Vortices in superconducting films - statistics and fractional quantum Hall effect

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November 6, 1995

Abstract

We present a new derivation of the Berry phase picked up during exchange of parallel vortices. This derivation is based on the Bogolubov - de Gennes formalism. The origin of the Magnus force is also critically reanalised. The Magnus force can be interpreted as interaction with effective magnetic field. The effective magnetic field may be even of the order $10^6 T/A$. We discuss a possibility of the FQHE in vortex systems. As the real magnetic field is varied to drive changes in vortex density, the vortex density will prefer to stay at some quantised values. The mere existence of the FQHE does not depend on vortex quantumstatistics although the pattern of the plateaux does. We also discuss how the density of anyonic vortices can lower the effective strength of the Magnus force, what might be observable in measurements of Hall resistivity.

To appear in Physical Review B.

cond-mat/9508140

1 Introduction

In our recent paper [1] we have pointed out the possibility that vortices in superconducting films might be anyons. There is a classic paper by Haldane and Wu [2], which demonstrates that vortices in superfluid helium layers are not anyons.
because there is no well defined dilute limit. We have reanalysed this question from the point of view of the phenomenological time-dependent Ginzburg-Landau models. It was shown that there are situations when one can go around the argument in [2]. A classic example are vortices in the Ginzburg-Landau models of the fractional quantum Hall effect [3, 4]. Chern-Simons interaction makes them well localised objects and the dilute limit can be defined. The main effect of the CS gauge field is to remove the divergent gradient energies present in the global model. These divergencies are also removed in the Ginzburg-Landau model for superconductors. We have considered a gauged nonlinear Schrödinger equation as a minimal version of time-dependent Ginzburg-Landau model. To model the structure of the vortex core we assumed it was filled with normal fluid so as to make the whole structure locally charge-neutral. The strenght of the statistical interaction was proportional to the net deficit of the superfluid replaced inside the core by the normal phase.

It is argued [5] that the time evolution of the condensate is much faster then that of the normal fluid. Thus in the static case the core is filled with the normal fluid but when the vortex moves fast enough the normal fluid can not adjust itself and does not follow the vortex motion. It is a crude approximation [6] and it does not invalidate our results. The statistical interaction shows up in the adiabatic approximation which is quite an opposite limit. In this limit we can assume the nonuniformity of the normal fluid does follow the vortex motion. The effective Lagrangian for vortex motion can be arranged term by term according to the powers of vortex velocity. The statistical interaction together with the term responsible for the Magnus force [7] are the first terms in this effective Lagrangian. The distortion of the normal fluid distribution from the static one, which is at least of the first order in velocities, can contribute but only to the term quadratic in velocities which is the next to leading term. Thus at least for slow motion as compared to characteristic velocities, when expansion in powers of velocities is justified, the Magnus force and statistical interaction are not altered.

One can rewrite the local BCS model in terms of the gap-function field [8, 9] but the effective theory appears to be nonlocal - it contains derivatives of arbitrary order. In this way one goes from description in terms of electronic degrees of freedom to description in terms of Cooper pairs. The latter is well suited for the bulk of the superconductor. However in the core of the vortex we can expect some decoherence, which phenomenologically might be described by normal fluid. Whenever degrees of freedom other than those of Cooper pairs come into play the description in terms of Cooper pairs may happen to be irrelevant. One way of dealing with the problem is to truncate the effective theory on the lowest order in derivatives and introduce more or less explicitly something like a normal component. This approach is limited by the poor knowledge about the nature of the normal fluid. Another approach is to take as many orders in derivatives in the effective theory as possible to describe also the normal fluid in terms of the gap function. The problem is that in practice
one would have to cut the expansion at certain order and there is no warranty that such a cut theory would be self-consistent.

To go around these problems we will derive the statistical interaction in the framework of the Bogolubov-de Gennes formalism for pure samples at zero temperature. We will work in the quasi two-dimensional regime of long parallel vortices. In the case of superconducting layers thicker than 100\AA the penetration length $\lambda$ is still very close to the penetration length $\lambda$ in the bulk superconductor. In this regime we expect modifications to be rather quantitative in nature then qualitative. An important first step was done by Gaitan \cite{Gaitan} in his derivation of the Berry phase responsible for the Magnus force. In this paper we are going to reanalyze his derivation and then to extend the method to the case of two well separated vortices. In the microscopic theory the language of the normal and superfluid is not very fruitful. The distinction of the states below the Fermi surface into localised bound states and scattering states appears to be more natural. The distinction does not influence the value of the Magnus force but it is crucial for the statistical interaction. The scattering states are common to all vortices while bound states can be identified with particular ones.

