THE SPACE OF STRICTLY-CONVEX REAL-PROJECTIVE STRUCTURES ON A CLOSED MANIFOLD

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ABSTRACT. We give an expository proof of the fact that, if $M$ is a compact $n$–manifold with no boundary, then the set of holonomies of strictly-convex real-projective structures on $M$ is a subset of $\text{Hom}(\pi_1 M, \text{PGL}(n+1, \mathbb{R}))$ that is both open and closed.

If $M$ is a compact $n$–manifold, then $\text{Rep}(M) = \text{Hom}(\pi_1 M, \text{GL}(n+1, \mathbb{R}))$ is a real algebraic variety. Let $\text{Rep}_p(M)$ and $\text{Rep}_s(M)$ be, respectively, the subsets of $\text{Rep}(M)$ of holonomies of properly-convex, and of strictly-convex structures on $M$. Then $\text{Rep}_s(M) \subset \text{Rep}_p(M)$. Throughout this paper we use the Euclidean topology everywhere, and not the Zariski topology. In the following, $M$ is closed and $n \geq 2$.

1. Open Theorem. $\text{Rep}_p(M)$ is open in $\text{Rep}(M)$.

2. Closed Theorem. $\text{Rep}_s(M)$ is closed in $\text{Rep}(M)$.

3. Clopen Theorem. $\text{Rep}_s(M)$ is a union of connected components of $\text{Rep}(M)$.

It follows from (1.11) that the holonomy of a strictly-convex structure determines a projective manifold up to projective isomorphism. The Open Theorem is due to Koszul [17, 18]. Our proof is distilled from his, and proceeds by showing that $M$ is properly-convex if and only if there is a projective $(n+1)$–manifold $N \cong M \times I$ with $M \times 0$ flat, and $M \times 1$ convex, and triangulated so that adjacent $n$-simplices are never coplanar. This type of convexity is easily shown to be preserved by small deformations.

The Closed Theorem is due to Choi and Goldman [7] when $n = 2$, to Kim [15] when $n = 3$, and to Benoist [3] in general. Our proof is new, and based on a geometric argument called the box estimate (4.3). This might be viewed as related to Benzecri’s compactness theorem [5] for a properly-convex domain $\Omega$, but pertaining to Aut($\Omega$). It also uses an elementary geometric fact (2.8) about an analogue of centroids for subsets of the sphere.

The Clopen Theorem follows from the Open Theorem and the fact, due to Benoist, that a properly-convex manifold that is homeomorphic to a strictly convex manifold is also strictly-convex (1.10). We have made an effort to make the paper self-contained by including proofs of all the foundational results needed in Sections 1 and 2. We have endeavoured to simplify these proofs as much as possible. See [4] for an excellent survey.

There are extensions of the Open and Closed Theorems when $M$ is the interior of a compact manifold with boundary, see [10, 11, 12]. Indeed, the techniques in this paper were developed to handle this more general situation for which the pre-existing methods do not suffice. But it seems useful to present some of the main ideas in the simplest setting. We have liberally borrowed from [11].

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1. PROPERLY AND STRICTLY-CONVEX

This section and the next review some well known results concerning properly and strictly-convex projective manifolds that are needed to prove the theorems. The results needed subsequently are (1.10), (1.11), (1.13), (1.14), and (2.5), (2.6). The only mild innovation is that, to avoid appealing to results about word-hyperbolic groups, a more direct approach was taken to the proof of (1.10). The early sections of [9] greatly expand on the background in this section.

If M is a manifold, the universal cover is \( \tilde{M} \rightarrow M \), and if \( g \in \pi_1 M \) then \( \tau_g : \tilde{M} \rightarrow \tilde{M} \) is the covering transformation corresponding to \( g \). A geometry is a pair \((G, X)\), where \( G \) is a group that acts analytically and transitively on a manifold \( X \). A \((G, X)\)-structure on a manifold \( M \) is determined by a development pair \((\text{dev}, \rho)\) that consists of the holonomy \( \rho \in \text{Hom}(\pi_1 M, G) \) and the developing map \( \text{dev} : \tilde{M} \rightarrow X \) which is a local homeomorphism. The pair satisfies for all \((x, \nu) \) transitively on a manifold \( X \).

In what follows \( V = \mathbb{R}^{n+1} \), and \( V^* = \text{Hom}(V, \mathbb{R}) \) is the dual vector space, and \( \mathbb{R}_0^{n+1} = V_0 = V \setminus \{0\} \).

Projective space is \( \mathbb{P}^n = \mathbb{R}^{n+1}/\mathbb{R}_0 \), and \( [a] \in \text{Aut}(\mathbb{P}^n) = \text{PGL}(V) \) acts on \( \mathbb{P}^n \) by \([a][x] = [ax]\). Projective geometry is \((\text{Aut}(\mathbb{P}^n), \mathbb{P}^n)\) and is also written \((\text{Aut}(\mathbb{RP}^n), \mathbb{RP}^n)\).

We use the notation \( \mathbb{R}_0^+ = (0, \infty) \). Positive projective space is \( \mathbb{P}_+ = \mathbb{P}^n_+ = \mathbb{V}_0/\mathbb{R}_0^+ \) and \([x]_+ = \{\lambda x : \lambda > 0\} \) for \( x \in \mathbb{V}_0 \). Sometimes we identify \( \mathbb{P}_+ \) with the unit sphere \( S^0 = \mathbb{R}^{n+1}_0 \) via \([x]_+ \equiv x/||x||\). The group \( \text{Aut}(\mathbb{P}_+, V) := \text{SL}(V) \subset \text{GL}(V) \) is the subgroup with \( \det = \pm 1 \). Positive projective geometry \((\text{Aut}(\mathbb{P}_+, V), \mathbb{P}_+, V)\) is the double cover of projective geometry. We will pass back and forth between projective geometry and positive projective geometry without mention, often omitting the term positive.

If \( U \) is a vector subspace of \( V \) then \( \mathbb{P}U \subset \mathbb{P}^n \) is a projective subspace of \( \mathbb{P}V \), and is a (projective) plane if \( \dim U = 2 \) and a (projective) hyperplane if \( \dim U = \dim V - 1 \). The dual of \( U \) is the projective subspace \( \mathbb{P}U^0 \subset \mathbb{P}V^* \) where \( U^0 = \{\phi \in V^*: \phi(U) = 0\} \). We use the same terminology in positive projective geometry.

By lifting developing maps one obtains:

1.1. Proposition. Every projective structure on \( M \) lifts to a positive projective structure.

The frontier of a subset \( X \subset Y \) is \( \text{Fr} X = \text{cl}(X) \setminus \text{int}(X) \), and the boundary is \( \partial X = X \cap \text{Fr} X \). A segment is a connected, proper subset of a projective line that contains more than one point. In what follows \( \Omega \subset \mathbb{RP}_n \).

If \( H \) is a hyperplane and \( x \in H \cap \text{Fr} \Omega \) and \( H \cap \text{int} \Omega = \emptyset \), then \( H \) is a called a supporting hyperplane (to \( \Omega \)) at \( x \). The set \( \Omega \)

- is convex if every pair of points in \( \Omega \) is contained in a segment in \( \Omega \).
- is properly-convex if it is convex, and \( \text{cl} \Omega \) does not contain a projective line.
- is strictly-convex it is properly-convex and \( \text{Fr} \Omega \) does not contain a segment.
- is flat if it is convex and \( \dim \Omega < n \).
- is a convex domain if it is a properly convex open set.
- is a convex body if it is the closure of a convex domain.
- is \( C^1 \) if for each \( x \in \text{Fr} \Omega \) there is a unique supporting hyperplane at \( x \).

If \( V = U \oplus W \), then projection (along \( \mathbb{P}W \)) onto \( \mathbb{P}U \) is \( \pi : \mathbb{P}V \setminus \mathbb{P}W \rightarrow \mathbb{P}U \) given by \( \pi[u + w] = [u] \).

Duality is the map that sends each point \( \theta = [\phi] \in \mathbb{P}V^* \) to the hyperplane \( H_\theta = \mathbb{P} \ker \phi \subset \mathbb{P}V \). If \( L \) is a line in \( \mathbb{P}V^* \) the hyperplanes \( H_\theta \) dual to the points \( \theta \in L \) are called a pencil of hyperplanes. Then \( Q = \cap H_\theta \) is the projective dual of \( L \) and is called the core of the pencil.

1.2. Lemma. A convex subset \( \Omega \) is properly-convex if and only if \( \text{cl} \Omega \) is disjoint from some hyperplane.

Proof. Without loss of generality, suppose \( \Omega \) is closed. The result is obvious if \( n = 1 \). Since \( \Omega \) is convex and contains no projective line, it is simply connected, and so lifts to \( \Omega' \subset \mathbb{RP}_n^+ \). Let \( H \subset \mathbb{RP}_n^+ \) be a hyperplane. Then \( H \cap \Omega' \) is empty or properly-convex. By induction on dimension, \( H \) contains a projective subspace \( Q \) with \( \dim Q = n - 2 \) that is disjoint from \( H \cap \Omega' \). There is a pencil of hyperplanes \( H_\theta \) with core \( Q \). Now \( \Omega' \cap H_\theta \) is contained in one of the two components of \( H_\theta \setminus Q \). As \( \theta \) moves half way round \( \mathbb{RP}_n^+ \), the component must change. Thus for some \( \theta \) the intersection is empty. \( \square \)
In what follows $\Omega \subset \mathbb{R}P^n$ is a properly-convex domain, and $\text{Aut}(\Omega) \subset \text{Aut}(\mathbb{R}P^n)$ is the subgroup that preserves $\Omega$. The Hilbert metric $d_\Omega$ on $\Omega$ is defined as follows. If $\ell \subset \mathbb{R}P^n$ is a line and $\alpha = \Omega \cap \ell \neq \emptyset$ then $\alpha$ is a proper segment in $\Omega$ and $\alpha = (a_-, a_+)$ with $a_\pm \in \text{Fr}\Omega$. There is a projective isomorphism $f : \alpha \to \mathbb{R}^+$ and
\[
d_\Omega(x,y) = \frac{1}{2} \left| \log \frac{f(x)}{f(y)} \right|
\]
for $x, y \in \alpha$. A geodesic in $\Omega$ is a curve whose length is the distance between its endpoints. It follows that line segments are geodesics. It is immediate that $\text{Aut}(\Omega)$ acts by isometries of $d_\Omega$.

Let $H_\pm$ be supporting hyperplanes at $a_\pm$ and $P = H_+ \cap H_-$. Projection (along $P$) onto $\alpha$ is the map $\pi : \Omega \to \alpha$ given by $\pi[u + v] = [v]$ where $[u] \in P$ and $[v] \in \ell$. The choice of $H_\pm$ is unique only if $a_\pm$ are $C^1$ points. In general $\pi x$ is not the closest point on $\alpha$ to $x$.

1.3. Lemma. (i) $d_\Omega(x,y) \geq d_\Omega(x,\pi y)$.

(ii) If $\Omega$ is strictly convex then geodesics are segments of lines.

Proof. Figure 1 represent two or more dimensions. Let $\delta = (b_-, b_+)$ be the proper segment in $\Omega$ containing $x$ and $y$, and $\pi : \Omega \to \alpha$ as above. Then $d_\Omega(x,y) = d_\delta(x,y)$. Since $(\pi|\delta) : \delta \to \alpha$ is a projective embedding $d_\delta(x,y) = d_\pi \delta(\pi x, \pi y)$. Also $d_{\pi \delta}(\pi x, \pi y) \geq d_{\alpha}(\pi x, \pi y)$ because $\pi \delta \subset \alpha$. Finally $d_{\alpha}(\pi x, \pi y) = d_\Omega(\pi x, \pi y)$, which proves (i).

