Deformation of the $\sigma_2$-curvature

Almir Silva Santos* and Maria Andrade†

Universidade Federal de Sergipe, Departamento de Matemática
São Cristóvão-SE, 49100-000, Brasil

Abstract

Our main goal in this work is to deal with results concern to the $\sigma_2$-curvature. First we find a symmetric 2-tensor canonically associated to the $\sigma_2$-curvature and we present an Almost Schur Type Lemma. Using this tensor we introduce the notion of $\sigma_2$-singular space and under a certain hypothesis we prove a rigidity result. Also we deal with the relations between flat metrics and $\sigma_2$-curvature. With a suitable condition on the $\sigma_2$-curvature we show that a metric has to be flat if it is close to a flat metric. We conclude this paper by proving that the 3-dimensional torus does not admit a metric with constant scalar curvature and non-negative $\sigma_2$-curvature unless it is flat.

1 Introduction

In 1975, Fischer and Marsden [11] studied deformations of the scalar curvature in a smooth manifold $M$. They proved several results concern the scalar curvature map $R : \mathcal{M} \to C^\infty(M)$ which associates to each metric $g \in \mathcal{M}$ its scalar curvature $R_g$. Here $\mathcal{M}$ is the Riemannian metric space and $C^\infty(M)$ is the smooth functions space in $M$. Using the linearization of the map $R$ at a given metric $g$ and its $L^2$-formal adjoint, they were able to prove local surjectivity of $R$ under certain hypothesis. Another interesting result concerns flat metrics. They proved that if $g$ has nonnegative scalar curvature and it is close to a flat metric, then $g$ is also flat. Recently, Lin and Yuan [16] extended many of the results contained in [11] to the $Q$-curvature context.

Our goal in this work is to study deformations of the $\sigma_2$-curvature and prove analogous results to those in [11] and [16] in this setting.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. The $\sigma_2$-curvature, which we will denote by $\sigma_2(g)$, is defined as the second elementary symmetric
function of the eigenvalue of the tensor $A_g = \text{Ric}_g - \frac{R_g}{2(n-1)} g$, where $\text{Ric}_g$ and $R_g$ are the Ricci and scalar curvature of the metric $g$, respectively. A simple calculation gives

$$\sigma_2(g) = -\frac{1}{2} |\text{Ric}_g|^2 + \frac{n}{8(n-1)} R_g^2,$$

(1)

See [20] and the reference contained therein, for motivation to study the $\sigma_2$-curvature and its generalizations, the $\sigma_k$-curvature. We notice that the definition (1) and the definition in [20] are the same, up to a constant which depends only on the dimension of the manifold.

In [16] the authors observed that if $\gamma_g$ is the linearization of the map $R$ and $\gamma_g^*$ is its $L^2$-formal adjoint then $\gamma_g^*(1) = -\text{Ric}_g$ and then $R_g = -\text{tr}_g \gamma_g^*(1)$.

In fact, it is well known that

$$\gamma_g h = -\Delta_g \text{tr}_g h + \delta_g^2 h - \langle \text{Ric}_g, h \rangle$$

and

$$\gamma_g^*(f) = \nabla_g^2 f - (\Delta_g f) g - f \text{Ric}_g,$$

where $\delta_g = -\text{div}_g$ and $h \in S_2(M)$. Here $S_2(M)$ is the space of symmetric 2-tensor on $M$.

It is natural to ask whether for each kind of curvature there is a 2-tensor canonically associated to it and if so which information about the manifold we can find through this tensor. In [16] and [17] the authors addressed this problem in the $Q$-curvature context and have found some interesting results.

Motivated by the works [11], [16] and [17] we consider the $\sigma_2$-curvature as a nonlinear map $\sigma_2 : M \to C^\infty(M)$. Let $\Lambda_g : S_2(M) \to C^\infty(M)$ be the linearization of the $\sigma_2$-curvature at the metric $g$ and $\Lambda_g^* : C^\infty(M) \to S_2(M)$ be its $L^2$-formal adjoint. After some calculations we obtain the explicit expression of $\Lambda_g$ and $\Lambda_g^*$, see Propositions [11] and [2]. We will show that the relation between $\Lambda_g^*(1)$ and the $\sigma_2$-curvature is similar to the relation between the Ricci and scalar curvature. As a first application we derive a Schur Type Lemma for the $\sigma_2$-curvature. Moreover, we derive an almost-Schur lemma similar to that for the scalar curvature and the $Q$-curvature, see [10] and [17].

We notice here that the $Q$-curvature is defined as

$$Q_g = -\frac{1}{n-2} \Delta_g \sigma_1(g) + \frac{4}{(n-2)^2} \sigma_2(g) + \frac{n-4}{2(n-2)^2} \sigma_1(g)^2,$$

(2)

where $\sigma_1(g) = \text{tr}_g A_g$. Although the $\sigma_2$-curvature appears in the $Q$-curvature definition, it is not immediate that the $Q$-curvature and $\sigma_2$-curvature share properties. In fact this is not true. In conformal geometry, for instance, while the $Q$-curvature satisfies a similar equation to the Yamabe equation,
the conformal change of the \( \sigma_2 \)-curvature is more complicated, see [12], [15] and [20].

In [5], Chang, Gursky and Yang have defined a \( Q \)-singular space as a complete Riemannian manifold which has the \( L^2 \)-formal adjoint of the linearization of the \( Q \)-curvature map with nontrivial kernel. This motivated us to define the following notion of \( \sigma_2 \)-singular space.

**Definition 1.** A complete Riemannian manifold \((M, g)\) is \( \sigma_2 \)-singular if

\[ \ker \Lambda_g^* \neq \{0\}, \]

where \( \Lambda_g^* : C^\infty(M) \to S_2(M) \) is the \( L^2 \)-formal adjoint of \( \Lambda_g \). We will call the triple \((M, g, f)\) as a \( \sigma_2 \)-singular space if \( f \) is a nontrivial function in \( \ker \Lambda_g^* \).

In Section 3 we will prove our first result using this notion.

**Theorem 1.** Let \((M^n, g, f)\) be a \( \sigma_2 \)-singular space.

(a) If \( \text{Ric}_g - \frac{1}{2} R_g g \) has a sign in the tensorial sense, then the \( \sigma_2 \)-curvature is constant.

(b) If \( f \) is a nonzero constant function, then the \( \sigma_2 \)-curvature is identically zero.

