Mass-imbalanced fermionic mixture in a harmonic trap

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The mass-imbalanced fermionic mixture is studied, where $N \leq 5$ identical fermions interact resonantly with an impurity, a distinguishable atom. The shell structure is explored, and the physics of a dynamic light-impurity is shown to be different from that of the static heavy-impurity case. The energies in a harmonic trap at unitarity are calculated and extrapolated to the zero-range limit. In doing so, the scale factor of the ground state, as well as of a few excited states, is calculated. In the $2 \leq N \leq 4$ systems, pure $(N + 1)$ Efimov states exist for large enough mass ratio. However, no sign for a six-body Efimov state in the $(5 + 1)$ system is found in the mass ratio explored, $M/m \leq 12$.

I. INTRODUCTION

The system of $N$ identical fermions interacting resonantly with a distinguishable impurity exhibits a rich and interesting physics, including universal phenomena and the celebrated Efimov physics. For a recent review see, e.g., Ref. [1].

An important parameter here is the ratio of the impurity mass $m$ and the identical fermions mass $M$. In the ultracold limit the interaction between identical fermions can be neglected, and therefore in the heavy impurity case $m \gg M$ the problem is decoupled to $N$ independent fermions interacting with a static impurity. The opposite limit, where $m \ll M$, corresponds to a dynamic impurity which induces interaction between the identical fermions.

The simplest non trivial example is the $(2 + 1)$ system, composed of two identical fermions of mass $M$ and a distinguishable atom of mass $m$, where different particles have zero-range resonant interaction while identical particles do not interact. Efimov has shown that when the mass ratio $\alpha = M/m$ is larger than the critical value $\alpha_c = 13.607$, an infinite tower of trimers with angular momentum and parity $L^\pi = 1^-$ is produced [2]. The $n$-th trimer energy is $E_n = E_0 e^{-2\pi n/|\alpha|}$, where $E_0$ is the trimer ground-state energy. The scale factor $s = s(\alpha)$ is a function of the mass ratio and vanishes at the Efimov threshold $s(\alpha_c) = 0$.

In the non-Efimovian regime $\alpha < \alpha_c$ the scale factor characterizes the short-distance (and large momenta) behavior of a universal trimer, which exists for $8.173 < \alpha < \alpha_c$ for finite positive scattering length [3].

The physical interpretation of the scale factor can be understood from the adiabatic hyperspherical formalism [4]. To see that, one rearranges the relative coordinates into the hyperradius $\rho$, the only coordinate with a dimension of length, and $3N - 1$ hyperangles. Here

$\rho \propto \sqrt{mr^2 + M \sum_{i=1}^{N} R_i^2}$

where $\mathbf{r}$ ($\mathbf{R}_i$) is the position of the distinguishable (identical) atom in the center-of-mass frame. At small $\rho$, where $E$ and $1/a$ can be neglected, the hyperradial motion separates from hyperangular degrees of freedom and is governed by

$$\left[ -\frac{\partial^2}{\partial \rho^2} - \frac{3N-1}{\rho} \frac{\partial}{\partial \rho} + \frac{s^2 - (3N/2 - 1)^2}{\rho^2} \right] \Psi(\rho) = 0,$$

where $s^2$ is the hyperangular eigenvalue. The general solution of Eq. $[1]$ is a linear combination of $\Psi_+(\rho) \propto \rho^{-3N/2 + 1 + s}$ and $\Psi_-(\rho) \propto \rho^{-3N/2 + 1 - s}$. The case $s^2 < 0$ ($s = is_0$) corresponds to the Efimovian regime, where this linear combination is an oscillating function, and a three-body parameter is required to fix the relative phase of $\Psi_+$ and $\Psi_-$. The non-Efimovian regime appears for $s^2 > 0$ ($s > 0$) where, far from few-body resonances, $\Psi(\rho)$ is dominated by $\Psi_+(\rho)$.