Equality implies $\alpha = \pi \delta$. Thus, after relabelling if needed, $b_\pm \in H_\pm$. Thus $[a_\pm, b_\pm] \subset H_\pm \cap \text{Fr}\Omega$. If $\Omega$ is strictly-convex it follows that $b_\pm = a_\pm$. This gives (ii).

In a strictly convex domain $\Omega$ will use the term geodesic to mean a proper segment in $\Omega$. A subset $\mathcal{C} \subset V_0$ is a cone if $t \cdot \mathcal{C} = \mathcal{C}$ for all $t > 0$. The dual cone $\mathcal{C}^* = \text{int}(\{ \phi \in V^* : \phi(\mathcal{C}) \geq 0 \})$ is convex. If $\Omega \subset \mathbb{P}^n V$ then $\mathcal{C} \Omega$ is the cone $\{ v \in V_0 : [v] \in \mathcal{C} \subset \Omega \}$. If $\Omega$ is open and properly-convex, then the dual domain $\Omega^* = \mathbb{P}(\mathcal{C}^* \Omega^*)$ is open, and properly-convex. Using the natural identification $V \cong V^{**}$ it is immediate that $(\Omega^*)^* = \Omega$.

1.4. Corollary. If $\Omega$ is open and properly-convex, then $\Omega$ is $C^1$ (resp. strictly convex) if and only if $\Omega^*$ is strictly-convex (resp. $C^1$).

Proof. We have $[\phi] \in \text{Fr}\Omega^*$ if and only if the dual hyperplane $H = [\ker\phi]$ supports $\Omega$. If $p \in \text{Fr}\Omega$, then there are two distinct supporting hyperplanes $H_0 = [\ker\phi_0]$ and $H_1 = [\ker\phi_1]$ for $\Omega$ at $p$ if and only if all the hyperplanes $H_t = [\ker\phi_t]$ contain $p$ and support $\Omega$, where $\phi_t = (1-t)\phi_0 + t\phi_1$ and $0 \leq t \leq 1$. This happens if
and only if the segment $\sigma = \{ \phi_t : 0 \leq t \leq 1 \}$ is in Fr$\Omega$. Hence $\Omega$ is $C^1$ if and only if $\Omega^*$ is strictly convex. Replacing $\Omega$ by $\Omega^*$ and using $\Omega^{**} = \Omega$ gives the other result. \hfill \square

If $\Gamma \subset \text{Aut}(\Omega)$ is a discrete and torsion-free subgroup, then $M = \Omega/\Gamma$ is a properly (resp. strictly) convex manifold if $\Omega$ is properly (resp. strictly) convex. Then $M = \Omega$ and there is a development pair $(\text{dev}, \rho)$, where dev is the inclusion map, and the holonomy $\rho : \pi_1 M \to \Gamma$ is an isomorphism.

1.5. Lemma. Suppose $M$ and $M'$ are properly-convex manifolds and $\pi_1 M \cong \pi_1 M'$. If $M$ is closed, then $M'$ is closed.

Proof. Let $\pi : \Omega \to M$ be the projection. This is the universal covering space of $M$, so $M$ is a $K(\pi_1 M, 1)$. The same holds for $M'$. Hence $M$ and $M'$ are homotopy equivalent. If $n = \dim M$, then $M$ is closed if and only if $H_n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since homology is an invariant of homotopy type the result follows. \hfill \square

A compactness argument shows that if $\Omega$ is the universal cover of a closed strictly-convex manifold, then projection (along a projective subspace) onto a geodesic in $\Omega$ is uniformly distance decreasing in the following sense.

1.6. Lemma. Suppose $M = \Omega/\Gamma$ is a strictly-convex closed manifold. Given $b > 1$ there is $R = R(b) > 0$ so the following holds. Suppose $\pi : \Omega \to \alpha$ is a projection onto a geodesic $\alpha$. Suppose that $\eta$ is a rectifiable arc in $\Omega \setminus N_b(\alpha)$ of length $\ell$ with endpoints $x$ and $y$. Then $d_\Omega(\pi x, \pi y) < 1 + (\ell/b)$

Proof. By decomposing $\eta$ into finitely many subarcs, with one of length at most $b$, and $[\ell/b]$ of length $b$, it suffices to prove if $\text{length}(\eta) \leq b$ then $d_\Omega(\pi x, \pi y) \leq 1$.

Write $d = d_\Omega$. If no such $R$ exists then there are sequences $x_k, y_k \in \Omega$ and projections $\pi_k : \Omega \to \alpha_k$ with $d(x_k, y_k) \leq b$ and $d(\pi_k x_k, \pi_k y_k) \leq b$. Then $\beta_k = [\pi_k x_k, \pi_k y_k]$ and $\gamma_k = [\pi_k y_k, \pi_k y_k]$ are geodesic segments and $\pi_k \beta_k = \pi_k x_k$ and $\pi_k \gamma_k = \pi_k y_k$.

There is a compact $W \subset \Omega$ such that $\Gamma \cdot W = \Omega$. By applying an element of $\Gamma$ we may assume that $\pi_k x_k \in W$. After subsequenceing we may assume $\beta = \lim \beta_k$ and $\gamma = \lim \gamma_k$ and $\alpha = \lim \alpha_k$ and $\pi = \lim \pi_k$ all exist and $\pi : \Omega \to \alpha$ is projection. Refer to Figure 1. Thus $\beta$ and $\gamma$ are geodesic rays in $\Omega$ that start on $\alpha$, and end on Fr$\Omega$, with $\pi(\beta) = \alpha \cap \beta \neq \alpha \cap \gamma = \pi(\gamma)$.

Since $\Omega$ is strictly-convex, and $d(x_k, y_k) \leq b$, it follows that $x_k$ and $y_k$ limit on the same point $p \in \text{Fr}\Omega$. Hence $p$ is the endpoint of both $\beta$ and $\gamma$ on Fr$\Omega$. Also $p \in P = H_\perp \cap H_\perp$, where $H_\perp$ are the supporting hyperplanes to $\Omega$ at the endpoints of $\alpha$. Since $\Omega$ is strictly-convex, $p \notin P$. It follows that $\pi(\beta) = \pi(p) = \pi(\gamma)$ which is a contradiction. \hfill \square

If $X$ is a subset of a metric space $Y$, the $r$-neighborhood of $X$ is $N_r(X) = N(X, r) = \{ y \in Y : d(y, x) \leq r \}$.

The following is due to Benoist [2]. A triangle is a disc $\Delta$ in a projective plane bounded by three segments. If $\Delta \subset \text{cl} \Omega$ and $\Delta \cap \text{Fr}\Omega = \partial \Delta$, then $\Delta$ is called a properly embedded triangle or PET. A properly-convex set $\Omega$ has thin triangles if there is $\delta > 0$ such that for every triangle $T$ in $\Omega$, each side of $T$ is contained in a $\delta$-neighborhood of the union of the other two sides with respect to the Hilbert metric.

1.7. Proposition. If $M = \Omega/\Gamma$ is a properly-convex closed manifold, then the following are equivalent:

1. $M$ is strictly-convex,
2. $\Omega$ does not contain a PET,
3. $\Omega$ has thin triangles.

Proof. If $\Omega$ is not strictly-convex, there is a maximal segment $\ell \subset \text{Fr}\Omega$. Choose $x \in \Omega$ and let $P \subset \Omega$ be the interior of the triangle that is the convex hull of $x$ and $\ell$. Choose a sequence $x_n$ in $P$ that limits on the midpoint of $\ell$. Then $d_\Omega(x_n, \text{Fr} P) \to \infty$ because $\ell$ is maximal. Since $M$ is compact there is compact set $W \subset \Omega$ such that $\Gamma \cdot W = \Omega$. Thus there is $y_n \in \Gamma$ with $y_n x_n \in W$. After choosing a subsequence $y_n x_n \to y \in W$, and $y_n P$ converges to the interior of a PET $\Delta \subset \Omega$ that contains $y$. Since $\Delta$ is flat $d_\Omega(\Delta) = d_\Delta$. Now PGL$(\Delta)$ contains a subgroup $G \cong \mathbb{R}^2$ that acts transitively on $\Delta$. Therefore $(\Delta, d_\Delta)$ is isometric to a normed vector-space, thus does not have thin triangles.
Conversely if $\Omega$ does not have thin triangles, then there is a sequence of triangles $T_k$ in $\Omega$ and points $x_k$ in $T_k$ with $d_{Q}(x_k, \partial T_k) > k$. As above, after applying elements of $\Gamma$, we may assume $T_k$ converges to a PET, so $\Omega$ is not strictly-convex.

A $(K,L)$-quasi-isometric embedding is a map $f : X \to Y$ between metric spaces $(X,d_X)$ and $(Y,d_Y)$ such that

$$K^{-1} d_X(x,x') - L \leq d_Y(f(x),f(x')) \leq K d_X(x,x') + L$$

for all $x,x' \in X$. If $X = [a,b] \subset \mathbb{R}$, then $f$ is called quasi-geodesic. The map $f$ is a quasi-isometry or QI if $Y \subset N(fX,L)$.

If $g : Y \to Z$ is a $(K',L')$-QI, then $g \circ f$ is a $(KK',KL+2L')$-QI. In particular, if $g$ is a QI and $f$ is a quasi-geodesic, then $g \circ f$ is a quasi-geodesic with QI constants that only depend on those of $f$ and $g$.

If $G$ is a finitely generated group then a choice of finite generating set gives a bi-Lipschitz equivalent metric. The Švarc-Milnor lemma in this setting is

1.8. Proposition. If $M = \Omega / \Gamma$ is a closed and properly-convex manifold, then $(\Omega, d_\Omega)$ is QI to $\pi_1 M$.

Proof. Fix $x \in \bar{M} \supset \Omega$. Then $f : \pi_1 M \to \bar{M}$ given by $f(g) = \tau_g(x)$ is a QI. □

A metric space $(Y,d_Y)$ is ML if it satisfies the Morse Lemma that for all $(K,L)$ there is $S = S(K,L) > 0$, called a tracking constant, such that if $\alpha$ and $\beta$ are $(K,L)$-quasi-geodesics in $Y$ with the same endpoints, then $\alpha \subset N_S(\beta)$ and $\beta \subset N_S(\alpha)$. Since quasi-geodesics are sent to quasi-geodesics by quasi-sometries, it follows that the property of ML is preserved by quasi-isometries.

Clearly $\mathbb{R}^2$ is not ML, and any norm on $\mathbb{R}^2$ is QI to the standard norm, so $\mathbb{R}^2$ with any norm is not ML. A geodesic in the Cayley graph picks out a sequence in $G$ that is a quasi-geodesic. It follows from (1.8) that whether or not a closed properly convex manifold is ML only depends on $\pi_1 M$.

1.9. Proposition. If $M = \Omega / \Gamma$ is a properly-convex closed manifold, then $M$ is strictly-convex if and only if $(\Omega, d_\Omega)$ is ML.

Proof. If $\Omega$ is not strictly-convex, then by (1.7) it contains a PET $\Delta$ which is isometric to a norm on $\mathbb{R}^2$, and therefore is not ML.

Now suppose $\Omega$ is strictly-convex. Given $(K,L)$ let $R = R(2K)$ be given by (1.6). Let $S = R + 4RK + K + L$. We show that $S = 2KS + 2L$ is a tracking constant. Refer to Figure 2. Let $\alpha'$ be a $(K,L)$-quasi-geodesic in $\Omega$ with endpoints $p$ and $q$, and $\beta' = [p,q]$. For simplicity we assume $\alpha'$ is continuous. Let $\tilde{\beta}'$ be the geodesic that contains $\beta'$ and $\pi : \Omega \to \tilde{\beta}'$ be the projection.

Claim 1) $\alpha' \subset N(\beta',S)$.
Claim 2) $\alpha' \subset N(\beta',S/2)$ and $\beta' \subset N(\alpha',S/2)$.
Assuming this, if \( \gamma \)' is another \((K,L)\)-quasi-geodesic with endpoints \( p \) and \( q \) then \( \beta' \subset N(\gamma', S/2) \) so
\[
\alpha' \subset N(\beta', S/2) = N(N(\gamma', S/2), S/2) = N(\gamma', S)
\]
which proves that \( S \) is a tracking constant.

