\textbf{△Y-exchanges and the Conway-Gordon theorems}

RYO NIKKUNI AND KOIKI TANIYAMA

\textit{Dedicated to Professor Shin’ichi Suzuki for his 70th birthday}

Abstract. Conway-Gordon proved that for every spatial complete graph on 6 vertices, the sum of the linking numbers over all of the constituent 2-component links is congruent to 1 modulo 2, and for every spatial complete graph on 7 vertices, the sum of the Arf invariants over all of the Hamiltonian knots is also congruent to 1 modulo 2. In this paper, we give a Conway-Gordon type theorem for any graph which is obtained from the complete graph on 6 or 7 vertices by a finite sequence of △Y-exchanges.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $f$ be an embedding of a finite graph $G$ into the 3-sphere. Then $f$ is called a spatial embedding of $G$ and $f(G)$ is called a spatial graph. We denote the set of all spatial embeddings of $G$ by $\text{SE}(G)$. We call a subgraph of $G$ which is homeomorphic to a circle a cycle of $G$, and a cycle of $G$ which contains exactly $k$ edges a $k$-cycle of $G$. For a positive integer $n$, $\Gamma^{(n)}(G)$ denotes the set of all cycles of $G$ if $n = 1$ and the set of all unions of mutually disjoint $n$ cycles of $G$ if $n \geq 2$. We denote the union of $\Gamma^{(n)}(G)$ over all positive integer $n$ by $\Gamma(G)$. In the case of $n = 1$, we denote $\Gamma^{(1)}(G)$ by $\Gamma(G)$ simply, and denote the subset of $\Gamma(G)$ consisting of all $k$-cycles of $G$ by $\Gamma_k(G)$. For an element $\gamma$ in $\Gamma^{(n)}(G)$ and an element $f$ in $\text{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n = 1$ and an $n$-component link in $f(G)$ if $n \geq 2$.

Let $K_n$ be the complete graph on $n$ vertices, namely the simple graph consisting of $n$ vertices in which every pair of distinct vertices is connected by exactly one edge. For spatial embeddings of $K_6$ and $K_7$, we recall the following, which are called the Conway-Gordon theorems.

\textbf{Theorem 1.1.} (Conway-Gordon \cite{1})

(1) For any element $f$ in $\text{SE}(K_6)$, it follows that

$$\sum_{\gamma \in \Gamma^{(2)}(K_6)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2},$$

where $\text{lk}$ denotes the linking number in the 3-sphere.

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(2) For any element \( f \) in \( \text{SE}(K_7) \), it follows that
\[
\sum_{\gamma \in \Gamma(K_7)} \text{Arf}(f(\gamma)) \equiv 1 \pmod{2},
\]
where \( \text{Arf} \) denotes the Arf invariant [7].

Theorem 1.1 implies that for any element \( f \) in \( \text{SE}(K_6) \), there exists an element \( \gamma \) in \( \Gamma^{(2)}(K_6) \) such that \( \text{lk}(f(\gamma)) \) is odd, and for any element \( f \) in \( \text{SE}(K_7) \), there exists an element \( \gamma \) in \( \Gamma_7(K_7) \) such that \( \text{Arf}(f(\gamma)) = 1 \). A graph is said to be intrinsically linked if for any element \( f \) in \( \text{SE}(G) \), there exists an element \( \gamma \) in \( \Gamma(G) \) such that \( f(\gamma) \) is a nonsplittable 2-component link, and to be intrinsically knotted if for any element \( f \) in \( \text{SE}(G) \), there exists an element \( \gamma \) in \( \Gamma(G) \) such that \( f(\gamma) \) is a nontrivial knot. Theorem 1.1 also implies that \( K_6 \) is intrinsically linked and \( K_7 \) is intrinsically knotted. Moreover, we can obtain another intrinsically linked (resp. knotted) graph from \( K_6 \) (resp. \( K_7 \)) in the following way. A \( \triangle Y \)-exchange is an operation to obtain a new graph \( G_Y \) from a graph \( G_\triangle \) by removing all edges of a 3-cycle \( \triangle \) of \( G_\triangle \) with the edges \( uv, vw \) and \( wu \), and adding a new vertex \( x \) and connecting it to each of the vertices \( u, v \) and \( w \) as illustrated in Fig. 1.1 (we often denote \( ux \cup vx \cup wx \) by \( Y \)). A \( Y \triangle \)-exchange is the reverse of this operation. Throughout this paper, the symbols \( G_\triangle, G_Y, u, v, w \) and \( x \) are used as in the sense of Fig. 1.1. Motwani-Raghunathan-Saran [4] showed that if \( G_\triangle \) is intrinsically linked (resp. knotted) then \( G_Y \) is also intrinsically linked (resp. knotted). Thus any graph which is obtained from \( K_6 \) (resp. \( K_7 \)) by a finite sequence of \( \triangle Y \)-exchanges is intrinsically linked (resp. knotted). The set of all graphs obtained from \( K_6 \) (resp. \( K_7 \)) by a finite sequence of \( \triangle Y \)-exchanges consists of six (resp. fourteen) graphs as illustrated in Fig. 1.2 (resp. Fig. 1.3).

