UNTWISTED GAIOTTO EQUIVALENCE

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Abstract. This is a successive paper of [5]. We prove an equivalence between the category of finite-dimensional representations of degenerate supergroup $GL(M|N)$ and the category of $(GL_M(O) \ltimes U_{M,N}(F), \chi_{M,N})$-equivariant D-modules on $Gr_N$. We also prove that we can realize the category of finite-dimensional representations of degenerate supergroup $GL(M|N)$ as a category of D-modules on the mirabolic subgroup $Mir_L(F)$ with certain equivariant conditions for any $L$ bigger than $N$ and $M$.

1. Introduction

1.1. Notations. We denote by $F = \mathbb{C}((t))$ the field of Laurent series and by $O = \mathbb{C}[[t]]$ the ring of formal power series. Given a group scheme $G$, we denote by $G(F)$ the loop group of $G$ such that $\mathbb{C}$-points of $G$ are given by $\mathbb{C}((t))$-points of $G$, and denote by $G(O)$ the arc group of $G$ such that $\mathbb{C}$-points of $G(O)$ are given by $\mathbb{C}[[t]]$-points of $G$. Set $Gr_G := G(F)/G(O)$ the affine Grassmannian of $G$. It is known that $Gr_G$ is a formally smooth ind-scheme.

Assume $G = GL_N$. In this case, $Gr_N := Gr_{GL_N}$ admits a left action of $GL_N(F)$ by multiplication. In particular, any subgroup $H$ of $GL_N(F)$ acts on $Gr_N$, and we can consider the corresponding derived equivariant category. Denote by $D^H(Gr_N)$ the DG-category of left $H$-equivariant D-modules on $Gr_N$.

1.2. Reminder on mirabolic Satake equivalence. Let $\text{Sym}^\bullet(gl_N[-2])$ denote the symmetric DG-algebra generated by $gl_N := \text{Lie}(GL_N)$, such that the generator is placed in degree 2 and the differential is 0. A famous result (derived Satake equivalence) of [4] says that there is an equivalence between the bounded derived category of locally compact $GL_N(O)$-equivariant D-modules on $Gr_N$ and the derived category of perfect $GL_N$-equivariant modules over $\text{Sym}^\bullet(gl_N[-2])$. Namely,

$$D^{GL_N(O), \text{loc.c.}}(Gr_N) \simeq D^{\text{perf}}(\text{Sym}^\bullet(gl_N[-2])).$$

Here, locally compact means compact when regarded as a plain D-module on $Gr_N$.

In [5] and [7], the authors considered the mirabolic version of the above equivalence. On the D-module side in the mirabolic case, instead of considering $GL_N(O)$-equivariant D-modules on the affine Grassmannian $Gr_N$, we need to consider $GL_N(O)$-equivariant D-modules on the mirabolic affine Grassmannian $Gr_N \times F^N$. Here, the action of $GL_N(O)$ is given by the diagonal action on $Gr_N \times F^N$.

The definition of the mirabolic Satake category $D^{GL_N(O)}(Gr_N \times F^N)$ needs to be taken care of since any compact object (also, locally compact object) of the

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The mirabolic Satake category is supported on an infinite-dimensional space. Indeed, any GL_N(O)-orbit in GR_N × (F^N − 0) is infinite-dimensional.

The authors of [5] defined three different versions of the mirabolic Satake category: D^{GL_N(O)},loc.c((GR_N × F^N), *), D^{GL_N(O)},loc.c((GR_N × F^N), ◦), and D^{GL_N(O)},loc.c((GR_N × F^N), ⋊), according to different renormalizations. They are equipped with monoidal structures *, ◦ and fusion product, respectively. Roughly speaking, * is given by the convolution product and ◦ is a combination of the convolution product on GR_N and the tensor product on F^N. For more details of the definitions, see Section 3.1.

In order to introduce the coherent side of the mirabolic Satake equivalence, we should introduce a certain Lie supergroup.

Set C_N|N to be the super vector space with even vector space C^N and odd vector space C^N. We denote by gl(N|N) the Lie superalgebra of endomorphisms of C_N|N. One can define a degenerate version of the (super) Lie bracket on gl(N|N) where the supercommutator of an even element with any element is the same as in gl(N|N), but the supercommutator of any two odd elements is zero. The resulting Lie superalgebra is isomorphic to gl(N|N), and the corresponding Lie supergroup (with the even part isomorphic to GR_N × GL_N) is denoted by GL(N|N).

It is proved in [5] that D^{GL_N(O)},loc.c((GR_N × F^N) is equivalent to Rep_{fin}(GL(N|N)), the bounded derived category of finite-dimensional representations of GL(N|N).

This equivalence is t-exact with respect to the natural t-structure of Rep_{fin}(GL(N|N)) with the heart Rep_{fin}(GL(N|N))^○ and the perverse t-structure of D^{GL_N(O)},loc.c((GR_N × F^N). Furthermore, it is an equivalence of monoidal categories with respect to the fusion product on D^{GL_N(O)},loc.c((GR_N × F^N) and the tensor product on Rep_{fin}(GL(N|N)).

Similar to the case of the classical Satake category, the category of finite-dimensional modules over the degenerate Lie supergroup admits a Koszul dual realization D^{perf}_{GL_N} × GL_N(B_{N|N}), where B_{N|N} is a DG-algebra on an affine space with an action of GL_N × GL_N. If we forget the degrees, B_{N|N} is isomorphic to functions on gl_N × gl_N. By imposing different gradings, there are three different versions of the symmetric algebras B^{2,0}_{N|N}, B^{1,1}_{N|N}, B^{0,2}_{N|N}.

In Section 2.4, we will recall two monoidal structures A and B of D^{GL_N × GL_N}_{perf}(B_{N|N}) which are defined in [5, Section 3.2].

The following Theorem is proved in [5].

**Theorem 1.2.1.** There are monoidal equivalences of DG-categories,

\[(D^{GL_N(O)},loc.c((GR_N × F^N), ◦)) \simeq (D^{GL_N × GL_N}_{perf}(B^{2,0}_{N|N}), A),\]

\[(D^{GL_N(O)},loc.c((GR_N × F^N), *) \simeq (D^{GL_N × GL_N}_{perf}(B^{0,2}_{N|N}), B).\]

which commute with left and right convolutions with Per_{GL_N(O)}(GR_N) \simeq Rep_{fin}(GL_N)^○.

The above equivalences are also known as the untwisted Gaiotto conjecture in the case of N = M.

1.3. **Untwisted Gaiotto conjecture in the case M = N − 1.** We denote by gl(N − 1|N) the Lie superalgebra of endomorphisms of C^{N−1}|N and by gl(N − 1|N)
Theorem 1.3.1. There is an equivalence of categories,

\[ D_{\text{perf}}^{\mathfrak{g}^{1\times\mathfrak{g}}} (\mathfrak{b}_{N-1}\text{-loc}) \simeq D_{\text{perf}}^{\mathfrak{g}^{2,0}} (\mathfrak{b}_{N-1\mid N}). \]

This equivalence is deduced in [5] from the mirabolic Satake equivalence recalled in §1.2. We can present Rep$^\text{fin}(\mathfrak{g}(N-1\mid N))$ as a colimit of a full subcategory of Rep$^\text{fin}(\mathfrak{g}(N\mid N))$ with the transition functor being tensoring with the determinant module. On the D-module side, $D^{\mathfrak{g}^{1\times\mathfrak{g}}}(\mathfrak{b}_{N-1\mid N})$ admits a colimit presentation given by $D^{\mathfrak{g}^{2,0}}(\mathfrak{b}_{N-1\mid N})$ that is a full subcategory of $D^{\mathfrak{g}^{1\times\mathfrak{g}}}(\mathfrak{b}_{N-1\mid N})$. It is proved in loc.cit. that there exists an equivalence between these two full subcategories and the equivalence is compatible with taking colimits.

1.4. Untwisted Gaiotto conjecture in general case. In order to state the general untwisted Gaiotto conjecture, we need to introduce some notations.

Given a character $\chi : H' \to \mathbb{A}^1$, we denote by $D^{H'\chi}(\mathfrak{b}_{N})$ the category of $(H',\chi')\text{-equivariant}$ D-modules on $\mathfrak{b}_{N}$. Here, exp is the exponential D-module on $\mathbb{A}^1$.

Assuming $M < N$, we denote by $P_{M,N}(\mathfrak{g})$ the parabolic subgroup of GL$_N(\mathfrak{g})$ corresponding to the partition $(M+1,1,1,...,1)$, and denote by $U_{M,N}(\mathfrak{g})$ the unipotent radical of $P_{M,N}(\mathfrak{g})$.

There is a group morphism from $U_{M,N}(\mathfrak{g})$ to $\mathfrak{g}$ which sends $(u_{i,j}) \in U_{M,N}(\mathfrak{g})$ to $\sum u_{i,i+1}$. We compose this group morphism with taking residue

\[ \mathfrak{g} \to \mathbb{C} \]
\[ \sum a_it^i \mapsto a_{-1}. \]

The resulting morphism is a character of $U_{M,N}(\mathfrak{g})$ denoted by $\chi_{M,N}$.

The conjugation action of GL$_M(\mathfrak{g})$ on $U_{M,N}(\mathfrak{g})$ preserves $U_{M,N}(\mathfrak{g})$. Furthermore, the character $\chi_{M,N}$ is stable under the conjugation action. Hence, $\chi_{M,N}$ gives a
character on

\[
\text{GL}_M(\mathcal{O}) \ltimes U_{M,N}(\mathcal{F}) = \begin{cases}
\text{GL}_M(\mathcal{O}) & M \\
0 & N - M
\end{cases}
\]

\[
\begin{array}{cccc}
0 & \ast & \ldots & \ast \\
0 & \ast & \ldots & \ast \\
1 & \ast & \ldots & \ast \\
0 & \ldots & 0 & 1
\end{array}
\]

Note that there is a forgetful functor from \(D^{\text{GL}_M(\mathcal{O}) \ltimes U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N)\) to \(D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N)\). We denote by \((D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N))^{\text{GL}_M(\mathcal{O}),\text{loc},c}\) the full subcategory of \(D^{\text{GL}_M(\mathcal{O}) \ltimes U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N)\) generated by the preimage of compact objects of \(D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N)\).

In this paper, we will prove the following Gaiotto conjecture:

**Theorem 1.4.1.** Assume that \(N > M\). Then, the untwisted Gaiotto category \((D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N))^{\text{GL}_M(\mathcal{O}),\text{loc},c}\) is equivalent to the bounded derived category of finite-dimensional modules over the degenerate supergroup \(\text{GL}(M|N)\). In other words, there is an equivalence

\[
(D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N))^{\text{GL}_M(\mathcal{O}),\text{loc},c} \simeq D_{\text{perf}}^{\text{GL}_M(\mathcal{O}) \ltimes \text{Gr}_N}(\mathcal{B}^{2,0}_{M|N}).
\]

We will also establish an equivalence about the category of compact objects of \(D^{\text{GL}_M(\mathcal{O}) \ltimes U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N)\). It corresponds to a full subcategory of \(\text{Rep}^{\text{fin}}(\text{GL}(M|N))\) with a nilpotent support condition, see Theorem 4.4.1.

### 1.5. Strategy of the proof.
Fix \(N\), the proof of Theorem 1.4.1 proceeds by descending induction in \(M\). In §2.4 and §3.5, we will recall the monoidal structures of \(D^{\text{GL}_M(\mathcal{O}),\text{loc},c}(\text{Gr}_M \times \mathcal{F}^M)\) and of the category of modules over \(\text{GL}(M|M)\). Furthermore, we will see that these categories act on \((D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_N))^{\text{GL}_M(\mathcal{O}),\text{loc},c}\) and the category of modules over \(\text{GL}(M|N)\) respectively.

Hence, we can divide the proof of Theorem 1.4.1 into three steps.

The first step: we present \(\text{Rep}^{\text{fin}}(\text{GL}(M - 1|N))\) as a colimit of subcategories. Namely, we show that \(\text{Rep}^{\text{fin}}(\text{GL}(M - 1|N))\) can be obtained from \(\text{Rep}^{\text{fin}}(\text{GL}(M - 1|M))\) and \(\text{Rep}^{\text{fin}}(\text{GL}(M|N))\) by taking tensor product and colimits. It is proved in Proposition 2.6.1.

The second step: in §3, we present \((D^{U_{M-1,N}(\mathcal{F}),\chi_{M-1,N}(\text{Gr}_{\text{GL}_N}))^{\text{GL}_{M-1}(\mathcal{O}),\text{loc},c}\) as a colimit of subcategories. We prove an analog of Proposition 2.6.1 on the D-module side. Namely, we prove that there is an equivalence between the category \((D^{U_{M-1,N}(\mathcal{F}),\chi_{M-1,N}(\text{Gr}_{\text{GL}_N}))^{\text{GL}_{M-1}(\mathcal{O}),\text{loc},c}\) and the tensor product of \(D^{\text{GL}_{M-1}(\mathcal{O}),\text{loc},c}(\text{Gr}_{\text{GL}_N})\) and \((D^{U_{M,N}(\mathcal{F}),\chi_M,N}(\text{Gr}_{\text{GL}_N}))^{\text{GL}_M(\mathcal{O}),\text{loc},c}\) over the mirabolic Satake category of rank \(M\). This step is the key point of the proof, and we will work it out using the Fourier transform. It is proved in Theorem 3.9.1.

In §4, to finish the proof, we only need to compare two tensor products of categories. We only need to show that the equivalence [5, Theorem 4.1.1]
$D^{GL_{M-1}(O),loc,c}_N (\text{Gr}_{GL_N}) \simeq \text{Rep}^{\text{fin}}(GL(M-1|M))$ is compatible with the actions of the mirabolic Satake categories (Proposition 4.0.1). To prove this statement, we will use colimit presentations of the mirabolic Satake category of rank $M-1$ and the category of finite-dimensional modules over $GL(M-1|M-1)$. Then the proof can be obtained from the monoidal structure of the equivalence of Theorem 1.2.1.

Using a similar method, we can prove Theorem 4.4.1.

1.6. Symmetric definition of $D^{GL(0)\times U_{M,N}(F),\chi_{M,N}}(\text{Gr}_N)$. In the statement of Theorem 1.4.1 and 4.4.1, we need to require $M < N$: otherwise, the subgroup $U_{M,N}(F) \subset GL_N(F)$ makes no sense.

However, the supergroup $GL(M|N)$ is well-defined for any $N$ and $M$. Hence, we hope to find a good definition of the corresponding category on the D-module side which does not need to require $M < N$.

We can fix this problem by choosing an integer $L$ which is strictly bigger than $N$ and $M$, and define a category of D-modules on the mirabolic subgroup $\text{Mir}_L(F)$ of $GL_L(F)$ with certain equivariant conditions. To be more precise, we can define $C_{M,N,L}$ to be the category of D-modules on $\text{Mir}_L(K)$ that are left $(GL_M(O) \times U_{M,L}, \chi_{M,L})$-equivariant and right $(GL_N(O) \times U_{N,L}, \chi_{N,L})$-equivariant.

In §5, we prove that the resulting category $C_{M,N,L}$ is independent of the choice of $L$ and is equivalent to the category of $D^{GL_M(O)\times U_{M,N}(F),\chi_{M,N}}(\text{Gr}_N)$ if $M < N$ and $D^{GL_N(O)/(\text{Gr}_N \times F^N)}(\text{Gr}_N)$ if $M = N$.

1.7. Iwahori version of the conjecture. Let $I_N$ be the Iwahori subgroup of $GL_N(O)$ and $F_l \simeq GL_N(F)/I_N$ the affine flag variety for $GL_N$. In [2], the author proved that there is an equivalence between $D^{I_N}(F_l)$ and the derived category of $GL_N$-equivariant D-modules on the DG version Steinberg variety. Motivated by this result, [5] proposed the mirabolic version of Bezrukavnikov’s equivalence, and [6] proposed the orthosymplectic version of it.

Mimicking [5, Conjecture 1.4.1], we propose the following conjecture.

For $M < N$, let $V_1$ be an $M$-dimensional vector space and $V_2$ be an $N$-dimensional vector space. Let $\mathcal{F}_{\ell_i}, i = 1, 2$ denote the variety of complete flags in $V_{\ell_i}$. We let $St_{Mir,M,N}$ be the subvariety of $\text{Hom}(V_1, V_2)[1] \times \text{Hom}(V_2, V_1)[1] \times \mathcal{F}_{\ell_1} \times \mathcal{F}_{\ell_2}$, such that $(A, B, F_1 = (F_1^{(1)} \subset F_1^{(2)} \ldots \subset F_1^{(M)} = V_1), F_2 = (F_2^{(1)} \subset F_2^{(2)} \ldots \subset F_2^{(N)} = V_2))$ belongs to $St_{Mir,M,N}$ if and only if $AB(F_2^{(i)}) \subset F_2^{(j)}$, $i = 1, 2, ..., N$, and $BA(F_1^{(j)}) \subset F_1^{(j)}$, $j = 1, 2, ..., M$.

Let $D^{I_M \times U_{M,N}(F),\chi_{M,N}}(F_l)$ be the derived category of $(I_M \times U_{M,N}(F), \chi_{M,N})$-equivariant sheaves on $F_l$. We propose the following conjecture.

**Conjecture 1.7.1.** There is an equivalence of categories,

\begin{equation}
D^{I_M \times U_{M,N}(F),\chi_{M,N}}(F_l) \simeq D^{GL_M \times GL_N}(St_{Mir,M,N}).
\end{equation}

2. Coherent side

2.1. Degenerate supergroup. Let us recall the degenerate Lie superalgebra $gl(M|N)$ from [5]. By definition, its underlying super vector space is the same as the underlying super vector space of $gl(M|N)$, and the supercommutators of even...
elements with any element in $\mathfrak{gl}(M|N)$ and $\mathfrak{gl}(M|N)$ are same. But the supercommutator of any two odd elements in $\mathfrak{gl}(M|N)$ is set to be zero.

The bounded derived category of finite-dimensional representations of the corresponding supergroup $GL(M|N)$ (with the even part isomorphic to $GL_M \times GL_N$) is denoted by $\text{Rep}^{\text{fin}}(GL(M|N))$.

2.2. Symmetric algebra realization. Assume that $V_1$ is an $M$-dimensional vector space with a basis $e_1, e_2, \ldots, e_M$ and $V_2$ is an $N$-dimensional vector space. Set the symmetric algebra in the category of complexes $\mathcal{B}^{1,1}_{M|N} := \text{Sym}\big((\text{Hom}(V_1, V_2)[−1]) \otimes \text{Sym}\big((\text{Hom}(V_2, V_1)[−1]) = \text{Sym}^\bullet\big(\mathfrak{gl}(M|N)[−1]\big) = \text{Sym}^\bullet\big(\Pi(\text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1))[−1]\big)$ where $\Pi$ stands for change of parity.

Let us denote by $\text{Rep}^{\text{loc.fin}}(GL(M|N))$ the ind-completion of $\text{Rep}^{\text{fin}}(GL(M|N))$. More precisely, the category $\text{Rep}^{\text{loc.fin}}(GL(M|N))$ is the unbounded derived category of locally finite representations of $\mathfrak{gl}(M|N)$, and any object of $\text{Rep}^{\text{loc.fin}}(GL(M|N))$ is a filtered colimit of objects in $\text{Rep}^{\text{fin}}(GL(M|N))$.

