THE FERMI FLOW AND ITS APPLICATION TO GEOMETRY

Knut Smoczyk

Mathematics Department, Harvard University, Cambridge, MA 02138, USA

May 02, 1996

ABSTRACT. We introduce the notion of Fermi flow for hypersurfaces in Riemannian manifolds. It turns out that this is a powerful tool to study the geometry of distance surfaces about a given initial hypersurface. Some of the results in this paper are known in one form or another, however the aim is to demonstrate how they can all be derived by the same method and proved in a very simple manner. In addition we obtain some new results and results that are stronger than those stated in the literature.

1. Introduction and preliminary results

Let $N^{n+1}$ be a Riemannian manifold and $M^n$ be an orientable manifold (possibly with boundary) that is smoothly immersed by a local diffeomorphism $F$ into $N^{n+1}$. Further assume that $\nu$ is a normal vector field on $M_0 := F(M^n)$. Then we say that $M^n$ evolves under the Fermi flow if there exists a smooth family of local diffeomorphisms $F_t : M^n \to \tilde{M}_t := F_t(M^n) \subset N^{n+1}$ such that

$$\frac{d}{dt} F_t(x) = \nu(x, t), \quad F_0 = F.$$  

If $M^n$ is compact, then a smooth solution of this nonlinear first order PDE exists at least on some short time interval. Indeed the solutions of (1) are simply given by the foliation arising from Fermi coordinates about $M_0$ since the surfaces are moving by constant speed in their normal direction (see section 2 for details).

Remark: A more familiar expression for Fermi coordinates in codimension 1 is Gaussian normal coordinates. But the notation “Gaussian flow” would be rather confusing, since this could be mistaken for the flow defined by the Gaussian curvature on $M$.

If $\tilde{F} : \tilde{M}^n \to N^{n+1}$ is another immersion such that $\tilde{F}(y) = F(x)$, $\tilde{\nu}(y) = \nu(x)$ for a fixed pair $(y, x) \in \tilde{M}^n \times M^n$ then $\tilde{F}_t(y) = F_t(x)$ as long as a smooth solution of (1)

---

1 Supported by a Feodor Lynen Fellowship of the Alexander von Humboldt Foundation

2 email: ksmoczyk@abel.math.harvard.edu
exists. Roughly speaking, this means that the solutions of (1) depend only on the exterior geometry. In a more general setting, in [S], evolution equations have been derived for various curvature flows given by speed functions $-f$. Here we have $f = -1$. Usually one allows $f$ to depend on the second fundamental form of $M_t$. Here it does not, so “curvature” function is somewhat misleading. However, this flow measures the effect of exterior curvature on the motion of hypersurfaces. In the forthcoming we will assume that $M^n$ is compact and we also set $r = t$, since

$$M_t = \{ y \in N | y \text{ can be joined with } M_0 \text{ by a geodesic of length } t \text{ with initial speed } \nu \}$$

as long as a smooth solution of (1) exists.

Let $\bar{g}_{\alpha\beta}$ be the metric on $N$, $\sigma_{ij}$, $\tau_{ij}$ be the pullbacks of the metric and second fundamental form on $M_r$ (e.g. $\sigma_{ij} = \bar{g}_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$). Latin indices range from 1 to $n$, greek from 0 to $n$ and we will always sum over repeated indices. In particular to avoid confusion let us denote the initial pullbacks of the induced metric and second fundamental form on $M_0$ with $\rho_{ij}$, $\lambda_{ij}$ resp. Upper indices are always meant (if not otherwise mentioned) with respect to the metric $\sigma_{ij}$, e.g. $\tau^i = \sigma^{ij} \tau_{jk}$ where as usual upper indices for metrics denote the inverse.

For $1 \leq k$ we define

$$A_k := \tau_{i_1}^{i_2} \cdots \tau_{i_{k-1}}^{i_k} \tau_{i_k}^{i_1}$$

Then the calculations in [S] immediatly imply the following evolution equations

**Lemma 1.1:**

(a) $\frac{d}{dr} \sigma_{ij} = 2 \tau_{ij}$

(b) $\frac{d}{dr} d\mu = \tau d\mu$

(c) $\frac{d}{dr} \nu = 0$

(d) $\frac{d}{dr} \tau_{ij} = \tau_i^k \tau_{kj} - \bar{R}_{0i0j}$

(e) $\frac{d}{dr} \tau^i_j = -\tau^i_j \tau^j_l - \bar{R}^i_{0j}$

(f) $\frac{d}{dr} A_k = -k A_{k+1} - k \tau_{i_2}^{i_1} \cdots \tau_{i_k}^{i_1} \bar{R}_{0i_k}$

$\tau$ is the mean curvature, $d\mu$ the volume form on $M_r$ and $\bar{R}$ is the Riemann curvature tensor on $N$.

