Conditional Distribution Model Specification Testing
Using Chi-Square Goodness-of-Fit Tests*

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September 25, 2023

Abstract

This paper introduces chi-square goodness-of-fit tests to check for conditional distribution model specification. The data is cross-classified according to the Rosenblatt transform of the dependent variable and the explanatory variables, resulting in a contingency table with expected joint frequencies equal to the product of the row and column marginals, which are independent of the model parameters. The test statistics assess whether the difference between observed and expected frequencies is due to chance. We propose three types of test statistics: the classical trinity of tests based on the likelihood of grouped data, and two statistics based on the efficient raw data estimator—namely, a Chernoff-Lehmann and a generalized Wald statistic. The asymptotic distribution of these statistics is invariant to sample-dependent partitions. Monte Carlo experiments demonstrate the good performance of the proposed tests.

Keywords: Conditional distribution specification testing; Rosenblatt transform; Pearson statistic; Trinity of chi-square tests, Generalized Wald statistic.

*Research funded by Ministerio Economía y Competitividad (Spain), grant PID2021-15178NB-100.
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1 Introduction

The $\chi^2$ goodness-of-fit tests are widely applied to check for model specification of distribution functions using grouped data. These test statistics evaluate whether the difference between the observed and expected frequencies in each cell is due to chance. The observed frequencies are distributed as a multinomial random vector, and the $\chi^2$ statistics correspond to the classical trinity of tests (LM, Wald, and LR) based on the grouped data likelihood for testing that the multinomial parameters satisfy the $M - 1$ linearly independent restrictions imposed by the specified model.

McFadden (1974) applied these tests to check the specification of multinomial regression models with fixed regressors using product-multinomial sampling: for each subpopulation defined by each of the $J$ possible values taken by the dependent variable, independent random samples of the $L \times 1$ multinomial random vector are taken out. The observed frequencies for each independent sample are distributed as an $L \times 1$ multinomial vector with $L - 1$ linearly independent parameters. The trinity of tests are asymptotically distributed as a $\chi^2_{L(J-1)}$ under simple hypotheses, and as a $\chi^2_{L(J-1)-p}$ under composite hypotheses when the specification is correct and $p$ unknown parameters are efficiently estimated using the grouped data. When testing the specification of conditional distributions, the expected frequencies are an unknown function of the parameters in the model, and the $\chi^2$ tests cannot be implemented. However, the expected frequencies can be estimated, given some preliminary parameter estimator, and the generalized Wald statistic — a quadratic form in the difference of observed and estimated expected frequencies — forms a basis for conditional distributions model checking. Tests of this type were introduced by Heckman (1984), Horowitz (1985), and Andrews (1988a,b) in this context, who extended the tests to check for marginal distribution specification introduced by Nikulin (1973) and Rao and Robson.
(1974) (see also Moore, 1977, for a survey).

In this paper, we propose a cross-classification rule such that the trinity of $\chi^2$ goodness-of-fit tests can be applied to check for conditional distribution model specification. The data is cross-classified according to the Rosenblatt (1952) transform of the dependent variable on one hand, and the vector of explanatory variables on the other. This results in a contingency table where the expected joint frequencies are independent of the parameters in the model and equal to the product of the marginals. The $\chi^2$ statistics based on the grouped data likelihood are asymptotically distributed as a $\chi^2_{L(J-1)-p}$ when $p$ unknown parameters are estimated by the grouped data conditional MLE. We also propose two tests based on the raw data MLE. One of them is a Chernoff and Lehmann (1954) statistic using the Pearson (LM) criteria, whose critical values are between those of a $\chi^2_{J(L-1)-p}$ and a $\chi^2_{J(L-1)}$, which can be approximated by a $\chi^2_{J(L-1)}$ when $J(L-1)$ is large, and the other is a generalized Wald statistic, whose critical values are those of a $\chi^2_{J(L-1)}$.

The rest of the article is organized as follows. Section 2 introduces the proposed cross-classification of the data and the corresponding trinity of tests under simple hypotheses. Section 3 presents the grouped data MLE, justifies the tests for composite hypotheses, and provides the asymptotic distribution of the $X^2$ statistic using the efficient raw data MLE, and the corresponding Wald statistic. Section 4 justifies the validity of the tests when the explanatory variables’ grouping is data-dependent and discusses algorithmic grouping rules. Section 5 provides the power of the tests under contiguous alternatives. Finite sample properties of the tests are studied using Monte Carlo experiments in Section 6. The last section is devoted to concluding remarks.
The data set consists of i.i.d. observations \( \{Y_i, X_i\}_{i=1}^n \) of an \( \mathbb{R}^{1+k} \)-valued random vector \((Y, X')\)' with distribution \( P \), where \( Y \) is the dependent variable taking values in \( \mathbb{R} \), and \( X \) is the vector of explanatory variables taking values in \( \mathbb{R}^k \). The conditional cumulative distribution function (CDF) of \( Y \) given \( X \) is denoted by \( F_{Y|X}(\cdot | X) \).

**Assumption 1.** \( F_{Y|X}(\cdot | X) \) is continuous a.s. on \( \mathbb{R} \).

The hypothesis of interest is

\[
H_0 : F_{Y|X} \in \mathcal{F},
\]

for a family of parametric continuous conditional CDF’s \( \mathcal{F} = \{F_{Y|X, \theta} : \theta \in \Theta\} \), given a proper parameter space \( \Theta \subset \mathbb{R}^p \). That is, under \( H_0 \), there exists a \( \theta_0 \in \Theta \) such that

\[
F_{Y|X}(y | X) = F_{Y|X, \theta_0}(y | X) \text{ a.s. for all } y \in \mathbb{R}.
\]

We propose parameter-dependent partitions

\[
D(\theta) = \{D_{\ell j}(\theta) : \ell = 1, \ldots, L, j = 1, \ldots, J \},
\]

where \( D_{\ell j}(\theta) = \{(y, x) \in \mathbb{R} \times \mathbb{R}^k : F_{Y|X, \theta}(y | x) \in U_\ell, x \in A_j\} \), \( U = \{U_\ell\}_{\ell=1}^L \) is a partition of the interval \([0, 1]\), \( U_\ell = (\vartheta_{\ell-1}, \vartheta_\ell) \), with \( 0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_L = 1 \), and \( A = \{A_j\}_{j=1}^J \) is a partition of \( \mathbb{R}^k \).

Notice that

\[
P((Y, X) \in D_{\ell j}(\theta)) = \int_{x \in A_j} \int_{F_{Y|X, \theta}^{-1}(y | x)}^1 F_{Y|X}(dy | x) F_X(dx),
\]

and under \( H_0 \) there is a \( \theta_0 \in \Theta \) such that

\[
P((Y, X) \in D_{\ell j}(\theta_0)) = v_\ell q_j, \quad \text{for all } \ell = 1, \ldots, L, \text{ and } j = 1, \ldots, J,
\]

where \( v_\ell = \vartheta_\ell - \vartheta_{\ell-1} \), and \( q_j = P(X \in A_j) \).
The $\chi^2$ tests using partitions $D$ are designed to detect alternatives

$$H_{1D} : P((Y, X) \in D_{\ell j}(\theta)) \neq v_\ell q_j \text{ for all } \theta \in \Theta \text{ and some } \ell = 1, \ldots, L, \ j = 1, \ldots, J.$$  

Each observation $(Y_i, X_i)$ is simultaneously cross-classified according to the Rosenblatt transform $V_i(\theta) = F_{Y|X,\theta}(Y_i | X_i)$ into one of the $L$ classes of $U$ on the rows, and according to the explanatory variables vector, $X_i$, into one of the $J$ classes of $A$ on the columns. The vector of observed frequencies in each of the $LJ$ cells, for a given $\theta \in \Theta$, is

$$\hat{O}(\theta) = \left(\hat{O}_{11}(\theta), \ldots, \hat{O}_{1J}(\theta), \ldots, \hat{O}_{L1}(\theta), \ldots, \hat{O}_{LJ}(\theta)\right)' ,$$

where $\hat{O}_{\ell j}(\theta) = \sum_{i=1}^n 1\{V_i(\theta) \in U_{\ell}, X_i \in A_j\}$, and $1_A$ is the indicator function of $A$. The resulting contingency table is Table 1, where $\hat{q}_j = n^{-1} \sum_{i=1}^n 1\{X_i \in A_j\}$, and $\hat{v}_\ell(\theta) = n^{-1} \sum_{i=1}^n 1\{V_i(\theta) \in U_{\ell}\}$.

|       | $X \in A_1$ | $\cdots$ | $X \in A_J$ | Sum |
|-------|-------------|----------|-------------|-----|
| $V(\theta) \in U_1$ | $\hat{O}_{D_{\ell 1}}(\theta)$ | $\cdots$ | $\hat{O}_{D_{\ell J}}(\theta)$ | $n\hat{v}_1(\theta)$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $V(\theta) \in U_L$ | $\hat{O}_{D_{L1}}(\theta)$ | $\cdots$ | $\hat{O}_{D_{LJ}}(\theta)$ | $n\hat{v}_L(\theta)$ |
| Sum   | $\hat{n}_q_1$ | $\cdots$ | $\hat{n}_q_J$ | 1   |

**Table 1:** Contingency table corresponding to $D(\theta)$.

Under $H_0$, $\{V_i(\theta_0)\}_{i=1}^n$ are distributed as uniform random variables on $[0, 1]$ independently of $\{X_i\}_{i=1}^n$. Thus, under $H_0$, $P(V_i(\theta_0) \in [\vartheta_{\ell-1}, \vartheta_\ell]) = v_\ell$ for $\ell = 1, \ldots, L$, and $E(\hat{O}_0) = n \cdot v \otimes q$, where, henceforth, $\hat{O}_0 = \hat{O}(\theta_0)$, $v = (v_1, \ldots, v_L)'$, and $q = (q_1, \ldots, q_J)'$.

The alternative of interest is

$$H_{1D} : E(\hat{O}(\theta)) \neq v \otimes q \text{ for all } \theta \in \Theta.$$  

This is, in fact, the alternative hypothesis for the independence between rows and columns in the contingency table, where the expected marginal frequencies on the rows, $v$, are known. Therefore, there are $LJ - 1$ linearly independent restrictions and, since $v$ is known,
$J - 1$ free parameters (the $J - 1$ linearly independent components of $q$) to be estimated.

The log-likelihood of $\pi = (\pi_{11}, \ldots, \pi_{1J}, \ldots, \pi_{L1}, \ldots, \pi_{LJ})'$ given $\hat{O}_0$ is

$$L(\pi | \hat{O}_0) \propto \sum_{\ell=1}^{L} \sum_{j=1}^{J} \hat{O}_{\ell j}(\theta_0) \ln \pi_{\ell j}. \quad (1)$$

The unrestricted MLE of $E(\hat{O}_0)$ is $n^{-1} \hat{O}_0$, and the restricted estimator is $v \otimes \hat{q}$ with $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_J)'$. Thus, the $X^2$ statistic is $\hat{X}^2(\theta_0)$, where

$$\hat{X}^2(\theta) = \sum_{\ell=1}^{L} \sum_{j=1}^{J} \left( \frac{(\hat{O}_{\ell j}(\theta) - n \cdot v_{\ell} \cdot \hat{q}_j)^2}{n \cdot v_{\ell} \cdot \hat{q}_j} \right) = \hat{\Phi}'(\theta) \hat{\Phi}(\theta),$$

and

$$\hat{\Phi}(\theta) = \text{diag}^{-1/2}(n(v \otimes \hat{q}) \left( \hat{O}(\theta) - n(v \otimes \hat{q}) \right)). \quad (2)$$

The information matrix of $\hat{O}_0$ under $H_0$ is

$$I = \frac{1}{n} \text{Var}(\hat{O}_0 - n(v \otimes q)) = (\text{diag}(v) - vv') \otimes \text{diag}(q).$$

Therefore, $\text{Avar}(\hat{\Phi}(\theta_0)) = Q$ for any grouping of the explanatory variables $A$, where

$$Q = \text{diag}^{-1/2}(v \otimes q) \cdot I \cdot \text{diag}^{-1/2}(v \otimes q) = (I_L - \sqrt{v} \sqrt{v'}) \otimes I_J \quad (3)$$

is known and idempotent with rank $(Q) = \text{tr}(Q) = J(L - 1)$.

The Wald statistic based on (1), with known $\theta_0$, is

$$\hat{W}(\theta_0) = \hat{\Phi}'(\theta_0) \cdot Q^- \cdot \hat{\Phi}(\theta_0),$$

where $A^-$ is the generalized inverse of $A$, which is identical to $\hat{X}^2(\theta_0)$ as stated in the following remark.

