Characterization of Vibrating Plates by Bi-Laplacian Eigenvalue Problems

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Abstract

In this paper we derive boundary integral identities for the bi-Laplacian eigenvalue problems under Dirichlet, Navier and simply-supported boundary conditions. By using these integral identities, we first obtain uniqueness criteria for the solutions of the bi-Laplacian eigenvalue problems, and then prove that each eigenvalue of the problem with simply-supported boundary conditions increases strictly with Poisson’s ratio, thereby showing that each natural frequency of a simply-supported plate increases strictly with Poisson’s ratio. In addition, we obtain boundary integral representations for the strain energies of the vibrating plates under the three boundary conditions.

Keywords: Bi-Laplacian eigenvalue problems, Dirichlet boundary conditions, Navier boundary conditions, Simply-supported boundary conditions, Vibrating plates, Poisson’s ratio, Natural frequency.

AMS subject classifications: 35B99, 35J40

1 Introduction

The bi-Laplacian boundary value problems with Dirichlet, Navier and simply-supported boundary conditions are classical boundary value problems. Among these problems, Dirichlet and simply-supported problems are of more importance in solid mechanics. In this paper we shall first study the eigenvalues of the three problems, and then pay special attention to the simply-supported problem, in contrast to many publications which only focused on the other two problems. In fact, the Navier boundary conditions are a special case of the simply-supported boundary conditions and the Dirichlet (clamped-edge) problem can be approximated by the simply-supported boundary value problem if the interior domain is far enough from the boundary.

It is known that resonance problems are a major concern in mechanics. Resonant frequency of a solid structure is an instrumental parameter that greatly affects the structural dynamic behaviors governed by the solid mechanics theory \[1\], \[2\]. In this section, we shall simplify the simply-supported boundary conditions and give the problem statements for the three types of the eigenvalue problems. F. Rellich introduced a trial function \[\sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k} = x \cdot \nabla u\], applied it to the Dirichlet Laplacian operator and obtained a boundary integral identity
for the Dirichlet Laplacian problem in 1940 [3]. This idea has been generalized and applied to elliptic PDE problems [4], [5], [6]. In order to show the dependence of the eigenvalues on the parameter in the boundary conditions of the vibrating plates, we shall follow Rellich’s idea to derive three boundary integral expressions for the Dirichlet, Navier and simply-supported bi-Laplacian problems and shall establish uniqueness criteria for the solutions of the eigenvalue problems in Section 2. Based on these integral identities for the bi-Laplacian problems we show how the eigenvalues of the simply-supported problem and the resonant frequencies of the vibrating plate are influenced by Poisson’s ratio through the new uniqueness theorems in Section 3. In addition, we derive three boundary integral expressions for the strain energies of the vibrating plates at resonance using the boundary integral identities. These boundary integral expressions can be used to calculate the total strain energies when only boundary value data of the plates are available.

1.1 Dirichlet and Navier Bi-Laplacian Problems

Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain having a $C^{4,\beta}$ boundary $\partial \Omega$ ($0 < \beta < 1$). Let $\Lambda$ be an eigenvalue for which the Dirichlet eigenvalue problem,

$$\begin{align*}
\triangle \triangle U &= \Lambda U \quad \text{in } \Omega, \\
U &= |\nabla U| = 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(1)

has a nontrivial solution, and let $\lambda$ be an eigenvalue for which the Navier eigenvalue problem,

$$\begin{align*}
\triangle \triangle V &= \lambda V \quad \text{in } \Omega, \\
V &= \triangle V = 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(2)

has a nontrivial solution, where $\triangle \triangle = \triangle^2$ denotes the n-dimensional bi-Laplacian and $U$ and $V$ are respectively the eigenfunctions of Problems (1) and (2).

1.2 Simply supported Boundary Conditions

The supported boundary conditions (short for simply supported boundary conditions) in their standard form obtained from their mechanical property and implementation manner have been applied to the bi-Laplacian problems [1], [2]. These conditions are natural, based on the physical ground. In this subsection, we shall first simplify the standard form, and then define the corresponding eigenvalue problems.

We consider a thin elastic plate which is assumed to be homogeneous and isotropic under the supported boundary conditions:

$$\begin{align*}
W|_{\partial \Omega} &= 0, \\
M_{\nu}|_{\partial \Omega} &= 0.
\end{align*}$$
where \( W \) is the deflection of the plate in the vertical direction and \( M_\nu \) is the bending moment with respect to the \( \nu \) direction. These boundary conditions in two dimensions can be expressed as (See [10], Expression (110), p.94.)

\[
M_\nu|_{\partial \Omega} = \mu \Delta W|_{\partial \Omega} + (1 - \mu) \left[ \cos^2 \theta \frac{\partial^2 W}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 W}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 W}{\partial y^2} \right]|_{\partial \Omega} = 0,
\]

where \( \mu \) is Poisson’s ratio, an elasticity constant (0 < \( \mu \) < 1), the normal unit vector \( \nu \) is directed outward from \( \Omega \) and \( \theta \) is the angle between the normal \( \nu \) and \( x \)-axis (\( \cos \theta = \cos(x, \nu) = \nu_x \)). It is shown that the factor in the square brackets of the above equality is equal to \( \Delta W|_{\partial \Omega} \) (See the appendix to this paper). This can be written as

\[
\cos^2 \theta \frac{\partial^2 W}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 W}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial \nu^2} \text{ on } \partial \Omega.
\]