For Claim 1 suppose that \( \alpha \) is the closure of a component of \( \alpha' \setminus N(\beta', R) \). The endpoints \( x \) and \( y \) of \( \alpha \) satisfy \( d(x, \beta^+) = d(y, \beta^+) = R \). Let \( x', y' \in \beta^+ \) be chosen so that \( d(x, x') = R = d(y, y') \) and let \( \beta = [\pi x, \pi y] \subset \beta^+ \). Then
\[
\text{length}(\beta) \leq 1 + \text{length}(\alpha) / 2K
\]
by definition of \( R \). Since \( \pi \) is distance non-increasing \( d(x', \pi x) \leq d(x', x) = R \), and similarly \( d(y', \pi y) \leq R \). Thus
\[
\text{length}(\alpha) \leq 2(4RK + K + L).
\]
It follows that the distance of every point on \( \alpha' \) from \( \beta^+ \) is at most
\[
R + (1/2) \text{length}(\alpha) = R + 4RK + K + L = S'
\]
This proves Claim 1.

For Claim 2, there is a maximal subarc \( \delta \subset \alpha' \) that starts at \( p \) and ends at a point \( w \in \alpha' \) such that \( \pi w = p \). Possibly the arc is trivial with \( w = p \). By the above \( d(w, \beta^+) \leq S' \). The arc \( \delta \) is a \((K,L)\)-quasi-geodesic because it is a subarc of \( \alpha \), so \( d(\pi \delta, \beta^+) \leq K^{-1}(\text{length}(\delta) - L) \). Thus \( \text{length}(\delta) \leq K \text{length}(\alpha) / 2K + L \). Using \( \pi \) is distance non-increasing, gives \( \text{length}(\pi \delta) \leq S/2 \). Now \( \pi \alpha' \) extends beyond \( \beta' \) by \( \pi \delta \) so \( \pi \alpha' \subset N(\beta', S/2) \). Also \( \beta' \subset \pi \alpha' \) so \( \beta' \subset N(\alpha', S') \) and \( K \geq 1 \), so \( \beta' \subset N(\alpha', KS' + L) \). This proves the claim. \( \square \)

1.10. Corollary. Suppose \( M \) and \( N \) are closed and properly-convex, and that \( \pi_1 M \cong \pi_1 N \). If \( M \) is strictly-convex, then \( N \) is strictly-convex.

Proof. Since \( \text{ML} \) is preserved by quasi-isometry, it follows from (1.9) and (1.8) that \( M \) is strictly convex if and only if \( \pi_1 M \) is \( \text{ML} \), and this is determined by \( \pi_1 M \). \( \square \)

1.11. Lemma. If \( M = \Omega / \Gamma \) is a closed properly-convex manifold and \( \Omega' \subset \Omega \) is a non-empty properly-convex subset that is preserved by \( \Gamma \), then \( \Omega' = \Omega \).

Proof. Otherwise the function \( F : \Omega \to \mathbb{R} \) given by \( F(x) = d_{\Omega}(x, \Omega') \) is continuous, unbounded, and \( \Gamma \)-invariant. Thus it covers a continuous unbounded function \( f : M \to \mathbb{R} \), contradicting the compactness of \( M \). \( \square \)

The displacement distance of \( \gamma \in \text{Aut}(\Omega) \) is \( t(\gamma) = \inf \{ d_{\Omega}(x, \gamma x) : x \in \Omega \} \). The element \( \gamma \) is hyperbolic if \( t(\gamma) > 0 \).

1.12. Lemma (Hyperbolics). Suppose \( \Omega \subset \mathbb{P}V \) and \( M = \Omega / \Gamma \) is a strictly-convex closed manifold and \( 1 \neq \gamma \in \Gamma \). Then \( \gamma \) is hyperbolic and there are \( a_\pm \in \text{Fr}(\Omega) \) such that for all \( x \in \mathbb{P}V \setminus (H_+ \cup H_-) \) we have
\[
\lim_{n \to \pm \infty} \gamma^n x = a_\pm
\]
where \( H_\pm \) is the supporting hyperplane to \( \Omega \) at \( a_\pm \).

Proof. Since \( M \) is compact, the Arzela-Ascoli Theorem implies there is a closed geodesic \( C \) in \( M \) that is conjugate to \( \gamma \) in \( \pi_1 M \) and \( t(\gamma) = \text{length}(C) > 0 \). Hence \( \gamma \) is hyperbolic. By (1.3) \( C \) is covered by a proper segment \( \alpha = (a_-, a_+) \) in \( \Omega \) that is preserved by \( \gamma \). By (2.6) \( M \) is \( C \)-so there are unique supporting hyperplanes \( H_\pm \) to \( c\Omega \) that contain \( a_\pm \) respectively. Then \( Q = H_+ \cap H_- \) is a codimension-2 subspace, and it is disjoint from \( \Omega \) by strict convexity. Moreover \( Q \) is preserved by \( \gamma \).
The pencil of hyperplanes \( \{ H_t : t \in L \} \) in \( \mathbb{P} V \) that contain \( Q \) is dual to a line \( L \subset \mathbb{P}(V^*) \). Since the dual of \( \gamma \) acts on \( L \cong \mathbb{R}^1 \) projectively and non-trivially, it only fixes the two points \( [H_+] \in L \). The other hyperplanes \( [H_t] \) are moved by \( \gamma \) away from \( [H_-] \) and towards \( [H_+] \). Since \( \Omega \) is strictly-convex, \( \gamma \) moves all points in \( \Omega \) towards \( a_+ \). Suppose \( \ell \) is a projective line that contains \( a_- \). If \( \ell \) is not contained in \( H_- \) then, since \( H_- \) is the unique supporting hyperplane at \( a_- \), it follows that \( \Omega \cap \ell \neq \emptyset \). Thus \( \gamma^k \ell \to \ell' \) where \( \ell' \) is the projective line containing \( a_\alpha \). Hence \( \gamma^k(\ell \setminus a_-) \to a_+ \) as \( k \to \infty \). This reasoning applied to \( \gamma^{-1} \) gives the corresponding statements for \( a_- \), and gives the second conclusion. \( \square \)

The point \( a_+ \) is called the attracting, and \( a_- \) is the repelling, fixed point of \( \gamma \), and \( (a_- , a_+) \subset \Omega \) is called the axis of \( \gamma \). This is the only proper segment in \( \Omega \) preserved by \( \gamma \). The attracting fixed point of \( \gamma^{-1} \) is the repelling fixed point of \( \gamma \).

1.13. **Proposition** (Unique domain). Let \( n \geq 2 \). Suppose \( \Omega \subset \mathbb{R}^p \) and \( M = \Omega / \Gamma \) is a strictly-convex, closed \( n \)-manifold. If \( \Omega' \) is open and properly-convex and preserved by \( \Gamma \), then \( \Omega' = \Omega \).

**Proof.** Let \( \Omega \) be union, over all \( 1 \neq \gamma \in \Gamma \), of attracting fixed points of \( \gamma \). Then \( \Omega \subset \text{Fr} \Omega \) by (1.12). Let \( W \) be the convex hull of \( X \) in \( \text{cl} \Omega \). Then \( U = W \cap \Omega \) contains the axis of each hyperbolic in \( \Gamma \), so \( U \) is non-empty, convex, \( \Gamma \)-invariant. Since \( U \subset \Omega \), we have \( \Omega = U \) by (1.11). Since \( \partial \Omega \) is strictly-convex it follows \( \text{Fr} \Omega = \text{cl} X \).

Since \( \Omega' \) is preserved by \( \Gamma \) it follows that \( \text{cl} \Omega' \subset \text{cl} X \). Now \( \text{Fr} \Omega \) is a convex hypersurface of dimensions \( n-1 > 0 \). Only one side of \( \text{Fr} \Omega \) is locally convex. Hence \( \Omega' \) contains points on the same side of \( \text{Fr} \Omega \) as \( \Omega \). Thus \( U = \Omega' \cap \Omega \neq \emptyset \) is properly convex and is preserved by \( \Gamma \). By (1.11) \( \Omega = U = \Omega' \). \( \square \)

1.14. **Corollary** (nilpotent subgroups). If \( M = \Omega / \Gamma \) is a closed, strictly-convex, projective manifold, then every nilpotent subgroup of \( \Gamma \) is cyclic.

**Proof.** By (1.12) every element of \( \Gamma \) is hyperbolic, and the axis is the only segment preserved by a non-trivial hyperbolic in \( \Gamma \). Suppose \( 1 \neq \alpha , \beta \in \Gamma \) and \( [\alpha , \beta , \beta] = 1 \). Let \( \ell \) be the axis of \( \beta \). Since \( [\alpha , \beta] \) commutes with \( \beta \), it preserves \( \ell \). Now \( \beta \) preserves \( \ell \) and \( [\alpha , \beta] = (\alpha \beta \alpha^{-1}) \beta^{-1} \) so \( \alpha \beta \alpha^{-1} \) also preserves \( \ell \). But \( \alpha \beta \alpha^{-1} \) is a hyperbolic that preserves \( \alpha \ell \), thus \( \alpha \ell = \ell \). If follows by induction on the length of the upper central series that if \( \Gamma' \subset \Gamma \) is nilpotent then \( \Gamma' \) preserves an axis.

The action of \( \Gamma \) on \( \Omega \) is free, so \( \Gamma' \) acts freely on \( \ell \). A discrete group acting freely by homeomorphisms on \( \mathbb{R} \) is cyclic. \( \square \)

2. **Convex Cones**

This section is based on work of Vinberg, as simplified by Goldman. Write \( V = \mathbb{R}^{n+1} \) and fix an inner product \( \langle \cdot , \cdot \rangle \) on \( V \). This determines a norm on \( V \), and induces a Riemannian metric and associated volume form on every smooth submanifold of \( V \).

Given \( \phi \in \mathcal{C} \), the centroid of \( \mathcal{C}_\phi = \phi^{-1}(1) \cap \mathcal{C} \Omega \) is a point \( \mu(\mathcal{C}_\phi) \in \mathcal{C} \Omega \). Define \( \Theta(\phi) = \mu(\mathcal{C}_\phi) \). We show this is a homeomorphism \( \Theta : \mathcal{C} \Omega \to \mathcal{C} \Omega \).

