We develop a general theory of fermion liquids in spatial dimensions greater than one. The principal method, bosonization, is applied to the cases of short and long range longitudinal interactions, and to transverse gauge interactions. All the correlation functions of the system may be obtained with the use of a generating functional. Short-range and Coulomb interactions do not destroy the Landau Fermi fixed point. Novel fixed points are found, however, in the cases of a super-long range longitudinal interaction in two dimensions and transverse gauge interactions in two and three spatial dimensions. We consider in some detail the 2+1-dimensional problem of a Chern-Simons gauge action combined with a longitudinal two-body interaction \( V(q) \propto |q|^{y-1} \) which controls the density, and hence gauge, fluctuations. For \( y < 0 \) we find that the gauge interaction is irrelevant and the Landau fixed point is stable, while for \( y > 0 \) the interaction is relevant and the fixed point cannot be accessed by bosonization. Of special importance is the case \( y = 0 \) (Coulomb interaction) which describes the Halperin-Lee-Read theory of the half-filled Landau level. We obtain the full quasiparticle propagator which is of a marginal Fermi liquid form. Using Ward Identities, we show that neither the inclusion of nonlinear terms in the fermion dispersion, nor vertex corrections, alters our results: the fixed point is accessible by bosonization. As the two-point fermion Green’s function is not gauge invariant, we also investigate the gauge-invariant density response function. Near momentum \( Q = 2k_F \), in addition to the Kohn anomaly we find singular behavior. In Appendices we present a numerical calculation of the spectral function for a Fermi liquid with Landau parameter \( f_0 \neq 0 \). We also show how Kohn’s theorem is satisfied within the bosonization framework.

1. INTRODUCTION

The peculiar nature of electrons in a half-filled Landau level, and anomalies in the normal state transport properties of the high-temperature cuprate superconductors, have stimulated great interest in systems of strongly interacting fermions moving in spatial dimensions greater than one. In addition to methods such as mean-field theory, RPA, and \( 1/N \) perturbation theory, a number of non-perturbative theoretical tools such as the renormalization group, the Eikonal approximation, and Ward identities, have been used to study fermion liquids. Bosonization is another powerful tool for understanding the effects of both regular and singular interactions on fermion liquids as the realm in which it is applicable, for low-energy excitations near the Fermi surface, is precisely where these interactions are most important. Bosonization does not rely upon a Fermi liquid form for the quasiparticle propagator; consequently non-trivial zero temperature quantum critical fixed points are accessible. In an appropriate limit, bosonization reduces complicated many-body Hamiltonians to quadratic forms which possess infinite \( U(1) \) symmetry and which can be diagonalized exactly.

A general theory of fermion liquids in two and three spatial dimensions requires the consideration of a variety of Fermi surface geometries and interactions. In this paper we study only the simplest Fermi surface geometries: a circle in spatial dimension \( D = 2 \) or a sphere in \( D = 3 \). We assume that the symmetries of these surfaces do not break spontaneously when the interactions between fermions are turned on. The bosonization formalism applies, without significant modification, to more complicated Fermi surfaces provided there are no Van Hove singularities and nesting does not occur. However, the added technical complications of describing these more complicated surfaces would make detailed discussion of their properties cumbersome and, given this caveat, would not lead to any qualitative changes in the physics. We do consider a variety of two-body interactions between the fermions. In accord with expectations, we find, barring superconducting instabilities, that short-range and Coulomb interactions do not destroy the Landau fixed point. Novel fixed points are found, however, in the cases of super-long range interactions and transverse gauge interactions. The latter case is particularly significant as gauge fields play an important role in the physics of the half-filled Landau level and may be important in the cuprate superconductors. The effects of transverse gauge fields, which are not screened, are particularly interesting as perturbative expansions in the coupling constant break down.
Earlier work on multidimensional bosonization showed that it is an accurate calculational framework for understanding Landau Fermi liquids. Here we address the question of whether the approximations inherent in bosonization are adequate for the method to apply to a wider range of fermion liquids with singular or transverse gauge interactions. Three questions in particular must be answered in order to check that bosonization does not break down. First, the assumed current algebra must be consistent: it is derived for the free system, and it must continue to hold when interactions are turned on. Second, we examine whether or not the neglect of quadratic terms in the fermion quasiparticle spectrum due to Fermi surface curvature is an innocuous approximation. Finally, we need to check whether certain vertex corrections are significant by returning to the fermion basis to examine their contribution. In this paper, we examine systems which meet all three of the above conditions. The fixed points of these systems therefore can be accessed and studied via bosonization.

The outline of this paper is as follows. In Sec. [II] we develop a bosonization formalism applicable in any spatial dimension which describes both longitudinal and transverse gauge interactions. A generating functional permits the easy computation of 2N-point boson and fermion correlation functions. Interactions of interest are simply special cases of this general formalism. In Sec. [III], we demonstrate how to incorporate any two-body density-density interaction by introducing longitudinal gauge fields which mediate the interaction. As examples, we reproduce the Luttinger liquid in one spatial dimension and solve the problem of a single, but general, Landau parameter \( f_n \) in two and three spatial dimensions. In Appendix A we use the numerical method of fast-Fourier-transforms (FFT) to extract the spectral function of a two-dimensional Fermi liquid with Landau parameter \( f_0 \). And in Appendix B we present details of our solution for more complicated Landau parameters. Then we turn to the problem of long-range longitudinal interactions. We show that screening of the Coulomb interaction occurs naturally within the bosonization framework. By considering fermions with spin, we show that spin-charge separation does not occur in \( D > 1 \). In the case of super-long range interactions in \( D = 2 \), however, the Landau fixed point is destroyed by the emission of plasmons by the quasiparticles. We show that bosonization accurately describes this breakdown.

The fact that bosonization can access non-Fermi liquid fixed points, both in one and two spatial dimensions, motivates Sec. [IV], in which we study a more physically relevant example of the breakdown of Fermi liquid behavior: the Chern-Simons theory of half-filled Landau level put forward by Halperin, Lee, and Read [II]. To investigate the range of validity of bosonization, we examine the general case of an interaction of the form \( V(q) = |q|^y \) with \(-1 < y \leq 1\) when the inverse gauge propagator is given by \( i\gamma \omega /|q| - \chi q |q|^{y+1} \). Aided by the Ward Identity approach, we find that for \( y < 0 \) the interaction is irrelevant and the Landau fixed point is stable, while for \( y > 0 \) the interaction is relevant and the fixed point cannot be directly studied by bosonization. We discuss in detail the special case \( y = 0 \) corresponding to the Coulomb interaction. In this case a marginal Fermi liquid (MFL) fixed point controls the low energy physics. (An brief account of this work has appeared in print [II]) However, as the two-point Green’s function is not gauge invariant, it is not directly related to experimentally observable quantities. The density response function, on the other hand, is gauge invariant and in Sec. [V] we study its behavior for \( Q \approx 0 \) and \( Q \approx 2k_F \). In the former case our results are identical to those found in RPA while in the latter case we find non-analytic structure stronger than that found by Altshuler, Ioffe and Millis [II]. In Sec. [VI], we apply the formalism developed in Secs. [II] and [V] to the case of the physical Maxwell electromagnetic interaction in three dimensions to obtain the quasiparticle Green’s function in the Coulomb gauge. Although the Landau fixed point is unstable at extremely low energies, in accord with expectations it controls the behavior at physically relevant temperatures. A brief summary of our work and conclusions are found in Sec. [VI]. In Appendix C, we include the Landau parameter \( f_1 \) in the HLR theory to show that Kohn’s theorem [II] is recovered automatically: the bare cyclotron frequency, not a renormalized value, appears in the collective mode spectrum.

II. BOSONIZATION

In this section we formulate the problem of an interacting fermion liquid, making use of a bosonization method applicable in any spatial dimension to study both longitudinal and transverse gauge interactions. A generating functional permits the ready computation of 2N-point boson and fermion correlation functions.

We start from the bare Hamiltonian for fermions interacting with non-compact longitudinal and transverse gauge fields \( A^\mu \) in \( D \)-spatial dimensions. For simplicity, we consider spinless fermions and spherical Fermi surfaces only. In the Coulomb gauge, \( \nabla \cdot A = 0 \), and when \( \hbar \) is set equal to one,

\[
H = \int d^Dx \ c^\dagger(x) \left[ \frac{(-i\nabla - A)^2}{2m} - A_0 \right] c(x) + \frac{1}{2} \int d^Dx \ d^Dy \ V(x-y) \ c^\dagger(x) c^\dagger(y) c(y) c(x) \tag{1}
\]

where \( c_k = k^2/2m \), and \( V(x-y) \) represents Coulomb or other two-body interactions. Next we make use of the renormalization group to integrate out the high-energy Fermi degrees of freedom. The resulting low-energy effective
We use the notation $q$ of the effective mass $m^*$ theory is expressed in terms of quasiparticles $\psi_k$ which obey canonical anticommutation relations and which are related to the bare fermion operators by

$$\psi_k = Z_k^{-1/2} c_k$$

for momenta $k$ which are restricted to a narrow shell of thickness $\lambda$ around the Fermi surface: $k_F - \lambda/2 < |k| < k_F + \lambda/2$. The wavefunction renormalization factor $Z_k$ rescales the discontinuity in the quasiparticle occupancy at the Fermi surface back to one. To be explicit, consider the partition function

$$Z = \int D\psi \ D\psi^* e^{i S[A, \psi, \psi^*]}$$

where we have separated the fermion fields into low and high energy components, $\psi(x)$ and $\psi^*(x)$ respectively. Integrating out the high energy $\psi^*$ fields, which contain all of the degrees of freedom except for the narrow shell around the Fermi surface, we obtain the effective action:

$$S[A, \psi] = \int d^D x \ dt \ [\psi^*(x) (i\partial_t + A_0)\psi(x) + \frac{1}{2m^*}\psi^*(x)(\nabla - i A)^2\psi(x)]$$

$$+ \sum_q \int \frac{d\omega}{2\pi} \mathcal{M}_{\mu\nu}(q) A^\mu(q) A^\nu(-q).$$

We use the notation $q \equiv (\omega, q)$. Renormalization of the Fermi velocity $v_F \to v_F^*$ is expressed through the appearance of the effective mass $m^*$. The last term in Eq. (4), which is generated by integrating out the $\psi^*$ fields deep inside the Fermi sea contributes to the total diamagnetic term, $-(\rho_f/2m^*)A(q)\cdot A(-q)$; $\rho_f$ is the fermion number density. Irrelevant operators consisting of higher powers and/or derivatives of the gauge and Fermi degrees of freedom are also generated. These operators may be safely neglected in the low-energy limit assuming that the anomalous dimension of the gauge fields is not too different from their engineering dimension. To check that this is the case, the effective action of the gauge fields must be obtained by integrating out the fermion degrees of freedom and examined for the case of interest. The effects of two-body fermion interactions will be considered later in the discussion.

To bosonize the effective action Eq. (4) we introduce the coarse grained charge current

$$J(S; q) \equiv \sum_k \theta(S; k + q) \theta(S; k) \{ \psi^*_k \psi_k - \delta^3_{q,0} n_k \}.$$

Here $S$ labels a patch $[S \equiv (\theta, \phi)$ in three dimensions] on the Fermi surface with momentum $k_S$, and $\theta(S; k) = 1$ if $k$ lies inside a squat box centered on $S$ with height $\lambda$ in the radial (energy) direction and area $\Lambda^{D-1}$ along the Fermi surface, and equals zero otherwise. These two scales must be small in the sense $k_F \gg \Lambda \gg \lambda \gg |q|$: we satisfy these limits by setting $\lambda \equiv k_F/N$ and $\Lambda \equiv k_F/N^\alpha$ where $0 < \alpha < 1$ and $N \to \infty$. The currents so defined satisfy the $U(1)$ current algebra

$$[J(S; q), J(T; p)] = \delta^{D-1}_{S, T} \delta^D_{q+p, 0} \Omega \ q \cdot \hat{n}_S.$$

Here $\Omega \equiv \Lambda^{D-1}(L/2\pi)^D$ is the number of states in the squat box divided by $\lambda$ and $\hat{n}_S$ is a unit normal to the Fermi surface at $S$. The charge currents also have a bosonic representation; if we set

$$J(S; x) = \sqrt{4\pi} n_S \cdot \nabla \phi(S; x)$$

where the bosonic charge fields satisfy the canonical commutation relations,

$$[\phi(S; x), \phi(T; y)] = \frac{i}{4} \Omega^2 \delta^{D-1}_{S, T} \epsilon(\hat{n}_S \cdot [x - y]) : |x_\perp - y_\perp| \Lambda \ll 1$$

$$= 0 : |x_\perp - y_\perp| \Lambda \gg 1$$

where $\epsilon(x) = 1$ for $x > 0$ and equals $-1$ otherwise. Eq. (9) follows immediately from the commutation relations. The key formula of the bosonization procedure expresses the fermion quasiparticle field $\psi$ in terms of the boson fields as

$$\psi(S; x, t) = \frac{1}{\sqrt{V}} \sqrt{\frac{\Omega}{a}} e^{i k_S \cdot x} \exp\{i \frac{\sqrt{4\pi}}{\Omega} \phi(S; x, t)\} \hat{O}(S).$$
Here $V$ is the volume of the system and $\alpha \equiv 1/\lambda$ is an ultraviolet cutoff. $\hat{O}(S)$ is an ordering operator introduced to maintain Fermi statistics in the angular direction along the Fermi surface. Anticommuting statistics are obeyed automatically in the direction normal to the Fermi surface, just as in one-dimensional bosonization. It is now an interesting exercise to use this representation of the fermion field operators to verify that

$$J(S; x) = V \lim_{\epsilon \to 0} \left\{ \psi^\dagger (S; x + \epsilon \hat{n}_S) \, \psi(S; x) - \langle \psi^\dagger (S; x + \epsilon \hat{n}_S) \, \psi(S; x) \rangle \right\}$$

$$\equiv \sqrt{4\pi} \, \hat{n}_S \cdot \nabla \phi(S; x) \, .$$

(10)

The currents are invariant under $U(1)$ phase rotations through an angle $\beta(S)$ in each patch: $\psi(S; x) \to e^{i\beta(S)} \psi(S; x)$. This infinite $U(1)$ symmetry reflects the fact that the current operator in a given patch does not scatter quasiparticles outside of that patch. The effective action Eq. (11) now can be expressed in terms of the charge currents $J$ and consequently reflects the underlying infinite $U(1)$ symmetry of the fixed point:

$$S[A, a] = \sum_S \sum_{q, q} \int \frac{d\omega}{2\pi} (\omega - v_F^* \cdot q) \, a^*(S; q) \, a(S; q)$$

$$+ \frac{1}{V} \sum_q \int \frac{d\omega}{2\pi} \left\{ \sum_S J(S; q) [A_0(-q) + v_S^* \cdot A(-q)] - \frac{\rho}{2m^*} A(q) \cdot A(-q) \right\} \, .$$

(11)

Here $v_S^* \equiv k_S/m^*$ is the renormalized Fermi velocity vector at patch $S$ and the charge currents are related to the canonical boson operators $a$ and $a^\dagger$ by:

$$J(S; q) = \sqrt{\Omega} \, |\hat{n}_S \cdot q| \, [a(S; q) \, \theta(\hat{n}_S \cdot q) + a^\dagger(S; -q) \, \theta(-\hat{n}_S \cdot q)] \, ,$$

(12)

where

$$[a(S; q), \, a^\dagger(T; p)] = \delta^{D-1}_{ST} \, \delta_{q-p} \, .$$

(13)

and $\theta(x) = 1$ if $x > 0$ and is zero otherwise. Exchange scattering, which involves high momentum gauge fields, has not been included in the action, as it is not singular in the low-momentum limit. The action Eq. (11) must be supplemented by the bare gauge action which we take to have the quadratic form:

$$S^0[A] = \frac{1}{2} \int \frac{d^Dq}{(2\pi)^D} \int \frac{d\omega}{2\pi} \, K^0_{\mu\nu}(q) \, A^\mu(q) \, A^\nu(-q) \, .$$

(14)

We fix the gauge with the supplemental condition $\nabla \cdot A = 0$ (ie. the Coulomb gauge). Particular examples considered in this paper are the physical electromagnetic gauge action in $D = 3$, and in $D = 2$ the Chern-Simons action, which breaks time-reversal and parity, but not charge-conjugation, symmetries.