Thus, we obtain

\[
M_\nu|_{\partial \Omega} = \mu \Delta W|_{\partial \Omega} + (1 - \mu) \frac{\partial^2 W}{\partial \nu^2} |_{\partial \Omega} = 0.
\]

Using the above definition and the relation \( \Delta W|_{\partial \Omega} = \frac{\partial^2 W}{\partial \nu^2} + (n - 1) \kappa \frac{\partial W}{\partial \nu} |_{\partial \Omega} \) for \( n = 2 \), we obtain

\[
\Delta W|_{\partial \Omega} = (1 - \mu) \kappa \frac{\partial W}{\partial \nu} |_{\partial \Omega} = c_0 \frac{\partial W}{\partial \nu} |_{\partial \Omega},
\]

where \( \kappa \) is the mean curvature of \( \partial \Omega \) and \( c_0 = (1 - \mu) \kappa \geq 0 \) for convex domains.

### 1.3 The Supported Bi-Laplacian Problem and Basic Identities

We are now in a position to define the supported bi-Laplacian eigenvalue problem. Let \( \Omega \subset \mathbb{R}^2 \) be defined as in Subsection 1.1 and let \( \gamma \) be an eigenvalue for which the bi-Laplacian eigenvalue problem under the supported boundary conditions,

\[
\begin{align*}
\Delta \Delta W &= \gamma W \quad \text{in } \Omega, \\
W &= 0 \quad \text{on } \partial \Omega, \\
\Delta W &= c_0 \frac{\partial W}{\partial \nu} \quad \text{on } \partial \Omega,
\end{align*}
\]

has a nontrivial solution. Elliptic regularity theorems ensure that Problems (1), (2) and (3) will respectively have nontrivial solution \( U, V \) and \( W \in C^4(\Omega) \). Assuming that \( \alpha \) and \( u \) are the eigenvalue and the corresponding eigenfunction of Problem (1) or (2) or (3), Green’s identities show that

\[
\int_{\Omega} (\alpha u) u d\Omega = \int_{\Omega} (\Delta \Delta u) u d\Omega = \int_{\partial \Omega} \frac{\partial \Delta u}{\partial \nu} u dS - \int_{\Omega} \nabla \Delta u \cdot \nabla u d\Omega \\
= \int_{\partial \Omega} \frac{\partial \Delta u}{\partial \nu} u dS - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \Delta u dS + \int_{\Omega} (\Delta u)^2 d\Omega. \tag{4}
\]
The common boundary condition of Problems (1), (2) and (3), \( U|_{\partial\Omega} = V|_{\partial\Omega} = W|_{\partial\Omega} = 0 \), leads to \( \frac{\partial U}{\partial \nu}|_{\partial\Omega} = \frac{\partial V}{\partial \nu}|_{\partial\Omega} = \frac{\partial W}{\partial \nu}|_{\partial\Omega} = 0 \). For Problems (1) and (2), both the first and second surface integrals on \( \partial\Omega \) vanish. Therefore, the eigenvalues of the two problems can be expressed by the same formula

\[
\Lambda \text{ or } \lambda = \frac{\int_{\Omega} (\triangle u)^2 d\Omega}{\int_{\Omega} u^2 d\Omega}\quad (5)
\]

where \( u \) can be \( U \) or \( V \). For Problem (3), the identities shown in (4) shows that

\[
\gamma = \frac{\int_{\Omega} (\triangle W)^2 d\Omega - \int_{\partial\Omega} c_0 (\frac{\partial W}{\partial \nu})^2 dS}{\int_{\Omega} W^2 d\Omega}\quad (6)
\]

The positiveness of the eigenvalues are proved by the theory of elliptic partial differential equations [6], [7], [8].

2 Boundary Integral Expressions for Bi-Laplacian Problems

**Theorem 2.1** Let \( \Omega \) be defined as in section 1.1 and let \( U \) be a nontrivial solution of Problems (1). Then, Let \( \Omega \) be defined as in section 1.1 and let \( U \) be a nontrivial solution of Problems (1). Then,

\[
\Lambda = \frac{\int_{\partial\Omega} (x \cdot \nu) (\frac{\partial^2 U}{\partial \nu^2})^2 dS}{4 \int_{\Omega} U^2 d\Omega}\quad (7)
\]

**Theorem 2.3** Let \( \Omega \) be defined as in subsection 1.2 and let \( V \) be a nontrivial solution of Problem (2). Then,

\[
\lambda = \frac{-\int_{\partial\Omega} (x \cdot \nabla V) \frac{\partial^2 V}{\partial \nu^2} dS}{2 \int_{\Omega} V^2 d\Omega}\quad (8)
\]

**Theorem 2.5** Let \( \Omega \) be defined as in subsection 1.3 and let \( W \) be a nontrivial solution of Problem (3). Then,

\[
\gamma = \frac{-\int_{\partial\Omega} (x \cdot \nu) (c_0 \frac{\partial W}{\partial \nu})^2 dS + 2 \int_{\partial\Omega} (x \cdot \nu) \frac{\partial c_0}{\partial \nu} (\frac{\partial W}{\partial \nu})^2 dS}{4 \int_{\Omega} W^2 d\Omega}\quad (9)
\]