Let \( S^n = \{ x \in V : ||x||^2 = 1 \} \). Suppose \( \Omega \) is an open, properly-convex subset of \( S^n \) and \( \mathcal{C} = \mathcal{C} \Omega \) is the corresponding cone in \( V \). The dual cone is
\[ \mathcal{C}^* = \text{int} \{ \phi \in V^* : \phi(\mathcal{C}) \geq 0 \} \]
The centroid of a bounded convex set \( K \) in \( V \) is the point \( \mu(K) \) in \( K \) given by\[ \mu(K) = \int_K x \, d\text{vol}_K / \int_K d\text{vol}_K \]
Here \( \dim K \leq \dim V \) and \( d\text{vol}_K \) is the induced volume form on \( K \). The centroid is independent of the inner product.
Given $\phi \in \mathcal{C}^*$, the set $\phi^{-1}(1)$ is an affine hyperplane in $V$, and $\mathcal{C}_\phi = \{ x \in \mathcal{C} : \phi(x) = 1 \}$ is a bounded subset of this hyperplane that separates $\mathcal{C}$. The subset of $\mathcal{C}$ below this hyperplane, $\mathcal{C} \cap \phi^{-1}(0, 1]$, has finite volume and boundary $\mathcal{C}_\phi$. The volume function $\mathcal{V} : \mathcal{C}^* \to \mathbb{R}$ is defined by

$$\mathcal{V}(\phi) = \text{vol}(\mathcal{C} \cap \phi^{-1}(0, 1]) = \int_{\mathcal{C} \cap \phi^{-1}(0, 1]} \text{dvol}$$

Given $\phi \in \mathcal{C}^*$, there is $v \in V$ such that $\phi(x) = \langle v, x \rangle$ for all $x \in V$. Let $\text{d}B$ be the volume form on $S^n$. We compute this integral in polar coordinates, so $\text{dvol} = r^n \text{d}r \wedge \text{d}B$. Given $x \in \mathcal{C}_\phi$, let $y = x/\|x\|$ be the corresponding point in $S^n$. Using $\phi(x) = 1$ gives

$$r(x) = \|x\| = \phi(x)/\phi(y) = \langle y, v \rangle^{-1} \quad \text{and} \quad x = \langle y, v \rangle^{-1}y$$

Using polar coordinates

$$\mathcal{V}(\phi) = \int_{\mathcal{C}_\phi} r^n \text{d}r \wedge \text{d}B = \int_{\Omega} \left( \int_0^{\langle y, v \rangle^{-1}} r^n \text{d}r \right) \text{d}B_y = (n+1)^{-1} \int_{\Omega} \langle y, v \rangle^{-n-1} \text{d}B_y$$

For $q \in \mathcal{C}$, the set $\mathcal{C}_q = \{ \phi \in \mathcal{C}^* : \phi(q) = 1 \}$ is the intersection of a hyperplane in $V^*$ with $\mathcal{C}^*$, and has compact closure. Property (iv) below is very useful later.

2.1. Proposition. The volume function has the following properties:

(i) $\mathcal{V}$ is smooth and strictly-convex.

(ii) $\mathcal{V}(\phi) \to \infty$ as $\phi \to \text{Fr}\mathcal{C}^*$.

(iii) $\mathcal{V}(t \cdot \phi) = t^{-n-1} \mathcal{V}(\phi)$ for all $t > 0$.

(iv) There is a unique $\phi \in \mathcal{C}_q^*$ at which $\mathcal{V}|\mathcal{C}_q^*$ attains a minimum, and $q = \mu(\mathcal{C}_q)$.

Proof. We use the inner product to identify $V$ with $V^*$ so that $\mathcal{C}^*$ is a subset of $V$, and $\mathcal{C}_q^* = \{ x \in \mathcal{C}^* : \langle x, q \rangle = 1 \}$. We also regard $\mathcal{V}(\phi)$ as the function $\mathcal{W}(v)$ given by Equation (3) and prove corresponding statements for $\mathcal{W}$.

For fixed $v$ the integrand $f(v) = \langle y, v \rangle^{-n-1}$ in Equation (3) is a smooth and convex function of $v$. It follows that $\mathcal{W}(v)$ is a smooth, strictly convex function of $v$. This proves (i).

If $0 \neq \psi \in \text{Fr}\mathcal{C}^*$, then there is $0 \neq x \in \text{Fr}\mathcal{C}$ with $\psi(x) = 0$. Thus $\mathbb{R}^+ \cdot x \subset \text{cl}\mathcal{C}_\psi$, so $\mathcal{C}_\psi$ is not compact. Moreover $\mathcal{C}_\psi$ is convex and has non-empty interior, so $\text{vol}(\mathcal{C}_\psi) = \infty$. It easily follows that $\mathcal{V}(\phi) \to \infty$ as $\phi \to \psi$. This proves (ii), and (iii) follows from Equation (1). It follows from convexity and (ii) that $\mathcal{V}|\mathcal{C}_q^*$ has a unique critical point, and it is a minimum.
The gradient of $\mathcal{W}(v)$ is

$$\nabla \mathcal{W} = -\int_\Omega \langle y, v \rangle^{-n-2} y \, dB_y$$

From the definition of $\mathcal{C}_q^*$ it follows that the condition for a critical point is that $\nabla \mathcal{W} \in \mathbb{R} \cdot q$.

Refer to Figure 4. Let $v \in V$ be dual to $\phi$, so $\phi(x) = \langle v, x \rangle$. Thus $\phi(v\|v\|^{-2}) = 1$ and $v\|v\|^{-2}$ is the point on $\phi^{-1}(1)$ closest to 0. Thus the distance of $\phi^{-1}(1)$ from 0 is $\|v\|^{-1}$. Let $\pi : \mathcal{C}_q \to \Omega = S^n \cap \mathcal{C}$ be the radial projection $\pi(x) = x/\|x\|$. Given $x \in \mathcal{C}_q$ set $y = \pi(x)$ so $\|y\| = 1$, and define $\cos \theta = \langle y, v \rangle/\|v\|$. Since $\langle x, v \rangle = 1$ it follows that $x = y/\langle y, v \rangle$ and so $\|x\| = 1/\langle y, v \rangle$.

Let $S_x \subset \mathbb{R}^{n+1}$ denote the sphere with center 0 and radius $\|x\|$. The volume form on $S_x$ is $dS_x = \|x\|^n \pi_x^* dB$, where $\pi_1 : S_x \to S^n$ is radial projection. Let $dA_x$ be the volume element on $\mathcal{C}_q$. Then $dA_x = (\cos \theta)^{-1} \pi_x^* dS_x$, where $\pi_2 : \mathcal{C}_q \to S_x$ is radial projection. Then $\pi = \pi_1 \circ \pi_2 : \mathcal{C}_q \to S^n$ is radial projection, and combining gives

$$dA_x = (\cos \theta)^{-1} \|x\|^n \pi_x^* dB = \|v\| \langle y, v \rangle^{-n-1} \pi_x^* dB_y$$

It follows from Equations (4) and (5) that

$$-\nabla \mathcal{W} = \|v\|^{-1} \int_{\mathcal{C}_q} x \, dA_x = \|v\|^{-1} \mu(\mathcal{C}_q)$$

Thus $\mathcal{W}$ has a minimum when

$$\|v\|^{-1} \mu(\mathcal{C}_q) \in \mathbb{R} \cdot q$$

Since the centroid $\mu(\mathcal{C}_q)$ is in $\mathcal{C}_q$, and $q \in \mathcal{C}_q$, it follows that at the minimum we have $\mu(\mathcal{C}_q) = q$. □

The volume function gives a natural, strictly-convex hypersurface in $\mathcal{C}$:

2.2. Definition. Suppose $\mathcal{C}$ is a properly-convex cone. Let $\mathcal{D} \subset \mathcal{C}$ be the intersection of all affine halfspaces with the property that the volume of the subset of $\mathcal{C}$ outside the halfspace equals one.

In particular $\partial \mathcal{D}$ is strictly-convex and meets every ray in $\mathcal{C}$ once. Clearly $\partial \mathcal{D}$ is preserved by $\text{SL}(\mathcal{C})$. If $q \in \partial \mathcal{D}$, then with $\phi$ as in (2.1)(iv) the tangent hyperplane to $\partial \mathcal{D}$ at $q$ is $\phi^{-1}(1)$, and $q = \mu(\mathcal{C}_q)$. In other words, the part inside $\mathcal{C}$ of the tangent plane to $\mathcal{D}$ at a point $q \in \partial \mathcal{D}$ has centroid $q$. 

![Figure 4. Volume forms on $\Omega \subset S^n$ and $\mathcal{C}_q$](image-url)
2.3. Corollary. If \( \Omega \) is properly-convex, then \( \Theta : \mathcal{C}^{\ast} \Omega \to \mathcal{C}^{\ast} \Omega \) is a homeomorphism that maps rays to rays, and \( \Theta : \Omega^\ast \to \Omega \) is a homeomorphism.

Proof. Given \( p \in \Omega \) there is a unique \( x \in \partial \mathcal{D} \) with \( p = [x] \). By (2.1)(iv) there is a unique \( \phi \in \mathcal{C}^{\ast} \) with \( \phi(x) = 1 \) and \( \nu(\phi) = 1 \). Moreover \( \mu(x, \phi) = x \) so \( \Theta(\phi) = p \). Hence \( \Theta \) is a bijection. Clearly \( \Theta \) is continuous.

By (2.1)(ii) \( \Theta \) is proper, and thus a homeomorphism. Since \( \Theta(tx) = t^{-n-1} \Theta(x) \), rays are mapped to rays. It follows that \( \Theta \) is also a homeomorphism. \( \square \)

The dual action of \( A \in \text{SL}(V) \) on \( \mathbb{P}^{\ast}V \) is given by \( A[\phi] = [\phi \circ A^{-1}] \). If \( \Gamma \subset \text{SL}(V) \) and preserves \( \Omega \), then the dual action of \( \Gamma \) preserves \( \Omega^\ast \). It is clear that \( \Theta \) is equivariant with respect to these actions. It directly follows from (1.4) and (2.3) that:

2.4. Corollary (Vinberg). If \( M = \Omega / \Gamma \) is a closed, properly-convex manifold, then \( M^\ast = \Omega^\ast / \Gamma \) is a properly-convex manifold that is homeomorphic to \( M \). Thus \( \pi_1 M \cong \pi_1 M^\ast \).

If \( M = \Omega / \Gamma \), then \( M \) is called \( C^1 \) if \( \Omega \) is \( C^1 \). It follows from (1.4) that:

2.5. Corollary. \( M \) is strictly-convex if and only if \( M^\ast \) is \( C^1 \).

2.6. Corollary. If \( M = \Omega / \Gamma \) is a closed, strictly-convex manifold, then \( M^\ast = \Omega^\ast / \Gamma \) is a closed, strictly-convex manifold. We have \( \pi_1 M \cong \pi_1 M^\ast \) and call \( M^\ast \) the dual of \( M \). Moreover, \( M \) is \( C^1 \).

Proof. Since \( M \) is strictly convex, it is properly convex, so \( M^\ast \) is a properly-convex manifold, and \( \pi_1 M \cong \pi_1 M^\ast \). By (1.10) \( M^\ast \) is closed, and strictly-convex. Thus \( M = (M^\ast)^\ast \) is \( C^1 \) by (1.4). \( \square \)

The centroid of a bounded open convex set \( \Omega \subset \mathbb{R}^n \) is a distinguished point in \( \Omega \). We wish to define something similar for certain subsets of the sphere \( S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \} \). Imagine a room that contains a transparent globe close to one wall, with a light source at the center of the globe. The shadow of Belgium appears on the wall. You can rotate the globe so that the centroid of this shadow is the point \( p \) on the wall closest to the center of the globe. The point on the globe that projects to \( p \) is called the (spherical) center of Belgium.

The open hemisphere that is the \( \pi/2 \) neighborhood of \( y \in S^n \) is \( U_y = \{ x \in S^n : \langle x, y \rangle > 0 \} \). Radial projection \( \pi_y : U_y \to T_y S^n \) from the origin onto the tangent space to \( \mathbb{R}P_n^+ \) at \( y \) is given by

\[
\pi_y(x) = \frac{x - \langle x, y \rangle y}{\langle x, y \rangle}
\]

Identify \( S^n \) with \( \mathbb{R}P_n^+ \) using \( \pi_x : S^n \to \mathbb{R}P_n^+ \).

2.7. Definition. If \( \Omega \subset \mathbb{R}P_n^+ \) is open, then \( y \in \Omega^\ast \) is called a (spherical) center of \( \Omega \) if \( \mu(y, \Omega^\ast) = \pi_x(y) \).

If \( A \in O(n+1) \), then \( A \) is an isometry of the inner product and so \( (A\Omega)^\ast = A(\Omega^\ast) \), and \( Ay \) is a center of \( A\Omega \) if \( y \) is a center of \( \Omega \). The following property of \( \partial \mathcal{D} \) is important for the proof of the Closed Theorem.

2.8. Theorem. If \( \Omega \subset S^n \) is properly-convex, then it has a unique spherical center \( [x] \). Moreover \( x \) is the point on \( \partial \mathcal{D} \) that minimizes \( \|x\| \).

