Automated Mechanism Design via Neural Networks

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Abstract

Using AI approaches to automatically design mechanisms has been a central research mission at the interface of AI and economics [Conitzer and Sandholm, 2002]. Previous approaches that attempt to design revenue optimal auctions for the multi-dimensional settings fall short in at least one of the three aspects: 1) representation — search in a space that probably does not even contain the optimal mechanism; 2) exactness — finding a mechanism that is either not truthful or far from optimal; 3) domain dependence — need a different design for different environment settings.

To resolve the three difficulties, in this paper, we put forward \textit{MENU} — a unified neural network based framework that automatically learns to design revenue optimal mechanisms. Our framework consists of a mechanism network that takes an input distribution for training and outputs a mechanism, as well as a buyer network that takes a mechanism as input and output an action. Such a separation in design mitigates the difficulty to impose incentive compatibility constraints on the mechanism, by making it a rational choice of the buyer. As a result, our framework easily overcomes the previously mentioned difficulty in incorporating IC constraints and always returns exactly incentive compatible mechanisms.

We then apply our framework to a number of multi-item revenue optimal design settings, for a few of which the theoretically optimal mechanisms are unknown. We then go on to theoretically prove that the mechanisms found by our framework are indeed optimal.

To the best of our knowledge, we are the first to apply neural networks to discover optimal auction mechanisms with provable optimality.

1 Introduction

Designing revenue optimal mechanisms in various settings has been a central research agenda in economics, ever since the seminal works of Vickrey [27] and Myerson [18] in single item auctions. Lately, designing optimal mechanisms for selling multiple items has also been established as an important research agenda at the interface of economics and computer sciences [6, 15, 14, 3, 4, 16, 29, 22, 30, 23, 24].

Due to diversity in the researchers’ backgrounds, there are a number of quite different angles to study this problem. The standard economics theme aims to understand the exact optimal mechanisms in various settings. To name a few, Armstrong [2] obtains the revenue optimal mechanisms of selling two items to one buyer, whose valuations of the two items are perfect positively correlated (a ray through the origin). Manelli and Vincent [17] obtains partial characterization of optimal mechanisms, in the form of extremely points in the mechanism spaces. Pavlov [20] derives optimal mechanisms for two items when the buyer has symmetric uniform distributions. Daskalakis et al. [8] characterizes sufficient and necessary conditions for a mechanism to optimal and derive optimal mechanisms for two items for several valuation distributions. Tang and Wang [24] obtain the revenue optimal mechanisms of selling two items, of which the valuations are perfect negatively correlated. Yao [30] obtains the revenue optimal mechanisms of selling two additive items to multiple buyers, whose valuation towards the items are binary and independent.

Another category of research rooted in the AGT community aims to resolve the difficulties of characterizing optimal mechanisms via the lens of algorithm design. Cai et al. [3] and Alaei et al. [1] gives algorithmic characterizations of the optimal BIC mechanisms on discrete distributions using linear programs. Hartline and Roughgarden [15], Yao [29], Hart and Nisan [14] find approximately optimal mechanisms in various settings. Carroll [5] shows that for a certain multi-dimensional screening problem, the worst-case optimal mechanism is simply to sell each item separately.

The third category, at the interface of AI and economics, aims to search for the optimal mechanisms via various AI approaches. Conitzer and Sandholm [6] model the problem of revenue and welfare maximization as an instance
of constraints satisfaction problem (CSP) through which the optimal mechanism may be found using various search techniques, despite its general computation complexity. Sandholm and Likhodedov [22] model a restricted revenue maximization problem (within affine maximizing auctions) as a parameter search problem in a multi-dimensional parameter space, they find several sets of parameters that yields good empirical revenue. Düsing et al. [9] aims to learn optimal mechanisms by repeatedly sampling from the distribution. They obtain mechanisms that are approximately optimal and have low incentive compatibility regret on average.

One advantage of these computational approaches is that most of them are constructive so that one can systematically and computationally generate optimal mechanisms. However, a difficulty for most existing works in computer science (the second and third categories) is that mechanisms obtained this way are either not optimal in the exact sense, or not truthful in the exact sense. As a result, a typical economist may have a hard time to appreciate this type of results. A more desirable approach would be constructive on one hand and be able return exact incentive compatible and (hopefully) exact optimal mechanisms on the other hand.

1.1 Our methodology

In this paper, motivated by the above observation, we aim to put forward a computational approach that can design or assist one to design exact IC and optimal mechanisms. We train a neural network that represents the optimal mechanism using the valuation distributions. However, unlike the approach in Düsing et al. [9], we introduce another neural network that represents buyer’s behavior. In particular, this network takes a mechanism as input, and output an action. Our network structure resembles that of the generative adversarial nets (GAN) [11] but is essentially different because we do not need to train the buyer’s network. This independent buyer network allows us to easily model the exact IC constraints (which has been a major difficulty in previous works) in our network and any behavior model of this form. In contrast, Düsing et al. [9] first propose to hardwire the IC constraints into the mechanism network, which requires a lot of domain knowledge and the structure of the networks has to be domain specific. As a result, their approach can only reproduce mechanisms in the domains where the form of the optimal mechanism is known. To circumvent this difficulty, they further propose to add IC as a soft constraint so that the training objective is to minimize a linear combination of revenue loss and the degree of IC violations. However, this would produce mechanisms that are not IC.

Another innovation of our framework, MenuNet, is that we represent a mechanism as a menu (a list of (valuation, outcome) tuples) in the single buyer case. According to the taxation principle [28], by simply letting the buyer do the selection, we get an IC mechanism. An additional merit of using a menu to represent a mechanism is that it enables explicit restrictions of the menu size of the mechanism, which measures the degree of complexity of a mechanism [13].

Under the guidance of the solutions from our neural networks, one may be able to guess the structure of the optimal solution and the prove its optimality. Although our neural network framework cannot directly help with the optimality proof, its high accuracy (see Table 1) can greatly reduce the tremendous efforts that one often needs to guess the optimal solution.

1.2 Our results

We then apply our learning-aided mechanism design framework to the domain where a seller sells two items to one buyer. In particular, we investigate the following problems.

- What is the revenue optimal mechanisms when the menu size is restricted to a constant? To the best of our knowledge, the optimal mechanism of this kind is previously unknown for our setting.
- What is optimal mechanism for the case where the valuation domain is a triangle? The previously studied cases on this domain all focuses on rectangle shaped valuation domain (except for Haghpanah and Hartline [12]).
- What is the revenue optimal deterministic mechanism?
- What is the revenue optimal mechanism when the buyer has combinatorial value?

Some of the experimental results we obtained is shown in Table 1 with comparison to the exact optimal mechanisms (some of them are previously known results, while the others are our new findings).
Inspired by these empirical findings, using the techniques by Daskalakis et al. [8] and Pavlov [21], we then prove the exact optimal mechanisms for the first two problems. To the best of our knowledge, this is the first time to find the exact optimal mechanisms in these domains, so they are of independent interests to the economics society as well.

**Theorem (Restricted Menu Size).** The optimal mechanism for an additive buyer, \( v \sim U[0, 1]^2 \), with menu size no more than 3 is to either sell the first item at price \( 2/3 \) or sell the bundle of two items at price \( 5/6 \), yielding revenue 59/108.

In particular, the optimal mechanism must be asymmetric even if the distribution is symmetric!

**Theorem (Uniform Distribution on a Triangle\(^2\)).** The optimal mechanism for an additive buyer with value uniformly distributed in \( \{(v_1, v_2) | v_1/c + v_2 \leq 1, v_1, v_2 \geq 0\} \) (hence a correlated distribution) is as follows:

- if \( c \in [1, 4/3) \), two menu items: \([0, 0), 0\] and \([(1, 1), \sqrt{c/3}]\);
- if \( c > 4/3 \), three menu items: \([(0, 0), 0\] and \([(1, 1), 2c/3 + \sqrt{c(c - 1)/3}]\), and \([(1/c, 1), 2/3]\).

## 2 Preliminaries

In this paper, we consider the automated mechanism design problem for the single-buyer multi-dimensional setting. In this section, we introduce the basic notions for optimal multidimensional mechanism design problem.

**Environment** The seller has \( m \) heterogeneous items for sale, and the buyer has different private values for receiving different bundles of the items. An allocation of the items is specified by a vector \( x \in X \subseteq [0, 1]^m \), where \( x_i \) is the probability of allocating the \( i \)-th item to the buyer. An allocation \( x \) is called a deterministic allocation, if \( x \in \{0, 1\}^m \); otherwise a randomized allocation or a lottery allocation.