We proceed with the evaluation of the boson correlation function which can be carried out exactly. Rather than integrate out the gauge fields to obtain an effective current-current interaction, we introduce a generating functional for the boson correlation function and first integrate out the boson fields $a$ and $a^\dagger$. In this way we avoid the intermediate step of summing an infinite perturbation series in the effective interaction to obtain the boson propagator. We construct the generating functional by coupling fields $\xi$ and $\xi^*$ to the boson fields $a$ and $a^\dagger$.

$$Z[\xi, \xi^*] = \int \mathcal{D}' A^\mu \, \mathcal{D} a \, \mathcal{D} a^\dagger \exp\{i[S[A, a] + S^0[A]]\} \, \times \exp\left\{ i \sum_S \sum_{q, q} \int \frac{d\omega}{2\pi} \, [\xi(S; q) \, a^*(S; q) + \xi^*(S; q) \, a(S; q)] \right\} \, .$$

(15)

Here the prime over the gauge measure indicates that the integration respects the Coulomb gauge fixing condition. Then by completing the square we integrate out the boson fields in each patch $S$. It is important to note that the entire Fermi surface participates: patches at every point on the Fermi surface contribute to the effective action.

$$Z[\xi, \xi^*] = \mathcal{N} \int \mathcal{D}' A^\mu \, \exp\{i \, S_G[A]\} \exp\left\{ - \frac{i}{2} \sum_S \int \frac{d^dq}{(2\pi)^D} \int \frac{d\omega}{2\pi} \, \theta(q \cdot \hat{n}_S) \right\} 

\times \left[ \sqrt{\Omega q \cdot \hat{n}_S} \, [\xi(S; q) \, (A_0(-q) + v_S^* \cdot A(-q)) + \xi^*(S; q) \, (A_0(q) + v_S^* \cdot A(q))] \right. 

+ \left. V \xi(S; q) \, \xi^*(S; q) \right\} \, ,$$

(16)
where $S_G[A]$ is an effective gauge action, which since the bosons were coupled linearly to the gauge fields is a quadratic form,

$$
S_G[A] = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d\omega}{2\pi} K_{\mu\nu}(q) A^\mu(q) A^\nu(-q) .
$$

The inverse of the gauge propagator satisfies the equation

$$
K_{\mu\nu}(q) A^\mu(q) A^\nu(-q) = K^0_{\mu\nu}(q) A^\mu(q) A^\nu(-q) + \chi^0(q) A_0(q) A_0(-q) + \chi^T(q) A(q) \cdot A(-q) ,
$$

where $\chi^0$ and $\chi^T$ are the longitudinal and transverse susceptibilities which for the cases of perfectly circular or spherical Fermi surfaces, are given by

$$
\chi^0(q) = N^*(0) \left[ 1 - \frac{\theta(x^2 - 1)}{\sqrt{x^2 - 1}} + i \frac{\theta(1 - x^2)}{\sqrt{1 - x^2}} \right] ; \quad D = 2
$$

$$
= N^*(0) \left[ 1 - \frac{x}{2} \ln \left( \frac{x + 1}{x - 1} \right) \right] ; \quad D = 3 ,
$$

and

$$
\chi^T(q) = v_F^2 N^*(0) \left[ -x^2 + \theta(x^2 - 1) \right] \left( x - \frac{x}{\sqrt{x^2 - 1}} + i \frac{1 - x^2}{\sqrt{1 - x^2}} \right) ; \quad D = 2
$$

$$
= \frac{1}{2} v_F^2 N^*(0) \left[ -x^2 - \frac{x}{2} \right] \ln \left( \frac{x + 1}{x - 1} \right) ; \quad D = 3 ,
$$

where $x = \omega/(v_F |q|)$, and $N^*(0)$, which is the quasiparticle density of states at the Fermi surface, equals $m^*/2\pi$ for spinless fermions in $D = 2$, and $m^* k_F^2/(2\pi^2)$ in $D = 3$. The integral over the gauge fields $A^\mu$ can be carried out now giving the generating functional as an explicit function of $\xi$ and $\xi^*$. The boson correlation functions are obtained by differentiating the logarithm of the generating functional with respect to $\xi$ and $\xi^*$. In particular the boson propagator defined by

$$
\langle a(S; q) a^\dagger(S; q) \rangle = -\frac{\delta^2 \ln Z[\xi, \xi^*]}{\delta \xi(S; q) \delta \xi^*(S; q)} |_{\xi = \xi^* = 0}
$$

is given in terms of the gauge propagator $D_{\mu\nu}(q) = [K(q)^{-1}]_{\mu\nu}$ as

$$
\langle a(S; q) a^\dagger(S; q) \rangle = \frac{i}{\omega - v_F^* q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)} + i \frac{\Lambda^{D-1}}{(2\pi)^D} q \cdot \hat{n}_S \frac{D_{\mu\nu}(q, \omega) \epsilon^\mu(S; q) \epsilon^\nu(S; -q)}{\omega - v_F^* q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)^2} ,
$$

where $\epsilon^\mu(S; q)$ is the D+1-vector $(1, v_F^* S)$. Note that the velocity of the interacting bosons differs from that of free bosons by at most order $v_F^* \Lambda/k_F$. Therefore velocity renormalization of the bosons due to interactions among the final shell of excitations with energy less than $v_F^* \Lambda$ is insignificant in the $\Lambda \rightarrow 0$ limit except in the special case of one spatial dimension. We will return to this observation below when we address the question of whether or not spin-charge separation occurs in two or higher spatial dimensions.

Now the bosonization formula Eq. 11 can be used to express the space-time fermion Green’s function in terms of the Fourier transform of the boson correlation function. In general terms the Green’s function for fermions in a single patch on the Fermi surface is given by

$$
G_F(S; x, t) = \frac{\Omega}{V} e^{i k_S \cdot x} \exp \left[ \int \frac{d^D q}{(2\pi)^D} \int \frac{d\omega}{2\pi} \left( e^{i(q \cdot x - \omega t)} - 1 \right) \frac{(2\pi)^D}{\Lambda^{D-1}} \hat{n}_S \cdot q \langle a(S; q) a^\dagger(S; q) \rangle \right] ,
$$

which becomes, upon using Eq. 13 and dropping the prefactor of $\Omega/V$ for brevity,

$$
G_F(S; x, t) = \frac{e^{i k_S \cdot x}}{x \cdot \hat{n}_S - v_F^* t} \exp \left[ i \int \frac{d^D q}{(2\pi)^D} \int \frac{d\omega}{2\pi} \left( e^{i(q \cdot x - \omega t)} - 1 \right) \frac{D_{\mu\nu}(q, \omega) \epsilon^\mu(S; q) \epsilon^\nu(S; -q)}{\omega - v_F^* q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)^2} \right] .
$$

for $x_\perp \equiv x \times \hat{n}_S = 0$. The condition $x_\perp = 0$ can be replaced by $|x_\perp| \Lambda < 1$ provided we control the logarithmic infrared divergence of the free boson correlation function by placing the system inside a large box. The weak divergence, which
is a consequence of treating the Fermi surface as locally flat inside each patch, presents no real difficulties in practice, and will be ignored in the following analysis. For $|x_L| \Lambda \gg 1$ the fermion Green’s function vanishes since bosons separated by large $x_L$ are uncorrelated. In the second line of Eq. (24) for convenience we have introduced the notation $\delta G_B$ for the modified Fourier transform of the additive correction to the free boson propagator due to interactions.

$$\delta G_B(S; q, \omega) = \frac{D_{\mu\nu}(q, \omega)\, e^{\mu}(S; q)\, e^{\nu}(S; -q)}{[\omega - v_F^2 q \cdot \hat{n}_S + i\eta \operatorname{sgn}(\omega)]^2},$$

where we can identify $\delta G_B(S; q, \omega)$ as:

$$\delta G_B(S; q, \omega) = D_{\mu\nu}(q, \omega)\, e^{\mu}(S; q)\, e^{\nu}(S; -q) \left[\omega - v_F^2 q \cdot \hat{n}_S + i\eta \operatorname{sgn}(\omega)\right]^2.$$

In Eq. (23), the momentum integral over $q$ ranges over $(-\Lambda/2, \Lambda/2)$ in each direction. In contrast, in the free part of the boson and fermion propagators, momenta in the perpendicular directions range over the much larger interval $(-\Lambda/2, \Lambda/2)$. The reason for the difference lies in the fact that interactions which couple different patches on the Fermi surface must be cutoff at momentum $\Lambda \ll \Lambda$ because in general the Fermi surface normal vector points in a different direction for each patch. Therefore, as the patches are squat pillboxes with dimensions $\Lambda D^{-1} \times \lambda$, only modes with $|q| < \lambda$ are permitted by the geometry of the construction to couple patches together.

To complete the program of computing the fermion two point function, the Green’s function in Eq. (24) is summed over all patches on the Fermi surface:

$$G_F(x, t) = \sum_S^\prime G_F(S; x, t).$$

The prime indicates that the sum is only over patches for which $|x \times \hat{n}_S| \Lambda < 1$. The expression for the fermion Green’s function given here coincides with that found by Castellani et al. who make use of Ward Identities derived in the fermion basis to find Eq. (24). Why these two different approaches should give the same result and the consequent implications are discussed in detail in Sec. IV D.

### III. LONGITUDINAL INTERACTIONS

In this section we show how the generating functional may be used to determine the correlation functions of a fermion liquid interacting via two-body forces in two important special cases, for short-range interactions of Landau Fermi liquid type, and for Coulomb or longer range longitudinal interactions. The procedure parallels that given in the previous section as we introduce fictitious gauge fields, which are linearly coupled to the bosons, to mediate the interaction. First, to illustrate the non-perturbative nature of the method, we determine the propagator of a fermion liquid in one spatial dimension, which has a Luttinger liquid form. In higher dimensions, we extend the discussion to general short-range interactions which as expected do not destroy the Fermi liquid state. The quasiparticle propagator for a system of fermions interacting via a short-range interaction described by a single Landau parameter $f_0$ had been derived in a previous work by a different technique. Here we give a detailed derivation of the propagator for the nontrivial case of a fermion liquid with an $f_1$ interaction. This result will be used later in the derivation of Kohn’s theorem which is applied in the discussion of the half-filled Landau level studied in the next section. The derivation for a general interaction specified by more than one Landau parameter and a proof of Kohn’s theorem are given in Appendices B and C respectively. Finally, we show that, due to screening, the ground state of a fermion system interacting via the Coulomb interaction is a Landau Fermi liquid in two and three dimensions. For a super-long-range interaction in $D = 2$, however, the emission of plasmons by quasiparticles destroys the Fermi liquid fixed point.

In one dimension, even a short-range interaction drives the fermion system to a different fixed point, which is a Landau Fermi liquid. The formalism developed in Sec. IV D can be used to diagonalize the interacting fermion problem exactly and to access the Luttinger liquid. As an example we consider an interaction which makes a contribution to the action of the form

$$S_{\text{int}} = -\frac{f_0}{2} \int \frac{d\omega}{2\pi} \frac{dq}{2\pi} \sum_{S,T=L,R} J(S; q) J(T; -q)$$

where $L$ and $R$ specify the left and right Fermi points. We introduce a fictitious field, $A_0(q)$, which generates this interaction when integrated out of the partition function.
\[ S'_{\text{int}} = \int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} \left\{ \frac{1}{2f_0} A_0(q) A_0(-q) + [J(L;q) + J(R;q)] A_0(-q) \right\}. \]  

Now the total action has the form discussed in Sec. II and we follow the procedure given there to determine the boson correlation function. We define \( \eta \) where \( \tanh^2 \frac{v}{2} = \frac{1}{\sqrt{1 - F_0^2/4}} \omega - q v' f_0 / \sqrt{1 - F_0^2/4} + \frac{1}{\sqrt{1 - F_0^2/4}} \omega + q v' f_0 / \sqrt{1 - F_0^2/4} \) 

where \( v' = v_F + f_0/(2\pi) \) is a renormalized Fermi velocity due to intrapatch interactions and \( F_0 = f_0/(\pi v_F) \). If we now define \( \eta \) such that

\[ \cosh^2 \eta = \frac{1}{2} \left[ 1 + \frac{1}{\sqrt{1 - F_0^2/4}} \omega + q v_F / \sqrt{1 - F_0^2/4} \right] \]

the boson propagator can be written as

\[ \langle a(R; q) a^\dagger(R; q) \rangle = \frac{i}{2} \left\{ \cosh^2 \eta \frac{1}{\omega - q v_F / \sqrt{1 - F_0^2/4} - \sinh^2 \eta \frac{1}{\omega + q v_F / \sqrt{1 - F_0^2/4}} \right\} \]

where \( \tanh 2\eta = F_0 \). The fermion propagator found from Eq. (23) is non-analytic in the coupling constant \( f_0 \) and cannot be obtained by perturbing in the coupling constant,

\[ G_F(R; x, t) \propto (x^2 + v_F^2 t^2)^{-\alpha} \]

where the anomalous exponent \( \alpha = \sinh^2 \eta \) and the Fermi velocity \( v_F^2 = v_F^2 / \sqrt{1 - F_0^2/4} \). The result is exact and identical to that found by diagonalizing the bosonic Hamiltonian with a Bogoliubov transformation.

As a second example, we consider a fermion liquid in two dimensions with short-range interactions. In earlier work we discussed, by a different but equivalent approach, the effect of an interaction which could be parameterized by a single Landau parameter, \( f_0 \). It was shown that a perturbative expansion in \( f_0 \) gave for the imaginary part of the self-energy a term proportional to \( f_0^2 \omega^2 \ln |\omega| \) in two dimensions and \( f_0^2 \omega^2 \) in three dimensions as in Fermi liquid theory. A non-perturbative treatment of this problem, involving a numerical computation of the spectral function, is presented in Appendix A. This calculation confirms that the low-order perturbative result is qualitatively correct, even at large \( f_0 \). Also in this Appendix we show how the Green’s functions of the \( f_0 \) problem are obtained within the present generating function approach. Here, as a further illustration of this approach, we consider in detail a Fermi liquid with a \( f_1 \) interaction. The bosonized \( f_1 \) interaction contributes to the action the following term:

\[ S'_{\text{FL}} = -\frac{f_1}{2V k_F^2} \int \frac{d\omega}{2\pi} \sum_{S, \ell, q} J(S; q) \mathbf{k}_S \cdot \mathbf{k}_T J(T; -q) \]  

which can be separated into two parts, one longitudinal and the other transverse:

\[ S'_{\text{FL}} = \frac{f_1}{2V k_F^2} \int \frac{d\omega}{2\pi} \sum_{S, \ell, q} J(S; q) \left\{ \frac{\mathbf{k}_S \cdot \mathbf{q}}{|\mathbf{q}|} \frac{\mathbf{k}_T \cdot (-\mathbf{q})}{|\mathbf{q}|} + \frac{\mathbf{k}_S \times \mathbf{q}}{|\mathbf{q}|} \frac{\mathbf{k}_T \times (-\mathbf{q})}{|\mathbf{q}|} \right\} J(T; -q). \]

Introducing two fictitious fields \( A_t \) and \( A_t \) which generate the interaction when integrated out of the partition function, the total action is given by:

\[ S[A_t, a] = \sum_S \sum_{\mathbf{q}, \mathbf{q}_0} \int \frac{d\omega}{2\pi} (\omega - v_F^t \cdot \mathbf{q} \cdot \mathbf{n}_S) a^*(S; q) a(S; q) + S'_{\text{FL}} \]

where

\[ S'_{\text{FL}} = -\frac{1}{2 f_1} \int \frac{d\omega}{2\pi} \int \frac{d^2 q}{(2\pi)^2} [A_i(q) A_i(-q) + A_i(q) A_i(-q)] + \frac{1}{k_F V} \int \frac{d\omega}{2\pi} \sum_S \sum_{\mathbf{q}} \left( \frac{\mathbf{k}_S \cdot \mathbf{q}}{|\mathbf{q}|} J(S; q) A_i(-q) + \frac{\mathbf{k}_S \times \mathbf{q}}{|\mathbf{q}|} J(S; q) A_i(-q) \right). \]
As usual, we contract the generating functional and integrate out the boson fields, $a$ and $a^*$, to obtain the effective action for the $A_{t,t}$ fields:

$$ S_{\text{eff}}[A; t] = \frac{1}{2} \int \frac{d\omega}{2\pi} \int \frac{d^2 q}{(2\pi)^2} \left\{ \left[ -\frac{1}{f_1} + \chi_l(q) \right] A_t(q) A_t(-q) + \left[ -\frac{1}{f_1} + \chi_t(q) \right] A_t(q) A_t(-q) \right\}, $$

where

$$ \chi_l(q) = N^*(0) \left[ -x^2 - \frac{1}{2} + \theta(x^2 - 1)|x| \frac{x^2}{\sqrt{x^2 - 1}} - i\theta(1 - x^2)|x| \frac{x^2}{\sqrt{1 - x^2}} \right], $$

$$ \chi_t(q) = N^*(0) \left[ x^2 - \frac{1}{2} - \theta(x^2 - 1)|x| \sqrt{x^2 - 1} - i\theta(1 - x^2)|x| \sqrt{1 - x^2} \right]. $$

The boson propagator is given by

$$ \langle a(S; q) a^\dagger(S; q) \rangle = \frac{i}{\omega - v_F q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)} - i \frac{\Lambda}{(2\pi)^2} \frac{q \cdot \hat{n}_S (q \cdot \hat{n}_S)^2 D_l(q) + (q \times \hat{n}_S) D_t(q)}{|q|^2 - \omega - v_F q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)}.$$

where the propagators of the mediating fields are given by

$$ D_l(q) = \frac{1}{-1/f_1 + \chi_l(q)}, $$

$$ D_t(q) = \frac{1}{-1/f_1 + \chi_t(q)}.$$

These results will be used in the discussion of the physics of the half-filled Landau level given in the next section and in Appendix C.

