THE PROBLEM OF LAGRANGE ON PRINCIPAL BUNDLES UNDER A SUBGROUP OF SYMMETRIES

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We dedicate this work to our friend
Darryl D. Holm on the occasion of his 70th birthday.

Abstract. Given a Lagrangian density $L^v$ defined on the 1-jet extension $J^1P$ of a principal $G$-bundle $\pi: P \to M$ invariant under the action of a closed subgroup $H \subset G$, its Euler-Poincaré reduction in $J^1P/H = C(P) \times_M P/H$ (where $C(P)$ is the bundle of connections of $P$ and $P/H \to M$ is the bundle of $H$-structures) induces a Lagrange problem defined in $J^1(C(P) \times_M P/H)$ by a reduced Lagrangian density $L^v$ together with the constraints $\text{Curv} \sigma = 0, \nabla^g \bar{s} = 0$, for $\sigma$ and $\bar{s}$ sections of $C(P)$ and $P/H$ respectively. We prove that the critical section of this problem are solutions of the Euler-Poincaré equations of the reduced problem. We also study the Hamilton-Cartan formulation of this Lagrange problem, where we find some common points with Pontryagin’s approach to optimal control problems for $\sigma$ as control variables and $\bar{s}$ as dynamical variables. Finally, the theory is illustrated with the case of affine principal fiber bundles and its application to the modelling of the molecular strands on a Lorentzian plane.

1. Introduction. In [4], a new formulation of the Euler-Poincaré reduction in principal bundles by a subgroup of the structure group was introduced in the following way. Given a Lagrangian density $L^v$, defined in the 1-jet extension $J^1P$ of a principal $G$-bundle $\pi: P \to M$ invariant under the action of a subgroup $H \subset G$, the corresponding variational problem projects to $(J^1P)/H$, which can be naturally identified with $C(P) \times_M P/H$ by the isomorphism

$$(J^1P)/H \to C(P) \times_M P/H$$

$$(j^1_x s)_H \to ([j^1_x s]_G, [s(x)]_H)$$

where $(J^1P)/G = C(P) \to M$ is the bundle of connections of $P$ and $P/H \to M$ the bundle of $H$-structures. This projected problem is defined on connections

2010 Mathematics Subject Classification. Primary: 58E30; Secondary: 70S05, 53C05.

Key words and phrases. Constraints, field theories, Lagrange multipliers, principal bundles, reduction, variational calculus.

This work has been partially supported by MICINN (Spain) under projects MTM2015-63612-P and PGC2018-098321-B-I00, as well as Consejería de Educación, Junta de Castilla-León (Spain) under project SA090G19.
σ ∈ Γ(C(P)) and $H$-structures $\bar{s} ∈ Γ(P/H)$ by a reduced Lagrangian density $lv$ (projection of $Lν$), the constraints Curv$σ = 0$ and $∇^\sigma \bar{s} = 0$, and the representation of infinitesimal gauge transformations $η ∈ Γ(\bar{g}$ as the set of admissible infinitesimal variations ($\bar{g} \to M$ being the adjoint bundle of $π$). Due to the gauge functoriality of the curvature and the covariant derivative, given an admissible section ($σ, \bar{s}$, of the reduced problem (with the notation introduced along the article).

Due to the gauge functoriality of the curvature and the covariant derivative, given an admissible section ($σ, \bar{s}$) ∈ $Γ(C(P) ×_M P/H)$, i.e., a section satisfying Curv$σ = 0$ $∇^\sigma \bar{s} = 0$, the 1-jet extension $j^1(δσ, δ\bar{s})$ of an infinitesimal variation ($δσ, δ\bar{s}$) of the reduced problem is tangent along $j^1(σ, \bar{s})$ to the submanifold

$$S = \{j^1_*(σ, \bar{s})|Curvσ = 0, \nabla^\sigma \bar{s} = 0\} ⊂ J^1(C(P) ×_M P/H).$$

Therefore, we obtain a subspace of the space of admissible infinitesimal variations along an admissible section of the Lagrange problem defined in $J^1(C(P) ×_M P/H)$ by the reduced density $lv$ and the constraint submanifold $S$ (see [8] and [9] for recent geometric descriptions of the Lagrange problem, and also [1] for the point of view of the vakonomic formulation of constrained Field Theories). As a nice consequence we get that critical sections of the Lagrange problem are also solutions of the Euler-Poincaré equations

$$\text{div}^\sigma \frac{δl}{δσ} - P^\delta_\sigma \frac{δl}{δ\bar{s}} = 0,$$

of the reduced problem (with the notation introduced along the article).

We first tackled this problem in [3] for $H = G$, where the reduction morphism is simply $J^1P → (J^1P)/G = C(P)$ and no $H$-structures occur. The reduced problem and its induced Lagrange problem with constraint Curv$σ = 0$ on connections $σ ∈ Γ(C(P))$ define the same set of solutions only under some cohomology condition. On the other hand, in [5] we studied the particular case with arbitrary $H ⊂ G$ but $M = ℝ$, where the condition Curv$σ = 0$ is now trivial and the only constraint is $∇^\sigma \bar{s} = 0$. In particular, from the local expression of $∇^\sigma \bar{s}$, the Lagrange problem can be regarded as an optimal control problem where the dynamic variable is $\bar{s}$ and the control variable is $σ$ (see [2] and [8] for a recent version of optimal control problems).

In this present article, we give a solution of the general case for arbitrary manifolds $M$ and any subgroup $H ⊂ G$.

The structure of the work is as follows. Section 2 gives the definitions and the local expressions of some basic operators appearing in the Euler-Poincaré formalism that will be used in the following. In Section 3 we define the problem of Lagrange associated to an Euler-Poincaré reduction on a principal bundle by a subgroup of the structure group. We also compare explicitly the space of admissible infinitesimal variations of both problems (Theorem 3.3 and Corollary 1). In section 4, the Euler-Lagrange equations of the Lagrange problem are obtained through the Lagrange multipliers rule and we compare them with the Euler-Poincaré equations. Section 5 provides the Hamilton-Cartan formulation of the Lagrange problem showing its similarities with the Pontryagin formulation of optimal control problems with connections as optimal variables and $H$-structures and the field variables. Finally, in Section 6 the theory is illustrated with the case of affine principal bundles with the linear component of the structure group of the affine group as the subgroup $H$. This framework is applied to the model of a molecular strand on a Lorentzian plane.

2. Some definitions and local formulas. Let $π : P → M$ be a $G$-principal bundle and let $H$ be a subgroup of $G$. On the one hand, we consider the bundle of connections $C(P) → M$, the sections $σ$ of which are identified to principal
G-connections of $P$. Recall that any automorphism (or in particular, gauge transformation) $\Phi : P \to P$ of $P$ induces a bundle morphism $\Phi_C : C(P) \to C(P)$ such that for a section $\sigma \in \Gamma(C(P))$, the connection of $P$ identified to $\Phi_C \circ \sigma$ is the push-forward by $\Phi$ of the connection identified to $\sigma$. On the other hand, we consider the quotient $P/H \to M$, which can be also seen as the associated bundle $P \times_G (G/H)$ where $G$ acts upon the homogenous space $G/H$ by the left, the identification being

\[ P/H \longrightarrow P \times_G (G/H) \]

\[ [u]_H \mapsto [u, [e]_H]_G. \]

Again, an automorphism of $P$ (or in particular, a gauge transformation) $\Phi : P \to P$ induces a bundle morphism $\Phi_{P/H} : P/H \to P/H$ as $\Phi_{P/H}(\{p\}_H) = \{\Phi(p)\}_H$.