**Proofs of theorems 2.1, 2.3 and 2.5**

We first multiply both sides of the bi-Laplacian eigen equation of Problems (1), (2) and (3) by the test function \( (x \cdot \nabla u) \), and integrate over \( \Omega \) to obtain

\[
\int_{\Omega} (\triangle \triangle u) (x \cdot \nabla u) d\Omega = \int_{\Omega} (x \cdot \nabla u) d\Omega\quad (10)
\]
Applying Green’s theorem to (10), the right-hand side of (10) becomes

\[ \int_{\Omega} \alpha u \frac{\partial^2 u}{\partial x_j^2} + \sum_{i,j} x_i \frac{\partial^2 u}{\partial x_i \partial x_j} d\Omega = \alpha \int_{\partial \Omega} u^2 d\Sigma + \frac{\alpha}{2} \int_{\partial \Omega} (x \cdot \nu) u^2 dS, \quad (11) \]

and the left-hand side of (10) becomes

\[ \int_{\Omega} (\Delta u)(x \cdot \nabla u) d\Omega = - \int_{\Omega} \nabla \Delta u \cdot \nabla (x \cdot \nabla u) d\Omega + \int_{\partial \Omega} \frac{\partial \Delta u}{\partial \nu} (x \cdot \nabla u) dS. \quad (12) \]

The (generalized) volume integral term of the right-hand side of (12) can be written as

\[ - \int_{\Omega} \nabla \Delta u \cdot \nabla (x \cdot \nabla u) d\Omega = - \int_{\Omega} \sum_{j=1}^{n} \frac{\partial \Delta u}{\partial x_j} \left( \frac{\partial u}{\partial x_j} + \sum_{i} x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) d\Omega \]

\[ = \int_{\partial \Omega} \sum_{j=1}^{n} \frac{\partial \Delta u}{\partial x_j} \left( \frac{\partial u}{\partial x_j} + \sum_{i} x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_j dS, \]

where \( \nu_j = \cos(x_j, \nu) \). The volume integral term of the right-hand side of the above equality can be further written as

\[ + \int_{\Omega} \Delta u \left( \frac{\partial^2 u}{\partial x_j^2} + \sum_{i,j} x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) d\Omega = \int_{\Omega} \left( 2(\Delta u)^2 + \frac{1}{2} \sum_{i} x_i \frac{\partial (\Delta u)^2}{\partial x_i} \right) d\Omega \]

\[ = 2 \int_{\Omega} (\Delta u)^2 d\Omega - \frac{n}{2} \int_{\Omega} (\Delta u)^2 d\Omega + \frac{1}{2} \int_{\partial \Omega} (\Delta u)^2 (x \cdot \nu) dS. \quad (13) \]

From (14), we have

\[ \alpha \int_{\Omega} u^2 d\Omega = \int_{\partial \Omega} \frac{\partial \Delta u}{\partial \nu} u dS - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \Delta u dS + \int_{\Omega} (\Delta u)^2 d\Omega. \]

Expressing \( \int_{\Omega} (\Delta u)^2 d\Omega \) in terms of the integrals in the above equality, (13) becomes

\[ (2 - \frac{n}{2}) \left( \alpha \int_{\Omega} u^2 d\Omega - \int_{\partial \Omega} \frac{\partial \Delta u}{\partial \nu} u dS + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \Delta u dS \right) + \frac{1}{2} \int_{\partial \Omega} (\Delta u)^2 (x \cdot \nu) dS \quad (14) \]
Substituting (14) into (12) and equating the result to (10), the following identity results

\[
(2 - \frac{n}{2}) \left(\alpha \int_\Omega u^2 d\Omega - \int_{\partial \Omega} \frac{\partial \triangle u}{\partial \nu} u dS + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \triangle u dS \right) + \frac{1}{2} \int_{\partial \Omega} (x \cdot \nu)(\triangle u)^2 dS
- \int_{\partial \Omega} \triangle u \left(\frac{\partial u}{\partial \nu} + \sum_{ij} x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS + \int_{\partial \Omega} \frac{\partial \triangle u}{\partial \nu} (x \cdot \nabla u) dS
= - \frac{n\alpha}{2} \int_\Omega u^2 d\Omega + \frac{\alpha}{2} \int_{\partial \Omega} (x \cdot \nu) u^2 dS,
\]

where the relation

\[
- \int_{\partial \Omega} \triangle u \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} + \sum_{i=1}^n x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_j dS = - \int_{\partial \Omega} \triangle u \left(\frac{\partial u}{\partial \nu} + \sum_{ij} x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS
\]

is used. After cancelation, the above identity can be written as

\[
2\alpha \int_\Omega u^2 d\Omega = - \frac{1}{2} \int_{\partial \Omega} (x \cdot \nu) (\triangle u)^2 dS + \int_{\partial \Omega} \triangle u \left(\frac{\partial u}{\partial \nu} + \sum_{ij} x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS
- \int_{\partial \Omega} \frac{\partial \triangle u}{\partial \nu} (x \cdot \nabla u) dS + \frac{\alpha}{2} \int_{\partial \Omega} (x \cdot \nu) u^2 dS + (2 - \frac{n}{2}) \int_{\partial \Omega} \left(\frac{\partial \triangle u}{\partial \nu} u - \triangle u \frac{\partial u}{\partial \nu} \right) dS
\]

We first consider the Dirichlet boundary conditions of Problem (1). Let \( u = U \), \( \alpha = \Lambda \) and \( U|_{\partial \Omega} = |\nabla U||_{\partial \Omega} = 0 \); then (13) becomes

\[
2\alpha \int_\Omega U^2 d\Omega = - \frac{1}{2} \int_{\partial \Omega} (\triangle U)^2 (x \cdot \nu) dS + \int_{\partial \Omega} \triangle U \left(\sum_{ij} x_i \nu_j \frac{\partial^2 U}{\partial x_i \partial x_j} \right) dS.
\]