Our purpose in this paper is to give a Conway-Gordon type theorem as Theorem 1.1 (1) (resp. (2)) for any graph which is obtained from \( K_6 \) (resp. \( K_7 \)) by a finite sequence of \( \triangle Y \)-exchanges. Let \( G_\triangle \) and \( G_Y \) be two graphs such that \( G_Y \) is obtained from \( G_\triangle \) by a single \( \triangle Y \)-exchange. We denote the set of all elements in \( \tilde{\Gamma}(G_\triangle) \) containing \( \triangle \) by \( \tilde{\Gamma}_\triangle(G_\triangle) \). Let \( \gamma' \) be an element in \( \tilde{\Gamma}(G_\triangle) \) which does not contain \( \triangle \). Then there exists an element \( \Phi(\gamma') \) in \( \tilde{\Gamma}(G_Y) \) such that \( \gamma' \setminus \triangle = \Phi(\gamma') \setminus Y \). It is easy to see that the correspondence from \( \gamma' \) to \( \Phi(\gamma') \) defines a surjective map
\[
\Phi = \Phi_{G_\triangle, G_Y} : \tilde{\Gamma}(G_\triangle) \setminus \tilde{\Gamma}_\triangle(G_\triangle) \rightarrow \tilde{\Gamma}(G_Y).
\]
Let \( A \) be an additive group. We say that an \( A \)-valued unoriented link invariant \( \alpha \) is compressible if \( \alpha(L) = 0 \) for any unoriented link \( L \) which have a component \( K \) bounding a disk \( D \) in the 3-sphere with \( D \cap L = \partial D = K \). Namely \( \alpha(L) = 0 \) if \( L \) contains a trivial knot as a split component. In particular \( \alpha(L) = 0 \) when \( L \) is a
trivial knot. Suppose that for each element $\gamma'$ in $\tilde{\Gamma}(G_\triangle)$, an $A$-valued unoriented link invariant $\alpha_{\gamma'}$ is assigned. Then for each element $\gamma$ in $\tilde{\Gamma}(G_Y)$, we define an $A$-valued unoriented link invariant $\tilde{\alpha}_\gamma$ by
\[
\tilde{\alpha}_\gamma(L) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \alpha_{\gamma'}(L).
\]

Then we have the following theorem.

**Theorem 1.2.** Suppose that $\alpha_{\gamma'}$ is compressible for each element $\gamma'$ in $\tilde{\Gamma}_\triangle(G_\triangle)$. Suppose that there exists a fixed element $c$ in $A$ such that
\[
\sum_{\gamma' \in \Gamma(G_\triangle)} \alpha_{\gamma'}(g(\gamma')) = c
\]
for any element $g$ in $\text{SE}(G_\triangle)$. Then we have
\[
\sum_{\gamma \in \Gamma(G_Y)} \tilde{\alpha}_\gamma(f(\gamma)) = c
\]
for any element $f$ in $\text{SE}(G_Y)$.

As an application of Theorem 1.2, we have the following.

**Theorem 1.3.**

1. Let $G$ be a graph which is obtained from $K_6$ by a finite sequence of $\triangle Y$-exchanges. Then there exist a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element $f$ in $\text{SE}(G)$, it follows that
\[
2 \sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \text{lk}(f(\gamma))^2 - 1,
\]
where $a_i$ denotes the $i$th coefficient of the Conway polynomial.

