On Local Behavior of Holomorphic Functions 
Along Complex Submanifolds of $\mathbb{C}^N$

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Abstract

In this paper we establish some general results on local behavior of holomorphic functions along complex submanifolds of $\mathbb{C}^N$. As a corollary, we present multi-dimensional generalizations of an important result of Coman and Poletsky on Bernstein type inequalities on transcendental curves in $\mathbb{C}^2$.

1. Formulation of the Main Result

1.1. In this paper we establish some general results on restrictions of holomorphic functions to complex submanifolds of $\mathbb{C}^N$. The subject pertains to the area of the, so-called, polynomial inequalities for analytic and plurisubharmonic functions that includes, in particular, Bernstein, Markov and Remez type inequalities. Recently there has been a considerable interest in such inequalities in connection with various problems of analysis. Let us recall that the classical univariate inequalities for polynomials have appeared in approximation theory and for a long time have been considered as technical tools for proofs of Bernstein type inverse theorems. At the present time polynomial type inequalities have been found a lot of important applications in areas which are well apart from approximation theory. We will only briefly mention several of these areas.

The papers [GM], [Bou] and [KLS] apply polynomial inequalities with different integral norms to study some problems of Convex Geometry (in particular, the famous Slice Problem).

In the papers [B1], [B2], [BB], [G], [P] and [PP] and books [DS] and [JW] Chebyshev-Bernstein and related Markov type inequalities are used to explore a wide range of properties of the classical spaces of smooth functions including Sobolev type embeddings and trace theorems, extensions and differentiability.

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The papers [FN1] and [FN2] on Bernstein type inequalities for traces of polynomials to algebraic varieties were inspired by and would have important applications to some basic problems of the theory of subelliptic differential equations.

The paper [BLMT] discovers a profound relation between the exponents in the tangential Markov inequalities for restrictions of polynomials to a smooth manifold $M \subset \mathbb{R}^N$ and the property of $M$ to be an algebraic manifold.

An application of polynomial inequalities to Cartwright type theorems for entire functions is presented in [Br1] and [Br2], see also [LL], [Lo], [K].

In [T1], [T2] Bernstein type inequalities are used to obtain new results in transcendental number theory.

Finally, we mention applications of polynomial inequalities to the second part of Hilbert’s sixteenth problem concerning the number of limit cycles of planar polynomial vector fields, see [I], [RY], [Br3] and [Br4].

In [CP] Coman and Poletsky obtained an important result on Bernstein type inequalities for restrictions of holomorphic polynomials to certain transcendental curves in $\mathbb{C}^2$. The main purpose of our paper is to present a general approach to such kind of inequalities. As an application, we obtain multi-dimensional generalizations of the result of [CP].

1.2. In what follows by $\mathbb{B}^n_r(z_0) \subset \mathbb{C}^n$ we denote the open Euclidean ball of radius $r$ centered at $z_0$. We set

$$\mathbb{B}^n_r := \mathbb{B}^n_r(0), \quad \mathbb{B}^n := \mathbb{B}^n_1, \quad \mathbb{D}^n_r(z_0) := \mathbb{B}^n_r(z_0), \quad \mathbb{D}^n := \mathbb{B}^n_1, \quad \mathbb{D} := \mathbb{B}^1.
$$

By $\overline{S}$ and $\partial S$ we denote the closure and the boundary of $S \subset \mathbb{C}^n$.

For a continuous function $f : \mathbb{B}^n_r(z_0) \to \mathbb{C}$ we define

$$M_f(r, z_0) := \sup_{\mathbb{B}^n_r(z_0)} |f|, \quad m_f(r, z_0) := \ln M_f(r, z_0).$$

If $z_0 = 0$ we set $M_f(r) := M_f(r, 0), m_f(r) := m_f(r, 0)$.

Next, by $\mathcal{L}_n$ we denote the family of complex lines $l \subset \mathbb{C}^n$ passing through the origin. For each $l \in \mathcal{L}_n$ we naturally identify $l \cap \mathbb{B}^n_r$ with $\mathbb{D}_r$.

The main result of the paper (Theorem 1.2) consists of two inequalities that express in a quantitative way the following simple fact.

Let $f$ be a nonconstant holomorphic function in $\mathbb{B}^n_r$ and $g$ be a holomorphic function in the domain $\Omega := \mathbb{B}^n_r \times \mathbb{D}_{M_f(r)} \subset \mathbb{C}^{n+1}$. Suppose that

1. There exist a complex line $l \in \mathcal{L}_n$, a vector $(0, v) \in \mathbb{C}^{n+1}$, $v \in \mathbb{C}$, and a positive number $N \in \mathbb{N}$ such that the affine line $l(v) := l + (0, v)$ intersects the graph $\Gamma_f := \{(z, f(z)) \in \mathbb{C}^{n+1} : z \in \mathbb{B}^n_r\}$ transversely in at least $N$ points;

2. The univariate holomorphic function $g|_{l(v) \cap \Omega}$ is nonconstant and has less than $N$ zeros.

Then the restriction $g|_{\Gamma_f}$ is not identically zero.

As a corollary of Theorem 1.2 we obtain Bernstein, Markov, Remez and Jensen type inequalities for $g|_{\Gamma_f}$, see section 2.1. Also, for $f$ being a nonpolynomial entire
function on \( \mathbb{C}^n \) our result leads to a generalization of one of the main results of [CP], see section 2.2.

To give the precise formulation of the result we introduce some notations.

Assume that \( f : \mathbb{D}_r(z_0) \to \mathbb{C} \) is holomorphic. By \( n_f(r, z_0) \) we denote the number of zeros of \( f \) in \( \mathbb{D}_r(z_0) \). (We write \( n_f(r, z_0) = -\infty \) if \( f \equiv 0 \).) Then the valency of \( f \) in \( \mathbb{D}_r(z_0) \) is defined by

\[
v_f(r, z_0) := \sup_{c \in \mathbb{C}} n_{f+c}(r, z_0).
\]

Also, the Bernstein index \( b_f \) of \( f \) is given by the formula

\[
b_f(r, z_0) := \sup \{ m_f(es, z) - m_f(s, z) \}
\]

where the supremum is taken over all \( \mathbb{D}_{es}(z) \subset \subset \mathbb{D}_r(z_0) \). (If \( f \equiv 0 \) we assume that \( b_f(r, z_0) = 0 \).)

Let us mention that the values of \( n_f, v_f \) and \( b_f \) are finite for a nonzero \( f \) defined in a neighbourhood of the closure of \( \mathbb{D}_r(z_0) \). We set for brevity

\[
n_f(r) := n_f(r, 0), \quad v_f(r) := v_f(r, 0), \quad b_f(r) := b_f(r, 0).
\]

1.3. We are ready to formulate the main result of the paper.

Suppose that \( f \) is a holomorphic function in \( \mathbb{B}_{tr}^n, \ r > 0, \ 1 < t \leq 9, \) satisfying

\[
M_f(r/t) \geq M_1, \quad M_f(tr) \leq M_2 \quad \text{and} \quad R_f(r, t, t^2) \geq t
\]

where

\[
R_f(r, t, s) := \frac{M_f(r/t)}{M_f(r/s)}, \quad t \leq s < \infty.
\]

(The last inequality in (1.1) is valid, e.g., if \( f(0) = 0 \).)

For every \( l \in \mathcal{L}_n \) we set

\[
f_l := f|_{l \cap \mathbb{B}_t^n},
\]

and determine positive numbers \( V_f(r, t) \) and \( N_f(r, t) \) by the formulas

\[
V_f(r, t) := \inf_l \{ v_f(r/\sqrt{t}) : f_l \neq \text{const} \},
\]

\[
N_f(r, t) := \begin{cases} 
   v_f(r/\sqrt{t}) & \text{if } n = 1 \\
   \max \left\{ \sup_{s \in [t, \infty)} \left\{ \frac{\ln(R_f(r, t, s)/\sqrt{t})}{k(t, s)} \right\}, V_f(r, t) \right\} & \text{if } n \geq 2
\end{cases}
\]

where

\[
k(t, s) := \ln \left( \frac{8e^{\pi^2}s\sqrt{t}}{(\sqrt{t} - 1)^2} \right).
\]

In Lemma 6.1 we show that if \( n = 1 \), then

\[
\frac{\ln R_f(r, t, s)}{k(t, s)} \leq v_f(r/\sqrt{t}) \quad \text{for each } s \in [t, \infty).
\]
This, in particular, implies that
\[
\inf_{t} v_{f}(r/\sqrt{t}) \leq N_{f}(r, t) \leq \sup_{t} v_{f}(r/\sqrt{t}).
\]

Let \( g \) be a holomorphic function in the domain \( \mathbb{B}_{tr}^{n} \times \mathbb{D}_{3M_{2}} \subset \mathbb{C}^{n+1} \). For every \( l \in \mathcal{L}_{n} \) we determine
\[
g_{l} := g|_{\Omega_{l}} \quad \text{where} \quad \Omega_{l} := (l \cap \mathbb{B}_{tr}^{n}) \times \mathbb{D}_{3M_{2}}.
\]

**Definition 1.1** We say that the function \( g \) belongs to the class \( \mathcal{F}_{p,q}(r; t; M_{2}) \) for some \( p, q \geq 0 \) if
\[
M_{g(z,w)}(tr) \leq e^{\rho} \cdot M_{g(z,w)}(r) \quad \text{for all} \quad l \in \mathcal{L}_{n}, \quad w \in \mathbb{D}_{3M_{2}} \quad \text{and} \quad z \in \mathbb{B}_{tr}^{n}.
\]

Set
\[
g_{f}(z) := g(z, f(z)), \quad z \in \mathbb{B}_{tr}^{n}.
\]
(This function is the restriction of \( g \) to the graph \( \Gamma_{f} \subset \mathbb{C}^{n+1} \) of \( f \).)

Our main result is

**Theorem 1.2** Assume that
\[
p \leq \ln \left( \frac{1 + t}{2\sqrt{t}} \right) \cdot N_{f}(r, t).
\]
Then there are positive constants \( a_{1}(t), a_{2}(t) \) such that for any \( g \in \mathcal{F}_{p,q}(r; t; M_{2}) \)
\[
\sup_{\mathbb{B}_{tr}^{n} \times \mathbb{D}_{3M_{2}}} |g| \leq \left( \frac{a_{1}(t)M_{2}}{M_{1}} \right)^{a_{2}(t)(p+q)} M_{g_{f}}(r),
\]
\[
M_{g_{f}}(tr) \leq \left( \frac{a_{1}(t)M_{2}}{M_{1}} \right)^{a_{2}(t)(p+q)} M_{g_{f}}(r)
\]
where
\[
a_{1}(t) \leq \frac{300(\sqrt{t} + 1)t^{3/2}}{(t - 1)^{2}}, \quad a_{2}(t) \leq \frac{36(\sqrt{t} + 1)^{2} + 162 \ln \left( \frac{108e}{\sqrt{t} - 1} \right)}{(t - 1)^{4}}.
\]

**Remark 1.3** (A) As we will see from the proof inequality (1.9) guarantees the fulfillment of conditions (1), (2) of the statement formulated in section 1.2.

(B) A similar to Theorem 1.2 result is valid for \( f \) satisfying \( R_{f}(r, t, t^{2}) < t \). In this case the function \( \tilde{f} := f - f(0) \) satisfies (1.1) with \( M_{1} := M_{f}(r/t) \) and \( M_{2} := M_{\tilde{f}}(tr) \). Thus if \( g \) is such that \( \tilde{g} \in \mathcal{F}_{p,q}(r; t; M_{\tilde{f}}(tr)) \) with \( p \leq \ln \left( \frac{1 + t}{2\sqrt{t}} \right) N_{f}(r, t) \), where \( \tilde{g}(z, w) := g(z, w + f(0)) \), \((z, w) \in \mathbb{B}_{tr}^{n} \times \mathbb{D}_{3M_{\tilde{f}}(tr)} \), then (since \( \tilde{g}_{f} = g_{f} \))
\[
\sup_{\mathbb{B}_{tr}^{n} \times \mathbb{D}_{M_{\tilde{f}}(tr)}} |\tilde{g}| \leq \left( \frac{a_{1}(t)M_{\tilde{f}}(tr)}{M_{\tilde{f}}(r/t)} \right)^{a_{2}(t)(p+q)} M_{g_{f}}(r),
\]
\[
M_{g_{f}}(tr) \leq \left( \frac{a_{1}(t)M_{\tilde{f}}(tr)}{M_{\tilde{f}}(r/t)} \right)^{a_{2}(t)(p+q)} M_{g_{f}}(r).
\]
The proof of Theorem 1.2 is based on Cartan type inequalities for univariate holomorphic functions along with some geometric arguments. It is presented in sections 4-6. In the next section we formulate several corollaries of Theorem 1.2 illustrated by some examples.

2. Applications and Examples

2.1. We set
\[ c(M_1, M_2, t) := a_2(t) \cdot \ln \left( \frac{a_1(t)M_2}{M_1} \right). \] (2.1)

As a corollary of inequality (1.11) we obtain the following inequalities.

(1) (Bernstein type inequality)
\[ \ln \left( \frac{M_{g_1}(ts)}{M_{g_1}(s)} \right) \leq c(M_1, M_2, t)(p + q), \quad 0 < s \leq r. \] (2.2)

(2) (Markov type inequality)
There is a constant \( c_1(t) > 0 \) such that
\[ M_{D_v(g_1)}(s) \leq \frac{c_1(t) c(M_1, M_2, t)(p + q)}{s} M_{g_1}(s), \] (2.3)
\[ 0 < s \leq r, \quad v \in \mathbb{C}^n, \quad ||v|| = 1. \]

Here \( || \cdot || \) is the \( l_2 \)-norm on \( \mathbb{C}^n \) and \( D_v \) is the derivative in the direction \( v \).

(3) (Remez type inequality)
Consider the function \( \Phi(x) := x + \sqrt{x^2 - 1}, \quad |x| \geq 1. \) Then there is a constant \( c_2(t) > 0 \) such that
\[ \ln M_{g_1}(s; z) \leq c_2(t) c(M_1, M_2, t)(p + q) \ln \left( \Phi \left( \frac{1 + 2\sqrt{1 - \lambda}}{1 - 2\sqrt{1 - \lambda}} \right) \right) + \]
\[ \sup_\omega \ln |g_f| \leq c_2(t) c(M_1, M_2, t)(p + q) \ln \left( \frac{8n}{\lambda} \right) + \sup_\omega \ln |g_f| \] (2.4)
for every Lebesgue measurable \( \omega \subset \mathbb{B}^n_s(z) \) with \( \lambda := \frac{\lambda_2n(\omega)}{\lambda_2n(\mathbb{B}^n_s(z))} \) and every ball \( \mathbb{B}^n_s(z) \subset \mathbb{B}^n_r \) (here \( \lambda_2n \) is the Lebesgue measure on \( \mathbb{C}^n \)).

(4) (Jensen type inequality)
If \( n = 1 \), then
\[ n_{g_f}(r) \leq \frac{c(M_1, M_2, t)(p + q)}{\ln \left( \frac{1 + t^2}{2t} \right)}. \] (2.5)
**Remark 2.1** It is well known how to derive inequalities (2.2)-(2.5) from (1.11). For instance, (2.2) and (2.3) are obtained by means of the Hadamard three circle inequality, see (3.1), and the Cauchy integral formula for holomorphic functions (see section 3). Inequality (2.5) is obtained by the Jensen type inequality for the number of zeros of a holomorphic function proved in [VP] (see (3.5)). Finally, to get inequality (2.4) one repeats literally the arguments of the proof of Theorem 1.2 of [Br5] replacing inequalities (2.31) and (2.26) of [Br5] by their sharp forms presented in [BG], see there Lemmas 3 and 1.

**Example 2.2** Assume that $f$ is a holomorphic homogeneous polynomial on $\mathbb{C}^n$ of degree $d \geq 1$. Then $f$ clearly satisfies conditions (1.1) for each $r$ with $t = 9$ and $M_1 := M_f(r/9)$, $M_2 := M_f(9r)$. Now, according to (2.1), (1.12) and (1.4) we have for some $c_1 < 87$, $c_2 < 510$,

$$c(M_1, M_2, 9) < c_1 \ln(c_2 \cdot (81)^d) \quad \text{and} \quad N_f(r, 9) = d. \tag{2.6}$$

Then for a function $g \in \mathcal{F}_{p,q}(r; 9; M_2)$ with $p \leq \ln(5/3) \cdot d$ Theorem 1.2 implies

$$\sup_{\mathbb{B}^p_r \times \overline{D}_M} |g| \leq (81)^{87(d+2)(p+q)} M_{g_f}(r), \quad M_{g_f}(9r) \leq (81)^{87(d+2)(p+q)} M_{g_f}(r). \tag{2.7}$$

In particular, if $g$ is a holomorphic polynomial of degrees $k$ in $z \in \mathbb{C}^n$ and $l$ in $w \in \mathbb{C}$, then by the classical Bernstein inequality we have $g \in \mathcal{F}_{p,q}(r; 9; M_2)$ with $p := k \ln 9$, $q := l$. In this case inequalities (2.7) are valid for all $k \leq \frac{\ln(5/3) \cdot d}{\ln 9} < \frac{1}{4}d$.

The last estimate is sharp up to an absolute factor. Indeed, for a univariate holomorphic polynomial $h$ of degree $l - 1$, the polynomial $g(z, w) := (w - f(z))h(w)$ of degree $d$ in $z$ belongs to the class $\mathcal{F}_{p,q}(r; 9; M_2)$ with $p = d \ln 9$, $q = l$. Since $g_f \equiv 0$, it does not satisfy the first inequality in (2.7). Thus inequalities (2.7) hold for all polynomials of degrees $k$ in $z$ and $l$ in $w$, if $k < d$.

Next, for the class $\mathcal{F}_{d,d}(r; 9; M_2)$ we determine the constant $\gamma_d(r; M_2)$ by the formula

$$\gamma_d(r; M_2) := \sup_{g \in \mathcal{F}_{d,d}(r; 9; M_2)} \left\{ \sup_{\mathbb{B}^p_r \times \overline{D}_M} \ln |g| - \ln M_{g_f}(r) \right\} \tag{2.8}$$

where the supremum is taken over all $g \neq 0$.