Solving for \( \Lambda \) using \( \frac{\partial^2 U}{\partial x_i \partial x_j} = \nu_i \nu_j \frac{\partial^2 u}{\partial \nu^2} \) on \( \partial \Omega \), then

\[
\Lambda = - \frac{\int_{\partial \Omega} (x \cdot \nu) (\triangle U)^2 dS + 2 \int_{\partial \Omega} \triangle U \left(\sum_{ij} x_i \nu_j \frac{\partial^2 U}{\partial x_i \partial x_j} \right) dS}{4 \int_\Omega U^2 d\Omega}
= - \frac{\int_{\partial \Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2}\right)^2 dS + 2 \int_{\partial \Omega} \frac{\partial^2 U}{\partial \nu^2} (x \cdot \nu) \left(\frac{\partial^2 u}{\partial \nu^2}\right) dS}{4 \int_\Omega U^2 d\Omega}
\]

Therefore,

\[
\Lambda = \frac{\int_{\partial \Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2}\right)^2 dS}{4 \int_\Omega U^2 d\Omega}.
\]

Based on the fact that the eigenvalues of Problems (1) are greater than zero, we obtain
Corollary 2.2 Problem (1) admits only a trivial solution \( U \in C^4(\bar{\Omega}) \) for \( \Omega \subset \mathbb{R}^n, n \geq 2 \) if

\[
U|_{\partial\Omega} = \frac{\partial U}{\partial \nu}|_{\partial\Omega} = 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial \nu^2}|_{\partial\Omega} = 0.
\]

Secondly, we impose the Navier boundary conditions of Problem (2). Let \( u = V, \alpha = \lambda \) and \( V|_{\partial\Omega} = \Delta V|_{\partial\Omega} = 0 \); then (15) becomes

\[
2\lambda \int_{\partial\Omega} V^2 d\Omega = -\int_{\partial\Omega} \frac{\partial \Delta V}{\partial \nu} (x \cdot \nabla V) dS.
\]

Solving for \( \lambda \), we obtain

\[
\lambda = -\frac{\int_{\partial\Omega} (x \cdot \nabla V) \frac{\partial \Delta V}{\partial \nu} dS}{2\int_{\Omega} V^2 d\Omega} = -\frac{\int_{\partial\Omega} (x \cdot \nu) \frac{\partial V}{\partial \nu} \frac{\partial \Delta V}{\partial \nu} dS}{2\int_{\Omega} V^2 d\Omega},
\]

Based on the fact that the eigenvalues of Problems (2) is greater than zero, we obtain

Corollary 2.4 Problem (2) admits only a trivial solution \( V \in C^4(\bar{\Omega}) \) for \( \Omega \subset \mathbb{R}^n, n \geq 2 \) if

\[
V|_{\partial\Omega} = \Delta V|_{\partial\Omega} = 0 \quad \text{and} \quad \left( \frac{\partial \Delta V}{\partial \nu} \text{ or } (x \cdot \nu) \frac{\partial V}{\partial \nu} \right)|_{\partial\Omega} = 0.
\]

Since \( \Delta V|_{\partial\Omega} = \frac{\partial^2 V}{\partial \nu^2} + (n-1)\kappa \frac{\partial V}{\partial \nu}|_{\partial\Omega} = 0 \), the if conditions in Corollary 2.4 can be written as

\[
V|_{\partial\Omega} = 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial \nu^2}|_{\partial\Omega} = -(n-1)\kappa \frac{\partial V}{\partial \nu}|_{\partial\Omega}, \quad \text{and}
\]

\[
\left( \frac{\partial^3 V}{\partial \nu^3} + (n-1)\kappa \frac{\partial^2 V}{\partial \nu^2} \right)|_{\partial\Omega} = 0 \quad \text{or} \quad (x \cdot \nu) \frac{\partial V}{\partial \nu}|_{\partial\Omega} = 0.
\]

Next, we impose the supported boundary conditions of Problem (3). We assume the domain \( \Omega \) is convex, which is the common case for plates. Letting \( u = W, \alpha = \gamma, n = 2 \) and inserting \( W = 0|_{\partial\Omega} \) and \( \Delta W = c_0 \frac{\partial W}{\partial \nu}|_{\partial\Omega} \) into (15), we obtain

\[
2\gamma \int_{\Omega} W^2 d\Omega = -\frac{1}{2} \int_{\partial\Omega} (x \cdot \nu)(c_0 \frac{\partial W}{\partial \nu})^2 dS + \int_{\partial\Omega} c_0 \frac{\partial W}{\partial \nu} \left( \sum_{ij} x_i \nu_j \frac{\partial^2 W}{\partial x_i \partial x_j} \right) dS
\]

\[
-\int_{\partial\Omega} \frac{\partial}{\partial \nu} \left( c_0 \frac{\partial W}{\partial \nu} \right) (x \cdot \nabla W) dS,
\]
with $c_0 > 0$. Solving for $\gamma$, we obtain