By the Koszul duality, we have a monoidal equivalence

(2.1) \[ \text{Rep}^{\text{fin}}(GL(M|N)) \simeq D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{1,1}_{M|N}). \]

Here $D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{1,1}_{M|N})$ denotes the derived category of $(GL_{V_1} \times GL_{V_2})$-equivariant perfect modules over $\mathcal{B}^{1,1}_{M|N}$.

Remark 2.2.1. The above equivalence induces an equivalence of their ind-completions

(2.2) \[ \text{Rep}^{\text{loc.fin}}(GL(M|N)) \simeq D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{1,1}_{M|N}) \simeq \text{Ind}(D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{1,1}_{M|N})). \]

Here $D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{1,1}_{M|N})$ denotes the derived category of $GL_{V_1} \times GL_{V_2}$-invariants modules over $\mathcal{B}^{1,1}_{M|N}$.

In addition to $\mathcal{B}^{1,1}_{M|N}$, we can consider the symmetric algebras:

$\mathcal{B}^{2,0}_{M|N} := \text{Sym}\big((\text{Hom}(V_1, V_2)[−2]) \otimes \text{Sym}\big((\text{Hom}(V_2, V_1)),

\mathcal{B}^{0,2}_{M|N} := \text{Sym}\big((\text{Hom}(V_1, V_2)) \otimes \text{Sym}\big((\text{Hom}(V_2, V_1)[−2]).

The categories of $(GL_M \times GL_N)$-equivariant modules over $\mathcal{B}^{2,0}_{M|N}$, $\mathcal{B}^{1,1}_{M|N}$, and $\mathcal{B}^{0,2}_{M|N}$ are equivalent via the functors of changing cohomological degrees, see [5, Section 3.6].

Remark 2.2.2. Up to the degree shearing operation (cf. [1, Appendix A]), the categories $D^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\mathcal{B}^{2,0}_{M|N})$ are equivalent to $\text{IndCol}^{GL_{V_1} \times GL_{V_2}}_{\text{perf}}(\text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1))$ via identifying $\text{Hom}(V_1, V_2)$ with $\text{Hom}(V_2, V_1)^*$ and $\text{Hom}(V_2, V_1)$ with $\text{Hom}(V_1, V_2)^*$.

Sometimes, in order to indicate the vector spaces used in the definition, we will denote $\mathcal{B}^{1,1}_{M|N}$ by $\mathcal{B}^{1,1}_{V_1|V_2}$. 


\textbf{Remark 2.2.3.} One can show that for any other choice \( V'_1, V'_2 \) of vector spaces of the same dimensions \( M, N \), there is a \textit{canonical} equivalence

\[
D_{\text{perf}}^{GL_{V_1}} \times GL_{V_2}(\mathfrak{B}_{V_i/V_j}^{1,2}) \simeq D_{\text{perf}}^{GL_{V_1'}} \times GL_{V_2'}(\mathfrak{B}_{V_i'/V_j'}^{1,2})(i, 2 - i) \in \{(0, 2), (1, 1), (2, 0)\}).
\]

(Due to the \((GL(V_1) \times GL(V_2))-\text{equivalence, it is independent of the choice of isomorphisms } V_1 \simeq V'_1, V_2 \simeq V'_2)\)

2.3. Nilpotent support. We introduce a nilpotent cone \( \text{Nilp}_{V_1/V_2} \) of \( \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1) \). We denote by

\[
q : \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1) \to \text{End}(V_2)
\]

the morphism sending \( A_{1,2}, A_{2,1} \in \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1) \) to \( A_{1,2}A_{2,1} \in \text{End}(V_2) \).

We denote by \( \text{Nilp}_{V_1/V_2} \) the subscheme of \( \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1) \) such that the image \( A_{1,2}A_{2,1} \) is nilpotent (equivalently, \( A_{2,1}A_{1,2} \in \text{End}(V_1) \) is nilpotent).

We denote by \( D_{\text{perf}}^{GL_{V_1}} \times GL_{V_2}(\mathfrak{B}_{M/N}^{1,2}) \) the full subcategory of \( D_{\text{perf}}^{GL_{V_1}} \times GL_{V_2}(\mathfrak{B}_{M/N}^{1,2}) \) the full subcategory of \( D_{\text{perf}}^{GL_{V_1}} \times GL_{V_2}(\mathfrak{B}_{M/N}^{1,2}) \) such that \( \chi \in \text{End}(V_2) \).

\textbf{Remark 2.3.1.} According to [1, Corollary 9.1.7], the set-theoretical nilpotent support condition is Koszul dual to the nilpotent singular support condition of ind-coherent sheaves. In particular, the category \( D^{GL_{V_1}} \times GL_{V_2}(\mathfrak{B}_{M/N}^{1,2}) \) corresponds to the full subcategory of \( \text{IndCoh}((\text{Hom}(V_1, V_2) \oplus \text{Hom}(V_1, V_2)[-1])/(GL_{V_1} \times GL_{V_2})) \) with the nilpotent singular support condition.

2.4. Convolution. In this section, let us recall the definitions of the convolution products given in [5, Section 3.4].

Let us denote by \( \mathcal{Q}_A \) (resp. \( \mathcal{Q}_B \)) the closed subvariety in \( \mathcal{H} := \text{Hom}(W_1, W_2) \times \text{Hom}(W_2, W_1) \times \text{Hom}(W_2, W_3) \times \text{Hom}(W_3, W_2) \times \text{Hom}(W_2, W_1) \times \text{Hom}(W_1, W_3) \), such that, for any point

\[
(A_{1,2}, A_{2,1}, A_{2,3}, A_{3,2}, A_{3,1}, A_{1,3}) \in \mathcal{Q}_A,
\]

we have

\[
A_{1,3} = A_{2,3}A_{1,2}, A_{2,1} = A_{3,1}A_{2,3}, \text{ and } A_{3,2} = A_{1,2}A_{3,1}.
\]

(2.4) \( A_{3,1} = A_{2,1}A_{3,2}, A_{1,2} = A_{3,3}A_{1,3}, \text{ and } A_{2,3} = A_{1,3}A_{2,1}. \)

(2.5)

Consider the following diagram

\[
\begin{array}{ccc}
\text{Hom}(W_1, W_2) \times \text{Hom}(W_2, W_1) & \xrightarrow{p_{1,2}} & \mathcal{Q}_A \\
\text{Hom}(W_1, W_3) \times \text{Hom}(W_3, W_2) & \xrightarrow{p_{1,3}} & \text{Hom}(W_2, W_3) \times \text{Hom}(W_3, W_2) \\
\text{Hom}(W_1, W_3) \times \text{Hom}(W_3, W_1) & \xrightarrow{p_{2,3}} &
\end{array}
\]
In [5], the authors consider convolution products

\[ A \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^2) \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^2) \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^2) \]

\[ M_{1,2} \star M_{2,3} := p_{1,3,\star}(p_{1,2,1,2}^\star M_{1,2} \otimes C[Q_A]_{2,0} \otimes p_{2,3,2,3}^\star M_{2,3})^{GL(W_2)} \]

(2.6)

\[ B \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^0) \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^0) \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^0) \]

\[ M_{1,2} \star M_{2,3} := p_{1,3,\star}(p_{1,2,1,2}^\star M_{1,2} \otimes C[Q_B]_{0,2} \otimes p_{2,3,2,3}^\star M_{2,3})^{GL(W_2)}. \]

Here, \( C[Q]_{2,0} = \text{Sym}(\text{Hom}(W_1, W_2)) \otimes \text{Sym}(\text{Hom}(W_2, W_1) [-2]) \otimes \text{Sym}(\text{Hom}(W_3, W_2)) \otimes \text{Sym}(\text{Hom}(W_2, W_1) [-2]) \),

\[ C[Q]_{0,2} = \text{Sym}(\text{Hom}(W_2, W_3) [-2]) \otimes \text{Sym}(\text{Hom}(W_1, W_3)) \otimes \text{Sym}(\text{Hom}(W_3, W_1)) \]

\[ \otimes \text{Sym}(\text{Hom}(W_2, W_3) [-2]) \otimes \text{Sym}(\text{Hom}(W_3, W_1)), \]

\[ C[Q_B]_{0,2} := \text{Sym}(\text{Hom}(W_1, W_2)) \otimes \text{Sym}(\text{Hom}(W_2, W_3) [-2]) \otimes \text{Sym}(\text{Hom}(W_3, W_1) [-2]). \]

The operation \( A \star \) (resp. \( B \star \)) gives the category \( D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^2) \) (resp. \( D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^0) \)) a monoidal category structure if \( W_1 \) and \( W_2 \) are of the same dimensions, and the resulting monoidal category can act on \( D^{GL(W_3) \times GL(W_1)}(B_{W_3|W_1}^2) \) (resp. \( D^{GL(W_3) \times GL(W_1)}(B_{W_3|W_1}^0) \)) from right, and act on \( D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^2) \) (resp. \( D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^0) \)) from left for any finite-dimensional vector space \( W_3 \).

**Remark 2.4.1.** By construction, the convolution products induce the following functors

(2.7)

\[ A \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^2) \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^2)_{\text{Nilp}} \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^2)_{\text{Nilp}}, \]

\[ A \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^2)_{\text{Nilp}} \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^2) \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^2)_{\text{Nilp}}, \]

\[ B \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^0) \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^0)_{\text{Nilp}} \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^0)_{\text{Nilp}}, \]

\[ B \star : D^{GL(W_1) \times GL(W_2)}(B_{W_1|W_2}^0)_{\text{Nilp}} \times D^{GL(W_2) \times GL(W_3)}(B_{W_2|W_3}^0) \rightarrow D^{GL(W_1) \times GL(W_3)}(B_{W_1|W_3}^0)_{\text{Nilp}}. \]

**Remark 2.4.2.** The convolution products \( A \star \) and \( B \star \) preserve compact objects.

2.5. **Endofunctor of** \( D^{GL_M \times GL_N}(B_M|N) \). From now to the end of this section, we assume that \( \dim W_1 = \dim W_2 = M \) and \( \dim W_3 = N \). Let us take a \( GL(W_1) \times GL(W_2) \times GL(W_3) \)-invariant subvariety \( Q^0_A \subset Q_A \) by requiring \( A_{1,2} \) to be non-invertible.

Consider the following diagram
Here, $B_{\lambda} \times \text{V}_{\lambda}$ also be regarded as a full cocomplete subcategory $D$ since convolution commutes with colimits, the full subcategory of perfect complexes with nilpotent support condition, given by left convolution with $\text{GL}_{M,N}$ and an endofunctor of its ind-completion $\text{GL}_{M,N}$.

By definition, the endofunctor $T_{M,N}$ is isomorphic to the endofunctor which is given by left convolution with $\tilde{c} \in D^{\text{GL}_{M,N}}$, i.e.,

\[
T_{M,N} : D^{\text{GL}_{M,N}}(\mathfrak{B}^{2,0}_{W_2|W_3}) \longrightarrow D^{\text{GL}_{W_1}}(\mathfrak{B}^{2,0}_{W_1|W_3})
\]

\[
M \mapsto \tilde{c}^A M.
\]

Here, $\tilde{c}$ is the structure sheaf on the locus such that $A_{1,2}$ is non-invertible.

According to Remark 2.4.1, the endofunctor $T_{M,N}$ induces an endofunctor of the subcategory of perfect complexes with nilpotent support condition,

\[
T_{M,N} : D^{\text{GL}_{W_1}}(\mathfrak{B}^{2,0}_{W_3}|W_3)_{\text{Nilp}} \longrightarrow D^{\text{GL}_{W_1}}(\mathfrak{B}^{2,0}_{W_1|W_3})_{\text{Nilp}},
\]

and an endofunctor of its ind-completion

\[
T_{M,N} : D^{\text{GL}_{W_1}}(\mathfrak{B}^{2,0}_{W_3}|W_3)_{\text{Nilp}} \longrightarrow D^{\text{GL}_{W_1}}(\mathfrak{B}^{2,0}_{W_1|W_3})_{\text{Nilp}}.
\]

By construction, $\tilde{c}^A \times \tilde{c} \simeq \tilde{c}$. That is to say, $\tilde{c}$ is an idempotent algebra object with respect to $^A$. In particular, the Kleisli category $D(T_{M,N})$ of $D^{\text{GL}_{M,N}}(\mathfrak{B}^{2,0}_{M|N})$ is the full subcategory of $D^{\text{GL}_{M-1,N}}(\mathfrak{B}^{2,0}_{M-1|N})$ spanned by the essential image of $T_{M,N}$. Since convolution commutes with colimits, $T_{M,N}$ preserves colimits. In particular, $D(T_{M,N})$ is cocomplete.

Using the method applied in [5, Section 4.3], we can prove that $D(T_{M,N})$ can also be regarded as a full cocomplete subcategory $D^{\text{GL}_{M-1}}(\mathfrak{B}^{2,0}_{M-1|N})$ of $D^{\text{GL}_{M-1}}(\mathfrak{B}^{2,0}_{M-1|N})$ which is compactly generated by the objects $\{V^\lambda_{\text{GL}_{M-1}} \otimes \mathfrak{B}^{2,0}_{M-1|N} \otimes V^\mu_{\text{GL}_{N}}, \lambda, \mu \text{ is a partition of length } M-1, \mu \text{ is a signature of length } N\}$. Here, $V^\lambda_{\text{GL}_{M-1}}$ and $V^\mu_{\text{GL}_{N}}$ are the irreducible representations of $\text{GL}_{M-1}$ and $\text{GL}_{N}$.

To be self-contained, let us briefly recall the proof. Note that $D(T_{M,N})$ and $D^{\text{GL}_{M-1,N}}(\mathfrak{B}^{2,0}_{M-1|N})$ are compactly generated. It is sufficient to prove that the full subcategories of their compact objects are equivalent.

**Lemma 2.5.1.** For $M \leq N$, there exists an equivalence of categories

\[
\Psi : D_{\text{perf}}(T_{M,N}) \simto D^{\text{GL}_{M-1,N}}_{\text{perf}}(\mathfrak{B}^{2,0}_{M-1|N}).
\]
which is compatible with the natural forgetful functors to $D_{perf}^{GL_{M-1} \times GL_N}(\mathfrak{B}_{M-1|N}^{2,0})$. Here $D_{perf}(T_{M,N})$ denotes the Kleisli subcategory of $D_{perf}^{GL_M \times GL_N}(\mathfrak{B}_{M|N}^{2,0})$ associated with $T_{M,N}$.

Proof. Denote by $\tilde{W}$ the subspace of $W_1$ spanned by $e_1, e_2, \ldots, e_{M-1}$. Let us consider the functor given by taking the convolution with $c_{1,1}$ which is the structure sheaf on $\text{Hom}(W_1, \tilde{W}) \times \text{Hom}(\tilde{W}, W_1)$.

$$D_{perf}^{GL_{W_1} \times GL_{W_3}}(\mathfrak{B}_{W_1|W_3}^{2,0}) \to D_{perf}^{GL_{\tilde{W}_1} \times GL_{W_3}}(\mathfrak{B}_{W_1|W_3}^{2,0})$$

$$M_{1,3} \mapsto c_{1,1} \ast M_{1,3}.$$ (2.13)

We denote by

$$\Psi : D_{perf}(T_{M,N}) \to D_{perf}^{GL_{\tilde{W}_1} \times GL_{W_3}}(\mathfrak{B}_{W_1|W_3}^{2,0})$$

the restriction of (2.13) to $D_{perf}(T_{M,N})$. We claim that $\Psi$ is an equivalence between $D_{perf}(T_{M,N})$ and $D_{perf}^{GL_{M-1} \times GL_N, \geq 0}(\mathfrak{B}_{M-1|N}^{2,0})$.

For essential surjectivity, we only need to notice that the image of $V_{GL_M}^\lambda \otimes \mathfrak{B}_{W_1|W_3}^{2,0} \otimes V_{GL_N}^\mu$ is isomorphic to $V_{GL_{M-1}}^\lambda \otimes \mathfrak{B}_{W_1|W_3}^{2,0} \otimes V_{GL_{M-1}}^\mu$. Here, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{M-1})$ if $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{M-1} \geq 0)$, and $V_{GL_{M-1}}^\lambda = 0$ otherwise.

The full faithfulness follows from the isomorphism

$$\text{Hom}_{D_{perf}(T_{M,N})}(V_{GL_M}^\lambda \otimes \mathfrak{B}_{W_1|W_3}^{2,0} \otimes V_{GL_N}^\mu, V_{GL_M}^{\lambda'} \otimes \mathfrak{B}_{W_1|W_3}^{2,0} \otimes V_{GL_N}^\mu') \approx (V_{GL_{M-1}}^{\lambda \ast} \otimes V_{GL_{M-1}}^{\lambda'} \otimes (C[Q_{\lambda}^0]) \otimes V_{GL_{W_3}}^\mu \otimes V_{GL_{W_3}}^{\mu'}).$$ (2.15)

The last isomorphism follows from [5, Lemma 3.13] □

As a corollary, we have

**Corollary 2.5.2.** For $M \leq N$, we have

$$D_{perf}(T_{M,N})_{\text{Nilp}} \simeq D_{perf}^{GL_{M-1} \times GL_N, \geq 0}(\mathfrak{B}_{M-1|N}^{2,0})_{\text{Nilp}}.$$ (2.16)

Here $D_{perf}(T_{M,N})_{\text{Nilp}}$ denotes the Kleisli category of $D_{perf}^{GL_M \times GL_N}(\mathfrak{B}_{M|N}^{2,0})_{\text{Nilp}}$ associated with $T_{M,N}$, and $D_{perf}^{GL_{M-1} \times GL_N, \geq 0}(\mathfrak{B}_{M-1|N}^{2,0})_{\text{Nilp}}$ denotes the full subcategory of $D_{perf}^{GL_{M-1} \times GL_N, \geq 0}(\mathfrak{B}_{M-1|N}^{2,0})$ with nilpotent support condition.

Proof. We only need to show the restriction of $\Psi$ of (2.14) to $D_{perf}(T_{M,N})_{\text{Nilp}}$ is essential surjective.

Note that pullback along $q$ in (2.3) gives rise to a homomorphism,

$$\text{Sym}(\mathfrak{gl}_N[-2]) \to \mathfrak{B}_{M|N}^{2,0}.$$ 

Under the functor $\Psi$ in (2.12), $V_{GL_M}^\lambda \otimes (\mathfrak{B}_{M|N}^{2,0} \otimes \text{Sym}(\mathfrak{gl}_N[-2]|GL_N \mathbb{C}) \otimes V_{GL_N}^\mu$ goes to

$$V_{GL_{M-1}}^\lambda \otimes (\mathfrak{B}_{M-1|N}^{2,0} \otimes \text{Sym}(\mathfrak{gl}_N[-2]|GL_N \mathbb{C}) \otimes V_{GL_N}^\mu.$$ 

We have

$$\mathfrak{B}_{M|N}^{2,0} \otimes \text{Sym}(\mathfrak{gl}_N[-2]|GL_N \mathbb{C}) \simeq \mathfrak{O}_{\text{Nilp}_{M|N}}.$$
and
\[ \mathcal{B}_{M-1|N}^{2,0} \otimes \text{Sym}(\text{gl}_n[-2])^{GL_N} \cong \mathcal{O}_{\text{Nilp}_{M-1|N}}. \]

In particular, the image of (2.16) contains the collection of objects \( \{ V^\lambda_{GL_{M-1}} \otimes \mathcal{O}_{\text{Nilp}_{M-1|N}} \otimes V^\mu_{GL_N} | \lambda \text{ is a partition of length } M-1, \mu \text{ is a signature of length } N \} \). Now the claim follows from the fact that the category \( D_{\text{perf}}^{GL_{M-1} \times GL_N, \geq 0}(\mathcal{B}_{M-1|N})_{\text{Nilp}} \) is generated by \( \{ V^\lambda_{GL_{M-1}} \otimes \mathcal{O}_{\text{Nilp}_{M-1|N}} \otimes V^\mu_{GL_N} | \lambda \text{ is a partition of } M-1, \mu \text{ is a signature of } N \} \). □

Let \( D(T_{M,N})_{\text{Nilp}} \) be the Kleisli subcategory of \( D^{GL_{M} \times GL_N}(\mathcal{B}_{M|N}^{2,0}) \) associated with the idempotent monad \( T_{M,N} \), and let \( D^{GL_{M-1} \times GL_N, \geq 0}(\mathcal{B}_{M-1|N})_{\text{Nilp}} \) be the ind-completion of \( D_{\text{perf}}^{GL_{M-1} \times GL_N, \geq 0}(\mathcal{B}_{M-1|N})_{\text{Nilp}} \).