Interestingly, the same factor determines the energy of the trapped system at unitarity [5, 6], namely

$$E = \hbar \omega (s + 2n + 1),$$

where $\omega$ is the trapping frequency, taken to be identical for all particles, $n$ is a non-negative integer and the center-of-mass zero-point energy is omitted. This is because the trapping potential is involved only in the hyperradial equation, while $s$ is determined by the hyperangular equation which is identical in free space and in a trap. For a recent review of the trapped few-body problem, see Ref. [7].

Following Efimov, the mass-imbalanced $(2+1)$ system has attracted wide attention (see, e.g., Refs. [2, 3, 8–21]). The scale factor of the $(2+1)$ system was first calculated for the equal-mass case to be $s(1) = 1.7727$ for the $1^-$ ground state and $s(1) = 2.1662$ for the $0^+$ excited state [10]. Later, the method was generalized to include any angular momentum and mass ratio [13]. The $1^-$ trimer energy crosses the dimer+atom energy in a trap at $\alpha = 8.6186$ [19]. An ultracold mixture of $^6$Li and $^{40}$K ($\alpha \approx 6.4$) was realized experimentally, and a strong atom-dimer attraction was observed. This attraction was interpreted as $p$-wave interaction between two heavy particles induced by the light atom [22].

The trend of moving from a non-Efimovian universal state to an Efimovian state with the same symmetry as the mass ratio increases was discovered also in the $(3+1)$ and $(4+1)$ systems [23, 24]. The mass-imbalanced $(3+1)$ system has been the subject of a few studies [21, 23, 26]. Here a tower of $1^+$
Efimovian tetramers exists above $\alpha_c = 13.384$ [24], and a universal non-Efimovian $1^+$ tetramer is bound in free space for $8.862 < \alpha < \alpha_c$ [24] [25]. The scale factor of the tetramer ground state has been calculated for a few mass ratios [26], while that of excited states is known only for the equal-mass case [27]. The tetramer energy crosses the trimer+atom energy in a trap at $\alpha = 8.918$ [23].

The mass-imbalanced $(4 + 1)$ system was studied in Refs. [25] [26]. A tower of $0^-$ Efimovian pentamers exists above $\alpha_c = 13.279$, while a universal $0^-$ pentamer is bound in free space for $9.672 < \alpha < \alpha_c$ [25]. Here the scale factor is known for equal mass [27], when the pentamer is bound in free space [25] and for few other mass ratios [26]. The pentamer energy crosses the tetramer+atom energy in a trap at $\alpha = 9.41$ [25].

The ground-state properties of the $(N + 1)$ systems are summarized in Table I.

| System | $L^*$ | Free crossing | Trap crossing | Efimov |
|--------|-------|--------------|--------------|--------|
| $2+1$ | $1^-$ | 8.173        | 8.619        | 13.607 |
| $3+1$ | $1^+$ | 8.862        | 8.918        | 13.384 |
| $4+1$ | $0^-$ | 9.672        | 9.41         | 13.279 |
| $5+1$ | $0^-$ |              |              |        |

Very little is known about the $(5 + 1)$ system. A simplified model explains the similar trends in the $(2 + 1)$, $(3 + 1)$, and $(4 + 1)$ systems as populating a $p$ shell atom by atom. The $(5 + 1)$ system, therefore, should be different, since the $p$ shell is now full and the additional atom has to open a new shell [25]. Intriguing open questions are thus the following: is there a non-Efimovian universal bound hexamer and does the six-body Efimov effect exist?

The extrapolation toward the case of fermionic polaron, corresponding to the $N \gg 1$ case, is of special interest. As a step in this direction the shell structure of the few-body systems is studied here. In contrast to the static heavy-impurity case, it is shown that non perturbative physics arise in the dynamic light-impurity case.

The goal of this work is to study the scale factor, or equivalently the energy in a trap, of the $(N + 1)$ ($N \leq 5$) fermionic mixtures few lowest states, and to identify their properties. Calculation are done for a wide range of mass ratios, from the static-impurity limit $m \gg M$ to the dynamic-impurity limit $m \ll M$.

