String diagrams for traced and compact categories are oriented 1-cobordisms

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

| Citation       | Spivak, David I., Patrick Schultz and Dylan Rupel. "String diagrams for traced and compact categories are oriented 1-cobordisms." Journal of Pure and Applied Algebra 221, no. 8 (2017): pp. 2064-2110. |
|----------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| As Published   | https://doi.org/10.1016/j.jpaa.2016.10.009                                                                                                                                                    |
| Publisher      | Elsevier                                                                                                                                                                                      |
| Version        | Author’s final manuscript                                                                                                                                                                        |
| Citable link   | https://hdl.handle.net/1721.1/121541                                                                                                                                                    |
| Terms of Use   | Creative Commons Attribution-NonCommercial-NoDerivs License                                                                                                                                 |
| Detailed Terms | http://creativecommons.org/licenses/by-nc-nd/4.0/                                                                                                                                               |
String diagrams for traced and compact categories are oriented 1-cobordisms

David I. Spivak* Patrick Schultz*
Massachusetts Institute of Technology, Cambridge, MA 02139

Dylan Rupel†,‡
Northeastern University, Boston, MA 02115

Abstract

We give an alternate conception of string diagrams as labeled 1-dimensional oriented cobordisms, the operad of which we denote by \( \text{Cob}_\mathcal{O} \), where \( \mathcal{O} \) is the set of string labels. The axioms of traced (symmetric monoidal) categories are fully encoded by \( \text{Cob}_\mathcal{O} \) in the sense that there is an equivalence between \( \text{Cob}_\mathcal{O} \)-algebras, for varying \( \mathcal{O} \), and traced categories with varying object set. The same holds for compact (closed) categories, the difference being in terms of variance in \( \mathcal{O} \). As a consequence of our main theorem, we give a characterization of the 2-category of traced categories solely in terms of those of monoidal and compact categories, without any reference to the usual structures or axioms of traced categories. In an appendix we offer a complete proof of the well-known relationship between the 2-category of monoidal categories with strong monoidal functors and the 2-category of monoidal categories whose object set is free with strict functors; similarly for traced and compact categories.

Keywords: Traced monoidal categories, compact closed categories, monoidal categories, lax functors, equipments, operads, factorization systems.

Contents

1 Introduction 2
  1.1 The main results ........................................ 5
  1.2 Plan of the paper ...................................... 7

2 Background on equipments 7
  2.1 Equipments ........................................... 7
  2.2 Internal copresheaves .................................. 11
  2.3 Monoids and bimodules ................................. 12
  2.4 Exact equipments and \( \mathcal{O} \), \( \mathcal{F} \) factorization .............. 14

*Supported by AFOSR grant FA9550–14–1–0031, ONR grant N000141310260, and NASA grant NNNH13ZEA001N.
†Corresponding author
‡Present address: University of Notre Dame, Notre Dame, IN 46556
Email addresses: dspivak@math.mit.edu, schultzp@mit.edu, drupel@nd.edu
1 Introduction

Traced (symmetric monoidal) categories have been used to model processes with feedback [1] or operators with fixed points [26]. A graphical calculus for traced categories was developed by Joyal, Street, and Verity [16] in which string diagrams of the form

represent compositions in a traced category $T$. That is, new morphisms are constructed from old by specifying which outputs will be fed back into which inputs. These are related to Penrose diagrams in $\text{Vect}$ and the word traced originates in this vector space terminology.

The string diagrams of [16] typically do not explicitly include the outer box $Y$. If we include it, as in (1), the resulting wiring diagram can be given a seemingly new interpretation: it represents a 1-dimensional cobordism between oriented 0-manifolds. Indeed, the objects in $\text{Cob}$ are signed sets $X = X^- \sqcup X^+$, each of which can be drawn as a box with input wires $X^-$ entering on the left and output wires $X^+$ exiting on the right.

Moreover, the wiring diagram itself in which boxes $X_1, \ldots, X_n$ are wired together inside a larger box $Y$ can be interpreted as an oriented cobordism from $X_1 \sqcup \cdots \sqcup X_n$ to $Y$. In
fact, this is more appropriately interpreted as a morphism in the (colored) operad \( \text{Cob} \) underlying the symmetric monoidal category of oriented 1-cobordisms. The following shows the two approaches to drawing a 2-ary morphism \( X_1, X_2 \to Y \) in \( \text{Cob}/\mathcal{O} \):

There is actually a bit more data in a string (or wiring) diagram for a traced category \( \mathcal{T} \) than in a cobordism. Namely, each input and output of a box must be labeled by an object of \( \mathcal{T} \) and the wires connecting boxes must respect the labels (e.g. in (1) objects \( 1c \) and \( 2b \) must be equal). We will thus consider the operad \( \text{Cob}/\mathcal{O} \) of oriented 1-dimensional cobordisms over a fixed set of labels \( \mathcal{O} \). We also write \( \text{Cob}/\mathcal{O} \) to denote the corresponding symmetric monoidal category.

In the table below, we record these two interpretations of a string diagram. Note the “degree shift” between the second and third columns.

| String diagram | Traced category \( \mathcal{T} \) | \( \text{Cob}/\mathcal{O} \) |
|----------------|----------------------------------|----------------------------------|
| Wire label set, \( \mathcal{O} \) | Objects, \( \mathcal{O} := \text{Ob}(\mathcal{T}) \) | Label set, \( \mathcal{O} \) |
| Boxes, e.g. \( \mathcal{O} \) | Morphisms in \( \mathcal{T} \) | Objects (oriented 0-mfds over \( \mathcal{O} \)) |
| String diagrams | Compositions in \( \mathcal{T} \) | Morphisms (cobordisms over \( \mathcal{O} \)) |
| Nesting | Axioms of traced cats | Composition (of cobordisms) |

In the last row above, each of the seven axioms of traced categories is vacuous from the cobordism perspective in the sense that both sides of the equation correspond to the same cobordism (up to diffeomorphism). For example, the axiom of superposition reads:

\[
\text{Tr}_{X,Y}^U \left[ f \right] \otimes g = \text{Tr}_{X \otimes W, Y \otimes Z}^U \left[ f \otimes g \right]
\]

for every \( f : U \otimes X \to U \otimes Y \) and \( g : W \to Z \), or diagrammatically:
To make precise the relationship between these interpretations of string diagrams, we fix the set 0 of labels. Let \( \text{TrCat} \) denote the 1-category of traced categories and traced strict monoidal functors. Write \( \text{TrCat}_0 \) for the subcategory of those traced categories \( \mathcal{T} \) for which the monoid of objects is free on the set 0, with identity-on-objects functors \( \mathcal{T} \to \mathcal{T}' \) between them.

**Theorem 0.** There is an equivalence of 1-categories

\[
\text{Cob}_0^{-\text{Alg}} \simeq \text{TrCat}_0,
\]

where, given any monoidal category \( \mathcal{M} \), we denote by \( \mathcal{M}^{-\text{Alg}} := \text{Lax}(\mathcal{M}, \text{Set}) \) the category of lax functors \( \mathcal{M} \to \text{Set} \) and monoidal natural transformations.

To build intuition for this statement note that the same data are required, and the same conditions are satisfied, whether one is specifying a lax functor \( P \in \text{Cob}_0^{-\text{Alg}} \) or a traced category \( \mathcal{T} \in \text{TrCat}_0 \) with objects freely generated by the set 0. First, for each box \( X = (X^-, X^+) \) that might appear in a string diagram, both \( P \) require a set, \( P(X) \) and \( \text{Hom}_\mathcal{T}(X^-, X^+) \), respectively. Second, for each string diagram, both \( P \) and \( \mathcal{T} \) require a function: an action on morphisms in the case of \( P \) and a formula for performing the required compositions, tensors, and traces in the case of \( \mathcal{T} \). The condition that \( P \) is functorial corresponds to the fact that \( \mathcal{T} \) satisfies the axioms of traced categories.

We will briefly specify how to construct a lax functor \( P \) from a traced category \( (\mathcal{T}, \otimes, I, \text{Tr}) \) whose objects are freely generated by 0. In what follows, we abuse notation slightly: given a relative set \( \iota : Z \to 0 \) we will use the same symbol \( Z \) to denote the tensor \( \bigotimes_{z \in Z} \iota(z) \) in \( \mathcal{T} \). For an oriented 0-manifold \( X = X^- \sqcup X^+ \) over 0, put \( P(X) := \text{Hom}_\mathcal{T}(X^-, X^+) \). Given a cobordism \( \Phi : X \to Y \), we need a function \( P(\Phi) : P(X) \to P(Y) \). To specify it, note that for any cobordism \( \Phi \) there exist \( A, B, C, D, E \in \text{Ob}(\mathcal{T}) \) such that \( X^- \cong C \otimes A, X^+ \cong C \otimes B, Y^- \cong A \otimes D, Y^+ \cong B \otimes D, \) and \( E \) is the set of floating loops in \( \Phi \); thus \( \Phi \) is essentially equivalent to the cobordism shown on the left side of (3).

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
A & \rightarrow & A \\
C & \rightarrow & C \\
B & \rightarrow & B \\
\end{array}
\]

With the above notation, for \( f \in P(X) \) we can follow the string diagram (right side of (3)) and define

\[
P(\Phi)(f) := \text{Tr}_A^C[f] \otimes D \otimes \text{Tr}_D^E[I],
\]

where we abuse notation and write \( D \) and \( E \) for the identity maps on these objects. One may easily check, using each axiom of the trace [16] in an essential way, that (4) defines an algebra over \( \text{Cob}_0 \). We will not prove Theorem 0 directly as indicated here but to specify our proof strategy we must introduce more notation.
1.1 The main results

The equivalence (2) has two significant conceptual drawbacks: the object set of the traced category is fixed, and it is assumed to be freely generated by some set under tensor products and functors are assumed strict; we refer to this latter condition using the term \textit{objectwise-free}. Much of the work in this paper goes towards relaxing these two conditions.

To overcome the use of a fixed object set, we first explain what kind of object variance is appropriate. There is an adjunction

\[ \mathbf{Set} \xleftrightarrow{F_T, U_T} \mathbf{TrCat} \]

inducing a monad $T_T$ on $\mathbf{Set}$, which is in fact isomorphic to the free monoid monad. Let $\mathcal{T}$ and $\mathcal{T}'$ be objectwise-free traced categories where $\text{Ob}(\mathcal{T})$ is the free monoid on a set $\emptyset$ and $\text{Ob}(\mathcal{T}')$ is the free monoid on a set $\emptyset'$. A strict (traced) monoidal functor $F : \mathcal{T} \to \mathcal{T}'$ induces a homomorphism $\text{Ob}(F) : \text{Ob}(\mathcal{T}) \to \text{Ob}(\mathcal{T}')$ between the free monoids, or equivalently a function $\text{Ob}F : \emptyset \to T_T(\emptyset')$ which can be identified with a morphism in the Kleisli category $\mathbf{Set}_{T_T}$ of this monad.

The compact category $\mathbf{Cob}/_\emptyset$ clearly varies functorially in $\emptyset \in \mathbf{Set}$, but it is not much harder to see that it is also functorial in $\emptyset \in \mathbf{Set}_{T_T}$, giving rise to a functor

\[ (\mathbf{Cob}/_\emptyset) : \mathbf{Set}_{T_T} \to \mathbf{CpCat} \]

to the category $\mathbf{CpCat}$ of compact categories and strict functors, under which $\emptyset$ is sent to the free compact category on $\emptyset$ (e.g. see [17, 2]). We can compose this with $\text{Lax}(\mathbf{Set})$ to obtain a functor which we denote

\[ (\mathbf{Cob}/_\emptyset)\text{-Alg} : \mathbf{Set}_{T_T}^{\text{op}} \to \mathbf{Cat}. \quad (5) \]

By applying the Grothendieck construction (denoted by $\int$ here) to (5), we obtain a fibration for which the fiber over a set $\emptyset$ is equivalent (by ??) to $\mathbf{TrCat}_\emptyset$. Let $\mathbf{TrFrObCat} \subset \mathbf{TrCat}$ denote the full subcategory spanned by the objectwise-free traced categories.

\textbf{Theorem A.} \textit{There is an equivalence of 1-categories}

\[ \int_{\emptyset \in \mathbf{Set}_{T_T}} (\mathbf{Cob}/_\emptyset)\text{-Alg} \to \mathbf{TrFrObCat}. \]

This result is proven in Section 3.5 together with an analogous statement for compact categories.

The fact that the traced categories appearing in \textbf{Theorem A} are assumed objectwise-free and the functors between them are strict is the second of two drawbacks mentioned above. To address it, we prove that the 2-category of traced categories and strong functors is biequivalent to that of objectwise-free traced categories and strict functors; see Corollary A.3.2. This result seems to be well-known to experts but is difficult to find in the literature.
Remark 1.1.1. Another way to address the condition of objectwise-freeness is to embrace it, and ask that functors between them be not only strict but also send generating objects to generating objects, i.e. to consider PROPs rather than monoidal categories. In fact, there is an analogue of Theorem A in which $\text{TrFrObCat}$ is replaced by $\text{TrPROP}$. We do not prove this result, but we will clarify it at an appropriate time; see Remark 3.5.4.

In the course of proving Theorem A, we will also establish generalizations characterizing lax functors out of arbitrary compact categories, and in particular lax functors out of $\text{Int}(\mathcal{F})$ for an arbitrary traced category $\mathcal{F}$. In order to state this characterization, we prove (Theorem 2.4.14 and Proposition 3.4.2) that the well-known $(\text{bo}, \text{ff})$ factorization system of $\text{Cat}$ restricts to a factorization system on $\text{TrCat}$; more precisely the left class consists of bijective-on-objects functors and the right class consists of fully faithful functors.

Write $\text{TrCat}^{\text{bo}}$ for the full subcategory of the arrow category $\text{TrCat} \to$ spanned by the bijective-on-objects functors. The existence of the factorization system implies that the domain functor

$$\text{dom} : \text{TrCat}^{\text{bo}} \to \text{TrCat}$$

is a fibration. For a fixed traced category $\mathcal{F}$, the fiber $\text{TrCat}^{\text{bo}}_{\mathcal{F}/} := \text{dom}^{-1}(\mathcal{F})$ is the category of strict monoidal, bijective-on-objects functors from $\mathcal{F}$ to another traced category, with the evident commutative triangles as morphisms. Note that we have an isomorphism $\text{TrCat}_{\mathcal{F}/0} \cong \text{TrCat}_{\text{FrObCat}^{\text{bo}}/0}$.

Recall from [16] that traced categories can be thought of as full subcategories of compact categories: the Int construction applied to a traced category $\mathcal{F}$ builds the smallest compact category $\text{Int}(\mathcal{F})$ of which $\mathcal{F}$ is a monoidal subcategory. Generalizing (2), we can give a complete characterization of lax functors out of such compact categories: for a fixed traced category $\mathcal{F}$ there is an equivalence of categories

$$\text{Lax}(\text{Int}(\mathcal{F}), \text{Set}) \simeq \text{TrCat}^{\text{bo}}_{\mathcal{F}/}.$$ 

In Section 3.4 we show that these equivalences glue together to form an equivalence of fibrations:

**Theorem B.** There is an equivalence of fibrations

$$\mathcal{F} \in \text{TrCat} \xrightarrow{\int} \text{Lax}(\text{Int}(\mathcal{F}), \text{Set}) \xrightarrow{\simeq} \text{TrCat}^{\text{bo}} \xrightarrow{\text{dom}} \text{TrCat}.$$ 

Our main tool in proving this result will be the 2-categorical notion of (proarrow) equipments, which we recall in Section 2. We will introduce what appears to be a new definition of monoidal profunctors, and the equipment thereof, in Section 3.

### 1.2 Plan of the paper

Make sure this plan is still correct.
Section 2.1 reviews the notion of an equipment (or framed bicategory [29]). In Section 2.2 we record the definition of copresheaves internal to an equipment which will allow us to reformulate and properly understand our main theorems. Section 2.3 recalls monoids and bimodules in an equipment and establishes basic facts and constructions needed for the main theorems; in particular, exact equipments [28] are defined using this language. In Section 3.1 we review monoidal, traced, and compact categories. We finally introduce various equipments of monoidal profunctors (\(\mathbf{MnProf}\), \(\mathbf{TrProf}\), and \(\mathbf{CpProf}\)) at the heart of the paper in Section 3.2. In Section 3.3 we prove the special properties about \(\mathbf{CpProf}\) which are at the core of our results. Indeed one might view the rest of the paper as a formal wrapper for the results in that section. In Section 3.4 we prove that the equipments of interest are exact and then apply the theory developed in Section 2 to deduce Theorem B. In Section 3.5 we deal with the freeness-on-objects needed for Theorem A.

The appendix contains material that is not necessary to the paper. Its function is to prove the biequivalence between the 2-category \(\tilde{\mathbf{MnCat}}\) of monoidal categories with arbitrary object set and strong functors, on the one hand, and the 2-category \(\mathbf{MnFrObCat}\) of monoidal categories with free object set and strict functors. We do the same for traced and compact categories, all in Corollary A.3.2.

Acknowledgments

Thanks go to Steve Awodey and Ed Morehouse for suggesting we formally connect the operad-algebra picture in [27] to string diagrams in traced categories. We also thank Mike Shulman for many useful conversations, and Tobias Fritz, Justin Hilburn, Dmitry Vagner, and Christina Vasilakopoulou for helpful comments on drafts of this paper. Finally, we thank our referee for many useful suggestions.

2 Background on equipments

This section introduces equipments, which we use to properly situate traced and compact categories. This tool will eventually allow us to clarify the relationship between strict monoidal functors between monoidal categories and lax monoidal functors to \(\mathbf{Set}\).

2.1 Equipments

A double category is a 2-category-like structure involving horizontal and vertical arrows, as well as 2-cells. An equipment (sometimes called a proarrow equipment or framed bicategory) is a double category satisfying a certain fibrancy condition. In this section, we will spell this out and give two relevant examples. An excellent reference is Shulman’s paper [29]; see also [32] and [33].

Definition 2.1.1. A double category\(^1\) \(\mathbf{D}\) consists of the following data:

\(^1\)We will use many flavors of category in this paper. To help distinguish, we denote named 1-categories, monoidal categories, and operads with bold roman letters, e.g. \(\mathbf{Cob}\), and unnamed 1-categories with script,
2.1. Equipments

- A category $D_0$, which we refer to as the vertical category of $D$. For any two objects $c, d \in D_0$, we will write $D_0(c, d)$ for the set of vertical arrows from $c$ to $d$. We refer to objects of $D_0$ as objects of $D$.

- A category $D_1$, equipped with two functors $L, R : D_1 \to D_0$, called the left frame and right frame functors. Given an object $M \in \text{Ob}(D_1)$ with $c = L(M)$ and $c' = R(M)$, we say that $M$ is a proarrow (or horizontal arrow) from $c$ to $c'$ and write $M : c \rightarrow c'$. A morphism $\phi : M \to N$ in $D_1$ is called a 2-cell, and is drawn as follows, where $f = L(\phi)$ and $f' = R(\phi)$:

$$
\begin{align*}
&c \xrightarrow{M} c' \\
f &\downarrow \Downarrow{\phi} \\
&d \xrightarrow{N} d'
\end{align*}
$$

- A unit functor $U : D_0 \to D_1$, which is a strict section of both $L$ and $R$, i.e. $L \circ U = id_{D_0} = R \circ U$. We will often abuse notation by writing $c$ for the unit proarrow $U(c) : c \rightarrow c$, and similarly for vertical arrows.

- A functor $\odot : D_1 \times_{D_0} D_1 \to D_1$, called horizontal composition, which is weakly associative and unital in the sense that there are coherent unitor and associator isomorphisms. See [29] for more details.

Given a double category $D$ there is a strict 2-category called the vertical 2-category, denoted $\text{Vert}(D)$, whose underlying 1-category is $D_0$ and whose 2-morphisms $f \Rightarrow f'$ are defined to be 2-cells (6) where $M = U(c)$ and $N = U(d)$ are unit proarrows. There is also a horizontal bicategory, denoted $\text{Hor}(D)$, whose objects and 1-cells are the objects and horizontal 1-cells of $D$, and whose 2-cells are the 2-cells of $D$ of the form (6) such that $f = id_c$ and $f' = id_{c'}$.

A strong double functor $F : C \to D$ consists of functors $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ commuting with the frames $L, R$, and preserving the unit $U$ and horizontal composition $\odot$ up to coherent isomorphism.

Recall that a fibration of categories $p : E \to B$ is a functor with a lifting property: for every $f : b' \to b$ in $B$ and object $e \in E$ with $p(e) = b$, there exists $e' \to e$ over $f$ that is cartesian, i.e. universal in an appropriate sense. We denote fibrations of 1-categories using two-headed arrows $\twoheadrightarrow$.

