, let $L_i = L_i + d_i$, where $L_i = \text{col}(L_{1i}, L_{2i}, L_{3i}) = \text{col}(3, -3, 1, 6)$ denotes the nominal value of $L_i$, and $d_i = \text{col}(d_{1i}, d_{2i}, d_{3i})$ represents the uncertainty of $L_i$. Assume $d = \text{col}(d_1, d_2, d_3) \in D = \{d \in \mathbb{R}^3 | |d_{ji}| \leq 0.2, i, j = 1, 2, 3\}$. Let $H_1 = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ which are both $\mathcal{M}$ matrices. Next, let a Lyapunov function candidate be $V_i(Z_i) = \frac{1}{4}z_{1i}^2 + \frac{1}{4}z_{2i}^2 + \frac{1}{2}z_{2i}^2$. Then it is possible to show that $V_i(Z_i) \leq -2.3z_{1i}^2 - 1.8z_{1i}^4 - 2.5z_{2i}^2 + 5.2e_i^2 + 26.5e_i^4 \leq -4.6V_i(Z_i) + 5.2e_i^2 + 26.5e_i^4$. Thus Assumption 1 is satisfied.

As a result, using the procedure introduced in this section, the cooperative robust stabilization problem for system (44) is solvable by a switched control law:

$$\bar{u}_i = -12(e_{vpi(t)}^i + 1)e_{vpi(t)}, \ i = 1, 2, 3,$$ for any $\sigma(t) \in S_{ave}[\tau_d, N_0]$ with $\tau_d > 2.72$ and arbitrary $N_0$. Simulation is conducted for the following specific switch-

![Fig. 1. State trajectories](image-url)
ing signal

\[
\sigma(t) = \begin{cases} 
1, & \text{if } sT \leq t < (s + \frac{1}{2})T \\
2, & \text{if } (s + \frac{1}{2})T \leq t < (s + 1)T 
\end{cases}
\] (45)

where \( s = 0, 1, 2, \ldots, \) and \( T = 2\tau_d = 6 \text{ sec} \). Other data for the simulation are \( d_i = [-0.2, 0.1, -0.2]^T \) for \( i = 1, 2, 3 \), \( Z_1(0) = [2.6, -0.7]^T \), \( Z_2(0) = [-2.8, 0.9]^T \), \( Z_3(0) = [-1.4, 1.8]^T \) and \( e(0) = [0.3, 0.2, -0.1]^T \). Figure 1 shows the state trajectories of the closed-loop system. Due to the space limit, the details for designing the controller are omitted.

4 An Application

In the past few years, the cooperative control problems for nonlinear multi-agent systems have been extensively studied for the static network case in [1,2,7,13,14,15], and for the switching network case in [9,16]. Note that the nonlinear systems considered in [16] contain no disturbance and uncertainty, and the nonlinear systems considered in [9] need to satisfy certain growth condition. Also, the switching network is assumed to be undirected in [9]. In this section, we will apply our main result to the cooperative global robust output regulation problem for a class of nonlinear multi-agent systems with switching network. The systems studied here contain both disturbance and parameter uncertainty, and do not need to satisfy the growth condition, moreover, the switching network is relaxed to be directed.

Consider the nonlinear multi-agent systems in normal form with unity relative degree as follows, which is the same as those studied in [2,9],

\[
\begin{align*}
\dot{z}_i &= f_i(z_i, y_i, v, w) \\
\dot{y}_i &= b_i(w)u_i + g_i(z_i, y_i, v, w) \\
e_i &= y_i - q(v, w), \quad i = 1, \ldots, N
\end{align*}
\] (46)

where, for \( i = 1, \ldots, N \), \( (z_i, y_i) \in \mathbb{R}^{n_z} \times \mathbb{R} \) is the state, \( u_i \in \mathbb{R} \) is the input, \( e_i \in \mathbb{R} \) is the error output, \( w \in \mathbb{R}^{n_w} \) is an uncertain parameter vector, and \( v(t) \in \mathbb{R}^{n_v} \) is an exogenous signal representing both reference input and disturbance. It is assumed that \( v(t) \) is generated by a linear system of the following form

\[
\dot{v} = Sv, \quad y_0 = q(v, w)
\] (47)

and all functions in (46) and (47) are globally defined, sufficiently smooth, and satisfy \( f_i(0, 0, 0, w) = 0 \), \( g_i(0, 0, 0, w) = 0 \) and \( q(0, w) = 0 \) for all \( w \in \mathbb{R}^{n_w} \).