In the following let $r_0$ denote the maximal distance (time) for which a smooth solution of (1) exists. We will abbreviate derivatives with respect to $r$ with a prime. Now for any fixed $x$ let $\eta_{ij}(x, r)$ be the solution of the pointwise second order ODE

$$\eta_{ij}'' + \bar{R}_{0j0k} \eta^{kl} \rho_{li} = 0$$

2
\[ \eta_{ij}(0) = \rho_{ij} ; \quad \eta_{ij}'(0) = \lambda_{ij} . \]

Here \( \eta^{kl} \) denotes the inverse of \( \eta_{ij} \). Note that there exists a unique, invertible solution up to some distance since \( \eta_{ij}(0) \) is invertible. Then we have

**Theorem 1.2:**

(a) \( \sigma_{ij} = \rho^{kl} \eta_{ki} \eta_{lj} \); \( \sigma^{mn} = \eta^{ms} \eta^{nt} \rho_{st} \)

(b) \( \tau_{ij} = \rho^{kl} \eta_{ki}' \eta_{lj} = \rho^{kl} \eta_{ki} \eta_{lj}' \)

(c) \( d\mu_t = \text{det}(\rho^{ik} \eta_{kj}) d\mu_0 \).

Proof: \( \sigma_{ij} \) and \( a_{ij} := \rho^{kl} \eta_{ki} \eta_{lj} \) both satisfy the pointwise second order ODE

\[ y_{ij}' = \frac{1}{2} y_{ik}' y_{kl} - 2\mathbf{R}_{0ij0j} \]

\[ y_{ij}(0) = \rho_{ij} ; \quad y_{ij}'(0) = 2\lambda_{ij} . \]

From the theory of ODE it follows therefore that \( \sigma_{ij} = a_{ij} \). For (b) we first look at the tensor \( \rho^{kl}(\eta_{ki} \eta_{lj} - \eta_{ki} \eta_{lj}') \) and by differentiating this we see that it vanishes identically. Thus \( \rho^{kl} \eta_{ki} \eta_{lj} \) is symmetric in \( i \) and \( j \), then (b) is obtained by differentiation of \( \sigma_{ij} \). (c) is a direct consequence of (a), since locally

\[ d\mu_t = \sqrt{\text{det}(\sigma_{ij})} dx = \sqrt{\det(\rho^{ik} \eta_{kj})^2 \det(\rho_{ij})} dx = \det(\rho^{ik} \eta_{kj}) d\mu_0 , \]

where the last equality is true since \( \eta_{ij} \) is invertible and therefore positive definite (\( \eta_{ij}(0) \) is positive definite). Furthermore \( \det(\rho^{ik} \eta_{kj}) \) does not depend on the choice of a coordinate system.

For a constant \( \kappa \) define the function

\[ s_\kappa := \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}r) & \kappa > 0 ; \\
r & \kappa = 0 ; \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}r) & \kappa < 0 .
\end{cases} \]

Let \( \text{sym}_k \) be the \( k \)-th elementary symmetric function, i.e. for all \( a_1, \ldots, a_n \in \mathbb{R} \)

\[ \text{sym}_k(a_1, \ldots, a_n) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k}, \quad 0 \leq k \leq n \]

and set

\[ S_k := \text{sym}_k(\lambda_1, \ldots, \lambda_n) , \]

where \( \lambda_1, \ldots, \lambda_n \) are the principal curvatures on \( M_0 \).

3
Corollary 1.3 (Steiner formula):

If the sectional curvature $\kappa$ is constant and if we denote the area of $M_r$ by $|M_r|$, then

$$
\eta_{ij} = s_\kappa \rho_{ij} + s_\kappa \lambda_{ij}
$$

$$
|M_r| = \sum_{l=0}^{n} s_\kappa^l s_{n-l} \int_{M^n} S_{n-l}(0) d\mu_0.
$$

Proof: That $\eta_{ij}$ satisfies the above equation follows from the fact that it satisfies the appropriate evolution equation. The area formula is then a consequence of Theorem 1.2 (c).

Remark: For an excellent book on Weyl’s tube formula and Steiner type formulas see [G].

Proposition 1.4:

(a) Assume that $\overline{R}_{0i0j} \leq \mu(r)\sigma_{ij}$, $\lambda_{ij} \geq b_1 \rho_{ij}$ and that $f_\mu$ is any function with $f_\mu'' + \mu f_\mu \leq 0$, $f_\mu(0) = 1$ and $f_\mu'(0) \leq b_1$. Then

$$
\tau_{ij} \geq \frac{f_\mu'}{f_\mu} \sigma_{ij},
$$

as long as $f_\mu > 0$.

(b) Assume that $\overline{R}_{0i0j} \geq \nu(r)\sigma_{ij}$, $\lambda_{ij} \leq b_1 \rho_{ij}$ and that $f_\nu$ is any function with $f_\nu'' + \nu f_\nu \geq 0$, $f_\nu(0) = 1$ and $f_\nu'(0) \geq b_1$. Then

$$
\tau_{ij} \leq \frac{f_\nu'}{f_\nu} \sigma_{ij},
$$

as long as $f_\nu > 0$.

