Nonmetricity and torsion induced by dilaton gravity in two dimension

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Abstract

We develop a theory in which there are couplings amongst Dirac spinor, dilaton and non-Riemannian gravity and explore the nature of connection-induced dilaton couplings to gravity and Dirac spinor when the theory is reformulated in terms of the Levi-Civita connection. After presenting some exact solutions without spinors, we investigate the minimal spinor couplings to the model and in conclusion we can not find any nontrivial dilaton couplings to spinor.
It is worthwhile to study dilaton gravity theories because they are connected with black holes in effective string models and may be related to higher dimensional gravity theories for some special choices of the dilaton and matter couplings. Two ways can be followed in order to write nontrivial dilaton gravity models in Riemannian spacetimes. In the first method, a simple Lagrangian \( D \)-form, \( L \), with Levi-Civita connection in \( D > 2 \) dimension is guessed and the new \((D-1)\)-form, \( L \), is obtained via Kaluza-Klein dimensional reduction procedure in which the Levi-Civita connection is calculated uniquely from a particular \( D \)-dimensional metric, \( G \), including \((D-1)\)-dimensional metric, \( g \), gauge potentials, \( A \), (components of vector fields) and dilatons, \( \phi \), (scalar fields) [1], [2]:

\[
(G) \quad G = g + f^2(\phi)A \otimes A + f^2(\phi)(dy \otimes A + A \otimes dy) + f^2(\phi)dy \otimes dy \quad \rightarrow \quad (L) \quad L = L \wedge dy
\]

where \( y \) is a coordinate function living only in \( D \)-dimensional manifold. This procedure can be applied successively several times for lower dimensions.

In the second approach, non-Riemannian geometry in which the full connection, \( \Lambda^a_{\ b} \), contains Levi-Civita, \( \omega^a_{\ b} \), torsion, \( T^a \), and nonmetricity, \( Q^a_{\ b} \), contributions is used. Here variational calculation with constraints on nonmetricity and torsion is needed. After writing a non-Riemannian Lagrangian \( D \)-form, firstly the full connection is calculated via the constraint equations as Levi-Civita plus dilaton terms, \( \Lambda \approx \omega + \phi \), and then by inserting the solved connection to the other variational equations, the equations with the standard Levi-Civita connection are obtained. Now one repeats the operations in reverse order, that is, firstly non-Riemannian Lagrangian is decomposed by inserting the solved connection, and then the field equations are derived from the new Lagrangian [3]. In this approach, it is seen that the theory can be rewritten in terms of Levi-Civita and thus torsion and nonmetricity tensors are interpreted as matter induced couplings for Riemannian gravity. This scheme may be applied to supergravity in which models generally includes complicated matter couplings with respect to Levi-Civita connection and these may have a much tidier form when they are reformulated in terms of the full connection with torsion and nonmetricity.

In this paper, by adding nonzero torsion we generalize the paper [3], in which authors showed how theories of dilaton gravity can be constructed in terms of a torsion-free nonmetric connection and found the corresponding renormalization of the connection induced couplings when the theory is reformulated in terms of the Levi-Civita connection. Then we discuss some solutions and finally investigate whether there are nontrivial dilaton couplings to spinor by treating Dirac Lagrangian.
We make use of the following conventions and notations. The signature of the 2-dimensional metric is assumed to be \((-+, +)\). While Latin indices, \(a, b, \cdots = 0, 1\), label the orthonormal frame components, Greek indices, \(\alpha, \beta, \cdots = \hat{0}, \hat{1}\), denote the coordinate frame components. Along with the orthonormal co-frame 1-forms \(e^a\) and the exterior product \(\wedge\), we will use the short hand notation \(e^a \wedge e^b = e^{ab}\). Furthermore, we define the spacetime orientation in terms of the Hodge dual such that \(*1 = e^{01}\) is the volume 2-form.

