The Theory of Discrete Extension
An Operator-Free Approach To Quantization

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Abstract

The theory of discrete extension, introduced here, demonstrates that various results from elementary, non-relativistic quantum mechanics can be obtained without recourse to eigenvalue problems, operators, or state vectors in Hilbert space. Using a mathematical structure similar to that of Newtonian mechanics, the theory generates correct, quasi-discrete values of action increments and energy levels for bound states of quantum-level systems. Quantization of the harmonic oscillator provides a simple illustration.

Like Newtonian mechanics the theory provides intuitive conceptualizations for basic entities and kinematics. However, in place of the classical concepts of particle and field, the non-classical, unitary concept of a discretely extended object emerges from the theory as an alternative characterization of the nature of quantum-level entities. Because such an object is well-defined and unitary, the theory avoids the dilemmas of wave-particle dualism and complementarity.

The theory offers insight into the nature of a number of quantum effects such as the zero-point energy of a harmonic oscillator. It also makes a number of testable predictions that distinguish it from standard quantum theory and from Bohmian mechanics.

Keywords: Bohmian clustor discrete extension discretely extended operator-free quasi-discrete

1 INTRODUCTION

1.1 Intent and Scope

The student of quantum theory is confronted, at the outset, with a number of enigmatic fundamentals such as wave-particle dualism, superposition of states, wave function collapse, indeterminism, and the substitution of mathematical operators for real-valued observables. Upon reflection, one also becomes aware that the standard interpretation

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of the theory points away from an instinctive, objective world-view and toward the somewhat unnatural philosophy of subjectivism. After sufficient exposure, one may come to accommodate these tenets, but the troubling mysteries remain, nevertheless. The intent of the theory of discrete extension is to make a small measure of progress against these mysteries.

A body of results has been obtained concerning the development, interpretation, and application of the theory and will be presented in this and subsequent papers. The principal goals of this initial paper are limited, however, to three. The first is to introduce the concept of a *discretely extended object* and to propose it as a characterization of the nature of quantum-level entities. Such an object will also be referred to here as a *clustor*. The paper offers this non-classical concept as a unitary alternative to the dual, classical concepts of wave and particle. The theory of discrete extension embodies this well-defined, unitary concept and thereby avoids the dilemmas of wave-particle dualism and complementarity, and the associated philosophical problem of subjectivism.

The second goal is to develop the mathematical theory of clustor dynamics, to show that the concept of a discretely extended object arises naturally within the theory, and to illustrate examples of clustor dynamics for simple systems. The paper shows that this dynamical theory is an extension of Newtonian mechanics that takes into account the nonzero value of Planck’s constant, $\hbar$. The paper proposes that the theory is valid in the microscopic domain.

The third goal is to show that the theory generates correct, quasi-discrete action increments and energy levels for bound-state systems without recourse to the operator formalism and eigenvalue problems of standard quantum theory. The paper illustrates this capability by applying the theory to the harmonic oscillator. This simple exercise is sufficient to show how the theory differs from standard quantum theory in physical, mathematical, interpretive, and predictive content. In particular, it shows how quantization can be achieved within a mathematical structure similar to that of Newtonian dynamics.

In its current form the theory deals with concepts and systems within the realm of elementary quantum mechanics only. This limited domain seems to be the appropriate one, however, when the fundamentals of quantum theory are being examined. Not only are the main mysteries of quantum theory already present at that level, but the elementary theory also serves as the foundation for the more advanced theories.

The principles of the theory of discrete extension have not yet been applied to relativistic systems, and its implications for quantum field theory have yet to be established. Nevertheless, the theory already offers insight into the nature of a number of quantum effects such as the zero-point energy of a harmonic oscillator, as shown below. It also makes a number of testable predictions that distinguish it from standard quantum theory and from Bohmian mechanics. As shown below, for example, it predicts that the frequency of a harmonic oscillator depends on the oscillator’s energy. It may be possible to look for this effect experimentally in the field of nanotechnology.

1.2 Additional Results

The following additional results have been obtained and will be presented in subsequent papers. Application of the theory to the double slit system shows that the associated
interference pattern is the cumulative result of a series of random, but deterministic events. An analysis of two coupled oscillators demonstrates the quasi-discrete transfer of energy between subsystems. An analysis of the hydrogen atom shows that the theory generates correct, quasi-discrete values of quantum numbers for a system with three degrees of freedom. An analysis of distant, entangled subsystems demonstrates the influence of one subsystem on another and introduces the concept of energy of entanglement. Finally, the theory shows that wave function collapse is associated with a continuous process that obeys the ordinary equations of clustor dynamics.

1.3 Discrete Extension and the Concept of a Clustor

In the 20th century a number of previously unknown physical effects were discovered such as the Compton effect and electron diffraction. These discoveries revealed that light and matter have both particle-like and wave-like aspects to their behaviors. Over the years, various attempts have been made to interpret these new phenomena in terms of classical wave and particle concepts that had been carried over from the 19th century.

The most widely accepted interpretation, the Copenhagen interpretation, incorporates these classical concepts through the notions of wave-particle dualism and complementarity [1]. Other interpretations, such as Schrödinger’s unitary wave theory [2], attempt to explain particle-like behavior on the basis of continuum dynamics. Conversely, still other interpretations, such as Landé’s unitary particle theory [3], attempt to explain wave-like interference phenomena on the basis of particle dynamics. Finally, interpretations such as the de Broglie-Bohm pilot wave theory [4] postulate a dual physical basis of particles together with an objectively real field.

These interpretations all rely on the classical wave and particle concepts. In contrast, the theory of discrete extension is based on the non-classical, unitary concept of a discretely extended object or clustor.

A classical particle is a point-like entity and can be described, therefore, as an unextended object. A classical wave or field is a space-filling entity and can be described as a continuously extended object. In contrast to these classical concepts, a clustor is a single, indivisible entity with multiple, spatially isolated, point-like manifestations. Therefore, with similar terminology, it can be described as a discretely extended object.

(In special cases a clustor can be unextended like a particle, as shown below.)

In standard quantum theory the notion of a particle trajectory is rejected in accordance with the uncertainty principle. A clustor, on the other hand, has a well-defined spatial trajectory and a continuous space-time world-line. Despite its partitioned appearance, a clustor is an undivided entity whose spatially isolated manifestations, called clustor points, have no individual existence apart from the clustor as a whole. An entity will be considered to be undivided or whole if its world-line is continuous.

At first, the characteristics of having a partitioned appearance and being undivided seem to be in conflict. There is no inconsistency, however, because, as shown below, clustor world-lines are, in general, oscillatory and will therefore have multiple intersections with lines of simultaneity.

Being both point-like and extended, a clustor is a hybrid object that combines certain attributes of both particles and waves. In particular, the point-like manifestations of a free clustor can have a spatial distribution that is quasi-periodic. Therefore, in
addition to its point-like aspect, such a clustor possesses an attribute that is analogous to the wavelength of a harmonic wave. In future publications it will be shown that the dynamics of this hybrid object can lead to phenomena that are normally attributed to the interference of physical waves.

Although it is possible for a clustor to exist in an unextended state, no such clustors are likely to be found in nature. Any clustor that is found in, or has been specially prepared in, an unextended state is nearly certain to become discretely extended upon interaction with the environment. This behavior will be illustrated below by an encounter between an unextended free clustor and a rectangular barrier. Thus, naturally occurring clustors are almost certain to be discretely extended due to a history of countless interactions with the environment.

2  THE MATHEMATICAL THEORY OF CLUSTOR DYNAMICS

2.1 Mathematical Framework

The mathematical theory of clustor dynamics is an extension of Newtonian mechanics that takes into account the non-zero value of Planck's constant, \( h \). As such, it is proposed here that the theory is applicable in the microscopic, quantum domain. The starting point for this extension is the Hamilton-Jacobi formulation of Newtonian theory, a choice that is motivated by the following considerations.

2.1.1 The Polar Decomposition of Schrödinger’s Equation

In standard quantum theory the wave function, \( \Psi \), for a one-particle system satisfies Schrödinger’s equation,

\[
\frac{\hbar^2}{2m} \nabla^2 \Psi - V(\mathbf{x}, t)\Psi + i\hbar \frac{\partial \Psi}{\partial t} = 0,
\]

where \( m \) is the particle’s mass, \( V(\mathbf{x}, t) \) is the potential function, and \( \hbar = h/2\pi \). The complex wave function can be expressed in the polar form \( \Psi = R \exp(iS/\hbar) \), and Schrödinger’s equation is then equivalent to the following coupled, real equations for the amplitude function, \( R \), and the phase function, \( S \):

\[
\nabla \cdot \left( \frac{R^2 \nabla S}{m} \right) + \frac{\partial}{\partial t} \left( R^2 \right) = 0 \quad (1)
\]

\[
\frac{(\nabla S)^2}{2m} + V(\mathbf{x}, t) + \frac{\partial S}{\partial t} - \hbar^2 \frac{\nabla^2 R}{2m R} = 0 \quad (2)
\]

This decomposition of Schrödinger’s equation is the basis of the WKB approximation scheme and is used in the development of both the de Broglie-Bohm pilot wave theory and the theory of discrete extension introduced here.

Equation (1) is a continuity equation that, in standard quantum theory, expresses the conservation of probability. The real quantities

\[
R^2 \frac{\nabla S}{m} = \frac{i\hbar}{2m} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = \jmath \quad \text{and} \quad R^2 = \Psi^* \Psi = \rho
\]
are interpreted there as a probability current density and a probability density, respectively. \( \Psi^* \) is the complex conjugate of \( \Psi \).

### 2.1.2 The Generalization of Hamilton’s Principal Function

In the classical limit with \( \hbar = 0 \), equation (2) is uncoupled from \( R \) and reduces to the time-dependent Hamilton-Jacobi equation of Newtonian theory. Any equation that is uncoupled from \( R \) will subsequently be called a direct equation. Thus, for \( \hbar = 0 \), equation (2) reduces to a direct equation for Hamilton’s principal function, \( S_N \), where the subscript, \( N \), indicates a quantity from Newtonian theory.

For stationary states of a conservative system with one degree of freedom, equations (1) and (2) can be uncoupled even when \( \hbar \neq 0 \). In that case the wave function has the form \( \Psi = R(x) \exp\left[i(W(x) - Et)/\hbar\right] \) where the constant \( E \) is the energy of the system. Then, \( \partial R/\partial t = 0 \) and \( \partial S/\partial t = -E \), and the function \( W(x) \) satisfies the following direct, third-order equation:

\[
\left( \frac{dW}{dx} \right)^2 \left[ \left( \frac{dW}{dx} \right)^2 - 2m[E - V(x)] \right] + \left( \frac{\hbar}{2} \right)^2 \left[ 2 \frac{dW}{dx} \frac{d^3W}{dx^3} - 3 \left( \frac{d^2W}{dx^2} \right)^2 \right] = 0. \tag{3}
\]

For \( \hbar = 0 \) this equation reduces to the time-independent Hamilton-Jacobi equation of Newtonian theory.

The fact that equations (2) and (3) from standard quantum theory reduce to Newtonian Hamilton-Jacobi equations for \( \hbar = 0 \) suggests that Newtonian Hamilton-Jacobi theory can be broadened into a quantum theory when \( \hbar \neq 0 \). Therefore, the theory of discrete extension adopts as its framework the entire Hamilton-Jacobi formalism of Newtonian theory but with the Schrödinger phase function, \( S \), playing the role of Hamilton’s principal function. The theory adopts this formalism not only for the one-particle systems discussed above, but for systems of arbitrary complexity. In all cases the generalized principal function, \( S \), will satisfy a direct, higher-order equation determined, implicitly, by Schrödinger’s equation.

### 2.1.3 The Generalized Hamilton-Jacobi Equation

At first this prescription seems impossible. On the one hand, Hamilton-Jacobi theory is based on the mathematical relation that exists between any first-order partial differential equation and its associated system of characteristic, first-order, ordinary differential equations (Hamilton’s canonical equations in the case of Newtonian theory). On the other hand, no such relation exists for higher-order partial differential equations such as the implicit, direct equation for \( S \).

This higher-order equation, however, may have an implicit, first-order, intermediary integral that shares solutions with its parent equation. In the theory of discrete extension an intermediary integral of this sort plays the role of a generalized Hamilton-Jacobi equation. Being of first-order, it is mathematically qualified for that role.