A convenient way to describe the system is the Skorniakov and Ter-Martirosian (STM) integral equation [28] [29], which deals directly with zero-range interaction by applying the Bethe-Peierls boundary condition when two different particles approach each other. One has to solve an integral equation in $3(N-1)$ dimensions, but utilizing the system symmetries the number of dimensions can be reduced further.

For $N = 2$, the STM equation for the scale factor is reduced to a transcendental equation which can be easily solved. For $N = 3$, it can be reduced to two dimensions, allowing the solution on a grid [23]. For $N = 4$, however, a five-dimensional equation makes a grid-based approach challenging if possible at all. A method based on a Monte-Carlo process to solve the STM equation was developed for this case in Ref. [25]. However, this method is limited to bound systems and therefore cannot be used to calculate the scale factor for all mass ratios. In addition, as a fermionic Monte-Carlo method it might suffer from a sign problem if the wave function has radial nodes.

Thus we take here another approach. We solve the Schrödinger equation for the trapped system with finite-range interspecies potential and then extrapolate to the zero-range limit. A similar method was applied in Refs. [26] [27].

Using this method we calculate the scale factor for $0 \leq \alpha \leq 12$ for the ground state, as well as for a few lowest excited states, of the $(N + 1)$ fermionic system up to $N \leq 5$. We set a simple model to understand the shell structure for the static-impurity case, and explore the effects of the dynamic impurity as the mass ratio increases.

We find that no $(5 + 1)$ Efimov states exist for $\alpha \leq 12$. As the mass ratio increases, finite-range corrections become significant and the extrapolation to the zero-range limit cannot be trusted anymore. A further study is therefore needed to explore such states for larger mass ratios, $12 < \alpha < 13.279$.

## II. METHODS

As we have explained, the zero-range limit is not directly used here; instead, a series of calculations with a finite-range potential with decreasing range is used to extrapolate the zero-range limit.

The Hamiltonian of the $(N + 1)$ system is

$$H = T + U + V,$$

where $T$ is the internal kinetic energy and $U$ is the confining harmonic potential. Here, $V$ is the interspecies attractive interaction, taken of the form

$$V = -V_0 \sum_{i=1}^{N} \exp \left( -\frac{(r - R_i)^2}{2R_0^2} \right),$$

where $V_0 > 0$ is the potential strength and $R_0$ is its range. We seek the limit of $R_0 \rightarrow 0$ while $V_0$ is tuned to keep the two-body system at unitarity.

To solve the few-body problem, we use the stochastic variational method (SVM) [30]. The wave function is expanded in an over-complete basis of correlated Gaussians, where the basis functions are chosen in a stochastic...
way utilizing the variational principle. The energies and the corresponding wave functions can be found then by solving a generalized eigenvalue problem.

The basis functions are chosen to have the necessary permutational symmetry, parity $\pi$, and angular momentum $L$ and its projection $M$,

$$\phi_{LM}^\pi(A, u; \eta) = \mathcal{A} e^{-\frac{i}{2} A \eta} \theta_{LM}^\pi(u; \eta)$$

(5)

where $\eta = \{ \eta_1, \ldots, \eta_N \}$ is a set of $N$ Jacobi coordinates, $\mathcal{A}$ is the appropriate anti-symmetrization operator, $A$ is an $N \times N$ real, symmetric, and positive definite matrix, and $\theta_{LM}^\pi(u; \eta)$ is the angular part. The $N(N+1)/2$ real numbers defining $A$ are optimized in a stochastic way such as the energy is minimized. Spin and isospin functions can be introduced but are not needed here.

The angular part is characterized by the global vector representation $\mathbf{Y}$. For a natural parity $\pi = (-1)^L$ it is

$$\theta_{LM}^\pi(u; \eta) = \mathbf{Y}_{LM}(\mathbf{v}),$$

(6)

where $\mathbf{Y}_{LM}$ is the regular solid harmonic and $\mathbf{v} = u^T \eta$ is a global vector, whose elements are also optimized in a stochastic way.

