Polynomiality of helicity off-forward distribution functions 
in the chiral quark-soliton model

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Abstract

The polynomiality condition – i.e. the property that Mellin moments of off-forward distribution functions are even polynomials in the skewedness parameter $\xi$ – is a demanding check of consistency for model approaches. We demonstrate that the helicity off-forward distribution functions in the chiral quark-soliton model satisfy the polynomiality property. The proof contributes to the demonstration that the description of off-forward distribution functions in the model is consistent.

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1 Introduction

Off-forward distribution functions (OFDFs) of partons [1] (see [2, 3, 4] for recent reviews) are a rich source of new information on the internal nucleon structure [5]. OFDFs can be accessed in a variety of hard exclusive processes [6] on which recently first data became available [7]. Many efforts have been devoted to predict and/or understand these experimental results [8], which – at these early stages – necessarily are based on physical intuition, educated guesses and model studies. Important insights into non-perturbative aspects of OFDFs are due to studies in the bag model [9] and the chiral quark-soliton model ($\chi$QSM) [10, 11, 12]. More recently also calculations in a light-front Hamiltonian approach [13] and in a constituent model [14] have been reported.

The $\chi$QSM is based on an effective relativistic quantum field-theory which was derived from the instanton model of the QCD-vacuum. The model describes in the limit of a large number of colours $N_c$ a large variety of nucleonic properties – among others form factors [15] and (anti)quark distribution functions [16] – typically within (10 – 30)$\%$ without adjustable parameters. One of the virtues of the $\chi$QSM is its field-theoretical character, which ensures the theoretical consistency of the approach. E.g., in Ref. [16] it was proven that the quark and antiquark distribution functions computed in the model satisfy all general QCD requirements (sum rules, inequalities, etc.). With the same rigour it was shown that the $\chi$QSM expressions for OFDFs reduce to usual parton distributions in the forward limit, and that their first moments yield form factors [10, 11].

An important property of OFDFs which puts a demanding check of the consistency of a model approach is the polynomiality property [2] (another equally important and demanding property is positivity, see [17] and references therein). In QCD it follows from hermiticity, time-reversal, parity and Lorentz-invariance that the $m^{th}$ moment in $x$ of an OFDF is at $t = 0$ an even polynomial in the skewedness variable $\xi$ (we use the notation of [2, 4]). The degree of the polynomial is less than or equal to $m$ in the case of the unpolarized OFDFs $H^q(x,\xi,t)$ and $E^q(x,\xi,t)$, and $(m - 1)$ in the case of the helicity OFDFs $\tilde{H}^q(x,\xi,t)$ and $\tilde{E}^q(x,\xi,t)$.

The effective low energy theory underlying the $\chi$QSM is hermitian, time-reversal, parity and Lorentz-invariant – like QCD – and so one can expect on general grounds that the OFDFs in the
\( \chi \) QSM satisfy the polynomiality property. In the model expressions, however, the manifestation of these properties becomes obscure and it is necessary to check explicitly the polynomiality property. In [10, 11] it was checked numerically that the unpolarized and helicity OFDFs in the model satisfy the polynomiality property. Of course, numerical checks have limitations due to numerical accuracy, and in [10, 11] the polynomiality property was practically verified for the lowest moments. However, in [12] it was shown explicitly that the unpolarized OFDFs in the \( \chi \) QSM studied in [10] satisfy the polynomiality property for arbitrary moments.

In this note we generalize the methods of Ref. [12] to the case of helicity OFDFs, and we explicitly prove that the model expressions derived in [11] satisfy the polynomiality property. The proof presented here contributes to the demonstration, that the description of OFDFs in the \( \chi \) QSM is theoretically consistent. The note is organized as follows. In Sec. 2 a brief introduction to the \( \chi \) QSM is given. In Sec. 3 the model expressions for the helicity OFDFs are presented. The proof of polynomiality is given in Secs. 4 for the non-spin-flip and in 5 for the spin-flip helicity OFDF. Finally, in Sec. 6 we summarize our results and conclude. The Appendix contains a discussion of the forward limit of \( (\tilde{H}^u - \tilde{H}^d)(x, \xi, t) \).

2 The chiral quark-soliton model (\( \chi \) QSM)

The \( \chi \) QSM [18] is based on the principles of chiral symmetry breaking and the limit of a large number of colours \( N_c \). The effective chiral relativistic quantum field theory underlying the \( \chi \) QSM is given by the partition function [19, 20, 21]

\[
Z_{\text{eff}} = \int D\psi D\bar{\psi} DU \exp \left( i \int d^4x \bar{\psi}(i\not{\partial} - M U^{\gamma_5})\psi \right), \quad U^{\gamma_5} = e^{i\gamma_5 \tau^a \pi^a},
\]

where \( M \) denotes the dynamical quark mass which is due to spontaneous breakdown of chiral symmetry and \( U = \exp(i\tau^a \pi^a) \) is the \( SU(2) \) chiral pion field. The effective theory (1) was derived from the instanton model of the QCD vacuum [21], which provides a mechanism of dynamical chiral symmetry breaking, and is valid at low energies below a scale of about \( \rho_{\text{av}}^{-1} \approx 600 \text{ MeV} \) where \( \rho_{\text{av}} \) is the average instanton size.