To complete this section we consider the effect of long-range longitudinal interactions such as the Coulomb interaction or the super long-range interaction of Bares and Wen. Following the procedure given in Sec. I we obtain the fermion Green’s function by setting $K_{00} = 1/V(q)$ in Eq. (18) and eliminating the transverse gauge fields. We briefly review what we find from this analysis and refer the reader to a previous paper [4] for details. The correction to the boson propagator is given by

$$ \delta G_B(S; q, \omega) = i \frac{V(q)}{1 + V(q)/\chi^2(q) \left( \omega - v_F q \cdot \hat{n}_S + i\eta \text{sgn}(\omega) \right)^2} $$

and the effect of screening is manifest. For example, in the case of the Coulomb interaction, $V(q) \propto e^2|q|^{1-D}$ and consequently, in dimensions greater than one, $\delta G_B$ is independent of the coupling constant $e^2$ in the limit of small $q$ and small $\omega$. We can calculate the correction to the fermion Green’s function by expanding in powers of $\delta G_B$. To first order, we obtain the well-known result that the imaginary part of the fermion self-energy is proportional to $(\omega^2/\epsilon_F) \ln|\omega|$ in $D = 2$, and $(\lambda/\epsilon_F k_F) \omega^2$ in $D = 3$, in agreement with Fermi liquid theory. Below we will show that in the case of Coulomb interactions the weight of the quasiparticle pole, $Z_F$, remains non-zero at the Fermi surface and the Landau Fermi liquid is stable as expected. It might be thought that this is the case for all types of longitudinal interactions, as it appears to be a consequence of screening. However, super long-range interactions can destroy the Fermi liquid state. Bares and Wen considered a system of fermions in two dimensions interacting via a logarithmic potential, $V(q) = g/q^2$, and showed that within the random phase approximation (RPA) $Z_F = 0$. Note that $g$ is an energy scale and the plasmon gap is non-zero due to the super-long-range nature of the interaction and is given by $\omega_p = v_F \sqrt{mg/4\pi}$. (By Kohn’s theorem, the gap depends on the bare fermion mass and Fermi velocity. See Appendix C for details on how Kohn’s theorem is automatically recovered within multidimensional bosonization.)

One might be tempted to compute $Z_F$ by employing the Kramers-Kronig relation to derive the real part of the self-energy from the imaginary part estimated here. This procedure, however, is unreliable, as the Kramers-Kronig relations involve an integral over all frequencies, whereas the above calculation ignores high-energy processes such as the emission of plasmons by quasiparticles. Instead we compute the real-space, equal-time, two-point fermion Green’s function directly. Because we wish to examine the effects of plasmons, we take the energy cutoff to be much larger than the plasmon energy, $v_F/\lambda \gg \omega_p$. Obviously, then, bosonization is less accurate for systems with a large plasmon gap. Nevertheless, we expect it to be qualitatively correct, as discussed below. Setting $t = 0$, $x_\perp \equiv x \times \hat{n}_S = 0$ and introducing $x_\parallel \equiv x \cdot \hat{n}_S$, $q_\parallel \equiv q \cdot \hat{n}_S$, and $q_\perp \equiv q \times \hat{n}_S$ we find:
\[ G_F(S; x_\parallel) \approx \frac{e^{ik_Fx_\parallel}}{x_\parallel} \exp \left\{ i \int_{-\lambda/2}^{\lambda/2} \frac{dq_\perp}{2\pi} \int_{-\lambda/2}^{\lambda/2} \frac{dq_\parallel}{2\pi} \exp(iq_\parallel x_\parallel) - 1 \right\} \]

\[ \times \int_{|q|}^{\lambda} \frac{d\omega}{\pi} \frac{1}{1/g - v_F^2 m/(4\pi\omega^2)} \frac{1}{|\omega - v_F^2 q_\parallel + i\eta \text{sgn}(|\omega)|^2} \]. \hspace{1cm} (44)

Here because plasmons are important only in the large-\(x\) limit, we have expanded the Lindhard function \(\chi^0\) in powers of \(1/x, \ x \equiv \omega/(v_F^2|q|)\):

\[ \chi^0(x) = -\frac{N^+(0)}{2x^2} + O(1/x^4) + i\eta \] \hspace{1cm} (45)

Kohn’s theorem is respected by replacing the effective mass and velocity in the plasmon pole by bare quantities. Now since \(\omega > v_F^2|q|\), in the frequency integral we may make the approximation of replacing \([|\omega - v_F^2 q_\parallel + i\eta \text{sgn}(|\omega)|^2\] by \(\omega + i\eta\) and carry out the integrations. The integral over \(\omega\) yields a constant due to the plasmon pole at non-zero frequency. Logarithmic dependence on \(x_\parallel\) then is due to the factor of \(q^2 = q_\perp^2 + q_\parallel^2\) in the denominator of the momentum integral. Upon exponentiating the boson Green’s function we obtain the fermion quasiparticle propagator in configuration space:

\[ G_F(S; x_\parallel) \approx \frac{e^{ik_Fx_\parallel}}{x_\parallel} \left( \frac{|\lambda|x_\parallel|^{-\zeta}}{x_\parallel} \right), \quad \lambda|x_\parallel| \gg 1 \]

\[ \approx \frac{e^{ik_Fx_\parallel}}{x_\parallel}, \quad \lambda|x_\parallel| \ll 1 \] \hspace{1cm} (46)

where

\[ \zeta = \frac{1}{2} \sqrt{\frac{g}{\pi v_F^2 k_F}}. \] \hspace{1cm} (47)

The non-analytic dependence on \(x_\parallel\) demonstrates explicitly the destruction of the quasiparticle pole by the emission of plasmons. Power law behavior with exponent \(\zeta\), rather than the logarithmic behavior found in RPA, is a consequence of bosonization which treats the interaction non-perturbatively. In fact, Bares and Wen\(^4\) used both a hydrodynamic calculation and a variational wavefunction to argue that the logarithm is promoted to a power law, with exponent \(\zeta/\sqrt{2}\), the difference by a factor of \(1/\sqrt{2}\) is due to spin degeneracy which is not taken into account here. This is further evidence that bosonization can accurately describe non-Fermi liquid systems. In the opposite limit of \(\omega_0|t| \gg 1\), however, an evaluation of the fermion propagator for a system of area \(L^2\) yields the usual \((x_\parallel - v_F^2 t)^{-1}\) Fermi liquid form, multiplied by a prefactor of \((\lambda L)^{-\zeta}\). As screening is effective at low frequencies, thermodynamic properties such as the specific heat which are defined in the equilibrium \(g\)-limit also retain the usual Fermi liquid form. Note that in either limit the propagator is odd under the combined CPT operation of \((x, t) \rightarrow -(x, t)\) and complex conjugation as demanded by Fermi statistics. The anomalous exponent \(\zeta\) in the equal-time two-point function shows that the discontinuity in the quasiparticle occupancy at the Fermi surface has been replaced by a continuous, though non-analytic, crossover. The quasiparticle occupancy decreases from 1 to 0 over a momentum scale of order \(\lambda\). Since the discontinuity has been smeared out, we might expect to encounter problems of consistency as the derivation of the current algebra Eq. (1) relies upon the existence of an abrupt change in the quasiparticle occupancy over a momentum scale much smaller than \(\lambda\). Nevertheless, as noted above, the form of the Green’s function agrees with that found by an independent argument using a variational wavefunction\(^5\). The approximations implicit in bosonization (restriction to low energy processes, the existence of the discontinuity, and assumed stability of the fixed point) are robust in this case.

It is instructive to repeat the calculation of the equal time Green’s function for the case of a Coulomb interaction in both two and three dimensions. In two dimensions plasmons are found at \(\omega \propto \sqrt{|q|}\). In this case the integral over \(\omega\) is no longer constant, as it was for the Bares and Wen interaction, but rather is proportional to \(\sqrt{|q|}\) and vanishes in the limit of small momentum. The momentum integral therefore no longer diverges logarithmically at large-\(x_\parallel\); rather the integral is proportional to \(\sqrt{\lambda/k_F} \rightarrow 0\). While there is a non-zero gap for plasmon excitations in three dimensions, the integral over \(q\) again is suppressed, in this case because of the reduced phase space at small momentum and consequently \(\delta G_\parallel \propto \lambda/k_F \rightarrow 0\). In either case the quasiparticle propagator has the standard Fermi liquid form. The weight of the pole undergoes no additional renormalization beyond the factor of \(Z_k\) due to integrating out the high-energy degrees of freedom, Eq. (2). This is as expected, since the renormalization of the quasiparticle pole in Landau Fermi liquids is due to high-energy processes, not the low-energy processes found near the Fermi surface.
In this section we examined the effects of both short- and long-range longitudinal interactions on fermion liquids. We find that the Fermi liquid state is the only solution to the problem of a degenerate gas of fermions interacting via short-range or Coulomb two-body interaction in two and three spatial dimensions. Bosonization allows us to go beyond an assumed Fermi liquid form for the quasiparticle propagator. Indeed, had we considered fermions with spin, we would have found that the bosonized Hamiltonian separates into charge and spin parts, leading to the possibility that, as in one dimension, the quasiparticle propagator would also exhibit spin-charge separation. Spin-charge separation in dimensions larger than one would, however, destroy the Fermi liquid, as the key element, the existence of a pole in the single-particle Green’s function with non-zero spectral weight, would be replaced by a branch cut and the pole would be destroyed. This does not occur because the location of the pole of the boson propagator is unchanged from its free value in the $\Lambda \to 0$ limit. Consequently the spin and charge velocities are equal and both degrees of freedom propagate together in the usual quasiparticle form. On the other hand, the Fermi liquid form is destroyed both in one dimension, where a Luttinger liquid occurs instead, and in the case of the super long-range interaction in two dimensions studied by Bares and Wen. The non-Fermi liquid fixed points of these two examples are accessible via bosonization. We therefore have reason to expect that bosonization can describe other non-Fermi liquid states of matter. The system we study in the next section is another such example.

IV. TRANSVERSE GAUGE FIELDS: THE HALF-FILLED LANDAU LEVEL

The two dimensional electron liquid in a magnetic field with even denominator filling fraction has been the subject of extensive studies. Experiments at even denominator fillings point to a novel state which is compressible unlike the incompressible states found at odd denominator fractional fillings. Attempts have been made to understand the new quantum state by constructing composite particles which are made of magnetic flux tubes attached to each electron. The composite particles see zero average magnetic field and in many respects behave like real particles. In this section we study the specific case of filling fraction $\nu = 1/2$.

A. Half-Filled Landau Level

The Hamiltonian for a two-dimensional electron gas of density $\rho_f$ in a perpendicular magnetic field $B$ is given by:

$$ H = \int d^2 x ~ c_i^\dagger(x) \frac{(-i \nabla + (\frac{\phi}{e}) A)}{2m} c_i(x) + H_{\text{Coulomb}}. $$

(48)

Here $B = \nabla \times A$; at a value $B = 4\pi \rho_f c/e$ half of the states in the first Landau level are filled. The HLR theory makes use of a local gauge transformation to describe this system as a collection of quasiparticles which obey Fermi statistics:

$$ c_i(x) = c_i^\dagger(x) \exp \left[ -i \phi \int d^2 y \arg(x - y) ~ c_i^\dagger(y) ~ c_i(y) \right]. $$

(49)

When $\phi$ is an even integer the quasiparticles are fermions. Each quasiparticle is a composite object consisting of the physical electron together with a flux tube of integer $\phi$ quanta. The transformed Hamiltonian is given by

$$ H = \int d^2 x ~ c_i^\dagger(x) \frac{(-i \nabla + (\frac{\phi}{e}) A - A')}{2m} c(x), $$

(50)

where

$$ A'(x) = \phi \int d^2 y \frac{\hat{\nabla} \times (x - y)}{(x - y)^2} ~ c_i^\dagger(y) ~ c(y). $$

(51)

The average field strength of the flux tubes can now be adjusted to cancel out the external magnetic field. At half-filling setting $\phi = 2$

$$ \nabla \times \left[ \left( \frac{e}{c} A - A' \right) \right] = \frac{e}{c} \left( B - \frac{2\pi \rho_f c \phi}{e} \right) = 0, $$

(52)

consequently, at the mean-field level, the quasiparticles behave as if they are free fermions in zero net field; the infinite Landau degeneracy is lifted and presumably the state is stable against two-body Coulomb interactions. Now
the important question is whether gauge fluctuations (or equivalently quasiparticle density fluctuations) modify or destroy this mean-field state. Unlike physical Maxwell electromagnetic gauge fields for which the gauge coupling is the fine structure constant \( \alpha \approx 1/137 \), the coupling constant between the composite fermions and the transverse component of the statistical gauge field is of order unity.

