A GOLOD COMPLEX WITH NON-SUSPENSION MOMENT-ANGLE COMPLEX

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Abstract. It could be expected that the moment-angle complex associated with a Golod simplicial complex is homotopy equivalent to a suspension space. In this paper, we provide a counter example to this expectation. We have discovered this complex through the studies of the Golod property of the Alexander dual of a join of simplicial complexes, and that of a union of simplicial complexes.

1. Introduction

The Stanley-Reisner ring (or face ring) of a simplicial complex $K$ over an index set $[m] = \{1, \cdots, m\}$ is defined as the quotient graded algebra

$$k[K] = k[v_1, \cdots, v_m]/I_K,$$

where $k$ is a commutative ring with unit and $I_K = (v_{i_1} \cdots v_{i_k} | \{i_1, \cdots, i_k\} \notin K)$ is the Stanley-Reisner ideal of $K$. $K$ is called Golod over a field $k$ if its Stanley-Reisner ring $k[K]$ is Golod over $k$. That is, the multiplication and all higher Massey products in

$$\text{Tor}^{k[v_1, \cdots, v_m]}_{k[K]}(k[K], k) = H(\Lambda[u_1, \cdots, u_m] \otimes k[K], d)$$

are trivial, where the Koszul differential algebra $(\Lambda[u_1, \cdots, u_m] \otimes k[K], d)$ is the bigraded differential algebra with $\deg u_i = (1, 2)$, $\deg v_i = (0, 2)$, and $du_i = v_i$ for $i = 1, \cdots, m$. Originally, the algebra $k[K]$ or the ideal $I_K$ was defined to be Golod if the following equation holds:

$$\sum_{i \geq 0; \ j \geq 0} \dim_k \text{Tor}^{k[K]}_{j,2i}(k, k) t^j z^i = \frac{(1 + tz)^m}{1 - t \sum_{i \geq 0; \ j \geq 1} \dim_k \text{Tor}^{k[v_1, \cdots, v_m]}_{j,2i}(k[K], k) t^j z^i},$$

where $\text{Tor}^{k[K]}_{j,2i}(k, k)$ and $\text{Tor}^{k[v_1, \cdots, v_m]}_{j,2i}(k[K], k)$ denote the homogeneous components of degree $2i$. Golod [8] proved the equivalence of the two conditions, and thereafter his name has been used to refer a ring that satisfies the condition. The reader may also refer to Gulliksen and Levin [11] or Avramov [1].

Baskakov, Buchstaber, and Panov [3] and Franz [7] independently demonstrated that the torsion algebra $\text{Tor}^{k[v_1, \cdots, v_m]}(k[K], k)$ is isomorphic to the cohomology ring of the moment-angle complex $Z_K$ associated with $K$.

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Theorem 1.1([3, 7]). For a commutative ring $k$ with unit, the following isomorphisms of algebras hold:

$$H^*(Z_K; k) \cong \text{Tor}_{v_1, \ldots, v_m}^k(k[K], k) \cong \bigoplus_{I \subset [m]} \tilde{H}^*(K_I; k),$$

where $\tilde{H}^*(K_I; k)$ denotes the reduced cohomology of the full subcomplex $K_I$ of $K$ on $I$, and $\tilde{H}^*(K_\emptyset; k) = 0$ for $* \neq -1$ and $= k$ for $* = -1$. The last isomorphism is the sum of isomorphisms given by

$$H^p(Z_K; k) \cong \bigoplus_{I \subset [m]} \tilde{H}^{p-|I|-1}(K_I; k),$$

and the ring structure is given by the maps

$$\tilde{H}^{p-|I|-1}(K_I; k) \otimes \tilde{H}^{q-|J|-1}(K_J; k) \to \tilde{H}^{p+q-|I|-|J|-1}(K_{I\cup J}; k)$$

that are induced by the canonical inclusion maps $\iota_{I,J}: K_{I\cup J} \to K_I \ast K_J$ for $I \cap J = \emptyset$ and zero otherwise, where $K_I \ast K_J$ denotes the join of two simplicial complexes $K_I$ and $K_J$.

Here, we recall that if the moment-angle complex $Z_K$ is homotopy equivalent to a suspension, then the multiplication and all higher Massey products in $H^*(Z_K; k)$ are trivial. For example, see Corollary 3.11 of [23]. That is, the following implication holds:

(1.1) $Z_K$ is homotopy equivalent to a suspension $\implies K$ is Golod,

where $K$ is Golod if $K$ is Golod over any field $k$. This observation enables us to investigate the Golod property through the study of moment-angle complexes. One of the first studies in this direction was introduced by Grbić and Theriault [10]. They demonstrated that the moment-angle complex associated with a shifted simplicial complex is homotopy equivalent to a wedge of spheres. In [15], Kishimoto and the first author extended this result to dual sequentially Cohen-Macaulay complexes, and provided some new Golod complexes. In these studies, the following theorem concerning the decomposition of polyhedral products (see Definition 2.1), as introduced by Bahri, Bendersky, Cohen, and Gitler [2], plays an essential role.

Theorem 1.2([2]). Let $K$ be a simplicial complex on $[m]$ and let $(CX, X) = \{(CX_i, X_i)\}_{i \in [m]}$, where each $X_i$ is a based space and $CX$ is the reduced cone of a based space $X$. Then, the following homotopy equivalence holds:

$$\Sigma Z_K(CX, X) \cong \Sigma \bigvee_{I \subset [m]} \Sigma|K_I| \land \hat{X}^I,$$

where $\hat{X}^I = \land_{i \in I} X_i$ and $\hat{X}^\emptyset = *$.

We call this decomposition of polyhedral products the BBCG decomposition for $K$. If this decomposition is desuspendable, i.e., if the homotopy equivalence

$$Z_K(CX, X) \cong \bigvee_{I \subset [m]} \Sigma|K_I| \land \hat{X}^I$$


holds for any sequence of based CW-complexes $X$, then we say that the BBCG decomposition is \textit{desuspendable} for $K$. In particular, this implies that $Z_K$ is homotopy equivalent to a suspension.

In this paper, we study the Golod properties of the Alexander dual of $K \ast L$ and $K \cup_\alpha L$, where $\alpha$ is a common face of $K$ and $L$. The precise statements of the results are given in the next section.

By Theorem 1.1, the multiplicative structure of $H^*(Z_K; k)$ is trivial if and only if the maps $\iota_{I,J} : K_{I \cup J} \to K_I \ast K_J$ for $I \cap J = \emptyset$ induce the trivial maps on the reduced cohomology theory. By strengthening this condition, $K$ is said to be \textit{(stably) homotopy Golod} [15] if the maps $|\iota_{I,J}| : |K_{I \cup J}| \to |K_I \ast K_J|$ for $I \cap J = \emptyset$ are (stably) null homotopic and $H^*(Z_K; k)$ has trivial higher Massey products for any fields $k$. By definition, the following implication holds:

\[
\text{stably homotopy Golod} \implies \text{Golod}.
\]

The second purpose of this paper is to prove that this implication is strict.

\textbf{Theorem 1.3.} There is a Golod simplicial complex $K$ such that $K$ is not stably homotopy Golod. Moreover, $Z_K$ can be chosen to be torsion free.

Here, a space or a simplicial complex $X$ is called \textit{torsion free} if its integral homology groups $H_*(X; \mathbb{Z})$ are torsion free. In general, $Z_K$ can have torsion in its integral homology groups even if $K$ is Golod. The 6 vertex triangulation of $\mathbb{R}P^2$ is such an example, see for example [9] or Example 10.10 of [15].

It could be expected that the converse of the implication (1.1) is also true. Theorem 1.3 provides a counter example to this expectation. In fact, if $Z_K$ is homotopy equivalent to a suspension then the fat wedge filtration of $Z_K$ is trivial, by Theorem 1.3 of [15]. By Theorem 6.9 of the same paper, we see that $K$ is stably homotopy Golod, which contradicts our result. Thus, $Z_K$ is not homotopy equivalent to a suspension.

In the next section, we state the main results of this paper. The subsequent sections are devoted to their proofs.

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2. Results

In this section, we state our main results. We begin by setting notation regarding simplicial complexes.

Let $K$ be a simplicial complex on an index set $V$. In this paper, we only consider finite simplicial complexes. A facet of a simplicial complex is a maximal face with respect to the inclusion, and a subset $\sigma \subset V$ is called a minimal non-face if $\sigma \notin K$ but $\sigma - v \subset K$ for every $v \in \sigma$. The subset $V(K)$ of $V$ defined by $V(K) = \cup_{\sigma \in K} \sigma$ is called the vertex set of $K$, and an element of $V - V(K)$ is called a ghost vertex. In this paper we allow a simplicial complex to have a ghost vertex. But we assume that a simplicial complex has the simplex $\emptyset$.

For a finite set $V$, we denote the full simplex on $V$ by $\Delta^V$. Its boundary is denoted by $\partial \Delta^V$. We also use the symbol $\Delta^n$ to denote an $n$-dimensional simplex. $|K|$ denotes a geometric realization of $K$. The link and star of a face $\sigma$ of $K$ are denoted by $\text{link}_K(\sigma)$ and $\text{star}_K(\sigma)$, respectively:

$$\text{link}_K(\sigma) = \{ \tau \subset \sigma^c = V - \sigma \mid \tau \cup \sigma \in K \}, \quad \text{star}_K(\sigma) = \{ \tau \subset V \mid \tau \cup \sigma \in K \}.$$  

For a subset $I \subset V$, $K_I$ denotes the full subcomplex of $K$ indexed by $I$; that is, $K_I = \{ \sigma \subset I \mid \sigma \in K \}$. Then, $K_I$ is called the restriction of $K$ to $I$. For a vertex $v$ of $K$, the deletion of $v$ from $K$ is denoted by $K - v = K_{V - \{v\}}$. For simplicial complexes $K$ and $L$ with disjoint index sets $V$ and $W$, the simplicial join $K \ast L$ on the index set $V \sqcup W$ is defined by $K \ast L = \{ \sigma \sqcup \tau \subset V \sqcup W \mid \sigma \in K, \tau \in L \}$, where $\sqcup$ always denotes the disjoint union of sets. We write $K \ast c$ to stand for $K \ast \Delta^{|c|}$. For disjoint subsets $I, J \subset V$ the canonical inclusion map $\iota_{I,J} : K_{I \sqcup J} \to K_I \ast K_J$ mentioned in Theorem 1.1 is defined as $\iota_{I,J}(\sigma) = (\sigma \cap I) \cup (\sigma \cap J)$.