In the same direction of the vacuum static spaces and the \( Q \)-singular spaces we expected to prove that the \( \sigma_2 \)-singular spaces give rise to a small set in the space of all Riemannian manifolds. For this purpose we prove that there is no \( \sigma_2 \)-singular Einstein manifold with negative scalar curvature.

**Theorem 2.** Let \((M^n, g)\) be a closed Einstein manifold with negative scalar curvature. Then

\[ \ker \Lambda_g^* = \{0\} \]

that is, \((M^n, g)\) cannot be a \( \sigma_2 \)-singular space.

On the other hand, if a \( \sigma_2 \)-singular Einstein manifold has positive \( \sigma_2 \)-curvature, then we obtain the following rigidity result.

**Theorem 3.** Let \((M^n, g, f)\) be a closed \( \sigma_2 \)-singular Einstein manifold with positive \( \sigma_2 \)-curvature. Then \((M^n, g)\) is isometric to the round sphere with radius

\[ r = \left( \frac{n(n-1)}{R_g} \right)^{\frac{1}{2}} \]

and \( f \) is an eigenfunction of the Laplacian associated to the first eigenvalue \( \frac{R_g}{n-1} \) on \( S^n(r) \).

If the manifold is not \( \sigma_2 \)-singular we can find a locally prescribed \( \sigma_2 \)-curvature problem.

**Theorem 4.** Let \((M^n, g_0)\) be a closed Riemannian manifold not \( \sigma_2 \)-singular and satisfying one of the conditions
(i) It is Einstein with positive $\sigma_2$-curvature; or

(ii) $\sigma_2(g) > \frac{1}{2(n-1)}|Ric_g|^2$.

Then, there is a neighborhood $U \subset C^\infty(M)$ of $\sigma_2(g_0)$ such that for any $\psi \in U$, there is a metric $g$ on $M$ close to $g_0$ with $\sigma_2(g) = \psi$.

While the condition (ii) in the theorem can be satisfied by manifolds of any dimension, we notice here that by the Cauchy-Schwarz inequality we have $R_g^2 \leq n|Ric_g|^2$. This implies that if a manifold satisfies the condition (ii), which is equivalent to $R_g^2 > 4|Ric_g|^2$, then its dimension has to be greater than 4. It should be an interesting problem to study what happens in dimension 3 and 4. Also, we do not know if the condition (ii) is needed or not; that is, if we can prove the result only with $\sigma_2(g) \geq 0$.

Finally, in Section 4 we study the relation between flat metrics and the $\sigma_2$-curvature which the first result reads as follows.

**Theorem 5.** Let $(M^n, g_0)$ be a closed flat Riemannian manifold with dimension $n \geq 3$. Let $g$ be a metric on $M$ with

$$\int_M \sigma_2(g) dv_g > \frac{1}{8n(n-1)} \int_M R_g^2 dv_g.$$

If $||g - g_0||_{C^2(M, g_0)}$ is sufficiently small, then $g$ is also flat.

Our last result is concerned with metrics in the 3-torus with constant scalar curvature and nonnegative $\sigma_2$-curvature. This result can be viewed as an extension to $\sigma_2$-curvature context of the result by Schoen-Yau [18].

**Theorem 6.** The 3-dimensional torus does not admit a metric with constant scalar curvature and nonnegative $\sigma_2$-curvature unless it is flat.

The organization of this paper is as follows. In Section 2 we do the calculations to find the linearization of the $\sigma_2$-curvature map and its $L^2$-formal adjoint. Then we find a symmetric 2-tensor canonically associated to the $\sigma_2$-curvature and we find an almost-Schur lemma in this context. In Section 3 we introduce the notion of $\sigma_2$-singular space and we prove some results. For instance, we prove that under certain hypothesis a $\sigma_2$-singular space has constant $\sigma_2$-curvature. We prove that there is no $\sigma_2$-singular Einstein manifold with negative scalar curvature. Another interesting result in this section is that the only $\sigma_2$-singular Einstein manifold with positive $\sigma_2$-curvature is the round sphere. Moreover, we study a local prescribed $\sigma_2$-curvature problem for non $\sigma_2$-singular manifolds. Finally, in Section 4 we study the relations between flat metrics and $\sigma_2$-curvature. Also we prove a non existence result in the 3-torus.
2 A symmetric 2-tensor canonically associated to the $\sigma_2$-curvature

In this section, we will find a symmetric 2-tensor canonically associated to the $\sigma_2$-curvature which plays an analogous role to that of the Ricci curvature. Then using the result in [6] we find directly an almost-Schur type lemma analogous to that in [10]. Some of the calculations in this section can be found in [4] and [13] (see also [14]). For the sake of the reader we do some of the calculations.

Let $(M,g)$ be an $n$-dimensional closed Riemannian manifold with $n \geq 3$. Consider a one-parameter family of metrics $\{g(t)\}$ on $M$ with $t \in (-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$ and $g(0) = g$. Define $h := \frac{\partial}{\partial t} \bigg|_{t=0} g(t)$. It is well-known that the evolution equations of the Ricci and the scalar curvature are

$$\frac{\partial}{\partial t} Ric_g = -\frac{1}{2} \left( \Delta_L h + \nabla^2 tr_g h + 2\delta^* \delta h \right)$$

and

$$\frac{\partial}{\partial t} R_g = -\Delta_g tr_g h + \delta^2 h - \langle Ric, h \rangle,$$

respectively, where

$$\Delta_L h := \Delta h + 2 \hat{R}(h) - Ric \circ h - h \circ Ric$$

is the Lichnerowicz Laplacian acting on symmetric 2-tensor. See [8], for instance. Here $\delta h = -\text{div}_g h$, $\delta^*$ is the $L^2$-formal adjoint of $\delta$, $\hat{R}(h)_{ij} = g^{kl} g^{st} R_{klst} h_{lt}$ and $(A \circ B)_{ij} = g^{kl} A_{ki} B_{lj}$ for any $A, B \in S^2(M)$.

First we have the following lemma which the proof is a direct calculation.

