Features of Hopping Integral and Unconventional Orbital Modes for Quantum Skyrmions in Heisenberg Ferromagnet

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We use the coherent state approximation for the skyrmion in the isotropic quantum Heisenberg ferromagnet to obtain analytical expression for skyrmion hopping integral as a function of distance. It appears the skyrmion hopping is restricted by distances larger than its effective diameter. Some puzzling properties of the orbital skyrmionic modes are discussed. Spin distribution in these states is obtained and effect of quantum spin contraction is demonstrated. The principal possibility of lowering the skyrmion energy due to a bonding-like state formation is illustrated.

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1 Introduction

Translational motion of the 2D topological excitations, considered as a particle-like object, is a problem of enhanced importance both in theoretical and practical sense. Classical approach implies making use of phenomenological equations of motion. A variational approach is one of the most attractive methods to study an appropriate quantum problem. The variational wave function of collective motions, like translations, rotations and oscillations, was initially proposed by Hill and Wheeler [1] to be

$$\Psi([r]) = \int d^2R f(R)\Psi_R([r]),$$

where $\Psi_R([r])$ is a particle-like wave function for the topological excitation centered at point $R = (R, \Theta)$, $f(R)$ a weighting function, and $[r]$ is a set of particle coordinates. For continuous 2D media the
angular dependence of the weighting function follows immediately from the cylindrical symmetry of the system, so,

\[ f(R) = f_l(R) \exp(i l \Theta) \]

with \( l \) being the angular momentum. By varying the weighting function to minimize the expectation value of the Hamiltonian

\[ E = \langle \Psi([r]) | \hat{H} | \Psi([r]) \rangle, \]

one obtains an integral equation for \( f(R) \),

\[ \int d^2R' K(R', R) f(R') = 0, \tag{2} \]

where the kernel is

\[ K(R', R) = \langle \Psi_{R'}([r]) | \hat{H} - E | \Psi_{R}([r]) \rangle = H(R', R) - ES(R', R), \tag{3} \]

where we have made use of matrix elements of Hamiltonian, and overlap integral \( S(R', R) \). The kernel \( K(R', R) \) may be considered as an effective transfer (hopping) integral for quasiparticle transfer between \( R' \) and \( R \) sites.

It should be noted however that full realization of above mentioned scheme (2,3) for the real topological defect on the lattice is a difficult task as the lattice has no authentic rotational symmetry unlikely the conventional case of the vortex in the superfluid [2]. Obviously, the first step to treat this problem implies using continual approximation for describing soliton in the lattice.

Below we calculate quantity \( K(R', R) \) for quantized skyrmion being peculiar topological defect described in the framework of the coherent state approximation and use continual approximation to treat skyrmion motion in a discrete lattice.

Skyrmions are general static solutions of 2D Heisenberg ferromagnet with isotropic spin-Hamiltonian

\[ H = -J \sum_{i,b} S_i S_{i+b}, \tag{4} \]

obtained by Belavin and Polyakov [3] from classical nonlinear sigma model. A renewed interest to these unconventional spin textures is stimulated by high-\( T_c \) problem in doped quasi-2D-cuprates and quantum Hall effect.
The skyrmion spin texture consists of a vortex-like arrangement of the in-plane components of spin with the \( z \)-component reversed in the centre of the skyrmion and gradually increasing to match the homogeneous background at infinity. The spin distribution within classical skyrmion is given as follows

\[
S_x = \frac{2r\lambda}{r^2 + \lambda^2} \cos \varphi, \quad S_y = \frac{2r\lambda}{r^2 + \lambda^2} \sin \varphi, \quad S_z = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}. \tag{5}
\]

In terms of the stereographic variables the skyrmion with radius \( \lambda \) and phase \( \varphi_0 \) centered at a point \( z_0 \) is identified with spin distribution \( w(z) = \frac{\Lambda}{z-z_0} \), where \( z = x + iy = re^{i\varphi} \) is a point in the complex plane, \( \Lambda = \lambda e^{i\theta} \), and characterized by three modes: translational, or positional \( z_0 \)-mode, ”rotational” \( \theta \)-mode and ”di- latational” \( \lambda \)-mode. Each of them corresponds to certain symmetry of the classical skyrmion configuration. For example, \( \theta \)-mode corresponds to combination of rotational symmetry and internal \( U(1) \) transformation.