In the large-\( N_c \) limit the nucleon can be viewed as a classical soliton of the pion field [22]. The \( \chi \) QSM provides a realization of this idea. In practice the large-\( N_c \) limit allows to solve the path integral over pion field configurations in (1) in the saddle-point approximation [18]. In leading order of the large-\( N_c \) expansion the pion field is static, and one can determine the spectrum of the effective one-particle Hamiltonian of the theory (1)

\[
\tilde{H}_{\text{eff}}[n] = E_n|n\rangle, \quad \tilde{H}_{\text{eff}} = -i\not{\gamma} \gamma^k \partial_k + \gamma^0 M U^{\gamma_5}.
\]

The spectrum consists of an upper and a lower Dirac continuum, which are distorted by the pion field as compared to continua of the free Dirac-Hamiltonian \( \tilde{H}_0 = -i\gamma^0 \gamma^k \partial_k + \gamma^0 M \), and of a discrete bound state level of energy \( E_{\text{lev}} \) for a strong enough pion field of unity winding number. By occupying the discrete level and the states of lower continuum each by \( N_c \) quarks in an antisymmetric colour state, one obtains a state with unity baryon number. The minimization of the soliton energy \( E_{\text{sol}} \) with respect to variations of the chiral field \( U \) yields the so called self-consistent pion field \( U_c \) and the mass of the nucleon

\[
M_N = E_{\text{sol}}[U_c] = \min_U E_{\text{sol}}[U], \quad E_{\text{sol}}[U] = \min_N N_c \left( E_{\text{lev}} + \sum_{E_n < 0} (E_n - E_{n_0}) \right).
\]

The momentum of the nucleon and the spin and isospin quantum numbers are described by quantizing the zero modes of the soliton solution. Corrections in \( 1/N_c \) can be included by considering time dependent pion field configurations [18]. The results of the \( \chi \) QSM respect all general counting rules of the large-\( N_c \) phenomenology.
For the following it is important to note that the effective Hamiltonian $\hat{H}_{\text{eff}}$ commutes with the parity operator $\hat{\Pi}$ and the grand-spin operator $\hat{K}$, defined as

$$\hat{K} = \hat{L} + \hat{S} + \hat{T},$$

(4)

where $\hat{L} = \vec{r} \times \vec{p}$ and $\hat{S} = \frac{1}{2} \gamma_5 \gamma_0 \gamma$ and $\hat{T} = \frac{1}{2} \tau$ are respectively the quark orbital angular momentum, spin and isospin operators. With the maximal set of commuting operators $\hat{H}_{\text{eff}}, \hat{\Pi}, \hat{K}^2$ and $\hat{K}^3$ the single quark states $|n\rangle$ in (2) are characterized by the quantum numbers parity $\pi$ and $K, M$

$$|n\rangle = |E_n, \pi, K, M\rangle.$$  

(5)

The $\chi$QSM allows to evaluate in a parameter-free way nucleon matrix elements of QCD quark bilinear operators as

$$\langle N'(P')|\tilde{\psi}(z_1)|[z_1, z_2]\Gamma \psi(z_2)|N(P)\rangle = A_{NN'}^f M_N N_c \int d^3 X e^{i(P'-P)X} \sum_{n,occ} \tilde{\Phi}_n(z_1 - X) \Gamma \Phi_n(z_2 - X) e^{iE_n(z_1^0 - z_2^0)} + \ldots$$

(6)

where $[z_1, z_2]$ is the gauge link and the dots denote terms subleading in the $1/N_c$ expansion, which will not be needed here. In (6) $\Gamma$ is some Dirac and flavour matrix, $A_{NN'}^f$ a constant depending on $\Gamma$ and the spin and flavour quantum numbers of the nucleon state $|N\rangle = |S_3, T_3\rangle$ and $\Phi_n(x) = \langle x|n\rangle$. The sum in Eq. (6) goes over occupied levels $n$ (i.e. $n$ with $E_n \leq E_{\text{lev}}$), and vacuum subtraction is implied for $E_n < E_{\text{lev}}$ as in Eq. (3). If in QCD the left hand side of (6) is scale dependent then the result in the model on right hand side corresponds to scale of about 600 MeV.

In the way sketched in Eq. (6) static nucleonic observables [15], twist-2 quark and anti-quark distribution functions of the nucleon at a low normalization point [16] have been computed in the $\chi$QSM and found to agree within (10 – 30)\% with experimental data or phenomenological parameterizations. In [10, 11] the approach has been generalized to describe OFDFs at a low normalization point.