## 2 Dilaton gravity in non-Riemannian geometry

A two dimensional spacetime consists of a differentiable manifold \(M\) equipped with a Lorentzian metric \(g\) and a linear connection \(\nabla\) that defines parallel transport of vectors (or tensors) and more generally spinors. Given an orthonormal basis \(\{X_a\}\), the metric reads

\[
g = \eta_{ab} e^a \otimes e^b = - e^0 \otimes e^0 + e^1 \otimes e^1
\]

where \(e^a\) is the orthonormal co-frame such that

\[
\iota_a e^b = e^b(X_a) = \delta_a^b.
\]

Here \(\iota_{X_a} \equiv \iota_a\) denotes interior product. The nonmetricity 1-forms, torsion 2-forms and curvature 2-forms are defined by the Cartan structure equations

\[
2Q_{ab} := -\nabla_\eta_{ab} = \Lambda_{ab} + \Lambda_{ba},
\]

\[
T^a := D e^a = d e^a + \Lambda^a_b \wedge e^b,
\]

\[
R^a_{\ b} := D \Lambda^a_{\ b} := d \Lambda^a_{\ b} + \Lambda^a_{\ c} \wedge \Lambda^c_{\ b}
\]

where \(d\), \(D\) denote the exterior derivative and the covariant exterior derivative, respectively. The linear connection \(\nabla\) is determined by the connection 1-forms \(\Lambda^a_{\ b}\) which can be decomposed in a unique way according to \([4]\):

\[
\Lambda^a_{\ b} = \omega^a_{\ b} + K^a_{\ b} + q^a_{\ b} + Q^a_{\ b}
\]

where \(\omega^a_{\ b}\) are the Levi-Civita connection 1-forms:

\[
\omega^a_{\ b} \wedge e^b = -d e^a \quad \text{or} \quad 2\omega_{ab} = -\iota_a(d e_b) + \iota_b(d e_a) + \iota_a \iota_b(d e_c) e^c
\]

\(K^a_{\ b}\) are the contortion 1-forms:

\[
K^a_{\ b} \wedge e^b = T^a \quad \text{or} \quad 2K_{ab} = \iota_a(T_b) - \iota_b(T_a) - \iota_a \iota_b(T_c) e^c
\]
and \( q^{a}{}_{b} \) are the anti-symmetric tensor 1-forms:

\[
q_{ab} = -(i_{a}Q_{bc})e^{c} + (i_{b}Q_{ac})e^{c} .
\] (9)

The literature on the most general non-Riemannian gravity in two-dimension may be found in [5] in which in spite of the inclusion of nonmetricity and torsion together, the author has not discussed the dilaton couplings to his model. Furthermore, while the simplest model of dilaton gravity can be written as the coupling of a dilaton scalar \( \phi \) to the curvature scalar in two-dimension, we will develop a theory with a connection including nonmetricity determined by (same as [3])

\[
Q^{b}{}_{a} = \delta^{b}{}_{a}(kd\phi + l \ast d\phi) .
\] (10)

and torsion determined by

\[
T^{a} = e^{a} \wedge (pd\phi + q \ast d\phi)
\] (11)

where \( k, l, p \) and \( q \) are fundamental coupling constants. In section 6 of the reference [6], the author proved that a general two-dimensional dilaton gravity is equivalent to two-dimensional gravity with torsion in terms of the first order Hamiltonian formulation. We notice that our \textit{ad hoc} anzatz for torsion is motivated by the equations of motion of that analysis. Our theory is based on the Lagrangian 2-form

\[
L = \frac{1}{2} \phi^{2} R^{a}{}_{b} \wedge \ast e^{a}{}_{b} + \frac{\alpha}{2} d\phi \wedge \ast d\phi + \frac{\beta}{2} \phi^{2} \ast 1 + \frac{\mu}{4} Q^{b}{}_{a} \wedge \ast Q^{b}{}_{a} + \frac{\nu}{2} T^{a} \wedge \ast T^{a}
\]

\[
+ \rho^{a}{}_{b} \wedge (Q^{b}{}_{a} - k\delta^{b}{}_{a}d\phi - l\delta^{b}{}_{a} \ast d\phi) + \lambda_{a}(T^{a} - pe^{a} \wedge d\phi - qe^{a} \wedge \ast d\phi)
\] (12)