2 Preliminaries on vortex solution in BCS theory

The problem of the vortex solution in the BCS theory can be conveniently posed within the Bogolubov-de Gennes formalism \cite{Bogolubov}. It has not been completely solved although some qualitative pictures of the solution are known \cite{Bogolubov}. We will restrict here to listing the basic ingredients of the formalism.

The Bogolubov equation, defined in the Nambu spinor space \cite{Nambu}, is

$$\begin{pmatrix} E_n - \hat{H}_{BOG} & u_n \\ v_n & v_n\star \end{pmatrix} = 0,$$

where $\hat{H}_{BOG}$ is the Bogolubov hamiltonian

$$\hat{H}_{BOG} = \begin{pmatrix} -\frac{1}{2}(\nabla - ieA)^2 - E_F & \Delta(x) \\ -\frac{1}{2}(\nabla + ieA)^2 + E_F & \Delta\star(x) \end{pmatrix}. $$

$E_F$ is the Fermi energy. $\Delta(x) = \Delta_0(r) \exp(-i\theta)$ denotes the gap function of vortex solution. $\Delta_0(r)$ interpolates between 0 at the origin and a constant, which we will call $\sqrt{\rho_0}$, at infinity. There are both positive and negative energy solutions. If $(u_n, v_n)$ is an eigenstate with energy $E_n > 0$, then $(-v_n\star, u_n\star)$ is a solution with energy $-E_n < 0$. The equations (1) have to be supplemented by a self-consistency condition

$$\Delta(x) = \sum_n u_n(x) v_n\star(x),$$

3
where $g$ is the BCS coupling constant, together with Maxwell equations determining the vector potential.

The field operator for Nambu quasiparticles can be expanded in terms of the solutions of Eq.(1).

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^*(x) \end{pmatrix} = \sum_n \left[ \gamma_n \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} + \gamma_n^\dagger \begin{pmatrix} -v_n^*(x) \\ u_n^*(x) \end{pmatrix} \right].$$  (4)

The negative energy states are occupied in the BCS ground state $|\text{BCS}\rangle = \prod_n \gamma_n^\dagger |0\rangle$.  (5)

The eigenstates satisfy the following orthogonality

$$\int d^3x \left[ u_n(x) u_m^*(x) + v_n(x) v_m^*(x) \right] = \delta_{nm},$$
$$\int d^3x \left[ u_n(x) v_m(x) - v_n(x) u_m(x) \right] = 0$$  (6)

and completeness relations

$$\sum_n \left[ u_n(x) u_n^*(x') + v_n(x) v_n^*(x') \right] = \delta(x - x'),$$
$$\sum_n \left[ u_n(x) v_n^*(x') - v_n(x) u_n(x') \right] = 0.$$  (7)

With the help of these relations the creation and annihilation operators can be expressed as

$$\gamma_n \downarrow = \int d^3x \left[ -\psi_\uparrow^*(x) v_n(x) + \psi_\downarrow(x) u_n^*(x) \right],$$
$$\gamma_n\uparrow = \int d^3x \left[ \psi_\uparrow^*(x) u_n(x) + \psi_\downarrow(x) v_n^*(x) \right].$$  (8)

Adiabatic vortex motion \cite{10} gives rise to a Berry phase in the solutions of Eq.(1), $(u_n, v_n) \rightarrow \exp[i\phi_n](u_n, v_n)$. These Berry phases sum up to the total Berry phase picked up by the ground state $|\text{BCS}\rangle \rightarrow \exp[i\Gamma]|\text{BCS}\rangle$ which with the help of Eqs.(8,5) can be established to be

$$\Gamma = -\sum_n \phi_n.$$  (9)

To pursue some of the questions we need more detailed knowledge about the eigenstates. The axially symmetric ansatz takes the form

$$\chi_n(x) = \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = e^{ik_z z} e^{i(\mu-\frac{1}{2}\sigma_z) \theta} f_n(r),$$  (10)
where $\hbar k_z$ is the $z$-component of the momentum. $\mu$ must be a half-integer for the expression to be single-valued. The functions $f_n(r)$ have been investigated in [11] with the help of WKB approximation. We will quote some more detailed results in the following.