The simplest wave function of the quantum spin system, which corresponds to classical skyrmion, is a product of spin coherent states \([4]\). In case of spin \( s = \frac{1}{2} \)

\[
\Psi_{sk}(0) = \prod_i \left[ \cos \frac{\theta_i}{2} e^{i\frac{\varphi_i}{2}} |\uparrow\rangle + \sin \frac{\theta_i}{2} e^{-i\frac{\varphi_i}{2}} |\downarrow\rangle \right], \tag{6}
\]

where \( \theta_i = \arccos \frac{\varphi^2_i - \lambda^2}{\varphi^2_i + \lambda^2} \). Coherent state implies a maximal equivalence to classical state with minimal uncertainty of spin components.

Classical skyrmions with different phases and radia have equal energy. Nevertheless, stationary state of quantum skyrmion is not a superposition of states with different phase and radius \([5]\) and has a certain distinct value of \( \lambda \). Fluctuations in the in-plane orientation and size \( \lambda \) for quantum skyrmion are intimately connected; rotating a skyrmion changes its size. This result follows from the commutation relations of quantum angular momentum operators \([6]\).

In our recent paper \([7]\) it has been obtained an analytical expression for overlap integral of skyrmion states

\[
S(R_1, R_2) = \exp\left[ -\frac{\pi S}{a^2} (R_{12}^2 - 4i\Delta_{12}) \right], \tag{7}
\]
if $R_{12} < 2\lambda$ or

$$S(R_1, R_2) = \exp\left[ -\frac{\pi S}{a^2} (R_{12}^2 - 4i\Delta_{12} \\
-R_{12}\sqrt{R_{12}^2 - 4\lambda^2} + 2\lambda^2 \ln\left( \frac{R_{12} + \sqrt{R_{12}^2 - 4\lambda^2}}{R_{12} - \sqrt{R_{12}^2 - 4\lambda^2}} \right) \right]$$

if $R_{12} > 2\lambda$. Here one introduces a quantity

$$\Delta_{12} = \frac{1}{2} \| R_1 \| \| R_2 \| \sin(\varphi_1 - \varphi_2)$$

being area of a sector skyrmion covers while moving in the plane, $a$ is a lattice parameter. Skyrmion, having wandered closed contour on the plane, acquires the phase $4\pi \frac{\Delta_{12}}{a^2}$, that is each spin lying inside the contour contributes $4\pi$ into the total phase of the skyrmion, so its motion appears to be likely the motion of a charged particle in magnetic field [8, 9, 10]. Extending this analogy one may introduce the effective magnetic length $d = (\frac{4\pi S}{a^2})^{-\frac{1}{2}}$. Note the dependence of the phase factor in the overlap integral on the origin of the vectors $R_1, R_2$. It should be noted that for large separation $R_{12} \gg \lambda$ the $R$ dependence of the overlap integral obeys the power law

$$S(R_1, R_2) \propto (\lambda/R_{12})^p$$

with $p = (\lambda/d)^2$.

2 Hopping integral for stiff quantum skyrmion

One of the main goals of the paper is a calculation of the nondiagonal matrix element of the Hamiltonian (4)

$$H(R_1, R_2) = \langle \Psi_{R_1}^{sk}(\{r\}) | \hat{H} | \Psi_{R_2}^{sk}(\{r\}) \rangle,$$

between the states of quantum skyrmions, located at points $R_1$ and $R_2$, respectively, or transfer integral, likely being the principal quantity defining the quasiparticle motion of this topological defect.

We use approximation of "stiff" skyrmions, which takes no account of internal skyrmion quantum fluctuations and is closely related to classical limit. Making use of the coherent states (6) we
write out the appropriate matrix element as follows:

\[ H(R_1, R_2) = -\frac{1}{2} J S_{12} \sum_{i,b} \frac{\langle \varsigma_i | S_i | \varsigma_{i+b} \rangle \langle S_{i+R_{12}} | \varsigma_{i+R_{12}} \rangle}{\langle \varsigma_i | \varsigma_{i+b} \rangle \langle \varsigma_{i+R_{12}} \rangle \langle \varsigma_{i+b+R_{12}} \rangle}. \]  

(9)