**Remark 1.** When using an arbitrary partition $C = \{C_m\}_{m=1}^{M}$ of $\mathbb{R}^{1+k}$ into $M$ cells, the vector of expected frequencies $\pi_C(\theta)$ can be estimated by $\hat{\pi}_C(\theta) = (\hat{\pi}_{C_1}(\theta), \ldots, \hat{\pi}_{C_M}(\theta))'$ with $\hat{\pi}_{C_m}(\theta) = n^{-1} \sum_{i=1}^{n} \int_{(y_i,X_i) \in C_m} F_{Y|X_i}(dy | X_i)$. Under $H_0$, $\hat{\Phi}_C(\theta_0) \overset{d}{\rightarrow} N_M(0, Q_C(\theta_0))$, where $Q_C(\theta_0)$ is the information matrix of $\hat{O}_0$ under $H_0$.
where $\hat{\Phi}_C(\theta) = \text{diag}^{-1/2}(\hat{\pi}_C(\theta)) \left( \hat{O} - n\hat{\pi}_C(\theta) \right)$, and

$$Q_C(\theta) = \text{diag}^{-1/2}(\pi_C(\theta)) \cdot \Sigma_C(\theta) \cdot \text{diag}^{-1/2}(\pi_C(\theta)),$$

with $\Sigma_C(\theta) = \mathbb{E}\left( \text{diag}(p_{C,\theta}(X)) - p_{C,\theta}(X)p'_{C,\theta}(X) \right)$, $p_{C,\theta}(X) = (p_{C_1,\theta}(X), \ldots, p_{C_M,\theta}(X))'$, and $p_{C_m,\theta}(X) = f_{y,X} F_{y|X,\theta}(dy | X)$. The Pearson-type statistic is given by $\hat{X}_C^2(\theta_0) = \Phi'_C(\theta_0)\hat{\phi}_C(\theta_0) \overset{d}{\rightarrow} \sum_{m=1}^{M} \alpha_m Z_{m}^2$, where the $\{Z_m\}_{m=1}^{M}$ are independent standard normal random variables, and $\{\alpha_m\}_{m=1}^{M}$ are the eigenvalues of $Q_C(\theta_0)$. However, the $\alpha_m$’s are usually not in $\{0, 1\}$, since they depend on $\theta_0$ and $F_X$, and $\hat{X}_C^2(\theta_0)$ is not asymptotically pivotal. Moreover, we can use the Wald statistic $\hat{W}_C(\theta_0) = \hat{\Phi}'_C(\theta_0)\hat{\Omega}_C(\theta_0)\hat{\phi}_C(\theta_0)$ with $\hat{\Omega}_C \overset{p}{\rightarrow} Q_C(\theta_0)$. In this case $\hat{W}_C(\theta_0) \overset{d}{\rightarrow} \frac{\chi^2_{\text{rank}(Q_C(\theta_0))}}{\text{Var}(\pi_C(\theta_0))}$. See Heckman (1984), Horowitz (1985), and Andrews (1988a,b). It is worth noticing that, because the generalized inverse is not a continuous function in its components, given a consistent estimator of $Q_C(\theta_0)$, say $\hat{Q}_C(\theta_0)$, there is no guarantee that the limiting distribution of $\hat{W}_C(\theta_0)$, with $\hat{\Omega}_C = \hat{Q}_C^{-}(\theta_0)$, is a $\chi^2_{\text{rank}(\text{Var}(\pi_C(\theta_0)))}$. See Andrews (1987, 1988a).

In order to derive the LM statistic, we need to obtain the efficiency bound of the grouped data $\hat{O}_0$. To this end, it suffices to consider $LJ - 1$ free parameters $\pi^* = (\pi_{1}^*, \ldots, \pi_{LJ-1}^*)'$, where we remove $\pi_{LJ}$ from $\pi$, i.e., $\pi_s^* = \pi_{\ell j}$ for $s = (\ell - 1)J + j$ with $(\ell, j) \in \{(1, \ldots, L) \times \{1, \ldots, J\} \setminus \{(L, J)\}$. Notice that the score vector is $\hat{\Psi}(\theta_0)$, where $\hat{\Psi}(\theta) = n^{-1} \left( \hat{\psi}_1(\theta), \ldots, \hat{\psi}_{LJ-1}(\theta) \right)'$ and

$$\hat{\psi}_s(\theta) = \frac{\partial}{\partial \pi_s} \mathcal{L}(\pi | \hat{O}(\theta)) \bigg|_{\pi = \pi^*} = \frac{\hat{O}_{\ell j}(\theta)}{v_{\ell j}q_j} - \frac{\hat{O}_{LJ}(\theta)}{v_{LJ}q_j}.$$

The information matrix inverse of $\hat{O}_0$ is

$$\mathcal{I}^{-1} = \text{Var} \left( \frac{\hat{O}_s^*}{\sqrt{n}} \right) = \text{diag}(\pi^*) - \pi^* \pi^{*'},$$

where $\hat{O}^*(\theta) = (\hat{O}_1^*(\theta), \ldots, \hat{O}_{LJ-1}^*(\theta))'$, $\hat{O}_s^*(\theta) = \hat{O}_{\ell j}(\theta)$, $\pi^* = (\pi_{1}^*, \ldots, \pi_{LJ-1}^*)'$, and $\pi_s^* = \frac{\partial}{\partial \pi_s} \mathcal{L}(\pi | \hat{O}(\theta)) \bigg|_{\pi = \pi^*}$.
\[ v \ell q \under H_0. \text{ Thus, } I^{-1} \text{ is estimated by } \hat{I}^{-1}(\theta) = \text{diag}(\hat{\pi}^*(\theta)) - \hat{\pi}^*(\theta)\hat{\pi}'(\theta), \text{ where } \hat{\pi}^* = (\hat{\pi}_1^*, \ldots, \hat{\pi}_{LJ}^*)' \text{ and } \hat{\pi}_s = v \ell q_j. \text{ The LM statistic is } \hat{LM}(\theta_0), \text{ with }\]

\[
\hat{LM}(\theta) = n \cdot \hat{\Psi}'(\theta) \cdot \hat{I}^{-1}(\theta) \cdot \hat{\Psi}(\theta).
\]

The LR statistic based on (1), with known \( \theta_0 \), is \( \hat{G}^2(\theta_0) \) with

\[
\hat{G}^2(\theta) = 2 \sum_{\ell=1}^L \sum_{j=1}^J \hat{O}_{\ell j}(\theta) \ln \frac{\hat{O}_{\ell j}(\theta)}{nv \ell q_j}.
\]

**Theorem 1.** Let Assumption 1 be satisfied. Then,

(i) \( \hat{X}^2(\theta_0) \overset{d}{\to} \chi^2_{J(L-1)}, \text{ under } H_0, \)

(ii) \( \hat{G}^2(\theta_0) = \hat{X}^2(\theta_0) + o_p(1), \text{ under } H_0, \)

(iii) \( \hat{X}^2(\theta_0) = \hat{W}(\theta_0) \text{ a.s.}, \)

(iv) \( \hat{X}^2(\theta) = \hat{LM}(\theta) \text{ a.s. for all } \theta \in \Theta. \)

### 3 Composite Hypotheses

We first derive the asymptotic distribution of \( \hat{\Phi}(\tilde{\theta}) \) for any \( \sqrt{n} \)-consistent estimator \( \tilde{\theta} \).

**Assumption 2.** \( \theta_0 \) is an interior point of \( \Theta \), which is a compact subset of \( \mathbb{R}^p \), \( F_{Y|X,\theta}(y \mid X) \) is continuously differentiable on \( \Theta \) for all \( y \in \mathbb{R} \) a.s.

Define

\[
B(\theta) = \text{diag}^{-1/2}(v \otimes q) \cdot \mathbb{E}(\tau(\theta) \otimes \zeta),
\]

where \( \zeta = (\zeta_1, \ldots, \zeta_J)' \) with \( \zeta_j = 1_{\{X \in A_j\}} \) and \( \tau(\theta) = (\tau_{\theta_1}(X)', \ldots, \tau_{\theta_L}(X)')' \) and

\[
\tau_{\theta_\ell}(X) = \int_{F^{-1}_{Y|X,\theta}(\theta_\ell \mid X)} \frac{\partial}{\partial \theta} F_{Y|X,\theta}(dy \mid X), \quad \ell = 1, \ldots, L.
\]

**Assumption 3.** \( \|\partial F_{Y|X,\theta}(F^{-1}_{Y|X,\theta}(\theta_\ell \mid X) \mid X) \| \leq m(X) \) a.s. such that \( \mathbb{E}(m(X)) < \infty \) for all \( \ell = 1, \ldots, L. \)
Theorem 2. Let Assumptions 1, 2, and 3 be satisfied, and \( \tilde{\theta} = \theta_0 + O_p\left(n^{-1/2}\right) \). Then, under \( H_0 \), \( \tilde{\Phi}(\tilde{\theta}) = \tilde{\Phi}(\theta_0) - B(\theta_0)\sqrt{n}(\tilde{\theta} - \theta_0) + o_p(1) \).

Most \( \sqrt{n} \)-consistent estimators of \( \theta_0 \) satisfy the following asymptotic representation.

**Assumption 4.** \( \tilde{\theta} = \theta_0 + n^{-1} \sum_{i=1}^n \ell_{\theta_0}(y_i, X_i) + o_p\left(n^{-1/2}\right) \), where \( \ell_{\theta_0}: \mathbb{R}^{1+k} \rightarrow \mathbb{R}^p \) is such that, for every \( \theta \in \Theta \),

\[
\int_\mathbb{R} \ell_\theta(y, X) f_{Y|X, \theta}(dy | X) = 0 \text{ a.s. and } \int_\mathbb{R} \ell_\theta(y, X) \ell'_\theta(y, X) f_{Y|X, \theta}(dy | X) < \infty \text{ a.s.}
\]

Define

\[
\Omega_{\tilde{\theta}} = Q + G_{\tilde{\theta}}
\]

with

\[
G_{\tilde{\theta}} = B(\theta_0)L(\theta_0)B'(\theta_0) - B(\theta_0)Y(\theta_0) - Y'(\theta_0)B'(\theta_0), \quad (4)
\]

and

\[
L(\theta) = \mathbb{E}(\ell_{\theta}(Y, X)\ell'_\theta(Y, X)), \quad Y(\theta) = \text{diag}^{-1/2}(v \otimes q) \mathbb{E}((\eta(\theta) - v) \otimes \zeta)\ell'_\theta(Y, X),
\]

where \( \eta(\theta) = (\eta_1(\theta), \ldots, \eta_L(\theta))' \) and \( \eta_\ell(\theta) = 1\{V_\ell(\theta) \in U_\ell\} \).

Theorem 3. Let Assumptions 1, 2, and 3 be satisfied and \( \tilde{\theta} \) be as in Assumption 4. Then, under \( H_0 \),

\[
\tilde{\Phi}(\tilde{\theta}) \overset{d}{\rightarrow} N_{LJ}(0, \Omega_{\tilde{\theta}}).
\]

For a given \( \theta_* \in \Theta \), the vector of observed frequencies \( \hat{O}(\theta_*) \) is distributed, conditional on \( \{X_i\}_{i=1}^n \), as a multinomial random vector with parameters \( \mathbb{E}\left(\hat{O}(\theta_*) \mid \{X_i\}_{i=1}^n\right) = \hat{\pi}_{\theta_*}(\theta_0) \) under \( H_0 \), where

\[
\hat{\pi}_{\theta_*}(\theta) = (\hat{\pi}_{11, \theta_*}(\theta), \ldots, \hat{\pi}_{1L, \theta_*}(\theta), \ldots, \hat{\pi}_{L1, \theta_*}(\theta), \ldots, \hat{\pi}_{LL, \theta_*}(\theta))'.
\]
and
\[ \hat{\pi}_{\ell j, \theta_0}(\theta) = \frac{1}{n} \sum_{i=1}^{n} I(x_i \in A_j) \int_{F_{Y|X, \theta_0}^{-1}(\theta_{\ell-1} | x_i)}^{F_{Y|X, \theta_0}^{-1}(\theta_{\ell} | x_i)} \int F_{Y|X, \theta_0}^{-1}(\theta_{\ell} | x_i) F_Y|X, \theta(dy | X_i). \]

Therefore, the (infeasible) conditional log-likelihood of \( \theta \) for given grouped data \( \hat{O}(\theta_0) \) is
\[
L^c \left( \theta \middle| \hat{O}(\theta_0) \right) \propto \sum_{\ell=1}^{L} \sum_{j=1}^{J} \hat{\pi}_{\ell j, \theta_0}(\theta_0) \ln \hat{\pi}_{\ell j, \theta_0}(\theta_0),
\]
and the (infeasible) conditional MLE, \( \hat{\theta} \), maximizes it over \( \Theta \). The next theorem provides its asymptotic distribution assuming the following identifiability condition.

**Assumption 5.** \( \text{rank}(B(\theta_0)) = p \) and, for any \( \delta > 0 \), there exists an \( \varepsilon > 0 \) such that
\[
\inf_{\|\theta - \theta_0\| \geq \delta} \sum_{\ell=1}^{L} \sum_{j=1}^{J} v_{\ell j} q_j \log \frac{v_{\ell j} q_j}{\pi_{\ell j}(\theta_0)} \geq \varepsilon.
\]

These assumptions are standard for the consistency of the MLE in multinomial models (see, e.g., Rao, 2002, Section 5.e). The information matrix of \( \hat{O}(\theta_0) \) is \( \Sigma(\theta_0) \) with \( \Sigma(\theta) = B'(\theta)B(\theta) \).