**Corollary 2.5.3.** For \( M \leq N \), we have
\begin{equation}
D(T_{M,N})_{\text{Nilp}} \cong D^{GL_{M-1} \times GL_N, \geq 0}(\mathcal{B}_{M-1|N})_{\text{Nilp}}.
\end{equation}

**Proof.** It follows from the fact that \( D(T_{M,N})_{\text{Nilp}} \) is equivalent to the ind-completion of \( c_{1,1} \ast D_{\text{perf}}^{GL_{M} \times GL_N}(\mathcal{B}_{M|N}^{2,0})_{\text{Nilp}} \cong D_{\text{perf}}(T_{M,N})_{\text{Nilp}}. \) □

### 2.6. Tensor product of categories of representations

All the categories that we consider are DG-categories\(^1\). Given a (resp. cocomplete) monoidal category \( \mathcal{A} \) and a right (resp. cocomplete) \( \mathcal{A} \)-module category \( \mathcal{M}_1 \) and a left (resp. cocomplete) \( \mathcal{A} \)-module category \( \mathcal{M}_2 \). We let the relative tensor product \( \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \) be the geometric realization of \( \mathcal{M}_1 \otimes \mathcal{A}^n \otimes \mathcal{M}_2 \) inside the category of DG-categories (resp. cocomplete DG-categories with continuous functors),
\[ \cdots \mathcal{M}_1 \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M}_2 \overset{\sim}{\longrightarrow} \mathcal{M}_1 \otimes \mathcal{A} \otimes \mathcal{M}_2 \overset{\sim}{\longrightarrow} \mathcal{M}_1 \otimes \mathcal{M}_2. \]

See [9, Section 4.4].

**Proposition 2.6.1.** For any \( M \leq N \), there is an equivalence induced by (2.7)
\begin{equation}
D^{GL_{M-1} \times GL_M}(\mathcal{B}_{M-1|M}^{2,0}) \otimes D^{GL_M \times GL_M}(\mathcal{B}_{M|M}^{2,0}) \overset{\sim}{\longrightarrow} D^{GL_{M} \times GL_N}(\mathcal{B}_{M|N}^{2,0}) \cong D^{GL_{M-1} \times GL_N}(\mathcal{B}_{M-1|N}^{2,0}).
\end{equation}

Similar claims hold for \( D_{\text{perf}}, D_{\text{perf}} \) with nilpotent support, and \( D \) with nilpotent support.

**Proof.** In [5, Section 4], the authors constructed an invertible endofunctor \( \eta \) of \( D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}_{M-1|N}^{2,0}) \) which corresponds to tensoring with the determinant module up to a cohomological shift. This functor restricts to an endofunctor of \( D_{\text{perf}}^{GL_{M-1} \times GL_N, \geq 0}(\mathcal{B}_{M-1|N}^{2,0}) \).

Given a stable category \( \mathcal{C} \) and an endofunctor \( F : \mathcal{C} \to \mathcal{C} \), we let \( \text{colim}_F \mathcal{C} \) denote the colimit of the sequence \( \mathcal{C} \overset{F}{\to} \mathcal{C} \overset{F}{\to} \mathcal{C} \overset{F}{\to} \cdots \) in the \((\infty, 1)\)-category of stable \((\infty, 1)\)-categories. Similarly, given a stable cocomplete category \( \mathcal{C} \) and a continuous endofunctor \( F : \mathcal{C} \to \mathcal{C} \), we let \( \text{colim}_F \mathcal{C} \) denote the colimit of the sequence \( \mathcal{C} \overset{F}{\to} \mathcal{C} \overset{F}{\to} \cdots \)

\(^1\)By DG-category, we mean a stable \((\infty, 1)\)-category enhanced over the bounded complexes of finite dimensional vector spaces; and by cocomplete stable DG-category, we mean a cocomplete stable \((\infty, 1)\)-category enhanced over the complexes of vector spaces.
Then, the equivalence (2.18) follows by taking colimits in the category of cocomplete categories with continuous functors. It is well-known that taking ind-completion preserves colimits.

For any object $M \in D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|N})$, $\eta^n(M) \in D_{\text{perf}}^{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N})$ for sufficient large $n$. Hence, the fully faithful embedding:

$$D_{\text{perf}}^{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N}) \longrightarrow D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|N})$$

induces an equivalence

$$\text{colim}_{\eta} D_{\text{perf}}^{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N}) \cong \text{colim}_{\eta} D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|N}) \cong D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|N}).$$

(2.19)

In particular, $\text{colim}_{\eta} D_{\text{perf}}^{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N}) \cong D_{\text{perf}}^{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|M})$.

By taking ind-completion, we obtain

$$\text{colim}_{\eta} D_{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N}) \cong D_{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|N}),$$

$$\text{colim}_{\eta} D_{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|M}) \cong D_{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M-1|M}).$$

To prove (2.18), it suffices to prove the following equivalence:

$$D_{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|M}) \otimes_{D_{GL_{M-1} \times GL_N}(\mathcal{B}^{2,0}_{M|M})} D_{GL_{M} \times GL_N}(\mathcal{B}^{2,0}_{M|N})$$

$$\cong D_{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N}).$$

(2.20)

Then, the equivalence (2.18) follows by taking colimits in the category of $(\infty, 1)$-stable cocomplete categories with continuous functors on both sides.

To prove the equivalence (2.20), it suffices to use the convolution product defined in §2.4 and Lemma 2.5.1.

Indeed, by Corollary 2.5.2,

$$\text{LHS of (2.20)} \cong D(T_{M,M}) \otimes_{D_{GL_{M} \times GL_N}(\mathcal{B}^{2,0}_{M|M})} D_{GL_{M} \times GL_N}(\mathcal{B}^{2,0}_{M|N})$$

$$\cong \tilde{c} \star_{T_M, M} D_{GL_{M} \times GL_N}(\mathcal{B}^{2,0}_{M|N})$$

$$\cong \tilde{c} \star D_{GL_{M} \times GL_N}(\mathcal{B}^{2,0}_{M|N})$$

$$\cong D(T_{M,N})$$

$$\cong D_{GL_{M-1} \times GL_N,\geq 0}(\mathcal{B}^{2,0}_{M-1|N})$$

$$\cong \text{RHS of (2.20)}.$$

The proof for other sheaf categories follows from the same proof and Lemma 2.5.1 and Corollary 2.5.3. \qed

3. D-MODULE SIDE

In this section, we will prove Theorem 3.9.1, which is the D-module side analog of Proposition 2.6.1.
3.1. **Definition of D-modules on ind-pro-schemes.** Since we need to handle with schemes of ind-pro-finite type, we need to take care when we define $D_{GLM}(O)(\text{Gr}_M \times F)$, $D_{GLM}(O)^{\times U_{M,N}}(\text{Gr}_M \times \text{Gr}_N)$, etc. In this section, let us recall the definition of the category of D-modules on ind-pro-schemes.

Assume $Y = \text{colim } Y_i$ is an ind-pro-scheme, and schemes $Y_i$ are of pro-finite type. Let us denote the transition map by $\iota_{i,i'}: Y_i \hookrightarrow Y_i'$. Since each scheme $Y_i$ is of pro-finite type, we can write $Y_i$ as a limit $Y_i = \lim_{\to} Y_{i,j}$ such that each $Y_{i,j}$ is of finite type and the transition functors $\pi_{i,j,j'}: Y_{i,j} \twoheadrightarrow Y_{i,j'}$ are smooth projections.

Let us define $D!\(Y\) := \lim_{\to} \iota^! \colim \pi^! D(Y_{i,j})$, and $D\(Y\) := \colim \pi_* \lim_{\to} \iota_* D(Y_{i,j})$.

There is a canonical equivalence $(D!\(Y\))^Y = D\(Y\)$.

Furthermore, choosing a dimension theory of $Y$ (see [3, Section 3.3.7]) gives rise to an equivalence between $D!\(Y\)$ and $D\(Y\)$. In particular, if $Y$ is an ind-scheme, there is a canonical identification of $D!\(Y\)$ and $D\(Y\)$, we denote it by $D\(Y\)$ for short.

**Lemma 3.1.1.** Assume $G$ is a group ind-pro-scheme, we denote by

\[ m: G \times G \rightarrow G \]

the multiplication map, and by

\[ \Delta: G \rightarrow G \times G \]

the diagonal embedding.

Then $D\(G\)$ is a Hopf algebra with the product $m_\ast$ and the coproduct $\Delta_\ast$. Similarly, $D!\(G\)$ is a Hopf algebra with the product $\Delta^!$ and the coproduct $m^!$.

3.2. **Strong $G$-invariants.** Assume that $\mathcal{C}$ is a category with an action of $D\(G\)$, we define the category of $G$-invariants of $\mathcal{C}$ as

\[ \mathcal{C}^G := \text{Hom}_{D\(G\)}(\text{Vect}, \mathcal{C}). \]

In other words, it is the totalization of

\[ \mathcal{C} \xrightarrow{\text{Hom}} \text{Hom}(D\(G\), \mathcal{C}) \xrightarrow{\text{Hom}} \text{Hom}(D\(G\) \otimes D\(G\), \mathcal{C}) \cdots. \]

If $\mu: G \rightarrow \mathbb{G}_a$ is an additive character, we denote by $\mu^!(\text{exp})$ the character D-module on $G$ corresponding to $\mu$. Let us denote by $\text{Vect}_\mu$ the $D\(G\)$-module category with the underlying category $\text{Vect}$ and the coaction corresponding to $\mu^!(\text{exp})$.

We define the category of $G$-invariants against $\mu$ of $\mathcal{C}$ as

\[ \mathcal{C}^{G,\mu} := \text{Hom}_{D\(G\)}(\text{Vect}_\mu, \mathcal{C}). \]
3.3. Locally compact objects and renormalization. Given a category \(\mathcal{C}\) with an action of \(G\), there is a natural forgetful functor
\[
\mathcal{C}^G \to \mathcal{C}.
\]
We denote by \(\mathcal{C}^{G,\text{loc.c}}\) the full subcategory of \(\mathcal{C}^G\) generated by the preimage of \(\mathcal{C}^c\). We denote its ind-completion by \(\mathcal{C}^{G,\text{ren}}\).

Now we introduce some basic properties of \(\mathcal{C}^{G,\text{ren}}\) and \(\mathcal{C}^{G,\text{loc.c}}\).

**Proposition 3.3.1.** If \(U\) is pro-unipotent, \(U\) acts on \(\mathcal{C}\), and \(\mathcal{C}^U\) is compactly generated, then
\[
\mathcal{C}^{U,\text{ren}} \simeq \mathcal{C}^U.
\]

**Proof.** Since both \(\mathcal{C}^{U,\text{ren}}\) and \(\mathcal{C}^U\) are compactly generated, we only need to verify the equivalence at compact objects level. Namely, we need to show that \(c \in \mathcal{C}^U\) is compact in \(\mathcal{C}\) if and only if \(c\) is compact in \(\mathcal{C}^U\).

Assume \(c \in \mathcal{C}^U\) is compact in \(\mathcal{C}\), i.e.,
\[
\text{Hom}_\mathcal{C}(\text{oblv}(c), \text{colim} d_i) \simeq \text{colim} \text{Hom}_\mathcal{C}(\text{oblv}(c), d_i)
\]
for any filtered colimit of \(d_i \in \mathcal{C}\). By adjointness,
\[
\text{Hom}_{\mathcal{C}^U}(c, \text{Av}_U^*(\text{colim} d_i)) \simeq \text{colim} \text{Hom}_{\mathcal{C}}(c, \text{Av}_U^*(d_i)).
\]
By the assumption on \(U\), \(\text{Av}_U^*\) is a continuous idempotent monad, so there is
\[
\text{Hom}_{\mathcal{C}^U}(c, \text{colim} \text{Av}_U^*(d_i)) \simeq \text{colim} \text{Hom}_{\mathcal{C}}(c, \text{Av}_U^*(d_i)).
\]
For any object \(d' \in \mathcal{C}^U\), we have \(\text{Av}_U^* \circ \text{oblv}(d') \simeq d'\). Combined with the above formula,
\[
\text{Hom}_{\mathcal{C}^U}(c, \text{colim} d'_i) \simeq \text{colim} \text{Hom}_{\mathcal{C}}(c, d'_i)
\]
for any \(d'_i \in \mathcal{C}^U\).

On the other hand, let \(c\) be compact in \(\mathcal{C}^U\). Then \(c\) is compact in \(\mathcal{C}\), since the forgetful functor \(\mathcal{C}^U \to \mathcal{C}\) admits a continuous right adjoint functor \(\text{Av}_U^*\). \(\square\)

**Remark 3.3.2.** The above proposition is false if \(U\) is not of pro-finite type. For example, if \(U\) is the loop group of a unipotent group, then the forgetful functor \(\mathcal{C}^U \to \mathcal{C}\) does not preserve compact objects.

The above proposition is also false for non-unipotent group (even in finite type case). For example, it is false in the case \(\mathcal{C} = \text{Vect}\) and \(U = \mathbb{G}_m\). In this case, the forgetful functor \(\mathcal{C}^U \to \mathcal{C}\) is conservative, so \(\text{Vect}^{\mathbb{G}_m, \text{loc.c}}\) is generated by the essential image of the left adjoint functor \(\text{Av}^{\mathbb{G}_m}_*\) of the forgetful functor. The constant sheaf \(\mathcal{C}\) is not in \(\text{Vect}^{\mathbb{G}_m, \text{loc.c}}\), but in \(\text{Vect}^{\mathbb{G}_m, \text{loc.c}}\).

Recall the following lemma about invariants.

**Lemma 3.3.3.** Assume \(G := H \rtimes S\) is a group ind-pro-scheme. Let us take a character \(\mu_1\) of \(S\) and a character \(\mu_2\) of \(H\). We assume \(\mu_1\) is stable under the conjugation action of \(G\) on \(S\), then \(\mu_1\) and \(\mu_2\) give rise to a character \(\mu\) of \(G\).

Let \(\mathcal{C}\) be a category admitting an action of \(G\), then the category \(\mathcal{C}^{S, \mu_1}\) admits an action of \(H\) and
\[
\text{(3.1)} \quad (\mathcal{C}^{S, \mu_1})^{H, \mu_2} \simeq (\mathcal{C})^G_{\mu}.
\]
Proof. It follows from [3, Lemma 6.5.4, Remark 6.5.6] that for any category $\mathcal{C}'$ admitting an action of $G$, we have

$$((\mathcal{C}')^S)^H \simeq (\mathcal{C}')^G.$$  

The lemma follows from the above equivalence immediately by letting $\mathcal{C}' := \mathcal{C} \otimes \text{Vect}_\mu$. \[\square\]

**Proposition 3.3.4.** In the above lemma, if we assume $S$ is pro-unipotent, then

$$\mathcal{C}^{G,\mu,\text{ren}} \simeq (\mathcal{C}^{S,\mu_1})^{H,\mu_2,\text{ren}}.$$  

Proof. With loss of generality, we only need to show the case without characters. The category $(\mathcal{C}^S)^H,\text{ren}$ is the ind-completion of $(\mathcal{C}^S)^H,\text{loc,c}$. By Proposition 3.3.1, an object in $(\mathcal{C}^S)^H,\text{loc,c}$ is compact if and only if it is compact when regarded as an object in $\mathcal{C}$. So, both $(\mathcal{C}^S)^H,\text{loc,c}$ and $\mathcal{C}^{G,\text{loc,c}}$ are full subcategories of $\mathcal{C}^G$ generated by the objects which are compact in $\mathcal{C}$.

Now, we consider categories over quotient stacks (ref. [3, Section 6.1]). Let $X$ be a scheme and $G$ be an algebraic group (or: a group scheme) which acts on $X$. By definition, a category $\mathcal{C}$ is over $X/G$ if it admits an action of the monoidal category $D(X) \rtimes D_*(G)$.

**Example 3.3.5.** Given a group $G$ and a subgroup $S \subset G$. Then for any category $\mathcal{C}$ admitting an action of $S$, the category $\mathcal{C} \otimes_{D_*(S)} D_*(G)$ is a category over $X/G$. Here $X := G/S$ is the quotient scheme.

The following lemma is from [3, Theorem 6.4.2].

**Lemma 3.3.6.** Let $\mathcal{C}$ be a category over $X/G$. Assume $X$ is a pro-scheme and $G$ is a pro-group scheme which acts transitively on $X$. Then

$$\mathcal{C}^G \simeq (\mathcal{C}_x)^S.$$  

Here $\mathcal{C}_x := \mathcal{C} \otimes_{D_!(X)} \text{Vect}$ denotes cofiber of $\mathcal{C}$ at $x \in X$, and $S = \text{Stab}_G(x)$ is the stabilizer of $x$.

**Proposition 3.3.7.** In the above situation, assume $\mathcal{C} \simeq \mathcal{C}_x \otimes_{D_*(S)} D_*(G)$. Then there is

$$\mathcal{C}^{G,\text{loc,c}} \subset (\mathcal{C}_x)^{S,\text{loc,c}}.$$  

Proof. We need to prove that if

$$\Hom_{\mathcal{C}'}(\text{obl}^G(c), \text{colim} d) \simeq \text{colim} \Hom_{\mathcal{C}'}(\text{obl}^G(c), d)$$

for any $d \in \mathcal{C}$, then

$$\Hom_{\mathcal{C}_x}(i_x^0 \text{obl}^G(c), \text{colim} d') \simeq \text{colim} \Hom_{\mathcal{C}_x}(i_x^0 \text{obl}^G(c), d')$$

for any $d' \in \mathcal{C}_x$. Here $i_x^0 : \mathcal{C} \to \mathcal{C}_x$ denotes the pullback of categories associated with $x \to X$. 


Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^G & \overset{(i_x^J)^G}{\longrightarrow} & (\mathcal{C}_x)^S \\
\mathcal{C} & \overset{i_x^J}{\longrightarrow} & (\mathcal{C}_x)_x,
\end{array}
\]

so that we have

\[
i_x^J \circ \text{obl}^G \simeq \text{obl}^S (i_x^J)^G.
\]

Note that by Lemma 3.3.6, \((i_x^J)^G\) is an equivalence, and therefore its inverse is its continuous right adjoint, denoted \((i_x^J)^{G,R}\).