To get the unnatural parity $\pi = (-1)^{L+1}$ for $L > 0$ one has to couple two global vectors,

$$\theta_{LM}^\pi(u; \eta) = [\mathbf{Y}_{LM}(\mathbf{v}_1) \otimes \mathbf{Y}_{LM}(\mathbf{v}_2)],$$

(7)

while three global vectors are needed to get the $0^-$ symmetry,

$$\theta_{00}^\alpha(u; \eta) = [\mathbf{Y}_L(\mathbf{v}_1) \otimes \mathbf{Y}_L(\mathbf{v}_2) \otimes \mathbf{Y}_L(\mathbf{v}_3)]_{00}.$$  

(8)

The overlap of such basis functions, as well as the matrix elements of the Hamiltonian, are known analytically $[27, 30, 33]$.

For a given number of particles, angular momentum, and parity, the ground-state energy is calculated for various potential ranges. From these energies, the zero-range limit is extrapolated.

Typical results for the $(2+1)$ ground state are shown in Fig. 4, where results calculated from finite-range potentials are compared to the zero-range results. The radius of convergence for the extrapolation is shown to be much larger for $\alpha = 4$ than for $\alpha = 12$. In the latter case, close to the Efimovian limit, the extrapolated value will be completely off if one uses, say, results with $R_0 > 0.03 \sqrt{\hbar/c \omega}$.

To estimate the extrapolation uncertainty, we fit the results with a few shortest $R_0$ with linear and parabolic curves and account for their differences. The error due to the finite basis set becomes significant for $N > 3$ and is also considered.

Taking the potential range to be smaller, the numerical calculation becomes harder. Therefore close to the Efimovian limit, where finite-range corrections become significant, the extrapolations can not be trusted anymore. To correctly treat this region one should use a method dealing with the zero-range limit directly. For example, one would like to solve the STM equation using a diffusion Monte-Carlo (DMC)-like approach $[25]$. This task is left for future work.

### III. RESULTS

#### A. The $\alpha = 0$ limit

We start to analyze the $\alpha = 0$ limit, where the impurity is infinitely heavy and therefore static. This case reduces to the problem of $N$ trapped fermions scattering on a zero-range potential at the trap center. The analytic solution for the two-body problem is known $[34]$, giving at unitarity an energy shift of $-\hbar \omega$ for the $s$ shell with respect to the non interacting case. The quantum numbers characterizing a shell are the radial number $n$ and the angular momentum $l$ and its projection; its energy is given by

$$E_{nl} = \hbar \omega (2n + l - \delta_{l,0} + 3/2)$$  

(9)

and the energy of the $(N+1)$ system is just a sum of $N$ single-particle energies. To ease comparison between

![Figure 1: Convergence of finite-range potentials toward the zero-range limit $R_0 \rightarrow 0$ for the $(2+1)$ ground state. (a) $\alpha = 4$, away from the Efimovian limit. (b) $\alpha = 12$, near the Efimovian limit. The zero-range result (red square) is the exact solution of Eq. (11).](image-url)
clusters with different particle numbers, the zero-point energy $\hbar\omega \cdot 3N/2$ is subtracted. Energy is measured in units of $\hbar\omega$ and with respect to the dimer energy, i.e.,

$$\epsilon = E/\hbar\omega - 3N/2 + 1. \quad (10)$$

Only interacting states, i.e., those states which have an atom in an s shell, are considered.

Applying the fermionic symmetry, the spectrum and properties of the $(N+1)$ systems can be calculated. Table [II] summarizes the ground-state properties of the $(N+1)$ systems. For completeness, the properties of the two lowest excited states are also tabulated in the Appendix. Here and thereafter we ignore the trivial $2L+1$ degeneracy due to different total angular momentum projections.