**Theorem 4.** Let Assumptions 1, 2, 3, and 5 be satisfied. Then, under \( H_0 \),
\[
\sqrt{n} \left( \hat{\theta} - \theta_0 \right) \overset{d}{\rightarrow} N_p \left( 0, \Sigma^{-1}(\theta_0) \right).
\]

We can use a feasible asymptotically equally efficient one-step ahead Gauss-Newton estimator starting from any preliminary \( \sqrt{n} \)-consistent estimator \( \tilde{\theta} \), i.e.,
\[
\hat{\theta}^{(1)} = \tilde{\theta} + \hat{\Sigma}^{-1}(\tilde{\theta}) \hat{B}'(\tilde{\theta}) n^{-1/2} \hat{\Phi}(\tilde{\theta}),
\]
where \( \hat{\Sigma}(\theta) = \hat{B}'(\theta)\hat{B}(\theta) \), and
\[
\hat{B}(\theta) = \text{diag}^{-1/2}(v \otimes q)n^{-1} \sum_{i=1}^{n} \tau_\theta(X_i) \otimes \zeta_i,
\]
and \( \{\zeta_i\}_{i=1}^{n} \) are the observations of \( \zeta \).
Corollary 1. Under the conditions in Theorem 4,

\[ \sqrt{n} \left( \hat{\theta}^{(1)} - \theta_0 \right) \xrightarrow{d} \mathcal{N}_p \left( 0, \Sigma^{-1}(\theta_0) \right). \]

In order to obtain the asymptotic variance of \( \hat{\Phi} \left( \hat{\theta} \right) \), and \( \hat{\Phi} \left( \hat{\theta}^{(1)} \right) \), note that in this case \( L(\theta) = \Sigma^{-1}(\theta) \) for all \( \theta \in \Theta \), and \( \Upsilon(\theta_0)B'(\theta) = B(\theta)\Sigma^{-1}(\theta)B'(\theta) \). Hence,

\[ \hat{\Phi} \left( \hat{\theta} \right) \xrightarrow{d} \mathcal{N}_{JL} \left( 0, \Omega_{\hat{\theta}} \right), \]

where \( \Omega_{\hat{\theta}} = Q - G_{\hat{\theta}} \) with \( G_{\hat{\theta}} = B'(\theta_0)\Sigma^{-1}(\theta_0)B(\theta_0) \), which is idempotent with rank \( J(L - 1) - p \).

The next theorem establishes the limiting distribution of the statistics under \( H_0 \).

Theorem 5. Let Assumptions 1, 2, 3, and 5 be satisfied. Then, under \( H_0 \), \( \hat{\chi}^2 \left( \hat{\theta} \right) \), \( \hat{G}^2 \left( \hat{\theta} \right) \), and \( \hat{W} \left( \hat{\theta} \right) \) are asymptotically distributed as a \( \chi^2_{J(L - 1) - p} \). The asymptotic distribution does not change when \( \hat{\theta} \) is replaced by \( \hat{\theta}^{(1)} \).

A sensible starting estimator in (5) is the MLE based on the raw data \( \{Y_i, X_i\}_{i=1}^n \),

\[ \hat{\theta}^{(0)} = \arg \max_{\theta \in \Theta} \sum_{i=1}^n \log f_{Y|X,\theta}(Y_i \mid X_i), \]

where \( f_{Y|X,\theta}(y \mid X) = \partial F_{Y|X,\theta}(y \mid X) / \partial y \). In this case, assuming that \( f_{Y|X,\theta}(y \mid X) \) is twice continuously differentiable on \( \Theta \) a.s. for all \( y \in \mathbb{R} \) and other regularity conditions,

\[ \ell_\theta(Y, X) = \mathbb{I}^{-1}(\theta) \frac{\partial}{\partial \theta} \ln f_{Y|X,\theta}(Y \mid X), \quad \text{where} \quad \mathbb{I}(\theta) = -\mathbb{E} \left( \frac{\partial^2}{\partial \theta \partial \theta'} \ln f_{Y|X,\theta}(Y \mid X) \right), \]

which is taken for granted to be positive definite uniformly in \( \Theta \), and \( \mathbb{I}(\theta_0) \) is the conditional information matrix of \( F_{Y|X} \) under \( H_0 \). Therefore, \( L(\theta) = \mathbb{I}^{-1}(\theta) \), and

\[ \mathbb{E} \left( \frac{\partial}{\partial \theta} \ln f_{Y|X,\theta}(Y \mid X) \right)_{\theta = \theta_0} = 0. \]
under \( H_0 \). Notice that, in this case, we can write
\[
B(\theta_0) = \text{diag}^{-1/2}(v \otimes q) \cdot \mathbb{E}
\left[(\eta(\theta_0) \otimes \zeta) \frac{\partial}{\partial \theta} \ln f_{Y|X,\theta}(Y | X) \bigg|_{\theta=\theta_0}\right].
\]
Therefore, \( \Upsilon(\theta_0) B'(\theta_0) = B(\theta_0) I(\theta_0) - B(\theta_0) B'(\theta_0) \), and \( \text{Avar} \left( \hat{\Phi} \left( \hat{\theta}^{(0)} \right) \right) = \Omega_{\hat{\theta}^{(0)}} = Q - G_{\hat{\theta}^{(0)}} \) with
\[
G_{\hat{\theta}^{(0)}} = B(\theta_0) I(\theta_0) - B(\theta_0) B'(\theta_0).
\] (8)

The next theorem extends the results in Chernoff and Lehmann (1954) to the conditional case. Let \( \{\lambda_j\}_{j=J(L-1)}^{J(L-1)} \) be the \( p \) roots of the determinantal equation
\[
|B'(\theta_0)B(\theta_0) - (1 - \lambda)I(\theta_0)| = 0,
\]
which always satisfy \( 0 \leq \lambda_i < 1 \), and \( 0 < \lambda_i < 1 \) when \( I(\theta_0) - B'(\theta_0)B(\theta_0) \) is positive definite.

**Assumption 6.** \( I(\theta) - B'(\theta)B(\theta) \) is positive definite for all \( \theta \in \Theta \).

This assumption holds unless the raw data contains no more information than the grouped data.

**Assumption 4’.** \( \hat{\theta}^{(0)} \) satisfies Assumption 4 with \( \ell_\theta \) defined in (7).

**Theorem 6.** Let Assumptions 1, 2, 3, and 4’ be satisfied. Then, under \( H_0 \),
\[
\hat{X}^2 \left( \hat{\theta}^{(0)} \right) \xrightarrow{d} \sum_{j=1}^{J(L-1)-p} Z_j^2 + \sum_{j=J(L-1)-p+1}^{J(L-1)} \lambda_j Z_j^2,
\]
where \( Z_j \)’s are i.i.d. standard normal random variables with \( 0 \leq \lambda_j < 1 \).

Then, using \( \hat{X}^2 \left( \hat{\theta}^{(0)} \right) \) requires using critical values between a \( \chi^2_{J(L-1)-p} \) and a \( \chi^2_{J(L-1)} \) as bounds of the unknown critical values. When \( J(L-1) \) is large, it can be implemented using critical values from a \( \chi^2_{J(L-1)} \).
Given any estimator $\hat{\theta}$ satisfying Assumption 4, the generalized Wald statistic, which is in the class of statistics introduced by Andrews (1988b), is

$$ W(\hat{\theta}) = \Phi'(\hat{\theta}) \hat{\Psi}_\theta \Phi'(\hat{\theta}) $$

with $\hat{\Psi}_\theta = \Omega^{-1} + o_p(1)$. Recall that, in (24), $\Omega(\hat{\theta}) = I_{JL} - \sqrt{v} \sqrt{v'} - G_{\hat{\theta}}$, with $G_{\hat{\theta}}$ defined in (8). Thus, by Theorem 3, under $H_0$, $W(\hat{\theta}) \overset{d}{\rightarrow} \chi^2_{\text{rank}(\Omega(\hat{\theta}))}$. If we know that rank $(\Omega(\hat{\theta})) = J(L - 1)$, then, since $\sqrt{v} \sqrt{v'}$ is orthogonal to $\Omega(\hat{\theta})$, it follows that rank $(I_{JL} - G_{\hat{\theta}})^{-1} = JL$ and $(I_{JL} - G_{\hat{\theta}})^{-1}$ is the generalized inverse of $\Omega(\hat{\theta})$. Now, the natural estimator of $G_{\hat{\theta}}$ is $\hat{G}(\theta)$ defined by replacing $B(\theta_0)$, $L(\theta_0)$, and $\Upsilon(\theta_0)$ by $\hat{B}(\theta)$ in (6), $\hat{L}(\hat{\theta}) = n^{-1} \sum_{i=1}^{n} \ell_{\theta}(Y_i, X_i) \ell'_{\theta}(Y_i, X_i)$, and

$$ \hat{\Upsilon}(\hat{\theta}) = \text{diag}^{-1/2}(v \otimes \hat{q}) n^{-1} \sum_{i=1}^{n} \left( (\eta_{i}(\hat{\theta}) - v) \otimes \zeta_{i} \right) \ell'_{\theta}(Y_i, X_i). $$

Since $\hat{G}_{\hat{\theta}}$ is the corresponding version of $G_{\hat{\theta}}$, by the same argument, rank $(I_{JL} - \hat{G}_{\hat{\theta}}) = JL$ and $\hat{\Omega}_{\hat{\theta}} = \left(I_{JL} - \hat{G}_{\hat{\theta}}\right)^{-1}$. When $\hat{\theta}$ is the conditional MLE using raw data $\hat{\theta}(0)$, $\Omega_{\hat{\theta}(0)}$ in (8) has rank $J(L - 1)$ if Assumption 6 is satisfied. The natural estimator of $G_{\hat{\theta}(0)}$ in (8) is

$$ \hat{G}_{\hat{\theta}(0)} = \hat{B}(\theta(0)) \hat{I}^{-1}(\hat{\theta}(0)) \hat{B}'(\theta(0)) $$

with

$$ \hat{I}^{-1}(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2}{\partial \theta \partial \theta'} \ln f_{Y|X,\theta}(Y_i | X_i). $$

Hence,

$$ \hat{\Omega}_{\hat{\theta}(0)} = \left(I_{JL} - \hat{G}_{\hat{\theta}(0)}\right)^{-1} = \left(I_{JL} - G_{\hat{\theta}(0)}\right)^{-1} + o_p(1), $$

and

$$ \hat{W}(\hat{\theta}(0)) = \Phi'(\theta(0)) \left(I_{JL} - \hat{G}_{\hat{\theta}(0)}\right)^{-1} \Phi'(\theta(0)). $$

We provide the asymptotic distribution of $\hat{W}(\hat{\theta}(0))$ as a corollary of Theorem 3.
Corollary 2. Let Assumptions 1, 2, 4′, and 6 be satisfied. Then, under $H_0$,

$$
\hat{W}(\hat{\theta}^{(0)}) \overset{d}{\to} \chi^2_{J(L-1)}.
$$

4 Sample-Dependent Grouping and Classification Algorithms

4.1 Asymptotics with sample-dependent grouping

In view of (2), the asymptotic distribution of the $\chi^2$ statistics is non-pivotal when we use a sample-dependent $U$. Since $V = F_{Y|X}(Y | X)$ is uniformly distributed on $[0, 1]$ independently of $X$, it is sensible to use partitions $U = \{([\vartheta_{\ell-1}, \vartheta_\ell])_{\ell=1}^L$ with $\vartheta_\ell = \ell/L$. However, partitions $A$ can be data-dependent. Using Pollard (1979) notation, the cells $A_j$ in $A$ are chosen from a class $\mathcal{A}$ of measurable cells, such that $A$ belongs to the class of partitions

$$
\mathcal{H} = \{A \in \mathcal{A}^J : A_1, \ldots, A_J \text{ disjoint and } \bigcup_{j=1}^J A_j = \mathbb{R}^k\}.
$$

Equip $\mathcal{H}$ with its product topology and Borel structure. A partition of $\mathbb{R}^k$ into data-dependent cells $\hat{A}_1, \ldots, \hat{A}_J$ determines a map $\hat{A}$ from the underlying probability space into $\mathcal{H}$. Call $\hat{A} = \{\hat{A}_j\}_{j=1}^J$ a random element of $\mathcal{H}$ if it is a measurable map. We assume that the set of random cells, $\hat{A}$, converges in probability (in the sense of the topology on $\mathcal{H}$) to a set of fixed cells $A \in \mathcal{H}$. That is, $F_X\left\{\hat{A}_j \Delta A_j\right\} \overset{p}{\to} 0$ for $j = 1, \ldots, J$, where $F_X\{S\} = \int_{x \in S} dF_X$, and $A \Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference.

Assumption 7. $A$ is a $P$-Donsker class of sets, and $\hat{A} = \{\hat{A}_j\}_{j=1}^J$ is a sequence of random elements of $\mathcal{H}$ converging in probability to a fixed $A = \{A_j\}_{j=1}^J \in \mathcal{H}$ such that for each component $A_j$, $q_j > 0$, $j = 1, \ldots, J$.