**Definition 2.1.2.** An equipment is a double category $D$ in which the frame functor

$$(L, R) : D_1 \to D_0 \times D_0$$

e.g. $\mathcal{E}$. For named 2-categories or bicategories we do almost the same, but change the font of the first letter to calligraphic, such as $\mathcal{T}r\mathcal{Cat}$; for unnamed 2-categories we use (unbold) calligraphic, e.g. $\mathcal{D}$. Finally, for double categories we make the first letter blackboard bold, whether named (e.g., $\mathbb{P}rof$) or unnamed (e.g., $\mathbb{D}$). The objects in a category, 2-category, or double category will be denoted with the usual math font (e.g. $T \in \text{Ob}\mathcal{T}r\mathcal{Cat}$ or $c \in \text{Ob}D$).
is a fibration. If \( f : c \to d \) and \( f' : c' \to d' \) are vertical morphisms and \( N : d \to d' \) is a proarrow, a cartesian morphism \( M \to N \) in \( \mathcal{D}_1 \) over \( (f, f') \) is a 2-cell

\[
\begin{array}{ccc}
c & \xrightarrow{M} & c' \\
f \downarrow & \text{cart} & \downarrow f' \\
d & \xrightarrow{N} & d'
\end{array}
\]

which we call a \textit{cartesian} 2-cell. We refer to \( M \) as the \textit{restriction of} \( N \) along \( f \) and \( f' \), written \( M = N(f', f) \).

For any vertical morphism \( f : c \to d \) in an equipment \( \mathcal{D} \), there are two canonical proarrows \( \hat{f} : c \to d \) and \( \check{f} : d \to c \), called respectively the \textit{companion} and \textit{conjoint} of \( f \), defined by restriction:

\[
\begin{array}{ccc}
c & \xrightarrow{\hat{f}} & d \\
f \downarrow & \text{cart} & \downarrow f \\
d & \xrightarrow{U(d)} & d \\
\end{array}
\quad
\begin{array}{ccc}
d & \xrightarrow{\check{f}} & c \\
n \downarrow & \text{cart} & \downarrow f \\
d & \xrightarrow{U(d)} & d.
\end{array}
\]

In \cite{29}, it is shown that all restrictions can be obtained by composing with companions and conjoints. In particular, \( N(f, g) \cong \hat{f} \circ N \circ \check{g} \) for any proarrow \( N \). Moreover, \( \hat{f} \) and \( \check{f} \) form an adjunction in \( \mathcal{H} \text{or}(\mathcal{D}) \); we write \( \eta_f : \hat{f} \circ \check{f} \to U(d) \) and \( \epsilon_f : U(c) \to \hat{f} \circ \check{f} \) for the unit and counit of this adjunction.

Recall that a pseudo-pullback of a cospan \( A_1 \xrightarrow{f_1} A \xleftarrow{f_2} A_2 \) is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g_1} & A_2 \\
g_2 \downarrow \quad \text{\( g_2 \) is universal, up to equivalence} \quad \downarrow f_2 \\
A_1 & \xrightarrow{f_1} & A
\end{array}
\]

where the tuple \( (X, g_1, g_2, \alpha) \) is universal, up to equivalence, for data of that shape. This definition makes sense in any 2-category \( \mathcal{C} \), though we will use it only in a special case, described in the next paragraph.

Suppose that \( \mathcal{C} = \mathcal{C} \text{at} \), that \( f_2 \) is a fibration, and that the square in (7) strictly commutes, i.e. that \( \alpha \) is the identity. It is a standard fact that a strict pullback of a Grothendieck fibration along an arbitrary functor is a fibration and that the strict pullback is also a pseudo-pullback. The upshot is that \( g_2 \) is a pseudo-pullback if and only if, for any strict pullback \( g_2' \) of \( f_2 \) along \( f_1 \), the induced map \( g_2 \to g_2' \) is an equivalence of fibrations.

**Definition 2.1.3.** By an \textit{equipment functor}, we simply mean a strong double functor between equipments. We refer to an equipment functor \( F : \mathcal{C} \to \mathcal{D} \) as a \textit{local equivalence} if the following (strictly commuting) square is a pseudo-pullback of categories:

\[
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \\
\downarrow (L,R) & \quad & \downarrow (L,R) \\
\mathcal{C}_0 \times \mathcal{C}_0 & \xrightarrow{F_0 \times F_0} & \mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
\]
2.1. Equipments

If moreover \( F_0 : C_0 \to D_0 \) is fully faithful, we say that \( F \) is a \textit{fully faithful local equivalence}.

\textbf{Remark 2.1.4.} As discussed above, if the square (8) is a strict pullback, it will be a pseudo-pullback, and hence a local equivalence. Any local equivalence can be replaced by an equivalent strict pullback, which we define in Definition 2.1.5.

Also note that the frame fibration for \( C \) is equivalent to the functor \( C_0 \times C_0 \to \text{Cat} \), the 2-category of small categories, sending \((c,d)\) to \( \mathcal{Hor}(C)(c,d) \) and similarly for \( D \). In this language, \( F \) is a local equivalence if and only if the induced functors \( \mathcal{Hor}(C)(c,d) \to \mathcal{Hor}(D)(F_0(c),F_0(d)) \) are equivalences of categories for every pair of objects \((c,d)\). The square (8) is a strict pullback precisely when these are isomorphisms of categories.

\textbf{Definition 2.1.5.} Let \( D \) be a double category and \( F_0 : C_0 \to D_0 \) be a functor. A strict pullback of the form (8) defines a double category \( C \), which we denote
\[
C := F_0^*(D).
\]

If \( D \) is an equipment, \( F_0^*(D) \) will be as well since fibrations are stable under pullback. In this case we call \( F_0^*(D) \) the \textit{equipment induced by} \( F_0 \). By Remark 2.1.4, the induced equipment functor \( F_0^*(D) \to D \) is a local equivalence.

Our main tool in this paper will be equipments in which the horizontal arrows are (generalizations of) profunctors, as in the following example.

\textbf{Example 2.1.6.} The equipment \( \text{Prof} \) is a double category whose vertical category \( \text{Prof}_0 = \text{Cat} \) is the category of small 1-categories and functors. Given categories \( \mathcal{E}, \mathcal{E}' \in \text{Prof}_0 \), a proarrow
\[
\mathcal{E} \xrightarrow{M} \mathcal{E}'
\]
in \( \text{Prof}_1 \) is a functor, i.e. a functor \( M : \mathcal{E}^{\text{op}} \times \mathcal{E}' \to \text{Set} \). The left and right frame functors are given by \( L(M) = \mathcal{E} \) and \( R(M) = \mathcal{E}' \). A 2-cell \( \phi \) in \( \text{Prof} \), as to the left, denotes a natural transformation, as to the right, in (9):

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{M} & \mathcal{E}' \\
F & \downarrow \phi & \downarrow F' \\
\mathcal{D} & \xrightarrow{N} & \mathcal{D}'
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{E}^{\text{op}} \times \mathcal{E}' & \xrightarrow{\mathcal{E}^{\text{op}} \times \phi} & \mathcal{D}^{\text{op}} \times \mathcal{D}' \\
\mathcal{E}^{\text{op}} \times \phi & \xrightarrow{F \times F'} & \mathcal{D}^{\text{op}} \times \mathcal{D}' \\
& \downarrow M & \downarrow N \\
\mathcal{D} & \xrightarrow{\phi} & \mathcal{D}'
\end{array}
\]

(9)

The unit functor \( U : \text{Cat} \to \text{Prof}_1 \) sends a category \( \mathcal{E} \) to the hom profunctor \( \text{Hom}_\mathcal{E} : \mathcal{E}^{\text{op}} \times \mathcal{E} \to \text{Set} \). Given two profunctors
\[
\mathcal{E} \xrightarrow{M} \mathcal{D} \xrightarrow{N} \mathcal{E},
\]
define the horizontal composition \( M \circ N \) on objects \( c \in \mathcal{E} \) and \( e \in \mathcal{E} \) as the coequalizer of the diagram
\[
\begin{array}{ccc}
& & M(c,d_1) \times \mathcal{D}(d_1,d_2) \times N(d_2,e) \\
\coprod_{d_1,d_2 \in \mathcal{D}} & \xrightarrow{\coprod_{d_1,d_2 \in \mathcal{D}}} & M(c,d) \times N(d,e)
\end{array}
\]

(10)
where the two maps are given by the right and left actions of \( \mathcal{D} \) on \( M \) and \( N \) respectively. Note that the coequalizer (10) is in fact a reflexive coequalizer, using \( \text{id}_d \in \mathcal{D}(d,d) \).

Given a profunctor \( M: \mathcal{C} \to \mathcal{C}' \) there are canonical isomorphisms \( \text{Hom}_\mathcal{C} \odot M \cong M \cong M \odot \text{Hom}_\mathcal{C}' \) which can be viewed as giving an action of \( \text{Hom}_\mathcal{C} \) and of \( \text{Hom}_\mathcal{C}' \) on \( M \), from the left and right respectively.

At this point we have given \( \text{Prof} \) the structure of a double category. To see that \( \text{Prof} \) is an equipment, note that from a pair of functors \( F: \mathcal{C} \to \mathcal{D}, F': \mathcal{C}' \to \mathcal{D}' \) and a profunctor \( N: \mathcal{D} \to \mathcal{D}' \) we may form the composite

\[
\mathcal{C}^{op} \times \mathcal{C}' \xrightarrow{F \times F'} \mathcal{D}^{op} \times \mathcal{D}' \xrightarrow{N} \text{Set},
\]

denoted \( N(F',F): \mathcal{C} \to \mathcal{C}' \), such that

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{N(F',F)} & \mathcal{C}' \\
F & \downarrow \phi & \downarrow F' \\
\mathcal{D} & \xrightarrow{N} & \mathcal{D}'
\end{array}
\]

is a cartesian 2-cell. A simple Yoneda lemma argument yields \( \text{Vert}(\text{Prof}) \cong \text{Cat} \).

Remark 2.1.7. There is a strong analogy relating profunctors between categories with bimodules between rings. Besides being a useful source of intuition, we can also exploit this analogy to provide a convenient notation for working with profunctors.

If \( M: \mathcal{C}^{op} \times \mathcal{D} \to \text{Set} \) is a profunctor, then for any element \( m \in M(c,d) \) and morphisms \( f: c' \to c \) and \( g: d \to d' \), we can write \( g \cdot m \in M(c,d') \) and \( m \cdot f \in M(c',d) \) for the elements \( M(\text{id}_c,g)(m) \) and \( M(f,\text{id}_d)(m) \) respectively. Thus we think of the functoriality of \( M \) as providing left and right actions of \( \mathcal{D} \) and \( \mathcal{C} \) on the elements of \( M \). The equations \((g \cdot m) \cdot f = g \cdot (m \cdot f), g' \cdot (g \cdot m) = (g' \circ g) \cdot m, \) and \((m \cdot f) \cdot f' = m \cdot (f \circ f')\) clearly hold whenever they make sense.

The coequalizer (10) can be easily expressed in this notation: the elements of \((M \odot N)(c,c')\) are pairs \( m \odot n \) of elements \( m \in M(c,d) \) and \( n \in N(d,e) \) for some \( d \in \mathcal{D} \) modulo the relation \((m \cdot f) \odot n = m \odot (f \cdot n)\), whenever this makes sense.

Finally, a 2-cell \( \phi \) of the form (9) is function sending elements \( m \in M(c,c') \) to elements \( \phi(m) \in N(Fc,F'c') \) such that the equation \( \phi(f \cdot m \cdot g) = F(f) \cdot \phi(m) \cdot F'(g) \) holds whenever it makes sense.

### 2.2 Internal copresheaves

Copresheaves on a category \( \mathcal{C} \) can be identified with profunctors \( 1 \to \mathcal{C} \) in \( \text{Prof} \). Motivated by this, we will think of proarrows \( 1 \to c \) in any equipment \( \mathcal{D} \) with a terminal object 1 as “internal copresheaves” on the object \( c \). For each object, there is a category of copresheaves \( \text{Hor}(\mathcal{D})(1,c) \). We can give a direct construction of the bifibration over \( \mathcal{D}_0 \) whose fiber over an object \( c \) is the category of copresheaves on \( c \):
Definition 2.2.1. Let \( \mathcal{D} \) be an equipment with a terminal object \( 1 \in \mathcal{D}_0 \).\(^2\) We define the category \( \text{CPsh}(\mathcal{D}) \), bifibered over \( \mathcal{D}_0 \), by the strict pullback of categories

\[
\begin{array}{ccc}
\text{CPsh}(\mathcal{D}) & \rightarrow & \mathcal{D}_1 \\
\downarrow & & \downarrow \mathcal{J} \\
1 \times \mathcal{D}_0 & \rightarrow & \mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
\]

Lemma 2.2.2. Let \( F: \mathcal{C} \rightarrow \mathcal{D} \) be an equipment functor. Suppose that \( \mathcal{C}_0 \) and \( \mathcal{D}_0 \) have terminal objects, preserved by \( F_0 \). Then there is an induced morphism of fibrations

\[
\begin{array}{ccc}
\text{CPsh}(\mathcal{C}) & \xrightarrow{\tilde{F}} & \text{CPsh}(\mathcal{D}) \\
\downarrow & & \downarrow \mathcal{J} \\
\mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0.
\end{array}
\]

Moreover, if \( F \) is a local equivalence, then (12) is a pseudo-pullback.

Proof. Consider the cube

\[
\begin{array}{ccc}
\text{CPsh}(\mathcal{C}) & \xrightarrow{\tilde{F}} & \text{CPsh}(\mathcal{D}) \\
\downarrow & & \downarrow \mathcal{J} \\
\mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \\
\downarrow & & \downarrow \mathcal{J} \\
1 \times \mathcal{C}_0 & \xrightarrow{1 \times F_0} & 1 \times \mathcal{D}_0 \\
\downarrow & & \downarrow \mathcal{J} \\
\mathcal{C}_0 \times \mathcal{C}_0 & \xrightarrow{F_0 \times F_0} & \mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
\]

Since \( F_0 \) preserves terminal objects, the bottom face of the cube commutes. The left and right faces of the cube are strict pullbacks by definition, hence there is a unique \( \tilde{F} \) making the cube commute.

If \( F \) is a local equivalence, then the front face is a pseudo-pullback. The left and right faces are strict pullbacks along fibrations, hence pseudo-pullbacks (see Remark 2.1.4). It follows that the back face is a pseudo-pullback as well. \( \square \)

2.3 Monoids and bimodules

Our eventual proofs of Theorem A and Theorem B will revolve around a careful understanding of internal monoids in an equipment \( \mathcal{D} \). In particular, following [28], the exactness of an equipment and the resulting \((bo, ff)\) factorization system, both given in Section 2.4, are built on notions related to monoids in \( \mathcal{D} \).

\(^2\)In fact, such a definition makes sense for any object of \( \mathcal{D}_0 \), but for our purposes we require the terminal object.
Definition 2.3.1. Denote by $\text{Mon}(\mathcal{D})$ the category of monoids in $\mathcal{D}$. More precisely, the objects are monoids: 4-tuples $(c, M, i_M, m_M)$ consisting of an object $c$ of $\mathcal{D}$ and a proarrow $M: c \to c$ together with unit and multiplication cells

$$
\begin{array}{cccc}
  & c & \xrightarrow{c} & c \\
\| & \| & \| & \| \\
  & \downarrow \psi_M & & \downarrow \psi_M \\
  & c & \xrightarrow{M} & c \\
\end{array}
$$

satisfying the evident unit and associativity axioms. The morphisms are monoid homomorphisms: pairs $(f, \tilde{f})$ consisting of a vertical arrow $f: c \to d$ in $\mathcal{D}$ and a cell

$$
\begin{array}{ccc}
  c & \xrightarrow{M} & c \\
  \downarrow f & \psi \tilde{f} & \downarrow f \\
  d & \xrightarrow{N} & d
\end{array}
$$

which respects the unit and multiplication cells of $M$ and $N$.

There is an evident forgetful functor $|\cdot|: \text{Mon}(\mathcal{D}) \to \mathcal{D}_0$ sending a monoid $M: c \to c$ to its underlying object $|M| := c$. The following result can also be found in [10].

Lemma 2.3.2. Let $\mathcal{D}$ be an equipment. The forgetful functor $|\cdot|: \text{Mon}(\mathcal{D}) \to \mathcal{D}_0$ is a fibration and there is a morphism of fibrations

$$
\begin{array}{ccc}
  \text{Mon}(\mathcal{D}) & \longrightarrow & \mathcal{D}_1 \\
  |\cdot| & \downarrow \quad & \downarrow (LR) \\
  \mathcal{D}_0 & \xrightarrow{\Lambda} & \mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
$$

Proof. Let $f: c \to d$ be a vertical morphism of $\mathcal{D}$ and $N: d \to d$ a monoid in $\mathcal{D}$. Since the 2-cell defining the restriction of $N$ along $f$ is cartesian, there is an induced monoid structure on $N(f, f)$ which in particular makes this cartesian 2-cell a monoid homomorphism. The result follows.

Lemma 2.3.3. For a local equivalence $F: \mathcal{C} \to \mathcal{D}$, the induced square

$$
\begin{array}{ccc}
  \text{Mon}(\mathcal{C}) & \xrightarrow{\text{Mon}(F)} & \text{Mon}(\mathcal{D}) \\
  |\cdot| & \downarrow & \downarrow |\cdot| \\
  \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0
\end{array}
$$

is a pseudo-pullback of categories.

Proof. By Remark 2.1.4, we may assume that the pullback in Definition 2.1.3 realizing $F: \mathcal{C} \to \mathcal{D}$ as a local equivalence is strict. It is then straightforward to check directly that the above square is again a strict pullback and hence a pseudo-pullback.
In all our cases of interest, \( \text{Mon}(C) \) will be the vertical part of an equipment. The following is a standard construction; see \[29\]

**Definition 2.3.4.** Let \( \mathcal{D} \) be an equipment with local coequalizers, i.e. such that each category \( \mathcal{H}(\mathcal{D})(c, d) \) has coequalizers and \( \otimes \) preserves coequalizers in each variable. The equipment \( \text{Mod}(\mathcal{D}) \) of monoids and bimodules is defined as follows:

- The vertical category \( \text{Mod}(\mathcal{D})_0 \) is the category \( \text{Mon}(\mathcal{D}) \) of monoids in \( \mathcal{D} \).
- The proarrows \( B : M \rightarrow N \) are bimodules: triples \((B, l_B, r_B)\) consisting of a proarrow \( B : c \rightarrow d \) in \( \mathcal{D} \) and cells

\[
\begin{array}{ccc}
c & \xrightarrow{M} & c \\
\| & \| & \| \\
\| & \| & \| \\
c & \xrightarrow{B} & d
\end{array}
\]

\[
\begin{array}{ccc}
c & \xrightarrow{B} & d \\
\| & \| & \| \\
\| & \| & \| \\
c & \xrightarrow{N} & d
\end{array}
\]

- The horizontal composition \( B_1 \otimes_{M'} B_2 \) of bimodules \( B_1 : M \rightarrow M' \) and \( B_2 : M' \rightarrow M'' \) is given by the coequalizer in \( \mathcal{H}(\mathcal{D})(M, M'') \)

\[
B_1 \otimes M' \otimes B_2 \rightrightarrows B_1 \otimes B_2 \rightarrow B_1 \otimes_{M'} B_2
\]

together with the evident left \( M \) and right \( M'' \) actions.

- The 2-cells are bimodule homomorphisms: cells in \( \mathcal{D} \)

\[
\begin{array}{ccc}
c & \xrightarrow{A} & c' \\
f \downarrow & \Downarrow{\phi} & \Downarrow{f'} \\
d & \xrightarrow{B} & d'
\end{array}
\]

which are compatible with the left and right actions of the bimodules.

We will write \( _M\text{Bimod}_N \) to denote the 1-category of \((M, N)\)-bimodules and bimodule morphisms.

The forgetful functor \( |\cdot| : \text{Mon}(\mathcal{D}) \rightarrow \mathcal{D}_0 \) extends to a forgetful equipment functor \( |\cdot| : \text{Mod}(\mathcal{D}) \rightarrow \mathcal{D} \). There is also a local equivalence \( U : \mathcal{D} \rightarrow \text{Mod}(\mathcal{D}) \) sending \( c \) to the trivial monoid structure. If \( F : \mathcal{C} \rightarrow \mathcal{D} \) is an equipment functor, then there is an evident equipment functor \( \text{Mod}(F) : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{D}) \). In fact, we have the following which is immediate from the definitions.

**Lemma 2.3.5.** For a local equivalence \( F : \mathcal{C} \rightarrow \mathcal{D} \) the induced functor \( \text{Mod}(F) : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{D}) \) is a local equivalence. If \( F \) is a fully faithful local equivalence, then so is \( \text{Mod}(F) \).

### 2.4 Exact equipments and \( bo, ff \) factorization

Patrick: introduce this section.
**Definition 2.4.1.** Let $M: c \rightarrow c$ be a monoid in an equipment $\mathcal{D}$. An embedding of $M$ into an object $x \in \mathcal{D}_0$ is a monoid homomorphism $(f, \hat{f})$ from $M$ to the trivial monoid on $x$:

$$
\begin{array}{c}
M \downarrow \hat{f} \\
\downarrow f \\
x \rightarrow x.
\end{array}
$$

We will sometimes write an embedding as $(f, \hat{f}): (c, M) \rightarrow x$, or even just $f: M \rightarrow x$ when clear from context. We will write $\text{Emb}(M, x)$ for the set of embeddings from $M$ to $x$. This defines a functor $\text{Emb}: \text{Mon}(\mathcal{D})^{\text{op}} \times \mathcal{D}_0 \rightarrow \text{Set}$.

