An Invariant of Symmetry Protected Topological Phases with On-Site Finite Group Symmetry for Two-Dimensional Fermion Systems

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Abstract: We consider SPT-phases with on-site finite group $G$ symmetry for two-dimensional Fermion systems. We derive an invariant of the classification.

1. Introduction

The notion of symmetry protected topological (SPT) phases was introduced by Gu and Wen [GW]. It is defined as follows: we consider the set of all Hamiltonians with some symmetry, which have a unique gapped ground state in the bulk, and can be smoothly deformed into a common trivial gapped Hamiltonian without closing the gap. We say two such Hamiltonians are equivalent, if they can be smoothly deformed into each other, without breaking the symmetry. We call an equivalence class of this classification, a symmetry protected topological (SPT) phase. In [BO] we derived $\mathbb{Z}_2 \times H^1(G, U(1)) \times H^2(G, U(1))$-valued invariant of one-dimensional Fermionic SPT-phases. In this paper, we derive an invariant of SPT-phases in two-dimensional Fermionic systems.

We start by summarizing standard setup of Fermionic systems on the two dimensional lattice $\mathbb{Z}^2$ [BR1, BR2, EK]. We will use freely the basic notation in Sect. A. We also use the facts and notation of graded $C^*$-algebra from [Bla] and [BO].

Let us first recall the definition of the self-dual CAR-algebra introduced by Araki [A]. (See also [EK] Chapter 6.)

Definition 1.1. For a Hilbert space $\mathcal{R}$ with a complex conjugation $\mathcal{C}$ (i.e., anti-unitary such that $\mathcal{C} = \mathcal{C}^*$), self-dual-CAR-algebra $\mathfrak{A}_{SDC}(\mathcal{R}, \mathcal{C})$ over $(\mathcal{R}, \mathcal{C})$ is defined as the universal enveloping $C^*$-algebra generated by $\{B(f) \mid f \in \mathcal{R}\}$ such that

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\[ \mathcal{A} \ni f \mapsto B(f), \quad \text{linear} \]
\[ \{ B(f), B(g) \} = \langle f, g \rangle \mathbb{I}, \]
\[ B(f)^* = B(Cf), \quad f, g \in \mathcal{A}. \] (1.1)

If \( u \in \mathcal{U}(\mathcal{A}) \) satisfies \( u\mathcal{C} = \mathcal{Cu} \), then there exists an automorphism \( \Xi_u \in \text{Aut}(\mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C})) \) such that \( \Xi_u(B(f)) = B(uf), f \in \mathcal{A} \). In particular, for \( u = \mathbb{I}_\mathcal{A}, \Theta_\mathcal{R} := \Xi_{-\mathbb{I}} \) defines an automorphism on \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \) satisfying \( \Theta_\mathcal{R}^2 = \text{id} \). Hence it defines a grading over \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \). In general, for a graded \( C^* \)-algebra \( \mathcal{B} \), we denote by \( \mathcal{B}^{(0)} \) its even part and by \( \mathcal{B}^{(1)} \) its odd part. Elements in \( \mathcal{B}^{(0)} \) or \( \mathcal{B}^{(1)} \) are said to be homogeneous. For a homogeneous element \( b \in \mathcal{B} \), we denote by \( \delta b \) the grading of \( b \). With this grading, \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \) has an odd self-adjoint unitary (consider \( B(f) \) for \( f \in \mathcal{A} \) with \( Cf = f, \| f \|^2 = 2 \).) We say a state \( \omega \) on \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \) is homogeneous if it is invariant under \( \Theta_\mathcal{R} \). When an automorphism \( \alpha \) on \( \text{Aut}(\mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C})) \) commutes with \( \Theta_\mathcal{R} \), we say that \( \alpha \) is graded. We denote by \( \text{Aut}^{(0)}(\mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C})) \) the set of all graded automorphisms on \( \text{Aut}(\mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C})) \).

A basis projection for \( (\mathcal{A}, \mathcal{C}) \) is an orthogonal projection on \( \mathcal{A} \) such that \( p + \mathcal{C} p \mathcal{C} = \mathbb{I}_\mathcal{A} \). Basis projection exists if \( \mathcal{A} \) is even or infinite dimensional. If \( p \) is a basis projection for \( (\mathcal{A}, \mathcal{C}) \), the self-dual CAR-algebra \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \) is \( * \)-isomorphic to the CAR-algebra \( \mathcal{A}_{CAR}(p\mathcal{A}) \) over \( p\mathcal{A} \) via a \( * \)-isomorphism \( \gamma_p : \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \to \mathcal{A}_{CAR}(p\mathcal{A}) \) such that

\[ \gamma_p(B(f)) = a^*(pf) + a(p\mathcal{C}f), \quad f \in \mathcal{A}. \] (1.2)

Here, \( a^*(g), a(g), g \in p\mathcal{A} \) denotes the creation and annihilation operators of the CAR-algebra \( \mathcal{A}_{CAR}(p\mathcal{A}) \). With the Fock state \( \omega \) over \( \mathcal{A}_{CAR}(p\mathcal{A}) \) (i.e., the state with \( \omega(a^*(g)a(g)) = 0 \) for any \( g \in p\mathcal{A} \)), via the \( * \)-isomorphism above, we can define a state \( \omega_p := \omega \gamma_p^{-1} \) over \( \mathcal{A}_{SDC}(\mathcal{A}, \mathcal{C}) \).

For Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \) with complex conjugations \( \mathcal{C}_1, \mathcal{C}_2 \), there is a \( * \)-isomorphism

\[ \gamma_{12} : \mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1) \otimes \mathcal{A}_{SDC}(\mathcal{H}_2, \mathcal{C}_2) \to \mathcal{A}_{SDC}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2) \] such that

\[ \gamma_{12}(B_1(f_1) \mathbb{I} + \mathbb{I} \hat{\otimes} B_2(f_2)) = B(f_1 \oplus f_2), \quad f_1 \in \mathcal{A}_1, \ f_2 \in \mathcal{A}_2. \] (1.3)

(Recall the graded tensor product \( \hat{\otimes} \) from [Bla] section 14.) Here we denoted the generators of \( \mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1) \) by \( B_1(f_1), f_1 \in \mathcal{A}_1 \). We identify \( \mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1) \otimes \mathcal{A}_{SDC}(\mathcal{H}_2, \mathcal{C}_2) \) and \( \mathcal{A}_{SDC}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2) \) via this isomorphism throughout this paper, without writing \( \gamma_{12} \) explicitly. In particular, for homogeneous states \( \varphi_1, \varphi_2 \) on \( \mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1), \mathcal{A}_{SDC}(\mathcal{H}_2, \mathcal{C}_2) \), we denote the state \( (\varphi_1 \hat{\otimes} \varphi_2) \circ \gamma_{12}^{-1} \) simply by \( \varphi_1 \hat{\otimes} \varphi_2 \). (Recall that \( \varphi_1 \hat{\otimes} \varphi_2 \) is a state on \( \mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1) \otimes \mathcal{A}_{SDC}(\mathcal{H}_2, \mathcal{C}_2) \) such that \( (\varphi_1 \hat{\otimes} \varphi_2)(a \hat{\otimes} b) = \varphi_1(a)\varphi_2(b) \).)

For graded automorphisms \( \alpha_1 \in \text{Aut}^{(0)}(\mathcal{A}_{SDC}(\mathcal{H}_1, \mathcal{C}_1)), \alpha_2 \in \text{Aut}^{(0)}(\mathcal{A}_{SDC}(\mathcal{H}_2, \mathcal{C}_2)) \), we denote the automorphism \( \gamma_{12}(\alpha_1 \hat{\otimes} \alpha_2) \circ \gamma_{12}^{-1} \) simply by \( \alpha_1 \hat{\otimes} \alpha_2 \).

Throughout this paper, we fix some \( d \in 2\mathbb{N} \). For each \( k \in \mathbb{Z} \), we set

\[ H_{U}^k := \mathbb{Z} \times \mathbb{Z}_{\geq k}, \quad H_{D}^k := \mathbb{Z} \times \mathbb{Z}_{\leq k}, \quad H_{L}^k := \mathbb{Z}_{\leq k} \times \mathbb{Z}, \quad H_{R}^k := \mathbb{Z}_{\geq k} \times \mathbb{Z}. \] (1.4)

In particular, left, right, upper, lower half planes are denoted by \( H_{L} := H_{U}^{-1}, H_{R} := H_{D}^{0}, H_{U}^0 := \mathbb{H}_{U}, H_{D}^0 := \mathbb{H}_{D}^{-1} \). We set \( \mathfrak{h} := l^2(\mathbb{Z}^2) \). Let \( \{ \delta_{(x,y)} \mid (x, y) \in \mathbb{Z}^2 \} \) be the standard basis of \( l^2(\mathbb{Z}^2) \), and \( \mathcal{C} \), the complex conjugation on \( l^2(\mathbb{Z}^2) \) with respect to it.

For each \( X \subset \mathbb{Z}^2 \), we set

\[ \mathcal{X} := \{(dx + j, y) \mid (x, y) \in X, \ j = 0, \ldots, d - 1 \}, \quad \mathfrak{h}_X := l^2(\mathcal{X}) \subset l^2(\mathbb{Z}^2). \] (1.5)
We denote the restriction $\mathcal{C}|_h$ of $\mathcal{C}$ on $h_X$ by $\mathcal{C}_X$, and set

$$A_X := \mathfrak{A}_{SDC}(h_X, \mathcal{C}_X), \quad \tilde{A}_X := \mathfrak{A}_{SDC}(l^2(X), \mathcal{C}|_{l^2(X)}) .$$

(1.6)

In particular, we set $A := A_{\mathbb{Z}^2}$. We denote by $\Theta_X := \Theta_{h_X}$ the grading automorphism on $A_X$. For $X = H_L, H_R$, we also set $\Theta_L := \Theta_{H_L}, \Theta_R := \Theta_{H_R}$.

Now we define a reference state. For each $(x, y) \in \mathbb{Z}^2$, $\{\delta_{(z,y)} \mid z = dx, dx + 1, \ldots, dx + d - 1\}$ is a CONS of $h_{(x,y)} := h_{(x,y)}$. One can decompose $h_{(x,y)}$ as

$$h_{(x,y)} = \bigoplus_{j=0}^{d/2-1} h^{(j)}_{(x,y)}$$

(1.7)

with its 2-dimensional subspaces

$$h^{(j)}_{(x,y)} := \mathbb{C} - \mathrm{span}\{\delta_{(dx+2j,y)}, \delta_{(dx+2j+1,y)}\}, \quad j = 0, \ldots, d/2 - 1.$$

(1.8)

Because each $h^{(j)}_{(x,y)}$ is invariant under $\mathcal{C}$, its restriction $\mathcal{C}^{(j)}_{(x,y)} := \mathcal{C}|_{h^{(j)}_{(x,y)}}$ gives a complex conjugation on $h^{(j)}_{(x,y)}$.

Let $p_{(x,y)}^{(j)}$ be the orthogonal projection on $h^{(j)}_{(x,y)}$ onto the one-dimensional subspace spanned by $\delta_{(dx+2j,y)} + i\delta_{(dx+2j+1,y)}$. By the definition, we see that $\mathcal{C}^{(j)}_{(x,y)}$ is a basis projection for $(h^{(j)}_{(x,y)}, \mathcal{C}^{(j)}_{(x,y)})$, i.e.,

$$\mathcal{C}^{(j)}_{(x,y)} p_{(x,y)}^{(j)} \mathcal{C}^{(j)}_{(x,y)} = \mathbb{I}_{h^{(j)}_{(x,y)}} - p_{(x,y)}^{(j)} .$$

(1.9)

Set projections $p_{(x,y)}$, $(x, y) \in \mathbb{Z}^2$, $p_X, X \subset \mathbb{Z}^2$ on $h_{(x,y)}, h_X$ respectively by

$$p_{(x,y)} := \bigoplus_{j=0}^{d/2-1} p_{(x,y)}^{(j)}, \quad p_X := \bigoplus_{(x,y) \in X} p_{(x,y)} .$$

(1.10)

By (1.9), we see that $p_X$ is a basis projection for $(h_X, \mathcal{C}_X)$

$$\mathcal{C}_X p_X \mathcal{C}_X = \mathbb{I}_{h_X} - p_X .$$

(1.11)

In particular, $p := p_{\mathbb{Z}^2}$ is a basis projection for $(h, \mathcal{C})$. From this basis projection $p$, we can construct a Fock state $\omega^{(0)} := \omega_p$ on $A$. Set $g_{j,(x,y)} := \delta_{(dx+2j,y)} + i\delta_{(dx+2j+1,y)}$, for $(x, y) \in \mathbb{Z}^2$ and $j = 0, \ldots, d/2 - 1$. With $\gamma_p$ in (1.2), we note that

$$\gamma_p^{-1} (a^*(pg_{j,(x,y)})a(pg_{j,(x,y)})) = B(g_{j,(x,y)})B(g_{j,(x,y)})^* .$$

(1.12)

This $\omega^{(0)}$ is our reference state. We also define Fock states on $A_{H_L}, A_{H_R}$ out of basis projections $p_{H_L}, p_{H_R}$ on $h_{H_L}, h_{H_R}$, $\omega_L := \omega_{p_{H_L}}, \omega_R := \omega_{p_{H_R}}$. From the structure we see that

$$\omega^{(0)} = \omega_L \otimes \omega_R .$$

(1.13)
Throughout this paper, we fix a finite group $G$ and a unitary representation $U$ on $\mathbb{C}^d$, commuting with the complex conjugation with respect to the standard basis of $\mathbb{C}^d$. Identifying the standard basis of $\mathfrak{h}_{(x,y)} \{ \delta_{(d(x+k,y)} \mid k = 0, \ldots, d-1 \}$ with that of $\mathbb{C}^d$, we write the copy of $U$ on $\mathfrak{h}_{(x,y)}$ as $U_{(x,y)}$. Note that $U_{(x,y)}(g), g \in G$ and $\mathfrak{c}_{(x,y)}$ commute, hence $U_X(g) := \bigoplus_{(x,y) \in X} U_{(x,y)}(g), g \in G$ and $\mathfrak{c}_X$ commute, for each $X \subset \mathbb{Z}^2$. As a result, for each $X \subset \mathbb{Z}^2$ and $g \in G$ there is an automorphism $\beta_g^X := \Xi_{U_X(g)}$ such that

$$\beta_g^X(B(f)) = B(U_X(g)f), \quad f \in \mathfrak{h}_X. \quad (1.14)$$

We set

$$\beta_g := \beta_g^{\mathbb{Z}^2}, \quad \beta_g^U := \beta_g^{H^1_U}, \quad \beta_g^D := \beta_g^{\mathbb{H}^1_D}, \quad \beta_g^{UR} := \beta_f^{H^1_U \cap H^1_R}, \quad \beta_g^{UL} := \beta_f^{H^1_U \cap H^1_L}, \quad g \in G. \quad (1.15)$$

A mathematical model on two dimensional Fermionic system is fully specified by its even interaction $\Phi$. We denote the set of all finite subsets of $\mathbb{Z}^2$ by $\mathcal{G}_{\mathbb{Z}^2}$. A uniformly bounded even interaction on $\mathcal{A}$ is a map $\Phi : \mathcal{G}_{\mathbb{Z}^2} \to \mathcal{A}^{(0)}$ such that

$$\Phi(X) = \Phi(X)^* \in \mathcal{A}_{X}^{(0)}, \quad X \in \mathcal{G}_{\mathbb{Z}^2}, \quad (1.16)$$

and

$$\sup_{X \in \mathcal{G}_{\mathbb{Z}^2}} \| \Phi(X) \| < \infty. \quad (1.17)$$

It is of finite range with interaction length less than or equal to $R \in \mathbb{N}$ if $\Phi(X) = 0$ for any $X \in \mathcal{G}_{\mathbb{Z}^2}$ whose diameter is larger than $R$. An on-site interaction, i.e., an interaction with $\Phi(X) = 0$ unless $X$ consists of a single point, is said to be trivial. An even interaction $\Phi$ is $\beta$-invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for any $X \in \mathcal{G}_{\mathbb{Z}^2}$. For a uniformly bounded and finite range even interaction $\Phi$ and $\Lambda \in \mathcal{G}_{\mathbb{Z}^2}$ define the local Hamiltonian

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X), \quad (1.18)$$

and denote the dynamics

$$\tau^{(\Lambda)\Phi}_t(A) := e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}. \quad (1.19)$$

By the uniform boundedness and finite rangeness of $\Phi$, for each $A \in \mathcal{A}$, the following limit exists

$$\lim_{\Lambda \to \mathbb{Z}^2} \tau^{(\Lambda)\Phi}_t(A) =: \tau^\Phi_t(A), \quad t \in \mathbb{R}, \quad (1.20)$$

and defines the dynamics $\tau^\Phi$ on $\mathcal{A}$. For a uniformly bounded and finite range even interaction $\Phi$, a state $\varphi$ on $\mathcal{A}$ is called a $\tau^\Phi$-ground state if the inequality $-i \varphi(A^* \delta_\Phi(A)) \geq 0$ holds for any element $A$ in the domain $\mathcal{D}(\delta_\Phi)$ of the generator $\delta_\Phi$. Let $\varphi$ be a $\tau^\Phi$-ground state, with a GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$. Then there exists a unique positive operator $H_{\varphi,\Phi}$ on $\mathcal{H}_\varphi$ such that $e^{itH_{\varphi,\Phi}} \pi_\varphi(A) \Omega_\varphi = \pi_\varphi(\tau^\Phi_t(A)) \Omega_\varphi$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi,\Phi}$ the bulk Hamiltonian associated with $\varphi$. 
**Definition 1.2.** We say that an interaction $\Phi$ has a unique gapped ground state if (i) the $\tau^{\Phi}$-ground state, which we denote as $\omega_{\Phi}$, is unique, and (ii) there exists a $\gamma > 0$ such that $\sigma(H_{\omega_{\Phi}, \Phi}) \setminus \{0\} \subset [\gamma, \infty)$, where $\sigma(H_{\omega_{\Phi}, \Phi})$ is the spectrum of $H_{\omega_{\Phi}, \Phi}$. We denote by $\mathcal{P}_{UG}$ the set of all uniformly bounded finite range even interactions, with unique gapped ground state. We denote by $\mathcal{P}_{UG0}$ the set of all uniformly bounded finite range $\beta$-invariant even interactions, with unique gapped ground state.

Set $\Phi_{p} : \mathcal{G}_{Z^{2}} \to \mathcal{A}$ by

$$
\Phi_{p}((x, y)) := \sum_{j=1}^{d-1} (B(g_{j,(x,y)})B(g_{j,(x,y)})^{*} - B(g_{j,(x,y)})^{*}B(g_{j,(x,y)})), \quad (x, y) \in Z^{2}
$$

and $\Phi_{p}(X) = 0$ otherwise. Then we see that $\Phi_{p}$ is a uniformly bounded on-site interaction with a unique gapped ground state $\omega^{(0)} := \omega_{p}[A]$.

In this paper we consider a classification problem of a subset of $\mathcal{P}_{UG0}$, To describe this subset we need to explain the classification problem of unique gapped ground state phases, without symmetry. For $\Gamma \subset Z^{2}$, we denote by $\Pi_{\Gamma} : \mathcal{A} \to \mathcal{A}_{\Gamma}$ the conditional expectation with respect to the trace state (see Theorem 4.7 [AM]). Note from the proof of Theorem 4.7 [AM] that $\Pi_{\Gamma}(a \hat{\otimes} b) = 0$ for any $a \in \mathcal{A}_{\Gamma}$ and $b \in \mathcal{A}_{\Gamma}^{(1)}$. Let $f : (0, \infty) \to (0, \infty)$ be a continuous decreasing function with $\lim_{t \to \infty} f(t) = 0$. For each $A \in \mathcal{A}$, let

$$
\|A\|_{f} := \|A\| + \sup_{N \in \mathbb{N}} \left( \|A - \Pi_{A_{N}}(A)\| / f(N) \right). \tag{1.22}
$$

We denote by $\mathcal{D}_{f}$ the set of all $A \in \mathcal{A}$ such that $\|A\|_{f} < \infty$. Here $A_{N} := [-N, N]^{2} \cap Z^{2}$.

The classification of unique gapped ground state phases $\mathcal{P}_{UG}$ without symmetry is the following.

**Definition 1.3.** Two interactions $\Phi_{0}, \Phi_{1} \in \mathcal{P}_{UG}$ are equivalent if there is a path of even interactions $\Phi : [0, 1] \to \mathcal{P}_{UG}$ satisfying the following:

1. $\Phi(0) = \Phi_{0}$ and $\Phi(1) = \Phi_{1}$.
2. For each $X \in \mathcal{G}_{Z^{2}}$, the map $[0, 1] \ni s \to \Phi(X; s) \in A_{X}^{(0)}$ is $C^{1}$. We denote by $\Phi_{s}$ the corresponding derivatives. The interaction obtained by differentiation is denoted by $\Phi(s)$, for each $s \in [0, 1]$.
3. There is a number $R \in \mathbb{N}$ such that $X \in \mathcal{G}_{Z^{2}}$ and $\text{diam}X \geq R$ imply $\Phi(X; s) = 0$, for all $s \in [0, 1]$.
4. Interactions are bounded as follows

$$
C_{b}^{\Phi} := \sup_{s \in [0,1]} \sup_{X \in \mathcal{G}_{Z^{2}}} (\|\Phi(X; s)\| + |X|\|\Phi(X; s)\|) < \infty. \tag{1.23}
$$

5. Setting

$$
b(\epsilon) := \sup_{Z \in \mathcal{G}_{Z^{2}}} \sup_{s, s_{0} \in [0,1], 0 < |s-s_{0}| < \epsilon} \left\| \Phi(Z; s) - \Phi(Z; s_{0}) - \Phi(Z; s_{0}) \right\| \tag{1.24}
$$

for each $\epsilon > 0$, we have $\lim_{\epsilon \to 0} b(\epsilon) = 0$. 

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We use the following Lemma below. The proof is straightforward from the definition.

Definition 1.4. We denote by $\Phi$ the symmetry protected topological (SPT)-phases.

Our invariant is given by

$$\omega(\Phi) = \frac{1}{2\pi} \int_{\gamma} \omega_\Phi,$$

for any $\gamma$ in $D_\Phi$. (Recall (1.22)).

We write $\Phi_0 \sim \Phi_1$ if $\Phi_0$ and $\Phi_1$ are equivalent. If $\Phi_0, \Phi_1 \in \mathcal{P}_U$ and if we can take the path in $\mathcal{P}_U$, i.e., so that $\beta_g(\Phi(X, s)) = \Phi(X, s), \forall g \in G$ for all $s \in [0, 1]$, then we say $\Phi_0$ and $\Phi_1$ are $\beta$-equivalent and write $\Phi_0 \sim_\beta \Phi_1$.

The object we classify in this paper is the following:

**Definition 1.4.** We denote by $\mathcal{P}_SL\beta$ the set of all $\Phi \in \mathcal{P}_U$ such that $\Phi \sim \Phi_p$ with $\Phi_p \in \mathcal{P}_G$ defined in (1.21). Connected components of $\mathcal{P}_SL\beta$ with respect to $\sim_\beta$ are the symmetry protected topological (SPT)-phases.

In this paper, we introduce an invariant of this classification.

### 2. Main Result

Our invariant is given by

$$\left( C^3(G, U(1) \oplus U(1)) \right) \times \left( H^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \right) \times \left( H^1(G, \mathbb{Z}_2) \right),$$

(2.1)

devided by some equivalence relation. First let us specify it.

For $A := \mathbb{Z}_2, U(1)$, we define a $\mathbb{Z}_2$-action on $A \oplus A$ by

$$\mathbb{Z}_2 \times (A \oplus A) \ni (a, x) \mapsto x^a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x \in A \oplus A.$$

(2.2)

We associate $A \oplus A$ the point-wise multiplication, i.e., for $x = (x_+, x_-), y = (y_+, y_-) \in A \oplus A$, we set $x \cdot y := (x_+ y_+, x_- y_-)$. For $x = (x_+, x_-) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we also set $(-1)^x := ((-1)^{x_+}, (-1)^{x_-}) \in U(1) \oplus U(1)$. For $x \in C^1(G, A \oplus A), y \in C^2(G, A \oplus A), z \in C^3(G, A \oplus A)$ and $a \in H^1(G, \mathbb{Z}_2)$, we set

$$d^1_a x(g, h) := \frac{(x^a(h)) \cdot x(g)}{x(gh)},$$

$$d^2_a y(g, h, k) := \frac{(y^a(h, k)) \cdot y(g, hk)}{y(gh, k) \cdot y(g, h)},$$

$$d^3_a z(g, h, k, f) := \frac{(z^a(h, k, f)) \cdot z(g, hk, f) \cdot z(g, h, k)}{z(gh, k, f) \cdot z(g, h, kf)}.$$

(2.3)

We use the following Lemma below. The proof is straightforward from the definition.
Lemma 2.1. For any \( m, x \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \), \( a \in H^1(G, \mathbb{Z}) \), setting \( \tilde{\sigma}(g, h) := (-1)^{x(g) \cdot m(a)(g)} \), we have

\[
(-1)d_a^1x(g,h) \cdot m^a(g)(k) + x(g) \cdot d_a^1m^a(g)(k) = d_a^2\tilde{\sigma}(g, h, k).
\] (2.4)

We denote by \( \bar{PD}(G) \) the pentad \((c, \kappa_R, \kappa_L, b, a)\) of

\[
c \in C^3(G, U(1) \oplus U(1)), \quad \kappa_R \in C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2), \quad \kappa_L \in C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2), \quad b \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2), \quad a \in H^1(G, \mathbb{Z}_2)
\] (2.5)
such that

\[
d_a^1b(g, h) = \kappa_L(g, h) + \kappa_R(g, h),
\] (2.6)

\[
d_a^2\kappa_R(g, h, k) = 0,
\] (2.7)

\[
d_a^2\kappa_L(g, h, k) = 0,
\] (2.8)

\[
d_a^3c(g, h, k, f) = (-1)^{\kappa_L(g, h)}(\kappa^a(g)(h, f)).
\] (2.9)

Lemma 2.2. On \( \bar{PD}(G) \), set

\[
(c^{(1)}, \kappa_R^{(1)}, \kappa_L^{(1)}, b^{(1)}, a^{(1)}) \sim_{PD(G)} (c^{(2)}, \kappa_R^{(2)}, \kappa_L^{(2)}, b^{(2)}, a^{(2)})
\]

for \((c^{(1)}, \kappa_R^{(1)}, \kappa_L^{(1)}, b^{(1)}, a^{(1)}), (c^{(2)}, \kappa_R^{(2)}, \kappa_L^{(2)}, b^{(2)}, a^{(2)}) \in \bar{PD}(G)\) if the following hold.

(i) \( a^{(1)}(g) = a^{(2)}(g) =: a(g) \) for any \( g \in G \), and

(ii) there exist an \( m \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \) and a \( \sigma \in C^2(G, U(1) \oplus U(1)) \) such that

\[
\kappa_R^{(2)}(g, h) = d_a^1m(g, h) + \kappa_R^{(1)}(g, h),
\] (2.10)

\[
\kappa_L^{(2)}(g, h) = d_a^1b^{(2)}(g, h) - d_a^1b^{(1)}(g, h) - d_a^1m(g, h) + \kappa_L^{(1)}(g, h),
\] (2.11)

\[
c^{(2)}(g, h, k) = (-1)^{\kappa_L^{(1)}(g, h) \cdot m(a)(k)} (-1)^{b^{(2)}(g) - b^{(1)}(g) - m(g)} \cdot (\kappa^{a(g)}_R(h, k)) d_a^2\sigma(g, h, k) c^{(1)}(g, h, k).
\] (2.12)

Then this \( \sim_{PD(G)} \) is an equivalence relation.

Definition 2.3. We denote the equivalence classes by \( PD(G) \). We also denote by \([(c, \kappa_R, \kappa_L, b, a)]_{PD(G)}\) the equivalence class containing \((c, \kappa_R, \kappa_L, b, a) \in \bar{PD}(G)\).

Proof. We show only the transitivity. The proof for the symmetry is analogous. Suppose that

\[
(c^{(1)}, \kappa_R^{(1)}, \kappa_L^{(1)}, b^{(1)}, a^{(1)}) \sim_{PD(G)} (c^{(2)}, \kappa_R^{(2)}, \kappa_L^{(2)}, b^{(2)}, a^{(2)})
\]

with \( m \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \) and a \( \sigma \in C^2(G, U(1) \oplus U(1)) \), and

\[
(c^{(2)}, \kappa_R^{(2)}, \kappa_L^{(2)}, b^{(2)}, a^{(2)}) \sim_{PD(G)} (c^{(3)}, \kappa_R^{(3)}, \kappa_L^{(3)}, b^{(3)}, a^{(3)})
\]
with \( l \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \) and a \( \sigma' \in C^2(G, U(1) \oplus U(1)) \). Then we have \( a^{(1)}(g) = a^{(2)}(g) = a^{(3)}(g) = a(g) \) and

\[
\kappa_R^{(3)}(g, h) = d_a^1(l + m)(g, h) + \kappa_R^{(1)}(g, h),
\]

\[
\kappa_L^{(3)}(g, h) = d_a^1\left(b^{(3)} - b^{(1)} - (l + m)\right)(g, h) + \kappa_L^{(1)}(g, h).
\]

(2.13)

(2.14)

Note that

\[
\begin{align*}
(1)\kappa_L^{(2)}(g, h) \cdot m^{a(g)}(h) & = (1)\left(b^{(2)}(g) - b^{(1)} - m(g)\right)\left(\kappa_R^{(2)}\right)^{a(g)}(h, k) \\
(1)\kappa_R^{(2)}(g, h) \cdot m^{a(g)}(h) & = (1)\left(b^{(3)}(g) - b^{(2)}(g) - l(g)\right)\left(\kappa_R^{(3)}\right)^{a(g)}(h, k) \\
& = (1)\kappa_L^{(1)}(g, h) \cdot (l + m)^{a(g)}(h) \cdot \left(\kappa_R^{(1)}\right)^{a(g)}(h, k) \\
& = (1)\kappa_L^{(1)}(g, h) \cdot (l + m)^{a(g)}(h) \cdot \left(\kappa_R^{(3)}\right)^{a(g)}(h, k) \\
& = (1)\kappa_L^{(1)}(g, h) \cdot (l + m)^{a(g)}(h) \cdot \left(\kappa_R^{(3)}\right)^{a(g)}(h, k)
\end{align*}
\]

(2.15)

with \( \sigma''(g, h) := (1)\left(b^{(2)} - b^{(1)} - m\right)(g) \cdot m^{a(g)}(h) \). Here we used Lemma 2.1 in the last equation. Setting \( \tilde{\sigma} := \sigma \sigma'' \), we get

\[
(c^{(1)}, \kappa_R^{(1)}, \kappa_L^{(1)}, b^{(1)}, a^{(1)}) \sim \mathcal{P}_D(G) (c^{(3)}, \kappa_R^{(3)}, \kappa_L^{(3)}, b^{(3)}, a^{(3)})
\]

with \( l + m \) and \( \tilde{\sigma} \).