Let \( \mathcal{N} \) be the number of degrees of freedom of a given system, and let \( n = 1, 2, \ldots, \mathcal{N} \) be an index for the coordinates, \( q_n \), in the configuration space of that system. Also, let \( \nu = 0, 1, 2, \ldots, \mathcal{N} \) be an index in the extended configuration space that includes time as the coordinate \( q_0 = t \).
Reduction of the higher-order equation for \( S \) to a first-order, intermediary integral will then result in a generalized Hamilton-Jacobi equation of the form

\[
K(q_\nu, \frac{\partial S}{\partial q_\nu}, \gamma) = 0 \tag{4}
\]

where \( \gamma \) is a set of arbitrary integration constants that are introduced in the order-reduction process. The function \( K(q_\nu, p_\nu, \gamma) \), which is identically zero, serves as an extended Hamiltonian function for the system. The quantities \( (q_\nu, p_\nu) \) in this Hamiltonian are canonical pairs of coordinates and momenta in the extended phase space of the system. They include the canonical pair \( (q_0, p_0) = (t, -E) \) where \( E = -\partial S / \partial t \) is the energy of the system (not necessarily constant).

The Hamilton-Jacobi formalism requires a complete solution of equation (4). In addition to an additive constant that can be ignored, such a solution, \( S(q_\nu, \alpha_n, \gamma) \), will contain \( N \) essential Hamilton-Jacobi constants, \( \alpha_n \), as well as the set, \( \gamma \), of order-reduction constants. It therefore involves two distinct sets of arbitrary constants, different from each other in both origin and mathematical status.

Since \( S \) depends on the arbitrary constants in the set \( \gamma \), the Hamilton-Jacobi motion equations, \( \partial S / \partial \alpha_n = \beta_n \), will also depend on them. These constants comprise non-Newtonian data that must be specified in addition to the usual Newtonian initial conditions in order to uniquely define the behavior of the system. The theory of discrete extension is, therefore, richer than Newtonian theory in that it permits infinite families of diverse cluster dynamics for any given set of Newtonian initial conditions.

2.1.4 The Roles of Schrödinger’s Equation and the Wave Function

In the theory of discrete extension Schrödinger’s equation is viewed only as an auxiliary mathematical tool for obtaining the real-valued function, \( S \), while avoiding the inherent difficulties in the derivation and solution of a non-linear, direct equation for \( S \). The following progression summarizes the logical steps that relate Schrödinger’s equation to the generalized Hamilton-Jacobi equation (4):

1. Schrödinger’s equation for the wave function, \( \Psi = R \exp(iS/\hbar) \), is second-order, linear, and complex. It can be decomposed into

2. Two coupled equations for \( R \) and \( S \) which are second-order, non-linear, and real. Elimination of \( R \) between them leads implicitly to

3. A direct equation for \( S \) which is higher-order, non-linear, and real. An implicit, first-order, intermediary integral of this implicit equation is then

4. The generalized Hamilton-Jacobi equation, \( K(q_\nu, \partial S / \partial q_\nu, \gamma) = 0 \), for \( S \).

In the theory of discrete extension the wave function, \( \Psi \), has neither the status of a physical field nor an interpretation as either a probability amplitude or a state vector. Its role is simply to lend its phase function, \( S \), for use as a generalized principal function in the Hamilton-Jacobi formalism.
2.1.5 Action Functions and Phase Functions

In Newtonian theory the solution, $S_N$, of the Hamilton-Jacobi equation is closely related to the Newtonian action function. Therefore, in the theory of discrete extension, Schrödinger’s phase function, $S$, which satisfies a generalized Hamilton-Jacobi equation, will be viewed as a generalized action function and will be referred to simply as the action. Recognition of a close connection between action functions and phase functions is, of course, not new. In the path integral formulation of standard quantum theory, for example, $1/h$ times the Newtonian action integral, $\int L dt$, over a given classical path is adopted as the phase of the contribution that the path makes to the total quantum amplitude. Thus, in that theory, the conceptual progression is from an action function to a phase function. In the theory of discrete extension, however, the progression is reversed: $h$ times the phase of the Schrödinger wave function is used in the Hamilton-Jacobi formalism as a non-classical action function for a theory of cluster dynamics.

2.2 Contrast with the de Broglie-Bohm Pilot Wave Theory

Both the de Broglie-Bohm theory and the theory of discrete extension have a connection with the Hamilton-Jacobi formulation of Newtonian mechanics, and both make use of the phase function, $S$, in the polar form of Schrödinger’s wave function. The two theories also have many differences. Two important areas of disagreement are quantization procedures and motion equations.

2.2.1 Quantization Procedures

The de Broglie-Bohm theory made an important conceptual advance by demonstrating that neither indeterminism nor subjectivism are necessary parts of a coherent interpretation of quantum theory. It can be criticized, however, for interpretive problems of its own. It postulates a dual physical basis consisting of point particles together with an “objectively real field” that guides the motions of the particles. Postulating that the pilot wave is objectively real provides the theory with a physical justification for requiring that the mathematical representation of the field be finite, continuous, and single-valued. That is, it justifies the use of eigenvalue problems to achieve quantization. This postulate ensures that the theory will generate the same quantization results as standard quantum mechanics.

There are strong arguments, however, against the plausibility of this postulate. Unlike a physical object such as the electromagnetic field, the pilot field is sourceless and is without an energy density. Furthermore, the influence of the pilot field is independent of its magnitude; it can be multiplied by an arbitrary constant without changing its effect on the motion of the particles. Finally, for a system of $N$ particles, it is represented mathematically as a function in the $3N$-dimensional configuration space of the particles rather than in physical space. For these reasons the pilot field should be viewed as nothing more than an auxiliary mathematical construct. As a result, there is no physical basis for requiring it to be finite, continuous, and single-valued. The de Broglie-Bohm theory, therefore, lacks a physical justification for using eigenvalue
problems to achieve quantization. (These same comments apply to the quantization procedure of standard quantum mechanics.)

The theory of discrete extension, on the other hand, is based on the unitary physical concept of a cluster and does not postulate the existence of an associated physical field. Therefore, in this theory, the mathematical requirements stated above are irrelevant. As shown below, quantization emerges naturally from the dynamical equations of the theory without the need for supplementary mathematical conditions.

2.2.2 Motion Equations

It can also be noted that the de Broglie-Bohm theory makes use of only half of the Hamilton-Jacobi formalism while disregarding the other half. In accordance with the formalism, it postulates that the momentum components of a particle are given by
\[ p_n = \frac{\partial S}{\partial q_n} \]
where the function \( S \) is adopted as a generalization of Hamilton’s principal function. It also postulates, however, that the velocity components of the particle are given by the Newtonian expression
\[ v_n = \frac{p_n}{m} \]
where \( m \) is the particle’s mass. The purpose of this second postulate is to preserve the standard probability interpretation of equation (1) for an ensemble of particles [8]. Taken together, these two postulates produce motions that are in conflict with the Hamilton-Jacobi motion equations,
\[ \frac{\partial S}{\partial \alpha_n} = \beta_n, \]
where \( \alpha_n \) and \( \beta_n \) are the usual Hamilton-Jacobi constants. The de Broglie-Bohm theory ignores this set of motion equations.

The theory of discrete extension, on the other hand, employs the entire Hamilton-Jacobi formalism, including the motion equations. In doing so, it leads to a new relation between momentum and velocity and a new expression for the probability density of position. Furthermore, the concept of a discretely extended object arises naturally from this mathematical structure due to the oscillatory nature of the world-lines produced by these motion equations.

2.3 Action, Momentum, and World-Line in Terms of the Wave Function

This section derives the action, momentum, and world-line functions for a given system in terms of a particular wave function, \( \Psi \), for that system. It will be seen in later sections that this particular wave function consists of a specific linear combination of linearly independent solutions of Schrödinger’s equation and that the coefficients of that linear combination are defined in terms of the non-Newtonian constants that comprise the set \( \gamma \) introduced in section 2.1.3.

2.3.1 The Action Function

For now, assume that the required linear combination for \( \Psi \) is given, and let its real and imaginary parts be \( \Psi_R \) and \( \Psi_I \), respectively. The action function, \( S(q_v, \alpha_n, \gamma) \), can then be derived in terms of the wave function, \( \Psi = \Psi_R + i\Psi_I = R \exp(i S/h) \), and its value, \( \Psi_0 = \Psi R_0 + i \Psi I_0 = R_0 \exp(i S_0/h) \), at a selected reference point, \( q_v = (q_v)_0 \), in the extended configuration space of the given system. Let \( S_0 = 0 \) by convention. The
wave function then has the form
\[ \Psi = \Psi_0 \left( \frac{R}{R_0} \right) \exp \left( \frac{iS}{\hbar} \right). \]  

(5)

Solving for \( S \) gives
\[ S = \hbar \tan \left( \frac{\Psi_I \Psi_R - \Psi_R \Psi_I}{\Psi_R^2 + \Psi_I^2} \right). \]

This equation is independent of \( R \) and is therefore a direct equation for \( S \). Applying the arctangent addition theorem leads to the simplified equation
\[ S = \hbar \left[ \tan \left( \frac{\Psi_I}{\Psi_R} \right) - \tan \left( \frac{\Psi_I}{\Psi_R} \right) \right]. \]  

(6)

For a stationary state of a conservative system with energy \( E \), the wave function has the form
\[ \Psi = \psi \exp \left( -\frac{iEt}{\hbar} \right) \]

where \( \psi = R \exp \left( i\frac{W}{\hbar} \right) \) is the time-independent wave function. The action function is then \( S = W - Et \). Substituting these expressions for \( \psi \) and \( S \) into equation (5) gives
\[ \psi = \psi_0 \left( \frac{R}{R_0} \right) \exp \left( \frac{iW}{\hbar} \right) \]

which is of the same form as equation (5). Therefore, solving for the abbreviated action, \( W \), gives
\[ W = \hbar \left[ \tan \left( \frac{\psi_I}{\psi_R} \right) - \tan \left( \frac{\psi_I}{\psi_R} \right) \right] \]  

(7)

which is identical to equation (6) except that \( \Psi \) has been replaced by \( \psi \).

2.3.2 Generalized Momentum Components

In the theory of discrete extension, as in Newtonian theory, the derivative of \( S \) with respect to a generalized coordinate, \( q_\nu \), is the generalized momentum, \( p_\nu \), conjugate to \( q_\nu \). In terms of the real and imaginary parts of the wave function the momentum components, \( p_\nu \), are given by
\[ p_\nu = \frac{\partial S}{\partial q_\nu} = \frac{\hbar}{2} \left( \frac{\psi_R \frac{\partial \psi_I}{\partial q_\nu} - \psi_I \frac{\partial \psi_R}{\partial q_\nu}}{\psi_R^2 + \psi_I^2} \right). \]

This expression is independent of the reference point.

In standard quantum theory the probability density for position and the probability current density in the \( q_n \) direction are
\[ \rho = \Psi R^2 + \Psi I^2 \quad \text{and} \quad j_n = \frac{\hbar}{m} \left( \frac{\psi_R \frac{\partial \psi_I}{\partial q_n} - \psi_I \frac{\partial \psi_R}{\partial q_n}}{\psi_R^2 + \psi_I^2} \right), \]

respectively. Furthermore, if \( \hbar \) is allowed to approach zero in the standard theory, the resultant classical limit leads to an interpretation in which \( \Psi \) describes an ensemble
of particles with momentum components \( p_n = m j_n / \rho \). In the theory of discrete extension the probability interpretations of \( \rho \) and \( j_n \) do not apply, but it can be seen that for each cluster point the equation \( p_n = m j_n / \rho \) is true even when \( \hbar \neq 0 \).

For a stationary state of a conservative system \( S = W - Et \), and the momentum components are

\[
p_n = \frac{\partial S}{\partial q_n} = \frac{\partial W}{\partial q_n} = \frac{\hbar}{\Psi R^2 + \Psi I^2} \left( \psi_R \frac{\partial \psi_I}{\partial q_n} - \psi_I \frac{\partial \psi_R}{\partial q_n} \right).
\]

### 2.3.3 Motion Equations

In the theory of discrete extension, as in Newtonian theory, the motion equations are obtained from the derivatives of \( S \) with respect to the parameters, \( \alpha_n \), in a complete solution, \( S(q_{\nu}, (q_{\nu})_0, \alpha_n, \gamma) \), of the (generalized) Hamilton-Jacobi equation.

The derivative of \( S \) with respect to the parameter \( \alpha_n \) is

\[
\frac{\partial S}{\partial \alpha_n} = \frac{\hbar}{\Psi R^2 + \Psi I^2} \left( \psi_R \frac{\partial \psi_I}{\partial \alpha_n} - \psi_I \frac{\partial \psi_R}{\partial \alpha_n} \right) - \frac{\hbar}{\Psi R_0^2 + \Psi I_0^2} \left( \psi_R_0 \frac{\partial \psi_I_0}{\partial \alpha_n} - \psi_I_0 \frac{\partial \psi_R_0}{\partial \alpha_n} \right).
\]

This derivative depends on the reference point only through an additive constant.