### 3 Origin of the Magnus force

Now we are going to rederive the Berry phase responsible for the Magnus force following the argument of Gaitan [10]. In comparison with [10] we clarify some points and remove some unnecessary assumptions.

The general form of the Berry phase in two dimensions is

$$\phi_n = i \int dt \int d^2x \chi_n^\dagger \left( \frac{d}{dt} + ie\hbar A_0 \right) \chi_n .$$

(11)

The time derivative is understood as a total derivative with respect to slow degrees of freedom. For an adiabatic motion of a single vortex the derivative has to be replaced by $\dot{r}_0 \nabla r_0$, where $r_0$ is a position of the vortex singularity. The scalar potential vanishes for the vortex solution so we will skip the second term in what follows. With the axially symmetric ansatz (10) the phase becomes

$$\phi_n = \int dt \int d^2x \dot{r}_0 \left[ (\chi_n^\dagger (-\mu + \frac{1}{2} \sigma_z)) \chi_n \right] \nabla r_0 \theta + f_n^\dagger(r) \nabla r_0 f_n(r) \]$$

(12)

The second term vanishes by symmetry arguments. The contribution of the first term to the total Berry phase is

$$\Gamma = - \sum_n \phi_n = - \int dt \int d^2x (\dot{r}_0 \nabla r_0 \theta) S(r)$$

(13)

where

$$S(r) = \sum_n |u_n(r)|^2 (-\mu + \frac{1}{2}) + |v_n(r)|^2 (-\mu - \frac{1}{2}) .$$

(14)

$\hbar S$ is minus the $z$-component of the canonical angular momentum density. It is not a gauge-invariant integral of motion. $S$ is the expectation value density of the operator $-i\hbar \frac{\partial}{\partial \theta}$ instead of the gauge-invariant $-i\hbar \frac{\partial}{\partial \theta} + \sigma_z A_\theta$.

$S(0) = 0$ because either $u_n$ ($v_n$) or the factor $(-\mu + \frac{1}{2})$ vanishes at the origin. To find out its asymptotic behavior at infinity we would need a much more detailed knowledge about the solutions. We go around this problem by resorting to the effective theory which is equivalent to the microscopic formalism. The general term linear in the covariant time derivative reads

$$\int dt \int d^2x i[\Delta^*(\hbar \partial_\tau + 2ieA_0)\Delta - c.c.][G(\Delta^* \Delta) + spatial \ derivative \ terms] .$$

(15)
By "spatial derivative terms" we mean terms which are of at least first order in the covariant spatial derivatives. $G$ is a function of $\Delta \star \Delta$ only which tends to $\rho_s/\rho_0$ as the gap function $\Delta$ approaches its asymptotic equilibrium value. $\rho_s$ is the equilibrium Cooper pairs’ density. Let us consider the adiabatic rotation of the vortex solution $\Delta = \Delta_0(r)e^{-i\theta}$ around its axis, $\theta \to \theta - \omega t$. The action picks up a term (to lowest order in $\omega$)

$$\omega \int dt \int d^2x \{-h\Delta^*\Delta[G(\Delta^*\Delta) + \text{spatial derivative terms}]\}. \quad (16)$$

The spatial integral is just the total angular momentum. For large $r$, where $A_0$ tends to zero, the density of this angular momentum is, by gauge invariance, equal to $\hbar$ times minus the bulk Cooper pairs’ density $\bar{\hbar}S \approx -\bar{\hbar}\rho_s = h\lim_{r \to \infty} \frac{\delta W}{\delta (2eA_0)}$, where $W$ is the effective action and $\rho_s > 0$. We have to stress that we make use of the effective theory only very far from the vortex core where it should be equivalent to the microscopic treatment. In particular in the distant asymptotic region there is no contribution from unpaired bound states which can not be described in terms of Cooper pairs.