\( \square \)

**Lemma 2.4.** For any \((c, \kappa_L, \kappa_R, b, a) \in \mathcal{P}_D(G)\), set

\[
\tilde{c}(g, h, k) := (1)\left(b^{(g)}\kappa_R^{(g)}(h, k)\right) c(g, h, k).
\]

(2.16)

Then we have \((\tilde{c}, \kappa_L, \kappa_R, 0, a) \in \mathcal{P}_D(G)\) and \((c, \kappa_L, \kappa_R, b, a) \sim \mathcal{P}_D(G) (\tilde{c}, \kappa_R, \kappa_R, 0, a)\).

**Proof.** Setting

\[
x(g, h, k) := (1)\left(b^{(g)}\kappa_R^{(g)}(h, k)\right),
\]

(2.17)

one can check

\[
d_a^3 x(g, h, k, f) = (1)d_a^1 h(g, h) \cdot \kappa_R^{(g)}(k, f),
\]

(2.18)

using the fact that \( d_a^2 \kappa_R = 0 \). From this, we have

\[
d_a^3 \tilde{c}(g, h, k, f) = d_a^3 x(g, h, k, f) d_a^3 c(g, h, k, f) = (1)d_a^1 h(g, h) \cdot \kappa_L^{(g)}(k, f)
\]

\[
= (1)\kappa_R^{(g, h)} \cdot \kappa_R^{(g)}(k, f).
\]

(2.19)

Combining this and

\[
d_a^1 0(g, h) = 0 = \kappa_R(g, h) + \kappa_R(g, h), \quad d_a^2 \kappa_R(g, h, k) = 0
\]

(2.20)

in \( \mathbb{Z}_2 \), we obtain \((\tilde{c}, \kappa_R, \kappa_R, 0, a) \in \mathcal{P}_D(G)\). Checking \((c, \kappa_L, \kappa_R, b, a) \sim \mathcal{P}_D(G) (\tilde{c}, \kappa_R, \kappa_R, 0, a)\) is immediate.

\( \square \)
Hence to consider \( \mathcal{PD}(G) \), it suffices to think of the following.

**Definition 2.5.** We set
\[
\mathcal{PD}_0(G) = \left\{ (c, \kappa, a) \mid (c, \kappa, 0, a) \in \mathcal{PD}(G) \right\}.
\]
(2.21)

We introduce the equivalence relation \( \sim_{\mathcal{PD}_0(G)} \) on \( \mathcal{PD}_0(G) \) by
\[
(c^{(1)}, \kappa^{(1)}, a^{(1)}) \sim_{\mathcal{PD}_0(G)} (c^{(2)}, \kappa^{(2)}, a^{(2)})
\]
if \((c^{(1)}, \kappa^{(1)}, \kappa^{(1)}, 0, a^{(1)}) \sim_{\mathcal{PD}(G)} (c^{(2)}, \kappa^{(2)}, \kappa^{(2)}, 0, a^{(2)}) \).
(2.22)

We denote the equivalence classes by \( \mathcal{PD}_0(G) \). We also denote by \([ (c, \kappa, a) ]_{\mathcal{PD}_0(G)} \) the equivalence class containing \((c, \kappa, a) \in \mathcal{PD}_0(G) \).

The main theorem of this paper is the following.

**Theorem 2.6.** There is a \( \mathcal{PD}_0(G) \)-valued index on \( \mathcal{PSL}_\beta \), which is an invariant of the classification \( \sim_\beta \) of \( \mathcal{PSL}_\beta \).

The proof follows the strategy of [O4] for the quantum spin system case. (See reviews and videos in [O2] [O3] [O5].) We consider the action of \( \beta_g^U \) on our ground state \( \omega_\Phi \) of \( \Phi \) in SPT-phase i.e., \( \omega_\Phi \circ \beta_g^U \). Due to the fact that \( \Phi \) is in the SPT-phase, \( \omega_\Phi \) can be written as \( \omega_\Phi = \omega(0) \alpha \) with some quasi-local automorphism \( \alpha \), which does not create the long range entanglement. From this and that \( \omega_\Phi \) is \( \beta_g \)-invariant, we see that the effective excitation caused by \( \beta_g^U \) on \( \omega_\Phi \) is localized around the \( x \)-axis. In particular, \( \omega(0) \circ \alpha \beta_g^U \alpha^{-1} \) satisfies the split property with respect to the cut \( H_L - H_R \). Namely, it is quasi-equivalent to a state of the form \( \varphi_L \hat{\otimes} \varphi_R \), with homogeneous states \( \varphi_L, \varphi_R \) on \( \mathcal{A}_H_L, \mathcal{A}_H_R \). The difference from the quantum spin case [O4] and our Fermionic case is that in Fermionic case when a state satisfies the split property, there are two possibilities. If the restriction of \( \omega(0) \circ \alpha \beta_g^U \alpha^{-1} \) to \( \mathcal{A}_H_R \) is a factor state, then it is equivalent to the state of the form \( \omega(0) \circ (\eta_{gL} \hat{\otimes} \eta_{gR}) \), with some graded automorphisms \( \eta_{gL}, \eta_{gR} \) on \( \mathcal{A}_{H_L}, \mathcal{A}_{H_R} \) localized around \( x \)-axis. If the restriction is not a factor state, then it is equivalent to the state of the form \( \omega(1) \circ (\eta_{gL} \hat{\otimes} \eta_{gR}) \), with some other reference state \( \omega(1) \). This state \( \omega(1) \) is of the form \( \omega(0) \circ \tau \), with some automorphism \( \tau \) on the \( x \)-axis representing the space translation on the \( x \)-axis. This dichotomy gives us the index taking value in \( H^1(G, \mathbb{Z}_2) \).

Some combination of automorphisms \( \eta_{gL} \) and \( \eta_{gR} \) is implementable by a unitary in the GNS representation of \( \omega_\Phi \) and it give us some indices in \( C^3(G, U(1) \oplus U(1)) \) (see Lemma 5.6) just like in the quantum spin case. The \( C^2(G, U(1) \oplus U(1)) \) part represents if this unitary is even or odd. Of course they are not independent to each other that result in the complication of our index in \( \mathcal{PD}_0(G) \). Without the doubled structure like \( U(1) \oplus U(1) \), result coincides with the one predicted in the quantum field theory [BM] [WG], Pontrjagin dual of 3-dimensional Spin Bordism.

The rest of the paper is organized as follows. In the analysis, the automorphic equivalence gets important. It is explained in Sect. 3. The \( \eta_{gL}, \eta_{gR} \) above are given from the classification of pure states satisfying the split property, analogous to that of [O1]. Its Fermionic version is given in Sect. 4. The \( \mathcal{PD}_0(G) \)-valued index is derived in Sect. 5. In Sect. 6, we show it is actually the invariant of our classification. Section 7 gives some convenient property of \( \mathcal{A}_{\text{SDC}}(\mathbb{R}, \mathcal{C}) \) which we use in the analysis. Notations from “Appendix A” are used freely. Some facts about graded von Neumann algebras are collected/ proven in “Appendix B”.

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3. Automorphic Equivalence

A very important fact about gapped ground state phases is the automorphic equivalence \[ \text{[BMNS]} \ \text{[NSY2]}, \ \text{[MO]}, \] which started as Hastings adiabatic Lemma \[ \text{[HW]} \].

First we introduce a class of paths of interactions. A norm-continuous interaction on \( A \) defined on an interval \([0, 1]\) is a map \( \Phi : \mathcal{S}_{\mathbb{Z}^2} \times [0, 1] \to A \) such that

(i) for any \( t \in [0, 1] \), \( \Phi(\cdot, t) : \mathcal{S}_{\mathbb{Z}^2} \to A \) is an even interaction, and

(ii) for any \( Z \in \mathcal{S}_{\mathbb{Z}^2} \), the map \( \Phi(Z, \cdot) : [0, 1] \to A_Z \) is piecewise norm-continuous.

Let \( F \) be an \( F \)-function on \((\mathbb{Z}^2, d)\). (See \[ \text{[NSY2]} \) or \[ \text{[O4]} \) Appendix C for the definition of \( F \)-functions.) We denote by \( \hat{B}_F([0, 1]) \) the set of all norm continuous interactions \( F \) on \( A \) defined on an interval \([0, 1]\) such that

\[
\|\Phi\|_F := \sup_{x, y \in \mathbb{Z}^2} \frac{1}{F(d(x, y))} \sum_{Z \in \mathcal{S}_{\mathbb{Z}^2}, Z \ni (x, y)} \sup_{t \in [0, 1]} ||\Phi(Z; t)|| < \infty. \tag{3.1}
\]

From \( \Psi \in \hat{B}_F([0, 1]) \), we can construct a path of automorphisms \( \tau_{t, s}^\Psi \) following the same argument as quantum spin case \([\text{BSP,NSY1,BO}]\). We can consider analogous path of interactions on \( \Gamma \subset \mathbb{Z}^2 \), which we denote by \( \hat{B}_{F, \Gamma}([0, 1]) \)

For a subset \( \Gamma \subset \mathbb{Z}^2 \), we set

\[
Q\text{Aut}(A_{\Gamma}) := \left\{ \alpha | \alpha = \tau_{s,t}^\Psi, \text{ for some } \Psi \in \hat{B}_{F, \Gamma}([0, 1]), \ F: F\text{-function } s, t \in [0, 1]\right\},
\]

\[
Q\text{Aut}_{\beta}(A_{\Gamma}) := \left\{ \alpha | \alpha = \tau_{s,t}^\Psi, \text{ for some } \beta\text{-invariant } \Psi \in \hat{B}_{F, \Gamma}([0, 1]), \ F: F\text{-function } s, t \in [0, 1]\right\}. \tag{3.2}
\]

They form subgroups of \( \text{Aut}^{(0)}(A_{\Gamma}) \). Note from the proof of Theorem 4.7 \[ \text{[AM]} \] that for the conditional expectation \( \Pi_{\Gamma} \), we have \( \Pi_{\Gamma}(a \hat{\otimes} b) = 0 \) for any \( a \in A_{\Gamma} \) and \( b \in A_{\Gamma}^{(1)} \).

Using this fact, just by following the argument in \[ \text{[MO]} \], we can show the following Fermionic version of automorphic equivalence.

**Theorem 3.1.** Let \( \Phi_0, \Phi_1 \in P_{UG} \) and \( \omega_{\Phi_0}, \omega_{\Phi_1} \) be their unique gapped ground states. Suppose that \( \Phi_0 \sim \Phi_1 \) holds, via a path \( \Phi : [0, 1] \to P_{UG} \). Then there exists some \( \alpha \in Q\text{Aut}(A) \) such that \( \omega_{\Phi_1} = \omega_{\Phi_0} \circ \alpha \). If \( \Phi_0, \Phi_1 \in P_{UG_{\beta}} \) and \( \Phi \sim_{\beta} \Phi_0 \), we may take \( \alpha \) from \( Q\text{Aut}_{\beta}(A) \).

Because of the theorem, \( Q\text{Aut}(A) \) plays an important role for us, and the ground states we consider belong to the following set.

\[
\text{SPT} := \left\{ \omega^{(0)} \circ \alpha | \alpha \in Q\text{Aut}(A), \ \omega^{(0)} \circ \alpha \circ \beta_{\mathcal{S}} = \omega^{(0)} \circ \alpha \right\}. \tag{3.3}
\]

For each \( \omega \in \text{SPT} \), by definition, the set

\[
\text{EAut}(\omega) = \left\{ \alpha \in Q\text{Aut}(A) | \omega = \omega^{(0)} \circ \alpha \right\} \tag{3.4}
\]

is non-empty.

From the fact they are given by local interactions, automorphisms in \( Q\text{Aut}(A) \) satisfy nice properties. We list up such properties for the rest of this section. Most of them can be proven using the same argument as that of quantum spin case \[ \text{[O4]} \] combined with the property of conditional expectations \( \Pi_{\Gamma} \) mentioned above, and we omit the proof.
For $0 < \theta < \frac{\pi}{2}$, we set
\[ C_\theta := \{(x, y) \mid |y| \leq \tan \theta \cdot |x|\}. \tag{3.5} \]

For $0 < \theta_1 < \theta_2 \leq \frac{\pi}{2}$, we use a notation $\mathcal{C}_{(\theta_1, \theta_2]} := C_{\theta_2} \setminus C_{\theta_1}$ and $\mathcal{C}_{[0, \theta_1]} := C_{\theta_1}$.

We also set
\[ c_L := -5, \quad c_R := 5. \tag{3.6} \]

An automorphism $\alpha \in \text{QAut}(A)$ satisfies a factorization property. It basically says that we can split $\alpha$ into two along any cut of the system modulo some error terms localized around the boundary. For example, if we cut the system along the $y$-axis, we obtain the following: for any $0 < \theta < \frac{\pi}{2}$, there are
\[ \alpha_L \in \text{QAut}(A_{H_L}), \quad \alpha_R \in \text{QAut}(A_{H_R}), \quad \Upsilon \in \text{QAut}(A_{\mathcal{C}_\theta}) \tag{3.7} \]

decomposing $\alpha$ as
\[ \alpha = (\text{inner}) \circ (\alpha_L \hat{\otimes} \alpha_R) \circ \Upsilon. \tag{3.8} \]

If we cut the system along the $x$-axis, we have the following: there are
\[ \alpha_U \in \text{QAut}(A_{H_U^1}), \quad \alpha_D \in \text{QAut}(A_{H_U^{-1}}), \quad \Xi_L \in \text{QAut}(A_{C_\theta \cap H_L}), \quad \Xi_R \in \text{QAut}(A_{C_\theta \cap H_R}) \tag{3.9} \]

such that
\[ \alpha = (\alpha_U \hat{\otimes} \alpha_D) \circ (\Xi_L \hat{\otimes} \Xi_R) \text{ (inner)}. \tag{3.10} \]

Hence for $\alpha \in \text{QAut}(A)$, and $0 < \theta < \frac{\pi}{2}$, the following sets are non-empty.

\[ \mathcal{D}^V(\alpha, \theta) := \left\{(\alpha_L, \alpha_R, \Upsilon) \mid \alpha_L \in \text{QAut}(A_{H_L}), \quad \alpha_R \in \text{QAut}(A_{H_R}), \quad \Upsilon \in \text{QAut}(A_{\mathcal{C}_\theta}) \right. \]
\[ \left. \quad \text{such that } \alpha = (\alpha_L \hat{\otimes} \alpha_R) \circ \Upsilon \text{ (inner)} \right\}. \tag{3.11} \]

\[ \mathcal{D}^H(\alpha, \theta) := \left\{(\alpha_U, \alpha_D, \Xi_L, \Xi_R) \mid \alpha_U \in \text{QAut}(A_{H_U^1}), \quad \alpha_D \in \text{QAut}(A_{H_U^{-1}}), \quad \Xi_L \in \text{QAut}(A_{C_\theta \cap H_L}), \quad \Xi_R \in \text{QAut}(A_{C_\theta \cap H_R}) \right. \]
\[ \left. \quad \text{such that } \alpha = (\alpha_U \hat{\otimes} \alpha_D) \circ (\Xi_L \hat{\otimes} \Xi_R) \text{ (inner)} \right\}. \tag{3.11} \]

We can consider finer factorization: for each
\[ 0 < \theta_{0,8} < \theta_1 < \theta_{1,2} < \theta_{1,8} < \theta_2 < \theta_{2,2} < \theta_{2,8} < \theta_3 < \theta_{3,2} < \frac{\pi}{2}, \tag{3.12} \]

$\alpha \in \text{QAut}(A)$ can be decomposed as
\[ \alpha = (\text{inner}) \circ \left( \alpha_{[0,\theta_1]} \otimes \alpha_{[\theta_1,\theta_2]} \otimes \alpha_{[\theta_2,\theta_3]} \otimes \alpha_{[\theta_3,\pi/2]} \right) \circ \left( \alpha_{[0.8,\theta_{1.2}]} \otimes \alpha_{[\theta_{1.8},\theta_{2.2}]} \otimes \alpha_{[\theta_{2.8},\theta_{3.2}]} \right) \]

(3.13)

with

\[ \alpha_X := \bigotimes_{\sigma=L,R,\zeta=D,U} \alpha_{X,\sigma,\zeta}, \quad \alpha_{[0,\theta_1]} := \bigotimes_{\sigma=L,\zeta=D,U} \alpha_{[0,\theta_1],\sigma}, \]

\[ \alpha_{[\theta_3,\pi/2]} := \bigotimes_{\zeta=D,U} \alpha_{[\theta_3,\pi/2],\zeta}, \]

\[ \alpha_{X,\sigma,\zeta} \in \text{QAut} \left( A_{C_X \cap H^\sigma_\zeta} \right), \quad \alpha_{X,\sigma} := \bigotimes_{\zeta=U,D} \alpha_{X,\sigma,\zeta}, \]

\[ \alpha_{X,\zeta} := \bigotimes_{\sigma=L,R} \alpha_{X,\sigma,\zeta}, \]

\[ \alpha_{[0,\theta_1],\sigma} \in \text{QAut} \left( A_{C_{[0,\theta_1]} \cap H^\sigma_{\pi/2}} \right), \quad \alpha_{[\theta_3,\pi/2],\zeta} \in \text{QAut} \left( A_{C_{[\theta_3,\pi/2]} \cap H^\zeta} \right), \]

(3.14)

for

\[ X = (\theta_1, \theta_2], (\theta_2, \theta_3], (\theta_{0.8}, \theta_{1.2}], (\theta_{1.8}, \theta_{2.2}], (\theta_{2.8}, \theta_{3.2}], \quad \sigma = L, R, \quad \zeta = D, U. \]

(3.15)

In order to define our index, we introduce an automorphism localized along \( x \)-axis. Let \( v_\tau \) be a unitary on \( h = l^2(\mathbb{Z}^2) \) such that

\[ v_\tau \delta(x,y) := \begin{cases} 
\delta(x,y), & y \neq 0 \\
\delta(x+1,0), & y = 0.
\end{cases} \]

(3.16)

Note that \( v_\tau \) commutes with the complex conjugation \( \mathcal{C} \). Therefore, it defines an automorphism \( \tau := \Xi_{v_\tau} \) on \( A \) such that

\[ \tau(B(f)) := B(v_\tau f), \quad f \in h. \]

(3.17)

Decompositions like above allow us to derive the following Lemma immediately.

**Lemma 3.2.** For any \( \alpha \in \text{QAut}(A_{\mathbb{Z}^2}) \), the following hold.

(i) For any \( 0 < \varphi < \pi/2 \) and \( a \in \mathbb{Z} \), there are automorphisms \( \zeta_\sigma \in \text{Aut}(A_{C_{\varphi} \cap H^\sigma_{\pi/2}}) \), \( \sigma = L, R \) such that

\[ \alpha \tau^a \alpha^{-1} = \tau^a \left( \zeta_\sigma \right) \circ \text{(inner)}. \]

(3.18)

(ii) For any \( 0 < \varphi' < \varphi'' < \pi/2 \) and \( X_\sigma \in \text{Aut}(A_{C_{\varphi'} \cap H^\sigma_{\pi/2}}) \), \( \sigma = L, R \) there are automorphisms \( \tilde{X}_\sigma \in \text{Aut}(A_{C_{\varphi''} \cap H^\sigma_{\pi/2}}) \), \( \sigma = L, R \) such that

\[ \alpha \left( X_L \hat{\otimes} X_R \right) \alpha^{-1} = \left( \tilde{X}_L \hat{\otimes} \tilde{X}_R \right) \circ \text{(inner)}. \]

(3.19)
Definition 3.3. For any $\Lambda \subset \mathbb{Z}^2$ and $X \in \text{Aut}^0(\tilde{A}_\Lambda)$ and $a \in \mathbb{Z}$, we denote by $X_a \in \text{Aut}^0(\tilde{A}_{(\Lambda_0+a\mathbb{Z}) \cup \Lambda_1})$ the automorphism such that $\tau_a X \tau_a^{-1} = X_a$. Here we decomposed $\Lambda$ as
\[ \Lambda = \Lambda_0 \cup \Lambda_1 := (\Lambda \cap (\mathbb{Z} \times \{0\})) \cup (\Lambda \cap (\mathbb{Z}^2 \setminus (\mathbb{Z} \times \{0\}))), \tag{3.20} \]
and set $e_0 := (1, 0) \in \mathbb{Z}^2$.

From Lemma 3.2 (i) and $\omega^{(0)} \circ \tau^{2a} = \omega^{(0)}$, we obtain the following:

Lemma 3.4. For any $\alpha \in \text{QAut}(A_{\mathbb{Z}^2})$ and $0 < \varphi < \frac{\pi}{2}$, $a \in \mathbb{Z}$ there are automorphisms $\xi_\sigma \in \text{Aut}^0(\tilde{A}_{\text{C}_\varphi \cap H_{\sigma}^a})$, $\sigma = L, R$ such that
\[ \omega^{(0)} \alpha \circ \tau^{2a} \simeq \omega^{(0)} \alpha (\xi_L \hat{\otimes} \xi_R). \tag{3.21} \]

4. Split Property of Pure States on Self-Dual-CAR-Algebras

Having the automorphic equivalence $\omega_\Phi = \omega^{(0)} \alpha$ with $\alpha \in \text{QAut}(A)$ and the $\beta_g$-invariance of $\omega_\Phi$, we expect that the effective excitation caused by $\beta_g^U$ on $\omega_\Phi$ is localized around the $x$-axis. It can be shown so, by observing that $\omega^{(0)} \circ \alpha \beta_g^U \alpha^{-1}$ satisfies the split property (Definition 4.3) with respect to the $H_L - H_R$ cut. In this section, as a preparation of our analysis, we investigate the split property of pure states on self-dual CAR-algebras. For Fermionic systems, the split property gives some dichotomy (Lemma 4.4). The main proposition of this section is Proposition 4.5, which states that two pure split states belonging to the same category of the dichotomy can be connected by automorphisms of graded tensor product form. We use definitions and facts from “Appendix A” and “Appendix B” freely.

We start by some basic fact we repeatedly use.

Lemma 4.1. Let $\mathfrak{h}_1, \mathfrak{h}_2$ be Hilbert spaces with complex conjugation $\mathfrak{c}_1, \mathfrak{c}_2$, respectively. Let $\varphi, \omega$ be homogeneous pure states on $\mathfrak{A}_\text{SDC}(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \mathfrak{c}_1 \oplus \mathfrak{c}_2)$. Suppose that $\varphi$ and $\omega$ are equivalent. Then their restrictions $\varphi|_{\mathfrak{A}_\text{SDC}(\mathfrak{h}_1, \mathfrak{c}_1)}$, $\omega|_{\mathfrak{A}_\text{SDC}(\mathfrak{h}_1, \mathfrak{c}_1)}$ onto $\mathfrak{A}_\text{SDC}(\mathfrak{h}_1, \mathfrak{c}_1)$ are quasi-equivalent.

Proof. We write $\mathfrak{A}_i := \mathfrak{A}_\text{SDC}(\mathfrak{h}_i, \mathfrak{c}_i)$, for $i = 1, 2$. Let $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ be a GNS representation of $\varphi$. Because $\varphi$ is homogeneous, there is a self-adjoint unitary $\Gamma_\varphi$ on $\mathcal{H}_\varphi$ such that $\Gamma_\varphi \pi_\varphi(A) \Omega_\varphi = \pi_\varphi \Theta(A) \Omega_\varphi$, $A \in \mathfrak{A}_\text{SDC}(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \mathfrak{c}_1 \oplus \mathfrak{c}_2)$, for the grading operator $\Theta$ on $\mathfrak{A}_\text{SDC}(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \mathfrak{c}_1 \oplus \mathfrak{c}_2)$. Let $p_{\varphi}$ be an orthogonal projection onto the subspace $\mathcal{K}_\varphi := \overline{\pi_\varphi(\mathfrak{A}_1) \Omega_\varphi}$. Because $p_{\varphi} \in \pi_\varphi(\mathfrak{A}_1)'$, $\rho_\varphi(A) := \pi_\varphi(A) p_{\varphi}$, $A \in \mathfrak{A}_1$ defines a representation of $\mathfrak{A}_1$ on $\mathcal{K}_\varphi$. Then $(\mathcal{K}_\varphi, \rho_\varphi, \Omega_\varphi)$ is a GNS representation of $\varphi|_{\mathfrak{A}_1}$. By the Kaplansky density theorem, we have $\rho_\varphi(\mathfrak{A}_1)'' = \pi_\varphi(\mathfrak{A}_1)'' p_{\varphi}$. Let us consider the map $\tau_\varphi : \pi_\varphi(\mathfrak{A}_1)'' \rightarrow \rho_\varphi(\mathfrak{A}_1)''$, defined by $\tau_\varphi(x) := x p_{\varphi}$, for $x \in \pi_\varphi(\mathfrak{A}_1)'$. It is a $*$-homomorphism onto $\rho_\varphi(\mathfrak{A}_1)''$. We claim that $\tau_\varphi$ is injective. To see this, we note $p_{\varphi}$ and $\Gamma_\varphi$ commute because $\mathcal{K}_\varphi$ is $\Gamma_\varphi$-invariant. As a result, we see that $\ker \tau_\varphi = \text{Ad} \left( \Gamma_\varphi \right) (\ker \tau_\varphi)$. Let $E$ be the central projection of $\pi_\varphi(\mathfrak{A}_1)''$ such that $\ker \tau_\varphi = \pi_\varphi(\mathfrak{A}_1)'' E$. From the above observation, we see that
\[ \pi_\varphi(\mathfrak{A}_1)'' \text{Ad} \left( \Gamma_\varphi \right) (E) = \text{Ad} \left( \Gamma_\varphi \right) (\pi_\varphi(\mathfrak{A}_1)'' \text{Ad} \left( \Gamma_\varphi \right) (E) = \text{Ad} \left( \Gamma_\varphi \right) (\ker \tau_\varphi) \]
Definition 4.3. \[ \tau = \ker \tau_\varphi = \pi_\varphi (\mathfrak{A}_1)^\prime \prime E. \] (4.1)

From this, we have Ad (\( \Gamma_\varphi \)) (E) = E. Hence we have

\[ E \in \mathbb{Z} (\pi_\varphi (\mathfrak{A}_1)^\prime) \cap \left( \pi_\varphi \left( \mathfrak{A}_1^{(0)} \right)^\prime \right) \subset \pi_\varphi (\mathfrak{A}_1) \cap \pi_\varphi (\mathfrak{A}_2) \]
\[ = \pi_\varphi (\mathfrak{A} (h_1 \oplus h_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2)) = \mathbb{C} I, \] (4.2)

because \( \varphi \) is pure. Then we conclude \( E = 0 \) and this proves the claim, and our \( \tau_\varphi \) is a *-isomorphism.

Hence for GNS representations \( \pi_\varphi, \rho_\varphi \) of \( \varphi, \varphi\mid_{\mathfrak{A}_1} \) respectively, there is a *-isomorphism \( \tau_\varphi : \pi_\varphi (\mathfrak{A}_1)^\prime \rightarrow \rho_\varphi (\mathfrak{A}_1)^\prime \) such that \( \tau_\varphi \circ \pi_\varphi (A) = \rho_\varphi (A) \), for each \( A \in \mathfrak{A}_1 \). Similarly, for GNS representations \( \pi_\omega, \rho_\omega \) of \( \omega, \omega\mid_{\mathfrak{A}_1} \) respectively, there is a *-isomorphism \( \tau_\omega : \pi_\omega (\mathfrak{A}_1)^\prime \rightarrow \rho_\omega (\mathfrak{A}_1)^\prime \) such that \( \tau_\omega \circ \pi_\omega (A) = \rho_\omega (A) \), for each \( A \in \mathfrak{A}_1 \). Because \( \varphi \) and \( \omega \) are equivalent, there is a *-isomorphism \( \tau : \pi_\varphi (\mathfrak{A} (h_1 \oplus h_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2))^\prime \rightarrow \pi_\omega (\mathfrak{A} (h_1 \oplus h_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2))^\prime \) such that \( \tau \circ \pi_\varphi (A) = \pi_\omega (A) \) for \( A \in \mathfrak{A} (h_1 \oplus h_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2) \). Restricting this \( \tau \) to \( \pi_\varphi (\mathfrak{A}_1)^\prime \), we obtain a *-isomorphism \( \tau_1 : \pi_\varphi (\mathfrak{A}_1)^\prime \rightarrow \pi_\omega (\mathfrak{A}_1)^\prime \) such that \( \tau_1 \circ \pi_\varphi (A) = \pi_\omega (A) \) for \( A \in \mathfrak{A}_1 \). Then we see that \( \tau_\omega \circ \tau_1 \circ \tau_\varphi^{-1} \) defines a *-isomorphism from \( \rho_\varphi (\mathfrak{A}_1)^\prime \) onto \( \rho_\omega (\mathfrak{A}_1)^\prime \) such that \( \tau_\omega \circ \tau_1 \circ \tau_\varphi^{-1} \circ \rho_\varphi (A) = \rho_\omega (A), A \in \mathfrak{A}_1 \). \( \square \)

We encounter the following situation as well.