The (simplicial) Alexander dual $K^*_V$ for a simplicial complex $K$ on an index set $V$ such that $K \neq \Delta^V$ is defined by $K^*_V = \{ \sigma \subset V \mid \sigma^c \notin K \}$. If $V = V(K)$ or $V$ is clear from the context, we simply write $K^*$ for $K^*_V$. It is easy to see that for a subset $I$ of $V$, $I$ is a facet of $K$ if and only if $I^c$ is a minimal-non face of $K^*$. The restriction of $K^*$ and the dual to a link of $K$ are related by the following formula: $(K^*)_I = (\text{link}_K(I^c))^*$. 

Next we review polyhedral products, which are a generalization of moment-angle complexes.

**Definition 2.1.** Let $K$ be a simplicial complex on $[m]$, and $(X, A)$ be a sequence of pairs of based spaces $\{(X_i, A_i)\}_{i \in [m]}$. The polyhedral product $Z_K(X, A)$ is defined by

$$Z_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma \quad (\subset X_1 \times \cdots \times X_m),$$

where $(X, A)^\sigma = Y_1 \times \cdots \times Y_m$, with $Y_i = X_i$ for $i \in \sigma$ and $A_i$ for $i \notin \sigma$. If $(X_i, A_i) = (X, A)$ for all $i \in [m]$, then we write $Z_K(X, A)$ for $Z_K(X, A)$.

The polyhedral product $Z_K(D^2, S^1)$ is the moment-angle complex of $K$, and is written simply as $Z_K$. We refer the reader to [15, 5] for further examples of polyhedral products. In this paper, we are interested in the homotopy types of polyhedral products $Z_K(CX, X)$. 

If $K$ has a ghost vertex, then
\[ Z_K(X, A) = Z_{K^V(K)}(X^V(K), A^V(K)) \times \prod_{i \in V(K)^c} A_i, \]
where $X_I$ is the sub-sequence of $X$ indexed by $I \subset [m]$. In particular, the moment-angle complex $Z_K = Z_{K^V(K)} \times (S^1)^{|V(K)^c|}$. If $|V(K)^c| > 1$ or $Z_{K^V(K)}$ is not contractible, $K$ is non-Golod. When we consider Golodness of simplicial complexes, therefore, we usually restrict ourselves to simplicial complexes without ghost vertices. Nevertheless, in this paper we allow a simplicial complex to have a ghost vertex. The reason for this is that we mainly consider dual simplicial complexes as in Theorems 2.3, 2.5 and 2.7. These dual simplicial complexes do not have ghost vertices even if the initial simplicial complexes have.

Example 2.2. In [22], Porter proved the following homotopy equivalence:
\[
Z_{\partial \Delta^m}(CX, X) = \bigcup_{i=1}^{m} CX_1 \times \cdots \times CX_{i-1} \times X_i \times CX_{i+1} \times \cdots \times CX_m \\
\simeq X_1 \ast \cdots \ast X_m \\
\simeq \Sigma^{m-1} X_1 \wedge \cdots \wedge X_m.
\]
He used this homotopy equivalence to define a higher order Whitehead product.

Theorem 2.3. Let $K$ and $L$ be simplicial complexes on disjoint index sets $V$ and $W$, respectively. If $K \neq \Delta^V$ and $L \neq \Delta^W$, then the BBCG decomposition is desuspendable for $(K \ast L)^*$. 

Corollary 2.4. Let $K$ and $L$ be simplicial complexes on disjoint index sets $V$ and $W$, respectively. If $K \neq \Delta^V$ and $L \neq \Delta^W$, then $(K \ast L)^*$ is Golod by (1.1).

Because the Stanley-Reisner ideal of $(K \ast L)^*$ is the product of those of $K^*$ and $L^*$, this corollary provides a topological proof of a classical result given in [13]. The reader may also refer to [6] and [17]. We also remark that $(\Delta^V \ast L)^*$ is Golod if and only if $L^*$ is Golod, because $(\Delta^V \ast L)^* = \Delta^V \ast L^*$ (see Lemma 3.1).

Theorem 2.5. Let $K$ and $L$ be simplicial complexes of non-negative dimension on index sets $V$ and $W$, respectively. Assume that $\alpha = V \cap W$ is a common face of $K$ and $L$. If $K \neq \Delta^V$ or $L \neq \Delta^W$, and $\alpha$ is neither a facet of $K$ nor $L$, then the BBCG decomposition is desuspendable for $(K \cup_{\alpha} L)^*$.

Corollary 2.6. Let $K$ and $L$ be simplicial complexes that satisfy the same conditions as stated in Theorem 2.5. If $\alpha$ is neither a facet of $K$ nor $L$, then $(K \cup_{\alpha} L)^*$ is Golod by (1.1).

It is natural to ask whether $(K \cup_{\alpha} L)^*$ is still Golod even when $\alpha$ is a facet of $K$ or $L$ in Corollary 2.6. In general, the answer is that this does not hold. In fact, we can construct many
non-Golod simplicial complexes \((\Delta^V \cup_a L)^*\), such as in Example 5.3.4. A necessary and sufficient condition for \((K \cup_a L)^*\) being non-Golod is given by Corollary 5.3.3. Incidentally, \((\Delta^V \cup_a \Delta^W)^*\) is a non-Golod simplicial complex if \(\alpha \neq V\) and \(\alpha \neq W\), since \((\Delta^V \cup_a \Delta^W)^* = \partial \Delta^V - \alpha \ast \Delta^a \ast \partial \Delta^W - \alpha\) (see Lemma 4.1).

To construct a simplicial complex satisfying Theorem 1.3, we first need to fix some notation.

Let \(K\) be a simplicial complex on an index set \(V\), with facets \(F_1, \ldots, F_k\). We take new vertices \(v_1, \ldots, v_k\), and define a new simplicial complex \(F(K)\) on the index set \(V \cup \{v_1, \ldots, v_k\}\) with facets \(F_1 + v_1, \ldots, F_k + v_k\), where \(F_i + v_i = F_i \cup \{v_i\}\) is the disjoint union of two sets. Then, \(K\) is a subcomplex of \(F(K)\) and \(|K|\) is a deformation retract of \(|F(K)|\).

For two simplicial complexes \(K\) and \(L\), we define a “product” \(K \boxtimes L\) as follows. Define a linear order \(\leq\) on the vertex sets of \(K\) and \(L\). The vertex set of \(K \boxtimes L\) is \(V(K) \times V(L)\). An \(n\)-simplex is a set \(\{(x_0, y_0), \ldots, (x_n, y_n)\}\) such that \(x_0 \leq \cdots \leq x_n, y_0 \leq \cdots \leq y_n, \{x_0, \ldots, x_n\}\) is a simplex of \(K\), and \(\{y_0, \ldots, y_n\}\) is a simplex of \(L\). It is well-known that \(|K \boxtimes L|\) is homeomorphic to \(|K| \times |L|\). If \(v\) is a vertex of \(L\), then the subcomplex \(K \boxtimes \Delta^k\) of \(K \boxtimes L\) is abbreviated as \(K \boxtimes L\).

By \(S^n_k\) we denote a triangulation of an \(n\)-sphere \(S^n\) with \(k\)-vertices. It follows from the simplicial approximation theorem that there is a simplicial map \(\eta_k : S^n_k \to S^n_4\) for sufficiently large \(k\) whose geometrical realization \(|\eta_k| : |S^n_k| \to |S^n_4|\) is homotopic to the Hopf map \(\eta : S^n_3 \to S^2\). In fact, we can choose \(k = 12\) in this case, by [19].

We consider the simplicial set \(\Delta^1\) as the full simplex on \([2]\). By \(S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^4_2\) we denote the simplicial complex obtained from the disjoint union of two simplicial complexes \(S^3_k \boxtimes \Delta^1\) and \(S^4_2\), given by identifying \((v, 2) \in V(S^3_k) \times [2]\) with \(\eta_k(v) \in V(S^4_2)\). We embed \(S^3_k\) into \(S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^4_2\) by applying the map \(v \mapsto (v, 1)\).

We set \(V\) to be the vertex set of the union of two simplicial complexes \(S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^4_2\) and \(F(S^3_k)\) along \(S^3_k\). Finally, we take new vertices \(v_0, w_1, w_2\) and set

\[
K = \Delta^{V+w_0} \cup ((S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^4_2) \cup F(S^3_k)) \ast \Delta^{\{w_1\}} \cup S^3_k \ast \Delta^{\{w_1,w_2\}},
\]

which is the union of \(\Delta^{V+w_0}\) and \(\Delta^V \cup ((S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^4_2) \cup F(S^3_k)) \ast \Delta^{\{w_1\}} \cup S^3_k \ast \Delta^{\{w_1,w_2\}}\).

**Theorem 2.7.** \(K^*\) is a Golod simplicial complex that is not stably homotopy Golod. Moreover, \(Z_{K^*}\) is torsion free if \(K\) is constructed from the map \(\eta_{12} : S^3_{12} \to S^2_4\) defined in [19].