As in [9], the plant (46) and the exosystem (47) together can be viewed as a multi-agent system of \( (N + 1) \) agents with (47) as the leader and the \( N \) subsystems of (46) as \( N \) followers. Given the plant (46), the exosystem (47), and a switching signal \( \sigma(t) \), we can define a time-varying digraph \( \mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t(t)) \), where \( \mathcal{V} = \{0, 1, \ldots, N\} \) with 0 associated with the leader system and with \( \bar{i} = 1, \ldots, N \) associated with the \( N \) followers, respectively, and \( \mathcal{E}_t(t) \subseteq \mathcal{V} \times \mathcal{V} \) for all \( t \geq 0 \). For all \( t \geq 0 \), each \( j = 0, 1, \ldots, N \), \( i = 1, \ldots, N, i \neq j \), \( (j, i) \in \mathcal{E}_t(t) \) if and only if the control \( u_i(t) \) can make use of \( y_j(t) - y_i(t) \) for feedback control. Let \( \mathcal{G}_t(t) = (\mathcal{V}, \mathcal{E}_t(t)) \) where \( \mathcal{V} = \{1, \ldots, N\} \), \( \mathcal{E}_t(t) \subseteq \mathcal{V} \times \mathcal{V} \) is obtained from \( \mathcal{E}_t(t) \) by removing all edges between the node 0 and the nodes in \( \mathcal{V} \) for all \( t \geq 0 \).

Let \( \bar{A}(t) = [\bar{a}_{ij}(t)] \in \mathbb{R}^{(N+1)\times(N+1)} \) be the adjacency matrix of the digraph \( \mathcal{G}_t(t) \), where \( \bar{a}_{ij}(t) = 0 \) and \( \bar{a}_{ij}(t) = 1 \iff (j, i) \in \mathcal{E}_t(t) \), \( i, j = 0, 1, \ldots, N \). Define the virtual regulated output as follows:

\[
e_{v\sigma(t)i} = \sum_{j=0}^{N} \bar{a}_{ij}(t)(y_i - y_j).
\] (48)

Let \( e_{v\sigma(t)i} = \col(e_{v\sigma(t)1}, \ldots, e_{v\sigma(t)N}) \), \( e = \col(e_1, \ldots, e_N) \), \( e_0 = 0 \) and \( H_{\sigma(t)} = [h_{ij}(t)]_{N \times N} \) with \( h_{ij}(t) = \sum_{j \geq 0} \bar{a}_{ij}(t) \) and \( h_{ij}(t) = -\bar{a}_{ij}(t) \) for \( i \neq j \). It can be verified that \( e_{v\sigma(t)} = H_{\sigma(t)}e \). Then our control law will be of the following form

\[
u_i = \hat{k}_i(\hat{\eta}_i, e_{v\sigma(t)i}), \quad \hat{\eta}_i = \hat{g}_i(\hat{\eta}_i, e_{v\sigma(t)i})
\] (49)

where the functions \( \hat{k}_i \) and \( \hat{g}_i \) are sufficiently smooth vanishing at the origin. A control law of the form (49) is called a distributed switched output feedback control law since \( e_{v\sigma(t)i} \) is a switching signal and depends on \( (y_i - y_j) \) if only if the node \( j \) is a neighbor of the node \( i \).