Infinitesimal gauge transformations are $\pi$-vertical $G$-invariant vector fields on $P$. They can be seen as sections of the adjoint bundle $\tilde{\mathfrak{g}} = P \times_G \mathfrak{g}$, where $G$ act upon $\mathfrak{g}$ by the adjoint action. Given a section $\eta \in \Gamma(\tilde{\mathfrak{g}})$, the identification is $\eta_u = d/d\varepsilon|_{\varepsilon=0} u \cdot \exp \varepsilon B \in T_u P$, where $\eta(\pi(u)) = [u, B]_G$, $u \in P$. Since the flow of an infinitesimal gauge transformation is a uniparametric subgroup of gauge transformation, the infinitesimal version of the representation $\Phi \mapsto \Phi_C$ and $\Phi \mapsto \Phi_{P/H}$ described above induces a representation of the gauge algebra $\Gamma(\tilde{\mathfrak{g}})$ into the set of vertical vector fields in $C(P)$ and $P/H$ respectively that we denote by

\[ \eta \mapsto (\eta_C, \eta_{P/H}) \in \mathfrak{x}^v(C(P)) \times \mathfrak{x}^v(P/H). \]

In the following, we will need this representation along sections. More precisely, given a section $(\sigma, \tilde{s})$ of the bundle $C(P) \times_M P/H \to M$, we define the operator

\[ P_{(\sigma, \tilde{s})} : \Gamma(\tilde{\mathfrak{g}}) \longrightarrow \Gamma(\sigma^*V(C(P))) \times \Gamma(\tilde{s}^*V(P/H)) \]

\[ \eta \mapsto (P_\sigma(\eta), P_{\tilde{s}}(\eta)) \]

as

\[ P_\sigma(\eta) = \eta_C|_\sigma = \nabla^\sigma \eta, \]

\[ P_{\tilde{s}}(\eta) = \eta_{P/H}|_{\tilde{s}}, \]

where we have taken into account that $\eta_C$ along the section $\sigma$ is exactly the covariant derivative of $\eta$ with respect to the connection defined by $\sigma$. Note that $\nabla^\sigma \eta$ is a 1-form on $M$ taking values in $\tilde{\mathfrak{g}}$, which is consistent with fact that $\sigma^*V(C(P)) \simeq T^*M \otimes \tilde{\mathfrak{g}}$ since the bundle of connections is an affine bundle modeled on that vector bundle. It is easy to see that the operator $P_{\tilde{s}}$ is surjective and its kernel is

\[ \ker P_{\tilde{s}} = \{ \eta \in \Gamma(\tilde{\mathfrak{g}}) | \eta(x) = [u, B]_G, \text{ with } [u]_H = \tilde{s}(x) \text{ and } B \in H \}, \]

which in addition is isomorphic to sections of $\tilde{s}^*\mathfrak{h}$, via the map

\[ \Gamma(\tilde{s}^*\mathfrak{h}) \longrightarrow \ker P_{\tilde{s}} \]

\[ [u, B]_H \mapsto [u, B]_G \]

where $\tilde{\mathfrak{h}} \to P/H$ is the adjoint bundle of the $H$-principal bundle $P \to P/H$.

In the following, we will need the notion of covariant derivative $\nabla^\sigma \tilde{s}$ of sections $\tilde{s}$ of the bundle $P/H \to M$ with respect to a connection $\sigma \in \Gamma(C(P))$ on $P$. The horizontal distribution of $\sigma$ on $P$ projects to a distribution on $P/H$ of subspaces complementary to the vertical distribution $V_y(P/H)$, $y \in P/H$. Then, the covariant derivative $\nabla^\sigma \tilde{s} \in \Gamma(T^*M \otimes \tilde{s}^*V(P/H))$ is a 1-form on $M$ taking values in $\tilde{s}^*V(P/H)$, and it is defined as

\[ \nabla^\sigma \tilde{s}(w) = d\tilde{s}(w) - w^h, \]
for any \( w \in TM \), where \( w^h \) stands for the horizontal lift of \( w \) with respect to the horizontal distribution on \( P/H \) just mentioned.

We now introduce a coordinate system \( (x^1, \ldots, x^n) \), \( n = \dim M \), over a domain \( U \subset M \) such that \( \pi \) is trivializable, that is \( \pi^{-1}(U) \simeq U \times G \). We also choose a basis \( \{B_1, \ldots, B_m\} \), \( m = \dim G \), of the Lie algebra \( \mathfrak{g} \). Given an element of the Lie algebra \( B \in \mathfrak{g} \), the vector field \( \tilde{B} \in \mathfrak{X}(\pi^{-1}(U)) \) defined as \( \tilde{B}_u = d/d\varepsilon|_{\varepsilon=0} (x, \exp \varepsilon B \cdot g) \), \( u = (x, g) \), is an infinitesimal gauge transformation. The bundle of connections is endowed with natural coordinates \( (x^i, A^a_i) \), \( i = 1, \ldots, m \), \( \alpha = 1, \ldots, m \), over \( U \) such that the horizontal lift with respect to a connection \( \sigma \in \Gamma(C(P)) \) at a point \( u = (x, g) \in \pi^{-1}(U) \) is

\[
\left( \frac{\partial}{\partial x^i} \right)^h = \frac{\partial}{\partial x^i} + A^a_i(\sigma(x)) (\tilde{B}_\alpha)_u \in T_u P, \quad i = 1, \ldots, n.
\]

With these coordinates, one easily finds that the expression of the curvature \( \Omega^\sigma \) of a connection \( \sigma \in \Gamma(C(P)) \) is

\[
\text{Curv}(\sigma) = \left( \frac{\partial (A^a_i \circ \sigma)}{\partial x^i} - \frac{\partial (A^a_j \circ \sigma)}{\partial x^j} + e^{\alpha}_{\beta \gamma}(A^a_i \circ \sigma) \right) dx^i \wedge dx^j \otimes \tilde{B}_\alpha.
\]

Any infinitesimal gauge transformation \( \eta \in \Gamma(\mathfrak{g}) \) on \( U \) can be written as \( \eta = \eta^a \tilde{B}_a \), for some functions \( \eta^a \in C^\infty(U) \). The covariant derivative of \( \eta \) with respect to a connection reads

\[
P_\sigma(\eta) = \nabla^\sigma \eta = \left( \frac{\partial \eta^a}{\partial x^i} - e^{\alpha}_{\beta \gamma}(A^a_i \circ \sigma) \eta^\gamma \right) dx^i \otimes \tilde{B}_\alpha.
\]

On the other hand, let \( U \times (G/H) \to U \) be the trivialization of the bundle \( P/H = P \times_G (G/H) \to M \) induced by \( \pi^{-1}(U) \simeq U \times G \). If \( (y^j) \), \( 1 \leq j \leq r = \dim G - \dim H \), are coordinates on \( G/H \), then the projection \( (\tilde{B}_\alpha)_{P/H} \) of the vector field \( \tilde{B}_\alpha \) can be expressed as

\[
(\tilde{B}_\alpha)_{P/H} = \Psi^j_\alpha \frac{\partial}{\partial y^j},
\]

for some functions \( \Psi^j_\alpha \in C^\infty(G/H) \). In this situation, for every \( \bar{s} \in \Gamma(U, G/H) \) and every \( \eta = \eta^a \tilde{B}_a \in \Gamma(\mathfrak{g}) \), we have

\[
P_{\bar{s}}(\eta) = (\eta_{P/H})|_{\bar{s}} = \eta^a \cdot \Psi^j_\alpha \circ \bar{s} \left( \frac{\partial}{\partial y^j} \right)_{\bar{s}}.
\]