3  Off-forward distribution functions in the $\chi$QSM

The helicity quark off-forward distribution functions are defined as

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P', S'_3|\tilde{\psi}_q(-\lambda n/2) \not\gamma_5(-\lambda n/2) \lambda n/2|\psi_q(\lambda n/2)|P, S_3\rangle = \hat{H}^q(x, \xi, t) \tilde{U}(P', S'_3) \not\gamma_5 \hat{U}(P, S_3) + \hat{E}^q(x, \xi, t) \tilde{U}(P', S'_3) \not\gamma_5 \hat{U}(P, S_3) + \ldots$$

(7)

where the dots denote higher-twist contributions. The normalization scale dependence of $\hat{H}^q(x, \xi, t)$ and $\hat{E}^q(x, \xi, t)$ is not indicated for brevity. The light-like vector $n^\mu$, the four-momentum transfer $\Delta^\mu$, the skewedness parameter $\xi$ and the Mandelstam variable $t$ are defined as

$$n^2 = 0, \quad n(P' + P) = 2, \quad \Delta^\mu = (P' - P)^\mu, \quad n\Delta = -2\xi, \quad t = \Delta^2.$$  

(8)

In (7) $x \in [-1, 1]$ with the understanding that for negative $x$ Eq. (7) describes the respective antiquark OFDF. In the $\chi$QSM we work in the large-$N_c$ limit where $M_N = \mathcal{O}(N_c)$, $|\Delta^i| = \mathcal{O}(N_c^0)$ and consequently $|\Delta^0| = \mathcal{O}(N_c^{-1})$. The variables $x$ and $\xi$ are of $\mathcal{O}(N_c^{-1})$. Choosing the 3-axis for the light-cone space direction, we have in the “large-$N_c$ kinematics”

$$n^\mu = (1, 0, 0, -1)/M_N = (1, -e^3)/M_N, \quad t = -\Delta^2, \quad \xi = -\Delta^2/(2M_N).$$  

(9)

Different flavour combinations of the OFDFs exhibit different behaviour in the large-$N_c$ limit [4]

$$\hat{H}^u - \hat{H}^d(x, \xi, t) = N_c^2 f(N_c x, N_c \xi, t), \quad (\hat{E}^u - \hat{E}^d)(x, \xi, t) = N_c^4 f(N_c x, N_c \xi, t),$$

(10)

$$\hat{H}^u + \hat{H}^d(x, \xi, t) = N_c f(N_c x, N_c \xi, t), \quad (\hat{E}^u + \hat{E}^d)(x, \xi, t) = N_c^3 f(N_c x, N_c \xi, t).$$  

(11)
The functions \( f(u,v,t) \) in Eqs. (10, 11) are stable in the large-\( N_c \) limit for fixed values of the \( \mathcal{O}(N_c^0) \) variables \( u = N_c x \), \( v = N_c \xi \) and \( t \), and of course different for the different OFDFs.

The model expressions for the leading OFDFs in (10) were derived in Ref. [11] and read

\[
(\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) = - (2T^3)^3 \frac{2M_N N_c}{3(\Delta^1)^2} \int d^3 x e^{ix} \sum_{n, occ} \int \frac{dz^0}{2\pi} e^{iz^0(\mathbf{x} \mathcal{M} - E_n)}
\]

\[
\times \Phi_n(\mathbf{x} + \frac{z^0}{2} \mathbf{e}^3) \left( 1 + \gamma^0 \gamma^3 \right) \gamma_5 \left( -t \frac{r^3}{2} + \xi M_N \Delta \tau \right) \Phi_n(\mathbf{x} - \frac{z^0}{2} \mathbf{e}^3) , \quad (12)
\]

\[
(\mathcal{E}^u - \mathcal{E}^d)(x, \xi, t) = - (2T^3)^3 \frac{2M^2 N_c}{3(\Delta^1)^2} \int d^3 x e^{ix} \sum_{n, occ} \int \frac{dz^0}{2\pi} e^{iz^0(\mathbf{x} \mathcal{M} - E_n)}
\]

\[
\times \Phi_n(\mathbf{x} + \frac{z^0}{2} \mathbf{e}^3) \left( 1 + \gamma^0 \gamma^3 \right) \gamma_5 \left( \tau^1 \Delta \tau \right) \Phi_n(\mathbf{x} - \frac{z^0}{2} \mathbf{e}^3) . \quad (13)
\]

The 3-axis singled out because of the choice in (9) and \( \Delta^1 = (\Delta^1, \Delta^2, 0) \), etc.