In the following we assume that despite strong gauge fluctuations the bosonic construction continues to apply. Later we will verify that this assumption is correct for the case of a Coulomb interaction. We determine the composite quasiparticle Green’s function and show that the physics of the \( \nu = 1/2 \) state is controlled by a marginal Fermi liquid fixed point. It should be noted that the longitudinal interaction plays a crucial role in the Chern-Simons gauge theory for it destroys this mean-field state. Unlike physical Maxwell electromagnetic gauge fields for which the gauge coupling is run over only the time and transverse directions, 0 and \( T \) are set equal to zero, they are described by the term \( -1/2 \phi^2 \).

\[
S = \int d^2x \frac{d}{dt} \left[ c^*(i \partial_t + A_0)c + \frac{1}{2m} \left( \nabla - \frac{e}{c} \right)^2 c + \frac{A_0}{2\pi\phi} (\nabla \times A') \right] \\
- \frac{1}{2} \int d^2x d^2y \left[ \nabla_x \times A'(x) \right] V(x - y) \left[ \nabla_y \times A'(y) \right],
\]  

The last term in the action is the longitudinal two-body Coulomb interaction which now is written in terms of the gauge field by making use of the constraint that a flux tube is attached to each electron, \( \nabla \times A' = -2\pi\phi \phi^* \). It is important to notice that the longitudinal interaction plays a crucial role in the Chern-Simons gauge theory for it suppresses fermion density fluctuations. Since density fluctuations are equivalent to gauge fluctuations in a Chern-Simons theory, repulsive long-range longitudinal interactions moderate and control the gauge interactions. After integrating out the high energy quasiparticles following the procedure of Sec. 11 and adopting a change of notation \( A' - (e/c)A \rightarrow A \), the HLR effective action reads:

\[
S_{FL} = -\frac{f_1}{2V k_F^2} \int \frac{d\omega}{2\pi} \sum_{S,T,q} J(S; q) \frac{k_s \cdot k_T}{J(T; -q)}
\]

which is made gauge covariant by the replacement \( k = -i \nabla \rightarrow -i \nabla - A \). The bare gauge action, Eq. 14, can now be read off from Eq. 13.

\[
S_G^0 = \frac{i}{4\pi\phi} \int \frac{d^2q}{2\pi^2} \int \frac{d\omega}{2\pi} \left\{ A_0(-q) [q \times A(q)] - A_0(q) [q \times A(-q)] \right\} \\
- \frac{1}{2} \int \frac{d^2q}{2\pi^2} \int \frac{d\omega}{2\pi} q^2 V(q) [A(q) \cdot A(-q)],
\]

here \( V(q) \) is the Fourier transform of \( V(x - y) \); for the Coulomb interaction, \( V(q) = 2\pi e^2/|q| \). For computational convenience, we work in the Coulomb gauge which eliminates the longitudinal component of the vector gauge field, only the transverse component \( A^T(q) = q \times A(q)/|q| \) remains.

**B. Bosonization of Half-Filled Landau Level**

We now use the bosonization method to determine the composite quasiparticle two-point Green’s function. As before, we bosonize the action Eq. 14, construct the generating functional \( Z[\xi, \xi^*] \), and integrate out the boson fields to calculate the effective gauge action. The components of the inverse gauge propagator \( K_{\mu\nu} = [D^{-1}]_{\mu\nu} \) (now the indices \( \mu \) and \( \nu \) run over only the time and transverse directions, 0 and \( T \)) are given by:
Again the additive correction to the boson propagator is given by Eq. (23) where \( \epsilon^\mu(S; q) \) is now the two-component vector \( (1, q \times \mathbf{v}_F^*/|q|) \). In the q-limit, \( \omega \ll v_F^*|q| \), the gauge propagator is dominated by the transverse components and

\[
D_{\mu\nu}(q) \ e^\nu(S; -q) \ e^\mu(S; -q) \approx \frac{|q \times \hat{n}_S|^2}{q^2} \frac{v_F^2 N^*(0)}{i\gamma N^*(0) \omega + \frac{q^2}{(2\pi\phi)^2}[1 + N^*(0) V(q)]};
\]  

(58)

here \( \gamma = \rho_f/\pi v_F^*N^*(0) \) following the notation of HLR\[2\]. In this limit the gauge propagator has an imaginary part which is a result of quasiparticle damping. For the Coulomb interaction,

\[
D_{\mu\nu}(q) \ e^\nu(S; q) \ e^\mu(S; -q) \approx v_F^2 \frac{|q \times \hat{n}_S|^2}{q^2} \frac{1}{i\gamma N^*(0) \omega - \chi|q|};
\]  

(59)

where \( \chi = 2\pi\epsilon^2/(4\pi^2\epsilon_0^2) \), \( \epsilon \) is the dielectric constant, and the pole is located at \( \omega \approx i\epsilon^2 q^2/k_F \). If, on the other hand, \( V(q) \) is a short-range interaction, the pole is at \( \omega \approx i v_F^*|q|^2/k_F^2 \). The pole location agrees with the RPA result\[19\]. In the opposite \( \omega \)-limit, \( \omega \gg v_F^*|q| \), we find

\[
D_{\mu\nu}(q) \ e^\nu(S; q) \ e^\mu(S; -q) \approx \frac{1}{2} \frac{\omega^2}{q^2} \frac{v_F^2 N^*(0) (2\pi\phi)^2}{\omega^2 - (2\pi\phi v_f)^2/m^*};
\]  

(60)

in this limit the poles which characterize collective excitations\[19\], the cyclotron modes of the composite fermions, are found at \( |\omega| = 2\pi\rho_f\phi/m^* \), the cyclotron frequency of a free particle of mass \( m^* \) in an external magnetic field of \( B = 2\pi\rho_f\phi/e \). As it stands this result violates Kohn’s theorem\[2\]. The correct result is obtained when the Fermi liquid interaction, Eq. (35), is included in the bosonized theory. The poles are then shifted to \( |\omega| = \omega_c = 2\pi\rho_f(1/m^* + f_1/4\pi\phi) \) in accord with Kohn’s theorem. The details of the proof of this result are given in Appendix C. The form of Eq. (60) is similar to that found for the super long range interaction of Bares and Wen\[3\] in the previous section. Since a projection onto the lowest Landau level is implicit in the HLR theory, however, we must assume \( v_F^*\lambda \ll \omega \) so that inter-Landau level transitions are not allowed. This restriction is an auxiliary condition which must be taken into account when defining the composite fermion Green’s function. Non-Fermi liquid behavior, if it exists, must have its origin in the behavior of the gauge propagator in the opposite q-limit. We may estimate the fermion quasiparticle self-energy\[2\] from Eq. (35) by expanding the fermion Green’s function in powers of \( D_{\mu\nu}; \) however it should be noted that there is no obvious small parameter to justify the expansion. Given this caveat, in the case of the Coulomb interaction at first order in the expansion we obtain the marginal Fermi liquid (MFL) form for the self-energy in agreement with the RPA calculation of HLR\[2\].

As the expansion in powers of the boson propagator is unreliable, it is more useful to determine the quasiparticle propagator directly in real space and time. We consider the fermion Green’s function, Eq. (24) and focus on the self-energy in agreement with the RPA calculation of HLR. In the q-limit, \( \omega \ll v_F^*|q| \), the gauge propagator is dominated by the transverse components and

\[
K_{00}(q) = \chi^0(q)
\]

\[
K_{0T}(q) = \frac{i}{2\pi\phi}|q| = K_{T0}(q)
\]

\[
K_{TT}(q) = \frac{q^2}{(2\pi\phi)^2} V(q) + \chi^T(q).
\]

(57)

Evaluation of the integral over \( q \) yields

\[
\delta G_B(S; x, t) = 2i|x||v_F^*| \int_0^{v_F^*\lambda/2} d\omega \ e^{i\omega \text{sgn}(v_F^*)(x_1/v_F^* - t)} \int \frac{dq}{q_F^0} \ d^2q_\perp \frac{q_\perp dq_\perp}{\omega^2 - i|\omega|^2/\chi} \approx i|x||v_F^*| \int_0^{v_F^*\lambda/2} d\omega \ e^{i\omega \text{sgn}(v_F^*)(x_1/v_F^* - t) - 2\omega/(\lambda v_F^*)} \ln \left( \frac{iv_F^*\delta + \omega}{\omega} \right);
\]

(62)

here
\[ \delta \equiv \frac{e^2}{4\epsilon \nu_F^* \phi^2} \frac{\lambda^2}{k_F} \]  

(63)

is a new momentum scale and

\[ \zeta \equiv \frac{\tilde{\delta}^2 \epsilon \nu_F^*}{2\pi e^2} \]  

(64)

is a dimensionless number. The final integration over frequency can be carried out using the integration formula

\[ \int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{1}{\mu} [\ln \beta - e^{\mu \beta} \text{Ei}(\mu \beta)] \]  

(65)

where Ei(x) is the exponential integral function. The result is

\[ \delta G_B(S; x, t) = -\frac{x_{||} \zeta}{x_{||} - \nu_F^* t + 2ia \, \text{sgn}(x_{||})} \left\{ \frac{1}{2} \ln \left[ (x_{||} - \nu_F^* t)^2 \delta^2 + 4a^2 \delta^2 \right] + \gamma 
- e^{(x_{||} - \nu_F^* t) \delta} \, \text{sgn}(x_{||}) + 2ia \delta \right\} \]  

(66)

which, upon making use of the asymptotic expansion of the exponential integral function, reduces to

\[ \delta G_B(S; x, t) \approx \zeta |x_{||} \delta| \ln |x_{||} - \nu_F^* t| \delta ; \quad |x_{||} - \nu_F^* t| \delta \ll 1 \]
\[ \approx -\frac{x_{||} \zeta}{x_{||} - \nu_F^* t + 2ia \, \text{sgn}(x_{||})} \left[ \gamma + \ln |x_{||} - \nu_F^* t| \delta \right] ; \quad |x_{||} - \nu_F^* t| \delta \gg 1 \].  

(67)

Here \( \gamma \approx 0.577216 \) is Euler’s constant and \( a \equiv 1/\lambda \).

For simplicity, first consider the equal-time quasiparticle propagator given at long distances \( |\hat{n}_S \cdot x| \gg 1/\delta \) by:

\[ G_F(S; x) \approx \frac{e^{ik_S \cdot x}}{\hat{n}_S \cdot x} (\delta |\hat{n}_S \cdot x|)^{-\zeta} \]  

(68)

and at short distances \( |\hat{n}_S \cdot x| \ll 1/\delta \) by the usual Fermi liquid form:

\[ G_F(S; x) \approx \frac{e^{ik_S \cdot x}}{\hat{n}_S \cdot x}. \]  

(69)

At momentum scales less than \( \delta \) the quasiparticle pole is destroyed by transverse gauge interactions suggesting that the \( \nu = 1/2 \) system is controlled by a novel non-Fermi liquid fixed point. Despite the appearance of the anomalous exponent \( \zeta \), the composite quasiparticle Green’s function differs in an important way from the Green’s function found for the super long-range longitudinal interaction in the previous section. The composite quasiparticle occupancy drops abruptly from one to zero at the Fermi surface over a momentum scale \( \delta \propto \lambda^2 \) which is much less than the cutoff \( \lambda \) as \( \lambda \to 0 \). As the basic consistency condition is satisfied in this limit, bosonization can provide a systematic approach to the study of the new fixed point.

For typical GaAs/AlGaAs samples the carrier density \( \rho_f \) is of order \( 10^{11}/\text{cm}^2 \), the electron band mass \( m_0 = 0.068m_e \), and the dielectric constant is given by \( \epsilon = 128 \). However, the effective mass of the composite fermion quasiparticle has to be estimated. Since electron states in the lowest Landau level are degenerate, the electron band mass does not play an important role. Instead, the energy scale is determined by the Coulomb energy at the magnetic length scale.

From dimensional analysis\(^{[3]} \) the effective mass of the quasiparticle must be therefore:

\[ m^* = \frac{\epsilon}{e^2} \sqrt{4\pi \rho_f} \frac{\sqrt{\lambda^2}}{C} \]  

(70)

where \( C \) is a dimensionless constant to be determined either numerically, or from experiment. Experiments at \( \nu = 1/2 \) suggest\(^{[4]} \) that \( m^* \approx 10m_0 \) and hence \( C \approx 0.10 \). For values in this range we find \( \zeta = C\tilde{\delta}^2/2\pi \approx 0.05 \). The small value of the exponent \( \zeta \) is consistent with the abrupt change in the occupancy inferred from the experimental observation of sharp cyclotron resonances\(^{[5]} \). For the purpose of estimating \( \delta \), we may take the cutoff to be of order the Fermi momentum \( \lambda \approx k_F \) and find \( \delta = \lambda^2/(4C\tilde{\delta}^2k_F) = O(k_F) \) as expected since \( k_F \) is the only physical momentum scale in the problem. Note that the Fermi energy, \( \epsilon_F = k_F^2/(2m^*) \), is still much smaller than the cyclotron energy, \( \omega_c = k_F^2\tilde{\delta}/(2m_0) \), and hence the requirement \( \nu_F^* \lambda << \omega_c \) is respected even for \( \lambda \approx k_F \). In the opposite limit \( |x| = 0 \).
the quasiparticle Green’s function is given by the free form \( G_F(0,t) = 1/(v_F^* t) \). Consequently the density of states is the same as that of non-interacting fermions with Fermi velocity \( v_F^* \).

We examine briefly longitudinal interactions other than Coulomb. Because longitudinal interactions control fermion density fluctuations and hence, by the constraint, gauge fluctuations, we expect repulsive interactions which are longer-ranged than Coulomb to stabilize the Fermi liquid fixed point. For a general interaction \( V(q) \propto 1/|q|^{1-y} \) this question has been addressed by Nayak and Wilczek using a renormalization group expansion for small \( y \). They concluded that for \( y \) positive (negative) the gauge interaction is relevant (irrelevant). The case \( y = 0 \) is marginal and they found logarithmic corrections to Fermi liquid behavior. Here we set \( V = \lambda/2 \) and repeat the calculation of the equal-time Green’s function. We obtain:

\[
\delta G_B(S;x) \approx i \int \frac{dq}{2\pi} \int_0^{\lambda/2} \frac{d\omega}{2\pi} \left( \frac{x}{v_F^*} \right) e^{i\omega x/\gamma} e^{i\delta x/\gamma} \frac{v_F^*}{\gamma + i\omega} |q_y|^{2+y} \approx \delta G(\omega) \tag{71}
\]

where \( \gamma \equiv g/(2\pi \tilde{\delta})^2. \) When \( y \neq 0 \), the correction to the boson propagator scales as \( \delta G_B \propto (|x|\delta')^{y/(2+y)} \) where \( \delta' \equiv (\gamma/v_F^* \gamma)(v_F^*/\chi)^{(2+y)}/y. \) Therefore, for \( y < 0 \), the gauge interaction is irrelevant and \( \delta G_B \to 0 \) as \( |x| \to \infty \), and the Landau fixed point is recovered. For \( y > 0 \), however, we expect pronounced deviations from the Landau fixed point when \( |\delta G_B| \gg 1 \), or equivalently when \( |x| \gg 1/\delta' \). The small momentum region is controlled by a non-Fermi liquid fixed point about which bosonization can tell us little as the deviations from Fermi liquid behavior are too large. For example, the discontinuity in the quasiparticle occupancy at the Fermi surface is completely eliminated by Lee and Nagaosa as a model for the cuprate superconductors. This is the special case \( y = 1 \). Again this system is outside the scope of bosonization.