Finally, the expression of the covariant derivative of a section \( \bar{s} \) of \( P/H \to M \) along \( U \) is

\[
\nabla^\sigma \bar{s} = \left( \frac{\partial \bar{s}}{\partial x^i} \circ \bar{s} - (A^a_i \circ \sigma)(\Psi^j_\alpha \circ \bar{s}) \right) dx^i \otimes \left( \frac{\partial}{\partial y^j} \right)_{\bar{s}}.
\]

Indeed, given the vector \( \partial/\partial x^i \in T_x M \), its horizontal lift to \( P \) is \( \partial/\partial x^i + A^a_i(\sigma(x)) \), \( (\tilde{B}_\alpha)_u \), \( \pi(u) = x \). From (4), the projection to \( P/H \) is

\[
\frac{\partial}{\partial x^i} + A^a_i(\sigma(x)) \Psi^j_\alpha \frac{\partial}{\partial y^j},
\]

and we get (5) from (2), since \( d\bar{s}(\partial/\partial x^i) = \partial/\partial x^i + \partial(y^j \circ \bar{s})/\partial x^i(\partial/\partial y^j) \).
3. Lagrange problem associated to an Euler-Poincaré reduction on a principal bundle by a subgroup of the structure group. For a $G$-principal bundle $P \to M$ and a subgroup $H \subset G$, we consider the bundle

$$C(P) \times_M (P/H) \to M.$$

In the jet bundle $J^1(C(P) \times_M (P/H))$ we define the submanifold

$$S = \{j^1_x(\sigma, \bar{s}) : (\nabla^\sigma \bar{s})_x = 0, \text{Curv}(\sigma)_x = 0 \} \subset J^1(C(P) \times_M (P/H)), \text{ (constraint)}$$

and the so-called set of admissible sections

$$\mathcal{S} = \{ (\sigma, \bar{s}) \in \Gamma(C(P) \times_M (P/H)) : j^1_x(\sigma, \bar{s}) \in S, \forall x \in M \},$$

that is, sections such that $\bar{s}$ is parallel with respect to $\sigma$ ($\nabla^\sigma \bar{s} = 0$) and $\sigma$ is flat ($\text{Curv} \sigma = 0$).

**Definition 3.1.** Given $(\sigma, \bar{s}) \in \mathcal{S}$, we define the tangent space $T_{(\sigma, \bar{s})}\mathcal{S}$ as the set of vertical vector fields $(\delta \sigma, \delta \bar{s})$ along the sections $\sigma$ and $\bar{s}$ (that is, $\delta \sigma \in \Gamma(\sigma^*V(C(P )))$, $\delta \bar{s} \in \Gamma(\bar{s}^*V(P/H)))$ such that their 1-jet lifts $(\delta \sigma)^{(1)}$ and $(\delta \bar{s})^{(1)}$ to $J^1(C(P) \times_M (P/H))$ are tangent to the submanifold $S \subset J^1(C(P) \times_M (P/H))$ along $j^1(\sigma, \bar{s})$.

We now prove that, although the computation of jet lifts $(\delta \sigma)^{(1)}$ and $(\delta \bar{s})^{(1)}$ as vertical vector fields on $J^1(C(P) \times_M (P/H))$ respectively requires an extension of $\delta \sigma$ and $\delta \bar{s}$ to vector field on the entire bundles $C(P)$ and $P/H$, the fact of being tangent to $S \subset J^1(C(P) \times_M (P/H))$ along $j^1 \sigma$ and $j^1 \bar{s}$ does not depend on the chosen extension, so that the definition above makes sense.

**Lemma 3.2.** Let $\sigma$ and $\bar{s}$ be sections of $C(P) \to M$ and $(P/H) \to M$ respectively, and $\delta \sigma \in \Gamma(\sigma^*V(C(P ))) = \Gamma(T^*M \otimes \mathfrak{g})$, and $\delta \bar{s} \in \Gamma(\bar{s}^*V(P/H))$ vertical vectors along them. Then, given any extension of $\delta \sigma$ and $\delta \bar{s}$ to vertical vector field on $C(P)$ and $P/H$ respectively, the restriction of its 1-jet lift is tangent to $S$ if and only if

$$\nabla^\sigma(\delta \sigma) = 0,$$

and

$$P_\bar{s}(\nabla^\sigma \eta - \delta \sigma) = (\nabla^\sigma \eta - \delta \sigma)_{P/H} \bar{s} = 0,$$  \hspace{1cm} (7)

where, $P_\bar{s}$ being surjective, $\eta \in \Gamma(\bar{s})$ is such that $P_\bar{s}(\eta) = (\eta_{P/H}) = \delta \bar{s}$.

**Proof.** Given flows $\{\Phi^s_t\}_{t \in \mathbb{R}} \subset \text{Diff}(C(P))$, $\{\Phi^\sigma_t\}_{t \in \mathbb{R}} \subset \text{Diff}(P/H)$ such that

$$d/dt|_{t=0} \Phi^s_t(\sigma(x)) = (\delta \sigma)_x, \hspace{1cm} d/dt|_{t=0} \Phi^\sigma_t(\bar{s}(x)) = (\delta \bar{s})_x,$$

we consider the uniparametric family of sections $\sigma_t = \Phi^\sigma_t \circ \sigma$ and $\bar{s}_t = \Phi^s_t \circ \bar{s}$. Then, the 1-jet lifts of the extensions defined by those flows are tangent to $S \subset J^1(C(P) \times_M (P/H))$ along $j^1(\sigma, \bar{s})$ if and only if

$$\left| \frac{d}{dt} \right|_{t=0} \text{Curv}(\sigma_t) = 0, \hspace{1cm} (8)$$

$$\left| \frac{d}{dt} \right|_{t=0} \nabla^\sigma \bar{s}_t = 0, \hspace{1cm} (9)$$

Since these conditions are local, we can work on a domain $U \subset M$ such that $\pi : P \to M$ is trivial. Under this trivialization, connections $\sigma$ and variations $\delta \sigma$ can be seen $\mathfrak{g}$-valued 1-forms on $M$. Hence

$$\left| \frac{d}{dt} \right|_{t=0} \text{Curv}(\sigma_t) = \left| \frac{d}{dt} \right|_{t=0} (d\sigma_t + [\sigma_t, \sigma_t]) = d\delta \sigma + [\sigma, \delta \sigma] = \nabla^\sigma \delta \sigma,$$
and (6) is proved. For the proof of (7) we also assume that \( U \) is a chart domain with coordinates \((x^1, \ldots, x^n)\) and we have coordinates \((y^1, \ldots, y^r)\) on \(G/H\). From the local expression (5), we have

\[
\frac{d}{dt} \bigg|_{t=0} \nabla^\sigma s_t = \left( \frac{\partial (\delta \tilde{s})^j}{\partial x^i} - (\delta \sigma)^i \psi^j - (A^i \circ \sigma) \frac{\partial \psi^j}{\partial y^k} (\delta \tilde{s})^k \right) dx^i \otimes \frac{\partial}{\partial y^j},
\]