The motion equations are obtained by setting the derivatives, \( \partial S / \partial \alpha_n \), equal to constants, \( \beta_n \). It can be seen from equation (8), however, that setting \( q_{\nu} = (q_{\nu})_0 \) in \( \partial S / \partial \alpha_n = \beta_n \) implies that \( \beta_n = 0 \) for each \( n \). Therefore, the motion equations are simply \( \partial S / \partial \alpha_n = 0 \) or, equivalently,

\[
\frac{\psi_R \frac{\partial \psi_I}{\partial \alpha_n} - \psi_I \frac{\partial \psi_R}{\partial \alpha_n}}{\Psi R^2 + \Psi I^2} = c_n = \frac{\hbar}{\Psi R_0^2 + \Psi I_0^2} \left( \psi_R_0 \frac{\partial \psi_I_0}{\partial \alpha_n} - \psi_I_0 \frac{\partial \psi_R_0}{\partial \alpha_n} \right)
\]

where the \( c_n \) are new constants that are determined by \( \Psi_0 \) and its derivatives.

For a stationary state of a conservative system, the energy, \( E \), may be taken as one of the constants, \( \alpha_n \), say \( \alpha_{n'} \). Then \( S = W(q_n, (q_n)_0, \alpha_1, \alpha_2, \ldots, \alpha_{n'-1}, E, \gamma) - Et \), and the motion equations are

\[
\frac{\partial W}{\partial \alpha_n} = \beta_n \quad \text{for} \quad n = 1, 2, \ldots, N - 1
\]

\[
\frac{\partial W}{\partial E} = 1 + \beta_{n'}. \]
In terms of the function $\psi$, these motion equations become

$$\frac{\psi_R}{\psi_R^2 + \psi_I^2} \frac{\partial \psi_I}{\partial \alpha_n} - \frac{\psi_I}{\psi_R^2 + \psi_I^2} \frac{\partial \psi_R}{\partial \alpha_n} = c_n \quad \text{for} \quad n = 1, 2, \ldots, N - 1$$

and

$$\frac{\psi_R}{\psi_R^2 + \psi_I^2} \frac{\partial \psi_I}{\partial E} - \frac{\psi_I}{\psi_R^2 + \psi_I^2} \frac{\partial \psi_R}{\partial E} = t + c_N.$$  

### 2.4 Conservative Systems with One Degree of Freedom

This section shows that conservative systems with one degree of freedom form a special class for which the theory of discrete extension provides simple, closed-form expressions for the action, momentum, and world-line functions. It also shows that various equations of the theory are generalizations of the corresponding equations in Newtonian theory.

#### 2.4.1 Dynamic Equations for Conservative Systems with One Degree of Freedom

If a conservative system is in a stationary state with energy $E$, the wave function has the form $\Psi(x, t, E) = \psi(x, E) \exp(-iEt/\hbar)$ where $\psi(x, E)$ satisfies the time-independent Schrödinger equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2} \left[ E - V(x) \right] \psi = 0. \quad (9)$$

A partial derivative is indicated here because the Hamilton-Jacobi formalism also involves derivatives with respect to the “constant” $E$. Since this equation is real, its most general complex solution has the form $\psi = C_1 \psi_1 + C_2 \psi_2$ where $\psi_1 = \psi_1(x, E)$ and $\psi_2 = \psi_2(x, E)$ are any two linearly independent, real solutions, and $C_1$ and $C_2$ are arbitrary complex constants. According to equation (7) the (abbreviated) action, $W$, depends only on the ratio of the real and imaginary parts of $\psi$. Therefore, as far as $W$ is concerned, $\psi$ can be divided by $C_1$ leaving a general solution of the form

$$\psi = \psi_1 + (\Lambda + iB) \psi_2 \quad (10)$$

where $\Lambda$ and $B$ are any two arbitrary, real constants. Substituting $\psi_R = \psi_1 + \Lambda \psi_2$ and $\psi_I = B \psi_2$ into equation (7) gives

$$W = \hbar \left[ \mathrm{atan} \left( \frac{B \psi_2}{\psi_1 + \Lambda \psi_2} \right) - \mathrm{atan} \left( \frac{B \psi_2}{\psi_1 + \Lambda \psi_2} \right) \right] \quad (11)$$

where $\psi_{10} = \psi_1(x_0, E), \psi_{20} = \psi_2(x_0, E)$, and $x_0$ is the reference point coordinate.

The momentum and its $x$-derivative are then

$$p = \frac{\partial W}{\partial x} = \hbar \left[ \frac{B \left[ \psi_1 \psi_2 x - \psi_2 \psi_1 x \right]}{\left( \psi_1 + \Lambda \psi_2 \right)^2 + \left( B \psi_2 \right)^2} \right]$$

and

$$W = \hbar \left[ \frac{B \left[ \psi_1 \psi_2 x - \psi_2 \psi_1 x \right]}{\left( \psi_1 + \Lambda \psi_2 \right)^2 + \left( B \psi_2 \right)^2} \right]$$
\[ p_x = \frac{\partial p}{\partial x} = \hbar \left[ \frac{2\mathbb{B} (\psi_2 \psi_1 - \psi_1 \psi_2) \left[ (\psi_1 + \lambda \psi_2)(\psi_1 + \lambda \psi_2) + \mathbb{B}^2 \psi_2 \psi_2 \right]}{\left[ (\psi_1 + \lambda \psi_2)^2 + (\mathbb{B} \psi_2)^2 \right]^2} \right] \]

where \( \psi_1 = \psi_1 / \partial x \) and \( \psi_2 = \psi_2 / \partial x \), and where Schrödinger’s equation (9) was used to eliminate the second derivatives, \( \psi_{1r} \) and \( \psi_{2r} \), from the \( p_x \) equation.

The constants \( \lambda \) and \( \mathbb{B} \) can now be evaluated in terms of pairs of physical constants such as the momenta, \( p_1 = p(x_1, E) \) and \( p_2 = p(x_2, E) \), at two reference locations, or the momentum and its \( x \)-derivative, \( p_0 = p(x_0, E) \) and \( (p_x)_0 = p_x(x_0, E) \), at one reference location. Choosing the second alternative, solving for \( \lambda \) and \( \mathbb{B} \) in terms of \( p_0 \) and \( (p_x)_0 \), and substituting into equations (10), (11), and (12) gives the general equations

\[ \psi = C \left[ \{ \hbar [2 p_0 (\psi_2 x_0) + (p_x)_0 \psi_1] - 2 i p_0^2 \psi_2 \} \psi_1 \right] - \{ \hbar [2 p_0 (\psi_1 x_0) + (p_x)_0 \psi_0] - 2 i p_0^2 \psi_1 \} \psi_2 \]  

\[ W = \frac{\hbar}{2 \pi} \text{catan} \left[ \frac{2 p_0^2 (\psi_1 x_0 - \psi_0 \psi_1)}{\hbar + \{ (p_x)_0 \psi_0 - 2 p_0 (\psi_2 x_0) \} \psi_1} \right] \]  

\[ p = \frac{4 p_0^2}{\hbar} \left[ \psi_1 (\psi_2 x_0 - \psi_0 \psi_2) - \psi_0 (\psi_2 x_0) \right] \left( \psi_1 (\psi_2 x_0 - \psi_0 \psi_2) - \psi_0 (\psi_2 x_0) \right)^2 \]  

The absolute square of equation (13) is

\[ \psi^* \psi = |C|^2 \left[ \hbar^2 \left[ \left( \frac{(p_x)_0 \psi_2 x_0 + 2 p_0 (\psi_2 x_0)}{\psi_1} \right)^2 \right] + \left[ \frac{2 p_0^2 (\psi_2 x_0 - \psi_0 \psi_2)}{\hbar} \right]^2 \right] \]  

While equation (14) expresses \( W \) in terms of solutions of Schrödinger’s equation, \( W \) itself, satisfies the third-order, non-linear equation (5). Therefore, the constants \( p_0 \) and \( (p_x)_0 \) together with a trivial additive constant can be viewed as three arbitrary integration constants for this third-order equation.

Two sets of arbitrary constants are discussed in section 2.1.3 the order-reduction constants, \( \gamma \), and the Hamilton-Jacobi constants, \( \alpha_n \). For the conservative systems with one degree of freedom considered here, \( p_0 \) and \( (p_x)_0 \) comprise the set \( \gamma \), and the energy, \( E \), comprises the set \( \alpha_n \).

In equations (13) and (16) \( C \) is an arbitrary complex constant that plays no role in the theory. In equation (14) the function \text{catan} is a continuous, cumulative version of the arctangent function. Thus, whereas the ordinary arctangent function jumps discontinuously from \( \pi/2 \) to \( -\pi/2 \) if its argument jumps from \( \infty \) to \( -\infty \), the \text{catan} function
continues to increase. To clarify this difference, consider the two functions

\[ f(x) = \tan\left[\tan(x)\right] \quad \text{and} \quad g(x) = \tan\left[x\right] = \int_0^x \left[\frac{2a}{1 + a^2 + (1 - a^2) \cos(2u)}\right] du. \]

These functions are identical in the interval \(-\pi/2 < x < \pi/2\), and their derivatives of all orders are identical everywhere except at discontinuities. In general, the function \(f(x)\) is discontinuous and periodic with period \(\pi\), whereas \(g(x)\) is continuous and monotonically increasing. For the special case \(a = 1\), however, \(f(x) = g(x) = x\).

Figures 1 and 2 illustrate these two functions for \(a = .02\) and \(a = 50\), respectively. The dashed and solid lines represent the functions \(f(x)\) and \(g(x)\), respectively. Although \(g(x)\) is continuous and smooth, it approaches a staircase function as \(a \to 0\). Similarly, as \(a \to \infty\), it approaches a shifted staircase function. In the theory of discrete extension the action function for a quantum system is a quasi-discrete function of this type, and a parameter like \(a\) characterizes a cluster’s level of quantum activation, a concept defined below. Such a parameter will be called an activation parameter.

In addition to the action and momentum functions, (14) and (15), the dynamical equations of the theory include the cluster’s world-line function. This function is given by the Hamilton-Jacobi relation \(t + \beta \cdot \psi = \partial W/\partial E\). In taking this derivative it must be observed that the “constants” \(\psi_{10}, \psi_{20}, (\psi_{1x})_0, (\psi_{2x})_0, p_{0}, \) and \((p_x)_0\) are all functions of \(E\). Therefore, differentiation of \(W\) with respect to \(E\) generates the following six new
The equation of the world-line is then
\[ (\psi_1, \psi_2, t_0, t_{xx}) = 0 \]
The dynamic variables, \( \psi \) clustor point at the reference point, where the time origin has been chosen so that \( \beta = v_E \).

The equation of the world-line is then
\[ (\psi_1, \psi_2, t_0, t_{xx}) = 0 \]
The momentum equation, (15), has the form
\[ p = f(x, E) \]
and the generalized Hamiltonian function, introduced in section 2.1.3, is therefore \( K(x, p, E) = p - f(x, E) = 0 \).

The Generalized Hamiltonian Function, Canonical Equations, and Hamilton-Jacobi Equation

The momentum equation, (15), has the form \( p = f(x, E) \), and the generalized Hamiltonian function, introduced in section 2.1.3, is therefore \( K(x, p, E) = p - f(x, E) = 0 \).

This equation also defines the system’s phase space trajectory for any fixed value of \( E \).
If the equation \( K(x, p, E) = 0 \) could be solved for \( E \), the result would be the Hamiltonian in its explicit, ordinary form, \( H = g(x, p) = E \). As it is, however, the equation \( K(x, p, E) = p - f(x, E) = 0 \) is an implicit equation for \( H \). Therefore, in terms of the functions \( f(x, E) \) and \( W(x, E) \), Hamilton’s canonical equations,

\[
\frac{dx}{dt} = \frac{\partial g}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial g}{\partial x},
\]

become

\[
\frac{dx}{dt} = \frac{1}{\left( \frac{\partial f}{\partial E} \right)} = \frac{1}{\left( \frac{\partial^2 W}{\partial x \partial E} \right)}, \quad \frac{dp}{dt} = \left( \frac{\partial f}{\partial x} \right) = \left( \frac{\partial^2 W}{\partial x^2} \right).
\]

Substituting \( \frac{\partial W}{\partial x} \) for \( p \) in the generalized Hamiltonian function gives the first-order, generalized Hamilton-Jacobi equation for the system,

\[
\frac{\partial W}{\partial x} - f(x, E) = 0,
\]

in terms of solutions of Schrödinger’s equation. In addition to the Hamilton-Jacobi constant, \( \alpha = E \), it contains two arbitrary, non-Newtonian constants, \( p_0 \) and \( (p_x)_0 \), which comprise the set of constants, \( \gamma \), associated with the reduction of the third-order equation (3) to a first order equation. In Newtonian theory the corresponding equation is, of course,

\[
\frac{\partial W_N}{\partial x} - \sqrt{2m(E - V(x))} = 0.
\]

If the generalized Hamiltonian function, \( K(x, p, E) \), could be derived from the physical properties of the system, as in Newtonian theory, then the wave function would not be needed, and \( W \) could be obtained from the integral

\[
W = \int p \, dx = \int f(x, E) \, dx.
\]

In the theory of discrete extension, however, that is impossible, and \( W \) must be obtained from solutions of Schrödinger’s equation.