3. **Proof of Theorem 2.3.**

In this section, we prove Theorem 2.3. We begin by stating some elementary lemmas, for which the proofs are omitted.

**Lemma 3.1.** Let \(K\) and \(L\) be simplicial complexes with disjoint index sets \(V\) and \(W\), respectively. Then, \((K \ast L)^*_{V \sqcup W} = K^*_V \ast \Delta^W \cup \Delta^V \ast L^*_W\).
Lemma 3.2. Let $K$ and $L$ be simplicial complexes with disjoint index sets, and let $K_i$ and $L_i$ for $i = 1, 2$ be subcomplexes of $K$ and $L$, respectively. Then,

$$(K_1 \ast L_1) \cap (K_2 \ast L_2) = (K_1 \cap K_2) \ast (L_1 \cap L_2).$$

Lemma 3.3. Let $K$ be a simplicial complex with two subcomplexes $K_1$ and $K_2$. If $K = K_1 \cup K_2$, then

$$Z_K(CX, X) = Z_{K_1}(CX, X) \cup Z_{K_2}(CX, X)$$

and

$$Z_{K_1}(CX, X) \cap Z_{K_2}(CX, X) = Z_{K_1 \cap K_2}(CX, X).$$

Lemma 3.4. Let $K$ and $L$ be simplicial complexes with disjoint index sets $V$ and $W$, respectively. Then,

$$Z_{K \ast L}(CX, X) = Z_K(CX_V, X_V) \times Z_L(CX_W, X_W).$$

Proposition 3.5. Let $K$ be a simplicial complex on $[m]$ and $X$ be a sequence of based CW-complexes. If $Z_K(CX, X)$ is a simply connected co-H-space, then

$$Z_K(CX, X) \simeq \bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I.$$  

Proof. First, we show that $\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I$ is also simply connected. By Theorem 1.2 we have that $H_1(\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I) \cong H_1(Z_K(CX, X)) = 0$. Because $\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I$ is a suspension, its fundamental group is a free group. Thus, $\pi_1(\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I) = 0$.

For a subset $I \subset [m]$, the canonical projection $p_I : \prod_{i \in [m]} CX_i \to \prod_{i \in I} CX_i$ induces a map $Z_K(CX, X) \to Z_{K_I}(CX_I, X_I)$, which is also denoted by $p_I$. Let $I = \{i_1, \ldots, i_k\} \subset [m]$. We define a subset $Z_{K_I}(CX_I, X_I)'$ of $Z_{K_I}(CX_I, X_I)$ by the equation

$$Z_{K_I}(CX_I, X_I)' = \{(x_{i_1}, \ldots, x_{i_k}) \in Z_{K_I}(CX_I, X_I) \mid x_{i_j} = \ast \text{ for some } j\}.$$  

In [15], it is shown that $Z_{K_I}(CX_I, X_I)/Z_{K_I}(CX_I, X_I)' \simeq \Sigma|K_I| \wedge \hat{X}^I$. Now, we consider the composite of maps

$$f : Z_K(CX, X) \xrightarrow{2^m} \bigvee_{I \subset [m]} Z_K(CX, X) \xrightarrow{\cup_{I \subset [m]} p_I} \bigvee_{I \subset [m]} Z_{K_I}(CX_I, X_I)$$

$$\xrightarrow{\cup_{I \subset [m]} p_I} \bigvee_{I \subset [m]} Z_{K_I}(CX_I, X_I)/Z_{K_I}(CX_I, X_I)' \simeq \bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{X}^I,$$
where the first map is the iterated co-multiplication of $Z_K(CX, X)$. Because $Z_K(CX, X)$ and $\bigvee_{I \subseteq [m]} \Sigma |K_I| \wedge X^I$ are simply connected CW-complexes, to prove that $f$ is homotopy equivalent it suffices to show that $f$ induces a homology isomorphism. In [16], it is shown that $\Sigma f$ is a homotopy equivalence. In particular, $f$ induces a homology isomorphism, and thus we complete the proof.

Proof of Theorem 2.3. By Lemmas 3.1 and 3.2, we have that $(K*L)^* = K^* \Delta^W \cup \Delta^V * L^*$ and $K^* \Delta^{V(L)} \cap \Delta^{V(\ast)} * L^* = K^* * L^*$. Therefore, from Lemma 3.3 we obtain the following push-out diagram of spaces:

$$
\begin{array}{ccc}
Z_K^* & \longrightarrow & Z_{K^* \Delta^w}(CX, X) \\
\downarrow & & \downarrow \\
Z_{\Delta^V L^*}(CX, X) & \longrightarrow & Z_{(K*L)^*}(CX, X).
\end{array}
$$

Here, we remark that $K^* * L^*$ and $K^* \Delta^W$ are non-void simplicial complexes, because we assume that $K \neq \Delta^V$ and $L \neq \Delta^W$. By Lemma 3.4, the above push-out diagram is equivalent to the following push-out diagram:

$$
\begin{array}{ccc}
Z_K^*(CX_V, X_V) \times Z_L^*(CX_W, X_W) & \longrightarrow & Z_K^*(CX_V, X_V) \times \prod_{w \in W} CX_w \\
\downarrow & & \downarrow \\
\prod_{v \in V} CX_v \times Z_L^*(CX_W, X_W) & \longrightarrow & Z_{(K*L)^*}(CX, X).
\end{array}
$$

Because $\prod_{v \in V} CX_v$ and $\prod_{w \in W} CX_w$ are contractible, the above diagram yields the following homotopy equivalences:

$$
Z_{(K*L)^*}(CX, X) \simeq Z_K^*(CX_V, X_V) * Z_L^*(CX_W, X_W) \\ \simeq \Sigma Z_K^*(CX_V, X_V) \wedge Z_L^*(CX_W, X_W).
$$

By Theorem 1.2, $\Sigma Z_K^*(CX_V, X_V)$ is a double suspension, which implies that $Z_{(K*L)^*}(CX, X)$ is also a double suspension. By invoking Proposition 3.5, we complete the proof.

4. Proof of Theorem 2.5.

In this section, we prove Theorem 2.5. Again, we begin by stating some elementary lemmas.

Lemma 4.1. Let $K$ and $L$ be simplicial complexes without ghost vertices on index sets $V$ and $W$, respectively. Let $\alpha = V \cap W$ be a common face of $K$ and $L$. Then,

$$(K \cup_\alpha L)^* = (\partial \Delta^{V-\alpha} * \Delta^\alpha * \partial \Delta^{W-\alpha}) \cup (K^* * \Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^*).$$

Proof. For $u \in V - \alpha$ and $v \in W - \alpha$, $\{u, v\}$ is a minimal non-face of $K \cup_\alpha L$, and a minimal non-face of $K$ or $L$ is also a minimal non-face of $K \cup_\alpha L$. For any $u \in V - \alpha$ and $v \in W - \alpha$, 


$V \cup W - \{u, v\}$ is a facet of $(K \cup \alpha L)^*$. Furthermore, for any minimal non-face $\sigma$ of $K$ or $L$, $V \cup W - \sigma$ is a facet of $(K \cup \alpha L)^*$. This implies the desired equality of the two complexes. □

Lemma 4.2. If $K$ is a simplicial complex with a ghost vertex, then the BBCG decomposition for $K^*$ is desuspendable. In particular, $K^*$ is Golod.

Proof. Let $K$ be a simplicial complex on $[m]$ and $v$ be a ghost vertex of $K$. Then, $[m] - v$ is a facet of $K^*$, and thus $\dim K^* \geq m - 2$. Then, it follows from Theorem 1.2 and Proposition 3.5 of [15] that the BBCG decomposition for $K^*$ is desuspendable. □

Lemma 4.3. Let $\alpha$ be a face of a simplicial complex $K$ on an index set $V$. If $K \neq \Delta^V$ and $\alpha$ is not a facet of $K$, then the inclusion map

$$Z_{K^*}(CX, X) \to Z_{(\Delta^\alpha)^*}(CX, X) = Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$$

is null homotopic.

Proof. Because $\alpha$ is not a facet of $K$, there is a face $\beta$ of $K$ such that $\alpha \not\subseteq \beta$. Then, $\Delta^\alpha \not\subseteq \Delta^\beta \subset K$, which implies that $K^* \subset (\Delta^\beta)^* \subset (\Delta^\alpha)^*$. That is, $K^* \subset \partial\Delta^V - \beta*\Delta^\beta \subset \partial\Delta^V - \alpha*\Delta^\alpha$. Therefore, the inclusion $Z_{K^*}(CX, X) \hookrightarrow Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$ factors as $Z_K(CX, X) \to Z_{\partial\Delta^V - \beta*\Delta^\beta}(CX, X) \to Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$. To show that the inclusion $Z_K(CX, X) \to Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$ is null homotopic, it is sufficient to show that $Z_{\partial\Delta^V - \beta*\Delta^\beta}(CX, X) \to Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$ is null homotopic. We have the following homotopy commutative diagram

$$Z_{\partial\Delta^V - \beta*\Delta^\beta}(CX, X) = Z_{\partial\Delta^V - \beta}(CX_{V - \beta}, X_{V - \beta}) \times \prod_{j \in \beta} CX_j \xrightarrow{\simeq} \prod_{i \in V - \beta} X_i,$$

$$Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X) = Z_{\partial\Delta^V - \alpha}(CX_{V - \alpha}, X_{V - \alpha}) \times \prod_{j \in \alpha} CX_j \xrightarrow{\simeq} \prod_{i \in V - \alpha} X_i \times \prod_{i \in \beta - \alpha} X_i,$$

where the horizontal maps are homotopy equivalences by Example 2.2 and the right vertical map is the canonical inclusion $X \to X \ast Y$. Since this map is null homotopic, $Z_{\partial\Delta^V - \beta*\Delta^\beta}(CX, X) \to Z_{\partial\Delta^V - \alpha*\Delta^\alpha}(CX, X)$ is also null homotopic, and thus we complete the proof. □

In addition to the above, we require the following lemma to prove Theorem 2.5.