We have all we need to calculate the Berry phase. Let us expand the integrand in Eq.(13) around the vortex position $r_0 = (X,Y)$ close to the origin, $(X,Y) = (0,0)$,

$$S = S(r) - S'(r)\left[ X\cos\theta + Y\sin\theta \right] + O(r_0^2),$$

$$\dot{r}_0 \nabla_{r_0} \theta = X\left(\frac{\sin\theta}{r} + \frac{X\sin 2\theta - Y\cos 2\theta}{r^2}\right) + Y\left(-\frac{\cos\theta}{r} - \frac{X\cos 2\theta + Y\sin 2\theta}{r^2}\right) + O(r_0^2).$$

A straightforward integration yields

$$\Gamma = -\pi [S(\infty) - S(0)] \int dt \varepsilon_{kl} X^k \dot{X}^l + O(r_0^2). \quad (18)$$

The expression $O(r_0^2)$ does vanish. We are considering single vortex in absence of any driven current. Such a system is translationally invariant and isotropic. The first term on the R.H.S. of Eq.(18) is already the most general term linear in velocity which is, up to a total time derivative, translationally invariant and isotropic. Thus we do not need to consider finite $r_0$ to obtain a generally valid expression. The phase (18) is remarkably simple to evaluate. For a vortex with winding number $-1$ it reads

$$\Gamma = \pi \rho_s \int dt \varepsilon_{kl} X^k \dot{X}^l. \quad (19)$$

From our derivation of the Magnus force it is clear that the Wess-Zumino term (in the gauge $A_0 = 0$)

$$\int dt \int d^2x [\rho_s \partial_\theta],$$

$$\int dt \int d^2x [\rho_s \partial_\theta] \quad (20)$$
with $\rho_s = \text{const}$, does not make much sense as it stands. $\rho_s$ is the same at the origin as at infinity so the Magnus force vanishes. The formula (20) is to be understood with an implicit assumption that a small area around the phase singularity is excluded from the spatial integration. In other words $\rho_s$ must be put equal to 0 in this area. It is not difficult to realise, by performing radial integration first and then integration over the angle around the singularity, that the way of regularisation does not matter. In particular it does not need to be rotationally symmetric. The only factors that determine the Magnus force are the two limit values of $\rho_s$. Thus vortices in a condensate will always feel the Magnus force. It is not the case for say Jackiw-Pi solitons [16], where $\rho_s$ is zero both at the origin and at infinity.

4 Mutual statistical interaction of vortices

Let us consider two vortices: "1" at the origin and "2" very far apart at $R(t)$. It is important to realise that the eigenstates of the Bogolubov hamiltonian can be divided into common scattering states, which we will still denote by just $u_n, v_n,$ and bound states which can be identified with a given vortex $u_n^{(1,2)}, v_n^{(1,2)}$. Vortices are very distant so there is no overlap between their localised bound states.

The bound states of the stationary vortex "1" feel what is going on around them through the pair potential $\Delta(t, x)$ inside and around the core. Vortices are well localised so a fairly good approximation to a two-vortex gap function is the product ansatz

$$\sqrt{\rho_0} \Delta(t, x) = \Delta_v(x) \Delta_v[x - R(t)] ,$$

where $\Delta_v(x) = \Delta_0(r) \exp(-i\theta)$ denotes the gap function of a single vortex centered at the origin. Close to $x = 0$ this expression can be further simplified

$$\Delta(t, x) = \Delta_v(x) e^{-i\theta|x - R(t)|} ,$$

Thus the bound states of the static vortex have to be modified as

$$\chi_n^{(1)}[x, R(t)] = e^{-i\frac{2\pi}{\theta}[x - R(t)]} \chi_n(x) .$$

Their contribution to the Berry phase is

$$- \int dt \int d^2x \left\{ \dot{R} \nabla R \theta[R| x - R(t)|] \right\} \sum_{\text{bound st.}} \left( \frac{1}{2} \left| u_n \right|^2 - \frac{1}{2} \left| v_n \right|^2 \right) \approx \left\{ \frac{1}{2} \int d^2x \sum_{\text{bound st.}} \left( \left| u_n \right|^2 - \left| v_n \right|^2 \right) \right\} \int dt \dot{R} \nabla R \theta[R(t)] ,$$