Observe that the polynomial $g(z, w) := w^d \in \mathcal{F}_{d,d}(r; 9; M_2)$ and satisfies

$$\sup_{\mathbb{B}^p_r \times \overline{D}_M} |g| := (M_f(9r))^d = 9^d M_{g_f}(r).$$

This and (2.7) imply that

$$\ln 9 \cdot d^2 \leq \gamma_d(r; M_2) \leq 348 \ln 9 \cdot (d + 2)d. \tag{2.9}$$

Thus in the case $p = q = d$ the logarithm of the constant in (2.7) up to an absolute factor coincides with the optimal constant $\gamma_d(r; M_2)$. 


One can also obtain analogs of inequalities (1.10), (1.11) for restrictions of holomorphic functions to complex submanifolds of $\mathbb{C}^N$ of codimension $\geq 2$. However, in general the application of Theorem 1.2 requires some additional conditions imposed on these submanifolds. In this paper we present only the case of complex curves in $\mathbb{C}^N$ for which no additional conditions are required.

Assume that holomorphic functions $f_i$ on $D_{tr}$ satisfy conditions (1.1) with bounds $M_{i1}$ and $M_{i2}$, $1 \leq i \leq k$. We fix a permutation $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, k\}$ such that

$$N_{f_{i_1}}(r, t) \leq N_{f_{i_2}}(r, t) \leq \cdots \leq N_{f_{i_k}}(r, t). \quad (2.10)$$

Let $g$ be a holomorphic function in the domain $D_{tr} \times D_{M_{i_2}} \times \cdots \times D_{M_{i_k}} \subset \mathbb{C}^{k+1}$. Suppose that for some nonnegative $p, q_1, \ldots, q_k$ and all $1 \leq i \leq k$

$$g(\cdot, w_1, \ldots, w_{i-1}, \cdot, w_{i+1}, \ldots, w_k) \in F_{p,q_i}(r; t; M_{i2}) \text{ for all } w_j \in D_{M_{j2}}, j \neq i. \quad (2.11)$$

We set

$$\Phi(z) := (f_1(z), \ldots, f_k(z)) \text{ and } g_\Phi(z) := g(z, \Phi(z)), \quad z \in D_{tr}. \quad (2.12)$$

Next, we determine the sequence of nonnegative numbers $p_0, p_1, \ldots, p_k$ by the formulas

$$p_0 := p \quad \text{and} \quad p_j := c(M_{j1}, M_{j2}, t)(p_{j-1} + q_{i_j}), \quad 1 \leq j \leq k, \quad (2.13)$$

where $c(M_{j1}, M_{j2}, t)$ are defined in (2.1).

**Theorem 2.3** Assume that

$$p_j \leq \ln \left( \frac{1 + t}{2\sqrt{t}} \right) \cdot N_{f_{i_j+1}}(r, t) \quad \text{for all} \quad 0 \leq j \leq k - 1.$$  

Then

$$\max_{D_{tr} \times D_{M_{i_2}} \times \cdots \times D_{M_{i_k}}} |g| \leq e^{p_1 + \cdots + p_k} M_{g_\Phi}(r) \quad \text{and} \quad M_{g_\Phi}(tr) \leq e^{p_k} M_{g_\Phi}(r). \quad (2.14)$$

**Remark 2.4** Since $N_{f_j}(r) := v_{f_j}(r/\sqrt{t}) \geq 1$ for all $1 \leq j \leq k$, inequalities (2.14) are always valid for functions $g$ with sufficiently small $p, q_0, \ldots, q_{k-1}$.

2.2. In [CP] Coman and Poletsky obtained an important result on polynomial type inequalities for restrictions of holomorphic polynomials to certain transcendental curves in $\mathbb{C}^2$. In this part we establish some multi-dimensional generalizations of their result that can be considered as corollaries of Theorem 1.2.

Let us recall that an entire function $f$ on $\mathbb{C}^n$ is of order $\rho \geq 0$ if

$$\rho = \limsup_{r \to \infty} \frac{\ln m_f(r)}{\ln r}. \quad (2.15)$$

If $\rho < \infty$, then $f$ is called of finite order.
The following result was proved in [CP, Theorem 1.1]:

**Theorem.** For any entire function $f$ on $\mathbb{C}$ of finite order $\rho > 0$, there exist sequences $\{n_j\} \subset \mathbb{N}$ convergent to $\infty$ and $\{\epsilon_j\} \subset \mathbb{R}_+$ convergent to 0 such that for every holomorphic polynomial $g$ on $\mathbb{C}^2$ of degree $n_j$ one has

$$
\sup_{D \times \overline{D}} |g| \leq e^{C_1n_j^2 \ln n_j} M_{g_r}(1), \quad M_{g_r}(r) \leq e^{C_2n_j^2 \ln r} M_{g_r}(1), \quad 1 \leq r \leq \frac{1}{2} n_j^{1/(\rho - \epsilon_j)}. \quad (2.16)
$$

For every $r \geq 1$ there exists an integer $j_r$ such that if $j \geq j_r$, then

$$
n_{g_r}(r) \leq C_3n_j^2, \quad \frac{M_{g_r}(2r)}{M_{g_r}(r)} \leq 2^\epsilon n_j^2, \quad M_{(g_r)'}(r) \leq C_4n_j^2 \frac{M_{g_r}(r)}{r}. \quad (2.17)
$$

Moreover, all the constants are effectively computed and depend only on $\rho$.

The proof of this theorem is based on the Ahlfors theory of coverings surfaces and certain results of Dufresnoy along with Cartan type estimates.

Let us present a multi-dimensional generalization of this result.

**Theorem 2.5** Let $f$ be a nonpolynomial entire function on $\mathbb{C}^n$ of order $\rho$. Then there exist sequences $\{n_j\}, \{r_j\} \subset \mathbb{R}_+$ convergent to $\infty$ and $\{\epsilon_j\} \subset \mathbb{R}_+$ convergent to 0 such that for every function $g \in F_{p,q}(er; e; M_f(e^2r_j))$ with $p \leq n_j$ and every $1 \leq r \leq r_j$ the following inequalities hold:

(a) $\sup_{\mathbb{B}^n \times \overline{D}} |g| \leq e^{C \rho n_j^{1+\epsilon_j} \ln r \max\{p,q\}} M_{g_r}(1)$;

(b) $M_{g_r}(r) \leq e^{C \rho n_j^{1+\epsilon_j} \ln r \max\{p,q\}} M_{g_r}(1)$;

(c) $\frac{M_{g_r}(er)}{M_{g_r}(r)} \leq e^{C \rho n_j^{1+\epsilon_j} \max\{p,q\}}$;

(d) $M_{D_v(g_r)}(r) \leq c_1C_\rho n_j^{1+\epsilon_j} \max\{p,q\} \frac{M_{g_r}(r)}{r}$, $v \in \mathbb{C}^n$, $||v|| = 1$;

(e) $\ln M_{g_r}(s, z) \leq c_2C_\rho n_j^{1+\epsilon_j} \max\{p,q\} \ln \left( \frac{8\lambda_{2n}(\mathbb{B}^n)}{\lambda_{2n}(\omega)} \right) + \sup_{\omega} \ln |g_r|$ for every Lebesgue measurable set $\omega \subset \mathbb{B}^n_s(z)$ and every ball $\mathbb{B}^n_s(z) \subset \mathbb{B}^n_{r_j}$;

(f) $n_{g_r}(r) \leq c_3C_\rho n_j^{1+\epsilon_j} \max\{p,q\}$ for $n = 1$.

Here $c_1 < 9$, $c_2$ and $c_3 < 5$ are absolute constants and $C_\rho$ depends on $\rho$ only. Moreover,

(1) If $\rho < \infty$, then all $\epsilon_j = 0$ and $r_j \geq n_j^{1/(\rho + \epsilon'_j)}$, $j \in \mathbb{N}$, for some sequence $\{\epsilon'_j\} \subset \mathbb{R}_+$ convergent to 0. Also, $C_\rho \leq c(\ln(\rho + 1) + 1)^2(\rho + 1)^7$ for an absolute constant $c > 0$. 


(2) If \( 0 < \rho < \infty \), then \( r_j \leq c_\rho n_j^{1/(\rho - \epsilon_j)} \), \( j \in \mathbb{N} \), for a sequence \( \{\epsilon_j\} \subset \mathbb{R}_+ \) convergent to 0. Here \( c_\rho \leq (\frac{1}{\rho^*})^{1/\rho^*} \) for an absolute constant \( \tilde{c} > 0 \), and \( \rho^* := \min\{1, \rho\} \).

(3) If \( \rho = \infty \), then \( r_j := \frac{1}{q} m_j^{-1}(n_j^{1+\epsilon_j''}) \), \( j \in \mathbb{N} \), for some sequence \( \{\epsilon_j''\} \subset \mathbb{R}_+ \) convergent to 0, and \( C_\infty = 1 \).

**Remark 2.6** In the case \( \rho = \infty \) we prove that \( \ln r_j \leq n_j^{\delta_j} \) for some \( \{\delta_j\} \subset \mathbb{R}_+ \) convergent to 0, see (8.36), (8.37). Thus one can replace \( n_j^{1+\epsilon_j} \ln r_j \) in inequality (a) by \( n_j^{1+\tilde{\epsilon}_j} \) for some \( \{\tilde{\epsilon}_j\} \subset \mathbb{R}_+ \) convergent to 0.

**Example 2.7** (A) If \( g \) is a holomorphic polynomial of degree \( \leq n_j \) on \( \mathbb{C}^{n+1} \), then by the classical Bernstein inequality \( g \in \mathcal{F}_{p,q}(er_j; e; M_f(e^2 r_j)) \) with \( p = q \leq n_j \). Thus Theorem 2.5 can be applied to such \( g \).

(B) Let \( f \) be an entire function on \( \mathbb{C}^n \) of order \( 1 < \rho < \infty \) and \( g \) be an exponential polynomial on \( \mathbb{C}^{n+1} \), that is,

\[
g(z, w) = \sum_{j=1}^{m} p_j(z, w)e^{l_j(z,w)}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},
\]

where \( p_j \) is a holomorphic polynomial on \( \mathbb{C}^{n+1} \) of degree \( d_j \) and \( l_j \) is a complex linear functional on \( \mathbb{C}^{n+1} \) of \( l_2 \)-norm \( v_j \), \( 1 \leq j \leq m \).

The expression

\[
m(g) := \sum_{j=1}^{m} (1 + d_j)
\]

is called the *degree* of \( g \). Also, the *exponential type* of \( g \) is defined by the formula

\[
\epsilon(g) := \max_{1 \leq j \leq m} v_j.
\]

Next, let \( l \in \mathcal{L}_n \) be a complex line passing through the origin. We naturally identify it with \( \mathbb{C} \) and define the exponential polynomial \( g_l \) on \( \mathbb{C} \times \mathbb{C} \) by the formula

\[
g_l(z, w) := g(z, w), \quad z \in l.
\]

Then by [VP, page 27, formula (21)] with \( S^* := e^2 r_j \), \( S := er_j \) we obtain

\[
M_{g_l(., w)}(e^2 r_j) \leq e^{m(g)+2e\epsilon(g)r_j} M_{g_l(., w)}(er_j) \quad \text{for all} \quad w \in \mathbb{C}.
\]

Similarly, by the same formula we have

\[
b_{g_l(., .)}(3M_f(e^2 r_j)) \leq m(g) + 6\epsilon(g)M_f(e^2 r_j) \quad \text{for all} \quad z \in l.
\]

Therefore, \( g \in \mathcal{F}_{p,q}(er_j; e; M_f(e^2 r_j)) \) with \( p := m(g) + 2e^2\epsilon(g)r_j \leq m(g) + 2e^2c_\rho\epsilon(g)n_j^{1/(\rho - \tilde{\epsilon}_j)} \) and \( q := m(g) + 6\epsilon(g)M_f(e^2 r_j) \), see Definition 1.1. Since \( \rho > 1 \), for all sufficiently large \( j \) we have \( p \leq n_j \). Hence for such \( j \) we can apply Theorem
2.5. Also, observe that $2e^2 r_j \leq 6 M_f(e^2 r_j)$ for all sufficiently large $j$. In particular, \( \max\{p, q\} = q \) for such $j$ and inequalities of Theorem 2.5 are valid with \( \max\{p, q\} \) substituted for $m(g) + 6 \epsilon(g) M_f(e^2 c\epsilon n_j^{1/(\rho - \epsilon_j^j)}) \leq m(g) + \epsilon(g) e^{n_j^{1+\epsilon_j}}$ for some \( \{\epsilon_j\} \subset \mathbb{R}_+ \) convergent to 0.

Suppose now that the functionals $l_j$ in the definition of $g$ do not depend on $w$. Thus, for a fixed $z \in \mathbb{C}^n$, the function $g(z, \cdot)$ is a polynomial of degree \( \leq d := \max_{1 \leq j \leq m} d_j \). In particular, instead of (2.20) we have in this case

$$b_{g(z, \cdot)}(3 M_f(e^2 r_j)) \leq d.$$ (2.21)

Therefore for all sufficiently large $j$ inequalities of Theorem 2.5 are valid with \( \max\{p, q\} \) substituted for $m(g) + 2e^2 c\rho \epsilon(g) n_j^{1/(\rho - \epsilon_j^j)}$.

Now, let us formulate some conditions under which inequalities of Theorem 2.5 are valid for all sufficiently large $n_j$ and $r_j$.

For a nonconstant entire function $f$ on $\mathbb{C}^n$ of order $\rho$ we set

$$\phi_f(t) := m_f(e^t), \quad t \in \mathbb{R}.$$ 
Then $\phi_f$ is a convex increasing function, and so the derivative $\phi'_f$ exists and is continuous outside a countable set $S \subset \mathbb{R}$. Also, $\phi'_f$ is a positive nondecreasing function on $\mathbb{R} \setminus S$ having singularities of the first kind at the points of $S$. We extend $\phi'_f$ to $S$ by the formula

$$\phi'_f(s) := \frac{\phi'_f(s+) + \phi'_f(s-)}{2}, \quad s \in S,$$
and call the extended function the derivative of $\phi_f$ on $\mathbb{R}$.

**Theorem 2.8** Assume that $f$ satisfies one of the following conditions

(I) If $\rho < \infty$,

$$\limsup_{t \to \infty} \frac{m_f(e^{\alpha_f t}) - m_f(e^{-\alpha_f t}) + \rho e^\rho t}{m_f(e^{-\alpha_f t}) - m_f(e^{-2\alpha_f t})} < A < \infty$$

where $\alpha_\rho := \min\{1, \ln(1 + 1/\rho)\}$.

(II) If $\rho = \infty$,

$$\lim_{t \to \infty} t^2 \left( \frac{1}{\ln \phi_f(t)} \right)' = 0.$$

Then there exist numbers $k_0, r_0 \geq 1$, a continuous increasing to $\infty$ function $r : [k_0, \infty) \to [r_0, \infty)$ and a continuous function $\epsilon : [k_0, \infty) \to \mathbb{R}_+$ decreasing to 0 as $k \to \infty$ such that for each $k \geq k_0$, $r(k) \geq r_0$, every $g \in \mathcal{F}_{p,q}(er(k); e; M_f(e^2 r(k)))$ with $p \leq k$ and every $1 \leq r \leq r(k)$ the following inequalities hold:

(a) $\sup_{\mathbb{R}^n \times \mathbb{D}} |g| \leq e^{Ck^{1+\epsilon(k)} \ln r(k) \max\{p, q\} M_{g_f}(1)}$.
(b) $M_{g_f}(r) \leq e^{C k^{1+\varepsilon(k)}} \ln r \max\{p,q\} M_{g_f}(1)$;

c) $\frac{M_{g_f}(er)}{M_{g_f}(r)} \leq e^{C k^{1+\varepsilon(k)} \max\{p,q\}}$;

d) $M_{P_c(g_f)}(r) \leq c_1 C k^{1+\varepsilon(k)} \max\{p,q\} \frac{M_{g_f}(r)}{r}$, $v \in \mathbb{C}^n$, $||v|| = 1$;

e) $\ln M_{g_f}(s; z) \leq c_2 C k^{1+\varepsilon(k)} \max\{p,q\} \ln \frac{8\lambda_{2n}(\mathbb{B}_n)}{\lambda_{2n}(\omega)} + \sup_{\omega} \ln |g_f|$ for every Lebesgue measurable set $\omega \subset \mathbb{B}_n(z)$ and every ball $\mathbb{B}_n(z) \subset \mathbb{B}_r$;

(f) $n_{g_f}(r) \leq c_3 C k^{1+\varepsilon(k)} \max\{p,q\}$ for $n = 1$.

Here $c_1 < 9$, $c_2$ and $c_3 < 5$ are absolute constants, for $\rho < \infty$ the constant $C$ depends on $A, \rho$ only and $C = 1$ for $\rho = \infty$.

Moreover,

(1) If $\rho < \infty$, then $\epsilon \equiv 0$ and $r(k) \geq k^{1/(\rho+\epsilon(k))}$, $k \geq k_0$, for some continuous function $\epsilon' : [k_0, \infty) \to \mathbb{R}_+$ decreasing to 0 as $k \to \infty$.

(2) If $0 < \rho < \infty$, then $r(k) \leq c k^{1/\rho}$, $k \geq k_0$, for some $c$ depending on $A, \rho$.

(3) If $\rho = \infty$, then $r(k) = \frac{1}{c_m} m_f^{-1}(k^{1+\epsilon''(k)})$, $k \geq k_0$, for some continuous function $\epsilon'' : [k_0, \infty) \to \mathbb{R}_+$ decreasing to 0 as $k \to \infty$.

Remark 2.9 (A) In the case $\rho = \infty$ we show that $\ln r(k) \leq k^{\delta(k)}$ for a continuous function $\delta : [k_0, \infty) \to \mathbb{R}_+$, decreasing to 0 as $k \to \infty$, see (9.11), (9.12). Thus one can replace $k^{1+\epsilon(k)} \ln r(k)$ in inequality (a) by $k^{1+\tilde{\epsilon}(k)}$ for some continuous function $\tilde{\epsilon} : [k_0, \infty) \to \mathbb{R}_+$ decreasing to 0 as $k \to \infty$.

(B) As an example of function $f$ satisfying condition (I) one can take, e.g.,

$$f(z) = \sum_{j=1}^{m} p_j(z) e^{q_j(z)}$$

where $p_j, q_j$ are holomorphic polynomials on $\mathbb{C}^n$. (In this case $\lim_{r \to \infty} m_{f(r)} = a > 0$.)

(C) As an example of function $f$ satisfying condition (II) one can take, e.g.,

$$f(z) = e^{h(z)} \quad \text{where} \quad h(z) = \sum_{j=1}^{m} p_j(z) e^{q_j(z)}$$

and $p_j, q_j$ are holomorphic polynomials with nonnegative coefficients on $\mathbb{C}^n$.