Lemma 4.4 (Lemma 3.2 of [14] with $B = \ast$). Define $Q$ as the push-out

$$\begin{array}{ccc}
A \times C & \xrightarrow{1 \times 1} & CA \times C \\
\downarrow 1 \times \ast & & \downarrow \\
A \times D & \xrightarrow{i} & Q,
\end{array}$$

where $i : A \to CA$ is the inclusion. Then, the homotopy equivalence

$$Q \xrightarrow{\simeq} (A \times D) \vee \Sigma(A \wedge C)$$
holds, which is natural with respect to \( A, C, \) and \( D \), where \( X \times Y = (X \times Y) / (X \times \ast) \). Moreover, the inclusion map \( j : A \times D \to Q \) is homotopic to the following composite of maps
\[
A \times D \to A \times D \to (A \times D) \vee \Sigma(A \wedge C) \simeq Q,
\]
where the first map is the collapsing map and the second map is the inclusion.

**Proof of Theorem 2.5.** If \( K \) or \( L \) has a ghost vertex, then \( K \cup_a L \) also has a ghost vertex. In this case, the BBCG decomposition is desuspendable, by Lemma 4.2. Therefore, we assume that \( K \) and \( L \) do not have any ghost vertices. In the following proof, \( Z_K(CX, X) \) is abbreviated as \( Z^K \).

First, we will show that if \( K = \Delta^V \) and \( L \neq \Delta^W \), then
\[
Z_{(\Delta^V \cup_a L)^*} \simeq (Z_{\partial \Delta^V \ast} \times Z_{\partial \Delta^W \ast}) \vee (Z_{\partial \Delta^V \ast} \wedge Z_{L^*}).
\]
Here we remark that we need the assumption that \( L \neq \Delta^W \) and that \( \alpha \) is not a facet of \( L \) to apply Lemma 4.3.

It follows from Lemma 4.1 that \( (\Delta^V \cup_a L)^* = (\partial \Delta^V \ast \Delta^a \ast \partial \Delta^W \ast) \cup (\Delta^V \ast L^*) \). Then, by Lemma 3.3 we have the push-out diagram of spaces
\[
\begin{array}{ccc}
Z_{\partial \Delta^V \ast} \times Z_{L^*} & \longrightarrow & Z_{\Delta^V \ast} \times Z_{L^*} \\
\downarrow & & \downarrow \\
Z_{\partial \Delta^V \ast} \times Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} & \longrightarrow & Z_{(\Delta^V \cup_a L)^*},
\end{array}
\]
which by Lemma 3.4 is equivalent to the following push-out diagram:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V \ast} \times Z_{L^*} & \longrightarrow & Z_{\Delta^V \ast} \times Z_{L^*} \\
\downarrow & & \downarrow \\
Z_{\partial \Delta^V \ast} \times Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} & \longrightarrow & Z_{(\Delta^V \cup_a L)^*}.
\end{array}
\]
Since we assumed that \( L \neq \Delta^W \) and that \( \alpha \) is not a facet of \( L \), by Lemma 4.3 the inclusion map \( Z_{L^*} \hookrightarrow Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} \) is null-homotopic. Therefore, \( Z_{(\Delta^V \cup_a L)^*} \) is homotopy equivalent to the push-out \( P \) of the following diagram:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V \ast} \times Z_{L^*} & \longrightarrow & Z_{\Delta^V \ast} \times Z_{L^*} \\
\downarrow & & \downarrow \\
Z_{\partial \Delta^V \ast} \times Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} & \longrightarrow & P,
\end{array}
\]
(4.1)
\[
\begin{array}{ccc}
Z_{\partial \Delta^V \ast} \times Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} & \longrightarrow & P.
\end{array}
\]
Since \( Z_{\Delta^V \ast} = CZ_{\partial \Delta^V \ast} \), by Lemma 4.4 \( P \) is homotopy equivalent to
\[
(Z_{\partial \Delta^V \ast} \times (Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast})) \vee (Z_{\partial \Delta^V \ast} \wedge Z_{L^*}) \simeq (Z_{\partial \Delta^V \ast} \times Z_{\partial \Delta^W \ast}) \vee (Z_{\partial \Delta^V \ast} \wedge Z_{L^*}),
\]
and \( j \) in the diagram (4.1) can be identified with the following composite of canonical maps:
\[
(4.2)
\begin{align*}
Z_{\partial \Delta^V \ast} \times Z_{\Delta^a \ast} \times Z_{\partial \Delta^W \ast} & \longrightarrow Z_{\partial \Delta^V \ast} \times Z_{\partial \Delta^W \ast} \\
Z_{\partial \Delta^V \ast} \times Z_{\partial \Delta^W \ast} & \hookrightarrow (Z_{\partial \Delta^V \ast} \times Z_{\partial \Delta^W \ast}) \vee (Z_{\partial \Delta^V \ast} \wedge Z_{L^*}).
\end{align*}
\]
Thus, the following homotopy equivalence holds:

\[ Z_{(\Delta V \cup_a L)^*} \simeq (Z_{\partial \Delta V-a} \times Z_{\partial \Delta W-a}) \lor \Sigma Z_{\partial \Delta V-a} \land Z_{L^*}. \]

This homotopy equivalence induces the following homotopy equivalences:

\[
Z_{(\Delta V \cup_a L)^*} \simeq (Z_{\partial \Delta V-a} \times Z_{\partial \Delta W-a}) \lor \Sigma Z_{\partial \Delta V-a} \land Z_{L^*} \\
\simeq Z_{\partial \Delta W-a} \lor (Z_{\partial \Delta V-a} \land Z_{\partial \Delta W-a}) \lor \Sigma Z_{\partial \Delta V-a} \land Z_{L^*} \\
\simeq Z_{\partial \Delta W-a} \lor (Z_{\partial \Delta V-a} \land Z_{\partial \Delta W-a}) \lor \sum_{J \subseteq W} \Sigma Z_{\partial \Delta V-a} \land \Sigma |(L)^*| \lor \hat{X}^{V-a+J} \\
\simeq \bigvee_{I \subseteq [m]} \Sigma |((\Delta V \cup_a L)^*)_I| \lor \hat{X}^I,
\]

where in the third homotopy equivalence we used the BBCG decomposition for \( \Sigma Z_{L^*} \). To show that the last homotopy equivalence holds we need the homotopy type of \(|((\Delta V \cup_a L)^*)_I| \) for \( I \subseteq [m] \). Since \( (\Delta V \cup_a L)^* = (\partial \Delta V-a \lor \Delta^a \lor \partial \Delta W-a) \lor (\Delta V-a \land L^*) \), we have the following push-out diagram for \( I = J' \cup J \) with \( J' \subset V - \alpha \) and \( J \subset W \):

\[
\begin{array}{ccc}
|\partial \Delta V-a|_{J'} \lor (L)^*_{J} & \longrightarrow & |\Delta V-a|_{J'} \lor (L)^*_{J} \\
\downarrow & & \downarrow \\
|\partial \Delta V-a|_{J'} \lor (\Delta^a \lor \partial \Delta W-a)_{J} & \longrightarrow & |((\Delta V \cup_a L)^*)_I|
\end{array}
\]

Since \( \alpha \) is not a facet of \( L \), there is a face \( \beta \) of \( L \) and we have inclusions \( \Delta^a \subseteq \Delta^\beta \subseteq L \). This implies that \( |(L)^*_{J}| \subseteq |(\Delta^\beta \lor \partial \Delta W-\beta)_{J}| \subseteq |(\Delta^a \lor \partial \Delta W-a)_{J}| \). Since the inclusion map \(|(\Delta^\beta \lor \partial \Delta W-\beta)_{J}| \hookrightarrow |(\Delta^a \lor \partial \Delta W-a)_{J}| \) is null homotopic, the left vertical map of the diagram above is null homotopic. Thus we have

\[
|((\Delta V \cup_a L)^*)_I| \simeq |\partial \Delta V-a|_{J'} \lor (\Delta^a \lor \partial \Delta W-a)_{J} \lor |(\Delta V-a)_{J'} \lor (L)^*_{J}| / |\partial \Delta V-a|_{J'} \lor (L)^*_{J}|
\]

If \( J' \neq \emptyset, V - \alpha \), then \(|\partial \Delta V-a|_{J'}| \) is contractible, and \(|((\Delta V \cup_a L)^*)_I| \) is also contractible. If \( J' = \emptyset \), then \(|((\Delta V \cup_a L)^*)_I| = |(\Delta^a \lor \partial \Delta W-a)_{J}| \) which is \(|\partial \Delta W-a| \) if \( J = W - \alpha \), and is contractible otherwise. Finally we obtain the following homotopy equivalence, where \(|\partial \Delta V-a \lor (\partial \Delta W-a)_{J}| \) is contractible unless \( J = W - \alpha \).

\[
|((\Delta V \cup_a L)^*)_I| \simeq \\
\begin{cases} 
|\partial \Delta W-a| & \text{for } I = W - \alpha, \\
|\partial \Delta V-a \lor (\partial \Delta W-a)_{J}| \lor \sum |\partial \Delta V-a \lor (L)^*_{J}| & \text{for } I = (V - \alpha) \cup J \text{ for some } J \subset W, \\
\end{cases}
\]