Proof. Let \( \phi \in V^\ast \) be given by \( \phi(y) = \langle y, x \rangle \). The tangent plane to \( \partial \mathcal{D} \) at \( x \) is \( \phi^{-1}(1) \) and is orthogonal to \( x \). Thus \( \pi_x(\Omega) = \mathcal{C}_\phi \) and \( \mu(\mathcal{C}_\phi) = x \). \( \square \)

2.9. Corollary. If \( \Omega \subset \mathbb{R}P^n \) is properly-convex and \( p \in \Omega \), then there is an affine patch \( \mathbb{R}^n \subset \mathbb{R}P^n \) such that \( p \) is the centroid of \( \Omega \) in \( \mathbb{R}^n \).

Proof. There is a unique \( x \in \partial \mathcal{D} \) with \( p = [x] \). Choose an inner product on \( V \) so that \( x \) is the closest point to \( 0 \) on \( \partial \mathcal{D} \). The required affine patch is \( \mathbb{R}P^n \setminus \mathbb{P}(x^\perp) \). \( \square \)
A set is \textit{locally convex} if every point has a convex neighborhood. There is a basic local-to-global principle for convexity.

3.1. \textbf{Proposition.} If $K$ is a closed, connected, locally convex subset of $\mathbb{R}^n$, then $K$ is convex.

If $K$ has non-empty interior, then local convexity only needs to be checked at points in $\partial K$. Suppose that $K$ is contained in the upper halfspace $x_1 \geq 0$. Then $\Omega = \text{int}(K \cap (0 \times \mathbb{R}^n))$ is a convex subset of $\mathbb{R}^n$ if the subset $S \subset \partial K$ where $x_1 > 0$ is a locally convex hypersurface in $\mathbb{R}^{n+1}$. Informally: the base of a convex mountain is convex. We now explain how to use this to show that a manifold is properly-convex.

A smooth hypersurface $S \subset \mathbb{R}^n$ is \textit{Hessian-convex} if the surface is locally the zero-set of a smooth, real-valued function with positive-definite Hessian. Suppose $\Omega \subset \mathbb{R}^n$ is properly-convex and $\mathcal{C} \Omega = \{v \in \mathbb{R}^{n+1}_0 : [v] \in \Omega\}$ is the corresponding convex cone. Suppose $M = \Omega/\Gamma$ is a compact, and properly-convex, $n$–manifold. Then $\tilde{W} = \mathcal{C} \Omega/\Gamma \cong M \times (0, \infty)$ is a properly-convex affine $(n+1)$–manifold. We may divide out by a homothety to obtain a compact affine $(n+1)$–manifold $W \cong M \times S^1$.

Suppose there is a hypersurface $S \subset \mathcal{C} \Omega$ that is $\Gamma$–invariant and Hessian-convex away from 0. Then $Q = S/\Gamma$ is a compact, Hessian-convex, codimension–1 submanifold of $W$. Let $U$ be the component of $\mathcal{C} \Omega \setminus S$ whose closure does not contain 0. Let $K \subset \mathbb{R}^{n+1}$ be the closure of $U$ and regard $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \cup \mathbb{R}^n_{\infty}$. The interior of $K \cap \mathbb{R}^n_{\infty}$ can be identified with $\Omega$. The existence of such $Q$ \textit{imply} the existence of $S \subset \mathbb{R}^{n+1}$, which implies $\Omega$ is properly-convex, by the reasoning above.

For general reasons (the Ehresmann-Thurston principle) deforming $\Gamma$ a small amount to $\Gamma'$ gives a new projective $n$–manifold $M' \cong M$, and a new affine $(n+1)$–manifold $W' \cong M \times S^1$. Now $W'$ contains a submanifold $Q'$ that is Hessian-convex, provided the deformation of the developing map is small enough in $C^2$. It then follows that $M'$ is properly-convex. This is the approach taken in [10] for non-compact manifolds.

Here, we work with a piecewise linear submanifold $Q$ in place of a smooth one. Hessian convexity is replaced by the condition that at each vertex of the triangulation the determinants, of certain matrices formed by the relative positions of vertices, are strictly positive. This ensures local convexity near the vertex. This version of convexity is preserved by small $C^{0,1}$-deformations of the developing map. We start by reviewing these ideas for hyperbolic manifolds.

The hyperboloid model of the hyperbolic plane is the action of $\text{SO}(2,1)$ on the surface $S \subset \mathbb{R}^3$ given by $x^2 + y^2 - z^2 = -1$. If we identify $\mathbb{R}^3$ with $\mathbb{R}P^3 \setminus \mathbb{R}P^2_\infty$ then we can regard this as an affine action on $\mathbb{R}P^3$ by using the embedding $\text{SO}(2,1) \oplus (1) \subset \text{SL}(4,\mathbb{R})$. The open disc

$$D = \{[x : y : z : 0] : x^2 + y^2 < z^2\} \subset \mathbb{R}P^2_\infty$$

has the same frontier as $S$ and $(\text{SO}(2,1), D)$ is the projective (Klein) model of the hyperbolic plane.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hyperboloid_klein_model.png}
\caption{The Hyperboloid and Klein models of $\mathbb{H}^2$}
\end{figure}

The surface $\text{cl}(S \cup D) \subset \mathbb{R}P^3$ bounds a closed 3-ball $B \subset \mathbb{R}P^3$. Let $\Omega = B \setminus \text{Fr} D \cong D \times I$. The fact that $S$ is a strictly-convex surface \textit{implies} $\text{cl}(S \cup D)$ is a convex surface in some affine patch which \textit{implies} $\Omega$ is properly-convex, and this \textit{implies} $D$ is properly-convex.
The action of $\text{SO}(2,1) \oplus (1)$ preserves $B$. Suppose $\Gamma \subset \text{SO}(2,1)$ and $\Sigma = D/\Gamma$ is a properly-convex projective (and hyperbolic) surface. Let $\Gamma' = \Gamma \oplus (1)$ then $N = B/\Gamma' \cong \Sigma \times [0,1]$ is a properly-convex manifold with one flat boundary component $\Sigma = D/\Gamma$ and one strictly-convex boundary component $M = S/\Gamma$. The fact that $M$ is a strictly-convex surface implies $\Sigma$ is properly-convex. We will generalize this construction to arbitrary properly-convex manifolds in place of $\Sigma$. But first we divide out by a homothety.

Consider the cone $\mathcal{C} = \{ \lambda \cdot x : x \in \Sigma, \lambda > 0 \}$. There is a product structure $\hat{\phi} : S \times \mathbb{R}^+ \to \mathcal{C}$ given by $\hat{\phi}(x, \lambda) = \lambda^{-1} \cdot x$ on $\mathcal{C}$ that is preserved by $\Gamma'$. Let $H \subset \text{GL}(4, \mathbb{R})$ be the cyclic group generated by $\exp(\text{Diag}(0,0,0,1))$. Then $\Gamma'$ centralizes $H$ and the group $\Gamma^+ \subset \text{GL}(4, \mathbb{R})$ generated by $\Gamma'$ and $H$ preserves $\mathcal{C}$ and $W := \mathcal{C}/\Gamma^+$ is a 3–manifold.

In what follows $\mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \}$, and $S^1 = \mathbb{R}^+ / \exp(\mathbb{Z})$, has universal cover $\mathbb{R}^+$. There is a product structure $\phi : \Sigma \times S^1 \to W$ covered by $\hat{\phi}$, and the surfaces $\phi(\Sigma, \theta)$ are convex. The reader might contemplate all this in the case of one dimension lower, where $\Gamma \subset \text{SO}(1,1)$ is generated by a hyperbolic and $\mathcal{C}/\Gamma^+$ is an affine structure on $S^1 \times S^1$.

We now return to the general setting. Regard $\mathbb{R}^+$ as a subgroup of $\text{GL}(1, \mathbb{R})$, and write an element as either $x$ or as $(x)$. Let $\text{FG}(n) = \text{SL}(n, \mathbb{R}) \oplus \mathbb{R}^+ \subset \text{GL}(n+1, \mathbb{R})$. We also use the notation $\mathbb{R}^+$ to denote the subgroup $1 \oplus \mathbb{R}^+ \subset \text{FG}(n)$. Flow geometry is the subgeometry $(\text{FG}(n), \mathbb{R}^+_0) \subset (\text{GL}(n+1, \mathbb{R}), \text{RP}^n)$ of projective geometry where $\mathbb{R}^+_0 = \{ [x : 1] : x \in \mathbb{R}^+_0 \} \subset \text{RP}^n$.

The subgroup $\mathbb{R}^+ \subset \text{FG}(n)$ is called the homothety flow and the action of $1 \oplus (t)$ on $\mathbb{R}^+_0$ is given by $(1 \oplus (t))[x : 1] = [x : t]$. In affine coordinates, this action is $(1 \oplus (t))x = t^{-1} \cdot x$. Thus points in $\mathbb{R}^+_0$ move towards 0 as $t$ increases. The projection $\pi_{\text{hor}} : \text{FG}(n) \to \text{SL}(n, \mathbb{R})$ is called the horizontal holonomy and $\pi_{\text{rad}} : \text{FG}(n) \to \mathbb{R}^+$ is called the radial holonomy.

**3.2. Definition.** If $M$ is a closed $n$–manifold, then a flow product structure on $M \times S^1$ is a flow geometry structure $(\text{dev}, \rho)$ such that:

- $\text{dev}(x,t) = t^{-1} \cdot \text{dev}(x,1)$ where $(x,t) \in \tilde{M} \times \mathbb{R}^+$
- $\pi_{\text{rad}}(\rho \pi_1 M) = 1$ regarding $\pi_1 M \subset \pi_1 (M \times S^1)$

It follows that $\pi_{\text{rad}}(\rho \pi_1 M \times S^1) = \exp \mathbb{Z}$. The action of $\mathbb{R}^+$ on the right factor of $\tilde{W} = \tilde{M} \times \mathbb{R}^+$ is conjugate, by the developing map, to the homothety action on $\mathbb{R}^+_0$. The action of $S^1$ on the right factor of $W = M \times S^1$ is covered by this action on $\tilde{W}$. We call all of these actions homothety. Flow geometry is also a subgeometry of affine geometry, so we may use affine notions.

Suppose $W$ is an affine $n$–manifold, and $S$ is a hypersurface in $W$. Then $S$ is a convex hypersurface, and it is locally convex, if there is a submanifold $V \cong S \times [0,1] \subset W$ with $\partial V = S$, and every point $p \in S$ has a small neighborhood $U \subset V$, such that $\text{dev}(U) \subset \mathbb{R}^n$ is convex.

If $S$ is locally convex and, in addition, every maximal flat subset of the universal cover $\tilde{S}$ is compact, then $S$ is called strongly locally convex. This condition automatically holds if $S$ contains no segment.

A hypersurface, $S \subset W$, is simplicial if it has a triangulation by flat simplices. Strongly locally convex simplicial surfaces are a substitute for strictly convex in this context.

If $W = M \times S^1$ has a flow product structure, then $S$ is outwards locally-convex if, in addition, $t \cdot \partial V \subset V$ for some $t > 1$. In other words $\text{dev} \tilde{S}$ is locally-convex away from 0 in $\mathbb{R}^{n+1}$.

**3.3. Definition.** A flow product structure $(\text{dev}, \rho)$ on $M \times S^1$ is flow convex if $M \times \theta$ is a strongly outwards convex hypersurface in $M \times S^1$ for some (and hence all) $\theta \in S^1$.

If $\sigma, \tau \subset W$ are flat $(n-1)$–simplices and $\sigma \cap \tau \neq \emptyset$ then they are coplanar if they have lifts $\tilde{\sigma}, \tilde{\tau} \subset \tilde{W}$ with $\tilde{\sigma} \cap \tilde{\tau} \neq \emptyset$ and $\text{dev} (\tilde{\sigma} \cup \tilde{\tau})$ is contained in a hyperplane.