**Lemma 2.4.2.** Suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is a local equivalence induced by $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$. Suppose $M \in \text{Mon}(\mathcal{C})$ is a monoid and $x \in \mathcal{C}_0$ is an object. For $N = \text{Mon}(F)(M)$ and $y = F_0(x)$ we have a pullback square in $\text{Set}$, natural in $M$ and $x$:

$$
\begin{array}{ccc}
\text{Emb}_\mathcal{C}(M, x) & \rightarrow & \text{Emb}_\mathcal{D}(N, y) \\
\downarrow & & \downarrow \\
\mathcal{C}_0(|M|, x) & \rightarrow & \mathcal{D}_0(|N|, y).
\end{array}
$$

**Definition 2.4.3.** Let $M: c \rightarrow c$ be a monoid in an equipment $\mathcal{D}$. The collapse of $M$ is defined to be a universal embedding of $M$. That is, the collapse of $M$ is an object $\langle M \rangle \in \mathcal{D}_0$ together with an embedding

$$
\begin{array}{c}
M \downarrow \hat{i}_M \\
\downarrow \hat{id}_f \\
\langle M \rangle \rightarrow \langle M \rangle
\end{array}
$$

such that any other embedding of $M$ factors uniquely through $\tilde{i}_M$:

$$
\begin{array}{c}
M \downarrow \hat{i}_M \\
\downarrow \hat{id}_f \\
\langle M \rangle \rightarrow \langle M \rangle
\end{array}
$$

In other words, $\langle M \rangle$ represents the functor $\text{Emb}(M, -): \mathcal{D}_0 \rightarrow \text{Set}$.

**Remark 2.4.4.** For any monoid $M: c \rightarrow c$, the companion $\tilde{i}_M: a \rightarrow b$ (resp. the conjoint $\tilde{i}_M: b \rightarrow a$) of the embedding $i_M: c \rightarrow \langle M \rangle$ has the structure of a left (resp. right) $M$-module. Indeed, these are given by rewriting the collapse 2-cell $\tilde{i}_M$ as follows:

$$
\begin{array}{c}
M \downarrow \hat{i}_M \\
\downarrow \hat{id}_f \\
\langle M \rangle \rightarrow \langle M \rangle
\end{array}
$$

Here is some additional text to fill out the page.
2.4. Exact equipments and bo, ff factorization

**Lemma 2.4.5.** Let $M: c \rightsquigarrow c$ and $N: d \rightsquigarrow d$ be monoids in an equipment $\mathcal{D}$ and assume they admit collapses $\langle M \rangle$ and $\langle N \rangle$, respectively. Then restriction induces a functor

$$\mathcal{H}or(\mathcal{D})(\langle M \rangle, \langle N \rangle) \rightarrow _M\text{Bimod}_N.$$  

**Proof.** For a proarrow $X: \langle M \rangle \rightarrow \langle N \rangle$ of $\mathcal{D}$, define $\hat{X}: c \rightarrow d$ by the cartesian 2-cell

$$c \xrightarrow{X} d$$

$$_{\mathcal{D}}\epsilon^{-1}$$

Imposing $i_M$ and $i_N$.

Then a 2-cell $X \Rightarrow Y$ immediately lifts to a 2-cell $\hat{X} \Rightarrow \hat{Y}$. Since (14) is cartesian, we obtain an equality

$$\begin{align*}
\begin{array}{ccc}
c & \xrightarrow{M} & c \\
i_M & \downarrow & \downarrow \mathcal{D} \epsilon^{-1} \ni_M \\
\langle M \rangle & \xrightarrow{\mathcal{D} \epsilon^{-1}} & \langle N \rangle
\end{array}
& \cong
\begin{array}{ccc}
c & \xrightarrow{X} & d \\
i_M & \downarrow & \downarrow \mathcal{D} \epsilon^{-1} \ni_M \\
\langle M \rangle & \xrightarrow{\mathcal{D} \epsilon^{-1}} & \langle N \rangle
\end{array}
\end{align*}$$

Then a 2-cell $X \Rightarrow Y$ immediately lifts to a 2-cell $\hat{X} \Rightarrow \hat{Y}$. Since (14) is cartesian, we obtain an equality

$$\begin{align*}
\begin{array}{ccc}
c & \xrightarrow{M} & c \\
i_M & \downarrow & \downarrow \mathcal{D} \epsilon^{-1} \ni_M \\
\langle M \rangle & \xrightarrow{\mathcal{D} \epsilon^{-1}} & \langle N \rangle
\end{array}
& \cong
\begin{array}{ccc}
c & \xrightarrow{X} & d \\
i_M & \downarrow & \downarrow \mathcal{D} \epsilon^{-1} \ni_M \\
\langle M \rangle & \xrightarrow{\mathcal{D} \epsilon^{-1}} & \langle N \rangle
\end{array}
\end{align*}$$

giving the action of $M$ on $\hat{X}$. The action $r_{\hat{X}}$ of $N$ on $\hat{X}$ is obtained similarly, and one easily checks the axioms making $\hat{X}$ an $(M,N)$-bimodule. \hfill \Box

**Definition 2.4.6.** [28, Proposition 5.4] An equipment $\mathcal{D}$ is exact if the following hold:

1. every monoid $M: c \rightsquigarrow c$ has a collapse $\langle M \rangle$ with $\mathcal{D} \epsilon^{-1}$ cartesian;
2. for every pair of monoids $M$ and $N$ the restriction functor

$$\mathcal{H}or(\mathcal{D})(\langle M \rangle, \langle N \rangle) \xrightarrow{\sim} _M\text{Bimod}_N$$

is an equivalence of categories.

**Example 2.4.7.** It was proven in [28, Proposition 5.2] that for any equipment $\mathcal{D}$, its equipment $\text{Mod}(\mathcal{D})$ of monoids and bimodules is exact. Thus $\text{Prof}$ is exact, since there is an equivalence $\text{Prof} \cong \text{Mod}(\text{Span})$, where $\text{Span}$ is the equipment of spans in $\text{Set}$; see [29].

Exact equipments arising in practice almost always have local coequalizers, and in this case it is possible to simplify the definition, as we show in Proposition 2.4.8. Recall from Remark 2.4.4 the natural $M$-module structures on the companion $\mathcal{D} \epsilon^{-1}$ and joint $\mathcal{D} \epsilon^{-1}$ of the collapse embedding $i_M: c \rightarrow \langle M \rangle$. Recall also the notation $Ua: a \rightarrow a$ for the unit map on an object $a$.

**Proposition 2.4.8.** Let $\mathcal{D}$ be an equipment with local coequalizers, and suppose $\mathcal{D}$ satisfies Condition 1 of Definition 2.4.6, giving $i_M: c \rightarrow \langle M \rangle$. Then $\mathcal{D}$ satisfies Condition 2 if and only if for every monoid $M: c \rightsquigarrow c$, the following diagram is a coequalizer in $\mathcal{H}or(\mathcal{D})(\langle M \rangle, \langle M \rangle)$:

$$\begin{align*}
\mathcal{D} \epsilon^{-1} & \circ \mathcal{D} \epsilon^{-1} \circ \mathcal{D} \epsilon^{-1} \\
\mathcal{D} \epsilon^{-1} \circ \mathcal{D} \epsilon^{-1} \circ \mathcal{D} \epsilon^{-1}
\end{align*}$$

$$\begin{align*}
\begin{array}{ccc}
\mathcal{D} \epsilon^{-1} & \circ \mathcal{D} \epsilon^{-1} & \circ \mathcal{D} \epsilon^{-1} \\
i_M \circ i_M \circ \mathcal{D} \epsilon^{-1} & = & \mathcal{D} \epsilon^{-1} \circ i_M \circ \mathcal{D} \epsilon^{-1}
\end{array}
\end{align*}$$

question

16
or, equivalently, $\tilde{i}_M \otimes_M \tilde{i}_M \cong U(M)$.

**Proof.** By Condition 1 of Definition 2.4.6, we have $M \cong \tilde{i}_M \otimes \tilde{i}_M$, so the final equivalence is just the definition of horizontal composition in $\mathbf{Mod}(D)$; see Definition 2.3.4 and Remark 2.4.4. Below we will use the fact that, by definition of $D$ having local coequalizers, $\otimes$ preserves coequalizers in each variable, and that $\otimes$ is defined as a coequalizer. Finally, note that the restriction functor (15) is isomorphic to the functor $X \mapsto \tilde{i}_M \otimes X \otimes \tilde{i}_N$, with the left and right actions given by the left action of $M$ on $\tilde{i}_M$ and right action of $N$ on $\tilde{i}_N$.

Assuming $\tilde{i}_M \otimes_M \tilde{i}_M \cong U(M)$, we can construct an inverse to this restriction functor, sending an $(M, N)$-bimodule $B$ to $\tilde{i}_M \otimes_M B \otimes_N \tilde{i}_N$. It is easy to check that this gives an equivalence of categories:

$$
\tilde{i}_M \otimes_M (\tilde{i}_M \otimes X \otimes \tilde{i}_N) \otimes_N \tilde{i}_N \cong (\tilde{i}_M \otimes_M \tilde{i}_M) \otimes X \otimes (\tilde{i}_N \otimes_N \tilde{i}_N)
\cong U(M) \otimes X \otimes U(N)
\cong X
$$

and

$$
\tilde{i}_M \otimes (\tilde{i}_M \otimes_M B \otimes_N \tilde{i}_N) \otimes_N \tilde{i}_N \cong (\tilde{i}_M \otimes \tilde{i}_M) \otimes_M B \otimes N (\tilde{i}_N \otimes \tilde{i}_N)
\cong M \otimes_M B \otimes N
\cong B.
$$

Conversely, assuming the functor (15) is an equivalence of categories, then we can prove that $\tilde{i}_M \otimes_M \tilde{i}_M \cong U(M)$ is an isomorphism by first applying the restriction functor:

$$
\tilde{i}_M \otimes (\tilde{i}_M \otimes_M \tilde{i}_M) \otimes \tilde{i}_M \cong (\tilde{i}_M \otimes \tilde{i}_M) \otimes_M (\tilde{i}_M \otimes \tilde{i}_M)
\cong M \otimes_M M
\cong M
\cong \tilde{i}_M \otimes \tilde{i}_M
\cong \tilde{i}_M \otimes U(M) \otimes \tilde{i}_M.
$$

**Example 2.4.9.** While the exactness of $\mathbf{Prof}$ follows from formal reasons, as we saw in Example 2.4.7, it will be helpful to understand collapse in $\mathbf{Prof}$ concretely.

Consider a monoid $M: \mathcal{C} \to \mathcal{C}$ in $\mathbf{Prof}$. The unit is a profunctor morphism $i: \text{Hom}_{\mathcal{C}} \to M$. So for any $f: c \to d$ in $\mathcal{C}$ there is an element $i(f) \in M(c, d)$, such that

$$
g \cdot i(f) \cdot h = i(g \circ f \circ h)
$$

(17)

whenever this makes sense.

The multiplication $M \odot M \to M$ is an operation assigning to any elements $m_1 \in M(c, d)$ and $m_2 \in M(d, e)$ an element $m_2 \bullet m_1 \in M(c, e)$, which is associative and satisfies the following equations whenever they make sense:

$$
(f \cdot m_2) \bullet (m_1 \cdot h) = f \cdot (m_2 \bullet m_1) \cdot h
$$

(18)

$$
(m_3 \cdot g) \bullet m_1 = m_3 \bullet (g \cdot m_1)
$$

(19)
Specifically, equations (18) and (19) simply say that \( \bullet \) is a well defined morphism \( M \circ M \to M \), while (20) says that \( \bullet \) is unital with respect to \( i \).

The collapse \( \langle M \rangle \) is then the category with the same objects as \( C \), with morphisms \( \langle M \rangle (c, d) = M(c, d) \), and with composition given by \( \bullet \). The unit \( i \) of \( M \) gives the functor \( i_M: \langle \mathcal{E} \rangle \to \langle M \rangle \).

Remark 2.4.10. The equations (17)–(20) are actually overdetermined. It is easy to see that equations (18) and (19) follow from (20) and the associativity of \( \bullet \). Thus, when proving that \( \bullet: M \circ M \to M \) and \( i: \text{Hom}_C \to M \) form a monoid, it suffices to prove (17), (20), and associativity of \( \bullet \). These observations will be used to slightly simplify the proof of Proposition 3.3.8.

Proposition 2.4.11. If \( D \) is an exact equipment with local coequalizers, then collapse induces an equipment functor \( \langle - \rangle: \text{Mod}(D) \to D \) which is a local equivalence.

Proof. It is easy to use the universal property of collapse to construct, from any monoid homomorphism \( (f, \vec{f}): (c, M) \to (d, N) \), a vertical morphism \( \langle f \rangle: \langle M \rangle \to \langle N \rangle \) in \( D \), thus defining a functor \( \text{Mon}(D) \to D_0 \).

The functor \( \langle - \rangle \) is defined on horizontal arrows and 2-cells using the inverse of the equivalence (15). The collapse 2-cell being cartesian implies that \( \langle - \rangle \) is a local equivalence. It is straightforward to verify that this is a strong double functor using the method of the proof of Proposition 2.4.8. \( \square \)

With these definitions in place we can now introduce two distinguished classes of vertical morphisms in an equipment \( D \). These will become the left and right classes in an orthogonal factorization system on \( \text{Vert}(D) \) for exact equipments.

Definition 2.4.12. [28, Definitions 4.3 and 4.5] Let \( D \) be an equipment and \( f: c \to d \) a vertical morphism of \( D \). Consider the restriction square and unit square shown below:

\[
\begin{array}{ccc}
c & \xrightarrow{d(f,f)} & c \\
\downarrow f & & \downarrow f \\
d & \xrightarrow{\text{cart}} & d
\end{array}
\quad
\begin{array}{ccc}
c & \xrightarrow{\xi} & c \\
\downarrow f & & \downarrow f \\
d & \xrightarrow{\text{id}_d} & d
\end{array}
\]

We say that \( f \) is bo if the restriction square, where \( d(f,f) \) has the induced monoid structure, is a collapse. We say that \( f \) is ff if the unit square is cartesian.

In Section 3.2 we will define equipments of profunctors on monoidal categories and verify their exactness directly in Section 3.4. The key ingredient in verifying that the equipment of traced profunctors is exact will be orthogonal factorization systems. Thus we briefly recall the notion of orthogonal factorization systems for 1-categories and strict 2-categories. Additional background on orthogonal factorization systems can be found in [6, Chapter 5.5]. The main result below is that exact equipments admit orthogonal factorization systems.
2.4. Exact equipments and $\mathsf{bo}$, $\mathsf{ff}$ factorization

**Definition 2.4.13.** Let $\mathcal{V}$ be either $\mathsf{Set}$ or $\mathsf{Cat}$, and suppose that $\mathcal{C}$ is a $\mathcal{V}$-enriched category. An *orthogonal factorization system* in $\mathcal{C}$ consists of two distinguished classes of morphisms, $(\mathcal{L}, \mathcal{R})$, with the following properties:

- Each morphism $f \in \mathcal{C}$ factors as $f = e \circ m$, where $m \in \mathcal{L}$ and $e \in \mathcal{R}$.
- If $m: a \to b$ in $\mathcal{L}$ and $e: c \to d$ in $\mathcal{R}$, then the left-hand square below is a pullback in $\mathcal{V}$:

$$
\begin{array}{ccc}
\mathcal{C}(b, c) & \to & \mathcal{C}(a, c) \\
\downarrow & & \downarrow_{\sim} \\
\mathcal{C}(b, d) & \to & \mathcal{C}(a, d)
\end{array}
\quad
\begin{array}{ccc}
a & \to & c \\
\downarrow & & \downarrow m \\
b & \to & d
\end{array}
$$

(21)

In particular, for all solid arrow squares, as in the right-hand diagram, there exists a unique diagonal filler. We say that $m$ is “left-orthogonal” to $e$, or that $e$ is “right-orthogonal” to $m$, and denote this relation as $m \sqsubset e$.

- If $m \sqsubset e$ for all $e \in \mathcal{R}$, then $m \in \mathcal{L}$. Likewise, if $m \sqsubset e$ for all $m \in \mathcal{L}$, then $e \in \mathcal{R}$.

As shown, we often indicate morphisms in $\mathcal{L}$ using a two-headed arrow and morphisms in $\mathcal{R}$ using a hooked arrow.\(^3\)

**Theorem 2.4.14.** [28, Theorem 4.17] If an equipment $\mathcal{D}$ is exact, then the vertical 2-category $\mathsf{Vert}(\mathcal{D})$ admits a 2-orthogonal factorization system $(\mathsf{bo}, \mathsf{ff})$ as in Definition 2.4.12. In particular, there is an orthogonal factorization system $(\mathsf{bo}, \mathsf{ff})$ on the vertical 1-category $\mathcal{D}_0$.

In fact, the orthogonal factorization system above has a more concrete description:

**Definition 2.4.15.** A morphism $f: a \to b$ in a 2-category $\mathcal{C}$ is *fully faithful* if the functor $\mathcal{C}(x, a) \to \mathcal{C}(x, b)$, induced by composition with $f$, is fully faithful for every $x$. That is, $f$ is fully faithful if, for every diagram

$$
\begin{array}{ccc}
x & \to & a \\
\downarrow_{u} & \searrow & \downarrow_{u'} \\
\downarrow_{v} & \swarrow \alpha & \downarrow_{v'} \\
x & \to & b
\end{array}
$$

such that $fu = u'$ and $fv = v'$, there exists a unique $\alpha: u \Rightarrow v$ such that $f\alpha = \alpha'$.

A morphism $f: a \to b$ in a 2-category $\mathcal{C}$ is *bijective-on-objects* if it is left orthogonal to every fully faithful morphism.

**Lemma 2.4.16.** Let $\mathcal{D}$ be an exact equipment. If a vertical morphism $f$ in $\mathcal{D}$ is $\mathsf{bo}$ (respectively $\mathsf{ff}$), then $f$ is a bijective-on-objects (respectively fully faithful) morphism in $\mathsf{Vert}(\mathcal{D})$ in the sense of Definition 2.4.15.

\(^3\) We sometimes also use the two-headed arrow symbol $\Rightarrow$ to indicate fibrations of categories (e.g., as we did in Theorem B or when defining the frame fibration for equipments, Definition 2.1.2). Whether we mean a bo map in an equipment or a fibration of categories should be clear from context.
2.4. Exact equipments and bo, ff factorization

\textbf{Proof.} It is easy to see that the condition of \( f \) being fully faithful in \( \text{Vert}(\mathcal{D}) \) is a special case of the condition of the unit 2-cell on \( f \) being cartesian in \( \mathcal{D} \). If \( f \) is bo, then Theorem 2.4.14 in particular implies that \( f \) is left orthogonal to every ff morphism, hence \( f \) is bijective-on-objects. \( \square \)

The bijective-on-objects vertical arrows of an exact equipment \( \mathcal{D} \) will play a particularly special role in our work.

\textbf{Definition 2.4.17.} Let \( \mathcal{D} \) be an exact equipment. We define the equipment \( \mathcal{D}^{\text{bo}} \) as follows: the vertical category \( \mathcal{D}^{\text{bo}}_0 \subseteq \mathcal{D}_0^\downarrow \) is the full subcategory of the arrow category of \( \mathcal{D}_0 \) spanned by the arrows in the class \( \text{bo} \). As such, we have functors \( \text{dom}, \text{cod} : \mathcal{D}^{\text{bo}}_0 \to \mathcal{D}_0 \). The rest of the structure of \( \mathcal{D}^{\text{bo}} \) is defined by setting \( \mathcal{D}^{\text{bo}} := \text{cod}^* \mathcal{D} \), i.e. by the strict pullback of categories

\[
\begin{array}{ccc}
\mathcal{D}^{\text{bo}}_1 & \longrightarrow & \mathcal{D}_1 \\
\downarrow & & \downarrow \\
\mathcal{D}^{\text{bo}}_0 \times \mathcal{D}^{\text{bo}}_0 & \underset{\text{cod} \times \text{cod}}{\longrightarrow} & \mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
\]

\textbf{Proposition 2.4.18.} Let \( \mathcal{D} \) be an exact equipment. There is an equivalence of fibrations on the left such that the triangle on the right also commutes:

\[
\begin{array}{ccc}
\text{Mon}(\mathcal{D}) & \overset{\sim}{\longrightarrow} & \mathcal{D}^{\text{bo}}_0 \\
\downarrow & & \downarrow \text{dom} \\
\mathcal{D}_0 & \longrightarrow & \mathcal{D}_0 \\
\end{array}
\qquad \quad
\begin{array}{ccc}
\text{Mon}(\mathcal{D}) & \overset{\sim}{\longrightarrow} & \mathcal{D}^{\text{bo}}_0 \\
\downarrow & & \downarrow \text{cod} \\
\mathcal{D}_0 & \longrightarrow & \mathcal{D}_0 \\
\end{array}
\]

\textbf{Proof.} The functor \( \text{dom} : \mathcal{D}^{\text{bo}}_0 \to \mathcal{D}_0 \) is a fibration via the factorization system in Theorem thm:orthogonal. The equivalence sends a monoid \((c, M)\) to the collapse morphism \( i_M : c \to \langle M \rangle \), which is in \( \text{bo} \) by the exactness of \( \mathcal{D} \). Since \( i_M \) is the universal embedding of \( M \), any monoid homomorphism \((f, \tilde{f})\) gives rise to a unique \( \tilde{f} \) such that

\[
\begin{array}{ccc}
c & \overset{M}{\longrightarrow} & c \\
\downarrow & \downarrow f \\
d & \overset{\langle N \rangle}{\longrightarrow} & \langle N \rangle \\
\end{array}
\quad = \quad
\begin{array}{ccc}
c & \overset{M}{\longrightarrow} & c \\
\downarrow & \downarrow i_M \\
\langle M \rangle & \overset{\langle M \rangle}{\longrightarrow} & \langle M \rangle \\
\end{array}
\]

Moreover, the pair \((f, \tilde{f})\) defines a morphism of arrows \( i_M \to i_N \) in \( \mathcal{D}^{\text{bo}}_0 \). By [28, Lemma 4.14], if \( \tilde{f} \) is cartesian then so is \( i_N \tilde{f} \), and clearly the converse also holds. It follows that the left triangle is a morphism of fibrations since \( \tilde{f} \) being cartesian over \( f \) implies \((f, \tilde{f})\) is as well.