In Ref. [11] it was demonstrated that in the forward limit \( (\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) \) in (12) reduces to the model expression for the helicity isovector distribution function (see also the App. A)

\[
\lim_{\Delta^1 \to 0} (\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) = (g_1^u - g_1^d)(x) \quad (14)
\]

and that the model expressions (12, 13) are correctly normalized to the axial vector and pseudoscalar form factor, respectively

\[
\int_{-1}^{1} dx (\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) = (G_1^u - G_1^d)(t) \quad , \quad \int_{-1}^{1} dx (\mathcal{E}^u - \mathcal{E}^d)(x, \xi, t) = (G_2^u - G_2^d)(t) . \quad (15)
\]

The purpose of this note is to demonstrate that the model expressions in Eqs. (12, 13) satisfy the polynomiality condition, i.e.

\[
M_H^q(m)(\xi, 0) \equiv \int_{-1}^{1} dx x^{m-1} \mathcal{H}^q(x, \xi, 0) = h_0^{q(m)} + h_2^{q(m)} \xi^2 + \ldots + \begin{cases} h_{m-2}^{q(m)} \xi^{m-2} \text{ for } m \text{ even} \\ h_{m-1}^{q(m)} \xi^{m-1} \text{ for } m \text{ odd} \end{cases} \quad (16)
\]

\[
M_E^{q(m)}(\xi, 0) \equiv \int_{-1}^{1} dx x^{m-1} \mathcal{E}^{q(m)}(x, \xi, 0) = e_0^{q(m)} + e_2^{q(m)} \xi^2 + \ldots + \begin{cases} e_{m-2}^{q(m)} \xi^{m-2} \text{ for } m \text{ even} \\ e_{m-1}^{q(m)} \xi^{m-1} \text{ for } m \text{ odd} \end{cases} \quad (17)
\]

In Ref. [11] \( (\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) \) and \( (\mathcal{E}^u - \mathcal{E}^d)(x, \xi, t) \) have been computed as functions of (physical values of) \( x, \xi \) and \( t \), and the polynomiality conditions, Eqs. (16, 17), have been checked by taking (numerically) moments \( M_H^q(m)(\xi, t) \), \( M_E^{q(m)}(\xi, t) \) and extrapolating (numerically) to the unphysical point \( t = 0 \). Of course, this only allows to check the polynomiality condition (for low moments) within the numerical accuracy. In the next two sections strict proofs will be given that all moments of the model expressions (12, 13) satisfy the polynomiality property.

4 Proof of polynomiality for \((\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t)\)

Let us denote by \( M_H^q(m)(\xi, t) \) the \( m \)-th moment of \( (\mathcal{H}^u - \mathcal{H}^d)(x, \xi, t) \) in Eq. (12). It is given by (cf. App. A of [12])

\[
M_H^q(m)(\xi, t) = - \frac{(2T^3)^3 2N_c}{3(\Delta^1)^2 M_N^{m-1}} \sum_{n, occ} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{E_n^{m-1-k}}{2^k} \sum_{j=0}^{k} \binom{k}{j} \times \langle n | (1 + \gamma^0 \gamma^3) \gamma_5 \left( -t \frac{r^3}{2} + \xi M_N \Delta \tau \right) (p^3)^j \exp(\mathbf{i} \Delta \mathbf{X}) (p^3)^{k-j} | n \rangle , \quad (18)
\]
where \( \hat{p} \) and \( \hat{X} \) mean the free-momentum operator and the position operator, respectively.

The next step is to take the limit \( t \to 0 \) in (18). The moments \( M_H^{(m)}(\xi, t) \) depend on \( \xi \) and \( t \) through \( \Delta \) in (9). Continuing analytically the operator \( \exp(i\Delta \hat{X}) \) to \( t = 0 \) one obtains [12]

\[
\lim_{t \to 0} \exp(i\Delta \hat{X}) = \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N|\hat{X}|)^{l_e}}{l_e!} \Phi_e(\cos \hat{\theta}) \, ,
\]

(19)

where the operator \( \cos \hat{\theta} \equiv \hat{X}^3/|\hat{X}| \). The prefactor \( 1/(\Delta^2)^2 \) \( = (-t - 4\xi^2 M^2_N)^{-1} \) with \( t \to 0 \). (We assume \( \xi \neq 0 \). However, the final expressions will be well defined also at \( \xi = 0 \), see below.) Thus, the first expression in the curly brackets in (18) vanishes like \( t \) in the limit \( t \to 0 \). Using (19) the result of analytical continuation \( t \to 0 \) of the operator \( (\Delta \tau) \exp(i\Delta \hat{X}) = [\tau \hat{p}, \exp(i\Delta \hat{X})] \) is given by

\[
\lim_{t \to 0} \tau \Delta \exp(i\Delta \hat{X}) = \sum_{l_e=1}^{\infty} \frac{(-2i\xi M_N)^{l_e}}{l_e!} \tau \hat{p}, \hat{X} \]^{l_e} \Phi_e(\cos \hat{\theta}) = \tau \hat{p}, \hat{X} \]^{l_e} \Phi_e(\cos \hat{\theta}) \, ,
\]

(20)

Thus we obtain for the moments (18) at \( t = 0 \) the result

\[
M_H^{(m)}(\xi, 0) = \frac{(-2T^3 N_c)}{3M_N^{m-1}} \sum_{n, occ} \sum_{k=0}^{m-1} \frac{c_{m-1-k}}{2k} \sum_{j=0}^{k} (j \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N)^{l_e}}{l_e!} \hat{X} \]^{l_e} \Phi_e(\cos \hat{\theta}) \, ,
\]