Then we describe our problem as follows:

**Problem 2** Given the multi-agent system (46), the exosystem (47), a group of digraphs \( \mathcal{G}_p = (\mathcal{V}, \mathcal{E}_p) \) with \( \mathcal{V} = \{0, 1, \ldots, N\}, p = 1, \ldots, n_0, \) and some compact subsets \( \mathcal{V} \subseteq \mathbb{R}^{n_z} \) and \( \mathcal{W} \subseteq \mathbb{R}^{n_w} \) with \( 0 \in \mathcal{W} \) and \( 0 \in \mathcal{V} \), find \( \tau_d, N_0 \), and a control law of the form (49) such that, for any \( v(t) \in \mathcal{V}, w \in \mathcal{W}, \) and any \( \sigma \in S_{\text{zero}}[\tau_d, N_0] \), the trajectory of the closed-loop system composed of (46) and (49) starting from any initial state \( z_i(0), y_i(0) \) and \( \eta_i(0) \) exists and is bounded for all \( t \geq 0 \), and \( \lim_{t \to \infty} e(t) = 0 \).

The above problem was studied recently in [9] where it was shown that this problem can be converted to the problem studied in Section 3 under the following assumptions:

**Assumption 2** The exosystem is neutrally stable, i.e., all the eigenvalues of \( S \) are semi-simple with zero real parts.
Assumption 3 For $i = 1, \cdots, N$, $|b_i(w)| > 0$ for all $w \in \mathbb{R}^{n_w}$.

Assumption 4 There exist globally defined smooth functions $z_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^{n_z}$ with $z_i(0, w) = 0$ such that
\[
\frac{\partial z_i(v, w)}{\partial v} S_v = f_i(z_i(v, w), q(v, w), v, w) \tag{50}
\]
for all $(v, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, $i = 1, \cdots, N$.

Assumption 5 Let $u_i(v, w) = b_i^{-1}(\frac{\partial g_i(v, w)}{\partial v} - g_i(z_i(v, w), q(v, w), v, w))$, $i = 1, \cdots, N$. Then $u_i(v, w)$ are polynomials in $v$ with coefficients depending on $w$.

Under Assumptions 2 to 5, there exist integers $s_i$, $i = 1, \cdots, N$, such that, for any Hurwitz matrices $M_i \in \mathbb{R}^{s_i \times s_i}$ and any column vector $N_i \in \mathbb{R}^{s_i \times 1}$ with $(M_i, N_i)$ controllable, the following linear dynamic compensator
\[
\dot{\eta}_i = M_i \eta_i + N_i u_i, \quad i = 1, \cdots, N \tag{51}
\]
is an internal model of system (46) [9]. Moreover, there exist some functions $\theta_i : \mathbb{R}^{n_x + n_w} \rightarrow \mathbb{R}^{s_i}$ vanishing at the origin, and some row vectors $\Psi_i \in \mathbb{R}^{s_i \times 1}$ such that the following coordinate and input transformation on the internal model (51) and the plant (46)
\[
\tilde{z}_i = z_i - z_i(v, w), \quad \tilde{\eta}_i = \eta_i - \theta_i(v, w) - N_i b_i^{-1} e_i \tag{52}
\]
gives rise to the so-called augmented system of the plant (46) and the exosystem (47) as follows.
\[
\dot{\tilde{z}}_i = f_i(\tilde{z}_i, e_i, d(t)) \tag{53}
\]
\[
\dot{\tilde{\eta}}_i = M_i \tilde{\eta}_i + N_i b_i^{-1} e_i - N_i b_i^{-1} \tilde{g}_i(\tilde{z}_i, e_i, d(t)) \tag{54}
\]
where $d(t) = (v, w)$, $f_i(z_i, e_i, d(t)) = f_i(\tilde{z}_i + z_i, e_i + q, v, w)$ and $\tilde{g}_i(\tilde{z}_i, e_i, d(t)) = g_i(\tilde{z}_i + z_i, e_i + q, v, w)$. Clearly, $f_i(0, 0, d(t)) = 0$ and $\tilde{g}_i(0, 0, d(t)) = 0$ for any $d(t) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$.