A possible outcome of the mechanism consists of a valid allocation vector \( x \in X \) and a monetary transfer amount \( p \in \mathbb{R}_+ \), called payment, from the buyer to the seller.

With the standard \textit{quasi-linear utility} assumption, the valuation function \( v : X \mapsto \mathbb{R}_+ \) describes the private preference of the buyer, i.e., an outcome \((x, p)\) is (weakly) preferred than another outcome \((x', p')\), if and only if:

\[
u(x, p; v) := v(x) - p \geq v(x') - p' = u(x', p'; v).
\]

In other words, the outcome with the highest utility is most preferred by the buyer.

**Mechanism** A naive mechanism (without applying the revelation principle) is defined by a set of actions and a mapping from the set of actions to the set of outcomes. Note that according to the taxation principle [28], simply letting the buyer do the selection, we get an incentive compatible mechanism. Formally,

\[\text{Table 1: Comparison with optimal mechanisms, where Optimality = Rev/OptRev.}\]

| Distributions          | Computed Mech Rev\(^1\) | Optimal Mech Rev | Optimality       |
|------------------------|--------------------------|------------------|------------------|
| \( U[0, 1]^2 \)        | 0.5491989                | \((12 + 2\sqrt{2})/27\) | \( \geq 99.996\% \) |
| \( U[0, 1] \times [0, 1, 5] \) | 0.6838542 | \((15 + 2\sqrt{3})/27\) | \( \geq 99.997\% \) |
| \( U[0, 1] \times [0, 1, 9] \) | 0.7888323 | \((17.4 + 2\sqrt{3.8})/27\) | \( \geq 99.988\% \) |
| \( U[0, 1] \times [0, 2] \) | 0.8148131 | \(22/27\) | \( \geq 99.997\% \) |
| \( U[0, 1] \times [0, 2, 5] \) | 0.9435182 | 1019/1080 | \( \geq 99.9996\% \) |
| \( U[0, 1]^2 \) menu size \( \leq 3 \) | 0.5462947 | 59/108 | \( \geq 99.997\% \) |
| \( U[0, 1]^2 \) menu size \( \leq 2 \) | 0.5443309 | 2\sqrt{6}/9 | \( \geq 99.999997\% \) |
| \( U[v_1, v_2 \geq 0 | v_1/2 + v_2 \leq 1 \) | 0.5491225 | \((12 + 2\sqrt{2})/27\) | \( \geq 99.9857\% \) |

\(^1\)In the recent versions since 2020, Dütting et al. [10] followed our methodology to discover and prove the optimal mechanisms for the uniform distribution on a shifted and scaled triangle, i.e., \( \{(v_1, v_2) | v_1/c + v_2 \leq 2, v_1 \geq 0, v_2 \geq 1\} \).

\(^2\)The computed revenue is NOT directly given by the loss of our network. Instead, we ignore the buyer network and compute the expected revenue according only to the menu given by our network.
**Definition 2.1 (Naïve Mechanism).** A naïve mechanism consists of an action set $\mathcal{A}$ and an associated mapping from any action to a possible outcome, i.e., $(x, p) : \mathcal{A} \mapsto X \times \mathbb{R}_+$. In particular, there exists a special action $\perp$ meaning "exiting the mechanism" such that 

$$x(\perp) = 0, p(\perp) = 0.$$  \hfill (Exit)

In such a naïve mechanism, a strategy of the buyer is then a mapping from the set of private valuation functions to the action set, i.e., $s : V \mapsto \mathcal{A}$. Furthermore, if the buyer is rational, then her strategy must maximize her utility:

$$s(v) \in \arg\max_{s' \in S} u(x(s'(v)), p(s'(v)); v).$$  \hfill (Rational)

The corresponding outcomes of the actions are also known as menu items. Throughout this paper, we use $[x, p]$ to denote a specific menu item, e.g., the zero menu item $[0, 0] = [(0, \ldots, 0), 0]$ is the corresponding menu item of the exiting action $\perp$. Note that the naïve mechanism with the menu presentation is a very general model of the mechanism design problem. In particular, even when the buyer is not fully rational, as long as a buyer behavior is available, the mechanism designer is still able to design the menus to maximize his objective assuming that the buyer responds according to the given behavior model. The robustness of naïve mechanisms is indeed critical to the flexibility and generality of our methodology.

**Direct Mechanism** With the above definition of naïve mechanisms, it is hard to characterize all the mechanisms with certain properties, because the design of the action set, at first glance, could be arbitrary. One critical step in the mechanism design theory is to applying the celebrating revelation principle [19, p.224] to restrict the set of naïve mechanisms to a considerably smaller set of mechanisms — the direct mechanisms. In a direct mechanism, the action set is restricted to be identical to the set of valuation functions and the identity mapping also is required to be an optimal strategy for any rational buyer. Formally,

**Definition 2.2 (Direct Mechanism).** A direct mechanism fixes the action set $\mathcal{A} = V$ and remains to specify the mapping from $V$ to the set of possible outcomes.

In addition, the identity mapping must be a utility-maximizing strategy for any rational buyer, which can be equivalently stated as the following incentive compatible (IC) and individually rational (IR) constraints:

$$v \in \arg\max_{v' \in V} u(x(v'), p(v'); v),$$  \hfill (IC)

$$u(x(v), p(v); v) \geq 0.$$  \hfill (IR)

In fact, the constraints (IC) and (IR) are deduced from the constraints (RATIONAL) and (EXIT).

**The Designer’s Goal** The goal of the mechanism designer is to maximize the expectation of his objective $r : \mathcal{X} \times \mathbb{R}_+ \mapsto \mathbb{R}$, where the expectation is taken over his prior knowledge about the buyer’s private valuation function, i.e., $v \sim \mathcal{F}$.

We emphasize that our methodology is not restricted to any specific objective. However, in this paper, we would focus on the setting with the seller’s revenue as the objective:

$$r(x, p) = p.$$  \hfill (Objective)

Because revenue-optimal mechanism design in multi-dimensional environment is a both challenging and widely studied problem. Hence applying our method in such a setting allows us to verify that (i) whether it can find the optimal or nearly optimal solution, and (ii) whether it can provide a simpler approach to a hard problem.

**Assumptions** In most sections of this paper, we will make to the following two assumptions (Assumption 2.3 and Assumption 2.4). As we just stated, we would first verify that our method can be used to recover the optimal solutions to some known problems and little exact optimal solution is actually discovered without these two assumptions.

**Assumption 2.3 (Additive Valuation Functions).** The buyer’s valuation function $v$ is additive, i.e., $v$ can be decomposed as follows:

$$v(x) = \sum_{i \in [m]} v_i x_i,$$

where $v_i \in \mathbb{R}_+$. 

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With the additive valuation assumption, we refer each $v_i$ as the value of the $i$-th item. Moreover, we can make the following independent value assumption in addition.

**Assumption 2.4** (Independent Values). The prior distribution $F$ is independent in each dimension and can be decomposed as $F = F_1 \times \cdots \times F_m$, where each $v_i$ is independently drawn from $F_i$, i.e., $v_i \sim F_i$.

In the meanwhile, to show that our method is not limited to these assumptions, in Section 5, we show how it can be applied to settings without these assumptions. In particular, with the help of the characterization results by Daskalakis et al. [8], we are able to verify the optimality of the solution to an instance with correlated value distribution (while still with additive valuation functions).

### 3 Problem Analysis

Although the revelation principle is widely adopted by the theoretical analysis of mechanism design problems to efficiently restrict the design spaces, we decided not to follow this approach when applying neural networks to solve such problems.

The main difficulty of directly following the traditional revelation principle based approach is two-fold:

- It is unclear that what network structure can directly encode the incentive compatible (IC) and individually rational (IR) constraints;
- Some of the characterization results for additive valuation setting can be cast to certain network structures, but such structures are restricted (to additive valuation assumption) and heavily rely on the domain knowledge of the specific mechanism design problem.

In fact, the above difficulties also limit the generality of the methods built on these elegant but specific characterizations. For example, there might be some fundamental challenges while generalizing such approaches to the settings where the buyer is risk-averse (risk-seeking) or has partial (or bounded) rationality, etc. Furthermore, in many real applications, the buyer behavior models may come from real data instead of pure theoretical assumptions.