For the spin coherent states the following relations exist [11]:

\[ \langle \varsigma | S_z | \mu \rangle = -S \frac{(1 - \varsigma \mu)(1 + \varsigma \mu)^{2S-1}}{(1 + | \varsigma |^2)^S(1 + | \mu |^2)^S}, \]

\[ \langle \varsigma | S_+ | \mu \rangle = 2S \frac{\mu(1 + \varsigma \mu)^{2S-1}}{(1 + | \varsigma |^2)^S(1 + | \mu |^2)^S}, \]

\[ \langle \varsigma | S_- | \mu \rangle = \frac{\langle \varsigma | S_z | \mu \rangle}{\langle \varsigma | \mu \rangle} = -S \frac{1 - \varsigma \mu}{1 + \varsigma \mu}, \]

(10)

\[ \langle \varsigma | S_\pm | \mu \rangle = 2S \frac{\mu}{1 + | \mu |^2}, \]

Taking account of the spin distribution in the skyrmion we write out the matrix element addressed as follows

\[ H(R_1, R_2) = -\frac{1}{2} J S^2 S_{12} \sum_{i,b} ((\varsigma_i - R_1)(\varsigma_i - R_2) - \lambda^2)((\varsigma_{i+b} - R_1) \times \times (\varsigma_{i+b} - R_2) - \lambda^2) + 2\lambda^2((\varsigma_{i+b} - R_1)(\varsigma_i - R_2) + 2\lambda^2((\varsigma_{i+b} - R_2)(\varsigma_i - R_1)) \times \times [(\lambda^2 + (\varsigma_i - R_1)(\varsigma_i - R_2))((\lambda^2 + (\varsigma_{i+b} - R_1)(\varsigma_{i+b} - R_2))]^{-1}. \]  

(11)

Below we are going to find the limit of the right hand side when \( a \to 0 \) with the constant \( a \) of lattice assumed to be square. To do it one has to make expansion in lattice constant \( a \) and to keep the three first terms, then to turn the sum into integral. The first (zero order in \( a \)) term in the expansion, as it is easy to see, is simply the ground state energy. The linear in \( a \) term in the expansion is zero due to the inversion symmetry. So, one has to define the second order term for a given site \( i \). Straightforward calculation gives for that second order term in \( a \) in the matrix element for scalar product \( \sum_b S_i S_{i+b} \) the following result

\[ \frac{-8\lambda^2}{(\lambda^2 + (\varsigma_i - R_2)(\varsigma_i - R_1))^2} a^2. \]
While coming to the limit $a \to 0$ the summation in (11) is replaced by the integration, and this term results in a nonvanishing contribution to the transfer integral:

$$H(R_1, R_2) = [E_0 + 4JS^2\lambda^2 \int_0^\infty r dr$$

\[
\int_0^{2\pi} d\varphi \frac{1}{(\lambda^2 + (re^{-i\varphi} - R_2)(re^{i\varphi} - R_1))^2}]S(R_1, R_2),
\]

(12)

where $E_0$ is a ground state energy of spin system. An angular integral is given by following expression

\[
\int_0^{2\pi} d\varphi \frac{1}{(\lambda^2 + (re^{-i\varphi} - R_2)(re^{i\varphi} - R_1))^2} = \frac{2\pi(x_1 + x_2)}{(r R_2^2)(x_1 - x_2)^3},
\]

so the radial integral is

\[
\int_0^\infty \frac{2\pi r(r^2 + R_1 R_2 + \lambda^2)dr}{\sqrt{((r^2 + R_1 R_2 + \lambda^2)^2 - 4r^2 R_1 R_2)^3}} = \frac{1}{\lambda^2},
\]

where

$$x_{1,2}(r, R_1, R_2) = \frac{\lambda^2 + r^2 + R_1 R_2 \pm K}{2r R_2},$$

$$K(r, R_1, R_2) = \sqrt{(\lambda^2 + r^2 + R_1 R_2)^2 - 4r^2 R_1 R_2},$$

where the complex root has positive real part.