### C. A Marginal Fermi Liquid

The real-space and time composite fermion Green’s function given in the preceding section by Eqs. (24, 27) is the starting point for the more detailed discussion of single-particle properties which we give in this section. In particular we show that the Green’s function is of a marginal Fermi liquid (MFL) type. In this section for simplicity we set \( v_F^* = 1 \) and denote \( x \) by \( x \). We first rewrite the Green’s function in the two limits \( |x - t|\delta \gg 1 \) and \( |x - t|\delta \ll 1 \):

\[
G_F(x,t) = \frac{1}{x - t + ia \text{sgn}(x)} \exp \left\{ \zeta |x|\delta \ln |x - t|\delta \right\} \; |x - t|\delta \ll 1 \approx \frac{1}{x - t + ia \text{sgn}(x)} \exp \left\{ -\zeta \frac{x \ln |x - t|\delta}{x - t} \right\} \; |x - t|\delta \gg 1. \tag{72}
\]

Ideally we would Fourier transform the Green’s function directly into momentum and frequency space to extract the self-energy. Unfortunately this task is quite difficult, so instead we begin with the observation that the pole of the free fermion Green’s function at \( x = t \) has been destroyed by the transverse interactions. As \( x \to t \), \( G_F(x,t) \to \frac{|x - t|^\zeta\delta}{x - t} \) so the pole at \( x = t \) has been eliminated. To ascertain where the spectral weight is concentrated now, we expand the Green’s function in the large-time limit \( |t| \gg |x + \zeta x \ln(|x - t|\delta)| \) and show later that corrections arising from contributions at shorter times \( |t| \ll |x + \zeta x \ln(|x - t|\delta)| \) contribute to the incoherent spectral weight. Since in the large-time limit \( |x - (x-t)| \ll 1 \) we may Taylor-series expand the exponential appearing in the second line of Eq. (72). Keeping the leading terms we obtain:

\[
G_F(x,t) = \frac{1}{x - t + ia \text{sgn}(x)} \frac{1}{1 + \frac{x}{x - t} \ln(|x - t|\delta) + \text{HOT}} \frac{1}{(x - t) + \zeta x \ln(|x - t|\delta) + ia \text{sgn}(x) + \text{HOT}} \tag{73}
\]

where \( \text{HOT} \) denotes higher-order terms which are negligible in the \( |t| \gg |x| \) limit. We may now Fourier transform this leading term to extract the self-energy. Upon making the approximation that \( \ln(|x - t|\delta) \approx \ln(|x|\delta) \) near the pole in Eq. (73), the integral over time can be done:

\[
G_F(k,\omega) = \int dx dt \frac{e^{i(\omega t - kx)}}{(x - t) + \zeta x \ln(|x - t|\delta) + ia \text{sgn}(x)}
\]
\[ \approx 2\pi i \int dx [\theta(-x)\theta(\omega) - \theta(x)\theta(-\omega)] e^{i(\omega-k)x} \exp[i \zeta \omega x \ln(|x|\delta)]. \]

\[ \approx 2\pi i \int dx [\theta(-x)\theta(\omega) - \theta(x)\theta(-\omega)] e^{i(\omega-k)x} \sum_{n=0}^{\infty} \frac{|i \zeta \omega x \ln(|x|\delta)|^n}{n!}. \quad (74) \]

Each term resulting from the Taylor series expansion of the exponential can be integrated separately. It is easy to show that:

\[ \int_{0}^{\infty} dx \ e^{i(k-\eta)x} |x \ln(x\delta)|^n = \frac{1}{(ik-\eta)^{n+1}} \left\{ n! \ln^n[(\eta - ik)/\delta] + O(\ln^{n-1}[(\eta - ik)/\delta]) \right\} \quad (75) \]

where \( \eta > 0 \) is a small term which ensures convergence. Up to subleading corrections, the Green’s function is:

\[ G_F(k, \omega) = \frac{2\pi}{\omega - k} \sum_{n=0}^{\infty} \left[ \frac{\zeta \omega}{\omega - k} \ln[i(\omega - k) \ sgn(\omega)/\delta] \right]^n \]

\[ = \frac{2\pi}{(\omega - k) - \zeta \omega \ \ln[i(\omega - k) \ sgn(\omega)/\delta]}. \quad (76) \]

Hence the leading contribution to the self-energy of the quasiparticles is:

\[ \Sigma(k, \omega) = \zeta \omega \ln(|\omega - k|/\delta) + \frac{i\pi}{2} \zeta \omega \ sgn((\omega - k) \ \omega). \quad (77) \]

At the pole, \( k \approx \omega - \zeta \omega \ln(|\omega|/\delta) \), \( |\omega| \ll |k| \), and to logarithmic accuracy, the self-energy is given by

\[ \Sigma(k, \omega) = \zeta \omega \ln(|\omega|/v_F^* \delta) - \frac{i\pi}{2} \zeta \omega, \quad (78) \]

the marginal Fermi liquid form. In Eq. (73) we have restored \( v_F^* \). Note that \( v_F^* = k_F/m^* \) is the Fermi velocity obtained earlier by integrating out the high-energy degrees of freedom. In a marginal Fermi liquid such as this, there is no definite velocity; rather the velocity is a logarithmic function of the energy scale and is of order, but not equal to, \( v_F^* \). At half-filling, \( \phi = 2 \) and \( \zeta = 2v_F^*/\pi e^2 \). Upon setting \( \lambda \approx k_F \) for the purpose of estimating the size of the self-energy we find \( v_F^* \delta O(\epsilon_F) \), the Fermi energy; hence Eq. (78) is identical to that derived by Ioffe et al. by a 1/N perturbation expansion [4].

The evaluation of the self-energy might be questioned, in particular because the time integral in Eq. (73) extends over the whole axis, whereas the approximate Green’s function is accurate only at large time. Indeed, \( G_F(x, t) \) does not have a pole at \( t = x + \zeta x \ln(|x|/\delta) \) as can be seen from inspection of Eq. (74). To see why nevertheless the calculation is qualitatively correct, consider what happens if we restrict the time integral to large times by excluding intermediate times \( 0 < t < 2t_p \) for which the expansion of Eq. (74) is unreliable. Here \( t_p = x + \zeta x \ln(|x|\delta) \) and we can assume \( x > 0 \) without loss of generality. The restricted integral over time may be evaluated giving:

\[ \int_{-\infty}^{0} e^{-i\omega t} dt + \int_{2t_p}^{\infty} e^{-i\omega t} dt = e^{(a-t_p)\omega} \left\{ 2\pi i \theta(-\omega) + Ei[(it_p - a)\omega] - Ei[-(it_p + a)\omega] \right\} \]

\[ = e^{(a-t_p)\omega} \left\{ 2\pi i \theta(-\omega) - \pi i + 2i\omega t_p + O(\omega^3 t_p^3) \right\} \quad (79) \]

where the second line follows from an expansion of the exponential integral about the origin \( Ei(z) = \gamma + \ln(z) + z + z^2/4 + O(z^3) \) with the assumption that \( t_p \gg a \). By neglecting the term proportional to \( 2i\omega t_p \) and higher order terms which lead only to subleading corrections, the integral over \( x \) can be done. The resulting spectral weight is identical to that found from the imaginary part of Eq. (74). Apparently, coherent spectral weight in the \((k, \omega)\) plane is found in the long-time tail of the real-space and time MFL Green’s function despite the absence of a pole in the \((x, t)\) plane.

In recent work Altshuler et al [1] have determined the quasiparticle propagator by a 1/N perturbation theory in the fermion basis. In this work it is claimed that, because nonlinear terms in the fermion dispersion due to curvature of the Fermi surface are neglected in the bosonization scheme, the scheme cannot be used with confidence to study a transverse gauge theory. On the contrary, in the problem of the half-filled Landau level with a Coulomb interaction, the self-energy Eq. (73) is identical to that found by Altshuler et al. Indeed, the only difference between the two calculations is the anomalous power law decay of the equal-time propagator discussed in the previous section which has its origin in the incoherent part of the quasiparticle spectrum and therefore should not be expected to appear in analyses such as that of Altshuler et al. which consider only the low frequency behavior of the Green’s function. In the case of Coulomb interactions \((y = 0)\) nonlinear terms in the fermion dispersion due to Fermi surface curvature do not affect the low-energy quasiparticle spectrum in any significant way. Why this should be the case is examined in the next subsection.
D. Consistency of Bosonization: Nonlinear Dispersion and Vertex Corrections

An important feature of the bosonization procedure is the linearization of the free fermion spectrum in the vicinity of the Fermi surface: \( c_{k_S+q} - c_{k_S} \approx v_S \cdot q \). Nonlinear terms in the dispersion due to curvature of the Fermi surface are neglected and within bosonization can only be accounted for perturbatively. This approximation might be questioned, particularly in the case of a mediating transverse gauge interaction, when the interaction vertex draws its energy from the linearized approximation, \( v_S q q \), might dominate the physics for \( y \geq 0 \). This question is best addressed in the fermion basis by the consideration of the Ward Identities derived by Castellani, \textit{et al.}\ These authors found that the Green’s function derived using the Ward Identities is identical to that found by bosonization provided that the relationship between density and current vertices, \( \Delta_S = v_S^2 \Lambda^0_3 \), which becomes exact in one dimension, holds approximately in dimensions greater than one. In the same context, we may ask also whether deviations from this relationship are so singular that they destroy the form of the propagator found both within bosonization and the Ward Identity approach. For the Chern-Simons gauge theory with \( y = 0 \), the physically relevant case of the long-range Coulomb interaction, we show that the bosonization scheme is internally self-consistent, and that the results derived earlier in this section continue to hold.

We start by deriving the Ward Identities which relate the density and current vertices of the bosonized interacting fermion system in two dimensions; for simplicity, we give the proof for a fermion liquid interacting via a transverse Coulomb interaction, we show that the bosonization scheme is internally self-consistent, and that the results derived earlier in this section continue to hold.

We start by deriving the Ward Identities which relate the density and current vertices of the bosonized interacting fermion system in two dimensions; for simplicity, we give the proof for a fermion liquid interacting via a transverse Coulomb interaction. The result is readily shown to be true in general. We introduce the time-ordered current-current correlation function between currents in different patches

\[
\Pi_{S;Q}(q) = -i \frac{1}{V} \langle T \, J(S; q) \, J(Q; -q) \rangle
\]

and the amputated vertex function

\[
\Gamma_{S;T}(p_1, p_2; q) = -\langle T \, J(S; -q) \, \psi(T; p_1) \, \psi^\dagger(T; p_2) \rangle_{\text{amp}}
\]

where \( q + p_1 - p_2 = 0 \). Then using the current algebra, we derive their equations of motion:

\[
(v - q \cdot v_S) \Gamma_{S;T}(p_1, p_2; q) = q \cdot \hat{n}_S \frac{\Omega}{V} \sum_R V_{S;R}(q) \Gamma_{R;T}(p_1, p_2; q)
\]

\[
+ \delta_{S,T} [G^{-1}_F(T; p_2) - G^{-1}_F(T; p_1)]
\]

and

\[
(v - q \cdot v_S) \Pi_{S;Q}(q) = q \cdot \hat{n}_S \frac{\Omega}{V} \sum_T \Pi_{T;Q}(q) V_{Q;T}(q) + q \cdot \hat{n}_S \frac{\Omega}{V} \delta_{S,Q}
\]

where

\[
V_{S;T}(q) = \left( \frac{q \times v_S^\perp}{|q|} \right) \frac{1}{K^0_{TT}(q) - \rho_f/m^*} \left( \frac{q \times v_T^\perp}{|q|} \right)
\]

and \( K^0_{TT}(q) \) is the kernel of the bare gauge action. In addition, we make use of the Dyson equation which relates the reducible current vertices, \( \Gamma_{S;T} \), and the irreducible current vertices, \( \Lambda_{S;T} \):

\[
\Gamma_{S;T}(p_1, p_2; q) = \Lambda_{S;T}(p_1, p_2; q) + \sum_{R,Q} \Pi_{S;R}(q) \Lambda_{R;T}(p_1, p_2; q) V_{Q;R}(q).
\]

In all the above equations, \( q \) which denotes \( (\nu, q) \) is an index appearing in the bosonic quantities and \( p_i \) denotes \( (\omega_i, p_i) \), an index for the fermion variables and for example, \( G_F(S; p) = G_F(\omega, k_S + p) \) with \( |k_S| = k_F \gg |p| \). Combining Eqs. (82), (83), (84) gives the familiar Ward Identity relating the current and density vertices

\[
\nu \, \Lambda^0_S(p; q) - q \cdot \Lambda_S(p; q) = G^{-1}_F(S; p) - G^{-1}_F(S; p - q)
\]

where
\[ \Lambda^0_S(p; q) = \sum_T A_{T,S}(p - q, p; q) \] (87)

and

\[ \Lambda_S(p; q) = \sum_T v^*_T A_{T,S}(p - q, p; q) . \] (88)

In addition, Eqs. (82), (83), (84) can be solved directly to determine \( \Pi_{S;Q}(q) \):

\[ \Pi_{S,Q}(q) = \frac{\Omega}{\nu - q \cdot v^*_S} \left[ \delta_{S,Q} + \frac{\Omega}{\nu - q \cdot v^*_Q} \right] \frac{1}{q \times v^*_S} \left[ \rho_f/m^* - v^*_F \chi_1(q) \right] \quad \text{for } \nu - q \cdot v^*_Q > 0 \] (89)

The diagonal elements of the correlation function are simply related to the boson Green’s function Eq. (22). These equations also provide an explicit calculation of the irreducible vertex of the bosonized interacting fermion system

\[ \Lambda_{S;T}(p_1, p_2; q) = \frac{\delta_{S,T}}{\nu - q \cdot v^*_S} \left[ G^1_F(S; p_2) - G^{-1}_F(S; p_1) \right] \] (90)

from which it follows immediately that

\[ \Lambda_{T}(p_2; q) = \sum_S v^*_S \Lambda_{S;T}(p_1, p_2; q) = v^*_T \Lambda^0_T(p_2; q) \] (91)

an exact result within bosonization but in general only approximately true. Introducing the quantity

\[ Y_S(p; q) = \frac{q \cdot [\Lambda_S(p; q) - v^*_S \Lambda^0_S(p; q)]}{\Lambda^0_S(p; q)} \] (92)

and making use of the Dyson equation for the fermion self-energy and Eq. (80), Castellani, et al. found the fermion Green’s function to be given by the solution of

\[ G_F(S; p) = G_{F0}(S; p) + iG_{F0}(S; p) \int \frac{d^2q}{(2\pi)^3} \frac{G_F(S; p - q)}{\nu - v^*_F q_\| - Y_S(p; q)} D_{\mu\nu}(q) \epsilon^\mu(S; q) \epsilon^\nu(S; -q) \] (93)

where \( G^{-1}_F(S; q) = \omega - p \cdot v^*_S \). When \( Y_S(p; q) = 0 \), Eq. (93) can be solved and the result is exactly that derived by bosonization.

Direct calculation of \( Y_S(p; q) \) allows the investigation of the quality of approximation implicit in the bosonization approach. We evaluate \( Y_S(p; q) \) in the fermion basis, to the lowest order, for a fermion liquid interacting via the Chern-Simons gauge field and a two-body longitudinal interaction of the form \( V(q) = g/|q|^{1-y} \) and hence

\[ D_{\mu\nu}(q) \epsilon^\mu(S; q) \epsilon^\nu(S; -q) \approx v^*_F \frac{1}{|q|} \frac{1}{i\gamma |q| - \chi |q|^{1+y}} \] (94)

where \( \gamma = \rho_f / \pi v^*_F N^*(0) \) and \( \chi = g/(2\pi \tilde{\phi})^2 \). When the quadratic terms in the free particle dispersion are retained, the bare single-particle Green’s function is given by \( G_{F0}(S; p) \equiv \omega - p \cdot v^*_S - p^2_\perp/2m^* \). Correspondingly, the bare vertex function is given by

\[ \Lambda^0_S(p; q) = 1 \]
\[ \Lambda_S(p; q) = v^*_S \frac{q}{2m^*} + \frac{p}{m^*} \] (95)

and therefore \( Y_S(p; q) \) is nonzero at leading order

\[ Y^0_S(p; q) = \frac{-q^2_{\perp}}{2m^*} + \frac{p \cdot q}{m^*} \] (96)

once curvature is accounted for. However, it is easily seen that these terms have no effect on the Green’s function determined from Eq. (93). Integrating over \( q_\parallel \), \( Y^0_S(p; q) \) exactly cancels the contributions due to curvature in the
Green’s functions of the internal loop and the solution is as before. It remains to be shown whether or not vertex corrections due to the transverse gauge interaction modify this result.

We follow Kim et al. and use a perturbative $1/N$ expansion, where $N$ is the number of fermion flavors. The diagrams giving the correction to the vertex function at leading order in $1/N$ are shown in Figs. I. At this order because the singular contributions arising from the two diagrams of Fig. I cancel each other by Furry’s theorem, the only singular correction to $Y_S(p; q)$ is due to the diagram of Fig. I as can be verified by direct calculation. Furry’s theorem states that for systems with charge-conjugation symmetry, all diagrams containing a fermion loop with an odd number of vertices may be omitted. Charge-conjugation (C) symmetry here is equivalent to particle-hole symmetry. The nonlinear pieces of the fermion dispersion, due to curvature of the Fermi surface, break this symmetry.