Following [CP] for an entire function $f$ on $\mathbb{C}^n$ we define

$$m_k(r, f) := \sup \{ \ln M_{g_f}(r) : g \in \mathcal{P}_{k,n+1}, M_{g_f}(1) \leq 1 \}, \quad r \geq 1. \quad (2.22)$$

where $\mathcal{P}_{k,n+1}$ is the space of holomorphic polynomials of degree $k$ on $\mathbb{C}^{n+1}$.
Next, we introduce the lower order of transcendence of \( f \) as
\[
\tau(f) = \sup \left\{ \tau : \liminf_{k \to \infty} \frac{m_k(e, f)}{k^\tau} > 0 \right\},
\] (2.23)
and the upper order of transcendence of \( f \) as
\[
\bar{\tau}(f) = \inf \left\{ \tau : \limsup_{k \to \infty} \frac{m_k(e, f)}{k^\tau} < \infty \right\}.
\] (2.24)

If \( f \) is a polynomial, then using the Bernstein inequality one can show that \( \tau(f) = \bar{\tau}(f) = 1 \). In the case \( n = 1 \) and \( f \) is an entire function of finite positive order it was proved in [CP] that \( \tau(f) = 2 \). Also, for each \( \tau \in [3, \infty] \) there were constructed some examples of entire functions \( f \) of finite positive order for which \( \tau - 1 \leq \bar{\tau}(f) \leq \tau \).

Now, as a corollary of Theorem 2.5 we obtain the following generalization of the above cited result of [CP].

**Corollary 2.10** If \( f \) is a nonpolynomial entire function on \( C^n \), then
\[
1 + \frac{1}{n} \leq \tau(f) \leq 2.
\]

**Remark 2.11** (1) Let us consider the function \( \bar{\tau} : \mathcal{E}_n \to [1 + 1/n, 2] \), \( f \to \bar{\tau}(f) \), defined on the set of all nonpolynomial entire functions on \( C^n \). Since for \( n = 1 \) the lower order of transcendence of any nonpolynomial function is 2, one can easily construct entire functions \( f \) on \( C^n \), \( n > 1 \), for which \( \bar{\tau}(f) = 2 \). Thus 2 belongs to the image of \( \bar{\tau} \). However, we do not know what other numbers from \([1 + 1/n, 2]\) belong to this image.

(2) If \( f \) satisfies conditions of Theorem 2.8, then \( \tau(f) = \tau(f) \).

In the next section we gather some auxiliary results used in the proof of Theorem 1.2. Sections 4-9 are devoted to proofs of our results.

### 3. Auxiliary Results

**3.1.** In our proofs we use the corollary of the classical Hadamard three circle inequality stating that for a holomorphic function \( h \) defined on \( B_{r_2} \), \( r_2 > 0 \),
\[
M_h(r_1) \leq (M_h(r_0))^{1-\theta} (M_h(r_2))^{\theta}, \quad r_0 \leq r_1 \leq r_2, \quad \theta := \frac{\ln(r_1/r_0)}{\ln(r_2/r_0)}.
\] (3.1)

This shows that if \( h \neq 0 \), then the function
\[
\phi_h(t) := m_h(e^t), \quad -\infty < t < \ln r_2,
\]
is convex and nondecreasing. In turn, the latter implies the following inequalities
(a) For each $0 < r < r_2$,
$$\frac{M_h(r)}{M_h(r/e)} \leq \frac{M_h(r_2)}{M_h(r_2/e)}.$$  

(b) If $r_2 > e$, then for each $1 \leq r \leq r_2/e$,
$$\frac{M_h(r)}{M_h(1)} \leq \left( \frac{M_h(\epsilon r)}{M_h(r)} \right)^{\ln r}.$$  

(c) If $1 < t \leq e$, then
$$\frac{M_h(r_2)}{M_h(r_2/e)} \leq \left( \frac{M_h(r_2)}{M_h(r_2/t)} \right)^{\frac{1}{\ln t}}.$$  

3.2. We also use Cartan type inequalities for univariate holomorphic functions.

Let $f$ be a nonzero holomorphic function in the disk $\mathbb{D}_R$. Fix positive $\alpha, \beta$ such that $\alpha < \beta < 1$.

**Theorem 3.1** Let $H$ be a positive number $\leq \beta e$ and $d > 0$. Then there is a family of open disks $\{D_j\}_{1 \leq j \leq k}$, $k \leq n_f(\beta R)$, with $\sum r_j^d \leq \frac{(2HR)^d}{d}$ where $r_j$ is the radius of $D_j$ such that

$$|f(z)| \geq M_f(\beta R) \left( \frac{M_f(\alpha R)}{M_f(\beta R)} \right)^{\left( \frac{\beta + \alpha}{\beta - \alpha} \right)^2} \cdot \left( \frac{H}{\beta e} \right)^{n_f(\beta R)} \geq$$

$$M_f(\beta R) \left( \frac{M_f(\alpha R)}{M_f(\beta R)} \right)^{\left( \frac{\beta + \alpha}{\beta - \alpha} \right)^2} \cdot \left( \frac{M_f(\beta R)}{M_f(R)} \right)^{\frac{\ln(\beta e)}{\ln(1 + \frac{\beta^2}{2\alpha})}} \geq \frac{\ln(\frac{\beta e}{R})}{\ln(\frac{\beta e}{R})^2}$$  

(3.2)

for any $z \in \mathbb{D}_{\alpha R} \setminus \bigcup_j D_j$.

**Proof.** We first prove the theorem for $g(z) := f(\beta Rz)$, $z \in \mathbb{D}_\delta$, and the disks $\mathbb{D}_\gamma \subset \mathbb{D} \subset \mathbb{D}_\delta$ where $\gamma := \alpha/\beta$, $\delta := 1/\beta$.

For $z \in \mathbb{D}$ we write
$$g(z) := B(z) \cdot h(z)$$

where $B$ is the Blaschke product whose zeros are the same as for $g$ (counted with their multiplicities) and $h$ has no zeros in $\mathbb{D}$. Let

$$\rho(z, w) := \frac{|z - w|}{1 - \overline{w}z}.$$
be the pseudohyperbolic metric in \( \mathbb{D} \). Applying to \( \rho \) and \( \log |B| \) the abstract Cartan estimates established in [Br6, Theorem 2.3] we have

Given \( H > 0, \ d > 0 \) there is a family of open \( \rho \)-balls \( \{B_j\}_{1 \leq j \leq k}, \ k \leq n_f(1) = n_f(\beta R) \), with \( \sum r_j^d \leq \frac{(2H)^d}{d} \) where \( r_j \) is the radius of \( B_j \) such that for any \( z \in \mathbb{D} \setminus \bigcup_j B_j \)

\[
|B(z)| \geq \left( \frac{H}{e} \right)^{n_f(\beta R)}.
\] (3.3)

Since each \( B_i \) is the subset of the Euclidean disk \( D_i \) centered at the same point and of the same radius, the above inequality is also valid for each \( z \in \mathbb{D} \setminus \bigcup_j D_j \).

Next, we have \( M_f(\beta R) = M_h(1) \) and \( M_f(\alpha R) = M_g(\gamma) \). These identities imply that \( M_h(\gamma) \geq M_f(\alpha R) \) and that the function \( u(z) := -\ln |h(z)| + \ln M_f(\beta R) \) is nonnegative harmonic in \( \mathbb{D} \). We will apply to \( u \) the classical Harnack inequality.

Take \( w = \gamma e^{i\phi} \) such that \( M_h(\gamma) = |h(w)| \) and let

\[
G(z) = \frac{z + w}{1 + \overline{w}z}
\]

be the Möbius transformation of \( \mathbb{D} \) sending 0 to \( w \). Then \( u_G(z) := u(G(z)) \) is a nonnegative harmonic function in \( \mathbb{D} \) and \( u_G(0) \leq \ln[M_f(\beta R)/M_f(\alpha R)] \). By the Harnack inequality we have

\[
u(0) = u_G(-w) \leq u_G(0)\left[ \frac{1 + |w|}{1 - |w|} \right] \leq \left[ \ln \left( \frac{M_f(\beta R)}{M_f(\alpha R)} \right) \right] \left( \frac{1 + \gamma}{1 - \gamma} \right).
\]

Applying again the Harnack inequality to \( u \) at the points 0 and \( y \) such that \( |y| = \gamma \) and \( u(y) = \sup_{\mathbb{D}_\gamma} u \) and using the previous estimate we have

\[
\sup_{\mathbb{D}_\gamma} u \leq \left[ \ln \left( \frac{M_f(\beta R)}{M_f(\alpha R)} \right) \right] \left( \frac{1 + \gamma}{1 - \gamma} \right)^2.
\]

From here and the definition of \( u \) it follows that for any \( z \in \mathbb{D}_\gamma \)

\[
|h(z)| \geq M_f(\beta R) \cdot \left( \frac{M_f(\alpha R)}{M_f(\beta R)} \right)^{\left( \frac{\beta + \alpha}{\beta - \alpha} \right)^2}.
\] (3.4)

Combining inequalities (3.3), (3.4) and going back to \( f \) we obtain the first inequality of (3.2). To obtain the second inequality we use the estimate from [VP, Lemma 1]:

\[
n_f(\beta R) \leq \frac{\ln \left( \frac{M_f(R)}{M_f(\beta R)} \right)}{\ln \left( \frac{1 + \beta^2}{2\beta} \right)}.
\] (3.5)

Finally, the third inequality is obtained by the application of the Hadamard three circle inequality estimating \( M_f(\beta R) \) by \( M_f(R) \) and \( M_f(\alpha R) \) to the second term of the second line of (3.2) and by replacing \( M_f(\beta R) \) to \( M_f(\alpha R) \) in the third term of that line. \( \square \)

Applying Theorem 3.1 to \( R := tr, \ \beta R := \sqrt{tr} \) and \( \alpha R := r, \ r > 0, \ 1 < t \leq 9, \) we obtain
Theorem 3.2 Let $f$ be a nonzero holomorphic function in $D_{tr}$, $r > 0$, $1 \leq t \leq 9$. Let $H$ be a positive number $\leq e/\sqrt{t}$. Then there is a family of open disks $\{D_j\}_{1 \leq j \leq k}$, $k \leq n_f(\sqrt{t}r)$, with $\sum r_j \leq 2Htr$ where $r_j$ is the radius of $D_j$ such that for each $z \in \mathbb{D}_r \cup \bigcup_j D_j$

$$|f(z)| \geq M_f(\sqrt{t}r) \left( \frac{M_f(r)}{M_f(tr)} \right)^{c(H)}$$ (3.6)

where

$$c(H) := \frac{(\sqrt{t} + 1)^4 + 18(\sqrt{t} + 1)^2 \ln \left( \frac{\sqrt{t}}{H} \right)}{2(t-1)^2}$$ (3.7)

and

$$n_f(\sqrt{t}r) \leq \frac{\ln \left( \frac{M_f(tr)}{M_f(\sqrt{t}r)} \right)}{\ln \left( \frac{1+\sqrt{t}}{2\sqrt{t}} \right)} \leq \frac{9(\sqrt{t} + 1)^2 \ln \left( \frac{M_f(tr)}{M_f(\sqrt{t}r)} \right)}{(t-1)^2}.$$ (3.8)

Proof. Inequality (3.6) follows directly from (3.2) with

$$\frac{1}{2} \left( \frac{\sqrt{t} + 1}{\sqrt{t} - 1} \right)^2 + \frac{\ln \left( \frac{\sqrt{t}r}{H} \right)}{\ln \left( \frac{1+\sqrt{t}}{2\sqrt{t}} \right)}$$

instead of $c(H)$ defined by (3.7). Here

$$\frac{(\sqrt{t} + 1)^2}{(t-1)^2} = \frac{(\sqrt{t} + 1)^4}{(t-1)^2}$$ and

$$\frac{1}{\ln \left( \frac{1+\sqrt{t}}{2\sqrt{t}} \right)} \leq \frac{1}{\ln \left( \frac{1+(\sqrt{t}-1)^2}{6} \right)} \leq \frac{9(\sqrt{t} + 1)^2}{(t-1)^2}.$$ (3.9)

We used that $\ln(1 + x) \geq \frac{2}{3}x$ for $0 \leq x \leq \frac{2}{3}$.

Now, from (3.9) we obtain (3.6) with $c(H)$ given by (3.7) and inequality (3.8).

□

Corollary 3.3 Under the assumptions of Theorem 3.2 there exists a circle $S_l := \{z \in \mathbb{C} : |z| = l\}$, $r/\sqrt{t} \leq l \leq r$, such that for each $z \in S_l$

$$|f(z)| \geq M_f(\sqrt{t}r) \left( \frac{M_f(r)}{M_f(tr)} \right)^{\gamma(t)}$$ (3.10)

where

$$\gamma(t) := \frac{(\sqrt{t} + 1)^4 + 18(\sqrt{t} + 1)^2 \ln \left( \frac{\sqrt{t}}{\sqrt{t}-1} \right)}{2(t-1)^2} > 0.$$ (3.11)

Proof. We apply Theorem 3.2 with $H = \frac{\sqrt{t}-1}{2\sqrt{t}r}$. Then the sum of radii of the disks $D_j$ is $\leq \frac{\sqrt{t}-1}{2\sqrt{t}r}$. In particular, the projection of $\bigcup_j D_j$ onto the radial axis (in polar coordinates of $\mathbb{C}$) is an open set of linear measure $\leq \frac{\sqrt{t}-1}{\sqrt{t}}r$. Therefore this set cannot
cover the closed interval \( \{ s \in \mathbb{R}_+ : r/\sqrt{t} \leq s \leq r \} \). This implies that there is a

\[ S_l = \{ r/\sqrt{t} \leq l \leq r \} \]

which does not intersect \( \bigcup_j D_j \). According to (3.6) \( f|_{S_l} \) satisfies the required estimate. □

3.3. In the proofs we use also the following Markov type inequality.

**Theorem 3.4** Assume that \( h \) is a holomorphic function in the ball \( B^n_{tR}, R > 0, 1 < t \leq 9 \), satisfying for some \( d \geq 0 \)

\[
M_h(tR) \leq e^d M_h(R). \quad (3.12)
\]

Then

\[
M_{D_v h}(R) \leq \frac{\kappa(d; t)}{R} M_h(R) \quad (3.13)
\]

where \( D_v \) is the derivative in the direction \( v \in \mathbb{C}^n, ||v|| = 1, \) and

\[
\kappa(d; t) := \begin{cases} 
\frac{e}{d \ln(1+1/d)} - 1 & \text{if } d \geq \frac{1}{e-1} \\
\frac{2d}{\sqrt{t} - 1} & \text{if } 0 \leq d < \ln \left( \frac{1+t}{2\sqrt{t}} \right) \\
\frac{e^d}{t-1} & \text{if } \ln \left( \frac{1+t}{2\sqrt{t}} \right) \leq d < \frac{1}{e-1}.
\end{cases} \quad (3.14)
\]

(Observe that \( \ln \left( \frac{1+9}{2\sqrt{9}} \right) < \frac{1}{e-1} \) so that formula (3.14) is correct.)

**Proof.** Without loss of generality we may assume that \( h \) is not identically zero. We will consider several cases.

(1) Assume that \( d \geq \frac{1}{e-1} \). Take \( x \in \partial B^n_R \) and let \( l = \{ x + zv : z \in \mathbb{C} \} \) be the complex line passing through \( x \). We set \( D_s = B^n_{sR} \cap l, 1 \leq s \leq t \). Then \( D_s \subset l \) is the disk of radius \( r_s \) centered at \( c \in B^n_R \) where \( c \) is such that \( h := \text{dist}(l, 0) = ||c|| \) and \( r_s := \sqrt{(sR)^2 - h^2} \). We will naturally identify \( D_s \) with \( \mathbb{D}_{r_s} \). It is easy to check that for all \( s \geq q \geq 1 \) the following inequalities hold:

\[
\frac{r_s}{r_q} \geq \frac{s}{q} \quad \text{and} \quad r_s - r_q \geq (s-q)R. \quad (3.15)
\]

We set \( s := t^{\ln(1+1/d)} \) so that \( 1 < s \leq t \). Then by means of the Hadamard three circle inequality, see (3.1), we obtain

\[
M_h(sR) \leq e^{(d\ln s)/(\ln t)} M_h(R) = e^{d\ln(1+1/d)} M_h(R) \leq eM_h(R). \quad (3.16)
\]

Consider the function \( \tilde{h} := h|_{D_s} \) and the disk \( D \subset l \) centered at \( x \in D_1 \) of radius \( (s-1)R \). By (3.15) we have

\[
r_1 + (s-1)R \leq r_s \leq s.
\]
Thus \( D \) belongs to \( D_s \subset \mathbb{B}^n_{sR} \). Now, from the Cauchy integral formula for the derivative of \( \tilde{h} \) in \( D \) by (3.16) we get

\[
| (D_v h)(x) | := | \tilde{h}'(x) | \leq \frac{1}{(s-1)R} M_h(sR) \leq \frac{e}{(t \ln(1 + 1/d) - 1) R} M_h(R). \tag{3.17}
\]

(2) Suppose now that \( d < \ln \left( \frac{1 + t}{2\sqrt{t}} \right) \). Let \( Z_h \) be the zero set of \( h \). We first prove

**Lemma 3.5** Under the above condition \( Z_h \cap \mathbb{B}^n_{\sqrt{t}R} = \emptyset \).

**Proof.** Assume, on the contrary, that there is \( y \in Z_h \cap \mathbb{B}^n_{\sqrt{t}R} \). Take \( v \in \mathbb{B}^n_R \) such that

\[
|h(v)| = \sup_{\mathbb{B}_h^R} |h|.
\]

Let \( l \) be a complex line passing through \( v \) and \( y \). As before we set \( D_s = \mathbb{B}^n_{sR} \cap l \) and identify it with \( \mathbb{D}_{r_s} \) with an appropriate definition \( r_s \) (see case (1)). Then for the function \( \tilde{h} := h|_{D_v} \) we have by [VP, Lemma 1]

\[
n_{\tilde{h}}(r_{\sqrt{t}}) \leq \frac{\ln \left( \frac{M_h(r_s)}{M_h(r_{\sqrt{t}})} \right)}{\ln \left( \frac{1 + (r_s/r_{\sqrt{t}})^2}{2(r_s/r_{\sqrt{t}})} \right)} \leq \frac{\ln \left( \frac{M_h(tR)}{M_h(R)} \right)}{\ln \left( \frac{1 + (r_s/r_{\sqrt{t}})^2}{2(r_s/r_{\sqrt{t}})} \right)} \leq \frac{\ln \left( \frac{1 + \frac{r_s}{r_{\sqrt{t}}}}{2} \right)}{\ln \left( \frac{1 + \frac{r_s}{r_{\sqrt{t}}}}{2} \right)} \leq 1.
\]

We used here that \( M_{\tilde{h}}(r_{\sqrt{t}}) \geq M_h(R) \) (by the choice of \( l \)), the function \( x \mapsto \ln \left( \frac{1 + x^2}{2x} \right) \) is increasing for \( x \geq 1 \) and \( \sqrt{t} \leq \frac{r_s}{r_{\sqrt{t}}} \), see (3.15).

