On the Densest Packing of Polycylinders in Any Dimension

Wöden Kusner

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Abstract Using transversality and a dimension reduction argument, a result of Bezdek and Kuperberg is applied to polycylinders, showing that the optimal packing density of $D_2 \times \mathbb{R}^n$ equals $\pi/\sqrt{12}$ for all natural numbers $n$.

Keywords Polycylinders · Packing · Density · Slicing

1 Introduction

Open and closed Euclidean unit $n$-balls will be denoted by $B^n$ and $D^n$ respectively. The closed unit interval is denoted by $I$. A general polycylinder $C$ is a set congruent to $\prod_{i=1}^{m} \lambda_i D^{k_i}$ in $\mathbb{R}^{k_1+\cdots+k_m}$, where $\lambda_i$ is in $[0, \infty]$. For this article, the term polycylinder refers to the special case of an infinite polycylinder over a two-dimensional disk of unit radius. A polycylinder is a set congruent to $D^2 \times \mathbb{R}^n$ in $\mathbb{R}^{n+2}$. A polycylinder packing of $\mathbb{R}^{n+2}$ is a family $\mathcal{C} = \{C_i\}_{i \in I}$ of polycylinders $C_i \subset \mathbb{R}^{n+2}$ with mutually disjoint interiors. The upper density $\delta^+(\mathcal{C})$ of a packing $\mathcal{C}$ of $\mathbb{R}^n$ is defined to be

$$\delta^+(\mathcal{C}) = \limsup_{r \to \infty} \frac{\text{Vol}(\mathcal{C} \cap rB^n)}{\text{Vol}(rB^n)}.$$

The upper packing density $\delta^+(C)$ of an object $C$ is the supremum of $\delta^+(\mathcal{C})$ over all packings $\mathcal{C}$ of $\mathbb{R}^n$ by $C$.

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Wöden Kusner
wkusner@gmail.com

1 Institute of Analysis and Number Theory, Graz University of Technology, Steyrergasse 30/II, 8010 Graz, Austria

Springer Graz, Austria
This article proves the following sharp bound for the packing density of infinite polycylinders:

**Theorem 1** \( \delta^+ (\mathbb{D}^2 \times \mathbb{R}^n) = \pi / \sqrt{12} \) for all natural numbers \( n \).

Theorem 1 generalizes a result of Bezdek and Kuperberg [1] and improves on results that may be computed using a method of Fejes Tóth and Kuperberg [3], cf. [2,5]; it gives some of the first sharp upper bounds for packing density in high dimensions.

### 2 Transversality

This section introduces the required transversality arguments from affine geometry. A \( d \)-flat is a \( d \)-dimensional affine subspace of \( \mathbb{R}^n \). The **parallel dimension** \( \dim \parallel \{F, \ldots, G\} \) of a collection of flats \( \{F, \ldots, G\} \) is the dimension of their maximal parallel sub-flats. The notion of parallel dimension can be interpreted in several ways, allowing a modest abuse of notation.

- For a collection of flats \( \{F, \ldots, G\} \), consider their tangent cones at infinity \( \{F_\infty, \ldots, G_\infty\} \). The parallel dimension of \( \{F, \ldots, G\} \) is the dimension of the intersection of these tangent cones. This may be viewed as the limit of a rescaling process \( \mathbb{R}^n \to r \mathbb{R}^n \) as \( r \) tends to 0, leaving only the scale-invariant information.

- For a collection of flats \( \{F, \ldots, G\} \), consider each flat as a system of linear equations. The corresponding homogeneous equations determine a collection of linear subspaces \( \{F_\infty, \ldots, G_\infty\} \). The parallel dimension is the dimension of their intersection \( F_\infty \cap \cdots \cap G_\infty \).

Two disjoint \( d \)-flats are **parallel** if their parallel dimension is \( d \), that is, if every line in one is parallel to a line in the other.

**Lemma 1** A pair of disjoint \( n \)-flats in \( \mathbb{R}^{n+k} \) with \( n \geq k \), has parallel dimension strictly greater than \( n - k \).