**3.4. Definition.** A simplicial hypersurface in an affine manifold is generic-convex if it is locally convex, and whenever two $(n-1)$-simplices intersect then they are not coplanar.
It is immediate that generic-convex implies strongly locally convex. Moreover a small movement of
the vertices of a generic-convex hypersurface produces a new generic-convex hypersurface. We state these
properties in the following:

3.5. Lemma. If $S \subset M$ is a generic-convex simplicial hypersurface, then $S$ is strongly locally convex. Moreover,
for each vertex $v$ of $S$, there is a neighborhood $U(v) \subset M$, such that the hypersurface $S'$ obtained by
moving each vertex $v$ inside $U(v)$ is generic-convex.

Proof. The first conclusion is immediate. The generic-convex condition is equivalent to the positivity of the
determinants of certain $n \times n$ matrices formed by vectors of the form $u - v$ for certain vertices, $u$, that are
adjacent to $v$ in $S$. If the vertices of $S$ are moved a small enough distance, these determinants remain positive.
Thus $S'$ is generic-convex. Now we explain the determinant condition.

Let $K$ be the union of the simplices in $S$ that contain $v$. Then $\dim K = n - 1$. Suppose $\sigma$ is a simplex in $K$
of dimension $(n - 1)$. Then $\sigma$ is contained in a hyperplane $H$. If $K$ is generic convex then the closure, $W$, of
one component of the complement of $H$ satisfies $W \cap K = \sigma$.

Conversely, $S$ is generic convex at $v$ if this condition holds for each such $\sigma$. Let $\{v_0, \cdots, v_{n-1}\}$ be the
vertices of $\sigma$. Let $f(u) = \det(v_0 - u : v_1 - u : \cdots : v_{n-1} - u)$. The condition for generic convexity is that the
sign of $f(v)$ is constant as $u$ ranges over all vertices in $K$ that are connected by an edge to $\sigma$ and that are not
in $\sigma$. This ensures the simplices adjacent to $\sigma$ in $K$ all lie on the same side of the hyperplane that contains
$\sigma$. \qed

The convex hull, $CH(Y)$, of $Y \subset \mathbb{R}^n$ is the intersection of all the closed convex sets that contain $Y$. Let
$\pi_x : V = \{v_i \in \mathbb{R}^n : x \cdot v_i > 0\}$ be the projection $\pi_x : x = [x]_+$. The next result can be proven by using the existence of a
characteristic hypersurface [16, 21], or of an affine sphere [6, 20].

3.6. Theorem. Suppose $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a convex cone, $\Omega = \pi_{\mathcal{C}}(C) \subset \mathbb{R}^p$ and $M = \Omega / \Gamma$ is a properly-convex
closed manifold.

Then there is a convex proper submanifold $P \subset \mathcal{C}$ with $t \cdot P \subset P$ for all $t \geq 1$, and $\pi_{\mathcal{C}}(\partial P) : \partial P \to \Omega$
is a homeomorphism. Moreover, $P$ can be chosen such that $\partial P / \Gamma \subset \mathcal{C} / \Gamma$ is a compact, generic-convex,
simplicial hypersurface.

Proof. Let $\mathcal{D} \subset \mathcal{C}$ be the closed convex subset given by (2.2). Then $N = \mathcal{D} / \Gamma$ is a convex submanifold of
$\mathcal{C} / \Gamma$ with boundary a strictly-convex hypersurface. Let $H \subset \mathbb{R}^{n+1}$ be an affine hyperplane that separates $\mathcal{D}$
to two components, such that the bounded component, $Y$, projects to a subset $\pi(Y) \subset N$ that is contained
in a small ball in $N$. Replace $N$ by $N \setminus \pi(Y)$. This produces a flat part in the boundary. This can be done
finitely many times, to produce a submanifold $Q \subset N$ with simplicial boundary. The preimage of $Q$ in $\mathcal{C}$ is
the required $P$. \qed

3.7. Corollary ([19]). A closed properly-convex manifold $\Omega / \Gamma$ has a convex polyhedral fundamental domain in $\Omega$.

Proof. Use the notation in the proof of (3.6). Let $H$ be a closed halfspace that contains $\mathcal{D}$ and such that
$(\partial \mathcal{D}) \cap \partial H$ is a single point. Let $Y = \cap(\gamma \cdot H)$ where the intersection is over all $\gamma \in \Gamma$. Then $\mathcal{D} \subset Y \subset \mathcal{C} \Omega$,
and $Q = Y \setminus \partial H$ is a convex polytope. Moreover $\partial Q = \Gamma \cdot Q$ is a locally finite union of images of $Q$. It
follows that $\pi_x(Q)$ is a convex polyhedron.

Suppose that $M = \Omega / \Gamma$ is properly-convex with $\Omega \subset \mathbb{R}P^n_+$ and $\Gamma \subset SU(n + 1, \mathbb{R})$. Then $\mathcal{C} = \{x \in \mathbb{R}^{n+1} : [x]_+ \in \Omega\}$ is a convex cone. Let $\Gamma' = \Gamma \oplus \exp \mathbb{Z} \subset FG(n + 1)$. Then $W = \mathcal{C} / \Gamma'$ is a flow product structure
on $M \times S^1$ by (3.6). Conversely if $(\text{dev}, \rho)$ is a flow product structure on $M \times S^1$ then $\pi_{\mathcal{C}}(\text{dev}, \rho)$ is a proper-convex structure on $M \times \theta$ because of:

3.8. Theorem. If $M \times S^1$ is compact and flow-convex, then $M \times S^1$ is properly-convex.

Proof. Let $N' = M \times S^1$ and $\pi_{\mathcal{C}'} : N' = \widetilde{M} \times \mathbb{R}^+ \to N'$ be the universal cover, and $\text{dev} : \widetilde{N}' \to \mathbb{R}^{n+1}$ the
developing map for $N'$. Let $R' = \widetilde{M} \times 1$ and choose a basepoint $p' \in R'$. Let $V \subset \mathbb{R}^{n+1}$ be a 2-dimensional

vector subspace that contains $p = \text{dev}(p')$. Then $\text{dev}^{-1} V = X \times \mathbb{R}^+$ for some 1-submanifold $X \subset \tilde{M}$. Let $C'$ be the component of $R' \cap \text{dev}^{-1} V$ that contains $p'$. Then $C'$ is a connected curve in $\text{dev}^{-1} V$ without endpoints that is transverse to the homothety flow and convex outwards.

The curve $C = \text{dev}(C')$ is immersed in $V_0$, and is everywhere transverse to the radial direction, and convex outwards: radially away from 0. Let $\pi : V_0 \to S^1$ be radial projection $\pi(x) = x/\|x\|$.

**Claim 1:** $\pi : C \to S^1$ is injective. Since $C'$ is transverse to the radial direction in $N'$, it follows that $\theta = \pi \circ \text{dev} : C \to S^1$ is an immersion. Let $\ell \subset V$ be the tangent line to $C$ at $p$. Suppose $q \in C \cap \ell$ is distinct from $p$. Then at some point $r$ in $C$ between $p$ and $q$, the distance of $r$ from $\ell$ is a maximum. This contradicts that $C$ is convex outwards at $r$. Hence $\pi(\ell)$ is an open semi-circle in $S^1$ that contains $\pi(C)$. Thus $\theta$ is an immersion of $C'$ into another arc, therefore $\theta$ is injective, and it follows that $\pi$ is injective. This proves Claim 1.

Define $R \subset \mathbb{R}^{n+1}$ to be the union, as $V$ varies, of all the curves $C$ above. Since the developing map is a local homeomorphism, each curve $C$ is an open arc, and $R$ is the developing image of a connected open subset of $\tilde{R}'$. Thus $R$ is a locally convex hypersurface without boundary.

**Claim 2:** $R$ is a closed subset of $\mathbb{R}^{n+1}$. Otherwise there is a sequence $r_k \in R$ that converges to a point $r \in \mathbb{R}^{n+1} \setminus R$. Define some two dimensional vector spaces of $\mathbb{R}^{n+1}$ by $V_k = \langle p, r_k \rangle$, and $V = \langle p, r \rangle$. Then $V_k$ converges to $V$. The segment $\gamma_k = [p, r_k]$ in $V_k$ is the developing image of a segment $\gamma_k' = [p', r_k']$ in $\tilde{N}'$. This is because $\text{dev}(\mathbb{R}^+ \cdot \tilde{R}')$ contains $\mathbb{R}^+ \cdot (R \cap V_k)$, and $\gamma_k$ is in the latter.

![Figure 6](image-url)  

**Figure 6.** $\kappa$ approaching $e^t C$ from above or below

Let $C = V \cap R$ and let $C' \subset \tilde{R}'$ be the curve with $\text{dev}(C') = C$. Let $C^+ \subset V$ be the Hausdorff limit of the curves $C_t = R \cap V_t$ in $\mathbb{R}^{n+1}$. Then $r \in C^+ \setminus C$. Let $w$ be the limit point of $C$ in $C^+$ closest along $C^+$ to $r$. Let $\kappa : [0,1) \to V$ be an affine homeomorphism with image $[p, w)$. There is an affine ray $\tilde{\kappa}' : [0,1) \to \tilde{N}'$ with $\kappa = \text{dev} \circ \tilde{\kappa}'$ that starts at $\tilde{\kappa}'(0) = p'$ and is the limit of segments $[p', c_n']$ in $\tilde{N}'$ where $c_n' \in C'$ and $\lim_{n \to \infty} \text{dev}(c_n) = w$. The points $c_n'$ in $\tilde{N}'$ leave every compact set. In $V$ the point $w$ is on $e^t C$ for some $t$, and $\kappa$ limits on $w$, either from above or from below. See Figure 6.

It follows that $\kappa' = \pi' \circ \tilde{\kappa}' : [0,1) \to N'$ is an affine ray that spirals inwards, and accumulates on, some subset $F$ of $M \times t \subset N'$. In other words $F$ is the forward limit set of $\kappa'$. Now $F$ is flat because $\kappa'$ is affine. Some component of $\text{dev}^{-1}(F)$ in $\tilde{N}'$ is not compact, otherwise $\kappa'$ converges to a limit point in $F$ that maps to $w$. This contradicts that $M \times t$ is strongly locally convex, and thus contradicts that $M \times S^1$ is flow-convex. This proves Claim 2.

Hence $R$ is a hypersurface that is properly embedded in $\mathbb{R}^{n+1}$ and that is transverse to the radial direction and convex outwards. Then $X = \bigcup_{t \geq 1} e^t \cdot R$ is the closure of the component of $\mathbb{R}^n \setminus R$ that does not contain 0. Thus $X$ is closed and has boundary $R$. Since $R = \partial X$ is a convex hypersurface, it follows that $X$ is convex, and the image of $X$ in $\mathbb{R}P^n$ is properly convex. Now $\mathbb{R}^+ \cdot \tilde{R}'$ is a clopen subset of $\tilde{N}'$, so these sets are equal. Hence $M \times S^1$ is also properly convex. \qed
Let $FC(M \times S^1) \subset \text{Hom}(\pi_1 M, \text{SL}(n+1, \mathbb{R}))$ consist of all $\sigma$ such that there is a flow convex structure $(\text{dev}, \rho)$ on $M \times S^1$ with $\sigma = \pi_{\text{hor}} \circ \rho$.

3.9. Theorem. If $M \times S^1$ is compact, then $FC(M \times S^1)$ is an open subset of $\text{Hom}(\pi_1 M, \text{SL}(n+1, \mathbb{R}))$ where $n = \dim M$.

Proof. Suppose $(\text{dev}_0, \rho_0)$ is a flow convex structure on $W = M \times S^1$. By (3.8) $W$ is properly-convex. Thus $M$ is properly-convex. Then by (3.6) there is a triangulation, $\mathcal{T}$, of $W$ such that $M \times 0$ is a generic-convex subcomplex of that which is outwards locally convex.