2. Let $G$ be a graph which is obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges. Then there exists a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element $f$ in $\text{SE}(G)$, it follows that
\[
\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = 2 \sum_{\gamma \in \Gamma^{(2)}(G)} \omega(\gamma)\text{lk}(f(\gamma))^2 - 21.
\]
Figure 1.3.
As we will say later in Theorem 1.3, Theorem 1.3 (1) and (2) has been already shown by the first author in the case $G = K_6$ and $K_7$, respectively [5]. Theorem 1.3 is shown by combining Theorem 1.2 and Theorem 3.3.

Note that the square of the linking number is congruent to the linking number modulo two, and the second coefficient of the Conway polynomial of a knot is congruent to the Arf invariant modulo two [3]. Thus by taking the modulo two reduction in Theorem 1.3, we have the following corollary.

**Corollary 1.4.**

(1) Let $G$ be a graph which is obtained from $K_6$ by a finite sequence of $\triangle Y$-exchanges. Then, for any element $f$ in $SE(G)$, it follows that

$$\sum_{\gamma \in \Gamma(G)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2}.$$ 

(2) Let $G$ be a graph which is obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges. Then there exists a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}_2$ such that for any element $f$ in $SE(G)$, it follows that

$$\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \equiv 1 \pmod{2}.$$ 

In other words, there exists a subset $\Gamma$ of $\Gamma(G)$ such that for any element $f$ in $SE(G)$, it follows that

$$\sum_{\gamma \in \Gamma} \text{Arf}(f(\gamma)) \equiv 1 \pmod{2}.$$ 

Note that Corollary 1.4 (1) has already pointed out by Sachs [8] and the second author-Yasuhara [9], but as far as the authors know, Corollary 1.4 (2) has not been known yet except the case $G$ is $K_7$, see also Remark 3.6.

**Remark 1.5.**

(1) The set of all graphs obtained from $K_6$ by a finite sequence of $\triangle Y$ and $Y \triangle$-exchanges is called the Petersen family. This family consists of six graphs of Fig. 1.2 and the complete tripartite graph $K_{3,3,1}$ which cannot be obtained from $K_6$ by a finite sequence of $\triangle Y$-exchanges ($K_{3,3,1}$ is obtained from $P_8$ by a single $Y \triangle$-exchange at marked $Y$ as illustrated in Fig. 1.2). It is known that $K_{3,3,1}$ is also intrinsically linked [8] and it follows that

$$\sum_{\gamma \in \Gamma(K_{3,3,1})} \text{lk}(f(\gamma)) \equiv 1 \pmod{2}$$

for any element $f$ in $SE(K_{3,3,1})$ [9]. Recently O’Donnel showed in [G] that there exist a map $\omega$ from $\Gamma(K_{3,3,1})$ to $\mathbb{Z}$ such that

$$2 \sum_{\gamma \in \Gamma(K_{3,3,1})} \omega(\gamma)a_2(f(\gamma)) = \sum_{\gamma \in \Gamma(K_{3,3,1})} \text{lk}(f(\gamma))^2 - 1$$

for any element $f$ in $SE(K_{3,3,1})$. Namely, an integral version of the Conway-Gordon type theorem as Theorem 1.3 (1) holds for any graph in the Petersen family.

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1For a graph $G$ which is obtained from $K_6$ by a finite sequence of $\triangle Y$ and $Y \triangle$-exchanges, Sachs showed that for every spatial embedding of $G$, the sum of certain geometric positive integer-valued invariants over all of the constituent 2-component links is odd. In fact, this geometric invariant is congruent to the linking number modulo two.
(2) The set of all graphs which is obtained from $K_7$ by a finite sequence of $\triangle Y$ and $Y \triangle$-exchanges is called the Heawood family. This family consists of fourteen graphs of Fig. 1.3 and the other six graphs which cannot be obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges. It is known that a graph in the Heawood family is intrinsically knotted if and only if the graph is obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges [2]. Namely, an integral version of the Conway-Gordon type theorem as Theorem 1.3 (2) holds for any graph in the Heawood family which is intrinsically knotted.

In the next section, we prove Theorem 1.2. In section 3, we prove Theorem 1.3.