When $R_{12} < 2\lambda$ we finally obtain for the transfer integral

$$H(R_1, R_2) = [E_0 + 4\pi J S^2]S(R_1, R_2).$$

(13)

As $R_{12} = 0$ $H(R_1, R_1) = [E_0 + 4\pi J S^2]$, and this is the classical skyrmion energy. The structure of the transfer integral $H(R_1, R_2)$ is such, that its contribution to the effective hopping integral (3) $K(R_1, R_2)$ at $R_{12} < 2\lambda$ is exactly compensated by the overlap contribution. Thus, effective hopping integral for the "stiff" skyrmion is zero for distances $R_{12} < 2\lambda$. 
When $R_{12} > 2\lambda$, the integral (12) contains two singular points and, respectively, it is ill-defined in that region. To avoid the contribution from these points and obtain result independent of choice of coordinates we calculate its principal value, that is remove the contribution of points, that lie inside the circles of small radius $\epsilon$ with centers in singular points. To this end we divide the plane into two equal parts drawing the border between halfplanes at the middle of the line connecting two singular points. The value of integral (12) is twice the value of integral over halfplane. Integral (12) being represented as halfplane integral in coordinate system with center in singular point takes the form

$$H(R_1, R_2) = [E_0 + 8JS^2\lambda^2 \int_c^\infty r d\varphi \times$$

$$\times \int_{\arccos \frac{c}{r}}^{2\pi - \arccos \frac{c}{r}} \frac{1}{r^2(r + (\frac{R_{12}}{2} - c)e^{-i\varphi} - (\frac{R_{12}}{2} + c)e^{i\varphi})^2} +$$

$$+ 8JS^2\lambda^2 \int_0^c r d\varphi \int_0^{2\pi} \frac{1}{r^2(r + (\frac{R_{12}}{2} - c)e^{-i\varphi} - (\frac{R_{12}}{2} + c)e^{i\varphi})^2} S(R_1, R_2),$$

where $c = \sqrt{\frac{R_{12}^2}{4} - \lambda^2}$ is a half-separation between singular points. Here we have restricted ourselves to the case of real $R_1, R_2$ to simplify calculations. It is easy to see that last term in (14) is zero as angular integral is zero no matter what the value of $r$ is. Then,

$$\int d\varphi \frac{1}{r + (\frac{R_{12}}{2} - c)e^{-i\varphi} - (\frac{R_{12}}{2} + c)e^{i\varphi})^2} =$$

$$\frac{1}{i(r^2 + 4\lambda^2)}[-e^{i\varphi} - e^{i\varphi} - x_- - x_+ \ln e^{i\varphi} - x_+],$$

where

$$x_{+, -} = \frac{r \pm \sqrt{r^2 + 4\lambda^2}}{2(\frac{R_{12}}{2} + c)}.$$

As a result, we obtain the following radial integral

$$\int_c^\infty r d\varphi \int_{\arccos \frac{c}{r}}^{2\pi - \arccos \frac{c}{r}} \frac{1}{r^2(r + (\frac{R_{12}}{2} - c)e^{-i\varphi} - (\frac{R_{12}}{2} + c)e^{i\varphi})^2} =$$
\[
\int_{c}^{\infty} \frac{dr}{r(r^2 + 4\lambda^2)^{1/2}} \times \frac{r}{\sqrt{(r^2 + 4\lambda^2)}} \times \arctan \sqrt{(r^2 + 4\lambda^2)(\frac{1}{c^2} - \frac{1}{r^2})} - \frac{\sqrt{r^2 - c^2(2\lambda^2 + r^2)}2c}{-4c^2\lambda^2 + r^2(r^2 + 4\lambda^2)} = 0.
\]

Thus, it appears that the effective hopping integral is nonzero in this regime and its value is simply proportional to the overlap integral.

So, the final result for effective hopping integral is
\[
K(R_1, R_2) = -4\pi S^2 J S(R_1, R_2)\theta(R_{12} - 2\lambda),
\]
where
\[
\theta(R_{12} - 2\lambda) = \begin{cases} 
1, & \text{if } R_{12} > 2\lambda \\
0, & \text{otherwise}
\end{cases}
\]
and \(S(R_1, R_2)\) is given by expressions (7), (8).

Thus, one might conclude that the skyrmion hopping is restricted by distances larger than \(2\lambda\). The nonzero effective hopping integral for distances larger than \(2\lambda\) allows to lower skyrmion energy by construction of appropriate combination of skyrmion states centered at different points of lattice and having definite value of angular moment. Energy of these states will depend on the value of the angular moment and on number of the levels as it is the case for the Landau levels in a magnetic field. Note also that transfer integral has the negative sign as must be the case.