### 2.4.3 Generalized Newtonian Energy Equation

Making the substitution \( dW/\partial x = p \) in equation (3) and solving for \( E \) leads to the energy equation for the theory of discrete extension, presented here in both expanded and compact forms:

\[
E = \frac{p_x^2}{2m} \left[ 1 + \left( \frac{\hbar}{2} \right) \left( 2 \frac{p_x}{p^4} \right) \right] + V(x) = T + V
\]

\[
E = \frac{p_x^2}{2m} \left[ 1 + \left( \frac{\hbar}{2} \right) \frac{1}{p_x} \frac{d}{dx} \left( \frac{p_x^3}{p^3} \right) \right] + V(x) = T + V.
\]

15
Converting the $x$-derivatives to $t$-derivatives gives

$$E = \frac{p^2}{2m} \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \left( \frac{2p(\dot{x} - \dot{p}) - 3\dot{p}^2}{p^2\dot{x}^3} \right) \right] + V(x) = T + V$$

$$E = \frac{p^2}{2m} \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \frac{1}{\dot{p}} \frac{d}{dt} \left( \frac{\dot{p}^2}{p^3\dot{x}^2} \right) \right] + V(x) = T + V \tag{19}$$

These equations reduce to the classical energy equation for $\hbar = 0$. As in Newtonian theory the total energy is the sum of a kinetic part, $T$, and a potential part, $V$.

In equation (19) both $p$ and $\dot{x}$ appear in the denominator of $T$ so that neither can become zero if the kinetic energy is to remain finite. (An exception can occur, however, if the motion is such that the numerator of the quantum term is zero.) Thus, in general, a cluster can never have zero momentum and can never come to rest, i.e., can never achieve $\dot{x} = 0$. It will be seen below that these restrictions are illustrated by the behavior of a harmonic oscillator near its classical turning points. On the contrary, for a Newtonian oscillator $p = \dot{x} = 0$ at a turning point.

### 2.4.4 Generalized Newtonian Equations of Motion

In the theory of discrete extension the dynamic behavior of a conservative system with one degree of freedom is given in closed form by equations (15) and (17). These equations are a result of the Hamilton-Jacobi approach to the problem. Alternatively, the theory can be formulated as a set of differential equations that are a generalization of the Newtonian equations of motion.

For a conservative system with one degree of freedom the Newtonian equations can be obtained by differentiating the Newtonian Hamilton-Jacobi equation,

$$\frac{1}{2m} \left( \frac{\partial}{\partial x} W_N(x, E) \right)^2 + V(x) - E = 0 \tag{20},$$

with respect to $E$ and with respect to $x$, making the substitutions $\frac{\partial W_N}{\partial x} = p$ and $\frac{\partial W_N}{\partial E} = t$, and converting $x$-derivatives to $t$-derivatives. Thus, differentiating (20) with respect to $E$ gives the first Newtonian equation, $p = m\dot{x}$. Differentiating (20) with respect to $x$ gives $p\dot{p} = -m\ddot{x}dV/dx$. Since $p = m\dot{x}$, however, this equation reduces to the second Newtonian equation, $\ddot{p} = -dV/dx = F$ where $F$ is the force. These two Newtonian equations comprise a linear system that is first-order in the time derivatives of both $x$ and $p$. The initial values $x_0$ and $p_0$ are required to determine a solution.

In the theory of discrete extension equation (3) is a generalization of the classical Hamilton-Jacobi equation (20). Differentiating it with respect to $E$ and $x$, converting $x$-derivatives to $t$-derivatives, eliminating $E$, and simplifying leads to the following
generalization of the Newtonian equations of motion:

\[
\begin{align*}
\rho \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \left( \frac{p \dot{x}^2}{\rho^4 x^4} \right) \right] &= m \ddot{x} \\
\rho \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \left( \frac{p \dot{x}^2}{m \rho^2 \dot{x}^4} \right) \right] &= -\frac{dV}{dx} = F.
\end{align*}
\]

A more compact form for these equations is

\[
\begin{align*}
\rho \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \frac{1}{2} \frac{d^2}{dt^2} \left( \frac{1}{p^2 x^2} \right) \right] &= m \ddot{x} \\
\rho \left[ 1 + \left( \frac{\hbar}{2} \right)^2 \frac{p^2}{m} \frac{d}{dt} \left( \frac{\dot{p}}{p^3 x^3} \right) \right] &= -\frac{dV}{dx} = F.
\end{align*}
\]

The de Broglie-Bohm theory postulates the classical relation, \( p = m \dot{x} \), between momentum and velocity. It can be seen that in the theory of discrete extension this simple relation does not hold.

Although these equations of motion reduce to Newton’s equations for \( \hbar = 0 \), they can be integrated only in patches if \( \hbar \neq 0 \). This restriction is due to the generally multi-valued nature of \( x(t) \) and \( p(t) \) and to the singularities at \( p = 0 \) and \( \dot{x} = 0 \).

The equations of motion (21) comprise a non-linear system that is third order in the time derivatives of both \( x \) and \( p \). Therefore, a set of six initial values are required to determine a solution. In addition to the two Newtonian initial values, \( x_0 \) and \( p_0 \), four non-Newtonian initial values, \( \dot{x}_0 \), \( \ddot{x}_0 \), \( \dot{p}_0 \), and \( \ddot{p}_0 \), are also required. Note that in the Hamilton-Jacobi formulation above, the world-line equation (17) also contains a set of six data elements: \( E, x_0, p_0, (p_x)_0, (t_x)_0, \) and \( (t_{xx})_0 \). With the help of the energy equation (18), the four non-Newtonian initial values can be expressed in terms of the Hamilton-Jacobi data as follows:

\[\dot{x}_0 = \frac{1}{(t_x)_0} \quad \ddot{x}_0 = -\frac{(t_{xx})_0}{(t_x)_0^3}\]

\[\dot{p}_0 = \frac{(p_x)_0}{(t_x)_0} \quad \ddot{p}_0 = \frac{p_0^2 (t_x)_0 \left[ 2m(E - V(x_0)) - p_0^2 \right]}{2 \left( \frac{\hbar}{2} \right)^2 \rho_0 (t_x)_0^3} + \frac{(h/2)^2 (p_x)_0 [3(p_x)_0 (t_x)_0 - 2 p_0 (t_{xx})_0]}{2 \left( \frac{\hbar}{2} \right)^2 \rho_0 (t_x)_0^3} .\]

2.4.5 Quasi-Newtonian Clusters

For any given potential function, \( V(x) \), there are special values of the reference point parameters, \( p_0, (p_x)_0, (t_x)_0, \) and \( (t_{xx})_0 \), for which the cluster’s behavior is particularly
simple. Let the reference point, \( x = x_0 \), be chosen so that \( V(x_0) = 0 \). These special values are then the values of the corresponding Newtonian parameters, \( p_N, (p_x)_N, (t_x)_N \), and \( (t_{xx})_N \) at that point. That is,

\[
\begin{align*}
p_0 &= p_N = \sqrt{2mE} \\
(p_x)_0 &= (p_x)_N = \frac{\dot{p}}{\dot{x}} = -\sqrt{\frac{m}{2E}} V'(x_0) \\
(t_x)_0 &= (t_x)_N = \frac{1}{\dot{x}} = \sqrt{\frac{m}{2E}} \\
(t_{xx})_0 &= (t_{xx})_N = -\frac{\ddot{x}}{\dot{x}^3} = \sqrt{\frac{m}{8E^3}} V'(x_0)
\end{align*}
\]

where \( V'(x) = dV/dx \) and where \( m \) and \( E \) are the mass and energy of the clustor. For these special values the clustor’s action function, phase-space trajectory, and worldline will resemble those of a Newtonian particle that is subject to the same potential. A clustor of this type will be called quasi-Newtonian.

It will be seen below that a quasi-Newtonian free clustor is an unextended object and that its behavior is identical to the behavior of a Newtonian free particle. It will also be seen that for the harmonic oscillator potential a quasi-Newtonian clustor is unextended and that its behavior is approximately Newtonian as long as its energy is high and it is not near a turning point.

If a reference point is chosen at which \( V'(x_0) = 0 \) in addition to \( V(x_0) = 0 \), then \( (p_x)_N = 0 \) and \( (t_{xx})_N = 0 \) at that point. Therefore, for a clustor to be quasi-Newtonian in such cases, it is required that \( (p_x)_0 = 0 \) and \( (t_{xx})_0 = 0 \). For a free clustor \( V = V' = 0 \) at all points. For the harmonic oscillator these conditions occur at the equilibrium point. For the Coulomb potential they occur at \( r = \infty \). In contrast, there is no point at which \( V' = 0 \) for the linear potential, \( V = cx \), of a uniform field.

3 THE FREE CLUSTOR

The simplest of all systems is the free clustor with one degree of freedom. The dynamic equations for that system are derived in this section, and the corresponding clustor behavior is illustrated. This section also introduces the concept of quantum activation, and presents parameters for quantifying an activation level.

Since a free clustor is not a bound state system, the theory places no restrictions on its allowed energy levels. For bound-state systems, however, the theory leads to quasi-discrete energy levels, a feature of clustor dynamics that will become apparent later in the analysis of the harmonic oscillator.

Let the mass and energy of the free clustor be \( m \) and \( E \) respectively, and let the reference point be at \( x_0 = 0 \). The potential function is \( V(x) = 0 \), and two linearly independent solutions of the time-independent Schrödinger equation \( \psi_1 \) are

\[
\psi_1(x, E) = \cos[k(E) x] \quad \text{and} \quad \psi_2(x, E) = \sin[k(E) x]
\]

where \( k(E) = \sqrt{2mE}/\hbar \). The associated wavelength is \( \lambda = 2\pi/k \). Since the reference point coordinate is \( x_0 = 0 \), the wave function constants needed for equations \( 13 \)
through (17) are

\[\psi_1 = 0 \quad \psi_2 = 0 \quad (\psi_1)_0 = 0 \quad (\psi_2)_0 = k\]

\[ (\psi_1 E)_0 = 0 \quad (\psi_2 E)_0 = 0 \quad (\psi_1 x)_0 = 0 \quad (\psi_2 x)_0 = \frac{m}{\hbar k}. \]

The following dimensionless constants are defined in terms of the reference point parameters, \( p_0, (p_x)_0, (t_x)_0, \) and \((t_{xx})_0\), and serve to simplify subsequent equations:

\[ A = \frac{p_0}{\hbar k} \quad B = \frac{(p_x)_0}{2kp_0} \quad C = \frac{\hbar^2 k^2 (t_x)_0}{mp_0} - 1 \quad D = \frac{\hbar^2 k [p_0 (t_{xx})_0 - 2(p_x)_0 (t_x)_0]}{2mp_0^2}. \]

### 3.1 The Wave Function and the Action, Momentum, and World-Line Functions

Substituting these expressions into equations (13), (14), (15), and (17) gives

\[
\begin{align*}
\psi &= \cos(kx) - B \sin(kx) + iA \sin(kx) \\
W &= \hbar \left[ \frac{1}{2\pi} \tan^{-1} \left( \frac{A \sin(kx)}{\cos(kx) - B \sin(kx)} \right) \right] \\
p &= \hbar k \left[ \frac{A}{[\cos(kx) - B \sin(kx)]^2 + [A \sin(kx)]^2} \right] \\
t &= \frac{m}{\hbar k^2} \left[ \frac{A [kx + C \sin(kx) \cos(kx) + D \sin(kx)^2]}{[\cos(kx) - B \sin(kx)]^2 + [A \sin(kx)]^2} \right].
\end{align*}
\]

The corresponding Newtonian functions are

\[
W_N = \hbar \left( \frac{kx}{2\pi} \right) \quad p_N = \hbar k \quad t_N = \frac{m}{\hbar k^2} (kx).
\]

If the reference point parameters for a given clustor have the Newtonian values,

\[ p_0 = \sqrt{2mE} = \hbar k \quad (p_x)_0 = 0 \quad (t_x)_0 = \sqrt{m/2E} = m/\hbar k \quad (t_{xx})_0 = 0, \]

then \( A = 1 \) and \( B = C = D = 0 \), and the clustor is quasi-Newtonian by definition. In equations (22) the wave function then becomes \( \psi = \exp(ikx) \), and the expressions for \( W, p, \) and \( t \) reduce to the corresponding Newtonian equations (23). Therefore, a quasi-Newtonian free clustor is unextended, and it behaves in every respect like a Newtonian free particle.