For this system bosonization provides a consistent calculational framework. We see from Eq. (98) that the most important effect of finite $Y_S(p; q)$ on $G_F$ comes from the pole at $\nu - v_F^* p_\parallel/q_\parallel^2/2m^* - p_\perp q_\perp/m^* = 0$. At the pole, the density vertex is uncorrected and the only correction is to the transverse component of the current vertex:

$$A_S(p; q)|_{\text{pole}} \approx v_F^* - \frac{q}{2m^*} + \frac{p}{m^*} + \Lambda \delta A_S(p; q)|_{\text{pole}}$$  \hspace{1cm} (97)

where

$$\Lambda \delta A_S(p; q)|_{\text{pole}} \approx i \frac{1}{q_\perp} \frac{m^* v_F^*}{\gamma^2} \int_{-\lambda/2}^{\lambda/2} dq' |q'| \left[ \text{sgn}(\omega - \nu) \ln \left( 1 + \frac{|\omega - \nu|}{|q'|^{2+\gamma}} \right) - \text{sgn}(\omega) \ln \left( 1 + \frac{|\omega|}{|q'|^{2+\gamma}} \right) \right].$$  \hspace{1cm} (98)

We see from Eq. (98) that $q_\perp \Lambda \delta A_S$ scales as $(|\omega|/\gamma)^{2/(2+\gamma)}$; substituted in Eq. (92), this results in subleading corrections to the bare external fermion frequency, $\omega$, if $y < 0$ and the fermion Green’s function is unchanged. On the other hand, if $y > 0$, the corrections dominate and the results derived from bosonization are not reliable, as we had concluded for different reasons in Sec. IV.

We now consider the marginal and physically interesting case of the longitudinal Coulomb interaction $V(q) = 2\pi e^2/q_\perp$, $y = 0$, in detail. Carrying out the integration in Eq. (98),

$$\Lambda \delta A_S(p; q)|_{\text{pole}} = \frac{1}{q_\perp} (4\zeta) \left[ \omega \ln \left( -i \frac{|\omega|}{v_F^*} \right) - (\omega - \nu) \ln \left( -i \frac{|\omega - \nu|}{v_F^*} \right) \right] + \text{subleading terms},$$  \hspace{1cm} (99)

substituting in Eq. (93) and integrating over $q_\parallel$ we find

$$G_F(S; p) = G_F(0; p) \left\{ 1 + \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\nu \int dq_\perp \theta(\nu) \text{sgn}(\nu) \frac{v_F^*}{|\nu|} \frac{1}{q_\perp + \gamma |q_\parallel|} \right\}$$

$$\times \frac{1}{G_F(0; p) + 4\zeta \left[ \omega \ln \left( -i \frac{|\omega|}{v_F^*} \right) - (\omega - \nu) \ln \left( -i \frac{|\omega - \nu|}{v_F^*} \right) \right] - \Sigma(\omega - \nu)}$$  \hspace{1cm} (100)

The $q_\perp$ integral over the gauge propagator is logarithmically singular at $\nu = 0$ and the $\nu$-dependence in the denominator may be safely neglected on evaluating the final frequency integral. We then recover

$$G_F(S; p) = G_F(0; p) \left[ 1 + \Sigma(S; p) \right]$$  \hspace{1cm} (101)

where

$$\Sigma(\omega) = -\zeta \omega \ln \left( \frac{iv_F^* + |\omega|}{|\omega|} \right),$$  \hspace{1cm} (102)

the marginal Fermi liquid form in Eq. (78).

In summary, for a fermion liquid interacting via the Chern-Simons gauge interaction and a long-range Coulomb interaction, the form of the propagator is not modified by quadratic terms in the quasiparticle dispersion due to curvature of the Fermi surface or by vertex corrections beyond those already incorporated in multidimensional bosonization. For this system bosonization provides a consistent calculational framework.
V. DENSITY RESPONSE FUNCTIONS

In previous sections we focused attention exclusively on the single-particle properties of the interacting fermion system. In particular we found evidence of non-Fermi liquid behavior in the single-particle Green’s function for the case of transverse gauge interactions. However, the single-particle Green’s function does not have clear physical meaning as it is not gauge invariant. Indeed, as shown by Kim et al., different types of singularities might appear depending on the choice of gauge. These authors used a perturbative expansion to study such gauge invariant quantities as the density and current response at small momentum transfer. They were able to show that these properties of the interacting gauge system were indeed Fermi liquid like: singular corrections to the quasiparticle self-energy were canceled by singular vertex corrections. However, the possibility that singular behavior could arise at large momentum transfer was not ruled out. In fact, other workers have found singular behavior at large momentum transfer.

In particular, Ioffe et al. have argued that the vertex correction is singular at momentum transfer $2k_F$ because the gauge fields mediate an attractive interaction between currents moving in the same direction. In this section, we calculate the gauge invariant response functions for both small momentum transfer $Q ≈ 0$ and for $Q ≈ 2k_F$. First we check that bosonization yields correct results for the free problem in both cases, and then we take into account the effects of interactions.

We begin by writing the fermion density-density correlation function in bosonized form. The fermion density fluctuation is given by:

$$\delta \rho(x, t) = : \psi_d^\dagger(x, t) \psi(x, t) : = \sum_{S, T} : \psi_d^\dagger(S; x, t) \psi(T; x, t) :$$  \hspace{1cm} (103)

where the colons denote normal ordering which in this case is equivalent to the subtraction of the uniform fermion background density. Then the density-density correlation function can be written as:

$$\langle \delta \rho(x, t) \delta \rho(0, 0) \rangle = \frac{1}{V^2} \sum_{S, T} \langle J(S; x, t) J(T; 0, 0) \rangle + \left( \frac{\Omega}{V_a} \right)^2 \sum_{S \neq T, U \neq V} e^{i(k_S - k_V) \cdot x} \times \left\{ \exp \left\{ i \frac{\sqrt{4\pi}}{\Omega} \left[ \phi(S; x, t) - \phi(T; x, t) \right] \right\} \exp \left\{ i \frac{\sqrt{4\pi}}{\Omega} \left[ \phi(U; 0, 0) - \phi(V; 0, 0) \right] \right\} \right\}$$  \hspace{1cm} (104)

For small momenta transfer, $|Q| < \lambda \ll k_F$, and only the first term in Eq. (104) is relevant. The second term is important for momentum transfers near $Q = 2k_F$. In subsections V A and V B we examine the first and the second terms respectively. We concentrate primarily on the case of two spatial dimensions.

A. Density Response at Small-$Q$

1. Free Fermions and Landau Fermi Liquids

For free fermions, currents in different patches are uncorrelated. In two dimensions, and for $|Q| \ll k_F$, the density response is given by the retarded correlation function

$$\Pi^0_{00}(Q, \omega) = -i \int d^2x \int_0^\infty dt \, e^{i(q \cdot x - \omega t)} \frac{1}{V^2} \sum_S \langle J(S; x, t) J(S; 0, 0) \rangle$$

$$= N^*(0) \left( \frac{x}{\sqrt{x^2 - 1}} - 1 \right),$$  \hspace{1cm} (105)

where $x \equiv \omega/(v_F |Q|)$. This is the well-known result for the density response of the free electron gas in the limit of low frequency and small momentum.

Including a general longitudinal interaction $V(Q)$, we use the generating functional to calculate the correlation between currents in different patches. Then the retarded correlation function which combines contributions from all possible two-point boson Green’s functions $\langle a(S; Q, \omega) \ a^\ast(T; Q, \omega) \rangle$, $\langle a(S; Q, \omega) \ a(T; - Q, - \omega) \rangle$, and $\langle a^\ast(S; Q, \omega) \ a^\ast(T; Q, \omega) \rangle$ is given by.
\[ \langle J(S; Q, \omega) J(T; -Q, -\omega) \rangle_{\text{Ret}} = \frac{i \Omega \hat{n}_S \cdot Q \delta_{S,T}}{\omega - v_F^2 \hat{n}_S \cdot Q + i\eta} \]

\[ + \frac{i \Omega}{(2\pi)^2} \frac{(\hat{n}_S \cdot Q) (\hat{n}_T \cdot Q)}{V(Q)} \left( \frac{\hat{n}_S \cdot Q + i\eta}{\omega - v_F^2 \hat{n}_S \cdot Q + i\eta} \right) \times \frac{\hat{n}_T \cdot Q + i\eta}{\omega - v_F^2 \hat{n}_T \cdot Q + i\eta} \]

\[ \times \frac{1}{1 - V(Q) \Pi^0_{00}(Q, \omega)}. \] (106)

Summing over \( S \) and \( T \), we obtain

\[ \Pi_{00}(Q, \omega) = \frac{\Pi^0_{00}(Q, \omega)}{1 - V(Q) \Pi^0_{00}(Q, \omega)}, \] (107)

identical to the RPA result for the density response which is exact in the limit \( Q, \omega \to 0 \).

2. Gauge Theory

At two-loop order in a 1/\( N \) perturbative expansion, Kim et al. found that beyond RPA there were only subleading corrections to the irreducible polarizability. We showed in Sec. IV D that implicit in the bosonization scheme is a relation between the density and current vertices, \( A_S = v_F^2 A^0_S \). This relation suggests that within bosonization there are no corrections to the polarizability \( \Pi \) beyond those of RPA. This supposition is correct as we will now demonstrate by explicit calculation for the case of the Chern-Simons gauge theory with a longitudinal interaction.

The correlation function between currents in different patches can be found using the generating functional and is given by

\[ \langle J(S; Q, \omega) J(T; -Q, -\omega) \rangle_{\text{Ret}} = \frac{i \Omega \hat{n}_S \cdot Q \delta_{S,T}}{\omega - v_F^2 \hat{n}_S \cdot Q + i\eta} \]

\[ + \frac{i \Omega}{(2\pi)^2} \frac{(\hat{n}_S \cdot Q) (\hat{n}_T \cdot Q)}{V(Q)} \left( \frac{\hat{n}_S \cdot Q + i\eta}{\omega - v_F^2 \hat{n}_S \cdot Q + i\eta} \right) \times \frac{\hat{n}_T \cdot Q + i\eta}{\omega - v_F^2 \hat{n}_T \cdot Q + i\eta} \]

\[ \times \frac{1}{1 - V(Q) \Pi^0_{00}(Q, \omega)} \] (108)

where \( D^R_{\mu} \) denotes the retarded part of the gauge propagator. Only the term proportional to \( D^R_{00}(Q, \omega) \) survives the sum over patches and we obtain for the density response:

\[ \Pi_{00}(Q, \omega) = \Pi^0_{00}(Q, \omega) \left[ 1 + D^R_{00}(Q, \omega) \Pi^0_{11}(Q, \omega) \right] \]

\[ = \frac{Q^2/(2\pi\phi)^2}{Q^2/(2\pi\phi)^2 - \Pi^0_{11}(Q, \omega) [\rho_f/m^* + Q^2 V(Q) - \Pi^0_{11}(Q, \omega)]} \] (109)

where \( \Pi^0_{11}(Q, \omega) \) is the current response function of the free electron gas

\[ \Pi^0_{11}(Q, \omega) = -v_F^2 N^*(0) \left[ x^2 - \frac{1}{2} - x \sqrt{x^2 - 1} \right]. \] (110)

In the same way, we may evaluate the current response function:

\[ \Pi_{11}(Q, \omega) = \Pi^0_{11}(Q, \omega) \left[ 1 - D^R_{11T}(Q, \omega) \Pi^0_{11}(Q, \omega) \right] \] (111)

where

\[ D^R_{11T}(Q, \omega) = \frac{\Pi^0_{00}(Q, \omega)}{\Pi^0_{00}(Q, \omega) [\rho_f/m^* + Q^2 V(Q) - \Pi^0_{11}(Q, \omega)] - Q^2/(2\pi\phi)^2} \] (112)

As stated above, we reproduce the RPA results, in agreement with the conclusion of Kim et al.
B. Density Response at $Q \approx 2k_F$

1. Free Fermions and Landau Fermi Liquids

It is important to check whether bosonization captures the physics of the Kohn anomaly. At first glance it is not obvious how bosonization, with the restriction $|q| < \lambda \ll k_F$, can possibly describe processes with large momentum transfer, especially those near $2k_F$. The reason why it can be made clear by observing that scattering between different patches is allowed, and that in fact the second term in Eq. (104) contains these processes explicitly. Recall that $2k_F$ processes are described correctly by standard one-dimensional bosonization. Bosonization in one and higher spatial dimensions correctly describes processes with either small or large momentum, provided they are of low energy. That is why the specific heat, computed within the bosonization framework, is exact in the low-temperature limit.

Since for free fermions the bosons in different patches are independent, the correlation function in the second term of Eq. (104) is zero except when $V = S$ and $U = T$, and the $Q \approx 2k_F$ contribution to the density response function is given by:

$$
\Pi^{0}_{2k_F}(\mathbf{x}, t) = -i \left( \frac{\Omega}{V} \right)^2 \sum_{S,T} e^{i k_S - k_T} \mathbf{x} \left< e^{i k_S \mathbf{x}} \left[ \phi(S; \mathbf{x}, t) - \phi(T; \mathbf{x}, t) \right] e^{i k_T \mathbf{x}} \left[ \phi(T; 0, t) - \phi(S; 0, t) \right] \right> \theta(t)
$$

$$
= i \left[ \frac{\Lambda^{-1}}{(2\pi)^D} \sum_{S} \hat{\mathbf{n}}_S \cdot \mathbf{x} - v_F^* t + i \eta \text{sgn}(t) \right] \left[ \frac{\Lambda^{-1}}{(2\pi)^D} \sum_{T} \hat{\mathbf{n}}_T \cdot \mathbf{x} - v_F^* t + i \eta \text{sgn}(t) \right] \theta(t) .
$$

(113)

Implicit in the last line is the restriction $|x| < \lambda$. As we are interested in the limit $Q \rightarrow 2k_F$, only patches which are nearly opposite to each other on the Fermi surface participate in the scattering, $\hat{\mathbf{n}}_S \approx -\hat{\mathbf{n}}_T$. Therefore we may carry over the sum over patches in each of the two bracketed terms in Eq. (113), keeping in mind that the spatial directions perpendicular to the Fermi surface normal are the same for both terms. To simplify the calculation, we consider only the static ($\omega = 0$) response and specialize to the case of a circular Fermi surface in two dimensions. Integrating the density response, Eq. (113), over positive time we obtain:

$$
\Pi^{0}_{2k_F}(R, \omega = 0) = - \frac{k_F^2 \pi}{v_F^* (2\pi)^4} \int_0^{2\pi} d\theta_1 d\theta_2 \exp[i k_F R \cos(\theta_1 - \theta_2)] \frac{\text{sgn}(\cos(\theta_1) - \text{sgn}(\cos(\theta_2))}{(\cos(\theta_1) - \cos(\theta_2)) R}
$$

(114)

where $\theta_1, \theta_2$ are the angles between $\mathbf{x}$ and $\mathbf{k}_S$ and $\mathbf{k}_F$ respectively and $R = |\mathbf{x}|$. Now we may focus on the case of $Q \approx 2k_F$ by limiting the range of integration over the angular variables $\theta_{1,2}$ to angles near $\theta_1 = 0$ and $\theta_2 = \pi$. In this limited region the cosines appearing in Eq. (114) may be expanded in Taylor-series as $\cos(\theta_{1,2}) = \pm(1 - \theta_{1,2}^2/2)$ and the Gaussian integrals over the angles can be carried out. Finally, the Fourier transform into $Q$-space is carried out. If $\theta$ is the angle between $\mathbf{Q}$ and $\mathbf{x}$, the density response function is given by:

$$
\Pi^{0}_{2k_F}(Q, \omega = 0) = \frac{2i k_F^2 \pi^2}{v_F^* (2\pi)^4} \int_0^{2\pi} d\theta \frac{\exp[i(2k_F - Q \cos(\theta)) R]}{R^2}.
$$

(115)

In the second line we have assumed $Q \geq 2k_F$ and have also introduced an ultraviolet cutoff $a$ to eliminate the artificial divergence at small-$R$. The nonanalytic term we seek comes from the large-$R$ region defined by $k_F R \gg 1$. Again in this limit a Taylor-series expansion of the cos $\theta$ term is justified and the resulting Gaussian integral in the $\theta$ variable has been carried out. For $Q < 2k_F$, obviously $Q \cos(\theta) < 2k_F$ and consequently the integral over $\theta$ contains only terms which oscillate rapidly with increasing $R$ and which therefore do not generate non-analytic behavior at large-$R$. Finally, asymptotic analysis of the radial integral gives the non-analytic term in the density response function:

$$
\Pi^{0}_{2k_F}(Q, \omega = 0) = \sqrt{\frac{2m^*}{4\pi}} \left( \frac{Q - 2k_F}{2k_F} \right)^{3/2} + C; \ Q > 2k_F
$$

$$
= C; \ Q < 2k_F .
$$

(116)

Here $C$ is a constant which can be determined by a more careful evaluation of the integrals in the small-$R$ region. Continuity across $Q = 2k_F$ ensures that the same constant appears in both limits of Eq. (116). The Kohn anomaly
of Eq. (116) agrees precisely with that found directly in the fermion basis. Repeating the calculation in $D = 3$ we easily show that the static response is given by:

$$
\Pi_{2k_F}^0(Q, \omega = 0) = \frac{k_F m^*}{2(2\pi)^2} \frac{2k_F - Q}{Q} \left[ \ln \left| \frac{Q - 2k_F}{2k_F} \right| + \gamma - 1 - \frac{i\pi}{2} \right] + C' 
$$

(117)

which again agrees with the exact result found in the fermion basis in the limit $Q \to 2k_F$. The Kohn anomaly persists even if longitudinal interactions are included. In fact, this has already been anticipated in Eq. (116) and Eq. (117) where the effective mass $m^*$ rather than the bare mass $m$ appears.