Proof: We prove only part (a) since the proofs for (a) and (b) are almost the same. Assume that for positive $\epsilon$, $f_\epsilon$ is the solution of the Jacobi equation

$$
f_\epsilon'' = -(\mu + \epsilon)f_\epsilon, \quad f_\epsilon(0) = 1, \quad f_\epsilon'(0) = b_1 - \epsilon,
$$

and let $c_\epsilon := \frac{f_\epsilon'}{f_\epsilon}$. Define the symmetric tensor $M_{ij} := \tau_{ij} - c_\epsilon \sigma_{ij}$. Assume that $r_0$ is the first distance where a zero eigenvector $v$ of $M_{ij}$ at one point $x \in M$ occurs and let us assume that at this point we have chosen normal coordinates with respect to
\[ \sigma_{ij} \] such that \( M_{ij} \) (and therefore also \( \tau_{ij} \)) becomes diagonal and we can also assume (after a rotation of the coordinate frame) that at this point \( v^i = \delta_1^i \). Now since \( r_0 \) is the first distance where \( M_{ij} v^i v^j \) vanishes we must have \( \frac{d}{dr} |_{r_0} M_{ij} v^i v^j \leq 0 \). But on the other hand we calculate from Lemma 1.1 that

\[
\frac{d}{dr} |_{r_0} M_{ij} v^i v^j = (\tau_{il} \sigma^{lm} \tau_{mj} - R_{0i0j} - 2c_{\epsilon} \tau_{ij} - c_{\epsilon}' \sigma_{ij}) v^i v^j
\]

\[
= \tau_{11}^2 - R_{0101} - 2c_{\epsilon} \tau_{11} - c_{\epsilon}'
= -(c_{\epsilon}' + c_{\epsilon}^2 + R_{0101}) > -(c_{\epsilon}' + c_{\epsilon}^2 + \mu + \epsilon) = 0 .
\]

This contradiction proves that \( \tau_{ij} > \frac{f_{\epsilon}}{f_{\epsilon}} \sigma_{ij} \) as long as \( f_{\epsilon} > 0 \). Since the solution of the above Jacobi equation depends continuously on \( \epsilon \) we have

\[
\tau_{ij} \geq \frac{f_{0}'}{f_{0}} \sigma_{ij} ,
\]

as long as \( f_0 > 0 \), where

\[
f_{0}' = - \mu f_{0} , \quad f_0(0) = 1 , \quad f_0'(0) = b_{l} .
\]

Now let \( \tau_{\mu} := \max\{r \in (0, R) | f_{\mu}(r) > 0\} \), \( \tau_0 := \max\{r \in (0, R) | f_{0}(r) > 0\} \). Define the function \( h := f_{\mu} f_{0}' - f_{\mu}' f_{0} \). Then we have \( h(0) \geq 0 \) and \( h' = f_{\mu}' f_{\mu} - f_{0} f_{\mu}'' \geq 0 \) for \( r \in \min\{\tau_0, \tau_{\mu}\} \). Thus \( \left( \frac{f_{0}}{f_{\mu}} \right)' = \frac{h}{f_{\mu}''} \geq 0 \) on \( [0, \min\{\tau_0, \tau_{\mu}\}] \). But since \( \frac{f_{0}}{f_{\mu}}(0) = 1 \), this implies that \( f_{0} \geq f_{\mu} \) on \( [0, \min\{\tau_0, \tau_{\mu}\}] \) and thus \( \tau_0 \geq \tau_{\mu} \). So \( h(r) \geq 0 \) for \( r \in [0, \tau_{\mu}] \) and consequently also \( \frac{f_{0}}{f_{\mu}} \geq \frac{f_{0}'}{f_{\mu}'} \) for \( r \in [0, \tau_{\mu}] \). This proves the proposition.

**Corollary 1.5:**

\[
f_{\nu}' \rho_{ij} \geq \sigma_{ij}
\]

as long as \( f_{\nu} > 0 \) and

\[
\sigma_{ij} \geq f_{\mu}' \rho_{ij}
\]

as long as \( f_{\mu} > 0 \).

Proof: We have \( \frac{d}{dr} \left( \frac{1}{f_{\mu}} \sigma_{ij} - \rho_{ij} \right) = \frac{2}{f_{\mu}'} \left( \tau_{ij} - \frac{f_{\mu}'}{f_{\mu}} \sigma_{ij} \right) \geq 0 \) and \( \frac{d}{dr} \left( \frac{1}{f_{\mu}} \sigma_{ij} - \rho_{ij} \right) \leq 0 \).

**Corollary 1.6 (Rauch comparison theorem):**

Let \( N_1, N_2 \) be two Riemannian manifolds with sectional curvatures \( \kappa_1, \kappa_2 \) and let \( c_1, c_2 \) be two geodesics parametrized by arclength and assume that along
these geodesics the inequality $\kappa_1 \geq \kappa_2$ holds. Assume $J_1$, $J_2$ are two Jacobi fields along these geodesics such that $\langle J_i(0), c_i'(0) \rangle = \langle J_i'(0), c_i(0) \rangle = 0$ and $|J_1(0)| = |J_2(0)|$, $|J_1'(0)| = |J_2'(0)|$. Then as long as there are no conjugate points on $c_1$ we have the estimate