To circumvent these difficulties and ensure the highest extendability, in this paper, we build up our method from the most basic naïve mechanisms — simply let the buyer choose her favorite option — which is even more close to the first principles of how people make decisions. Interestingly, via this approach, our method will automatically produce an exactly incentive compatible and individually rational mechanism. To the best of our knowledge, this is the first neural network based approach that outputs an both exactly incentive compatible and exactly individually rational mechanism under multi-dimensional settings.

#### 3.1 Revisiting the Naïve Mechanism

We then briefly explain how the naïve mechanism helps us to formulate a neural network based approach for mechanism design.

Intuitively, the naïve mechanism in our context simply provides the buyer various menu items, i.e., allocations associated with different prices, and lets her choose the most preferred one. In this case, once a buyer utility function is specified (either by assumption or learnt from data), the choice of the buyer is simply an argmax of the utility function. As long as the utility function could be encoded via neural network, which is a mild assumption, the buyer’s behavior model can encoded as a neural network with an additional argmax layer.

**High-level sketch of the network structure** For now, we can think the encoded mechanism as a black-box that outputs a set of allocation-payment pairs (see Figure 1(a)). These pairs then are feeded into many “buyer networks”, each with different private valuation functions (hence different choices). Finally, the “buyer networks” output their choices and the choices are used to evaluate the expected objective of the mechanism designer, where the choices are weighted according to the probabilities of the corresponding private valuation functions and the training loss is simply the negative of the expected objective.

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3Such as Myerson’s virtual value for single-dimension and Rochet’s increasing, convex and Lipschitz-1 buyer utility function for multi-dimension [9].

4Even if the buyer utility function is not available, such a gadget could be replaced by any buyer behavior model (either given or learnt from data), which is encoded as a neural network.
One key advantage of formulating the network as a naïve mechanism rather than a direct mechanism is that no additional constraints (such as IC and IR) are required for the former. In fact, the difficulty of optimizing the direct mechanism network (see Figure 1(b)) is that the violations of IC or IC constraints are not directly reflected in the designer’s objective. Hence the standard optimization methods for neural networks do not directly apply. In contrast, in the naïve mechanism network, the effect of any mechanism outcome mutations on the buyer preferences is reflected in the designer’s objective via the “buyer networks”. Such properties facilitate the optimization in standard training methods of neural networks.

4 Network Structure of MENU\textsc{Net}

Our MENU\textsc{Net} structure contains two networks: the mechanism network and the buyer network. Since the networks represent a naïve mechanism, the output of the mechanism network is a set of choices along with different prices (or menu items) and the buyer network takes the set of menu items as input and outputs its choice. The overall network structure is shown in Figure 2.

4.1 Mechanism Network

In most applications, a neural network usually takes a possible input $x$ and then outputs a possible output $y$. However, our mechanism network is different from most neural networks in the sense that its output is a set of menu items, which already represents the entire mechanism. Therefore, our mechanism network does not actually need to take an input to give an output.

However, in order to fit in with most neural network frameworks, we use a one dimensional constant 1 as the input of our mechanism network. The output of the network consists of two parts. The first part is an allocation matrix $X$ of $m$ rows and $k$ columns, where $m$ is the number of items and $k$ is the number of menu items. Each column of the allocation matrix contains the allocation of all $m$ items. The second part is a payment vector $p$ of length $k$, representing $k$ different prices for the $k$ menu items. The last column of the allocation matrix and the last element of the payment vector is always set to be 0. This encodes the “exit” choice of the buyer and ensures that the buyer can always choose this menu item to guarantee individual rationality.

The structure of the mechanism network is simple enough. The constant input 1 goes through a 1 fully connected layer to form each row $X_i$ (except the last column, which is always 0) of the allocation matrix. We choose the sigmoid function as the activation function since the allocation of each item is always inside the interval $[0, 1]$. The payment vector is even simpler. Each element $p_i$ of the payment vector is formed by multiplying the input constant by a scalar parameter. Therefore, the training of our network is very fast, since the network structure is very simple.
4.2 Buyer Network

The buyer network is a function that maps a mechanism to the buyer’s strategy $s(v)$ (a distribution over all possible menu items) for each value profile $v = (v_1, v_2, \ldots, v_m)$, where each $v_i$ is the value of getting the $i$-th item. The output of the mechanism network (the allocation matrix $X$ and the payment vector $p$) is taken as the input of the buyer network. To define the output of the buyer network, suppose that each $v_i$ is bounded and $0 \leq v_i \leq \bar{v}_i$. We discretize the interval $[0, \bar{v}_i]$ to $d_i$ discrete values. Let $V_i$ be the set of possible discrete values of $v_i$ and define $V = \prod_{i=1}^{m} V_i$.

The output of the buyer network is a $m + 1$ dimensional tensor, with the first $m$ dimension corresponding to the buyer’s $m$ dimensional value, and the last dimension representing the probability of choosing each menu item. Therefore, the $i$-th ($i \leq m$) dimension of the tensor has length $d_i$ and the last dimension has length $k$.

Although here we use the same notation as in Assumption 2.3, this notation does not lose generality since we do not make any assumption about the buyer’s valuation of obtaining multiple items or only a fraction of an item. It is also worth mentioning that the buyer’s utility function is not necessary to build the buyer network, since the network only outputs buyer’s strategy, which may not even be consistent with any utility function.

The buyer network can be any type of network that has the same format of input and output as described above. When we do not know the buyer’s exact utility function but have plenty of interaction data (e.g., the sponsored search setting), we can train the buyer network with the interaction data.

When the buyer’s utility function is known, we can manually design the buyer network structure so that the network outputs the buyer’s strategy more accurately. For example, when Assumption 2.3 and Assumption 2.4 holds, we know that the buyer always choose the menu item that maximizes his additive valuation with probability 1. We can construct $m$ tensors $V_1, V_2, \ldots, V_m$, with size $d_1 \times d_2 \times \cdots \times d_m$. In $V_i$, an element’s value is only determined by its $i$-th dimensional index in the tensor, and it equals the $j$-th discretized value of the interval $[0, \bar{v}_i]$, if its $i$-th dimensional index is $j$. Recall that the $i$-th row of the allocation matrix $X_i$ represents different allocations of the $i$-th item in different menu items. We then multiply the $i$-th tensor with the $X_i$ to get an $m + 1$ dimensional tensor $V_i$ with size $d_1 \times d_2 \times \cdots \times d_m \times k$.

We also construct a payment tensor $p$ with size $d_1 \times d_2 \times \cdots \times d_m \times k$, where an element equals to the $p_i$ if its index for the last dimension is $j$. 
Finally, we compute the utility tensor $\mathcal{U}$ by

$$\mathcal{U} = \left( \sum_{i \in [m]} \mathcal{X}_i \right) - \mathcal{P}. $$

And then apply the softmax function to the last dimension of the utility tensor $\mathcal{U}$ to produce the output $\mathcal{S}$, which is an aggregation of $s(v)$, $\forall v \in V$. One can easily verify that for each value profile, the menu with the largest utility has the highest probability of being chosen. Of course, we also multiply the utility tensor by a large constant to make the probability of the best menu item close enough to 1.

### 4.3 Loss Function

The loss function can be any function specified according the mechanism designer’s objective. However, in this paper, we mainly focus on how to optimize the revenue of the mechanism and set the loss function to be the negative revenue.

Recall that the output of the buyer network is the buyer’s strategy $s(v)$ for each value profile $v$. Then the loss function of the networks is

$$\text{Loss} = -\text{REV} = - \sum_{v \in V} \Pr [v] p^T s(v)$$

where $\Pr [v]$ is the probability that $v$ appears, which can be easily computed from the joint value distribution $\mathcal{F}$.

Note that in the above loss function, we do not make any assumption about the probability distribution $\Pr [v]$. Our networks are able to handle any joint distribution, including correlated ones.

### 5 Experiments and Analysis

In this section, we first list some results of our neural networks in Section 5.1. Inspired by these results, we are able to prove the closed-form optimal mechanisms in some cases where the exact optimal solutions are previously unknown: i) the setting with correlated triangle distribution (Section 5.1.2) and ii) the setting with uniform square distribution but restricted menu size (Section 5.1.3). The theoretical analysis and proofs of our newly discovered optimal mechanisms are presented in Section 5.2.