(10)

where \(\delta \sigma = (\delta \sigma)^i dx^i \otimes B^i, \delta \tilde{s} = (\delta \tilde{s})^k \partial \tilde{y}^k\). We now take \(\eta \in \Gamma(\tilde{\mathfrak{g}}), \eta = \eta^\alpha \tilde{B}_\alpha\), as in the statement. From (4), its local expression is

\[
\delta \tilde{s} = (\eta_{P/H}) \tilde{s} = \eta^\alpha \tilde{B}_\alpha \left( \frac{\partial}{\partial \tilde{y}^j} \right) s_t.
\]

Putting this expression in (10), we get

\[
\frac{d}{dt} \bigg|_{t=0} \nabla^\sigma s_t = \left( \frac{\partial \eta^\alpha}{\partial x^i} \psi^j + \eta^\beta \frac{\partial \psi^j}{\partial y^k} (A^i \circ \sigma) \psi^k - (A^i \circ \sigma) \frac{\partial \psi^j}{\partial y^k} \eta^\beta \psi^k - (\delta \sigma)^i \psi^j \right) dx^i \otimes \frac{\partial}{\partial y^j},
\]

where in the last step we have taken into account that \(\nabla^\sigma \tilde{s} = 0\). Hence

\[
\frac{d}{dt} \bigg|_{t=0} \nabla^\sigma s_t = \left( \frac{\partial \eta^\alpha}{\partial x^i} \psi^j + \eta^\beta \frac{\partial \psi^j}{\partial y^k} (A^i \circ \sigma) \psi^k - (A^i \circ \sigma) \frac{\partial \psi^j}{\partial y^k} \eta^\beta \psi^k - (\delta \sigma)^i \psi^j \right) dx^i \otimes \frac{\partial}{\partial y^j} + c^\alpha_{\beta \gamma} \eta^\beta \frac{\partial}{\partial y^j} dx^i \otimes \frac{\partial}{\partial y^j},
\]

where we have used the following easy relation

\[
c^\alpha_{\beta \gamma} \frac{\partial}{\partial y^j} = c^\alpha_{\beta \gamma} (\pi_H)_* [\tilde{B}^\alpha] = (\pi_H)_* [\tilde{B}^\alpha] - \frac{\partial}{\partial y^j}.
\]

and \(c^\alpha_{\beta \gamma}\) are the structure constants of the basis \(\{B_1, \ldots, B_m\}\). Hence, the local expression of \(d/dt|_{t=0} \nabla^\sigma s_t\) is precisely that of \(\nabla^\sigma \eta - \delta \sigma\) \(P/H\) along \(s\) and we conclude (7).

\[\square\]

Not only is this last Lemma necessary for Definition 3.1 above, but also the fundamental of the following characterization of tangents elements to \(\mathcal{S}\), as the following results shows.

**Theorem 3.3.** The set of elements of the tangent space \(T_{(\sigma, \tilde{s})} \mathcal{S}\) of a constrained section \((\sigma, \tilde{s}) \in \mathcal{S}\) are of the type

\[
\delta \sigma = \nabla^\sigma \eta + \omega, \quad \delta \tilde{s} = (\eta_{P/H}) \tilde{s},
\]

(11)

where \(\eta \in \Gamma(\tilde{\mathfrak{g}})\) is any section of the adjoint and \(\omega \in \Gamma(T^* M \otimes \tilde{s}^* \mathfrak{h})\) with \(\nabla^\sigma \omega = 0\), is any parallel 1-form taking values in \(\Gamma(\tilde{s}^* \mathfrak{h})\) = \(\text{ker} P_s\) (see the identification (1)).
Proof. From the previous Lemma, \((\delta \sigma, \delta \bar{s})\) belongs to \(T_{(\sigma, \bar{s})}\mathcal{S}\) if and only if \(\nabla^\sigma (\delta \sigma) = 0\) and there exists \(\eta \in \Gamma(\mathfrak{g})\) such that \((\eta_{P/H})_{\bar{s}} = \delta \bar{s}\) and \(P_s(\nabla^\eta \eta - \delta \sigma) = 0\). The last condition means that \(\omega = \delta \sigma - \nabla^\sigma \eta \in \Gamma(T^*M \otimes \mathfrak{s}^*\mathfrak{h})\), and the first that \(\nabla^\sigma \omega\) vanishes, since \(\sigma\) is flat. The converse is immediate.

**Corollary 1.** If the first cohomology group \(H^1(M, \mathfrak{s}^*\mathfrak{h})\) of the chain \((\Omega^*(M, \mathfrak{s}^*\mathfrak{h}), d = \nabla^\sigma)\) vanishes and the bundle \(P \to M\) is trivial, then the set of elements of the tangent space \(T_{(\sigma, \bar{s})}\mathcal{S}\) of a constrained section \((\sigma, \bar{s})\in \mathcal{S}\) are of the type
\[
\delta \sigma = \nabla^\sigma \eta, \quad \delta \bar{s} = (\eta_{P/H})_{\bar{s}},
\]
for any section \(\eta \in \Gamma(\mathfrak{g})\).

**Proof.** According to Theorem 3.3, \(\delta \sigma = \nabla^\sigma \eta + \omega\) and \(\delta \bar{s} = \eta_{P/H}\) with \(\nabla^\sigma \omega = 0\). Then, since \(H^1(M, \mathfrak{s}^*\mathfrak{h}) = 0\), \(\sigma\) is flat and the bundle \(P \to M\) is trivial there exists \(\zeta \in \Gamma(\mathfrak{s}^*\mathfrak{h})\) such that \(\nabla^\sigma \zeta = \omega\). Then we consider \(\eta' = \eta + \zeta\) and we get
\[
(\eta'_{P/H})_{\bar{s}} = P_s(\eta') = P_s(\eta) + P_s(\zeta) = P_s(\eta) = \delta \bar{s},
\]
\[
\nabla^\sigma \eta' = \nabla^\sigma \eta + \nabla^\sigma \nabla^\sigma \zeta = \nabla^\sigma \eta + [\text{Curv}(\sigma), \zeta] = \nabla^\sigma \eta = \delta \sigma,
\]
as desired. \(\square\)

**Definition 3.4.** Given a volume form \(v \in \Omega^n(M)\) and a zero-order Lagrangian function \(l : C(P) \times_M (P/H) \to \mathbb{R}\), a section \((\sigma, \bar{s})\in \mathcal{S}\) is said to be critical for the constrained variational problem defined by \(\mathcal{S}\) if and only if the variation of the action
\[
\mathcal{A}(\sigma, \bar{s}) = \int_M l(\sigma, \bar{s})v
\]
vanishes for any compact supported variation \((\delta \sigma, \delta \bar{s})\in T_{(\sigma, \bar{s})}\mathcal{S}\).