Lemma 4.2. Let \( \mathfrak{K}_1, \mathfrak{K}_2 \) be Hilbert spaces with complex conjugation \( \mathfrak{C}_1, \mathfrak{C}_2 \), respectively. Let \( \omega \) be a homogeneous pure states on \( \mathfrak{A}_{SDC} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2) \), and \( \varphi_1, \varphi_2 \) homogeneous states on \( \mathfrak{A}_{SDC} (\mathfrak{K}_1, \mathfrak{C}_1), \mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2) \) respectively. If \( \omega \sim_{q.e.} \varphi_1 \otimes \varphi_2 \), then we have \( \omega\mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_1, \mathfrak{C}_1)} \sim_{q.e.} \varphi_1 \) and \( \omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \sim_{q.e.} \varphi_2. \)

Proof. By the proof of Lemma 4.1, we have \( \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \sim_{q.e.} \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \). (Note that \( \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \) is the restriction of the GNS representation of \( \omega \) while \( \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \) is the GNS representation of the restriction of \( \omega \).) Because of \( \omega \sim_{q.e.} \varphi_1 \otimes \varphi_2 \), we have \( \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \sim_{q.e.} \pi_\varphi_2 \). Hence we have \( \pi_\omega \mid_{\mathfrak{A}_{SDC} (\mathfrak{K}_2, \mathfrak{C}_2)} \sim_{q.e.} \pi_\varphi_2 \). The same for \( \mathfrak{A}_{SDC} (\mathfrak{K}_1, \mathfrak{C}_1) \). This proves the Lemma. \( \square \)

The situation in the previous Lemma has a name.

Definition 4.3. Let \( \mathfrak{K}_i \) be an infinite dimensional Hilbert space with a complex conjugation \( \mathfrak{C}_i \), for \( i = 1, 2 \). We say a homogeneous pure state \( \omega \) on \( \mathfrak{A}_{SDC} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2) \) satisfies the split property if there are homogeneous states \( \varphi_i \) on \( \mathfrak{A}_{SDC} (\mathfrak{K}_i, \mathfrak{C}_i) \), \( i = 1, 2 \) such that \( \omega \sim_{q.e.} \varphi_1 \otimes \varphi_2 \).

The following is a refinement of the dichotomy introduced in [M4].

Lemma 4.4. Let \( \mathfrak{K}_i \) be an infinite dimensional Hilbert space with a complex conjugation \( \mathfrak{C}_i \), for \( i = 1, 2 \). Let \( \omega \) be a homogeneous pure state on \( \mathfrak{A}_{SDC} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2) \), satisfying the split property. Let \( \Theta_1, \Theta_2 \) be automorphisms on \( \mathfrak{A}_{SDC}^{(0)} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2) \) such that \( \Theta_1 (B (f_1 \oplus f_2)) = B (-f_1 \oplus f_2), \Theta_2 (B (f_1 \oplus f_2)) = B (f_1 \oplus (-f_2)) \), for \( f_i \in \mathfrak{K}_i, i = 1, 2 \). Then one of the following occurs.

(i) We have \( \omega \mid_{\mathfrak{A}_{SDC}^{(0)} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2)} \Theta_2 \sim_{q.e.} \omega \mid_{\mathfrak{A}_{SDC}^{(0)} (\mathfrak{K}_1 \oplus \mathfrak{K}_2, \mathfrak{C}_1 \oplus \mathfrak{C}_2)} \). The state \( \omega \) has a GNS representation of the form \( (\mathcal{H}_1 \otimes \mathcal{H}_2, \pi_1 \otimes \pi_2, \Omega) \) with \( (\mathcal{H}_i, \pi_i) \) an irreducible
representation of $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$ for $i = 1, 2$. There is a self-adjoint unitary $\Gamma_{i}$ on $\mathcal{H}_{i}$ implementing $\Theta_{i}$, i.e., $\text{Ad}(\Gamma_{i})\pi_{i} = \pi_{i}\Theta_{i}$ for $i = 1, 2$. Decomposing $\mathcal{H}_{i}$ with $\mathcal{H}_{i} \pm := \frac{1 \pm \Gamma_{i}}{2} \mathcal{H}_{i}$, as $\mathcal{H}_{i} = \mathcal{H}_{i+} \oplus \mathcal{H}_{i-}$, we have $\pi_{i}^{(0)}_{\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})} = \pi_{i+} \oplus \pi_{i-}$ with $\pi_{i\pm}$ mutually singular irreducible representations on $\mathcal{H}_{i\pm}$ of $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$.

(ii) The states $\omega|_{\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})}$ and $\omega|_{\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})}$ are mutually singular. The state $\omega$ has a GNS representation of the form $(\mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathbb{C}^{2}, \pi, \Omega)$. There are irreducible representations $\pi_{i}$ of $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$, on $\mathcal{H}_{i}$, $i = 1, 2$ such that

$$
\pi(a \hat{\otimes} b) = \pi_{1}(a) \otimes \pi_{2}(b) \otimes I_{\mathbb{C}^{2}}, \quad a \in \mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}), \quad b \in \mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}) .
$$

(4.3)

We have

$$
\pi(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})) = B(\mathcal{H}_{1}) \otimes \mathbb{C}I_{\mathcal{H}_{2}} \otimes (\mathbb{C}\sigma_{2} + \mathbb{C}I) ,
$$

$$
\pi(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})) = \mathbb{C}I_{\mathcal{H}_{1}} \otimes B(\mathcal{H}_{2}) \otimes (\mathbb{C}\sigma_{2} + \mathbb{C}I) .
$$

(4.4)

We also have $\text{Ad}(I_{\mathcal{H}_{1}} \otimes I_{\mathcal{H}_{2}} \otimes \sigma_{y}) \circ \pi = \pi \circ \Theta$.

If $\omega$ satisfies (i) (resp. (ii)), then $\omega \circ (\eta_{1} \hat{\otimes} \eta_{2})$ also satisfies (i) (resp. (ii)), for any $\eta_{1} \in \text{Aut}^{(0)}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})))$, $\eta_{2} \in \text{Aut}^{(0)}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})))$.

Proof. Let $\varphi_{i}$ be homogeneous states on $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$, $i = 1, 2$ such that $\omega \sim_{q.e.} \varphi_{1} \hat{\otimes} \varphi_{2}$. Let $\omega_{i} := \omega|_{\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})}$, $i = 1, 2$, on $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$ be the restriction of $\omega$, which is homogeneous. By Lemma 4.2, we have $\omega_{i} \sim_{q.e.} \omega_{i} \hat{\otimes} \omega_{2}$.

Because $\omega$ is a pure state, $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a type I factor. From Lemma B.2, we conclude that both of $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ and $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ are type I. From Lemma B.1, they are either a type I factor or a direct sum of two type I factors. If $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a factor but not $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$, then by Lemma A.2 of [BO] $\{ \pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})) \}$ has a self-adjoint odd unitary $b$ while $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ includes a even self-adjoint unitary $\theta_{1}$ implementing the grading on $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$. Note that $\theta_{1} \hat{\otimes} b$ belongs to the center of $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})) \hat{\otimes} \pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$. This contradicts to the fact that the latter algebra is a factor. Hence if $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a factor, then $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a factor as well. Similarly, if $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a factor then $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a factor as well. As a result, either both of $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$, $i = 1, 2$ are type I factors or (ii) both of $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$, $i = 1, 2$ are direct sum of two type I factors. Note from Lemma 6.23 of [EK] that $\omega|_{\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})}$ is pure.

(i) If both of $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$, $i = 1, 2$ are type I factors, then from Lemma 5.5 of [BO] and its proof, the state $\omega$ has a GNS representation of the form $(\mathcal{H}_{1} \otimes \mathcal{H}_{2}, \pi_{i} \otimes \pi_{2}, \Omega)$ with $(\mathcal{H}_{i}, \pi_{i})$ an irreducible representation of $\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C})$ for $i = 1, 2$. Because $\pi_{\omega}(\mathfrak{A}_{SDC}(\mathfrak{K}, \mathfrak{C}))$ is a type I factor, there is a self-adjoint unitary $\Gamma_{i}$ on $\mathcal{H}_{i}$ implementing $\Theta_{i}$, i.e., $\text{Ad}(\Gamma_{i}) \pi_{i} = \pi_{i} \Theta_{i}$ for $i = 1, 2$. From this, $\Gamma_{1} \otimes \Gamma_{2}$ implements $\Theta$ in $\pi_{1} \otimes \pi_{2}$. Decomposing $\mathcal{H}_{i}$ with $\mathcal{H}_{i} \pm := \frac{1 \pm \Gamma_{i}}{2} \mathcal{H}_{i}$, as $\mathcal{H}_{i} = \mathcal{H}_{i+} \oplus \mathcal{H}_{i-}$, we have
\( \pi_i|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}, \xi)} = \pi_{i+} \oplus \pi_{i-} \) with \( \pi_{i\pm} \) mutually singular irreducible representation of \( \mathcal{A}_{\text{SDC}}(\mathfrak{A}, \xi) \). This last property follows from Lemma 6.24 of [EK].

Because \( \mathbb{I}_{\mathcal{H}_i} \otimes \Gamma_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \left( (\pi_1 \otimes \pi_2)\left( \mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2) \right) \right)'' \) is even with respect to \( \text{Ad}(\Gamma_1 \otimes \Gamma_2) \), we have \( \mathbb{I}_{\mathcal{H}_1} \otimes \Gamma_2 \in \left( (\pi_1 \otimes \pi_2)\left( \mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2) \right) \right)'' \). Therefore,

\[
\omega \circ \Theta_2|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)} = \left( \mathbb{I}_{\mathcal{H}_1} \otimes \Gamma_2 \right) \Omega \cdot \left( \pi_1 \otimes \pi_2 \right)|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)} \left( \mathbb{I}_{\mathcal{H}_1} \otimes \Gamma_2 \right) \Omega \tag{4.5.1} \]

is quasi-equivalent to \( \omega|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)} \).

(ii) If both of \( \pi_\omega(\mathcal{A}_{\text{SDC}}(\mathfrak{A}, \xi))'' \), \( i = 1, 2 \) are summation of two type I factors, then by Lemma B.3, the state \( \omega \) has a GNS representation of the form \( (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2, \pi, \Omega) \). There are irreducible representations \( \pi_i \) of \( \mathcal{A}_{\text{SDC}}(\mathfrak{A}, \xi_i) \), on \( \mathcal{H}_i, i = 1, 2 \) such that

\[
\pi(a \hat{\otimes} b) = \pi_1(a) \otimes \pi_2(b) \otimes \mathbb{I}_{\mathbb{C}^2}, \quad a \in \mathcal{A}_{\text{SDC}}(\mathfrak{A}_1, \xi_1), \quad b \in \mathcal{A}_{\text{SDC}}(\mathfrak{A}_2, \xi_2). \tag{4.6} \]

We have

\[
\pi(\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1, \xi_1))'' = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C}[\mathbb{H}_2 \otimes (\mathbb{C}\sigma_z + \mathbb{C})],
\]

\[
\pi(\mathcal{A}_{\text{SDC}}(\mathfrak{A}_2, \xi_2))'' = \mathbb{C}[\mathcal{H}_1] \otimes \mathcal{B}(\mathcal{H}_2) \otimes (\mathbb{C}\sigma_x + \mathbb{C}). \tag{4.7} \]

We also have \( \text{Ad}(\mathbb{I}_{\mathcal{H}_1} \otimes \mathbb{I}_{\mathcal{H}_2} \otimes \sigma_y) \circ \pi = \pi \Theta_2 \). Set \( \Gamma_2 := \mathbb{I}_{\mathcal{H}_1} \otimes \mathbb{I}_{\mathcal{H}_2} \otimes \sigma_z \). Note that \( \text{Ad}(\Gamma_2) \circ \pi = \pi \Theta_2 \) and

\[
\pi\left(\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)\right)'' = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \otimes (\mathbb{C}[\mathbb{C}]) \tag{4.8} \]

Hence the center of \( \pi\left(\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)\right)'' \) is \( \mathcal{C}[\mathcal{H}_1] \otimes \mathcal{C}[\mathcal{H}_2] \otimes (\mathbb{C}\sigma_x + \mathbb{C}) \) with \( r_{\pm} := \mathbb{I}_{\mathcal{H}_1} \otimes \mathbb{I}_{\mathcal{H}_2} \otimes \frac{\mathbb{I} \pm \sigma_y}{2} \), and \( \text{Ad}(\Gamma_2) \) flips \( r_+ \) and \( r_- \). From this, \( \pi_{0, \pm} := \pi|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)}(\cdot)|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)}(\cdot) \) defines mutually singular irreducible representations of \( \mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2) \). Because \( \omega|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)} \) is pure, there is some \( \zeta = \pm \) such that \( \Omega = r_{\zeta} \Omega \). Note that

\[
\Gamma_2 \Omega = \Gamma_2 r_\zeta \Omega = \Gamma_2 r_\zeta \Gamma_2 \Omega = r_{-\zeta} \Gamma_2 \Omega. \tag{4.9} \]

From these, we have

\[
\omega|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)}(\Omega, \pi_{0, \zeta}(\cdot) \Omega) = \left( \Gamma_2 \Omega, \pi_{0, -\zeta}(\cdot) \Gamma_2 \Omega \right), \tag{4.10} \]

Because \( \pi_{0, \pm} \) are mutually singular, \( \omega|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)}(\cdot) \) and \( \omega|_{\mathcal{A}_{\text{SDC}}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \xi_1 \oplus \xi_2)}(\cdot) \circ \Theta_2 \) are disjoint.
The last statement comes from the fact that $\pi_\omega \circ (\eta_1 \hat{\otimes} \eta_2)$ is a GNS representation of \( \omega \circ (\eta_1 \hat{\otimes} \eta_2) \) and

$$\pi_\omega \circ (\eta_1 \hat{\otimes} \eta_2) \left( \mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2) \right)'' = \pi_\omega \left( \mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2) \right)''$$

where the right hand side is a factor if and only if the left hand side is. \( \square \)

Here is the main Proposition of this section.

**Proposition 4.5.** Let \( \Lambda_\sigma \subset \mathbb{Z}^2, \sigma = L, R \) be mutually disjoint infinite subsets of \( \mathbb{Z}^2 \). Set \( \mathfrak{h}_{\Lambda_\sigma} := \mathfrak{h}_{\Lambda_\sigma} \) with the complex conjugation \( \mathcal{C}_\sigma := \mathcal{C}_{\Lambda_\sigma} \), for \( \sigma = L, R \). Let \( \omega_0, \omega_1 \) be homogeneous pure states on \( \mathfrak{A}_{\text{SDC}} (\mathfrak{h}_L \oplus \mathfrak{h}_R, \mathcal{C}_L \oplus \mathcal{C}_R) \), satisfying the split property. Note from Lemma 4.4, either (i) or (ii) of Lemma 4.4 occurs. If

(a) (i) occurs for both of \( \omega_0, \omega_1 \), or
(b) (ii) occurs for both of \( \omega_0, \omega_1 \),

then there are automorphisms \( \eta_\sigma \in \text{Aut}^{(0)} (\mathfrak{A}_{\text{SDC}} (\mathfrak{h}_{\sigma}, \mathcal{C}_\sigma)) \) \( \sigma = L, R \) satisfying

$$\omega_1 \simeq \omega_0 \left( \eta_L \hat{\otimes} \eta_R \right).$$

(4.11)

Furthermore, if none of (a), (b) occurs, then \( \omega_1 \) and \( \omega_0 \) are not quasi-equivalent.

The proof is a deformation of that of [O1]. As we are considering non-twisted crossed product here, it is even simpler than the case of [O1]. For the rest of this section, \( \Lambda \) indicates an infinite subset of \( \mathbb{Z}^2 \). We recall notations from [O1] adapted to our current setting. An irreducible covariant representation of \( \Sigma_\Lambda := (\mathcal{H}, \pi, \Gamma) \) is a triple \((\mathcal{H}, \pi)\) where \((\mathcal{H}, \pi)\) is an irreducible representation of \( \mathcal{A}_\Lambda \) and \( \Gamma \) a self-adjoint unitary on \( \mathcal{H} \) satisfying \( \text{Ad} \Gamma \circ \pi = \pi \circ \Theta_\Lambda \).

Let \( C(\mathbb{Z}_2, \mathcal{A}_\Lambda) \) be the linear space of \( \mathcal{A}_\Lambda \)-valued functions on \( \mathbb{Z}_2 \). We equip \( C(\mathbb{Z}_2, \mathcal{A}_\Lambda) \) with a product and *-operation as follows:

$$f_1 \ast f_2 (h) := \sum_{g \in \mathbb{Z}_2} f_1 (g) \cdot \Theta^{g \Lambda}_\Lambda \left( f_2 (g^{-1} h) \right), \quad h \in \mathbb{Z}_2,$$

(4.12)

$$f^* (h) := \Theta^{h \Lambda}_\Lambda \left( f (h^{-1})^* \right), \quad h \in \mathbb{Z}_2,$$

(4.13)

for \( f_1, f_2, f \in C(\mathbb{Z}_2, \mathcal{A}_\Lambda) \). The linear space \( C(\mathbb{Z}_2, \mathcal{A}_\Lambda) \) which is a *-algebra with these operations is denoted by \( C(\Sigma_\Lambda) \).

For a covariant representation \((\mathcal{H}, \pi, \Gamma)\) of \( \Sigma_\Lambda \), we may introduce a *-representation \((\mathcal{H}, \pi \times \Gamma)\) of \( C(\Sigma_\Lambda) \) by

$$\left( \pi \times \Gamma \right) (f) := \pi \left( f (0) \right) + \pi \left( f (1) \right), \quad f \in C(\Sigma_\Lambda).$$

(4.14)

The twisted crossed product of \( \Sigma_\Lambda \), denoted \( C^* (\Sigma_\Lambda) \) is the completion of \( C(\Sigma_\Lambda) \) with respect to the norm

$$\| f \|_\nu := \sup \left\{ \| (\pi \times \Gamma) (f) \| \mid (\pi, \Gamma) \text{ : covariant representation} \right\}, \quad f \in C(\Sigma_\Lambda).$$

(4.15)

Because \( \mathbb{Z}_2 \) is finite, we actually have \( C(\Sigma_\Lambda) = C^* (\Sigma_\Lambda) \). It also coincides with the reduced crossed product \( C^*_r (\Sigma_\Lambda) \) and simple because \( \Theta \) is properly outer [E]. As \( \mathcal{A}_\Lambda \) is unital, we have unitaries \( \lambda_1 \in C^* (\Sigma_\Lambda) \), such that

$$\lambda_1 a \lambda_1^* = \Theta_\Lambda (a), \quad a \in \mathcal{A}_\Lambda.$$

(4.16)

For the rest of this section, we use the following notations.
Notation 4.1. For each finite subset \( \Lambda_0 \) of \( \Lambda \), let \( \{ e_{IJ}^{(\Lambda_0)} \}_I \) be a system of matrix units spanning \( \mathcal{A}_{\Lambda_0} \) with \( \Theta(e_{IJ}^{(\Lambda_0)}) = (-1)^{|I|+|J|} e_{IJ}^{(\Lambda_0)} \). Because \( d \) is even, \( \mathfrak{h}_{\Lambda_0} \) is even-dimensional and our \( \mathcal{A}_{\Lambda_0} = \mathfrak{A}_{SDC}(\mathfrak{h}_{\Lambda_0}) \) is isomorphic to a finite dimensional CAR-algebra. Therefore, such system of matrix units exists. Fix some \( \mathfrak{A}(\Lambda_0) \).

Proposition 4.6. Let \( \omega \) be any finite set \( \Lambda_0 \). Let \( \tilde{\omega} \subseteq \omega \) be the decomposition associated to Proposition 4.2 \([O1]\) which relies on the techniques developed in \( \mathcal{A}(\Lambda) \) [KOS] [F].

Lemma 4.7. Set \( \mathcal{G}_{\Lambda_0} := \left\{ \frac{1}{\sqrt{2}} e_{11}^{(\Lambda_0)} + \frac{1}{\sqrt{2}} \lambda e_{10}^{(\Lambda_0)} \right\}_I, \mathcal{G}_{\Lambda_0}^{(0)} := \{ e_{10}^{(\Lambda_0)} \}_I \).

Proof. This follows by the same argument as that of Lemma 4.6 \([O1]\).
Lemma 4.8. For any finite subset $\Lambda_0 \subset \Lambda$ and $\epsilon > 0$, there exists $\delta_1(\epsilon, \Lambda_0) > 0$ satisfying the followings: For any irreducible covariant representation $(\mathcal{H}, \pi, \Gamma)$ of $\Sigma_\Lambda$ and unit vectors $\xi, \eta \in \mathcal{H}$ satisfying

\[
\left\langle (\pi \times \Gamma) (x^*) \xi, \eta \right\rangle \leq (\pi \times \Gamma) (xy^*) \eta, \quad \left\| (\pi \times \Gamma) (x^*) \xi - (\pi \times \Gamma) (xy^*) \eta \right\| < \delta_1(\epsilon, \Lambda_0),
\]  

for any $x, y \in \mathcal{G}_{\Lambda_0}$,

there exists an even positive element $\hat{h} \in \mathcal{A}_{\Lambda \setminus \Lambda_0, 1}$ such that

\[
\left\| e^{i\pi\hat{h}\xi} - \eta \right\| < \frac{\delta_{2,a}(\epsilon)}{4\sqrt{2}}.
\]  

Proof. With $\delta_{3,a}(\epsilon, n)$ given in Lemma B.6 [O1], we set $\delta_1(\epsilon, \Lambda_0) := \delta_{3,a}(\epsilon, 2^{d(\Lambda_0) + 1}) > 0$. Suppose that $(\mathcal{H}, \pi, \Gamma), \xi, \eta$ satisfy the condition (4.22) for this $\delta_1(\epsilon, \Lambda_0)$. Then by Lemma B.6 of [O1], there exists $h \in (C^*(\Sigma_\Lambda))_{+,1}$ such that

\[
\left\| (\pi \times \Gamma) (\hat{h}) (\xi + \eta) \right\|, \left\| (\hat{h} - (\pi \times \Gamma) (\hat{h}) \right\| < \frac{\delta_{2,a}(\epsilon)}{4\sqrt{2}},
\]

for

\[
\hat{h} := \frac{1}{2} \sum_{\sigma = 0, 1} \sum_I \lambda_1^\sigma \epsilon_{I,0} \epsilon_{I,0} \in C^*(\Sigma).
\]

By the definition, we see that $\hat{h}$ commutes with both of $\lambda_1$ and elements in $\mathcal{A}_{\Lambda_0}$. From Lemma 5.5 and its proof of [BO], there are irreducible covariant representations $(\mathcal{H}_1, \pi_1, \Gamma_1), (\mathcal{H}_2, \pi_2, \Gamma_2)$ of $\mathcal{A}_{\Lambda_0}, \mathcal{A}_{\Lambda \setminus \Lambda_0}$ and a unitary $W : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\text{Ad} W \circ \pi = \pi_1 \otimes \pi_2$ and $\text{Ad} (W) (\Gamma) = \Gamma_1 \otimes \Gamma_2$. Because $\text{Ad} W (\pi (\hat{h}))$ commutes with $\text{Ad} W (\pi (\mathcal{A}_{\Lambda_0})) = B(\mathcal{H}_1) \otimes I$ and $\text{Ad} (W) (\Gamma) = \Gamma_1 \otimes \Gamma_2$, $\text{Ad} W (\pi (\hat{h}))$ is of the form $\mathbb{I} \otimes x$ with $x \in B(\mathcal{H}_2)_{+,1}$ such that $\text{Ad} \Gamma_2 (x) = x$. Because $(\mathcal{H}_2, \pi_2, \Gamma_2)$ is an irreducible covariant representation, as in the proof of Lemma 4.10 [O1], there is some $\hat{h} \in \left( \mathcal{A}_{\Lambda \setminus \Lambda_0}^0 \right)_{1,+}$ such that $\left\| \left( \mathbb{I} \otimes \pi_2 (\hat{h}) - \mathbb{I} \otimes \pi_2 (x) \right) W (\xi + \eta) \right\|$ and $\left\| \left( \mathbb{I} \otimes \pi_2 (\hat{h}) - \mathbb{I} \otimes \pi_2 (x) \right) W (\xi - \eta) \right\|$ are small enough so that

\[
\left\| (\hat{h} (\xi + \eta) \right\| \leq \left\| (\hat{h} - (\pi \times \Gamma) (\hat{h}) (\xi + \eta) \right\| + \left\| (\pi \times \Gamma) (\hat{h}) (\xi + \eta) \right\| < \frac{\delta_{2,a}(\epsilon)}{4\sqrt{2}},
\]

\[
\left\| \left( \mathbb{I} \otimes \pi_2 (\hat{h}) - \mathbb{I} \otimes \pi_2 (x) \right) W (\xi + \eta) \right\| < \frac{\delta_{2,a}(\epsilon)}{4\sqrt{2}},
\]

\[
\left\| \left( \mathbb{I} \otimes \pi_2 (\hat{h}) - \mathbb{I} \otimes \pi_2 (x) \right) W (\xi - \eta) \right\| < \frac{\delta_{2,a}(\epsilon)}{4\sqrt{2}},
\]

hold. Then by the argument in Lemma 4.10 of [O1] (eq. (91)), for this $\hat{h}$, we obtain (4.23).
Lemma 4.9. For any $\epsilon > 0$ and a finite subset $\mathcal{F}$ of $A_\Lambda$, set $\delta_2(\epsilon, \mathcal{F}) := \frac{1}{2}\delta_1\left(\frac{\epsilon}{4}, \Lambda(\epsilon, \mathcal{F})\right)$ and $\tilde{G}_1(\epsilon, \mathcal{F}) := G_{\Lambda(\epsilon, \mathcal{F})}$, where $\delta_1(\epsilon, \Lambda_0)$ is defined in Lemma 4.8. Then for any irreducible covariant representation $(\mathcal{H}, \pi, \Gamma)$ of $\Sigma_\Lambda$ with the associated decomposition $(\mathcal{H}_+, \pi_+)$ and unit vectors $\xi, \eta \in \mathcal{H}_+$ such that
\[
\left| \left\langle \xi, (\pi \times \Gamma) (xy^*) \xi \right\rangle - \left\langle \xi, (\pi \times \Gamma) (xy^*) \eta \right\rangle \right| < \delta_2(\epsilon, \mathcal{F}), \quad x, y \in \tilde{G}_1(\epsilon, \mathcal{F}) \tag{4.27}
\]
for $\tilde{\xi} := \xi \oplus 0, \tilde{\eta} := \eta \oplus 0 \in \mathcal{H}_+ \oplus \mathcal{H}_-$, there is a continuous map $v : [0, 1] \rightarrow \mathcal{U}(A_\Lambda^{(0)})$ satisfying
\[
\tilde{\eta} = \pi(v(1)) \tilde{\xi}, \quad v(0) = I, \quad \sup_{t \in [0, 1]} \|\operatorname{Ad}v(t)(a) - a\| < \epsilon, \quad a \in \mathcal{F}. \tag{4.28}
\]
Proof. Let $K$ be a finite dimensional subspace of $\mathcal{H}$ spanned by
\[
\left\{ (\pi \times \Gamma) (xy^*) \xi, (\pi \times \Gamma) (xy^*) \eta \mid x, y \in \tilde{G}_1(\epsilon, \mathcal{F}) \right\}.
\]
Then because $C^*(\Sigma_\Lambda)$ is simple, by Glimm's Lemma (Lemma 5.2.5 [F]), there exists a unit vector $\zeta \in K^\perp$ such that
\[
\left| \left\langle \tilde{\xi}, (\pi \times \Gamma) (xy^*) \tilde{\xi} \right\rangle - \left\langle \xi, (\pi \times \Gamma) (xy^*) \zeta \right\rangle \right| < \delta_2(\epsilon, \mathcal{F}) < \delta_1\left(\frac{\epsilon}{4}, \Lambda(\epsilon, \mathcal{F})\right), \quad x, y \in \tilde{G}_1(\epsilon, \mathcal{F}). \tag{4.29}
\]
Combining this with (4.27), we also have
\[
\left| \left\langle \tilde{\eta}, (\pi \times \Gamma) (xy^*) \tilde{\eta} \right\rangle - \left\langle \xi, (\pi \times \Gamma) (xy^*) \zeta \right\rangle \right| < 2\delta_2(\epsilon, \mathcal{F}) = \delta_1\left(\frac{\epsilon}{4}, \Lambda(\epsilon, \mathcal{F})\right), \quad x, y \in \tilde{G}_1(\epsilon, \mathcal{F}). \tag{4.30}
\]
Because we have
\[(\pi \times \Gamma) (y^*) \tilde{\xi}, (\pi \times \Gamma) (y^*) \tilde{\eta} \perp (\pi \times \Gamma) (x^*) \zeta \quad \text{for any} \quad x, y \in \tilde{G}_1(\epsilon, \mathcal{F}), \tag{4.31}
\]
from Lemma 4.8, there are $h_1, h_2 \in \left(\mathcal{A}_\Lambda^{(0)}\right)_{1+}$ such that
\[
\left\| e^{i\pi \pi_+(h_1)} \tilde{\xi} - \zeta \right\|, \left\| e^{i\pi \pi_+(h_2)} \tilde{\eta} - \zeta \right\| < \frac{1}{4\sqrt{2}} \delta_2(a) \left(\frac{\epsilon}{32}\right). \tag{4.32}
\]
From this, recalling that $h_1$ and $h_2$ are even, we have
\[
\left\| e^{i\pi \pi_+(h_1)} \tilde{\xi} - e^{i\pi \pi_+(h_2)} \tilde{\eta} \right\|_{\mathcal{H}_+} = \left\| e^{i\pi \pi_+(h_1)} \tilde{\xi} - e^{i\pi \pi_+(h_2)} \tilde{\eta} \right\| < \frac{1}{2\sqrt{2}} \delta_2(a) \left(\frac{\epsilon}{32}\right). \tag{4.33}
\]
Here $\left\| \cdot \right\|_{\mathcal{H}_+}$ means the norm associated to $\mathcal{H}_+$. Because $\pi_+$ is irreducible, by the argument in the proof of Lemma 4.13 in [O1] (around equation (103)), there is some self-adjoint $K \in \mathcal{A}(0)$ such that
\[
e^{i\pi \pi_+(K)} e^{i\pi \pi_+(h_1)} \tilde{\xi} = e^{i\pi \pi_+(h_2)} \tilde{\eta},
\]
\[
\left\| K \right\| < \delta_{1,a} \left(\frac{\epsilon}{32}\right), \quad \sup_{\theta \in [0, 1]} \left\| e^{iK\theta} - I \right\| \leq \frac{\epsilon}{32}. \tag{4.34}
\]
Here, $\delta_{1,a}(\epsilon)$ is given in Notation B.3 of [O1]. From this, as in the proof of Lemma 4.13 [O1] (from equation (105)), we can find $v(t)$ satisfying the condition (4.28). \qed
Lemma 4.10. Let $(\mathcal{H}_i, \pi_i, \Gamma_i)$, $i = 0, 1$ be irreducible covariant representations of $\Sigma$. Let $(\mathcal{H}_{i\pm}, \pi_{i\pm})$ be the decomposition associated to $(\mathcal{H}_i, \pi_i, \Gamma_i)$, and $\xi_i \in \mathcal{H}_{i\pm}$ a unit vector. Set $\tilde{\xi}_i := \xi_i \oplus 0 \in \mathcal{H}_{i\pm} \oplus \mathcal{H}_{i\mp}$. Suppose that for $\varepsilon > 0$ and a finite set $\mathcal{F} \subset (\mathcal{A}_\Lambda)_1,$

$$\left\| \left( \tilde{\xi}_0, (\pi_0 \times \Gamma_0)(xy^*)\tilde{\xi}_0 \right) - \left( \tilde{\xi}_1, (\pi_1 \times \Gamma_1)(xy^*)\tilde{\xi}_1 \right) \right\| < \frac{1}{2} \delta_2(\varepsilon, \mathcal{F}), \quad x, y \in \tilde{\mathcal{G}}_1(\varepsilon, \mathcal{F})$$

(4.35)

hold. Then for any $\varepsilon' > 0$ and finite set $\mathcal{F}' \subset C^*(\Sigma)$, there exists a norm continuous path $v : [0, 1] \to \mathcal{U}(\mathcal{A}_\Lambda^{(0)})$ such that

$$\left\| \left( \tilde{\xi}_0, (\pi_0 \times \Gamma_0)(a)\tilde{\xi}_0 \right) - \left( \tilde{\xi}_1, (\pi_1 \times \Gamma_1) \circ \Ad(v(1))(a)\tilde{\xi}_1 \right) \right\| < \varepsilon', \quad a \in \mathcal{F}',$$

$$\left\| \Ad(v(t))(y) - y \right\| < \varepsilon, \quad y \in \mathcal{F}, \quad t \in [0, 1].$$

(4.36)

**Proof.** This corresponds to Lemma 4.15 of [O1]. By Lemma 4.7, there exists a self-adjoint $h \in \mathcal{A}_\Lambda^{(0)}$ such that

$$\left\| \left( \tilde{\xi}_0, (\pi_0 \times \Gamma_0)(a)\tilde{\xi}_0 \right) - \left( \tilde{\xi}_1, (\pi_1 \times \Gamma_1) \circ \Ad(e^{ih})(a)\tilde{\xi}_1 \right) \right\| < \min\{\varepsilon', \frac{1}{2} \delta_2(\varepsilon, \mathcal{F})\}, \quad a \in \mathcal{F}' \cup \tilde{\mathcal{G}}_1(\varepsilon, \mathcal{F}) \left( \tilde{\mathcal{G}}_1(\varepsilon, \mathcal{F}) \right)^*.$$

(4.37)

Combining this with (4.35), we obtain

$$\left\| \left( \tilde{\xi}_1, (\pi_1 \times \Gamma_1) \circ \Ad(e^{ih})(xy^*)\tilde{\xi}_1 \right) - \left( \tilde{\xi}_1, (\pi_1 \times \Gamma_1)(xy^*)\tilde{\xi}_1 \right) \right\| < \delta_2(\varepsilon, \mathcal{F}), \quad x, y \in \tilde{\mathcal{G}}_1(\varepsilon, \mathcal{F}).$$

(4.38)

Here, because $h$ is even, we have

$$(\pi_1 \times \Gamma_1) \left( e^{-ih} \right) \tilde{\xi}_1 = e^{-i\pi_{1+}(h)}\tilde{\xi}_1 \oplus 0.$$  

(4.39)

From this, applying Lemma 4.9, we obtain the Lemma.  