2. Proof of Theorem 1.2

Let $f$ be a spatial embedding of $G_Y$ and $D$ a 2-disk in the 3-sphere such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $\varphi(f)$ be a spatial embedding of $G_\triangle$ such that $\varphi(f)(x) = f(x)$ for $x \in G_\triangle \setminus \triangle = G_Y \setminus Y$ and $\varphi(f)(G_\triangle) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi : \text{SE}(G_Y) \longrightarrow \text{SE}(G_\triangle).$$

Then we immediately have the following.

**Proposition 2.1.** Let $f$ be an element in $\text{SE}(G_Y)$ and $\gamma$ an element in $\bar{\Gamma}(G_Y)$. Then, $f(\gamma)$ is ambient isotopic to $\varphi(f)(\gamma')$ for each element $\gamma'$ in the inverse image of $\gamma$ by $\bar{\Phi}$.

Suppose that for each element $\gamma'$ in $\bar{\Gamma}(G_\triangle)$, an $A$-valued unoriented link invariant $\alpha_{\gamma'}$ is assigned. Then we have the following lemma.

**Lemma 2.2.** If $\alpha_{\gamma'}$ is compressible for any element $\gamma'$ in $\bar{\Gamma}_\triangle(G_\triangle)$, then we have

$$\sum_{\gamma \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_\gamma(f(\gamma)) = \sum_{\gamma' \in \bar{\Gamma}_\triangle(G_\triangle)} \alpha_{\gamma'}(\varphi(f)(\gamma'))$$

for any element $f$ in $\text{SE}(G_Y)$.

**Proof.** For an element $\gamma'$ in $\bar{\Gamma}_\triangle(G_\triangle)$, we see that $\varphi(f)(\gamma')$ is the trivial knot if $\gamma'$ belongs to $\Gamma(G_\triangle)$ and a link containing a trivial knot as a split component if $\gamma'$ belongs to $\bar{\Gamma}(G_\triangle) \setminus \Gamma(G_\triangle)$. Since $\alpha_{\gamma'}$ is compressible for any element $\gamma'$ in $\bar{\Gamma}(G_\triangle)$, we see that

$$\sum_{\gamma' \in \bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)} \alpha_{\gamma'}(\varphi(f)(\gamma')) = \sum_{\gamma' \in \bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)} \alpha_{\gamma'}(\varphi(f)(\gamma')).$$

Note that

$$\bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle) = \bigcup_{\gamma \in \bar{\Gamma}(G_Y)} \bar{\Phi}^{-1}(\gamma).$$
Then, by Proposition 2.1 we see that
\[
\sum_{\gamma' \in \bar{\Gamma}(G \Delta)} \alpha_{\gamma'}(\varphi(f)(\gamma')) = \sum_{\gamma' \in \bar{\Gamma}(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \alpha_{\gamma'}(\varphi(f)(\gamma')) \right)
\]
\[
= \sum_{\gamma' \in \bar{\Gamma}(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \alpha_{\gamma'}(f(\gamma)) \right)
\]
\[
= \sum_{\gamma' \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_\gamma(f(\gamma)).
\]
Thus we have the result. □

Proof of Theorem 1.2. Suppose that there exists a fixed element \( c \) in \( A \) such that

\[
\sum_{\gamma' \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_\gamma(f(\gamma)) = c
\]

for any element \( f \) in \( SE(G_Y) \). Then by Lemma 2.2 and (2.2), we have

\[
\sum_{\gamma' \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_\gamma(f(\gamma)) = c
\]

for any element \( f \) in \( SE(G_Y) \). □

3. Proof of Theorem 1.3

Let \( \gamma \) be an element in \( \bar{\Gamma}(G_Y) \). Then we see that the inverse image of \( \gamma \) by \( \bar{\Phi} \) contains at most two elements in \( \bar{\Gamma}(G_Y) \), see Fig. 3.1. Moreover, we also see the following.