**B. The (2+1) case**

We move now to the general mass-imbalanced case and start with two identical fermions interacting with a distinguishable atom.

For the natural parity case, the scale factor $s$ corresponding to a total angular momentum $L$ is the solution of a transcendental equation,

$$\frac{2}{\Gamma(a - 1/2)\Gamma(b - 1/2)} \frac{(\gamma)^L}{\sqrt{\pi\Gamma(c)}} \, _2F_1(a, b; c; \gamma^2) = 0 \quad (11)$$

where $a = 1 + (L - s)/2$, $b = 1 + (L + s)/2$, $c = L + 3/2$, $\, _2F_1$ is the hypergeometric function, and $\gamma = \alpha/(\alpha + 1)$ [13].

Unnatural parity means here that both identical fermions are excited to $l > 0$ shell, resulting in a non-interacting case that will be ignored here.

For $\alpha = 0$ the ground state has two degenerate states, $1^-$ and $0^+$, where in the first case the additional atom populates a $p$ shell while in the latter it sits in an excited $s$ shell. The energy degeneracy is lifted for $\alpha > 0$, where the dynamic impurity induces interaction between the identical fermions, which is attractive (repulsive) for an odd (even) angular momentum. Hence, the $1^-$ state becomes the ground state.

This behavior can be understood in the Born-Oppenheimer (BO) approximation, which holds for $\alpha \gg 1$ [8]. Utilizing the mass difference, the distance between heavy particles $R = R_1 - R_2$ can be treated as a parameter in the light-particle equation, which becomes simply the double-well potential problem, with the known eigenvalues $\epsilon_{\pm}(R)$. In the heavy-particle equation, $\epsilon_{\pm}(R)$ has the meaning of an effective potential and is attractive or repulsive, depending on the parity. Applying the fermionic symmetry for heavy particles’ permutation, the effective potential for odd-$L$ states is found to be attractive and goes like $-1/mR^2$ for $R \ll a$, while the effective potential for even-$L$ states is repulsive.

For the attractive channel, the mass ratio governs the competition between the centrifugal barrier $\propto L(L + 1)/MR^2$ and the effective attraction. Increasing $\alpha$ tips the scales in favor of the attraction; hence the trimer energy decreases. In a trap the trimer energy crosses the dimer+atom energy ($\epsilon = 0$ in our conventions) for $\alpha$ slightly larger than needed in free space. Increasing $\alpha$ further the effective interaction becomes purely attractive and the system becomes Efimovian. In the $(2+1)$ system, the $1^-$ symmetry is the only symmetry where this phenomenon occurs.

To benchmark our method, we calculate the unitary $(2+1)$ trapped system energy by extrapolating finite-range results to the zero-range limit. The scale factor can be easily calculated from Eq. (11) and is connected to the energy in a trap by Eq. (2), giving here (for $n = 0$) $s = \epsilon + 1$. Hence, the Efimovian limit $s = 0$ corresponds here to $\epsilon = -1$. Our results are plotted in Fig. 2, showing a nice agreement with the solutions of Eq. (11). The limit of $\alpha = 0$ from Tables [II], [VII] and [VIII] is also reproduced.

Note that in a trap, each solution of Eq. (11) starts a ladder of solutions, corresponding to hyperradial excitations and giving an additional $2\hbar\omega$ for each hyperradial node. The first excited state of the $1^-$ symmetry is also shown in Fig. 2.

**C. The (3+1) case**

We now add another identical particle and move to the $(3+1)$ system. For $\alpha = 0$, the ground state has two degenerate states, $1^+$ and $1^-$, both with $\epsilon = 2$. These states have different atomic configurations: while in the $1^-$ state the additional atom sits in a $p$ shell, the $1^+$ state corresponds to atom-trimer $s$-wave scattering. $d$-wave atom-trimer scattering states, corresponding to $1^-$, $2^-$, and $3^-$ symmetries, have higher energy in this limit, $\epsilon = 3$.
The energy degeneracy is lifted for $\alpha > 0$, where the $1^+$ state energy becomes lower than the $1^-$ state energy, in qualitative agreement with the BO picture where the interaction induced by the impurity is attractive in a $p$ wave and repulsive in an $s$ wave.