Let $\mathcal{T}$ be the lifted triangulation on the universal cover $\widetilde{W}$. Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be the image of $\text{dev}_0$ and $\Gamma = \rho_0(\pi_1 W)$. Then $\mathcal{C}$ is triangulated by the $\text{dev} \mathcal{T}$ and $\mathcal{C}$ is compact. Let $D$ be a finite subcomplex that contains $\mathcal{K}$ such that $\Gamma \cdot D = \mathcal{C}$. Let $\mathcal{P}$ be the (finite) set of vertices of $D$ and choose a subset $\mathcal{P} \subset \mathcal{P}$ that consists of one point in each $\Gamma$-orbit.

For each $v \in \mathcal{P}$ there is a unique $g_v \in \pi_1 M$ such that $ho_0(g_v)v \in \mathcal{P}^-$. Given $v \in \mathcal{P}$ define $f_v : \text{Rep}(M) \to \mathbb{R}^{n+1}$ by

$$f_v(\rho) = \rho(g_v^{-1})\rho_0(g_v)v$$

for $\rho \in \text{Rep}(M)$. This is continuous and $f_v(\rho_0) = v$. Define $\mathcal{P}(\rho) = \{ f_v(\rho) : v \in \mathcal{P} \}$, then $\mathcal{P}(\rho_0) = \mathcal{P}$. If $\rho$ is close enough to $\rho_0$, there is a simplicial complex $D(\rho) \subset \mathbb{R}^{n+1}$ with vertex set $\mathcal{P}(\rho)$, and a simplicial homeomorphism $F : D \to D(\rho)$ defined as follows.

Given $X \subset D$ a map $F : X \to \mathbb{R}^{n+1}$ is called equivariant if whenever $a, b \in X$ and $g \in \pi_1 M$ and $\rho_0(g)a = b$ then $\rho(g)F(a) = F(b)$. Since the action of $\Gamma$ is free there is at most one such $g$.

Define $F : \mathcal{P} \to \mathcal{P}(\rho)$ by $F(v) = f_v(\rho)$. Two vertices $u, v \in \mathcal{P}$ have the same image in $\mathcal{P}$ if and only if $\rho_0(g_v)v = \rho_0(g_u)u \in \mathcal{P}^-$. Since $\rho_0(g_v)v = \rho(g_v)f_v(\rho)$ it follows from Equation (6) that $\rho(g_v)f_v(\rho) = \rho(g_v)f_v(\rho)$, thus $\rho(g_v)F(v) = \rho(g_u)F(u)$. Hence $F$ is equivariant. Extend $F$ over each simplex $\sigma \in D$ using the affine map determined by the images of the vertices. This extension is equivariant. If $\rho$ is close enough to $\rho_0$ then $F$ is a homeomorphism.

Whenever $g \in \pi_1 M$, and $\rho_0(g)$ identifies two simplices in $\partial D$, we use $\rho(g)$ to identify the corresponding simplices of $D(\rho)$. Set $M = D/\sim$ and $N = D(\rho)/\sim$. Then $F$ covers a simplicial homeomorphism $f : M \to N$. Using $M = \mathcal{C}/\Gamma$, then $M = \mathcal{C}$, and $D \subset M$. It follows that $N$ is an affine structure on $M$ for some developing pair $(\text{dev}, \rho)$ with $\text{dev} D = F$. Since $K \subset D$, by choosing $\rho$ close enough to $\rho_0$, then $\text{dev} K$ is close to $\text{dev} K$. By choosing $K$ large enough, it follows from (3.5) that the subcomplex $W \times 0$ is generic-convex, and thus locally convex. Thus $(\text{dev}, \rho)$ is a flow convex structure on $M$.

3.1. Proof of (1. Open). Suppose $\rho_0 \in \text{Rep}_0(M)$ and $(\text{dev}, \rho_0)$ is a development pair for $M$. By (3.6) there is a flow convex structure on $M \times S^1$ with horizontal holonomy $\rho_0$. By (3.9) if $\rho$ is close to $\rho_0$ there is a flow convex structure on $M \times S^1$ with horizontal holonomy $\rho$. By (3.8) this is a properly-convex structure. Thus $M$ has a properly-convex structure with holonomy $\rho$. ☐

4. CLOSED

We first give an outline of the proof of the Closed Theorem. Suppose $\rho_k$ is a sequence of holonomies of properly-convex real projective structures on $M$, so that $M \equiv \Omega_k/\rho_k(\pi_1 M)$ with $\Omega_k$ properly-convex. Suppose the holonomies converge pointwise to $\lim \rho_k = \rho_\infty$. If $M$ is strictly-convex, a special case of Chuckrow’s theorem (4.2) implies $\rho_\infty$ is discrete and faithful; in general $\rho_\infty$ is neither. After taking a subsequence we may assume $\Omega_\infty = \lim \Omega_k \subset \mathbb{RP}^n$ exists. If $\Omega_\infty$ is properly-convex then $\Omega_\infty/\rho_\infty(\pi_1 M)$ is a properly-convex structure on $M$. But $\Omega_\infty$ might have smaller dimension, or it might not be properly-convex. We describe this by saying the domain has degenerated.

The box estimate (4.3) implies that one may replace the original sequence $\rho_k$ by conjugates $\rho'_k$ which preserve domains $\Omega'_k$ and $|\lim \rho'_k = \sigma|$, and $|\lim \Omega'_k = \Omega|$ is properly-convex. Then $N = \Omega/\sigma(\pi_1 M)$ is a properly-convex manifold homotopy equivalent to $M$. Hence $N$ is closed and $\pi_1 M \cong \pi_1 N$. It then follows from Gromov-Hausdorff convergence that $M$ is homeomorphic to $N$. Also $N$ is strictly-convex by (1.10). Finally,
\( \sigma \) is irreducible by (4.1), which implies that, in fact, the original domains \( \Omega_k \) did not degenerate. Thus \( \rho_{\alpha} \) is the holonomy of a strictly-convex structure on \( M \).

We supply the missing details, starting with the algebraic preliminaries.

4.1. Theorem (Irreducible). Suppose \( M = \Omega / \Gamma \) is a strictly-convex closed manifold and \( \dim M \geq 2 \). Then \( \Gamma \) does not preserve any proper projective subspace.

Proof. Suppose \( \mathbb{R}^{n+1} = V + U \) with \( U \neq 0 \neq V \) and \( \Gamma \) preserves \( U \). Then \( \Gamma \) preserves \( Y = \text{cl} \Omega \cap \mathbb{P}^n \cup U \). Now \( Y \cap \Omega = \emptyset \) by (1.11). Suppose \( Y = \emptyset \). By the hyperplane separation theorem, there is \( \phi \in (\mathbb{R}^{n+1})^* \) such that \( \text{ker} \phi \) contains \( U \), and \( \phi(\text{cl} \Omega) > 0 \). Thus \( \{ \phi \} \in \mathbb{P}^n, U^0 \cap \Omega^\prime \). By (2.6) the dual manifold \( M^* = \Omega^\prime / \Gamma \) is compact and strictly-convex, and the dual action of \( \Gamma \) preserves \( U^0 \), which contradicts (1.11). Thus \( Y = \emptyset \), so \( Y \subset \text{Fr} \Omega \). Since \( M \) is strictly-convex, \( Y = c \) is a single point that is fixed by \( \Gamma \).

Thus every non-trivial element of \( \Gamma \) has an axis with one endpoint at \( Y \). If \( \ell \) and \( \ell' \) are the axes of \( \gamma, \gamma' \in \Gamma \) then they limit on \( Y \). Now \( \Omega \) is \( C^1 \) at \( Y \) by (2.5) so \( d\Omega(p, \ell') \to 0 \) as the point \( p \) on \( \ell \) approaches \( c \). Let \( \pi : \Omega \to M \) be the projection. Then \( C = \pi(\ell) \) and \( C' = \pi(\ell') \) are closed geodesics that become arbitrarily close to each other. It follows that \( C = C' \), so \( \ell = \ell' \). Thus \( \Gamma \) preserves \( \ell \) which contradicts (1.11) unless \( \Omega = \ell \), but then \( \dim M = 1 \).

The following is a special case of Chuckrow’s theorem (see [8] or [14](8.4)).

4.2. Lemma. If \( M \) is a closed, strictly-convex, projective manifold and \( \dim M \geq 2 \), then the closure of \( \text{Rep}_g(M) \) in \( \text{Rep}(M) \) consists of discrete faithful representations.

Proof. Let \( d \) be the metric on \( G = \text{GL}(m+1, \mathbb{R}) \) given by \( d(g,h) = \max_1 \| (g-h)_{ij} \| \) and set \( \| g \| = d(I,g) \). Let \( \mathcal{W} \subset \pi_1 M \) be a finite generating set. Suppose the sequence \( \rho_n \in \text{Rep}_g(M) \) converges to \( \rho_{\infty} \in \text{Rep}(M) \). Then there is a compact set \( K \subset G \) such that \( \rho_n(\mathcal{W}) \subset K \) for all \( n \).

The map \( \theta : G \times G \to G \) given by \( \theta(g,h) = [(g,h),h] \) has zero derivative on \( G \times I \). Thus there is a neighborhood \( U \subset G \) of the identity such that if \( k \in K \) and \( u \in U \), then \( \| [k,u],u \| \leq \| u \|/2 \).

Since \( \rho = \rho_{\infty} \) is discrete, there is \( 1 \neq \alpha \in \pi_1 M \) that minimizes \( \| \rho(\alpha) \| \). Suppose that \( \rho(\alpha) \in U \). If \( \beta \in \mathcal{W} \), then \( \rho(\beta) \in K \), so \( \| \rho(\beta, \alpha, \alpha) \| \leq \| \rho(\alpha) \|/2 \). By minimality, \( \rho(\beta, \alpha, \alpha) = 1 \), and since \( \rho \) is injective, \( [\beta, \alpha, \alpha] = 1 \). By (1.14) \( \alpha \) and \( \beta \) commute. Since \( \mathcal{W} \) is a generating set, \( \alpha \) is central in \( \pi_1 M \). Thus the entire group preserves the axis of \( \alpha \). This contradicts (1.11) because \( \dim M \geq 2 \). Thus \( \rho(\alpha) \notin U \). Since \( U \) is an open neighborhood of the identity, it contains an open metric ball \( \mathcal{U} \), and since \( \rho(\alpha) \) is of minimal norm, we have \( \rho(\gamma) \notin U' \) for all \( 1 \neq \gamma \in \pi_1 M \). Since \( U \) and \( U' \) are independent of \( n \), we have \( \rho_{\infty}(\alpha) \notin U \) for all \( 1 \neq \gamma \in \pi_1 M \). This implies that \( \rho_{\infty} \) is discrete and faithful. \( \square \)

Let \( \mathcal{B} = \prod_{i=1}^n [-1, 1] \subset \mathbb{R}^n \subset \mathbb{R}^{p^n} \). For each \( K > 0 \), the set \( K \cdot \mathcal{B} = \prod_{i=1}^n [-K,K] \) is called a box.

4.3. Lemma (Box estimate). If \( A = (A_{ij}) \in \text{GL}(n+1, \mathbb{R}) \) and \( K \geq 1 \) and \( |A(\mathcal{B}) \subset K \cdot \mathcal{B} \) then

\[
|A_{ij}| \leq 2K \cdot |A_{n+1,n+1}|
\]

Proof. Set \( \alpha = A_{n+1,n+1} \). Using the standard basis we have

\[
[x_1 e_1 + \cdots + x_n e_n + e_{n+1}] = [x_1 x_2 : \cdots : x_n : 1] = (x_1, \ldots, x_n) \in \mathcal{B} \iff \max_i |x_i| \leq 1
\]

First consider the entries \( A_{i,n+1} \) in the last column of \( A \). Since \( e_{n+1} = 0 \in \mathcal{B} \) we have

\[
[A_{e_{n+1}}] = [A_{1,n+1} e_1 + A_{2,n+1} e_2 + \cdots + A_{n,n+1} e_n + \alpha e_{n+1}] \in K \cdot \mathcal{B}
\]

It follows that \( |A_{i,n+1}|/|\alpha| \leq K \). This establishes the bound when \( j = n+1 \) and \( i \leq n \).