The inverse equivalence \( \mathcal{D}^{\text{bo}}_0 \to \text{Mon}(\mathcal{D}) \) sends a bo map \( f : c \to d \) to the restriction \( d(f, f) \) with its induced monoid structure. \( \square \)
Theorem 2.4.19. Let $\mathcal{D}$ be an exact equipment with local coequalizers. There is an equivalence of equipments

\[
\begin{align*}
\text{Mod}(\mathcal{D}) & \xrightarrow{\sim} \mathcal{D}^{\text{bo}} \\
\langle - \rangle \xrightarrow{\sim} \mathcal{D} & \xrightarrow{\text{cod}} \mathcal{D}
\end{align*}
\]

Proof. By Proposition 2.4.11 the equipment functor $\langle - \rangle : \text{Mod}(\mathcal{D}) \to \mathcal{D}$ is a local equivalence, and $\text{cod} : \mathcal{D}^{\text{bo}} \to \mathcal{D}$ is a local equivalence by definition of $\mathcal{D}^{\text{bo}}$. It follows that the equivalence of fibrations from Proposition 2.4.18 extends to an equivalence of equipments. \qed

3 Equipments of monoidal profunctors

In this section we set up the necessary equipment to prove our main results, Theorem A and Theorem B. The high-level view of the argument runs as follows.

For any compact category $\mathcal{C}$, there is an equivalence of categories (Proposition 3.3.8) between the lax functors $\mathcal{C} \to \text{Set}$ and the monoids on $\mathcal{C}$ in the equipment $\mathcal{C}_{\mathbf{Prof}}$. Because $\mathcal{C}_{\mathbf{Prof}}$ is an exact equipment (Proposition 3.4.2), the monoids on $\mathcal{C}$ can be identified with the bijective-on-objects functors out of $\mathcal{C}$ by Theorem 2.4.19; this establishes the equivalence $\mathbb{CPsh}(\mathcal{C}_{\mathbf{Prof}}) \simeq \mathcal{C}_{\mathbf{Prof}}^{\text{bo}}$. Similar results hold for traced categories; see Theorem 3.3.1 and Corollary 3.3.2. These results suffice to prove Theorem B.

The remaining difficulty is dealing with freeness-on-objects, which we need for Theorem A. This is the purpose of Section 3.5.

3.1 Monoidal, Compact, and Traced Categories

We begin by reminding the reader of some categorical preliminaries: basic definitions and facts about monoidal, traced, and compact categories, lax and strong functors, and the Int construction. Standard references include [17], [15], and [16].

A strict monoidal category $\mathcal{M}$ is a category equipped with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and an object $I \in \mathcal{M}$, satisfying the usual monoid axioms.\footnote{We also used the notation $\otimes$ to denote bimodule composition in Definition 2.3.4; hopefully the intended meaning of the symbol will be clear from context.} In other words, a strict monoidal category is a monoid object in the category $\text{Cat}$. Such a category $\mathcal{M}$ is symmetric if there are in addition natural isomorphisms

\[
\sigma_{X,Y} : X \otimes Y \to Y \otimes X
\]

satisfying equations $\sigma_{X,Y \otimes Z} = (\text{id}_X \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes \text{id}_Z)$ and $\sigma_{Y,X} \circ \sigma_{X,Y} = \text{id}_{X \otimes Y}$.

Warning 3.1.1. Aside from the appendix, whenever we discuss monoidal categories in this article, we will mean symmetric strict monoidal categories.
Let \( \mathcal{M} \) and \( \mathcal{N} \) be monoidal categories. A functor \( F: \mathcal{M} \to \mathcal{N} \) is called lax monoidal if it is equipped with coherence morphisms

\[
I_N \xrightarrow{\mu} F(I_M) \quad \text{and} \quad F(X) \otimes_Y F(Y) \xrightarrow{\mu_{XY}} F(X \otimes Y)
\]
satisfying certain compatibility equations (see, e.g. [22, 7]). If all coherence morphisms are identities (resp. isomorphisms), then \( F \) is strict (resp. strong). Let \( \text{Lax}(\mathcal{M}, \mathcal{N}) \) denote the category of lax monoidal functors and monoidal transformations from \( \mathcal{M} \) to \( \mathcal{N} \).

Write \( \mathcal{MnCat} \) for the 2-category of strict symmetric monoidal categories, strict symmetric monoidal functors, and monoidal transformations. Let \( \mathcal{MnCat} \) denote the underlying 1-category.

A compact category is a (symmetric) monoidal category \( \mathcal{C} \) with the property that for every object \( X \in \mathcal{C} \) there exists an object \( X^* \) and morphisms \( \eta_X: I \to X^* \otimes X \) and \( \epsilon_X: X \otimes X^* \to I \) such that the following diagrams commute:

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow & & \downarrow \\
X \otimes I & \xrightarrow{\eta_X \otimes 1} & I \otimes X \\
\downarrow & & \downarrow \\
X \otimes (X^* \otimes Y) & \xrightarrow{(X^* \otimes X) \otimes X} & (X \otimes X^*) \otimes X
\end{array}
\quad \quad
\begin{array}{ccc}
X^* & \xrightarrow{id_{X^*}} & X^* \\
\downarrow & & \downarrow \\
I \otimes X^* & \xrightarrow{\epsilon_X \otimes X^*} & X^* \otimes I \\
\downarrow & & \downarrow \\
(X^* \otimes X) \otimes X & \xrightarrow{X^* \otimes \epsilon_X} & X^* \otimes (X \otimes X^*)
\end{array}
\]

We will denote by \( \text{CpCat} \) the full sub-2-category of \( \mathcal{MnCat} \) spanned by the compact categories and write \( \text{Ucm}: \text{CpCat} \to \mathcal{MnCat} \) for the corresponding forgetful functor. Let \( \text{CpCat} \) denote the underlying 1-category.

Given a morphism \( f: X \to Y \) in \( \mathcal{C} \), we denote by \( f^*: Y^* \to X^* \) the composite

\[
Y^* \xrightarrow{\eta_Y} X^* \otimes X \otimes Y^* \xrightarrow{f} X^* \otimes Y \otimes Y^* \xrightarrow{\epsilon_Y} X^*.
\]

It is easy to check that a strong functor \( F: \mathcal{C} \to \mathcal{M} \) to a monoidal category preserves all duals that exist in \( \mathcal{C} \), i.e. there is a natural isomorphism \( F(c^*) \cong F(c)^* \). From this, it follows that if \( F, G: \mathcal{C} \to \mathcal{C} \) are functors between compact categories, then any natural transformation \( \alpha: F \to G \) is a natural isomorphism. Indeed, for any object \( c \in \mathcal{C} \), the inverse of the \( c \)-component \( \alpha_c: Fc \to Gc \) is given by the dual morphism \( (\alpha_c)^*: Gc \to Fc \) to the dual component. Thus all 2-cells in \( \text{CpCat} \) are invertible.

A trace structure on a (symmetric) monoidal category \( \mathcal{T} \) is a collection of functions

\[
\text{Tr}_{X,Y}^U: \text{Hom}_\mathcal{T}(U \otimes X, U \otimes Y) \to \text{Hom}_\mathcal{T}(X, Y)
\]

satisfying seven equational axioms, we refer the reader to [16] for more details. If \( \mathcal{T} \) and \( \mathcal{U} \) are traced categories, then a (strict) traced functor is simply a strict symmetric monoidal functor which commutes with the trace operation.

In [16], it is shown that every traced category \( \mathcal{T} \) embeds as a full subcategory of a compact category \( \text{Int}(\mathcal{T}) \) whose objects are pairs \((X^-, X^+) \in \text{Ob}(\mathcal{T}) \times \text{Ob}(\mathcal{T})\) with morphisms given by

\[
\text{Hom}_{\text{Int}(\mathcal{T})}(X^-, X^+), (Y^-, Y^+) = \text{Hom}_\mathcal{T}(X^+ \otimes Y^+, X^+ \otimes Y^-)
\]

and compositions computed using the trace of \( \mathcal{T} \).
Remark 3.1.2. Traced categories were first defined in [16], which defines the 2-morphisms
between traced functors to simply be monoidal transformations. However, this choice
does not behave appropriately with the \textbf{Int} construction (for example \textbf{Int} would not be
2-functorial). The error was corrected in [13], where it was shown that the appropriate
2-morphisms between traced functors are natural isomorphisms.

We denote by \textbf{TrCat} the corrected 2-category of traced categories (where 2-cells are
invertible), and we denote its underlying 1-category by \textbf{TrCat}. Write \( U_{TM} : \textbf{TrCat} \to \textbf{MnCat} \) for the forgetful functor.

Every compact category \( C \) has a canonical trace structure, defined on a morphism
\( f : U \otimes X \to U \otimes Y \) morally (up to symmetries and identities) to be \( \epsilon_U \circ f \circ \eta_U \). More
precisely, if \( \sigma_{A,B} \) is the symmetry isomorphism, one defines \( \text{Tr}_{U^*}X,Y[f] \) to be the composite
\[
X \xrightarrow{\eta_{U^*} \otimes X} U^* \otimes U \otimes X \xrightarrow{U^* \otimes f} U^* \otimes U \otimes Y \xrightarrow{\sigma_{U^*,U \otimes Y}} U \otimes U^* \otimes Y \xrightarrow{\epsilon_{U \otimes Y}} Y
\]
Thus we have a functor \( U_{CT} : \textbf{CpCat} \to \textbf{TrCat} \). It is shown in [16] and [13] that this
functor is the right half of the 2-adjunction
\[
\textbf{TrCat} \overset{\text{Int}}{\xleftarrow{U_{CT}}} \textbf{CpCat}.
\]
Note that \( U_{CM} = U_{CT}U_{TM} \). In Section 3.6 we will be able to formally define the 2-
category \( \textbf{TrCat} \) without mentioning the trace structure (22) or the usual seven axioms,
but instead via the relationship between compact and monoidal categories.

Remark 3.1.3. We record the following facts, which hold for any traced category \( \mathcal{T} \); each
is shown in, or trivially derived from, [16]:

i. The component \( \mathcal{T} \to \text{Int}(\mathcal{T}) \) of the unit of the adjunction (23) is fully faithful. It
follows that \( \text{Int} : \textbf{TrCat} \to \textbf{CpCat} \) is locally fully faithful.

ii. If \( M \) is a monoidal category and \( F : M \to \mathcal{T} \) is a fully faithful symmetric monoidal
functor, then \( M \) has a unique trace for which \( F \) is a traced functor.

iii. If \( \mathcal{T} \) is compact then the counit \( \text{Int}(\mathcal{T}) \xrightarrow{\sim} \mathcal{T} \) is an equivalence.

iv. Suppose that \( \mathcal{T}' \) is a traced category and that \( F : \mathcal{T} \to \mathcal{T}' \) is a traced functor. Then
\( F \) is bijective-on-objects (resp. fully faithful) if and only if \( \text{Int}(F) \) is.

### 3.2 Monoidal profunctors

Mention that we may be introducing these for the first time?

Suppose \( \mathcal{C} \) and \( \mathcal{D} \) are monoidal categories. A \textit{monoidal profunctor} \( M \) from \( \mathcal{C} \)
onto \( \mathcal{D} \) is an ordinary profunctor (see Example 2.1.6) \( M : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \textbf{Set} \) in \( \textbf{MnCat} \), i.e.
\( M : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \textbf{Set} \) is equipped with a lax-monoidal structure where \textbf{Set} is endowed
with the cartesian monoidal structure. In the bimodule notation, this means that there is
an associative operation assigning to any elements \( m_1 \in M(c_1,d_1) \) and \( m_2 \in M(c_2,d_2) \)
an element \( m_1 \boxtimes m_2 \in M(c_1 \otimes c_2,d_1 \otimes d_2) \) such that
\[
(f_1 \cdot m_1 \cdot g_1) \boxtimes (f_2 \cdot m_2 \cdot g_2) = (f_1 \otimes f_2) \cdot (m_1 \boxtimes m_2) \cdot (g_1 \otimes g_2),
\]
as well as a distinguished element \( I_M \in M(I, I) \) such that \( I_M \otimes m = m = m \otimes I_M \) for any \( m \in M(c, d) \). If moreover \( m_2 \otimes m_1 = \sigma_{d_1, d_2} \cdot (m_1 \otimes m_2) \cdot \sigma_{c_1, c_2}^{-1} \), then one says \( M \) is symmetric monoidal.\(^5\)

A monoidal profunctor morphism \( \phi: M \to N \) is simply a monoidal transformation. Spelling this out in bimodule notation, \( \phi \) is an ordinary morphism of profunctors such that \( \phi(m_1 \otimes m_2) = \phi(m_1) \otimes \phi(m_2) \) and \( \phi(I_M) = I_N \).

We define a double category \( \mathrm{MnProf} \) whose objects are (symmetric) monoidal categories, vertical arrows are strict (symmetric) monoidal functors, horizontal arrows are (symmetric) monoidal profunctors, and 2-cells are defined as in (9), requiring \( \phi \) to be a monoidal transformation. It remains to check that the horizontal composition of monoidal profunctors is monoidal. This follows from the fact that reflexive coequalizers—namely the ones from (10)—commute with products in \( \mathrm{Set} \). Note that \( \mathrm{MnProf} \) is in fact an equipment since the cartesian 2-cell (11) is a monoidal transformation if \( N, F, \) and \( F' \) are monoidal functors.

The fully faithful functors \( U_{\mathrm{CM}}: \mathrm{CpCat} \to \mathrm{MnCat} \) and \( \mathrm{Int}: \mathrm{TrCat} \to \mathrm{CpCat} \), defined above, induce equipments \( \mathrm{CpProf} := U_{\mathrm{CM}}^*(\mathrm{MnProf}) \) and \( \mathrm{TrProf} := \mathrm{Int}^*(\mathrm{CpProf}) \) as in Definition 2.1.5. In particular, the vertical 1-categories of these equipments are given by

\[
\mathrm{MnProf}_0 = \mathrm{MnCat}, \quad \mathrm{CpProf}_0 = \mathrm{CpCat}, \quad \mathrm{TrProf}_0 = \mathrm{TrCat}.
\]

It may seem strange at first to define a profunctor \( \mathcal{T} \mapsto \mathcal{T}' \) between traced categories to be a profunctor \( \mathrm{Int}(\mathcal{T}) \to \mathrm{Int}(\mathcal{T}') \). The next proposition serves as a first sanity check on this definition, and the remainder of this paper provides further support.

**Proposition 3.2.1.** There is an isomorphism of 2-categories, \( \mathrm{Vert}(\mathrm{TrProf}) \cong \mathrm{TrCat} \).

**Proof.** Clearly these two 2-categories have the same underlying 1-category, so it suffices to show that there is a bijection \( \mathrm{Vert}(\mathrm{TrProf})(F, G) \cong \mathrm{TrCat}(F, G) \) for any traced functors \( F, G: \mathcal{T} \to \mathcal{T}' \) which preserve units and composition. By the definition of \( \mathrm{TrProf} \), we have \( \mathrm{Vert}(\mathrm{TrProf})(F, G) = \mathrm{CpCat}(\mathrm{Int}(F), \mathrm{Int}(G)) \). The result then follows since \( \mathrm{Int} \) is locally fully faithful [16]. \( \square \)

Thus from the definitions and Proposition 3.2.1, we see that the vertical 2-categories of these equipments are as expected:

\[
\mathrm{Vert}(\mathrm{MnProf}) \cong \mathrm{MnCat}, \quad \mathrm{Vert}(\mathrm{CpProf}) \cong \mathrm{CpCat}, \quad \mathrm{Vert}(\mathrm{TrProf}) \cong \mathrm{TrCat}.
\]

### 3.3 Special properties of \( \mathrm{CpProf} \)

This section aims to establish an equivalence between copresheaves on, and monoids in, compact (resp. traced) categories. See Definitions 2.2.1 and 2.3.1.

---

\(^5\) We will generally suppress the word *symmetric* since all monoidal categories and monoidal profunctors are symmetric by assumption; see Warning 3.1.1.
### 3.3. Special properties of \( \text{CpProf} \)

**Theorem 3.3.1.** There is an equivalence of fibrations

\[
\begin{align*}
\text{CPsh}(\text{CpProf}) & \xrightarrow{\simeq} \text{Mon}(\text{CpProf}) \\
\downarrow \text{Id} & \downarrow \text{Id}
\end{align*}
\]

\( \text{CpCat.} \)

We prove this result by factoring the desired equivalence as a composition of two more manageable equivalences in Proposition 3.3.6 and Proposition 3.3.8 below.

**Corollary 3.3.2.** There is an equivalence of fibrations

\[
\begin{align*}
\text{CPsh}(\text{TrProf}) & \xrightarrow{\simeq} \text{Mon}(\text{TrProf}) \\
\downarrow \text{Id} & \downarrow \text{Id}
\end{align*}
\]

\( \text{TrCat.} \)

**Proof.** We have that \( \text{Int}: \text{TrProf} \rightarrow \text{CpProf} \) is a local equivalence that preserves the terminal object. Thus using Lemma 2.3.3 and Lemma 2.2.2 we construct the desired equivalence \( \text{CPsh}(\text{TrProf}) \rightarrow \text{Mon}(\text{TrProf}) \) as the pullback along \( \text{Int} \) of the equivalence \( \text{CPsh}(\text{CpProf}) \rightarrow \text{Mon}(\text{CpProf}) \) from Theorem 3.3.1.

To prove Theorem 3.3.1 we introduce a third fibration, that of pointed endo-proarrows, and establish its equivalence with each of \( \text{CPsh}(\text{CpProf}) \) and \( \text{Mon}(\text{CpProf}) \).

**Definition 3.3.3.** Given an equipment \( \mathcal{D} \), we define the fibration of endo-proarrows by the strict pullback

\[
\begin{array}{c}
\text{End}(\mathcal{D}) \\
\downarrow \text{Id}
\end{array}
\begin{array}{c}
\xrightarrow{\Delta} \\
\downarrow (L,R)
\end{array}
\begin{array}{c}
\mathcal{D}_1 \\
\mathcal{D}_0 \\
\downarrow \text{Id}
\end{array}
\begin{array}{c}
\xrightarrow{\Delta} \\
\end{array}
\begin{array}{c}
\mathcal{D}_0 \times \mathcal{D}_0.
\end{array}
\]

We also define a fibration \( \text{Ptd}(\mathcal{D}) \rightarrow \mathcal{D}_0 \) whose objects are pointed endo-proarrows, i.e. endo-proarrows \( M: c \rightarrow c \) in \( \mathcal{D} \) equipped with a unit \( i_M: c \rightarrow M \) as in (13) (but not a multiplication), and whose morphisms are 2-cells which preserve the units.

**Lemma 3.3.4.** Let \( \mathcal{C} \) be a compact category. For any pointed endo-profunctor \( i: \text{Hom}_{\mathcal{C}} \rightarrow N \) in \( \text{Ptd}(\text{CpProf}) \), there is a natural bijection \( N(a,b) \cong N(I,a^* \otimes b) \) for any objects \( a,b \in \mathcal{C} \).

**Proof.** Given \( n \in N(a,b) \), we can construct an element

\[
(i(id_a) \otimes n) \cdot \eta_a \in N(I,a^* \otimes b).
\]

Conversely, given \( n' \in N(I,a^* \otimes b) \), we can construct an element

\[
(e_a \otimes id_b) \cdot (i(id_a) \otimes n') \in N(a,b).
\]

It is simple to check that this defines a natural bijection. \( \square \)
With the fibration \( \text{End}(\text{CpProf}) \to \text{CpCat} \) from Definition 3.3.3, we can define the functors

\[
\begin{array}{ccc}
\text{CPsh}(\text{CpProf}) & \xrightarrow{F} & \text{End}(\text{CpProf}) \\
| & | & | \\
\text{CpCat} & \xleftarrow{U} & \text{End}(\text{CpProf})
\end{array}
\]

where \( FM : \mathcal{E}^{op} \times \mathcal{E} \to \text{Set} \) is defined by \( FM(a,b) := M(a^* \otimes b) \) while \( UN : \mathcal{E} \to \text{Set} \) is given by \( UN(a) := N(I, a) \). It is simple to check that these are morphisms of fibrations, i.e. that they preserve cartesian morphisms.