We now think of \(l\) as the reduction of an \(H\)-invariant first order Lagrangian \(L : J^1P \to \mathbb{R}\) as considered in the Introduction, for which we consider the Euler-Poincaré reduction picture. From Theorem 3.3, critical sections with respect to Definition 3.4 are also solutions to the Euler-Poincaré equations, since Euler-Poincaré variations \(\delta \sigma = \nabla^\sigma \eta, \delta \bar{s} = \eta_{P/H}\), are special cases of constrained variations \((\delta \sigma, \delta \bar{s})\in T_{(\sigma, \bar{s})}\mathcal{S}\). Under the topological assumptions of Corollary 1, this inclusion between sets of solutions is a bijection since in that case the set of variations are the same for both problems. However, if these cohomology condition is not satisfied, this equivalence needs not be true and we have proper inclusion of Euler-Poincaré solution into the set of critical solutions. In the next section we explore with more detail this fact about the equations on the set of critical solutions \((\sigma, \bar{s})\in \mathcal{S}\) through the method of Lagrange multipliers.

4. **The Lagrange multiplier rule.** To study the variational problem defined by a volume form \(v\) and a Lagrangian
\[
l : C(P) \times (P/H) \to \mathbb{R},
\]
that is constrained by the set \(\mathcal{S}\), that is, with sections \((\sigma, \bar{s})\) such that
\[
\text{Curv}(\sigma) = 0, \quad \nabla^\sigma \bar{s} = 0,
\]
and variations \((\delta \sigma, \delta \bar{s})\in T_{(\sigma, \bar{s})}\mathcal{S}\), we consider the first order Lagrangian
\[
\ell : J^1C(P) \times J^1(P/H) \times (\bigwedge^2 TM \otimes \mathfrak{g}^*) \times (TM \otimes V^*(P/H)) \to \mathbb{R} \quad (12)
\]
\[
\ell(j^1\sigma, j^1\bar{s}, \chi, \lambda) = l(\sigma, \bar{s}) + \langle \chi, \text{Curv}(\sigma) \rangle + \langle \lambda, \nabla^\sigma \bar{s} \rangle
\]
as a free variational problem with no constraints for either the sections \((\sigma, \bar{s}, \chi, \lambda)\) or their variations.

We first need to introduce the duals of two objects of this work. For the operator \(P_{\bar{s}} : \Gamma(\tilde{g}) \to \Gamma(\bar{s}^*V(P/H))\) its adjoint

\[
P_{\bar{s}}^+ : \Gamma(\bar{s}^*V(P/H)) \to \Gamma(\tilde{g}^*)
\]

is defined as the only operator such that

\[
\int_M \left< P_{\bar{s}}^+ (\varpi), \eta \right> v = \int_M \left< P_{\bar{s}}(\eta), \varpi \right> v
\]

for any compact supported sections \(\varpi \in \Gamma(\bar{s}^*V(P/H))\) and \(\eta \in \Gamma(\tilde{g})\). Moreover, for a connection \(\sigma\) in \(P\), the divergence operator

\[
\text{div}^\sigma : \Gamma(TM \otimes \tilde{g}^*) \to \Gamma(\tilde{g}^*)
\]

is defined as the only operator such that

\[
\left< \text{div}^\sigma (\xi), \eta \right> = \text{div} \left< \xi, \eta \right> - \left< \xi, \nabla^\sigma \eta \right>, \quad \forall \xi \in \Gamma(TM \otimes \tilde{g}^*), \eta \in \Gamma(\tilde{g})
\]

where \(\text{div}\) is the standard divergence operator defined by \(v\) in \(M\).

**Theorem 4.1.** The variational equations of the Lagrangian \(\ell\) defined above are

\[
\begin{bmatrix}
\frac{\delta \ell}{\delta \sigma} - \text{div}^\sigma \chi - P_{\bar{s}}^+ \lambda = 0 \\
P_{\bar{s}}^+ \frac{\delta \ell}{\delta \bar{s}} - \text{div}^\sigma P_{\bar{s}}^+ \lambda = 0 \\
\text{Curv}(\sigma) = 0 \\
\nabla^\sigma \bar{s} = 0
\end{bmatrix}
\]

(13)

**Proof.** The vanishing of the variation of the action is

\[
0 = \int_M \left( \left< \frac{\delta \ell}{\delta \sigma}, \delta \sigma \right> + \left< \frac{\delta \ell}{\delta \bar{s}}, \delta \bar{s} \right> + \left< \frac{\delta \ell}{\delta \chi}, \delta \chi \right> + \left< \frac{\delta \ell}{\delta \lambda}, \delta \lambda \right> \right) v,
\]

(14)

for arbitrary variations \(\delta \sigma, \delta \bar{s}, \delta \chi\) and \(\delta \lambda\). Variations on \(\chi\) and \(\lambda\) give \(\text{Curv}(\sigma) = 0\), and \(\nabla^\sigma \bar{s} = 0\) respectively. On the other hand, for any variation \(\delta \sigma \in \Gamma(\sigma^*V(C(P)) \cong \Gamma(T^*M \otimes \tilde{g}))\), we have

\[
\left< \frac{\delta \ell}{\delta \sigma}, \delta \sigma \right> = \frac{d}{d\varepsilon} \left|_{\varepsilon=0} \ell(j^1 \sigma, j^1 \bar{s}, \chi, \lambda) \right.
\]

\[
= \frac{d}{d\varepsilon} \left|_{\varepsilon=0} \ell(\sigma, \bar{s}) + \frac{d}{d\varepsilon} \left|_{\varepsilon=0} \left< \chi, \text{Curv}(\sigma_{\varepsilon}) \right> + \frac{d}{d\varepsilon} \left|_{\varepsilon=0} \left< \lambda, \nabla^\sigma \bar{s} \right> \right.ight),
\]

where \(\sigma_{\varepsilon} = \sigma + \varepsilon \delta \sigma\). For the derivative of the curvature, we easily have \(d/d\varepsilon|_{\varepsilon=0} \text{Curv}(\sigma_{\varepsilon}) = \nabla^\sigma \delta \sigma\). On the other hand, from Lemma 3.2 we have

\[
\frac{d}{d\varepsilon} \left|_{\varepsilon=0} \left< \lambda, \nabla^\sigma \bar{s} \right> = - \left< \lambda, P_{\bar{s}}(\delta \sigma) \right> \right),
\]

being this formula a special case of the computation of equation (10) for \(\eta = 0\). Then

\[
\left< \frac{\delta \ell}{\delta \sigma}, \delta \sigma \right> = \left< \frac{\delta \ell}{\delta \bar{s}}, \delta \sigma \right> + \left< \chi, \nabla^\sigma \delta \sigma \right> - \left< \lambda, P_{\bar{s}}(\delta \sigma) \right>.
\]

and, taking into account the adjoint operator \(P_{\bar{s}}^+\) and the covariant divergence \(\text{div}^\sigma\), the variation of the action (14) for \(\delta \sigma\) reads

\[
0 = \int_M \left< \frac{\delta \ell}{\delta \sigma} - \text{div}^\sigma \chi - P_{\bar{s}}^+ \lambda, \delta \sigma \right> v + \int_M \text{div} \left< \chi, \delta \sigma \right> v.
\]
The last term of the expression above vanishes by Stokes theorem, since the variations are compact supported. Hence, as \( \delta \sigma \) is free, we get the first equation in the statement.