The states considered have definite phase and radius. In fact phase and radius can not be treated separately as radius is determined by spin of the skyrmion and spin in turn is canonically conjugate to the phase. It is easy to show that states of definite phase and radius are not mixed by the Hamiltonian. This is related to the fact that we deal with the skyrmion of charge \(q = 1\), which has infinite spin. As the spin and phase operators do not commute, the spin must have infinite value for the phase to be definite. In the case of higher topological charges there would be mixing of the states with different phases and radius. The similar conclusions were drawn by Stern [5].
3 Orbital motion of skyrmions. Small and large orbital $l$-skyrmions

Making use of the above derived results we may exploit particular analogy between skyrmion and particle in zero Landau level to construct a subset of orthogonal states with definite value of orbital momentum, or $l$-skyrmions. Consider firstly the following orbital skyrmionic modes being linear combinations of single skyrmion states

$$
\Psi_{l}^{sk}(R) = \int_{0}^{2\pi} \exp(\imath l \Theta) \Psi_{R \exp(\imath \Theta)}^{sk} d\Theta,
$$

(18)

where $\Psi_{R \exp(\imath \Theta)}^{sk}$ is a single skyrmion state centered at the point $Z = R \exp(\imath \Theta)$. Below, we will distinguish small and large $l$-skyrmions depending on whether $R < \lambda$ or $R > \lambda$, respectively.

3.1 Small orbital $l$-skyrmions

It seems, any two equally centered $\Psi_{l}^{sk}(R_1)$ and $\Psi_{l}^{sk}(R_2)$ states with commonly different $R_1 < \lambda$ and $R_2 < \lambda$, but the same orbital momentum $l$ are linearly dependent. Indeed, for small $l$-skyrmions one might use a simple Gaussian form (7) for the overlap integral of single skyrmionic states, so the corresponding unnormalized overlap integral is given by a rather simple analytical expression

$$
S(R_1, R_2) = \int_{0}^{2\pi} \exp[\imath l (\Theta_1 - \Theta_2)] \exp\left[-\frac{1}{4d^2} (R_1^2 + R_2^2 - 2R_1 R_2)ight] \exp\left[-\frac{1}{4d^2} (R_1^2 + R_2^2) \frac{R_1 R_2}{2d^2}\right] d\Theta_1 d\Theta_2 = \frac{4\pi^2}{l!} \exp\left[-\frac{1}{4d^2} (R_1^2 + R_2^2) \frac{R_1 R_2}{2d^2}\right].
$$

(19)

It is easy to see that after normalization the overlap integral for two $l$-skyrmions with different $R_1$ and $R_2$

$$
\frac{S(R_1, R_2)}{\sqrt{S(R_1, R_1) S(R_2, R_2)}} = 1
$$

does not depend on $R_1, R_2$ at all, which means at least that the corresponding functions are not independent. So, these states are distinguished by a single quantum number $l$ which is an orbital
momentum and are degenerate in energy in full analogy with a zero Landau level. It should be emphasized that for \( l \)-skyrmion \( \Psi_l \) with \( R < \lambda \) the orbital moment is restricted by non-negative values \( l \geq 0 \), due to the definite sign of the Berry phase in the overlap integral for single skyrmion. To describe a spatial distribution of a "single skyrmionic (quasiparticle) density" for the orbital \( l \)-skyrmion \( \Psi_l \) one might use an overlap integral of the orbital state (18) with the simple skyrmion state in the point \( z = r \exp i\theta \) which has a rather simple form for \( r < \lambda \)

\[
S_{sk}^l(r, R) = \int_0^{2\pi} \exp[il\theta] \exp[-\frac{1}{4d^2}(r^2+R^2-2rR\exp[-i(\theta-\Theta)])]d\Theta = \frac{2\pi}{l!} \exp[il\theta] \exp[-\frac{1}{4d^2}(r^2 + R^2)l(rR/2d^2)^l, \tag{20}
\]

that with account for normalization conditions gives the quantity

\[
S_{sk}^l(r) = \frac{S_{sk}(r, R)}{\sqrt{S(R, R)}} = \frac{1}{\sqrt{l!}} \exp[il\theta] \left( \frac{r^2}{2d^2} \right)^l \exp[-r^2/2d^2], \tag{21}
\]

which does not depend at all on \( R \) value. This distribution function has maximum at \( r = r_l \), where

\[
\frac{2\pi Sr_l^2}{a^2} = l, \tag{22}
\]

so, the orbital momentum \( l \) defines the characteristic "orbital area" \( \pi r_l^2 \) measured in terms of a "flux quantum" \( a^2/2\pi \). This relation looks like a specific quantization condition for the "orbital area". The \( r_l \) value defined by expression (22) may be called as a characteristic "internal" radius of the \( \Psi_l(R) \) orbital states which magnitude does not depend on \( R \) value. One should note that for "s-skyrmion" \( r_0 = 0 \) as it should be for typical quasiparticle orbital s-states.