Proposition 3.1. Let \( \gamma \) be an element in \( \bar{\Gamma}(G_Y) \). Then, the inverse image of \( \gamma \) by \( \bar{\Phi} \) consists of exactly one element if and only if \( \gamma \) contains \( u, v, w \) and \( x \), or \( \gamma \) does not contain \( x \). □

![Figure 3.1.](image_url)

Note that if \( \gamma' \) is an element in \( \Gamma^{(n)}(G_Y) \) then \( \bar{\Phi}(\gamma') \) is an element in \( \Gamma^{(n)}(G_Y) \). This implies that the restriction map of \( \bar{\Phi} \) on \( \Gamma^{(n)}(G_Y) \) induces a surjective map

\[
\Phi^{(n)} = \Phi^{(n)}_{G_Y} : \Gamma^{(n)}(G_Y) \to \Gamma^{(n)}(G_Y). 
\]
In particular, we denote $\Phi^{(1)}$ by $\Phi$ simply. The surjectivity of $\Phi^{(n)}$ implies that if $\Gamma^{(n)}(G_{\Delta})$ is an empty set then $\Gamma^{(n)}(G_Y)$ is also an empty set for $n \geq 2$. Since both $\Gamma^{(n)}(K_6)$ and $\Gamma^{(n)}(K_7)$ are the empty sets for $n \geq 3$, we have the following.

**Proposition 3.2.** Let $G$ be a graph which is obtained from $K_6$ or $K_7$ by a finite sequence of $\Delta Y$-exchanges. Then $\Gamma^{(n)}(G)$ is an empty set for $n \geq 3$. \hfill $\square$

Now we prove Theorem 1.3. First we recall a refinement of the Conway-Gordon theorems which was shown by the first author.

**Theorem 3.3.** (5)\[ (1) \text{For any element } f \text{ in } SE(K_6), \text{ it follows that} \]
\[ 2 \sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(K_6)} \text{lk}(f(\gamma))^2 - 1. \]

(2) For any element $f$ in $SE(K_7)$, it follows that
\[ 7 \sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 6 \sum_{\gamma \in \Gamma_6(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma)) \]
\[ = 2 \sum_{\gamma \in \Gamma^{(2)}(K_7)} \text{lk}(f(\gamma))^2 - 21, \]

where $\Gamma^{(2)}(K_7)$ denotes the set of all unions of two disjoint cycles of $K_7$ consisting of a $k$-cycle and an $l$-cycle. \hfill $\square$

Note that Theorem 1.3 can be obtained from Theorem 3.3 by taking the modulo two reduction.

**Proof of Theorem 1.3.** First we show (1). We define a map $\omega$ from $\bar{\Gamma}(K_6)$ to $\mathbb{Z}$ by
\[ \omega(\gamma') = \begin{cases} 1 & \text{if } \gamma' \in \Gamma_6(K_6) \cup \Gamma^{(2)}(K_6) \\ -1 & \text{if } \gamma' \in \Gamma_5(K_6) \\ 0 & \text{otherwise} \end{cases} \]

for an element $\gamma'$ in $\bar{\Gamma}(K_6)$. Then by Theorem 3.3 (1), it follows that
\[ (3.1) \]
\[ 2 \sum_{\gamma' \in \Gamma(K_6)} \omega(\gamma') a_2(g(\gamma')) = \sum_{\gamma' \in \Gamma^{(2)}(K_6)} \omega(\gamma') \text{lk}(g(\gamma'))^2 - 1 \]

for any element $g$ in $SE(K_6)$. For each element $\gamma'$ in $\bar{\Gamma}(K_6)$, we define an integer-valued unoriented link invariant $\alpha_{\gamma'}$ of an unoriented link $L$ as follows. If $\gamma'$ is an element in $\Gamma(K_6)$, then $\alpha_{\gamma'}(L) = 2 \omega(\gamma') a_2(L)$ if $L$ is a knot and 0 if $L$ is not a knot. If $\gamma'$ is an element in $\Gamma^{(2)}(K_6)$, then $\alpha_{\gamma'}(L) = -\omega(\gamma') a_1(L)^2$ if $L$ is a 2-component link and 0 if $L$ is not a 2-component link. Note that $a_1(L)^2 = \text{lk}(L)^2$ if $L$ is a 2-component link. Then by (3.1), we have
\[ (3.2) \]
\[ \sum_{\gamma' \in \Gamma(K_6)} \alpha_{\gamma'}(g(\gamma')) = -1. \]