Thus \( \tilde{h} \) has no zeros in \( D_{\sqrt{t}} \). This contradicts to the assumption \( y \in Z_h \cap D_{\sqrt{t}} \).

Continuing the proof of the theorem consider the line \( l \) as in the proof of case (1). Then according to the lemma the corresponding function \( \tilde{h} := h|_{D_v} \) has no zeros on \( \mathbb{D}_{r_{\sqrt{t}}} \). In particular, the holomorphic function \( g := \ln(\tilde{h}/M_h(R)) \) is well defined there (for some choice of the branch of the logarithm). Also, the function \( g + \overline{g} = \ln |\tilde{h}/M_h(R)|^2 \) is harmonic on \( \mathbb{D}_{r_{\sqrt{t}}} \). Now from the Cauchy integral formula in the disk centered at \( x \in \mathbb{D}_{r_1} \) of radius \( (\sqrt{t} - 1)R \) we obtain

\[
\frac{|\tilde{h}'(x)|}{|\tilde{h}(x)|} = |g'(x)| \leq \frac{\sup_{r_{\sqrt{t}}} |g + \overline{g}|}{(\sqrt{t} - 1)R} \leq \frac{\sup_{r_{\sqrt{t}}} |\ln |\tilde{h}/M_h(R)||^2}{(\sqrt{t} - 1)R} \leq \frac{2d}{(\sqrt{t} - 1)R}.
\]

(We used here that \( r_1 + (\sqrt{t} - 1)R \leq r_{\sqrt{t}} \leq \sqrt{t}R \), see (3.15).)

This implies

\[
| (D_v h)(x) | := | \tilde{h}'(x) | \leq \frac{2d}{(\sqrt{t} - 1)R} M_h(R). \tag{3.18}
\]

(3) Finally, assume that \( \ln \left( \frac{1 + t}{2\sqrt{t}} \right) \leq d < \frac{1}{e - 1} \). Applying to \( \tilde{h} := h|_{D_v} \) the Cauchy integral formula for \( x \in D_1 \subset l \) (with \( l \) as in case (1)) we have:

\[
| (D_v h)(x) | := | \tilde{h}'(x) | \leq \frac{1}{(t - 1)R} M_{\tilde{h}}(r_1) \leq \frac{e^d}{(t - 1)R} M_h(R) \leq \frac{e^{3/(e - 1)d}}{(t - 1) \ln \left( \frac{1 + t}{2\sqrt{t}} \right) R} M_h(R). \tag{3.19}
\]
Inequalities (3.17), (3.18), (3.19) imply inequality (3.13). □

Remark 3.6 (1) If $t = e$, then
\[
k(\sqrt{d}; e) \leq \max \left\{ ed, \frac{2d}{\sqrt{e-1}}, \frac{e^{1/e-1}d}{(e-1)\ln\left(\frac{1+e}{2\sqrt{e}}\right)} \right\} < 9d. \tag{3.20}\]

(2) For $d \geq \frac{1}{e-1}$ we have by the mean-value inequality for $f(x) := x^{\ln t},$
\[
k(d; t) := \frac{e}{t^{\ln(1+1/d)} - 1} = \frac{e}{(1+1/d)^{\ln t} - 1} \leq \begin{cases} \frac{ed}{\ln t} & \text{if } t \geq e \\ \frac{e^2d}{t\ln t} & \text{if } t < e. \end{cases} \tag{3.21}\]

4. A Geometric Result

4.1. The proof of Theorem 1.2 is based on the following result.
Let $F$ be a nonconstant holomorphic function in $\mathbb{D}_t$, $1 < t \leq 9$, satisfying
\[M_F(t) \leq 1, \quad M_F(1/t) \geq M \quad \text{and} \quad \frac{M_F(1/t)}{M_F(1/t^2)} \geq \sqrt{t}. \tag{4.1}\]
We set
\begin{align*}
N_F(t) &= N_F(1, t) := v_F(1/\sqrt{t}), \quad \lambda(t) := \frac{9(\sqrt{t}+1)^2}{(t-1)^2} \ln\left(\frac{2(\sqrt{t}+1)}{M(t-1)^2}\right), \\
\gamma(t) &= \frac{(\sqrt{t}+1)^4 + 18(\sqrt{t}+1)^2 \ln\left(\frac{4t^{3/2}}{\sqrt{t}-1}\right)}{2(t-1)^2}, \quad r_0(t) := \left(\frac{M(t-1)}{4(\sqrt{t}+1)}\right)^{\gamma(t)+1}. \tag{4.2}\end{align*}

Theorem 4.1 There is a number $c \in \mathbb{C}, \ |c| < 1$, and for each $y \in \mathbb{C}, \ |y| \leq r_0(t)$, and $s \in (0, r_0(t)/3]$ there is $c_{y,s} \in \mathbb{C}, \ |c_{y,s}| < s$, such that the set of zeros of the function $F - c - y - c_{y,s}$ in $\mathbb{D}$ contains at least $N_F(t)$ points with pairwise distances greater than $\frac{s(t-1)}{\sqrt{\lambda(t)}}$.

This result can be reformulated as follows.

Let $\Gamma := \{(z, F(z)) \in \mathbb{C}^2 : z \in \mathbb{D}_t\}$ be the graph of $F$. There is a number $c \in \mathbb{C}, \ |c| < 1$, such that for each point $v = (x, c + y) \in \mathbb{D}_t \times \mathbb{D}_{r_0(t)}(c)$ and every $s \in (0, r_0(t)/3)$, there is a point $v' = (x, c + y + c_{y,s}) \in \{x\} \times \mathbb{C}, \ |v' - v| < s$, such that the complex line $l := \{(z, w) \in \mathbb{C}^2 : w - c - y - c_{y,s} = 0\}$, parallel to the $z$-axis and passing through $v'$, intersects the graph $\Gamma$ over $\mathbb{D}$ in at least $N_F(t)$ points with pairwise distances greater than $\frac{s(t-1)}{\sqrt{\lambda(t)}}$.

In sections 4.2-4.4 we formulate some auxiliary results used in the proof of Theorem 4.1.
4.2. Applying the Hadamard three circle inequality (3.1) to our function $F$ with $r_0 := 1/t^2$, $r_1 := 1/t$ and $r_2 := 1$ from (4.1) we obtain

$$\frac{M_F(1)}{M_F(1/t)} \geq \frac{M_F(1/t)}{M_F(1/t^2)} \geq \sqrt{t}. \quad (4.3)$$

Then applying (3.1) with $r_0 := 1/t$, $r_1 := 1$ and $r_2 := t$ from (4.3) we obtain

$$\frac{1}{M} \geq \frac{M_F(t)}{M_F(1/t)} \geq t. \quad (4.4)$$

We use this estimate to prove

**Lemma 4.2** For $F' := \frac{dF}{dz}$ we have

$$n_{F'}(1) < \lambda(t). \quad (4.5)$$

**Proof.** For $z \in \mathbb{D}$, using the mean-value inequality $|F(z) - F(0)| \leq M_F(1)$, we obtain

$$M_F'(1) \geq M_F(1) - M_F(1/t). \quad (4.6)$$

Also, by the Cauchy integral formula for $F'$ we get

$$M_F'(\sqrt{t}) \leq \sup_{z \in \mathbb{D}_t} \left\{ \frac{1}{2\pi i} \int_{S_t} |F'(|z|)| \frac{d\xi}{|\xi - z|^2} \right\} \leq \frac{M_F(t)}{\sqrt{t}(\sqrt{t} - 1)} < \frac{2}{t - 1}. \quad (4.7)$$

Here $S_{t-\sqrt{t}}(z)$ stands for the boundary of the disk $\mathbb{D}_{t-\sqrt{t}}(z)$.

Finally, we apply the Jensen inequality for the number of zeros of a holomorphic function proved in [VP]. Then from (4.6), (4.7), (4.1), (4.3) and (3.9) we obtain

$$n_{F'}(1) \leq \frac{1}{\ln \left( \frac{4 + \sqrt{t}}{2\sqrt{t}} \right)} \ln \left( \frac{M_F'(\sqrt{t})}{M_F'(1)} \right) < \frac{9(\sqrt{t} + 1)^2}{(t - 1)^2} \ln \left( \frac{2}{(t - 1)(M_F(1) - M_F(1/t))} \right) \leq$$

$$\frac{9(\sqrt{t} + 1)^2}{(t - 1)^2} \ln \left( \frac{2(\sqrt{t} + 1)}{M(t - 1)^2} \right). \quad \square$$

4.3. By the definition of $N_F(t)$ there is a number $c \in \mathbb{C}$, $|c| \leq M_F(1/\sqrt{t})$, such that the function $F_c := F - c$ has $N_F(t)$ zeros in $\mathbb{D}_{1/\sqrt{t}}$. For this function we have

$$M_{F_c}(t) \leq M_F(t) + M_F(1/\sqrt{t}) < 2. \quad (4.8)$$

Further, $M_{F_c}(1) > M_{F_c}(1/t) \geq |c| - M_F(1/t)$. Assuming, first, that $|c| \geq \frac{1 + \sqrt{t}}{2\sqrt{t}} M_F(1)$ and using (4.3) we get from here

$$M_{F_c}(1) > \frac{1 + \sqrt{t}}{2\sqrt{t}} M_F(1) - M_F(1/t) \geq \frac{\sqrt{t} - 1}{2} M_F(1/t) \geq \frac{\sqrt{t} - 1}{2} M.$$
Assume now that $|c| < \frac{1 + \sqrt{t}}{2\sqrt{t}} M_F(1)$. Then

$$M_{F_c}(1) \geq M_F(1) - |c| > \frac{\sqrt{t} - 1}{2\sqrt{t}} M_F(1) \geq \frac{\sqrt{t} - 1}{2} M_F(1/t) \geq \frac{\sqrt{t} - 1}{2} M.$$  

Thus we have

$$M_{F_c}(1) > \frac{\sqrt{t} - 1}{2} M.$$ \hspace{1cm} (4.9)

In particular, from (4.8) and (4.9) we obtain

$$\frac{M_{F_c}(t)}{M_{F_c}(1)} \leq \frac{4(\sqrt{t} + 1)}{M(t - 1)}. \hspace{1cm} (4.10)$$

From here and the Jensen inequality of [VP] we get (recall that $1 < t \leq 9$)

$$N_F(t) \leq n_{F_c}(1) \leq \frac{1}{\ln \left(\frac{1 + \sqrt{t}}{2t}\right)} \ln \left(\frac{M_{F_c}(t)}{M_{F_c}(1)}\right) \leq \frac{9(\sqrt{t} + 1)^2}{2(t - 1)^2} \ln \left(\frac{4(\sqrt{t} + 1)}{M(t - 1)}\right) =: \delta(t).$$ \hspace{1cm} (4.11)

4.4. We will also use Corollary 3.3. According to this corollary for $f := F_c$ and $r := 1$ using (4.9), (4.10) we obtain that there is a circle $S_l$ with $1/\sqrt{t} \leq l \leq 1$ such that

$$|F_c(z)| > 2 \left(\frac{M(t - 1)}{4(\sqrt{t} + 1)}\right)^{\gamma(t)+1} =: 2r_0(t) \quad \text{for all} \quad z \in S_l. \hspace{1cm} (4.12)$$

4.5. Proof of Theorem 4.1. Let $c \in \mathbb{C}$, $|c| < 1$, be the number introduced in section 3.3. We will prove that for each $y \in \mathbb{C}$, $|y| \leq r_0(t)$, and $s \in (0, r_0(t)/3]$ there is $c_{y,s} \in \mathbb{C}$, $|c_{y,s}| < s$, such that the set of zeros of the function $F - c - y - c_{y,s}$ in $\mathbb{D}$ contains at least $N_F(t)$ points (see (4.2)) with pairwise distances greater than $\frac{s(t-1)}{2\sqrt{\lambda(t)}}$

First, from inequality (4.12) by the Rouché theorem we deduce that

$$n_{F_{c-a}}(l) = n_{F_c}(l) \geq N_F(t) \quad \text{for all} \quad a \in \mathbb{C}, \quad |a| \leq 2r_0(t). \hspace{1cm} (4.13)$$

This is valid, in particular, for $a := y + b$ with $|b| \leq r_0(t)$.

Let $C_F \subset F_c(\mathbb{D}) \subset \mathbb{C}$ be the set of critical values of $F_c|_{\mathbb{D}}$.

Lemma 4.3 For each $s \in (0, r_0/3]$ there is $c_{y,s} \in \mathbb{C}$, $|c_{y,s}| < s$, such that

$$\text{dist}(y + c_{y,s}, C_F) > \frac{s}{\sqrt{\lambda(t)}}.$$

(Observe that by (4.4) and (4.2), $\sqrt{\lambda(t)} > \frac{1}{2}$ for $1 < t \leq 9$.)

Proof. We will assume that $C_F \neq \emptyset$. For otherwise, we set $c_{y,s} = 0$. 

20
By Lemma 4.2 the number of critical points of $F_c$ in $\mathbb{D}$ is $< \lambda(t)$. Since $N_F(t) \geq 1$, (4.13) implies that $\mathbb{D}_s(y) \subset F_c(\mathbb{D})$. In particular, $\mathbb{D}_s(y) \cap C_F$ contains $< \lambda(t)$ points. Thus there is $c_{y,s} \in \mathbb{D}_s$ such that $\text{dist}(y + c_{y,s}, C_F) > \frac{s}{\sqrt{\lambda(t)}}$. Indeed, for otherwise, the closed disks of radius $\frac{s}{\sqrt{\lambda(t)}}$ centered at the points of $\mathbb{D}_s(y) \cap C_F$ cover $\mathbb{D}_s(y)$. Comparing the areas of $\mathbb{D}_s(y)$ and of this cover we obtain a contradiction:

$$\pi s^2 < \lambda(t) \cdot \pi \cdot \left(\frac{s}{\sqrt{\lambda(t)}}\right)^2 = \pi s^2. \quad \Box$$

Now, by Lemma 4.3 we obtain that $\mathbb{D}_{r_1}(y + c_{y,s}) \cap C_F = \emptyset$, $r_1 := \frac{s}{\sqrt{\lambda(t)}}$. Moreover, by (4.13) we have $\mathbb{D}_{r_1}(y + c_{y,s}) \subset F_c(\mathbb{D}_t)$ because $|c_{y,s}| + r_1 < 3s \leq r_0(t)$. Thus from (4.13) for $X := F_c^{-1}(\mathbb{D}_{r_1}(y + c_{y,s})) \cap \mathbb{D}_t$ we obtain that $F_c : X \to \mathbb{D}_{r_1}(y + c_{y,s})$ is a proper conformal map and $\#\{F_c^{-1}(z) \cap \mathbb{D}_t\}$ is the same for any $z \in \mathbb{D}_{r_1}(y + c_{y,s})$. Hence $F_c : X \to \mathbb{D}_{r_1}(y + c_{y,s})$ is an unbranched covering of $\mathbb{D}_{r_1}(y + c_{y,s})$ consisting of at least $N_F(t)$ sheets. In particular, $X$ is biholomorphic to the disjoint union of $k$ copies of $\mathbb{D}_{r_1}(y + c_{y,s})$ where $k$ is the number of sheets of $F_c|_X$.

We set $Y = \{y_1, \ldots, y_k\} := F_c^{-1}(y + c_{y,s}) \cap \mathbb{D}_t$. Assume that for some $i \neq j$ we have $|y_i - y_j| \leq (t - 1)r_1$. Then by the mean-value theorem for each $z$ from the interval $\gamma := [y_i, y_j]$ we have

$$|F_c(z) - F_c(y_i)| \leq M_{F'}(t)|y_i - y_j| \leq \frac{M_F(t)}{t - 1}(t - 1)r_1 \leq r_1.$$

Thus $F_c(z) \in \mathbb{D}_{r_1}(y + c_{y,s})$. In particular, $F_c(\gamma) \subset \mathbb{D}_{r_1}(y + c_{y,s})$ is a closed curve. Then since $F_c : X \to \mathbb{D}_{r_1}(y + c_{y,s})$ is an unbranched covering, $\gamma$ should be a closed curve, as well. This contradiction shows that $|y_i - y_j| > (t - 1)r_1$ for all $i \neq j$.

The proof of Theorem 4.1 is complete. \quad \Box

5. Proof of Theorem 1.2: Case $n = 1$

5.1. First, we will prove Theorem 1.2 for the functions $f$ and $g$ satisfying the assumptions of the theorem for $n = 1$ and such that in (1.1)

$$R_f(r, t, t^2) := \frac{M_f(r/t)}{M_f(r/t^2)} \geq \sqrt{t}. \quad (5.1)$$

We also set

$$M := \frac{M_1}{M_2}. \quad (5.2)$$

Next, we define new functions $F$ and $G$ by the formulas

$$F(z) = \frac{f(rz)}{M_2}, \quad z \in \mathbb{D}_t, \quad \text{and} \quad \quad (5.3)$$

$$G(z, w) := g(rz, M_2w), \quad (z, w) \in \mathbb{D}_t \times \mathbb{D}_3.$$
Then \( F \) satisfies conditions (4.1) and \( G \) satisfies the conditions

\[
M_{G(\cdot, w)}(t) \leq e^p \cdot M_{G(\cdot, w)}(1) \quad \text{for all } \ w \in \mathbb{D}_3; \tag{5.4}
\]

\[
b_{G(z, \cdot)}(3) \leq q \quad \text{for all } \ z \in \mathbb{D}_t. \tag{5.5}
\]

Now, we will prove the following version of Theorem 1.2.