For variations \( \delta \bar{s} \in \Gamma(\bar{s}^* V(P/H)) \), we have

\[
\left\langle \frac{\delta \ell}{\delta \bar{s}}, \delta \bar{s} \right\rangle = \left\langle \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \ell(j^1 \sigma, j^1 \bar{s}_\varepsilon, \chi, \lambda) \right\rangle
\]

where \( \bar{s}_\varepsilon \) is a family of sections with \( \bar{s}_0 = \bar{s} \) and \( d/d\varepsilon|_{\varepsilon=0} \bar{s}_\varepsilon = \delta \bar{s} \). Again a particular case of formula (10) in Lemma 3.2, now for \( \delta \sigma = 0 \), gives

\[
\left\langle \frac{\delta \ell}{\delta \bar{s}}, \delta \bar{s} \right\rangle = \left\langle \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \ell(\sigma, \bar{s}_\varepsilon) + \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \langle \lambda, \nabla^\sigma \bar{s}_\varepsilon \rangle, \right\rangle
\]

and the action principle (14) for variations \( \delta \bar{s} \) is

\[
0 = \int_M \left( \left\langle P^+_\bar{s} \frac{\delta \ell}{\delta \bar{s}}, \eta \right\rangle + \langle \lambda, P^+_\bar{s} (\nabla^\sigma \eta) \rangle \right) \textbf{v} + \int_M \text{div}(P^+_\bar{s} \lambda, \eta) \textbf{v}
\]

The second equation in the statement is obtained since \( \eta \) is free and compact supported.

**Remark 1.** It is important to recall the bundles where each of the equations in (13) takes place. The first equation is defined in \( TM \otimes \bar{g}^* \), the second in \( \bar{g}^* \), the third in \( \bigwedge^2 TM \otimes \bar{g}^* \) and the forth in \( TM \otimes V^*(P/H) \). It is easy to check that every term of these equations, along \( (\sigma, \bar{s}) \), is a section of the corresponding bundle.

**Corollary 2.** Every solution of the Lagrange problem defined by \( \ell \) is a solution of the Euler-Poincaré equations defined by \( l \) (or by \( L \)).

**Proof.** Given a solution \( (\sigma, \bar{s}) \) of the Lagrange problem, the divergence of the first equation in Theorem 4.1 reads

\[
0 = \text{div}^\sigma \frac{\delta \ell}{\delta \sigma} - \text{div}^\sigma \text{div}^\sigma \chi - \text{div}^\sigma P^+_\bar{s} \lambda
\]

where we have taken into account that \( \text{div}^\sigma \text{div}^\sigma \chi = \langle \text{Curv}^\sigma, \chi \rangle = 0 \) as \( \sigma \) is flat. We then recover the Euler-Poincaré equations for \( L \).

**Remark 2.** As we already pointed out in the previous section, the converse of the previous Corollary needs not be true for manifolds \( M \) with \( H^1(M, \bar{g}^* \mathfrak{h}) \neq 0 \), since there might be non-vanishing sections \( \tau \) of \( TM \otimes \bar{g}^* \) with \( \text{div}^\sigma \tau = 0 \). Solutions of the equation

\[
\frac{\delta \ell}{\delta \sigma} - \text{div}^\sigma \chi - P^+_\bar{s} \lambda = \tau
\]
could provide solutions of the Euler-Poincaré problem that are not solutions of the Lagrange problem.

5. **Hamilton-Cartan formulation.** We consider a trivializable open domain \( U \subset M \) with coordinates \((x^1, \ldots, x^n)\) such that \( \mathbf{v} = dx^1 \wedge \ldots \wedge dx^n \), as well as the following induced coordinate systems: \((x^i, A^\alpha_i)\) in \(C(P)\); \((x^i, y^j)\) in \(P/H\); \((x^i, \chi^i_j)\), \(i < j\), in \(\bigwedge^2 TM \otimes \tilde{\mathbb{R}}^*\); and \((x^i, \lambda^i_j)\) in \(TM \otimes V^*(P/H)\). The coordinates on jet bundles are \((x^i, A^\alpha_i, A^\alpha_{i,j})\) in \(J^1C(P)\) and \((x^i, y^j, y^i_j)\) in \(J^1(P/H)\) respectively. From formulas (3) and (5), we have that the local expression of the Lagrangian (12) in these coordinate systems is

\[
\ell = l(x^i, A^\alpha_i, y^j) + \sum_{i < j} \chi^i_j A^\alpha_i \left( A^\alpha_{i,j} - A^\alpha_{j,i} + c^\alpha_{\beta\gamma} A^\beta_i A^\gamma_j \right) + \lambda^i_j \left( y^i_j - A^\alpha_i \Psi^j_\alpha \right) \tag{15}
\]

From this formula, the Poincaré-Cartan form (see [7]) of the Lagrangian density \(\ell \mathbf{v}\) has the following local expression

\[
\Theta_\ell = \frac{\partial \ell}{\partial A^\alpha_{i,j}} (dA^\alpha_i - A^\alpha_{i,k} dx^k) \wedge \mathbf{v}_j + \frac{\partial \ell}{\partial y^i_j} (dy^i_j - y^i_j dx^k) \wedge \mathbf{v}_j + \ell \mathbf{v}
\]

\[
= \frac{\partial \ell}{\partial A^\alpha_{i,j}} dA^\alpha_i \wedge \mathbf{v}_j + \frac{\partial \ell}{\partial y^i_j} dy^i_j \wedge \mathbf{v}_j + H \mathbf{v},
\]

where \(\mathbf{v}_i = i_{\partial/\partial x^i} \mathbf{v}\) and

\[
H = \ell - \frac{\partial \ell}{\partial A^\alpha_{i,j}} A^\alpha_{i,j} - \frac{\partial \ell}{\partial y^i_j} y^i_j.
\]

Since \(\partial \ell / \partial A^\alpha_{i,j} = \chi^i_j\), for \(i < j\), \(\partial \ell / \partial A^\alpha_{i,j} = -\chi^i_j\), for \(i > j\) and \(\partial \ell / \partial y^i_j = \lambda^i_j\), we have

\[
\Theta_\ell = \sum_{i < j} \chi^i_j (dA^\alpha_i \wedge \mathbf{v}_j - dA^\alpha_j \wedge \mathbf{v}_i) + \lambda^i_j dy^i_j \wedge \mathbf{v}_j + H \mathbf{v},
\]

with

\[
H = \ell - \sum_{i < j} \chi^i_j (A^\alpha_{i,j} - A^\alpha_{j,i}) - \lambda^i_j y^i_j.
\]

Therefore, the local expression of the multisymplectic form associated to the problem defined by \(\ell\), that is, the differential of the Poincaré-Cartan form, is

\[
\Omega_\ell = d\Theta_\ell = \sum_{i < j} d\chi^i_j \wedge (dA^\alpha_i \wedge \mathbf{v}_j - dA^\alpha_j \wedge \mathbf{v}_i) + d\lambda^i_j \wedge dy^i_j \wedge \mathbf{v}_j + dH \wedge \mathbf{v}. \tag{16}
\]

The Hamilton-Cartan equations defined by this form for sections \((\sigma, s, \chi, \lambda)\) are given by the condition

\[
(\sigma, s, \chi, \lambda)^* i_A \Omega_\ell = 0,
\]

for all vector field \(A\) vertical with respect to the fibration onto \(M\). From the expression (16), the local expression of these equations are

\[
(\sigma, s, \chi, \lambda)^* i_A \omega = \left( \frac{\partial A^\alpha_{i,j}}{\partial x^j} - \frac{\partial A^\alpha_{j,i}}{\partial x^i} - \frac{\partial H}{\partial \chi^i_j} \right), \mathbf{v} = 0 \quad i < j,
\]

\[
(\sigma, s, \chi, \lambda)^* i_A \omega = \left( -\sum_{i < j} \frac{\partial \chi^i_j}{\partial x^j} + \sum_{j < i} \frac{\partial \chi^i_j}{\partial x^i} - \frac{\partial H}{\partial A^\alpha_i} \right) \mathbf{v} = 0,
\]

for all vector field \(A\) vertical with respect to the fibration onto \(M\). From the expression (16), the local expression of these equations are
\[(\sigma, \tilde{s}, \chi, \lambda)^* i \circ \Omega_\ell = \left( \frac{\partial y^i}{\partial x^j} - \frac{\partial H}{\partial \lambda_i^j} \right) v, \quad (\sigma, \tilde{s}, \chi, \lambda)^* i \circ \Omega_\ell = \left( \frac{\partial \lambda_i^j}{\partial x^j} - \frac{\partial H}{\partial y^i} \right) v = 0. \]

As we know from the general theory on Hamiltonian formulation of variational problems (see, for example [7]), these equations are equivalent to the equations (13).