**Proposition 3.3.5.** The functor \( F : \text{CPsh}(\text{CpProf}) \to \text{End}(\text{CpProf}) \) factors through \( \text{Ptd}(\text{CpProf}) \).

**Proof.** Let \( M : \mathcal{E} \to \text{Set} \) be an object in \( \text{CPsh}(\text{CpProf}) \). Since \( M \) is a monoidal profunctor, there is a given unit element \( I_M \in M(I) \). Thus given any \( f : c \to d \) in \( \mathcal{E} \), we can define the element \( i(f) \in FM(c,d) = M(c^* \otimes d) \) via

\[
i(f) := \left( (id_c \otimes f) \circ \eta_c \right) \cdot I_M.
\]

It is easy to check that this construction of a unit \( i \) is functorial. \( \Box \)

Thus, we have induced functors \( F, U : \text{CPsh}(\text{CpProf}) \leftrightarrows \text{Ptd}(\text{CpProf}) \) giving the diagram

\[
\begin{array}{ccc}
\text{CPsh}(\text{CpProf}) & \xrightarrow{F} & \text{Ptd}(\text{CpProf}) \\
| & | & | \\
\text{CpCat} & \xleftarrow{U} & \text{End}(\text{CpProf})
\end{array}
\]

in which the triangle involving the \( F \)'s and the triangle involving the \( U \)'s both commute.

**Proposition 3.3.6.** The functors \( F \) and \( U \) from (25) form an equivalence of fibrations

\[
\begin{array}{ccc}
\text{CPsh}(\text{CpProf}) & \xrightarrow{\simeq} & \text{Ptd}(\text{CpProf}) \\
| & | & | \\
\text{CpCat} & \xleftarrow{U} & \text{End}(\text{CpProf})
\end{array}
\]

**Proof.** If \( M \in \text{CPsh}(\text{CpProf}) \), i.e. \( M \) is a lax functor \( \mathcal{E} \to \text{Set} \) for some compact \( \mathcal{E} \), then \( (U(FM))(a) = (FM)(I, a) = M(I^* \otimes a) \cong M(a) \) for any \( a \in \mathcal{E} \). On the other hand, given \( N \in \text{Ptd}(\text{CpProf}) \), we have \( (F(UN))(a,b) = N(I, a^* \otimes b) \), and the equivalence follows from Lemma 3.3.4. \( \Box \)

As preparation for the proof of Proposition 3.3.8 below, we work out what a monoid in \( \text{MnProf} \) looks like using the bimodule notation for profunctors. A unit for a monoidal profunctor \( M : \mathcal{E} \to \mathcal{E} \) is a unit \( i : \text{Hom}_\mathcal{E} \to M \) as in Example 2.4.9 where

\[
i(id_{I_\mathcal{E}}) = I_M \quad \text{and} \quad i(f \otimes g) = i(f) \boxtimes i(g)
\]
for any morphisms \( f \) and \( g \) in \( \mathcal{C} \). Similarly, the multiplication \( \bullet \) on \( M \) must satisfy

\[
I_M \bullet I_M = I_M \\
(m_2 \boxtimes m'_2) \bullet (m_1 \boxtimes m'_1) = (m_2 \bullet m_1) \boxtimes (m'_2 \bullet m'_1)
\]

for any \( m_1 \in M(c,d) \), \( m'_1 \in M(c',d') \), \( m_2 \in M(d,e) \), and \( m'_2 \in M(d'e') \), in addition to the requirements from Example 2.4.9.

Remark 3.3.7. Equation (27) follows immediately from (20) and the identification \( i(\text{id}_{I_C}) = I_M \). Thus, to prove that \( i \) and \( \bullet \) form a monoid in \( \text{MnProf} \), it suffices to show (26) and (28), in addition to the requirements discussed in Remark 2.4.10.
3.3. Special properties of \( \text{CpProf} \)

\[
m \cdot f
\]

\[
g \cdot m
\]

\[
m_2 \cdot m_1
\]

\[
i(f)
\]

\[
m \cdot i(f) = m \cdot f :
\]

\[
(f \cdot m_2) \cdot (m_1 \cdot h) = f \cdot (m_2 \cdot m_1) \cdot h
\]

\[
(m_3 \cdot g) \cdot m_1 = m_3 \cdot (g \cdot m_1)
\]

\[
I_M = i(id_1)
\]

\[
I \boxtimes m = m :
\]

\[
(g_1 \cdot m_1 \cdot f_1) \boxtimes (g_2 \cdot m_2 \cdot f_2) = (g_1 \otimes g_2) \cdot (m_1 \boxtimes m_2) \cdot (f_1 \otimes f_2)
\]
3.3. Special properties of $\text{CpProf}$

\[ m_1 \boxtimes m_2 = \sigma_{d_1,d_2} \cdot (m_2 \boxtimes m_1) \cdot \sigma_{c_1,c_2}^{-1} : \]

\[ i(f_1 \otimes f_2) = i(f_1) \boxtimes i(f_2) \]

\[ (m_2 \boxtimes m'_2) \bullet (m_1 \boxtimes m'_1) = (m_2 \bullet m_1) \boxtimes (m'_2 \bullet m'_1) \]
Proposition 3.3.8. The forgetful functor \( \text{Mon}(\text{CpProf}) \to \text{Ptd}(\text{CpProf}) \) is an equivalence of fibrations over \( \text{CpCat} \).

Proof. It is clear that this forgetful functor, which we refer to as \( U \) in the proof, is a morphism of fibrations, so we must show that \( U \) is an equivalence of categories.

To define an inverse functor \( U^{-1} \), consider an object of \( \text{Ptd}(\text{CpProf}) \), i.e., a profunctor \( N: c \leftrightarrow c \) with basepoint \( i: \text{Hom}_c \to N \). We can define a multiplication on \( N \) by the formula

\[
 n_2 \bullet n_1 := (\epsilon_d \otimes \text{id}_c) \cdot (n_1 \boxtimes i(\text{id}_{d'}) \boxtimes n_2) \cdot (\text{id}_c \otimes \eta_d)
\]

for any \( n_1 \in N(c,d) \) and \( n_2 \in N(d,e) \), or in picture form:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) at (0,0) {$n_1$};
\node (n2) at (2,0) {$n_2$};
\draw[->] (m) to (n2);
\end{tikzpicture}
\end{array}
\]

It is straightforward to check that this multiplication is associative. Remark 3.3.7 says that, in order to show that \( N \) together with \( i \) and \( \bullet \) define an object in \( \text{Mon}(\text{CpProf}) \), we must additionally show that this multiplication satisfies the equations (20) and (28). We will begin by showing that \( n \bullet i(f) = n \cdot f \) for any \( n \in N(d,e) \) and \( f: c \to d \):

\[
 n \bullet i(f) = (\epsilon_d \otimes \text{id}_c) \cdot (i(f) \boxtimes i(\text{id}_{d'}) \boxtimes n) \cdot (\text{id}_c \otimes \eta_d)
\]

\[
 = (\epsilon_d \otimes \text{id}_c) \cdot (i(f \otimes \text{id}_{d'}) \boxtimes (n \cdot \text{id}_d)) \cdot (\text{id}_c \otimes \eta_d)
\]

\[
 = (i(\epsilon_d \circ (f \otimes \text{id}_{d'})) \boxtimes (n \cdot \text{id}_d)) \cdot (\text{id}_c \otimes \eta_d)
\]

\[
 = (i(\text{id}_f) \boxtimes n) \cdot (i(\epsilon_d \circ (f \otimes \text{id}_{d'})) \otimes \text{id}_d) \circ (\text{id}_c \otimes \eta_d))
\]

\[
 = (I_N \boxtimes n) \cdot ((\epsilon_d \otimes \text{id}_d) \circ (\text{id}_d \otimes \eta_d)) \circ (f \otimes \text{id}_f))
\]

\[
 = (I_N \boxtimes n) \cdot (f \otimes \text{id}_f)
\]

\[
 = n \cdot f.
\]

The equation \( i(f) \bullet n = f \cdot n \) follows similarly, so we have verified (20).

Finally, we must check (28). Recall that this says

\[
(n_2 \boxtimes n'_2) \bullet (n_1 \boxtimes n'_1) = (n_2 \bullet n_1) \boxtimes (n'_2 \bullet n'_1)
\]
for any \( n_1 \in N(c,d), n_1' \in N(c',d'), n_2 \in N(d,e), \) and \( n_2' \in N(d',e') \), which we prove below:

\[
(n_2 \boxtimes n_2') \cdot (n_1 \boxtimes n_1') \\
= (e_d \otimes \id \otimes \id \otimes \id) \cdot ((n_1 \boxtimes n_1') \boxtimes \id \cdot (\id \otimes \id) \boxtimes (n_2 \boxtimes n_2')) \cdot (\id \otimes \id \otimes \id)
\]

Thus we have shown that the multiplication \( \bullet \) defines a monoid \( U^{-1}(N) \).

To define \( U^{-1} \) on morphisms, suppose that \( M \in \text{MnProf}(\mathcal{C}, \mathcal{C}) \) is another monoidal profunctor with unit, and that \( \phi : M \to N \) is a monoidal profunctor morphism which preserves units. Then \( \phi \) also preserves the canonical multiplications:

\[
\phi(n_2 \bullet n_1) = \phi\left[ (e_d \otimes \id) \cdot (n_1 \boxtimes i(id_{d'}) \boxtimes \id) \cdot (\id \otimes \id) \right]
\]

Clearly \( U \circ U^{-1} = \id_{\text{Prof}(\mathcal{C})} \). For the other direction, consider a monoid \( M : \mathcal{C} \rightarrow \mathcal{C} \) with unit \( i \) and multiplication \( \ast \). Then the multiplication \( \bullet \) defined above in fact coincides with \( \ast \):

\[
n_2 \ast n_1 = n_2 \ast \left[ ((e_d \otimes \id) \circ (\id \otimes \id)) \cdot n_1 \right]
\]
3.4. MnProf, CpProf, and TrProf are exact

Thus $U^{-1} \circ U = \text{id}_{\text{Mon}(\text{CpProf})}$, and $U$ is an equivalence (in fact, isomorphism) of categories.

Remark 3.3.9. One can think of compact categories as a categorification of groups, where duals of objects act like inverses of group elements. From this perspective, the results of this section can be seen as categorifications of basic facts from group theory.

We can think of profunctors between compact categories as playing the role of relations between groups which are stable under multiplication. Pointed endo-profunctors act like reflexive relations, and monoids in profunctors act like reflexive and transitive relations. In fact, one can define an equipment of groups, group homomorphisms, and equivariant relations, in which monoids are precisely reflexive transitive relations. It is easy to see that copresheaves, i.e. equivariant relations $\rightarrow G$, are the same as subgroups of $G$.

In this way, the equivalence $\text{CPsh}(\text{CpProf}) \simeq \text{Mon}(\text{CpProf})$ categorifies the standard fact that a subgroup determines, and is determined by, the conjugacy congruence. The equivalence $\text{Ptd}(\text{CpProf}) \simeq \text{Mon}(\text{CpProf})$ would seem to be saying that every reflexive relation (stable under multiplication) on a group is in fact transitive, which while true is perhaps less familiar than the conjugacy relation. But note, in the definition of a Mal’cev category (see [8]) this property is singled out as characterizing categories in which some amount of classical group theory can be developed. By analogy, we might think of this section as proving that CpProf is a “Mal’cev equipment”.

3.4 MnProf, CpProf, and TrProf are exact

As we saw throughout Section 2.3 the exactness of an equipment gives rise to many equivalences of fibrations. Thus to establish our main theorems we aim first to show that the equipments from Section 3.2 are all exact, as in Definition 2.4.6.

Proposition 3.4.1. The equipment MnProf is exact.

Proof. Suppose that $M: \mathcal{C} \rightarrow \mathcal{C}$ is a monoid in MnProf. One uses $M$ to construct a category $\langle M \rangle$ with the same objects as $\mathcal{C}$, and with hom sets defined by $\langle M \rangle(c,d) := M(c,d)$ for any pair of objects $c,d \in \text{Ob}(\mathcal{C})$. For any object $c$, the identity is provided by $i(id_c)$, while the multiplication $\bullet$ on $M$ provides the composition of $\langle M \rangle$.

The unit of $M$ can also be used to construct an identity-on-objects functor $i_M: \mathcal{C} \rightarrow \langle M \rangle$ and an embedding 2-cell $\eta_M$ sending any element of $M$ to itself as a morphism of
\( \langle M \rangle \). It is easy to see that \( \tilde{\iota}_M \) is cartesian and that \( (i_M, \tilde{\iota}_M) \) is a collapse. The category \( \langle M \rangle \) has a canonical monoidal structure, which on objects is just that of \( \mathcal{C} \) and on morphisms is induced by the monoidal profunctor structure of \( M \). It is also simple to verify the second part of Definition 2.4.6: an \( (M, N) \)-bimodule is precisely the data of a profunctor \( \langle M \rangle \to \langle N \rangle \).

**Proposition 3.4.2.** The equipment \( \mathcal{C}_\mathcal{PProf} \) is exact.

**Proof.** We can consider a monoid \( M : \mathcal{C} \to \mathcal{C} \) in \( \mathcal{C}_\mathcal{PProf} \) as a monoid in \( \mathcal{M}_\mathcal{PProf} \), which has a collapse embedding \( i_M : M \to \langle M \rangle \) by Proposition 3.4.1. The collapse \( \langle M \rangle \) is a monoidal category and, by Theorem 2.4.19, and \( i_M \) is a (strict symmetric monoidal) bifunctor. But any strong monoidal functor preserves duals, so every object of \( \langle M \rangle \) has a dual, hence \( \langle M \rangle \) is compact. The map \( U_{CM} : \mathcal{C}_\mathcal{PProf} \to \mathcal{M}_\mathcal{PProf} \) is a fully faithful local equivalence and so \( \langle M \rangle \) being a collapse in \( \mathcal{M}_\mathcal{PProf} \) implies it is a collapse in \( \mathcal{C}_\mathcal{PProf} \).

We record the following consequence of Theorem 2.4.14 in the current notation.

**Corollary 3.4.3.** The 2-categories \( \mathcal{M}_\mathcal{NCat} \) and \( \mathcal{C}_\mathcal{PCat} \) admit 2-orthogonal factorization systems.

We abuse notation slightly and denote each of the factorization systems above by \((\text{bo}, \text{ff})\). Recall the adjunction (23) and write \( \eta_\mathcal{T} : \mathcal{T} \to U_{\mathcal{CT} \mathcal{Int}(\mathcal{T})} \) for the unit component on \( \mathcal{T} \in \mathcal{T}_\mathcal{TrCat} \).

**Lemma 3.4.4.** Let \( \mathcal{T} \) be a traced category, \( \mathcal{C} \) a compact category, and \( F : \mathcal{Int}(\mathcal{T}) \to \mathcal{C} \) a bijective-on-objects monoidal functor. Consider any orthogonal factorization in \( \mathcal{M}_\mathcal{NCat} \) as follows:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\exists H} & U_{\mathcal{CM}} \mathcal{C} \\
\downarrow & & \downarrow U_{\mathcal{CM} F} \\
\exists G & & \\
U_{\mathcal{TM} \mathcal{T}} & \xleftarrow{U_{\mathcal{TM} \eta_\mathcal{T}}} & U_{\mathcal{CM} \mathcal{Int}(\mathcal{T})}
\end{array}
\]

There is a unique trace structure on \( \mathcal{M} \), i.e. a unique traced category \( \mathcal{T}' \) with \( U_{\mathcal{TM} \mathcal{T}} \mathcal{T}' = \mathcal{M} \), such that the factorization lifts to \( \mathcal{T}_\mathcal{TrCat} \):

\[
\begin{array}{ccc}
\mathcal{T} & \xleftarrow{\eta_\mathcal{T}} & U_{\mathcal{CT} \mathcal{Int}(\mathcal{T})} \\
\downarrow G & & \downarrow U_{\mathcal{CT} F} \\
\mathcal{T}' & \xleftarrow{H} & U_{\mathcal{CT} \mathcal{C}}
\end{array}
\]

Moreover, there is an isomorphism \( \alpha : \mathcal{Int}(\mathcal{T}') \cong \mathcal{C} \) such that \( U_{\mathcal{CT} \alpha} \circ \eta_\mathcal{T}' = H \) and \( \alpha \circ \mathcal{Int}(G) = F \).

**Proof.** This derives mainly from basic properties of the \( \mathcal{Int} \) construction; see Remark 3.1.3. Since \( H : \mathcal{M} \to U_{\mathcal{CM} \mathcal{C}} \) is fully faithful, the trace on \( U_{\mathcal{CT} \mathcal{C}} \) uniquely determines the desired trace structure on \( \mathcal{T}' \) by which \( H \) is a traced functor. It also follows that \( G \) respects the trace in \( \mathcal{T} \) since \( U_{\mathcal{CT} F} \circ \eta_\mathcal{T} \) does.
For the final claim, consider the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\eta_{\mathcal{F}}} & U_{CT}\text{Int}(\mathcal{F}) \\
G & \downarrow & \downarrow U_{CT}\text{Int}(G) \\
\mathcal{F}' & \xrightarrow{\eta_{\mathcal{F}'}} & U_{CT}\text{Int}(\mathcal{F}')
\end{array}
\]

where \(\alpha: \text{Int}(\mathcal{F}') \to \mathcal{C}\) is the adjunct of \(H\), which is fully faithful since \(H\) is. Since \(G\) is bo, \(U_{CT}\text{Int}(G)\) will be bo as well. But \(U_{CT}F\) is bo, so \(U_{CT}\alpha\) and hence \(\alpha\) must be bo also.

Since \(\alpha\) is both ff and bo, it is an isomorphism, completing the proof.

\[\square\]

**Proposition 3.4.5.** The equipment \(\text{TrProf}\) is exact.

**Proof.** Let \(M: \mathcal{F} \to \mathcal{F}\) be a monoid in \(\text{TrProf}\). By definition of \(\text{TrProf}\) this is a monoid \(M: \text{Int}(\mathcal{F}) \to \text{Int}(\mathcal{F})\) in \(\text{CpProf}\), so \(M = \text{Mon}(\text{Int})(M)\) (in the language of Lemma 2.4.2). Define \((M)_C\) and \((i_M, i'_M): (\text{Int}(\mathcal{F}), M) \to (M)_C\) to be the collapse embedding of \(M\) in \(\text{CpProf}\).

Then applying Lemma 3.4.4 with \(F = i_M\) gives a traced category \((M)_T\) and a bo functor \(i'_M: \mathcal{F} \to (M)_T\) as in the diagram below:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\eta_{\mathcal{F}}} & U_{CT}\text{Int}(\mathcal{F}) \\
\langle M \rangle_T & \xleftarrow{i'_M} & U_{CT} \langle M \rangle_C \\
\end{array}
\]

To see that \((M)_T\) is a collapse in \(\text{TrProf}\) we must establish the bijection \(\text{Emb}_{\mathcal{F}}(M, \mathcal{F}') \cong \text{TrCat}(\langle M \rangle_T, \mathcal{F}')\), natural in the traced category \(\mathcal{F}'\).

Using the adjunction bijection and precomposition with the inverse of \(\alpha: \text{Int}(\langle M \rangle_T) \cong \langle M \rangle_C\) from Lemma 3.4.4, we get an isomorphism

\[
\text{TrCat}(\langle M \rangle_T, U_{CT}\text{Int}(\mathcal{F}')) \cong \text{CpCat}(\text{Int}(\langle M \rangle_T), \text{Int}(\mathcal{F}')) \cong \text{CpCat}(\langle M \rangle_C, \text{Int}(\mathcal{F}')).
\]

Since \(i_M = \alpha \circ \text{Int}(i'_M)\), and by the naturality of the \((\text{Int}, U_{CT})\) adjunction, these morphisms become the right-hand square of the following commutative diagram:

\[
\begin{array}{ccc}
\text{TrCat}(\langle M \rangle_T, \mathcal{F}') & \xrightarrow{\eta_{\mathcal{F}'\circ \text{Int}}} & \text{TrCat}(\langle M \rangle_T, U_{CT}\text{Int}(\mathcal{F}')) \cong \text{CpCat}(\langle M \rangle_C, \text{Int}(\mathcal{F}')) \\
\downarrow -\circ i'_M & & \downarrow -\circ i'_M \\
\text{TrCat}(\mathcal{F}, \mathcal{F}') & \xrightarrow{\eta_{\mathcal{F}'\circ \text{Int}}} & \text{TrCat}(\mathcal{F}, U_{CT}\text{Int}(\mathcal{F}')) \cong \text{CpCat}(\text{Int}(\mathcal{F}), \text{Int}(\mathcal{F}')) \end{array}
\]

The left square is a pullback, by the orthogonality of \(i'_M\) \in bo and \(\eta_{\mathcal{F}'}\) \in ff, and the right square is a pullback because the top and bottom maps are isomorphisms. Hence the outer square is a pullback as well. Since \((M)_C\) is a collapse in \(\text{CpProf}\), there is a
bijection $\text{Emb}_{\mathcal{C}_p}(M, \text{Int}(\mathcal{J}')) \cong \mathcal{C}_p\text{Cat}(\langle M \rangle_\mathcal{C}, \text{Int}(\mathcal{J}'))$ so by Lemma 2.4.2, the outer pullback produces the desired natural isomorphism $\text{Emb}_{\mathcal{T}_r}(M, \mathcal{J}) \cong \mathcal{T}_r\text{Cat}(\langle M \rangle_\mathcal{T}, \mathcal{J}')$.