Here, we remark that we can apply Proposition 3.5 if \(|W - \alpha| \geq 3|.
Similarly, if $K \neq \Delta^V$ and $L = \Delta^W$ then we have
\[ Z_{(K \cup \partial \Delta^W)} \simeq (Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{K^*} \times Z_{\partial \Delta^W}) \simeq \bigvee_{I \subseteq [m]} \Sigma((K \cup \alpha \Delta^W)^* I) \land \hat{X}^I. \]

Next, we consider the case that $K \neq \Delta^V$ and $L \neq \Delta^W$. Then, we have the push-out diagram
\[
\begin{array}{ccc}
(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) & \longrightarrow & (\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (\Delta^V \times L^*) \\
\downarrow & & \downarrow \\
(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (K^* \times \Delta^W) & \longrightarrow & (K \cup \alpha L)^*
\end{array}
\]
which induces the following push-out diagram of spaces by Lemmas 3.3 and 3.4:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V} \times Z_{\Delta^a} \times Z_{\partial \Delta^W} & \longrightarrow & Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (\Delta^V \times L^*)} \\
\downarrow & & \downarrow \\
Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (K^* \times \Delta^W)} & \longrightarrow & Z_{(K \cup \alpha L)^*}.
\end{array}
\]
Because the top arrow of the diagram above is a closed cofibration, $Z_{(K \cup \alpha L)^*}$ is homotopy equivalent to the homotopy push-out of the following diagram:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V} \times Z_{\Delta^a} \times Z_{\partial \Delta^W} & \longrightarrow & Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (\Delta^V \times L^*)} \\
\downarrow & & \downarrow \\
Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (K^* \times \Delta^W)} & \longrightarrow & Z_{(K \cup \alpha L)^*}
\end{array}
\]
Since we have the following homotopy equivalences
\[
\begin{align*}
Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (\Delta^V \times L^*)} \simeq & \ (Z_{\partial \Delta^V} \times (Z_{\Delta^a} \times Z_{\partial \Delta^W})) \sqcup \Sigma(Z_{\partial \Delta^V} \land Z_{L^*}) \\
\simeq & \ (Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{\partial \Delta^V} \land Z_{L^*}), \\
Z_{(\partial \Delta^V \times \Delta^a \times \partial \Delta^W) \cup (K^* \times \Delta^W)} \simeq & \ ((Z_{\partial \Delta^V} \times Z_{\Delta^a}) \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{K^*} \land Z_{\partial \Delta^W}) \\
\simeq & \ (Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{K^*} \land Z_{\partial \Delta^W}),
\end{align*}
\]
and homotopy push-out preserves homotopy equivalences, $Z_{(K \cup \alpha L)^*}$ is homotopy equivalent to the homotopy push-out of the following diagram:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V} \times Z_{\Delta^a} \times Z_{\partial \Delta^W} & \longrightarrow & (Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{\partial \Delta^V} \land Z_{L^*}) \\
\downarrow_{j_1} & & \downarrow \\
(Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}) \sqcup \Sigma(Z_{K^*} \land Z_{\partial \Delta^W})
\end{array}
\]
where $j_1$ and $j_2$ are as in (4.2). It follows that
\[
Z_{(K \cup \alpha L)^*} \simeq Q \sqcup \Sigma(Z_{\partial \Delta^V} \land Z_{L^*}) \sqcup \Sigma(Z_{K^*} \land Z_{\partial \Delta^W}),
\]
where $Q$ is the homotopy push-out of the following diagram:
\[
\begin{array}{ccc}
Z_{\partial \Delta^V} \times Z_{\partial \Delta^W} & \longrightarrow & Z_{\partial \Delta^V} \times Z_{\partial \Delta^W} \\
\downarrow & & \downarrow \\
Z_{\partial \Delta^V} \times Z_{\partial \Delta^W}
\end{array}
\]
where the vertical and horizontal maps are the canonical collapsing maps. Then, it is easy to see that $Q \simeq Z_{\partial \Delta^v} \wedge Z_{\partial \Delta^w}$, and we have obtained a homotopy equivalence

$$(4.3) \quad Z_{(K \cup_a L)^*} \simeq (Z_{\partial \Delta^v} \wedge Z_{\partial \Delta^w}) \vee \Sigma(Z_{\partial \Delta^v} \wedge Z_{L^*}) \vee \Sigma(Z_{K^*} \wedge Z_{\partial \Delta^w}).$$

To complete the proof, we now apply Proposition 3.5. Clearly, (4.3) implies that $Z_{(K \cup_a L)^*}$ is a suspension space. Because $\alpha$ is neither a facet of $K$ nor $L$, we have that $|V - \alpha| \geq 2$ and $|W - \alpha| \geq 2$. Then, by Example 2.2 we have that $Z_{\partial \Delta^v} = Z_{\partial \Delta^v}(C_X, X)$ and $Z_{\partial \Delta^w} = Z_{\partial \Delta^w}(C_X, X)$ are suspensions. Thus, (4.3) implies that $Z_{(K \cup_a L)^*}$ is a double suspension, and by Proposition 3.5 we have the desired homotopy equivalence

$$Z_{(K \cup_a L)^*}(C_X, X) \simeq \bigvee_{I \subseteq [m]} \Sigma((K \cup_a L)^*)_I \wedge \hat{X}^I$$

and we complete the proof. \qed

5. Preliminaries for the proof of Theorem 2.7

In this section, we review the Alexander duality and elementary collapses of simplicial complexes, and the Massey products of the Koszul homology.

5.1. Alexander duality. First, we review the Alexander duality of simplicial complexes. Although an elementary proof has been derived [4], we follow the classical argument presented in chapter six of Spanier’s book [24].

Let $M$ be a simplicial complex that is a subcomplex of $\partial \Delta^{n+1}$, and let $L$ be a subcomplex of $M$. We denote the Alexander dual of $L$ and $M$ in $V(\partial \Delta^{n+1})$ by $L^*$, $M^*$. Then, $M^*$ is a subcomplex of $L^*$, and the following duality is well-known, where we write $S^n = |\partial \Delta^{n+1}|$:

$$\gamma: H_q(|L^*|, |M^*|) \cong H_q(|\text{Sd}L^*|, |\text{Sd}M^*|) \cong H_q(|\bar{L}|, |\bar{M}|) \cong H_q(S^n - |L|, S^n - |M|) \cong H^{n-q}(|M|, |L|)$$

where $\text{Sd}K$ denotes the barycentric subdivision of $K$, $\bar{L}$ is the supplement of $L$ in $\partial \Delta^{n+1}$ defined in Definition 2.5.18 of [20], and the last isomorphism is the (topological) duality

$$\gamma_U: H_q(S^n - |L|, S^n - |M|) \cong H^{n-q}(|M|, |L|)$$

induced by an orientation class $U \in H^n(S^n \times S^n, S^n \times S^n - \Delta(S^n))$ of $S^n$, where $\Delta(S^n)$ is the diagonal set of $S^n \times S^n$.

Now, we consider the cohomology theory with coefficient $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and the relation between the duality map $\gamma: H_q(|L^*|, |M^*|; \mathbb{Z}_2) \to H^{n-q}(|M|, |L|; \mathbb{Z}_2)$ and the Steenrod squaring operations $Sq^i$. For a pair of finite complexes $(X, A)$, the Steenrod squaring operations $Sq^i$ on the homology theory are defined as follows:

$$< x, Sq^i a > = < Sq^i x, a > \quad \text{for} \ x \in H^q(X, A; \mathbb{Z}_2) \text{and} \ a \in H_{q+i}(X, A; \mathbb{Z}_2),$$

where $< -, - >$ denotes the Kronecker product.
Lemma 5.1.1. For $a \in H_q(|L^*|, |M^*|)$ and $i > 0$, we have that
\[
\gamma(Sq^i a) = \sum_{k=0}^{i-1} Sq^{i-k}(\gamma(Sq^k a)).
\]

Proof. In this proof, we omit the coefficient ring $\mathbb{Z}_2$ in the (co)homology theory. First, we remark that $Sq^i(U) = 0$ for $i > 0$. This follows from the fact that the natural restriction map $H^n(S^n \times S^n, S^n \times S^n - \Delta(S^n)) \to \tilde{H}^n(S^n \times S^n)$ is monomorphic. In the definition of the duality map $\gamma$, all maps except for $\gamma_U$ commute with the squaring operations. Therefore, it suffices to prove the corresponding formula for $\gamma_U$.

Because $\gamma_U(Sq^i a) \in H^{n-q+i}(|M|, |L|)$, we take an element $b \in H_{n-q+i}(|M|, |L|)$ and compute the Kronecker product $\langle \gamma_U(Sq^i a), b \rangle$. Let
\[
j : (|M|, |L|) \times (S^n - |L|, S^n - |M|) \to (S^n \times S^n, S^n \times S^n - \Delta(S^n))
\]
be the inclusion map. Then,
\[
\langle \gamma_U(Sq^i a), b \rangle = \langle j^*(U)/Sq^i a, b \rangle = \langle j^*(U), b \times Sq^i a \rangle
\]
\[
= \langle j^*(U), Sq^i(b \times a) - \sum_{k=0}^{i-1} Sq^{i-k}b \times Sq^k a \rangle
\]
\[
= \langle Sq^i j^*(U), b \times a \rangle + \sum_{k=0}^{i-1} \langle j^*(U), Sq^{i-k}b \times Sq^k a \rangle
\]
\[
= \sum_{k=0}^{i-1} \langle Sq^i j^*(U)/Sq^k a, Sq^{i-k}b \rangle
\]
\[
= \langle \sum_{k=0}^{i-1} Sq^{i-k}(\gamma_U(Sq^k a)), b \rangle,
\]
where in the fifth equation we use the fact that $Sq^i j^*(U) = j^* Sq^i(U) = 0$. Thus, the proof is completed. \( \square \)

5.2. Elementary collapse. In this subsection, we briefly review the notion of elementary collapse, which is necessary to prove Theorem 2.7.