6. **Application to affine principal bundles: the molecular strands.** Given a Lie group \( G \) acting linearly upon a vector space \( V \), we consider \( G_{aff} = G \rtimes V \) as the Lie group defined by the (semidirect) product

\[(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 \cdot v_2).\]

Let \( P \to M \) be a principal \( G \)-bundle. The bundle \( P_{aff} = P \times_M E \to M \), where \( E = P \times_G V \to M \) is the associated vector bundle to the representation of \( G \), is a principal bundle with structure group \( G_{aff} \). In the particular (but essential) case where \( P = LM \) is the frame bundle of \( M \), \( G = Gl(n, \mathbb{R}) \), \( n = \dim M \), and \( V = \mathbb{R}^n \) with the natural action, the bundle \( P_{aff} \to M \) is the standard bundle of affine frames, a fact that justifies the aff-notation used for the general case.

We now explore the constructions of previous sections for \( P_{aff} \to M \), taking \( H = G \) as the subgroup of the structure group \( G_{aff} \). First, we note that the bundle of connections \( C(P_{aff}) \) of \( P_{aff} \) naturally splits as

\[C(P_{aff}) = C(P) \times_M (T^*M \otimes E),\]

since connections \( \sigma_{aff} \) in \( P_{aff} \) split as

\[\sigma_{aff} = (\sigma, h), \quad \sigma \in C(P), \quad h \in T^*M \otimes E. \quad (17)\]

In particular, we have

\[\frac{\delta L}{\delta \sigma_{aff}} = \left( \begin{array}{c} \frac{\delta L}{\delta \sigma} \\ \frac{\delta L}{\delta h} \end{array} \right).\]

On the other hand,

\[P_{aff}/G = (P_{aff} \times (G_{aff}/G))/G = (P_{aff} \times V)/G = E.\]

Therefore, the starting point for the Lagrange study in the context of affine bundles is a Lagrangian

\[l : C(P) \times_M (T^*M \otimes E) \times_M E \to \mathbb{R},\]

so that the Lagrange multipliers of the corresponding Lagrangian \( \ell \) in (12) are

\[\lambda \in \Gamma(TM \otimes V^*(P_{aff}/G)) = \Gamma(TM \otimes E^*),\]

and

\[\chi_{aff} \in \bigwedge^2 TM \otimes \tilde{g}_{aff}^* = \bigwedge^2 TM \otimes \tilde{g}^* \oplus \bigwedge^2 TM \otimes E^*\]

\[\chi_{aff} = (\chi, \zeta).\]

Finally, the expression of the adjoint operator

\[P^+_\delta : \Gamma(E^*) \to \Gamma(\tilde{g}_{aff}^*) = \Gamma(\tilde{g}^*) \oplus \Gamma(E^*)\]

is

\[P^+_\delta (\omega) = (\omega \otimes \tilde{s}, \omega),\]
where $E \otimes E^*$ is immersed into $\tilde{\mathfrak{g}}^*$ via the mapping
\[(e \otimes e^*)(\eta) = e^*(\eta \cdot e), \quad e \in E, e^* \in E^*, \eta \in \tilde{\mathfrak{g}}.\] (18)
The system (13), that is, the equations
\[
\begin{align*}
\delta l \delta \sigma \text{aff} - \text{div} \sigma \text{aff} \chi \text{aff} - P_\tilde{s}^+ \lambda &= 0, \\
P_\tilde{s}^- \frac{\delta l}{\delta \tilde{s}} - \text{div} \sigma \text{aff} P_\tilde{s}^+ \lambda &= 0, \\
\text{Curv}(\sigma \text{aff}) &= 0, \\
\nabla^{\sigma \text{aff}} \tilde{s} &= 0
\end{align*}
\]
now take the following form:
- One can easily check that $\text{Curv}(\sigma \text{aff}) = (\text{Curv}(\sigma), \nabla \sigma \text{aff})$ and $\nabla \sigma \text{aff} \tilde{s} = \nabla \sigma \tilde{s} + h$.
- Hence, the constraints (that is, the last two equations) amount to
\[
\text{Curv}(\sigma) = 0, \quad \nabla^{\sigma} \tilde{s} = -h, \quad (19)
\]
since $\nabla^{\sigma} h = -\nabla^{\sigma} \nabla^{\sigma} \tilde{s}$ identically vanishes as $\sigma$ is flat.
- With respect to the first equation, taking into account the decomposition of $\sigma \text{aff} = (\sigma, h)$ given in (17), it splits as
\[
\frac{\delta L}{\delta \sigma} - (\text{div}^{\sigma} \chi + h \hat{\otimes} \zeta) - \lambda \otimes \tilde{s} = 0, \quad (20)
\]
\[
\frac{\delta L}{\delta h} - \text{div}^{\sigma} \zeta - \lambda = 0, \quad (21)
\]
where
\[
\text{div}^{\sigma \text{aff}} \chi \text{aff} = (\text{div}^{\sigma} \chi + h \hat{\otimes} \zeta, \text{div}^{\sigma} \zeta).
\]
The notation $h \hat{\otimes} \zeta \in T^*M \otimes \tilde{\mathfrak{g}}$ is defined as
\[
\langle h \hat{\otimes} \zeta, \eta \rangle = \langle \eta \hat{\otimes} h, \zeta \rangle
\]
where
\[
\hat{\otimes} : (T^*M \otimes \tilde{\mathfrak{g}}) \times (T^*M \otimes E) \to T^*M \otimes E
\]
is the wedge product twisted with the action of $\tilde{\mathfrak{g}}$ on $E$. If one solves (21) for $\lambda$ and put it in (20), then
\[
\frac{\delta L}{\delta \sigma} - (\text{div}^{\sigma} \chi + h \hat{\otimes} \zeta) - \left(\frac{\delta L}{\delta h} - \text{div}^{\sigma} \zeta\right) \otimes \tilde{s} = 0. \quad (22)
\]

**Lemma 6.1.** We have that
\[
\text{div}^{\sigma}(\tilde{s} \otimes \zeta) = \nabla^{\sigma} \tilde{s} \hat{\otimes} \zeta + \text{div}^{\sigma} \zeta \otimes \tilde{s}.
\]