where the approximate equality is valid for small $\frac{r_c}{R}$, where $r_c$ is a radius of the core. The equality $\left| u_n \right|^2 = \left| v_n \right|^2$ holds for the bound states, at least up to the WKB approximation [11], so their contribution to the Berry phase vanishes.
Now as the vortex "2" moves its bound states follow its trajectory $\chi^{(2)}[r - R(t)]$, similarly as in the single vortex case considered in the previous section. In addition, as an effect due to the vortex "1" (21), their components perform the relative phase rotation

$$\chi_n^{(2)}[x, R(t)] \approx e^{-i\frac{\pi}{2}\theta_{(x)}} \chi_n[x - R(t)] .$$  \hspace{0.5cm} (25)

The contribution from this relative phase rotation is once again zero. The bound states contribute but only to the Magnus term (19) just as in the single vortex case.

The bound states do not give rise to any new effects so let us consider scattering states common to both vortices. In the vortex core region the asymptotes of the scattering states must be close to those of the scattering states for a single vortex but the phase has to be replaced by the asymptote of the phase in the product ansatz (22). Close to the origin

$$\chi_n[x, R(t)] \approx e^{ik_z x} e^{i(\mu - \frac{1}{2})\theta(x + \theta[x - R(t)])} f_n(r) .$$  \hspace{0.5cm} (26)

The contribution from around the stationary vortex is

$$-\int dt \int d^2x \dot{R} \nabla R \theta |x - R(t)| \bar{S}(r) ,$$  \hspace{0.5cm} (27)

where $\bar{S}$ is a part of the canonical angular momentum due to the scattering states

$$\bar{S}(r) = \sum_{\text{scatt. st.}} \left[ |u_n(r)|^2 (-\mu + \frac{1}{2}) + |v_n(r)|^2 (-\mu - \frac{1}{2}) \right] .$$  \hspace{0.5cm} (28)

Far from the core $\bar{S} \approx S \approx -\rho_s$. If $\bar{S}$ were equal to $-\rho_s$ also in the core, the contribution to the Berry phase from (27) would be just the same as to the Magnus term. Thus we are interested only in the effects due to deviations of $\bar{S}$ from its asymptotic value $-\rho_s$. Inside the core $|u_n|^2$ and $|v_n|^2$ are changed by a factor which is $> 1$ for the states with $\mu$ negative and $< 1$ for $\mu$ positive [11]. The net deviation $\delta \bar{S}(r) = S(r) + \rho_s$ is positive. At the very origin $\delta \bar{S}(0) = \rho_s$.

The total change in the Berry phase is twice that in (27), as there are two vortices, and amounts to

$$\delta \Gamma = [2 \int d^2x \delta \bar{S}(r) \int dt \dot{R} \nabla R \theta(R) .$$  \hspace{0.5cm} (29)

Thus the total Berry phase for a dilute vortex system is

$$\Gamma = \int dt \left[ -\pi \rho_s \sum_p n_p e^{iX_{(p)}^k X_{(p)}^k} + \alpha \sum_{p < q} n_p n_q \frac{d}{dt} \Theta_{(p,q)} \right] ,$$  \hspace{0.5cm} (30)

where the indices $p, q$ run over vortices, $n$’s are their winding numbers, $\Theta_{(p,q)}$ is the angle between the $p$-th and $q$-th vortex and the numerical factor $\alpha$ can be
α is roughly the number of electrons inside the core and as such it can range from \( \sim 1 \) for high \( T_c \) superconductors to \( \sim 10^5 \) for some conventional type \( II \) superconductors.

### 5 Vortex statistics within variational wave-function approach

Once we have derived statistical interaction in the microscopic setting it may be worthwhile to reanalyse some earlier approaches to similar problems. In the paper by Ao and Thouless \(^7\) the Magnus force was derived with the help of the variational many-electron vortex wave-function

\[
\psi_v[z] = \exp\left[i \frac{1}{2} \sum_k \theta(z_k - z_0)]\right] \psi_0[z],
\]

where \( z_0 \) is a complex vortex position, \( z_k \)'s are positions of electrons and \( \psi_0[z] \) is an antisymmetric variational function. The phase factors in the wave-function are determined by the demand of correct electronic quantum statistics and by topological properties. There are variational profile functions in \( \psi_0 \) which can not be established without dynamical considerations. One can consider an adiabatic vortex motion along some trajectory and calculate the Berry phase picked up by the wave-function. This Berry phase coincides with Eq.(19). For a closed path the Berry phase is proportional to the number of electrons enclosed by the trajectory.