$$
\gamma = -\frac{\int_{\partial \Omega} (x \cdot \nu) (c_0 \frac{\partial W}{\partial \nu})^2 dS + 2 \int_{\partial \Omega} c_0 \frac{\partial W}{\partial \nu} \left( \sum_{ij} x_i \nu_j \frac{\partial^2 W}{\partial x_i \partial x_j} \right) dS}{4 \int_\Omega W^2 d\Omega} + \frac{2 \int_{\partial \Omega} (x \cdot \nabla W) dS}{4 \int_\Omega W^2 d\Omega} - \frac{\int_{\partial \Omega} (x \cdot \nu) (c_0 \frac{\partial W}{\partial \nu})^2 dS - 2 \int_{\partial \Omega} (x \cdot \nu) \frac{\partial c_0}{\partial \nu} (\frac{\partial W}{\partial \nu})^2 dS}{4 \int_\Omega W^2 d\Omega},
$$

where the relations $\frac{\partial^2 u}{\partial x_i \partial x_j} = \nu_i \nu_j \frac{\partial^2 u}{\partial \nu^2}$ and $x \cdot \nabla W = \frac{\partial W}{\partial \nu} (x \cdot \nu)$ on $\partial \Omega$ are used. This completes the proof of Theorem 2.5.

Based on the fact that the eigenvalues of Problems (3) are greater than zero, we obtain

**Corollary 2.6** Problem (3) admits only a trivial solution $W \in C^4(\bar{\Omega})$ for $\Omega \subset \mathbb{R}^2$ if

$$
W |_{\partial \Omega} = \frac{\partial W}{\partial \nu}|_{\partial \Omega} = 0 \text{ and } \frac{\partial^2 W}{\partial \nu^2}|_{\partial \Omega} = \Delta W|_{\partial \Omega} = c_0 \frac{\partial W}{\partial \nu}|_{\partial \Omega}.
$$

**Corollary 2.7** Corollaries 2.2, 2.4, 2.6 illustrate that a nontrivial eigenfunction cannot simultaneously be an eigenfunction of the Dirichlet and supported (or Navier) plate of the same shape.

### 3 Application to Solid Mechanics

Plates (short for elastic uniform-thin plates) are a particular two-dimensional representation of a three-dimensional solid, which have a much smaller thickness in comparison with the in-plane dimensions [1], [2]. In the following, we will apply our theorems and corollaries to vibrating plate problems.

#### 3.1 Effect of Poisson’s Ratio on Eigenvalues of Supported Problem

**Theorem 3.1** Let $\Omega$ be a convex domain defined as in section 1.3 with $\kappa(x) > 0, \forall x \subset \partial \Omega$ and let $W(x, \mu) \in C^{4,\beta}_1 (\bar{\Omega}, (0, 1))$ be a nontrivial solution of the supported eigenvalue problem. Then,

$$
\frac{\partial \gamma}{\partial \mu} > 0,
$$

where $W(x, \mu) \in C^{4,\beta}_1 (\bar{\Omega}, (0, 1))$ denotes $W(x, \cdot) \in C^{4,\beta}(\bar{\Omega})$ and $W(\cdot, \mu) \in C^4(0, 1)$.

**Proof of Theorem 3.1** By using $\Delta \Delta W = \gamma W$ of Problem (3) and differentiating the identity with respect to $\mu$, we have

$$
\frac{\partial \Delta W}{\partial \mu} = \gamma \frac{\partial W}{\partial \mu} + W \frac{\partial \gamma}{\partial \mu}.
$$
Since $\frac{\partial W}{\partial \mu}$ is continuously differentiable, we obtain
\[
\frac{\partial \triangle W}{\partial \mu} = \triangle \frac{\partial W}{\partial \mu} = \gamma \frac{\partial W}{\partial \mu} + W \frac{\partial \gamma}{\partial \mu}.
\] (16)

Therefore,
\[
W \triangle \frac{\partial W}{\partial \mu} = \gamma W \frac{\partial W}{\partial \mu} + W^2 \frac{\partial \gamma}{\partial \mu}.
\] (17)

Integrating both sides of the above identity in $\Omega$ and using the Green’s identities, the left-hand side of (17) becomes
\[
\int_{\Omega} W \triangle \frac{\partial W}{\partial \mu} d\Omega = -\int_{\partial \Omega} \frac{\partial W}{\partial \nu} \left( -\kappa \frac{\partial W}{\partial \nu} + c_0 \frac{\partial^2 W}{\partial \nu \partial \mu} \right) dS + \int_{\partial \Omega} c_0 \frac{\partial W}{\partial \nu} \frac{\partial^2 W}{\partial \nu \partial \mu} dS
+ \int_{\partial \Omega} \gamma W \frac{\partial W}{\partial \mu} dS = \int_{\partial \Omega} \kappa \frac{\partial W}{\partial \mu}^2 dS + \int_{\Omega} \gamma W \frac{\partial W}{\partial \mu} d\Omega. \tag{18}
\]

Using $\triangle W = \gamma W$ and identity $\underline{17}$, and imposing the supported boundary conditions $W|_{\partial \Omega} = 0$ and $\triangle W|_{\partial \Omega} = (1 - \mu)\kappa \frac{\partial W}{\partial \nu} |_{\partial \Omega} = c_0 \frac{\partial W}{\partial \nu} |_{\partial \Omega}$, we obtain
\[
\int_{\Omega} W \triangle \frac{\partial W}{\partial \mu} d\Omega = -\int_{\partial \Omega} \frac{\partial W}{\partial \nu} \left( -\kappa \frac{\partial W}{\partial \nu} + c_0 \frac{\partial^2 W}{\partial \nu \partial \mu} \right) dS + \int_{\partial \Omega} c_0 \frac{\partial W}{\partial \nu} \frac{\partial^2 W}{\partial \nu \partial \mu} dS
+ \int_{\partial \Omega} \gamma W \frac{\partial W}{\partial \mu} dS = \int_{\partial \Omega} \kappa \frac{\partial W}{\partial \mu}^2 dS + \int_{\Omega} \gamma W \frac{\partial W}{\partial \mu} d\Omega. \tag{18}
\]