**Proof of Proposition 4.6.** Having Lemmas 4.7 and 4.10, the proof of Proposition 4.6 is the same as that of Proposition 4.2 of [O1].  

Next we prepare a proposition needed for the case (b) of Proposition 4.5.

**Proposition 4.11.** Let $(\mathcal{H}_i, \pi_i), i = 0, 1$ be irreducible representations of $\mathcal{A}_\Lambda$ such that $\pi_i \left( \mathcal{A}_\Lambda^{(0)} \right)^\prime\prime = B(\mathcal{H}_i).$ Let $\xi_i \in \mathcal{H}_i, i = 0, 1$ be unit vectors and set

$$\omega_i := \langle \xi_i, \pi_i(\cdot)\xi_i \rangle.$$  

(4.40)

Then there is an automorphism $\eta \in \Aut(\mathcal{A}_\Lambda^{(0)})$ such that

$$\omega_1 = \omega_0 \eta.$$  

(4.41)

The proof follows the argument in [KOS], choosing the unitary there each time from $\mathcal{A}_\Lambda^{(0)}$, which is possible because of $\pi_i \left( \mathcal{A}_\Lambda^{(0)} \right)^\prime\prime = B(\mathcal{H}_i).$
Lemma 4.12. Let \((\mathcal{H}_i, \pi_i), i = 0, 1\) be irreducible representations \(A_{\Lambda}\) such that \(\pi_i \left( A_{\Lambda}^{(0)} \right)'' = B(\mathcal{H}_i)\). Let \(\xi_i \in \mathcal{H}_i, i = 0, 1\) be unit vectors. Then for any finite subset \(F\) of \(A_{\Lambda}\) and \(\varepsilon > 0\), there exists a self-adjoint element \(h \in A_{\Lambda}^{(0)}\) such that

\[
\left| \langle \xi_0, \pi_0(a)\xi_0 \rangle - \left\langle \xi_1, \pi_1 \circ \text{Ad}(e^{ih})(a)\xi_1 \right\rangle \right| < \varepsilon, \quad a \in F.
\]  (4.42)

Proof. Let \(\omega_i := \langle \xi_i, \pi_i(\cdot)\xi_i \rangle\) \(i = 0, 1\) be pure states on \(A_{\Lambda}\). By Lemma B.1 [O1], there exist \(f \in (A_{\Lambda})_{+1}\) and a unit vector \(\xi'_1 \in \mathcal{H}_1\) such that

\[
\pi_1(f)\xi'_1 = \xi'_1, \quad \| f (a - \omega_0(a)f) \| < \varepsilon, \quad a \in F.
\]  (4.43)

Because we now have \(\pi_1 \left( A_{\Lambda}^{(0)} \right)'' = B(\mathcal{H}_1)\), for unit vectors \(\xi_1, \xi'_1 \in \mathcal{H}_1\) there exists a self-adjoint element \(h \in A_{\Lambda}^{(0)}\) such that \(e^{i\pi_1(h)}\xi'_1 = \xi_1\) by the Kadison transitivity. For this \(h\) we have

\[
\left| \langle \xi_0, \pi_0(a)\xi_0 \rangle - \left\langle \xi_1, \pi_1 \circ \text{Ad}(e^{ih})(a)\xi_1 \right\rangle \right| = \left| \left\langle \xi'_1, \pi_1(f)(\omega_0(a) - \pi_1(a))\pi_1(f)\xi'_1 \right\rangle \right| < \varepsilon, \quad a \in F.
\]  (4.44)

\[\square\]

Lemma 4.13. For any \(\varepsilon > 0\) and a finite subset \(\Lambda_0 \subset \Lambda\), there exists \(\delta_3(\varepsilon, \Lambda_0) > 0\) satisfying the following: For any \((\mathcal{H}, \pi)\) an irreducible representation of \(A_{\Lambda}\) with \(\pi \left( A_{\Lambda}^{(0)} \right)'' = B(\mathcal{H})\) and unit vectors \(\xi, \eta \in \mathcal{H}\) satisfying

\[
\pi(x^*)\xi \perp \pi(y^*)\eta, \\
\left| \left| \langle \xi, \pi(xy^*)\xi \rangle - \langle \eta, \pi(xy^*)\eta \rangle \right| \right| < \delta_3(\varepsilon, \Lambda_0),
\]  (4.45)

for any \(x, y \in (e_{1I_0}^{(\Lambda_0)})_I\),

there exist self-adjoint operators \(h \in \left( A_{\Lambda}^{(0)} \right)_{+1}\) and \(k \in A_{\Lambda}^{(0)}\) such that

\[
h := \sum_I e_{1I_0}^{(\Lambda_0)}he_{I_0I}^{(\Lambda_0)} \in \left( A_{\Lambda_0}^{(0)} \right)_{+1},
\]

\[
\left\| e^{i\pi(h)}\xi - \eta \right\| < \frac{1}{\sqrt{2}} \delta_{2,a} \left( \frac{\varepsilon}{8} \right)
\]

\[
\left\| e^{itk} - 1 \right\| \leq \frac{\varepsilon}{8}, \quad t \in [0, 1],
\]

\[
e^{i\pi(k)}e^{i\pi(h)}\xi = \eta.
\]  (4.46)

Proof. With \(\delta_{3,a}(\varepsilon, \eta)\) given in Lemma B.6 [O1], we set \(\delta_3(\varepsilon, \Lambda_0) := \delta_{3,a}(\varepsilon, 2^{d(\Lambda_0)})\). Suppose that \((\mathcal{H}, \pi)\) is an irreducible representation of \(A_{\Lambda}\) satisfying the conditions

...
above with respect to this \( \delta_3(\epsilon, \Lambda_0) \). Applying Lemma B.6 \([O1]\), we obtain \( h \in (A_\Lambda)_+, \) such that
\[
\| \pi(h)(\xi + \eta) \| < \frac{1}{4\sqrt{2}} \delta_{2,a} \left( \frac{\epsilon}{8} \right) e^{-\pi},
\]
\[
\| (I - \pi(h))(\xi - \eta) \| < \frac{1}{4\sqrt{2}} \delta_{2,a} \left( \frac{\epsilon}{8} \right) e^{-\pi}
\]
holds for
\[
h := \sum I e^{(\Lambda_0)I} h e^{(\Lambda_0)I} .
\]

This \( h \) is obtained via the Kadison transitivity, using the irreducibility of \( \pi \) (see section 3 of [KOS], also section 5.6.1 of [F]). Because we have \( \pi(A(0))'' = B(\mathcal{H}) \), in fact we can choose this \( h \) from \( (A_\Lambda(0))_+, \) Doing so, we have
\[
h := \sum I e^{(\Lambda_0)I} h e^{(\Lambda_0)I} \in (A_\Lambda(0))_+ \cap A_\Lambda(0) \subset (A_\Lambda(0)_+, 1) .
\]
The last inclusion comes from Lemma 4.15 \([AM]\). For this \( \tilde{h} \), by the argument in Lemma 4.10 of \([O1]\) (eq. (91)), we have
\[
\| e^{i\pi(\tilde{h})}\xi - \eta \| < \frac{1}{\sqrt{2}} \delta_{2,a} \left( \frac{\epsilon}{8} \right) .
\]

Because we have \( \pi(A_\Lambda(0))'' = B(\mathcal{H}) \), by the Kadison transitivity, (Theorem B.4 \([O1]\)) we get some self-adjoint \( k \in A_\Lambda(0) \) such that
\[
e^{i\pi(k)} e^{i\pi(\tilde{h})}\xi = \eta , \quad \sup_{t \in [0,1]} \left\| e^{ikt} - I \right\| \leq \frac{\epsilon}{8} .
\]

\[\square\]

**Lemma 4.14.** For any \( \epsilon > 0 \) and a finite subset \( \mathcal{F} \) of \( (A_\Lambda)_1 \), set \( \delta_4(\epsilon, \mathcal{F}) := \frac{1}{2} \delta_3(\epsilon, \Lambda(\epsilon, \mathcal{F})) \) and \( \tilde{G}_2(\epsilon, \mathcal{F}) : = G_{A(0)}^{(0)}(\epsilon, \mathcal{F}) \), where \( \delta_3(\epsilon, \Lambda_0) \) is defined in Lemma 4.13. Then for any irreducible representation \( (\mathcal{H}, \pi) \) of \( A_\Lambda \) with \( \pi(A_\Lambda(0))'' = B(\mathcal{H}) \), and unit vectors \( \xi, \eta \in \mathcal{H} \) such that
\[
\left| \langle \xi, \pi(xy^*)\xi \rangle - \langle \eta, \pi(xy^*)\eta \rangle \right| < \delta_4(\epsilon, \mathcal{F}), \quad x, y \in \tilde{G}_2(\epsilon, \mathcal{F}) ,
\]
there exists a norm-continuous path \( v : [0, 1] \to \mathcal{U}(A_\Lambda(0)) \) such that
\[
\eta = \pi(v(1))\xi, \quad v(0) = I , \quad \sup_{t \in [0,1]} \| \text{Ad}(v(t))(a) - a \| < \epsilon , \quad a \in \mathcal{F} .
\]
Proof. Because $\mathcal{A}_\Lambda$ is simple, by Glimm’s Lemma, there exists a unit vector $\zeta$ such that

$$\zeta \perp \pi(xy^*)\xi, \quad x, y \in \tilde{G}_2(\varepsilon, \mathcal{F}),$$

$$\left|\langle \xi, \pi(xy^*)\xi \rangle - \langle \zeta, \pi(xy^*)\zeta \rangle\right| < \delta_4(\varepsilon, \mathcal{F}) < \delta_3(\varepsilon, \Lambda(\varepsilon, \mathcal{F})), \quad x, y \in \tilde{G}_2(\varepsilon, \mathcal{F}).$$

(4.54)

From the latter inequality combined with (4.52), we have

$$\left|\langle \eta, \pi(xy^*)\eta \rangle - \langle \zeta, \pi(xy^*)\zeta \rangle\right| < \delta_3(\varepsilon, \Lambda(\varepsilon, \mathcal{F})).$$

(4.55)

By Lemma 4.13, there exist $h_1, h_2 \in \left(\mathcal{A}_\Lambda^{(0)} \setminus \Lambda(\varepsilon, \mathcal{F})\right)_{+1}$ such that

$$\left\| e^{i\pi(h_1)}\xi - \xi \right\| < \frac{1}{\sqrt{2}}\delta_{2,a}\left(\frac{\varepsilon}{8}\right), \quad \left\| e^{i\pi(h_2)}\eta - \zeta \right\| < \frac{1}{\sqrt{2}}\delta_{2,a}\left(\frac{\varepsilon}{8}\right),$$

and self-adjoint $k_1, k_2 \in \mathcal{A}_\Lambda^{(0)}$ such that

$$e^{i\pi(k_1)}e^{i\pi(h_1)}\xi = \zeta, \quad e^{i\pi(k_2)}e^{i\pi(h_2)}\eta = \zeta.$$

Out of these $h_1, h_2, k_1, k_2$, as in the proof of Lemma 4.13 of [O1] (around (105)), we can obtain the desired $v(t)$.

Lemma 4.15. Let $\varepsilon > 0$ and $\mathcal{F}$ a finite subset of $(\mathcal{A}_\Lambda)_1$. Let $(\mathcal{H}_i, \pi_i), \ i = 0, 1$ be irreducible representations of $\mathcal{A}_\Lambda$ such that $\pi_i \left(\mathcal{A}_\Lambda^{(0)}\right)^\prime\prime = \mathcal{B}(\mathcal{H}_i)$. Let $\xi_i \in \mathcal{H}_i, \ i = 0, 1$ be unit vectors satisfying

$$\left|\langle \xi_0, \pi_0(xy^*)\xi_0 \rangle - \langle \xi_1, \pi_1(xy^*)\xi_1 \rangle\right| < \frac{1}{2}\delta_4(\varepsilon, \mathcal{F}), \quad x, y \in \tilde{G}_2(\varepsilon, \mathcal{F}),$$

(4.58)

where $\delta_4(\varepsilon, \mathcal{F}), \tilde{G}_2(\varepsilon, \mathcal{F})$ are defined in Lemma 4.14. Then for any $\varepsilon' > 0$, finite subset $\mathcal{F}' \subset \mathcal{A}_\Lambda$, there exists a norm-continuous path $v : [0, 1] \to \mathcal{U}(\mathcal{A}_\Lambda^{(0)})$ such that

$$v(0) = I,$$

$$\left|\langle \xi_0, \pi_0(a)\xi_0 \rangle - \langle \xi_1, \pi_1 \circ \text{Ad}(v(1))(a)\xi_1 \rangle\right| < \varepsilon', \quad a \in \mathcal{F}',$$

$$\left\| \text{Ad}(v(t))(y) - y \right\| < \varepsilon, \quad y \in \mathcal{F}, \ t \in [0, 1].$$

(4.59)

Proof. The proof of this Lemma is the same as that of Lemma 4.10, using Lemmas 4.12 and 4.14.

Proof of Proposition 4.11. Having Lemmas 4.12 and 4.15, the proof of Proposition 4.11 is the same as that of Proposition 4.2 of [O1].

Now we are ready to prove the main proposition of this section, Proposition 4.5.
Proof of Proposition 4.5. Let $\Theta_\sigma := \Theta_{\Lambda_\sigma}$ be the grading operator on $A_{\Lambda_\sigma} = \mathcal{A}_{SDC} (\mathfrak{k}_\sigma, \mathcal{C}_\sigma)$ for $\sigma = L, R$. Case (a)

In this case, $\omega_i, i = 0, 1$ has a GNS representation of the form $(H_{L,i} \otimes H_{R,i} \otimes \pi_{L,i} \otimes \pi_{R,i}, \Omega_i)$ with irreducible covariant representations $(H_{\sigma,i}, \pi_{\sigma,i}, \Gamma_{\sigma,i})$ of $\Sigma_{\sigma,i} := (\mathbb{Z}_2, A_{\Lambda_\sigma}, \Theta_{\Lambda_\sigma}), \sigma = L, R$. Let $(H_{\sigma,i,\pm}, \pi_{\sigma,i,\pm})$ be the decomposition associated to $(H_{\sigma,i}, \pi_{\sigma,i}, \Gamma_{\sigma,i})$. Choose and fix unit vectors $\xi_{\sigma,i} \in H_{\sigma,i,\pm}$, $i = 0, 1, \sigma = L, R$, and set $\xi_{\sigma,i} := \xi_{\sigma,i} \otimes 0 \in H_{\sigma,i,\pm} \oplus H_{\sigma,i,-} = H_{\sigma,i}$. Then $\psi_{\sigma,i} := \langle \xi_{\sigma,i}, \pi_{\sigma,i}(\cdot)\xi_{\sigma,i} \rangle, i = 0, 1$ defines a pure homogeneous state on $A_{\Lambda_\sigma}$, $\sigma = L, R$ such that

$$\omega_i \simeq \langle \psi_{L,i} \otimes \psi_{R,i} \rangle, \quad i = 0, 1.$$  

(4.60)

By Proposition 4.6, there exist $\eta_{\sigma} \in \text{Aut}(0) \left( A_{\Lambda_\sigma} \right)$ such that $\psi_{1} = \psi_{0} \circ \eta_{\sigma}, \sigma = L, R$. From this we have $\omega_1 \simeq \omega_0 \circ (\eta_1 \otimes \eta_R)$.

Case (b)

In this case $\omega_i, i = 0, 1$ has a GNS representation of the form $(H_{L,i} \otimes H_{R,i} \otimes \mathbb{C}^2, \pi_i, \Omega_i)$. There are irreducible representations $\pi_{\sigma,i}$ of $A_{\Lambda_\sigma}$ on $H_{\sigma,i}, \sigma = L, R, i = 0, 1$ such that

$$\pi_i (a \otimes b) = \pi_{L,i}(a) \otimes \pi_{R,i}(b) \otimes \mathbb{C}^2, \quad a \in A_{\Lambda_L}, \quad b \in A_{\Lambda_R}.$$  

(4.61)

We have

$$\pi_i \left( A_{\Lambda_L} \right)^\prime = B(H_{Li}) \otimes \mathbb{C}_L \otimes (C_{\sigma}\mathbb{I} + \mathbb{C}I), \quad \pi_i \left( A_{\Lambda_R} \right)^\prime = \mathbb{C}_L \otimes B(H_{Ri}) \otimes (C_{\sigma}\mathbb{I} + \mathbb{C}I).$$  

(4.62)

From this form, there are representations $\rho^\sigma_i$ of $A_{\Lambda_\sigma}$ on $H_{\sigma,i} \otimes \mathbb{C}^2$ such that

$$\pi_i(a) = \rho^L_i(a) \otimes \mathbb{I}_{H_Ri}, \quad a \in A_{\Lambda_L}, \quad \pi_i(b) = \mathbb{I}_{H_Li} \otimes \rho^R_i(b), \quad b \in A_{\Lambda_R}.$$  

(4.63)

Note that $\mathbb{I}_{H_Li} \otimes \mathbb{I}_{H_Ri} \otimes r^L \otimes r^L, \mathbb{I}_{H_Li} \otimes \mathbb{I}_{H_Ri} \otimes r^R \otimes r^R$ with $r^L := \frac{1 + \sigma}{\sqrt{2}}, r^R := \frac{1 - \sigma}{\sqrt{2}}$ commute with $\pi_i \left( A_{\Lambda_L} \right)^\prime$ and $\pi_i \left( A_{\Lambda_R} \right)^\prime$, respectively. Therefore,

$$\rho^L_i(a) \otimes r^L \otimes r^L := \rho^L_i(a) \left( \mathbb{I}_{H_Li} \otimes r^L \otimes r^L \right), \quad a \in A_{\Lambda_L}, \quad \rho^R_i(b) \otimes r^R \otimes r^R$$

$$:= \rho^R_i(b) \left( \mathbb{I}_{H_Ri} \otimes r^R \otimes r^R \right), \quad b \in A_{\Lambda_R}.$$  

(4.64)

define irreducible representations $\rho^\sigma_i$ of $A_{\Lambda_\sigma}$ on $H_{\sigma,i}$, for $i = 0, 1, \sigma = L, R$. Note also that

$$\rho^\sigma_i \left( A_{\Lambda_\sigma} \right)^\prime = B(H_{\sigma,i}).$$  

(4.65)

Because $\mathbb{I}_{H_Li} \otimes \mathbb{I}_{H_Ri} \otimes \sigma_j$ flips $\mathbb{I}_{H_Li} \otimes \mathbb{I}_{H_Ri} \otimes r^\sigma_i$, we have

$$\rho^L_i \circ \Theta_L(a) = \rho^L_i(a), \quad a \in A_{\Lambda_L},$$

$$\rho^R_i \circ \Theta_R(b) = \rho^R_i(b), \quad b \in A_{\Lambda_R}.$$  

(4.66)
Choose unit vectors $\xi_{\sigma i} \in \mathcal{H}_{\sigma i}$ and define pure states

$$
\psi_{\sigma i} := \{\xi_{\sigma i}, \rho_{\sigma i}^0 (\cdot ) \xi_{\sigma i}\}, \quad i = 0, 1, \quad \sigma = L, R
$$

(4.67)
on $\mathcal{A}_{\Lambda_\sigma}$ Hence applying Proposition 4.11, there exist automorphisms $\eta_{\sigma} \in \text{Aut}^{(0)} (\mathcal{A}_{\Lambda_\sigma})$ such that $\psi_{\sigma 1} = \psi_{\sigma 0} \circ \eta_{\sigma}$. Then both of $(\mathcal{H}_{\sigma 1}, \rho_{\sigma 1}^0, \xi_{\sigma 1})$ and $(\mathcal{H}_{\sigma 0}, \rho_{\sigma 0}^0 \circ \eta_{\sigma}, \xi_{\sigma 0})$ are GNS triple of $\psi_{\sigma 1}$. Therefore, there are unitaries $W_{\sigma} : \mathcal{H}_{\sigma 0} \to \mathcal{H}_{\sigma 1}$ such that $\text{Ad}(W_{\sigma}) \circ \rho_{0\sigma}^0 \circ \eta_{\sigma} = \rho_{1\sigma}^0, \sigma = L, R$. Combining this and (4.66), we have $\text{Ad}(W_{\sigma}) \circ \rho_{0\sigma}^0 \circ \eta_{\sigma} = \rho_{1\sigma}^0$. Then we have

$$
\text{Ad}\left(W_L \otimes \mathbb{I}_{\mathcal{C}^2}\right) \left(\rho_{0L}^0 \circ \eta_L (a)\right) = \rho_{1L}^0 (a), \quad a \in \mathcal{A}_{\Lambda L},
$$

$$
\text{Ad}\left(W_R \otimes \mathbb{I}_{\mathcal{C}^2}\right) \left(\rho_{0R}^0 \circ \eta_R (b)\right) = \rho_{1R}^0 (b), \quad b \in \mathcal{A}_{\Lambda R}.
$$

(4.68)

We claim $\pi_1 = \text{Ad}\left(W_L \otimes W_R \otimes \mathbb{I}_{\mathcal{C}^2}\right) \circ \pi_0 \left(\eta_L \otimes \eta_R\right)$. In fact we have

$$
\text{Ad}\left(W_L \otimes W_R \otimes \mathbb{I}_{\mathcal{C}^2}\right) \circ \pi_0 \left(\eta_L \otimes \eta_R\right) (a) = \rho_{1L}^0 (a) \otimes I_{\mathcal{H}_{L_1}} = \pi_1 (a), \quad a \in \mathcal{A}_{\Lambda L},
$$

$$
\text{Ad}\left(W_L \otimes W_R \otimes \mathbb{I}_{\mathcal{C}^2}\right) \circ \pi_0 \left(\eta_L \otimes \eta_R\right) (b) = I_{\mathcal{H}_{L_1}} \otimes \rho_{1R}^0 (b) = \pi_1 (b), \quad b \in \mathcal{A}_{\Lambda R}.
$$

(4.69)

Hence $\pi_1$ and $\pi_0 \left(\eta_L \otimes \eta_R\right)$ are unitarily equivalent, and $\omega_1 \cong \omega_0 \circ \left(\eta_L \otimes \eta_R\right)$.

Suppose that none of (a), (b) occurs. In this case, one of $\pi_{\omega_1} \left(\mathcal{A}_{\Lambda R}\right)^{''}, \pi_{\omega_0} \left(\mathcal{A}_{\Lambda R}\right)^{''}$ is a factor and the other is not. Therefore, $\omega_1$ and $\omega_0$ are not quasi-equivalent. □

We conclude this section with examples of the dichotomy of Lemma 4.4. It is clear that $\omega^{(0)}$ satisfies (i) of Lemma 4.4.

Recall the basis projection $p (1.10)$ we considered in Sect. 1, defining $\omega^{(0)}$. Because $v_\tau$ commutes with $\mathcal{C}, q := v_\tau^* p v_\tau$ is also a basis projection of $(h, \mathcal{C})$. We denote by $\omega^{(1)}$ the Fock state on $\mathcal{A}$ given by this basis projection $q = v_\tau^* p v_\tau$. Note that $\omega^{(1)} = \omega^{(0)} \circ \tau$, and $\omega^{(1)} \circ \tau = \omega^{(0)}$. To describe $v_\tau^* p v_\tau$, set for each $x \in \mathbb{Z}$ two dimensional space $K_{(x,0)}$ spanned by $\delta_{(2x-1,0)}, \delta_{(2x,0)}$. Let $q_x$ be an orthogonal projection on $K_{(x,0)}$ onto the one-dimensional space spanned by $\delta_{(2x-1,0)} + i \delta_{(2x,0)}$. By the definition, we have

$$
q = v_\tau^* p v_\tau = Q_L^{(1)} \oplus Q_R^{(1)} \oplus q_0,
$$

(4.70)

with

$$
Q_L^{(1)} := p_{\mathbb{Z}_{\leq -1} \times (\mathbb{Z} \setminus \{0\})} \bigoplus_{x \in \mathbb{Z}_{\leq -1}} q_x, \quad Q_R^{(1)} := p_{\mathbb{Z}_{\geq 0} \times (\mathbb{Z} \setminus \{0\})} \bigoplus_{x \in \mathbb{Z}_{\geq 1}} q_x,
$$

(4.71)

which are projections on $L^2 (\mathbb{Z}_{\leq -1} \times (\mathbb{Z} \setminus \{0\})) \oplus L^2 (\mathbb{Z}_{\leq -2} \times \{0\}) \subset h_{H_L}$ and $L^2 (\mathbb{Z}_{\geq 0} \times (\mathbb{Z} \setminus \{0\})) \oplus L^2 (\mathbb{Z}_{\geq 1} \times \{0\}) \subset h_{H_R}$ respectively. Noting that $q_0$ is a projection on a finite dimensional Hilbert space $K_{(0,0)}$, we see that there are homogeneous states $\varphi_L, \varphi_R$ on $\mathcal{A}_{\Lambda L}, \mathcal{A}_{\Lambda R}$ such that $\omega^{(1)}$ is quasi-equivalent to $\varphi_L \otimes \varphi_R$. Hence $\omega^{(1)}$ satisfies the split property for $H_L - H_R$-cut. Now we would like to show (ii) of Lemma 4.4 occurs. With a unitary $\vartheta := \Pi_{h_{H_L}} \ominus \Pi_{h_{H_R}}$, we note that $\omega^{(1)} \circ \vartheta_R$ is a Fock state on $\mathcal{A}$ given by the basis projection $\vartheta q \vartheta^*$. (Note that $\vartheta q \vartheta^*$ is a basis projection because $\vartheta$ and $\mathcal{C}$ commute.) From the definition, this projection $\vartheta q \vartheta^*$ can be decomposed as

$$
\vartheta q \vartheta^* = Q_L^{(1)} \oplus Q_R^{(1)} \oplus (\Pi_{K_{(0,0)}} - q_0).
$$

(4.72)
Comparing this and (4.70),
\[ q \land (\mathbb{I}_q - \mathcal{Q} q \mathcal{Q}^*) = q_0. \quad (4.73) \]
Noting \( q_0 \) is a one-rank projection, from Theorem 6.30 of [EK], \( \omega(1)|_{\mathcal{A}(0)} \) and \( \omega(1)|_{\mathcal{A}(0)} \Theta R \) are not equivalent. Hence (ii) of Lemma 4.4 holds for \( \omega(1) \).
Combining this with Proposition 4.5, we conclude that any homogeneous pure state satisfying the split property with respect to the \( H_L - H_R \)-cut can be obtained out of either \( \omega(0) \) or \( \omega(1) \) via an automorphism of the form \( \eta_L \hat{\otimes} \eta_R \).