This setting is convenient to analyse what is the dependence of the Magnus force on impurities \(^7\). An impurity can be viewed as an attractive potential which traps some of electrons in localised bound states. The trapped electrons disappear from the ansatz (32). The Berry phase is still proportional to the area enclosed by the trajectory but this time the area should not be multiplied by the total density of electrons but rather by the total density minus the density of electrons trapped by impurities. Impurities lower the value of the Magnus force.

Now let us consider the effect of an exchange of two vortices. More precisely, let us fix the position of one vortex and consider another distant vortex moving around it. One could argue there is no special effect because the net charge of any vortex must be zero. Provided the trajectory is large enough, there is no change in the number of enclosed electrons due to the enclosed vortex. The last sentence is certainly true but the example with impurities taught us that it is not the total number of electrons that really matters but rather the number of electrons in the coherent state described by the wave-function (32). We know
from the discussion in the previous sections that inside vortex core the scattering or continuum states are replaced by bound states. Thus vortex can be viewed as a kind of impurity, which traps some of electrons into localised bound states with energies within the energy gap band. The localised electrons are removed from the wave-function \( \langle \psi | \rangle \). There is an additional Berry phase proportional to the number of vortices enclosed by the trajectory. Each enclosed vortex contributes a term proportional to the number of electrons trapped inside its core.

We can consider a path for a chosen vortex in a more or less uniform distribution of vortices. If we neglect possible intervortex correlation effects, the background vortices could be regarded as uniform distribution of impurities lowering the density of electrons in the coherent state \( \langle \psi | \rangle \). In this mean-field approximation the Magnus force acting on a chosen vortex is lowered by the presence of another vortices. This approximation is nothing else but the delocalisation procedure so often applied to anyonic systems. The Magnus force can be interpreted as Lorenz force due to interaction of effectively charged vortices with some uniform effective magnetic field. The statistical interaction can be seen as Aharonov-Bohm effect due to the fluxes attached to vortices. In the mean-field approximation the fluxes, which are opposite to the external flux, are delocalised and they lower the net uniform flux. In the same way the real impurities can be interpreted as localised fluxes, opposite to the external field, randomly distributed over the plane. If the M-F approximation appears to work for real impurities, it will also work for vortices.

6 Hall angle and vortex density

The fact that the value of the Magnus force can be lowered with increasing density of vortices can, in principle, be observable in Hall experiments \[14\]. The vortex density should increase and the M-F Magnus force should decrease with increasing real magnetic field. This should manifest itself in the changes of the measured Hall angle. Vortex equation of motion takes the form \[15\]

\[
m_{\text{eff}} \ddot{\mathbf{r}} = \frac{\rho_s h d}{2} (\dot{\mathbf{r}} - \mathbf{v}_s) \times \hat{z} - \eta d \dot{\mathbf{r}} + F_{\text{pin}} + \mathbf{f},
\]

(33)

where \( m_{\text{eff}} \) is a small effective vortex mass, \( \eta \) is a vortex viscosity, \( F_{\text{pin}} \) is a pinning force, \( f \) is a fluctuating force, \( d \) is a sample thickness and \( \mathbf{v}_s \) is a driven uniform superfluid velocity. When we neglect pinning and average over fluctuations the stationary state motion will be determined by the equation

\[
(\dot{\mathbf{r}} - \mathbf{v}_s) \times \hat{z} = \tan(\theta_H) \dot{\mathbf{r}},
\]

(34)

where \( \tan(\theta_H) = \frac{2\eta}{\rho_s h} \). The solution is

\[
\dot{\mathbf{r}} = \frac{\mathbf{v}_s + (\hat{z} \times \mathbf{v}_s) \tan(\theta_H)}{1 + \tan^2(\theta_H)}.
\]

(35)
\( \theta_H \) is the angle between the superfluid velocity \( \mathbf{u}_s \) and the stationary vortex velocity. The angle is the larger the weaker is the Magnus force. If the effective Magnus force is lowered with increased vortex density the angle should also grow with external magnetic field, which drives the rise in vortex density. The changes of the Magnus force due to changes in vortex density should be the more rapid the larger is the number of electrons trapped in the vortex core. For this reason we would recommend experiments on mildly type \( II \) conventional superconductors with large vortex cores (large correlation length \( \xi \)). Rather strong viscosity should be preferred for the angle to be more sensitive to the strength of the Magnus force. The sample should be pure of pinning centers to avoid obscure pinning effects.