$$|J_1(r)| \leq |J_2(r)|.$$ 

Proof: First assume that $w_i := J_i(0) \neq 0$. Let $M_i$ be a small hypersurface, normal to the geodesic $c_i$ such that $\nabla_{w_i}^\top u_i = v_i := J_i'(0)$ (it is clear that such surfaces exist). Furthermore let $\gamma_i(s)$ be curves on $M_i$ such that $\gamma_i(0) = c_i(0)$, $\gamma_i'(0) = w_i$ and define a family of geodesics $c_i(s, r) := \exp_{\gamma_i(s)} r v_i(s)$, where $v_i(s)$ is that unit normal vector field along $\gamma_i$ with $v_i(0) = c_i'(0)$. Then we have $J_i(r) = \frac{\partial}{\partial s}|_{s=0} c_i(s, r)$. Now choose $\kappa$ such that $\kappa_1 \geq \kappa \geq \kappa_2$ and solve the Fermi flow for both $M_i$. Using Corollary 1.5 we see that $|J_1(r)|^2 \leq f_\kappa^2|J_1(0)|^2 = f_\kappa^2|J_2(0)|^2 \leq |J_2(r)|^2$ as long as both solutions exist. That shows that no conjugate point on $c_2$ can occur before one occurs on $c_1$. If we now choose the initial hypersurfaces smaller and smaller, we see that the maximal time for which both solution exist tends to the maximal time for which there are no conjugate points on $c_1$.

In the case where $w_i = 0$ we set $\tilde{w}_i := \epsilon z_i$ for some arbitrary fixed unit vectors $z_i$ and let $\epsilon$ tend to zero. This gives the result since the solution of the Jacobi equation depends continuously on initial data.

The evolution equations for the mean curvature and the volume form immediately imply an estimate for the volume of a geodesic ball, if the Ricci curvature is bounded below. This well known result is due to Heintze and Karcher (see for example [GHL]). Unfortunately, almost no results are known when the Ricci curvature is bounded above. We are able to prove such a result under an additional assumption on the curvature tensor.

**Definition 1.7:**

We say that the sectional curvature $\sigma$ is $\epsilon$–Ricci pinched, if for any vector $X$ and any two-plane $X \wedge Y$ the estimate $|\sigma(X \wedge Y) - \frac{1}{n} Ric(X, X)| \leq \epsilon$ holds, where the dimension of $N$ is $n + 1$.

We are now going to show, that the $\epsilon$–Ricci pinching is strongly related to the umbilicness of a convex surface.

**Proposition 1.8:**
Assume $\sigma$ is $\epsilon$–Ricci pinched and $M_0$ is the immersion of a convex hypersurface such that $|\tau_{ij} - \frac{\tau}{n}\sigma_{ij}| \leq c\sigma_{ij}$. Then as long as $M_r$ stays convex, we have the estimate

$$|\tau_{ij} - \frac{\tau}{n}\sigma_{ij}| \leq (c + \epsilon r)\sigma_{ij}.$$ 

Proof: Define the tensor

$$M_{ij} := \tau_{ij} - \frac{\tau}{n}\sigma_{ij} + (c + \tilde{\epsilon}r)\sigma_{ij}$$

and

$$N_{ij} := \tau_{ij} - \frac{\tau}{n}\sigma_{ij} - (c + \tilde{\epsilon}r)\sigma_{ij},$$

where $\tilde{\epsilon} > \epsilon$. As in the proof of Proposition 1.4 we look at the first time where a zero eigenvector of $M_{ij}$ or $N_{ij}$ occurs. With the conventions as above assume that $v$ is a zero eigenvector of $N_{ij}$ (the case $M_{ij}$ will be analogous). Then we have

$$\frac{d}{dr}|_{r_0} N_{ij} v^i v^j = (\tau_{il}\sigma_{lm}\tau_{mj} - \overline{R}_{0i0j} - 2(\frac{\tau}{n} + c + \tilde{\epsilon}r)\tau_{ij} + \frac{1}{n}(|\tau_{ij}|^2 + \overline{\text{Ric}}(\nu, \nu))\sigma_{ij} - \tilde{\epsilon}\sigma_{ij}) v^i v^j$$

$$= -\tau_{11}^2 + \frac{1}{n}|\tau_{ij}|^2 - \overline{R}_{0101} + \frac{1}{n}\overline{\text{Ric}}(\nu, \nu) - \tilde{\epsilon}$$

If $M_r$ is still convex, we have $|\tau_{ij}|^2 \leq n\tau_{11}^2$ and therefore using the $\epsilon$–Ricci umbilicness

$$\frac{d}{dr}|_{r_0} N_{ij} v^i v^j \leq \epsilon - \tilde{\epsilon} < 0$$

which is a contradiction. For $\tilde{\epsilon} \to \epsilon$ we get the desired result.

