Kronecker-Weber via Stickelberger

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RéSUMÉ. Nous donnons une nouvelle démonstration du théorème de Kronecker et Weber fondée sur la théorie de Kummer et le théorème de Stickelberger.

Abstract. In this note we give a new proof of the theorem of Kronecker-Weber based on Kummer theory and Stickelberger’s theorem.

Introduction

The theorem of Kronecker-Weber states that every abelian extension of \( \mathbb{Q} \) is cyclotomic, i.e., contained in some cyclotomic field. The most common proof found in textbooks is based on proofs given by Hilbert [2] and Speiser [7]; a routine argument shows that it is sufficient to consider cyclic extensions of prime power degree \( p^m \) unramified outside \( p \), and this special case is then proved by a somewhat technical calculation of differentials using higher ramification groups and an application of Minkowski’s theorem, according to which every extension of \( \mathbb{Q} \) is ramified. In the proof below, this not very intuitive part is replaced by a straightforward argument using Kummer theory and Stickelberger’s theorem.

In this note, \( \zeta_m \) denotes a primitive \( m \)-th root of unity, and “unramified” always means unramified at all finite primes. Moreover, we say that a normal extension \( K/F \)

- is of type \((p^a, p^b)\) if \( \text{Gal}(K/F) \simeq (\mathbb{Z}/p^a\mathbb{Z}) \times (\mathbb{Z}/p^b\mathbb{Z}) \);
- has exponent \( m \) if \( \text{Gal}(K/F) \) has exponent \( m \).

1. The Reduction

In this section we will show that it is sufficient to prove the following special case of the Kronecker-Weber theorem (it seems that the reduction to extensions of prime degree is due to Steinbacher [8]):

**Proposition 1.1.** The maximal abelian extension of exponent \( p \) that is unramified outside \( p \) is cyclic: it is the subfield of degree \( p \) of \( \mathbb{Q}(\zeta_{p^2}) \).

The corresponding result for the prime \( p = 2 \) is easily proved:

**Proposition 1.2.** The maximal real abelian 2-extension of \( \mathbb{Q} \) with exponent 2 and unramified outside 2 is cyclic: it is the subfield \( \mathbb{Q}(\sqrt{2}) \) of \( \mathbb{Q}(\zeta_8) \).
Proof. The only quadratic extensions of \( \mathbb{Q} \) that are unramified outside 2 are \( \mathbb{Q}(i) \), \( \mathbb{Q}(\sqrt{-2}) \), and \( \mathbb{Q}(\sqrt{2}) \). \( \square \)

The following simple observation will be used repeatedly below:

**Lemma 1.3.** If the compositum of two cyclic \( p \)-extensions \( K, K' \) is cyclic, then \( K \subseteq K' \) or \( K' \subseteq K \).

Now we show that Prop. 1.1 implies the corresponding result for extensions of prime power degree:

**Proposition 1.4.** Let \( K/\mathbb{Q} \) be a cyclic extension of odd prime power degree \( p^m \) and unramified outside \( p \). Then \( K \) is cyclotomic.

**Proof.** Let \( K' \) be the subfield of degree \( p^m \) in \( \mathbb{Q}(\zeta_{p^m+1}) \). If \( K'K \) is not cyclic, then it contains a subfield of type \( (p, p) \) unramified outside \( p \), which contradicts Prop. 1.1. Thus \( K'K \) is cyclic, and Lemma 1.3 implies that \( K = K' \). \( \square \)

Next we prove the analog for \( p = 2 \):

**Proposition 1.5.** Let \( K/\mathbb{Q} \) be a cyclic extension of degree \( 2^m \) and unramified outside 2. Then \( K \) is cyclotomic.

**Proof.** If \( m = 1 \) we are done by Prop. 1.2. If \( m \geq 2 \), assume first that \( K \) is nonreal. Then \( K(i)/K \) is a quadratic extension, and its maximal real subfield \( M \) is cyclic of degree \( 2^m \) by Prop. 1.2. Since \( K/\mathbb{Q} \) is cyclotomic if and only if \( M \) is, we may assume that \( K \) is totally real.

Now let \( K' \) be the maximal real subfield of \( \mathbb{Q}(\zeta_{2^{m+2}}) \). If \( K'K \) is not cyclic, then it contains three real quadratic fields unramified outside 2, which contradicts Prop. 1.2. Thus \( K'K \) is cyclic, and Lemma 1.3 implies that \( K = K' \). \( \square \)

Now the theorem of Kronecker-Weber follows: first observe that abelian groups are direct products of cyclic groups of prime power order; this shows that it is sufficient to consider cyclic extensions of prime power degree \( p^m \). If \( K/\mathbb{Q} \) is such an extension, and if \( q \neq p \) is ramified in \( K/\mathbb{Q} \), then there exists a cyclic cyclotomic extension \( L/\mathbb{Q} \) with the property that \( KL = FL \) for some cyclic extension \( F/\mathbb{Q} \) of prime power degree in which \( q \) is unramified. Since \( K \) is cyclotomic if and only if \( F \) is, we see that after finitely many steps we have reduced Kronecker-Weber to showing that cyclic extensions of degree \( p^m \) unramified outside \( p \) are cyclotomic. But this is the content of Prop. 1.4 and 1.5.