Let us consider the graph $Q_7$ which is obtained from $K_6$ by a single $\Delta Y$-exchange. Note that $\alpha_{\gamma'}$ is compressible for any element $\gamma'$ in $\bar{\Gamma}(K_6)$. Thus by Theorem 1.2 and (3.2), we have
\[ (3.3) \]
\[ \sum_{\gamma \in \Gamma(Q_7)} \hat{\alpha}_{\gamma}(f(\gamma)) = -1 \]
for any element \( f \) in \( \text{SE}(Q_7) \). Now we define a map \( \tilde{\omega} \) from \( \tilde{\Gamma}(Q_7) \) to \( \mathbb{Z} \) by

\[
(3.4) \quad \tilde{\omega}(\gamma) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')
\]

for an element \( \gamma \) in \( \tilde{\Gamma}(Q_7) \). Then we have

\[
(3.5) \quad \tilde{\alpha}_\gamma(L) = 2 \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')a_2(L) = 2\tilde{\omega}(\gamma)a_2(L)
\]

for an element \( \gamma \) in \( \Gamma(Q_7) \), and

\[
(3.6) \quad \tilde{\alpha}_\gamma(L) = -\sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')a_1(L)^2 = -\tilde{\omega}(\gamma)a_1(L)^2
\]

for any element \( \gamma \) in \( \Gamma'(Q_7) \). Recall \( \tilde{\Gamma}(Q_7) = \Gamma(Q_7) \cup \Gamma'(Q_7) \) by Proposition 3.2. Thus by combining (3.3), (3.5) and (3.6), we have

\[
(3.7) \quad 2 \sum_{\gamma \in \Gamma(Q_7)} \tilde{\omega}(\gamma)a_2(f(\gamma)) - \sum_{\gamma \in \Gamma'(Q_7)} \tilde{\omega}(\gamma)\text{lk}(f(\gamma))^2 = -1.
\]

It can be checked directly that each union of mutually disjoint two cycles of a graph in the Petersen family contains all of the vertices of the graph. Thus the map

\[
\Phi^{(2)} : \Gamma^{(2)}(K_6) \setminus \tilde{\Gamma}(K_6) \to \Gamma^{(2)}(Q_7)
\]

is bijective by Proposition 3.4 and therefore

\[
(3.8) \quad \tilde{\omega}(\gamma) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma') = 1
\]

for any element \( \gamma \) in \( \Gamma^{(2)}(Q_7) \). Thus by (3.7) and (3.8), we have

\[
2 \sum_{\gamma \in \Gamma(Q_7)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma \in \Gamma'(Q_7)} \text{lk}(f(\gamma))^2 - 1.
\]

By repeating this argument, we have the desired conclusion.

Next we show (2). We define a map \( \omega \) from \( \tilde{\Gamma}(K_7) \) to \( \mathbb{Z} \) by

\[
\omega(\gamma') = \begin{cases} 
7 & \text{if } \gamma' \in \Gamma_7(K_7) \\
-6 & \text{if } \gamma' \in \Gamma_6(K_7) \\
-2 & \text{if } \gamma' \in \Gamma_5(K_7) \\
1 & \text{if } \gamma' \in \Gamma_{4,3}(K_7) \\
0 & \text{otherwise}
\end{cases}
\]

for an element \( \gamma' \) in \( \tilde{\Gamma}(K_7) \). Then by Theorem 3.3 (2), it follows that

\[
(3.9) \quad \sum_{\gamma \in \Gamma(K_7)} \omega(\gamma)a_2(g(\gamma)) = 2 \sum_{\gamma \in \Gamma^{(2)}(K_7)} \omega(\gamma)\text{lk}(g(\gamma))^2 - 21
\]

for any element \( g \) in \( \text{SE}(K_7) \). For each element \( \gamma' \) in \( \tilde{\Gamma}(K_7) \), we define an integer-valued unoriented link invariant \( \alpha_{\gamma'} \) of an unoriented link \( L \) as follows. If \( \gamma' \) is an element in \( \Gamma(K_7) \), then \( \alpha_{\gamma'}(L) = \omega(\gamma')a_2(L) \) if \( L \) is a knot and 0 if \( L \) is not a knot. If \( \gamma' \) is an element in \( \Gamma^{(2)}(K_7) \), then \( \alpha_{\gamma'}(L) = -2\omega(\gamma')a_2(L)^2 \) if \( L \) is a 2-component link and 0 if \( L \) is not a 2-component link. Then by (3.4), we have