**Theorem 5.1** Assume that \( p \leq \ln \left( \frac{1+\delta}{2\sqrt{t}} \right) \cdot N_F(t) \). Then for \( G_F(z) := G(z, F(z)) \)

\[
\sup_{\mathbb{D} \times \mathbb{D}} |G| \leq \left( \frac{c_1(t)}{M} \right)^{c_2(t)(p+q)} M_{G_F}(1)
\]

where

\[c_1(t) := \frac{50(\sqrt{t}+1)}{(t-1)^2}, \quad c_2(t) := \frac{18(\sqrt{t}+1)^2 + 81 \ln \left( \frac{108e}{\sqrt{t}-1} \right)}{(\sqrt{t}-1)^4}. \tag{5.6}\]

Going back to the functions \( f \) and \( g \) and noticing that \( G_F(z) = g_F(rz), \ z \in \mathbb{D}_t, \)

\[
\sup_{\mathbb{D} \times \mathbb{D}} |G| = \sup_{\mathbb{D} \times \mathbb{D}} |g|, \ M_{G_F}(1) = M_{g'_f}(r) \text{ and } N_F(t) = N_f(r; t) \text{ we obtain from this theorem inequality (1.10) in the case } n = 1.
\]

### 5.2. Proof of Theorem 5.1.

We retain the notations of section 4.1. Also, without loss of generality we may and will assume that \( G \) is nonconstant and \( p, q > 0 \).

Let us consider the open set \( \mathbb{D} \times \mathbb{D}_{r_0(t)}(c) \subset \mathbb{C}^2 \). By \( v = (x, c + y) \in \mathbb{D} \times \mathbb{D}_{r_0(t)}(c), \)

\[|x| = 1, |y| = r_0(t)\], we denote a point such that

\[|G(v)| = \sup_{\mathbb{D} \times \mathbb{D}_{r_0(t)}(c)} |G|. \]

We set

\[s := \frac{r_0(t)e^{-\max\{p,q\}/p}}{12\delta(t)}. \tag{5.7}\]

(Observe that \( 12\delta(t) > 12, 1 < t \leq 9 \), see (4.11), (4.4). Hence, \( s < r_0(t)/12 \).)

Consider the point \( v' = (x, c + y + c_{u,s}) \) with \( c_{u,s} \) as in Theorem 4.1. Then \( v' \) belongs to the disk \( \{x\} \times \mathbb{D}_{r_1}(c) \) of radius \( r_1 := r_0(t) + s \). Applying (5.5) to \( h := G|_{\{x\} \times \mathbb{D}_{r_0(t)}(c)} \) from Hadamard’s three circle inequality (see (3.1)) for disks centered at \( (x, c) \) of radii \( r_0(t), r_1 \) and \( er_0(t) \) (observe that \( er_1 < 2 \), see (4.2)), so that \( \mathbb{D}_{er_0(t)}(c) \subset \mathbb{D}_{er_1}(c) \subset \mathbb{D}_3 \) we obtain that

\[
\sup_{\{x\} \times \mathbb{D}_{r_1}(c)} |h| < \left( 1 + \frac{s}{r_0(t)} \right)^q \sup_{\{x\} \times \mathbb{D}_{r_0(t)}(c)} |h|. \tag{5.8}
\]

Also, by Theorem 3.4 and (3.20) with \( R := r_1, t = e \) we get

\[
\sup_{\{x\} \times \mathbb{D}_{r_1}(c)} |h'| \leq \frac{9q}{r_1} \sup_{\{x\} \times \mathbb{D}_{r_1}(c)} |h|. \tag{5.9}
\]
Using (5.8) and (5.9) we can estimate \(|h(v')|\) by the mean-value inequality:

\[
|h(v)| - |h(v')| \leq |h(v) - h(v')| \leq \sup_{\{z\} \times \mathbb{D}_r} |h'| \cdot |v - v'| \leq
\]

\[
\frac{9qs}{r_0(t) + s} \left(1 + \frac{s}{r_0(t)}\right)^q |h(v)| \leq \frac{9q}{(e^{\max\{p,q\}}/2)/(2\delta(t))} \cdot \left(1 + \frac{e^{-\max\{p,q\}}/2}{2\delta(t)}\right)^q |h(v)|
\]

\[
\leq \frac{9e^{-p}}{12\delta(t)} e^{q(e^{-\max\{p,q\}}/2)/(2\delta(t))} |h(v)| \leq \frac{p}{3\delta(t)} e^{q(e^{-p}/2)/(2\delta(t))} |h(v)| \leq
\]

\[
\frac{p}{3\delta(t)} e^{q/(18e)} |h(v)| < \frac{2p}{5\delta(t)} |h(v)| < |h(v)|.
\]

(We used here the following inequalities: \(\delta(t) \geq N(t) \geq \frac{1}{\ln \left(\frac{1+4}{2\sqrt{7}}\right)} p \geq \frac{2\sqrt{7}}{(\sqrt{t-1})^2} p \geq \frac{3}{7} p, 1 < t \leq 9, \) see (4.11), \(\max\{p,q\}/p \geq 1\) and \(xe^{-x} \leq e^{-1}\) for \(x \geq 1\).) Hence

\[
|h(v')| > \left(1 - \frac{2p}{5\delta(t)}\right) |h(v)|. \quad (5.10)
\]

Next, let us consider the line \(l := \{(z, w) \in \mathbb{C}^2 : w - c - y - c_y, s = 0\}\). Then \(l\) passes through \(v'\) and according to Theorem 4.1 intersects the graph \(\Gamma = \{(z, F(z)) : z \in \mathbb{D}\}\) in a set \(Y\) containing at least \(N_F(t)\) points with pairwise distances greater than \(\frac{s(t-1)}{\sqrt{\lambda(t)}}\). For \(R > 0\) we set

\[
l_R := \{(z, w) \in l : |z| < R\}.
\]

We naturally identify \(l_R\) with disk \(\mathbb{D}_R\). Now let us apply Theorem 3.2 to the univariate function \(g_l := G|_l\) with \(r = 1, H := \frac{s(t-1)}{4\sqrt{\lambda(t)}}\). According to this theorem and inequality (3.8) in the disk \(l\) outside the union of open disks \(\{D_j\}_{1 \leq j \leq k} \subset l, k < \frac{1}{\ln \left(\frac{1+4}{2\sqrt{7}}\right)} \cdot p\), with the sum of radii \(\sum_{k=1}^n r_k \leq 2Ht \leq \frac{s(t-1)}{2\sqrt{\lambda(t)}}\),

\[
|g_l(z)| \geq M_{g_l}(1) \left(\frac{M_{g_l}(1)}{M_{g_l}(t)}\right)^{c(H)} \geq M_{g_l}(1) e^{-c(H)p} \quad (5.11)
\]

where

\[
c(H) := \frac{(\sqrt{t} + 1)^4 + 18(\sqrt{t} + 1)^2 \ln \left(\frac{e}{H}\right)}{2(t - 1)^2}. \quad (5.12)
\]

**Remark 5.2** We single out that the inequality for \(k\) is strict. Indeed, if \(g_l \equiv \text{const}\), then \(k = 0\) and the conclusion is obvious because \(p > 0\). If \(g_l \not\equiv \text{const},\) then according to (3.8)

\[
k \leq \frac{\ln \left(\frac{M_{g_l}(tr)}{M_{g_l}(\sqrt{tr})}\right)}{\ln \left(\frac{1+4}{2\sqrt{7}}\right)} < \frac{\ln \left(\frac{M_{g_l}(tr)}{M_{g_l}(\sqrt{tr})}\right)}{\ln \left(\frac{1+4}{2\sqrt{7}}\right)} \leq \frac{p}{\ln \left(\frac{1+4}{2\sqrt{7}}\right)}.
\]

23
Now, by the definition of $H$ we have, see (4.2), (4.11),
\[
\frac{1}{H} = \frac{4t\sqrt{\lambda(t)}}{s(t-1)} = \frac{48t^6(t)\sqrt{\lambda(t)}}{s(t-1)} \left(\frac{4(\sqrt{t} + 1)}{M(t-1)}\right)^{\gamma(t)+1} e^{\max\{p,q\}/p} = \\
\frac{648t(\sqrt{t} + 1)^3}{(t-1)^4} \left(\frac{4(\sqrt{t} + 1)}{M(t-1)}\right)^{\gamma(t)+1} \ln \left(\frac{4(\sqrt{t} + 1)}{M(t-1)}\right) \sqrt{\ln \left(\frac{2(\sqrt{t} + 1)}{M(t-1)}\right)} \cdot e^{\max\{p,q\}/p}
\]
\[
< \frac{648t(\sqrt{t} + 1)^3}{(t-1)^4} \left(\frac{32(\sqrt{t} + 1)}{M(t-1)^2}\right)^{\gamma(t)+2} e^{\max\{p,q\}/p} < \left(\frac{32(\sqrt{t} + 1)}{M(t-1)^2}\right)^{\gamma(t)+5} e^{\max\{p,q\}/p}.
\]
(We used here that \(\frac{4(\sqrt{t} + 1)}{M(t-1)} > 10\), \(1 < t \leq 9\), see (4.4), \(\ln x < \sqrt{x}\) for \(x \geq 10\) and \(\ln x < \sqrt{x}\) for all \(x > 1\).)

Hence, see (4.2),
\[
e^{c(H)} = e^{\tilde{a}_1(t)} \left(\frac{32(\sqrt{t} + 1)}{M(t-1)^2}\right)^{\tilde{a}_2(t)\tilde{a}_3(t)} e^{\frac{\tilde{a}_3(t)\gamma(p)}{p}},
\]
where
\[
\tilde{a}_1(t) := \frac{(\sqrt{t} + 1)^4}{2(t-1)^2}, \quad \tilde{a}_2(t) := \frac{(\sqrt{t} + 1)^4 + 18(\sqrt{t} + 1)^2 \ln \left(\frac{\deg^{3/2}}{\sqrt{t-1}}\right)}{2(t-1)^2} + 5, \quad (5.13)
\]
\[
\tilde{a}_3(t) := \frac{9(\sqrt{t} + 1)^2}{(t-1)^2}.
\]

Since by our assumption \(p \leq \ln \left(\frac{1+\delta}{2\sqrt{t}}\right) \cdot N_F(t)\), there exists a point \(a \in Y\) such that \(a \notin \cup_j D_j\). Actually, by our choice of \(H\) and the definition of \(Y\) we obtain that every \(D_j\) can contain at most one point of \(Y\). But \(#\{Y\} \geq N_F(t)\) and the number of the disks \(k < \frac{1}{\ln \left(\frac{1+\delta}{2\sqrt{t}}\right)} \cdot p \leq N_F(t)\). This gives the required result.

From (5.11) we obtain
\[
\sup_{z \in \mathbb{D}} |G_F(z)| \geq |g_1(a)| \geq M_{g_1}(1)e^{-c(H)p}. \quad (5.14)
\]

Using that \(M_{g_1}(1) \geq |h(v')|\), \(\tilde{a}_2(t) > 10\), \(\tilde{a}_3(t) > 1\), \(\frac{\tilde{a}_1(t)}{\tilde{a}_3(t)} < 1\), and \(\frac{2p}{5\delta(t)} \leq \frac{4}{15}, \delta(t) \geq 1\) for \(1 < t \leq 9\), we obtain directly from (5.14), (5.13) and (5.10) by the choice of \(v\):
\[
\sup_{\mathbb{D} \times \mathbb{D}_{0}(t)(c)} |G| \leq \frac{e^{c(H)p}}{1 - (2p/(5\delta(t)))} M_{G_F}(1) \leq e^{(c(H)+6/11)p} M_{G_F}(1) < \\
e^{\tilde{a}_3(t)q} \left(\frac{50(\sqrt{t} + 1)}{M(t-1)^2}\right)^{\tilde{a}_2(t)\tilde{a}_3(t)p} M_{G_F}(1). \quad (5.15)
\]
(We also used the inequality \(-\ln(1-s) \leq \frac{15}{11}s, 0 \leq s \leq \frac{4}{15}\), applied to \(s := \frac{2p}{5\delta(t)}\).)
Finally let us consider the polydisk $D := \mathbb{D} \times \mathbb{D}_2(c)$. Since $|c| < 1$, $\mathbb{D} \times \mathbb{D} \subset D \subset \mathbb{D} \times \mathbb{D}_3$. Then from the fact that the Bernstein index of $G$ over vertical disks in $D$ is $\leq q$ we easily obtain

$$
\sup_{D \times \mathbb{D}_2(c)} |G| \leq e^q \sup_{D \times \mathbb{D}_2(c)} |G|, \quad j = 0, 1, \ldots, \lfloor \ln(2/r_0(t)) \rfloor.
$$

From here, (4.2) and (5.15) we deduce that

$$
\sup_{D \times \mathbb{D}} |G| \leq e^{2q} \left( \frac{4(\sqrt{t} + 1)}{M(t - 1)} \right)^{q(\gamma(t) + 1)} \sup_{D \times \mathbb{D}_1(c)} |G| \leq \left( \frac{c_1(t)}{M} \right)^{c_2(t)(p+q)} M_G \gamma \quad (5.16)
$$

where

$$
c_1(t) := \frac{50(\sqrt{t} + 1)}{(t - 1)^2}, \quad c_2(t) := \frac{18(\sqrt{t} + 1)^2 + 81 \ln\left( \frac{108e}{\sqrt{t} - 1} \right)}{(\sqrt{t} - 1)^4}. \quad (5.17)
$$

(We used that $\tilde{a}_3(t) + 2 < \gamma(t) + 1$ and $\max\{\gamma(t) + 1, \tilde{a}_2(t)\tilde{a}_3(t)\} < c_2(t)$ for $1 < t \leq 9$)

The proof of Theorem 5.1 is complete. \[\square\]

6. Proof of Theorem 1.2: Case $n \geq 2$

6.1. In the proof we use the following estimate.

Lemma 6.1 Let $h$ be a nonconstant holomorphic function in the disk $\mathbb{D}_{R/\sqrt{t}}$, $R > 0$, $t > 1$. Then for each $s \in [t, \infty)$,

$$
v_h(R/\sqrt{t}) \geq \max \left\{ \frac{1}{k(t, s)} \ln \left( \frac{M_h(R/t)}{M_h(R/s)} \right), 1 \right\}
$$

where

$$
k(t, s) := \ln \left( \frac{8e^2s\sqrt{t}}{(\sqrt{t} - 1)^2} \right).
$$

Proof. We make use of the following result proved in [JO]:

Let $h(z) = \sum_{k=0}^{\infty} a_k z^k$ be a $p$-valent holomorphic function in the disk $\mathbb{D}_{s_1}$, $\mu_p(s_1) = \max_{0 \leq k \leq p} |a_k|s_1^k$ and $0 < s_2 < s_1$. Then

$$
M_h(s_2) \leq A(p)\mu_p(s_1)(1 - s_2/s_1)^{-2p}
$$

where $A(p) = (p + 2)2^{3p-1}e^{p^2/12}$.

Applying this result to a function $h$ with $s_1 = R/\sqrt{t}$, $s_2 = R/t$ and $p = v_h(R/\sqrt{t})$ from the Cauchy inequality for coefficients of the Taylor series of $h$ we obtain

$$
M_h(R/t) \leq A(p)\mu_p(R/\sqrt{t}) \left( \frac{\sqrt{t}}{\sqrt{t} - 1} \right)^{2p} \leq A(p)\mu_p(R/s) \left( \frac{s}{\sqrt{t}} \right)^p \left( \frac{\sqrt{t}}{\sqrt{t} - 1} \right)^{2p}
$$

$$
\leq A(p) \left( \frac{s\sqrt{t}}{(\sqrt{t} - 1)^2} \right)^p M_h(R/s), \quad s \in [t, \infty).
$$

\[\text{(6.1)}\]
Apply now (6.1) to the functions $h^k$, $k \in \mathbb{N}$. Since

$$v_h(R/\sqrt{t}) \leq kp, \quad M_h(R/t) = (M_h(R/t))^k, \quad M_h(R/s) = (M_h(R/s))^k,$$

inequality (6.1) in this case implies

$$M_h(R/t) \leq \left( \lim_{k \to \infty} (A(kp))^{1/k} \right) \left( \frac{s\sqrt{t}}{\sqrt{t} - 1} \right)^p M_h(R/s) \leq \left( \frac{8e\pi^2 s\sqrt{t}}{(\sqrt{t} - 1)^2} \right)^p M_h(R/s).$$

This gives the required inequality. \(\square\)

6.2. Let $f$ and $g$ satisfy conditions of Theorem 1.2 for $n \geq 2$. As in section 5.2 we will assume without loss of generality that $r = 1, M_1 = M$ and $M_2 = 1$.

By $l_v \in \mathcal{L}_n$ we denote the complex line \{\(vz \in \mathbb{C}^n : z \in \mathbb{C}, v \in \mathbb{C}^n, ||v|| = 1\}\}. We set

$$f_v(z) := f(vz), \quad g_v(z, w) := g(vz, w), \quad z \in \mathbb{D}_t, w \in \mathbb{D}_3. \quad (6.2)$$

(In notations of section 1.3 $f_v := f_{l_v}$ and $g_v := g_{l_v}$.) According to assumptions of Theorem 1.2, every $g_v \in \mathcal{F}_{p,q}(1; t; 1)$ with $p \leq \ln (\frac{1+t}{2\sqrt{t}}) N_f(1, t)$.