(21)

Next we use symmetries of the model to show that certain operators in (21) vanish. Consider the unitary matrix given by \( G_5 = \tau^2 \gamma_1 \gamma_3 \) in the standard representation of \( \gamma \) and \( \tau \)-matrices. It has the property \( G_5^\gamma \) \( = (\gamma)^T \) and \( G_5^\tau = - (\tau)^T \). In coordinate space \( \tilde{p}^j = - (\tilde{p})^T \) holds formally, and one finds that \( G_5 \) transforms in the coordinate-space representation the Hamiltonian \( \hat{H}_{eff} \) and single quark states, Eq. (2), as \( G_5 \hat{H}_{eff} G_5^{-1} = \hat{H}_{eff} \) and \( G_5 \Phi_n(x) = \Phi_n(x) \) [15]. Applying the \( G_5 \)-transformation to the matrix elements in (21) we find (cf. [12])

\[
M_H^{(m)}(\xi, 0) = \frac{(-2T^3 N_c)}{3M_N^{m-1}} \sum_{n, occ} \sum_{k=0}^{m-1} \frac{c_{m-1-k}}{2k} \sum_{j=0}^{k} (j \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N)^{l_e}}{l_e!} \hat{X} \]^{l_e} \Phi_e(\cos \hat{\theta}) \, ,
\]

(22)

The use of the \( G_5 \)-transformation corresponds to exploring hermiticity and time reversal invariance. Note that \( (\gamma \gamma_3)^k \) is equal to \( \gamma_3 \gamma_3 \) (unity) for odd (even) \( k \) and introduced in (22) for notational convenience.

Next we use the parity transformation \( \tilde{\Pi} = \gamma^0 \tilde{\gamma} \), where \( \tilde{\gamma} \gamma \tilde{\gamma}^{-1} = - \hat{X} \) and \( \tilde{\gamma} \tilde{\gamma} \tilde{\gamma}^{-1} = - \tilde{\gamma} \), which acts on the single quark states (5) as \( \tilde{\Pi} |n \rangle = \pi |n \rangle \). Applying the parity transformation in (22) we obtain

\[
M_H^{(m)}(\xi, 0) = \frac{(-2T^3 N_c)}{3M_N^{m-1}} \sum_{n, occ} \sum_{k=0}^{m-1} \frac{c_{m-1-k}}{2k} \sum_{j=0}^{k} (j \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N)^{l_e}}{l_e!} \hat{X} \]^{l_e} \Phi_e(\cos \hat{\theta}) \, ,
\]

(23)

Thus the moments of \( M_H^{(m)}(\xi, 0) \) contain only even powers of \( \xi \) and what remains to be done is to demonstrate that the infinite series over \( l_e \) in (23) terminates at an appropriate \( l_e^{max} \) such that the polynomiality condition (16) holds.

\footnote{More precisely, the \( G_5 \)-transformation is the “standard” time reversal operation and a simultaneous flavour-SU(2) rotation, and not – as mentioned in [12] – “non-standard” time reversal. For a thorough discussion of time-reversal in chiral models (and an explanation of the notions of standard/non-standard) see Ref. [23] (and references therein).}
For that we observe that the operators in the matrix elements in (23) transform as irreducible tensor operators $T_M^L$ of rank $L$ and $M = 0$ under simultaneous rotations in space and isospin-space. More precisely, with the unitary operator $\hat{U}(n, \alpha) = \exp(-i \alpha n \cdot \hat{K})$ – where the axis $n$ and angle $\alpha$ characterize the rotation and $\hat{K}$ as given in (4) – irreducible tensor operators are defined as

$$\hat{U}(n, \alpha) T_M^L \hat{U}^\dagger(n, \alpha) = \sum_{M'=-L}^{L} D_{MM'}^{(L)}(n, \alpha) T_{M'}^{L}.$$  \hfill (24)

Under (24) $\gamma_5$, $[\hat{X}, \tau \hat{p}]$ transform as rank zero, $\gamma_0 \gamma_3$, $\hat{p}^3$ as rank 1, and $P_{1c}(\cos \theta)$ as rank $l_e$ operators (cf. [12]). The product of several irreducible tensor operators is generally not an irreducible tensor operator but it can be decomposed into a sum of such. Considering the quantum numbers of the single quark states $|n\rangle$ in Eq. (5) we see that (23) is a trace\(^2\) of those irreducible tensor operators. Such a trace vanishes unless the operator has rank zero [24]. Thus, what we are interested in is to construct out of the operators in (23) those irreducible tensor operators in which $l_e$ takes its largest possible value $l_e^{\max}$.