**Lemma 1.**

$$\frac{\partial}{\partial t} \left| Ric_g(t) \right|^2 = 2 \left\langle Ric_g, \frac{\partial}{\partial t} Ric_g \right\rangle - 2\langle Ric_g \circ Ric_g, h \rangle$$

and

$$\frac{\partial^2}{\partial t^2} \left| Ric_g(t) \right|^2 = 4 \langle h \circ h, Ric \circ Ric \rangle + 2 |Ric \circ h|^2 - 8 \left\langle Ric \circ h, \frac{\partial}{\partial t} Ric \right\rangle + 2 |\partial_t Ric|^2 + 2 \left\langle Ric, \frac{\partial^2}{\partial t^2} Ric \right\rangle.$$ 

Thus we can find the linearization of the $\sigma_2$-curvature.

**Proposition 1.** The linearization of the $\sigma_2$-curvature is given by

$$\Lambda_g(h) = \frac{1}{2} \left\langle Ric_g, \Delta_g h + \nabla^2 tr_g h + 2\delta^* \delta h + 2\hat{R}(h) \right\rangle$$

$$- \frac{n}{4(n-1)} R_g \left( \Delta_g tr_g h - \delta^2 h + \langle Ric, h \rangle \right).$$
Proof. Let \( \{g(t)\} \) be a family of metrics as before. By (1), (3), (4) and Lemma 1 we get
\[
\frac{\partial}{\partial t} \sigma_2(g) = \langle Ric_g \circ Ric_g, h \rangle + \frac{1}{2} \langle Ric_g, \Delta_L h + \nabla^2 tr_g h + 2\delta^* \delta h \rangle - \frac{n}{4(n-1)} R_g (\Delta_g tr_g h - \delta^2 h + \langle Ric, h \rangle)
\]
and then, by the definition of the Lichnerowicz Laplacian (5), we obtain the result.

Therefore we can find the expression of \( \Lambda_g^* \) which the proof is a simple calculation using integration by parts and taking compactly supported symmetric 2-tensor \( h \in S_2(M) \).

**Proposition 2.** The \( L^2 \)-formal adjoint of the operator \( \Lambda_g : S_2(M) \to C^\infty(M) \) is the operator \( \Lambda_g^* : C^\infty(M) \to S_2(M) \) given by
\[
\Lambda_g^*(f) = \frac{1}{2} \Delta_g(f Ric_g) + \frac{1}{2} \delta^2(f Ric_g)g + \delta^* \delta(f Ric_g) + f \hat{R}(Ric_g) - \frac{n}{4(n-1)} (\Delta_g(f R_g)g - \nabla^2(f R_g)g + f R_g Ric_g).
\]

This implies that
\[
tr_g \Lambda_g^*(f) = \frac{2-n}{4} R_g \Delta_g f + \frac{n-2}{2} \langle \nabla^2 f, Ric_g \rangle - 2\sigma_2(g)f. \quad (6)
\]

For any smooth vector field \( X \in \mathcal{X}(M) \), if we denote by \( X^* \) its dual form, that is, \( X^*(Y) = g(X, Y) \) for all \( Y \in \mathcal{X}(M) \), then for any \( f \in C^\infty(M) \) we have
\[
\int_M \langle X^*, \delta \Lambda_g^*(f) \rangle dv_g = \int_M \langle \delta^* X^*, \Lambda_g^*(f) \rangle dv_g = \frac{1}{2} \int_M f \Lambda_g(\mathcal{L}_X g) dv_g,
\]
since \( \delta^* X^* = \frac{1}{2} \mathcal{L}_X g \). Let \( \varphi_t \) be the one-parameter group of diffeomorphisms generate by \( X \) such that \( \varphi_0 = id_M \). Since the \( \sigma_2 \)-curvature is invariant by diffeomorphisms, we have \( \sigma_2(\varphi_t^* g) = \varphi_t^* (\sigma_2(g)) \). This implies that
\[
\Lambda_g(\mathcal{L}_X g) = D\sigma_2(g)(\mathcal{L}_X g) = \langle d\sigma_2(g), X^* \rangle
\]
and so
\[
\int_M \langle X^*, \delta \Lambda_g^*(f) \rangle dv_g = \frac{1}{2} \int_M \langle fd\sigma_2(g), X^* \rangle dv_g.
\]
Therefore
\[
\delta \Lambda_g^*(f) = \frac{1}{2} f d\sigma_2(g). \quad (7)
\]
By (6) and (7) we obtain
\[
tr_g \Lambda_g^*(1) = -2\sigma_2(g). \quad (8)
\]
and
\[ \text{div}_g \Lambda^*_g(1) = -\frac{1}{2}d\sigma_2(g). \] (9)

The relations (8) and (9) are similar to the relations between the Ricci tensor and the scalar curvature, namely \( R_g = \text{tr}_g \text{Ric}_g \) and \( \text{div}_g \text{Ric}_g = \frac{1}{2}dR_g \).

In [6] and [7], Cheng generalized the so-called almost-Schur lemma by De Lellis and Topping [10] which is close related to the classical Schur’s Lemma for Einstein manifolds. As a consequence of (8), (9) and the Theorem 1.8 in [7] we find an almost-Schur type lemma for the \( \sigma_2 \)-curvature.

**Theorem 7.** Let \((M,g)\) be a closed Riemannian manifold of dimension \(n \neq 4\). Suppose that \(\text{Ric}_g \geq -(n-1)K\) for some constant \(K \geq 0\). Then
\[
\int_M (\sigma_2(g) - \overline{\sigma_2(g)})^2 \leq \frac{8n(n-1)}{(n-4)^2} \left( 1 + \frac{nK}{\lambda_1} \right) \int_M \left| \Lambda^*_g(1) + \frac{2}{n}\overline{\sigma_2(g)}g \right|^2,
\]
or equivalently
\[
\int_M \left| \Lambda^*_g(1) + \frac{2}{n}\overline{\sigma_2(g)}g \right|^2 \leq \left( 1 + \frac{16(n-1)}{(n-4)^2} \left( 1 + \frac{K}{\lambda_1} \right) \right) \int_M \left| \Lambda^*_g(1) + \frac{2}{n}\overline{\sigma_2(g)}g \right|^2
\]
where \(\overline{\sigma_2(g)}\) denotes the average of \(\sigma_2(g)\) over \(M\), \(\lambda_1\) denotes the first nonzero eigenvalue of the Laplace operator on \((M,g)\). If \(\text{Ric}_g > 0\) then the equality holds if and only if \(\Lambda^*_g(1) = -\frac{2}{n}\overline{\sigma_2(g)}g\), with constant \(\sigma_2\)-curvature.