Let $v_\ast \in \mathbb{C}^n, ||v_\ast|| = 1$, be such that

$$M_f(1/t) = M_{f_{v_\ast}}(1/t). \quad (6.3)$$

**Lemma 6.2** Suppose that

$$||v - v_\ast|| \leq \gamma := \frac{(\sqrt{t} - 1) \ln t}{9e \sqrt{t} \max\{t, \ln(1/M)\}}. \quad (6.4)$$

Then

$$M_{f_v}(1/s) \leq M_f(1/s), \quad s \in [t, \infty), \quad M_{f_v}(1/t) \geq \frac{1}{\sqrt{t}} M_f(1/t) \geq \frac{1}{\sqrt{t}} M.$$

Observe that from condition (1.1) similarly to inequality (4.4) we deduce

$$\frac{1}{M} \geq t^2. \quad (6.5)$$

**Proof.** The first inequality is obvious. So, let us prove the second one.

Due to inequalities (3.13), (3.21) we have (recall that $1 < t \leq 9$)

$$M_{D_{s,f}}(1/t) \leq \frac{9e}{\ln t} \max\{t, \ln(1/M)\} M_f(1/t) \quad \text{for all} \quad s \in \mathbb{C}^n, ||s|| = 1. \quad (6.6)$$

From (6.6) by the mean-value inequality we get for $|z| \leq 1/t$,

$$|f_v(z) - f_{v_\ast}(z)| \leq \frac{9e}{\ln t} \max\{t, \ln(1/M)\} M_f(1/t)||v - v_\ast|| \leq \frac{\sqrt{t} - 1}{\sqrt{t}} M_f(1/t).$$
This and (6.3) imply the required inequality of the lemma:

\[ M_{f_v}(1/t) \geq \frac{1}{\sqrt{t}} M_f(1/t) \geq \frac{1}{\sqrt{t}} M. \quad \square \]

Thus for each \( v \) satisfying (6.4) we get from Lemmas 6.1, 6.2 (see (1.3), (1.4)):

\[ R_{f_v}(1,t,t^2) := \frac{M_{f_v}(1/t^2)}{M_{f_v}(1/t)} \geq \frac{1}{\sqrt{t}} R_f(1,t,t^2) \geq \sqrt{t} \quad \text{and} \quad (6.7) \]

\[ N_{f_v}(1,t) := v_{f_v}(1/\sqrt{t}) \geq \max \left\{ \sup_{s \in [t,\infty)} \left\{ \frac{\ln R_{f_v}(1,t,s)}{k(t,s)} \right\}, V_f(1,t) \right\} \geq \max \left\{ \sup_{s \in [t,\infty)} \left\{ \frac{\ln R_f(1,t,s)/(\sqrt{t})}{k(t,s)} \right\}, V_f(1,t) \right\} =: N_f(1,t). \quad (6.8) \]

Inequalities (6.7), (6.8) show that we can apply to the functions \( f_v \) and \( g_v \in \mathcal{F}_{p,q}(1; t, t^2) \) the inequality of Theorem 1.2 for \( n = 1 \) (proved in section 4) with \( M_1 := \frac{1}{\sqrt{t}} M, M_2 := 1 \). Then we obtain

**Proposition 6.3** For \( v \) satisfying (6.4) and \( (g_f)_v := g_v(z, f_v(z)) \) the inequality

\[ \sup_{D \times D} |g_v| \leq \left( \frac{c_1(t)\sqrt{t}}{M} \right)^{c_2(t)(p+q)} M_{(g_f)_v}(1) \]

holds with \( c_1(t), c_2(t) \) defined by (5.6). \( \square \)

Let \( K \subset \mathbb{B}^n \) be the convex body determined by the formula

\[ K := \{ zv \in \mathbb{B}^n : (z, v) \in \mathbb{C} \times \mathbb{C}^n, |z| < 1, ||v|| = 1, ||v - v_*|| \leq \gamma \}. \quad (6.9) \]

Then from Proposition 6.3 we obtain the following statement:

**Under the assumptions of Theorem 1.2,**

\[ \sup_{K \times \mathbb{D}} |g| \leq \left( \frac{c_1(t)\sqrt{t}}{M} \right)^{c_2(t)(p+q)} \sup_K |g_f|. \quad (6.10) \]

We deduce from here the required inequality of the theorem.

**Lemma 6.4** For every boundary point \( z \) of \( \mathbb{B}^n \) there is a real straight line \( l_z \subset \mathbb{C}^n \) passing through \( x \) such that \( l_z \) intersects \( K \) in an interval \( I_z \) of length

\[ |I_z| > \gamma_1 := \frac{9}{10} \gamma. \]

**Proof.** Without loss of generality we may assume that \( z \) does not belong to \( K \) (for otherwise, choose \( l_z \) joining \( z \) with \( 0 \) so that \( |I_z| = 1 > \gamma \)). Then as the \( l_z \) we will take the line passing through \( z \) and \( v_* \). Considering the real two-dimensional plane \( P \) containing \( 0, z \) and \( v_* \) we reduce the question on the bound of \( |I_z| \) to the
two-dimensional case. Without loss of generality we may assume that \( P = \mathbb{R}^2 \)
and \( v_* = (1, 0) \in \mathbb{R}^2 \). In this case \( K_P := K \cap P \) is a convex set defined in polar
coordinates \((r, \phi)\) by the inequalities
\[
|2 \sin(\phi/2)| \leq \gamma, \quad |r| < 1.
\]

Also, we may assume that \( z \) belongs to the upper semicircle \( S_+ \) of the unit disk.
Now, from (6.4) and the inequality \( 1 < t \leq 9 \) we obtain that \( K_P \cap S_+ \subset \{(r, \phi) : 0 \leq \phi < \pi/3\} \). Thus \( |I_z| \) is the distance from \( v_* \) to the line \( \{(r, \phi) : \sin(\phi/2) = \gamma/2\} \).

\[
|I_z| > \gamma \sqrt{1 - \left(\frac{2 \ln 3}{9 e}\right)^2} > \frac{9}{10} \gamma. \quad \Box
\]

Let \((x, y)\) be the boundary point of \( \mathbb{B}^n \times \mathbb{D} \) such that
\[
|g(x, y)| = \sup_{\mathbb{B}^n \times \mathbb{D}} |g|.
\]

According to Lemma 6.4 there is a straight line \( l \subset \mathbb{C}^n \times \{y\} \) passing through \((x, y)\)
and intersecting \( K \times \{y\} \) in the interval \( I \) of length \( > \gamma_1 \). Let \( l^c \) be the complex line
containing \( l \). We set
\[
D_1 := l^c \cap (\mathbb{B}^n \times \{y\}) \quad \text{and} \quad D_2 := l^c \cap (\mathbb{B}^n \times \{y\})).
\]

We can naturally identify \( D_1 \) and \( D_2 \) we the disks centered at the point \( o \in l^c \) such
that \( d := ||o - (0, y)|| := \text{dist}(0, l^c - (0, y)) \) of radii \( r_1 := \sqrt{1 - d^2} \) and \( r_2 := \sqrt{t^2 - d^2} \).
Observe also that \( tr_1 \leq r_2 \). Thus \( tD_1 \subset D_2 \) where \( kD_1, k > 0 \), denotes the dilation
of \( D_1 \) in \( k \) times with respect to \( o \).

Further, by (6.11) we have for \( \tilde{g} := g|_{D_2} \):
\[
M_{\tilde{g}}(r_1; o) = M_{g,\{,y\}}(1) \quad \text{and} \quad M_{\tilde{g}}(tr_1; o) \leq M_{\tilde{g}}(r_2; o) \leq M_{g,\{,y\}}(t).
\]

Also, the first condition in (1.7) implies easily that
\[
M_{g,\{,y\}}(t) \leq e^p \cdot M_{g,\{,y\}}(1).
\]

By the definition the interval \( I \) of length \( > \gamma_1 \) is contained in \( D_1 \). Assuming
without loss of generality that \( \tilde{g} \) is nonconstant, we apply to the triple \( \tilde{g}, D_1, tD_1 \)
Theorem 3.2 with \( H := \gamma_1/4t \). According to this theorem in the disk \( D_1 \) outside the
union of a finite number of disks with the sum of radii \( < (\gamma_1 r_1)/2 \) we have
\[
\frac{|	ilde{g}(z)|}{M_{\tilde{g}}(r_1; o)} > \left(\frac{M_{\tilde{g}}(r_1; o)}{M_{\tilde{g}}(tr_1; o)}\right)^{c(H)}
\]
where
\[
c(H) := \frac{(\sqrt{t} + 1)^4 + 18(\sqrt{t} + 1)^2 \ln \left(\frac{4t}{\gamma_1}\right)}{2(t - 1)^2}.
\]
Since $r_1 \leq 1$ and $|I| > \gamma_1$, the union of such disks cannot cover $I$. In particular, there is a point $a \in I$ at which inequality (6.14) holds. Now, from this inequality and (6.11), (6.12), (6.13), (6.10) we obtain

$$\sup_{\mathbb{R}^n \times \mathbb{D}} |g| \leq e^{c(H)p} \sup_{K \times \mathbb{D}} |g| \leq e^{c(H)p} \left( \frac{c_1(t) \sqrt{t}}{M} \right)^{c_2(t)(p+q)} M g_f(1).$$

(6.15)

Observe that

$$e^{c(H)} = e^{\tilde{a}_1(t)} \cdot \left( \frac{40e^2 t^{3/2} \max\{t, \ln(1/M)\}}{(\sqrt{t} - 1) \ln t} \right) \tilde{a}_3(t)$$

where

$$\tilde{a}_1(t) = \frac{(\sqrt{t} + 1)^4}{2(t - 1)^2}, \quad \tilde{a}_3(t) = \frac{9(\sqrt{t} + 1)^2}{(t - 1)^2}.$$ 

(6.16)

Also, recall that

$$c_1(t) := \frac{50(\sqrt{t} + 1)}{(t - 1)^2}, \quad c_2(t) := \frac{18(\sqrt{t} + 1)^2 + 81 \ln \left( \frac{108e}{\sqrt{t} - 1} \right)}{(\sqrt{t} - 1)^4}.$$ 

We consider two cases:

1. If $\max\{t, \ln(1/M)\} = t$, then (because $1/M \geq t^2$)

$$\frac{40e^2 t^{3/2} t}{(\sqrt{t} - 1) \ln t} \leq \frac{40e^2 (\sqrt{t} + 1) \sqrt{t}}{M(t - 1) \ln t} = \frac{4e^2}{5} \cdot \frac{t - 1}{t \ln t} \cdot \frac{c_1(t) t^{3/2}}{M} \leq \frac{4e^2}{5} \cdot \frac{1}{t \ln t} \cdot \frac{c_1(t) t^{3/2}}{M} < \frac{6c_1(t) t^{3/2}}{M}.$$ 

(We used that the function $t \mapsto \frac{t - 1}{t \ln t}$, $t > 1$, is decreasing.)

Also, since $c_1(t) > e$, for $1 < t \leq 9$ we have

$$e^{\tilde{a}_1(t)+1} < (c_1(t) t^{3/2})^{\tilde{a}_1(t)+1} < (c_1(t) t^{3/2})^{\tilde{a}_1(t)+1}.$$ 

Combining together these inequalities we get in this case

$$e^{c(H)+1} \leq \left( \frac{6c_1(t) t^{3/2}}{M} \right)^{c_2(t)}. \quad (6.17)$$

2. Assume now that $\max\{t, \ln(1/M)\} = \ln(1/M)$. Then we obtain as before

$$\frac{40e^2 t^{3/2} \max\{t, \ln(1/M)\}}{(\sqrt{t} - 1) \ln t} = \frac{4e^2}{5} \cdot M \ln(1/M) \cdot \frac{t - 1}{t \ln t} \cdot \frac{c_1(t) t^{3/2}}{M} \leq \frac{4e^2}{5} \cdot t^2 e^{-t} \cdot \frac{t - 1}{t \ln t} \cdot \frac{c_1(t) t^{3/2}}{M} \leq \frac{16}{5} \cdot \frac{c_1(t) t^{3/2}}{M}.$$ 

29
(We used that \( M \ln(1/M) \leq te^{-t} \), because \( M \leq e^{-t} < e^{-1} \), and that \( t^2 e^{-t} \leq 4e^{-2} \).

Thus in this case
\[
e^{c(H)+1} \leq \left( \frac{4c_1(t)t^{3/2}}{M} \right)^{c_2(t)}.
\]

(6.18)

From (6.15) and (6.17), (6.18) we conclude that
\[
\sup_{B^n \times \mathbb{D}} |g| \leq e^{-p} \left( \frac{a_1(t)}{M} \right)^{a_2(t)(p+q)} M_{g_1}(1),
\]

(6.19)

where
\[
a_1(t) := 6c_1(t)t^{3/2}, \quad a_2(t) := 2c_2(t).
\]

This completes the proof of Theorem 1.2.

\[\Box\]

7. Proof of Theorem 2.3

Choosing a suitable permutation of coordinates on \( \mathbb{C}^k \) without loss of generality we may assume that \( i_j = j, 1 \leq j \leq k \). For a fixed \((w_2, \ldots, w_k) \in \mathbb{D}_{3M_{k2}} \times \cdots \times \mathbb{D}_{3M_{k2}}\) from the conditions of the theorem for \( p_0 \) by Theorem 1.2 we have:

\[
\sup_{(z,w_1) \in \mathbb{D}_r \times \mathbb{D}_{M_{k2}}} |g(z, w_1, w_2, \ldots, w_k)| \leq e^{p_1} \sup_{z \in \mathbb{D}_r} |g(z, f_1(z), w_2, \ldots, w_k)|
\]

Further, for \((w_3, \ldots, w_k) \in \mathbb{D}_{3M_{k2}} \times \cdots \times \mathbb{D}_{3M_{k2}}\) from the conditions of the theorem for \( p_1 \) we have by Theorem 1.2 and the previous inequality:

\[
\sup_{(z,w_2) \in \mathbb{D}_r \times \mathbb{D}_{M_{k2}}} |g(z, f_1(z), w_2, \ldots, w_k)| \leq e^{p_2} \sup_{z \in \mathbb{D}_r} |g(z, f_1(z), f_2(z), w_3 \ldots, w_k)|
\]

Continuing this process we finally obtain

\[
\sup_{(z,w_k) \in \mathbb{D}_r \times \mathbb{D}_{M_{k2}}} |g(z, f_1(z), \ldots, f_{k-1}(z), w_k)| \leq e^{p_k} \sup_{z \in \mathbb{D}_r} |g(z, f_1(z), \ldots, f_k(z))|
\]

Combining together all previous inequalities we get the required:

\[
\max_{\mathbb{D}_r \times \mathbb{D}_{M_{k2}} \times \cdots \times \mathbb{D}_{M_{k2}}} |g| \leq e^{p_1+\cdots+p_k} M_{g_k}(r) \quad \text{and} \quad M_{g_k}(tr) \leq e^{p_k} M_{g_k}(r). \quad \Box
\]
8. Proof of Theorem 2.5

8.1. We first prove the theorem for nonpolynomial entire functions on $\mathbb{C}^n$ of order $\rho < \infty$. We use the following result established in [L, Theorem I.16].

Suppose that $\theta(x)$, $x > x_+ > 0$, is a positive function with

$$\rho = \limsup_{x \to \infty} \frac{\ln \theta(x)}{\ln x} < \infty.$$  

Then $\theta$ has a proximate order $\rho(x)$ with the following properties:

(i) $\lim_{x \to \infty} \rho(x) = \rho$;

(ii) $\theta(x) \leq x^{\rho(x)}$ and $\theta(x_j) = x_j^{\rho(x_j)}$ for some sequence $x_j \to \infty$;

(iii) the function $\psi(x) = x^{\rho(x)-\rho}$ is slowly increasing, i.e.,

$$\lim_{x \to \infty} \frac{\psi(kx)}{\psi(x)} = 1$$

uniformly on each interval $0 < a \leq k \leq b < \infty$;

(iv) $\rho(x)$ is a smooth function satisfying

$$\lim_{x \to \infty} x^{\rho'(x)} \ln x = 0.$$  

In fact, condition (iii) follows from condition (iv). Also, if $x^{\rho(x)-\rho}$ is a slowly increasing function, then for every $\epsilon > 0$ and every $0 < a < b < \infty$ there is $x_0$ such that

$$(1 - \epsilon)k^\rho x^{\rho(x)} < (kx)^{\rho(kx)} < (1 + \epsilon)k^\rho x^{\rho(x)}$$  

for $a \leq k \leq b$ and $x \geq x_0$.

Now, let $f$ be a nonpolynomial entire function on $\mathbb{C}^n$ of order $\rho < \infty$. As before, we define

$$\phi_f(t) := m_f(e^t), \quad t \in \mathbb{R},$$  

where $m_f(r) := \ln M_f(r)$, $r > 0$. Then $\phi_f$ is a convex increasing function. In particular, $\phi_f'$ is a positive nondecreasing function on $\mathbb{R}$ (here $\phi_f'$ is defined before the formulation of Theorem 2.8). Let $\rho_f$ be the proximate order of $m_f$. We define

$$\tilde{\rho}_f(t) := \rho_f(e^t).$$  

Lemma 8.1 Set

$$\alpha_\rho := \min\{1, \ln(1 + 1/\rho)\}.$$  

Then

$$\liminf_{t \to \infty} \frac{\phi_f'(t + \alpha_\rho) + \rho e^{\tilde{\rho}_f(t)}}{\phi_f'(t - 2\alpha_\rho)} \leq \epsilon^3 + \epsilon^2.$$
Proof. Assume, on the contrary, that the statement of the lemma is wrong. Then there are positive numbers $t_0$, $a$ such that for any $t \geq t_0$

$$\phi'_f(t + \alpha\rho) + \rho e^{t\tilde{\rho}_f(t)} > (e^3 + e^2 + a)\phi'_f(t - 2\alpha\rho). \quad (8.3)$$

We will assume without loss of generality that $\rho > 0$. (The arguments in the case $\rho = 0$ are similar.) Then conditions (i) and (iv) in the definition of $\rho_f$ yield

$$\lim_{t \to \infty} \frac{\rho}{(t\rho_f(t))'} = 1.$$  

From here and (8.3) for sufficiently large $t_0$ we obtain

$$\phi'_f(t + \alpha\rho) + b(e^{t\tilde{\rho}_f(t)})' > (e^3 + e^2 + a)\phi'_f(t - 2\alpha\rho)$$

where $b := 1 + ae^{-3}$.