Actually, the maximal value of orbital momentum for the small \( l \)-skyrmion is restricted by the size of the bare skyrmion and, in general, can not significantly exceed \( l_\lambda \), because of the number of translational degrees of freedom is restricted by the "skyrmionic area" \( \pi \lambda^2 \). If we consider values of \( l \) much larger than \( l_\lambda \) we would obtain meaningless result for the spin distribution in the \( l \)-skyrmion...
because in this case continual approximation for the skyrmion overlap breaks down and expression (7) that neglects discretness effects is not adequate.

3.2 Quantum spin contraction for small radius \( l \)-skyrmions

Spin distribution in the puzzling "\( l \)-skyrmionic" \( \Psi_1(R) \) states with small \( R < \lambda \) can also be shown to be independent on radius \( R \). Besides, the corresponding spin texture is characterized by a quantum spin contraction, that is the mean spin length is less than \( S \). It can be calculated with making use of relations (10), so that we have for the spin averages

\[
S_+ (z) = \frac{1}{N} \int_0^{2\pi} \exp [il(\theta_1 - \theta_2)] \frac{\langle (z_{-1}) \mid S_+ \mid (z_{-2}) \rangle}{\langle (z_{-1}) \mid (z_{-2}) \rangle} S_{12} (z_1 - z_2) d\theta_1 d\theta_2 =
\]

\[
= \frac{1}{N} \int_0^{2\pi} \exp [il(\theta_1 - \theta_2)] S \frac{2\lambda (z - r \exp (i\theta_1))}{\lambda^2 + (z - r \exp (i\theta_1))(\bar{z} - r \exp (-i\theta_2))} \times
\]

\[
\times \exp \left[-\frac{1}{2d^2} (r^2 - r^2 \exp [-i(\theta_1 - \theta_2)]) \right] d\theta_1 d\theta_2,
\]

where \( z_1 = r \exp i\theta_1, z_2 = r \exp i\theta_2 \) are locations of distinct skyrmion states on the circle of radius \( r \),

\[
N = S(r, r) = \frac{(2\pi)^2}{l!} \frac{r^2}{(2d^2)^l} \exp \left[-\frac{r^2}{2d^2}\right]
\]

is a normalization integral. Introducing complex variables \( \hat{z}_1 = \exp (i\theta_1), \hat{z}_2 = \exp (-i\theta_2) \) and taking integral in \( \hat{z}_1 \), and then in \( \hat{z}_2 \) we obtain

\[
S_+ (z) = \frac{1}{N} \int \int \hat{z}_1^{-l+1} \hat{z}_2^{-l+1} \frac{2\lambda (z - r \hat{z}_1)}{\lambda^2 + (z - r \hat{z}_1)(\bar{z} - r \hat{z}_2)} \exp \left[-\frac{1}{2d^2} (r^2 - r^2 \hat{z}_1 \bar{z}_2) \right] d\hat{z}_1 d\hat{z}_2
\]

\[
= \frac{1}{l!} \left[ \frac{2d^2}{r^2} \right]^l \lim_{\hat{z}_2 \to 0} d\hat{z}_2 \lim_{\hat{z}_1 \to 0} d\hat{z}_1 \frac{2\lambda (z - r \hat{z}_1)}{\lambda^2 + (z - r \hat{z}_1)(\bar{z} - r \hat{z}_2)} \exp \left[-\frac{r^2}{2d^2} \hat{z}_1 \hat{z}_2 \right].
\]

This is an exact expression as integrals in (24) have no other poles but those of \((l + 1)\) order in \( \hat{z}_1 = 0 \) and \( \hat{z}_2 = 0 \), as long as \( r < l \). For the \( S_2 \) we have a similar expression
\[ S_z(z) = \frac{1}{l!} \left( \frac{2d^2}{r^2} \right)^l \lim_{\hat{z}_2 \to 0} \frac{d^l}{d\hat{z}_2^l} \lim_{\hat{z}_1 \to 0} \frac{d^l}{d\hat{z}_1^l} \lambda^2 - (z - r\hat{z}_1)(\bar{z} - r\hat{z}_2) \exp \left[ \frac{1}{2d^2} r^2 \hat{z}_1 \hat{z}_2 \right]. \]  