where \( \alpha, \beta, \mu, \nu \) are coupling constants, \( \rho^{a}{}_{b} \) symmetric Lagrange multiplier 1-forms constraining the nonmetricity to (10) and \( \lambda_{a} \) Lagrange multiplier 0-forms ensuring the torsion constraint (11). In the following \( \rho = \rho^{a}{}_{a} \) and \( \lambda = \lambda_{a}e^{a} \). Variations with respect to \( \Lambda^{a}{}_{b}, \phi \) and \( e^{a} \) give the field equations:

\[
\frac{1}{2} d\phi^{2} \wedge \ast e^{a}{}_{b} + \frac{\phi^{2}}{2}(2Q^{bc} \wedge \ast e_{ac} - Q \wedge \ast e^{a}{}_{b}) + \frac{\mu}{2} \ast Q^{b}{}_{a} - \rho^{b}{}_{a} + \lambda_{a}e^{b} + \nu e^{b} \ast T^{a} = 0
\] (13)

\[
\phi R^{a}{}_{b} \wedge \ast e^{a}{}_{b} - \alpha d \ast d\phi + \beta \phi \ast 1 - k d\rho + l d \ast \rho - p d\lambda + q d \ast \lambda = 0
\] (14)

\[
-\frac{\alpha}{2} \tau_{a}[\phi] + \frac{\beta}{2} \phi^{2} \ast e_{a} - \frac{\mu}{4} \tau_{a}[Q] + l[\ast (i_{a}d\phi) \ast \rho + \ast (i_{a}d\phi) \rho] + \nu D \ast T_{a}
\]

\[-\frac{\nu}{2} (i_{a}T^{b}) \ast T_{b} + D\lambda_{a} - p\lambda_{a}d\phi - q\lambda_{a} \ast d\phi + q[\ast (i_{a}d\phi) \ast \lambda + \ast (i_{a}d\phi) \lambda] = 0
\] (15)

where the stress forms are

\[
\tau_{a}[\phi] = (i_{a}d\phi) \ast d\phi + (i_{a}d\phi) d\phi
\] (16)
while

\[ \tau_a[Q] = (\imath_a Q^b_c) Q^c_b + (\imath_a * Q^c_b) Q^b_c . \]  

(17)

We can solve the multipliers by first inserting (10) and (11) into (13)

\[ \frac{1}{2} d\phi^2 \wedge *e^b_a + \frac{\mu}{2} \eta_{ab}(k * d\phi + l d\phi) - \nu e_b \wedge [p(\imath_a * d\phi + q(\imath_a d\phi))] + \lambda_a e_b - \rho_{ab} = 0 . \]  

(18)

Now after contracting (18) with \( \imath^a \) and with \( \imath^b \) we subtract these equations side by side after relabeling the indices in one of them

\[ \lambda_a = \imath_a (2 \phi * d\phi + \nu p * d\phi + \nu q d\phi) \]  

(19)

and

\[ \lambda = 2 \phi * d\phi + \nu p * d\phi + \nu q d\phi . \]  

(20)

Inserting (19) in (18) gives

\[ \rho = \mu l d\phi + \mu k * d\phi + 2 \phi * d\phi . \]  

(21)

Putting these results for the multipliers into (14) and (15) yields the field equations for the dilaton and metric from the Lagrangian (12).

3 Reduction to a theory with the Levi-Civita connection

By inserting (10) and (11) in (6) the full connection 1-forms can be written in terms of the Levi-Civita 1-forms and dilaton contributions

\[ \Lambda^a_b = \omega^a_b + e^a_b [(k + p) * d\phi + (l + q) d\phi] + \delta^a_b (kd\phi + l * d\phi) \]  

(22)

In the same manner the curvature 2-forms can be decomposed as follows

\[ R^a_b(\Lambda) = R^a_b(\omega) + [(k + p) e^a_b + l \delta^a_b] d * d\phi . \]  

(23)