where $\frac{\partial W}{\partial \nu} |_{\partial \Omega} = 0$ and $\frac{\partial^2 W}{\partial \mu \partial \nu} |_{\partial \Omega} = (\kappa \frac{\partial W}{\partial \nu} |_{\partial \Omega} + c_0 \frac{\partial^2 W}{\partial \nu \partial \mu}|_{\partial \Omega})$ is used. By using the identity $\int_{\Omega} W \triangle \frac{\partial W}{\partial \mu} d\Omega = \int_{\Omega} \left( W^2 \frac{\partial^2 W}{\partial \mu \partial \nu} + \gamma W \frac{\partial W}{\partial \mu} \right) d\Omega$, (18) can be written as
\[
\int_{\partial \Omega} \kappa \frac{\partial W}{\partial \mu}^2 dS = \int_{\Omega} W^2 \frac{\partial \gamma}{\partial \mu} d\Omega.
\]

Hence,
\[
\frac{\partial \gamma}{\partial \mu} = \frac{\int_{\partial \Omega} \kappa \frac{\partial W}{\partial \mu}^2 dS}{\int_{\Omega} W^2 d\Omega} \geq 0, \tag{19}
\]

which shows that $\gamma$ is a nondecreasing function of $\mu$.

Furthermore, for those domains with $\kappa > 0$ on $\partial \Omega$, from (19), we obtain that if $\frac{\partial \gamma}{\partial \mu} = 0$, then $\frac{\partial W}{\partial \mu} = 0$ on $\partial \Omega$. Applying Corollary 2.6, if $\Omega \subset \mathbb{R}^2$ with $\kappa(x) > 0$ on $\partial \Omega$, then $W \equiv 0$ in $\Omega$. This contradicts the assumption that $W$ is a nontrivial solution of Problem 3. Hence, $\frac{\partial \gamma}{\partial \mu} \neq 0$. Therefore,
\[
\frac{\partial \gamma}{\partial \mu} > 0,
\]

for all convex domains with $\kappa > 0$ on $\partial \Omega$. This completes the proof of theorem 3.1.
Corollary 3.2 Each eigenvalue of simply-supported convex plates with $\kappa > 0$ is strictly monotonic in $\mu$. Equivalently, a nontrivial eigenfunction cannot simultaneously be an eigenfunction of two simply-supported plates of the same shape with different values of $\mu$ if the boundary of the plate is convex and there is no straight segment on the boundary ($\kappa > 0$).

It is known that a natural (intrinsic) frequency of a vibrating plate $\omega = \sqrt{\frac{\gamma D}{\bar{m}m}}$, where $\bar{m}$ denotes the equivalent mass per unit area of the plate, $D = \frac{1}{12} \left( \frac{\kappa}{1-\mu^2} \right)$, $E$ denotes the modulus of elasticity and $h$ is the thickness of the plate. Therefore, Theorem 3.1 leads to the following corollary.

Corollary 3.3 Each natural frequency of simply-supported convex plates with $\kappa > 0$ increases strictly with Poisson’s ratio.

The property of the internal parts of the plates far enough from the boundary can be approximated by the clamped boundary conditions, and thus the result is applicable to the Dirichlet problem in the internal parts.

3.2 Boundary Integral Identities for the Eigen-Problems

The Strain energy of a mechanical structure provides a good measure for exceeded stresses and strains or exceeded strain energy in the structure. It is often used, with certain failure criteria of materials, to evaluate if a structure under certain loads is in a safe condition. Identities representing the strain energy conservation of elastostatic problems have been developed and used to prove the uniqueness of the equilibrium solutions in elastostatics [12, [14].

For the Dirichlet Problem (1), applying Theorem 2.1 and (5), we have

$$\frac{\int_{\Omega} (\triangle U)^2 d\Omega}{\int_{\Omega} U^2 d\Omega} = -\frac{\int_{\partial\Omega} (x \cdot \nu)(\frac{\partial U}{\partial \nu})^2 dS}{4 \int_{\Omega} U^2 d\Omega}$$

Multiplying both sides of the above identity by $\int_{\Omega} U^2 d\Omega$, the following identity yields

$$\int_{\Omega} (\triangle U)^2 d\Omega = \frac{1}{4} \int_{\partial\Omega} (x \cdot \nu)(\frac{\partial U}{\partial \nu})^2 dS$$  \hspace{1cm} (20)

For the Navier Problem (2), applying Theorem 2.2 and (5), we have

$$\frac{\int_{\Omega} (\triangle V)^2 d\Omega}{\int_{\Omega} V^2 d\Omega} = -\frac{\int_{\partial\Omega} (x \cdot \nabla V)(\frac{\partial \triangle V}{\partial \nu}) dS}{2 \int_{\Omega} V^2 d\Omega}$$

Multiplying both sides of the above identity by $\int_{\Omega} V^2 d\Omega$, the following identity yields

$$\int_{\Omega} (\triangle V)^2 d\Omega = -\frac{1}{2} \int_{\partial\Omega} (x \cdot \nabla V)(\frac{\partial \triangle V}{\partial \nu}) dS$$  \hspace{1cm} (21)