In [7] the author has proved an almost-Schur lemma for the \(\sigma_k\)-curvature in locally conformally flat manifold. In this case, he used the Newton Transformations associated with \(A_g\) and a result by Viaclovsky [22] to get a divergence free tensor and the identity (9).

### 3 \(\sigma_2\)-singular space

In this section we use the definition of the \(\sigma_2\)-singular space, Definition 11 to prove some interesting results. First we obtain that under a certain hypothesis a \(\sigma_2\)-singular space has constant \(\sigma_2\)-curvature. Then we classify all closed \(\sigma_2\)-singular Einstein manifold with \(\sigma_2\)-curvature. Finally in the spirit of Fischer-Marsden [11] we prove the local surjectivity of the \(\sigma_2\)-curvature.

Our first result in this section reads as follows.

**Theorem 8 (Theorem 11).** Let \((M^n, g, f)\) be a \(\sigma_2\)-singular space.

(a) If \(\text{Ric}_g - \frac{1}{2}R_g g\) has a sign in the tensorial sense, then the \(\sigma_2\)-curvature is constant.

(b) If \(f\) is a nonzero constant function, then the \(\sigma_2\)-curvature is identically zero.
Proof. In the equation (7) we obtained that

\[ \text{div } \Lambda_g^*(f) = -\frac{1}{2} f d\sigma_2(g). \]

Since \((M^n, g, f)\) is a \(\sigma_2\)-singular space, then \(\Lambda_g^*(f) = 0\) and so \(f d\sigma_2(g) = 0\). Suppose that there exists an \(x_0 \in M\) with \(f(x_0) = 0\) and \(d\sigma_2(g)_{x_0} \neq 0\). By taking derivatives, we can see that \(\nabla^m f(x_0) = 0\) for all \(m \geq 1\). Moreover, note that by (6) the function \(f\) satisfies

\[ \frac{n-2}{2} \left( \nabla^2 f, \text{Ric}_g - \frac{1}{2} R_g g \right) - 2\sigma_2(g)f = 0. \]  

(10)

By hypothesis \(\text{Ric}_g - \frac{1}{2} R_g g\) has a sign in the tensorial sense, which implies that (10) is an elliptic equation. By results in [1] and [9] we can conclude that \(f\) vanishes identically in \(M\). But this is a contradiction. Therefore, \(d\sigma_2(g)\) vanishes in \(M\) and thus \(\sigma_2(g)\) is constant.

Now, if \(f\) is a nonzero constant, then we can suppose that the constant is equal to 1. Therefore, by (8) we obtain

\[ \text{tr}_g \Lambda_g^*(1) = -2\sigma_2(g) = 0. \]

\[ \square \]

Lemma 2. Let \((M^n, g)\) be an Einstein manifold with dimension \(n \geq 3\), then

\[ \Lambda_g^*(f) = \frac{(n - 2)^2}{4n(n - 1)} \left( R_g \nabla^2 f - R_g (\Delta_g f) g - \frac{R_g^2}{n} g f \right). \]  

(11)

Moreover, if \(f \in \ker \Lambda_g^*\) then

\[ R_g \nabla^2 f_g + \frac{1}{n(n - 1)} R_g^2 f g = 0. \]  

(12)

Proof. Since \((M, g)\) is Einstein, then \(\text{Ric}_g = \frac{R_g}{n} g\), \(R_g\) is constant and

\[ \sigma_2(g) = \frac{(n - 2)^2}{8n(n - 1)} R_g^2. \]  

(13)
By Proposition 2 we get
\[ \Lambda^*_g(f) = \frac{1}{2} \Delta_g(f \text{Ric}_g) + \frac{1}{2} \text{div}(\text{div}(f \text{Ric}_g))g + \delta \delta(f \text{Ric}_g) + f \hat{R}(\text{Ric}_g) \]
\[ - \frac{n}{4(n-1)} (\Delta_g(f R_g)g - \nabla^2(f R_g) + f R_g \text{Ric}_g) \]
\[ = \frac{R_g}{n}(\Delta_g f)g - \frac{R_g}{n} \nabla^2 f + f \frac{R^2_g}{n^2}g \]
\[ - \frac{n}{4(n-1)} \left( R_g(\Delta_g f)g - R_g \nabla^2 f + f \frac{R^2_g}{n}g \right) \]
\[ = -R_g \Delta_g f g \left( \frac{(n-2)^2}{4n(n-1)} \right) + R_g \nabla^2 f \left( \frac{(n-2)^2}{4n(n-1)} \right) \]
\[ - R^2_g f \left( \frac{(n-2)^2}{4n^2(n-1)} \right). \]

Therefore we get (14). Now, if \( f \in \ker \Lambda^*_g \), then by (6) and (13) we get
\[ R_g \Delta_g f = -\frac{R^2_g}{n-1} f. \] Thus we obtain (12). \( \square \)

The trivial examples of \( \sigma_2 \)-singular space are Ricci flat spaces, because by Proposition 2 we have \( \Lambda^*_g \equiv 0 \).

Let \( \mathbb{M} \) be the round sphere \( S^n \subset \mathbb{R}^{n+1} \) or the hyperbolic space
\[ \mathbb{H}^n = \{ (x', x_{n+1}) \in \mathbb{R}^{n+1}; |x'|^2 - |x_{n+1}|^2 = -1, x' \in \mathbb{R}^n, x_{n+1} > 0 \}. \]

It is well known that if we take the function \( f \) defined in \( \mathbb{M} \) by \( f(x) = x_k \) for some \( k \in \{1, \ldots, n+1\} \), then
\[ \nabla^2 f + \delta f = 0, \]
where \( \delta = +1 \) in the sphere and \( \delta = -1 \) in the hyperbolic space. Since the scalar curvature of \( \mathbb{M} \) is \( \delta n(n-1) \), then by Lemma 2 we have \( \Lambda^*_g(f) = 0 \). We point out that these examples are also consequence of Tashiro’s work [21].

Up to isometries, we will prove in Theorem 10 that the round sphere is the only closed \( \sigma_2 \)-singular Einstein manifold with positive \( \sigma_2 \)-curvature.

**Lemma 3.** If \( \sigma_2(g) > 0 \) and \( (M^n, g, f) \) is a closed \( \sigma_2 \)-singular Einstein manifold, then \( R_g > 0 \).