2. Gauge Theory

Now we consider the real-space density response function in two dimension at momentum transfer $2k_F$ for the Chern-Simons theory. With the gauge interaction, currents moving in the same direction are attracted to each other so we might anticipate a singularity in $\Pi_{2k_F}$. This singularity has a different physical origin than the Kohn anomaly of the previous section, which was not singular, just nonanalytic. The Kohn anomaly is due to the reduced availability of the phase space at low energies. In contrast, the singularity, if it exists, should be apparent even if we consider only scattering between two patches at exact opposite points of the Fermi surface. Therefore we need consider only terms with $k_S = -k_T$ in the $2k_F$ response function, Eq. (113).

$$
\Pi_{2k_F}(x, t) = -i \frac{(\Omega \omega)^2}{V a^2} \sum_S \exp \left\{ \frac{4\pi}{\Omega^2} \left[ \langle \phi(S; x, t)\phi(S; 0, 0) \rangle + \langle \phi(-S; x, t)\phi(-S; 0, 0) \rangle 
- \langle \phi(S; x, t)\phi(-S; 0, 0) \rangle - \langle \phi(-S; x, t)\phi(S; 0, 0) \rangle \right] 
- \frac{4\pi}{\Omega^2} \text{[terms with } (x, t) \to (0, 0)] \right\} \theta(t). 
$$

(118)

The in-patch boson correlation function is given by Eq. (12):

$$
\langle \phi(S; x, t)\phi(S; 0, 0) - \phi^2(S; 0, 0) \rangle = \frac{\Omega^2}{4\pi} \ln \left( \frac{i\hat{n}_S - v_F t}{x \cdot \hat{n}_S - v_F t} \right) 
+ \frac{\Omega^2}{4\pi} i \int \frac{d^2 q}{(2\pi)^2} \int \frac{d\omega}{2\pi} [e^{i(q \cdot x - \omega t)} - 1] \frac{1}{(\omega - v_F q \cdot \hat{n}_S)^2} 
\times \left\{ D_{00}^R(q) - v_F^2 |q \times \hat{n}_S|^2 \frac{D_T^R(q)}{|q|^2} \right\}. 
$$

(119)

The correlation function between bosons at opposite points of the Fermi surface contains the new physics. The correlation function $\langle \phi(S; x, t)\phi(-S; 0, 0) - \phi(S; 0, 0)\phi(-S; 0, 0) \rangle$ can be found using the generating functional and is given by an integral over frequency and momenta of:

$$
\frac{\theta(q)}{q} \left\{ (e^{i(q_{x1} \cdot -\omega t)} - 1) \langle a(S; q) a(-S; -q) \rangle 
+ (e^{-i(q_{x1} \cdot -\omega t)} - 1) \langle a(S; q) a(-S; -q) \rangle \right\} 
= \Omega \frac{\theta(q)}{v_F^2 q^2 - \omega^2} \left\{ (e^{i(q_{x1} \cdot -\omega t)} - 1) \left[ D_{00}^R(q) + v_F^2 \frac{q^2}{|q|^2} \right] 
+ (e^{-i(q_{x1} \cdot -\omega t)} - 1) \left[ D_{00}^R(q) + v_F^2 \frac{q^2}{|q|^2} \right] \right\}. 
$$

(120)

As in the case of the two-point function, the most singular contribution comes from the $q$-limit. Inserting Eqs. (119, 120) into Eq. (118) and computing the integrals in $q$-limit we obtain

$$
\Pi_{2k_F}(x, t) = \sum_S \left[ x \cdot \hat{n}_S - v_F t + i\eta \text{sgn}(x \cdot \hat{n}_S) \right] \frac{-i}{(|x| - v_F t + i\eta)} \exp \left[ \mathcal{E}(x \cdot \hat{n}_S, t) \right] 
$$

(121)
where the prime on the above sum over patches indicates that the sum is only over patches for which $|x \times \hat{n}_S| \Lambda < 1$ and where

$$E(x,t) \approx -\zeta \left\{ \frac{x}{x-v_F^* t + i \eta \text{sgn}(x)} \ln |x-v_F^* t\delta| + \frac{x}{x+v_F^* t + i \eta \text{sgn}(x)} \ln |x+v_F^* t\delta| 
- \ln^2 |(x-v_F^* t\delta)| - \ln^2 |(x+v_F^* t\delta)| \right\},$$

(122)

for $|x| - v_F^* t\delta \gg 1$ and $|x| + v_F^* t\delta \gg 1$. The log-squared terms appearing in $E$ have their origin in the correlations induced by the gauge interaction between the bosons in opposite hemispheres. Evidently a log-squared singularity at momentum transfer $2k_F$ exists in the real-space response function. Although it is technically difficult to Fourier transform it into frequency space, we can estimate the leading singular factor of $\Pi_{2k_F}(\omega)$ with the use of a scaling argument and find:

$$\Pi_{2k_F}(\omega) \propto \exp(2\zeta \ln^2 |\omega/(v_F^* \delta)|),$$

(123)

where again $v_F^* \delta = O(\epsilon_F)$. This singularity is similar to, though stronger than, that found by Altshuler et al.

VI. TRANSVERSE GAUGE FIELDS: FERMIONS IN THREE DIMENSIONS

In this section, we discuss the most familiar physical example of fermions interacting via gauge interactions: electrons in three spatial dimensions with the full Maxwell electromagnetic field. The longitudinal Coulomb interaction is screened, and the transverse gauge interaction is suppressed by a factor of fine structure constant multiplied by the ratio of the Fermi velocity to the speed of light. For these reasons we expect the behavior to be controlled by the Landau Fermi liquid fixed point except at extremely low temperatures. Nevertheless it is an interesting exercise to study the crossover to the non-Fermi liquid fixed point.

An expansion in the fine structure constant leads to a modified quasiparticle spectrum $\epsilon(p)$ near the Fermi surface:

$$\epsilon(p) = v_F(|p| - p_F) - a\epsilon^2(v_F/c)^2(|p| - p_F) \ln |p| - p_F|,$$

(124)

here $a$ is a positive constant. It has been conjectured that the logarithmic term exponentiates to give a power law spectrum $\epsilon(p)$.

$$\epsilon(p) = v_F(|p| - p_F)|p| - p_F|^{-a\epsilon v_F/c}.$$

(125)

Here we investigate the fixed point using the formalism developed in previous sections. However, we do not find a power law, rather, a logarithmic spectrum of marginal Fermi liquid type characterizes the fixed point.

We consider the following bare gauge action in the Coulomb gauge:

$$S^0_{\text{g}}[A] = \frac{1}{8\pi(e/c)^2} \int d^3x \, dt \left[ \frac{1}{c^2} \left( \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right],$$

(126)

where the longitudinal interaction is neglected since it is screened. Once the bare gauge action, Eq.(14), is specified we can follow the prescription given in Sec. I to obtain:

$$D_{\mu
u}(q) \, \epsilon^{\mu}(S) \, \epsilon^{\nu}(S) = \frac{|q \times \hat{n}_S|^2}{|q|^2} \frac{4\pi v_F^2 e^2}{4\pi e^2 \chi^T(q) + \omega^2 - c^2 |q|^2} +$$

(127)

where the transverse susceptibility $\chi^T(q)$ was defined in Eq.(20). This result is in agreement with the RPA calculation of the gauge propagator. In $\omega$-limit, $\omega > v_F^* |q|$, the poles of Eq. (127) are at $\omega^2 = 4\pi\rho_F e^2/m + c^2 |q|^2$, the plasma frequency. In the opposite $q$-limit, $\omega < v_F^* |q|$, we obtain:

$$D_{\mu\nu}(q) \, \epsilon^{\mu}(S) \, \epsilon^{\nu}(S) = v_F^2 \frac{|q \times \hat{n}_S|^2}{|q|^2} \frac{1}{i \gamma |q| - \omega} \chi|q|^2,$$

(128)

where now $\gamma = \pi v_F^* N^*(0)/4$ and $\chi = e^2/4\pi e^2$. Inserting this expression into the boson propagator Eq.(22), we obtain the imaginary part of the boson self-energy due to quasiparticle damping. As in the case of the half-filled Landau level, the $q$-limit is the more singular of the two limits, and is the origin of the non-Fermi liquid fixed point. From Eqs.(29, 23, 124), we can now compute the real space fermion Green’s function.
function into a product of the free term $G^0_f(S; x, t) = (x \cdot \hat{n}_S - v_F^* t)^{-1}$ and the correction term $\exp[\delta G_B(S; x, t)]$, we have

$$
\delta G_B(S; x, t) = \int \frac{d^2 q_\perp}{(2\pi)^2} \int \frac{dq_\parallel}{2\pi} \int \frac{d\omega}{2\pi} e^{i(q \cdot x - \omega t)} \left[ D_{\mu\nu}(q, \omega) e^{i(S; q)} e^{i(S; -q)} \right] \omega - v_F^* q \cdot \hat{n}_S + i\eta \text{sgn}(\omega) \right]^2}
$$

$$
i \int \frac{d^2 q_\perp}{(2\pi)^2} \int \frac{d\omega}{2\pi} e^{i\omega \text{sgn}(x_\parallel)} (x_1/v_F^* - t) \frac{|q_\parallel|}{x_\parallel^3 + i\gamma\omega}
$$

$$\approx \frac{4\pi i e^2}{3c^2} |x_\parallel| \int_0^{\lambda\pi/2} \frac{d\omega}{2\pi} e^{i\omega \text{sgn}(x_\parallel)} (x_1/v_F^* - t) \ln \left( \frac{\lambda^2/8\gamma - i\omega}{-i\omega} \right),
$$

(129)

in the second line we use the fact that $|q_\parallel| \gg |q_\parallel|$ in the region of integration where the integrand is the most singular, and perform a contour integration around the pole located at $q_\parallel = \omega/v_F^*$. The frequency integration in the last line of Eq. (129) is identical in form to Eq. (62). Introducing a new momentum scale $\delta = \lambda^2 c^2 / 4e^2 v_F^* k_F^2$ and dimensionless constant $\zeta = 2e^2 v_F^* / 3c^2$, we obtain:

$$
\delta G_B(S; x, t) \approx -\zeta |x_\parallel| \ln |\delta(x_\parallel - v_F^* t)| : |x_\parallel - v_F^* t| \delta \ll 1
$$

$$\approx -\zeta \frac{x_\parallel}{x_\parallel - v_F^* t + 2i\alpha \text{sgn}(x_\parallel)} \left[ \gamma + \ln [(x_\parallel - v_F^* t)\delta \text{sgn}(x_\parallel)] \right] : |x_\parallel - v_F^* t| \delta \gg 1
$$

(130)

As in the case of the half-filled Landau level, the Green’s function has the marginal Fermi liquid form with self-energy given by Eq. (84). A similar result was found in a fermion renormalization group calculation by Chakravarty et. al. However, because $\zeta \approx 10^{-5}$ for a typical metal, the crossover to MFL behavior occurs at an exceedingly low energy scale $\omega$ set by

$$
\zeta \log (E/\omega) = 1
$$

(131)

where $E \equiv (c^2 k_F^2)/(4e^2)$. Including nonlinear terms due to curvature of the Fermi surface in the fermion dispersion and vertex corrections due to the gauge interactions, we may apply the arguments presented in Sec. [V D]. It is straightforward task to show that bosonization is consistent and provides an adequate description of the three dimensional gauge-fermion liquid.

VII. CONCLUSIONS

In multidimensional bosonization, Feynman diagrams are replaced by geometrical and algebraic ways of understanding strongly correlated fermions. The generating functional approach is a convenient way to compute correlation functions of bosons and hence, fermions, interacting via short- and long-range longitudinal, and transverse gauge interactions. In one dimension well-known results for the Luttinger liquid are recovered. Short-range and Coulomb interactions in two or higher spatial dimensions do not destroy the Landau Fermi liquid fixed point; spin-charge separation does not occur. The Kohn anomaly in the density response function near momentum $2k_F$ has the standard nonanalytic form, showing that all low energy processes, even those with large-momentum, are correctly described by multidimensional bosonization. Likewise, the specific heat has the correct linear dependence on temperature with subleading corrections which are easily computed. Collective modes in the charge and spin sectors follow directly from the equations of motion for the abelian $U(1)$ and non-abelian $SU(2)$ currents. The stability of the Landau fixed point against superconducting BCS, CDW, and SDW instabilities also can be ascertained using the renormalization-group in the boson basis. Novel fixed points in higher dimension arise in several cases. These examples demonstrate that bosonization is more encompassing than Landau Fermi liquid theory as non-Fermi liquid fixed points are accessible. For a super-long range longitudinal interaction in two dimensions, a system first studied by Bares and Wen we find an anomalous exponent in the equal-time Green’s function. For transverse gauge interactions we find marginal Fermi liquid (MFL) fixed points in two instances: for the HLR theory of the half-filled Landau level with the Coulomb interaction and for ordinary Maxwell gauge interactions in three dimensions.

We studied the gauge-invariant density response function at the half-filled Landau level fixed point. At low momentum and frequencies we obtain the same result as the RPA calculation, and the response function is indistinguishable from that of a Landau Fermi liquid. Deviations from Fermi liquid behavior appear near momentum transfers of $2k_F$. Since odd-denominator fractional fillings are equivalent, via Jain’s construction to the half-filled Landau level in an additional external magnetic field $\Delta B$, it would be interesting to examine questions of the divergent renormalization of the composite fermion mass $2B$ and the associated compressibility as a function of $\Delta B$ within the bosonization framework.
The intimate connection between multidimensional bosonization and the Ward Identity method of Castellani, Di Castro, and Metzner enables us to go outside the framework of bosonization to check its consistency and accuracy. Neither non-linear terms in the fermion dispersion due to Fermi surface curvature, nor vertex corrections due to gauge fields, change our results in any qualitative way. All of the fixed points we found possess much higher symmetry than the bare interacting problem. This infinite \(U(1)\) symmetry reflects vanishingly small large-angle scattering at such zero temperature quantum critical fixed points. Multidimensional bosonization is a good description of Fermi and nearly-Fermi liquids.

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APPENDIX A: NUMERICAL COMPUTATION OF THE SPECTRAL FUNCTION

Small-angle scattering between the fermion quasiparticles generates a subleading additive correction to the free boson propagator. We use the bosonization formula Eq. (24) to compute the fermion spectral function. Since bosonization is carried out in \((x, t)\) space we must perform three operations. First we inverse Fourier transform the boson propagator into real space and time. Next the exponential of the resulting expression yields the fermion propagator. Finally a second Fourier transform of the fermion propagator back into momentum space allows us to extract the spectral function.

It is a difficult technical problem to carry out these three steps analytically. In this Appendix we resort to the numerical method of fast Fourier transforms (FFTs) to compute the full spectral function. The problem is still challenging since we seek to extract sub-leading corrections to scaling. For example, the imaginary part of the fermion self-energy is of order \((\omega^2/\epsilon_F) \ln|\omega|\) and therefore much smaller than the quasiparticle energy at small frequencies. In fact, for spectral broadening to show up in the numerical calculation, it is necessary to set \(\lambda = O(k_F)\). As we shall see, setting \(\lambda = O(k_F)\) forces us to treat velocity renormalization carefully.