Definition 5.2.1. A non-empty face $\sigma$ in a simplicial complex $K$ is a free face if it is not a facet of $K$ and is contained in exactly one facet of $K$.

An elementary collapse of $K$ is a simplicial complex $K'$ obtained from $K$ by the removal of a free face $\sigma$, along with all faces that contain $\sigma$. If there is a sequence of elementary collapses leading from $K$ to $K'$, we say that $K$ is collapsible onto $K'$, and we use the notation $K \searrow K'$. Then $|K'|$ is a deformation retract of $|K|$.
Example 5.2.2. Let $\Delta^m$ denote the full simplex on the vertex set $[m+1]$. There exists a sequence of elementary collapses of $\Delta^m \times \Delta^1$ given by removing pairs of faces from the top to bottom in the following list:

$$\{1 \cdots m+1, 1 \cdots m+1 \overline{m+1}\},$$

$$\cdots$$

$$\{1 \cdots \overline{i} + 1 \cdots m+1, 1 \cdots \overline{i} + 1 \cdots m+1\},$$

$$\cdots$$

$$\{1^{\overline{2}} \cdots m+1, 1^{\overline{2}} \cdots m+1\},$$

where $\overline{i} = (i, 1)$, $\overline{i} = (i, 2)$, and $1 \cdots \overline{i} + 1 \cdots m+1$ denotes, for example, the face $\{1, \cdots, \overline{i}, \overline{i} + 1, \cdots, m+1\}$ of $\Delta^m \times \Delta^1$. Thus, we see that $\Delta^m \times \Delta^1 \setminus (\partial \Delta^m \times \Delta^1) \cup (\Delta^m \times 2)$. By repeating the same process for smaller simplicies, we see that $\Delta^m \times \Delta^1 \setminus \Delta^m \times 2$. This can be applied to any simplicial complex $K$, and we see that $K \times \Delta^1$ is collapsible onto $K \times 2$.

5.3. Massey product in $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k)$. Finally, we review the Massey products in $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k)$, according to Section 3.2 of Buchstaber and Panov’s book [5]. Here, we remark that we adopt the different conventions for the grading of $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k)$.

Recall that the torsion algebra $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k)$ is defined as the homology group of the Koszul differential algebra $(\Lambda[u_1, \cdots, u_m] \otimes k[K], d)$, where their bigrading and the differentials are defined by $\deg u_i = (1, 2)$, $\deg v_i = (0, 2)$, and $du_i = v_i$, for $i = 1, \cdots, m$.

A multigrading of the torsion algebra $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k)$ is defined by setting

$$m \deg u_1^{\varepsilon_1} \cdots u_m^{\varepsilon_m} v_1^{i_1} \cdots v_m^{i_m} = (\varepsilon_1 + \cdots + \varepsilon_m, 2(i_1 + \varepsilon_1), \cdots, 2(i_m + \varepsilon_m)),$$

and it is easy to see that

$$\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k) = \bigoplus_{a \in \mathbb{N}^m} \text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k).$$

A subset $I \subset [m]$ may be viewed as a $(0, 1)$-vector in $\mathbb{N}^m$, whose $i$-th coordinate is 1 if $i \in I$ and 0 otherwise. Then, the following multigraded version of Hochster’s formula holds.

Theorem 5.3.1 (Theorem 3.2.9 of [5]). For any subset $I \subset [m]$, we have that

$$\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k) \cong \tilde{H}^{[I]}[-i-1](K_I; k),$$

and $\text{Tor}^k_{[v_1, \cdots, v_m]}(k[K], k) = 0$ unless $a$ is a $(0, 1)$-vector.

We recall the definition of the Massey product. Let $n$ be an integer greater than 1. Define $\tilde{a}$ to be $(-1)^{\deg a+1}a$. Then, the $n$-fold Massey product $\langle a_1, \cdots, a_n \rangle$ is defined to be the set of all homology classes represented by elements of the form

$$a_{1,n} = \tilde{a}_{1,1}a_{2,n} + \tilde{a}_{1,2}a_{3,n} + \cdots + \tilde{a}_{1,n-1}a_{n,n},$$
for all solutions of the equations
\[
a_i = [a_{i,i}] \quad \text{for } 1 \leq i \leq n,
\]
\[
da_{i,j} = \bar{a}_{i,i} a_{i+1,j} + \bar{a}_{i,i+1} a_{i+2,j} + \cdots + \bar{a}_{i,j-1} a_{j,j} \quad \text{for } 1 \leq i < j \leq n, (i, j) \neq (1, n).
\]

By inductive argument it is easy to see that if \(a_i \in \text{Tor}_{k_i, 2a_i}^{[v_1, \ldots, v_m]}(k[K], k)\), then
\[
a_{i,j} \in \text{Tor}_{k_i + \cdots + k_{j+(j-i)}, 2(a_i + \cdots + a_j)}^{[v_1, \ldots, v_m]}(k[K], k) \quad \text{for } 1 \leq i \leq j \leq n, (i, j) \neq (1, n),
\]
and therefore,
\[
\langle a_1, \ldots, a_n \rangle \subset \text{Tor}_{k_1 + \cdots + k_n + (n-2), 2(a_1 + \cdots + a_n)}^{[v_1, \ldots, v_m]}(k[K], k).
\]
The Massey product \(\langle a_1, \ldots, a_n \rangle\) is said to be trivial if it contains the zero-element.

**Proposition 5.3.2.** If an \(n\)-fold Massey product \(\langle a_1, \ldots, a_n \rangle\) is defined for \(a_i \in \text{Tor}_{k_i, 2a_i}^{[v_1, \ldots, v_m]}(k[K], k)\), and \(I_k \cap I_\ell \neq \emptyset\) for some \(k \neq \ell\), then \(\langle a_1, \ldots, a_n \rangle\) is trivial.

**Proof.** Let \(n \geq 2\), and consider an \(n\)-fold Massey product. As is observed above we have
\[
\langle a_1, \ldots, a_n \rangle \subset \text{Tor}_{k_i + \cdots + k_{n+(n-2)}, 2(a_1 + \cdots + a_n)}^{[v_1, \ldots, v_m]}(k[K], k).
\]
If \(I_k \cap I_\ell \neq \emptyset\) for some \(k \neq \ell\), then \(I_1 + \cdots + I_n\) is not a \((0, 1)\)-vector. Therefore, it follows from Theorem 5.3.1 that \(\text{Tor}_{k_i, 2(a_1 + \cdots + a_n)}^{[v_1, \ldots, v_m]}(k[K], k) = 0\). Thus, we complete the proof.

**Corollary 5.3.3.** Let \(K = K_1 \cup_0 K_2\) be a simplicial complex on \([m]\), which is obtained from two simplicial complexes \(K_1\) and \(K_2\) by gluing along a common simplex \(\alpha\) that is neither equal to \(V(K_1)\) nor \(V(K_2)\). Then, \(K^*\) is non-Golod over a field \(k\) if and only if there are faces \(\sigma\) and \(\tau\) of \(K\) satisfying the following conditions:

1. \(V(K_1) - \alpha \subset \sigma, V(K_2) - \alpha \subset \tau,\) and \(\sigma \cup \tau = [m]\),
2. the inclusion map
\[
\text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \sigma \cap \tau) \rightarrow \text{link}_K(\sigma \cap \tau)
\]
induces a non-trivial map in the homology theory with coefficients \(k\).

**Proof.** First, we show that Massey products for \(n > 2\) in \(\text{Tor}_{k_i, 2a_i}^{[v_1, \ldots, v_m]}(k[K^*], k)\) are trivial for any simplicial complex \(K = K_1 \cup_0 K_2\). This reduces the problem to proving that a non-trivial cup product exists if and only if the conditions (1) and (2) hold.

Let \(a_i \in \text{Tor}_{k_i, 2a_i}^{[v_1, \ldots, v_m]}(k[K^*], k)\) for \(i = 1, \ldots, n\), where \(n > 2\). Then, we want to prove that the \(n\)-fold Massey product \(\langle a_1, \ldots, a_n \rangle\) is trivial. If \(J\) is a face of \(K^*\), then
\[
\text{Tor}_{k_i, 2a_i}^{[v_1, \ldots, v_m]}(k[K^*], k) \cong \check{H}^*(K^*; k) = 0.
\]
By this fact and the proposition above, we may assume that \(I_i = \sigma_i^c\) for some simplex \(\sigma_i \in K\) for \(i = 1, \ldots, n\), and \(I_i \cap I_j = \emptyset\) for \(i \neq j\). This implies that \(\sigma_i \cup \sigma_j = [m]\). Thus, we may also assume that \(v_0 \in I_1\), where \(v_0 \in V(K_1) - \alpha\). Then, \(v_0 \notin I_j = \sigma_j^c\), i.e., \(v_0 \in \sigma_j\) for \(j > 1\). Because a face of \(K\) containing the vertex \(v_0\) is a subset of \(V(K_1)\), we see that \(V(K_2) - \alpha = V(K_1)^c \subset \sigma_j^c = I_j\) for \(j > 1\), which contradicts the
assumption that $I_j \cap I_k = \emptyset$ for $j \neq k > 1$. Thus, we have proved that Massey products for $n > 2$ in $\text{Tor}^*_k[e_1, \ldots, e_m](k[K^*], k)$ are trivial.