Since the trivial monoid on $\langle M \rangle_\mathcal{T}$ in $\mathcal{T}_r\text{Prof}$ is by definition the trivial monoid on $\text{Int}(\langle M \rangle_\mathcal{T}) \cong \langle M \rangle_\mathcal{C}$ in $\mathcal{C}_p\text{Prof}$, the collapse embedding $M \Rightarrow \langle M \rangle_\mathcal{T}$ is (after composition with the isomorphism $\alpha$) just the collapse $(i_M, \vec{i}_M)$ in $\mathcal{C}_p\text{Prof}$, and hence is cartesian. Since the inclusion $\mathcal{T}_r\text{Prof} \rightarrow \mathcal{C}_p\text{Prof}$ is a local equivalence and $\mathcal{C}_p\text{Prof}$ is exact, the second condition of Definition 2.4.6 follows immediately.

As a corollary we obtain our second main theorem. Recall that we use the notation $\int$ to denote the Grothendieck construction.

**Theorem B.** There are equivalences of fibrations

\[
\begin{array}{ccc}
\int \text{Lax}(\mathcal{C}, \text{Set}) & \cong & \text{CpCat} \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{CpCat} & & \text{TrCat}
\end{array}
\]

\[
\begin{array}{ccc}
\int \text{Lax}(\text{Int}(\mathcal{J}), \text{Set}) & \cong & \text{TrCat} \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{TrCat} & & \text{TrCat}
\end{array}
\]

**Proof.** Essentially by definition, we have isomorphisms of fibrations

\[
\begin{array}{ccc}
\int \text{Lax}(\mathcal{C}, \text{Set}) & \cong & \text{CPsh}(\mathcal{C}_p\text{Prof}) \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{CpCat} & & \text{TrCat}
\end{array}
\]

\[
\begin{array}{ccc}
\int \text{Lax}(\text{Int}(\mathcal{J}), \text{Set}) & \cong & \text{CPsh}(\mathcal{T}_r\text{Prof}) \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{TrCat} & & \text{TrCat}
\end{array}
\]

Since $\mathcal{C}_p\text{Prof}$ and $\mathcal{T}_r\text{Prof}$ are exact by Proposition 3.4.2 and 3.4.5, we may apply Proposition 2.4.18 to get equivalences of fibrations

\[
\begin{array}{ccc}
\text{Mon}(\mathcal{C}_p\text{Prof}) & \cong & \text{CpCat}^{\text{bo}} \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{CpCat} & & \text{TrCat}
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mon}(\mathcal{T}_r\text{Prof}) & \cong & \text{TrCat}^{\text{bo}} \\
\downarrow \text{dom} & & \downarrow \text{dom} \\
\text{TrCat} & & \text{TrCat}
\end{array}
\]

The result then follows from Theorem 3.3.1 and Corollary 3.3.2.

3.5 Objectwise-freeness

If we momentarily denote the free traced category on a set $\emptyset$ as $F(\emptyset)$, a corollary of Theorem B is an isomorphism $\text{Lax}(\text{Int}(\mathcal{J}), \text{Set}) \cong \mathcal{T}_r\text{Cat}(\emptyset)$, where the latter is the category of traced categories with fixed object set $\emptyset$. This was called Theorem 0 in the introduction. Our goal in the present section is to prove Theorem A, for which we must formalize what it means for an object in an equipment to itself be objectwise-free.

Consider an equipment $\mathcal{D}$, and let $\text{dom}: \mathcal{D}_0^{\text{bo}} \rightarrow \mathcal{D}_0$ denote the domain fibration. Suppose we are given an adjunction to a category $\mathcal{J}$:

\[
\mathcal{J} \xleftarrow{F} \xrightarrow{U} \mathcal{D}_0.
\]
Let $T = UF$ be the monad on $\mathcal{S}$ corresponding to this adjunction, and write $\mathcal{S}_T$ for the Kleisli category for $T$, i.e. the full subcategory of free objects $Fs$ in $\mathcal{D}_0$. Let $k_T: \mathcal{S}_T \to \mathcal{D}_0$ denote the inclusion, and define $k^{bo}_T$ to be the strict pullback of $k_T$ along $\text{dom}$:

$$
\begin{array}{c}
(\mathcal{D}^{bo}_T)_0 \xrightarrow{k^{bo}_T} \mathcal{D}^{bo}_0 \\
\downarrow \quad \downarrow \text{dom} \\
\mathcal{S}_T \xrightarrow{k_T} \mathcal{D}_0
\end{array}
$$

**Definition 3.5.1.** The fully faithful functors $k_T: \mathcal{S}_T \to \mathcal{D}_0$ and $k^{bo}_T: (\mathcal{D}^{bo}_T)_0 \to \mathcal{D}^{bo}_0$ induce equipments $\mathcal{D}_T := k^*_T \mathcal{D}$ and $\mathcal{D}^{bo}_T := (k^{bo}_T)^* \mathcal{D}^{bo}$ (as in Definition 2.1.5), as well as fully faithful local equivalences, which we denote

$$
\varphi_T: \mathcal{D}_T \to \mathcal{D} \quad \text{and} \quad \varphi^{bo}_T: \mathcal{D}^{bo}_T \to \mathcal{D}^{bo}.
$$

**Proposition 3.5.2.** With the setup as in Definition 3.5.1, suppose also that $\mathcal{D}$ is exact and has local coequalizers. There is a commutative diagram of equipments, in which the vertical functors are equivalences and the horizontal functors are local equivalences:

$$
\begin{array}{rcl}
\text{Mod}(\mathcal{D}_T) & \xrightarrow{\text{Mod}(\varphi_T)} & \text{Mod}(\mathcal{D}) \\
\downarrow \varphi^{bo}_T & & \downarrow \varphi^{bo}\text{coeq} \\
\mathcal{D}^{bo}_T & \xrightarrow{\varphi^{bo}_T} & \mathcal{D}^{bo}.
\end{array}
$$

Suppose moreover that $U(\text{bo}) \subseteq \text{iso}(\mathcal{S})$, the set of isomorphisms in $\mathcal{S}$. Then the following composite is a fully faithful local equivalence:

$$
\begin{array}{rcl}
\text{Mod}(\mathcal{D}_T) & \xrightarrow{\text{Mod}(\varphi_T)} & \text{Mod}(\mathcal{D}) \\
\downarrow \sim & & \downarrow \sim \\
\mathcal{D}^{bo}_T & \xrightarrow{\sim} & \mathcal{D}^{bo}.
\end{array}
$$

**Proof.** By Lemma 2.3.5, $\text{Mod}(\varphi_T): \text{Mod}(\mathcal{D}_T) \to \text{Mod}(\mathcal{D})$ is a fully faithful local equivalence. The remainder of the first claim follows from Theorem 2.4.19 and the definition of $\mathcal{D}^{bo}_T$.

For the second claim, assume $U(\text{bo}) \subseteq \text{iso}(\mathcal{S})$. From Theorem 2.4.19 and the first part of the proposition, it suffices to consider the composition

$$
\begin{array}{rcl}
\mathcal{D}^{bo}_T & \xrightarrow{\varphi^{bo}_T} & \mathcal{D}^{bo} \\
& \xrightarrow{\text{cod}} & \mathcal{D}.
\end{array}
$$

By definition, both $\varphi^{bo}_T$ and cod are local equivalences, hence the composition is also.

To see that $(\text{cod} \varphi^{bo}_T)_0$ is fully faithful, consider a pair of objects $p: Fs \to D$ and $p': Fs' \to D'$ in $(\mathcal{D}^{bo}_T)_0$, and a vertical morphism $f: D \to D'$ in $\mathcal{D}_0$. In the square

$$
\begin{array}{c}
\mathcal{D}_0(Fs, Fs') \xrightarrow{p'\circ} \mathcal{D}_0(Fs, D') \\
\downarrow \cong \quad \downarrow \cong \\
\mathcal{S}(s, UF s') \xrightarrow{U p'\circ} \mathcal{S}(s, UD')
\end{array}
$$
which commutes by naturality of the adjunction bijection, the bottom function is a bijection since \( U(p') \) is an isomorphism for any \( p' \in \text{bo} \). Hence the top function is a bijection, which shows that there exists a unique lift of \( f \) to a morphism in \((D_{T})_{0}\):

\[
\begin{array}{ccc}
F_{S} & \xrightarrow{f} & F_{S'} \\
\downarrow^{p} & & \downarrow^{p'} \\
D & \xrightarrow{f} & D'
\end{array}
\]
as desired. \( \square \)

We apply the above work to define equipments of objectwise-free monoidal, compact, and traced categories and conclude by addressing Theorem A. The idea is that an objectwise-free monoidal category can be identified with a \( \text{bo} \)-map out of the free monoidal category on a set. Thus we begin by defining the latter.

Consider the free-forgetful adjunctions\(^{6}\)

\[
\begin{array}{ccc}
\text{Set} & \xleftarrow{U_{M}} & \text{MnCat} \\
\text{Set} & \xleftarrow{U_{C}} & \text{CpCat} \\
\text{Set} & \xleftarrow{U_{T}} & \text{TrCat}
\end{array}
\]

and write \( T_{M}, T_{C}, \) and \( T_{T} \) for the corresponding monads on \( \text{Set} \). Note that \( T_{M} \) and \( T_{T} \) are both isomorphic to the free monoid monad, while \( T_{C} \) is isomorphic to the free monoid-with-involution monad.\(^{7}\) Following Definition 3.5.1, we have equipments:

\[
\begin{array}{ccc}
\text{FMnProf} := \text{MnProf}_{T_{M}} & & \text{FCpProf} := \text{CpProf}_{T_{C}} & & \text{FTrProf} := \text{TrProf}_{T_{T}}
\end{array}
\]

In other words, \( \text{FMnProf} \rightarrow \text{MnProf} \) is a fully faithful local equivalence, meaning it can be identified with the full sub-equipment of \( \text{MnProf} \) spanned by the monoidal categories which are \textit{free on a set}; similarly for \( \text{FCpProf} \) and \( \text{FTrProf} \). We will write

\[
\begin{array}{ccc}
\text{FMnCat} := \text{FMnProf}_{0} = \text{Set}_{T_{M}} & & \text{FCpCat} := \text{FCpProf}_{0} = \text{Set}_{T_{C}} & & \text{FTrCat} := \text{FTrProf}_{0} = \text{Set}_{T_{T}}
\end{array}
\]

for the vertical 1-categories. Note that each of these categories has a terminal object.

**Definition 3.5.3.** A monoidal (resp. compact or traced) category \( \mathcal{M} \) is \textit{objectwise-free} if there is a set \( \emptyset \) and a bijective-on-objects functor \( F_{M}(\emptyset) \rightarrow \mathcal{M} \), (resp. \( F_{C}(\emptyset) \rightarrow \mathcal{C} \) or \( F_{T}(\emptyset) \rightarrow \mathcal{T} \)). Denote by

\[
\begin{array}{ccc}
\text{MnFrObCat} \subseteq \text{MnCat} & & \text{CpFrObCat} \subseteq \text{CpCat} & & \text{TrFrObCat} \subseteq \text{TrCat}
\end{array}
\]

the full 2-subcategories spanned by the objectwise-free monoidal (resp. compact or traced) categories. In other words, using Definition 3.5.1 we may write

\[
\begin{array}{ccc}
\text{MnFrObCat} := \text{Vert} \left( \text{MnProf}_{T_{M}}^{\text{bo}} \right) & & \text{CpFrObCat} := \text{Vert} \left( \text{CpProf}_{T_{C}}^{\text{bo}} \right) & & \text{TrFrObCat} := \text{Vert} \left( \text{TrProf}_{T_{T}}^{\text{bo}} \right)
\end{array}
\]

\(^{6}\)These three adjunctions in fact extend to 2-adjunctions; see Corollary A.1.4.

\(^{7}\)Note that \( T_{M} \) is not the free-commutative-monoid monad, even though the objects of \( \text{MnCat} \) are \textit{symmetric} monoidal categories, because the symmetries are encoded by natural isomorphisms, not equalities.
Remark 3.5.4. Definition 3.5.3 defines objectwise-free monoidal categories, which are also known as (colored) PROPs (see, e.g. [12] for more on PROPs). However, the morphisms between PROPs are more restrictive than those defined above, because they must "send colors to colors". To define an equipment of PROPs, consider the functor $F_M: \text{Set} \to \text{MnCat}$ and let $\text{PROP} := F_*^\text{MnProf}$ be the induced equipment. Similarly, one can define traced and compact (colored) PROPs as $F_*^\text{TrProf}$ and $F_*^\text{CpProf}$ respectively.

Although we will not prove it here, one can prove a variant of Theorem A, namely that there is an equivalence of categories

$$\int_{O \in \text{Set}} (\text{Cob}$/O)–\text{Alg} \to \text{CpPROP} \quad \text{and} \quad \int_{O \in \text{Set}} (\text{Cob}$/O)–\text{Alg} \to \text{TrPROP}. $$

See [14] for another approach to compact PROPs.

As a consequence of Proposition 3.5.2 we obtain the following.

Corollary 3.5.5. There are fully faithful local equivalences of equipments, in the left column, and equivalences of 2-categories, in the right column:

| $\text{Mod}(F\text{MnProf}) \to \text{MnProf}$ | $\text{Mon}(F\text{MnProf}) \simeq \text{MnFrObCat}$ |
| $\text{Mod}(F\text{TrProf}) \to \text{TrProf}$ | $\text{Mon}(F\text{TrProf}) \simeq \text{TrFrObCat}$ |
| $\text{Mod}(F\text{CpProf}) \to \text{CpProf}$ | $\text{Mon}(F\text{CpProf}) \simeq \text{CpFrObCat}$ |

Proof. The left column comes from the second part of Proposition 3.5.2, while the right column follows by applying $\text{Vert}$ to the equivalence $\text{Mod}(\mathcal{D}_T) \simeq \mathcal{D}_T^\text{b}$ in the first part of Proposition 3.5.2. $\square$

Lemma 3.5.6. There are equivalences of fibrations

$$\text{CPsh}(F\text{CpProf}) \sim \text{Mon}(F\text{CpProf}) \quad \text{and} \quad \text{CPsh}(F\text{TrProf}) \sim \text{Mon}(F\text{TrProf})$$

Proof. The equipment functor $\varphi_C: F\text{CpProf} \to \text{CpProf}$ (resp. $\varphi_T: F\text{TrProf} \to \text{TrProf}$) is by definition a local equivalence, and it preserves terminal objects in the vertical category. Thus using Lemma 2.3.3 and Lemma 2.2.2 we construct the desired equivalence $\text{CPsh}(F\text{CpProf}) \to \text{Mon}(F\text{CpProf})$ as the pullback along $\varphi_C$ of the equivalence $\text{CPsh}(\text{CpProf}) \to \text{Mon}(\text{CpProf})$ from Theorem 3.3.1 (resp. for the traced case). $\square$

We never said that $\text{Cob}/_O$ is the free compact category on $O$!!! Did’nt we? See comment in intro.

Theorem A. There are equivalences of 1-categories

$$\int_{O \in \text{Set}_C} (\text{Cob}$/O)–\text{Alg} \to \text{CpFrObCat} \quad \text{and} \quad \int_{O \in \text{Set}_T} (\text{Cob}$/O)–\text{Alg} \to \text{TrFrObCat}. $$

38
3.6. Axiom-free definition of traced categories

Proof. First note that, essentially by definition, there are isomorphisms of fibrations

\[ \int \left( \text{Cob}/_O \right) \rightarrow \text{Alg} \cong \text{CPsh}(\text{FTrProf}) \]

and

\[ \int \left( \text{Cob}/_O \right) \rightarrow \text{Alg} \cong \text{CPsh}(\text{FCpProf}) \]

By Lemma 3.5.6, we have equivalences of 1-categories

\[ \text{CPsh}(\text{FTrProf}) \cong \text{Mon}(\text{FTrProf}) \quad \text{and} \quad \text{CPsh}(\text{FCpProf}) \cong \text{Mon}(\text{FCpProf}). \]  

The result now follows from Corollary 3.5.5, which provides equivalences of 2-categories:

\[ \text{Mon}(\text{FTrProf}) \cong \text{TrFrObCat} \quad \text{and} \quad \text{Mon}(\text{FCpProf}) \cong \text{CpFrObCat}. \]

\[ \square \]

3.6 Axiom-free definition of traced categories

The forgetful functors between the categories of structured monoidal categories commute with the underlying set functors, i.e. the following diagram commutes:

\[ \begin{array}{ccc}
\text{TrCat} & \xrightarrow{U_T} & \text{MnCat} \\
\text{CpCat} & \xrightarrow{U_C} & \text{MnCat} \\
\text{Set} & \xrightarrow{U_M} & \text{Set}
\end{array} \]

(33)

Because the functor \( U_{CM} : \text{CpCat} \rightarrow \text{MnCat} \) commutes with the right adjoints of the adjunctions to \( \text{Set} \), i.e. \( U_M U_{CM} = U_C \), it induces a monad morphism \( \alpha : T_M \rightarrow T_C \) (i.e. a natural transformation \( \alpha : T_M \rightarrow T_C \) compatible with the units and multiplications), given by the composition of the natural transformations

\[
\begin{align*}
T_M & \xrightarrow{U_M F_M} U_M F_M U_C F_C \\
T_C & \xleftarrow{U_M U_{CM} F_C} U_M F_M U_M U_{CM} F_C \\
\end{align*}
\]

The component \( \alpha_0 \) of this transformation is simply the evident inclusion of the free monoid on a set \( 0 \) into the free monoid-with-involution on \( 0 \). The monad map \( \alpha \) induces a functor between the Kleisli categories:

\[ F_{MC} : \text{Set}_{T_M} \rightarrow \text{Set}_{T_C}. \]
Because the monads $T_M$ and $T_T$ are in fact isomorphic, we have

$$\text{FTrCat} = \text{Set}_T \cong \text{Set}_M = \text{FMnCat}.$$ 

The following proposition defines the 2-category $\text{TrCat}$ of traced categories purely in terms of $\text{CpProf}$, $\text{MnProf}$, and the adjunctions $F_M \dashv U_M$ and $F_C \dashv U_C$. In particular, it does not involve any explicit mention of the trace structure defined in [16].

**Proposition 3.6.1.** Consider the functor $\text{FMnCat} \xrightarrow{FMC} \text{FCpCat} \xrightarrow{kC} \text{CpCat}$ and the induced equipment $\mathcal{F} := (kC \circ FMC)^*(\text{CpProf})$. The equipment $\text{Mod}(\mathcal{F})$ is equivalent to $\text{TrProf}$, and in particular there is an equivalence of 2-categories $\text{Mon}(\mathcal{F}) \simeq \text{TrCat}$.

**Proof.** By combining the definitions of $\text{FTrProf}$ and $\text{TrProf}$, it is easy to see that the following square is a pullback:

\[
\begin{array}{ccc}
\text{FTrProf}_1 & \rightarrow & \text{CpProf}_1 \\
\downarrow & & \downarrow \\
\text{FMnCat} \times \text{FMnCat} & \rightarrow & \text{CpCat} \times \text{CpCat}
\end{array}
\]

Thus we have an equivalence $\mathcal{F} \simeq \text{FTrProf}$, and the result follows by Corollary 3.5.5.  

### Appendix

This section is mostly independent from the rest of the paper. It is really only used to prove Corollary A.3.2, the three biequivalences

$$\text{MnFrObCat} \rightarrow \tilde{\text{MnCat}} \quad \text{TrFrObCat} \rightarrow \tilde{\text{TrCat}} \quad \text{CpFrObCat} \rightarrow \tilde{\text{CpCat}}.$$ 

Here, $\text{MnFrObCat}$ (resp. $\text{TrFrObCat}$ and $\text{CpFrObCat}$) is the 2-category of objectwise-free monoidal (resp. traced and compact) categories and strict functors between them, whereas $\tilde{\text{MnCat}}$ (resp. $\tilde{\text{TrCat}}$ and $\tilde{\text{CpCat}}$) is the 2-category of monoidal (resp. traced and compact) categories with arbitrary objects and strong functors between them. This result will not be new to experts, but we found it difficult to find in the literature.