Since \(G\) and \(S\) are of pro-finite type, \(\text{obl}^G\) and \(\text{obl}^S\) admit continuous right adjoint functors \(\text{Av}_*^G\) and \(\text{Av}_*^S\). So (3.4) can be rewritten as follows:

\[
\text{Hom}_{eq}(c, \underset{\mathcal{C}}{\text{colim}} \text{Av}_*^G(d)) \simeq \text{colim} \text{Hom}_{eq}(c, \text{Av}_*^G(d)),
\]

and similarly, (3.5) becomes

\[
\text{Hom}_{eq}(c, \underset{\mathcal{C}}{\text{colim}} \text{Av}_*^S(d')) \simeq \text{colim} \text{Hom}_{eq}(c, \text{Av}_*^S(d')).
\]

By passing to the right adjoint of (3.7), we have

\[
\text{Av}_*^G (i_x^J) \overset{R}{\simeq} (i_x^J)^{G,R} \text{Av}_*^S.
\]

Let \(d = (i_x^J)(d')\), the left-hand side of (3.8) is

\[
\text{Hom}_{eq}(c, \underset{\mathcal{C}}{\text{colim}} \text{Av}_*^G (i_x^J) d') \simeq \text{Hom}_{eq}(c, \text{colim} (i_x^J)^{G,R} \text{Av}_*^S (d'))
\]

(3.11)

\[
\simeq \text{Hom}_{eq}(c, (i_x^J)^{G,R} \underset{\mathcal{C}}{\text{colim}} \text{Av}_*^S (d')),
\]

and the right-hand side of (3.8) is

\[
\underset{\mathcal{C}}{\text{colim}} \text{Hom}_{eq}(c, \text{Av}_*^G (i_x^J) d') \simeq \text{colim} \text{Hom}_{eq}(c, (i_x^J)^{G,R} \text{Av}_*^S (d')).
\]

By adjointness, it implies

\[
\text{Hom}_{eq}(c, \text{Av}_*^S (d')) \simeq \text{colim} \text{Hom}_{eq}(c, (i_x^J)^{G,R} \text{Av}_*^S (d')).
\]

Proposition 3.3.8. In the above assumption, if \(G\) and \(S\) are of finite type, then \(\mathcal{C}^{G,loc,c} \supseteq (\mathcal{C}_x)^{S,loc,c}\). In particular, we have

\[
\mathcal{C}^{G,loc,c} \simeq (\mathcal{C}_x)^{S,loc,c},
\]

and

\[
\mathcal{C}^{G,ren} \simeq (\mathcal{C}_x)^{S,ren}.
\]

Proof. We only need to show that if the restriction of a \(G\)-equivariant object \(c\) of \(\mathcal{C} \simeq \mathcal{C}_x \otimes_{D_*(S)} D_*(G)\) to \(\mathcal{C}_x \simeq \mathcal{C}_x \otimes_{D_*(S)} D_*(S)\) is compact, then \(c\) is compact.

Assume \(U' := \coprod U_i'\) is an étale cover of \(G\), such that each \(U_i'\) is isomorphic to \(S \times U_i\), the map \(j_{U' \rightarrow G}: U' = S \times U := S \times \coprod U_i \rightarrow G\) is \(S\)-invariant and \(S \times U_i\) is finite on its image. We claim that the restriction of \(c\) to \(\mathcal{C}_x \otimes_{D_*(S)} D_*(S \times U)\) is compact.
Indeed, since the $S$-invariant map $U' \xrightarrow{\bar{j}_{U' \rightarrow G}} G \xrightarrow{p_G} \text{pt}$ factors through the $S$-invariant map $U' \xrightarrow{p_{U' \to S}} S \xrightarrow{p_S} \text{pt}$, the restriction of $c$ to $\mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(U')$ is isomorphic to the pullback of $c' \in \mathcal{C}_x \otimes_{D_\ast(S)} \text{Vect}$ along
\[
\mathcal{C}_x \otimes_{D_\ast(S)} \text{Vect} \rightarrow \mathcal{C}_x \simeq \mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(S) \rightarrow \mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(U').
\]
Here $c'$ is descended from $c$, i.e., $p_G^!(c') \simeq c$.

By our assumption, $p_S^!(c')$ is compact in $\mathcal{C}_x$. Furthermore, since $p_{U' \to S}$ is smooth, $S \times U_i$ and $S$ are of finite type, the $*$-pullback functor $p_{U' \to S}^!$ is well-defined and equals $p_{U' \to S}^!$ up to a cohomology shift. In particular, $p_{U' \to S}^!$ preserves compactness. So the restriction of $c$ to $\mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(U')$ is compact.

Consider the Čech complex associated with $\{U'_i\}$ removing diagonals in the relative Cartesian products. Since finite limit commutes with filtered colimit, compactness of $\bar{j}_{U' \rightarrow G}^!(c)$ implies the compactness of $c$. \hfill \Box

As a corollary of Proposition 3.3.8, we have

**Corollary 3.3.9.** If $X$ and $G$ are of pro-finite type, and $G = G_r \ltimes G_u$, $S = S_r \ltimes S_u$. Here $S_r \subset G_r$ are reductive and $S_u \subset G_u$ are pro-unipotent. Assume $\mathcal{C} \simeq \mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(G)$. Then there is
\[
\mathcal{C}^{G, \text{ren}} \simeq (\mathcal{C}_x)^{S, \text{ren}}.
\]

**Proof.** By Proposition 3.3.4, we have
\[
(\mathcal{C})^{G, \text{ren}} \simeq (\mathcal{C}^{G_u})^{G_r, \text{ren}},
\]
and
\[
(\mathcal{C}_x)^{S, \text{ren}} \simeq (\mathcal{C}^{S_u})^{S_r, \text{ren}}.
\]

Since $\mathcal{C} \simeq \mathcal{C}_x \otimes_{D_\ast(S)} D_\ast(G)$, there is
\[
\mathcal{C}^{G_u} \simeq \mathcal{C}_x \otimes_{D_\ast(S)} D(G/G_u) \simeq \mathcal{C}_x \otimes_{D_\ast(S_r) \otimes D_\ast(S_u)} D_\ast(G_r).
\]

Since $S_u \subset G_u$ and $G_u$ is a normal subgroup of $G$, the action of $S_u$ on $G/G_u$ is trivial. In particular, $D_\ast(S_u)$ acts on $D_\ast(G_r)$ trivially. There is
\[
\mathcal{C}_x \otimes_{D_\ast(S_r) \otimes D_\ast(S_u)} D_\ast(G_r) \simeq (\mathcal{C}_x)^{S_u} \otimes_{D_\ast(S_r)} D_\ast(G_r).
\]

Note that $(\mathcal{C}_x)^{S_u} \otimes_{D_\ast(S_r)} D_\ast(G_r)$ is a category over $X_r/G_r$. Here $X_r := G_r/S_r$. By Proposition 3.3.8, we have
\[
((\mathcal{C}_x)^{S_u} \otimes_{D_\ast(S_r)} D_\ast(G_r))^{G_r, \text{ren}} \simeq (\mathcal{C}_x^{S_u})^{S_r, \text{ren}}.
\]

Apply Proposition 3.3.4 again, we obtain
\[
(\mathcal{C}_x^{S_u})^{S_r, \text{ren}} \simeq (\mathcal{C}_x)^{S, \text{ren}}.
\]

Now, by (3.17) (3.18) and (3.19), we have
\[
\mathcal{C}^{G, \text{ren}} \simeq (\mathcal{C}_x^{S_u})^{S_r, \text{ren}} \simeq (\mathcal{C}_x)^{S, \text{ren}}.
\]
\hfill \Box
3.4. Conventions. Similar to the coherent side, there are four categories on the D-module side corresponding to $D_{\text{perf}}$, $D$, $D_{\text{perf}}$ with nilpotent condition, and $D$ with nilpotent condition.

In order to simplify the notation, let us make the following conventions.

If $M < N$, we denote by $C_{M|N}$ the DG category of $(GL^{}_{M}(O) \ltimes U_{M,N}(F), \chi_{M,N})$-equivariant D-modules on $Gr_{N}$; if $M > N$, we denote by $C_{M|N}$ the DG category of right $(GL^{}_{N}(O) \ltimes U_{N,M}(F), \chi_{N,M})$-equivariant D-modules on $GL^{}_{M}(O) \backslash GL^{}_{M}(F)$; and we denote by $C_{M|M}$ the DG category of $GL^{}_{M}(O)$-equivariant D-modules on $Gr_{M} \times F^{M}$.\footnote{It does not matter if we let $C_{M|M}$ be $D_{\text{gen}}^{GL^{}_{M}(O)}(Gr_{M} \times F^{M})$ or $D_{\text{gen}}^{GL^{}_{M}(O)}(Gr_{M} \times F^{M})$, since there is a (monoidal) equivalence between them.} The categories of their compacts are denoted by $C_{c}^{\text{ren}}_{M|N}$, $C_{c}^{\text{ren}}_{M|M}$ and $C_{c}^{\text{ren}}_{M|M,\ast}$, respectively.

If $M < N$, there is a forgetful functor from $C_{M|N}$ to the category of $(U_{M,N}(F), \chi_{M,N})$-equivariant D-modules on $Gr_{N}$. We denote by $C_{\text{loc},c}^{\text{ren}}_{M|N}$ the full subcategory of $C_{M|N}$ spanned by the preimage of compact $(U_{M,N}(F), \chi_{M,N})$-equivariant D-modules. Furthermore, we denote by $C_{\text{ren}}^{\text{loc}}_{M|N}$ the ind-completion of $C_{\text{loc},c}^{\text{ren}}_{M|N}$. Similarly for $M > N$ and $M = N$.

In this paper, we will prove that

1. $C_{M|N}^{\text{ren}}$ is equivalent to $D_{\text{gen}}^{GL^{}_{M}(O) \times GL^{}_{N}(F)}(\mathfrak{B}_{M|N}^{2,0})$,
2. $C_{\text{loc},c}^{\text{ren}}_{M|N}$ is equivalent to $D_{\text{gen}}^{GL^{}_{M}(O) \times GL^{}_{N}(F)}(\mathfrak{B}_{M|N}^{2,0})$,
3. $C_{M|N}$ is equivalent to $D_{\text{gen}}^{GL^{}_{M}(O) \times GL^{}_{N}(F)}(\mathfrak{B}_{M|N}^{2,0})_{\text{Nilp}}$,
4. $C_{c}^{\text{ren}}_{M|N}$ is equivalent to $D_{\text{gen}}^{GL^{}_{M}(O) \times GL^{}_{N}(F)}(\mathfrak{B}_{M|N}^{2,0})_{\text{Nilp}}$.

The following easy lemma establishes an equivalence between $C_{M|N}$ and $C_{N|M}$ (similarly for $C_{c}$, $C_{\text{loc},c}$ and $C_{\text{ren}}$).

**Lemma 3.4.1.** Pushforward along the map $g \in GL^{}_{N}(F) \longrightarrow g^{-1} \in GL^{}_{N}(F)$ induces an equivalence functor $C_{M|N} \simeq C_{N|M}$.\footnote{Replace $\chi_{N,M}$ by $-\chi_{N,M}$ in the definition of $C_{N|M}$.}

**Proof.** Pushforward along the map $g \in GL^{}_{N}(F) \longrightarrow g^{-1} \in GL^{}_{N}(F)$ sends left $GL^{}_{M}(O)$-equivariant sheaves to right $GL^{}_{M}(O)$-equivariant sheaves, and sends left $(U_{M,N}(F), \chi_{M,N})$-equivariant sheaves to right $(U_{N,M}(F), -\chi_{N,M})$-equivariant sheaves. It induces a functor $C_{M|N} \longrightarrow C_{N|M}$. Similarly, pushforward along the map $g \in GL^{}_{N}(F) \longrightarrow g^{-1} \in GL^{}_{N}(F)$ induces a functor $C_{N|M} \longrightarrow C_{M|N}$. One checks easily that these two functors are inverse to each other. \hfill $\Box$

3.5. Convolution monoidal structure of $C_{M|M}$. If $M = N$, we can define convolution monoidal structures for any sheaf theory of $C_{\text{gen}}^{\text{ren}}_{M|M}$, $C_{\text{loc},c}^{\text{ren}}_{M|M}$, $C_{M|M}$ and $C_{c}^{\text{ren}}_{M|M}$. For simplicity, we only demonstrate the case of $C_{M|M}$ in Section 3.5-3.8. In these sections, let us recall the definitions of the monoidal category structures of $C_{M|M}$, and the actions of $C_{M|M}$ on $C_{M|N}$ (from left) and $C_{M|M-1|M}$ (from right).

The monoidal category structures of $C_{M|M}$ are defined in [5, Section 3.2]. Actually, there are two different monoidal category structures on $C_{M|M}$, which are related by the Fourier transform.
The first one is the naïve convolution product induced by the multiplication. For groups \( G \subset H \), there is a convolution product functor

\[
(3.20) \quad D(G \setminus H/G) \times D(G \setminus H/G) \longrightarrow D(G \setminus H^G) \longrightarrow D(G \setminus H/G).
\]

Let \( H \) be the transpose of the mirabolic subgroup \( \text{Mir}_{M+1}(F) = \begin{pmatrix} \GL_M(F) & * \\ 0 & 1 \end{pmatrix} \) of \( \GL_{M+1}(F) \), and \( G \) be \( \GL_M(O) = \begin{pmatrix} \GL_M(O) & 0 \\ 0 & 1 \end{pmatrix} \subset \GL_{M+1}(O) \subset \GL_{M+1}(F) \).

Applying (3.20), we thus obtain the convolution product on

\[
C_{M|M} = \GL_M(O)(\Gr_M \times F^M) = D_*^{\GL_M(O)}(\Mir_M^t(F)/\GL_M(O)) = D_*(G \setminus H/G).
\]

To be more precise, consider the following diagram,

\[
\begin{array}{ccc}
H \times H^G & \xrightarrow{q} & H^G \\
\downarrow p \times \text{id} & & \downarrow m \\
H/H \times H/G & & H/G,
\end{array}
\]

where \( p, q \) denote the projection maps, \( m \) denotes the multiplication map.

Given \( \mathcal{F}_1, \mathcal{F}_2 \in D^{\GL_M(O)}(\Gr_M \times F^M) \) (here \( ?, =, !, * \)), the pullback of \( \mathcal{F}_1 \boxtimes \mathcal{F}_2 \) along \( p \times \text{id} \) descends to a sheaf \( \mathcal{F}_1 \boxtimes \mathcal{F}_2 \) on \( \Mir_M^t(F) \times \Mir_{M+1}(F)/\GL_M(O) \). We set \( \mathcal{F}_1 \boxtimes \mathcal{F}_2 := m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \).

**Remark 3.5.1.** We can also use the (non-transposed) mirabolic subgroup \( \Mir_{M+1}(F) = \begin{pmatrix} \GL_M(F) & 0 \\ 0 & 1 \end{pmatrix} \) to define the above monoidal structure. Indeed, we only need to notice that the composition of taking transpose and taking inverse induces a monoidal equivalence between \( D(\GL_M(O)\setminus \Mir_M^t(F)/\GL_M(O)) \) and \( D(\GL_M(O)\setminus \Mir_{M+1}(F)/\GL_M(O)) \).

**3.6. The second monoidal structure.** There is another monoidal category structure on \( D^{\GL_M(O)}(\Gr_M \times F^M) \) defined in [5].

We note that \( \Gr_M \times F^M = \GL_M(F) \times^{\GL_M(O)} F^M \). For \( g \in \GL_M(F), v \in F^M \), we denote by \([g, v]\) the corresponding point in \( \Gr_M \times F^M \). For \( h \in \GL_M(F) \), \( h \cdot [g, v] = [h \cdot g, v] \). Under the isomorphism \( \Gr_M \times F^M = \Mir_M^t(F)/\GL_M(O) \), \([g, v]\) is the image of \((g, v)\) under the map \( \GL_M(F) \times F^M = \Mir_M^t(F) \longrightarrow \Mir_{M+1}(F)/\GL_M(O) \longrightarrow \Gr_M \times F^M \).

Consider the following maps

\[
\begin{array}{ccc}
\GL_M(F) \times (\Gr_M \times F^M) & \xrightarrow{q} & \GL_M(F) \times^{\GL_M(O)} \Gr_M \times F^M \\
\downarrow p & & \downarrow m \\
(\Gr_M \times F^M) \times (\Gr_M \times F^M) & & \Gr_M \times F^M.
\end{array}
\]

Here,

\[
p(g_1, [g_2, v]) = ([g_1, g_2 v], [g_2, v]),
\]
Given \( \mathcal{F}_1, \mathcal{F}_2 \in D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \), the pullback of \( \mathcal{F}_1 \boxtimes \mathcal{F}_2 \) along \( p \) descends to a sheaf \( \mathcal{F}_1 \boxtimes \mathcal{F}_2 \) on \( \mathrm{GL}_M(F) \times \mathbf{Gr}_M \times F^M \). We set \( \mathcal{F}_1 \circledast \mathcal{F}_2 := m_!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = m_!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \).

**Remark 3.6.1.** Up to formal issues related to infinite-dimensionality, the \( \circledast \) monoidal structure is a particular case of the monoidal structure on \( D(H \backslash (G \times X)/H) \) where \( X \) is a variety acted on by an algebraic group \( G \) with a subgroup \( H \subset G \), the right \( H \)-action on \( G \times X \) is along the \( G \) factor, while the left \( H \)-action is diagonal. (This is applied to \( G = \mathrm{GL}_M(F), H = \mathrm{GL}_M(O) \).

**Remark 3.6.2.** On the subcategory of \( D \)-modules supported on \( \mathbf{Gr}_M \times \{0\} \subset \mathbf{Gr}_M \times F^M \), the functors

\[
\otimes, \ast : D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \times D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \longrightarrow D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M)
\]

restrict to the usual convolution product of the spherical Hecke category.

On the subcategory of constructible sheaves supported on \( \{1\} \times F^M \subset \mathbf{Gr}_M \times F^M \) (i.e., require \( g = 1 \)), the functor

\[
\otimes : D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \times D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \longrightarrow D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M)
\]

restricts to the tensor product of \( D^\mathrm{GL}_M(O)(F^M) \), and the functor

\[
\ast : D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \times D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \longrightarrow D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M)
\]

restricts to the convolution product of \( D^\mathrm{GL}_M(O)(F^M) \).

**Remark 3.6.3.** \( \otimes, \ast \) are related by the Fourier transform. That is to say, the Fourier transform functor

\[
(3.21) \quad FT : D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \longrightarrow D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times (F^M)^\vee)
\]

intertwines \( \otimes \) and \( \ast \).

By a choice of basis, we can identify \( F^M \) with \( (F^M)^\vee \). Accordingly, the action of \( \mathrm{GL}_M(O) \) becomes the original action composed with \( g \to (g')^{-1} \). For more details, see [5, Section 3.15].

Regarding Theorem 1.1, later on, when we use the monoidal structure \( \otimes \), we identify \( C_{M|M} \) as \( D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \), and when we use \( \ast \), we identify \( C_{M|M} \) as \( D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M) \).

### 3.7 Left action of \( C_{M|M} \).

In this section, we will define the left action of \( C_{M|M} \) on \( C_{M|N} \). Since \( FT \) is an equivalence of monoidal categories, the category of the categories admitting an action of \( (D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M), \ast) \) is equivalent to the category of the categories admitting an action of \( (D^\mathrm{GL}_M(O)(\mathbf{Gr}_M \times F^M), \otimes) \). Hence, we only need to construct such an action for \( (C_{M|M}, \ast) \).
Assume $N > M$, let us consider the following diagram,

$$
\begin{array}{c}
\text{Mir}_{M+1}^t(F) \times \text{Gr}_N \\
p \times \text{id} \\
\text{Mir}_{M+1}^t(F)/\text{GL}_M(O) \times \text{Gr}_N
\end{array}
\xrightarrow{\text{p \times \text{id}}} \begin{array}{c}
\text{Mir}_{M+1}^t(F) \times \text{Gr}_N \\
m_N
\end{array}
$$

Given $\mathcal{F}_1 \in D^\text{GL}_M(O)(\text{Gr}_M \times F^M)$ and $\mathcal{F}_2 \in D^\text{GL}_M(O) \times U_{M,N}(F), \chi_{M,N}(\text{Gr}_N)$, the pullback of $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ along $p \times \text{id}$ descends to a sheaf $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ on $\text{Mir}^t_{M+1}(F) \times \text{Gr}_N$. We define $\mathcal{F}_1 \ast \mathcal{F}_2 := m_{N, *} (\mathcal{F}_1 \boxtimes \mathcal{F}_2)$.