It follows from Theorem 1.1 that $K^*$ is non-Golod if and only if there are disjoint subsets $I$ and $J$ of $[m]$ such that

$$\iota_{I,J} : (K^*)_{I \cup J} = (\text{link}_K(I^c \cap J^c))_{I \cup J} \rightarrow (K^*)_I \ast (K^*)_J = (\text{link}_K(I^c))_I \ast (\text{link}_K(J^c))_J$$

induces a non-trivial map in the cohomology theory. Then, $I^c$ and $J^c$ must be faces of $K$, and $I^c \cup J^c = [m]$, because $I \cap J = \emptyset$. Thus, we may assume that $V(K_1) - \alpha \subset I^c = \sigma$ and $V(K_2) - \alpha \subset J^c = \tau$. By the Alexander duality and its naturality, it follows that $\iota_{I,J}$ induces a non-trivial map in the homology theory if and only if the dual map

$$\iota^*_{I,J} : (\text{link}_K(\sigma))^* \ast (\text{link}_K(\tau))^*_{I \cup J} = \text{link}_K(\sigma) \ast \Delta^I \cup \Delta^J \ast \text{link}_K(\tau)$$

$$\quad = \text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \tau \cap \tau)$$

$$\quad \rightarrow \text{link}_K(\sigma \cap \tau)$$

induces a non-trivial map in the homology theory, where the first equality follows from Lemma 3.1. Thus, we complete the proof. \hfill \Box

**Example 5.3.4.** Let $L$ be a simplicial complex on $\{2, 3, 4, 5, 6, 7\}$ with facets 267, 367, 467, 567, 236, 456, 2345.

Put $\alpha = 2345$ and consider the simplicial complex $K$ obtained from $\Delta^{a+1}$ and $L$ by gluing along a common simplex $\alpha$. Then, the Alexander dual of $K = \Delta^{a+1} \cup_\alpha L$ is not Golod. In fact, for $\sigma = 12345$ and $\tau = 67$, the inclusion map

$$\text{star}_K(\sigma) \cup \text{star}_K(\tau) = \Delta^{a+1} \cup \{\emptyset, 2, 3, 4, 5\} \ast \Delta^{(6,7)} \rightarrow K$$

induces a non-trivial map in one dimensional homology groups. Needless to say, $\alpha$ is a facet of $L$.

6. Proof of Theorem 2.7

In this section, we prove Theorem 2.7. The proof is divided into three parts. In the first part, we show that $K^*$ is Golod. Next, we show that $K^*$ is not stably homotopy Golod. Finally, we demonstrate that $Z_{K^*}$ is torsion free for a particular $K$.

Before beginning the proof, we study the topology of the space $|(S^3_k \times \Delta^1) \cup_{\eta_k} S^2_4|$. As remarked at the end of Example 5.2.2, there exists a sequence of elementary collapses that collapses $S^3_k \times \Delta^1$ onto $S^3_k \times 2 = S^3_k$. This collapsin induces a deformation retraction $|S^3_k \times \Delta^1|$ onto $|S^3_k \times 2| = |S^3_k|$, and the composite

$$|S^3_k| = |S^3_k \times 1| \rightarrow |S^3_k \times \Delta^1| \rightarrow |S^3_k \times 2| = |S^3_k|$$

is homotopic to the identity map. This deformation retraction induces a deformation retraction

$$\pi : |(S^3_k \times \Delta^1) \cup_{\eta_k} S^2_4| \rightarrow |S^2_4|$$
such that
\[ \pi \circ |i| \simeq |\eta_k| : |S^2_k| \to |(S^2_k \boxtimes \Delta^1) \cup_{\eta_k} S^2_4| \to |S^2_4|, \]
where \( i \) is the inclusion map.

6.1. \( K^* \) is Golod. Recall that
\[ K = \Delta^{V+v_0} \cup ((S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^2_4) \cup F(S^3_k)) \ast \Delta\{w_1\} \cup S^3_k \ast \Delta\{w_1,w_2\}, \]
where \( F(S^3_k) \) is the simplicial complex defined in §2. We apply Corollary 5.3.3 to \( K = K_1 \cup K_2 \), where
\[ K_1 = \Delta^{V+v_0}, \quad K_2 = \Delta^V \cup ((S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^2_4) \cup F(S^3_k)) \ast \Delta\{w_1\} \cup S^3_k \ast \Delta\{w_1,w_2\}, \]
\( \alpha = V, \ V(K_1) = V + v_0 \), and \( V(K_2) = V \cup \{w_1, w_2\} \). Thus, by Corollary 5.3.3 we only have to prove that the map
\[ \text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \sigma \cap \tau) \to \text{link}_K(\sigma \cap \tau) \]
duces the trivial map in the homology theory, where \( \sigma \) and \( \tau \) are faces of \( K \) such that \( v_0 \in \sigma \), \( \{w_1, w_2\} \subset \tau \), and \( \sigma \cap \tau = V(K) \).

First, we consider the case with \( \sigma \cap \tau = \emptyset \). That is, we prove that the map
\[ \text{star}_K(\sigma) \cup \text{star}_K(\tau) = \Delta^{V+v_0} \cup \text{star}_K(\tau) \to K \]
duces the trivial map in the homology theory. Because this map factors as
\[ \Delta^{V+v_0} \cup \text{star}_K(\tau) \to \Delta^{V+v_0} \cup S^3_k \ast \Delta\{w_1,w_2\} \to K, \]
it suffices to prove that the inclusion map
\[ \Delta^{V+v_0} \cup S^3_k \ast \Delta\{w_1,w_2\} \to K \]
duces the trivial map in the homology theory. Because \( |\Delta^{V+v_0} \cup S^3_k \ast \Delta\{w_1,w_2\}| \simeq |S^3_k| = S^4 \) and \( |K| \simeq \Sigma |S^2_4| = S^3 \), the above inclusion map induces the trivial map in the homology theory.

Next, we consider the case that \( \rho = \sigma \cap \tau \) is a non-empty face of \( S^3_k \). Let \( L = L_1 \cup_\alpha L_2 \) be a simplicial complex obtained from \( L_1 \) and \( L_2 \) by gluing along a common face \( \alpha \), and let \( \beta \) be a face of \( \alpha \). Then \( \text{link}_{L_1}(\beta) = \text{link}_{L_1}(\beta) \cup_{\alpha-\beta} \text{link}_{L_2}(\beta) \). By this observation we see that
\[ \text{link}_K(\rho) = \Delta^{(V+v_0)-\rho} \cup (\text{link}_{S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^2_4}(\rho) \cup F(\text{link}_{S^3_k}(\rho)) \ast \Delta^{v_1} \cup \text{link}_{S^3_k}(\rho) \ast \Delta\{w_1,w_2\}). \]
The situation is, therefore, similar to the case with \( \rho = \emptyset \). Here, we are only required to prove that the map
\[ f : |\text{link}_{S^3_k}(\rho)| \to |\text{link}_{S^3_k \boxtimes \Delta^1 \cup_{\eta_k} S^2_4}(\rho)| \]
induced by the inclusion induces the trivial map in the homology theory. \( f \) factors through a contractible space \( |\text{link}_{S^3_k \boxtimes \Delta^1}(\rho)| \). In fact, because \( S^3_k \boxtimes \Delta^1 \) is a triangulation of \( S^3 \times I \) and \( \rho \) is a non-empty simplex in the boundary, it follows that \( \text{link}_{S^3_k \boxtimes \Delta^1}(\rho) \) is a triangulation of a hemisphere. Thus, \( f \) clearly induces the trivial map in the homology theory, and we complete the proof. \( \square \)
6.2. \( K^* \) is not stably homotopy Golod. Next, we show that the map

\[
(6.1) |t_{w_1,w_2},v+v_0| : |K^*| \to |(K^*)_{w_1,w_2} \ast (K^*)_{v+v_0}|
\]

is stably non-trivial. To show this, we consider the mapping cone of \(|t_{w_1,w_2},v+v_0|\), which is denoted by \(C|t_{w_1,w_2},v+v_0|\), and show that \(Sq^2\) acts non-trivially on its mod-2 cohomology groups. Let \(m = |V + \{v_0, w_1, w_2\}|. \) Then we have the following isomorphisms of (co)homology groups:

\[
\tilde{H}^p(C|t_{w_1,w_2},v+v_0|) \cong H^p(|(K^*)_{w_1,w_2} \ast (K^*)_{v+v_0}|, |K^*|) \\
\cong H_{m-2-p}(K, \Delta^{v+v_0} \cup S^3_k \ast \Delta^{\{w_1,w_2\}}) \\
\cong H_{m-2-p}(\Delta^{v+v_0} \cup (\Sigma S^3_k \ast \Delta^1 \cup \eta S^2_4) \ast \Delta^{\{w_1\}}, \Delta^{v+v_0} \cup S^3_k \ast \Delta^{\{w_1\}}) \\
\cong H_{m-2-p}(\Sigma S^3_k \ast \Delta^1 \cup \eta S^2_4, \Sigma S^3_k) \\
\cong \tilde{H}_{m-2-p}(\Sigma S^3_k \ast \Delta^1 \cup \eta S^2_4),
\]

where the second isomorphism follows from the Alexander duality and the fact that

\[
((K^*)_{w_1,w_2} \ast (K^*)_{v+v_0})^* = ((K^*)_{w_1,w_2})^* \ast (K^*)_{v+v_0} \ast \Delta^{v+v_0} \cup \Delta^{\{w_1,w_2\}} \ast ((K^*)_{v+v_0})^* \\
= \text{link}_K(V + v_0) \ast \Delta^{v+v_0} \cup \Delta^{\{w_1,w_2\}} \ast \text{link}_K(\{w_1, w_2\}) \\
= \Delta^{v+v_0} \cup S^3_k \ast \Delta^{\{w_1,w_2\}}.
\]

In the third isomorphism we deformed \(S^3_k \ast \Delta^{\{w_1,w_2\}}\) onto \(S^3_k \ast \Delta^{\{w_1\}}\), and after that we deformed \(F(S^3_k) \ast \Delta^{\{w_1\}}\) onto \(S^3_k \ast \Delta^{\{w_1\}}\), in the fourth isomorphism we collapses \(|\Delta^{v+v_0}|\) to the point, and in the fifth isomorphism we used the fact that \(|(S^3_k \ast \Delta^1) \cup \eta S^2_4|\) is the mapping cylinder of \(\eta : |S^3_k| \to |S^2_4|\) and \(\mathbb{C}P^2 \cong S^2 \cup \eta e^4\) where \(e^4\) denotes a 4-dimensional cell. Since \(Sq^2_k\) acts non-trivially on \(\tilde{H}_m(\Sigma S^3_k \ast \Delta^1 \cup \eta S^2_4)\), it follows from Lemma 5.1.1 that \(Sq^2\) acts non-trivially on \(\tilde{H}^{m-7}(C|t_{w_1,w_2},v+v_0|)\), which implies that the map (6.1) is stably non-trivial. \(\square\)

6.3. \( Z_{K^*} \) is torsion free. Finally, we will demonstrate that \( Z_{K^*} \) is torsion free if \( K \) is constructed using the simplicial map \( \eta_{12} : S^3_{12} \to S^2_4 \) described in [19]. We believe that \( Z_{K^*} \) is torsion free in general, but we are currently unable to prove this.