Consider even $k$ in (23). Then there are $\gamma_0 \gamma_3$ and $k$-times the operator $\hat{p}^3$ which can combine maximally to an operator of rank $(k+1)$. This rank has to be compensated by the rank of $P_{1c}(\cos \theta)$ to yield finally an operator of rank zero. Thus $l_e^{\max} = k + 1$. For odd $k$ in (23) no $\gamma_0 \gamma_3$-operator appears, and we obtain $l_e^{\max} = k$. The summation index $k$ goes from zero to $(m - 1)$, i.e. $l_e^{\max}$ is constrained by

$$l_e^{\max}(m) = \begin{cases} m - 1 & \text{for } m \text{ even} \\ m & \text{for } m \text{ odd.} \end{cases} \hfill (25)$$

Inserting this result into (23) we obtain the desired result

$$M_{H}^{(m)}(\xi, 0) = -\frac{(2T^3) N_c}{3 M_{\text{occ}}^{m-1}} \sum_{n, \text{occ}} \sum_{k=0}^{m-1} \binom{m - 1}{k} \frac{E_{n}^{m-1-k}}{2k} \sum_{j=0}^{k} \binom{k}{j} \sum_{l=0}^{\max(m)} \frac{(-2i \xi M_N)^l}{l!} \sum_{l=0}^{\max(m)} \frac{(-2i \xi M_N)^l}{l!} \times<n | (\gamma_0 \gamma_3)^{k+1} \gamma_5 (\hat{p}^3)^j i[\tau \hat{p}, [\hat{X}, P_{1c}(\cos \theta)] (\hat{p}^3)^{k-j} | n \rangle. \hfill (26)$$

By renaming $l_e \rightarrow l + 1$ the above result can be written as

$$M_{H}^{(m)}(\xi, 0) = -\frac{(2T^3) N_c}{3 M_{\text{occ}}^{m-1}} \sum_{n, \text{occ}} \sum_{k=0}^{m-1} \binom{m - 1}{k} \frac{E_{n}^{m-1-k}}{2k} \sum_{j=0}^{k} \binom{k}{j} \sum_{l=0}^{\max(m)} \frac{(-2i \xi M_N)^l}{(l + 1)!} \times<n | (\gamma_0 \gamma_3)^{k+1} \gamma_5 (\hat{p}^3)^j i[\tau \hat{p}, [\hat{X}, P_{1c}(\cos \theta)] (\hat{p}^3)^{k-j} | n \rangle. \hfill (27)$$

## 5 Proof of polynomiality for $(\tilde{E}^u - \tilde{E}^d)(x, \xi, t)$

The $m^{\text{th}}$ moment $M_{E}^{(m)}(\xi, t)$ of $(\tilde{E}^u - \tilde{E}^d)(x, \xi, t)$ in Eq. (13) is given by (cf. [12])

$$M_{E}^{(m)}(\xi, t) = -\frac{(2T^3) 2 N_c}{3 \xi (\Delta^\perp)^2 M_{\text{occ}}^{m-2}} \sum_{n, \text{occ}} \sum_{k=0}^{m-1} \binom{m - 1}{k} \frac{E_{n}^{m-1-k}}{2k} \sum_{j=0}^{k} \binom{k}{j} \times<n | (1 + \gamma_0 \gamma_3) \gamma_5 (\Delta^\perp \tau^\perp)(\hat{p}^3)^j \exp(i \Delta \hat{X}) (\hat{p}^3)^{k-j} | n \rangle. \hfill (28)$$

In order to continue analytically to $t \rightarrow 0$ we can use the result in (20) (with the difference that here the sum starts with $l = 2$ (since for $l = 1$ we have $i[\hat{p}^\perp, [\hat{X}, P_{1c}(\cos \theta)] = \nabla^\perp \hat{X}^3 = 0$), i.e.

$$M_{E}^{(m)}(\xi, 0) = \frac{(2T^3) 4 N_c}{3 M_{\text{occ}}^{m-1}} \sum_{n, \text{occ}} \sum_{k=0}^{m-1} \binom{m - 1}{k} \frac{E_{n}^{m-1-k}}{2k} \sum_{j=0}^{k} \binom{k}{j} \sum_{l=2}^{\infty} \frac{(-2i \xi M_N)^{l} \hat{e}^{2l}}{l!} \times<n | (1 + \gamma_0 \gamma_3) \gamma_5 (\hat{p}^3)^j i[\tau^\perp \hat{p}^\perp, [\hat{X}, P_{1c}(\cos \theta)] (\hat{p}^3)^{k-j} | n \rangle. \hfill (29)$$

$\sum_{\text{occ}} = \sum \delta_n \leq E_{\text{lev}}, \pi, K, M$ is a sum over matrix elements diagonal in $K$ and $M$, i.e. a trace.
The work reported here supplements the proof given in [12] that also the unpolarized OFDFs in condition, which for model approaches is one of the most demanding properties of OFDFs to fulfil.

expressions for the helicity OFDFs (π-pion pole contribution in (π^3) j i[τ^⊥ p^⊥ |X]|l_e = (cos ̂θ) (p^3) k-j|n) . (30)