**Proposition 1.9:**

Let $N$ be $\epsilon$–Ricci pinched with $\overline{\text{Ric}} \leq d$. Let $m := \frac{d}{n} - \epsilon$ and define $R_0 := \frac{\pi}{2\sqrt{d}}$, if $\frac{d}{n} + \epsilon > 0$ and otherwise set $R_0 := \infty$. Then any geodesic sphere of radius smaller than $R_0$ is convex, and if we assume that $S_{r_1}$, $S_{r_2}$ are two geodesic spheres with $0 < r_1 < r_2 < R_0$ such that on $S_{r_1}$ $\tau > \tau_-$ and $|\tau_{ij} - \frac{\tau}{n}\sigma_{ij}| \leq c\sigma_{ij}$, then we can estimate the enclosed volume between these spheres, as long as the function $u(r) := (\frac{\tau}{n} + c)s_m(r) + s'_m(r)$ is positive, by

$$V(r_1, r_2) \geq |S_{r_1}| \int_{r_0}^{r_2-r_1} u^n(r)e^{-ncr-\frac{m}{d}r^2} dr.$$  

Proof: The assumptions on the curvature imply that the sectional curvature is bounded between $\frac{d}{n} - \epsilon < \kappa < \frac{d}{n} + \epsilon$. Using Proposition 1.4 with $\mu = \frac{d}{n} + \epsilon$ one easily
checks that any geodesic sphere of radius smaller than $R_0$ is convex since all principal curvatures tend to infinity as the radius tends to zero, and therefore $b_l$ can be chosen to be arbitrarily large. Now we use Proposition 1.8 and obtain

$$|\tau_{ij}|^2 \leq n(\frac{\tau}{n} + c + \epsilon r)^2.$$ 

Define $y := \frac{\tau}{n} + c + \epsilon r$. Then we calculate

$$y' = -\frac{1}{n}(|\tau_{ij}|^2 + \nabla \tau) + \epsilon \geq -y^2 - m.$$ 

Let $f$ be the solution of $f' = -f^2 - m$, $f(0) = \frac{\tau}{n} + c$, i.e. $f = (\ln u)'$ with $u$ as above. Then $(y - f)' \geq -(y + f)(y - f)$ and consequently $y \geq f$ as long as $r_1 + r < R_0$ and $f$ is defined, i.e. $u$ is positive. This implies

$$|S_{r_1+r}|' = \int \tau d\mu \geq n|S_{r_1+r}|(\ln u - cr - \frac{\epsilon}{2}r^2)' ,$$

thus

$$|S_{r_1+r}| \geq |S_{r_1}|u^n(r)e^{-n\epsilon r - \frac{n}{2}r^2}.$$ 

Integration from 0 to $r_2 - r_1$ gives the result.

2. Short- and longtime existence results for the Fermi flow

So far we have not proven that a solution exists for some short time interval. In this section we prove the existence of a unique solution and also derive estimates for the maximal time interval on which it is admitted. In particular we show that in some cases we obtain eternal solutions.

**Proposition 2.1 :**

Let $F_0 : M^n \to N^{n+1}$ be a smooth immersion of an orientable and compact manifold and let $\nu$ be one of the unit normal vector fields on $M$. Then there exists an $\epsilon > 0$, a unique family of smooth submanifolds $M_r$ and a smooth family of local diffeomorphisms $F_r : M^n \to N^{n+1}$ such that

$$\frac{dF_r}{dr} = \nu , \quad M_r = F_r(M^n) \quad \forall r \in [0, \epsilon)$$

**Proof:** Let $c_x(r) := \exp_{F_0(x)} r \nu(F_0(x))$ and define $F_r(x) := c_x(r)$. This is well defined since $M^n$ is compact and therefore the geodesic equation can be solved uniformly
for all \( x \in M^n \) on some short time interval \([0, \epsilon)\). Moreover, \( F_r \) is smooth since the solution of the geodesic equation depends smoothly on the initial conditions. If we assume \( \epsilon \) to be so small that the exponential map at each point on the initial hypersurface is injective, then \( F_r \) becomes a local diffeomorphism. From the Gauss lemma it follows that \( \frac{dF_r}{dr} = \frac{d\nu}{dr} \) is a unit normal to the hypersurfaces defined by \( F_r \). This proves existence. Uniqueness follows from Lemma 1.1(c) and the fact that the solution of the geodesic equation is unique.

Remark: We have just proved that it is possible to reduce the nonlinear system of first order PDE’s that is defined through (1) to an n-parameter family of nonlinear systems of second order ODE’s modeled over \( M^n \).

Definition 2.2:

Let \( M \) be an embedded, orientable hypersurface with unit normal vector field \( \nu \) and let \( \Gamma \) be the set of all geodesics \( \gamma \) parametrized by arclength with \( \gamma(0) \in M, \gamma'(0) = \nu \). The self distance \( d_{\nu} \) of \( M \) with respect to \( \nu \) is defined as

\[
d_{\nu} := \sup \{ \rho | \Gamma_{|(0,\rho)} \cap M = \emptyset \} .
\]

Lemma 2.3:

Assume that \( M_0 \) is the embedding of a closed, orientable hypersurface in a Riemannian manifold \( N \) such that \( N \) is connected and \( N-M_0 \) is disconnected and consists of two components \( A \) and \( B \) and let \( d_{\nu} \) be the self distance of \( M_0 \) with respect to the normal vector field that points in direction of \( A \). Then \( \frac{d_{\nu}}{2} \) is the first time where \( M_t \) fails to be an embedding (provided the flow exists this far) and for all \( t \leq \frac{d_{\nu}}{2} \) and all \( p \in M_t \) we have \( d(p, M_0) = t \).

Proof: Let us first remark that since \( M_0 \) is compact and smooth, it is clear that there exists an \( \epsilon > 0 \) such that \( d(p, M_0) = t \) for all \( t \leq \epsilon \) and all \( p \in M_t \).