To the best of our knowledge, we are the first to discover exact optimal mechanisms under the help with the neural network based approach. Although one still need to tolerate the complexity of the theoretical proof (mostly on constructing the matching dual solution), the neural network can greatly help on guessing the structure of the optimal primal solution. We believe the methodology is of its own interests. As one followup, in a recent version, Dütting et al. [10] followed this methodology and discovered the optimal mechanisms for some different correlated triangle distributions.

### 5.1 Experiment results

#### 5.1.1 Uniform $[0,c] \times [0,1]$

The optimal mechanism for this setting is already known [25]. We draw both the optimal mechanism and our experiments results together in Figure 3. The color blocks represents the mechanism given by our network, where each color corresponds to a different menu item. The dashed line represents the optimal mechanism (they are NOT drawn according to the color blocks). The two mechanisms are almost identical except for the slight difference in Figure 3(c).

#### 5.1.2 Correlated Distribution: Uniform Triangle

Suppose that the buyer’s value $v = (v_1, v_2)$ is uniformly distributed among the triangle described by $\frac{v_1}{c} + v_2 \leq 1$, $v_1 \geq 0$, $v_2 \geq 0$, where $c \geq 1$. The color blocks in Figure 4 show the mechanisms given by our network. Note that in our framework, the joint value distribution is only used to compute the objective function. So our framework can handle arbitrary value distributions.
In fact, guided by these experiment results, we are able to find the closed-form optimal mechanism for this kind of value distributions. In particular, there are two possible cases for this problem. When \( c \) is large, the optimal mechanism contains two menu items. And when \( c \) is small, the optimal contains only two menus, i.e., use a posted price for the bundle of the items. Formally, we have

**Theorem 5.1.** When \( c > \frac{2}{3} \), the optimal menu for the uniform triangle distribution contains the following items: \((0, 0), (\frac{\sqrt{c}}{2}, \frac{\sqrt{c}}{2}), (1, 1), \frac{2}{3} c - \frac{1}{3} \sqrt{c(c - 1)}\).

When \( c \leq \frac{2}{3} \), the optimal menu for the uniform triangle distribution contains the following items: \((0, 0), 0 \) and \((1, 1), \sqrt{\frac{c}{2}}\).

The proof is deferred to Section 5.2. In a recent version, Dütting et al. [10] followed our approach and also gave the optimal mechanisms for similar triangle distributions. But the support of the distribution in their case is different from ours by a constant translation.

**5.1.3 Restricted Menu Size**

The output of our mechanism network is a set of menus. Thus we can control the menu size by directly setting the output size of the network.

Restricting the menu size results in simpler mechanisms. It is known that the optimal menu for some distributions
contains infinitely many items [8]. Such results directly motivates the study of simple mechanisms, since they are easier to implement and optimize in practice.

We consider the case where the buyer’s value is uniformly distributed in the unit square \([0, 1]^2\). It is known that the optimal mechanism contains 4 menu items. When the menu can only contain at most 2 items, the optimal mechanism is to trivially set a posted price for the bundle. The experiment results are shown in Figure 5.

Surprisingly, when the menu can have at most 3 items, our network gives an asymmetric menu, despite that the value distribution is symmetric. In fact, we can also find the optimal menu with at most 3 items analytically. Our analysis shows that the optimal menu is indeed asymmetric. The intuition is that, if we add a symmetry constraint to the solution, then the optimal menu degenerates to a 2-item one. We provide the theoretical result here, but defer the proof to Section 5.2.

**Theorem 5.2.** The optimal at-most-three-menu mechanism for two additive items with \(v \sim U[0, 1]^2\) is to sell the first item at price \(2/3\) or the bundle of two items at price \(5/6\), yielding revenue \(59/108 \approx 0.546296\).

By symmetry, the mechanism could also be selling the second item at price \(2/3\) or the bundle of two items at price \(5/6\). In particular, these is no other at-most-three-menu mechanisms could generate as much revenue as they do.

### 5.1.4 Unit-Demand Buyer

The unit-demand setting is also intensively studied in the literature. In this setting, the allocation must satisfy \(x_1 + x_2 \leq 1\). [26] provides detailed analysis and closed-form solutions on the unit-demand setting. With slight modifications, our mechanism network can also produce feasible allocations in this setting. Instead of applying the sigmoid function to each element of the allocation matrix, we apply a softmax function to each column (representing each menu item) of the allocation matrix. However, with such a modification, the allocation satisfies \(x_1 + x_2 = 1\) rather than \(x_1 + x_2 \leq 1\). The solution is to add an extra dummy element to each column before applying the softmax function.

The experiment results are shown in Figure 6(a).

Figure 6: Empirical results.
When \( c > \frac{4}{3} \), suppose that the buyer's type is uniformly distributed among the set \( T = \{ (v_1, v_2) | \frac{2}{c} + v_2 \leq 1, v_1 \geq 0, v_2 \geq 0 \} \). Then the optimal menu contains the following items: \( (0, 0), (\frac{1}{c}, 1), \frac{2}{3}, and (1, 1), \frac{2}{3}c - \frac{1}{3}\sqrt{c(c - 1)} \).

**Remark 5.4.** Note that the condition \( c > \frac{4}{3} \) guarantees that the price of the third menu item is positive.

To prove Theorem 5.3, we apply the duality theory in [8, 7] to our setting. We provide a brief description here and refer readers to [8] and [7] for details.

Let \( f(v) \) be the joint value distribution of \( v = (v_1, v_2) \), and \( V \) be the support of \( f(v) \). Define measures \( \mu_0, \mu_\partial, \mu_s \) as follows:

- \( \mu_0 \) has a single point mass at \( v = 0 \), i.e., \( \mu_0(V) = \hat{I}(v \in A) \), where \( \hat{I}(\cdot) \) is the indicator function, and \( v \in A \) is the smallest type in \( V \).
- \( \mu_\partial \) is only distributed along the boundary of \( V \), with a density \( f(v)(v \cdot \eta(v)) \), where \( \eta(v) \) is the outer unit normal vector at \( v \).
• \( \mu_s \) is distributed in \( V \) with a density \( \nabla f(v) \cdot v + (n + 1) f(v) \), where \( n \) is the number of items.

Let \( \mu = \mu_0 + \mu_\partial - \mu_s \). Define \( \mu_+ \) and \( \mu_- \) to be two non-negative measures such that \( \mu = \mu_+ - \mu_- \). Let \( V_+ \) and \( V_- \) be the support sets of \( \mu_+ \) and \( \mu_- \). [8, 7] shows that designing an optimal mechanism for selling \( n \) items to 1 buyer is equivalent to solving the following program:

\[
\begin{align*}
\sup \int_V u \, d\mu_+ - \int_V u \, d\mu_- \\
\text{s.t. } u(v) - u(v') &\leq \|v - v'\|_1, \forall v \in V_+, v' \in V_- \\
 u &\text{ is convex, } u(v) = 0 \tag{P}
\end{align*}
\]

where \( u(v) \) is the utility of the buyer when his value is \( v \), and \( \|v - v'\|_1 = \sum_{i=1}^n \max(0, v_i - v'_i) \).

Relax the above program by removing the convexity constraint and write the dual program of the relaxed program:

\[
\begin{align*}
\inf \int_{V \times V} &\|v - v'\|_1 \, dy \\
\text{s.t. } \gamma &\in \Gamma(\mu_+, \mu_-) \tag{D}
\end{align*}
\]

where \( \Gamma(\mu_+, \mu_-) \) is the set of non-negative measures \( \gamma \) defined over \( V \times V \) such that, for any \( V' \subseteq V \), the following equations hold:

\[
\int_{V \times V'} dy = \mu_+(V') \quad \text{and} \quad \int_{V \times V'} dy = \mu_-(V')
\]

**Lemma 5.5** (Daskalakis et al. [8]). (D) is a weak dual of (P).

We omit the proof here but refer readers to [8] and [7] for details. The dual program (D) has an optimal transport interpretation. We "move" the mass from \( \mu_+ \) to other points to form \( \mu_- \) and the measure \( \gamma \) corresponds to the amount of mass that goes from each point to another in \( V \).

Although (D) is only a weak dual of (P), we can still use it to certify the optimality of a solution. We already give a menu in Theorem 5.3. Therefore, the relaxed convexity constraint is automatically satisfied if the buyer always choose the best menu item.