By the internal model principle as can be found from [9] or [2], if a control law of the following form
\[
\tilde{u}_i = k_i(e_{\sigma(t)}, t), \quad i = 1, \cdots, N \tag{54}
\]
solves the cooperative global robust stabilization problem of (53), then the cooperative global robust output regulation of system (46) is solved by the following distributed switched output feedback controller:
\[
u_i = k_i(e_{\sigma(t)}, t) + \Psi_i \eta_i \tag{55}
\]
\[
\dot{\eta}_i = M_i \eta_i + N_i u_i, \quad i = 1, \cdots, N.
\]

It was further shown in [9] that the cooperative global robust stabilization problem of (53) was solvable by a control law of the form (54) under the following three assumptions:

Assumption 6 For the given compact subset $\mathcal{D} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, the subsystem $\tilde{z}_i = f_i(\tilde{z}_i, e_i, d(t))$ admits a $C^1$ function $V_i(\tilde{z}_i)$ such that, for any $d(t) \in \mathcal{D}$,
\[
\alpha_{1i}(\|\tilde{z}_i\|) \leq V_i(\tilde{z}_i) \leq \alpha_{2i}(\|\tilde{z}_i\|), \quad \forall \tilde{z}_i \tag{56}
\]
\[
\tilde{V}_i(\tilde{z}_i) \leq -\lambda_1 V_i(\tilde{z}_i) + \beta_1(e_i), \quad \forall \tilde{z}_i, e_i \tag{57}
\]
for some class $\mathcal{K}_\infty$ functions $\alpha_{1i}(\cdot)$ and $\alpha_{2i}(\cdot)$ with $\alpha_{1i}(\cdot)$ satisfying $\lim_{\|z\| \to \infty} \sup_{t \geq 0} \frac{\beta_1(z)}{\|z\|} < \infty$, some positive real number $\lambda_1$, and some smooth positive definite function $\beta_1(\cdot)$.

Assumption 7 For any $p \in \mathcal{P}$, $\mathcal{G}_p$ is undirected, and the node 0 can reach every other node of the digraph $\mathcal{G}_p$.

To introduce the last assumption, note that, since $\tilde{g}_i(\tilde{z}_i, e_i, d(t))$ is smooth and $\tilde{g}_i(0, 0, d(t)) = 0$ for all $d(t) \in \mathcal{D}$, by Lemma 7.8 of [6], there exist some smooth positive definite functions $\delta_i(\tilde{z}_i)$ such that, for all $\tilde{z}_i \in \mathbb{R}^{n_i}$, $e_i \in \mathbb{R}$ and $d(t) \in \mathcal{D}$,
\[
|\tilde{g}_i(\tilde{z}_i, e_i, d(t))|^2 \leq \delta_i(\tilde{z}_i) + l_i(e_i). \tag{58}
\]

Assumption 8 For some real number $c > 0$, $c \delta_i(\tilde{z}_i) \leq V_i(\tilde{z}_i)$.

Assumption 8 is called a growth condition on the nonlinear function $\tilde{g}_i(\tilde{z}_i, e_i, d(t))$ which is quite restrictive, and Assumption 7 requires the graph $\mathcal{G}_p(t)$ to be undirected for all $t \geq 0$ which may also be restrictive. By making use of Theorem 3.1 of this paper, it is possible to remove Assumption 8 and significantly relax this. For this purpose, let $Z_i = \det(\tilde{z}_i, \eta_i)$, $G_i(\tilde{z}_i, e_i, d(t)) = g_i(\tilde{z}_i, e_i, d(t)) + b_i \Psi_i \eta_i + \Psi_i N_i e_i$ and $F_i(\tilde{z}_i, e_i, d(t)) = \det(f_i(\tilde{z}_i, e_i, d(t)), M_i \tilde{\eta}_i + N_i b_i^{-1} e_i - N_i b_i^{-1} \tilde{g}_i(\tilde{z}_i, e_i, d(t)))$. Then the system (53) can be put in exactly the same form as (1). By Theorem 3.1, Assumption 7 can be relaxed to the following

Assumption 9 For any $p \in \mathcal{P}$, every node $i = 1, \cdots, N$ of the digraph $\mathcal{G}_p$ is reachable from the node 0.

Remark 4.1 Under Assumption 9, by lemma 4 in [5], $H_p$ is an $M$ matrix for any $p \in \mathcal{P}$.