Integrating this inequality from $t_0$ to $t$ we get

$$\phi_f(t + \alpha\rho) - \phi_f(t_0 + \alpha\rho) + be^{t\tilde{\rho}_f(t)} - be^{t_0\tilde{\rho}_f(t_0)} >$$

$$(e^3 + e^2 + a)(\phi_f(t - 2\alpha\rho) - \phi_f(t_0 - 2\alpha\rho)). \quad (8.4)$$

Next, by $t_j := \ln x_j$, $j \in \mathbb{N}$, we denote the sequence from condition (ii) of the definition of the proximate order $\rho_f$. Now, for every $\epsilon > 0$ there is $t_\epsilon > 0$ such that for all $t \geq t_\epsilon$ we have from (8.1)

$$(1 - \epsilon)e^{3\rho_\alpha\rho} e^{(t-2\alpha\rho)\tilde{\rho}_f(t-2\alpha\rho)} < e^{(t+\alpha\rho)\tilde{\rho}_f(t+\alpha\rho)} < (1 + \epsilon)e^{3\rho_\alpha\rho} e^{(t-2\alpha\rho)\tilde{\rho}_f(t-2\alpha\rho)} \quad (8.5)$$

Moreover, according to condition (ii) for $\rho_f$,

$$\phi_f(t) \leq e^{t\tilde{\rho}_f(t)}, \quad \phi_f(t_j) = e^{t_j\tilde{\rho}_f(t_j)}.$$  

From here and inequalities (8.5) for all $j \geq j_\epsilon$ we obtain

$$\phi_f(t_j + 3\alpha\rho) \leq (1 + \epsilon)e^{3\rho_\alpha\rho}\phi_f(t_j). \quad (8.6)$$

Using (8.5), (8.6) and condition (iii) for $\rho_f$ we derive from (8.4) with $t := t_j + 2\alpha\rho$

$$e^3 + e^2 + a \leq$$

$$\liminf_{j \to \infty} \frac{\phi_f(t_j + 3\alpha\rho) - \phi_f(t_0 + \alpha\rho) + be^{(t_j+2\alpha\rho)\tilde{\rho}_f(t_j+2\alpha\rho)} - be^{t_0\tilde{\rho}_f(t_0)}}{\phi_f(t_j) - \phi_f(t_0 - 2\alpha\rho)} =$$

$$\liminf_{j \to \infty} \frac{\phi_f(t_j + 3\alpha\rho) + be^{(t_j+2\alpha\rho)\tilde{\rho}_f(t_j+2\alpha\rho)}}{\phi_f(t_j)} \leq e^{3\rho_\alpha\rho} + be^{2\rho_\alpha\rho} \leq e^3 + e^2 + ae^{-1}.$$  

This gives a contradiction. \qed

As a corollary we obtain
Proposition 8.2 There is a sequence \( \{ \tilde{r}_j \} \subset \mathbb{R}_+ \) convergent to \( \infty \) such that
\[
\frac{m_f(e^{\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}) + \ln(a_1(e^{\alpha}))}{m_f(e^{-\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}/4) - 1} \leq 83,
\]
\[
\frac{\rho \tilde{r}_j^\rho f(\tilde{r}_j)}{m_f(e^{-\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}/4) - 1} \leq \frac{28}{\alpha}, \quad j \in \mathbb{N}.
\]
Here \( a_1(t) \) is defined in (1.12).

Proof. As the \( \tilde{r}_j \) we will take \( e^{s_j} \) where \( \{ s_j \} \subset \mathbb{R}_+ \) is such that
\[
\frac{\phi_f'(s_j + \alpha) + \ln(a_1(e^{\alpha}))}{\phi_f'(s_j - 2\alpha) - 1/\alpha} \leq e^3 + e^2 + 1/10, \quad (8.8)
\]
\[
\frac{\rho e^{s_j} \tilde{r}_j}{\phi_f'(s_j - 2\alpha) - 1/\alpha} \leq e^3 + e^2 + 1/10, \quad j \in \mathbb{N}.
\]
Observe that \( \phi_f'(t) \) tends to \( \infty \) as \( t \to \infty \). (For otherwise, \( m_f(r) \leq A \ln r \) so that \( f \) is a polynomial.) Thus such a sequence exists by Lemma 8.1.

Since the function \( \phi_f' \) is nondecreasing from (8.8) we get
\[
\frac{m_f(e^{\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}) + \ln(a_1(e^{\alpha}))}{m_f(e^{-\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}) - 1} = \frac{\phi_f(s_j + \alpha) - \phi_f(s_j - 2\alpha) + \ln(a_1(e^{\alpha}))}{\phi_f(s_j - \alpha) - \phi_f(s_j - 2\alpha) - 1} \leq (3\alpha)\frac{\phi_f'(s_j + \alpha) + \ln(a_1(e^{\alpha}))}{(\alpha)\phi_f'(s_j - 2\alpha) - 1} \leq 3(e^3 + e^2 + 1/10) < 83.
\]
Similarly,
\[
\frac{\rho \tilde{r}_j^\rho f(\tilde{r}_j)}{m_f(e^{-\alpha \tilde{r}_j}) - m_f(e^{-2\alpha \tilde{r}_j}/4) - 1} \leq \frac{\rho e^{s_j} \tilde{r}_j}{(\alpha)\phi_f'(s_j - 2\alpha) - 1} \leq \frac{e^3 + e^2 + 1/10}{\alpha} \leq \frac{28}{\alpha}.
\]

Now, let us prove the theorem in the case \( \rho < \infty \).
For the sequence \( \{ \tilde{r}_j \} \) satisfying Proposition 8.2 we have, see (1.4) and the defi-
nition of \( \alpha \rho \),

\[
N_f(\tilde{r}_j, e^{\alpha \rho}) \geq \frac{m_f(e^{-\alpha \rho} \tilde{r}_j) - m_f(e^{-2\alpha \rho} \tilde{r}_j) - \alpha \rho/2}{k(e^{\alpha \rho}, e^{2\alpha \rho})} > \\
\frac{m_f(e^{-\alpha \rho} \tilde{r}_j) - m_f(e^{-2\alpha \rho} \tilde{r}_j) - 1}{\ln 8 + \pi^2 + (5\alpha \rho)/2 + 2 \ln(\sqrt{e} + 1) + 2 \ln(\rho + 1/(e - 1))} \geq (8.11)
\]

Set

\[
n_j := \frac{m_f(e^{-\alpha \rho} \tilde{r}_j) - m_f(e^{-2\alpha \rho} \tilde{r}_j) - 1}{9(\sqrt{e} + 1)^2(\rho^2 + 1)(17 + 2 \ln(\rho + 1))}, \quad j \in \mathbb{N}. (8.12)
\]

Then using inequality (3.9) we obtain

\[
n_j \leq \ln \left( \frac{1 + e^{\alpha \rho}}{2e^{\alpha \rho/2}} \right) N_f(\tilde{r}_j, e^{\alpha \rho}). \quad (8.13)
\]

Also, according to Proposition 8.2 we have

\[
m_f(e^{\alpha \rho} \tilde{r}_j) - m_f(e^{-\alpha \rho} \tilde{r}_j) + \ln(a_1(e^{\rho})) \leq c(\rho)n_j, \quad i \in \mathbb{N}, \quad (8.14)
\]

where

\[
c(\rho) := 747(\sqrt{e} + 1)^2(\rho^2 + 1)(17 + 2 \ln(\rho + 1)) \leq c(\rho + 1)^2(1 + \ln(\rho + 1)) \quad (8.15)
\]

for an absolute constant \( c > 0 \).

Observe that \( n_j \to \infty \) as \( j \to \infty \) because \( f \) is not a polynomial. In addition, we have the following statement.

**Proposition 8.3** For all sufficiently large \( j \) the inequality

\[
n_j^{1/(\rho + \epsilon_j)} \leq \tilde{r}_j \leq \left( \frac{A}{\rho_*} \right)^{1/\rho_*} n_j^{1/(\rho - \epsilon'_j)}
\]

holds for an absolute constant \( A > 0 \) and some sequences \( \{\epsilon_j\}, \{\epsilon'_j\} \subset \mathbb{R}_+ \) convergent to 0. Here \( \rho_* := \min\{1, \rho\} \).

(In the case \( \rho = 0 \) we assume that the right-hand side is \( \infty \).)

**Proof.** According to formula (8.12) and the definition of the order \( \rho \), for all sufficiently large \( j \) the inequality

\[
n_j \leq \frac{83m_f(\tilde{r}_j)}{c(\rho)} \leq \frac{83\tilde{r}_j^{\rho + \epsilon_j}}{c(\rho)} \leq \tilde{r}_j^{\rho + \epsilon_j}
\]

holds for some sequences \( \{\tilde{\epsilon}_j\}, \{\epsilon_j\} \subset \mathbb{R}_+ \) convergent to 0. This proves the left-hand side inequality of the proposition.

34
Further, according to Proposition 8.2 and definitions of \( c(\rho), \alpha_\rho \) and \( \rho_f \) we have

\[
\tilde{r}_j \leq \left( \frac{28c(\rho)}{33\rho_\alpha} \right)^{1/\rho_f(\tilde{r}_j)} n_j^{1/\rho_f(\tilde{r}_j)} \leq \left( \frac{A}{\rho_*} \right)^{1/\rho_*} n_j^{1/\rho(\rho-\epsilon'_j)}
\]

for an absolute constant \( A > 0 \) and a sequence \( \{\epsilon'_j\} \subset \mathbb{R}_+ \) convergent to 0. □

Suppose now that \( g \in \mathcal{F}_{p,q}(\tilde{r}_j; e; M_f(e\tilde{r}_j)) \) with \( p \leq n_j \). Then inequality (8.13) implies that \( g \) satisfies conditions of Theorem 1.2 with \( r = \tilde{r}_j, t = e^{\alpha_\rho} \). From this theorem we obtain, see (1.12),

\[
\sup_{\mathbb{B}^{\alpha_\rho} \times \mathcal{D}_{M_f(e^{\alpha_\rho} \tilde{r}_j)}} \ln |g| \leq a_2(e^{\alpha_\rho})(p + q)(\ln a_1(e^{\alpha_\rho}) + m_f(e^{\alpha_\rho} \tilde{r}_j) - m_f(e^{-\alpha_\rho} \tilde{r}_j)) + \ln M_{g_f}(\tilde{r}_j),
\]

where \( M_{g_f}(e^{\alpha_\rho} \tilde{r}_j) \leq a_2(e^{\alpha_\rho})(p + q)(\ln a_1(e^{\alpha_\rho}) + m_f(e^{\alpha_\rho} \tilde{r}_j) - m_f(e^{-\alpha_\rho} \tilde{r}_j) + \ln M_{g_f}(\tilde{r}_j)). \)

Estimating \( a_2(e^{\alpha_\rho}) \) by (1.12) from here, (8.14) and (8.15) we deduce that

\[
\sup_{\mathbb{B}^{\alpha_\rho} \times \mathcal{D}_{M_f(e^{\alpha_\rho} \tilde{r}_j)}} |g| \leq e^{C(\rho)n_j \max\{p,q\}} M_{g_f}(\tilde{r}_j),
\]

where \( C(\rho) = C(\rho + 1)^6(1 + \ln(\rho + 1))^2 \) for an absolute constant \( C > 0 \).

Finally, as the sequence \( \{r_j\} \) of the theorem we will take \( \{\tilde{r}_j\} \) where \( \{\tilde{r}_j\} \) satisfies Propositions 8.2 and 8.3. Then by the Hadamard three circle inequality, see section 3.1, we have

\[
\frac{M_{g_f}(r_j)}{M_{g_f}(\tilde{r}_j)} \leq \frac{M_{g_f}(e^{\alpha_\rho} \tilde{r}_j)}{M_{g_f}(e^{\alpha_\rho} \tilde{r}_j/e)} \leq \left( \frac{M_{g_f}(e^{\alpha_\rho} \tilde{r}_j)}{M_{g_f}(\tilde{r}_j)} \right)^{1/\alpha_\rho} \leq e^{C_\rho n_j \max\{p,q\}}
\]

where

\[
C_\rho := \frac{C(\rho)}{\alpha_\rho} \leq c(\rho + 1)^7(1 + \ln(\rho + 1))^2
\]

for and absolute constant \( c > 0 \).

From here, the first inequality (8.16) and the results of sections 3.1, 3.3, see also Remark 2.1, we obtain straightforwardly the required inequalities of the theorem for \( \rho < \infty \). Observe also, that if \( \rho > 0 \), then Proposition 8.3 implies that

\[
\frac{1}{n_j^{1/(\rho+\epsilon'_j)}} \leq r_j \leq \frac{1}{e} \left( \frac{A}{\rho_*} \right)^{1/\rho_*} n_j^{1/(\rho-\epsilon'_j)}
\]

for an absolute constant \( A > 0 \) and sequences \( \{\epsilon'_j\}, \{\tilde{\epsilon}'_j\} \subset \mathbb{R}_+ \) convergent to 0. This gives the inequalities of statements (1) and (2) of the theorem.

Thus the proof of the theorem for \( \rho < \infty \) is complete.

8.2. Let us prove now the theorem for \( \rho = \infty \). For \( t \geq t_0 \) with a sufficiently large \( t_0 \) we set

\[
\psi_f(t) := \frac{1}{\ln \phi_f(t)}.
\]

35
The function $\psi_f$ is decreasing and differentiable outside a countable set $S \subset \mathbb{R}$ and its derivative $\psi'_f$ is continuous outside $S$ and has discontinuities of the first kind at the points of $S$. We extend $\psi'_f$ to $S$ in the same way as we extended $\phi'_f$. Then we have
\[
\psi'_f(t) = \frac{-\phi'_f(t)}{\phi_f(t)(\ln \phi(t))^2}, \quad t \geq t_0.
\]

**Lemma 8.4**
\[
\lim_{t \to \infty}(-t^2\psi'_f(t)) = 0.
\]

**Proof.** Since the order of $f$ is $\infty$, there are sequences $\{t_j\}, \{a_j\} \subset \mathbb{R}_+$ convergent to $\infty$ such that
\[
\ln \phi_f(t_j) \geq a_j t_j, \quad j \in \mathbb{N}.
\]
This implies
\[
\psi_f(t_j) \leq \frac{\epsilon_j}{t_j}, \quad j \in \mathbb{N}, \quad \text{where} \quad \epsilon_j := \frac{1}{a_j} \to \infty \quad \text{as} \quad j \to \infty. \tag{8.18}
\]
Assume, on the contrary, that there is some $a > 0$ such that
\[
\lim_{t \to \infty}(-t^2\psi'_f(t)) > a.
\]
Then there is $t_* \geq t_0$ such that
\[
-\psi'_f(t) > \frac{a}{t^2} \quad \text{for all} \quad t \in [t_*, \infty). \tag{8.19}
\]
Integrating this inequality from $t \geq t_*$ to $\infty$ we get
\[
\psi_f(t) > \frac{a}{t}, \quad t \in [t_*, \infty). \tag{8.20}
\]
From here and (8.18) we obtain for all sufficiently large $j$,
\[
\frac{a}{t_j} \leq \frac{\epsilon_j}{t_j},
\]
a contradiction. \(\square\)

According to this lemma there is a sequence $\{v_j\} \subset \mathbb{R}_+$ convergent to $\infty$ such that
\[
\lim_{j \to \infty} v_j^2 \psi'_f(v_j) = 0 \tag{8.21}
\]
and $\psi'_f$ is continuous at each $v_j$.

For each sufficiently large $j$ by $s_j > 0$ we denote a number such that
\[
\frac{\phi_f(v_j)}{\phi_f(v_j - 2s_j)} = e. \tag{8.22}
\]
Since $\phi_f$ is an increasing function and $\phi'_f$ is a positive nondecreasing function, from (8.22) we obtain
\[
1 = \ln \phi_f(v_j) - \ln \phi_f(v_j - 2s_j) \leq 2s_j \frac{\phi'_f(v_j)}{\phi_f(v_j - 2s_j)} = 2e s_j \frac{\phi'_f(v_j)}{\phi_f(v_j)}.
\]
Hence,
\[
\frac{1}{s_j} \leq \frac{2e\phi'_f(v_j)}{\phi_f(v_j)}. \tag{8.23}
\]

Set
\[
\tilde{s}_j := \min\{s_j, 1\}, \quad j \in \mathbb{N}. \tag{8.24}
\]

Using the inequality \(\ln(1/s) \leq \frac{\pi}{\sqrt{s}} - 2, s > 0\), from (8.23) we have for all sufficiently large \(j\) (for which, in particular, \(\ln \phi_f(v_j) > 0\))
\[
\frac{\ln(1/\tilde{s}_j)}{\ln \phi_f(v_j)} \leq \frac{2/\sqrt{\tilde{s}_j}}{\ln \phi_f(v_j)} \leq 2\sqrt{2e} \left( \frac{\phi'_f(v_j)}{\phi_f(v_j)(\ln \phi_f(v_j))^2} \right)^{1/2} = 2\sqrt{2e}(-\psi'_f(v_j))^{1/2}. \tag{8.25}
\]

From here and (8.21) we obtain that there is a sequence \(\{\epsilon_j\} \subset \mathbb{R}_+\) convergent to 0 such that for all sufficiently large \(j\)
\[
\frac{\ln(1/\tilde{s}_j)}{\ln \phi_f(v_j)} \leq \frac{\epsilon_j}{v_j}, \quad \text{equivalently,} \quad \frac{1}{s_j} \leq (m_f(e^{v_j}))^{\frac{\epsilon_j}{v_j}}. \tag{8.26}
\]

Next, we set
\[
t_j := e^{\tilde{s}_j}, \quad \tilde{r}_j := e^{v_j - \tilde{s}_j}, \quad j \in \mathbb{N}, \tag{8.27}
\]
and apply Theorem 1.2 to \(f\) defined on \(\mathbb{B}_t \tilde{r}_j\).

According to this theorem for every function \(g \in \mathcal{F}_{p,q}(\tilde{r}_j; t_j; M_f(t_j \tilde{r}_j))\) with \(p \leq \ln \left(\frac{1+t_j}{2\sqrt{t_j}}\right) N_f(\tilde{r}_j, t_j)\) and all sufficiently large \(j\) we have
\[
\sup_{\mathbb{B}_t \tilde{r}_j \times \mathbb{D}_{M_f(t_j \tilde{r}_j)}} |g| \leq (a_1(t_j)M_f(t_j \tilde{r}_j))^{a_2(t_j)(p+q)} M_g(\tilde{r}_j); \tag{8.28}
\]
\[
M_g(\tilde{r}_j) \leq (a_1(t_j)M_f(t_j \tilde{r}_j))^{a_2(t_j)(p+q)} M_g(\tilde{r}_j). \nonumber
\]

(We used here that \(M_f(\tilde{r}_j/t_j) \geq 1\) for all sufficiently large \(j\) because \(t_j \leq e\) and \(\{\tilde{r}_j\}\) tends to \(\infty\).)