Since $U_{M,N}(F)$ is a normal subgroup of $\text{GL}_M(O) \times U_{M,N}(F)$ and $\chi_{M,N}$ is stable under the conjugation action of $\text{GL}_M(O)$, the resulting sheaf $\mathcal{F}_1 \ast \mathcal{F}_2$ is still $(\text{GL}_M(O) \times U_{M,N}(F), \chi_{M,N})$-equivariant. We get an action of $C_{M|M}$ on $C_{M-1|M}$.

### 3.8 Right action on $C_{M-1|M}$

In this section, we will construct a right action of $C_{M|M}$ on $C_{M-1|M}$.

Let us denote by $j_* \in D^\text{GL}_M(O)(\text{Gr}_M \times F^M)$ the $*$-extension of the constant sheaf on $1 \times (O^M \times tO^M)$ along the locally closed embedding

$$j: 1 \times (O^M \times tO^M) \hookrightarrow \text{Gr}_M \times F^M,$$

then taking the convolution with $j_*$ gives an endofunctor of $C_{M|M}$,

$$j_* \otimes -: C_{M|M} \to C_{M|M}$$

(3.22)

\[ \mathcal{F} \mapsto j_* \otimes \mathcal{F}. \]

It is easy to see that $j_*$ is an idempotent algebra object of $(C_{M|M}, \otimes)$, i.e., $j_* \otimes j_* \simeq j_*$ with the associativity. In particular, the Kleisli category $j_* \otimes C_{M|M}$ is a full subcategory of $C_{M|M}$.

By definition, $j_* \otimes C_{M|M}$ is equivalent to $D^\text{Mir}(O)(\text{Gr}_M)$. Indeed, since $\text{Stab}_{\text{GL}_M(O)}(e_M) = \text{Mir}_M(O)$, we have

$$D^\text{Mir}(O)(\text{Gr}_M) \simeq D^\text{Mir}(O)(\text{Gr}_M \times e_M) \simeq D^\text{GL}_M(O)(\text{Gr}_M \times (O^M \times tO^M)).$$

The second isomorphism is by [3, Corollary 6.2.5].

Let $\xi_M$ be the automorphism of $\text{Gr}_M$ which is induced by sending $e_M$ to $e_M$ and $e_i$ to $te_i$ for any $i = 1, 2, \ldots, M - 1$. Pushforward along $\xi_M$ induces an endofunctor of $D^\text{Mir}(O)(\text{Gr}_M)$.

By [5, Section 4], we have

$$C_{M-1|M} \simeq \text{colim}_{\xi_M,*} D^\text{Mir}(O)(\text{Gr}_M).$$

We denote by

$$\xi_M: j_* \otimes C_{M|M} \to j_* \otimes C_{M|M}$$

(3.24)

the corresponding transition functor corresponding to $\xi_M,*$ under the equivalence $D^\text{Mir}(O)(\text{Gr}_M) \simeq j_* \otimes C_{M|M}$.

Note that the transition functor $\xi_M,*$ is given by left multiplication with a diagonal element $D_M = \text{diag}(t, t, \ldots, t, 1)$ in $\text{GL}_M(F)$, hence the right action of $C_{M|M}$ on $j_* \otimes C_{M|M}$ (resp. $D^\text{Mir}(O)(\text{Gr}_M)$) is compatible with respect to the transition
functor $\zeta_M$ (resp. $\xi_{M,s}$). By taking the colimit of the action of $C_{M|M}$ on $j_* \oplus C_{M|M}$, we obtain an action of $C_{M|M}$ on colim $j_* \oplus C_{M|M} = C_{M-1|M}$.

**Remark 3.8.1.** We note that we can also describe $j_* \oplus C_{M|M}$ as a $\text{Mir}^t(M)(O)$-equivariant category instead of a $\text{Mir}(M)(O)$-equivariant category. Namely, by $g \rightarrow g^{t, -1}$, there is an equivalence $D^{\text{Mir}^t(M)(O)}(\text{Gr}_M) \simeq D^{\text{Mir}(M)(O)}(\text{Gr}_M) \simeq j_* \oplus C_{M|M}$.

If we use the identification $j_* \oplus C_{M|M} \simeq D^{\text{Mir}^t(M)(O)}(\text{Gr}_M)$, we need to replace the transition functor $\xi_{M,s}$ by $\xi_{M,s}^{-1}$ in (3.24), i.e.,

$$C_{M-1|M} \simeq \text{colim}_{\xi_{M,s}^{-1}} D^{\text{Mir}^t(M)(O)}(\text{Gr}_M).$$

### 3.9. Tensor product of Gaiotto categories.

Now we are going to prove the $D$-module side analog of (2.18).

**Theorem 3.9.1.** For $M \leq N$, we have

$$C^{\text{ren}}_{M-1|M} \otimes^{C^{\text{ren}}_{M|M}} C^{\text{ren}}_{M|N} \simeq C^{\text{ren}}_{M-1|N}.$$

Similar claim holds for other sheaf theories, e.g.,

$$C_{M-1|M} \otimes^{C_{M|M}} C_{M|N} \simeq C_{M-1|N}.$$

### 3.10. Mirabolic equivariant category.

To prove Theorem 3.9.1, we need to consider the mirabolic equivariant category. Let us denote by $D^{\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N)$ the category of left $(\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F), \chi_{M,N})$-equivariant $D$-modules on $\text{Gr}_N$. This category is well-defined because $\chi_{M,N}$ is stable under the conjugation action of $\text{Mir}^t_{M+1}(O)$. Similarly, we can define the renormalized mirabolic equivariant category. Namely, we denote by $(D^{U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^t_{M+1}(O)\text{ren}}$ the ind-completion of $(D^{U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^t_{M+1}(O), \text{loc.c}}$. Here $(D^{U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^t_{M+1}(O), \text{loc.c}}$ is the category of locally compact left $(\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F), \chi_{M,N})$-equivariant $D$-modules on $\text{Gr}_N$.

Since $\text{GL}_M(O)$ is a subgroup of $\text{Mir}^t_{M+1}(O)$, there is a forgetful functor

$$\text{Res}_{M,N} : (D^{U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^t_{M+1}(O)} \rightarrow (D^{U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O)}.$$

Denote by $\xi_{M,N}$ the automorphism of $\text{Gr}_N$ which sends $e_i$ to $t \cdot e_i$, if $1 \leq i \leq M$, and $e_i$ to $e_i$, if $M + 1 \leq i \leq N$. The action of $\xi_{M,N}$ on $\text{Gr}_N$ is given by left multiplication with the $N \times N$ diagonal matrix $D_{M,N} = (t, t, ..., t, 1, 1, ..., 1)$. By a simple calculation,

$$\text{GL}_M(O) \otimes U_{M,N}(F) = D_{M,N}^{-1}(\text{GL}_M(O) \otimes U_{M,N}(F))D_{M,N}$$

(resp. $\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F) \subset D_{M,N}^{-1}(\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F))D_{M,N}$).

Furthermore, the character $\chi_{M,N}$ of $U_{M,N}(F)$ is stable under the conjugation by $D_{M,N}^{-1}$, hence the functor $\xi_{M,N,s}$ induces an auto-equivalence (resp. endofunctor) of $D^{\text{GL}_M(O) \otimes U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N)$ (resp. $D^{\text{Mir}^t_{M+1}(O) \otimes U_{M,N}(F), \chi_{M,N}}(\text{Gr}_N)$).
colim is an equivalence. In particular, we have
\[ \xi_{M,N,*}^{-1} : (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O)} \to (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O)}, \]
\[ \xi_{M,N,*}^{-1} : (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^{t}_{M+1}(O)} \to (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^{t}_{M+1}(O)}. \]

We will prove the following Proposition after a short preparation.

**Proposition 3.10.1.** Taking colimit of $\text{Res}_{M,N}$ with respect to the transition functor $\xi_{M,N,*}^{-1}$ induces an equivalence of categories. That is to say, the functor
\[ \text{colim}_{\xi_{M,N,*}^{-1}} (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^{t}_{M+1}(O)} \to \text{colim}_{\xi_{M,N,*}^{-1}} (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O)} \]
is an equivalence. In particular, we have
\[ \text{colim}_{\xi_{M,N,*}^{-1}} (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{Mir}^{t}_{M+1}(O)} \simeq (D^{U_{M,N}(F)}_{\chi_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O)}. \]

Note that $\text{Mir}^{t}_{M+1}(O)$ is a pro-group scheme, we denote by $D_{*}(\text{Mir}^{t}_{M+1}(O))$ the category of D-modules defined in §3.1. It is a monoidal category with the convolution monoidal structure.

Conjugating by $D_{*}(\text{Mir}^{t}_{M+1}(O))$ induces a monoidal functor
\[ D_{*}(\text{Mir}^{t}_{M+1}(O)) \to D_{*}(\text{Mir}^{t}_{M+1}(O)) \]
which factors through $D_{*}(\text{GL}_M(O) \ltimes tO^M)$.

In particular, it induces a functor
\[ F_{M+1} : \mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} \to \mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} \]
for any category $\mathcal{C}$ with an action of $\text{Mir}^{t}_{M+1}(O)$. It factors through $\mathcal{C}^{\text{GL}_M(O) \ltimes tO^M}$.

**Lemma 3.10.2.** There is an equivalence of categories
\[ \text{colim}_{F_{M+1}} \mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} \simeq \mathcal{C}^{\text{GL}_M(O)}. \]

**Proof.** Note that there is a commutative diagram,
\[ \begin{array}{ccc}
\mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} & \xrightarrow{F_{M+1}} & \mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} \\
\downarrow & & \downarrow \text{oblv} \\
\mathcal{C}^{\text{GL}_M(O) \ltimes tO^M} & \xrightarrow{\text{oblv}} & \mathcal{C}^{\text{GL}_M(O) \ltimes tO^M}. \\
\end{array} \]

So, we have
\[ \text{colim}_{F_{M+1}} \mathcal{C}^{\text{Mir}^{t}_{M+1}(O)} \simeq \text{colim}_{\text{oblv}} \mathcal{C}^{\text{GL}_M(O) \ltimes tO^M}. \]

Here, the transition functors of the right-hand side are forgetful functors.

By Proposition 3.3.4, we have an equivalence
\[ \mathcal{C}^{\text{GL}_M(O) \ltimes tO^M} \simeq (\mathcal{C}^{tO^M})^{\text{GL}_M(O)}. \]

Since $\text{GL}_M(O)$ is a pro-group scheme, taking $\text{GL}_M(O)$-invariants commutes with colimit. So we have
\[ (\text{colim} \mathcal{C}^{tO^M})^{\text{GL}_M(O)} \simeq \text{colim} ((\mathcal{C}^{tO^M})^{\text{GL}_M(O)}). \]

Now the claim follows from [3, Proposition 4.3.8]
\[ \text{colim}_{\text{oblv}} \mathcal{C}^{tO^M} \simeq \mathcal{C}. \]
Proof of Proposition 3.10.1. We let \( \mathcal{C} := D^{U_{M,N}(F),X_{M,N}}(\text{Gr}_N) \) in the above lemma, now it is sufficient to notice that \( F^{M+1} \) coincides with \( \xi^{1}_{M,N,*} \). \( \square \)

By definition, \( \text{C}^{\text{Mir}_{M+1}^t(O)} \) is a full subcategory of \( \text{C}^{\text{GL}_M(O)} \). The endofunctor \( F_{M+1} \) of \( \text{C}^{\text{Mir}_{M+1}^t(O)} \) comes from the restriction of the auto-equivalence of \( \text{C}^{\text{GL}_M(O)} \) induced by conjugating with \( D_{M+1} \). In particular, \( F_{M+1} \) preserves local compactness on \( \text{C}^{\text{Mir}_{M+1}^t(O)} \). Hence \( F_{M+1} \) induces a functor

\[
F_{M+1}^{\text{ren}} : \text{C}^{\text{Mir}_{M+1}^t(O)}^{\text{ren}} \rightarrow \text{C}^{\text{Mir}_{M+1}^t(O)}^{\text{ren}}.
\]

It is expected that there is an equivalence

\[
(3.37) \quad \text{colim} F_{M+1}^{\text{ren}} : \text{C}^{\text{Mir}_{M+1}^t(O)}^{\text{ren}} \simeq \text{C}^{\text{GL}_M(O)}^{\text{ren}}.
\]

In other words, there is an equivalence

\[
(3.38) \quad \text{colim}_{\text{oblv}} \text{C}^{\text{GL}_M(O) \times t^n O^M, \text{ren}} \simeq \text{C}^{\text{GL}_M(O) \times \text{ren}}.
\]

Since taking ind-completion commutes with taking colimit, we only need to show

\[
(3.39) \quad \text{colim}_{\text{oblv}} \text{C}^{\text{GL}_M(O) \times t^n O^M, \text{loc}, \text{c}} \simeq \text{C}^{\text{GL}_M(O) \times \text{loc}, \text{c}}.
\]

It is not known if (3.39) is true for general \( \mathcal{C} \). However, it is true in the following case.

**Corollary 3.10.3.** There is an equivalence of categories

\[
(3.40) \quad \text{colim}_{\text{Mann}} (D^{U_{M,N}(F),X_{M,N}}(\text{Gr}_N))^{\text{Mir}_{M+1}^t(O), \text{loc}, \text{c}} \simeq (D^{U_{M,N}(F),X_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O), \text{loc}, \text{c}}.
\]

In particular,

\[
(3.41) \quad \text{colim}_{\text{Mann}} (D^{U_{M,N}(F),X_{M,N}}(\text{Gr}_N))^{\text{Mir}_{M+1}^t(O), \text{ren}} \simeq (D^{U_{M,N}(F),X_{M,N}}(\text{Gr}_N))^{\text{GL}_M(O), \text{ren}}.
\]

**Proof.** It is sufficient to show that any \( \text{GL}_M(O) \times U_{M,N}(F) \)-orbit \( O \) of \( \text{Gr}_N \) is \( (\text{GL}_M(O) \times t^n O^M) \times U_{M,N}(F) \)-invariant for sufficiently large \( n \). Note that \( (\text{GL}_M(O) \times t^n O^M) \times U_{M,N}(F) \) is generated by \( (\text{GL}_M(O) \times U_{M,N}(F)) \) and \( t^n O^M \), so it is sufficient to prove that \( O \) is \( t^n O^M \)-invariant for sufficiently large \( n \). Here \( t^n O^M \) is regarded as a subgroup of \( \text{GL}_N(F) \) as follows:

\[
t^n O^M = \begin{pmatrix} 1 & t^n O & & \cdots \ 1 & \ddots & \ddots \ & & 1 \ M & & N-M \end{pmatrix} \subset \text{GL}_N(F).
\]

Fix \( g \in \text{GL}_N(F) \). We assume that entries of both \( g \) and \( g^{-1} \) belong to \( t^{-m} O \). We claim that the orbit passing through \( g \) is \( t^{2m} O^M \)-invariant.

Indeed, we have

\[
\text{GL}_M(O) \times U_{M,N}(F) \cdot g \cdot \text{GL}_N(O) = \text{GL}_M(O) \times U_{M,N}(F) \cdot g \text{GL}_N(O) g^{-1} \cdot g \cdot \text{GL}_N(O),
\]

which is equivalent to

\[
\text{GL}_M(O) \times U_{M,N}(F) \cdot t^{-m} O = \text{GL}_M(O) \times U_{M,N}(F) \cdot t^{2m} O^M.
\]
\[ t^{2m}O^M \cdot GL_M(O) \times U_{M,N}(F) = GL_M(O) \times U_{M,N}(F) \cdot t^{2m}O^M, \]

and

\[ t^{2m}O^M \subset gGL_N(O)g^{-1}. \]

### 3.11 Fourier transform on Gaiotto category

By Proposition 3.10.1, \( C_{M-1|N} \) can be written as a colimit of \( (D^{U_{M-1,N}(F) \times M-1,N(G_N)})^{Mir_M(O)} \) with the transition functor \( \xi_M^{1,-1,N,*} \).

As a result, we can write both sides of (3.27) as colimits. Namely,

(3.42) \[ \text{LHS of (3.27)} \simeq \text{colim}_{s_{M}} \, D^{Mir_M(O)}(Gr_M) \otimes C_{M|N}, \]

and

(3.43) \[ \text{RHS of (3.27)} \simeq \text{colim}_{s_{M-1,N,*}} \, (D^{U_{M-1,N}(F) \times M,N(G_N)})^{Mir_M(O)}. \]

Similarly for (3.28). To prove Theorem 3.9.1, it suffices to show the following proposition.

**Proposition 3.11.1.** For \( M \leq N \), we have

a).

\[ D^{Mir_M(O),\text{ren}}(Gr_M) \otimes C^{\text{ren}}_{M|N} \simeq (D^{U_{M-1,N}(F) \times M,N(G_N)})^{Mir_M(O),\text{ren}}. \]

b).

\[ D^{Mir_M(O)}(Gr_M) \otimes C_{M|N} \simeq (D^{U_{M-1,N}(F) \times M,N(G_N)})^{Mir_M(O)}. \]

Let us first prove (3.45).

Recall the definition of \((C_{M|M}, \otimes)\). The pushforward along the closed embedding

\[ 1 \times F^M \rightarrow Gr_M \times F^M \]

gives a monoidal fully faithful embedding

(3.46) \[ D^{GL_M(O)}_1(F^M) \rightarrow C_{M|M}. \]

Note that the image of this embedding contains the object \( j_* \), so below we use the same notation for the corresponding object in the source category.

**Lemma 3.11.2.** For any category \( \mathcal{D} \) admitting a left action of \((C_{M|M}, \otimes)\), we have

(3.47) \[ (j_* \otimes C_{M|M}) \otimes \mathcal{D} \simeq D^{GL_M(O)}_1(O^M \otimes tO^M) \otimes D^{GL_M(O)}_1(F_M) \otimes \mathcal{D}. \]

**Proof.** It follows from the fact that:

\[
\begin{align*}
(j_* \otimes C_{M|M}) & \otimes \mathcal{D} \simeq (j_* \otimes D^{GL_M(O)}_1(F^M)) \otimes D^{GL_M(O)}_1(F_M) \otimes \mathcal{D} \\
& \simeq D^{GL_M(O)}_1(O^M \otimes tO^M) \otimes D^{GL_M(O)}_1(F_M) \otimes \mathcal{D} \\
& \simeq D^{GL_M(O)}_1(O^M \otimes tO^M) \otimes D^{GL_M(O)}_1(F_M) \otimes \mathcal{D},
\end{align*}
\]

\[ \square \]
Remark 3.11.3. Both sides of (3.47) are equivalent to the image of the action functor of $j_*$ on $\mathcal{D}$.