Since this argument can be found in all the proofs based on the Hilbert-Speiser approach (see e.g. Greenberg [1] or Marcus [6]), we need not repeat the details here.
2. Proof of Proposition 1.1

Let $K/Q$ be a cyclic extension of prime degree $p$ and unramified outside $p$. We will now use Kummer theory to show that it is cyclotomic. For the rest of this article, set $F = Q(\zeta_p)$ and define $\sigma_a \in G = \text{Gal}(F/Q)$ by $\sigma_a(\zeta_p) = \zeta_p^a$ for $1 \leq a < p$.

**Lemma 2.1.** The Kummer extension $L = F(\sqrt[p]{\mu})$ is abelian over $Q$ if and only if for every $\sigma_a \in G$ there is a $\xi \in F^\times$ such that $\sigma_a(\mu) = \xi^p \mu^a$.

For the simple proof, see e.g. Hilbert [3 Satz 147] or Washington [9 Lemma 14.7].

Let $K/Q$ be a cyclic extension of prime degree $p$ and unramified outside $p$. Put $F = Q(\zeta_p)$ and $L = KF$; then $L = F(\sqrt[p]{\mu})$ for some nonzero $\mu \in O_F$, and $L/F$ is unramified outside $p$.

**Lemma 2.2.** Let $q$ be a prime ideal in $F$ with $(\mu) = q^r a$, $q \nmid a$; if $p \nmid r$ and $L/Q$ is abelian, then $q$ splits completely in $F/Q$.

**Proof.** Let $\sigma$ be an element of the decomposition group $Z(q|q)$ of $q$. Since $L/Q$ is abelian, we must have $\sigma(\mu) = \xi^p \mu^a$. Now $\sigma_a(\mu) = q$ implies $q^r \mid \xi^p \mu^a$, and this implies $r \equiv ar \mod p$; but $p \nmid r$ show that this is possible only if $a = 1$. Thus $\sigma_a = 1$, and $q$ splits completely in $F/Q$. □

In particular, we find that $(1 - \zeta) \nmid \mu$. Since $L/F$ is unramified outside $p$, prime ideals $p \nmid p$ must satisfy $p^{bp} \mid \mu$ for some integer $b$. This shows that $(\mu) = a^p$ is the $p$-th power of some ideal $a$. From $(\mu) = a^p$ and the fact that $L/Q$ is abelian we deduce that $\sigma_a(a)^p = a^{pa} \xi^p$, where $\sigma_a(\zeta_p) = \zeta_p^a$. Thus $\sigma_a(c) = c^a$ for the ideal class $c = [a]$ and for every $a$ with $1 \leq a < p$. Now we invoke Stickelberger’s Theorem (cf. [4 or 5 Chap. 11]) to show that $a$ is principal:

**Theorem 2.3.** Let $F = Q(\zeta_p)$; then the Stickelberger element

$$\theta = \sum_{a=1}^{p-1} a\sigma_a^{-1} \in \mathbb{Z}[\text{Gal}(F/Q)]$$

annihilates the ideal class group $\text{Cl}(F)$.

From this theorem we find that $1 = \theta^p = \prod_{a=1}^{p-1} \sigma_a^{-1}(a)^a = \varepsilon^{p-1} = c^{-1}$, hence $c = 1$ as claimed. In particular $a = (\alpha)$ is principal. This shows that $\mu = \alpha^p \eta$ for some unit $\eta$, hence $L = F(\sqrt[p]{\eta})$. Now write $\eta = \zeta^t \varepsilon$ for some unit $\varepsilon$ in the maximal real subfield of $F$. Since $\varepsilon$ is fixed by complex conjugation $\sigma_{-1}$ and since $L/Q$ is abelian, we see that $\zeta^{-t} \varepsilon = \sigma_{-1}(\mu) = \xi^p \mu^{-1}$, hence $\zeta^{-t} \varepsilon = \xi^p \zeta^{-t} \varepsilon^{-1}$. Thus $\varepsilon$ is a $p$-th power, and we find $\mu = \zeta^t$. But this implies that $L = Q(\zeta_p^2)$, and Prop. 1.1 is proved.

Since every cyclotomic extension is ramified, we get the following special case of Minkowski’s theorem as a corollary:
Corollary 2.4. *Every solvable extension of \( \mathbb{Q} \) is ramified.*

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