Now let \( T \) be the first time where this property fails to be true, i.e. the first time such that there exists at least one point \( p \in M_T \) and two distance-minimizing geodesics \( \gamma_1, \gamma_2 \) between \( M_T \) and \( M_0 \), parametrized by arclength with \( \gamma_i(0) = p \) and \( \gamma_i(T) \in M_0 \). These geodesics are both normal to both surfaces and moreover \( \gamma_i'(T) = -\nu(\gamma_i(T)) \). This follows from the above remark and the fact the \( N-M_0 \) is disconnected; therefore,

\[
d(p, M_0) = \inf \{ l(\gamma) | \gamma \subset N \text{ connects } p \text{ with } M_0 \}
\]
Thus $\gamma_1 \cup \gamma_2$ is a smooth geodesic arc connecting $M_0$ with itself. Therefore $d_\nu \leq 2T$. Since both geodesics $\gamma_i$ satisfy $\dot{\gamma}_i(T) = -\nu(\gamma_i(T))$, $\gamma_i(0) = p$, this also shows that $M_T$ touches itself and cannot be more than immersed. On the other hand if $T < T$ were the first time where $M_T$ becomes immersed, then $M_T$ would touch itself at some point $p \in M_T$, and from the definition of $T$ we would obtain a contradiction.

**Proposition 2.4:**

Let $N^{n+1}$ be geodesically complete and connected and let $F_0 : M^n \to N^{n+1}$ be a smooth embedding of an orientable closed manifold such that $N^{n+1} - M^n$ is disconnected and consists of two components $A$ and $B$. Assume that $\nu$ is a unit normal vector field on $M_0$ that points in direction of $A$, that $d_\nu$ is the self distance of $M_0$, and that at any point $y \in A$ the sectional curvature $\sigma(y)$ is bounded above or below by functions $\mu(d(y)) = \frac{c_\mu}{(1 + a_\nu d(y))^2}$, or $\nu(d(y)) = \frac{c_\nu}{(1 + a_\mu d(y))^2}$ respectively, where $a_\mu, a_\nu > 0$, $c_\mu, c_\nu$ are fixed constants and $d(y)$ is the distance to $M_0$. Further, let $b_1, b_u$ be bounds for the second fundamental form on $M_0$, i.e. $b_\mu \rho_{ij} \geq \lambda_i j \geq b_i \rho_{ij}$, and let $m_i := \frac{4c_i}{a_i^2} - 1$ for $i = \mu, \nu$. Then we have, with $[a, b] := \min\{a, b\}$,

(a) $m_\mu \leq 0$ and $\frac{2}{a_\mu} b_1 - 1 \geq -\sqrt{-m_\mu} \Rightarrow r_0 = \frac{d_\nu}{2}, \infty$

(b) $m_\mu = 0$ and $\frac{2}{a_\mu} b_1 - 1 < -\sqrt{-m_\mu} \Rightarrow r_0 \geq \frac{d_\nu}{2}, \frac{1}{a_\mu}(e^{\frac{2a_\mu}{a_\mu^2 - 2b_1}} - 1)$

(c) $m_\mu < 0$ and $\frac{2}{a_\mu} b_1 - 1 < -\sqrt{-m_\mu} \Rightarrow r_0 \geq \frac{d_\nu}{2}, \frac{1}{a_\mu}(\sqrt{a_\mu - a_\mu b_1 + a_\mu \sqrt{-m_\mu}} - 1)$

(d) $m_\mu > 0$ and $\frac{2}{a_\mu} b_1 - 1 = 0 \Rightarrow r_0 \geq \frac{d_\nu}{2}, \frac{1}{a_\mu}(\sqrt{-m_\mu} - 1)$

(e) $m_\mu > 0$ and $\frac{2}{a_\mu} b_1 - 1 < 0 \Rightarrow r_0 \geq \frac{d_\nu}{2}, \frac{1}{a_\mu}(\sqrt{-m_\mu} \arctan \frac{a_\mu \sqrt{-m_\mu}}{a_\mu^2 - 2b_1} - 1)$

(f) $m_\mu > 0$ and $\frac{2}{a_\mu} b_1 - 1 > 0 \Rightarrow r_0 \geq \frac{d_\nu}{2}, \frac{1}{a_\mu}(\sqrt{-m_\mu} (\arctan \frac{a_\mu \sqrt{-m_\mu}}{a_\mu^2 - 2b_1}) - 1)$

(g) $m_\nu = 0$ and $\frac{2}{a_\nu} b_u - 1 < -\sqrt{-m_\nu} \Rightarrow r_0 \geq \frac{1}{a_\nu}(\sqrt{a_\nu - 2b_u + a_\nu b_u - a_\nu \sqrt{-m_\nu}} - 1)$

(h) $m_\nu < 0$ and $\frac{2}{a_\nu} b_u - 1 < -\sqrt{-m_\nu} \Rightarrow r_0 \geq \frac{1}{a_\nu}(\sqrt{-m_\nu} \arctan \frac{a_\nu \sqrt{-m_\nu}}{a_\nu^2 - 2b_u} - 1)$