(25)

Surprisingly, for the case \( l = 0 \) we obtain the same spin distribution as in the simple single skyrmion state. In other words, in the framework of a classical approach the s-skyrmion is equivalent to a single skyrmion. For the ”p-skyrmion” with \( l = 1 \) the straightforward calculation gives

\[ S_+(z) = \frac{2\lambda z}{\lambda^2 + z\bar{z}} - \frac{8\lambda d^2 z}{(\lambda^2 + z\bar{z})^2} + \frac{8\lambda d^2 z^2 \bar{z}}{(\lambda^2 + z\bar{z})^3} = \frac{2\lambda z}{\lambda^2 + z\bar{z}} \left[ 1 - \frac{4d^2 \lambda^2}{(\lambda^2 + z\bar{z})^2} \right]. \]

\[ S_z(z) = \frac{\lambda^2 - z\bar{z}}{\lambda^2 + z\bar{z}} \left[ 1 - \frac{4d^2 \lambda^2}{(\lambda^2 + z\bar{z})^2} \right], \]  

(26)

where all values are measured in lattice constant. Here, we deal with a spin length contraction, compared with the case of classical skyrmion, specified by an universal function of skyrmion radius and \( r = |z| \)

\[ F(\lambda, r) = \left[ 1 - \frac{4d^2 \lambda^2}{(\lambda^2 + z\bar{z})^2} \right]. \]

Increasing the skyrmion radius entails decreasing the spin contraction. Minimal spin contraction takes place for large \( r \)'s and \( \lambda \)'s, while the maximal spin contraction does in the center of the spin texture for small \( \lambda \)'s. Interestingly, that for small skyrmion with \( \lambda = 2d \) the effective spin length in the core of the orbital p-skyrmion turns to zero: \( S(r = 0) = 0 \). However, as it is easy to see for very small bare skyrmions with \( \lambda \ll d \) the expression (26) leads to unphysical results. Indeed, the continuous approximation used above appears to be irrelevant for small skyrmions with radius \( \lambda \) less than the effective magnetic length \( d \).

The case \( l > 1 \) can not be represented in the simple form like (26), however, with increase of \( l \) the value of spin contraction in the center of spin distribution also increases. It should be noted, that for the orbital skyrmions with \( l = l_\lambda = \frac{2\pi \lambda^2 S}{a^2} = \frac{\lambda^2}{2d^2} \) (which is just a ratio of ”skyrmionic area” \( \pi \lambda^2 \) to the area corresponding to the quantum of flux \( \frac{\lambda^2}{2S} \)), the mean spin length in the center of spin distribution
turns to zero, that is the spin contraction appears to be maximal. This is due to the fact that at \( l = l_\lambda \) the characteristic internal radius \( r_{l=l_\lambda} \) of the \( \Psi_{l_\lambda} \) orbital skyrmion defined by expression (22) coincides with skyrmion radius \( \lambda \), so in the \( l_\lambda \)-skyrmion core we deal with a superposition of vectors with zero \( z \)-projection and all possible phases.

### 3.3 Large radius orbital skyrmions

Above we addressed to the particular type of the small orbital \( l \)-skyrmion with bare radius \( R < \lambda \). Unfortunately, for large \( l \) strongly exceeding \( l_\lambda \) continual approximation breaks down so we can not use expressions (24,25) to calculate spin distribution in the \( l \)-skyrmions with large effective radius. But this problem may be solved by taking bare radius \( R \) in the \( l \)-skyrmion equal to corresponding effective radius \( r_{l} = \frac{la}{2\pi S} \). However, in this case we cannot make use of advantages provided by a simple Gaussian form for the overlap integral of single skyrmionic states available only for \( R_{12} < 2\lambda \) (see (7)). The orbital motion with large bare radius acquires some unconventional properties compared to the case of "small orbits". Spin distribution in these states can be well approximated by the following expressions

\[
S_z(z) = -1 + \frac{2\lambda^2}{\sqrt{(\lambda^2 + z\bar{z} + R^2)^2 - 4z\bar{z}R^2}} \quad (27)
\]

\[
S_+(z) = \frac{\lambda}{z}(1 + \frac{z\bar{z} - \lambda^2 - R^2}{\sqrt{(\lambda^2 + z\bar{z} + R^2)^2 - 4z\bar{z}R^2}}). \quad (28)
\]