Thus Einstein-Hilbert term takes the form

\[ R^a_b \wedge *e^a_b = R^a_b(\omega) \wedge *e^a_b - 2(k + p) d * d\phi \]  

(24)

\(^1\)Last two terms of equation (44) in Ref.[3] were mistyped. Fortunately, this mistyping does not change the results.
and similarly
\[
\tau_a[Q] = 2(k^2 + l^2)\tau_a[\phi] + 4kl \ast \tau_a[\phi] \quad (25)
\]
\[
(i_a T^b) \ast T_b = (p^2 - q^2)(i_a d\phi) \ast d\phi - (p^2 - q^2)(i_a * d\phi) d\phi \quad (26)
\]
\[
D \ast T_a = D(\omega) \ast T_a + (pk + ql)\tau_a[\phi] + (kq + pq + pl) \ast \tau_a[\phi] + p^2(i_a d\phi) \ast d\phi + q^2(i_a \ast d\phi) d\phi \quad (27)
\]
\[
D\lambda_a = D(\omega)\lambda_a - (2k\phi + \nu pk + \nu q\tau_a[\phi]) - (2l\phi + \nu pq + \nu pl + \nu qk) \ast \tau_a[\phi] - (2p\phi + \nu p^2)(i_a d\phi) \ast d\phi - \nu q^2(i_a \ast d\phi) d\phi - 2q\phi(i_a d\phi) d\phi \quad (28)
\]
where \(D(\omega)\) denotes the covariant exterior derivative with respect to Levi-Civita. By using these expressions in (14) and (15) we obtain that the field equations for the dilaton and metric are rewritten just in terms of the Levi-Civita connection.
\[
\phi R^a_b(\omega) \ast e_a b - 4(k + p)\phi d \ast d\phi - 2(k + p) d\phi \wedge \ast d\phi
\]
\[
- \left[ \alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2) \right] d \ast d\phi + \beta \phi \ast 1 = 0 \quad (29)
\]
\[
D(\omega)(i_a \ast d\phi^2) + \frac{\beta}{2} \phi^2 \ast e_a - 2(k + p)\phi \tau_a[\phi] - \frac{1}{2} \left[ \alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2) \right] \tau_a[\phi] = 0 \quad (30)
\]
Now we decompose the followings by using the results above
\[
\phi^2 R^a_b(\omega) \wedge \ast e_a b = \phi^2 R^a_b(\omega) \wedge \ast e_a b + 4(k + p)\phi d\phi \wedge \ast d\phi 
\]
\[
Q^a_b \wedge \ast Q^b_a = 2(k^2 - l^2)d\phi \wedge \ast d\phi 
\]
\[
T^a \wedge \ast T_a = (p^2 - q^2)d\phi \wedge \ast d\phi .
\]
Inserting these results in (12) yields the new Lagrangian with Levi-Civita connection.
\[
L = \frac{1}{2} \phi^2 R^a_b(\omega) \wedge \ast e_a b + \frac{\beta}{2} \phi^2 \ast 1 + \lambda_a T^a + \rho_a \wedge Q^b_a
\]
\[
+ \left\{ 2(k + p)\phi + \frac{1}{2} \left[ \alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2) \right] \right\} d\phi \wedge \ast d\phi \quad (34)
\]
We have also verified that the new Lagrangian gives rise to the field equations (29) and (30). We point out the observation of how the nonmetricity and torsion tensors cause to kinetic terms of the scalar field and then rescale the stress forms of the scalar field and finally yield an original derivative scalar interaction for \( k + p \neq 0 \). Besides, we notice that our model is a subcase of the one discussed in [7], which is formulated just in terms of Riemannian geometry, under the following definitions of the potentials
\[
U(X) = -\frac{k + p}{\sqrt{X}} - \frac{\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)}{4X} \quad (35)
\]
\[
V(X) = \frac{\beta}{2} X \quad (36)
\]
where our dilaton is related to theirs via $\phi^2 = X$. Furthermore, it seems that one can extend our model by allowing $k, l, p, q, \alpha, \beta, \mu, \nu$ to be not just constants, but rather arbitrary functions of the dilaton field. Then, (34) will be essentially equivalent to (1.1) in ref. [7]. For the further references concerning the dilaton gravity theories and their applications to black hole physics and string models one can consult to ref.[8]-[11].