(i) $m_\nu > 0$ and $\frac{2}{a_\nu} b_u - 1 = 0 \Rightarrow r_0 \leq \frac{1}{a_\nu}(\sqrt{-m_\nu} - 1)$

(j) $m_\nu > 0$ and $\frac{2}{a_\nu} b_u - 1 < 0 \Rightarrow r_0 \leq \frac{1}{a_\nu}(\sqrt{-m_\nu} \arctan \frac{a_\nu \sqrt{-m_\nu}}{a_\nu^2 - 2b_u} - 1)$

(k) $m_\nu > 0$ and $\frac{2}{a_\nu} b_u - 1 > 0 \Rightarrow r_0 \leq \frac{1}{a_\nu}(\sqrt{-m_\nu} (\arctan \frac{a_\nu \sqrt{-m_\nu}}{a_\nu^2 - 2b_u}) - 1)$.

**Proof:** First we obtain from lemma 2.3 that $d(F_r(x)) = r$, $\forall x \in M^n$, as long as $r \leq \frac{d_\nu}{2}$. Now we can use Proposition 1.4 with $\mu(r) = \frac{c_\mu}{(1 + a_\mu r)^2}$, or $\nu(r) = \frac{c_\nu}{(1 + a_\nu r)^2}$ resp. Since
is geodesically complete, the maps \( F_r(x) := \exp_{F_0(x)}^{r\nu(F_r(x))} \) are welldefined, and as in the proof of proposition 2.1 it follows that this is a smooth family of maps and a family of diffeomorphisms provided \( r \) is small enough. So the only thing that can go wrong is that \( \sigma_{ij} \) could degenerate, i.e. the eigenvalues of \( \sigma_{ij} \) could tend to infinity or zero. Using Proposition 1.4 and the fact that the solution of the Jacobi equation \( f'' + \frac{c}{(1+ar)} f = 0, f(0) = 1, f'(0) = b \) is explicitly given by

\[
f(r) = \sqrt{1+ar} \left( \frac{2}{a} b - 1 \right) s_m \ln \sqrt{1+ar} + s_m \ln \sqrt{1+ar},
\]

with \( m := \frac{4c}{a^2} - 1 \), we see that no eigenvalue of the metric can tend to infinity in finite time. So an estimate for the distance, where \( f_\mu, f_\nu \) become zero for the first time, gives lower and upper bounds for \( r_0 \). One easily checks that these estimates are given by those stated in the proposition.

**Proposition 2.5:**

Under the assumption of Proposition 2.4 assume that a lower bound for the sectional curvature is now given by \( \nu(r) = \frac{c}{(1+ar)^{2+\epsilon}} \), where \( a, c, \epsilon > 0 \). Then there exists no eternal solution, i.e. \( r_0 < \infty \).

Proof: Assume that \( r_0 = \infty \) and choose \( R_0 \) so that \( 4c(1+aR_0)^\epsilon > a^2 \). Then we define \( \tilde{c} := c(1+aR_0)^\epsilon \) and conclude that for all \( r \geq R_0 \)

\[
\nu(r) = \frac{c}{(1+ar)^2} (1+ar)^\epsilon \geq \frac{\tilde{c}}{(1+ar)^2}.
\]

Using Proposition 2.4 we see that since \( \tilde{m}_\nu > 0 \), a further extension of the flow can only be made for some finite time, contradicting to \( r_0 = \infty \).

**Remark:** Proposition 2.5 means that a manifold cannot admit an asymptotically flat end if the curvature does not decay fast enough. One could also try to give estimates for \( r_0 \) in the case, where the functions \( \mu, \nu \) are given by \( \frac{c}{(1+ar)^\alpha} \) with some constant \( \alpha \). However, it turns out that the corresponding Jacobi equation admits a much more complicated solution (see [K], 2·14). By Proposition 1.4 it would be enough to give estimates for sub- or supersolutions of the Jacobi equation.

Let us finally prove a statement concerning the gradient growth of the second fundamental form.

**Proposition 2.6:**
Assume that $\frac{f_\mu'}{f_\mu} > 0$ on $[0,R)$ and that there exist positive constants $c_1$, $c_2$ such that $|\nabla_k R_{0i0j}|^2 \leq c_1 \left( \frac{f_\mu'}{f_\mu} \right)^2$ and $|\langle \tau_{kp}, \mathcal{R}_{0jpq} \rangle|^2 \leq c_2 \left( \frac{f_\mu'}{f_\mu} \right)^2$. Then for any $\epsilon > 0$ there exist constants $a_1$, $a_2$ such that for all $r \in [0,R)$

$$|\nabla_k \tau_{ij}|^2 \leq a_1 f_\mu^{5\epsilon - 6} + a_2,$$

with $a_1$ and $a_2$ given by

$$a_2 = \frac{c_1 + 4c_2}{\epsilon(6 - 5\epsilon)}, \quad a_1 = \min_{r=0} \left| \nabla_k \tau_{ij} \right|^2 - a_2.$$