7 Vortices’ fractional quantum Hall effect

To summarise our knowledge about the dynamics of planar vortices, let us write down an effective Lagrangian for diluted vortices with topological charge \(-1\)

\[
L_{\text{eff}} = \sum_p \left[ \frac{1}{2} m_{\text{eff}} \dot{X}_p \dot{X}_p + \hbar \pi \rho_s X_p \times \dot{X}_p \right] + \sum_{p<q} \left[ \hbar \alpha \frac{d}{dt} \Theta(p,q) - V_{\text{eff}}(|X_p - X_q|) \right].
\]

The indices \( p, q \) run over vortices. \( m_{\text{eff}} \) is the effective vortex mass. It is usually estimated to be around \( 10^8 \frac{m_e}{m} = 10^{-2} \frac{m_e}{\text{Å}} \).

The second term in Eq. (36) describes interaction of vortices with effective magnetic field. We stress that this field has nothing to do with the real magnetic field \( B_{\text{ext}} \), which in this case is just a device to drive the changes of vortex density. If we assumed the density of electrons to be \( \sim 10^{30} \text{m}^{-3} = 1 \text{Å}^{-3} \) the effective magnetic field defined by \( \frac{\varepsilon B_{\text{eff}}}{2} = \pi \hbar \rho_s \) would turn out to be \( \sim 10^6 \text{T/Å} \). When compared with the effective vortex mass per \( 1 \text{Å} \) the magnetic field turns out to be incredibly strong. Its effect on a vortex should be the same as that of the \( 10^6 \text{T} \) magnetic field on an electron. Vortices can be expected to be confined to the lowest Landau level (LLL).

Now let us consider a single vortex at \( z = X_1 + iX_2 \). What is the magnetic length \( l \) which determines the size of the LLL wavepacket

\[
\psi_0(z, \bar{z}) = \frac{1}{\sqrt{2\pi l^2}} e^{-\frac{\bar{z}^2}{4l^2}} ?
\]

The magnetic length is determined by the strength of the exactly known Magnus force \( l^2 = \frac{\hbar}{2\pi \rho_s} \). The area over which the center of such a vortex fluctuates can be estimated to be \( \frac{\rho_s}{\rho_s} \), which is of the same order as the area per one electron. This effect is significant for extremely type \( II \) superconductors where the core is very thin.

Now we are prepared to address the question of the fractional Hall effect. Vortices are anyons with a statistical parameter \( \alpha \). A nontrivial statistics would
be sufficient to prevent them from overlapping if the third term in (36) were not regularised at short distances and replaced by mutual charge-flux interaction \[17\]. Fortunately we also have an effective short range mutual repulsion \(V_{eff}\) which at low temperatures may be sufficient to keep vortices at a distance. On the other hand the potential is very weak as compared to the Landau energy so the mixing with higher LL’s should be in any case negligible. Following Laughlin \[18\], the trial wave-function for a many vortex state can be written

\[
\psi_m[z] = \prod_{p<q} (z_p - z_q)^{\alpha+2m} \prod_r \exp(-|z_r|^2/4l^2), \tag{38}
\]

where \(z_p\)'s are positions of vortices. \(m\) is a nonnegative integer so that the exponent \((\alpha + 2m)\) provides a correct quantumstatistics. The density \(n\) and the filling factor \(\nu\) in such a state are

\[
\nu_m = 2\pi l^2 n_m = \frac{1}{\alpha + 2m}. \tag{39}
\]

For \(m = 0\) the density is just \(n_0 = 1/2\pi\alpha l^2 = \rho_s/\alpha\). \(\alpha\) is roughly equal to the number of electrons inside the core so in the \(m = 0\) state vortex cores would have to overlap slightly. Certainly this density is close to that of the Abrikosov lattice \[19\]. For larger \(m\) the density is smaller and finally we should get outside of the crystalline regime. Then as the density of vortices is driven to change by the changes of the real magnetic field \(B_{ext}\) we should observe some plateaux at the densities \(n_m\) and maybe also at some other quantised filling factors. For conventional superconductors, which are mildly type II, \(\alpha\) is large, so the difference between \(n_m\) and \(n_{m+1}\) is too small for the plateaux to be distinguishable experimentally. For extremaly type II superconductors or high \(T_c\) superconductors we can expect \(\alpha\) to be even \(\sim \frac{1}{3}\) and \(\sim \frac{1}{5}\) of the Abrikosov lattice density.