5. The \( \mathcal{PD}_0(G) \)-Valued Index

Now we derive the \( \mathcal{PD}_0(G) \)-valued index out of SPT-phases.

5.1. A brief overview. A brief description of the derivation is as follows. From Theorem 3.1, the ground states in SPT-phases are of the form \( \omega = \omega(0) \circ \alpha \) with \( \alpha \in \text{QAut}(\mathcal{A}) \). Using the factorization property (3.9), we can show that the difference between \( \omega(0) \alpha \circ \beta_g U \alpha^{-1} \) and \( \omega(0) \) is localized around \( x \)-axis, and in particular, \( \omega(0) \alpha \circ \beta_g U \alpha^{-1} \) satisfies the split property. Recall that then there is a dichotomy given by Lemma 4.4 for this \( \omega(0) \alpha \circ \beta_g U \alpha^{-1} \). Let us set \( a_\omega(g) = 0 \) if (i) of Lemma 4.4 occurs and \( a_\omega(g) = 1 \) if (ii) of Lemma 4.4 occurs. Applying the result Proposition 4.5 from the previous section, it allows us to show \( \omega \circ \beta_g U \overset{\epsilon}{=} \frac{\omega \otimes \tau^{a_\omega(g)}}{\eta_{g,L} \hat{\otimes} \eta_{g,R}^\epsilon} \) for \( \epsilon = \pm, \) with \( \eta_{g,\sigma} \) automorphism localized in \( C_\sigma \cap H_\sigma, \sigma = L, R. \) (Proposition 5.2, Remark 5.4.)

Next we decompose \( \alpha \) as (3.8) where \( \alpha_L, \alpha_R \) are automorphisms localized to the left and right infinite planes \( H_L, H_R \) while \( \Upsilon \) is localized in \( (C_\sigma)^c \). We then have \( \omega \overset{\epsilon}{=} (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \) with pure states \( \omega_L, \omega_R \) on the left and right infinite planes. Hence we obtain \( (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \overset{\epsilon}{=} (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \otimes \tau^{a_\omega(g)} \left( \eta_{g,L} \hat{\otimes} \eta_{g,R}^\epsilon \right) \).

Setting \( \gamma^g = \beta_g \left( \eta_{g,L} \hat{\otimes} \eta_{g,R}^\epsilon \right)^{-1} \otimes \tau^{-a_\omega(g)} \) for \( \epsilon = \pm, g \in G \), we have \( (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \gamma^g \overset{\epsilon}{=} (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \). Repeated use of this gives us
\[
(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \gamma^g \overset{(-1)^{a_\omega(g)}}{=} \gamma^h \gamma^g \overset{(-1)^{a_\omega(g)}}{=} (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon.
\]

But one can see that
\[
\gamma^g \gamma^h \gamma^{(-1)^{a_\omega(g)}} \gamma^{h^{-1}} \overset{\epsilon}{=} \bigotimes_{\sigma = L,R} \zeta_{g,h,\sigma}^\epsilon
\]
with \( \zeta_{g,h,\sigma}^\epsilon \in \text{Aut}(\mathcal{A}_{\sigma \sigma \cap H_\sigma}) \) (5.34), and it commutes with \( \Upsilon \). Hence we obtain \( \omega_L \alpha_L \zeta_{g,h,\sigma}^\epsilon \hat{\otimes} \omega_R \alpha_R \zeta_{g,h,\sigma}^\epsilon \overset{\epsilon}{=} \omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R \), which implies \( \omega_\sigma \alpha_\sigma \zeta_{g,h,\sigma}^\epsilon \overset{\epsilon}{=} \omega_\sigma \alpha_\sigma \). This equivalence gives us a unitary \( u^g (g, h) \) in Lemma 5.6, implementing \( \zeta_{g,h,\sigma}^\epsilon \) in the GNS-representation of \( \omega_\sigma \alpha_\sigma \). Note also that \( (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \gamma^g \overset{\epsilon}{=} (\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R) \circ \Upsilon \) means there is a unitary \( W^g \) in the GNS-representation of \( \omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R \) implementing \( \Upsilon \gamma^g \Upsilon^{-1} \). The unitaries \( W^g \) and \( u^h \) are homogeneous with respect to the grading and we denote their grade by \( b^g, k^g \). Using the associativity of automorphisms, it turns out that \( W^g \) and \( u^h \) satisfy some non-trivial relation (5.46) Lemma 5.10, with some phase factor \( c^g (g, h, k) \in \mathbb{U}(1) \). Considering the grading of this equation,
we obtain some relation between \( b^x, \kappa^x(g, h) \). Juggling the relation (5.46), we obtain relation between \( \kappa^x(g, h) \) and \( \epsilon^x(g, h, k) \in U(1) \). This gives an element of \( \mathcal{PD}_0(G) \).

The difference of the choice of objects we introduced to define the index can be implemented by some unitary. Writing such relation down brings us to the equivalence relation we introduced to define \( \mathcal{PD}_0(G) \).

As we saw in Lemma 3.2, our translation \( \tau \) shifts supports of automorphism which is sometimes inconvenient. However, Proposition 7.1 proved in Sect. 7 allow us to modify them suitably. We occasionally use this Proposition 7.1 in this section.

5.2. Effective excitation caused by \( \beta^U_g \). In this subsection, we investigate the effective excitation caused by \( \beta^U_g \) on our SPT-ground state \( \omega \in \text{SPT} \). We denote by \( (\mathcal{H}_\sigma, \pi_\sigma, \Omega_\sigma) \) a GNS triple of \( \omega_\sigma \) for \( \sigma = L, R \). Because \( \omega_\sigma \) is homogeneous, there is a self-adjoint unitary \( \Gamma_\sigma \) on \( \mathcal{H}_\sigma \) such that \( \Gamma_\sigma \pi_\sigma(A) \Omega_\sigma = \pi_\sigma (\Theta_\sigma(A)) \Omega_\sigma \) for all \( A \in \mathcal{A}_{H_\sigma} \). From (1.13), \( (\mathcal{H}_L \otimes \mathcal{H}_R, \pi_L \otimes \pi_R, \Omega_L \otimes \Omega_R) \) is a GNS representation of \( \omega^{(0)} \).

**Lemma 5.1.** For \( \omega \in \text{SPT} \), \( \alpha \in \text{EAut}(\omega) \), \( 0 < \theta < \theta' < \frac{\pi}{2} \), \( (\omega_U, \alpha_D, \Xi_L, \Xi_R) \in \mathcal{D}^H(\alpha, \theta) \) and \( g \in G \), there are homogeneous states \( \varphi_L \varphi_R \) on \( \mathcal{A}_{H_L \cap H_U \cap C_\theta}, \mathcal{A}_{H_R \cap H_U \cap C_{\theta'}} \) such that

\[
\omega_{pH_U} \alpha_U \beta^U_g \omega_{pU}^{-1} \sim_{q.e.} \varphi_L \hat{\otimes} \varphi_R \omega_{pH_U \cap C_{\theta'}},
\]

\[
\omega_{pH_U} \alpha_U \beta^U_g \omega_{pU}^{-1} \big|_{\mathcal{A}_{H_U \cap C_{\theta'}}} \sim_{q.e.} \varphi_L \hat{\otimes} \varphi_R,
\]

\[
\omega_{pH_U} \alpha_U \beta^U_g \omega_{pU}^{-1} \big|_{\mathcal{A}_{H_U \cap C_{\theta'}}} \sim_{q.e.} \omega_{pH_U \cap C_{\theta'}}.
\]

**Proof.** Set \( c_U := 1 \) and \( c_D := -1 \). Because \( \alpha_\zeta \in \text{QAut}(\mathcal{A}_{H_\zeta}) \) for \( \zeta = U, D \) we can decompose \( \alpha_\zeta^{-1} \) as

\[
\alpha_\zeta^{-1} = (\alpha_{C_\theta, L, \zeta} \hat{\otimes} \alpha_{C_\theta, R, \zeta}) \circ \alpha_{C_\theta, \zeta} \circ (\alpha_{C_\theta, L, \zeta} \hat{\otimes} \alpha_{C_\theta, R, \zeta}) \circ \text{(inner)},
\]

where

\[
\alpha_{C_\theta, \sigma, \zeta} \in \text{QAut}(\mathcal{A}_{C_\theta \cap H_\sigma \cap H_\zeta}), \quad \alpha_{C_\theta, \zeta} \in \text{QAut}(\mathcal{A}_{C_\theta \cap H_\zeta}), \quad \alpha_{C_\theta, \sigma, \zeta}
\]

\[
\in \text{QAut}(\mathcal{A}_{C_\theta \cap H_\sigma \cap H_\zeta}),
\]

for \( \sigma = L, R, \zeta = U, D \). Using the decomposition (5.4) and the support of the automorphisms there in (5.5), we rewrite \( \alpha \beta_g \alpha^{-1} \) as

\[
\alpha \beta_g \alpha^{-1} = (\alpha_U \hat{\otimes} \alpha_D) \circ \beta_g (\alpha_U \hat{\otimes} \alpha_D)^{-1} (Y_L \hat{\otimes} Y_R) \circ \text{(inner)}
\]

with

\[
Y_{\sigma} := (\alpha_{C_\theta, \sigma, U}^{-1} \alpha_{C_\theta, \sigma, U}^{-1} \alpha_{C_\theta, \sigma, U}^{-1}) \hat{\otimes} (\alpha_{C_\theta, \sigma, D}^{-1} \alpha_{C_\theta, \sigma, D}^{-1} \alpha_{C_\theta, \sigma, D}^{-1})(\beta_{H_\sigma})^{-1} \Xi_{\sigma} \beta_{H_\sigma} \Xi_{\sigma}^{-1} (\alpha_{C_\theta, \sigma, U} \alpha_{C_\theta, \sigma, U})
\]

\[
\hat{\otimes} (\alpha_{C_\theta, \sigma, D} \alpha_{C_\theta, \sigma, D}) \in \text{QAut}(\mathcal{A}_{C_\theta \cap H_\sigma}), \quad \sigma = L, R.
\]
Substituting this, from the $\beta_g$-invariance of $\omega$, we have

$$\omega^{(0)} = \omega^{(0)} \alpha \beta \alpha^{-1} = \omega^{(0)} (\alpha_U \hat{\otimes} \alpha_D) \circ \beta_g (\alpha_U \hat{\otimes} \alpha_D)^{-1} (Y_L \hat{\otimes} Y_R) \circ (\text{inner}).$$

(5.8)

From this, we have

$$\left(\omega_{PH_U} \alpha_U \beta_g \alpha_U^{-1}\right) \hat{\otimes} \left(\omega_{PH_D} \alpha_D \beta_g \alpha_D^{-1}\right) \cong \left(\omega_{PH_U \cap C_0} Y_L^{-1}\right) \hat{\otimes} \left(\omega_{PH_R \cap C_0} Y_R^{-1}\right) \hat{\otimes} \omega_{PCA}.$$

(5.9)

Here, $Y'_\sigma \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_0 \cap C_0'}\right)$ are automorphisms such that

$$Y_L^{-1} \hat{\otimes} Y_R^{-1} = \left(\omega_{L \hat{\otimes} R}^{-1}\right) \hat{\otimes} \left(\omega_{L \hat{\otimes} R}^{-1}\right) \hat{\otimes} \left(\alpha_U \hat{\otimes} \alpha_D\right)^{-1} \hat{\otimes} \left(\alpha_U \hat{\otimes} \alpha_D\right)^{-1} \hat{\otimes} \omega_{PCA}.$$

(5.10)

Note that $\varphi_L := \omega_{PH_U \cap C_0} \mid \mathcal{A}_{H_U \cap H_L \cap C_0'}$, $\varphi_R := \omega_{PH_R \cap C_0} \mid \mathcal{A}_{H_U \cap H_R \cap C_0'}$ are homogeneous states on $\mathcal{A}_{H_U \cap H_L \cap C_0'}$, $\mathcal{A}_{H_U \cap H_R \cap C_0'}$ respectively.

Note that both sides of (5.9) are pure and homogeneous state on $\mathcal{A}$. Restricting (5.9) to $\mathcal{A}_{H_U}$, from Lemma 4.1 we obtain (5.1). If we restrict (5.9) to $\mathcal{A}_{H_U \cap C_0}$, then we have (5.2) from Lemma 4.1. If we restrict (5.9) to $\mathcal{A}_{H_U \cap C_0'}$, then we have (5.3) from Lemma 4.1.

The following Proposition gives the effective excitation caused by $\beta_g^U$ on $\omega \in \text{SPT}$.

**Proposition 5.2.** Let $\omega \in \text{SPT}$. Then there is a group homomorphism $a_\omega : G \to \{0, 1\} = \mathbb{Z}_2$ satisfying the following.

(i) For any $0 < \theta < \frac{\pi}{2}$, and $g \in G$ there are $\eta_{g,L} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_U \cap C_0'}\right)$, $\eta_{g,R} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_R \cap C_0'}\right)$ such that

$$\omega \circ \beta_g^U \cong \omega \circ \tau^{a_\omega(g)} \left(\eta_{g,L} \hat{\otimes} \eta_{g,R}\right).$$

(5.11)

(ii) If there is $g \in G$, $0 < \theta < \frac{\pi}{2}$, $a' \in \{0, 1\}$ and $\eta'_{g,L} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_U \cap C_0'}\right)$, $\eta'_{g,R} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_R \cap C_0'}\right)$ such that

$$\omega \circ \beta_g^U \cong \omega \circ \tau^{a'} \left(\eta'_{g,L} \hat{\otimes} \eta'_{g,R}\right),$$

(5.12)

then $a' = a_\omega(g)$.

**Definition 5.3.** Proposition 5.2 defines a $\ast$-homomorphism $a_\omega : G \to \{0, 1\} = \mathbb{Z}_2$ for each $\omega \in \text{SPT}$, which we will use this symbol henceforth.

**Proof.** Existence: Fix any $0 < \theta < \frac{\pi}{2}$. We show that for each $g \in G$, there are $a_\omega(g) = 0, 1$, $\eta_{g,L} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_U \cap C_0'}\right)$, $\eta_{g,R} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_R \cap C_0'}\right)$ satisfying (5.11). Choose some $0 < \theta' < \theta'' < \theta$, $\alpha \in \text{EAut}(\omega)$, and $(\alpha_U, \alpha_D, \mathbb{Z}_L, \mathbb{Z}_R) \in \mathcal{D}(\alpha, \theta')$. Then by Lemma 5.1, there are homogeneous states $\varphi_L, \varphi_R$ on $\mathcal{A}_{H_U \cap H_L \cap C_0}$, $\mathcal{A}_{H_U \cap H_R \cap C_0''}$ such that

$$\omega_{PH_U} \alpha_U \beta_g^U \alpha_U^{-1} \sim_{\text{q.e.}} \varphi_L \hat{\otimes} \varphi_R \hat{\otimes} \omega_{PH_U \cap C_0''},$$

(5.13)
\[ \omega_{PH_U} \alpha U \beta_S U^{-1} \big|_{A_{HU \cap C_{\theta \eta}}} \sim_{q.e.} \varphi_L \hat{\otimes} \varphi_R, \]  
(5.14)

\[ \omega_{PH_U} \alpha U \beta_S U^{-1} \big|_{A_{HU \cap C_{\theta \eta}}} \sim_{q.e.} \omega_{PH_U \cap C_{\theta \eta}}. \]  
(5.15)

Note that \( h_{HU \cap C_{\theta \eta}}, h_{HU \cap C_{\theta \eta}}, \omega_{PH_U} \alpha U \beta_S U^{-1} \) satisfy the conditions of \( \mathfrak{S}_1, \mathfrak{S}_2, \omega \) in Lemma 4.4 respectively. Applying Lemma 4.4, either (i) or (ii) of Lemma 4.4 occurs. From the quasi-equivalence (5.13), the von Neumann algebras \( \pi \omega_{PH_U} \alpha U \beta_S U^{-1} (A_{HU \cap C_{\theta \eta}})'' \) and \( \pi \omega_{PH_U \cap C_{\theta \eta}} (A_{HU \cap C_{\theta \eta}})'' \) are \( \ast \)-isomorphic. As \( \pi \omega_{PH_U \cap C_{\theta \eta}} (A_{HU \cap C_{\theta \eta}})'' \) is a type I factor, (because \( \omega_{PH_U \cap C_{\theta \eta}} \) is a pure state), it means \( \pi \omega_{PH_U} \alpha U \beta_S U^{-1} (A_{HU \cap C_{\theta \eta}})'' \) is also a type I factor. It means from Lemma 4.4 that Lemma 4.4 (i) occurs. Hence for both of pure homogeneous states \( \omega_{PH_U} \alpha U \beta_S U^{-1} \) and \( \omega_{PH_U} \), (i) of Lemma 4.4 occurs, and we may apply Proposition 4.5. Applying Proposition 4.5, there are automorphisms \( S \in \text{Aut}^0 \left( A_{HU \cap C_{\theta \eta}} \right), T \in \text{Aut}^0 \left( A_{HU \cap C_{\theta \eta}} \right) \) such that

\[ \omega_{PH_U} \alpha U \beta_S U^{-1} \simeq \left( \omega_{PH_U \cap C_{\theta \eta}} S \right) \hat{\otimes} \left( \omega_{PH_U \cap C_{\theta \eta}} T \right). \]  
(5.16)

Note that both sides are pure homogeneous states. Therefore, applying Lemma 4.2, and (5.15), we obtain

\[ \omega_{PH_U \cap C_{\theta \eta}} \sim_{q.e.} \omega_{PH_U} \alpha U \beta_S U^{-1} \big|_{A_{HU \cap C_{\theta \eta}}} \sim_{q.e.} \omega_{PH_U \cap C_{\theta \eta}} T. \]  
(5.17)

Because both of \( \omega_{PH_U \cap C_{\theta \eta}} \) and \( \omega_{PH_U \cap C_{\theta \eta}} T \) are pure, we conclude that they are equivalent. Substituting this to (5.16), we obtain

\[ \omega_{PH_U} \alpha U \beta_S U^{-1} \simeq \left( \omega_{PH_U \cap C_{\theta \eta}} S \right) \hat{\otimes} \left( \omega_{PH_U \cap C_{\theta \eta}} T \right) \simeq \left( \omega_{PH_U \cap C_{\theta \eta}} S \right) \hat{\otimes} \left( \omega_{PH_U \cap C_{\theta \eta}} T \right). \]  
(5.18)

Because both sides of this equation are homogeneous and pure, applying Lemma 4.1, and combining it with (5.14) we obtain

\[ \varphi_L \hat{\otimes} \varphi_R \sim_{q.e.} \omega_{PH_U} \alpha U \beta_S U^{-1} \big|_{A_{HU \cap C_{\theta \eta}}} \sim_{q.e.} \omega_{PH_U \cap C_{\theta \eta}} S. \]  
(5.19)

Hence \( h_{HU \cap H_L \cap C_{\theta \eta}}, h_{HU \cap H_R \cap C_{\theta \eta}}, \omega_{PH_U \cap C_{\theta \eta}}, S \) satisfies the condition of \( \mathfrak{S}_1, \mathfrak{S}_2, \omega \) in Lemma 4.4 holds and (i) or (ii) of Lemma 4.4 holds for \( \omega_{PH_U \cap C_{\theta \eta}}, S \).

If \( \omega_{PH_U \cap C_{\theta \eta}}, S \) satisfies (i) of Lemma 4.4, we set \( a_\omega(g) := 0 \), and if \( \omega_{PH_U \cap C_{\theta \eta}}, S \) satisfies (ii) of Lemma 4.4, we set \( a_\omega(g) := 1 \).

If \( a_\omega(g) = 0 \), because \( \omega_{PH_U \cap C_{\theta \eta}} \) also satisfies (i) of Lemma 4.4, from Proposition 4.5, there are \( \tilde{\eta}_g, L \in \text{Aut}^0 \left( A_{HU \cap H^L \cap C_{\theta \eta}} \right), \tilde{\eta}_g, R \in \text{Aut}^0 \left( A_{HU \cap H^R \cap C_{\theta \eta}} \right) \) such that

\[ \omega_{PH_U \cap C_{\theta \eta}} S \simeq \omega_{PH_U \cap C_{\theta \eta}} \circ \left( \tilde{\eta}_g, L \hat{\otimes} \tilde{\eta}_g, R \right). \]  
(5.20)
Here, we used Proposition 7.1 in Sect. 7 to take $\tilde{\eta}_{g,\sigma}$ with support in $H_U \cap H_{\sigma}^{C_\sigma} \cap C_{\theta''}$, not just $H_U \cap H_{\sigma} \cap C_{\theta''}$.

If $a_\omega(g) = 1$, because $\omega_{pH_U \cap C_{\theta''}} \tau$ also satisfies (ii) of Lemma 4.4 from the same argument as the last paragraph of Sect. 4, there are $\tilde{\eta}_{g,L} \in \mathrm{Aut}^{(0)} \left( A_{H_U \cap H_{\sigma}^{C_\sigma}} \right)$, $\tilde{\eta}_{g,R} \in \mathrm{Aut}^{(0)} \left( A_{H_U \cap H_{\sigma}^{C_\sigma}} \right)$ such that

$$\omega_{pH_U \cap C_{\theta''}} S \simeq \omega_{pH_U \cap C_{\theta''}} \circ \tau \circ \left( \tilde{\eta}_{g,L} \hat{\otimes} \tilde{\eta}_{g,R} \right). \quad (5.21)$$

(Again we used Proposition 4.5 and Proposition 7.1.) Combining these with (5.18), we obtain

$$\omega_{pH_U} \alpha_U^g \beta_U^g \alpha_U^{-1} \simeq \omega_{pH_U \cap C_{\theta''}} \circ \tau^{a_\omega}(g) \circ \left( \tilde{\eta}_{g,L} \hat{\otimes} \tilde{\eta}_{g,R} \right) \hat{\otimes} \left( \omega_{pH_U \cap C_{\theta''}} \right).$$

(5.22)

As in the proof of Lemma 5.1 (5.6), there are $Z_{\sigma} \in \mathrm{Aut}^{(0)} \left( A_{H_{\sigma} \cap C_{\theta''}} \right)$, $\sigma = L, R$ such that

$$\alpha \beta_g^U \alpha^{-1} = \alpha_U \beta_U^g \alpha_U^{-1} \circ \left( Z_L \hat{\otimes} Z_R \right) \circ \text{(inner)}. \quad (5.23)$$

From this and (5.22), we have

$$\omega^{(0)} \alpha \beta_g^U \alpha^{-1} \simeq \left( \omega_{pH_U} \right) \circ \left( Z_L \hat{\otimes} Z_R \right) = \omega^{(0)} \tau^{a_\omega}(g) \left( \tilde{\eta}_{g,L} \hat{\otimes} \tilde{\eta}_{g,R} \right). \quad (5.24)$$

Note from Lemma 3.2 that there are automorphisms $\eta_{g,L} \in \mathrm{Aut}^{(0)} \left( A_{C_0 \cap H_L} \right)$, $\eta_{g,R} \in \mathrm{Aut}^{(0)} \left( A_{C_0 \cap H_R} \right)$ such that

$$\alpha^{-1} \tau^{a_\omega}(g) \left( \tilde{\eta}_{g,L} \hat{\otimes} \tilde{\eta}_{g,R} \right) \alpha = \tau^{a_\omega}(g) \left( \eta_{g,L} \hat{\otimes} \eta_{g,R} \right) \circ \text{(inner)}. \quad (5.25)$$

From Lemma 7.1, we may assume that $\eta_{g,L} \in \mathrm{Aut}^{(0)} \left( A_{H_{\sigma}^{C_\sigma}} \right)$, $\eta_{g,R} \in \mathrm{Aut}^{(0)} \left( A_{H_{\sigma}^{C_\sigma}} \right)$. Substituting this to (5.24), we obtain

$$\omega \beta_g^U = \omega^{(0)} \alpha \beta_g^U \simeq \omega^{(0)} \alpha \tau^{a_\omega}(g) \left( \eta_{g,L} \hat{\otimes} \eta_{g,R} \right) = \omega \tau^{a_\omega}(g) \left( \eta_{g,L} \hat{\otimes} \eta_{g,R} \right). \quad (5.26)$$

Uniqueness: Suppose for $i = 1, 2$, there are $0 < \theta_i < \frac{\pi}{2}$, $a_1(g), a_2(g) = 0, 1$, $\eta^{(i)}_{g,L} \in \mathrm{Aut}^{(0)} \left( A_{H_{\sigma}^{C_\sigma}} \right)$, $\eta^{(i)}_{g,R} \in \mathrm{Aut}^{(0)} \left( A_{H_{\sigma}^{C_\sigma}} \right)$ satisfying

$$\omega_{\beta_g^U} \simeq \omega \tau^{a_1} \left( \eta^{(1)}_{g,L} \hat{\otimes} \eta^{(1)}_{g,R} \right) \simeq \omega \tau^{a_2} \left( \eta^{(2)}_{g,L} \hat{\otimes} \eta^{(2)}_{g,R} \right). \quad (5.27)$$

We set $\theta := \max\{\theta_1, \theta_2\}$. Let $\alpha \in \mathrm{EAut}(\omega)$ and $0 < \theta' < \frac{\pi}{2}$. Then from (5.27), Lemma 3.2 there are some automorphisms $\xi_{\sigma} \in \mathrm{Aut}^{(0)} \left( A_{C_\sigma \cap H_\sigma} \right)$ such that

$$\omega^{(0)} \simeq \omega^{(0)} \circ \alpha \tau^{a_1} \left( \eta^{(1)}_{g,L} \hat{\otimes} \eta^{(1)}_{g,R} \right) \simeq \omega^{(0)} \tau^{-a_2} \alpha^{-1} \simeq \omega^{(0)} \tau^{a_1-a_2} \left( \xi_L \hat{\otimes} \xi_R \right). \quad (5.28)$$
By Lemma 4.4, if $a_1 - a_2 = -1$, 1, then $\omega^{(0)}(\tau^{a_1 - a_2}) (\xi_L \otimes \xi_R)$ satisfies Lemma 4.4 (ii) and from Proposition 4.5 it cannot be equivalent to $\omega^{(0)}$. Therefore, from (5.28), $a_1 = a_2$.

Group homomorphism:
Let $g, h \in G$ and $\eta_{g, \sigma}, \eta_{h, \sigma} \in \text{Aut}^{(0)}(A_{C_{\emptyset} \cap H_0})$ satisfying (5.11) for $g, h$. Then we have

$$\omega_{gh}^{\mu} = \omega_{g}^{\mu} \rho_{h}^{\mu} \approx \omega_{\tau^{a_\omega(g)}}(\eta_{g}, L \hat{\otimes} \eta_{g}, R) \rho_{h}^{\mu}$$

$$= \omega_{g}^{\mu} \left( \rho_{h}^{U} \right)^{-1} \tau^{a_\omega(h)}(\eta_{h}, L \hat{\otimes} \eta_{h}, R) \rho_{h}^{U}$$

$$\approx \omega_{\tau^{a_\omega(h) + a_\omega(g)}}(\eta_{gh}, L \hat{\otimes} \eta_{gh}, R),$$

(5.29)

with some $\eta_{gh, \sigma} \in \text{Aut}^{(0)}(A_{H^{c_{\sigma}}_{g} \cap C_{\emptyset}})$, using Proposition 7.1. If $a_\omega(h) + a_\omega(g) = 0, 1$, then from the uniqueness, we have $a_\omega(g h) = a_\omega(g) + a_\omega(h)$. If $a_\omega(h) + a_\omega(g) = 2$, then combining with Lemma 3.4, we have $\omega_{g}^{\mu} \rho_{h}^{\mu} \approx \omega_{(\eta_{gh}, L \hat{\otimes} \eta_{gh}, R)}$, with some $\eta_{gh, \sigma}$ in $\text{Aut}^{(0)}(A_{H^{c_{\sigma}}_{g} \cap C_{\emptyset}})$ and we get $a_\omega(g h) = 0$. Hence identifying $\{0, 1\} = \mathbb{Z}_2$, we see that $a_\omega : G \rightarrow \{0, 1\} = \mathbb{Z}_2$ is a group homomorphism.

**Remark 5.4.** If $a_\omega(g) = 1$, then from Lemma 3.4, for any $0 < \theta < \frac{\pi}{2}$ we have

$$\omega_{\hat{\mu}}^{\mu} \approx \omega_{\hat{\tau}^{-1}}^{} \left( \eta_{g, L}^{(-1)} \hat{\otimes} \eta_{g, R}^{(-1)} \right),$$

(5.30)

for some $\eta_{g, \sigma}^{(-1)} \in \text{Aut}^{(0)}(A_{H^{c_{\sigma}}_{g} \cap C_{\emptyset}})$.

Hence for each $\omega \in \text{SPT}$, the following set is not empty.