The $P$-Donsker classes include the Vapnik-Chervonenkis classes, among others. The
next theorem states that using sample-dependent partitions

$$\hat{D}(\theta) = \left\{ \hat{D}_{ij}(\theta) : \ell = 1, \ldots, L, j = 1, \ldots, J \right\},$$

with $$\hat{D}_{ij}(\theta) = \left\{ (y, x) \in \mathbb{R}^{1+k} : F_{Y|X,x}(y | x) \in U_\ell, x \in \hat{A}_j \right\}$$ does not have any effect on the asymptotic distribution of the different statistics under the null. Let $$\hat{\theta}^{(1)}, \hat{X}^2, \hat{W},$$ and $$\hat{G}^2$$ be the analogs of $$\hat{\theta}(1), X^2, W,$$ and $$G^2$$ using $$\hat{D}$$ rather than $$D$$.

**Theorem 7.** Let Assumptions 1, 2, 3, 4, and 7 hold. Then, under $$H_0$$, $$\hat{X}^2(\hat{\theta}) = \hat{X}^2(\hat{\theta}) + o_p(1)$$ and $$\hat{G}^2(\hat{\theta}) = \hat{X}^2(\hat{\theta}) + o_p(1)$$ for any $$\hat{\theta} = \theta_0 + O_p(n^{-1/2})$$. If, in addition, Assumption 4' holds, then $$\hat{W}^2(\hat{\theta}) = \hat{W}^2(\hat{\theta}) + o_p(1)$$.

### 4.2 Algorithms

In this section, we propose a partitioning algorithm allowing to control the number of points in its cells and, hence, amenable to use with large $$k$$.

Gessaman (1970) has proposed a simple, deterministic rule to obtain $$\hat{A}_j$$ containing approximately the same number of points $$\{X_i\}_{i=1}^n$$. Let an integer $$T \geq 2$$ such that $$n \geq T^k$$ be given, and let $$X = (X_1, \ldots, X_k)'$$. Start by splitting $$\{X_i\}_{i=1}^n$$ into $$T$$ sets having the same number of points, except for perhaps the boundary sets, using hyperplanes perpendicular to the axis of $$X_1$$. If $$k = 1$$, the process terminates here. Otherwise, proceed recursively by next partitioning each of the obtained $$T$$ cylindrical sets. The procedure yields $$\hat{A} = \left\{ \hat{A}_j \right\}_{j=1}^J$$ with $$J = T^k$$ so that $$n \geq T^k$$ is required to make sure that the cells are non-empty. It can be shown that

$$\max_{j=1,\ldots,J} \hat{N}_j - \min_{j=1,\ldots,J} \hat{N}_j \leq 1,$$

where $$\hat{N}_j = \sum_{i=1}^n 1_{\{X_i \in \hat{A}_j\}}$$ is the number of points in $$\hat{A}_j$$, $$j = 1, \ldots, J$$. One can expect cells with approximately the same number of points to improve finite sample behavior. However, even with $$T = L = 2$$, the total number of cells, $$LJ = 2^{k+1}$$, is rapidly increasing.
in $k$ and demands an alternative approach for relatively large values of $k$.

We propose a Random Tree Partition (RTP) resulting into significantly fewer than $T^k$ cells while preserving the possibility of controlling the number of points per cell. Let $T \geq 2$ and $r \geq 1$ be integers. Let $\mathcal{I} \leftarrow (1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k)'$ be a $kr \times 1$ vector repeating each of $1, \ldots, k$ for $r$ times. Start with a tree containing a single node (the root) associated with $A \leftarrow \mathbb{R}^k$.

1. Uniformly at random select $j \in \mathcal{I}$.

2. Using $T - 1$ hyperplanes perpendicular to the $X_j$ axis, partition $A$ into $T$ cells $\hat{A}_1, \ldots, \hat{A}_T$ such that each cell contains approximately the same number of points $\{X_i\}_{i=1}^n$. Split the current node associated with $A$ by adding $T$ child nodes associated with $\hat{A}_1, \ldots, \hat{A}_T$.

3. Remove one instance of $j$ from $\mathcal{I}$.

4. If $\mathcal{I}$ has no elements left, the procedure terminates here. Otherwise, select a terminal node associated with $\hat{A}$ that contains the highest number of points and repeat Step 1 with $A \leftarrow \hat{A}$.

Each $j \in \mathcal{I}$ induces a single split, resulting in $kr$ splits in total. As we start with a single-node tree and each split increases the number of terminal nodes by $T - 1$, the final tree has $J = 1 + kr(T - 1)$ terminal nodes. To be able to perform all $kr$ splits and avoid empty terminal nodes, we require $n \geq 1 + kr(T - 1)$ hereafter. It can be shown that

$$\max_{j=1,\ldots,J} \hat{N}_j - \min_{j=1,\ldots,J} \hat{N}_j \leq 1 + (T - 1) \min_{j=1,\ldots,J} \hat{N}_j \iff \frac{\max_{j=1,\ldots,J} \hat{N}_j}{\min_{j=1,\ldots,J} \hat{N}_j} \leq T + \frac{1}{\min_{j=1,\ldots,J} \hat{N}_j}.$$ 

That is, the number of points in any two cells, including those with the largest and smallest number of points, differs by at most a little more than $T$ times. For simplicity and to avoid
high values of $J$ with large $k$, in Section 6 we consider binary trees with median splits so that $T = 2$, and the number of cells equals $J = 1 + kr$.

We emphasize that, for $n \geq J = 1 + kr(T - 1)$, the algorithm guarantees that each $\hat{A}_j$, $j = 1, \ldots, J$, will be nonempty, which is crucial as then $\hat{q}_j > 0$, $j = 1, \ldots, J$. As a result, the $X^2$ and Wald statistics are always computable when using an RTP. On the other hand, $G^2$ additionally requires that $\hat{O}_{\ell j}(\theta) > 0$ for all $j = 1, \ldots, J$ and $\ell = 1, \ldots, L$. The latter, however, cannot be guaranteed by any procedure without making $U$ data dependent, unless $L = J = JL = 1$.

5 Power

Consider contiguous alternatives, $H_{1n}: F_{Y | X}(dy | X) = 1 + \frac{t_{n\theta_0}(y, X)}{\sqrt{n}}$ a.s. for some $\theta_0 \in \Theta$ and all $y \in \mathbb{R}$, where $t_{n\theta}$ and $t_\theta$ are such that

$$\int_{\mathbb{R}} t_{n\theta}(y, X)F_{Y | X, \theta}(dy | X) = 0, \quad \int_{\mathbb{R}} (t_{n\theta} - t_\theta)^2(y, X)F_{Y | X, \theta}(dy | X) = o(1) \text{ as } n \to \infty \text{ a.s.}$$

This allows modeling departures from $H_0$ that are proper conditional CDF's. See Delgado and Stute (2008) for examples and discussion. Let us consider first the limiting distribution of $\hat{\Phi}(\tilde{\theta})$ under $H_{1n}$ for any $\sqrt{n}$-consistent estimator $\tilde{\theta}$. Define $T^{(1)} = [T^{(1)}]_{\ell j}$, $\ell = 1, \ldots, L$, $j = 1, \ldots, J$, with

$$[T^{(1)}]_{\ell j} = \frac{1}{v_{\ell j}q_j} \int_{x \in A_j} \int_{F_{Y | X, \theta_0}^{-1}(\theta_{\ell - 1} | x)}^{F_{Y | X, \theta_0}^{-1}(\theta_{\ell} | x)} t_{\theta_0}(y, x)F_{Y | X, \theta_0}(dy | x)F_X(dx).$$

Theorem 8. Let Assumptions 1, 2, 4, and 7. Then, under $H_{1n}$,

$$\hat{\Phi}(\tilde{\theta}) = \hat{\Phi}(\theta_0) - B(\theta_0)\sqrt{n}(\tilde{\theta} - \theta_0) + T^{(1)} + o_p(1).$$
Under $H_{1n}$, the $\ell_{\theta_0}$ term in Assumption 4 is not centered anymore (see Behnen and Neuhaus, 1975). Now $\sqrt{n} \left( \hat{\theta} - \theta_0 \right)$ has asymptotic mean

$$\delta = \int_{\mathbb{R}^{1+k}} \ell_{\theta_0}(y, x) t_{\theta_0}(y, x) F_{Y|X, \theta_0}(dy | x) F_X(dx).$$

This results in the additional shift $T^{(2)} = B(\theta_0) \cdot \delta$. Define $T = T^{(1)} - T^{(2)}$. Then $\sqrt{n} \hat{\Phi} \left( \hat{\theta} \right) - T$ under $H_{1n}$ has the same asymptotic distribution as $\sqrt{n} \hat{\Phi} \left( \hat{\theta} \right)$ under $H_0$.

The next result provides the limiting distribution of the different statistics under $H_{1n}$. Henceforth, $\chi^2_N(\Lambda) \overset{d}{=} \sum_{j=1}^{N} (Z_j + \omega_j)^2$ is a non-central chi-squared random variable with $N$ degrees of freedom and non-centrality parameter $\Lambda = \sum_{j=1}^{N} \omega_j^2$.

**Corollary 3.** Let $H_{1n}$ hold. If Assumptions 1, 2, 5, and 7 are satisfied, then $\hat{X}^2 \left( \hat{\theta}^{(1)} \right)$, $\hat{W} \left( \hat{\theta}^{(1)} \right)$, and $\hat{G}^2 \left( \hat{\theta}^{(1)} \right)$ are asymptotically distributed as a $\chi^2_{J(L-1)-p}(T'T)$. Under Assumptions 1, 4', and 7, $\hat{W} \left( \hat{\theta}^{(0)} \right) \overset{d}{\rightarrow} \chi^2_{J(L-1)}(T'T)$.

### 6 Monte Carlo

We consider the null hypothesis

$$H_0: Y \mid X \sim \mathcal{N} \left( \rho(X), \sigma^2 \right),$$

where the regression model is multivariate linear, i.e., $\rho(X) = \beta_0 + X'\beta_0$, with $\beta_0 = (\beta_{00}, \ldots, \beta_{k0})'$. We set $\beta_{00} = \cdots = \beta_{k0} = \sigma^2 = 1$. Data consists of $\{Y_i, X_i\}_{i=1}^{n}$, where $\{X_i\}_{i=1}^{n}$ are i.i.d. observations of $X$, a $k \times 1$ vector of independent random variables uniformly distributed on $[0, 1]$, and

$$Y_i = \rho(X_i) + \delta(X_i) + \sigma(X_i) \cdot \varepsilon_i,$$
where \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. with mean zero and variance one,

\[
\delta(x) = a \cdot \sum_{i=1}^{k} \ln(x_i), \quad \text{and} \quad \sigma(x) = \exp \left( -b \cdot \sum_{i=1}^{k} x_i \right) \frac{\sqrt{(2b)^ke^{2bk}}}{\sqrt{(e^{2b} - 1)^k}},
\]

where \( \sigma(\cdot) \) is such that \( \text{Var}[\sigma(X)\varepsilon_i] = 1 \) and \( a, b \geq 0 \). Under \( H_0 \), \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. standard normal and \( a = b = 0 \). Let \( \mu_k = \mathbb{E}[\varepsilon_i^k] \). We consider the following alternatives.

1. From linear regression specification: \( a = 1, b = 0 \), and \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. \( N(0, 1) \).

2. From conditional homoskedasticity: \( a = 0, b = \frac{2}{3} \), and \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. \( N(0, 1) \).

3. From conditional symmetry: \( a = b = 0 \), and \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. and follow a skewed generalized \( t \) (SGT) distribution proposed by Theodossiou (1998), with parameters \((\lambda, p, q)\). This allows to generate asymmetric distributions (with \( \mu_3 > 0 \)) but without an excess of kurtosis (\( \mu_4 = 0 \)). We report results of mild (SGT\(_{1/3}\)) and strong (SGT\(_{1/3}\)) asymmetry using the following parameter values.

   (i) \( (\lambda, p, q) = (0.253708, 2.97692, 3.84001) \) with \( \mu_1 = 0, \mu_2 = 1, \mu_3 = 1/3, \mu_4 = 3 \).

   (ii) \( (\lambda, p, q) = (0.998878, 3.27329, 8.6073) \) with \( \mu_1 = 0, \mu_2 = 1, \mu_3 = 2/3, \mu_4 = 3 \).

4. From conditional mesokurtosis: \( a = b = 0 \), and standardized \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. and follow a \( t \) distribution. We provide two cases, \( t_5 \) and \( t_{2.1} \), with the latter one having heavier tails and, unlike \( t_5 \), infinite variance.

   (i) \( \sqrt{\frac{5}{3}} \cdot \varepsilon_i \sim t_5, i = 1, \ldots, n \), with \( \mu_1 = 0, \mu_2 = 1, \mu_3 = 0, \mu_4 = 9 \).

   (ii) \( \sqrt{21} \cdot \varepsilon \sim t_{2.1}, i = 1, \ldots, n \), with \( \mu_1 = 0, \mu_2 = 1, \mu_3 \) undefined, \( \mu_4 = \infty \).