In our setting, \( f(v) = \frac{2}{c} \), and we have that \( V = T, v = (0, 0), \mu_\partial \) has a constant line density of \( \frac{2}{c} \) along the segment \( \frac{v_1}{c} + v_2 = 1, 0 \leq v_2 \leq 1 \), and \( \mu_s \) has a constant density of \( \frac{\delta}{c} \) over \( T \).

Let \( R_i \) be the region of \( T \) such that for any \( v \in R_i \), choosing menu item \( i \) maximizes the buyer’s utility.

It is straightforward to verify that the measures \( \mu_+ \) and \( \mu_- \) are balanced inside each region, i.e., \( \mu_+(R_i) = \mu_-(R_i), \forall i \). Therefore, the transport of mass only happens inside each region.

We construct the transport in \( R_1 \) and \( R_2 \) as follows:

• \( R_1 \): \( \mu_+ \) is concentrated on a single point 0. We move the mass at 0 uniformly to all points in \( R_1 \);

• \( R_2 \): \( \mu_+ \) is only distributed along the upper boundary of \( R_2 \). For each point \( v \) at the upper boundary, we draw a vertical line \( l \) through it, and move the mass at \( v \) uniformly to the points in \( L \cap R_2 \).

However, for \( R_3 \), \( \mu_+ \) is also only distributed along the upper boundary, but there is no easy transport as for \( R_1 \) and \( R_2 \). We provide the following Lemma 5.6.

**Lemma 5.6.** For \( R_3 \), there exists a transport of mass, such that for any two points \( v, v' \), if there is non-negative transport from \( v \) to \( v' \), then \( v_i \geq v'_i, \forall i \).

The proof of Lemma 5.6 is deferred to Appendix A.1. With this lemma, we can simplify our proof of Theorem 5.3, and do not need to construct the measure \( \gamma \) explicitly.

**Proof of Theorem 5.3.** Point \( D \) in Figure 10 has coordinates \((x_D, y_D)\), where \( x_D = \frac{2}{3} c - \frac{1}{3} \sqrt{c(c-1)} - \frac{1}{3} \sqrt{\frac{c}{c-1}} \) and \( y_D = \frac{1}{3} \sqrt{\frac{c^2}{c-1}} \). Therefore,

\[
\Pr(\text{The buyer chooses menu item 2}) = f(v) \cdot S(YCDI) = \frac{2}{c} \cdot \frac{1}{3} x_D
\]

\[
\Pr(\text{The buyer chooses menu item 3}) = f(v) \cdot S(CDEX) = \frac{2}{c} \left[ \frac{c}{2} \left( \frac{1}{3} + y_D \right) - \frac{1}{2} y_D^2 \right]
\]
Thus the revenue of the menu provided in Theorem 5.3 is:

\[
\text{Rev} = \frac{2}{3} \cdot \Pr\{\text{The buyer chooses menu item 2}\} + \frac{2}{3} \cdot \frac{1}{\sqrt{3c(c-1)}} \cdot \Pr\{\text{The buyer chooses menu item 3}\} = \frac{2}{27} \left[ 4 + \sqrt{c(c-1)} \right]
\]

Now we compute the objective of the dual program (D). And to prove the optimality of the menu, it suffices to show that the objective of (D) is equal to Rev.

Note that in our construction of the transport in \( R_1 \) and \( R_2 \), we only allow transport inside each region. In \( R_1 \), we transport mass from point 0 to other points. So it does not contribute to the objective of (D), and we can just ignore \( R_1 \). In \( R_2 \), the mass is always moved vertically down. Therefore, for any \( v, v' \), such that there is positive mass transport from \( v \) to \( v' \), we have \( v_i \geq v'_i, \forall i \) and \( \|v - v'\|_1 = \sum_j \max(0, v_i - v'_i) = \sum_j (v_i - v'_i) = \sum_j (v_i - 0) - \sum_j (v'_i - 0) \).

Therefore,

\[
\int_{R_2 \times R_2} \|(v - v')_+\|_1 \, dy = \int_{R_2 \times R_2} \|v - 0\|_1 \, dy - \int_{R_2 \times R_2} \|v' - 0\|_1 \, dy \quad (4)
\]

For the first term, we have:

\[
\int_{R_2 \times R_2} \|v - 0\|_1 \, dy = \int_{R_2} \|v - 0\|_1 \, d\mu_+
\]

\[
= \int_0^{\sigma x D} \left( v_1 + 1 - \frac{v_1}{c} \right) \frac{2}{\sqrt{1 + c^2}} \frac{\sqrt{1 + c^2}}{c} \, dv_1 = \frac{1}{9} \left( 8 - 6 \sqrt{\frac{c}{c-1}} + 5c - 4\sqrt{c(c-1)} \right)
\]

Similarly, the second term of Equation (4) is:

\[
\int_{R_2 \times R_2} \|v' - 0\|_1 \, dy = \int_{R_2} \|v' - 0\|_1 \, d\mu_- = \left( 2\sqrt{c-1} - \sqrt{c} \right) \left( 3 + 2c - \sqrt{c(c-1)} \right) \frac{9\sqrt{c-1}}{9c-1}
\]

For \( R_3 \), according to Lemma 5.6, it is also true that when there is positive mass transport from \( v \) to \( v' \), we always have \( v_i \geq v'_i, \forall i \). Therefore,

\[
\int_{R_3 \times R_3} \|(v - v')_+\|_1 \, dy = \int_{R_3 \times R_3} \|v - 0\|_1 \, dy - \int_{R_3 \times R_3} \|v' - 0\|_1 \, dy
\]

For the first term,

\[
\int_{R_3 \times R_3} \|v - 0\|_1 \, dy = \int_{R_3} \left( v_1 + 1 - \frac{v_1}{c} \right) \frac{2}{c} \, dv_1 = \frac{1}{9} \left( 1 + 4c + 2\sqrt{\frac{c}{c-1}} \right)
\]

Similarly, for the second term,

\[
\int_{R_3 \times R_3} \|v' - 0\|_1 \, dy = \int_{R_3} \frac{6}{c} (v_1 + v_2) \, dv = \frac{1}{27} \left( 1 + 5\sqrt{\frac{c}{c-1}} + 10c + 10c\sqrt{\frac{c}{c-1}} \right)
\]

Therefore, the objective of the dual program (D) is:

\[
\int_{T \times T} \|(v - v')_+\|_1 \, dy = \int_{R_2 \times R_2} \|(v - v')_+\|_1 \, dy + \int_{R_3 \times R_3} \|(v - v')_+\|_1 \, dy = \frac{2}{27} \left[ 4 + \sqrt{c(c-1)} \right] = \text{Rev}
\]

The above equation shows that the dual objective is equal to the actual revenue, which certifies that the menu is optimal.
When \( c \leq \frac{4}{7} \), the optimal mechanism only has two menu items.

**Theorem 5.7.** For any \( 1 \leq c \leq \frac{4}{7} \), suppose that the buyer’s type is uniformly distributed among the set \( T = \{(v_1, v_2) \mid \frac{v_1}{c} + v_2 \leq 1, v_1 \geq 0, v_2 \geq 0\} \). Then the optimal menu contains the following two items: \((0, 0)\) and \((1, 1)\).

One can prove Theorem 5.7 with the same trick in Lemma 5.6. We omit the proof of this theorem since it is easier compared to the other case described in Theorem 5.3.

### 5.2.2 Optimal mechanisms under limited menu size constraints

In this section, we consider the optimal 3-Menu Mechanisms for value distribution \( U[0, 1]^2 \).

**Theorem 5.8.** The optimal symmetric at-most-three-menu mechanism for two additive items with \( v \sim U[0, 1]^2 \) is to sell the bundle of two items at price \( \sqrt{6}/3 \), yielding revenue \( 2\sqrt{6}/9 \approx 0.54433 \).

We defer the proof to Appendix A.

**Theorem 5.9.** The optimal at-most-three-menu mechanism for two additive items with \( v \sim U[0, 1]^2 \) is to sell the first item at price \( 2/3 \) or the bundle of two items at price \( 5/6 \), yielding revenue \( 59/108 \approx 0.546296 \).

By symmetry, the mechanism could also be selling the second item at price \( 2/3 \) or the bundle of two items at price \( 5/6 \). In particular, there is no other at-most-three-menu mechanisms could generate as much revenue as they do.