**Definition 5.5.** For each $\omega \in \text{SPT}$ and $0 < \theta < \frac{\pi}{2}$, we set

$$I(\omega, \theta) := \left\{ (\eta_{g, \sigma}^{e}) \in G, \sigma = L, R, e = \pm \in \eta_{g, L}^{\sigma} \in \text{Aut}^{(0)}(A_{H^{c_{\sigma}}_{\emptyset} \cap C_{\emptyset}}), \eta_{g, R}^{\sigma} \in \text{Aut}^{(0)}(A_{H^{c_{\sigma}}_{R} \cap C_{\emptyset}}) \right\}.$$  \hspace{1cm} (5.31)

For $\left( \eta_{g, \sigma}^{e} \right) \in I(\omega, \theta)$, we set $\eta_{g, \sigma}^{e} \approx \tau^{a_\omega(g)}(\eta_{g, L}^{e} \hat{\otimes} \eta_{g, R}^{e})$.

5.3. Derivation of an element in $\mathcal{PD}(G)$. Using the elements in $I(\omega, \theta)$, we derive some element in $\tilde{\mathcal{PD}}(G)$ out of SPT-ground states. We set

$$c_L^{(1)} := -4 = c_L + 1, \quad c_R^{(1)} := 4 = c_R - 1, \quad c_L^{(2)} := -3 = c_L + 2, \quad c_R^{(2)} := 3 = c_R - 2. \hspace{1cm} (5.32)$$

The conjugation by $\tau$ or $\tau^{-1}$ maps an automorphism supported at $C_{\emptyset} \cap H^{c_{\sigma}}_{0}$ to an automorphism supported at $C_{\emptyset} \cap H^{c_{\sigma}}_{1}$. The conjugation by $\tau^2$ or $\tau^{-2}$ maps an automorphism supported at $C_{\emptyset} \cap H^{c_{\sigma}}_{0}$ to an automorphism supported at $C_{\emptyset} \cap H^{c_{\sigma}}_{2}$. (Lemma 3.2). Recall $\omega_L, \omega_R$ in Introduction. We denote by $(\mathcal{H}_\sigma, \pi_\sigma, \Omega_\sigma)$ GNS representation of $\omega_\sigma$. Because $\omega_\sigma$ is homogeneous, there exists a self-adjoint unitary $\Gamma_\sigma$ on $\mathcal{H}_\sigma$ with $\text{Ad}(\Gamma_\sigma) \pi_\sigma = \pi_\sigma \circ \Theta_\sigma$. 
Lemma 5.6. For any \( \omega \in \text{SPT} \), \( \alpha \in \text{EAut}(\omega) \), \( 0 < \theta < \frac{\pi}{2} \), \( (\eta_{g,\alpha}^\epsilon) \in I(\omega, \theta) \), \( (\alpha_L, \alpha_R, \Upsilon) \in \mathcal{D}^V(\alpha, \theta) \), there are unitaries \( u_{\sigma}^\epsilon(g, h) \in \mathcal{U}(\mathcal{H}_{\sigma}) \), \( g, h \in G, \sigma = L, R, \epsilon = \pm 1 \) such that

\[
\text{Ad} \left( u_{\sigma}^\epsilon(g, h) \right) \pi_{\sigma}^\epsilon = \pi_{\sigma}^\epsilon \alpha_{\sigma}^{U^\sigma} \left( \eta_{g,\alpha}^\epsilon \right)^{-1} \beta_{\sigma}^{U^\sigma} \left( \eta_{h,\sigma}^{(-1)^{a_\omega(g)} \epsilon} \right)^{-1} \eta_{g,\alpha \sigma}^\epsilon \left( \beta_{\sigma}^{U^\sigma g} \right)^{-1}.
\]

(5.33)

Proof. Set

\[
\zeta_{g,\epsilon,\sigma}^\epsilon := \beta_{\sigma}^{U^\sigma} \left( \eta_{g,\alpha}^\epsilon \right)^{-1} \beta_{\sigma}^{U^\sigma} \left( \eta_{h,\sigma}^{(-1)^{a_\omega(g)} \epsilon} \right)^{-1} \eta_{g,\epsilon,\sigma}^\epsilon \left( \beta_{\sigma}^{U^\sigma g} \right)^{-1}
\]

such that

\[
\zeta_{g,\epsilon,\sigma}^\epsilon \in \text{Aut}^{(0)} \left( \mathcal{A}_{C_0 \cap H^\sigma_{\theta}} \right).
\]

(5.34)

for \( g, h \in G, \sigma = L, R, \epsilon = \pm 1 \). Note from \( (\eta_{g,\alpha}^\epsilon) \in I(\omega, \theta) \) and

\[
a_\omega(g) \epsilon + a_\omega(h) (-1)^{a_\omega(g) \epsilon} = a_\omega(gh) \epsilon
\]

(5.35)

that

\[
\omega \circ \left( \zeta_{g,\epsilon,\sigma}^\epsilon \hat{\otimes} \zeta_{g,\epsilon,\sigma}^\epsilon \right) \simeq \omega.
\]

(5.36)

Substituting \( \omega \simeq \omega^{(0)} (\alpha_L \hat{\otimes} \alpha_R) \Upsilon \), from the commutativity of \( \Upsilon \) and \( \zeta_{g,\epsilon,\sigma}^\epsilon \), we obtain

\[
\omega \circ \left( \zeta_{g,\epsilon,\sigma}^\epsilon \hat{\otimes} \zeta_{g,\epsilon,\sigma}^\epsilon \right) \simeq \omega \circ \left( \zeta_{g,\epsilon,\sigma}^\epsilon \hat{\otimes} \zeta_{g,\epsilon,\sigma}^\epsilon \right).
\]

(5.37)

Because both sides are pure homogeneous states, Lemma 4.1 implies

\[
\omega \circ \left( \zeta_{g,\epsilon,\sigma}^\epsilon \hat{\otimes} \zeta_{g,\epsilon,\sigma}^\epsilon \right) \simeq \omega \circ \left( \zeta_{g,\epsilon,\sigma}^\epsilon \hat{\otimes} \zeta_{g,\epsilon,\sigma}^\epsilon \right).
\]

(5.38)

Hence we complete the proof. \( \square \)

Lemma 5.7. For any \( \omega \in \text{SPT} \), \( \alpha \in \text{EAut}(\omega) \), \( 0 < \theta < \frac{\pi}{2} \), \( (\eta_{g,\alpha}^\epsilon) \in I(\omega, \theta) \), \( (\alpha_L, \alpha_R, \Upsilon) \in \mathcal{D}^V(\alpha, \theta) \), there are unitaries \( W^\epsilon_g \in \mathcal{U}(\mathcal{H}_L \otimes \mathcal{H}_R) \) such that

\[
\text{Ad} \left( W^\epsilon_g \right) \circ \left( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \right) = \left( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \right) \circ \Upsilon \beta_{g}^{U^\sigma} \left( \eta_{g,\sigma}^\epsilon \right)^{-1} \tau^{-a_\omega(g) \epsilon} \Upsilon^{-1}.
\]

(5.39)

Proof. This is immediate from \( \omega \simeq \omega^{(0)} (\alpha_L \hat{\otimes} \alpha_R) \Upsilon \), and \( (\eta_{g,\alpha}^\epsilon) \in I(\omega, \theta) \). \( \square \)

Definition 5.8. For any \( \omega \in \text{SPT} \), \( \alpha \in \text{EAut}(\omega) \), \( 0 < \theta < \frac{\pi}{2} \), \( (\eta_{g,\alpha}^\epsilon) \in I(\omega, \theta) \), \( (\alpha_L, \alpha_R, \Upsilon) \in \mathcal{D}^V(\alpha, \theta) \), we denote by \( \text{IP} \left( \omega, \alpha, \theta, (\eta_{g,\alpha}^\epsilon), (\alpha_L, \alpha_R, \Upsilon) \right) \) the set of

\[
\left( W^\epsilon_g, (u^\epsilon_{\sigma}(g, h)) \right)
\]

satisfying (5.39) and (5.33). From Lemma C.1, there are \( b^\epsilon_g, \kappa^\epsilon_{\sigma}(g, h) \in \{0, 1\} \) such that

\[
\text{Ad} \left( \Gamma_L \otimes \Gamma_R \right) \left( W^\epsilon_g \right) = (-1)^{b^\epsilon_g} W^\epsilon_g, \quad \text{Ad} \Gamma_{\sigma} \left( u^\epsilon_{\sigma}(g, h) \right) = (-1)^{\kappa^\epsilon_{\sigma}(g, h)} u^\epsilon_{\sigma}(g, h).
\]

(5.40)
Because $\pi_L, \pi_R$ are irreducible, $b_\epsilon, \kappa_\sigma^\epsilon (g, h) \in \{0, 1\}$ are independent of the choice of $\left( (W_\theta^\epsilon), (u_\sigma^\epsilon (g, h)) \right)$. The independence of them from the choice of $\left( (W_\theta^\epsilon), (u_\sigma^\epsilon (g, h)) \right)$ is because the ambiguity of $\left( (W_\theta^\epsilon), (u_\sigma^\epsilon (g, h)) \right)$ are just $U(1)$-phases. Define $b \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2), \kappa_\sigma \in C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ by

$$b(g) := \left( b_g^{+1}, b_g^{-1} \right), \; \kappa_\sigma(g, h) := \left( \kappa^{-1}_\sigma (g, h), \kappa^{-1}_\sigma (g, h) \right) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2, \; g, h \in G. \tag{5.41}$$

We denote $b \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2), \kappa_\sigma \in C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ defined in (5.41) by

$$b \left( \omega, \alpha, \theta, (\eta_{g_\sigma}^\epsilon), (\alpha_L, \alpha_R, \Upsilon) \right), \kappa_\sigma \left( \omega, \alpha, \theta, (\eta_{g_\sigma}^\epsilon), (\alpha_L, \alpha_R, \Upsilon) \right), \tag{5.42}$$

respectively.

In the following theorem, we used notation from Definition 5.8.

**Lemma 5.9.** For any $\omega \in \text{SPT}, \alpha \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\eta_{g_\sigma}^\epsilon) \in I(\omega, \theta), (\alpha_L, \alpha_R, \Upsilon) \in \mathcal{D}^V(\alpha, \theta), \left( (W_\theta^\epsilon), (u_{\sigma}^\epsilon (g, h)) \right) \in \text{IP} \left( \omega, \alpha, \theta, (\eta_{g_\sigma}^\epsilon), (\alpha_L, \alpha_R, \Upsilon) \right)$ the following holds.

(i) $u_{\sigma}^\epsilon (g, h) \in \left( \pi_\alpha \alpha, A_{C_0 \cap H_\alpha^\epsilon (1) \cap H_\sigma}^c \right)'.

(ii) For any $x \in \mathcal{B} (\mathcal{H}_L \otimes \mathcal{H}_R)$,

$$\text{Ad} \left( W_\theta^\epsilon W_h^{(1)_{\text{SPT}}(g, h)} W_{\theta h}^\epsilon \right) (x) = \text{Ad} \left( u_{\sigma}^\epsilon (g, h) \otimes u_{\sigma}^\epsilon (g, h) \Gamma_{\alpha R}^{\kappa_\sigma^\epsilon (g, h)} \right) (x). \tag{5.43}$$

(iii)

$$\text{Ad} \left( W_\theta^\epsilon \right) \left( \mathfrak{C} \mathcal{H}_L \otimes \left( \pi_\alpha \alpha, A_{C_0 \cap H_\alpha^\epsilon (2) \cap H_\sigma}^c \right) \right)' \subset \mathfrak{C} \mathcal{H}_L \otimes \mathcal{B} (\mathcal{H}_R) \tag{5.44}$$

(iv)

$$\text{Ad} \left( W_\theta^\epsilon W_h^{(1)_{\text{SPT}}(g, h)} \right) \left( \mathfrak{H}_L \otimes u_{\alpha R}^\epsilon \right)(k, f) = \text{Ad} \left( \mathfrak{H}_L \otimes u_{\alpha R}^\epsilon (g, h) \Gamma_{\alpha R}^{\kappa_\sigma^\epsilon (g, h)} \right) W_{\theta h}^\epsilon (\mathfrak{H}_L \otimes u_{\alpha R}^\epsilon (k, f)) \tag{5.45}$$

**Proof.** The proof is the same as that of Lemma 2.3 of [O4], using Lemma C.1. We just need to care about the fact that $\tau$ can shift the support of automorphism according to Lemma 3.3. \qed
Lemma 5.10. Let \( \omega \in \text{SPT}, \alpha \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\eta^{\epsilon}_{g\sigma}) \in I(\omega, \theta), (\alpha_L, \alpha_R, \Upsilon) \in D^V(\alpha, \theta), \left( (W^\epsilon_g), (u^\epsilon_{g}(g, h)) \right) \in \text{IP} \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right). \) Then there is some \( c_{\sigma}^\epsilon(g, h, k) \in U(1) \) for each \( g, h, k \in G \) and \( \epsilon = \pm 1, \sigma = L, R \) such that

\[
W^\epsilon_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^{-1}_{R}(-1)^{a_\omega(g)}\epsilon (g, h) \right) W^\epsilon_g^* \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^\epsilon_R(g, hk) \right) = c_R^\epsilon(g, h, k) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^\epsilon_R(g, h)u^\epsilon_R(g'h, k) \right)
\]

\[
W^\epsilon_g \left( u^{-1}_{L}(-1)^{a_\omega(g)}\epsilon (h, k) \otimes \Gamma^{(-1)^{a_\omega(g)}\epsilon}_{R} \right) W^\epsilon_g^* \left( u^\epsilon_L(g, h) \otimes \Gamma^{\epsilon}_{R} \right)
\]

\[
= c_L^\epsilon(g, h, k)u^\epsilon_L(g, h)u^\epsilon_L(g'h, k) \otimes \Gamma^{\epsilon}_{R}.
\]

Definition 5.11. We define \( c \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon), (W^\epsilon_g), (u^\epsilon_{g}(g, h)) \right) \in C^3(\mathcal{G}, U(1) \oplus U(1)) \) by a map \( c : G^{\times 3} \rightarrow U(1) \oplus U(1) \) such that

\[
c(g, h, k) := \left( c_R^+\epsilon(g, h, k), c_R^{-1}\epsilon(g, h, k) \right) \in U(1) \oplus U(1), \quad g, h, k \in G
\]

with \( c_R^\epsilon(g, h, k) \) in Lemma 5.10.

Proof. The existence of \( c_{\sigma}^\epsilon(g, h, k) \in U(1) \) satisfying first equation of (5.46) follows from

\[
\text{Ad} \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^\epsilon_R(g, h)u^\epsilon_R(g'h, k) \right) \left( \pi_L\alpha_L \hat{\otimes} \pi_R\alpha_R \right) = \frac{U_R}{h_k} \left( \eta^{\epsilon}_{g'h, k,R} \right)^{-1} \tau^{-a_\omega(g'hk)}\epsilon
\]

\[
\left( \alpha_L \hat{\otimes} \pi_R\alpha_R \right) = \frac{U_R}{h_k} \left( \eta^{\epsilon}_{g'h, k,R} \right)^{-1} \tau^{-a_\omega(h)}\epsilon
\]

\[
= \text{Ad} \left( W^\epsilon_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^{-1}_{R}(-1)^{a_\omega(g)}\epsilon (h, k) \right) W^\epsilon_g^* \left( \mathbb{I}_{\mathcal{H}_L} \otimes u^\epsilon_R(g, h) \right) \right) \left( \pi_L\alpha_L \hat{\otimes} \pi_R\alpha_R \right) \frac{U_R}{h_k} \left( \eta^{\epsilon}_{g'h, k,R} \right)^{-1} \tau^{-a_\omega(g'hk)}\epsilon
\]

The second one can be obtained analogously.

Lemma 5.12. Let \( \omega \in \text{SPT}, \alpha \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\eta^{\epsilon}_{g\sigma}) \in I(\omega, \theta), (\alpha_L, \alpha_R, \Upsilon) \in D^V(\alpha, \theta), \left( (W^\epsilon_g), (u^\epsilon_{g}(g, h)) \right) \in \text{IP} \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right). \) Recall \( a_\omega \in H^1(G, \mathbb{Z}_2) \) and

\[
c \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon), (W^\epsilon_g), (u^\epsilon_{g}(g, h)) \right) =: c \in C^3 \left( \mathcal{G}, U(1) \oplus U(1) \right)
\]

\[
b \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right) =: b \in C^1 \left( G, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right),
\]

\[
\kappa_{\sigma} \left( \omega, \alpha, \theta, (\eta^{\epsilon}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right) =: \kappa_{\sigma} \in C^2 \left( G, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right), \quad \sigma = L, R
\]

(5.49)
defined in Definition 5.3 and Definition 5.11, Definition 5.8 respectively. Then we have

\[(c, \kappa_R, \kappa_L, b, a_\omega) \in \widehat{\mathcal{PD}}(G). \quad (5.50)\]

If \(\left( (W^{\xi}_g), (u^{\sigma}_\alpha (g, h)) \right)\) is another choice from \(\text{IP} \left( \omega, \alpha, \theta, (\eta^{\xi}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right)\)
and \(c \left( \omega, \alpha, \theta, (\eta^{\xi}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon), (W^{\xi}_g), (u^{\sigma}_\alpha (g, h)) \right) \):= \(c'\), then we have

\[(c, \kappa_R, \kappa_L, b, a_\omega) \sim_{\mathcal{PD}}(c', \kappa_R, \kappa_L, b, a_\omega). \quad (5.51)\]

**Definition 5.13.** From Lemma 5.12, for each \(\omega \in \text{SPT}, \alpha \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\eta^{\xi}_{g\sigma}) \in I(\omega, \theta), (\alpha_L, \alpha_R, \Upsilon) \in \mathcal{D}^V(\alpha, \theta)\), we may define

\[h^{(1)} \left( \omega, \alpha, \theta, (\eta^{\xi}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right) := \left( c, \kappa_R, \kappa_L, b, a_\omega \right) \in \mathcal{PD}(G). \quad (5.52)\]

with \((c, \kappa_R, \kappa_L, b, a_\omega)\) in Lemma 5.12, independent of the choice of \(\left( (W^{\xi}_g), (u^{\sigma}_\alpha (g, h)) \right) \in \text{IP} \left( \omega, \alpha, \theta, (\eta^{\xi}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right)\).

**Proof.** (ii) of Lemma 5.9 means that \(W^\xi_g, W^\xi_g = (\omega)^{\eta^{\xi}_{g\sigma}} W^\xi_g W^\xi_g \) and \(u^{\xi}_\alpha (g, h) \otimes u^{\xi}_R (g, h) \Gamma^{\xi}_R \) are proportional. Taking Ad \((\Gamma_L \otimes \Gamma_R)\) of them, we obtain (2.6). Taking Ad \((\Gamma_L \otimes \Gamma_R)\) of (5.46), we obtain (2.7) and (2.8). The condition (2.9) can be checked by the repeated use of (5.46) as in the proof of Lemma 2.4 in [O4]. Hence we get \((c, \kappa_R, \kappa_L, b, a_\omega) \in \widehat{\mathcal{PD}}(G)\).

To see the last claim (5.51), recall that the ambiguity of \(\left( (W^{\xi}_g), (u^{\sigma}_\alpha (g, h)) \right)\) are just U(1)-phases. Therefore, the difference of \(c\) and \(c'\) is of the form \(d^2 \sigma\), with some \(\sigma \in C^2(G, U(1) \oplus U(1))\). This proves (5.51).

\[\square\]

### 5.4. Well-defined-ness of the index

In this subsection, we show that the index we derived in the previous subsection does not depend on the choice of the objects we used.

**Lemma 5.14.** Let \(\omega \in \text{SPT}, \alpha^{(1)}, \alpha^{(2)} \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\eta^{\xi}_{g\sigma}) \in I(\omega, \theta), (\alpha^{(i)}_L, \alpha^{(i)}_R, \Upsilon^{(i)}) \in \mathcal{D}^V(\alpha^{(i)}, \theta), i = 1, 2.\) Then we have

\[h^{(1)} \left( \omega, \alpha^{(1)}, \theta, (\eta^{\xi}_{g\sigma}), (\alpha^{(1)}_L, \alpha^{(1)}_R, \Upsilon^{(1)}) \right) := h^{(1)} \left( \omega, \alpha^{(2)}, \theta, (\eta^{\xi}_{g\sigma}), (\alpha^{(2)}_L, \alpha^{(2)}_R, \Upsilon^{(2)}) \right) \in \mathcal{PD}(G). \quad (5.53)\]

**Definition 5.15.** From 5.14, for each \(\omega \in \text{SPT}, 0 < \theta < \frac{\pi}{2}, (\eta^{\xi}_{g\sigma}) \in I(\omega, \theta)\) we may define

\[h^{(2)} \left( \omega, \theta, (\eta^{\xi}_{g\sigma}) \right) := h^{(1)} \left( \omega, \alpha, \theta, (\eta^{\xi}_{g\sigma}), (\alpha_L, \alpha_R, \Upsilon) \right) \in \mathcal{PD}(G), \quad (5.54)\]

independent of the choice of \(\alpha, (\alpha_L, \alpha_R, \Upsilon)\).
Proof. Because
\[ \omega^{(0)} \left( \alpha^{(1)}_L \hat{\otimes} \alpha^{(1)}_R \right) \Gamma^{(1)} \simeq \omega \simeq \omega^{(0)} \left( \alpha^{(2)}_L \hat{\otimes} \alpha^{(2)}_R \right) \Gamma^{(2)}, \] (5.55)
there is a unitary \( V \in \mathcal{U}(\mathcal{H}_L \otimes \mathcal{H}_R) \) such that
\[ \text{Ad} V \circ (\pi_L \hat{\otimes} \pi_R) \circ \left( \alpha^{(1)}_L \hat{\otimes} \alpha^{(1)}_R \right) \Gamma^{(1)} = (\pi_L \hat{\otimes} \pi_R) \circ \left( \alpha^{(2)}_L \hat{\otimes} \alpha^{(2)}_R \right) \Gamma^{(2)}. \] (5.56)
As all the automorphisms in the equations are graded, from Lemma C.1, \( V \) is graded with respect to \( \Gamma_L \otimes \Gamma_R \). Let \( \left( (W^g), (u^e_{(g, h)}) \right) \in \text{IP} \left( \omega, (\alpha^{(1)}), (\eta_{g\sigma}^{(1)}), (\alpha^{(1)}_L, \alpha^{(1)}_R, \Gamma^{(1)}) \right) \). As in Lemma 2.11 of [O4], we have
\[ \text{Ad} \left( V W^g V^* \right) \circ \left( \pi_L \alpha^{(2)}_L \hat{\otimes} \pi_R \alpha^{(2)}_R \right) = \left( \pi_L \alpha^{(2)}_L \hat{\otimes} \pi_R \alpha^{(2)}_R \right) \Gamma^{(2)} \beta^U_{g} \eta^e_{g} \tau^{\epsilon_a(g)} \Gamma^{(2)} - 1. \] (5.57)
We also can check
\[ \text{Ad} \left( V (\mathbb{I}_{\mathcal{H}_L} \otimes u^e_R(g, h)) V^* \right) \circ \left( \alpha^{(2)}_L \hat{\otimes} \alpha^{(2)}_R \right) \] \[ = \pi_L \alpha^{(2)}_L \hat{\otimes} \pi_R \alpha^{(2)}_R \beta^U_{g} \left( \eta_{gR}^e \right)^{-1} \beta^U_{h} \left( \left( \eta_{hR}^{(1)\omega(g)} \right)^{-1} \eta_{gh,R}^e \beta^U_{g} \right)^{-1}. \] (5.58)
From Lemma C.1, this means that there is some \( u^{(2)e}_R(g, h) \in \mathcal{U}(\mathcal{H}_R) \) such that
\[ V (\mathbb{I}_{\mathcal{H}_L} \otimes u^{(2)e}_R(g, h)) V^* = \mathbb{I}_{\mathcal{H}_L} \otimes u^{(2)e}_R(g, h). \] Similarly, there is some \( u^{(2)e}_L(g, h) \in \mathcal{U}(\mathcal{H}_L) \) such that
\[ V (u^{(2)e}_L(g, h) \otimes \Gamma^{(2)}_R) V^* = u^{(2)e}_L(g, h) \otimes \Gamma^{(2)}_R. \] From (5.57), (5.58) and its analog for \( u^{e}_R(g, h), \) we can see that \( (V W^e V^*, (u^{(2)e}_\sigma)) \) belongs to \( \text{IP} \left( \omega, (\alpha^{(2)}), (\eta_{g\sigma}^{(2)}), (\alpha^{(2)}_L, \alpha^{(2)}_R, \Gamma^{(2)}) \right) \). As in Lemma 2.11 (2.90) of [O4], one can check that
\[ c \left( \omega, (\alpha^{(1)}), (\eta_{g\sigma}^{(1)}), (\alpha^{(1)}_L, \alpha^{(1)}_R, \Gamma^{(1)}), (W^e), (u^e(g, h)) \right) \] \[ = c \left( \omega, (\alpha^{(2)}), (\eta_{g\sigma}^{(2)}), (\alpha^{(2)}_L, \alpha^{(2)}_R, \Gamma^{(2)}), ((V W^e V^*), (u^{(2)e}_\sigma)) \right). \] (5.59)
It is also clear that the grading of \( V W^e V^*, u^{(2)e}_\sigma(g, h) \) are equal to that of \( W^e, u^{(2)e}_\sigma(g, h) \). Hence we obtain the claim. \( \square \)

**Lemma 5.16.** For any \( \omega \in \text{SPT} \), \( 0 < \theta < \frac{\pi}{2} \), \( (\eta_{g\sigma}^{(1)}), (\eta_{g\sigma}^{(2)})) \in I(\omega, \theta) \), we have
\[ h^{(2)}(\omega, \theta, (\eta_{g\sigma}^{(1)})) = h^{(2)}(\omega, \theta, (\eta_{g\sigma}^{(2)})). \] (5.60)

**Definition 5.17.** From Lemma 5.16, for each \( \omega \in \text{SPT} \), \( 0 < \theta < \frac{\pi}{2} \), we can define
\[ h^{(3)}(\omega, \theta) := h^{(2)}(\omega, \theta, (\eta^e_{g\sigma})) \in \mathcal{P\mathcal{D}}(G), \] (5.61)
independent of the choice of \( (\eta_{g\sigma}^{(2)}). \)
Proof. Fix some $\alpha \in \mathrm{EAut}(\omega)$ and $(\alpha_L, \alpha_R, \Upsilon) \in D^V(\alpha, \theta)$. Note that $K^{\epsilon, \sigma}_{g, \alpha} \in \mathrm{Aut}^0 \left( \mathcal{A}_C \cap \mathcal{H}^c_R \right)$. Set $K^{\epsilon}_{g, \alpha} = K^{\epsilon}_{g, L} \hat{\otimes} K^{\epsilon}_{g, R}$. Because $\omega \beta^U_g \simeq \omega \tau^{a_\omega(g)\epsilon} \eta^{\epsilon_1}_{g}$, we have $\omega \left( K^{\epsilon}_{g, \alpha} \right)_{a_\omega(g)\epsilon} \simeq \omega$. On the other hand, we have $\omega \simeq \omega^0(\alpha_L \hat{\otimes} \alpha_R)$ $\Upsilon$. Combining these, we obtain $\omega^0(\alpha_L \hat{\otimes} \alpha_R) \Upsilon \left( K^{\epsilon}_{g, \alpha} \right)_{a_\omega(g)\epsilon} \simeq \omega^0(\alpha_L \hat{\otimes} \alpha_R)$ $\Upsilon$. Because $K^{\epsilon}_{g, \alpha} \in \mathrm{Aut}^0 \left( \mathcal{A}_C \right)$ and $\Upsilon$ commute, we then obtain

$$\omega^0(\alpha_L \hat{\otimes} \alpha_R) \left( K^{\epsilon}_{g, \alpha} \right)_{a_\omega(g)\epsilon} \simeq \omega^0(\alpha_L \hat{\otimes} \alpha_R).$$

Because both sides of this equation are pure and homogeneous, from Lemma 4.1, we have $\omega \alpha \alpha \sigma \left( K^{\epsilon}_{g, \alpha} \right)_{a_\omega(g)\epsilon} \simeq \omega \alpha \alpha \sigma$. Hence there is a unitary $V^{\epsilon}_{g, \alpha}$ on $\mathcal{H}$ such that

$$\mathrm{Ad} \left( V^{\epsilon}_{g, \alpha} \right) \pi_{\sigma} = \pi_{\sigma} \alpha \sigma \left( K^{\epsilon}_{g, \alpha} \right)_{a_\omega(g)\epsilon} \alpha^{-1}. \tag{5.62}$$

It is homogeneous because of Lemma C.1 and we denote by $m^{\epsilon}_{g, \alpha}$ the grading of $V^{\epsilon}_{g, \alpha}$, i.e., $\mathrm{Ad} \Gamma_{\sigma} \left( V^{\epsilon}_{g, \alpha} \right) = (-1)^{m^{\epsilon}_{g, \alpha}} V^{\epsilon}_{g, \alpha}$. Fix some $\left( \left( W^{\epsilon}_{g, \alpha} \right), \left( u^{\epsilon}_{\sigma}(g, h) \right) \right)$ in $\mathrm{IP} \left( \omega, \alpha, \theta, \left( \eta^{\epsilon_1}_{g, \alpha} \right) \right)$, $(\alpha_L, \alpha_R, \Upsilon)$. We denote by $\kappa^{\epsilon}_{g, \alpha}(g, h)$ the grading of $u^{\epsilon}_{\sigma}(g, h)$. Because $K^{\epsilon}_{h, \alpha} \left( a(h) e' - a(g) \right)$ $\in \mathrm{Aut}^0 \left( \mathcal{A}_C \cap \mathcal{H}^c_R \right)$ and $\Upsilon$ commute, one can show that

$$\mathrm{Ad} \left( W^{\epsilon}_{g} \left( \mathbb{I}_{\mathcal{H}} \otimes V^{\epsilon}_{h, \alpha} \right) W^{\epsilon\ast}_{g} \right) \left( \pi_{\sigma} \alpha \sigma \right) \left( K^{\epsilon}_{h, \alpha} \right)_{a(h) e' - a(g) \epsilon} \simeq \left( \pi_{\sigma} \alpha \sigma \right) \left( K^{\epsilon}_{h, \alpha} \right)_{a(h) e' - a(g) \epsilon} \eta^{\epsilon_1}_{g, \alpha} \beta^U_{g} \eta^{-1}_{g, \alpha}. \tag{5.63}$$