By Theorem 1.1 and the Alexander duality, we have the following isomorphisms:

\[
H^p(Z_{K^*}; \mathbb{Z}) \cong \bigoplus_{I \subset [m]} \tilde{H}^{p-|I|-1}((K^*)_I ; \mathbb{Z}) \cong \bigoplus_{I \subset [m]} \tilde{H}_{2|I|-p-2}(\text{link}_K(I^c) ; \mathbb{Z}),
\]

where we have used the fact that \((K^*)_I = (\text{link}_K(I^c))^*\). Thus, to show that \( Z_{K^*} \) is torsion free, it suffices to show that every \( \text{link}_K(\sigma) \) is torsion free for all faces \( \sigma \) of \( K \). The longest part of the computation is the following, and the remaining parts are omitted.

**Lemma 6.3.1.** \( \text{link}_{S^3_{12} \ast \Delta^1 \cup \eta_{12} S^2_4}(\sigma) \) is torsion free for any simplex \( \sigma \) in \( S^3_{12} \ast \Delta^1 \cup \eta_{12} S^2_4 \).

**Proof.** Set \( L = S^3_{12} \ast \Delta^1 \cup \eta_{12} S^2_4 \). If \( \sigma = \emptyset \), then clearly \( \text{link}_L(\sigma) = L \) is torsion free.
Let \( v \) be a vertex of \( S^3_{12} \). The Mayer-Vietoris sequence associated with the decomposition 
\( L = (L - v) \cup \operatorname{star}_L(v) \) reduces to the long exact sequence
\[
\cdots \to \tilde{H}_i(\operatorname{link}_L(v)) \to \tilde{H}_i(L - v) \to \tilde{H}_i(L) \to \cdots.
\]
The sequence of elementary collapses of \( S^3_{12} \times \Delta^1 \) onto \( S^3_{12} \times 2 \) given in Example 5.2.2 induces a sequence of elementary collapses of \( S^3_{12} \times \Delta^1 \cup_{\eta_{12}} S^2_1 \) onto \( S^2_1 \). Moreover, this sequence of elementary collapses induces a collapse of \( L - v \) onto \( S^2_1 \). This means that the inclusion map 
\( |L - v| \to |L| \) is homotopy equivalent, and that \( \tilde{H}_*(\operatorname{link}_L(v)) = 0 \).

To proceed further with the computation, we need to know the concrete structure of \( L \). \( S^3_{12} \) has twelve vertices \( \{a_i, b_i, c_i, d_i\}_{i=0,1,2} \), and the following is the list of its facets:

\[
\begin{align*}
 &a_0b_0c_0c_1 \quad a_0b_0b_1c_1 \quad a_0a_1b_1c_1 \quad a_1a_2b_1c_1 \quad a_2b_1c_1c_2 \quad a_2b_1b_2c_2 \quad a_2b_0b_2c_2 \quad a_0a_2b_0c_2 \quad a_0b_0c_0c_2 \quad \\
 &a_0a_2b_0d_1 \quad a_0b_0b_1d_1 \quad b_1c_1c_2d_1 \quad b_1c_1c_2d_1 \quad a_0a_2c_2d_1 \quad a_0b_1d_0d_1 \quad b_1c_2d_0d_1 \quad b_0c_0c_1d_2 \quad b_0c_0c_2d_2 \quad \\
 &a_0b_1d_0d_2 \quad b_1c_2d_0d_2 \quad a_0a_1b_1d_2 \quad a_1a_2b_1d_2 \quad a_2b_1b_2d_2 \quad a_2b_0b_2d_2 \quad b_1c_2d_2d_2 \quad b_0c_0c_2d_2 \quad b_0c_0c_2d_2 \quad \\
 &b_0c_0c_2d_2 \quad b_0c_0c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2 \quad a_0a_1c_1d_2
\end{align*}
\]

We order the vertices as \( a_0 < a_1 < a_2 < b_0 < \cdots < c_0 < \cdots < d_0 < d_1 < d_2 \). The vertices of \( S^2_1 \) are \( a, b, c, d \), and are ordered such that \( a < b < c < d \). Furthermore, \( \eta_{12} : S^3_{12} \to S^2_1 \) is the map defined by the mappings \( a_i \mapsto a, b_i \mapsto b, c_i \mapsto c, \) and \( d_i \mapsto d \), for \( i = 0, 1, 2 \). Then, the list of the facets of \( L \) is as follows:

\[
\begin{align*}
 &a_0b_0c_0c_1c \quad a_0a_2b_0d_1d \quad a_0a_2b_0bd \quad a_0a_2abd \quad a_0b_1d_0d_2d \quad \\
 &a_0b_0b_1c_1c \quad a_0b_0b_1bc \quad a_0a_1b_1c_1c \quad a_0a_1b_1d_1d \quad a_0a_1b_1bd \quad a_0a_1abd \quad \\
 &a_0a_1b_1c_1c \quad a_0a_1b_1bc \quad a_0a_1b_1c_1c \quad a_0a_1b_1d_1d \quad a_0a_1b_1bd \quad a_0a_1abd \quad \\
 &a_1a_2b_1c_1c \quad a_1a_2b_1bc \quad a_1a_2b_1c_1c \quad a_1a_2b_1d_1d \quad a_1a_2b_1bd \quad a_1a_2abd \quad \\
 &a_2b_1c_1c_2c \quad a_2b_1c_1c_2c \quad a_2b_1c_1c_2c \quad a_2b_1c_1c_2d \quad a_2b_1c_1c_2d \quad a_2b_1c_1c_2d \quad \\
 &a_2b_1b_2c_1c \quad a_2b_1b_2bc \quad a_2b_1b_2c_1c \quad a_2b_1b_2d_1d \quad a_2b_1b_2bd \quad a_2b_1b_2bd \quad \\
 &a_2b_0b_2c_1c \quad a_2b_0b_2bc \quad a_2b_0b_2c_1c \quad a_2b_0b_2d_1d \quad a_2b_0b_2bd \quad a_2b_0b_2bd \quad \\
 &a_0a_2b_0c_1c \quad a_0a_2b_0bc \quad a_0a_2b_0c_1c \quad a_0a_2b_0d_1d \quad a_0a_2b_0bd \quad a_0a_2b_0bd \quad \\
 &a_0b_0c_0c_2c \quad a_0b_0c_0c_2c \quad a_0b_0c_0c_2c \quad a_0b_0c_0c_2c \quad a_0b_0c_0c_2c \quad a_0b_0c_0c_2c
\end{align*}
\]

We see that \( L = L_1 \cup L_2 \), where \( L_1 \) is the subcomplex generated by the facets on the first to third columns from the left in the above list, and \( L_2 \) is the subcomplex generated by the other facets. Then, \( L = \operatorname{star}_L(c) \cup \operatorname{star}_L(d) \). If \( \sigma \cap \{c, d\} = \emptyset \), then \( \operatorname{link}_L(\sigma) = \operatorname{link}_{L_1}(\sigma + c) \ast c \cup \operatorname{link}_{L_2}(\sigma + d) \ast d \), which implies that \( |\operatorname{link}_L(\sigma)| \) is homotopy equivalent to a suspension space. Therefore, if \( \operatorname{link}_L(\sigma) \leq 2 \), i.e., \( \dim \sigma \geq 1 \), then it follows that \( \operatorname{link}_L(\sigma) \) is torsion free, by dimensional
reasoning. We consider a face with $\dim \sigma \leq 0$. For a vertex or the empty face in $S^3_{12}$, we have already proved that $\text{link}_L(\sigma)$ is torsion free. If $v = a$ or $v = b$, then we see that $|\text{link}_L(v)|$ is homotopy equivalent to $S^3$. Thus, we have completed the proof in this case.

If $c \in \sigma$ or $d \in \sigma$, then we see by direct computation that $|\text{link}_L(c)|$ and $|\text{link}_L(d)|$ are homotopy equivalent to $S^3$. Moreover, $\text{link}_L(ac)$, $\text{link}_L(bc)$, $\text{link}_L(cd)$, $\text{link}_L(xc)$, $\text{link}_L(ad)$, $\text{link}_L(bd)$, and $\text{link}_L(xd)$ are triangulations of $S^2$, where $x$ is a vertex of $S^3_{12}$. If $\dim \text{link}_L(\sigma) \leq 1$, then $\text{link}_L(\sigma)$ is torsion free by dimensional reasoning, and the proof is complete for all cases.

□

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