In the next step, however, the above-mentioned difference matters because τ^⊥ p^⊥ = τ p - τ^3 p^3 can be decomposed into a rank zero and rank two operator with respect to simultaneous rotations in space and flavour-space. Apparently the rank-two piece in τ^⊥ p^⊥ allows for a larger l^max_e and we consider in the following this piece only. Consider even k in the matrix elements in (30). Then γ^0 γ^3, τ^⊥ p^⊥ and the k-times appearing operator p^3 allow to construct irreducible tensor operators of maximally rank k + 3. Thus l_e can take at most the value k + 3. For odd k there is no γ^0 γ^3-operator and the highest possible value of l_e is k + 2. Since 0 ≤ k ≤ m − 1 in (30) we obtain

l^max_e(m) = \begin{cases} 
  m + 1 & \text{for } m \text{ even} \\
  m + 2 & \text{for } m \text{ odd}.
\end{cases} (31)

Inserting this result into (30) and renaming the summation label l_e → l + 3 we obtain the desired result

M^{(m)}_E(ξ, 0) = \frac{(2T^3)^4 N_e}{3 M^{m-1}_N} Σ_{n,occ} \sum_{k=0}^{m-1} (m − 1) \frac{E_m^{m-1-k}}{2k} \sum_{j=0}^{k} \frac{(k)}{j} \sum_{l=0}^{m-1} \frac{(-2iξ M_N)^l}{l!} \langle n |(γ^0 γ^3)^{k+1} γ_5 \langle p^3^3 \rangle j i[τ^⊥ p^⊥ |X]|l_e = (cos ̂θ) (p^3) k-j|n) . (32)

6 Summary

The recently reported first measurements of deeply virtual Compton scattering began an exciting era which will reveal novel properties of the nucleon. At the early stage of art non-perturbative model calculations play an important role as a source of inspiration for phenomenological modelling of OFDFs. In this context it is important to ensure the theoretical consistency which provides a base for the reliability of the non-perturbative model results.

In this note we have presented a study of the helicity OFDFs in the χQSM – in which such prominent observations have been made like the D-term [25] in the unpolarized OFDFs [10] or the pion pole contribution in (E^u − E^d)(x, ξ, t) [11]. Here we have shown explicitly that the χQSM expressions for the helicity OFDFs (H^u − H^d)(x, ξ, t) and (E^u − E^d)(x, ξ, t) satisfy the polynomiality condition, which for model approaches is one of the most demanding properties of OFDFs to fulfill. The work reported here supplements the proof given in [12] that also the unpolarized OFDFs in the χQSM satisfy the polynomiality condition.

This note makes a further contribution to the demonstration that the description of OFDFs in the χQSM is theoretically consistent, and helps to increase the confidence into predictions and estimates based on or inspired by the results from that model.

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A The forward limit of $(\tilde{H}^u - \tilde{H}^d)(x, \xi, t)$

In Ref. [11] it was checked that in the forward limit $\Delta^i \to 0$ the model expression for the helicity OFDF $(\tilde{H}^u - \tilde{H}^d)(x, \xi, t)$ reduces to the helicity distribution function $(g^u_1 - g^d_1)(x)$, see Eq. (14). In the $\chi$QSM $(g^u_1 - g^d_1)(x)$ is given in leading order of the large-$N_c$ limit by

$$ (g^u_1 - g^d_1)(x) = - (2T^3 \frac{M_{SN} N_c}{3}) \sum_{n, occ} \langle n | (1 + \gamma^0 \gamma^3) \gamma_5 \tau^3 \delta(x M_N - E_n - \vec{p}^3) | n \rangle. \quad (33) $$

Here we will repeat this check and explicitly demonstrate that the correct forward limit is obtained – irrespective the way one takes it. In particular we shall see that

$$ (g^u_1 - g^d_1)(x) = \lim_{\Delta^i \to 0} (\tilde{H}^u - \tilde{H}^d)(x, \xi, t) \quad (34) $$

$$ = \lim_{t \to 0} \left[ \lim_{\xi \to 0} (\tilde{H}^u - \tilde{H}^d)(x, \xi, t) \right] \quad (35) $$

$$ = \lim_{\xi \to 0} \left[ \lim_{t \to 0} (\tilde{H}^u - \tilde{H}^d)(x, \xi, t) \right]. \quad (36) $$

Note that Eq. (36) provides a cross check for model expression analytically continued to $t = 0$.

**Limit $\Delta^i \to 0$.** Let us introduce $\Delta_i = \epsilon (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)$ where $\epsilon > 0$ and $\alpha \in [0, 2\pi]$ and $\beta \in [0, \pi]$ are arbitrary angles. (However, we exclude that $\alpha$ is an integer multiple of $\pi$. This restriction can be dropped in the final expression.) Then $t_\epsilon = -\epsilon^2$, $(\Delta^i_\epsilon)^2 = \epsilon^2 \sin^2 \alpha$, $2\xi_\epsilon M_N = -\epsilon \cos \alpha$. The limit $\Delta^i \to 0$ will be taken by letting $\epsilon \to 0$. With the function $f(\alpha, \beta, X) = \Delta_i X / \epsilon$ which does not depend on $\epsilon$ we obtain from (12)