From this, by Lemma C.1, there are unitaries $x^{\epsilon, e'}_{g, h, \alpha} \in \mathcal{U}(\mathcal{H})$ such that

$$W^{\epsilon}_{g} \left( \mathbb{I}_{\mathcal{H}} \otimes V^{\epsilon}_{h, \alpha} \right) W^{\epsilon\ast}_{g} = \mathbb{I}_{\mathcal{H}} \otimes x^{\epsilon, e'}_{g, h, \alpha}. \tag{5.64}$$

We also set $v^{\epsilon}_{g, \alpha} := x^{\epsilon, e'}_{g, h, \alpha}$. Note that this unitary $v^{\epsilon}_{g, \alpha}$ having the grading $m^{\epsilon}_{g, \alpha}$ as $V^{\epsilon}_{g, \alpha}$ with respect to $\mathrm{Ad} \Gamma_{\alpha}$. Because $\beta^{U}_{g} \eta^{\epsilon_1}_{g, \alpha} \left( K^{\epsilon}_{h, \alpha} \right)_{a(h) e' - a(g) \epsilon} \eta^{\epsilon_1}_{g, \alpha} \beta^{U}_{g} \eta^{-1}_{g, \alpha}$ belongs to

$$\mathrm{Aut}^0 \left( \mathcal{A}_{C_0 \cap H^c_R} \right),$$

we have

$$x^{\epsilon, e'}_{g, h, \alpha} \in \pi \alpha \left( \mathcal{A}_{C_0 \cap H^c_R} \right)^c \cap \mathcal{H}. \tag{5.65}$$

From Lemma 5.9 (iii), we have

$$\mathrm{Ad} \left( W^{\epsilon}_{g} \right) \left( \mathbb{I}_{\mathcal{H}} \otimes x^{\epsilon, e'}_{g, h, \alpha} \right) \in \mathbb{I}_{\mathcal{H}} \otimes \mathcal{B}(\mathcal{H}). \tag{5.66}$$
Therefore, from Lemma 5.9 (ii), we have
\[
\text{Ad} \left( W_k^e W_f^{(1)a_{(k)}^e} \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes \chi_{g,h,R}^{e,e'} \right) = \text{Ad} \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R^{e''} (k, f) \Gamma_L^{e''} \right) \text{Ad} \left( W_k^e \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes \chi_{g,h,R}^{e,e'} \right). \tag{5.67}
\]

Similarly, recalling Lemma C.1, we have
\[
\text{Ad} \left( W_g^e \left( V_{hL}^e \otimes \Gamma_R^{m_L^e} \right) W_g^* \right) \left( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \right) = \pi_L \alpha_L \beta_g^U \eta_{gL}^{\epsilon (1)^{-1}} \left( K_{hL}^e \right)_{\alpha(h) e \beta(g)} \eta_{gL}^{\epsilon (1)} \beta_g^U \hat{\otimes} \pi_R \alpha_R. \tag{5.68}
\]

Furthermore, by the same argument as (5.63), one can also check
\[
\text{Ad} \left( W_g^e \left( V_{gL}^e \otimes V_{gR}^e \Gamma_R^{m_L^e} \right) W_g^* \right) \left( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \right) = \left( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \right) \gamma \beta_g^U \eta_g^{\epsilon (2)^{-1}} \tau_{-a_{(g)}} \gamma^{-1}. \tag{5.69}
\]

Using (5.35), we have
\[
\beta_g^U \left( \eta_{g\sigma}^{\epsilon (2)^{-1}} \right) \beta_h^U \left( \eta_{h\sigma}^{\epsilon (1)a_{(g)}(g)^e (2)} \right) \beta_g^{\gamma \gamma \gamma} \left( \eta_{g\sigma}^{\epsilon (2)} \right) \beta_h^U \beta_g^U \eta_{g\sigma}^{\epsilon (1)} \left( K_{h\sigma}^e \right)_{\alpha(h) e \beta(g)} \eta_{g\sigma}^{\epsilon (1)} \beta_g^U \beta_g^U = \beta_g^U \eta_{g\sigma}^{\epsilon (1)^{-1}} \left( K_{h\sigma}^e \right)_{\alpha(h) e \beta(g)} \eta_{g\sigma}^{\epsilon (1)^{-1}} \beta_g^U \beta_g^U. \tag{5.70}
\]

Note from (5.63) that each lines on the right hand side can be implemented by some unitary in \( \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \), and we get
\[
\text{Ad} \left( W_g^e \left( \mathbb{I}_{\mathcal{H}_L} \otimes V_{gR}^e \right) \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R^{e''} (g, h) \right) W_g^* \left( \mathbb{I}_{\mathcal{H}_L} \right) = \pi_L \alpha_L \hat{\otimes} \pi_R \alpha_R \beta_g^U \eta_{gR}^{\epsilon (2)^{-1}} \beta_h^U \left( \eta_{hR}^{\epsilon (1)a_{(g)}(g)^e (2)} \right) \beta_g^{\gamma \gamma \gamma} \left( \eta_{gR}^{\epsilon (2)} \right) \beta_h^U \beta_g^U \beta_h^U \eta_{gR}^{\epsilon (1)} \left( \mathbb{I}_{\mathcal{H}_L} \otimes V_{gR}^e \right) \left( \mathbb{I}_{\mathcal{H}_R} \right). \tag{5.71}
\]

By Lemma C.1, this means that there is a homogeneous unitary \( u_R^{e''} (g, h) \) on \( \mathcal{H}_R \) such that
\[
\text{Ad} \left( u_R^{e''} (g, h) \right) \pi_R \alpha_R = \pi_R \alpha_R \beta_g^U \eta_{gR}^{\epsilon (2)^{-1}} \beta_h^U \left( \eta_{hR}^{\epsilon (1)a_{(g)}(g)^e (2)} \right) \beta_g^{\gamma \gamma \gamma} \left( \eta_{gR}^{\epsilon (2)} \right) \beta_h^U \beta_g^U \beta_h^U \eta_{gR}^{\epsilon (1)} \left( \mathbb{I}_{\mathcal{H}_L} \otimes V_{gR}^e \right) \left( \mathbb{I}_{\mathcal{H}_R} \right). \tag{5.72}
\]
and

\[
W_{gh}^e \left( \mathbb{H}_L \otimes V_{s g R}^e \right) W_{gh}^e \left( \mathbb{H}_L \otimes u^e_R(g, h) \right) W_{gh}^e \left( \mathbb{H}_L \otimes V_{h R}^{(1)\mu_\omega(g)} \right) W_{gh}^e
\]

Using \( v^e_R \) from (5.64), and (5.67), the latter equation can be written

\[
\mathbb{H}_L \otimes u^e_R(g, h) = \left( -1 \right)^{k^e_L(g, h)k^e_R(g, h)} \left( \mathbb{H}_L \otimes v^e_R(g, h) \right) Ad \left( \left( \mathbb{H}_L \otimes u^e_R(g, h) \right) \mathbb{H}_L \otimes v^e_R(g, h) \right)
\]

Similarly we get unitaries \( u^e_L(g, h) \) on \( \mathcal{H}_L \), satisfying

\[
Ad \left( u^e_L(g, h) \right) \pi_L \alpha_L
\]

and

\[
W_{s g}^e \left( V_{s g L}^e \otimes \Gamma_{L R}^{m_{\mu_\omega}} \right) W_{s g}^e \left( u^e_L(g, h) \otimes \Gamma_{L R}^{m_{\mu_\omega}} \right) W_{s g}^e \left( V_{h R}^{(1)\mu_\omega(g)} \right) W_{s g}^e
\]

Hence we obtain

\[
\left( W_{s g}^e \left( V_{s g L}^e \otimes \Gamma_{L R}^{m_{\mu_\omega}} \right) W_{s g}^e \left( u^e_L(g, h) \otimes \Gamma_{L R}^{m_{\mu_\omega}} \right) \right) \in IP \left( \omega, \alpha, \theta, (\eta_{g \sigma}^{e(2)}), (\alpha_L, \alpha_R, \gamma) \right)
\]

We set

\[
b^{(i)}(g, h) = \left( \omega, \alpha, \theta, \eta_{g \sigma}^{(i)}(g, h), (\alpha_L, \alpha_R, \gamma) \right) \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2),
\]

for \( i = 1, 2 \). (Recall Definitions 5.11, 5.8.) Note that \( \kappa^{(1)}_{\sigma} = \kappa_{\sigma}(g, h) \). We also set

\[
c^{(i)}(g, h) = \left( \omega, \alpha, \theta, \eta_{g \sigma}^{(i)}(g, h), (\alpha_L, \alpha_R, \gamma), (W_{s g}^{e(i)}), (u_{\sigma}^{e(i)}(g, h)) \right)
\]
for \( i = 1, 2 \).

We also define \( m \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \) by

\[
m(g) := \left( \frac{m_{g_R} + 1}{m_{g_R} - 1}, \frac{m_{g_R} - 1}{m_{g_R} + 1} \right), \quad g \in G.
\] (5.80)

Note from the definition and (5.73), (5.76) that

\[
\begin{align*}
\kappa_R^{(2)}(g, h) &= m_{g_R}^e + m_{h_R}^{(-1)^{a_0(g) e}} + m_{g h R}^e + \kappa_R^{(1)}(g, h), \\
\kappa_L^{(2)}(g, h) &= m_{g L}^e + m_{h L}^{(-1)^{a_0(g) e}} + m_{g h L}^e + \kappa_L^{(1)}(g, h).
\end{align*}
\] (5.81)

in \( \mathbb{Z}_2 \).

Next we would like to derive \( c^{(2)} \in C^3(G, U(1) \oplus U(1)) \). To do so, we first calculate \( u_R^{(2)}(g, h)u_R^{(2)}(gh, k) \) as follows. From (5.74)

\[
\begin{align*}
\mathbb{I}_L \otimes u_R^{(2)}(g, h)u_R^{(2)}(gh, k) &= (-1)^{\kappa_L^{(1)}(g, h) - \kappa_L^{(1)}(gh, k)} (-1)^{\kappa_L^{(1)}(gh, k) - m_{k_R}^{(-1)^{a_0(gh) e}}} \left( \mathbb{I} \otimes \left( v_g^e \right)^* \right) \\
&= \text{Ad} \left( W_{g h}^e \right) \left( \mathbb{I} \otimes \left( v_{h R}^{(-1)^{a_0(g) e}} \right)^* \right) \left( \mathbb{I} \otimes u_R^{(2)}(g, h) \right) \\
&= \text{Ad} \left( W_{g h}^e \right) \left( \mathbb{I} \otimes \left( v_{h R}^{(-1)^{a_0(g) e}} \right)^* \right) \left( \mathbb{I} \otimes u_R^{(2)}(gh, k)v_{g h k, R}^e \right) \\
&= (-1)^{\kappa_L^{(1)}(g, h) - m_{h_R}^{(-1)^{a_0(g) e}} + m_{k_R}^{(-1)^{a_0(gh) e}}} \left( \mathbb{I} \otimes \left( v_{g R}^e \right)^* \right) \\
&= \text{Ad} \left( W_{g h}^e \right) \left( \mathbb{I} \otimes \left( v_{h R}^{(-1)^{a_0(g) e}} \right)^* \right) \\
&= \text{Ad} \left( \left[ \mathbb{I} \otimes u_R^{(2)}(g, h) \Gamma_R^{(1)}(g, h) \right] W_{g h}^e \right) \left( \mathbb{I} \otimes \left( v_{h R}^{(-1)^{a_0(g) e}} \right)^* \right) \\
&= \left[ \mathbb{I} \otimes u_R^{(2)}(g, h) \right] \left( \mathbb{I} \otimes v_{g h k, R}^e \right).
\end{align*}
\] (5.82)

Here in the last equality, we got an extra sign \((-1)^{\kappa_L^{(1)}(g, h) - m_{k_R}^{(-1)^{a_0(gh) e}}} \) because we inserted \( \Gamma_R^{(1)}(g, h) \) in the middle. Applyin Lemma 5.10 to \([\cdot] \) part,
\[
\begin{align*}
(5.82) & \quad \kappa_L^{(1)}(g,h) \left( \left( m^k_R - 1 \right)^{\alpha(g)} \left( m_h^R - 1 \right)^{\alpha(g)} + m^k_R \right) \left( m^k_R - 1 \right)^{\alpha(g)} e \right) \kappa_L^{(1)}(g,h,k) \left( m^k_R - 1 \right)^{\alpha(g)} e \right) c^R(1)(g, h, k) \\
& \quad \left( \mathbb{I} \otimes \left( v^e_{g R} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{g h} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \left( \mathbb{I} \otimes u^e_R \left( g, h \right) \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \\
& \quad \left( \mathbb{I} \otimes \left( v^e_{g h} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \\
\end{align*}
\]

In the second equality, we used (5.67) to [] part. Now we recall the relation between $u^e_R$ and $u^e_R$ (5.74) and substitute it to the equation above to obtain

\[
(5.83) = \begin{align*}
& \quad \kappa_L^{(1)}(g,h) \left( \left( m^k_R - 1 \right)^{\alpha(g)} \left( m_h^R - 1 \right)^{\alpha(g)} + m^k_R \right) \left( m^k_R - 1 \right)^{\alpha(g)} e \right) \kappa_L^{(1)}(g,h,k) \left( m^k_R - 1 \right)^{\alpha(g)} e \right) c^R(1)(g, h, k) \\
& \quad \left( \mathbb{I} \otimes \left( v^e_{g R} \right)^* \right) \left( \mathbb{I} \otimes u^e_R \left( g, h \right) \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \\
& \quad \left( \mathbb{I} \otimes \left( v^e_{g h} \right)^* \right) \left( \mathbb{I} \otimes \left( v^e_{gh k, R} \right)^* \right) \\
\end{align*}
\]

(5.84)
In the second equality, we substituted the relation between $W^e_g$ and $W^e_g^{(2)}$ (5.77). Now setting $\tilde{\sigma} \in C^2(G, \mathbb{U}(1) \oplus \mathbb{U}(1))$ as

$$\tilde{\sigma}^e(g, h) := (-1)^{e_L^{(1)}(g, h)}m_{hR}^{(-1)^{\mu_0}(g)}$$

with (5.81) the phase factor in the last part of equation can be written as

$$\begin{align*}
&\equiv \frac{\tilde{\sigma}^e(h, k) \cdot \tilde{\sigma}^e(g, h k) \cdot (-1)^{e_L^{(1)}(g, h)} \cdot (m_{a_0}(g, h))}{(5.86)}
\end{align*}$$

Hence we get

$$\begin{align*}
&\mathbb{I}_L \otimes u_R^{e(2)}(g, h) u_R^{e(2)}(g, h, k) \\
&= \begin{pmatrix}
d_\alpha^2 \tilde{\sigma}(g, h, k) \cdot (-1)^{e_L^{(1)}(g, h)} \cdot (m_{a_0}(g, h)) \\
(-1)^{\left(\begin{array}{c}
\kappa_R^{(2)}(g, h, k)
\end{array}\right)} \cdot (b^{(2)}(g) - b^{(1)}(g) - m(g))
\end{pmatrix}
\begin{pmatrix}
c_R^{(1)}(g, h, k)
\end{pmatrix}
\end{align*}$$

Hence we have

$$c_R^{(2)}(g, h, k) = d_\alpha^2 \tilde{\sigma}(g, h, k) \cdot (-1)^{e_L^{(1)}(g, h)} \cdot (m_{a_0}(g, h))$$

Combining this with (5.81), we have

$$\begin{align*}
(c^{(1)}, \kappa_R^{(1)}, \kappa_L^{(1)}, b^{(1)}, a_\omega) \sim \mathcal{P}D(G, (c_R^{(2)}, \kappa_R^{(2)}, \kappa_L^{(2)}, b^{(2)}, a_\omega))
\end{align*}$$

with $m \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ in (5.80) and $\tilde{\sigma} \in C^2(G, \mathbb{U}(1) \oplus \mathbb{U}(1))$ in (5.85). This proves the Lemma.

**Definition 5.18.** Let $\omega \in \text{SPT}$. By the same simple argument as Lemma 2.17 in [O4], one can show that $h^{(3)}(\omega, \theta) \in \mathcal{P}D(G)$ is independent of the choice of $0 < \theta < \frac{\pi}{2}$. Recall from Lemma 2.4 that $\mathcal{P}D(G)$ and $\mathcal{P}D_0(G)$ are actually the same. Hence we can define an index $h(\omega) = h^{(3)}(\omega, \theta) \in \mathcal{P}D_0(G)$, independent of the choice of $\theta$. For $\Phi \in \mathcal{P}_{SL\beta}$ with the unique gapped ground state $\omega_\Phi$, we set $h_{\Phi} := h(\omega_\Phi)$.
6. Stability

Now, in order to prove Theorem 2.6, it suffices to prove the following.

**Theorem 6.1.** For any \( \omega \in \text{SPT} \) and \( \gamma \in Q\text{Aut}_\beta(\mathbb{A}_{\mathbb{Z}^2}) \), we have \( h(\omega) = h(\omega \circ \gamma) \). In particular, \( h_\Phi \) is an invariant of \( \mathcal{P}_{S\text{L}^\beta} \) for \( \sim_\beta \).

**Proof.** Let \( \alpha \in E\text{Aut}(\omega) \), \( 0 < \theta_0 < \theta < \frac{\pi}{2} \), \( (\eta^\alpha_{g_\sigma}) \in I(\omega, \theta_0), (\alpha_L, \alpha_R, \gamma) \in D^V(\alpha, \theta) \). Set \( \theta_2 := \theta \) and choose

\[
0 < \theta_0 < \theta_{0.8} < \theta_1 < \theta_{1.2} < \theta_{1.8} < \theta_2 < \theta_{2.2} < \theta_{2.8} < \theta_3 < \theta_{3.2} < \frac{\pi}{2}.
\] (6.1)

Because \( \gamma \in Q\text{Aut}_\beta(\mathbb{A}_{\mathbb{Z}^2}) \), by the same proof as Theorem 5.2 [O4], we can decompose \( \gamma \) as

\[
\gamma = \gamma_C \circ \gamma_H.
\] (6.2)

Here, automorphisms \( \gamma_H \) and \( \gamma_C \) belong to \( Q\text{Aut}(\mathbb{A}_{\mathbb{Z}^2}) \) and allow the following decompositions: 1. \( \gamma_H \) is decomposed as

\[
\gamma_H = (\text{inner}) \circ (\gamma_{H,L} \hat{\otimes} \gamma_{H,R}) = (\text{inner}) \circ \gamma_0
\] (6.3)

with some \( \gamma_{H,\sigma} \in \text{Aut}^{(0)} \left( \mathcal{A}_{(C_{\theta_0}) \cap H^\sigma_{\sigma}} \right) \), \( \sigma = L, R \). Here set \( \gamma_0 := \gamma_{H,L} \hat{\otimes} \gamma_{H,R} \in \text{Aut}^{(0)} \left( \mathcal{A}_{C_{\theta_0}} \right) \), and \( c_R^{(-1)} = 6, c_L^{(-1)} := -6 \). We have chosen the support of \( \gamma_{H,\sigma} \) as \( C_{\theta_0} \cap H^\sigma_{\sigma} \) so that \( \tilde{\eta}^\sigma_{g_\sigma} \) in (6.14) below has a support in \( C_{\theta_0} \cap H^\sigma_{\sigma} \).

2. The automorphism \( \gamma_C \) allows a decomposition

\[
\gamma_C = (\text{inner}) \circ \gamma_{CS}
\]

\[
\gamma_{CS} = \left( \gamma[0,\theta_1] \hat{\otimes} \gamma(\theta_1,\theta_2) \hat{\otimes} \gamma(\theta_2,\theta_3) \hat{\otimes} \gamma(\theta_3,\frac{\pi}{2}) \right) \circ \left( \gamma(\theta_{0.8},\theta_{1.2}) \hat{\otimes} \gamma(\theta_{1.8},\theta_{2.2}) \hat{\otimes} \gamma(\theta_{2.8},\theta_{3.2}) \right)
\] (6.4)

with

\[
\gamma_X := \bigotimes_{\sigma = L,R, \xi = D,U} \gamma_X, \sigma, \xi, \quad \gamma[0,\theta_1] := \bigotimes_{\sigma = L,R} \gamma[0,\theta_1], \sigma,
\]

\[
\gamma(\theta_3,\frac{\pi}{2}) := \bigotimes_{\xi = D,U} \gamma(\theta_3, \frac{\pi}{2}), \xi,
\]

\[
\gamma_{X,\sigma,\xi} \in \text{Aut}^{(0)} \left( \mathcal{A}_{C_X \cap H^\sigma_{\sigma} \cap H^\xi} \right), \quad \gamma_X, \sigma := \bigotimes_{\xi = U, D} \gamma_{X, \sigma, \xi},
\]

\[
\gamma_X, \xi := \bigotimes_{\sigma = L,R} \gamma_X, \sigma, \xi,
\]

\[
\gamma[0,\theta_1], \sigma \in \text{Aut}^{(0)} \left( \mathcal{A}_{C[0,\theta_1] \cap H^\sigma_{\sigma}} \right), \quad \gamma(\theta_3, \frac{\pi}{2}), \xi \in \text{Aut}^{(0)} \left( \mathcal{A}_{C[\theta_3, \frac{\pi}{2}] \cap H^\xi} \right),
\]

for

\[
X = (\theta_1, \theta_2), (\theta_2, \theta_3), (\theta_{0.8}, \theta_{1.2}), (\theta_{1.8}, \theta_{2.2}), (\theta_{2.8}, \theta_{3.2}), \quad \sigma = L, R, \quad \xi = D, U.
\] (6.6)
These automorphisms satisfy
\[ \gamma_I \circ \beta_g^U = \beta_g^U \circ \gamma_I \quad \text{for all} \quad g \in G, \quad \text{and} \quad \gamma_I \circ \tau = \tau \circ \gamma_I \]
(6.7) for any
\[ I = [0, \theta_1], (\theta_1, \theta_2], (\theta_2, \theta_3], \left(\theta_3, \frac{\pi}{2}\right], (\theta_{0.8}, \theta_{1.2}], (\theta_{1.8}, \theta_{2.2}], (\theta_{2.8}, \theta_{3.2}]. \]
(6.8)

Now we set
\[ \hat{\gamma} := \gamma \circ \left(\gamma(\theta_2, \theta_3] \otimes \gamma(\theta_3, \frac{\pi}{2}]\right) \circ \left(\gamma(\theta_{1.8}, \theta_{2.2}] \otimes \gamma(\theta_{2.8}, \theta_{3.2}]\right) \]
\[ \in \text{Aut}^{(0)} \left(\mathcal{A}_{C^E_{\theta_{1.8}}}\right) \subset \text{Aut}^{(0)} \left(\mathcal{A}_{C^E_{\theta_{2.2}}}\right), \]
(6.9)
and
\[ \hat{\alpha}_\sigma := \alpha \circ \left(\gamma(0, \theta_{0.1}) \otimes \gamma(\theta_{0.1}, \theta_{0.2}]\right) \circ \gamma(\theta_{0.8}, \theta_{1.2}] \circ \gamma_{H, \sigma} \in \text{Aut}^{(0)} \left(\mathcal{A}_{H_n}\right), \quad \sigma = L, R. \]
(6.10)

Then as in the proof of Theorem 3.1 of [O4] Step 1, we can check
\[ \alpha \circ \gamma = (\text{inner}) \circ (\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\gamma}. \]
(6.11)

Therefore, we have \((\hat{\alpha}_L, \hat{\alpha}_R, \hat{\gamma}) \in D^V (\alpha \gamma, \theta_{1.2}).\) Because \(\tau \) and \(\gamma_{CS}\) commute, we also have
\[ \gamma^{-1} \tau^a \gamma = \tau^a \left(\gamma_{HL}^{-1} \hat{\gamma}_{HL} \otimes \gamma_{HR}^{-1} \hat{\gamma}_{HR}\right) \circ (\text{inner}). \]
(6.12)

As in the proof of Theorem 3.1 of [O4] Step 2 we have
\[ \gamma^{-1} \eta^e_{g} \gamma = \gamma_0^{-1} \gamma(\theta_{0.8}, \theta_{1.2}] \gamma(\theta_{0.1}) \gamma(\theta_{0.8}, \theta_{1.2}] \gamma_0 \circ (\text{inner}), \]
\[ \gamma^{-1} \beta_g^{-1} \gamma \beta_g^U = \gamma_0^{-1} \beta_g^{-1} \gamma_0 \beta_g^U \circ (\text{inner}). \]
(6.13)

Set
\[ \tilde{\eta}^e_{\sigma} := (\gamma_{H_n})^{-1} \gamma(0, \theta_{0.1}) \gamma_{H_n} \gamma(\theta_{0.8}, \theta_{1.2}] \gamma_0 \circ (\text{inner}), \]
\[ \omega \gamma \beta_g^U = \omega \beta_g^U \beta_g^{-1} \gamma \beta_g^{-1} \gamma_0 \beta_g^U \circ (\text{inner}) \]
(6.14)

We claim \((\eta^e_{\sigma}) \in I (\omega \gamma, \theta_{1.2}).\) It can be seen as follows using (6.12), (6.13):
\[ \omega \gamma \beta_g^U = \omega \beta_g^U \beta_g^{-1} \gamma \beta_g^{-1} \gamma_0 \beta_g^U \circ (\text{inner}) \]
\[ = \omega \gamma^{-1} \tau^{a_{\omega}(g)} \gamma \circ \gamma^{-1} \left(\eta^e_{gL} \otimes \eta^e_{gR}\right) \beta_g^{-1} \gamma \beta_g^U \circ (\text{inner}) \]
\[ = \omega \gamma \tau^{a_{\omega}(g)} \left(\eta^e_{gL} \otimes \eta^e_{gR}\right) \circ (\text{inner}). \]
(6.15)

From this equality, we can also see that \(a_{\omega}(g) = a_{\omega \gamma}(g)\) for any \(g \in G.\)
Because \((\hat{\alpha}_L, \hat{\alpha}_R, \hat{\gamma}) \in D^V (\alpha \gamma, \theta_{1.2})\) and \((\tilde{\eta}^e_{\sigma}) \in I (\omega \gamma, \theta_{1.2}),\) in order to prove the Theorem, it suffices to show
\[ \text{IP} \left(\omega, \alpha, \theta_2, \left(\tilde{\eta}^e_{\sigma}\right), (\alpha_L, \alpha_R, \gamma)\right) = \text{IP} \left(\omega \gamma, \alpha \gamma, \theta_{1.2}, \left(\tilde{\eta}^e_{\sigma}\right), (\hat{\alpha}_L, \hat{\alpha}_R, \hat{\gamma})\right). \]
(6.16)
In other words, it suffices to show the following:

\[
(\hat{\alpha}_L \otimes \hat{\alpha}_R) \hat{\gamma} \beta^U_k \left( \tilde{\eta}^\epsilon_{g \sigma} \right)^{-1} \tau^{-a_{w_0}(g)\epsilon} \hat{\gamma}^{-1} (\hat{\alpha}_L \otimes \hat{\alpha}_R)^{-1} = (\alpha_L \otimes \alpha_R) \gamma \beta^U_k \left( \eta^\epsilon_{g \sigma} \right)^{-1} \tau^{-a_{w_0}(g)\epsilon} \gamma^{-1} (\alpha_L \otimes \alpha_R)^{-1},
\]

\[
\hat{\alpha}_\sigma \beta^U_{\sigma g} \left( \tilde{\eta}^\epsilon_{g \sigma} \right)^{-1} \beta^U_{\sigma h} \left( \left( \eta^\epsilon_{\sigma h} \right)^{-1} \left( \left( \eta^\epsilon_{(1)a_{w_0}(g)\epsilon} \right)_{-a_{w_0}(g)\epsilon} \right)^{-1} \eta^\epsilon_{\sigma h g} \beta^U_{\sigma g} \right)^{-1} = \alpha \beta^U_{\sigma g} \left( \eta^\epsilon_{g \sigma} \right)^{-1} \beta^U_{\sigma h} \left( \left( \eta^\epsilon_{\sigma h} \right)^{-1} \left( \left( \eta^\epsilon_{(1)a_{w_0}(g)\epsilon} \right)_{-a_{w_0}(g)\epsilon} \right)^{-1} \eta^\epsilon_{\sigma h g} \beta^U_{\sigma g} \right)^{-1}.
\]  

(6.17)

This can be shown as in Step 3. of Theorem 3.1 [O4], noting the commutativity (6.7) and (5.35).

Hence we completed the proof of Theorem 2.6.

7. Automorphisms on $\mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C})$

In this section we show the following proposition. It allows us to reduce the support of homogeneous automorphisms by finite portion.

**Proposition 7.1.** Let $\mathcal{R}_1$ be a finite even dimensional Hilbert space with a complex conjugation $\mathcal{C}_1$. Let $\mathcal{R}_2$ be an infinite dimensional Hilbert space with a complex conjugation $\mathcal{C}_2$. Set $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ with a complex conjugation $\mathcal{C} := \mathcal{C}_1 \oplus \mathcal{C}_2$. Then for any $\alpha \in \text{Aut}^{(0)} (\mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C}))$, there are even unitary $u \in \mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C})$ and an automorphism $\alpha_2 \in \text{Aut}^{(0)} (\mathcal{A}_{\text{SDC}} (\mathcal{R}_2, \mathcal{C}_2))$ such that

\[
\text{Ad}_u \circ \alpha = \text{id}_{\mathcal{A}_{\text{SDC}} (\mathcal{R}_1, \mathcal{C}_1)} \hat{\otimes} \alpha_2.
\]  

(7.1)

We start by several Lemmas.