We demonstrate the proof through the basic parametric method. Note that there must be a zero menu \( Z = [(0, 0), 0] \), and hence we have two menus to determine. Suppose that the remaining two menus are \( A = [(\alpha, \beta), p] \) and \( B = [(\gamma, \delta), q] \). We then solve the following problem:

\[
\text{maximize } \text{Rev}(A, B, Z) \\
\text{subject to } \alpha, \beta, \gamma, \delta \in [0, 1], \ p, q \geq 0. 
\]  

(3Menu)

To establish the connection between the menus and the revenue, let \( S_A \) be the set of values that menu \( A \) is the most preferred:

\[ S_A = \{(v_1, v_2) \in [0, 1]^2 \mid (v_1, v_2) \cdot (\alpha, \beta) - p \geq (v_1, v_2) \cdot (\gamma, \delta) - q \land (v_1, v_2) \cdot (\alpha, \beta) - p \geq 0\}. \]

Similarly, we define \( S_B \) and \( S_Z \) be the set of values where menu \( B \) and menu \( Z \) are the most preferred, respectively:

\[ S_B = \{(v_1, v_2) \in [0, 1]^2 \mid (v_1, v_2) \cdot (\gamma, \delta) - q \geq (v_1, v_2) \cdot (\alpha, \beta) - p \land (v_1, v_2) \cdot (\alpha, \beta) - p \geq 0\}, \]

\[ S_Z = \{(v_1, v_2) \in [0, 1]^2 \mid 0 \geq (v_1, v_2) \cdot (\alpha, \beta) - p \land 0 \geq (v_1, v_2) \cdot (\gamma, \delta) - q\}. \]

For any measurable set \( S \subseteq [0, 1]^2 \), let \( |S| = \Pr((v_1, v_2) \in S) \) be the probabilistic measure of \( S \). Then the revenue of the mechanism with menus \( A, B, \) and \( Z \) is

\[ \text{Rev}(A, B, Z) = |S_A| \cdot p + |S_B| \cdot q. \]

(3MenuRev)

With the above formulation, there are two major challenges to solve the program (3Menu):

- There are too many possible cases with different formulas of \(|S_A|\) and \(|S_B|\), hence the formula of \( \text{Rev}(A, B, Z) \).

  In particular, there are 4 possible intersection patterns between the boundary of the square \([0, 1]^2\) and the intersection of each two of the menus \((S_A \cap S_B, S_B \cap S_Z, S_Z \cap S_A)\). Hence roughly \(4^3 = 64\) different cases.

- Even within each specific case, the revenue \( \text{Rev} \) is still a high-order function with 6 variables. In general, there is no guarantee for closed-form solutions.

To overcome these two challenges, the following two lemmas are critical to reducing both the number of different cases and free variables:

**Lemma 5.10.** Without loss of generality, we can assume that the optimal at-most-three-menu mechanism includes bundling, \((1, 1)\), as one of its menu.
Similarly, by Lemma 5.11, we can fix one of $\gamma$ and $\delta$ to be 1, without loss of generality, $\gamma = 1$. Note that in the case with $(\gamma, \delta) = (0, 0)$, menu $B$ will be dominated by menu $Z$, hence reduced to a two-menu mechanism again.

Therefore, we remain to solve (3Menu) with additional constraints: $\alpha = \beta = 1$ and $p > q$.

Now consider the values $v = (v_1, v_2)$ in $S_A \cap S_B$, which must satisfy:

$$S_A \cap S_B : (v_1, v_2) \cdot (1, 1) - p = (v_1, v_2) \cdot (1, \delta) - q.$$

Similarly, $S_A \cap S_Z : (v_1, v_2) \cdot (1, 1) = p$, $S_B \cap S_Z : (v_1, v_2) \cdot (1, \delta) = q$, and hence $S_A \cap S_B \cap S_Z : v_1^* = \frac{q - \delta p}{1 - \delta}, v_2^* = \frac{p - q}{1 - \delta}$. Note that if $S_A$ or $S_B$ is empty, there would be only two menus and the revenue cannot be more than $2\sqrt{6}/9$.

Otherwise:

- For $S_A$ not being empty, we must have $v_2^* < 1$, hence:

  $$\frac{p - q}{1 - \delta} < 1; \quad \text{(NonEmptyA)}$$

- For $S_B$ not being empty, we must have $v_1^* < 1$, hence:

  $$\frac{q - \delta p}{1 - \delta} < 1. \quad \text{(NonEmptyB)}$$

Based on the constraints (NonEmptyA) and (NonEmptyB), there are three possible cases (see Figure 8). The solutions under these cases are summarized by the following lemmas.

**Lemma 5.12** (Case 1). Conditional on $p \leq 1$, the optimal mechanism consists of asymmetric three menus $A : [(1, 1), 5/6]$, $B : [(1, 0), 2/3]$, $Z : [(0, 0), 0]$, and yields revenue 59/108.
Lemma 5.13 (Case 2). Conditional on $p \geq 1 > q$, the optimal mechanism yields revenue $14/27$.

Lemma 5.14 (Case 3). Conditional on $p > q > 1$, the revenue of the mechanism is not more than $1/2$.

In summary, the optimal mechanism with at most 3 menus is to sell the first item at price $2/3$ or the bundle of two items at price $5/6$, yielding revenue $59/108$.

6 Performance

Setup As our method is very efficient, we were able to perform our experiments on a laptop (13-inch MacBook Pro, with 2.5 GHz Intel Core i7 CPU, 16 GB RAM) using TensorFlow. To solve the problems with continuous value distributions in finite neural networks, we simply discretize the value space. In particular, the discretization is parameterized by $N$, which is the number of the intervals (with length $1/N$) in unit length. In other words, there are $N^2$ squares of size $1/N$ by $1/N$ in any unit square. By default, we set $N = 100$.

6.1 Efficiency and Accuracy: Compared with Linear Programs

We compare the running time of our method and the straightforward linear program approach for the $U[0,1]^2$ setting. In the linear program, the variables are the allocation $x_1, x_2$ and payment $p$ of the values on each discretized grid (hence $O(N^2)$ variables) and the constraints are the IC and IR constraints (hence $O(N^4)$ constraints). We use the basic PuLP package in Python to solve the linear programs. In Figure 9(a), we compared the execution time of solving the linear programs with specific $N$'s ($N = 10, 15, 20, 25, 30$) and the execution time of training our neural
network to (i) achieve a mechanism with at least the same level of accuracy as the one given by the linear program (for \( N \leq 30 \)), and (ii) converge (for \( N = 40, 50, 200 \)). Note that the running time of the linear program approach grows very rapidly: for \( N = 30 \), it takes 51 mins and we are not able to apply it to \( N \geq 40 \). In contrast, the training time of our neural network grows much slower (less than 5 mins for \( N = 200 \), i.e., buyer distribution support of size 40000).

One key advantage of our approach over the linear program is that our problem size grows linearly in terms of the support size of the buyer's distribution (i.e., \( O(N^2) \)), while the size of the linear program grows quadratically in terms of the support size (i.e., \( O(N^4) \)). In Figure 9(b), we also plot the average training time for each iteration, which is in \( 1 \sim 30 \) milliseconds.

Figure 9(c) and Figure 9(d) illustrates that our method converges to the optimal very fast. The relative error also drops very fast even in the log-scale plot. In particular, \( \text{Rev} \) is evaluated on the original continuous distribution \( U[0, 1]^2 \). Hence the gap between \( \text{Rev} \) and \( \text{OptRev} \) cannot drop to zero as we discretized the value distribution.

**Conclusion** So far, we have shown that our approach is much more efficient than the linear program approach and hence much stronger scalability as well. To complement the time efficiency, we also show in Appendix B that our method also dominates the linear program approach in terms of accuracy.

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A Missing Proofs

A.1 Proof of Lemma 5.6

Proof of Lemma 5.6. Denote the upper boundary of $R_3$ by $B$. For each $v \in B$, define

$$R_L = \{ v' \in R_3 \mid v_1' \leq v_1 \} \quad \text{and} \quad R_U = \{ v' \in R_3 \mid v_1' \geq v_2 \}.$$

For any line $l_v$ through $v$ with a non-negative slope (or infinity), denote the part of $R_3$ that is above the line by $R_v$. It is easy to verify that $\mu_+(R_U) \geq \mu_-(R_U)$ and $\mu_+(R_L) \leq \mu_-(R_L)$. Thus there exists a line $l_v^*$ such that the corresponding $R_v^*$ satisfies $\mu_+(R_v^*) = \mu_-(R_v^*)$.