4 Discussion on solutions

The model is dependent of the eight real parameters $\{\alpha, \beta, \mu, \nu, k, l, p, q\}$ more general than [3]. Firstly we investigate the static solutions:

$$e^0 = F(x)dt, \quad e^1 = \frac{dx}{F(x)}, \quad \phi = \phi(x) \quad (37)$$

In this case, (29), zeroth and first components of (30) read explicitly; respectively

$$\phi(F^2)'' + \{2(k + p)(\phi')' + [\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\phi'(F^2)'
+\{2(k + p)(\phi')^2 + 4(k + p)\phi\phi'' + [\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\phi''\}F^2 = \beta\phi \quad (38)$$

$$\frac{1}{2}(\phi')'(F^2)' + [2(\phi')^2 + 2\phi\phi'']F^2
-\{2(k + p)\phi + \frac{1}{2}[\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\}(\phi')^2F^2 = \frac{\beta}{2}\phi^2 \quad (39)$$

$$\frac{1}{2}(\phi')'(F^2)' + \{2(k + p)\phi + \frac{1}{2}[\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\}(\phi')^2F^2 = \frac{\beta}{2}\phi^2 \quad (40)$$

where prime denotes the derivative with respect to $x$. First by adding (39) and (40)

$$(\phi')^2F^2 = \frac{\beta}{2}\phi^2 - \phi\phi''F^2 - \frac{1}{2}(\phi')'(F^2)' \quad (41)$$

and then by inserting this in (38) one obtains

$$\phi(F^2)'' + \{(k + p)(\phi')' + [\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\phi'(F^2)'
+\{2(k + p)\phi\phi'' + [\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\phi''\}F^2 = \beta\phi \quad (42)$$

This is the our most general equation and it seems impossible to solve generally it. Therefore, we look at some special cases.

Special Case:1 $k + p = 0$ and $\alpha + \mu(k^2 - l^2) + \nu(k^2 - q^2) = 4$

Here we first drop all the nonlinear terms in $\phi$ and find the solution

$$\phi(x) = e^{c_1x}, \quad F^2(x) = \frac{\beta + 4c_1^2c_2\phi^{-2}(x)}{4c_1^2} \quad (43)$$
where $c_1, c_2$ are constants. This is known as dilaton black hole. Our constraints on the parameters comprise the related cases discussed in [3] and more.

**Special Case:** 2 \( k + p = 0 \) and \( F^2 = 1 \)

One has to pay attention for interpreting this case. At first glance it seems that we work in Minkowski spacetime. This is true if we write the theory in terms of only Levi-Civita without thinking of nonmetricity and torsion like [31], but if we take nonmetricity and torsion into account the situation becomes quite different. In this case, although Riemannian curvature is zero, the non-Riemannian curvature in general is not zero because of (23) as long as \( l \neq 0 \) and thus we are still in a spacetime with curvature, torsion and nonmetricity. Technically speaking; while the worldlines of test particles coincide with the geodesics in the limit of \( k = l = p = q = 0 \) couplings, autoparallels of the full connection deviate from geodesics in the cases of nonzero \( k, l, p, q \) constants, and then we interpret these deviations as nonmetricity and torsion. Now we write down the solution as

\[
F^2(x) = 1, \quad \phi(x) = c_1 e^{\sqrt{-\beta} x} + c_2 e^{-\sqrt{-\beta} x}
\]  

(44)

where \( c_1, c_2 \) are arbitrary constants and \( c_3 = \alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2) \). Here it is interesting to observe that while if \( \beta \) and \( c_3 \) have the same sign, the solution is periodic, it is hyperbolic when they have the opposite signs.