Furthermore, if we assume that $\mathcal{D} = (\mathcal{D}')^{GL_M(\mathcal{O})}$ and the action of $D_i^{GL_M(\mathcal{O})}(\mathcal{F}^M)$ on $(\mathcal{D}')^{GL_M(\mathcal{O})}$ is given by the action of $D_i(\mathcal{F}^M)$ on $\mathcal{D}'$, we have

\begin{equation}
D_i^{GL_M(\mathcal{O})}(\mathcal{O}^M \otimes t\mathcal{O}^M)_{D_i^{GL_M(\mathcal{O})}(\mathcal{F}^M)} \simeq (D_i(\mathcal{O}^M \otimes t\mathcal{O}^M) \otimes (\mathcal{D}')^{GL_M(\mathcal{O})}).
\end{equation}

Now by Lemma 3.3.6, there is an equivalence

\begin{equation}
(D_i(\mathcal{O}^M \otimes t\mathcal{O}^M) \otimes (\mathcal{D}')^{GL_M(\mathcal{O})}) \simeq (\text{Vect}_{e_M} \otimes (\mathcal{D}')^{\text{Mir}_M(\mathcal{O})}).
\end{equation}

Here, $D_i(\mathcal{F}^M)$ acts on $\text{Vect}_{e_M}$ by pulling back along $\{e_M\} \rightarrow \mathcal{F}^M$.

Applying the isomorphism (3.49) to $\mathcal{D} = C_{M|N}$ and $\mathcal{D}' = D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)}$, we get

\begin{equation}
(j_* \otimes C_{M|N}) \otimes C_{M|N} \simeq (\text{Vect}_{e_M} \otimes D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)})^{\text{Mir}_M(\mathcal{O})}.
\end{equation}

To prove (3.45), we have to prove:

\begin{equation}
(\text{Vect}_{e_M} \otimes D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)})^{\text{Mir}_M(\mathcal{O})} \simeq (D^{U_{M-1,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)})^{\text{Mir}_M(\mathcal{O})}.
\end{equation}

3.11.1.

Proof of Proposition 3.11.1 b). By the Fourier equivalence of [3, Theorem 5.2.11], we have

\begin{equation}
\text{Vect}_{e_M} \otimes D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)} \simeq (D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)})^{\mathcal{F}^M \chi_{e_M}}.
\end{equation}

Here, $\chi_{e_M}$ denotes the character of taking residue of the coefficients of $e_M$, and $\mathcal{F}^M$ acts on $\text{Gr}_N$ via

\[
\mathcal{F}^M = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
* & 1 & \cdots & 1 \\
1 & \cdots & * & \cdots & \shoveleft{\cdots} \\
\shoveleft{\cdots} & \shoveleft{\cdots} & \shoveleft{\cdots} & \shoveleft{\cdots} & \shoveleft{\cdots} \\
M & N-M & M & N-M & \\
\end{pmatrix} \subset \text{GL}_N(\mathcal{F}).
\]

We have $U_{M-1,N}(\mathcal{F}) = \mathcal{F}^M \ltimes U_{M,N}(\mathcal{F})$, and the conjugation action of $\mathcal{F}^M$ on $U_{M,N}(\mathcal{F})$ preserves $\chi_{M,N}$. Hence, by Lemma 3.3.3,

\begin{equation}
(D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)})^{\mathcal{F}^M \chi_{e_M}} \simeq D^{U_{M-1,N}(\mathcal{F}) \chi_{M-1,N}(\text{Gr}_N)}.
\end{equation}

Combining (3.52) and (3.53), we have

\begin{equation}
\text{Vect}_{e_M} \otimes D^{U_{M,N}(\mathcal{F}) \chi_{M,N}(\text{Gr}_N)} \simeq D^{U_{M-1,N}(\mathcal{F}) \chi_{M-1,N}(\text{Gr}_N)}.
\end{equation}
Since (3.54) is induced by Fourier transform, we see that the action of the group \(\text{Mir}_M(O)\) on \(\text{Vect}_{e_M} \otimes D^U_{M,N}(\mathcal{F})_{X,M,N}(\mathcal{G}_N)\) becomes the action of
\[(\text{Mir}_M(O))^{t^{-1}} = \text{Mir}^t_M(O)\].

3.11.2. Then, we prove (3.44). Let \(\mathcal{D}' = D^U_{M,N}(\mathcal{F})_{X,M,N}(\mathcal{G}_N)\). To prove the renormalized version (3.44), we only need to notice that
\[D_*(\text{GL}_M(O)) \otimes D_*(\text{Mir}_M(O)) (\text{Vect}_{e_M} \otimes \mathcal{D}') \simeq D_*(O^M < tO^M) \otimes \mathcal{D}',\]
and \(\text{GL}_M(O) = \text{GL}_M \ltimes \text{GL}_{M,1}(O), \text{Mir}_M(O) = \text{Mir}_M \ltimes \text{Mir}_{M,1}(O)\). Here \(\text{GL}_{M,1}(O)\) and \(\text{Mir}_{M,1}(O)\) denote the corresponding congruence subgroups.

Applying Corollary 3.3.9, we obtain that there is an equivalence
\[(3.55) \quad (D_*(O^M < tO^M) \otimes \mathcal{D}')^{\text{GL}_M(O),\text{ren}} \simeq (\text{Vect}_{e_M} \otimes \mathcal{D}')^\text{Mir}_M(O),\text{ren} \simeq (D^U_{M,-1,N}(\mathcal{F})_{X,M,N}(\mathcal{G}_N))^{\text{Mir}_M(O),\text{ren}}.
\]
To prove (3.44), we only need to prove:
\[(3.56) \quad (\text{Vect}_{e_M} \otimes D^U_{M,N}(\mathcal{F})_{X,M,N}(\mathcal{G}_N))^{\text{Mir}_M(O),\text{ren}} \simeq (D^U_{M,-1,N}(\mathcal{F})_{X,M,N}(\mathcal{G}_N))^{\text{Mir}_M(O),\text{ren}},\]
which follows exactly from the same argument of §3.11.1.

4. Compatibility of actions

Theorem 1.3.1 implies that there is an equivalence of categories for any \(M\),
\[(4.1) \quad C_{M^-1|M}^\text{ren} \simeq D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M^-1|M}^{2,0}).\]

By §3.7, we have a left action of \(C_{M^-1|M}^\text{ren}\) on \(C_{M^-1|M}^\text{ren}\) and a right action of \(C_{M|M}^\text{ren}\) on \(C_{M^-1|M}^\text{ren}\). Similarly, by §2.4, we have a left action of \(D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M^-1|M}^{2,0})\) and a right action of \(D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M^-1|M}^{2,0})\). In this section, we will prove that the equivalence (4.1) is not only an equivalence of categories, but also an equivalence of module categories.

**Proposition 4.0.1.** a) The equivalence (4.1) is compatible with the left action of
\[C_{M^-1|M}^\text{ren} \simeq D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M^-1|M}^{2,0}).\]

b) The equivalence (4.1) is compatible with the right action of
\[C_{M|M}^\text{ren} \simeq D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M|M}^{2,0}).\]

**Proof of Proposition 4.0.1 b).** Let us first check the compatibility of the right actions.

Since the equivalence \(C_{M|M}^\text{ren} \simeq D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M|M}^{2,0})\) comes from the ind-completion of a monoidal equivalence functor \(C_{M|M}^\text{loc} \simeq D_{\text{perf}}^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M|M}^{2,0})\), it is monoidal. The right convolution always commutes with left convolution, so the equivalence \(j_* \otimes C_{M|M}^\text{ren} \simeq \tilde{c}^A \ast D^{\text{GL}_M \times \text{GL}_M}(\mathfrak{B}_{M|M}^{2,0})\) is compatible with the right action of (1.1).

Furthermore, note that the transition functor \(\xi_{M,*}\) is given by left multiplication with a diagonal element \(\text{diag}(t, t, \ldots, t, 1)\) in \(\text{GL}_M(\mathcal{F})\). In particular, the transition functor commutes with right action of \(C_{M|M}^\text{ren}\). Hence, the action of \(C_{M|M}^\text{ren}\) on
\[(4.2) \quad C_{M^-1|M}^\text{ren} \simeq \text{colim}_{\xi_{M,*}} D^{\text{Mir}_M(O),\text{ren}}(\mathcal{G}_M) \simeq \text{colim}_{\xi_M} j_* \otimes C_{M|M}^\text{ren}\]
is given by taking colimit of the action of $C_{\tilde{M}|M}^{\text{ren}}$ on $j_* \otimes C_{\tilde{M}|M}^{\text{ren}}$.

Similarly, the action of $D^{\text{GL},M \times \text{GL},M}(\mathfrak{B}_{M|M}^{2,0})$ on $D^{\text{GL},M-1 \times \text{GL},M}(\mathfrak{B}_{M-1|M}^{2,0})$ is given by taking the colimit of the action of $D^{\text{GL},M \times \text{GL},M}(\mathfrak{B}_{M|M}^{2,0})$ on $c^A \star D^{\text{GL},M \times \text{GL},M}(\mathfrak{B}_{M|M}^{2,0})$.

Now Proposition 4.0.1 a) follows from the fact that the following diagram

\[
\begin{array}{ccc}
j_* \otimes C_{\tilde{M}|M}^{\text{ren}} & \xrightarrow{\zeta_M} & j_* \otimes C_{\tilde{M}|M}^{\text{ren}} \\
\downarrow t & & \downarrow t \\
c^A \star D^{\text{GL},M \times \text{GL},M}(\mathfrak{B}_{M|M}^{2,0}) & \xrightarrow{\eta} & c^A \star D^{\text{GL},M \times \text{GL},M}(\mathfrak{B}_{M|M}^{2,0})
\end{array}
\]

commutes.

\[\square\]

In order to check (4.1) is compatible with left actions of $C_{\tilde{M}-1|M-1}^{\text{ren}} \simeq D^{\text{GL},M-1 \times \text{GL},M-1}(\mathfrak{B}_{M-1|M-1}^{2,0})$, we need the following lemma.

**Lemma 4.0.2.** There is an equivalence of categories

\[C_{\tilde{M|-1}|M|M}^{\text{ren}} \otimes C_{\tilde{M}|M|M}^{\text{ren}} \simeq C_{\tilde{M}-1|M-1}^{\text{ren}}.\]

**Proof.** By Lemma 3.4.1, $g \mapsto g^{-1}$ induces an equivalence $C_{\tilde{M}|M-1}^{\text{ren}} \simeq C_{\tilde{M}-1|M}^{\text{ren}}$. Furthermore, taking inverse induces a monoidal self-anti-equivalence of $C_{\tilde{M}|M}^{\text{ren}}$,

\[t : C_{\tilde{M}|M}^{\text{ren}} \longrightarrow C_{\tilde{M}|M}^{\text{ren}}
\]

such that,

\[t(\mathcal{F}) \otimes t(\mathcal{G}) \simeq t(\mathcal{S} \otimes \mathcal{F}).\]

The right action of $C_{\tilde{M}|M}^{\text{ren}}$ on $C_{\tilde{M}-1|M}^{\text{ren}}$ induces a left action $C_{\tilde{M}|M}^{\text{ren}}$ on $C_{\tilde{M}-1|M}^{\text{ren}}$.

In particular, $C_{\tilde{M}|M-1}^{\text{ren}} \simeq \text{colim}_{\tilde{M}} D(\text{Gr}_M)^{\text{Mir}^t_O} \simeq \text{colim}_{\tilde{M}} D(\text{Gr}_M)^{\text{Mir}^t_O} \simeq \text{colim}_{\tilde{M}} C_{\tilde{M}|M}^{\text{ren}} \otimes j_*$. Here $D(\text{Gr}_M)^{\text{Mir}^t_O}$ denotes the ind-completion of the category of locally compact right $\text{Mir}^t_O$-equivariant D-modules on $\text{GL}_M(O) \backslash \text{GL}_M(F)$, the transition functors $\xi_{\tilde{M},*}$ is given by taking pushforward along the map of right multiplication with the diagonal element $D_M = \text{diag}(t, t, \cdots, t, 1)$, and $\xi_{\tilde{M}}$ is the corresponding transition functor under the equivalence $D(\text{Gr}_M)^{\text{Mir}^t_O} \simeq C_{\tilde{M}|M}^{\text{ren}} \otimes j_*$.

Applying the above formula to the left-hand side of (4.3), we obtain

\[C_{\tilde{M}-1|M}^{\text{ren}} \otimes C_{\tilde{M}|M}^{\text{ren}} \simeq \text{colim}_{\tilde{M}} C_{\tilde{M}|M}^{\text{ren}} \otimes j_*
\]

\[\simeq \text{colim}_{\tilde{M}} C_{\tilde{M}-1|M}^{\text{ren}} \otimes C_{\tilde{M}|M}^{\text{ren}} \otimes j_*
\]

\[\simeq \text{colim}_{\tilde{M}} C_{\tilde{M}-1|M}^{\text{ren}} \otimes j_*
\]

By Lemma 4.0.3 below, we have

\[\text{colim}_{\tilde{M}} C_{\tilde{M}-1|M}^{\text{ren}} \otimes j_* \simeq \text{colim}_{\tilde{M}} D(\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}^t_O}.
\]

Here $D(\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}^t_O}$ denotes the ind-completion of the category of locally compact right $\text{Mir}^t_O$-equivariant D-modules on $\text{GL}_{M-1}(O) \backslash \text{Mir}^t_O(F) \simeq \text{Gr}_{M-1} \times F^{M-1}$.
By construction, we have

\[ D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}_{M}^!(O)}_{\text{loc.c}}) \simeq D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{GL}_{M-1}^!(O)}_{\text{ren}}). \]

Here \( D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{GL}_{M-1}^!(O)}_{\text{ren}}) \) denotes the ind-completion of the category of locally compact right \( \text{GL}_{M-1}^!(O) \)-equivariant D-modules on \( \text{GL}_{M-1}^!(O) \setminus \text{Mir}_{M}^!(F) \simeq \text{Gr}_{M-1} \times F^{M-1} \).

We only need to show that

\[ \text{colim}'_{M,*} D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}_{M}^!(O)}_{\text{loc.c}}) \simeq D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{GL}_{M-1}^!(O)}_{\text{loc.c}}). \]

It is equivalent to

\[ \text{colim}'_{M,*} D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{GL}_{M-1}^!(O) \times O^{M-1}}_{\text{loc.c}}) \simeq D_!((\text{Gr}_{M-1} \times F^{M-1})^{\text{GL}_{M-1}^!(O)}_{\text{loc.c}}). \]

By a left-right symmetry, we should prove the equivalence for the corresponding categories of left equivariant D-modules,

\[ \text{colim}'_{M,*} D_!^{\text{GL}_{M-1}^!(O) \times O^{M-1}}_{\text{loc.c}}((\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}_{M}^!(O)}_{\text{loc.c}}) \simeq D_!^{\text{GL}_{M-1}^!(O) \times O^{M-1}}_{\text{loc.c}}((\text{Gr}_{M-1} \times F^{M-1}). \]

By the construction of [5, Section 3.2], \( D_!^{\text{GL}_{M-1}^!(O) \times O^{M-1}}_{\text{loc.c}}((\text{Gr}_{M-1} \times F^{M-1}) \) is generated by IC-sheaves on \( \text{GL}_{M-1}^!(O) \)-orbits in \( \text{Mir}_{M}^!(F)/\text{GL}_{M-1}^!(O) \). Here \( \text{Mir}_{M}^!(F) := \{ (h,v) \in \text{Mir}_{M}^!(F) \mid v \neq 0 \} \). So we only need to prove that any \( \text{GL}_{M-1}^!(O) \)-orbit \( (h,g) \) in \( \text{Mir}_{M}^!(F)/\text{GL}_{M-1}^!(O) \) is \( t^n O^{M-1} \)-invariant for sufficiently large \( n \).

Assume \( 0 = \{ [h,g,v] \in \text{Gr}_{M-1} \times (F^{M-1} \setminus 0) \mid h \in \text{GL}_{M-1}^!(O) \} \) (see §3.6 for the definition of notation). Let \( m \) be a positive integer such that

\[ \text{GL}_{M-1,m}^!(O) \subset \bigcap_{h \in \text{GL}_{M-1}^!(O)} hg \text{GL}_{M-1}^!(O) g^{-1} h^{-1}. \]

Here \( \text{GL}_{M-1,m}^!(O) := 1 + t^m \text{End}_{M-1}^!(O) \) denotes the \( m \)-th congruence subgroup of \( \text{GL}_{M-1}^!(O) \). It acts trivially on the affine Grassmannian part of \( \text{Gr}_{M-1}^!(O) \).

To show that this positive integer \( m \) exists, assume \( g, g^{-1} \in t^{-1} \text{End}_{M-1}^!(O) \) for some \( t > 0 \). Then (4.11) is satisfied for any \( m \geq 2t \).

For any point \( [h,g,v] \in 0, \text{GL}_{M-1,m}^!(O) \subset \text{GL}_{M-1}^!(O) \), we have \( \text{GL}_{M-1,m}^!(O) hg, v \subset 0 \). We only need to prove that there exists a positive integer \( m_{[g,v]} \) which does not depend on \( h \), such that

\[ \text{GL}_{M-1,m}^!(O) hg v \supset h v + t^n O^{M-1}, \]

\[ \text{End}_{M-1}^!(O) hg v \supset t^n O^{M-1}, \]

i.e., \( \text{End}_{M-1}^!(O) hg v \supset t^n O^{M-1}, \) for any \( n \geq m_{[g,v]} \).

We can choose \( h' \in \text{GL}_{M-1}^!(O) \), such that \( h' h g v = \sum_{i=1}^{M-1} t^m e_i \). Assume \( m_i \) is the smallest number among \( \{ m_1, m_2, \ldots, m_{M-1} \} \), it is independent of \( h \) and \( h' \). We have \( \text{End}_{M-1}^!(O) hg v \supset t^{m_i} O^{M-1} \), and thus we can take \( m_{[g,v]} = m_i \).

\[ \square \]

**Lemma 4.0.3.** There is an equivalence

\[ \text{C}_{M-1|M}^\text{ren} \otimes j_* \simeq D((\text{Gr}_{M-1} \times F^{M-1})^{\text{Mir}_{M}^!(O)}_{\text{ren}}). \]

**Proof.** By construction, we have

\[ \text{C}_{M-1|M}^\text{ren} \otimes j_* \simeq \text{Ind}(\text{C}_{M-1|M}^\text{loc.c} \otimes j_*). \]
Hence, it is sufficient to prove that
\[ C_{M-1|M}^{\text{loc,c}} \otimes j_* \simeq D(\text{Gr}_{M-1} \times F^{M-1}\text{Mir}_M^t(O),\text{loc,c}). \]
Note that there is an equivalence
\[ C_{M-1|M} \otimes j_* \simeq D(\text{GL}_{M-1}(O)\setminus \text{GL}_M(F)/\text{GL}_M(O)) \otimes j_* \]
(4.14)
\[ \simeq D(\text{GL}_{M-1}(O)\setminus S'/\text{GL}_M(O)) \]
\[ \simeq D(\text{GL}_{M-1}(O)\setminus \text{Mir}_M^t(F)/\text{Mir}_M^t(O)). \]
Here \( S' \) denotes the subscheme of \( \text{GL}_M(F) \) consisting of \( g \in \text{GL}_M(F) \) such that \( g^t \cdot e_M \in O^M \setminus tO^M \). We should prove that two full subcategories
\[ C_{M-1|M}^{\text{loc,c}} \otimes j_* \subset C_{M-1|M} \otimes j_* \]
and
\[ D(\text{Gr}_{M-1} \times F^{M-1}\text{Mir}_M^t(O),\text{loc,c}) \subset D(\text{GL}_{M-1}(O)\setminus \text{Mir}_M^t(F)/\text{Mir}_M^t(O)) \]
coincide.