We have run simulations for tests based on \( \hat{X}_D^2(\hat{\theta}^{(1)}) \), \( \hat{G}_D^2(\hat{\theta}^{(1)}) \), \( \hat{W}_D(\hat{\theta}^{(1)}) \), and \( \hat{W}_D(\hat{\theta}^{(0)}) \). We use \( \hat{\theta}^{(1)} \) iterated until convergence as it significantly improves size accuracy. We also compare these tests with the omnibus conditional Kolmogorov-Smirnov (KS) bootstrap test proposed by Andrews (1997), which is based on the difference between
the sample joint distribution and its restricted version imposing the conditional CDF specification under $H_0$. Results are based on 4000 Monte Carlo iterations, and the KS test is based on 2000 resamples. We report results for $n = 50, 100, \text{ and } 500$.

We first consider balanced partitions $U$, i.e., partitions with $\vartheta_\ell = \ell/L$, $\ell = 1, \ldots, L$, which do not favor, in principle, any alternative. Next we consider unbalanced partitions, with small cells on the tails. This has been proposed in the classical literature to favor alternatives with heavy tails (see, e.g., Kallenberg et al., 1985). The power of $\chi^2$ tests, particularly $X^2$, improves for any of the considered alternatives, not only leptokurtic ones. In all the tables we consider $k = 1, 5, 10$ to assess the curse of dimensionality effect on the different tests and employ the RTP algorithm with $T = 2$ to partition $\mathbb{R}^k$.

Balanced partitions are considered with $L = 3, 6, 12, 24$, where $L = 3, 6$ are combined with $r = 1, 2, 5, 10$, and $L = 12, 24$ are used only with $r = 1$. Additionally, we consider $J = 1$, i.e., without partitioning $\mathbb{R}^k$ at all, with $L = 2, 3$, which cannot be used with the $X^2$ statistic, because $\text{rank}(B(\theta_0)) < p$ when $L = 2$ or $J = 1 < k + 1$. The rejection rates are not provided when $J = 1 + kr > n$.

Recall that $G^2$ can be computed only when $\hat{O}_{\ell j}(\theta) > 0$ for all $j = 1, \ldots, J$ and $\ell = 1, \ldots, L$. For a given $n$, after a certain point, the probability of no empty cells rapidly decreases as $JL$ increases. For example, with equiprobable cells and $n = 100$, it is approximately 98.5% under $LJ = 3 \cdot 5$ and 33.4% under $LJ = 3 \cdot 10$. As a result, $G^2$ cannot be computed in many cases. When available, the results for $G^2$ are almost identical to those of $X^2$ and, hence, they will not be reported. The results for $\hat{W}_D(\hat{\theta}^{(1)})$ are also very similar. Thus, we only report results for $\hat{X}_D^2(\hat{\theta}^{(1)})$ and $\hat{W}_D(\hat{\theta}^{(0)})$, which will be referred to as $X^2$ and $W$, respectively.

Table 2 reports the results under $H_0$. All the tests exhibit excellent size accuracy. It is worth noticing that the $\chi^2$ tests are almost as accurate as the KS bootstrap test.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$L$ & $r$ & $J$ & $k$ & $n = 50$ & $n = 100$ & $n = 500$ \\
\hline
 & & & & $0.01$ & $0.05$ & $0.10$ & $0.01$ & $0.05$ & $0.10$ & $0.01$ & $0.05$ & $0.10$ \\
\hline
2 & – & 1 & 1 & 0.01 & 0.03 & 0.10 & 0.01 & 0.06 & 0.13 & 0.01 & 0.04 & 0.08 \\
3 & 1 & 1 & 1 & 0.01 & 0.05 & 0.11 & 0.01 & 0.05 & 0.11 & 0.01 & 0.06 & 0.10 \\
3 & 2 & 1 & 1 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 3 & 1 & 1 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 4 & 1 & 1 & 0.01 & 0.05 & 0.09 & 0.01 & 0.05 & 0.10 & 0.01 & 0.04 & 0.10 \\
3 & 5 & 1 & 1 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 6 & 1 & 1 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 & 0.01 & 0.06 & 0.11 \\
3 & 7 & 1 & 1 & 0.00 & 0.04 & 0.08 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 \\
3 & 8 & 1 & 1 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 9 & 1 & 1 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 10 & 1 & 1 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 11 & 1 & 1 & 0.01 & 0.05 & 0.09 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 12 & 1 & 1 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
24 & 1 & 1 & 1 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
2 & – & 2 & 10 & 0.01 & 0.03 & 0.10 & 0.01 & 0.07 & 0.14 & 0.01 & 0.04 & 0.09 \\
3 & 3 & 1 & 10 & 0.01 & 0.05 & 0.11 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.09 \\
3 & 4 & 1 & 10 & 0.01 & 0.05 & 0.09 & 0.01 & 0.06 & 0.12 & 0.01 & 0.05 & 0.10 \\
3 & 5 & 2 & 10 & 0.00 & 0.05 & 0.11 & 0.01 & 0.04 & 0.10 & 0.01 & 0.06 & 0.11 \\
3 & 6 & 5 & 10 & – & – & – & 0.00 & 0.02 & 0.07 & 0.01 & 0.05 & 0.10 \\
3 & 7 & 101 & 10 & – & – & – & – & – & – & – & 0.01 & 0.04 & 0.10 \\
3 & 8 & 101 & 10 & – & – & – & – & – & – & 0.00 & 0.02 & 0.10 \\
3 & 9 & 11 & 10 & 0.01 & 0.04 & 0.09 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 \\
3 & 10 & 11 & 10 & 0.00 & 0.03 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 11 & 1 & 10 & 0.01 & 0.04 & 0.08 & 0.01 & 0.05 & 0.09 & 0.01 & 0.05 & 0.10 \\
3 & 12 & 1 & 10 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 \\
3 & 13 & 1 & 10 & 0.01 & 0.06 & 0.10 & 0.01 & 0.06 & 0.10 & 0.01 & 0.05 & 0.09 \\
3 & 14 & 1 & 10 & 0.01 & 0.05 & 0.11 & 0.01 & 0.06 & 0.11 & 0.01 & 0.06 & 0.11 \\
3 & 15 & 1 & 10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.06 & 0.10 \\
3 & 16 & 1 & 10 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 17 & 1 & 10 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 18 & 1 & 10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 19 & 1 & 10 & 0.01 & 0.05 & 0.09 & 0.01 & 0.04 & 0.10 & 0.01 & 0.05 & 0.10 \\
3 & 20 & 1 & 10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.05 & 0.10 \\
24 & 1 & 1 & 10 & 0.01 & 0.05 & 0.10 & 0.01 & 0.04 & 0.09 & 0.01 & 0.05 & 0.10 \\
\hline
\end{tabular}
\caption{Size accuracy using balanced partitions ($k = 1, 10$).}
\end{table}

The rejection rates under $H_1$ are reported in Table 3. The performance of all the tests is similar under the nonlinear regression ($a = 1, b = 0$) for $k = 1$, although the $X^2$ test appears to be more sensitive to the partition choice than $W$. Both the $X^2$ and KS tests suffer from the curse of dimensionality going from simple ($k = 1$) to multiple ($k = 5, 10$) regression, while $W$ is more robust to it. The $\chi^2$ tests perform very well under
Table 3: Power under deviations in mean, variance, skewness, and kurtosis using balanced partitions ($k = 1, 10$ and $\alpha = 5\%$).
conditional heteroskedasticity for any \( k \), while the KS test exhibits very poor power under this alternative, but the proportion of rejections clearly increases with the sample size when \( k = 1 \). The results under asymmetry (SGT\(_{1/3}\) and SGT\(_{2/3}\)) are mixed. Obviously, all the tests detect the stronger asymmetry \( (\mu_3 = 2/3) \) more easily than the mild one \( (\mu_3 = 1/3) \), particularly the \( X^2 \) test, which exhibits trivial power with many partitions in the latter case. Even though the \( W \) test performs much better under \( \mu_3 = 2/3 \), there are still some partitions that lead to trivial power. It seems preferable to choose large \( L \) for these asymmetric alternatives, though the \( X^2 \) test’s power is still very poor with \( \mu_3 = 1/3 \) even with this choice. Similar comments can be made for the leptokurtic alternatives. In this case, the \( W \) test exhibits excellent power, much better than the \( X^2 \) and KS tests. The \( X^2 \) test also exhibits almost trivial power for most partitions under the milder leptokurtic alternative \( (t_5) \) but performs much better with stronger leptokurtosis \( (t_{2.1}) \). The \( W \) test outperforms the KS test under all partitions and leptokurtosis alternatives. The KS test is much more sensitive to the curse of dimensionality than any of the \( \chi^2 \) tests.

Next we present results for unbalanced partitions \( U \) with \( \vartheta_\ell \neq \ell/L \) and small cells on the tails in the hope of closing the gap between the two \( \chi^2 \) tests. We report results for the following partitions \( (\vartheta_0, \vartheta_1, \ldots, \vartheta_L) \):

(i) \( 6^* \) with \( (0, 0.01, 0.25, 0.5, 0.75, 0.99, 1) \),

(ii) \( 6^{**} \) with \( (0, 0.05, 0.1, 0.5, 0.9, 0.95, 1) \),

(iii) \( 8^* \) with \( (0, 0.01, 0.06, 0.16, 0.5, 0.84, 0.94, 0.99, 1) \).

(iv) \( 8^{**} \) with \( (0, 0.01, 0.10, 0.33, 0.5, 0.66, 0.90, 0.99, 1) \).

These unbalanced partitions are intended, in principle, to improve the power in the direction of heavy-tailed alternatives. They have also proven to improve other alternatives in the classical case (Kallenberg et al., 1985).
In Table 4 we observe that for \( k = 1 \) the \( \chi^2 \) tests exhibit a similar size accuracy to that of the omnibus bootstrap test but suffer size distortions for the smaller sample sizes when \( k = 10 \), except when \( n = 500 \).

Table 5 reports the proportion of rejections under the alternative for the chosen unbalanced partitions. The power of the \( X^2 \) test improves when the partitions are unbalanced and is similar to that of the \( W \) test. Both \( \chi^2 \) tests are much more robust to the curse of dimensionality than the omnibus KS test with most partitions.

| \( L \) | \( r \) | \( J \) | \( k \) | \( n = 50 \) | \( n = 100 \) | \( n = 500 \) |
|-------|------|------|------|---------|---------|---------|
|       |      |      |      | 0.01   | 0.05   | 0.10   |
|       |      |      |      | 0.01   | 0.05   | 0.10   |
|       |      |      |      | 0.01   | 0.05   | 0.10   |

\( \chi^2 \) tests using \( \hat{\theta}^{(0)} \)

| \( L \) | \( r \) | \( J \) | \( k \) | \( n = 50 \) | \( n = 100 \) | \( n = 500 \) |
|-------|------|------|------|---------|---------|---------|
|       |      |      |      | 0.02   | 0.07   | 0.11   |
|       |      |      |      | 0.01   | 0.05   | 0.10   |
|       |      |      |      | 0.01   | 0.05   | 0.10   |

\( \chi^2 \) tests using Iterated \( \hat{\theta}^{(1)} \)

| \( L \) | \( r \) | \( J \) | \( k \) | \( n = 50 \) | \( n = 100 \) | \( n = 500 \) |
|-------|------|------|------|---------|---------|---------|
|       |      |      |      | 0.32   | 0.40   | 0.45   |
|       |      |      |      | 0.28   | 0.38   | 0.46   |
|       |      |      |      | 0.29   | 0.37   | 0.42   |