**Proof.** Since \((M, g)\) is an Einstein manifold, then by (13) we obtain that \( R_g \neq 0 \). By Theorem 8 the function \( f \) is not constant, and by (12) in Lemma 2 it satisfies
\[ \nabla^2 f + \frac{1}{n(n-1)} R_g f = 0. \]
Taking trace, we obtain
\[ \Delta_g f + \frac{R_g}{n-1} f = 0. \]

Thus
\[ \frac{R_g}{n-1} \int_M f^2 \, dv_g = - \int_M f \Delta_g f \, dv_g = \int_M |\nabla_g f|^2 \, dv_g > 0, \]
since \( f \) is not a constant function. This implies that \( R_g > 0 \).

By (13) we see that does not exist Einstein manifold \((M^n, g)\) with negative \(\sigma_2\)-curvature. Using Lemma 3 we get by contradiction the

**Theorem 9.** Let \((M^n, g)\) be a closed Einstein manifold with negative scalar curvature. Then
\[ \ker \Lambda_g^* = \{0\} \]
that is, \((M^n, g)\) cannot be a \(\sigma_2\)-singular space.

Now we will show a rigidity result for closed \(\sigma_2\)-singular Einstein manifold with positive \(\sigma_2\)-curvature.

**Theorem 10 (Theorem 3).** Let \((M^n, g, f)\) be a closed \(\sigma_2\)-singular Einstein manifold with positive \(\sigma_2\)-curvature. Then \((M^n, g)\) is isometric to the round sphere with radius \( r = \left( \frac{n(n-1)}{R_g} \right)^{\frac{1}{2}} \) and \( f \) is an eigenfunction of the Laplacian associated to the first eigenvalue \( \frac{R_g}{n-1} \) on \( S^n(r) \). Hence \( \dim \ker \Lambda_g^* = n + 1 \) and \( \int_M f = 0 \).

**Proof.** By Lemma 3 the scalar curvature \( R_g \) is a positive constant and by Theorem 8 the function \( f \) is not constant. As in the proof of Lemma 3 we obtain
\[ \Delta_g f + \frac{R_g}{n-1} f = 0, \] (14)
that is, \( f \) is an eigenfunction of the Laplacian associated to the eigenvalue \( \frac{R_g}{n-1} \). On the other hand, by Lichnerowicz-Obata’s Theorem, see [19], the first nonzero eigenvalue of the Laplacian satisfies the inequality
\[ \lambda_1 \geq \frac{R_g}{n-1}. \] (15)
Moreover, the equality holds in (15) if and only if \((M^n, g)\) is isometric to the round sphere with radius \( r = \left( \frac{n(n-1)}{R_g} \right)^{\frac{1}{2}} \).

Therefore, by (14) the \( \ker \Lambda_g^* \) can be identified with the eigenspace associated to the first nonzero eigenvalue \( \lambda_1 > 0 \) of Laplacian in the round sphere with radius \( r \). But, it is well known that this space has dimension \( n + 1 \). Hence \( \dim \ker \Lambda_g^* = n + 1 \).
Since the complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric is an Einstein manifold with positive $\sigma_2$-curvature, then we see that the condition on the singularity cannot be dropped. Also, since the hyperbolic space is a $\sigma_2$-singular space then the theorem is false if the manifold is only complete. We also can use the Theorem 2 in [21] and (14) to prove the Theorem [10].

An immediate consequence is the following

**Corollary 1.** Let $(M^n, g)$ be a closed Einstein manifold with positive $\sigma_2$-curvature. If $(M^n, g)$ is not isometric to the round sphere, then $(M^n, g)$ is not $\sigma_2$-singular.

### 3.1 Local Surjectivity of $\sigma_2$

In this section we prove a local surjective result to the $\sigma_2$-curvature. To achieve this goal we need of the Splitting Theorem and Generalized Inverse Function Theorem which can be found in [11]. The main result in this section reads as follows.

**Theorem 11** (Theorem [11]). Let $(M^n, g_0)$ be a closed Riemannian manifold not $\sigma_2$-singular. Suppose that

(i) $g_0$ is an Einstein metric with positive $\sigma_2$-curvature; or

(ii) 
\[
\sigma_2(g_0) > \frac{1}{8(n-1)} R_{g_0}^2.
\]

Then, there is a neighborhood $U \subset C^\infty(M)$ of $\sigma_2(g_0)$ such that for any $\psi \in U$, there is a metric $g$ on $M$ close to $g_0$ with $\sigma_2(g) = \psi$.

**Proof.** The principal symbol of $\Lambda^*_g$ is

\[
\sigma_\xi(\Lambda^*_g) = \frac{1}{2} |\xi|^2 R_{g_0} + \frac{1}{2} \langle \xi \otimes \xi, Ric_{g_0} \rangle g_0 - \frac{1}{2} g_0^{ij} \langle \xi^i R_{kj} + \xi_k R_{ij} \rangle
\]

\[- \frac{n}{4(n-1)} R_{g_0} |\xi|^2 g_0 + \frac{n}{4(n-1)} R_{g_0} \xi \otimes \xi.
\]

Taking trace, we get

\[
tr(\sigma_\xi(\Lambda^*_g)) = \frac{n - 2}{2} \left( -\frac{|\xi|^2}{2} R_{g_0} + \langle \xi \otimes \xi, Ric_{g_0} \rangle g_0 \right).
\]

Thus, $tr(\sigma_\xi(\Lambda^*_g)) = 0$ implies that

\[
\frac{|\xi|^2}{2} R_{g_0} = \langle \xi \otimes \xi, Ric_{g_0} \rangle g_0.
\] (17)

So, we have two cases.
Case 1: If \((M^n, g)\) is Einstein with \(\sigma_2(g_0) \neq 0\), then by (13) we get that \(R_{g_0} \neq 0\). Using the equation (17) and \(\text{Ric}_{g_0} = \frac{R_{g_0}}{n} g\) we get that \(\xi \equiv 0\).

Case 2: If \((M^n, g)\) is not Einstein, then by (17) we get
\[
\left| |\xi| \right|^2 \left| R_{g_0} \right| - \left| \text{Ric}_{g_0} \right| g_0 \leq \left| |\xi| \right|^2 \left( \frac{1}{2} \left| R_{g_0} \right| - \left| \text{Ric}_{g_0} \right| g_0 \right) \leq 0.
\]
The inequality (16) implies that \(\xi \equiv 0\).