This expressions can be easily obtained if one neglects overlap of skyrmions in different points at all. Here, \( R \) equals to \( r_{l} = \frac{la}{2\pi S} \) as was explained before. Numerical calculations show that if we take account of the skyrmion overlap the effective radius of the state with bare radius \( R \) may be changed but orbital momentum can not strongly deviate from \( \frac{2\pi SR^2}{a^2} \) because otherwise discretness effects become again significant. Expression (27) shows that maximal spin contraction occurs not in the origin as is the case for small skyrmions but close to the points with \(| z | = R \).
Strictly speaking, the correct description of large skyrmion implies that one has keep both bare radius $R$ and orbital momentum $l$ to characterize it because matrix element of the Hamiltonian and overlap integral are nontrivial when $R_{12} > \lambda$. But for large skyrmion radia such a description is redundant at least when we are interested in the properties of the zero Landau level only. The corresponding effects would be negligible.

As to the energy of these states its difference from the classical skyrmion energy is negligible for large skyrmion radia so both large and small skyrmions may be said to constitute zero Landau level of skyrmion motion. In principle, it is possible to obtain higher Landau levels but we think it requires detailed account of discretness effects and is beyond the scope of this paper. The calculation of the fine structure of the skyrmion energy due to translational effects is prevented by discretness effects in the framework of the present continual approximation since it does not allow to have arbitrary values of bare radius $R$ for a given value of $l$ so one can not realize variational procedure (2,3) similarly to the case of a vortex in superfluid [2].

Both types of orbital skyrmions represent very interesting objects for further investigations.

4 Conclusions

Making use of the coherent state approximation allows to obtain an analytical form for an effective hopping integral for single stiff Belavin-Polyakov skyrmion in 2D isotropic ferromagnet. It appears, the skyrmion hopping is restricted by distances larger than its effective diameter, thus resulting in a very unusual quasiparticle properties. We introduce two types of orbital skyrmionic modes, the small and large $l$-skyrmions, respectively. For the former it is found both the spatial distribution of a single skyrmionic density, and the mean spin density distribution. The small $l$-skyrmions with nonzero orbital momentum are characterized by a strong quantum spin contraction effect. Translational symmetry of the Heisenberg Hamiltonian and topological Berry phase in overlap integral implies that there exists highly degenerate set of skyrmion states in analogy
with a zero Landau level. Based on the calculated expression for the transfer integral of the skyrmion we confirm existence of this highly degenerate set.

It should be emphasized that the energy of the above addressed quantum small $l$-skyrmions with bare radius $R < \lambda$ is always equal to the classical single skyrmion energy independently both of $R$ and $l$ value. However, by forming the appropriate combinations of single skyrmion states centered at different space points with separation $R_{12} \geq 2\lambda$ the energy of the classical state can be lowered. Addressing for example the combinations of two different states with separation between them exceeding $2\lambda$ we obtain usual bonding and antibonding states. Energy of such a bonding state is

$$\frac{4S^2 J\pi}{1-|S_{12}(R)|}$$

and has a minimum at $R_{12} = 2\lambda$. This implies that skyrmion states prefer to be separated a distance $2\lambda$ from each other to lower the energy. Naturally, this effect will lower the energy of the orbital skyrmions with rather large bare radius $R > \lambda$ as compared to the energy of classical skyrmion, but for large skyrmion radii $\lambda$ this lowering is obviously negligible. Thus we may say that both large and small $l$-skyrmions form unified zero Landau level.

The interaction of orbital skyrmions having the same orbital momentum but different bare radius will result in a complex skyrmionic energy spectrum, which analysis is a challenge for a further study. However, continual approximation used here is inadequate for calculation of that spectrum because real lattice on which skyrmion moves has no true rotational symmetry.

A further elaboration of the quasiparticle approach to skyrmions implies a detailed description of the internal structure of the orbital skyrmions without any constraint on its bare radius, and realization of variational procedure \[2\]. An appropriate work is in progress.

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