### 3.2 Quantum Activation and Activation Parameters

This section introduces the concept of quantum activation in order to quantify the deviation of a free clustor’s state from the quasi-Newtonian state. Since the behaviors of a Newtonian free particle and a quasi-Newtonian free clustor are identical, such a clustor
will be said to have a quantum activation level of zero. Deviation of the value of any of the constants, \( A, B, C, \) or \( D, \) from its Newtonian value corresponds to non-Newtonian behavior of the clustor. In that case the clustor will be said to possess some level of quantum activation, and it will be useful to find parameters that quantify that level.

Toward that end the following four new constants are introduced:

\[
\alpha_1 = \sqrt{1 - \left(\frac{2A}{1+A^2+B^2}\right)^2} \quad \phi_1 = \frac{1}{2} \arctan(2B, 1-A^2-B^2)
\]

\[
\alpha_2 = \frac{1}{2} \sqrt{C^2+D^2} \quad \phi_2 = \frac{1}{2} \arctan(C, D).
\]

In terms of these new parameters the dynamic equations in (22) become:

\[
W = h\left[\frac{1}{2\pi} \text{cata}n\left(\pm\sqrt{1-\alpha_1^2} \left[\tan(kx+\phi_1) - \tan(\phi_1)\right]\right)\right]\]

(24)

\[
p = \hbar k\left[\pm\sqrt{1-\alpha_1^2} \frac{1+\alpha_1\cos(2(kx+\phi_1))}{1+\alpha_1\cos(2(kx+\phi_1))}\right]
\]

(25)

\[
t = \frac{m}{\hbar k^2}\left[\pm\sqrt{1-\alpha_1^2} \left[\cos(2(kx+\phi_2)) - \cos(2\phi_2)\right]\right].
\]

(26)

where the \( \pm \) sign on the radicals is determined by the sign of \( p_0. \)

The world-line equation (26) contains the product of two bracketed factors each with its own \( \alpha \) and \( \phi \) parameters. The first of these factors has the same periodic form as the momentum equation (25).

The constants \( \alpha_1 \) and \( \alpha_2 \) quantify the clustor’s level of activation and will be called the primary and secondary activation parameters, respectively. The phase angles \( \phi_1 \) and \( \phi_2, \) however, play no role in determining the activation level. Equations (24), (25), and (26) correspond to quasi-Newtonian behavior if and only if \( \alpha_1 = \alpha_2 = 0. \) In that case the clustor will be said to be unactivated.

### 3.3 The de Broglie Relation and Average Momentum

Oscillations of the wave function, \( \psi(x), \) in (22) have the spatial period \( \lambda = 2\pi/k. \) Oscillations of \( \rho = \psi^*(x)\psi(x), \) however, have the spatial period \( \lambda/2 \) as do oscillations of the momentum function, \( p(x). \)

Integrating the momentum equation (25) over one spatial period produces an average value, \( \overline{p} = \hbar k, \) independent of the values of \( p_0 \) and \( (p_x)_0 \) and independent of time. Therefore, the theory of discrete extension leads to the following relation that resembles the de Broglie relation of standard quantum theory:

\[
\lambda = \frac{2\pi}{k} = \frac{\hbar}{\overline{p}} = \frac{\hbar}{\overline{\rho}}.
\]

Note that while this equation relates \( \overline{p} \) to the wavelength of \( \psi(x), \) the wavelength of \( p(x) \) itself is half that value.
3.4 Illustrations of Free Cluster Dynamics

Figures 3 through 8 illustrate the behavior of a free cluster for two different levels of quantum activation. The mass and energy of the cluster are the same in both cases. In the first case the cluster is moderately activated with \( A = 0.378 \), i.e., with \( \alpha_1 = 0.75 \). In the second case the activation level is significantly higher with \( A = 0.022 \), i.e., with \( \alpha_1 = 0.999 \). In both cases \( B = C = D = \phi_1 = \alpha_2 = 0 \).

In figures 3 and 4 the solid curve represents the action function for the activated cluster. The dashed line represents the action function for the corresponding Newtonian particle or unactivated cluster. In figure 3 the action function for the moderately activated cluster follows the Newtonian function closely with only minor excursions above and below the Newtonian line. In figure 4 the activation level is significantly higher. In that case the action function for the activated cluster still follows the Newtonian line, but it now approaches a staircase function with steps of height \( h/2 \) and spatial period \( \lambda/2 \). Thus, rather than increasing smoothly with position, the action of a highly activated cluster increases in an alternating series of plateaus and quasi-discrete steps.

Figures 5 and 6 show that for an activated cluster the momentum is a periodic function of position. The dashed lines in the figures show the average momentum value, \( \hbar k \), which is independent of the level of quantum activation. This average equals the constant value of the momentum of the corresponding Newtonian particle or unactivated cluster. When the activation level is high, as in figure 6 the momentum is nearly zero except at a set of quasi-discrete, spatially periodic positions.

Figure 7 shows the world-line for the moderately activated cluster. The dots indicate the positions of the five cluster points that exist at time \( t = 4.5 \, m/\hbar k^2 \). These points are determined by the intersections of the world-line and the dashed line of simultaneity. It can be seen that as time advances, the cluster, as a whole, moves to the right as new cluster points are created on the right and old cluster points are annihilated on the left. It is also apparent that the number of cluster points increases or decreases in steps of two and that the total number is always odd. Figure 8 shows the world-line for the more highly activated cluster along with the first few cluster points that exist at time \( t = 320 \, m/\hbar k^2 \). The positions of the cluster points are distributed in a nearly periodic pattern.

It can be seen in figures 7 and 8 that, at any given moment, a cluster’s various cluster points are distributed over a range of different spatial positions. Therefore, in general, the action and momentum values associated with distinct cluster points will be unequal.

3.5 The Fully Activated Free Cluster

It is easily shown that \( \alpha_1 \) is real for all values of \( A \) and \( B \) or, equivalently, for all values of \( p_0 \) and \( (p_x)_0 \). Its range is \( 0 \leq \alpha_1 \leq 1 \). If \( \alpha_1 = 1 \) the cluster will be said to be fully activated. In that case the bracketed factor common to equations (25) and (26) reduces to the sum of an infinite number of equally-spaced Dirac delta functions:

\[
\lim_{\alpha_1 \to 1} \left[ \frac{\sqrt{1 - \alpha_1^2}}{1 + \alpha_1 \cos(2(kx + \phi_1))} \right] = \sum_{n = -\infty}^{\infty} \pi \delta[kx + \phi_1 - (n + 1/2) \pi].
\]
Fig. 3 Action function for a moderately activated free clustor. The dashed line is the action function for the corresponding Newtonian particle.

Fig. 4 At higher activation levels the action function approaches a staircase function with steps of height \( \hbar / 2 \) and spatial period \( \lambda / 2 \).

Fig. 5 The momentum of an activated free clustor varies periodically with position. The dashed line shows its average value, \( \hbar k \).

Fig. 6 At higher activation levels the momentum function approaches a series of delta functions. The average value remains \( \hbar k \).

Fig. 7 The world-line of an activated clustor is oscillatory. The number of clustor points remains odd while changing with time in steps of 2.

Fig. 8 At higher activation levels the world-line approaches a series of delta functions, and the clustor point locations become spatially periodic.
Certain aspects of a fully activated free clustor resemble those of a free particle as described in standard quantum theory. Specifically, as $\alpha_1 \to 1$ the dynamic activity of the clustor vanishes, and its clustor points become motionless. Thus, the clustor approaches a stationary state like that of a free particle in the standard theory. Like a free particle in the standard theory, there are an infinite number of positions on the real line at which a clustor point could be found. Furthermore, the probability of finding either object at a particular position is uniform over the set of the object’s possible positions. The sets of possible positions for the two objects are not the same, however. According to the standard theory the possible positions at which a free particle could be found comprise a continuous set that covers the entire line. For a fully activated clustor, on the other hand, the possible positions at which a clustor point could be found comprise only a discrete set of equally spaced points. It can be seen from the world-lines in figures 7 and 8 that as the activation level of a clustor is raised, its properties approach those just described for a fully activated clustor.

### 3.6 Two Types of Quantum Activation

Clustor behavior that is associated with activation due to $\alpha_1 \neq 0$ differs from the behavior that is associated with $\alpha_2 \neq 0$. This difference can be understood best by considering the two extreme cases ($\alpha_1 \neq 0$, $\alpha_2 = 0$) and ($\alpha_1 = 0$, $\alpha_2 \neq 0$). Clustor motion for the first case is illustrated in figures 7 and 8. For increasing values of $x$ the heights of successive peaks in these world-lines increase faster than the heights of successive troughs. Therefore, as time advances, more and more clustor points are generated, and the clustor spreads out spatially to ever greater widths.

In the second case, illustrated in figure 9, the values of the activation parameters are $\alpha_1 = 0$ and $\alpha_2 = 5$, and the world-line is a linearly rising sinusoidal curve. Therefore, as time advances, there is no net change in the number of clustor points. Furthermore, as the clustor moves, the clustor points do not spread out, but remain within a fixed spatial width, about $1.3\lambda$ in this example. Note that in this second case the action and momentum functions assume their Newtonian forms in (23) even though the world-line is non-Newtonian and the clustor is discretely extended.

### 4 THE INTERACTION OF A FREE CLUSTOR WITH A BARRIER — THE EMERGENCE OF DISCRETE EXTENSION

This section analyzes the interaction between a clustor and a rectangular barrier or well. It is shown that an unactivated clustor becomes activated as a result of the interaction.

A rectangular barrier or well is defined by a potential with a constant value, $V$, in the interval between two points, $x = x_1$ and $x = x_2$, and with the value zero elsewhere. For a barrier $V > 0$, and for a well $V < 0$. The region $x < x_1$ will be referred to as region 1; the region $x_1 \leq x \leq x_2$ will be referred to as region 2; and the region $x_2 < x$ will be referred to as region 3.

A free clustor with mass $m$ and energy $E > 0$ approaches the barrier or well from $x = x_0$ in region 1. In general, this clustor may be activated and may have arbitrary values for the reference point parameters, $p_0$, $(p_x)_0$, $(t_s)_0$, and $(t_{xx})_0$ (or equivalently...
Fig. 9 In this example the number of cluster points never exceeds seven. As the cluster moves, its width never exceeds a value of about $1.3\lambda$.

Fig. 10 An unactivated free cluster is activated by a barrier encounter. The resultant primary activation level in region 3 is given by $a1 = .998$

$A, B, C,$ and $D$) at $x_0$. The wave functions in the various regions have wave numbers that are functions of energy and are given by

$$k = \frac{\sqrt{2mE}}{\hbar} \quad \text{in regions 1 and 3} \quad \text{and} \quad \kappa = \frac{\sqrt{2m(V - E)}}{\hbar} = rk \quad \text{in region 2}.$$  

The wave number ratio, $r = \kappa/k$, is associated with the energy ratio $V/E = 1 + r^2$.

All of the results of the following analysis are valid whether $E < V$ so that $\kappa$ is real or $V < E$ so that $\kappa$ is imaginary. Certain functions that are hyperbolic in the first case automatically become trigonometric in the second. In both cases all action, momentum, and world-line functions are real.