**Special Case:** 3 \( \alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2) = 0 \) and \( F^2 = 1 \)

In this case we obtain the solution

\[
F^2(x) = 1, \quad \phi(x) = -\frac{1}{k + p} + c_1 e^{\sqrt{-\beta} x} + c_2 e^{-\sqrt{-\beta} x}
\]  

(45)

for arbitrary constants \( c_1, c_2 \). We point out that the solution depends crucially on \( k + p \) and periodicity is dependent of the sign of \( \beta \). Here while we are again in Minkowski spacetime in the Riemannian sense, we are in a curved spacetime with nonmetricity and torsion in the non-Riemannian sense.

Finally we examine the cosmological type solutions:

\[
e^0 = dt, \quad e^1 = F(t) dx, \quad \phi = \phi(t)
\]  

(46)

This time, (29), zeroth and first components of (30) read explicitly;

\[
2\dot{\phi}\dot{F} + \{4(k + p)\dot{\phi}\dot{\phi} + [\alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2)]\ddot{\phi}\}
\]

\[
+ \{4(k + p)\dot{\phi}\dot{\phi} + [\alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2)]\ddot{\phi} + 2(k + p)\dot{\phi}^2 + \beta\phi\} F = 0
\]  

(47)

\[
2\dot{\phi}\dot{F} + \{2(k + p)\dot{\phi} + \frac{1}{2}[\alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2)]\} \dot{\phi}^2 F + \frac{\beta}{2}\phi^2 F = 0
\]  

(48)

\[
2\ddot{\phi} + 2\dot{\phi}^2 - \{2(k + p)\dot{\phi} + \frac{1}{2}[\alpha + \mu (k^2 - l^2) + \nu (p^2 - q^2)]\} \dot{\phi}^2 + \frac{\beta}{2}\phi^2 = 0
\]  

(49)
where dot denotes the derivative with respect to \( t \). After multiplying (49) by \( F \), addition of the equations (48) and (49) yields

\[
\dot{\phi} \dot{F} = -[\phi \ddot{\phi} + (\dot{\phi})^2 + \frac{\beta}{2} \phi^2]F
\]

(50)

and then inserting this into (47) gives

\[
2\phi \ddot{F} + [\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\dot{\phi} \dot{F}
+ \{[\alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2)]\dot{\phi} - 2(k + p)(\dot{\phi})^2 - 2(k + p)\beta \phi^2 + \beta \phi\} F = 0
\]

(51)

If we take \( \alpha + \mu(k^2 - l^2) + \nu(p^2 - q^2) = 6 \), then

\[
\phi(t) = \frac{\beta}{4(k + p)} t^2 , \quad F(t) = c e^{\frac{\beta t^2}{4 k^3}}
\]

(52)

for the integration constant \( c \). Finally, for \( k + p = 0 \) we have the inflationary type solution

\[
\phi(t) = e^{-\frac{\gamma}{2} t} , \quad F(t) = e^{c_1 t}
\]

(53)

where

\[
c_1 = \pm \sqrt{\frac{4\beta}{\alpha + \mu(k^2 - l^2) + \nu(k^2 - q^2) - 8}}.
\]

(54)

5 Discussion on spinor couplings to the model

In this section we investigate if there are nontrivial couplings of spinor to the dilaton field. We use the following Dirac matrices as the generators of the Clifford algebra \( \mathcal{C}_{\ell,1} \)

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(55)

which they satisfy the anticommutation relation

\[
\{\gamma_a, \gamma_b\} = 2\eta_{ab}
\]

(56)

and the commutation relation

\[
[\gamma_a, \gamma_b] = 4\sigma_{ab}
\]

(57)

where \( \sigma_{ab} \) are the generators of the Lorentz group. Since we use the \( 2 \times 2 \) matrix representations of \( \mathcal{C}_{\ell,1} \), we represent the Dirac spinor \( \psi \) as a 2-component complex valued column matrix whose covariant exterior derivative is given explicitly by [12]

\[
D\psi = d\psi + \frac{1}{2} \Lambda^{[ab]} \sigma_{ab} \psi + \frac{1}{4} Q \psi
\]

(58)
where the Weyl 1-form $Q = Q^a_a$. Finally, the curvature of the spinor bundle is given by

$$D(D\psi) = \frac{1}{2} R^{[ab]}_{\ c} \sigma_{ab} \psi - \frac{1}{2} Q^a_{\ c} \wedge Q^b_{\ ab} \sigma_{ab} \psi + \frac{1}{4} dQ \psi.$$ (59)

In the Weyl geometry, i.e. $Q_{ab} = \eta_{ab} d\phi$ where $\phi$ any scalar field, the last two terms vanish.