#### A.1 Arrow objects and mapping path objects

**Definition A.1.1.** Let $a$ be an object in a 2-category $\mathcal{C}$. An **arrow object** of $a$ is an object $a^2$ together with a diagram

\[
\begin{array}{ccc}
a^2 & \xrightarrow{\text{dom}} & a \\
\downarrow_{\downarrow \kappa} & & \downarrow_{\downarrow \kappa} \\
\text{cod} & & \text{cod}
\end{array}
\]

which is universal among such diagrams: any diagram as on the left below factors uniquely as on the right

\[
\begin{array}{ccc}
x & \xrightarrow{\downarrow \kappa} & a \\
\downarrow_{\downarrow \kappa} & & \downarrow_{\downarrow \kappa} \\
x & \xrightarrow{\downarrow \kappa} & a^2 \\
\downarrow_{\downarrow \kappa} & & \downarrow_{\downarrow \kappa} \\
\end{array}
\]
Moreover, given a commutative square in \( C(x,a) \), i.e. another \( d': x \to a, c': x \to a \), \( a': d' \Rightarrow c' \) as on the left above, and 2-cells \( \beta: d \Rightarrow d' \) and \( \gamma: c \Rightarrow c' \) such that \( a' \circ \beta = \gamma \circ a \), there is a unique \( (\beta, \gamma): \hat{a} \Rightarrow \hat{a'} \) such that \( \text{dom}(\beta, \gamma) = \beta \) and \( \text{cod}(\beta, \gamma) = \gamma \).

We say that \( C \) has arrow objects if an arrow object \( c^2 \) exists for each object \( c \in C \).

**Example A.1.2.** The 2-categories \( \text{Cat}, \text{Cat}_\cong, \text{MnCat}, \text{TrCat}, \) and \( \text{CpCat} \) have arrow objects. Clearly for an object \( A \in \text{Cat} \), the usual arrow category \( A^2 \) of arrows and commutative squares, has the necessary universal property. Similarly, the arrow category of \( A \) in \( \text{Cat}_\cong \) is the category whose objects are isomorphisms in \( A \), and whose morphisms are commutative squares (in which the other morphisms need not be isomorphisms).

Arrow objects in \( \text{MnCat} \) are preserved by the forgetful functor to \( \text{Cat} \). If \( (\mathcal{M}, I, \otimes) \) is a monoidal category then the arrow object \( \mathcal{M}^2 \) (in \( \text{Cat} \)) has a natural monoidal product

\[
\mathcal{M}^2 \times \mathcal{M}^2 \cong (\mathcal{M} \times \mathcal{M})^2 \overset{\otimes_2}{\longrightarrow} \mathcal{M}^2,
\]

and monoidal unit given by the identity map \( \text{id}_I \) on the unit of \( \mathcal{M} \). The maps \( \text{dom}, \text{cod}: \mathcal{M}^2 \to \mathcal{M} \) are strict monoidal functors, and the transformation \( \kappa: \text{dom} \to \text{cod} \) is monoidal as well. Suppose given a diagram of strong monoidal functors:

\[
\begin{array}{ccc}
\mathcal{X} & \overset{\alpha}{\triangleright} & \mathcal{M} \\
\downarrow^{d} & & \downarrow^{c} \\
\mathcal{Y} & \overset{\gamma}{\triangleright} & \mathcal{M}
\end{array}
\]

The universal properties of the arrow object \( \mathcal{M}^2 \) in \( \text{Cat} \) guarantee that the induced functor \( \hat{\alpha}: \mathcal{X} \to \mathcal{M}^2 \) is strong monoidal. Note that if \( d, c \) are strict monoidal functors then \( \hat{\alpha} \) will be as well.

The 2-category \( \text{CpCat} \) has arrow objects, which are preserved by the functor \( \text{CpCat} \to \text{Cat}_\cong \). Recall from Section 3.1 that every natural transformation between compact categories is an isomorphism. Thus for a compact category \( \mathcal{C} \), the arrow category \( \mathcal{C}^2 \) has as objects the isomorphisms \( a \Rightarrow b \) in \( \mathcal{C} \), and as morphisms the commuting squares. This is compact: the dual of \( a \Rightarrow b \) is \( a^* \Rightarrow (f^{-1})^* b^* \).

The 2-morphisms between traced categories are also defined to be isomorphisms (see Remark 3.1.2). For a traced category \( \mathcal{T} \in \text{TrCat} \), the arrow object \( \mathcal{T}^2 \) has the isomorphisms in \( \mathcal{T} \) as objects and commuting squares as morphisms; i.e. it too is preserved by the 2-functor \( \text{TrCat} \to \text{Cat}_\cong \). To see the traced structure of \( \mathcal{T}^2 \), suppose given objects \( a: A \Rightarrow A', b: B \Rightarrow B', \) and \( u: U \Rightarrow U' \), as well as a morphism \( (f, g): a \otimes u \to b \otimes u \) as in the diagram to the left

\[
\begin{align*}
A \otimes U & \xrightarrow{f} B \otimes U \\
A' \otimes U' & \xrightarrow{g} B' \otimes U'
\end{align*}
\]

\[
\begin{align*}
A & \xrightarrow{Tr_{A,B}(f)} B \\
A' & \xrightarrow{Tr_{A',B'}(g)} B'
\end{align*}
\]
Composing with \(\text{id}_{B'} \otimes u^{-1}\), we have
\[
(id_{B'} \otimes u^{-1}) \circ (b \otimes u) \circ f = (id_{B'} \otimes u^{-1}) \circ g \circ (a \otimes u)
\]
as morphisms \(A \otimes U \to B' \otimes U\). The commutativity of the right-hand diagram in (35) follows from this equation and the axioms of traced categories [16].

**Lemma A.1.3.** Let \(R : \mathcal{C} \to \mathcal{D}\) be a 2-functor, and suppose that \(\mathcal{C}\) has arrow objects. Then \(R\) has a left 2-adjoint if and only if \(R\) has a left 1-adjoint and \(R\) preserves arrow objects.

**Proof.** First suppose \(R\) has a left 1-adjoint \(L\) and preserves arrow objects. We want to show that given morphisms \(f, g : D \to RC\) in \(\mathcal{D}\) and a 2-cell \(\alpha : f \Rightarrow g\), there is a unique \(\alpha'\) in \(\mathcal{C}\) such that \(R(\alpha')\eta_D = \alpha\). From the 1-adjunction, we know there are unique \(f', g' : LD \to C\) such that \(Rf' \circ \eta_D = f\) and \(Rg' \circ \eta_D = g\). Using the arrow object \(R(C^2) = (RC)^2\), there is a unique morphism \(\hat{\alpha} : D \to RC^2\) such that \(\kappa_{RC} \hat{\alpha} = \alpha\). Using the 1-adjunction again, there is a unique \(\hat{\alpha}' : RLD \to C^2\) such that \(R\hat{\alpha}' \circ \eta_D = \hat{\alpha}\). Finally, we let \(\alpha' := \kappa_C \hat{\alpha}'\), and check
\[
R(\alpha')\eta_D = R(\kappa_C)R(\hat{\alpha}')\eta_D = \kappa_{RC} \hat{\alpha} = \alpha.
\]
It is clear that this \(\alpha'\) is the unique such 2-cell.

Conversely, it is easy to check that if \(R\) has a left 2-adjoint, then \(R\) preserves arrow objects (right adjoints preserve limits).

The following result was promised above; see (31).

**Corollary A.1.4.** There are 2-adjunctions
\[
F_M : \mathcal{C} \leftrightarrow \mathcal{MnCat} : U_M \quad F_T : \mathcal{C} \leftrightarrow \mathcal{T rCat} : U_T \quad F_C : \mathcal{C} \leftrightarrow \mathcal{C pCat} : U_C
\]
that extend the 1-adjunctions constructed in [2].

**Proof.** Let \(R\) be either \(U_M\), \(U_T\), or \(U_C\). Its underlying 1-functor has a left adjoint, constructed in [2]. We showed in Example A.1.2 that \(\mathcal{MnCat}\), \(\mathcal{T rCat}\), and \(\mathcal{C pCat}\) have arrow objects, which are preserved by \(R\). The result follows by Lemma A.1.3.

**Definition A.1.5.** Let \(f : a \to b\) be a morphism in a 2-category \(\mathcal{C}\). A **mapping path object** of \(f\) is an object \(P(f)\) together with a diagram

\[
\begin{array}{c}
P(f) \\
\pi_a \\
\pi_b \\
\end{array} \quad \quad \begin{array}{c}
\pi_b \\
\rho \\
\pi_a \\
\end{array}
\]

\[
\begin{array}{c}
a \\
\rho \cong \\
b \\
f
\end{array}
\]
where $\rho$ is an isomorphism, which is universal among such diagrams: any diagram as on the left below, in which $\alpha$ is an isomorphism, factors uniquely as on the right

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{g} & & \downarrow{h} \\
\alpha \equiv & & \\
\downarrow{f} & & \downarrow{b}
\end{array}
= \begin{array}{ccc}
A & \xrightarrow{P(f)} & B \\
\downarrow{\pi_x} & & \downarrow{\pi_y} \\
\pi \equiv & & \\
\downarrow{\rho} & & \downarrow{\beta}
\end{array}
\]

Moreover, given another $g': x \to a, h': x \to b, \alpha': fg \equiv h'$ as on the left above, and isomorphisms $\beta: g \equiv g'$ and $\gamma: h \equiv h'$ such that $\alpha' \circ f \beta = \gamma \circ \alpha$, there is a unique isomorphism $(\beta, \gamma): \hat{\alpha} \equiv \hat{\alpha}'$ such that $\pi_x(\beta, \gamma) = \beta$ and $\pi_y(\beta, \gamma) = \gamma$.

We say that $\mathcal{C}$ has mapping path objects if a mapping path object $P(f)$ exists for each morphism $f: a \to b$ in $\mathcal{C}$.

**Example A.1.6.** The 2-categories $\mathbf{Cat}, \mathbf{Cat}_{\equiv}, \mathbf{MnCat}, \mathbf{TrCat}$, and $\mathbf{EpCat}$ have mapping path objects. For a morphism $F: \mathcal{A} \to \mathcal{B}$ in $\mathbf{Cat}$, the mapping path category $P(F)$ is a cousin to the comma category $(F \downarrow \text{id}_\mathcal{B})$: the objects are triples $\text{Ob}(P(F)) := \{(A, B, i) \mid A \in \text{Ob}(\mathcal{A}), B \in \text{Ob} (\mathcal{B}), i: F(A) \cong B \text{ is an isomorphism}\}$

and a morphism $(A, B, i) \to (A', B', i')$ in $P(F)$ consists of a pair of morphisms $A \to A'$ in $\mathcal{A}$ and $B \to B'$ in $\mathcal{B}$ such that the evident diagram commutes. The 2-category $\mathbf{Cat}_{\equiv}$ has exactly the same mapping path objects as $\mathbf{Cat}$.

The mapping path object of a strong functor $F: \mathcal{A} \to \mathcal{B}$ between monoidal, traced, or compact categories exists and is preserved by the forgetful functors to $\mathbf{Cat}$ and $\mathbf{Cat}_{\equiv}$. In the monoidal case, the mapping path object $P(F)$ of the functor between underlying categories has a canonical monoidal structure, e.g.,

\[(A, B, i) \otimes (A', B', i') := (A \otimes A', B \otimes B', (i \otimes i') \circ \mu_{A,A'}^{-1})\]

where $\mu_{A,A'}$ is the coherence isomorphism for $F$. The projection functors $A \xleftarrow{\pi_A} P(F) \xrightarrow{\pi_B} B$ are strict. Given a diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{G} & \mathcal{X} \\
\downarrow{\alpha} & & \downarrow{H} \\
\mathcal{F} & \xrightarrow{\beta} & \mathcal{B}
\end{array}
\]

in which $G$ and $H$ are strong (resp. strict) monoidal functors, the induced functor $\hat{\alpha}: \mathcal{X} \to P(F)$, given on objects by $x \mapsto (G(x), H(x), \alpha_x)$, will be strong (resp. strict) as well.

If $\mathcal{A}$ and $\mathcal{B}$ are traced categories and $F$ is a traced functor, one obtains a canonical trace structure on the monoidal category $P(F)$ in a manner similar to that shown in Example A.1.2. If $\mathcal{A}$ and $\mathcal{B}$ are compact categories, then the mapping path monoidal category $P(F)$ is naturally compact: the dual of $(A, B, i)$ is $(A^*, B^*, (i^{-1})^*)$. 

43
Remark A.1.7. The arrow objects and mapping path objects for the 2-categories $\tilde{\mathcal{M}}\text{ncat}$, $\tilde{\mathcal{T}}\text{rcat}$, and $\tilde{\mathcal{C}}\text{pcat}$ were discussed in Examples A.1.2 and A.1.6. Each has a notion of cone, in fact a certain weighted limit cone in $\mathcal{C}\text{at}$, though we will not discuss that notion here. We mentioned in passing that the structure morphisms for that cone are strict monoidal functors and that they “preserve and jointly detect” strictness in the sense of Definition A.2.1 below. In particular, the 2-categories $\mathcal{M}\text{ncat}$, $\mathcal{T}\text{rcat}$, and $\mathcal{C}\text{pcat}$ (objectwise-free monoidal categories and strict functors) also have arrow objects and mapping path objects, and the inclusions of strict-into-strong (e.g., $\mathcal{M}\text{ncat} \to \tilde{\mathcal{M}}\text{ncat}$) preserve them. Looking back at Examples A.1.2 and A.1.6, we see that the forgetful functors

$$U_M : \mathcal{M}\text{ncat} \to \mathcal{C}\text{at} \quad U_T : \mathcal{T}\text{rcat} \to \mathcal{C}\text{at} \cong \quad U_C : \mathcal{C}\text{pcat} \to \mathcal{C}\text{at} \cong$$

preserve arrow objects and mapping path objects.

Definition A.1.8. Say that a morphism $f : a \to b$ in a 2-category $\mathcal{C}$ is a *surjective equivalence* if it can be extended to an adjoint equivalence $g \dashv f$ in which the unit is the identity. That is, there is a morphism $g : b \to a$ and 2-cell $\epsilon : gf \cong 1_a$ such that $f g = 1_b$, $eg = 1_g$, and $fe = 1_f$.

Lemma A.1.9. Let $f : a \to b$ and $g : b \to a$ be morphisms in a 2-category such that $f g = 1_b$. Then $f$ (together with $g$) is a surjective equivalence if and only if $f$ is fully faithful in the sense of Definition 2.4.15.

Proof. Suppose $g \dashv f$ is a surjective equivalence. Then for any $x, f^* : \mathcal{C}(x, a) \to \mathcal{C}(x, b)$ is an equivalence of categories, hence fully faithful. Thus $f$ is fully faithful.

Conversely suppose $f$ is fully faithful. Then because $f g f = f = f 1_a$, there is a unique $e : g f \Rightarrow 1_a$ such that $fe = 1_f$. It is easy to check that $e$ is an isomorphism, and that $eg = 1_g$. \qed

Lemma A.1.10. For any morphism $f : a \to b$ with a mapping path object $P(f)$, the projection $\pi_a : P(f) \to a$ is a surjective equivalence, hence fully faithful.

Proof. By the universal property of $P(f)$ there is a unique morphism $s : a \to P(f)$ such that

$$\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow{1_f} & \searrow{1_f} & \\
a & \xrightarrow{f} & b
\end{array} = \begin{array}{ccc}
a & \xrightarrow{s} & P(f) \\
\downarrow{\pi_a} & \searrow{\rho} & \\
a & \xrightarrow{f} & b
\end{array}$$

Because $\pi_a s \pi_a = \pi_a$ and $\pi_b s \pi_a = f \pi_a \cong \pi_b$, we can use the 2-dimensional universality of $P(f)$ to obtain a unique isomorphism $e : s \pi_a \cong 1_{P(f)}$ such that $\pi_a e = 1_{\pi_a}$ and $\pi_b e = \rho$. 44
By 2-dimensional universality once more, we obtain $\epsilon s = 1_s$ from the following facts

$$\pi a \epsilon s = 1_{\pi a s} \quad \text{and} \quad \pi b \epsilon s = \rho s = 1_f = \pi b 1_s.$$  

It follows from Lemma A.1.9 that $\pi a$ is fully faithful.

### A.2 Strict vs. strong morphisms

Between monoidal categories, there are several notions of functor: strict, strong, lax, and colax. While researchers tend to be most interested in the 2-category $\tilde{\mathcal{M}}$ of monoidal categories and strong functors, and similarly $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{C}}$, the strict functors are theoretically important. In this section, we will present a formal framework which abstracts our examples of interest, and which provides tools for working with and connecting strict and strong morphisms.

In the case of monoidal categories, there is an inclusion $\iota: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ as well as a forgetful functor $\tilde{\mathcal{M}} \rightarrow \mathcal{C}$. The cases of traced and compact monoidal categories are similar, except there we can factor the forgetful functor through the 2-category (or, if one prefers, the (2,1)-category) $\mathcal{C} \simeq \mathcal{C}$ of categories, functors, and natural isomorphisms. In these examples, we will want to be able to represent strong functors in terms of strict ones, by means of a left adjoint to the inclusion of strict into strong. In Definition A.2.2 we will enumerate properties which are sufficient to prove the existence of this left adjoint, and which are satisfied by all of our motivating examples; see Example A.2.4.

**Definition A.2.1.** Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be 2-categories and let $\iota: \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ be a 2-functor that is identity-on-objects, faithful, and locally fully faithful. We say that the triple $(\mathcal{D}, \tilde{\mathcal{D}}, \iota)$ has mapping path objects if $\tilde{\mathcal{D}}$ has mapping path objects as in Definition A.1.5 such that

1. for any $f: a \rightarrow b$ in $\tilde{\mathcal{D}}$, the structure morphisms $a \xleftarrow{\pi a} P(f) \xrightarrow{\pi b} b$ are in $\mathcal{D}$, and
2. the pair $(\pi a, \pi b)$ preserves and jointly detects morphisms in $\mathcal{D}$ in the following sense: for any morphism $\ell: x \rightarrow P(f)$ in $\tilde{\mathcal{D}}$, we have that $\ell$ is in $\mathcal{D}$ if and only if the compositions $\pi a \circ \ell$ and $\pi b \circ \ell$ are in $\mathcal{D}$.

We say that the triple $(\mathcal{D}, \tilde{\mathcal{D}}, \iota)$ has arrow objects if the analogous conditions hold.

For the following definition, one may keep in mind the case $\mathcal{D} = \mathcal{M}$, $\tilde{\mathcal{D}} = \tilde{\mathcal{M}}$, and $\mathcal{C} = \mathcal{C}$. See Example A.2.4 below.

**Definition A.2.2.** Let $\mathcal{D}, \tilde{\mathcal{D}}$, and $\mathcal{C}$ be 2-categories, and let $U: \tilde{\mathcal{D}} \rightarrow \mathcal{C}$ and $\iota: \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ be 2-functors. We say that the collection $(\mathcal{D}, \tilde{\mathcal{D}}, \mathcal{C}, U, \iota)$ admits strong morphism classifiers if it satisfies the following properties:

1. The 2-category $\mathcal{D}$ has a bijective-on-objects/fully faithful factorization.
2. The functor $\iota$ is identity-on-objects, faithful, and locally fully faithful.
3. The triple \((\mathcal{D}_s, \tilde{\mathcal{D}}, \iota)\) has both arrow objects and mapping path objects (Definition A.2.1).

4. The functor \(U\iota: \mathcal{D}_s \to \mathcal{C}\) has a left 2-adjoint \(F\).

5. The functor \(U\iota\) preserves fully faithful morphisms (equivalently, \(F\) preserves bijective-on-objects morphisms).

6. The functor \(U\) preserves mapping path objects.

7. The functor \(U\) reflects identity 2-cells.

8. The pair \((U\iota, U)\) creates surjective equivalences: given any morphism \(f: A \to B\) in \(\mathcal{D}_s\) and surjective equivalence \(g \dashv U\iota(f)\) in \(\mathcal{C}\), there is a unique surjective equivalence \(\tilde{g} \dashv \iota f\) in \(\tilde{\mathcal{D}}\) such that \(U\tilde{g} = g\).

Remark A.2.3. Other than those involving bijective-on-objects or fully faithful morphisms, all of the properties enumerated in Definition A.2.2 (namely, Properties 2, 3, 4, 6, 7, and 8) hold whenever \(\mathcal{D}_s\) is the 2-category of strict algebras and strict morphisms for a 2-monad on \(\mathcal{C}\), and \(\tilde{\mathcal{D}}\) is the 2-category of strict algebras and pseudo-morphisms. While our main examples can be seen to be algebras for some 2-monad, we have found it easier to isolate just those properties we needed to prove Theorem A.2.5.

This section was strongly inspired by [9] and [21].