The wave function in the various regions is obtained in the usual way by matching the wave function and its derivative at the boundaries of the regions. The action function in the various regions is then obtained from this wave function by using equation [14] and requiring the action to be continuous at the boundaries.
4.1 The Action Function and the Change in Activation Level

The following dimensionless “constants” (actually functions of \(E\)) serve to simplify the expression for the action function:

\[
\begin{align*}
    a_1 &= 0 \quad a_2 = r \begin{cases}
        a \cos[k(x_1 - x_0)] + b \sin[k(x_1 - x_0)] & \text{if } x < x_1 \\
        a \cosh[rk(x_2 - x_1)] + b \sinh[rk(x_2 - x_1)] & \text{if } x_1 \leq x \leq x_2 \\
        a \cosh[rk(x_2 - x_1)] + b \sinh[rk(x_2 - x_1)] & \text{if } x_2 < x.
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    b_1 &= A \quad b_2 = \begin{cases}
        b \cos[k(x_1 - x_0)] - a \sin[k(x_1 - x_0)] & \text{if } x < x_1 \\
        b \cosh[rk(x_2 - x_1)] + a \sinh[rk(x_2 - x_1)] & \text{if } x_1 \leq x \leq x_2 \\
        b \cosh[rk(x_2 - x_1)] + a \sinh[rk(x_2 - x_1)] & \text{if } x_2 < x.
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    c_1 &= 1 \quad c_2 = r \begin{cases}
        c \cos[k(x_1 - x_0)] & \text{if } x < x_1 \\
        c \cos[k(x_1 - x_0)] & \text{if } x_1 \leq x \leq x_2 \\
        c \cos[k(x_1 - x_0)] & \text{if } x_2 < x.
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    d_1 &= B \quad d_2 = \begin{cases}
        d \cos[k(x_1 - x_0)] + c \sin[k(x_1 - x_0)] & \text{if } x < x_1 \\
        d \cos[k(x_1 - x_0)] + c \sin[k(x_1 - x_0)] & \text{if } x_1 \leq x \leq x_2 \\
        d \cos[k(x_1 - x_0)] + c \sin[k(x_1 - x_0)] & \text{if } x_2 < x.
    \end{cases}
\end{align*}
\]

Using these constants the action function is then

\[
W = \begin{cases}
    \frac{h}{2\pi} \text{c atan} \left( \frac{a \cos[k(x-x_0)] + b \sin[k(x-x_0)]}{c \cos[k(x-x_0)] - d \sin[k(x-x_0)]} \right) & \text{if } x < x_1 \\
    \frac{h}{2\pi} \text{c atan} \left( \frac{a \cosh[rk(x-x_1)] + b \sinh[rk(x-x_1)]}{c \cosh[rk(x-x_1)] - d \sinh[rk(x-x_1)]} \right) & \text{if } x_1 \leq x \leq x_2 \\
    \frac{h}{2\pi} \text{c atan} \left( \frac{a \cosh[rk(x-x_1)] + b \sinh[rk(x-x_1)]}{c \cosh[rk(x-x_1)] - d \sinh[rk(x-x_1)]} \right) & \text{if } x_2 < x.
\end{cases}
\]

In regions 1 and 3 the cluster’s primary activation parameter, \(\alpha_1\), is given by

\[
\alpha_1 = \begin{cases}
    \sqrt{1 - \frac{2(a_1 d_1 + b_1 c_1)}{a_1^2 + b_1^2 + c_1^2 + d_1^2}} & \text{if } x < x_1 \\
    \sqrt{1 - \frac{2(a_3 d_3 + b_3 c_3)}{a_3^2 + b_3^2 + c_3^2 + d_3^2}} & \text{if } x_2 < x.
\end{cases}
\]

Deriving an equation for the secondary activation parameter, \(\alpha_2\), in region 3 is a straightforward task, but the result is too lengthy to present here.

4.2 An Example of a Barrier Encounter

As an example of an encounter between a cluster and a barrier, consider the case of an unactivated, free cluster approaching the barrier from \(x_0 = 0\) in region 1. The wavelength of the associated wave function is \(\lambda = 2\pi/k\), and the boundaries of the barrier are at \(x_1 = 2\lambda\) and \(x_2 = 3\lambda\). The energy ratio in region 2 is \(V/E = .99\). The action
function for this encounter is shown in figure 10 where the staircase function in region 3 shows that the clustor has been activated by the interaction.

The behavior of the incident clustor is indistinguishable from that of a Newtonian particle. After its encounter with the barrier, however, the clustor is discretely extended. The primary activation parameter has changed from $\alpha_1 = 0$ in region 1 to $\alpha_1 = .998$ in region 3. There is also a change in the secondary activation parameter from $\alpha_2 = 0$ to $\alpha_2 = 294.9$.

This process of activation through interaction is quite general. A clustor's activation level almost always increases as the result of an interaction. It therefore seems evident that any clustor found in nature is likely to be highly activated due to numerous interactions with the environment throughout the clustor's history. As seen above in section 3.5, the characteristics of a highly activated free clustor are similar to those of a free particle as described in standard quantum theory. A world filled with naturally occurring clustors can, therefore, be expected to resemble the world described by standard quantum theory.

It is extremely unlikely, but not impossible, for a clustor's activation level to decrease during an interaction. For the activation level of a given clustor to decrease, the interaction parameters (barrier height, width, and position) would need to be finely tuned to the specific requirements of that clustor. It is expected that such fine tuning could be accomplished in the laboratory and that a clustor with a low or zero activation level could therefore be artificially generated. Since objects of that type do not exist in standard quantum theory, experimental evidence for them would lend support to the theory of discrete extension.

5 THE HARMONIC OSCILLATOR — AN EXAMPLE OF OPERATOR-FREE QUANTIZATION

In standard quantum theory the harmonic oscillator provides a simple example of a bound-state system with a discrete set of allowed energy levels. In the theory of discrete extension these same energy levels can be determined without recourse to the operator formalism of the standard theory. The analysis below shows that quantization of an oscillator's action increments and energy levels can be achieved within a mathematical structure similar to that of Newtonian mechanics. Specifically, it will be seen that the action generated by an oscillator remains nearly constant as the oscillator's energy is increased except at energy values equal to the eigenvalues of the standard theory. At these special values the action in a full cycle of the oscillator rises abruptly by an amount $\hbar$ from a lower to a higher plateau.

Let the mass, spring constant, and energy of the oscillator be $m$, $k$, and $E$ respectively, and let $\omega_N = \sqrt{k/m}$ be the angular frequency of the corresponding Newtonian oscillator. The Newtonian period is $\tau_N = 2\pi/\omega_N$. It will be seen that in the theory of discrete extension the angular frequency, $\omega$, of the oscillator depends not only on $k$ and $m$, but also on $E$ and on the clustor's level of quantum activation.

Let the reference point, $x_0 = 0$, be at the oscillator's equilibrium point. The system's potential function is then $V(x) = m \omega_N^2 x^2/2$, and two linearly independent solutions of
Schrödinger’s equation (9) are

\[ \psi_1(x, E) = M \left[ \frac{1}{4} \left( 1 - \frac{2E}{\hbar \omega_N} \right) - \frac{1}{2}, \frac{m \omega_N}{\hbar} x^2 \right] \exp \left( -\frac{m \omega_N}{2\hbar} x^2 \right) \]

and

\[ \psi_2(x, E) = \sqrt{\frac{m \omega_N}{\hbar}} x M \left[ \frac{1}{4} \left( 3 - \frac{2E}{\hbar \omega_N} \right), \frac{3}{2}, \frac{m \omega_N}{\hbar} x^2 \right] \exp \left( -\frac{m \omega_N}{2\hbar} x^2 \right) \]

where

\[ M(a, b, z) = \sum_{n=0}^{\infty} \frac{\Gamma(b) \Gamma(a+n) z^n}{\Gamma(a) \Gamma(b+n) n!} \]

is Kummer’s confluent hypergeometric function.

The derivatives of \( M(a, b, z) \) with respect to \( z \) and \( a \) are

\[ M_a(a, b, z) = \frac{\partial M}{\partial z} = a \frac{M(a+1, b+1, z)}{b} \]

\[ M_a(a, b, z) = \frac{\partial M}{\partial a} = \sum_{n=0}^{\infty} \left[ \Psi(a+n) - \Psi(a) \right] \frac{\Gamma(b) \Gamma(a+n) z^n}{\Gamma(a) \Gamma(b+n) n!} \]

and

\[ M_{aa}(a, b, z) = \frac{\partial^2 M}{\partial z \partial a} = a \frac{M(a+1, b+1, z) + \frac{1}{b} M(a+1, b+1, z)}{b} \]

where \( \Psi(z) \) is the digamma function. The values of these functions at \( z = 0 \) are

\[ M(a, b, 0) = 1 \]

\[ M_a(a, b, 0) = \frac{a}{b} \]

\[ M_{aa}(a, b, 0) = 0 \]

Therefore, at \( x_0 = 0 \) the wave function constants for equations (13) through (17) are

\[ \langle \psi_1 \rangle = 1 \quad \langle \psi_2 \rangle = 0 \quad \langle \psi_1^\prime \rangle = 0 \quad \langle \psi_2^\prime \rangle = 0 \]

Let units of mass, length, and time be defined by \( M = m \), \( L = \sqrt{\hbar/m \omega_N} \), and \( T = 1/\omega_N \), respectively. Let \( \mu = 1/L \). The unit of momentum is then \( \sqrt{\hbar m \omega_N} = \hbar \mu \).

In addition, the following dimensionless constants are defined in terms of the reference point parameters, \( p_0 \), \( (p_x)_0 \), \( (t_x)_0 \), and \( (t_{xx})_0 \), and serve to simplify subsequent equations:

\[ A = \frac{p_0}{\hbar \mu} \quad B = \frac{(p_x)_0}{2 \mu p_0} \quad C = \frac{2 \hbar \omega_N (t_x)_0}{p_0} \quad D = \frac{\hbar \omega_N [p_0 (t_{xx})_0 - 2(p_x)_0 (t_x)_0]}{\mu p_0^2} \]

In order to facilitate comparisons with standard quantum theory, it will be advantageous to adopt the parameter \( \eta = E/\hbar \omega_N - 1/2 \) as a dimensionless energy parameter. It is a continuous version of the energy eigenvalue, \( n = 0, 1, 2, \ldots \), of standard quantum
The corresponding Newtonian functions are

$$\text{have the Newtonian values}$$

moment the free cluster reaches the oscillator’s equilibrium point.

clustor in an oscillator potential, i.e., by “turning on” the oscillator potential at the

It is anticipated that they could be created artificially by capturing an unactivated free

This section illustrates the characteristics and the behavior of a quasi-Newtonian oscil-

5.2 The Quasi-Newtonian Oscillator

The Quasi-Newtonian Oscillator

5.1 Action, Momentum, and World-Line Functions

Substituting the expressions above into equations (14), (15), and (17) leads to the fol-

The corresponding Newtonian functions are

5.2 The Quasi-Newtonian Oscillator

$$\text{theory. Its range is } -1/2 \leq \eta < \infty \text{ in which the negative values correspond to energies}$$

below the standard zero-point energy, $E_0 = \hbar \omega_N / 2$.

The following abbreviated notation for the hypergeometric functions will also be

used to simplify the expressions for the action, momentum, and world-line functions:

$$\begin{align*}
M_{ij}(z, \eta) &= M\left(\frac{i-2\eta-1}{4}, \frac{j}{2}, z^2\right) \quad \text{and} \quad M_{ja}(z, \eta) = M_a\left(\frac{i-2\eta-1}{4}, \frac{j}{2}, z^2\right).
\end{align*}$$

$$\text{An oscillator is quasi-Newtonian, by definition, if its reference point parameters}$$

are quasi-Newtonian, by definition, if its reference point parameters

$$\begin{align*}
W &= h \left[ \frac{1}{2} \pi \arctan \left( \frac{A \mu x M33(\mu x, \eta)}{M11(\mu x, \eta) - B \mu x M33(\mu x, \eta)} \right) \right] \quad (27) \\
p &= \hbar \mu \left[ \frac{A[M11(\mu x, \eta) M31(\mu x, \eta) + 2\eta \mu^2 x^2 M33(\mu x, \eta) M53(\mu x, \eta)]}{[M11(\mu x, \eta) - B \mu x M33(\mu x, \eta)]^2 + [A \mu x M33(\mu x, \eta)]^2} \right] \quad (28) \\
t &= \tau_N \left[ \frac{A \mu x}{4 \pi} \left[ M33(\mu x, \eta) M11_a(\mu x, \eta) - M11(\mu x, \eta) M33_a(\mu x, \eta) \right] + C M11(\mu x, \eta) M33(\mu x, \eta) + D \mu x M33(\mu x, \eta)^2 \right] \quad (29)
\end{align*}$$

The following action, momentum, and world-line functions for the harmonic oscillator:

$$W_N = h \left[ \frac{1}{4} \pi \left( \mu x \sqrt{2\eta + 1} - \mu^2 x^2 + (2\eta + 1) \arcsin \left( \frac{\mu x}{\sqrt{2\eta + 1}} \right) \right) \right]$$

$$p_N = \hbar \sqrt{2\eta + 1} - \mu^2 x^2$$

$$t_N = \tau_N \left[ \frac{1}{2} \pi \arcsin \left( \frac{\mu x}{\sqrt{2\eta + 1}} \right) \right]. \quad (30)$$

Even though quasi-Newtonian oscillators are not expected to be found in nature, it is antici-

pated that they could be created artificially by capturing an unactivated free clustor in an oscillator potential, i.e., by “turning on” the oscillator potential at the moment the free clustor reaches the oscillator’s equilibrium point.