Now we show that for any two $v$ and $v'$, the intersection point of $l_v^*$ and $l_v'^*$ is not inside $R_3$. In Figure 10, the three regions $R_1, R_2, R_3$ are the quadrangles $OIDE, YCDI$ and $CDEX$, respectively. Let points $A, B$ correspond to the value profiles $v$ and $v'$. Assume, on the contrary, that the intersection point of $l_v^*$ (line $AA'$) and $l_v'^*$ (line $BB'$) is inside $R_3$. Then we have:

$$\mu_+(ACDA') = \mu_-(ACDA') \quad \text{and} \quad \mu_+(BCDB') = \mu_-(BCDB'). \quad (1)$$

![Figure 10](image)

Figure 10: The intersection point of $l_v^*$ and $l_v'^*$.

Note that $\mu_+$ is only distributed along the line $CX$ inside $R_3$. Thus

$$\mu_+(BCDA') = \mu_+(BCDB') = \mu_-(BCDB'). \quad (2)$$

However, $\mu_-$ has a positive density inside $R_3$. Therefore, we have

$$\mu_-(BCDA') > \mu_-(BCDB'). \quad (3)$$

Combining equations (1), (2) and (3), we obtain:

$$\mu_+(BAA') < \mu_-(BAA'). \quad (*)$$

Since $\mu_+$ is uniformly distributed along the line $CX$ with density, we have that $\mu_+(BAA') = \frac{2}{1 + c^2} \cdot l(AB)$, where $l(\cdot)$ denotes the length of a segment. Similarly, $\mu_-(BAA') = \frac{6}{c} \cdot S(BAA')$, where $S(BAA')$ is the area of triangle

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Let \( h \) be the altitude of the triangle \( BAA' \) with respect to the base \( AB \). So
\[
S(BAA') = \frac{1}{2} l(AB) \cdot h \\
\leq \frac{1}{2} l(AB) \cdot l(EF) \\
= \frac{1}{2} l(AB) \cdot \frac{c - \left( \frac{3}{2}c - \frac{1}{2}\sqrt{c(c-1)} \right)}{\sqrt{1 + c^2}} \\
= l(AB) \cdot \frac{c + \sqrt{c(c-1)}}{6\sqrt{1 + c^2}}
\]
where line \( EF \) is perpendicular to line \( CX \). Then
\[
\mu_-(BAA') = \frac{6}{c} \cdot S(BAA') \\
\leq \frac{6}{c} \cdot l(AB) \cdot \frac{c + \sqrt{c(c-1)}}{6\sqrt{1 + c^2}} \\
= l(AB) \cdot \frac{c + \sqrt{c(c-1)}}{c\sqrt{1 + c^2}} \\
\leq l(AB) \cdot \frac{2}{\sqrt{1 + c^2}} \\
= \mu_+(BAA')
\]
which contradicts to Equation (\(^*\)).

Consider the set of lines \( K = \{ l' \mid v \in B \} \). Since we have already shown that no two of these lines have an intersection point inside \( R_3 \), the line set \( K \) actually cuts the region \( R_3 \) into "slices". And for any "slice" \( s \), we have \( \mu_+(s) = \mu_-(s) \). Therefore, for each point in \( B \), we can find its corresponding "slice" and move its mass uniformly to all the points inside the "slice". And since \( l'_v \) always has a non-negative (or infinite) slope, we conclude that whenever there is a mass transport from \( v \) to \( v' \), we have \( v_i \geq v'_i \forall i \).

\[\square\]

### A.2 Proof of Theorem 5.8

**Proof of Theorem 5.8.** Since \((0, 0), 0\) must be one of the three symmetric menus, the other two must have the form of \((\alpha, \beta), p \) and \((\beta, \alpha), p \).

Without loss of generality, assume that \( \alpha \geq \beta \) and \( \alpha > 0 \) (otherwise \( \alpha = \beta = 0 \), yielding 0 revenue). Therefore, if \( \alpha v_1 + \beta v_2 < p \) and \( \beta v_1 + \alpha v_2 < p \), the buyer will choose the zero menu, \((0, 0), 0\).

Consider the following two cases: (i) \( p \leq \alpha \) and (ii) \( p \geq \alpha \).

If \( p \leq \alpha \):
\[
\Pr[\alpha v_1 + \beta v_2 < p \land \beta v_1 + \alpha v_2 < p] = 2 \Pr[\alpha v_1 + \beta v_2 < p \land v_1 \geq v_2] = \frac{p}{\alpha} \cdot \frac{p}{\alpha + \beta}.
\]
Then the revenue is
\[
\text{REV} = p \cdot \Pr[\text{the buyer didn't choose the zero menu}] = \left( 1 - \frac{p}{\alpha} \right) \cdot p.
\]
Hence
\[
\text{REV} = \left( 1 - \frac{p^2}{\alpha^2} \cdot \frac{1}{1 + \beta/\alpha} \right) \cdot p \leq (1 - p^2/2) \cdot p = \sqrt{(1 - p^2/2) \cdot (1 - p^2/2) \cdot p^2} \leq 2\sqrt{6}/9,
\]
where the first inequality is reached if and only if \( \alpha = \beta = 1 \) and the second inequality is reached if and only if \( 1 - p^2/2 = p^2 \iff p = \sqrt{2/3} \).
If \( p \geq \alpha \):

\[
\text{Pr}[\alpha v_1 + \beta v_2 < p \land \beta v_1 + \alpha v_2 < p] = 2 \text{Pr}[\alpha v_1 + \beta v_2 < p \land v_1 \geq v_2] = \frac{p}{\alpha} \cdot \frac{p}{\alpha + \beta} + \left( \frac{p}{\alpha} - 1 \right)^2.
\]

Hence

\[
\text{REV} = \left( 1 - \frac{p^2}{\alpha^2} \cdot \frac{1}{1 + \beta/\alpha} + \left( \frac{p}{\alpha} - 1 \right)^2 \right) \cdot p \leq \left( 1 - \frac{p^2}{\alpha^2} \cdot \frac{1}{2} + \left( \frac{p}{\alpha} - 1 \right)^2 \right) \cdot p = \frac{p}{2} \left( \frac{p^2}{\alpha^2} - 4 \cdot \frac{p}{\alpha} + 4 \right)
\]

Let \( x = p/\alpha \geq 1 \), the right-hand-side becomes

\[
\frac{\alpha}{2} (x^3 - 4x^2 + 4x) \leq \frac{1}{2} (x^3 - 4x^2 + 4x).
\]

Then consider the first order derivative of \( x^3 - 4x^2 + 4x \):

\[
(x^3 - 4x^2 + 4x)' = 3x^2 - 8x + 4 = (3x - 2)(x - 2),
\]

the local maximum is reached at \( x = 2/3 \). Note that in this case, \( x = p/\alpha \geq 1 \). Hence the maximum revenue conditional on \( p \geq \alpha \) is reached when \( p = \alpha = \beta = 1 \), where \( \text{REV} = 1/2 < 2\sqrt{6}/9 \).

A3.3 Proofs for Theorem 5.9

A3.3.1 Proof of Lemma 5.12

Proof of Lemma 5.12. When \( p \leq 1 \), consider:

\[
S_B \cap S_Z \cap \{v_2 = 0\} : v_1 = q, v_2 = 0
\]

\[
S_A \cap S_B \cap \{v_1 = 1\} : v_1 = 1, v_2 = \frac{p - q}{1 - \delta}.
\]

Note that \( q \leq p \leq 1 \):

\[
|S_B| = \frac{1}{2} \cdot \left( (1 - q) \cdot v_2^2 + \frac{p - q}{1 - \delta} \cdot (1 - v_1^3) \right)
\]

\[
|S_Z| = \frac{1}{2} \cdot (p^2 - (p - q) \cdot v_2^2)
\]

\[
|S_A| = 1 - |S_B| - |S_Z|
\]