The main observation of this section is that if the Magnus force is translated into the language of interaction with some effective uniform magnetic field, the field appears to be even \(\sim 10^6 T/A\). It has nothing to do with the real magnetic field which is at best \(\sim 10 T\). The FQHE is a result of this huge effective magnetic field and of repulsive intervortex potential but its existence does not depend on the vortex statistics as the FQHE is possible even for bosons \[20\]. However the pattern of the FQHE plateaux can help to identify the quantumstatistics.

An experimental setup to detect the FQHE would consist of a planar sample of some pure superconducting material in external uniform perpendicular magnetic field. One would have to measure the total flux \(\Phi\), which penetrates through the sample. This flux gives the actual number of vortices in the sample because each vortex carries one flux quantum. Just after continuously turning on weak magnetic field the field lines would be pushed out of the sample. Only above some threshold value of the flux the energy of the field could be
minimised by creating a vortex line. The story would repeat until some stable FQHE plateau were reached. The plateau is a manifestation of a very stable many-vortex state so an addition of one vortex to this state would cost more energy then to a state far from the plateau. When the stable state is approached from low densities the addition of one more vortex should be much easier than usually. Quite opposite, when we lower the external magnetic field it is easier to remove one flux quantum from the sample just above the stable state and more difficult to remove it just below the state, see the figure. The moduli of the change in the driving flux $\Delta \Phi$ (or some equivalent quantity) necessary to change the flux through the sample by one quantum will develop a characteristic pattern around the stable density $n_0$. Two measurements, one with adiabatically increasing and one with adiabatically decreasing external magnetic field, would give two curves with opposite polarisation. Taking their difference will amplify the effects due to the plateau and remove the not necessarily constant bias.

It should be stressed here that our results are not in contradiction with predictions of a bosonic Hall effect in Josephson junction arrays [21]. The arrays are strictly planar devices. The penetration length $\Lambda$ is likely to be greater than the sample size. What is more, it seems to be possible to excite a vortex without any bound states in the core, which indeed might be a boson. Our results are exact in the limit of long straight-linear parallel vortices. In practice it is sufficient that the sample is thick enough for the penetration length $\Lambda$ to be close to that in the bulk. The sample thickness would have to be a bit more then $d = 100\,\text{Å}$. For such a $d$ there is still no space for vortex entanglement. In the absence of entanglement vortices can be uniquely projected on a plane. In nonzero temperature one should expect transversal modes to be excited. These excitations do not affect the Berry phase picked up during vortices exchange as it depends on the overall topological properties of vortices. Vortices excited to different transversal states although in principle distinguishable interact statistically. It is an example of mutual statistical interaction [22] introduced first to describe interlayer phase correlations in the double-layer Hall effect. The plateaux pattern for our anyonic vortices is given by (39). In distinction to bosonic Hall effect the constant $\alpha$ is nonzero.

8 Summary

We have given two new arguments why vortices in superconducting films should be anyons. In addition we have discussed two different experiments where this theoretical prediction could be verified.

Acknowledgement. I would like to thank Prof. B.Halperin for stimulating comments on Ref.[1]. This research was supported in part by the KBN grant No. 2 P03B 085 08 and in part by Foundation for Polish Science fellowship.
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FIGURE CAPTION. Detection of the Hall plateau. $\Delta \Phi$ is the moduli of the change in the external magnetic flux necessary to drive the change in the number of vortices by one. $\Delta \Phi_0$ is its average value far from the plateau. $n$ is the density of vortices and $n_0$ its value at the plateau. The curve "a" corresponds to decreasing while "b" to increasing number of vortices. The curve "c" is the difference between "a" and "b".