\[
(3.10) \quad \sum_{\gamma' \in \Gamma(K_7)} \alpha_{\gamma'}(g(\gamma')) = -21.
\]
Let us consider the graph $H_8$ which is obtained from $K_7$ by a single $\Delta Y$-exchange. Note also that $\alpha_\gamma$ is compressible for any element $\gamma'$ in $\Gamma(K_7)$. Thus by Theorem 1.2 and (3.2), we have

$$\sum_{\gamma \in \Gamma(H_8)} \tilde{\alpha}_\gamma(f(\gamma)) = -21$$

for any element $f$ in $\text{SE}(H_8)$. Now we define a map $\tilde{\omega}$ from $\tilde{\Gamma}(H_8)$ to $\mathbb{Z}$ by

$$\tilde{\omega}(\gamma) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')$$

for an element $\gamma$ in $\tilde{\Gamma}(H_8)$. Then we can see that

$$\sum_{\gamma \in \Gamma(H_8)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = 2 \sum_{\gamma \in \Gamma(H_8)} \tilde{\omega}(\gamma)\text{lk}(f(\gamma))^2 - 21$$

for any element $f$ in $\text{SE}(H_8)$ in the same way as the proof of (1). By repeating this argument, we have the desired conclusion. \qed

**Example 3.4.** Let $\tilde{\omega}$ be the map from $\tilde{\Gamma}(Q_7)$ to $\mathbb{Z}$ as in (3.11). In the following, let us determine $\tilde{\omega}(\gamma)$ for each element $\gamma$ in $\Gamma(Q_7)$. If $\gamma$ is an element in $\Gamma_7(Q_7)$, then there uniquely exists an element $\gamma'$ in $\Gamma_6(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma') = 1$. If $\gamma$ is an element in $\Gamma_6(Q_7)$, we divide our situation into the following three cases. If $\gamma$ does not contain $x$, then there uniquely exists an element $\gamma'$ in $\Gamma_6(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma') = 1$. If $\gamma$ contains $x$ and does not contain $u, v$ or $w$, then there exists an element $\gamma'_1$ in $\Gamma_5(K_6)$ and an element $\gamma'_2$ in $\Gamma_5(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma'_1) + \omega(\gamma'_2) = 0$. If $\gamma$ contains $u, v, w$ and $x$, then there uniquely exists an element $\gamma'$ in $\Gamma_5(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma') = -1$. If $\gamma$ is an element in $\Gamma_5(Q_7)$, we divide our situation into the following two cases. If $\gamma$ does not contain $x$, then there uniquely exists an element $\gamma'$ in $\Gamma_5(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma') = -1$. If $\gamma$ contains $x$, then note that $\gamma$ does not contain $u, v$ or $w$. Then there exists an element $\gamma'_1$ in $\Gamma_4(K_6)$ and an element $\gamma'_2$ in $\Gamma_5(K_6)$ such that $\Phi_{K_6,Q_7}^{-1}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Thus we have $\tilde{\omega}(\gamma) = \omega(\gamma'_1) + \omega(\gamma'_2) = -1$. Finally, if $\gamma$ is an element in $\Gamma(Q_7) \setminus \Gamma_5(Q_7) \cup \Gamma_6(Q_7) \cup \Gamma_7(Q_7)$, we have $\tilde{\omega}(\gamma) = 0$. In conclusion, we see that

$$\tilde{\omega}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_7(Q_7) \cup \{\delta \in \Gamma_6(Q_7) \mid \delta \not\supseteq x\} \\ -1 & \text{if } \gamma \in \{\delta \in \Gamma_6(Q_7) \mid \delta \supseteq x, u, v, w\} \cup \Gamma_5(Q_7) \\ 0 & \text{otherwise} \end{cases}$$

for an element $\gamma$ in $\Gamma(Q_7)$.