**Lemma 7.2.** For each $n \in 2\mathbb{N}$, there is some $\delta_n > 0$ satisfying the following condition.:

Let $\mathcal{R}_i, i = 1, 2, 3$ be Hilbert spaces with complex conjugations $\mathcal{C}_i$. Suppose that $\mathcal{R}_1$ is $n$-dimensional. Suppose that $\mathcal{R}_3$ is even-finite-dimensional, and $\mathcal{R}_2$ is infinite dimensional. Set $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3$ with a complex conjugation $\mathcal{C} := \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3$. Let $\alpha \in \text{Aut}^{(0)} (\mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C}))$. Let $\{e_{IJ}\}_{I \subset \{1, \ldots, \dim \mathcal{R}_1\} \wedge J \subset \{1, \ldots, \dim \mathcal{R}_2\}}$ be a system of matrix units spanning $\mathcal{A}_{\text{SDC}} (\mathcal{R}_1, \mathcal{C}_1)$ with grading of $e_{IJ}$ being $|I| + |J|$ mod 2. Suppose that

\[
d (\alpha (e_{IJ}), \mathcal{A}_{\text{SDC}} (\mathcal{R}_1 \oplus \mathcal{R}_3, \mathcal{C}_1 \oplus \mathcal{C}_3)) < \delta_n.
\]  

(7.2)

Here $d(X, Y)$ means the distance between the two subsets $X, Y$ of $\mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C})$, with respect to the $C^*$-norm. Then there is an even unitary $u \in \mathcal{A}_{\text{SDC}} (\mathcal{R}, \mathcal{C})$ such that

\[
\text{Ad}_u (\alpha (\mathcal{A}_{\text{SDC}} (\mathcal{R}_1, \mathcal{C}_1))) \subset \mathcal{A}_{\text{SDC}} (R_1 \oplus R_3, \mathcal{C}_1 \oplus \mathcal{C}_3).
\]  

(7.3)

**Proof.** This holds directly from the proof of Lemma III 3.2 of [D]. It is easy to see that the unitary obtained by the argument there is even. 

\[\square\]
Lemma 7.3. Let $\mathcal{H}_i$, $i = 1, 2$ be finite even dimensional Hilbert spaces with complex conjugations $\mathcal{C}_i$. Let $\{e_{IJ}\}_{I,J \in \{1, \ldots, \dim \mathcal{H}_1\}}$ be a system of matrix units spanning $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$ with grading of $e_{IJ}$ being $|I| + |J|$ mod 2. Let $\{f_{IJ}\}_{I,J \in \{1, \ldots, \dim \mathcal{H}_2\}}$ be a system of matrix units in $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ with grading of $f_{IJ}$ being $|I| + |J|$ mod 2. Suppose that there is a self-adjoint unitary $u \in \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ such that $\text{Ad} (\gamma)(f_{IJ}) = (-1)^{|I|+|J|} f_{IJ}$. Then there is an even unitary $U \in (\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ such that $f_{IJ} = \text{Ad} (U) (e_{IJ})$.

Proof. Let $(\mathcal{H}_1, \pi_1)$ be a Fock representation of $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$ with grading unitary $\tilde{\Gamma}_1$. For each $I \in \{1, \ldots, \dim \mathcal{H}_1\}$, note that $\pi_1 (e_{II}) \mathcal{H}_1$ is one-dimensional space because $\pi_1 (e_{II})$ is a minimal projection of $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$ and $\pi_1$ is irreducible. Let $e_1$ be a unit vector in $\pi_1 (e_{II}) \mathcal{H}_1$. Because $\tilde{\Gamma}_1 \pi_1 (e_{II}) \mathcal{H}_1 = \pi_1 (e_{II}) \tilde{\Gamma}_1 \mathcal{H}_1 = \pi_1 (e_{II}) \mathcal{H}_1$, $e_1$ is eigenvector of $\Gamma_1$ with eigenvalue $\pm 1$. If $\Gamma_1 e_{II} = e_{II}/\Gamma_1 e_{II} = -e_{II}$, set $I_1 := \Gamma_1 / \Gamma_1 := -\Gamma_1$, so that $\Gamma_1 e_{II} = e_{II}$. For general $I \subset \{1, \ldots, \dim \mathcal{H}_1\}$, we have $e_{I} = c_I \pi_1 (e_{II}) e_0$, with some $c_I \in U(1)$. Hence we have $\Gamma_1 e_{I} = c_I \Gamma_1 \pi_1 (e_{II}) e_0 = (-1)^{|I|} c_I \pi_1 (e_{II}) e_0 = (-1)^{|I|} e_{I}$. Let $(\mathcal{H}_2, \pi_2)$ be Fock representations of $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2)$ with a grading unitary $\Gamma_2$. Then $(\mathcal{H}, \pi) := (\mathcal{H}_1 \otimes \mathcal{H}_2, \pi_1 \otimes \pi_2)$ is a Fock representation of $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ with a grading unitary $\Gamma := \Gamma_1 \otimes \Gamma_2$.

Setting $\mathcal{K} := \pi (f_{00}) \mathcal{H}$,
\begin{equation}
W \xi := \sum_{I \subset \{1, \ldots, \dim \mathcal{H}_1\}} e_I \otimes \pi (f_{00}) \xi, \quad \xi \in \mathcal{H}
\end{equation}
defines a unitary from $\mathcal{H}$ to $\mathcal{H}_1 \otimes \mathcal{K}$ satisfying
\begin{equation}
\text{Ad} (W) (\pi (f_{IJ})) = \pi_1 (e_{IJ}) \otimes \mathbb{I}.
\end{equation}
We also have
\begin{equation}
W \Gamma = (\Gamma_1 \otimes \Gamma_2) (f_{00}) W.
\end{equation}
Because $f_{00}$ is even, $\mathcal{K}$ is invariant under $\Gamma$ and $\Gamma_2 (f_{00})$ is a self-adjoint unitary on $\mathcal{K}$. We have $\text{Ad} (\pi (u)) (\Gamma_2 (f_{00})) = -\Gamma_2 (f_{00})$, with the unitary $v$. It means the eigenvalue 1 of $\Gamma_2 (f_{00})$ and that of $-1$ have the same degeneracy. From this, we may find a unitary $u : \mathcal{H}_2 \to \mathcal{K}$ such that $\text{Ad} u (\Gamma_2) = \Gamma_2 (f_{00})$. Setting $\tilde{U} := (\mathbb{1} \otimes u^*) W \in U(\mathcal{H})$, we have $\tilde{U} \Gamma = \Gamma \tilde{U}$. Then there is an even unitary $U \in \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ such that $\tilde{U} = \pi (U^*)$. From the definition, we can check that $f_{IJ} = \text{Ad} (U) (e_{IJ})$. \hfill \Box

Lemma 7.4. Let $\mathcal{H}$ be a finite even dimensional Hilbert space with a complex conjugation $\mathcal{C}_1$. Let $\mathcal{H}_2$ be an infinite dimensional Hilbert space with a complex conjugation $\mathcal{C}_2$. Set $\mathcal{K} := \mathcal{H}_1 \oplus \mathcal{H}_2$ with a complex conjugation $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$. Let $\alpha \in \text{Aut}^{(0)} (\mathfrak{A}_{\text{SDC}} (\mathcal{H}, \mathcal{C}))$ such that $\alpha | \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1) = \text{id} | \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$. Then there is an automorphism $\alpha_2 \in \text{Aut}^{(0)} (\mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2))$ such that $\alpha = \text{id} | \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1) \otimes \alpha_2$.

Proof. It suffices to show $\alpha (\mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2)) = \mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2)$. Because $\mathcal{H}_1$ is of finite even dimensional and $\mathcal{H}_2$ is of infinite dimensional they are *-isomorphic to some CAR-algebras. Let $v_{\mathcal{H}_1} \in \mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$ be the grading operator of $\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)$. It is an even self-adjoint unitary. From [AM] Lemma 4.15, we have
\begin{equation}
\mathfrak{A}_{\text{SDC}} (\mathcal{H}_1, \mathcal{C}_1)' \cap \mathfrak{A}_{\text{SDC}} (\mathcal{H}, \mathcal{C}) = \mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2) (0) + v_{\mathcal{H}_1} \mathfrak{A}_{\text{SDC}} (\mathcal{H}_2, \mathcal{C}_2) (1).
\end{equation}
For any $f \in \mathcal{R}_2$, we claim $\alpha(B(f))$ belongs to $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2)$. To see this, note because $\alpha$ is graded, that $\alpha(B(f))$ is odd. Furthermore, any homogeneous $a \in \mathfrak{A}_{\text{SDC}}(\mathcal{R}_1, \mathcal{C}_1)$, we have

$$a\alpha(B(f)) - (-1)^{\beta a} \alpha(B(f))a = a \left( aB(f) - (-1)^{\beta a}B(f)a \right) = 0,$$

because $\alpha(a) = a$ from the assumption. Hence we see that $\alpha(B(f))u_{\mathcal{R}_1}$ is an odd element in $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1, \mathcal{C}_1) \cap \mathfrak{A}_{\text{SDC}}(\mathcal{R}, \mathcal{C})$. From (7.7), we conclude $\alpha(B(f)) \in \mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2)^{(1)}$, proving the claim.

As this holds for any $f \in \mathcal{R}_2$, we conclude $\alpha(\mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2)) \subset \mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2)$. The same argument for $\alpha^{-1}$ implies the opposite inclusion and we obtain $\alpha(\mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2)) = \mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2).

**Proof of Proposition 7.1.** Let $\mathcal{R}_0$ be a 2-dimensional $\mathcal{C}_2$-invariant subspace of $\mathcal{R}_2$ with the complex conjugation $\mathcal{C}_0 := \mathcal{C}_2|_{\mathcal{R}_0}$. Let $\{e_{IJ}\}_{I,J \subseteq \{1, \ldots, \dim \mathcal{R}_1 + 1\}}$ be a system of matrix units spanning $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0, \mathcal{C}_1 \oplus \mathcal{C}_0)$ with grading of $e_{IJ}$ being $|I| + |J|$ mod 2. Let $\{f_{IJ}\}_{I,J \subseteq \{1, \ldots, \dim \mathcal{R}_1 + 1\}}$ be a system of matrix units spanning $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1, \mathcal{C}_1)$ with grading of $f_{IJ}$ being $|I| + |J|$ mod 2.

We claim that there is an even finite dimensional $\mathcal{C}_2$-invariant subspace $\mathcal{R}_3$ of $\mathcal{R}_2 \oplus \mathcal{R}_0$ with the complex conjugation $\mathcal{C}_3 := \mathcal{C}_2|_{\mathcal{R}_3}$, and a even unitary $U \in \mathcal{U}(\mathfrak{A}_{\text{SDC}}(\mathcal{R}, \mathcal{C}))$ such that

$$\text{Ad}U \circ \alpha(\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0, \mathcal{C}_1 \oplus \mathcal{C}_0)) \subset \mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_3, \mathcal{C}_1 \oplus \mathcal{C}_0 \oplus \mathcal{C}_3).$$

To show this, let $\delta_{\dim \mathcal{R}_1 + 2}$ be the number given in Lemma 7.2. Then there exists an even finite dimensional $\mathcal{C}_2$-invariant subspace $\mathcal{R}_3$ of $\mathcal{R}_2 \oplus \mathcal{R}_0$ such that

$$d(\alpha(e_{IJ}), \mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_3, \mathcal{C}_1 \oplus \mathcal{C}_0 \oplus \mathcal{C}_3)) < \delta_{\dim \mathcal{R}_1 + 2}.\quad (7.10)$$

Applying Lemma 7.2 with $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ replaced by $\mathcal{R}_1 \oplus \mathcal{R}_0, \mathcal{R}_2 \oplus (\mathcal{R}_0 \oplus \mathcal{R}_3), \mathcal{R}_3$ we obtain an even $U \in \mathcal{U}(\mathfrak{A}_{\text{SDC}}(\mathcal{R}, \mathcal{C}))$ satisfying (7.9).

Next we apply Lemma 7.3 with $\mathcal{R}_1$, $\mathcal{R}_2$, $E_{IJ}$ replaced by $\mathcal{R}_1 \mathcal{R}_0 \oplus \mathcal{R}_3$, $E_{IJ}$ $\text{Ad}U \circ \alpha(E_{IJ})$. Note that because $U$ is even and $\alpha$ is homogeneous, $\text{Ad}U \circ \alpha(E_{IJ})$ has a degree $|I| + |J|$ mod 2. The algebra $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_0, \mathcal{C}_0)$ has an odd self-adjoint unitary $v_0$. Because $U$ is even and $\alpha$ is homogeneous, $v := \text{Ad}U \circ \alpha(v_0)$ is an odd self-adjoint unitary in $\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_3, \mathcal{C}_1 \oplus \mathcal{C}_0 \oplus \mathcal{C}_3)$. It satisfies

$$\text{Ad}v(\text{Ad}U \circ \alpha(E_{IJ})) = \text{Ad}U \circ \alpha(\text{Ad}v(E_{IJ})) = (-1)^{|I| + |J|} \text{Ad}U \circ \alpha(E_{IJ}).\quad (7.11)$$

Hence the required conditions of Lemma 7.3 are satisfied. Applying Lemma 7.3, we obtain an even unitary $V \in \mathfrak{A}_{\text{SDC}}(\mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_3, \mathcal{C}_1 \oplus \mathcal{C}_0 \oplus \mathcal{C}_3)$ such that $\text{Ad}V(E_{IJ}) = \text{Ad}U \circ \alpha(E_{IJ})$. Setting $u := V^*U \in \mathfrak{A}_{\text{SDC}}(\mathcal{R}, \mathcal{C})$, $u$ is a even unitary such that $\text{Ad}u \circ \alpha(E_{IJ}) = E_{IJ}$ for all $I, J \subseteq \{1, \ldots, \dim \mathcal{R}_1\}$. It means $\text{Ad}u \circ \alpha|_{\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1, \mathcal{C}_1)} = \text{id}_{\mathfrak{A}_{\text{SDC}}(\mathcal{R}_1, \mathcal{C}_1)}$. By Lemma 7.4 it means an automorphism $\alpha_2 \in \text{Aut}^{(0)}(\mathfrak{A}_{\text{SDC}}(\mathcal{R}_2, \mathcal{C}_2))$ satisfying (7.1) exists.

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A. Basic Notations

For a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$, while $\mathcal{U}(\mathcal{H})$ denotes the set of all unitaries on $\mathcal{H}$. If $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear map from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$, then $\text{Ad}(V) : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ denotes the map $\text{Ad}(V)(x) := VxV^*$, $x \in \mathcal{B}(\mathcal{H}_1)$. Occasionally we write $\text{Ad}_V$ instead of $\text{Ad}(V)$. For a $C^*$-algebra $\mathcal{B}$ and $v \in \mathcal{B}$, we set $\text{Ad}(v)(x) := vxv^*$, $x \in \mathcal{B}$.

For a state $\omega$ on a $C^*$-algebra $\mathcal{B}$, we denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ its GNS triple. For a $C^*$-algebra $\mathcal{B}$, we denote by $\mathcal{U}(\mathcal{B})$ the set of all unitaries in $\mathcal{B}$. For a $C^*$-algebra $\mathcal{B}$, $\mathcal{B}_{+,1}$ denotes the set of all positive elements of $\mathcal{B}$ with norm less than or equal to 1. For states $\omega, \varphi$ on a $C^*$-algebra $\mathcal{B}$, we write $\omega \simeq \varphi$ if they are equivalent and $\omega \sim_{q.e.} \varphi$ if they are quasi-equivalent. We denote by $\text{Aut}\mathcal{B}$ the group of automorphisms on a $C^*$-algebra $\mathcal{B}$. The group of inner automorphisms on a unital $C^*$-algebra $\mathcal{B}$ is denoted by $\text{Inn}\mathcal{B}$. For $\gamma_1, \gamma_2 \in \text{Aut}(\mathcal{B}), \gamma_1 = (\text{inner}) \circ \gamma_2$ means there is some unitary $u$ in $\mathcal{B}$ such that $\gamma_1 = \text{Ad}(u) \circ \gamma_2$.

The center of a von Neumann algebra $\mathcal{M}$ is denoted by $Z(\mathcal{M})$. To denote the composition of automorphisms $\alpha_1, \alpha_2$, all of $\alpha_1 \circ \alpha_2, \alpha_1 \alpha_2, \alpha_1 \cdot \alpha_2$ are used. Frequently, the first one serves as a bracket to visually separate a group of operators.

B. Graded von Neumann Algebras

In this section we collect facts we use about graded von Neumann algebras. See [BO] for further explanation. A graded von Neumann algebra is a pair $(\mathcal{M}, \theta)$ with $\mathcal{M}$ a von Neumann algebra and an involutive automorphism on $\mathcal{M}$, $\theta^2 = \text{Id}$. The even/odd part of $\mathcal{M}$ with respect to the grading is denoted by $\mathcal{M}^{(0)}/\mathcal{M}^{(1)}$. If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and there is a self-adjoint unitary $\Gamma$ on $\mathcal{H}$ such that $\text{Ad}_\Gamma|_{\mathcal{M}} = \theta$, then we call $(\mathcal{M}, \theta)$ a spatially graded von Neumann algebra with grading operator $\Gamma$. We say $(\mathcal{M}, \theta)$ is balanced if $\mathcal{M}$ contains an odd self-adjoint unitary. If $Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = C\mathbb{I}$ for the center $Z(\mathcal{M})$ of $\mathcal{M}$, we say $(\mathcal{M}, \theta)$ is central.

For a homogeneous state $\omega$ on a graded $C^*$-algebra $\mathcal{B}$, there is a self-adjoint unitary $\Gamma_\omega$ implementing the grading with respect to $\pi_\omega$. As a result, the grading extends to the von Neumann algebra $\pi_\omega(\mathcal{B})''$ by $\text{Ad}_{\Gamma_\omega}$. We always consider this extension without mentioning explicitly.

Lemma B.1. Let $(\mathcal{M}, \Gamma)$ be a balanced central graded von Neumann algebra on a Hilbert space $\mathcal{H}$. Then both of $\mathcal{M}$ and $\mathcal{M}^{(0)}$ are either a factor or a direct sum of two factors of the same type.

Proof. The proof is basically the same as part of Proposition 2.9 of [BO]. From Lemma A.2 of [BO], if $\mathcal{M}$ is not a factor, it is a direct sum of two factors of the same type which maps to each other by $\text{Ad}\Gamma$.

Let $U$ be a self-adjoint odd unitary in $\mathcal{M}$. Suppose that $\mathcal{M}^{(0)}$ is not a factor. Then there exists a projection $z$ in $Z(\mathcal{M}^{(0)})$ which is not 0 nor $\mathbb{I}$. For such a projection, we have
Hence we have $Z(\mathcal{M}) \cap (\mathcal{M})' \cap \{U\}' = Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}I$, which then implies that $Z + \text{Ad}_U(z) = I$. (We note that for orthogonal projections $p, q$ satisfying $p + q = I$ with $t \in \mathbb{R}$, either $p + q = I$ or $p = 0$, $I$ holds, by considering the spectrum of $p = tI - q$.) We claim $Z(\mathcal{M}^{(0)}) = \mathbb{C}z + \mathbb{C}I$. Now, for any projection $s$ in $Z(\mathcal{M}^{(0)})$, $zs$ is a projection in $Z(\mathcal{M}^{(0)})$. Therefore either $zs = 0$ or $zs + \text{Ad}_U(zs) = I$. The latter is possible only if $zs = z$ because $z + \text{Ad}_U(z) = I$. Similarly, we have $(I - z)s = 0$ or $(I - z)s = I - z$. Hence we have $Z(\mathcal{M}^{(0)}) = \mathbb{C}z + \mathbb{C}I$, proving the claim.

Hence $\mathcal{M}^{(0)}$ is a summation of two factors $\mathcal{M}^{(0)} = \mathcal{M}^{(0)}z \oplus \mathcal{M}^{(0)}(I - z)$. Because $\text{Ad}U(z) = I - z$, these factors $\mathcal{M}^{(0)}z, \mathcal{M}^{(0)}(I - z)$ are isomorphic. In particular, they have the same type. \hfill $\square$

Let $(\mathcal{M}_1, \text{Ad}_{\Gamma_1})$ and $(\mathcal{M}_2, \text{Ad}_{\Gamma_2})$ be spatially graded von Neumann algebras acting on $\mathcal{H}_1, \mathcal{H}_2$ with grading operators $\Gamma_1, \Gamma_2$. We define a product and involution on the algebraic tensor product $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{a_1b_1}a_2b_1(a_1a_2 \hat{\otimes} b_1b_2),$$

$$(a \hat{\otimes} b)^* = (-1)^{a*b}a^* \hat{\otimes} b^*.$$  \hspace{1cm} (B.1)

for homogeneous elementary tensors. The algebraic tensor product with this multiplication and involution is a $*$-algebra, denoted $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$. On the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\pi(a \hat{\otimes} b) := a \Gamma_1^{\hat{\otimes}} \otimes b$ \hspace{1cm} (B.2)

for homogeneous $a \in \mathcal{M}_1, b \in \mathcal{M}_2$ defines a faithful $*$-representation of $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$.

We call the von Neumann algebra generated by $\pi(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2)$ the graded tensor product of $(\mathcal{M}_1, \mathcal{H}_1, \Gamma_1)$ and $(\mathcal{M}_2, \mathcal{H}_2, \Gamma_2)$ and denote it by $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$. It is simple to check that $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is a spatially graded von Neumann algebra with a grading operator $\Gamma_1 \otimes \Gamma_2$.

For $a \in \mathcal{M}_1$ and homogeneous $b \in \mathcal{M}_2$, we denote $\pi(a \hat{\otimes} b)$ by $a \hat{\otimes} b$, embedding $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ in $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$. Note that $\partial(a \hat{\otimes} b) = \partial(a) + \partial(b)$ for homogeneous $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$.

**Lemma B.2.** For each $i = 1, 2$, let $(\mathcal{M}_i, \Gamma_i)$ be balanced central graded von Neumann algebras on a Hilbert space $\mathcal{H}_i$. Suppose that $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is of type I factor. Then both of $\mathcal{M}_1$ and $\mathcal{M}_2$ are type I.

**Proof.** By Lemma B.1, all of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1^{(0)}, \mathcal{M}_2^{(0)}$ are either a factor or a direct sum of two factors of the same type. Then by Lemma A.1 of [BO], the type of $\mathcal{M}_i$ and $\mathcal{M}_i^{(0)}$ are the same for each $i = 1, 2$.

Because $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is a type I factor, it has a faithful semifinite normal trace $\tau$ whose restriction to $\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)}$ is also a faithful semifinite normal trace. Therefore, from Theorem 2.15 of [T], $\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)}$ is semifinite. From Theorem 2.30, it means both of $\mathcal{M}_1^{(0)}$ and $\mathcal{M}_2^{(0)}$ are semifinite. Because $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is a type I factor, for the set of orthogonal projections $\mathcal{P}(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2)$ of $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$, $\tau(\mathcal{P}(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2))$ is a countable set. It means that for the set of orthogonal projections $\mathcal{P}(\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)})$ of $\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)}$, $\tau(\mathcal{P}(\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)}))$ is also countable. It means that $\mathcal{M}_1^{(0)} \hat{\otimes} \mathcal{M}_2^{(0)}$ is not of type II. It means both of $\mathcal{M}_1^{(0)}$ and $\mathcal{M}_2^{(0)}$ are type I, hence both of $\mathcal{M}_1$ and $\mathcal{M}_2$ are type I. \hfill $\square$
Lemma B.3. Let $\mathcal{H}$ be a Hilbert space with a self-adjoint unitary $\Gamma$ that gives a grading for $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be $\text{Ad}_\Gamma$-invariant type I von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ with $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{B}(\mathcal{H})$. Suppose with respect to the grading given by $\text{Ad}_\Gamma$ that both of $\mathcal{M}_1$ and $\mathcal{M}_2$ have a center of the form $Z(\mathcal{M}_i) = \mathbb{C}I + \mathbb{C}V_i$ with a self-adjoint odd unitary $V_i$. Suppose further that

$$ab - (-1)^a b a = 0, \quad \text{for homogeneous } \ a \in \mathcal{M}_1, \ b \in \mathcal{M}_2. \tag{B.3}$$

Then there are Hilbert spaces $\mathcal{K}_1$, $\mathcal{K}_2$ and a unitary $W : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathbb{C}^2$ such that

$$\text{Ad}W(\mathcal{M}_1) = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{C}I_{\mathcal{K}_2} \otimes \mathbb{C}I_{\mathcal{C}_2},$$

$$\text{Ad}W(\mathcal{M}_2) = \mathbb{C}I_{\mathcal{K}_1} \otimes \mathcal{B}(\mathcal{K}_2) \otimes \mathbb{C}I_{\mathcal{C}_2}, \tag{B.4}$$

$$\text{Ad}W(V_1) = I_{\mathcal{K}_1} \otimes I_{\mathcal{K}_2} \otimes \sigma_z,$$

$$\text{Ad}W(V_2) = I_{\mathcal{K}_1} \otimes I_{\mathcal{K}_2} \otimes \sigma_x.$$

Proof. By Proposition 2.9 of [BO] (with $G$ a trivial group), $\mathcal{M}_i^{(0)} i = 1, 2$ are type I factors. By the assumption (B.3), $\mathcal{M}_1^{(0)}$ and $\mathcal{M}_2^{(0)}$ commute. From (B.3) and the fact that $V_i$ belongs to center of $\mathcal{M}_i$, we know that both of $V_1$ and $V_2$ commute with $\mathcal{M}_1^{(0)}$ and $\mathcal{M}_2^{(0)}$. As a result, there are Hilbert spaces $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and a unitary $\hat{W} : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{K}_3$ such that

$$\text{Ad}\hat{W}(\mathcal{M}_1) = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{C}I_{\mathcal{K}_2} \otimes \mathbb{C}I_{\mathcal{K}_3},$$

$$\text{Ad}\hat{W}(\mathcal{M}_2) = \mathbb{C}I_{\mathcal{K}_1} \otimes \mathcal{B}(\mathcal{K}_2) \otimes \mathbb{C}I_{\mathcal{K}_3}, \tag{B.5}$$

$$\text{Ad}\hat{W}(V_1) = I_{\mathcal{K}_1} \otimes I_{\mathcal{K}_2} \otimes y_1,$$

$$\text{Ad}\hat{W}(V_2) = I_{\mathcal{K}_1} \otimes I_{\mathcal{K}_2} \otimes y_2,$$

with $y_1, y_2$ self-adjoint unitaries on $\mathcal{K}_3$. Because $V_2 V_1 V_2^* = -V_1$, we have $y_2 y_1 y_2^* = -y_1$. With $y_1 = r_{1+} - r_{1-}$ as a spectral projection, this means that $y_2 r_{1\pm} y_2 = r_{1\mp}$, and $u := r_{1+} y_2 r_{1-} : r_{1-} \mathcal{K}_3 \to r_{1+} \mathcal{K}_3$ is a unitary. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$ and set $v : \mathcal{K}_3 \to \mathbb{C}^2 \otimes r_{1+} \mathcal{K}$ be a unitary given by

$$v \xi := e_1 \otimes r_{1+} \xi + e_2 \otimes u r_{1-} \xi, \quad \xi \in \mathcal{K}_3. \tag{B.6}$$

It is then straightforward to show that

$$\text{Ad}(y_1) = \sigma_z \otimes I_{r_{1+} \mathcal{K}},$$

$$\text{Ad}(y_2) = \sigma_x \otimes I_{r_{1+} \mathcal{K}}. \tag{B.7}$$

From this, (B.5) and $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{B}(\mathcal{H})$, we see that $r_{1+} \mathcal{K}$ is one-dimensional. Hence $W := (I_{\mathcal{K}_1} \otimes I_{\mathcal{K}_2} \otimes v)\hat{W} : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathbb{C}^2$ defines a unitary satisfying (B.4). \qed
C. Miscellaneous Lemmas

It is elementary to show the following Lemma. We omit the proof.

Lemma C.1. For $\sigma = L, R$, let $\mathcal{B}_\sigma$ be a graded $C^*$-algebra with a grading automorphism $\Theta_\sigma$. Let $(\mathcal{H}_\sigma, \pi_\sigma)$ be an irreducible representation of $\mathcal{B}_\sigma$ with a self-adjoint unitary $\Gamma_\sigma$ implementing $\Theta_\sigma$. Let $\xi_\sigma \in \text{Aut}(0)(\mathcal{B}_\sigma)$. Then the followings hold.

(i) If each $\xi_\sigma$ is implemented by a unitary $u_\sigma$ on $(\mathcal{H}_\sigma, \pi_\sigma)$, i.e., $\text{Ad}(u_\sigma) \circ \pi_\sigma = \pi_\sigma \circ \xi_\sigma$, then $u_\sigma$ is homogeneous with respect to $\text{Ad}(\Gamma_\sigma)$ and

$$
\text{Ad}(u_R \otimes \pi_R) \circ (\pi_L \hat{\otimes} \pi_R) = \left( \pi_L \hat{\otimes} \pi_R \right) \circ \text{id}_{\mathcal{B}_L} \hat{\otimes} \xi_\sigma,
$$

$$
\text{Ad}(u_L \otimes \Gamma_{RL}^\partial) \circ (\pi_L \hat{\otimes} \pi_R) = \left( \pi_L \hat{\otimes} \pi_R \right) \left( \xi_\sigma \hat{\otimes} \text{id}_{\mathcal{B}_R} \right).
$$

(C.1)

(ii) Suppose that there are unitaries $U_\sigma, \sigma = L, R$ on $\mathcal{H}_L \otimes \mathcal{H}_R$ such that

$$
\text{Ad}(U_R) \circ (\pi_L \hat{\otimes} \pi_R) = \left( \pi_L \hat{\otimes} \pi_R \right) \circ \text{id}_{\mathcal{B}_L} \hat{\otimes} \xi_\sigma,
$$

$$
\text{Ad}(U_L) \circ (\pi_L \hat{\otimes} \pi_R) = \left( \pi_L \hat{\otimes} \pi_R \right) \left( \xi_\sigma \hat{\otimes} \text{id}_{\mathcal{B}_R} \right).
$$

(C.2)

Then there are unitaries $u_\sigma \in \mathcal{U}(\mathcal{H}_\sigma)$ such that $\text{Ad}(u_\sigma) \circ \pi_\sigma = \pi_\sigma \circ \xi_\sigma$, and

$$
\mathbb{I} \otimes u_R = U_R, \quad u_L \otimes \Gamma_{RL}^\partial = U_L.
$$

(C.3)

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