Therefore, in any case \(\Lambda_{g_0}^*\) has an injective principal symbol. By the Splitting Theorem, see Corollary 4.2 in [3], we obtain that
\[
C^\infty(M) = \text{Im} \Lambda_{g_0} \oplus \ker \Lambda_{g_0}^*.
\]
Since we assume that \((M, g_0)\) is not \(\sigma_2\)-singular, then \(\ker \Lambda_{g_0}^* = \{0\}\), which implies that \(\Lambda_{g_0}\) is surjective.

Therefore, applying the Generalized Implicit Function Theorem, \(\sigma_2\) maps a neighborhood of \(g_0\) to a neighborhood of \(\sigma_2(g_0)\) in \(C^\infty(M)\).

Note that if \((M^n, g)\) is an Einstein manifold with dimension \(n > 4\), then (16) holds. Since the round sphere \(S^n\) is Einstein, then a metric in the unit sphere close to the round metric satisfies the condition (16) and is not \(\sigma_2\)-singular. Also we notice here that for any metric we have \(R_g^2 \leq n|\text{Ric}_g|^2\), thus if the inequality (16) is satisfied then \(n > 4\).

As an immediate consequence of the Corollary 1 and the Theorem 11 we obtain the next corollary.

Corollary 2. Let \((M^n, g_0)\) be a closed Einstein manifold with positive \(\sigma_2\)-curvature. Assume that \((M^n, g_0)\) is not isometric to the round sphere. Then, there is a neighborhood \(U \subset C^\infty(M)\) of \(\sigma_2(g_0)\) such that for any \(\psi \in U\), there is a metric \(g\) on \(M\) closed to \(g_0\) with \(\sigma_2(g) = \psi\).

4 Flat Metrics and the \(\sigma_2\)-curvature

The main goal of this section is to prove the Theorems 5 and 6.

Let \((M, g_0)\) be a closed Riemannian manifold. For each \(\varepsilon > 0\) define the functional
\[
\mathcal{F}_\varepsilon(g) = \int_M \sigma_2(g)dv_{g_0} - \left( \frac{1}{8n(n - 1)} + \varepsilon \right) \int_M R_g^2 dv_{g_0},
\]
which is defined in the space \(\mathcal{M}\) of all Riemannian metric in \(M\). Note that the volume element is with respect to the fixed metric \(g_0\). Next we find the first and second variation of the functional (18) under a special condition.
Lemma 4. Let \((M, g_0)\) be a closed flat Riemannian manifold. Let \(h\) be a symmetric 2-tensor with \(\text{div}(h) = 0\). Then the first variation of \(\mathcal{F}_\varepsilon\) at \(g_0\) is identically zero and the second variation is given by

\[
D^2 \mathcal{F}_\varepsilon(g_0)(h, h) = -2\varepsilon (\Delta tr(h))^2 + \frac{1}{4} |\Delta \hat{h}|^2 \ dv_{g_0},
\]

where \(\hat{h} = h - \frac{\text{tr}(h)}{n} g\) is the traceless part of \(h\).

Proof. The first variation of \(\mathcal{F}_\varepsilon\) is identically zero because of its definition \((18)\) and the metric is flat.

Now consider \(g(t) = g_0 + th\) for \(t\) small enough. Note that by \((3)\) and \((4)\) we get

\[
\frac{\partial}{\partial t} \text{Ric} = -\frac{1}{2} \Delta h - \frac{1}{2} \nabla^2 tr(h),
\]

and

\[
\frac{\partial}{\partial t} R_g = -\Delta tr(h). \quad (19)
\]

Next, by Lemma \((1)\) we get

\[
\frac{\partial^2}{\partial t^2} |\text{Ric}_{g(t)}|^2 = 2 \left| \frac{\partial}{\partial t} \text{Ric}_g \right|^2_{g_0}.
\]

Using that the metric is flat, we get that \(\Delta \text{div} = \text{div} \Delta\) and \(\text{div} \nabla^2 = \Delta \nabla\). Thus, by \((2)\), \((5)\) and the fact that \(\text{div}(h) = 0\) we obtain that

\[
\int_M \frac{\partial^2}{\partial t^2} |\text{Ric}_{g(t)}|^2 \ dv_{g_0} = \frac{1}{2} \int_M |\Delta h| + \nabla^2 tr(h)|^2 \ dv_g
\]

\[
= \frac{1}{2} \int_M \left( |\Delta h|^2 + 2\langle \Delta h, \nabla^2 tr(h) \rangle + |\nabla^2 tr(h)|^2 \right) \ dv_g
\]

\[
= \frac{1}{2} \int_M \left( |\Delta h|^2 - \langle \text{div} \nabla^2 tr(h), \nabla tr(h) \rangle \right) \ dv_g
\]

\[
= \frac{1}{2} \int_M \left( |\Delta h|^2 + (\Delta tr(h))^2 \right) \ dv_g.
\]

If \(\hat{h} = h - \frac{\text{tr}(h)}{n} g\) is the traceless part of \(h\), then

\[
|\Delta \hat{h}|^2 = |\Delta h|^2 - \frac{(\Delta tr(h))^2}{n}. \quad (20)
\]

Thus

\[
\int_M \frac{\partial^2}{\partial t^2} |\text{Ric}_{g(t)}|^2 \ dv_{g_0} = \frac{1}{2} \int_M \left( |\Delta \hat{h}|^2 + \frac{n+1}{n} (\Delta tr(h))^2 \right) \ dv_g. \quad (21)
\]

Using \((19)\) we get

\[
\frac{\partial^2}{\partial t^2} R_{g(t)}^2 = 2 (\Delta tr(h))^2. \quad (22)
\]
Finally, by (21) and (22) at \( t = 0 \) we have

\[
\frac{\partial^2}{\partial t^2} F_\varepsilon(g(t)) = \int_M \left( \frac{n+1}{8n} + \varepsilon \right) \frac{\partial^2}{\partial t^2} R_g - \frac{1}{2} \frac{\partial^2}{\partial t^2} |Ric_g|^2 \right) dv_{g_0}
\]

\[
= - \int_M \left( 2\varepsilon (\Delta tr(h))^2 + \frac{1}{4} |\Delta h|^2 \right) dv_{g_0}.
\]

\[\square\]

In the next result we need the following theorem (See [11], [16] and references contained therein).

**Theorem 12.** Let \((M, g_0)\) be a Riemannian manifold. For \( p > n \), let \( g \) be a Riemannian metric on \( M \) such that \( \|g - g_0\|_{W^{2,p}(M,g_0)} \) is sufficiently small.