Kolmogorov-Smirnov test using \( \hat{\theta}^{(0)} \)

| \( L \) | \( r \) | \( J \) | \( k \) | \( n = 50 \) | \( n = 100 \) | \( n = 500 \) |
|-------|------|------|------|---------|---------|---------|
| $L$ | $r$ | $J$ | $k$ | $a = 1$ | $b = 2/3$ | $\text{SGT}_{1/3}$ | $\text{SGT}_{2/3}$ | $t_5$ | $t_{2.1}$ |
|---|---|---|---|---|---|---|---|---|---|
|   |   |   |   | 50 | 100 | 500 | 50 | 100 | 500 | 50 | 100 | 500 | 50 | 100 | 500 |
| 6* | 1 | 2 | 1 | 0.10 | 0.16 | 0.65 | 0.14 | 0.21 | 0.84 | 0.07 | 0.09 | 0.45 | 0.17 | 0.38 | 1.00 | 0.21 | 0.30 | 0.84 | 0.67 | 0.91 | 1.00 |
| 6**| 1 | 2 | 1 | 0.07 | 0.13 | 0.54 | 0.08 | 0.17 | 0.89 | 0.08 | 0.10 | 0.43 | 0.27 | 0.70 | 1.00 | 0.08 | 0.15 | 0.65 | 0.39 | 0.68 | 1.00 |
| 8* | 1 | 2 | 1 | 0.11 | 0.16 | 0.67 | 0.12 | 0.20 | 0.91 | 0.06 | 0.09 | 0.42 | 0.20 | 0.63 | 1.00 | 0.20 | 0.30 | 0.86 | 0.62 | 0.87 | 1.00 |
| 8**| 1 | 2 | 1 | 0.11 | 0.17 | 0.65 | 0.13 | 0.21 | 0.90 | 0.06 | 0.09 | 0.38 | 0.20 | 0.39 | 0.99 | 0.21 | 0.31 | 0.87 | 0.70 | 0.92 | 1.00 |
| Best balanced | 1 | 0.20 | 0.44 | 1.00 | 0.10 | 0.18 | 0.81 | 0.09 | 0.16 | 0.66 | 0.28 | 0.54 | 1.00 | 0.19 | 0.31 | 0.90 | 0.69 | 0.92 | 1.00 |
| 6* | 1 | 11 | 10 | 0.24 | 0.26 | 0.77 | 0.49 | 0.71 | 1.00 | 0.16 | 0.10 | 0.14 | 0.17 | 0.18 | 0.87 | 0.33 | 0.35 | 0.69 | 0.57 | 0.80 | 1.00 |
| 6**| 1 | 11 | 10 | 0.09 | 0.07 | 0.52 | 0.05 | 0.06 | 0.96 | 0.11 | 0.09 | 0.15 | 0.12 | 0.14 | 1.00 | 0.06 | 0.04 | 0.22 | 0.03 | 0.20 | 0.99 |
| 8* | 1 | 11 | 10 | 0.15 | 0.16 | 0.72 | 0.29 | 0.49 | 1.00 | 0.10 | 0.07 | 0.14 | 0.13 | 0.13 | 0.99 | 0.21 | 0.21 | 0.63 | 0.39 | 0.69 | 1.00 |
| 8**| 1 | 11 | 10 | 0.19 | 0.21 | 0.76 | 0.42 | 0.67 | 1.00 | 0.12 | 0.08 | 0.12 | 0.14 | 0.13 | 0.75 | 0.27 | 0.30 | 0.70 | 0.53 | 0.81 | 1.00 |
| Best balanced | 10 | 0.09 | 0.24 | 0.93 | 0.20 | 0.59 | 1.00 | 0.08 | 0.13 | 0.61 | 0.14 | 0.39 | 1.00 | 0.10 | 0.21 | 0.78 | 0.40 | 0.81 | 1.00 |
| 6* | 1 | 12 | 10 | 0.12 | 0.16 | 0.58 | 0.15 | 0.23 | 0.89 | 0.04 | 0.05 | 0.22 | 0.09 | 0.08 | 0.61 | 0.23 | 0.31 | 0.81 | 0.63 | 0.85 | 1.00 |
| 6**| 1 | 12 | 10 | 0.07 | 0.08 | 0.28 | 0.10 | 0.19 | 0.92 | 0.10 | 0.12 | 0.42 | 0.37 | 0.79 | 1.00 | 0.05 | 0.04 | 0.11 | 0.06 | 0.12 | 0.78 |
| 8* | 1 | 12 | 10 | 0.10 | 0.14 | 0.53 | 0.13 | 0.22 | 0.93 | 0.06 | 0.09 | 0.47 | 0.23 | 0.68 | 1.00 | 0.14 | 0.19 | 0.69 | 0.39 | 0.66 | 1.00 |
| 8**| 1 | 12 | 10 | 0.11 | 0.15 | 0.54 | 0.15 | 0.23 | 0.91 | 0.06 | 0.08 | 0.40 | 0.18 | 0.38 | 0.99 | 0.19 | 0.25 | 0.74 | 0.55 | 0.78 | 1.00 |
| Best balanced | 10 | 0.20 | 0.43 | 1.00 | 0.10 | 0.20 | 0.81 | 0.07 | 0.08 | 0.31 | 0.21 | 0.47 | 1.00 | 0.08 | 0.09 | 0.17 | 0.20 | 0.33 | 0.93 |
| 6* | 1 | 13 | 10 | 0.38 | 0.27 | 0.77 | 0.44 | 0.55 | 1.00 | 0.40 | 0.27 | 0.09 | 0.40 | 0.27 | 0.27 | 0.41 | 0.38 | 0.64 | 0.59 | 0.68 | 1.00 |
| 6**| 1 | 13 | 10 | 0.35 | 0.16 | 0.51 | 0.41 | 0.26 | 0.87 | 0.41 | 0.24 | 0.20 | 0.40 | 0.27 | 0.85 | 0.40 | 0.24 | 0.06 | 0.53 | 0.27 | 0.15 |
| 8* | 1 | 13 | 10 | 0.30 | 0.19 | 0.76 | 0.42 | 0.51 | 1.00 | 0.36 | 0.27 | 0.16 | 0.36 | 0.29 | 0.78 | 0.40 | 0.36 | 0.48 | 0.61 | 0.57 | 0.98 |
| 8**| 1 | 13 | 10 | 0.30 | 0.22 | 0.78 | 0.41 | 0.54 | 1.00 | 0.34 | 0.25 | 0.13 | 0.34 | 0.24 | 0.59 | 0.37 | 0.38 | 0.55 | 0.59 | 0.62 | 0.99 |
| Best balanced | 10 | 0.33 | 0.14 | 0.23 | 0.32 | 0.35 | 0.89 | 0.28 | 0.13 | 0.10 | 0.29 | 0.13 | 0.52 | 0.29 | 0.17 | 0.08 | 0.34 | 0.36 | 0.41 |

$W$ test using $\hat{\theta}^{(0)}$

$X^2$ test using Iterated $\hat{\theta}^{(1)}$

Kolmogorov-Smirnov test using $\hat{\theta}^{(0)}$

**Table 5**: Power under deviations in mean, variance, skewness, and kurtosis using unbalanced partitions ($k = 1, 10$ and $\alpha = 5\%$).
7 Concluding Remarks

This paper has shown that the classical trinity of goodness-of-fit tests and the Chernoff-Lehmann statistic, based on the conditional MLE, can be used to check the specification of continuous conditional distributions using a type of partitions, possibly sample dependent, that involve a Rosenblatt transformation of the dependent variable. These tests have the advantage of using chi-square critical points, and they do not require matrix inversion. The corresponding generalized Wald statistic based on the conditional MLE also relies on chi-square critical points but requires matrix inversion. In both cases the tests are easy to use. We have found that the generalized Wald statistic performs, in general, significantly better than the trinity of tests. This aligns with the conclusion of the simulation study by Rao and Robson (1974) in the context of the classical specification testing of marginal distributions. In this context, Spruill (1976) for approximate Bahadur efficiency and Moore and Spruill (1975) for Pitman efficiency, showed that the efficiency of the trinity of tests can be either superior or inferior to the Wald test.

Appendix

Proof of Theorem 1. By the law of large numbers (LLN) and the central limit theorem (CLT), under $H_0$,

$$
\hat{\Phi}(\theta_0) = \left[\text{diag}^{-1/2}(v \otimes q) + o_p(1)\right] \frac{1}{\sqrt{n}} \sum_{i=1}^{n} ((\eta_i(\theta_0) - v) \otimes \zeta_i) \xrightarrow{d} N_{L,J}(0, Q),
$$

with $Q$ defined in (3). Since $Q$ is idempotent and

$$\text{rank}(Q) = \text{tr}(Q) = \text{tr}\left(\left(I_L - \sqrt{v} \sqrt{v}'\right) \otimes I_J\right) = J(L - 1),$$

26
after applying the continuous mapping theorem (CMT),

$$\hat{X}^2(\theta) = \hat{\Phi}'(\theta) \hat{\Phi}(\theta) \xrightarrow{d} \chi^2_{\text{f}(L-1)}.$$ 

which proves (i). To simplify notation, let \( \hat{\ell}_j = \hat{\ell}_j(\theta) \) until stated otherwise.

Thus, applying a Taylor expansion \( \ln(x+1)=x-x^2/2+o(x^2) \)

$$\ln \left( \frac{\hat{\ell}_j}{nv_\ell q_j} \right) = \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} - \frac{1}{2} \left( \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} \right)^2 + o \left( \left( \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} \right)^2 \right) \quad \text{a.s.}$$

so that

$$\hat{G}^2(\theta) = 2 \sum_{\ell=1}^L \sum_{j=1}^J \hat{\ell}_j \ln \left( \frac{\hat{\ell}_j}{nv_\ell q_j} \right)$$

$$= 2 \sum_{\ell=1}^L \sum_{j=1}^J \left( \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} \right) \ln \left( \frac{\hat{\ell}_j}{nv_\ell \hat{q}_j} \right) + \frac{1}{2} \left( \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} \right)^2 + o \left( \left( \frac{\hat{\ell}_j - nv_\ell \hat{q}_j}{nv_\ell \hat{q}_j} \right)^2 \right)$$

$$= \hat{X}^2(\theta) + o_p(1)$$

applying (9) and noticing that \( \sum_{\ell=1}^L \sum_{j=1}^J \hat{\ell}_j = n \) and \( \sum_{\ell=1}^L v_\ell = \sum_{j=1}^J \hat{q}_j = 1 \), which shows (ii).

Define \( v^\dagger = (v_1, \ldots, v_{L-1})' \), and let \( 1_R \) be an \( R \times 1 \) vector of ones. Applying Sherman and Morrison (1950) formula, \( (\text{diag} \left( v^\dagger \right) - v^\dagger v^\dagger) = \text{diag}^{-1} \left( v^\dagger \right) + 1_{L-1} 1_{L-1}' / v_L \). Hence,

\[
[(\text{diag}(v) - vv') \otimes \text{diag}(q)]^{-1} = \begin{pmatrix}
\text{diag}^{-1} \left( v^\dagger \right) + 1_{L-1} 1_{L-1}' / v_L & 0_{LJ-1} \\
0_{LJ-1} & 0
\end{pmatrix} \otimes \text{diag}^{-1}(q),
\]

and,

\[\hat{W}(\theta) = \hat{\Phi}'(\theta) Q^- \hat{\Phi}(\theta)\]
\[
\frac{1}{n} \left( \hat{O}(\theta_0) - nv \otimes \hat{q} \right) \left[ \left( \text{diag}^{-1}(\mathbf{v}) + 1_{L-1}1_{L-1}/v_L \ 0_{LJ-1} \right) \otimes \text{diag}^{-1}(\hat{q}) \right] \\
\times \left( \hat{O}(\theta_0) - nv \otimes \hat{q} \right)
\]

\[
= \frac{1}{n} \sum_{l=1}^{L-1} \sum_{j=1}^{J} \left( \hat{O}_{lj} - n v_l \hat{q}_j \right)^2 + \frac{1}{n} \sum_{j=1}^{J} \left( \sum_{l=1}^{L-1} \left( \hat{O}_{lj} - n v_l \hat{q}_j \right) \right)^2
\]

\[
= \frac{1}{n} \sum_{l=1}^{L-1} \sum_{j=1}^{J} \left( \hat{O}_{lj} - n v_l \hat{q}_j \right)^2 + \frac{1}{n} \sum_{j=1}^{J} \left( \hat{O}_{LJ} - n v_L \hat{q}_j \right)^2
\]

\[
= \hat{X}^2(\theta_0),
\]

which shows (iii). Notice that \( \hat{\Phi}'(\theta)Q - \hat{\Phi}(\theta) = \hat{X}^2(\theta) \) for all \( \theta \in \Theta \).

Now define \( \hat{O}_{lj} = \hat{O}_{lj}(\theta) \). Then, for all \( \theta \in \Theta \),

\[
\tilde{M}(\theta) = n \sum_{r=1}^{LJ-1} \left( \hat{O}_r^* \frac{n \hat{\pi}_r^* - \hat{O}_{LJ}}{n \hat{\pi}_{LJ}} \right)^2 \hat{\pi}_r^* - n \sum_{r=1}^{LJ-1} \sum_{s=1}^{LJ-1} \left( \hat{O}_r^* \frac{n \hat{\pi}_r^* - \hat{O}_{LJ}}{n \hat{\pi}_{LJ}} \right) \left( \hat{O}_s^* \frac{n \hat{\pi}_s^* - \hat{O}_{LJ}}{n \hat{\pi}_{LJ}} \right) \hat{\pi}_r^* \hat{\pi}_s^*.
\]

The second term in (10) equals

\[
- n \left[ \sum_{r=1}^{LJ-1} \left( \hat{O}_r^* \frac{n \hat{\pi}_r^* - \hat{O}_{LJ}}{n \hat{\pi}_{LJ}} \right) \hat{\pi}_r^* \right]^2 = - n \left( \hat{O}_{LJ} \frac{n \hat{\pi}_{LJ}}{n \hat{\pi}_{LJ}} - 1 \right)^2
\]

using the fact that \( \sum_{r=1}^{LJ-1} \hat{O}_r^* = n - \hat{O}_{LJ} \) and \( \sum_{r=1}^{LJ-1} \hat{\pi}_r^* = 1 - \hat{\pi}_{LJ}^* \). To simplify the first sum in (10) we write

\[
\frac{\hat{O}_r^*}{n \hat{\pi}_r^*} - \frac{\hat{O}_{LJ}}{n \hat{\pi}_{LJ}} = \left\{ \hat{\pi}_r^* \left( \hat{O}_r^* \frac{n \hat{\pi}_r^* - \hat{O}_{LJ}}{n \hat{\pi}_{LJ}} \right) - \hat{\pi}_r^* \left( \hat{O}_{LJ} \frac{n \hat{\pi}_{LJ}}{n \hat{\pi}_{LJ}} - 1 \right) \right\} \frac{1}{\hat{\pi}_r^* \hat{\pi}_{LJ}}.
\]