For a larger mass ratio, the $1^+$ state becomes bound in free space, then crosses the trimer+atom threshold in a trap, and eventually reaches the Efimov threshold, corresponding here to $\epsilon = -2.5$. States of other symmetries, nevertheless, does not reach the Efimov limit for any mass ratio smaller than the $(2 + 1)$ Efimov threshold.

The $1^+$ ground-state scale factor has been calculated in Ref. 29 using a grid-based method, similar to that of Ref. 23. That method is more accurate than our current method and can be used up to, and even beyond, the Efimov limit. For a benchmark, we compare in Fig. 3 the results of both methods, which are in nice agreement. The $\alpha = 0$ limit from Table III is also reproduced. For this symmetry the calculations for $\alpha > 10$ become sensitive, signaling a non universal resonance, identified in Ref. 26 to occur at $\alpha = 10.4(2)$ for a Gaussian interaction.

The scale factor of the $1^-$ lowest excited state has been calculated for an equal-mass system only. Our results are tabulated in Table III and shown in Fig. 3 agreeing well with the $\alpha = 0$ limit and with the $\alpha = 1$ result of Ref. 27.

The bending in the $1^-$ energy around $\alpha = 2$ is to be understood as level repulsion with an excited $1^-$ state. To make this point clear, the energies of a few lowest $1^-$ states are shown in Fig. 3. The atomic configurations for $\alpha = 0$ are the following. The state with $\epsilon = 2$ corresponds to the configuration $0s0p1s$, i.e. an atom-trimer $s$-wave state, while for $\epsilon = 3$ it is $0s0p0d$, i.e. an atom-trimer $d$-wave state. A clear avoided crossing between these states is seen around $\alpha = 2$.

Note, however, that the crossing of levels with different quantum numbers is allowed. States with different hyperradial quantum number $n$ can therefore cross, and are also shown in Fig. 4.

The next state, with $3^-$ symmetry, is also shown in Fig. 3. It moves closer to the $1^-$ state as the mass ratio increases. Since the lowest $1^-$ for large $\alpha$ is dominated by a $d$-wave atom-trimer state, like the $3^-$ state, this similarity makes sense. As we show later, this phenomena also exists, and is even stronger, for larger $N$.

Table III: The energies of the trapped tetramer lowest $1^-$ state.

| $M/m$ | This work | Ref. 27 | $M/m$ | This work |
|-------|-----------|---------|-------|-----------|
| 0     | 2         | 1.613(1)| 6     | 1.428(1)  |
| 1     | 2.183(2)  | 2.077(4)| 7     | 1.428(1)  |
| 2     | 2.433(2)  | 2.723(4)| 8     | 1.232(1)  |
| 3     | 2.115(2)  | 2.723(4)| 9     | 1.232(1)  |
| 4     | 1.959(1)  | 1.024(1)| 10    | 0.805(2)  |
| 5     | 1.791(1)  | 0.569(3)|       |           |

D. The $(4+1)$ case

Adding another identical particle, we now consider the $(4 + 1)$ system.

For $\alpha = 0$, two states are degenerate at $\epsilon = 3$, with symmetries $0^-$ and $1^+$. In the $0^-$ state the additional atom populates the last place in the $p$ shell, while the $1^+$ state corresponds to atom-tetramer $s$-wave scattering.
The degeneracy is lifted for $\alpha > 0$, where the $0^{-}$ state energy becomes lower than the $1^{+}$ energy. For larger mass ratios, the $0^{-}$ state crosses the tetramer+atom energy in a trap, becomes bound in free space, and eventually reaches the Efimov threshold, corresponding here to $\epsilon = -4$ [25].