The Dirac Lagrangian 2-form is written in terms of $C_{\ell_{1,1}}$-valued 1-forms $\gamma = \gamma^a e_a$ and the inverse of the Compton wavelength $M = \frac{mc}{\hbar}$ as follows

$$L_D = i \frac{1}{2}(\bar{\psi} \gamma \wedge D\psi + \bar{D\psi} \wedge \gamma \psi) + iM \bar{\psi} \psi * 1$$ (60)

where the Dirac adjoint of a spinor is defined $\bar{\psi} = \psi^\dagger \gamma_0$. While variation with respect to $\Lambda^a_{\ b}$ gives no contribution to (13) since $\gamma_5 \gamma_a + \gamma_a \gamma_5 = 0$ with the definition $\gamma_5 = \gamma_0 \gamma_1$, one obtains the following contribution to (15) after $e^a$ variation

$$\tau^a \psi = \frac{i}{2} \bar{\psi} (i_a \gamma) \wedge D\psi - \bar{D\psi} \wedge (i_a \gamma) \psi + iM \bar{\psi} \psi * e_a.$$ (61)

Finally, $\bar{\psi}$ variation gives the Dirac equation:

$$* \gamma \wedge D\psi + M \psi * 1 = \frac{1}{2} * \gamma \wedge T \psi - \frac{3}{4} * \gamma \wedge Q \psi - \frac{1}{2} \gamma_a Q^{ab} \wedge * e_b \psi = 0$$ (62)

where $T = i_a T^a$. When we insert (22) into (58), we obtain the decomposed covariant derivative of spinor

$$D\psi = D(\omega)\psi - \frac{1}{2} ((k + p) * d\phi + (l + q) d\phi) \gamma_5 \psi + \frac{1}{2} (kd\phi + l * d\phi) \psi.$$ (63)

Substitution of this result into (60) yields

$$L_D = i \frac{1}{2}(\bar{\psi} \gamma \wedge D(\omega)\psi + \bar{D(\omega)\psi} \wedge * \gamma \psi) + iM \bar{\psi} \psi * 1$$ (64)

which gives rise to the variational field equation

$$* \gamma \wedge D(\omega)\psi + M \psi * 1 = 0.$$ (65)

We have also verified that this equation is obtained when (63) is put into (62). Thus we showed that the interpretations of matter-dilaton couplings of nonmetricity and torsion do not produce any nontrivial spinor-dilaton couplings in this model.

### 6 Conclusions

We have generalized the theory [3], in which the authors developed a 2-dimensional theory of dilaton gravity in terms of a zero-torsion and nonmetric-compatible connection determined by
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Nonmetricity and torsion induced by dilaton gravity in two dimension

a two parameter nonmetricity tensor, by adding non-zero torsion determined by a further two parameters. After solving the Lagrange multipliers algebraically from the field equations we discussed some exact classical solutions for static and cosmological types. For more classical solutions of these kinds of models one can consult [7] in that the authors use a different gauge from the diagonal gauge. Basically, this kind of torsion contribution to the model shifts simply the coupling constants. Besides, we rewrote the theory in terms of the Levi-Civita connection in order to see the form of the dilaton couplings induced by the non-Riemannian formulation. Thus, we interpreted the deviations of the world lines of test particles from geodesics of the Levi-Civita connection as nonmetricity and torsion related effects. Finally, we examined the possible spinor couplings to the model and found that there is no nontrivial dilaton couplings to Dirac spinors in two dimension. People interested in nontrivial dilaton couplings to spinors in two dimensions have to consult [2] in which the authors used the Kaluza-Klein dimensional reduction scheme.

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