**Proof.** From the definition, given $\eta \in \Gamma(T^*M \otimes \tilde{\mathfrak{g}})$
\[
\langle \text{div}^{\sigma}(\tilde{s} \otimes \zeta), \eta \rangle = \text{div}(\tilde{s} \otimes \zeta, \eta) - \langle \tilde{s} \otimes \zeta, \text{div}^{\sigma} \eta \rangle
\]
\[
= \text{div}(\eta \cdot \tilde{s}, \zeta) - \langle (\text{div}^{\sigma} \eta) \cdot \tilde{s}, \zeta \rangle
\]
\[
= \text{div}(\eta \cdot \tilde{s}, \zeta) - \langle \nabla^{\sigma}(\eta \cdot \tilde{s}) - \eta \hat{\nabla}^{\sigma} \tilde{s}, \zeta \rangle
\]
\[
= \langle \text{div}^{\sigma} \zeta, \eta \cdot \tilde{s} \rangle + \langle \nabla^{\sigma} \tilde{s} \hat{\otimes} \zeta, \eta \rangle
\]
\[
= \langle \text{div}^{\sigma} \zeta \otimes \tilde{s}, \eta \rangle + \langle \nabla^{\sigma} \tilde{s} \hat{\otimes} \zeta, \eta \rangle. \quad \Box
\]

With this result and (19), we have that (22) gives
\[
\frac{\delta L}{\delta \sigma} - \frac{\delta L}{\delta h} \otimes \tilde{s} - \text{div}^{\sigma}(\chi + \tilde{s} \otimes \zeta) = 0.
\]
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With respect to the second equation we have
\[
\left( \frac{\delta L}{\delta s} \otimes \bar{s}, \frac{\delta L}{\delta s} \right) - \left( \text{div}^\sigma (\lambda \otimes \bar{s}) + h \otimes \lambda, \text{div}^\sigma \lambda \right) = 0.
\]

The first component is
\[
\frac{\delta L}{\delta s} \otimes \bar{s} - \text{div}^\sigma (\lambda \otimes \bar{s}) - h \otimes \lambda = 0,
\]
where \( h \otimes \bar{s} \in \Gamma(E \otimes E^*) \) is obtained by coupling the \( TM \) and \( T^*M \) parts of \( h \) and \( \bar{s} \) and \( E \otimes E^* \) is immersed into \( \tilde{g}^* \) as in (18). The second component, taking into account (21), is
\[
\frac{\delta L}{\delta h} - \text{div}^\sigma \lambda = 0.
\]

Collecting all these considerations, we have:

**Proposition 1.** The Lagrange problem for affine bundles is given by the following equations
\[
\begin{align*}
\frac{\delta l}{\delta \sigma} - \frac{\delta l}{\delta h} \otimes \bar{s} - \text{div}^\sigma (\chi + \bar{s} \otimes \zeta) = 0, \\
\frac{\delta l}{\delta h} - \text{div}^\sigma \zeta = 0, \\
\frac{\delta l}{\delta \bar{s}} \otimes \bar{s} - \text{div}^\sigma (\lambda \otimes \bar{s}) - h \otimes \lambda = 0, \\
\frac{\delta l}{\delta \bar{s}} - \text{div}^\sigma \frac{\delta l}{\delta h} = 0, \\
\text{Curv}(\sigma) = 0, \\
\nabla^\sigma \bar{s} = -h.
\end{align*}
\]

Reduction in affine principal bundles models molecular strands. For more details on this example, specially the physical meaning of the reduced variables, we refer the reader to [6] (see also [4]). We can consider in this case \( P = \mathbb{R}^2 \times SO(3) \to \mathbb{R}^2 \), \( G = SO(3) \) and \( V = \mathbb{R}^3 \), with the natural action upon it. Then \( E = \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2 \) and \( P_{\text{aff}} = \mathbb{R}^2 \times SE(3) \to \mathbb{R}^2 \). Since all the bundles are trivial, sections are understood as functions from \( \mathbb{R}^2 \) to their respective fibers. If we take coordinates \((x,t)\) in \( \mathbb{R}^2 \), \((x\) for the parameter of the strand and \(t\) for time), we can write
\[
\sigma = \Omega dx + \omega dt, \quad h = \Gamma dx + \gamma dt, \quad \lambda = \lambda_x \frac{\partial}{\partial x} + \lambda_t \frac{\partial}{\partial t}
\]
for certain \( \Omega, \omega \in C^\infty(\mathbb{R}^2, \mathfrak{s}\mathfrak{o}(3)), \Gamma, \gamma, \lambda_x, \lambda_t \in C^\infty(\mathbb{R}^2, \mathbb{R}^3) \). In the following, we will identify \( \mathfrak{s}\mathfrak{o}(3) \) with \( \mathbb{R}^3 \) and its Lie bracket with the cross product. We consider the Lagrangian defined on a Lorentzian plane \((\mathbb{R}^2, g), g = dx \otimes dx - v^2 dt \otimes dt\), defined as
\[
l(t, x, \Omega, \omega, \Gamma, \gamma, \bar{s}) = \langle \Gamma, \Gamma \rangle - \frac{1}{v^2} \langle \gamma, \gamma \rangle - U((\bar{s}, \bar{s})) + \bar{l}(t, x, \Omega, \omega),
\]
where \( U \) is a smooth function defining a (central) potential, and \( \bar{l} \) is a Lagrangian for the connection \( \Omega \) and \( \omega \). This Lagrangian \( l \) can be understood as a model for the filament of the strand \( \bar{s} = \bar{s}(x,t) \), interpreted as a kind of “vectorial wave” with \( v \) as propagation speed. In this situation, it is a matter of checking that equations
(23) take the following form

\[
\begin{align*}
\delta l & - 2\Gamma \otimes \bar{s} - \left( \frac{\partial}{\partial t} + \Omega \times \right) (\chi + \bar{s} \otimes \zeta) = 0 \\
\frac{\delta l}{\delta \omega} & + \frac{2}{\mu^2} \gamma \otimes \bar{s} - \left( \frac{\partial}{\partial t} + \omega \times \right) (\chi + \bar{s} \otimes \zeta) = 0 \\
2\Gamma & - \left( \frac{\partial}{\partial t} + \Omega \times \right) \zeta - \lambda_x = 0 \\
\frac{2}{\sigma^2} \gamma & - \left( \frac{\partial}{\partial t} + \omega \times \right) \zeta - \lambda_t = 0 \\
2U'(|\bar{s}, \bar{s}|) & \bar{s} \otimes \bar{s} - 2 \left( \frac{\partial}{\partial t} + \Omega \times \right) \lambda_x \otimes \bar{s} \\
& - \left( \frac{\partial}{\partial t} + \omega \times \right) (\lambda_t \otimes \bar{s}) - \Gamma \otimes \lambda_x - \gamma \otimes \lambda_t = 0 \\
2U'(|\bar{s}, \bar{s}|) & \bar{s} - 2 \left( \frac{\partial}{\partial t} + \Omega \times \right) \Gamma + \frac{2}{\sigma^2} \left( \frac{\partial}{\partial t} + \omega \times \right) \gamma = 0 \\
\frac{\partial \Omega}{\partial t} & - \frac{\partial \omega}{\partial x} + \Omega \times \omega = 0 \\
\left( \frac{\partial}{\partial t} + \Omega \times \right) \bar{s} & = -\Gamma \\
\left( \frac{\partial}{\partial t} + \omega \times \right) \bar{s} & = -\gamma 
\end{align*}
\]

where each of the braces corresponds to each of the lines in (23).

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Received April 2018; revised May 2019.

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