For Problem (3), applying Theorem 3.5 and (5), we have

$$\frac{\int_{\Omega} (\triangle W)^2 d\Omega - \int_{\partial\Omega} c_0(\frac{\partial W}{\partial \nu})^2 dS}{\int_{\Omega} W^2 d\Omega} = -\frac{\int_{\partial\Omega} (x \cdot \nu)(c_0 \frac{\partial W}{\partial \nu})^2 dS + 2 \int_{\partial\Omega} (x \cdot \nu)(\frac{\partial \triangle W}{\partial \nu})^2 dS}{4 \int_{\Omega} W^2 d\Omega}.$$
where \( c_0(x) = \kappa(x)(1 - \mu) \). Multiplying both sides of the above identity by \( \int_\Omega W^2d\Omega \), the following identity results

\[
\int_\Omega (\Delta W)^2d\Omega = - \frac{1}{4} \int_{\partial\Omega} (x \cdot \nu)(c_0\frac{\partial W}{\partial \nu})^2dS + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu)\frac{\partial c_0}{\partial \nu} (\frac{\partial W}{\partial \nu})^2dS \\
+ \int_{\partial\Omega} c_0(\frac{\partial W}{\partial \nu})^2dS \tag{22}
\]

### 3.3 Boundary Integral Expressions for Stain Energies

In the following we shall derive the boundary-integral expressions for the strain energies of the vibrating thin plates.

The general expression for the strain energy of a bent thin plate is given as (See [10], p.95.)

\[
E_s = \frac{1}{2}D \int_\Omega \left\{ \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 - 2(1 - \mu) \left[ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] \right\} d\Omega, \tag{23}
\]

where \( u \), is the deflection of the plate in the vertical direction and the integration is extended over the entire surface of the plate. Here \( u \) can be the solutions of Problems (1), (2) and (3). In the derivation of the strain energy expressions the following identity is needed (See [13], p.87)

\[
2 \int_\Omega \left[ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = \int_{\partial\Omega} \left[ 2 \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial s^2} + \frac{1}{\kappa} (\frac{\partial u}{\partial \nu})^2 + \frac{1}{\kappa} (\frac{\partial u}{\partial s})^2 \right] dS, \tag{24}
\]

with \( \kappa > 0 \). Under the Dirichlet boundary conditions this identity reduces to

\[
\int_\Omega \left[ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = 0, \tag{24}
\]

Under the Navier or supported boundary conditions this identity reduces to

\[
\int_\Omega \left[ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = \frac{1}{2} \int_{\partial\Omega} \frac{1}{\kappa} (\frac{\partial u}{\partial \nu})^2 dS. \tag{25}
\]

For the Dirichlet Problem, setting \( n = 2 \) and substituting (20) and (24) into (23) we obtain

\[
E^D_s = \frac{1}{8}D \int_{\partial\Omega} (x \cdot \nu)(\frac{\partial^2 U}{\partial \nu^2})^2dS, \tag{26}
\]

where \( E^D_s \) is the strain energy of the resonant plate under the Dirichlet boundary conditions.

For the Navier Problem, setting \( n = 2 \) and substituting (21) and (25) into (23) we obtain

\[
E^N_s = - \frac{D}{2} \int_{\partial\Omega} \left[ \frac{1}{2}(x \cdot \nu)\frac{\partial V}{\partial \nu} \frac{\partial \Delta V}{\partial \nu} + (1 - \mu) \frac{1}{\kappa} (\frac{\partial V}{\partial \nu})^2 \right] dS \tag{27}
\]
where $E^N_s$ is the strain energy of the resonant plate under the Navier boundary conditions. For the supported plate problem, setting $n = 2$ and substituting (22) and (25) into (23) we obtain

$$
E^S_s = -\frac{D}{8} \int_{\partial\Omega} (x \cdot \nu) (c_0 \frac{\partial W}{\partial \nu})^2 dS - \frac{D}{4} \int_{\partial\Omega} (x \cdot \nu) \frac{\partial c_0}{\partial \nu} (\frac{\partial W}{\partial \nu})^2 dS
$$

$$
+ \frac{D}{2} \int_{\partial\Omega} c_0 (\frac{\partial W}{\partial \nu})^2 dS - \frac{D}{2} (1 - \mu) \int_{\partial\Omega} \frac{1}{\kappa} (\frac{\partial W}{\partial \nu})^2 dS
$$

where $E^S_s$ is the strain energy of the resonant plate under the supported boundary conditions. The supported problem can be modeled as a hinged plate attached to the boundary with a torsion spring. When the spring constant goes to infinite, the problem becomes a clamped boundary plate.

**Remark 4.1** The above three forms for $E^D_s$, $E^N_s$ and $E^S_s$ are the boundary integral expressions for the strain energies, which simplify the calculation of the strain energy in a plane area to the boundary of the area at resonance. These expressions can be used to simplify the calculation of the strain energies if only boundary data are available.

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## 4 Appendix 1

We shall prove the equality, $\cos^2 \theta \frac{\partial^2 W}{\partial x^2} + 2\sin \theta \cos \theta \frac{\partial^2 W}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial \nu^2}$ on $\partial\Omega$, used in simplifying the supported boundary conditions.