**Example 3.5.** Let $\tilde{\omega}$ be the map from $\tilde{\Gamma}(H_8)$ to $\mathbb{Z}$ as in (3.12). Then we see that

$$\tilde{\omega}(\gamma) = \begin{cases} 7 & \text{if } \gamma \in \Gamma_8(H_8) \cup \{\delta \in \Gamma_7(H_8) \mid \delta \not\supseteq x\} \\ 1 & \text{if } \gamma \in \{\delta \in \Gamma_7(H_8) \mid \delta \supseteq x, \delta \not\supseteq \{u, v, w\}\} \\ -6 & \text{if } \gamma \in \{\delta \in \Gamma_7(H_8) \mid \delta \supseteq x, u, v, w\} \cup \{\delta \in \Gamma_6(H_8) \mid \delta \not\supseteq x\} \\ -8 & \text{if } \gamma \in \{\delta \in \Gamma_6(H_8) \mid \delta \supseteq x, \delta \not\supseteq \{u, v, w\}\} \\ -2 & \text{if } \gamma \in \{\delta \in \Gamma_6(H_8) \mid \delta \supseteq x, u, v, w\} \cup \Gamma_5(H_8) \\ 0 & \text{otherwise} \end{cases}$$
for an element $\gamma$ in $\Gamma(H_8)$ in a similar way as in Example 3.4 and also see that

$$\tilde{\omega}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma^{(2)}(H_8) \cup \Gamma^{(2)}(H_8) \cup \{ \mu \in \Gamma^{(2)}(H_8) \mid \mu \not\supset \{x, u, v, w\} \} \\ 0 & \text{otherwise} \end{cases}$$

for an element $\gamma$ in $\Gamma^{(2)}(Q_7)$. Now we denote the set $\Gamma(H_8) \cup \{ \delta \in \Gamma_7(H_8) \mid \delta \not\supset \{x, u, v, w\} \}$ by $\Gamma$. Then it follows that

$$\sum_{\gamma \in \Gamma} \text{Arf}(f(\gamma)) \equiv 1 \pmod{2}$$

for any element $f$ in $\text{SE}(H_8)$. We remark here that the restricted map of $\Phi_{K_7,H_8}$ on $\Gamma_7(K_7)$ is a bijection from $\Gamma_7(K_7)$ to $\Gamma$. On the other hand, the restricted map of $\Phi_{H_8,F_9} \circ \Phi_{K_7,H_8}$ on $\Gamma_7(K_7)$ is not injective, see Figure 3.2.

**Figure 3.2.**

Remark 3.6. Let $G$ be a graph which is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges. Then by Corollary 1.4 (2), for any element $f$ in $\text{SE}(G)$ there exists an element $\gamma$ in $\Gamma(G)$ such that $\text{Arf}(f(\gamma)) = 1$. This fact is also shown by applying Theorem 1.1 (2) and Proposition 2.1 directly as follows. It is sufficient to show that if for any element $g$ in $\text{SE}(G_\Delta)$ there exists an element $\gamma'$ in $\Gamma(G_\Delta)$ such that $\text{Arf}(g(\gamma')) = 1$, then for any element $f$ in $\text{SE}(G_Y)$ there exists an element $\gamma$ in $\Gamma(G_Y)$ such that $\text{Arf}(f(\gamma)) = 1$. Let $f$ be an element in $\text{SE}(G_Y)$. Then there exists an element $\gamma'$ in $\Gamma(G_\Delta)$ such that $\text{Arf}(\varphi(f)(\gamma')) = 1$. Note that $\gamma' \neq \Delta$ because $\varphi(f)(\Delta)$ is a trivial knot. Let $\gamma$ be the image of $\gamma'$ by $\Phi$. Then we have $\text{Arf}(f(\gamma)) = \text{Arf}(f(\Phi(\gamma'))) = \text{Arf}(\varphi(f)(\gamma')) = 1$. Corollary 1.4 (2) insists on the result that is stronger than the fact above. Namely, there exists a subset $\Gamma$ of $\Gamma(G)$ which depends on only $G$ such that the sum of the Arf invariants over all of the images of the elements in $\Gamma$ by $f$ is odd for any element $f$ in $\text{SE}(G)$.

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**Department of Mathematics, School of Arts and Sciences, Tokyo Woman’s Christian University, 2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585, Japan**

*E-mail address: nick@lab.twcu.ac.jp*

**Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku-ku, Tokyo, 169-8050, Japan**

*E-mail address: taniyama@waseda.jp*