$$ (\tilde{H}^u - \tilde{H}^d)(x, \xi, t_\epsilon) = - (2T^3 \frac{M_{SN} N_c}{3}) \int d^3 X e^{i\epsilon |X| f(\alpha, \beta, X)} \sum_{n, occ} \int \frac{dz}{2\pi} e^{iz^0(x M_N - E_n)} $$

$$ \times \Phi_n^*(X + \frac{z^0}{2} \epsilon^3) (1 + \gamma^0 \gamma^3) \gamma_5 \left( \tau^3 - (\tau^1 \cos \beta + \tau^2 \sin \beta) \cot \alpha \right) \Phi_n(X - \frac{z^0}{2} \epsilon^3). \quad (37) $$

Taking the limit $\epsilon \to 0$ in (37) and using $\Phi_n(X - \frac{z^0}{2} \epsilon^3) = \langle X | \exp(i \frac{z^0}{2} \epsilon^3) | n \rangle$ (analogue for $\Phi_n^*(X + \frac{z^0}{2} \epsilon^3)$) and $\int d^3 X |X\rangle \langle X| = 1$ and integrating over $z^0$ we obtain

$$ (\tilde{H}^u - \tilde{H}^d)(x, 0, 0) = - (2T^3 \frac{M_{SN} N_c}{3}) \sum_{n, occ} \langle n | (1 + \gamma^0 \gamma^3) \gamma_5 \delta(x M_N - E_n - \vec{p}^3) $$

$$ \times \left\{ \tau^3 - (\tau^1 \cos \beta + \tau^2 \sin \beta) \cot \alpha \right\} | n \rangle. \quad (38) $$

Let us now consider a simultaneous rotation in space and isospin space about the 3-axis around the angle $\pi$. The net effect of this rotation is to leave the contribution of $\tau^3$ in the curly brackets in (38) invariant but to change the signs of the contributions of $\tau^1$ and $\tau^2$. This shows that these contributions are strictly zero and that (38) coincides with the model expression (33) for $(g^u_1 - g^d_1)(x)$ – which verifies Eq. (34).

**Limit $\xi \to 0$ for $t \neq 0$ and subsequent $t \to 0$.** Taking $\xi \to 0$ in Eq. (12) we obtain

$$ (\tilde{H}^u - \tilde{H}^d)(x, 0, t) = - (2T^3 \frac{M_{SN} N_c}{3}) \sum_{n, occ} \langle n | (1 + \gamma^0 \gamma^3) \gamma_5 \delta(x M_N - E_n - \vec{p}^3) e^{i \Delta^i \cdot \hat{x}^i} | n \rangle. \quad (39) $$

\footnote{More precisely in the model one has to take the limit $\Delta^i \to 0$ ($i = 1, 2, 3$) because $|\Delta^0| = O(N_c^{-1}) \ll |\Delta^i| = O(N_c^0)$, see Eq. (9) and the text above it.}
In (39) we have used \( t = -(\Delta^\perp)^2 \) for \( \xi = 0 \) and performed steps similar to those leading to Eq. (38) (benefiting from \([\hat{p}^3, \hat{X}^\perp] = 0\)). The right-hand-side of (39) depends on \( t \) through \( \Delta^\perp \).

We introduce \( \Delta^\perp \epsilon = \epsilon (\sin \alpha, \cos \alpha) \) with an arbitrary angle \( \alpha \) such that \( t \epsilon = -\epsilon^2 \). The limit \( t \to 0 \) can be taken by letting \( \epsilon \to 0 \). In this limit (39) goes into (33) – which verifies Eq. (35).

**Limit \( t \to 0 \) for \( \xi \neq 0 \) and subsequent \( \xi \to 0 \).** We recall that \((\Delta^\perp)^2 = -t - (2\xi M_N)^2\).

Taking first the limit \( t \to 0 \) we obtain upon use of (20)

\[
(\tilde{H}^u - \tilde{H}^d)(x, \xi, 0) = -\frac{(2T^3)}{3} \frac{M_N N_c}{e^3} \int d^3X \sum_{n, occ} \int \frac{dz_0}{2\pi} e^{iz_0(x M_N - E_n)} \sum_{l=e}\infty \frac{(-2i\xi M_N)^{l-1}}{l!} \\
\times \Phi^*_n(X + \frac{z_0^0 e^3}{2}) (1 + \gamma^0 \gamma^3) \gamma^5 i[\tau\hat{p}, |X|P_3 e^3 (\cos \hat{\theta})] \Phi_n(X - \frac{z_0^0 e^3}{2}).
\] (40)

For \( \xi \to 0 \) in (40) only \( l_e = 1 \) in the sum over \( l_e \) contributes. Considering \( i[\tau\hat{p}, |X|P_1 e^3 (\cos \hat{\theta})] = \tau^3 \) we recover from (40) the expression for \((g_1^u - g_1^d)(x)\) in (33) – which verifies Eq. (36).

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