**Proof** Let \( F \) and \( G \) be such a pair. By homogeneity of \( \mathbb{R}^{n+k} \), let \( F = F_\infty \). As \( F_\infty \) and \( G \) are disjoint, \( G \) contains a non-trivial vector \( v \) such that \( G = G_\infty + v \) and \( v \) is not in \( F_\infty + G_\infty \). It follows that

\[
\dim(\mathbb{R}^{n+k}) \geq \dim(F_\infty + G_\infty + \text{span}(v)) > \dim(F_\infty + G_\infty) = \dim(F_\infty) + \dim(G_\infty) - \dim(F_\infty \cap G_\infty).
\]

Count dimensions to find \( n + k > n + n - \dim \parallel (F_\infty, G_\infty) \). \( \Box \)

**Corollary 1** A pair of disjoint \( n \)-flats in \( \mathbb{R}^{n+2} \) has parallel dimension at least \( n - 1 \).

### 3 Dimension Reduction

#### 3.1 Pairwise Foliations

The **core** \( a_i \) of a polycylinder \( C_i \) congruent to \( \mathbb{D}^2 \times \mathbb{R}^n \) in \( \mathbb{R}^{n+2} \) is the distinguished \( n \)-flat defining \( C_i \) as the set of points at most distance 1 from \( a_i \). In a packing \( \mathcal{C} \) of
\( \mathbb{R}^{n+2} \) by polycylinders, Corollary 1 shows that, for every pair of polycylinders \( C_i \) and \( C_j \), one can choose parallel \((n - 1)\)-dimensional subflats \( b_i \subset a_i \) and \( b_j \subset a_j \) and define a product foliation

\[
\mathcal{F}^{b_i, b_j} : \mathbb{R}^{n+2} \to \mathbb{R}^{n-1} \times \mathbb{R}^3
\]

with \( \mathbb{R}^3 \) leaves that are orthogonal to \( b_i \) and to \( b_j \). Given a point \( x \) in \( a_i \), there is a distinguished \( \mathbb{R}^3 \) leaf \( F^{b_i, b_j}_{x, b_j} \) that contains the point \( x \). The foliation \( \mathcal{F}^{b_i, b_j} \) restricts to foliations of \( C_i \) and \( C_j \) with right-circular-cylinder leaves.

### 3.2 The Dirichlet Slice

In a packing \( \mathcal{C} \) of \( \mathbb{R}^{n+2} \) by polycylinders, the Dirichlet cell \( D_i \) associated with a polycylinder \( C_i \) is the set of points in \( \mathbb{R}^{n+2} \) which lie no further from \( C_i \) than from any other polycylinder in \( \mathcal{C} \). The Dirichlet cells of a packing partition \( \mathbb{R}^{n+2} \), as \( C_i \subset D_i \) for all polycylinders \( C_i \). To bound the density \( \delta^+(\mathcal{C}) \), it is enough to fix an \( i \) in \( I \) and consider the density of \( C_i \) in \( D_i \).

Consider the following slicing of the Dirichlet cell \( D_i \). Given a fixed polycylinder \( C_i \) in a packing \( \mathcal{C} \) of \( \mathbb{R}^{n+2} \) by polycylinders and a point \( x \) on the core \( a_i \), the plane \( p_x \) is the 2-flat orthogonal to \( a_i \) and containing the point \( x \). The Dirichlet slice \( d_x \) is the intersection of \( D_i \) and \( p_x \).

Note that \( p_x \) is a sub-flat of \( F^{b_i, b_j}_{x, b_j} \) for all \( j \) in \( I \).

### 3.3 Bezdek–Kuperberg Bound

For any point \( x \) on the core \( a_i \) of a polycylinder \( C_i \), the results of Bezdek and Kuperberg [1] apply to the Dirichlet slice \( d_x \).