As a illustrative example of the numerical method, we consider the problem of a two-dimensional spinless Fermi liquid with only a single non-zero Landau parameter, \(f_0\). In this case we keep only the temporal component of the gauge fields appearing in Eq. (25) and set \(K_{00}(q) = 1/f_0\). Then the gauge propagator appearing in Eq. (24) is given by:

\[
D_{\mu\nu}(q, \omega) \epsilon^\mu(S; q) \epsilon^\nu(S; -q) = \frac{f_0}{1 + f_0 \chi^0(q)}.
\]  

(A1)

To continue our analysis, we now set \(x_\perp = 0\) in Eq. (24) as the spectral function has only weak and uninteresting dependence on transverse momentum. With this simplification, the integral over \(q_\perp\) in Eq. (24) may be performed analytically, yielding a rather complicated expression which, for the sake of brevity, we do not present here. Instead we denote the integral by:

\[
D(q_\parallel, \omega) \equiv \int_{-\lambda/2}^{\lambda/2} \frac{dq_\perp}{2\pi} \frac{f_0}{1 + f_0 \chi^0(q)}
\]  

(A2)

which, multiplied by the double pole \([\omega - v_F^* q_\parallel + i\eta \text{sgn}(\omega)]^{-2}\), is the quantity to be Fourier transformed into \((x_\parallel, t)\) space. Here we distinguish between \(v_F^*\), which is the velocity of the non-interacting bosons, and \(v_F^*\), which is the final velocity of the bosons (and fermions) after the effect of the current-current interaction \(f_0\) on the velocity has been accounted for.

For Landau Fermi liquids, the fermion velocity renormalizes from its free value when high-energy processes, those with energy greater than the scale set by the cutoff \(\lambda\), are integrated out. As long as \(\lambda \ll k_F\), further renormalization of the velocity due to the remaining low-energy processes, the ones to be bosonized, is insignificant. Velocity renormalization cannot, however, be neglected when \(\lambda\) is comparable to the Fermi momentum. To incorporate this renormalization directly into the Green’s function, from \(D(q_\parallel, \omega)\) we subtract its value at \(q_\parallel = \omega = 0, D(0, 0)\). Setting
\[ F_0 \equiv f_0 N'(0), \text{ where } N'(0) = k_F/2\pi v_F', \] 
we find \[ D(0,0) = \frac{v_F'}{k_F} \frac{F_0}{1 + F_0}. \] 
The final renormalized Fermi velocity is then given by: 

\[ v'_{F} = (1 + \frac{\lambda}{2\pi k_F} \frac{F_0}{1 + F_0}) v'_F. \]  

(A3)

The difference \( D(q||, \omega) - D(0,0) \), which is small for small frequencies and momenta, now appears in the numerator of the remaining integral over \( \omega \) and \( q|| \) in Eq. (24): 

\[ G_F(S; x||, t) = \frac{e^{ik_F x_l}}{x_l - v_F t} \exp \left\{ i \int \frac{dq||}{2\pi} \int d\omega \frac{[e^{i(q|| \cdot \omega t)} - 1]}{[\omega - v'_F q|| + i\eta \sgn(\omega)]^2} \right\}. \]  

(A4)

These two integrals may now be performed numerically by a two-dimensional (inverse) FFT. In practice it is simplest to study either the advanced or retarded Green's function rather than the full time-ordered Green's function; hence we restrict the FFT to the half plane \( t > 0 \). Upon subtracting the \( x_l = t = 0 \) value, exponentiating the difference, and multiplying by the free Green's function, the full fermion Green's function is obtained in real space and time. This result may then be converted back to \( (k, \omega) \) space by the second FFT. The spectral function is then extracted by taking the imaginary part. Displayed in Fig. 3 are cross-sectional plots of the spectral function at different frequencies computed on a \( 2048 \times 2048 \) lattice. The peak has been centered in each plot. The width of the peak at zero frequency is due solely to finite-size effects. As the frequency increases from zero, the width grows rapidly as expected. The nonperturbative results demonstrate that low-order perturbative expansions in the Landau parameters are qualitatively correct as there is no breakdown in Landau Fermi liquid behavior.

**APPENDIX B: GENERAL LANDAU PARAMETERS IN TWO AND THREE DIMENSIONS**

It is straightforward to include higher order Landau parameters in the computation of the 2N-point boson correlation functions. Extension to three spatial dimensions is straightforward. Finally, Landau parameters in the spin sector may be incorporated in a perturbative way with the use of Abelian bosonization. In two dimensions the boson propagator for a single \( f_n \) interaction in the charge sector is given by:

\[ \langle a(S; q) a^\dagger(S; q) \rangle = \int \frac{i}{\omega - v'_F q \cdot \hat{n}_S + i\eta \sgn(\omega)} \right\}
\]

where

\[ D_i^{(n)}(q) = \begin{cases} 1, & -1/f_n + \chi^{(n)}_i(q) \end{cases}, \]

(B2)

\[ D_i^{(n)}(q) = \begin{cases} 1, & -1/f_n + \chi^{(n)}_i(q) \end{cases}, \]

(B3)

and

\[ \chi^{(n)}_i(q) = N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta \cos^2(n \theta)}{x - \cos \theta + i\eta \sgn(x)} \]

\[ \chi^{(n)}_i(q) = N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta \sin^2(n \theta)}{x - \cos \theta + i\eta \sgn(x)} \].

(B4)

The function \( \beta(S; q) \) in Eq. (B1) denotes the angle between \( S \) and \( q \). In three dimension, the inclusion of \( l \)th Landau interaction can be performed using the addition theorem. The boson propagator is then:

\[ \langle a(S; q) a^\dagger(S; q) \rangle = \int \frac{i}{\omega - v'_F q \cdot \hat{n}_S + i\eta \sgn(\omega)} \right\}
\]

\[ + \frac{2\Lambda^2}{(2\pi)^3} \frac{q \cdot \hat{n}_S}{[\omega - v'_F q \cdot \hat{n}_S + i\eta \sgn(\omega)]^2} \sum_m \frac{Y^m_l(\Omega(S; q)) Y^m_l(\Omega(S; q))}{(\omega_F^m - \chi^{(n)}_i(q))} \]

(B5)
Appendix C for the case of the Chern-Simons gauge interaction plus the Fermi Liquid parameter then treat the gauge fields and the mediating fields on the same footing. An illustrative calculation is presented in with the Fermi Liquid interactions, we first construct a gauge-covariant form for the Fermi liquid interaction and interaction. The propagators of the mediating fields are then matrices. Upon the inclusion of gauge interactions along

\[ (2l + 1) f_l \text{P}_l(\cos \theta) = 4\pi f_l \sum_{m=-l}^{l} (-1)^m Y_l^m(\Omega) Y_l^{-m}(\Omega) \]  

and introduce a mediating field \( A^m_l(q) \) for each spherical harmonic, amounting to \( 2l + 1 \) fields for each Landau interaction. The propagators of the mediating fields are then matrices. Upon the inclusion of gauge interactions along with the Fermi Liquid interactions, we first construct a gauge-covariant form for the Fermi liquid interaction and then treat the gauge fields and the mediating fields on the same footing. An illustrative calculation is presented in Appendix C for the case of the Chern-Simons gauge interaction plus the Fermi Liquid parameter \( f_1 \).

**APPENDIX C: HOW KOHN’S THEOREM IS SATISFIED**

In this appendix, we prove the result quoted in Sec. [11] that the collective excitations of the interacting quasi-particles of the half-filled Landau level occur at a frequency \( \omega = 2\pi\rho_F \hat{\phi}/m \), the cyclotron frequency of a particle with bare mass \( m \) in a magnetic field \( B = 2\pi\rho_F c\hat{c}/e \). To prove this, we add to the action of the Chern-Simons gauge theory a Fermi liquid interaction in which all the coefficients except \( f_1 \) have been set equal to zero. The contribution to the action Eq. (B6) is given in real space and time by

\[ S_{FL} = \frac{f_1}{2V k_F^2} \int d^2x \, dt \, \psi^\dagger \nabla \psi \nabla \psi \]  

which is made gauge covariant by replacing \( \nabla \) by \( \nabla - iA \). On Fourier transforming it is given in terms of the currents as

\[ S_{FL} \approx -\frac{f_1}{2V k_F^2} \int \frac{d\omega}{2\pi} \sum_{S,T} J(S; q) \, k_S \cdot k_T \, J(T; -q) \]

\[ + \frac{f_1}{2V k_F^2} \sum_{S,q} \rho_f \left[ J(S; q) \, k_S \cdot A(-q) + J(S; -q) \, k_S \cdot A(q) \right] - \frac{f_1}{2V k_F^2} \sum_{q} \rho_f^2 \, A(q) \cdot A(-q) \]  

As in Sec. [11] the term quadratic in \( J \) is decoupled by introducing additional transverse \( A_t \) and longitudinal \( A_l \) fictitious fields.

Including the Fermi liquid interaction the complete action of the theory is given by

\[ S[A^\mu, a, \xi, \xi^*] = \sum_{S} \sum_{q,\tilde{q},m>0} \int \frac{d\omega}{2\pi} \left\{ (\omega - v_F^2 \, q \cdot \tilde{q} - \mathbf{\hat{n}}_S) \, a^\dagger(S; q) \, a(S; q) + \xi(S; q) \, a^\dagger(S; q) + \xi^*(S; q) \, a(S; q) \right\} \]

\[ + \frac{1}{V} \sum_{S,q} \int \frac{d\omega}{2\pi} J(S; q) \left\{ A_0(-q) + \left( v_F^2 + \frac{f_1 \rho_f}{k_F} \right) \mathbf{\hat{n}}_S \times \frac{q}{|q|} \right\} \]

\[ + \frac{\mathbf{\hat{n}}_S \cdot \mathbf{A}_l(-q)}{|q|} \]  

\[ + \frac{1}{V} \sum_{q} \int \frac{d\omega}{2\pi} \left\{ \left( \frac{\rho_f^2}{k_F^2} + \frac{q^2 \, V(q)}{(2\pi\phi)^2} \right) A_T(q) \, A_T(-q) + \frac{2i|q|}{2\pi\phi} \mathbf{A}_0(q) \, A_T(-q) \right\} \]  

\[ - \frac{1}{f_1} \mathbf{A}_l(q) \, A_l(-q) \]  

where \( \mu \) runs over 0, \( T, l \) and \( t \) and we couple the boson fields to \( \xi, \xi^* \) to construct a generating functional as in Eq. (B6). After integrating out the boson fields \( a, a^\dagger \), we obtain an effective action:

\[ 27 \]
\[ S_{\text{eff}}[A^\alpha, \xi, \xi^*] = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d\omega}{2\pi} \left\{ \frac{\rho_f}{m_F} + \frac{f_1 \rho_f}{k_F} + \frac{q^2 V(q)}{(2\pi)^2} + (v_F^* + \frac{f_1 \rho_f}{k_F})^2 \chi_i(q) \right\} A_T(q) A_T(-q) + \frac{2i|q|}{2\pi\phi} A_0(q) A_T(-q) - \frac{1}{f_1} + \chi_i(q) \right\} A_T(q) A_T(-q) + \frac{\chi_0(q)}{A_0(q) A_T(-q)} \right\} - \frac{1}{V} \sum_{q,q_1,q_2} \int \frac{d\omega}{2\pi} \sum_S \left\{ \frac{\chi(S;q) \sqrt{q \cdot \hat{n}_S}}{(\omega - v_F^* q \cdot \hat{n}_S)} A_0(q) \epsilon^\alpha(S;q) + \frac{\xi^*(S;q) \chi(S;q)}{\omega - v_F^* q \cdot \hat{n}_S} \right\}, \quad \text{(C4)} \]

here

\[ \epsilon^\alpha(S;q) = (1, [v_F^* + \frac{f_1 \rho_f}{k_F}], \hat{n}_S \times q, \hat{n}_S \cdot q, \hat{n}_S \times q) \quad \text{(C5)} \]

and the susceptibilities \( \chi_\alpha(q) \) are defined by:

\[ \chi_0(q) = -N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta}{x - \cos \theta + i\eta \text{sgn}(x)} = N^*(0) \left[ 1 - \theta(x^2 - 1) \frac{|x|}{\sqrt{x^2 - 1}} + i\theta(1 - x^2) \frac{|x|}{\sqrt{1 - x^2}} \right] \]
\[ \chi_i(q) = N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta}{x - \cos \theta + i\eta \text{sgn}(x)} = N^*(0) \left[ x^2 - \frac{1}{2} - \theta(x^2 - 1)|x| \sqrt{x^2 - 1} - i\theta(1 - x^2)|x| \sqrt{1 - x^2} \right] \]
\[ \chi_i(q) = N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta}{x - \cos \theta + i\eta \text{sgn}(x)} = N^*(0) \left[ -x^2 - \frac{1}{2} + \theta(x^2 - 1)|x| \sqrt{x^2 - 1} - i\theta(1 - x^2)|x| \sqrt{1 - x^2} \right] \]
\[ \chi_0(q) = -\chi_0(-q) = N^*(0) \int \frac{d\theta}{2\pi} \frac{\cos \theta}{x - \cos \theta + i\eta \text{sgn}(x)} = N^*(0) \left[ -x + \theta(x^2 - 1)|x| \frac{x}{\sqrt{x^2 - 1}} - i\theta(1 - x^2)|x| \frac{x}{\sqrt{1 - x^2}} \right], \quad \text{(C6)} \]

and \( x = \omega/(v_F^* |q|) \). The boson propagator is obtained in the usual way

\[ \langle a(S;q) \rangle_{\text{A}(S;q)} = \frac{i}{\omega - v_F^* q \cdot \hat{n}_S + i\eta \text{sgn}(\omega)} + i \frac{\Lambda}{(2\pi)^2} q \cdot \hat{n}_S D_{\alpha\beta}(q) \epsilon^\alpha(S;q) \epsilon^\beta(S;-q) \quad \text{(C7)} \]

and the collective modes are given by the poles of the gauge propagator. The inverse gauge propagator can be read off from the action Eq. (C4) and in the limit of interest \( \omega \gg v_F^* |q| \) is given by

\[ [K(q)]_{\alpha\beta} = \begin{pmatrix} N^*(0) \frac{2\pi}{x^2} & \frac{i|q|}{2\pi\phi} & 0 & 0 \\ \frac{i|q|}{2\pi\phi} & \frac{N^*(0)}{x^2} & 0 & 0 \\ 0 & 0 & -\frac{N^*(0)}{x^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{f_1} \end{pmatrix} \quad \text{(C8)} \]

The poles of \( D_{\alpha\beta} \) are found by setting \( \det[K_{\alpha\beta}] = 0 \) which gives \( \omega^2 = [2\pi\phi \rho_f (1/m_F \pm f_1/4\pi)]^2 = (2\pi\phi^2 \rho_f/m_0)^2 \). The second equality follows from the application of Galilean invariance to a Fermi liquid. The collective mode appears at the cyclotron frequency determined by the bare electron band mass in agreement with Kohn’s theorem.
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A copy of the “C”-code executing these operations is available from the authors.

FIG. 1. The first of two 1/N vertex correction, $\Lambda^{(1)}(p; q)$. The symbol X indicates that external lines are to be amputated.

FIG. 2. The two diagrams which give the second 1/N vertex corrections, $\Lambda^{(2)}(p; q)$. The leading divergences in these two diagrams cancel via Furry’s theorem (see text) as only the nonlinear terms in the fermion dispersion break charge-conjugation symmetry.

FIG. 3. Spectral function $\text{Im}G(k, \omega)$ in two dimensions for the case $F_0 = 2$. Slices at three different frequencies $\omega = 0, k_F/8$, and $k_F/4$ are shown respectively in (a), (b) and (c). Here $\lambda = k_F$, $v_F^* = 1$, and a lattice of 2048 $\times$ 2048 points is used to compute the Fourier transforms. The peak has been centered in each plot around the point $k = \omega$. 
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Figure 1
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Figure 3(a)
Figure 3(b)
Figure 3(c)
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