Then there exists a diffeomorphism \( \varphi \) of \( M \) such that \( h := \varphi^* g - g_0 \) satisfies that \( \text{div}_{g_0}(h) = 0 \) and

\[
\|h\|_{W^{2,p}(M,g_0)} \leq c \|g - g_0\|_{W^{2,p}(M,g_0)},
\]

where \( c \) is a positive constant which only depends on \((M, g_0)\).

Now we are ready to prove the Theorem 5.

**Proof of the Theorem 5.** By Theorem 12 there exists a diffeomorphism \( \varphi \) of \( M \) such that if we define \( h := \varphi^* g - g_0 \) then \( \text{div}_{g_0}(h) = 0 \) and

\[
\|h\|_{C^2(M,g_0)} \leq c \|g - g_0\|_{C^2(M,g_0)},
\]

where the positive constant \( c \) depends only on \((M, g_0)\).

Expanding \( F_\varepsilon(\varphi^* g) = F_\varepsilon(g_0 + h) \) at \( g_0 \) and using Lemma 4 we obtain

\[
F_\varepsilon(\varphi^* g) = F_\varepsilon(g_0) + D F_\varepsilon(g_0)(h) + \frac{1}{2} D^2 F_\varepsilon(g_0)(h,h) + E_3
\]

\[
= - \int_M \left( 2\varepsilon (\Delta tr(h))^2 + \frac{1}{4} |\Delta h|^2 \right) dv_{g_0} + E_3,
\]

where \(|E_3| \leq C \int_M |h|\nabla^2 h|^2 dv_{g_0}\) for some constant \( C = C(n, M, g_0) > 0 \).

Besides, by hypothesis we have

\[
F_\varepsilon(\varphi^* g) > 0,
\]

for \( \varepsilon > 0 \) small enough.
Now choose \( \varepsilon_n > 0 \) such that \( \varepsilon_n < \min\{2n\varepsilon, 1/4\} \). Thus using (20) and (23) we get
\[
\varepsilon_n \int_M |\Delta h|^2 dv_{g_0} \leq (2\varepsilon - \frac{\varepsilon_n}{n}) \int_M (\Delta tr(h))^2 dv_{g_0} + \left(\frac{1}{4} - \varepsilon_n\right) \int_M |\Delta h|^2 dv_{g_0}
\]
\[
+ \varepsilon_n \int_M |\Delta h|^2 dv_{g_0} = 2\varepsilon \int_M (\Delta tr(h))^2 dv_{g_0} + \frac{1}{4} \int_M |\Delta h|^2 dv_{g_0}
\]
\[
= -\mathcal{F}(\varphi^* g) + E_\delta \leq |E_\delta|.
\]
\[
\leq C_0 \int_M |h|\nabla^2 h||^2 dv_{g_0}.
\]
Suppose \( g \) is a Riemannian metric in \( M \) such that \( ||g - g_0||_{C^2(M,g_0)} < \frac{\varepsilon_n}{2C_0} \). The Theorem 12 implies that for \( \varepsilon_n > 0 \) small enough, there exists a diffeomorphism \( \varphi \) of \( M \) such that taking \( h := \varphi^* g - g_0 \) we have \( \text{div}_g(\varphi^* g) = 0 \) and
\[
||h||_{C^0(M,g_0)} \leq \frac{c|g - g_0||_{C^2(M,g_0)}}{2C_0}.
\]
Therefore
\[
\varepsilon_n \int_M |\nabla^2 h|^2 dv_{g_0} = \varepsilon_n \int_M |\Delta h|^2 dv_{g_0} \leq C_0 \int_M |h|\nabla^2 h||^2 dv_{g_0}
\]
\[
\leq \frac{\varepsilon_n}{2} \int_M |\nabla^2 h|^2 dv_{g_0},
\]
which implies that \( \nabla^2 h = 0 \) on \( M \). On the other hand,
\[
\int_M |\nabla h|^2 dv_{g_0} = -\int_M h\Delta hdv_{g_0} = 0
\]
and this implies that \( \nabla h = 0 \), that is, \( h \) is parallel with respect to \( g_0 \).

Since \( g_0 \) is flat, then given \( p \in M \) we can find local coordinates at \( p \) such that \( (g_0)_{ij} = \delta_{ij} \) and \( \partial_k(g_0)_{ij} = 0 \), for all \( i, j, k \in \{1, \ldots, n\} \), in some neighborhood \( U_p \). In these coordinates the Christoffel symbols of \( \varphi^* g = g_0 + h \) are
\[
\Gamma^k_{ij}(\varphi^* g) = \frac{1}{2}(\varphi^* g)^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) = 0.
\]

Therefore, the Riemann curvature tensor is identically zero. \( \square \)

As a consequence of the previous result we get the following corollary.

**Corollary 3.** Let \( U \subset \mathbb{R}^n \) be a bounded open set. Let \( \delta \) be the canonical metric on \( \mathbb{R}^n \). Let \( g \) be a metric on \( \mathbb{R}^n \) such that

(i) \( g = \delta \) in \( \mathbb{R}^n \setminus U \);

(ii) \( ||g - \bar{g}||_{C^2(\mathbb{R}^n,\delta)} \) is sufficiently small;

15
\[(iii) \int_M \sigma_2(g)dv_g > \frac{1}{8n(n-1)} \int_M R_g^2 dv_g.\]

Then \(g\) is a flat metric.

**Proof.** Since \(U\) is a bounded open set, then we can find a closed rectangle \(R \subset \mathbb{R}^n\) such that \(U \subset R\). Thus \(g = \delta\) in \(\mathbb{R}^n \setminus R\). Identifying the boundary of \(R\) properly we obtain the torus \(\mathbb{T}^n\) with one metric satisfying \((iii)\) and \(C^2\)-close to the flat metric. By Theorem 5 we have that the metric \(g\) is flat.

Now we will show that in the 3-dimensional torus \(\mathbb{T}^3\) does not exist a metric with constant scalar curvature and nonnegative \(\sigma_2\)-curvature, unless it is flat.

**Proof of the Theorem 6** Suppose that \(g\) is a metric with constant scalar curvature and nonnegative \(\sigma_2\)-curvature. Then by Theorem 5.2 in [18] we obtain that the scalar curvature has to be non positive and is zero if the metric is flat.