Next consider the entries \( A_{n+1,j} \) in the bottom row with \( j \leq n \). Observe that

\[
|p = [te_j + e_{n+1}]| \in \mathcal{B} \iff |t| \leq 1
\]

Then \( |A|p = [A(t e_j + e_{n+1})] \in K \cdot \mathcal{B} \). This is in \( \mathbb{R}^n \) so the \( e_{n+1} \) component is not zero. Hence \( tA_{n+1,j} + \alpha \neq 0 \) whenever \( |t| \leq 1 \) and it follows that \( |A_{n+1,j}| < |\alpha| \). Since \( K \geq 1 \) the required bound follows when \( i = n+1 \) and \( j \leq n \).
The remaining entries are \(1 \leq i, j \leq n\). Since \(p = [p_1; \cdots; p_{n+1}] = [A(te_j + e_{n+1})] \in K \cdot \mathfrak{B}\) it follows that

\[
|\alpha| \leq 1 \Rightarrow \left| \frac{p_i}{p_{n+1}} \right| = \left| \frac{A_{i,n+1} + tA_{i,j}}{\alpha + tA_{n+1,j}} \right| \leq K
\]

For all \(|\alpha| \leq 1\) the denominator is not zero hence \(|A_{n+1,j}| < |\alpha|\). It follows that

\[
|\alpha + tA_{n+1,j}| \leq 2|\alpha|
\]

Thus

\[
|\alpha| \leq 1 \Rightarrow |A_{i,n+1} + tA_{i,j}| \leq 2K |\alpha|
\]

We may choose the sign of \(t = \pm 1\) so that \(A_{i,n+1} + tA_{i,j}\) have the same sign. Then

\[
|A_{i,j}| \leq |A_{i,n+1} + tA_{i,j}|
\]

This gives the result \(|A_{i,j}| \leq 2|\alpha| \cdot K\) in this remaining case. \(\Box\)

If \(\Omega \subset \mathbb{R}^n\) has finite positive Lebesgue measure and the centroid of \(\Omega\) is \(\hat{\mu}(\Omega) = 0\), then

\[
Q_\Omega(y) = \int_K \left( \|x\|^2 \|y\|^2 - \langle x, y \rangle^2 \right) \, d\text{vol}_x
\]

is a positive definite quadratic form on \(\mathbb{R}^n\) called the inertia tensor.

4.4. Lemma (Uniform estimate). For each dimension \(n\) there is \(K = K(n) > 1\) such if \(\Omega \subset \mathbb{R}^n\) is an open bounded convex set with inertia tensor \(Q_\Omega = x_1^2 + \cdots + x_n^2\) and centroid at the origin, then \(K^{-1} \mathfrak{B} \subset \Omega \subset K \cdot \mathfrak{B}\).

Moreover, if \(A \in \mathrm{GL}(\Omega)\), then \(|A_{ij}| \leq K |A_{n+1,n+1}|\).

Proof. The first conclusion follows from the theorem of Fritz John [13], see also [1]. Let \(D \in \mathrm{GL}(n + 1, \mathbb{R})\) be the diagonal matrix \(\text{Diag}(K, \cdots, K, 1)\) then \(\mathfrak{B} \subset D(\Omega) \subset K^2 \mathfrak{B}\). Set \(A' = D \cdot A \cdot D^{-1}\) then \(A' \in \mathrm{GL}(D(\Omega))\), thus \(|A'_{ij}| \leq 2K^2 |A'_{n+1,n+1}|\) by (4.3). Now \(|A'_{n+1,n+1}| = |A_{n+1,n+1}|\) and \(|A_{ij}| \leq K^2 |A'_{ij}|\) thus \(|A_{ij}| \leq 2K^4 \cdot |A_{n+1,n+1}|\). The result now holds using the constant \(2K^4\). \(\Box\)

4.1. Proof of (2. Closed). Suppose \(\rho \in \text{Rep}(M)\) is the limit of the sequence \(\rho_k \in \text{Rep}_1(M)\). Let \(\Omega_k \subset \mathbb{R}^{p'}\) be the properly-convex open set preserved by \(\Gamma_k = \rho_k(\pi_1(M))\), then \(M \equiv M_k = \Omega_k/\Gamma_k\). Choose an affine patch \(\mathbb{R}^n \subset \mathbb{R}^{p'}\). Then by (2.8) there is \(\alpha_k \in \text{PO}(n + 1)\) such that \(\alpha_k(\Omega_k) \subset \mathbb{R}^n\) has center \(0 \in \mathbb{R}^n\). We may choose \(\alpha_k\) so that the interia tensor \(Q_k = Q(\alpha_k(\Omega_k))\) is diagonal in the standard coordinates on \(\mathbb{R}^n\), and the entries on the main diagonal of \(Q_k\) are non-increasing going down the diagonal. Since \(\text{PO}(n + 1)\) is compact, after subsecquenting we may assume the conjugates of \(\rho_k\) by \(\alpha_k\) converge. We now replace the original sequence of representations and domains by this new sequence.

Let \(K = K(n)\) be given by (4.4). There is a unique positive diagonal matrix \(D_k\) such that \(Q_k = D_k^{-2}\). Set \(\Omega'_k = D_k \Omega_k\), then \(Q_{\Omega'_k} = x_1^2 + \cdots + x_n^2\). By (4.4), there is \(K > 1\) depending only on \(n\), such that

\[
K^{-1} \mathfrak{B} \subset \Omega'_k \subset K \cdot \mathfrak{B}
\]

Given \(g \in \pi_1 M\), then \(A = A(k, g) = \rho_k(g) \in \text{SL}(n + 1, \mathbb{R})\) preserves \(\Omega_k\). The matrix \(B = B(k, g) = D_k A(k, g) D_k^{-1}\) preserves \(\Omega'_k\). By (4.4)

\[
\forall i, j \quad |B_{i,j}| \leq K \cdot |B_{n+1,n+1}|
\]

Since \(D_k\) is diagonal it follows that

\[
B_{n+1,n+1} = \alpha_{n+1,n+1}
\]

Now \(A_{n+1,n+1} = A(k, g)_{n+1,n+1}\) converges as \(k \to \infty\) for each \(g\). Hence the entries of \(B(k, g)\) are uniformly bounded for fixed \(g\) as \(k \to \infty\). Thus we may pass to a subsequence where \(B(k, g) = D_k \rho_k(g) D_k^{-1}\) converges for every \(g \in \pi_1 M\), and this gives a limiting representation \(\sigma = \lim D_k \rho_k D_k^{-1}\).
The space consisting of properly-convex open sets $\Omega$ with $K^{-1} \cdot \mathcal{B}_1 \subset \Omega \subset K \cdot \mathcal{B}_1$ is compact. Therefore there is a subsequence so that $\Omega = \lim x_i$ exists. Then $K^{-1} \mathcal{B}_1 \subset \Omega \subset K \cdot \mathcal{B}_1$, hence $\Omega$ is open and properly-convex. Then $\sigma$ is discrete and faithful by (4.2), and $\Gamma = \sigma(\pi_1 M)$ preserves $\Omega$, so $N = \Omega/\Gamma$ is a properly-convex manifold. Since $M$ is closed and $\pi_1 M \cong \pi_1 N$, then by (1.5) $N$ is also closed. Since $M$ is strictly-convex, (1.10) implies $N$ is strictly-convex.

Since $\text{Rep}_p(N)$ is open, for $D_k \rho_k(g) D_k^{-1}$ close enough to $\sigma$, there is a properly-convex open set $\Omega'_k \subset \mathbb{R}^m$ that is preserved by $\Gamma_k$ and $N_k = \Omega'_k/\Gamma_k$ is homeomorphic to $N$. By (1.10) $N_k$ is strictly-convex. By (1.13) $\Omega'_k = \Omega_k$ so $N \cong N_k = M_k \cong M$. Thus $\sigma \in \text{Rep}_p(M)$.

If $D_k$ does not remain bounded, since the entries are non-increasing going down the diagonal, and $\rho_k(g)$ remains bounded, $\sigma(g)$ is block upper triangular and therefore $\sigma$ is reducible. Since $N$ is strictly-convex, (4.1) implies $\sigma$ is irreducible, therefore $D_k$ stays bounded. Hence we may subsequence the $D_k$ so they converge. Then $\sigma = \rho$, so $\rho \in \text{Rep}_p(M)$.

\section*{References}

[1] Keith Ball. Ellipsoids of maximal volume in convex bodies. \textit{Geom. Dedicata}, 41(2):241–250, 1992.
[2] Yves Benoist. Convexes divisibles. I. In \textit{Algebraic groups and arithmetic}, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004.
[3] Yves Benoist. Convexes divisibles. III. \textit{Ann. Sci. École Norm. Sup.} (4), 38(5):793–832, 2005.
[4] Yves Benoist. A survey on divisible convex sets. In \textit{Geometry, analysis and topology of discrete groups}, volume 6 of \textit{Adv. Lect. Math. (ALM)}, pages 1–18. Int. Press, Somerville, MA, 2008.
[5] Jean-Paul Benzécri. Sur les variétés localement affines et localement projectives. \textit{Bull. Soc. Math. France}, 88:229–332, 1960.
[6] Shiu Yuen Cheng and Shing Tung Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$. \textit{Comm. Pure Appl. Math.}, 30(1):41–68, 1977.
[7] Suhyoung Choi and William Goldman. Convex real projective structures on closed surfaces are closed. \textit{Proc. Amer. Math. Soc.}, pages 657–661, 1993.
[8] Vicki Chuckrow. Schottky groups and limits of Kleinian groups. \textit{Bull. Amer. Math. Soc.}, 73:139–141, 1967.
[9] Daryl Cooper, Darren D. Long, and Stephan Tillmann. On convex projective manifolds and cusps. \textit{Adv. Math.}, 277:181–251, 2015.
[10] Daryl Cooper, Darren D. Long, and Stephan Tillmann. Deforming convex projective manifolds. \textit{Geom. Topol.}, 22(3):1349–1404, 2018.
[11] Daryl Cooper and Stephan Tillmann. On properly convex real-projective manifolds with generalized cusps. arXiv 2009.06569, 2020.
[12] Daryl Cooper and Stephan Tillmann. The space of properly-convex structures. arXiv 2009.06568, 2020.
[13] Fritz John. Extremum problems with inequalities as subsidiary conditions. In \textit{Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948}, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.
[14] Michael Kapovich. \textit{Hyperbolic manifolds and discrete groups}. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 2001 edition.
[15] Inkang Kim. Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds. \textit{J. Differential Geom.}, pages 189–2181, 2001.
[16] Max Koecher. Positivitätssbereiche im $R^n$. \textit{Amer. J. Math.}, 79:575–596, 1957.
[17] Jean-Louis Koszul. Variétés localement plates et convexité. \textit{Osaka J. Math.}, 2:285–290, 1965.
[18] Jean-Louis Koszul. Déformations de connexions localement plates. \textit{Ann. Inst. Fourier (Grenoble)}, 18(fasc. 1):103–114, 1968.
[19] Jaejeong Lee. A convexity theorem for real projective structures. \textit{Geom. Dedicata}, 182:1–41, 2016.
[20] John Loftin. Survey on affine spheres. In \textit{Handbook of geometric analysis, No. 2}, volume 13 of \textit{Adv. Lect. Math. (ALM)}, pages 161–191. Int. Press, Somerville, MA, 2010.
[21] Ernest B. Vinberg. The theory of homogeneous convex cones. \textit{Trudy Moskov. Mat. Obšč.}, 12:303–358, 1963.