**Lemma 2** A Dirichlet slice is convex and, if bounded, a parabola-sided polygon.

**Proof** Construct the Dirichlet slice \( d_x \) as an intersection. Define \( d^j \) to be the set of points in \( p_x \) which lie no further from \( C_i \) than from \( C_j \). Then the Dirichlet slice \( d_x \) is realized as

\[
d_x = \left\{ \bigcap_{j \in I} d^j \right\}.
\]

Each arc of the boundary of \( d_x \) in \( p_x \) is given by an arc of the boundary of some \( d^j \) in \( p_x \). The boundary of \( d^j \) in \( p_x \) is the set of points in \( p_x \) equidistant from \( C_i \) and \( C_j \). Since the foliation \( \mathcal{F}^{b_i, b_j} \) is a product foliation, the arc of the boundary of \( d^j \) in \( p_x \) is also the set of points in \( p_x \) equidistant from the leaf \( C_i \cap F^{b_i, b_j}_{x, b_j} \) of \( \mathcal{F}^{b_i, b_j} \) and the leaf \( C_j \cap F^{b_i, b_j}_{x, b_j} \) of \( \mathcal{F}^{b_i, b_j} \). This reduces the analysis to the case of a pair of cylinders in \( \mathbb{R}^3 \). From [1], it follows that \( d^j \) is convex and the boundary of \( d^j \) in \( p_x \) is a parabola; the intersection of such sets \( d^j \) in \( p_x \) is convex, and a parabola-sided polygon if bounded. \( \square \)
Let $S_x(r)$ be the circle of radius $r$ in $p_x$ centered at $x$.

**Lemma 3** The vertices of $d_x$ are not closer to $S_x(1)$ than the vertices of a regular hexagon circumscribed about $S_x(1)$.

**Proof** A vertex of $d_x$ occurs where three or more polycylinders are equidistant, so the vertex is the center of a $(n+2)$-ball $B$ tangent to three polycylinders. Thus $B$ is tangent to three disjoint unit $(n+2)$-balls $B_1, B_2, B_3$. By projecting into the affine hull of the centers of $B_1, B_2, B_3$, it is immediate that the radius of $B$ is no less than $2/\sqrt{3} - 1$. ∎

**Lemma 4** Let $y$ and $z$ be points on the circle $S_x(2/\sqrt{3})$. If each of $y$ and $z$ is equidistant from $C_i$ and $C_j$, then the angle $yxz$ is smaller than or equal to $2 \arccos(\sqrt{3} - 1) = 85.8828\ldots ^\circ$.

**Proof** Following [1,4], the existence of a supporting hyperplane of $C_i$ that separates $\text{int}(C_i)$ from $\text{int}(C_j)$ suffices. ∎

In [1], it is shown that planar objects satisfying Lemmas 2, 3 and 4 have area no less than $\sqrt{12}$. As the bound holds for all Dirichlet slices, it follows that $\delta^+ (\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ in $\mathbb{R}^{n+2}$. The product of the dense disk packing in the plane with $\mathbb{R}^n$ gives a polycylinder packing in $\mathbb{R}^{n+2}$ that achieves this density. Combining this with the result of Thue [6] for $n = 0$ and the result of Bezdek and Kuperberg [1] for $n = 1$, Theorem 1 follows.

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**References**

1. Bezdek, A., Kuperberg, W.: Maximum density space packing with congruent circular cylinders of infinite length. Mathematika 37(1), 74–80 (1990)
2. Blichfeldt, H.F.: The minimum value of quadratic forms, and the closest packing of spheres. Math. Ann. 101(1), 605–608 (1929)
3. Tóth, G.F., Kuperberg, W.: Blichfeldt’s density bound revisited. Math. Ann. 295(1), 721–727 (1993)
4. Kusner, W.: Upper bounds on packing density for circular cylinders with high aspect ratio. Discrete Comput. Geom. 51(4), 964–978 (2014)
5. Rankin, R.A.: On the closest packing of spheres in $n$ dimensions. Ann. Math. 48, 228–229 (1947)
6. Thue, A.: On the densest packing of congruent circles in the plane. Skr. Vidensk-Selsk, Christiania 1, 3–9 (1910)