Then, because \( \sum_{r=1}^{LJ-1} \left( \frac{\hat{O}_r^*}{n \hat{\pi}_r^*} - \hat{\pi}_r^* \right) = - \left( \frac{\hat{O}_{LJ}}{n \hat{\pi}_{LJ}} - \hat{\pi}_{LJ}^* \right) \), the first term in (10) becomes

\[
n \left\{ \sum_{r=1}^{LJ-1} \frac{1}{\hat{\pi}_r^*} \left( \frac{\hat{O}_r^*}{n \hat{\pi}_r^*} \right)^2 \right. + \frac{2}{\hat{\pi}_{LJ}^*} \left( \frac{\hat{O}_{LJ}}{n \hat{\pi}_{LJ}} - \hat{\pi}_{LJ}^* \right)^2 + \frac{1}{\hat{\pi}_{LJ}^*} (1 - \hat{\pi}_{LJ}^*) \left( \frac{\hat{O}_{LJ}}{n \hat{\pi}_{LJ}} - \hat{\pi}_{LJ}^* \right)^2 \left\}}
\]

\[
= n \left\{ \sum_{l=1}^{L} \sum_{j=1}^{J} \frac{1}{\hat{\pi}_{LJ}^*} \left( \frac{\hat{O}_{lj}}{n \hat{\pi}_{LJ}} - \hat{\pi}_{LJ}^* \right)^2 + \frac{1}{\hat{\pi}_{LJ}^*} \left( \frac{\hat{O}_{LJ}}{n \hat{\pi}_{LJ}} - \hat{\pi}_{LJ}^* \right)^2 \right\},
\]

which proves (iv).
The next lemma, which is an application of Lemma 1 by Delgado and Stute (2008), is used in several places below. Define, for \( V_i = F_{Y_i|X_i} (Y_i | X_i) \), real \( \kappa_1, \ldots, \kappa_n \) that range in a compact interval, and, for a given \( A \in \mathbb{R}^k \),

\[
\hat{\beta}(\kappa_1, \ldots, \kappa_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \mathbb{1}_{\{V_i \leq \theta + \kappa_in^{-1/2}\}} - \mathbb{1}_{\{V_i \leq \theta\}} - \kappa_in^{-1/2} \right] \mathbb{1}_{\{X_i \in A\}}.
\]

**Lemma 1.** For each finite \( K \),

\[
\sup_{\{\kappa_i \leq K, i=1, \ldots, n\}} \left| \hat{\beta}(\kappa_1, \ldots, \kappa_n) \right| = o_p(1), \quad \ell = 1, \ldots, L \text{ and } j = 1, \ldots, J.
\]

Define \( \zeta_{ji} = \mathbb{1}_{\{X_i \in A_j\}} \) and \( \alpha_{\ell i}(\theta_1, \theta_2) = \partial F_{Y_i|X_i} \left( F_{Y_i|X_i}^{-1}(\partial \ell | X_i) \right) / \partial \theta \big|_{\theta = \theta_1} \).

**Proof of Theorem 2.** Notice that under \( H_0 \), for all \( j = 1, \ldots, J \) and \( \ell = 1, \ldots, L \),

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i \leq \theta_\ell\}} - \partial \ell \right) \zeta_{ji} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i \leq \theta_\ell\}} - \partial \ell \right) \mathbb{1}_{\{V_i \leq \theta_\ell\} - \partial \ell} \zeta_{ji}, \quad (11)
\]

Applying the mean value theorem (MVT), for all \( i = 1, \ldots, n \), under \( H_0 \),

\[
F_{Y_i|X_i} \left( F_{Y_i|X_i}^{-1}(\partial \ell | X_i) \right) = \partial \ell + \alpha_{\ell i}(\bar{\theta}_i, \tilde{\theta}) (\theta_0 - \tilde{\theta}) \quad (12)
\]

for some \( \bar{\theta}_i \in \mathbb{R}^p \) such that \( \|\bar{\theta}_i - \theta_0\| \leq \|\theta - \theta_0\| \). Since \( \alpha_{\ell i}(\bar{\theta}_i, \tilde{\theta}) \) is bounded in a neighborhood of \( \theta_0 \) for \( i = 1, 2, \ldots, n \), \( \bar{\kappa}_i \ell = \alpha_{\ell i}(\bar{\theta}_i, \tilde{\theta}) \sqrt{n} (\theta_0 - \tilde{\theta}) \) range, with high probability, in a possibly large but compact set. Then (11) equals to

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i \leq \theta_\ell\}} - \partial \ell \right) + \frac{1}{n} \sum_{i=1}^{n} \alpha_{\ell i}(\bar{\theta}_i, \tilde{\theta}) \zeta_{ji} (\theta_0 - \tilde{\theta})
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \mathbb{1}_{\{V_i \leq \theta_\ell + \bar{\kappa}_i \ell n^{-1/2}\}} - \mathbb{1}_{\{V_i \leq \theta_\ell\}} - \bar{\kappa}_i \ell n^{-1/2} \right] \zeta_{ji},
\]

and the last term is \( o_p(1) \) by Lemma 1. Then, applying Lemma 1, the LLN, and noticing that \( \tau_{\ell i}(X_i) = \alpha_{\ell i}(\theta, \theta) - \alpha_{(\ell-1)i}(\theta, \theta) \) under \( H_0 \),

\[
\hat{\Phi} (\tilde{\theta}) = \text{diag}^{-1/2} \left( v \otimes \frac{1}{n} \sum_{i=1}^{n} \zeta_i \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\eta_i(\tilde{\theta}) - v) \otimes \zeta_i
\]

29
\[
= \text{diag}^{-1/2}(v \otimes \hat{q}) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (\eta_i(\theta_0) - v) \otimes \zeta_i - (\tau_{\theta_0}(X_i) \otimes \zeta_i) (\hat{\theta} - \theta_0) \right] + o_p(1)
\]
\[
= \Phi(\theta_0) - B(\theta_0) \sqrt{n} (\hat{\theta} - \theta_0) + o_p(1).
\]

**Proof of Theorem 3.** Define \( \ell_{\theta_0} = \ell_0(Y, X_i) \). Notice that by the LLN and the CLT,

\[
\left( \begin{array}{c}
\hat{\Phi}(\theta_0) \\
B(\theta_0) \sqrt{n} (\hat{\theta} - \theta_0)
\end{array} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \begin{array}{c}
\text{diag}^{-1/2}(v \otimes q)(\eta_i(\theta_0) - v) \otimes \zeta_i \\
B(\theta_0) \ell_{\theta_0}
\end{array} \right) + o_p(1)
\]

\[
\Rightarrow \quad \frac{d}{dL, L \to p} \left( \begin{array}{c}
0, \\
Q B(\theta_0) \mathcal{Y}(\theta_0)
\end{array} \right).
\]

Then, apply Theorem 2.

**Proof of Theorem 4.** First, \( \hat{\theta} = \theta_0 + o_p(1) \) by Assumption 5 and Rao (2002, Section 5.e.2 (i)). Recall that \( \hat{\theta} \) solves the equations

\[
0 = \sum_{\ell=1}^{L} \sum_{j=1}^{J} \hat{\ell}_{\ell j}(\theta_0) \frac{\partial \hat{\pi}_{\ell j, \theta_0}(\theta)}{\partial \theta} \text{evaluated at } \hat{\theta}
\]

and \( \hat{\pi}(\theta) = v \otimes \hat{q} \) for all \( \theta \in \Theta \). Taking into account that

\[
\sum_{\ell=1}^{L} \frac{\partial}{\partial \theta} \hat{\pi}_{\ell j, \theta_0}(\theta) = \frac{\partial}{\partial \theta} \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{L} \int_{\mathcal{Y}^{-1}(X_{\theta_0} \theta_{i-1} \mid X_i)} F_{\mathcal{Y} \mid X_{\theta}}(d\theta \mid X_i) = 0,
\]

we can express (13) as

\[
0 = \sum_{\ell=1}^{L} \sum_{j=1}^{J} \left( \frac{\hat{\ell}_{\ell j}(\theta_0)}{n \cdot \hat{\pi}_{\ell j, \theta_0}(\hat{\theta})} - 1 \right) \frac{\partial}{\partial \theta} \hat{\pi}_{\ell j, \theta_0}(\theta) \bigg|_{\theta = \hat{\theta}}
\]

\[
= n^{-1/2} \cdot \text{diag}^{-1} \left( \hat{\pi}_{\theta_0}(\hat{\theta}) \right) \cdot \text{diag}(v \otimes \hat{q}) \cdot \hat{B}'(\hat{\theta})
\]

\[
\times \left[ \text{diag}^{-1/2}(n \cdot v \otimes \hat{q}) \cdot (\hat{\Phi}(\theta_0) - n \cdot v \otimes \hat{q}) - \text{diag}^{-1/2}(v \otimes \hat{q}) \cdot \sqrt{n} \left( \hat{\pi}_{\theta_0}(\hat{\theta}) - v \otimes \hat{q} \right) \right].
\]

Hence, under \( H_0 \), applying the LLN, the CMT, and a MVT argument,

\[
0 = B'(\theta_0) \left[ \Phi(\theta_0) - B(\theta_0) \sqrt{n} (\hat{\theta} - \theta_0) \right] (1 + o_p(1))
\]
so that \( \sqrt{n}(\hat{\theta} - \theta_0) = [B'(\theta_0)B(\theta_0)]^{-1}B'(\theta_0)\hat{\Phi}(\theta_0)(1 + o_p(1)) \). Now, apply arguments from the proof of Theorem 1, taking into account that \( B'(\theta_0)Q = B'(\theta_0) \) by (14).

Proof of Corollary 1. By the LLN and CMT, \( \hat{\Phi}(\hat{\theta}) = B(\theta_0) + o_p(1) \) and, hence,

\[
\sqrt{n}(\hat{\theta}^{(1)} - \theta_0) = \sqrt{n}(\bar{\theta} - \theta_0) + \hat{\Sigma}^{-1}(\bar{\theta}) \hat{B}'(\hat{\theta}) \hat{\Phi}(\hat{\theta})
\]

\[
= \sqrt{n}(\bar{\theta} - \theta_0) + \left[ \Sigma^{-1}(\theta_0)B'(\theta_0) + o_p(1) \right] \hat{\Phi}(\hat{\theta})
\]

Then, apply Theorem 3.

Define \( R(\theta_0) = B(\theta_0)\Sigma^{-1}(\theta_0)B^{-1}(\theta_0) \).

Proof of Theorem 5. Apply Theorem 4 with \( \bar{\theta} = \hat{\theta} \). Notice that in this case, \( L(\theta) = \Sigma^{-1}(\theta) \) for all \( \theta \in \Theta \), and \( Y(\theta) = B(\theta_0)\Sigma^{-1}(\theta_0)B'(\theta_0) \). Thus, \( \hat{\Phi}(\hat{\theta}) \rightarrow \mathcal{N}_{LJ}(0, \Omega_{\hat{\theta}}) \), where \( \Omega_{\hat{\theta}} = Q - R(\theta_0) \) is idempotent with rank \( (\Omega_{\hat{\theta}}) = \text{tr}(\Omega_{\hat{\theta}}) = J(L - 1) - p \). This proves that \( \hat{X}^2(\hat{\theta}), \hat{G}^2(\hat{\theta}) \), and \( \hat{W}(\hat{\theta}) \) share the same limiting distribution under \( H_0 \), a \( \chi^2_{J(L-1)-p} \).

Proof of Theorem 6. Since \( \mathbb{I}(\theta_0) \) is positive definite, and all the components of \( \sqrt{v}v^T \otimes I_J \) are nonnegative, there exists an orthonormal matrix \( C \) that simultaneously diagonalizes \( \sqrt{v}v^T \otimes I_J, R(\theta_0) \) and \( S(\theta_0) = B(\theta_0)\mathbb{I}^{-1}(\theta_0)B'(\theta_0) \), and which satisfies

\[
[C \left( \sqrt{v}v^T \otimes I_J \right) C']_{1_LJ} = \begin{cases} 
0 & \text{if} \ j = 1, \ldots, J(L - 1), \\
1 & \text{if} \ j = J(L - 1) + 1, \ldots, JL,
\end{cases}
\]

(15)

\[
[CR(\theta_0)C']_{1_LJ} = \begin{cases} 
0 & \text{if} \ j = 1, \ldots, J(L - 1) - p, J(L - 1) + 1, \ldots, JL, \\
1 & \text{if} \ j = J(L - 1) - p + 1, \ldots, J(L - 1),
\end{cases}
\]

(16)

\[
[CS(\theta_0)C']_{1_LJ} = \begin{cases} 
0 & \text{if} \ j = 1, \ldots, J(L - 1) - p, J(L - 1) + 1, \ldots, JL, \\
1 - \lambda_j & \text{if} \ j = J(L - 1) - p + 1, \ldots, J(L - 1),
\end{cases}
\]
It follows reasoning as in the proof of Moore and Spruill (1975, Lemma 5.1). Since $\sqrt{v} \sqrt{v} \otimes I_J$ and $S(\theta_0)$ are commuting matrices, there exists an orthonormal matrix $C$ that diagonalizes them. Moreover, $\sqrt{v} \sqrt{v} \otimes I_J$ and $S(\theta_0)$ are orthogonal projections having ranks $J$ and $p$, respectively, so that by proper choice of basis we can take $C$ to satisfy (15) and (16). $S(\theta_0)$ has rank $p$ and range contained in the range of $R(\theta_0)$. It follows that the eigenvalues of $S(\theta_0)$ are $0$ except for those associated with the eigenvectors in the range of $R(\theta_0)$. That these are $1 - \lambda_j$ follows from the fact that they are roots of the determinantal equation $|S(\theta_0) - \beta I_{JL}| = 0$. Hence, we have

$$[C \Omega_{\theta(0)} C' 1_{LJ}]_j = \begin{cases} 1 & \text{if } j = 1, \ldots, J(L - 1) - p, \\ \lambda_j & \text{if } j = J(L - 1) - p + 1, \ldots, J(L - 1), \\ 0 & \text{if } j = J(L - 1) + 1, \ldots, LJ. \end{cases}$$

Proof of Corollary 2. Under Assumption 6, $0 < \lambda_j < 1$, $j = 1, \ldots, LJ$. Therefore, $\text{rank} \left( \text{Avar} \left( \Phi \left( \theta^{(0)} \right) \right) \right) = J(L - 1)$, and the result follows taking into account that $(I_{JL} - \hat{G}_{\theta(0)})^{-1} = (I_{JL} - G_{\theta(0)})^{-1} + o_p(1)$ and $\Omega_{\theta(0)} = (I_{JL} - G_{\theta(0)})^{-1}$.