Let $(s_1, \ldots, s_{n-1})$ be a local coordinates on $\partial\Omega$, i.e., $\partial\Omega$ is locally represented by

$$
\tilde{X} : R^{n-1} \rightarrow \partial\Omega
$$

Let

$$
\nu(s_1, \ldots, s_{n-1})
$$

be its unit outer normal of $\partial\Omega$ at $\tilde{X}(s_1, \ldots, s_{n-1})$. We also define a local coordinate system near $\partial\Omega$ by

$$
X(s_1, \ldots, s_{n-1}, t) = \tilde{X}(s_1, \ldots, s_{n-1}) - t\nu(s_1, \ldots, s_{n-1}).
$$

In two dimensional case, for any $u$ defined near $\partial\Omega$, we choose the coordinate system $(s, t)$, such that $t$ is the distance from a point $x$ in $\Omega$ to $\partial\Omega$ and $s$ is the arc-length parameter on $\partial\Omega$ in the counter-clockwise direction. On $\partial\Omega$ we write $\nu = (\nu_x, \nu_y)$ and have

$$
\nabla t = -\nu \quad and \quad \nabla s \cdot \nabla t = 0.
$$
Hence on $\partial \Omega$,

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t \partial s} \left( \frac{\partial t}{\partial x} \frac{\partial s}{\partial x} + \frac{\partial t}{\partial x} \frac{\partial s}{\partial x} \right) + \frac{\partial^2 u}{\partial t^2} \frac{\partial t}{\partial x} \frac{\partial t}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 t}{\partial x^2} = 2 \nu_x \nu_y \frac{\partial^2 u}{\partial t \partial s} + \nu_x^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial x^2} \frac{\partial u}{\partial x}.
\]

Similarly,

\[
\frac{\partial^2 u}{\partial y^2} = -2 \nu_x \nu_y \frac{\partial^2 u}{\partial t \partial s} + \nu_y^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial y^2} \frac{\partial u}{\partial y},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = (-\nu_x^2 + \nu_y^2) \frac{\partial^2 u}{\partial t \partial s} + \nu_x \nu_y \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial x \partial y} \frac{\partial u}{\partial t}.
\]

Let $\cos \theta = \cos(x, \nu) = \nu_x$, $\sin \theta = \cos(y, \nu) = \nu_y$ and define

\[
Gu = \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta.
\]

Substituting (28) into the above expression for $G$, then

\[
Gu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \nu^2},
\]

where we have used the fact that

\[
0 = \frac{\partial^2 t}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial t}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial t}{\partial y} \frac{\partial y}{\partial t} \right) = (\nu_x^2 \frac{\partial^2 t}{\partial x^2} + 2 \nu_x \nu_y \frac{\partial^2 t}{\partial x \partial y} + \nu_y^2 \frac{\partial^2 t}{\partial y^2}) = Gt
\]

This completes the proof of the equality.

References

[1] Timoshenko, S., D.H. Young and W. Weaver, JR. Vibration Problems in Engineering, John Wilier & Sons, 1974.

[2] Timoshenko, S., Theory of Plates and Shells, (second edition), Maple Press Company, York, PA, 1979.

[3] Rellich, F., Darstellung der Eigenwerte $\Delta u + \lambda u$ durch ein Randintegral, Math. Zeit., Vol. 46, 1940, pp. 635-646.

[4] Payne, L.E., Inequalities for Eigenvalues of Supported and Free Plates, J. Rational Mech. Anal. 4, (1955), pp. 517-529.

[5] Payne, L.E., Inequalities for Eigenvalues of Membrane and Plates, Quart. Appl. Math. 16, 1958, pp. 111-120.

[6] Drabek, P., Global Bifurcation Result for the $p$-biharmonic Operator, Electron. J. Differential Equations, 2008, No.48, pp. 19-23.
[7] Wang, M., Fu, A. and Han, Y., *Lower Bounds for Dirichlet Eigenvalues of Higher-order Elliptic Operators*, MATHEMATICA APPLICATA, 2000, 13(4), pp. 21-24.

[8] Payne, L.E. and Weinberger, H. F., *Lower Bounds for Vibration Frequencies of Elastically Supported Membranes and Plates*, J. Soc. Indus. Appl. Math. Vol. 5, No. 4, December, 1955, pp. 171-182.

[9] Rauch, J., *Partial Differential Equations*, Springer-Verlag, New York, 1991.

[10] Timoshenko, S., *Theory of Plates and Shells*, (First Edition), Maple Press Company, York, PA, 1940.

[11] Mazya, V.G. and Nazarov, S. A., *Paradoxes of Limit Passage in Solutions of Boundary Value Problems in Solving the Approximation of Smooth Domains By Polygonal Domains*, Math. USSR Izvestiya, Vol. 29, No. 3, 1987, pp. 511-533.

[12] Knops, R.J., *Uniqueness and Continuous Data Dependence in the Elastic Cylinder*, International Journal of Non-Linear Mechanics, 36, 2001, pp. 489-499.

[13] Birman, M. sh. *Variational Methods for Solving Boundary Value Problems Analogous to Trefftz’ Method*, Leningrad University, A.A. Zhdanova, 1956, pp. 69-89.

[14] Knops, R.J. and Stuart, C.A. *Quasiconvexity and Uniqueness of Equilibrium Solutions in Nonlinear Elasticity*, Arch. Rational Mech. Anal. 86/3, 1984, pp. 233-249.

[15] Friedlander, L. *Remarks on the Membrane and Buckling Eigenvalues for Planar Domains*, Moscow Mathematical Journal, vol. 4, no. 2, April-June, 2004, pp. 369-375.

[16] Robert, van der Vorst, *Fourth order elliptic equations with critical growth*, C.R. Acad. Sci. Paris, t. 320, Serie I, 1995, pp. 295-299.