Then the revenue is

\[
\text{REV} = (1 - |S_Z|) \cdot p - |S_B| \cdot (p - q)
\]

\[
= \frac{1}{2} \left( 2p - p^3 + (p - q) p \cdot v_2^2 - (1 - q)(p - q) \cdot v_2^2 - \frac{(p - q)^2}{1 - \delta} \cdot (1 - v_1^3) \right)
\]

\[
= \frac{1}{2} \left( 2p - p^3 - \frac{(p - q)^3}{(1 - \delta)^2} + \frac{(p - q)^2(2p + q - 2)}{1 - \delta} \right)
\]

\[
= \frac{1}{2} \left( 2p - p^3 + (p - q)^3 \cdot \left( \frac{1}{1 - \delta} - \frac{2p + q - 2}{2(p - q)} \right)^2 + \left( \frac{2p + q - 2}{2(p - q)} \right)^2 \right)
\]

\[
\leq \frac{1}{2} \left( 2p - p^3 + (p - q)^3 + q(2p + q - 2)^2 \right),
\]

where the upper bound is reached if and only if: (i) \( p = q \), or (ii) \( 1 - \delta = 2(p - q)/(2p + q - 2) \). Remember that we have shown that \( p \neq q \), hence we must have \( 1 - \delta = 2(p - q)/(2p + q - 2) \) and

\[
2\text{REV} = 2p - p^3 + (p - q)(p + q/2 - 1)^2 = 2p - p^3 + (p - q) \cdot (p + q/2 - 1) \cdot (p + q/2 - 1)
\]

\[
\leq 2p - p^3 + \left( \frac{(p - q) + (p + q/2 - 1) + (p + q/2 - 1)}{3} \right)^3 = 2p - p^3 + (p - 2/3)^3,
\]
where the upper bound is reached if and only if \( p - q = p + q/2 - 1 \) or equivalently \( q = 2/3 \). Substituting \( q \) with \( 2/3 \), we have
\[
\text{Rev} = -p^2 + 5/3p - 4/27 \leq -(p - 5/6)^2 + 25/36 - 4/27,
\]
and its local maximum is reached when \( p = 5/6 \), hence
\[
\text{Rev} = 59/108 \approx 0.546296 > 2\sqrt{6}/9 \approx 0.54433,
\]
and the menus are:
\[
A : [(1, 1), 5/6] \quad B : [(1, 0), 2/3] \quad Z : [(0, 0), 0].
\]

A.3.2 Proof of Lemma 5.13

Proof of Lemma 5.13. When \( p > 1 \geq q \), consider:
\[
S_B \cap S_Z \cap \{v_2 = 0\} : v_1 = q, v_2 = 0 \quad S_A \cap S_B \cap \{v_1 = 1\} : v_1 = 1, v_2 = p - q / 1 - \delta.
\]
Hence
\[
|S_B| = \frac{1}{2} \cdot \left( (1 - q) \cdot v_2^* + \frac{p - q}{1 - \delta} \cdot (1 - v_1^*) \right) \quad |S_Z| = 1 - |S_A| - |S_B| \quad |S_A| = \frac{1}{2} \cdot \left( (2 - p)^2 - \left( \frac{p - q}{1 - \delta} - (p - 1) \right) \cdot (1 - v_1^*) \right).
\]
Then the revenue is
\[
\text{Rev} = |S_A| \cdot p + |S_B| \cdot q = \frac{1}{2} \left( 3p - 2p^2 + (p - q)^2 \cdot \left( -\frac{p - q}{(1 - \delta)^2} + \frac{2p + q - 2}{1 - \delta} \right) \right)
\]
\[
= \frac{1}{2} \left( 3p - 2p^2 + (p - q)^3 \cdot \left( -\frac{1}{1 - \delta} - \frac{2p + q - 2}{2(p - q)} \right) + \left( 2p + q - 2 \right)^2 \right)
\]
\[
\leq \frac{1}{2} \left( 3p - 2p^2 + (p - q)(p + q/2 - 1)^2 \right) \leq 3p/2 - p^2 + (p - 2/3)^3/2
\]
\[
= \frac{1}{54} \left( 27p^3 - 108p^2 + 117p - 8 \right),
\]
where the two inequalities are reached if and only if \( 1 - \delta = 2(p - q)/(2p + q - 2) \) and \( q = 2/3 \).

Note that we have to ensure \( v_2^* \leq 1 \iff (p - q)/(1 - \delta) \leq 1 \), in other words,
\[
\frac{p - 2/3}{(p - 2/3)/(p + 1/3 - 1)} \leq 1 \iff p \leq 5/3.
\]

Now consider the maximum of \( (27p^3 - 108p^2 + 117p - 8)/54 \) with \( p \in [1, 5/3] \), by the first order condition, the local maximum and minimum are reached at \( p = (4 - \sqrt{5})/3 \approx 0.75598 < 1 \) and \( p = (4 + \sqrt{5})/3 \approx 1.91068 > 5/3 \), respectively. Therefore, in this case, the revenue is decreasing in \( p \) and hence the maximum revenue is reached at \( p = 1 \): \( \text{Rev}(p = 1) = 14/27 < 59/108 \). □

A.3.3 Proof of Lemma 5.14

Proof of Lemma 5.14. When \( p > q > 1 \), consider:
\[
S_B \cap S_Z \cap \{v_1 = 1\} : v_1 = 1, v_2 = \frac{q - 1}{\delta} \quad S_A \cap S_B \cap \{v_1 = 1\} : v_1 = 1, v_2 = \frac{p - q}{1 - \delta}.
\]
Hence

\[ |S_B| = \frac{1}{2} \cdot \left( \frac{p-q}{1-\delta} - \frac{q-1}{\delta} \right) \cdot (1 - v^*_1) \]

\[ |S_Z| = 1 - |S_A| - |S_B| \]

\[ |S_A| = \frac{1}{2} \cdot \left( (2-p)^2 - \left( \frac{p-q}{1-\delta} - (p-1) \right) \cdot (1 - v^*_1) \right). \]

Note that \((q\delta - p)/(1-\delta) < 1\) by (NonEmptyB), hence

\[ \frac{p-q}{1-\delta} = p - \frac{q\delta - p}{1-\delta} > p - 1. \]

Then the revenue is

\[ \text{REV} = |S_A| \cdot p + |S_B| \cdot q \]

\[ = \frac{1}{2} \left( (2-p)^2 \cdot p + \left( \frac{p-q}{1-\delta} - (p-1) \right) \cdot p + \left( \frac{p-q}{1-\delta} - \frac{q-1}{\delta} \right) \cdot q \right) \cdot (1 - v^*_1) \]

\[ \leq \frac{1}{2} \left( (2-p)^2 \cdot p + \left( \frac{p-q}{1-\delta} - (p-1) \right) \cdot q + \left( \frac{p-q}{1-\delta} - \frac{q-1}{\delta} \right) \cdot q \right) \cdot (1 - v^*_1) \]

\[ = \frac{1}{2} \left( (2-p)^2 \cdot p + \left( p - 1 - \frac{q-1}{\delta} \right) \cdot \left( 1 - \frac{q\delta - p}{1-\delta} \right) \cdot q \right). \]

In the meanwhile, note that by (NonEmptyA), \(v^*_2 = (p-q)/(1-\delta) < 1\), hence \(1 - v^*_1 = 1 - (p - v^*_2) < 2 - p\).

Therefore, we have

\[ \text{REV} = \frac{1}{2} \left( (2-p)^2 \cdot p + \left( p - 1 - \frac{q-1}{\delta} \right) \cdot \left( 1 - \frac{q\delta - p}{1-\delta} \right) \cdot q \right) \]

\[ \leq \frac{1}{2} \left( (2-p)^2 \cdot p + \left( p - 1 - \frac{q-1}{1} \right) \cdot (2-p) \cdot q \right) \]

\[ = (p-1/2)(-3p^2/4 + 2p - (q-p/2)^2). \]

Since \(p > 1 > 1/2\) and \(p/2 < 1 < q\), the supremum with \(q \in (1, 2]\) is reached when \(q = 1\):

\[ \text{REV} \leq (p-1/2)(-3p^2/4 + 2p - (1-p/2)^2) = (p^3 - 5p^2 + 7p - 2)/2. \]

According to the first order condition, the local maximum and local minimum of the right-hand-side is reached when \(p = 1\) and \(p = 7/3\), respectively. In other words, the supremum with \(p \in (1, 2]\) is reached when \(p = 1\):

\[ \text{REV} < (1^3 - 5 \cdot 1^2 + 7 \cdot 1 - 2)/2 = 1/2 < 59/108. \]

\[ \square \]

### B Comparison of Accuracy
Figure 11: Uniform $[0, 1]^2$ with discretization $N = 10$.

Figure 12: Uniform $[0, 1]^2$ with discretization $N = 20$. 
Figure 13: Uniform $[0, 1]^2$ with discretization $N = 30$. 

(a) Mechanism via LP

(b) Mechanism via our method