Define the empirical measure $P_n = n^{-1} \sum_{i=1}^{n} \delta_{(Y_i, X_i)}$, where $\delta_{(Y, X)}$ is the Dirac measure. So, for any Borel set $A$ in $\mathbb{R}^{1+k}$, $P_n(A) = n^{-1} \sum_{i=1}^{n} 1_{\{(Y_i, X_i) \in A\}}$ and for any function $f : \mathbb{R}^{1+k} \to \mathbb{R}$, $P_n f = n^{-1} \sum_{i=1}^{n} f(Y_i, X_i)$; we also use the notation $P f = \int f(y, x) dP(y, x)$.

Proof of Theorem 7. First we show that

$$\hat{q} = \tilde{q} + o_p(1),$$
$$\hat{B} (\tilde{\theta}) = \tilde{B} (\tilde{\theta}) + o_p(1),$$
$$\hat{\Phi} (\tilde{\theta}) = \tilde{\Phi} (\tilde{\theta}) + o_p(1).$$
To prove (17), notice that \( \hat{q} - \check{q} = P_n(1_A - 1_A) \) with \( 1_A(x) = (1_{A_1}(x), \ldots, 1_{A_J}(x))' \), \( 1_{A_j}(x) = 1_{\{x \in A_j\}} \), and

\[
\left| P_n(1_{A_j} - 1_{A_j}) \right| = (P_n - P)1_{A_j \triangle A_j} + P1_{A_j \triangle A_j} = o_p(1), \quad j = 1, \ldots, J. \tag{20}
\]

The first term on the right hand side of (20) is \( o(1) \) a.s. because the class of functions \( \mathcal{F} = \{1_A : A \in \mathcal{A}\} \) is \( P \)-Donsker by Assumption 7, and, hence, \( P \)-Glivenko-Cantelli, i.e., \( \sup_{A \in \mathcal{A}} |(P_n - P)1_A| = o(1) \) a.s. Thus, \( (P_n - P)1_{A_j \triangle A_j} = o(1) \) a.s. The second term in (20) is \( o_p(1) \) by Assumption 7, since \( \hat{A}_j \triangle A_j \in \mathcal{A} \), which proves (17).

In order to prove (18), apply (17) to write

\[
\hat{B}(\check{\theta}) - \hat{B}(\check{\theta}) = \left[ \text{diag}^{-1/2}(\nu \otimes \check{q}) + o_p(1) \right] P_n(\tau_{\check{\theta}} \otimes (1_A - 1_A)) = o_p(1),
\]

because, for \( \ell = 1, \ldots, L \) and \( j = 1, \ldots, L \),

\[
\left| P_n(\tau_{\theta \ell} \cdot (1_A - 1_A)) \right| = (P_n - P)\left(\tau_{\theta \ell} \cdot 1_{A \triangle A}\right) + P\tau_{\theta \ell} \cdot 1_{A \triangle A} = o_p(1) \tag{21}
\]

To prove that the first term on the right hand side of (21) is \( o_p(1) \), take into account that the functions \( \tau_{\theta \ell} \) are continuous on the compact \( \Theta \). Thus, the class of functions \( \{\tau_{\theta \ell} : \theta \in \Theta\} \) has an integrable envelope function (Assumption 3) and, hence, is a \( P \)-Glivenko-Cantelli class (e.g., van der Vaart, 1998, Example 19.8). Therefore, the class of functions \( \{\tau_{\theta \ell} \cdot 1_A : \theta \in \Theta, A \in \mathcal{A}\} \) is also \( P \)-Glivenko-Cantelli (e.g., van der Vaart and Wellner, 2000, Theorem 3); i.e., \( \sup_{\theta \in \Theta, A \in \mathcal{A}} \| (P_n - P)\tau_{\theta \ell} \cdot 1_A \| = o_p(1) \). The second term on the right hand side of (21) is \( o_p(1) \) because

\[
\left\| P\tau_{\theta \ell} \cdot 1_{\hat{A}_j \triangle A_j} \right\| \leq 2 \cdot \mu(\hat{A}_j \triangle A_j) = o_p(1), \tag{22}
\]

where \( \mu(A) = \int_{x \in A} m(x) \, dP(x) \), by Assumptions 3 and 7, after taking into account that \( \mu \) is an absolutely continuous (signed) measure with respect to \( P \).
In order to prove (19), notice that the \( \ell_j \)-th component of \( \hat{\Phi}(\hat{\theta}) - \Phi(\hat{\theta}) \) is

\[
\hat{\phi}_{\ell_j}(\hat{\theta}) - \phi_{\ell_j}(\hat{\theta}) = \frac{1}{\sqrt{n}v_{\ell济 j}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i(\hat{\theta}) \in U_i\}} - v_{\ell_j} \right) \mathbb{1}_{A_j}(X_i)
\]

\[
- \frac{1}{\sqrt{n}v_{\ell济 j}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i(\hat{\theta}) \in U_i\}} - v_{\ell_j} \right) \mathbb{1}_{A_j}(X_i)
\]

\[
= \left( \frac{1}{\sqrt{n}v_{\ell济 j}} + o_p(1) \right) \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i(\hat{\theta}) \in U_i\}} - v_{\ell_j} \right) \mathbb{1}_{A_j \Delta A_j}(X_i)
\]

by (17). Define \( \tilde{\kappa}_n(X_i) = \partial F_{Y|X,\theta} \left( \left( \mathbb{F}_{Y|X,\theta}(\theta_0 | X_i) \right) / \partial \theta' \right) \theta = \hat{\theta}_{\ell_j} \) for some \( \hat{\theta}_{\ell_j} \in \mathbb{R}^p \) such that \( \|\hat{\theta}_{\ell_j} - \theta_0\| \leq \|\theta - \theta_0\| \). Thus, using (11), (12) and the MVT, (19) follows from

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{1}_{\{V_i \leq \theta_{\ell_j} + n^{-1/2} \tilde{\kappa}_n(X_i)\}} - \theta_{\ell_j} \right) \mathbb{1}_{A_j \Delta A_j}(X_i) = \sqrt{n}P_n \left( \hat{f}_{n,A_j \Delta A_j} - f_{A_j \Delta A_j}^0 \right) + \sqrt{n}P_n f_{A_j \Delta A_j}^0
\]

(23)

being \( o_p(1) \), where \( \hat{f}_{n,A}(y, x) = \left( \mathbb{1}_{\{F_{Y|X}(y | x) \leq \theta_{\ell_j} + n^{-1/2} \tilde{\kappa}_n(x)\}} - \theta_{\ell_j} \right) \mathbb{1}_{A}(x) \) and \( f_{A}^0(y, x) = \left( \mathbb{1}_{\{F_{Y|X}(y | x) \in U_i\}} - v_{\ell_j} \right) \mathbb{1}_{A}(x) \). Write the first term on the right hand side in (23) as

\[
\mathbb{G}_n \left( \hat{f}_{n,A_j \Delta A_j} - f_{A_j \Delta A_j}^0 \right) + \sqrt{n}P \left( \hat{f}_{n,A_j \Delta A_j} - f_{A_j \Delta A_j}^0 \right),
\]

(24)

with \( \mathbb{G}_n = \sqrt{n}(P_n - P) \). To show that (24) is \( o_p(1) \), first note that \( f_{A}^0 \) is in the \( P \)-Donsker class of functions \( \mathcal{G} = \mathcal{G} \cdot \mathcal{F} \), where \( \mathcal{G} \) is a fixed bounded function (cf. van der Vaart and Wellner, 1988, Example 2.10.10), and \( \hat{f}_{n,A} \) takes values in \( \mathcal{G} \). Second,

\[
\int \left( \hat{f}_{n,A} - f_{A}^0 \right)^2 (y, x) dP(y, x)
\]

\[
= \int_{x \in A} \int_{\mathbb{R}} \left( \mathbb{1}_{\{F_{Y|X}(y | x) \leq \theta_{\ell_j} + n^{-1/2} \tilde{\kappa}_n(x)\}} - \mathbb{1}_{\{F_{Y|X}(y | x) \in U_i\}} \right)^2 \mathbb{1}_{A}(x) dP(y, x)
\]

\[
= 2\theta_{\ell_j} + \int_{x \in A} \left( n^{-1/2} \tilde{\kappa}_n(x) - 2 \min \left( \theta_{\ell_j} + n^{-1/2} \tilde{\kappa}_n(x), \theta_{\ell_j} \right) \right) dF_X(x) = o_p(1),
\]

by Assumption 3, \( \hat{\theta} = O_p(n^{-1/2}) \), and dominated convergence. Thus, applying van der Vaart (1998) Lemma 19.24, \( \sup_{A \in A} \left| \mathbb{G}_n \left( \hat{f}_{n,A} - f_{A}^0 \right) \right| \xrightarrow{P} 0 \), which implies that the first term
in (24) is $o_p(1)$. In order to prove that the second term in (24) is $o_p(1)$, notice that,

$$\sqrt{n}P \left( f_{n, \hat{A}_j \Delta A_j} - f^0_{\hat{A}_j \Delta A_j} \right) = \int_{\hat{A}_j \Delta A_j} \tilde{k}_x(x) F_X(dx) \leq \sqrt{n}||\hat{\theta} - \theta_0|| \cdot \mu(\hat{A}_j \Delta A_j) = O_p(1) \cdot o_p(1)$$

by the $\sqrt{n}$-consistency of $\tilde{\theta}$ and (22). This proves that the first term in (23) is $o_p(1)$. Now

$$E \left( \sqrt{n}P^{\frac{1}{2}} f_{\hat{A}_j \Delta A_j} \right)^2 = \int_{\hat{A}_j \Delta A_j} \left( \mathbb{1}_{\{F_{Y|X}(y | x) \notin U_\ell\}} - v_\ell \right)^2 dP(y, x) \leq P(\hat{A}_j \Delta A_j) = o_p(1)$$

by Assumption 7. Therefore, the second term in (23) is $o_p(1)$.

Notice that, by (17), (18), and (19), $\hat{\Sigma} \left( \tilde{\theta} \right) = \bar{\Sigma} \left( \tilde{\theta} \right) + o_p(1)$, and $\hat{\theta}^{(1)} = \bar{\theta}^{(1)} + o_p(1)$. Hence, since $\hat{\Phi} \left( \tilde{\theta} \right) = o_p(1)$, $\hat{X} \left( \hat{\theta}^{(1)} \right) = \bar{X} \left( \hat{\theta}^{(1)} \right) + o_p(1)$. Using the same arguments as in the proof of Theorem 1, but applying (18), $\hat{\theta}^2 \left( \hat{\theta}^{(1)} \right) = \bar{\theta}^2 \left( \hat{\theta}^{(1)} \right) + o_p(1)$. \hfill \QED

**Proof of Theorem 8.** Recall that $F_{Y|X}(dy | x) = \left[ 1 + n^{-1/2} t_{n\theta_0}(y, x) \right] F_{Y|X, \theta_0}(dy | x)$. Then, compared to the proof of Theorem 2, we have to add another term, namely

$$\frac{1}{\sqrt{n}}\int_{F_{Y|X, \theta}^0(\theta_{i-1} | X_i)}^{F_{Y|X, \theta}^{-1}(\theta_i | X_i)} t_{n\theta_0}(y, X_i) F_{Y|X, \theta_0}(dy | X_i),$$

to the right hand side of (12). Then, the theorem follows mimicking the arguments in Theorem 2’s proof and using the properties of $t_{n\theta_0}$. \hfill \QED

**Proof of Corollary 3.** This is an immediate consequences of Theorems 7 and 8. \hfill \QED

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