The $0^{-}$ scale factor has been calculated for a few mass ratios using finite-range models [26]. For $\alpha > 9.672$, when the pentamer is bound in free space, it was calculated by fitting the wave-function high-momentum tail to $F \propto Q^{-3N/2+1-s}$, where $Q$ is the hypermomentum conjugate to the hyperradius $\rho$ and $F$ is the momentum-space wave-function calculated in the STM-DMC method [25]. Our results are tabulated in Table IV and shown in Fig. 5.

The $1^{+}$ scale factor has been calculated only for the equal-mass case [27]. Our results are tabulated in Table V and shown in Fig. 5. Since for large mass ratio the zero-range extrapolation is not conclusive, we cannot work close to the Efimov threshold. However, no sign for an Efimov state with any symmetry other than $0^{-}$ is found in the explored mass ratios.

Similar to the (3+1) case, the bending in the $1^{+}$ energy results from avoided crossing around $\alpha = 1$ with another $1^{+}$ state (not shown). The latter state has $\epsilon = 4$ in the $\alpha = 0$ limit and corresponds to the $d$-wave atom-tetramer state. The same is true for the $2^{-}$ and $3^{+}$ states, also shown in Fig. 5, and indeed the energies of these states are close apart from the avoided crossing region.

### Table IV: The energies of the trapped pentamer $0^{-}$ state for various mass ratios.

| $M/m$ | This work | Ref. [26] | $M/m$ | This work | Ref. [25] |
|-------|-----------|-----------|-------|-----------|-----------|
| 0     | 3         | 6         | 1.01(1) |
| 1     | 2.42(1)   | 2.45      | 0.77(1) |
| 2     | 2.11(1)   | 2.15      | 0.44(1) |
| 3     | 1.83(1)   | 9         | 0.26(3) |
| 4     | 1.57(1)   | 1.68      | -0.2(1) | -0.41(1) |
| 5     | 1.28(1)   | 11        | -0.5(1) | -0.90(1) |

### Table V: The energies of the trapped pentamer $1^{+}$ state for various mass ratios.

| $M/m$ | This work | Ref. [27] | $M/m$ | This work |
|-------|-----------|-----------|-------|-----------|
| 0     | 3         | 6         | 2.01(2) |
| 1     | 3.19(1)   | 3.155     | 1.77(1) |
| 2     | 3.05(1)   | 8         | 1.56(3) |
| 3     | 2.85(1)   | 9         | 1.19(4) |
| 4     | 2.56(1)   | 10        | 0.99(1) |
| 5     | 2.31(1)   |           |         |

E. The (5+1) case

Adding another atom, we now move to the (5 + 1) system. Since no room is left in the $p$ shell, the additional atom can populate an excited $s$ shell, keeping the $0^{-}$ symmetry of the (4 + 1) core, or a $d$ shell, resulting in a $2^{-}$ state.

The energies of these states in a trap are tabulated in Table VI and plotted in Fig. 6. As the mass ratio becomes larger, the $0^{-}$ and $2^{-}$ states becomes degenerate within our error bars.
IV. CONCLUSION

We study mass-imbalanced mixtures of $N$ identical fermions interacting resonantly with a distinguishable atom. The scale factor, or the energy of the unitary system in a harmonic trap, was calculated for a few lowest states of the $N \leq 5$ systems. We solve the trapped few-body system with finite-range inter-species potentials using the stochastic variational method. The zero-range limit is then extrapolated. The shell structure of the system is explored and the effect of level repulsion is shown, revealing the significant change from the static-impurity case to the dynamic-impurity case. A series of Efimov states with $N = 2, 3, \text{ and } 4$ exist for large enough mass ratio. Nevertheless, no sign for the existence of a $(5 + 1)$ Efimov effect is shown in the mass ratios explored here, $\alpha \leq 12$. Further studies that would deal directly with the zero-range limit should be carried out to check the validity of this statement for mass ratios up to the $(4 + 1)$ Efimovian threshold.