Proof: First we calculate

$$\frac{d}{dr} \nabla_k \tau_{ij} = \nabla_k (\tau_{iu} \sigma^{uv} \tau_{vj} - \mathcal{R}_{0i0j}) - \tau_{pj} \sigma^{pq} (\nabla_k \tau_{qi} + \nabla_i \tau_{qk} - \nabla_q \tau_{ki})$$

$$- \tau_{pi} \sigma^{pq} (\nabla_k \tau_{qj} + \nabla_j \tau_{qk} - \nabla_q \tau_{kj})$$

$$= -\nabla_k \mathcal{R}_{0i0j} - \tau_{kp} \sigma^{pq} \mathcal{R}_{0q0j} - \tau_{kp} \sigma^{pq} \mathcal{R}_{0i0q}$$

where we have used the Codazzi equations $\nabla_i \tau_{kj} - \nabla_j \tau_{ki} = \mathcal{R}_{0kqi}$ (Note that $\frac{d}{dr} \nabla_k \tau_{ij} = 0$ in the case where $N$ is a space form), and then

$$\frac{d}{dr} \left| \nabla_k \tau_{ij} \right|^2 = -2\sigma^{ks} \sigma^{lt} \sigma^{ijm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm} - 4\sigma^{kl} \sigma^{is} \sigma^{nt} \sigma^{jm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm}$$

$$+ 2\sigma^{kl} \sigma^{is} \sigma^{jm} \nabla_l \tau_{mn} \frac{d}{dr} \nabla_k \tau_{ij}$$

$$= -2\sigma^{ks} \sigma^{lt} \sigma^{ijm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm} - 4\sigma^{kl} \sigma^{is} \sigma^{nt} \sigma^{jm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm}$$

$$- 2(\nabla_k \mathcal{R}_{0i0j} + 2\tau_{kp} \sigma^{pq} \mathcal{R}_{0q0j} + 2\tau_{kp} \sigma^{pq} \mathcal{R}_{0kqi}) \sigma^{kl} \sigma^{is} \sigma^{jm} \nabla_l \tau_{mn}.$$

Using Schwartz’s inequality we obtain, for any positive $\eta$ ,

$$\frac{d}{dr} \left| \nabla_k \tau_{ij} \right|^2 \leq -2\sigma^{ks} \sigma^{lt} \sigma^{ijm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm} - 4\sigma^{kl} \sigma^{is} \sigma^{nt} \sigma^{jm} \tau_{st} \nabla_k \tau_{ij} \nabla_l \tau_{nm}$$

$$+ \eta^{-1} (|\nabla_k \mathcal{R}_{0i0j}|^2 + 4|\langle \tau_{kp}, \mathcal{R}_{0jpq} \rangle|^2) + 5\eta \left| \nabla_k \tau_{ij} \right|^2.$$

Now we use Proposition 1.4 (a) and obtain

$$\frac{d}{dr} \left| \nabla_k \tau_{ij} \right|^2 \leq -\frac{f_\mu'}{f_\mu} \left| \nabla_k \tau_{ij} \right|^2 + \eta^{-1} (|\nabla_k \mathcal{R}_{0i0j}|^2 + 4|\langle \tau_{kp}, \mathcal{R}_{0jpq} \rangle|^2) + 5\eta \left| \nabla_k \tau_{ij} \right|^2,$$

12
and if we choose \( \eta = \epsilon \frac{f'}{f_\mu} \), then we get

\[
\frac{d}{dr} |\nabla_k \tau_{ij}|^2 \leq (5\epsilon - 6) \frac{f'}{f_\mu} |\nabla_k \tau_{ij}|^2 + \frac{c_1 + 4c_2 f'}{\epsilon f_\mu}.
\]

Thus we have shown

\[
\frac{d}{dr} \left( \ln \left( |\nabla_k \tau_{ij}|^2 - a_2 \right) f_\mu^{6-5\epsilon} \right) \leq 0,
\]

and consequently the result.

The same calculations prove the following

**Proposition 2.7:**

Assume that \( \frac{f'}{f_\mu} < 0 \) on \([0,R)\) and that there exist positive constants \( c_1, c_2 \) such that \( |\nabla_k R_{0i0j}|^2 \leq c_1 (\frac{f'}{f_\mu})^2 \) and \( |\langle \tau_{kp}, R_{0jpi} \rangle|^2 \leq c_2 (\frac{f'}{f_\mu})^2 \). Then for any \( \epsilon < 0 \) there exist constants \( a_1, a_2 \) such that for all \( r \in [0,R) \)

\[
|\nabla_k \tau_{ij}|^2 \leq a_1 f_\mu^{5\epsilon - 6} + a_2,
\]

with \( a_1 \) and \( a_2 \) given by

\[
a_2 = \frac{c_1 + 4c_2}{\epsilon (6 - 5\epsilon)}, \quad a_1 = \min_{r=0} |\nabla_k \tau_{ij}|^2 - a_2.
\]

References

[G] Gray, A.; *Tubes*, Addison Wesley, 1990

[GHL] Gallot, S.; Hulin, D.; Lafontaine, J; *Riemannian Geometry*, Springer Verlag, Berlin, New York, 1987

[K] Kamke; *Differentialgleichungen, Lösungsmethoden und Lösungen*, Chelsea Publishing Company, New York

[S] Smoczyk, K.; *Symmetric hypersurfaces in Riemannian manifolds contracting to Lie groups by their mean curvature*, Calc. Var. 1996, 4 (02), P. 155