An oscillator is quasi-Newtonian, by definition, if its reference point parameters

have the Newtonian values

$$\begin{align*}
p_0 &= \sqrt{2mE} = \sqrt{m \hbar \omega_N (2\eta + 1)} \quad (p_0) = 0 \\
(t_x)_0 &= \sqrt{m/2E} = \sqrt{m/\hbar \omega_N (2\eta + 1)} \quad (t_{xx}) = 0.
\end{align*}$$
The corresponding values of \( A, B, C, \) and \( D \) are

\[
A = \sqrt{2\eta + 1} \quad B = 0 \quad C = \frac{2}{\eta} + 1 \quad D = 0. \tag{31}
\]

5.2.1 **A Comparison of Newtonian and Quasi-Newtonian Dynamics**

Figures 11 through 16 compare the action, momentum, and motion functions of a quasi-Newtonian oscillator with those of the corresponding Newtonian oscillator. These comparisons are made for two different energy values, \( \eta = 0 \) and \( \eta = 12 \). In each figure the dashed and solid curves are associated with the Newtonian and quasi-Newtonian oscillators, respectively. The turning points of the Newtonian oscillator are indicated by vertical dashed lines.

It was seen above in section 3.1 that the behaviors of a Newtonian free particle and a quasi-Newtonian free cluster are identical in all respects. In contrast, it will be seen here that while Newtonian and quasi-Newtonian oscillators are both unextended objects with periodic motions, their behaviors differ in important ways. A significant difference concerns the amplitude of the respective oscillations. While the motion of a Newtonian oscillator is bounded by its turning points, that of a quasi-Newtonian oscillator is unbounded. In each figure the quasi-Newtonian curve continues beyond the classical turning points and is defined over the range \(-\infty < \mu x < \infty\). The possibility that a quasi-Newtonian oscillator can be found anywhere in this infinite range is consistent with standard quantum theory in which the probability density, \( \psi^*\psi \), for an oscillator’s position is non-zero at arbitrarily distant points.

The oscillator action functions for the lower and higher energies are shown in figures 11 and 12, respectively. In both figures the quasi-Newtonian action function continues to increase for cluster point positions beyond the classical turning points. For the higher energy case the Newtonian and quasi-Newtonian functions are nearly equal between the turning points.

The oscillator momentum functions (i.e., the phase space trajectories) for the two energies are shown in figures 13 and 14. It can be seen that the motions of Newtonian and quasi-Newtonian oscillators are both periodic. The system point of the Newtonian oscillator moves continuously around its circular trajectory at constant speed. In contrast, as the momentum of the quasi-Newtonian oscillator approaches zero, its system point accelerates and escapes, momentarily, to spatial infinity before returning for the second half of its cycle. The momentum is never zero at any finite spatial position, but as it approaches zero, the velocity, \( 1/tx \), approaches infinity.

The oscillator motion functions for the two energies are shown in figures 15 and 16. They were generated by inverting the world-line equations, (29) and (30), for the quasi-Newtonian and Newtonian systems, respectively. The motions of both systems are clearly seen to be periodic, and the periodic escape of the cluster point to infinity is evident. It is also evident that at low energies the periods of the two oscillators can differ significantly, while at higher energies they are nearly equal in accordance with the correspondence principle.
Energy Parameter, $\eta = 0$

Fig. 11 Newtonian and quasi-Newtonian action functions for low energy oscillators. The dashed vertical lines are at the classical turning points.

Energy Parameter, $\eta = 12$

Fig. 12 At higher energies the Newtonian and quasi-Newtonian action functions are approximately equal between the classical turning points.

Energy Parameter, $\eta = 0$

Fig. 13 Newtonian and quasi-Newtonian phase space trajectories. Beyond the classical turning points the clustor escapes momentarily to infinity.

Energy Parameter, $\eta = 12$

Fig. 14 At higher energies the Newtonian and quasi-Newtonian phase space trajectories are approximately equal except near the turning points.

Energy Parameter, $\eta = 0$

Fig. 15 Newtonian and quasi-Newtonian motions. At low energies the frequencies of the oscillators can differ significantly.

Energy Parameter, $\eta = 12$

Fig. 16 At higher energies the frequencies of Newtonian and quasi-Newtonian oscillators are approximately equal.
5.2.2 Probability Density for Position

The probability density function for the position of a quasi-Newtonian oscillator is derived in this section and is shown to differ from the density function, \( \psi^* \psi \), of standard quantum theory. In principle, this difference provides one of the ways the two theories can be distinguished experimentally.

Quasi-Newtonian and Newtonian oscillators are similar in that both are unextended objects, and both have world-lines that are monotonic between two successive turning points. As a result of these shared characteristics, the probability density functions for the two oscillators can be derived by similar arguments. Due to the periodicity and symmetry of an oscillator’s motion, its density function can be found by considering only a half-cycle of that motion.

Consider a series of trials in which random times, \( \tilde{t} \), are chosen uniformly randomly in the half-period \([t_1, t_2]\) where \( t_1 \) and \( t_2 \) are the times at which the oscillator arrives at two successive turning points. The density function for these random times is a constant, \( P_t(t) = 1/(t_2 - t_1) \). At each random time the oscillator will be located at some random position, \( \tilde{x} \). The density function for \( \tilde{x} \) can be determined from the standard theory of functions of a random variable. Thus, since \( x(t) \) is monotonic in \([t_1, t_2]\), the density function for \( \tilde{x} \) is simply \( P_x(x) = |t'(x)|P_t(t(x)) \) where \( t'(x) = dt/dx \) is the derivative of the cluster’s world-line function (29). The function \( P_t(t(x)) = 1/(t_2 - t_1) \) is a constant, independent of \( x \).

Since successive turning points of the quasi-Newtonian oscillator are at \( x = -\infty \) and \( x = \infty \), the half-period, \( t_2 - t_1 \), is equal to \( \int_{-\infty}^{\infty} |t'(x)| \, dx \). The probability density function for position is therefore

\[
P_x(x) = \frac{|t'(x)|}{\int_{-\infty}^{\infty} |t'(x)| \, dx}.
\]

For a Newtonian oscillator whose motion is given by \( x(t) = X \sin(\omega_N t) \), a similar argument leads to the density function

\[
(P_x)_N(x) = \begin{cases} 
1/\pi \sqrt{X^2 - x^2} & \text{if } -X < x < X \\
0 & \text{otherwise}
\end{cases}
\]

Figures 17 and 18 show these probability density functions for two different energies, \( \eta = 0 \) and \( \eta = 12 \), respectively. In both figures the thick, dashed curves and the solid curves are associated with the Newtonian and quasi-Newtonian oscillators, respectively. The vertical dashed lines are located at the classical turning points. For a comparison with standard quantum theory, figure 17 also contains a thin, dashed curve which represents the standard density function, \( \psi^* \psi \), for an energy of \( \eta = 0 \):

\[
\psi^* \psi = \sqrt{m \omega_N / \hbar \pi} \exp \left( -\frac{m \omega_N}{\hbar} x^2 \right).
\]

There are clear differences in the density functions predicted by the theory of discrete extension and by standard quantum theory.
For a quasi-Newtonian oscillator the probability density is non-zero beyond the classical turning points as in standard quantum theory.

At higher energies the probability density function for a quasi-Newtonian oscillator approximates the Newtonian density function.

It can be seen in both figures that the probability density for the quasi-Newtonian oscillator is non-zero beyond the Newtonian turning points. Therefore, as in standard quantum theory, the oscillator may occasionally be found at arbitrarily large distances from its equilibrium position.

Figure 18 shows that for higher energies the quasi-Newtonian curve approaches the Newtonian curve in accordance with the correspondence principle.

5.2.3 Oscillator Frequency and Full-Cycle Action as Functions of Energy

The motion of a quasi-Newtonian oscillator from \( x = 0 \) to \( x = \infty \) constitutes one quarter of a cycle. Therefore, using the world-line equation (29), the oscillator’s period as a function of its energy is \( \tau(\eta) = 4t(x = \infty, \eta) \) where the dependence of the world-line function on \( \eta \) is shown explicitly. From the previous section an alternative expression for the period is

\[
\tau(\eta) = 2 \int_{\infty}^{-\infty} |t'(x)| \, dx.
\]

In figure 19 the solid curve shows the frequency ratio, \( \omega(\eta)/\omega_N \), as a function of energy where \( \omega(\eta) = 2 \pi/\tau(\eta) \) is the angular frequency of the quasi-Newtonian oscillator. The dashed line shows the constant frequency ratio associated with the Newtonian oscillator. It can be seen that \( \omega(\eta)/\omega_N \approx .9 \) at the standard zero-point energy, \( \eta = 0 \).

It can also be seen that the behaviors of Newtonian and quasi-Newtonian oscillators differ markedly at low energies. If the energy of a Newtonian oscillator is decreased toward zero, its frequency, \( \omega_N \), remains constant. In contrast, the frequency of a quasi-Newtonian oscillator is energy-dependent and approaches zero as the oscillator’s energy approaches zero, i.e., as \( \eta \to -1/2 \). Therefore, if a quasi-Newtonian oscillator’s energy is made smaller and smaller, its period increases without bound as does the time required to make a frequency-based energy measurement.

Figure 19 also shows that, at higher energies, \( \omega(\eta) \) approaches the Newtonian frequency in accordance with the correspondence principle.
The action generated in one full cycle of the oscillator is

\[ J(\eta) = \int p(x, \eta) \, dx. \]

Like the oscillator’s frequency, the full-cycle action, \( J(\eta) \), is a function of energy. The solid curve in figure 20 shows this function for a quasi-Newtonian oscillator. The dashed line shows the corresponding Newtonian function.

In both the Newtonian and quasi-Newtonian cases \( J(\eta) \) approaches zero as the oscillator’s energy approaches zero. The Newtonian action approaches zero because the amplitude of the oscillations approaches zero while their frequency remains constant. In contrast, the quasi-Newtonian action approaches zero because the frequency of the oscillations approaches zero while their amplitude remains infinite.

For energies above the standard zero-point energy, \( E_0 \), the quasi-Newtonian action exceeds the Newtonian action by approximately \( \hbar/2 \).

5.3 Quantum Mechanical Activation — A Basis for Energy Quantization

In this section it is shown that if an oscillator has a high degree of quantum activation, then it is discretely extended, and, furthermore, there are quasi-discrete aspects to its dynamics. It will be seen, for example, that if the energy of the oscillator is smoothly increased, the full-cycle action, \( J(\eta) \), generated by the oscillator remains essentially constant except at integer values of the energy parameter, \( \eta \). At each of these special energy values the action makes a quasi-discrete transition to a new plateau.

For the quasi-Newtonian oscillator discussed above, the parameters \( A, B, C, \) and \( D \) have the Newtonian values \( [31] \). To demonstrate the emergence of quantization due to quantum activation, it is sufficient to consider activation due only to the deviation...
of the parameter $A$ from its Newtonian value. Thus, let $A = a\sqrt{2\eta + 1}$ where the activation parameter, $a$, will be used to raise or lower $A$ from the Newtonian value. The parameters $B$, $C$, and $D$ will retain their Newtonian values in the following discussion.

5.3.1 Dynamics of an Activated Oscillator

The solid curves in figures 21 through 26 illustrate the dynamics of oscillators for both moderate and higher levels of quantum activation. In both cases $a < 1$ so that the value of $A$ is less than the Newtonian value. For the moderate level $a = .8$, and for the higher level $a = .1$. In all six figures the energy parameter has the value $\eta = 12$ as in figures 12, 14 and 16 for the quasi-Newtonian oscillator. The dashed curves in the figures show the behavior of the corresponding Newtonian oscillators. In the interest of visual clarity, however, the Newtonian curves are omitted from figures 22 and 26. In figure 26, for example, the picture would have been densely overlaid with twenty Newtonian sinusoidal cycles. As before, the dashed vertical lines represent the classical turning points.

Figure 21 shows that for the moderate activation level the action function differs little from that of the quasi-Newtonian oscillator shown in figure 12. In figure 22, however, the activation level is higher, and the action function approaches a staircase function, a characteristic of discretely extended systems.

Figures 23 and 24 show the effect of quantum activation on the oscillator’s phase space trajectory. It can be seen that beyond the turning points the trajectories extend to infinity as in the quasi-Newtonian case shown in figure 14. Between the turning points, however, quantum activation causes the trajectories to oscillate around the corresponding Newtonian trajectory. At higher activation levels as in figure 24, the oscillations become high and narrow and approach a series of delta functions.

The motion functions in figure 16 show that at the higher energy level, $\eta = 12$, the frequencies of Newtonian and quasi-Newtonian oscillators are nearly equal. In contrast, the motion functions in figures 25 and 26 show that while quantum activation preserves periodicity, it generates a frequency shift away from the Newtonian frequency even at the higher energy level, $\eta = 12$. Higher activation levels produce greater frequency shifts, and as $a \to 0$ the system approaches a stationary state with an infinitely long period.

In both figures 25 and 26 there exist vertical lines of simultaneity that intersect the clustor motion functions at multiple points. These intersections demonstrate the existence of discrete extension for both activation levels, but show that it is much more pronounced for the higher activation level. If $a \to 0$, the system approaches a stationary state, and the peaks in the world-line function approach a series of delta functions. In the limit, the height of these peaks is infinite, but so is the spatial period that contains them. Therefore, in the limit, all vertical lines of simultaneity generate the same static set of clustor points.