Table VI: The energies of the two lowest $(5 + 1)$ hexamer states in a trap, with $0^-$ and $2^-$ symmetries, for various mass ratios.

| $M/m$ | $0^-$ | $2^-$ | $M/m$ | $0^-$ | $2^-$ |
|-------|-------|-------|-------|-------|-------|
| 0     | 4     | 5     | 6     | 2.71(1) | 2.73(4) |
| 1     | 4.23(1) | 4.34(1) | 7     | 2.31(1) | 2.44(6) |
| 2     | 3.89(3) | 3.96(2) | 8     | 2.41(1) | 2.20(3) |
| 3     | 3.52(3) | 3.63(2) | 9     | 1.81(1) | 1.81(1) |
| 4     | 3.19(3) | 3.31(2) | 10    | 1.83(1) | 1.51(1) |
| 5     | 2.87(4) | 2.99(3) | 11    | 1.33(3) | 1.22(2) |

The Efimov limit corresponds here to $\epsilon = -5.5$. Our results show no sign for a $(5 + 1)$ Efimov state for any symmetry up to $\alpha \leq 12$. As was have claimed, a different method would be probably needed to extend this conclusion up the the $(4 + 1)$ Efimovian threshold.

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Appendix: Excited states in the $\alpha = 0$ limit

For completeness, we list here the properties of the two lowest excited states in the $\alpha = 0$ limit. The properties of the lowest-excited state are tabulated in Table VII, while those of the next-to-lowest excited state are tabulated in Table VIII.

| System | $\epsilon$ | $\pi$ | $L$ | Configuration |
|--------|------------|-------|-----|---------------|
| 1+1    | 2          | 0     | 0   | 1s            |
| 2+1    | 2          | 2     | 0   | 0s 1s 0d      |
| 3+1    | 3          | 2     | 0   | 1,2,3 0s 0p 0d |
| 4+1    | 4          | 2,3   | 0   | 0s 0p$^2$ 0d  |
| 5+1    | 5          | 2,3   | 0   | 0s 1s 0p$^2$ 0d |

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Table VIII: The next-to-lowest excited state properties for $\alpha = 0$. Shown are the energy, the angular momentum, the parity, and the shell configuration for the ($N+1$) mixtures.

| System | $\epsilon$ | $\pi$ | $L$ | Configuration |
|--------|------------|-------|-----|---------------|
| 1+1    | 4          | +     | 0   | 2s            |
|        |            | +     | 0   | 0s 2s         |
| 2+1    | 3          | -     | 1   | 0s 1p         |
|        |            | +     | 1   | 1s 0p         |
|        |            |       | 3   | 0s 0f         |
|        | 0,1,2      |       | 0s 0p 1p |
|        | 1,3        |       | 0s 0d 2 |
|        | +          |       | 2s 3,4 | 0s 0p 0f |
| 3+1    | 4          |       | 1   | 1s 0p 2s     |
|        |            | -     | 1   | 0s 1s 0p     |
|        |            |       | 3   | 0s 1s 0f     |
|        | 0,1,2      |       | 0s 1s 0p 1p |
|        | 1,3        |       | 0s 2s 0p 2 |
|        | +          |       | 2s 3,4 | 0s 0p 0f    |
| 4+1    | 5          |       | 0,1,2,3,4 | 0s 0p 0d 2 |
|        | 0,1,2      |       | 0s 0p 2 1p |
|        |            | -     | 1   | 0s 1s 2s 0p 2 |
|        |            |       | 2s 3,4 | 0s 0p 0f 0f |
|        | 0,1,2      |       | 0s 0p 2 0d 2 |
|        |            | +     | 1   | 0s 0p 3 1p   |
|        |            |       | 1   | 0s 1s 2s 0p 2 |
|        |            |       | 3   | 0s 0p 2 0f   |
| 5+1    | 6          |       | 0   | 0s 2s 0p 3   |
|        |            | -     | 0,1,2 | 0s 1s 0p 2 1p |
|        |            |       | 2s 3,4 | 0s 1s 0p 0d 2 |

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