We now further show that, by making use of Lemma 2.1 and Remark 2.1, Assumption 8 can be removed. For this purpose, it suffices to show the following lemma.

Lemma 4.1 Under Assumption 6, the subsystem $\hat{Z}_i = F_i(Z_i, e_i, d(t))$ admits a strong exp-ISS Lyapunov function $V_i(Z_i)$ such that, for any $d(t) \in \mathcal{D}$,
\[
\alpha_{1i}(\|Z_i\|) \leq \hat{V}_i(Z_i) \leq \alpha_{2i}(\|Z_i\|), \quad \forall Z_i \tag{59}
\]
\[ \dot{V}_i(Z_i) \leq -\left( \lambda_0 \dot{V}_i(Z_i) + c_0 \gamma_i(Z_i) \right) + \bar{P}(\epsilon_i), \forall Z_i, \epsilon_i \] (60)

for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1(\cdot)$ and $\bar{\alpha}_2(\cdot)$, some positive real numbers $\lambda_0$ and $c_0$, and some positive definite smooth function $\bar{P}(\cdot)$.

**Proof:** By Lemma 2.1 and Remark 2.1, under Assumption 6, the subsystem $\dot{z}_i = f_1(\tilde{z}_i, \epsilon_i, d(t))$ admits a $C^1$ strong exp-ISS function $\tilde{V}_{\bar{z}_i}(\tilde{z}_i)$ such that, for any $d(t) \in \mathbb{D}$,

\[ \bar{\alpha}_1(\|\tilde{z}_i\|) \leq \tilde{V}_{\bar{z}_i}(\tilde{z}_i) \leq \bar{\alpha}_2(\|\tilde{z}_i\|), \forall \tilde{z}_i \] (61)

\[ \dot{\tilde{V}}_{\bar{z}_i}(\tilde{z}_i) \leq -\left( \bar{\lambda}_1 \tilde{V}_{\bar{z}_i}(\tilde{z}_i) + \bar{c}_1 \delta_i(\tilde{z}_i) \right) + \bar{\beta}_1(\epsilon_i), \forall \tilde{z}_i, \epsilon_i \] (62)

for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1(\cdot)$ and $\bar{\alpha}_2(\cdot)$, some positive real numbers $\bar{\lambda}_1$ and $\bar{c}_1$, and some positive definite smooth function $\bar{\beta}_1(\cdot)$.

Let $V_{\bar{h}_i} = \bar{\eta}_i^T P \bar{h}_i$, $\dot{V}_i(Z_i) = \tilde{V}_{\bar{h}_i} + V_{\bar{h}_i}$, and $\gamma_i(Z_i) = \delta_i(\tilde{z}_i) + \|\bar{h}_i\|^2$, where $P$ is a symmetric positive definite matrix. Then, by Lemma 3.1 and Remark 3.3 of [9], $V_i(Z_i)$ satisfies both (59) and (60). \qed

It is noted that conditions (59) and (60) are the same as conditions (30) and (31)) of Theorem 3.1. Thus, combining Theorem 3.1, Remark 3.2, and Lemma 4.1 gives the following result.

**Theorem 4.1** Under Assumptions 2-6, and 9, for every $\sigma \in S_{\text{ave}}[\tau_d, N_0]$ with $\tau_d > \frac{\ln \mu}{\lambda_0}$ and arbitrary $N_0$, the cooperative global robust output regulation problem of system (46) with the directed switching graph $\bar{G}_\sigma(t)$ is solvable by the distributed switched output feedback control law of the form (55).

## 5 Conclusion

In this paper, we have established a specific changing supply pair technique to analyze the exponential input to state stability for nonlinear systems. Then, combining this technique with multiple Lyapunov functions and the average dwell time method, we have solved the cooperative global robust stabilization problem for a class of nonlinear multi-agent systems by a distributed switched output feedback control law. Finally, we have applied this result to the cooperative global robust output regulation problem for nonlinear multi-agent systems in normal form with unity relative degree under directed switching network.

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