The category \( D(\text{GL}_{M-1}(O)\setminus S'/\text{GL}_M(O)) \) can be regarded as the category of right \( \text{GL}_M(O) \)-equivariant D-modules on \( \text{GL}_{M-1}(O)\setminus S' \). The category \( C_{M-1|M}^{\text{loc,c}} \otimes j_* \), is generated by the IC (intersection cohomology) D-modules on \( \text{GL}_M(O) \)-orbits of \( \text{GL}_{M-1}(O)\setminus S' \). Under the equivalence \( D(\text{GL}_{M-1}(O)\setminus S'/\text{GL}_M(O)) \simeq D(\text{GL}_{M-1}(O)\setminus \text{Mir}_M^t(F)/\text{Mir}_M^t(O)), C_{M-1|M}^{\text{loc,c}} \otimes j_* \) corresponds to the full subcategory generated by the IC D-modules on \( \text{Mir}_M^t(O) \)-orbits of \( \text{GL}_{M-1}(O)\setminus \text{Mir}_M^t(F) \). It is exactly the category \( D(\text{Gr}_{M-1} \times F^{M-1}\text{Mir}_M^t(O),\text{loc,c}) \). \( \square \)

### 4.1. Colimit description of monoidal structures.

Note that Lemma 4.0.2 presents \( C_{M-1|M-1}^{\text{ren}} \) as a colimit of monoidal categories. Namely,
\[ C_{M-1|M}^{\text{ren}} \otimes C_{M|M-1}^{\text{ren}} C_{M|M}^{\text{ren}} \]
\[ \simeq C_{M-1|M}^{\text{ren}} \otimes C_{M|M}^{\text{ren}} C_{M|M-1}^{\text{ren}} \]
\[ \simeq (\text{colim}_{\zeta_M\zeta_M} j_* \otimes C_{M|M}^{\text{ren}}) \otimes C_{M|M}^{\text{ren}} C_{M|M}^{\text{ren}} \text{colim}_{\zeta_M} C_{M|M}^{\text{ren}} j_* \]
\[ \simeq \text{colim}_{\zeta_M\zeta_M} j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \text{colim}_{\zeta_M} C_{M|M}^{\text{ren}} j_* \text{colim}_{\zeta_M} j_* \text{colim}_{\zeta_M} j_* \]
Denote by \( \tilde{\zeta}_M \) the composition of \( \zeta_M \) and \( \zeta_M \). Since the index set \( \{(x,x), x \in \mathbb{N}\} \) is cofinal in \( \mathbb{N} \times \mathbb{N} \), we have
\[ \text{colim}_{\zeta_M\zeta_M} j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \simeq \text{colim}_{\zeta_M} j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \text{colim}_{\zeta_M} j_* \text{colim}_{\zeta_M} j_* \text{colim}_{\zeta_M} j_* \]

In this section, we will describe the monoidal structure * (equivalently, \( \otimes \)) on \( C_{M-1|M-1}^{\text{ren}} \) as a colimit of the restricted monoidal structure on \( j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \).

#### 4.1.1. Colimit description of *.

We note that \( j_* \) is an idempotent algebra object of \( C_{M|M}^{\text{ren}} \), so the monoidal structure on \( C_{M|M}^{\text{ren}} \) induces a monoidal structure of \( j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \).

\[ \otimes : (j_* \otimes C_{M|M}^{\text{ren}} \otimes j_*) \otimes (j_* \otimes C_{M|M}^{\text{ren}} \otimes j_*) \rightarrow j_* \otimes C_{M|M}^{\text{ren}} \otimes j_* \]

It is obtained from the ind-completion of
\[ \otimes : (j_* \otimes C_{M|M}^{\text{loc,c}} \otimes j_*) \otimes (j_* \otimes C_{M|M}^{\text{loc,c}} \otimes j_*) \rightarrow j_* \otimes C_{M|M}^{\text{loc,c}} \otimes j_* \]
which is the restriction of
\[ \oplus : (j_* \otimes C_{M|M} \otimes j_*) \otimes (j_* \otimes C_{M|M} \otimes j_*) \to j_* \otimes C_{M|M} \otimes j_* . \]

Note that there is an equivalence
\[ (j_* \otimes C_{M|M} \otimes j_*) \simeq D(M \mathcal{M}_M) \mathcal{M}_M(F)/\mathcal{M}_M(O). \]
Indeed, by the definition of \( \oplus \), we have
\[ (j_* \otimes C_{M|M} \otimes j_*) \simeq j_* \otimes D(GM(O)) \mathcal{M}_M+1(F)/GLM(O) \otimes j_* \]
where \( S := \{ (g, v) \in \mathcal{M}_M+1(F) = GLM(F) \times FM | v \in O^M \setminus tO^M, gv \in O^M \setminus tO^M \} \).
Note that we have the following isomorphism
\[ GLM(O)\mathcal{M}_M(F)/GLM(O) \simeq (O^M \setminus tO^M)/GLM(O) \]
\[ = (O^M \setminus tO^M)/GLM(O) \]
\[ = \text{pt}/MIRM(O) \times \text{pt}/MIRM(O) \]
\[ = \text{pt}/MIRM(O)/\mathcal{M}_M(F)/MIRM(O). \]

We obtain a monoidal equivalence,
\[ (D(\mathcal{M}_M(O)/S/\mathcal{M}_M(O)), \oplus) \simeq (D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O), *), \]
where the monoidal structure \( * \) of the right-hand side is given by the restriction of the convolution monoidal of \( C_{M-1|M-1} \) to the full subcategory \( D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O)) \).

The transition functor of \( D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O)) \) corresponding to \( \tilde{\zeta}_M \)
is given by pushforward along
\[ \tilde{\zeta}_M^{-1} : \mathcal{M}_M(F) \to \mathcal{M}_M(F) \]
\[ g \mapsto D_M g D_M^{-1}. \]

We have
\[ \text{colim}_{\mathcal{M}_M, *} D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O)) \simeq D(GLM-1(O)\mathcal{M}_M(F)/GLM-1(O)). \]

**Remark 4.1.1.** We can also give a description of the above monoidal structure in terms of the transposed mirabolic subgroup. Namely, by taking transpose inverse
\[ \mathcal{M}_M(F) \to \mathcal{M}_M(F) \]
\[ g \mapsto g^{t-1}, \]
we can obtain a monoidal equivalence
\[ (D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O), *), \) \]
\[ (D(\mathcal{M}_M(F)\mathcal{M}_M(F)/\mathcal{M}_M(O), *), \)

The transition functor of \( D(\mathcal{M}_M(O)\mathcal{M}_M(F)/\mathcal{M}_M(O)) \) corresponding to \( \tilde{\zeta}_M \)
is given by pushforward along
\[ \tilde{\zeta}_M : \mathcal{M}_M(F) \to \mathcal{M}_M(F) \]
\[ g \mapsto D_M^{-1} g D_M. \]
It is easy to see that $\xi_{M,*}$ is an auto-equivalence of monoidal category. That is to say, the monoidal structure on $D(M\text{Ir}_M^t(O)\setminus D(M\text{Ir}_M^t(F)/\text{Mir}_M^t(O)))$ commutes with the transition functors, i.e., the following diagram commutes
\[
\begin{array}{ccc}
D(M(O)\setminus M(F)/M(O)) & \otimes & D(M(O)\setminus M(F)/M(O)) \\
\xi_{M,*}\otimes \xi_{M,*} & \Rightarrow & \xi_{M,*}
\end{array}
\]

Here $M(O) := \text{Mir}_M^t(O)$ and $M(F) := \text{Mir}_M^t(F)$. In particular, the convolution product of $D(M\text{Ir}_M^t(O)\setminus M\text{Ir}_M^t(F)/\text{Mir}_M^t(O))$ gives a monoidal structure $\ast'$ on $C_{M-1|M-1} \simeq \text{colim}_{\xi_{M,*}} D(M\text{Ir}_M^t(O)\setminus M\text{Ir}_M^t(F)/\text{Mir}_M^t(O))$.

Its restriction to $C_{M-1|M-1}^{\text{loc.c}}$ induces a monoidal structure $\ast'$ on $C_{M-1|M-1}^{\text{loc.c}}$. Taking its ind-completion, we extend $\ast'$ to $C_{M-1|M-1}^{\text{ren}}$.

Similarly, taking colimit of the action of $j_* \otimes C_{M|M}^\text{ren} \otimes j_*$ on $j_* \otimes C_{M|M}^\text{ren}$ gives rise to an action of $(C_{M-1|M-1}^{\text{ren}}, \ast')$ on $C_{M-1|M-1}^{\text{ren}}$.

(4.22) $C_{M-1|M-1}^{\text{ren}} \otimes C_{M-1|M}^{\text{ren}} \rightarrow C_{M-1|M}^{\text{ren}}$

Now we claim

**Proposition 4.1.2.** The monoidal structure $\ast$ of $C_{M-1|M-1}^{\text{ren}}$ is isomorphic to $\ast'$. Furthermore, the action (4.22) of $C_{M-1|M-1}^{\text{ren}}$ on $C_{M-1|M}^{\text{ren}}$ is isomorphic to the action in §3.7.

**Proof.** We only prove the first claim, the second claim follows from the same argument.

Note that $C_{M-1|M-1}^{\text{ren}}$ is the ind-completion of $C_{M-1|M-1}^{\text{loc.c}}$ and both monoidal structures $\ast$ and $\ast'$ are induced from $C_{M-1|M-1}^{\text{loc.c}}$, so we only need to compare the monoidal structures of $C_{M-1|M-1}^{\text{loc.c}}$, which are obtained from the restriction from monoidal structures of $C_{M-1|M-1}^{\text{loc.c}}$.

Just as the proof of Lemma 3.10.2, we can rewrite $C_{M-1|M-1}$ as the colimit of $D((\text{GL}_{M-1}(O) \ltimes t'\text{O}^{M-1}) \setminus \text{Mir}_M^t(F)/(\text{GL}_{M-1}(O) \ltimes t'\text{O}^{M-1}))$, and the transition functors are forgetful functors. Now the claim follows from the fact that the restrictions of both $\ast'$ and $\ast$ to $D((\text{GL}_{M-1}(O) \ltimes t'\text{O}^{M-1}) \setminus \text{Mir}_M^t(F)/(\text{GL}_{M-1}(O) \ltimes t'\text{O}^{M-1}))$ are given by the usual convolution.

**Remark 4.1.3.** In terms of the monoidal structure $\otimes$, by the analysis above, the analog of the above proposition says that the monoidal structure $\otimes$ of $C_{M-1|M-1}$ is the colimit of the monoidal structure $\otimes$ of $j_* \otimes C_{M|M} \otimes j_*$ and the action of $(C_{M-1|M-1}, \otimes)$ on $C_{M-1|M} \simeq \text{colim} j_* \otimes C_{M|M}$ is the colimit of the action of $j_* \otimes C_{M|M} \otimes j_*$ on $j_* \otimes C_{M|M}$.

### 4.2. Compatibility of left actions.

**Proof of Proposition 4.0.1 a).** Note that the equivalence of Theorem 1.2.1 is monoidal, and $j_*$ goes to $\tilde{c}$ under the equivalence. So there is an equivalence of
monoidal categories

\[(4.23) \quad j_* \otimes C_{M|M}^\text{ren} \otimes j_* \simeq \tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}) \star \tilde{c}.\]

By the same reason, we have an equivalence

\[(4.24) \quad j_* \otimes C_{M|M}^\text{ren} \simeq \tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}),\]

which is compatible with the action of \((4.23)\).

Note that \(\tilde{c}\) is compatible with the left action of \(C\) and \((4.27)\) over \((1.1)\). By Proposition 4.0.1, we obtain an equivalence

\[(4.25) \quad D^{GL_{M-1} \times GL_{M-1}}(\mathcal{B}^{2,0}_{M-1|M-1}) \simeq \colim j_* \otimes C_{M|M}^\text{ren} \otimes j_* \simeq \colim \tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}) \star \tilde{c},\]

and

\[(4.26) \quad D^{GL_{M-1} \times GL_{M}}(\mathcal{B}^{2,0}_{M-1|M}) \simeq \colim \tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}).\]

So the monoidal structure of \(D^{GL_{M-1} \times GL_{M-1}}(\mathcal{B}^{2,0}_{M-1|M-1})\) also comes from the colimit of \(\tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}) \star \tilde{c}\), and the action of \(D^{GL_{M-1} \times GL_{M-1}}(\mathcal{B}^{2,0}_{M-1|M-1})\) on \(D^{GL_{M-1} \times GL_{M}}(\mathcal{B}^{2,0}_{M-1|M})\) comes from the colimit of action of \(\tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}) \star \tilde{c}\) on \(\tilde{c}^A D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}).\)

Now the claim follows from the fact that the equivalence \((4.23)\) is monoidal. \(\Box\)

4.3. Proof of the main theorem. Finally, we can give a proof of Theorem 1.4.1.

Proof of Theorem 1.4.1. We will prove a stronger theorem: for \(N \geq M\), we have a left action of \(C_{M|N}^\text{ren}\) and a right action of \(C_{N|N}^\text{ren}\) on \(C_{M|N}^\text{ren}\), such that there is an equivalence:

\[(4.27) \quad C_{M|N}^\text{ren} \simeq D^{GL_M \times GL_N}(\mathcal{B}^{2,0}_{M|N}),\]

which is compatible with the left action of

\[C_{M|M}^\text{ren} \simeq D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}),\]

and the right action of

\[C_{N|N}^\text{ren} \simeq D^{GL_N \times GL_N}(\mathcal{B}^{2,0}_{N|N}).\]

Fix \(N \geq 1\), let us prove the theorem using the descending induction on \(M\).

If \(M = N\) or \(M = N - 1\), the theorem has already been proved in [5] and Proposition 4.0.1.

We assume that the theorem holds for \(M\). We take the tensor product of \((4.1)\) and \((4.27)\) over \((1.1)\). By Proposition 4.0.1, we obtain an equivalence

\[(4.28) \quad C_{M-1|M}^\text{ren} \otimes C_{M|M}^\text{ren} \simeq D^{GL_{M-1} \times GL_{M}}(\mathcal{B}^{2,0}_{M-1|M}) \otimes D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M}) D^{GL_M \times GL_N}(\mathcal{B}^{2,0}_{M|N})\]

which is compatible with the left action of

\[C_{M|M}^\text{ren} \simeq D^{GL_M \times GL_M}(\mathcal{B}^{2,0}_{M|M})\]
and the right action of
\[ C_{N[N]}^{ren} \simeq D^{GL_N \times GL_N}(\mathfrak{B}_{N[N]}^{2,0}). \]

By Proposition 2.6.1 and Theorem 3.9.1, the left-hand side of \((4.28)\) is equivalent to \(C_{M-1[N]}^{ren}\), and the right-hand side is equivalent to \(D^{GL_{M-1} \times GL_N}(\mathfrak{B}_{M-1[N]}^{2,0})\). Hence, we obtain the claim for \(M-1\).

Taking subcategories of compact objects, we obtain Theorem 1.4.1. \(\square\)

4.4. Restricted equivalence. In this section, we will prove the following theorem.

**Theorem 4.4.1.** For \(M \leq N\), there is an equivalence
\[(4.29)\]
\[ C_{M[N]} \simeq D^{GL_M \times GL_N}(\mathfrak{B}_{M[N]}^{2,0})_{\text{Nilp}}. \]

Before we prove the above theorem, let first check the above theorem in the case \(M = N - 1\) or \(M = N\).

**Proposition 4.4.2.** There are equivalences of categories
\[(4.30)\]
\[ C_{M|M} \simeq D^{GL_M \times GL_M}(\mathfrak{B}_{M|M}^{2,0})_{\text{Nilp}}, \]
and
\[(4.31)\]
\[ C_{M-1|M} \simeq D^{GL_{M-1} \times GL_M}(\mathfrak{B}_{M-1|M}^{2,0})_{\text{Nilp}}. \]

**Proof.** We only prove the statement
\[(4.32)\]
\[ C_{M|M}^{c} \simeq D^{GL_M \times GL_M}_{\text{perf}}(\mathfrak{B}_{M|M}^{2,0})_{\text{Nilp}}, \]
then \((4.30)\) follows from taking ind-completion and \((4.31)\) follows from a similar argument.

Let \(D^{GL_M(O)-\text{mon}}(Gr_M \times F^M)\) be the full subcategory of \(D(Gr_M \times F^M)\) generated by locally compact \(GL_M(O)\)-equivariant \(D\)-modules on \(Gr_M \times F^M\). Since objects in \(D^{GL_M(O)-\text{mon}}(Gr_M \times F^M)\) are ind-holonomic, the left adjoint functor
\[ Av^{GL_M(O)}: D^{GL_M(O)-\text{mon}}(Gr_M \times F^M) \rightarrow C_{M|M} \]
of the forgetful functor is well-defined. In addition, since the forgetful functor
\[ C_{M|M} \rightarrow D^{GL_M(O)-\text{mon}}(Gr_M \times F^M) \]
is conservative, the essential image of compact objects of \((D^{GL_M(O)-\text{mon}}(Gr_M \times F^M))^c\) under \(Av^{GL_M(O)}\) generates \(C_{M|M}^c\) ([8, Lemma 5.4.3]).

Let \(IC_{0,0} \in C_{M|M}^{loc,c}\) be the IC-extension of the perverse constant sheaf on \(1 \times O^M\).
It is known from the construction in [5] that \(F_{M,M}(IC_{0,0}) = \mathfrak{B}_{M,M}^{2,0}\). Here \(F_{M,M}: C_{M|M}^{loc,c} \rightarrow D^{GL_M \times GL_M}_{\text{perf}}(\mathfrak{B}_{M|M}^{2,0})\) denotes the equivalence functor in Theorem 1.2.1. Note that \(IC_{0,0}\) is the pullback of \(\mathbb{C}\) along \(1 \times O^M \rightarrow \text{pt}\). By [1, Lemma 12.6.5] and the fact that \(!\)-averaging functor commutes with pullback, there is
\[ F_{M,M}(Av^{GL_M(O)}(IC_{0,0})) \simeq F_{M,M}(IC_{0,0}) \otimes_{\text{Sym}(\mathfrak{g}_M[-2])^{GL_M}} \mathbb{C}. \]

Since
\[ \mathfrak{B}_{M|M}^{2,0} \otimes_{\text{Sym}(\mathfrak{g}_M[-2])^{GL_M}} \mathbb{C} \simeq \mathcal{O}_{\text{Nilp}_M|M}, \]

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there is
\[ F_{M,M}(\text{Av}^{\text{GL}_M(O)}_!(\text{IC}_{0,0})) \simeq \mathcal{O}_{\text{Nilp}_M|M} \]
under the equivalence of Theorem 1.2.1.

The equivalence in Theorem 1.2.1 commutes with the convolution with
\( \text{Perv}_{\text{GL}_M(O)}(\text{Gr}_M) \simeq \text{Rep}^{\text{fin}}(\text{GL}_M) \), so \( \text{Av}^{\text{GL}_M(O)}_!(\text{IC}_{\lambda} \star \text{IC}_{0,0} \star \text{IC}_\mu) \) corresponds to
\( V_{\text{GL}_M}^\lambda \otimes \mathcal{O}_{\text{Nilp}_M|M} \otimes V_{\text{GL}_M}^\mu \) under the equivalence \( F_{M,M} \). Now the claim follows from the
fact that the objects \( V_{\text{GL}_M}^\lambda \otimes \mathcal{O}_{\text{Nilp}_M|M} \otimes V_{\text{GL}_M}^\mu \) generate \( D_{\text{perf}}^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{M,N}\text{Nilp}) \).