Suppose, without loss of generality, that the constant is \(-1\). Then by (I) we have

\[\sigma_2(g) = -\frac{1}{2} |Ric_g|^2_g + \frac{3}{16} \geq 0.\]

This implies that \(|Ric_g|^2_g \leq 3/8\).

Let \(p \in M\) be a fixed point. Choose an orthonormal basis \(\{e_1, e_2, e_3\}\) for \(T_pM\) such that the Ricci tensor at \(p\) is diagonal. Let \(\{\lambda_1, \lambda_2, \lambda_3\}\) be the eigenvalues of \(Ric_g(p)\). Then we have

\[\lambda_1 + \lambda_2 + \lambda_3 = -1\]

and

\[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq \frac{3}{8}.\]

Thus, for \(i \neq j\), we have

\[0 \geq \lambda_i^2 + \lambda_j^2 + \lambda_3^2 - \frac{3}{8} = \lambda_i^2 + \lambda_j^2 + (1 + \lambda_i + \lambda_j)^2 - \frac{3}{8} = (\lambda_i + \lambda_j)^2 + 2(\lambda_i + \lambda_j) + \lambda_i^2 + \lambda_j^2 + \frac{5}{8} \geq \frac{3}{2}(\lambda_i + \lambda_j)^2 + 2(\lambda_i + \lambda_j) + \frac{5}{8},\]

where in the last inequality we used the inequality \(2(\lambda_i^2 + \lambda_j^2) \geq (\lambda_i + \lambda_j)^2\).

This implies that

\[12(\lambda_i + \lambda_j)^2 + 16(\lambda_i + \lambda_j) + 5 \leq 0.\]
Since the roots of the equation $12x^2 + 16x + 5 = 0$ are $-5/6$ and $-1/2$, then $-5/6 \leq \lambda_i + \lambda_j \leq -1/2$.

Since the Weyl tensor vanishes in dimension 3, we obtain that the decomposition of the curvature tensor, see [8], is given by

$$R_{ijkl} = R_{il} g_{jk} + R_{jk} g_{il} - R_{ik} g_{jl} - R_{jl} g_{ik} - \frac{1}{12} R_g (g_{il} g_{jk} - g_{ik} g_{jl}).$$

Therefore, we obtain that the sectional curvature of the plane spanned by $e_i$ and $e_j$ satisfies

$$K(e_i, e_j) = R_{ijji} = R_{ii} g_{jj} + R_{jj} g_{ii} - \frac{1}{2} R_g g_{ii} g_{jj} = \lambda_i + \lambda_j + \frac{1}{2} \leq 0.$$

Thus $g$ has nonpositive sectional curvature. However, the torus does not admit a metric with nonpositive sectional curvature, see Corollary 2 in [2].

Lin and Yuan [16] have proved an analogous result for the $Q$-curvature. In dimension 3 if the scalar curvature is constant by (1) and (2) we have

$$Q_g = 4\sigma_2(g) - \frac{1}{2} \sigma_1(g)^2 = \frac{23}{32} R_g^2 - 2|\text{Ric}_g|^2.$$  \hspace{1cm} (23)

In this case if the $Q$-curvature is nonnegative, then the $\sigma_2$-curvature is nonnegative as well. But, by (23) we see that the sign of the $\sigma_2$-curvature does not determine the sign of the $Q$-curvature. In particular, the Theorem 6 is an extension of the Proposition 5.13 in [16].

References

[1] N. Aronszajn, Sur l’unicit´ e du prolongement des solutions des ´ equations aux d´ eriv´ ees partielles elliptiques du second ordre, C. R. Acad. Sci. Paris 242 (1956), 723–725.

[2] W. Ballmann, M. Brin and R. Spatzier, Structure of manifolds of non-positive curvature. II, Ann. of Math. (2) 122 (1985), no. 2, 205–235.

[3] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, J. Differential Geometry 3 (1969), 379–392.

[4] G. Catino, Some rigidity results on critical metrics for quadratic functionals, Calc. Var. Partial Differential Equations 54 (2015), no. 3, 2921–2937.

[5] S.-Y. A. Chang, M. Gursky and P. Yang, Remarks on a fourth order invariant in conformal geometry, Asp. Math. HKU, 353–372.
[6] X. Cheng, A generalization of almost-Schur lemma for closed Riemannian manifolds, Ann. Global Anal. Geom. 43 (2013), no. 2, 153–160.

[7] X. Cheng, An almost-Schur type lemma for symmetric (2, 0) tensors and applications, Pacific J. Math. 267 (2014), no. 2, 325–340.

[8] B. Chow, P. Lu and L. Ni, Hamilton’s Ricci flow, xxxvi+608. American Mathematical Society, Providence, RI; Science Press, New York (2006)

[9] H. O. Cordes, Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1956 (1956), 239–258.

[10] C. D. Lellis and P. M. Topping, Almost-Schur lemma, Calc. Var. Partial Differential Equations 43 (2012), no. 3-4, 347–354.

[11] A. E. Fischer and J. E. Marsden, Deformations of the scalar curvature, Duke Math. J. 42 (1975), no. 3, 519–547.

[12] M. J. Gursky and A. Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the $Q$-curvature, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 9, 2137–2173.

[13] M. J. Gursky and J. A. Viaclovsky, Rigidity and stability of Einstein metrics for quadratic curvature functionals, J. Reine Angew. Math. 700 (2015), 37–91.

[14] K. Kröncke, Stability of Einstein manifolds. Dissertation. Universität Potsdam, 2013.

[15] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37–91.

[16] Y.-J. Lin and W. Yuan, Deformations of $Q$-curvature I, Calc. Var. Partial Differential Equations 55 (2016), no. 4, Paper No. 101, 29 pp.

[17] Y.-J. and W. Yuan, A symmetric 2-tensor canonically associated to $Q$-curvature and its Applications, arXiv:1602.01212v2. 2016.

[18] R. Schoen and S. T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. of Math. (2) 110 (1979), no. 1, 127–142.

[19] R. Schoen and S. T. Yau, Lectures on differential geometry, v+235. International Press, Cambridge, MA (1994)

[20] A. Silva Santos, Solutions to the singular $\sigma_2$-Yamabe problem with isolated singularities, Indi. Univ. Math. J., Vol. 66 (3), pp. 741–790, 2017.
[21] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Amer. Math. Soc. 117 (1965), 251–275.

[22] J. A. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, Duke Math. J. 101 (2000), no. 2, 283–316.