Quantum activation also arises if $a > 1$ so that the value of $A$ is greater than the Newtonian value. In the limit as $a \to \infty$ the oscillator again approaches a stationary state, but in this case the period is infinitesimal. Again, the peaks in the world-line function approach a series of delta functions.
Fig. 21 At low activation levels the action function differs very little from the corresponding quasi-Newtonian function of figure 12.

Fig. 22 At higher activation levels the action function approaches a staircase function with steps of height $h/2$.

Fig. 23 The phase space trajectory oscillates around the Newtonian trajectory. The clusters escape to infinity as in the quasi-Newtonian case.

Fig. 24 At higher activation levels the oscillations become high and narrow and approach a series of delta functions.

Fig. 25 Activation causes the frequency to shift away from the Newtonian value even at high energy levels. Discrete extension begins to emerge.

Fig. 26 At higher activation levels discrete extension is pronounced. The frequency shift grows, and a stationary state is approached as $a \to 0$. 
Whether $a \to 0$ or $a \to \infty$, it is evident that the associated probability density for cluster point position is sharply peaked around the delta function locations.

5.3.2 Quantized Action Increments and Energy Levels

Figures 20, 27, and 28 show an oscillator’s full-cycle action, $J(\eta)$, as a function of its energy for three different levels of quantum activation. Figure 20 is associated with the extreme case of a quasi-Newtonian oscillator. Figure 28 is associated with the opposite extreme case of a highly activated oscillator. Figure 27 corresponds to an oscillator with an intermediate level of activation. In all three figures the dashed straight line represents the full-cycle action of the corresponding Newtonian oscillator.

Any value of the activation parameter other than $a = 1$ is associated with some level of quantum activation. The values of $a$ used in figure 27 for intermediate activation levels are $a = .25$ and $a = 4$. The values of $a$ used in figure 28 for high activation levels are $a = .005$ and $a = 200$. It can be seen from these figures that as the oscillator’s activation level is increased, the quasi-Newtonian curve of figure 20 evolves toward a quasi-discrete form in two different ways depending on whether $a \to 0$ or $a \to \infty$.

The quasi-discrete curves in figure 28 show that as the energy of a highly activated oscillator is raised, the full-cycle action increases by quasi-discrete increments of magnitude $\hbar$ at successive integer values of the energy parameter, $\eta$. For values of $\eta$ between successive integers, the action remains on a plateau with a nearly constant value. These integer values of the continuous variable $\eta$ are, of course, the energy eigenvalues of standard quantum theory, and the action increments are those specified in the old quantum theory. Thus, the Bohr-Sommerfeld quantization rule,

$$\oint p\, dq = nh \quad n = 1, 2, 3, \ldots,$$

is reflected in the theory of discrete extension, not as an ad hoc rule, but as a deduction from the theory.

The action increments belong to alternating sequences of even and odd parity. When $0 < a \ll 1$, quasi-discrete increments of size $2\hbar$ occur at even values of $\eta$, and when $1 \ll a < \infty$ they occur at odd values of $\eta$.

In addition to these action increments at integer values of $\eta$, a special action increment of magnitude $\hbar$ occurs at the unique half-integer value, $\eta = -1/2$. It is part of the odd-parity sequence and is discussed in the following section in connection with the nature of zero-point energy.

These quasi-discrete action increments can be understood in terms of the behavior of the oscillator’s action function (27) as the energy of the system is raised through a transition point. Figures 29 through 32 illustrate the evolution of this function around the transition point, $\eta = 4$, in the even-parity sequence. For this example the oscillator is highly activated with $a = .0001$. In each of the figures the solid curve shows the oscillator’s action as a function of the cluster point coordinate, $x$, where $-\infty < x < \infty$. The dashed curve is the action function for the corresponding Newtonian oscillator, and the dashed vertical lines represent the classical turning points.

In figure 29 the energy value, $\eta = 3.99$, is just below the transition point. At this energy level quasi-discrete action increments of size $\hbar/2$ occur at four cluster point
Fig. 27 The total action generated in a full cycle in units of $h$. At a moderate activation level the functions begin to take on quasi-discrete forms.

Fig. 28 At higher activation levels quasi-discrete action increments of size $h$ occur at the quantized energy values of standard quantum theory.

Fig. 29 Action as a function of cluster point position within a half cycle. Quasi-discrete action increments of size $h/2$ occur at four positions.

Fig. 30 If the energy is increased slightly to the eigenvalue $\eta = 4$, the action function is found to be in the midst of an abrupt transition.

Fig. 31 After another small increase in energy the transition is complete, and there are now six action increments of size $h/2$.

Fig. 32 No additional action increments occur as the energy is increased toward a new transition point at the next even eigenvalue, $\eta = 6$. 
positions. In figure 30 the energy has been raised slightly to the integer value \( \eta = 4 \).
The action function is now in the midst of an abrupt transition that will lead to the appearance of a new, higher plateau. In figure 31 the energy has been raised slightly again to the value \( \eta = 4.01 \), just above the transition point. The transition is now complete, and action increments of size \( h/2 \) occur at six cluster point positions. Taken together, the two new action increments add an additional action amount, \( h \), to the half-cycle shown. Therefore, in the even-parity sequence, the full-cycle action, \( J \), increases abruptly by \( 2h \) as the energy passes through the integer value, \( \eta = 4 \).

Finally, in figure 32 the energy has been raised from \( \eta = 4.01 \) to \( \eta = 5.99 \). It can be seen that throughout the energy range \( 4 < \eta < 6 \) no additional action increments occur in the even-parity sequence. The oscillator, however, is now on the verge of a new transition at the next even integer, \( \eta = 6 \).

5.3.3 The Nature of Zero-Point Energy

According to standard quantum theory the lowest possible energy of an oscillator is the zero-point energy, \( E_0 = \hbar \omega N/2 \). Furthermore, the experimental evidence for the apparent existence of this lower bound is extensive and diverse [10, 11]. Energies below this value are allowed, however, in the theory of discrete extension, and the theory must explain why oscillators with these lower energies have not been recognized experimentally. This lack of recognition can be understood if it is assumed that naturally-occurring oscillators have a high degree of quantum activation due to prior interactions with their environment.

Figure 28 from the theory of discrete extension shows the full-cycle action of a highly activated oscillator as a function of the oscillator’s energy. The odd-parity sequence reveals a quasi-discrete action increment of size \( h \) near \( \eta = -1/2 \), i.e., near \( E = 0 \). The existence of this action increment leads to the following explanation for the apparent lower bound on energy at \( E = E_0 \).

If the energy of a highly activated oscillator is gradually reduced to \( E_0 \) from a higher value, the full-cycle action decreases in a series of quasi-discrete steps as shown in figure 28. If the energy is then decreased further, the action remains nearly constant at the value \( h \) and drops to zero suddenly only when \( E \) reaches a value very near to zero. Therefore, judged on the basis of its ability to generate action, the oscillator’s behavior at energies in the range \( 0 < E < E_0 \) cannot be distinguished from its behavior at \( E = E_0 \). For a highly activated oscillator the existence of these lower energies is, therefore, hidden. Figures 20 and 27 show, however, that, for an oscillator with a low or moderate level of quantum activation, the full-cycle action falls to zero smoothly as \( E \to 0 \), and energies below \( E_0 \) may then be evident experimentally.

6 MICROSCOPIC TIME AND MACROSCOPIC TIME

The question arises as to how the behavior of a cluster appears to a macroscopic observer in a Newtonian world. The theory of discrete extension specifies the behavior of a cluster by providing an equation (17) for the cluster’s world-line. The time variable in this equation is specific to the theory of discrete extension and has no a priori connection to the time variable of Newtonian theory. Furthermore, this world-line inhabits
the passive, static realm of space-time and therefore does not, by itself, reveal the active, unfolding dynamics of the clustor. An active view of the clustor’s behavior must be generated by sweeping a line of simultaneity through space-time and noting the progression of its intersections with the clustor’s world-line. To determine the unfolding dynamics as seen by a macroscopic observer, the line of simultaneity must be swept at a rate that represents the uniform flow of Newtonian time. To accomplish this goal, a relation must be found between the time variable of the theory of discrete extension and the time variable of Newtonian theory. Such a relation can be determined as follows.

According to the Newtonian law of inertia, equal increments of classical time are associated with equal changes in the position of a Newtonian free particle. The motion of such a particle can therefore be used to define the uniform flow of Newtonian time. Similarly, the motion of an unextended free clustor can be used to define the uniform flow of time in the theory of discrete extension. As seen above in section 3.1, however, the equations that describe the behaviors of an unextended free clustor and a Newtonian free particle are identical. Therefore, it will be assumed that Newtonian time and the time defined by the motion of an unextended free clustor are identical. The observed changes, $\delta x$, in the position of an unextended free clustor thus provide the theory of discrete extension with a self-contained standard for the uniform progression of Newtonian time, namely $\delta t = \delta x \sqrt{m/2E}$. Therefore, if a line of simultaneity sweeps through space-time according to this standard of uniformity and at this rate, the progression of its intersections with the world-line of any clustor, free or not, will reveal that clustor’s behavior as seen by a macroscopic observer.

7 THE DISTRIBUTION OF MASS AMONG THE CLUSTOR POINTS

At any given moment a clustor will, in general, have a multitude of clustor points. The question arises as to how the clustor’s mass, $m$, is distributed among its various points at that moment. A hypothesis can be made based on the following considerations:

1. At each moment the various clustor points can be divided into two classes based on the sign of the world-line derivative, $dt/dx$. If $dt/dx > 0$ at the location of a given clustor point, that point will be called a positive point. If $dt/dx < 0$, it will be called a negative point. Since all clustor points in a given class have the same character and status, it is natural to assume that they all have the same mass.

2. As time goes on, clustor points may appear or disappear, but they always do so in pairs consisting of one point from each class.

3. The number of clustor points is always odd with one excess positive point in addition to a number of pairs of positive and negative points.

4. In a non-relativistic theory the conservation of mass requires that the total mass of all the clustor points remains constant even as the number of clustor points increases or decreases.

5. Negative points move in the direction of decreasing $x$, and yet the dynamic equations of the theory generate positive values for the momenta of such points.
Based on these considerations, the theory of discrete extension tentatively assigns a mass value of \( m \) to each positive clustor point and \(-m\) to each negative clustor point. The combined mass of each positive/negative pair is then zero, and the mass of the one unpaired, positive point is \( m \). As a result, the total mass of the clustor remains constant in time at the correct value, \( m \), even as the number of clustor points rises or falls. Furthermore, assigning a negative mass value to the negative points is consistent with the positive momentum values generated for such points by the theory.

In order for this assignment of positive and negative mass values to be consistent with the theory’s dynamical equations, it is also necessary to assign a positive energy value, \( E \), to positive clustor points and a negative value, \(-E\), to negative clustor points. This conclusion can be made plausible by the following considerations:

1. At negative clustor points the mass is \(-m\), and the momentum factor, \( \sqrt{2mE} \), would be imaginary unless the energy is also negative.

2. Consider potential functions, \( V(x) \), such as the harmonic oscillator potential, that are proportional to \( m \). For such potentials both terms, \( T \) and \( V \), on the right side of the energy equation (18) are negative when the mass is negative. Therefore this equation yields a negative energy, \( E \), for negative clustor points.

3. For this same class of potential functions, Schrödinger’s equation (9) and its solutions are invariant with respect to simultaneous sign changes of \( m \) and \( E \).

Similar considerations with regard to the Coulomb potential and a clustor’s electric charge imply that charge should be distributed among the clustor points in the same way that mass is distributed, i.e., with opposite signs at positive and negative clustor points.

It is hoped that the concepts of negative mass and energy introduced here will find a natural interpretation in a relativistic version of the theory of discrete extension. Just as negative energy electron states in the Dirac electron theory were able to be re-interpreted as positive energy anti-particle states, it is anticipated that a similar re-interpretation for negative clustor points will be possible. Furthermore, given the equivalence of mass and energy in a relativistic theory, the issues of negative mass and negative energy would be expected to be resolved simultaneously.

In the theory of discrete extension, space is occupied by countless, far-flung pairs of positive and negative clustor points associated with a myriad of highly activated clustors, free and otherwise. As illustrated in figure 7 these pairs spontaneously appear and disappear from space as prescribed by the equations of clustor dynamics. This activity resembles the activity of virtual particle/antiparticle pairs in relativistic quantum field theory. Therefore, it is conjectured that clustor point pairs and virtual particle/antiparticle pairs would have similar implications regarding the nature of the quantum vacuum.
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