The equivalence (4.30) is an ind-completion of a monoidal equivalence functor, so it is monoidal. In addition, the equivalence (4.31) is compatible with the left action and right action of (4.30).

**Corollary 4.4.3.** a) The equivalence (4.31) is compatible with the left action of
\[ C_{M-1|M-1} \simeq D^{\text{GL}_M \times \text{GL}_N}_{M-1|M-1}(\mathfrak{B}^{2,0}_{M-1|M-1}\text{Nilp}). \]
b) The equivalence is compatible with the right action of
\[ C_{M|M} \simeq D^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{M|M}\text{Nilp}). \]

Now let us repeat the argument of the proof of Theorem 1.4.1 to prove Theorem 4.4.1.

**Proof.** (of Theorem 4.4.1.)

By Proposition 4.4.2, Theorem 4.4.1 holds in the case \( M = N \) or \( M = N - 1 \).

Fix \( N \geq 1 \), assume that we have already proved Theorem 4.4.1 for \( M \). We take
the tensor product of (4.31) and
\[ C_{M|N} \simeq D^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{M|N}\text{Nilp}) \]
over (4.30). By Proposition 4.4.2, Corollary 4.4.3, and our assumption, we obtain an equivalence
(4.33)
\[ C_{M-1|M} \otimes C_{M|N} \simeq D^{\text{GL}_M \times \text{GL}_N}_{M-1|M}(\mathfrak{B}^{2,0}_{M-1|M}\text{Nilp}) \otimes D^{\text{GL}_M \times \text{GL}_N}_{M|M}(\mathfrak{B}^{2,0}_{M|N}\text{Nilp}) D^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{M|M}\text{Nilp}) \]
which is compatible with the left action of
\[ C_{M|M} \simeq D^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{M|M}\text{Nilp}) \]
and the right action of
\[ C_{N|N} \simeq D^{\text{GL}_M \times \text{GL}_N}(\mathfrak{B}^{2,0}_{N|N}\text{Nilp}). \]

By Proposition 2.6.1 and Theorem 3.9.1, the left-hand side of (4.33) is equivalent to \( C_{M-1|N} \), and the right-hand side is equivalent to \( D^{\text{GL}_M \times \text{GL}_N}_{M-1,N}(\mathfrak{B}^{2,0}_{M-1,N}\text{Nilp}) \). Hence, we obtain the claim for \( M - 1 \).

**Remark 4.4.4.** Using the equivalence interchanging left and right actions, we can also prove Theorem 1.4.1 and Theorem 4.4.1 for \( N \leq M \).
5. A SYMMETRIC DEFINITION OF $C_{M,N}$

In the definition of $D^{\text{GL}_M \times \text{GL}_N}_{(\mathfrak{B}_{M,N}^{2,0})}$ and $D^{\text{GL}_M \times \text{GL}_N}_{(\mathfrak{B}_{M,N}^{2,0})_{\text{Par}}}$, we do not need to require $N \leq M$ or $M \leq N$. But in the definition of $C_{M,N}$, we need to require $M \leq N$ or $N \leq M$, or the definition does not make sense. Although there is a canonical equivalence between $C_{N|M} \simeq C_{M,N}$, we want to introduce a more 'symmetric' definition of $C_{M,N}$ for any $M, N$.

Furthermore, in (1.4), it seems unreasonable to require that the embedding of $\text{GL}_M(\mathcal{O})$ into $\text{GL}_N(\mathbf{F})$ concentrates in the left top part. Indeed, in this section we will also study the corresponding equivariant category if we move $\text{GL}_M(\mathcal{O})$ to other columns and rows, and we will see that the resulting category does not depend on where we put $\text{GL}_M(\mathcal{O})$. It is expected that the choice of columns and rows corresponds to the choice of a Borel subgroup of the supergroup.

5.1. Definition. Assume $L > N$, $M$ and $r + s = L - M - 1$, we denote by

$$U^{r,s}_{M,L} = \begin{pmatrix} U_r & * & * \\ 1_{M+1} & * & U_s \end{pmatrix},$$

the unipotent radical of the parabolic subgroup $P_{r,s}(\mathbf{F})$ corresponding to the partition $(1, 1, ..., 1, M + 1, 1, ... , 1)$. We denote by $\chi_{M,L}^{r,s(k)}$ the character

$$(u_{ij}) \in U^{r,s}_{M,L} \longrightarrow \text{Res}_{t=0}(\sum_{i=1}^{r-1} u_{i,i+1} + u_{r,k} + u_{k,L-s+1} + \sum_{i=L-s+1}^{L-1} u_{i,i+1})$$

for any choice of $k \in \{r + 1, ..., L - s\}$. If $r = 0$ or $s = 0$, one of $u_{r,k}$ and $u_{k,L-s-1}$ should be omitted.

We embed $\text{GL}_M(\mathcal{O})$ into $P_{r,s}(\mathbf{F})$ along the columns and rows $r + 1, r + 2, ..., k - 1, k + 1, ..., L - s$. Since $\text{GL}_M(\mathcal{O})$ belongs to the normalizer group of $U^{r,s}_{M,L}$, we can form the semidirect product $\text{GL}_M(\mathcal{O}) \rtimes U^{r,s}_{M,L}$. Furthermore, the conjugation action of $\text{GL}_M(\mathcal{O})$ on $U^{r,s}_{M,L}$ preserves $\chi_{M,L}^{r,s(k)}$. In particular, $(\text{GL}_M(\mathcal{O}) \rtimes U^{r,s}_{M,L}, \chi_{M,L}^{r,s(k)})$-invariants are well-defined.

Remark 5.1.1. Assume $k_1, k_2 \in \{r + 1, r + 2, ..., L - s\}$. Let $s_{k_1,k_2} \in \text{GL}_L(\mathbf{F})$ be the permutation matrix which permutes $k_1$ and $k_2$. Left multiplication with $s_{k_1,k_2}$ gives an equivalence:

$$C^{\text{GL}_M(\mathcal{O}) \rtimes U^{r,s}_{M,L}, \chi_{M,L}^{r,s(k_1)}} \simeq C^{\text{GL}_M(\mathcal{O}) \rtimes U^{r,s}_{M,L}, \chi_{M,L}^{r,s(k_2)}}$$

for any category $C$ which admits an action of $\text{GL}_L(\mathbf{F})$. Hence, we may omit $k$ in the definition of $\chi_{M,L}^{r,s(k)}$.

Definition 5.1.2. We define

$$C_{M,N,r,s,r',s',L} := D(\text{GL}_M(\mathcal{O}) \rtimes U^{r,s}_{M,L}, \chi_{M,L}^{r,s} \backslash \text{Mir}_L^t(\mathbf{F})/\text{GL}_N(\mathcal{O}) \rtimes U^{r',s'}_{N,L}, \chi_{N,L}^{r',s'})$$

The main goal of this section is to prove the following theorem.

Theorem 5.1.3. The category $C_{M,N,r,s,r',s',L}$ only depends on $M, N$. That is to say, for any

$$r_1 + s_1 = L_1 - M - 1, \quad r_2 + s_2 = L_2 - M - 1,$$

$$r'_1 + s'_1 = L_1 - N - 1, \quad r'_2 + s'_2 = L_2 - N - 1,$$
we have

(5.2) \[ C_{M,N,r_1,s_1,r_1',s_1',L_1} \simeq C_{M,N,r_2,s_2,r_2',s_2',L_2}. \]

In particular, \( C_{M,N,r,s,r',s',L} \simeq C_{M,N} \).

5.2. Examples.

(1) If \( M = N = L - 1 \), \( r = r' = 0 \), \( s = s' = 0 \), then \( GL_M(O) \ltimes U_{M,L}^{r,s} = GL_N(O) \ltimes U_{M,L}^{r',s'} = GL_M(O) \) and the characters are trivial. Hence, \( C_{M,N,r,s,r',s',L} \simeq D(GL_M(O) \backslash \text{Mir}_M \ltimes (F)/GL_M(O)) \simeq C_{M,M} \).

(2) If \( M = L - 2 \), \( N = L - 1 \), \( r = r' = 0 \), \( s = 1 \), and \( s' = 0 \). Then, \( GL_M(O) \ltimes U_{M,L}^{r,s} = GL_M(O) \ltimes F^L \), and \( GL_N(O) \ltimes U_{M,L}^{r',s'} = GL_N(O) \). Here, \( \chi_{r,s}^{L,M} \) equals taking residue of coefficients of \( e_{M+1} \), while \( \chi_{r,s}^{L,M} \) is trivial. We have

\[
C_{M,N,r,s,r',s',L} \simeq D(GL_M(O) \ltimes F^{M+1} \times \chi_{r,s}^{L,M} \ltimes \text{Mir}_L(F)/GL_N(O))
\]

\[
\simeq D/GL_{M+1}(\text{Gr}_{M+1} \ltimes F^{M+1})
\]

\[
\simeq D/GL_{M}(\text{Gr}_{M+1}).
\]

Here, the last equivalence is given by Fourier transform.

(3) If \( M = 0 \) and \( L = N + 1 \), in this case, \( C_{M,N,r,s,r',s',L} \simeq \text{Whit}(\text{Gr}_N) \). Here, \( \text{Whit}(\text{Gr}_N) \) denotes the Whittaker model of \( D(\text{Gr}_N) \). The rest of this section is devoted to the proof of (5.2).

5.3. Independence of \( s,r \). In this section, we will prove if \( L_1 = L_2 \), then (5.2) holds. By induction, we only need to show,

(5.3) \[ C_{M,N,r,s,r',s',L} \simeq C_{M,N,r+1,s-1,r',s',L}. \]

It is easy to see that the above statement follows from the following lemma,

Lemma 5.3.1. For any category \( C \) admitting an action of \( GL_L(F) \), we have

(5.4) \[ C_{U_{M,L}^{r,s}}^{U_{M,L}^{r+1,s-1}} = C_{U_{M,L}^{r+1,s-1}}^{U_{M,L}^{r,s}} \]

Proof. Consider the following groups,

\[
\begin{array}{c}
U_{M+1,L}^{r,s} \\
U_{M+1,L}^{r+1,s-1}
\end{array}
\]

\[
\begin{array}{c}
U_{M,L}^{r,s-1} \\
U_{M,L}^{r+1,s-1}
\end{array}
\]

By definition and Lemma 3.3.3, we have

\[
C_{U_{M,L}^{r,s}}^{U_{M,L}^{r,s-1}} = \left( C_{U_{M+1,L}^{r,s-1}}^{U_{M+1,L}^{r,s}} \right)^{U_{M,L}^{r,s-1}/U_{M+1,L}^{r,s-1}}
\]

\[
C_{U_{M,L}^{r+1,s-1}}^{U_{M,L}^{r,s}} = \left( C_{U_{M+1,L}^{r+1,s-1}}^{U_{M+1,L}^{r,s-1}} \right)^{U_{M,L}^{r+1,s-1}/U_{M+1,L}^{r+1,s-1}}
\]
We note that \( \chi_{M,L}^{r,s(r+1)} |_{U_{M+1,L}^{r,s-1}} = \chi_{M,L}^{r+1,s-1,(L-s+1)} |_{U_{M+1,L}^{r,s-1}} \), so we should prove that for any category \( \mathcal{C}' \) admitting an action of \( U_{M-1,L}^{r,s+1} / U_{M+1,L}^{r,s-1}, \) we have

\[
(\mathcal{C}')_{U_{M,L}^{r,s} / U_{M+1,L}^{r,s-1}, \chi_{M,L}^{r,s(r+1)}} \simeq (\mathcal{C}')_{U_{M,L}^{r,s} / U_{M+1,L}^{r,s-1}, \chi_{M,L}^{r+1,s-1,(L-s+1)}}.
\]

We can identify \( U_{M,L}^{r,s} / U_{M+1,L}^{r,s-1} \) with a vector space \( V \oplus F \) for a vector space \( V \), identify \( U_{M,L}^{r,s} / U_{M+1,L}^{r,s-1} \) as \( V^* \oplus F \), and identify \( U_{M-1,L}^{r,s+1} / U_{M+1,L}^{r,s-1} \) with the Heisenberg group \( \text{Heis} := V \oplus V^* \oplus F \). The group structure of \( \text{Heis} \) is given by

\[
\text{Heis} \times \text{Heis} \rightarrow \text{Heis}
\]

\[
(v, v^*, c), (v', v'^*, c') \mapsto (v \oplus v', v^* \oplus v'^*, c + c' + \langle v, v'^* \rangle).
\]

Taking restriction along \( V^* \rightarrow \text{Heis} \) (resp. \( V \rightarrow \text{Heis} \)) defines an equivalence

\[
D(\text{Heis}/(V \oplus F), \chi) \simeq D(V^*).
\]

The action of \( V \) on \( D(V^*) \) is given by shifts and the action of \( v^* \in V^* \) on \( D(V^*) \) is given by tensoring with the local system \( e^{v^*} \) on \( V \). In the \( D(V^*) \) realization, it is the other way around, and the Fourier transform intertwines these two actions.

Hence, we obtain an equivalence of left \( \text{Heis} \)-module categories by re-averaging:

\[
D^{V \oplus F, \chi}(\text{Heis}) \simeq D(\text{Heis}/(V \oplus F), \chi) \simeq D(\text{Heis}/(V^* \oplus F), \chi) \simeq D^{V^* \oplus F, \chi}(\text{Heis}).
\]

Now, the equivalence (5.5) follows from the facts:

\[
\text{Funct}_{D(\text{Heis})}(D^{V \oplus F, \chi}(\text{Heis}), \mathcal{C}') \simeq (\mathcal{C}')^{V \oplus F, \chi}
\]

and

\[
\text{Funct}_{D(\text{Heis})}(D^{V^* \oplus F, \chi}(\text{Heis}), \mathcal{C}') \simeq (\mathcal{C}')^{V^* \oplus F, \chi}.
\]

5.4. Independence of \( L \). In this section, we will show that the definition of \( C_{M,N,L,r,s,r',s'} \) is independent of the choice of \( L \).

\textbf{Remark 5.4.1.} Since we have already seen that \( C_{M,N,L,r,s,r',s'} \) is independent of the choice of \( r, s, r', s' \), we could just denote it by \( C_{M,N,L} \). We denote by \( \chi_{M,L} \) the character \( \chi_{M,L}^{r,s} \) for any choice of \( r, s \).

We are going to prove the following lemma first,

\textbf{Lemma 5.4.2.} We have

\[
D(\text{Mir}_L^L(F)) \simeq D(F^L, \chi_{L-1,L+1} \setminus \text{Mir}_{L+1}^L(F) / F^L, \chi_{L-1,L+1})
\]

\textbf{Proof.} By the Fourier transform,

\[
D(F^L, \chi_{L-1,L+1} \setminus \text{Mir}_{L+1}^L(F) / F^L, \chi_{L-1,L+1})
\]

\[
\simeq \text{Vect}_{\epsilon.L} \otimes_{D(F^L)} D(\text{Mir}_{L+1}^L(F)) \otimes_{D(F^L)} \text{Vect}_{\epsilon.L}.
\]
Here, both functors $D(F^L) \to \text{Vect}_{e_L} \cong \text{Vect}$ are given by taking restriction at $e_L \in O^L \setminus tO^L$. The left (resp. right) action of $D(F^L)$ on $D(\text{Mir}_{L+1}^L(F))$ is given by the pullback along

$$\text{Mir}_{L+1}^L(F) \cong F^L \rtimes \text{GL}_L(F) \to F^L$$

Hence,

$$\text{Vect}_{e_L} \otimes_{D(F^L)} D(\text{Mir}_{L+1}^L(F)) \otimes_{D(F^L)} \text{Vect}_{e_L} \cong D(H''),$$

where $H'' := \{(g, v) \in \text{GL}_L(F) \rtimes F^L \mid v = e_L, gv = e_L\} \cong \text{Mir}_L(F)$. We proved the lemma. \hfill \square

Now let us take $(\text{GL}_M(O) \ltimes U_{M,L}(F), \chi_{M,L})$-invariants on the left and $(\text{GL}_N(O) \ltimes U_{N,L}(F), \chi_{N,L})$-invariants on the right (5.12). We have

$$C_{M,N,L} \cong D(\text{GL}_M(O) \ltimes U_{M,L}(F)) / \text{Mir}_{L+1}^L(F) / \text{GL}_M(O) \ltimes U_{N,L}(F), \chi_{N,L})$$

In particular, the category $C_{M,N,L}$ does not depend on $L$.

**Remark 5.4.3.** The Gaiotto meta conjecture claimed that the Langlands dual of the category $\hat{\text{gl}}(M|N) \mod$ is the Whittaker bi-invariants of the category of D-modules on $\text{Mir}_L^L(F)$ for any $L > M, N$. The lemma above asserts that this Whittaker bi-invariants category is independent of $L$. In fact, taking $(U_{M,L}(F), \chi_{M,L})$-invariants on the left and $(U_{N,L}(F), \chi_{N,L})$-invariants on the right of (5.12), we obtain that

$$D(U_{M,L}(F), \chi_{M,L}) \cong D(U_{N,L+1}(F), \chi_{N,L+1}).$$

**Remark 5.4.4.** From the construction of the equivalence $C_{M,N,L} \cong C_{M,N,L+1}$, the equivalence is compatible with the left action of $C_{N|M}$ and the right action of $C_{M|M}$.

As a corollary,

**Corollary 5.4.5.** If $M' \leq M$ and $N' \leq N$, we have

$$C_{M'|M} \otimes_{C_{M|M}} C_{M,N,L} \otimes_{C_{M|M}} C_{N'|N} \cong C_{M',N',L},$$

which is compatible with the left action of $C_{M'|M'}$ and the right action of $C_{N'|N'}$.

**Corollary 5.4.6.** The left action of $C_{M|M}$ on $C_{M|N}$ is equivalent to the left action of $C_{M,M,L} \cong C_{M|M}$ on $C_{M,N,L} \cong C_{M|N}$.

Similarly for the right action.
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References

[1] D. Arinkin and D. Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture, Selecta Mathematica 21(1). doi:10.1007/s00029-014-0167-5. arXiv:1201.6343 [math.AG]
[2] R. Bezrukavnikov, On two geometric realizations of an affine Hecke algebra, Publ. Math. Inst. Hautes Etudes Sci. 123 (2016), 1-67.
[3] D. Beraldo, Loop Group Actions on Categories and Whittaker Invariants, Advances in Mathematics 322 (2017) 565-636. arXiv:1310.5127[math.RT].
[4] R. Bezrukavnikov and M. Finkelberg, Equivariant Satake category and Kostant Whittaker reduction, Moscow Math. Journal 8 (2008), no. 1, 39-72.
[5] A. Braverman, M. Finkelberg, V. Ginzburg, and R. Travkin, Mirabolic Satake equivalence and supergroups, Compositio Mathematica, 157(8), 1724-1765. doi:10.1112/S0010437X21007387
[6] A. Braverman, M. Finkelberg, and R. Travkin, Orthosymplectic Satake equivalence, arXiv:1912.01930 [math.RT]
[7] M. Finkelberg, V. Ginzburg, and R. Travkin, Mirabolic affine Grassmannian and character sheaves, Selecta Mathematica volume 14, pages 607-628 (2009)
[8] D. Gaitsgory and N. Rozenblyum, A study in derived algebraic geometry Volume I: Correspondences and duality, Vol. 221. American Mathematical Society, 2019. https://people.math.harvard.edu/~gaitsgde/GL/Vol1.pdf
[9] J. Lurie, Higher algebra., https://people.math.harvard.edu/~lurie/papers/HAb.pdf (2017).

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