Example A.2.4. Suppose that the collection \((\mathcal{D}_s, \tilde{\mathcal{D}}, \iota, U)\) is defined as in one of the following cases:

- \(\mathcal{D}_s = \text{MnCat}, \quad \tilde{\mathcal{D}} = \tilde{\text{MnCat}}, \quad \mathcal{C} = \text{Cat}, \) where \(\iota: \mathcal{D}_s \to \tilde{\mathcal{D}}\) is the inclusion and \(U: \tilde{\mathcal{D}} \to \mathcal{C}\) is the forgetful functor;
- \(\mathcal{D}_s = \text{TrCat}, \quad \tilde{\mathcal{D}} = \tilde{\text{TrCat}}, \quad \mathcal{C} = \text{Cat}_{\sim}, \) where \(\iota: \mathcal{D}_s \to \tilde{\mathcal{D}}\) is the inclusion and \(U: \tilde{\mathcal{D}} \to \mathcal{C}\) is the forgetful functor; or
- \(\mathcal{D}_s = \text{CpCat}, \quad \tilde{\mathcal{D}} = \tilde{\text{CpCat}}, \quad \mathcal{C} = \text{Cat}_{\sim}, \) where \(\iota: \mathcal{D}_s \to \tilde{\mathcal{D}}\) is the inclusion and \(U: \tilde{\mathcal{D}} \to \mathcal{C}\) is the forgetful functor.

We will now show that in each case the collection admits strong morphism classifiers.

Property 1 is proved as Lemma 2.4.16 and the exactness of \(\text{MnProf}, \text{TrProf}, \) and \(\text{CpProf}\); see Section 3.4. Property 2 is obvious for \(\text{MnCat}\) and \(\text{CpCat}\), and by definition (see Remark 3.1.2) for \(\text{TrCat}\). Property 3 is shown in Remark A.1.7. Property 4 is shown in Corollary A.1.4. Property 5 is a consequence of Lemma 2.4.16 and Propositions 3.4.1, 3.4.2, and 3.4.5. Property 6 is shown in Remark A.1.7. Property 7 is obvious: if \(\alpha: F \to G\) is a 2-cell in \(\mathcal{D}_s\) whose underlying natural transformation (in \(\text{Cat}\)) is the identity then it is the identity. It remains to prove Property 8; we first treat the case \(\mathcal{D}_s = \text{MnCat}\).

Suppose that \(F: A \to B\) is a strict monoidal functor and that there is a surjective equivalence \(g \dashv U\iota(F)\) in \(\text{Cat}\). Let \(f: a \to b\) denote \(U\iota(F)\), so \(g: b \to a\). By Definition A.1.8, we have a 2-cell \(\epsilon: gf \cong 1_a\) and equalities \(1_b = fg, \epsilon g = 1_g\) and \(f \epsilon = 1_f\).
A strong functor $G: B \to A$ with $UG = g$ of course acts the same as $g$ on objects and morphisms. Thus it suffices to give the coherence isomorphisms $\mu: I_A \congto G(I_B)$ and $\mu_{x,y}: G(x) \otimes G(y) \congto G(x \otimes y)$ for objects $x, y \in B$, which satisfy the required equations. Define $\mu$ to be the composite $I_A \xrightarrow{\epsilon^{-1}} g f(I_A) = g(I_B)$, and define $\mu_{x,y}$ to be the composite $g x \otimes g y \xrightarrow{\epsilon^{-1}} g(f g x \otimes f g y) = g(x \otimes y)$.

The requisite equations can be checked by direct computation, though they actually follow from a more general theory (doctrinal adjunctions); see [18].

Property 8 holds for the case $D_s = C_p\text{Cat}$ because it is a full subcategory of $\text{MnCat}$. For the case $D_s = \text{TrCat}$, suppose given a strict traced functor $F: A \to B$, and let $G: B \to A$ be the associated monoidal functor constructed above. To see that it is traced, note that $B \xrightarrow{G} A \xrightarrow{F} B$ is the identity, so $G$ is fully faithful, and the result follows from Remark 3.1.3.

Since $\iota$ is identity on objects, we often suppress it for convenience. We draw ordinary arrows $\cdot \to \cdot$ for morphisms in $D_s$ and snaked arrows, $\cdot \xrightarrow{\sim} \cdot$ for morphisms in $\bar{D}$.

**Theorem A.2.5.** Suppose that $(D_s, \bar{D}, \iota, U, \iota)$ admits strong morphism classifiers. Then the functor $\iota$ has a left 2-adjoint $Q: \bar{D} \to D_s$. The counit $q_A: QA \to A$ of this adjunction is given by factoring the counit $\epsilon_A$ of the $F \dashv U \iota$ adjunction:

\[ \begin{array}{ccc}
FU A & \xrightarrow{r_A} & QA \\
\downarrow \epsilon_A & & \downarrow q_A \\
A & & 
\end{array} \]  

(36)

**Proof.** Define $Q$, $r$, and $q$ as in (36). We will begin by showing that $q_A$ is a surjective equivalence for any $A$, whose inverse $p_A: A \xrightarrow{\sim} QA$ will become the unit of the $Q \dashv \iota$ adjunction. We write $U$ to denote $U \iota$, in a minor abuse of notation. Because $U$ creates surjective equivalences, it suffices to show that $Uq_A$ is a surjective equivalence. But $q_A$ is fully faithful by construction, so $Uq_A$ is fully faithful, hence by Lemma A.1.9 it suffices to construct a section of $Uq_A$. We can easily check that $Ur_A \circ \eta_{UA}$ is such a section:

\[ \begin{array}{ccc}
UA & \xrightarrow{\eta_{UA}} & UFUA \\
\downarrow \iota_{UA} & & \downarrow U r_A \\
UQA & & U A \\
\downarrow U q_A & & \\
U A & & 
\end{array} \]

Thus there is a unique surjective equivalence $p_A \dashv q_A$ in $\bar{D}$ such that $Up_A = Ur_A \circ \eta_{UA}$.

We next must show that the morphism $p_A: A \xrightarrow{\sim} QA$ has the following universal property: for any morphism $f: A \xrightarrow{\sim} B$ in $\bar{D}$, there is a unique morphism $f': QA \to B$ in $D_s$ for which $f = f' \circ p_A$. From this, it follows that $Q$ extends to a 1-functor which is left adjoint to $\iota$. It then follows from Lemma A.1.3 that $Q$ extends to a 2-functor which is left 2-adjoint to $\iota$, completing the proof of the theorem.
A.2. Strict vs. strong morphisms

First, given an \( f: A \rightarrow B \), define a morphism \( \tilde{f}: FA \rightarrow P(f) \) in \( D_s \) as the adjoint of the section \( \tilde{s}_{uf}: UA \rightarrow U(P(f)) = P(Uf) \) defined as in Lemma A.1.10, i.e., \( \tilde{f} := \epsilon_{P(f)} \circ F(s_{uf}) \). It follows by adjointness that the following diagram in \( D_s \) commutes:

\[
\begin{array}{ccc}
FA & \xrightarrow{\tilde{f}} & FUB \\
\downarrow{\epsilon_A} & & \downarrow{\epsilon_B} \\
A & \xrightarrow{\pi_A} & P(f) & \xrightarrow{\pi_B} & B
\end{array}
\]

Then by orthogonality there is a unique morphism \( \tilde{f} \) in the diagram

\[
\begin{array}{ccc}
FA & \xrightarrow{\tilde{f}} & P(f) & \xrightarrow{\pi_B} & B \\
AQ & \xrightarrow{\tilde{f}} & A & \xrightarrow{\pi_A} & QA
\end{array}
\]

making the square commute, and we define \( f' := \pi_B \circ \tilde{f} \).

We next must check that our definition of \( f' \) satisfies \( f = f' \circ p_A \). We can construct an isomorphism 2-cell \( f \cong f' \circ p_A \):

\[
\begin{array}{ccc}
QA & \xrightarrow{\tilde{f}} & P(f) \\
\downarrow{p_A} & & \downarrow{\pi_B} \\
A & \xrightarrow{\pi_A} & \cong & \xrightarrow{\pi_B} & B
\end{array}
\]

We can check directly that the underlying 2-cell of \( \rho_f \tilde{f} p_A \) is the identity on \( Uf \),

\[
U(\rho_f \tilde{f} p_A) = \rho_{Uf} U(\tilde{f}) U(r_A) \eta_{UA} \\
= \rho_{Uf} U(\tilde{f}) \eta_{UA} \\
= \rho_{Uf} s_{uf} \\
= 1_{Uf}.
\]

Since \( U \) reflects identity 2-cells, it follows that \( \rho_f \tilde{f} q_A \) is the identity, \( f = f' \circ p_A \).

Finally, we need to verify that if \( f'' : QA \rightarrow B \) is any other strict morphism such that \( f = f'' \circ p_A \), then \( f'' = f' \). We begin by factoring \( f'' = \pi_B \circ \tilde{f}'' \):

\[
\begin{array}{ccc}
QA & \xrightarrow{\tilde{f}''} & P(f) & \xrightarrow{\pi_B} & B \\
\downarrow{p_A} & & \downarrow{\pi_B} & \cong & \xrightarrow{p_A} \\
A & \xrightarrow{f} & B
\end{array}
\]
A.2. Strict vs. strong morphisms

It will then suffice to show that the diagram

\[
\begin{array}{ccc}
FUA & \xrightarrow{f} & P(f) \\
\downarrow r_A & & \downarrow \pi_A \\
QA & \xrightarrow{q_A} & A
\end{array}
\]

commutes, as then \( f'' = \hat{f} \) by orthogonality, and \( f'' = \pi_B \hat{f}'' = \pi_B \hat{f} = f' \). The lower triangle \( \pi_A \circ \hat{f}'' = q_A \) follows directly from (37). To show that the upper triangle \( \hat{f} = \hat{f}'' \circ r_A \) commutes, it suffices to check equality of the adjoints \( s_{UF} = U(\hat{f}'' \circ r_A) \circ \eta_{UA} \). We will check this using the universal property of the mapping path object \( UP(f) = P(Uf) \) by showing that \( \rho_{UF} U(\hat{f}'' r_A) \eta_{UA} = \rho_{UF} s_{UF} \):

\[
\begin{array}{ccc}
UA & \xrightarrow{\eta_{UA}} & UFUA \\
\downarrow u_{FA} & & \downarrow u_{FA} \\
UQA & \xrightarrow{u_{QA}} & UB \\
\downarrow \pi_{UA} & & \downarrow \pi_{UB} \\
UA & \xrightarrow{u_{uf}} & UB
\end{array}
= \begin{array}{ccc}
UA & \xrightarrow{Uf} & UB \\
\downarrow u_{uf} & & \downarrow u_{uf} \\
UA & \xrightarrow{Uf} & UB
\end{array}
\]

Example A.2.6. Consider the case of Theorem A.2.5 applied to the case where \( \mathcal{D} = \text{MnCat}, \mathcal{D} = \text{MnCat}, \text{and C = Cat}. \) Given any monoidal category \( A \), we construct the monoidal category \( QA \) by factoring the counit:

\[
F_M U_M A \overset{r_A}{\longrightarrow} QA \overset{q_A}{\longrightarrow} A
\]

Concretely, this says that the underlying monoid of objects of \( QA \) is the free monoid on \( \text{Ob}(A) \), and that given two elements \([x_1, \ldots, x_n]\) and \([y_1, \ldots, y_m]\) of the free monoid, the hom set is defined

\[
QA([x_1, \ldots, x_n], [y_1, \ldots, y_m]) := A(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m)
\]
Theorem A.2.5 then says that strong monoidal functors out of $A$ are the same as strict monoidal functors out of $QA$, or more precisely that for any monoidal category $B$ there is an isomorphism of categories $\overline{\text{MnCat}}(A, B) \cong \text{MnCat}(QA, B)$.

The cases of $\text{TrCat}$ and $\text{CpCat}$ are analogous.

### A.3 Objectwise-free monoidal, traced, and compact categories

Our next goal is to show, continuing the assumptions of the Theorem A.2.5, that $\tilde{D}$ is 2-equivalent to the full subcategory of $D_s$ spanned by those objects which are "objectwise-free". To make this precise, we will further assume we have a 1-category $\mathcal{F}$, together with a fully faithful functor $\text{Disc}: \mathcal{F} \to \mathcal{C}_0$ into the underlying category of $\mathcal{C}$ with right adjoint $\text{Ob}$, such that a morphism $f$ in $\mathcal{C}_0$ is bo if and only if $\text{Ob}f$ is an isomorphism. The reader may recognize this situation from Proposition 3.5.2. We will write $D_{\text{FrOb}}$ for the full sub-2-category of $D_s$ spanned by those objects $A$ for which there exists an object $s \in \mathcal{F}$ and a bo morphism $F(\text{Disc}(s)) \to A$. Then we have that:

**Theorem A.3.1.** The composition

$$D_{\text{FrOb}} \hookrightarrow D_s \xrightarrow{\iota} \tilde{D}$$

is a biequivalence of 2-categories.

**Proof.** We first need to show that $\iota$ induces equivalences of categories $D_s(A, B) \cong \tilde{D}(\iota A, \iota B)$ for any $A$ and $B$ which are objectwise-free. In fact, this will hold as long as $A$ is objectwise-free.

If $A \in D_s$ is objectwise-free, then there exists an object $s \in \mathcal{F}$ and a bo morphism $f: F(\text{Disc}(s)) \to A$ in $D_s$. By the $F \dashv U$ adjunction\(^8\), there is a unique morphism $\tilde{f}: \text{Disc}(s) \to UA$ such that $f = \epsilon_A \circ F \tilde{f}$. Factoring $\epsilon_A = q_A \circ r_A$ as in Theorem A.2.5, we obtain by orthogonality a unique lift $p'_A \in D_s$ in the square

$$
\begin{array}{ccc}
F(\text{Disc}(s)) & \xrightarrow{f} & FU A \\
\downarrow f & & \downarrow q_A \\
A & \xrightarrow{p'_A} & A
\end{array}
$$

By Lemma A.1.9, it follows that $q_A$ is an equivalence in $D_s$. Hence composition with $q_A$ induces the left equivalence in

$$D_s(A, B) \xrightarrow{\cong} D_s(Q\iota A, B) \xrightarrow{\cong} \tilde{D}(\iota A, \iota B)$$

and it is easy to check that the composition is precisely $\iota$ on hom categories.

\(^8\)Note that we continue to commit the abuse of notation writing $U$ for $U\iota$. 

50
Finally, to prove essential surjectivity, consider an object $A \in \tilde{D}$. We know that in the factorization

\[
\begin{array}{c}
FUA \\
\downarrow r_A \\
QA \\
\uparrow q_A
\end{array}
\begin{array}{c}
\epsilon_A \\
\end{array}
\begin{array}{c}
A.
\end{array}
\]

$q_A$ is an equivalence in $\tilde{D}$. We will be done if we can show that $QA$ is objectwise-free.

Consider the counit $\epsilon_{UA} : \text{Disc}(\text{Ob}(UA)) \to UA$. Because Disc is fully faithful, it follows that $\text{Ob}(\epsilon_{UA})$ is an isomorphism, hence $\epsilon_{UA} \in \text{bo}$, and therefore $F(\epsilon_{UA}) \in \text{bo}$ as well. Thus we can take the composition

\[
F(\text{Disc}(\text{Ob}(UA))) \xrightarrow{F(\epsilon_{UA})} FUA \xrightarrow{r_A} QA
\]

showing that $QA$ is objectwise-free.

\[\textbf{Corollary A.3.2.} \] The canonical inclusions

\[
\begin{align*}
\text{MnFrObCat} & \rightarrow \tilde{\text{MnCat}} \\
\text{TrFrObCat} & \rightarrow \tilde{\text{TrCat}} \\
\text{CpFrObCat} & \rightarrow \tilde{\text{CpCat}}
\end{align*}
\]

are biequivalences of 2-categories.

\[\textbf{Proof.} \] Let $\mathcal{S} = \text{Set}$, and let $\text{Disc} : \mathcal{S} \rightleftarrows \text{Cat} : \text{Ob}$ be the discrete adjunction. Note that a morphism $f$ in $\text{Cat}$ is $\text{bo}$ if and only if $\text{Ob}(f)$ is an isomorphism. The result follows by Example A.2.4 and Theorem A.3.1.

\[\text{Bibliography}\]

[1] S. Abramsky, Retracing some paths in process algebra, CONCUR’96: Concurrency Theory (1996), pp. 1–17.

[2] S. Abramsky, Abstract scalars, loops, and free traced and strongly compact closed categories, Proceedings of CALCO (2005), pp. 1–31.

[3] S. Abramsky, R. Blute, P. Panangaden, Nuclear and Trace Ideals in Tensored $*$-Categories, Preprint: http://arxiv.org/abs/math/9805102 (1998).

[4] J. Adamek, H. Herrlich, and G.E. Strecker, Abstract and Concrete Categories: The joy of cats, Dover Publications (2009).

[5] J.C. Baez, J. Dolan, Higher-dimensional Algebra and Topological Quantum Field Theory, Journal of Mathematical Physics, 36 (1995), no. 11, pp. 6073–6105.
[6] F. Borceux, Handbook of Categorical Algebra: Volume 1, Basic Category Theory, Cambridge University Press (1994).

[7] F. Borceux, Handbook of Categorical Algebra: Volume 2, Categories and Structures, Cambridge University Press (1994).

[8] F. Borceux, D. Bourn, Mal’cev, protomodular, homological and semi-abelian categories, Mathematics and Its Applications, 566, Kluwer (2004).

[9] J. Bourke, Two-dimensional monadicity (English summary), Advances in Mathematics, 252 (2014), pp. 708–747.

[10] T.M. Fiore, N. Gambino, J. Kock, Monads in double categories (English summary), J. Pure Appl. Algebra, 215 (2011), no. 6, pp. 1174–1197.

[11] P. Freyd, D.N. Yetter, Braided compact closed categories with applications to low-dimensional topology, Advances in Mathematics, 77 (1989), no. 2, pp. 156–182.

[12] P. Hackney, M. Robertson, On the category of props, Preprint: http://arxiv.org/abs/1207.2773v2 (2012).

[13] M. Hasegawa, S-Y. Katsumata, A note on the biadjunction between 2-categories of traced monoidal categories and tortile monoidal categories, Mathematical Proceedings of the Cambridge Philosophical Society, 148 (2010), no. 1.

[14] A. Joyal; J. Kock, Feynman graphs, and nerve theorem for compact symmetric multicategories, Preprint: http://arxiv.org/abs/0908.2675 (2009).

[15] A. Joyal, R. Street, Braided tensor categories, Adv. Math. 102 (1993), no. 1, 200–78.

[16] A. Joyal, R. Street, and D. Verity, Traced monoidal categories, Mathematical Proceedings of the Cambridge Philosophical Society, 119 (1996), no. 3, pp. 447–468.

[17] G.M. Kelly, M.L. Laplaza, Coherence for compact closed categories, Journal of Pure and Applied Algebra, 19 (1980), pp. 193–213.

[18] G.M. Kelly, Doctrinal adjunction, Category Seminar (Proc. Sem., Sydney, 1972/1973), Lecture Notes in Math., 420 (1974) Springer, Berlin, pp. 257–280.

[19] J. Kock, Frobenius algebras and 2D topological quantum field theories, London Mathematical Society Student Texts, 59, Cambridge University Press, Cambridge (2004).

[20] S. Lack, A 2-categories companion (English summary), Towards higher categories, pp. 105–191, IMA Vol. Math. Appl., 152 (2010), Springer, New York.
[21] S. Lack, Homotopy-theoretic aspects of 2-monads (English summary), Journal of Homotopy and Related Structures, 2 (2007), no. 2, pp. 229–260.

[22] T. Leinster, Higher Operads, higher categories, London Mathematical Society Lecture Note Series, 298, Cambridge University Press, Cambridge (2004).

[23] S. MacLane, Categories for the Working Mathematician, Springer Science & Business Media (1978).

[24] S. MacLane, Natural Associativity and Commutativity, Rice University Studies, 49 (1963), no. 4.

[25] G. Miller, The magical number seven, plus or minus two: Some limits on our capacity for processing information, Psychological review, 63 (1956), no. 2, pp. 81.

[26] K. Ponto and M. Shulman, Traces in symmetric monoidal categories, Expositiones Mathematicae, 32 (2014), no. 3, pp. 248–273.

[27] D. Rupel and D. I. Spivak, The operad of temporal wiring diagrams: Formalizing a graphical language for discrete-time processes, Preprint: http://arxiv.org/abs/1307.6894 (2013).

[28] P. Schultz, Regular and exact (virtual) double categories, Preprint: http://arxiv.org/abs/1505.00712 (2015)

[29] M. Shulman, Framed bicategories and monoidal fibrations, Theory and Applications of Categories, 20 (2008), no. 18, pp. 650–738.

[30] D. I. Spivak, The operad of wiring diagrams: Formalizing a graphical language for databases, recursion, and plug-and-play circuits, Preprint: http://arxiv.org/abs/1305.0297 (2013).

[31] D. Vagner, D. I. Spivak, and E. Lerman, Algebras of open dynamical systems on the operad of wiring diagrams, Preprint: http://arxiv.org/abs/1408.1598 (2014).

[32] R. J. Wood, Abstract proarrows. I. Cahiers Topologie Géom. Différentielle, 23 (1982), no. 3, pp. 279–290.

[33] R. J. Wood. Proarrows. II. Cahiers Topologie Géom. Différentielle, 26 (1985), no. 2, pp. 135–168.