Further, according to formulas (1.12), (8.26) we have
\[
a_1(t_j) \leq \frac{300(\sqrt{e} + 1)e^{3/2}}{(e^{s_j} - 1)^2} \leq \frac{3562}{s_j^2} \leq 3562 (m_f(t_j \tilde{r}_j))^{\frac{2s_j}{v_j}}. \tag{8.29}
\]
\[
a_2(t_j) \leq \frac{36(\sqrt{e} + 1)^2 + 162 \ln \left(\frac{108}{e^{s_j/2} - 1}\right)}{(e^{s_j/2} - 1)^4} \leq \frac{4042 + 2592 \ln \left(\frac{216e}{s_j}\right)}{s_j^4} \leq \frac{20567 + 2592 \ln \left(\frac{1}{s_j}\right)}{s_j^4} \leq \frac{20567}{s_j^5} \leq 20567 (m_f(t_j \tilde{r}_j))^{\frac{5s_j}{v_j}}. \nonumber
\]

(We used that \(\ln x \leq x - 1\) for \(x > 0\).)
Thus from (8.28), (8.29) we obtain for all sufficiently large \( j \)

\[
\ln \left( \frac{\sup_{B_{r_j}(t_j, r_j)} |g|}{M_{g_f}(r_j)} \right) \leq 41134 \cdot (m_f(t_j \bar{r}_j))^{1+\frac{5}{v_j}} \max\{p, q\},
\]

(8.30)

\[
\ln \left( \frac{M_{g_f}(t_j \bar{r}_j)}{M_{g_f}(\bar{r}_j)} \right) \leq 41134 \cdot (m_f(t_j \bar{r}_j))^{1+\frac{5}{v_j}} \max\{p, q\}.
\]

Let us estimate from below the expression \( \ln \left( \frac{1+t_j}{2\sqrt{r_j}} \right) N_f(\bar{r}_j, t_j) \).

**Lemma 8.5** There is a sequence \( \{\delta_j\} \subset \mathbb{R}_+ \) convergent to 0 such that for all sufficiently large \( j \)

\[
\ln \left( \frac{1+t_j}{2\sqrt{r_j}} \right) N_f(\bar{r}_j, t_j) \geq (m_f(t_j \bar{r}_j))^{1-\delta_j}.
\]

**Proof.** Using (3.9) we obtain

\[
\frac{1}{\ln \left( \frac{1+t_j}{2\sqrt{r_j}} \right)} \leq \frac{9(\sqrt{r_j} + 1)^2}{(t_j - 1)^2} \leq \frac{9(\sqrt{v} + 1)^2}{s^2_j} \leq 64(m_f(t_j \bar{r}_j))^{\frac{2v_j}{v_j}}. \tag{8.31}
\]

Also, for all sufficiently large \( j \) by (1.4), (1.5), (8.22) we have

\[
N_f(\bar{r}_j, t_j) \geq \ln(M_f(\bar{r}_j/t_j)/\sqrt{M_f(1)}) \geq \frac{m_f(t_j \bar{r}_j)}{3k(t_j, \bar{r}_j)}. \tag{8.32}
\]

In turn, for all sufficiently large \( j \),

\[
k(t_j, \bar{r}_j) \leq \ln \left( \frac{8e^{-\frac{r_j}{2}} \sqrt{v} \sqrt{v + 1}}{s^2_j} \right) \leq \ln \bar{r}_j + 15 + \frac{2\bar{r}_j}{v_j} \ln m_f(t_j \bar{r}_j). \tag{8.33}
\]

Next, we estimate \( \ln \bar{r}_j \).

Since the function \( \phi_f' \) is nondecreasing,

\[
\frac{\phi_f(v_j) - \phi_f(t_0)}{(v_j - t_0)\phi_f(v_j)(\ln \phi_f(v_j))} \leq \frac{\phi_f'(v_j)}{\phi_f(v_j)(\ln \phi_f(v_j))} := \psi_f'(v_j) \leq \frac{\bar{r}_j}{v_j^2} \tag{8.34}
\]

for some \( \{\bar{r}_j\} \subset \mathbb{R}_+ \) convergent to 0.

Also,

\[
\lim_{j \to \infty} \frac{v_j(\phi_f(v_j) - \phi_f(t_0))}{(v_j - t_0)\phi_f(v_j)} = 1. \tag{8.35}
\]

From (8.34) and (8.35) for all sufficiently large \( j \) and some sequence \( \{\epsilon_j'\} \subset \mathbb{R}_+ \) convergent to 0 we obtain

\[
\ln \bar{r}_j := v_j - \bar{s}_j < v_j \leq \epsilon_j' (\ln \phi_f(v_j))^2 = \epsilon_j' (\ln m_f(t_j \bar{r}_j))^2. \tag{8.36}
\]

38
Hence, for all sufficiently large \( j \)

\[
3k(t_j, \tilde{r}_j) \leq \epsilon''_j (\ln m_f(t_j \tilde{r}_j))^2
\]

for some \( \{\epsilon''_j\} \subset \mathbb{R}_+ \) convergent to 0.

From here, (8.32) and (8.31) we obtain for all sufficiently large \( j \)

\[
\ln \left( \frac{1 + t_j}{2 \sqrt{t_j}} \right) N_f(\tilde{r}_j, t_j) \geq (m_f(t_j \tilde{r}_j))^{1-\delta_j}
\]

for some \( \{\delta_j\} \subset \mathbb{R}_+ \) convergent to 0. \( \Box \)

We set

\[
n_j := (m_f(t_j \tilde{r}_j))^{1-\delta_j}
\]

with \( \{\delta_j\} \) satisfying Lemma 8.5.

Then from (8.30) we get for every \( g \in \mathcal{F}_{p,q}(\tilde{r}_j; t_j; M_f(t_j \tilde{r}_j)) \) with \( p \leq n_j \)

\[
\sup_{B^\circ_{r_j} \times \mathbb{D}_{M_f(t_j \tilde{r}_j)}} |g| \leq e^{n_j^{1+\epsilon_j}} \max\{p,q\} M_{g_f}(\tilde{r}_j),
\]

(8.38)

for some \( \{\epsilon_j\} \) convergent to 0.

Finally, we define

\[
r_j := \frac{t_j \tilde{r}_j}{e^2}.
\]

Observe that \( \mathcal{F}_{p,q}(er_j; e; M_f(e^2 r_j)) \subset \mathcal{F}_{p,q}(\tilde{r}_j; t_j; M_f(t_j \tilde{r}_j)) \), because \( t_j \leq e \). Then by the Hadamard three circle inequality, see section 3.1, using the estimate for \( 1/\ln t_j \), see (8.26), we have for all sufficiently large \( j \) and \( g \in \mathcal{F}_{p,q}(er_j; e; M_f(e^2 r_j)) \), \( p \leq n_j \),

\[
\frac{M_{g_f}(er_j)}{M_{g_f}(r_j)} \leq \frac{M_{g_f}(t_j \tilde{r}_j)}{M_{g_f}(t_j \tilde{r}_j/e)} \leq \left( \frac{M_{g_f}(t_j \tilde{r}_j)}{M_{g_f}(\tilde{r}_j)} \right)^{1/t_j} \leq e^{n_j^{1+\epsilon_j}} \max\{p,q\}
\]

for some \( \{\epsilon_j\} \subset \mathbb{R}_+ \) convergent to 0.

From here, the first inequality (8.38), definition (8.37) and the results of sections 3.1, 3.3, see also Remark 2.1, we obtain straightforwardly the required inequalities of the theorem for \( \rho = \infty \).

The proof of Theorem 2.5 is complete. \( \Box \)

9. Proofs of Theorem 2.8 and Corollary 2.10

9.1. Proof of Theorem 2.8.

(1) Assume that an entire function \( f \) on \( \mathbb{C}^n \) of order \( \rho < \infty \) satisfies

\[
\limsup_{t \to \infty} \frac{m_f(e^{\rho t}) - m_f(e^{-\rho t}) + \rho e^t}{m_f(e^{-\rho t}) - m_f(e^{-2\rho t})} < A < \infty
\]

(9.1)
where \( \alpha_\rho := \min \{1, \ln(1 + 1/\rho)\} \).

Then for a sufficiently large \( r_0 \) and all \( r \geq r_0 \) we have
\[
\frac{m_f(e^{\alpha_\rho r}) - m_f(e^{-\alpha_\rho r}) + \ln(a_1(e^{\alpha_\rho}))}{m_f(e^{-\alpha_\rho r}) - m_f(e^{-2\alpha_\rho r}) - 1} < A,
\]
(9.2)
\[
\frac{\rho e^{\rho t}}{m_f(e^{-\alpha_\rho r}) - m_f(e^{-2\alpha_\rho r}) - 1} < A.
\]

Similarly to the definition of \( n_\rho \) in section 8.1, see (8.12), we determine
\[
k(r) := \frac{m_f(e^{-\alpha_\rho r}) - m_f(e^{-2\alpha_\rho r}) - 1}{9(\sqrt{e} + 1)^2(\rho^2 + 1)(17 + 2\ln(\rho + 1))}, \quad r \geq r_0.
\]
(9.3)

Then
\[
k(r) \leq \ln \left( \frac{1 + e^{\alpha_\rho}}{2e^{\alpha_\rho/2}} \right) N_f(r, e^{\alpha_\rho}).
\]
(9.4)

Also, according to (9.2)
\[
m_f(e^{\alpha_\rho r}) - m_f(e^{-\alpha_\rho r}) + \ln(a_1(e^{\alpha_\rho})) \leq A \tilde{c}(\rho) k(r), \quad r \geq r_0,
\]
(9.5)
where \( \tilde{c}(\rho) := 9(\sqrt{e} + 1)^2(\rho^2 + 1)(17 + 2\ln(\rho + 1)) \).

Further, as in the proof of Proposition 8.3, using the definition of the order \( \rho \), by (9.2) we obtain for a sufficiently large \( r_0 \) and all \( r \geq r_0 \),
\[
(k(r))^{1/(\rho + \epsilon'(r))} \leq r \leq \left( \frac{\tilde{c}}{\rho_*} \right)^{1/\rho} (k(r))^{1/\rho},
\]
(9.6)
for some \( \tilde{c} \) depending on \( A \); here \( \rho_* := \min \{1, \rho\} \) and \( \epsilon' : [r_0, \infty) \rightarrow \mathbb{R} \) is a continuous function decreasing to 0 as \( r \rightarrow \infty \).

As in section 8.1 inequalities (9.4), (9.5), (9.6) imply the fulfillment of Theorem 2.5 for functions \( g \in \mathcal{F}_{\rho_0}(er; e; M_f(e^2r)) \) with \( p \leq k(r) \) in which \( n_j \) is substituted for \( k(r) \), \( r \geq r_0 \), \( r_j \) is substituted for \( r \geq r_0 \), \( \varepsilon_j \) is substituted for 0 and \( \varepsilon_j' \) is substituted for \( \varepsilon'(r) \). The constants in these inequalities depend on \( A \) and \( \rho \) only.

Finally, observe that the continuous function \( k(r), r \geq r_0 \), is positive nondecreasing, tending to \( \infty \) as \( r \rightarrow \infty \) (because \( f \) is not a polynomial). In particular, we can determine its right inverse by the formula
\[
r(l) := \inf \{ s : k(s) = l \}, \quad l \geq k(r_0) := k_0.
\]
(9.7)

Thus \( r : [k_0, \infty) \rightarrow [r_0, \infty) \) is a continuous increasing function tending to \( \infty \) as \( k \rightarrow \infty \) and such that \( k \circ r = id \). Substituting in the obtained inequalities and inequality (9.6) \( k \) instead of \( k(r) \) and \( r(k) \) instead of \( r \) we obtain the required statements of Theorem 2.8 for \( \rho < \infty \).

(2) Assume now that \( \phi_f(t) := m_f(e^t) \) satisfies
\[
\lim_{t \rightarrow \infty} t^2 \left( \frac{1}{\ln \phi_f(t)} \right)' = 0.
\]
(9.8)
For each sufficiently large \( v \in \mathbb{R} \) by \( s(v) \in \mathbb{R}_+ \) we denote a number such that
\[
\frac{\phi_f(v)}{\phi_f(v - 2s(v))} = e.
\]
Since \( \phi_f \) is a continuous increasing function,
\[
s(v) = \frac{1}{2} \left( v - \phi_f^{-1}(\phi_f(v)/e) \right),
\]
i.e., \( s(v) \) is a continuous in \( v \) function.

Then similarly to (8.23) we obtain
\[
\frac{1}{s(v)} \leq \frac{2e\phi_f(v)}{\phi_f(v)}. \quad (9.9)
\]
From here arguing as in section 8.2 and using (9.8) for \( \tilde{s}(v) := \min\{s(v), 1\} \) we get
\[
\frac{1}{\tilde{s}(v)} \leq \left( m_f(e^v) \right)^{\epsilon(v)}, \quad v \geq v_0,
\]
where \( \epsilon(v) \) is a positive continuous function in \( v \) tending to 0 as \( v \to \infty \) and \( v_0 \in \mathbb{R} \) is sufficiently large.

Next, we determine continuous in \( v \) functions
\[
t(v) := e^{\tilde{s}(v)}, \quad \tilde{r}(v) := e^{v-\tilde{s}(v)}.
\]
Then as in section 8.2 we obtain
\[
\ln \tilde{r}(v) \leq \epsilon'(v)\left[ \ln m_f(t(v)\tilde{r}(v)) \right]^2, \quad v \geq v_0,
\]
where \( \epsilon'(v) \) is a positive continuous function in \( v \) such that \( \lim_{v \to \infty} \epsilon'(v) = 0 \). Also, in this case instead of Lemma 8.5 by similar arguments we deduce that
\[
\ln \left( \frac{1 + t(v)}{2\sqrt{t(v)}} \right) N_f(\tilde{r}(v), t(v)) \geq (m_f(t(v)\tilde{r}(v)))^{1-\delta(v)} =: k(v), \quad v \geq v_0,
\]
for some positive continuous function \( \delta(v) \) tending to 0 as \( v \to \infty \). Diminishing, if necessary, \( 1 - \delta(v) \) we can assume that this function is increasing. In particular, the function \( k(v), \) \( v \geq v_0 \), is increasing because \( t(v)\tilde{r}(v) = e^v \).

From inequalities (9.10), (9.11), (9.12) arguing as in section 8.2 we obtain inequalities for \( g \in \mathcal{F}_{p,q}(\tilde{r}(v); t(v); M_f(t(v)\tilde{r}(v))) \) with \( p \leq k(v), \) \( v \geq v_0 \), similar to (8.38) in which \( n_{j+\epsilon_j}^{1+\epsilon_j} \) is substituted for \( (k(v))^{1+\epsilon_j} \) for some continuous nonnegative function \( \tilde{\epsilon}(v) \), tending to 0 as \( v \to \infty \). As before, these inequalities give rise to inequalities of Theorem 2.5 with \( \rho = \infty \) in which \( n_{j+\epsilon_j}^{1+\epsilon_j} \) is substituted for \( (k(v))^{1+\epsilon(v)} \), \( v \geq v_0 \), for some continuous nonnegative function \( \epsilon(v) \) tending to 0 as \( v \to \infty \), and \( r_j \) is substituted for \( \frac{t(v)\tilde{r}(v)}{e^v} := e^{v-2}, \) \( v \geq v_0 \).
Finally, we write in the latter inequalities \( k \) instead of \( k(v) \). Then since the function \( k \) has a continuous increasing inverse \( s \), instead of \( v \) and \( e^{v-2} \) we obtain continuous increasing functions \( s(k) \) and \( r(k) := e^{s(k)-2} \), \( k \geq k_0 := k(v_0) \). This gives the required statements of the theorem (with \( r_0 := r(k_0) \)). \( \square \)

### 9.3. Proof of Corollary 2.10.

Inequality \( \tau(f) \leq 2 \) follows directly from Theorem 2.5 (c) applied to restrictions to the graph of \( f \) of polynomials \( g \) of degrees \([n_j] \). (In this case \( g \in \mathcal{F}_{p,q}(er_j; e; M_f(e^2r_j)) \) with \( p = q = [n_j] \leq n_j, j \in \mathbb{N} \).

Now, let us prove the lower bound for \( \tau(f) \).

By the definition the dimension of the space \( P_{k,n}^{k+1} \) of holomorphic polynomials of degree \( k \) on \( \mathbb{C}^{n+1} \) is \( d_{k,n+1} := \frac{(n+k+1)k!}{(n+1)!k!} \). Hence,

\[
\lim_{k \to \infty} \frac{d_{k,n+1}}{k^{n+1}} = \frac{1}{(n+1)!} \tag{9.13}
\]

Let

\[
f(z) := \sum_{\alpha=0}^{\infty} f_\alpha z^\alpha, \quad z \in \mathbb{C}^n,
\]

be the Taylor series of \( f \) at 0. Here \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \) and \( z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \).

The number of coefficients of the series at monomials of degree \( \leq p_k := \left\lfloor \frac{k^{1+1/n}}{(n+2)^{1/n}} \right\rfloor \) is \( d_{p_k,n} \). In particular,

\[
\lim_{k \to \infty} \frac{d_{p_k,n}}{k^{n+1}} = \frac{1}{(n+2)n!} \tag{9.14}
\]

Comparing (9.13) and (9.14) we conclude that there is \( k_0 \in \mathbb{N} \) such that for each \( k \geq k_0 \) we have \( d_{p_k,n} < d_{k,n+1} \). Thus for each \( k \geq k_0 \) there is \( g \in P_{k,n+1} \) such that the Taylor series of \( g_f(z) := g(z, f(z)) \) has the form

\[
g_f(z) = \sum_{\alpha=p_k+1}^{\infty} [g_f]_\alpha z^\alpha, \quad z \in \mathbb{C}^n. \tag{9.15}
\]

(Indeed, the linear map \( \pi : P_{k,n+1} \to P_{p_k,n} \), \( \pi(g) := \sum_{\alpha=p_k}^{p_k+1} [g_f]_\alpha z^\alpha \), has a nonzero kernel because \( d_{p_k,n} < d_{k,n+1} \).) Also, since \( f \) is not a polynomial, \( g_f \not\equiv 0 \).

Let \( l \in \mathcal{L}_n \) be a complex line passing through 0 such that

\[
\sup_{l \cap \mathbb{B}^n} |g_f| = M_{g_f}(1)
\]

Let us identify \( l \) with \( \mathbb{C} \). Then for the holomorphic function \( h := g_f|_l \) on \( \mathbb{C} \) by (9.15) we deduce that the function

\[
\tilde{h}(z) := \frac{h(z)}{z^{p_k+1}}
\]

is holomorphic and \( M_{\tilde{h}}(1) = M_h(1) \not\equiv 0 \). This implies

\[
M_{g_f}(1) = M_{\tilde{h}}(1) \leq M_{\tilde{h}}(e) \leq \frac{M_h(e)}{e^{p_k+1}} \leq \frac{M_{g_f}(e)}{e^{p_k+1}}.
\]
From here we have, see (2.22),

\[ m_k(e, f) \geq p_k + 1. \]

Therefore

\[ \liminf_{k \to \infty} \frac{m_k(e, f)}{k^{1+1/n}} \geq \liminf_{k \to \infty} \frac{p_k + 1}{k^{1+1/n}} = \frac{1}{(n+2)^{1/n}} > 0. \]

This shows that \( \tau(f) \geq 1 + 1/n. \) \( \square \)

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